Lab 2: Minecrafting

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1 Introduction

The current mining project involves a 4 km deep mine shaft at the equator. A gravitational force will act on objects within the mine shaft based on their depth. This gravitational force can be considered as constant throughout the mineshaft, or as varying linearly, or according to some other function. The Coriolis force due to the Earth's rotation will also act on the object. The time it takes for an object to fall through the shaft, determined by the gravitational force, has been proposed as a method of determining shaft depth. This report will use several models of gravitational force to analyze the behavior of an object in free fall in the mine shaft and its fall time.

2 Calculation of Fall Time

The simplest model of gravitational force is to consider a constant acceleration. This provides an accurate description of the behavior of objects at or near the surface of the Earth. In this case the acceleration due to gravity has been measured to be $g = 9.81\ m/s$. According to this model, if an object is dropped from the top of the mine shaft, it will fall with an acceleration of exactly g for its entire descent. The fall time was calculated by solving the corresponding second order differential equation:

$$\frac{d^2y}{dt^2} = -g$$

The solution to this equation was calculated numerically using a numerical differential equation solver that implements the Runge-Kutta method to solve ordinary differential equations with high accuracy. This tool has a built in functionality allowing for a time to be recorded when a certain event is reached. In this case the time at which the object reached the bottom of the shaft was recorded by the solver. The calculated fall time for constant g is 28.6 s. The fall time can also be calculated analytically from the kinematic equations for an object undergoing constant acceleration. Such an approach yields a value of time consistent with the numerical one, with an effectively negligible difference of less than 10^{-15} %.

This model of gravitational force is only accurate close to the surface of the Earth. As the object falls through the mine shaft, the gravitational force will vary according to depth. One of the simplest models of a changing force is to consider the acceleration as linear, given by:

$$g(r) = g_0 \left(\frac{r}{R_\oplus} \right)$$

where $g_0 = 9.81\ m/s$, r is the distance from the center of the Earth, and $R_\oplus$ is the radius of the Earth. Thus, adjusting for depth measured from the surface, a new differential equation of motion can be written as:

$$\frac{d^2y}{dt^2} = -g_0 \left(\frac{R_\oplus + y}{R_\oplus} \right)$$

This equation was numerically solved using the same technique, and the fall time was found to be 28.6 s. The change compared to constant gravity is only 0.005%, and thus the effect of linear gravity on the fall time in the mine shaft is effectively negligible.

To further increase the accuracy of the calculation, a drag force due to air resistance can also be considered. A drag force opposes the direction of motion, and is modeled here as being proportional to some power γ of velocity. This gives a new differential equation:

$$\frac{d^2y}{dt^2} = -g_0 \left(\frac{R_\oplus + y}{R_\oplus} \right) - \alpha \left(\frac{dy}{dt} \right)^\gamma$$

Here γ is chosen as 2, and α is the drag coefficient with units $s^{\gamma-2} m^{1-\gamma}$. As the object falls, it will eventually reach a terminal velocity, when it will stop accelerating. This terminal velocity is known from previous experiments to be approximately 50.0 m/s.

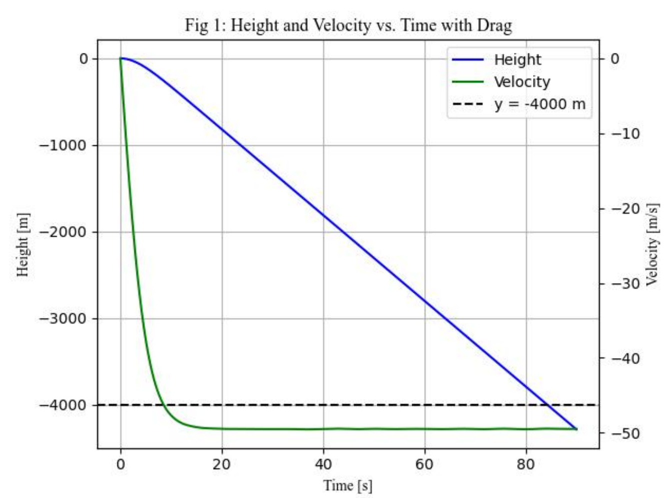

Figure 1: Solution to differential equation for height and velocity vs. time of dropped object with drag

Various values of α between 0 and 1 were tested to empirically determine an optimum value of α, found to be 0.004 m^{-1}. The resulting differential equation was solved in the same way, giving a fall time of 84.3 s, a 195% increase from the model without drag. Figure 1 shows the graph of the object's height as a function of time while falling with drag in the negative y direction starting from 0 m.

Thus, while the effect of a linear model of gravity did not significantly impact fall time, the consideration of drag is highly influential, increasing the fall time by a factor of 2.95.

3 Feasibility of depth measurement approach with Coriolis force

The Coriolis force acts on objects due to the rotation of the Earth. Because the mine shaft is located at the equator, the effect of the Earth's rotation can be large, and the Coriolis force on the object in the shaft must be considered. The Coriolis force is given as:

$$\vec{F_c} = -2m\left(\vec{\Omega} \times \vec{v}\right),$$

where Ω is the angular speed of the Earth. If a 3 axis coordinate system is considered with $\hat{y}$ pointing up the mine shaft, $\hat{x}$ pointing East, and $\hat{z}$ pointing north, then the Coriolis force in each direction is:

$$F_x = 2m\Omega v_y$$

$$F_y = -2m\Omega v_x$$

$$F_z = 0$$

In this analysis, these forces were factored into the previous differential equation with drag, giving a new differential equation for the acceleration in the x and y directions. This equation was solved using the same numerical tools, and a new fall time was determined to be 84.3 s. However, considering a 5 m wide mine shaft, the object will hit the wall after 29.6 s, and will thus not reach the bottom before colliding with the side. Even if drag is neglected, the object hits the side at 21.9 s, before it would hit the bottom at 28.6 s. Figure 2 shows the path of an object falling with the Coriolis force with and without drag.

Thus, calculating fall time is not a feasible method of determining depth of the described shaft, as the object will collide with the side before hitting the bottom. If it is necessary that the object reach the bottom of the shaft without hitting the side, the mineshaft must be at least 24.3 m in radius, the distance the object will drift. The mine shaft could also instead be dug further from the equator, which will decrease the Coriolis force.

4 Crossing times for homogeneous and non-homogeneous Earth

In all of the above models, the density of the Earth was considered constant. In reality, the density of the earth varies, with density decreasing with distance

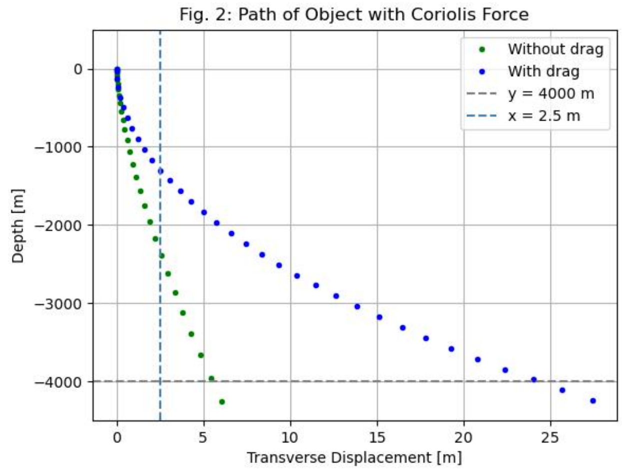

Figure 2: Depth and transverse displacement of dropped object affected by Coriolis force, from the numerical solution to the differential equation of motion

from the Earth's center. Density can be modeled as a function of radius r by:

$$\rho(r) = \rho_n\left(1 - \frac{r}{R_\oplus}\right)^n$$

Here n is some exponent, and ρ_n is a normalizing constant corresponding to n. The simplest case of $n = 0$ is the homogeneous Earth. The cases of $n = 0, 1, 2, 9$ were calculated. The corresponding constants ρ_n were calculated using the fact that:

$$M_E = \iiint_V \rho(r)dV = 4\pi\rho_n \int_0^{R_\oplus}\left(1 - \frac{r}{R_\oplus}\right)^n dr$$

For each n, the mass of the Earth, M_E, is divided by the right integral and constant of 4π, to give ρ_n. After calculating ρ_n, a density function for each n could be used to give force as a function of radius for each value of n:

$$F(r) = -\frac{4\pi G m_1}{r^2}\int_0^r \rho(r) r^2 dr$$

Here G is the universal gravitational constant, and m_1 is some arbitrary test mass. Here the the density function is integrated from 0 to r to calculate the mass below the reference point r, and then Newton's equation for gravitational force is used to calculate the total force. All integrals were calculated using a numerical integrator that relies on the QUADPACK Fortran library to numerically integrate functions of a single variable.

This force divided by the test mass gives acceleration as a function of r. This can be solved as a differential equation corresponding to some given n. In this case drag was ignored, and the mine shaft was considered to be a hole drilled through the entire Earth starting at a pole, such that Coriolis forces can be ignored. These differential equations were similarly solved, giving sinusoidal solutions with the object oscillating between sides of the Earth. The 'crossing times' at which the object reached the other side of the Earth were recorded. For the case $n = 0$, the crossing time is 2534.1 s, and for $n = 9$, the crossing time is 1881.5 s, a decrease of 25.5%. Figure 3 shows the height and velocity vs. time for the extreme cases $n = 0$ and $n = 9$ of an object dropped from the surface of the Earth. Thus for a trans-planetary mineshaft, higher order polynomial models of density give lower crossing times. Such models describe situations where density is increasingly concentrated near the Earth's core, and thus have greater forces deeper in the shaft, decreasing the time by rapidly increasing acceleration in such regions.

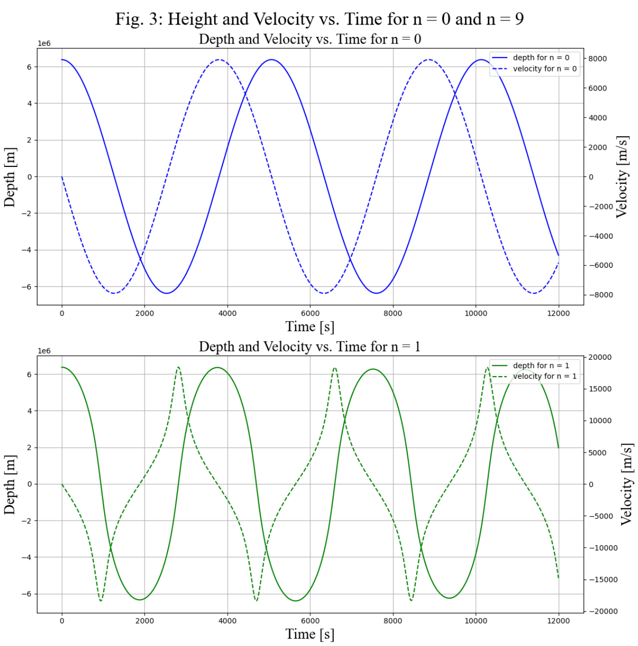

Figure 3: Solutions to differential equations for height and velocity vs. time of density distributions n = 0 and n = 9

The crossing time for a trans-lunar mine shaft was also calculated using the same method from mass and radius of the Earth's Moon, assuming a homogeneous mass distribution. The crossing time for a trans-lunar mine shaft was calculated to be $t_M = 3249.8\ s$. This gives a crossing time ratio of $\frac{t_E}{t_M} = 0.78$

In addition, the average densities of the Earth and the Moon were compared. The average density of the Earth, was calculated as 5494.9 kg/m^3, and the average density of the Moon to be 3341.8 kg/m^3. This gives a ratio of $\frac{\rho_E}{\rho_M} = 1.64$.

A relationship can be derived between the crossing time, and density. From a balance of forces it for an object in circular motion around a mass, it follows that:

$$\frac{GM_E}{R^2} = \frac{G\rho V}{R^2} = \frac{v^2}{R} = \left(\frac{2\pi R}{T}\right)^2 \frac{1}{R}$$

Where T is orbital period, which is thus proportional to the inverse of the square root of density. The calculated crossing times were empirically determined to be half of the orbital period for a planet of the same mass. Thus, the crossing time is also proportional to the inverse of the square root of density. That is:

$$\frac{t_E}{t_M} = \sqrt{\frac{\rho_M}{\rho_E}}$$

This holds for the calculated times and densities:

$$\sqrt{\frac{\rho_M}{\rho_E}} = \frac{t_E}{t_M} \approx \sqrt{\frac{1}{1.64}} = \frac{2534.1\ s}{3249.8\ s} = 0.78$$

This relationship allows for a greater understanding of the effect of density on mining large celestial bodies.

5 Conclusion

This report provides a brief overview of constant acceleration, linear gravity, drag, Coriolis forces, more complex density distributions in the Earth, and their effects on an object in free fall. From these models a fall time based depth measurement has been found ineffective under current conditions. While this analysis offers approximations of gravitational force that may be sufficient, more accurate models could be created.

The true density of the Earth cannot be described by a smooth curve. A more accurate model of the Earth's density can be created from geological data, which would provide more accurate descriptions of forces and motion. Drag was also simply approximated, and not considered for the trans-planetary mine shafts. Further research can investigate drag in a more detailed way and in more cases. Additionally, the construction of a mine shaft directly on a pole or the equator is nearly impossible, and further research analyzing other placements should be performed. Further analysis could investigate better methods of determining shaft properties, and improve mining operations through a better understanding of the involved physics. This allows for more accurate descriptions of mine shafts and thus safer and more efficient mining.