Drude Model

Further A University

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In this question, we explore a set of assumptions which lead to a useful model of electrons in a conductor, including their collisions with ion cores in the metal lattice, which enables us to understand the resistance of metals.

Part A Probability of collision in a time

When an electron moves for a very short period of time δt, the probability of a collision with an ion core in the metal is proportional to δt. We write that the probability in this time as $p\ \delta t$ where p is a constant.

Throughout this question, you may assume that δt is sufficiently small that the fraction of the electrons which will collide in this time is a very small proportion of the total. You may also assume that δt is sufficiently small that no electron will collide twice during this time.

How many electrons will collide during a time interval δt if there were n electrons at the start of the interval?

The following symbols may be useful: `deltat, n, p, t`

How many electrons remain uncollided after the time interval δt?

The following symbols may be useful: `deltat, n, p, t`

Part B Differential equation for n

Write a differential equation for the number n of electrons which have not yet collided since $t = 0$. Enter your equation in the form $\frac{\mathrm{d}n}{\mathrm{d}t} = \cdots$.

The following symbols may be useful: `Derivative(_, t), n, p, t`

Solve the differential equation above. Then write an equation for the number n of electrons which haven't yet collided since $t = 0$. Write your answer in terms of p, t and the initial number of electrons n_0. Enter your answer in the form $n = \cdots$.

The following symbols may be useful: `e, n, n_0, p, t`

Part C Probability of collision

Write an expression for the number of electrons you expect to collide (for the first time since $t = 0$) between time t and time $t + \delta t$. You may assume that δt is short enough that the number colliding per unit time does not change during this interval. Hint: what does $\frac{\mathrm{d}n}{\mathrm{d}t}$ represent?

The following symbols may be useful: `deltat, e, n_0, p, t`

Consider one of the electrons. Write an expression for the probability that it makes its first collision (after $t = 0$) between times t and $t + \delta t$.

The following symbols may be useful: `deltat, e, p, t`

Part D Relaxation time

If the probability that an electron's first collision after $t = 0$ happens between t and $t + \delta t$ is written $q(t)\,\delta t$, then the average time between collisions τ is given by the integral

$$\tau = \int_{t=0}^{\infty} t\, q(t)\, \mathrm{d}t.$$

You can read about the use of integrals to work out the average value of a random variable at our concept page on Continuous probability distributions.

Use your answer to the previous question to write an expression for the average time between collisions in terms of p.

The following symbols may be useful: `p`

Part E Distance between collisions (steady speed)

If all electrons move at a steady speed v between collisions, how far on average do they travel between collisions? Give your answer in terms of the average time between collisions τ and the speed v.

The following symbols may be useful: `tau, v`

Part F Distance between collisions (accelerated motion)

If an electron accelerates from rest (at $t = 0$) with constant acceleration a, give an expression for $s(t)$, the distance it has moved by time t.

The following symbols may be useful: `a, t`

You can use your expression for $s(t)$ to derive an expression for the average distance moved by an electron which accelerates from rest until it next collides. To do this, you need to use your expression for the probability for a collision between t and $t + \delta t$ derived in Part C. The average distance moved is then given by

$$S = \int_{t=0}^{\infty} s(t)\, q(t)\, dt.$$

where $q(t)\,\delta t$ is the probability of a collision between time t and $t + \delta t$.

Give your expression for S in terms of the constant acceleration a and the average time between collisions τ.

The following symbols may be useful: `a, tau`

Part G Drift velocity

Using your previous answer, without performing further integrals, write an expression for the average velocity of the electrons in the metal in terms of the constant acceleration a and the average time between collisions τ. You may assume that electrons stop whenever they collide.

The following symbols may be useful: `a, tau`

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Deriving Complex Impedances

Further A

P P P

This question is best done using complex numbers. If you have not come across complex impedances, you may wish to look at our concept page for that topic.

Complex numbers are used by engineers to describe the way alternating currents flow through circuits and components. This is relevant not only for power engineering but also signal processing.

In this section, we will assume that the voltage across a component is given by the real part of

$$V = V_0 \mathrm{e}^{i\omega t}.$$

We will also assume that the current through the component is given by the real part of

$$I = I_0 \mathrm{e}^{i\omega t}.$$

Note that V_0 and I_0 will not necessarily be real. The complex impedance Z is defined as

$$Z = \frac{V}{I} = \frac{V_0}{I_0}.$$

We use a sign convention so that the impedance of a resistor $Z_R = R$. This is a real number equal to its resistance. This means that a current is thought of as positive if it flows from a place of high electrical potential to a place with a lower electrical potential.

Part A Inductor

Using our sign convention, the voltage across an inductor is given by the equation

$$V = L\frac{\mathrm{d}I}{\mathrm{d}t}$$

where L is its inductance. Inductance is measured in henries (H).

If the current through the inductor is $I = I_0 \mathrm{e}^{i\omega t}$, write an expression for the voltage across it in terms of I, ω and L.

The following symbols may be useful: `I, L, V, i, omega`

Part B Complex impedance of inductor

Use your answer to part A to write an expression for the complex impedance Z_{L} of the inductor. Give your answer in terms of ω and L.

The following symbols may be useful: `L, i, omega`

Part C Thinking about the inductor

Just like resistance, when the impedance Z is very large, the component will obstruct or limit the flow of charge. Similarly, when Z is low, the charge flows easily.

Current flows most easily through an inductor when the frequency is ______.

Current is effectively blocked by an inductor when the frequency is ______.

Items:

high low

Part D Capacitor

The charge stored on a capacitor is given by the equation $Q = CV$ where C is its capacitance measured in farads (F).

Using our sign convention,

$$I = \frac{\mathrm{d}Q}{\mathrm{d}t}.$$

If the voltage across the capacitor is $V = V_0 \mathrm{e}^{i\omega t}$, write an expression for the current through it in terms of V, ω and C.

The following symbols may be useful: `C, I, V, i, omega`

Part E Complex impedance of capacitor

Use your answer to part D to write an expression for the complex impedance Z_C of the capacitor. Give your answer in terms of ω and C.

The following symbols may be useful: `C`, `i`, `omega`

Part F Thinking about the capacitor

Just like resistance, when the impedance Z is very large, the component will obstruct or limit the flow of charge. Similarly, when Z is low, the charge flows easily.

Current flows most easily through a capacitor when the frequency is ______.

Current is effectively blocked by a capacitor when the frequency is ______.

Items:

low high

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Factory Capacitor

Further A

C C C

This question is best done using complex numbers. If you have not come across complex impedances, you may wish to look at our concept page for that topic.

A factory contains a great deal of high current electrical equipment including motors with coils of wire in them. The factory's equipment, as a whole, is equivalent to a resistance R in series with an inductor of inductance L.

The inductance causes the alternating current and voltage to become out of phase, which makes the factory liable for punitive charges from the electricity supply company.

The factory electrician installs a heavy duty capacitor of capacitance C in parallel with the factory's equipment to fix the problem.

The complex impedance of a capacitor is $Z_{\mathrm{C}} = \dfrac{1}{\mathrm{i}\omega C}$ and the complex impedance of an inductor is $Z_{\mathrm{L}} = \mathrm{i}\omega L$.

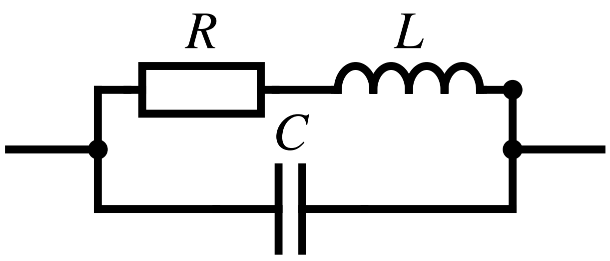

Figure 1: A circuit representing the factory equipment and the installed heavy duty capacitor.

Part A Negligible resistance

Calculate the capacitance C needed to ensure that the overall current taken from the supply by the factory as a whole (including the capacitor) is in phase with the supply voltage $V = V_0 \cos(\omega t)$.

For this part of the question, you may assume that the resistance is very low and can be neglected.

Give your answer in terms of L and the angular frequency of the supply ω.

The following symbols may be useful: `C`, `L`, `omega`

Part B Don't neglect the resistance this time

Calculate the capacitance C needed to ensure that the overall current taken from the supply by the factory as a whole (including the capacitor) is in phase with the supply voltage $V = V_0 \cos(\omega t)$. This time, you may not assume that $R = 0$.

Give your answer in terms of R, L and the angular frequency of the supply ω.

The following symbols may be useful: `C`, `L`, `R`, `omega`

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Gameboard:

Using complex numbers with electrical components and waves

Superposition with Complex Numbers

Further A

P P P

This question is best done using complex numbers. If you have not come across complex impedances, you may wish to look at our concept page for that topic.

In this question, we will use complex numbers to work out the amplitude at a point where two waves meet.

One of the waves arriving at point P can be described using $y_1 = A\cos(\omega t)$ where A is its amplitude, t is the time and $\omega = 2\pi f$ is the angular frequency of the wave in $\mathrm{rad\,s^{-1}}$.

At point P, this wave meets a wave of the same frequency and amplitude which travelled along a longer route. As such this wave is at an earlier stage of oscillation. We write this as $y_2 = A\cos(\omega t - \phi)$ where ϕ is the phase difference between the two waves caused by the difference in length of the routes.

Our aim is to find the amplitude of the combined wave, and also to relate this to the difference in distance.

Part A Phase angle

Suppose that the difference in the lengths of the routes by which the waves arrived at P is L.

Write an expression for the phase angle ϕ (in radians) in terms of L and the wavelength λ.

You may wish to remember that a distance of λ corresponds to a phase change of one whole oscillation, which is $2\pi\,\mathrm{rad}$.

The following symbols may be useful: `L`, `lambda`, `phi`, `pi`

Part B Complex oscillation

To express our oscillations using complex numbers, we write y_1 and y_2 as the real parts of complex displacements z_1 and z_2.

For the first wave $y_1 = A\cos(\omega t)$, we write $y_1 = \mathrm{Re}\, z_1$, where $z_1 = Ae^{i\omega t}$.

For the second wave $y_2 = A\cos(\omega t - \phi)$, and we write $y_2 = \mathrm{Re}\, z_2$.

Write an expression for z_2 in terms of A, ω, t and ϕ.

The following symbols may be useful: `A, e, i, omega, phi, t, z_2`

Part C Superposition

The total displacement due to both waves at point P is given by $y_T = y_1 + y_2$.

We write $y_T = \mathrm{Re}\, z_T$ where $z_T = z_1 + z_2$.

The total displacement can be written in the form $z_T = f(\phi)\, e^{i\omega t}$.

Give an expression for $f(\phi)$ in terms of A and ϕ.

The following symbols may be useful: `A, e, f, i, phi`

Part D Intensity

The intensity of the combined wave is proportional to the modulus square of z_T. This is the quantity $|z_T|^2$, which is equal to $z_T^{\star} z_T$.

Given that $z_T = f(\phi)\, e^{i\omega t}$, express the modulus square of z_T in terms of the modulus square of $f(\phi)$.

Now use your previous expression for $f(\phi)$ to work out $z_T^{\star} z_T$.

Give your answer (which should be a real number) in terms of A and $\cos(\phi)$. Both A and ϕ are real.

The following symbols may be useful: `A, cos(), phi`

Part E Constructive interference

Constructive interference is when the amplitude of the combined wave is as large as it can be.

What is the smallest positive, non-zero value of ϕ (in radians) which would give rise to constructive interference?

The following symbols may be useful: `pi`

What is the value of $z_T^{\star}\, z_T$ at constructive interference?

The following symbols may be useful: `A, pi`

Thinking back to your answer to the first part of this question, where you related ϕ to the path difference L, what is the smallest positive, non-zero value of L at which you will have constructive interference?

Give your answer in terms of the wavelength λ.

The following symbols may be useful: `L, lambda`

Part F Quarter wavelength

Evaluate ϕ when $L = \frac{\lambda}{4}$.

Give your answer in radians.

The following symbols may be useful: `phi, pi`

Evaluate $z_T^\star\, z_T$ when $L = \frac{\lambda}{4}$.

Give your answer in terms of A.

The following symbols may be useful: `A`

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