Question

The Newton-Raphson Method

A-level Maths Topic Summaries - Numerical Methods

Subject & topics: Maths | Number **Stage & difficulty:** A Level P3

Fill in the boxes to complete the notes on the Newton-Raphson method.

Part A

How the method works

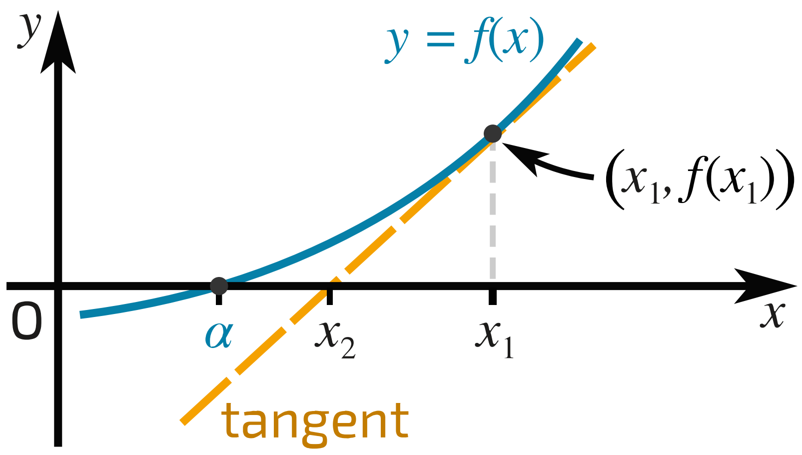

Figure 1: Illustrating the Newton-Raphson method.

The **Newton-Raphson method** is a numerical method for finding the [ ] of equations.

Figure 1 illustrates how the method works. A curve $y = f(x)$ has a root α. We start with an initial approximation for the root, x_1. The place where the tangent to the curve at x_1 cuts the x-axis is closer to the root than x_1. Therefore, this location is a better approximation to the root, and hence is our next approximation, x_2.

The gradient of the tangent is [ ], and the tangent meets the curve at [ ]. Hence, the gradient of the tangent is related to the distance $x_1 - x_2$ by

$$f'(x_1) = \frac{f(x_1)}{\boxed{}}$$

Rearranging to get an expression for x_2, we get

$$x_2 = \boxed{}$$

We can then repeat the procedure to find even better approximations. The formula we use is

$$x_{r+1} = \boxed{}$$

Items:

$x_1 - x_2$ | $(x_1, f(x_1))$ | $f'(x_1)$ | $x_1 - \frac{f(x_1)}{f'(x_1)}$ | $x_r - \frac{f(x_r)}{f'(x_r)}$ | roots

Part B

Limitations of the method

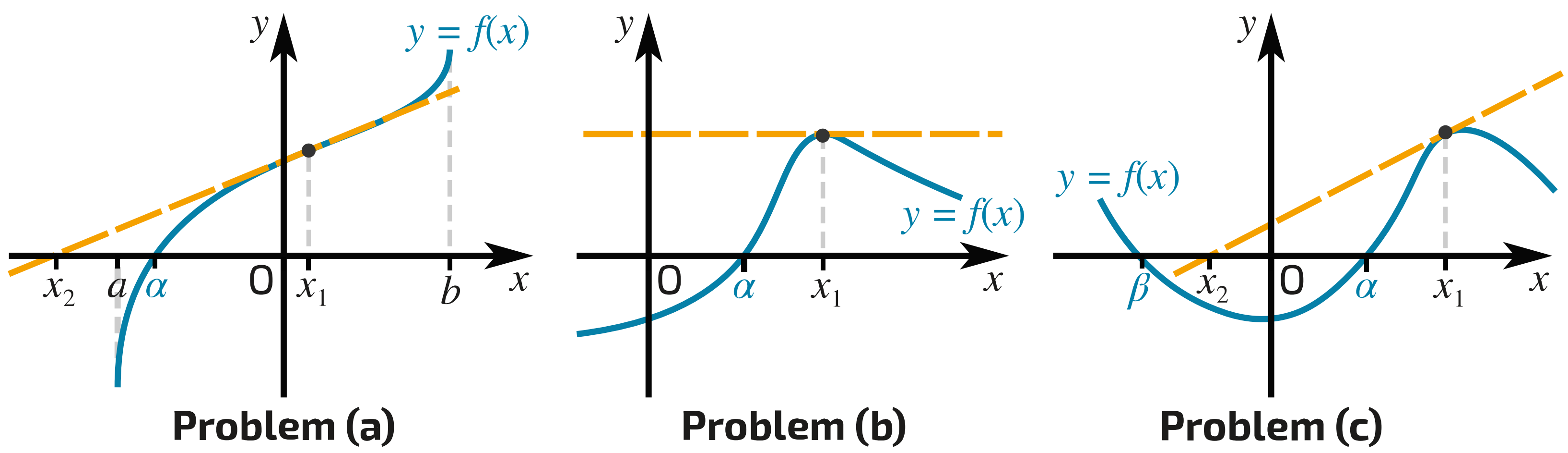

Figure 2: Three problems that can occur.

Figure 2 illustrates three problems that can occur when using the Newton-Raphson method.

Problem (a): The diagram shows a function $y = f(x)$ that is defined for $a \leq x \leq b$ and has a root α. The tangent to the function at x_1 intercepts the x-axis at x_2, which is _____ the interval $a \leq x \leq b$. Hence, we _____ the function or its gradient at x_2, and we cannot find further approximations to α.

Problem (b): The diagram shows a function $y = f(x)$ with a root α. The initial approximation to the root, x_1, is at a _____ of the function. The gradient of the tangent is _____. Hence, the tangent _____ the x-axis, and so we never get a second approximation x_2.

Problem (c): The diagram shows a function $y = f(x)$ with two roots α and β. x_1, an initial approximation to the root α, is _____ a turning point of the function. The gradient is _____ close to a turning point, so the location where the tangent crosses the x-axis is a long way from x_1. For this function it is closer to β than α. Repeated application of the Newton-Raphson method will converge to _____, not α.

Items:

0 | β | cannot evaluate | close to | never crosses | outside | small | turning point

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Fixed Point Iteration

A-level Maths Topic Summaries - Numerical Methods

Subject & topics: Maths | Number **Stage & difficulty:** A Level P3

Fill in the boxes to complete the notes on fixed point iteration.

Part A

How the method works

Fixed point iteration is a numerical method for finding a [ ] of an equation. We will illustrate the method by finding a root of the equation $x^3 - 3x - 1 = 0$ close to -1. First we rearrange the equation into the form [ ]. One rearrangement is $x = \sqrt[3]{1+3x}$. We use this to create a sequence,

$$\square = \sqrt[3]{1+3\square}$$

Next, we choose a starting value. We will use [ ] $= -1$. We use the formula for the sequence to find $x_2, x_3, x_4, \ldots$ and see if the sequence converges. If the sequence converges, the value we put in on the right hand side of the sequence formula is the same as the value we get out on the left hand side, and we have a solution to our original equation. For $x_1 = -1$, the sequence above converges to the solution [ ] to 3 sf.

An equation can often be rearranged into the form $x = \ldots$ in [ ] way. Alternative rearrangements may be useful for finding [ ] roots. For example, the equation above can also be rearranged into $x = \frac{x^3-1}{3}$, from which we get the sequence

$$x_{r+1} = \frac{x_r^3+1}{3}$$

For $x_1 = -1$, this sequence converges to the solution [ ] to 3 sf.

Items:

$x = \ldots$ | x_r | x_{r+1} | x_1 | $x = -1.53$ | $x = -0.347$ | different | more than one | root

Part B

Convergence and divergence

In fixed point iteration, we start with an equation which we are trying to solve, $f(x) = 0$. We rearrange this equation so that there is an x on its own on the left hand side, and create an iterative sequence which has the form $x_{r+1} = g(x_r)$.

Graphically, we are looking for where the lines $y = x$ and $y = g(x)$ meet. The value of x where this occurs is a root of the original equation, α. **Figure 1** illustrates the types of convergence and divergence patterns that can occur near the root.

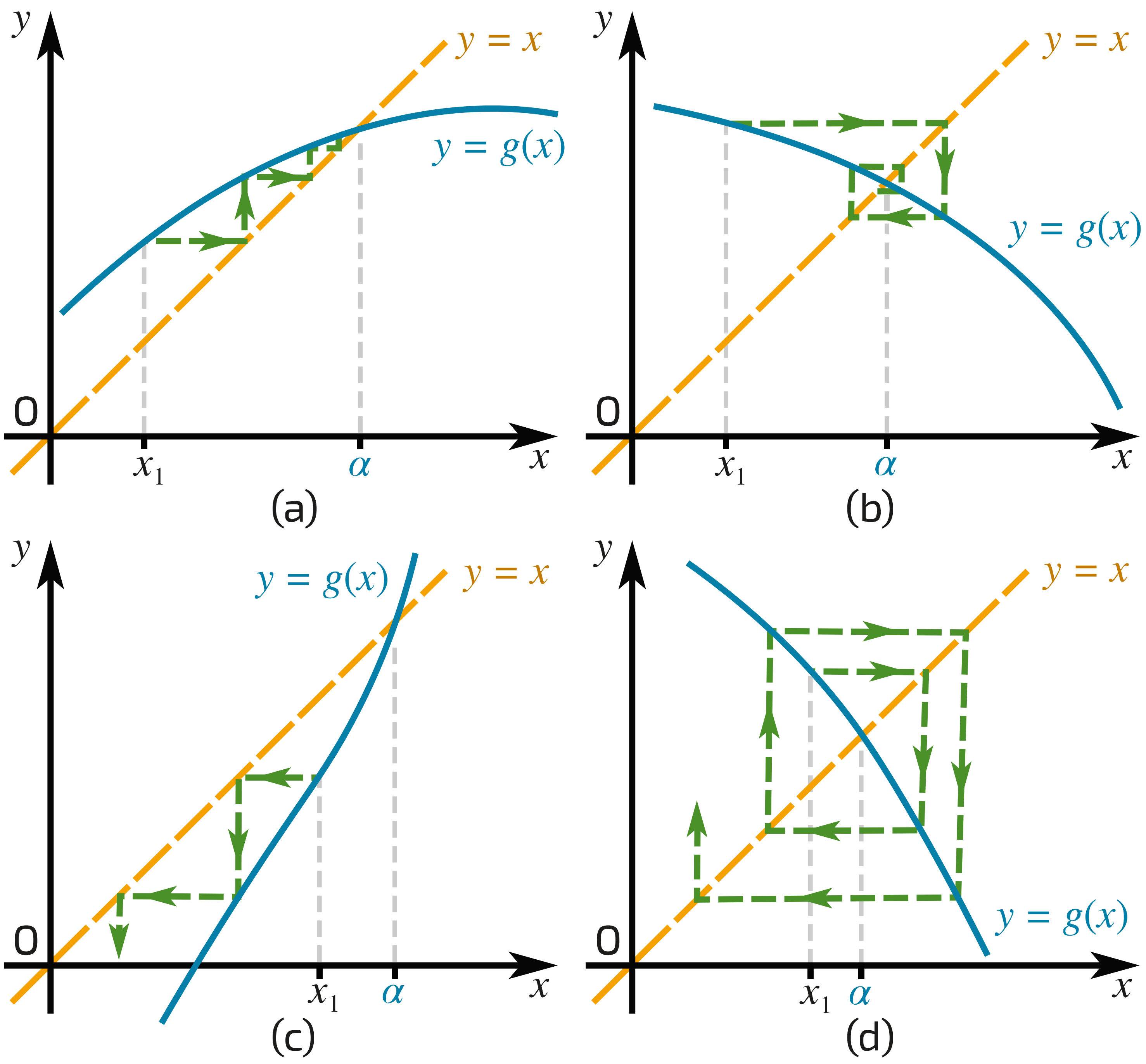

Figure 1: Illustrating convergence to, or divergence from, a root. (a) and (b) convergence, (c) and (d) divergence.

Whether an iterative sequence converges or diverges near a root depends on the ______ of the ______ of $g(x)$ ______ the root. There are four cases. These are illustrated in **Figure 1**.

- If ______ near the root, then if the starting value is sufficiently close to the root the sequence will ______ to the root. (a) illustrates convergence for a positive gradient, and (b) illustrates convergence for a negative gradient.
- If ______ near the root, the sequence will ______ away from the root. (c) illustrates divergence for a positive gradient, and (d) illustrates divergence for a negative gradient.

(a) and (c) are ______ diagrams, and (b) and (d) are ______ diagrams.

Items:

$|g'(x)| < 1$ $|g'(x)| > 1$ cobweb converge diverge gradient magnitude near staircase

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The Trapezium Rule

A-level Maths Topic Summaries - Numerical Methods

Subject & topics: Maths | Number **Stage & difficulty:** A Level P3

Fill in the boxes to complete the notes on the trapezium rule.

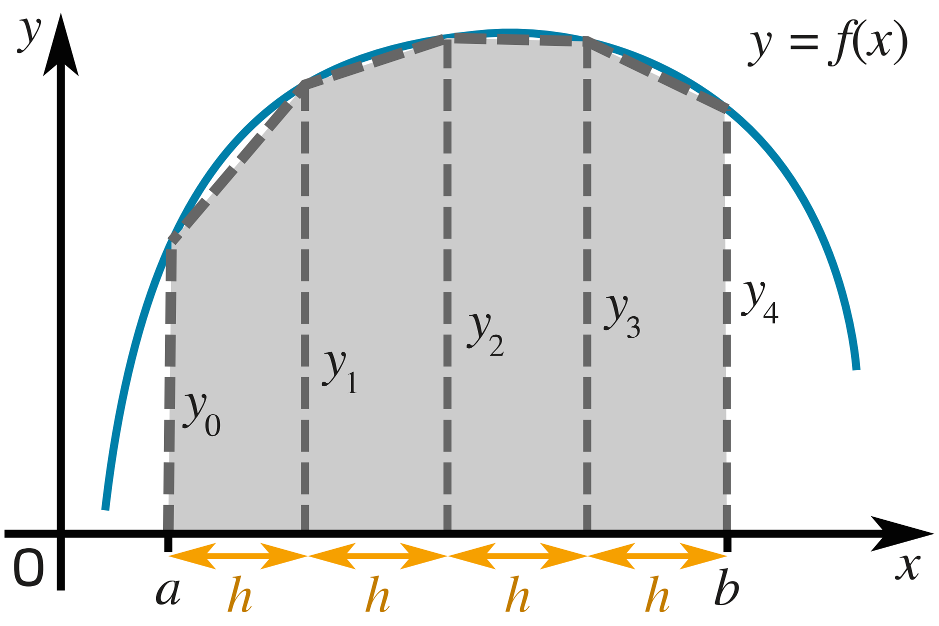

Figure 1: Estimating an area using four trapezia.

The **trapezium rule** is a numerical method for estimating the ______ under a curve.

Figure 1 illustrates how the method works. The area we wish to estimate is the area bounded by the curve, the x-axis, and the lines $x = a$ and $x = b$. We model the area under the curve using trapezia of equal width. If we use n trapezia, the width of one trapezium h is

$$h = \frac{\boxed{}}{n}$$

The area under the curve is given by the trapezium rule formula,

$$\text{Area} \approx \frac{h}{2}\left(\boxed{} + 2(y_1 + y_2 + ... + y_{n-2} + \boxed{})\right)$$

In this formula there is a 2 inside the brackets as y_1 to y_{n-1} are all side lengths for ______ trapezia, while y_0 and y_n are each only side lengths for the trapezia at ______ end.

To get a better approximation, we use a ______ number of trapezia. We get a better approximation as the tops of the trapezia are a ______ to the curve.

Items:

area | $b-a$ | better fit | larger | one | two | $y_0 + y_n$ | y_{n-1}

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Newton-Raphson Method 3ii

Subject & topics: Maths **Stage & difficulty:** A Level P3

It is given that $f(x) = x^2 - \arctan x$ and that $\frac{d}{dx}(\arctan x) = \frac{1}{1+x^2}$.

Part A
Interval containing the root

Explain why the equation $f(x) = 0$ has a root in the interval $0.8 < x < 0.9$.

The value of $f(x)$ when $x = 0.8$ is ____, and the value of $f(x)$ when $x = 0.9$ is ____. These values of $f(x)$ have ____. Hence, as $f(x)$ is a continuous function, there is a value of x in the interval $0.8 < x < 0.9$ for which $f(x) = 0$.

A root of an equation is a value of x for which $f(x) =$ ____. Hence, there is a root of $f(x)$ in the interval $0.8 < x < 0.9$.

Items:

1 | 0.0772 | 1.348 | 0 | the same sign | 0.0771 | -0.0347 | different signs

Part B
Find the root

Use the Newton-Raphson method, with a first approximation 0.8 , to find the next approximation to this root. Give your answer correct to 3 significant figures.

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Newton-Raphson Method 1ii

Subject & topics: Maths **Stage & difficulty:** A Level P3

The diagram shows the curve with equation $y = xe^{-x} + 1$. The curve crosses the x-axis at $x = \alpha$.

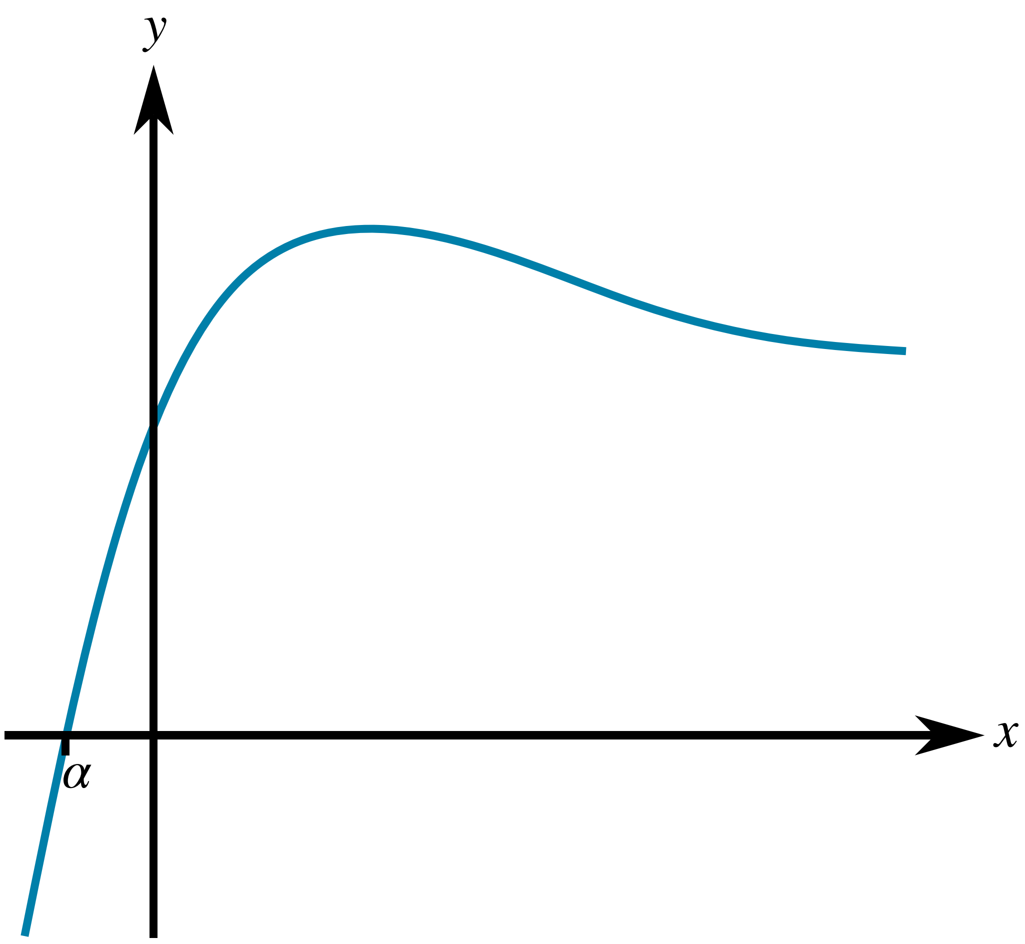

Figure 1: A sketch of the curve $y = xe^{-x} + 1$.

Part A

x-coordinate of stationary point

Use differentiation to calculate the x-coordinate of the stationary point.

The following symbols may be useful: x

Part B

Explain

α is to be found using the Newton-Raphson method, with $f(x) = xe^{-x} + 1$.

Explain why this method will not converge to α if an initial approximation x_1 is chosen such that $x_1 > 1$.

The iterative formula for the Newton-Raphson method is $x_{n+1} = x_n - \frac{f(x)}{f'(x)}$. For all values of x greater than 1, $f(x)$ is positive, and the [ ] of $f(x)$ is negative (and close to [ ]). Hence, $-\frac{f(x)}{f'(x)}$ is positive and so x_{n+1} is larger than x_n. Visually, the x-intercepts of [ ] at successive approximations will reach progressively [ ] x-values and, hence, move further away from α.

Items:

tangents | value | intercept | 0 | 1 | -1 | normals | gradient | larger | smaller

Part C

Values

α is to be found using the Newton-Raphson method, with $f(x) = xe^{-x} + 1$.

Use this method, with a first approximation $x_1 = 0$, to find the next three approximations x_2, x_3, x_4. Give your answers to 4 sf where necessary.

$x_2 =$ [ ]

$x_3 =$ [ ]

$x_4 =$ [ ]

Find α correct to 3 significant figures.

$\alpha =$ [ ]

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Roots and Iteration 3i

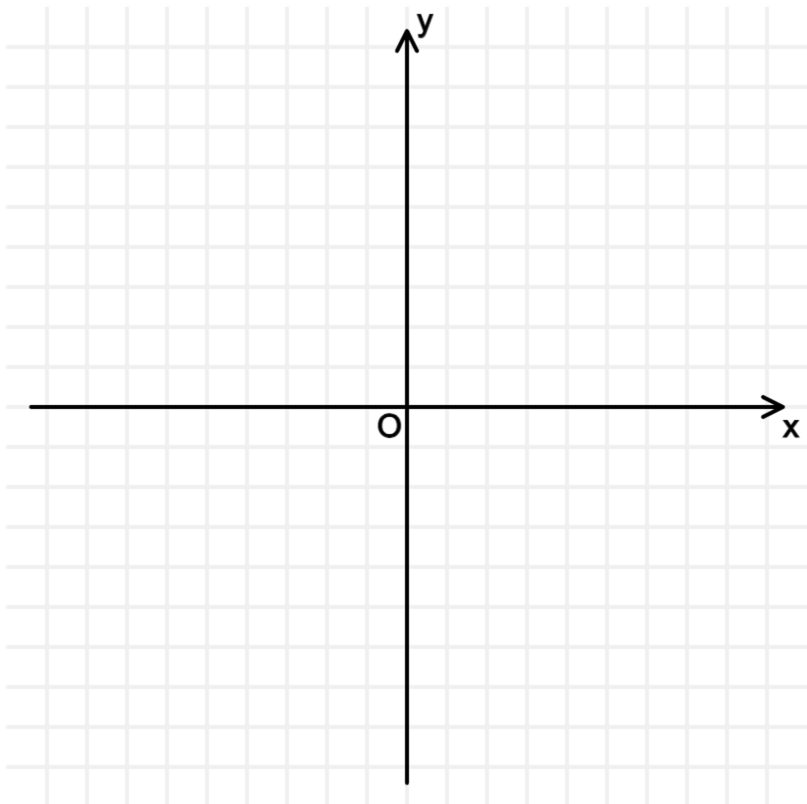
Subject & topics: Maths **Stage & difficulty:** A Level P2

Part A

Sketch

By sketching two suitable graphs on a single diagram, find the number of roots to the equation

$$14 - x^2 = 3\ln x.$$



From your sketch, state how many roots there are to the equation

$$14 - x^2 = 3\ln x$$

Part B

Integer below α

Find by calculation the largest integer which is less than the root α.

Part C

Iteration

Use the iterative formula $x_{n+1} = \sqrt{14 - 3\ln x_n}$, with a suitable starting value to find α correct to 3 significant figures.

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Question

Roots and Iteration 1i

Subject & topics: Maths **Stage & difficulty:** A Level P2

It is required to solve the equation $f(x) = \ln(4x - 1) - x = 0$.

Part A

Root existence

Show that the equation $f(x) = 0$ has two roots, α and β, such that $0.5 < \alpha < 1$ and $1 < \beta < 2$.

We find that $f(0.5) =$ ____, $f(1) =$ ____ and $f(2) =$ ____.

Since there is a ____ between $f(0.5)$ and $f(1)$, there must be a root α such that $0.5 < \alpha < 1$. As there is also a ____ between $f(1)$ and $f(2)$, there must be a root β such that $1 < \beta < 2$.

Items:

1.099 | 1 | 1.61 | 0.5 | -0.0541 | difference | -0.303 | 1.95 | change of sign | -0.5 | 0.109 | 0.0986 | change of value

Part B

Iteration with $g(x)$

Let $g(x) = \ln(4x - 1)$. Use the iterative formula $x_{r+1} = g(x_r)$ with $x_0 = 1.8$ to find x_1, x_2, and x_3, correct to 5 decimal places.

$x_1 =$

$x_2 =$

$x_3 =$

Continue the iterative process with $x_{r+1} = g(x_r)$ to find β correct to 3 decimal places.

$\beta =$

Part C

New rearrangement $h(x)$

The equation $f(x) = 0$ can be rearranged into the form

$$x = h(x) = \frac{e^{ax} + b}{c}$$

where a, b and c are constants. Find $h(x)$.

The following symbols may be useful: `e, h, x`

Part D

Iteration with $h(x)$

Use the iterative formula $x_{r+1} = h(x_r)$ with $x_0 = 0.8$ to find α correct to 4 decimal places.

Part E

Root finding analysis

Show that the iterative formula $x_{r+1} = g(x_r)$ will not find the value of α. Similarly, determine whether the iterative formula $x_{r+1} = h(x_r)$ will find the value of β.

The iterative formula $x_{r+1} = g(x_r)$ will not converge to a root if [ ] near that root.

For $g(x)$, differentiating we find that $g'(x) =$ [ ]. Using the value for α calculated in Part D, this gives $g'(\alpha) =$ [ ] > 1. Therefore the iterative formula $x_{r+1} = g(x_r)$ will not converge to α.

For $h(x)$, differentiating we find that $h'(x) =$ [ ]. Using the value for β calculated in Part B, $h'(\beta) =$ [ ] > 1. Therefore the iterative formula $x_{r+1} = h(x_r)$ will not converge to β.

Items:

0.307 | e^x | $\frac{1}{x}$ | 0.443 | 6.47 | $\frac{e^x + 1}{4}$ | $\frac{1}{4x}$ | $g'(x) > 1$ | 1.87 | $g'(x) < 1$ | $|g'(x)| < 1$ | 1.23 | $|g'(x)| > 1$

1.62 | 1.77 | $\frac{1}{4x - 1}$ | $\frac{e^x}{4}$ | $\frac{4}{4x - 1}$

Part F

Staircase diagrams

From the figures below, select the two figures that illustrate the iterations for $x_{r+1} = g(x_r)$ and $x_{r+1} = h(x_r)$.

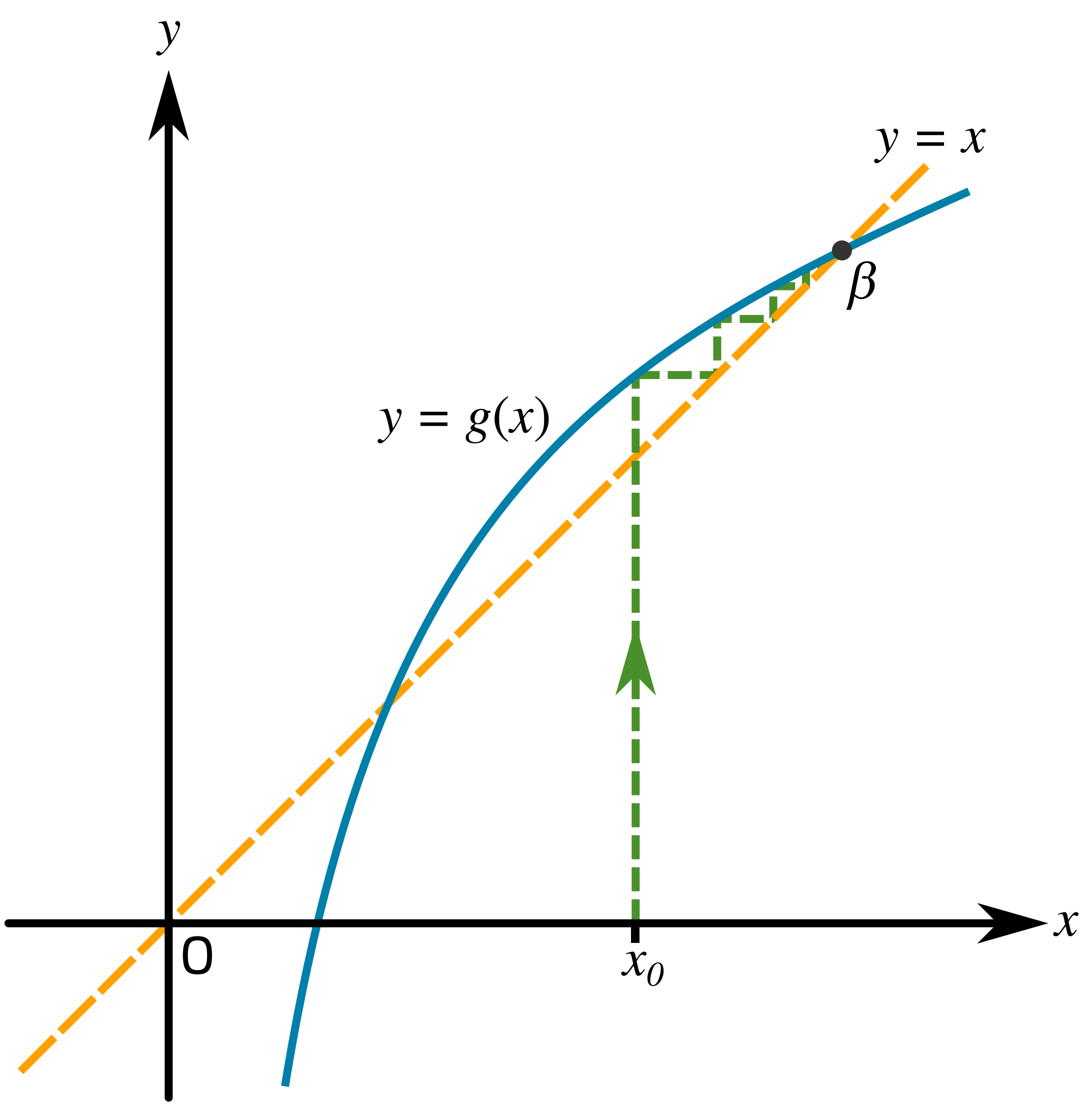

Figure 1: Graph of the iterative process for $x_{r+1} = g(x_r)$ towards β.

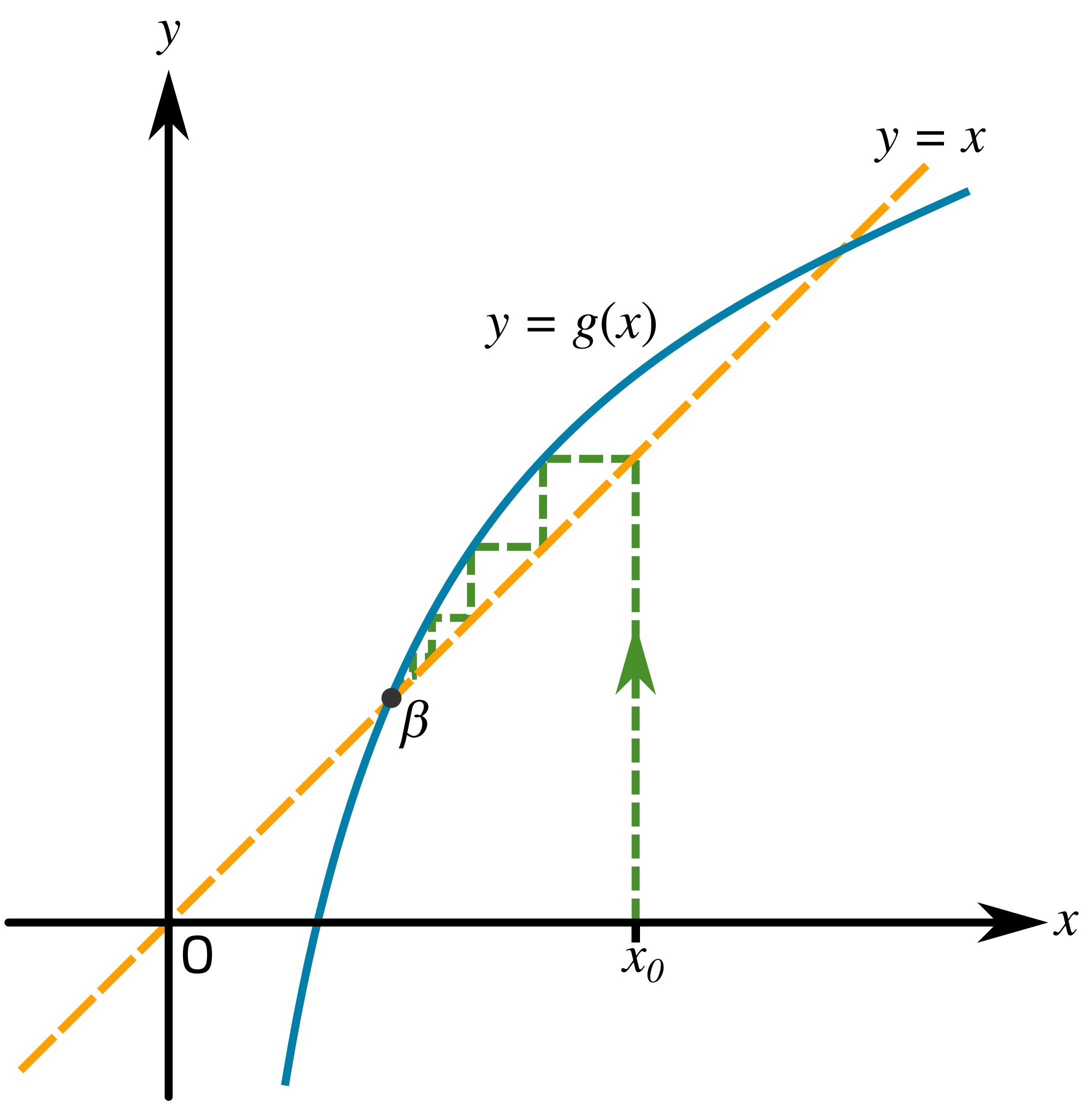

Figure 2: Graph of the iterative process for $x_{r+1} = g(x_r)$ towards β.

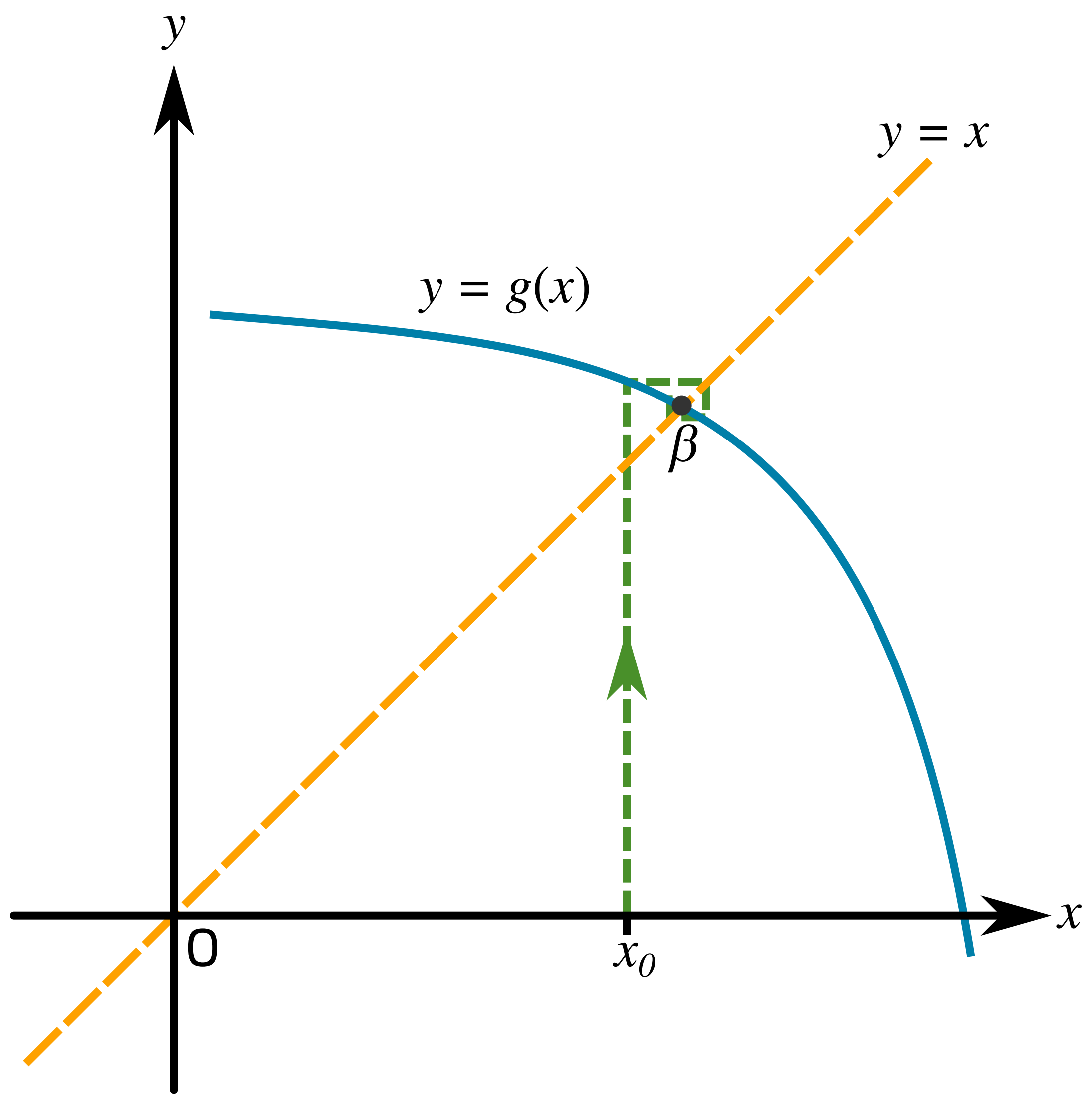

Figure 3: Graph of the iterative process for $x_{r+1} = g(x_r)$ towards β.

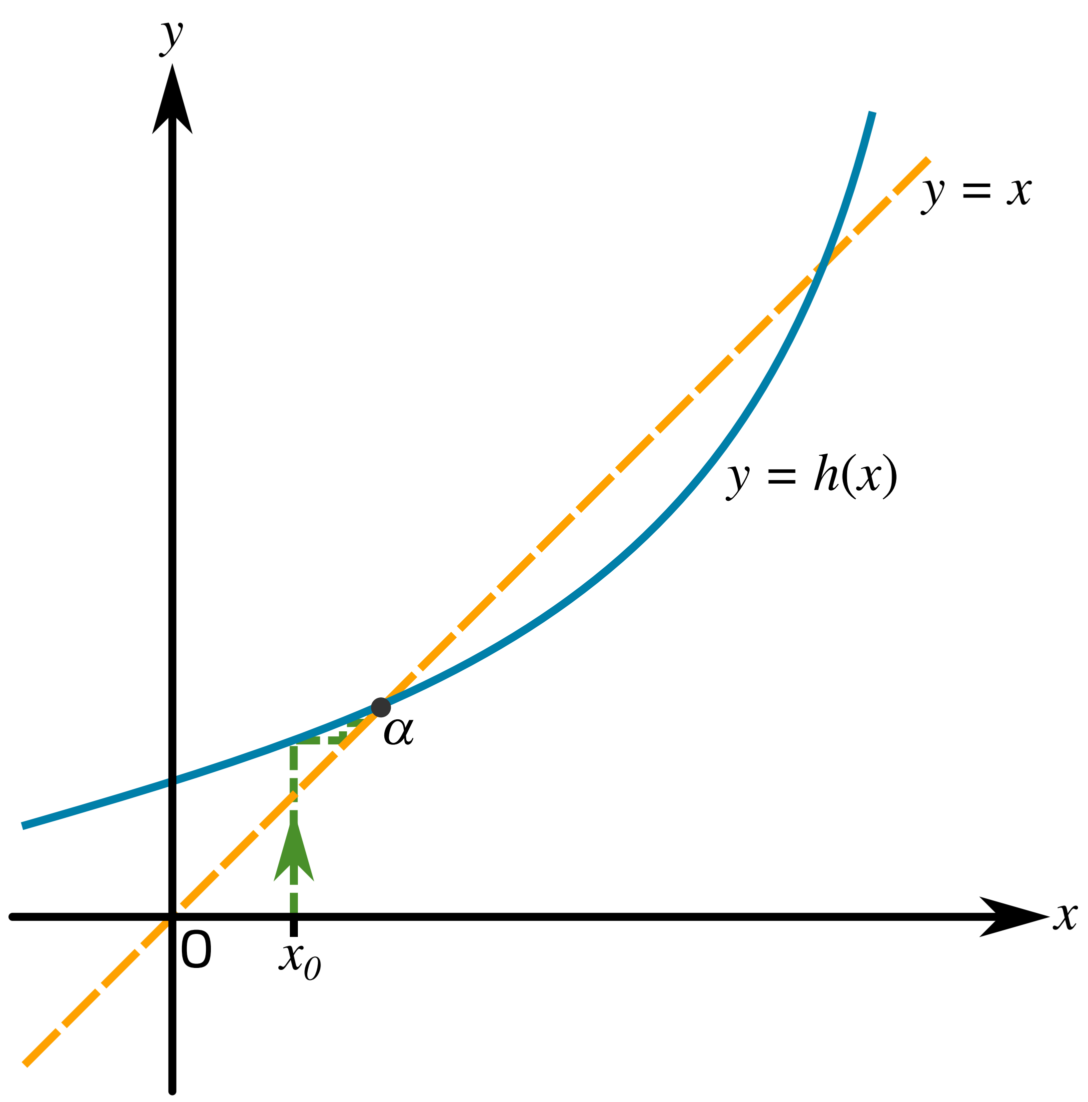

Figure 4: Graph of the iterative process for $x_{r+1} = h(x_r)$ towards α.

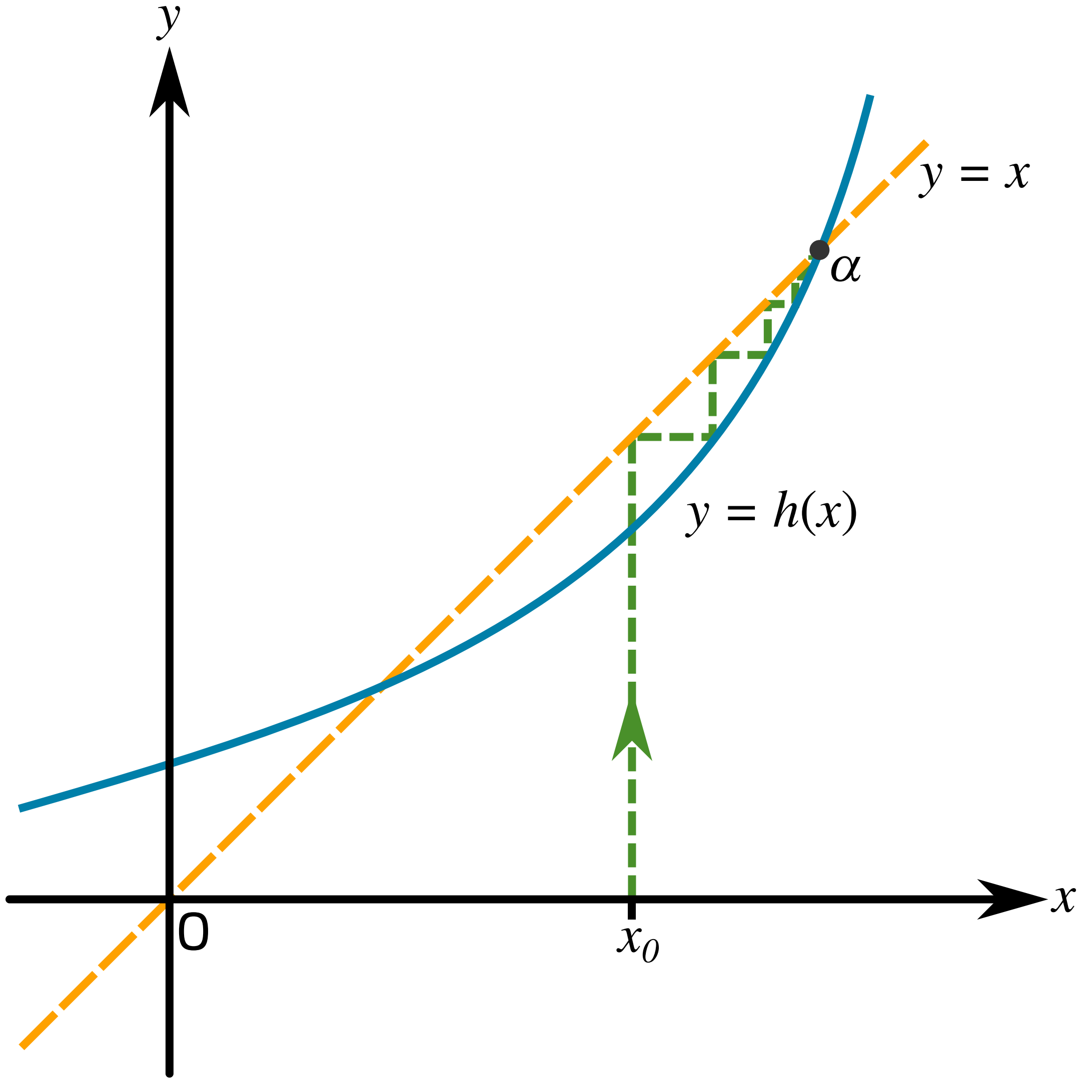

Figure 5: Graph of the iterative process for $x_{r+1} = h(x_r)$ towards α.

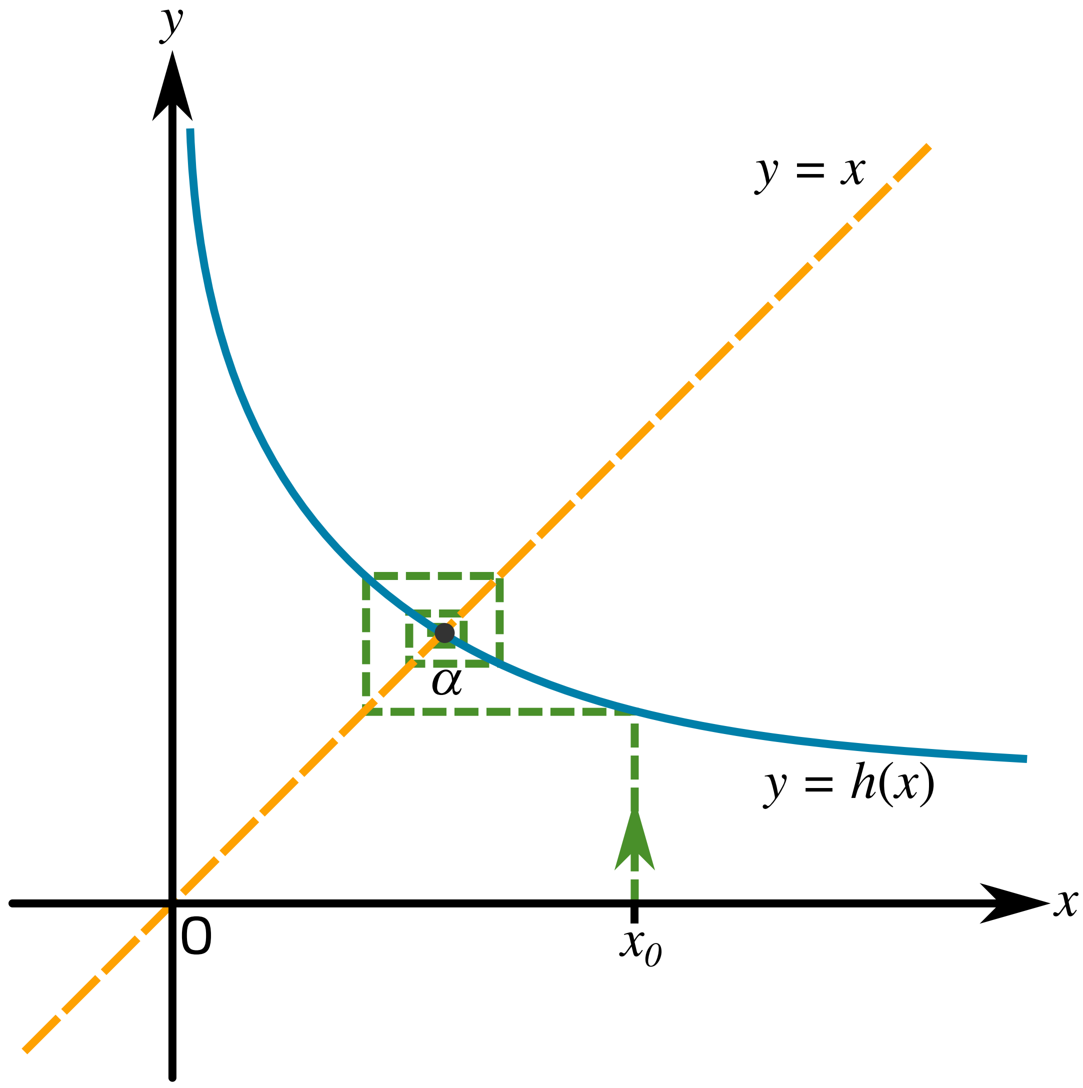

Figure 6: Graph of the iterative process for $x_{r+1} = h(x_r)$ towards α.

- [ ] Figure 1
- [ ] Figure 2
- [ ] Figure 3

- [ ] **Figure 4**
- [ ] **Figure 5**
- [ ] **Figure 6**

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Trapezium Rule 2ii

Subject & topics: Maths **Stage & difficulty:** A Level P3

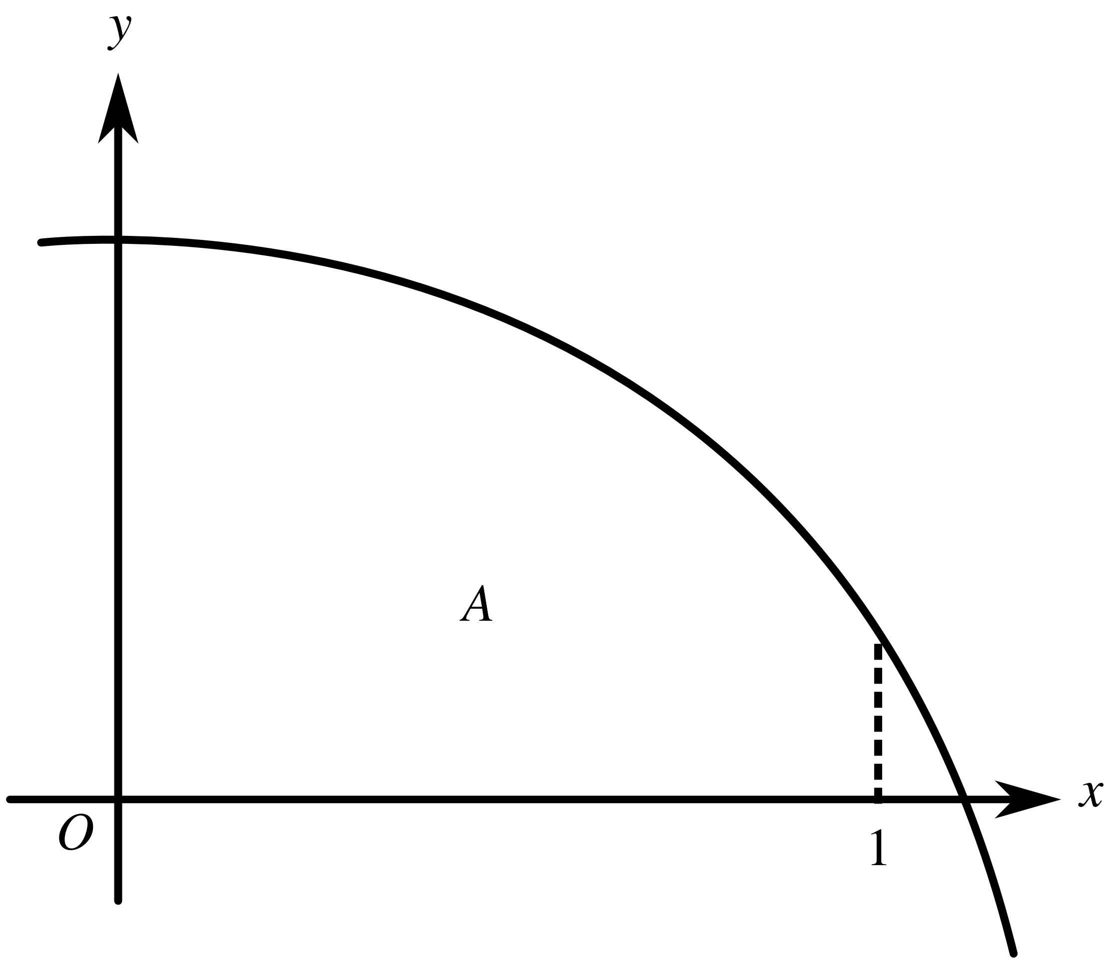

Figure 1: The diagram of the curve $y = \ln(16 - 12x^2)$.

Figure 1 shows part of the curve $y = \ln(16 - 12x^2)$. The region A is bounded by the curve and the lines $x = 0$, $x = 1$ and $y = 0$.

Part A

Trapezium Rule

Find an approximate value for A by using the trapezium rule, with two strips each of width $\frac{1}{2}$. Give your answer in the form $a \ln b$.

Part B

Overestimate or underestimate

Explain, using the diagram, whether the value obtained in Part A is an underestimate or overestimate for the area of A.

The diagram shows that for $0 \le x \le 1$ the value of y is [ ] and the curve has a [ ] shape (the gradient of the curve is becoming more negative). Hence, the tops of the trapezia used in part A all lie [ ] the curve, and so the area of the trapezia is an [ ] of the area of A.

Items:

under | negative | concave | above | underestimate | convex | positive | overestimate

Part C

Improving the approximation

Which of these options would improve the estimate of the area of A?

- Use the same number of trapezia, but double their height.
- Use 4 trapezia of width $\frac{1}{4}$.
- Use a larger number of trapezia with the same width, $\frac{1}{2}$.
- Use 4 trapezia of width $\frac{1}{8}$.

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Question

Trapezium Rule 3i

Subject & topics: Maths **Stage & difficulty:** A Level P3

The value of $\int_0^8 \ln\left(3+x^2\right) \mathrm{d}x$ obtained by using the trapezium rule with four strips is denoted by A.

Part A

Trapezium Rule

Find the value of A correct to 3 significant figures.

Part B

Approximation of $\int_0^8 \ln\left(9+6x^2+x^4\right) \mathrm{d}x$

Write, in terms of A, an expression for an approximate value of $\int_0^8 \ln\left(9+6x^2+x^4\right) \mathrm{d}x$.

The following symbols may be useful: A

Part C

Approximation of $\int_0^8 \ln\left(3e + ex^2\right)\mathrm{d}x$

Write, in terms of A, an expression for an approximate value of $\int_0^8 \ln\left(3e + ex^2\right)\mathrm{d}x$.

The following symbols may be useful: `A`

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Question

Area: Numerical Integration 2ii

Subject & topics: Maths **Stage & difficulty:** A Level P3

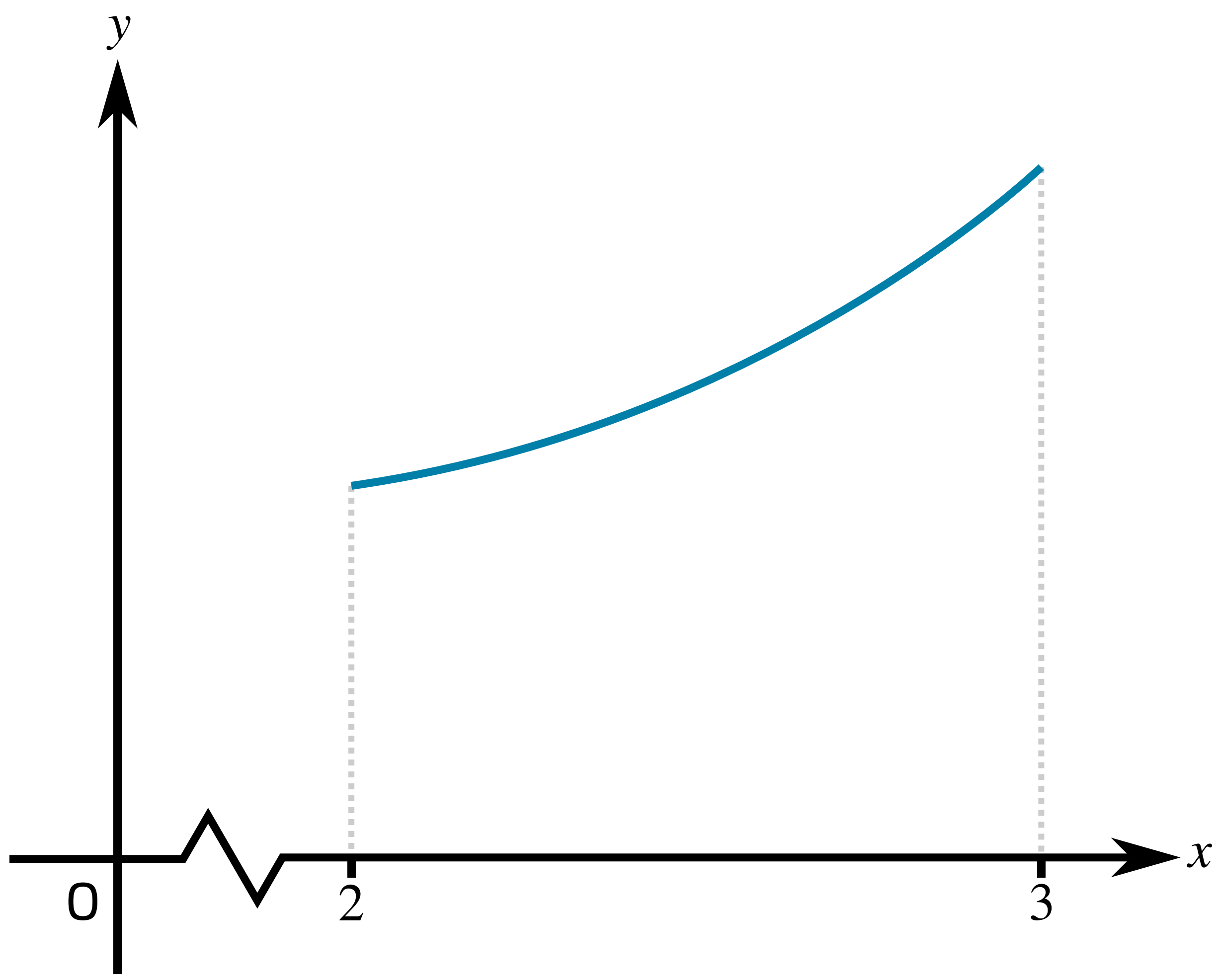

Figure 1: The curve with equation $y = \sqrt{1 + x^3}$, for $2 \leqslant x \leqslant 3$.

Figure 1 shows the curve with equation $y = \sqrt{1 + x^3}$, for $2 \leqslant x \leqslant 3$. The region under the curve between these limits has area A.

Part A

Bounding A

Using the figure below, fill in the blanks to explain why $3 < A < \sqrt{28}$.

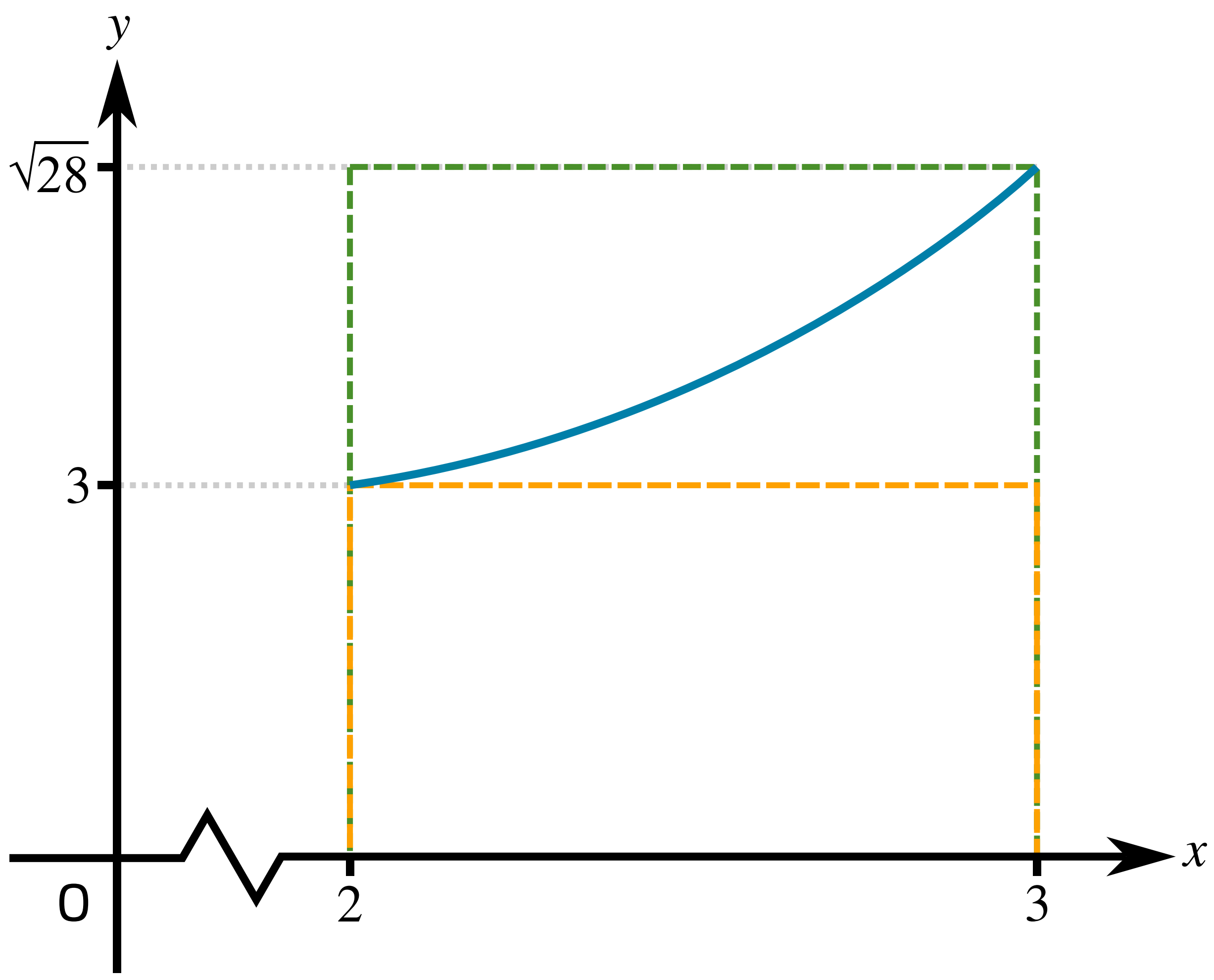

Figure 2: A diagram showing rectangles with areas which bound A.

Two rectangles are shown in **Figure 2**. Both rectangles begin on the x-axis and have width one. The area of the smaller rectangle, which lies ______ the curve, is ______. The area of the second rectangle, the top of which lies ______ the curve, is ______. The rectangles have areas which bound A, and hence:

$$3 < A < \sqrt{28}$$

Items:

3 6 $\sqrt{28}$ above below $3\sqrt{28}$

Part B

Improved bounds

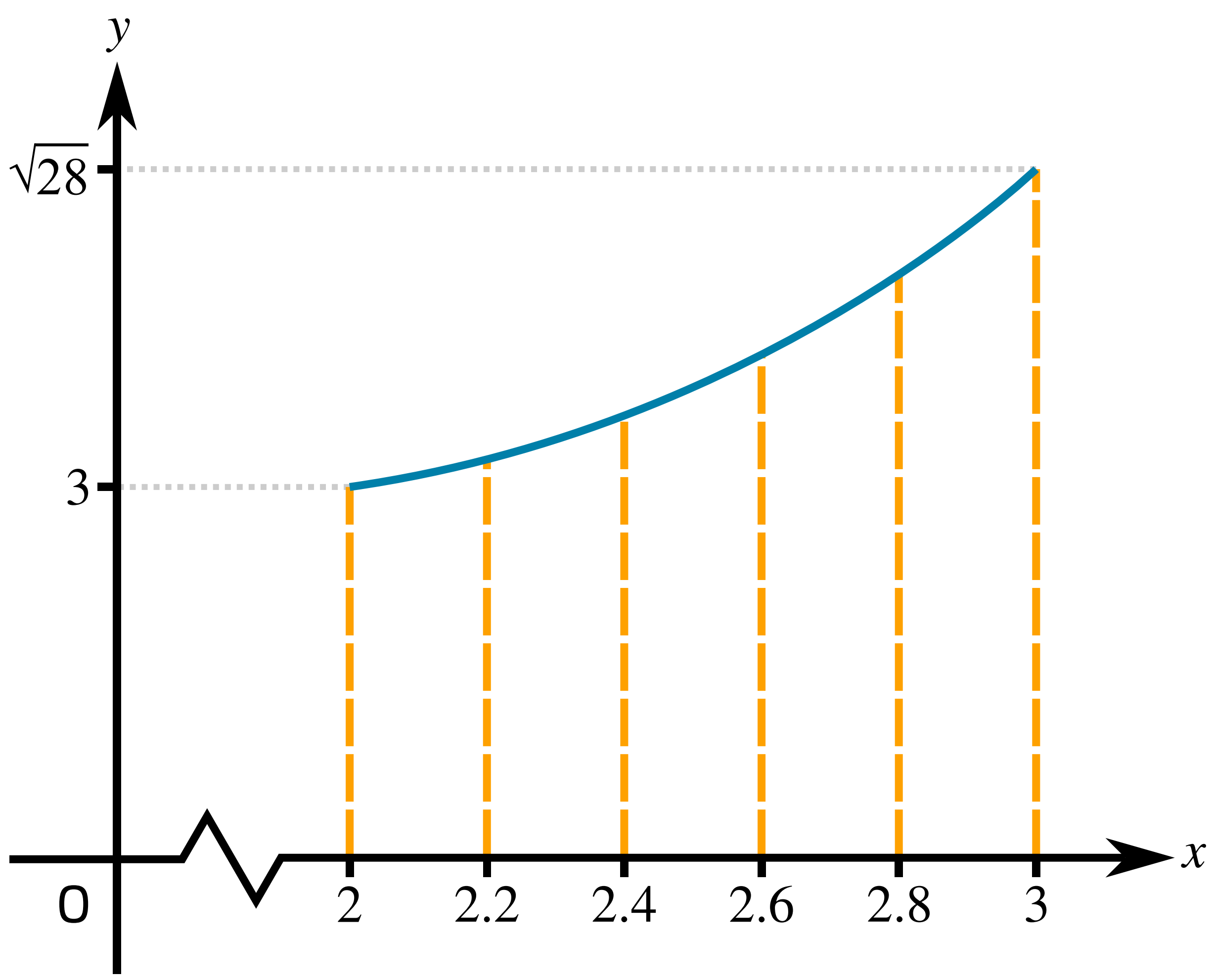

Figure 3: The curve with equation $y = \sqrt{1+x^3}$, for $2 \leqslant x \leqslant 3$, divided into 5 strips of equal width.

The region is divided into 5 strips, each of width 0.2. Use suitable rectangles with these strips to find improved lower and upper bounds for A. Give your answers to 3 significant figures.

lower bound for A:

upper bound for A:

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