Differentiation from First Principles 2

A Level

Differentiating a function $f(x)$ from first principles involves taking a limit. The derivative of $f(x)$ is given by the expression

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}.$$

Part A Differentiate $4x^2+2x+7$ from first principles

Differentiate $f(x) = 4x^2+2x+7$ from first principles. Drag and drop options into the spaces below.

$f(x+h) = 4(x+h)^2+2(x+h)+7$. Substituting this into the expression for $f'(x)$,

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$f'(x) = \lim_{h\to 0} \frac{(4(x+h)^2+2(x+h)+7)-(4x^2+2x+7)}{h}.$$

Next, expanding the brackets in the numerator and simplifying gives

$$f'(x) = \lim_{h\to 0} \frac{\boxed{}-(4x^2+2x+7)}{h}$$

$$f'(x) = \lim_{h\to 0}(\boxed{}+(\boxed{})h).$$

Finally, take the limit. As $h\to 0$, the terms containing h tend to 0. Therefore,

$$f'(x) = \boxed{}.$$

Items:

$4x^2+4h^2$ | $8x+2$ | $4x^2+2x+7+4hx+2h+4h^2$ | 4 | $8x+4$ | 7 | $4x^2+2x+7+8hx+2h+4h^2$

Part B Differentiate $ax^2 + bx + c$ from first principles

Differentiate $f(x) = ax^2 + bx + c$, where a, b and c are constants, from first principles.

$f(x+h) = a(x+h)^2 + b(x+h) + c$. Substituting this into the expression for $f'(x)$,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$f'(x) = \lim_{h \to 0} \frac{(a(x+h)^2 + b(x+h) + c) - (ax^2 + bx + c)}{h}.$$

Next, expanding the brackets in the numerator and simplifying gives

$$f'(x) = \lim_{h \to 0} \frac{\boxed{} + (\boxed{})h + (\boxed{})h^2}{h}$$

$$f'(x) = \lim_{h \to 0} (\boxed{} + (\boxed{})h).$$

Finally, take the limit. As $h \to 0$, the terms containing h tend to 0. Therefore,

$$f'(x) = \boxed{}.$$

Items:

$b + ah$ | 1 | ab | $2a$ | $ax^2 + 2ahx + ah^2$ | $2ax + b$ | a | 0

Created for isaacphysics.org by Jonathan Waugh

Calculus

A Level

C C C

Part A Integrating a factorised expression

Find $\int (x^2 + 9)(x - 4)\mathrm{d}x$.

The following symbols may be useful: `c, x`

Part B Differentiation

A curve has the equation $y = \frac{1}{3}x^3 - 9x$.

Find $\frac{\mathrm{d}y}{\mathrm{d}x}$.

The following symbols may be useful: `Derivative(y, x), x, y`

Part C Stationary points

Find the coordinates of the stationary points of the curve $y = \frac{1}{3}x^3 - 9x$. Enter the x and y coordinates of the stationary point with the largest x coordinate.

Enter the x-coordinate of the stationary point with the largest (most positive) x:

The following symbols may be useful: `x`

Enter its corresponding y coordinate:

The following symbols may be useful: `y`

Part D Nature of stationary point

Determine the nature of the stationary point with the largest x-coordinate.

- Minimum
- Maximum
- Neither/Inconclusive

Part E Tangent to the curve

Given that $24x + 3y + 2 = 0$ is the equation of the tangent to the curve $y = \frac{1}{3}x^3 - 9x$ at the point (p, q), find the values of p and q.

(i) Enter value of p:

The following symbols may be useful: p

(ii) Enter value of q:

The following symbols may be useful: q

Part F Normal to the curve

Find the equation of the normal to the curve $y = \frac{1}{3}x^3 - 9x$ at the point (p, q) you found in Part E.

Give your answer in the form $ax + by + c = 0$, where a, b, and c are integers

The following symbols may be useful: x, y

Modified by Sally Waugh with permission from UCLES, A Level, June 2005, Paper 4721, Question 10.

Gameboard:
[STEM SMART Double Maths 16 - Calculus Revision]

Area of Isosceles Triangle

A Level C C C Further A P P P

The isosceles triangle shown in **Figure 1** has a base of length $2b$ and perpendicular height h. The length p of the perimeter of the triangle is fixed. Find an expression in terms of p for the value of b which will maximise the area A of the triangle. Find an expression for this maximum area.

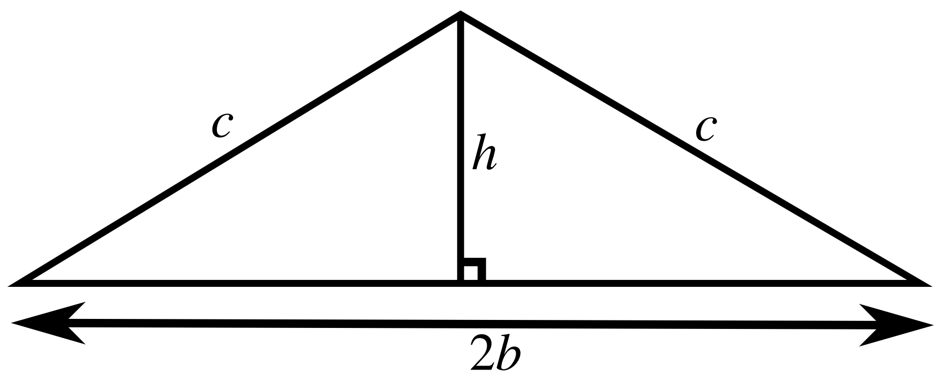

Figure 1: An isosceles triangle with a base of length $2b$, perpendicular height h and sides of length c.

Part A   Area A and perimeter p

Write down the equation for the area A of the triangle in terms of b and h.

The following symbols may be useful: `A, b, h`

Find the equation for the perimeter p of the triangle in terms of b and h.

- $p = 2b + \sqrt{4b^2 + h^2}$
- $p = 2b + 2\sqrt{4b^2 + h^2}$
- $p = b + 2\sqrt{b^2 + h^2}$
- $p = 2\left(b + \sqrt{b^2 + h^2}\right)$
- $p = 2b + \sqrt{b^2 + h^2}$
- $p = b + \sqrt{b^2 + h^2}$

Using the above, obtain an equation for A in terms of p and b.

The following symbols may be useful: `A, b, p`

Part B Expressions for b and h

Using the equation for A you found in Part A, find an **expression** in terms of p for the value of b which will maximise the area A of the triangle. (Since p is fixed you may treat it as a constant.)

Hint: you may not know how to differentiate the expression for A, but note that since A is positive it will be a maximum when A^2 is a maximum.

The following symbols may be useful: p

Find, in terms of p, the expression for h corresponding to this value of b.

The following symbols may be useful: p

Part C The maximum area

Using your result from Part B, find an expression for the maximum area in terms of p.

The following symbols may be useful: p

Part D Check that the area is a maximum

Find, at the value of b deduced above, an expression in terms of p for the second derivative of A^2 with respect to b; convince yourself that the value of the second derivative indicates that the value of A^2, and hence of A, is a maximum.

The following symbols may be useful: p

Created for isaacphysics.org by Julia Riley

Gameboard:
STEM SMART Double Maths 16 - Calculus Revision

Differentiation and Gradients: Beyond Polynomials 4ii

A Level

Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ in each of the following cases.

Part A Derivative 1

$$y = x^3 e^{2x}$$

The following symbols may be useful: `Derivative(y, x), e, ln(), log(), x, y`

Part B Derivative 2

$$y = \ln\left(3 + 2x^2\right)$$

The following symbols may be useful: `Derivative(y, x), e, ln(), log(), x, y`

Part C Derivative 3

$$y = \frac{x}{2x+1}$$

The following symbols may be useful: `Derivative(y, x), e, ln(), log(), x, y`

Used with permission from UCLES A-level Maths papers, 2003-2017.

Gameboard:

STEM SMART Double Maths 16 - Calculus Revision

Differentiating Trig Functions 2

A Level P P P Further A P P P

Part A Differentiate $s = r\sin(\alpha\theta)$

Find $\frac{\mathrm{d}s}{\mathrm{d}\theta}$ if $s = r\sin(\alpha\theta)$ and r and α are constants.

The following symbols may be useful: `alpha`, `r`, `theta`

Part B Differentiate $q = l\cos(\alpha - 2\beta\theta)$

Find $\frac{\mathrm{d}q}{\mathrm{d}\theta}$ if $q = l\cos(\alpha - 2\beta\theta)$ and l, α and β are constants.

The following symbols may be useful: `alpha`, `beta`, `l`, `theta`

Created for isaacphysics.org by Julia Riley

Gameboard:
STEM SMART Double Maths 16 - Calculus Revision

Differentiation: Chain Rule 4ii

A Level

P P P

The volume, V cubic metres, of water in a reservoir is given by

$$V = 3(2+\sqrt{h})^6 - 192,$$

where h metres is the depth of the water. Water is flowing into the reservoir at a constant rate of 150 cubic metres per hour. Find the rate at which the depth of water is increasing at the instant when the depth is 1.4 metres, to three significant figures, in metres per hour.

Gameboard:

STEM SMART Double Maths 16 - Calculus Revision

Area Between Two Curves 4ii

A Level

P P P

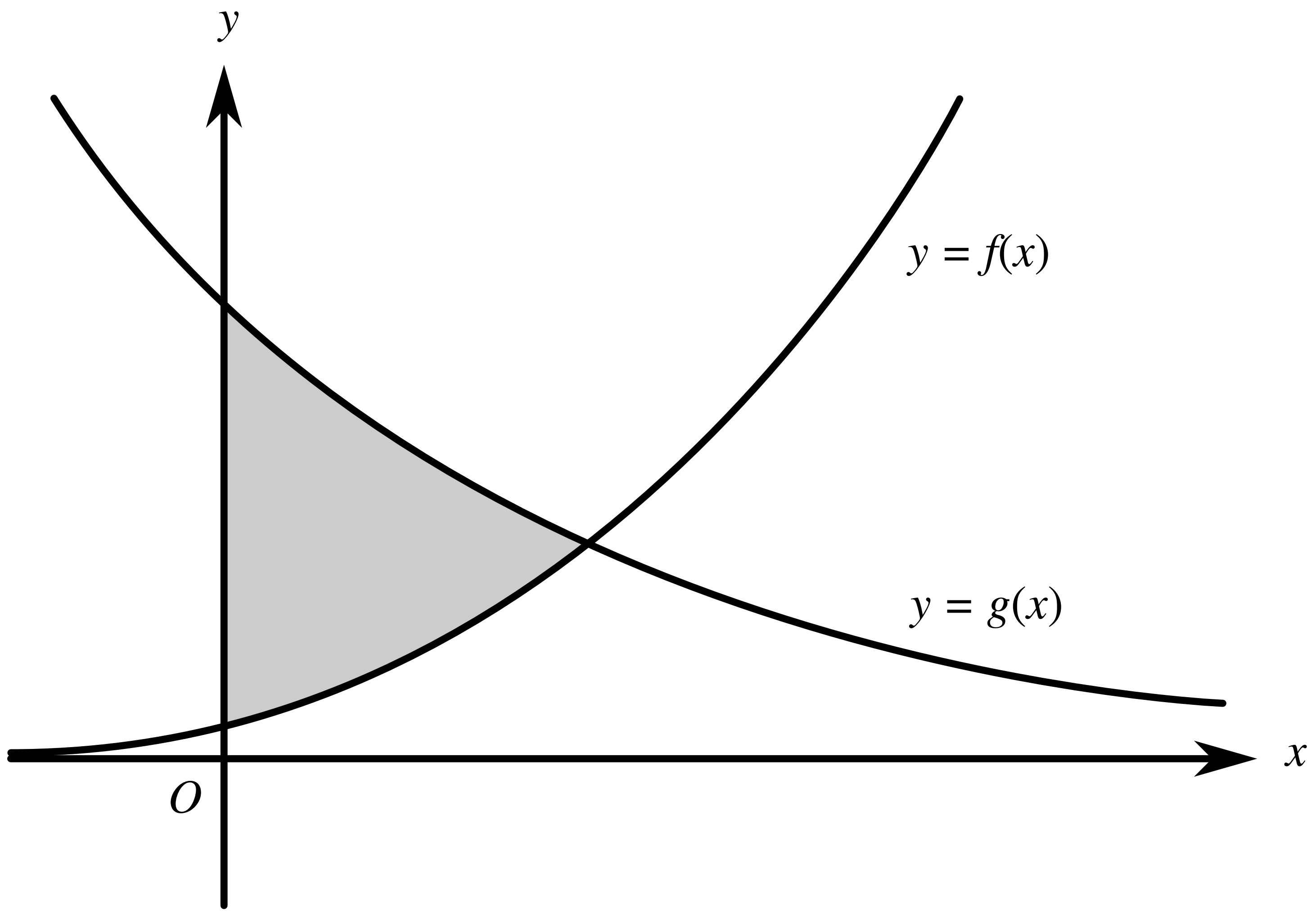

Figure 1: The curves $y = f(x)$ and $y = g(x)$.

Figure 1 shows the curves $y = f(x)$ and $y = g(x)$, where

$$f(x) = \mathrm{e}^{2x}$$

$$g(x) = 8\mathrm{e}^{-x}$$

The shaded region is bounded by the curves and the y-axis.

Part A x-coordinate

Find the exact x-coordinate of the point of intersection of the curves.

The following symbols may be useful: x

Part B Integral

Find the exact area of the shaded region. Give your answer as a fraction.

Gameboard:

STEM SMART Double Maths 16 - Calculus Revision

Integration by Substitution 3i

A Level

Part A Substitution

Find the expression that appears to the right of the integral sign after the substitution $u = \mathrm{e}^x + 1$ has been applied to $\int \frac{\mathrm{e}^{2x}}{\mathrm{e}^x + 1}\mathrm{d}x$. Include $\mathrm{d}u$ in your answer.

The following symbols may be useful: `du`, `u`

Part B Integral

Hence find the exact value of $\int_0^1 \frac{\mathrm{e}^{2x}}{\mathrm{e}^x + 1}\mathrm{d}x$.

The following symbols may be useful: `e`

Gameboard:

STEM SMART Double Maths 16 - Calculus Revision

Home Gameboard Maths Integration by Parts 2ii

Integration by Parts 2ii

A Level

Find the exact value of $\int_1^8 \frac{1}{\sqrt[3]{x}} \ln(x) \mathrm{d}x$, giving your answer in the form $A \ln(2) + B$, where A and B are constants to be found.

Used with permission from UCLES A-level Maths papers, 2003-2017.