Section 1

$\boxed{\textbf{Exercise 2.1.i}}$ The question asks us to the consider natural transformations between $\mathbf{Cat}(\mathbf{1}, -)$ and $\mathbf{Cat}(\mathbf{2}, -)$ induced by functors between $\mathbf{1}$ and $\mathbf{2}$.

For the collapsing functor $! : \mathbf{2} \to \mathbf{1}$, we have the following naturality square:

$$\begin{tikzcd}
\mathbf{Cat}(\mathbf{1}, x) \arrow[r, "!^*"] \arrow[d, "f_*"] & \mathbf{Cat}(\mathbf{2}, x) \arrow[d, "f_*"] \\
\mathbf{Cat}(\mathbf{1}, y) \arrow[r, "!^*"] & \mathbf{Cat}(\mathbf{2}, y)
\end{tikzcd}$$

This diagram commutes because for any suitable input x, $f_*!^*(x) = !xf = !^*f_*(x)$.

We achieve similar results for 0 and 2 by drawing a similar diagram for natural transformations from $\mathbf{Cat}(\mathbf{2}, -) \to \mathbf{Cat}(\mathbf{1}, -)$.

what else should be said about these natural transformations?

$\boxed{\textbf{Exercise 2.1.ii}}$ If F is representable then we have a natural isomorphism α and the following commutative square:

$$\begin{tikzcd}
Fx \arrow[r, leftrightarrow, "\alpha_x"] \arrow[d, "Ff"'] & C(c, x) \arrow[d, "f_*"] \\
Fy \arrow[r, leftrightarrow, "\alpha_y"] & C(c, y)
\end{tikzcd}$$

Now suppose f is monic and pick two morphisms $g, h : W \rightrightarrows C(c, x)$ for some set W, such that $f_*g = f_*h$. This is equivalent to saying $\forall w \in W, f_*(g(w)) = f_*(h(w)) \implies f \circ (g(w)) = f \circ (h(w)) \implies g(w) = h(w) \implies g = h$ by the fact that f is monic. Hence f_* is monic.

Now, by the square, we have $\alpha_y Ff = f_* \alpha_x \implies Ff = \alpha_y^{-1} f_* \alpha_x$. Hence Ff is a composition of three injective maps which together are also injective. Therefore F preserves monomorphisms.

As for part two, pick a functor $\mathbf{2} \to \mathbf{Set}$ that takes the morphism to a non-injective function. By the contrapositive of the above statement, the functor is not representable because it does not preserve monomorphisms. ///

Section 2

$\boxed{\textbf{Exercise 2.2.i}}$ To reach the dual of the Yoneda Lemma, consider the category in question to be C^{op}. Then for a functor $F : C^{\mathrm{op}} \to \mathbf{Set}$, there is a bijection

$$\mathrm{Hom}(C^{\mathrm{op}}(c, -), F) \cong Fc$$

for any $c \in C^{\mathrm{op}}$. Now realize that $C(a, b) = C^{\mathrm{op}}(b, a)$, so we have

$$\mathrm{Hom}(C(-, c), F) \cong Fc$$

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$\boxed{\textbf{Exercise 2.2.ii}}$ It does not dualize to that because that is just not the dual statement. ///

$\boxed{\textbf{Exercise 2.2.iii}}$ Pick two objects $c, d \in \omega$ and consider $\omega(c, d)$ and $\mathbf{Set}^{\omega^{\mathrm{op}}}(yc, yd) = \mathbf{Set}^{\omega^{\mathrm{op}}}(\omega(-, c), \omega(-, d))$. Because ω is a poset, there is at most one morphism in $\omega(c, d)$

$\boxed{\textbf{Exercise 2.2.iv}}$ Suppose $f : x \to y$ is an isomorphism. Then $f_* : C(-, x) \to C(-, y)$ defines a natural transformation one direction and $(f^{-1})_* = f_*^{-1}$ defines the other direction.

Suppose f_* is a natural isomorphism. Then f_* has an inverse, f_*^{-1}. This means that we have $f_* \circ f_*^{-1} = \mathrm{id}_y$ and $f_*^{-1} \circ f_* = \mathrm{id}_x$. So, for suitable morphisms, h, k, we have $f_*^{-1}(f_*(h)) = f_*^{-1}(fh) = h$ and $f_*(f_*^{-1}(k)) = ff_*^{-1}(k) = k$

(INSERT PROOF) hence f is an isomorphism.

The same reasoning applies for f^*.