Equilibrium Gerrymandering

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May 26, 2020

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Abstract

Through gerrymandering, a state drawing congressional districts can have a large effect on who gets elected. This in turn affects the policy chosen by elected representatives. This paper studies the optimal gerrymandering in an equilibrium of the fifty states electing members of the United States House of Representatives. First I find the optimal districting strategy when a party seeks to maximize the expected number of seats they win. This strategy always employs "cracking" (splitting up the opponent's base to spread them out in many districts) and it sometimes employs "packing" (cramming one district full of exlusively the opponent's base.) The optimal strategy can be found using techniques from information design. When the district drawer seeks to maximize the welfare of the state's citizens the care not just about the average seats won by each party, but the entire seat-vote curve. A seat-vote curve is a graph of the fraction of seats in congress that go to a political party against the fraction of votes obtained by that party. The national social optimal is for each state to have a seat-vote curve that is less responsive (flatter) than proportional (45 degree line). However, each state has an incentive individually to choose a highly responsive seat-vote curve to disproportionately swing policy in their favor. In equilibrium each state chooses an extreme seat-vote curve close to a winner-take-all election. This is a prisoner's dilemma situation where every state is worse off in equilibrium, but it is the dominant strategy of each state to choose a highly responsive seat-vote curve. I then empirically estimate the seat-vote curve for each state and observe a few motivating facts. First, seat-vote curves are highly responsive. Every state's seat-vote curve has a slope much steeper than one (the "proportional" seat-vote curve). Second, the size of the state is predictive of the responsiveness. Smaller states have steeper curves.

1 Introduction

Through gerrymandering, the group drawing politcal districts can affect who gets elected to congress. Getting different people elected will change the policy chosen and have a national impact.

The House of Representatives of the United States Congress has 435 members. Each state is allocated some number of these representatives to elect based on its population. The state of Minnesota gets to elect eight representatives. The state is divided up into eight districts and each of the districts elects one of the representatives by popular vote. How the state is divided into districts will have a large impact on who will win the elections.

For example, suppose you could perfectly predict how everyone would vote and the state population is 51 percent Democrat and 49 percent Republican. You could put all the Republicans in district 1 through 4 and put all the Democrats in district 5 through 8. Then the representatives from the state would be half Republicans and half Democrats. Another possible districting would be to make each of the eight districts perfectly representative of the whole state. If each district was 51 percent Democrat and 49 percent Republican, the all of the elected representatives would be Democrat.

I take a simple model of voters (largely the same as Coate and Knight (2007)) and find the optimal districting strategies. There are three types of citizens in the state with policy preference distributed on the interval $[0, 1]$: Democrats (0), Republicans (1), and Independents $\in (0, 1)$. The Independents' preferences follow some distribution, but the mean of the distribution in a given election is unknown. Thus the districter doesn't know how many Independents will vote for each party. The objective of the districter plays a large role in designing the optimal districts.

In many states the districts are chosen by the state congress. Here the majority party can attempt to maximize the number of seats their party will win in the next election. This is the controversial gerrymandering that is frequently seen in court cases on the news. In section three, I will show the optimal way to do this partisan gerrymandering in this model.

The optimal districting plan has three types of districts. First, there are districts the party is garunteed to win. The districter fill these districts fifty percent (plus ϵ) of the way with voters from their own party to garuntee the win. Any more voters from their own party would be wasteful. The rest of this district is filled up with the opposition party's voters (if there are enough. If not, use Independents after.) The second type of district are the ones the party is garunteed to lose. The districter only need to have any of these if the opposing party has a larger base than their own party. Since these districts are going to be lost, they might as well be filled 100 percent with the opposition party. The third type of district are the competitive ones. These districts will be filled exclusively with Independent voters and thus will be equally likely to vote for either party. I will show in section 3 that this districting scheme maximizes the expected number of seats the districter's party will win.

However, in many states, the districts are chosen by a non-partisan committee or judicial branch. For these states, I model the districter as trying to maximize social welfare in the state. The policy that maximizes welfare (and thus the represetatives you want elected) depend on the particular realization of the Independent distribution in each election. If more of the Independents are Democrats this year, the districter would like to elect more Democrats and have a policy that is more Democrat. For this reason, the districter doesn't care only about the expected number of seats won by each party but rather about the entire seat-vote curve.

A seat-vote curve is the expected fraction of elected representatives that come from the Democratic party graphed against the fraction of the state-wide vote that was for the Democratic party. It is the proportion of seats the party wins graphed against their vote share. An intuitive example of a seat-vote curve is the 45 degree line. That is, a line with an intercept of zero and a slope of one. With this curve, if the Democratic party wins x percent of the votes in the states they will win that same x percent of the seats in congress from the state. Generally, the curve may be non-linear, steeper, flatter, or even biased toward one party or the other.

In section 4, I solve for the equilibrium seat-vote curves. While the preferred policy reacts modestly to changes in the vote share, the equilibrium effects lead to highly responsive seat-vote curves. Essentially, when a state wants policy to move by one percent to the left they need one percent more of congress to be on the left. But, the state only chooses one-fiftieth of congress on average. Thus, they need to elect fifty percent more of their state's representatives from the left.

This leads to every state having a highly responsive seat-vote curve. In section 5, I show that this is not what will be socially optimal for the country as a whole. Political districting is a prisoners dilemma between the states. Total welfare is optimized when all states have modest seat-votes curves. However, each individual state has an incentive to deviate to a highly responsive curve. When each state chooses highly responsive curves, everyone is made worse off.

In section 6, I estimate the seat-vote curves in each of the 50 states and present a few stylized facts. The first fact is that seat-vote curves highly responsive. Rather than having a slope of one, on the interval 45 percent to 55 percent of the vote, the average slope is around four. If one percent more of the state votes Democrat, about four percent more of the elected representatives will be Democrats. The second fact is that the responsiveness has a strong negative correlation with the size of the state. Smaller states have especially steep seat-vote curves. I show that the inverse of the number of representatives from a state is a good predicter of seat-vote curve responsiveness, The responsiveness is not predicted by how much control a political party has in the state. The responsiveness is partially explained by who was in charge of drawing the districts (state congress, bipartisan committee, courts, etc.) I take these facts as validation of the model.

2 Model

Here a present a model of voters help understand optimal gerrymandering and explain the stylized facts in the data presented in section 6. The model is very similar to Coate and Knight (2007), but extended to have many states.

2.1 Model Setup

The payoff relevant object of interest is the policy chosen. A policy is a number that lies in the interval $[0, 1]$. Think of 0 as the preferred policy of Democratic party and 1 as the preferred policy of the Republican party. After the representatives from all states are elected, the policy will be the average of the representatives' preferences. In section 7, I explore the model where the policy chosen is equal to that of the median representative and the main results are qualitatively similar.

The strategic players in the game are the states' district designers. There are fifty states indexed by $i \in \{1, 2, \ldots, M\}$. Each state is characterized by the distribution of voters in the state. The districter will choose how to split the distribution of voters into different districts. Through this, they affect who gets elected and what policy is chosen. Two different objectives for the state district designer will e treated seperately. In section three, I will study a districter that seeks to maximize the expected number of seats for a given party. In section four, I will study a districter that seeks to maximize the aggregate welfare of the citizens in the state.

Every voter has a private preference over the policy chosen. In each state, there are three different groups of voters. There is a mass π_{Di} for state i of Democrats with a preferred policy

$\theta = 0$. There is a mass π_{Ri} for state i of Republicans with a preferred policy $\theta = 1$. There is a mass π_{Ii} for state i of Independents with preferred policy distributed over $\theta \in [0, 1]$. Call m_i the mean preference among the independent voters, and call $2\tau_i$ the width. The independent voters have a preferred policy uniformly distributed on the interval $[m_i - \tau_i, m_i + \tau_i]$. This mean, m_i, is unknown for every to the districter. The districter is also unable to distinguish between independent voters that lie in different parts of the interval. At the time of the election the mean is drawn from a uniform distribution $m_i \sim U\left(\left[\frac{1}{2} - \tau_i, \frac{1}{2} + \tau_i\right]\right)$. This means that the fraction of independents that will vote Democrat in a given election is uniformly distributed over $[0, 1]$.

Each voter wants the policy to be as close as possible to their preferred policy. They face a quadratic loss funtion. If a voter's preferred policy is $\hat{\theta}$ and the policy chosen is θ, the voter recieves a payoff of $-\left(\theta - \hat{\theta}\right)^2$. The independent voters are not strategic in their choice. A voter will vote Democrat if their preferred policy is less than or equal to $\frac{1}{2}$. Otherwise, they will vote Republican.

3 Partisan Districting Plans

In many states, the party in control of state congress can draw the districts. In this section, I will study what a districter would do if they wanted to maximize the expected number of seats that a given party will win. This isn't really an equilibrium problem. What districts other states draw, and who they elect doesn't enter into the objective in any way. We can simply solve a state's districting problem in isolation.

3.1 Example

Consider a state that has 50 percent Democrats, 25 percent Republicans, and 25 percent Independents. The intuitive outcome is that the Democrats should get 50 to 75 percent of the seats in congress and the Republicans should get 25 to 50 percent of the seats, depending on how the Independents vote. If the districts are drawn to maximize the expected number of seats for either party, the outcome will look very different. First, if the Democrats are in charge of drawing the districts, they can win all the seats in this model. All they need to do is make every district look just like the state as a whole. Each district will be 50 percent Democrat, 25 percent Republican, and 25 percent Independent. It doesn't even seem like an extreme gerrymender on the face of it. They simply make every district identical and representative of the state. However, since the Democrats are now garunteed to have at least 50 percent of the vote in every district, they will win all the congressional seats.

What if the Republicans are in charge of drawing the districts. They can use a concept called "packing" to win some of the seats. "Packing" is when you group together supporters of the opposition into a single district. If you're going to lose a district, you might as well lose big. They can simply puting all the Democrats together in the first half of the districts. Then in the other half of the districts they can employ a strategy called "cracking". "Cracking" is when you split up your own party's supporters so you can win more districts. If you're already going to win a district, each additional vote you get in that district is wasted. In the second half of the districts they can make each district an equal mix of Republicans and Independents. Since they have garunteed at least half the votes in each of these districts, they will win them all. This districting strategy is simple and it gets the Republicans half the seats.

However, the Republicans can do even better than this. It might seem like since half the state is certainly going to vote Democrat, Republicans can never win more than half the seats, but they can. For simplicity, assume that ties are broken in the Republican's favor. Throughout the section, I will always assume that ties are broken in favor of the party drawing the districts. Otherwise they would simply need to put one additional voter from their party in each of this districts to insure a win.

Consider the following districting. In one quarter of the districts they follow a packing strategy. One quarter of the districts are made up entirely of Democrats. Then in half of the districts they use a cracking strategy. They do it a little differently than before. If you're going to win a district for sure, you might as well make sure the rest of the votes are against you. You don't want to waste any votes. So, in this half of the districts there will be an equal mix of Republicans and Democrats. The Republicans will win all of these districts. Finally, the remaining quarter of the districts are made up entirely of Independents. On average Republicans will win half of these districts.

This plan gives the Republicans five-eighths of the seats in congress on average. This is the optimal districting plan in this example. If a state used to be heavily Republican, they may have a majority of Republicans in their state congress. This districting shows how they can maintain their majority in congress even if the population in the state changes to be heavily Democrat.

3.2 Optimal Districting

Generally, I will show that the optimal districting for maximizing the expected number of seats for a given party will have the same form as the example.

There are three types of voters. The distributions of these voters can be represented by a simplex as in figure **??**. This problem is very similar to a Bayesian Persuasion problem. In Bayesian persuasion, the sender is looking at ways to split a up distribution (the prior beliefs) into multiple other distributions (posterior beliefs) that must average out to the first distribution (by Bayes Rule). In this gerrymandering problem, the districter is looking at ways to split up a distribution (the population of voters in the state) into multiple other distributions (district populations) that add up to the state population. Just like in Bayesian Persuasion, the optimal value can be thought of as a "concavification" on this simplex.

Suppose the districter wants to maximize the expected number of seats for the Republican party. The distributions in figure **??** can be broken up into a few segments. The lower left triangle in the figure (red) are distributions where the Republicans have at least 50 percent of the population. They would win these districts with certainty. The upper triangle (blue)

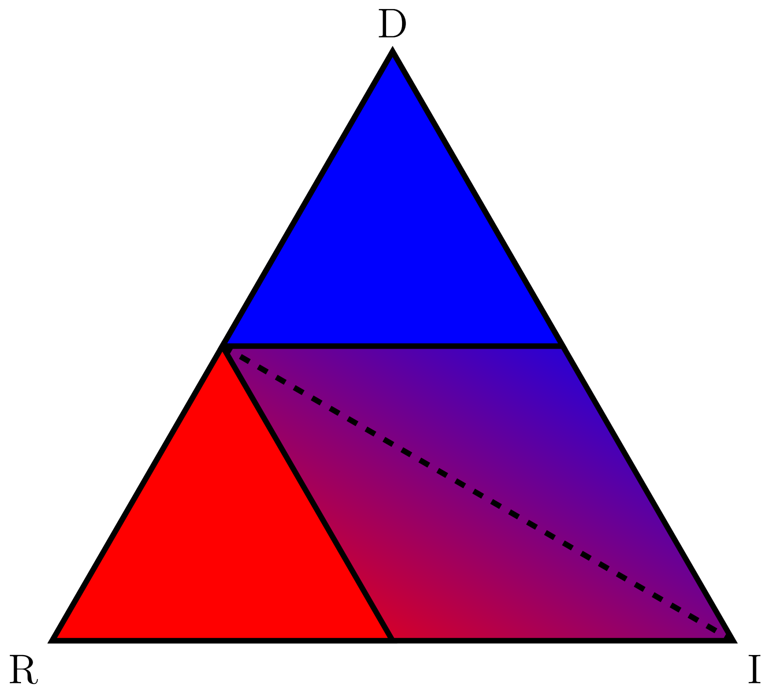

Figure 1: It's a triangle.

are distributions where Democrats have at least 50 percent of the population. They would lose these districts with certainty. The rhombus of remaining distributions (purple) are distributions where either party could win depending on how the Independents vote. The distributions along the dashed line are equally likely to be won by Republicans or Democrats. The likelihood of Republicans winning transitions from one to zero as you move up and to the right through this region.

To think about concavification, picture this triangle as a 3D object with the colors representing height. The lower left triangle (red) has a height of one. The upper triangle (blue) has a height of zero. The rhombus (purple) linearly connects the two triangles. Now imagine taking a cloth, laying it on top of the object, and pulling the edges down tight. This is the concavification of the function. The height of this concavification is the maximum expected number of seats Republicans can win with the optimal districting. The points where the concavification is equal to the original function (where the cloth is touching the 3D object) are the distributions that are used in the optimal districting schemes. If the state's population

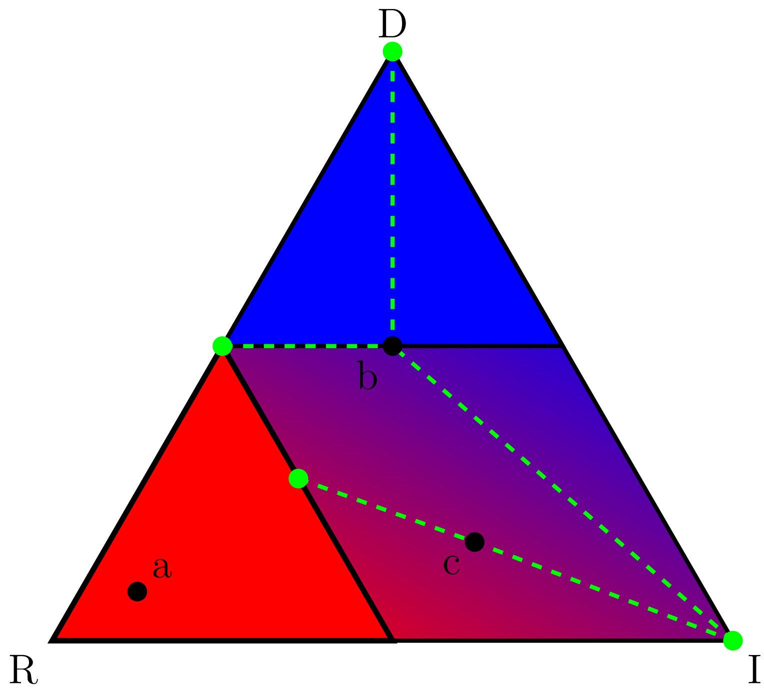

Figure 2: It's a triangle.

is on one of those points, it is optimal to make every district have a distribution identical to the state as a whole. If the state's population is not one of those points, the optimal districts with break up the state into different districts that are all among those points.

These optimal points consist of the entire lower left triangle (Republicans greater than 50 percent), and the other two corners of the simplex (Democrats have 100 percent and Independents have 100 percent). Any other distribution is dominated by a combination of these distributions. If your state is in the lower left triangle (Republicans have more than 50 percent), you don't have to do much of anything. The optimal districting will have all districts identical to the state population. The Republicans will win every district. If your state population is in the purple rhombus below the dashed line (more Republicans than Democrats), the optimal splitting is a combination of the bottom right corner (100 percent Independents) and the point on the edge of the lower left triangle (50 percent Republicans) that is straight across the prior from the corner. Republicans will win all of the latter districts and have a 50 percent chance of wining the former districts. If your state population is anywhere above the dashed line (more Democrats than Republicans) the optimal splitting is a combination of the point where the lower triangle meets the upper triangle (50 percent Republican and 50 percent Democrat), the bottom right corner (100 percent Independent), and the top corner (100 percent Democrat). Republicans will win all of the first type of districts, have a 50 percent chance in the second type of districts, an lose all of the last type of district. Each of these scenarios is graphed is shown in figure **??**.

The optimal districting strategy can be stated simply as follows. You start with the districts you are going to win for sure. Fill as many districts as possible half full with Republicans. Fill the other half of those districts with Democrats. If there aren't enough Democrats, continue filling with Independents. If there aren't enough Independents, fill remaineder with Republicans. After filling these districts entirely, all remaining Democrats and Independents are separated into the reset of the districts. So, there are districts that are 50 percent Republican, and 50 percent Democrat or Independent (Democrat is preferred). There are districts that are 100 percent Democrat. Finally, there are districts that are 100 percent Independent.

Proposition 1. *The maximum expected fraction of seats Republicans can win is*

$$v_R = \begin{cases} 1 & \text{if } \pi_R \geq \frac{1}{2} \\ 2\pi_R + \frac{1}{2}\left(\pi_I - (\pi_R - \pi_D)\right) & \text{if } \pi_D < \pi_R < \frac{1}{2} \\ 2\pi_R + \frac{1}{2}\pi_I & \text{if } \pi_R \leq \pi_D. \end{cases} \tag{1}$$

This can be rewritten as

$$v_R = \min\left\{1\ ,\ 2\pi_R + \frac{1}{2}\left(\pi_I - \max\left\{0,\ \pi_R - \pi_D\right\}\right)\right\}. \tag{2}$$

If there are fewer Republicans than Democrats, you fill as many districts as possible with half Republicans and half Democrats. You will win all these. There are $2\pi_R$ of these districts. Then you have all the Independents in their own districts. You win half of these. There are π_I of these districts. Finally, the rest of the Democrats are in their won districts and you don't win any of those. Hence, your expected number of seats is $2\pi_R + \frac{1}{2}\pi_I$.

If there are more Republicans than Democrats, you do the same strategy. The difference is that you now have to put some Independents in the first type of district because there aren't enough districts. From the payoff you need to subtract off the Independents put in the first type of district so they aren't double counted.

This will continue until you are able to fill all the districts half way with Republicans. Then the payoff is flat at one. Any additional Republicans are superfluous.

4 Non-Partisan Districting Plans

In many states, the districts are not drawn by the state congress. Rather a non-patisan committee or even the judicial branch may be the ones drawing the districts. In these cases we wouldn't expect the districts to be maximizing the expected seats of either party. For this section, consider what the optimal districting strategy would be if the districter seeks only to maximize the utility of the state's citizens.

This problem needs to be solved very differently from that of the partisan districter. All that the partisan districter cared about in the previous section was an average. The non-partisan districter will care about all the outcomes. I will show you why. The non-partisan districter cares about the preferred policy of all of the citizens. If the median independent voter, m_i, turns out to be very high in a given election, this means that the citizens prefer more of a Democrat policy. Thus, the districter would like to have more Democrats elected. Whereas if m_i is very low, the districter would like to have more Republicans elected. The districter's optimal fraction of seats is different for each election outcome.

The point of different districting strategies is to induce a certain seat-vote curve. The seat-vote curve is the expected fraction of congressional seats a party wins as a function of the party's vote share in the state. Consider a state in which 40 percent are Democrats, 40 percent are Republicans, and 20 percent are Independents. You could stick all the Democrats together in the first 40 percent of the districts, all the Republicans in the next 40 percent of districts, and the Independents in the last 20 percent of districts. Then regardless of what the independents do, at least 40 percent of seats will be won be Democrats and at least 40 percent won by Republicans. This means the seat-vote curve will start with a high intercept (.4), increase only very modestly, and max out at a low height (.6). This seat-vote curve is not very responsive. Alternatively, you could make every district look demographically like the state as a whole. That is, each district would be made up of 40 percent Democrats, 40 percent Republicans, and 20 percent Independents. Then, all the districts could potentially be flipped based on what happens to the Independents. This seat-vote curve would have a minimum of 0, a maximum of 1, and be very steep in the middle region. This is a highly responsive seat-vote curve. The most intuitive idea people think of is that congressional seats should match the distribution of the voters. So, if Democrats get 54 percent of the vote they should get 54 percent of the seats in congress. This represented by a seat-vote curve that is simply a 45 degree line. The curve has a slope of one everywhere and an intercept of zero.

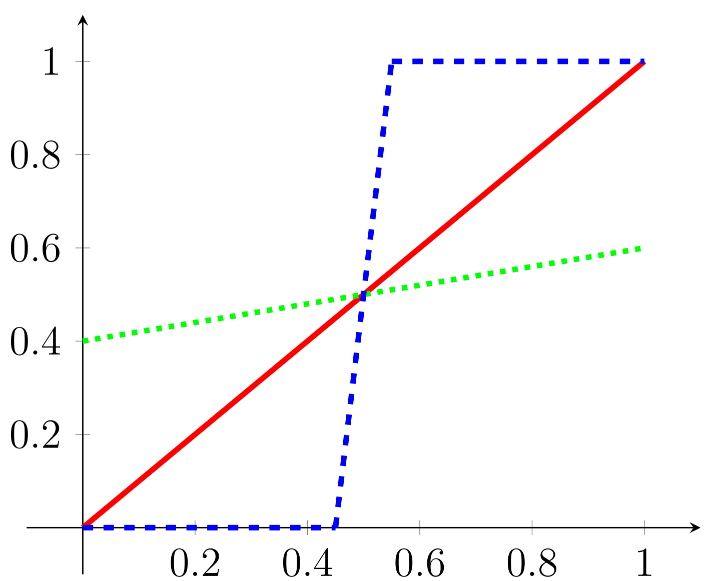

Figure 3: Three different seat-vote curves implemented by gerrymandering.

Rather than working directly with how to split the distribution into districts, we will assume that the state simply chooses a seat-vote curve. We won't worry for right now about what districting is used to implement that curve.

Call v_i the fraction of voters that vote Democrat in state i. This is a combination of how many Democrats, Republicans, and Independents are in the state, as well as where the median Independent voter is.

$$v_i = \pi_{Di} + \pi_{Ii} \left(\frac{\frac{1}{2} - (m_i - \tau_i)}{2\tau_i} \right) \tag{3}$$

The seat-vote curve $S : [0,1] \to [0,1]$ is a mapping from the fraction of votes that Democrats won, v_i, to the fraction of representatives Democrats get in congress, $S_i(v_i)$.

After the election, the policy is chosen collectively by the representatives from all 50 states. Let N_i be the number of representatives assigned to state i. The fraction of congress comming from state i is $n_i = \frac{N_i}{\sum_{j=1}^{M} N_j}$. If each state, i, elected $S_i(v_i)$ Democrats, then congress will have a fraction $S = \sum_{i=1}^{M} n_i S_i(v_i)$. The members of congress choose the policy that maximizes their welfare. Policy is set equal to the average of all the representatives preferences. So, the chosen policy, θ is equal to $1 - S$.

This is a simulateous move game where each state chooses a seat-vote curve, $S_i(v_i) \in [0,1]$ $\forall v_i \in [0,1]$, to maximize the welfare of its voters,

$$\max_{S_i(v_i)} \quad - \mathbb{E}_{m_1, m_2, \ldots, m_M} \left[\pi_{Di} \left(\theta\right)^2 + \pi_{Ri} \left(1-\theta\right)^2 + \pi_{Ii} \int_{m_i - \tau_i}^{m_i + \tau_i} \left(\theta - x\right)^2 \frac{dx}{2\tau} \right] \tag{4}$$

where θ, the policy, is equal to one minus the average fraction of seats chosen by the states for Democrats.

$$\theta = 1 - \sum_{i=1}^{M} n_i S_i(v_i) \tag{5}$$

We will find the unique Nash equilibrium of this game.

4.1 Optimal Seat-Vote Curve

Before solving explicitly for the solution of this game, let us breifly examine the desired policy of a state. If the voters of a state have individual preferences, $\tilde{\theta}$, distributed according to G_i, then the state's preferred policy solves

$$\max_{\hat{\theta}} \; -\int \left(\tilde{\theta} - \hat{\theta}\right)^2 dG_i(\tilde{\theta}). \tag{6}$$

So, the state would like the policy to be equal to the average preference in the state.

$$\Rightarrow \hat{\theta}^* = \mathbb{E}_{G_i}\left[\theta\right] \tag{7}$$

If every voter was either a pure Democrat or pure Republican living on the extreme, $\pi_{Ii} = 0$, then the average preference is exactly equal to the vote share. This is where the idea of a proportional seat-vote curve with a slope of one comes from. However, a one percent increase in the population voting Democrat is not from one percent of the population that were staunch Republicans and are now suddenly staunch Democrats. The one percent increase in votes comes from one percent of the population that was near the center but leaning slightly Republican and is now near the center and leaning slightly Democrat. It is a much smaller shift in aggregate preference. Thus, optimal policy would shift toward the Democrats by less than one percent. This is a motivation for a flatter seat-vote curve.

In this model, the state's preferred policy is equal to the seat-vote curve derived in Coate and Knight (2007).

$$\hat{\theta}^* = \mathbb{E}_{m_i}\left[\theta\right] \tag{8}$$

$$= \frac{1}{2} + (\pi_{Di} - \pi_{Ri})\left(\frac{1}{2} - \tau_i\right) + 2\tau_i\left((v_i - \frac{1}{2}\right) \tag{9}$$

Even though the shift in desired policy is mild (slope of $2\tau_i$ in the above equation), the strategy a state must take to impliment that shift is extreme. We can see this by solving for state i's best response function in the game. Once the state sees the vote share, v_i, they no longer face any uncertainty about their own voters' preferences. This means that the optimal seat-vote curve can be solved pointwise.

$$\max_{S_i(v_i)} -\mathbb{E}_{v_{-i}} \left[\pi_{Di} \left(1 - \sum_{j=1}^{M} n_j S_j(v_j) \right)^2 + \pi_{Ri} \left(\sum_{j=1}^{M} n_j S_j(v_j) \right)^2 \right. \\ \left. + \ \pi_{Ii} \int_{m_i - \tau_i}^{m_i + \tau_i} \left(1 - \sum_{j=1}^{M} n_j S_j(v_j) - x \right)^2 \frac{dx}{2\tau_i} \right| v_i \Bigg] \quad (10)$$

Now we differentiate the equation to find the local maximum.

$$\frac{\partial W_i(v_i)}{\partial S_i(v_i)} = 2n_i \mathbb{E} \left[\pi_{Di} \left(1 - \sum_{j=1}^{M} n_j S_j(v_j) \right) \right. \quad (11)$$

$$\left. -\pi_{Ri} \sum_{j=1}^{M} n_j S_j(v_j) + \pi_{Ii} \left(1 - \sum_{j=1}^{M} n_j S_j(v_j) \right) - \pi_{Ii} \int_{m_i - \tau_i}^{m_i + \tau_i} \frac{x}{2\tau_i} dx \right| v_i \Bigg] \quad (12)$$

$$= 2n_i \left(\pi_{Di} + \pi_{Ii}(1 - m_i) - \sum_{j=1}^{M} n_j \mathbb{E}_{v_j} [S_j(v_j) | v_i] \right). \quad (13)$$

Setting this equation equal to zero gives us the welfare maximizing share of seats for Democrats given the realization of the Independent voters.

$$\hat{S}_i(v_i) = \frac{1}{n_i}(\pi_{Di} + \pi_{Ii}(1 - m_i)) - \sum_{j \neq i} \frac{n_j}{n_i} \mathbb{E}_{v_j} [S_j(v_j) | v_i]. \quad (14)$$

Here the seats are a function of m_i. Of course, we would like the seats to be a function explicitly of v_i to get our seat-vote curve. We can write m_i as a function of the vote share simply by inverting equation (1).

$$m_i = \frac{1}{2} + \tau_i \left(\frac{\pi_{Ii} + 2\pi_{Di} - 2v_i}{\pi_{Ii}} \right) \quad (15)$$

Plugging this in, we can get the seats as a function of the vote share.

$$\hat{S}_i(v_i) = \frac{1}{n_i}\left(\frac{1}{2} + (\pi_{Di} - \pi_{Ri})\left(\frac{1}{2} - \tau_i\right) + 2\tau_i\left(v_i - \frac{1}{2}\right) - \sum_{j\neq i} n_j \mathbb{E}_{v_j}[S_j(v_j)|v_i]\right). \tag{16}$$

If the state could choose any values for their seat-vote curve, this is what they would choose. However, they are restricted to choose a share between 0 and 1. This equation does not always lie in the interval $[0, 1]$. In fact, it is usually outside the interval. Since the objective is quadratic, the solution for the best response function will still be very simple. With a quadratic objective, the optimum will just be at the boundary closer to the unconstrained optimum.

$$S_i^*(v_i) = \begin{cases} 0 & \text{if } \hat{S}_i(v_i) < 0 \\ \hat{S}_i(v_i) & \text{if } \hat{S}_i(v_i) \in [0, 1] \\ 1 & \text{if } \hat{S}_i(v_i) > 0. \end{cases} \tag{17}$$

$S_i^*(v_i)$ is the optimal seat-vote curve for state i.

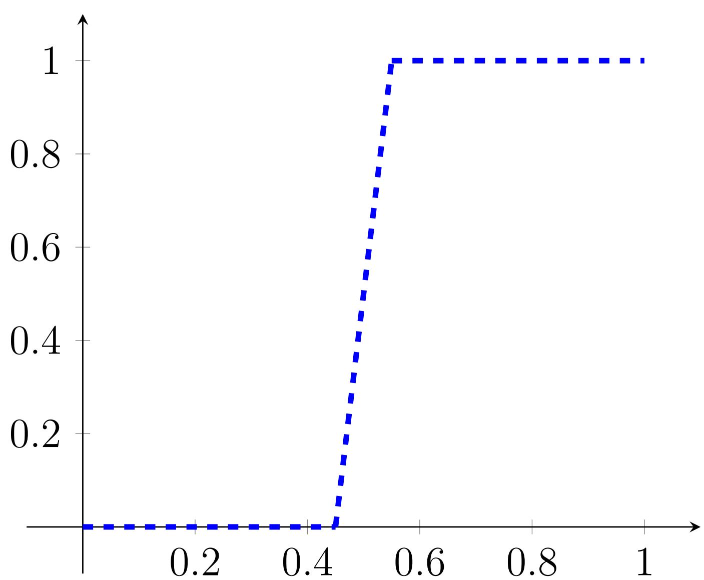

Figure 4: The optimal seta-vote curve.

Assume for now that m_i is drawn independently from m_j for every state j. We can see already that the seat-vote curve for state i is very steep in the middle.

$$\frac{\partial S_i^*}{\partial v_i} = \frac{2\tau_i}{n_i} \tag{18}$$

The first component, $2\tau_i < 1$, comes from the fact that a change in vote share represents a smaller change in average political preference in the state and the seat-vote curve should be less responsive. The second component, $\frac{1}{n_i}$, comes from the fact that state i is only a small part of the national congress and to get congress to shift by 1 percent the state needs to shift their representatives by an average of 50 percent. This is the same non-linear relationship we will see fits well in the empirical section. This is the main takeaway from this section. Equilibrium seat-vote curves are very steep and have a responsiveness proportional to the inverse of the number of representatives the state is alloted.

4.2 Equilibrium

I have not yet completely charactarized the equilibrium of the game. What we have is the best response of state i as a function of the seat-vote curves of all the other states. The equilibrium is the fixed point of all these functions. Notice in equation (14) that the other states' chosen curves only enter the best response in a very simple way. State i only cares about the expected fraction of seats the other states will elect from the Democratic Party, $\sum_{j \neq i} n_j E\left[S_j^*\right]$. This makes the system easy to solve.

For notational simplicity, let's call $s_i = \mathbb{E}[S_i^*(v_i)]$. Using equation (12) we see that the state will be in the interior portion of their seat-vote curve whenever

$$0 < \frac{1}{n_i}(\pi_{Di} + \pi_{Ii}(1 - m_i) - \sum_{j \neq i} n_j s_j) < 1. \tag{19}$$

Simplifying, the state will be in the highly responsive portion of the seat-vote curve as long as the median independent voter isn't too far to either extreme.

$$1 - \frac{1}{\pi_{Ii}}(n_i + \sum_{j \neq i} n_j s_j - \pi_{Di}) \;<\; m_i \;<\; 1 - \frac{1}{\pi_{Ii}}(\sum_{j \neq i} n_j s_j - \pi_{Di}). \tag{20}$$

Also, $S_i^*(v_i)$ will be equal to 1 whenever the median independent, m_i, is below that lower cutoff.

Now the goal is to take the expected value of equation (15) to get the average fraction of seats that go to Democrats in state i. There are three segments to the best response function we need to average over. On the first segment, Democrats get zero seats. So, this drops out of the equation. In the middle segment we integrate over the likelihood of each value. On the third segment, Democrats get all the seats. So, the contribution to the expectation is just the probability of being in this segment times 1. The likelihood of this is equal to the cdf of m_i at the cutoff. Since m_i is uniformly distributed on $[\frac{1}{2} - \epsilon_i, \frac{1}{2} + \epsilon_i]$, the probability that $S_i^*(v_i) = 1$ equals the following.

$$p_i = \frac{1 - \frac{1}{\pi_{Ii}}(n_i + \sum_{j \neq i} n_j s_j - \pi_{Di}) - (\frac{1}{2} - \epsilon)}{2\epsilon_i}. \tag{21}$$

Now we can compute the expected number of seats Democrats will win in state i.

$$s_i = \mathbb{E}[S_i^*(m_i)] \tag{22}$$

$$= \int_{1-\frac{1}{\pi_{Ii}}(n_i + \sum_{j \neq i} n_j s_j - \pi_{Di})}^{1-\frac{1}{\pi_{Ii}}(\sum_{j \neq i} n_j s_j - \pi_{Di})} \frac{1}{n_i}(\pi_{Di} + \pi_{Ii}(1-m) - \sum_{j \neq i} n_j s_j)\,\frac{dm}{2\epsilon_i} \tag{23}$$

$$+ \frac{1 - \frac{1}{\pi_{Ii}}(n_i + \sum_{j \neq i} n_j s_j - \pi_{Di}) - (\frac{1}{2} - \epsilon_i)}{2\epsilon_i} \tag{24}$$

$$= \frac{\pi_{Di} + \pi_{Ii} - \sum_{j \neq i} n_j s_j}{2\pi_{Ii}\epsilon_i} - \frac{\pi_{Ii}}{2\epsilon_i} \int_{1-\frac{1}{\pi_{Ii}}(n_i + \sum_{j \neq i} n_j s_j - \pi_{Di})}^{1-\frac{1}{\pi_{Ii}}(\sum_{j \neq i} n_j s_j - \pi_{Di})} m\;dm \tag{25}$$

$$+ \frac{1 - \frac{1}{\pi_{Ii}}(n_i + \sum_{j \neq i} n_j s_j - \pi_{Di}) - (\frac{1}{2} - \epsilon_i)}{2\epsilon_i} \tag{26}$$

$$= \frac{\pi_{Di} + \pi_{Ii} - \sum_{j \neq i} n_j s_j}{2\pi_{Ii}\epsilon_i} + \frac{n_i}{2\pi_{Ii}\epsilon_i}(1 - \frac{1}{\pi_{Ii}}(n_i + \sum_{j \neq i} n_j c_j - \pi_{Di})) \tag{27}$$

$$+\frac{1-\frac{1}{\pi_{Ii}}(n_i+\sum_{j\neq i} n_j s_j-\pi_{Di})-(\frac{1}{2}-\epsilon_i)}{2\epsilon_i} \tag{28}$$

This gives a system of 50 equations and 50 unknowns. Solving for the 50 s_i terms that satisfy this system will finish the construction of the equilibrium. While this initially looks messy, we now have s_i written as a linear function of all s_j with $j \neq i$. We can write the equation simply as

$$As = b \tag{29}$$

where b is an $M \times 1$ vector with

$$b_i = \frac{1}{2} + \frac{2\pi_{Di} + \frac{3}{2}\pi_{Ii} - n_i^2 - n_i + n_i\pi_{Ii} - n_i\pi_{Ii}\pi_{Di}}{2\pi_{Ii}\epsilon_i} \tag{30}$$

and A is an $M \times M$ matrix with

$$A_{ij} = \begin{cases} \frac{\left(\frac{n_i}{\pi_{Ii}}+2\right)n_j}{2\pi_{Ii}\epsilon_i} & \text{if } i \neq j \\ 1 & \text{if } i = j. \end{cases} \tag{31}$$

This A matrix is always full rank. This means that there always exists a unique solution. The solution gives us the final constants we needed in the best response function. Thus, it completes the equilibrium.

5 Welfare

Even though the described strategies connstitute the only equilibrium, they don't give the highest total payoffs in the game. They aren't socially optimal. Let us construct the socially optimal policy if a planner was in charge of all 50 states. The objective function would look similar, but it would account for the citizens in every state.

$$\max_{\theta} \quad -\sum_{i=1}^{M} n_i \mathbb{E}_v \left[\pi_{Di} \left(\theta\right)^2 + \pi_{Ri} \left(1-\theta\right)^2 + \pi_{Ii} \int_{m_i-\tau_i}^{m_i+\tau_i} \left(\theta - x\right)^2 \frac{dx}{2\tau_i}\right] \tag{32}$$

We can differentiate this with respect to S to solve for the optimum.

$$\frac{\partial W}{\partial \theta} = 2\sum_{i=1}^{M} n_i \pi_{Di}(\theta) - 2\sum_{i=1}^{M} n_i \pi_{Ri}(1-\theta) - 2\sum_{i=1}^{M} n_i \pi_{Ii} \int_{m_i-\tau_i}^{m_i+\tau_i} \left(\theta - x\right) \frac{dx}{2\tau_i} \tag{33}$$

Evaluating the uniform integral and simplifying,

$$= 2\sum_{i=1}^{M} n_i \pi_{Di} + 2\sum_{i=1}^{M} n_i \pi_{Ii}(1 - \mathbb{E}[m_i]) - 2(1-\theta) \tag{34}$$

since $\sum_{i=1}^{M} n_i = 1$.

Putting the planner on the same footing as the states in the game, let them choose an entire seat-vote curve for the nation. This makes the expectation go away. Now, we can substitute in for m_i as before to get an expression in terms of vote shares, v_i. Setting this equal to zero gives the optimal policy.

$$\begin{aligned}
1 - \theta^* &= \sum_{i=1}^{M} n_i \left(\pi_{Di} + \pi_{Ii}(1 - m_i)\right) && (35)\\
&= \sum_{i=1}^{M} n_i \left(\pi_{Di} + \pi_{Ii} \left(\frac{1}{2} - \tau_i \frac{\pi_{Ii} + 2\pi_{Di} - 2v_i}{\pi_{Ii}}\right)\right) && (36)\\
&= \sum_{i=1}^{M} n_i \left(\left(\frac{1}{2} - \tau_i\right) \pi_{Ii} + \left(1 - 2\tau_i\right) \pi_{Di} + 2\tau_i v_i\right) && (37)\\
&= \frac{1}{2} + \sum_{i=1}^{M} n_i \left(\left(\pi_{Di} - \pi_{Ri}\right) \left(\frac{1}{2} - \tau_i\right) + 2\tau_i \left(v_i - \frac{1}{2}\right)\right) && (38)
\end{aligned}$$

Assuming τ_i is the same in each state, this would be a linear function of the aggregate vote share with a slope equal to 2τ. It's also easy to impliment. Since it's linear, we would get this policy by each state having a seat-vote curve equal to

$$S_i(v_i) = \frac{1}{2} + (\pi_{Di} - \pi_{Ri}) \left(\frac{1}{2} - \tau \right) + 2\tau \left(v_i - \frac{1}{2} \right). \tag{39}$$

with that same low slope. Notice that this is the same as equation (6), which is the same as in Coate and Knight (2007) (CK). This is where each state simply chooses their most preferred policy. If a state wants the policy to be 60 percent Democrat, then they just elect 60 percent Democrats.

Take the following example. Imagine that everyone in the room was to write down what temperature they would like room. Then the thermastat is set to the average of all the votes. The social optimum is obtained if everyone just honestly writes their most preferred temperature. However, you may have an incentive to write something different. If you like to the room to be 67 degrees and you think that's a little colder than most people like, you have an incentive write a vote that is much colder. You want the average of the votes to be 67. So, you might write something more extreme like 60 degrees.

In equilibrium, political districting is the same way. This mild seat-vote curve is what will maximize national welfare, but each state has an incentive to deviate. When a state gets a lot of votes for Democrats, they think their state is likely more Democrat than the average state. Then to make the policy a little more Democrat the state wants to elect a lot more Democrats. Regardless of what the other states are doing, each state has an incentive to deviate by playing a highly responsive seat-vote curve.

This is a prisoner's dilemma situation though, because in equilibrium every state is worse off. No state prefers the equilibrium to the collusive outcome where all states choose modest seat-vote curves.

The smallest states have the largest incentive to deviate. We saw in equation (14) that the smaller the state, the steeper they would like their seat-vote curve. In the limit, as a state grows in size its best response would approach the socially optimal curve. The state

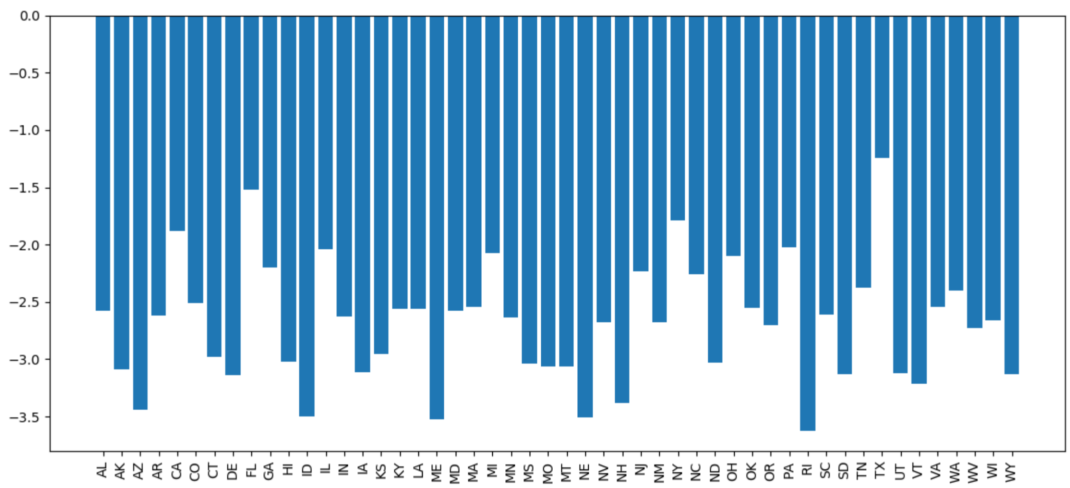

Figure 5: The loss each state faces in equilibrium from their expected utility in the social optimum.

would play the social optimum if their own elected congress members could move policy one for one. States only play the more extreme curves because the comgress members they elect are only a small part of making the policy. This becomes a larger and larger factor as a state shrinks in size.

6 Empirical Analysis

In this section, I validate my model by documenting a few empirical facts about seat-vote curves across states.

6.1 Estimation

First I do a simple estimation of the seat-vote curves in each state. I use data from the Cooperative Congressional Election Study. From this I aggregate across each of the 435 congressional districts what fraction report as Democrat, Republican, or Independent/Not Sure. Suppose that the self reported Democrats will vote for the Deomract candidate, the reported Republicans will vote for the Republicans candidate, and that the Independents and Not Sure respondents could vote either way.

We want to estimate the share of seats Democrats would win across all possible vote shares. To do this, I randomly draw the fraction of Independents to vote Democrat in each district 10,000 times. In each draw, we can compute how many districts in the state Democrats won and what fraction of the overall vote in the state Democrats won. Then I take the average fraction of state's seats earned for each level of the state vote share to be the seat-vote curve.

From election results in each district, we can estimate the distribution of Independents that vote Democrat and the correlation of this draw across districts within a state. In constructing the following motivational facts, I drew the fraction of Independents to vote Democrat from a Uniform distribution over $[0, 1]$ with a correlation of .7 between districts of a state. This appears to match election results fairly well.

6.2 Stylized Facts

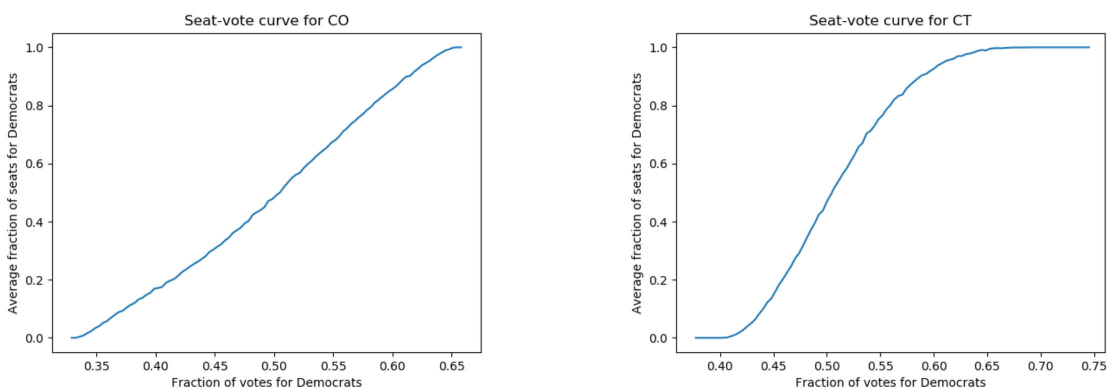

Figure 6: The estimated seat-vote curves for Colorado and Connecticut. Connecticut's curve is much steeper in the middle than Colorado's. Connecticut has the more responsive curve.

Figure 1 shows the estimated seat-vote curves for two states. Call the responsiveness of a seat-vote curve the average slope of the curve on the interval $[.45, .55]$. Over the middle ranges of votes shares, this is how much the congressional seats respond to changes in the vote on average.

The intuitive proportional seat-vote curve would be a linear function with a slope of one. We can see in the graph that the seat-vote curve in Colorado is nearly linear, but with a slope closer to three for intermediate values of the vote share. Connecticut's seat-vote curve has more of a slanted "S" shape. The is a very high slope in the intermediate values of the vote share. Between vote shares of 45 percent and 55 percent the slope averages about 6.5 or a little more than twice as steep as Colorado.

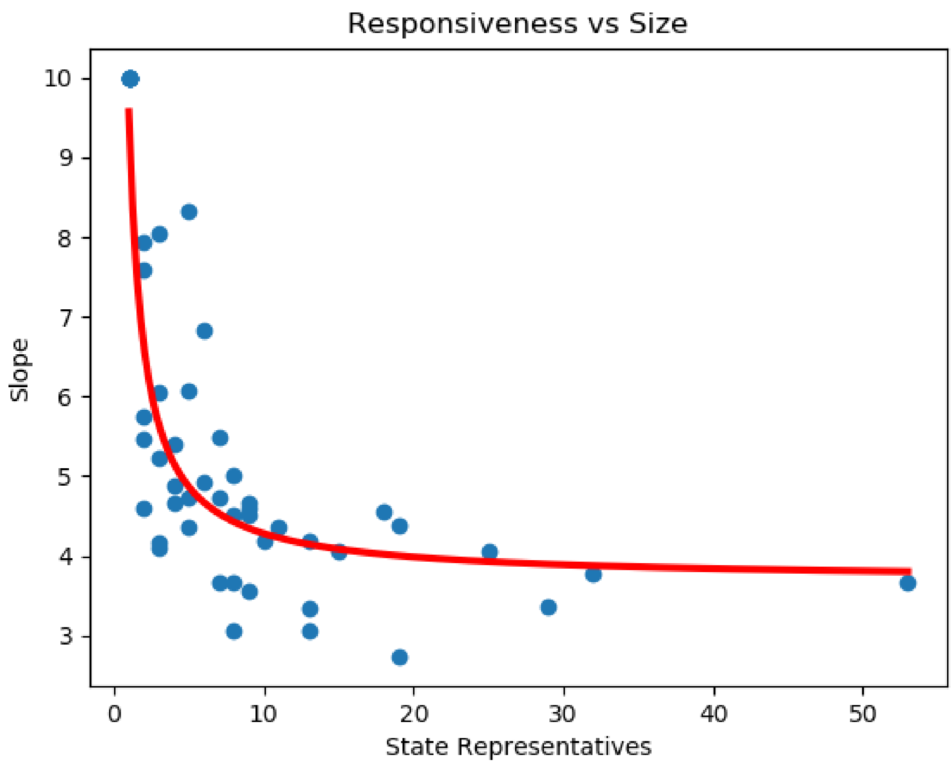

Figure 2 shows a scatter plot of the estimated slopes on the 45 percent to 55 percent interval for each state graphed against the number of representatives allocated to the state. Remember that the number of representatives a state has is approximately proportional to the population size of the state. The first observation is that all the slopes are well above one. The flattest state seat-vote curve is about 2.8 while the steepest curve has 10 as its average slope. Second, notice a very clear non-linear negative relationship between the responsiveness of a state's seat-vote curve and its size.

A simple regression confirms what we see in the scatter plot. *Representatives* is the number of representatives a state has. *Reps^2* is the square of *Representatives* to pick up the non-linear relationship. *Slope* is the slope of the estimated seat-vote curve. *Dem_control* is the fraction of the state that are self reported Democrats minus the fraction that are self reported Republicans. *Party_control* is the absolute value of *Dem_control*. *n_inverse* is one divided by the number of representatives alloted to the state.

VARIABLES	(1) Model1	(2) Model2	(3) Model3	(4) Model3
Representatives	-0.375***	-0.335***		
	(0.0613)	(0.0649)		
Reps^2	0.00613***	0.00560***		
	(0.00136)	(0.00137)		
Dem_control		-3.805	-3.201**	-3.573**
		(2.319)	(1.316)	(1.378)
n_inverse			5.912***	5.817***
			(0.459)	(0.471)
Party_control				2.017
				(2.172)
Constant	7.777***	7.648***	3.849***	3.702***
	(0.406)	(0.407)	(0.209)	(0.262)
Observations	50	50	50	50
R-squared	0.492	0.520	0.826	0.829

Standard errors in parentheses
*** p<0.01, ** p<0.05, * p<0.1

There is always a large significant relationship between *Slope* and the number of representatives. Smaller states have much steeper seat-vote curves. This relationship persist when controlling for which party is in control or how much control a party has in a state.

We can also see that the non-linearity of the relationship is picked up well by an inverse proportionality to the size of the state.

$$\frac{\partial Seats}{\partial votes} \approx a\frac{1}{n_i} + \epsilon_i \tag{40}$$

where n_i is the number of representatives (size) of state i.

7 Extension

7.1 Winner-Take-All

It may be that the optimal seat-vote curve in the model is difficult to implement in the world, requires a lot of information, or that the model assumptions don't line up exactly with your view of the world. There is a very simple seat-vote curve that can always be implemented, relies on no model assumptions, requires no information, and is very close to the optimal value. It is a winner-take-all election.

Take a seat-vote curve equal to

$$S_i(v_i) = \begin{cases} 0 & \text{if } v_i < .5 \\ 1 & \text{if } v_i \geq .5 \end{cases} \tag{41}$$

That is, if the Democrats get more than 50 percent of the vote in the state, all of the state's representatives will be Democrats. Otherwise, all the representatives will be Republicans.

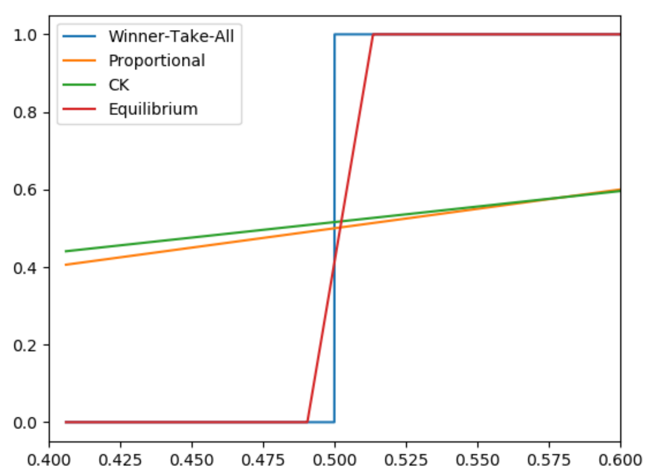

Figure 7: This is a graph of several seat-vote curves for the state of Minnesota. The "proportional" curve is a linear seat-vote curve with a slope of one. The "CK" curve is the socially optimal seat-vote curve. "Equilibrium" is the curve Minnesota would choose as their best response in equilibrium play. "Winner-take-all" is simply an indicator function for vote shares above 50 percent.

This doesn't require any knowledge about the voting population in the state. It is also extremely simple and easy to understand. It could be done without creating any districts. If districts are desired, all the state needs to do is randomly assign each citizen a district number regardless of geography. Each district in the United States is composed of about 700,000 people. By the law of large numbers, each district would then have pretty close to the same fraction voting Democrat in each election. This would impliment the winner-take-all seat-vote curve.

While winner-take-all is not the best response, it is very close to the best response. Every state's seat-vote curve is flat at zero, then increases rapidly to one, then flat at one. The increase isn't infinitely steep like a winner-take-all, but the average slope is about 50. The graph shows these two curves for Minnesota.

In fact, nearly all the gains from deviating to the best response from the previous section can be obtained from deviating to a winner-take-all function.

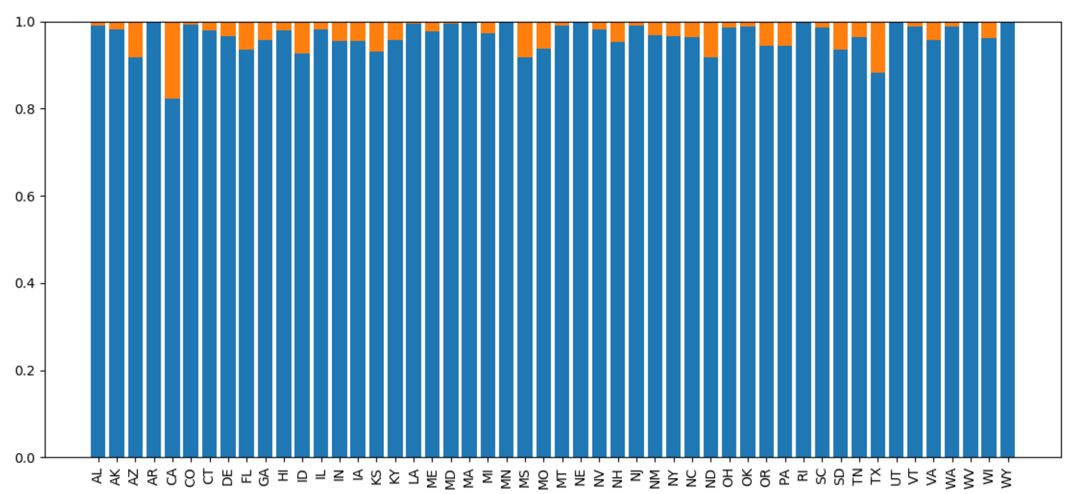

Figure 8: The gain from deviating from the social optimal to winner-take-all as a percentage of the gain from deviating to the best response.

In fact the strategies are almost always the same. In the best response, the fraction of Democrats elected is either 0 or 1 more than 80 percent of the time in each state already.

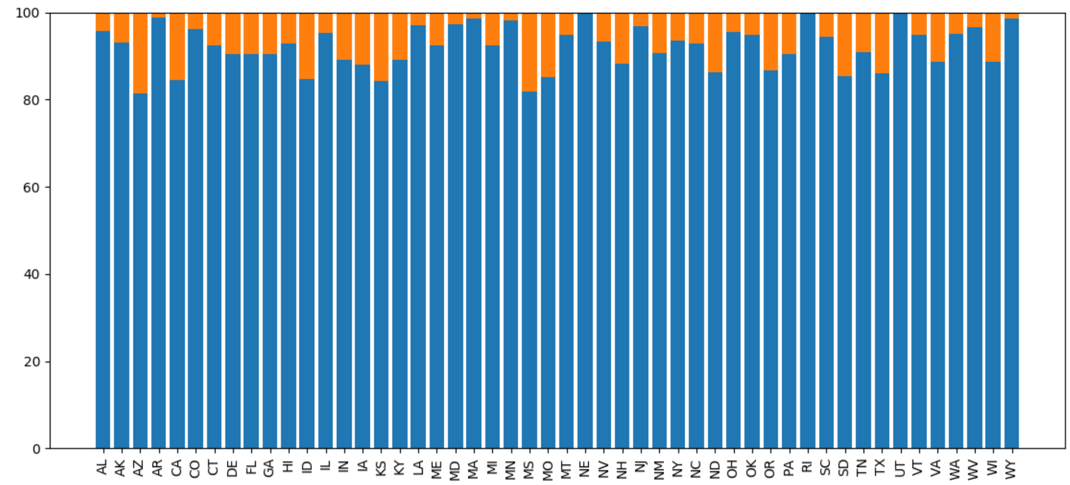

Figure 9: The fraction of the time the best response and a simple winner-take-all choose the same number of Democrats.

7.2 Median Member of Congress

One assumption of the model up to this point has been that the policy chosen is equal to the mean preference of all the elected congress members. One might not believe that all the members of congress are so readily willing to compromise. Perhaps, the majority party can disproportionately pull policy in their favor. In this section, I consider the opposite extreme from the rest of the paper. Suppose that the policy chosen is equal to that of the median member of congress. This would be if the majority party has complete control over choosing policy and doesn't need to compromise to please the minority. The optimal seat-vote curves will be even simpler than before but not that different intuitively.

Now, there are only two real outcomes of the game. Either the Democrats win the majority or the Republicans win the majority. State i first needs to determine which of those it prefers. State i will prefer Democrats win the majority as long as

$$\pi_{Ri} + \pi_{Ii} \int_{m_i-\tau_i}^{m_i+\tau_i} x \frac{dx}{2\tau_i} \geq \pi Di + \pi_{Ii} \int_{m_i-\tau_i}^{m_i+\tau_i} (1-x)^2 \frac{dx}{2\tau_i}. \tag{42}$$

Computing the integral and simplifying yields a simple equation. The state will prefer a

Democrat majority as long as

$$v_i \geq \frac{1}{2} + (\pi_{Ri} - \pi_{Di}) \left(\frac{1}{4\tau_i} - \frac{1}{2} \right). \tag{43}$$

The unique best response is for the state to give all its representatives to the Democratic Party when the vote share is above that cutoff and give all representatives to the Republican Party when it is below the cutoff. This is the case regardless of what any other state is doing. The optimal seat-vote curve for every state is a winner-take-all election. The difference from a standard winner-take-all though is that the cutoff for the Democrats to win my not be at exactly 50 percent. This is because not every vote is created equal. A firm member of the Democrat party counts for more than an independent voter that is leaning slightly Republican. Thus, the cutoff will be at a point that favors the party that has a larger political base in the state.

In this game, the socially optimal policy is not unique. Any congress with a Democratic majority is the same regardless of how strong that majority after all. The socially optimal policy from the previous sections in fact remains socially optimal in this version of the game.

Note that the equilibrium strategy for each state no longer depends on the size of the state. Even if a state had complete control over the policy chosen, they would still find it optimal to choose this same seat-vote curve.

8 Conclusion

8.1 Reltated Literature

The closest paper to mine is Coate and Knight (2007). In their paper, they seek the socially optimal seat-vote curve for a state. This is done for elections to the state congress instead of the national congress. In the state congress, the seat-vote curve determines all members of congress and thus has complete control over the policy. My model of a state is the same as theirs but I include it as one of 50 states that need to choose policy together. In mine the motive arises for a state to have a highly responsive curve to balance with the expected representatives from the other states.

Another paper that compliments mine well is Carson and Crespin (2004). Across states different groups are put in charge of drawing the congressional districts. They show that in states where a bipartison committee or a court is in charge of redistricting elections are more competitive than when state legislature does the redistricting. My model is closer to the motivation of a bipartison committee or court. Since having more competitive elections translates to a steeper seat-vote curve, this evidence is supportive of my model.

Also talk about Friedman and Holden (2008) and Bergmann et al. (2015).

8.2 Conclusion

Gerrymandering is a topic fiercely debated every ten years when congressional districts are redrawn. It is important because it can have a significant impact on who gets elected and ultimately on what policies are put into place. A fundamental question we should ask about gerrymandering is what seat-vote curves would we expect to see in equilibrium and what are the socially optimal seat-vote curves.

In this paper, I first estimate the seat-vote curves for each of the 50 states. I find that seat vote curves are highly responsive. I also find that small states choose steeper seat-vote curves than larger states. The slope of the seat-vote curve in a state is approximately proportional to one over the number of representatives that state has in congress. This relationship holds after controlling for the politcal party in power.

I then present a model of states choosing seat-vote curves to compete with each other over the policy that is passed. I find that in equilibrium, every state chooses an extreme seat-vote curve to disproportionately affect policy in their favor. This motive is especially strong in smaller states. In equilibrium, the slope of each state's seat-vote curve is proportional to one over the number of representatives they have. This is the same as in the empirical section.

I show that there is a deadweight loss to society in equilibrium. If every state were to commit to playing a more modest seat-vote curve, a Pareto improvement could be had. However, each state has an incentive to deviate, making this a Prisoner's Dilemma.

I also find that the optimal seat-vote curve can be approximately implimented with a very simple rule. This is a winner-take-all election for the state's representatives. If policy is chosen entirely by the majority party in congress, then this winner-take-all election becomes exactly the unique optimal rule for each state.

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