Lab 2: Minecrafting

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1 Introduction

For the current mining proposal, a 4 km deep mine shaft is being dug at the equator. Gravitational force will act on objects within the mine shaft based on depth. This gravitational force can be considered as constant throughout the mineshaft, varying either linearly, or according to some other function. The Coriolis force due to the Earth's rotation can also be considered. The fall time based on the gravitational force has been proposed as a method of determining shaft depth. This report will analyze the effects of these models of forces on an object in free fall in the mine shaft and its fall time.

2 Calculation of Fall Time

The simplest model of gravitational force is to consider a constant acceleration. This provides an accurate description of the behavior of objects at the surface of the Earth. In this case, the acceleration due to gravity, is measured to be $g = 9.81\ m/s$. If an object is dropped from the top of the mine shaft, it will fall with an acceleration of exactly g for its entire descent. The fall time was calculated by solving the second order differential equation:

$$\frac{d^2y}{dt^2} = -g$$

The solution to this equation was calculated numerically using a numerical differential equation solver that uses the Runge-Kutta method to accurately solve ordinary differential equations. This tool has a built in functionality allowing for a time to be recorded when a certain event is reached. In this case the time at which the object reached the bottom of the shaft was recorded by the solver. The calculated fall time for constant g is 28.6 s.

This model of gravitational force is only accurate while close to the surface of the Earth. As the object falls through the mine shaft, the gravitational force will vary according to depth. One of the simplest models of this changing force is to consider the acceleration as linear, given by:

$$g(r) = g_0 \left(\frac{r}{R_\oplus} \right),$$

where $g_0 = 9.81\ m/s$, r is the distance from the center of the Earth, and $R_\oplus$ is the radius of the Earth. Thus, adjusting for depth measured from the surface, a new differential equation of motion can be written as

$$\frac{d^2y}{dt^2} = -g_0 \left(\frac{R_\oplus + y}{R_\oplus} \right)$$

This equation was numerically solved using the same numerical technique, and the fall time was found to be 28.6 s. The change compared to constant gravity is only 0.005%, and thus the effect of non-constant gravity on the fall time in the mine shaft is effectively negligible. Fig. 2 shows the path of the object with and without drag, moving in the negative y and positive x directions.

To further increase the accuracy of the calculation, a drag force due to air resistance can be considered. A drag force opposes the direction of motion, and is modeled here as being proportional to the square of the velocity. This gives a new differential equation:

$$\frac{d^2y}{dt^2} = -g_0 \left(\frac{R_\oplus + y}{R_\oplus} \right) - \alpha \left(\frac{dy}{dt} \right)^2$$

Here α is the drag coefficient. As the object falls, it will eventually reach a terminal velocity, when it will stop accelerating. This terminal velocity is known from previous experiments to be approximately 50.0 m/s.

Varying values of α between 0 and 1 were tested to empirically determine an optimum value of $alpha$, found to be 0.004. The resulting differential equation was solved in the same way, giving a fall time of 84.3 s, a 195% increase from the model without drag. Figure 1 shows the graph of the object falling with drag in the negative y direction

Thus, while the effect of a linear model of gravity did not significantly impact fall time, the consideration of drag is highly influential, increasing the fall time by a factor of 2.95.

3 Feasibility of depth measurement approach

The Coriolis force acts on objects due to the rotation of the Earth. Because the mine shaft is located at the equator, the effect of the Earth's rotation can be large, and the Coriolis force on the object must be

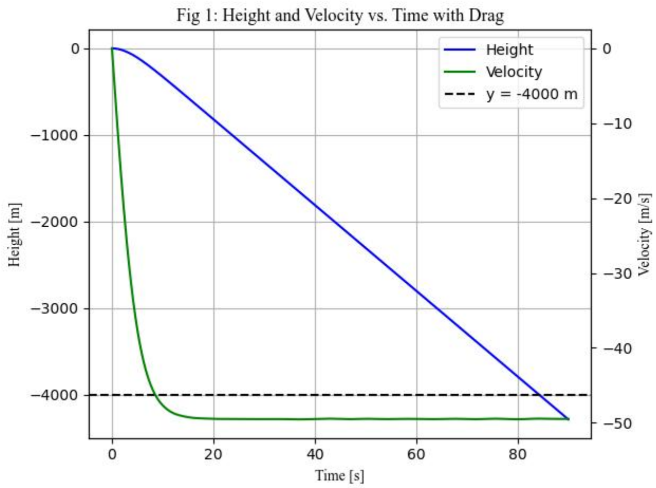

Figure 1: Height and velocity of dropped object with drag

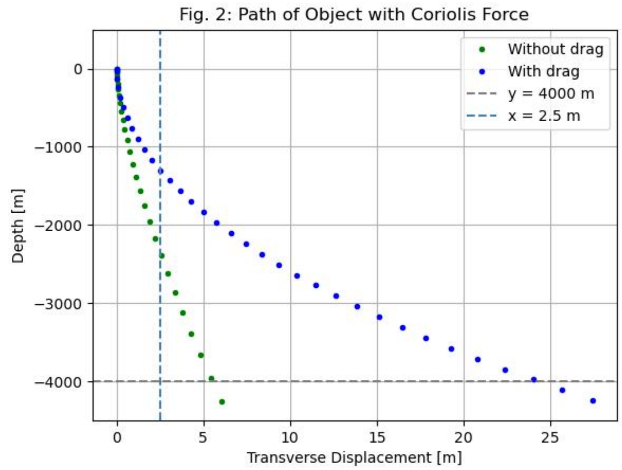

Figure 2: Depth and transverse displacement of dropped object with drag and the Coriolis force

considered. The Coriolis force is given as:

$$\vec{F_c} = -2m\left(\vec{\Omega} \times \vec{v}\right),$$

where Ω is the angular speed of the Earth. If a 3 axis coordinate system is considered with $\hat{y}$ pointing down the mine shaft, $\hat{x}$ pointing East, and $\hat{y}$ pointing north, then the Coriolis force in each direction is:

$$F_x = 2m\Omega v_y$$

$$F_y = -2m\Omega v_x$$

$$F_z = 0$$

In this analysis, These forces were factored into the previous differential equation with drag, giving a new second order differential equation for the acceleration in the x and y directions. This differential equation was solved using the same numerical tools, and a fall time was calculated as 84.3 s. However, considering a 5 m wide mine shaft, the object will hit the wall after 29.6 s, and will thus not reach the bottom before colliding with the side. Even if drag is neglected, the object hits the side at 21.9 s, before it would hit the bottom at 28.6 s

Thus, calculating drop time is not a feasible method of determining depth of the described shaft, as the object will collide with the side before hitting the bottom. If it is necessary that the object reach the bottom of the shaft without hitting the side, the mineshaft must be at least 24.3 m in radius, the distance the object will drift. The mine shaft could also instead be dug further from the equator, which will decrease the Coriolis force.

4 Crossing times for homogeneous and non-homogeneous Earth

In all of the above models, the density of the Earth was considered constant. In reality, the density of the earth varies, with density decreases with distance from the center. Density can be modeled as a function of radius r by:

$$\rho(r) = \rho_n \left(1 - \frac{r}{R_\oplus}\right)^n$$

Here n is some exponent, and ρ_n is a normalizing constant corresponding to n. The simplest case of $n = 0$ is the homogeneous Earth. The cases of $n = 0, 1, 2, 9$ were calculated. The corresponding constants ρ_n were calculated using the fact that:

$$M_E = \iiint_V \rho(r)dV = 4\pi\rho_n \int_0^{R_\oplus} \left(1 - \frac{r}{R_\oplus}\right)^n dr$$

For each n, the mass of the Earth, M_E, is divided by the right integral, to give ρ_n. After calculating ρ_n, each density function could be used to give force as a function of radius:

$$F(r) = -\frac{4\pi G m_1}{r^2}\int_0^r \rho(r)r^2 dr,$$

Where G is the gravitational constant, and m_1 is some arbitrary test mass. The the density function is integrated from 0 to r to calculate the mass below the reference point r, and then Newton's equation for gravitational force is used to calculate the total force. These integrals were calculated using scipy.integrate.quad, a numerical integrator that relies on the QUADPACK Fortran library to numerically integrate functions of a single variable.

This force divided by the test mass gives acceleration as a function of r. This can be solved as a differential equation corresponding to the given n. In this case drag was ignored, and the mine shaft is considered to be a hole drilled through the entire Earth starting at a pole, such that the Coriolis forces can be ignored. These differential equations were similarly solved, and this time the 'crossing times' at which the object reached the other side of the Earth were recorded. For the case $n = 0$, the crossing time is 2534.1 s, and for the case where $n = 9$, the crossing time is 1881.5 s, a decrease of 25.5%. Figure 3 shows the height and velocity vs. time for the extreme cases $n = 0$ and $n = 9$. It is thus seen that for a trans-planetary mineshaft, density functions of higher order polynomials give lower crossing times. Such polynomials describe situations where density is increasingly concentrated near the Earth's core, and are indicative of a greater crossing time.

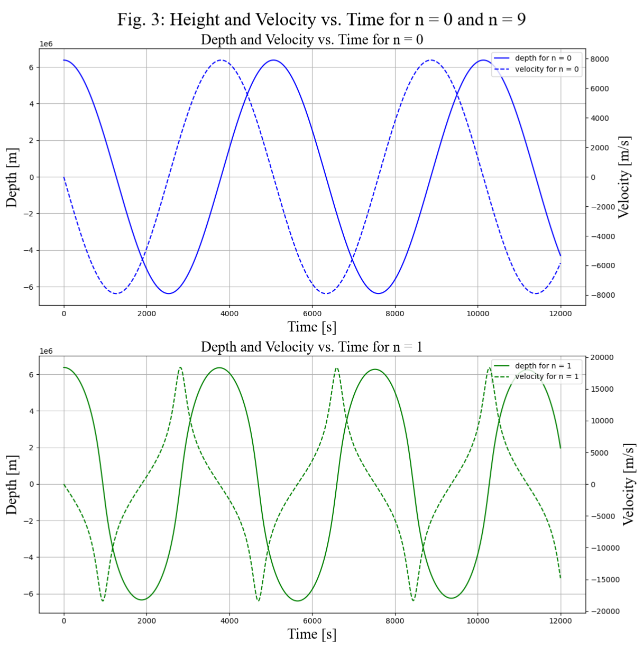

Figure 3: Height and velocity vs. time for density distributions corresponding to n = 0, and n = 9

The crossing time for a trans-lunar mine shaft was also calculated using the same method, but with the mass and radius of the moon. The crossing time for a trans-lunar mine shaft through a moon with homogeneously distributed mass was calculated to be 3249.8 s. This gives a ratio of $\frac{t_E}{t_M} = 0.78$

In addition, the average densities of the Earth and the moon were compared. The average density of the Earth, was calculated as 5494.9 kg/m^3, and the average density of the moon was calculated to be 3341.8 kg/m^3. This gives a ratio of $\frac{\rho_E}{\rho_M} = 1.64$.

A relationship can be derived between the crossing time, and density. From Kepler's laws it follows that:

$$T^2 = \frac{4\pi r^3}{Gm} = \frac{4}{3G\rho}$$

Where T is orbital period, which is thus proportional to the inverse of the square root of density. The numerically calculated crossing times, were seen to be half of the orbital period for a planet of the same mass. That is, the crossing time for Earth t_E obeys:

$$t_E = 2\sqrt{\frac{Gm_E}{r^3}}$$

Thus, the crossing time is also proportional to the inverse of the square root of density. That is:

$$\frac{t_E}{t_M} = \sqrt{\frac{\rho_M}{\rho_E}}$$

This holds for the calculated times and densities, with:

$$\sqrt{\frac{\rho_M}{\rho_E}} = \frac{t_E}{t_M} = \sqrt{\frac{1}{1.64}} = \frac{2534.1\ s}{3249.8\ s} = 0.78$$

. This relationship allows for a greater understanding of the effect of density on mining large celestial bodies.

5 Conclusion

This report provides a brief overview of constant acceleration, linear gravity, drag, Coriolis forces, and more complex density distributions in the Earth and the effect on an object in free fall. While these offer approximations of gravitational force that are sufficient for many applications, more accurate models could be created. A more accurate description of the Earth's density can be created from geological data, which would provide a more accurate description of the mine shaft. Further research can also investigate drag in more detailed way than simply being proportional to velocity. The construction of a mine shaft directly on a pole or the equator is nearly impossible. Further research that considers the Coriolis force at other positions may be beneficial. Such further research would allow for more accurate descriptions of mine shafts and thus safer and more efficient mining.