Area Between Two Curves 3ii

A Level

P P P

Figure 1 shows part of the curve $y = \frac{1}{x}$.

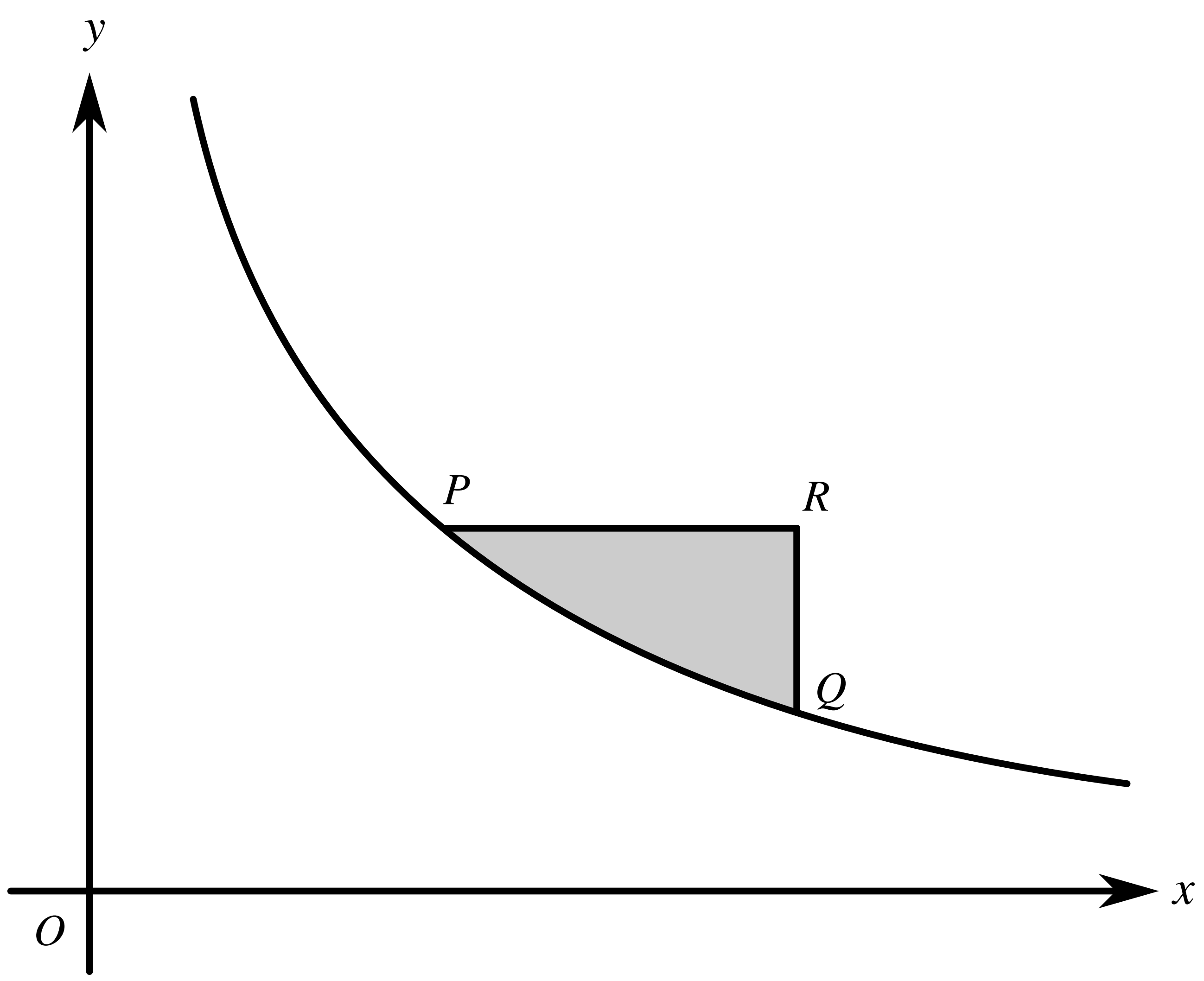

Figure 1: Part of the curve $y = \frac{1}{x}$.

The point P has coordinates $\left(a, \frac{1}{a}\right)$ and the point Q has coordinates $\left(2a, \frac{1}{2a}\right)$, where a is a positive constant. The point R is such that PR is parallel to the x-axis and QR is parallel to the y-axis. The region shaded in the diagram is bounded by the curve and by the lines PR and QR.

Find the area of the shaded region.

The following symbols may be useful: `a`, `e`

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Integration by Substitution 1

A Level Further A

P P P P P P

Part A Integrate $\sin(c\theta)$

Find $\int \sin(c\theta)\,\mathrm{d}\theta$, where c is a constant.

The following symbols may be useful: `C, c, k, theta`

Part B Integrate $e^{\alpha v}$

Find $\int \mathrm{e}^{\alpha v}\,\mathrm{d}v$, where α is a constant.

The following symbols may be useful: `alpha, e, k, v`

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Area Between Two Curves 1ii

A Level
P P P

Figure 1 shows the curve $y = \mathrm{e}^{3x} - 6\mathrm{e}^{2x} + 32$.

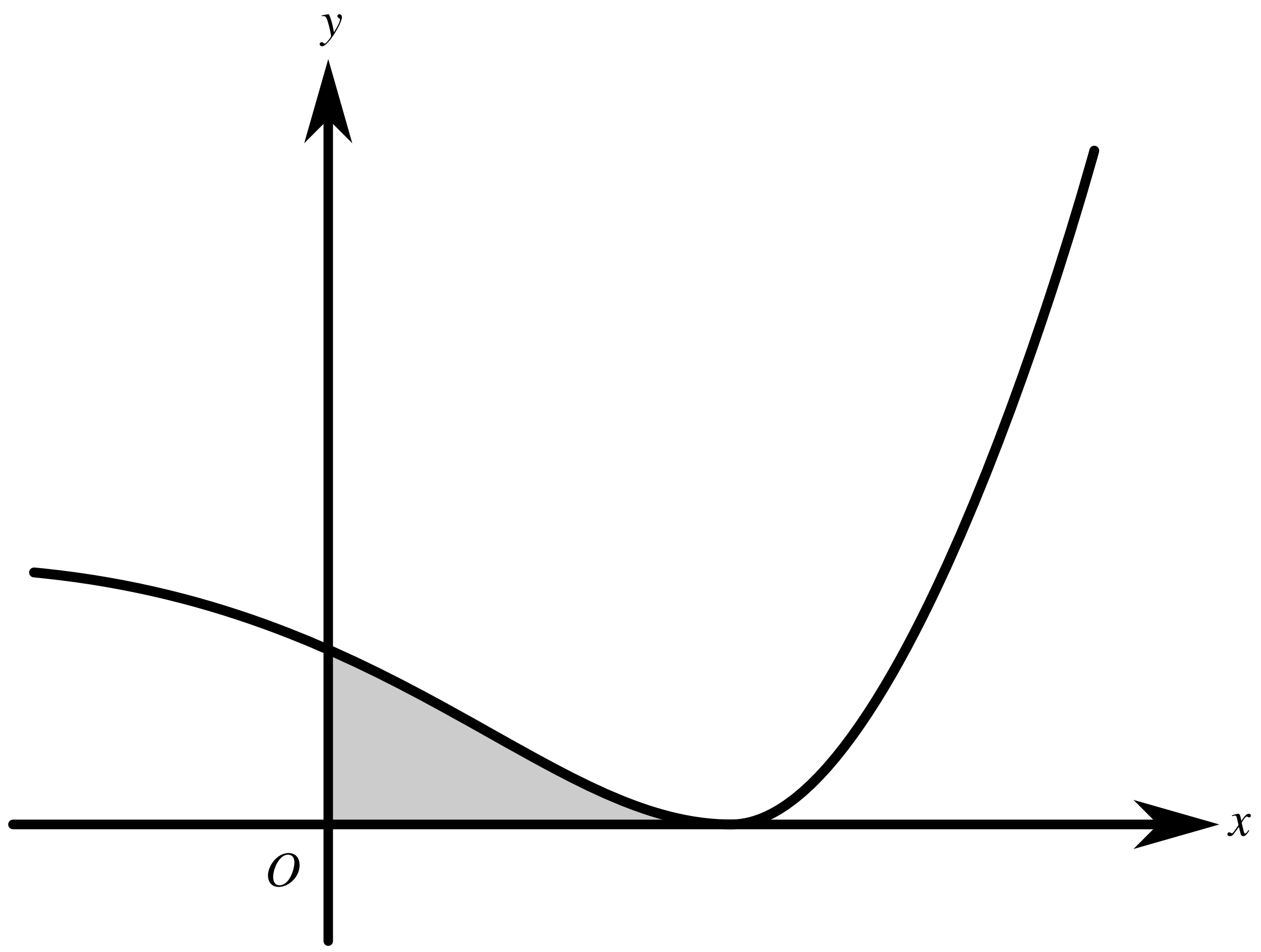

Figure 1: The curve $y = \mathrm{e}^{3x} - 6\mathrm{e}^{2x} + 32$.

Part A x-coordinate

Give the exact x-coordinate of the minimum point and verify that the y-coordinate of the minimum point is 0.

The following symbols may be useful: x

Part B Area of shaded region

Find the exact area of the shaded region enclosed by the curve and the coordinate axes.

The following symbols may be useful: e

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Physics. You work it out.

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Integration by Substitution 3

A Level Further A

P P P P P P

Part A Integrate $\frac{3}{(z+1)^2}$

Find $\int_0^2 \frac{3}{(z+1)^2}\mathrm{d}z$.

Part B Integrate $\frac{e^{-\alpha x}}{(1+e^{-\alpha x})^4}$

Find $\int \frac{\mathrm{e}^{-\alpha x}}{(1+\mathrm{e}^{-\alpha x})^4}\mathrm{d}x$, where α is a constant.

The following symbols may be useful: `alpha`, `e`, `k`, `x`

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Integration by Substitution 4

A Level Further A

P P P P P P

Part A Integrate $\frac{1}{b(x+a)}$

Find $\int_0^a \frac{1}{b(x+a)}\mathrm{d}x$, where a and b are constants.

The following symbols may be useful: `a, b, k, x`

Part B Integrate $\frac{x}{1+x^2}$

Find $\int_0^1 \frac{x}{1+x^2}\mathrm{d}x$.

The following symbols may be useful: `k, x`

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Integration by Substitution 2ii

A Level

P P P

Use the substitution $u = 1 + \ln(x) + x$ to find $\displaystyle\int \frac{3(x+1)(1-\ln(x)-x)}{x(1+\ln(x)+x)}\mathrm{d}x.$

The following symbols may be useful: `I, c, x`

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Integration by Parts 1

A Level Further A

P P P P P P

Part A Integrate $(s+1)e^{-\frac{s}{\alpha}}$

Find $\int_0^{\alpha}(s+1)e^{-\frac{s}{\alpha}}\,ds$, where α is a constant.

The following symbols may be useful: `alpha, e`

Part B Integrate $x\sin x$

Find $\int_0^{\frac{\pi}{2}} x\sin x\,dx$.

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Integration by Parts 2

A Level Further A

P P P P P P

Part A Integrate $z^2 \cos z$

Find $\int z^2 \cos z \mathrm{d}z$.

The following symbols may be useful: `k, z`

Part B Integrate $3t^2 \ln t$

Find $\int_1^2 3t^2 \ln t \mathrm{d}t$.

The following symbols may be useful: `ln(), t`

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Integration by Parts 4

A Level Further A

P P P P P P

Find $\int \frac{\ln(\sin x)}{\cos^2 x} \mathrm{d}x$.

The following symbols may be useful: `k, x`

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Home Gameboard Maths Calculus Integration Integrating 1/x

Integrating 1/x

A Level

The usual rule for integrating polynomials $\int x^n \, \mathrm{d}x = \frac{x^{n+1}}{n+1} + C$ breaks down for x^{-1}. In this question we explore the properties of this integral. This will give us insight into common $(\ln x)$ logarithms and also the exponential function (e^x).

Part A Logarithms

The fundamental property of a logarithm is that $\log ab = \log a + \log b$ regardless of your choice of base. Here we will define

$$L(y) = \int_1^y \frac{1}{x} \mathrm{d}x$$

and show that our function $L(y)$ does indeed have the property of a logarithm.

To work out $L(ab)$ we will break the integral into two sections

$$L(ab) = \int_1^{ab} \frac{1}{x} \mathrm{d}x = \int_1^a \frac{1}{x} \mathrm{d}x + \int_a^{ab} \frac{1}{x} \mathrm{d}x$$

so

$$L(ab) = L(a) + \int_a^{ab} \frac{1}{x} \mathrm{d}x.$$

Which substitution will be most suitable to express $\int_a^{ab} \frac{1}{x} \mathrm{d}x$ in terms of our function L?

- $z = \frac{x}{a}$
- $z = \frac{x}{b}$
- $z = ax$
- $z = bx$

Once the appropriate substitution has been made, we find that $\int_a^{ab} \frac{1}{x} \mathrm{d}x$ is equal to

- $\int_1^a \frac{1}{z} \mathrm{d}z = L(a)$
- $\int_1^b \frac{1}{z} \mathrm{d}z = L(b)$
- $\int_1^{ab} \frac{1}{z} \mathrm{d}z = L(ab)$
- $\int_1^b \frac{a}{z} \mathrm{d}z = a\, L(b)$

You have shown that $L(ab) = L(a) + L(b)$ and therefore that our function $L(z) = \int_1^z \frac{1}{z} \mathrm{d}z$ is some kind of logarithm.

Part B Logarithm base

As in the previous section, we write $L(a) = \int_1^a \frac{1}{x} \mathrm{d}x$. By definition, this means that $\frac{\mathrm{d}L}{\mathrm{d}x} = \frac{1}{x}$.

We already know that $L(x)$ has the properties of a logarithm. This means that if $y = L(x)$, then $x = g^y$ for some unknown constant g, which will be the base of the logarithms.

Remembering that $\frac{\mathrm{d}x}{\mathrm{d}y} = \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^{-1}$, we can combine the information above to show that $\frac{\mathrm{d}\,g^y}{\mathrm{d}y}$ is equal to

- y
- g^y
- $\frac{1}{g^y}$
- $\frac{1}{y}$

One of the defining features of the exponential function e^x is that $\frac{\mathrm{d}\,\mathrm{e}^x}{\mathrm{d}x} = \mathrm{e}^x$. The number e is also the base of the natural logarithms $\ln(x)$.

It follows that $g^x = \mathrm{e}^x$ and that accordingly $L(x) = \log_e x = \ln x$.

We therefore know (at least for positive x) that $\int \frac{1}{x} \mathrm{d}x = \ln x + C$.

Part C Expansion first term

In this section, we investigate the exponential function and how it might be evaluated.

We make an assumption that the function $y = \mathrm{e}^x$ can be written as polynomial with fixed co-efficients a_n:

$$\mathrm{e}^x = a_0 + a_1\,x + a_2\,x^2 + a_3\,x^3 + a_4\,x^4 + \cdots + a_n\,x^n + \cdots$$

What must be the value of a_0?

Part D Exponential expansion co-efficient

In this section, we continue to investigate the exponential function and how it might be evaluated.

We assume that the function $y = e^x$ can be written as polynomial with fixed co-efficients a_n:

$$e^x = a_0 + a_1\,x + a_2\,x^2 + a_3\,x^3 + a_4\,x^4 + \cdots + a_n\,x^n + \cdots$$

One property of the exponential function e^x is that $\dfrac{\mathrm{d}\,\mathrm{e}^x}{\mathrm{d}x} = \mathrm{e}^x$.

Using this information, write an expression for $\dfrac{a_n}{a_{n-1}}$.

The following symbols may be useful: `n`

Part E Exponential expansion first terms

In this section, we continue to investigate the exponential function and how it might be evaluated.

We assume that the function $y = \mathrm{e}^x$ can be written as polynomial with fixed co-efficients a_n:

$$\mathrm{e}^x = a_0 + a_1\,x + a_2\,x^2 + a_3\,x^3 + a_4\,x^4 + \cdots + a_n\,x^n + \cdots$$

Use the answers to the previous questions to write the expansion of e^x up to and including the x^4 term. Do **not** use factorial notation in your answer (write 24 rather than $4!$).

The following symbols may be useful: `e`, `x`

Part F Value of e

Use your expansion up to and including the x^4 term from the last question to calculate the value of e to three significant figures.

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