Complex Numbers: Equations to Quartics 3i

Further A

P P P

Part A $z^4 = 16$

The roots of the equation $z^4 = 16$ can be written in the form $x + iy$.

Give the solution with positive x.

The following symbols may be useful: `i`

Give the solution with negative x.

The following symbols may be useful: `i`

Give the solution with positive y.

The following symbols may be useful: `i`

Give the solution with negative y.

The following symbols may be useful: `i`

Part B **Another quartic**

The previous part can be used to solve the equation $w^4 = 16(1-w)^4$. The solutions can be written in the form $x + iy$.

Of the two roots that are purely real - that is, $\mathrm{Im}(w) = 0$ - give the larger root.

Of the two roots that are purely real - that is, $\mathrm{Im}(w) = 0$ - give the smaller root.

The other two roots are complex, and can be written in the form $x + iy$.

Give the complex root with positive y.

The following symbols may be useful: `i`

Give the complex root with negative y.

The following symbols may be useful: `i`

Adapted with permission from UCLES, A Level, Jan 2010, Paper 4727, Question 4.

Physics. You work it out.

Argand Diagrams: Solving Inequalities 3i

Further A

P P P

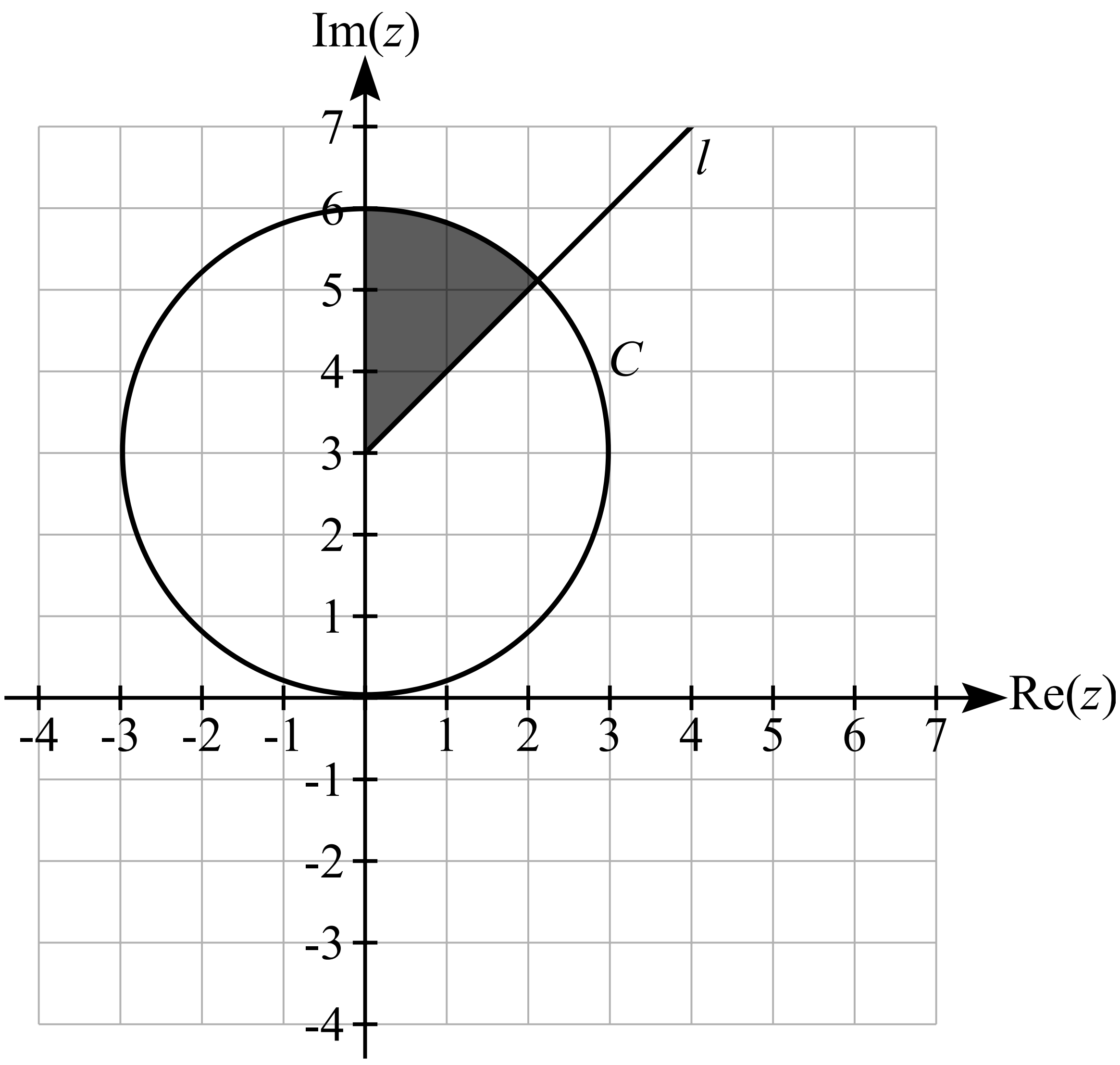

Figure 1: C and l are shown on a single Argand diagram.

The Argand diagram above shows a half-line l and a circle C. The circle has centre $3i$ and passes through the origin.

Part A Equation of C

Give the equation of C in the form

$$|z - z_1| = a$$

where z_1 is complex and a is real.

The following symbols may be useful: `i`, `z`

Part B Equation of l

The equation of l can be written in the form

$$\arg(z - z_1) = a\pi$$

where z_1 is complex and a is real.

Give z_1 in the form $x + iy$.

The following symbols may be useful: `i, pi, z_2`

Give the value of a in exact form.

The following symbols may be useful: `a`

Part C Inequalities

The shaded region is defined by two inequalities. [The shaded region includes the boundaries.]

They are in the form

$$|z - z_1| \text{ inequality } a$$

and

$$b\pi \leq \arg(z - z_2)$$

$$\arg(z - z_2) \leq c\pi$$

where inequality indicates either $<$, $>$, $\geq$ or $\leq$.

Give the full expression for the first inequality.

The following symbols may be useful: `<`, `<=`, `>`, `>=`, `i`, `z`

Give the value of b in the second inequality.

The following symbols may be useful: `b`

Give the value of c in the third inequality.

The following symbols may be useful: `c`

Adapted with permission from UCLES, A Level, June 2013, Paper 4725, Question 6.

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Complex Numbers: De Moivre 4ii

Further A
P P P

Part A $\cos^6\theta$

By expressing $\cos\theta$ in terms of $\mathrm{e}^{\mathrm{i}\theta}$, write $\cos^6\theta$ in terms of $\cos(n\theta)$.

Give your answer in the form

$$\cos^6\theta = \mathrm{f}(\cos 6\theta, \cos 4\theta, \cos 2\theta)$$

The following symbols may be useful: `cos()`, `theta`

Part B Solutions to an equation

Hence solve, for $0 \leqslant \theta \leqslant \pi$,

$$\cos 6\theta + 6\cos 4\theta + 2\cos 2\theta = 3.$$

Give your solutions in increasing order, in radians, to 3 significant figures.

What is θ_1?

What is θ_2?

What is θ_3?

Algebra and Roots: Cubics with Substitution 3i

Further A

The cubic equation $x^3 + 2x^2 + 3x + 4 = 0$ has roots α, β and γ.

Part A Substitution

Use the substitution $x = \frac{1}{u+1}$ to find a cubic equation in u in the form $au^3 + bu^2 + cu + d = 0$ where a, b, c and d are integers.

The following symbols may be useful: u

Part B $\left(\frac{1}{\alpha} - 1\right)\left(\frac{1}{\beta} - 1\right)\left(\frac{1}{\gamma} - 1\right)$

Hence, find the value of $\left(\frac{1}{\alpha} - 1\right)\left(\frac{1}{\beta} - 1\right)\left(\frac{1}{\gamma} - 1\right)$ as a single fraction.

Adapted with permission from UCLES, A Level, June 2018, Paper 4725, Question 5.

Matrices - Linear Equations 1

Further A University

Use matrix notation to solve the following set of three equations:

$$\begin{aligned} x+ky-z&=3\\ 3x+ky&=1\\ -x+4y+z&=3. \end{aligned}$$

where k is a constant.

Part A Matrix form

Write these equations in matrix form $\mathbf{A}\mathbf{x}=\mathbf{b}$.

If the matrix $\mathbf{A}$ is written in the form

$$\mathbf{A}=\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

give the values of the following elements of the matrix $\mathbf{A}$.

Give the value of a_{12}.

The following symbols may be useful: k

Give the value of a_{21}.

Give the value of a_{23}.

Give the value of a_{32}.

Part B No unique solution

Find an expression for the condition that there is no unique solution to this set of equations.

The following symbols may be useful: k

Part C The inverse matrix

Find $\mathbf{A}^{-1}$, the inverse of $\mathbf{A}$.

If the matrix $\mathbf{A}^{-1}$ is written in the form

$$\mathbf{A}^{-1} = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix}$$

give expressions for the following elements of $\mathbf{A}^{-1}$.

Give an expression for α_{13}.

The following symbols may be useful: k

Give an expression for α_{21}.

The following symbols may be useful: k

Give an expression for α_{31}.

Give an expression for α_{32}.

The following symbols may be useful: k

Part D Solution to the set of equations

Using $\mathbf{A}^{-1}$, find the solutions for x, y and z in terms of k.

Find an expression for x.

The following symbols may be useful: k

Find an expression for y.

The following symbols may be useful: k

Find an expression for z.

The following symbols may be useful: k

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Matrices - Transformations 2

Further A University
P P P P P P

$\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ are 3×3 matrices such that $\mathbf{C} = \mathbf{BA}$ and

$$\mathbf{B} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and

$$\mathbf{C} = \begin{pmatrix} p & 0 & q \\ 0 & r & 0 \\ s & 0 & t \end{pmatrix}$$

Part A Matrix $\mathbf{B}^{-1}$

Find $\mathbf{B}^{-1}$. Give expressions for the elements of $\mathbf{B}^{-1}$ on the leading diagonal i.e. β_{11}, β_{22} and β_{33}.

Give an expression for β_{11}.

The following symbols may be useful: k

Give an expression for β_{22}.

The following symbols may be useful: k

Give an expression for β_{33}.

Part B Matrix $\mathbf{A}$

Use $\mathbf{B}^{-1}$ to deduce the form of the matrix $\mathbf{A}$.

Give your answer by writing the elements in each row in brackets in the form (a_{m1},a_{m2},a_{m3}) where $m = 1, 2$ or 3. Thus, if $a_{21} = 1$, $a_{22} = 2$ and $a_{23} = 0$, type: (1,2,0) with no spaces.

Give the elements in the top row ($m = 1$) of the matrix, writing them in the form indicated above.

Give the elements in the second row ($m = 2$) of the matrix, writing them in the form indicated above.

Give the elements in the bottom row ($m = 3$) of the matrix, writing them in the form indicated above.

Part C Transformation produced by $\mathbf{A}$

If the matrix $\mathbf{A}$ represents rotation anticlockwise about the y-axis through an angle D deduce the expressions for p, q, r, s and $\frac{r}{k}$.

Deduce the expression for p.

The following symbols may be useful: `D, cos(), k, sin(), tan()`

Deduce an expression for q.

The following symbols may be useful: `D, cos(), k, sin(), tan()`

Deduce an expression for $\frac{r}{k}$.

The following symbols may be useful: `cos(), k, r, sin(), tan()`

Deduce an expression for s.

The following symbols may be useful: `D, cos(), k, sin(), tan()`

Deduce an expression for t.

The following symbols may be useful: `D, cos(), k, sin(), tan()`

Part D Reflection in the z=0 plane

If $\mathbf{C}$ represents reflection in the $z = 0$ plane deduce the values of r and D.

Deduce the value of r.

Deduce the value of the angle D. Give your answer in radians and assume $0 \leq D < 2\pi$.

The following symbols may be useful: `pi`

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Polar Coordinates: General 3i

Further A

The equation of a curve, in polar coordinates, is

$$r = \sqrt{3} + \tan\theta, \quad \text{for } -\frac{1}{3}\pi \leqslant \theta \leqslant \frac{1}{4}\pi.$$

Part A Tangent at the pole

Find the equation of the tangent at the pole in the form $\theta = \alpha$.

The following symbols may be useful: `pi`, `theta`

Part B Greatest value of r

State the greatest value of r.

Part C Corresponding value of θ

State the value of θ at which r takes its greatest value.

The following symbols may be useful: `pi`

Part D Sketch the curve

Sketch the curve.

Which curve in **Figure 1** most resembles your sketch?

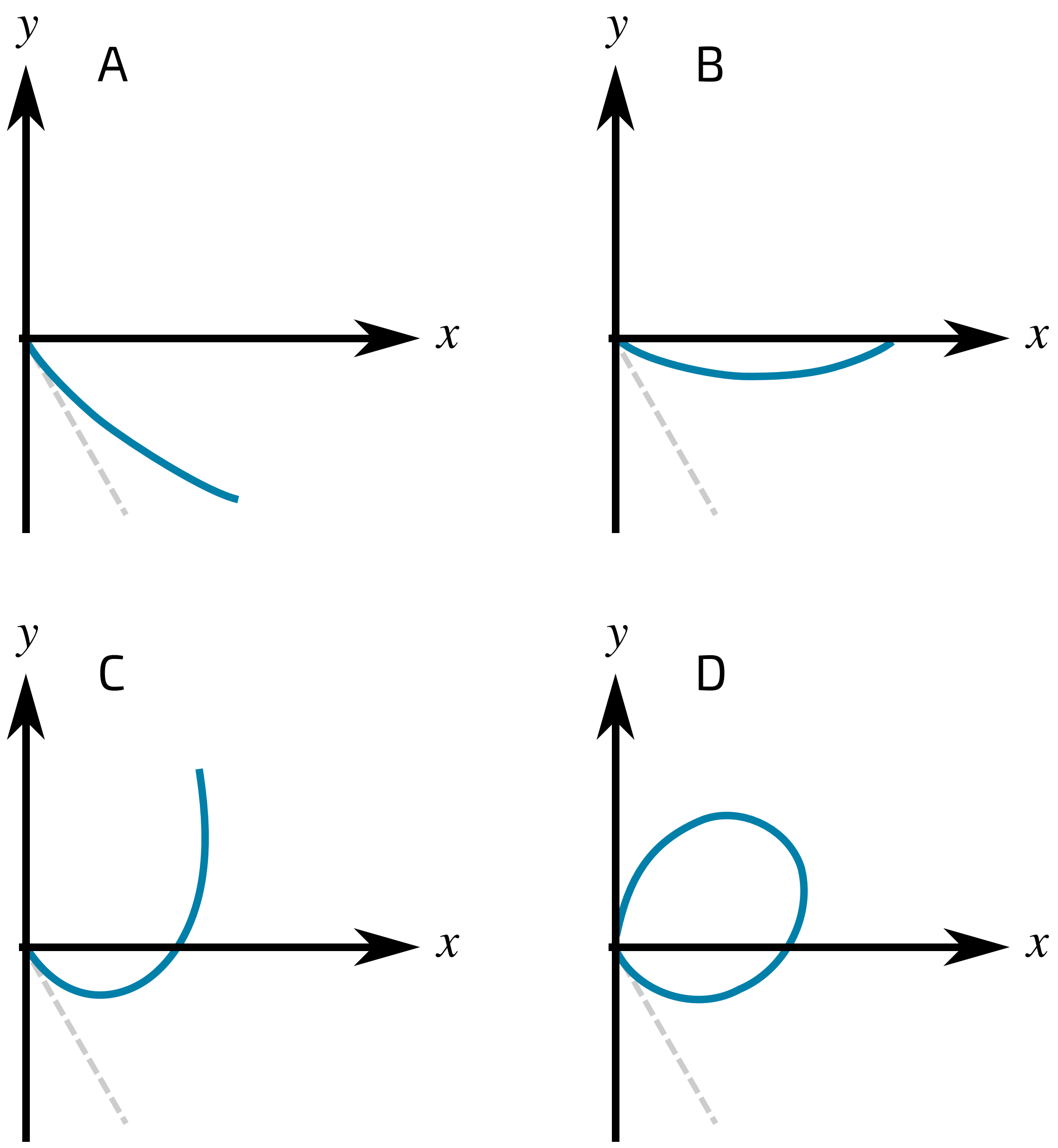

Figure 1: Four curves.

- ○ Curve A
- ○ Curve B
- ○ Curve C
- ○ Curve D

Part E Area of region

Given that

$$\int \tan x \,\mathrm{d}x = \ln|\sec x| + C,$$

find the exact area of the region enclosed by the curve and the lines $\theta = 0$ and $\theta = \frac{1}{4}\pi$.

The following symbols may be useful: `ln()`, `log()`, `pi`

Adapted with permission from UCLES, A Level, June 2006, Paper 4726, Question 7.

Vectors: Lines and Planes 3i

Further A

The plane Π passes through the points $(1, 2, 1)$, $(2, 3, 6)$ and $(4, -1, 2)$.

Part A Cartesian equation of Π

Find a cartesian equation of the plane Π.

Give your answer in the form $ax + by + cz = 19$.

The following symbols may be useful: `x`, `y`, `z`

Part B Intersection of l and Π

The line l has equation $r = \begin{pmatrix} -1 \\ -2 \\ 6 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ 3 \\ -2 \end{pmatrix}$.

Find the value of λ at the point of intersection of Π and l.

Part C Angle between l and Π

Find the acute angle between Π and l.

Give your answer in degrees to 3 significant figures.

Adapted with permission from UCLES, A Level, June 2015, Paper 4727, Question 3.

Vectors: Geometry 2i

Further A

P P P

(In this question, the notation $\triangle ABC$ denotes the area of the triangle ABC.)

The vector product of two vectors $\underline{\boldsymbol{p}}$ and $\underline{\boldsymbol{q}}$ is given by $\underline{\boldsymbol{p}} \times \underline{\boldsymbol{q}} = |\underline{\boldsymbol{p}}||\underline{\boldsymbol{q}}| \sin\theta \underline{\hat{\boldsymbol{n}}}$ where θ is the angle between $\underline{\boldsymbol{p}}$ and $\underline{\boldsymbol{q}}$, with $0 \leq \theta \leq \pi$, and $\underline{\hat{\boldsymbol{n}}}$ is a unit vector perpendicular to both $\underline{\boldsymbol{p}}$ and $\underline{\boldsymbol{q}}$ in the right-handed sense.

The points P, Q and R have position vectors $p\underline{\boldsymbol{i}}, q\underline{\boldsymbol{j}}$ and $r\underline{\boldsymbol{k}}$ respectively, relative to the origin O, where p, q and r are positive. The points O, P, Q and R are joined to form a tetrahedron.

Part A Sketch tetrahedron

Draw a sketch of the tetrahedron.

Which of these sketches is correct?

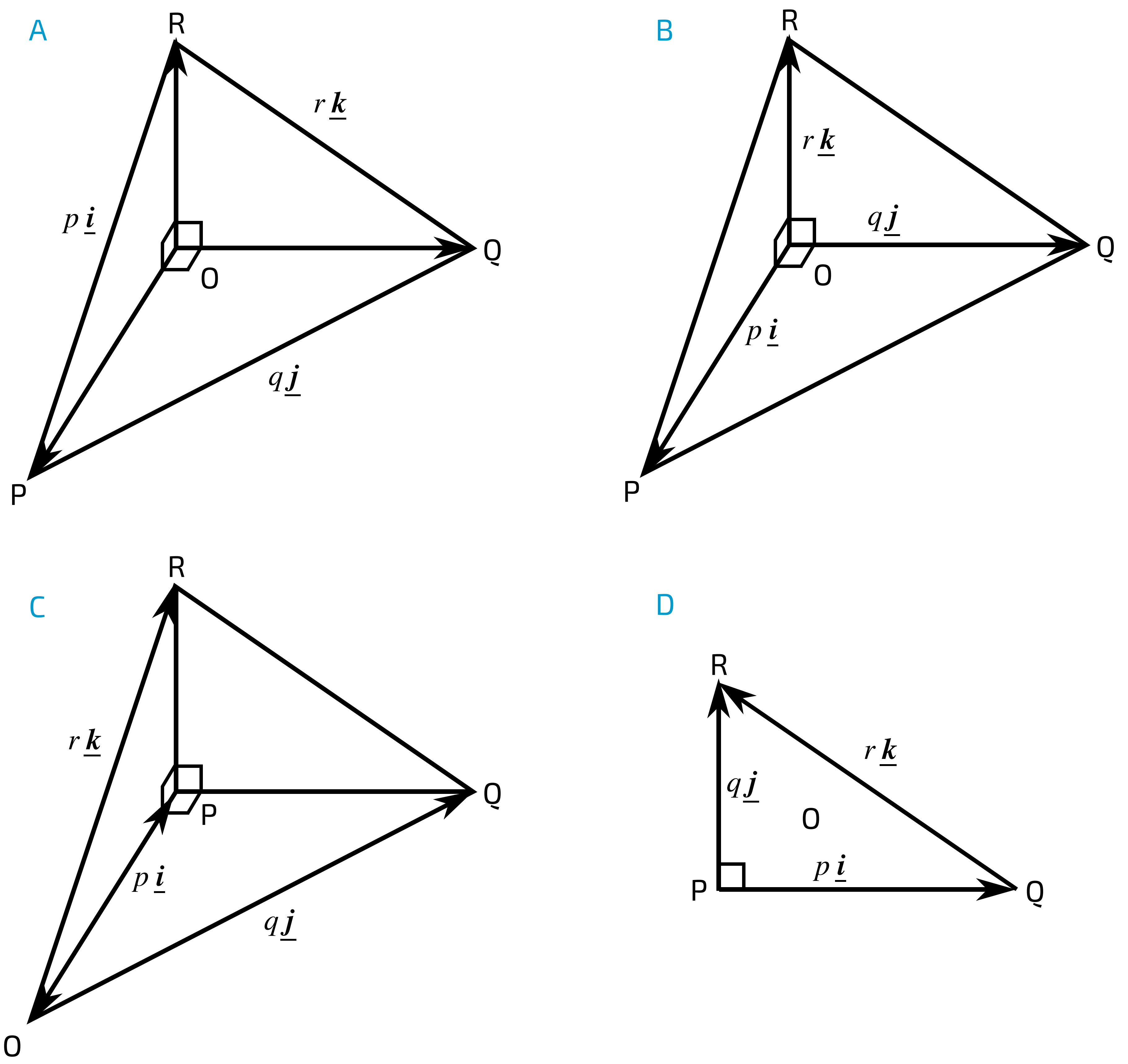

Figure 1: Four sketches.

- ○ Sketch A
- ○ Sketch B
- ○ Sketch C
- ○ Sketch D

Part B Triangle areas

Write down the values of $\triangle OPQ, \triangle OQR$ and $\triangle ORP$.

What is $\triangle OPQ$?

The following symbols may be useful: `p, q, r`

What is $\triangle OQR$?

The following symbols may be useful: `p, q, r`

What is $\triangle ORP$?

Part C Vector product and area

Use the definition of the vector product to show that $k|\overrightarrow{RP} \times \overrightarrow{RQ}|$ is equal to the area of one of the tetrahedron's faces, where k is a constant to be found.

Which area is $k|\overrightarrow{RP} \times \overrightarrow{RQ}|$ equal to?

- $\triangle OQR$
- $\triangle OPQ$
- $\triangle ORP$
- $\triangle PQR$

What is the value of k?

Part D Relationship between areas

Show that we can find an equation of the form

$$(\triangle OPQ)^2 + (\triangle OQR)^2 + (\triangle ORP)^2 = \alpha(\triangle PQR)^2$$

where α is a constant to be found.

What is α?

Adapted with permission from UCLES, A Level, June 2011, Paper 4727, Question 7.