Probability 5.5

A Level P P P Further A P P P

Assuming they all follow a normal distribution find the following probabilities.

Part A $X \sim \mathrm{N}(10, 4)$

If $X \sim \mathrm{N}(10, 4)$ find the following probabilities, giving your answers to 3 sf:

$\mathrm{P}(X \leq 9) =$

$\mathrm{P}(X \geq 15) =$

Part B $X \sim \mathrm{N}(15, 0.3)$

If $X \sim \mathrm{N}(15, 0.3)$ find the following probabilities, giving your answers to 3 sf:

$\mathrm{P}(X < 14) =$

$\mathrm{P}(14 < X < 16) =$

Part C S normally distributed

Assuming that the variable S follows a normal distribution with mean 40 and variance 16 find the following probabilities, giving your answers to 3 sf:

$\mathrm{P}(S > 44) =$

$\mathrm{P}(|S - 40| < 5) =$

Part D Points of inflection

The points of inflection of a normal distribution curve occur at -2 and 6. Deduce the mean and variance of the distribution.

The mean is and the variance is .

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Probability 5.8

A Level P P P Further A P P P

The number of gamma-rays emitted by a radioactive sample in a given time period follows a normal distribution and has a mean of 500.8 and standard deviation of 7.1. Find the following.

Part A 30% of the measurements $< p$

30% of the measurements have a value less than p; find the value of p. Give your answer to 4 sf.

Part B 15% of the measurements $> q$

15% of the measurements have values greater than q; find the value of q. Give your answer to 4 sf.

Part C Six successive measurements

Six measurements are made successively.

Find the probability that none of the 6 measurements are greater than 515. Give your answer to 3 sf.

Find the probability that exactly 4 of the 6 are less than 510. Give your answer to 3 sf.

Probability 5.9

A Level P P P Further A P P P

Answer the following questions. You may assume in all cases that the values of the quantities follow a normal distribution.

Part A A batch of batteries

A batch of batteries has a mean voltage of $1.50\,\mathrm{V}$ and standard deviation σ. It is found that 20% have a voltage less than $1.47\,\mathrm{V}$. Deduce the value of σ, giving your answer to 3 sf.

Part B Falling times

A number of students measure the time it takes for an object to fall a certain distance. The distribution of times has a standard deviation of $0.095\,\mathrm{s}$. It is found that 10% of the measurements exceed $5.80\,\mathrm{s}$. Find the mean time taken, giving your answer to 3 sf.

Part C A batch of lenses

An experimenter has a batch of 150 lenses. They find that 10 of them have focal lengths less than $14.7\,\mathrm{cm}$ and 6 have focal lengths greater than $15.6\,\mathrm{cm}$.

Find the mean and the variance of the focal lengths, giving your answers to 3 sf.

The mean is ____ and the variance is ____.

Data Analysis 5.4

A Level
C C C

The distribution of the velocities of gas molecules in one dimension is given by a normal distribution with mean zero and variance which is proportional to the temperature of the gas.

At room temperature, $T_1 = 300\,\mathrm{K}$, 20% of the molecules have speeds greater than $370\,\mathrm{m\,s^{-1}}$. At a higher temperature, T_2, 30% of the molecules have speeds greater than $370\,\mathrm{m\,s^{-1}}$. Answer the following questions and hence deduce the value of T_2.

Part A σ_1 at T_1

At room temperature, $T_1 = 300\,\mathrm{K}$, 20% of the molecules have speeds greater than $370\,\mathrm{m\,s^{-1}}$; find the value of the standard deviation in the velocities σ_1. Give your answer to 2 s.f.

Part B σ_2 at T_2

At T_2, 30% of the molecules have speeds greater than $370\,\mathrm{m\,s^{-1}}$; find the value of the standard deviation in the velocities σ_2. Give your answer to 2 s.f.

Part C The value of T_2

Use your answers to parts A and B and the information given above to deduce the temperature T_2. Give your answer to 2 s.f.

Hypothesis Testing: Normal Distribution 1

A Level
P P P

A species of chicken is known to produce eggs which have a mass that follows a normal distribution with a mean mass of $55.0\,\mathrm{g}$ and a variance of $5.00\,\mathrm{g}^2$.

A chicken-keeper has 1000 birds of this species. Each bird lays one egg per day, and on one particular day the chicken-keeper collects the eggs from 125 birds and finds a mean mass of $54.0\,\mathrm{g}$. The chicken-keeper wants to know whether their birds produce eggs with a below average mass.

Test, at the 5% significance level, whether this is the case.

Part A Assumptions

In order to carry out a hypothesis test, which of the following do you need to assume?

- The variance of the masses of the eggs produced by the chicken-keeper's birds is the same as that of the whole chicken population.
- The variance of the masses of the eggs produced by the chicken-keeper's birds is 1000 times smaller than that of the population as the chicken keeper has 1000 birds.
- The variance of the masses of the eggs produced by the chicken-keeper's birds is 1000 times that of the population as the chicken keeper has 1000 birds.

Part B The null and alternative hypotheses

Drag and drop into the spaces below to define the variables and state the null and alternative hypotheses for this test.

Let M be the mass of an egg produced by the chicken-keeper's birds, and let μ be the mean mass of the eggs produced by the chicken-keeper's birds. Then $M \sim$ ____ $(\mu,$ ____ $)$.

H_0: ____ ____ ____

H_1: ____ ____ ____

Items:

$=$ | $55.0\,\mathrm{g}$ | Po | $>$ | $<$ | $54.0\,\mathrm{g}$ | B | μ | 5.00 | N

Part C The distribution of the sample means

Fill in the blanks below to complete the description of the distribution of the sample means.

The masses of the eggs produced by the chicken-keeper's birds have a ____ distribution. Therefore, if samples of these eggs are taken, the mean values of these samples, $\overline{M}$, have a ____ distribution.

The masses of the eggs produced by the chicken-keeper's birds have a variance of ____ g^2. For samples containing 125 eggs, the variance of the sample is therefore ____ g^2.

Hence, under the assumption that the null hypothesis is true, the sample distribution is $\overline{M} \sim$ ____ $($ ____ $,$ ____ $)$.

Items:

55.0 | 5.00 | **normal** | N | B | 0.005 | **binomial** | 0.04

Part D Carrying out the test

Choose three options and put them into order to complete the hypothesis test.

Available items

$z = \dfrac{\overline{M} - \mu}{\sqrt{\dfrac{\sigma^2}{n}}}$. Hence, the boundary of the critical region is at $\overline{M} = 55.0 + 0.2 \times 1.645 = 55.33$. The critical region is therefore $\overline{M} < 55.33$.

$z = \dfrac{\overline{M} - \mu}{\sqrt{\dfrac{\sigma^2}{n}}}$. Hence, the boundary of the critical region is at $\overline{M} = 55.0 + 0.2 \times -1.645 = 54.67$. The critical region is therefore $\overline{M} < 54.67$.

Therefore, as the value of the sample mean $(54.0\,\mathrm{g})$ lies in the acceptance region, we do not reject the null hypothesis. There is no significant evidence that the chicken-keeper's birds produce eggs with a below average mass.

We need to carry out a one-tailed test at the 5% significance level. Using the inverse normal distribution, the z-value at the upper boundary of the critical (rejection) region is $z = \Phi^{-1}(0.05) = -1.645$.

We need to carry out a one-tailed test at the 5% significance level. Using the inverse normal distribution, the z-value at the upper boundary of the critical (rejection) region is $z = \Phi^{-1}(0.95) = 1.645$.

Therefore, as the value of the sample mean $(54.0\,\mathrm{g})$ lies in the critical region, we reject the null hypothesis. There is evidence that the chicken-keeper's birds produce eggs with a below average mass.

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Hypothesis Testing: Normal Distribution 2

A Level

P P P

Using a single black ink cartridge, a specific type of large printer is known to print a mean of $10\,750$ pages of black and white text, with a variance of $90\,000\,\mathrm{pages}^2$.

The company that makes the printer changes some of its software in an attempt to make the use of ink by their printers more efficient. They perform 80 tests on printers using the updated software and calculate a mean of $10\,834$ pages per ink cartridge.

Test at the 2% significance level whether the new software has improved the ink efficiency.

Part A Assumptions

Which of the following options do you need to assume in order to perform a hypothesis test? Select all that apply.

- [ ] That the number of pages printed using one ink tank has a normal distribution.
- [ ] That the variance of the sample obeys a Poisson distribution.
- [ ] That the variance in the number of pages printed in the sample of 80 tests is much smaller than the variance when using the old software, because 80 is much smaller than the number of printers using the old software.
- [ ] That the variance in the number of pages printed using the new software is the same as the variance when using the old software.

Part B Carrying out the test

Fill in the blanks to complete the description of the hypothesis test.

Let X be the number of pages printed using one cartridge with the new software. Then $X \sim \mathrm{N}(\mu, \boxed{})$, where μ is the mean.

The null hypothesis is that the new software does not improve ink efficiency, and the mean number of pages per cartridge is the same as before. The alternative hypothesis is that the new software improves ink efficiency.

$$\mathrm{H}_0 : \mu = 10\,750 \qquad \mathrm{H}_1 : \boxed{}$$

Let $\overline{X}$ be the mean number of pages printed using one cartridge with the new software. Assuming that the null hypothesis is true, then $\overline{X} \sim \mathrm{N}(10\,750, \frac{\boxed{}}{80})$.

Using the z-statistic $z = \dfrac{\overline{X} - \mu}{\sqrt{\frac{\sigma^2}{n}}}$, the p-value for a sample mean of $10\,834$ pages is found to be

$$p = \mathrm{P}(\overline{X} \geq 10\,834) = \mathrm{P}\left(z \geq \frac{10\,834 - 10\,750}{\sqrt{\frac{\boxed{}}{80}}}\right) = \boxed{}$$

For a one-tailed test at the 2% significance level, p-values of less than 0.02 are in the critical (rejection) region. The calculated p value for the sample is $\boxed{}$ 0.02.

Therefore, $\boxed{}$ the null hypothesis. There $\boxed{}$ significant evidence that the new software improves ink efficiency.

Items:

is not | less than | do not reject | $\mu > 10\,750$ | 0.3897 | $90\,000$ | $\mu < 10\,750$ | is | greater than | reject | 300 | 0.006133

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Gameboard:

STEM SMART Double Maths 34 - The Normal Distribution & Combining Variables

Expectation and Variance

Further A

P P P

A discrete random variable X has the following probability distribution:

x	0	1	2	3	4
$\mathsf{P}(X=x)$	$\frac{1}{6}$	$\frac{1}{6}$	k	$2k$	$\frac{1}{3}-k$

Part A Find k

Find the value of k.

The following symbols may be useful: `k`

Part B Expectation of X

Find $\mathrm{E}(X)$.

The following symbols may be useful: `E, X`

Part C Variance of X

Find $\mathrm{Var}(X)$.

Part D Expectation of Y

Another discrete random variable Y is defined as $Y = 2X - 3$.

Find $\mathrm{E}(Y)$.

The following symbols may be useful: `E`, `Y`

Part E Variance of Y

Find $\mathrm{Var}(Y)$.

Part F Probability that $X > Y$

Find $\mathrm{P}(X > Y)$.

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Gameboard:
STEM SMART Double Maths 34 - The Normal Distribution & Combining Variables

Combining Random Variables

Further A

C C C

R is a discrete random variable. S is another discrete random variable and is defined by $S = \frac{3R+2}{4}$. It is given that $\mathrm{E}(S) = -2$ and $\mathrm{Var}(S) = 9$.

Part A Expectation of R

Find $\mathrm{E}(R)$.

The following symbols may be useful: `E`, `R`

Part B Variance of R

Find $\mathrm{Var}(R)$.

Part C Expectation of S and T

The random variable T is such that $\mathrm{E}(T) = 5$ and $\mathrm{Var}(T) = 21$.

Find $\mathrm{E}(3S - 2T)$.

The following symbols may be useful: `E`, `S`, `T`

Part D Variance of S and T

Find $\mathrm{Var}(3S - 2T)$.

Part E Expectation of R and T

Find $\mathrm{E}(R_1 + R_2 + R_3 + \frac{T}{3})$, where R_1, R_2 and R_3 indicate independent readings of R.

Part F Variance of R and T

Find $\mathrm{Var}(R_1 + R_2 + R_3 + \frac{T}{3})$, where R_1, R_2 and R_3 indicate independent readings of R.

Part G Variance of $3R$ and T

Find $\mathrm{Var}(3R + \frac{T}{3})$.

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Combining Variables: Laminate

Further A P P P

A laminate consists of 4 layers of material C and 3 layers of material D. The thickness of a layer of material C has a normal distribution with mean $1\,\mathrm{mm}$ and standard deviation $0.1\,\mathrm{mm}$, and the thickness of a layer of material D has a normal distribution with mean $8\,\mathrm{mm}$ and standard deviation $0.2\,\mathrm{mm}$. The layers are independent of one another.

Part A Mean

Find the mean of the total thickness of the laminate.

Part B Variance

Find the variance of the total thickness of the laminate.

Part C 1% of the laminates

What total thickness is exceeded by 1% of the laminates? Give your answer to 3 s.f.

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Combining Normal Variables

Further A

P P P

The independent random variables X and Y have distributions $\mathrm{N}(30, \sigma^2)$ and $\mathrm{N}(20, \sigma^2)$ respectively. The random variable $aX + bY$, where a and b are constants, has the distribution $\mathrm{N}(410, 130\sigma^2)$.

Part A Values of a and b

It is given that a and b are integers. Find the values of a and b.

$a =$

$b =$

Part B Value of σ^2

Given that P$(X > Y) = 0.966$, find σ^2. Give your answer to 3 sf.

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