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Home Maths Newton-Raphson Method 3ii

Newton-Raphson Method 3ii

A Level

P P P

It is given that $f(x) = x^2 - \arctan x$ and that $\frac{d}{dx}(\arctan x) = \frac{1}{1+x^2}$.

Part A Interval containing the root

Explain why the equation $f(x) = 0$ has a root in the interval $0.8 < x < 0.9$.

The value of $f(x)$ when $x = 0.8$ is ____, and the value of $f(x)$ when $x = 0.9$ is ____. These values of $f(x)$ have ____. Hence, as $f(x)$ is a continuous function, there is a value of x in the interval $0.8 < x < 0.9$ for which $f(x) = 0$.

A root of an equation is a value of x for which $f(x) =$ ____. Hence, there is a root of $f(x)$ in the interval $0.8 < x < 0.9$.

Items:

1.348 | **different signs** | **the same sign** | 0.0772 | -0.0347 | 0.0771 | 1 | 0

Part B Find the root

Use the Newton-Raphson method, with a first approximation 0.8 , to find the next approximation to this root. Give your answer correct to 3 significant figures.

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Home Maths Newton-Raphson Method 1ii

Newton-Raphson Method 1ii

A Level
P P P

The diagram shows the curve with equation $y = xe^{-x} + 1$. The curve crosses the x-axis at $x = \alpha$.

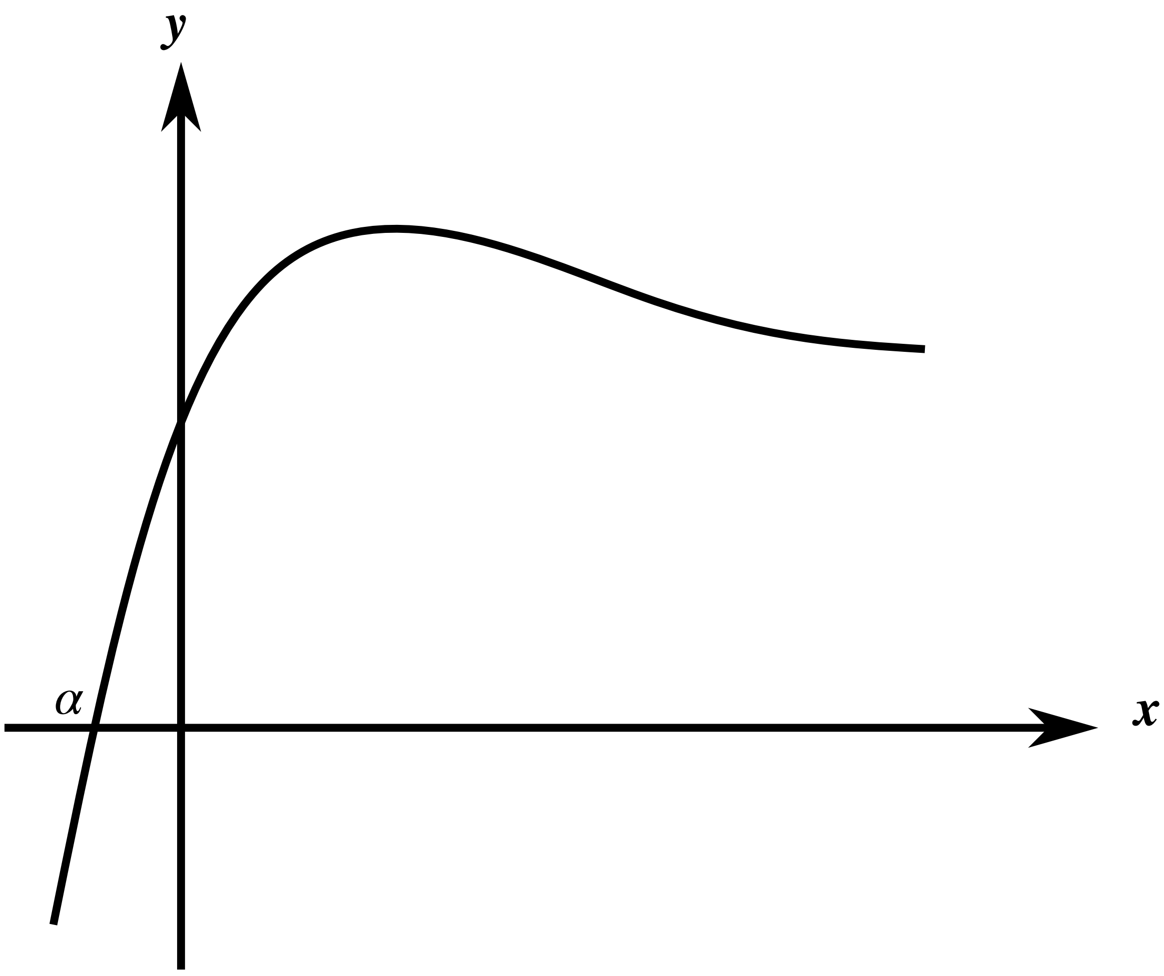


Figure 1: A sketch of the curve $y = xe^{-x} + 1$.

Part A x-coordinate of stationary point

Use differentiation to calculate the x-coordinate of the stationary point.

The following symbols may be useful: x

Part B Explain

α is to be found using the Newton-Raphson method, with $f(x) = xe^{-x} + 1$.

Explain why this method will not converge to α if an initial approximation x_1 is chosen such that $x_1 > 1$.

The iterative formula for the Newton-Raphson method is $x_{n+1} = x_n - \dfrac{f(x)}{f'(x)}$. For all values of x greater than 1, $f(x)$ is positive, and the ____ of $f(x)$ is negative and close to ____. Hence, $-\dfrac{f(x)}{f'(x)}$ is positive and so x_{n+1} is larger than x_n. Visually, the x-intercepts of ____ at successive approximations will reach progressively ____ x-values and, hence, move further away from α.

Items:

larger | gradient | normals | value | 0 | 1 | intercept | smaller | -1 | tangents

Part C Values

α is to be found using the Newton-Raphson method, with $f(x) = xe^{-x} + 1$.

Use this method, with a first approximation $x_1 = 0$, to find the next three approximations x_2, x_3, x_4. Find α correct to 3 significant figures.

Write down x_2.

Write down x_3, correct to 4 significant figures.

Write down x_4, correct to 4 significant figures.

Find α correct to 3 significant figures.

Physics. You work it out.

Home Maths Roots and Iteration 3i

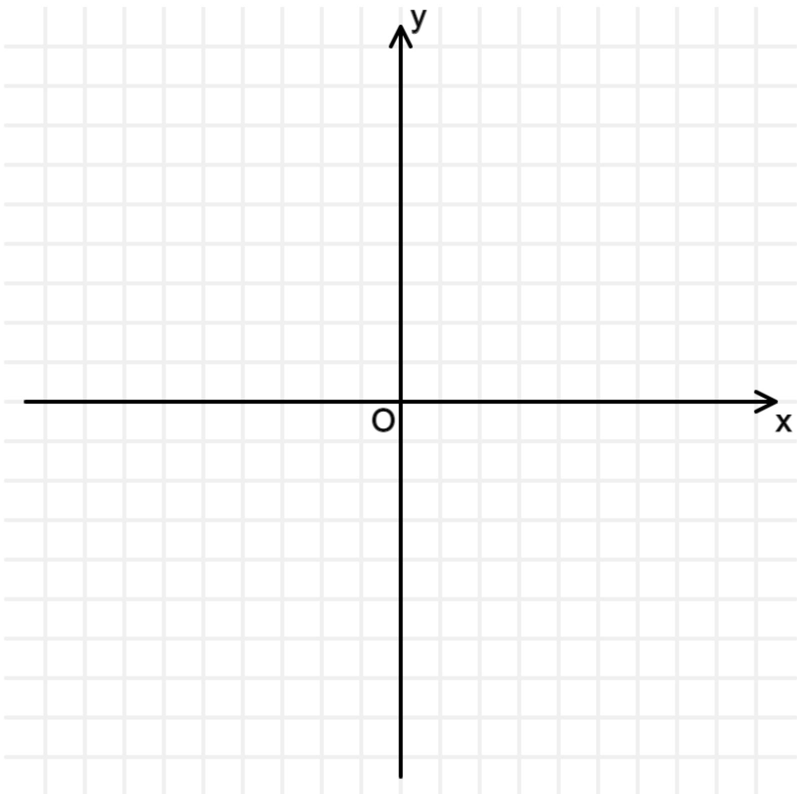
Roots and Iteration 3i

A Level
P P P

Part A Sketch

By sketching two suitable graphs on a single diagram, find the number of roots to the equation

$$14 - x^2 = 3\ln x.$$



From your sketch, state how many roots there are to the equation

$$14 - x^2 = 3\ln x$$

Part B Integer below α

Find by calculation the largest integer which is less than α.

Part C Iteration

Use the iterative formula $x_{n+1} = \sqrt{14 - 3\ln x_n}$, with a suitable starting value to find α correct to 3 significant figures.

Roots and Iteration 1i

A Level

It is required to solve the equation $f(x) = \ln(4x - 1) - x = 0$.

Part A Root existence

Show that the equation $f(x) = 0$ has two roots, α and β, such that $0.5 < \alpha < 1$ and $1 < \beta < 2$.

We find that $f(0.5) =$ ____, $f(1) =$ ____ and $f(2) =$ ____.

Since there is a ____ between $f(0.5)$ and $f(1)$, there must be a root α such that $0.5 < \alpha < 1$. As there is also a ____ between $f(1)$ and $f(2)$, there must be a root β such that $1 < \beta < 2$.

Items:

0.109 | 0.5 | 0.0986 | 1.61 | 1 | 1.099 | -0.0541 | 1.95 | **difference** | **change of sign** | **change of value** | -0.5 | -0.303

Part B Iteration with $g(x)$

Let $g(x) = \ln(4x - 1)$. Use the iterative formula $x_{r+1} = g(x_r)$ with $x_0 = 1.8$ to find x_1, x_2, and x_3, correct to 5 decimal places.

Give x_1

Give x_2

Give x_3

Continue the iterative process with $x_{r+1} = g(x_r)$ to find β correct to 3 decimal places.

Part C New rearrangement $h(x)$

The equation $f(x) = 0$ can be rearranged into the form

$$x = h(x) = \frac{e^{ax} + b}{c}$$

where a, b and c are constants. Find $h(x)$.

The following symbols may be useful: `e, h, x`

Part D Iteration with $h(x)$

Use the iterative formula $x_{r+1} = h(x_r)$ with $x_0 = 0.8$ to find α correct to 4 decimal places.

Part E Root finding analysis

Show that the iterative formula $x_{r+1} = g(x_r)$ will not find the value of α. Similarly, determine whether the iterative formula $x_{r+1} = h(x_r)$ will find the value of β.

The iterative formula $x_{r+1} = g(x_r)$ will not converge to a root if ______ near that root.

For $g(x)$, differentiating we find that $g'(x) =$ ______. Using the value for α calculated in Part D, this gives $g'(\alpha) =$ ______ > 1. Therefore the iterative formula $x_{r+1} = g(x_r)$ will not converge to α.

For $h(x)$, differentiating we find that $h'(x) =$ ______. Using the value for β calculated in Part B, $h'(\beta) =$ ______ > 1. Therefore the iterative formula $x_{r+1} = h(x_r)$ will not converge to β.

Items:

$\frac{1}{x}$ | 0.443 | 1.62 | $g'(x) < 1$ | $g'(x) > 1$ | 0.307 | 1.87 | $|g'(x)| > 1$ | 6.47 | $\frac{4}{4x-1}$ | $\frac{e^x}{4}$ | $\frac{1}{4x}$ | $\frac{1}{4x-1}$ | $\frac{e^x+1}{4}$ | $|g'(x)| < 1$ | e^x | 1.77 | 1.23

Part F Staircase diagrams

From the figures below, select the two figures that illustrate the iterations for $x_{r+1} = g(x_r)$ and $x_{r+1} = h(x_r)$.

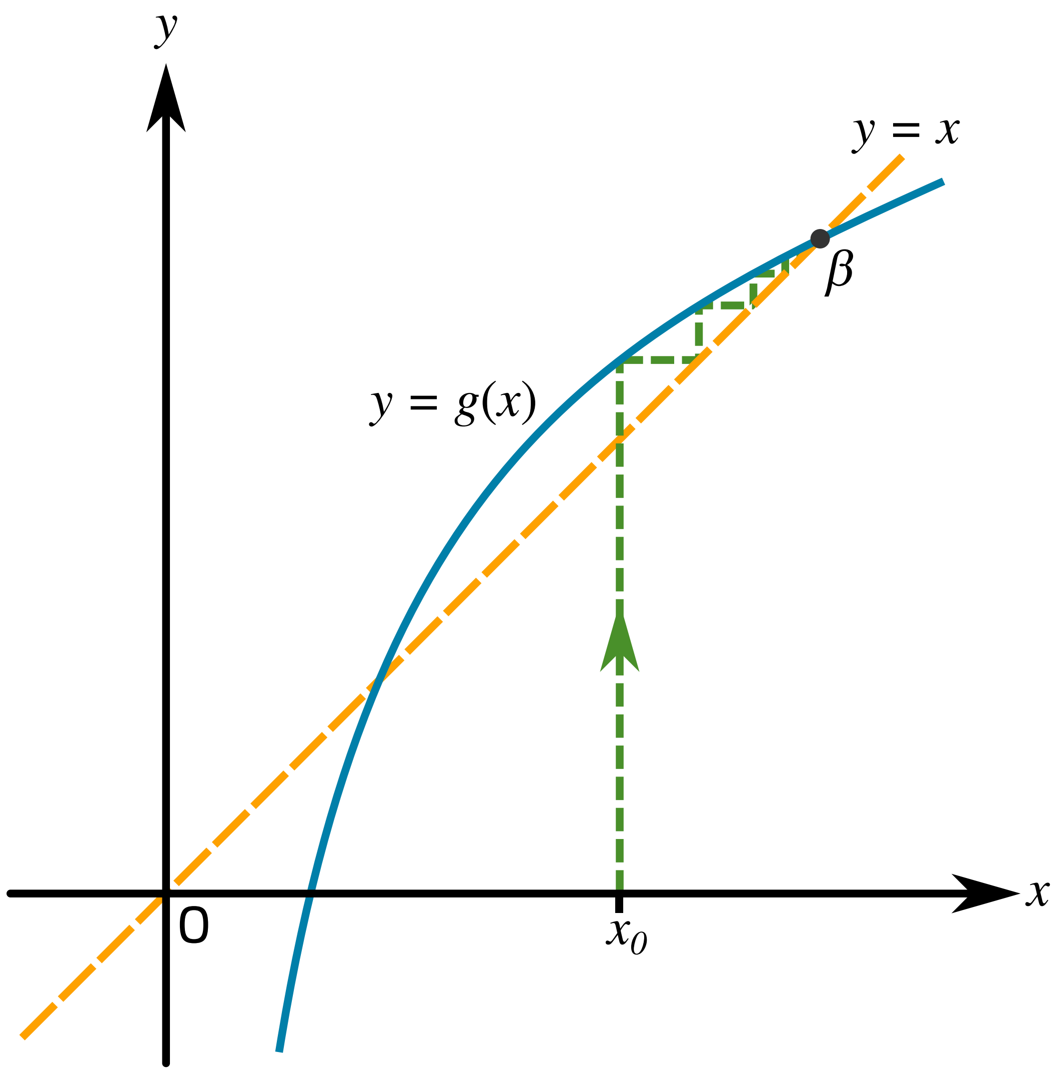


Figure 1: Graph of the iterative process for $x_{r+1} = g(x_r)$ towards β.

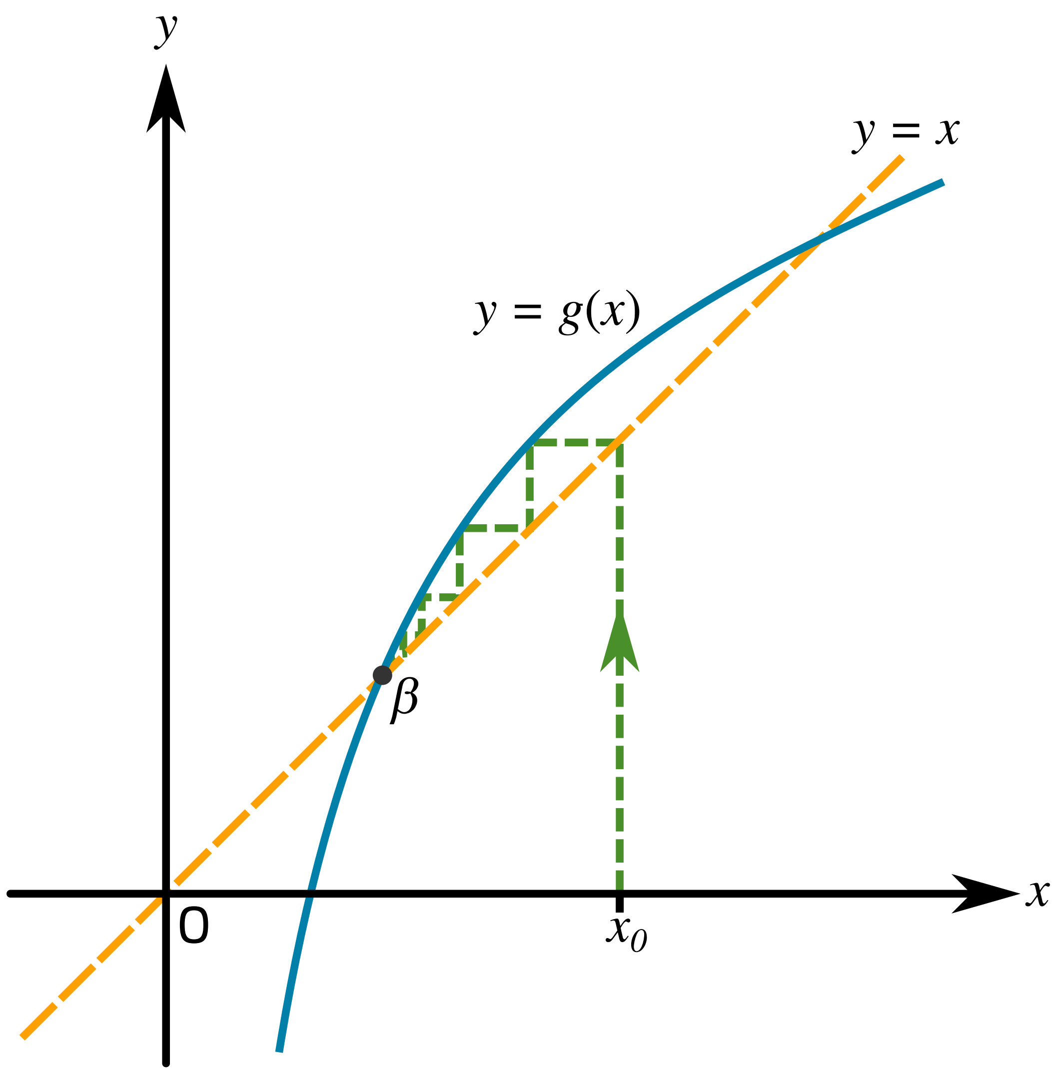


Figure 2: Graph of the iterative process for $x_{r+1} = g(x_r)$ towards β.

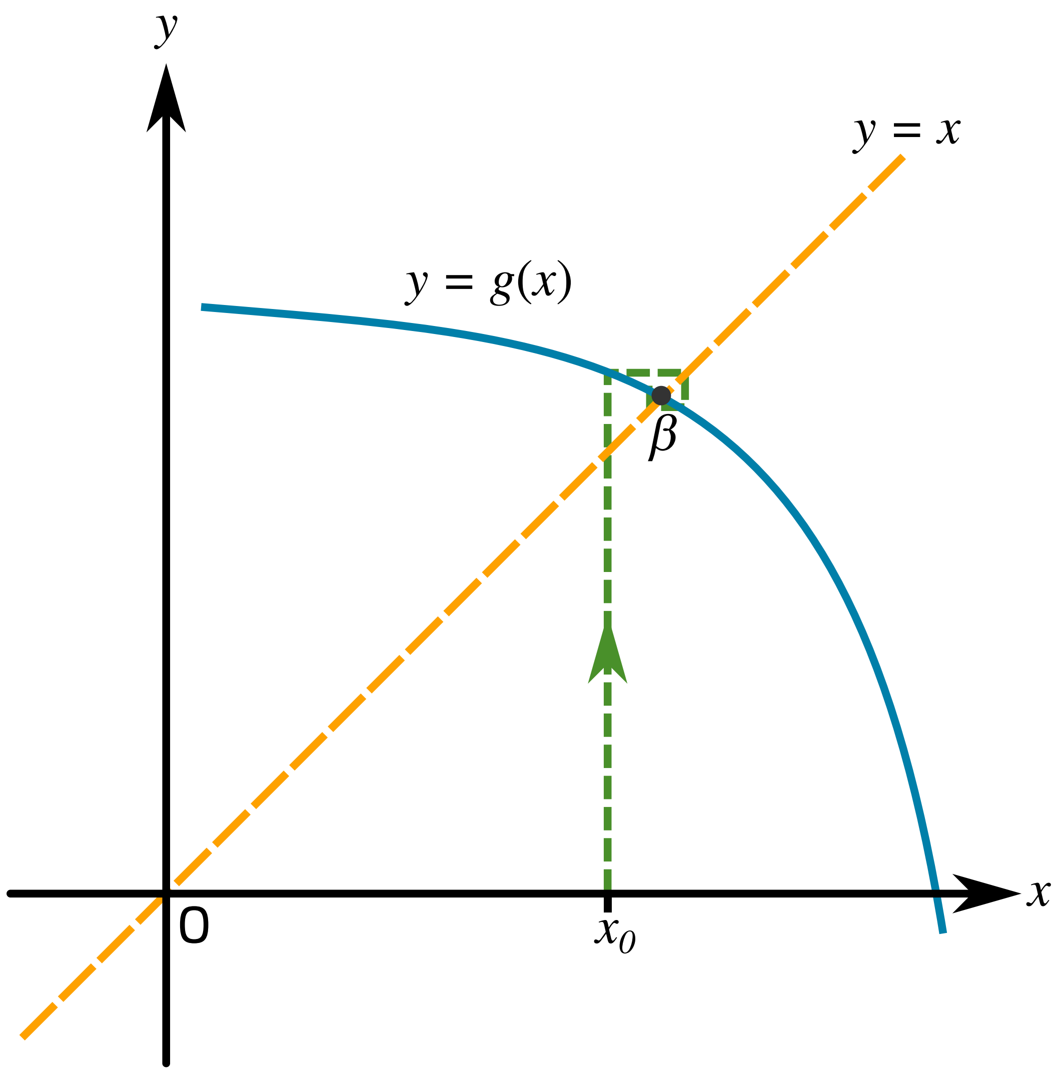


Figure 3: Graph of the iterative process for $x_{r+1} = g(x_r)$ towards β.

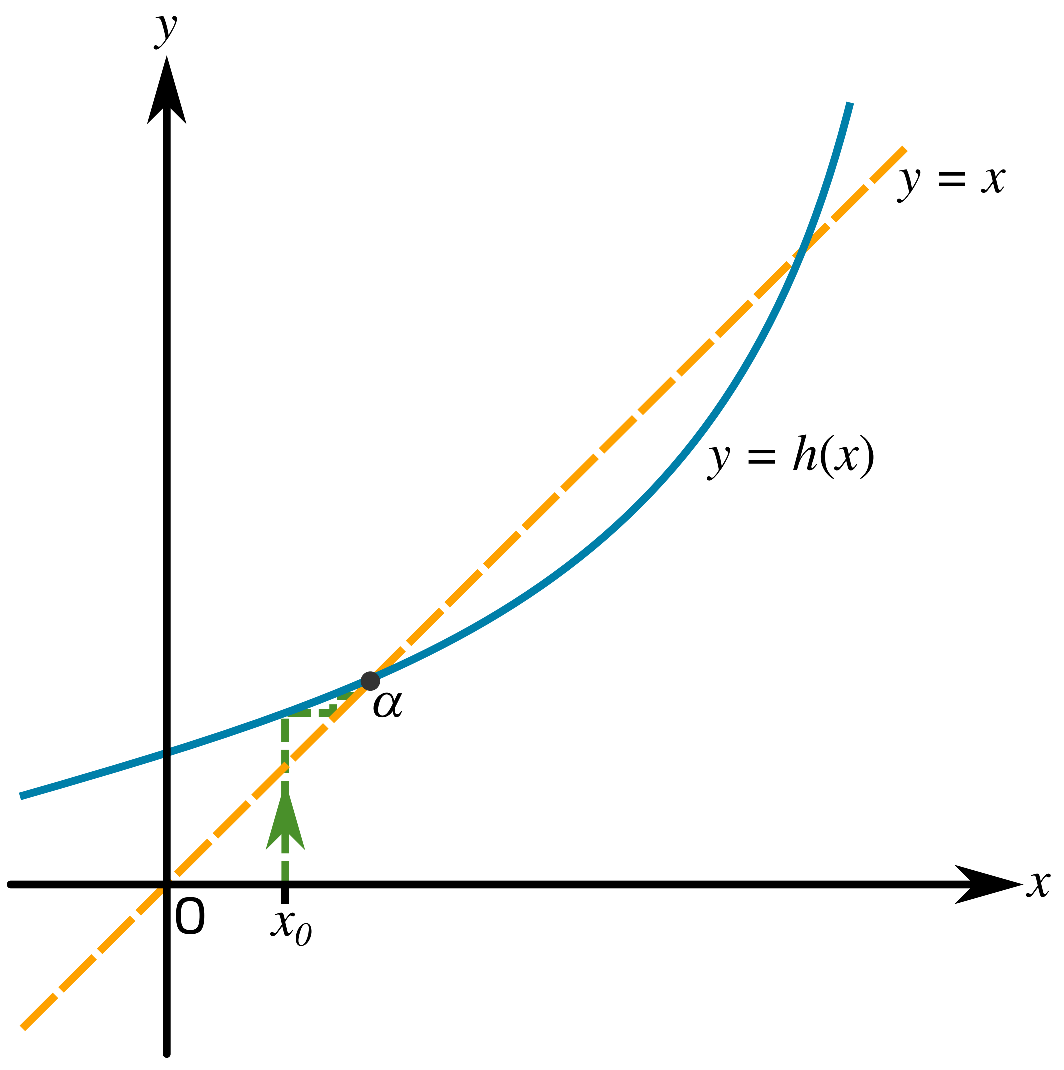


Figure 4: Graph of the iterative process for $x_{r+1} = h(x_r)$ towards α.

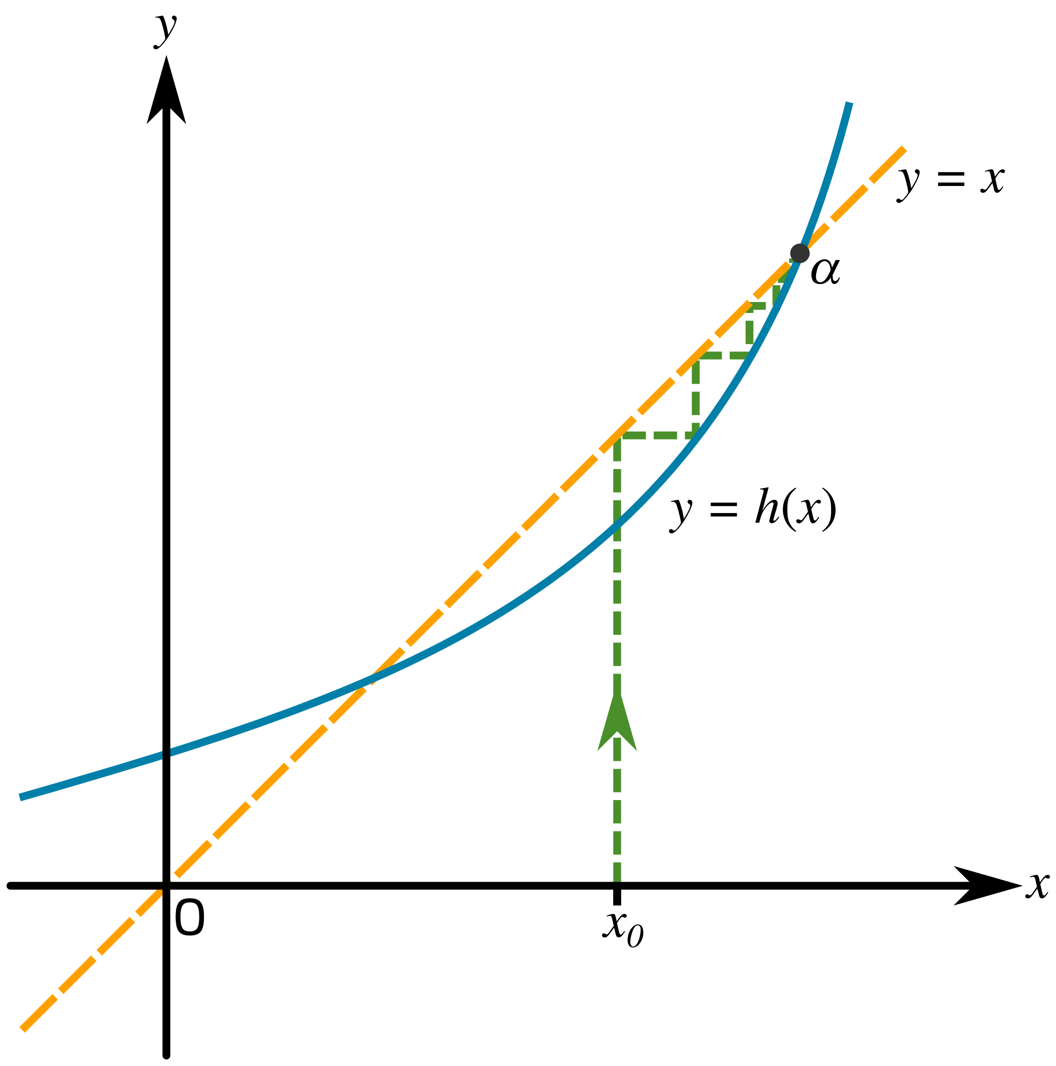


Figure 5: Graph of the iterative process for $x_{r+1} = h(x_r)$ towards α.

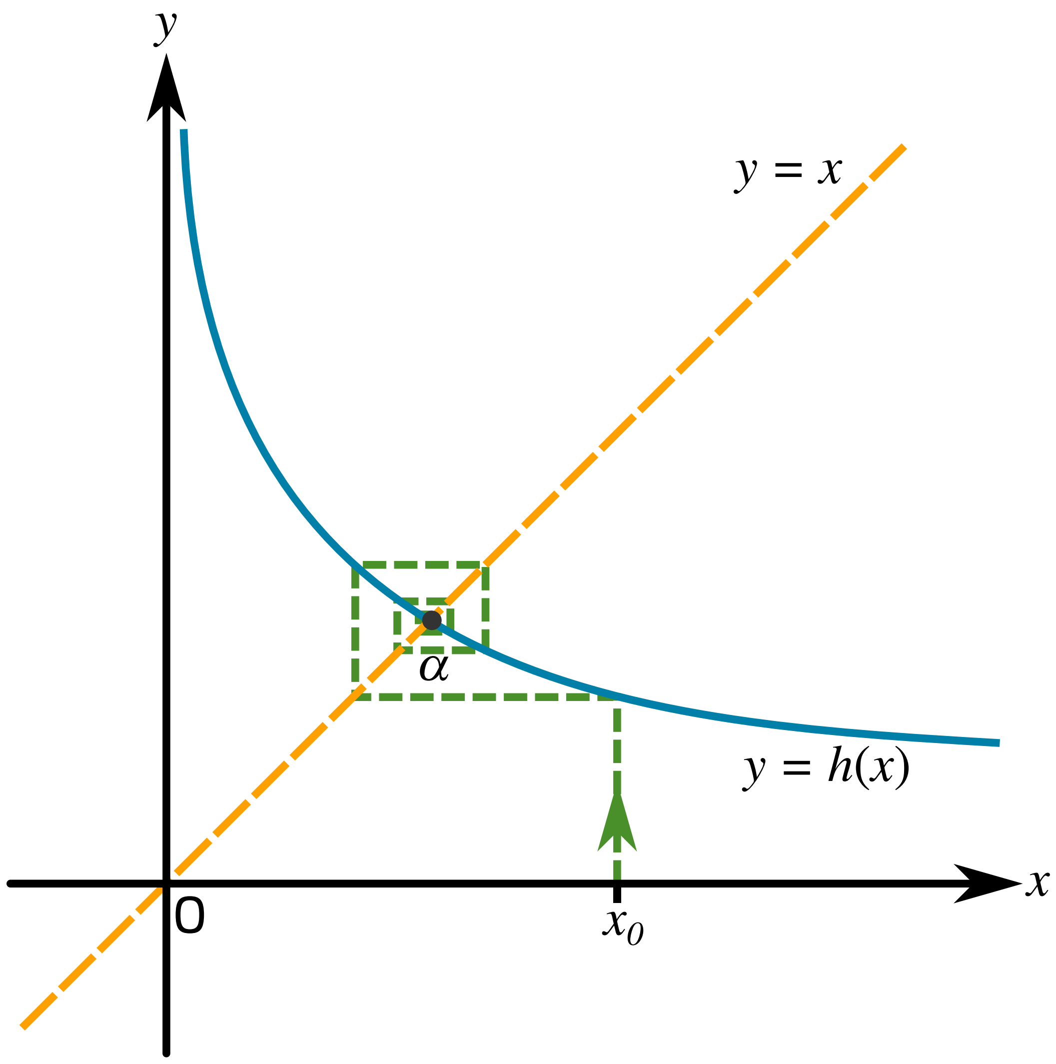


Figure 6: Graph of the iterative process for $x_{r+1} = h(x_r)$ towards α.

- [ ] Figure 1
- [ ] Figure 2
- [ ] Figure 3
- [ ] Figure 4
- [ ] Figure 5

Figure 6

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Area: Numerical Integration 2ii

A Level

P P P

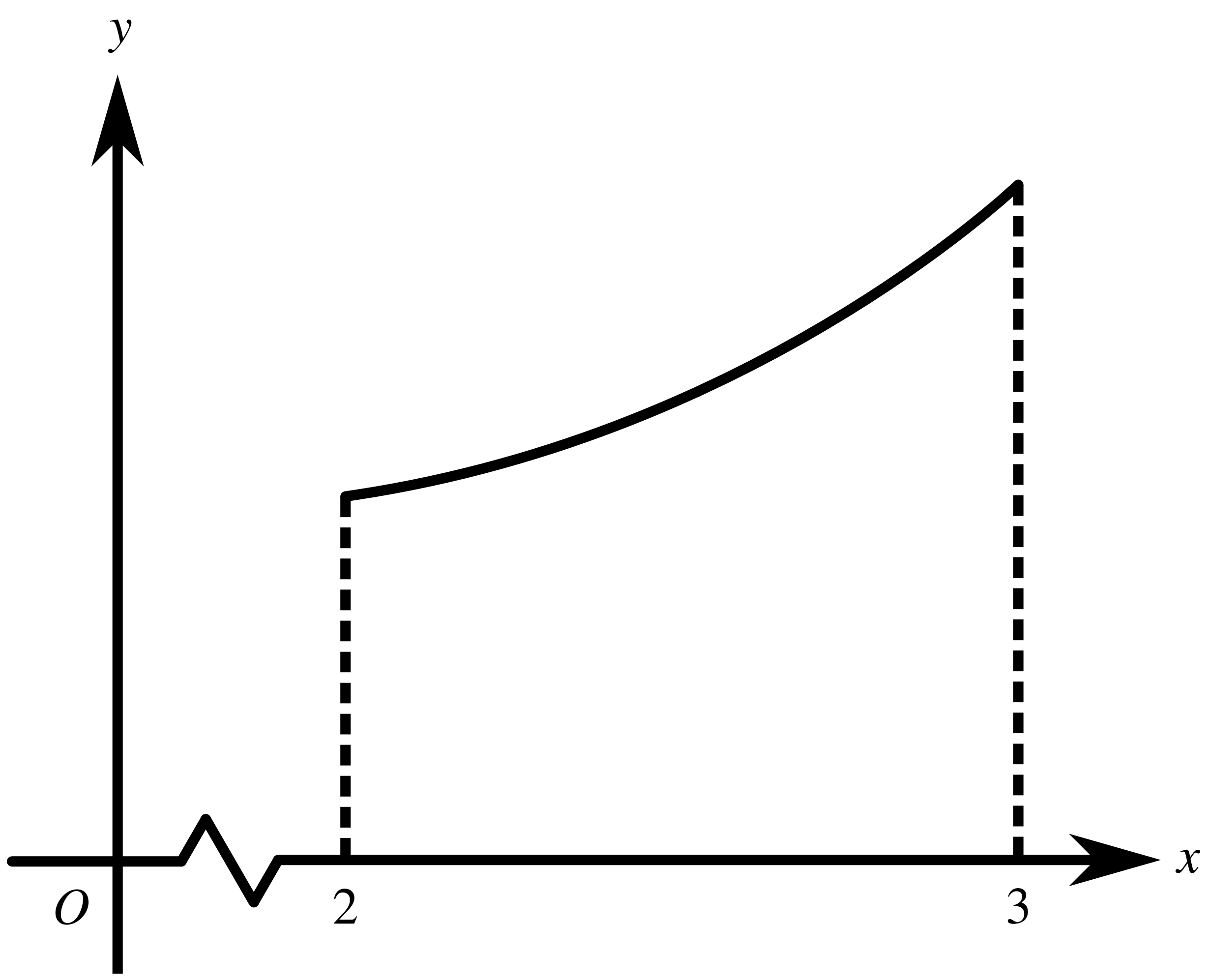

Figure 1: The curve with equation $y = \sqrt{1+x^3}$, for $2 \leqslant x \leqslant 3$.

Figure 1 shows the curve with equation $y = \sqrt{1+x^3}$, for $2 \leqslant x \leqslant 3$. The region under the curve between these limits has area A.

Part A Bounding A

Using the figure below, fill in the blanks to explain why $3 < A < \sqrt{28}$.

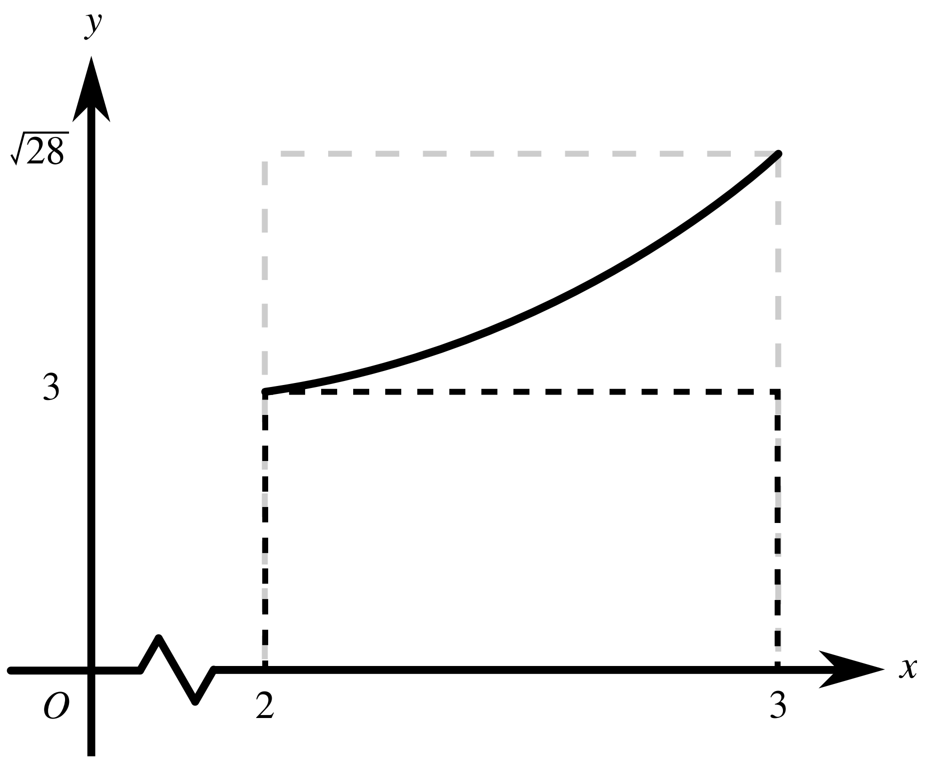

Figure 2: A diagram showing rectangles with areas which bound A.

Two rectangles are shown in **Figure 2**. Both rectangles begin on the x-axis and have width one. The area of the smaller rectangle, which lies ______ the curve, is ______. The area of the second rectangle, the top of which lies ______ the curve, is ______. The rectangles have areas which bound A, and hence:

$$3 < A < \sqrt{28}$$

Items:

$3\sqrt{28}$ | 6 | **above** | **below** | 3 | $\sqrt{28}$

Part B Improved bounds

The region is divided into 5 strips, each of width 0.2. Use suitable rectangles within these strips to find improved lower and upper bounds for A. Give your answers to 3 significant figures.

Give the lower bound for A.

Give the upper bound for A.

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Home Maths Area: Numerical Integration 3i

Area: Numerical Integration 3i

A Level

P P P

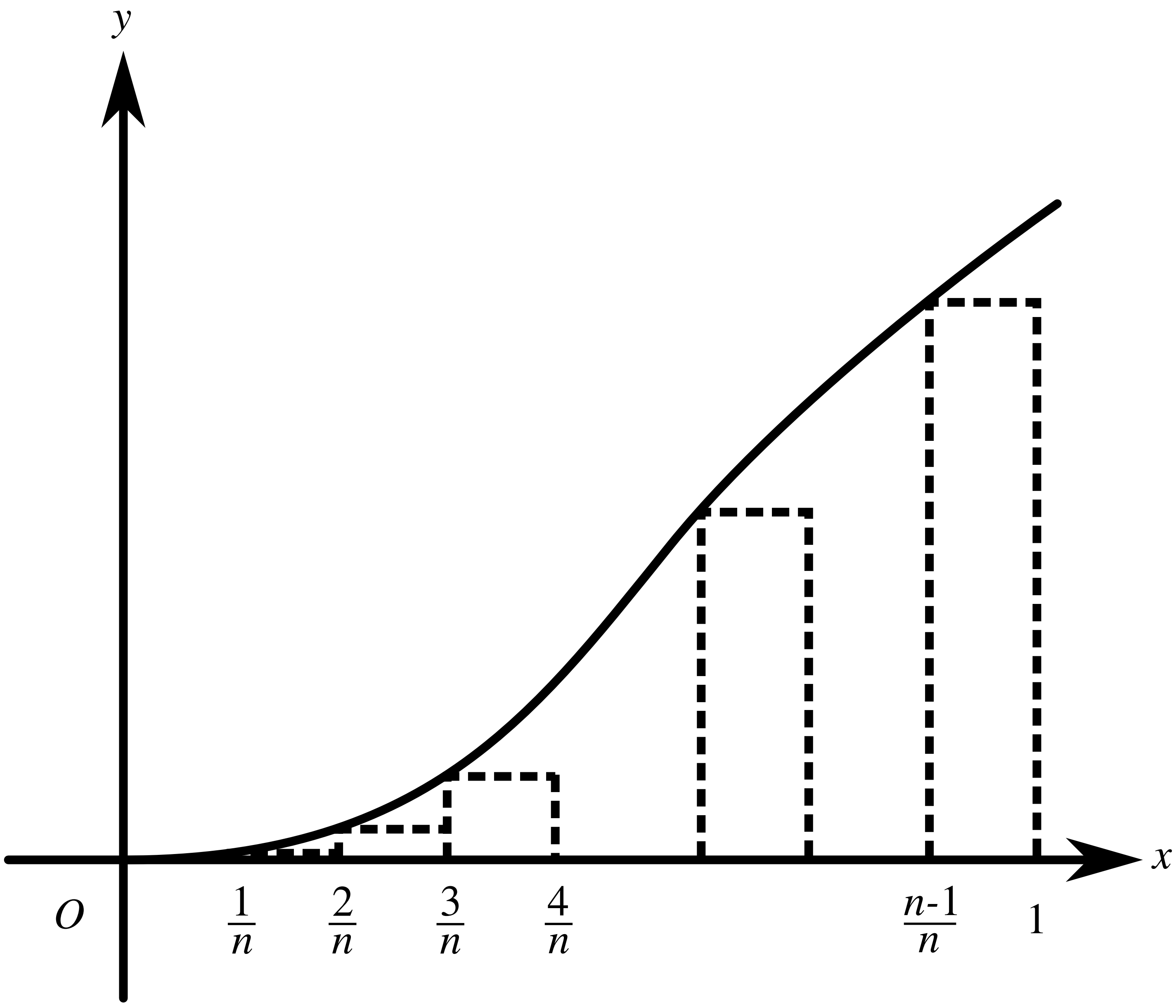

Figure 1: The diagram shows the curve $y=\mathrm{e}^{-\frac{1}{x}}$ for $0<x\leqslant 1$.

Figure 1 shows the curve $y=\mathrm{e}^{-\frac{1}{x}}$ for $0<x\leqslant 1$. A set of $(n-1)$ rectangles is drawn under the curve as shown.

Part A Lower bound

Fill in the blanks below to explain why a lower bound for $\int_0^1 \mathrm{e}^{-\frac{1}{x}} \mathrm{d}x$ can be expressed as:

$$\frac{1}{n} \times (\mathrm{e}^{-n} + \mathrm{e}^{-\frac{n}{2}} + \mathrm{e}^{-\frac{n}{3}} + \ldots + \mathrm{e}^{-\frac{n}{n-1}})$$

The integral $\int_0^1 \mathrm{e}^{-\frac{1}{x}} \mathrm{d}x$ is the area enclosed between the curve and the x-axis between $x = 0$ and $x = 1$.

The area under the curve completely covers the rectangles, so the total area of the rectangles, each of width [ ], is [ ] bound for $\int_0^1 \mathrm{e}^{-\frac{1}{x}} \mathrm{d}x$. The $(n-1)$ rectangles have heights [ ], $\mathrm{e}^{-\frac{n}{2}}$, ... $\mathrm{e}^{-\frac{n}{n-1}}$, and the total area of the rectangles is the sum of the areas of each individual rectangle. Therefore:

$$\frac{1}{n} \times (\mathrm{e}^{-n} + \mathrm{e}^{-\frac{n}{2}} + \mathrm{e}^{-\frac{n}{3}} + \ldots + \mathrm{e}^{-\frac{n}{n-1}}) \;[\;]\; \int_0^1 \mathrm{e}^{-\frac{1}{x}} \mathrm{d}x$$

Items:

a lower | n | $\frac{1}{n}$ | $=$ | $<$ | e^{-n} | **an upper** | $>$ | $\mathrm{e}^{-\frac{1}{n}}$

Part B Upper bound

Using a set of 3 rectangles, write down a similar expression for an upper bound for $\int_0^1 \mathrm{e}^{-\frac{1}{x}} \mathrm{d}x$.

The following symbols may be useful: e

Part C Evaluate bounds

Evaluate these bounds using $n = 4$, giving your answers correct to 3 significant figures.

Give the lower bound

Give the upper bound

Part D Difference between bounds

When $n \geqslant N$, the difference between the upper and lower bounds is less than 0.001. By expressing this difference in terms of n, find the least possible value of N.

Physics. You work it out.

Home Maths Trapezium Rule 2ii

Trapezium Rule 2ii

A Level

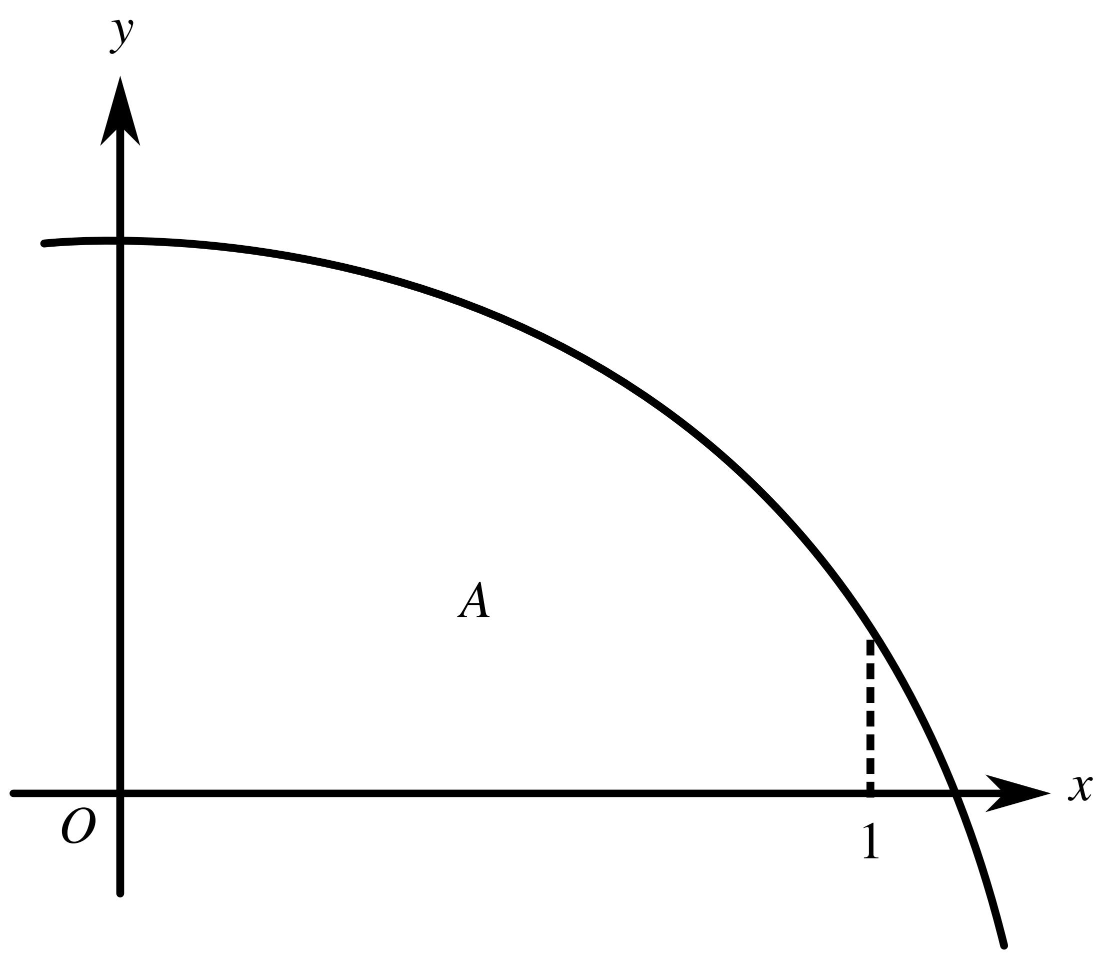

Figure 1: The diagram of the curve $y = \ln(16 - 12x^2)$.

Figure 1 shows part of the curve $y = \ln(16 - 12x^2)$. The region A is bounded by the curve and the lines $x = 0$, $x = 1$ and $y = 0$.

Part A Trapezium Rule

Find an approximate value for A by using the trapezium rule, with two strips each of width $\frac{1}{2}$. Give your answer in the form $a \ln b$.

Part B Overestimate or underestimate

Explain, using the diagram, whether the value obtained in Part A is an underestimate or overestimate for the area of A.

The diagram shows that for $0 \leq x \leq 1$ the value of y is ______ and has a ______ shape (the gradient of the curve is becoming more negative). Hence, the tops of the trapezia used in part A all lie ______ the curve, and so the area of the trapezia is an ______ of the area of A.

Items:

positive **convex** **overestimate** **under** **negative** **underestimate** **above** **concave**

Part C Improving the approximation

Which of these options would improve the estimate of the area of A?

- Use 4 trapezia of width $\frac{1}{8}$.
- Use the same number of trapezia, but double their height.
- Use a larger number of trapezia with the same width, $\frac{1}{2}$.
- Use 4 trapezia of width $\frac{1}{4}$.

Trapezium Rule 3i

A Level

P P P

The value of $\int_0^8 \ln(3+x^2)\,\mathrm{d}x$ obtained by using the trapezium rule with four strips is denoted by A.

Part A Trapezium Rule

Find the value of A correct to 3 significant figures.

Part B Approximation of $\int_0^8 \ln(9+6x^2+x^4)\,\mathrm{d}x$

Write, in terms of A, an expression for an approximate value of $\int_0^8 \ln(9+6x^2+x^4)\,\mathrm{d}x$.

The following symbols may be useful: A

Part C Approximation of $\int_0^8 \ln(3e+ex^2)\,\mathrm{d}x$

Write, in terms of A, an expression for an approximate value of $\int_0^8 \ln(3e+ex^2)\,\mathrm{d}x$.

The following symbols may be useful: A

Gameboard:

Pure Maths Practice: Trapezium Rule