

## Section 1

### Exercise 1.1.i

- (i) Let  $f : x \rightarrow y$  be a morphism and  $g, g' : y \rightarrow x$  be inverses. Then  $fg = \text{id}_y$  and  $gf = \text{id}_x$  (resp  $g'$ ). Hence  $gf = g'f$  so

$$\begin{aligned} gfg &= g'fg \\ g &= g' \end{aligned}$$

Therefore any morphism has at most one inverse. ///

- (ii) Suppose there are morphisms  $g, h : y \rightrightarrows x$  such that  $fg = \text{id}_y$  and  $hf = \text{id}_x$ . Then

$$\begin{aligned} hfg &= h \\ g &= h \end{aligned}$$

Hence  $fg = \text{id}_y$  and  $gf = \text{id}_x$  so  $f$  is an isomorphism. ///

**Exercise 1.1.ii** Consider a category  $C$  and the subset of isomorphisms it contains. The subset contains the identity morphisms for each object because the identity is an isomorphism. The subset inherits associativity from  $C$ . Now pick two isomorphisms  $f : x \rightarrow y$  and  $g : y \rightarrow z$  and consider  $gf$ . We have  $(f^{-1}g^{-1})gf = f^{-1}g^{-1}gf = f^{-1}f = \text{id}_x$  and  $gf(f^{-1}g^{-1}) = \text{id}_y$  so  $gf$  is also an isomorphism. Hence  $C$  contains a maximal groupoid. ///

### Exercise 1.1.iii

- (i) Observe that for any object  $f : c \rightarrow x$  in  $c/C$ , the morphism  $\text{id}_x : x \rightarrow x$  is the identity morphism for  $f$ . Now pick two morphisms in  $c/C$ ,  $h : x \rightarrow y$  and  $k : y \rightarrow z$  and consider  $kh$ .

$$\begin{array}{ccccc} & & c & & \\ & f \swarrow & \downarrow g & \searrow b & \\ x & \xrightarrow{h} & y & \xrightarrow{k} & z \end{array}$$

Because  $hf = g$  and  $kg = b$  we have  $khf = b$  so  $kh$  is a morphism in  $c/C$  which is enough to show it is a category. ///

- (ii) A similar argument as above shows that  $C/c$  is a category. ///

## Section 2

**Exercise 1.2.iii** Let  $f : x \hookrightarrow y$  and  $g : y \hookrightarrow z$  be monomorphisms and  $h, k : w \rightarrow x$  be morphisms such that  $gh = gfk$ . Because  $g$  is monic, it follows that  $fh = fk$ . Similarly, because  $f$  is monic we have  $h = k$ . Hence  $gf$  is monic.

Now suppose  $f$  and  $g$  are simply morphisms and that  $gf$  is monic. BWO suppose that  $f$  is not monic. Then there exist morphisms  $h, k$  such that  $fh = fk$  but  $h \neq k$ . Hence

we have  $gfh = gfk$  this is a contradiction because  $gf$  being monic implies that  $h = k$ . Hence  $f$  is monic.

Now observe that the monomorphisms in a category form a subcategory. Every object has an identity morphism because the identity is monic. Associativity is inherited from the larger category and composition is preserved because the composition of two monomorphisms is also a monomorphism as shown above.

By duality the epimorphisms also form a subcategory. ///

**Exercise 1.2.iv** Question: What are the monomorphisms in Field?

**Exercise 1.2.v** In **Ring**, consider the inclusion map  $i : \mathbb{Z} \hookrightarrow \mathbb{Q}$ . Now pick two morphisms,  $f, g : R \rightrightarrows \mathbb{Z}$  such that  $if = ig$  where  $R$  is some ring. Now BWOC suppose that  $f \neq g$ . Then there exists an  $r \in R$  such that  $f(r) \neq g(r)$ . Hence  $i(f(r)) \neq i(g(r))$  so  $if \neq ig$  which is a contradiction. Hence the above implies that  $f = g$  so  $i$  is monic.

Now pick morphisms  $h, k : \mathbb{Q} \rightrightarrows R$  such that  $hi = ki$ . This means that for all integers  $z$ ,  $h(z) = k(z)$ . Each rational  $q$  may be written canonically as  $nd^{-1}$  for  $n, d \in \mathbb{Z}$ . Hence  $h(q) = h(nd^{-1}) = h(n)h(d^{-1})$ . Further, because every element in  $\mathbb{Q}$  is a unit,  $h(n)h(d^{-1}) = h(n)h(d)^{-1}$ . Because  $h$  and  $k$  agree on the integers, we have  $h(n)h(d)^{-1} = k(n)k(d)^{-1}$ . Hence  $h = k$  and  $i$  is an epimorphism.

$\mathbb{Q}$  and  $\mathbb{Z}$  are not isomorphic in **Ring** because  $\mathbb{Q}$  is a field while  $\mathbb{Z}$  is not. Hence in **Ring**, monic and epic does not imply isomorphism. ///

**Exercise 1.2.vii** For a given subset of objects  $A \subseteq P$ , consider  $M_A = \{x \in \text{obj}P \mid \forall a \in A, \exists f \in P(a, x)\}$ , i.e. the set of all elements in  $P$  targeted by a morphism from every element in  $A$ . If it exists (when could the supremum not exist?), the supremum of  $A$  is the least element in  $M_A$  w.r.t.  $\leq$ .

By duality...

## Section 3

**Exercise 1.3.i** Given two groups  $G$  and  $H$ , a functor  $BG \rightarrow BH$  is a group homomorphism. Given such a functor  $F$ , and elements  $x, y \in G$ , that is morphisms in  $BG$ , we have  $F(x)F(y) = F(xy)$  which is exactly a group homomorphism. ///

**Exercise 1.3.iii** Question: Give an example of a functor such that its image is not a subcategory of the codomain.

**Exercise 1.3.iv** First observe that  $C(c, -)$  and  $C(-, c)$  both preserve identities. Consider the action of  $C(c, -)$  on  $\text{id}_x : x \rightarrow x$ .  $\text{id}_x$  maps to post composition by the identity on  $C(c, x)$  which is the identity morphism for  $C(c, x)$ . Similar reason applies for  $C(-, c)$ .

Now pick two morphisms  $f : x \rightarrow y$  and  $g : y \rightarrow z$  in  $C$ . Then  $C(c, -)g \cdot C(c, -)f = g_*f_* = (gf)_* = C(c, -)(gf)$ . Hence  $C(c, -)$  is in fact a functor. A similar argument shows the same holds for  $C(-, c)$ .

Is this by duality? ///

**Exercise 1.3.v** I'm not sure I understand the question but I think the pairs of functors are duals: A functor  $F : C^{\text{op}} \rightarrow D$  sends a morphism  $f : x \rightarrow y$  to  $Ff : Fy \rightarrow Fx$ . We may construct a new functor  $F' : C \rightarrow D^{\text{op}}$  such that  $f : y \rightarrow x \mapsto F'f : F'x \rightarrow F'y$ .

**Exercise 1.3.viii** Consider the trivial Functor  $\mathbf{0} : \text{Set} \rightarrow \text{Set}$  that maps every set to  $\emptyset$  and every morphism to  $\text{id}_{\emptyset}$ . Observe that this is indeed a functor because  $\mathbf{0}(\text{id}_x) = \text{id}_{\mathbf{0}x} = \text{id}_{\emptyset}$  and  $\mathbf{0}(gf) = \mathbf{0}(g)\mathbf{0}(f) = \text{id}_{\emptyset}$ . Now pick any morphism in  $\text{Set}$  that is not an isomorphism. It is mapped to  $\text{id}_{\emptyset}$  which is an isomorphism. Hence functors do not reflect isomorphisms. ///

**Exercise 1.3.vi** Consider the map  $\text{dom} : F \downarrow G \rightarrow D$  that takes an object  $(d, e, f) \mapsto d$  and a morphism  $(h, k) \mapsto h$ . Now pick an identity morphism  $(\text{id}_d, \text{id}_e)$ ; clearly this functor preserves identity. Now pick a composable pair of morphisms  $(h, k)$  and  $(h', k')$ . We have  $\text{dom}((h', k')(h, k)) = \text{dom}(h'h, k'k) = h'h = \text{dom}(h', k')\text{dom}(h, k)$ . Hence  $\text{dom}$  is a functor. Define  $\text{cod}$  analogously. ///

**Exercise 1.3.x** Let  $\text{cl}(g)$  be the conjugacy class of  $g$  and  $\text{Cl}(G)$  be the set of all conjugacy classes of  $G$  (I don't know what the standard notation for this is). Consider the following mapping:

$$\begin{aligned} \text{Conj} : \text{Group} &\longrightarrow \text{Set} \\ G &\longmapsto \text{Cl}(G) \\ (f : G \rightarrow H) &\longmapsto (f' : \text{Cl}(G) \rightarrow \text{Cl}(H)) \end{aligned}$$

Where  $f'$  maps  $\text{cl}(x)$  to  $\text{cl}(f(x))$ . Note that this is indeed a well defined map: pick an element  $b$  in  $G$ . Then  $f'(b) = \text{cl}(f(b))$ . For any other element  $a \in \text{cl}(b)$ , we have  $f'(a) = \text{cl}(f(a)) = \text{cl}(f(g)f(b)f(g)^{-1}) = \text{cl}(f(b))$ .

Observe that  $\text{Conj}$  preserves identities. Now pick  $h : G \rightarrow H$  and  $k : H \rightarrow I$ . Then  $\text{Conj}(kh) = f' : \text{Cl}(G) \rightarrow \text{Cl}(I) = \text{Conj}(k)\text{Conj}(h)$ . Hence  $\text{Conj}$  is a functor.

Now pick a pair of groups such that the cardinalities of their sets of conjugacy classes differ. (How do I conclude that they are not isomorphic?).