Dynamics of Price Discovery

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Prices Reflect Information

> A market in which prices always "fully reflect" available information is called "efficient."
>
> *Eugene F. Fama (1970)*

Prices come from the market's expectation of future payouts.

$$p_0 = \mathbb{E}\left[\sum_{t=0}^{\infty} m_t x_t \;|\mathcal{I}_0\right]$$

where x_t is the payoff and m_t is the stochastic discount factor.

This is especially pertinent in financial markets.

Prices Changes from Information

In a well functioning market, price dynamics should be driven by the arrival of new information.

Is this driven by information?

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Is this driven by information?

Dynamics from Strategic Information

Question 1:
What price dynamics arise from strategically released information?

Price Manipulation

- If you can move prices, you can manipulate prices.
- If you can profit from releasing true information, you can profit from releasing false information.

Question 2:

How is misleading information and price manipulation optimally mixed with true information to maximize profits? How does this affect price dynamics and price discovery?

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Literature Review

Foundation of Prices and Efficient Markets

- Grossman (1976), Samuelson (1965), Fama (1970)

Asymmetric Information in Prices

- Kyle (1985), Glosten and Milgrom (1985), Hellwig (1980)

Information Design

- Cheap Talk
  - Static: Crawford and Sobel (1982), Lipnowski and Ravid (WP)
- Bayesian Persuasion
  - Static: Kamenica and Gentzkow (2011), Kolotilin (2017), Aumann and Maschler (1995)
  - Dynamic: Ely (2017), Ely Kamenica and Frankel (2015), Renault Solan and Vieille (2017)

Setup

There is an asset that will pay zero or one with equal probability at the end of the game.

- $\omega \in \{0, 1\}$ and $Prob\{\omega = 1\} = \frac{1}{2}$

There is an efficient highly risk-neutral market. (Competitive fringe)

- At any period, you can buy or sell the asset at the posted price.
- Price is equal to expected payout, $P_t = \mathbb{E}[\omega | \mathcal{I}_t]$ ($P_1 = \frac{1}{2}$)

If you are privately informed of ω, what is the best you can do?

- You see ω at the start of the game
- You have two periods to trade
- You can take a long or short position, but face a capacity constraint, $x_t \in [-1, 1]$

Simple Strategy

First consider the intuitive "honest" strategy: buy if the state is good and sell if the state is bad.

$$x_1 = \begin{cases} 1 & \text{if } \omega = 1 \\ -1 & \text{if } \omega = 0 \end{cases}$$

In the second period, you are already at you capacity constraint. $x_2 = x_1$. Let's examine the payoff of such strategy.

If $\omega = 1$:

$$Payoff = -p_0 + \omega = -\frac{1}{2} + 1 = \frac{1}{2}$$

If $\omega = 0$:

$$Payoff = p_0 - \omega = \frac{1}{2} - 0 = \frac{1}{2}$$

You get a payoff of $\frac{1}{2}$ regardless of the state.

Optimal Strategy

While the previous strategy is intuitive, if the trader can commit to a randomized strategy they can do better.

- Trade multiple times to take advantage of price dynamics
- Manipulate the market with misleading information
- Rothschild and the Battle of Waterloo

Consider the following candidate strategy.

$$\text{If } \omega = 1 : \quad \begin{cases} \text{Buy } (x_1 = 1) & \text{with probability } \frac{3}{4} \\ \text{Sell } (x_1 = -1) & \text{with probability } \frac{1}{4} \end{cases}$$

Then in period 2, do the same as the "honest" strategy: $x_2 = 1$.
If $\omega = 0$ play the symmetric trading strategy.

Optimal Strategy

To calculate the payoffs, we first need to know the prices in each period.
The price in the first period is the same as before.

$$P_1 = \frac{1}{2}$$

In the second period, the market has seen the first period trades and can update their beliefs.

$$P_2 = \mathbb{E}[\omega | x_1]$$

This is computed simply with Bayes rule.

$$P_{2|buy} = \frac{\frac{1}{2}\frac{3}{4}}{\frac{1}{2}\frac{3}{4} + \frac{1}{2}\frac{1}{4}} = \frac{3}{4}; \qquad P_{2|sell} = \frac{1}{4}$$

Optimal Strategy

We can now compute the expected payoff to the trader in each state.

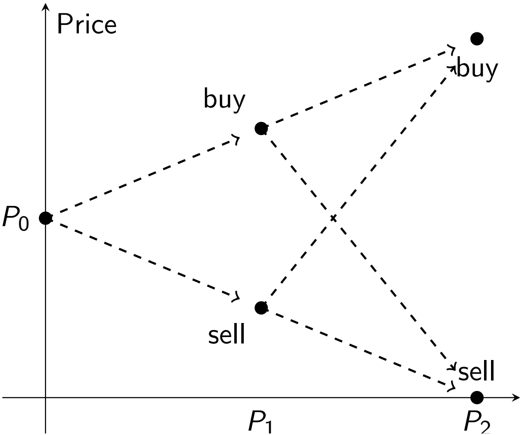
If $\omega = 1$:

$$\begin{aligned} \mathit{Payoff} &= \frac{3}{4}\underbrace{\left(-\frac{1}{2}+0+1\right)}_{\text{honest payoff}} + \frac{1}{4}\underbrace{\left(\frac{1}{2}-2\frac{1}{4}+1\right)}_{\text{manipulative payoff}} \\ &= \frac{3}{4}\qquad\frac{1}{2} \qquad\qquad + \frac{1}{4}\qquad 1 \\ &= \frac{5}{8} > \frac{1}{2} \end{aligned}$$

We get the mirrored equation if $\omega = 0$:

$$\begin{aligned} \mathit{Payoff} &= \frac{3}{4}\left(\frac{1}{2}+0+0\right) + \frac{1}{4}\left(-\frac{1}{2}+2\frac{3}{4}+0\right) \\ &= \frac{5}{8} \end{aligned}$$

Price Dynamics



Takeaways

Commitment

- The trader commits to randomizing over trades
- Higher profit is obtained by sometimes moving prices in the "wrong" direction
- Prices bounce up and down across time

Without commitment

The strategy described only works if the trader is able to commit ex ante to a randomized trading strategy.
When the state is good,

- Play honest strategy $\frac{3}{4}$ of the time, receive payoff of $\frac{1}{2}$
- Play manipulative strategy $\frac{1}{4}$ of the time, receive payoff of 1.

The trader would like to deviate to always manipulating.

Without commitment, the best the trader can do is the naive honest strategy.

Plan

I'm going to generalize the example somewhat

- Infinite number of periods, with discounting
- Price is general function of beliefs

Intuitively, the results will be similar

- Without commitment, all information will be released at once but at a randomized time
- With commitment, prices will bounce back and forth every period and drift toward the full information value

Overview

Three key elements of the model

- Uncertainty
- A strategic informed trader
- Prices and liquidity

Overview

- Two possible states

There is no commitment.

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- Two possible states
- Two players
  - Informed trader buys/sells shares to maximize expected profit
  - Uninformed market maker sets price and offers some liquidity
- They take turns for an infinite number of periods

There is no commitment.

Informed Trader

- The informed trader knows the state, $\omega \in \{0, 1\}$
- They choose how much to buy/sell each period (potentially a mixed strategy), holdings $x_t \in [-1, 1]$
- Maximize expected profit

$$\mathbb{E}\left[\int_0^\infty e^{-rt} x_t \, dP_t\right]$$

- No dividends or final value important to trader, only capital gains
    - Think foreign currency, gold, Amazon stock, penny stocks, etc.
- Usually work with discrete version ($\delta = e^{-r\Delta t}$)

$$V(\mu_0) = \max_{\{x_t\}_{t=0}^\infty} \mathbb{E}\left[\sum_{t=0}^\infty \delta^t (P_{t+1} - P_t) x_t\right]$$

Market Maker

The market maker has their beliefs about the state from the previous period, μ_{t-1}.
They observe the action of the informed trader, x_{t-1}, and update their beliefs using Bayes rule to get μ_t.
The market maker then chooses a price to optimize some flow utility.

$$P(\mu_t) = \operatorname*{argmax}_{p \in \mathbb{R}} U(p, \mu_t)$$

Assume $P(\mu_t)$ exists and is single valued.

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Examples

- Price is an expectation: $U(p, \mu_t) = -\mathbb{E}_{\mu_t}\left[(p-\omega)^2\right]$
- Price can be any general function of beliefs: $P(\mu_t) = \mathbb{E}_{\mu_t}\left[m(\omega)z(\omega)\right]$

Equilibrium

We will look for Perfect Bayesian Equilibria (focus on Markov Equilibria).

- The informed trader maximizes: Holding fixed prices and beliefs at all histories, trades $(x_t : \mathcal{H}_t \to [-1, 1])$ maximize

$$\mathbb{E}\left[\sum_{t=0}^{\infty} \delta^t (P_{t+1} - P_t) x_t\right]$$

- The market maker maximizes: Holding fixed trades and beliefs at all histories, prices (p_t) maximize

$$U(p, \mu_t)$$

- Beliefs satisfy Bayes' rule from the informed trader's strategy where possible.

The focus is on the equilibrium that maximizes the informed trader's profit.

Maximization Problem

We will find the equilibrium that maximizes the profit to the trader by choosing distributions of prices $\{p_t\}$, trades $\{x_t\}$, and beliefs $\{\mu_t\}$ to maximize

$$\mathbb{E}\left[\sum_{t=0}^{\infty} \delta^t (P_{t+1} - P_t) x_t\right]$$

subject to the prices, trades, and beliefs constituting an equilibrium.

There are three sets of constraints to insure we have an equilibrium.

- Incentive compatability for the market maker
- Incentive compatability for the trader
- Bayes' rule to make sure beliefs are consistent

Simplifying

First note that incentive compatability for the market maker is simply

$$p_t = P(\mu_t)$$

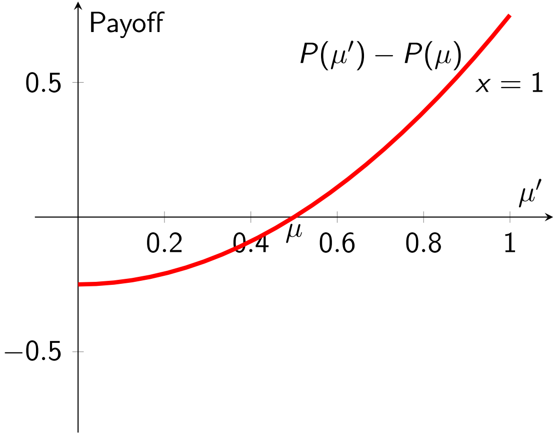
Now see that the problem can be rewritten recursively as

$$V(\mu) = \max_{\mu', x} \left(P(\mu') - P(\mu) \right) x + \delta V(\mu')$$

subject to incentive compatability for the trader and Bayes' rule.

Graph

Consider the relationship between x and μ'.



The informed trader's payoff if they buy, $x = 1$, graphed against the posterior induced.

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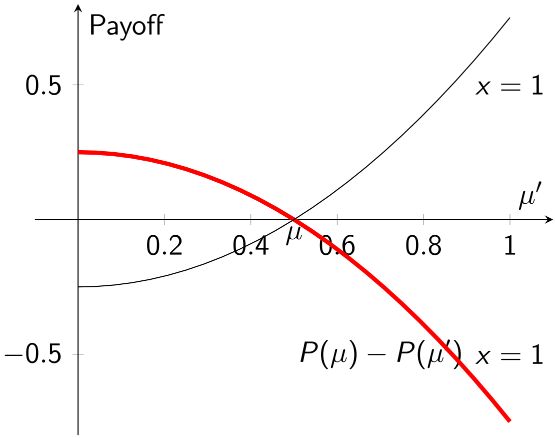
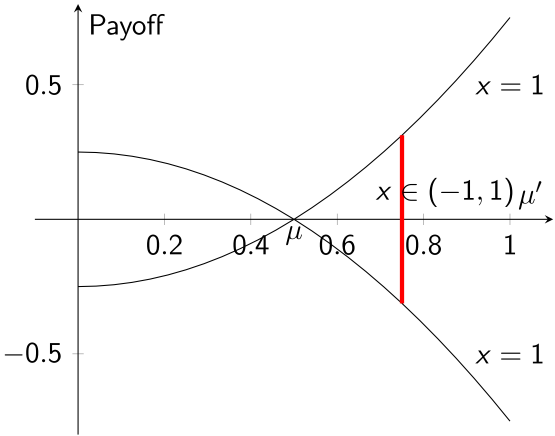
Consider the relationship between x and μ'.



The informed trader's payoff if they sell, $x = -1$, graphed against the posterior induced.

Graph

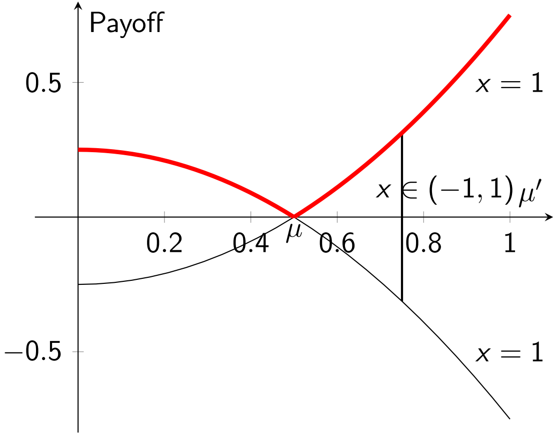
Consider the relationship between x and μ'.



Given the chosen posteriors, what must x be?

Graph

Consider the relationship between x and μ'.



The flow payoff is the upper envelope.

Reformulated Problem

The problem can now be written simply as one of choosing a distribution of beliefs subject to constraints.

$$V(\mu) = \max_{\Delta(\mu') \in \Delta[0,1]} \mathbb{E}\left[|P(\mu') - P(\mu)| + \delta V(\mu')\right]$$
$$\text{s.t. } \mathbb{E}[\mu'] = \mu \qquad (BayesPlausibility)$$
$$I.C.$$

The incentive compatability constraint takes a particularly simple form here.

$$|P(\mu') - P(\mu)| + \delta V(\mu')$$

must be equal for all posteriors μ' chosen with positive probability.

Value Function

The value function ends up taking a particularly simple form.

Proposition

If $P(\cdot)$ is continuous and monotone, regardless of δ,

$$V(\mu) = \min\{P(\mu) - P(0),\ P(1) - P(\mu)\}$$

Call μ^* the posterior where the minimum switches arguements in the above equation.

$$P(\mu^*) - P(0) = P(1) - P(\mu^*)$$

Proof of Proposition 1

Call $W_\mu(\mu')$ the unmaximized Bellman equation.

$$W_\mu(\mu') = |P(\mu') - P(\mu)| + \delta V(\mu')$$

Suppose $\mu < \mu^*$. Plug in the conjectured value function.

$$W_\mu(\mu') = \begin{cases} P(\mu) - \delta P(0) - (1-\delta)P(\mu') & \text{if } \mu' \leq \mu \\ (1+\delta)P(\mu') - P(\mu) - \delta P(0) & \text{if } \mu < \mu' \leq \mu^* \\ \delta P(1) + (1-\delta)P(\mu') - P(\mu) & \text{if } \mu' > \mu^* \end{cases}$$

Note that this is decreasing for all $\mu' \in [0, \mu)$ and increasing for $\mu' \in (\mu, 1]$.

Main Result 1

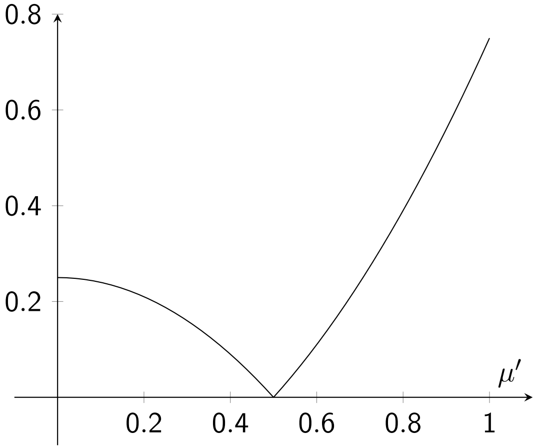
Even though the value doesn't depend on δ, the equilibrium strategy that achieves that value does vary with δ.

Theorem

For any differentiable strictly monotone price function, $P(\mu)$, as δ goes to one the price proccess converges to a Poisson process.

Think of this as the continuous time limit of the discrete game.

Strategy Depends on δ



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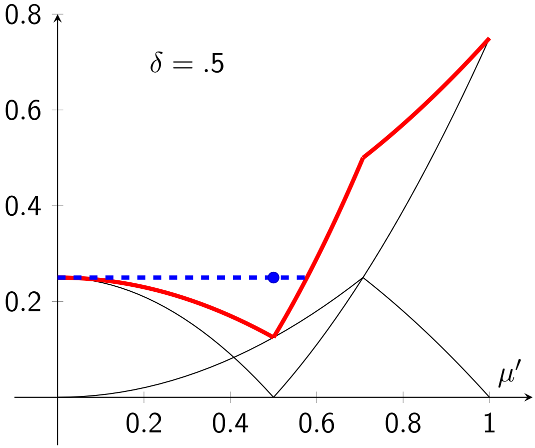

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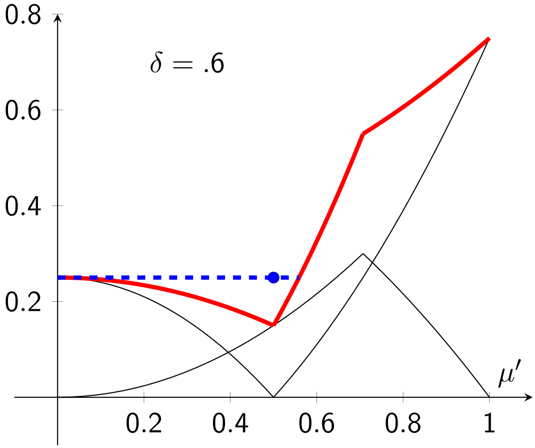

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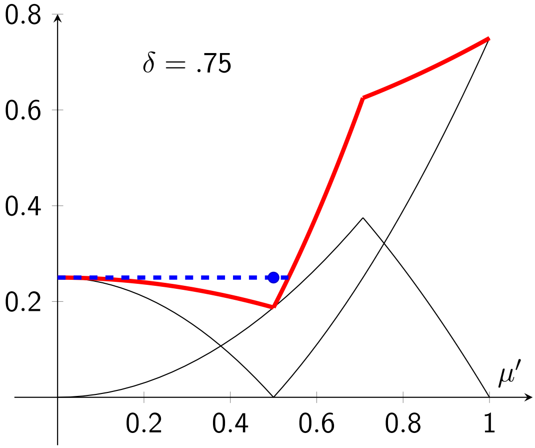

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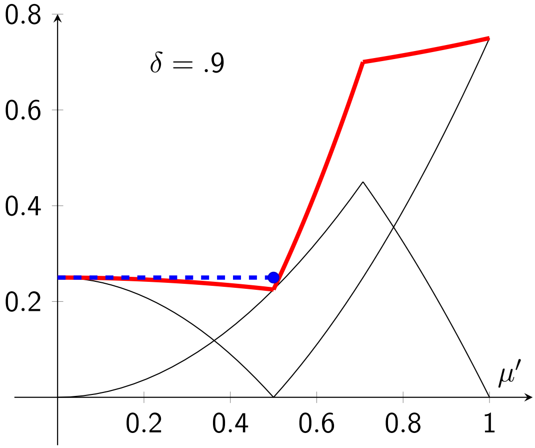

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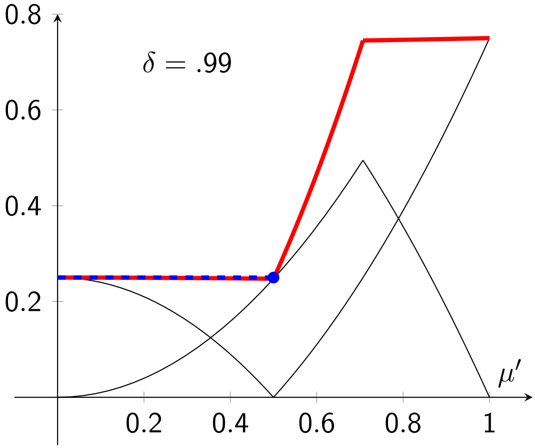


Functional Form

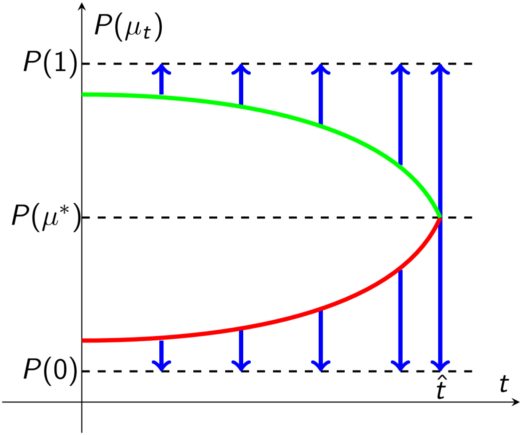
The functional forms depend on the price function, $P(\mu)$.

- If $\mu_t < \mu^*$,

$$dP(\mu_t) = \frac{r}{2}(P(\mu_t) - P(0))dt - (P(\mu_{t-}) - P(0))dN_t$$

  N_t is a standard Poisson with arrival rate $\lambda = \frac{r}{2}\frac{P(\mu_t)-P(0)}{\mu_t P'(\mu_t)}$.
- If $\mu_t > \mu^*$, price follows a symmetric process.
- If $\mu_t = \mu^*$, all information is revealed immediately and the price jumps to either $P(1)$ or $P(0)$.

Price Dynamics



The optimal strategy employs pump-and-dump and short-and-distort schemes.

Explain Theorem

The informed trader needs to be indifferent between revealing the asset to be bad today, and waiting to reveal it to be bad tomorrow. Call λ the percentage drift in beliefs.

$$\underbrace{P(\mu_t) - P(0)}_{\text{reveal today}} \approx \underbrace{P'(\mu_t)\mu_t\lambda_t}_{\text{drift today}} + \delta(\underbrace{P'(\mu_t)\mu_t\lambda_t + P(\mu_t) - P(0)}_{\text{reveal tomorrow}})$$

As δ gets large, this gives a linear relationship between λ and $1-\delta$.

$$\Rightarrow \lambda_t \approx \frac{P(\mu_t) - P(0)}{2\mu_t P'(\mu_t)}(1-\delta)$$

Arrival Rate

Consider the Taylor series expansion.

$$P(0) = P(\mu_t) - P'(\mu_t)\mu_t + \frac{1}{2}P''(\mu_t)\mu_t^2 + \dots$$

This gives another expression for the approximate arrival rate.

$$\lambda(\mu_t) = \frac{r}{2}\frac{P(\mu_t) - P(0)}{P'(\mu_t)\mu_t} \approx \frac{r}{2}\left(1 - \frac{1}{2}\mu_t\frac{P''(\mu_t)}{P'(\mu_t)}\right)$$

If $P(\mu_t)$ is concave, the arrival rate is greater than $\frac{r}{2}$ and increasing. If convex, it is smaller and decreasing.

Full Revelation

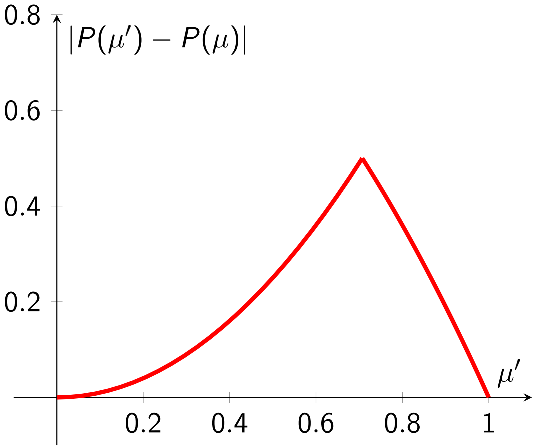
All information is revealed in finite time.

$$t^{max} = \frac{2}{r} \log \left(\frac{P(\mu^*) - P(0)}{P(\mu_0) - P(0)} \right)$$

Commitment

Consider if the informed trader could commit ex ante to a strategy.

The value function without commitment.

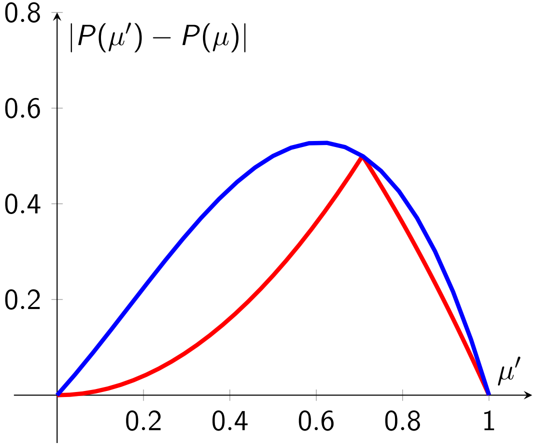


$$V(\mu) = \min\{P(\mu) - P(0),\ P(1) - P(\mu)\}$$

Commitment

Consider if the informed trader could commit ex ante to a strategy.

Buy if good and sell if bad does better.
Why wasn't it incentive compatible before?



$$\tilde{V}(\mu) \geq (1-\mu)(P(\mu) - P(0)) + \mu(P(1) - P(\mu)) \geq V(\mu)$$

Informed Trader's Problem and Main Result 2

Commitment power removes the incentive compatability constraint on the informed trader.

Theorem

For any monotone $\mathcal{C}^2$ function $P(\mu)$, as δ goes to one the price process converges to an Itô process.

Price dynamics are driven by a Brownian motion with a drift and variance term.

Proof Intuition

Let $\bar{\epsilon} \in [0, 1-\mu]$ and $\underline{\epsilon} \in [0, \mu]$ be the change in beliefs after buying or selling. The objective becomes

$$\tilde{V}(\mu) = \max_{\bar{\epsilon},\underline{\epsilon}} \left(P(\mu+\bar{\epsilon}) - P(\mu) + \delta \tilde{V}(\mu+\bar{\epsilon})\right) \frac{\underline{\epsilon}}{\bar{\epsilon}+\underline{\epsilon}} + \left(P(\mu) - P(\mu-\underline{\epsilon}) + \delta \tilde{V}(\mu-\underline{\epsilon})\right) \frac{\bar{\epsilon}}{\bar{\epsilon}+\underline{\epsilon}}. \quad (1)$$

For small ϵ, $P(\mu+\epsilon) - P(\mu) \approx P'(\mu)\epsilon$ and $\tilde{V}(\mu+\epsilon) \approx \tilde{V}(\mu) + \tilde{V}'(\mu)\epsilon + \frac{1}{2}\tilde{V}''(\mu)\epsilon^2$.

$$(1-\delta)\tilde{V}(\mu) = \max_{\bar{\epsilon},\underline{\epsilon}} \underbrace{2|P'(\mu)|\frac{\bar{\epsilon}\underline{\epsilon}}{\bar{\epsilon}+\underline{\epsilon}}}_{\text{current gain}} + \underbrace{\delta \tilde{V}''(\mu)\bar{\epsilon}\underline{\epsilon}}_{\text{future loss}}$$

More Proof Intuition

Optimizing gives

$$\bar{\epsilon} = \underline{\epsilon} = \frac{|P'(\mu)|}{2\delta \tilde{V}''(\mu)}.$$

Putting this back into the objective gives

$$\tilde{V}(\mu)\tilde{V}''(\mu) = -\frac{|P'(\mu)|^2}{2\delta(1-\delta)}.$$

Letting $\hat{V}(\mu) = \tilde{V}(\mu)\sqrt{2\delta(1-\delta)}$, $\sigma(\mu) = \frac{\sqrt{2}|P'(\mu)|}{\hat{V}''(\mu)}$, and $r\Delta t = \frac{1-\delta}{\delta}$, gives

$$\mu' - \mu = \begin{cases} \sigma(\mu)\sqrt{\Delta t} & \text{with probability } \frac{1}{2} \\ -\sigma(\mu)\sqrt{\Delta t} & \text{with probability } \frac{1}{2}. \end{cases}$$

Strategy With Commitment

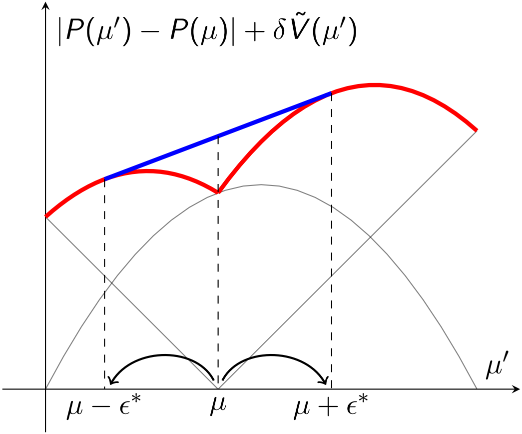


Figure: The blue segment gives the optimal policy and value. Beliefs either jump up or down by step size ϵ^* each period.

Functional Forms

Generally, the exact process can't be solved analytically, but the form of the solution is known.

$$dP(\mu_t) = \frac{r}{2}P''(\mu_t)\sigma^2(\mu_t)dt + \sqrt{r}P'(\mu_t)\sigma(\mu_t)dB_t$$

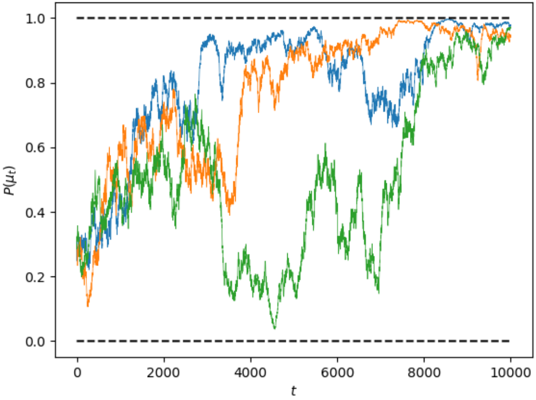
If the price function is linear with slope p, then

$$dP(\mu_t) = \sqrt{2r}p\phi(\mu_t)dB_t$$

where $\phi(\mu_t)$ is the normal pdf evaluated at the μ_t quantile.

Solution Explanation

- Gradual, continuous information release
- Spiky, random price movements
- Conditional on ω, price drifts toward full information value
- Nearly equal amount of information and misinformation



Importance

Endogenous dynamics

- Option pricing
- Information acquisition

Dynamic information disclosure

- Cheap talk vs Bayesian persuasion

Price Discovery

- Rumors
- Price volatility

Persistence

Suppose that ω_t follows a Markov process.

- Call $\pi_1 = 1 - \lambda_1 \Delta t$ and $\pi_0 = \lambda_0 \Delta t$ the probability that $\omega_{t+1} = 1$ given that $\omega_t = 1$ or 0 respectively.
- Take price function to be linear, $P(\mu) = \mu$.
- State observed by insider only in date 0 or every period

Objective

$$\max \ \mathbb{E}\left[\sum_{t=0}^{\infty} \delta^t |\mu_t' - \mu_t|\right]$$

where $\mu_{t+1} = \pi_0 + (\pi_1 - \pi_0)\mu_t'$.

One Time Information

Call $\tilde{\mu}_t$ the beliefs about ω_0 in date t.

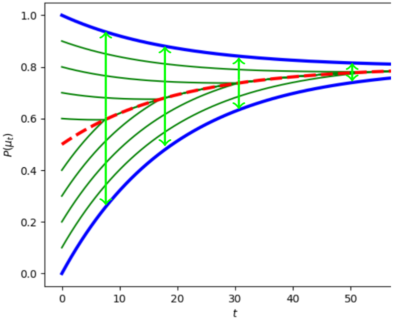
$$\begin{aligned} \mu_0 &= \tilde{\mu}_0 \\ \mu_1 &= \pi_0 + (\pi_1 - \pi_0)\tilde{\mu}_1 \\ &\vdots \\ \mu_t &= \sum_{\tau=0}^{t-1} \pi_0(\pi_1 - \pi_0)^{\tau} + (\pi_1 - \pi_0)^t \tilde{\mu}_t \end{aligned}$$

Simply puts bounds on where beliefs can be sent.

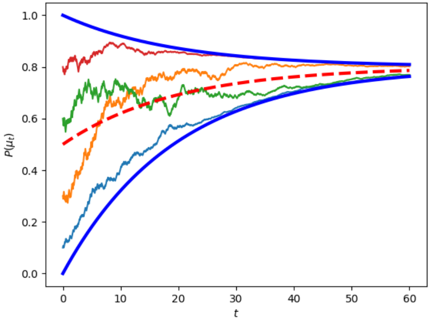
$$\mathbb{E}\left[\sum_{t=0}^{\infty} \delta^t |\mu_t' - \mu_t|\right] = \mathbb{E}\left[\sum_{t=0}^{\infty} (\delta(\pi_1 - \pi_0))^t |\tilde{\mu}_t' - \tilde{\mu}_t|\right]$$

Drift and arrival rate are higher, $\tilde{\delta} \approx 1 - (r + \lambda_1 + \lambda_0)\Delta t$.

Persistence



(a) Without commitment



(b) With commitment

Figure: Sample paths for prices.

Information Flow Every Period

Bellman equation becomes

$$V(\mu) = \max_{x} \mathbb{E}\left[(P(\mu') - P(\mu))x + \delta V(\pi_0 + (\pi_1 - \pi_0)\mu')\right].$$

Flow payoff is unchanged, but continuation value function is flatter. This implies the highest flat line still always hits the boundary.

$$V(\mu) = \min\left\{P(\mu) - P(0) + \frac{P(\pi_0) - P(0)}{1-\delta},\ P(1) - P(\mu) + \frac{P(1) - P(\pi$$

Information Flow Every Period

Take $\mu < \mu^*$ and $P(\mu) = \mu$.
The left endpoint of the strategy is 0 and the right endpoint is

$$\mu^{up} = \frac{2\mu + (1-\delta)\pi_0}{1 + \delta(\pi_1 - \pi_0)}.$$

The probability of jumping to zero is then equal to

$$\frac{\mu^{up} - \mu}{\mu^{up}\Delta t} \approx \frac{\mu(r + \lambda_1 + \lambda_0)}{2\mu + \Delta t}.$$

This give the same arrival rate as in the previous problem.

$$\lambda(\mu_t) = \lim_{\Delta t \to 0} \frac{p^{jump}}{\Delta t} = \frac{r + \lambda_1 + \lambda_0}{2}$$

Conclusion

Thank you.