Equilibrium Political Redistricting (Equilibrium Seat-Vote Curves)

Isaac Swift

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Motivation

What is redistricting?

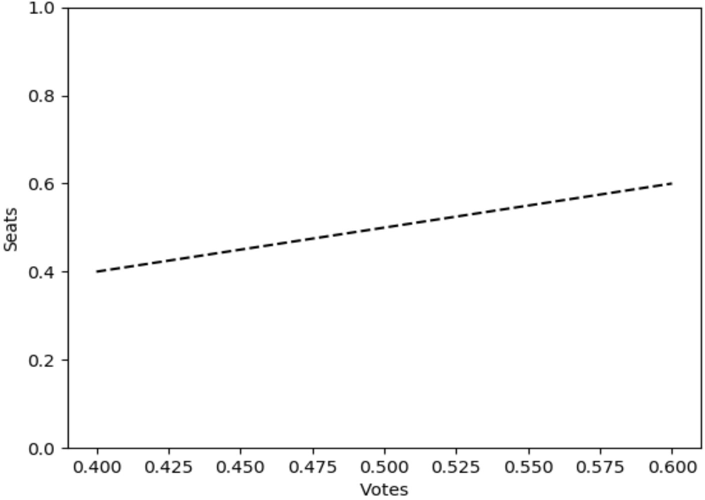
- The House of Representatives has 435 representatives from all 50 states.
- Each state gets to elect a number of representatives based on their population.
- They do this by dividing the state into districts and allowing each district to elect one representative.
- How the districts are drawn has a large impact on the representatives chosen by the state.

What is the goal when drawing districts?

- Pennsylvania supreme court - "closely reflects the partisan composition of the state"
- Arizona commission - "electorally competitive"
- Controlling party - maximize number of representatives the party wins

Seat-vote curve

Each redistricting map induces a seat-vote curve.



Seat-vote curve

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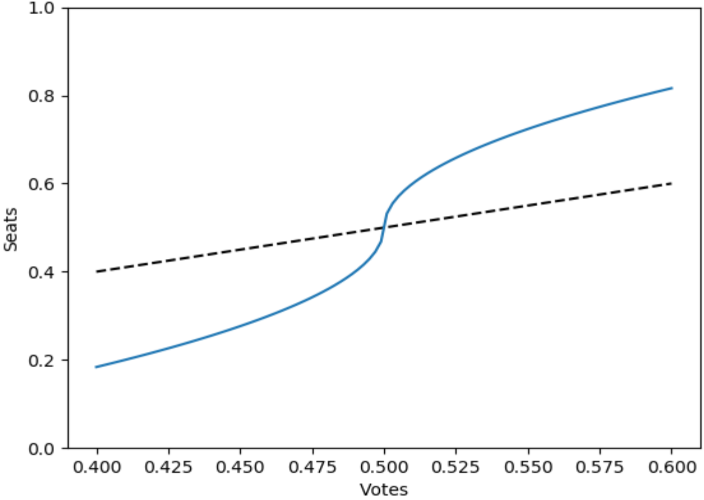
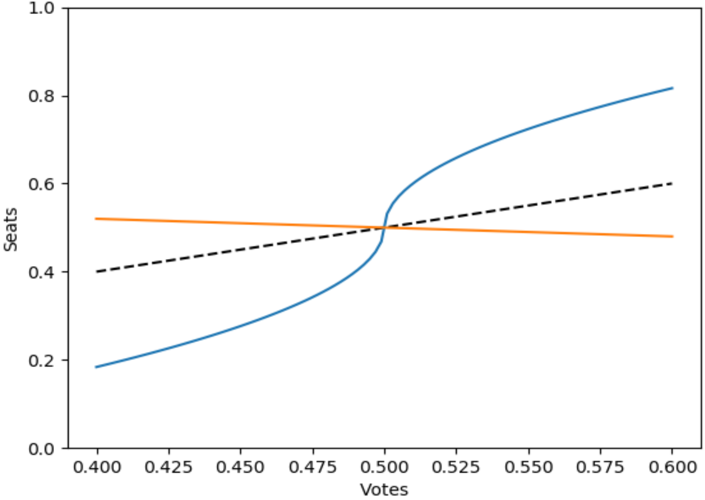
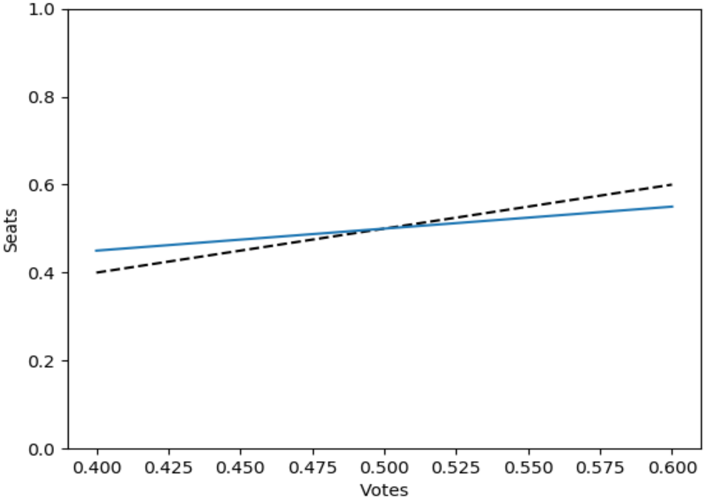


Seat-vote curve

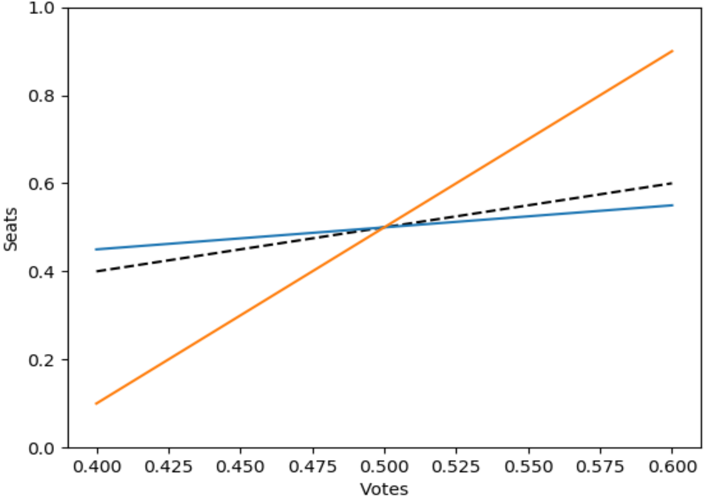
Each redistricting map induces a seat-vote curve.



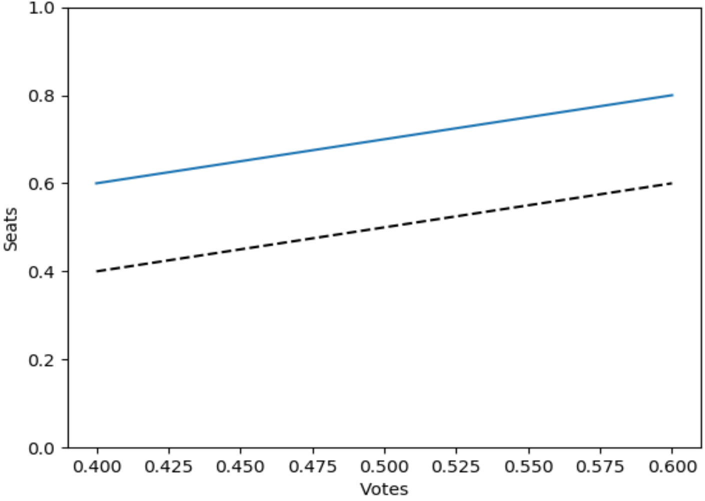
Responsiveness



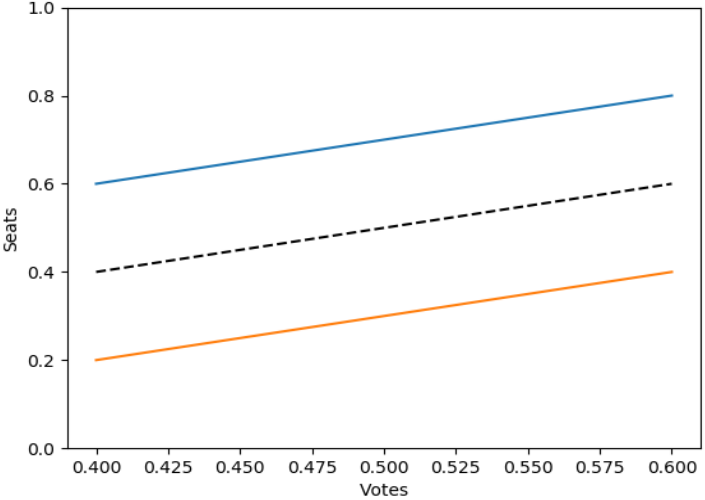
Responsiveness



Bias


1.0
0.8
0.6
0.4
0.2
0.0
Seats
0.400
0.425
0.450
0.475
0.500
0.525
0.550
0.575
0.600
Votes

Bias



safe seats

Preview Results

Coate and Knight (2007)

- One state in isolation

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- Linear seat vote curve with low responsiveness

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- Winner-take-all election is nearly optimal

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This paper

- Many states in equilibrium
- Very high responsiveness
- Winner-take-all election is nearly optimal

Voters

- Policies lie in the interval $[0, 1]$
- There are three type of voters that vary in their prefered policy
    - Democrats $\theta = 0$
    - Republicans $\theta = 1$
    - Independents $\theta \in (0, 1)$
- Independents are uniformly distributed along $[m - \tau, m + \tau]$
- The median independent, m, is unknown and drawn from a uniform distribution on $[.5 - \epsilon, .5 + \epsilon]$
- $\epsilon < \tau$ and $\epsilon + \tau < \frac{1}{2}$
- Voter payoffs are a quadratic loss function $-(x - \theta)^2$
- There is a mass of voters in state i of π_{Di}, π_{Ri}, π_{Ii} that sum to one
- Voters are not strategic

Districts

Label each district $j \in \left\{ \frac{1}{N_i}, \frac{2}{N_i}, \ldots, 1 \right\}$.
A districting is $\delta = (\pi_{Di}(j), \pi_{Ri}(j), \pi_{Ii}(j))_{j=\frac{1}{N_i}}^{1}$ such that the average $\pi_{Di}(j)$ equals π_{Di}.
The Democrat vote share in district j is then

$$v_i(j; m_i) = \pi_{Di}(j) + \pi_{Ii}(j) \left(\frac{\frac{1}{2} - (m_i - \tau)}{2\tau} \right). \quad (1)$$

The state wide vote share looks the same.

$$v_i(m_i) = \pi_{Di} + \pi_{Ii} \left(\frac{\frac{1}{2} - (m_i - \tau)}{2\tau} \right). \quad (2)$$

Seat-Vote Curve

Take the inverted function.

$$m_i(v_i) = \frac{1}{2} + \tau \left(\frac{\pi_{Ii} + 2\pi_{Di} - 2v_i}{\pi_{Ii}} \right) \tag{3}$$

We can use this to relate the district vote share to the state wide vote share.

$$v_i(j; m_i(v_i)) = \pi_{Di}(j) + \pi_{Ii}(j) \left(\frac{v_i - \pi_{Di}}{\pi_{Ii}} \right) \tag{4}$$

Then we can define the seat-vote curve as

$$S(v_i|\delta) = \max \left\{ j \mid v_i(j) > \frac{1}{2} \right\}. \tag{5}$$

Policy

- Let $n^i = \frac{N_i}{\sum_{k=1}^{K} N_k}$ be the fraction of national representatives that are from state i
- When v_i is drawn, state i elects $S_i(v_i)$ Democrats
- Nationally, $\sum_{k=1}^{K} n_k S_k(v_k)$ fraction of representatives are Democrats
- The policy is chosen as the mean preference of the representatives
- The representatives choose the policy to maximize their welfare, $1 - \sum_{k=1}^{K} n_k S_k(v_k)$

State's Problem

The state now looks to choose the seat-vote curve subject to it being implementable by some districting.

Let Ω_i be the set of feasible districtings. Taking the other state's seat-vote curves as given, state i's best response is

$$\underset{S_i(v_i)}{\operatorname{argmax}} -E_v\left[\pi_{Di}\left(1-\sum_{k=1}^{K} n_k S_k(v_k)\right)^2 + \pi_{Ri}\left(\sum_{k=1}^{K} n_k S_k(v_k)\right)^2 \right. \tag{6}$$

$$\left. + \pi_{Ii}\int_{m_i(v_i)-\tau}^{m_i(v_i)+\tau} \frac{\left(1-\sum_{k=1}^{K} n_k S_k(v_k) - x\right)^2}{2\tau}\, dx\right] \tag{7}$$

$$\text{s.t. } S_i(v_i) = S_i(v_i|\delta) \text{ for some } \delta \in \Omega_i \tag{8}$$

Relaxed Solution

For now, let's ignore the implementability constraint and solve the relaxed problem. We can write the problem as

$$\max_{S_i(v_i)} \quad -\mathbb{E}_{v_i}\left[\mathbb{E}_{v_{-i}}[W(S_i(v_i), S_{-i}(v_{-i}))]\right] \tag{9}$$

where W is the big welfare function on the previous slide.
Now it can be solved pointwise for each v_i.
The derivative of the objective function is

$$2n_i\left(\pi_{Di} + \pi_{Ii}(1 - m_i(v_i)) - \sum_{k=1}^{K} n_k \mathbb{E}_{v_k}[S_k(v_k)]\right). \tag{10}$$

Solution

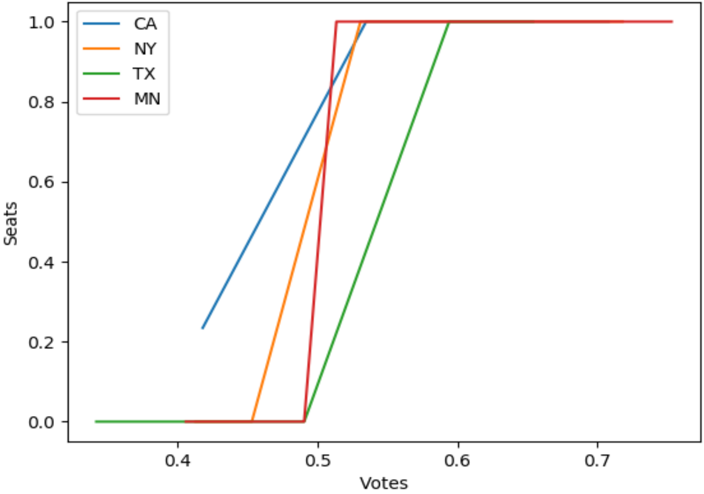
Setting this equal to zero and adding in the equation for $m_i(v_i)$ gives

$$\hat{S}_i(v_i) = \frac{1}{n_i}\left(\frac{1}{2} + (\pi_{Di} - \pi_{Ri})(.5 - \tau) + 2\tau(v_i - .5) - \sum_{k \neq i} n_k \mathbb{E}_{v_k}[S_k(v_k)]\right) \tag{11}$$

The best response is then

$$S_i^*(v_i) = \begin{cases} 0 & \text{if } \hat{S}_i(v_i) < 0 \\ \hat{S}_i(v_i) & \text{if } \hat{S}_i(v_i) \in [0,1] \\ 1 & \text{if } \hat{S}_i(v_i) > 1. \end{cases} \tag{12}$$

Graphs



Socially Optimal

What does this mean nationally?

- Is equilibrium play socially optimal nationally?
- Is it Pareto optimal?
- No.

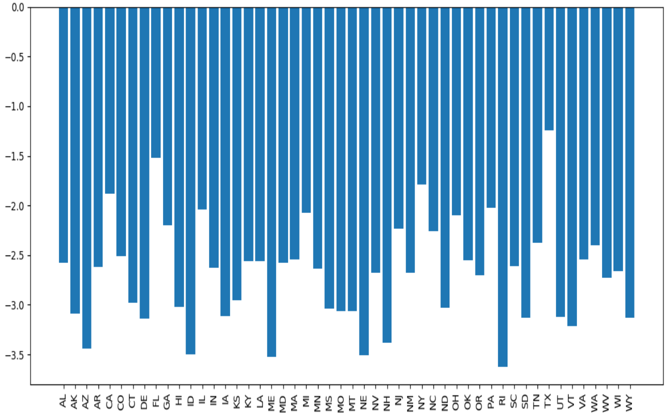
The socially optimal strategy is

$$S_i^{CK}(v_i) = \frac{1}{2} + (\pi_{Di} - \pi_{Ri})\left(\frac{1}{2} - \tau\right) + 2\tau\left(v_i - \frac{1}{2}\right) \qquad (13)$$

for every state.
Moreover, this is a Pareto improvement.

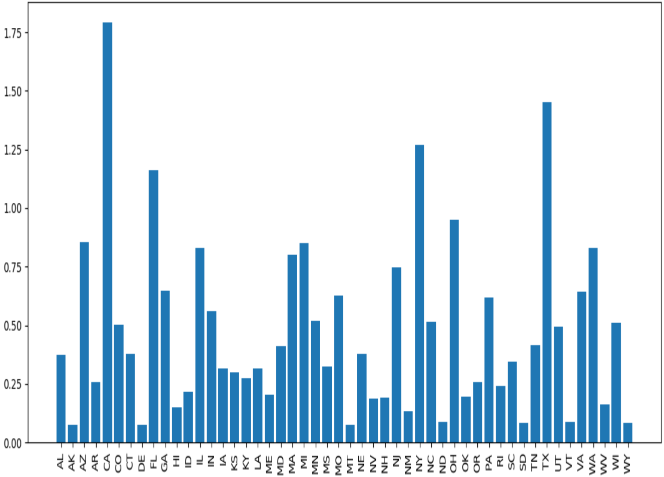
Socially Optimal

The average payoff in equilibrium minus the average payoff in the CK strategy (as a percentage of you CK payoff).



Socially Optimal

The average gain of best response if all other states play CK.



Implementability

Let's talk about districting this best response for moment.

- Requires you to know the party affiliation of everyone in the state
- Ignores geography
- It's weird
- It's not always implementable

Winner-Take-All

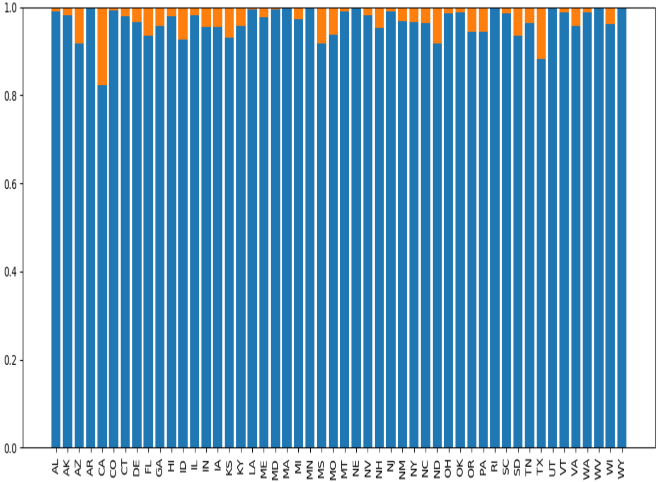
Consider a particular very simple seat-vote curve.

$$S(v) = \begin{cases} 1 & \text{if } v \geq \frac{1}{2} \\ 0 & \text{if } v < \frac{1}{2} \end{cases} \tag{14}$$

This is a winner-take-all election.
This is easily understood by the public (or policy maker), and it is always implementable.

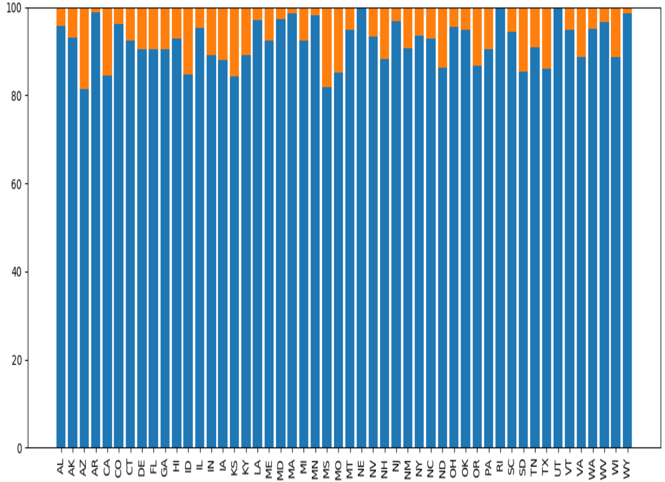
Welfare

It's almost just as good for the state's payoff.



Strategy

It's almost the same strategy.



Majority-Ruled Congress

How is policy chosen by the legislators?

- Maximize their welfare (mean of their preferences)
- Majority chooses
- A mix of the two

Let's now consider the other extreme, where the majority chooses the policy.

Single State

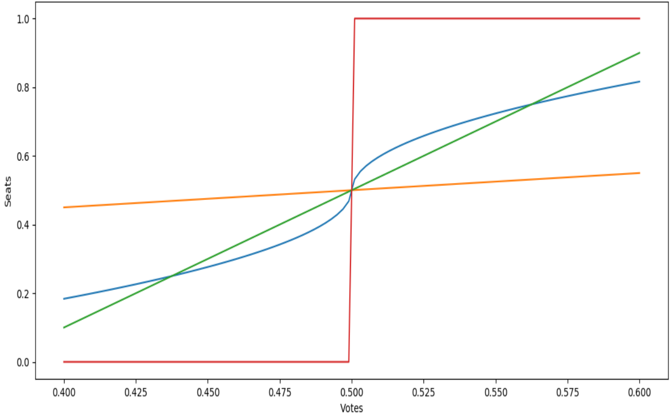
There are now really only two outcomes you choose between. Either the Democrats win a majority, or the Republicans do. You only need to check if a policy of 0 or 1 has higher welfare.
The Republican majority is prefered when,

$$v \leq \frac{1}{2} + (\pi_R - \pi_D)\left(\frac{1}{4\tau} - \frac{1}{2}\right). \tag{15}$$

There are many seat-votes curves that give this.

Single State

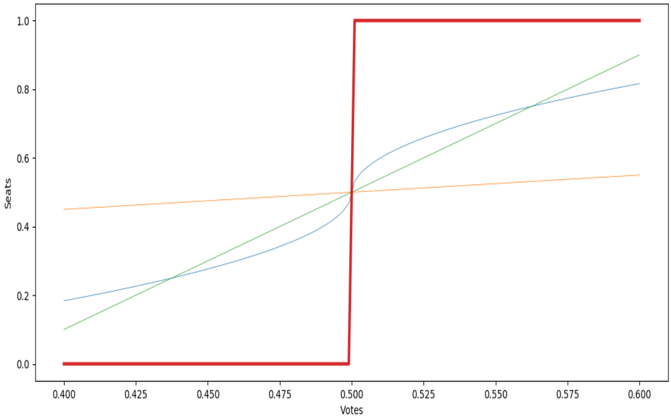
You are indifferent between all these seat-vote curves.



CK is in this set.

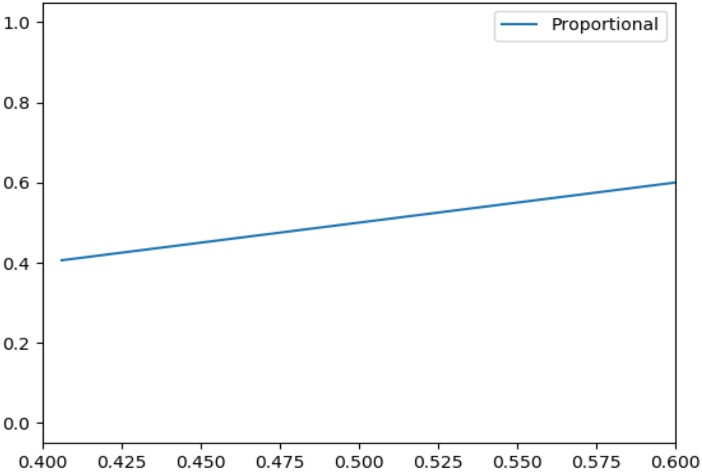
Equilibrium

If there is any probability of being pivotal, your best response becomes single valued at the most extreme one.



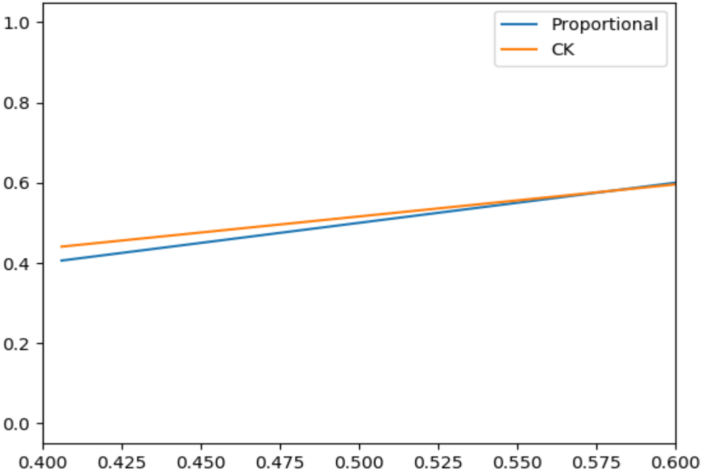
Comparison (for Minnesota)

Here is proportional.



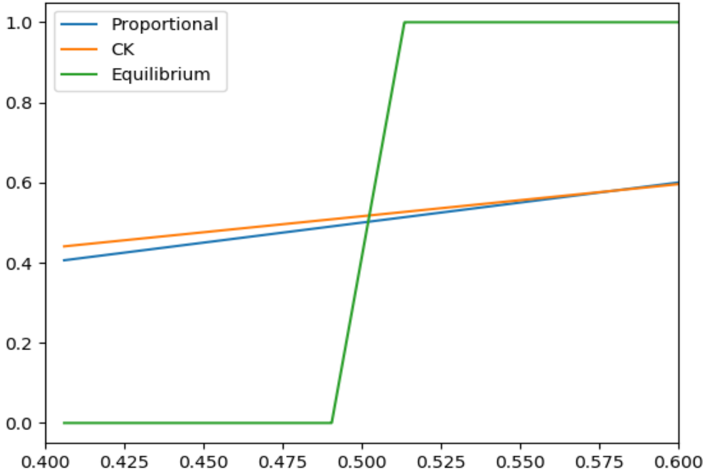
Comparison (for Minnesota)

Here is socially optimal.



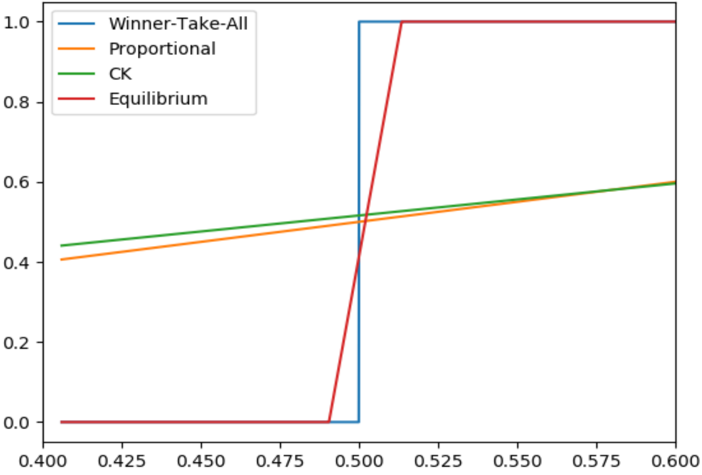
Comparison (for Minnesota)

Here is equilibrium.



Comparison (for Minnesota)

Here is winner-takes-all.



Conclusion

What redistricting policies should a state adopt?

- Winner-take-all
- Allow (encourage) gerrymandering if you're an extreme state
- Encourage competitive elections if you're a moderate state
- Don't try to make it more proportional

What redistricting policies should the country adopt?

- Strictly proportional