Differentiation from First Principles 1

A Level

P P P

Pre-Uni Maths for Sciences J3.1 & J3.2

To differentiate a function $f(x)$ from first principles involves taking a limit. The derivative of $f(x)$ is given by the expression

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}.$$

Part A Differentiate x^3 from first principles

Differentiate x^3 from first principles. Drag and drop options into the spaces below.

In this question $f(x) = x^3$. Therefore, $f(x+h) = \square$. Substituting this into the expression for $f'(x)$,

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\square - x^3}{h}.$$

Next, expand the brackets in the numerator and simplify:

$$f'(x) = \lim_{h\to 0} \frac{(x^3 + 3x^2h + 3xh^2 + h^3) - x^3}{h}$$

$$f'(x) = \lim_{h\to 0} \frac{\square}{h} = \lim_{h\to 0} \square.$$

Finally, take the limit. As $h \to 0$, the term containing x^2 is unchanged (because it does not depend on h), but the terms containing xh and h^2 tend to 0. Therefore,

$$f'(x) = \square.$$

Items:

x^3h^3 | $x^2h + xh^2 + h^3$ | $(x+h)^3$ | $3x^2h + 3xh^2 + h^3$ | $3x$ | $3x^2$ | $x^2 + xh$ | $3x^2 + 3xh + h^2$ | $2x^2h + 2xh^2 + h^3$

Part B Differentiate $2x^3 + 5$ from first principles

Differentiate $2x^3 + 5$ from first principles. Drag and drop options into the spaces below.

In this question $f(x) = 2x^3 + 5$. Therefore, $f(x+h) =$ [ ]. Substituting this into the expression for $f'(x)$,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\boxed{} - (2x^3 + 5)}{h}.$$

Next, just as in part A, expand the brackets in the numerator. After simplification, this produces:

$$f'(x) = \lim_{h \to 0} \boxed{}.$$

Finally, take the limit. As $h \to 0$, the term containing x^2 is unchanged (because it does not depend on h), but the terms containing xh and h^2 tend to 0. Therefore,

$$f'(x) = \boxed{}.$$

Items:

$2x^3h^3 + 5$ | $6x^2$ | $2(x+h)^3 + 5$ | $2x^3 + 5h$ | $6x^2 + 5$ | $6x^2 + 6xh + 2h^2 + 5$ | $6x^2 + 6xh + 2h^2$

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Home Gameboard Maths Calculus Differentiation Differentiation from First Principles 3

Differentiation from First Principles 3

A Level
P P P

Pre-Uni Maths for Sciences J3.5

Differentiating from first principles involves taking a limit. The derivative of y with respect to x is given by

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x}.$$

In this expression δy is the small change in y produced by δx, a small change in x.

The value of $\frac{\mathrm{d}y}{\mathrm{d}x}$ at a point on a curve is the gradient of the tangent to the curve at that point.

Part A Expand $(x+a)^4$

Expand $(x+a)^4$ and simplify as far as possible.

The following symbols may be useful: `a, x`

Part B Differentiate $y = 9x^4 - 8x$ from first principles

Differentiate $y = 9x^4 - 8x$ from first principles. Drag and drop options into the spaces below.

Consider the coordinates (x, y) of a point on the curve $y = 9x^4 - 8x$. When x increases by δx to $x + \delta x$, y changes to $y + \delta y =$ [ ]. Substituting this into the expression for the derivative,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \lim_{\delta x \to 0} \frac{(y + \delta y) - y}{\delta x} = \frac{\mathrm{d}y}{\mathrm{d}x} = \lim_{\delta x \to 0} \frac{(\boxed{}) - (9x^4 - 8x)}{\delta x}.$$

Using the answer to part A gives

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \lim_{\delta x \to 0} \frac{\boxed{} - (9x^4 - 8x)}{\delta x}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \lim_{\delta x \to 0} (\boxed{} + \boxed{}\delta x + 36x(\delta x)^2 + 9(\delta x)^3).$$

Finally, take the limit. As $\delta x \to 0$, the terms containing δx tend to 0. Therefore,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \boxed{}.$$

Items:

- $9(x + \delta x)^4 - 8(x + \delta x)$
- $x^4 - 8x + 4x^3(\delta x) - 8(\delta x) + 6x^2(\delta x)^2 + 4x(\delta x)^3 + (\delta x)^4$
- $9x^4 + 36x^3(\delta x)$
- $54x^2$
- $36x^3 - 8$
- $9x^4 - 8x + 36x^3(\delta x) - 8\delta x + 54x^2(\delta x)^2 + 36x(\delta x)^3 + 9(\delta x)^4$
- $9x^4 + 9(\delta x)^4 - 8x - 8(\delta x)$
- $9x^3 - 8$

Part C Gradient of tangent

Find the gradient of the tangent to the curve at the point $(1, 1)$.

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Differentiation (powers of x) 3ii

A Level
P P P

Find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ in each of the following cases.

Part A Algebraic fraction

$$y = \frac{(3x)^2 \times x^4}{x}.$$

The following symbols may be useful: x

Part B Cube root

$$y = \sqrt[3]{x}.$$

The following symbols may be useful: x

Part C Reciprocal

$$y = \frac{1}{2x^3}.$$

The following symbols may be useful: x

Gameboard:

STEM SMART Double Maths 10 - Differentiation

Differentiating Powers 4

A Level P P P

Pre-Uni Maths for Sciences J1.6

Part A Derivative of $v = Bu^{-3}$

Find $\frac{\mathrm{d}v}{\mathrm{d}u}$ if $v = Bu^{-3}$, where B is a constant.

The following symbols may be useful: `B, u`

Part B Force if potential $V = \frac{q^2}{4\pi\epsilon_0 r}$

The electrostatic potential energy V of two equal charges q a distance r apart is given by $V = \frac{q^2}{4\pi\epsilon_0 r}$, where ϵ_0 and q are constants. The force between the two charges is given by $-\frac{\mathrm{d}V}{\mathrm{d}r}$; find an expression for this force.

The following symbols may be useful: `epsilon_0, pi, q, r`

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Gameboard:
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Gradient Function: Tangents and Normals 1i

A Level
P P P

A curve has equation $y = x^2 + x$.

Part A Gradient

Find the gradient of the curve at the point where $x = 2$.

Part B Normal

Find the equation of the normal to the curve at the point for which $x = 2$, giving your answer in the form $ax + by + c = 0$, where a, b and c are integers.

The following symbols may be useful: x, y

Part C Find k

Find the smallest value of k for which the line $y = kx - 4$ is a tangent to the curve.

The following symbols may be useful: k

Used with permission from UCLES, A level, January 2005, Question

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STEM SMART Double Maths 10 - Differentiation

Stationary Points 2ii

A Level

P P P

Part A Find coordinate

Find the coordinates of the stationary points on the curve $y = x^3 - 3x^2 + 4$.

Enter the x- and y-coordinates of the stationary point with the greatest x-coordinate. If a value is not a whole number, enter the value as a decimal.

(,)

Part B Stationary point

Determine whether the stationary point whose coordinates you entered is a maximum point or a minimum point.

- Maximum
- Inconclusive
- Minimum

Part C Range of x

For what range of values of x does $x^3 - 3x^2 + 4$ decrease as x increases?

Construct your answer from the items below.

Items:

$<$ $>$ x $< x <$ $\leq x \leq$ $< x$ or $x <$ $\leq x$ or $x \leq$ $\leq$ $\geq$ -2 -1 0 1 2 3 4

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Stationary Points 3

A Level

C C C

Pre-Uni Maths for Sciences J2.5

Part A Maximum height of a projectile

A particle is fired upwards into the air with a initial speed w and moves subsequently under the influence of gravity with an acceleration g downwards, such that its height h at time t is given by $h = wt - \frac{1}{2}gt^2$, where w and g are constants. Find an expression for its maximum height above its initial position.

The following symbols may be useful: `g`, `h`, `w`

Part B Potential energy of two molecules

The potential energy of two molecules separated by a distance r is given by

$$U = U_0\left(\left(\frac{a}{r}\right)^{12} - 2\left(\frac{a}{r}\right)^{6}\right)$$

where U_0 and a are positive constants. The equilibrium separation of the two molecules occurs when the potential energy is a minimum.

Find an expression for the equilibrium separation of the molecules.

The following symbols may be useful: `U`, `U_0`, `a`, `r`

Find an expression for the potential energy when the molecules are at their equilibrium separation.

The following symbols may be useful: `U`, `U_0`, `a`, `r`

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STEM SMART Double Maths 10 - Differentiation

Maxima and Minima: Problems 1ii

A Level

P P P

Figure 1 shows a rectangular enclosure, with a wall forming one side. A rope, of length 20 metres, is used to form the remaining three sides. The width of the enclosure is x metres, and the area of the enclosure is $A\,\mathrm{m}^2$.

Figure 1: The rectangular enclosure.

Part A Express as equation

Show that A can be expressed in the form $px - qx^2$, and find this expression.

The following symbols may be useful: `A, x`

Part B Use differentiation

Use differentiation to find the maximum value of the area of the enclosure, $A\,\mathrm{m}^2$.

Enter your value of A:

The following symbols may be useful: `A`

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STEM SMART Double Maths 10 - Differentiation

Physics. You work it out.

Home Gameboard Maths Calculus Differentiation Minimising the Area

Minimising the Area

A Level

P P P

Pre-Uni Maths for Sciences J2.2

A rectangular cuboid has a base with sides of length a and b and a height c. Its volume V and height c are fixed.

Part A Volume V and surface area A

Write down the equation for the volume V of the rectangular cuboid in terms of a, b and c.

The following symbols may be useful: `V, a, b, c`

Write down the equation for the surface area A of the rectangular cuboid in terms of a, b and c.

The following symbols may be useful: `A, a, b, c`

From your equation for V deduce an expression for b in terms of V, a and c. Hence, by substitution, obtain an equation for A in terms of V, a and c.

The following symbols may be useful: `A, V, a, c`

Part B Expressions for a and b

Differentiate with respect to a the expression for A you found in Part A (since V and c are fixed you may treat them as constants). Hence find in terms of V and c an expression for the value of a for which the surface area A is minimised.

The following symbols may be useful: V, c

Find, in terms of V and c, the expression for b corresponding to this value of a.

The following symbols may be useful: V, c

Part C The minimum area

Find an expression for the minimum surface area in terms of V and c.

The following symbols may be useful: V, c

Part D Check that the area is a minimum

Find, at the value of a deduced in Part B, an expression in terms of V and c for the second derivative of A with respect to a; convince yourself that the value of the second derivative indicates that the value of A is a minimum at this point.

The following symbols may be useful: V, c

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