Polynomials, Factors and Roots 4i

A Level
P P P

The polynomial $f(x)$ is given by $f(x) = 2x^3 + 9x^2 + 11x - 8$.

Part A Factors

Using the factor theorem decide whether $(2x - 1)$ is a factor of $f(x)$ or not.

- $(2x - 1)$ is not a factor of $f(x)$
- $(2x - 1)$ is a factor of $f(x)$

Part B Find quadratic factor

Express $f(x)$ as a product of a linear factor and a quadratic factor.

The following symbols may be useful: x

Part C Real roots

State the number of real roots to the equation $f(x) = 0$.

Algebraic Division 5ii

A Level
P P P

Part A Quotient and Remainder 1

Find the quotient and remainder when $3x^4 - x^3 - 3x^2 - 14x - 8$ is divided by $x^2 + x + 2$.

Give the quotient.

The following symbols may be useful: x

Give the remainder.

The following symbols may be useful: x

Part B Quotient and Remainder 2

Find the quotient and remainder when $4x^3 + 8x^2 - 5x + 12$ is divided by $2x^2 + 1$.

Give the quotient.

The following symbols may be useful: x

Give the remainder.

The following symbols may be useful: x

Algebraic Division 5i

A Level
P P P

Part A Quotient and Remainder

Find the quotient and remainder when $x^4 + 1$ is divided by $x^2 + 1$.

State the quotient.

The following symbols may be useful: x

State the remainder.

Part B Find $f(x)$

When the polynomial $f(x)$ is divided by $x^2 + 1$, the quotient is $x^2 + 4x + 2$ and the remainder is $x - 1$. Find $f(x)$, simplifying your answer.

The following symbols may be useful: x

Used with permission from UCLES A-level Maths papers, 2003-2017.

Gameboard:
STEM SMART Double Maths 18 - Polynomials, Proof & Induction

Algebraic Division 3ii

A Level

P P P

The cubic polynomial $ax^3 - 4x^2 - 7ax + 12$ is denoted by $f(x)$.

Part A Value of a

Given that $(x - 3)$ is a factor of $f(x)$, find the value of the constant a.

The following symbols may be useful: a

Part B Remainder

Using this value of a, find the remainder when $f(x)$ is divided by $(x + 2)$.

Gameboard:

STEM SMART Double Maths 18 - Polynomials, Proof & Induction

Proof and Hollow Pyramids

A Level

A hollow pyramid shape can be made by stacking identical spheres.

Part A Square-based pyramids

The diagram below shows the first three pyramids in a sequence of square-based hollow pyramids.

Figure 1: The first three square-based hollow pyramids, with sides made up of 4, 5 and 6 identical spheres. These are the pyramids for $k = 1$, $k = 2$ and $k = 3$.

Let the number of spheres that make up the kth pyramid in the sequence be S_k. From the list below, choose the correct expression for S_k.

- $8k + 21$
- $4k + 5$
- $8k + 13$
- $16k - 11$

Part B Triangle-based pyramids

The diagram below shows the first three pyramids in a sequence of triangle-based hollow pyramids.

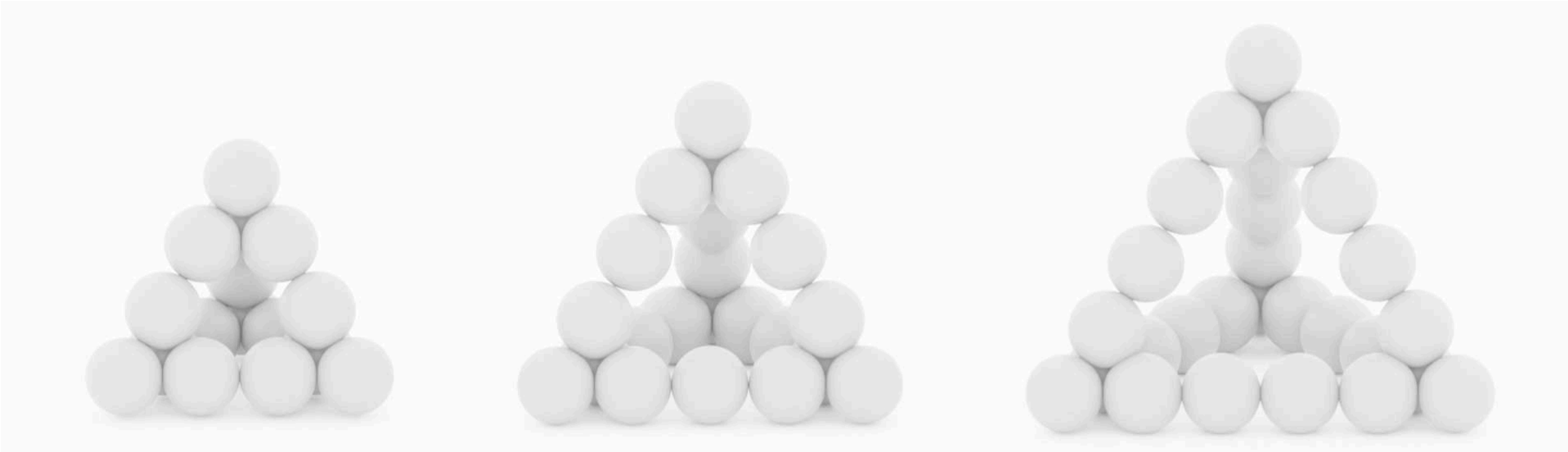

Figure 2: The first three triangle-based hollow pyramids, with sides made up of 4, 5 and 6 identical spheres. These are the pyramids for $n = 1$, $n = 2$ and $n = 3$.

Find an expression for T_n, the number of spheres that make up the nth pyramid in this sequence.

The following symbols may be useful: `T_n, n`

Part C Is rearrangement possible?

Prove that it is not possible to rearrange the spheres making up any square-based pyramid to produce a triangle-based pyramid (of any size) without either having spheres left over or needing extra spheres.

Once you have worked out a proof yourself, consider the model proof below. Drag and drop answers into the blank spaces to complete the proof. You may use the same answer more than once.

We will use proof by deduction.

Reasoning:

The number of spheres making up the kth hollow square-based pyramid is given by $8k + 13$. For any positive ______ value of k, $8k$ is ______. Hence, $8k + 13$ is always ______.

The number of spheres making up the nth hollow triangle-based pyramid is given by ______. For any positive ______ value of n, $6n$ is ______. Hence, ______ is always even.

Therefore, the number of spheres required to make a hollow square-based pyramid ______ the same as the number of spheres required to make a hollow triangle-based pyramid.

Conclusion:

Hence, it is not possible to rearrange the spheres making up any square-based pyramid to produce a triangle-based pyramid (of any size) without either having spheres left over or needing extra spheres.

Items:

rational **even** **integer** $10n + 6$ $6n + 10$ **is always** **fractional** **odd** **can never be**

Created for isaacphysics.org by J. Waugh

Gameboard:

STEM SMART Double Maths 18 - Polynomials, Proof & Induction

Induction: Sequences 1i

Further A

P P P

The sequence $u_1, u_2, u_3 \ldots$ is defined by $u_1 = 2$ and $u_{n+1} = \frac{u_n}{1+u_n}$ for $n \geq 1$.

Part A u_2, u_3, and u_4

Find u_2, u_3 and u_4, giving your answers as simplified fractions.

$u_2 =$

$u_3 =$

$u_4 =$

Part B u_n in terms of n

Hence, suggest an expression for u_n in terms of n.

The following symbols may be useful: `n`, `u_n`

Part C Induction

Use induction to prove that your suggested expression in part B is correct for all positive integers n.

Base case

When $n = 1$, the expression gives

$$\begin{aligned} u_1 &= \frac{2}{\boxed{}} \\ &= \boxed{} \end{aligned}$$

Hence the expression works for $n = 1$.

Assumption

Let's assume the expression works when $n = k$, giving us $u_k = \boxed{}$.

Inductive step

Let's check if our assumption implies that the expression also works for $n = k + 1$. Using the recurrence relation,

$$\begin{aligned} u_{k+1} &= \frac{u_k}{1 + u_k} \\ &= \frac{\boxed{}}{1 + \boxed{}} \\ &= \boxed{} \\ &= \frac{2}{2(\boxed{}) - 1} \end{aligned}$$

Hence our assumption implies that the expression also works for $n = k + 1$.

Conclusion

We have shown that the expression is true for $n = 1$ and that if it is true for $n = k$ then it must also be true for $n = k + 1$. Therefore, by $\boxed{}$, the expression must be true for all integers $n \geq 1$.

Items:

$\frac{2}{1-2k}$ $\frac{2}{3}$ $2k+3$ $\frac{2}{2k+1}$ $\frac{2}{5}$ 2 $2(1)+1$ **induction** $\frac{2}{2k-1}$ **deduction** $k-1$ $2(1)-1$ $k+1$ $2k+1$ **exhaustion**

Adapted with permission from UCLES, A Level, Jan 2013, Paper 4725, Question 10.

Induction: Divisibility 1i

Further A

P P P

The sequence $u_1, u_2, u_3 \ldots$ is defined by $u_n = 5^n + 2^{n-1}$.

Part A u_1, u_2 and u_3

Find u_1, u_2 and u_3.

$u_1 =$

$u_2 =$

$u_3 =$

Part B Divisibility

Hence, suggest a positive integer, other than 1, which divides exactly into every term of the sequence.

Part C Induction

By considering $u_{n+1} + u_n$, prove by induction that the suggested integer in part B will divide every term in the sequence.

Base case

When $n = 1$, we see that

$$\begin{aligned} u_1 &= 5^1 + 2^{1-1} \\ &= 5 + \boxed{} \\ &= \boxed{} \times 2 \end{aligned}$$

Hence the integer divides u_1.

Assumption

Let's assume that the integer divides u_k, giving us

$$\begin{aligned} u_k &= 5^k + 2^{k-1} \\ &= \boxed{} A \end{aligned}$$

for some integer A.

Inductive step

Let's check if our assumption implies that the integer also divides u_{k+1}. Considering $u_{k+1} + u_k$, we find

$$\begin{aligned} u_{k+1} + u_k &= 5^{k+1} + 2^k + 5^k + 2^{k-1} \\ &= 5^k(\boxed{}) + 2^{k-1}(\boxed{}) \\ &= \boxed{}(2 \times 5^k + 2^{k-1}) \\ u_{k+1} + \boxed{} A &= \boxed{}(5^k + \boxed{} A) \\ u_{k+1} &= \boxed{}(5^k + \boxed{} A) \end{aligned}$$

Hence our assumption implies that the integer also divides u_{k+1}.

Conclusion

We have shown that the integer divides u_1 and that if it divides u_k then it must also divide u_{k+1}. Therefore, by $\boxed{}$, the integer divides every term in the sequence.

Items:

$\boxed{1}$ $\boxed{2}$ $\boxed{3}$ $\boxed{4}$ $\boxed{5}$ $\boxed{6}$ $\boxed{7}$ $\boxed{8}$ $\boxed{9}$ **induction** **deduction** **exhaustion**

Induction: Matrices 2i

Further A

P P P

The matrix $\mathbf{M}$ is given by $\mathbf{M} = \begin{pmatrix} 3 & 0 \\ 2 & 1 \end{pmatrix}$.

Part A $\mathbf{M}^4$

Find the matrix $\mathbf{M}^4$.

$$\begin{pmatrix} \square & \square \\ \square & \square \end{pmatrix}$$

Part B $\mathbf{M}^n$

Hence, suggest a suitable form for $\mathbf{M}^n$ in terms of n.

$$\mathbf{M}^n = \begin{pmatrix} \square & \square \\ \square & \square \end{pmatrix}$$

Items:

0 1 2 3 2^{n-1} $2^n - 1$ 2^n $2^n + 1$ 2^{n+1} 3^{n-1} $3^n - 1$ 3^n $3^n + 1$ 3^{n+1}

Part C Induction

Use induction to prove that your suggested form for $\mathbf{M}^n$ in part B is correct for all positive integers n.

Base case

When $n = 1$, the form gives

$$\begin{aligned}\mathbf{M}^1 &= \begin{pmatrix} \square & 0 \\ \square & 1 \end{pmatrix} \\ &= \mathbf{M}\end{aligned}$$

Hence the form works for $n = 1$.

Assumption

Let's assume the form works when $n = k$, giving us $\mathbf{M}^k = \begin{pmatrix} \square & 0 \\ \square & 1 \end{pmatrix}$.

Inductive step

Let's check if our assumption implies that the form also works for $n = k + 1$. We see that,

$$\begin{aligned}\mathbf{M}^{k+1} &= \mathbf{M} \times \mathbf{M}^k \\ &= \begin{pmatrix} 3 & 0 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} \square & 0 \\ \square & 1 \end{pmatrix} \\ &= \begin{pmatrix} \square & 0 \\ \square & 1 \end{pmatrix} \\ &= \begin{pmatrix} 3^{\square} & 0 \\ 3^{\square} - 1 & 1 \end{pmatrix}\end{aligned}$$

Hence our assumption implies that the form also works for $n = k + 1$.

Conclusion

We have shown that the form is true for $n = 1$ and that if it is true for $n = k$ then it must also be true for $n = k + 1$. Therefore, by $\square$, the form must be true for all integers $n \geq 1$.

Items:

$3^1 - 1$ | 3^1 | $3^1 + 1$ | $3^k - 1$ | 3^k | $3^k + 1$ | $2(3^k) - 1$ | $2(3^k)$ | $2(3^k) + 1$ | $3(3^k) - 1$ | $3(3^k)$ | $3(3^k) + 1$ | $k - 1$ | $k + 1$ | **induction** | **deduction** | **exhaustion**

Divisibility by Exhaustion

A Level C C C Further A C C C

A sequence u_n is defined by $u_n = n^7 - n$, where $n \in \mathbb{N}$. The first four terms of this sequence are

$$0, 126, 2184, 16380, \ldots$$

What is the largest integer that will divide every term of this sequence?

Part A Factorise u_n

Factorise u_n completely.

The following symbols may be useful: n

Part B Divisibility by 2

Using your expression from part A, prove that every term in the sequence is divisible by 2.

Once you have worked out a proof yourself, consider the model proof below. Drag and drop answers into the blank spaces to complete the proof.

We know that the factors of u_n from part A.

When n is even, it is divisible by 2 and we can see that ______ is a factor of u_n, so u_n is divisible by 2.

When n is odd, we can write $n =$ ______ in terms of k, where $k \in \mathbb{Z}$. Then the factor ______ $=$ ______ in terms of k, so the factor ______ is divisible by 2, and hence u_n is divisible by 2.

Therefore, u_n is divisible by 2 for any value of n. So every term in the sequence is divisible by 2.

Items:

$n^2 - n + 1$ | $n^2 + n + 1$ | $n - 1$ | $2k + 1$ | $2k$ | $n + 1$ | n

Part C Divisibility by 3

Using your expression from part A, prove that every term in the sequence is divisible by 3.

Once you have worked out a proof yourself, consider the model proof below. Drag and drop answers into the blank spaces to complete the proof.

We know that the factors of u_n from part A.

When n is a multiple of 3, it is divisible by 3 and we can see that [ ] is a factor of u_n, so u_n is divisible by 3.

When $n = 3k + 1$, where $k \in \mathbb{Z}$, then the factor [ ] $=$ [ ] in terms of k, so the factor [ ] is divisible by 3, and hence u_n is divisible by 3.

When $n = 3k + 2$, where $k \in \mathbb{Z}$, then the factor [ ] $=$ [ ] in terms of k, so the factor [ ] is divisible by 3, and hence u_n is divisible by 3.

Therefore, u_n is divisible by 3 for any value of n. So every term in the sequence is divisible by 3.

Items:

$n^2 + n + 1$ $3k + 3$ $3k + 1$ n $n - 1$ $3k$ $3k + 2$ $n + 1$ $n^2 - n + 1$ $3k - 3$

Part D Divisibility by 7

Using your expression from part A, prove that every term in the sequence is divisible by 7.

Once you have worked out a proof yourself, consider the model proof below. Drag and drop answers into the blank spaces to complete the proof. You may use the same answer more than once.

We know that the factors of u_n from part A.

When n is a multiple of 7, it is divisible by 7 and we can see that ______ is a factor of u_n, so u_n is divisible by 7.

When $n = 7k + 1$, where $k \in \mathbb{Z}$, then the factor ______ $=$ ______ in terms of k, so the factor ______ is divisible by 7, and hence u_n is divisible by 7.

When $n = 7k + 2$, where $k \in \mathbb{Z}$, then the factor ______ $=$ ______ in terms of k, so the factor ______ is divisible by 7, and hence u_n is divisible by 7.

When $n = 7k + 3$, where $k \in \mathbb{Z}$, then the factor ______ $=$ ______ in terms of k, so the factor ______ is divisible by 7, and hence u_n is divisible by 7.

When $n = 7k + 4$, where $k \in \mathbb{Z}$, then the factor ______ $=$ ______ in terms of k, so the factor ______ is divisible by 7, and hence u_n is divisible by 7.

When $n = 7k + 5$, where $k \in \mathbb{Z}$, then the factor ______ $=$ ______ in terms of k, so the factor ______ is divisible by 7, and hence u_n is divisible by 7.

When $n = 7k + 6$, where $k \in \mathbb{Z}$, then the factor ______ $=$ ______ in terms of k, so the factor ______ is divisible by 7, and hence u_n is divisible by 7.

Therefore, u_n is divisible by 7 for any value of n. So every term in the sequence is divisible by 7.

Items:

$\boxed{7k+7}$ $\boxed{49k^2+35k+7}$ $\boxed{n}$ $\boxed{7k}$ $\boxed{n^2+n+1}$ $\boxed{49k^2+63k+21}$ $\boxed{n-1}$ $\boxed{n^2-n+1}$ $\boxed{n+1}$

Part E Largest Divisor

Prove that 42 is the largest integer that will divide every term of u_n.

Once you have worked out a proof yourself, consider the model proof below. Drag and drop answers into the blank spaces to complete the proof. You may use the same answer more than once.

We know from earlier that u_n is divisible by 2, 3 and 7. So we know that $2 \times 3 \times 7 =$ ______ will divide u_n. Are there any larger integers that can do so?

Let's consider the first non-zero term, 126. We find that $126 \div 42 =$ ______. This shows that the prime factorisation of 126 is ______. Hence, the only larger factors of 126 are (in order of increasing size) ______ and ______. Will these divide any other terms of u_n?

Looking at the next term, we find that $2184 \div$ ______ $= \frac{104}{3}$, so ______ does not divide 2184. Considering our other factor, we find that $2184 \div$ ______ $= \frac{52}{3}$, so ______ does not divide 2184 either.

Therefore, 42 is the largest integer that will divide every term of u_n.

Items:

45 | $2 \times 3^2 \times 5$ | $2^2 \times 3 \times 7$ | 42 | 3 | $2 \times 3^2 \times 7$ | 2 | 7 | 5 | 63 | 126 | 18

Created for isaacphysics.org by Matthew Rihan