Proof Set #1

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0.3.1 : *Show* $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$.

Proof:

Let us prove $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$. First, we must show that if $x \in A \setminus (B \cap C)$, then $x \in (A \setminus B) \cup (A \setminus C)$. Second, we must show that if $x \in (A \setminus B) \cup (A \setminus C)$, then $x \in A \setminus (B \cap C)$.

So let us assume $x \in A \setminus (B \cap C)$. Then $x \in A$, but $x \notin B \cap C$. Since $x \notin B \cap C$, then either $x \notin B$ or $x \notin C$. Thus we have two cases, where $x \in A$ and $x \notin B$ or where $x \in A$ and $x \notin C$. In the first case, $x \in A \setminus B$ and in the second case $x \in A \setminus C$. Thus, $x \in (A \setminus B) \cup (A \setminus C)$.

On the other hand, suppose $x \in (A \setminus B) \cup (A \setminus C)$. Thus $x \in A \setminus B$ or $x \in A \setminus C$. In the first case, $x \in A$ and $x \notin B$, and in the second case $x \in A$ and $x \notin C$. In both cases, $x \in A$. Also, $x \notin B$ or $x \notin C$, so $x \notin B \cap C$. Hence, $x \in A \setminus (B \cap C)$. This completes the proof.

(Not in Text) : Prove $2^n < n!$ for $n > 3$ by mathematical induction.

Proof:

Let us prove this by mathematical induction. We must first prove the base case. Our base case is when n = 4. We have $2^4 = 16 < 24 = 4!$. Thus the base case is proven.

We must now prove the induction step. Let n_0 be a natural number where $n_0 > 3$. We assume $2^{n_0} < n_0!$. We must prove $2^{n_0+1} < (n_0 + 1)!$. We have $2^{n_0+1} = 2^{n_0} * 2 < n_0! * 2 < n_0! * (n_0 + 1) = (n_0 + 1)!$. The first inequality follows from the induction hypothesis, and the second inequality follows from $n_0 > 3$. This completes the induction step, and thus completes the proof.

1.2.12 : If $S \subset \mathbb{R}$ is a nonempty set, bounded from above, then for every $\varepsilon > 0$ there exists $x \in S$ such that $(sup\ S) - \varepsilon < x \leq sup\ S$.

Proof:

Let us prove this by contradiction. Let us assume that there does not exist $x \in S$ such that $(sup\ S) - \varepsilon < x \leq sup\ S$ for every $\varepsilon > 0$. This would mean that $(sup\ S) - \varepsilon$ is an upper bound for S, as there is no $x \in S$ where $(sup\ S) - \varepsilon < x$. This is a contradiction, as $sup\ S$ is less than or equal to all upper bounds of S, but $(sup\ S) - \varepsilon < sup\ S$. This completes the proof.

2.3.9 : If $S \subset \mathbb{R}$ is a set, then $x \in \mathbb{R}$ is a cluster point if for every $\varepsilon > 0,$ the set $(x - \varepsilon, x + \varepsilon) \cap S \setminus \{x\}$ is not empty. That is, if there are points of S arbitrarily close to x. For example, $S := \{1/n : n \in \mathbb{N}\}$ has a unique (only one) cluster point 0, but 0 $\notin S$. Prove the following version of the Bolzano - Weierstrass theorem:

Theorem: Let $S \subset \mathbb{R}$ be a bounded infinite set, then there exists at least one cluster point of S.

Proof:

We know that S is infinite, so S contains a countably infinite subset. That is, there is a sequence $\{x_n\}$ of distinct numbers in $S,$ and since S is bounded, $\{x_n\}$ is bounded. By the Bolzano-Weierstrass Theorem, **Theorem 2.3.8** in the Jirka text, we know that there is a convergent subsequence of $\{x_n\},$ which we will call $\{x_{n_i}\}$. If we let $lim\ \{x_{n_i}\} = x,$ we can specify that if x is in $\{x_{n_i}\},$ we create a new sequence, $\{x_{n_j}\}$ which is equal to $\{x_{n_i}\}$ except with x removed. If x is not in $\{x_{n_i}\},$ let $\{x_{n_j}\} = \{x_{n_i}\}.$ By **Proposition 3.1.2** in the Jirka text, x is a cluster point of S if and only if there is a convergent sequence of numbers $\{x_n\}$ such that $x_n \neq x,\ x_n \in S,$ and $lim\ x_n = x$. We let $lim\ \{x_{n_j}\} = x.$ Thus, we know that x is a cluster point of $S.$ This completes the proof.

2.4.8 : True/False prove or find a counterexample: If $\{x_n\}$ is a Cauchy sequence then there exists an M such that for all $n \geq M$ we have $|x_{n+1} - x_n| \leq |x_n - x_{n-1}|$

Proof:

This statement is false. I will find a counterexample to show this. The counterexample must take the following form: it must be a sequence $\{x_n\}$ that is a Cauchy sequence but where there does not exist an M such that for all $n \geq M$ we have $|x_{n+1} - x_n| \leq |x_n - x_{n-1}|$.

Let us present the counterexample. Let our sequence $\{x_n\}$ be equal to $1/n$ if n is even and $1/(n-1)$ if n is odd. Thus, part of our sequence would be $\{...1/100,\ 1/100,\ 1/102,\ 1/102,\ ...\}$. We will first show that this sequence is a Cauchy sequence. Given $\varepsilon > 0$, find an M such that $M > 2/\varepsilon\ + 1$. Thus, $M - 1 > 2/\varepsilon$. So $1/(M-1) < \varepsilon/2$. Then for $n, k \geq M$, $1/(n-1) < \varepsilon/2$ and $1/(k-1) < \varepsilon/2$. It follows that $1/n < \varepsilon/2$ and $1/k < \varepsilon/2$. Therefore, for $n, k \geq M$, regardless of the parity of n and m, $x_n < \varepsilon/2$ and $x_k < \varepsilon/2$. Therefore, $|x_n - x_k| \leq |x_n| + |x_k| < \varepsilon/2 + \varepsilon/2 = \varepsilon$. Thus we have proven that $\{x_n\}$ is a Cauchy sequence.

We must now show that it is not true that for our counterexample sequence $\{x_n\}$, we have $|x_{n+1} - x_n| \leq |x_n - x_{n-1}|$. We can see that for all n, if n is odd, $x_n = 1/(n-1)$ and $x_{n-1} = 1/(n-1)$ so $|x_n - x_{n-1}| = 0$. But when n is odd, $x_{n+1} = 1/(n+1)$, so $|x_{n+1} - x_n| = |1/(n+1) - 1/(n-1)| = |-2/(n^2-1)| = 2/(n^2+1) > 0$. When n is odd, $|x_{n+1} - x_n| > |x_n - x_{n-1}|$. This shows that regardless of how big M is, there will be an n where n is odd and $n > M$. This completes the proof.

3.1.10 : Let c be a cluster point of $A \subset \mathbb{R}$, and $f: A \to \mathbb{R}$ be a function. Suppose for every sequence $\{x_n\}$ in A, such that $lim\, x_n = c$, the sequence $\{f(x_n)\}_{n=1}^{\inf}$ is Cauchy. Prove that $\lim_{x \to c} f(x)$ exists.

Proof:

We are proving that $\lim_{x \to c} f(x)$ exists given that for every sequence $\{x_n\}$ in A, such that $lim\, x_n = c$, the sequence $\{f(x_n)\}_{n=1}^{\inf}$ is a Cauchy sequence. We first note that by **Theorem 2.4.5** from the Jirka Text, *a sequence of real numbers is a Cauchy sequence if and only if it converges.* We now know that for every sequence $\{x_n\}$ in A, such that $lim\, x_n = c$, the sequence $\{f(x_n)\}_{n=1}^{\inf}$ converges.

We can now show that for every sequence $\{x_n\}$ in A, such that $lim\, x_n = c$, the sequence $\{f(x_n)\}_{n=1}^{\inf}$ converges to the same value L. We will show this by a proof by contradiction. Suppose we have two sequences $\{x_n\}$ and $\{y_n\}$, where $lim\, x_n = c$ and $lim\, y_n = c$, and $\{f(x_n)\}_{n=1}^{\inf}$ converges to L_1 and $\{f(y_n)\}_{n=1}^{\inf}$ converges to L_2 where $L_1 \neq L_2$. Now take the sequence $\{z_n\} = \{x_1, y_1, x_2, y_2, \ldots x_n,\ y_n,\ \ldots\}$. We know that $lim\, Z_n = c$, and thus $\{f(z_n)\}_{n=1}^{\inf}$ is convergent. However, $\{f(z_n)\}_{n=1}^{\inf}$ is not convergent because the sequences $\{f(x_n)\}_{n=1}^{\inf}$ and $\{f(y_n)\}_{n=1}^{\inf}$ converge to different values, L_1 and L_2. This is a contradiction.

Thus, we now know that for every sequence $\{x_n\}$ in A, such that $lim\, x_n = c$, the sequence $\{f(x_n)\}_{n=1}^{\inf}$ converges to L. Note that every sequence every sequence $\{x_n\}$ in $A \setminus \{c\}$, such that $lim\, x_n = c$ can also be viewed as $\{x_n\}$ in A, such that $lim\, x_n = c$. Thus, by **Lemma 3.1.7** from the Jirka Text, $f(x) \to L$ as $x \to c$. This proves that $\lim_{x \to c} f(x)$ exists and completes the proof.