STEM SMART Double Maths 26 - Partial Fractions, Proof & Method of Differences

Partial Fractions 2

Pre-Uni Maths for Sciences A5.2

Subject & topics: Maths | Algebra | Manipulation **Stage & difficulty:** A Level P3

The function $\dfrac{w+2}{(w-1)(w+1)(2w+1)}$ can be written as $\dfrac{A}{(w-1)}+\dfrac{B}{(w+1)}+\dfrac{C}{(2w+1)}$. Using the substitution method find the constants A, B and C.

Part A

Find A

Find the constant A.

The following symbols may be useful: A

Part B

Find B

Find the constant B.

The following symbols may be useful: B

Part C

Find C

Find the constant C.

The following symbols may be useful: C

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Improper Partial Fractions 2

Pre-Uni Maths for Sciences A5.4

Subject & topics: Maths | Algebra | Manipulation **Stage & difficulty:** A Level P3

Express $\dfrac{16x^3+36x^2+2x-25}{4x^2+12x+9}$ as partial fractions.

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Quadratic Partial Fractions 1

Subject & topics: Maths | Algebra | Manipulation **Stage & difficulty:** Further A P2

Express $\dfrac{5x^2-7x+8}{(x-2)(x^2+3)}$ as partial fractions.

The following symbols may be useful: x

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Proof and Odd Perfect Numbers

Subject & topics: Maths | Number | Arithmetic **Stage & difficulty:** A Level P3

The **proper divisors** of a number are those factors which are not equal to the number itself. For the number 6,

- The divisors of 6 are 1, 2, 3 and 6.
- The proper divisors of 6 are 1, 2 and 3.

The number 6 is an example of a **perfect number**. A perfect number is a number for which the sum of its proper divisors is equal to the number itself. For the number 6,

$$1 + 2 + 3 = 6$$

In this question you will use proof by contradiction to show that an odd perfect number cannot be a square number.

Assumption:

We will assume that there is an odd perfect number, n, that is also a square number. Then $n = m^2$, where m is an integer.

Part A
Reasoning: odd and even factors

- An even number multiplied by an even number is always an ______ number.
- An even number multiplied by an odd number is always an ______ number.
- An odd number multiplied by an odd number is always an ______ number.

Therefore, as n is an ______ number, the factors of n can only be ______ numbers.

Items:

odd even

Part B

Reasoning: sum of proper divisors

As $n = m^2$, m is a factor of n.

Consider another factor of n. Call this factor p. As p is a factor of n, $q =$ [ ] is also a factor of n. As $n = m^2$, $q = \frac{m^2}{p}$. Hence,

- If $p < m$, q [ ] m.
- If $p > m$, q [ ] m.

Therefore, with the exception of m, the factors of n occur in pairs. One factor in the pair is smaller than m, and the other factor is larger than m. Including m, the total number of factors of n is therefore an [ ] number.

For any value of n, one of the factor pairs is 1 and n. The number of proper divisors (factors other than n itself) is therefore an [ ] number. As we have shown in part A that all of the factors of n are [ ] numbers, the sum of the proper divisors of n is therefore an [ ] number.

Items:

odd $>$ $\frac{p}{n}$ $\frac{n}{p}$ **even** $<$

Part C
Conclusion

Our starting assumption was that n is an odd perfect number and also a square number.

The definition of a perfect number means that the sum of the proper divisors of n is equal to _____. The sum of the proper divisors must therefore be an _____ number.

However, in part B we have shown that if n is an odd number which is also a square number, the sum of the proper divisors has to be an _____ number.

Therefore, we have arrived at a contradiction. We conclude that there are no odd perfect numbers that are also square numbers.

Items:

n $2n$ n^2 **even** **odd**

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Proof Applied to Surface Areas

Subject & topics: Maths | Number | Arithmetic **Stage & difficulty:** A Level P3

Consider a sphere with a radius r cm, where r is a rational number. Using proof by contradiction, show that the side length of a cube with the same surface area cannot also be a rational number of cm.

Once you have worked out a proof yourself, consider the model proof below. Drag and drop answers into the blank spaces to complete the proof. You may use the same answer more than once.

Assumption:
Consider a sphere of radius r cm, where r is a rational number. Let the side length of a cube with the same surface area as the sphere be a cm. Assume that a is a rational number, in which case $a = \frac{b}{c}$, where b and c are integers with no common factor.

Reasoning:
The surface area of the sphere is [ ]. Because r is a rational number, $r = \frac{p}{q}$, where p and q are integers with no common factor. Hence, the surface area of the sphere may be written as [ ].

The surface area of the cube is [ ]. Using $a = \frac{b}{c}$, the surface area may be written as [ ].

The surface area of the sphere and the cube are equal. Hence, $4\pi\left(\frac{p}{q}\right)^2 = 6\left(\frac{b}{c}\right)^2$. Re-arranging this equation to give an expression for [ ] produces [ ].

As b, c, p and q are all integers, this expression implies [ ] must be [ ] number. However, π is not [ ] number.

Conclusion:
The assumption that a is rational has resulted in a contradiction. Hence, the assumption cannot be true. Therefore, the side length of a cube with the same surface area as a sphere of radius r cm, where r is a rational number, cannot be a rational number of cm.

Items:

π | $4\pi\left(\frac{p}{q}\right)^2$ | a^3 | $6a^2$ | a real | $\pi = \frac{3b^2q^2}{2c^2p^2}$ | an irrational | $6\left(\frac{b}{c}\right)^2$ | $\frac{3b^2q^2}{2c^2p^2}$ | $\pi = \frac{3b^2p^2}{2c^2q^2}$ | $4\pi r^2$ | a rational

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Partial Fractions Applied to Other Functions

Pre-Uni Maths for Sciences A5.5, A5.6 & A5.7

Subject & topics: Maths | Algebra | Manipulation **Stage & difficulty:** A Level C3

Express the following functions in partial fraction form.

Part A

A trigonometric function

Express the function $\dfrac{\cos y}{(\cos y + 1)(2\cos y + 1)}$ in the form $\dfrac{A}{\cos y + 1} + \dfrac{B}{2\cos y + 1}$, where A and B are constants.

The following symbols may be useful: `cos()`, `sin()`, `tan()`, `y`

Part B

An exponential function

Express the function $\dfrac{\mathrm{e}^{2x} + 5}{(\mathrm{e}^x - 1)(\mathrm{e}^x - 2)(\mathrm{e}^x - 3)}$ in the form $\dfrac{A}{\mathrm{e}^x - 1} + \dfrac{B}{\mathrm{e}^x - 2} + \dfrac{C}{\mathrm{e}^x - 3}$, where A, B and C are constants.

The following symbols may be useful: `e`, `x`

Part C

A logarithmic function

Express the function $\dfrac{5\ln z+20}{(\ln z)^2+\ln(z^2)+1}$ in the form $\dfrac{A}{(\ln z+1)^2}+\dfrac{B}{\ln z+1}$, where A and B are constants.

The following symbols may be useful: `ln()`, `log()`, `z`

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Force From Electric Dipole

Subject & topics: Physics | Fields | Electric Fields **Stage & difficulty:** A Level C3

An electric dipole consists of two charges that are equal in size but opposite in sign, with a separation between them. The diagram below shows an electric dipole PQ. P has charge $-q$ and Q has charge $+q$, and the separation between P and Q is $2a$. Another charge, S, is near to the dipole. S is in line with the axis of the dipole and a distance r from the dipole's centre.

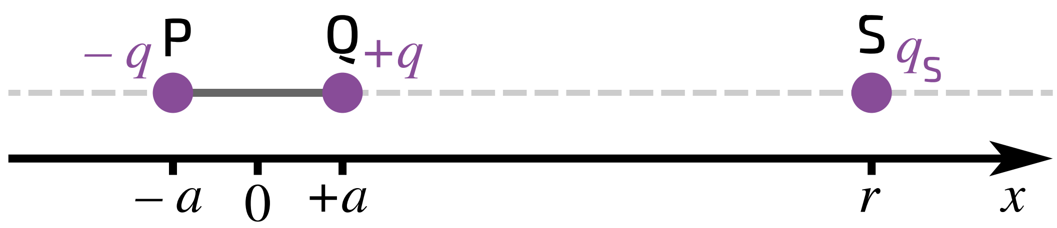

Figure 1: An electric dipole PQ and a charge S.

The resultant force on charge S is the sum of the force on S from P and the force on S from Q. For a particular value of q_S, the resultant force is given by the expression

$$F_{\text{res}} = \frac{-3q^2}{4\pi\varepsilon_0}\frac{ar}{(r^2-a^2)^2}$$

where ε_0 is a constant. In this question you will use partial fractions to split F_{res} into two terms, and hence find an expression for q_S in terms of q.

Part A

Splitting into terms - A

In general, a rational function with a denominator of $4\pi\varepsilon_0(r^2 - a^2)^2$ would produce four terms when written in terms of partial fractions:

$$\frac{A}{4\pi\varepsilon_0\,(r+a)^2} + \frac{B}{4\pi\varepsilon_0\,(r-a)^2} + \frac{C}{4\pi\varepsilon_0\,(r+a)} + \frac{D}{4\pi\varepsilon_0\,(r-a)}$$

However, if the expression for F_{res} is written in terms of partial fractions, it turns out that two of the coefficients (C and D) are both 0.

Write the expression for F_{res} in the form $\dfrac{A}{4\pi\varepsilon_0(r+a)^2} + \dfrac{B}{4\pi\varepsilon_0(r-a)^2}$, where A and B are constants which depend on q.

Enter your expression for A.

The following symbols may be useful: `A, a, pi, q, varepsilon_0`

Part B

Splitting into terms - B

Write the expression for F_{res} in the form $\dfrac{A}{4\pi\varepsilon_0(r+a)^2} + \dfrac{B}{4\pi\varepsilon_0(r-a)^2}$, where A and B are constants which depend on q.

Enter your expression for B.

The following symbols may be useful: `B, a, pi, q, varepsilon_0`

Part C

Finding q_S

The force between two particles with electric charges q_1 and q_2 separated by a distance d is given by

$$F = \frac{q_1 \, q_2}{4\pi\varepsilon_0 \, d^2}$$

where ε_0 is a constant.

Using your answers to parts A and B, or otherwise, find an expression for the charge on S, q_S, in terms of q.

The following symbols may be useful: `a, pi, q, q_S, varepsilon_0`

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Series: Method of Differences 2i

Subject & topics: Maths **Stage & difficulty:** Further A P2

Part A

$(r+2)! - (r+1)!$

Show that $(r+2)! - (r+1)! = f(r) \times r!$ where $f(r)$ is a function to be found.

What is $f(r)$?

The following symbols may be useful: r

Part B

Expression for a series

Hence find an expression, in terms of n, for

$$2^2 \times 1! + 3^2 \times 2! + 4^2 \times 3! + \ldots + (n+1)^2 \times n!$$

Your answer can be written as $g(n)! - 2$.

What is $g(n)$?

The following symbols may be useful: n

Part C

Convergence

State, giving a brief reason, whether the series

$$2^2 \times 1! + 3^2 \times 2! + 4^2 \times 3! + \dots$$

converges. Fill in the gaps in the argument (you can use an item more than once).

We can express this series as a summation as [ ]. This is the limit of the partial sum [ ] as $n \to$ [ ].

From Part A we can write the partial sum as [ ], and from Part B we know that the partial sum evaluates to [ ].

As $n \to \infty$, the partial sum tends to [ ], so the series [ ] converge.

Items:

$\sum_{r=1}^{\infty} (r+1)^2 r!$ | $\sum_{r=1}^{\infty} r^2 (r+1)!$ | $\sum_{r=1}^{n} (r+1)^2 r!$ | $\sum_{r=1}^{n} r^2 (r+1)!$ | 0 | 1 | ∞ | $\sum_{r=1}^{n} [(r+2)! - (r+1)!]$ | $\sum_{r=1}^{n} [(r+1)! - r!]$ | $(n+2)! - 2$ | $(n+1)! - 1$ | **does** | **does not**

Adapted with permission from UCLES, A Level, January 2007 , Paper 4725, Question 8.

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Series: Method of Differences 1i

Subject & topics: Maths **Stage & difficulty:** Further A P2

Part A

Rewriting a fraction

Express $\frac{2}{r} - \frac{1}{r+1} - \frac{1}{r+2}$ as a single fraction.

The following symbols may be useful: r

Part B

Sum of a series

Hence find an expression, in terms of n, for

$$\sum_{r=1}^{n} \frac{3r+4}{r(r+1)(r+2)}.$$

The following symbols may be useful: n

Part C

Limit as $n \to \infty$

Hence write down the value of

$$\sum_{r=1}^{\infty} \frac{3r+4}{r(r+1)(r+2)}.$$

Part D

Solve for N

Given that

$$\sum_{r=N+1}^{\infty} \frac{3r+4}{r(r+1)(r+2)} = \frac{7}{10}$$

find the value of N.