Roots and Iteration 1ii

A Level
P P P

It is given that $F(x) = 2 + \ln x$. The iteration $x_{n+1} = F(x_n)$ is to be used to find a root, α, of the equation $x = 2 + \ln x$.

Part A First 3 Terms

Taking $x_1 = 3.1$, find x_2, and x_3, giving your answers correct to 6 significant figures.

$x_2 =$

$x_3 =$

Part B Error

The error e_n is defined by $e_n = \alpha - x_n$. Given that $\alpha = 3.14619$ correct to 5 decimal places, and that $F'(\alpha) \approx \frac{e_3}{e_2}$, use the values of e_2 and e_3 to make an estimate of $F'(\alpha)$ correct to 3 significant figures. State the true value of $F'(\alpha)$ correct to 4 significant figures.

Give the estimate of $F'(\alpha)$ correct to 3 significant figures.

$F'(\alpha) \approx$

State the true value of $F'(\alpha)$ correct to 4 significant figures.

$F'(\alpha) =$

Part C Convergence

Illustrate the iteration by drawing a sketch of $y = x$ and $y = F(x)$, showing how the values of x_n approach α. State whether the convergence is of the 'staircase' or 'cobweb' type.

- Cobweb
- Staircase

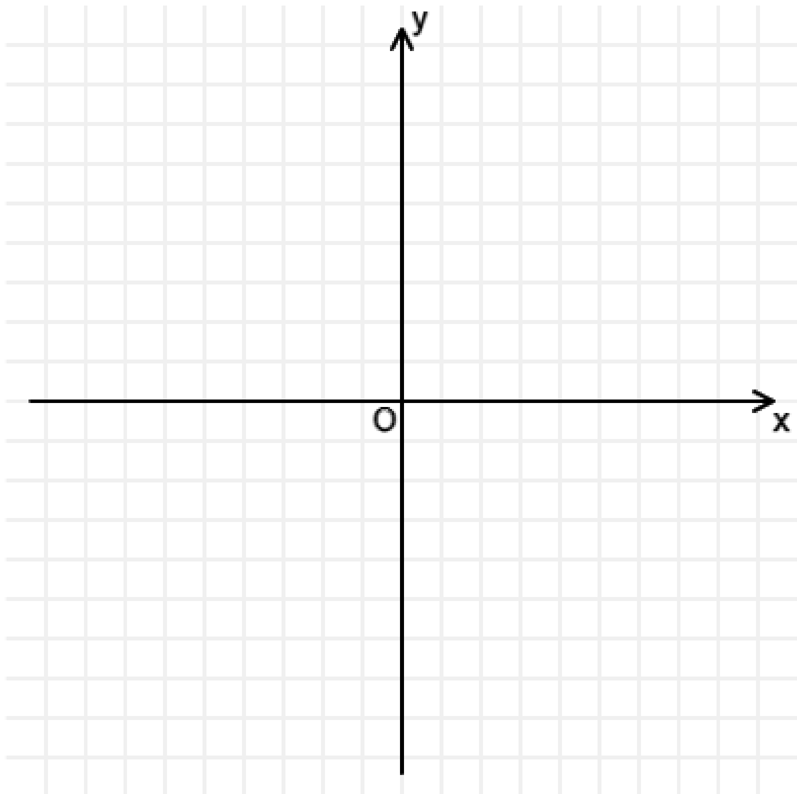
Roots and Iteration 3i

A Level
P P P

Part A Sketch

By sketching two suitable graphs on a single diagram, find the number of roots to the equation

$$14 - x^2 = 3\ln x.$$



From your sketch, state how many roots there are to the equation

$$14 - x^2 = 3\ln x$$

Part B Integer below α

Find by calculation the largest integer which is less than the root α.

Part C Iteration

Use the iterative formula $x_{n+1} = \sqrt{14 - 3\ln x_n}$, with a suitable starting value to find α correct to 3 significant figures.

Gameboard:

STEM SMART Double Maths 19 - Numerical Methods & Integration

Roots and Iteration 1i

A Level

P P P

It is required to solve the equation $f(x) = \ln(4x - 1) - x = 0$.

Part A Root existence

Show that the equation $f(x) = 0$ has two roots, α and β, such that $0.5 < \alpha < 1$ and $1 < \beta < 2$.

We find that $f(0.5) =$ ____, $f(1) =$ ____ and $f(2) =$ ____.

Since there is a ____ between $f(0.5)$ and $f(1)$, there must be a root α such that $0.5 < \alpha < 1$. As there is also a ____ between $f(1)$ and $f(2)$, there must be a root β such that $1 < \beta < 2$.

Items:

1.61 | -0.303 | 0.5 | -0.0541 | difference | change of sign | 1.95 | change of value | 0.0986 | -0.5 | 1.099 | 0.109 | 1

Part B Iteration with $g(x)$

Let $g(x) = \ln(4x - 1)$. Use the iterative formula $x_{r+1} = g(x_r)$ with $x_0 = 1.8$ to find x_1, x_2, and x_3, correct to 5 decimal places.

$x_1 =$ ☐

$x_2 =$ ☐

$x_3 =$ ☐

Continue the iterative process with $x_{r+1} = g(x_r)$ to find β correct to 3 decimal places.

$\beta =$ ☐

Part C New rearrangement $h(x)$

The equation $f(x) = 0$ can be rearranged into the form

$$x = h(x) = \frac{e^{ax} + b}{c}$$

where a, b and c are constants. Find $h(x)$.

The following symbols may be useful: `e, h, x`

Part D Iteration with $h(x)$

Use the iterative formula $x_{r+1} = h(x_r)$ with $x_0 = 0.8$ to find α correct to 4 decimal places.

Part E Root finding analysis

Show that the iterative formula $x_{r+1} = g(x_r)$ will not find the value of α. Similarly, determine whether the iterative formula $x_{r+1} = h(x_r)$ will find the value of β.

The iterative formula $x_{r+1} = g(x_r)$ will not converge to a root if [ ] near that root.

For $g(x)$, differentiating we find that $g'(x) =$ [ ]. Using the value for α calculated in Part D, this gives $g'(\alpha) =$ [ ] > 1. Therefore the iterative formula $x_{r+1} = g(x_r)$ will not converge to α.

For $h(x)$, differentiating we find that $h'(x) =$ [ ]. Using the value for β calculated in Part B, $h'(\beta) =$ [ ] > 1. Therefore the iterative formula $x_{r+1} = h(x_r)$ will not converge to β.

Items:

$g'(x) > 1$ $g'(x) < 1$ $\frac{1}{4x}$ $\frac{e^x}{4}$ e^x $\frac{4}{4x-1}$ $\frac{1}{x}$ $|g'(x)| < 1$ $|g'(x)| > 1$ 0.443 1.62 1.87 6.47

1.23 $\frac{1}{4x-1}$ 1.77 $\frac{e^x+1}{4}$ 0.307

Part F Staircase diagrams

From the figures below, select the two figures that illustrate the iterations for $x_{r+1} = g(x_r)$ and $x_{r+1} = h(x_r)$.

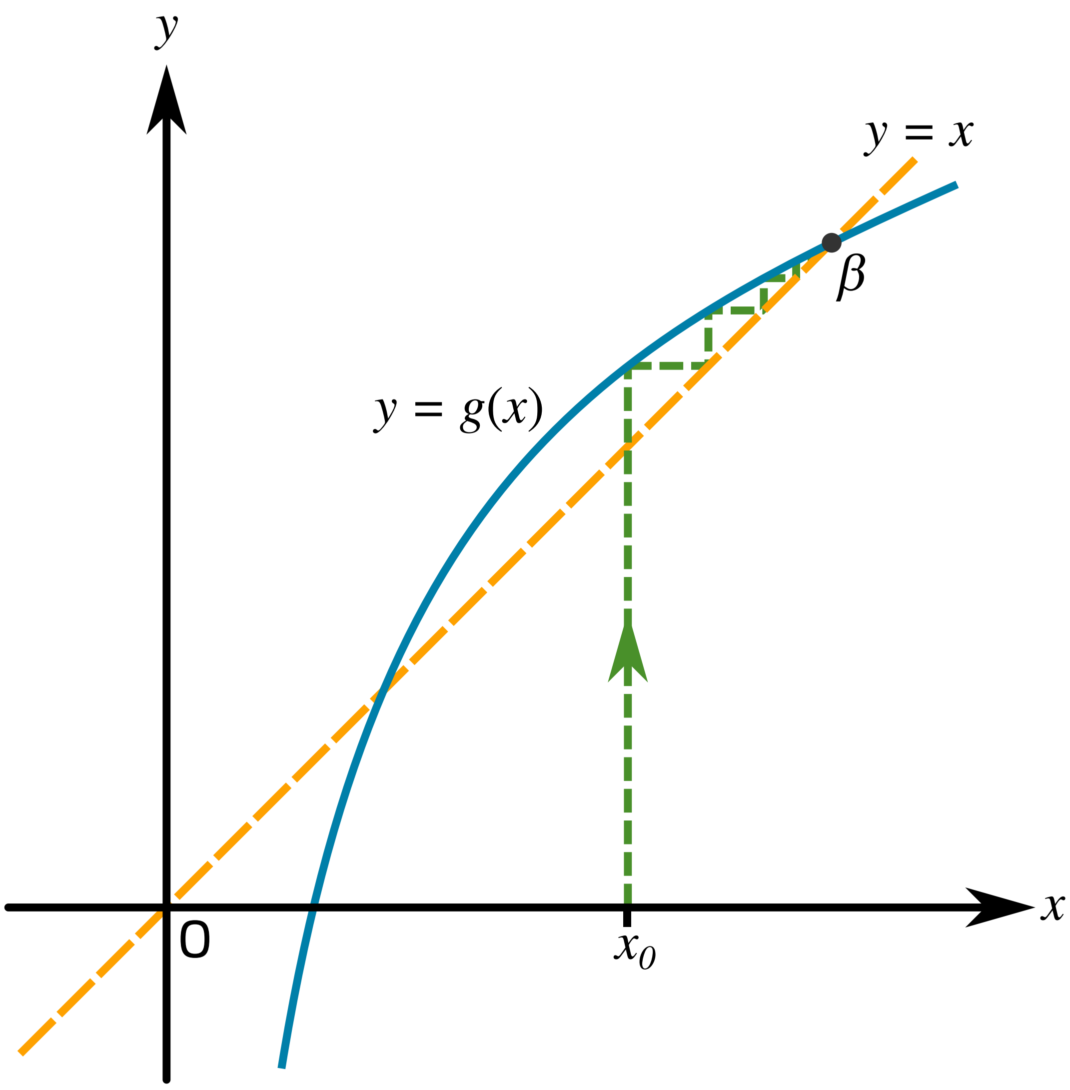


Figure 1: Graph of the iterative process for $x_{r+1} = g(x_r)$ towards β.

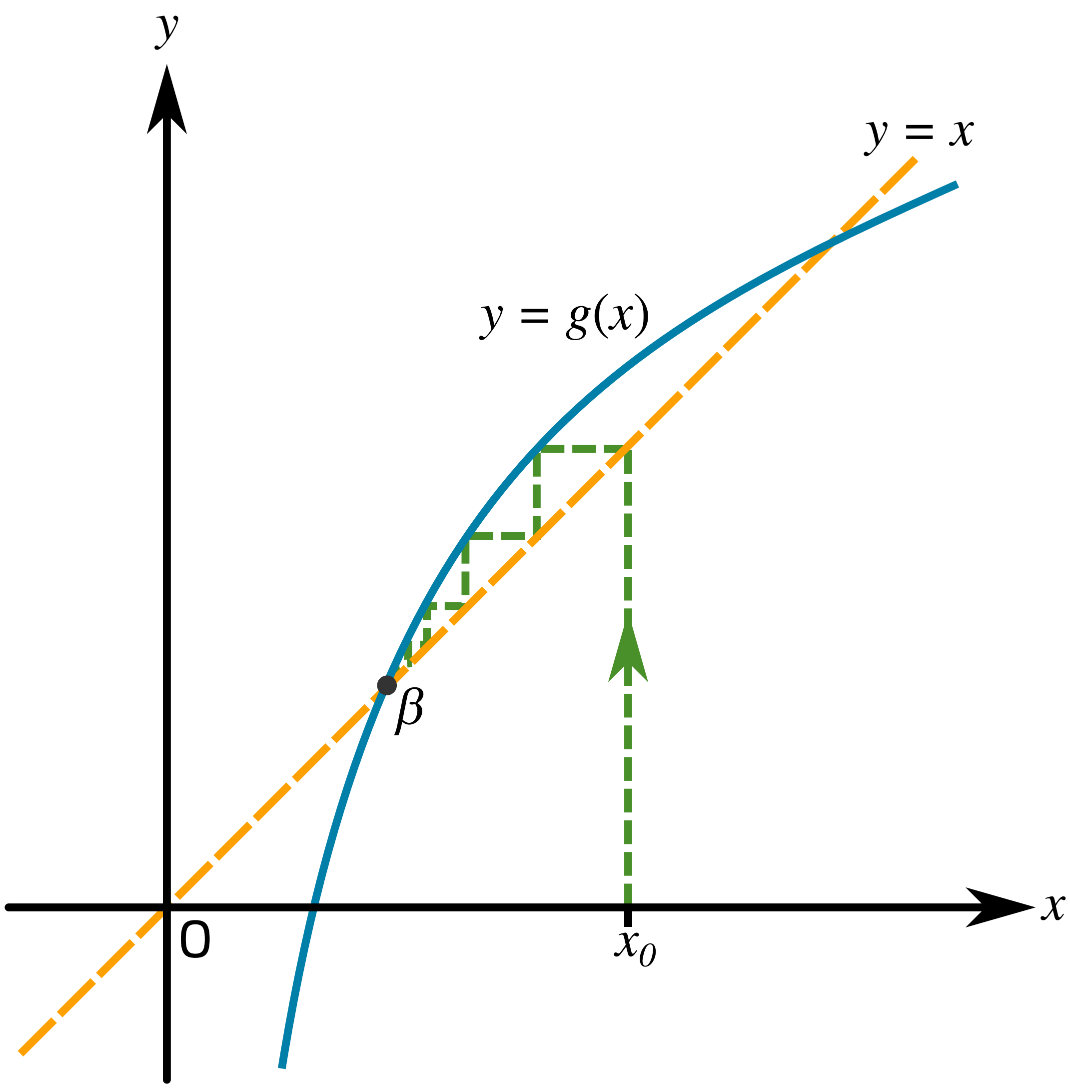


Figure 2: Graph of the iterative process for $x_{r+1} = g(x_r)$ towards β.

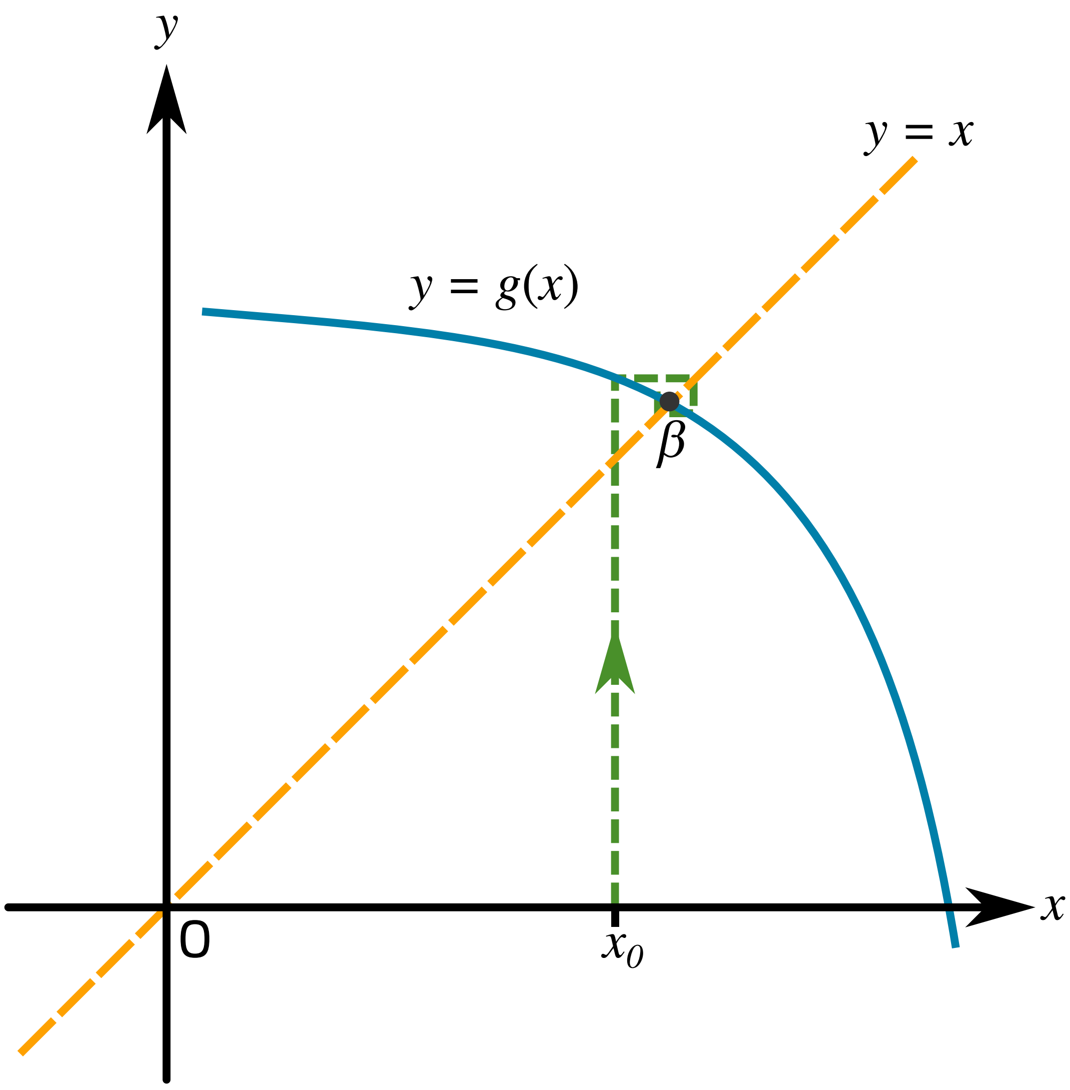


Figure 3: Graph of the iterative process for $x_{r+1} = g(x_r)$ towards β.

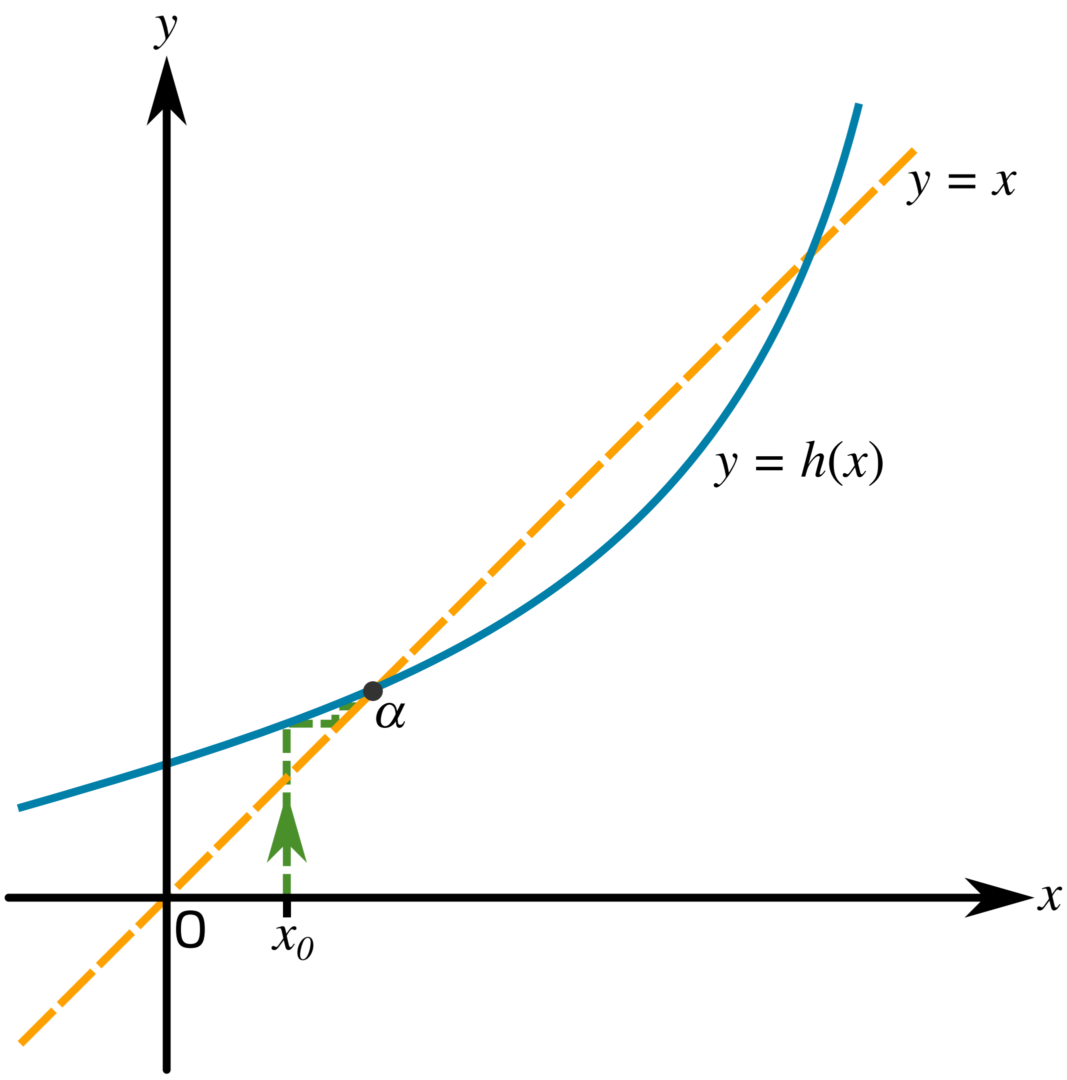


Figure 4: Graph of the iterative process for $x_{r+1} = h(x_r)$ towards α.

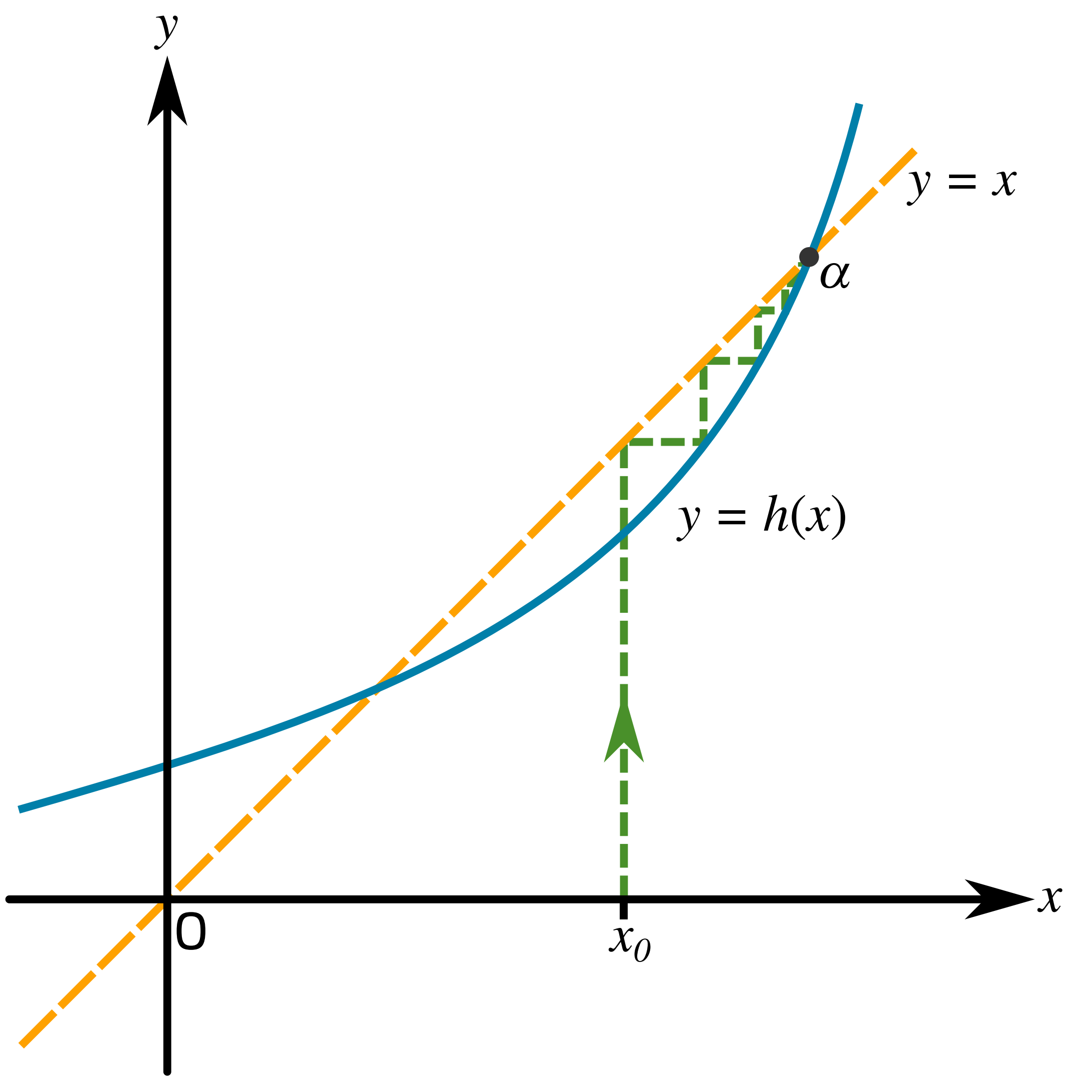


Figure 5: Graph of the iterative process for $x_{r+1} = h(x_r)$ towards α.

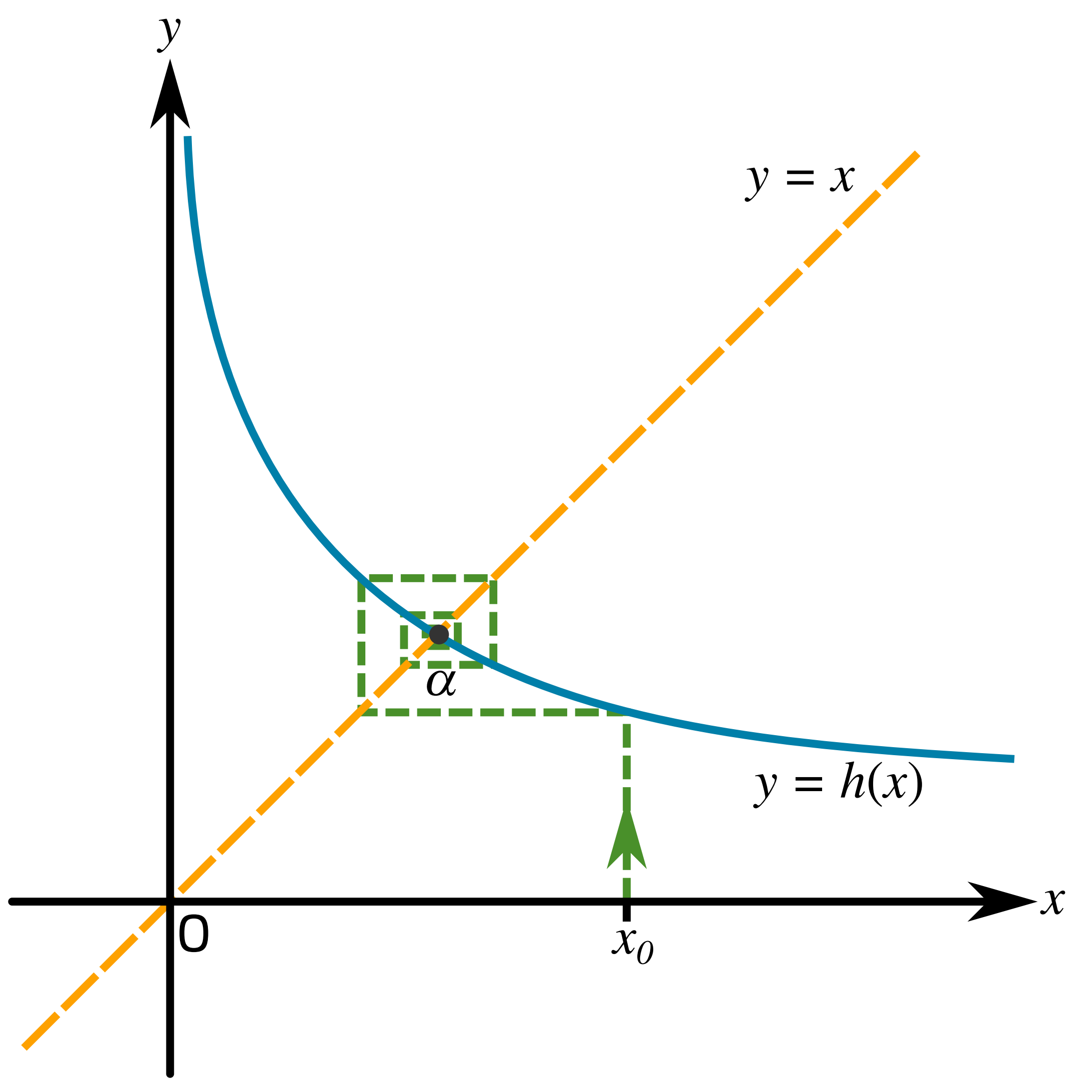


Figure 6: Graph of the iterative process for $x_{r+1} = h(x_r)$ towards α.

- [ ] **Figure 1**
- [ ] **Figure 2**
- [ ] **Figure 3**

- [ ] Figure 4
- [ ] Figure 5
- [ ] Figure 6

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Newton-Raphson Method 1ii

A Level
P P P

The diagram shows the curve with equation $y = xe^{-x} + 1$. The curve crosses the x-axis at $x = \alpha$.

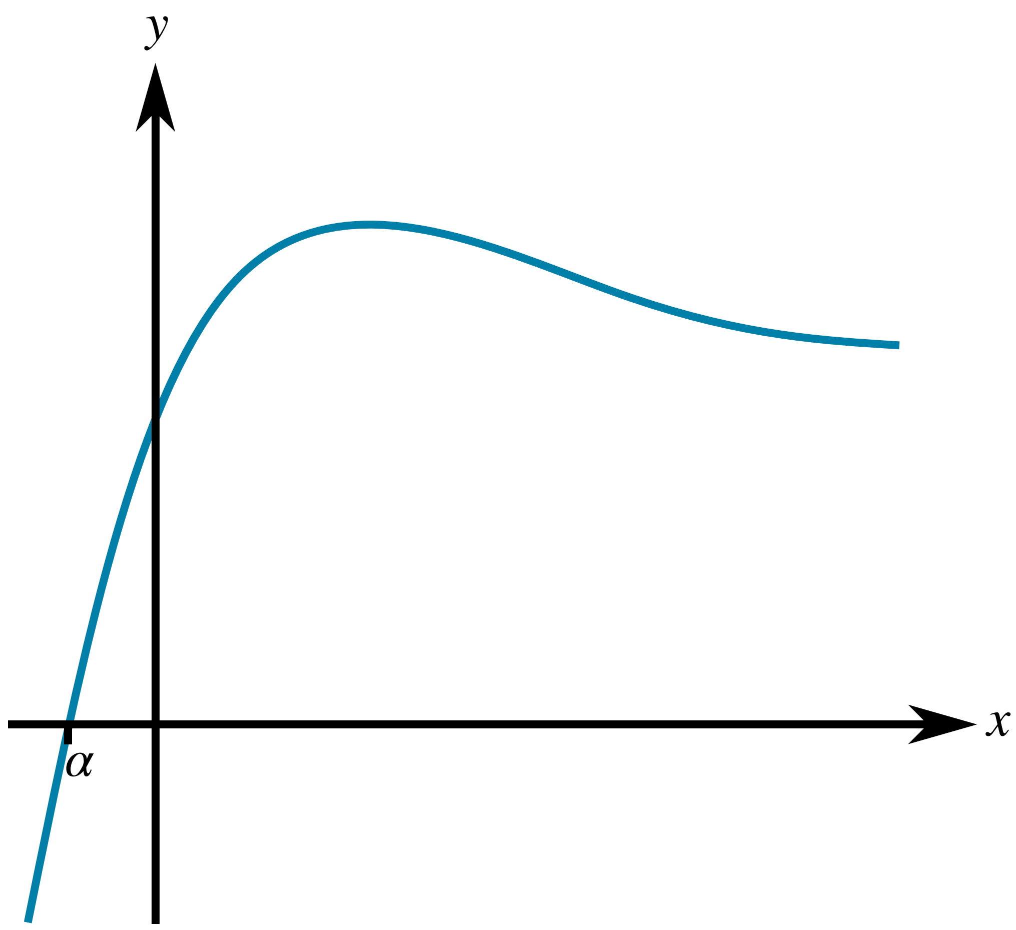

Figure 1: A sketch of the curve $y = xe^{-x} + 1$.

Part A x-coordinate of stationary point

Use differentiation to calculate the x-coordinate of the stationary point.

The following symbols may be useful: x

Part B Explain

α is to be found using the Newton-Raphson method, with $f(x) = xe^{-x} + 1$.

Explain why this method will not converge to α if an initial approximation x_1 is chosen such that $x_1 > 1$.

The iterative formula for the Newton-Raphson method is $x_{n+1} = x_n - \dfrac{f(x)}{f'(x)}$. For all values of x greater than 1, $f(x)$ is positive, and the ______ of $f(x)$ is negative (and close to ______). Hence, $-\dfrac{f(x)}{f'(x)}$ is positive and so x_{n+1} is larger than x_n. Visually, the x-intercepts of ______ at successive approximations will reach progressively ______ x-values and, hence, move further away from α.

Items:

1 | tangents | smaller | value | gradient | larger | 0 | -1 | intercept | normals

Part C Values

α is to be found using the Newton-Raphson method, with $f(x) = xe^{-x} + 1$.

Use this method, with a first approximation $x_1 = 0$, to find the next three approximations x_2, x_3, x_4. Give your answers to 4 sf where necessary.

$x_2 =$ ______

$x_3 =$ ______

$x_4 =$ ______

Find α correct to 3 significant figures.

$\alpha =$ ______

Gameboard:

STEM SMART Double Maths 19 - Numerical Methods & Integration

Newton-Raphson Method 4ii

A Level
P P P

It is given that $f(x) = 1 - \dfrac{7}{x^2}$.

Part A Approximations

Use the Newton-Raphson method, with a first approximation $x_1 = 2.5$, to find the next approximations x_2 and x_3 to a root of $f(x) = 0$. Give the answers correct to 7 significant figures.

$x_2 =$

$x_3 =$

Part B Root

The root of $f(x) = 0$ for which x_1, x_2, and x_3 are approximations is denoted by α. Write down the exact value of α.

The following symbols may be useful: `alpha`

Part C Error Function

The error function e_n is defined by $e_n = \alpha - x_n$. Find e_1, e_2 and e_3, giving your answers to 5 decimal places.

$e_1 =$

$e_2 =$

$e_3 =$

Part D Ratio $\frac{e_2^3}{e_1^2}$

Calculate $\frac{e_2^3}{e_1^2}$, giving your answer to 5 decimal places.

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Newton-Raphson Method 3i

A Level
P P P

The equation $x^3 - 5x + 3 = 0$ may be solved by the Newton-Raphson method. Successive approximations to the root are denoted by $x_1, x_2, \ldots, x_n, \ldots$

Part A Newton-Raphson Formula

Find the Newton-Raphson formula in the form $x_{n+1} = F(x_n)$, where $F(x_n)$ is a single fraction in its simplest form.

Give an expression for $F(x_n)$.

The following symbols may be useful: `x_n`

Part B The derivative $F'(x)$

Give an expression for $F'(x)$.

The following symbols may be useful: `Derivative(F, x)`, `x`

Part C $F'(x)$ when $x = \alpha$

Show that $F'(\alpha) = 0$, where α is any one of the roots of equation $x^3 - 5x + 3 = 0$. Then, fill in the blanks to complete the argument below.

To say that α is a root of the equation $x^3 - 5x + 3 = 0$ means that α is a value of x which satisfies this equation, i.e. $\alpha^3 - 5\alpha + 3 =$ [ ].

In part B it was found that $F'(x) =$ [ ]. Hence, we can write $F'(x) = g(x) \times$ [ ], where $g(x) = \frac{6x}{(3x^2-5)^2}$. When $x = \alpha$, this means $F'(\alpha) = g(\alpha) \times$ [ ]. Hence, as we know $\alpha^3 - 5\alpha + 3 = 0$, $F'(\alpha) = 0$.

Items:

$(x^3 - 5x + 3)$ $(\alpha^3 - 5\alpha + 3)$ $6x\frac{(x^3-5x+3)}{(3x^2-5)^2}$ x 0 $\frac{(3x^2-5)^2}{6x}$ 1 $\frac{6x}{(3x^2-5)^2}$

Part D Finding a root

Use the Newton-Raphson method to find the root of equation $x^3 - 5x + 3 = 0$ which is close to 2. Write down sufficient approximations to find the root correct to 5 significant figures.

Gameboard:

STEM SMART Double Maths 19 - Numerical Methods & Integration

Area: Numerical Integration 2ii

A Level

P P P

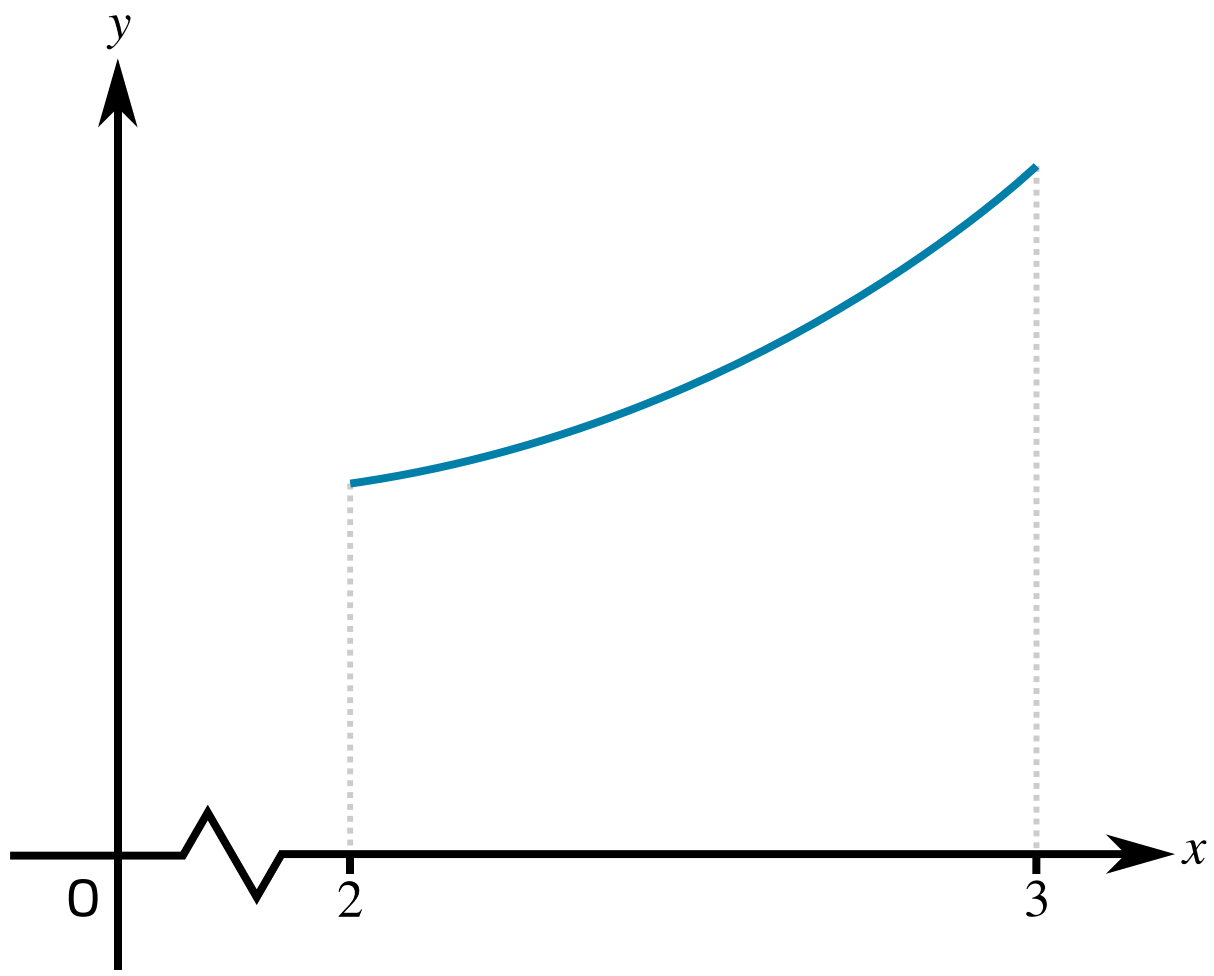

Figure 1: The curve with equation $y=\sqrt{1+x^3}$, for $2 \leqslant x \leqslant 3$.

Figure 1 shows the curve with equation $y=\sqrt{1+x^3}$, for $2 \leqslant x \leqslant 3$. The region under the curve between these limits has area A.

Part A **Bounding A**

Using the figure below, fill in the blanks to explain why $3 < A < \sqrt{28}$.

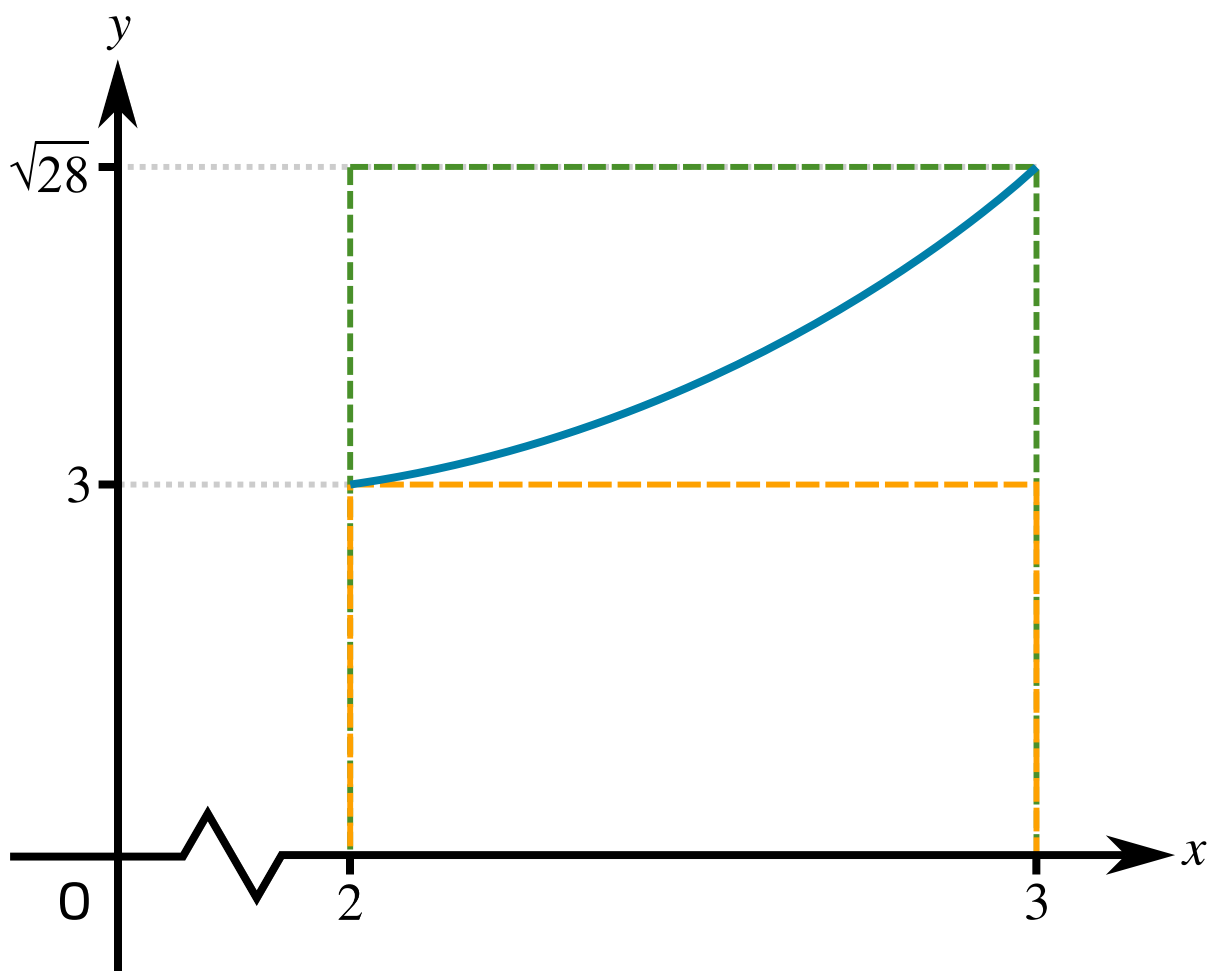

Figure 2: A diagram showing rectangles with areas which bound A.

Two rectangles are shown in **Figure 2**. Both rectangles begin on the x-axis and have width one. The area of the smaller rectangle, which lies ______ the curve, is ______. The area of the second rectangle, the top of which lies ______ the curve, is ______. The rectangles have areas which bound A, and hence:

$$3 < A < \sqrt{28}$$

Items:

below 3 $3\sqrt{28}$ 6 **above** $\sqrt{28}$

Part B Improved bounds

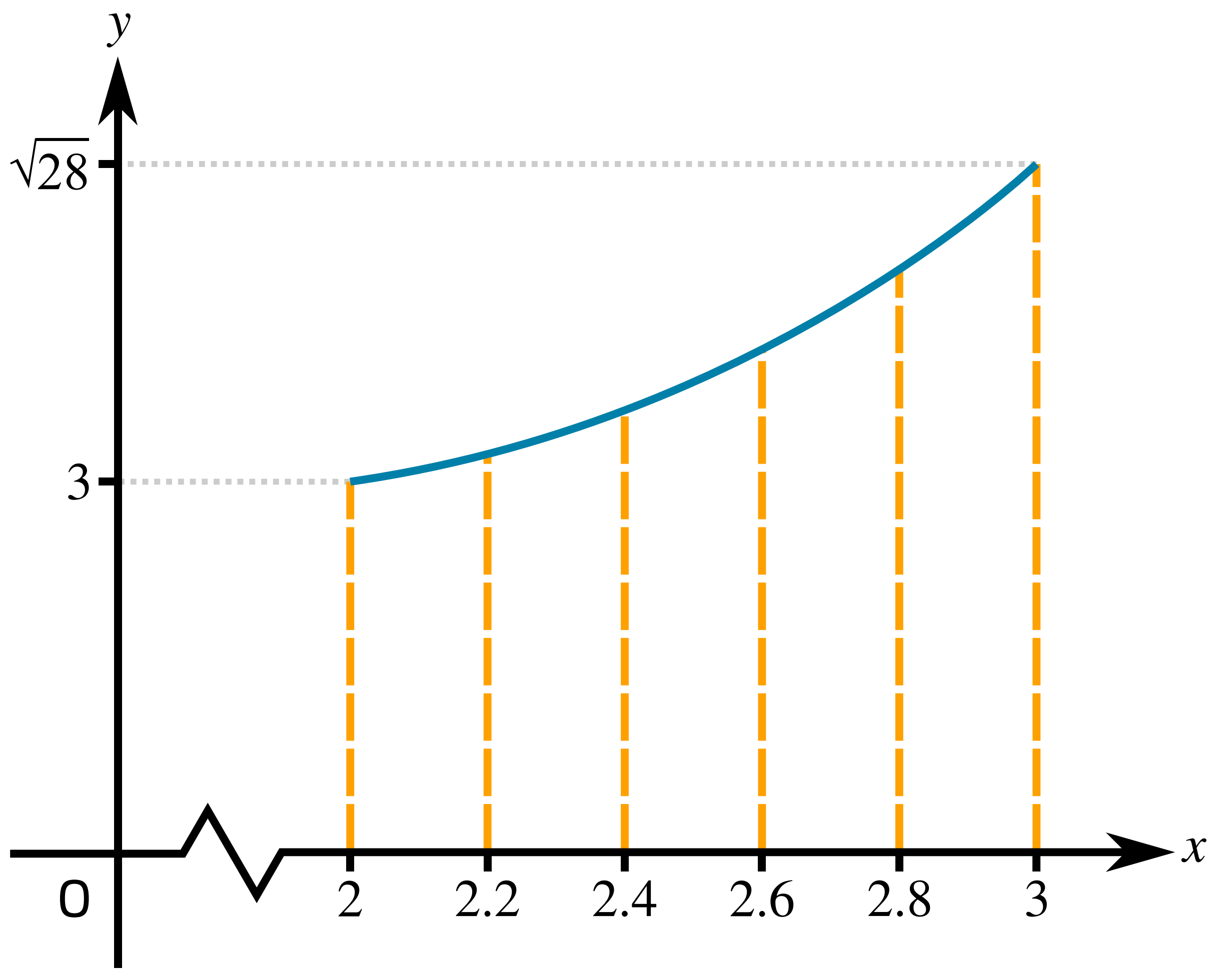

Figure 3: The curve with equation $y = \sqrt{1+x^3}$, for $2 \leqslant x \leqslant 3$, divided into 5 strips of equal width.

The region is divided into 5 strips, each of width 0.2. Use suitable rectangles with these strips to find improved lower and upper bounds for A. Give your answers to 3 significant figures.

lower bound for A:

upper bound for A:

Gameboard:

STEM SMART Double Maths 19 - Numerical Methods & Integration

Area: Numerical Integration 3i

A Level
P P P

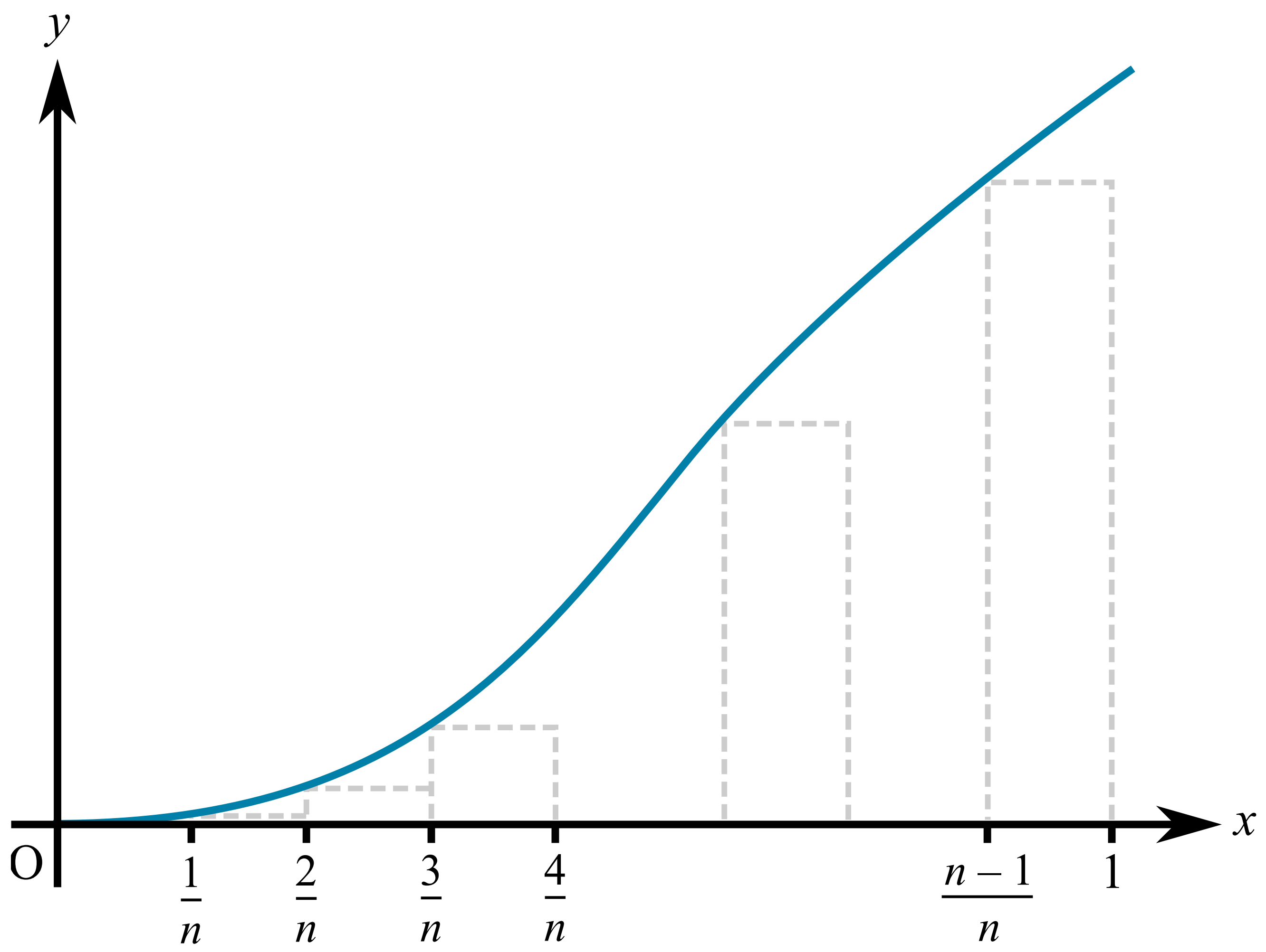

Figure 1: The diagram shows the curve $y = \mathrm{e}^{-\frac{1}{x}}$ for $0 < x \leqslant 1$.

Figure 1 shows the curve $y = \mathrm{e}^{-\frac{1}{x}}$ for $0 < x \leqslant 1$. A set of $(n-1)$ rectangles is drawn under the curve as shown.

Part A Lower bound

Fill in the blanks below to explain why a lower bound for $\int_0^1 \mathrm{e}^{-\frac{1}{x}}\mathrm{d}x$ can be expressed as:

$$\frac{1}{n} \times (\mathrm{e}^{-n} + \mathrm{e}^{-\frac{n}{2}} + \mathrm{e}^{-\frac{n}{3}} + \ldots + \mathrm{e}^{-\frac{n}{n-1}})$$

The integral $\int_0^1 \mathrm{e}^{-\frac{1}{x}}\mathrm{d}x$ is the area enclosed between the curve and the x-axis between $x = 0$ and $x = 1$.

The area under the curve completely covers the rectangles, so the total area of the rectangles, each of width _____, is _____ bound for $\int_0^1 \mathrm{e}^{-\frac{1}{x}}\mathrm{d}x$. The $(n - 1)$ rectangles have heights _____, $\mathrm{e}^{-\frac{n}{2}}$, ... $\mathrm{e}^{-\frac{n}{n-1}}$, and the total area of the rectangles is the sum of the areas of each individual rectangle. Therefore:

$$\frac{1}{n} \times (\mathrm{e}^{-n} + \mathrm{e}^{-\frac{n}{2}} + \mathrm{e}^{-\frac{n}{3}} + \ldots + \mathrm{e}^{-\frac{n}{n-1}}) \;\_\_\_\_\_\; \int_0^1 \mathrm{e}^{-\frac{1}{x}}\,\mathrm{d}x$$

Items:

a lower | an upper | $<$ | $=$ | $>$ | $\frac{1}{n}$ | n | $\mathrm{e}^{-\frac{1}{n}}$ | e^{-n}

Part B Upper bound

Using a set of 3 rectangles, write down a similar expression for an upper bound for $\int_0^1 \mathrm{e}^{-\frac{1}{x}}\mathrm{d}x$.

The following symbols may be useful: e

Part C Evaluate bounds

Evaluate the lower and upper bounds using $n = 4$, giving your answers correct to 3 significant figures.

lower bound for A: _____

upper bound for A: _____

Part D Difference between bounds

When $n \geqslant N$, the difference between the upper and lower bounds is less than 0.001. By expressing this difference in terms of n, find the least possible value of N.

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Trapezium Rule 3i

A Level

P P P

The value of $\int_0^8 \ln\left(3+x^2\right) \mathrm{d}x$ obtained by using the trapezium rule with four strips is denoted by A.

Part A Trapezium Rule

Find the value of A correct to 3 significant figures.

Part B Approximation of $\int_0^8 \ln\left(9+6x^2+x^4\right) \mathrm{d}x$

Write, in terms of A, an expression for an approximate value of $\int_0^8 \ln\left(9+6x^2+x^4\right) \mathrm{d}x$.

The following symbols may be useful: A

Part C Approximation of $\int_0^8 \ln\left(3e+ex^2\right) \mathrm{d}x$

Write, in terms of A, an expression for an approximate value of $\int_0^8 \ln\left(3e+ex^2\right) \mathrm{d}x$.

The following symbols may be useful: A

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Trapezium Rule 4i

A Level

P P P

Figure 1 shows the curve $y = 1.25^x$.

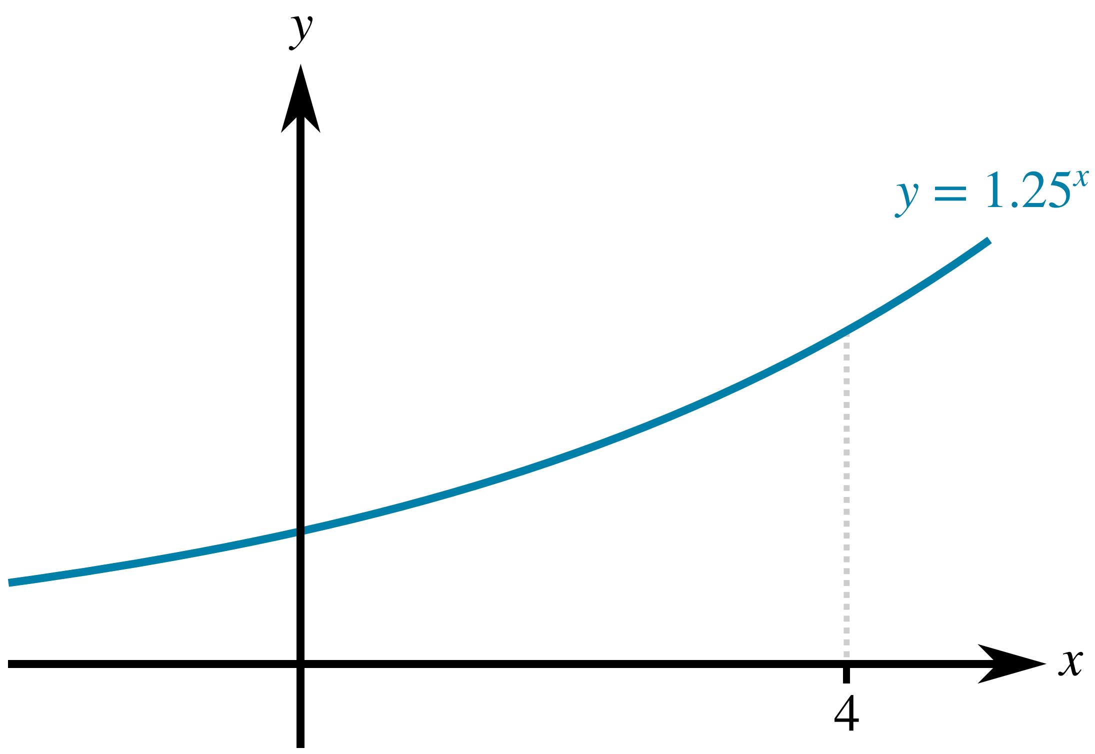

Figure 1: The curve $y = 1.25^x$.

Part A x-Coordinate

A point on the curve has y-coordinate 2 , calculate its x-coordinate, giving your answer to 3 significant figures.

Part B Derivative of y

Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of x.

The following symbols may be useful: `Derivative(y, x)`, `e`, `ln()`, `log()`, `x`

Part C Trapezium Rule

Use the trapezium rule with 4 intervals to estimate the area of the region bounded by the curve, the axes and the line $x = 4$. Give your answer to three significant figures.

Part D Overestimate or Underestimate?

Is the estimate found in part C an overestimate or an underestimate?

- Underestimate
- Overestimate

Part E More Accurate Estimates

How could the trapezium rule could be used to find a more accurate estimate of the shaded region?

- Use the same number of trapezia, but reduce the width of the trapezia. Narrower trapezia are a better fit to the curve as they reduce the surplus area between the tops of the trapezia and the curve, and so will yield a better approximation to the area.
- Double the number of trapezia, keeping their width the same. Using more trapezia always results in a better approximation.
- Use rectangles instead of trapezia. Their shape will better fit this particular curve, and so give a more accurate approximation.
- Use a larger number of (narrower) trapezia over the same interval. This will reduce the surplus area between the tops of the trapezia and the curve, and so give a more accurate approximation.