Dynamics of Price Discovery

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Motivation

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- How do prices come to incorporate the private information of traders?

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- How much variation in prices can come from information alone?

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- What is the optimal trading strategy of a strategic informed trader?
- What is the role of rumors and false information?
- What do the price dynamics look like?
- How much variation in prices can come from information alone?
- Endogenous formation of common price processes.

Table of Contents

Literature Review

Information Design

- Crawford and Sobel (1982), Aumann and Maschler (1995), Kamenica and Gentzkow (2011), Lipnowski and Ravid (WP)

Dynamic Information

- Ely (2017), Ely, Frankel, and Kamenica (2015), Orlev, Skrzypacz, and Zryumov (WP), Hörner and Skrzypacz (2016)

Price Discovery (add a bunch more here)

- Kyle (1985), Van Bommel (2003), De Meyer (2003)

Overview

Three key elements of the model

- Uncertainty
- A strategic informed trader
- Prices and liquidity

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  - Focus on Markov equilibria

Informed Trader

- The informed trader knows the state
- They choose how much to buy/sell each period (potentially a mixed strategy), holdings $x_t \in [-1, 1]$
- Maximize expected profit

$$\mathbb{E}\left[\int_0^\infty e^{-rt} x_t \, dP_t\right] \tag{1}$$

- No dividends or final value important to trader, only capital gains
  - Think foreign currency, gold, Amazon stock, etc.
- Usually work with discrete version ($\delta = e^{-r\Delta t}$)

$$V(\mu_0) = \max_{\{x_t\}_{t=0}^\infty} \mathbb{E}\left[\sum_{t=0}^\infty \delta^t (P_{t+1} - P_t) x_t\right] \tag{2}$$

Market Maker

The market maker has their beliefs about the state from the previous period, μ_{t-1}.
They observe the action of the informed trader, x_{t-1}, and update their beliefs using Bayes rule to get μ_t.
The market maker then chooses a price to optimize some flow utility.

$$P(\mu_t) = \operatorname*{argmax}_{p \in \mathbb{R}} U(p, \mu_t) \tag{3}$$

Assume $P(\mu_t)$ exists and is single valued.

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Examples

- Price is an expectation: $U(p, \mu_t) = -\mathbb{E}_{\mu_t}\left[(p - z(\omega))^2\right]$
- Stochastic discount factor: $P(\mu_t) = \mathbb{E}_{\mu_t}\left[m(\omega)z(\omega)\right]$

Informed Trader's Problem

The problem can be reformulated as one where the informed trader chooses beliefs subject to incentive compatability constraints.

$$V(\mu) = \max_{\mu' \in \Delta[0,1]} \mathbb{E}\left[(P(\mu') - P(\mu))x + \delta V(\mu')\right] \tag{4}$$

$$\text{subject to} \quad \mathbb{E}[\mu'] = \mu \qquad (BayesPlausibility) \tag{5}$$

$$I.C. \tag{6}$$

Incentive compatability takes a simple form in this problem. The value obtained at some posterior, μ',

$$(P(\mu') - P(\mu))x(\mu') + \delta V(\mu') \tag{7}$$

must be equal for all μ' chosen with positive probability.

Value Function

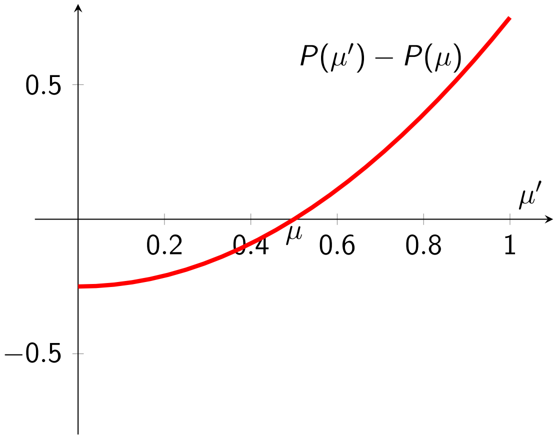
The value function takes a particularly simple form.

Proposition

Let $P(\mu)$ be continuous and monotone. For any discount factor, $\delta \in [0, 1)$, the value is

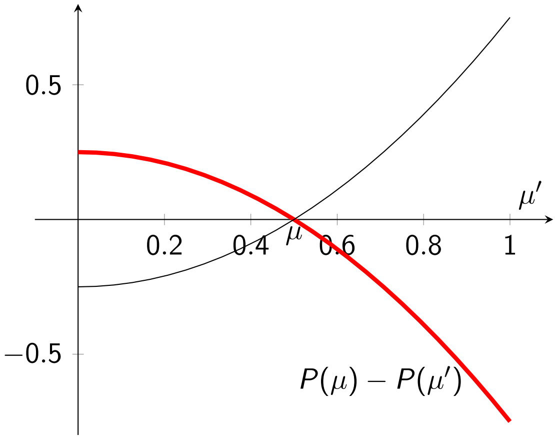
$$V(\mu) = \min\{P(\mu) - P(0),\ P(1) - P(\mu)\} \tag{8}$$

Value Function



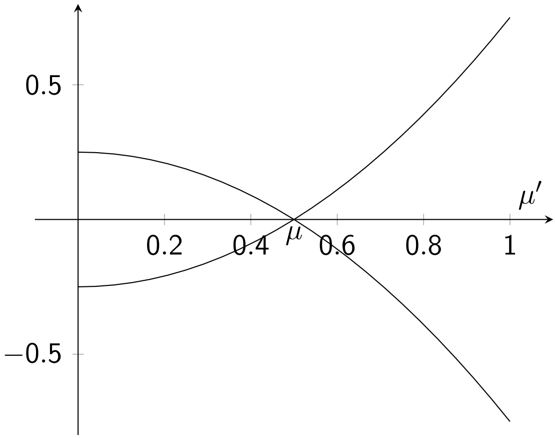
The informed trader's payoff if they buy, $x = 1$, graphed against the posterior induced.

Value Function



The informed trader's payoff if they sell, $x = -1$, graphed against the posterior induced.

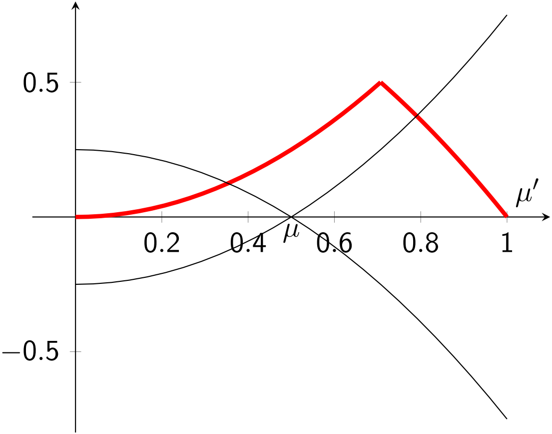
Value Function



Given the chosen posteriors, what must x be?

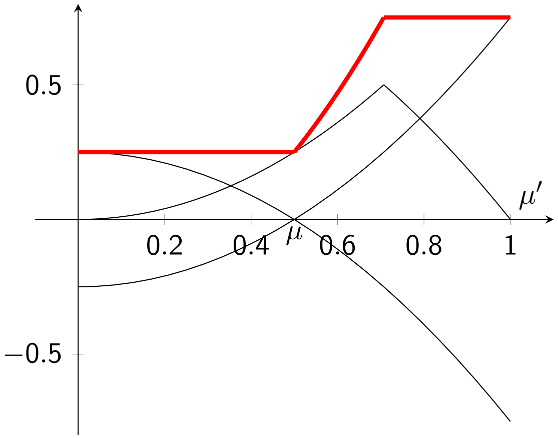
The flow payoff is the upper envelope.

Value Function



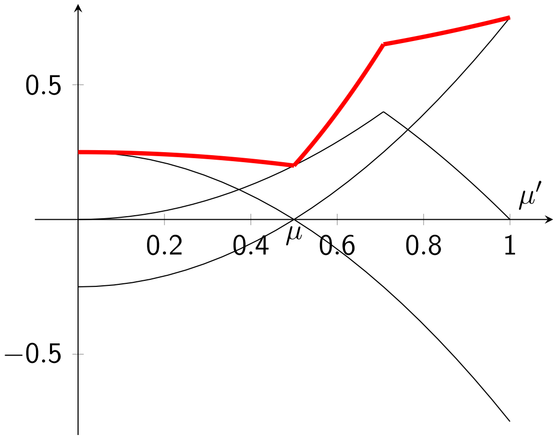
The conjectured value function.

Value Function



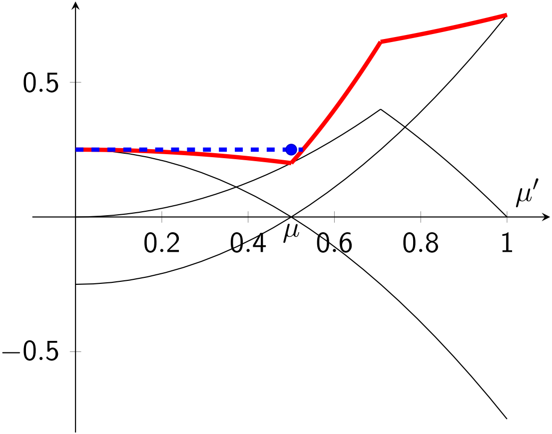
Take the sum of the flow payoff and continuation.

Value Function



With $\delta < 1$, the total payoff must be decreasing below the prior and increasing after.

Value Function



Incentive compatability means that the optimal payoff is the highest flat line on this curve.

Main Result 1

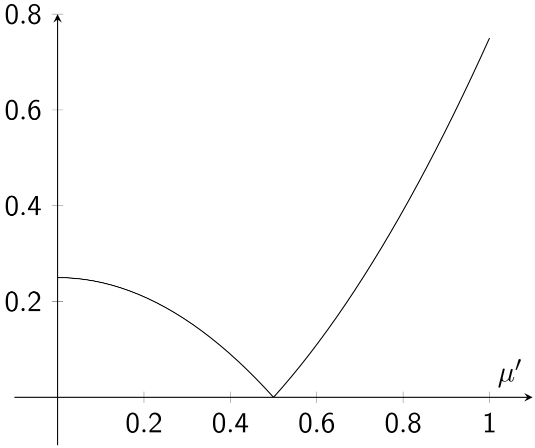
Even though the value doesn't depend on δ, the equilibrium strategy that achieves that value does vary with δ.

Theorem

For any differentiable strictly monotone price function, $P(\mu)$, as δ goes to one the price proccess converges to a Poisson process.

Think of this as the continuous time limit of the discrete game.

Strategy Depends on δ



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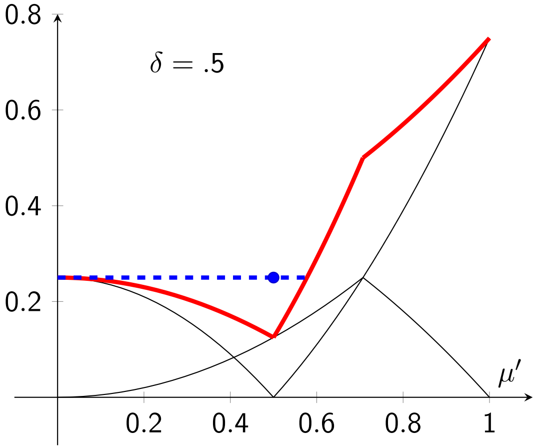

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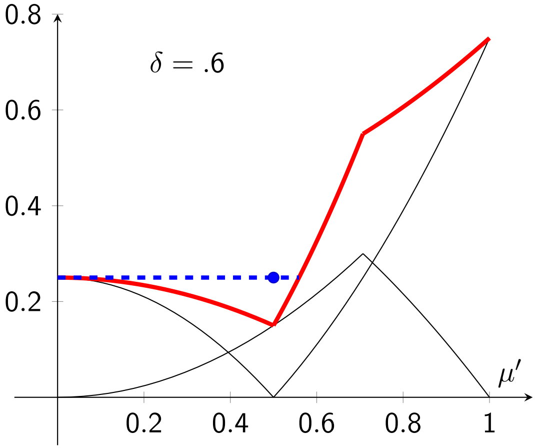

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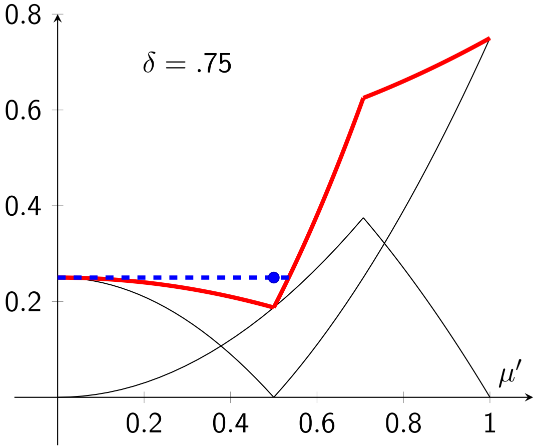

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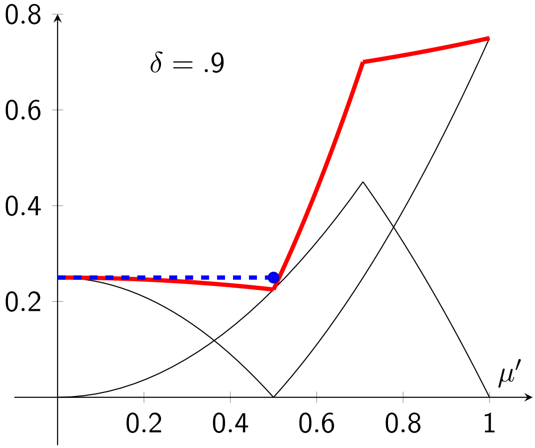

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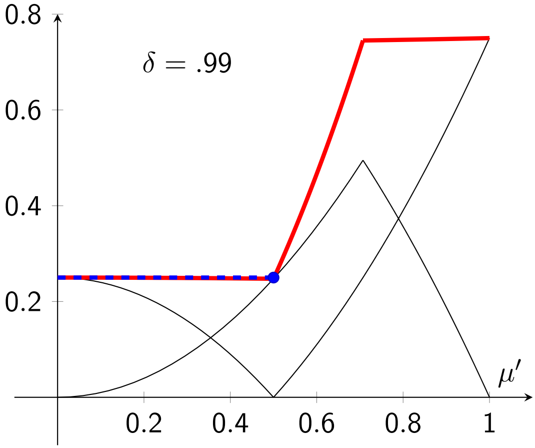


Functional Form

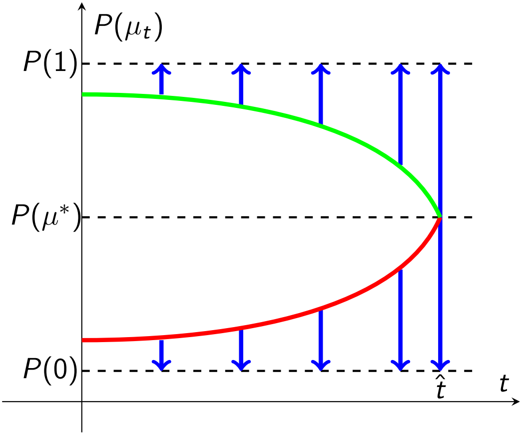
The functional forms depend on the price function, $P(\mu)$.

- If $\mu_t < \mu^*$,

$$dP(\mu_t) = \frac{r}{2}(P(\mu_t) - P(0))dt - (P(\mu_t) - P(0))dN_t \qquad (9)$$

  N_t is a standard Poisson with arrival rate $\lambda = \frac{r}{2}\frac{P(\mu_t)-P(0)}{\mu_t P'(\mu_t)}$.
- If $\mu_t > \mu^*$, price follows a symmetric process.
- If $\mu_t = \mu^*$, all information is revealed immediately and the price jumps to either $P(1)$ or $P(0)$.

Price Dynamics



The optimal strategy employs pump-and-dump and short-and-distort schemes.

Explain Theorem

The informed trader needs to be indifferent between revealing the asset to be bad today, and waiting to reveal it to be bad tomorrow. Call λ the percentage drift in beliefs.

$$\underbrace{P(\mu_t) - P(0)}_{\text{reveal today}} \approx \underbrace{P'(\mu_t)\mu_t\lambda_t}_{\text{drift today}} + \delta(\underbrace{P'(\mu_t)\mu_t\lambda_t + P(\mu_t) - P(0)}_{\text{reveal tomorrow}}) \tag{10}$$

As δ gets large, this gives a linear relationship between λ and $1-\delta$.

$$\Rightarrow \lambda_t \approx \frac{P(\mu_t) - P(0)}{2\mu_t P'(\mu_t)}(1-\delta) \tag{11}$$

Arrival Rate

Consider the Taylor series expansion.

$$P(0) = P(\mu_t) - P'(\mu_t)\mu_t + \frac{1}{2}P''(\mu_t)\mu_t^2 + \dots \tag{12}$$

This gives another expression for the approximate arrival rate.

$$\lambda(\mu_t) = \frac{r}{2}\frac{P(\mu_t) - P(0)}{P'(\mu_t)\mu_t} \approx \frac{r}{2}\left(1 - \frac{1}{2}\mu_t \frac{P''(\mu_t)}{P'(\mu_t)}\right) \tag{13}$$

If $P(\mu_t)$ is concave, the arrival rate is greater than $\frac{r}{2}$ and increasing. If convex, it is smaller and decreasing.

Full Revelation

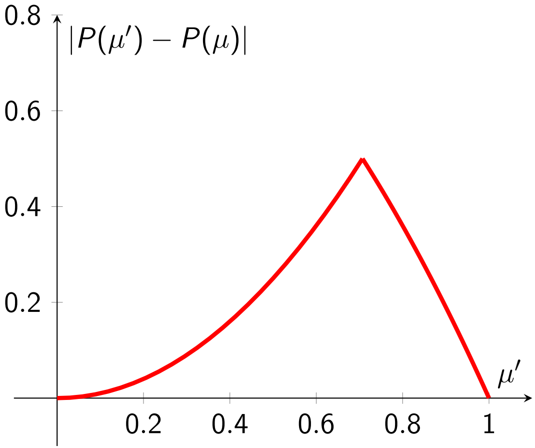
All information is revealed in finite time.

$$t^{max} = \frac{2}{r} \log \left(\frac{P(\mu^*) - P(0)}{P(\mu_0) - P(0)} \right) \tag{14}$$

Commitment

Consider if the informed trader could commit ex ante to a strategy.

The value function without commitment.

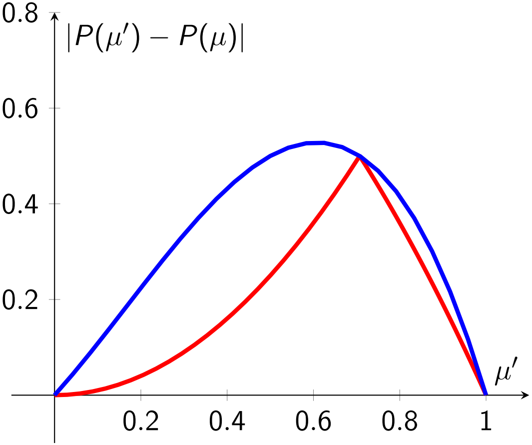


$$V(\mu) = \min\left\{P(\mu) - P(0),\ P(1) - P(\mu)\right\} \tag{15}$$

Commitment

Consider if the informed trader could commit ex ante to a strategy.

Buy if good and sell if bad does better.
Why wasn't it incentive compatible before?



$$\tilde{V}(\mu) \geq (1-\mu)(P(\mu) - P(0)) + \mu(P(1) - P(\mu)) \geq V(\mu). \quad (15)$$

Informed Trader's Problem and Main Result 2

Commitment power removes the incentive compatability constraint on the informed trader.

Theorem

For any $\mathcal{C}^2$ function $P(\mu)$, as δ goes to one the price process converges to an Itô process.

Price dynamics are driven by a Brownian motion with a drift and variance term.

Strategy With Commitment

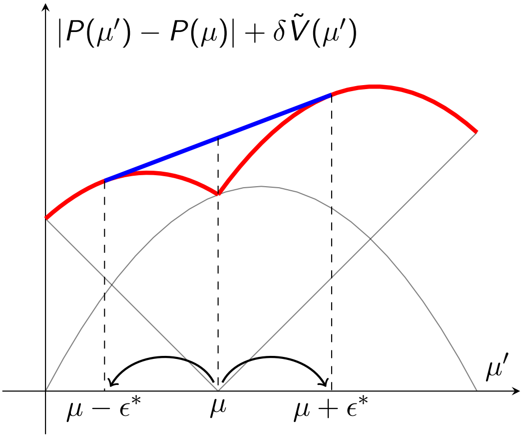


Figure: The blue segment gives the optimal policy and value. Beliefs either jump up or down by step size ϵ^* each period.

Proof Intuition

Let $\bar{\epsilon} \in [0, 1-\mu]$ and $\underline{\epsilon} \in [0, \mu]$ be the change in beliefs after buying or selling. The objective becomes

$$\tilde{V}(\mu) = \max_{\bar{\epsilon},\underline{\epsilon}} \left(P(\mu+\bar{\epsilon}) - P(\mu) + \delta \tilde{V}(\mu+\bar{\epsilon})\right) \frac{\underline{\epsilon}}{\bar{\epsilon}+\underline{\epsilon}} + \left(P(\mu) - P(\mu-\underline{\epsilon}) + \delta \tilde{V}(\mu-\underline{\epsilon})\right) \frac{\bar{\epsilon}}{\bar{\epsilon}+\underline{\epsilon}}. \quad (16)$$

For small ϵ, $P(\mu+\epsilon) - P(\mu) \approx P'(\mu)\epsilon$ and $\tilde{V}(\mu+\epsilon) \approx \tilde{V}(\mu) + \tilde{V}'(\mu)\epsilon + \frac{1}{2}\tilde{V}''(\mu)\epsilon^2$.

$$(1-\delta)\tilde{V}(\mu) = \max_{\bar{\epsilon},\underline{\epsilon}} \underbrace{2|P'(\mu)|\frac{\bar{\epsilon}\underline{\epsilon}}{\bar{\epsilon}+\underline{\epsilon}}}_{\text{current gain}} + \underbrace{\delta \tilde{V}''(\mu)\bar{\epsilon}\underline{\epsilon}}_{\text{future loss}} \quad (17)$$

More Proof Intuition

Optimizing gives

$$\bar{\epsilon} = \underline{\epsilon} = \frac{|P'(\mu)|}{2\delta \tilde{V}''(\mu)}. \tag{18}$$

Putting this back into the objective gives

$$\tilde{V}(\mu)\tilde{V}''(\mu) = -\frac{|P'(\mu)|^2}{2\delta(1-\delta)}. \tag{19}$$

Letting $\hat{V}(\mu) = \tilde{V}(\mu)\sqrt{2\delta(1-\delta)}$, $\sigma(\mu) = \frac{\sqrt{2}|P'(\mu)|}{\hat{V}''(\mu)}$, and $r\Delta t = \frac{1-\delta}{\delta}$, gives

$$\mu' - \mu = \begin{cases} \sigma(\mu)\sqrt{\Delta t} & \text{with probability } \frac{1}{2} \\ -\sigma(\mu)\sqrt{\Delta t} & \text{with probability } \frac{1}{2}. \end{cases} \tag{20}$$

Functional Forms

Generally, the exact process can't be solved analytically, but the form of the solution is known.

$$dP(\mu_t) = \frac{r}{2}P''(\mu_t)\sigma^2(\mu_t)dt + \sqrt{r}P'(\mu_t)\sigma(\mu_t)dB_t \tag{21}$$

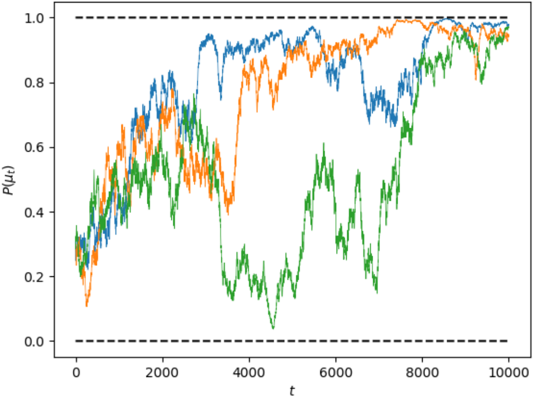
If the price function is linear with slope p, then

$$dP(\mu_t) = \sqrt{2r}p\phi(\mu_t)dB_t \tag{22}$$

where $\phi(\mu_t)$ is the normal pdf evaluated at the μ_t quantile.

Solution Explanation

- Gradual, continuous information release
- Spiky, random price movements
- Conditional on ω, price drifts toward full information value
- Nearly equal amount of information and misinformation



Importance

Endogenous dynamics

- Option pricing
- Information acquisition

Dynamic information disclosure

- Cheap talk vs Bayesian persuasion

Price Discovery

- Rumors
- Price volatility

Persistence

Suppose that ω_t follows a Markov process.

- Call $\pi_1 = 1 - \lambda_1 \Delta t$ and $\pi_0 = \lambda_0 \Delta t$ the probability that $\omega_{t+1} = 1$ given that $\omega_t = 1$ or 0 respectively.
- Take price function to be linear, $P(\mu) = \mu$.
- State observed by insider only in date 0 or every period

Objective

$$\max \ \mathbb{E}\left[\sum_{t=0}^{\infty} \delta^t |\mu_t' - \mu_t|\right] \tag{23}$$

where $\mu_{t+1} = \pi_0 + (\pi_1 - \pi_0)\mu_t'$.

One Time Information

Call $\tilde{\mu}_t$ the beliefs about ω_0 in date t.

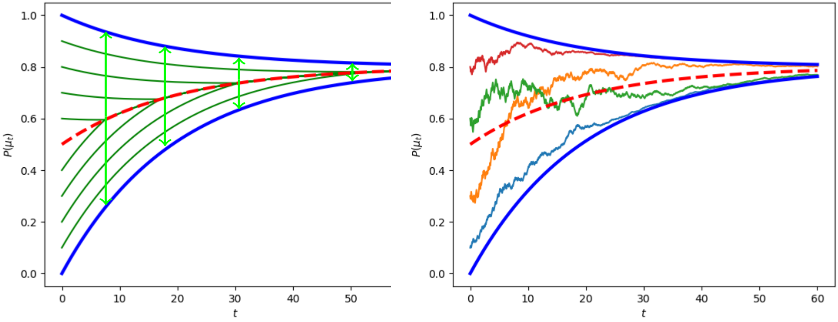
$$
\begin{aligned}
\mu_0 &= \tilde{\mu}_0 \\
\mu_1 &= \pi_0 + (\pi_1 - \pi_0)\tilde{\mu}_1 \\
&\vdots \\
\mu_t &= \sum_{\tau=0}^{t-1} \pi_0(\pi_1 - \pi_0)^\tau + (\pi_1 - \pi_0)^t \tilde{\mu}_t
\end{aligned}
$$

Simply puts bounds on where beliefs can be sent.

$$
\mathbb{E}\left[\sum_{t=0}^{\infty} \delta^t |\mu_t' - \mu_t|\right] = \mathbb{E}\left[\sum_{t=0}^{\infty} (\delta(\pi_1 - \pi_0))^t |\tilde{\mu}_t' - \tilde{\mu}_t|\right] \tag{24}
$$

Drift and arrival rate are higher, $\tilde{\delta} \approx 1 - (r + \lambda_1 + \lambda_0)\Delta t$.

Persistence



(a) Without commitment (b) With commitment

Figure: Sample paths for prices.

Information Flow Every Period

Bellman equation becomes

$$V(\mu) = \max_x \ \mathbb{E}\left[(P(\mu') - P(\mu))x + \delta V(\pi_0 + (\pi_1 - \pi_0)\mu')\right]. \quad (25)$$

Flow payoff is unchanged, but continuation value function is flatter. This implies the highest flat line still always hits the boundary.

$$V(\mu) = \min\left\{P(\mu) - P(0) + \frac{P(\pi_0) - P(0)}{1-\delta},\ P(1) - P(\mu) + \frac{P(1) - P(\pi}{1-\delta} \quad (26)$$

Information Flow Every Period

Take $\mu < \mu^*$ and $P(\mu) = \mu$.
The left endpoint of the strategy is 0 and the right endpoint is

$$\mu^{up} = \frac{2\mu + (1-\delta)\pi_0}{1 + \delta(\pi_1 - \pi_0)}. \tag{27}$$

The probability of jumping to zero is then equal to

$$\frac{\mu^{up} - \mu}{\mu^{up}\Delta t} \approx \frac{\mu(r + \lambda_1 + \lambda_0)}{2\mu + \Delta t}. \tag{28}$$

This give the same arrival rate as in the previous problem.

$$\lambda(\mu_t) = \lim_{\Delta t \to 0} \frac{p^{jump}}{\Delta t} = \frac{r + \lambda_1 + \lambda_0}{2} \tag{29}$$

Conclusion

Thank you.