Equilibrium Gerrymandering

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Abstract

I present a game theoretic model of non-partisan gerrymandering between many states electing members to the U.S. House of Representatives. States choose seat-vote curves to affect policy through the makeup of Congress. A seat-vote curve is the fractions of a state's congressional seats won by a party as a function of that party's statewide vote share. In equilibrium, every state chooses a highly responsive seat-vote curve (the slope in the middle is much steeper than one.) The responsiveness is inversely proportional to the number of representatives that state gets to elect. I then estimate seat-vote curves in each state and show that these two equilibrium facts are true empirically. I show that there is a deadweight loss, and that universally switching to proportional representation would benefit every state. However, this is a prisoner's dilemma situation where each state has an incentive to deviate to a highly responsive seat-vote curve regardless of other states' actions. I conclude by solving for the optimal partisan gerrymandering of a state using information design techniques. I show that the optimal policy always employs "cracking" and sometimes employs "packing."

1 Introduction

In 2020, all seats of the U.S. House of Representatives were up for re-election. The state of Massachusetts elected nine of these representatives. The state was divided up into districts and each district elected one representative by a majority vote of the population. Despite 25 percent of the state voting for Republicans, Democrats won all nine districts. Conversely, Oklahoma and Utah elected five and four representatives respectively. Despite over 30 percent of the population voting for Democrats in each of these states, all districts were won by Republicans. If the districts had been drawn differently, the minority party in these states could have won some of the congressional seats. Clearly, how the state is divided into districts will impact the fraction of congressional seats that are won by each party. In turn, these districts affect policy.

I seek to understand the incentives for gerrymandering. While there is a large literature on gerrymandering, previous papers study a single state choosing districts in isolation. I study states in an equilibrium model and new motives arise leading to predictions about the districts chosen. I test these predictions empirically and document their existence in the data.

To understand the model used and predictions made, a few terms must be explained the way they are to be used in this paper. I will refer throughout this paper to the practice of choosing how to divide a state into districts to achieve some aim as gerrymandering and the person or group choosing the districts as the designer. Often the term gerrymandering has a negative connotation and implies designing districts to help one party, but I will use it more broadly to be designing districts for any objective of the designer. This objective may be helping a political party or anything else.

The agents in the model don't care directly about districts, but about policy. Since policy is chosen by the elected representatives, the object of interest is what is called a seat-vote curve. A state's seat-vote curve is is simply the fraction of seats won by a political party as a function of the fraction of statewide votes received for that party. The slope of the seat-vote curve in the middle (for vote shares between 45 and 55 percent) is known as the responsiveness. Seat-vote curves become the objective of a district designer. This paper's less catchy (though likely more accurate) title could be "Equilibrium Seat-Vote Curves."

The designer's gerrymandering problem has two parts. The first part is choosing an optimal seat-vote curve. Given any way the population votes, how many Democrats do you want in Congress? This is simple if the designer is the agent of a political party. The objective is just to make the seat-vote curve as high or low as possible. Assuming the designer is non-partisan and values only the welfare of the citizens, the optimal seat-vote curve may be more complicated. The second part of the problem is finding how to divide the state into districts to implement the desired seat-vote curve. Even for a simple seat-vote curve, this can be challenging.

There are many papers about designing districts to implement an objective seat-vote curve but relatively fewer about finding the optimal seat-vote curve. Most papers studying gerrymandering also focus on a district designer maximizing representation by a given party. However, several states that have a bipartisan committee select the districts. Bipartisan motivations have not been as well studied. I am not aware of any paper addressing the problem of finding the optimal seat-vote curves for a non-partisan designer in an equilibrium setting. This problem is the focus of the current paper.

In the next section, I construct an equilibrium model of bipartisan district designers from each state and derive the first main result. I show that in contrast to single state models where the responsiveness of the optimal seat-vote curve is low (less than one) and independent of state size, in an equilibrium model the responsiveness of seat-vote curves is high (much larger than one) and inversely related to the state's size. That is, smaller states have more responsive seat-vote curves. There is a very simple intuition for these results and why the presence of other states matters. Policy is chosen by the elected representatives. If x percent of your state population is Democrat and $1 - x$ is Republican, you typically want x percent of the representatives to be Democrats and $1 - x$ to be Republicans. Then, the policy that maximizes the welfare of the representatives is the same as the policy that maximizes the welfare of the state population. If a model with only one state, this proportional representation is obtain by a seat-vote curve with a responsiveness of one. If one percent more of the state is Democrat, one percent more or the representatives are Democrats. When their are many states, a more extreme seat-vote curve is optimal. The intuition is the same, but policy is chosen by the representatives from every state, not just your own. If one percent more of the state is Democrat, you would like one percent more of the representatives in Congress to be Democrats. One percent of the representatives in Congress is a lot more than one percent of the representatives from your state (fifty times more on average). Thus, the optimal seat-vote curve for your state has a very high responsiveness. This logic applies even stronger for smaller states than for larger states. If a state wants Congress to be one percent more Democrat, they need to elect about four more Democrats. For a state like Iowa that's all of their allotted representatives while for California it is about seven percent of their allotted representatives. Thus, we see that smaller states have an incentive to choose more highly responsive seat-vote curves in equilibrium than large states.

In section three, I empirically test these predictions. Using the Cooperative Congressional Election Survey, I estimate the seat-vote curve in each state in the last two redistricting cycles (redistricting is done every ten years). I find that the average responsiveness of these seat-vote curves is significantly higher than one. I also find that the inverse of the state's size is highly predictive of seat-vote curve responsiveness. Smaller states have higher responsiveness. I show that this relationship does not arise mechanically from the fact that smaller states have fewer districts.

While in equilibrium each state chooses a highly responsive seat-vote curve, I show in section four that this is not socially optimal. In fact, the gerrymandering problem is a prisoner's dilemma game. The socially optimal strategy is for each state to choose a seat-vote curve with a lower responsiveness. However, each state individually has a dominant strategy of choosing a high responsiveness. In equilibrium, every state is made worse off.

In section five, I show that a winner-take-all election in every state is approximately optimal. In section six, I give extensions of the model. I first explain the difference if policy is chosen by the median member of Congress rather than the mean member of Congress. The winner-takes-all election then becomes exactly optimal for every state. I next show the optimal gerrymandering strategy in this model for a state designer that wishes to maximize the number of representatives from a particular political party. This can be solved for using information design techniques. The optimal strategy always involves "cracking" and sometimes involves "packing". The final section discusses related literature and concludes.

2 Model

The model is simply the model of Coate and Knight (2007) extended to have several states competing in equilibrium. Section six will have extensions of this model. It is clear that the qualitative results hold much more generally, but I don't find any gain from presenting the model more generally. The simple model makes the insights sufficiently clear.

2.1 Model Setup

Picture the policy chosen as a point lying in the interval from zero to one. Zero represents the policy most preferred by the Democrat party and one is the policy most preferred by the Republican party. Anything in between is a compromise. The policy is chosen by the representatives in Congress elected by each state. State $i \in \{1, 2, \ldots, M\}$ is given N_i representatives to elect to Congress with $n_i = \frac{N_i}{\sum_{i=1}^{M} N_i}$ denoting the fraction of Congress elected by state i. The state needs to choose how many of their N_i seats they give to Democrats and how many they give to Republicans. I'll assume that the seats are perfectly divisible (electing .5 Democrats could be thought of as electing a moderate representative that is halfway between the Democrat and Republican parties). The information the state has to base this decision off of is the votes and the distribution of preferences among the state population. That is to say, the state designer is choosing a seat-vote curve.

The point of different districting strategies is to induce a certain seat-vote curve. The seat-vote curve is the expected fraction of congressional seats a party wins as a function of the party's vote share in the state. Consider a state in which 40 percent are Democrats, 40 percent are Republicans, and 20 percent are Independents. You could have half of the districts be made up only of Democrats and Independents (4 Democrats for each Independent) and the other half of the districts be only Republicans and Independents (4 Republicans for each Independent). Then regardless of what the independents do, half seats will be won by Democrats and half won by Republicans. This means the seat-vote curve will completely horizontal with a height of one half. This seat-vote curve has a responsiveness of zero. Alternatively, you could make every district look demographically like the state as a whole. That is, each district would be made up of 40 percent Democrats, 40 percent Republicans, and 20 percent Independents. Then, all the districts could potentially be flipped based on what happens with the Independents. This seat-vote curve would have a minimum of 0, a maximum of 1, and be very steep in the middle region. This is a highly responsive seat-vote curve. The most intuitive idea people think of is that congressional seats should match the distribution of the voters. So, if Democrats get 54 percent of the vote they should get 54 percent of the seats in congress. This represented by a seat-vote curve that is simply a 45 degree line. The curve has a slope of one everywhere and an intercept of zero.

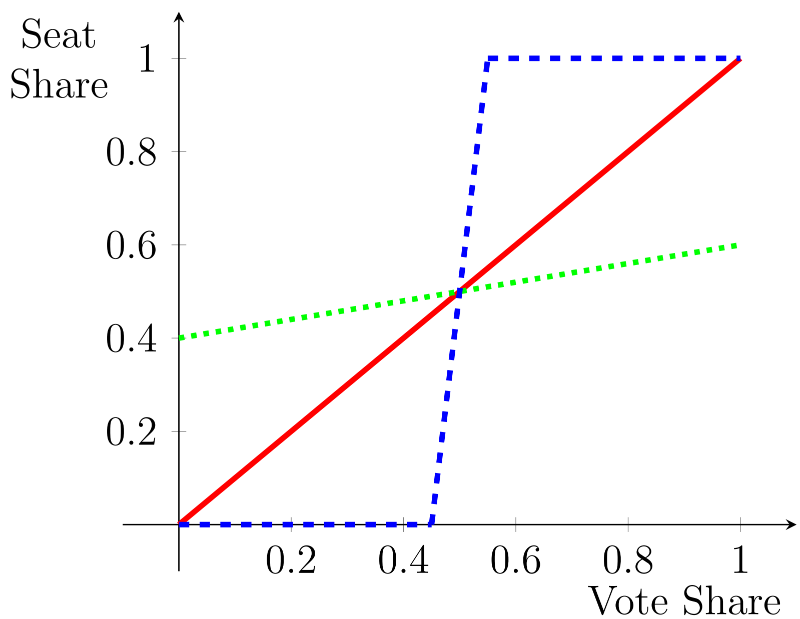

Figure 1: Three different seat-vote curves implemented by gerrymandering. The solid line is the “proportional” seat-vote curve. The dotted line is a less responsive seat-vote curve with safe seats for both parties. The dashed line is a highly responsive seat-vote curve.

Each state has a unit mass of voters. Every voter has a private preference over the policy chosen. In each state, there are three different groups of voters. There is a mass π_{Di} of Democrats with a preferred policy $\theta = 0$. There is a mass π_{Ri} of Republicans with a preferred policy $\theta = 1$. There is a mass π_{Ii} of Independents with preferred policy distributed over $\theta \in [0, 1]$. Call m_i the mean preference among the independent voters, and call $2\tau_i$ the width. The independent voters have a preferred policy uniformly distributed on the interval $[m_i - \tau_i, m_i + \tau_i]$. This mean, m_i, is unknown at the time the designer chooses a seat-vote curve. The unknown mean is what makes a seat-vote curve necessary in the first place. If the designer fully knew the distribution of preferences in the state, they could simply compute the welfare maximizing policy and and choose the corresponding fraction of congressional seats for Democrats. With an unknown mean, the optimal fraction of seats to give to each party is also known. When m_i is high (low), more of your state likes the Republican (Democrat) policy and you would like to give more seats to Republicans (Democrats). At the time of the election the mean is drawn from a uniform distribution $m_i \sim U\left(\left[\frac{1}{2}-\tau_i, \frac{1}{2}+\tau_i\right]\right)$. This means that the fraction of independents that will vote Democrat in a given election is uniformly distributed over $[0,1]$.

Each voter wants the policy to be as close as possible to their preferred policy. They face a quadratic loss function. If a voter's preferred policy is $\hat{\theta}$ and the policy chosen is θ, the voter receives a payoff of $-\left(\theta-\hat{\theta}\right)^2$. The voters are not strategic in their choice. A voter will vote Democrat if their preferred policy is less than or equal to $\frac{1}{2}$. Otherwise, they will vote Republican. A slightly Democrat leaning voter may still want to vote Republican if they think the rest of the state leans even more Democrat and they want to balance it out. However, with a continuum of voters, an individual vote doesn't affect the outcome and thus such strategic motivations don't arise. The voters are therefore kept simple.

The strategic players of the game are each state's district designer. They each choose a seat-vote curve. Call v_i the fraction of voters that vote Democrat in state i. This is a combination of how many Democrats, Republicans, and Independents are in the state, as well as where the median Independent voter is.

$$v_i = \pi_{Di} + \pi_{Ii}\left(\frac{\frac{1}{2}-(m_i-\tau_i)}{2\tau_i}\right) \tag{1}$$

The seat-vote curve $S:[0,1]\rightarrow[0,1]$ is a mapping from the fraction of votes that Democrats won, v_i, to the fraction of representatives Democrats get in congress, $S_i(v_i)$.

After the election, the policy is chosen collectively by the representatives from all 50 states. Recall that n_i is the fraction of representatives that come from state i. If each state, i, elected $S_i(v_i)$ Democrats, then congress will have a fraction $S=\sum_{i=1}^{M} n_i S_i(v_i)$. The members of congress choose the policy that maximizes their welfare. Policy is set equal to the average of all the representatives preferences. So, the chosen policy, θ is equal to $1-S$ (because a Democrat's preferred policy is 0).

This is a simultaneous move game where each state chooses a seat-vote curve, $S_i(v_i)\in$

$[0,1]\ \forall v_i \in [0,1]$, to maximize the welfare of its voters,

$$\max_{S_i(v_i)} \quad -\mathbb{E}_{m_1,m_2,\ldots,m_M}\left[\pi_{Di}(\theta)^2 + \pi_{Ri}(1-\theta)^2 + \pi_{Ii}\int_{m_i-\tau_i}^{m_i+\tau_i}(\theta-x)^2\frac{dx}{2\tau}\right] \tag{2}$$

where θ, the policy, is equal to one minus the average fraction of seats chosen by the states for Democrats.

$$\theta = 1 - \sum_{i=1}^{M} n_i S_i(v_i) \tag{3}$$

This game has a unique Nash equilibrium.

2.2 Single State

Before solving explicitly for the solution of this game, let us briefly examine the desired policy of a state. This can be found by looking at the special case where $M = 1$. The case of only a single state is what the literature has studied previously. Call G the distribution of voters' individual preferences in the state. Since there is only one state, the policy chosen is completely determined by the fraction of Democrats the state chooses to elect. Thus, the problem can be written simply.

$$\max_{\hat{\theta}} \; -\int\left(\tilde{\theta}-\theta\right)^2 dG(\tilde{\theta}) \tag{4}$$

This problem has a simple solution. The state would like the policy to be equal to the average preference in the state.

$$\Rightarrow \theta^* = \mathbb{E}_G[\theta] \tag{5}$$

Equation (5) most of out intuition about seat-vote curves to this point. Plugging the assumptions about the distribution G, this equation gives us the optimal seat-vote curve derived by Coate and Knight (2007).

$$\theta^* = \mathbb{E}_m[\theta] \tag{6}$$

$$= \frac{1}{2} + (\pi_D - \pi_R)\left(\frac{1}{2}-\tau\right) + 2\tau\left(v - \frac{1}{2}\right) \tag{7}$$

This equation also shows the motivation supporting the proportional seat-vote curve. If every voter was either a pure Democrat or pure Republican living on the extreme, $\pi_I = 0$, then the average preference is exactly equal to the vote share. This can be seen from equation (6) using the facts that $\pi_R = 1 - \pi_D - \pi_I$ that $\pi_I = 0$ implies $v = \pi_D$. Since there are no independents, the fraction of votes for Democrats is just the fraction of Democrats in the voting population.

$$\theta^* = \frac{1}{2} + (\pi_D - \pi_R)\left(\frac{1}{2} - \tau\right) + 2\tau\left(v - \frac{1}{2}\right) \tag{8}$$

$$= \frac{1}{2} + (2\pi_D - 1)\left(\frac{1}{2} - \tau\right) + 2\tau\left(\pi_D - \frac{1}{2}\right) \tag{9}$$

$$= \pi_D \tag{10}$$

When the state is one percent more Democrat, you want policy to be one percent more Democrat. Thus, you elect one percent more Democrats. Essentially the same result ($\theta^* = v$ which is proportional representation) obtains when $\tau = \frac{1}{2}$. However, when $\pi_I > 0$ and $\tau < \frac{1}{2}$, you do not get proportional representation. A one percent increase in the population voting Democrat is not from one percent of the population that were staunch Republicans and are now suddenly staunch Democrats. The one percent increase in votes comes from one percent of the population that was near the center but leaning slightly Republican and is now near the center and leaning slightly Democrat. It is a much smaller shift in aggregate preference. Thus, optimal policy would shift toward the Democrats by less than one percent. This is a motivation for a flatter seat-vote curve. This implies that optimal seat-vote curves should have a low responsiveness, a fact not verified in the data.

2.3 Many States

Taking the single state case as a baseline for comparison, return now to the case of many states, $M = 50$. Even though the shift in desired policy is mild (slope of $2\tau_i$ in the above equation), the strategy a state must take to implement that shift is extreme. We can see this by solving for state i's best response function in the game. Once the state sees the vote share, v_i, they no longer face any uncertainty about their own voters' preferences. This means that the optimal seat-vote curve can be solved for point-wise. The setup is nearly the same as the single state case.

$$\max_{S_i(v_i)} \quad -\mathbb{E}_{v_{-i}}\left[\int \left(\tilde{\theta} - \theta\right)^2 dG_i(\tilde{\theta}) \,\middle|\, v_i\right] \tag{11}$$

$$\text{s.t.} \quad \theta = 1 - \sum_{j=1}^{M} n_j S_j(v_j) \tag{12}$$

We can plug in for G_i and θ to get the whole messy objective.

$$
\begin{aligned}
\max_{S_i(v_i)} -\mathbb{E}_{v_{-i}} \Bigg[\pi_{Di} \left(1 - \sum_{j=1}^{M} n_j S_j(v_j)\right)^2 &+ \pi_{Ri} \left(\sum_{j=1}^{M} n_j S_j(v_j)\right)^2 \\
&+ \pi_{Ii} \int_{m_i-\tau_i}^{m_i+\tau_i} \left(1 - \sum_{j=1}^{M} n_j S_j(v_j) - x\right)^2 \frac{dx}{2\tau_i} \Bigg| v_i \Bigg] \quad (13)
\end{aligned}
$$

Now to find the maximum, we differentiate and simplify considerably.

$$
\begin{aligned}
\frac{\partial W_i(v_i)}{\partial S_i(v_i)} &= 2n_i \mathbb{E}\Bigg[\pi_{Di}\left(1 - \sum_{j=1}^{M} n_j S_j(v_j)\right) && (14)\\
&\quad -\pi_{Ri} \sum_{j=1}^{M} n_j S_j(v_j) + \pi_{Ii}\left(1 - \sum_{j=1}^{M} n_j S_j(v_j)\right) - \pi_{Ii} \int_{m_i-\tau_i}^{m_i+\tau_i} \frac{x}{2\tau_i} dx \Bigg| v_i \Bigg] && (15)\\
&= 2n_i \left(\pi_{Di} + \pi_{Ii}(1 - m_i) - \sum_{j=1}^{M} n_j \mathbb{E}_{v_j}[S_j(v_j)|v_i]\right). && (16)
\end{aligned}
$$

Setting this equation equal to zero gives us the welfare maximizing share of seats for Democrats given the realization of the Independent voters.

$$
\hat{S}_i(v_i) = \frac{1}{n_i}(\pi_{Di} + \pi_{Ii}(1 - m_i)) - \sum_{j \neq i} \frac{n_j}{n_i} \mathbb{E}_{v_j}[S_j(v_j)|v_i]. \quad (17)
$$

Here the seats are a function of m_i. Of course, we would like the seats to be a function explicitly of v_i to get our seat-vote curve. You can write m_i as a function of the vote share simply by inverting equation (1).

$$
m_i = \frac{1}{2} + \tau_i \left(\frac{\pi_{Ii} + 2\pi_{Di} - 2v_i}{\pi_{Ii}}\right) \quad (18)
$$

Plugging this in, you can get the seats as a function of the vote share, call it $\hat{S}_i(v_i)$.

$$
\hat{S}_i(v_i) = \frac{1}{n_i}\left(\frac{1}{2} + (\pi_{Di} - \pi_{Ri})\left(\frac{1}{2} - \tau_i\right) + 2\tau_i\left(v_i - \frac{1}{2}\right) - \sum_{j \neq i} n_j \mathbb{E}_{v_j}[S_j(v_j)|v_i]\right). \quad (19)
$$

$\hat{S}_i(v_i)$ would be the best response for state i if they were not constrained to choose seat shares between 0 and 1. This equation, however, will frequently lie outside the interval $[0, 1]$. Since the objective is quadratic, the solution for the best response function will still be very simple. With a quadratic objective, the optimum will just be at the boundary closer to the unconstrained optimum.

Proposition 1. *The equilibrium seat-vote curve for state i is*

$$S_i^*(v_i) = \begin{cases} 0 & \text{if } \hat{S}_i(v_i) < 0 \\ \hat{S}_i(v_i) & \text{if } \hat{S}_i(v_i) \in [0,1] \\ 1 & \text{if } \hat{S}_i(v_i) > 0. \end{cases} \tag{20}$$

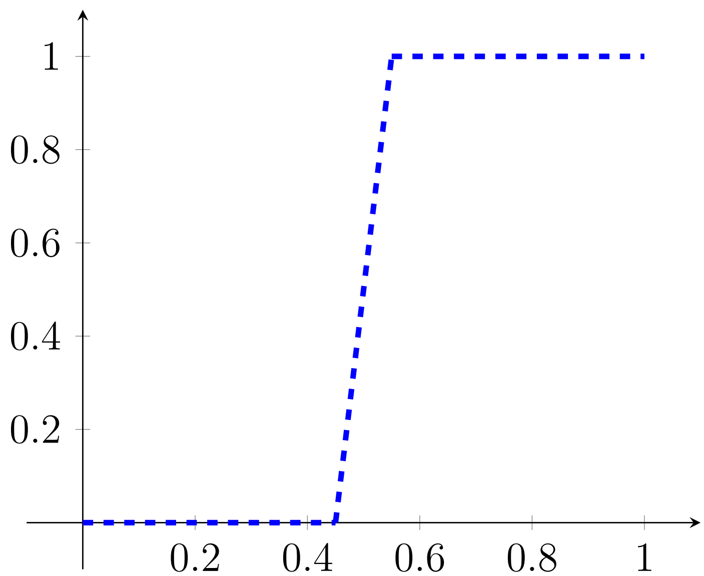

Figure 2: The optimal seat-vote curve.

The shape of a typical equilibrium seat-vote curve can be seen in figure (**??**). In contrast to the single state setup, seat-vote curves are S-shaped in equilibrium with high responsiveness. This can be seen from equation (19). Assume that m_i is drawn independently from m_j for every state j. You can see already that the seat-vote curve for state i is very steep in the middle.

$$\frac{\partial S_i^*}{\partial v_i} = \frac{2\tau_i}{n_i} \tag{21}$$

The first component, $2\tau_i < 1$, comes from the fact that a change in vote share represents a smaller change in average political preference in the state and the seat-vote curve should be less responsive. The second component, $\frac{1}{n_i}$, comes from the fact that state i is only a small part of the national congress and to get congress to shift by 1 percent the state needs to shift their representatives by an average of 50 percent. This is the key difference in the equilibrium model. States have highly responsive seat-vote curves because they know they are small and need a dramatic action the make any sizable impact on a national scale. The same insight is frequently heard when discussing the U.S. electoral college. Nearly every state choose to allocate their electoral college votes according to a winner-take-all vote.

Notice that the incentive to act dramatically (have a highly responsive seat-vote curve) is greater for small states than for large states. California can make a large impact on Congress flipping only a fifth of their representatives, while Minnesota needs to flip all of them to have a similar impact. The responsiveness of the equilibrium seat-vote curve is thus inversely proportional to the size of the state (measured in terms of the number of representatives allocated to the state).

Another difference from the single state model comes from and additional term in equation (19).

$$-\sum_{j\neq i} n_j \mathbb{E}_{v_j}[S_j(v_j)] \tag{22}$$

This term is the only way that states directly respond to the the action of other states. You want to adjust how many Democrats you elect to account for what you think all the other states are going to do. If you think the other states are going to elect a lot of Democrats, you elect fewer. If you think the other states will elect a lot of Republicans, you elect more Democrats. Even a state that leans Democrat could find it optimal to elect more Republicans than Democrats if they believe the rest of the country leans significantly more Democrat than they do. Notice that this incentive does not impact the responsiveness of the equilibrium seat-vote curve. This only shifts the level of the curve, not the slope.

Proposition 1 describes only the best response function of each state. In the appendix I solve this system to equations to show existence and completeness of an equilibrium. That analysis isn't important for the main results though because we see that the slope of a state's seat-vote curve is independent of the other states' actions. Thus, we already know everything about equilibrium responsiveness.

3 Empirical Results

We would now like to empirically confirm the existence of the main insights from the theoretical model, namely that seat-vote curves are highly responsive and that this responsiveness is inversely proportional to the state's size.

3.1 Methodology

First I do a simple estimation of the seat-vote curves in each state. I use data from the Cooperative Congressional Election Study from 2006 to 2019. This give me survey data from two different redistricting cycles. From this I aggregate across each of the 435 congressional districts what fraction self report as Democrat, Republican, or Independent/Not Sure. The goal is to estimate seat-vote curves by simulating many election results. As in the model, I suppose that the self reported Democrats will vote for the Democrat candidate, the reported Republicans will vote for the Republican candidate, and that the Independents and Not Sure respondents could vote either way.

We want to estimate the share of seats Democrats would win across all possible vote shares. To do this, I randomly draw the fraction of Independents to vote Democrat in each district 10,000 times. In each draw, you can compute how many districts in the state Democrats won and what fraction of the overall vote in the state Democrats won. This gives 10,000 simulated elections. The smoothed version of these points is the estimated seat-vote curve . I take the average fraction of state's seats earned for each level (1 percent window) of the state vote share.

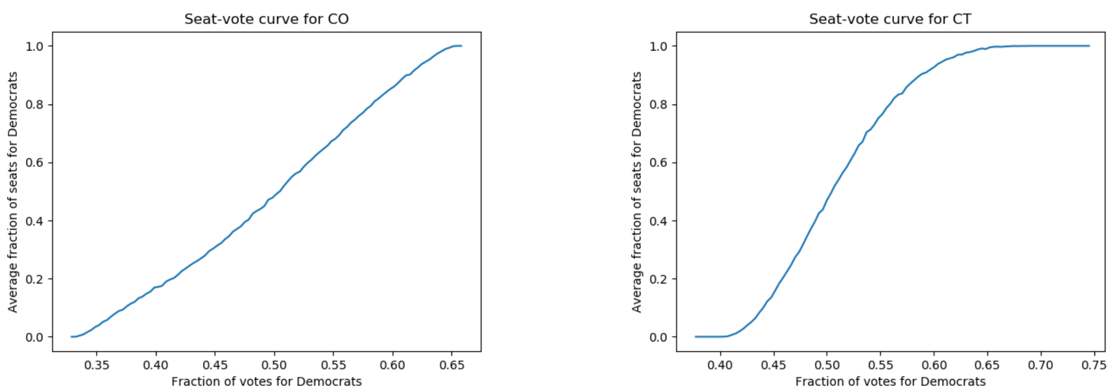

Figure 3: The estimated seat-vote curves for Colorado and Connecticut. Connecticut's curve is much steeper in the middle than Colorado's. Connecticut has the more responsive curve.

Figure 3 shows the estimated seat-vote curves for two states. Call the responsiveness of a seat-vote curve the average slope of the curve on the interval $[.45, .55]$. Over the middle ranges of votes shares, this is how much the congressional seats respond to changes in the vote on average.

The intuitive proportional seat-vote curve would be a linear function with a slope of one. From the graph, the seat-vote curve in Colorado is nearly linear, but with a slope closer to three for intermediate values of the vote share. Connecticut's seat-vote curve has more of a slanted "S" shape. The is a very high slope in the intermediate values of the vote share. Between vote shares of 45 percent and 55 percent the slope averages about 6.5 or a little more than twice as steep as Colorado's.

3.2 Results

Now we want to find the correlation between the responsiveness of a state's seat-vote curve and the inverse of the state's size. I use the average slope of the estimated seat-vote curve between vote shares of 45 and 55 percent as the measure of responsiveness. The size is just the number of representatives allocated to that state. Figure 4 shows a scatter plot of responsiveness versus size. The main theoretical results we wanted to verify jump out immediately. First note the high responsiveness. Every state has a responsiveness well above one. Next see the negative correlation between responsiveness and size. Furthermore, the non-linear relationship appears to be well approximated by the solid line which is one over the state's size (and constants).

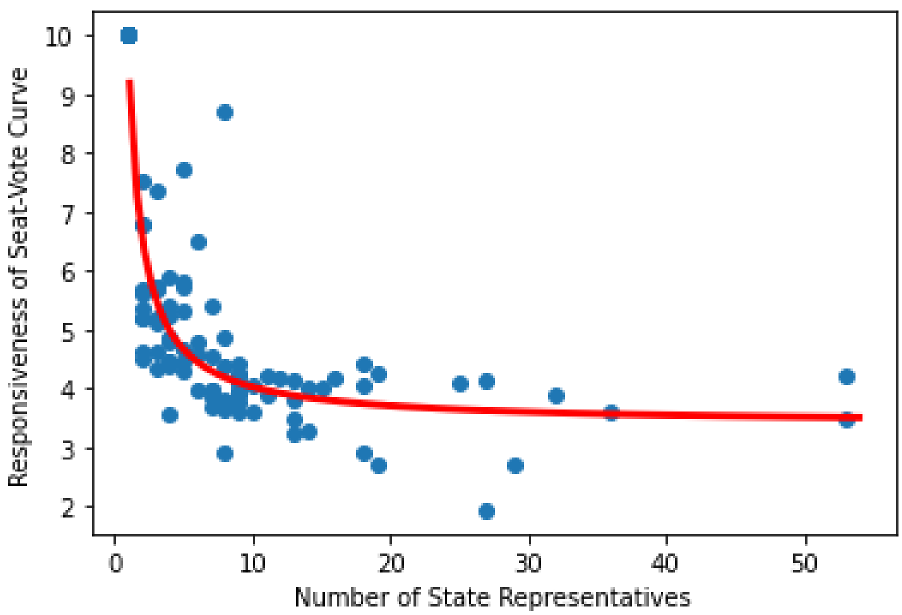

Figure 4: A scatter plot of seat-vote curve responsiveness verses the number of representatives allocated to a state (size).

A simple regression can be run to confirm the significance of the apparent relationship. The dependent variable is the responsiveness of each state. I estimate

$$y = \beta_0 + \beta_1 \frac{1}{N_i} + \beta_2 X_i + \epsilon_i \tag{23}$$

where N_i is the number of representatives allocated to the state and X_i are controls for year, political party in power, method of choosing districts, and the fraction of independents in the state. The smallest seven states were only allocated one representative each. Those states don't have a districting decision and were omitted from the regression.

Variables	(1)	(2)	(3)	(4)
N	-.0577***	-.2005***		
	(.012)	(.032)		
N^2		.0031***		
		(.001)		
$\frac{1}{N}$			4.8010***	6.1199***
			(.714)	(.819)
Constant	5.9964***	7.3417***	3.6451***	6.0947***
	(1.040)	(.962)	(.169)	(.874)
Controls	Yes	Yes	No	Yes
Observations	86	86	86	86
R-squared	.260	.428	.350	.442

Table 1: Regression output showing the relationship between the responsiveness of a state's seat-vote curve and its size.

There is always a large significant relationship between the responsiveness and the number of representatives. Smaller states have much steeper seat-vote curves. This relationship persist and is unaffected by which party is in control or how much control a party has in a state. In fact, but coefficients are larger when the controls are included.

We can also see that the non-linearity of the relationship is picked up well by an inverse proportionality to the size of the state. This is the same equation we derived from the theoretical model.

$$\frac{\partial Seats}{\partial votes} \approx a + b\frac{1}{n_i} + \epsilon_i \tag{24}$$

where N_i is the number of representatives (size) of state i.

3.3 Simulation

While it's clear that responsiveness is well predicted by the inverse of a state's size, is this a strategic choice from the states or just something that arises automatically from the way the system is set up? Consider the states that have one one representative for a moment (even though they were omitted from the regression). Their seat-vote curve is just a winner-take-all function. If the Democrats get 45 percent of the vote, they win 0 percent of the seats. If they get 55 percent of the vote, they win 100 percent of the seats. Thus, increasing the vote share from .45 to .55 increases the seat share from 0 to 1. This makes the responsiveness of these states equal to 10 which is higher than any other state. The smallest states (with only 1 district) have the highest responsiveness (10), but this wasn't due to any strategic choice. This just happened automatically. Is the entire trend automatic? Will states with fewer representatives have higher responsiveness automatically or is this a quirk specific to states with only one representative?

I run simulations of randomly assigned districts to find if the trend arises automatically, and I am able to reject the hypothesis that the trend in the data is driven by circumstance. For every state size (from 2 to 53 districts), I randomly assigned the fraction Democrats, Republicans, and Independents in each district. With the randomly assigned districts, I estimated the seat-vote curve and responsiveness by simulating a large number of elections as in the previous section. I did this 100 times for each possible state size. Finally, I regress the responsiveness on the inverse of the number of districts in the randomly simulated data.

Variables	Coeff.	Std. err.
$\frac{1}{N}$	2.2930	0.160
$Constant$	1.4008	0.016
Observations	2686	
R-squared	.071	

Table 2: Regression output showing the relationship between the responsiveness of a state's seat-vote curve and its size in randomly generated districts.

You can see from the regression output that some effect does arise from the setup of the districting problem. The average responsiveness in the randomly assigned districts is greater than 1, the coefficient on the inverse of the number of districts is greater than 0 (2.2930), and these values are both statistically significant. However, we also see that both coefficients are significantly smaller than in the actual data where they were 3.6451 and 4.8010. Since those coefficients had a standard error of .714 and .169 respectively, we can clearly reject the null hypothesis that the high average and the relationship between responsiveness and state size are generated by the setup of the problem. Additionally, The R-squared in the regression with simulated data is only .071, compared with .350 in the regression with the real data without controls. To further illustrate the point, I've included a plot where you can see that the same trends exist in the simulated data but at a smaller order of magnitude.

4 Welfare

Even though the described strategies constitute the only equilibrium, they don't give the highest total payoffs in the game. They aren't socially optimal. Let us construct the socially optimal policy if a planner was in charge of all 50 states. The objective function would look

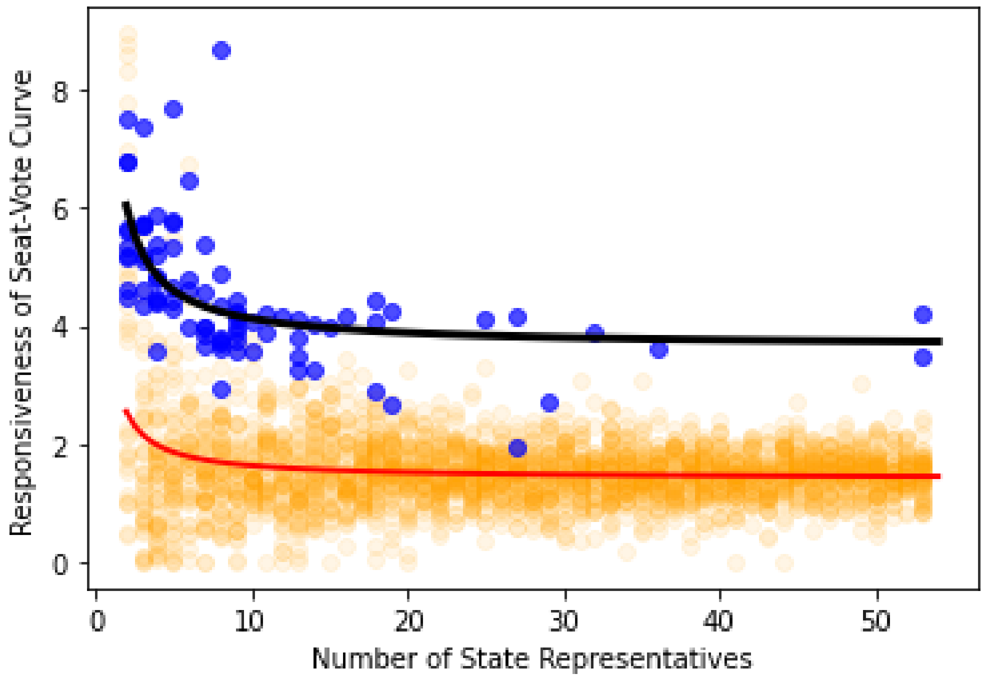

Figure 5: The lighter dots (orange) are from the simulated data. The darker dots (blue) are from the actual data. Each also has a regression line.

similar, but it would account for the citizens in every state.

$$\max_{\theta} \quad -\sum_{i=1}^{M} n_i \mathbb{E}_v \left[\pi_{Di} \left(\theta\right)^2 + \pi_{Ri} \left(1-\theta\right)^2 + \pi_{Ii} \int_{m_i-\tau_i}^{m_i+\tau_i} \left(\theta - x\right)^2 \frac{dx}{2\tau_i} \right] \tag{25}$$

Differentiate this with respect to S to solve for the optimum.

$$\frac{\partial W}{\partial \theta} = 2\sum_{i=1}^{M} n_i \pi_{Di}(\theta) - 2\sum_{i=1}^{M} n_i \pi_{Ri}(1-\theta) - 2\sum_{i=1}^{M} n_i \pi_{Ii} \int_{m_i-\tau_i}^{m_i+\tau_i} \left(\theta - x\right) \frac{dx}{2\tau_i} \tag{26}$$

Evaluating the uniform integral and simplifying,

$$= 2\sum_{i=1}^{M} n_i \pi_{Di} + 2\sum_{i=1}^{M} n_i \pi_{Ii} (1 - \mathbb{E}[m_i]) - 2(1-\theta) \tag{27}$$

since $\sum_{i=1}^{M} n_i = 1$.

Putting the planner on the same footing as the states in the game, let them choose an entire seat-vote curve for the nation. This makes the expectation go away. Now, substitute in for m_i as before to get an expression in terms of vote shares, v_i. Setting this equal to zero gives the optimal policy.

$$
\begin{aligned}
1-\theta^* &= \sum_{i=1}^{M} n_i \left(\pi_{Di} + \pi_{Ii}(1-m_i)\right) && (28)\\
&= \sum_{i=1}^{M} n_i \left(\pi_{Di} + \pi_{Ii}\left(\frac{1}{2} - \tau_i \frac{\pi_{Ii} + 2\pi_{Di} - 2v_i}{\pi_{Ii}}\right)\right) && (29)\\
&= \sum_{i=1}^{M} n_i \left(\left(\frac{1}{2} - \tau_i\right)\pi_{Ii} + (1-2\tau_i)\,\pi_{Di} + 2\tau_i v_i\right) && (30)\\
&= \frac{1}{2} + \sum_{i=1}^{M} n_i \left((\pi_{Di} - \pi_{Ri})\left(\frac{1}{2} - \tau_i\right) + 2\tau_i\left(v_i - \frac{1}{2}\right)\right) && (31)
\end{aligned}
$$

Assuming τ_i is the same in each state, this would be a linear function of the aggregate vote share with a slope equal to 2τ. It's also easy to impliment. Since it's linear, we would get this policy by each state having a seat-vote curve equal to

$$
S_i(v_i) = \frac{1}{2} + (\pi_{Di} - \pi_{Ri})\left(\frac{1}{2} - \tau\right) + 2\tau\left(v_i - \frac{1}{2}\right). \tag{32}
$$

with that same low slope. Notice that this is the same as equation (6), which is the same as in Coate and Knight (2007) (CK). This is where each state simply chooses their most preferred policy. If a state wants the policy to be 60 percent Democrat, then they just elect 60 percent Democrats.

Take the following example. Imagine that everyone in the room was to write down what temperature they would like room. Then the thermostat is set to the average of all the votes. The social optimum is obtained if everyone just honestly writes their most preferred temperature. However, you may have an incentive to write something different. If you like to the room to be 67 degrees and you think that's a little colder than most people like, you have an incentive write a vote that is much colder. You want the average of the votes to be 67. So, you might write something more extreme like 60 degrees.

In equilibrium, political districting is the same way. This mild seat-vote curve is what will maximize national welfare, but each state has an incentive to deviate. When a state gets a lot of votes for Democrats, they think their state is likely more Democrat than the average state. Then to make the policy a little more Democrat the state wants to elect a lot more Democrats. Regardless of what the other states are doing, each state has an incentive to deviate by playing a highly responsive seat-vote curve.

This is a prisoner's dilemma situation though, because in equilibrium every state is worse off. No state prefers the equilibrium to the collusive outcome where all states choose modest

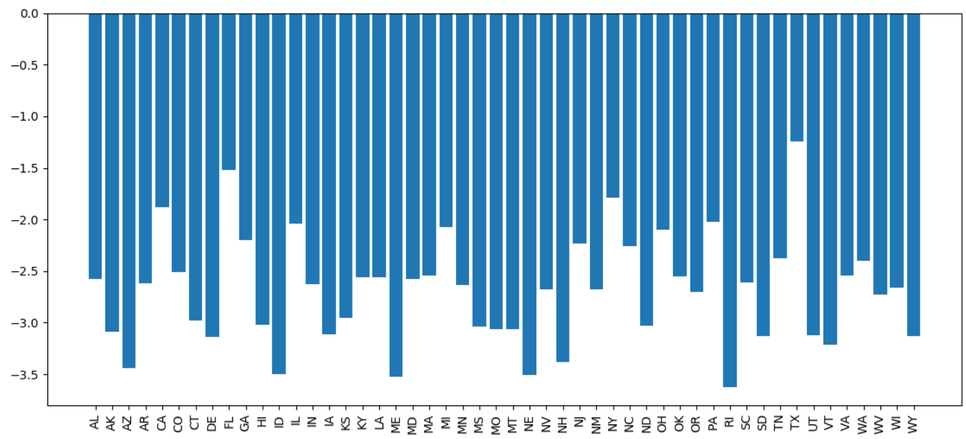

Figure 6: The loss each state faces in equilibrium from their expected utility in the social optimum.

seat-vote curves.

The smallest states have the largest incentive to deviate. We saw in equation (19) that the smaller the state, the steeper they would like their seat-vote curve. In the limit, as a state grows in size its best response would approach the socially optimal curve. The state would play the social optimum if their own elected congress members could move policy one for one. States only play the more extreme curves because the congress members they elect are only a small part of making the policy. This becomes a larger and larger factor as a state shrinks in size.

5 Winner-Take-All

It may be that the optimal seat-vote curve in the model is difficult to implement in the world, requires a lot of information, or that the model assumptions don't line up exactly with your view of the world. There is a very simple seat-vote curve that can always be implemented, relies on no model assumptions, requires no information, and is very close to the optimal value. It is a winner-take-all election.

Take a seat-vote curve equal to

$$S_i(v_i) = \begin{cases} 0 & \text{if } v_i < .5 \\ 1 & \text{if } v_i \geq .5 \end{cases} \tag{33}$$

That is, if the Democrats get more than 50 percent of the vote in the state, all of the state's representatives will be Democrats. Otherwise, all the representatives will be Republicans.

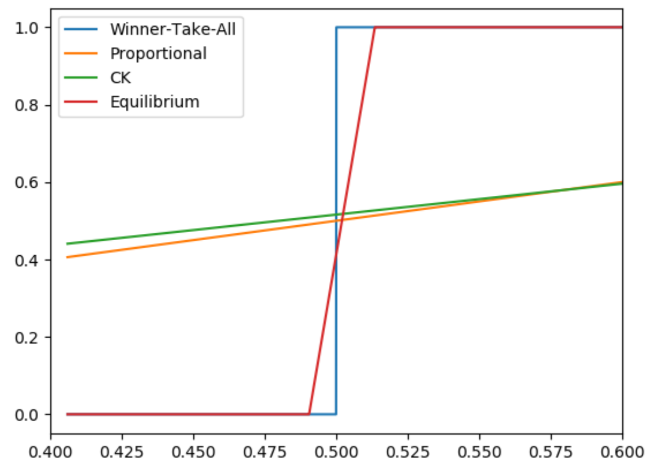

Figure 7: This is a graph of several seat-vote curves for the state of Minnesota. The "proportional" curve is a linear seat-vote curve with a slope of one. The "CK" curve is the socially optimal seat-vote curve. "Equilibrium" is the curve Minnesota would choose as their best response in equilibrium play. "Winner-take-all" is simply an indicator function for vote shares above 50 percent.

This doesn't require any knowledge about the voting population in the state. It is also extremely simple and easy to understand. It could be done without creating any districts. If districts are desired, all the state needs to do is randomly assign each citizen a district number regardless of geography. Each district in the United States is composed of about 700,000 people. By the law of large numbers, each district would then have pretty close to the same fraction voting Democrat in each election. This would implement the winner-take-all seat-vote curve.

While winner-take-all is not the best response, it is very close to the best response. Every state's seat-vote curve is flat at zero, then increases rapidly to one, then is flat at one. For the true best response the increase isn't infinitely steep like a winner-take-all, but the average slope is about 50. The graph shows these two curves for Minnesota.

In fact, nearly all the gains from deviating to the best response from the previous section can be obtained from deviating to a winner-take-all function.

In fact the strategies are almost always the same. In the best response, the fraction of Democrats elected is either 0 or 1 more than 80 percent of the time in each state already.

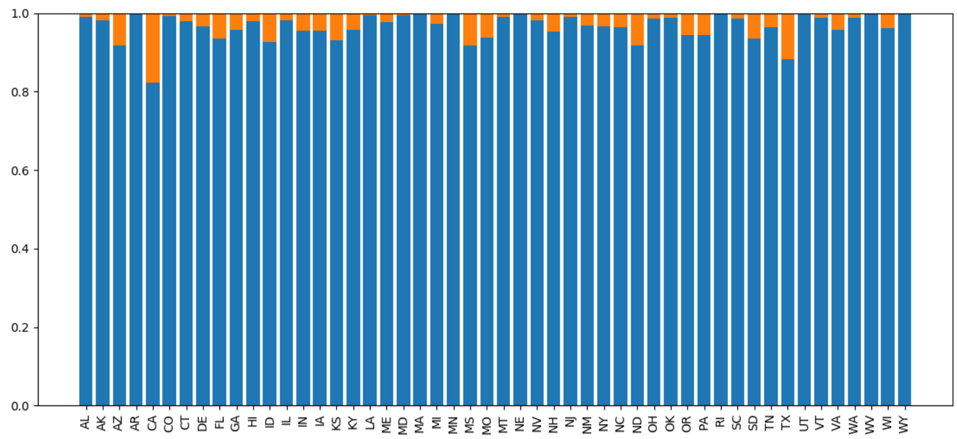

Figure 8: The gain from deviating from the social optimal to winner-take-all as a percentage of the gain from deviating to the best response.

6 Extensions

Here I go through a few major extensions of the model. The extensions aren't to show robustness of the main results. Rather, these are changes to the model that are frequently asked for and I believe contain important insights in their own right.

6.1 Median Congress Member

One assumption of the model up to this point has been that the policy chosen is equal to the mean preference of all the elected congress members. One might not believe that all the members of congress are so readily willing to compromise. Perhaps, the majority party can disproportionately pull policy in their favor. In this section, I consider the opposite extreme from the rest of the paper. Suppose that the policy chosen is equal to that of the median member of congress. This would be if the majority party has complete control over choosing policy and doesn't need to compromise to please the minority. The optimal seat-vote curves will be even simpler than before but not that different intuitively.

Now, there are only two real outcomes of the game. Either the Democrats win the majority or the Republicans win the majority. State i first needs to determine which of those it prefers. State i will prefer Democrats win the majority as long as

$$\pi_{Ri} + \pi_{Ii} \int_{m_i-\tau_i}^{m_i+\tau_i} x \frac{dx}{2\tau_i} \geq \pi_{Di} + \pi_{Ii} \int_{m_i-\tau_i}^{m_i+\tau_i} (1-x)^2 \frac{dx}{2\tau_i}. \tag{34}$$

Computing the integral and simplifying yields a simple equation. The state will prefer a

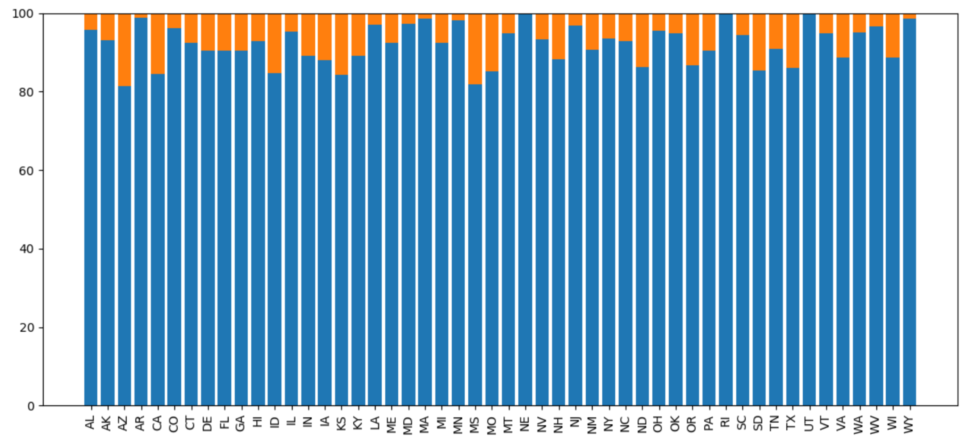

Figure 9: The fraction of the time the best response and a simple winner-take-all choose the same number of Democrats.

Democrat majority as long as

$$v_i \leq \frac{1}{2} + (\pi_{Ri} - \pi_{Di}) \left(\frac{1}{4\tau_i} - \frac{1}{2} \right). \tag{35}$$

The unique best response is for the state to give all its representatives to the Democratic Party when the vote share is above that cutoff and give all representatives to the Republican Party when it is below the cutoff. This is the case regardless of what any other state is doing. The optimal seat-vote curve for every state is a winner-take-all election. The difference from a standard winner-take-all though is that the cutoff for the Democrats to win my not be at exactly 50 percent. This is because not every vote is created equal. A firm member of the Democrat party counts for more than an independent voter that is leaning slightly Republican. Thus, the cutoff will be at a point that favors the party that has a larger political base in the state.

In this game, the socially optimal policy is not unique. Any congress with a Democratic majority is the same regardless of how strong that majority after all. The socially optimal national seat-vote curve will be any curve that is above one half if inequality 35 is satisfied and is below one half if inequality 35 is not satisfied. This could be a winner take all election with that cutoff point. This could also be the socially optimal seat-vote curve from the previous section when policy was decided by the mean congress member. This can still be implemented by every state playing the moderate “CK” seat-vote curve as before. However, every state doing a winner take all election does not implement a socially optimal national seat-vote curve.

Note that the equilibrium strategy for each state no longer depends on the size of the state. The winner-take-all election would look the same for Wyoming and Texas or for Rhode Island and California. Even if a state had complete control over the policy chosen, they would still find it optimal to choose this same seat-vote curve. That is to say, this curve is a dominant strategy for all states regardless of size and regardless of what every other state might be doing.

In truth, policy is not controlled entirely by the median congress member or by the mean congress member. It is likely something in between those two. In this model I found the optimal seat-vote curve for each state to be similar for each of the two extremes, and thus expect it to be similar for any imtermediate policy selection method.

6.2 Partisan Gerrymandering

In many states, the party in control of state congress can draw the districts. In this section, I will study what a districter would do if they wanted to maximize the expected number of seats that a given party will win. This isn't really an equilibrium problem. What districts other states draw, and who they elect doesn't enter into the objective in any way. We can simply solve a state's districting problem in isolation.

6.2.1 Example

Consider a state that has 50 percent Democrats, 25 percent Republicans, and 25 percent Independents. The intuitive outcome is that the Democrats should get 50 to 75 percent of the seats in congress and the Republicans should get 25 to 50 percent of the seats, depending on how the Independents vote. If the districts are drawn to maximize the expected number of seats for either party, the outcome will look very different. First, if the Democrats are in charge of drawing the districts, they can win all the seats in this model. All they need to do is make every district look just like the state as a whole. Each district will be 50 percent Democrat, 25 percent Republican, and 25 percent Independent. It doesn't even seem like an extreme gerrymender on the face of it. They simply make every district identical and representative of the state. However, since the Democrats are now garunteed to have at least 50 percent of the vote in every district, they will win all the congressional seats.

What if the Republicans are in charge of drawing the districts. They can use a concept called "packing" to win some of the seats. "Packing" is when you group together supporters of the opposition into a single district. If you're going to lose a district, you might as well lose big. They can simply puting all the Democrats together in the first half of the districts. Then in the other half of the districts they can employ a strategy called "cracking". "Cracking" is when you split up your own party's supporters so you can win more districts. If you're already going to win a district, each additional vote you get in that district is wasted. In the second half of the districts they can make each district an equal mix of Republicans and Independents. Since they have garunteed at least half the votes in each of these districts, they will win them all. This districting strategy is simple and it gets the Republicans half the seats.

However, the Republicans can do even better than this. It might seem like since half the state is certainly going to vote Democrat, Republicans can never win more than half the seats, but they can. For simplicity, assume that ties are broken in the Republican's favor. Throughout the section, I will always assume that ties are broken in favor of the party drawing the districts. Otherwise they would simply need to put one additional voter from their party in each of this districts to insure a win.

Consider the following districting. In one quarter of the districts they follow a packing strategy. One quarter of the districts are made up entirely of Democrats. Then in half of the districts they use a cracking strategy. They do it a little differently than before. If you're going to win a district for sure, you might as well make sure the rest of the votes are against you. You don't want to waste any votes. So, in this half of the districts there will be an equal mix of Republicans and Democrats. The Republicans will win all of these districts. Finally, the remaining quarter of the districts are made up entirely of Independents. On average Republicans will win half of these districts.

This plan gives the Republicans five-eighths of the seats in congress on average. This is the optimal districting plan in this example. If a state used to be heavily Republican, they may have a majority of Republicans in their state congress. This districting shows how they can maintain their majority in congress even if the population in the state changes to be heavily Democrat.

6.2.2 Optimal Districting

Generally, I will show that the optimal districting for maximizing the expected number of seats for a given party will have the same form as the example.

There are three types of voters. The distributions of these voters can be represented by a simplex as in figure 10. This problem is very similar to a Bayesian Persuasion problem. In Bayesian persuasion, the sender is looking at ways to split a up distribution (the prior beliefs) into multiple other distributions (posterior beliefs) that must average out to the first distribution (by Bayes Rule). In this gerrymandering problem, the districter is looking at ways to split up a distribution (the population of voters in the state) into multiple other distributions (district populations) that add up to the state population. Just like in Bayesian

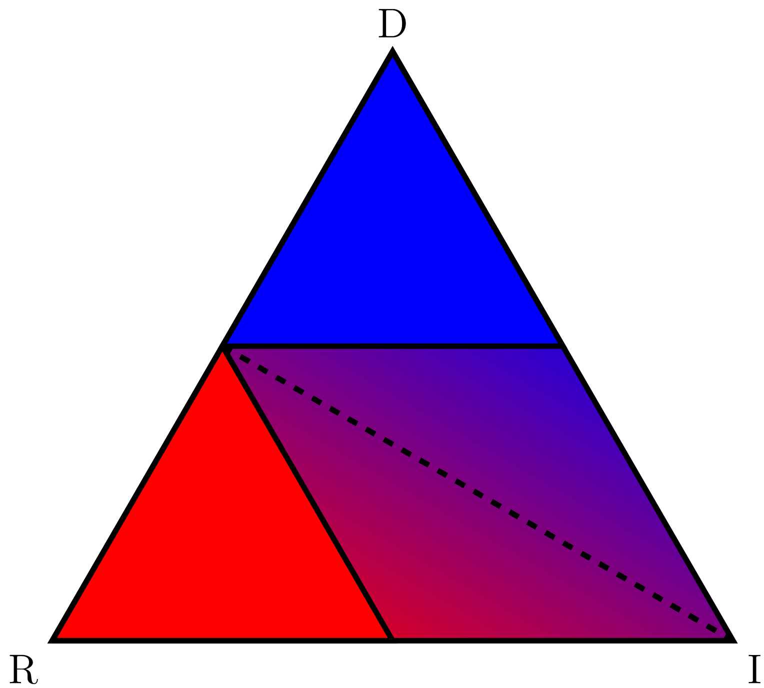

Figure 10: The distribution of voters in the state.

Persuasion, the optimal value can be thought of as a "concavification" on this simplex.

Suppose the districter wants to maximize the expected number of seats for the Republican party. The distributions in figure 10 can be broken up into a few segments. The lower left triangle in the figure (red) are distributions where the Republicans have at least 50 percent of the population. They would win these districts with certainty. The upper triangle (blue) are distributions where Democrats have at least 50 percent of the population. They would lose these districts with certainty. The rhombus of remaining distributions (purple) are distributions where either party could win depending on how the Independents vote. The distributions along the dashed line are equally likely to be won by Republicans or Democrats. The likelihood of Republicans winning transitions from one to zero as you move up and to the right through this region.

To think about concavification, picture this triangle as a 3D object with the colors representing height. The lower left triangle (red) has a height of one. The upper triangle (blue) has a height of zero. The rhombus (purple) linearly connects the two triangles. Now imagine taking a cloth, laying it on top of the object, and pulling the edges down tight. This is the concavification of the function. The height of this concavification is the maximum expected number of seats Republicans can win with the optimal districting. The points where the concavification is equal to the original function (where the cloth is touching the 3D object) are the distributions that are used in the optimal districting schemes. If the state's population is on one of those points, it is optimal to make every district have a distribution identical to the state as a whole. If the state's population is not one of those points, the optimal districts with break up the state into different districts that are all among those points.

Let $U \subset \Delta^2$ be the subset of distributions containing all distributions with $\pi_R \geq \frac{1}{2}$ and the other two extreme points, $\pi_D = 1$ and $\pi_I = 1$. A district is dominated if it can be segmented into two or more districts to get a strictly higher payoff.

Lemma 1. *A district is undominated if and only if its distribution of voters is in U.*

First see that each of these points in U is undominted. If you are at one of the corners, all voters are the same, so there is no way to split the voters into different districts. If $\pi_R \geq \frac{1}{2}$ then your party will already win with certainty and you get the maximal payoff of one.

Now see that every other distribution is dominated by some splitting. First, distributions where $\pi_D \geq \frac{1}{2}$ (upper blue triangle) would get a payoff of zero. Any splitting with a strictly positive payoff will dominate. For example placing all the Democrats in their own districts and all the Independents and Republicans in another district. There is a positive probability of winning the districts without any Democrats.

Next consider the distributions with $\pi_R < \frac{1}{2}$ and $\pi_D < \frac{1}{2}$ (the purple rhombus). The probability that the Republicans will win a district with a distribution of voters in this area is

$$Pr\{winning\} = \frac{\frac{1}{2} - \pi_D}{1 - \pi_D - \pi_R}. \tag{36}$$

Along horizontal lines, this is a convex function. In other words, holding fixed π_D, $Pr\{winning\}$ is a convex function of π_R. Since these horizontal bars are convex, splitting to the endpoints (distributions with $\pi_R = \frac{1}{2}$ and distributions with $\pi_R = 0$) will give a strictly higher payoff.

The only distributions left to check are the boundary where $\pi_R = 0$ and $\pi_D \leq \frac{1}{2}$.

$$\begin{aligned} Pr\{winning\} &= \frac{\frac{1}{2} - \pi_D}{1 - \pi_D} && (37)\\ &\leq \frac{\frac{1}{2} - \pi_D + \frac{1}{2}\pi_D^2}{1 - \pi_D} && (38)\\ &= \frac{1}{2}\frac{(1 - \pi_D)^2}{1 - \pi_D} && (39)\\ &= \frac{1 - \pi_D}{2} && (40) \end{aligned}$$

Note that $\frac{1-\pi_D}{2}$ is the payoff from splitting the Democrats and Independents into their own separate districts. Thus, distributions on this boundary are dominated by splitting into distributions where $\pi_D = 1$ and $\pi_I = 1$.

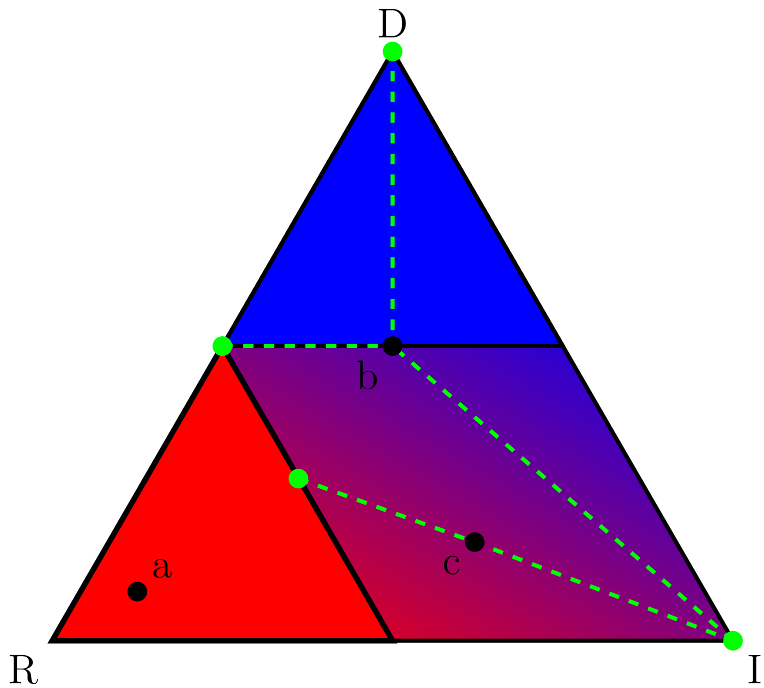

Figure 11: Splitting the state's population into optimal districts.

The optimal strategy requires splitting your state population up into only undominated districts. If your state population is in U, you simply make every district identical. If your state population is in the purple rhombus below the dashed line (more Republicans than Democrats), the optimal splitting is a combination of the bottom right corner (100 percent Independents) and the point on the edge of the lower left triangle (50 percent Republicans) that is straight across the prior from the corner. Republicans will win all of the latter districts and have a 50 percent chance of wining the former districts. If your state population is anywhere above the dashed line (more Democrats than Republicans) the optimal splitting is a combination of the point where the lower triangle meets the upper triangle (50 percent Republican and 50 percent Democrat), the bottom right corner (100 percent Independent), and the top corner (100 percent Democrat). Republicans will win all of the first type of districts, have a 50 percent chance in the second type of districts, an lose all of the last type of district. Each of these scenarios is graphed is shown in figure 11.

The optimal districting strategy can be stated simply as follows. You start with the districts you are going to win for sure. Fill as many districts as possible half full with Republicans. Fill the other half of those districts with Democrats (cracking). If there aren't enough Democrats, continue filling with Independents. If there aren't enough Independents, fill remainder with Republicans. After filling these districts entirely, all remaining Democrats are placed into their own districts (packing). Then, all remaining Independents are placed into their own districts. So, there are districts that are 50 percent Republican, and 50 percent Democrat or Independent (Democrat is preferred). There are districts that are 100 percent Democrat. Finally, there are districts that are 100 percent Independent.

Proposition 2. *The maximum expected fraction of seats Republicans can win is*

$$v_R = \begin{cases} 1 & \text{if } \pi_R \geq \frac{1}{2} \\ 2\pi_R + \frac{1}{2}\left(\pi_I - (\pi_R - \pi_D)\right) & \text{if } \pi_D < \pi_R < \frac{1}{2} \\ 2\pi_R + \frac{1}{2}\pi_I & \text{if } \pi_R \leq \pi_D. \end{cases} \tag{41}$$

This can be rewritten as

$$v_R = \min\left\{1\ ,\ 2\pi_R + \frac{1}{2}\left(\pi_I - \max\left\{0,\ \pi_R - \pi_D\right\}\right)\right\}. \tag{42}$$

If there are fewer Republicans than Democrats, you fill as many districts as possible with half Republicans and half Democrats. You will win all these. There are $2\pi_R$ of these districts. Then you have all the Independents in their own districts. You win half of these. There are π_I of these districts. Finally, the rest of the Democrats are in their own districts and you don't win any of those. Hence, your expected number of seats is $2\pi_R + \frac{1}{2}\pi_I$.

If there are more Republicans than Democrats, you do the same strategy. The difference is that you now have to put some Independents in the first type of district (the guaranteed wins) because there aren't enough Democrats to fill them. From the payoff you need to subtract off the Independents put in the first type of district so they aren't double counted.

This will continue until you are able to fill all the districts half way with Republicans. Then the payoff is flat at one. Any additional Republicans are superfluous.

7 Conclusion

Literature Review (if necessary)

The closest paper to mine is Coate and Knight (2007). In their paper, they seek the socially optimal seat-vote curve for a state. This is done for elections to the state congress instead of the national congress. In the state congress, the seat-vote curve determines all members of congress and thus has complete control over the policy. My model of a state is the same as theirs but I include it as one of 50 states that need to choose policy together. In mine the motive arises for a state to have a highly responsive curve to balance with the expected representatives from the other states.

Another paper that compliments mine well is Carson and Crespin (2004). Across states different groups are put in charge of drawing the congressional districts. They show that in states where a bipartisan committee or a court is in charge of redistricting elections are more competitive than when state legislature does the redistricting. My model is closer to the motivation of a bipartisan committee or court. Since having more competitive elections translates to a steeper seat-vote curve, this evidence is supportive of my model.

Bergmann et al. (2015) looks at a monopolist problem. They consider ways to divide up the population for price discrimination. Mathematically, it is very similar to how I divide up the population into districts.

Another closely related paper that deals with gerrymandering is Friedman and Holden (2008). In their paper, the districts are drawn to maximize the expected number of seats a given party wins. They find that you should sometimes "pack", but never "crack". In their paper you only see a signal of how each voter will vote. Different strength of signals lead to a whole spectrum of voter types.

7.1 Conclusion

Gerrymandering is a topic fiercely debated every ten years when congressional districts are redrawn. It is important because it can have a significant impact on who gets elected and ultimately on what policies are put into place. A fundamental question we should ask about gerrymandering is what seat-vote curves would we expect to see in equilibrium and what are the socially optimal seat-vote curves.

In this paper, I present a model of states choosing seat-vote curves to compete with each other over the policy that is passed. I first solve for the optimal districting strategy of one trying to maximize the expected number of seats won by a given party. I showed that the party drawing the districts can win a significantly higher fraction of seats than their share of supporters in the population. The optimal districting strategy involved both "packing" and "cracking".

I then consider a district designer that cares only about maximizing the welfare of their state's citizens. I find that in equilibrium, every state chooses an extreme seat-vote curve to disproportionately affect policy in their favor. This motive is especially strong in smaller states. In equilibrium, the slope of each state's seat-vote curve is proportional to one over the number of representatives they have.

I show that there is a deadweight loss to society in equilibrium. If every state were to commit to playing a more modest seat-vote curve, a Pareto improvement could be had. However, each state has an incentive to deviate, making this a Prisoner's Dilemma.

I also find that the optimal seat-vote curve can be approximately implemented with a very simple rule. This is a winner-take-all election for the state's representatives. If policy is chosen entirely by the majority party in congress, then this winner-take-all election becomes exactly the unique optimal rule for each state.

I then estimate the seat-vote curves for each of the 50 states. I find that seat vote curves are highly responsive. I also find that small states choose steeper seat-vote curves than larger states. The slope of the seat-vote curve in a state is approximately proportional to one over the number of representatives that state has in congress. This is the same relationship found in the equilibrium model. This relationship holds after controlling for the political party in power.

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8 Appendix A

I have not yet completely characterized the equilibrium of the game. What we have is the best response of state i as a function of the seat-vote curves of all the other states. The equilibrium is the fixed point of all these functions. Notice in equation (19) that the other states' chosen curves only enter the best response in a very simple way. State i only cares about the expected fraction of seats the other states will elect from the Democratic Party, $\sum_{j\neq i} n_j E\left[S_j^*\right]$. This makes the system easy to solve.

For notational simplicity, let's call $s_i = \mathbb{E}[S_i^*(v_i)]$. Using equation (17) we see that the state will be in the interior portion of their seat-vote curve whenever

$$0 < \frac{1}{n_i}(\pi_{Di} + \pi_{Ii}(1 - m_i) - \sum_{j\neq i} n_j s_j) < 1. \tag{43}$$

Simplifying, the state will be in the highly responsive portion of the seat-vote curve as long as the median independent voter isn't too far to either extreme.

$$1 - \frac{1}{\pi_{Ii}}(n_i + \sum_{j\neq i} n_j s_j - \pi_{Di}) \ < \ m_i \ < \ 1 - \frac{1}{\pi_{Ii}}(\sum_{j\neq i} n_j s_j - \pi_{Di}). \tag{44}$$

Also, $S_i^*(v_i)$ will be equal to 1 whenever the median independent, m_i, is below that lower cutoff.

Now the goal is to take the expected value of equation (20) to get the average fraction of seats that go to Democrats in state i. There are three segments to the best response function we need to average over. On the first segment, Democrats get zero seats. So, this drops out of the equation. In the middle segment we integrate over the likelihood of each value. On the third segment, Democrats get all the seats. So, the contribution to the expectation is just the probability of being in this segment times 1. The likelihood of this is equal to the cdf of m_i at the cutoff. Since m_i is uniformly distributed on $[\frac{1}{2} - \epsilon_i, \frac{1}{2} + \epsilon_i]$, the probability that $S_i^*(v_i) = 1$ equals the following.

$$p_i = \frac{1 - \frac{1}{\pi_{Ii}}(n_i + \sum_{j\neq i} n_j s_j - \pi_{Di}) - (\frac{1}{2} - \epsilon)}{2\epsilon_i}. \tag{45}$$

Now we can compute the expected number of seats Democrats will win in state i.

$$s_i = \mathbb{E}[S_i^*(m_i)] \tag{46}$$

$$= \int_{1-\frac{1}{\pi_{Ii}}(n_i + \sum_{j\neq i} n_j s_j - \pi_{Di})}^{1-\frac{1}{\pi_{Ii}}(\sum_{j\neq i} n_j s_j - \pi_{Di})} \frac{1}{n_i}(\pi_{Di} + \pi_{Ii}(1 - m) - \sum_{j\neq i} n_j s_j)\ \frac{dm}{2\epsilon_i} \tag{47}$$

$$+\frac{1-\frac{1}{\pi_{Ii}}(n_i+\sum_{j\neq i}n_js_j-\pi_{Di})-(\frac{1}{2}-\epsilon_i)}{2\epsilon_i} \tag{48}$$

$$=\frac{\pi_{Di}+\pi_{Ii}-\sum_{j\neq i}n_js_j}{2\pi_{Ii}\epsilon_i}-\frac{\pi_{Ii}}{2\epsilon_i}\int_{1-\frac{1}{\pi_{Ii}}(n_i+\sum_{j\neq i}n_js_j-\pi_{Di})}^{1-\frac{1}{\pi_{Ii}}(\sum_{j\neq i}n_js_j-\pi_{Di})} m\ dm \tag{49}$$

$$+\frac{1-\frac{1}{\pi_{Ii}}(n_i+\sum_{j\neq i}n_js_j-\pi_{Di})-(\frac{1}{2}-\epsilon_i)}{2\epsilon_i} \tag{50}$$

$$=\frac{\pi_{Di}+\pi_{Ii}-\sum_{j\neq i}n_js_j}{2\pi_{Ii}\epsilon_i}+\frac{n_i}{2\pi_{Ii}\epsilon_i}(1-\frac{1}{\pi_{Ii}}(n_i+\sum_{j\neq i}n_jc_j-\pi_{Di})) \tag{51}$$

$$+\frac{1-\frac{1}{\pi_{Ii}}(n_i+\sum_{j\neq i}n_js_j-\pi_{Di})-(\frac{1}{2}-\epsilon_i)}{2\epsilon_i} \tag{52}$$

This gives a system of 50 equations and 50 unknowns. Solving for the 50 s_i terms that satisfy this system will finish the construction of the equilibrium. While this initially looks messy, we now have s_i written as a linear function of all s_j with $j \neq i$. We can write the equation simply as

$$As = b \tag{53}$$

where b is an $M \times 1$ vector with

$$b_i = \frac{1}{2}+\frac{2\pi_{Di}+\frac{3}{2}\pi_{Ii}-n_i^2-n_i+n_i\pi_{Ii}-n_i\pi_{Ii}\pi_{Di}}{2\pi_{Ii}\epsilon_i} \tag{54}$$

and A is an $M \times M$ matrix with

$$A_{ij} = \begin{cases} \frac{\left(\frac{n_i}{\pi_{Ii}}+2\right)n_j}{2\pi_{Ii}\epsilon_i} & \text{if } i \neq j \\ 1 & \text{if } i = j. \end{cases} \tag{55}$$

This A matrix is always full rank. This means that there always exists a unique solution. The solution gives us the final constants we needed in the best response function. Thus, it completes the equilibrium.