STEM SMART Double Maths 32 - Complex Exponentials & Hyperbolic Functions

Complex Numbers: Polar Form 3ii

Subject & topics: Maths **Stage & difficulty:** Further A P2

Part A

Expression for $z_1 z_2$

Given that $z_1 = 2\mathrm{e}^{\frac{1}{6}\pi\mathrm{i}}$ and $z_2 = 3\mathrm{e}^{\frac{1}{4}\pi\mathrm{i}}$, express $z_1 z_2$ in the form $r\mathrm{e}^{\mathrm{i}\theta}$.

$r > 0$ and $0 \leqslant \theta < 2\pi$.

The following symbols may be useful: `e`, `i`, `pi`

Part B

Expression for $\frac{z_1}{z_2}$

Express $\frac{z_1}{z_2}$ in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where $r > 0$ and $0 \leqslant \theta < 2\pi$.

The following symbols may be useful: `e`, `i`, `pi`

Part C

Expression for w^{-5}

Given that $w = 2\left(\cos\frac{1}{8}\pi + \mathrm{i}\sin\frac{1}{8}\pi\right)$, express w^{-5} in the form $r(\cos\theta + \mathrm{i}\sin\theta)$, where $r > 0$ and $0 \leqslant \theta < 2\pi$.

The following symbols may be useful: `cos()`, `i`, `pi`, `sin()`

Adapted with permission from UCLES, A Level, June 2006, Paper 4727, Question 2.

STEM SMART Double Maths 32 - Complex Exponentials & Hyperbolic Functions

Complex Numbers: Polar Form 1i

Subject & topics: Maths **Stage & difficulty:** Further A P2

Part A

$z^6 = 1$

Solve the equation $z^6 = 1$, giving your answers in the form $r\mathrm{e}^{\mathrm{i}\theta}$ where $0 \leq \theta < 2\pi$ and $r > 0$.

Write your answer in terms of k where $k = 0, 1, 2, 3, 4, 5$.

The following symbols may be useful: `e, i, k, pi, z`

Part B

Argand diagram

Sketch an Argand diagram showing the solutions to $z^6 = 1$.

When you have made your sketch, answer this question to see an example sketch: Which of the four sketches in **Figure 1** is most accurate?

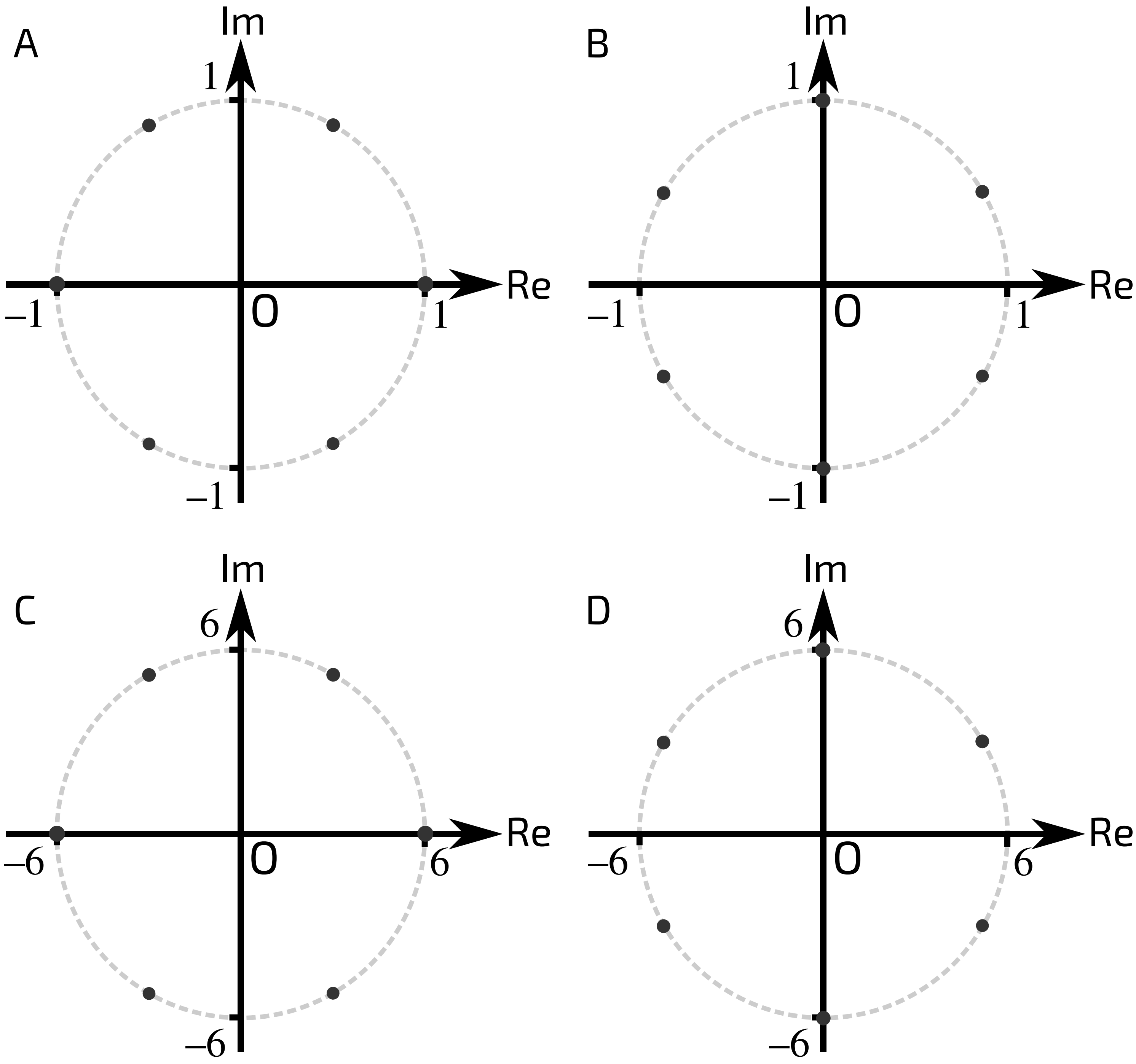

Figure 1: Four Argand diagram sketches.

- Sketch A
- Sketch B
- Sketch C
- Sketch D

Part C
$(1+\mathrm{i})^6$

Evaluate $(1+\mathrm{i})^6$.

Give your answer in the form $x+\mathrm{i}y$.

The following symbols may be useful: `i`

Part D
$z^6+8\mathrm{i}=0$

Hence, or otherwise, solve the equation $z^6+8\mathrm{i}=0$, giving your answers in the form $r\mathrm{e}^{\pi\mathrm{i}(a+bk)}$ where a and b are rational, $r>0$, $0\leq \arg z<2\pi$ and $k=0,1,2,3,4,5$.

The following symbols may be useful: `e, i, k, pi, z`

Adapted with permission from UCLES, A Level, June 2014, Paper 4727, Question 3.

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STEM SMART Double Maths 32 - Complex Exponentials & Hyperbolic Functions

STEM SMART Double Maths 32 - Complex Exponentials & Hyperbolic Functions

Complex Numbers: De Moivre 3ii

Subject & topics: Maths **Stage & difficulty:** Further A P2

Part A

$\cos 5\theta$

Use de Moivre's theorem to show that $\cos 5\theta \equiv \mathrm{f}(\cos\theta)$.

What is $\mathrm{f}(\cos\theta)$?

The following symbols may be useful: `cos()`, `theta`

Part B

Quartic roots

Hence find the roots of $16x^4 - 20x^2 + 5 = 0$ in the form $\cos k\pi$ where $0 \leq k \leq 1$.

Give the values of k for each root as a decimal.

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Part C

$\cos \frac{1}{10}\pi$

Hence find the exact value of $\cos \frac{1}{10}\pi$.

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Complex Numbers: De Moivre 1i

Subject & topics: Maths **Stage & difficulty:** Further A P2

The series C and S are defined for $0 < \theta < \pi$ by

$$\begin{aligned} C &= 1 + \cos\theta + \cos 2\theta + \cos 3\theta + \cos 4\theta + \cos 5\theta, \\ S &= \sin\theta + \sin 2\theta + \sin 3\theta + \sin 4\theta + \sin 5\theta. \end{aligned}$$

Part A
$C + \mathrm{i}S$

Write $C + \mathrm{i}S$ in terms of exponentials.

The following symbols may be useful: `e`, `i`, `theta`

Part B
Expression for C

Deduce that C can be written as a product of trigonometric functions of the form $\sin a\theta \cos b\theta \operatorname{cosec} c\theta$ where a, b and c are rational numbers. Write down that expression for C.

The following symbols may be useful: `cos()`, `cosec()`, `sin()`, `theta`

Part C

Expression for S

Write down a corresponding expression for S as a product of trigonometric functions.

The following symbols may be useful: `cosec()`, `sin()`, `theta`

Part D

Solving $C = S$

Hence find the values of θ, in the range $0 < \theta < \pi$, for which $C = S$. Give your answers to 3 sf.

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Adapted with permission from UCLES, A Level, June 2006, Paper 4727, Question 7.

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Complex Numbers: De Moivre 5i

Subject & topics: Maths **Stage & difficulty:** Further A P2

Part A

$\sin^6\theta$

By expressing $\sin\theta$ in terms of $e^{i\theta}$ and $e^{-i\theta}$, show that

$$\sin^6\theta \equiv f(\cos 6\theta, \cos 4\theta, \cos 2\theta).$$

What is $f(\cos 6\theta, \cos 4\theta, \cos 2\theta)$?

The following symbols may be useful: `cos(), theta`

Part B

$\cos^6\theta$

Replace θ by $\left(\frac{1}{2}\pi - \theta\right)$ in the identity in part A to obtain a similar identity for $\cos^6\theta$ of the form

$$\cos^6\theta = g(\cos 6\theta, \cos 4\theta, \cos 2\theta).$$

What is $g(\cos 6\theta, \cos 4\theta, \cos 2\theta)$?

The following symbols may be useful: `cos(), theta`

Part C

Value of an integral

Hence find the exact value of

$$\int_0^{\frac{1}{4}\pi} \left(\sin^6\theta - \cos^6\theta\right) \, \mathrm{d}\theta.$$

Adapted with permission from UCLES, A Level, January 2009, Paper 4727, Question 8.

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STEM SMART Double Maths 32 - Complex Exponentials & Hyperbolic Functions

STEM SMART Double Maths 32 - Complex Exponentials & Hyperbolic Functions

Hyperbolic Functions: Manipulations 1ii

Subject & topics: Maths **Stage & difficulty:** Further A P2

Part A
$\cosh x \cosh y - \sinh x \sinh y$

Using the definitions of hyperbolic functions in terms of exponentials, prove that

$$\cosh x \cosh y - \sinh x \sinh y = \cosh(f(x,y)).$$

What is $f(x,y)$?

The following symbols may be useful: x, y

Part B
Solving for y

Given that $\cosh x \cosh y = 9$ and $\sinh x \sinh y = 8$, write an expression for y in terms of x.

The following symbols may be useful: x, y

Part C

Possible values of x and y

Hence find the values of x and y which satisfy the equations given in part B, giving your answers to 3 sf.

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Adapted with permission from UCLES, A Level, June 2007 , Paper 4726, Question 7.

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STEM SMART Double Maths 32 - Complex Exponentials & Hyperbolic Functions

Hyperbolic Functions: Manipulations 3i

Subject & topics: Maths **Stage & difficulty:** Further A P2

Part A
Defining $\tanh y$

Write an expression for $\tanh y$ in terms of e^{y} and e^{-y}.

The following symbols may be useful: `e`, `y`

Part B
Log form of $\operatorname{artanh} x$

Given that $y = \operatorname{artanh} x$, where $-1 < x < 1$, write an expression for y as a logarithm in terms of x.

The following symbols may be useful: `ln()`, `log()`, `x`

Part C
Solve $3\cosh x = 4\sinh x$

Find the exact solution of the equation $3\cosh x = 4\sinh x$, giving the answer in terms of a logarithm.

The following symbols may be useful: `ln()`, `log()`

Part D

Solve $\operatorname{artanh} x + \ln(1-x) = \ln\left(\frac{4}{5}\right)$

Solve the equation

$$\operatorname{artanh} x + \ln(1-x) = \ln\left(\frac{4}{5}\right).$$

You may wish to use the $\pm$ symbol.

Adapted with permission from UCLES, A Level, January 2007 , Paper 4726, Question 8.

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STEM SMART Double Maths 32 - Complex Exponentials & Hyperbolic Functions

STEM SMART Double Maths 32 - Complex Exponentials & Hyperbolic Functions

Hyperbolic Functions: Differentiation 2ii

Subject & topics: Maths **Stage & difficulty:** Further A P2

The equation of a curve is $y = \cosh x - 2\sinh 2x$.

Part A

$\frac{\mathrm{d}y}{\mathrm{d}x}$

Find an expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of hyperbolic functions.

The following symbols may be useful: `cosech()`, `cosh()`, `coth()`, `sech()`, `sinh()`, `tanh()`

Part B

Turning points

Hence, explain why the curve has no turning points.

Drag six of the items to the right-hand column, and order them correctly to make an example proof.

Available items

- We begin by stating that at a turning point, $\frac{\mathrm{d}y}{\mathrm{d}x} > 0$.
- We begin by stating that at a turning point, $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$.
- The discriminant is < 0, therefore the equation has no real roots.
- $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ for all x, so we have no turning points. QED
- For the function in this question, the equation for the x coordinate of a turning point is: $\sinh x + 4\cosh 2x = 0$
- $\frac{\mathrm{d}y}{\mathrm{d}x} \neq 0$ for all x, so we have no turning points. QED
- This equation rearranges to $8\sinh^2 x - \sinh x - 4 = 0$.
- This is a quadratic equation in $\cosh x$.
- This equation rearranges to $8\sinh^2 x - \sinh x + 4 = 0$.
- The discriminant is > 0, therefore the equation has no real roots.
- For the function in this question, the equation for the x coordinate of a turning point is: $\sinh x - 4\cosh 2x = 0$.
- This is a quadratic equation in $\sinh x$.

Adapted with permission from UCLES, A Level, June 2018, Paper 4726, Question 3.

Question deck:

STEM SMART Double Maths 32 - Complex Exponentials & Hyperbolic Functions

STEM SMART Double Maths 32 - Complex Exponentials & Hyperbolic Functions

Hyperbolic Functions: Integration 1ii

Subject & topics: Maths **Stage & difficulty:** Further A P2

Part A

Definition of $\cosh x$

Using the definition of $\cosh x$ in terms of e^x and e^{-x}, write $\cosh 2x$ in terms of $\cosh^2 x$.

Give your answer in the form $\cosh 2x = f(\cosh^2 x)$

The following symbols may be useful: `cosech()`, `cosh()`, `coth()`, `sech()`, `sinh()`, `tanh()`, `x`

Part B

$\int_0^1 \cosh^2 3x \, \mathrm{d}x$

Find

$$\int_0^1 \cosh^2 3x \, \mathrm{d}x,$$

giving your answer in the form $A + B \sinh C$, where A, B and C are constants to be found.

The following symbols may be useful: `cosech()`, `cosh()`, `coth()`, `sech()`, `sinh()`, `tanh()`

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STEM SMART Double Maths 32 - Complex Exponentials & Hyperbolic Functions

Hyperbolic Functions: Integration 2i

Subject & topics: Maths **Stage & difficulty:** Further A P2

By first completing the square, find

$$\int_0^1 \frac{1}{\sqrt{x^2+4x+8}}\,\mathrm{d}x$$

giving your answer in an exact form.

The following symbols may be useful: `arccosech()`, `arccosh()`, `arccoth()`, `arcsech()`, `arcsinh()`, `arctanh()`, `ln()`, `log()`

Adapted with permission from UCLES, A Level, January 2013 , Paper 4726, Question 6.