Differentiation from First Principles 1

Pre-Uni Maths for Sciences J3.1 & J3.2

A Level

P P P

To differentiate a function $f(x)$ from first principles involves taking a limit. The derivative of $f(x)$ is given by the expression

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}.$$

Part A Differentiate x^3 from first principles

Differentiate x^3 from first principles. Drag and drop options into the spaces below.

In this question $f(x) = x^3$. Therefore, $f(x+h) = \boxed{}$. Substituting this into the expression for $f'(x)$,

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\boxed{} - x^3}{h}.$$

Next, expand the brackets in the numerator and simplify:

$$f'(x) = \lim_{h\to 0} \frac{(x^3 + 3x^2h + 3xh^2 + h^3) - x^3}{h}$$

$$f'(x) = \lim_{h\to 0} \frac{\boxed{}}{h} = \lim_{h\to 0} \boxed{}.$$

Finally, take the limit. As $h \to 0$, the term containing x^2 is unchanged (because it does not depend on h), but the terms containing xh and h^2 tend to 0. Therefore,

$$f'(x) = \boxed{}.$$

Items:

$3x$ | $3x^2h + 3xh^2 + h^3$ | $x^2 + xh$ | $(x+h)^3$ | $x^2h + xh^2 + h^3$ | $3x^2 + 3xh + h^2$ | $2x^2h + 2xh^2 + h^3$ | $3x^2$ | x^3h^3

Part B Differentiate $2x^3 + 5$ from first principles

Differentiate $2x^3 + 5$ from first principles. Drag and drop options into the spaces below.

In this question $f(x) = 2x^3 + 5$. Therefore, $f(x+h) =$ ☐. Substituting this into the expression for $f'(x)$,

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\square - (2x^3+5)}{h}.$$

Next, just as in part A, expand the brackets in the numerator. After simplification, this produces:

$$f'(x) = \lim_{h\to 0} \square.$$

Finally, take the limit. As $h \to 0$, the term containing x^2 is unchanged (because it does not depend on h), but the terms containing xh and h^2 tend to 0. Therefore,

$$f'(x) = \square.$$

Items:

$2x^3 + 5h$ | $2x^3h^3 + 5$ | $6x^2 + 6xh + 2h^2$ | $6x^2 + 5$ | $6x^2 + 6xh + 2h^2 + 5$ | $2(x+h)^3 + 5$ | $6x^2$

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Differentiation from First Principles 3

A Level

P P P

Pre-Uni Maths for Sciences J3.5

Differentiating from first principles involves taking a limit. The derivative of y with respect to x is given by

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x}.$$

In this expression δy is the small change in y produced by δx, a small change in x.

The value of $\frac{\mathrm{d}y}{\mathrm{d}x}$ at a point on a curve is the gradient of the tangent to the curve at that point.

Part A Expand $(x+a)^4$

Expand $(x+a)^4$ and simplify as far as possible.

The following symbols may be useful: `a`, `x`

Part B Differentiate $y = 9x^4 - 8x$ from first principles

Differentiate $y = 9x^4 - 8x$ from first principles. Drag and drop options into the spaces below.

Consider the coordinates (x, y) of a point on the curve $y = 9x^4 - 8x$. When x increases by δx to $x + \delta x$, y changes to $y + \delta y =$ _____. Substituting this into the expression for the derivative,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \lim_{\delta x \to 0} \frac{(y+\delta y) - y}{\delta x} = \frac{\mathrm{d}y}{\mathrm{d}x} = \lim_{\delta x \to 0} \frac{(\_\_\_\_\_) - (9x^4 - 8x)}{\delta x}.$$

Using the answer to part A gives

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \lim_{\delta x \to 0} \frac{\_\_\_\_\_ - (9x^4 - 8x)}{\delta x}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \lim_{\delta x \to 0} (\_\_\_\_\_ + \_\_\_\_\_ \delta x + 36x(\delta x)^2 + 9(\delta x)^3).$$

Finally, take the limit. As $\delta x \to 0$, the terms containing δx tend to 0. Therefore,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \_\_\_\_\_.$$

Items:

- $9(x+\delta x)^4 - 8(x+\delta x)$
- $9x^4 + 9(\delta x)^4 - 8x - 8(\delta x)$
- $9x^4 + 36x^3(\delta x)$
- $9x^4 - 8x + 36x^3(\delta x) - 8\delta x + 54x^2(\delta x)^2 + 36x(\delta x)^3 + 9(\delta x)^4$
- $54x^2$
- $9x^3 - 8$
- $x^4 - 8x + 4x^3(\delta x) - 8(\delta x) + 6x^2(\delta x)^2 + 4x(\delta x)^3 + (\delta x)^4$
- $36x^3 - 8$

Part C Gradient of tangent

Find the gradient of the tangent to the curve at the point $(1, 1)$.

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Gameboard:

STEM SMART Double Maths 10 - Differentiation

Differentiation (powers of x) 3ii

A Level

P P P

Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ in each of the following cases.

Part A Algebraic fraction

$$y = \frac{(3x)^2 \times x^4}{x}.$$

The following symbols may be useful: x

Part B Cube root

$$y = \sqrt[3]{x}.$$

The following symbols may be useful: x

Part C Reciprocal

$$y = \frac{1}{2x^3}.$$

The following symbols may be useful: x

Physics. You work it out.

Home Gameboard Maths Calculus Differentiation Differentiating Powers 3

Differentiating Powers 3

A Level P P P Further A P P P

Pre-Uni Maths for Sciences 2.6.6

Part A Derivative of $v = Bu^{-3}$

Find $\frac{\mathrm{d}v}{\mathrm{d}u}$ if $v = Bu^{-3}$, where B is a constant.

The following symbols may be useful: `B, u`

Part B Force if potential $V = \frac{q^2}{4\pi\epsilon_0 r}$

The electrostatic potential energy V of two equal charges q a distance r apart is given by $V = \frac{q^2}{4\pi\epsilon_0 r}$, where ϵ_0 and q are constants. The force between the two charges is given by $-\frac{\mathrm{d}V}{\mathrm{d}r}$; find an expression for this force.

The following symbols may be useful: `epsilon_0, pi, q, r`

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Gameboard:
STEM SMART Double Maths 10 - Differentiation

Gradient Function: Tangents and Normals 1i

A Level

A curve has equation $y = x^2 + x$.

Part A Gradient

Find the gradient of the curve at the point where $x = 2$.

Part B Normal

Find the equation of the normal to the curve at the point for which $x = 2$, giving your answer in the form $ax + by + c = 0$, where a, b and c are integers.

The following symbols may be useful: x, y

Part C Find k

Find the smallest value of k for which the line $y = kx - 4$ is a tangent to the curve.

The following symbols may be useful: k

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Physics. You work it out.

Home Gameboard Maths Stationary Points 2ii

Stationary Points 2ii

A Level
P P P

Part A Find coordinate

Find the coordinates of the stationary points on the curve $y = x^3 - 3x^2 + 4$. Enter the x and y coordinates of the stationary point with the greatest x coordinate.

Enter the x-coordinate:

The following symbols may be useful: x

Enter the y-coordinate:

The following symbols may be useful: y

Part B Stationary point

Determine whether the stationary point whose coordinates you entered is a maximum point or a minimum point.

- Minimum
- Maximum
- Inconclusive

Part C Range of x

For what range of values of x does $x^3 - 3x^2 + 4$ decrease as x increases?

What form does your answer take? Choose from the list below, where a and b are constants and $a < b$, and then find a and/or b.

- $x < a$
- $x \leq a$
- $x > a$
- $x \geq a$
- $a < x < b$
- $a \leq x \leq b$
- $x < a$ or $x > b$
- $x \leq a$ or $x \geq b$

Write down the value of a.

Write down the value of b (or if your chosen form has no b, write "n").

The following symbols may be useful: n

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Gameboard:
STEM SMART Double Maths 10 - Differentiation

Stationary Points 3

A Level C C C Further A P P P

Pre-Uni Maths for Sciences 2.6.10

Part A Find the maximum height of a projectile

A particle is fired upwards into the air with a initial speed w and moves subsequently under the influence of gravity with an acceleration g downwards, such that its height h at time t is given by $h = wt - \frac{1}{2}gt^2$, where w and g are constants. Find an expression for its maximum height above its initial position.

The following symbols may be useful: `g`, `h`, `w`

Part B Examine the potential energy of two molecules

The potential energy of two molecules separated by a distance r is given by

$$U = U_0\,((\frac{a}{r})^{12} - 2\,(\frac{a}{r})^{6})$$

where U_0 and a are positive constants. The equilibrium separation of the two molecules occurs when the potential energy is a minimum; find expressions for the equilibrium separation and the value of the potential energy at this separation.

(a) Find an expression for the equilibrium separation of the molecules.

The following symbols may be useful: `U, U_0, a, r`

(b) Find an expression for the potential energy when the molecules are at their equilibrium separation.

The following symbols may be useful: `U, U_0, a, r`

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Gameboard:

STEM SMART Double Maths 10 - Differentiation

Maxima and Minima: Problems 1ii

A Level

P P P

Figure 1 shows a rectangular enclosure, with a wall forming one side. A rope, of length 20 metres, is used to form the remaining three sides. The width of the enclosure is x metres, and the area of the enclosure is $A\,\mathrm{m}^2$.

Figure 1: The rectangular enclosure.

Part A Express as equation

Show that A can be expressed in the form $px - qx^2$, and find this expression.

The following symbols may be useful: `A`, `x`

Part B Use differentiation

Use differentiation to find the maximum value of the area of the enclosure, $A\,\mathrm{m}^2$.

Enter your value of A:

The following symbols may be useful: `A`

Physics. You work it out.

Home Gameboard Maths Calculus Differentiation Minimising the Area

Minimising the Area

A Level

P P P

Pre-Uni Maths for Sciences 3.4.1

A rectangular cuboid has a base with sides of length a and b and a height c. Its volume V and height c are fixed. By following the steps below find expressions in terms of V and c for the values of a and b which will minimise the surface area A of the cuboid, find an expression for this minimum surface area and check that this is indeed a minimum.

Part A Volume V and surface area A

Write down the equation for the volume V of the rectangular cuboid in terms of a, b and c.

The following symbols may be useful: `V, a, b, c`

Write down the equation for the surface area A of the rectangular cuboid in terms of a, b and c.

The following symbols may be useful: `A, a, b, c`

From your equation for V deduce an expression for b in terms of V, a and c. Hence, by substitution, obtain an equation for A in terms of V, a and c.

The following symbols may be useful: `A, V, a, c`

Part B Expressions for a and b

Differentiate with respect to a the expression for A you found in Part A (since V and c are fixed you may treat them as constants). Hence find in terms of V and c an expression for the value of a for which the surface area A is minimised.

The following symbols may be useful: V, c

Find, in terms of V and c, the expression for b corresponding to this value of a.

The following symbols may be useful: V, c

Part C The minimum area

Find an expression for the minimum surface area in terms of V and c.

The following symbols may be useful: V, c

Part D Check that the area is a minimum

Find, at the value of a deduced in Part B, an expression in terms of V and c for the second derivative of A with respect to a; convince yourself that the value of the second derivative indicates that the value of A is a minimum at this point.

The following symbols may be useful: V, c

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