Series: Summation - Standard Results 2i

Further A

P P P

Use the standard results for $\sum_{r=1}^{n} r$, $\sum_{r=1}^{n} r^2$ and $\sum_{r=1}^{n} r^3$ to express

$$\sum_{r=1}^{n} \left(8r^3 - 6r^2 + 2r\right)$$

in terms of n, in a fully factorised form.

The following symbols may be useful: n

Adapted with permission from UCLES, A Level, January 2006, Paper 4725, Question 5.

Series: Induction 1i

Further A

P P P

Prove by induction that, for $n \geqslant 1$,

$$\sum_{r=1}^{n} \frac{1}{(2r-1)(2r+1)} = \frac{n}{2n+1}$$

by filling in the gaps below.

Proposition

We claim that $\sum_{r=1}^{n} \frac{1}{(2r-1)(2r+1)} = \frac{n}{2n+1}$ for $n \geqslant 1$.

Base case

Our claim holds for $n =$ ☐, since both sides of the equation are equal to ☐.

Assumption

Assume that for $n =$ ☐, where $k \geqslant 1$ is an integer,

☐ ☐ = ☐.

Induction

Now consider $n =$ ☐.

$$\square \frac{1}{(2r-1)(2r+1)} = \square \frac{1}{(2r-1)(2r+1)} + \square = \square$$

Conclusion

So, if our claim holds for $n =$ ☐, then it must hold for $n =$ ☐. Since we have shown that it holds for $n =$ ☐, by ☐, our claim holds for all ☐ $n \geqslant 1$.

Items:

$\frac{k+1}{(2(k+1)+1)}$ | $\frac{k}{2k+1}$ | n | contradiction | integers | 0 | real numbers | $\frac{1}{(2k+1)(2k+3)}$ | $\frac{1}{(2r+1)(2r+3)}$ | k | $\frac{1}{(2r-1)(2r+1)}$ | 1 | $\frac{1}{(2k-1)(2k+1)}$ | $\sum_{r=1}^{k+1}$ | $k+1$ | $\frac{1}{3}$ | $\sum_{r=1}^{k}$ | induction

Adapted with permission from UCLES, A Level, June 2017, Paper 4725, Question 4.

Series: Method of Differences 4i

Further A

P P P

Part A Rearrange $\frac{1}{2r+1}-\frac{1}{2r+3}$

Write $\dfrac{1}{2r+1}-\dfrac{1}{2r+3}$ as a single fraction.

The following symbols may be useful: r

Part B Sum to n

Hence find

$$\sum_{r=1}^{n}\frac{1}{(2r+1)(2r+3)}$$

giving your answer as a single fraction.

The following symbols may be useful: n

Part C Sum from n to infinity

Find

$$\sum_{r=n}^{\infty}\frac{1}{(2r+1)(2r+3)}$$

giving your answer as a single fraction.

The following symbols may be useful: n

isaac Physics. You work it out.

Home Gameboard Maths Algebra Series Maclaurin Series - Cos & Sin 2

Maclaurin Series - Cos & Sin 2

Further A University

Part A Expand $\sin(4\theta)$

Write down the third non-zero term in the expansion of $\sin(4\theta)$.

The following symbols may be useful: `alpha`, `theta`

Part B Expand $\cos((\pi/3) - \alpha)$

Using the standard trig formula for the cosine of the sum of two angles write $\cos((\pi/3) - \alpha)$ in terms of $\cos\alpha$ and $\sin\alpha$. Hence find the first 5 terms in the Maclaurin expansion of $\cos((\pi/3) - \alpha)$.

The following symbols may be useful: `alpha`, `theta`

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Calculus: Volume of Revolution 5i

Further A

P P P

The diagram shows the curve with equation $y = \frac{1}{4}\ln x$.

The region R is bounded by the curve and the lines $x = 0, y = 0$ and $y = \frac{1}{4}\ln 3$.

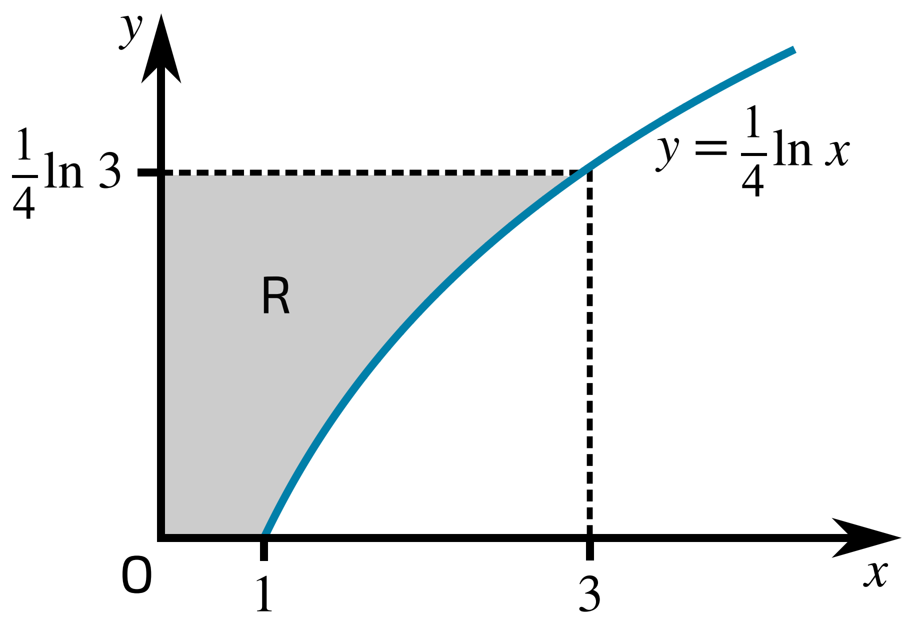

Figure 1: The curve $y = \frac{1}{4}\ln x$ and the region R.

The region R is rotated through four right angles about the y-axis. Find the volume of the solid generated.

The following symbols may be useful: `pi`

Adapted with permission from UCLES, A Level, June 2003, Paper 2632, Question 8.

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Calculus: Inverse Trigonometry 1i

Further A

P P P

Part A Derivative of $\arccos x$

Find the derivative of $\arccos x$.

The following symbols may be useful: x

Part B Gradient of a curve

A curve has equation $y = \arccos\left(1 - x^2\right)$, for $0 < x < \sqrt{2}$.

Find and simplify $\frac{\mathrm{d}y}{\mathrm{d}x}$.

The following symbols may be useful: x

Part C Finding $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$

Hence show that

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = f(x)\frac{\mathrm{d}y}{\mathrm{d}x}.$$

Where $f(x)$ is some function of x to be found.

Find $f(x)$.

The following symbols may be useful: x

Adapted with permission from UCLES, A Level, January 2012, Paper 4726, Question 6.

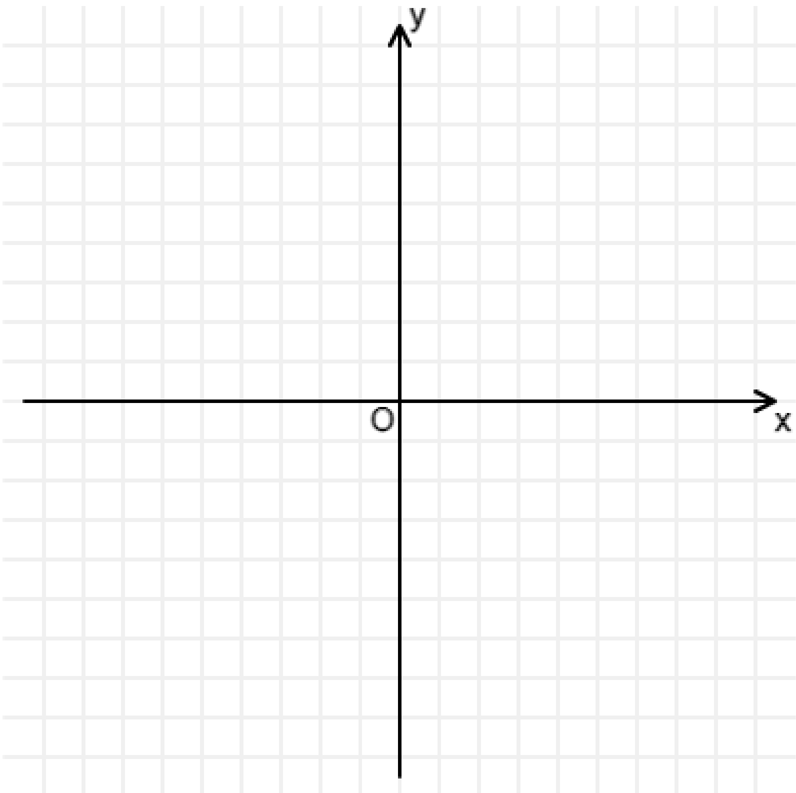
Hyperbolic Functions: Integration 1i

Further A

This question is about properties of the $\tanh$ function and its inverse.

Part A Sketch $y = \tanh x$

Sketch the graph of $y = \tanh x$.



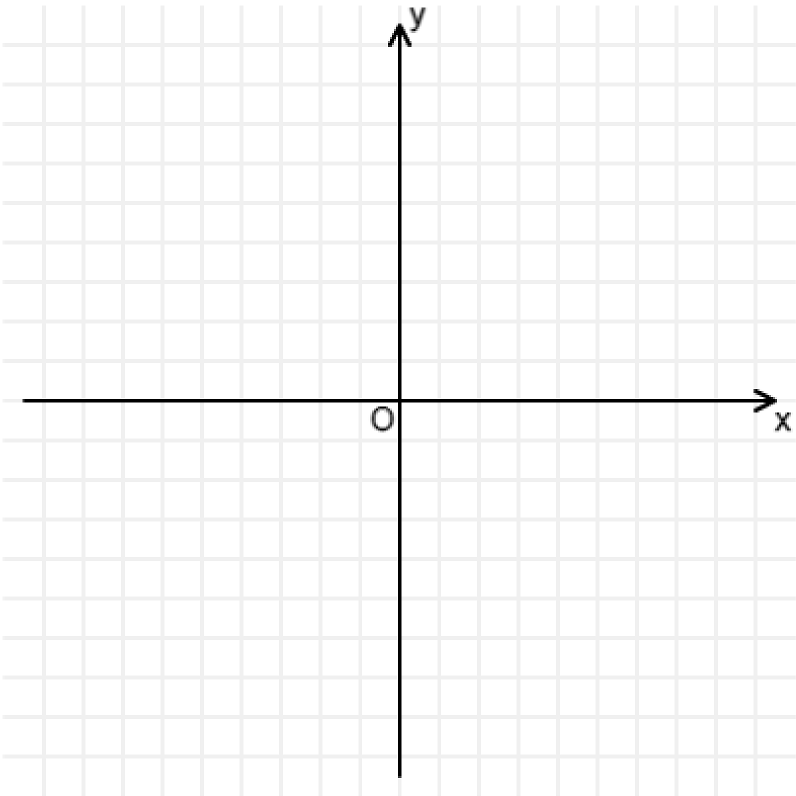
State the value of the gradient of $y = \tanh x$ when $x = 0$.

What can you say about the asymptotes of $y = \tanh x$?

- There are two asymptotes at $y = -1$ and $y = 1$
- There are two asymptotes at $x = -1$ and $x = 1$
- There is one asymptote at $x = 0$
- There is one asymptote at $y = 0$

Part B **Sketch $y = \operatorname{artanh} x$**

Sketch the graph of $y = \operatorname{artanh} x$.



How are the graphs of $y = \tanh x$ and $y = \operatorname{artanh} x$ related?

- Reflection in the line $y = x$
- Rotation of π radians about the origin
- Reflection in the line $y = 0$
- Reflection in the line $x = 0$

What can you say about the asymptotes of $y = \operatorname{artanh} x$?

- There is one asymptote at $x = 0$
- There are two asymptotes at $y = -1$ and $y = 1$
- There is one asymptote at $y = 0$
- There are two asymptotes at $x = -1$ and $x = 1$

Part C Integrate $\tanh x$

Find $\int_0^k \tanh x \, \mathrm{d}x$, where $k > 0$.

The following symbols may be useful: `cosech()`, `cosh()`, `coth()`, `k`, `ln()`, `log()`, `sech()`, `sinh()`, `tanh()`

Part D Deduce the area

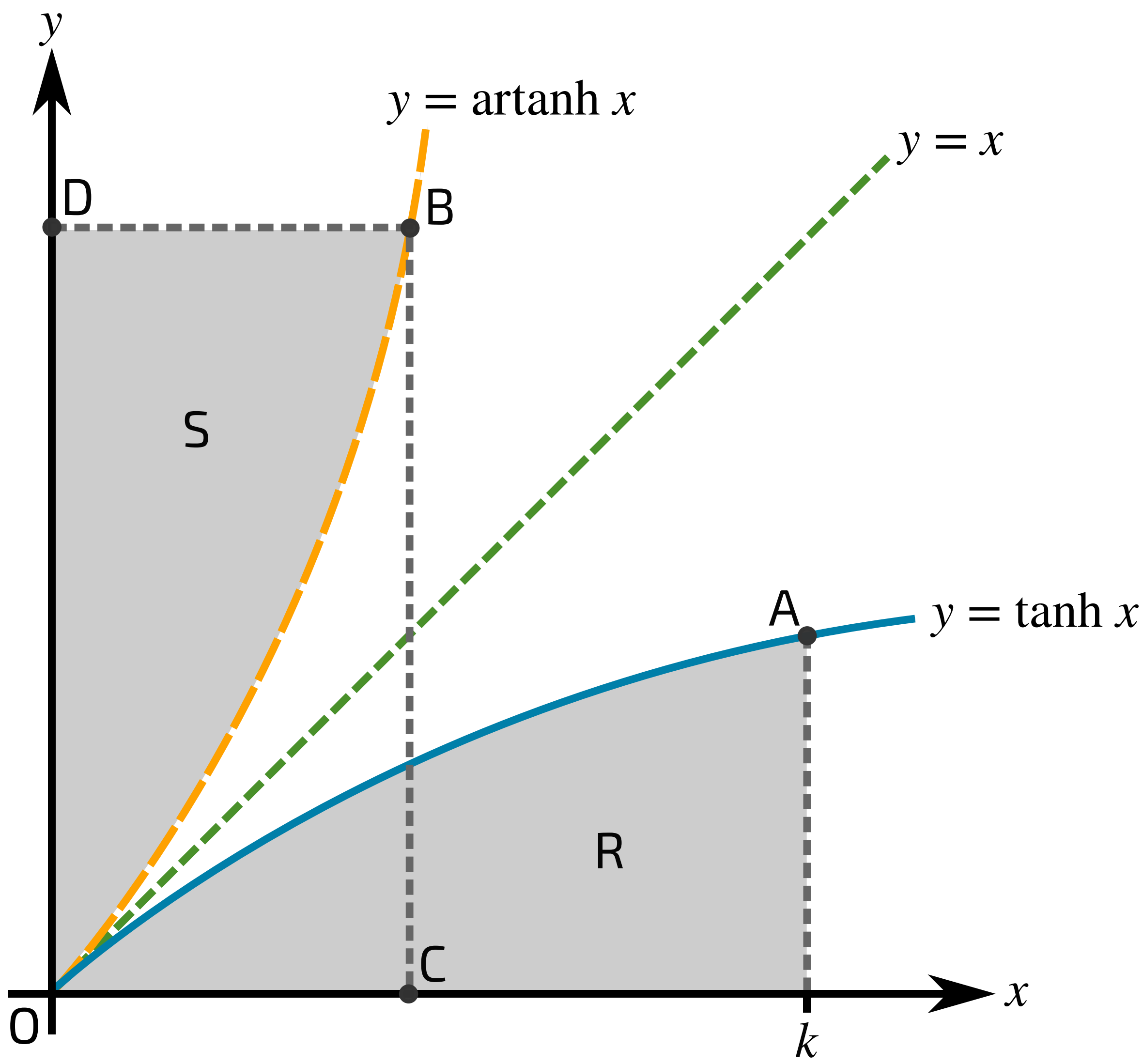


Figure 1: A plot of $\tanh x$ and its inverse. Point A has x coordinate k.

Deduce the value of $\int_0^{\tanh k} \operatorname{artanh} x \, dx$ by filling in the gaps below. Refer to **Figure 1** in your answer.

Point A has coordinates $(k, \square)$. Point B is the reflection of A in the line $y = x$. Its coordinates are $(\square, \square)$.

By symmetry the area of S is equal to the area of R, which we worked out in Part C to be $\int_0^k \tanh x \, dx = \square$.

Therefore, we can deduce that the area represented by $\int_0^{\tanh k} \operatorname{artanh} x \, dx$ is given by:

$$\text{Area of rectangle OCBD} - \text{Area of S} = \square - \square$$

Items:

$\ln(\cosh k)$ $k\ln(\cosh k)$ $\cosh k$ $k\tanh k$ k k^2 $k\ln(\sinh k)$ $\operatorname{artanh} k$ $\ln(\sinh k)$ $\tanh k$

Adapted with permission from UCLES, A Level, June 2011, Paper 4726, Question 7.

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Coupled Differential Equations

Further A

Two species of insect, X and Y, compete for survival on an island. After t decades, the populations of the species are x and y respectively. The situation is modelled by the simultaneous differential equations

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 2x + 2y$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = 6y - 4x.$$

Part A Second order differential equation

Eliminate y to obtain a second order differential equation for x in terms of t. Give your answer in the form $\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + P\frac{\mathrm{d}x}{\mathrm{d}t} + Qx = 0$, where P and Q are integers.

The following symbols may be useful: `Derivative(_, t)`, `Derivative(_, t, t)`, `t`, `x`

Part B General solution for x

Hence, using the differential equation found in part A, find the general solution for x in terms of t and constants A and B.

The following symbols may be useful: `A`, `B`, `cos()`, `e`, `sin()`, `t`, `tan()`, `x`

Part C General solution for y

Find the corresponding general solution for y in terms of t and the same constants A and B used in part B.

The following symbols may be useful: `A`, `B`, `cos()`, `e`, `sin()`, `t`, `tan()`, `y`

Part D Particular solutions

When $t = 0$, $\frac{\mathrm{d}x}{\mathrm{d}t} = 10$ and the population of species Y is k times the population of species X, where k is a positive constant.

Find the particular solution for x, in terms of t and k.

The following symbols may be useful: `cos()`, `e`, `k`, `sin()`, `t`, `tan()`, `x`

Find the particular solution for y, in terms of t and k.

The following symbols may be useful: `cos()`, `e`, `k`, `sin()`, `t`, `tan()`, `y`

Part E Species dying out

Consider the case where $k = 6$.

Determine whether the model predicts that species X or species Y dies out first.

- ○ Species Y dies out first.
- ○ Species X dies out first.
- ○ Neither of the species die out.
- ○ Both species die out at the same time.

After how many years does this first species die out? Give your answer to 2 s.f.

Part F Reliability of model

Comment on why the time predicted by the model for the second species to die out is unreliable. Fill in the gaps below.

Once one species has died out, the model will start to give [ ] values for that species' population, so the model [ ]. Hence the model's prediction for when the second species dies out is [ ].

Items:

negative | positive | constant | no longer holds | reliable | unreliable | still holds

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Differential Equations

Further A

C C C

Part A Integrating factor

The differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{1}{1-x^2}y = (1-x)^{\frac{1}{2}}, \qquad \text{where } |x| < 1,$$

can be solved by the integrating factor method.

Write an expression for the integrating factor in the form $(g(x))^{\frac{1}{2}}$ where $g(x)$ is a fraction.

The following symbols may be useful: x

Part B Particular solution 1

$$\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{1}{1-x^2}y = (1-x)^{\frac{1}{2}}, \quad |x| < 1$$

Hence find the solution of the differential equation in Part A for which $y = 2$ when $x = 0$, giving your answer in the form $y = f(x)$.

The following symbols may be useful: x, y

Part C Finding k

Find the value of the constant k such that $y = kx^2\mathrm{e}^{-2x}$ is a particular integral of the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 4\frac{\mathrm{d}y}{\mathrm{d}x} + 4y = 2\mathrm{e}^{-2x}.$$

Part D Particular solution 2

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 4\frac{\mathrm{d}y}{\mathrm{d}x} + 4y = 2\mathrm{e}^{-2x}$$

Find the solution of the differential equation in Part C for which $y = 1$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ when $x = 0$. Give your answer in the form $y = f(x)$.

The following symbols may be useful: `e`, `x`, `y`

Part E Finding $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 4\frac{\mathrm{d}y}{\mathrm{d}x} + 4y = 2\mathrm{e}^{-2x}$$

Use the differential equation in Part C to determine the value of $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ when $x = 0$.

Part F Inequality for y

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 4\frac{\mathrm{d}y}{\mathrm{d}x} + 4y = 2\mathrm{e}^{-2x}$$

Hence prove that $a < y \leqslant b$ for $x \geqslant 0$, where a and b are constants to be found.

State the value of a.

State the value of b.

Part G General solution

Find the general solution of the differential equation

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + 4y = 0.$$

Give your answer in the form $y = f(x)$ and in terms of the sum of two single trigonometric functions. Use the letters A and B as your constants.

The following symbols may be useful: `A, B, cos(), sin(), tan(), x, y`

Adapted with permission from UCLES, A Level, Specimen Paper, Paper 4727, Question 4. Part F created for Isaac Physics by Sally A Waugh.