Infinity and Beyond

June 28, 2023

Isaac Van Doren

Finite Sets

Finite Sets

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Finite Sets

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Finite Sets

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{1,2,3,4}

Finite Sets

$\{\}$

$\{$  $\}$

$\{1,2,3,4\}$

$\{890, \text{“foo”}, \pi\}$

Infinite Sets

Infinite Sets

$$\mathbb{N} = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, \ldots\}$$

Infinite Sets

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$$\mathbb{Z} = \{0, -1, 1, -2, 2, -3, 3, -4, 4, -5, 5, \ldots\}$$

Infinite Sets

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$$\mathbb{Z} = \{0, -1, 1, -2, 2, -3, 3, -4, 4, -5, 5, \ldots\}$$
$$\mathbb{Q} = \{0, 1, 2, \tfrac{1}{2}, 3, 4, \tfrac{3}{2}, \tfrac{2}{3}, \tfrac{1}{4}, \tfrac{1}{5}, 5, 6, \tfrac{5}{2}, \ldots\}$$

Infinite Sets

Infinite Sets

{ every possible book }

Infinite Sets

{ every possible book }

{ every possible book that starts with “supercalifragilisticexpialidocious”}

Infinite Sets

{ every possible book }

{ every possible book that starts with “supercalifragilisticexpialidocious”}

{ every point on a circle }

Infinite Sets

{ every possible book }
{ every possible book that starts with “supercalifragilisticexpialidocious”}
{ every point on a circle }
{ every possible board game }

Infinite Sets

{ every possible book }
{ every possible book that starts with “supercalifragilisticexpialidocious”}
{ every point on a circle }
{ every possible board game }
{ every color }

Infinite Sets

{ every possible book }
{ every possible book that starts with “supercalifragilisticexpialidocious”}
{ every point on a circle }
{ every possible board game }
{ every color }
{ every triangle }

Subsets

Subsets

$$\{1, 2, 3\} \subseteq \{1, 2, 3, 4, 5\}$$

Subsets

$$\{1, 2, 3\} \subseteq \{1, 2, 3, 4, 5\}$$
$$\{700, 543, \tfrac{1}{2}\} \not\subseteq \{1, 3, 5, 700\}$$

Subsets

$$\{1, 2, 3\} \subseteq \{1, 2, 3, 4, 5\}$$
$$\{700, 543, \tfrac{1}{2}\} \not\subseteq \{1, 3, 5, 700\}$$
$$\{\} \subseteq \{0, 1\}$$

Size

How can we tell if two sets are the same size?

Size

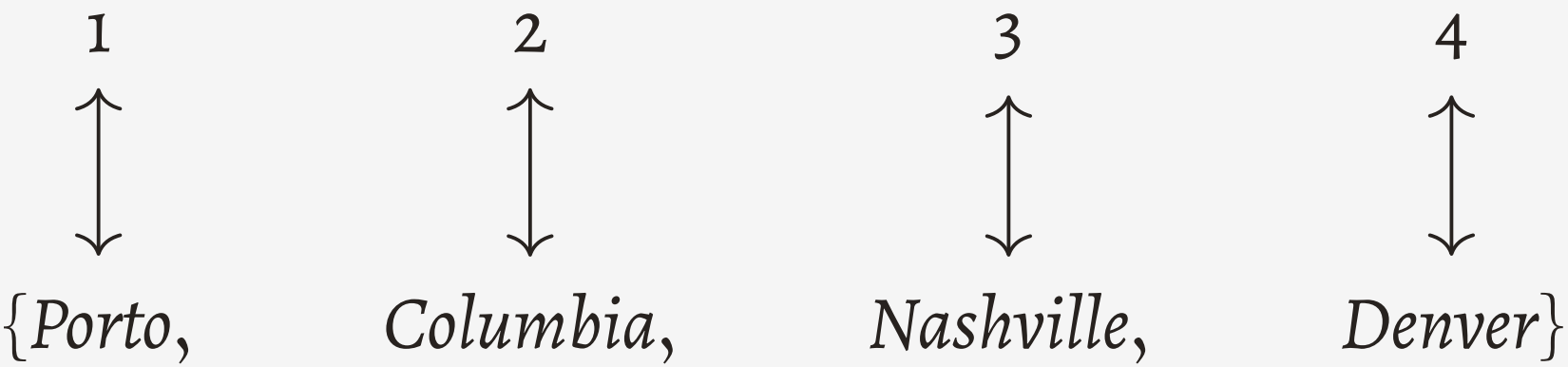
How can we tell if two sets are the same size?

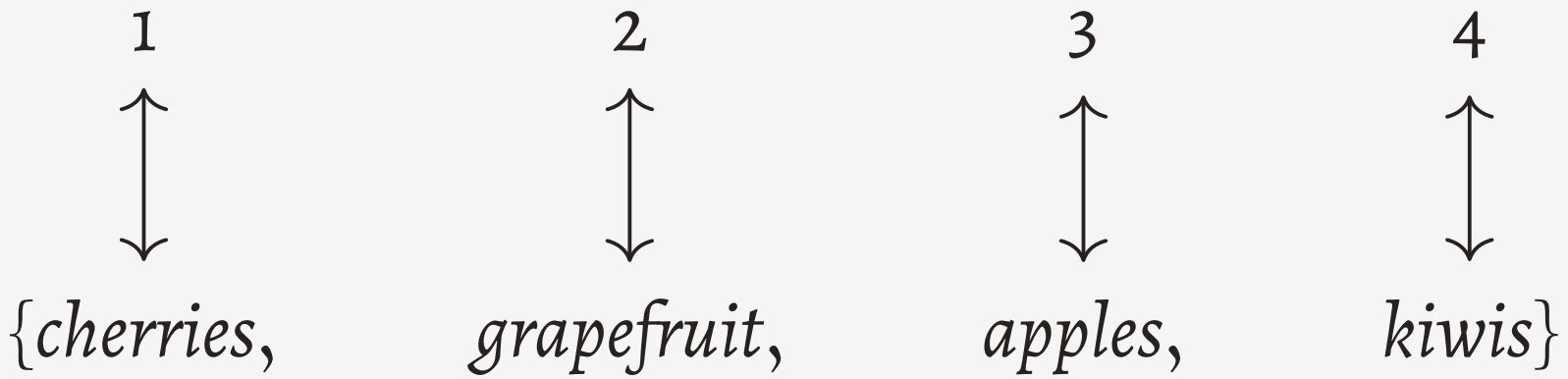
Just count them!

Size

{Porto, Columbia, Nashville, Denver}
{ cherries, grapefruit, apples, kiwis }

Size


1
2
3
4
{Porto,
Columbia,
Nashville,
Denver}


1
2
3
4
{cherries,
grapefruit,
apples,
kiwis}

Size

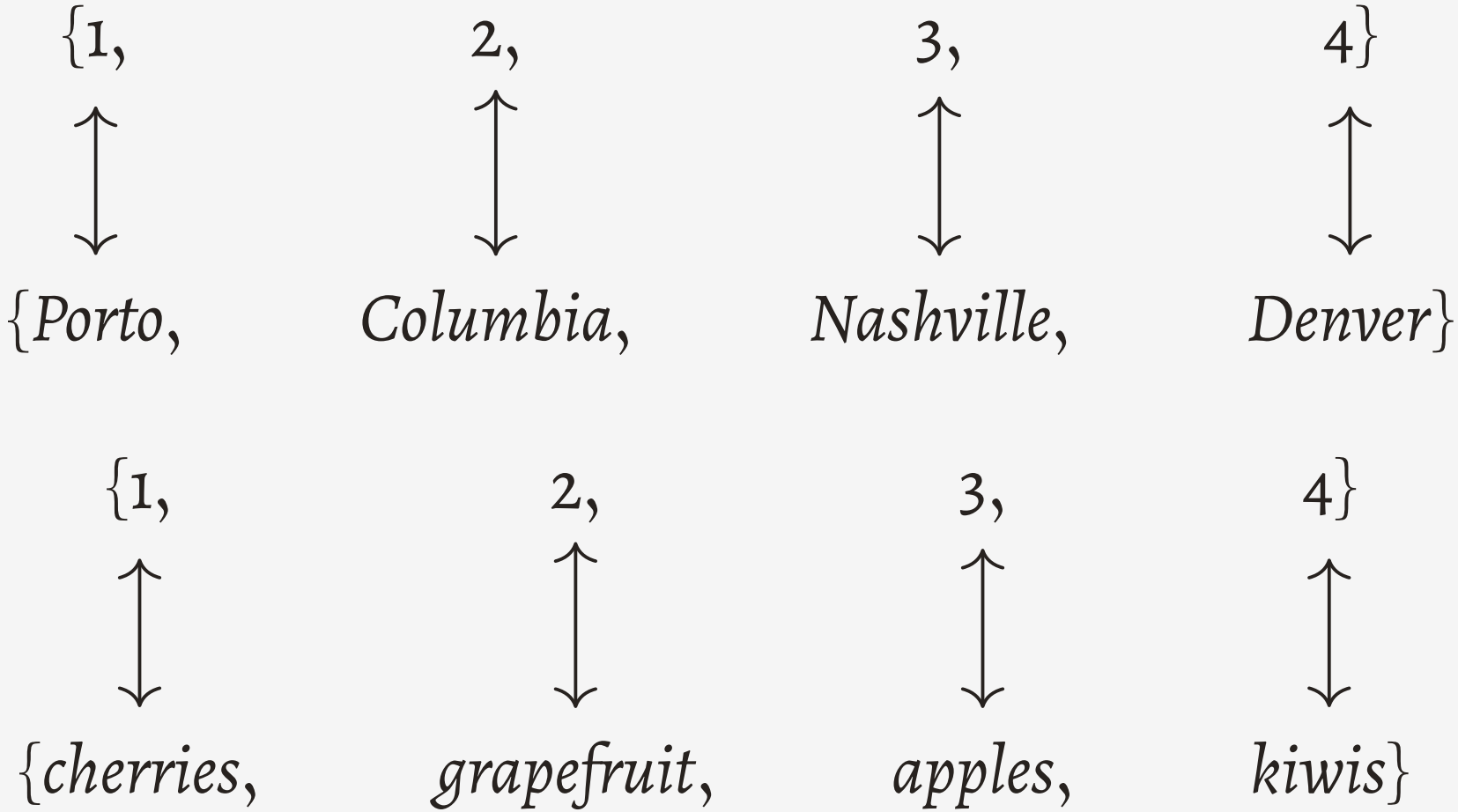
$$4 = 4$$

Size

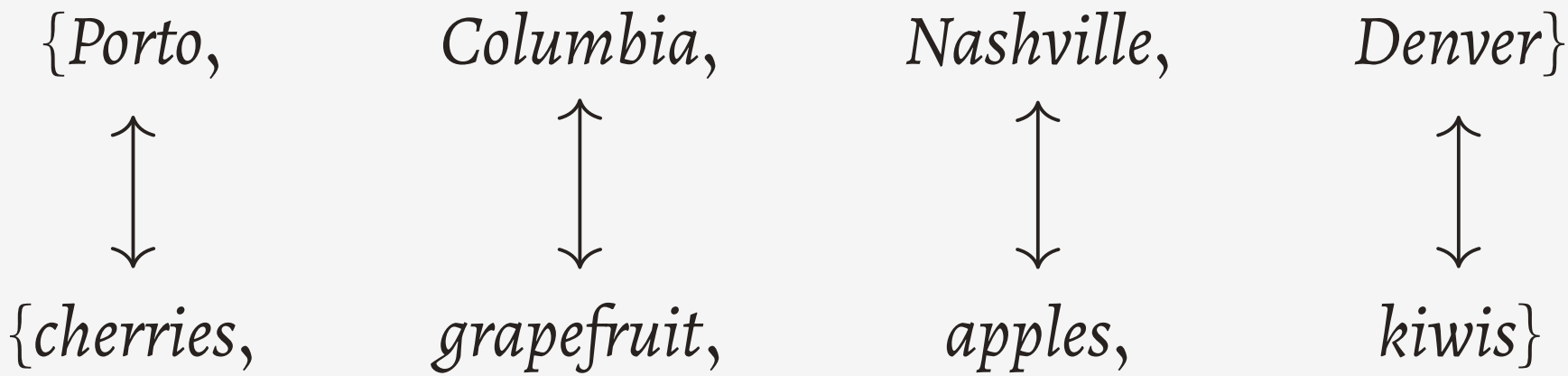
$4 = 4$

$\#\ \text{cities} = \#\ \text{fruits}$

Size

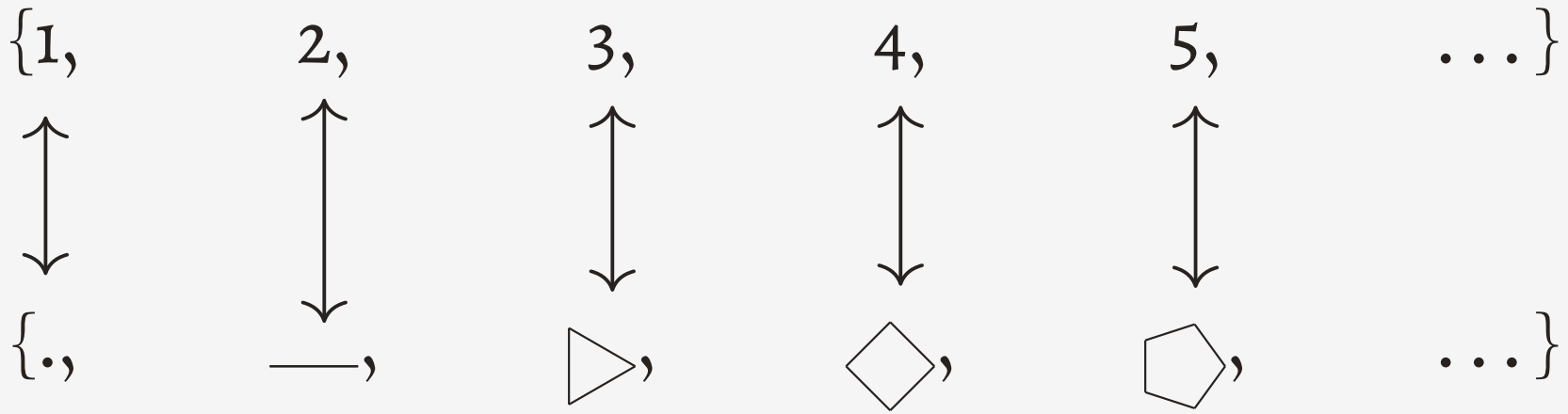

{1, 2, 3, 4}
{Porto, Columbia, Nashville, Denver}
{1, 2, 3, 4}
{cherries, grapefruit, apples, kiwis}

Size


{Porto,
Columbia,
Nashville,
Denver}
{cherries,
grapefruit,
apples,
kiwis}

Generalization

Generalization



Surprise #1 - Subsets

$$\{1, 2\} \subseteq \{1, 2, 3, 4\}$$

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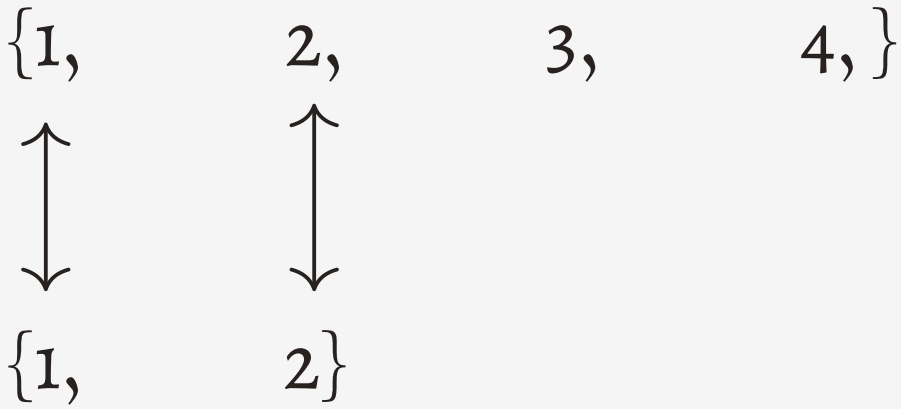
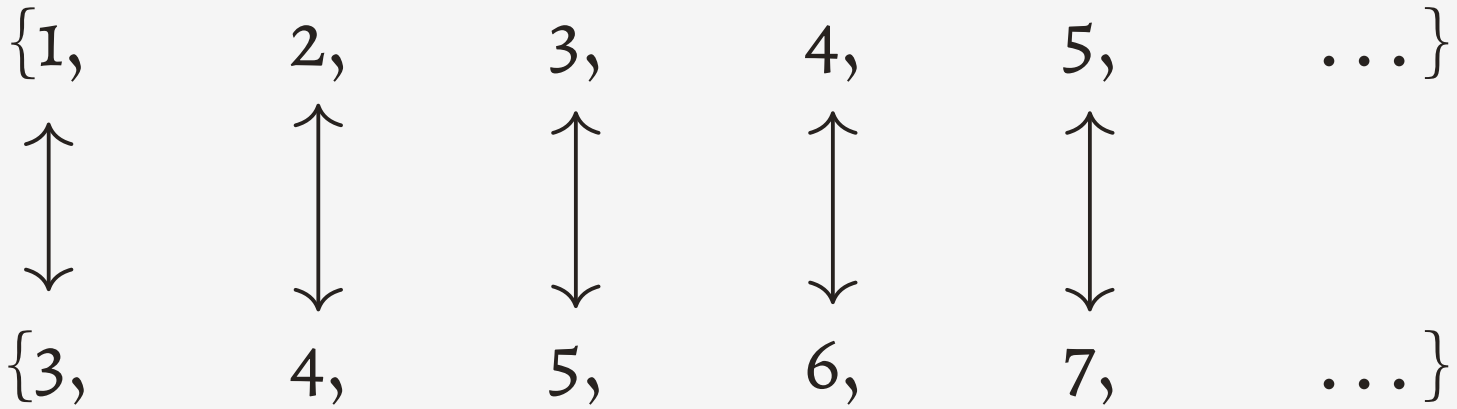


Surprise #1 - Subsets

$$\{3, 4, 5, 6, 7, \ldots\} \subseteq \{1, 2, 3, 4, 5, \ldots\}$$

Surprise #1 - Subsets

$$\{3, 4, 5, 6, 7, \ldots\} \subseteq \{1, 2, 3, 4, 5, \ldots\}$$



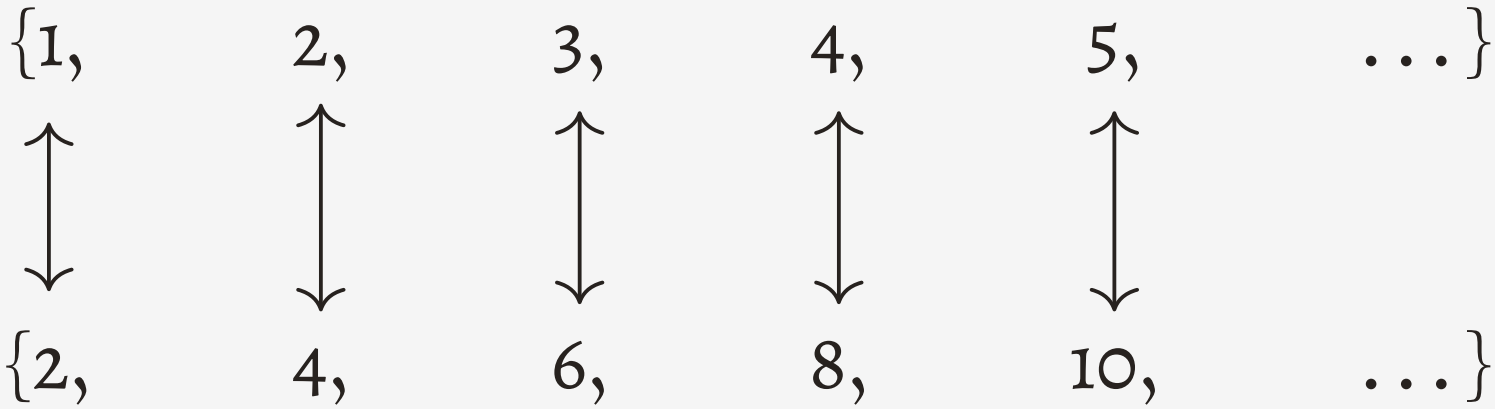
Surprise #1 - Subsets

Surprise #1 - Subsets

$$\{2, 4, 6, 8, 10, \ldots\} \subseteq \{1, 2, 3, 4, 5, \ldots\}$$

Surprise #1 - Subsets

$$\{2, 4, 6, 8, 10, \ldots\} \subseteq \{1, 2, 3, 4, 5, \ldots\}$$

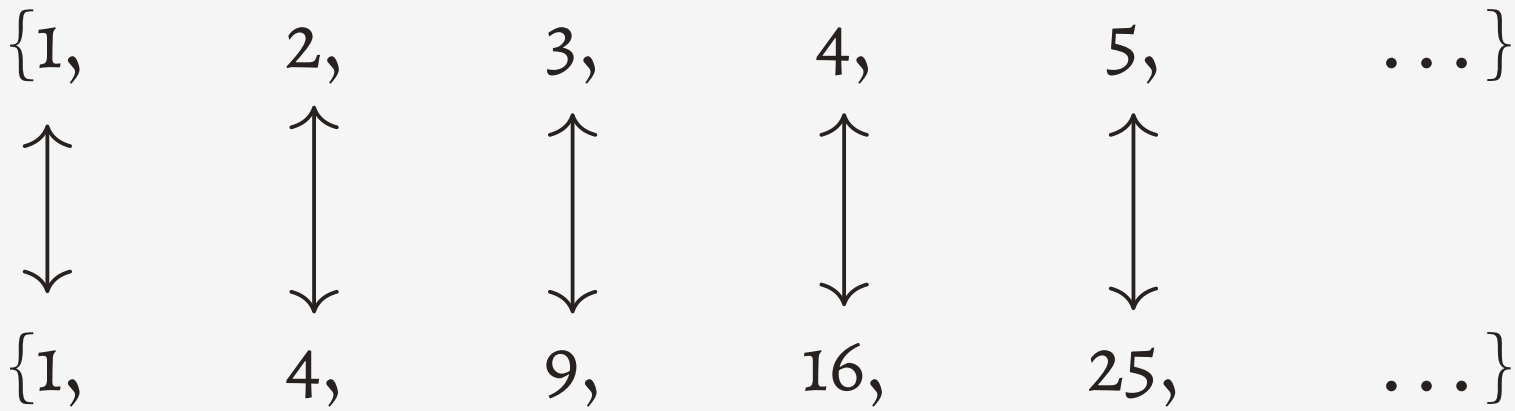


Surprise #1 - Subsets

$$\{1, 4, 9, 16, 25, \ldots\} \subseteq \{1, 2, 3, 4, 5, \ldots\}$$

Surprise #1 - Subsets

$$\{1, 4, 9, 16, 25, \ldots\} \subseteq \{1, 2, 3, 4, 5, \ldots\}$$



Surprise #1 - Subsets

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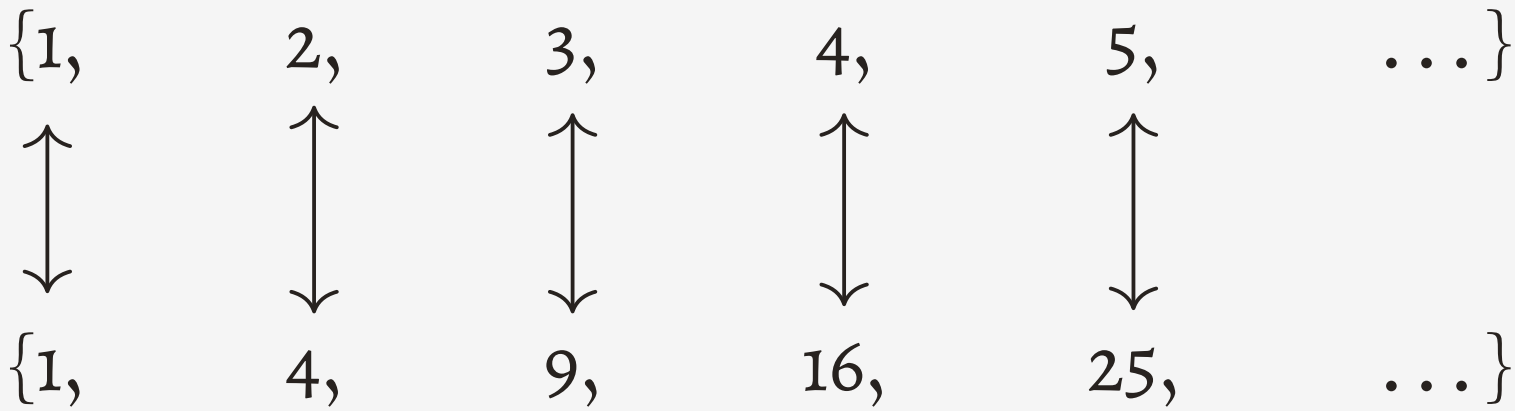
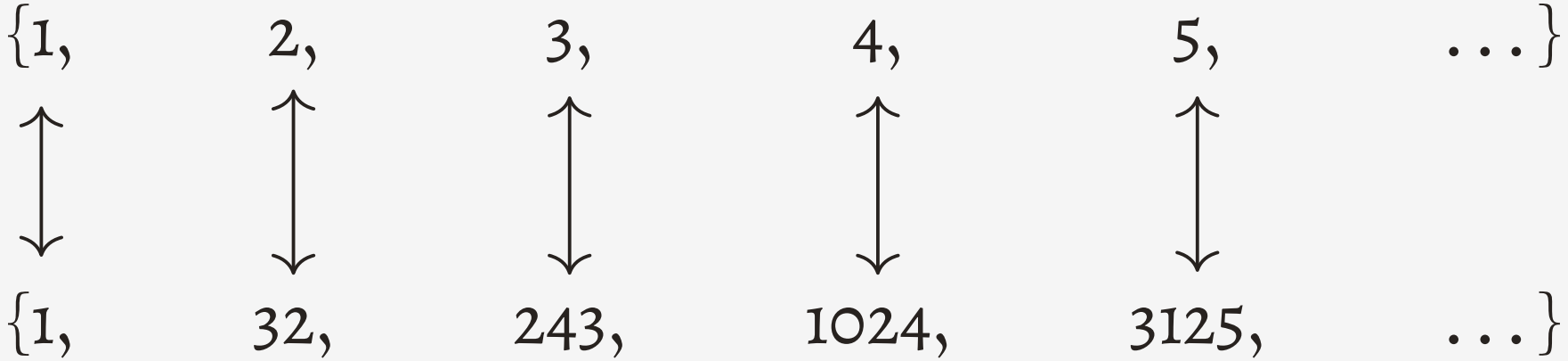


Surprise #1 - Subsets

$$\{1, 32, 243, 1024, 3125, \ldots\} \subseteq$$
$$\{1, 2, 3, 4, 5, \ldots\}$$

Surprise #1 - Subsets

$$\{1, 32, 243, 1024, 3125, \ldots\} \subseteq$$
$$\{1, 2, 3, 4, 5, \ldots\}$$



Surprise #1 - Subsets

So infinity is infinity is infinity then… right?

Surprise #1 - Subsets

So infinity is infinity is infinity then… right?

No!

There are different sized infinities!

Surprise #2 - Sizes

The Cantor Set

$2^{\omega} = \{00000000\ldots, 11111111\ldots, 10101010\ldots,$
$00110100\ldots, 11111010\ldots, \ldots\}$

= all infinite binary sequences

Surprise #2 - Sizes

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Can we count it?

Surprise #2 - Sizes

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The Cantor Set

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$00110100\ldots, 11111010\ldots, \ldots\}$

= all infinite binary sequences

Can we count it? No, we can't?!

Cantor's Diagonal Argument

	1	2	3	4	5	6	7	⋯
1	1	0	0	1	0	0	0	⋯
2	0	0	0	1	1	1	1	⋯
3	0	1	1	1	0	0	1	⋯
4	0	0	1	1	0	0	0	⋯
5	1	1	1	0	1	0	0	⋯
6	0	1	0	1	0	1	0	⋯
7	0	0	0	0	1	0	0	⋯
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	

	0							
	1	2	3	4	5	6	7	⋯
1	1	0	0	1	0	0	0	⋯
2	0	0	0	1	1	1	1	⋯
3	0	1	1	1	0	0	1	⋯
4	0	0	1	1	0	0	0	⋯
5	1	1	1	0	1	0	0	⋯
6	0	1	0	1	0	1	0	⋯
7	0	0	0	0	1	0	0	⋯
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	

	0	1					
	1	2	3	4	5	6	7	⋯
1	1	0	0	1	0	0	0	⋯
2	0	0	0	1	1	1	1	⋯
3	0	1	1	1	0	0	1	⋯
4	0	0	1	1	0	0	0	⋯
5	1	1	1	0	1	0	0	⋯
6	0	1	0	1	0	1	0	⋯
7	0	0	0	0	1	0	0	⋯
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	

	0	1	0					
	1	2	3	4	5	6	7	⋯
1	1	0	0	1	0	0	0	⋯
2	0	0	0	1	1	1	1	⋯
3	0	1	1	1	0	0	1	⋯
4	0	0	1	1	0	0	0	⋯
5	1	1	1	0	1	0	0	⋯
6	0	1	0	1	0	1	0	⋯
7	0	0	0	0	1	0	0	⋯
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	

	0	1	0	0				
	1	2	3	4	5	6	7	⋯
1	1	0	0	1	0	0	0	⋯
2	0	0	0	1	1	1	1	⋯
3	0	1	1	1	0	0	1	⋯
4	0	0	1	1	0	0	0	⋯
5	1	1	1	0	1	0	0	⋯
6	0	1	0	1	0	1	0	⋯
7	0	0	0	0	1	0	0	⋯
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	

	0	1	0	0	0			
	1	2	3	4	5	6	7	⋯
1	1	0	0	1	0	0	0	⋯
2	0	0	0	1	1	1	1	⋯
3	0	1	1	1	0	0	1	⋯
4	0	0	1	1	0	0	0	⋯
5	1	1	1	0	1	0	0	⋯
6	0	1	0	1	0	1	0	⋯
7	0	0	0	0	1	0	0	⋯
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	

	0	1	0	0	0	0		
	1	2	3	4	5	6	7	⋯
1	1	0	0	1	0	0	0	⋯
2	0	0	0	1	1	1	1	⋯
3	0	1	1	1	0	0	1	⋯
4	0	0	1	1	0	0	0	⋯
5	1	1	1	0	1	0	0	⋯
6	0	1	0	1	0	1	0	⋯
7	0	0	0	0	1	0	0	⋯
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	

	0	1	0	0	0	0	1	
	1	2	3	4	5	6	7	⋯
1	1	0	0	1	0	0	0	⋯
2	0	0	0	1	1	1	1	⋯
3	0	1	1	1	0	0	1	⋯
4	0	0	1	1	0	0	0	⋯
5	1	1	1	0	1	0	0	⋯
6	0	1	0	1	0	1	0	⋯
7	0	0	0	0	1	0	0	⋯
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	

	0	1	0	0	0	0	1	
	1	2	3	4	5	6	7	⋯
1	1	0	0	1	0	0	0	⋯
2	0	0	0	1	1	1	1	⋯
3	0	1	1	1	0	0	1	⋯
4	0	0	1	1	0	0	0	⋯
5	1	1	1	0	1	0	0	⋯
6	0	1	0	1	0	1	0	⋯
7	0	0	0	0	1	0	0	⋯
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	