Equilibrium Seat-Votes Curves

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Abstract

Through gerrymandering, a state drawing congressional districts can have a large effect on who gets elected. This in turn affects the policy chosen by elected representatives. This paper studies the seat-votes curves from an equilibrium of the fifty states electing members of the United States House of Representatives. A seat-vote curve is a graph of the fraction of seats in congress that go to a political party against the fraction of votes obtained by that party. I first estimate the seat-vote curve for each state and observe a few motivating facts. First, seat-vote curves are highly responsive. Every state's seat-vote curve has a slope much steeper than one (the "proportional" seat-vote curve). Second, the size of the state is predictive of the responsiveness. Smaller states have steeper curves. Third, amount of control a political party has in the state is uncorrelated with its responsiveness. I then propose a game theoretic model that explains these facts. Each state has a distribution of citizens' preferred policy in an interval. A state chooses a seat-vote curve to minimize the welfare cost to its citizens. The national social optimal is for each state to have a seat-vote curve that is less responsive (flatter) than proportional (45 degree line). However, each state has an incentive individually to choose a highly responsive seat-vote curve to disproportionately swing policy in their favor. In equilibrium each state chooses an extreme seat-vote curve close to a winner-take-all election. This is a prisoner's dilemma situation where every state is worse off in equilibrium, but it is the dominant strategy of each state to choose a highly responsive seat-vote curve.

1 Introduction

Through gerrymandering, the group drawing politcal districts can affect who gets elected to congress. Getting different people elected will change the policy chosen and have a national impact.

The House of Representatives of the United States Congress has 435 members. Each state is allocated some number of these representatives to elect based on its population. The state of Minnesota gets to elect eight representatives. The state is divided up into eight districts and each of the districts elects one of the representatives by popular vote. How the state is divided into districts will have a large impact on who will win the elections.

For example, suppose you could perfectly predict how everyone would vote and the state population is 51 percent Democrat and 49 percent Republican. You could put all the Republicans in district 1 through 4 and put all the Democrats in district 5 through 8. Then the representatives from the state would be half Republicans and half Democrats. Another possible districting would be to make each of the eight districts perfectly representative of the whole state. If each district was 51 percent Democrat and 49 percent Republican, the all of the elected representatives would be Democrat.

There are two parts to gerrymandering. The first is choosing a seat-vote curve for the state. The second is drawing the district boundaries to try to implement the chosen seat-vote curve. In this paper, I will focus on the first.

A seat-vote curve is the expected fraction of elected representatives that come from the Democratic party graphed against the fraction of the state-wide vote that was for the Democratic party. It is the proportion of seats the party wins graphed against their vote share. An intuitive example of a seat-vote curve is the 45 degree line. That is, a line with an intercept of zero and a slope of one. With this curve, if the Democratic party wins x percent of the votes in the states they will win that same x percent of the seats in congress from the state. Generally, the curve may be non-linear, steeper, flatter, or even biased toward one party or the other.

In the next section I estimate the seat-vote curves in each of the 50 states and present a few stylized facts. The first fact is well documented and it is that seat-vote curves highly responsive. Rather than having a slope of one, on the interval 45 percent to 55 percent of the vote, slopes are around four. If one percent more of the state votes Democrat, about four percent more of the elected representatives will be Democrats. The second fact is that the responsiveness has a strong negative correlation with the size of the state. Small states have very steep seat-vote curves. The responsiveness is not predicted by how much control a political party has in the state. The responsiveness is partially explained by who was in charge of drawing the districts (state congress, bipartisan committee, courts, etc.)

I present a model that can explain these facts. The key to the model is that each state is not acting in isolation, but that the representatives elected from each state collectively choose the policy that affects everyone. Consider an example of a very simple game that gives the necessary intuition of my results.

1.1 Example

You and your roommate meet to decide at what temperature you will keep the thermastat. The two of you come up with a way to settle the matter. You will both simulateously write down your preferred temperature. Then, you'll reveal what you wrote and the thermastat will be set to the average of the votes.

You have a preferred temperature of 68 degrees. You would like the temperature to be as close as possible to 68 and face a quadradic loss funtion. That is, if the temperature is t, your utility is

$$U(t) = -(t - 68)^2. \tag{1}$$

The problem is that you don't know your roommate's preferred temperature for certain. Say your beliefs about your roommate's preferred temperature are uniformly distributed on the interval $[66, 76]$. What do you write on the paper to maximize your expected utility?

If you and your roommate vote $\hat{t}_1$ and $\hat{t}_2$, you can write your objective as

$$\mathbb{E}\left[-\left(\frac{\hat{t}_1+\hat{t}_2}{2}-68\right)^2\right]. \tag{2}$$

Then you get your optimal strategy by simply differentiating and finding the max.

$$\hat{t}_1^* = 68 + \left(68 - \mathbb{E}\left[\hat{t}_2\right]\right) \tag{3}$$

Rather than simply writing down your prefered temperature, the optimal strategy has you adjust you vote based on what you think your roommate is going to write. The average preferred temperature your roommate might have is 71, and if they are playing a symmetric strategy their expected vote will also be 71. This means that your optimal vote is 65 rather than 68.

Let's look at the ex ante welfare of the players. Say that each player draws their preferred temperature independently from a uniform distribution on the interval $[66, 76]$, then they play the game as described. The equilibrium is where each player writes the vote $\hat{t}_i$ equal to

$$\hat{t}_i = 71 + 2\left(t_i - 71\right) \tag{4}$$

where t_i is the preferred temperature of player i. Your vote is a linear function of your type with a slope of two. In other words, your optimal vote is twice as responsive to your type as simply writing down your preference would be.

We can see right away that the temperature chosen will not always be socially optimal or even Pareto optimal ex post. The lowest temperature anyone ever wants is 66, but if the players each draw a preference of 66 they will be writing lower votes down and the realized temperature will end up being 61 and both roommates will freeze. The ex ante welfare to player 1 is,

$$W_1 = -\frac{1}{10}\frac{1}{10}\int_{66}^{76}\int_{66}^{76}\left(\frac{\hat{t}_1+\hat{t}_2}{2}-t_1\right)^2 \ dt_1 dt_2 \tag{5}$$

$$= -\frac{1}{100}\int_{66}^{76}\int_{66}^{76}\left(\frac{71+2(t_1-71)+71+2(t_2-71)}{2}-t_1\right)^2 \ dt_1 dt_2 \tag{6}$$

$$= -\frac{1}{100}\int_{66}^{76}\int_{66}^{76}(t_2-71)^2 \ dt_1 dt_2 \tag{7}$$

$$= -\frac{25}{3}. \tag{8}$$

If both players were to simply write down their most preferred temperature, we similar computation gives the ex ante welfare for each player.

$$\tilde{W}_1 = -\frac{1}{10}\frac{1}{10}\int_{66}^{76}\int_{66}^{76}\left(\frac{t_1+t_2}{2}-t_1\right)^2 \ dt_1 dt_2 \tag{9}$$

$$= -\frac{1}{400}\int_{66}^{76}\int_{66}^{76}(t_2-t_1)^2 \ dt_1 dt_2 \tag{10}$$

$$= -\frac{1}{1200}\int_{66}^{76}\left((t_2-76)^3-(t_2-66)^3\right) \ dt_2 \tag{11}$$

$$= -\frac{25}{6}. \tag{12}$$

Thus we would have a Pareto improvement. The expected loss for each player is twice as big in equilibrium than it would be if both roommates simply wrote their honest preferrence. This isn't due to any irrationality. Even if the other player is writing their honest preference for their vote, player i's best response is still

$$\hat{t}_i^* = 71 + 2(t_i - 71). \tag{13}$$

Thus, we have a prisoner's dilemma.

Adding more roommates into the mix would only make the problem worse. With three people voting the equilibrium strategy is three times as responsive to a player's type.

$$\hat{t}_i^* = 71 + 3(t_1 - 71) \tag{14}$$

As a player gets smaller relative to the total size of the house, they choose a more highly responsive strategy.

2 Empirical Motivation

In this section, I motivate my model by documenting a few empirical facts about seat-vote curves across states.

2.1 Estimation

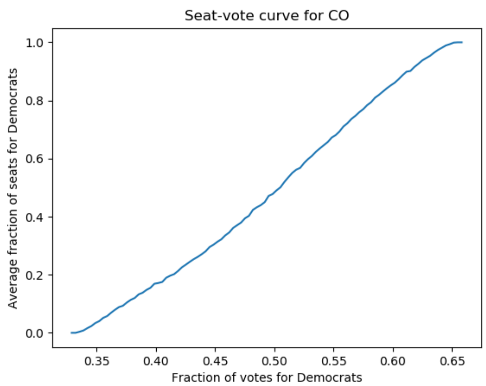
First I do a simple estimation of the seat-vote curves in each state. I use data from the Cooperative Congressional Election Study. From this I aggregate across each of the 435 congressional districts what fraction report as Democrat, Republican, or Independent/Not Sure. Suppose that the self reported Democrats will vote for the Deomract candidate, the reported Republicans will vote for the Republicans candidate, and that the Independents and Not Sure respondents could vote either way.

We want to estimate the share of seats Democrats would win across all possible vote shares. To do this, I randomly draw the fraction of Independents to vote Democrat in each district 10,000 times. In each draw, we can compute how many districts in the state Democrats won and what fraction of the overall vote in the state Democrats won. Then I take the average fraction of state's seats earned for each level of the state vote share to be the seat-vote curve.

From election results in each district, we can estimate the distribution of Independents that vote Democrat and the correlation of this draw across districts within a state. In constructing the following motivational facts, I drew the fraction of Independents to vote

Democrat from a Uniform distribution over $[0, 1]$ with a correlation of .7 between districts of a state. This appears to match election results fairly well.

2.2 Stylized Facts



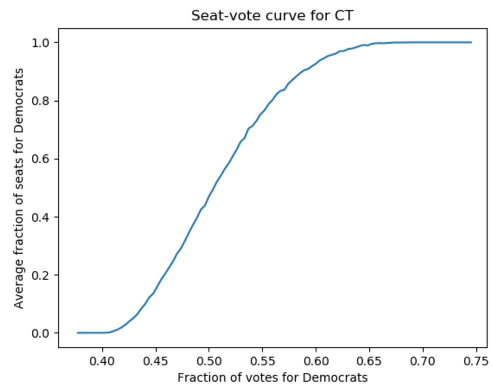
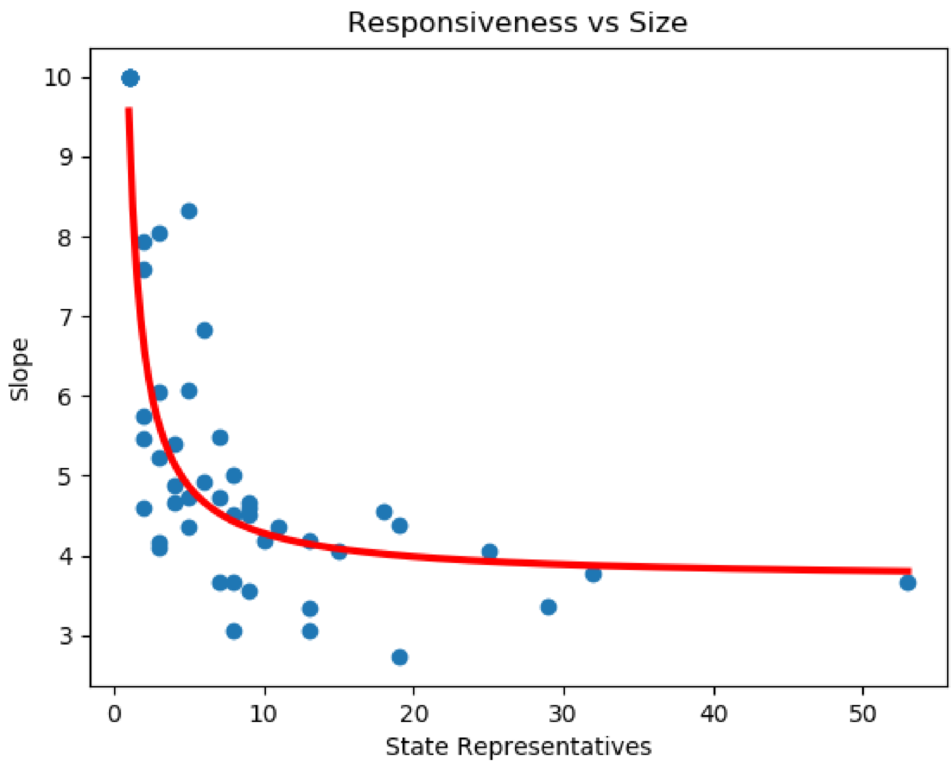


Figure 1: The estimated seat-vote curves for Colorado and Connecticut. Connecticut's curve is much steeper in the middle than Colorado's. Connecticut has the more responsive curve.

Figure 1 shows the estimated seat-vote curves for two states. Call the responsiveness of a seat-vote curve the average slope of the curve on the interval $[.45, .55]$. Over the middle ranges of votes shares, this is how much the congressional seats respond to changes in the vote on average.

The intuitive proportional seat-vote curve would be a linear function with a slope of one. We can see in the graph that the seat-vote curve in Colorado is nearly linear, but with a slope closer to three for intermediate values of the vote share. Connecticut's seat-vote curve has more of a slanted "S" shape. The is a very high slope in the intermediate values of the vote share. Between vote shares of 45 percent and 55 percent the slope averages about 6.5 or a little more than twice as steep as Colorado.

Figure 2 shows a scatter plot of the estimated slopes on the 45 percent to 55 percent interval for each state graphed against the number of representatives allocated to the state. Remember that the number of representatives a state has is approximately proportional to the population size of the state. The first observation is that all the slopes are well



above one. The flattest state seat-vote curve is about 2.8 while the steepest curve has 10 as its average slope. Second, notice a very clear non-linear negative relationship between the responsiveness of a state's seat-vote curve and its size.

A simple regression confirms what we see in the scatter plot. *Representatives* is the number of representatives a state has. *Reps^2* is the square of *Representatives* to pick up the non-linear relationship. *Slope* is the slope of the estimated seat-vote curve. *Dem_control* is the fraction of the state that are self reported Democrats minus the fraction that are self reported Republicans. *Party_control* is the absolute value of *Dem_control*. *n_inverse* is one divided by the number of representatives alloted to the state.

There is always a large significant relationship between *Slope* and the number of representatives. Smaller states have much steeper seat-vote curves. This relationship persist when controlling for which party is in control or how much control a party has in a state.

We can also see that the non-linearity of the relationship is picked up well by an inverse

VARIABLES	(1) Model1	(2) Model2	(3) Model3	(4) Model3
Representatives	-0.375***	-0.335***		
	(0.0613)	(0.0649)		
Reps^2	0.00613***	0.00560***		
	(0.00136)	(0.00137)		
Dem_control		-3.805	-3.201**	-3.573**
		(2.319)	(1.316)	(1.378)
n_inverse			5.912***	5.817***
			(0.459)	(0.471)
Party_control				2.017
				(2.172)
Constant	7.777***	7.648***	3.849***	3.702***
	(0.406)	(0.407)	(0.209)	(0.262)
Observations	50	50	50	50
R-squared	0.492	0.520	0.826	0.829

Standard errors in parentheses
*** p<0.01, ** p<0.05, * p<0.1

proportionality to the size of the state.

$$\frac{\partial Seats}{\partial votes} \approx a\frac{1}{n_i} + \epsilon_i \tag{15}$$

where n_i is the number of representatives (size) of state i.

3 Equilibrium Seat-Vote Curves

Here a present a model of states choosing seat-vote curves to maximize the welfare of their citizens that will help explain the stylized facts above. The model is very similar to Coate and Knight (2007), but extended to have many states.

3.1 Model Setup

The payoff relevant object of interest is the policy chosen. A policy is a number that lies in the interval $[0, 1]$. Think of 0 as the preferred policy of Democratic party and 1 as the preferred policy of the Republican party.

The strategic players in the game are the states. There are fifty states indexed by $i \in \{1, 2, \ldots, M\}$. Each state is characterized by the distribution of voters in the state. The state will choose a seat-vote curve to maximize the welfare of its voters.

Every voter has a private preference over the policy chosen. In each state, there are three different groups of voters. There is a mass π_{Di} for state i of Democrats with a preferred policy $\theta = 0$. There is a mass π_{Ri} for state i of Republicans with a preferred policy $\theta = 1$. There is a mass π_{Ii} for state i of Independents with preferred policy distributed over $\theta \in [0, 1]$. Call m_i the mean preference among the independent voters, and call $2\tau_i$ the width. The independent voters have a preferred policy uniformly distributed on the interval $[m_i - \tau_i, m_i + \tau_i]$. This mean, m_i, is unknown for every i. At the time of the election the mean is drawn from some distribution $m_i \sim F_i([0, 1])$. The voters are not strategic. A voter will vote Democrat if their preferred policy is less than or equal to $\frac{1}{2}$. Otherwise, they will vote Republican. Call v_i the fraction of voters that vote Democrat in state i.

$$v_i = \pi_{Di} + \pi_{Ii} \left(\frac{\frac{1}{2} - (m_i - \tau_i)}{2\tau_i} \right) \tag{16}$$

The state chooses a seat-vote curve. The seat-vote curve $S : [0, 1] \to [0, 1]$ is a mapping from the fraction of votes that Democrats won, v_i, to the fraction of representatives Democrats get in congress, $S_i(v_i)$. We will allow any function for this seat-vote curve for now. Particularly, the state is not constrained by the discrete number of representatives they have to elect or by whether or not disctricts can be drawn geographically to impliment the curve.

After the election, the policy is chosen collectively by the representatives from all 50 states. Let N_i be the number of representatives assigned to state i. The fraction of congress comming from state i is $n_i = \frac{N_i}{\sum_{j=1}^{M} N_j}$. If each state, i, elected $S_i(v_i)$ Democrats, then congress will have a fraction $S = \sum_{i=1}^{M} n_i S_i(v_i)$. The members of congress choose the policy that maximizes their welfare. Policy is set equal to the average of all the representatives preferences. So, the chosen policy is equal to $1 - S$.

This is a simulateous move game where each state chooses a seat-vote curve, $S_i(v_i) \in [0,1]$ $\forall v_i$, to maximize the welfare of its voters,

$$\max_{S_i(v_i)} \quad -\mathbb{E}_{m_1,m_2,\ldots,m_M}\left[\pi_{Di}\left(1-S\right)^2 + \pi_{Ri}\left(S\right)^2 + \pi_{Ii}\int_{m_i-\tau_i}^{m_i+\tau_i}\left(1-S-x\right)^2\frac{dx}{2\tau}\right] \tag{17}$$

where S is the average fraction of seats chosen by the states.

$$S = \sum_{i=1}^{M} n_i S_i(v_i) \tag{18}$$

We will find the unique Nash equilibrium of this game.

3.2 Optimal Seat-Vote Curve

Before solving explicitly for the solution of this game, let us breifly examine the desired policy of a state. If the voters of a state are distributed according to G_i, then the state's preferred policy solves

$$\max_{\hat{\theta}} \quad -\int\left(\theta-\hat{\theta}\right)^2 dG_i(\theta). \tag{19}$$

So, the state would like the policy to be equal to the average preference in the state.

$$\Rightarrow \hat{\theta} = \mathbb{E}_{G_i}\left[\theta\right] \tag{20}$$

If every voter was either a pure Democrat or pure Republican living on the extreme, $\pi_{Ii} = 0$, then the average preference is exactly equal to the vote share. This is where the idea of a proportional seat-vote curve with a slope of one comes from. However, a one percent increase in the population voting Democrat is not from one percent of the population that were staunch Republicans and are now suddenly staunch Democrats. The one percent increase in votes comes from one percent of the population that was near the center but leaning slightly Republican and is now near the center and leaning slightly Democrat. It is a much smaller shift in aggregate preference. Thus, optimal policy would shift toward the Democrats by less than one percent.

Even though the shift in desired policy is mild, the strategy a state must take to impliment that shift is extreme. We can see this by solving for state i's best response function. Once the state sees the vote share, v_i, they no longer face any uncertainty about their own voters' preferences. This means that the optimal seat-vote curve can be solved pointwise.

$$\max_{S_i(v_i)} -\mathbb{E}_{v_{-i}} \left[\pi_{Di} \left(1 - \sum_{j=1}^{M} n_j S_j(v_j)\right)^2 + \pi_{Ri} \left(\sum_{j=1}^{M} n_j S_j(v_j)\right)^2 + \pi_{Ii} \int_{m_i-\tau_i}^{m_i+\tau_i} \left(1 - \sum_{j=1}^{M} n_j S_j(v_j) - x\right)^2 \frac{dx}{2\tau_i} \,\middle|\, v_i \right] \quad (21)$$

Differentiating this equation gives,

$$\frac{\partial W_i(v_i)}{\partial S_i(v_i)} = \quad (22)$$

$$2n_i \mathbb{E} \left[\pi_{Di} \left(1 - \sum_{j=1}^{M} n_j S_j(v_j)\right) - \pi_{Ri} \sum_{j=1}^{M} n_j S_j(v_j) + \pi_{Ii} \left(1 - \sum_{j=1}^{M} n_j S_j(v_j)\right) - \pi_{Ii} \int_{m_i-\tau_i}^{m_i+\tau_i} \frac{x}{2\tau_i} dx \,\middle|\, v_i \right] \quad (23)$$

$$= 2n_i \left(\pi_{Di} + \pi_{Ii}(1 - m_i) - \sum_{j=1}^{M} n_j \mathbb{E}_{v_j}[S_j(v_j)|v_i] \right). \quad (24)$$

Setting this equal to zero gives,

$$\hat{S}_i(v_i) = \frac{1}{n_i}(\pi_{Di} + \pi_{Ii}(1 - m_i)) - \sum_{j \neq i} \frac{n_j}{n_i} \mathbb{E}_{v_j}[S_j(v_j)|v_i]. \quad (25)$$

Here the seats are a function of m_i. We can write m_i as a function of the vote share.

$$m_i = \frac{1}{2} + \tau_i \left(\frac{\pi_{Ii} + 2\pi_{Di} - 2v_i}{\pi_{Ii}} \right) \tag{26}$$

Plugging this in, we can get the seats as a function of the vote share.

$$\hat{S}_i(v_i) = \frac{1}{n_i} \left(\frac{1}{2} + (\pi_{Di} - \pi_{Ri}) \left(\frac{1}{2} - \tau_i \right) + 2\tau_i \left(v_i - \frac{1}{2} \right) - \sum_{j \neq i} n_j \mathbb{E}_{v_j}[S_j(v_j)|v_i] \right). \tag{27}$$

This does not always lie in the interval $[0, 1]$, but since the objective is quadratic, the solution for the best response function is still very simple.

$$S_i^* = \begin{cases} 0 & \text{if } \hat{S}_i(v_i) < 0 \\ \hat{S}_i(v_i) & \text{if } \hat{S}_i(v_i) \in [0,1] \\ 1 & \text{if } \hat{S}_i(v_i) > 0. \end{cases} \tag{28}$$

Assume for now that m_i is drawn independently from m_j for every state j. We can see already that the seat-vote curve for state i is very steep in the middle.

$$\frac{\partial S_i^*}{\partial v_i} = \frac{2\tau_i}{n_i} \tag{29}$$

The first component, $2\tau_i < 1$, comes from the fact that a change in vote share represents a smaller change in average political preference in the state and the seat-vote curve should be less responsive. The second component, $\frac{1}{n_i}$, comes from the fact that state i is only a small part of the national congress and to get congress to shift by 1 percent the state needs to shift their representatives by an average of 50 percent. This is the same non-linear relationship we saw fit well in the empirical section.

3.3 Equilibrium

Thus far, we have not completely charactarized the equilibrium of the game. What we have is simply the best response of state i as a function of the seat-vote curves of all the other states. However, notice in equation (27) that the other states' chosen curves only enter the best response in a very simple way. State i only cares about the expected fraction of seats the other states will elect from the Democratic Party, $\sum_{j\neq i} n_j E\left[S_j^*\right]$. This makes the system easy to solve.

For notational simplicity, let's call $c_i = \mathbb{E}[S_i^*(v_i)]$. Using equation (27) we see that the solution will be interior whenever

$$0 < \frac{1}{n_i}(\pi_{Di} + \pi_{Ii}(1 - m_i) - \sum_{j\neq i} n_j c_j) < 1 \tag{30}$$

or

$$1 - \frac{1}{\pi_{Ii}}(n_i + \sum_{j\neq i} n_j c_j - \pi_{Di}) \ < \ m_i \ < \ 1 - \frac{1}{\pi_{Ii}}(\sum_{j\neq i} n_j c_j - \pi_{Di}). \tag{31}$$

S_i^* will be equal to 1 whenever the median independent, m_i, is below that lower cutoff. The likelihood of this is equal to the cdf of m_i at the cutoff. Since m_i is uniformly distributed on $[\frac{1}{2} - \epsilon_i, \frac{1}{2} + \epsilon_i]$, the probability that $S_i^* = 1$ equals the following.

$$p_i = \frac{1 - \frac{1}{\pi_{Ii}}(n_i + \sum_{j\neq i} n_j c_j - \pi_{Di}) - (\frac{1}{2} - \epsilon)}{2\epsilon_i}. \tag{32}$$

Now the goal is to take the expected value of equation (28) to get the average fraction of seats that go to Democrats in state i. There are three segments to the best response function we need to average over. On the first segment, Democrats get zero seats. So, this drops out of the equation. In the middle segment we integrate over the likelihood of each value. On the third segment, Democrats get all the seats. So, the contribution to the expectation is just the probability of being in this segment times 1.

$$c_i = \mathbb{E}[S_i^*(m_i)] \tag{33}$$

$$= \int_{1-\frac{1}{\pi_{Ii}}(n_i+\sum_{j\neq i} n_j c_j - \pi_{Di})}^{1-\frac{1}{\pi_{Ii}}(\sum_{j\neq i} n_j c_j - \pi_{Di})} \frac{1}{n_i}(\pi_{Di} + \pi_{Ii}(1-m) - \sum_{j\neq i} n_j c_j)\ \frac{dm}{2\epsilon_i} \tag{34}$$

$$+ \frac{1 - \frac{1}{\pi_{Ii}}(n_i + \sum_{j\neq i} n_j c_j - \pi_{Di}) - (\frac{1}{2} - \epsilon_i)}{2\epsilon_i} \tag{35}$$

$$= \frac{\pi_{Di} + \pi_{Ii} - \sum_{j\neq i} n_j c_j}{2\pi_{Ii}\epsilon_i} - \frac{\pi_{Ii}}{2\epsilon_i} \int_{1-\frac{1}{\pi_{Ii}}(n_i+\sum_{j\neq i} n_j c_j - \pi_{Di})}^{1-\frac{1}{\pi_{Ii}}(\sum_{j\neq i} n_j c_j - \pi_{Di})} m\ dm \tag{36}$$

$$+ \frac{1 - \frac{1}{\pi_{Ii}}(n_i + \sum_{j\neq i} n_j c_j - \pi_{Di}) - (\frac{1}{2} - \epsilon_i)}{2\epsilon_i} \tag{37}$$

$$= \frac{\pi_{Di} + \pi_{Ii} - \sum_{j\neq i} n_j c_j}{2\pi_{Ii}\epsilon_i} + \frac{n_i}{2\pi_{Ii}\epsilon_i}(1 - \frac{1}{\pi_{Ii}}(n_i + \sum_{j\neq i} n_j c_j - \pi_{Di})) \tag{38}$$

$$+ \frac{1 - \frac{1}{\pi_{Ii}}(n_i + \sum_{j\neq i} n_j c_j - \pi_{Di}) - (\frac{1}{2} - \epsilon_i)}{2\epsilon_i} \tag{39}$$

While this initially looks messy, we have c_i written as a linear function of all c_j with $j \neq i$. We can write the equation simply as

$$Ac = b \tag{40}$$

where b is an $M \times 1$ vector with

$$b_i = \frac{1}{2} + \frac{2\pi_{Di} + \frac{3}{2}\pi_{Ii} - n_i^2 - n_i + n_i\pi_{Ii} - n_i\pi_{Ii}\pi_{Di}}{2\pi_{Ii}\epsilon_i} \tag{41}$$

and A is an $M \times M$ matrix with

$$A_{ij} = \begin{cases} \frac{\left(\frac{n_i}{\pi_{Ii}} + 2\right) n_j}{2\pi_{Ii}\epsilon_i} & \text{if } i \neq j \\ 1 & \text{if } i = j. \end{cases} \tag{42}$$

This A matrix is always full rank. This means that there always exists a unique solution.

The solution gives us the final constant we needed in the best response function. Thus, it completes the equilibrium.

4 Welfare

Even though the described strategies connstitute the only equilibrium, they don't give the highest payoffs attainable. They aren't socially optimal. Next we construct the socially optimal policy if a planner was in charge of all 50 states. The objective function would look similar, but it would account for the citizens in every state.

$$\max_{S} \quad -\sum_{i=1}^{M} n_i \mathbb{E}_v \left[\pi_{Di} \left(1-S\right)^2 + \pi_{Ri} \left(S\right)^2 + \pi_{Ii} \int_{m_i-\tau_i}^{m_i+\tau_i} \left(1-S-x\right)^2 \frac{dx}{2\tau_i} \right] \tag{43}$$

We can differentiate this with respect to S to solve for the optimum.

$$\frac{\partial W}{\partial S} = 2\sum_{i=1}^{M} n_i \pi_{Di}(1-S) - 2\sum_{i=1}^{M} n_i \pi_{Ri} S - 2\sum_{i=1}^{M} n_i \pi_{Ii} \int_{m_i-\tau_i}^{m_i+\tau_i} \left(1-S-x\right) \frac{dx}{2\tau_i} \tag{44}$$

Evaluating the uniform integral and simplifying,

$$= 2\sum_{i=1}^{M} n_i \pi_{Di} + 2\sum_{i=1}^{M} n_i \pi_{Ii}(1-\mathbb{E}[m_i]) - 2S \tag{45}$$

since $\sum_{i=1}^{M} n_i = 1$.

Putting the planner on the same footing as the states in the game, let them observe the vote shares before choosing the policy. This makes the expectation go away. Now, we can substitute in for m_i as before to get an expression in terms of vote shares, v_i. Setting this equal to zero gives the optimal policy.

$$
\begin{aligned}
S^* &= \sum_{i=1}^{M} n_i \left(\pi_{Di} + \pi_{Ii}(1 - m_i)\right) && (46) \\
&= \sum_{i=1}^{M} n_i \left(\pi_{Di} + \pi_{Ii} \left(\frac{1}{2} - \tau_i \frac{\pi_{Ii} + 2\pi_{Di} - 2v_i}{\pi_{Ii}}\right)\right) && (47) \\
&= \sum_{i=1}^{M} n_i \left(\left(\frac{1}{2} - \tau_i\right) \pi_{Ii} + (1 - 2\tau_i)\, \pi_{Di} + 2\tau_i v_i\right) && (48) \\
&= \frac{1}{2} + \sum_{i=1}^{M} n_i \left((\pi_{Di} - \pi_{Ri}) \left(\frac{1}{2} - \tau_i\right) + 2\tau_i \left(v_i - \frac{1}{2}\right)\right) && (49)
\end{aligned}
$$

Assuming τ_i is the same in each state, this would be a linear function of the aggregate vote share with a slope equal to $2\tau_i$. It's also easy to impliment. Since it's linear, we would get this policy by each state having a seat-vote curve equal to

$$
S_i(v_i) = \frac{1}{2} + (\pi_{Di} - \pi_{Ri}) \left(\frac{1}{2} - \tau_i\right) + 2\tau \left(v_i - \frac{1}{2}\right). \tag{50}
$$

with that same low slope.

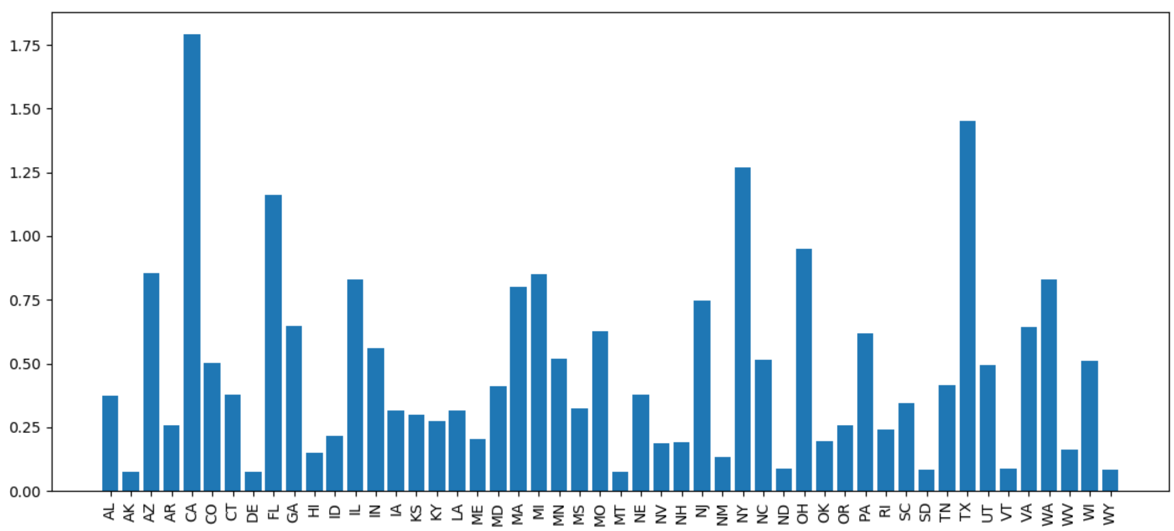


Figure 2: The expected utility gain for each state from deviating from this social optimal to their best response.

This is similar to the thermastat example where welfare is maximized if everyone just reports their most preferred temperature. This is the seat-vote curve a state would choose if they had complete control over the policy. However, every state has an incentive to deviate from this outcome. The large states are able to profit the most from this deviation.

This is a prisoner's dilemma situation though, because in equilibrium every state is worse off. Particularly the small states are worse off because they don't have as much sway.

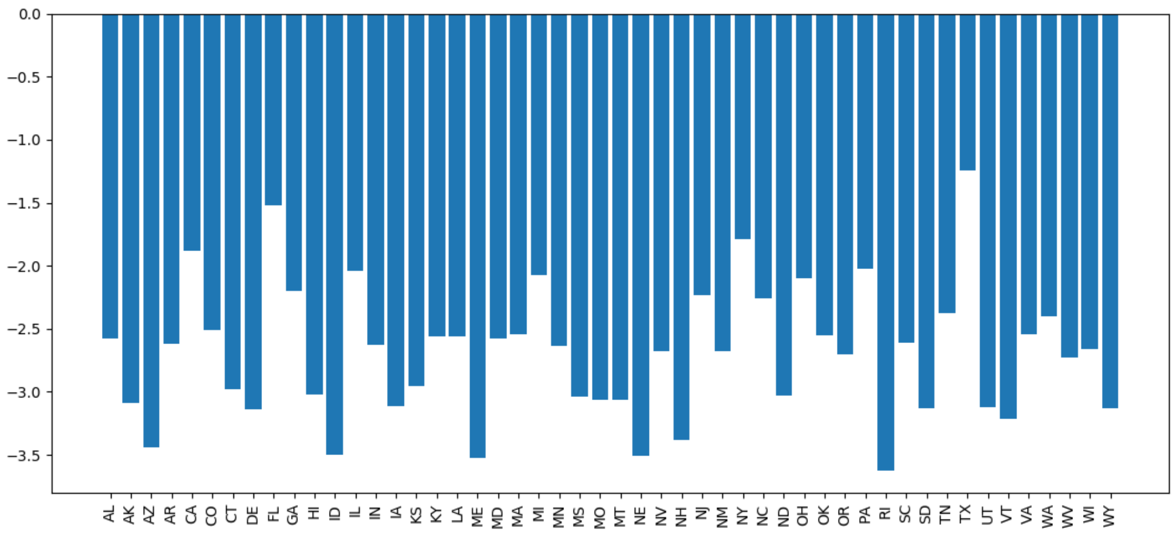


Figure 3: The loss each state faces in equilibrium from their expected utility in the social optimum.

4.1 Winner-Take-All

It may be that the optimal seat-vote curve in the model is difficult to implement in the world, requires a lot of information, or that the model assumptions don't line up exactly with your view of the world. There is a very simple seat-vote curve that can always be implemented, relies on no model assumptions, requires no information, and is very close to the optimal value. It is a winner-take-all election.

Take a seat-vote curve equal to

$$S_i(v_i) = \begin{cases} 0 & \text{if } v_i < .5 \\ \\ 1 & \text{if } v_i \geq .5 \end{cases} \tag{51}$$

That is, if the Democrats get more than 50 percent of the vote in the state, all of the state's representatives will be Democrats. Otherwise, all the representatives will be Republicans.

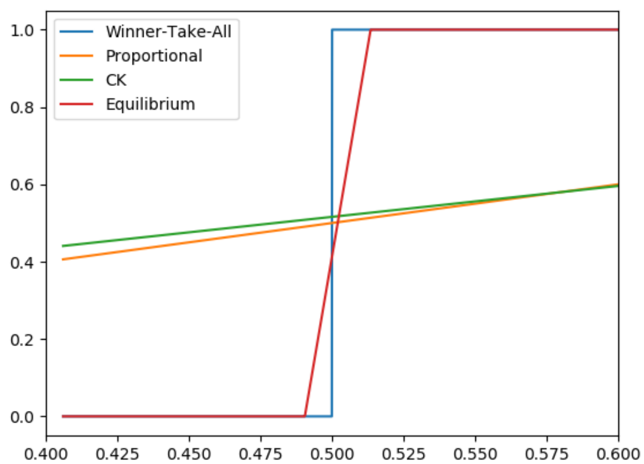


Figure 4: This is a graph of several seat-vote curves for the state of Minnesota. The "proportional" curve is a linear seat-vote curve with a slope of one. The "CK" curve is the socially optimal seat-vote curve. "Equilibrium" is the curve Minnesota would choose as their best response in equilibrium play. "Winner-take-all" is simply an indicator function for vote shares above 50 percent.

This doesn't require any knowledge about the voting population in the state. It is also extremely simple and easy to understand. It could be done without creating any districts. If districts are desired, all the state needs to do is randomly assign each citizen a district number regardless of geography. Each district in the United States is composed of about 700,000 people. By the law of large numbers, each district would then have pretty close to the same fraction voting Democrat in each election. This would impliment the winner-take-all seat-vote curve.

While winner-take-all is not the best response, it is very close to the best response. Every state's seat-vote curve is flat at zero, then increases rapidly to one, then flat at one. The increase isn't infinitely steep like a winner-take-all, but the average slope is about 50. The graph shows these two curves for Minnesota.

In fact, nearly all the gains from deviating to the best response from the previous section can be obtained from deviating to a winner-take-all function.

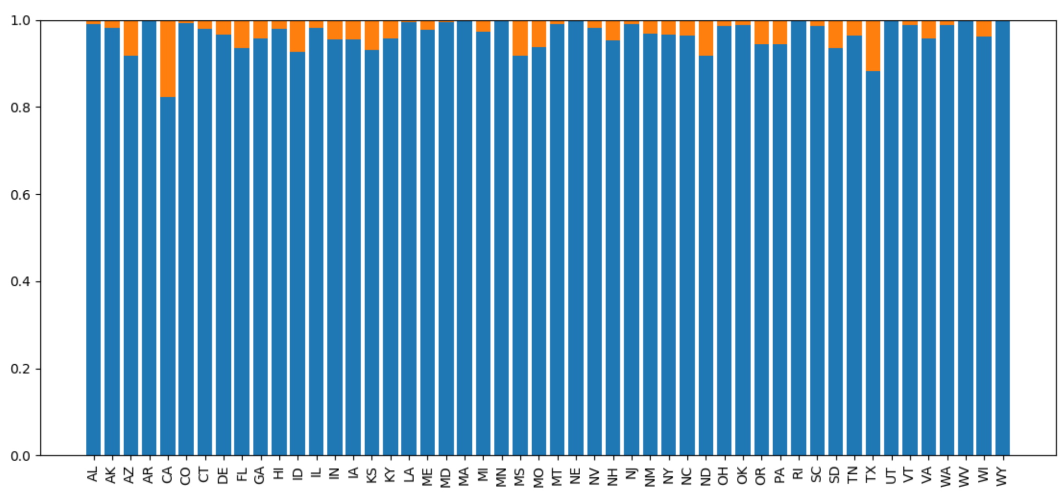


Figure 5: The gain from deviating from the social optimal to winner-take-all as a percentage of the gain from deviating to the best response.

In fact the strategies are almost always the same. In the best response, the fraction of Democrats elected is either 0 or 1 more than 80 percent of the time in each state already.

5 Median Member of Congress

What if the policy chosen is equal to the preference of the median member of congress rather than the mean? Now it's even more like winner-take-all. You now only care about which party wins the majority. The motivation to choose a modest slope to match the slow changing political ideology of your state goes away. Your best response, regardless of what other states do is always to elect all your representatives from one party. I can show this in the model.

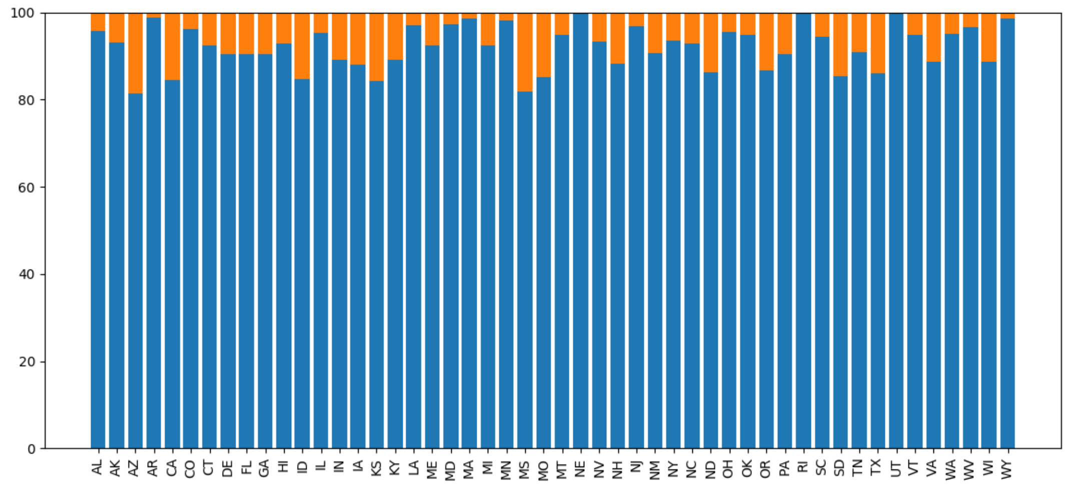


Figure 6: The fraction of the time the best response and a simple winner-take-all choose the same number of Democrats.

6 Conclusion

nothing.

References

Coate, S. and Knight, B. (2007). Socially optimal redistricting: A theoretical and empirical exploration. *Quarterly Journal of Economics*, 122(4):1409–1471.