Section 1

Exercise 1

(i) Let $f : x \to y$ be a morphism and $g, g' : y \to x$ be inverses. Then $fg = \mathrm{id}_y$ and $gf = \mathrm{id}_x$ (resp g'). Hence $gf = g'f$ so

$$\begin{aligned} gfg &= g'fg \\ g &= g' \end{aligned}$$

Therefore any morphism has at most one inverse. ///

(ii) Suppose there are morphisms $g, h : y \rightrightarrows x$ such that $fg = \mathrm{id}_y$ and $hf = \mathrm{id}_x$. Then

$$\begin{aligned} hfg &= h \\ g &= h \end{aligned}$$

Hence $fg = \mathrm{id}_y$ and $gf = \mathrm{id}_x$ so f is an isomorphism. ///

Exercise 2 Consider a category C and the subset of isomorphisms it contains. The subset contains the identity morphisms for each object because the identity is an isomorphism. The subset inherits associativity from C. Now pick two morphisms $f : x \to y$ and $g : y \to z$ and consider gf. Because f and g are isomorphisms, f^{-1} and g^{-1} exist and are isomorphisms so are also in the set. Hence