Incorporating Maths in Question-Writing

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What is the Question for?

- What purpose is the question for?
  - Assessment
  - Practice to develop a skill
  - Extension to combine different skills
  - To highlight a particular case

- What audience is it for?

- These inform the skills required

Considerations for Maths Skills

- What skills are required?
  - Differentiation, chain rule, product rule, integration, partial fractions
- Are there multiple different methods?
- Do extra skills make an interesting extension?
  - Quadratics, indices, logarithms, binomial distribution
- Do extra skills add too much challenge?
  - Integration, double angle formulae, t-formulae
- How to sequence and develop the problem?
- Do we want to model an approach?
- Work through the problem to check.

Three Examples

- **Pure:** Divisibility by Exhaustion
- **Mechanics:** Pedal Power
- **Statistics:** t-tests: Oak Leaves

Divisibility by Exhaustion

What Maths skills does it need?

https://isaacphysics.org/questions/proof_divisibility_exhaustion

Divisibility by Exhaustion

A Level C C C
Further A C C C

A sequence u_n is defined by $u_n = n^7 - n$, where $n \in \mathbb{N}$. The first four terms of this sequence are

$$0, 126, 2184, 16380, \ldots$$

What is the largest integer that will divide every term of this sequence?

Part A **Factorise u_n**

Factorise u_n completely.

Divisibility by Exhaustion - A

Divisibility by Exhaustion

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A sequence u_n is defined by $u_n = n^7 - n$, where $n \in \mathbb{N}$. The first four terms of this sequence are

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What is the largest integer that will divide every term of this sequence?

Part A **Factorise u_n**

Factorise u_n completely.

$$u_n = n^7 - n$$
$$= n(n^6 - 1)$$
$$= n(n^3 - 1)(n^3 + 1)$$

What **Maths skills** does it need?

- Factorising
- Difference of two squares
- Factor theorem
- Diff/sum of two cubes

Divisibility by Exhaustion – B, C, D

Part B **Divisibility by 2**

Using your expression from part A, prove that every term in the sequence is divisible by 2.

Once you have worked out a proof yourself, consider the model proof below. Drag and drop answers into the blank spaces to complete the proof.

We know that the factors of u_n from part A.

When n is even, it is divisible by 2 and we can see that ______ is a factor of u_n, so u_n is divisible by 2.

When n is odd, we can write $n =$ ______ in terms of k, where $k \in \mathbb{Z}$. Then the factor ______ $=$ ______ in terms of k, so the factor ______ is divisible by 2, and hence u_n is divisible by 2.

Therefore, u_n is divisible by 2 for any value of n. So every term in the sequence is divisible by 2.

Part C **Divisibility by 3**

Part D **Divisibility by 7**

What Maths skills does it need?

- Proof by exhaustion
- Split context into cases
- Represent numbers as a multiple, or off by an amount
- Substitution
- Divisibility

Divisibility by Exhaustion – E

Part E **Largest Divisor**

Prove that 42 is the largest integer that will divide every term of u_n.

Once you have worked out a proof yourself, consider the model proof below. Drag and drop answers into the blank spaces to complete the proof. You may use the same answer more than once.

We know from earlier that u_n is divisible by 2, 3 and 7. So we know that $2 \times 3 \times 7 =$ will divide u_n. Are there any larger integers that can do so?

Let's consider the first non-zero term, 126. We find that $126 \div 42 =$. This shows that the prime factorisation of 126 is . Hence, the only larger factors of 126 are (in order of increasing size) and . Will these divide any other terms of u_n?

Looking at the next term, we find that $2184 \div$ $= \frac{104}{3}$, so does not divide 2184. Considering our other factor, we find that $2184 \div$ $= \frac{52}{3}$, so does not divide 2184 either.

Therefore, 42 is the largest integer that will divide every term of u_n.

What Maths skills does it need?

- Proof by exhaustion
- Prime factorisation
- Divisibility

Divisibility by Exhaustion Summary

- Helps to teach concept
- Sequencing establishes the steps required
- Format helps to model writing a proof
- Multiple methods can be used

Pedal Power

What Maths skills does it need? https://isaacphysics.org/questions/pedal_power_r1

Pedal Power

A Level

C C C

A $75\,\mathrm{kg}$ cyclist on a $15\,\mathrm{kg}$ bicycle pedals against a backwards resistive force that is proportional to the square of their speed. On a flat road, they can travel at a steady speed of $10.0\,\mathrm{m\,s^{-1}}$. While cycling up an incline, they produce the same power, but their steady speed is only $5.0\,\mathrm{m\,s^{-1}}$.

Part A **Coasting down**

At what speed could the cyclist coast down the incline, if they do not pedal?

Pedal Power - A

Pedal Power

A Level

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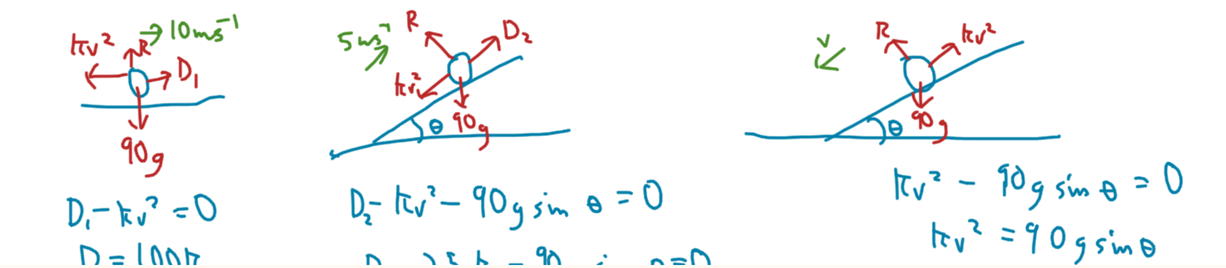
Part A **Coasting down**

At what speed could the cyclist coast down the incline, if they do not pedal?



What **Maths skills** does it need?

- Modelling
- Draw diagrams
- Resolve forces
- Newton’s laws
- Relate power and velocity
- Substitution

Pedal Power - B

Part B **Head down**

The cyclist knows that, regardless of their speed, they can reduce the resistive force by 20% by putting their head down. This allows them to travel at a higher steady speed.

By how much does their speed increase, if they put their head down:

(i) while coasting down the incline?

Unit...

(ii) while pedalling on a flat road?

Unit...

k becomes $0.8k$

$$0.8kv^2 - 90g\sin\theta = 0$$
$$0.8kv^2 = 90g\sin\theta$$
$$0.8kv^2 = 175k$$

What Maths skills does it need?

- Percentages
- Rearrange equations
- Substitution

Pedal Power – B (iii)

Part B **Head down**

The cyclist knows that, regardless of their speed, they can reduce the resistive force by 20% by putting their head down. This allows them to travel at a higher steady speed.

By how much does their speed increase, if they put their head down:

(i) while coasting down the incline?

Unit...

(ii) while pedalling on a flat road?

Unit...

(iii) while pedalling up the incline?

$$\frac{1000k}{v} - 0.8kv^2 - 90g\sin\theta = 0$$

$$1000kv^{-1} - 0.8kv^2 - 175k = 0$$

$$1000\cancel{k} - 0.8\cancel{k}v^3 - 175\cancel{k}v = 0$$

$$-0.8v^3 - 175v + 1000 = 0$$

Part (iii) required solving a cubic in v with an irrational root. This was a skill we weren't intending to assess here.

Could use numerical methods, but we chose to remove it.

Pedal Power - C

Part C **Angle of the incline**

When the cyclist puts their head down while cycling on a flat road at a steady speed, their initial acceleration is $0.050\,\mathrm{m\,s^{-2}}$.

What is the angle of the incline from earlier in the question?

What Maths skills does it need?

- Newton's laws
- Rearrange equations
- Substitution
- Trigonometry

Pedal Power Summary

- Challenging question bringing together several skills
- Working through revealed a skill that was too much challenge
- Sequencing uses prior equations and asks what happens if this changes?

Example: t-tests: Oak Leaves

What Maths skills does it need?

https://isaacphysics.org/questions/t_test_oak_leaves_r1

t-tests: Oak Leaves

Further A

P P P

The leaves from oak trees growing in two different areas A and B are being measured. The lengths, in cm, of a random sample of 7 oak leaves from area A are:

$$6.2, \quad 8.3, \quad 7.8, \quad 9.3, \quad 10.2, \quad 8.4, \quad 7.2$$

Part A Confidence interval

Assuming that the distribution is normal, find a 95% confidence interval for the mean length of oak leaves from area A.

(Unit... , Unit...)

Example: t-tests: Oak Leaves - A

t-tests: Oak Leaves

Further A

The leaves from oak trees growing in two different areas A and B are being measured. The lengths, in cm, of a random sample of 7 oak leaves from area A are:

$$6.2, \quad 8.3, \quad 7.8, \quad 9.3, \quad 10.2, \quad 8.4, \quad 7.2$$

Part A **Confidence interval**

Assuming that the distribution is normal, find a 95% confidence interval for the mean length of oak leaves from area A.

([ ] Unit... , [ ] Unit...)

CI $\bar{x} \pm t \frac{s}{\sqrt{n}}$

$$\bar{x} = \frac{\Sigma x}{7}$$

$$= 8.2$$

$$s_x^2 = \frac{1}{6}\left(\Sigma x^2 - 7\bar{x}^2\right)$$

What **Maths skills** does it need?

- Mean
- Estimate population variance from sample
- Critical values for t-distribution
- Confidence intervals

Example: t-tests: Oak Leaves - B

Part B **Assumptions**

The lengths, in cm, of a random sample of 5 oak leaves from area B are:

$$5.9, \quad 7.4, \quad 6.8, \quad 8.2, \quad 8.7$$

State two assumptions needed to carry out a suitable t-test on the difference between the mean lengths of oak leaves from areas A and B.

- [ ] The lengths of oak leaves from areas A and B have equal means.
- [ ] The length of oak leaves from area B follow a uniform distribution.
- [ ] The length of oak leaves from area B follow a geometric distribution.
- [ ] The lengths of oak leaves from areas A and B have equal variances.
- [ ] The length of oak leaves from area B follow a normal distribution.

What Maths skills does it need?

- Assumptions for two-sample t-test
- Requirements for t-distribution

Example: t-tests: Oak Leaves - C

Part C **Hypothesis test**

Test, at the 5% significance level, whether the mean length of oak leaves from area A is greater than the mean length of oak leaves from area B.

The null and alternative hypotheses are:

$$\mathrm{H}_0 : \mu_a \; \square \; \mu_b \qquad \mathrm{H}_1 : \mu_a \; \square \; \mu_b$$

Calculating the difference as $\bar{a} - \bar{b}$, the test statistic, $t =$ ______. The critical value is $t_{\text{crit}} =$ ______.

Comparing these, we find that t ______ t_{crit}.

Therefore we ______ H_0 at the 5% level. There ______ evidence to suggest that the mean length of oak leaves from area A is greater than the mean length of oak leaves from area B.

$H_0 : \mu_a = \mu_b \qquad H_1 : \mu_a > \mu_b$

From earlier

$\bar{a} = 8.2$

$s_a^2 = \frac{521}{300}$

$\bar{b} = \frac{37}{5}$

$= 7.4$

$s_b^2 = \frac{1}{4}(\Sigma y^2 - 5\bar{y}^2)$

$= \frac{1}{4}(278.74 - 5(7.4)^2)$

$= \frac{247}{200}$

$= 1.235$

$s_p^2 = \frac{6s_a^2 + 4s_b^2}{7+5-2}$

What Maths skills does it need?

- Hypothesis tests
- Two-sample t-test
- Mean
- Estimate variance from sample
- Pooled estimate of variance

Summary Oak Leaves

- Standard question on the subject.
- Sequencing provides information as it becomes relevant, building from one-sample to two.
- Checks modelling assumptions.
- Format helps to model writing a hypothesis test.
- Multiple methods available, finding statistics manually or using a calculator.

Summary

- Establish the skills to assess and any related required skills.
- Consider whether there are multiple different methods.
- Consider whether extra skills make for interesting extension or too much challenge.
- Use sequencing to establish required steps and present information as it becomes relevant
- Can use to model an approach.
- Work through the problem to check.