Dynamics of Price Discovery

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Abstract

Changes in the price of a financial asset represent learning as the market updates its expectation about fundamentals. In this paper I characterize what price dynamics are possible when the information is being released strategically by a profit maximizing trader and market participants are Bayesian. I study how information is incorporated into prices over time in model with general trading strategies that allow for the spread of false information and price manipulation. Every period an informed trader reveals their information by buying or selling an asset. After observing the trade, beliefs and prices are updated. The informed trader's prefered equilibrium is characterized with and without commitment leading to starkly different results. Regardless of how beliefs impact prices, the optimal strategy for the informed trader is to release their information gradually mixed with a nearly equal amount of misinformation. This strategy leads to volatile price paths that bounce back and forth each period. In the continuous time limit, the price process converges to a Brownian motion. Moving prices back and forth in this way hinges critically on the informed traders ability to commit ex ante to a strategy. Without such commitment power, the optimal strategy is to

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release nearly all information suddenly at randomized times. The optimum resembles a pump-and-dump price manipulation scheme and can lead to sudden crashes or spikes in the price of the asset. In the limit, the price converges to a Poisson process. This paper gives a micro-foundation to price processes commonly assumed in the literature.

JEL Classification: D82, D83, G14

1 Introduction

In an efficient well functioning financial market an asset's price reflects the market's beliefs about fundamentals. Dynamics of prices are then driven by learning as the market updates its expectation of these fundamentals. Does supposing the market to be rational and use Bayesian updating put any restriction on the dynamics we would expect to see in asset prices? Going even further, suppose that the new information that leads the the market learning doesn't just arrive exogenously. If the new informataion is being released by a strategic trader maximizing their profit, can we infer anything about what price dynamics should be possible? In fact, the informed agent behaving optimally along with the market being Bayesian implies a strict characterization of possible price dynamics. The subject of this paper is deriving these dynamics.

In this paper I use a simple model to study how information gets incorporated into prices. Begin with a trader holding some information not commonly known to the rest of the market. As the informed trader acts over time they reveal something about their information. The market price will continuously adjust as the market updates its beliefs to reflect the information revealed. I study the dynamics of prices through this process.

To understand how prices move from revealed information, we need to understand what information is being revealed by trades. We must first know how an informed trader optimally incorporates their information into their trading strategy. The informed trader buys or sells an asset primarily based on what they think the market price will do in the future.

How the market price responds to their trade is of fundamental importance in creating the optimal trading stratey. Thus, the price process and optimal trading strategy are equilibrium objects that must be determined jointly.

I will use a simpler model of market structure to allow for more general trading strategies and a few incentives that are typically ignored in the literature. If the informed agent's trades can move prices, there will always be an incentive to manipulate those prices. The informed trader may potentially choose to buy the asset today, try to convince people the state is good to drive up the price, then just sell it tomorrow. This paper shows how rumors and misinformation can interact with incentives to reveal true information and how much it can hinder price discovery.

In an efficient market, prices change only as new information is incorporated. Observed stock prices jump around continually throughout the trading day, seemingly too much the be justified by new information alone (see Shiller (1981) among others). In this paper, I find the maximum amount of variation that can be caused solely by information.

I address these questions using a dynamic information disclosure game in which information is the only factor moving prices. Each period an informed trader buys or sells an asset and their actions reveal their information to the market maker over time. The informed trader profits from having a long position when the price goes up and a short position when the price goes down. I then take the limit to go to continuous time and characterize the stochastic process the asset price follows.

I find that regardless on how the information affects the asset's price, the optimal way for the informed trader to release the information is Brownian. The informed trader gradually releases the information by giving small pieces of true or false news with nearly equal frequency each period. This leads to prices following an Itó process. The asset has continuous but spiky price paths. This leads to the maximal amount of variation over time. Only asymptotically is all information revealed.

However, implimenting this policy requires a strong amount of commitment on the side of the informed trader. The informed trader must commit to a distribution of actions before observing the state. This is the same type of commitment used in the Bayesian persuasion literature. Without this commitment ability, the optimal policy looks very different. The informed agent is no longer able to break up the information and release it gradually in a profitable way, but they retain a lot of control over the timing of information release. The optimal policy becomes a Poisson process with drift that looks a lot like a pump-and-dump scheme. This leads to a price path with sudden discontinuous crashes or jumps.

The Itó and Poisson processes arise completely endogenously as the optimal strategy of an informed trader. Nothing in the setup of the model is Normal or Poisson or game to lead to these distributions. Throughout finance and particularly option pricing, an Itó process possibly with a Poisson process added on form the dynamics of a stock price. This is typically assumed for tractability, but my model gives a micro-foundation for these to arise as the natural stochastic process for prices. In papers of learning and information aquisition the arrival of information is typically modeled to be either Brownian or Poisson. My model shows that such information flows may be motivated by the information being obtained from a strategic player.

1.1 Example

Consider a very simple example that will allow me to fix ideas and explain the general concepts of the model. While the model can be much more general than this example, the intuition is similar.

There is an asset with a payoff, ω, equal to either zero or one with equal probability. This asset is just a one dollar bet. This payoff is received once at the end of the game. Suppose further that this asset is highly liquid, all agents are risk neutral, and there is no discounting. What I mean by these is that at any time, you can walk down to the market and buy or sell a share of this asset at the posted price, which is equal to its expected value. Since either state is equally likely, the initial price is $\frac{1}{2}$.

If a trader privately learns what the payoff is going to be, how can she best profit from this information? For simplicity, say that there are two trading periods before the asset payoffs and that the trader can take a long or short position, but faces a capacity constraint of one share (holdings $x \in [-1, 1]$). The obvious candidate strategy is that if she learns the asset's payoff will be high ($\omega = 1$) she should buy the asset and if she learns that the asset's payoff will be low ($\omega = 0$) she should sell the asset. This strategy would earn her a total payoff of $\frac{1}{2}$. If she realizes the asset is good, she will buy it in the first period for a price of $\frac{1}{2}$. She then doesn't need to do anything in the second period because she is already holding her max amount. Then the payoffs from the asset realize and she gets 1. If the asset is actually bad, her payoffs are just the mirror. She sells the asset in the first period for a gain of $\frac{1}{2}$, then at the end of the game doesn't need to pay back anything ($\omega = 0$). Her payoff is the same in either state.

Now I'm going to propose a candidate strategy that can do better than simply buying if the asset is good and selling if it is bad. In this strategy the trader will utilize the fact that there are multiple periods of trading by misleading the market in the first period and manipulating prices to get a higher return in the second period. Consider the following randomized strategy. In the first period if the trader observes the asset is good, she will buy with probability $\frac{3}{4}$ and sell with probability $\frac{1}{4}$. If she observes the asset is bad she will do the opposite, buy with probability $\frac{1}{4}$ and sell with probability $\frac{3}{4}$. Then in the second period she will do the simple strategy of holding a long position if the asset is good and a short position if the asset is bad, just like the previously proposed strategy above.

This strategy will give a higher payoff. To see this, we first need to know what the prices will be in the second period. Since the price is always equal to the expected payoff, this is computed using Bayes rule. If she buys the asset in the first period, the price in the second period will be $\frac{\frac{1}{2}\frac{3}{4}}{\frac{1}{2}\frac{3}{4}+\frac{1}{2}\frac{1}{4}} = \frac{3}{4}$. If she sells in the first period, the price move the opposite way, $\frac{\frac{1}{2}\frac{1}{4}}{\frac{1}{2}\frac{1}{4}+\frac{1}{2}\frac{3}{4}} = \frac{1}{4}$.

Let's now compute the trader's expected payoff. First take the good state ($\omega = 1$). Three quarters of the time she will buy in the first period at a price of one half, in the secon period maintains that long position with no cost, then receives a payment of one at the end of the game. The other one quarter of the time she will sell at a price of one half in the first period, buy at a price of one quarter in the second period, and receive a payoff of one at the end of the game. This gives an expected payoff of

$$\frac{3}{4}\left(-\frac{1}{2}+1\right)+\frac{1}{4}\left(\frac{1}{2}-2\frac{1}{4}+1\right)=\frac{5}{8}$$

which is larger than $\frac{1}{2}$. If the state is good we can get the payoff with a similar calculation.

$$\frac{3}{4}\left(\frac{1}{2}-0\right)+\frac{1}{4}\left(-\frac{1}{2}+2\frac{3}{4}-0\right)=\frac{5}{8}$$

Most of the time the trader does the usual strategy of buying when the asset is good and selling when it is bad to get a payoff of $\frac{1}{2}$. Occasionally, she trades opposite of her information in the first period. This allows her to transact at a more favorable price in the second period. This manipulation earns a higher payoff whether the asset is good or bad.

If there were three periods of trading, the trader would be able to manipulate prices for two periods before taking the obvious trade in the last period. If there are many trading periods, the optimal strategy involves the trader potentially moving prices back and forth between higher and lower levels many times before the end. In fact, in the limit as you took an infinite number of trading periods the price would approach a Brownian motion. Every period she would choose to buy or sell with nearly equal probability. This will cause the price to continue to bounce up or down by infinitesimal amounts.

In turns out that $\frac{5}{8}$ is the highest payout the trader can garuntee herself in this two period game. Notice, however, that acheiving this requires a strong amount of commitment on the part of the trader. If not committed ex ante, the trader has an incentive to deviate from the outlined strategy some of the time. When the asset is good, the trader mixes between buying and selling in the first state. When she buys in the first stage the trader gets a payoff of $\frac{1}{2}$, but when she sells in the first stage the trader gets a payoff of 1. Recall that the market doesn't obvserve the state (ω), only the trade. Hold fixed the market prices($p_1 = \frac{1}{2}$, $p_2 = \frac{3}{4}$ if buy is observed in the first period, and $p_2 = \frac{1}{4}$ if sell is observed in the first period). When the asset is good, the trader will always prefer to sell and manipulate the price because it gives a higher payoff. However, the manipulation strategy was able to effectively move prices simply because it is done infrequently.

Thus, without commitment power this equilibrium would fall apart completely. The best the trader can do in an equilibrium without commitment is exactly the simple strategy of buying if the asset is good and selling if the asset is bad. This is done in one period (the last one). In the more general form of the model with infinite periods, it is still the case that the best the trader can do without commitment is to reveal nearly all their information at once. The trader still will maintain a lot of power over the timing of this information dump. The profit is maximized by randomizing of the timing of the information release. In the continuous time limit the price will converge to a Poisson process.

2 Model

In this section, I present the model in a much more general form than the example, but I still strive for simplicity. I've stripped away all aspects that distract from the main result. To see how it generalizes or how it relates to more standard models, see sections five and six respectively. What is essential for the results is simply a strategic informed trader that profits from price changes and a rule for how beliefs are converted to prices. I'll first solve the model here without commitment. After examining the solution briefly, I take the continuous time limit and arrive at a Poisson process. Then in section four I will solve the model with commitment. Again the continuous time limit will be studied and a Brownian motion will arise. In section five, I will give extensions of this model.

There is a permanent state of nature that takes one of two possible values, $\omega \in \{0, 1\}$.

I'll sometimes refer to the state as the asset being bad or good. The probability that $\omega = 1$ is denoted by μ. There is a long lived asset whose value is affected by the state.

There are discrete time periods $t = 1, 2, \ldots, T$ and usually T will be taken to be infinite. There are two players. Each period the market maker chooses a price at which they are willing to buy or sell. Then the informed trader that knows the value of ω chooses to buy or sell the asset.

2.1 Informed Trader

The informed trader is risk neutral chooses how much of the asset to hold each period, x_t, to maximize expected discounted profit. Because the objective is linear, a bang-bang solution obtains and the insider would like to hold a infinite number of shares of the asset. I restrict the number of shares that can be held to $x_t \in [-1, 1]$. When T is finite, I will force $x_T = 0$ so that the payoffs to the insider don't include any final value of holding the asset at the end of the game.

The payoff to the trader is the following.

$$V(\mu_0) = \mathbb{E}\left[\int_0^T e^{-rt} x_t dp_t\right] \tag{1}$$

Though the main results are continuous processes, the model is in discrete time and the main theorems are about the limits as the time periods get smaller. This means that through most of the paper, I will be using a discrete approximation to this payoff function. Call $\delta = e^{-r\Delta t}$. The discrete payoff used is

$$V(\mu_0) = \mathbb{E}\left[\sum_{t=0}^{T} \delta^t (p_{t+1} - p_t) x_t\right]. \tag{2}$$

If the price increases by three dollars today, the trader receives three times the number of shares they are holding.

The expectation won't be important at this point of the model. The only natural uncertainty in the model is ω, which is known to the informed trader. There will be further uncertainty due to the fact that the informed trader will randomize, but again that is known to the trader. The only time the expectation would be meaningful would be if the market maker is randomizing their price, but that will be ruled out in equilibrium.

The asset pays no dividends (Japanese yen, gold, Amazon stock, etc.). This means that the only profit to the informed trader comes from changes in the price. You can also think of the next dividend of the asset being far enough in the future to be beyond the horizon of the game.

2.2 Market Maker

The market maker is meant as a stand in for whatever market process determines the prices and creates liquidity. What is important is simply that for any beliefs the market will arrive at some price and there will be traders willing to buy or sell.

For concreteness sake, I will say that the market maker chooses a price each period to maximize some flow payoff and stands willing to buy or sell up to one share at that price. Call $P(\mu_t)$ the optimal price given beliefs μ_t. I will assume this function to always have a unique value.

$$P(\mu_t) = \operatorname*{argmax}_{p_t \in \mathbb{R}} U(p_t, \mu_t) \tag{3}$$

for some period payoff function $U(p_t, \mu_t)$.

You may think of $U(p_t, \mu_t)$ as capturing profit from unspecified liquidity traders in the market.

The simplest example is thinking of a robot market maker that simply sets the price equal to some expected value like in Kyle (1985).

$$P(\mu_t) = \mathbb{E}[z(\omega)] \tag{4}$$

where z represents the value of the asset to the market maker (resale value) in each state of the world.

Another example is to consider a large market where $P(\mu_t)$ is a standard price equation for stochastic discount factor m and asset payoffs z by having

$$U(p_t, \mu_t) = \mathbb{E}_{\mu_t}\left[(p_t - m(\omega)v(\omega))^2\right]. \tag{5}$$

All that matters for the results is that there is some function $P(\mu)$ that gives the price for any beliefs and is single valued.

2.3 Equilibrium

I will solve for the Perfect Bayesian Equilibrium that gives the highest payoff to the informed trader. Given the market maker's pricing strategy, the informed trader will choose asset holdings each period to maximize their expected discounted profit. Given the informed trader's trading strategy and beliefs about the state, the market maker will choose the price to maximize period utility $U(p_t, \mu_t)$. Beliefs at all nodes will be obtained using Bayes rule from the informed trader's strategy when possible.

I will focus throughout the paper on Markov equilibria only. Other equilibria of the game will be discussed in the appendix.

3 Profit Maximizing Equilibrium

The strategy to solving for the informed trader's maximal equilibrium profit is as follows. Write the recursive formulation of the problem. Translate the problem into one of choosing posteriors rather than trades. Assume the function $P(\mu)$ is continuous and monotone. Do value function iteration by hand with an initial guess of $V(\mu) = 0$. We will see that our iteration will converge to the optimum in only two steps. After studying the solution, I take the limit to continuous time and further analyze the dynamics of prices.

Similar to a mechanism design problem, I will set this up as the the trader choosing all variables (prices, beliefs, and trades) subjects to contraints for incentive compatability and Bayesian updating to insure that it is an equilibrium of the game. The incentive constraint for the market maker to be maximizing is simply $p_t = P(\mu_t)$ here. We can plug this straight into the problem. This will give the following recursive formulation.

$$V(\mu) = \max_{x\in[-1,1]} \mathbb{E}\left[(P(\mu') - P(\mu))\, x + \delta V(\mu')\right] \tag{6}$$

where μ' is obtained using Bayes rule.

Take any two posteriors on opposite sides of the prior, $\bar{\mu} \geq \mu \geq \underline{\mu}$. By rearranging Bayes rule, we can see that there always exists a mixed strategy for x such that these posteriors would be induced after buying and selling.

$$\pi(x = 1|\omega = 1) = \frac{\bar{\mu}(\mu - \underline{\mu})}{\mu(\bar{\mu} - \underline{\mu})}; \qquad \pi(x = 1|\omega = 0) = \frac{(1-\bar{\mu})(\mu - \underline{\mu})}{(1-\mu)(\bar{\mu} - \underline{\mu})} \tag{7}$$

So, the problem can be written as one of choosing two posteriors on opposite sides of the prior subject to incentive compatability. It will be made clear in the next section that incentive compatability here reduces to the two posteriors giving the same payoff.

3.1 Iteration

Assume $P(\mu)$ is continuous and monotone. The usual arguements apply for the Contraction Mapping Theorem, so we can iterate to find the value function. Take an initial guess of $V(\mu) = 0$ and consider the first iteration.

$$V_1(\mu) = \max_{x\in[-1,1]} \mathbb{E}[(P(\mu') - P(\mu))x] \tag{8}$$

Before telling you what is an equilibrium, it will be illustrative to tell you what isn't an equilibrium. It would seem natural to think that the informed trader should buy if the state is good and sell if the state is bad. This won't be supported in any equilibrium. In this candidate equilibrium, after observing a buy beliefs would go to one and the price would rise to $P(1)$. After observing a sell price would fall to $P(0)$. Hold the updated prices after observing buy or sell fixed and consider the best response of the informed trader. The traders gets a profit of $P(1)-P(\mu)$ if they buy and $P(\mu)-P(0)$ if they sell. Since price is continuous and monotone, there is one knife edge case where these will be equal. Call that belief μ^*.

$$\mu^* = P^{-1}\left(\frac{P(1) - P(0)}{2}\right) \tag{9}$$

If the price is currently lower than that, buying is more profitable than selling. Holding fixed the market maker's strategy, always buying is a profitable deviation.

In essance, it isn't credible for the informed trader to completely reveal the state because revealing the good state is more valuable that revealing the bad state. They have an incentive to lie when the state is bad.

However, always buying (or always selling) won't be profitable for the informed trader. The action will be uninformative. This means that beliefs, and therefore prices, won't change. This gives zero profit to the informed trader.

Any equilibrium with positive profits must then have the informed trader playing a mixed strategy. For them to be willing to play a mixed strategy, it must be that they are indifferent between buying and selling. Consider two posteriors on opposite sides of the prior that give the same payoff.

$$P(\bar{\mu}) - P(\mu) = P(\mu) - P(\underline{\mu}) \tag{10}$$

If the market maker is playing $P(\bar{\mu})$ after observing a buy and $P(\underline{\mu})$ after observing sell, then the informed trader is indifferent between the two actions. In fact, the informed trader's best response contains any mixed strategy of the two actions. As we saw above, there exists a mixed strategy such that $\bar{\mu}$ and $\underline{\mu}$ are the correct Bayesian updates. Thus, both players are playing a best response. These strategies constitutes an equilibirium. If we draw a graph of the the profit to the informed trader against the posterior of the market maker, any flat line on the graph connecting two points on opposite sides of the prior is an equilibrium profit.

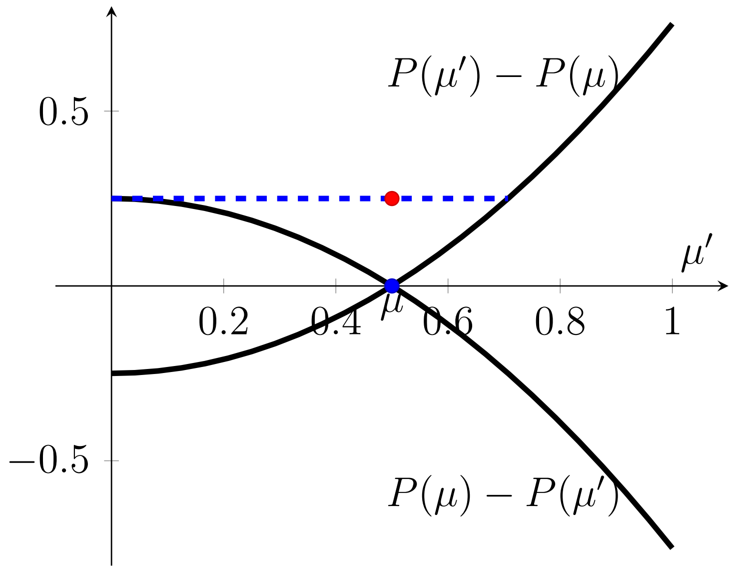


Figure 1: The upward sloping solid line is the profit from buying the asset graphed against the posterior induced. The downward sloping line is the profit from selling the asset. An equilibrium is a flat line on this graph that crosses the prior. The highest equilibrium profit is shown by the dashed line.

Consider the highest profit the informed trader can obtain in equilibrium. This would be the highest flat line on such a graph. If $P(\mu)$ is monotone, then the highest payoff on either side of the posterior is at the boundary. Consider the boundary with the lower payoff of the two. Clearly the informed trader cannot get a higher payoff than that. If $P(\mu)$ is continuous, the Intermediate Value Theorem says there must be a posterior on the other side of the prior that gives the same payoff.

This gives the new value function after one iteration.

$$V_1(\mu) = \min\{P(\mu) - P(0), P(1) - P(\mu)\} \tag{11}$$

3.2 Solution

To do the next iteration, all we need to do is find the highest flat line on

$$W(\mu') = |P(\mu') - P(\mu)| + \delta V_1(\mu') \tag{12}$$

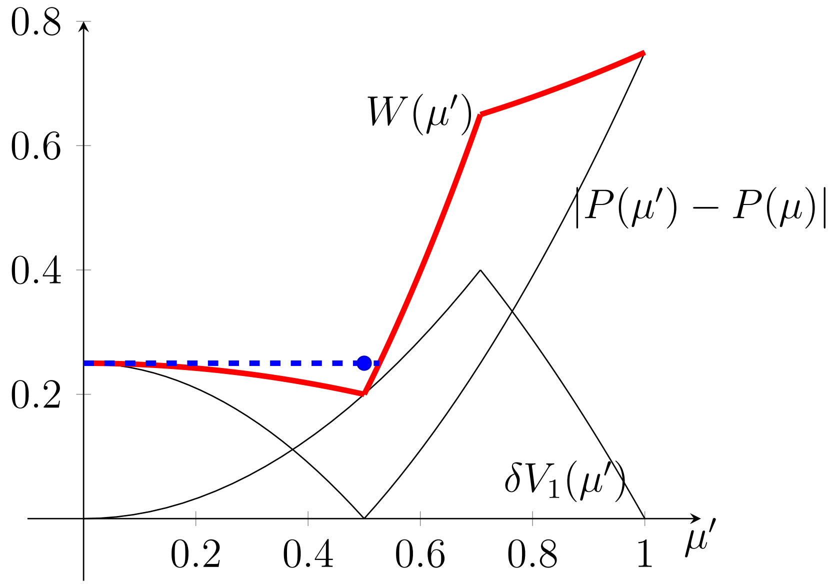
graphed against μ'.

Take $\mu < \mu^*$. For any $\delta < 1$, $W(\mu')$ is decreasing on $[0, \mu]$. Take $\hat{\mu} \in [0, \mu]$.

$$W(\hat{\mu}) = P(\mu) - P(\hat{\mu}) + \delta(P(\hat{\mu}) - P(0)) \tag{13}$$

$$< P(\mu) - P(0) \tag{14}$$

So, the highest payoff to the left is still at the boundary. Similarly, the highest payoff to the right is at the other boundary. Again, by continuity the minimum of these can be obtained by an equilibrium. This gives our value function. This is the same value we had in the previous iteration. Thus, we have found a fixed point.



Proposition 1. *Let $P(\mu)$ be continuous and monotone. For any discount factor, $\delta \in [0, 1)$,*

the value is

$$V(\mu) = \min\{P(\mu) - P(0),\ P(1) - P(\mu)\} \tag{15}$$

3.3 Price Dynamics

The endpoints of the flat line giving the value are the posteriors that are induced by the optimal strategy. We saw that one endpoint is always at the boundary, but the other is generally interior. Take $\mu < \mu^*$. When the informed trader sells, this perfectly reveals the state to be bad and beliefs fall to zero. When the insider buys, beliefs increase to an interior point just high enough to make the informed trader indifferent between buying and selling.

The payoff from inducing a posterior μ' can be obtained by putting the the value function from the previous proposition.

$$W(\mu') = \begin{cases} P(\mu) - \delta P(0) - (1-\delta)P(\mu') & \text{if } \mu' \leq \mu \\ (1+\delta)P(\mu') - P(\mu) - \delta P(0) & \text{if } \mu < \mu' \leq \mu^* \\ (1-\delta)P(\mu') - P(\mu) + \delta P(1) & \text{if } \mu' > \mu^* \end{cases} \tag{16}$$

To the left of the prior we have a strictly decreasing function of μ', so the left endpoint will be zero. The right endpoint could be on the second or third segment of the curve $W(\mu')$.

$$\bar{\mu} = \begin{cases} P^{-1}\left(\frac{2P(\mu)-(1-\delta)P(0)}{1+\delta}\right) & \text{if } \mu \leq P^{-1}\left(\frac{P(1)(1+\delta)+P(0)(3-\delta)}{4}\right) \\ P^{-1}\left(\frac{2P(\mu)-P(0)-\delta P(1)}{1-\delta}\right) & \text{otherwise} \end{cases} \tag{17}$$

This equation doesn't have a nice intuition to it, but it is important to note that unlike the left endpoint, the right endpoint does depend on δ.

3.3.1 Continuous Time Limit

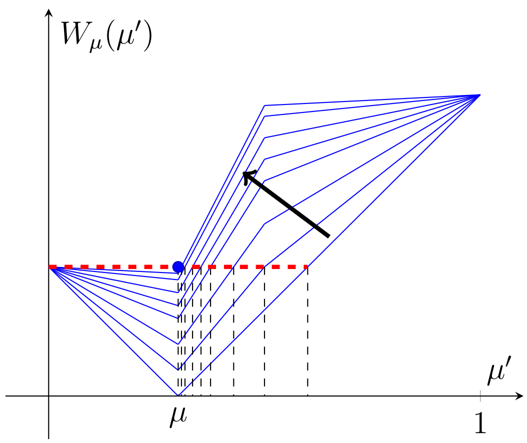


Figure 2: The blue lines are $W(\mu')$ for $\delta = \{0, .2, .4, .6, .7, .8, .9, .95\}$. The dashed line for each one is $\bar{\mu}$.

Consider the limit as δ goes to 1. After selling beliefs still drop to 0. After buying beliefs move up to the right endpoint described above. This right endpoint is moving closer and closer to the prior as δ goes to 1. Since beliefs are a martingale, this means that the frequency of the jumps down to $P(0)$ needs to be going to 0. The likelihood that beliefs just drift up the small amount in a given period goes to 1. Though not obvious from looking at it, in the appendix I show that the probability of a jump is shrinking to 0 at a linear rate in $1-\delta$. This gives use the first main result of the paper.

Theorem 1. *For any differentiable strictly monotone price function, $P(\mu)$, as δ goes to one the price proccess converges to a Poisson process.*

- *If $\mu_t < \mu^*$,*
$$dP(\mu_t) = \frac{1}{2}(P(\mu_t) - P(0))dt - (P(\mu_t) - P(0))dN_t \tag{18}$$
where N_t is a standard Poisson process with arrival rate $\lambda = \frac{P(\mu_t)-P(0)}{2\mu_t P'(\mu_t)}$.

- *If $\mu_t > \mu^*$,*
$$dP(\mu_t) = -\frac{1}{2}(P(1) - P(\mu_t))dt + (P(1) - P(\mu_t))dN_t \tag{19}$$
where N_t is a standard Poisson process with arrival rate $\lambda = \frac{P(1)-P(\mu_t)}{2(1-\mu_t)P'(\mu_t)}$.

- *If $\mu_t = \mu^*$, all information is revealed immediately and the price jumps to either $P(1)$ or $P(0)$.*

Consider a low initial price ($\mu < \mu^*$). With a Poisson arrival rate the informed trader will sell the asset and completely reveal that the state is bad. This makes the price crash to $P(0)$. While this Poisson information hasn't arrived, the informed trader will hold a long

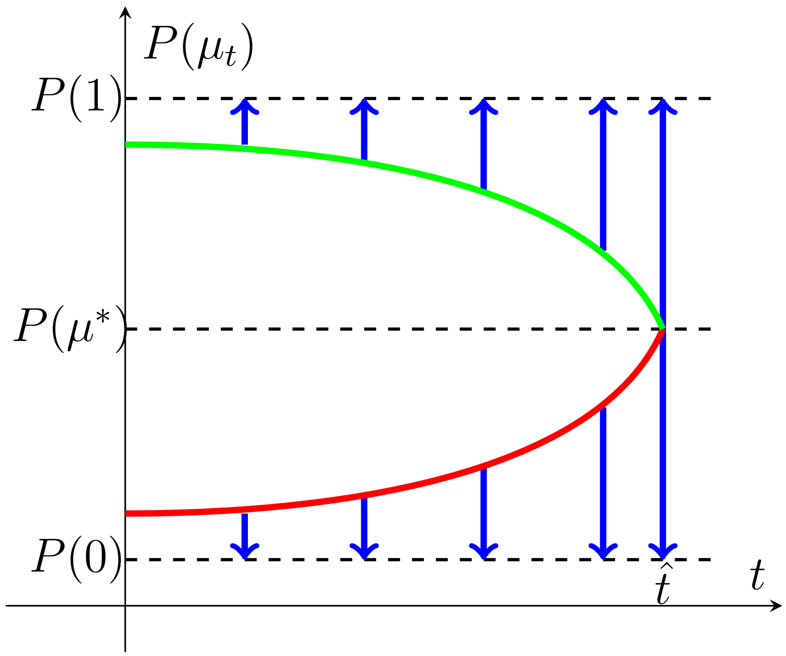


Figure 3: Sample price paths when the price starts high (green) and when the price starts low (red).

position in the asset and the price will slowly drift upward. If the price starts high, the dynamics are simply the mirror image. With a Poisson arrival rate the informed trader will buy and the price will spike to $P(1)$. At all other times the informed trader holds a short position and the price drifts down. If the beliefs ever reach μ^*, the informed trader perfectly reveals the state, good or bad.

The arrival rate can be obtained intuitively by considering the informed trader's incentive compatability constraint. The informed trader must be indifferent between revealing the state to be bad today or letting the price drift up a little and revealing the state to be bad tomorrow. Call λ the arrival rate of the Poisson process. Once it arrives, beliefs jump down to zero. In order for beliefs to be a martingale, the drift must equal $\lambda_t \mu_t$. The drift in price is then equal to $\lambda_t \mu_t P'(\mu_t)$.

$$\underbrace{P(\mu_t) - P(0)}_{\text{reveal today}} \approx \underbrace{P'(\mu_t)\mu_t\lambda_t}_{\text{drift today}} + \delta(\underbrace{P'(\mu_t)\mu_t\lambda_t + P(\mu_t) - P(0)}_{\text{reveal tomorrow}}) \tag{20}$$

As δ gets large, this gives a linear relationship between λ and $1 - \delta$.

$$\Rightarrow \lambda_t \approx \frac{P(\mu_t) - P(0)}{2\mu_t P'(\mu_t)}(1 - \delta) \tag{21}$$

3.4 Intuition

If the price starts low, the dynamics look like a pump-and-dump scheme. The informed trader buys every period and spreads good information to pump up the value of the asset. Then, at a random arrival date they dump all their shares and reveal the asset to be bad. This causes the price to crash suddenly. If the asset actually is good that crash date never comes, hence the positive drift.

If the price starts high we have the mirroring dynamics. The informed trader sells each period causing the price to drift down. If the asset is actually good, then at a random arrival rate they buy back all the shares and reveal the state causing a sudden spike in price. This is a short-and-distort scheme. This matches the empirical fact that pump-and-dumps are usually done on cheap stocks and short-and-distorts on more expensive stocks.

The form of the optimal strategy comes from the intuition in the one period model for why buy when good sell when bad isn't an equilibrium. If the price is initially low, the informed trader can't credibly reveal the state because revealing the good state is better than revealing the bad state. They'd like to lie and always say it's the good state.

The dynamics give them that credibility. When the informed trader says the state is bad, the market maker believes them and moves the price all the way to $P(0)$. When the informed trader says the state is good, the market maker mostly doesn't believe it because there is a much bigger potential gain from the state being good. Price increases only a small amount. After many periods of repeatedly saying the state is good, beliefs eventually drift up to μ^* which is the cutoff point for when the good state can be credibly revealed. They small price jump can be done immediately, but the informed trader needs to spend time to build credibility before they can get the big price jump.

The arrival rate of the Poisson process depends on the function $P(\mu)$. If $P(\mu)$ is linear then is simplifies to $\lambda = \frac{1}{2}$. If $P(\mu)$ is concave, then the arrival rate will be bigger than $\frac{1}{2}$ and it will be increasing over time. If $P(\mu)$ is convex, then the arrival rate will be smaller than $\frac{1}{2}$ and it will be decreasing over time. You can see this by writing the Taylor series for $P(0)$.

$$P(0) = P(\mu_t) - P'(\mu_t)\mu_t + \frac{1}{2}P''(\mu_t)\mu_t^2 + \dots \tag{22}$$

This gives us an approximate equation for the arrival rate.

$$\lambda(\mu_t) = \frac{P(\mu_t) - P(0)}{2\mu_t P'(\mu_t)} \approx \frac{1}{2} - \frac{1}{4}\mu_t \frac{P''(\mu_t)}{P'(\mu_t)} \tag{23}$$

We can see there how the arrival rate depends on the concavity of $P(\mu)$.

However, regardless of the curvature of $P(\mu)$, the price will always reach $P(\mu^*)$ and then jump to the boundaries in finite time. We can see this because the magnitude of the drift is increasing in time.

Proposition 2. *The maximum time to full information revelation is*

$$t^{max} = 2\log\left(\frac{P(\mu^*) - P(0)}{P(\mu_0) - P(0)}\right) \tag{24}$$

if $\mu_0 \leq \mu^*$ *and*

$$t^{max} = 2\log\left(\frac{P(1) - P(\mu^*)}{P(1) - P(\mu_0)}\right) \tag{25}$$

if $\mu_0 > \mu^*$.

The proof of this result is in the appendix.

4 Commitment

In the previous section buy if the state is good sell if the state is bad was not an equilibrium because the informed trader could not credibly commit to that strategy. In this section I'll derive the optimal policy when they do have that commitment power. The informed trader is allowed to commit ex ante to their mixed strategy.

Rather than choose an action (buy or sell) each period the informed trader can choose a distribution of actions contingent on the state. For example they can buy when good and sell when bad, or when the state is good randomize fifty-fifty between buying and selling. This is the same type of commitment power that is assumed in the Bayesian persuasion literature. We could equivalently think of commitment as saying the informed trader has verifiable information to reveal. This essentially allows us to ignore the incentive compatability constraint in the previous problem. This unconstrained problem is much more difficult to solve analytically, but I can still characterize the continuous time limit as in the previous section.

Even though buy when the state is good sell when the state is bad is clearly going to be optimal in the one period model in the dynamic model the solution is nearly the exact opposite.

4.1 Setup

As before the solution will require the informed trader to mix, but they can now mix between any two points on the payoff graph, not just ones that are equal.

I can write the problem as choosing the posteriors that will be induced after buying and selling subject to the constraint that beliefs are a martingale. If I write the posteriors as

$$\mu_{buy} = \mu + \bar{\epsilon}, \quad \text{and} \quad \mu_{sell} = \mu - \underline{\epsilon} \tag{26}$$

then the martingale requirement stipulates the probabilities must be

$$p(buy) = \frac{\underline{\epsilon}}{\bar{\epsilon} + \underline{\epsilon}} \quad \text{and} \quad p(sell) = \frac{\bar{\epsilon}}{\bar{\epsilon} + \underline{\epsilon}}. \tag{27}$$

The constraints $\bar{\epsilon} \in [0, 1-\mu]$ and $\underline{\epsilon} \in [0, \mu]$ ensure that beliefs stay between zero and one.

The Bellman equation can then be written simply.

$$V(\mu) = \max_{\bar{\epsilon},\underline{\epsilon}} \ (|P(\mu+\bar{\epsilon}) - P(\mu)| + \delta V(\mu+\bar{\epsilon}))\, \frac{\underline{\epsilon}}{\bar{\epsilon}+\underline{\epsilon}} + (|P(\mu) - P(\mu-\underline{\epsilon})| + \delta V(\mu-\underline{\epsilon}))\, \frac{\bar{\epsilon}}{\bar{\epsilon}+\underline{\epsilon}} \tag{28}$$

$$= \max_{\bar{\epsilon},\underline{\epsilon}} \ \mathbb{E}_{\mu'}\left[|P(\mu') - P(\mu)| + \delta V(\mu')\right] \tag{29}$$

We'll characterize the the approximate solution when it's assumed $\bar{\epsilon}$ and $\underline{\epsilon}$ are small. Then when we take the limit to continuous time, the approximation will give us the exact solution. First notice that $|P(\mu+\bar{\epsilon}) - P(\mu)| \approx |P'(\mu)|\bar{\epsilon}$. Now take a second order approximation to $V(\mu+\bar{\epsilon})$.

$$V(\mu+\bar{\epsilon}) \approx V(\mu) + V'(\mu)\bar{\epsilon} + \frac{1}{2}V''(\mu)\bar{\epsilon}^2 \tag{30}$$

Put these into equation (28).

$$(1-\delta)V(\mu) = \max_{\bar{\epsilon},\underline{\epsilon}} \ 2|P'(\mu)|\frac{\bar{\epsilon}\underline{\epsilon}}{\bar{\epsilon}+\underline{\epsilon}} + \frac{1}{2}\delta V''(\mu)\bar{\epsilon}\underline{\epsilon} \tag{31}$$

The main tradeoff can be seen in the two right hand side terms. Giving away more information gives a higher payoff this period because it leads to larger price changes. On the other hand since $V(\mu)$ is concave, the more information I give away the worse my expected continuation value.

4.2 Price Dynamics

Holding the product $\bar{\epsilon}\underline{\epsilon}$ fixed, we'd like to minimize the sum $\bar{\epsilon}+\underline{\epsilon}$. This is always accomplished when $\bar{\epsilon} = \underline{\epsilon}$. So, this is only a problem of one variable when we have an interior solution.

$$(1-\delta)V(\mu) = \max_{\epsilon} \ |P'(\mu)|\epsilon + \frac{1}{2}\delta V''(\mu)\epsilon^2 \tag{32}$$

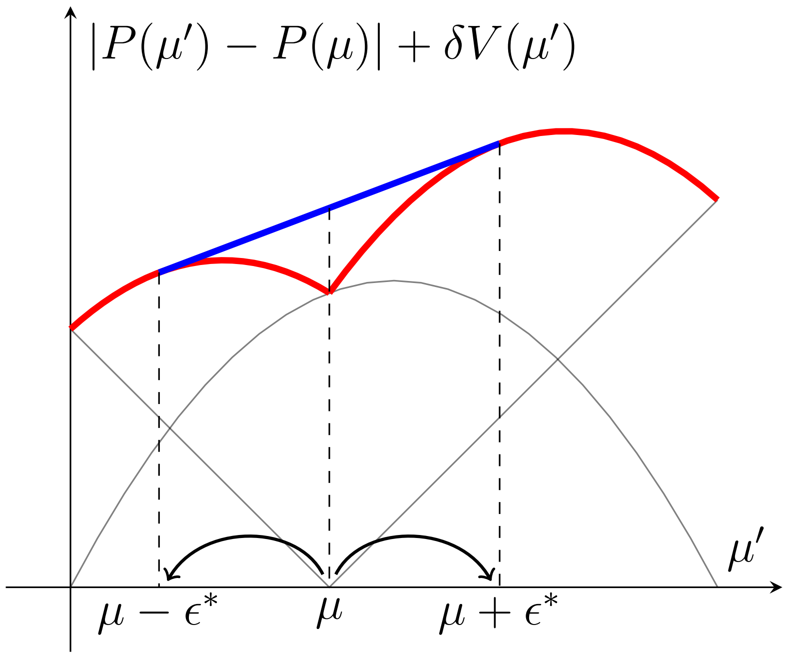


Figure 4: The faded black lines are the one period value, $|P(\mu')-P(\mu)|$, and the continuation value, $\delta V(\mu')$. The thick red line is the sum of those two. The blue segment connecting it gives the optimal policy and value. Beliefs either jump up or down by step size ϵ^* each period.

Taking the derivative and setting it equal to zero yields the optimal ϵ.

$$\epsilon^* = -\frac{|P'(\mu)|}{\delta V''(\mu)} \tag{33}$$

Beliefs now follow a random walk. Each period beliefs either jump up or down by a small step of size ϵ^*. As δ goes to one, $V(\mu)$ and $V''(\mu)$ both grow in magnitude toward infinity. This means that the step size, ϵ^*, is going to zero. The key is that it is going to zero slowly (at a rate of $\sqrt{1-\delta}$). As we take the step size shrinking to zero, this converges to a Brownian Motion for beliefs. Itó's Lemma gives us the process for prices which are a smooth function of beliefs. Thus, we have the second main result. Details are given in the appendix.

Theorem 2. *Let $P(\mu)$ be any $\mathcal{C}^2$ function. As δ goes to one, the price converges to an Itó*

Process over time.

$$dP(\mu_t) = \frac{1}{2}P''(\mu_t)\sigma^2(\mu_t)dt + P'(\mu_t)\sigma(\mu_t)dB_t \tag{34}$$

The B_t here is a standard Brownian Motion.

4.3 Intuition

In this section I ellaborate on the form of the solution and give intuition.

Even though optimal information release is Brownian, there is still a non-constant drift and variance term. The function

$$\sigma(\mu_t) = \frac{|P'(\mu_t)|}{\hat{V}''(\mu_t)} \tag{35}$$

is the standard deviation multiplying the Brownian increments in beliefs. $\hat{V}(\mu_t)$ is the value function after rescaling for the discount factor. Itó's Lemma tells us that the standard deviation multiplying the Brownian increment on prices is then $P'(\mu_t)\sigma(\mu_t)$. There is higher variance when the price is more sensitive to information.

The drift in prices is also pinned down. Since beliefs need to be a martingale, they must have zero drift. This doesn't mean that prices won't have a drift. In fact, Jensen's inquality tells us that the drift needs to be positive (negative) when price is a convex (concave) function of beliefs. Itó's Lemma confirms that the drift is $\frac{1}{2}P''(\mu_t)\sigma^2(\mu_t)dt$.

This is not a complete solution because $\sigma(\mu_t)$ was defined in terms of the value function for which a complete analytical solution cannot always be given. In the special case of a linear price function, $P(\mu_t) = \mu_t$, the analytic solution can be written. We then have that

$$\sigma(\mu_t) = n(N^{-1}(\mu_t)) \tag{36}$$

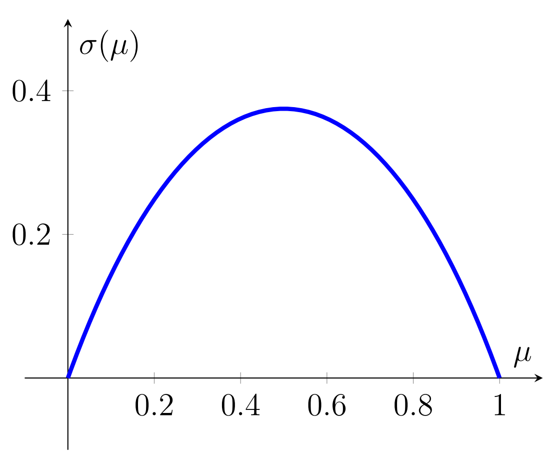


Figure 5: The standard deviation of prices as a function of beliefs.

where $n(\cdot)$ is the normal distribution pdf and $N(\cdot)$ is the normal distribution cdf.

The function $\sigma(\mu_t)$ is the normal distribution evaluated at the μ_t quantile. It is similar to a geometric Brownian motion in that the standard deviation goes to zero linearly in the price. This ensures that prices can never drop below zero. Prices are the most volatile when there is the most uncertainty.

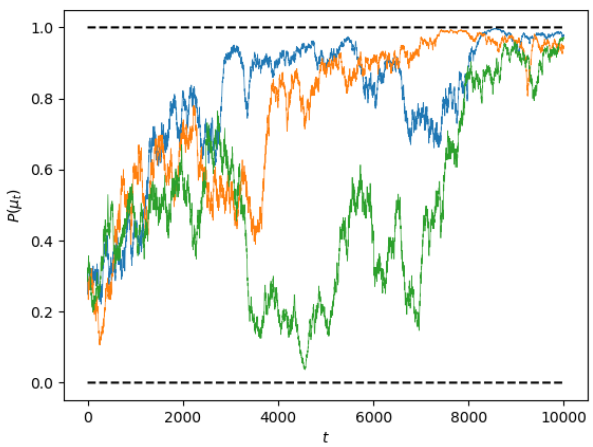


Figure 6: Three sample price paths when the true state is good.

From the perspective of the informed trader, the price still follows and Itó process. The variance is the same, but the drift is different. With the linear price function, the drift conditional on the state being good is $\frac{\sigma^2(\mu_t)}{\mu_t}$. When the state is bad, the drift is $-\frac{\sigma^2(\mu_t)}{1-\mu_t}$. Beliefs always drift toward the true state. When beliefs are far from the true state the drift is large, and when they are close to the truth the drift is small. Beliefs converge to the truth over time in the weak$-^*$ topology.

4.4 Comparison

In this section I discuss the relation between the solutions with and without commitment and their importance for finance and economics.

The price paths induced with and without commitment are intuitively the opposite of each other. With commitment, the information release happens gradually and all information is revealed only in the limit. The price paths are continuous and highly volatile. This represents inside information being slowly leaked to the public and being encorporated into the price. Without commitment, all information is released suddenly in finite time. The price paths are smooth until a discontinuous jump. This is a pump-and-dump scheme to manipulate the price of the asset.

The dramatically different price dynamics can inform us on whether observed information leaks are likely to be strategic. Consider the commitment assumption to be about whether the informed trader can generate verifiable evidence or not. A strategic insider releases information with verifiable evidence gradually, but unverifiable information is released suddenly.

Together these form a micro-foundation of the price processes commonly assumed throughout finance.

5 Extensions

5.1 Persistence

Say that the state is not permanent. Assume for this section that the state follows a Markov process. Call π_1 and π_0 the probability that $\omega_{t+1} = 1$ conditional on $\omega_t = 1$ or 0 respectively.

The timing of the game requires a bit more care in this section. At the beginning of period t beliefs are μ_t. Then, the informed trader can choose to buy or sell the asset at price $P(\mu_t)$. The market maker immediately observes the trade and updates beliefs according to Bayes rule to μ'_t. The price is updated right away and the informed trader closes their position at price $P(\mu'_t)$. Then, after trading is done ω_{t+1} is drawn from a Markov process. Beliefs at this point are updated for the next period, $\mu_{t+1} = \pi_0 + (\pi_1 - \pi_0)\mu'_t$. In the model with a permanent state, $\mu_{t+1} = \mu'_t$. It wasn't important at that point to say that the informed trader closes their position at the end of each period, because the price at the end of each period was the same as the price at the start of the next period. That is no longer the case. Between periods, the state could change. Thus, beliefs and prices will also change between each period.

There are two different ways we can think about private information of a persistent state. The first is that the informed trader is able to see ω_t every period. The second is that the informed trader is able to see the state only in the first period, ω_0. In the first, the fact that the state is persistent rather than permanent is a good thing for the informed trader. It means that there is more information flowing to them each period. The informed trader then has more opportunity for profit. In the second, the fact that the state is persistent rather than permanent is a bad thing for the informed trader. It means that their information has less predictive power of the state as time passes. The informed trader's information is becoming less valuable each period.

Interestingly, both cases incentivize the informed trader to reveal information at a faster rate. I will show in this section that even though the value and the price process will look very different in the two cases, the optimal strategy is identical. I am not aware of any other paper that shows this kind of relationship between the two types of private information of a persistent state.

5.1.1 One Time Information

The informed trader observes ω_0 but not ω_t for $t > 0$. In this section, i will assume a linear price function for simplicity. $P(\mu) = \mu$. It is still the case that the informed trader can choose to buy when the price is about to go up and sell when the price is about to go down. The objective is

$$\mathbb{E}\left[\sum_{t=0}^{\infty} \delta^t |\mu_t' - \mu_t|\right] \tag{37}$$

where $\mu_{t+1} = \pi_0 + (\pi_1 - \pi_0)\mu_t'$.

The main difference here is that the informed trader no longer has complete control over beliefs in every period. μ_t is the beliefs about the state in period t, but the informed trader only knows the state in period 0. Call $\tilde{\mu}_t$ the beliefs in date t about ω_0. These are the beliefs that the informed trader can control.

$$\begin{aligned}
\mu_0 &= \tilde{\mu}_0 \\
\mu_1 &= \pi_0 + (\pi_1 - \pi_0)\tilde{\mu}_1 \\
&\vdots \\
\mu_t &= \sum_{\tau=0}^{t-1} \pi_0(\pi_1 - \pi_0)^\tau + (\pi_1 - \pi_0)^t \tilde{\mu}_t
\end{aligned}$$

In equation 37 the informed trader is constrained by Bayes plausibility, incentive compatability, and shrinking bounds on where beliefs can be sent due to the informativeness of their signal deteriorating. The level of persistence puts an upper and lower bound on beliefs each period.

$$\sum_{\tau=0}^{t-1} \pi_0(\pi_1 - \pi_0)^\tau \leq \mu_t \leq \sum_{\tau=0}^{t-1} \pi_0(\pi_1 - \pi_0)^\tau + (\pi_1 - \pi_0)^t \tag{38}$$

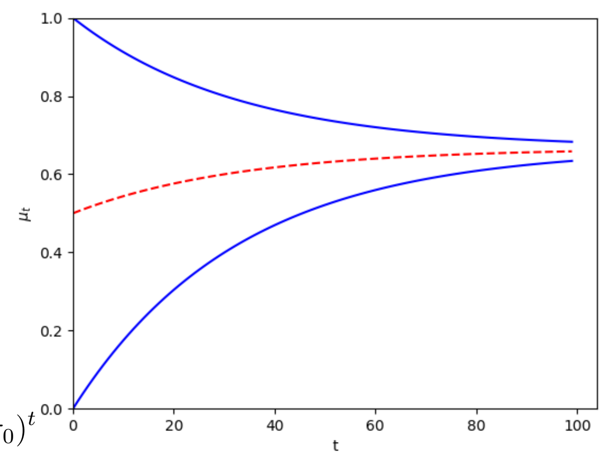


Figure 7: The solid lines are the upper and lower bounds on beliefs. The dashed line shows the drift. $\pi_1 = .99$, $\pi_0 = .02$

As time goes on, beliefs must ultimately converge to $\frac{\pi_0}{1-\pi_1+\pi_0}$ regardless of the informed trader's actions. The martingale condition only holds within each period. Between periods the beliefs have a drift determined solely by the persistence of the states. Intuitively, the size of the game is just shrinking over time. We can see this precisely by rewriting the problem in terms of $\tilde{\mu}_t$.

$$\mathbb{E}\left[\sum_{t=0}^{\infty} \delta^t |\mu_t' - \mu_t|\right] = \mathbb{E}\left[\sum_{t=0}^{\infty} (\delta(\pi_1 - \pi_0))^t |\tilde{\mu}_t' - \tilde{\mu}_t|\right] \tag{39}$$

The only effect of persistence on the strategy is to reduce the discount factor. To take limits set $\delta = 1 - r\Delta t$, $\pi_1 = 1 - \lambda_1 \Delta t$, and $\pi_0 = \lambda_0 \Delta t$. For small time periods, $\tilde{\delta} = \delta(\pi_1 - \pi_0) \approx 1 - (r + \lambda_1 + \lambda_0)\Delta t$. The arrival rate of state switches simply adds on the the discount rate. The rest of the calculations from the proofs of theorem 1 and 2 go through the same as before.

Without commitment beliefs follow,

$$d\tilde{\mu}_t = \frac{r + \lambda_1 + \lambda_0}{2}\tilde{\mu}_t dt - \tilde{\mu}_t dN_t \tag{40}$$

if $\tilde{\mu}_t < \frac{1}{2}$, the symmetric equation if $\tilde{\mu}_t > \frac{1}{2}$, or jump immediately to 0 or 1 if $\tilde{\mu}_t = \frac{1}{2}$. The arrival rate of the Poisson process is $\lambda(\tilde{\mu}_t) = \frac{r+\lambda_1+\lambda_0}{2}$.

With commitment beliefs follow

$$d\tilde{\mu}_t = \frac{r + \lambda_1 + \lambda_0}{2}\phi(\tilde{\mu}_t) dB_t. \tag{41}$$

Prices are not based on beliefs about what the state was in date zero, $\tilde{\mu}_t$, but on beliefs about the current state, μ_t. In the limit, μ_t can still be written as a function of $\tilde{\mu}_t$ and time.

$$\mu_t = \frac{\lambda_0}{\lambda_1 + \lambda_0} + (1 - \lambda_1 - \lambda_0)^t \left(\tilde{\mu}_t - \frac{\lambda_0}{\lambda_1 + \lambda_0}\right) \tag{42}$$

Prices without commitment must then follow

$$dP(\mu_t) = \left(\frac{r + \lambda_1 + \lambda_0}{2} + \log(1 - \lambda_1 - \lambda_0)\left(\tilde{\mu}_t - \frac{\lambda_0}{\lambda_1 + \lambda_0}\right)\right)(1-\lambda_1-\lambda_0)^t dt - \tilde{\mu}_t (1-\lambda_1-\lambda_0)^t dN_t \tag{43}$$

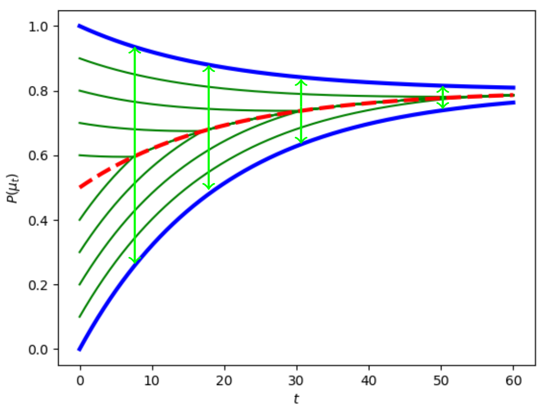
whenever $\mu_t < \mu_t^*$, the symmetric equation when $\mu_t > \mu_t^*$, and jump to the shrinking boundaries immediately when $\mu_t = \mu_t^*$. Once the price hits a boundary, it remains on the boundary for the rest of the game and continues to drift toward $P\left(\frac{\lambda_0}{\lambda_1+\lambda_0}\right)$. The midpoint, μ_t^* is also changing over time now.

$$\mu_t^* = \frac{1}{2}(1-\lambda_1-\lambda_0)^t + \left(1-(1-\lambda_1-\lambda_0)^t\right)\frac{\lambda_0}{\lambda_1+\lambda_0} \tag{44}$$

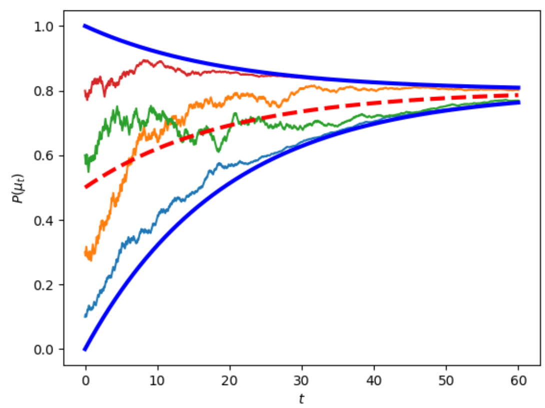
This is the dashed line in the previous figure.

With commitment prices follow

$$dP(\mu_t) = \log(1-\lambda_1-\lambda_0)\left(\tilde{\mu}_t - \frac{\lambda_0}{\lambda_1+\lambda_0}\right)(1-\lambda_1-\lambda_0)^t dt + (1-\lambda_1-\lambda_0)^t \frac{r+\lambda_1+\lambda_0}{2}\phi(\mu_t)dB_t. \tag{45}$$



(a) Without commitment



(b) With commitment

Figure 8: Sample paths for prices.

5.1.2 Information Flows Every Period

Basically, the persistence flattens out the continuation value function, but the current period payoff slope remains the same. This means you still always hit a boundary. The difference is that when you hit the boundary, the game doesn't end. Posteriors drift back interior because the state may switch and you can trade again. It is easy to verify that the value function is now

$$V(\mu) = \min\left\{P(\mu) - P(0) + \frac{P(\pi_0)-P(0)}{1-\delta}, P(1) - P(\mu) + \frac{P(1)-P(\pi_1)}{1-\delta}\right\}. \tag{46}$$

Assume the price function is linear to solve for the policy function and take the limit. Take $\mu < \mu^*$ (μ^* is not equal to $\frac{1}{2}$ anymore). The left endpoint of the strategy jumps to zero still. Some algebra reveals that the right endpoint is

$$\mu' = \frac{2\mu + (1-\delta)\pi_0}{1 + \delta(\pi_1 - \pi 0)}. \tag{47}$$

This approaches the prior, μ, as δ goes to one. This is approximately equal to

$$\mu' \approx \frac{2\mu}{2 - (r - \lambda_1 - \lambda_0)\Delta t}. \tag{48}$$

Write out the probability of jumping down to zero and simplify.

$$\frac{\mu' - \mu}{\mu' \Delta t} = \frac{\mu(r + \lambda_1 + \lambda_0)}{2\mu + \Delta t} \tag{49}$$

Then we have our arrival rate.

$$\lambda(\mu_t) = \lim_{\Delta t \to 0} \frac{prob_{sell}}{\Delta t} = \frac{r + \lambda_1 + \lambda_0}{2} \tag{50}$$

Notice that this is exactly the same strategy we had for $\tilde{\mu}_t$ when the informed trader only knew the state in period zero. The prices and value function are different, however. The value is scaled up by a factor depending on the persistence of the state. A less persistent state leds to a higher payoff, because the informed trader is getting more information. The price process is derived using Itò's Lemma. The drift comes from the potentially switching state and from the information conveyed by the Poisson event not arriving. There are Poisson jumps that send beliefs all the way to the boundary.

If $\mu_t < \mu^*$,

$$dP(\mu_t) = P'(\mu_t)(\lambda_0 + \mu_t(1 + \lambda(\mu_t) - \lambda_1 - \lambda_0))dt - (P(\mu_t) - P(0))dN_t. \tag{51}$$

The process is the reverse if $\mu_t \mu^*$.

5.2 N states

Say that there are N possible states of nature, $\omega \in \{0, 1, \dots, N-1\}$. If the informed trader is choosing $x_t \in [-1, 1]$, it is no longer the case that they will always choose only $x = -1$, or $x = 1$ in the best equilibrium. In fact, the best equilibrium is not going to exist. Since there are N states, the informed trader could generally benefit from sending N messages. That means choosing orders of N different sizes. The informed trader would like to place orders as large as possible accompanying each message that makes price go up, but cannot have the order sizes be equal. The message that makes price increase the most will be accompanied with buying $x = 1$ shares. The informed trader would like the message that makes price increase the second most to be as high as possible without being equal to 1. Such a number does not exist without some constraint forcing discrete increments on orders. In this section I will focus on a the asset holdings of the insider being chosen from the discrete set, $x_t \in \{-1, 1\}$.

5.2.1 Lack of Commitment

Suppose there are N possible states of the world, $\omega \in \{\omega_1, \omega_2, \dots, \omega_N\}$. You can relabel the states so they are in order of lowest price to highest price. Call μ_i the probability of state i. Let the price be linear function of the probabilities.

$$P(\mu) = \sum_{i=1}^{N} p_i \mu_i \tag{52}$$

By linearity, the support of the distribution of posteriors induced will still always contain a boundary point. The boundary point will either be one where the probability $\omega = \omega_1$ is zero or where the probability $\omega = \omega_N$ is zero.

In the limit the value is obtained with a series of Poisson processes all with arrival rate

$\lambda = \frac{1}{2}$. If the price is low today, at a Poisson rate the price will jump down to the boundary where $\mu_N = 0$ (the trader sells) and the rest of the time the price will slowly drift up (the trader buys). Once it is revealed that the state is not $\omega = \omega_N$, the process starts over as before but with only $N-1$ remaining possible states. The informed trader reveals the state of the world piece by piece with a series of Poisson processes. Each time a Poisson arrival happens, the trader is revealing one of the states that is not the true state.

5.2.2 Commitment

The problem is still equivalent to one where the informed trader chooses two posteriors rather than choosing trades directly. With two states the informed trader only had to choose how much beliefs increase after a buy and how much they decrease after a sell. With many states, the informed trader is still choosing the size of the increase after buying and decrease after selling, but they also need to choose a direction in $\mathbb{R}^{N-1}$. Let μ be an $N-1$ length vector representing the belief probabilities for all but the last state. The informed trader chooses $\tilde{\epsilon}$ from among unit vectors in $\mathbb{R}^{N-1}$. The informed trader also chooses $\underline{c}$, $\overline{c}$ from $\mathbb{R}_+$. Beliefs will move to $\mu + \overline{c}\tilde{\epsilon}$ after observing a buy and to $\mu - \underline{c}\tilde{\epsilon}$ afterobserving a sell. The martingale condition on beliefs will ensure that the two posteriors induced need to be in exactly opposite directions of each other. This means that the insider does not get to choose to directional vectors, only two magnitudes on one direction.

The setup of the problem still looks similar.

$$V(\mu) = \max_{\overline{c},\underline{c},\tilde{\epsilon}} \quad (P(\mu + \overline{c}\tilde{\epsilon}) - P(\mu) + \delta V(\mu + \overline{c}\tilde{\epsilon})) \frac{\underline{c}}{\overline{c} + \underline{c}} + (P(\mu) - P(\mu - \underline{c}\tilde{\epsilon}) + \delta V(\mu - \underline{c}\tilde{\epsilon})) \frac{\overline{c}}{\overline{c} + \underline{c}} \tag{53}$$

There rest looks very similar to the proof of theorem 2. Taking the same approximations as in the two state case, we can simplify the problem.

$$(1-\delta)V(\mu) = \max_{\overline{c},\underline{c},\tilde{\epsilon}} \quad 2\nabla P(\mu)^T \tilde{\epsilon} \frac{\overline{c}\underline{c}}{\overline{c} + \underline{c}} + \delta \tilde{\epsilon}^T \mathbf{H}_V(\mu) \tilde{\epsilon} \; \overline{c}\underline{c} \tag{54}$$

We can again see that in the optimum $\overline{c} = \underline{c}$ since $\mathbf{H}_V$ is negative definite. Define $\epsilon = c\tilde{\epsilon}$. The problem is again simplified.

$$(1-\delta)V(\mu) = \max_{\epsilon}\ \nabla P(\mu)^T \epsilon + \delta \epsilon^T \mathbf{H}_V(\mu)\epsilon \tag{55}$$

subject to the constraint that the vectors $\mu+\epsilon$ and $\mu-\epsilon$ each have elements that are positive and sum to less than one.

The maximum is obtain by differentiating.

$$\epsilon^* = \frac{1}{\delta}\mathbf{H}_V(\mu)^{-1}\ \nabla P(\mu) \tag{56}$$

We can now put ϵ^* into the objective function and get

$$(1-\delta)V(\mu) = -\frac{1}{\delta}\nabla P(\mu)^T \mathbf{H}_V(\mu)^{-1}\nabla P(\mu). \tag{57}$$

From this differential equation, we see that $\hat{V}(\mu) = V(\mu)\sqrt{\delta(1-\delta)}$ is constant in δ. If we substitute $\mathbf{H}_V(\mu) = \frac{\mathbf{H}_{\hat{V}}(\mu)}{\sqrt{\delta(1-\delta)}}$ into the equation for ϵ^* we get an easy expression.

$$\epsilon^* = \mathbf{H}_{\hat{V}}(\mu)^{-1}\ \nabla P(\mu)\sqrt{\frac{1-\delta}{\delta}} \tag{58}$$

With $\Delta t = \frac{1-\delta}{\delta}$ we see that beliefs converge to a Brownian motion. More precisely, we see that there is one Brownian motion, and the probabilities of each of the N states are driven by different weights of these states. Price is a smooth function of beliefs, so price also follows a Brownian motion. Explain more here later.

5.3 Multiple Informed Traders

In this section, I will study the game when there are multiple informed traders with the same information.

5.3.1 Pump-and-Dump

If there are two traders with the same information, the pump-and-dump strategy will still be an equilibrium.

Consider the same indifference condition when the beleifs are low. The trader needs to be the same between revealing the state to be bad today (by selling the asset), and claiming the state to be good today (by buying) to let the price drift up and revealing the state to be bad tomorrow. Call π_1 and π_2 the probability with which each player sells when the state is bad. Suppose that the market takes either trader selling to be fully revealing that the asset is bad.

$$P(\mu)-P(0) = (1-\pi_2)\left(P'(\mu)\mu(\pi_1+\pi_2) + \delta\left(P'(\mu)\mu(\pi_1+\pi_2) + P(\mu) - P(0)\right)\right) - \pi_2\left(P(\mu) - P(0)\right) \tag{59}$$

On the left hand side, the value of revealing the state to be bad is the same, $P(\mu) - P(0)$. The value of waiting one more period now had two parts. There is a chance the other player will reveal the state today and you will lose money because you are long, $-\pi_2\left(P(\mu) - P(0)\right)$. If the other trader doesn't reveal the state first $(1 - \pi_2)$, then the payoff is the same. You get the positive drift today which depends on the probabilities of revealing a bad state $(P'(\mu)\mu(\pi_1 + \pi_2))$, plus you get the payoff from revealing tomorrow $\delta\left(P(\mu) - P(0)\right)$.

$$\left(P(\mu) - P(0)\right)\left(1 - \delta + \pi_2(1-\delta)\right) = (1-\pi_2)(1+\delta)P'(\mu)\mu(\pi_1+\pi_2) \tag{60}$$

Rearranging a little, we can get a more convinient expression.

$$\pi_1 + \pi_2 = \frac{P(\mu) - P(0)}{P'(\mu)\mu}\frac{1-\delta+\pi_2(1-\delta)}{(1-\pi_2)(1+\delta)} \tag{61}$$

Now, suppose that player 2 is using a Poisson pump-and-dump strategy as before. We will see that it will also be optimal for player 1 to use a Poisson process. Let $\pi_2 = \lambda_2(1-\delta)$.

Now we can see that for large δ, π_1 will also be linear in $1-\delta$.

$$\pi_1 + \pi_2 \approx \frac{P(\mu) - P(0)}{P'(\mu)\mu}(1 + 2\lambda_2)(1-\delta) \tag{62}$$

Call $\pi_1 = \lambda_1(1-\delta)$. Now we can solve for the equilibrium arrival rate. We can find the symmetric equilibrium by letting $\lambda_1 = \lambda_2$.

$$2\lambda\left(1 - \frac{P(\mu) - P(0)}{2P'(\mu)\mu}\right) = \frac{P(\mu) - P(0)}{2P'(\mu)\mu} \tag{63}$$

Solving for λ gives the arrival rate of the Poisson procces.

$$\lambda = \frac{1}{2}\frac{\frac{P(\mu)-P(0)}{2P'(\mu)\mu}}{1 - \frac{P(\mu)-P(0)}{2P'(\mu)\mu}} \tag{64}$$

5.3.2 More Revealing Equilibria

Even though the pump-and-dump scheme is an equilibrium just as in the one trader model, it is no longer the case that it will be the best equilibrium. This equilibrium gives a payoff to the informed traders of $\min\{P(\mu) - P(0), P(1) - P(\mu)\}$ as before.

Consider another equilibrium. If the state is good, both informed players will buy the asset in the first period. If the state is bad, both informed players will sell the asset in the first period. When both traders buy the market maker's beliefs move to one. When both traders sell the market maker's beliefs move to zero. If the two informed traders do different actions (off path), the market maker's beliefs go to zero if $P(\mu) - P(0) \leq P(1) - P(\mu)$ and one otherwise.

Rather than the informed traders getting the minimum of the value of sending beliefs to zero or one, they get the average in expectation. This wasn't an equilibrium when there was only one informed player. With only one informed player if the price was low today, when the state is bad the trader would like to deviate to buying and pretending the state is good. The price is already low, so revealing the state to be bad only moves price a little but revealing the state to be good moves the price a lot.

This deviation is no longer profitable when there are multiple informed traders. When the state is bad, you know the other informed trader is going to reveal the state to be bad. So if you try to deviate by buying, the price will still fall to $P(0)$ because the other informed trader revealed it to be bad and you will lose money because you are long. Essentially, since the other informed trader is revealing information that will move the price, you always want to match what they are doing. You need to buy when they buy (and make the price go up) and sell when they sell (and make the price go down). This gives to strictly positive profits where any deviation is going to give you negative profits.

In fact, this arguement reveals that there are many move equilibria that can be supported. Many of these give an even higher payoff to the informed traders. If beliefs move up to $\mu_{up}(\mu)$ after both informed traders buy, down to $\mu_{down}(\mu)$ after both traders sell, and they move to whichever of those beliefs is farther from μ^* when the informed traders take different actions (off path), the an informed trader always has an incentive to match whatever action the other informed trader is doing. There will be no profitable way to deviate. If the traders are playing a mixed strategy, this requires them both to be able to see one randomization device. The informed trader needs to know what the other informed trader is doing (or supposed to do in equilibrium). The most profitable equilibrium would of course then be the same strategy and payoff the single informed trader used whith commitment.

They informed traders aren't colluding. There are no trigger strategies to prevent deviations. They are simply coordinating, and it's never profitable to go against the grain of what the other informed traders are doing. We can see that if there are any number of informed traders greater than one, and they can coordinate (see a common randomization device), the best payoff is going to be the same as with commitment in the single player case and lead to prices following a Brownian motion. Coordiantion becomes a substitute for commitment.

5.4 Other Information

A natural question would be how this model could generate both the Brownian motion and Poisson jumps at the same time. After all, this is what we seem to see in the data and option pricing models with jumps typically have a Brownian motion with jumps not just the jumps by themselves. This is achieved by having multiple informed traders with independent pieces of information. Say the asset is Amazon stock. There may be one trader that can obtain verifiable (commitment) information about the aquisition of Whole Foods, and another trader with unverifiable (no commitment) information about the cloud computing services. Both of these pieces of information may be relevent to the value of Amazon stock, but they don't necessarily need to be correlated.

I will again explain this using a linear price function. Suppose that there are N independent pieces of information, $\omega_i \in \{0, 1\}$, relevent to the value of the asset.

$$P(\mu_1, \mu_2, \ldots, \mu_N) = \sum_{i=1}^{N} \mu_i z_i \tag{65}$$

Let's solve the problem of some trader that knows the value of ω_j, when there may potentially be other traders that know the other pieces of information.

Trader j holds fixed the stochastic processes for μ_i for all $i \neq j$, and chooses the optimal process for μ_j.

$$V_j(\mu_1, \mu_2, \ldots, \mu_N) = \max_{\Delta(\mu_j') \in \Delta([0,1])} \mathbb{E}\left[\left(P(\mu_1', \mu_2', \ldots, \mu_N') - P(\mu_1, \mu_2, \ldots, \mu_N)\right) x_j + \delta V_j(\mu_1', \mu_2', \ldots, \mu_N')\right] \tag{66}$$

subject to Bayes plausibility and incentive compatability if solving the non-commitment problem. Trader j is also not allowed to correlate their strategy for μ_j with any other μ_i as they don't know those pieces of information and they are independent.

With the linear price function and independence, the problem simplifies.

$$V_j(\mu_1, \mu_2, \ldots, \mu_N) = \max_{\Delta(\mu'_j)\in\Delta([0,1])} \mathbb{E}[(\mu'_j - \mu_j)z_j x_j] + \sum_{i\neq j}(\mathbb{E}[\mu'_i] - \mu_i)z_i x_j + \delta\mathbb{E}\left[V_j(\mu'_1, \mu'_2, \ldots, \mu'_N)\right] \tag{67}$$

The beliefs about each piece of information must be a martingale. Regardless of what strategy the other players use, each term in the sum is zero from trader j's perspective.

$$V_j(\mu_1, \mu_2, \ldots, \mu_n) = \max_{\Delta(\mu_j)\in\Delta([0,1])} \mathbb{E}[|\mu'_j - \mu_j|z_j] + \delta\mathbb{E}[V(\mu_1, \mu_2, \ldots, \mu_N)] \tag{68}$$

Hence while prices and profits are quite different and will move randomly outside the control of player j, the strategy and expected profits of player j remain the same as in the version with only one strategic informed trader. This holds rather the other $N-1$ pieces of information are being release by other strategic players of if they are just randomly arriving by some exogenous process.

6 Conclusion

6.1 Literature Review

The topic of how prices move in an efficient capital market goes back to early work by Paul Samuelson and greatly expanded with Eugene Fama's papers, see Samuelson (1965) and Fama (1970). I study this by merging the literature of strategic informed trading with that of information design.

Questions of strategic informed trading often studied using a variant of Kyle (1985). Many of these study whether the trader's private information gets full incorporated into the price. Some examples of this are Holden and Subrahmanyam (1992), Foster and Viswanathan (1996), Ostrovsky (2012), and many others. Other models of prices in financial markets with informed strategic trading that don't use Kyle (1985) usually derive from

Glosten and Milgrom (1985), Hellwig (1980), or Hanson (2003, 2007). Van Bommel (2003) extends a Kyle model to allow for false information and rumors.

My model solves for the maximum amount of volatility in prices cause solely by information. Shiller (1981) shows empirically how prices are seemingly too volatile to be explained by information alone.

My model without commitment becomes a model of cheap talk popularized in Crawford and Sobel (1982). I use techniques for solving this like those in Lipnowski and Ravid (2019). They are able to simplify cheap talk games considerably when the sender's preference does not depend on the state.

My model with commitment is akin to a Bayesian persuasion model like Kamenica and Gentzkow (2011) and going back to Aumann and Maschler (1995). Dynamic information design models (like mine) are popular recently, Ely (2017), Renault et al. (2017), Orlov et al. (2019), and Hörner and Skrzypacz (2016). My model with commitment generalizes the results of Ely et al. (2015) where they study how much an informed agent can surprise and uninformed player.

I use tools developed by Mertens and Zamir (1977) who study the maximal variation of a bounded martingale. This gives results to the model with commitment similar to De Meyer and Saley (2003); De Meyer (2010) who seek a strategic foundation for Brownian motion in finance.

My results from the model without commitment look a lot like the solution of Zhong (2019), who studies dynamic information aquisition. They show how a Poisson process gives the most uncertainty across time, and that eponentially discounting agents are risk loving over lotteries across time. This same logic applies to the strategy of my trader without commitment power.

7 Appendix

7.1 Theorem 1

Here I give the remaining details needed for the proof of theorem 1. Take $\mu < \mu^*$.

We know that each period beliefs jump down to zero or go up to an interior point that is indifferent.

$$P(\mu) - P(0) = P(\mu_{buy}) - P(\mu) + \delta V(\mu_{buy}) \tag{69}$$

Putting the value function in, we can solve for the posterior induced after buying.

$$\mu_{buy} = P^{-1}\left(\frac{2P(\mu) - (1-\delta)P(0)}{1+\delta}\right) \tag{70}$$

from prior μ.

Beliefs being a martingale pins down what the probability of buying and selling must be.

$$p_{sell} = \frac{P^{-1}\left(\frac{2P(\mu)-(1-\delta)P(0)}{1+\delta}\right) - \mu}{P^{-1}\left(\frac{2P(\mu)-(1-\delta)P(0)}{1+\delta}\right)}. \tag{71}$$

Taking limits gives,

$$\lim_{\delta\to 1} \mu_{buy} = \mu \tag{72}$$

and therefore

$$\lim_{\delta\to 1} p_{sell} = 0. \tag{73}$$

Link the discount factor to the length of each period with $\Delta t = 1 - \delta$. We now need to show that $\lim_{\delta\to 1} \frac{p_{sell}}{\Delta t}$ converges to a finite positive number. That number is the arrival rate of the Poisson process.

$$\frac{p_{sell}}{\Delta t} = \frac{P^{-1}\left(\frac{2P(\mu)-(1-\delta)P(0)}{1+\delta}\right) - \mu}{P^{-1}\left(\frac{2P(\mu)-(1-\delta)P(0)}{1+\delta}\right)(1-\delta)} \tag{74}$$

Clearly, the numerator and demoninator are both going to zero as δ goes to one. We need to use L'Hôpital's rule.

First the derivative of the numerator.

$$\frac{\partial top}{\partial \delta} = P^{-1'}\left(\frac{2P(\mu)-(1-\delta)P(0)}{1+\delta}\right)\left(\frac{P(0)}{1+\delta} - \frac{2P(\mu)-(1-\delta)P(0)}{(1+\delta)^2}\right) \tag{75}$$

The limit of this is non-zero.

$$\lim_{\delta\to 1}\frac{\partial top}{\partial \delta} = \frac{1}{2}P^{-1'}\left(P(\mu)\right)\left(P(0)-P(\mu)\right) \tag{76}$$

Now the derivative of the denominator.

$$\begin{aligned}\frac{\partial bottom}{\partial \delta} = P^{-1'}\left(\frac{2P(\mu)-(1-\delta)P(0)}{1+\delta}\right)\left(\frac{P(0)}{1+\delta} - \frac{2P(\mu)-(1-\delta)P(0)}{(1+\delta)^2}\right)(1-\delta) \\ - P^{-1}\left(\frac{2P(\mu)-(1-\delta)P(0)}{1+\delta}\right)\end{aligned} \tag{77}$$

This limit is even simpler.

$$\lim_{\delta\to 1}\frac{\partial bottom}{\partial \delta} = -\mu \tag{78}$$

We now have our arrival rate.

$$\begin{aligned}\lambda(\mu) &= \frac{p_{sell}}{\Delta t} &(79)\\ &= \frac{P^{-1'}(P(\mu))}{2\mu}(P(\mu)-P(0)) &(80)\\ &= \frac{P(\mu)-P(0)}{2P'(\mu)\mu} &(81)\end{aligned}$$

The last equality follows from the inverse function theorem.

This gives a Poisson term on beliefs equal to

$$-\mu_t dN_t \tag{82}$$

where N_t is a standard Poisson process with arrival rate λ. The drift must be such that beliefs are a martingale.

$$drift dt = \mathbb{E}[\mu_t dN_t] \tag{83}$$

$$= \mu_t \lambda(\mu_t) dt \tag{84}$$

Beliefs follow

$$d\mu_t = \mu_t \lambda(\mu_t) dt - \mu_t dN_t. \tag{85}$$

Since the price is a differentiable function of beliefs, this gives us the process for prices.

$$dP(\mu_t) = P'(\mu_t)\mu_t \lambda(\mu_t) dt - (P(\mu_t) - P(0)) dN_t \tag{86}$$

Putting in our equation for $\lambda(\mu_t)$ gives the result.

$$dP(\mu_t) = \frac{1}{2}(P(\mu_t) - P(0)) dt - (P(\mu_t) - P(0)) dN_t \tag{87}$$

A symmetric arguement holds for $\mu > \mu^*$.

7.2 Proposition 2

The maximal time to full information revelation is the amount of time it take beliefs to drift to μ^* if the Poisson jump doesn't arrive. After that, beliefs immediately move to the boundary. Take $\mu_0 \leq \mu^*$. We saw from theorem 1 that

$$dP(\mu_t) = \frac{1}{2}(P(\mu_t) - P(0)) - (P(\mu_t) - P(0)) dN_t. \tag{88}$$

When the Poisson jump doesn't arrive, the price follows a smooth differentiable path.

This gives a linear differential equation for prices as a function of time.

$$\frac{dP(\mu_t)}{dt} = \frac{1}{2}(P(\mu_t) - P(0)) \tag{89}$$

This implies that the price must be an exponential function in time.

$$P(\mu_t) = ce^{\frac{1}{2}t} - P(0) \tag{90}$$

We get the constant from the initial condition that the price at time 0 equals $P(\mu_0)$.

$$P(\mu_t) = (P(\mu_0) - P(0))e^{\frac{1}{2}t} - P(0) \tag{91}$$

We then solve for the time at which $P(\mu_t) = P(\mu^*)$. This gives the result.

$$t^{max} = 2\log\left(\frac{P(\mu^*) - P(0)}{P(\mu_0) - P(0)}\right) \tag{92}$$

If we start with a high initial belief, $\mu_0 > \mu^*$, the only change in the equations is the use of $P(1)$ in place of $P(0)$.

7.3 Theorem 2

Here I give the remaining details needed for the proof of theorem 2. Once you write the problem as choosing posteriors subject to Bayes' plausibility, the problem becomes

$$\begin{aligned} V(\mu) = \max_{\bar{\epsilon},\underline{\epsilon}} \ & |P(\mu+\bar{\epsilon}) - P(\mu)|\,\frac{\underline{\epsilon}}{\bar{\epsilon}+\underline{\epsilon}} + |P(\mu-\underline{\epsilon}) - P(\mu)|\frac{\bar{\epsilon}}{\bar{\epsilon}+\underline{\epsilon}} \\ & + \delta\left(V(\mu+\bar{\epsilon})\frac{\underline{\epsilon}}{\bar{\epsilon}+\underline{\epsilon}} + V(\mu-\underline{\epsilon})\frac{\bar{\epsilon}}{\bar{\epsilon}+\underline{\epsilon}}\right). \end{aligned} \tag{93}$$

First note that for small values of $\bar{\epsilon}$

$$|P(\mu + \bar{\epsilon}) - P(\mu)| \approx |P'(\mu)|\bar{\epsilon} \tag{94}$$

and the same for $\underline{\epsilon}$.

Now take a second order approximation to $V(\mu')$.

$$V(\mu + \bar{\epsilon}) \approx V(\mu) + V'(\mu)\bar{\epsilon} + \frac{1}{2}V''(\mu)\bar{\epsilon}^2 \tag{95}$$

Then

$$V(\mu + \bar{\epsilon})\frac{\underline{\epsilon}}{\bar{\epsilon} + \underline{\epsilon}} + V(\mu - \underline{\epsilon})\frac{\bar{\epsilon}}{\bar{\epsilon} + \underline{\epsilon}} = V(\mu) + \frac{1}{2}V''(\mu)\bar{\epsilon}\underline{\epsilon} \tag{96}$$

since the first order term cancels out.

Now (93) becomes

$$(1 - \delta)V(\mu) = \max_{\bar{\epsilon},\underline{\epsilon}} \; 2|P'(\mu)|\frac{\bar{\epsilon}\underline{\epsilon}}{\bar{\epsilon} + \underline{\epsilon}} + \frac{1}{2}\delta V''(\mu)\bar{\epsilon}\underline{\epsilon} \tag{97}$$

subject to the posterior remaining between zero and one.

The first term is positive, so for any given level of the product $\bar{\epsilon}\underline{\epsilon}$ you would like to minimize the denominator $\bar{\epsilon} + \underline{\epsilon}$. The way to minimize the sum of two variables given a fixed level of their product is always $\bar{\epsilon} = \underline{\epsilon}$. Call it ϵ.

$$(1 - \delta)V(\mu) = \max_{\epsilon} \; |P'(\mu)|\epsilon + \frac{1}{2}\delta V''(\mu)\epsilon^2 \tag{98}$$

Taking the derivative of the right hand side and setting it equal to zero yields

$$\epsilon^* = -\frac{|P'(\mu)|}{\delta V''(\mu)}. \tag{99}$$

Note that this is positive because $V(\mu)$ is concave.

Putting the solution for ϵ^* into the objective gives a second order differential equation.

$$V''(\mu)V(\mu) = -\frac{|P'(\mu)|^2}{2\delta(1-\delta)} \tag{100}$$

Define

$$\hat{V}(\mu) = V(\mu)\sqrt{2\delta(1-\delta)}. \tag{101}$$

This gives a simpler differential equation.

$$\hat{V}''(\mu)\hat{V}(\mu) = -|P'(\mu)|^2 \tag{102}$$

While this differential equation is not generally solvable for any function $P(\mu)$ we can see that $\hat{V}(\mu)$ is constant in δ.

This implies that $V''(\mu)$ is going to minus infinity at rate $\frac{1}{\sqrt{1-\delta}}$. Link the discount factor to the width of each time interval, $\Delta t = \frac{1-\delta}{\delta}$.

We now have beliefs following a binomial model,

$$\mu' - \mu = \begin{cases} \sigma(\mu)\sqrt{\Delta t} & \text{with probability } \frac{1}{2} \\ -\sigma(\mu)\sqrt{\Delta t} & \text{with probability } \frac{1}{2} \end{cases} \tag{103}$$

where

$$\sigma(\mu) = \frac{\sqrt{2}|P'(\mu)|}{\hat{V}''(\mu)}. \tag{104}$$

This gives convergence of beliefs to a Brownian motion.

$$d\mu_t = \sigma(\mu)dB_t \tag{105}$$

Itô's Lemma gives the price process.

$$dP(\mu_t) = \frac{1}{2}P''(\mu_t)\sigma^2(\mu_t)dt + P'(\mu_t)\sigma(\mu_t)dB_t \tag{106}$$

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