

Infinity and Beyond

June 28, 2023

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Finite Sets

Finite Sets

$\{\}$

Finite Sets

$\{\}$

$\{\text{red}, \text{blue}, \text{green}\}$

Finite Sets

$\{\}$

$\{\text{red}, \text{blue}, \text{green}\}$

$\{1, 2, 3, 4\}$

Finite Sets

$\{\}$

$\{\text{red}, \text{blue}, \text{green}\}$

$\{1, 2, 3, 4\}$

$\{890, \text{"foo"}, \pi\}$

Infinite Sets

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$$\mathbb{N} = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, \dots\}$$

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$$\mathbb{Z} = \{0, -1, 1, -2, 2, -3, 3, -4, 4, -5, 5, \dots\}$$

Infinite Sets

$$\mathbb{N} = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, \dots\}$$

$$\mathbb{Z} = \{0, -1, 1, -2, 2, -3, 3, -4, 4, -5, 5, \dots\}$$

$$\mathbb{Q} = \{0, 1, 2, \frac{1}{2}, 3, 4, \frac{3}{2}, \frac{2}{3}, \frac{1}{4}, \frac{1}{5}, 5, 6, \frac{5}{2}, \dots\}$$

Infinite Sets

Infinite Sets

$\{ \text{every possible book} \}$

Infinite Sets

{ every possible book }

{ every possible book that starts with
“supercalifragilisticexpialidocious” }

Infinite Sets

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{ every point on a circle }

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{ every triangle }

Subsets

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$$\{1, 2, 3\} \subseteq \{1, 2, 3, 4, 5\}$$

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$$\{700, 543, \tfrac{1}{2}\} \not\subseteq \{1, 3, 5, 700\}$$

Subsets

$$\{1, 2, 3\} \subseteq \{1, 2, 3, 4, 5\}$$

$$\{700, 543, \tfrac{1}{2}\} \not\subseteq \{1, 3, 5, 700\}$$

$$\{\} \subseteq \{0, 1\}$$

Size

How can we tell if two sets are the same size?

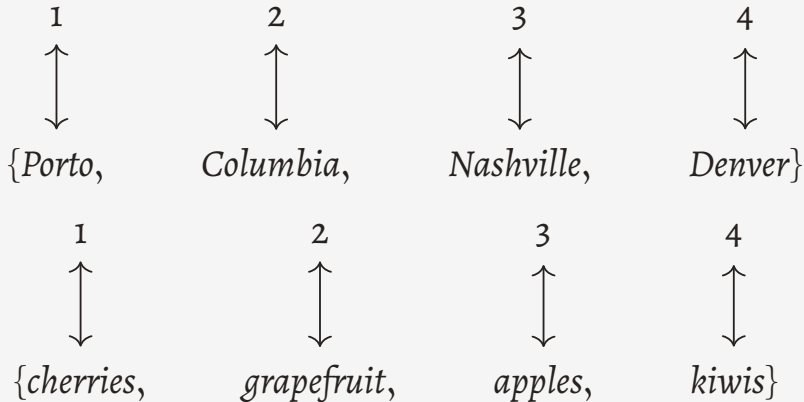
Size

How can we tell if two sets are the same size?
Just count them!

Size

{ Porto, Columbia, Nashville, Denver }
{ cherries, grapefruit, apples, kiwis }

Size



Size

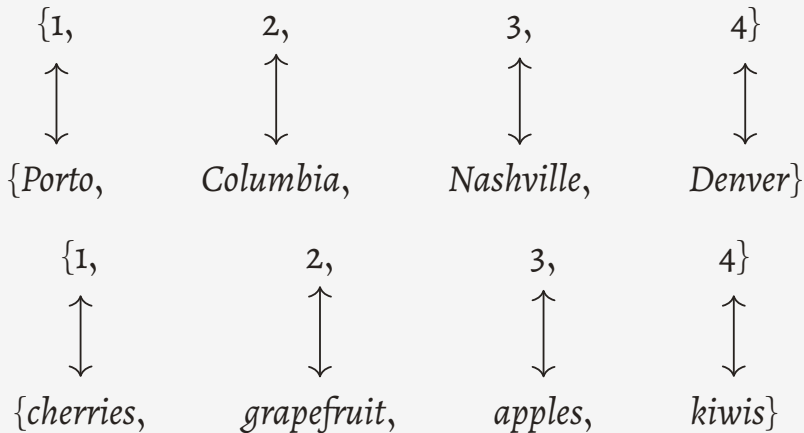
$$4 = 4$$

Size

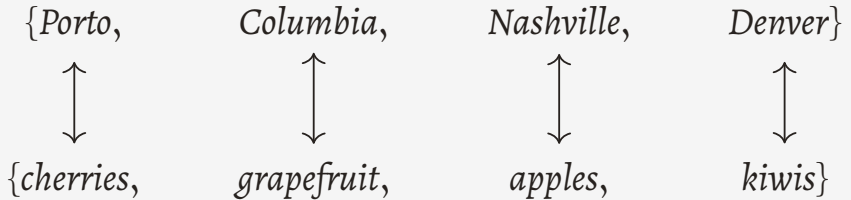
$$4 = 4$$

cities = # fruits

Size

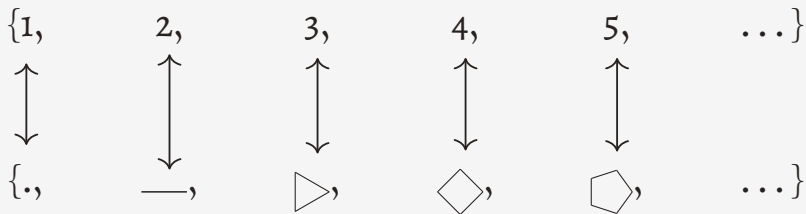


Size



Generalization

Generalization



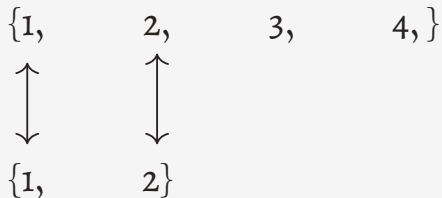
Surprise #1 - Subsets

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$$\{1, 2\} \subseteq \{1, 2, 3, 4\}$$

Surprise #1 - Subsets

$$\{1, 2\} \subseteq \{1, 2, 3, 4\}$$

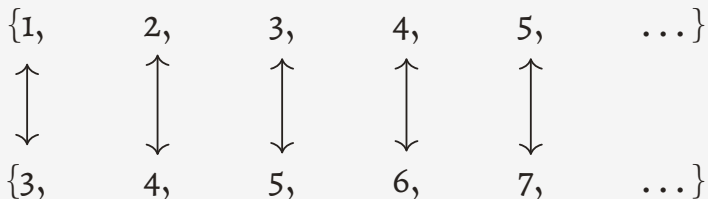


Surprise #1 - Subsets

$$\{3, 4, 5, 6, 7, \dots\} \subseteq \{1, 2, 3, 4, 5, \dots\}$$

Surprise #1 - Subsets

$$\{3, 4, 5, 6, 7, \dots\} \subseteq \{1, 2, 3, 4, 5, \dots\}$$

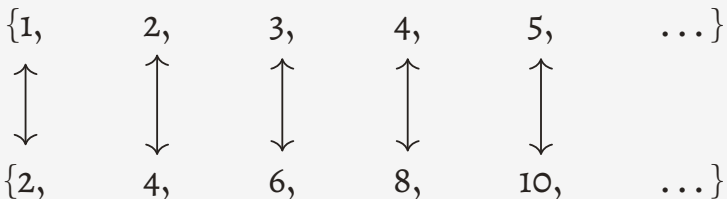


Surprise #1 - Subsets

$$\{2, 4, 6, 8, 10, \dots\} \subseteq \{1, 2, 3, 4, 5, \dots\}$$

Surprise #1 - Subsets

$$\{2, 4, 6, 8, 10, \dots\} \subseteq \{1, 2, 3, 4, 5, \dots\}$$

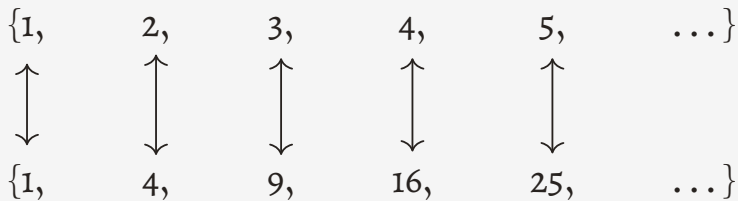


Surprise #1 - Subsets

$$\{1, 4, 9, 16, 25, \dots\} \subseteq \{1, 2, 3, 4, 5, \dots\}$$

Surprise #1 - Subsets

$$\{1, 4, 9, 16, 25, \dots\} \subseteq \{1, 2, 3, 4, 5, \dots\}$$

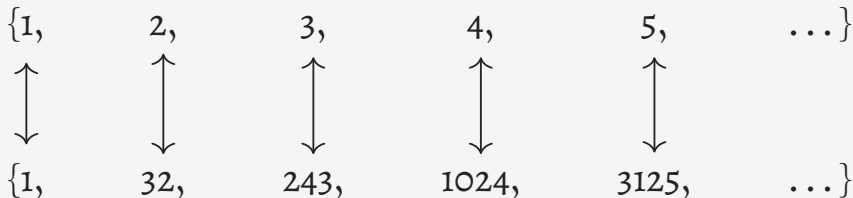


Surprise #1 - Subsets

$$\{1, 32, 243, 1024, 3125, \dots\} \subseteq \{1, 2, 3, 4, 5, \dots\}$$

Surprise #1 - Subsets

$$\{1, 32, 243, 1024, 3125, \dots\} \subseteq \{1, 2, 3, 4, 5, \dots\}$$



Surprise #1 - Subsets

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So infinity is infinity is infinity then... right?

Surprise #1 - Subsets

So infinity is infinity is infinity then... right?

No!

There are different sized infinities!

Surprise #2 - Sizes

The Cantor Set

Surprise #2 - Sizes

The Cantor Set

$$\begin{aligned} 2^{\omega} &= \{\text{all infinite binary sequences}\} \\ &= \{00000000\dots, 11111111\dots, 10101010\dots, \\ &\quad 00110100\dots, 11111010\dots, \dots\} \end{aligned}$$

Surprise #2 - Sizes

The Cantor Set

$$\begin{aligned} 2^{\omega} &= \{\text{all infinite binary sequences}\} \\ &= \{00000000\dots, 11111111\dots, 10101010\dots, \\ &\quad 00110100\dots, 11111010\dots, \dots\} \end{aligned}$$

Can we count it?

Cantor's Diagonal Argument

	0							
1	1	0	0	1	0	0	0	...
2	0	0	0	1	1	1	1	...
3	0	1	1	1	0	0	1	...
4	0	0	1	1	0	0	0	...
5	1	1	1	0	1	0	0	...
6	0	1	0	1	0	1	0	...
7	0	0	0	0	1	0	0	...
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	

	0	1						
1	1	0	0	1	0	0	0	...
2	0	0	0	1	1	1	1	...
3	0	1	1	1	0	0	1	...
4	0	0	1	1	0	0	0	...
5	1	1	1	0	1	0	0	...
6	0	1	0	1	0	1	0	...
7	0	0	0	0	1	0	0	...
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	

	0	1	0					
1	1	0	0	1	0	0	0	...
2	0	0	0	1	1	1	1	...
3	0	1	1	1	0	0	1	...
4	0	0	1	1	0	0	0	...
5	1	1	1	0	1	0	0	...
6	0	1	0	1	0	1	0	...
7	0	0	0	0	1	0	0	...
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	

	0	1	0	0	0	0	
1	1	0	0	1	0	0	0 ...
2	0	0	0	1	1	1	1 ...
3	0	1	1	1	0	0	1 ...
4	0	0	1	1	0	0	0 ...
5	1	1	1	0	1	0	0 ...
6	0	1	0	1	0	1	0 ...
7	0	0	0	0	1	0	0 ...
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮

	0	1	0	0	0	0	1	
1	1	0	0	1	0	0	0	...
2	0	0	0	1	1	1	1	...
3	0	1	1	1	0	0	1	...
4	0	0	1	1	0	0	0	...
5	1	1	1	0	1	0	0	...
6	0	1	0	1	0	1	0	...
7	0	0	0	0	1	0	0	...
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	

Surprise #2 - Sizes

We can't pair up each element of \mathbb{N} with 2^ω .

Surprise #2 - Sizes

We can't pair up each element of \mathbb{N} with 2^ω .

$$2^\omega > \mathbb{N}!$$

Infinitely Many Infinities

There are infinitely many sizes of infinities!

Infinitely Many Infinities

$$\mathcal{P}(S) = \{\text{all subsets of } S\}$$

Infinitely Many Infinities

$$\mathcal{P}(S) = \{\text{all subsets of } S\}$$

$$\mathcal{P}(\{1, 2, 3\}) =$$

$$\{\{\}, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$$

Infinitely Many Infinities

$$\mathbb{N} < \mathcal{P}(\mathbb{N}) < \mathcal{P}(\mathcal{P}(\mathbb{N})) < \mathcal{P}(\mathcal{P}(\mathcal{P}(\mathbb{N}))) < \dots$$

Infinitely Many Infinities

$$\mathbb{N} < \mathcal{P}(\mathbb{N}) < \mathcal{P}(\mathcal{P}(\mathbb{N})) < \mathcal{P}(\mathcal{P}(\mathcal{P}(\mathbb{N}))) < \dots$$

$$\aleph_0 < \aleph_1 < \aleph_2 < \aleph_3 < \dots$$

Infinitely Many Infinities

It keeps on going!

Infinitely Many Infinities

It keeps on going!

$$\aleph_\omega$$

Further Exploration

- How To Count Past Infinity - Vsauce
- The Mystery of the Aleph - Amir D. Aczel
- Infinitely More - Joel David Hamkins