ASSIGNMENT 1:

SCALING & DIMENSIONAL ANALYSIS

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We are trying to find the relation between the cooking time and size (mass) of a turkey. While the *best* way to cook a turkey is clearly by deep frying it, it may be useful to some to understand this relationship for use in cooking turkeys in the oven. Betty Crocker claims that $t \propto M$ is the optimal rule.

We are given the variables which are relevant to this problem. We list these along with their dimensions.

$[\varepsilon] \rightarrow \frac{M \times L^2}{T^2} \times \frac{1}{L^3} = \frac{M}{T^2 \times L}$	Temperature of Oven (Energy per unit volume sent into the bird)
$[\rho] \rightarrow \frac{M}{L^3}$	Mass Density of Turkey
$[l] \rightarrow L$	Diameter of Turkey
$[\kappa] \rightarrow \frac{L^2}{T}$	Thermal Conductivity of Turkey
$[t] \rightarrow T$	Time

1. PI THEROEM

We start with $t = f(l, \rho, \varepsilon, \kappa)$ and plan to arrive at the functional form $t = \frac{l^2}{\kappa} F(\frac{\varepsilon l^2}{\rho\kappa^2})$.

We will use the Pi Theorem to do this.

From the dimensions listed above, we know we have *n=5* variables $(\varepsilon, \rho, l, \kappa, t)$ and *m=3* dimensions (*L,M,T*). Thus we have *n-m= 5-3 = 2* dimensionless Pi groups.

These groups are

$\pi_1 = \rho^a l^b t^c \kappa$ and $\pi_2 = \rho^a l^b t^c \varepsilon$

We will first solve π_1

$\pi_1 = \rho^a l^b t^c \kappa$

$= (\frac{M}{L^3})^a (L)^b (T)^c (\frac{L^2}{T})$

$M = a = 0$

$T = c - 1 = 0$

$L = -3a + b + 2 = 0$

$a = 0$

$c = 1$

$-3(0) + b + 2 = 0$

$b = -2$

$\pi_1 = l^{-2} t^1 \kappa$

$\pi_1 = \frac{t\kappa}{l^2}$

We then solve π_2

$\pi_2 = \rho^a l^b t^c \varepsilon$

$= (\frac{M}{L^3})^a (L)^b (T)^c (\frac{M}{T^2 \times L})$

$M = a + 1 = 0$

$T = c - 2 = 0$

$L = -3a + b - 1 = 0$

$a = -1$

$c = 2$

$-3(-1) + b - 1 = 0$

$b = -2$

$\pi_2 = \rho^{-1} l^{-2} t^2 \varepsilon$

$\pi_2 = \frac{t^2 \varepsilon}{\rho l^2}$

From this, we can set $\pi_1 = F(\pi_2)$

$$\frac{t\kappa}{l^2} = F(\frac{t^2\varepsilon}{\rho l^2})$$

$$t = \frac{l^2}{\kappa} F(\frac{t^2\varepsilon}{\rho l^2})$$

This is close to our formula, except we are left with a t^2 inside of the function.

From our dimensional analysis, we know that

$$[\kappa] \rightarrow \frac{L^2}{T}$$

$$[l] \rightarrow L$$

and

$$[t] \rightarrow T$$

We can thus show that

$\frac{l^4}{\kappa^2}$ is dimensionally the same as t^2

Thus, we maintain our dimensionless pi groups and find

$$t = \frac{l^2}{\kappa} F(\frac{l^4\varepsilon}{\rho l^2 \kappa^2})$$

$$t = \frac{l^2}{\kappa} F(\frac{\varepsilon l^2}{\rho \kappa^2})$$

2. ASSUMPTIONS

We are now asked to asses the validity of the assumptions

1. Temperature of the oven is fixed

This is a reasonable assumption. If we have a high quality oven that maintains its temperature throughout the entire cooking time, we can safely assume that the temperature of the oven is fixed.

2. Thermal conductivity and density of turkeys are pretty much fixed

This is again a reasonable assumption. As the turkey cooks, some fat will drip out, some water will evaporate out, but these will be small amounts, and thus the density will remain pretty much the same. The thermal conductivity might change small amounts as the turkey cooks, but again this amount will be negligible and we can safely assume it is "pretty much fixed."

We now must discuss what would happen if these were not valid. If they were not valid, we would have several problems. Firstly, we would no longer have a linear equation (in terms of l) as we would have several input variables that are changing and affecting the output. These variable would also be changing throughout the cooking time (as opposed to l, which is constant) and thus would have to introduce differential equations, as the rate of change and value of the variable would change in response to the value of the variable. This would complicate our function immensely and make this problem much more difficult to model.

3. TRANSFORM THE FUNCTION

We can transform the previous functional form into $t = M^{2/3}\,\widehat{F}(M^{2/3})$.

We start with $t = \frac{l^2}{\kappa} F(\frac{\varepsilon l^2}{\rho\kappa^2})$

We know that $M = \rho \times l^3$ (Mass equals density times volume)

Thus we know that $l^2 = \frac{M^{2/3}}{\rho^{2/3}}$

We can substitute this into our equation to get

$t = \frac{M^{2/3}}{\kappa\rho^{2/3}} F(\frac{\varepsilon M^{2/3}}{\rho\kappa^2\rho^{2/3}})$

We can now absorb the proportionality constants into the function $\widehat{F}$ and get

$t = M^{2/3}\,\widehat{F}(M^{2/3})$

(Note: what absorbing the proportionality constants means is not initially clear to everyone. What it means is that we transform the function F into $\widehat{F}$ by including the constants in the definition of $\widehat{F}$. An easy to understand example of this is with the equation $t = 5H(x)$, where $H(x) = 3x^2$. To simplify this, we can transform *H(x)* into $\widehat{H}(x) = 5 \times 3x^2 = 15x^2$ and can express our equation as $t = \widehat{H}(x)$.)

4. ALTERNATE PI THEOREM

We are now supposed to obtain $t = l\sqrt{\frac{\rho}{\varepsilon}}G(\frac{\kappa}{l}\sqrt{\frac{\rho}{\varepsilon}})$ from our initial $t = f(l, \rho, \varepsilon, \kappa)$.

There are many ways to approach the Buckingham Pi theorem, and we don't want to spend a lot of time choosing the wrong variables and coming to the wrong conclusion. Thus we should do some reverse engineering of the goal function to understand which variables we should be our 3 variables with exponents to compare to the two remaining variables.

From $t = l\sqrt{\frac{\rho}{\varepsilon}}G(\frac{\kappa}{l}\sqrt{\frac{\rho}{\varepsilon}})$

We can get

$\frac{t}{l}\sqrt{\frac{\varepsilon}{\rho}} = G(\frac{\kappa}{l}\sqrt{\frac{\rho}{\varepsilon}})$

We can see from this two π groups.

We know that $l, \varepsilon,$ and ρ are the variables that we need to select, as they are all in both π groups and both t and κ have an exponent of 1 in π groups.

From this we can create two π groups.

$\pi_1 = \rho^a \varepsilon^b l^c \kappa$ and $\pi_2 = \rho^a \varepsilon^b l^c t$

We will first solve π_1

$$\pi_1 = \rho^a \varepsilon^b l^c \kappa$$

$$= (\tfrac{M}{L^3})^a (\tfrac{M}{T^2 \times L})^b (L)^c (\tfrac{L^2}{T})$$

$$M = a + b = 0$$

$$T = -3a - b + c + 2 = 0$$

$$L = -2b - 1 = 0$$

$$a = -b$$

$$-2b - 1 = 0$$

$$b = -1/2$$

$$a = 1/2$$

$$-3(1/2) - (-1/2) + c + 2 = 0$$

$$c = -1$$

$$\pi_1 = \rho^{1/2} \varepsilon^{-1/2} l^{-1} \kappa$$

$$\pi_1 = \sqrt{\tfrac{\rho}{\varepsilon}}\ \tfrac{\kappa}{l}$$

And then solve π_2

$$\pi_2 = \rho^a \varepsilon^b l^c t$$

$= (\frac{M}{L^3})^a (\frac{M}{T^2 \times L})^b (L)^c (T)$

$M = a + b = 0$

$T \; = \; -3a - b + c = 0$

$L = -2b + 1 = 0$

$a = -b$

$-2b + 1 = 0$

$b = 1/2$

$a = -1/2$

$-3(-1/2) \; - (1/2) + c =$

$c = -1$

$\pi_2 = \rho^{-1/2} \varepsilon^{1/2} l^{-1} t$

$\pi_2 = \sqrt{\frac{\varepsilon}{\rho}} \frac{t}{l}$

Taking $\pi_2 = \mathrm{G}(\pi_1)$ we get

$\sqrt{\frac{\varepsilon}{\rho}} \frac{t}{l} = G(\sqrt{\frac{\rho}{\varepsilon}} \frac{\kappa}{l})$

$t = l \sqrt{\frac{\rho}{\varepsilon}} G(\frac{\kappa}{l} \sqrt{\frac{\rho}{\varepsilon}})$

From this, we show that if we again absorb the constants we get to $t = M^{1/3}\widehat{G}(M^{1/3})$

We know that $M = \rho l^3$

And thus

$l = (\frac{M}{\rho})^{1/3}$

$t = (\frac{M}{\rho})^{1/3}\sqrt{\frac{\rho}{\varepsilon}}G(\frac{\kappa\rho^{1/3}}{M^{1/3}}\sqrt{\frac{\rho}{\varepsilon}})$

We absorb the constants into $\widehat{G}$ and get

$t = M^{1/3}\widehat{G}(M^{-1/3})$

We can then absorb the ^(-1) term in $\widehat{G}$ as well and get

$t = M^{1/3}\widehat{G}(M^{1/3})$

5. COMPARING THE FUNCTIONS

We make the simplest assumption and assume that $\widehat{F}$ and $\widehat{G}$ are constants.

We thus have to choose between $t \propto M^{2/3}$ and $t \propto M^{1/3}$. We use the compendium of cooking times and turkey weights for the tastiest of turkeys and fit two models (one with $M^{1/3}$ and one with $M^{2/3}$) and see which one is best.

I used python with the given data to find which linear approximation was more accurate.

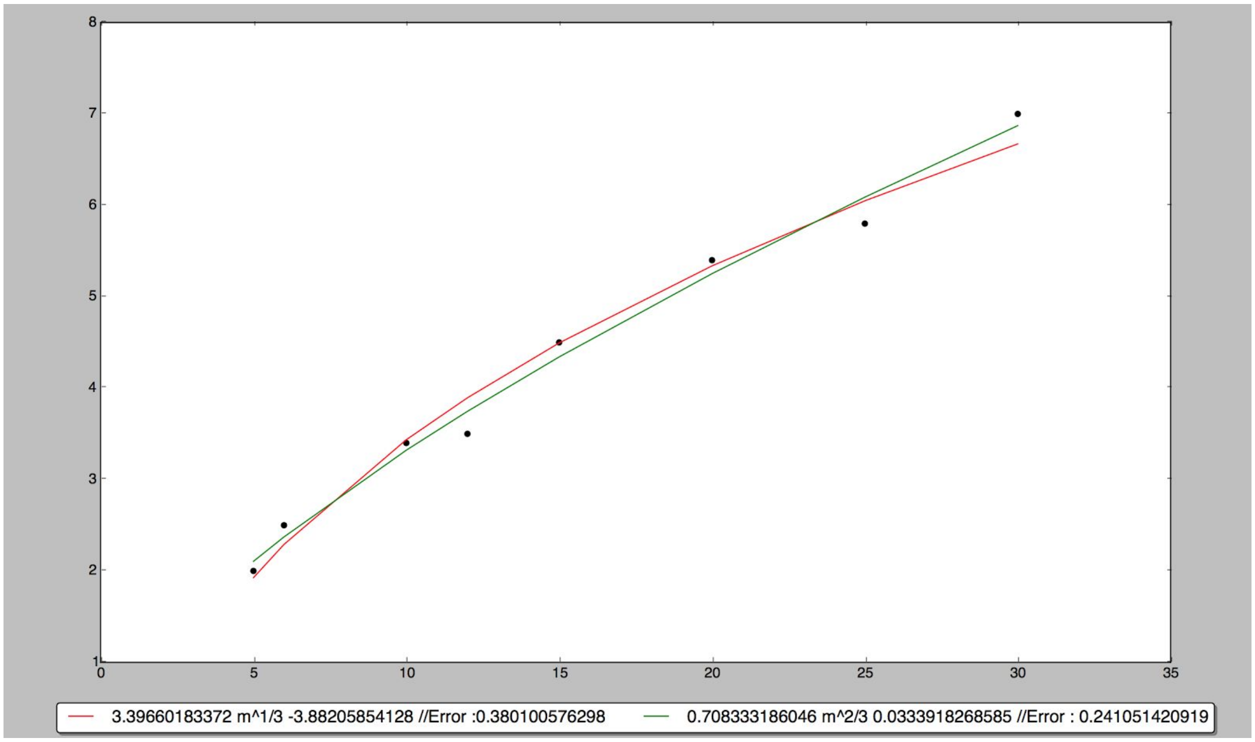
I used polyfit to create the a linear regression with regards to our linear relationship. This used the linear relationship and adjusted the constants (a,b) of the equation $t = am + b$ to find the best fit. This was graphed, and the residual output of polyfit was also used to calculate the MSE of both regressions.

The polyfit code and resulting graph are below.

```
# To find which linear relation works better, we use polyfit

# polyfit creates a linear fit to the data in accordance with our specified M^__ term
# It outputs an array [a,b] where t = am + b
# and the Residuals of the least-squares fit


cf1,residual1,a1,b1,c1 = np.polyfit(m**(1/3.0), t, 1, full=True)
cf2,residual2,a2,b2,c2 = np.polyfit(m**(2/3.0), t, 1, full=True)
```



The mean squared error for the $t \propto M^{1/3}$ relationship was 0.38, while for $t \propto M^{2/3}$ it was 0.24 . We can see from the above graph (the green being $t \propto M^{2/3}$) and from the lower MSE that the $t \propto M^{2/3}$ is superior to $t \propto M^{1/3}$. This makes sense, as Betty Crocker stated that there was a $t \propto M$ relationship, which is closer to $t \propto M^{2/3}$.