Fluid Mechanics & Aviation Physics

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Isaac Physics Project

Contents

Conversion Factors

1 ft	=	1 foot
1 ml	=	1 statute (ordinary) mile
1 nm	=	1 nautical mile
1 mph	=	1 [statute] mile per hour
1 kt	=	1 knot = 1 nautical mile per hour

Distances

1 metre	=	0.001 km	=	3.281 ft	=	0.00054 nm	=	0.0006214 ml
1 km	=	1000 m	=	3281 ft	=	0.5400 nm	=	0.6214 ml
1 ft	=	0.3048 m	=	0.0003048 km	=	0.0001646 nm	=	0.0001894 ml
1 nm	=	1852 m	=	1.852 km	=	6076 ft	=	1.151 ml
1 mile	=	1609 m	=	1.609 km	=	5280 ft	=	0.8690 nm

Speeds

1 m/s	=	1.944 kt	=	2.237 mph	=	3.600 km/h	=	196.9 ft/min
1 kt	=	0.5144 m/s	=	1.151 mph	=	1.852 km/h	=	101.3 ft/min
1 mph	=	0.4470 m/s	=	0.8690 kt	=	1.609 km/h	=	88.00 ft/min
1 km/h	=	0.2778 m/s	=	0.5400 kt	=	0.6214 mph	=	54.68 ft/min
100 ft/min	=	0.5080 m/s	=	0.9875 kt	=	1.136 mph	=	1.829 km/h

Acknowledgements

It is a particular pleasure, when coming near to the end of a writing assignment, to offer thanks to a number of individuals and organizations who have been of particular help. The Preacher of Ecclesiastes wrote *there is nothing new under the sun*, and that is certainly going to be true of a small book outlining some of the basics of fluid mechanics and aerodynamics. Therefore, I am going to be particularly endebted to those who have taught me, and whose works I have found to be very helpful.

Firstly, I would like to thank the authors of the following books which are, in my view, excellent. Should the reader wish to explore further, I would encourage them to start here:

- C. Carpenter *Flightwise: Principles of Aircraft Flight* Airlife, 1996
- C. Carpenter *Flightwise: Aircraft Stability and Control* Airlife, 1997
- R.V. Churchill *Complex Variables and Applications* McGraw-Hill, 1960
- R.E. Collins *Mathematical Methods for Physicists and Engineers* Dover, 1999
- A.C. Kermode *Flight without Formulae* Pitman, 1970
- A.C. Kermode *Mechanics of Flight* Longman, 1987
- B.S. Massey *Mechanics of Fluids* Van Nostrand, 1979
- B.W. McCormick *Aerodynamics, Aeronautics and Flight Mechanics* Wiley, 1995
- C.B. Millikan *Aerodynamics of the Airplane* Dover, 2018
- R. von Mises *Theory of Flight* Dover, 1959
- T. Thom *The Air Pilot's Manual Vol.4: The Aeroplane — Technical* Airlife, 1988
- F.M. White *Fluid Mechanics* McGraw Hill, 1999

I would also like to thank my undergraduate tutors for their inspirational teaching all those years ago, Professors P. Read and J.S. Wark. Peter Read deserves particular thanks for introducing me to Fluid Mechanics when it was not part of the required core course.

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Finally, it is my pleasure as always to acknowledge the delight of sharing life with Helen Machacek, and to thank her for her willingness for me to spend time studying and writing, and indeed for all of her love.

I hope you find this useful, and that it whets your appetite for the more technical, more useful and more accurate material you will find in the flight manuals and formal textbooks of aerodynamics.

Soli Deo Gloria,

ACM
Westcliff-on-Sea, 2020

Chapter 1

Units and Dimensional Analysis

1.1 Aviation Units and Unit Conversion

Physics has become much easier to understand now that there is one system of units used by all physicists — the Système International (SI). In the 'physics' sections of this booklet, SI units will be used throughout. However, pilots and aviation authorities use a mixed set of units, and we need to be familiar with these as well. This section will familiarize you with the units needed, while at the same time giving you the tools needed to convert units from one system to another. This has usefulness which goes beyond aviation.

Distances cause the most trouble, especially in the United Kingdom. While visibility and runway length are given in metres, vertical distances are given in feet (1ft = 0.305m), and extended horizontal distances (such as journey lengths) are given in nautical miles (n.m.) where 1nm = 1852m. Vertical velocities are given in hundreds of feet per minute. Speeds along a journey are given in nautical miles per hour. A speed of 90 nautical miles per hour is called 90 knots and written 90kt. Most aeroplane speedometers (known as air speed indicators or ASI) are calibrated in knots, although you do find the odd one marked in miles per hour.

Vertical Distances

When giving a vertical distance, great importance is given to the distinction between *height* and *altitude*. Height is how far above the ground you are, whereas altitude is your distance above mean sea level. The height measuring instrument (the altimeter) in an aircraft can be set to either. When taking off or landing, it is customary to set the altimeter to what is known as QFE, whereupon it reads the height. While flying a route the altimeter is set to QNH, and it reads the altitude. The pilot keeps an eye on the map (called a chart) and makes sure that the altitude is significantly higher than any hills or tall buildings in the vicinity, the altitudes of which are given on the chart. When flying at high altitudes, such as a commercial airliner at 37 000ft where terrain avoidance is not going to be an issue, a different system is used, and this will be explained later on page 28.

The instrument panel of an aircraft will not only contain an altimeter, it will also usually have a vertical speed indicator (VSI) which records the vertical component of velocity — the rate of change of altitude. British and Northern American VSIs are calibrated in hundreds of feet per minute, whereas Continental European altimeters are calibrated in metres with their VSIs reading metres per second.

A typical rate of descent when approaching an airport is 500 feet/min. What would this be in metres per second? Here are two methods for working it out.

The first is to switch the units as we go:

$$\frac{500\text{ft}}{1\text{min}} = \frac{500 \times 0.305\text{m}}{60\text{s}} = \frac{152.5\text{m}}{60\text{s}} = 2.54\text{m}/\text{s}. \tag{1.1}$$

The rule used by pilots is to take the vertical velocity in hundreds of feet per minute and halve it to convert to metres per second.

Another method which can be useful when dealing with more complicated conversions is to 'keep multiplying by one'. As you will know, multiplying a number by one will not change its value. As 1 foot and 0.305m are the same thing, $(1\text{ft}/0.305\text{m})$ is the same thing as 1. If we were to take a distance in metres, say 560m, and multiply it by $(1\text{ft})/(0.305\text{m})$, we get $560\text{m} \times 1\text{ft}/0.305\text{m}$. The metres cancel out, and we are left with $560\text{ft}/0.305 = 1836\text{ft}$. We can convert the 500ft/min in the same way:

$$\frac{500\text{ft}}{1\text{min}} = \frac{500\text{ft}}{1\text{min}} \times \frac{1\text{min}}{60\text{s}} \times \frac{0.305\text{m}}{1\text{ft}} = \frac{500 \times 0.305\text{m}}{60\text{s}} = \frac{2.54\text{m}}{\text{s}}. \tag{1.2}$$

Altimeters measure vertical distance using air pressure. As we will see on page 24 the pressure of the air reduces as we climb as there is less air above us requiring support. The drop in pressure as an aircraft climbs is measured by the altimeter, which tells us how high we have climbed. The fly in the ointment is weather, which causes the pressure of the air to change from one day to the next. Altimeters have to be adjusted to compensate for this. We turn a control on the altimeter to set the pressure. If we want the meter to show our height, we dial in the pressure of the air at ground level. The altimeter then shows us how high we are relative to the ground. If we want altitude, we dial in the pressure of air at mean sea level, and the altimeter shows us our height above mean sea level.

In Europe (including Britain), these pressures are given in hectopascals (hPa) which is the same thing as millibar (mbar). One hectopascal means 100Pa or 100N/m^2. In North America, pressures are given in inches of mercury, where standard atmospheric pressure 1013hPa is equal to 29.92inHg.

Question 1.1 *You fly an aeroplane from the United States to a customer in Europe. The British air traffic controller tells you that the QNH (the pressure setting for altitude) is 997hPa. What pressure do you need to set on your American altimeter?*

Question 1.2 *You are flying a British aeroplane in France. You are told to arrive 400m above the airfield and to descend at 3.0m/s. What is the arrival height in feet, and what is the descent rate in feet per minute?*

Question 1.3 *Which is more dangerous, thinking your altimeter is set for QNH when it is actually set for QFE or vice-versa?*

Question 1.4 *Tyre pressures are often given in 'psi' or 'pounds per square inch'. 1 psi means the pressure you would have if a pound (0.454kg) mass were supported by a column one inch wide and one inch deep. One inch is equal to 0.0254m. If a tyre requires a pressure of 24psi, what is that in pascals?* HINT: remember that the pound will have a weight of 0.454kg $\times$ 9.81N/kg.

Angles

You will be used to measuring angles in degrees $(^\circ)$ where there are 360° in a full circle. For accurate work, such as when specifying position using longitude and latitude, each degree is broken into sixty minutes ($1^\circ = 60'$) and each minute is made up of sixty seconds ($1' = 60''$).

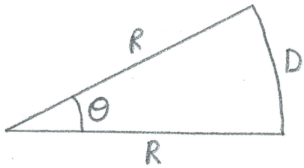

Figure 1.1: Measuring angles in radians

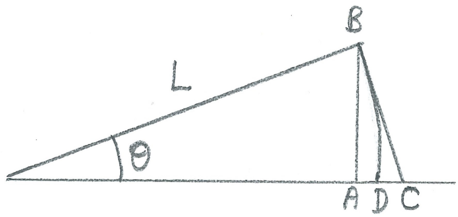

Figure 1.2: The small angle approximations. The distance between A and B is given by $L \sin \theta$, which is very similar to the distance between C and B given by $L \tan \theta$. Both will be similar to the distance along the arc from D to B which is equal to $L\theta$ where the angle θ is measured in radians.

Another unit used is the radian. Measuring angles in radians is illustrated in figure 1.1. The power of this unit is that if you move on a circular path of radius R such that you travel a distance D, the angle in radians is D/R. Equally, if the angle is θ, then the distance is given by $R\theta$, which is much simpler than the equivalent formula in degrees, which is $2\pi R \times (\theta/360)$. If you go round a full circle, then D is the circumference of the circle $2\pi R$. So a full circle represents an angle of $D/R = 2\pi R/R = 2\pi \,\mathrm{rad}$, and this is equivalent to $360°$.

Given that $360° = 2\pi \,\mathrm{rad}$, it follows that $1\mathrm{rad} = 360°/(2\pi) \approx 60°$. Using this approximation, an object moving on a circular path of radius R by angle θ_D in degrees will cover a distance of $R\theta_{\mathrm{rad}} \approx R\theta_D/60$.

As shown in figure 1.2, for small angles, the sine and tangent are ap-

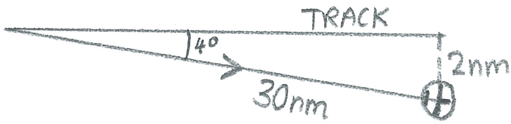

Figure 1.3: An aircraft flying an incorrect heading. After flying 30nm it is 2nm to the side of the intended route (the *track*).

proximately equal to the angle in radians (or one sixtieth of the angle in degrees). If, as shown in figure 1.3 after flying for 30nm, you find yourself 2nm off course, then your path must make an angle of about $\frac{2}{30}\mathrm{rad}$ to the proper course, which is about $\frac{2}{30} \times 60^\circ \approx 4^\circ$. This calculation is the first stage in working out what you need to do to correct the error.

Question 1.5 *What is 90° in radians?*

Question 1.6 *What is 0.24 rad in degrees?*

Question 1.7 *An unexpected extra wind from the side blows you 1nm to the East for every 8nm you fly North. What angle does your route make to the direction you intended to fly (ie. North)? Give your answer first in radians, then in degrees.*

Question 1.8 *You wish to fly a 30nm journey to the West. After flying for 10nm, you find yourself 1.5nm North of your intended route. By what angle do you need to turn the aircraft so that you intercept the intended route 20nm from the starting point?*

Question 1.9 *Repeat question 1.8, but this time work out the angle you would need to turn the aircraft to fly straight from its position to the intended destination.*

Distances Travelled

The circumference of the Earth is about 40 000km. If this (as a circle) is broken down into degrees, each degree of latitude (or longitude at the equator) is equivalent to about 111km. As this is a large distance, degrees of latitude and longitude are divided into sixty minutes. One minute is equivalent to 1.852km. This is one nautical mile, which you will notice is about 15% larger than a 'statute' or ordinary mile (which is 1.609km). If an aircraft were flying at 100kt (100 nautical miles per hour) around the equator, it would cover 100 minutes of arc (that is $\frac{100}{60}^\circ = 1\frac{2}{3}^\circ$) of longitude each hour.

Question 1.10 *An aeroplane has a take off speed of 50kt, but its airspeed indicator (ASI) is calibrated in miles per hour (mph). What will the ASI read at take off speed?*

Question 1.11 *Commercial airliners arriving at many airports are required to use a 3° descent path. This means that their velocity is at 3° to the horizontal. If the aircraft has a speed of 150kt over the ground, what will its vertical velocity in feet/min be?*

1.2 Dimensional Analysis

The full equations of fluid mechanics are difficult to solve. In many cases it is not possible to solve them algebraically. We introduce dimensional analysis as a way of predicting the equations we are going to need to study fluids. It turns out that this tool has other uses too such as simplifying experiments and enabling scale models to be used accurately as representations of full sized prototypes.

We start with the simplest type of problem. We have a variable X which we think depends on other variables a, b, c, d.... Accordingly we write an equation of the form

$$X = Pa^\alpha b^\beta c^\gamma d^\delta, \tag{1.3}$$

where P is some kind of constant of proportionality, and the indices α, β, γ and δ are numbers yet to be worked out. We then study the units of X, a, b, c and d to find the values of α, β, γ and δ.

Here is an example. Suppose that we think that the force needed to make something go round in a circle F depends on the mass m of the object, the speed v it is going and the radius r of the turn, we write

$$F = Pm^{\alpha} v^{\beta} r^{\gamma}. \tag{1.4}$$

The units of both sides of the equation must be the same, and we insist that the constant P has no units (it is a constant of proportionality). To avoid possible confusions, we limit our choice of units to the SI fundamentals of metres, seconds and kilograms. This ensures that there is only one way of expressing any particular quantity. Here F is measured in N, which is the same thing as a kgms^{-2}, v is measured in m/s = ms^{-1} and r is measured in metres m. For the units to balance, we therefore must have

$$\begin{aligned} F &= Pm^{\alpha} v^{\beta} r^{\gamma} \\ \mathrm{kgms}^{-2} &= (\mathrm{kg})^{\alpha} \left(\mathrm{ms}^{-1}\right)^{\beta} (\mathrm{m})^{\gamma} \\ \mathrm{kgms}^{-2} &= \mathrm{kg}^{\alpha} \mathrm{m}^{\beta+\gamma} \mathrm{s}^{-\beta}. \end{aligned} \tag{1.5}$$

Considering the kilograms, on the left we have just kg with no power — that is kg^1. On the right, the kilogram is to the power α. Therefore $\alpha = 1$. Similarly, equating powers of metres, we must have $1 = \beta + \gamma$; and for the seconds $-2 = -\beta$. To make the last equation work, we choose $\beta = 2$, and then our equation for metres becomes $1 = 2 + \gamma$ and so $\gamma = -1$.

After this algebra, we are confident that $F = Pm^1 v^2 r^{-1}$ and so we expect F to be proportional to mv^2/r, and this is indeed found to be the case.

Question 1.12 *The time period of a simple pendulum T in seconds might depend on the mass m of the pendulum bob in kilograms, the length L of the pendulum string in metres and the local gravitational field strength g in ms^{-2}. By guessing $T = Pm^{\alpha} L^{\beta} g^{\gamma}$ and solving for α, β and γ, work out the form of the equation for T. Note that the units on the left hand side will be s which is the same as $\mathrm{kg}^0\mathrm{m}^0\mathrm{s}^1$. HINT: if one of the powers is zero, then that quantity does not affect the time period.*

In this last question, we hope that you were able to ascertain that T would be proportional to $\sqrt{L/g}$, however the full equation is $T = 2\pi\sqrt{L/g}$. Dimensional analysis can predict the form of the equation, but not the value of any constants of proportionality. These must either be worked out using a more complicated mathematical proof or by experiment.

Question 1.13 *If we assume that the force of drag on an object in a flow of air depends only on the density ρ of the air (in $\mathrm{kgm^{-3}}$), the cross sectional area of the object A (in $\mathrm{m^2}$) and the speed v of the object (in $\mathrm{ms^{-1}}$), work out the form the equation must take.*

In this case, the constant of proportionality P is related to a number called the drag co-efficientC_D, and is equal to $\frac{1}{2}C_D$. The value of C_D depends on the shape of the object and the way it is aligned to the airflow. There is a very similar equation for the lift force on a wing but it has a different co-efficient C_L, known as a lift co-efficient. These co-efficients and their importance are explained further on page 49.

As you can imagine, the real situation with fluids is not as straightforward as this. Fluids also have viscosity μ. A viscous fluid like glycerol or engine oil does not flow as easily as water does. Viscosity is introduced properly on page 29, but it is measured in kg/(ms) and is larger for 'thicker' fluids.

If we want to evaluate the drag on an object in a viscous fluid, we have a more complicated equation:

$$
\begin{aligned}
F &= P\rho^{\alpha} A^{\beta} v^{\gamma} \mu^{\delta} \\
\mathrm{kgms^{-2}} &= \left(\mathrm{kgm^{-3}}\right)^{\alpha} \left(\mathrm{m^2}\right)^{\beta} \left(\mathrm{ms^{-1}}\right)^{\gamma} \left(\mathrm{kgm^{-1}s^{-1}}\right)^{\delta} \\
\mathrm{kgms^{-2}} &= \mathrm{kg}^{\alpha+\delta} \mathrm{m}^{-3\alpha+2\beta+\gamma-\delta} \mathrm{s}^{-\gamma-\delta}. \qquad (1.6)
\end{aligned}
$$

It follows that the kilogram term gives $1 = \alpha + \delta$, the metre term gives $1 = 2\beta + \gamma - 3\alpha - \delta$ and the term in seconds gives $-2 = -\gamma - \delta$. The difficulty here is that we have three equations in four unknowns and accordingly can not work out individual values for our numbers α, β, γ and δ. The best we can do is to express three of our numbers in terms of the fourth. The most helpful form of final answer is obtained if we choose δ. Doing this, our first equation tells us that $\alpha = 1 - \delta$, while our last gives $\gamma = 2 - \delta$. Finally, our middle equation $1 = 2\beta + \gamma - 3\alpha - \delta$ is now $1 = 2\beta + (2 - \delta) - 3(1 - \delta) - \delta$ and so $1 = 2\beta - 1 + \delta$ and $\beta = 1 - \frac{1}{2}\delta$.

Our equation for the force is therefore

$$
\begin{aligned}
F &= P\rho^{\alpha} A^{\beta} v^{\gamma} \mu^{\delta} \\
&= P\rho^{1-\delta} A^{1-\frac{\delta}{2}} v^{2-\delta} \mu^{\delta}
\end{aligned}
$$

$$= \quad P \times (\rho A v^2) \times \left(\frac{\mu}{\rho \sqrt{A} v} \right)^{\delta}. \qquad (1.7)$$

Notice that this equation will work regardless of the value of δ, as the term in brackets $\mu / (\rho v \sqrt{A})$ has no units. This term is said to be 'dimensionless'. In practice there need not be a particular value of δ which fits with experiment, it is more the case that the force will depend on a function of this term, and we write

$$F = \rho A v^2 \times f \left(\frac{\mu}{\rho v \sqrt{A}} \right). \qquad (1.8)$$

The reciprocal of the term in brackets, namely $\rho v \sqrt{A} / \mu$, is known as the Reynolds number, and its value is crucial in identifying the way in which the fluid will flow. We will find out on page 33 that it determines the relative importance of the viscosity, and thus whether the flow will be laminar (ordered and regular) or turbulent.

Question 1.14 *The rate of flow of a fluid down a pipe Q measured in $\mathrm{m^3 s^{-1}}$ depends on the diameter of the pipe D and the speed v of the fluid. Predict the form of the equation which links Q to v and D.*

Question 1.15 *To force fluid down a pipe, a pressure difference is needed across its two ends. The pressure difference per metre of pipe P' depends upon the speed of the fluid v, its viscosity μ and the diameter of the pipe D providing that the speed is not too high. P' is measured in $\mathrm{Pa/m}$ which is the same as $\mathrm{Nm^{-3}} = \mathrm{kgm^{-2}s^{-2}}$. Use dimensional analysis to write an equation for P' in terms of v, D and μ.*

Question 1.16 *Combine your answers to questions 1.14 and 1.15 to derive an equation for the flow rate Q in terms of the pressure gradient P', the diameter D and the viscosity μ.*

Question 1.17 *At higher speeds, the pressure drop in the pipe also depends upon the density ρ. Write a new equation for P' which takes this into account. Begin with the assumption that P' depends upon v, μ, D and ρ. You should find that your final answer contains the product of a term like the one in question 1.15 and a function of the Reynolds Number $\mathrm{Re} = \rho v D / \mu$.*

While the ability to predict the form of a fluid equation is very useful, this is not the only reason for learning this technique. Suppose you were investigating the force on an object in a fluid. Were it not for dimensional analysis, you would have to conduct an experiment for each combination of the relevant variables (A, ρ, v, μ and so on). This would require thousands if not millions of separate experiments. As it is, dimensional analysis has simplified the matter. We just need to measure the values of $F/\rho A v^2$ for different values of the Reynolds Number $\mathrm{Re} = \rho v \sqrt{A}/\mu$ to establish the form of the function f, which would require tens of measurements instead of millions. This is simpler, quicker and cheaper.

Our tool also enables us to use scale models with confidence. Clearly it is cheaper to build a scale model of a new wing or aeroplane rather than a full size prototype. Equally the model does not need such a large wind tunnel for the tests, and a small wind tunnel is much cheaper. To make the test valid, we do need to ensure that the Reynolds number is the same for the model in the tunnel as it would be for the full size aircraft. As a model will have a smaller cross sectional area A than the real aircraft, we are likely to need either faster air, or to conduct the test in a pressurized tank to increase the density of the air in order to keep Re the same. However if that has been done, we know that the force F will be proportional to $\rho A v^2$ from equation 1.8. Let us suppose that we used a model which was one tenth of the size of the real aircraft, and used air going at ten times the normal speed to keep the Reynolds number the same without needing to compress the air. It will follow that the $F_m/\rho A_m v_m^2$ for the model will equal $F_a/\rho A_a v_a^2$ for the real aircraft, and accordingly

$$
\begin{aligned}
\frac{F_m}{\rho A_m v_m^2} &= \frac{F_a}{\rho A_a v_a^2} \\
F_a &= F_m \times \frac{\rho A_a v_a^2}{\rho A_m v_m^2} \\
&= F_m \times \frac{A_a}{A_m} \times \left(\frac{v_a}{v_m}\right)^2 \qquad (1.9) \\
&= F_m \times 100 \times (0.1)^2 = F_m
\end{aligned}
$$

the force on the real aircraft will be the same as the force on the model. This is not a terribly realistic scenario, but it indicates the power of the methodology.

Chapter 2

Fundamentals of Fluids

2.1 Pressure

Pressure measures how 'concentrated' a force is over an area. You calculate it using the formula

$$\text{Pressure} = \frac{\text{Force}}{\text{Area}}. \tag{2.1}$$

Given that we measure force in newtons (N) and area in square metres (m^2), pressure is measured in newtons per square metre (N/m^2), and this unit is also called a pascal (Pa). Larger pressures might be measured in hPa (1hPa = 100Pa), kPa (1 kPa = 1000Pa) or MPa (1 Mpa = 1000 000Pa). Sometimes it is easier to give a pressure in newtons per square centimetre (N/cm^2), but be careful when converting to pascals:

$$\frac{1\mathrm{N}}{\mathrm{cm}^2} = \frac{1\mathrm{N}}{(10^{-2}\mathrm{m})^2} = \frac{1\mathrm{N}}{10^{-4}\,\mathrm{m}^2} = \frac{10^4\mathrm{N}}{\mathrm{m}^2}. \tag{2.2}$$

This comes about as one square metre has 100 square centimetres along each edge, and therefore contains $100 \times 100 = 10000$ square centimetres in total.

Atmospheric pressure varies from day to day depending on the weather, however at sea level it is usually within 2kPa of 101.3kPa which is known as standard atmospheric pressure and can also be written as 1013hPa. If people quote pressure in 'atmospheres' (atm), they are referring to this standard pressure. So a pressure of 5atm = 506.5kPa. Another unit in frequent use is the bar. One bar is the same as 10^5Pa, and is slightly less than 1atm.

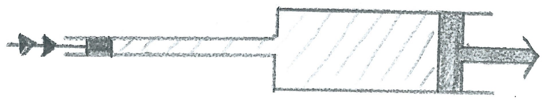

Figure 2.1: Two pipes of different diameters joined together. Despite the different forces on the pistons, the fluid is in equilibrium because the pressure caused by each piston is the same.

Question 2.1 *The window in a space station is circular with a diameter of 10cm. Assuming that the air inside the station is at a pressure of 1bar, what is the outwards force on the window?*

Unlike solids, which sit still in equilibrium if the forces are in balance, a fluid (a liquid or gas) will be in equilibrium if the pressure on either side is equal. Therefore if a narrow pipe is joined to a wider one, as in Figure 2.1, a relatively small force on the narrow piston can cause a much larger one on the wider piston.

Where the pressures are different, there will be a force on the fluid and anything in the fluid. For example, the blockage in the pipe shown in figure 2.2 has a net force to the left.

The diagram makes another important point about fluids. When a fluid touches a surface, it applies a force outwards onto the surface (at right angles to the surface) regardless of the direction of the original force which created the pressure.

Before moving on, there is one last point about pressure which will make subsequent explanations much easier, and it relates to energy. Energy transfer (in joules) equals the force applied (in newtons) multiplied by the distance moved (in metres). Therefore the newton is the same thing as the joule per metre. It follows that the unit of pressure, the newton per m^2 is the same thing as the joule per m^3. Pressure can be therefore be regarded as the 'energy content' of a cubic metre of fluid due to its pressure.

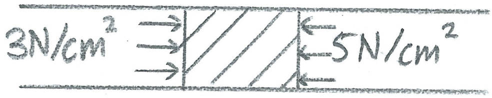

Figure 2.2: A pipe with a blockage. The pipe has a cross sectional area of $4\mathrm{cm}^2$. The force on the left of the block (pushing it to the right) will be $F = pA = 3\mathrm{N}/\mathrm{cm}^2 \times 4\mathrm{cm}^2 = 12\mathrm{N}$. Similarly, the force on the right of the block (pushing it to the left) will be $F = pA = 5\mathrm{N}/\mathrm{cm}^2 \times 4\mathrm{cm}^2 = 20\mathrm{N}$. The net (or resultant) force on the block will therefore be $20 - 12 = 8$N to the left.

A more detailed explanation is given using figure 2.3. A section of fluid in a pipe is removed after pistons have been put either side of it. We hold the right hand piston stationary, but allow the left piston to move. The fluid to the left of this piston pushes it to the right until it meets the other piston. The energy given to us via the piston is then calculated using

$$\text{Energy} = \text{Force} \times \text{Distance} = \text{Pressure} \times \text{Cross sectional Area} \times \text{Distance}. \tag{2.3}$$

However the volume of the fluid removed is equal to the Cross sectional area of the pipe multiplied by the length of the original section of fluid. The length of the section is the same as the distance the piston moved. Therefore the distance moved by the piston is equal to the volume of fluid divided by the cross sectional area of the pipe, and so

$$\begin{aligned} \text{Energy} &= \text{Pressure} \times \text{Area} \times \frac{\text{Volume}}{\text{Area}} \\ &= \text{Pressure} \times \text{Volume}. \end{aligned} \tag{2.4}$$

We see that the pressure is the energy released per unit volume of fluid drained.

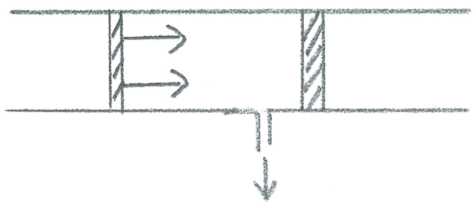

Figure 2.3: We work out the energy of a fluid under pressure by enclosing a section of fluid in a pipe between two pistons. With the pistons in place, the fluid is drained, and the fluid in the pipe pushes the pistons together. The energy released can be calculated from the work done on the moving piston.

2.2 Continuity

If a fluid is incompressible then the volume of a particular 'packet' or parcel of fluid will not change as it moves around a system. This is called the principle of continuity — new fluid can't just appear from nowhere, nor can old fluid vanish. Let us imagine that our packet of fluid has a volume V, and starts off in a pipe of cross sectional area A_1. It will therefore have a length of $L_1 = V/A_1$. Suppose the parcel then moves from that pipe to a narrower one with cross sectional area A_2. It will need to take up more length in the new pipe, and its new length will be $L_2 = V/A_2$.

Figure 2.4 shows the parcel of fluid passing from the wider pipe into the narrower one over the course of a period of time t. During this time, the back of the parcel moves a distance L_1, while the front moves a distance L_2. This tells us that the fluid in the first pipe has a speed $v_1 = L_1/t$ and the fluid in the second pipe has a higher speed $v_2 = L_2/t$. Since $L = V/A$, we have

$$
\begin{aligned}
v_1 &= \frac{L_1}{t} = \frac{V/A_1}{t} = \frac{V}{A_1 t} \\
v_1 A_1 &= \frac{V}{t} \qquad (2.5)
\end{aligned}
$$

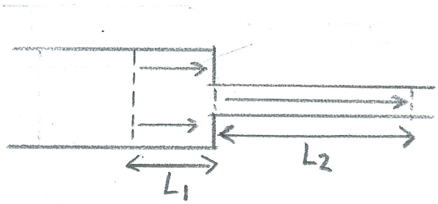

Figure 2.4: An incompressible fluid passing from a wider to a narrower pipe. As the same amount of fluid will take up a longer length of pipe after passing the narrowing, it has to speed up to prevent a 'traffic jam'.

$$\begin{aligned} v_2 &= \frac{L_2}{t} = \frac{V/A_2}{t} = \frac{V}{A_2 t} \\ v_2 A_2 &= \frac{V}{t} = v_1 A_1. \end{aligned} \tag{2.6}$$

In short, the product of the speed and the cross sectional area gives the flow rate (the number of cubic metres of fluid flowing each second) and as such will be the same at all points in the system. So forcing a fluid into a pipe with half the cross sectional area will require the speed of the fluid to double as it crosses from one pipe into the other.

Question 2.2 *A river is 2.1m wide and has an average depth of 32cm and average speed of 0.75m/s. Work out the flow rate of the river as a number of cubic metres per second.*

Question 2.3 *An incompressible hydraulic fluid passes from a pipe with a 1.2cm diameter to one of 4.3mm diameter. It speed in the wider pipe was 0.26m/s. What will its speed be in the narrower pipe?*

If the fluid is compressible, then things are more difficult, as a cubic metre of fluid will not remain a cubic metre as it passes around the system. However the mass of a particular parcel of fluid still can not change. The mass of one cubic metre of fluid is called its density (ρ), and is measured in kg/m^3. The mass is therefore equal to density $\times$ volume. As the fluid is compressed, its density will increase.

Using the fact that $V = m/\rho$, we re-write equation 2.5 as

$$v_1 A_1 = \frac{m}{\rho_1 t} \quad (2.7)$$

$$v_1 A_1 \rho_1 = \frac{m}{t}. \quad (2.8)$$

Since the mass flow per second m/t must be the same for both widths of pipe, the equivalent equation to 2.6 for a compressible fluid is

$$v_2 A_2 \rho_2 = \frac{m}{t} = v_1 A_1 \rho_1. \quad (2.9)$$

Question 2.4 *Rework question 2.3 for the situation where the fluid expands as it passes the join in the pipe to twice its original volume. HINT Doubling the volume means that the density has halved.*

2.3 Bernoulli's Principle

In the last section, we explained that when an incompressible fluid moves from a wide pipe to a narrower one, it has to speed up to prevent it clogging the pipe. However, you will know that nothing speeds up of its own accord — it needs a force to accelerate it. In this case, where is the force coming from to speed up the fluid? The answer is that it will not work unless the pressure in the narrower tube is lower. In this way there is an unbalanced force on the fluid in the 'funnel' between the wide and narrow pipes speeding it up as it goes into the narrow pipe. Similarly, the pressure of an incompressible fluid will rise as it slows down on passing from a narrow to a wide pipe. This is Bernoulli's principle.

Mathematically, Bernoulli's principle can be written down using the idea of the Conservation of Energy — the idea that energy can not be created nor destroyed, but can be stored in various different ways. We already know that the 'pressure energy' of a parcel of fluid per unit volume is its pressure. Accordingly, the 'pressure energy' of a particlar packet of fluid is given by $E_p = pV$ where p is the pressure.

The kinetic (motion) energy of the same packet of fluid is equal to $E_k = \frac{1}{2}mv^2$ where m is its mass and v its speed. Here $m = \rho V$ and so $E_k = \frac{1}{2}\rho V v^2$.

If there are no other ways of storing the energy involved, then the total energy $E_p + E_k$ must remain the same as the packet of fluid travels round the system. It follows that

$$E_p + E_k = pV + \frac{\rho V v^2}{2} = \left(p + \frac{\rho v^2}{2}\right) V \tag{2.10}$$

must remain the same too. The volume V is arbitrary (we could have chosen a cubic centimetre, or millimetre or anything) and so the term in brackets must remain constant too. Thus we often see Bernoulli's principle stated as

$$p + \frac{\rho v^2}{2} = \text{constant.} \tag{2.11}$$

From this equation we can see that increases in pressure must be accompanied by decreases in fluid speed and vice-versa. Accordingly, as soon as someone can convince you that air flows faster over the top of a wing than under it (not as easy as it sometimes sounds), it automatically follows that there will be higher pressure under the wing than over it, and the wing will lift up your aeroplane.

Question 2.5 *By how much will the pressure of air at 101kPa drop if it accelerates from rest to 50m/s? Assume that the density of air is 1.2kg/m*^{3}.

On page 7 it was mentioned that the altimeter in an aircraft measures the pressure of the air outside in order to work out how high it is flying. This is done using a small hole in the side of the aircraft which the air flows past (but not into). The sensor inside the hole measures the external air pressure p. This is also called the *static pressure*.

There is another hole facing the air flow in a protruberance called the pitot. The air which goes into this hole can't get out again, and accordingly has to slow down. As it does so its pressure rises from p to $p + \frac{1}{2}\rho v^2$, and it is this higher pressure (called the *total pressure*) can be passed on to other flight instruments. The difference between the static and total pressures, given by $\frac{1}{2}\rho v^2$ is called the *dynamic pressure*.

Question 2.6 *What is the dynamic pressure of air it is travelling at 150m/s? Assume that the density of air is 1.2kg/m*^{3}.

Question 2.7 *How fast are you travelling if the total pressure of the air is 630Pa higher than the static pressure? Give your answer in m/s and also in knots (nautical miles per hour). Assume that the density of air is 1.2kg/m^3, and remember that there are 1850m in one nautical mile.*

The airspeed indicator (ASI) is connected to both holes, and accordingly can measure the difference between the static and total pressures. It therefore measures the dynamic pressure $\frac{1}{2}\rho v^2$. From this it can display the speed of the aircraft. When the weather changes, and the density of the air changes as well, the same speed would cause a different reading on the air speed indicator as $\frac{1}{2}\rho v^2$ will be different. Pilots speak of the actual speed of the aircraft relative to the surrounding air as the true airspeed (TAS) and the speed shown on the dial of the ASI as the indicated air speed (IAS). You can accordingly convert IAS to TAS using this reasoning:

$$\begin{aligned} p + \frac{\rho_{\text{Actual}} \text{TAS}^2}{2} &= p + \frac{\rho_{\text{Standard}} \text{IAS}^2}{2} \\ \rho_{\text{Actual}} \text{TAS}^2 &= \rho_{\text{Standard}} \text{IAS}^2 \\ \frac{\text{TAS}}{\text{IAS}} &= \sqrt{\frac{\rho_{\text{Standard}}}{\rho_{\text{Actual}}}}, \end{aligned} \tag{2.12}$$

where ρ_{Standard} is the average density of the atmosphere which was assumed when the ASI was designed.

Question 2.8 *At a steady temperature, the density of air is proportional to its pressure. Air speed indicators (ASI) are calibrated assuming that the pressure is 1013hPa. If the ASI is indicating a speed of 90 knots on a day when the pressure is 980hPa, how fast are you going relative to the surrounding air?*

2.4 Height and Pressure

In the previous section, the principle of the Conservation of Energy was used to derive Bernoulli's equation. However an important kind of energy was neglected — gravitational potential energy. The gravitational potential energy of an object is given by $E_g = mgh$ where m is its mass, h is its height above some convenient reference point and $g = 9.81$N/kg is the Earth's gravitational field strength. As before, $m = \rho V$, and accordingly

$E_g = \rho V g h$. Adding this to our collection of ways of storing energy, equation 2.10 becomes

$$E_p + E_k + E_g = pV + \frac{\rho V v^2}{2} + \rho V g h = \left(p + \frac{\rho v^2}{2} + \rho g h\right) V \tag{2.13}$$

while equation 2.11 becomes

$$p + \frac{\rho v^2}{2} + \rho g h = \text{constant.} \tag{2.14}$$

It follows that as you rise higher in a stationary fluid (including the atmosphere), and the gravitational potential energy gets larger, the pressure must reduce to keep the total energy the same.

Equally, as you descend into a fluid and the gravitational energy gets less, the pressure will increase. The classic example of this is the pressure under water. At the surface, the pressure will be atmospheric pressure p_{atm}. We work out what will happen if you descend from the height of the surface h_0 to a lower height $h_0 - d$ where d is the depth. To conserve energy we must have

$$\begin{aligned} p_{\text{atm}} + \rho g h &= p_{\text{depth}} + \rho g (h - d) \\ p_{\text{atm}} &= p_{\text{depth}} - \rho g d \\ p_{\text{depth}} &= p_{\text{atm}} + \rho g d. \end{aligned} \tag{2.15}$$

Question 2.9 *Assume that $p_{\text{atm}} = 101$kPa, that $g = 9.81$N/kg and that the density of water $\rho_w = 1000$kg/m^3. How deep do you have to go before you reach a pressure twice that at the surface?*

Question 2.10 *Mercury (a liquid metal with a density of $\rho_{Hg} = 13500$kg/m^3) is in a beaker on the floor with the surface of the mercury exposed to the air. A long glass tube is held vertically, with the bottom end submerged in the mercury and the top end connected to a vacuum pump. How high does the mercury rise up the tube? HINT The pressure above the level of the mercury in the beaker will be atmospheric pressure. The pressure above the level of the mercury in the evacuated tube will be zero.*

Question 2.11 *Archimedes' Principle states that the upthrust on (or buoyancy of) an object immersed in a fluid is equal to the weight of fluid displaced. Let* *us check that this idea is consistent with equation 2.14. Imagine that a prism of length L and cross-sectional area A is immersed in a fluid of density ρ with its axis vertical, so that its top is at a depth d below the surface of the fluid. Its base will, accordingly, be at a depth $d + L$ below the surface of the fluid. Atmospheric pressure is p_{atm}. Calculate the pressure of the liquid in contact with the top face, and thereby give an equation for the downwards force on the prism. Then calculate the pressure of the liquid in contact with the bottom face, and write an equation for the upwards force on the prism. Finally, write an equation for the resultant force on the prism due to the liquid, and check that it is equal to $\rho g A L$, which is the weight of fluid displaced by the prism.*

The gases in our atmosphere are very well described by an equation called the Ideal Gas Law. It has been verified experimentally (as well as having its roots in sound physical theories), and states that the pressure p and volume V of a gas depend on the number of molecules N in the gas such that

$$pV = NkT \tag{2.16}$$

where $k = 1.38 \times 10^{-23}$J/K is a constant called the Boltzmann constant and T is the absolute temperature of the gas measured in kelvins. The kelvin temperature is equal to the temperature in celsius added to 273, and is set up so that the coldest possible temperature (called absolute zero, at $-273°\mathrm{C}$) is zero kelvins. The same constant k is used for all gases — it is a fundamental property of the Universe which you can find out more about by studying Statistical Mechanics.

We can use this equation to work out the expected density of the atmosphere. If we take the average mass of a molecule in the air as $m = 4.78 \times 10^{-26}$kg, the pressure as atmospheric and the temperature as $15°$C = 288K, then the density of air will be given by the mass of all of the molecules (mN) divided by the volume $V = NkT/p$:

$$\begin{aligned} \text{density} &= \frac{\text{mass}}{V} = \frac{Nm}{V} \\ &= \frac{pm}{kT} = \frac{1.01 \times 10^5 \times 4.78 \times 10^{-26}}{1.38 \times 10^{-23} \times 288} = 1.21\mathrm{kg/m^3} \end{aligned} \tag{2.17}$$

Question 2.12 *Work out the density of air at 980hPa at a temperature of $30°$C. HINT Remember to change the temperature into kelvins by adding 273.*

Question 2.13 *If the ASI in an aircraft reads 110kt on a day when the pressure is 980hPa and the temperature is* $30°\mathrm{C}$*, how fast is the aircraft actually travelling through the air?*

Question 2.14 *Work out the upthrust (buoyancy force) on an airship with a volume of 10 000m*^{3} *of helium and the weight of the helium. Assume a temperature of* $15°\mathrm{C}$ *at atmospheric pressure. The mass of a helium atom is* $6.66 \times 10^{-27}\mathrm{kg}$*. Finally, work out the maximum mass of equipment (including gas bag skins, engines, structural parts and payload) which can be carried by this airship.*

Now that we know the density of air, we can work out how its pressure will decrease as we rise through the atmosphere. For now, we assume that the density will not change as we rise. If we rise by 1m, h will become $h+1$, and so ρgh will become $\rho g\,(h+1) = \rho gh + \rho g$. Thus ρgh increases by ρg. We know that $p + \rho gh$ must stay the same, so the pressure must decrease by the same amount as ρgh increases: $\rho g = 1.21 \times 9.81 = 11.9\mathrm{Pa}$. An ascent of 100ft ($100 \times 0.305$m = 30.5m) will cause a pressure drop of $30.5 \times 11.9\mathrm{Pa} = 364\mathrm{Pa} = 3.64$hPa.

Question 2.15 *Calculate the gain in height needed for the atmospheric pressure to drop by 1hPa. How close is your answer to the pilots' way of estimating pressure differences which is 'a drop of 1hPa for each 30 feet'?*

Question 2.16 *You are flying, and have set your altimeter to the local QNH of 1024hPa (ie the pressure at mean sea level). This means that the altimeter is showing your height above mean sea level. You are about to land at an aifield which is 500 feet above sea level. What pressure will you need to set the altimeter to in order that it will give you your height above the airfield? HINT Work out the pressure expected 500 feet above sea level. Note that in practice, a pilot would be given this pressure setting over the radio. However if landing at an airfield without a radio operator, they would have to work it out themselves.*

Question 2.17 *Normal air at 101kPa contains approximately 20% oxygen. This means that the 'partial pressure' of the oxygen is 101kPa* $\times$ *20% = 20.2kPa. You begin to have difficulty breathing and working well if the partial pressure of oxygen falls to about two-thirds of this value. How high (in feet) can you climb* *in normal air without difficulties without an extra oxygen supply? Assume that the fraction of oxygen in the air is the same at all altitudes.*

The altimeters in aircraft use this principle. However, as we will find using calculus on page 72, the true situation is more complicated than this. Air is compressible, and so its density drops as you reach the lower pressures at higher altitudes, so the pressure falls less quickly with height the further up you go. In addition, the temperature drops as you go higher (pilots assume it starts at $15°$C at ground level and then it drops by $2°$C for every thousand feet, and will express the actual temperatures relative to these assumed values) which makes the mathematics even more complicated, so there is much more to learn...

2.5 Flight Levels

On page 7 you first encountered the difference between QFE (the setting on an altimeter enabling it to read the height above the ground) and QNH (the setting so that it reads altitude above sea level). Pilots flying within 2000 feet of the ground will set QNH on their altimeters, and by planning their flights well, and keeping an eye on the map, they can make sure that they fly over any hills or radio masts even if they can not see them.

When airliners cruise, they do so at much greater heights, above the storms and where the engines are much more efficient. It is not uncommon for an airliner to exceed an altitude of 40 000ft. At these heights, there is no danger of flying into terrain, however it is vital that all aircraft in the region have the same setting on their altimeters, otherwise there is a risk of collision between aircraft whose pilots thought they were at different heights when they weren't. Given the speed at which these airliners fly, and the way in which air pressure changes from place to place because of the weather, if airline pilots wished to fly on QNH, they would have to keep changing the settings on their altimeters, and there is the danger that two planes with different QNHs set might collide. Accordingly, once aircraft are above a particular altitude called the transition altitude (3000 feet for most of the UK), their pilots set 'standard pressure' 1013hPa on their altimeters. While the readings will approximate to the aircraft's altitude, they will actually measure the height above the air that is at 1013hPa. Aircraft flying in this way are said to be at a 'flight level'. The flight level is the height above the 1013hPa reference level given in hundreds of feet. So flight level 75 (also written FL75 or F075 on a flight plan) means flying at an indicated altitude of 7500 feet with your altimeter set to 1013hPa. Many countries (or groups of countries) have rules mandating particular flight levels when flying in different directions to help prevent collisions in the air.

Question 2.18 *An aircraft is flying at FL35 on a day when the QNH pressure (the pressure at sea level) is 980hPa. How high is it above sea level in feet? Assume that pressures drop by 1hPa for each 30 feet of extra altitude.*

Question 2.19 *A pilot in a region where the transition altitude is 3000 feet would be wary of flying at FL35 if the pressure at sea level was significantly different from 1013hPa. After all, they might risk a collision with an aircraft flying at 3000 feet altitude using the QNH. However would they be more wary of a sea level pressure of 995hPa or 1031hPa?*

2.6 Viscosity

The last property of a fluid we consider in this introductory chapter is viscosity. The great aeronautical engineer and expositor A.C. Kermode gave an excellent introduction:

> It is rather difficult to explain what this term means except by saying that *treacle is very viscous*. It is the tendency of one layer of the fluid to "stick" to the next layer and to prevent relative movement between the two. One can feel this in treacle, one can imagine it in water; but one would hardly think of air being "sticky" — yet sticky it is, though of course to a much less degree than water, let alone treacle.[1]

Viscosity is one of the principal causes of drag on an object moving through a fluid (the other being the inertia of the fluid in front which has to get out of the way). It turns out that no matter how fast anything moves in a fluid, the layer of fluid touching the object moves at the same speed as the object. It is as if that layer were glued to the moving object. This layer is called a *boundary layer*. Therefore if your aeroplane (and the air next to it) is moving at 100kt, while the rest of the air is not, there will be viscous forces trying to pull this boundary layer air back and make it stop.

[1] p68 of A.C. Kermode *Flight Without Formulae*, Pitman, London 1970

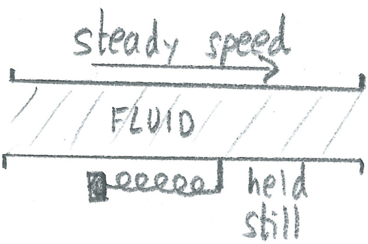

Figure 2.5: The way viscosity is measured. The top plate is moved at a steady speed to the right, and the bottom plate is pulled to the right by a force caused by the viscosity. The size of the force is measured by seeing how far the springs are stretched.

The way viscosity is measured is shown in figure 2.5 where fluid is in a narrow gap between two large parallel plates. The top plate is moving, the bottom one is stationary, but is held in place with springs. The viscosity of the fluid causes the bottom plate to be pulled along, and the amount by which the springs stretch before equilibrium has been reached can be used to measure the force attempting to pull the bottom plate.

It turns out that the force F is dependent on the area of the bottom plate A, the speed of the top plate v and the distance between the plate and the fluid d. Algebraically we write $F \propto Av/d$. The constant of proportionality, that is the ratio between F and Av/d, will be different for different fluids and tells us how "thick" or "sticky" the fluid is. This number measures viscosity (and is also called the *dynamic viscosity*) and is usually given the symbol μ. Re-arranging the equation $\mu = Fd/(Av)$, we can therefore work out that its units should be $\mathrm{Nm}/(\mathrm{m}^2 \times \mathrm{ms}^{-1}) = \mathrm{Ns}/\mathrm{m}^2$. Given that 1N is the same as 1 kgms^{-2} it also follows that the unit of viscosity is equivalent to $\mathrm{kg}/(\mathrm{ms})$.

When you see a measurement of viscosity, please be aware that engineers also define *kinematic viscosity* as the ratio $\nu = \mu/\rho$ of ordinary viscosity to density, thereby giving a quantity measured in m^2/s. The only way you can be sure that you have correctly understood which kind of viscosity is being given is to check the units.

A particularly important result is the force of fluid resistance (drag) D on a sphere of radius r which is falling at speed v through a fluid of viscosity μ. This is known as Stokes Law:

$$D = 6\pi\mu v r. \tag{2.18}$$

Question 2.20 *Using dimensional analysis, verify that both sides of equation 2.18 have the same units.*

Question 2.21 *According to Stokes Law, what is the drag on a raindrop of radius 0.15mm if it is falling at 1.1m/s? The viscosity of air is* $1.85 \times 10^{-5}\mathrm{kg}/(\mathrm{m\,s})$. *Taking the density of water to be 1000kg/m^3, work out the weight of the drop. How fast would the drop need to be falling for the drag to balance the weight? This speed is known as the terminal velocity of the drop.*

Question 2.22 *In question 1.16 on page 15, you determined that the flow rate of fluid down a pipe Q was proportional to R^4P'/μ where R is the radius of the pipe and P' is the pressure drop per unit length along the pipe. Measurements and theory agree that the constant of proportionality is $\pi/8$. Thus $Q = \pi R^4 P'/(8\mu)$. The viscosity of water is* $1.10 \times 10^{-3}\mathrm{kg}/\mathrm{m\,s}$. *Work out the pressure drop needed along an 8m hosepipe with a diameter of 2cm if you wish to be able to fill an 8 litre bucket in 30s. HINT One litre is the same as 10^{-3} m^3.*

2.7 Inertia, Drag and Reynolds Number

After reading the previous section, it might be tempting to believe that viscosity is the main cause of drag, and as air has such a low viscosity, anything travelling through air would not be affected much by drag. However viscosity is not the only factor in determining drag, as you found when working towards equation 1.7. Indeed, in question 1.13 on page 14, before the idea of viscosity had been introduced, you worked out that drag would be proportional to $\rho A v^2$ where ρ is the density of the fluid, v the speed of the object and A the area of the object facing the fluid. It is now time to explain why density can cause drag even in the absence of viscosity.

In order for an object to pass through a fluid, the fluid in front of it has to get out of the way. This requires a force. We will estimate the size of the force using a simplified situation. In figure 2.6, a fluid is being pushed aside

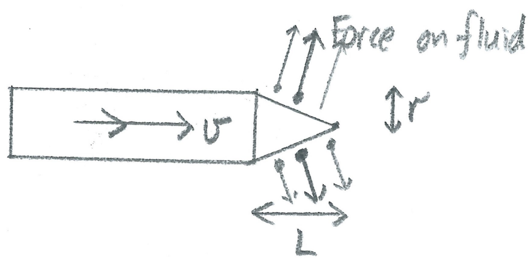

Figure 2.6: A block moving to the right must push fluid out of the way in order to move. In order to get out of the way, the fluid must be pushed sideways and also in the direction of motion of the block.

by a cone stuck to the front of a moving cylinder. Looking at the faces of the cone, you see that the fluid is not only pushed outwards, it is also pushed to the right as it is forced out of the way. If the fluid is being pushed to the right by the cone, it follows by Newton's third law that the cone must be pushed backwards by the fluid. This kind of drag does not require any viscosity in order to work.

The distance travelled by the block in one second is equal to v. The volume of fluid which needs to get out of the way each second is given by Av, and the mass of this fluid is ρAv. If the fluid follows the nosecone (the cone), which has length L, any given molecule of air needs to take a time of no more than L/v to move sideways. If the radius of the object is r, the necessary sideways speed is going to be $r/t = r/(L/v) = rv/L$. Therefore each second a mass of ρAv needs to be given a speed of rv/L. This is a momentum change each second of $\rho Av \times rv/L = \rho Arv^2/L$. For a given shape of object, r will be proportional to L, and so r/L will be a constant number. Therefore we have a momentum change each second proportional to ρAv^2. By Newton's second law, the force is the momentum change each second, so the drag force will be proportional to $\rho Av^2 = \rho\pi r^2 v^2$.

We showed in the last section that the viscous drag force is proportional to μ, r and v. The drag due to inertia is different — it is proportional to ρ,

r^2 and v^2. Therefore the ratio

$$\frac{\text{Inertia Drag}}{\text{Viscous Drag}} \propto \frac{\rho r^2 v^2}{\mu r v} = \frac{\rho r v}{\mu}. \tag{2.19}$$

As has been mentioned after equation 1.8 on page 15 this dimensionless quantity is called the Reynolds Number. If it is very large (such as when an aeroplane wing goes through the air), inertia effects are more important. In these cases, drag will be proportional to the square of the speed. If it is small (as when raindrops fall), viscous effects are going to be more important. Here, drag will be proportional to speed.

It should also be noted that viscosity, like friction, is a force which has a tendency to reduce the kinetic energy of a fluid, but warms it up. It accordingly has a tendency to damp down extreme motions and keep things orderly in streamlines. This is known as *laminar* flow — the fluid behaves as a pile of separate rigid plates sliding over each other. However when the energy of a moving object has been passed into its surrounding fluid through inertia drag, the fluid will move in a wide variety of ways, including whirls (also known as vortices), and while this energy lost by the object will eventually end up warming the fluid, it will cause a lot of chaotic motion first. This is *turbulence*. It was Reynolds' experiments which first taught us that as you increase the Reynolds number (named in his honour) there comes a critical value where turbulence begins. Once this has happened, the drag increases drastically.

To give an example, when fluid flows down a pipe, the Reynolds number is defined as $\mathrm{Re} = \rho v D/\mu$, where D is the diameter of the pipe and v is the mean speed. The mean speed is equal to the flow rate (in m^3/s) divided by the cross sectional area. For this case, experiments have shown that the flow is turbulent if $\mathrm{Re} > 2300$.

Question 2.23 *Water has a density of* $1000\mathrm{kg/m^3}$ *and a viscosity of* $1.1 \times 10^{-3}\mathrm{kg/m\,s}$. *How fast will the water need to travel down a 2cm diameter pipe before the flow becomes turbulent? What is the critical speed for a 10cm diameter pipe?*

Question 2.24 *If an aeroplane has a roughly square cross-section of side length 2.0m, and moves through air with a density of* $1.2\mathrm{kg/m^3}$ *and dynamic viscosity* $1.85 \times 10^{-5}\mathrm{kg/(ms)}$ *at a speed of 50m/s, calculate the Reynolds number. Then comment on whether you think that inertia or viscosity drag will be more* *important when modelling the drag force on the aircraft, and whether you expect the drag to be proportional to the speed or the square of the speed.*

Question 2.25 *Traffic flow on a motorway has much in common with fluid flow in a pipe. When the flow is particularly heavy, traffic jams seem to start for no reason, and the flow becomes much less efficient. Increasing the number of vehicles on the road effectively increases ρ, thus increasing the Reynolds number and making turbulent flow more likely. The same can be said for increasing the speed of the vehicles. Using this analogy, traffic engineers have been able to give instructions to drivers (and modifications to the design of roads and their signs) to make the flow more laminar. How should drivers on a motorway drive if they want to lower their Reynolds number and make traffic jams less likely?*

Chapter 3

Things called Wings

> If you can't explain it simply then you don't understand it well enough. *Attributed to A. Einstein*

It is highly unusual to begin a chapter of a book with an apology. However, unfortunately, I believe that I am compelled to do so here. One of the first questions which comes to mind when people think about flight is, "How does an aeroplane stay in the air?" Given that this is the primary function of the wings, the questioner is asking how the wing works. And yet, you may find my answer to this question here inadequate. I shall do my best. We shall lay a good foundation for your further studies in aeronautics, as well as introducing practical rules which help pilots understand their aircraft better. However if you are looking for a handy one liner which

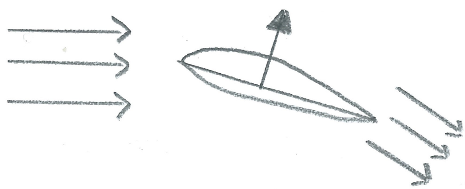

Figure 3.1: The airflow round a wing oversimplified

makes immediate sense to you, but is also fully accurate once expressed in mathematical language, I fear I may well disappoint you.

This does not mean that we, as a scientific community, are ignorant of the working of wings. The motion of air around a wing has been comprehensively studied. The equations are well known, and they agree perfectly with experiments conducted in wind tunnels. In fact our understanding is sufficiently good that new wing shapes can be critiqued by a computer as soon as they have been drawn. It is perfectly possible for modern aeronautical engineers to design a wing without recourse to wind tunnel tests in full confidence that it will work as planned. Accordingly, in addition to giving an overview of the science involved in lift, I do also wish to show you how engineers go about performing the calculations.

In this chapter, we take two different approaches to tease out the working of a wing. The first is based on Newton's Laws, and at a first glance is highly intuitive. If you put your hand out of the window of a car when it is moving, and you tilt your hand so that the side in front, or 'leading edge' is higher than the back edge, the air hits your palm rather than the back of your hand, and as it bounces off, it pushes your hand upwards and backwards. Surely, something similar is happening with a wing? It is true that the air behind a wing is measured to be flowing downwards as shown in figure 3.1. This explains why when birds fly in formation the rear birds do not fly directly behind the leader - they would have extra work trying to fly on descending air. It would be like trying to climb a downwards-moving escalator. This means that somehow, the wing has pushed the air in such a way that it continues moving downwards after it has passed behind the aircraft. If the wing pushed the air down, then the air must have pushed the wing up by Newton's Third Law , which states that when one object forces another, the forces on each object are equal and opposite to each other.

We study a simple model based on this idea and find that it predicts certain aspects of lift correctly, but struggles to explain other observations. In addition, not all of its predictions are accurate.

We then start afresh using a different approach — that of Bernoulli. If the air is travelling faster over the top of the wing than it flows under the base, then the pressure above the wing will be lower than the pressure below it, and the wing will experience a net upwards force. This is found to be true, and is described by aeronautical engineers as *circulation*, which incidentally does *not* mean that any of the molecules are actually rotating

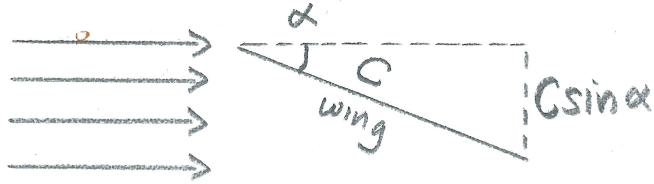

Figure 3.2: A simple model to estimate the force of air on a wing.

hundreds of times around the wing during a flight from London to New York. Our challenge here is to explain what causes this speed difference.

The explanation here is more rigorous, and does accurately predict the lift. However it does rely on a few assumptions. The simplified version I use here can only predict the lift for very simple wing shapes, but we will also have a brief glimpse of the method used to perform the calculations for general wing shapes. The approach taken in this section is largely qualitative and descriptive, however reference is made to the relevant parts of chapter 6 where the formulae are given and worked through.

Having established the nature of lift, physically and mathematically, the remainder of the chapter uses these results to analyse practical aspects of flight such as the best speed to fly, as well as how to climb, descend and turn.

3.1 Lift from Momentum Change — à la Newton

We shall use a very simple model to estimate the force of the air on the wing, as illustrated in figure 3.2. The wing would not push any air downwards unless it were diagonally aligned to the airflow. The angle it makes to the oncoming air is called the *angle of attack* α. Aeroplanes are constructed so that even when the aeroplane is flying with its body (fuselage) horizontal, the wing has a slightly positive angle of attack. The angle of attack can be varied by the pilot by pointing the nose of the aircraft up or down using controls called elevators.

The volume of air reaching the wing each second equals the area of

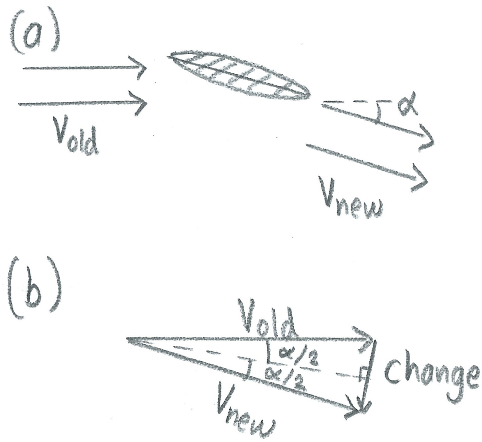

Figure 3.3: The change in velocity of air needed so that air can pass the wing. As shown in (a), initially it is flowing horizontally. Afterwards it is travelling downwards at an angle α to the horizontal. Diagram (b) shows the vector triangle of the velocities before and after, together with the change in velocity Δv. Splitting the diagram into two right angled triangles each with hypotenuse v allows us to see that $\Delta v = 2v \sin \frac{1}{2}\alpha$. We are using a simplifying assumption that the air does not slow down.

wing "presented to the air" multiplied by the speed of the air. Although the wing has an area of $A = Cb$ where C is shown in figure 3.2 and b is the wingspan, the air would only face all of this if $\alpha = 90°$. As it is, the "height" of this wing as seen by the air is the vertical component of C, namely $C \sin \alpha$. Thus the area of wing presented to the airflow is $b \times C \sin \alpha = A \sin \alpha$. Using the methodology found on page 21, the volume of air passing the wing each second is therefore $Av \sin \alpha$, and its mass is $\rho A v \sin \alpha$.

In order for the air to pass the wing, it must be given a downwards velocity at an angle of at least α. As shown in figure 3.3 this means that its velocity must be changed by at least $\Delta v = 2v \sin \frac{1}{2}\alpha$.

Putting two pieces of information together, the force on the wing will be the momentum change of the air each second (by Newton's second law).

This is given by

$$
\begin{aligned}
F &= \text{mass} \times \text{velocity change} \\
&= \rho A v \sin\alpha \times 2v \sin\frac{\alpha}{2} \\
&= 2\rho A v^2 \sin(\alpha) \sin\frac{\alpha}{2} && (3.1) \\
&\approx \rho A v^2 \alpha^2 && (3.2)
\end{aligned}
$$

where we have used the approximations $\sin\alpha \approx \alpha$ and $\sin\frac{1}{2}\alpha \approx \frac{1}{2}\alpha$ in the final line. This is true as long as we measure α in radians (one radian is about 57°) and the angles are not too large, as explained on page 10. The vertical component of this force (the component perpendicular to the airflow) is called the *lift* L. Given that α is small, $L = F\cos\alpha \approx F$, and we expect $L \approx \rho A v^2 \alpha^2$. The horizontal component, parallel to the airflow, is called *drag* D. Correspondingly we would expect $D = F\sin\alpha \approx F\alpha = \rho A v^2 \alpha^3$.

It turns out that these formulae are not too far from the truth. The forces are proportional to A, ρ and to v^2, as we would have expected anyway from our work on pages 14 and 31. In fact, the lift and drag formulae are usually written as

$$L = C_L \frac{\rho A v^2}{2}, \tag{3.3}$$

$$D = C_D \frac{\rho A v^2}{2}, \tag{3.4}$$

where C_L and C_D are called the lift and drag *co-efficients*, and will depend on the shape of the wing and the angle of attack. A factor of one half is included to show that lift is proportional to the dynamic pressure of the air $\rho v^2/2$, upon which it clearly depends.

So far, so good. However, we have predicted that lift will be proportional to α^2 and drag to α^3, but when we measure the force we find that the lift is more-or-less proportional to α and the drag is more complicated. Put simply: we have got it wrong. This does not mean that Newton's laws don't work with wings, however our analysis is clearly missing out some important complicating detail. Our simple model also fails to account for another measured fact: when tested in wind tunnels, the bulk of the lift force on a wing comes from the front part, and the top surface. If our simple 'air pushing' model were the whole truth, the force would come mainly from the lower surface (the one being hit by the air).

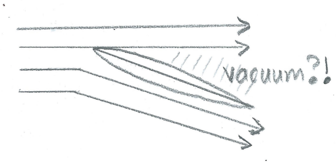

Figure 3.4: A simple explanation of lift from the upper surface of a wing. If the air flowing over the top of the wing experienced no force, it would continue backwards in a straight line. This would lead to a vacuum in the wedge shape above the wing and behind it. In practice, this is not going to happen, however when the air above the wing expands to fill the extra space, there will be a reduction in pressure.

It is worth exploring whether a simple Newtonian model could be used to predict the lift generated from the top surface. Unless acted upon by a force, an object (such as a particle in a fluid) will continue in a straight line. If we followed this principle, we might expect air passing over the top of the front (or *leading*) edge of a wing to continue moving horizontally backwards, as it would if the wing weren't there, as shown in figure 3.4. However, in reality, we expect the air passing over the wing to follow the shape of the wing downwards. If it didn't, it would leave a vacuum in the wedge shape above the wing as shown in figure 3.4. So, the air above will expand to fill the gap, and as it does so there must be a downwards force on it, coming from the higher pressure air above it. However, as this air 'spreads out' its pressure will drop. This will cause a pressure difference between upper and lower surfaces which could explain lift. Furthermore the low pressure air level with the upper surface behind the wing will cause the air above the wing to speed up as it goes over, which fits in with the Bernoulli explanation we shall encounter in our next section.

Question 3.1 *If the lift force from a pair of wings on a light aircraft with an area of 15m^2 is 7500N when flying at 45m/s, calculate the lift co-efficient for these wings. Take the density of air as 1.2kg/m^3.*

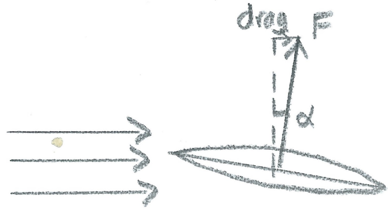

Figure 3.5: The force on a wing due to the air is broken into two components. The component parallel to the initial air flow is the drag D, while the component perpendicular is the lift L.

Question 3.2 *The pilot of the aeroplane in question 3.1 wishes to fly at 40m/s instead. The lift needs to be the same as before (to balance the weight, which has not changed), and to prevent a loss of height, the pilot raises the nose of the aircraft, increasing the angle of attack. Calculate the new angle of attack on the assumption that C_L is proportional to α if the original angle of attack was $4°$.*

Question 3.3 *Calculate the lift expected from a wing of area A if the air contains N molecules per cubic metre, if the molecules each have mass m, and if these molecules bounce off the wing like billiard balls. Take the speed of the wing as v and the angle of attack as α. HINT Calculate how many molecules are going to hit the wing each second, and then calculate the momentum change when one molecule bounces.*

3.2 Lift from Circulation — à la Bernoulli

Measurements made in air tunnels conclusively show that air flowing over the top of a wing travels faster than the air going underneath. A very simplified diagram of this is shown in figure 3.6. To work out the lift, we need to calculate the effect this speed difference has on the pressure of the air above and below the wing, using equation 2.11.

Let us suppose that the original speed was U, but this rose to $U + w$, and that this causes a change in pressure p so that the new pressure is $P + p$, where P was the original pressure. We use Bernoulli's equation (which

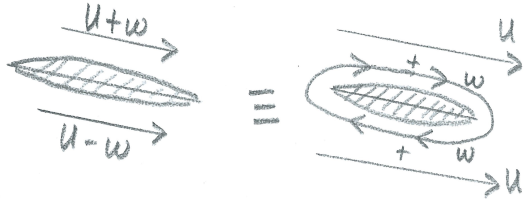

Figure 3.6: The meaning of circulation. If air is going over the wing faster than the air underneath, then we can write the air flow over the top as having a speed backwards of $U+w$ relative to the wing, while the air underneath has a speed of $U-w$ backwards relative to the wing. This is equivalent to the addition of two air flows — a uniform flow backwards of U and a circulating air flow of speed w.

tells us that $P+\frac{1}{2}\rho v^2$ is a constant) to tell us

$$\begin{aligned}
P+\frac{\rho U^2}{2} &= (P+p)+\frac{\rho\,(U+w)^2}{2} \\
\frac{\rho U^2}{2} &= p+\frac{\rho\left(U^2+2Uw+w^2\right)}{2} \\
0 &= p+\rho U w+\frac{\rho w^2}{2} \\
p &\approx -\rho U w, \qquad (3.5)
\end{aligned}$$

where in the last line we have made the assumption that w is significantly less than U (physicists write $w \ll U$) and thus that the term in w^2 will be small enough to be ignored.

For our wing, the speed above it is $U+w$, so the pressure change $p_\uparrow = -\rho U w$ is negative, telling us that the pressure has dropped. Similarly below the wing, where the speed is $U-w$, the pressure difference will be $p_\downarrow = +\rho U w$. The pressure difference between the top and bottom of the wing is therefore $p_\downarrow - p_\uparrow = 2\rho U w$. If the wing has area A, the lift force on the wing will be equal to pressure $\times$ area, so

$$L = \text{Pressure Difference} \times A = 2\rho U w A. \qquad (3.6)$$

The distance between the front and the back of the wing (ie from the *leading edge* to the *trailing edge*) is called the *chord* C. If the wingspan is b, then

$A = Cb$, and the lift force is given by

$$L = 2\rho UwA = 2\rho UwbC. \tag{3.7}$$

This leads us to the aeronautical engineer's fundamental principle that lift is caused by the speed change. In equation 3.7 if there is no speed difference between the upper and lower surfaces, $w = 0$, and there will be no lift. In other words, if you manage to achieve a difference in speed, you will get lift. Aeronautical engineers call this difference in speed a *circulation*.

It must be stressed that circulation does not mean that some of the molecules go backwards across the top surface, turn the corner at the back (or *trailing edge*) of the wing, and then push their way to the front of the wing against the oncoming torrent of airflow before beginning the process all over again. Rather, if there is a difference in speed above and below the wing, this can be thought of as the sum of a uniform front-to-back air flow and a rotating air flow, as shown in figure 3.6.

In fact, for this chapter to make any sense we need to give a formal definition of a circulation.

Circulation equals the speed of a small parcel of air over a route round an object multiplied by the distance that the air moves along this route, providing that the air is moving in the same direction as our journey around the object. We typically start our route at the front (leading) edge of the wing and work our way backwards along the upper surface. To complete our route, we must then pass along the underside of the wing from the back to the front. In this section of the journey, we travel around the wing in the opposite direction to the motion of the air. The speed $\times$ distance for this section is accordingly counted negatively.

For our wing, shown in figure 3.6 the route goes backwards over the top surface of the wing, where the speed is $U + w$, and the distance travelled is C. This contributes $(U + w)\,C$ to the circulation. We then bend the corner at the back of the wing (which in our model takes no distance, so there is no contribution to the circulation as our route flips past the trailing edge), and proceed forwards along the underside of the wing. Here the speed is $U - w$ and the distance is C again, but now the air is travelling in oppsition to our route, so we count the speed negatively, and accordingly the contribution to the circulation is $-(U - w)\,C$. The route then flips round the leading edge to join up with itself and make a complete path. The total circulation is

$$\Gamma = (U + w)\,C - (U - w)\,C = UC + wC - UC + wC = 2wC. \quad (3.8)$$

We combine this with equation 3.7 and find that the lift is given by

$$L = 2\rho UwCb = \rho Ub\ (2wC) = \rho Ub\Gamma. \quad (3.9)$$

This puts the mathematical link between lift and circulation on a more definite and clear footing. If you know the circulation, you just multiply by the density, speed of the aircraft and the wingspan to get the lift. We are, at this stage (and throughout this book) ignoring any asymmetries caused by the wing narrowing or changing shape as you get further from the fuselage, however even here the equation works if you say that per metre of wingspan the lift of a particular section is $L/b = \rho U\Gamma$. This is known as the *Kutta-Joukowski Law* and is a fundamental relationship in aerodynamics.

So, circulation causes lift, but how do you get circulation?

One method, beloved of footballers and cricket bowlers, is to spin the flying object. Kicking a football to the side leads to it rotating about a vertical axis as it travels through the air. This rotates the air around it, and leads to a 'lift' force to one side or the other causing the ball's trajectory to curve, or to "bend it like Beckham." A golfer will often try to impart "backspin" to the ball, where the the ball rotates about a horizontal axis such that the top of the ball is moving backwards, while the base of the ball is moving forwards. This increases the speed of the air passing over the top of the ball (lowering its pressure) while doing the opposite to the bottom. The ball accordingly has lift, and travels up higher than expected. These forces are often referred to as resulting from the Magnus effect.

Having linked lift and circulation, we transfer our attention to the airflow itself, and we will consider a simplified wing — a flat sheet, as found in nearly every paper aeroplane or dart. The flow of air around such a sheet is shown in figure 3.7, where we initially neglect viscosity and the onset of turbulent flow. When a wing is working properly, there should be little turbulence. As far as viscosity is concerned, its main role here is to keep a very thin *boundary layer* of air next to the wing stationary, which does not affect these diagrams. While these flows can be proved mathematically, they are also intuitive: the air generally goes round the nearer edge. Notice the line separating the air which goes above the wing from that which goes below it. The two points where this line meets the wing are called *stagnation*

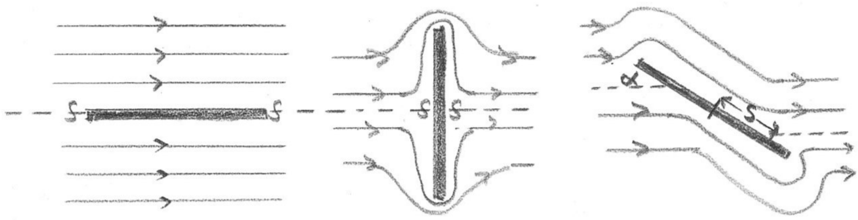

Figure 3.7: The airflow around a flat sheet of width $2b$. The dotted line shows the division between the air which will flow above and below the sheet. The *stagnation points* , marked S, show where this boundary meets the sheet itself. Diagram (c) shows a sheet meeting the airflow at an angle α.

points as the model predicts that air in contact with the wing near this part (above the boundary layer) will not be moving along the wing.

It gets much more interesting when the sheet is at a diagonal angle to the airflow, as show figure 3.7. Our simplified model predicts that the stagnation points move to reduce the distance air will be travelling in opposition to its original direction (from back to front, as with the air just below the leading edge of the wing). If we call the distance from the centre of the plate to the stagnation point s then we see that when $\alpha = 0$, $s = C/2$ and the stagnation points are at the edge of the plate, as in figure 3.7 (a). When $\alpha = 90^\circ$, the stagnation points are at the centre and $s = 0$ as in part (b) of the diagram. In the third case (c), the points are in between such that s gets smaller as α increases from 0° to 90°. A full analysis gives $s = \frac{1}{2}C\cos\alpha$, and can be seen on page 115.

The symmetry of these situations means that there is no circulation here. Accordingly, if the airflow remained as in figure 3.7, where the flows around leading and trailing edges of the wing were effectively the same, there would be no lift.

However, when an aeroplane accelerates along a runway, something happens to change the pattern. As shown in figure 3.8, the air passing under the trailing edge of the wing has sufficient inertia it does not manage to follow the surface of the wing round to the upper side. As shown in part

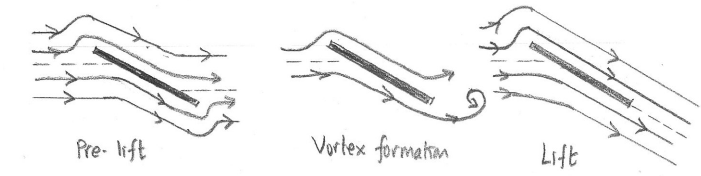

Figure 3.8: The change in stagnation points when a wing starts to generate lift.

(b) of the diagram, it curls around as best it can, and in doing so, forms a vortex which is blown backwards by the air. After this, the boundary between the air which passes over the wing and the air which passes under the wing takes on a different shape as shown in (c), starting at the trailing edge itself and running parallel to the wing's chord from the wing in the direction of the shed vortex.

The observation that the rear stagnation point is at the trailing edge, which is confirmed by wind tunnel experiments, is known as the *Kutta condition*. Any numerical model used to describe airflow around a wing is adapted to make sure that it is compatible with the Kutta condition.

Notice that the air's direction has now been permanently affected by the wing (it has been given, not loaned, downwards momentum by the wing). This means that we can use Newton's third law to justify an equal and opposite upwards force on the wing.

To calculate the force on the wing, we return to ideas of circulation.

Moving the rear stagnation point has broken the symmetry of figure 3.7, and we now have a net circulation. For the flat plate, this circulation can be calculated as shown on page 116 to give

$$\Gamma = \pi C U \sin\alpha. \tag{3.10}$$

We now use the Kutta-Joukowski Law to calculate the lift

$$\begin{aligned} L &= \rho U b \Gamma = \rho U b \times \pi C U \sin\alpha \\ &= \pi \sin\alpha \times \rho \times Cb \times U^2 \\ &= (2\pi \sin\alpha) \times \frac{\rho A U^2}{2}, \end{aligned} \tag{3.11}$$

where we remember that the area A of the wing is the product of its span b and its chord length C.

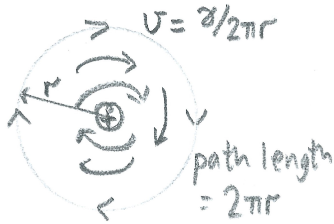

Figure 3.9: A path round a vortex adding up the fluid speeds on the way.

The final line has been arranged in the same manner as equation 3.3 to show that we have achieved a result in agreement with experiment and practice. Furthermore, we have correctly obtained the lift co-efficient for a wing made of a flat plate $C_L = 2\pi \sin \alpha$, which approximates to $2\pi\alpha$ if α is measured in radians. We have succeeded with our circulation model where our simplified Newtonian explanation failed — we have explained that the lift will be proportional to the angle of attack (at least while the assumptions of laminar flow over the wing hold).

While this reasoning is satisfying, aeronautical engineers can not leave it there. Real wings are not made of flat sheets of metal, as curved shapes are more effective. Unfortunately, the methods used here for the flat plate can not be used in all cases. Accordingly, the engineers have to use a different method.

Mathematicians can show that any changes to the flow pattern of an incompressible fluid caused by an object can be built up of little whirls (or vortices). They are the 'building bricks' of more complicated flow. In the same way as a very complicated structure can be made of simple building bricks (think of Lego or Minecraft, for example), we can build our flow using a set of vortices of different strengths in specified locations to mirror the actual flow round the object. The most useful building brick is a vortex of circulation γ. If we draw a route round the vortex at a distance r from its centre, the distance round the route is $2\pi r$, so we must have $\gamma = v \times 2\pi r$, so $v = \gamma/2\pi r$, and the speed of these little whirls gets less as you move further from them. If you double your distance from the centre, the speed halves, as shown in figure 3.9.

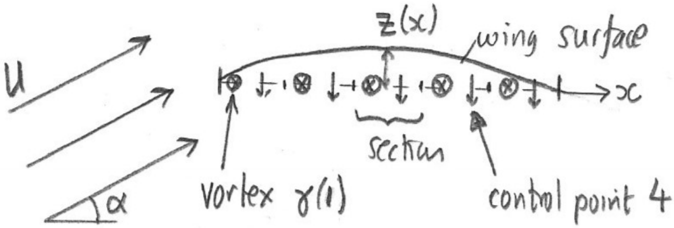

Figure 3.10: A wing modelled as a row of vortices. In the model, the strength of each vortex can be varied until the overall flow of air correctly flows over the surface of the wing without penetrating it.

In our computer (or using algebra on our sheet of paper), we draw a wing as a line of vortices as shown in figure 3.10. In the model, the whirls' strengths are individually adjusted to ensure that no air passes through the surface of the wing, thus mirroring reality where the molecules simply can't do it. Once the strength of the model whirls have been accordingly adjusted, the total circulation of the wing is given by the total of the circulations of all of the vortices, and the lift can be calculated using the Kutta-Joukowski Law. A more detailed description of the procedure is given on page 116.

After two sections of text looking at the origin of lift, you could easily be forgiven for wishing a summary. For me, the best is that of Carpenter[1], which I will paraphrase (although I shall omit the part about viscosity, vital as it is, as we have not been able to cover that):

> Lift is generated by the production of circulation around the wings. By circulation, we mean that the air flows faster over the top of the wing than underneath. By the conservation of energy, this results in a lower pressure above the wing than below it, and thus a net upward force. The circulation is generated when the point at which the air leaves the top surface

[1] p178 of C. Carpenter *Flightwise: Principles of Aircraft Flight*, Airlife, Shrewsbury 1996

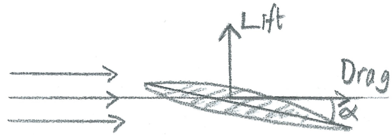

Figure 3.11: Definitions of lift, drag and angle of attack for a wing.

of the wing moves back to the trailing edge, because the air passing underneath can not turn the corner rapidly enough.

3.3 Lift and Drag co-efficients

Having explained as best as I can the nature of aerodynamic lift, we now turn to the practical business of using our knowledge to best advantage as it affects practical flight.

For the purposes of our ongoing analysis, all we need to remember of the earlier part of the chapter is that lift is the force on an object at right angles to the direction of fluid flow, and is given by $L = \frac{1}{2}C_L \rho A v^2$. Similarly we take drag as the force on an object parallel to the direction of fluid flow given by $D = \frac{1}{2}C_D \rho A v^2$. We treat these separately because the physical principles causing them are so different. These are shown illustrated in figure 3.11. Note that drag co-efficients plotted on graphs usually refer to the drag caused by the wing alone, and do not include drag from the fuselage, tail and other parts of the aircraft.

Typical values for C_L and C_D for a wing are shown in figure 3.12. Let us look at lift first.

We see that the co-efficients depend on the angle of attack α. Something very interesting happens to the lift as the angle of attack increases. When it gets too large, the air flow over the wing becomes much more turbulent, and the lift decreases rapidly once the angle of attack is increased further. This is the condition known as a stall, and it occurs whenever the wing's angle of attack is made too large. As you have already noted in question 3.2 a higher angle of attack is needed in order to fly slowly. However there comes a point where any reduction in speed would require an angle of attack which would result in a stall. This is not possible without losing

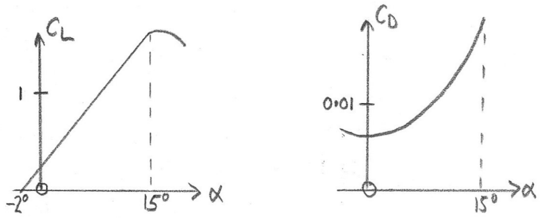

Figure 3.12: Graphs showing the values of lift and drag co-efficients for the wing of a typical light aircraft as a function of the angle of attack α. Notice that the graph of C_L is more or less a straight line until it peaks.

much height, and this speed is known as the 'stalling speed' - the speed below which a pilot must not attempt to fly. However as we will see, the stalling speed depends on many things.

Below the stalling angle of attack, the graph of C_L is a straight line, however it does not go through the origin. This is because the wing has an asymmetrical shape. The upper surface is more curved than the lower one and accordingly some circulation is induced even when the trailing edge is at the same height as the leading edge.

The angle of attack at which you gain zero lift is written α_0 and is usually about $-2°$. The coefficient of lift C_L is more or less proportional to the angle of attack measured from this datum: that is, C_L is proportional to $\alpha' = \alpha - \alpha_0$. Accordingly, when we plot a graph of C_L against α', it goes through the origin. In this section we will write $C_L = k_L\,\alpha'$ to include the angle of attack in our formula explicitly (where k_L is a constant).

On page 47, we showed for a flat plate (where $\alpha_0 = 0$) that $C_L = 2\pi\alpha$. This would mean that $k_L = C_L/\alpha = 2\pi$ if we measure α in radians. On graphs of lift and drag co-efficients it is more common to use degrees. To convert, we note that if k_L is 2π per radian and a radian is $(180/\pi)°$, $k_L = 2\pi/(180/\pi) = \pi^2/90 \approx 0.1$ per degree. We expect the gradient of our graph of C_L against α to be of this order. To give an example, the NACA 2412 aerofoil used on the Cessna 152 training aircraft has a k_L of 0.07 per degree.

Using the notation that $C_L = k_L\alpha'$, we may write our equation for the lift as

$$L = \frac{k_L \rho A \alpha' v^2}{2} = \frac{k_L \rho A (\alpha - \alpha_0) v^2}{2}. \tag{3.12}$$

If we neglect the fact that the air must be pushed downwards by the wing, and will not continue horizontally, then this lift force will be vertical. If the mechanism of thrust (e.g. a propeller) is only providing a horizontal force, level flight will occur if the lift and the weight balance. Thus

$$W = \frac{k_L \rho A \alpha' v^2}{2}, \tag{3.13}$$

and as the weight of an aeroplane does not change rapidly, steady flight requires

$$\alpha' = \frac{2W}{k_L \rho} \frac{1}{v^2} = \frac{\text{constant}}{v^2}, \tag{3.14}$$

and accordingly the required angle of attack (relative to α_0) is inversely proportional to the square of the speed of flight.

Question 3.4 *Data for the NACA 2412 aerofoil indicates that* $\alpha_0 = -2.1°$ *and* $k_L = 0.072$ *per degree. The wing area of the aircraft is* 14.8m^2*, and we shall assume an air density of* $\rho = 1.2\text{kg}/\text{m}^3$*. If the total loaded mass of the aircraft is 750kg, calculate the angles of attack needed to fly level at 50kt and 90kt. Hint: first calcluate the weight, which must equal the lift, then the* C_L *values needed for each speed.*

We now consider *drag* — that is aerodynamic forces acting parallel to the airflow. There are two parts to this drag. The part caused by the wing's cross section, negecting any effects occurring at the wing tip, is called the *parasitic drag*. This includes *form drag* caused by any whorls or vortices of air set up by the motion of air over the wing, and also *skin friction* whereby the layer of air nearest the wing 'sticks' to it, and viscous effects then resist the passage of air flowing over the boundary layer. These types of drag have nothing to do with the production of lift.

Our next type of drag is different, and is an essential side-effect of lift. A real wing has tips, and air will pass from below to above round the edge. This sets up vortices which force the air in the vicinity of the wing itself downwards, as explained mathematically on page 119. Thus the air passing the wings is now flowing downwards as it passes the wings, as shown in figure 3.13. The lift is perpendicular to this airflow, and correspondingly

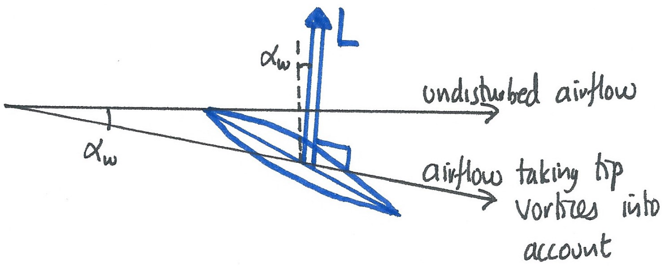

Figure 3.13: The effect of downwash caused by wingtip vortices on the angle of attack. We notice that the lift, which is perpendicular to the air as it actually flows diagonally downwards past the wing now has a horizontal component.

is no longer perpendicular to the completely undisturbed air far from the wing. The backwards horizontal component relative to the undisturbed air is a type of drag known as *induced drag*.

We see in figure 3.13 that the horizontal component of the lift will be $L \sin \alpha_w$ where α_w is the angle by which the air is deflected by the wingtip vortices. As we shall see on page 124, this angle is proportional to the lift co-efficient C_L. Thus the induced drag will be $D_i = L \sin \alpha_w \approx L\alpha_w$ providing α_w is small, and will therefore be proportional to LC_L. For an aircraft in level flight, the lift must balance the weight $L = W$, and so the induced drag will be proportional to WC_L. Given that we already know that C_L is proportional to α' and hence to $1/v^2$ in level flight, it follows that the induced drag will also be proportional to $1/v^2$, the weight being constant.

The two forms of drag are given their own coefficients, so the parasitic drag is given by $D_p = \frac{1}{2} C_{Dp} \rho A v^2$, while the induced part is given by $D_i = \frac{1}{2} C_{Di} \rho A v^2$. The total drag will equal the sum of these two parts $D = \frac{1}{2} C_D \rho A v^2$, where $C_D = C_{Dp} + C_{Di}$. It should be noted that graphs of C_D against α usually consider the aerofoil (wing) alone. If you are analysing the motion of a whole aircraft, then the drag from the fuselage (body) and tail also need to be added.

The dependence of any form of drag on speed in the case where the angle of attack is constant is simple. In this case drag is proportional to

v^2 because C_D does not change unless α does. However, when an aircraft flies at different speeds, the angle of attack will be changed to keep the lift equal to the weight, which changes the values of C_L and C_D.

We have already seen that in this case, the induced drag D_i is proportional to $1/v^2$. We can also determine how the induced drag co-efficient C_{Di} depends on the angle of attack and the speed. As we saw earlier, D_i is proportional to LC_L, and hence to $\frac{1}{2}C_L\rho Av^2 \times C_L = \frac{1}{2}C_L^2\rho Av^2$. Thus, at a given speed, C_{Di} must be proportional to C_L^2. We already know that C_L is proportional to $\alpha - \alpha_0$, which in turn is proportional to $1/v^2$ for level flight. Thus C_{Di} is proportional to $(\alpha - \alpha_0)^2$, and so to $1/v^4$ in this case.

The parasitic drag of a wing is more complicated. For many wings (and indeed the simplified models used in some books), the parasitic drag co-efficient varies little between different angles of attack actually used in flight, and accordingly the parasitic drag $D_p = \frac{1}{2}C_{Dp}\rho Av^2$ will be proportional to v^2. However, as shown in figure 3.12 some wings (particularly of older aircraft) have a component of their wing's drag co-efficient which is proportional to the square of the angle of attack, not all of which is due to induced drag.

The fuselage and other parts of the aircraft will also contribute to the parasitic drag. The total parasitic drag for the aeroplane is usually written $\rho f v^2/2$. In this equation, f is the sum of $C_D \times A$ for parasitic drag for all components of the aircraft (excepting any wing parasitic drag component with C_{Dp} not constant). Using this, together with equation 6.30 on page 125 for the induced drag for a simplified scenario, gives a total drag on the aircraft of

$$D = \frac{fv^2}{2} + \frac{gL^2}{v^2} + \frac{2L^2}{\pi\rho b^2 v^2}, \tag{3.15}$$

where b is the wingspan, L is the lift (which in level flight must equal the weight) and the second term in the equation represents the components of wing parasitic drag which depend on $(\alpha - \alpha_0)^2$. Often this middle term can be ignored.

As shown in figure 3.14 drag is accordingly high at low speeds where there is lots of induced drag, and also high at high speeds where the parasitic drag predominates. Somewhere in the middle is the speed of lowest drag. This is the speed you fly at if you wish to minimize the force needed from your engine. We will refer to it as $v_{\min\mathrm{F}}$. The total work done against

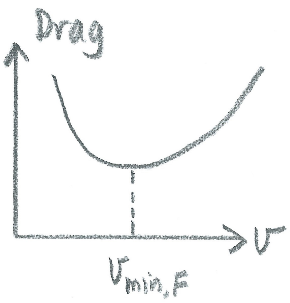

Figure 3.14: Graph showing the total drag on an aircraft flying level at different speeds.

this force when you fly a distance D is given by $F \times D$, and this matches the energy requirement you have from your fuel. Assuming that the distance D you need to travel is fixed, you use the least fuel if you fly at the speed which minimizes the total drag force. Alternatively, if you have a fixed amount of fuel, this is the speed which allows you to travel the greatest distance — it gives you your maximum *range*.

The calculation is different if you are a recreational pilot and wish to spend half an hour in the air and don't mind how far you go. In this case, the energy needed E is equal to the power of the engine P multiplied by the time airborne t. We gain maximum fuel economy in this case (or, alternatively, the maximum time aloft for a given amount of fuel) if we fly with the lowest power $P = E/t$. Now, if $E = FD$, and $D = vt$, it follows that $E = Fvt$, and so the power $P = E/t = Fv$. We maximize our time airborne when we fly at the speed which makes $P = F \times v$ as small as possible. This is flying for maximum *endurance* at a slightly slower speed which we shall write as $v_{\mathrm{min\,P}}$.

In propeller aircraft, the setting of the throttle in the cockpit broadly controls the power P of the engine. In normal flight, where we are at or above $v_{\mathrm{min\,P}}$, higher power settings are needed the faster we fly. However there is nothing stopping you flying slower than this, providing that you don't stall. However at these slow speeds, you actually need a greater power the slower you go (to combat the fact that you have to point the nose up further up as you slow down). This is known as flying on the 'back of the drag curve' and is not instinctive.

Question 3.5 *Assuming that, following the form of equation 3.15, the drag force* $D = a/v^2 + bv^2$ *where* a *and* b *are co-efficients representing the strengths of the induced and parasitic drag, work out the speed for least drag in terms of* a *and* b*. (This question requires calculus or the ability to complete the square.)*

Question 3.6 *Assuming that the drag force* $D = a/v^2 + bv^2$ *where* a *and* b *are co-efficients representing the strengths of the induced and parasitic drag, work out the speed for flight at least power* $P = Dv$ *in terms of* a *and* b*. (This question requires calculus .)*

Question 3.7 *The Information Manual of the Cessna 152 (a very popular training aircraft) gives the fuel consumption for the aircraft at different speeds. At standard temperature at an altitude of 2000ft, it quotes a usage of 23.8 litres per hour at 102kt, and 16.2 litres per hour at 81kt. Assuming that fuel consumption is proportional to power* $P = Dv$ *and that* $D = a/v^2 + bv^2$*, calculate the speeds needed for maximum range (i.e. flying distance) and maximum endurance (i.e. flying time). In practice, pilots of light aircraft do not fly this slowly as it is impractical. However the recommended speed when gliding after an engine failure (60kt) is reasonably close to the answer you should get for the speed for maximum endurance. Hint: write the fuel consumption as* $F_c = A/v + Bv^3$ *and work out the values of* A *and* B *needed to fit the data. Then use calculus or completing the square to work out the speed needed to minimise* F_c *(for greatest endurance) or use calculus to work out the speed needed to minimise* F_c/v *(for greatest range).*

3.4 Mini Wings

Our discussion so far has focused on the ways in which the wings of an aeroplane (and, by extension, the rotor blades of helicopter or gyroplane) enable humanity to defy gravity and join the birds in the enjoyment of flight. This is a remarkable discovery, and is the main focus of this chapter. However, as a masterful stage actor might acknowledge the vital support of the unassuming stage manager, so we must not ignore the other vital ingredient for safe flight — stability. It was the insight of the Wright brothers in the department of producing a flying machine with sufficient stability in flight and controllability for the pilot which set them apart from their contemporaries to the greatest degree.

An aeroplane usually has a tail with a horizontal stabilizer (tailplane) and a vertical stabilizer (fin). The same physics which applies to wings also applies to these surfaces. There is an important difference between these stabilizers and the wings in that they are usually constructred symmetrically (with the same amount of curvature on both sides). As a result, there is no lift when the angle of attack is zero and we can write $\alpha_0 = 0$. Their short, stubby, nature also leads to less lift being produced for a given area than we would expect for our ideal, infinitely long, wing. Thus typical values for k_L are of order 1.5 per radian, or 0.03 per degree.

In most designs of aeroplane, the tailplane and fin are fixed to the aircraft's body, however they each have a moveable rear section which is adjustable by the pilot. The adjustable part of a tailplane is called an elevator, while the equivalent on the fin is known as the rudder. The angle of attack of the tailplane can be controlled in flight to produce either upwards or downwards 'lift' to cause the tail to rise or fall in relation to the centre of gravity. This is how a pilot controls the aircraft's *pitch*. Similarly, if the rudder is moved, a 'lift' force will act sideways causing the tail to move left or right. This is known as *yaw*.

Even without pilot input, both surfaces enable stable flight. Suppose a gust suddenly caused the aircraft to pitch up (nose high). The increased lift on the tailplane due to its increased angle of attack, together with the considerable leverage of the tail given its location far from the centre of gravity, will bring the nose back down again. Equally if an aircraft began to yaw, such that the airflow was striking the aeroplane from the side, the fin would have a non-zero angle of attack, and its lift force would move the tail sidways pointing the nose back into the flow of the oncoming air.

To ensure this stability, aircraft are designed to have their centres of gravity in front of the point where the wings' lift acts as if it is generated. (This point is known as the *centre of pressure*.) A gust causing a nose-high attitude will be corrected as the extra lift from the wings will tend to point the nose down. This does have the interesting and counter-intuitive consequence that left to itself, a wing would cause the aircraft to pitch down, and to balance this, the tail usually needs to be set to give a downwards force.

There are also adjustable surfaces on the trailing (rear) edges of the outer sections of the wings called *ailerons*. When the pilot wishes to bank left, the left aileron is moved up (reducing the angle of attack and the lift

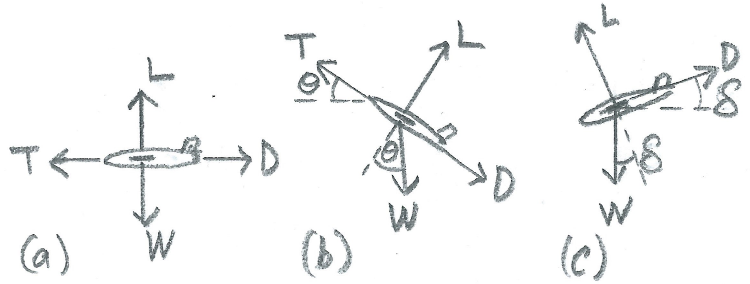

Figure 3.15: The four forces on an aeroplane when (a) in level flight, (b) climbing, and (c) gliding.

from the left wing) while the right aileron is moved down (increasing the angle of attack and the lift from the right wing).

In modern aeroplanes, the controls are set out ergonomically and intuitively. Ailerons are controlled by turning the control wheel, elevators by pushing the control wheel forwards or backwards (forwards to pitch down), and the rudder is connected to foot pedals (pressing the left pedal causes the rudder to move left, thereby producing a yaw to the left). This, together with the inherent stability of all aeroplanes flown manually, means that anyone can fly a plane.

Early aircraft were more tricky - they were not as stable, and the controls were not laid out as conveniently. The Wright brothers Flyer required the pilot to lie down, and shift their hips from side to side to control wing warp (roll) and rudder, while a lever held in the left hand was linked to the elevators.

3.5 Four Forces

So far we have given attention to two of the forces on an aeroplane in flight, namely lift (the force due to airflow over the aircraft which is perpendicular to airflow) and drag (the force due to airflow which is in opposition to airflow). The other forces of concern to us are weight, which always points downwards, and thrust (the forwards force caused by the engine).

In level flight at steady speed, show in figure 3.15 (a), we must have $T = D$ horizontally and $L = W$ vertically. If we were to suddenly open the throttle, this would increase T, and the aircraft would accelerate forwards. The new, higher, speed would lead to greater lift (so the aeroplane would climb even if the pilot did not 'pull up' on the controls) and greater drag leading to a reduction in the acceleration. If the engine were suddenly to fail, the aircraft would decelerate, leading to a reduction in lift, and a resulting loss of altitude.

The situation is slightly more complicated when an aircraft is climbing with its nose raised. This is shown in figure 3.15 (b). Notice that while T, L and D point in directions determined by the motion of the air over the aircraft, the weight is still downwards. If the aircraft is climbing at an angle θ then it follows that for steady speed, the forces must balance.

It is best to analyse this situation by resolving parallel and perpendicular to the airflow. The weight W can be broken down into two parts. There is a component $W\cos\theta$ opposing the lift L and a component 'backwards' $W\sin\theta$ opposing the thrust T. For the forces to balance, we must have

$$\begin{aligned} L &= W\cos\theta && (3.16) \\ T &= D + W\sin\theta. && (3.17) \end{aligned}$$

We can work out the angle of climb θ from its sine if we rearrange equation 3.17:

$$\sin\theta = \frac{T - D}{W}. \tag{3.18}$$

This leads to the strange conclusion that the lift force on an aircraft climbing with its nose up is less than the lift force on an aeroplane in level flight.

We finally consider the case where the engine has been turned off, or has failed. As mentioned above, if unchecked by the pilot, the aircraft would slow down, the lift would reduce, and there is a risk of a stall. Under these circumstances, the pilot will push the control wheel forward to lower the nose. Figure 3.15 (c) shows the situation, where the nose has been lowered by an angle δ. The forces will be balanced if

$$\begin{aligned} L &= W\cos\delta && (3.19) \\ W\sin\delta &= D. && (3.20) \end{aligned}$$

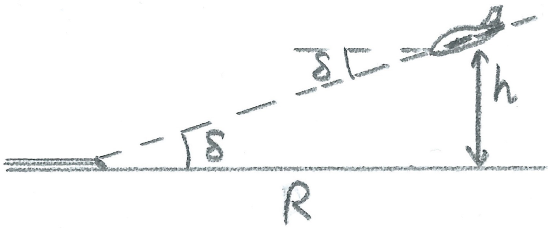

Figure 3.16: The distance an aircraft can glide.

Notice that this means that

$$\frac{D}{L} = \frac{W \sin\delta}{W \cos\delta} = \frac{\sin\delta}{\cos\delta} = \tan\delta. \tag{3.21}$$

The quotient L/D is known as the lift:drag ratio, and for a light aeroplane of traditional construction its maximum value is approximately 9. For $L/D = 9$, we have $\tan\delta = 1/9$ and accordingly, $\delta \approx 6^\circ$.

The value of δ affects how far an aeroplane can glide from a starting height h as shown in figure 3.16. We see there that $h/R = \tan\delta = D/L$. Thus

$$R = \frac{h}{\tan\delta} = \frac{h\,L}{D} \approx 9h \tag{3.22}$$

for a light aircraft.

Question 3.8 *Assuming an L/D ratio of 9, what is the gliding distance R in nautical miles (1 nm = 1852m) from a starting height of 1000ft (1ft = 0.305m)?*

Question 3.9 *Supposing that the speed which optimizes the ratio L/D is 65kt relative to the surrounding air, work out how your answer to question 3.8 would change if the aircraft were facing a headwind of 10kt.*

Any pilot preparing to fly a new type of aircraft for the first time will read the manual carefully, and will memorize relevant speeds. One such speed is the speed for the best glide. This is the speed at which the lift and drag forces have the largest ratio and it occurs at an angle of attack of about 10°. It is worth noting that the angle of attack for maximum C_L/C_D for a wing is much lower (usually about 3°), however this doesn't take the drag from the fuselage into account. The pilot will use the elevators to ensure that the aircraft flies at this speed if the engine fails. If the speed is higher or lower, the aircraft will not glide as far from that height before reaching the ground.

3.6 Turning

While you may not change speed, it still takes a force to turn. On the road, this is provided by friction between the tyres and the ground. A physicist would say that the force is necessary because although you are not changing your *speed*, you are changing your *velocity*. Velocity is a combination of the information of how fast *and which way* you are going. Any change of velocity represents an acceleration, and accordingly requires a force. When you whirl a stone around on the end of a string, the string provides the force. Given that the string links the stone with you (in the centre of the circle), centripetal forces and accelerations always point towards the centre of the circle or turn.

The acceleration of a turning object is called a *centripetal* acceleration. The size of the acceleration depends on the speed of the object and also the rate of turning. Remembering that we like to measure angles in radians (introduced on page 9), the rate of turning is measured in radians per second. This tells us how far the turn progresses each second, and is also known as the *angular velocity*. It is given by the symbol ω, and is equal to the angle turned divided by the time taken.

We already know that the distance travelled while something moves on a circular path of radius R is given by $D = R\theta$, where θ (the angle turned) is in radians. The speed of the object will be the distance divided by the time taken, so

$$v = \frac{D}{t} = \frac{R\theta}{t} = R\frac{\theta}{t} = R\omega, \tag{3.23}$$

where in the final stage we have equated $\omega = \theta/t$.

So, if a car, a stone on a string, a planet in an orbit, or an aeroplane is moving at speed v and turning at rate ω, how large is its acceleration? The

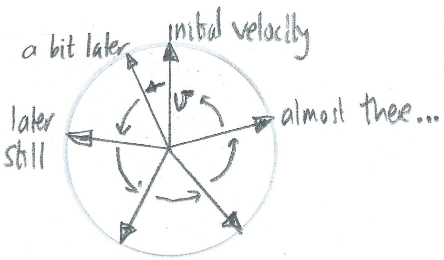

Figure 3.17: The change in velocity during a full turn.

answer is

$$a = \omega\, v. \tag{3.24}$$

In flying, when an air traffic controller tells a pilot flying under instrument flight conditions to turn, the assumed turn is said to be a *Rate 1* turn, in which the aircraft will complete a 'U' turn in one minute. At this rate, the angular velocity will be $\omega = \theta/t = \pi\,\mathrm{rad}/60\,\mathrm{s}$. A Rate 2 turn would have twice this angular velocity. This is approximately the rate of turn used by pilots when flying by what they see out of the window, and involves banking the aircraft to an angle of $30°$.

Question 3.10 *Calculate the centripetal acceleration during a Rate 1 turn at a speed of 200m/s.*

Question 3.11 *An aerobatic pilot is flying at 60m/s when they do a 'U' turn in 5 seconds. Calculate their centripetal acceleration.*

The rest of this section is broken into three parts. Firstly we justify the equation $a = \omega v$, then we study how an aeroplane turns at a constant height, and finally we look at gliding turns.

Centripetal Acceleration

Suppose that we make a full turn in time T. A full turn is an angle of $2\pi\,\mathrm{rad}$, and so $\omega = 2\pi/T$. Therefore $T = 2\pi/\omega$.

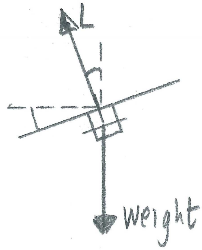

Figure 3.18: An aeroplane (viewed from the rear) in a banked turn with a bank angle β. The lift force is now diagonal.

If we draw the velocity $\mathbf{v}$ as a vector at different stages in the turn, as shown in figure 3.17, the vector itself rotates once - just like the second hand of a clock which is watched for one minute. The total amount by which the velocity has changed is the 'distance' moved by the arrow head of this vector, namely the circumference of this circle. The radius of the circle is v (the magnitude of the vector), so its circumference will be $2\pi v$.

The acceleration is the rate of change of velocity — namely the change in velocity divided by the time taken. In this case

$$a = \frac{\text{Velocity Change}}{\text{Time}} = \frac{2\pi v}{T} = \frac{2\pi v}{2\pi/\omega} = v\omega. \tag{3.25}$$

Level Turns

It was mentioned earlier that in a car, friction between the tyres and the road provide the force and thus the acceleration needed to turn the car when you turn the steering wheel. In the air, there is no road and there are no tyres, however the 'steering wheel' is still vital. It controls the *ailerons* which *roll* the aeroplane, banking it to one side or the other. Once the aircraft is in a bank, the lift force no longer points upwards, but diagonally to the side. It is the horizontal part (or *component*) of this force which we use to turn.

The situation is shown in figure 3.18 with an aeroplane banked by angle β. The vertical component of the lift $L\,\cos\beta$ balances the weight, while the horizontal component $L\ \sin\beta$ is used to provide the centripetal acceleration.

There are two values we need to calculate. The first is to work out the bank angle required for a given turn. The second is to evaluate the lift needed to enable the turn to proceed without loss of height.

To evaluate the bank angle, we combine the formulae involving the two components of the lift L. Firstly, the vertical component must balance weight

$$L_v = L\cos\beta = mg. \tag{3.26}$$

Secondly, the horizontal component must provide the centripetal acceleration, so we use Newton's Second Law ($F = ma$ here) to write

$$\begin{aligned} F &= ma \\ L_h = L\sin\beta &= m\omega v. \end{aligned} \tag{3.27}$$

We divide equation 3.27 by equation 3.26 to give

$$\begin{aligned} \frac{L\ \sin\beta}{L\ \cos\beta} &= \frac{m\omega v}{mg} \\ \tan\beta &= \frac{\omega v}{g}. \end{aligned} \tag{3.28}$$

Question 3.12 *Work out the bank angle to produce a Rate 1 turn at a speed of 90kt.*

Question 3.13 *When recreational pilots perform ordinary turns in open skies, they usually bank the aircraft to approximately* 30° *and fly at about 90kt. If initially flying East, how long will it take to turn North?*

When turning, the weight of the aeroplane will not change. We see from equation 3.26 that in order for the aircraft not to descend, the lift will need to exceed its normal straight–flight value. For gentle turns, the pilot increases the lift by pulling back slightly on the controls so that the aircraft's nose rises, thereby increasing the angle of attack of the wings and providing the extra lift. There will come a point, however, where this is not enough, and accordingly for steeper turns, the pilot will also open the throttle so that the vertical component of the thrust can help out too. The extra engine power is also needed to counter the additional drag produced as we increase the angle of attack.

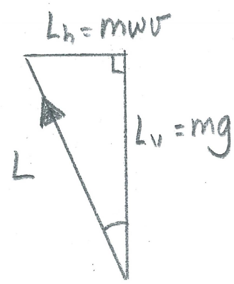

Figure 3.19: Lift and its components drawn as a vector triangle.

Let us now see how much lift we need. To evaluate this, look at figure 3.19, where the lift, together with its vertical L_v and horizontal L_h components are drawn as a triangle. We can use Pythagoras' Theorem to work out the total lift:

$$\begin{aligned} L^2 &= L_v^2 + L_h^2 \\ &= (mg)^2 + (m\omega v)^2 \\ &= m^2 g^2 + m^2 \omega^2 v^2 \\ L &= mg\sqrt{1 + \frac{\omega^2 v^2}{g^2}}. \end{aligned} \tag{3.29}$$

The expression $L/mg = \sqrt{1 + \omega^2 v^2 / g^2}$ is referred to by pilots as the *load factor*. You will notice that for a particular wing, there will be a maximum value which L can reach (just before the stall), and accordingly, there is a maximum bank angle for a level turn of steady speed.

Question 3.14 *Combine equations 3.29, 3.26 and 3.27, or alternatively use figure 3.19 to show that the number of 'g's pulled during a turn (L/mg) is equal to $1/\cos\beta$.*

Question 3.15 *Use your answer to question 3.14 to work out the bank angle needed to pull $2g$.*

Question 3.16 *Use the information given in question 3.4 on page 51, together with your answer, and also your answer to question 3.14, to work out the angle of attack needed to fly that same aircraft at a constant height in a* 30° *banked turn at 90kt. Also work out the angle of attacks needed for* 45° *and* 60° *banked turns.*

On page 49 we wrote that the lift $L = C_L \rho A v^2/2$. When the aircraft is at its stalling speed v_s, the maximum lift available for that configuration of the aircraft only just balances weight so $mg = C_{L,s}\rho A v_s^2/2$. Here we use $C_{L,s}$ to refer to the lift co-efficient at the stalling angle of attack. This means, by dividing these two equations, we get

$$\frac{L}{mg} = \frac{C_L \rho v^2/2}{C_{L,s}\rho v_s^2/2} = \frac{C_L}{C_{L,s}}\left(\frac{v}{v_s}\right)^2. \tag{3.30}$$

.

This equation, when combined with equation 3.29, enables us to work out the fastest rate of turn possible in an aircraft if we know the current speed and the stalling speed. Note that if we bank the aircraft at speed v and raise the nose to give the necessary extra lift to prevent descent, eventually we will reach the stalling angle of attack at which $C_L = C_{L,s}$ and therefore $L/mg = (v/v_s)^2$.

Question 3.17 *Suppose you wish to turn at 45m/s in an aircraft which (in its current configuration) has a stalling speed of 25m/s. Work out the number of 'g's pulled and the maximum bank angle you can use in a level turn without stalling or losing height.*

Sideslipping

In this section, we consider a bank that does not lead to a turn, which is useful when landing at times when the wind direction is not parallel with the runway (a *cross wind*).

Suppose a pilot is approaching an East-West runway from the East, but there is a wind blowing from the South. If they point the aircraft westwards, the wind will blow them off course. The usual remedy is to point the nose slightly south of west, and rely on the wind to blow the aircraft back onto a westwards course. However the pilot wishes to land pointing along the runway. Otherwise, as soon as the wheels make contact with the ground,

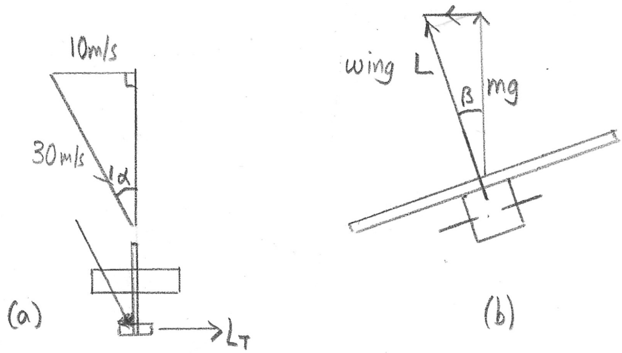

Figure 3.20: Calculating the bank angle needed to offset drift when approaching a runway in a crosswind.

the aeroplane will roll off the side of the runway. They could maintain the south-of-west heading until the moment before landing, then use the rudder to turn west (this is called 'crabbing'). However there is an alternative, known as the 'wing-down method', which is a form of side slipping.

In this, the pilot points the aircraft West. The lift force on the fin will then tend to push the aircraft North, and also cause it to turn left. The former is counteracted by banking left (so that the lift force from the wings now partially points left), while the latter is prevented by using the rudder. Effectively, these forces on tail and rudder are of very similar strength given that the rudder has prevented the aircraft from turning into the wind. Drag from the fuselage, however, will still endeavour to push the aircraft North, and accordingly, a left bank (or roll) is still needed in order to fly West.

Let us calculate the bank angle needed for the case where a 750kg aircraft is flying in $\rho = 1.2\text{kg}/\text{m}^3$ air westwards at 30m/s with a 6m/s wind from the South, with a fuselage area of $A_F = 6\text{m}^2$ and a fuselage $C_D = 0.5$. As shown in figure 3.20 (a) the air reaches the fuselage at an angle of $\alpha = \sin^{-1}(6/30) = 11.5°$, and so the area of fuselage presented to the airflow will be $A_F \times \sin\alpha = 1.2\text{m}^2$. The drag force on the fuselage D can be calculated

$$D = \frac{C_D \rho A_F v^2 \sin\alpha}{2} = \frac{0.5 \times 1.2 \times 6 \times 30^2 \times 0.2}{2} = 320\text{N}. \tag{3.31}$$

To balance this force with a sideways component of lift, while maintaining sufficient vertical lift to counteract weight, as shown in figure 3.20 (b) requires a bank angle of

$$\beta = \tan^{-1}\left(\frac{D}{W}\right) = \tan^{-1}\left(\frac{320}{750 \times 9.81}\right) = 2.5^\circ. \tag{3.32}$$

Gliding Turns

In the previous section, we noticed that during a turn, less of the lift is available to keep the aircraft in the air, as some of it is needed to provide the turning (or *centripetal* acceleration). Accordingly, the pilot will use the control column to keep the nose up and possibly add more power to ensure that it does not descend. During a gliding turn, the engine is not available, and descent is inevitable. Here it is necessary for the pilot to ensure that the nose is *not* pulled up to ensure that the aircraft maintains sufficient speed to remain airborne. In other words, a descent may be tolerated, but a stall must be avoided.

Using the terminology of the previous two sections, we represent the angle of bank by β and the angle of descent by δ.

As in equation 3.20, the component of the weight pointing in the direction of flight balances the drag:

$$W \sin\delta = D. \tag{3.33}$$

However equation 3.19 needs to be rewritten, as not all of the lift L is available to counter the other component of the weight ($W \cos\delta$), but only $L \cos\beta$ as in equation 3.26, thus we have

$$W \cos\delta = L \cos\beta. \tag{3.34}$$

Finally, as in equation 3.27, the sideways component of lift $L \sin\beta$ provides the centripetal acceleration:

$$m\omega v = L \sin\beta. \tag{3.35}$$

Dividing equation 3.35 by equation 3.34 gives a formula for the angle of bank:

$$\tan\beta = \frac{m\omega v}{W\,\cos\delta} = \frac{\omega v}{g\,\cos\delta}, \tag{3.36}$$

where we remember that $W = mg$. We see that we will need a slightly larger angle of bank β for a given speed and rate of turn as $\cos\delta < 1$.

Next, we divide equation 3.33 by equation 3.34 to give an expression for δ:

$$\tan\delta = \frac{D}{L\,\cos\beta} = \frac{C_D}{C_L\,\cos\beta}. \tag{3.37}$$

Notice that we are going to need a larger angle δ than in a straight glide on account of the angle of bank: $\cos\beta < 1$, and this makes $\tan\delta$ larger. This is the mathematical way of expressing the pilot's adage that they must not pull back too hard on the controls when turning in a glide otherwise they will slow and stall - the speed must be maintained, and there is no engine to save the day.

Question 3.18 *For a rate 1 turn (that is, π radians in one minute) at 40m/s with $C_L/C_D = 9$ work out the angles δ and β needed. Hint: algebraically this is difficult because of the nature of equations 3.36 and 3.37 — it is hard to eliminate either β or δ. The simplest method is to initially estimate δ by assuming we are not turning (using equation 3.21), then use this value of δ in equation 3.36 to gain an initial estimate for β. We then use the new value of β in equation 3.37 to get a better value of δ, then use this improved δ to get a better β using equation 3.36. We then use this newest value of β in equation 3.37 to get an even better value of δ, then use this even better δ to get a still better β using equation 3.36. This process can be repeated until the values* converge, *that is, the repetition does not change the values significantly. Alternatively, if you insist on an algebraic expression for the final answer, try squaring equations 3.36 and 3.37 and remember the trigonometric identity $1 + \tan^2\theta = 1/\cos^2\theta$ to help you perform the elimination.*

Question 3.19 *A particularly interesting gliding turn which is useful to pilots in the event of engine failure requiring them to land a light (single-engine) aeroplane in a field, is to fly until the desired landing point is to the side of the aeroplane at a particular angle below the horizon ϵ (when viewed in the window), and then to turn to keep it at that angle of depression until the aircraft is lined up ready to land. The aircraft then descends on a sort of spiral drawn on a cone* *with its apex at the landing point and its axis vertical. Given that angles of depression ϵ can be difficult to judge, it turns out that during the descent, the path can be maintained by keeping the landing point a certain angle ψ to the side of the nose. Calculate this angle as a function of the descent angle δ and the angle of depression ϵ.*

Hints: assume a speed v and use r to denote the distance of the aircraft from the cone's axis. Calculate, in turn (i) the vertical velocity, (ii) the rate of decrease of r using a knowledge of ϵ which is the cone's angle above the horizontal, (iii) the horizontal component of velocity, (iv) the component of the horizontal velocity which is directed towards the aim point in terms of ψ, and then equate your answers to (ii) and (iv).

Solutions: Vertical velocity $=\ v\,\sin\delta$, thus rate of decrease of r will be $v\,\sin\delta/\tan\epsilon$. Horizontal velocity $= v\,\cos\delta$, which has a component $v\,\cos\delta\,\cos\psi$ towards the aim point, which must be the rate of decrease of r. Thus $v\,\sin\delta/\tan\epsilon = v\,\cos\delta\,\cos\psi$, and hence $\cos\psi = \tan\delta/\tan\epsilon = C_D/(C_L\,\tan\epsilon)$.

With Calculus

Chapter 4

Up, Up and Away

So far, we have looked at the motion of fluids, particularly with relevance to how they affect flight without using calculus. However our analysis becomes much more powerful when we do. In this chapter, we study in greater depth what happens in our atmosphere as we rise up through it. The pressure reduces, as we have already seen, but it does so more slowly as we rise. This is because air is compressible, and as we rise, the density decreases as well as the pressure. We solve this algebraically, and find that the pressure reduces exponentially. This gives us our first taste of a general principle called the Boltzmann Law.

We also find that as the pressure drops, the temperature does as well, and work out the rate at which it does so. Moisture in the air has a particular importance for our atmosphere, and we study how much water vapour a particular parcel of air can hold. A further application of the Boltzmann Law enables us to work out the temperature drop in saturated air, and we can study the height at which clouds are likely to form (and the height of the cloud top too). Finally in this chapter, we introduce an ingenioius diagram used by meteorologists to chart the atmosphere.

The next chapter uses methods of vector calculus to write Newton's Laws for a fluid (the *Navier Stokes equation*), and to point out how the effects described in earlier chapters flow naturally from this formulation. This will enable you to progress to more advanced texts.

4.1 Pressure at Height

In equation 2.14 on page 25, we saw that the Conservation of Energy required that

$$p + \frac{\rho v^2}{2} + \rho g h \tag{4.1}$$

remained constant. Accordingly, as the height rises, the pressure reduces. In this chapter, we consider the effect of this reduced pressure on the density. We begin by tightening up some of the notation. This equation only works completely when the density is constant. We can prepare for a variable density by considering only small changes in heights δh which lead to small changes in pressure δp, as over this small distance, the density will not change, and the constant will not change either (by definition). This gives us

$$\delta p + \delta \frac{\rho v^2}{2} + \rho g \, \delta h = 0. \tag{4.2}$$

If we assume we are dealing with stationary air, then $v = 0$, and we can then write

$$\delta p = -\rho g \, \delta h, \tag{4.3}$$

and so in the limit of very small changes in height we have

$$\frac{dp}{dh} = \lim_{\delta h \to 0} \frac{\delta p}{\delta h} = -\rho g. \tag{4.4}$$

Equation 2.14 on page 25 gives the density of air as

$$\rho = \frac{pm}{kT} \tag{4.5}$$

where m is the mass of a molecule, and k is the Boltzmann constant.

It follows that Bernoulli's equation 4.4 becomes

$$\frac{dp}{dh} = -\rho g = -\frac{mg}{kT} \, p. \tag{4.6}$$

This is a first order differential equation of the form $dy/dx = By$ where B is a constant. The solution of equations such as this is $y = y_0 e^{Bx} \equiv y_0 \exp(Bx)$ where y_0 is another constant. In our case, equation 4.6 solves to give

$$p = p_0 \exp\left(-\frac{mgh}{kT}\right). \tag{4.7}$$

The constant p_0 in this case is the pressure at sea level (where $h = 0$).

Before moving on to a more practical way of calculating with this equation, I wish to point out an interesting feature of this equation. The pressure at height h is proportional to the number of molecules at this height, and accordingly is linked to the probability of finding a given molecule at this height. Inside the exponent, the numerator is mgh which you will recognize as the gravitational potential energy of a molecule at this height. This is our first indication of a general principle known as the Boltzmann Law which states

$$\text{Probability of molecule having energy } E \propto \exp\left(-\frac{E}{kT}\right). \tag{4.8}$$

We shall return to this idea when we consider moisture in the air.

As molecules are so small, we shall find it more convenient in this chapter to deal in a fixed large number of them to make the masses and volumes more easy to comprehend. This fixed amout is called a *mole* and contains $N_A = 6.02 \times 10^{23}$ particles. The mass of all of these molecules is called the *molar mass* and is written as $M_r = N_A\, m$. The equivalent to the Boltzmann constant for a mole of molecules $R = N_A\, k$ is called the gas constant, and in our units is equal to $8.31\,\mathrm{J}/(\mathrm{mol\,K})$.

Working in moles rather than molecules, our equation for density becomes

$$\rho = \frac{pM_r}{RT}, \tag{4.9}$$

and our relationship for pressure at height becomes

$$p = p_0 \exp\left(-\frac{M_r gh}{RT}\right). \tag{4.10}$$

It is also worth pointing out that the Ideal Gas Law 2.16 on page 26 can be written for n moles of gas as

$$pV = nRT. \tag{4.11}$$

Question 4.1 *Work out the molar mass of air if you assume that it is made of 79% nitrogen (N_2) with a molar mass of 0.0280kg and 21% oxygen (O_2) with a molar mass of 0.0320kg.*

Question 4.2 *Work out the expected pressure at 1000ft, 5000ft, 10 000ft and 40 000ft using our original assumption that pressure drops by 364Pa for each 100ft increase in altitude, and also using equation 4.10 taking the temperature as 288K and the molar mass of air as calculated in question 4.1.*

Question 4.3 *Assuming that equation 4.10 were true, how high would you have to go for the atmospheric pressure to halve? Remember that if $y = A\ \exp(x)$, then it follows that $x = \ln(y/A)$.*

4.2 Dry Adiabatic Lapse Rate

The results of the last section show that the dependence of pressure on height is more complicated than we previously thought. Now we turn to another complication — the change of temperature as we climb through the air. As a parcel of air ascends into regions of lower pressure, it will surely expand. In pushing other air out of the way as it does so, it will expend energy, and therefore cool down. In air at equilibrium, the air already at that height will be at the same temperature as this freshly-cooled air. We now wish to work out the rate at which the temperature falls with height in dry air. This is called the *Dry Adiabatic Lapse Rate* or DALR for short, *adiabatic* meaning that the parcel of air does not lose or gain heat energy as it rises. If the air contains moisture there is an additional complication in that as the air rises, some of the water vapour will condense giving out latent heat reducing the temperature drop overall. We shall continue this special case on page 83.

The Lapse Rate is effectively $-dT/dh$, and using the chain rule we work this out as

$$-\frac{dT}{dh} = -\frac{dT}{dp}\frac{dp}{dh}. \tag{4.12}$$

We begin by working out dT/dh.

The First Law of Thermodynamics states that the change in the internal energy U of an object is equal to the heat Q given it and the work W done on it. We already know from equation 2.4 on page 19 that at a steady pressure p, the energy contained in a volume V of fluid is pV. If the pressure changes, then we limit ourselves to small changes in volume δV which will not affect the pressure too much. Energy will be given to the fluid if the pressure increases, which means that the fluid is being compressed. The work done on the fluid is therefore $\delta W = -p\delta V$. The internal energy, on the other hand is related to the temperature T. We define the energy needed to raise the temperature of one mole of the gas by 1K as the molar heat capacity at constant volume C_V. Accordingly, $\delta U = C_V\,\delta T$ where we shall assume from now on that we are dealing with one mole of gas.

The First Law of Thermodynamics states $\delta U = \delta Q + \delta W$ and therefore that

$$C_V\,\delta T = \delta Q - p\,\delta V. \tag{4.13}$$

Before proceding, it turns out to be more helpful if we set up our equation to have δp terms rather than δV. By the product rule, $p\,\delta V = \delta\,(pV) - V\,\delta p$. Furthermore $pV = RT$ for one mole (see equation 4.11), and accordingly, we replace the $p\,\delta V$ with $R\,\delta T - V\,\delta p$ and arrive at

$$\begin{aligned} C_V\,\delta T &= \delta Q - R\,\delta T + V\,\delta p \\ (C_V + R)\,\delta T &= \delta Q + V\,\delta p \end{aligned} \tag{4.14}$$

In the adiabatic case, $\delta Q = 0$ and so

$$\frac{\delta T}{\delta p} = \frac{V}{C_V + R}, \tag{4.15}$$

and in the limit of very small changes to the pressure we have the differential equation

$$\frac{dT}{dp} = \frac{V}{C_V + R}, \tag{4.16}$$

However, the Gas Law 4.11 reminds us that $pV = nRT$, and therefore for one mole $n = 1$ and we have $V = RT/p$. Substituting this into equation 4.16 we have

$$\frac{dT}{dp} = \frac{RT}{p\,(C_V + R)} = \frac{R}{C_V + R}\,\frac{T}{p}. \tag{4.17}$$

We now have a value for dT/dp as required by equation 4.12, but proceding it is useful go derive an equation for the way in which the pressure and temperature are linked. Remember that when differentiating $y = A\,x^n$ we get $dy/dx = nA\,x^{n-1} = nA\,x^n/x$ and accordingly that if y is proportional to some power of x, $dy/dx = n\,y/x$. Looking at equation 4.17 we see that dT/dp is proportional to T/p with the constant of proportionality $R/(C_V + R)$. It follows that for a gas unable to exchange heat with the surroundings,

$$T \propto p^{R/(C_V+R)}. \tag{4.18}$$

It also follows that

$$T^{C_V+R} \propto p^R. \tag{4.19}$$

We next look up $dp/dh = -M_r gp/(RT)$ from equation 4.6 after conversion to mole form, which enables us to evaluate, finally,

$$\begin{aligned} -\frac{dT}{dh} &= -\frac{dT}{dp}\frac{dp}{dh} \\ &= -\frac{R}{C_V+R}\frac{T}{p} \times \frac{-M_r gp}{RT} \\ &= \frac{R}{C_V+R} \times \frac{M_r g}{R}. \end{aligned} \tag{4.20}$$

The theorem of equipartition teaches that C_V is equal to $R/2$ multiplied by the number of *degrees of freedom* of the system. Here this means $R/2$ needs to be multiplied by the number of axes about which the molecule can store energy. For a monatomic molecules like helium, there would be three — it can move in any of three perpendicular directions (up, forward and left) — and so for helium we have $C_V = 3R/2$. Air principally comprises nitrogen and oxygen which are diatomic. These molecules also have two axes about which the molecule's rotation could be seen (perpendicular to the bond), and so a nitrogen molecule has five degrees of freedom, and accordingly has $C_V = 5R/2$, and $C_V + R = 7R/2$ (a constant also known as C_P as it corresponds to the energy needed to raise the temperature of a mole of gas by 1K at a constant pressure).

For diatomic molecules, equation 4.20 becomes

$$-\frac{dT}{dh} = \frac{2}{7}\frac{M_r g}{R}, \tag{4.21}$$

and if we take the molar mass of air as M_r=0.0288kg as a weighted average of 79% of 0.028kg for the nitrogen and 21% of 0.032kg for the oxygen, and perform the calculation, we expect air to get 9.7K (or 9.7°C) colder for every kilometre we ascend. Returning to pilot units, this corresponds to 0.00973×305 = 2.97°C for each thousand feet. The established answer is very close to this, at 9.8°C for each kilometre, and the difference is due to our simplification in taking air as a simple mix of oxygen and nitrogen.

Question 4.4 *Given what we know about C_V and R, we could have simplified this working from equation 4.17 onwards by writing $R/(C_V + R) = 2/7$.*

Follow the algebra from there, and make sure that the final answer for the DALR comes out as in equation 4.21.

Question 4.5 *If the atmosphere were made up of monatomic gases like helium, then we would have $C_V = 3R/2$. What would the equation for the DALR be in this case?*

Question 4.6 *In standard texts on thermal physics, the ratio $(C_V + R)/C_V$ is written γ. Show that using this notation, equation 4.18 could be written $p \propto V^{-\gamma}$. This is usually written $pV^\gamma = \text{constant}$.*

Question 4.7 *Starting with the First Law of Thermodynamics $\delta U = \delta Q - p\,\delta V$, and using the Gas Law for one mole $pV = RT$, the fact that $\delta U = C_V\,\delta T$ and the product rule $\delta(pV) = p\,\delta V + V\,\delta p$, show that $\delta Q = (C_V + R)\,\delta T - V\,\delta p$ and accordingly that the heat needed to raise the temperature of the mole of gas by unit temperature with constant pressure ($\delta p = 0$) is given by $C_P = C_V + R$. Note that this means that the γ constant of question 4.6 can also be written as $\gamma = C_P/C_V$.*

4.3 International Standard Atmosphere

Given that the temperature drops as we rise in the atmosphere, equation 4.7 is only an approximation to the pressure as a function of height. To make a better approximation to the usual state of our atmosphere, meteorogists have devised a set of assumptions. This is known as the International Standard Atmosphere, or ISA for short. It begins with the equations 4.4 and 2.14 as we used on page 72 to derive equation 4.6. The additional assumption is to take $T = T_0 - \Gamma h$. In other words, we assume a particular temperature at the ground $T_0 = 15°\text{C} = 288\text{K}$ and a lapse rate $\Gamma = 2\text{K}/1000\text{ft} = 0.00656\text{K}/\text{m}$. We take the pressure at sea level $h = 0$ to be $p_0 = 1.013 \times 10^5 \text{N}/\text{m}^2$.

Starting with equation 4.6 and substituting $T = T_0 - \Gamma h$, we solve the equation by separating the variables:

$$\begin{aligned}
\frac{dp}{dh} &= -\frac{mgp}{k(T_0 - \Gamma h)} \\
\int \frac{dp}{p} &= \frac{mg}{k\Gamma} \int \frac{dh}{h - T_0/\Gamma}
\end{aligned}$$

$$
\begin{aligned}
\ln p &= \frac{mg}{k\Gamma} \ln \left| h - \frac{T_0}{\Gamma} \right| + C \\
&= \frac{mg}{k\Gamma} \ln \left(\frac{T_0}{\Gamma} - h \right) + C,
\end{aligned} \tag{4.22}
$$

where C is the constant of integration, and we assume that $h < T_0/\Gamma$. We can evaluate the constant C by noting that $p = p_0$ when $T = T_0$ and $h = 0$. This gives

$$
\ln p_0 = \frac{mg}{k\Gamma} \ln \left(\frac{T_0}{\Gamma} \right) + C. \tag{4.23}
$$

Combining these equations to eliminate C gives

$$
\begin{aligned}
\ln p - \ln p_0 &= \frac{mg}{k\Gamma} \left\{ \ln \left(\frac{T_0}{\Gamma} - h \right) - \ln \left(\frac{T_0}{\Gamma} \right) \right\} \\
\ln \left(\frac{p}{p_0} \right) &= \frac{mg}{k\Gamma} \ln \left(\frac{T_0/\Gamma - h}{T_0/\Gamma} \right) \\
&= \frac{mg}{k\Gamma} \ln \left(1 - \frac{h\Gamma}{T_0} \right).
\end{aligned} \tag{4.24}
$$

Remembering that $\ln x^a = a \ln x$, we can derive an equation for p in terms of h as follows:

$$
\begin{aligned}
\frac{p}{p_0} &= \left(1 - \frac{h\Gamma}{T_0} \right)^{mg/k\Gamma} \\
p &= p_0 \left(1 - \frac{h\Gamma}{T_0} \right)^{mg/k\Gamma} = p_0 \left(1 - \frac{h\Gamma}{T_0} \right)^{1/\mu},
\end{aligned} \tag{4.25}
$$

where we define $\mu = k\Gamma/mg$. Using $\Gamma = 0.00656\,\mathrm{K/m}$, and taking $m = 4.78 \times 10^{-26}\mathrm{kg}$ as is typical for an air molecule, we have $\mu = 0.193$, however a more accurate evaluation gives an accepted value of $\mu = 0.190284$.

Pressure Altitude

The subject of equation 4.25 can be changed to give an expression for the height h:

$$
\left(1 - \frac{h\Gamma}{T_0} \right)^{1/\mu} = \frac{p}{p_0}
$$

$$1 - \frac{h\Gamma}{T_0} = \left(\frac{p}{p_0}\right)^{\mu} \tag{4.26}$$

$$h = \frac{T_0}{\Gamma}\left\{1 - \left(\frac{p}{p_0}\right)^{\mu}\right\}. \tag{4.27}$$

This equation enables us to calculate the height in the standard atmosphere at which we would expect to find pressure p. This is known as the *pressure altitude* — a measure of air pressure in units of height. The United States National Oceanic and Atmospheric Administration (NOAA) gives the formula as

$$h = 145366\left\{1 - \left(\frac{p}{1013.25\,\mathrm{hPa}}\right)^{0.190284}\right\}, \tag{4.28}$$

which is in line with our equation 4.27, albeit with the constant T_0/Γ divided by 0.305 to give the height in feet rather than metres. Altimeters in aircraft are set to give height readings in accordance with this formula, and if the altimeter is set to measure heights relative to a pressure of $1013\mathrm{hPa}$, then they read the pressure altitude directly. Otherwise, their reading will be offset from the pressure altitude by an amount to take into account the actual pressure at sea level that day (QNH).

Density Altitude

Just as aeronautical engineers and pilots find it more convenient at times to refer to the altitude at which you would expect a particular pressure rather than the pressure itself, similarly there is an equivalent for density. This is the *density altitude*, and is the altitude in the International Standard Atmosphere at which you would usually find air of the density specified.

Remembering from equation 2.14 that the density $\rho = mp/kT$, and using our assumed temperature $T = T_0 - \Gamma h$, we may write $p = k\rho\left(T_0 - \Gamma h\right)/m$. The density expected at sea level with standard pressure p_0 and temperature T_0 will be $\rho_0 = mp_0/kT_0$. Thus $p_0 = kT_0\rho_0/m$, and we can combine these equations to give a general expression for the pressure

$$\frac{p}{p_0} = \frac{\rho}{\rho_0}\left(1 - \frac{\Gamma h}{T_0}\right). \tag{4.29}$$

If this equation for p/p_0 is substituted into equation 4.26 we can derive an expression for the density altitude:

$$\begin{aligned} 1 - \frac{h\Gamma}{T_0} &= \left\{ \frac{\rho}{\rho_0} \left(1 - \frac{\Gamma h}{T_0} \right) \right\}^{\mu} \\ \left(1 - \frac{h\Gamma}{T_0} \right)^{1-\mu} &= \left(\frac{\rho}{\rho_0} \right)^{\mu} \\ 1 - \frac{h\Gamma}{T_0} &= \left(\frac{\rho}{\rho_0} \right)^{\mu/(1-\mu)} \\ h &= \frac{T_0}{\Gamma} \left\{ 1 - \left(\frac{\rho}{\rho_0} \right)^{\mu/(1-\mu)} \right\}. \end{aligned} \tag{4.30}$$

Remembering that $\rho = mp/kT$ and $\rho_0 = mp_0/kT_0$, it follows that

$$\frac{\rho}{\rho_0} = \frac{pT_0}{p_0T}, \tag{4.31}$$

and so the density altitude can be written in terms of a measured pressure and temperature as

$$h = \frac{T_0}{\Gamma} \left\{ 1 - \left(\frac{pT_0}{p_0T} \right)^{\mu/(1-\mu)} \right\}. \tag{4.32}$$

Correction from Density to Pressure Altitude

While these formulae are the accepted method of calculating a pressure or density altitude, pilots need a quick method of converting a pressure altitude (as read from their altimeter, for example) to a density altitude for checking the runway length needed or a stalling speed. To see if there is such a method, let us use equation 4.32 to determine the density altitude of sea level $h = 0$ if the pressure $p = p_0$, but the temperature is higher than expected $T = T_0 + \tau$. The equation gives

$$\begin{aligned} h &= \frac{T_0}{\Gamma} \left\{ 1 - \left(\frac{T_0}{T_0 + \tau} \right)^{\mu/(1-\mu)} \right\} \\ &= \frac{T_0}{\Gamma} \left\{ 1 - \left(\frac{T_0 + \tau}{T_0} \right)^{-\mu/(1-\mu)} \right\} \end{aligned}$$

$$
\begin{aligned}
&= \frac{T_0}{\Gamma}\left\{1-\left(1+\frac{\tau}{T_0}\right)^{-\mu/(1-\mu)}\right\} \\
&\approx \frac{T_0}{\Gamma}\left\{1-\left(1+\frac{\tau}{T_0}\times\frac{-\mu}{1-\mu}\right)\right\} \\
&= \frac{\tau}{\Gamma}\times\frac{\mu}{1-\mu},
\end{aligned}
\tag{4.33}
$$

where we have used the first term of the Binomial expansion in the third line.

Using our value of $\Gamma = 0.00656\,\mathrm{K/m}$, and taking $\mu = 0.1902$, we have $h = 36.4\tau$ in metres, and therefore $h = 36.4\tau/0.305 = 120\tau$ in feet.

The simplified correction, therefore, is that to get the density altitude, you take the pressure altitude and add 120 multiplied by the number of degrees by which the temperature is higher than you would expect.

4.4 Humidity

Air contains water vapour, and can also contain water droplets and ice particles too. At any given temperature there is a maximum quantity of water vapour which can 'dissolve' in the air, which rises as the temperature increases. Accordingly if you take moist air and cool it, there will come a point where the air can no longer hold all of its moisture as a vapour and condensation will occur. This is the simplest way of explaining why clouds have a base. Below the cloudbase, air is warm enough to contain all of the moisture present in it as a vapour. Above the cloudbase, the air is not warm enough, and the excess water condenses, thereby making a cloud.

We begin by considering how much water air can hold. When water evaporates or boils, the molecules get further apart. Given that they attract, causing this separation requires energy known as *latent heat of vaporization*. We will write the latent heat of vaporization of water per mole (0.018kg) as L. At typical temperatures it takes a value of approximately 44 000J/mol. Using the Boltzmann Law (equation 4.8) it follows that the fraction of water molecules able to vaporize at a given temperature will be

$$
f_w \propto \exp\left(-\frac{L}{RT}\right). \tag{4.34}
$$

The number of molecules will be proportional to this fraction, and as we know that pressure is proportional to the number of molecules, it follows that the pressure caused by the water molecules in the vapour will be given by a similar equation:

$$p_{w,\mathrm{sat}} = Q \exp\left(-\frac{L}{RT}\right), \tag{4.35}$$

where Q is a constant we haven't worked out yet. This is called the *saturated vapour pressure* being the pressure of the water vapour if there is a ready supply of water and therefore the air is holding as much vapour as it can. The vapour pressure can be lower than this if there is insufficient water to saturate the air. By definition, the boiling temperature of a liquid is the temperature at which the saturated vapour pressure becomes equal to atmospheric pressure. At standard atmospheric pressure $p_0 = 1013\mathrm{hPa}$, water boils at 100°C which is the same as $T_b = 373\mathrm{K}$. This means that

$$p_0 = Q \exp\left(-\frac{L}{RT_b}\right), \tag{4.36}$$

and so we can evaluate the constant Q as

$$Q = p_0 \exp\left(\frac{L}{RT_b}\right). \tag{4.37}$$

This means that equation 4.35 can be rewritten as

$$p_{w,\mathrm{sat}} = p_0 \exp\left(\frac{L}{RT_b}\right) \exp\left(-\frac{L}{RT}\right) = p_0 \exp\left(\frac{L}{RT_b} - \frac{L}{RT}\right). \tag{4.38}$$

This gives the *saturated vapour pressure* for air at a given temperature — the pressure caused by the water vapour when the air is holding the maximum amount of water vapour.

Question 4.8 *Use equations 4.10 and 4.38 to work out the temperature at which water will boil when you make a cup of tea at the top of a 3km high mountain. Make the simplifying assumption that the temperature in the atmosphere is 288K at all altitudes. Remember that at the boiling temperature the saturated vapour pressure will equal the local atmospheric pressure.*

In practice, air will not normally be saturated. We refer to the *dew point* T_d of air as the temperature you would have to cool it down to before it became saturated and condensation began to occur. If the air is fully saturated, then T_d will equal the local temperature. The dew point is not to be confused with the 'wet bulb' thermometer temperature which can also be used to calculate humidity. The wet bulb temperature is the temperature the air falls to if in contact with a reservoir of water so that it can reach saturation, but in doing so loses heat to evaporate the water, thus falling in temperature.

Often, rather than give T and T_d, it is common for weather reports to contain the *relative humidity* which is the water vapour content of the air as a fraction of the maximum it will hold. We can work out the relative humidity approximately by using equation 4.35 twice: once using the current temperature T which will give the saturated vapour pressure $p_{w,\text{sat}}$ (the vapour pressure if the air held the maximum amount of water vapour); and once using the dew point T_d where it will give the actual vapour pressure p_w. The relative humidity is the actual vapour pressure as a fraction of the saturated vapour pressure:

$$\text{R.H.} = \frac{p_w}{p_{w,\text{sat}}} = \frac{Q\,\exp\left(-L/RT_d\right)}{Q\,\exp\left(-L/RT\right)} = \exp\left(\frac{L}{RT} - \frac{L}{RT_d}\right). \tag{4.39}$$

Question 4.9 *Work out the relative humidity of air at 10°C when the dew point is 7°C.*

Question 4.10 *Work out the saturated vapour pressure of water in air at normal atmospheric pressure at 15°C. Remember to convert the temperature to kelvins (K). From this, use the Gas Law $p_{w,\text{sat}}V = n_w RT$ to calculate n_w/V the maximum number of moles of water vapour we will have in each cubic metre of air. Next, work out the number of moles of regular air in one cubic metre (n/V) at atmospheric pressure and 288K. Finally, give the number of moles of water vapour as a fraction of the number of moles of air molecules. This is approximately equal to the fraction of the molecules which are water molecules in the air at saturation at this temperature.*

4.5 Saturated Adiabatic Lapse Rate

We now turn our attention to saturated water vapour, such as that found in clouds, and wish to work out the temperature drop per unit gain in altitude

— the Saturated Adiabatic Lapse Rate or SALR. For this situation, we can not strictly write $\delta Q = 0$ in the First Law of Thermodynamics , as latent heat is released as the pressure falls and some of the water vapour condenses. We continue to work with a mole of air, and write n_w as the number of moles of water vapour contained within it. As we rise, n_w will decrease, and the heat liberated and given to the air will be equal to $\delta Q = -L\,\delta n_w$ (the heat given out is positive when n_w is getting smaller and $\delta n_w < 0$). With this extra term, equation 4.13 becomes

$$-L\,\delta n_w = \delta U + p\,\delta V. \tag{4.40}$$

As before, we write $\delta U = C_V\,\delta T$ and use the product rule to rewrite $p\,\delta V = \delta(pV) - V\,\delta p = R\,\delta T - V\,\delta p$ and obtain

$$-L\,\delta n_w = (C_V + R)\;\delta T - V\,\delta p = C_P\,\delta T - V\,\delta p, \tag{4.41}$$

where we have used the result from question 4.7 that $C_P = C_V + R$.

We next need to work out what to do with the δn_w term. This is the number of moles of water vapour in one mole of air, and accordingly represents the fraction of molecules in the air which are of water vapour. As the pressure due to a gas is proportional to the number of moleculess (or moles), δn_w will also equal the pressure due to the water vapour as a fraction of the gas pressure. As this air is saturated, the pressure due to the water vapour will be the saturated vapour pressure as written in equation 4.35, and accordingly

$$n_W = \frac{p_{w,\mathrm{sat}}}{p} = \frac{Q\,\exp\left(-L/RT\right)}{p}. \tag{4.42}$$

Note that we can evaluate

$$\begin{aligned} \frac{dn_w}{dp} &= -\frac{Q\,\exp\left(-L/RT\right)}{p^2} = -\frac{n_w}{p} && (4.43)\\ \frac{dn_w}{dT} &= \frac{L}{RT^2}\frac{Q\,\exp\left(-L/RT\right)}{p} = \frac{Ln_w}{RT^2} && (4.44) \end{aligned}$$

Writing equation 4.41 using

$$\begin{aligned} \delta n_w &= \frac{dn_w}{dp}\,\delta p + \frac{dn_w}{dT}\,\delta T \\ &= -\frac{n_w}{p}\,\delta p + \frac{Ln_w}{RT^2}\,\delta T && (4.45) \end{aligned}$$

gives us

$$\begin{aligned}
L\frac{n_w}{p}\,\delta p - L\frac{Ln_w}{RT^2}\,\delta T &= C_P\,\delta T - V\,\delta p \\
\left(V + \frac{Ln_w}{p}\right)\,\delta p &= \left(C_P + \frac{L^2 n_w}{RT^2}\right)\,\delta T \\
\frac{\delta T}{\delta p} &= \frac{V + \frac{Ln_w}{p}}{C_P + \frac{L^2 n_w}{RT^2}} \\
&= \frac{\frac{RT}{p} + \frac{Ln_w}{p}}{C_P + \frac{L^2 n_w}{RT^2}} \\
&= \frac{T}{p}\,\frac{R}{C_P}\,\frac{1 + \frac{Ln_w}{RT}}{1 + \frac{L^2 n_w}{C_P RT^2}}, \qquad (4.46)
\end{aligned}$$

where on the fourth line we have replaced V for RT/p using the Gas Law. If we then follow the reasoning we adopted to derive the Dry Adiabatic Lapse Rate (equation 4.20), we find

$$\begin{aligned}
-\frac{dT}{dh} &= -\frac{dT}{dp}\,\frac{dp}{dh} \\
&= -\frac{T}{p}\,\frac{R}{C_P}\,\frac{1 + \frac{Ln_w}{RT}}{1 + \frac{L^2 n_w}{C_P RT^2}} \times \frac{-M_r g p}{RT} \\
&= \frac{R}{C_P}\,\frac{1 + \frac{Ln_w}{RT}}{1 + \frac{L^2 n_w}{C_P RT^2}} \times \frac{M_r g}{R} \\
&= \frac{R}{C_V + R}\,\frac{M_r g}{R}\,\frac{1 + \frac{Ln_w}{RT}}{1 + \frac{L^2 n_w}{C_P RT^2}} \qquad (4.47) \\
&= \mathrm{DALR} \times \frac{1 + \frac{Ln_w}{RT}}{1 + \frac{L^2 n_w}{C_P RT^2}}. \qquad (4.48)
\end{aligned}$$

This can be evaluated for different temperatures and pressures (note that n_w does have to be calculated separately for each one). At very cold temperatures, hardly any water evaporates, $n_w \approx 0$ and the SALR is approximately the same as the DALR. However for warm air such as atmospheric pressure air at 20°C, the SALR is about half of the DALR.

4.6 Tephigrams

Calculations of the type explained in this chapter can leave us failing to see the wood for the trees when it comes to understanding the conditions in our atmosphere. To help us, meteorologists have devised clever diagrams and charts which enable the situations to be described visually. Probably the most famous is the *tephigram*, which as the name suggests plots T and Φ. It enables the pressure, height, temperature and quantity of water vapour in air to be read off easily, and enables dry and saturated adiabats to be followed with ease.

In studies of Thermodynamics, it is quite common to see plots of engine cycles given on graphs where the entropy is plotted on the vertical and temperature on the horizontal. Isothermal changes are shown as vertical lines, while adiabats are shown horizontally. The Carnot cycle, for example, then appears as a simple rectangle. A similar thing is done for the atmosphere, with two subtle changes.

The first is that instead of plotting entropy per se, the *potential* is plotted. For a given parcel of air the potential is the kelvin temperature it would have if it were expanded or compressed adiabatically until it reached a pressure of 1000hPa. The potential therefore has units K, and is written θ.

To evaluate the potential of air of a given temperature and pressure, we use equation 4.18. This can be written $T = B\,p^{R/(C_V+R)}$ where B is a constant. Given that we know from page 76 that for air, $C_V = 5R/2$, it follows that this equation can be written $T = B\,p^{2/7}$. The constant will take the same value B when the air is changed adiabatically to a pressure of $p_0 = 1000\text{hPa}$, at which point the new temperature will be θ. This means that $\theta = B\,p_0^{2/7}$. Combining these equations gives

$$B = \frac{\theta}{p_0^{2/7}} = \frac{T}{p^{2/7}}$$

$$\theta = T\left(\frac{p_0}{p}\right)^{2/7}. \tag{4.49}$$

Notice that if θ is plotted on the vertical and T on the horizontal, the line $\theta = T$ will be straight with unit gradient. This refers to all the points with a pressure of 1000hPa. Lines representing other pressures will also be straight, and will have a gradient of $(p_0/p)^{2/7}$. Usually only values of T and θ corresponding to those experienced in regular air (from about $-50^\circ\mathrm{C}$ to

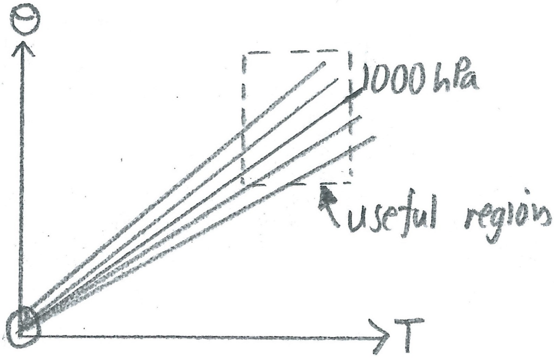

Figure 4.1: When plotting θ against T for practical temperatures, the isobars (lines of equal pressure) appear almost parallel with a gradient of approximately 1.

about $30°\mathrm{C}$ are plotted, and air pressures rarely vary by more than 5% of 1000hPa. This means that on our plot the other lines of equal pressure are almost parallel to the 1000hPa line, as shown in figure 4.1, and the lines representing 1010hPa, 1020hPa, 1030hPa and so on are almost equally spaced, with the lower pressures having steeper lines.

The pressure of air represented by a given pair of (T, θ) values can be calculated by re-arranging equation 4.49 to make p the subject:

$$p = p_0 \left(\frac{T}{\theta}\right)^{7/2}. \tag{4.50}$$

Usually the tephigram is plotted after a $45°$ rotation clockwise so that the 1000hPa line is now aligned with the x-axis. Temperatures increase as you pass to the bottom right, while potentials increase to the top right. The other isobars are also more or less horizontal, and this layout has the convenience that high altitudes (low pressures) are reached by moving up the diagram. This rotation is shown in figure 4.2.

Lines followed by a parcel of dry air expanding or being compressed without heat loss are equipotentials — straight lines of gradient -1 on the tephigram (to keep θ constant), while isothermal expansion or compression is represented by motion along a straight line of gradient $+1$ to keep

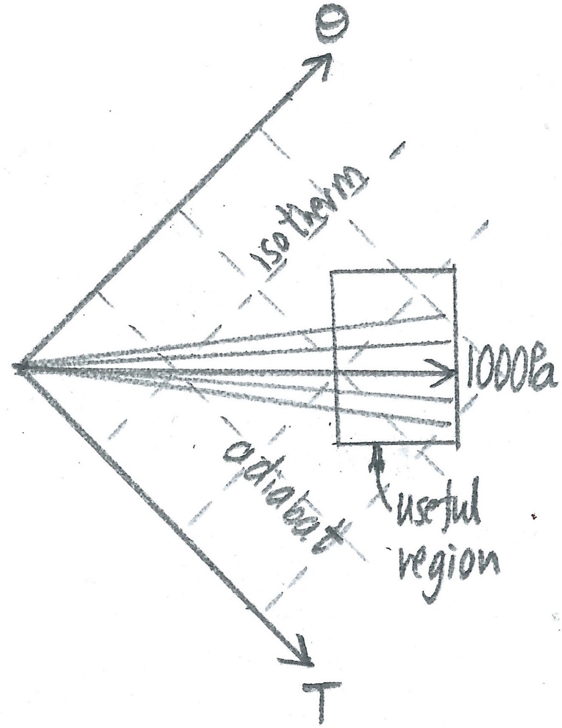

Figure 4.2: Tephigrams are plots of θ vs T rotated by 45° clockwise so that the 1000hPa isobar now forms the x-axis.

T constant. The lines followed by saturated air expanding or being compressed adiabatically (following an SALR) can also be plotted. These are curves, but by the nature of the graph, end up being almost vertical on the plot as they pass the 1000hPa isobar. They are labelled with the temperature the air would attain on reaching 1000hPa.

There is one further type of line marked on the graph — lines joining points of equal saturated water vapour content. These are also more or less straight as they pass along the graph. We can work out the form of these lines as follows. We start, as explained on page 84 with the observation that

$n_w = p_{w,\mathrm{sat}}/p$ and use equation 4.38 to write

$$n_w = \frac{p_{w,\mathrm{sat}}}{p} = \frac{p_0 \exp\left(\frac{L}{RT_b} - \frac{L}{RT}\right)}{p}. \tag{4.51}$$

We then use equation 4.50 to substitute for p in terms of T and θ. This gives

$$n_w = \frac{p_0 \exp\left(\frac{L}{RT_b} - \frac{L}{RT}\right)}{p_0 \left(T/\theta\right)^{7/2}}. \tag{4.52}$$

Finally, we re-arrange to make θ the subject to gain

$$\begin{aligned}\theta &= T \left\{ \frac{n_w}{\exp\left(\frac{L}{RT_b} - \frac{L}{RT}\right)} \right\}^{2/7} \\ &= T\, n_w^{2/7}\, e^{2L/7RT}\, e^{-2L/7RT_b}\end{aligned} \tag{4.53}$$

We can calculate the gradient using the derivative of equation 4.53:

$$\frac{d\theta}{dT} = \left(1 - \frac{2L}{7RT}\right) n_w^{2/7}\, e^{2L/7RT}\, e^{-2L/7RT_b}, \tag{4.54}$$

and then use equation 4.51 to express this in terms of the pressure p:

$$\frac{d\theta}{dT} = \left(1 - \frac{2L}{7RT}\right) \left(\frac{p_0}{p}\right)^{2/7}. \tag{4.55}$$

This shows that for typical air temperatures and pressures, where $(p_0/p)^{2/7} \approx 1$, lines of equal saturated water content are nearly straight and have gradients of approximately -4 on a plot of θ against T.

It is also interesting to see what happens to the dew point as the pressure decreases with altitude. For a constant $n_w = p_{w,\mathrm{sat}}/p$ it follows that $dp/dT = n_w^{-1} \times dp_{w,\mathrm{sat}}/dT$. We therefore have, with the help of equation 4.38,

$$\begin{aligned}\frac{dp}{dT} &= \frac{d}{dT}\left\{ \frac{p_0}{n_w} \exp\left(\frac{L}{RT_b} - \frac{L}{RT}\right) \right\} \\ &= \frac{L}{RT^2} \frac{p_0}{n_w} \exp\left(\frac{L}{RT_b} - \frac{L}{RT}\right) \\ &= \frac{Lp}{RT^2}.\end{aligned} \tag{4.56}$$

The change of the dew point with height will therefore be

$$\begin{aligned}\frac{dT_d}{dh} &= \frac{dT_d}{dp}\frac{dp}{dh} \\ &= \frac{RT^2}{Lp} \times \frac{-M_r g p}{RT} \\ &= -\frac{M_r g T}{L}\end{aligned} \tag{4.57}$$

Using a temperature of 288K, this gives a drop of 1.85K for each kilometre of altitude, which we will refer to as the Constant Saturation Lapse Rate (CSLR).

4.7 Cloud Height

One practical use of the physics in this chapter is to estimate the height of the cloudbase. Suppose that at ground level, the temperature is T and the dew point is T_d. If you rise to an altitude h, non-saturated air will be expected to cool at the DALR, and will reach a temperature of $T - \mathrm{DALR} \times h$. The dew point, on the other hand, will drop to $T_d - \mathrm{CSLR} \times h$. Clouds will begin to form at the height where the temperature of the air reaches the dew point. This is the height at which

$$\begin{aligned}T - \mathrm{DALR} \times h &= T_d - \mathrm{CSLR} \times h \\ T - T_d &= h\,(\mathrm{DALR} - \mathrm{CSLR})\,.\end{aligned} \tag{4.58}$$

For our values of DALR of 9.8K/km and CSLR of 1.9K/km, the cloud base will be at a height of about $0.12\,(T - T_d)$km, which is about $400\,(T - T_d)$ft.

The dew point is given in aeronautical weather briefings and reports. However if you are restricted to the weather reports commonly available in the media, you can work it out from the relative humidity using equation 4.39. On the other hand, you might have a wet and dry bulb thermometer. In this case, you have a different calculation to perform. The wet bulb temperature falls with altitude at the SALR. Accordingly if you have a wet bulb temperature of T_w, the altitude of the cloud base will be (by analogy with equation 4.58)

$$T - T_w = h\,(\mathrm{DALR} - \mathrm{SALR})\,. \tag{4.59}$$

Often an approximation can be made of the height of the top of the cloud by working out the height above the base at which the temperature will fall to $0°C$ using the Saturated Adiabatic Lapse Rate.

Chapter 5

The Honourable Company of Navier & Stokes

In this chapter, we seek to put much of the fundamental analysis of fluids on a more mathematical footing. Our ultimate aim here is to write Newton's Second Law for a fluid, which is known as the Navier Stokes equation.

Before we can do this, and explore the power of the equation, we need to introduce extra notation in relation to vectors. Any undergraduate reading this can miss this section out, as it will be familiar to them already.

5.1 Primer in Vector Calculus

Partial Differentiation

In this section, we shall be dealing with functions of more than one variable. For example, the velocity at a point in a fluid depends on the time of day t and also on the exact position $\mathbf{r} = (x, y, z)$. Furthermore this velocity itself is a vector with three components $\mathbf{v} = (v_x, v_y, v_z)$. If v_x is a function of x, y, z and t and we wish to differentiate it with respect to x, say, we need to specify what we should do with the other variables (such as y). The most frequent meaning of the derivative in this context is, "how much does v_x change as we increase x assuming we keep everything else constant?" This is the partial derivative of v_x with respect to x and is written $\partial v_x / \partial x$. To give an example, if $v_x = x^2 y + z$ then $\partial v_x / \partial x = 2xy$. When doing the differentiation we treat all other variables (here y and z) as constants.

grad and ∇

Suppose that we have a function $z(x, y)$ which gives us the height z of each point of some terrain above sea level depending on its grid reference on a map (specified by the Easting x and the Northing y). If we ask "how steep is the land at (4,6)?" the answer depends on the direction we walk as we pass the point. If we wish to give as much information as possible, we specify how steep the path is when walking North ($\partial z/\partial y$) and also the steepness when walking East ($\partial z/\partial x$). The two can be written as a vector to give all of the information (providing that we assume that the ground close to the point is not curved). This is the two dimensional gradient function

$$\operatorname{grad} z = \left(\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}\right). \tag{5.1}$$

Suppose that we were walking in a direction making an angle θ to North. After moving a distance δs, our height would have changed by $\partial z/\partial x \times \delta s \sin\theta$ owing to the motion East, and would have also changed by $\partial z/\partial y \times \delta s \cos\theta$ due to the motion North. Our total change in height would be

$$\begin{aligned} \delta z &= \partial z/\partial x\, \delta x + \partial z/\partial y\, \delta y \\ &= \partial z/\partial x\, \delta s \sin\theta + \partial z/\partial y\, \delta s \cos\theta \\ &= (\partial z/\partial x\,,\, \partial z/\partial y) \cdot (\delta s \sin\theta\,,\, \delta s \cos\theta) \end{aligned} \tag{5.2}$$

where we use the $\cdot$ symbol to represent the *scalar product* of the two vectors (defined as the sum of the products of the components $\mathbf{a}\cdot\mathbf{b} \equiv a_x b_x + a_y b_y$.

Let us now imagine that the vector $\operatorname{grad} z$ (which is also written ∇z) has a magnitude D and a direction which makes an angle ϕ with respect to North. It follows that $\partial z/\partial x = D\sin\phi$ and $\partial z/\partial y = D\cos\phi$. Equation 5.2 then becomes

$$\begin{aligned} \delta z &= D\sin\phi \times \delta s \sin\theta + D\cos\phi \times \delta s\cos\theta \\ &= D\,\delta s\,(\sin\phi\sin\theta + \cos\phi\cos\theta) \\ &= D\,\delta s\cos(\phi - \theta)\,. \end{aligned} \tag{5.3}$$

Accordingly, we will get the largest δz or gain in height if we walk with $\theta = \phi$. Thus ϕ, the direction of ∇z corresponds to the steepest uphill direction. Notice that in this case, the gain in height $\delta z = D\,\delta s$ and accordingly the magnitude of ∇z (which we call D) is the gradient you experience if you walk this way. If we walk in a direction at right angles to this, we will not climb or descend, as $\cos(\phi - \theta) = 0$, and we are walking along the contour.

We now extend this to a three-dimensional situation. For example, the pressure $p(x, y, z)$ at a point in a fluid depends on its co-ordinates, and we can define a three-dimensional, vector derivative of this scalar function using the grad function:

$$\operatorname{grad} p \equiv \nabla p = \left(\frac{\partial p}{\partial x}, \frac{\partial p}{\partial y}, \frac{\partial p}{\partial z} \right). \tag{5.4}$$

We will explore this further in our next section on pressure. Suffice it to say that if we move a short distance $\delta \mathbf{s}$, the change in pressure will be (by analogy with equation 5.2)

$$\delta p = \nabla p \cdot \delta \mathbf{s}. \tag{5.5}$$

Laplacian or ∇^2

Another function we may encounter is the Laplacian operator $\nabla^2 \equiv \nabla \cdot \nabla$. For a scalar function $\phi(x, y, z)$, this is defined as

$$\begin{aligned} \nabla^2 \phi &= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \cdot \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \phi \\ &= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \phi \\ &= \left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} \right). \end{aligned} \tag{5.6}$$

You can also apply the Laplacian operator to a vector. In this case, the result will also be a vector.

Divergence, div or $\nabla \cdot$

Suppose we have a vector $\mathbf{v}(x, y, z)$ which depends upon position. The divergence (or div for short) is obtained by calcuating

$$\operatorname{div} \mathbf{v} \equiv \nabla \cdot \mathbf{v} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \cdot (v_x, v_y, v_z)$$

$$= \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}. \tag{5.7}$$

It is easiest to visualize what is going on by thinking of $\mathbf{v}$ as a measure of the volume of an incompressible fluid moving each second (per unit area) and the direction it is travelling. Now think of a cube of side δs aligned with x, y and z axes centred on our point of interest. Now let us consider

$$\nabla \cdot \mathbf{v}\, \delta s = \frac{\partial v_x}{\partial x}\, \delta s + \frac{\partial v_y}{\partial y}\, \delta s + \frac{\partial v_z}{\partial z}\, \delta s. \tag{5.8}$$

The first term in equation 5.8 accordingly represents how much extra fluid flows out of the $+x$ face compared with the amount flowing in at $-x$. The other two terms represent similar quantities for the other faces of the cube. There is nothing stopping more fluid coming out of the $+x$ face of the cube than flows into $-x$ providing that some extra fluid is flowing in through some of the remaining four faces. However as the fluid is incompressible, the total outflow from the cube must be zero, and we write this mathematically as

$$\operatorname{div} \mathbf{v} \equiv \nabla \cdot \mathbf{v} = 0. \tag{5.9}$$

Such a flow is known as *divergence free*, and for such a flow, equation 5.9 is the equivalent of the one dimensional continuity equation 2.6 on page 21. For a compressible fluid, we define a density function $\rho(x, y, z)$. If $\nabla \cdot \mathbf{v} > 0$ then there is a net outflow of fluid. This must correspond to a reduction in the density of the fluid in our 'cube', so $\partial\rho/\partial t < 0$. Continuity in this case is expressed by the equation

$$\nabla \cdot \mathbf{v} = -\frac{\partial \rho}{\partial t}, \tag{5.10}$$

which is the three-dimensional equivalent of equation 2.9 on page 22.

Curl, curl or $\nabla\times$

This function is evaluated, by analogy with the div function, by taking a product of ∇ with a vector $\mathbf{v}$, but now we take the vector (or cross) product:

$$\operatorname{curl} \mathbf{v} \equiv \nabla \times \mathbf{v} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \times (v_x, v_y, v_z)$$

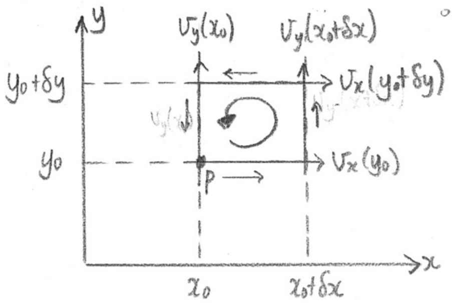

Figure 5.1: Evaluating the circulation around a small square in the xy plane. Starting at the point P (x_0, y_0), we move around the square in an anticlockwise direction.

$$= \begin{pmatrix} \frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z} \\ \frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x} \\ \frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} \end{pmatrix}. \tag{5.11}$$

This measures the degree (and direction) of any whorls or vortices at a point in the fluid. To see this, we calculate the circulation around a small square in the xy plane as shown in figure 5.1. The total circulation, where we use the procedure outlined on page 43, will be

$$\begin{aligned} \gamma &= v_x(y_0)\,\delta x + v_y(x_0+\delta x)\,\delta y - v_x(y_0+\delta y)\,\delta x - v_y(x_0)\,\delta y \\ &= \{v_y(x_0+\delta x) - v_y(x_0)\}\ \delta y - \{v_x(y_0+\delta y) - v_x(y_0)\}\ \delta x \\ &\approx \frac{\partial v_y}{\partial x}\,\delta x\delta y - \frac{\partial v_x}{\partial y}\,\delta y\delta x = \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\,\delta x\delta y, \end{aligned} \tag{5.12}$$

where the approximation tends to equality as δx and δy tend to zero. Comparing this with equation 5.11, we see that this is the z component of $\nabla \times \mathbf{v}$ multiplied by the area enclosed within our integration route.

The component of the curl in a particular direction therefore gives the circulation per unit area measured in a plane perpendicular to that direction.

For larger routes, which may not be rectangular, the total circulation is the integral of the curl over the area bounded by the route.

If there are no vortices, then $\nabla \times \mathbf{v} = 0$. Looking at the x component of $\nabla \times \mathbf{v}$ in equation 5.11 we see that in this case,

$$\frac{\partial v_z}{\partial y} = \frac{\partial v_y}{\partial z}. \tag{5.13}$$

Notice that if we found a scalar function ϕ such that $\mathbf{v} = \nabla\phi$, then equation 5.13 would automatically be true as

$$\frac{\partial v_z}{\partial y} = \frac{\partial^2 \phi}{\partial y\, \partial z}, \tag{5.14}$$

which is going to be the same as

$$\frac{\partial v_y}{\partial z} = \frac{\partial^2 \phi}{\partial z\, \partial y}. \tag{5.15}$$

It therefore follows that any flow which can be written $\mathbf{v} = \nabla\phi$ has no vortices and so $\nabla \times \mathbf{v} = 0$.

5.2 Pressure

We now begin our task to write Newton's Second Law for a fluid. Given that the mass of each particle of fluid will not be changing, we may take this law as $\mathbf{F} = m\mathbf{a}$. We begin by exploring two sources of force on a fluid. In this section we consider variations in fluid pressure, in the next we consider viscosity.

On page 94, equation 5.4 was used to indicate how the grad function ∇p would give us information on the way in which the pressure changed as you moved around in a fluid:

$$\text{grad}\, p \equiv \nabla p = \left(\frac{\partial p}{\partial x}, \frac{\partial p}{\partial y}, \frac{\partial p}{\partial z} \right). \tag{5.16}$$

Following on from the work of that section, ∇p is a vector which represents the direction you would go if you wanted the pressure to increase as quickly as possible, and accordingly any forces which result from uneven pressure distributions will point in the opposite way. In short $\mathbf{F}$ points in the direction of $-\nabla p$.

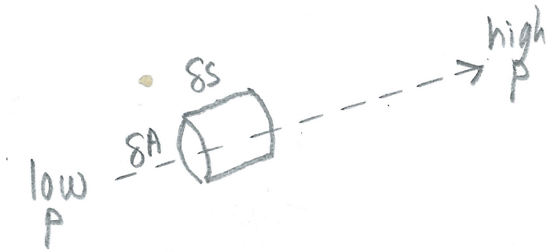

Figure 5.2: A parcel of fluid being pushed by a pressure gradient

We now aim to write Newton's Second Law for this situation, using the layout shown in figure 5.2. A small volume of fluid presents an area δA to the pressure gradient, and has a thickness δs in the direction of the pressure gradient. The difference in pressure across the two faces of the parcel will be $\delta p = \nabla p \cdot$ **ffis** using equation 5.5, which in our case will be equal to $|\nabla p|\ \delta s$ as we have arranged δs to be parallel to the pressure gradient, and so the force on the parcel will be

$$F = \delta A \times \delta p = \delta A \times |\nabla p|\ \delta s = |\nabla p|\ \delta A\, \delta s. \tag{5.17}$$

The mass of the parcel will be $m = \rho\, \delta A\, \delta s$, and accordingly we can write Newton's Second Law $F = ma$ as

$$|\nabla p| = \rho\, a. \tag{5.18}$$

Now adding in the directionality, where the force and hence the acceleration must be in opposition to ∇p we arrive at

$$\nabla p = -\rho\, \mathbf{a}. \tag{5.19}$$

5.3 Viscosity

Next we consider the viscosity, using the geometry of figure 5.3, where we will make the simplifying assumption that the fluid is not compressible. Here we have a fluid moving the $+x$ direction at a speed determined by the value of y. Our velocity function is therefore $(v_x(y), 0, 0)$, and we will

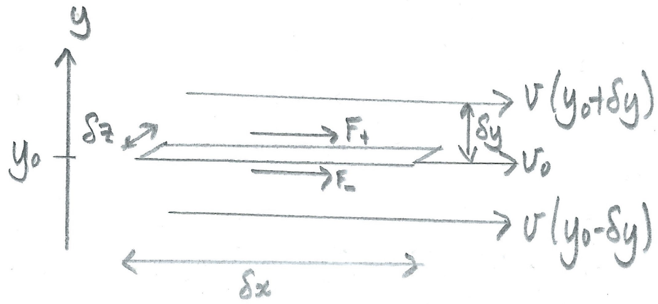

Figure 5.3: For fluid all flowing in the $+x$ direction at a velocity dependent on y, we estimate the force on the place in the middle at $y = y_0$.

use v_0 to represent the speed of the fluid at $y = y_0$. Let us consider the viscous drag on a slab (with area $A = \delta x\, \delta z$) of fluid at $y = y_0$ due to that above at $y = y_0 + \delta y$ and that below at $y = y_0 - \delta y$.

If we use a Taylor expansion, the velocity of the fluid at $y + \delta y$ will be given by

$$v\left(y + \delta y\right) = v_0 + \frac{\partial v}{\partial y}\delta y + \frac{1}{2}\frac{\partial^2 v}{\partial y^2}\delta y^2 + \cdots, \tag{5.20}$$

so the velocity gradient between y_0 and $y_0 + \delta y$ will be

$$\frac{v\left(y + \delta y\right) - v_0}{\delta y} = \frac{\partial v}{\partial y} + \frac{1}{2}\frac{\partial^2 v}{\partial y^2}\delta y + \cdots. \tag{5.21}$$

Using the definition of viscosity on page 30, the force on the plate because of the fluid above it will be

$$F_+ = \mu A\left(\frac{\partial v}{\partial y} + \frac{1}{2}\frac{\partial^2 v}{\partial y^2}\delta y + \cdots\right). \tag{5.22}$$

Similarly, the velocity below the plate at $y_0 - \delta y$ will be

$$v\left(y - \delta y\right) = v_0 - \frac{\partial v}{\partial y}\delta y + \frac{1}{2}\frac{\partial^2 v}{\partial y^2}\delta y^2 + \cdots, \tag{5.23}$$

and the velocity gradient between $y_0 - \delta y$ and y_0 will be

$$\frac{v\,(y-\delta y) - v_0}{\delta y} = -\frac{\partial v}{\partial y} + \frac{1}{2}\,\frac{\partial^2 v}{\partial y^2}\delta y + \cdots. \tag{5.24}$$

This will cause force on the bottom surface of the plate

$$F_- = \mu A \left(-\frac{\partial v}{\partial y} + \frac{1}{2}\,\frac{\partial^2 v}{\partial y^2}\delta y + \cdots \right). \tag{5.25}$$

And so, we can now conclude that the total force on the plate is

$$\begin{aligned} F = F_+ + F_- &= \mu A\,\frac{\partial^2 v}{\partial y^2}\,\delta y \\ &= \mu\,\frac{\partial^2 v}{\partial y^2}\,\delta y\,\delta x\,\delta z. \end{aligned} \tag{5.26}$$

The mass of the plate equals its density multiplied by volume $m = \rho\,\delta x\,\delta y\,\delta z$, and accordingly Newton's Second Law $F = ma$ can be written

$$\mu\frac{\partial^2 v}{\partial y^2} = \rho\,a. \tag{5.27}$$

We now wish to write this using vectors. The acceleration must be in the same direction as the velocities. In addition, note that had there also been a velocity gradient in the z direction, we would just have had to add a $\mu\partial^2 v/\partial z^2$ term, and the same for x. The general expression can be written using the Laplacian operator as in equation 5.6:

$$\mu\nabla^2\mathbf{v} = \rho\mathbf{a}. \tag{5.28}$$

5.4 Eulerian Formulation

We may now combine equations 5.19 and 5.28 and the effect of gravity $\mathbf{F} = m\mathbf{g} = \rho\mathbf{g}\,\delta V$ to give Newton's law in an incompressible fluid:

$$\rho\mathbf{a} = \rho\mathbf{g} + \mu\nabla^2\mathbf{v} - \nabla p. \tag{5.29}$$

As far as it goes, this equation is very good. However we do not find this to be a helpful formulation for many problems we face. This equation has a function $\mathbf{v}$ which contains information about the speed of a parcel of fluid which can speed up and slow down at will - it is the speed of the parcel and is fixed to it. However suppose we want $\mathbf{v}(x, y, z, t)$ to mean the velocity of the fluid which happens to be at that point at that time? In other words we want the $\mathbf{v}$ function to be fixed to the point in space not the fluid. In this formulation, consider a situation where the flow is steady. In short $\mathbf{v}(x, y, z)$ is not time dependent. In such a scenario the fluid can still accelerate as the $v_x(x + \delta x)$ might well be higher than $v_x(x)$, however the values of the speed at the two positions do not change. We need a way of writing the acceleration in such a situation.

The acceleration has two parts. One is the explicit time dependence $\partial \mathbf{v}/\partial t$ which would be zero in a steady flow situation. The other part is the acceleration experienced by a parcel of fluid as it moves from one place to another. Assuming for the moment that speeds only change in the x direction, the acceleration can be calculated between x_0 to $x_0 + \delta x$ as

$$\begin{aligned}
\mathbf{a} &= \lim_{\delta t \to 0} \frac{\delta \mathbf{v}}{\delta t} \\
&= \lim_{\delta t \to 0} \frac{\mathbf{v}(x_0 + \delta x) - \mathbf{v}(x_0)}{\delta t} \\
&= \lim_{\delta t \to 0} \frac{\partial \mathbf{v}/\partial x \, \delta x}{\delta t} \\
&= \lim_{\delta t \to 0} \frac{\partial \mathbf{v}}{\partial x} \frac{\delta x}{\delta t} \\
&= \frac{\partial \mathbf{v}}{\partial x} v_x. \qquad (5.30)
\end{aligned}$$

Adding in the other two dimensions gives (for steady flow)

$$\begin{aligned}
\mathbf{a} &= \frac{\partial \mathbf{v}}{\partial x} v_x + \frac{\partial \mathbf{v}}{\partial y} v_y + \frac{\partial \mathbf{v}}{\partial z} v_z \\
&= v_x \frac{\partial \mathbf{v}}{\partial x} + v_y \frac{\partial \mathbf{v}}{\partial y} + v_z \frac{\partial \mathbf{v}}{\partial z} \\
&= \left(v_x \frac{\partial}{\partial x} + v_y \frac{\partial}{\partial y} + v_z \frac{\partial}{\partial z} \right) \mathbf{v} \\
&= (\mathbf{v} \cdot \nabla) \mathbf{v}. \qquad (5.31)
\end{aligned}$$

Adding this in to our explicit time dependence $\mathbf{a} = \partial \mathbf{v}/\partial t$ gives

$$\mathbf{a} = \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v}$$

$$= \left(\frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla\right)\mathbf{v}, \tag{5.32}$$

and allows us, finally, to write the Eulerian formulation of Newton's Laws for a fluid:

$$\rho\left(\frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla\right)\mathbf{v} = \rho\mathbf{g} + \mu\nabla^2\mathbf{v} - \nabla p, \tag{5.33}$$

or as it is more commonly written,

$$\rho\left(\frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla\right)\mathbf{v} + \nabla p - \mu\nabla^2\mathbf{v} = \rho\mathbf{g}, \tag{5.34}$$

which is also known as the Navier–Stokes equation for an incompressible fluid.

5.5 Bernoulli Revisited

We first put the Navier–Stokes equation 5.34 through its paces by seeing if it can predict Bernoulli's principle. We envisage a steady state situation (where $\partial\mathbf{v}/\partial t = 0$) in the absence of viscosity. In this case, the equation becomes

$$\rho\left(\mathbf{v}\cdot\nabla\right)\mathbf{v} + \nabla p = \rho\mathbf{g}. \tag{5.35}$$

Let us now multiply all terms in the equation with a short distance $\delta\mathbf{s}$ in the direction of the fluid's motion using the dot (or scalar) product. In the first term, the vector derivative ∇ will only produce a result in the direction in which anything changes — namely the direction of the fluid's motion. Similarly the dot product of $\mathbf{v}\cdot\delta\mathbf{s} = v\,\delta s$ since we have already agreed that ffis is in the same direction as the velocity. Thus

$$\begin{aligned}\left(\mathbf{v}\cdot\nabla\right)\mathbf{v}\cdot\delta\mathbf{s} &= v\,\frac{dv}{ds}\,\delta s \\ &= v\,\delta v = \delta\left(\frac{v^2}{2}\right).\end{aligned} \tag{5.36}$$

In turn $\nabla p\ \cdot\delta\mathbf{s}$ in the direction of the motion (and hence the changing pressure) will simply equal $dp/ds\,\delta s = \delta p$, namely the change in pressure. For our final term $\mathbf{g}\cdot\delta\mathbf{s} = g\,\delta s\,\cos\theta$ where θ is the angle between ffis and the downwards vertical. However $\delta s\cos\theta$ is the distance moved downwards, and thefore equals $-\delta h$ the drop in height.

Putting these changes together gives

$$\rho\delta\left(\frac{v^2}{2}\right)+\delta p=-\rho g\,\delta h, \tag{5.37}$$

which can also be written as

$$\delta\left(\frac{\rho v^2}{2}+p+\rho g h\right)=0, \tag{5.38}$$

which agrees with equation 2.14 on page 25 as a statement of the conservation of energy à la Bernoulli.

Question 5.1 *The Navier-Stokes equation for the steady flow of an incompressible fluid in the absence of viscosity can be written:*

$$\rho\left(\mathbf{v}\cdot\nabla\right)\mathbf{v}+\nabla p+\nabla\left(\rho g z\right)=0,$$

where the fact that the flow is steady means that the partial derivative with respect to time $\partial\mathbf{v}/\partial t=0$, and we note that $\nabla\left(-\rho g z\right)$ gives a vector of magnitude ρg pointing downwards.

Show that if you assume that there are no vortices, and hence $\nabla\times\mathbf{v}=0$, the Navier-Stokes equation can be written

$$\nabla\left(\frac{\rho v^2}{2}+p+\rho g z\right)=0,$$

and accordingly, that $\rho v^2/2+p+\rho g z$ must be a constant. Given that z is the height, we have derived the Bernoulli equation by a different method.

5.6 Reynolds Revisited

In the final section of this chapter, we see where Reynolds number fits in to all of this. The Navier–Stokes equation is almost impossible to solve without making simplifying assumptions in nearly all cases of practical interest. Accordingly, what we often do to solve a situation is to ignore the smallest term in the equation, solve what remains, and then fiddle about with the result (usually using methods called perturbation theories) to work out and correct the error made by missing the term out.

The two terms which cause the most grief are $\rho\left(\mathbf{v}\cdot\nabla\right)\mathbf{v}$ (the inertia term) and $\mu\nabla^2\mathbf{v}$ (the viscosity term). The inertia term is particularly annoying for mathematicians as it is non linear which makes it harder to solve.

To help us decide which one to get rid of, let us work out the size of the inertia term in relation to the viscosity term. We simplify the vector terms by writing them as regular derivatives along the direction of motion. The ratio is

$$\frac{\text{inertia}}{\text{viscosity}} = \frac{\rho v\frac{dv}{ds}}{\mu\frac{d^2v}{ds^2}}. \tag{5.39}$$

We now integrate both terms with respect to s to give

$$\frac{\text{inertia}}{\text{viscosity}} = \frac{\rho v^2}{\mu\frac{dv}{ds}}, \tag{5.40}$$

and finally assume that a change in velocity from v to 0 would happen over a distance of approximately D. It follows that $dv/ds = v/D$. This makes

$$\frac{\text{inertia}}{\text{viscosity}} = \frac{\rho v^2}{\mu\frac{v}{D}} = \frac{\rho v D}{\mu}, \tag{5.41}$$

which is exactly the same thing as the Reynolds number stated in equation 2.19 on page 33.

What this shows is that when solving the Navier–Stokes equation, it pays to know your Reynolds number, as this tells you which term is more important. Our work on page 33 tells us that it is when the inertia term dominates (high Reynolds number) that we have very complex flows, which is not a surprise to us now, as this is the situation where the non-linear term in the equation is also the largest one. There be dragons.

Question 5.2 *Imagine that you have a mathematical function* $\mathbf{v}$ *which satisfies the incompressible Navier-Stokes equation 5.34. Now let us imagine that you change the scaling of the arrangement such that all distances are stretched* $L \to aL$*, all velocities increased* $\mathbf{v} \to w\mathbf{v}$*, the density increases* $\rho \to r\rho$*, the pressure increases* $p \to qp$ *and the dynamic viscosity increases* $\mu \to m\mu$*.*

Show that the original flow pattern will still satisfy the Navier-Stokes equation provided that $q = rw^2$ *and that* $m = rwa$*. This means that as long as the ratios* $p/\rho v^2$ *and* $\mu/\rho v L$ *do not change, the new flow will satisfy the same equation.*

Our studies in dimensional analysis have convinced us that pressures will indeed be proportional to ρv^2. *The second condition states that the flows will be the same providing that the Reynolds' number has not changed. This is, effectively, a justification of the validity of scaled-down fluid experiments in wind tunnels, as mentioned on page* *16*.

Chapter 6

Mathematical Treatment of Lift

In this short chapter we give a more complete description of the mathematics used in our treatment of wings. The first part shows how fluid flow in a two-dimensional situation can be very conveniently modelled in the complex plane. This is a form of two dimensional number line, and we show how this enables us to predict the flow of air over a cylinder and also a flat plate, which serves as a model for a very simple wing. This enables the derivation of the co-efficient of lift for a flat plate used as a wing. An introduction to complex variable analysis has to be given before we can reach the results of most interest to us.

We then introduce the technique and recipies used to find the co-efficients of lift for practical aerofoils.

Finally, we consider the effect of the finite length of real wings and how this affects the analysis. This leads to a better understanding of induced drag (that is, the drag which is a necessary consequence of lift).

6.1 Describing Fluid Flow using the Complex Plane

Essentials of Complex Numbers

We shall require the following facts about complex numbers in our work in this section. If there is anything on this list which you have not met before, and which you wish to understand rather than accepting on face value, fuller explanations can be found in any textbook used for teaching A-level (or equivalent pre-university) Pure Mathematics.

- Complex numbers are written $z = 3 + 2i$, where the 3 is called the *real* part $\mathrm{Re}z = 3$, and 2 is called the *imaginary* part $\mathrm{Im}z = 2$. Real and imaginary parts can be positive, zero, or negative.

- Complex numbers are conviently plotted on a diagram called the *complex plane* or *Argand diagram* where the real part of the number is plotted on the horizontal axis, and the imaginary part is plotted on the vertical axis. There is therefore a resemblance between an x, y graph with a vector $(4, -1)$, for example, and a complex plane with $4 - i$ plotted.

- Complex numbers are added or subtracted using the normal rules of algebra. Thus, if $a = 3 + 9i$ and $b = 4 - 5i$ then $a + b = 7 + 4i$ and $a - b = -1 + 14i$. This means that complex numbers on the complex plane add in the same manner as vectors.

- Complex numbers are multiplied and divided using the same rules as for normal algebra, with the extra fact that $i^2 = -1$. This also means that since $i \times i = -1$, that $-i \times i = 1$ and $1/i = -i$. Thus

$$(3 + 9i)(4 - 5i) = 12 + 36i - 15i - 45i^2 = -33 + 21i.$$

- The complex conjugate of $3 + 2i$ is $3 - 2i$ (the sign of the imaginary part is switched). The complex congugate of z is written z^*. Note that when a number is multiplied by its complex conjugate, the product is always real. For example, suppose $z = x + iy$, and so $z^* = x - iy$, we find

$$zz^* = (x + iy)\ (x - iy) = x^2 + iyx - xiy - i^2y^2 = x^2 + y^2.$$

- The 'magnitude' of a complex number (that is the distance of its point on the complex plane from the origin) is known as its *modulus*. The modulus of z is written $|z|$ and is equal to the square root of $z \times z^*$.

- The 'bearing' of a complex number (that is the angle made by a line joining its point on the complex plane to the origin and the $+x$ axis) is known as its *argument*. Arguments are always given in radians. Thus $\mathrm{Arg}(1 + i) = \pi/4$.

- It can be proved that $e^{i\theta} = \cos\theta + i\sin\theta$. Thus $e^{i\theta}$ is like a unit vector pointing in a direction which makes angle θ with the $+x$ axis. This means that a complex number with a modulus r and argument θ can be written $re^{i\theta}$.

- If we multiply two complex numbers $re^{i\theta}$ and $se^{i\phi}$ we get $rs\, e^{i(\theta+\phi)}$. Thus the modulus of the product is the product of the moduli, and the argument of the product is the sum of the arguments.

- The reciprocal of a complex number $z^{-1} = \left(re^{i\theta}\right)^{-1} = r^{-1}e^{-i\theta}$.

- Given that $e^{i\theta} = \cos\theta + i\sin\theta$, $e^{i\theta} + e^{-i\theta} = 2\cos\theta$. It is also true that $e^{i\theta} - e^{-i\theta} = 2i\sin\theta$.

- Calculus can be performed on complex numbers using the normal rules, treating i as a constant.

- Given that i is a complex number with modulus 1 and argument $\pi/2$ it follows that $i = e^{i\pi/2}$. Thus, multiplying a number by i rotates its location on the complex plane anticlockwise by a right angle. Multiplication by $-i$ performs a rotation by a right angle clockwise.

Functions in the Complex Plane

In this section, we think about a complex number $z = x + iy$, and a function which acts upon it F. Let us suppose that when F acts on z, the result, which will be another complex number, is $w = u + iv$. Thus $w = F(z)$, or if you prefer to look at it with real and imaginary parts, we could write $u + iv = F(x + iy)$.

This function can be differentiated unambiguously. For example, suppose $w = z^2$, then $dw/dz = 2z$. This is in contrast to the situation with vectors, where we have different derivatives in different directions. While this is much simpler than vector analysis, it does mean that only certain patterns can be described in the complex plane. Any function which can be written in terms of z such as z^2, e^z or $3z^* + 4/z^6$ will work, however not all functions written specifically in terms of the real parts and imaginary parts, say $F(z) = u(x, y) + iv(x, y)$ where u and v are two completely different functions, will be suitable.

Suppose we make a small change to z, which we call δz. Initially let us just change the real part of z, so the change $\delta z = \delta x$. The change in $w = u + iv$ will be

$$\begin{aligned}\delta w &= \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\delta x \\ &= \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\delta z. \end{aligned} \tag{6.1}$$

Now let us imagine changing the imaginary part of z, so $\delta z = i\delta y$. This means that $\delta y = \delta z / i = -i\delta z$. The change in $w = u + iv$ will now be

$$\begin{aligned}\delta w &= \left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)\delta y \\ &= -i\left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)\delta z \\ &= \left(-i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}\right)\delta z. \end{aligned} \tag{6.2}$$

However equations 6.1 and 6.2 both give a value for δw, and therefore they need to be equal. This means that, equating the real and imaginary parts, we must have

$$\begin{aligned}\frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \\ \frac{\partial v}{\partial x} &= -\frac{\partial u}{\partial y}. \end{aligned} \tag{6.3}$$

These are known as the *Cauchy-Riemann Conditions*, and a function $F(z) = u(z) + iv(z)$ can not be described adequately using the complex plane unless it is satisfied. When it is, the two parts of the function $u(x, y)$ and $v(x, y)$ are called *conjugate harmonics*.

The final point we will make about the complex plane requires us to draw two sets of real and imaginary axes — one labelled z with x as its real part and y as its imaginary part, and a second labelled w with u as its real part and v its imaginary part. The function $w = f(z)$ then takes you from a co-ordinate on the x, y graph to a co-ordinate on the u, v graph. This is called a mapping.

Mappings using complex functions have a special property built in — they are said to be *conformal*. This means that if you zoom in on a point z_0 and its equivalent $w_0 = f(z_0)$, in the immediate vicinity of the two points any patterns will look the same (albeit that they could be magnified and rotated as a whole). To prove this, we use Taylor's theorem, and evaluate $w = w_0 + \delta w$ if $z = z_0 + \delta z$ if $\delta z = e^{i\theta}\,\delta r$:

$$\begin{aligned} w = w_0 + \delta w &= f\,(z + \delta z) \\ &\approx f\,(z_0) + \frac{df}{dz}\,\delta z \\ &= w_0 + \frac{df}{dz}\,e^{i\theta}\,\delta r \\ \delta w &\approx \frac{df}{dz}\,e^{i\theta}\,\delta r. \end{aligned} \tag{6.4}$$

When evaluated at the point z_0, the derivative $df\,/\,dz$ will have a specific complex value. Suppose that this is $T\,e^{i\tau}$. Then $\delta w = e^{i(\theta+\tau)}\,T\,\delta r$. Notice that if we change θ, the argument of δw increases in step. Thus two lines, defined by functions of z which met at 30° in the z plane, when transformed by $f(z)$ also meet at 30° in the w plane. And if two complex functions are evaluated with the same value of z, say $a(z)$ and $b(z)$ and we find that $a(z)$ changes twice as much as $b(z)$ when we move from z_0 to $z_0 + \delta z$, then we will similarly find that $a(w)$ will change twice as much as $b(w)$ when we move from w_0 to $w_0 + \delta w$.

Fluids described in the Complex Plane

Fluids are particularly well suited to analysis using this technique. Consider a complex function $F(z) = \phi + i\psi$, where ϕ and ψ are functions of z which satisfy the Cauchy-Rieman conditions. It will follow that

$$\begin{aligned} \frac{\partial^2\phi}{\partial x^2} &= \frac{\partial}{\partial x}\frac{\partial\phi}{\partial x} = \frac{\partial}{\partial x}\frac{\partial\psi}{\partial y} = \frac{\partial^2\psi}{\partial x\partial y} \\ \frac{\partial^2\phi}{\partial y^2} &= \frac{\partial}{\partial y}\frac{\partial\phi}{\partial y} = -\frac{\partial}{\partial y}\frac{\partial\psi}{\partial x} = -\frac{\partial^2\psi}{\partial y\partial x} \\ 0 &= \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} \end{aligned}$$

Now imagine that we have a fluid, and we manage to come up with a function $\phi(x,y)$ such that the components of the fluid's velocity at a point (x,y)

are given by $V_x = \partial\phi/\partial x$ and $V_y = \partial\phi/\partial y$. This funtion is called a *velocity potential*. It follows from our last equation, that

$$\frac{\partial}{\partial x} V_x + \frac{\partial}{\partial y} V_y = 0.$$

As shown on page 94, this is equivalent to writing $\nabla \cdot \mathbf{V} = 0$, which is the condition for continuity of an incompressible fluid.

Thus functions in the complex plane naturally describe the motion of fluids.

If ϕ describes the velocity potential at a point, its conjugate harmonic function ψ is called the stream function, for a reason we are about to discover. Its derivatives, $\partial\psi/\partial x$ and $\partial\psi/\partial y$, are equivalent to the $\partial v/\partial x$ and $\partial v/\partial y$ of equation 6.3 where $u = \phi$ and $v = \psi$. We therefore have

$$\begin{aligned} \frac{\partial\psi}{\partial x} &= -\frac{\partial\phi}{\partial y} = -V_y \\ \frac{\partial\psi}{\partial y} &= \frac{\partial\phi}{\partial x} = +V_x. \end{aligned}$$

If we write these two derivatives together in the form of a complex number (just as we did with $V = \partial\phi/\partial x + i\partial\phi/\partial y$), it becomes $\partial\psi/\partial x + i\partial\psi/\partial y = -V_y + iV_x$, which is the same as $i\left(V_x + iV_y\right)$. This points in a direction at a right angle to $V = V_x + iV_y$, as multiplication by i always rotates the point by a right angle. This means that the value of ψ changes most rapidly when we move at right angles to the flow V, and accordingly does not change if we move in the direction of the flow. This is analagous to a contour being at right angles to the direction of ∇z on page 94. Thus the value of ψ stays the same as you follow the fluid along its path. We call this path a *stream line*, and accordingly ψ is called the stream function as if we write $\psi = 5$ (or any other constant number) this will specify a particular stream line.

The combined function $F = \phi + i\psi$ is known as the *complex potential* of the flow.

Not only can the complex potential F give you the potential and the stream function, there is a simple method for working out the velocity V of the fluid. We note that

$$V = \frac{\partial\phi}{\partial x} + i\frac{\partial\phi}{\partial y}$$

$$
\begin{aligned}
&= \frac{\partial \phi}{\partial x} - i\frac{\partial \psi}{\partial x} \\
&= \frac{\partial}{\partial x}\left(\phi - i\psi\right) \\
&= \frac{d}{dz}\, F^*. && (6.5)
\end{aligned}
$$

Before leaving this section, let us write the complex potentials $F = \phi + i\psi$ for two simple examples of fluid flow.

Our first example is uniform flow of speed s in a direction α. In this case, we want $V = s\cos\alpha + is\sin\alpha = s\,e^{i\alpha}$. Using equation 6.5, we integrate

$$
\begin{aligned}
V = s\,e^{i\alpha} &= \frac{d}{dz}\, F^* \\
sz\,e^{i\alpha} &= F^* \\
sz\,e^{-i\alpha} &= F. && (6.6)
\end{aligned}
$$

Our second example is the flow for a vortex of circulation γ, as described on page 47. If we place the centre of the vortex at the origin, then at a position z, the fluid will be flowing at a right angle to z in an anticlockwise direction at a speed $s = \gamma/2\pi|z|$. The direction of the flow will therefore be iz, so V must be in the iz direction of modulus $\gamma/2\pi|z|$. A complex number of unit modulus in the right direction will be $iz/|z|$, and so we write

$$
\begin{aligned}
V = \frac{d}{dz}\, F^* &= \frac{iz\gamma}{2\pi|z|^2} \\
&= \frac{iz\gamma}{2\pi z z^*} \\
&= \frac{i\gamma}{2\pi z^*} \\
F^* &= \frac{i\gamma}{2\pi}\,\ln\, z^* \\
F &= -\frac{i\gamma}{2\pi}\,\ln\, z^*. && (6.7)
\end{aligned}
$$

This completes the toolkit we need in the complex plane for solving the lift equation for a straight wing.

Flow round a cylinder

To investigate the flow round a cylinder, we use a very useful conformal mapping $w = z + z^{-1}$, where we are only interested in z above the real axis (ie. we are only interested in $z = x + iy$ if $y \geq 0$). If we write $z = r\,e^{i\theta}$, it follows that

$$\begin{aligned}
w &= r\,e^{i\theta} + \frac{1}{r\,e^{i\theta}} \\
&= r\,e^{i\theta} + \frac{e^{-i\theta}}{r} \\
u = \operatorname{Re} w &= r\,\cos\theta + \frac{\cos\theta}{r} \\
&= \left(r + \frac{1}{r}\right)\,\cos\theta, \text{ and} \qquad (6.8) \\
v = \operatorname{Im} w &= r\,\sin\theta - \frac{\sin\theta}{r} \\
&= \left(r - \frac{1}{r}\right)\,\sin\theta. \qquad (6.9)
\end{aligned}$$

If $r > 1$ then $r > r^{-1}$, and $v > 0$. If $r < 1$ then $r < r^{-1}$, and $v < 0$. Therefore the points which were within the unit circle in our original z space above the real axis have been moved below the real axis in w space, while those originally outside the circle and above the real axis have occupied the whole of the region above the real axis in w space. The points on the circle itself $r = 1$ are on the real axis in w with $w = 2\cos\theta$.

We start by considering an easy flow pattern — that in w space parallel to the real axis. So $V = s$ where s is some steady speed. The complex potential of this flow will be $F(w) = sw$ using equation 6.6 with $\alpha = 0$ as we wish horizontal flow.

To obtain the flow pattern in the z plane, we rewrite the function F replacing each w with $z + z^{-1}$ and this gives

$$F(z) = s\left(z + \frac{1}{z}\right). \qquad (6.10)$$

If we are interested in a flow past a cylinder of radius R, we scale up so that the unit circle becomes a circle of radius R. In this way, the z of equation 6.10 needs to become the new z/R which will still be 1 on the

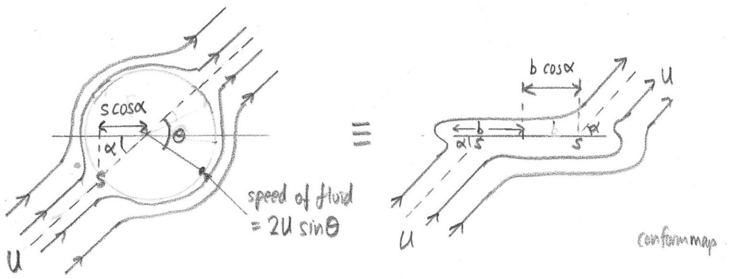

Figure 6.1: The flow of air over a cylinder, showing the stagnation points. The equivalent points on the plate are marked.

cylinder itself. Similarly, if we want to simulate a flow in a direction α to the real axis, we replace z in our original equation with $z\,e^{-i\alpha}$, which will line up with the real axis when the argument of the new z is α. In short we are swapping z for $ze^{-i\alpha}/R$. With these changes we get

$$F = s\left(\frac{z\,e^{-i\alpha}}{R} + \frac{R}{z\,e^{-i\alpha}}\right). \tag{6.11}$$

The velocity function can be evaluated using our normal rule (equation 6.5):

$$\begin{aligned} V &= \frac{d}{dz}F^* \\ &= \frac{d}{dz}s\left(\frac{z^*\,e^{i\alpha}}{R} + \frac{R}{z^*\,e^{i\alpha}}\right) \\ &= s\left(\frac{e^{i\alpha}}{R} - \frac{R}{z^{*2}\,e^{i\alpha}}\right) \\ &= \frac{s\,e^{i\alpha}}{R}\left(1 - \frac{R^2}{z^{*2}\,e^{2i\alpha}}\right). \end{aligned} \tag{6.12}$$

Notice that the speed at great distance from the cylinder (when $|z| \to \infty$) is s/R, which we shall refer to as U, just as we did in the first chapter on wings. The situation is also shown in figure 6.1.

We are particularly interested in the speed of the flow past the cylinder itself $|z| = R$. We therefore write $z = Re^{i\theta}$ and find

$$\begin{aligned}
V &= U\,e^{i\alpha}\left(1 - \frac{1}{e^{-2i\theta}\,e^{2i\alpha}}\right) \\
&= Ue^{-i\alpha}\,e^{2i\theta}\,\left(e^{2i(\alpha-\theta)} - 1\right) \\
&= Ue^{i\theta}\,\left(e^{i(\alpha-\theta)} - e^{-i(\alpha-\theta)}\right) \\
&= 2iU\,e^{i\theta}\,\sin\left(\alpha - \theta\right). \qquad (6.13)
\end{aligned}$$

The stagnation points on the cylinder, where $V = 0$ are therefore at $\theta = \alpha$ and $\theta = \alpha - \pi$, matching the direction from which the flow comes, and to which the flow travels. On the cylinder itself, the fluid flows past tangentially to the cylinder at a speed $2U\,\sin\left(\theta - \alpha\right)$. On the real axis, where $\theta = 0$ we have a speed of magnitude $2U \sin\alpha$. We shall need that speed later.

What we now do is very cunning. Having calculated the flow past the cylinder, we now throw away the lower half of the diagram (below the real axis), as we can work this out by symmetry anyway, and put the new flow through our conformal mapping $w = z + z^{-1}$ backwards, to work out a new flow in w. Our cylinder has now been morphed back onto the plate which runs from $w = -2R$ (when $z = -R$) to $w = 2R$ (when $z = R$). Thus the width of the plate, which is equivalent to the *chord length* of the wing, that is the distance from the leading edge to the trailing edge on a straight line, is $C = 4R$.

If we wanted to know the flow in detail, we could work it out, and this would predict the flow of air over the flat-plate wing in the theoretical case shown in figure 3.7. However, all we wish to know is where the upper stagnation point is. On the cylinder, it was at $R\,e^{i\alpha}$, so on the flat-plate wing it will be at

$$\begin{aligned}
w &= z + \frac{1}{z} \\
&= R\left(e^{i\alpha} + e^{-i\alpha}\right) \\
&= 2R\cos\alpha. \qquad (6.14)
\end{aligned}$$

For true lift, the stagnation point moves to the back edge of the wing, where $w = 2R$, and this must mean $z = R$. So we now want to modify our cylinder model so that the fluid is stationary at $z = R$ (on the real axis). We do this by adding a clockwise rotation to the cylinder γ.

Before the stagnation point moved back, the speed of the fluid here was $2U \sin\alpha$ as shown above. Therefore we need a circulation which will give a rotational speed of $2U \sin\alpha$ at a radius R. So, the circulation needed, remembering that $\gamma = 2\pi v R$ as defined on page 47, is

$$\begin{aligned}\gamma &= 2\pi R v \\ &= 2\pi R\, 2U \sin\alpha \\ &= 4\pi R U \sin\alpha. \end{aligned} \tag{6.15}$$

In the terminology of the plate wing, where $C = 4R$, the circulation is $\gamma = \pi C U \sin\alpha$.

Using equation 3.9 on page 44, and noting that $bC = A$ the area of the wing, the lift will therefore be

$$\begin{aligned} L &= \rho b U \gamma \\ &= \rho b U \times \pi C U \sin\alpha \\ &= \rho\, bC\, U^2\, \pi\, \sin\alpha \\ &= \rho A U^2\, \pi\, \sin\alpha \\ &= \frac{\rho A U^2}{2} \times 2\pi \sin\alpha, \end{aligned} \tag{6.16}$$

where we have ensured that the equation is in the same form as 3.3 on page 39, thereby indicating that not only does our model give a plausible answer for the lift of w simple wing, it also predicts a value for the lift co-efficient $C_L = 2\pi \sin\alpha$, which for small angles will approximate to $C_L = 2\pi\alpha$.

6.2 The Lift for more Realistic Aerofoils

In this section, we give more detail regarding the method outlined on page 47 for calculating the lift derived from a wing with a more complicated cross section (or *aerofoil*). In this section, we work out the lift from a curved, although thin, wing.

Figure 6.2 shows that we put a row of vortices in a straight line joining the leading edge of the wing to the trailing edge. The line from leading edge to trailing edge (the *chord*) is split into N sections of equal length. One

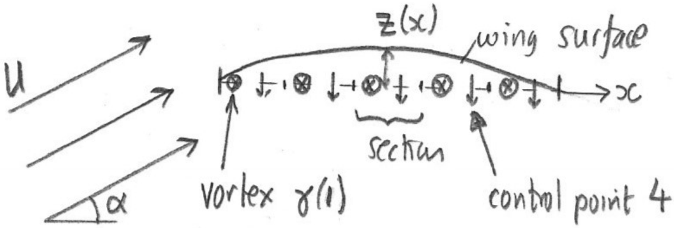

Figure 6.2: Modelling a thin wing as a set of vortices in a straight line. The strength of the i-th vortex is labelled $\gamma(i)$, and it is at a distance $x_v(i)$ behind the leading edge of the wing. We are particularly interested in the flow predicted by the model at the *control points* which are in between the vortices at distances $x_c(i)$ behind the wing.

vortex is put in each section — usually one quarter of the way back. The flow is monitored at control points which are placed three-quarters of the way back in each section. We are particularly interested in the component of the velocity perpendicular to the chord at each control point.

The speed of flow due to a vortex is $\gamma/2\pi s$ where s is the distance between the vortex and the point whose flow we are interested in. Given that if vortices γ are defined to be positive if clockwise (the direction needed to give lift), then the speed of the flow will be upwards if the control point is to the left of the vortex, and downwards if to the right. If we regard an upwards flow as positive, then we write the speed at a position x_c on the line due to a vortex at x_v as $\gamma/2\pi\,(x_v - x_c)$.

Therefore, if we add up the contributions from all of the vortices (which we number j, where j runs from 1 to N), then the upwards perpendicular flow at point $x_c(i)$, which we call $w_c(i)$ will be given by

$$w_c(i) = \sum_{j=1}^{N} \frac{1}{2\pi} \frac{\gamma(j)}{x_v(j) - x_c(i)}. \qquad (6.17)$$

To solve this, we arrange the values of $\gamma(j)$ such that the flow is in the right direction at each control point — namely parallel to the wing's surface. Suppose that the height of the wing above the chord line at position x is written $z(x)$, where here z is not a complex number. Then the upward gradient of the wing at this point is given by dz/dx. The air must therefore be travelling in this direction too. Let us suppose that the air's speed is not changed much by the wing, then it has magnitude U, and upward component w, so its gradient will be approximately w/U. The model will therefore fit if at each control point,

$$\frac{w_c(i)}{U} = \left(\frac{dz}{dx}\right)_{x_c(i)}. \tag{6.18}$$

The situation is more complicated once we take angles of attack into account. The sum of the vortices predicts the effect of the wing on the air — in other words the difference the wing makes to the uniform front-to-back flow we would have if the wing were not there. If there is an angle of attack α, then when the vortices are added to this flow, if equation 6.18 were true, the air would still be flowing at an 'uphill' angle of α through each of the sections of the wing. We can compensate for this by forcing our vortex model to predict values of w at each location which correspond to a fluid motion downwards at angle α through the wing's surface. Then when the vortex flow is added to the uniform flow U, we will have air passing along the wing not through it.

This means that the gradient of the vortex flow at a position now needs to be $dz/dx - \alpha$ not just dz/dx, and we modify equation 6.18 accordingly:

$$\frac{w_c(i)}{U} = \left(\frac{dz}{dx}\right)_{x_c(i)} - \alpha. \tag{6.19}$$

Putting equations 6.17 and 6.19 together, we obtain

$$U\left(\frac{dz}{dx}_{x_c(i)} - \alpha\right) = \sum_{j=1}^{N} \frac{1}{2\pi} \frac{\gamma(j)}{x_v(j) - x_c(i)}. \tag{6.20}$$

Although this was written on one line, it is in fact a set of N linear simultaneous equations. The equations are linear, because each one only contains the unknown values $\gamma(i)$, and not non-linear functions of them. If we define

$$\begin{aligned} A_i &= U\left(\frac{dz}{dx}_{x_c(i)} - \alpha\right) \\ B_{ij} &= \frac{1}{2\pi} \frac{1}{x_v(j) - x_c(i)}, \end{aligned}$$

then equation 6.20 becomes

$$A_i = \sum_{j=1}^{N} B_{ij}\,\gamma(j), \tag{6.21}$$

where A and B are known and we wish to find γ. This set of linear equations can be solved numerically by a computer using standard numerical techniques. If we are familiar with matrix notation, then the vector $\mathbf{A}$ is equal to the product of the matrix B with the vector **fl**, and so $\mathbf{fl} = B^{-1}\,\mathbf{A}$.

Real wings are not thin. To modify this procedure for thicker wings, we double the number of vortices and put them in two lines — one along the top of a polygon representing the wing, and one along the bottom. We double the number of control points so that they are present along the upper and lower surfaces of the wing, and then we have double the number of simultaneous equations to make sure that air does not flow through either the upper or lower surface of the wing.

The total circulation of the wing is given by the sum $\sum_i \gamma(i)$, and the lift co-efficient can then be evaluated using the Kutta-Joukowski condition.

6.3 Finite Span Wings

Our analysis so far has been two-dimensional. We have analyzed the flow assuming that each metre of the wing's width acts the same as each other. In practice, this is far from the truth, even for an aircraft like the original Piper Cherokee with its 'Hershey bar' wing which is rectangular when viewed from above. Simply put, the wing has a tip around which air can pass. The degree to which the wing tip affects the flow will be greater for short, stubby wings, and the parameter used to measure 'stubbiness' is called the *aspect ratio* and is defined as b^2/A where b is the wingspan, and A is the area of the wing. For a rectangular wing, $A = bc$ where c is the chord length (the straight distance from leading to trailing edge), $R = b/c$. The ideal wing with no edge would have an infinite aspect ratio.

We find that wings of finite length create vortices about axes stretching backwards from the wing which push the air down. This creates the downwash we encountered in our first chapter on wings, and enables us to explain the induced drag created by a wing in a more rigorous manner.

To summarize the reason for this — consider the wing tip. We have high pressure under the wing, and low pressure above it. Therefore there must

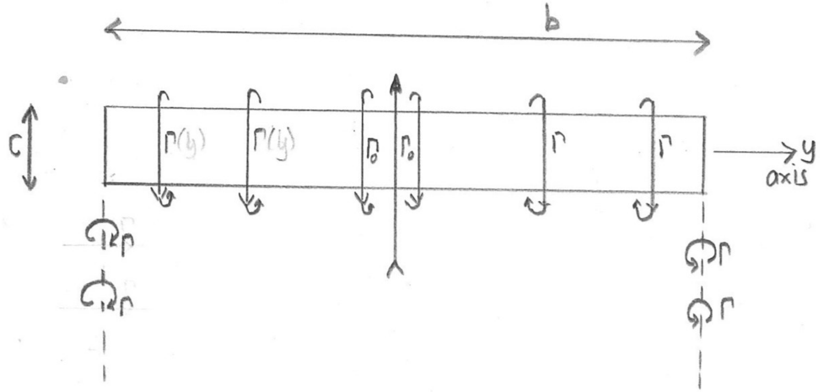

Figure 6.3: The wingtip vortex as an extension of the circulation of air over the main body of the wing.

be air passing from the bottom to the top side by going round the edge on the side. This is another kind of whorl or vortex, and is known as a wing-tip vortex. Recent airliners have had small fins on the end of their wings which have improved the lift of the aeroplane partly by obstructing this flow.

This situation, in a very simplified form, is shown in figure 6.3. It is as if the axis of the circulation around the wing, which in our earlier models stretched out from the fuselage to infinity, turns a corner backwards when it reaches the wing's edge. If this were the case, then because none of the circulation has 'leaked' backwards as you travel from the wing root (next to the fuselage) to the tip, we would have equal circulation, and hence lift, from every part of the wing.

The truth is not so simple. We find that the circulation, and hence the lift, of the wing is greatest at its root, and drops until it reaches zero at the wing tip. For a wing of span b, which accordingly sticks out by a distance of $b/2$ each side of the fuselage, a good approximation is that the circulation (and hence the lift) when plotted as a function of distance out from the centre (we shall refer to this as y) makes an ellipse.

Let us write the circulation as

$$\Gamma(y) = \Gamma_0 \sqrt{1 - \left(\frac{2y}{b}\right)^2}, \tag{6.22}$$

where Γ_0 is the circulation at the wing root where $y = 0$, and accordingly, the circulation Γ drops to zero at $y = b/2$.

Using the statement of the Kutta-Joukowski condition in equation 3.9 on page 44, it follows that the lift derived from the part of the wing between y and $y + \delta y$ will be

$$\delta L = \Gamma(y)\, \rho U\, \delta y. \tag{6.23}$$

It follows that the total lift, if we integrate along the whole wing, will be

$$L = \Gamma_0 \rho U \int_{-b/2}^{b/2} \sqrt{1 - \left(\frac{2y}{b}\right)^2}\, dy.$$

To perform the integral, we make the substitution $y = (b/2)\cos\phi$, so $dy = -(b/2)\sin\phi\, d\phi$:

$$\begin{aligned} L &= -\frac{\Gamma_0 \rho U b}{2} \int_{\pi}^{0} \sin^2\phi\, d\phi \\ &= \frac{\Gamma_0 \rho U b}{2} \int_{0}^{\pi} \sin^2\phi\, d\phi \\ &= \frac{\Gamma_0 \rho U b}{2} \int_{0}^{\pi} \frac{1 - \cos 2\phi}{2}\, d\phi \\ &= \frac{\Gamma_0 \rho U b}{4} \left[\phi - \frac{\sin 2\phi}{2}\right]_0^{\pi} \\ &= \frac{\Gamma_0 \rho U b \pi}{4}. \end{aligned} \tag{6.24}$$

The circulation drops as you pass along the wing at a rate given by the derivative of equation 6.22:

$$\frac{d\Gamma}{dy} = \frac{\Gamma_0}{\sqrt{1 - \left(\frac{2y}{b}\right)^2}} \frac{4y}{b^2}.$$

As shown in figure 6.4, this enables us to work out the vertex 'shed' per unit length of wingspan. These vortices cause a circulation in the air which

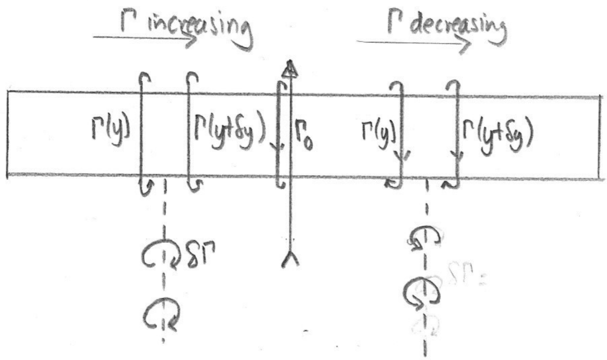

Figure 6.4: The reduction in circulation around a wing $\Gamma(y)$ as we travel along the wing from root to tip corresponds to a shedding of vortices behind the wing. If the loss in circulation around the wing between y and $y + \delta y$ is $\delta\Gamma$, then the strength of the vortex shed behind the wing is also $\delta\Gamma = d\Gamma/dy\, \delta y$.

flows up at the end near the wing tip and down at the end near the wing root. Up until now, we have used the equation $w = \gamma/2\pi r$ for the speed caused by a simple vortex. However the vortices we have studied up until now have been in a two-dimensional analysis. This means, that when extended to three dimensions, they assume that the axis of the vortex is infinite in both directions. The shed vortices here start at the wing and go backwards, but do not go forwards. Thus the speed generated by the vortex is half of the two-dimensional value, and we write $w = \gamma/4\pi r$.

If we wish to calculate the total speed of air moving downwards at a point y_0 on the wing, it will depend on the shed vortices $d\Gamma/dy$ all along the wing. Suppose we consider the shed vortex at y. This will cause a downwash, and a positive value of w when $y < y_0$, whereas it will cause an upwards current (and a negative value of w) when $y < y_0$. The total downwash will therefore be

$$W = \frac{1}{4\pi} \int_0^b \frac{d\Gamma/dy}{y_0 - y}\, dy$$

$$= \frac{1}{4\pi} \int_0^b \frac{\Gamma_0}{\sqrt{1-\left(\frac{2y}{b}\right)^2}} \frac{4y}{b^2} \frac{1}{y_0 - y} \, dy$$

$$= \frac{\Gamma_0}{\pi b^2} \int_0^b \frac{y}{\sqrt{1-\left(\frac{2y}{b}\right)^2}} \frac{1}{y_0 - y} \, dy. \tag{6.25}$$

We complete the integral with the substitution $y = (b/2)\cos\phi$, so $dy = -(b/2)\sin\phi \, d\phi$ as before:

$$\begin{aligned} W &= \frac{\Gamma_0}{2\pi b} \int_0^\pi \frac{\cos\phi \, \sin\phi}{\sin\phi \, (\cos\phi_0 - \cos\phi)} \, d\phi \\ &= \frac{\Gamma_0}{2\pi b} \int_0^\pi \frac{\cos\phi}{\cos\phi_0 - \cos\phi} \, d\phi \\ &= \frac{\Gamma_0}{2b}, \end{aligned} \tag{6.26}$$

where in the final stage, we have looked up the value of the definite integral, which is π[1]

This calculation has told us that the downwash caused by the shed vortices has the same speed at all points along the wing, as our equation for W has no dependence on y or ϕ. Given that the speed of the air over the wing will be close to U, the downwash W causes the air to move downwards at an angle

$$\alpha_w = \frac{W}{U} = \frac{\Gamma_0}{2bU}. \tag{6.27}$$

The effect of this on the angle of attack is shown in figure 6.5. The true angle of attack α is equal to the angle of attack as it appears at the wing relative to the downflowing air added to α_w. In other words, this finite-span wing, with angle of attack α, ends up producing the same lift as an infinite-span wing (which caused no vortices) would produce with angle of attack $\alpha_\infty = \alpha - \alpha_w$.

Next, we remember from equation 6.24 that $L = \Gamma_0 \rho U b \pi/4$, and accordingly, $\Gamma_0 = 4L/\rho U b \pi$. Accordingly

[1] A proof of this result for a suite of similar integrals is given on p208 of R. von Mises *Theory of Flight* Dover, 1959; however other proofs can be found by looking up Poisson Integrals on the web or in a Mathematical reference book.

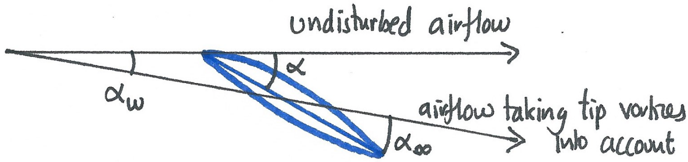

Figure 6.5: The effect of downwash caused by vortices on the angle of attack. We notice that for a 'real' angle of attack α, the downwash causes the air to meet the wing with an effective angle of attack $\alpha - \alpha_w$.

$$\alpha_w = \frac{\Gamma_0}{2bU} = \frac{4L}{2bu \times \rho U b \pi} = \frac{2L}{\pi \rho b^2 U^2}. \tag{6.28}$$

Given that $L = \frac{1}{2} C_L \rho A U^2$, this equation can be simplified

$$\alpha_w = \frac{C_L \rho A U^2}{2} \times \frac{2}{\pi \rho b^2 U^2} = \frac{C_L A}{\pi b^2} = \frac{C_L}{\pi R}, \tag{6.29}$$

where we remember that the aspect ratio $R = b^2/A$. Please note that equations 6.27 – 6.29 give the angle α_w in radians. The angle in degrees will be larger by a factor of $180/\pi$. We therefore expect a wing of finite span to have a larger angle of attack for a given lift co-efficient. This correction needs to be taken into account when working out the lift co-efficient for a finite wing from calculations such as those earlier in this chapter, which treated airflow in a two-dimensional manner.

Question 6.1 *The NACA* $65_2 - 415$ *aerofoil section was tested and found to have* $\alpha_0 = -3.0^\circ$ *and* $k_L = 0.11/^\circ$ *using the notation on page* 50. *What would you expect the values of* α_0 *and* k_L *to be for a wing made using this section with an aspect ratio of* $R = 6.0$?

Given that lift is defined as the aeronautical force on the wing perpendicular to the flow of the air, the fact that the air is now flowing at a downwards angle means that the lift force will tip back by the same angle, and will now be at angle $\alpha_w = W/U$ to the vertical. This in turn means that the lift itself will cause drag, as there is a component of the lift pointing backwards when we view it from the perspective of level flight rather than the airflow. The magnitude of this drag will be $D = L \sin \alpha_w \approx L\alpha_w$. Thus

$$D = L\,\frac{\Gamma_0}{2bU}.$$

Using again the fact that $\Gamma_0 = 4L/\rho U b \pi$, we calculate that the drag is equal to

$$D = L\,\frac{4L}{2bU \times \rho U b \pi} = \frac{2L^2}{\pi \rho b^2 U^2}. \tag{6.30}$$

For level flight, the lift L must equal the weight W, and so we see that the strength of this particular type of drag is proportional to U^{-2}. In other words, it gets smaller the faster we fly. This may seem counter-intuitive for drag, however the slower we go, the more circulation we need to provide the same lift, the greater the shed vortices, the greater the downwash, and the more the lift vector points backwards. This special kind of drag which is a necessary consequence of generating lift is known as *induced drag*, and was introduced in our first chapter on wings.

Our final piece of analysis seeks to evaluate a drag co-efficient for the induced drag. By analogy with lift, which is written in terms of a co-efficient C_L thus: $L = C_L \rho A U^2/2$, we have induced drag $D_i = C_{Di} \rho A U^2/2$. Substituting for L and D in equation 6.30 we have

$$\begin{aligned} C_{Di}\,\frac{\rho A U^2}{2} &= \frac{2}{\pi \rho b^2 U^2}\left(\frac{C_L \rho A U^2}{2}\right)^2 \\ C_{Di} &= \frac{2}{\pi \rho b^2 U^2}\, C_L^2\, \frac{\rho A U^2}{2} \\ &= \frac{A C_L^2}{\pi b^2}. \end{aligned} \tag{6.31}$$

Remembering that $b^2/A = R$, the aspect ratio, we find that the drag co-efficient due to induced drag is given by

$$C_{Di} = \frac{C_L^2}{\pi R}. \tag{6.32}$$

Question 6.2 *The NACA $65_2 - 415$ aerofoil mentioned earlier has a drag co-efficient $C_D = 0.004 C_L$ for angles of attack $-3^\circ < \alpha < 5^\circ$. What would you* *expect the drag co-efficient for a finite wing with aspect ratio* $R = 6$ *to be, as a function of* C_L*, over this range? Evaluate* C_D *for the infinite and finite (*$R = 6$*) wing for* $C_L = 0.6$*.*

Chapter 7

Solutions to Exercises

7.1 Units & Dimensional Analysis

Solution 1.1 *We convert the pressure from hPa to inHg using the information given that 1013hPa is the same as 29.92inHg:*

$$997\text{hPa} \times \frac{29.92\text{inHg}}{1013\text{hPa}} = 29.44\text{inHg}$$

Solution 1.2 *One foot is 0.305m. The height is easier to convert:* $400\text{m} = 400/0.305\text{ft} = 1311\text{ft}$. *The vertical speed takes a bit more work:*

$$\frac{3.0\text{m}}{1\text{s}} \times \frac{1\text{ft}}{0.305\text{m}} \times \frac{60\text{s}}{1\text{min}} = \frac{3.0 \times 60\text{ft}}{0.305\text{min}} = 590\text{ft}/\text{min}$$

Solution 1.3 *We shall assume that you are flying in a region where the ground is above mean sea level. This means that QFE readings will be less than QNH readings. If your altimeter were set to QFE when you thought it was set to QNH you would be much lower than you thought, and accordingly could crash into the ground. For example, suppose you were flying in an area where the ground were 500 feet above sea level, and you had QFE set, but thought your altimeter were reading the height above mean sea level, then you would hit the ground when you thought there was still 500 feet of clear air below you.*

Solution 1.4 *We convert the unit as shown below. The symbol for 'one pound' is 1lb.*

$$\frac{24\text{lb}}{1\text{in}^2} = \frac{24 \times 0.454\text{kg} \times 9.81\text{N}/\text{kg}}{(0.0254\text{m})^2} = \frac{24 \times 0.454 \times 9.81\text{N}}{0.0006452\text{m}^2} = \frac{1.657 \times 10^5\text{N}}{\text{m}^2}$$

Solution 1.5 *To convert degrees to radians, we multiply by* $\pi\,\mathrm{rad}/180^\circ$*, giving* $\pi/2 = 1.57$ *rad.*

Solution 1.6 *To convert radians to degrees, we multiply by* $180^\circ/\pi\mathrm{rad}$*, giving* $0.24 \times 180/\pi = 13.8^\circ$.

Solution 1.7 *The angle in radians is* $1/8 = 0.125$*, which in degrees is* $180/(8\pi) = 7.2^\circ$.

Solution 1.8 *Your angle to the intended direction is* $1.5/10 = 0.15$ *rad, which is* 8.6°*. If you turn* 8.6° *to the left, you would now fly West. If you turn an additional* 8.6°*, you will fly towards your original planned course such that you meet it after the same distance as you have already flown, namely 20nm from the start.*

Solution 1.9 *We use the same method as the previous question. We need to turn* 8.6° *to fly West. However if we only want to reach the original course at the destination, the extra angle we need to turn is to cover 1.5nm South in 20nm, namely an angle of* $1.5/20 = 0.075$ *radians, or* 4.3°*. The total angle turned is* $8.6 + 4.3 = 12.9^\circ$.

Solution 1.10 *We convert the units as shown below:*

$$\frac{50\mathrm{nm}}{\mathrm{hour}} \times \frac{1852\mathrm{m}}{1\mathrm{nm}} \times \frac{1\mathrm{mile}}{1609\mathrm{m}} = \frac{50 \times 1852}{1609}\mathrm{mph} = 57.6\mathrm{mph}$$

Solution 1.11 *An angle of* 3° *is the same as* $3 \times \pi/180 = 0.0524\mathrm{rad}$*. This will be the ratio of the vertical speed to the horizontal speed. The horizontal speed, in m/s, is*

$$\frac{150\mathrm{nm}}{1\mathrm{hr}} = \frac{150 \times 1852\mathrm{m}}{3600\mathrm{s}} = 77.17\mathrm{m/s}$$

For a 3° *glidepath, the vertical speed will be* $77.17\mathrm{m/s} \times 0.0524 = 4.04\mathrm{m/s}$*. We now convert this into feet/min:*

$$\frac{4.04\mathrm{m}}{1\mathrm{s}} \times \frac{1\mathrm{ft}}{0.305\mathrm{m}} \times \frac{60\mathrm{s}}{1\mathrm{min}} = \frac{4.04 \times 60\mathrm{ft}}{0.305\mathrm{min}} = 795\mathrm{ft/min}$$

Solution 1.12 *Swapping the symbols for units, we have*

$$\mathrm{s} = (\mathrm{kg})^{\alpha}\ (\mathrm{m})^{\beta}\ \left(\mathrm{ms}^{-2}\right)^{\gamma}$$

Equating powers of kg gives $0 = \alpha$, *equating powers of m gives* $0 = \beta + \gamma$, *while powers of s gives* $1 = -2\gamma$. *It therefore follows that* $\alpha = 0$, $\gamma = -1/2$ *and* $\beta = 0 - \gamma = 1/2$. *Thus the equation becomes* $T = Pm^0 L^{1/2} g^{-1/2}$. *As* $L^{1/2}$ *is the same thing as* $\sqrt{L}$, *and* $m^0 = 1$, *it follows that* T *is proportional to* $\sqrt{L/g}$.

Solution 1.13 *We start by writing an equation for the drag* $D = P\rho^{\alpha} A^{\beta} v^{\gamma}$. *The drag will be in newtons N where we remember that 1N = 1kgms*^{2}. *Writing the equation in terms of units, we get*

$$\mathrm{kgms}^{-2} = \left(\mathrm{kgm}^{-3}\right)^{\alpha}\ \left(\mathrm{m}^2\right)^{\beta}\ \left(\mathrm{ms}^{-1}\right)^{\gamma}$$

Equating the power of units gives for kg: $1 = \alpha$, *for m:* $1 = -3\alpha + 2\beta + \gamma$ *and for s:* $-2 = -\gamma$. *Thus* $\alpha = 1$, $\gamma = 2$, *and so* $2\beta = 1 + 3\alpha - \gamma = 2$, *thus* $\beta = 1$. *Our equation is thus* $D = P\rho^1 A^1 v^2 = P\rho A v^2$.

Solution 1.14 *Let us write the equation* $Q = PD^{\alpha} v^{\beta}$. *In terms of units, we have*

$$\mathrm{m}^3\mathrm{s}^{-1} = \mathrm{m}^{\alpha}\ \left(\mathrm{ms}^{-1}\right)^{\beta}.$$

It follows that for m: $3 = \alpha + \beta$, *while for s:* $-1 = -\beta$. *Thus* $\beta = 1$, *and* $\alpha = 3 - \beta = 2$. *We therefore have* $Q = PD^2 v^1$. *Thus* Q *is proportional to* $D^2 v$.

Solution 1.15 *We write the equation* $P' = kv^{\alpha}\mu^{\beta}D^{\gamma}$ *where we use* k *to represent the constant of proportionality as we don't want any confusion which might be caused by using* P *to represent a constant when we are working with a very much non-constant pressure* P'. *In terms of units, we have*

$$\mathrm{kgm}^{-2}\mathrm{s}^{-2} = \left(\mathrm{ms}^{-1}\right)^{\alpha}\ \left(\mathrm{kgm}^{-1}\mathrm{s}^{-1}\right)^{\beta}\ \mathrm{m}^{\gamma}.$$

Equating the powers of kg gives $1 = \beta$, *equating powers of m gives* $-2 = \alpha - \beta + \gamma$, *while the powers of s give* $-2 = -\alpha - \beta$. *Therefore* $\beta = 1$, *so* $\alpha = 2 - \beta = 1$ *and* $\gamma = -2 - \alpha + \beta = -2 - 1 + 1 = -2$. *The equation becomes* $P' = kv^1\mu^1 D^{-2}$ *and so the pressure drop per metre is proportional to* $v\mu/D^2$.

Solution 1.16 *We know that* $Q \propto D^2 v$ *where the symbol* $\propto$ *means 'proportional to'. Using our equation from the last question,* $P' \propto v\mu/D^2$*, and so* $v \propto P'D^2/\mu$*. This means that* $Q \propto D^2 \times \left(P'D^2/\mu\right)$*, so* $Q \propto P'D^4/\mu$*. It turns out that the actual equation is* $Q = \pi D^4 P'/128\mu$*.*

Solution 1.17 *Here we write the equation* $P' = kv^{\alpha}\mu^{\beta}D^{\gamma}\rho^{\delta}$*. Written in terms of units this becomes*

$$\mathrm{kgm^{-2}s^{-2}} = \left(\mathrm{ms^{-1}}\right)^{\alpha} \left(\mathrm{kgm^{-1}s^{-1}}\right)^{\beta} \mathrm{m}^{\gamma} \left(\mathrm{kgm^{-3}}\right)^{\delta}.$$

The term in kg tells us $1 = \beta + \delta$*, the term in m gives us* $-2 = \alpha - \beta + \gamma - 3\delta$*, and the term in s is* $-2 = -\alpha - \beta$*. We are going to express* α*,* β *and* γ *in terms of* δ*. We have* $\beta = 1 - \delta$*,* $\alpha = 2 - \beta = 2 - 1 + \delta = 1 + \delta$ *and* $\gamma = -2 - \alpha + \beta + 3\delta = -2 - (1+\delta) + (1-\delta) + 3\delta = -2 + \delta$*. Our equation then becomes* $P' = kv^{1+\delta}\mu^{1-\delta}D^{-2+\delta}\rho^{\delta}$*. Grouping the terms produces*

$$P' = k\frac{v\mu}{D^2}\left(\frac{vD\rho}{\mu}\right)^{\delta}.$$

and we do indeed have a part similar to question 1.15 multiplied by some function of the Reynolds number $\rho v D/\mu$*.*

7.2 Fundamentals of Fluids

Solution 2.1 *The force is given by* $F = pA$*. The radius of the window is 5cm = 0.05m, so the area is* $\pi r^2 = \pi \times 0.05^2$*. Therefore the force is* $F = pA = 10^5 \times \pi \times 0.05^2 = 785\mathrm{N}$.

Solution 2.2 *The flow rate is* $V/t = vA = 0.75 \times (2.1 \times 0.32) = 0.504\mathrm{m^3s^{-1}}$.

Solution 2.3 *The product of* vA *must be the same for the inflow and outflow pipe. Therefore we have*

$$\begin{aligned} v_1 A_1 &= v_2 A_2 \\ v_1 \pi R_1^2 &= v_2 \pi R_2^2 \\ v_2 &= v_1 \left(\frac{R_1}{R_2}\right)^2 \\ &= 0.26 \times \left(\frac{1.2}{0.43}\right)^2 = 2.02\mathrm{m/s}. \end{aligned}$$

Solution 2.4 *The product of $\rho v A$ must be the same for the inflow and outflow pipe. Therefore we have*

$$\begin{aligned}
\rho_1 v_1 A_1 &= \rho_2 v_2 A_2 \\
\rho_1 v_1 \pi R_1^2 &= \rho_2 v_2 \pi R_2^2 \\
v_2 &= v_1 \left(\frac{\rho_1}{\rho_2}\right)\left(\frac{R_1}{R_2}\right)^2 \\
&= 0.26 \times 2 \times \left(\frac{1.2}{0.43}\right)^2 = 4.05\mathrm{m/s}.
\end{aligned}$$

Solution 2.5 *The value of $p + \frac{1}{2}\rho v^2$ remains the same. The value of $\frac{1}{2}\rho v^2$ rises from 0 to $\frac{1}{2} \times 1.2 \times 50^2 = 1500\mathrm{Pa}$. Thus the pressure p must fall by the same amount.*

Solution 2.6 *Dynamic pressure is given by $\frac{1}{2}\rho v^2 = \frac{1}{2} \times 1.2 \times 150^2 = 13500\mathrm{Pa}.$*

Solution 2.7 *The difference between the total and static pressures (630Pa) is the dynamic pressure, which is equal to $\frac{1}{2}\rho v^2$. Thus $630 = \rho v^2/2$ so $v = \sqrt{630 \times 2/\rho} = \sqrt{630 \times 2/1.2} = 32.4\mathrm{m/s}$. We now convert this speed to knots, remembering that there are 1850m in one nautical mile. Therefore*

$$\frac{32.4\mathrm{m}}{\mathrm{s}} \times \frac{3600\mathrm{s}}{1\mathrm{hr}} \times \frac{1\mathrm{nm}}{1850\mathrm{m}} = 63\mathrm{nm/hr} = 63\mathrm{kt}.$$

Solution 2.8 *To answer this question we need to know the ratio of the density of the air in this situation, and density of air assumed when the instrument was made. The question tells us that the density is proportional to the pressure. Thus $\rho_{\mathrm{Std}}/\rho_{\mathrm{Act}} = p_{\mathrm{Std}}/p_{\mathrm{Act}} = 1013/980 = 1.0337$. It follows, using equation 2.12 that TAS = IAS $\times \sqrt{1.0337} = 90 \times \sqrt{1.0337} = 91.5\mathrm{kt}.$*

Solution 2.9 *The pressure at depth will be 202kPa, and so 202kPa $+\rho g d = p_{\mathrm{atm}}$, and so $\rho g d$ =101kPa. Thus $d = p_{\mathrm{atm}}/\rho g$. This evaluates to $1.01 \times 10^5/(1000 \times 9.81) = 10.3\mathrm{m}.$*

Solution 2.10 *We take $v = 0$, and therefore $p + \rho g h$ must be the same in both columns. This gives us*

$$\begin{aligned}
p_{\mathrm{atm}} &= p_{\mathrm{vac}} + \rho g h \\
p_{\mathrm{atm}} &= 0 + \rho g h \\
h &= \frac{p_{\mathrm{atm}}}{\rho g} = \frac{1.01 \times 10^5}{13500 \times 9.8} = 0.763\mathrm{m}.
\end{aligned}$$

Solution 2.11 *The pressure on the top face (at depth d) will be $p_{\text{atm}} + \rho g d$, and so the force downwards on the top face will be $pA = p_{\text{atm}} A + \rho g d A$. The pressure on the bottom face (at depth $d + L$) will be $p_{\text{atm}} + \rho g (d + L)$, and so the force upwards on the bottom face will be $pA = p_{\text{atm}} A + \rho g A (d + L)$. The difference in forces (the unbalanced force) is $\rho g A L$ upwards, which is equal to $\rho g V$ where V is the volume. We remember that $m = \rho V$, so the force equals mg where m is the mass of water which would be needed to fill the volume, and thus the force equals the weight of water which would have occupied the space if the prism were not there. This agrees with Archimedes' Principle.*

Solution 2.12 *We use equation 2.17, and remembering that the kelvin temperature $T = 30 + 273 = 303$ we calculate*

$$\text{density} = \frac{pm}{kT} = \frac{9.80 \times 10^4 \times 4.78 \times 10^{-26}}{1.38 \times 10^{-23} \times 303} = 1.12\text{kg}/\text{m}^3.$$

Solution 2.13 *We use the density we have calculated in the previous question. So $\rho_{\text{Act}} = 1.12\text{kg}/\text{m}^3$. We then finish off the question by using equation 2.12 to give*

$$\text{TAS} = \text{IAS} \times \sqrt{\frac{\rho_{\text{Std}}}{\rho_{\text{Act}}}} = 110 \times \sqrt{\frac{1.21}{1.12}} = 114\text{kt}.$$

Solution 2.14 *Firstly, we need the density of the air at $15°C = 15 + 273 = 288K$ and 1013hPa. We use equation 2.17 and calculate*

$$\text{density} = \frac{pm}{kT} = \frac{1.013 \times 10^5 \times 4.78 \times 10^{-26}}{1.38 \times 10^{-23} \times 288} = 1.218\text{kg}/\text{m}^3.$$

From this, we can now work out the weight of the air which would usually occupy the volume the airship now takes up:

$$W = mg = \rho V g = 1.218 \times 10^4 \times 9.81 = 1.195 \times 10^5\text{N}.$$

This will be the upwards (buoyancy) force on the airship. Next, we work out the density of the helium:

$$\text{density} = \frac{pm}{kT} = \frac{1.013 \times 10^5 \times 6.66 \times 10^{-27}}{1.38 \times 10^{-23} \times 288} = 0.1698\text{kg}/\text{m}^3,$$

and then the weight of the helium:

$$W = mg = \rho V g = 0.1698 \times 10^4 \times 9.81 = 1.665 \times 10^4\text{N}.$$

The net force on the airship will therefore be $119500 - 16650 = 102850$*N. This will be able to support the airship as long as the mass of its structure and payload is less than*

$$m = \frac{W}{g} = \frac{102850\text{N}}{9.81\text{N/kg}} = 10480\text{kg}.$$

To the accuracy supported by the data, we would say that the structure must have a mass of no more than 10.5 tonnes (1 tonne = 1000 kg).

Solution 2.15 *As the pressure drop for each metre is given by* $\rho g = 11.9$*Pa/m, the height needed for a drop of 1hPa = 100Pa will be* $100/11.9 = 8.40\text{m}$. *As 1 foot = 0.305m, this corresponds to* $8.40/0.305 = 27.6$*ft. This is within 10% of the pilots' estimate of 30ft.*

Solution 2.16 *A change of pressure of 1hPa corresponds to 27.6 feet, as shown in the last question. Thus a change of 500 feet will cause a change of* $500/27.6 = 18.1$*hPa. If the pressure at sea level is 1024hPa, the pressure at 500 feet will be 18hPa lower, namely 1006hPa. The pilot should accordingly set their altimeter to 1006hPa to read the height above the airfield. If you solve the question using the pilots' estimate of 30 feet for a 1hPa change, then the answer comes out as* $1024 - 500/30 = 1027 - 17 = 1010$*hPa.*

Solution 2.17 *The full pressure of air falls by 11.9Pa for each metre you rise. This means that the partial pressure of oxygen will fall by 20% of this, namely 2.38Pa. The partial pressure at which you would begin to have trouble is* $20200 \times 2/3 = 13470$*Pa. This is a drop of* $20000 - 13470 = 6530$*Pa compared with the partial pressure at ground level. The height needed to cause this drop will be* $6530/2.38 = 2740$*m, which is equivalent to* $2740/0.305 = 9000$*feet. You would be very unwise to fly much higher than this in a non-pressurized aircraft without your own supply of oxygen. The rules governing flight specify that special precautions must be taken over 10 000 feet.*

Solution 2.18 *If the sea-level pressure is 980hPa, then under normal circumstances (if the ground didn't get in the way) we would expect standard pressure of 1013hPa to occur below sea level at a depth of* $(1013 - 980) \times 30 = 990$ *feet. The altimeter here is measuring the height of the aircraft above this point. So FL35, which is 3500 feet above the standard point will actually be* $3500 - 990 = 2510$ *feet above the ground.*

Solution 2.19 *They would be more wary of a QNH of 995hPa. As this is lower than standard pressure of 1013hPa, the height of the 1013hPa reference level will be below ground, and accordingly the actual altitude will be less than the flight level would suggest. A sea level pressure of 995hPa corresponds to a depth of the reference level of* $(1013 - 995) \times 30 = 540$ *feet. Thus an aircraft flying at FL35 would be flying at an altitude of* $3500 - 540 = 2960$ *feet, and would be unacceptably close to anything flying at an altitude of 3000 feet.*

Solution 2.20 *The left hand side is a force in newtons N. 1N is the same as 1 kgms*$^{-2}$*. The units of the right hand side are*

$$\begin{aligned} & \left(\mathrm{kgm^{-1}s^{-1}}\right)\left(\mathrm{ms^{-1}}\right)\mathrm{m} \\ = \ & \mathrm{kg^1\, m^1\, s^{-2}}, \end{aligned}$$

which matches, and therefore the equations agree. Note that the 6π *has no unit, and does not need to be included in the analysis.*

Solution 2.21 *Using equation 2.18 we calculate*

$$D = 6\pi\mu v r = 6\pi \times 1.85 \times 10^{-5} \times 1.1 \times 0.15 \times 10^{-3} = 5.75 \times 10^{-8}\mathrm{N}.$$

The weight of the drop is $W = mg = \rho V g = \frac{4}{3}\pi\rho g r^3 = 1.39 \times 10^{-7}$*N. To work out the weight of the drop, we substitute this weight for* D *in the Stokes Law equation, and solve for* v*:*

$$W = 6\pi\mu v r \Rightarrow v = \frac{W}{6\pi\mu r} = \frac{1.39 \times 10^{-7}}{6\pi \times 1.85 \times 10^{-5} \times 0.15 \times 10^{-3}} = 2.65\mathrm{m/s}.$$

Solution 2.22 *The flow rate (or discharge)* $Q = 8 \times 10^{-3}/30 = 2.667 \times 10^{-4}\mathrm{m^3/s}$*. Re-arranging the equation gives a value for* P'*:*

$$P' = \frac{8\mu Q}{\pi R^4} = \frac{8 \times 1.10 \times 10^{-3} \times 2.667 \times 10^{-4}}{\pi \times 0.01^4} = 74.7\mathrm{Pa/m}.$$

It follows that the pressure difference needed along an 8m pipe is $8 \times 74.7 =$ *598Pa. Unfortunately, as we shall find out in our next question, things are not as simple. To fill the bucket at this rate requires a speed of 0.85m/s in the hose (given by* $Q/A = Q/\pi r^2$*), which is above the speed at which turbulence will occur.*

Solution 2.23 *Re-arranging the equation for the Reynolds number to make the speed the subject gives*

$$v = \frac{\text{Re} \times \mu}{\rho D} = \frac{2300 \times 1.1 \times 10^{-3}}{1000 \times 0.02} = 0.127\text{m/s}.$$

A 10cm pipe has five times the diameter, so will have a speed five times less, namely 0.0253m/s.

Solution 2.24 *We calculate*

$$\text{Re} = \frac{\rho r v}{\mu} = \frac{1.2 \times 2.0 \times 50}{1.85 \times 10^{-5}} = 6.5 \times 10^{6}.$$

Given that this Reynolds number is much larger than 1, inertia drag will be far more significant than viscosity drag, and we can expect the drag to be proportional to the square of the speed. The viscosity is still important in flight, as it creates a boundary layer of air stationary relative to the aircraft, and determines the exact points where orderly flow separates from the surface of a wing.

Solution 2.25 *In order to make Re smaller without reducing the number of cars on the road ρ, we need to reduce speed v and increase the viscosity μ. Speed is reduced on 'smart motorways' by using variable speed limits, which are reduced at the onset of likely congestion. Cars will behave like a more viscous fluid if they do not change lane (unless essential) and do not accelerate or brake too rapidly. Instructions not to change lane are usually given to drivers in congestion, as is advice not to accelerate or brake severely.*

7.3 Things called Wings

Solution 3.1 *Using equation 3.3, we re-arrange to make C_L the subject:*

$$C_L = \frac{2L}{\rho A v^2} = \frac{2 \times 7500}{1.2 \times 15 \times 45^2} = 0.412.$$

Solution 3.2 *The new co-efficient of lift will be*

$$C_L = \frac{2L}{\rho A v^2} = \frac{2 \times 7500}{1.2 \times 15 \times 40^2} = 0.521.$$

We are allowed to assume that C_L is proportional to α, the angle of attack, and therefore C_L/α will be the same for both situations. Thus

$$\frac{0.412}{4} = \frac{0.521}{\alpha} \Rightarrow \alpha = \frac{0.521 \times 4}{0.412} = 5.06^\circ.$$

Solution 3.3 *For full reflection, the air molecule approaches the wing at angle α, and will bounce off at the same angle. Thus the angle of deflection is 2α, and the velocity change is given by a modified form of figure 3.3 where the angles are twice as large. The velocity change is thus $\Delta v = 2v \sin \alpha$, and the momentum change per molecule will be $\Delta mv = 2mv \sin \alpha$. The mass hitting the wing each second is still $\rho Av \sin \alpha$ as shown on page 38, and so the number of molecules hitting will be the mass divided by the mass of one molecule: $\rho Av \sin \alpha / m$. The force is the momentum change per second, and is these two expressions multiplied together. This gives $L = \rho Av \sin \alpha \times 2v \sin \alpha = 2\rho Av^2 \sin^2 \alpha$.*

Solution 3.4 *As the aeroplane is in level flight, the lift must equal the weight $= 750\text{kg} \times 9.8\text{N/kg} = 7360\text{N}$. Given that $L = \rho AC_L v^2/2$, it follows that (for 90kt = $90 \times 1852/3600$ = 46.3m/s)*

$$C_L = \frac{2L}{\rho Av^2} = \frac{2 \times 7360}{1.2 \times 14.8 \times 46.3^2} = 0.39.$$

A similar calculation for 50kt = 25.7m/s gives $C_L = 1.26$. The lift co-efficient C_L is related to the angle of attack by $C_L = k_L\ (\alpha - \alpha_0)$. It follows that (for 90kt)

$$\alpha = \alpha_0 + \frac{C_L}{k_L} = -2.1 + \frac{0.39}{0.072} = 3.3^\circ,$$

while for 50kt a similar calculation gives $\alpha = 15^\circ$. The wing on this particular aircraft is mounted so that when the aircraft is at its cruising speed of 90kt, the fuselage is horizontal. As you will see, this means that the wing has to be set at about 3° to the fuselage (so a horizontal fuselage gives the wing the angle of attack it needs for normal flight). Flying at the slower speed of 50kt would require the nose to be pitched up about 12° above the horizontal.

Solution 3.5 *We shall use completing the square to solve this question, and calculus the next. Either approach is possible in this question - however the calculus method is more straightforward. So, if you do know how to differentiate, please use calculus. We start by writing $D = a/v^2 + bv^2$ as a quadratic* *in $V = v^2$. This gives $D = a/V + bV$ and so $DV = a + bV^2$, and thus $V^2 - DV/b = -a/b$. Completing the square gives $(V - D/2b)^2 = -a/b + D^2/4b^2$. The term in brackets, once squared, has a minimum value of zero (if something is squared, it can not be negative). Looking at the right hand side, it must follow that the smallest value of D we can have is when $D^2/4b^2 = a/b$ and so $D^2 = 4ab$. The term in brackets will be zero when $V = D/2b$, so $D = 2bV$ and $D^2 = 4b^2V^2$. Putting our two equations for D^2 together gives $4ab = 4b^2V^2$, so $V^2 = a/b$. Thus $v = \sqrt{V} = \sqrt[4]{a/b}$.*

Solution 3.6 *Given that $D = a/v^2 + bv^2$, the power will be $P = Dv = a/v + bv^3$. It follows that $dP/dv = -a/v^2 + 3bv^2$. When P is a minimum, $dP/dv = 0$, and so $a/v^2 = 3bv^2$ and $v^4 = a/3b$.*

Solution 3.7 *Given that we are assuming that fuel consumption will be proportional to $P = Fv = a/v + bv^3$, let us write Consumption $= A/v + Bv^3$. It follows that we have two simultaneous equations:*

$$\begin{aligned} 23.8 &= \frac{A}{102} + B \times 102^3 \\ 16.2 &= \frac{A}{81} + B \times 81^3. \end{aligned}$$

If we multiply all terms in the first equation by 102, multiply all terms in the second by 81, and then subtract our two new equations, we get

$$23.8 \times 102 - 16.2 \times 81 = B\left(102^4 - 81^4\right).$$

It follows that $B = 1.711 \times 10^{-5}$. Using the first equation, we can substitute this value for B to get a value for $A = 102 \times 23.8 - B \times 102^4 = 575.7$. We know from the solution to question 3.4 that the least drag (greatest range) is when $v = \sqrt[4]{a/b} = \sqrt[4]{A/B} = 76.1$kt. The greatest endurance, as shown in the solution to question 3.5 is when $v = \sqrt[4]{a/3b} = \sqrt[4]{A/3B} = 57.9$kt.

Solution 3.8 $R = 9h = 9 \times 1000 \times 0.305 = 2745\text{m}$. *This is equal to* $2745/1852 = 1.48\text{nm}$.

Solution 3.9 *Using the answers to question 3.7, the gliding time is 1.48nm / 65kt, which is 0.0228hr. The groundspeed with the wind will be $65 - 10 = 55\text{kt}$, so the new distance travelled will be* $55\text{kt} \times 0.0228\text{hr} = 1.25\text{nm}$.

Solution 3.10 *For rate 1, the aircraft will turn through* $180^\circ = \pi\mathrm{rad}$ *in 60s, so* $\omega = \pi/60 = 0.0524\mathrm{rad/s}$. *Therefore, using equation 3.24,* $a = \omega v = 0.105 \times 200 = 20.9\mathrm{m/s^2}$.

Solution 3.11 *Here* $\omega = \pi/5 = 0.628\mathrm{rad/s}$*, so* $a = \omega v = 0.628 \times 60 = 37.7\mathrm{m/s^2}$.

Solution 3.12 *For rate 1, the aircraft will turn through* $180^\circ = \pi\,\mathrm{rad}$ *in 60s, so* $\omega = \pi/60 = 0.0524\mathrm{rad/s}$. *We next convert the speed* $90\mathrm{kt} = 90 \times 1852\mathrm{m}/3600\mathrm{s} = 46.3\mathrm{m/s}$. *Using equation 3.28,* $\tan\beta = \omega v/g = 0.0524 \times 46.3/9.81 = 0.247$. *Thus the bank angle is* $\tan^{-1} 0.247 = 13.9^\circ$.

Solution 3.13 *As in the last question, 90kt is the same as 46.3m/s. Using equation 3.28 it follows that* $\omega = g\tan\beta/v = 9.81 \times \tan 30^\circ/46.3 = 0.122\,\mathrm{rad/s}$. *A turn of* 90° *is equivalent to* $\pi/2\,\mathrm{rad}$*, and so the time taken for the turn will be given by* $\frac{1}{2}\pi/\omega = 0.25\pi/0.122 = 12.8\mathrm{s}$.

Solution 3.14 *The easiest is to use figure 3.19 on page 64.* L *is the hypotenuse, and* mg *is the side adjacent to the angle. Thus* $\cos\beta = mg/L$*, so* $L/mg = 1/\cos\beta$.

Solution 3.15 *If* $L/mg = 2$ *then, using the relationship from the last question,* $1/\cos\beta = 2$*, so* $\cos\beta = 0.5$. *It follows that* $\beta = 60^\circ$.

Solution 3.16 *In the solution to question 3.4, we see that 90kt is the same as 46.3m/s, and an angle of attack of* 3.3° *was calculated for straight-and-level flight. Using the result from question 3.14, we can work out the lift needed in the* 30° *turn:*

$$L = \frac{mg}{\cos\beta} = \frac{750 \times 9.81}{\cos(30^\circ)} = 8500\mathrm{N}$$

We now use equation 3.12 to work out the angle of attack:

$$\alpha = \alpha_0 + \frac{2L}{k_L \rho A v^2} = -2.1 + \frac{2 \times 8500}{0.072 \times 1.2 \times 14.8 \times 46.3^2} = -2.1 + 6.2 = 4.1^\circ.$$

Repeating this calculation for 45° *and* 60° *banked turns gives lift values of 10410N and 14720N respectively. This leads to angles* $(\alpha - \alpha_0)$ *of* 7.60° *and* 10.74°*, and hence angles of attack* α *of* 5.5° *and* 8.6°. *Thus when entering the turn from straight and level flight, the pilot would need to raise the nose of the aircraft by about* 1° *for the* 30° *bank, by about* 2° *for the* 45° *bank and about* 5° *for the* 60° *banked turn.*

Solution 3.17 *Using equation 3.30,* $L/mg = (v/v_s)^2 = (45/25)^2 = 3.24$. *Using our answer to question to 3.13, it follows that* $1/\cos\beta = 3.24$, *so* $\cos\beta = 0.309$ *and accordingly,* $\beta = 72°$.

Solution 3.18 *Using the recursive method as detailed in the text, we start with equation 3.21* $\tan\delta = 1/9$ *which gives* $\delta = 6.34°$. *Putting this into equation 3.36 with* $\omega = \pi/60 = 0.0524\,\mathrm{rad/s}$ *and* $v = 40\mathrm{m/s}$ *gives a value of* $\beta = 12.1°$, *which if put into equation 3.37 gives* $\delta = 6.48°$, *and we can repeat the cycle until the answers converge.*

Alternatively, we can solve algebraically. To simplify the algebra, we shall write $L/D = \lambda$, *and* $g/\omega v = \Gamma$. *Given that* $1/\cos^2\delta = \tan^2\delta + 1$, *it follows that equation 3.36 can be written*

$$
\begin{aligned}
\tan^2\beta &= \frac{\omega^2 v^2}{g^2}\left(\tan^2\delta + 1\right) \\
\Gamma^2 \tan^2\beta &= \tan^2\delta + 1 = \frac{1}{\lambda^2\cos^2\beta} + 1 \\
&= \frac{\tan^2\beta + 1}{\lambda^2} + 1 \\
\left(\Gamma^2 - \frac{1}{\lambda^2}\right)\tan^2\beta &= \frac{1}{\lambda^2} + 1 \\
\tan^2\beta &= \frac{1+\lambda^2}{\lambda^2\Gamma^2 - 1},
\end{aligned}
$$

where we used equation 3.37 on the second line. From this, with $\omega = \pi/60\,\mathrm{rad/s}$, $v = 40\mathrm{m/s}$ *and* $\lambda = 9$, *we have* $\Gamma = 4.684$, *so* $\tan^2\beta = 0.04616$, *so* $\beta = 12.1°$, *which means that* $\tan\delta = 1/(\lambda\cos\beta) = 0.114$, *so* $\delta = 6.48°$.

Solution 3.19 *A method of solution for this question was provided in the question itself.*

7.4 Up, Up and Away

Solution 4.1 *The fractions shown means that one mole of air will contain 0.79 moles of nitrogen and 0.21 moles of oxygen. The mass of this air will be* $0.79 \times 0.0280 + 0.21 \times 0.0320$ *which gives 0.0288kg/mol. When people quote the fractions of different gases in the atmosphere, the fractions are given by volume, which matches (by the Ideal Gas Law) with the fractions per mole. The* *fractions are not done by mass, so 21% of the mass of a cubic metre of air is not the mass of the oxygen in it.*

Solution 4.2 *Using the original assumption of a drop of 364Pa = 3.64hPa for each 100 feet, we have a pressure at 1000ft of* $1013 - 10 \times 3.64 = 977$*hPa, for 5000ft we have 831hPa, for 10 000ft we have 649hPa and for 40 000ft we have* -443*hPa, and answer which is plainly ridiculous as pressures can not be negative.*

Now we use equation 4.10, where we set $p_0 = 1013$*hPa. We calculate*

$$\frac{M_r g}{RT} = \frac{0.0288 \times 9.81}{8.31 \times 288} = 1.18 \times 10^{-4},$$

and convert the heights to metres. A height of 100ft is equivalent to $100 \times 0.305 = 30.5$*m, so the heights we need are 305m, 1525m, 3050m and 12 200m. Putting these into the equation, where we give the first as a fully worked solution, we have*

$$p = p_0 \exp\left(-\frac{M_r g h}{RT}\right) = 1013 \times \exp\left(-1.18 \times 10^{-4} \times 305\right) = 977\text{hPa}.$$

The other pressures come out as 846hPa, 707hPa and 240hPa. We see that there is almost perfect agreement up to 1000 feet, but then the differences get larger.

Solution 4.3 *The question tells us that if* $y = A\exp(x)$*, then* $x = \ln(y/A)$*. If we want the pressure to halve, then* $p = p_0/2$*. It follows that*

$$\frac{p_0}{2} = p_0 \exp\left(-\frac{M_r g h}{RT}\right),$$

and so

$$-\frac{M_r g h}{RT} = \ln\left(\frac{p_0/2}{p_0}\right) = \ln 0.5.$$

Re-arranging to make h *the subject gives*

$$h = -\frac{RT}{M_r g} \times \ln 0.5 = -\frac{1}{1.18 \times 10^{-4}} \times \ln 0.5,$$

which evaluates to 5870m. This is the same as 5870/0.305 = 19 300 feet.

Solution 4.4 *Given that we have replaced the* $R/(C_V + R)$ *in equation 4.17 with* $2/7$*, it also follows that we can do the same in equation 4.20 which then matches with equation 4.21 exactly.*

Solution 4.5 *In this case,* $C_V + R = 5R/2$*, and so* $R/(C_V + R)$ *becomes* $2/5$*. Thus equation 4.21 would give the DALR as* $2M_r g/5R$*.*

Solution 4.6 *We reason as follows, where we remember that by the Ideal Gas Law,* $T \propto pV$*:*

$$\begin{array}{rcl} T & \propto & p^{R/(C_V+R)} \\ T^{C_V+R} & \propto & p^R \\ (pV)^{C_V+R} & \propto & p^R \\ V^{C_V+R} & \propto & p^{R-(C_V+R)} \\ V^{C_V+R} & \propto & p^{-C_V} \\ V^{(C_V+R)/C_V} & \propto & p^{-1} \\ V^\gamma & \propto & p^{-1}, \end{array}$$

which means that pV^γ *will be a constant.*

Solution 4.7 *Most of the hints are already in the question. We begin as follows, remembering that we are working with one mole here* $n = 1$*, so* $pV = RT$*.*

$$\begin{array}{rcl} \delta Q & = & \delta U + p\delta V \\ & = & C_V\delta T + p\delta V \\ & = & C_V\delta T + p\delta V + V\delta p - V\delta p \\ & = & C_V\delta T + \delta(pV) - V\delta p \\ & = & C_V\delta T + \delta(RT) - V\delta p \\ & = & (C_V + R)\,\delta T - V\delta p \end{array}$$

Thus, if we heat the gas without changing the pressure, so $\delta p = 0$*, we have* $\delta Q = (C_V + R)\,\delta T$*. This means that the heat needed to increase* T *by one is* $C_V + R$*, which is why this is known as the heat capacity at constant pressure and given the symbol* C_P*. So* $C_P = C_V + R$*.*

Solution 4.8 *We start by working out the pressure of the air at the top of the mountain using equation 4.10.*

$$\begin{aligned}
p &= p_0 \exp\left(-\frac{M_r g h}{RT}\right) \\
&= 1.013 \times 10^5\mathrm{Pa} \times \exp\left(-\frac{0.0288 \times 9.81 \times 3000}{8.31 \times 288}\right) \\
&= 7.101 \times 10^4\mathrm{Pa}
\end{aligned}$$

We now use equation 4.38 setting the saturation pressure $p_{w,\mathrm{sat}}$ *to be* $7.101 \times 10^4\mathrm{Pa}$*, given that for the water to boil at the top of the mountain, we raise the temperature* T *until the saturation pressure equals the local atmospheric pressure. We now solve the equation for* T*, remembering that the boiling temperature of water, in kelvins, is* $100 + 273 = 373$*:*

$$\begin{aligned}
\frac{p_{w,\mathrm{sat}}}{p_0} &= \exp\left(\frac{L}{RT_b} - \frac{L}{RT}\right) \\
\ln\left(\frac{p_{w,\mathrm{sat}}}{p_0}\right) &= \frac{L}{RT_b} - \frac{L}{RT} \\
\frac{L}{RT} &= \frac{L}{RT_b} - \ln\left(\frac{p_{w,\mathrm{sat}}}{p_0}\right) \\
\frac{1}{T} &= \frac{1}{T_b} - \frac{R}{L}\ln\left(\frac{p_{w,\mathrm{sat}}}{p_0}\right) \\
&= \frac{1}{373} - \frac{8.31}{44000}\ln\left(\frac{7.101 \times 10^4}{1.013 \times 10^5}\right) \\
&= 2.748 \times 10^{-3} \\
T &= \left(2.748 \times 10^{-3}\right)^{-1} = 363.9\mathrm{K} = 90.9^\circ\mathrm{C}
\end{aligned}$$

Solution 4.9 *To solve this, we put the appropriate values into equation 4.39:*

$$\begin{aligned}
\mathrm{R.H.} &= \exp\left(\frac{L}{RT} - \frac{L}{RT_d}\right) \\
&= \exp\left\{\frac{L}{R}\left(\frac{1}{T} - \frac{1}{T_d}\right)\right\} \\
&= \exp\left\{\frac{44000}{8.31}\left(\frac{1}{283} - \frac{1}{280}\right)\right\} \\
&= \exp\left(-0.2005\right) = 0.818 = 81.8\%
\end{aligned}$$

Solution 4.10 *We follow the stages given in the question. We start by using equation 4.38 to work out the saturated vapour pressure*

$$\begin{aligned} p_{w,\text{sat}} &= p_0 \exp\left(\frac{L}{RT_b} - \frac{L}{RT}\right) \\ &= 1.013 \times 10^5 \text{Pa} \times \exp\left(\frac{44000}{8.31 \times 373} - \frac{44000}{8.31 \times 288}\right) \\ &= 1535 \text{Pa}. \end{aligned}$$

We next use this to calculate:

$$\frac{n_w}{V} = \frac{p_{w,\text{sat}}}{RT} = \frac{1535}{8.31 \times 288} = 0.6414.$$

For normal air, the number of moles per cubic metre is

$$\frac{n}{V} = \frac{p_0}{RT} = \frac{1.013 \times 10^5}{8.31 \times 288} = 42.33,$$

so the number of moles of water vapour per mole of air will be $0.6414/42.33$ *which is 0.0152, so slightly more than one in a hundred of the molecules are water.*

7.5 Navier-Stokes

Solution 5.1 *As we are told that there are no vortices, it follows by equation 5.13 that*

$$\frac{\partial v_y}{\partial z} = \frac{\partial v_z}{\partial y}.$$

Let us look at the x *component of* $(\mathbf{v} \cdot \nabla)\, \mathbf{v}$*. It is*

$$v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} + v_z \frac{\partial v_x}{\partial z},$$

which will therefore be equal to

$$v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_y}{\partial x} + v_z \frac{\partial v_z}{\partial x},$$

which in turn is equal to

$$\frac{1}{2} \frac{\partial}{\partial x} \left(v_x^2 + v_y^2 + y_z^2\right),$$

and this is the x component of

$$\frac{1}{2}\nabla\left(v_x^2+v_y^2+y_z^2\right)=\frac{\nabla v^2}{2}.$$

It follows that we can write our Navier-Stokes equation as

$$\frac{\rho}{2}\nabla v^2+\nabla p+\nabla\left(\rho g z\right)=0,$$

and so

$$\nabla\left(\frac{\rho v^2}{2}+p+\rho g z\right)=0.$$

Solution 5.2 *Given the changes stated, note that a time $t = L/v$ will become $t \to aL/wv = (a/w)t$. Making these replacements in the Navier-Stokes equation gives*

$$r\rho\left(\frac{w\,\partial}{a\,\partial t}+w\mathbf{v}\cdot\frac{\nabla}{a}\right)w\mathbf{v}+q\frac{\nabla}{a}p-m\mu\frac{\nabla^2}{a^2}w\mathbf{v}=r\rho\frac{a}{a^2/w^2}\mathbf{g},$$

where we have noted that ∇ is a derivative with respect to a distance, so scales as $1/L$, and that $\mathbf{g}$ is an acceleration, and accordingly scales as a distance divided by the square of a time. Collecting the terms gives

$$\frac{rw^2}{a}\rho\left(\frac{\partial}{\partial t}+\mathbf{v}\cdot\nabla\right)\mathbf{v}+\frac{q}{a}\nabla p-\frac{mw}{a^2}\mu\nabla^2\mathbf{v}=\frac{rw^2}{a}\rho\mathbf{g},$$

and hence

$$\rho\left(\frac{\partial}{\partial t}+\mathbf{v}\cdot\nabla\right)\mathbf{v}+\frac{q}{rw^2}\nabla p-\frac{m}{rwa}\mu\nabla^2\mathbf{v}=\rho\mathbf{g}.$$

This reduces to the original equation if $q = rw^2$ and $m = rwa$ as required.

7.6 Mathematical Lift

Solution 6.1 *Let us use the notation of page 50 where $\alpha' = \alpha - \alpha_0$. Furthermore we will use α_∞ to refer to the angle of attack relative to the aerofoil section and $\alpha_F = \alpha_\infty + \alpha_w$ to refer to the angle of attack of the finite wing of aspect ratio R. Notice that when $\alpha = \alpha_0$, there is no lift, $C_L = 0$, and accordingly*

$\alpha_w = 0$, so $\alpha_{0,\infty} = \alpha_{0,F}$. It follows that $\alpha'_F = \alpha'_\infty + \alpha_w$. Now $\alpha' = C_L/k_L$. Thus

$$\begin{aligned}
\alpha'_F &= \alpha'_\infty + \alpha_w \\
\frac{C_L}{k_{L,F}} &= \frac{C_L}{k_{L,\infty}} + \frac{180\,C_L}{\pi^2 R} \\
\frac{1}{k_{L,F}} &= \frac{1}{k_{L,\infty}} + \frac{180}{\pi^2 R},
\end{aligned} \tag{7.1}$$

where we use equation 6.29 to give the value for α_w, but multiply it by $180/\pi$ to convert it to degrees. We now substitute $R = 6.0$ and $k_{L,\infty} = 0.11$ into our equation to give $k_{L,F} = 0.082/^\circ$. Notice that the finite span of the wing has reduced the lift available for a given angle of attack, as we might expect.

Solution 6.2 *We use equation 6.32 to evaluate the drag co-efficient due to induced drag $C_{Di} = C_L^2/\pi R$. The parasitic drag is given in the question as $C_{Dp} = 0.004C_L$. This means that for the finite wing we have a total drag co-efficient of*

$$\begin{aligned}
C_D &= C_{Dp} + C_{Di} \\
&= 0.004C_L + \frac{C_L^2}{\pi R}.
\end{aligned}$$

For the case when $C_L = 0.6$ and $R = 6$, we have $C_{Dp} = 0.004C_L = 0.0024$ and $C_{Di} = C_L^2/\pi R = 0.019$, so the total drag co-efficient is $C_D = 0.021$.

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