



Physics. *You work it out.*

Waves and Complex numbers

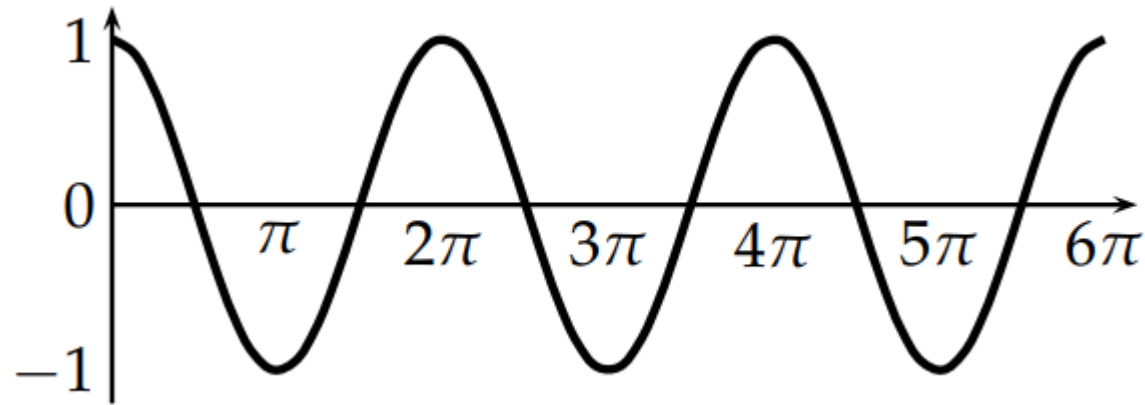
Overture for Senior Physics Challenge

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Describing a wave

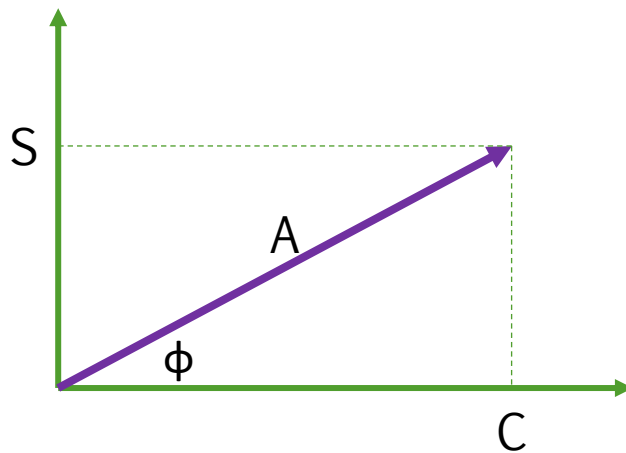
- › Phase gives 'location' within wave 0=peak.
- › Angles are always measured in radians ($1 \text{ rad} = 180/\pi^\circ$)



- › We describe the wave $y = A \cos \omega t$
- › Angular frequency $\omega = 2\pi f = 2\pi/T$ measured in rad/s

Wave with initial phase

- › A wave may not have a peak at $t=0$.
- › $y = A \cos(\omega t + \phi_0) = A \cos \omega t \cos \phi_0 - A \sin \omega t \sin \phi_0$
- › $y = C \cos \omega t - S \sin \omega t$, where $C = A \cos \phi_0$, $S = A \sin \phi_0$
- › Overall amplitude $A = \sqrt{C^2 + S^2}$, $\tan \phi_0 = S/C$

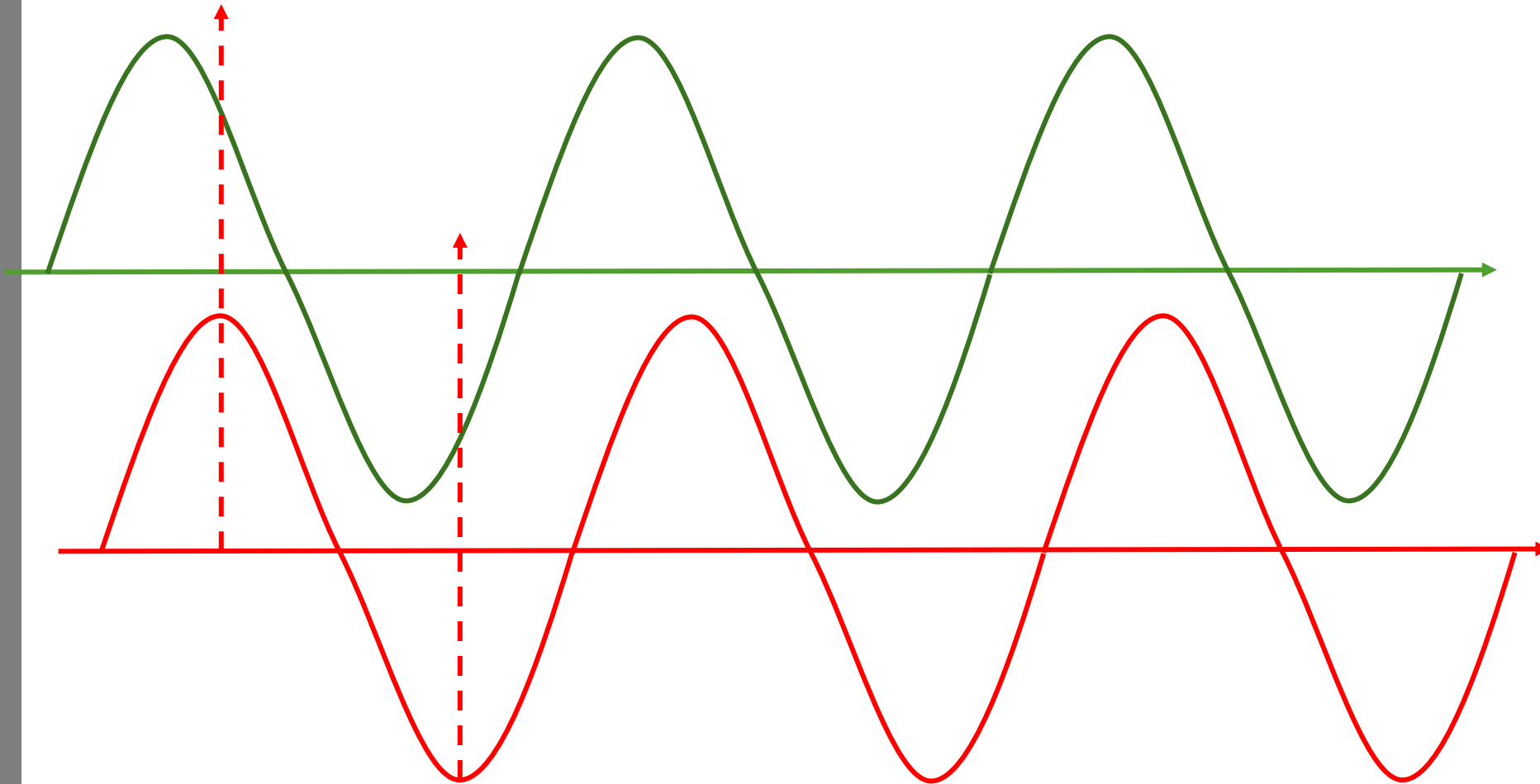


It is often helpful to draw the amplitude in two dimensions to show its phase, its cosine (even) and sine (odd) components.



Phasor with single wave

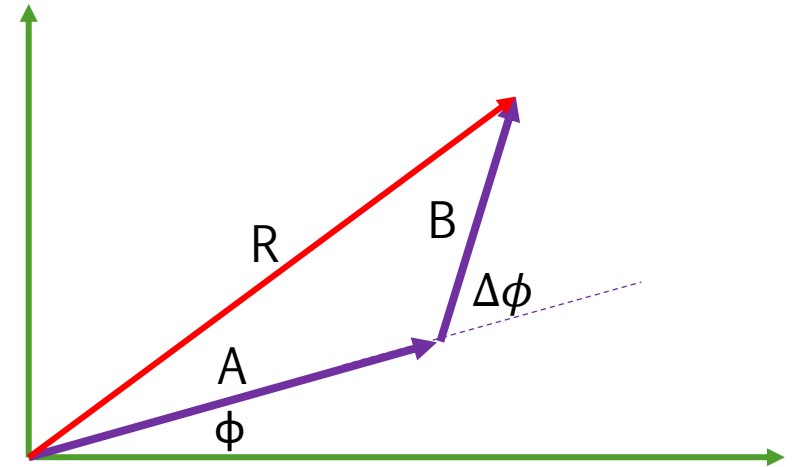
Phase difference



Phase difference is constant if both waves have same ω

Phase difference and interference

- › Trigonometry:
- › $A \cos(\omega t + \phi) + B \cos(\omega t + \theta)$
- › $= A \cos \omega t \cos \phi - A \sin \omega t \sin \phi + B \cos \omega t \cos \theta - B \sin \omega t \sin \theta$
- › $C = A \cos \phi + B \cos \theta ,$
- › $S = A \sin \phi + B \sin \theta$





Phasor with two waves

Waves

- › Oscillation ($x=0$) described as $y = A \cos(\omega t)$
- › Wave travelling to right at speed c
- › $y_{\rightarrow}(x, t) = y\left(0, t - \frac{x}{c}\right) = A \cos\left(\omega t - \frac{\omega x}{c}\right) = A \cos(\omega t - kx)$
- › $k = \frac{\omega}{c} = \frac{2\pi f}{c} = \frac{2\pi}{\lambda}$ is known as the wavenumber

You work it out!

- › Write a wave travelling left y_{\leftarrow} with the same ω, k, c
- › Add the two waves, and see what you get.



Is this a vector space?

Our diagrams look like vectors, but

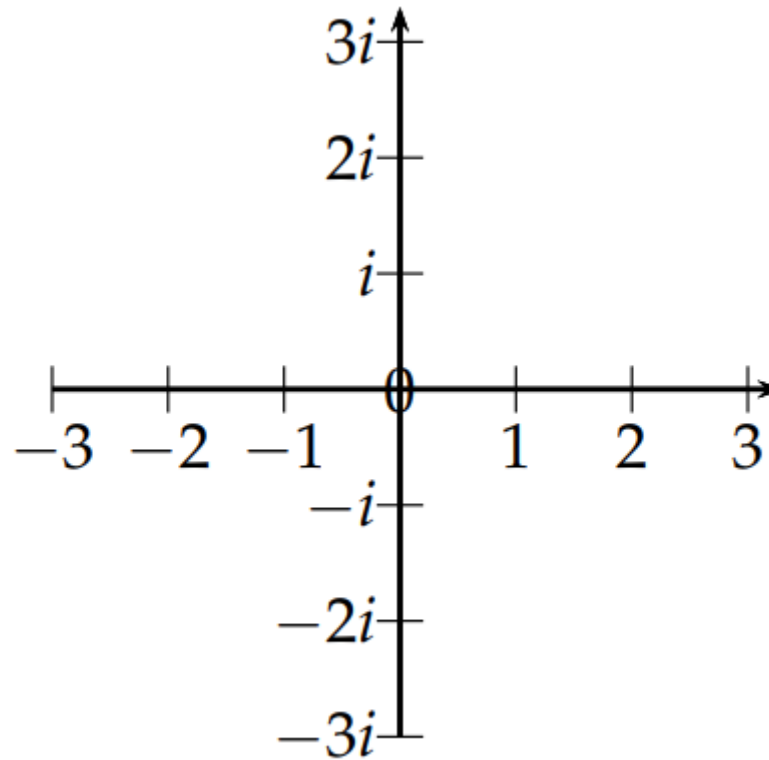
- › only the x-component really means anything
- › the two components are not independent (odd and even parts of the same wave)

Added complications

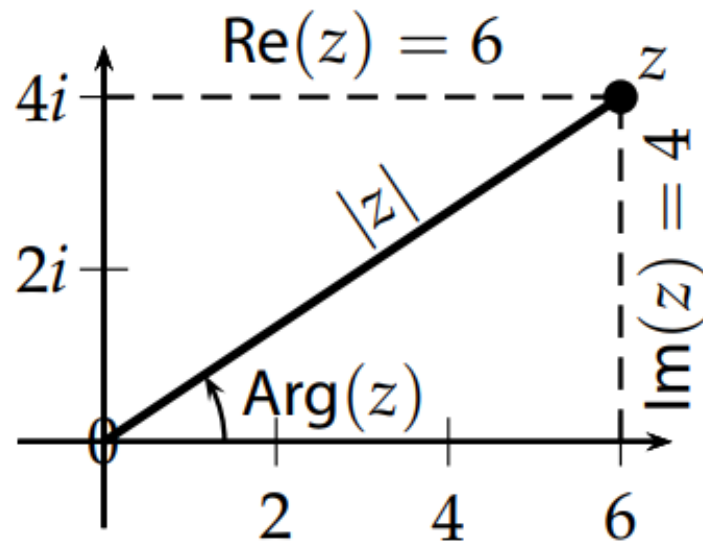
- › not easy to see overall amplitude when using \cos & \sin
- › differential equations tough since $\cos \rightarrow -\sin \rightarrow -\cos \rightarrow \dots$

Solution: use complex numbers and all our problems go away!

The Argand diagram



A complex number



This number is $z = 6 + 4i$

Two complex numbers are added with usual algebra

$$3 + 2i + 6 - 3i = 9 - i$$

Multiplication by i represents rotation by $\frac{\pi}{2}$ anticlockwise.

$$i^2 = -1, \quad -i \times i = 1, \quad \frac{1}{i} = -i$$

Complex conjugate $z^* = 6 - 4i$, $z^* z = |z|^2$

Euler: $z = r \cos \theta + ir \sin \theta = r e^{i\theta}$, so $re^{i\theta} we^{i\phi} = rwe^{i(\theta+\phi)}$

Oscillations with complex numbers

› $y = A \cos(\omega t + \phi) = \text{Re } Ae^{i\phi} e^{i\omega t}$

› Adding two oscillations:

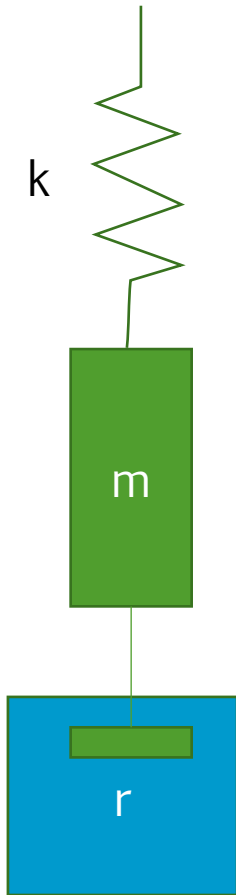
›
$$\begin{aligned} y &= \text{Re} (Ae^{i\phi} e^{i\omega t} + Ae^{i\theta} e^{i\omega t}) \\ &= \text{Re} \{e^{i\omega t} (Ae^{i\phi} + Ae^{i\theta})\} \\ &= \text{Re} \{e^{i\omega t + i(\phi+\theta)/2} (Ae^{i(\phi-\theta)/2} + Ae^{i(\theta-\phi)/2})\} \\ &= 2A \cos\left(\frac{\phi - \theta}{2}\right) \text{Re} \{e^{i\omega t + i(\phi+\theta)/2}\} \\ &= 2A \cos\left(\frac{\phi - \theta}{2}\right) \cos\left(\omega t + \frac{\phi + \theta}{2}\right) \end{aligned}$$

› Remember $e^{i\theta} = \cos \theta + i \sin \theta$ so $e^{i\theta} + e^{-i\theta} = 2 \cos \theta$

Forced, damped harmonic oscillator

$$F = m\ddot{x} = -kx - r\dot{x} + F_0 \cos \omega t$$

Try solution $z = Ae^{i\omega t}$ to $m\ddot{z} = -kz - r\dot{z} + F_0e^{i\omega t}$

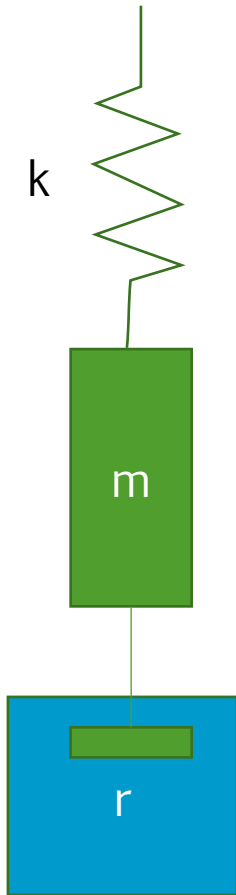


$$\frac{d}{dt}e^{i\omega t} = i\omega e^{i\omega t} \quad \frac{d^2}{dt^2}e^{i\omega t} = -\omega^2 e^{i\omega t}$$

Forced, damped harmonic oscillator

$$F = m\ddot{x} = -kx - r\dot{x} + F_0 \cos \omega t$$

Try solution $z = Ae^{i\omega t}$ to $m\ddot{z} = -kz - r\dot{z} + F_0e^{i\omega t}$



$$\frac{d}{dt}e^{i\omega t} = i\omega e^{i\omega t} \quad \frac{d^2}{dt^2}e^{i\omega t} = -\omega^2 e^{i\omega t}$$

$$-m\omega^2 Ae^{i\omega t} = -kAe^{i\omega t} - ri\omega Ae^{i\omega t} + F_0e^{i\omega t}$$

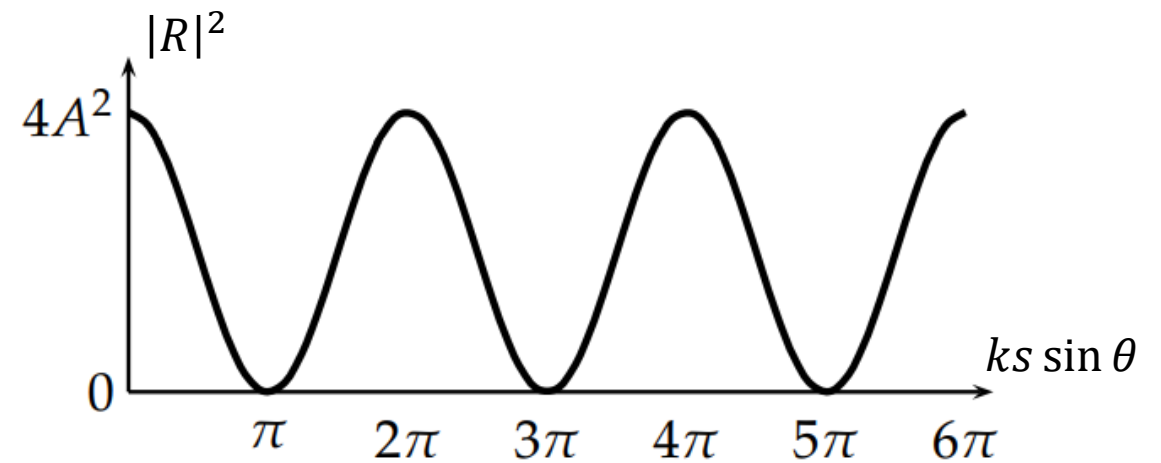
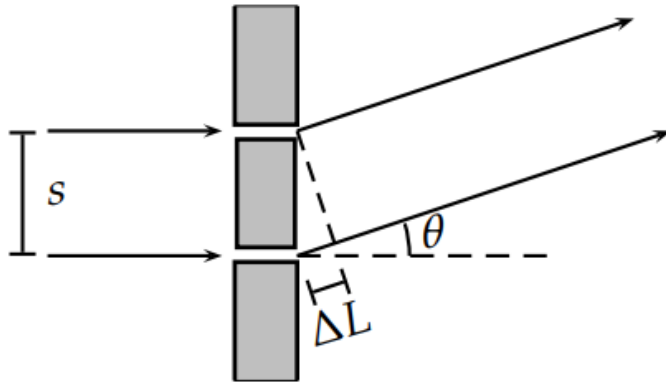
$$(k - m\omega^2 + ir\omega)A = F_0$$

Waves with complex numbers

- › $y = A \cos(\omega t - kx) = \text{Re } Ae^{i\omega t} e^{-ikx}$
- › Adding two waves travelling different distances:
- ›
$$\begin{aligned} y &= \text{Re} \left(Ae^{i\omega t} e^{-ikx} + Ae^{i\omega t} e^{-ik(x+\Delta)} \right) \\ &= \text{Re} \left\{ e^{i\omega t} \left(Ae^{-ikx} + Ae^{-ik(x+\Delta)} \right) \right\} \\ &= \text{Re} \left\{ e^{i\omega t - ikx} \left(A + Ae^{-ik\Delta} \right) \right\} \end{aligned}$$
- › Modulus of $R = A + Ae^{-ik\Delta}$ gives overall amplitude

Two slits

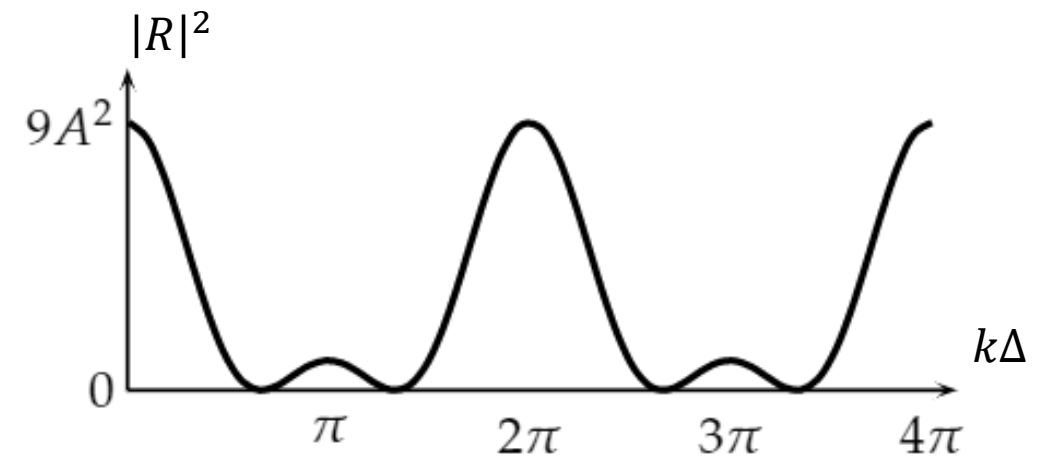
$$\triangleright R = A + Ae^{-ik \Delta L} = A(1 + e^{-ik \Delta L}) = A(1 + e^{-iks \sin \theta})$$





3 slits

› $R = A + Ae^{-ik\Delta} + Ae^{-2ik\Delta}$ Hint: $R = e^{-ik\Delta}(e^{ik\Delta} + 1 + e^{-ik\Delta})$



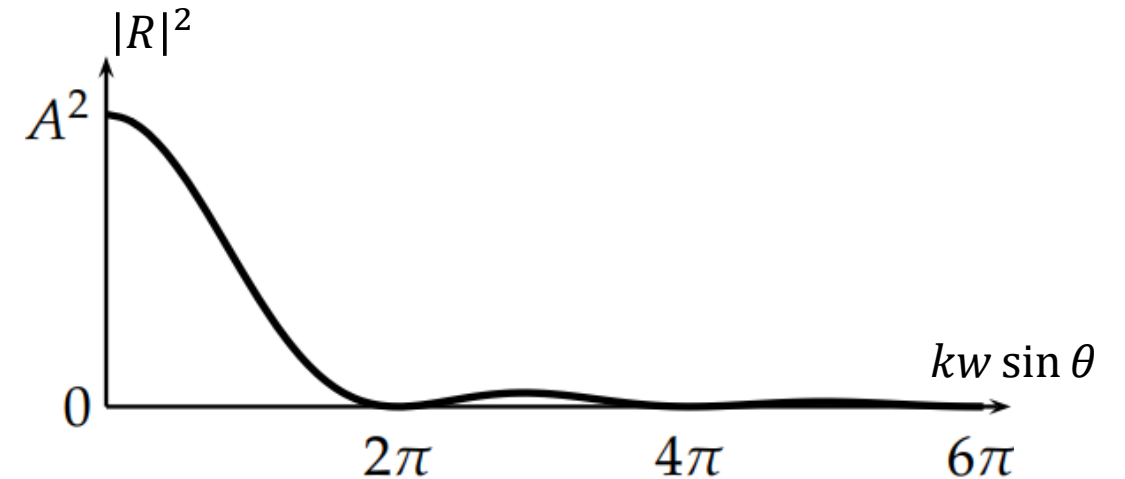
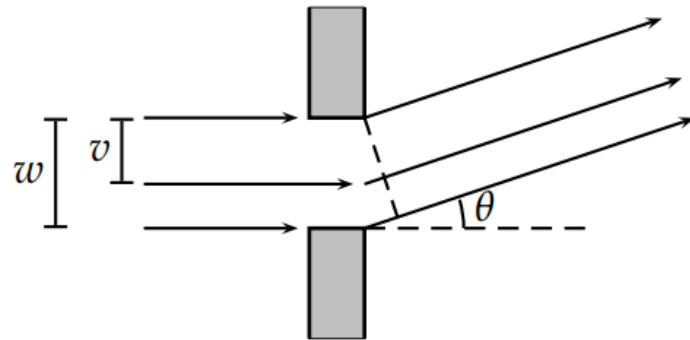


N slits

$$\triangleright R = A + Ae^{-ik\Delta} + Ae^{-2ik\Delta} + \dots + Ae^{-(N-1)ik\Delta}$$

A single, wide slit

$$\triangleright R = \int_{v=0}^w \frac{Ae^{-ikv \sin \theta}}{w} dv$$



QM and normal wave differences

- › Normal wave – only the real part is meaningful
- › Intensity $\propto \frac{1}{2} |A|^2$ as $\overline{\cos^2 \omega t} = \frac{1}{2}$
- › Intensity at a place in space pulses as time progresses

- › QM wave – really isn't real
- › Probability density $\propto |\psi|^2$. No factor of $\frac{1}{2}$.
- › Intensity at a place in space does not pulse



Eigenfunctions

› We write $\psi = Ae^{i(kx-\omega t)}$

› Work out $\frac{\partial \psi}{\partial t}$ $\frac{\partial^2 \psi}{\partial t^2}$

› Work out $\frac{\partial \psi}{\partial x}$ $\frac{\partial^2 \psi}{\partial x^2}$

› Wave equation $\frac{\partial^2 \psi}{\partial x^2} = ? \frac{\partial^2 \psi}{\partial t^2}$

› $\hbar = \frac{h}{2\pi}$. Now write $p = \frac{h}{\lambda}$ in terms of k, \hbar

› $k\psi = \dots$ $p\psi = \dots$ $\frac{p^2}{2m} \psi = \dots$

Eigenfunctions

› We write $\psi = Ae^{i(kx-\omega t)}$

› Work out $\frac{\partial \psi}{\partial t} = -i\omega\psi$ $\frac{\partial^2 \psi}{\partial t^2} = -\omega^2\psi$

› Work out $\frac{\partial \psi}{\partial x} = ik\psi$ $\frac{\partial^2 \psi}{\partial x^2} = -k^2\psi$

› Wave equation $\frac{\partial^2 \psi}{\partial x^2} = \frac{k^2}{\omega^2} \frac{\partial^2 \psi}{\partial t^2} = = \frac{1}{c^2} \frac{\partial^2 \psi}{\partial t^2}$

› $\hbar = \frac{h}{2\pi}$. Now write $p = \frac{h}{\lambda} = \frac{2\pi\hbar}{2\pi/k} = \hbar k$

› $k\psi = -i \frac{\partial}{\partial x} \psi$ $p\psi = -i\hbar \frac{\partial}{\partial x} \psi$ $\frac{p^2}{2m} \psi = \frac{1}{2m} \left(-i\hbar \frac{\partial}{\partial x} \right)^2 \psi$