Waves and Complex numbers

Overture for SPC BPhO Summer School

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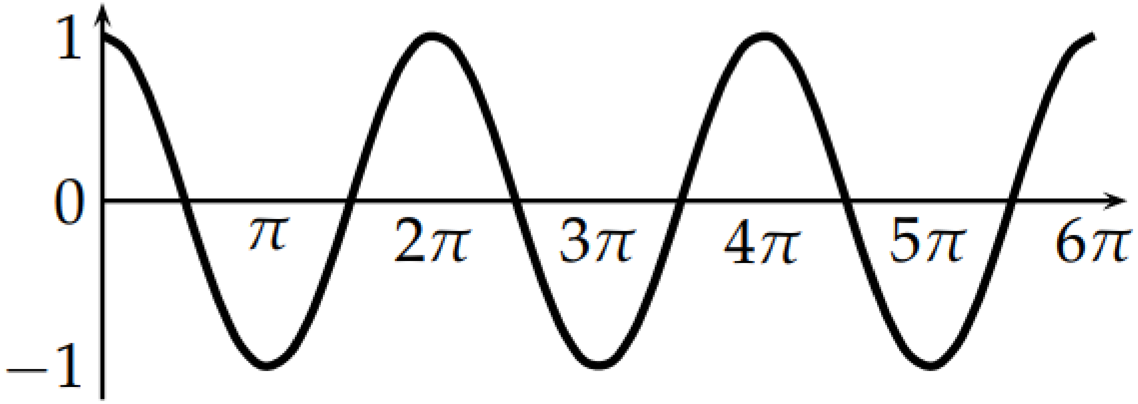
Describing a wave

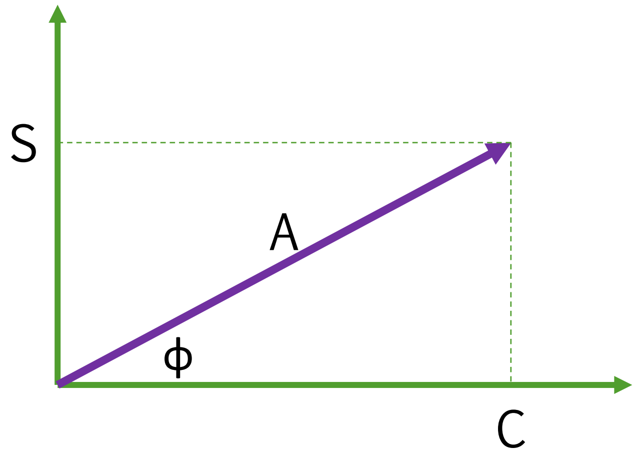
- Phase gives 'location' within wave 0=peak.
- Angles are always measured in radians (1 rad = 180/π°)



- We describe the wave $y = A\cos\omega t$
- Angular frequency $\omega = 2\pi f = 2\pi/T$ measured in rad/s

Wave with initial phase

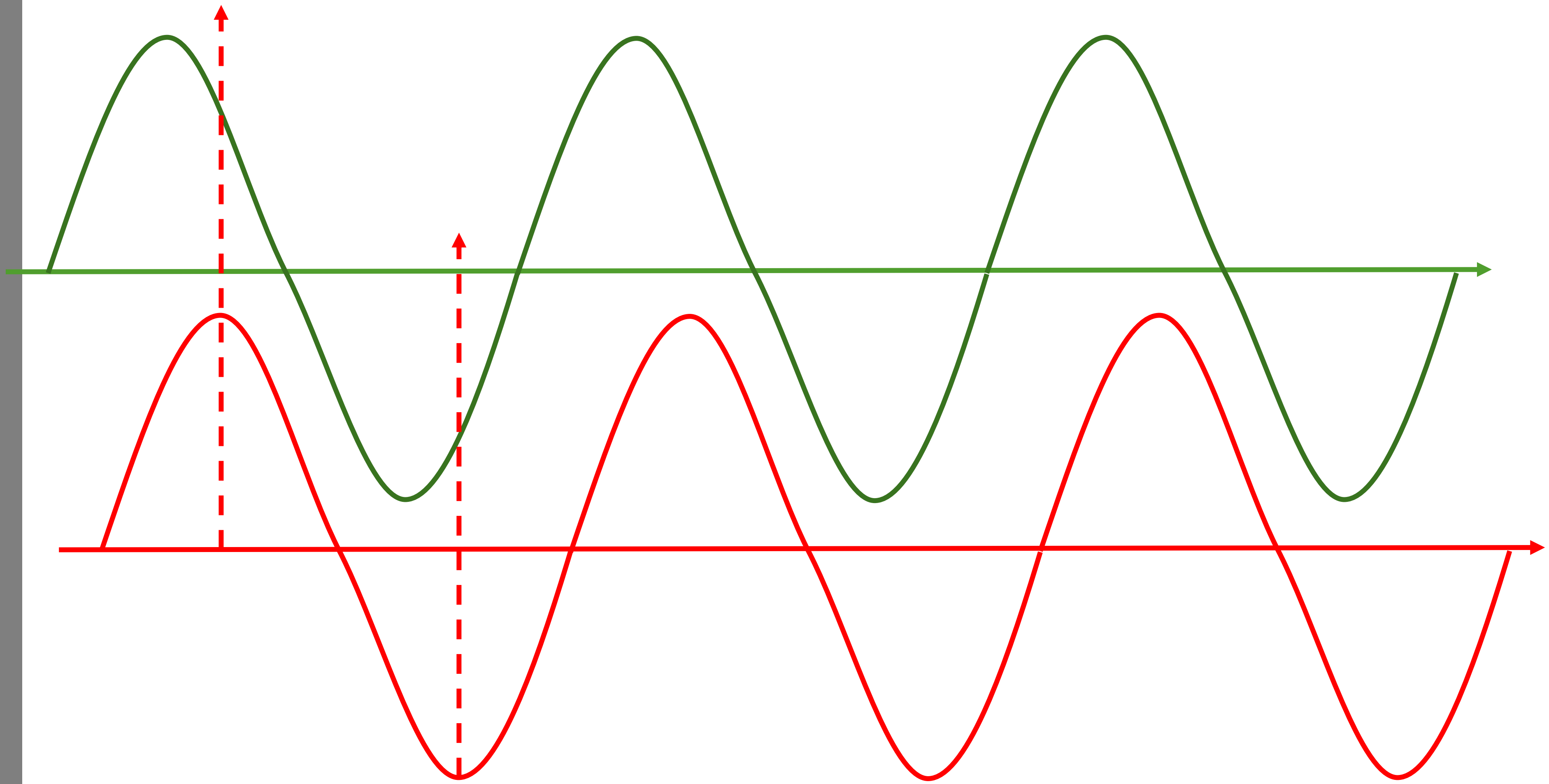
- A wave may not have a peak at *t=0.*
- $y = A\cos(\omega t + \phi_0) = A\cos\omega t\cos\phi_0 - A\sin\omega t\sin\phi_0$
- $y = C\cos\omega t \; - S\sin\omega t$, where $C = \mathrm{A}\cos\phi_0\,, S = \mathrm{A}\sin\phi_0$
- Overall amplitude $A = \sqrt{C^2 + S^2}$, $\tan\phi_0 = S/C$



It is often helpful to draw the amplitude in two dimensions to show its phase, its cosine (even) and sine (odd) components.

Phasor with single wave

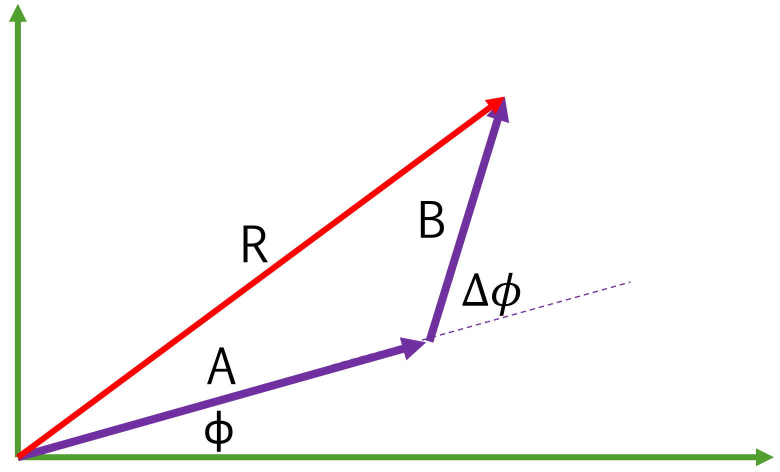
Phase difference



Phase difference is constant if both waves have same ω

Phase difference and interference

- Trigonometry:
- $A\cos(\omega t+\phi)+B\cos(\omega t+\theta)$
- $= A\cos\omega t\cos\phi - A\sin\omega t\sin\phi\ +$
  $B\cos\omega t\cos\theta - B\sin\omega t\sin\theta$
- $C\ = A\cos\phi + B\cos\theta\,,$
- $S\ = A\sin\phi + B\sin\theta$



Phasor with two waves

Beats – You work it out!

- Two sounds have the same amplitude, but very slightly different frequencies ω and $\omega + \Delta\omega$.
- $y = A\cos\omega t + A\cos(\omega + \Delta\omega)t$
- What do you hear?

Waves

- Oscillation (x=0) described as $y = A\cos(\omega t)$
- Wave travelling to right at speed *c*
- $y_{\rightarrow}(x,t) = y\left(0, t - \frac{x}{c}\right) = A\cos(\omega t\ - \frac{\omega x}{c}) = A\cos(\omega t - kx)$
- $k = \frac{\omega}{c} = \frac{2\pi f}{c} = \frac{2\pi}{\lambda}$ is known as the wavenumber

<u>You work it out!</u>

- Write a wave travelling left $y_{\leftarrow}$ with the same ω, k, c
- Add the two waves, and see what you get.

Is this a vector space?

Our diagrams look like vectors, but

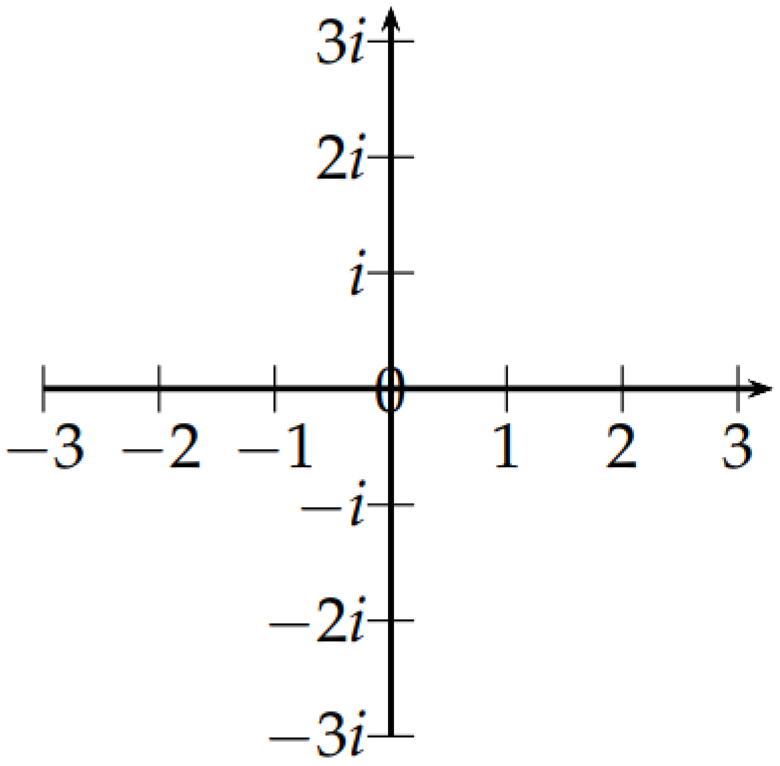
- only the x-component really means anything
- the two components are not independent (odd and even parts of the same wave)

Added complications

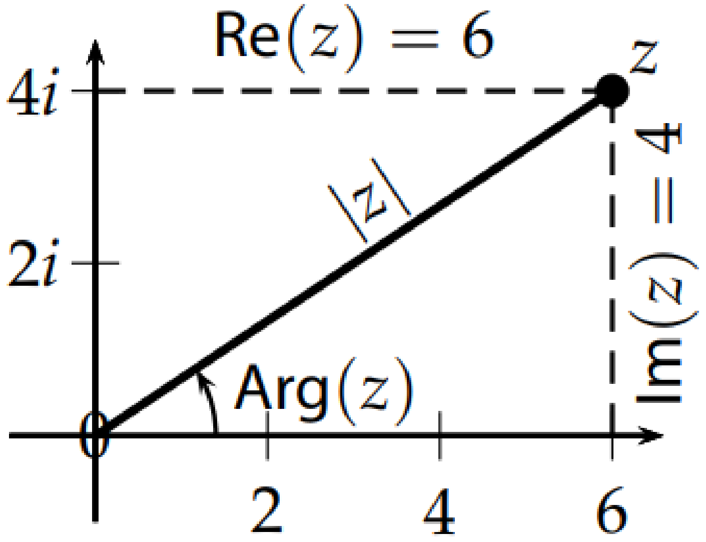
- not easy to see overall amplitude when using cos & sin
- differential equations tough since $\cos \rightarrow -\sin \rightarrow -\cos$ ...

Solution: use complex numbers and all our problems go away!

The Argand diagram



A complex number



This number is $z = 6 + 4i$

Two complex numbers are added with usual algebra

$$3 + 2i + 6 - 3i = 9 - i$$

Multiplication by i represents rotation by $\frac{\pi}{2}$ anticlockwise.

$$i^2 = -1, \qquad -i \times i = 1, \qquad \frac{1}{i} = -i$$

Complex conjugate $z^\star = 6 - 4i, \quad z^\star z = |z|^2$

Euler: $z = r\cos\theta + ir\sin\theta = r\,e^{i\theta}$, so $re^{i\theta}we^{i\phi} = rwe^{i(\theta+\phi)}$

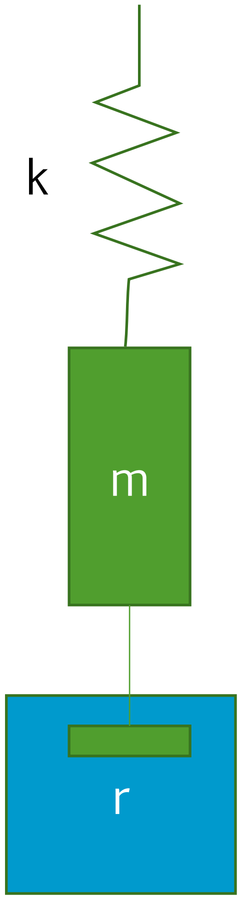
Using complex numbers

- Using $e^{i\theta} = \cos\theta + i\sin\theta$, work out
- $e^{i\theta} + e^{-i\theta}$
- $e^{i\theta} - e^{-i\theta}$
- sin (a+b)
- sin a + sin b
- Find the modulus of $ae^{i\phi} + be^{i\theta}$

Oscillations with complex numbers

- $y = A\cos(\omega t + \phi) = \mathrm{Re}\, Ae^{i\phi}e^{i\omega t}$
- Adding two oscillations:
- $$\begin{aligned} y = \mathrm{Re}\left(Ae^{i\phi}e^{i\omega t} + Ae^{i\theta}e^{i\omega t}\right) &= \mathrm{Re}\left\{e^{i\omega t}\left(Ae^{i\phi} + Ae^{i\theta}\right)\right\} \\ &= \mathrm{Re}\left\{e^{i\omega t + i(\phi+\theta)/2}\left(Ae^{i(\phi-\theta)/2} + Ae^{i(\theta-\phi)/2}\right)\right\} \\ &= 2A\cos\left(\frac{\phi-\theta}{2}\right)\mathrm{Re}\left\{e^{i\omega t + i(\phi+\theta)/2}\right\} \\ &= 2A\cos\left(\frac{\phi-\theta}{2}\right)\cos\left(\omega t + \frac{\phi+\theta}{2}\right) \end{aligned}$$
- Remember $e^{i\theta} = \cos\theta + i\sin\theta$

Forced, damped harmonic oscillator

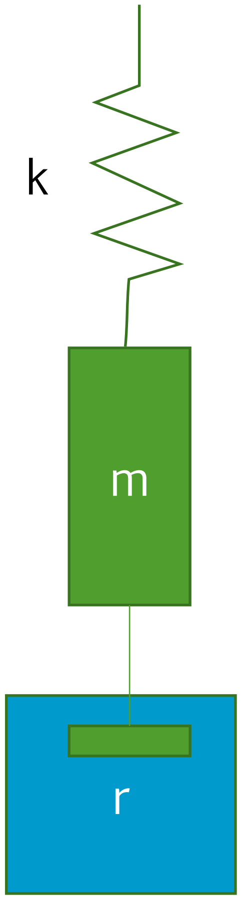


$$F = m\ddot{x} = -kx - r\dot{x} + F_0 \cos \omega t$$

Try solution $z = Ae^{i\omega t}$ to $m\ddot{z} = -kz - r\dot{z} + F_0 e^{i\omega t}$

$$\frac{d}{dt} e^{i\omega t} = i\omega e^{i\omega t} \qquad \frac{d^2}{dt^2} e^{i\omega t} = -\omega^2 e^{i\omega t}$$

Forced, damped harmonic oscillator



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Try solution $z = Ae^{i\omega t}$ to $m\ddot{z} = -kz - r\dot{z} + F_0 e^{i\omega t}$

$$\frac{d}{dt}e^{i\omega t} = i\omega e^{i\omega t} \qquad \frac{d^2}{dt^2}e^{i\omega t} = -\omega^2 e^{i\omega t}$$

$$-m\omega^2 Ae^{i\omega t} = -kAei^{\omega t} - ri\omega Aei^{\omega t} + F_0 e^{i\omega t}$$

$$(k - m\omega^2 + ir\omega)A = F_0$$

Waves with complex numbers

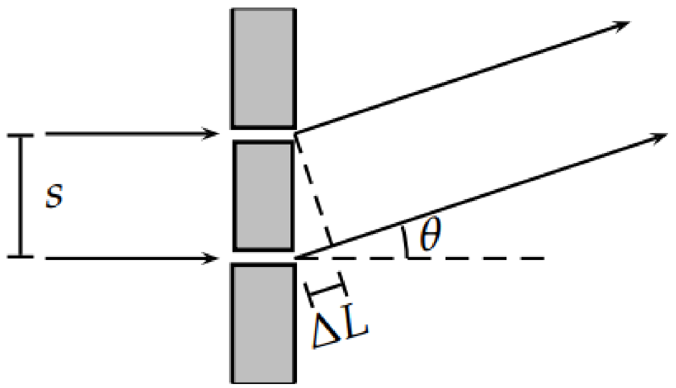
- $y = A\cos(\omega t - kx) = \mathrm{Re}\, Ae^{i\omega t}e^{-ikx}$
- Adding two waves travelling different distances:
- $$\begin{aligned} y = \mathrm{Re}\left(Ae^{i\omega t}e^{-ikx} + Ae^{i\omega t}e^{-ik(x+\Delta)}\right) \\ = \mathrm{Re}\left\{e^{i\omega t}\left(Ae^{-ikx} + Ae^{-ik(x+\Delta)}\right)\right\} \\ = \mathrm{Re}\left\{e^{i\omega t-ikx}\left(A + Ae^{-ik\Delta}\right)\right\} \end{aligned}$$
- Modulus of $\mathrm{R} = A + Ae^{-ik\Delta}$ gives overall amplitude

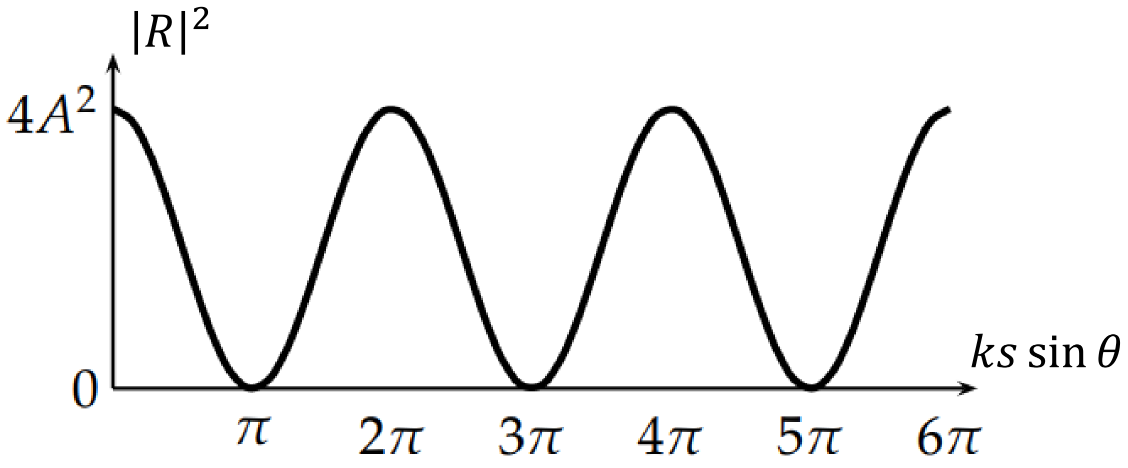
Waves with path difference

- $y = Ae^{i(\omega t - kL_1)} + Ae^{i(\omega t - kL_2)}$
- work out overall amplitude in terms of $\Delta L = L_2 - L_1$

Two slits

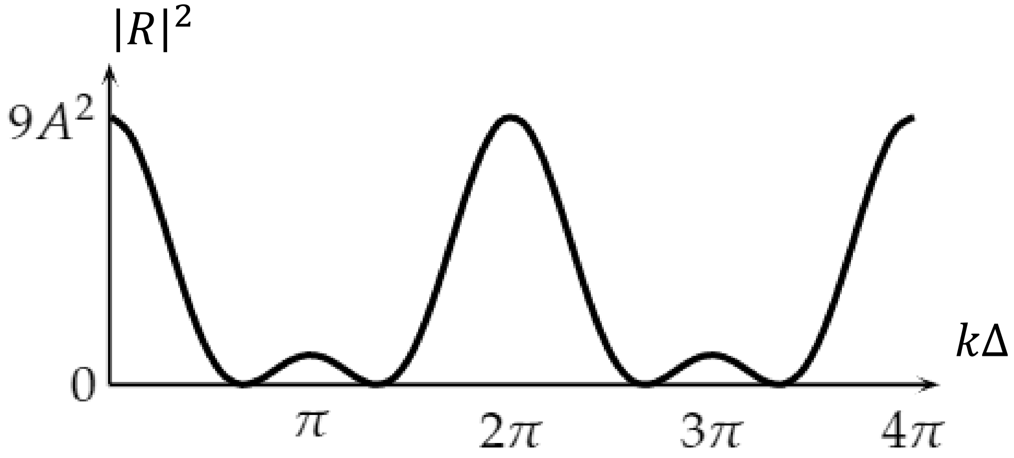
› $R = A + Ae^{-ik\,\Delta L} = A\left(1 + e^{-ik\,\Delta L}\right) = A\left(1 + e^{-iks\sin\theta}\right)$





3 slits

- $R = A + Ae^{-ik\Delta} + Ae^{-2ik\Delta}$ Hint: $R = e^{-ik\Delta}\left(e^{ik\Delta} + 1 + e^{-ik\Delta}\right)$

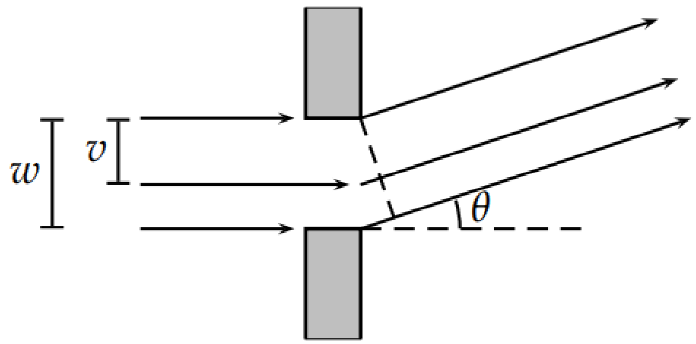


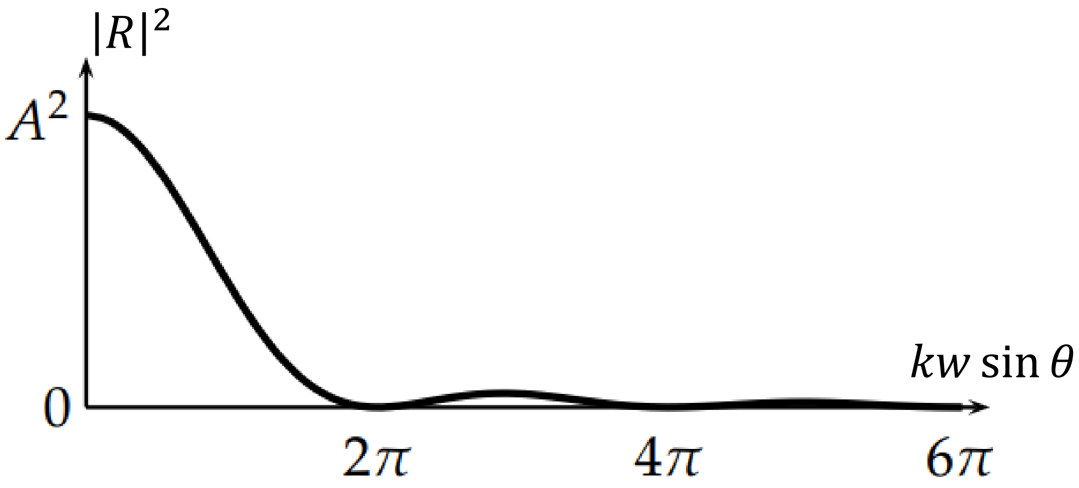
N slits

- $R = A + Ae^{-ik\Delta} + Ae^{-2ik\Delta} + \cdots + Ae^{-(N-1)ik\Delta}$

A single, wide slit

- $R = \int_{v=0}^{w} \frac{Ae^{-ikv\sin\theta}}{w} dv$





QM and normal wave differences

- Normal wave – only the real part is meaningful
- Intensity $\propto \frac{1}{2}|A|^2$ as $\overline{\cos^2 \omega t} = \frac{1}{2}$
- Intensity at a place in space pulses as time progresses

- QM wave – really isn't real
- Probability density $\propto |\psi|^2$. No factor of ½.
- Intensity at a place in space does not pulse

Eigenfunctions

- We write $\psi = Ae^{i(kx-\omega t)}$
- Work out $\frac{\partial \psi}{\partial t}$ $\qquad \frac{\partial^2 \psi}{\partial t^2}$
- Work out $\frac{\partial \psi}{\partial x}$ $\qquad \frac{\partial^2 \psi}{\partial x^2}$
- Wave equation $\frac{\partial^2 \psi}{\partial x^2} = ?\,\frac{\partial^2 \psi}{\partial t^2}$
- $\hbar = \frac{h}{2\pi}$. Now write $p = \frac{h}{\lambda}$ in terms of $k, \hbar$
- $\mathrm{k}\psi =$ ... $\qquad \mathrm{p}\psi =$ ... $\qquad \frac{\mathrm{p}^2}{2\mathrm{m}}\,\psi =$ ...

Eigenfunctions

- We write $\psi = Ae^{i(kx-\omega t)}$
- Work out $\frac{\partial\psi}{\partial t} = -i\omega\psi$ $\frac{\partial^2\psi}{\partial t^2} = -\omega^2\psi$
- Work out $\frac{\partial\psi}{\partial x} = ik\psi$ $\frac{\partial^2\psi}{\partial x^2} = -k^2\psi$
- Wave equation $\frac{\partial^2\psi}{\partial x^2} = \frac{k^2}{\omega^2}\frac{\partial^2\psi}{\partial t^2} = = \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2}$
- $\hbar = \frac{h}{2\pi}$. Now write $p = \frac{h}{\lambda} = \frac{2\pi\hbar}{2\pi/k} = \hbar k$
- $k\psi = -i\frac{\partial}{\partial x}\psi$ $\mathrm{p}\psi = -i\hbar\frac{\partial}{\partial x}\psi$ $\frac{\mathrm{p}^2}{2\mathrm{m}}\psi = \frac{1}{2m}\left(-i\hbar\frac{\partial}{\partial x}\right)^2\psi$

An important reminder

- Differential equation with negative sign
  - $\frac{\partial^2\psi}{\partial x^2} = -k^2\psi$
  - Solution $\psi = A\cos kx + B\sin kx$ or $\psi = Ae^{ikx} + Be^{-ikx}$
  - Oscillatory with wavelength $\lambda = 2\,\pi/k$
- Differential equation with positive sign
  - $\frac{\partial^2\psi}{\partial x^2} = +\mu^2\psi$
  - Solution $\psi = Ae^{\mu x} + Be^{-\mu x}$
  - Evanescent (decaying) with halving distance $\ln 2/\mu$