Section 7

Exercise 1.7.i Let $F, G : C \rightrightarrows D$ be functors. Observe that for any $c \in C$, $D(Fc, Gc)$ is a set because D is locally small. Now consider

$$A = \bigcup_{c \in C} D(Fc, Gc)$$

A is a set because it is a union of sets indexed by another set.

Now, every natural transformation is uniquely defined by its components. Each component is a map $\alpha_c : Fc \to Gc$ for some c. Hence, each natural transformation corresponds to an element in $\mathcal{P}(A)$ that contains all of its components.

Thus the collection of natural transformations between F, G is a set because it is in bijection with a subset of $\mathcal{P}(A)$ which is a set.

Hence There are set-many morphisms between any two objects in D^C so it is locally small. ///

Exercise 1.7.ii The following diagram commutes because it is the commutativity diagram for β for a morphism $Ff : Fc \to Fc'$.

$$\begin{tikzcd} H(Fc) \arrow[r, "\beta_{Fc}"] \arrow[d, "H(Ff)"] & K(Fc) \arrow[d, "K(Ff)"] \\ H(Fc') \arrow[r, "\beta_{Fc'}"] & K(Fc') \end{tikzcd}$$

Hence we may apply L to the common composite and by functoriality we have the following.

$$L(\beta_{Fc'} \circ HFf) = L(KFf \circ \beta_{Fc})$$
$$L(\beta_{Fc'})L(HFf) = L(KFf)L(\beta_{Fc})$$

Thus the diagram for $L\beta F$ commutes so it is natural. ///

Exercise 1.7.v Define the monoid operation to be vertical composition of natural transformations $\mathbf{1}_C \Rightarrow \mathbf{1}_C$. The identity natural transformation serves as the identity. Further, this operation is associative because the composition of components is associative.

Exercise 1.7.vii For each $c \in C$, $F(c, -)$ inherits functoriality from F, hence F determines a functor for each c. F also determines a natural transformation $F(f, -) : F(c, -) \Rightarrow F(c', -)$ as follows. For some morphism $g : x \to y$ in D, we have

$$\begin{tikzcd} F(c,x) \arrow[r, "F(f,x)"] \arrow[d, "F(c,g)"] & F(c',x) \arrow[d, "F(c',g)"] \\ F(c,y) \arrow[r, "F(f,y)"] & F(c',y) \end{tikzcd}$$

The square commutes because f and g each act independently on the left and right components respectively.

Now, for each $c \in C$, pick a functor $K_c : D \to E$ and for each morphism $f : c \to c'$, pick a natural transformation $\alpha : K_c \Rightarrow K_{c'}$. Then we may define $F : C \times D \to E$ for objects as $(c, d) \mapsto K_c(d)$. For a morphism $(f, g) : (c, d) \to (c', d')$, we need to construct a mapping $F(f, g) : F(c, d) \to F(c', d')$. By the definition of F on objects, this is $F(f, g) : K_c(d) \to K_{c'}(d)$. Hence define $F(f, g) = K_{c'}(g) \circ \alpha_d$.

We have $F(\mathrm{id}_c, \mathrm{id}_d) = K_{c'}(\mathrm{id}_d) \circ \alpha_d = \mathrm{id}_{K_c(d)} \circ \mathrm{id}_{K_c(d)} = \mathrm{id}_{K_c(d)}$ by functoriality of K_c and (I'm not sure this works because α_d need not be the identity)

how is the above equivalent to showing this? $\to$ Hence there is a bijection between $C \times D \to E$ and $C \to E^D$.