Polynomials, Factors and Roots 1i

A Level
P P P

The polynomial $f(x)$ is defined by

$$f(x) = x^3 + px + q,$$

where p and q are constants. It is given that $x + 1$ and $x - 3$ are factors of $f(x)$.

Part A Values of p and q

Find the value of p.

The following symbols may be useful: p

Find the value of q.

The following symbols may be useful: q

Part B The equation $f(x) = 0$

Solve the equation $f(x) = 0$, and state the greatest value of x for which $f(x) = 0$.

The following symbols may be useful: x

Part C Simplify

Simplify $(x-5)(x^2+3)-(x+4)(x-1)$. Give your answer as a polynomial with the highest power of x first.

The following symbols may be useful: x

Used with permission from UCLES, A Level, June 2009, Paper 4721, Question 5.

Polynomials, Factors and Roots 4i

A Level

P P P

The polynomial $f(x)$ is given by $f(x) = 2x^3 + 9x^2 + 11x - 8$.

Part A Factors

Using the factor theorem decide whether $(2x - 1)$ is a factor of $f(x)$ or not.

- $(2x - 1)$ is not a factor of $f(x)$
- $(2x - 1)$ is a factor of $f(x)$

Part B Find quadratic factor

Express $f(x)$ as a product of a linear factor and a quadratic factor.

The following symbols may be useful: x

Part C Real roots

State the number of real roots to the equation $f(x) = 0$.

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Algebraic Division 5ii

A Level

P P P

Part A Quotient and Remainder 1

Find the quotient and remainder when $3x^4 - x^3 - 3x^2 - 14x - 8$ is divided by $x^2 + x + 2$.

Give the quotient.

The following symbols may be useful: x

Give the remainder.

The following symbols may be useful: x

Part B Quotient and Remainder 2

Find the quotient and remainder when $4x^3 + 8x^2 - 5x + 12$ is divided by $2x^2 + 1$.

Give the quotient.

The following symbols may be useful: x

Give the remainder.

The following symbols may be useful: x

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Physics. You work it out.

Home Gameboard Maths Algebraic Division 5i

Algebraic Division 5i

A Level
P P P

Part A Quotient and Remainder

Find the quotient and remainder when $x^4 + 1$ is divided by $x^2 + 1$.

State the quotient.

The following symbols may be useful: x

State the remainder.

Part B Find $f(x)$

When the polynomial $f(x)$ is divided by $x^2 + 1$, the quotient is $x^2 + 4x + 2$ and the remainder is $x - 1$. Find $f(x)$, simplifying your answer.

The following symbols may be useful: x

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Physics. You work it out.

Algebraic Division 3ii

A Level

P P P

The cubic polynomial $ax^3 - 4x^2 - 7ax + 12$ is denoted by $f(x)$.

Part A Value of a

Given that $(x - 3)$ is a factor of $f(x)$, find the value of the constant a.

The following symbols may be useful: a

Part B Remainder

Using this value of a, find the remainder when $f(x)$ is divided by $(x + 2)$.

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Disproof by Counter-example

A Level

In each part, choose the numerical value or expression from the list which can be used to show that the general statement is NOT true.

Part A Inequality

Choose the numerical value which can be used to show that this statement is NOT true:

"*The solution to the inequality* $x^2 + 5x + 7 \geq 9x + 4$ *is* $1 \leq x \leq 3$."

- $x = 2$
- $x = 1$
- $x = 3$

Part B Multiples of 11

Choose the numerical value for k in the list which can be used to show that this statement is NOT true:

"$10^k + 1$, *where* k *is a positive integer, is always a multiple of eleven.*"

- $k = 1$
- $k = 3$
- $k = 6$
- $k = 7$

Part C Integrating powers

Choose the numerical values for a and b, or the mathematical expression, which can be used to show that this statement is NOT true:

"*For all real values of a and b, $\int \frac{x^a}{x^b} dx = \frac{x^{a-b+1}}{a-b+1} + c$, where c is a constant.*"

- $a = 3, b = 2$
- $b - a = 1$
- $b = a + \frac{1}{2}$
- $a = 4, b = 6$

Part D Triangular and Fibonacci Numbers

The sequence of triangular numbers can be defined by this term-to-term relationship:

$T_1 = 1, T_n = T_{n-1} + n$ for $n > 1$.

The Fibonacci sequence can be defined by this term-to-term relationship:

$F_1 = 0, F_2 = 1, F_k = F_{k-1} + F_{k-2}$ for $k > 2$.

Choose the integer from the list which can be used to show that this statement is NOT true:

"*There is no integer greater than 3 which is both a triangular number and a Fibonacci number.*"

- 21
- 13
- 120
- 14

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Proof and Hollow Pyramids

A Level

A hollow pyramid shape can be made by stacking identical spheres.

Part A Square-based pyramids

The diagram below shows the first three pyramids in a sequence of square-based hollow pyramids.

Figure 1: The first three square-based hollow pyramids, with sides made up of 4, 5 and 6 identical spheres.

Let the number of spheres that make up the k^{th} pyramid in the sequence be S_k. From the list below, choose the correct expression for S_k.

- $8k + 21$
- $4k + 5$
- $8k + 13$
- $16k - 11$

Part B Triangle-based pyramids

The diagram below shows the first three pyramids in a sequence of triangle-based hollow pyramids.

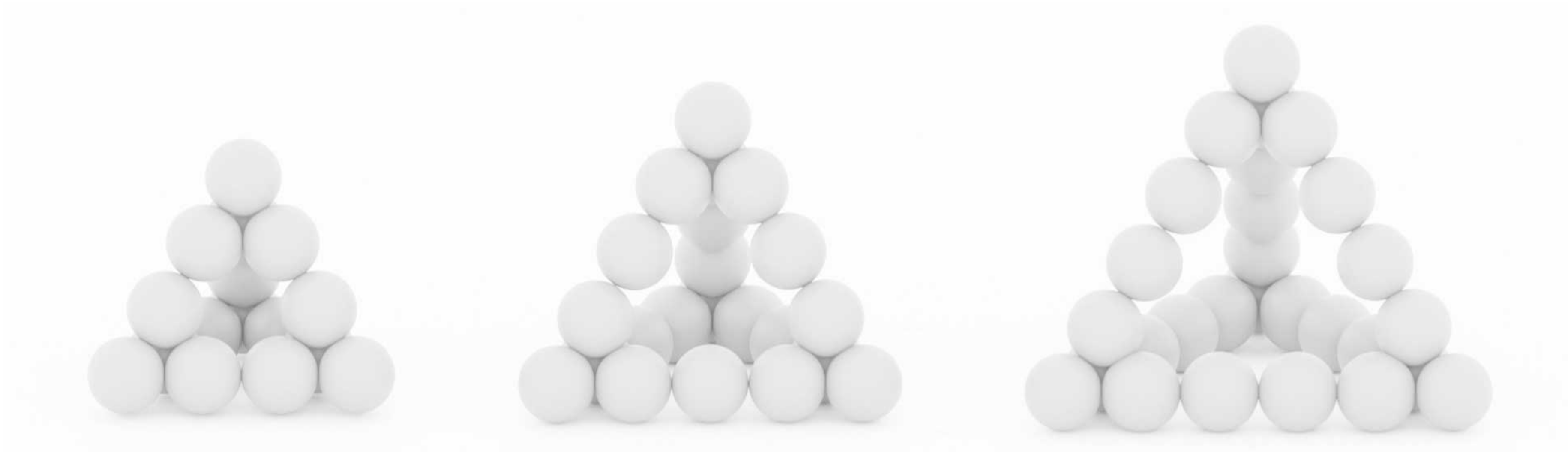

Figure 2: The first three triangle-based hollow pyramids, with sides made up of 4, 5 and 6 identical spheres.

Find an expression for T_n, the number of spheres that make up the n^{th} pyramid in this sequence.

The following symbols may be useful: n

Part C Is rearrangement possible?

Prove that it is not possible to rearrange the spheres making up any square-based pyramid to produce a triangle-based pyramid (of any size) without either having spheres left over or needing extra spheres.

Once you have worked out a proof yourself, consider the model proof below. Drag and drop answers into the blank spaces to complete the proof. You may use the same answer more than once.

We will use proof by deduction.

Reasoning:

The number of spheres making up the k^{th} hollow square-based pyramid is given by $8k + 13$. For any positive ______ value of k, $8k$ is ______. Hence, $8k + 13$ is always ______.

The number of spheres making up the n^{th} hollow triangle-based pyramid is given by ______. For any positive ______ value of n, $6n$ is ______. Hence, ______ is always even.

Therefore, the number of spheres required to make a hollow square-based pyramid ______ the same as the number of spheres required to make a hollow triangle-based pyramid.

Conclusion:

Hence, it is not possible to rearrange the spheres making up any square-based pyramid to produce a triangle-based pyramid (of any size) without either having spheres left over or needing extra spheres.

Items:

even | **integer** | **odd** | $6n + 10$ | **can never be** | **is always** | **rational** | $10n + 6$ | **fractional**

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STEM SMART Single Maths 19 - Polynomials and Proof

Divisibility by Exhaustion

A Level Further A

A sequence u_n is defined by $u_n = n^7 - n$, where $n \in \mathbb{N}$. The first four terms of this sequence are

$$0, 126, 2184, 16380, \ldots$$

What is the largest integer that will divide every term of this sequence?

Part A Factorise u_n

Factorise u_n completely.

The following symbols may be useful: n

Part B Divisibility by 2

Using your expression from part A, prove that every term in the sequence is divisible by 2.

Once you have worked out a proof yourself, consider the model proof below. Drag and drop answers into the blank spaces to complete the proof.

We know that $u_n = (n-1)n(n+1)(n^2+n+1)(n^2-n+1)$.

When n is even, it is divisible by 2 and we can see that [ ] is a factor of u_n, so u_n is divisible by 2.

When n is odd, we can write $n =$ [ ] in terms of k, where $k \in \mathbb{Z}$. Then the factor [ ] $=$ [ ] in terms of k, so the factor [ ] is divisible by 2, and hence u_n is divisible by 2.

Therefore, u_n is divisible by 2 for any value of n. So every term in the sequence is divisible by 2.

Items:

$n-1$ | $n+1$ | n | n^2-n+1 | n^2+n+1 | $2k$ | $2k+1$

Part C Divisibility by 3

Using your expression from part A, prove that every term in the sequence is divisible by 3.

Once you have worked out a proof yourself, consider the model proof below. Drag and drop answers into the blank spaces to complete the proof.

We know that $u_n = (n-1)n(n+1)(n^2+n+1)(n^2-n+1)$.

When n is a multiple of 3, it is divisible by 3 and we can see that ______ is a factor of u_n, so u_n is divisible by 3.

When $n = 3k+1$, where $k \in \mathbb{Z}$, then the factor ______ $=$ ______ in terms of k, so the factor ______ is divisible by 3, and hence u_n is divisible by 3.

When $n = 3k+2$, where $k \in \mathbb{Z}$, then the factor ______ $=$ ______ in terms of k, so the factor ______ is divisible by 3, and hence u_n is divisible by 3.

Therefore, u_n is divisible by 3 for any value of n. So every term in the sequence is divisible by 3.

Items:

$3k$ | n^2+n+1 | $3k+3$ | $n-1$ | $3k-3$ | n^2-n+1 | $n+1$ | n | $3k+2$ | $3k+1$

Part D Divisibility by 7

Using your expression from part A, prove that every term in the sequence is divisible by 7.

Once you have worked out a proof yourself, consider the model proof below. Drag and drop answers into the blank spaces to complete the proof. You may use the same answer more than once.

We know that $u_n = (n-1)n(n+1)(n^2+n+1)(n^2-n+1)$.

When n is a multiple of 7, it is divisible by 7 and we can see that ____ is a factor of u_n, so u_n is divisible by 7.

When $n = 7k+1$, where $k \in \mathbb{Z}$, then the factor ____ $=$ ____ in terms of k, so the factor ____ is divisible by 7, and hence u_n is divisible by 7.

When $n = 7k+2$, where $k \in \mathbb{Z}$, then the factor ____ $=$ ____ in terms of k, so the factor ____ is divisible by 7, and hence u_n is divisible by 7.

When $n = 7k+3$, where $k \in \mathbb{Z}$, then the factor ____ $=$ ____ in terms of k, so the factor ____ is divisible by 7, and hence u_n is divisible by 7.

When $n = 7k+4$, where $k \in \mathbb{Z}$, then the factor ____ $=$ ____ in terms of k, so the factor ____ is divisible by 7, and hence u_n is divisible by 7.

When $n = 7k+5$, where $k \in \mathbb{Z}$, then the factor ____ $=$ ____ in terms of k, so the factor ____ is divisible by 7, and hence u_n is divisible by 7.

When $n = 7k+6$, where $k \in \mathbb{Z}$, then the factor ____ $=$ ____ in terms of k, so the factor ____ is divisible by 7, and hence u_n is divisible by 7.

Therefore, u_n is divisible by 7 for any value of n. So every term in the sequence is divisible by 7.

Items:

n^2+n+1 | n | $n-1$ | $n+1$ | $49k^2+35k+7$ | $7k$ | n^2-n+1 | $7k+7$ | $49k^2+63k+21$

Part E Largest Divisor

Prove that 42 is the largest integer that will divide every term of u_n.

Once you have worked out a proof yourself, consider the model proof below. Drag and drop answers into the blank spaces to complete the proof. You may use the same answer more than once.

We know that u_n is divisible by 2, 3 and 7. So we know that $2 \times 3 \times 7 =$ ☐ will divide u_n. Are there any larger integers that can do so?

Let's consider the first non-zero term, 126. We find that $126 \div 42 =$ ☐. This shows that the prime factorisation of 126 is ☐. Hence, the only larger factors of 126 are (in order of increasing size) ☐ and ☐. Will these divide any other terms of u_n?

Looking at the next term, we find that $2184 \div$ ☐ $= \frac{104}{3}$, so ☐ does not divide 2184. Considering our other factor, we find that $2184 \div$ ☐ $= \frac{52}{3}$, so ☐ does not divide 2184 either.

Therefore, 42 is the largest integer that will divide every term of u_n.

Items:

42 | 5 | $2 \times 3^2 \times 7$ | 3 | 7 | 126 | $2^2 \times 3 \times 7$ | 18 | 2 | 63 | 45 | $2 \times 3^2 \times 5$

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