Infinity and Beyond

June 29, 2023

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Finite Sets

Finite Sets

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Finite Sets

$\{\}$

$\{\textcolor{red}{\bullet}, \textcolor{blue}{\bullet}, \textcolor{green}{\bullet}\}$

Finite Sets

$\{\}$

$\{\textcolor{red}{\bullet}, \textcolor{blue}{\bullet}, \textcolor{green}{\bullet}\}$

$\{1,2,3,4\}$

Finite Sets

$\{\}$

$\{\textcolor{red}{\bullet},\textcolor{blue}{\bullet},\textcolor{green}{\bullet}\}$

$\{1,2,3,4\}$

$\{890, \text{“foo”}, \pi\}$

Infinite Sets

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$$\mathbb{N} = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, \ldots\}$$

Infinite Sets

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$$\mathbb{Z} = \{0, -1, 1, -2, 2, -3, 3, -4, 4, -5, 5, \ldots\}$$

Infinite Sets

$$\mathbb{N} = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, \dots\}$$
$$\mathbb{Z} = \{0, -1, 1, -2, 2, -3, 3, -4, 4, -5, 5, \dots\}$$
$$\mathbb{Q} = \{0, 1, 2, \tfrac{1}{2}, 3, 4, \tfrac{3}{2}, \tfrac{2}{3}, \tfrac{1}{4}, \tfrac{1}{5}, 5, 6, \tfrac{5}{2}, \dots\}$$

Infinite Sets

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{ every possible book }

Infinite Sets

{ every possible book }

{ every possible book that starts with "supercalifragilisticexpialidocious"}

Infinite Sets

{ every possible book }

{ every possible book that starts with "supercalifragilisticexpialidocious"}

{ every point on a circle }

Infinite Sets

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{ every triangle }

Subsets

Subsets

$$\{1, 2, 3\} \subseteq \{1, 2, 3, 4, 5\}$$

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$$\{1, 2, 3\} \subseteq \{1, 2, 3, 4, 5\}$$
$$\{700, 543, \tfrac{1}{2}\} \not\subseteq \{1, 3, 5, 700\}$$

Subsets

$$\{1, 2, 3\} \subseteq \{1, 2, 3, 4, 5\}$$
$$\{700, 543, \tfrac{1}{2}\} \not\subseteq \{1, 3, 5, 700\}$$
$$\{\} \subseteq \{0, 1\}$$

Size

How can we tell if two sets are the same size?

Size

How can we tell if two sets are the same size?

Just count them!

Size

{Porto, Columbia, Nashville, Denver}
{ cherries, grapefruit, apples, kiwis }

Size

$$\begin{array}{cccc} 1 & 2 & 3 & 4 \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow \\ \{\textit{Porto}, & \textit{Columbia}, & \textit{Nashville}, & \textit{Denver}\} \end{array}$$

$$\begin{array}{cccc} 1 & 2 & 3 & 4 \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow \\ \{\textit{cherries}, & \textit{grapefruit}, & \textit{apples}, & \textit{kiwis}\} \end{array}$$

Size

$$4 = 4$$

Size

4 = 4

cities = # fruits

Size

$$\begin{array}{cccc} \{1, & 2, & 3, & 4\} \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow \\ \{\mathit{Porto}, & \mathit{Columbia}, & \mathit{Nashville}, & \mathit{Denver}\} \end{array}$$

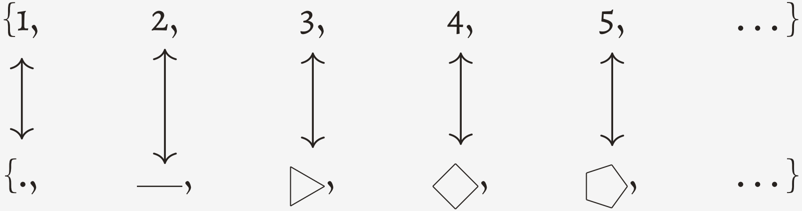
$$\begin{array}{cccc} \{1, & 2, & 3, & 4\} \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow \\ \{\mathit{cherries}, & \mathit{grapefruit}, & \mathit{apples}, & \mathit{kiwis}\} \end{array}$$

Size

$$\begin{array}{cccc} \{\textit{Porto}, & \textit{Columbia}, & \textit{Nashville}, & \textit{Denver}\} \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow \\ \{\textit{cherries}, & \textit{grapefruit}, & \textit{apples}, & \textit{kiwis}\} \end{array}$$

Generalization

Generalization



Surprise #1 - Subsets

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$$\{1, 2\} \subseteq \{1, 2, 3, 4\}$$

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$$\{1, 2\} \subseteq \{1, 2, 3, 4\}$$

$$\begin{array}{cccc}
\{1, & 2, & 3, & 4, \} \\
\updownarrow & \updownarrow & & \\
\{1, & 2\} & &
\end{array}$$

Surprise #1 - Subsets

$$\{3, 4, 5, 6, 7, \ldots\} \subseteq \{1, 2, 3, 4, 5, \ldots\}$$

Surprise #1 - Subsets

$$\{3, 4, 5, 6, 7, \dots\} \subseteq \{1, 2, 3, 4, 5, \dots\}$$

$$\begin{array}{cccccc}
\{1, & 2, & 3, & 4, & 5, & \dots\} \\
\updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \\
\{3, & 4, & 5, & 6, & 7, & \dots\}
\end{array}$$

Surprise #1 - Subsets

$$\{2, 4, 6, 8, 10, \ldots\} \subseteq \{1, 2, 3, 4, 5, \ldots\}$$

Surprise #1 - Subsets

$$\{2, 4, 6, 8, 10, \ldots\} \subseteq \{1, 2, 3, 4, 5, \ldots\}$$

$$\begin{array}{cccccc}
\{1, & 2, & 3, & 4, & 5, & \ldots\} \\
\updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \\
\{2, & 4, & 6, & 8, & 10, & \ldots\}
\end{array}$$

Surprise #1 - Subsets

$$\{1, 4, 9, 16, 25, \ldots\} \subseteq \{1, 2, 3, 4, 5, \ldots\}$$

Surprise #1 - Subsets

$$\{1, 4, 9, 16, 25, \ldots\} \subseteq \{1, 2, 3, 4, 5, \ldots\}$$

$$\begin{array}{cccccc}
\{1, & 2, & 3, & 4, & 5, & \ldots\} \\
\updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \\
\{1, & 4, & 9, & 16, & 25, & \ldots\}
\end{array}$$

Surprise #1 - Subsets

$$\{1, 32, 243, 1024, 3125, \ldots\} \subseteq \{1, 2, 3, 4, 5, \ldots\}$$

Surprise #1 - Subsets

$$\{1, 32, 243, 1024, 3125, \ldots\} \subseteq \{1, 2, 3, 4, 5, \ldots\}$$

$$\begin{array}{cccccc}
\{1, & 2, & 3, & 4, & 5, & \ldots\} \\
\updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \\
\{1, & 32, & 243, & 1024, & 3125, & \ldots\}
\end{array}$$

Surprise #1 - Subsets

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So infinity is infinity is infinity then… right?

Surprise #1 - Subsets

So infinity is infinity is infinity then… right?

No!

There are different sized infinities!

Surprise #2 - Sizes

The Cantor Set

Surprise #2 - Sizes

The Cantor Set

$2^\omega = \{\text{all infinite binary sequences}\}$

$= \{00000000\ldots, 11111111\ldots, 10101010\ldots,$

$00110100\ldots, 1111010\ldots, \ldots\}$

Surprise #2 - Sizes

The Cantor Set

$2^\omega = \{\text{all infinite binary sequences}\}$

$= \{00000000\ldots, 11111111\ldots, 10101010\ldots,$
$00110100\ldots, 1111010\ldots, \ldots\}$

Can we count it?

Cantor’s Diagonal Argument

1	1	0	0	1	0	0	0	⋯
2	0	0	0	1	1	1	1	⋯
3	0	1	1	1	0	0	1	⋯
4	0	0	1	1	0	0	0	⋯
5	1	1	1	0	1	0	0	⋯
6	0	1	0	1	0	1	0	⋯
7	0	0	0	0	1	0	0	⋯
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	

	0							
1	1	0	0	1	0	0	0	...
2	0	0	0	1	1	1	1	...
3	0	1	1	1	0	0	1	...
4	0	0	1	1	0	0	0	...
5	1	1	1	0	1	0	0	...
6	0	1	0	1	0	1	0	...
7	0	0	0	0	1	0	0	...
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	

	0	1						
1	1	0	0	1	0	0	0	...
2	0	0	0	1	1	1	1	...
3	0	1	1	1	0	0	1	...
4	0	0	1	1	0	0	0	...
5	1	1	1	0	1	0	0	...
6	0	1	0	1	0	1	0	...
7	0	0	0	0	1	0	0	...
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	

	0	1	0					
1	1	0	0	1	0	0	0	…
2	0	0	0	1	1	1	1	…
3	0	1	1	1	0	0	1	…
4	0	0	1	1	0	0	0	…
5	1	1	1	0	1	0	0	…
6	0	1	0	1	0	1	0	…
7	0	0	0	0	1	0	0	…
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	

	0	1	0	0				
1	1	0	0	1	0	0	0	$\cdots$
2	0	0	0	1	1	1	1	$\cdots$
3	0	1	1	1	0	0	1	$\cdots$
4	0	0	1	1	0	0	0	$\cdots$
5	1	1	1	0	1	0	0	$\cdots$
6	0	1	0	1	0	1	0	$\cdots$
7	0	0	0	0	1	0	0	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	

	0	1	0	0	0			
1	1	0	0	1	0	0	0	...
2	0	0	0	1	1	1	1	...
3	0	1	1	1	0	0	1	...
4	0	0	1	1	0	0	0	...
5	1	1	1	0	1	0	0	...
6	0	1	0	1	0	1	0	...
7	0	0	0	0	1	0	0	...
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	

	0	1	0	0	0	0		
1	1	0	0	1	0	0	0	⋯
2	0	0	0	1	1	1	1	⋯
3	0	1	1	1	0	0	1	⋯
4	0	0	1	1	0	0	0	⋯
5	1	1	1	0	1	0	0	⋯
6	0	1	0	1	0	1	0	⋯
7	0	0	0	0	1	0	0	⋯
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	

	0	1	0	0	0	0	1	
1	1	0	0	1	0	0	0	…
2	0	0	0	1	1	1	1	…
3	0	1	1	1	0	0	1	…
4	0	0	1	1	0	0	0	…
5	1	1	1	0	1	0	0	…
6	0	1	0	1	0	1	0	…
7	0	0	0	0	1	0	0	…
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	

	0	1	0	0	0	0	1	…
1	1	0	0	1	0	0	0	…
2	0	0	0	1	1	1	1	…
3	0	1	1	1	0	0	1	…
4	0	0	1	1	0	0	0	…
5	1	1	1	0	1	0	0	…
6	0	1	0	1	0	1	0	…
7	0	0	0	0	1	0	0	…
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	

Surprise #2 - Sizes

We can't pair up each element of $\mathbb{N}$ with 2^{ω}.

Surprise #2 - Sizes

We can't pair up each element of $\mathbb{N}$ with 2^{ω}.

$$2^{\omega} > \mathbb{N}!$$

Infinitely Many Infinities

There are infinitely many sizes of infinities!

Infinitely Many Infinities

$$\mathcal{P}(S) = \{\text{all subsets of } S\}$$

Infinitely Many Infinities

$$\mathcal{P}(S) = \{\text{all subsets of } S\}$$

$$\mathcal{P}(\{1, 2, 3\}) =$$

$$\{\{\}, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$$

Infinitely Many Infinities

$$\mathbb{N} < \mathcal{P}(\mathbb{N}) < \mathcal{P}(\mathcal{P}(\mathbb{N})) < \mathcal{P}(\mathcal{P}(\mathcal{P}(\mathbb{N}))) < \cdots$$

Infinitely Many Infinities

$$\mathbb{N} < \mathcal{P}(\mathbb{N}) < \mathcal{P}(\mathcal{P}(\mathbb{N})) < \mathcal{P}(\mathcal{P}(\mathcal{P}(\mathbb{N}))) < \cdots$$
$$\aleph_0 < \aleph_1 < \aleph_2 < \aleph_3 < \cdots$$

Infinitely Many Infinities

$\aleph_0$ - $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, {even numbers}, {all possible books}, …

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$\aleph_0$ - $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, {even numbers}, {all possible books}, …
$\aleph_1$ - 2^ω, $\mathbb{R}$, {points on a circle}, {points on a line}, …

Infinitely Many Infinities

$\aleph_0$ - $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, {even numbers}, {all possible books}, …
$\aleph_1$ - 2^ω, $\mathbb{R}$, {points on a circle}, {points on a line}, …
$\aleph_2$ - {functions $\mathbb{R} \to \mathbb{R}$}, $\mathcal{P}(2^\omega)$, …
$\vdots$

Infinitely Many Infinities

It keeps on going!

Infinitely Many Infinities

It keeps on going!

$\aleph_\omega$

Further Exploration

- How To Count Past Infinity - Vsauce
- The Mystery of the Aleph - Amir D. Aczel
- Infinitely More - Joel David Hamkins