# Dynamics of Price Discovery

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**Abstract**

Changes in the price of a financial asset represent learning as the market updates its expectation about fundamentals. In this paper I characterize what price dynamics are possible when the information is being released strategically by a profit maximizing trader and market participants are Bayesian. I study how information is incorporated into prices over time in a model with general trading strategies that allow for the spread of false information and price manipulation. Every period an informed trader reveals their information by buying or selling an asset. After observing the trade, beliefs and prices are updated. The informed trader's preferred equilibrium is characterized with and without commitment leading to starkly different results. Regardless of how beliefs impact prices, the optimal strategy for the informed trader is to release their information gradually mixed with nearly an equal amount of misinformation. This strategy leads to volatile price paths that bounce back and forth each period. In the continuous time limit, the price process converges to a Brownian motion. Moving prices back and forth in this way hinges critically on the informed traders ability to commit ex ante to their strategy. Without such commitment power, the optimal strategy is to release nearly all information suddenly at

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randomized times. The optimum resembles a pump-and-dump price manipulation scheme and can lead to sudden crashes or spikes in the price of the asset. In the limit, the price converges to a Poisson process. This paper gives a micro-foundation to price processes commonly assumed in the literature.

JEL Classification: D82, D83, G14

# 1 Introduction

Information is at the core of financial markets. In a frictionless market an asset's price is a reflection of the market's beliefs about fundamentals. Dynamics of prices are therefore driven by learning as the market updates its expectation about these fundamentals. Does supposing the market to be rational and use Bayesian updating put any restriction on the dynamics we would expect to see in asset prices? Going even further, suppose that the new information that leads to the market learning doesn't just arrive exogenously. If the new information is being released by a strategic trader maximizing their profit, can we infer anything about what price dynamics should be possible? In fact, the informed agent behaving optimally along with the market being Bayesian and efficient implies a strict characterization of possible price dynamics. The subject of this paper is deriving these dynamics.

In this paper I use a theoretical model to study how information gets incorporated into prices. Begin with a trader holding some information not commonly known to the rest of the market. As the informed trader acts over time their actions reveal information. The market price will continuously adjust as the market updates its beliefs to reflect the information revealed. I study the dynamics of prices through this process.

I present a dynamic trading game between a large informed trader and a competitive fringe of uninformed traders. I allow for general trading strategies, beyond most microstructure models. I show that rather than focusing on trades, the problem can be rewritten to focus on the beliefs induced after each trade. I am then able to use information design techniques to solve for the equilibrium. The informed trader is going to profit by being long when the price of the asset goes up and short when the price goes down. Since the price is driven by beliefs, the informed trader needs to use their trades to try to persuade the Bayesian market of uninformed traders.

A famous story of Nathan Rothschild gives us an example of such persuasion. On the 18th of June 1815 the British allies defeated Napoleon's army in the Battle of Waterloo. This victory effectively ended the Napoleonic Wars. Now, this isn't a paper about wars and history. Information and finance come into this story because banker Nathan Rothschild was the first person in London to hear the news of the British victory. He may have gotten the news through a series of horseback couriers, or through his prized racing pigeons, but it is believed that Rothschild heard the news a full 48 hours before others. Rothschild predicted that the price of British consols would rise with this news. However, Rothschild didn't use this knowledge to buy consols, at least, not initially. Rothschild began to publicly sell large amounts of consols. Word of his trading activity spread quickly. Everyone assumed that Rothschild knew the outcome of the battle. His selling was interpreted to to mean that the British must have lost the battle. The price of consols dropped dramatically. Shortly before news of the battle arrives, Rothschild then buys a large quantity of consols at this lower price. The correct news of the victory causes the price to rise. Rothschild was able to use his information to profit from both the downswing and the upswing in price. For more on this story, see Morton (1962).

This story raises a lot of questions. How can such a profitable strategy be possible? Does it rely on non-rational agents? If all the market participants are fully rational, shouldn't they foresee that Rothschild has an incentive to manipulate them in this way? If the market sees this incentive, they won't be convinced when Rothschild sells. The price doesn't fall, then the strategy is no longer profitable. The main question arises. Is information based price manipulation of this form possible when market participants are fully rational.

I show that manipulation is possible with rational traders, and it is optimal. The equilibrium will feature the informed trader trading against their information to mislead markets often. The equilibrium strategy is much like the strategy of Nathan Rothschild. Suppose the information is bad and the price today is low. The informed trader will almost always choose to be long in the asset for a time to drive the price up before selling the asset and revealing the state to be bad. This is a strategy seen in practice. The SEC prosecutes such "pump-and-dump" schemers every year. If the price is high today and the information is bad, the reverse strategy, known as a "short-and-distort" will be employed. When the price is low, the market knows the informed trader has an incentive the manipulate by buying. The market knows in equilibrium that the trader will almost always buy. This knowledge makes the act of buying not very informative. Thus, the informed trader will initially buy and the price will only slowly move upward. The informed trader is randomizing between almost always buying and the price drifting up a small amount, and with low probability selling and causing the price to crash as they reveal the information to be bad. If you take the continuous time limit of the game, the price process will converge exactly to a Poisson process with a drift. This leads to a price path with sudden discontinuous crashes or jumps. All information is revealed in finite time.

If the informed trader was able to commit themselves ex ante to a trading strategy, they could generate much higher profits than they get from the simple "pump-and-dump" scheme. It is natural to ask why the trader doesn't do a Rothschild style manipulation scheme more than once. Perhaps the trader could sell the asset to drive the price down, then buy the asset and try to drive the price up, before finally selling again. It turns out that this wouldn't normally be possible in equilibrium with rational agents. If the market price was responding to both buy and sell actions from the informed trader multiple times, the informed trader would have an incentive to continue manipulating the price back and forth indefinitely without revealing any information. Since information isn't being revealed, rational players won't respond at all to these trades and the price won't move. The manipulation will fail to be convincing and therefore fail to be profitable. However, if the informed trader could commit to a trading strategy, suddenly the impossible becomes possible. The trader could commit to only manipulate with some frequency and to reveal the true information over time. Then the trades will still be informative and prices will still respond. This commitment assumption is akin to the common commitment assumption in the Bayesian Persuasion literature.

In the optimal strategy, the trader will manipulate the price as many times as possible. This will make the size of the market response to each trade increasingly small. In each period the trader will buy or sell the asset, effectively announcing good or bad news, with nearly equal frequency. This strategy will cause the price to move a very small amount up or down each period. As we again take the continuous time limit, we will see that the price process will now converge to one driven by a Brownian motion. The equilibrium price is now a strictly positive Itó process with drift and variance depending on the what the information means for the asset's value. The asset will have continuous but spiky price paths. These characteristics are completely opposite the the equilibrium prices when the informed trader did not have commitment. This Itó process turns out to be the most variation in price that is possible due to information alone. Only asymptotically is all of the informed trader's information revealed.

The Itó and Poisson processes arise completely endogenously as the optimal strategy of an informed trader. Nothing in the setup of the model is Normal or Poisson to lead to these distributions. Throughout finance and particularly fields like option pricing, an Itó process possibly with a Poisson process added on form the dynamics of a stock price. This is typically assumed for tractability, but my model gives a micro-foundation for these to arise as the natural stochastic process for prices. In papers of learning and information acquisition the arrival of information is typically modeled to be either Brownian or Poisson. My model shows that such information flows may be motivated by the information being obtained from a strategic player.

The setup of the model is intentionally kept simple. The purpose of this paper is to study the natural dynamics that arise from price discovery. This paper is not meant to focus on dynamics that are driven by particular frictions or inefficiencies or by the way these frictions are modeled. The whole point is to strip away all these frictions get to the core dynamics of what price discovery looks like in an efficient market. All other models with frictions will be introducing wedges that distort the market from these dynamics that have a intrinsic strategic motive to arise. Several extensions of the model with multiple informed traders or with more complex information are presented in the appendix.

## 1.1 Literature Review

The theoretical study of how prices move in an efficient capital market goes back for several decades. See seminal works such as Samuelson (1965), Fama (1970), and Grossman and Stiglitz (1980). I am going to approach this topic with a model merging the literatures on strategic informed trading and dynamic information design.

Questions of strategic informed trading often studied using a variant of Kyle (1985). Many of these study whether the trader's private information gets full incorporated into the price. Some examples of this are Holden and Subrahmanyam (1992), Foster and Viswannathan (1996), Ostrovsky (2012), and many others. Other models of prices in financial markets with informed strategic trading that don't use Kyle (1985) usually derive from Glosten and Milgrom (1985), Hellwig (1980), or Hanson (2003, 2007). My model is the limiting case of all of these models as the asset becomes more liquid. Thus mine will be studying strategic trading in a frictionless efficient market. Also, most strategic trading models don't allow for manipulative trading strategies. A few papers that do study manipulative trading strategies include Van Bommel (2003), Benabou and Laroque (1992), and John and Narayanan (1997). This paper will extend their work by allowing for a completely general set of trading strategies.

I show that the model can be solved using information design techniques. Even though no messages are being sent directly, the model can be transformed to a cheap talk model as popularized by Crawford and Sobel (1982). In cheap talk games with “transparent motives” Lipnowski and Ravid (2020) develop techniques for solving for the sender’s preferred equilibrium. I take their techniques to a dynamic setting. When my trader has the ability to commit, the model can be transformed to a Bayesian persuasion model. Bayesian persuasion models of communication with commitment were developed in Kamenica and Gentzkow (2011) and build from Aumann and Maschler (1995). It has been shown that these can be extended to dynamic settings in several recent papers including Ely (2017), Renault et al. (2017), Orlov et al. (2020), and Hörner and Skrzypacz (2016).

The closest paper to mine in the information design literature is Ely et al. (2015). They characterize the most an informed sender can surprise an uninformed receiver. This is close to my paper because the informed trader wants to profit by moving prices and does this in effect by surprising the market. If the price of the asset was taken to be a simple expected value, my model with commitment would then reduce to theirs. So, this paper extends their results to more general measures of surprise and computes the surprise optimal policy without commitment. A mathematical paper Mertens and Zamir (1977) computes the maximal variation a bounded martingale can have. Since beliefs are a bounded martingales and prices are a function of beliefs, the surprise model of Ely et al. (2015), the commitment model of this paper, and De Meyer and Saley (2003) and De Meyer (2010) that look for a strategic foundation for the Brownian motion, can all be thought of as building off of Mertens and Zamir (1977). This paper also addresses the question of how much volatility in price can be caused by information alone, see Shiller (1981).

My results from the model without commitment look a lot like the solution of Zhong (2022), who studies dynamic information acquisition. They show how a Poisson process gives the most uncertainty across time, and that exponentially discounting agents are risk loving over lotteries across time. This same logic applies to the strategy of my trader without commitment power. The trader receives a large payout when they reveal the true state, and they have an incentive to disperse the timing of that payoff.

# 2 Model

In this section I present the main model, extensions can be found in the appendix. All that is needed is a strategic trader with private information, and prices.

## 2.1 Informed Trader

There is a permanent unknown state of the world $\omega \in \{0, 1\}$ that is relevant to the price of the asset. The informed trader knows the value of $\omega$ and the rest of the market has prior probability $\mu_0$ that $\omega = 1$. The risk neutral informed trader chooses how much of the asset to hold at each point in time, $x_t$, to maximize expected discounted profit. Because the objective is linear, a bang-bang solution obtains and the insider would like to hold an infinite number of shares of the asset. I restrict the number of shares that can be held to $x_t \in [-1, 1]$. The trader has a capacity constraint, not a trading constraint. When $T$ is finite, I will force $x_T = 0$ so that the payoffs to the insider don't include any final value of holding the asset at the end of the game.

The payoff to the trader is the following.

$$V(\mu_0) = \max_{x_t} \quad \mathbb{E}\left[\int_0^T e^{-rt} x_t dp_t\right] \tag{1}$$

However, the plan is to use a discretization of the payoff and only look at the continuous-time game as the limit of the discrete version. Call $\delta = e^{-r\Delta t}$. The payoff function will then be

$$V(\mu_0) = \max_{x_t} \quad \mathbb{E}\left[\sum_{t=0}^{T} \delta^t (p_{t+1} - p_t) x_t\right]. \tag{2}$$

Each morning, the trader wakes up, sees the price, and decides how much to buy or sell of the asset. If the price increases by three dollars today, the trader receives three times the number of shares they are holding. Note that the payoff from a price change happens immediately rather than when the asset position is closed. This could be for psychological reasons or because the asset is marked-to-market.

The expectation won't be important at this point of the model. The only natural uncertainty in the model is $\omega$, which is known to the informed trader. There will be further uncertainty due to the fact that the informed trader will randomize, but again that is known to the trader. The only time the expectation would be meaningful would be if the market price is random given $\omega$, but that won't be the case in equilibrium.

As you can see from the objective function, the asset pays no dividends during the game. This means that the only profit to the informed trader comes from changes in the price. You can also think of the next dividend of the asset being far enough in the future to be beyond the horizon of the game (Amazon stock etc.) or think of the asset as having intrinsic value to others but not the insider (foreign currency, commodities, etc.).

## 2.2 Competitive Fringe

There is a perfectly competitive fringe of uninformed traders that provide liquidity to the market. The fringe is a stand-in for the rest of the market, and will be left largely unmodeled. Each member myopically chooses prices at which to stand willing to buy and sell the asset. Being uninformed, the liquidity traders will obviously lose money in their trades with the informed trader, but will presumably make up for it by profiting off of all their other trades with uninformed traders. As the number of uninformed trades grows relative to the size of the informed trader, the market becomes more liquid. As an approximation to a highly liquid well-functioning market, we'll suppose that competition has driven the bid-ask spread to zero. Thus, each member of the competitive fringe is choosing a single price to maximize their payoffs from exogenous factors such as noise traders. The problem each period is

$$\max_{p_t \in \mathbb{R}} \quad U(p_t, \mu_t) \tag{3}$$

with $U(p_t, \mu_t)$ left general. $\mu_t$ is the competitive fringe's belief that $\omega = 1$ given information up to period $t$. Each period the fringe will observe the trades made by the informed trader and from that observation make inferences about the state. The competitive fringe is comprised of fully rational Bayesians.

The simplest example is thinking of a robot market maker that simply sets the price equal to some expected value, like in Kyle (1985) and others. Another example is to consider a large market where the price follows a standard equation for stochastic discount factor $m$ and asset payoffs $z$ by having

$$U(p_t, \mu_t) = \mathbb{E}_{\mu_t}\left[(p_t - m(\omega)z(\omega))^2\right]. \tag{4}$$

The point is that a price is chosen every period according to some myopic process, and that the price incorporates information that is revealed by the informed trader's actions.

## 2.3 Equilibrium

I will solve for the Perfect Bayesian Equilibrium that gives the highest payoff to the informed trader. Call $h^t$ the publicly observable history up to time $t$ (all the prices and trades). The equilibrium consists of a trading strategy for the informed trader, $X_t(\omega, h^t)$, a pricing strategy for the competitive fringe, $P_t(h^t)$, and beliefs, $\mu_t(h^t)$, that satisfy the following three conditions.

- The informed trader optimizes: holding fixed $P_t(h^t)$ and $\mu_t(h^t)$, $X_t(\omega, h^t)$ solves
$$X_t(\omega, h^t) \in \underset{x_t\in[-1,1]}{argmax}\ \mathbb{E}\left[\sum_{t=0}^{T}\delta^t(p_{t+1} - p_t)x_t\right] \tag{5}$$
- The competitive fringe optimizes: holding fixed $X_t(\omega, h^t)$ and $\mu_t(h^t)$, $P_t(h^t)$ solves
$$P_t(h^t) \in \underset{p_t\in\mathbb{R}}{argmax}\ U(p_t, \mu_t) \qquad \forall t \tag{6}$$
- Consistency: $\mu_t(h^t)$ is obtained from Bayes's rule using $X_t(\omega, h^t)$ where possible.

There are infinitely many such equilibrium. I will focus on finding the one that gives the highest payoff to the informed trader.

# 3 Profit Maximizing Equilibrium

The strategy to solving for the informed trader's maximal equilibrium profit is as follows. Write the recursive formulation of the problem. Translate the problem into one of choosing posteriors rather than trades. Assume the function $P(\mu)$ is single-valued, continuous, and monotone. Do value function iteration by hand with an initial guess of $V(\mu) = 0$. We will see that our iteration will converge to the optimum in only two steps. After studying the solution, I take the limit to continuous time and further analyze the dynamics of prices.

Similar to a mechanism design problem, I will set up the problem as one of choosing all variables (prices, beliefs, and trades) subjects to constraints for incentive compatibility and Bayesian updating to insure that it is an equilibrium of the game. So, I maximize expected discounted profit subject to the three equilibrium conditions listed in the previous section (5, 6, and Bayes' rule). As the competitive fringe is myopic, the price will not depend on $t$ or even $h^t$ beyond the information conveyed in $\mu_t$. Suppose that $P(\mu_t)$ is single valued for all values of $\mu_t$ and monotone. The incentive constraint for the competitive fringe to be maximizing is then simply $p_t = P(\mu_t)$. We can plug this straight into the problem setup.

Notice that the profit made in each period is bounded by $P(1) - P(0)$. Now when $T = \infty$, we can write the following recursive formulation.

$$V(\mu) = \max_{x \in [-1,1]} \mathbb{E}\left[\left(P(\mu') - P(\mu)\right) x + \delta V(\mu')\right] \tag{7}$$

subject to $\mu'$ being derived by Bayes' rule and incentive compatibility for the informed trader (5).

We are trying to find both trades, $x$, and posteriors, $\mu'$, to maximize our objective. Since these are linked by Bayes' rule, we can maximize over either one and take the other as a function of that variable. Take any two posteriors on opposite sides of the prior, $\bar{\mu} \geq \mu \geq \underline{\mu}$. By rearranging Bayes rule, we can see that there always exists a mixed strategy for $x$ such that these posteriors would be induced after buying ($x = 1$) and selling ($x = -1$).

$$\pi(x = 1|\omega = 1) = \frac{\bar{\mu}(\mu - \underline{\mu})}{\mu(\bar{\mu} - \underline{\mu})}; \qquad \pi(x = 1|\omega = 0) = \frac{(1 - \bar{\mu})(\mu - \underline{\mu})}{(1 - \mu)(\bar{\mu} - \underline{\mu})} \tag{8}$$

In fact, any distribution of posteriors that averages out to the prior can be induced by some trading strategy, but in this two state model we won't need more than two posteriors in equilibrium. It will become clear that even if we write the problem as one of choosing a general distribution of posteriors, the optimum will always only use two. Thus, we can plug in the (potentially mixed) strategy for $x$ directly as a function of the distribution of posteriors, $\mu'$. Altneratively, we can working with optimizing over $x$ and plugging in the posteriors as solved for by Bayes Rule. Typically, the former will be easier. In doing so, we reduce the problem of three groups of variables ($p_t$, $x_t$, and $\mu_t$) and three constraints to one of a single variable ($\mu'$) and a single constraint (5). So, the problem can be written as one of choosing two posteriors on opposite sides of the prior subject to incentive compatibility. It will be made clear in the next section that incentive compatibility here reduces to the two posteriors giving the same payoff.

I will note that Bayes Rule by itself doesn't completely pin down the trades, $x$, given the posteriors. Two posteriors opposite the prior can be induced by mixing with the proper probabilities ($\pi$ specified above) over any two trade volumes (say values $x_1$ and $x_2$). However, $x_1$ is going to induce the belief and in turn the price to rise and $x_2$ will induce the belief and price to fall. Holding fixed the beliefs, you would like to be long as much as possible when the price rises and short as much as possible when the price falls. Thus, the profit maximizing strategy must have $x_1 = 1$ and $x_2 = -1$.

## 3.1 Iteration

Assume $P(\mu)$ is continuous, and increasing. The usual arguments apply for the Contraction Mapping Theorem, so we can iterate to find the value function. Take an initial guess of $V(\mu) = 0$ and consider the first iteration.

$$V_1(\mu) = \max_{x \in [-1,1]} \mathbb{E}[(P(\mu') - P(\mu))x] \tag{9}$$

subject to Bayes Rule and incentive compatability.

Before telling you what is an equilibrium, it will be illustrative to tell you what isn't an equilibrium. It would seem natural to think that the informed trader should buy if the state is good and sell if the state is bad. This won't be supported in any equilibrium. In this candidate equilibrium, after observing a buy beliefs would go to one and the price would rise to $P(1)$. After observing the trader sell price would fall to $P(0)$. Hold the updated prices after observing buy or sell fixed and consider the best response of the informed trader. The trader gets a profit of $P(1) - P(\mu)$ if they buy and $P(\mu) - P(0)$ if they sell. Since price is continuous and monotone, there is one knife edge case where these will be equal. Call that belief $\mu^*$.

$$\mu^* = P^{-1}\left(\frac{P(1) - P(0)}{2}\right) \tag{10}$$

If the price is currently lower than that, buying is more profitable than selling. Holding fixed the competitive fringe's strategy, always buying is a profitable deviation.

In essence, it isn't credible for the informed trader to completely reveal the state because revealing the good state is more valuable that revealing the bad state. They have an incentive to lie when the state is bad. However, always buying (or always selling) won't be in a profitable equilibrium for the informed trader either. The action will be uninformative. This means that beliefs, and therefore prices, won't change. This gives zero profit to the informed trader.

Therefore, any equilibrium with positive profits must the have the informed trader playing a mixed strategy. For them to be willing to play a mixed

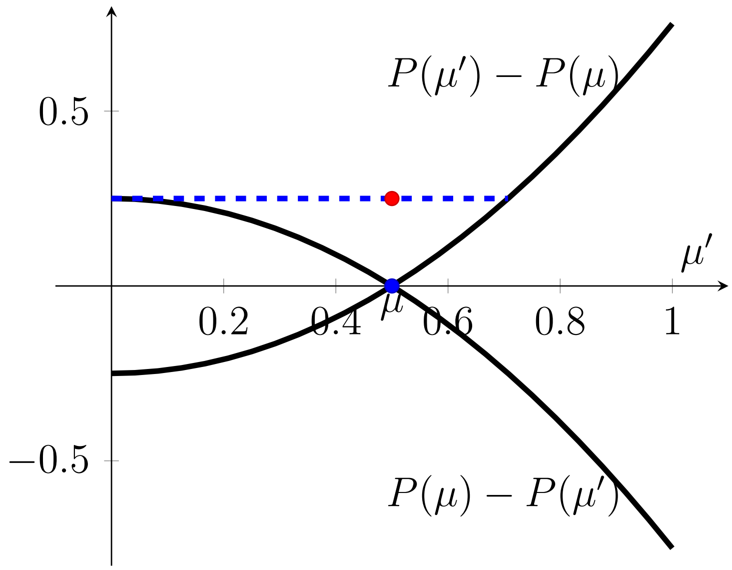


Figure 1: The upward sloping solid line is the profit from buying the asset graphed against the posterior induced. The downward sloping line is the profit from selling the asset. An equilibrium is a flat line on this graph that crosses the prior. The highest equilibrium profit is shown by the dashed line.

strategy, it must be that they are indifferent between buying and selling. This is the incentive compatibility constraint for the informed trader, simply that the induced posteriors must give the same payoff. Consider two posteriors on opposite sides of the prior that give the same payoff.

$$P(\bar{\mu}) - P(\mu) = P(\mu) - P(\underline{\mu}) \tag{11}$$

If the fringe is playing $P(\bar{\mu})$ after observing a buy and $P(\underline{\mu})$ after observing sell, then the informed trader is indifferent between the two actions. In fact, the informed trader's best response contains any mixed strategy of the two actions. As we saw above, there exists a mixed strategy such that $\bar{\mu}$ and $\underline{\mu}$ are the correct Bayesian updates. Thus, both players are playing a best response. These strategies constitute an equilibrium. If we draw a graph of the the profit to the informed trader against the posterior of the fringe, any flat line on the graph connecting two points on opposite sides of the prior is an equilibrium profit level.

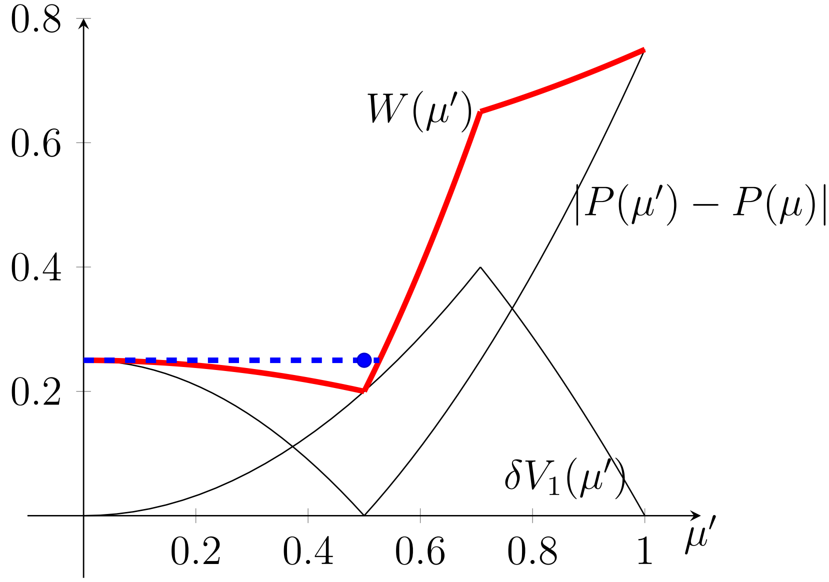


Figure 2: $W(\mu')$ is the unmaximized value of the current period payoff, $|P(\mu')-P(\mu)|$, plus the continuation value $V(\mu')$

Consider the highest profit the informed trader can obtain in equilibrium. This would be the highest flat line on such a graph. If $P(\mu)$ is monotone, then the highest payoff on either side of the posterior is at the boundary (0 or 1). Take the boundary with the lower payoff of the two. Clearly the informed trader cannot get a higher payoff than that. If $P(\mu)$ is continuous, the Intermediate Value Theorem says there must be a posterior on the other side of the prior that gives the same payoff.

This gives the new value function after one iteration.

$$V_1(\mu) = \min\{P(\mu) - P(0), P(1) - P(\mu)\} \tag{12}$$

## 3.2 Solution

To do the next iteration, all we need to do is find the highest flat line on

$$W(\mu') = |P(\mu') - P(\mu)| + \delta V_1(\mu'), \tag{13}$$

the one period payoff and the continuation value function from the previous iteration.

Just as in the previous iteration, $W(\mu')$ is decreasing for $\mu' < \mu^*$ and increasing afterwards. Take $\mu < \mu^*$. For any $\delta < 1$, $W(\mu')$ is decreasing on $[0, \mu]$. Take $\hat{\mu} \in [0, \mu]$.

$$W(\hat{\mu}) = P(\mu) - P(\hat{\mu}) + \delta(P(\hat{\mu}) - P(0)) \tag{14}$$

$$< P(\mu) - P(0) \tag{15}$$

So, the highest payoff to the left of the prior is still at the boundary. Similarly, the highest payoff to the right is at the other boundary. Again, by continuity the minimum of these can be obtained by an equilibrium. This gives our value function. This is the same value we had in the previous iteration. Thus, we have found a fixed point.

**Proposition 1.** *Let $P(\mu)$ be continuous and monotone. For any discount factor, $\delta \in [0, 1)$, the value is*

$$V(\mu) = \min\{P(\mu) - P(0),\ P(1) - P(\mu)\} \tag{16}$$

## 3.3 Price Dynamics

The endpoints of the flat line giving the value are the posteriors that are induced by the optimal strategy. We saw that one endpoint is always at the boundary (0 or 1), but the other is generally interior. Take $\mu < \mu^*$. When the informed trader sells, this perfectly reveals the state to be bad and beliefs fall to zero. When the insider buys, beliefs increase to an interior point just high enough to make the informed trader indifferent between buying and selling.

To see the optimal strategy used, we plot the unmaximized objective function, $W(\mu') = |P(\mu') - P(\mu)| + \delta V(\mu')$ (rememeber that $x$ is plus or minus one, giving the absolute value). We want to find the highest flat line on the one period payoff plus the the discounted continuation value as found in the previous proposition. The absolute value in the current period payoff and the max in the value function can be written out as cases.

$$W(\mu') = |P(\mu') - P(\mu)| + \delta V(\mu') = \begin{cases} P(\mu) - \delta P(0) - (1-\delta)P(\mu') & \text{if } \mu' \leq \mu \\ (1+\delta)P(\mu') - P(\mu) - \delta P(0) & \text{if } \mu < \mu' \leq \mu^* \\ (1-\delta)P(\mu') - P(\mu) + \delta P(1) & \text{if } \mu' > \mu^* \end{cases} \tag{17}$$

To the left of the prior we have a strictly decreasing function of $\mu'$, so the left endpoint will be zero. To the right of the prior, the function is increasing and continuous. Thus there will be a posterior to the right of the prior giving the same value as the left extreme ($\mu' = 0$). The right endpoint could be on the second or third segment of the curve $W(\mu')$.

$$\bar{\mu} = \begin{cases} P^{-1}\left(\frac{2P(\mu)-(1-\delta)P(0)}{1+\delta}\right) & \text{if } \mu \leq P^{-1}\left(\frac{P(1)(1+\delta)+P(0)(3-\delta)}{4}\right) \\ P^{-1}\left(\frac{2P(\mu)-P(0)-\delta P(1)}{1-\delta}\right) & \text{otherwise} \end{cases} \tag{18}$$

This equation doesn't look pretty or have a nice intuition to it, but it is important to note that unlike the height of the line or the left endpoint, the right endpoint does depend on $\delta$.

### 3.3.1 Continuous Time Limit

We want to study the dynamics of pricing in the continuous time limit of the game. To do this, think about the periods shrinking. Consider the limit as $\delta$ goes to 1. After the informed trader sells beliefs still drop all the way to 0 regardless of $\delta$. After the trader buys beliefs move up to the right endpoint described above. This right endpoint is moving closer and closer to the prior as $\delta$ goes to 1. You can see this because the slope of the one period payoff and the slope of the value function are excatly opposite. This means that as $\delta$ goes to one, the sum is approaching a constant (at least on the first and third segments of the function $W(\mu')$). Since $\underline{\mu}$ is constant, $\bar{\mu}$ is shrinking toward the prior, and beliefs are a martingale, the frequency of the jumps down to $P(0)$ needs to be going to 0. The likelihood that beliefs just drift up the small

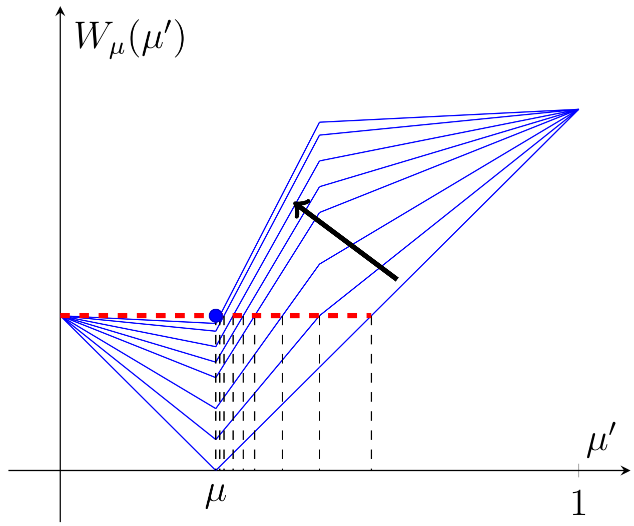


Figure 3: $W(\mu')$ for $\delta = \{0, .2, .4, .6, .7, .8, .9, .95\}$.

amount in a given period goes to 1. Though not obvious from looking at the graph, in the appendix I show that the probability of a jump is shrinking to 0 at a linear rate in $1-\delta$. This gives us the first main result of the paper.

**Theorem 1.** *For any differentiable strictly monotone price function, $P(\mu)$, as $\delta$ goes to one the price process converges to a Poisson process.*

- *If $\mu_t < \mu^*$,*

$$dP(\mu_t) = \frac{r}{2}(P(\mu_t) - P(0))dt - (P(\mu_t) - P(0))dN_t \tag{19}$$

  *where $N_t$ is a standard Poisson process with arrival rate $\lambda = \frac{r}{2}\frac{P(\mu_t)-P(0)}{\mu_t P'(\mu_t)}$.*

- *If $\mu_t > \mu^*$,*

$$dP(\mu_t) = -\frac{r}{2}(P(1) - P(\mu_t))dt + (P(1) - P(\mu_t))dN_t \tag{20}$$

  *where $N_t$ is a standard Poisson process with arrival rate $\lambda = \frac{r}{2}\frac{P(1)-P(\mu_t)}{(1-\mu_t)P'(\mu_t)}$.*

- *If $\mu_t = \mu^*$, all information is revealed immediately and the price jumps to either $P(1)$ or $P(0)$.*

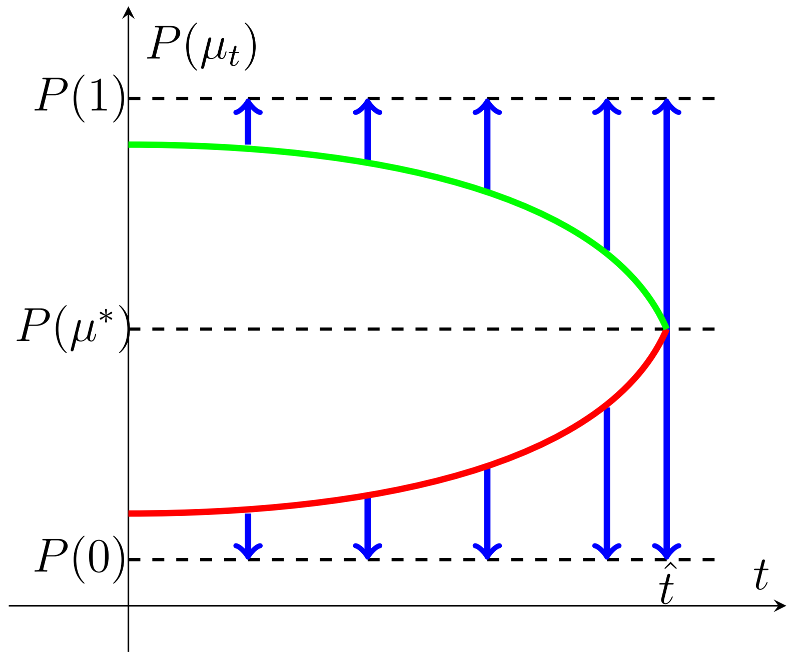


Figure 4: Sample price paths when the price starts high (green) and when the price starts low (red).

Consider a low initial price ($\mu < \mu^*$). With a Poisson arrival rate the informed trader will sell the asset and completely reveal that the state is bad. This makes the price crash to $P(0)$. While this Poisson information hasn't arrived, the informed trader will hold a long position in the asset and the price will slowly drift upward. If the price starts high, the dynamics are simply the mirror image. With a Poisson arrival rate the informed trader will buy and the price will spike to $P(1)$. At all other times the informed trader holds a short position and the price drifts down. If the beliefs ever reach $\mu^*$, the informed trader perfectly reveals the state, good or bad.

The arrival rate can be obtained intuitively by considering the informed trader's incentive compatibility constraint. The informed trader must be indifferent between revealing the state to be bad today or letting the price drift up a little and revealing the state to be bad tomorrow. Call $\lambda$ the arrival rate of the Poisson process. Once it arrives, beliefs jump down to zero. In order for beliefs to be a martingale, the drift must equal $\lambda_t \mu_t$. The drift in price is then equal to $\lambda_t \mu_t P'(\mu_t)$.

$$\underbrace{P(\mu_t) - P(0)}_{\text{reveal today}} \approx \underbrace{P'(\mu_t)\mu_t\lambda_t}_{\text{drift today}} + \delta\underbrace{\left(P'(\mu_t)\mu_t\lambda_t + P(\mu_t) - P(0)\right)}_{\text{reveal tomorrow}} \tag{21}$$

As $\delta$ gets large, this gives a linear relationship between $\lambda$ and $1-\delta$.

$$\Rightarrow \lambda_t \approx \frac{P(\mu_t) - P(0)}{2\mu_t P'(\mu_t)}(1-\delta) \tag{22}$$

## 3.4 Intuition

If the price starts low, the dynamics look like a pump-and-dump scheme. The informed trader buys every period and spreads good information to pump up the value of the asset. Then, at a random arrival date they dump all their shares and reveal the asset to be bad. This causes the price to crash suddenly. If the asset actually is good that crash date never comes, hence the positive drift.

If the price starts high we have the mirroring dynamics. The informed trader sells each period causing the price to drift down. If the asset is actually good, then at a random arrival rate they buy back all the shares and reveal the state causing a sudden spike in price. This is a short-and-distort scheme. This matches the empirical fact that pump-and-dumps are usually done on cheap stocks and short-and-distorts on more expensive stocks.

The form of the optimal strategy comes from the intuition in the one period model for why buy when good sell when bad isn't an equilibrium. If the price is initially low, the informed trader can't credibly reveal the state because revealing the good state is better than revealing the bad state. They'd like to lie and always say it's the good state.

The dynamics give them that credibility. When the informed trader says the state is bad, the fringe believes them and moves the price all the way to $P(0)$. When the informed trader says the state is good, the fringe mostly doesn't believe it because there is a much bigger potential gain from the state being good. Price increases only a small amount. After many periods of repeatedly saying the state is good, beliefs eventually drift up to $\mu^*$ which is the cutoff point for when the good state can be credibly revealed. The small price jump can be done immediately, but the informed trader needs to spend time to build credibility before they can get the big price jump.

The arrival rate of the Poisson process depends on the function $P(\mu)$. If $P(\mu)$ is linear then it simplifies to $\lambda = \frac{1}{2}$. If $P(\mu)$ is concave, then the arrival rate will be bigger than $\frac{1}{2}$ and it will be increasing over time. If $P(\mu)$ is convex, then the arrival rate will be smaller than $\frac{1}{2}$ and it will be decreasing over time. You can see this by writing the Taylor series for $P(0)$.

$$P(0) = P(\mu_t) - P'(\mu_t)\mu_t + \frac{1}{2}P''(\mu_t)\mu_t^2 + \dots \tag{23}$$

This gives us an approximate equation for the arrival rate.

$$\lambda(\mu_t) = \frac{r}{2}\frac{P(\mu_t) - P(0)}{\mu_t P'(\mu_t)} \approx \frac{r}{2} - \frac{r}{4}\mu_t \frac{P''(\mu_t)}{P'(\mu_t)} \tag{24}$$

We can see there how the arrival rate depends on the concavity of $P(\mu)$.

However, regardless of the curvature of $P(\mu)$, the price will always reach $P(\mu^*)$ and then jump to the boundaries in finite time. We can see this because the magnitude of the drift is increasing over time. Even though the arrival rate of price jumps may be increasing or decreasing, the size of the jumps is always increasing more than enough to make up for it as can be seen in the *dt* term of the price dynamics.

**Proposition 2.** *The maximum time to full information revelation is*

$$t^{max} = \frac{2}{r}\log\left(\frac{P(\mu^*) - P(0)}{P(\mu_0) - P(0)}\right) \tag{25}$$

*if* $\mu_0 \leq \mu^*$ *and*

$$t^{max} = \frac{2}{r}\log\left(\frac{P(1) - P(\mu^*)}{P(1) - P(\mu_0)}\right) \tag{26}$$

*if* $\mu_0 > \mu^*$.

When the price reaches $P(\mu^*)$, which it always does in finite time, the trader is finally able to credibly reveal all their information regardless of the state. At this point, the trader makes the same amount of profit from revealing good news or bad news. Thus, revealing the truth can now be in their best response.

# 4 Commitment

In the previous section buy if the state is good sell if the state is bad was not an equilibrium because the informed trader could not credibly commit to that strategy. In this section I'll derive the optimal policy when they do have such commitment power. Let us suppose now that the informed trader is allowed to commit ex ante to any (potentially mixed) strategy.

Rather than choose an action (buy or sell) each period the informed trader can choose a distribution of actions contingent on the state. For example they can buy when good and sell when bad, or when the state is good randomize fifty-fifty between buying and selling. This is the same type of commitment power that is assumed in the Bayesian persuasion literature. We could equivalently think of commitment as saying the informed trader has verifiable information to reveal. This essentially allows us to ignore the incentive compatibility constraint in the previous problem. This unconstrained problem is much more difficult to solve analytically, but I can still characterize the continuous time limit as in the previous section.

Even though "buy when the state is good sell when the state is bad" is clearly going to be optimal in the one period model in the dynamic model the solution is nearly the exact opposite.

## 4.1 Example

Consider a very simple example that will allow me to fix ideas and explain the general concepts of the model. While the model can be much more general than this example, the intuition is similar.

There is an asset with a payoff, $\omega$, equal to either zero or one with equal probability. This asset is just a one dollar bet. This payoff is received once at the end of the game. Suppose further that this asset is highly liquid, all agents are risk neutral, and there is no discounting. What I mean by these is that at any time, you can walk down to the market and buy or sell a share of this asset at the posted price, which is equal to its expected value. Since either state is equally likely, the initial price is $\frac{1}{2}$.

If a trader privately learns what the payoff is going to be, how can she best profit from this information? For simplicity, say that there are two trading periods before the asset pays out and that the trader can take a long or short position, but faces a capacity constraint of one share (holdings $x \in [-1, 1]$). The obvious candidate strategy is that if she learns the asset's payoff will be high ($\omega = 1$) she should buy the asset and if she learns that the asset's payoff will be low ($\omega = 0$) she should sell the asset. This strategy would earn her a total payoff of $\frac{1}{2}$. If she realizes the asset is good, she will buy it in the first period for a price of $\frac{1}{2}$. She then doesn't need to do anything in the second period because she is already holding her max amount. Then the payoffs from the asset realize and she gets 1. If the asset is actually bad, her payoffs are just the mirror. She sells the asset in the first period for a gain of $\frac{1}{2}$, then at the end of the game doesn't need to pay back anything ($\omega = 0$). Her payoff is the same in either state.

Now I'm going to propose a candidate strategy that can do better than simply buying if the asset is good and selling if it is bad. In this strategy the trader will utilize the fact that there are multiple periods of trading by misleading the market in the first period and manipulating prices to get a higher return in the second period. Consider the following randomized strategy. In the first period if the trader observes the asset is good, she will buy with probability $\frac{3}{4}$ and sell with probability $\frac{1}{4}$. If she observes the asset is bad she will do the opposite, buy with probability $\frac{1}{4}$ and sell with probability $\frac{3}{4}$. Then in the second period she will do the simple strategy of holding a long position if the asset is good and a short position if the asset is bad, just like the previously proposed strategy above.

This strategy will give a higher payoff. To see this, we first need to know what the prices will be in the second period. Since the price is always equal to the expected value, this is computed using Bayes rule. If she buys the asset in the first period, the price in the second period will be $\frac{\frac{1}{2}\frac{3}{4}}{\frac{1}{2}\frac{3}{4}+\frac{1}{2}\frac{1}{4}} = \frac{3}{4}$. If she sells in the first period, the price move the opposite way, $\frac{\frac{1}{2}\frac{1}{4}}{\frac{1}{2}\frac{1}{4}+\frac{1}{2}\frac{3}{4}} = \frac{1}{4}$.

Let's now compute the trader's expected payoff. First take the good state ($\omega = 1$). Three quarters of the time she will buy in the first period at a price of one half, in the second period maintains that long position with no cost, then receives a payment of one at the end of the game. The other one quarter of the time she will sell at a price of one half in the first period, buy at a price of one quarter in the second period, and receive a payoff of one at the end of the game. This gives an expected payoff of

$$\frac{3}{4}\left(-\frac{1}{2}+1\right)+\frac{1}{4}\left(\frac{1}{2}-2\frac{1}{4}+1\right)=\frac{5}{8}$$

which is larger than $\frac{1}{2}$. If the state is bad we can get the payoff with a similar calculation.

$$\frac{3}{4}\left(\frac{1}{2}-0\right)+\frac{1}{4}\left(-\frac{1}{2}+2\frac{3}{4}-0\right)=\frac{5}{8}$$

Most of the time the trader does the usual strategy of buying when the asset is good and selling when it is bad to get a payoff of $\frac{1}{2}$. Occasionally, she trades opposite of her information in the first period. This allows her to transact at a more favorable price in the second period. This manipulation earns a higher payoff whether the asset is good or bad.

If there were three periods of trading, the trader would be able to manipulate prices for two periods before taking the obvious trade in the last period. If there are many trading periods, the optimal strategy involves the trader potentially moving prices back and forth between higher and lower levels many times before the end. In fact, in the limit as you took an infinite number of trading periods the price would approach a Brownian motion. Every period she would choose to buy or sell with nearly equal probability. This will cause the price to continue to bounce up or down by infinitesimal amounts.

In turns out that $\frac{5}{8}$ is the highest payout the trader can guarantee herself in this two period game. Notice, however, that achieving this requires a strong amount of commitment on the part of the trader. If not committed ex ante, the trader has an incentive to deviate from the outlined strategy some of the time. When the asset is good, the trader mixes between buying and selling in the first state. When she buys in the first stage the trader gets a payoff of $\frac{1}{2}$, but when she sells in the first stage the trader gets a payoff of 1. Recall that the market doesn't observe the state ($\omega$), only the trade. Hold fixed the market prices($p_1 = \frac{1}{2}$, $p_2 = \frac{3}{4}$ if buy is observed in the first period, and $p_2 = \frac{1}{4}$ if sell is observed in the first period). When the asset is good, the trader will always prefer to sell and manipulate the price because it gives a higher payoff. However, the manipulation strategy is able to effectively move prices precisely because it is done infrequently.

Thus, without commitment power this equilibrium would fall apart completely. The best the trader can do in an equilibrium without commitment is exactly the simple strategy of buying if the asset is good and selling if the asset is bad. This is done in one period (the last one). In the more general form of the model with infinite periods, it is still the case that the best the trader can do without commitment is to reveal nearly all their information at once. The trader still will maintain a lot of power over the timing of this information dump. The profit is maximized by randomizing of the timing of the information release. In the continuous time limit the price will converge to a Poisson process as computed in the previous section.

## 4.2 Setup

The setup of the game is entirely the same as in the previous section. The only difference here is that the informed trader can commit to any distribution of trades. The competitive fringe sets the market price. The trader chooses a (possibly degenerate) distribution of trades over the interval $[-1, 1]$. A trade realizes. The competitive fringe observes the trade and updates their beliefs. Then, the same thing repeats in the next period. The payoffs are as before, except with the potential randomization of trades, the expectation operator becomes more meaningful.

As before, we can write the problem as choosing the posteriors that will be induced after buying and selling subject to the constraint that beliefs are a martingale. The only difference is that the trader does not need to be indifferent between the selected posteriors as they can commit to going through with it even if they prefer a different trade. If we write the posteriors as

$$\mu_{buy} = \mu + \bar{\epsilon}, \quad \text{and} \quad \mu_{sell} = \mu - \underline{\epsilon} \tag{27}$$

then the martingale requirement stipulates the probabilities must be

$$p(buy) = \frac{\underline{\epsilon}}{\bar{\epsilon} + \underline{\epsilon}} \quad \text{and} \quad p(sell) = \frac{\bar{\epsilon}}{\bar{\epsilon} + \underline{\epsilon}}. \tag{28}$$

The constraints $\bar{\epsilon} \in [0, 1 - \mu]$ and $\underline{\epsilon} \in [0, \mu]$ ensure that beliefs stay between zero and one.

The Bellman equation can then be written simply.

$$V(\mu) = \max_{\bar{\epsilon},\underline{\epsilon}} \left(|P(\mu + \bar{\epsilon}) - P(\mu)| + \delta V(\mu + \bar{\epsilon})\right) \frac{\underline{\epsilon}}{\bar{\epsilon} + \underline{\epsilon}} \tag{29}$$

$$+ \left(|P(\mu) - P(\mu - \underline{\epsilon})| + \delta V(\mu - \underline{\epsilon})\right) \frac{\bar{\epsilon}}{\bar{\epsilon} + \underline{\epsilon}} \tag{30}$$

$$= \max_{\bar{\epsilon},\underline{\epsilon}} \mathbb{E}_{\mu'} \left[|P(\mu') - P(\mu)| + \delta V(\mu')\right] \tag{31}$$

As the beliefs are a bounded martingale, the solution of this equation closely follows Mertens and Zamir (1977). We'll characterize the the approximate solution when it's assumed $\bar{\epsilon}$ and $\underline{\epsilon}$ are small. Then when we take the limit to continuous time, the approximation will give us the exact solution. First notice that $|P(\mu + \bar{\epsilon}) - P(\mu)| \approx |P'(\mu)|\bar{\epsilon}$. Now take a second order approximation to $V(\mu + \bar{\epsilon})$.

$$V(\mu + \bar{\epsilon}) \approx V(\mu) + V'(\mu)\bar{\epsilon} + \frac{1}{2}V''(\mu)\bar{\epsilon}^2 \tag{32}$$

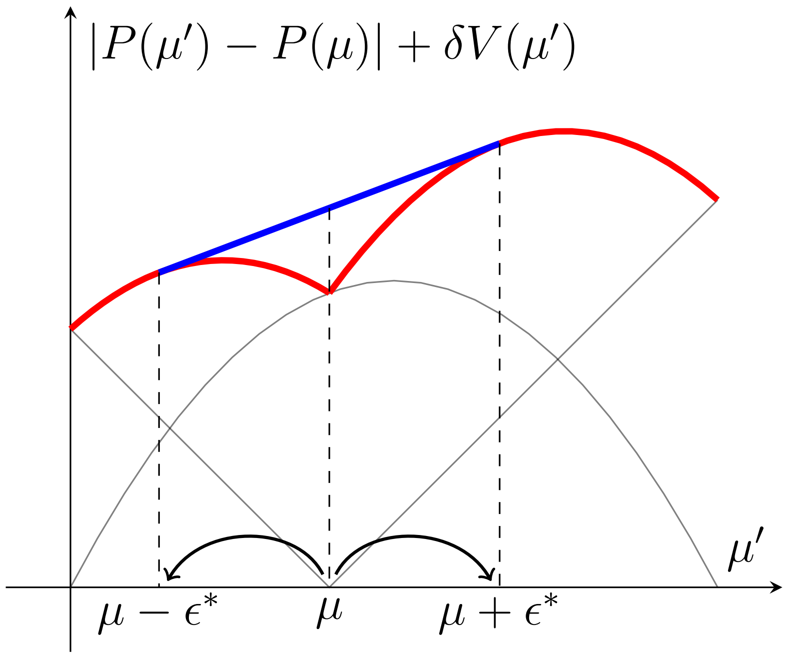


Figure 5: The faded black lines are the one period value, $|P(\mu') - P(\mu)|$, and the continuation value, $\delta V(\mu')$. The thick red line is the sum of those two. The blue segment connecting it gives the optimal policy and value. Beliefs either jump up or down by step size $\epsilon^*$ each period.

Put these into equation (29).

$$(1-\delta)V(\mu) = \max_{\bar{\epsilon},\underline{\epsilon}} \; 2|P'(\mu)|\frac{\bar{\epsilon}\underline{\epsilon}}{\bar{\epsilon}+\underline{\epsilon}} + \frac{1}{2}\delta V''(\mu)\bar{\epsilon}\underline{\epsilon} \tag{33}$$

The main trade-off can be seen in the two right hand side terms. Giving away more information gives a higher payoff this period because it leads to larger price changes. On the other hand since $V(\mu)$ is concave, the more information I give away the worse my expected continuation value.

## 4.3 Price Dynamics

Holding the product $\bar{\epsilon}\underline{\epsilon}$ fixed, we'd like to minimize the sum $\bar{\epsilon} + \underline{\epsilon}$. This is always accomplished when $\bar{\epsilon} = \underline{\epsilon}$. So, this is only a problem of one variable when we have an interior solution.

$$(1-\delta)V(\mu) = \max_{\epsilon} \ |P'(\mu)|\epsilon + \frac{1}{2}\delta V''(\mu)\epsilon^2 \tag{34}$$

Taking the derivative and setting it equal to zero yields the optimal $\epsilon$.

$$\epsilon^* = -\frac{|P'(\mu)|}{\delta V''(\mu)} \tag{35}$$

Beliefs now follow a random walk. Each period beliefs either jump up or down by a small step of size $\epsilon^*$. As $\delta$ goes to one, $V(\mu)$ and $V''(\mu)$ both grow in magnitude toward infinity. This means that the step size, $\epsilon^*$, is going to zero. The key is that it is going to zero slowly (at a rate of $\sqrt{1-\delta}$). As we take the step size shrinking to zero, this converges to a Brownian Motion for beliefs. Itó's Lemma gives us the process for prices which are a smooth function of beliefs. Thus, we have the second main result. More details are given in the appendix.

**Theorem 2.** *Let $P(\mu)$ be any $\mathcal{C}^2$ function. As $\delta$ goes to one, the price converges to an Itó Process over time.*

$$dP(\mu_t) = \frac{1}{2}P''(\mu_t)\sigma^2(\mu_t)dt + P'(\mu_t)\sigma(\mu_t)dB_t \tag{36}$$

The $B_t$ here is a standard Brownian Motion.

## 4.4 Intuition

Even though optimal information release is Brownian, there is still a non-constant drift and variance term. The function

$$\sigma(\mu_t) = \frac{\sqrt{2r}|P'(\mu_t)|}{\hat{V}''(\mu_t)} \tag{37}$$

is the standard deviation multiplying the Brownian increments in beliefs. $\hat{V}(\mu_t)$ is the value function after rescaling for the discount factor. Itó's Lemma tells us that the standard deviation multiplying the Brownian increment on prices is then $P'(\mu_t)\sigma(\mu_t)$. There is higher variance when the price is more sensitive to information.

The drift in prices is also pinned down. Since beliefs need to be a martingale, they must have zero drift. This doesn't mean that prices won't have a drift. In fact, Jensen's inequality tells us that the drift needs to be positive (negative) when price is a convex (concave) function of beliefs. Itó's Lemma confirms that the drift is $\frac{1}{2}P''(\mu_t)\sigma^2(\mu_t)dt$.

This is not a complete solution because $\sigma(\mu_t)$ was defined in terms of the value function for which a complete analytical solution cannot always be given. In the special case of a linear price function, $P(\mu_t) = \mu_t$, the analytic solution can be written. We then have that

$$\sigma(\mu_t) = n(N^{-1}(\mu_t))r \tag{38}$$

where $n(\cdot)$ is the normal distribution pdf and $N(\cdot)$ is the normal distribution cdf.

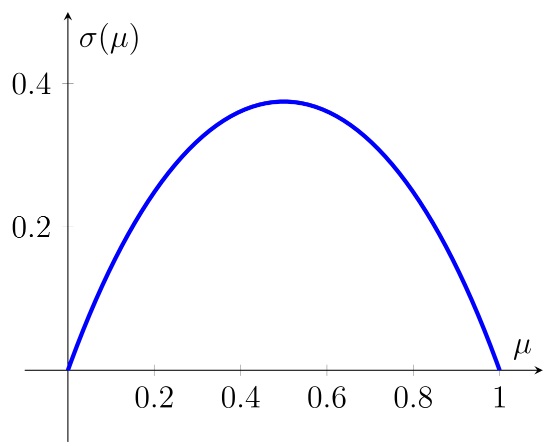


Figure 6: The standard deviation of prices as a function of beliefs.

The function $\sigma(\mu_t)$ is the normal distribution evaluated at the $\mu_t$ quantile. It is similar to a geometric Brownian motion in that the standard deviation goes to zero linearly in the price. This ensures that prices can never drop below zero. Prices are the most volatile when there is the most uncertainty.

From the perspective of the informed trader, the price still follows an Itó process. The variance is the same, but the drift is different. With the linear price function, the drift conditional on the state being good is $\frac{\sigma^2(\mu_t)}{\mu_t}$. When the state is bad, the drift is $-\frac{\sigma^2(\mu_t)}{1-\mu_t}$. Beliefs always drift toward the true state. When beliefs are far from the true state the drift is large, and when they are close to the truth the drift is small. Beliefs converge to the truth over time in the weak* topology.

### 4.5 Comparison

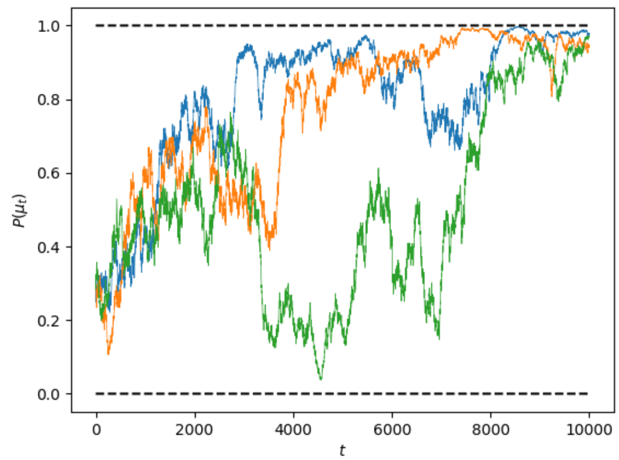


Figure 7: Three sample price paths when the true state is good.

The price paths induced with and without commitment are intuitively the opposite of each other. With commitment, the information release happens gradually and all information is revealed only in the limit. The price paths are continuous and highly volatile. This represents inside information being slowly leaked to the public and being incorporated into the price. Without commitment, all information is released suddenly in finite time. The price paths are smooth until a discontinuous jump. This is a pump-and-dump scheme to manipulate the price of the asset.

The dramatically different price dynamics can inform us on whether observed information leaks are likely to be strategic. Consider the commitment assumption to be about whether the informed trader can generate verifiable evidence or not. A strategic insider releases information with verifiable evidence gradually, but unverifiable information is released suddenly.

Together these form a micro-foundation of the price processes commonly assumed throughout finance.

# 5 Conclusion

Related to a model like Kyle (1985), this model is meant to be more efficient and have less friction. It is more efficient because all information from trades is incorporated into the price very quickly, whereas in a Kyle model only information from the aggregate level of trade is revealed. This model has less friction because it is highly liquid. The trader can buy or sell the asset at the stated price. There is no slope to the supply of limit orders on the book (at least over the amounts the trader is able to buy).

In slightly more detail, consider a Kyle model where the trader faces a capacity constraint on how much they can buy or sell such as Van Bommel (2003). Rather than simply looking at the Nash equilibrium, consider finding the communication equilibrium that gives the highest payoff (see Forges (1986)). As long as the amount of liquidity trading is large, the informed trader will always be against their capacity constraint (see Van Bommel (2003)). Since the trader has already bought (or sold) as much as they can, they are happy to fully reveal their previous action. The price in the next period will be as if the trade was observable. Now when you take the limit as the amount of noise trading goes to infinity, you get my model. As the market becomes perfectly liquid, you reach the posted price.

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# 6 Appendix: Extensions

## 6.1 Persistence

Say that the state is not permanent. Assume for this section that the state follows a Markov process. Call $\pi_1$ and $\pi_0$ the probability that $\omega_{t+1} = 1$ conditional on $\omega_t = 1$ or 0 respectively.

The timing of the game requires a bit more care in this section. At the beginning of period $t$ beliefs are $\mu_t$. Then, the informed trader can choose to buy or sell the asset at price $P(\mu_t)$. The fringe immediately observes the trade and updates beliefs according to Bayes rule to $\mu_t'$. The price is updated right away and the informed trader closes their position at price $P(\mu_t')$. Then, after trading is done $\omega_{t+1}$ is drawn from a Markov process. Beliefs at this point are updated for the next period, $\mu_{t+1} = \pi_0 + (\pi_1 - \pi_0)\mu_t'$. In the model with a permanent state, $\mu_{t+1} = \mu_t'$. It wasn't important at that point to say that the informed trader closes their position at the end of each period, because the price at the end of each period was the same as the price at the start of the next period. That is no longer the case. Between periods, the state could change. Thus, beliefs and prices will also change between each period.

There are two different ways we can think about private information of a persistent state. The first is that the informed trader is able to see $\omega_t$ every period. The second is that the informed trader is able to see the state only in the first period, $\omega_0$. In the first, the fact that the state is persistent rather than permanent is a good thing for the informed trader. It means that there is more information flowing to them each period. The informed trader then has more opportunity for profit. In the second, the fact that the state is persistent rather than permanent is a bad thing for the informed trader. It means that their information has less predictive power of the state as time passes. The informed trader's information is becoming less valuable each period.

Interestingly, both cases incentivize the informed trader to reveal information at a faster rate. I will show in this section that even though the value and the price process will look very different in the two cases, the optimal strategy is identical. I am not aware of any other paper that shows this kind of relationship between the two types of private information of a persistent state.

### 6.1.1 One Time Information

The informed trader observes $\omega_0$ but not $\omega_t$ for $t > 0$. In this section, I will assume a linear price function for simplicity. $P(\mu) = \mu$. It is still the case that the informed trader can choose to buy when the price is about to go up and sell when the price is about to go down. The objective is

$$\mathbb{E}\left[\sum_{t=0}^{\infty} \delta^t |\mu_t' - \mu_t|\right] \tag{39}$$

where $\mu_{t+1} = \pi_0 + (\pi_1 - \pi_0)\mu_t'$.

The main difference here is that the informed trader no longer has complete control over beliefs in every period. $\mu_t$ is the public belief about the state in period $t$, but the informed trader only knows the state in period 0. Call $\tilde{\mu}_t$ the public belief in date $t$ about the initial state $\omega_0$. This is the belief that the informed trader can control and it is related to the belief about the current period state.

$$\begin{aligned}
\mu_0 &= \tilde{\mu}_0 \\
\mu_1 &= \pi_0 + (\pi_1 - \pi_0)\tilde{\mu}_1 \\
&\vdots \\
\mu_t &= \sum_{\tau=0}^{t-1} \pi_0(\pi_1 - \pi_0)^\tau + (\pi_1 - \pi_0)^t \tilde{\mu}_t
\end{aligned}$$

In equation 39 the informed trader is constrained by Bayes plausibility, incentive compatability, and shrinking bounds on where beliefs can be sent due to the informativeness of their signal deteriorating. The level of persistence puts an upper and lower bound on beliefs each period.

$$\sum_{\tau=0}^{t-1} \pi_0(\pi_1 - \pi_0)^\tau \leq \mu_t \leq \sum_{\tau=0}^{t-1} \pi_0(\pi_1 - \pi_0)^\tau + (\pi_1 - \pi_0)^t \tag{40}$$

As time goes on, beliefs must ultimately converge to $\frac{\pi_0}{1-\pi_1+\pi_0}$ regardless of the informed trader's actions. The martingale condition only holds within each period. Between periods the beliefs have a drift determined solely by the persistence of the states. Intuitively, the size of the game is just shrinking over time. We can see this precisely by rewriting the problem in terms of $\tilde{\mu}_t$.

$$\mathbb{E}\left[\sum_{t=0}^{\infty} \delta^t |\mu_t' - \mu_t|\right] = \mathbb{E}\left[\sum_{t=0}^{\infty} (\delta(\pi_1 - \pi_0))^t |\tilde{\mu}_t' - \tilde{\mu}_t|\right] \tag{41}$$

The only effect of persistence on the strategy is to reduce the discount factor. Recall $\delta = \exp^{-r\Delta t}$, and set $\pi_1 = 1 - \lambda_1 \Delta t$, and $\pi_0 = \lambda_0 \Delta t$. For small time periods, $\tilde{\delta} = \delta(\pi_1 - \pi_0) \approx 1-(r+\lambda_1+\lambda_0)\Delta t$. The arrival rate of state switches simply adds on the the discount rate. The rest of the calculations from the proofs of theorem 1 and 2 go through the same as before.

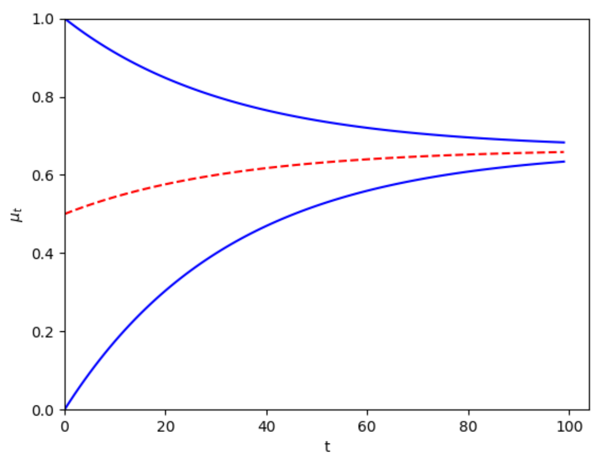



Figure 8: The solid lines are the upper and lower bounds on beliefs. The dashed line shows the drift. $\pi_1 = .99$, $\pi_0 = .02$

Without commitment beliefs follow,

$$d\tilde{\mu}_t = \frac{r + \lambda_1 + \lambda_0}{2}\tilde{\mu}_t dt - \tilde{\mu}_t dN_t \tag{42}$$

if $\tilde{\mu}_t < \frac{1}{2}$, the symmetric equation if $\tilde{\mu}_t > \frac{1}{2}$, or jump immediately to 0 or 1 if $\tilde{\mu}_t = \frac{1}{2}$. The arrival rate of the Poisson process is $\lambda(\tilde{\mu}_t) = \frac{r+\lambda_1+\lambda_0}{2}$.

With commitment beliefs follow

$$d\tilde{\mu}_t = \frac{r+\lambda_1+\lambda_0}{2}\phi(\tilde{\mu}_t)dB_t. \tag{43}$$

Prices are not based on beliefs about what the state was in date zero, $\tilde{\mu}_t$, but on beliefs about the current state, $\mu_t$. In the limit, $\mu_t$ can still be written as a function of $\tilde{\mu}_t$ and time.

$$\mu_t = \frac{\lambda_0}{\lambda_1+\lambda_0} + (1-\lambda_1-\lambda_0)^t\left(\tilde{\mu}_t - \frac{\lambda_0}{\lambda_1+\lambda_0}\right) \tag{44}$$

Prices without commitment must then follow

$$dP(\mu_t) = \left(\frac{r+\lambda_1+\lambda_0}{2}\tilde{\mu}_t + \log(1-\lambda_1-\lambda_0)\left(\tilde{\mu}_t - \frac{\lambda_0}{\lambda_1+\lambda_0}\right)\right)(1-\lambda_1-\lambda_0)^t dt - \tilde{\mu}_t(1-\lambda_1-\lambda_0)^t dN_t \tag{45}$$

whenever $\mu_t < \mu_t^*$, the symmetric equation when $\mu_t > \mu_t^*$, and jump to the shrinking boundaries immediately when $\mu_t = \mu_t^*$. Once the price hits a boundary, it remains on the boundary for the rest of the game and continues to drift toward $P\left(\frac{\lambda_0}{\lambda_1+\lambda_0}\right)$. The midpoint, $\mu_t^*$ is also changing over time now.

$$\mu_t^* = \frac{1}{2}(1-\lambda_1-\lambda_0)^t + \left(1-(1-\lambda_1-\lambda_0)^t\right)\frac{\lambda_0}{\lambda_1+\lambda_0} \tag{46}$$

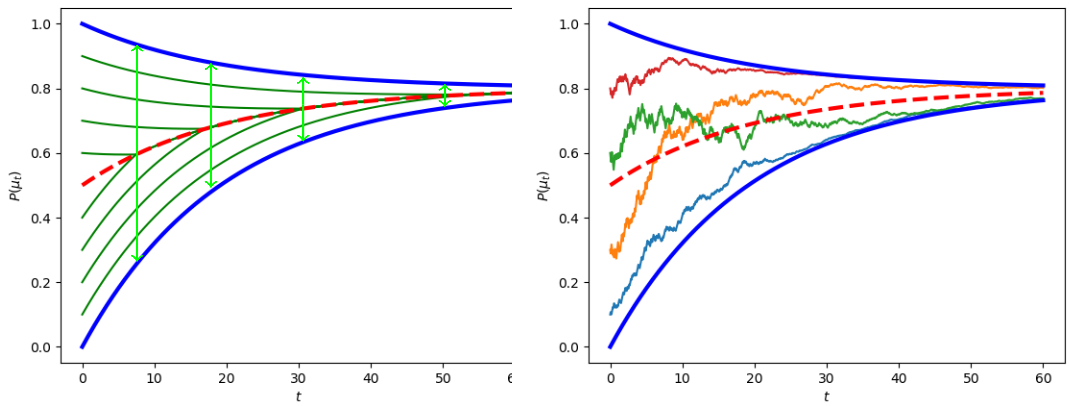
This is the dashed line in the previous figure.

With commitment prices follow

$$dP(\mu_t) = \left(\mu_t - \frac{\lambda_0}{\lambda_1+\lambda_0}\right)\log(1-\lambda_0-\lambda_1)dt + (1-\lambda_1-\lambda_0)^t\frac{r+\lambda_1+\lambda_0}{2}\phi(\mu_t)dB_t. \tag{47}$$

### 6.1.2 Information Flows Every Period

Basically, the persistence flattens out the continuation value function, but the current period payoff slope remains the same. This means you still always hit a boundary. The difference is that when you hit the boundary, the game doesn't end. Posteriors drift back interior because the state may switch and




(a) Without commitment (b) With commitment

Figure 9: Sample paths for prices.

you can trade again. It is easy to verify by iterating that the value function is now

$$V(\mu) = \min\left\{P(\mu) - P(0) + (P(\pi_0) - P(0))\frac{\delta}{1-\delta}, P(1) - P(\mu) + (P(1) - P(\pi_1))\frac{\delta}{1-\delta}\right\}. \tag{48}$$

Assume the price function is linear to solve for the policy function and take the limit. Take $\mu < \mu^*$ ($\mu^*$ is not equal to $\frac{1}{2}$ anymore). The left endpoint of the strategy jumps to zero still. Some algebra reveals that the right endpoint is

$$\mu' = \frac{2\mu}{1 + \delta(\pi_1 - \pi_0)}. \tag{49}$$

This approaches the prior, $\mu$, as $\delta$ goes to one. This is approximately equal to

$$\mu' \approx \frac{2\mu}{2 - (r + \lambda_1 + \lambda_0)\Delta t}. \tag{50}$$

Write out the probability of jumping down to zero and simplify.

$$\frac{\mu' - \mu}{\mu'} = \frac{\mu(r + \lambda_1 + \lambda_0)\Delta t}{2\mu} \tag{51}$$

Then we have our arrival rate.

$$\lambda(\mu_t) = \lim_{\Delta t \to 0} \frac{prob_{sell}}{\Delta t} = \frac{r + \lambda_1 + \lambda_0}{2} \tag{52}$$

Notice that this is exactly the same strategy we had for $\tilde{\mu}_t$ when the informed trader only knew the state in period zero. The prices and value function are different, however.

$$dP(\mu_t) = \left(\lambda_0 + \mu_t \frac{r - \lambda_0 - \lambda_1}{2}\right) dt - \mu_t dN_t \tag{53}$$

## 6.2 Multiple Informed Traders

A Brownian motion and/or a Poisson process will still arrive if there are multiple informed traders.

### 6.2.1 Independent Information

A natural question would be how this model could generate both the Brownian motion and Poisson jumps at the same time. After all, this is what we seem to see in the data and option pricing models with jumps typically have a Brownian motion with jumps not just the jumps by themselves. This is achieved by having multiple informed traders with independent pieces of information. Say the asset is Amazon stock. There may be one trader that can obtain verifiable (commitment) information about the acquisition of Whole Foods, and another trader with unverifiable (no commitment) information about the cloud computing services. Both of these pieces of information may be relevant to the value of Amazon stock, but they don't necessarily need to be correlated.

I will again explain this using a linear price function. Suppose that there are $N$ independent pieces of information, $\omega_i \in \{0, 1\}$, relevant to the value of the asset.

$$P(\mu_1, \mu_2, \ldots, \mu_N) = \sum_{i=1}^{N} \mu_i z_i \tag{54}$$

Let's solve the problem of some trader that knows the value of $\omega_j$, when there may potentially be other traders that know the other pieces of information.

Trader $j$ holds fixed the stochastic processes for $\mu_i$ for all $i \neq j$, and chooses the optimal process for $\mu_j$.

$$V_j(\mu_1, \ldots, \mu_N) = \max_{\Delta(\mu_j') \in \Delta([0,1])} \mathbb{E}\left[\left(P(\mu_1', \ldots, \mu_N') - P(\mu_1, \ldots, \mu_N)\right) x_j + \delta V_j(\mu_1', \ldots, \mu_N')\right] \tag{55}$$

subject to Bayes plausibility and incentive compatibility if solving the non-commitment problem. Trader $j$ is also not allowed to correlate their strategy for $\mu_j$ with any other $\mu_i$ as they don't know those pieces of information and they are independent.

With the linear price function and independence, the problem simplifies.

$$V_j(\mu_1, \ldots, \mu_N) = \max_{\Delta(\mu_j') \in \Delta([0,1])} \mathbb{E}[(\mu_j' - \mu_j) z_j x_j] + \sum_{i \neq j} (\mathbb{E}[\mu_i'] - \mu_i) z_i x_j + \delta \mathbb{E}\left[V_j(\mu_1', \ldots, \mu_N')\right] \tag{56}$$

The beliefs about each piece of information must be a martingale. Regardless of what strategy the other players use, each term in the sum is zero from trader $j$'s perspective.

$$V_j(\mu_1, \ldots, \mu_n) = \max_{\Delta(\mu_j) \in \Delta([0,1])} \mathbb{E}[|\mu_j' - \mu_j| z_j] + \delta \mathbb{E}[V_j(\mu_1, \ldots, \mu_N)] \tag{57}$$

Hence while prices and profits are quite different and will move randomly outside the control of player $j$, the strategy and expected profits of player $j$ remain the same as in the version with only one strategic informed trader. This holds whether the other $N-1$ pieces of information are being released by other strategic players of they are just randomly arriving by some exogenous process.

### 6.2.2 The Same Information

Suppose now that there are two identical traders that are informed of the state. In each period the price is set, then the informed traders simultaneously submit their orders. The equilibrium price process and strategies are exactly the same as in the one trader case, except the arrival rate on the Poisson process may be different. Start with the case where the price is low (below $P(\mu^*)$). Suppose that informed trader number 2 is going to buy with a high probability and sell with a low probability (this probability is linear in time and approaching a Poisson process). Suppose also that the market adjust beliefs and prices up a small amount if both informed traders buy and down to 0 and $P(0)$ if either informed trader sells. If the upward drift in price is just the right size, informed trader number 1 will be indifferent between buying, selling, or randomizing with any probability (including a probability linear in time) just as they were in the single informed trader case. Similarly, given trader 1 using a Poisson (in the limit) strategy there is a drift in price that would make trader 2 indifferent between buying, selling, and randomizing. By applying Bayes rule, we can see that the amount of upward drift in price needed to make the traders indifferent only comes from a Poisson process.

Let's consider the same condition as in the single informed trader case to find the arrival rate. Call $\pi_2$ the probability that trader 2 sells and suppose it's proportional to time ($\approx \lambda_2 \Delta t$). Call $\pi_1$ the probability that trader 1 sells. For the sake of comparison, call $\lambda_0$ the arrival rate in the single informed trader case,

$$\lambda_0 = r\frac{P(\mu) - P(0)}{2P'(\mu)\mu} \tag{58}$$

and let's assume the price function isn't too concave ($\lambda_0 < 1$) so everything remains well defined. It needs to be that trader 1 is indifferent between selling to day to reveal the state is bad and buying today to get a small drift and selling tomorrow even with the now added risk that trader 2 might sell and reveal the state to be bad.

$$P(\mu) - P(0) \approx (1-\pi_2)\left(P'(\mu)\mu(\pi_1+\pi_2) + \delta\left(P'(\mu)\mu(\pi_1+\pi_2) + P(\mu) - P(0)\right)\right) - \pi_2\left(P(\mu) - P(0)\right) \tag{59}$$

Once we simplify this equation and cancel higher order terms, we can see that the frequency with which trader 1 sells is also linear in time.

$$\pi_1 + \pi_2 \approx r\frac{P(\mu) - P(0)}{2P'(\mu)\mu}(1 + 2\lambda_2)(1 - \delta) \tag{60}$$

Now we can use the symmetry to find the arrival rate, $\lambda_1 = \lambda_2 = \lambda$ where $\lambda_1 = \frac{\pi_1}{\Delta t}$. Substituting in $\lambda_0$ and simplifying yields,

$$\lambda = \frac{1}{2}\frac{\lambda_0}{1 - \lambda_0}. \tag{61}$$

The presence of a second informed trader may increase or decrease the first trader's arrival rate for revealing their information depending on whether the price function is concave or convex. If the price function is linear, the arrival rate is still equal to $\frac{1}{2}$ as it was in the case with a single informed trader. If the price function is concave, the arrival rate was higher than $\frac{1}{2}$ originally. With multiple informed traders, not only is it higher than $\frac{1}{2}$, but it is higher than in the single informed trader cases as well. When the price function is convex, the arrival rate is lower than in the single informed trader case, which is lower than $\frac{1}{2}$. So, when the arrival rate is high, adding another informed trader increases it further, and when the arrival rate is low adding another informed trader decreases it further. However, adding a second informed trader does not decrease the speed with which information is revealed in the market. We can see from the equation that $\lambda$ is always greater than $\frac{\lambda_0}{2}$. So even though a trader may slow down their own information revelation, the total information revealed by all traders must go up.

### 6.2.3 Coordination

In the equilibrium described in the previous section, both the informed traders are randomizing over buying and selling in each period. It is assumed that such randomization needs to be independent of each other. If the informed traders had access to a joint randomization device and could coordinate their mixed strategy, this would lead to an equilibrium more profitable for both of them. In fact, such coordination would be sufficient to to get the informed traders the full commitment payoff despite the lack of commitment. The reasoning for this is the same as would apply for any cheap talk game with multiple identical senders. Suppose that the other senders are playing the full commitment strategy. You have no incentive to deviate because if you send any other message, you will be contradicted by the other senders and you won't be believed. Thus there is no profit. So even though a sender is not committed to the optimal full commitment strategy, the fact that all the other senders are going through with it removes any profit from deviating.

The argument is clear for cases with three or more senders. If you deviate, you will be outnumbered. However, the result is the same with only two senders. Simply set the off-path beliefs when the senders give conflicting messages equal to the sender's less preferred of the two options. Allow me to explain further in the context of my model.

Suppose there are to informed traders identical to that in the baseline model. If they cannot coordinate their mixed strategies, the equilibrium price will follow a Poisson process similar to the single informed trader case. If they can coordinate their mixed strategies, the equilibrium strategy with the highest payoff is exactly the same as the strategy described with a single informed trader that has commitment power. Thus, in the limit prices will follow a Brownian motion. Suppose the informed traders are randomizing with the same probabilities as in the commitment section (chapter 4). When the market sees both traders buy (sell) they will update their beliefs and prices to $\mu_{buy}$ and $P(\mu_{buy})$ ($\mu_{sell}$ and $P(\mu_{sell})$). These must be the same levels as chapter 4 because the randomization probabilities are the same. To set off-path beliefs, if one informed trader buys while the other sells define the beliefs in two cases. If $\mu \leq \mu^*$, update the beliefs to $\mu_{sell}$ and if $\mu > \mu^*$ update the beliefs to $\mu_{buy}$. Now consider an informed trader's best response when the other informed trader is playing the full commitment strategy. When the price is low ($\mu < \mu^*$), you aren't indifferent between buying to drive the price up to $P(\mu_{buy})$ and selling to drive the price down to $P(\mu_{sell})$. You prefer the buying option because it moves beliefs toward more uncertainty and thus higher future profit. That is why this strategy doesn't work in the baseline model with one uncommitted trader. However the choice is quite different now with a second trader that is following the commitment strategy. When the randomization device says to sell the choice is to sell and drive the price down to $P(\mu_{sell})$ or to buy and see the price fall to $P(\mu_{sell})$ anyway because of the conflicting messages from the two traders. If you try to deviate by buying, the price still falls to the less preferred level. The only difference by deviating is that you will lose money in the current period by being long as the price falls. Thus, you will always want to match what the other trader is doing.

The two informed traders are not colluding and they are not committed. All they need is coordination to get the full commitment profit levels. The incentive to match what the other trader is doing is enough to eliminate deviations even without commitment.

# 7 Appendix: Proofs

## 7.1 Theorem 1

Here I give the remaining details needed for the proof of theorem 1. Take $\mu < \mu^*$.

We know that each period beliefs jump down to zero or go up to an interior point that is equally good for the trader.

$$P(\mu) - P(0) = P(\mu_{buy}) - P(\mu) + \delta V(\mu_{buy}) \tag{62}$$

Putting the value function in, we can solve for the posterior induced after buying.

$$\mu_{buy} = P^{-1}\left(\frac{2P(\mu) - (1-\delta)P(0)}{1+\delta}\right) \tag{63}$$

from prior $\mu$.

Beliefs being a martingale pins down what the probability of buying and selling must be.

$$p_{sell} = \frac{P^{-1}\left(\frac{2P(\mu)-(1-\delta)P(0)}{1+\delta}\right) - \mu}{P^{-1}\left(\frac{2P(\mu)-(1-\delta)P(0)}{1+\delta}\right)}. \tag{64}$$

Taking limits gives,

$$\lim_{\delta \to 1} \mu_{buy} = \mu \tag{65}$$

and therefore

$$\lim_{\delta \to 1} p_{sell} = 0. \tag{66}$$

Recall that $\delta = \exp^{-r\Delta t} \approx 1 - r\Delta t$. We now need to show that $\lim_{\Delta t \to 1} \frac{p_{sell}}{\Delta t}$ converges to a finite positive value. That value is the arrival rate of the Poisson process.

$$\frac{p_{sell}}{\Delta t} = \frac{P^{-1}\left(\frac{2P(\mu)-(1-\delta)P(0)}{1+\delta}\right) - \mu}{P^{-1}\left(\frac{2P(\mu)-(1-\delta)P(0)}{1+\delta}\right)\frac{1-\delta}{r}} \tag{67}$$

Clearly, the numerator and denominator are both going to zero as $\delta t$ goes to one. We need to use L'Hôpital's rule.

First the derivative of the numerator.

$$\frac{\partial top}{\partial \delta} = P^{-1\prime}\left(\frac{2P(\mu)-(1-\delta)P(0)}{1+\delta}\right)\left(\frac{P(0)}{1+\delta} - \frac{2P(\mu)-(1-\delta)P(0)}{(1+\delta)^2}\right) \tag{68}$$

The limit of this is non-zero.

$$\lim_{\delta \to 1} \frac{\partial top}{\partial \delta} = \frac{1}{2}P^{-1\prime}\left(P(\mu)\right)\left(P(0) - P(\mu)\right) \tag{69}$$

Now the derivative of the denominator.

$$\begin{aligned}\frac{\partial bottom}{\partial \delta} = P^{-1\prime}\left(\frac{2P(\mu)-(1-\delta)P(0)}{1+\delta}\right)\left(\frac{P(0)}{1+\delta} - \frac{2P(\mu)-(1-\delta)P(0)}{(1+\delta)^2}\right)\frac{1-\delta}{r} \\ - \frac{1}{r}P^{-1}\left(\frac{2P(\mu)-(1-\delta)P(0)}{1+\delta}\right)\end{aligned} \tag{70}$$

This limit is even simpler.

$$\lim_{\delta \to 1} \frac{\partial bottom}{\partial \delta} = -\frac{\mu}{r} \tag{71}$$

We now have our arrival rate.

$$\begin{aligned}
\lambda(\mu) &= \lim_{\Delta t \to 0} \frac{p_{sell}}{\Delta t} && (72)\\
&= \frac{P^{-1\prime}(P(\mu))}{2\mu}(P(\mu) - P(0))r && (73)\\
&= \frac{P(\mu) - P(0)}{2P'(\mu)\mu} r && (74)
\end{aligned}$$

The last equality follows from the inverse function theorem.

This gives a Poisson term on beliefs equal to

$$-\mu_t dN_t \tag{75}$$

where $N_t$ is a standard Poisson process with arrival rate $\lambda$. The drift must be such that beliefs are a martingale.

$$\begin{aligned}
drift dt &= \mathbb{E}[\mu_t dN_t] && (76)\\
&= \mu_t \lambda(\mu_t) dt && (77)
\end{aligned}$$

Beliefs follow

$$d\mu_t = \mu_t \lambda(\mu_t) dt - \mu_t dN_t. \tag{78}$$

Since the price is a differentiable function of beliefs, this gives us the process for prices.

$$dP(\mu_t) = P'(\mu_t)\mu_t \lambda(\mu_t) dt - (P(\mu_t) - P(0)) dN_t \tag{79}$$

Putting in our equation for $\lambda(\mu_t)$ gives the result.

$$dP(\mu_t) = \frac{r}{2}(P(\mu_t) - P(0)) dt - (P(\mu_t) - P(0)) dN_t \tag{80}$$

A symmetric argument holds for $\mu > \mu^*$.

## 7.2 Proposition 2

The maximal time to full information revelation is the amount of time it take beliefs to drift to $\mu^*$ if the Poisson jump doesn't arrive. After that, beliefs immediately move to the boundary. Take $\mu_0 \leq \mu^*$. We saw from theorem 1 that

$$dP(\mu_t) = \frac{r}{2}(P(\mu_t) - P(0))dt - (P(\mu_t) - P(0))dN_t. \tag{81}$$

When the Poisson jump doesn't arrive, the price follows a smooth differentiable path. This gives a linear differential equation for prices as a function of time.

$$\frac{dP(\mu_t)}{dt} = \frac{r}{2}(P(\mu_t) - P(0)) \tag{82}$$

This implies that the price must be an exponential function in time.

$$P(\mu_t) = ce^{\frac{r}{2}t} - P(0) \tag{83}$$

We get the constant from the initial condition that the price at time 0 equals $P(\mu_0)$.

$$P(\mu_t) = (P(\mu_0) - P(0))e^{\frac{r}{2}t} - P(0) \tag{84}$$

We then solve for the time at which $P(\mu_t) = P(\mu^*)$. This gives the result.

$$t^{max} = \frac{2}{r}\log\left(\frac{P(\mu^*) - P(0)}{P(\mu_0) - P(0)}\right) \tag{85}$$

If we start with a high initial belief, $\mu_0 > \mu^*$, the only change in the equations is the use of $P(1)$ in place of $P(0)$.

## 7.3 Theorem 2

Here I give the remaining details needed for the proof of theorem 2. The proof is largely the same as the proof of the main result in Mertens and Zamir (1977). Once you write the problem as choosing posteriors subject to Bayes' plausibility, the problem becomes

$$V(\mu) = \max_{\bar{\epsilon},\underline{\epsilon}} \ |P(\mu+\bar{\epsilon}) - P(\mu)| \frac{\underline{\epsilon}}{\bar{\epsilon}+\underline{\epsilon}} + |P(\mu-\underline{\epsilon}) - P(\mu)| \frac{\bar{\epsilon}}{\bar{\epsilon}+\underline{\epsilon}} + \delta\left(V(\mu+\bar{\epsilon})\frac{\underline{\epsilon}}{\bar{\epsilon}+\underline{\epsilon}} + V(\mu-\underline{\epsilon})\frac{\bar{\epsilon}}{\bar{\epsilon}+\underline{\epsilon}}\right). \quad (86)$$

First note that for small values of $\bar{\epsilon}$

$$|P(\mu+\bar{\epsilon}) - P(\mu)| \approx |P'(\mu)|\bar{\epsilon} \quad (87)$$

and the same for $\underline{\epsilon}$.

Now take a second order approximation to $V(\mu')$.

$$V(\mu+\bar{\epsilon}) \approx V(\mu) + V'(\mu)\bar{\epsilon} + \frac{1}{2}V''(\mu)\bar{\epsilon}^2 \quad (88)$$

Then

$$V(\mu+\bar{\epsilon})\frac{\underline{\epsilon}}{\bar{\epsilon}+\underline{\epsilon}} + V(\mu-\underline{\epsilon})\frac{\bar{\epsilon}}{\bar{\epsilon}+\underline{\epsilon}} = V(\mu) + \frac{1}{2}V''(\mu)\bar{\epsilon}\underline{\epsilon} \quad (89)$$

since the first order term cancels out.

Now (86) becomes

$$(1-\delta)V(\mu) = \max_{\bar{\epsilon},\underline{\epsilon}} \ 2|P'(\mu)|\frac{\bar{\epsilon}\underline{\epsilon}}{\bar{\epsilon}+\underline{\epsilon}} + \frac{1}{2}\delta V''(\mu)\bar{\epsilon}\underline{\epsilon} \quad (90)$$

subject to the posterior remaining between zero and one.

The first term is positive, so for any given level of the product $\bar{\epsilon}\underline{\epsilon}$ you would like to minimize the denominator $\bar{\epsilon}+\underline{\epsilon}$. The way to minimize the sum of two variables given a fixed level of their product is always $\bar{\epsilon} = \underline{\epsilon}$. Call it $\epsilon$.

$$(1-\delta)V(\mu) = \max_{\epsilon} \ |P'(\mu)|\epsilon + \frac{1}{2}\delta V''(\mu)\epsilon^2 \quad (91)$$

Taking the derivative of the right hand side and setting it equal to zero yields

$$\epsilon^* = -\frac{|P'(\mu)|}{\delta V''(\mu)}. \quad (92)$$

Note that this is positive because $V(\mu)$ is concave.

Putting the solution for $\epsilon^*$ into the objective gives a second order differential equation.

$$V''(\mu)V(\mu) = -\frac{|P'(\mu)|^2}{2\delta(1-\delta)} \tag{93}$$

Define

$$\hat{V}(\mu) = V(\mu)\sqrt{2\delta(1-\delta)}. \tag{94}$$

This gives a simpler differential equation.

$$\hat{V}''(\mu)\hat{V}(\mu) = -|P'(\mu)|^2 \tag{95}$$

While this differential equation is not generally solvable for any function $P(\mu)$ we can see that $\hat{V}(\mu)$ is constant in $\delta$.

This implies that $V''(\mu)$ is going to minus infinity at rate $\frac{1}{\sqrt{\delta(1-\delta)}} \approx \frac{1}{\sqrt{r\Delta t}}$.

We now have beliefs following a binomial model,

$$\mu' - \mu = \begin{cases} \sigma(\mu)\sqrt{\Delta t} & \text{with probability } \frac{1}{2} \\ -\sigma(\mu)\sqrt{\Delta t} & \text{with probability } \frac{1}{2} \end{cases} \tag{96}$$

where

$$\sigma(\mu) = \frac{\sqrt{2r}|P'(\mu)|}{\hat{V}''(\mu)}. \tag{97}$$

This gives convergence of beliefs to a Brownian motion.

$$d\mu_t = \sigma(\mu)dB_t \tag{98}$$

Itô's Lemma gives the price process.

$$dP(\mu_t) = \frac{1}{2}P''(\mu_t)\sigma^2(\mu_t)dt + P'(\mu_t)\sigma(\mu_t)dB_t \tag{99}$$