Newton-Raphson Method 3ii

A Level
P P P

It is given that $f(x) = x^2 - \arctan x$ and that $\frac{d}{dx}(\arctan x) = \frac{1}{1+x^2}$.

Part A Interval containing the root

Explain why the equation $f(x) = 0$ has a root in the interval $0.8 < x < 0.9$.

The value of $f(x)$ when $x = 0.8$ is ______, and the value of $f(x)$ when $x = 0.9$ is ______. These values of $f(x)$ have ______. Hence, as $f(x)$ is a continuous function, there is a value of x in the interval $0.8 < x < 0.9$ for which $f(x) = 0$.

A root of an equation is a value of x for which $f(x) =$ ______. Hence, there is a root of $f(x)$ in the interval $0.8 < x < 0.9$.

Items:

1 | the same sign | 0.0771 | -0.0347 | different signs | 1.348 | 0.0772 | 0

Part B Find the root

Use the Newton-Raphson method, with a first approximation 0.8, to find the next approximation to this root. Give your answer correct to 3 significant figures.

Newton-Raphson Method 1ii

A Level
P P P

The diagram shows the curve with equation $y = xe^{-x} + 1$. The curve crosses the x-axis at $x = \alpha$.

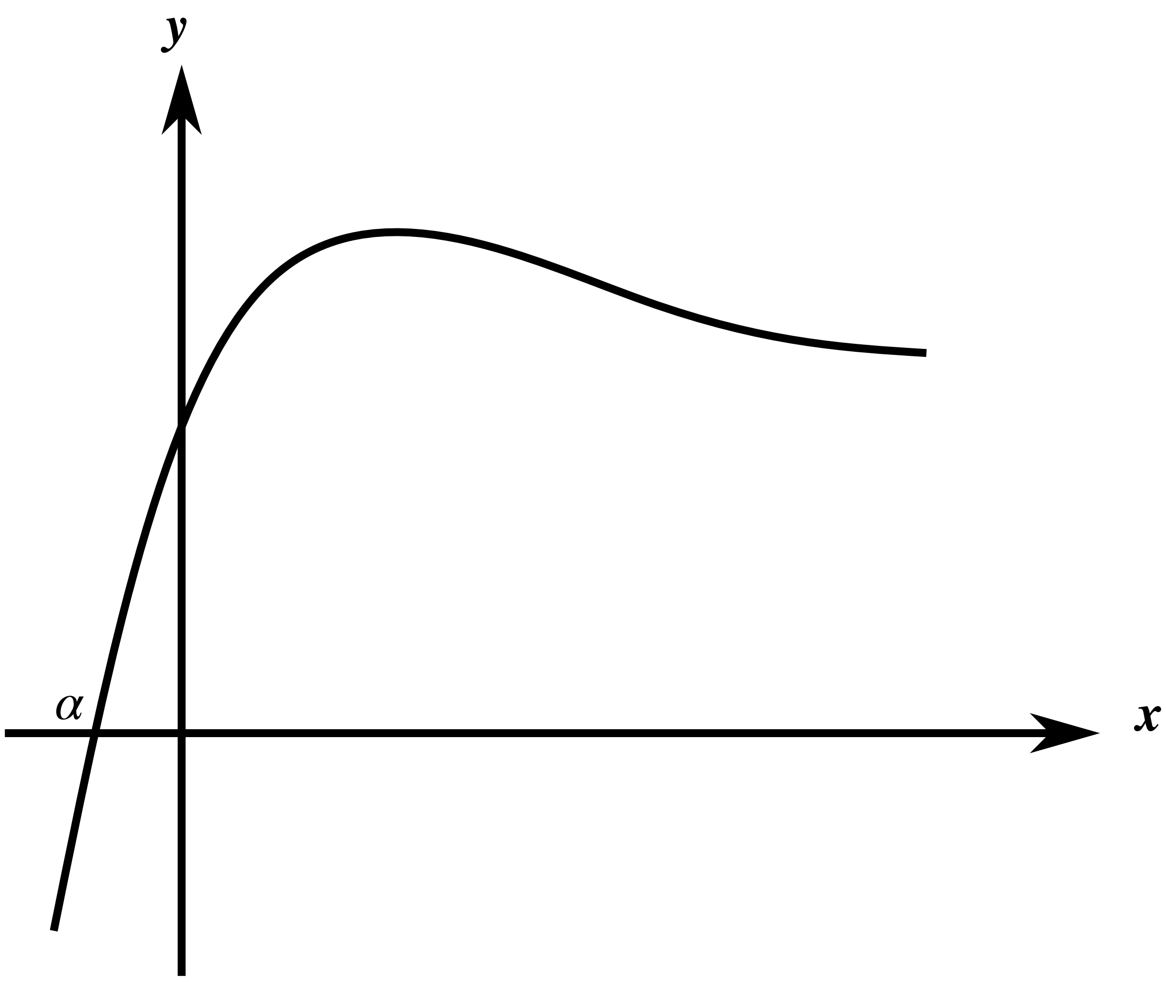

Figure 1: A sketch of the curve $y = xe^{-x} + 1$.

Part A x-coordinate of stationary point

Use differentiation to calculate the x-coordinate of the stationary point.

The following symbols may be useful: x

Part B Explain

α is to be found using the Newton-Raphson method, with $f(x) = xe^{-x} + 1$.

Explain why this method will not converge to α if an initial approximation x_1 is chosen such that $x_1 > 1$.

The iterative formula for the Newton-Raphson method is $x_{n+1} = x_n - \dfrac{f(x)}{f'(x)}$. For all values of x greater than 1, $f(x)$ is positive, and the _____ of $f(x)$ is negative (and close to _____). Hence, $-\dfrac{f(x)}{f'(x)}$ is positive and so x_{n+1} is larger than x_n. Visually, the x-intercepts of _____ at successive approximations will reach progressively _____ x-values and, hence, move further away from α.

Items:

larger | normals | value | gradient | 0 | -1 | smaller | 1 | tangents | intercept

Part C Values

α is to be found using the Newton-Raphson method, with $f(x) = xe^{-x} + 1$.

Use this method, with a first approximation $x_1 = 0$, to find the next three approximations x_2, x_3, x_4. Give your answers to 4 sf where necessary.

$x_2 =$ _____

$x_3 =$ _____

$x_4 =$ _____

Find α correct to 3 significant figures.

$\beta =$ _____

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Physics. You work it out.

Home Gameboard Maths Roots and Iteration 3i

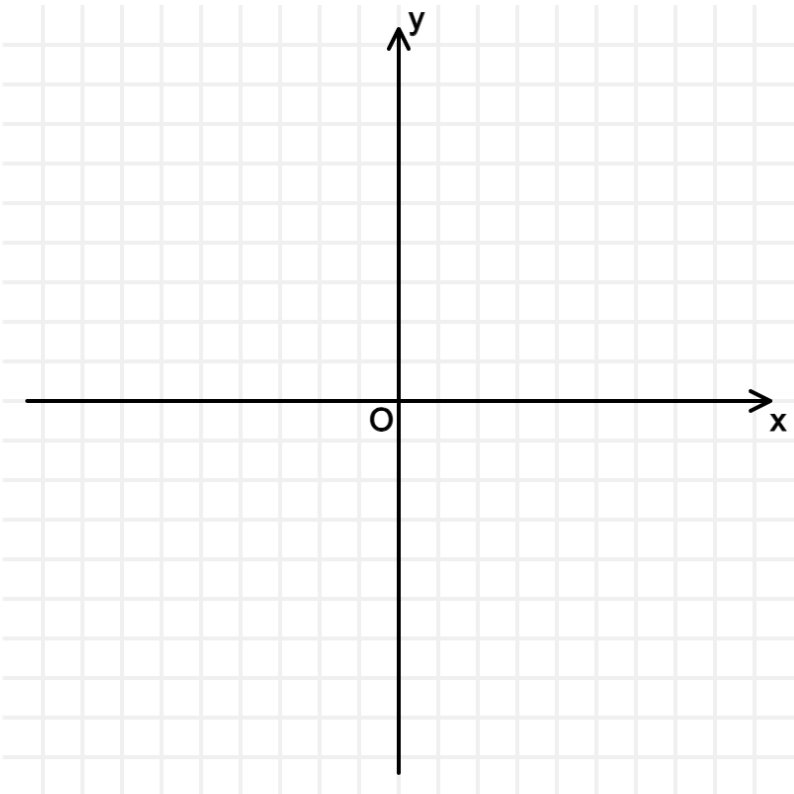
Roots and Iteration 3i

A Level
P P P

Part A Sketch

By sketching two suitable graphs on a single diagram, find the number of roots to the equation

$$14 - x^2 = 3\ln x.$$



From your sketch, state how many roots there are to the equation

$$14 - x^2 = 3\ln x$$

Part B Integer below α

Find by calculation the largest integer which is less than the root α.

Part C Iteration

Use the iterative formula $x_{n+1} = \sqrt{14 - 3\ln x_n}$, with a suitable starting value to find α correct to 3 significant figures.

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Roots and Iteration 1i

A Level
P P P

It is required to solve the equation $f(x) = \ln(4x - 1) - x = 0$.

Part A Root existence

Show that the equation $f(x) = 0$ has two roots, α and β, such that $0.5 < \alpha < 1$ and $1 < \beta < 2$.

We find that $f(0.5) =$ ______, $f(1) =$ ______ and $f(2) =$ ______.

Since there is a ______ between $f(0.5)$ and $f(1)$, there must be a root α such that $0.5 < \alpha < 1$. As there is also a ______ between $f(1)$ and $f(2)$, there must be a root β such that $1 < \beta < 2$.

Items:

-0.0541 | change of value | 1 | change of sign | 0.0986 | -0.303 | 0.109 | difference | -0.5 | 1.099 | 1.61 | 1.95 | 0.5

Part B Iteration with $g(x)$

Let $g(x) = \ln(4x - 1)$. Use the iterative formula $x_{r+1} = g(x_r)$ with $x_0 = 1.8$ to find x_1, x_2, and x_3, correct to 5 decimal places.

$x_1 =$

$x_2 =$

$x_3 =$

Continue the iterative process with $x_{r+1} = g(x_r)$ to find β correct to 3 decimal places.

$\beta =$

Part C New rearrangement $h(x)$

The equation $f(x) = 0$ can be rearranged into the form

$$x = h(x) = \frac{e^{ax} + b}{c}$$

where a, b and c are constants. Find $h(x)$.

The following symbols may be useful: `e`, `h`, `x`

Part D Iteration with $h(x)$

Use the iterative formula $x_{r+1} = h(x_r)$ with $x_0 = 0.8$ to find α correct to 4 decimal places.

Part E Root finding analysis

Show that the iterative formula $x_{r+1} = g(x_r)$ will not find the value of α. Similarly, determine whether the iterative formula $x_{r+1} = h(x_r)$ will find the value of β.

The iterative formula $x_{r+1} = g(x_r)$ will not converge to a root if [ ] near that root.

For $g(x)$, differentiating we find that $g'(x) =$ [ ]. Using the value for α calculated in Part D, this gives $g'(\alpha) =$ [ ] > 1. Therefore the iterative formula $x_{r+1} = g(x_r)$ will not converge to α.

For $h(x)$, differentiating we find that $h'(x) =$ [ ]. Using the value for β calculated in Part B, $h'(\beta) =$ [ ] > 1. Therefore the iterative formula $x_{r+1} = h(x_r)$ will not converge to β.

Items:

$\frac{1}{x}$ | e^x | 6.47 | $\frac{4}{4x-1}$ | $\frac{e^x+1}{4}$ | 0.307 | $g'(x) > 1$ | $\frac{1}{4x}$ | 1.23 | 1.87 | $g'(x) < 1$ | $\frac{e^x}{4}$ | $|g'(x)| < 1$ | 0.443 | 1.77 | $|g'(x)| > 1$ | 1.62 | $\frac{1}{4x-1}$

Part F Staircase diagrams

From the figures below, select the two figures that illustrate the iterations for $x_{r+1} = g(x_r)$ and $x_{r+1} = h(x_r)$.

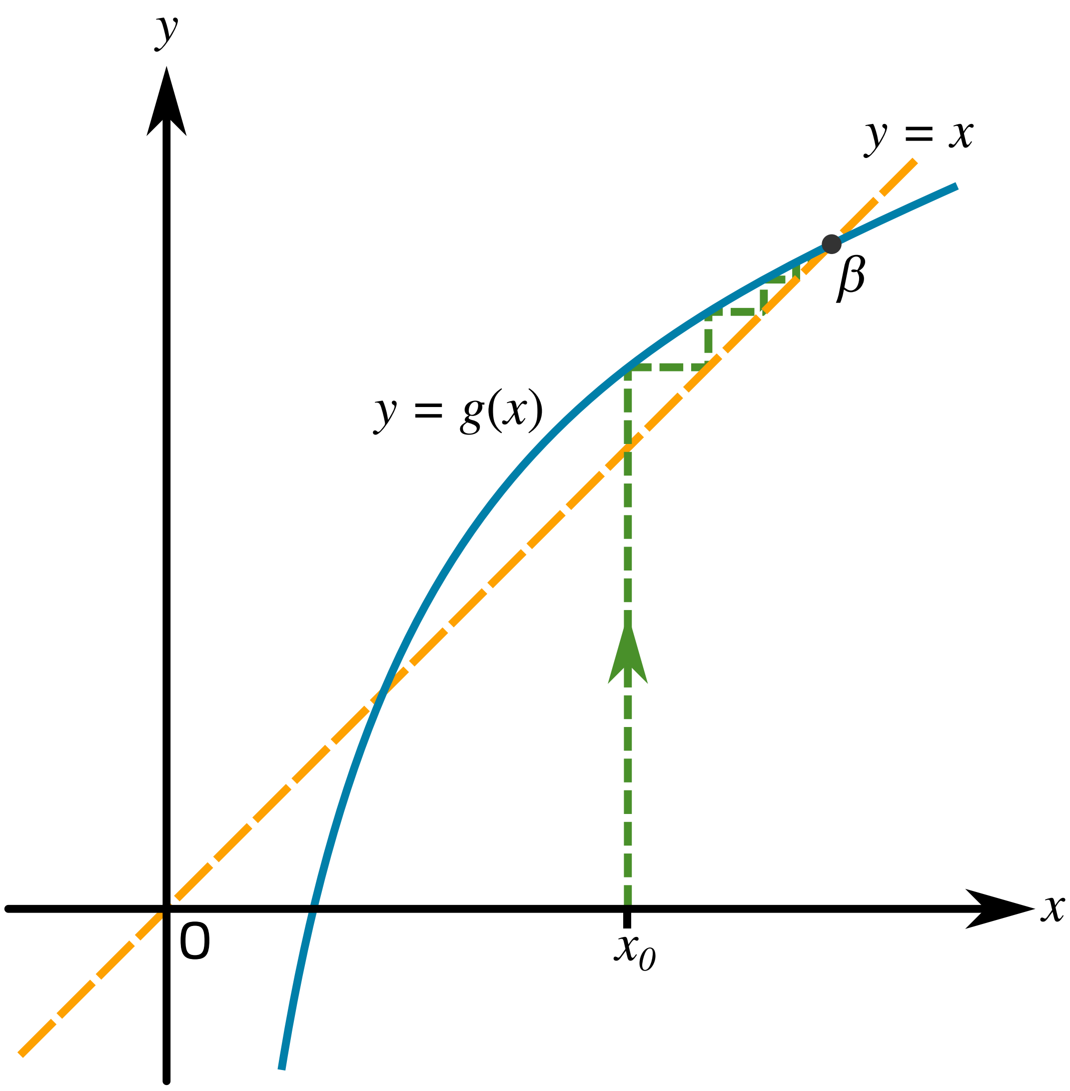

Figure 1: Graph of the iterative process for $x_{r+1} = g(x_r)$ towards β.

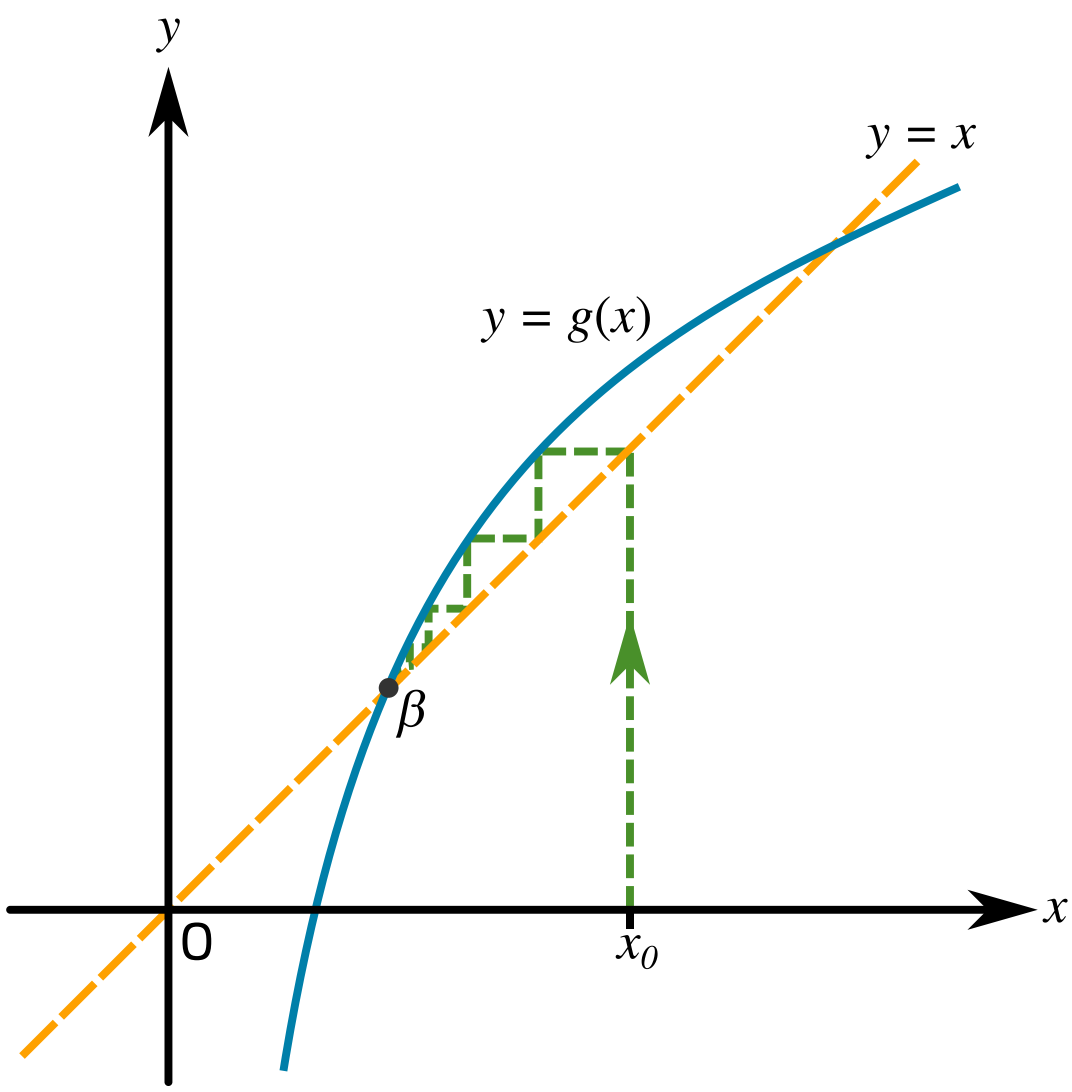

Figure 2: Graph of the iterative process for $x_{r+1} = g(x_r)$ towards β.

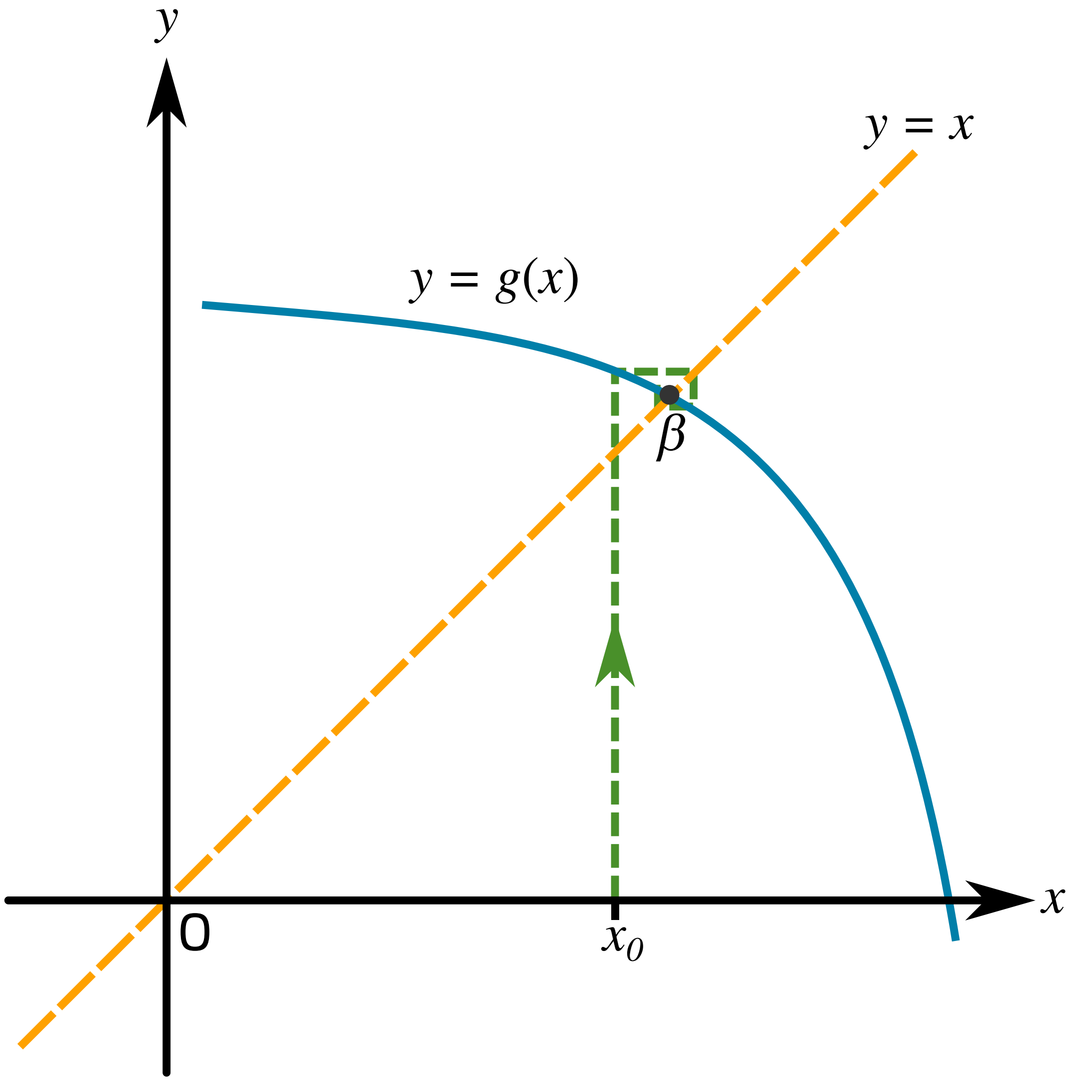

Figure 3: Graph of the iterative process for $x_{r+1} = g(x_r)$ towards β.

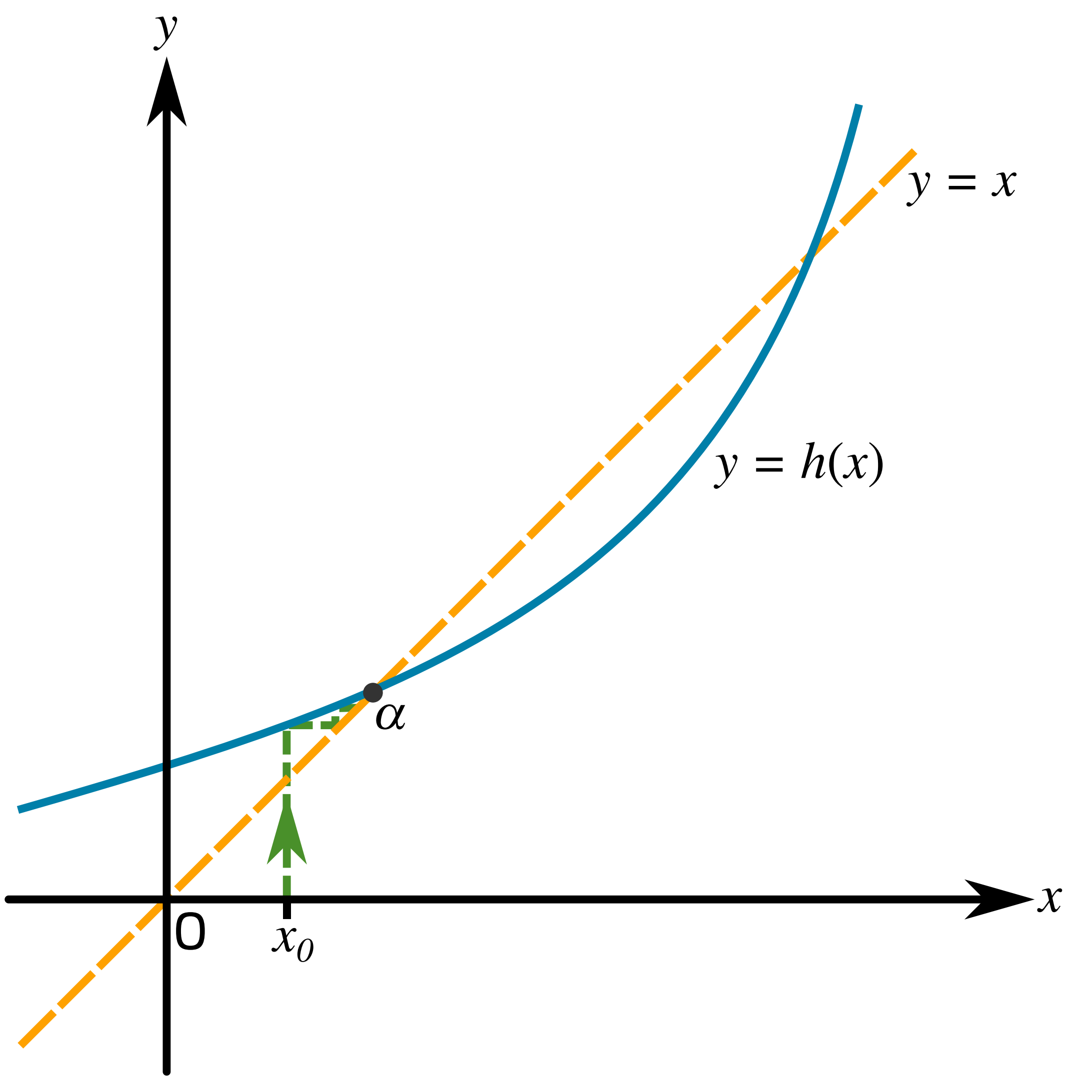

Figure 4: Graph of the iterative process for $x_{r+1} = h(x_r)$ towards α.

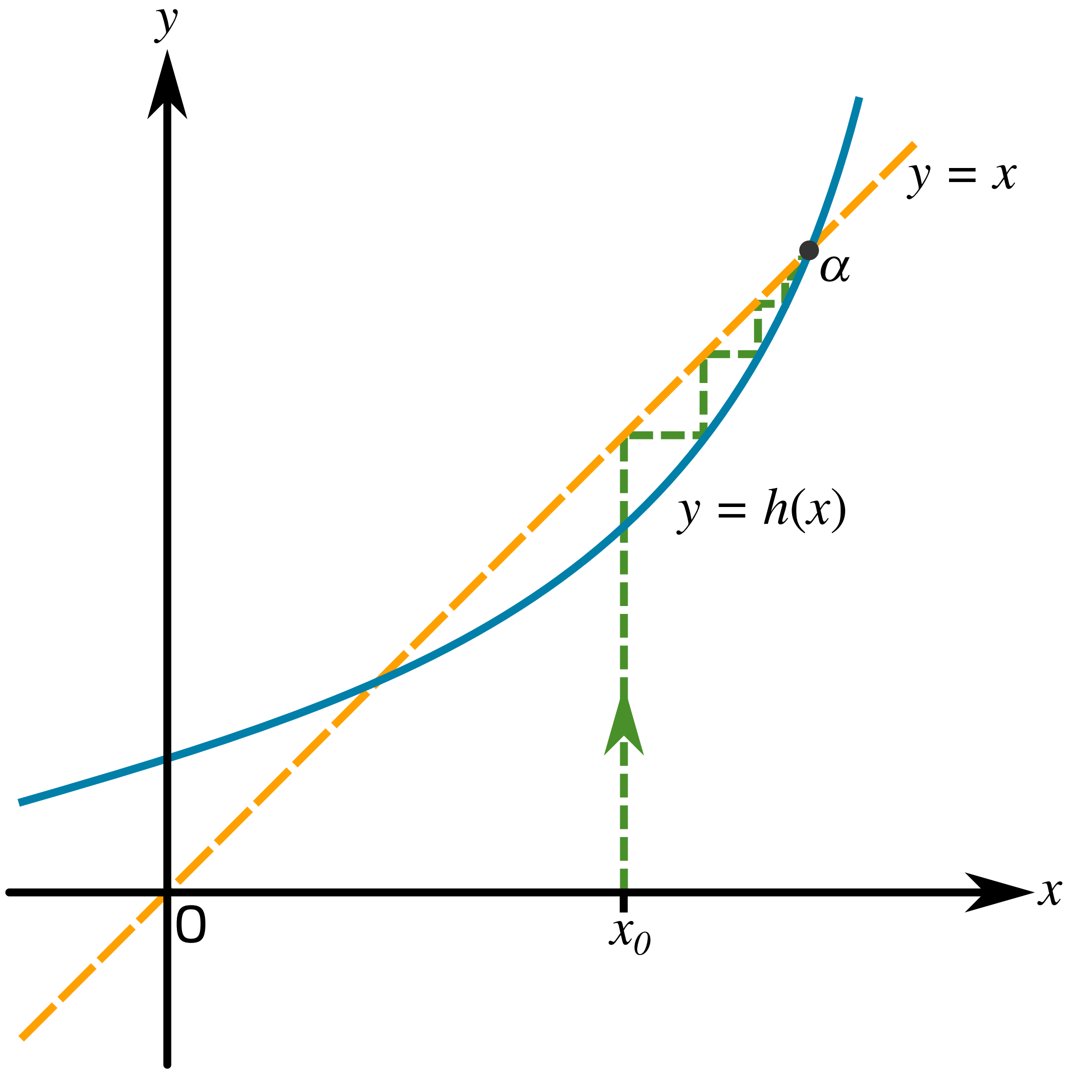

Figure 5: Graph of the iterative process for $x_{r+1} = h(x_r)$ towards α.

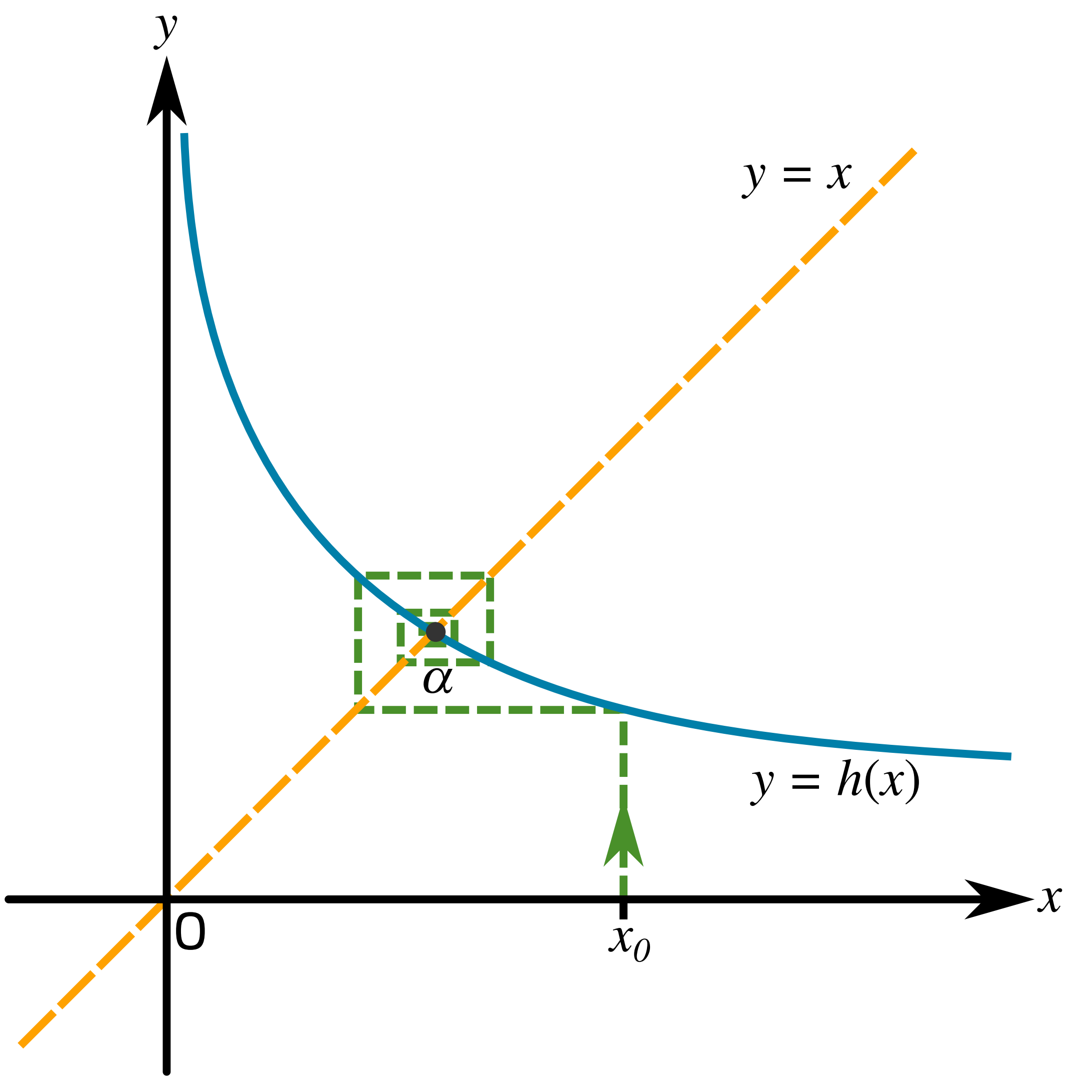

Figure 6: Graph of the iterative process for $x_{r+1} = h(x_r)$ towards α.

- [ ] **Figure 1**
- [ ] **Figure 2**
- [ ] **Figure 3**

- [ ] **Figure 4**
- [ ] **Figure 5**
- [ ] **Figure 6**

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Trapezium Rule 2ii

A Level
P P P

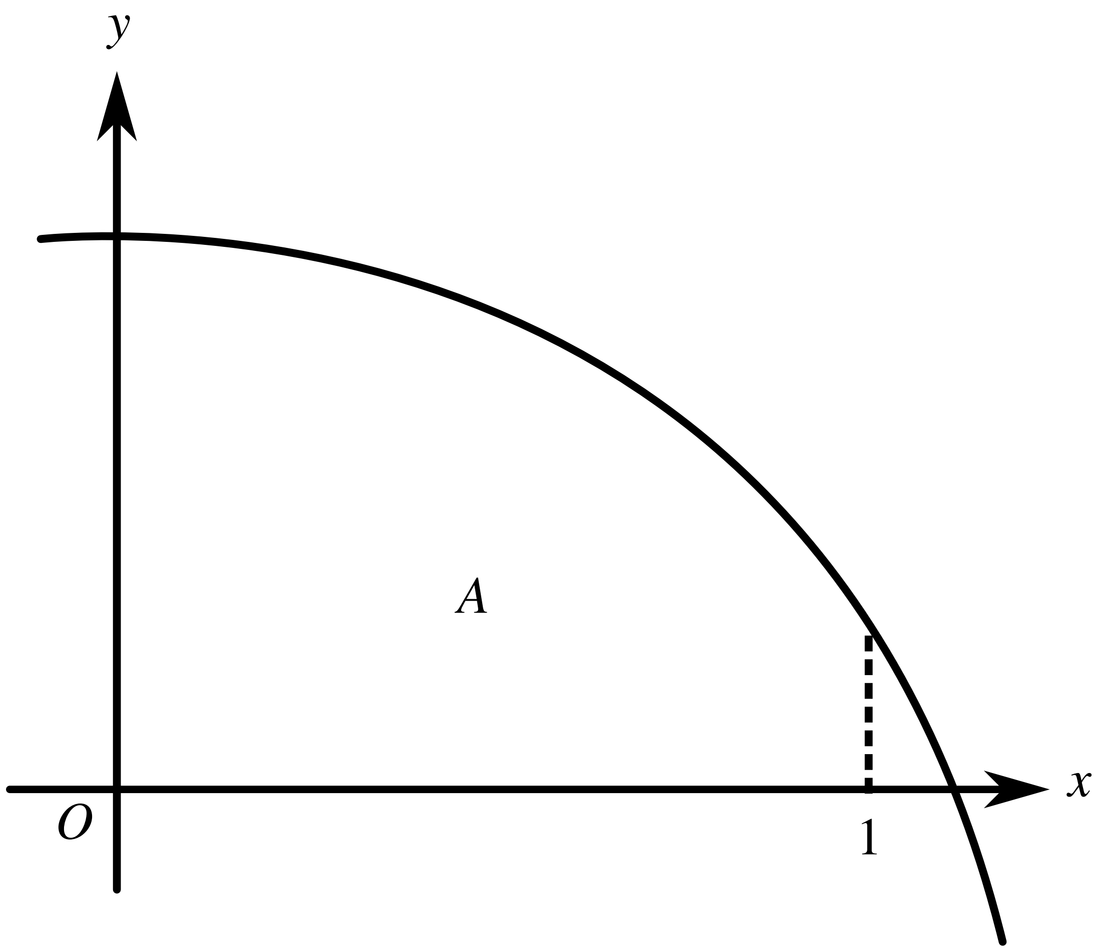

Figure 1: The diagram of the curve $y = \ln(16 - 12x^2)$.

Figure 1 shows part of the curve $y = \ln(16 - 12x^2)$. The region A is bounded by the curve and the lines $x = 0$, $x = 1$ and $y = 0$.

Part A Trapezium Rule

Find an approximate value for A by using the trapezium rule, with two strips each of width $\frac{1}{2}$. Give your answer in the form $a \ln b$.

Part B Overestimate or underestimate

Explain, using the diagram, whether the value obtained in Part A is an underestimate or overestimate for the area of A.

The diagram shows that for $0 \leq x \leq 1$ the value of y is ______ and the curve has a ______ shape (the gradient of the curve is becoming more negative). Hence, the tops of the trapezia used in part A all lie ______ the curve, and so the area of the trapezia is an ______ of the area of A.

Items:

negative | under | concave | above | positive | overestimate | convex | underestimate

Part C Improving the approximation

Which of these options would improve the estimate of the area of A?

- Use 4 trapezia of width $\frac{1}{8}$.
- Use 4 trapezia of width $\frac{1}{4}$.
- Use the same number of trapezia, but double their height.
- Use a larger number of trapezia with the same width, $\frac{1}{2}$.

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Trapezium Rule 3i

A Level

P P P

The value of $\int_0^8 \ln(3 + x^2)\,\mathrm{d}x$ obtained by using the trapezium rule with four strips is denoted by A.

Part A Trapezium Rule

Find the value of A correct to 3 significant figures.

Part B Approximation of $\int_0^8 \ln(9 + 6x^2 + x^4)\,\mathrm{d}x$

Write, in terms of A, an expression for an approximate value of $\int_0^8 \ln(9 + 6x^2 + x^4)\,\mathrm{d}x$.

The following symbols may be useful: A

Part C Approximation of $\int_0^8 \ln(3e + ex^2)\,\mathrm{d}x$

Write, in terms of A, an expression for an approximate value of $\int_0^8 \ln(3e + ex^2)\,\mathrm{d}x$.

The following symbols may be useful: A

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Area: Numerical Integration 2ii

A Level

P P P

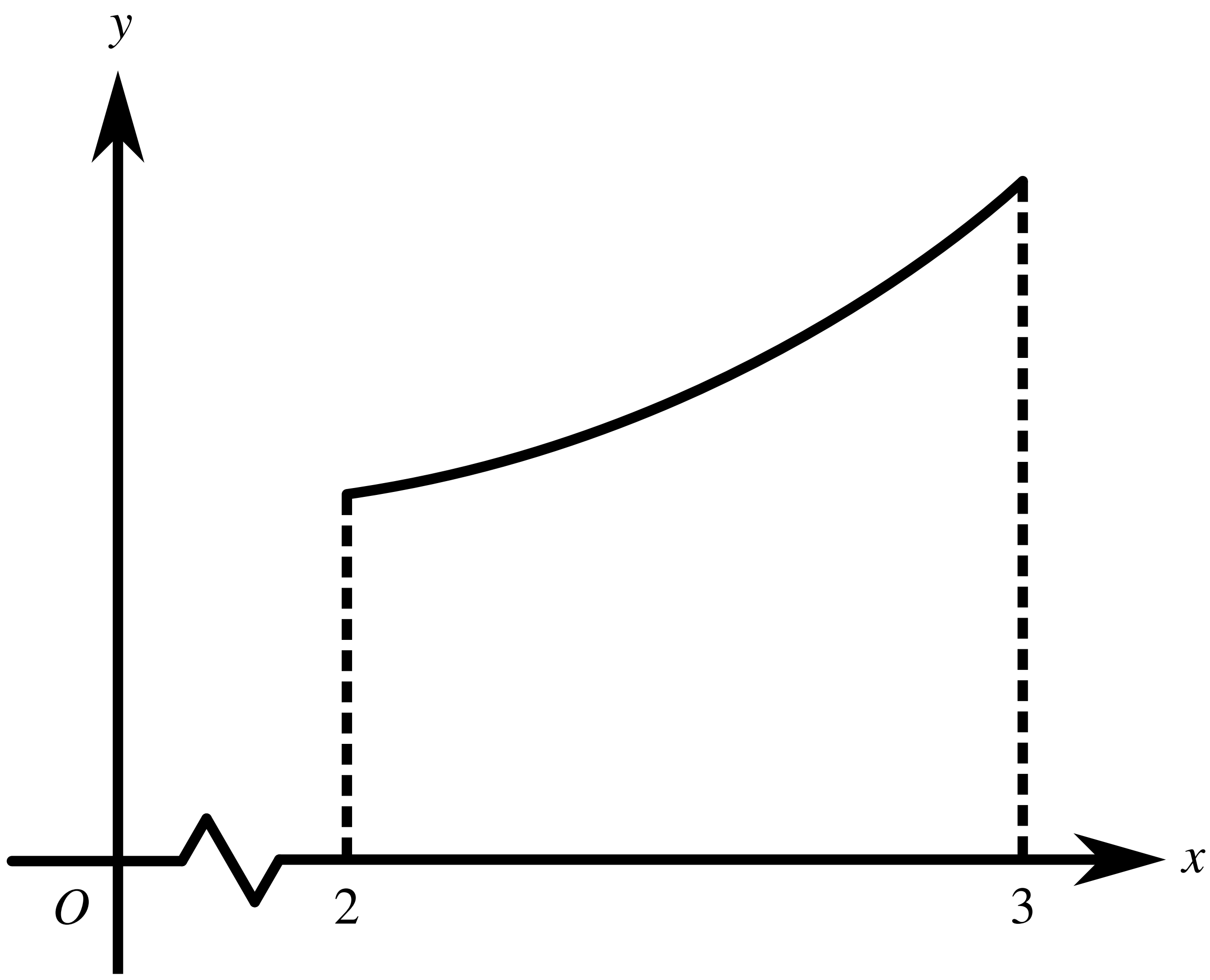

Figure 1: The curve with equation $y = \sqrt{1+x^3}$, for $2 \leqslant x \leqslant 3$.

Figure 1 shows the curve with equation $y = \sqrt{1+x^3}$, for $2 \leqslant x \leqslant 3$. The region under the curve between these limits has area A.

Part A Bounding A

Using the figure below, fill in the blanks to explain why $3 < A < \sqrt{28}$.

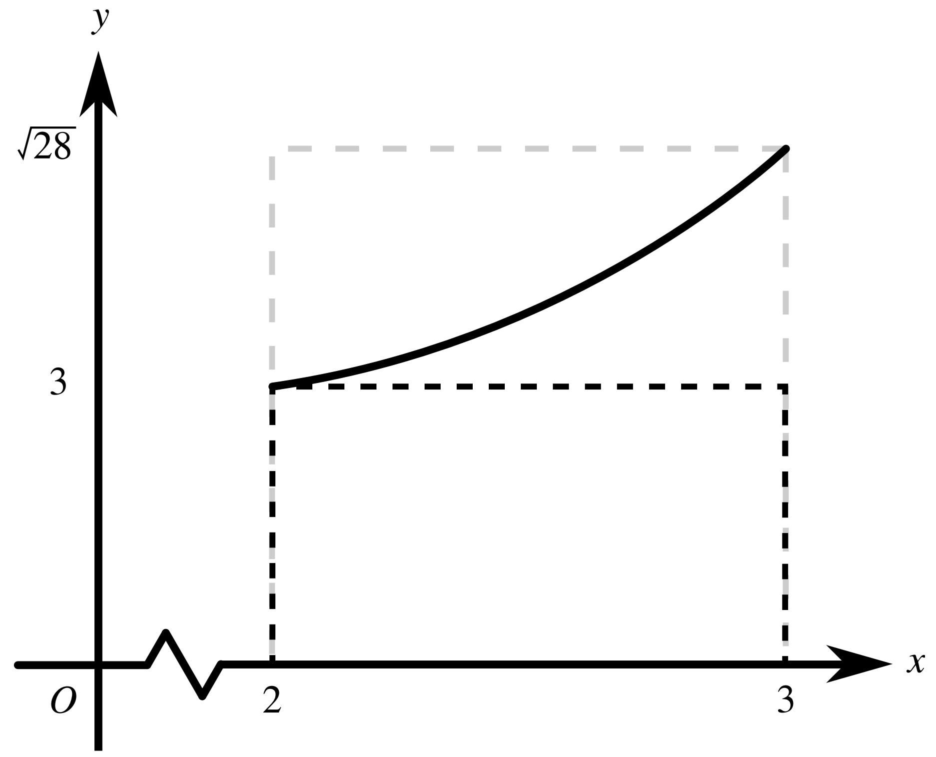

Figure 2: A diagram showing rectangles with areas which bound A.

Two rectangles are shown in **Figure 2**. Both rectangles begin on the x-axis and have width one. The area of the smaller rectangle, which lies [ ] the curve, is [ ]. The area of the second rectangle, the top of which lies [ ] the curve, is [ ]. The rectangles have areas which bound A, and hence:

$$3 < A < \sqrt{28}$$

Items:

above | $\sqrt{28}$ | $3\sqrt{28}$ | 3 | 6 | below

Part B Improved bounds

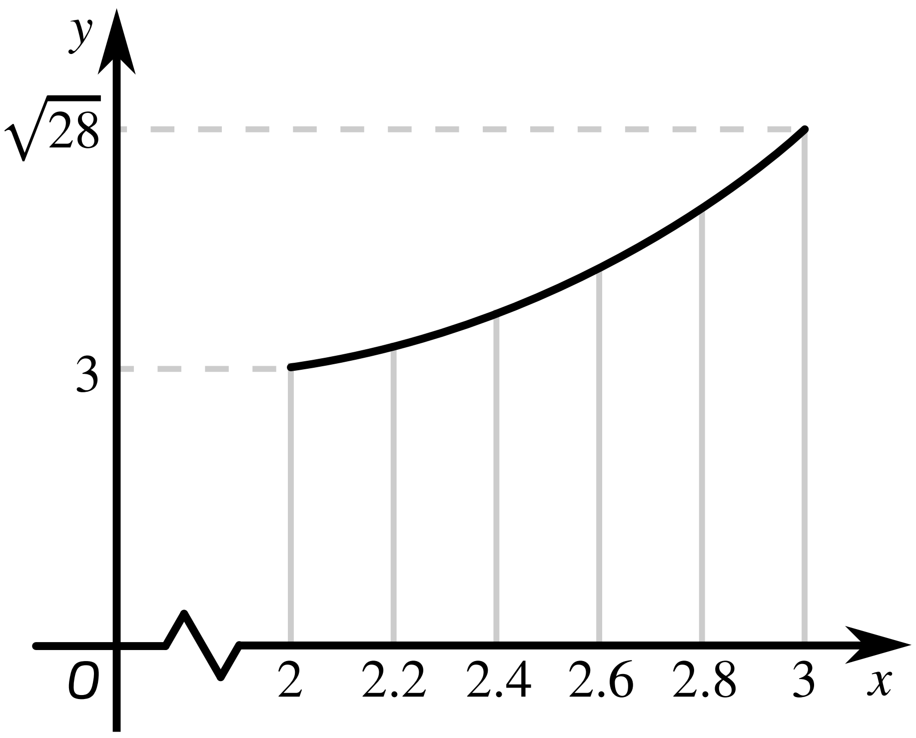

Figure 3: The curve with equation $y = \sqrt{1+x^3}$, for $2 \leqslant x \leqslant 3$, divided into 5 strips of equal width.

The region is divided into 5 strips, each of width 0.2. Use suitable rectangles with these strips to find improved lower and upper bounds for A. Give your answers to 3 significant figures.

lower bound for A:

upper bound for A: