

## Section 1

### Exercise 1.1.i

- (i) Let  $f : x \rightarrow y$  be a morphism and  $g, g' : y \rightarrow x$  be inverses. Then  $fg = \text{id}_y$  and  $gf = \text{id}_x$  (resp  $g'$ ). Hence  $gf = g'f$  so

$$\begin{aligned} gfg &= g'fg \\ g &= g' \end{aligned}$$

Therefore any morphism has at most one inverse. ///

- (ii) Suppose there are morphisms  $g, h : y \rightrightarrows x$  such that  $fg = \text{id}_y$  and  $hf = \text{id}_x$ . Then

$$\begin{aligned} hfg &= h \\ g &= h \end{aligned}$$

Hence  $fg = \text{id}_y$  and  $gf = \text{id}_x$  so  $f$  is an isomorphism. ///

**Exercise 1.1.ii** Consider a category  $C$  and the subset of isomorphisms it contains. The subset contains the identity morphisms for each object because the identity is an isomorphism. The subset inherits associativity from  $C$ . Now pick two isomorphisms  $f : x \rightarrow y$  and  $g : y \rightarrow z$  and consider  $gf$ . We have  $(f^{-1}g^{-1})gf = f^{-1}g^{-1}gf = f^{-1}f = \text{id}_x$  and  $gf(f^{-1}g^{-1}) = \text{id}_y$  so  $gf$  is also an isomorphism. Hence  $C$  contains a maximal groupoid. ///

### Exercise 1.1.iii

- (i) Observe that for any object  $f : c \rightarrow x$  in  $c/C$ , the morphism  $\text{id}_x : x \rightarrow x$  is the identity morphism for  $f$ . Now pick two morphisms in  $c/C$ ,  $h : x \rightarrow y$  and  $k : y \rightarrow z$  and consider  $kh$ .

$$\begin{array}{ccccc} & & c & & \\ & f \swarrow & \downarrow g & \searrow b & \\ x & \xrightarrow{h} & y & \xrightarrow{k} & z \end{array}$$

Because  $hf = g$  and  $kg = b$  we have  $khf = b$  so  $kh$  is a morphism in  $c/C$  which is enough to show it is a category. ///

- (ii) A similar argument as above shows that  $C/c$  is a category. ///

## Section 2

**Exercise 1.2.iii** Let  $f : x \hookrightarrow y$  and  $g : y \hookrightarrow z$  be monomorphisms and  $h, k : w \rightarrow x$  be morphisms such that  $gh = gfk$ . Because  $g$  is monic, it follows that  $fh = fk$ . Similarly, because  $f$  is monic we have  $h = k$ . Hence  $gf$  is monic.

Now suppose  $f$  and  $g$  are simply morphisms and that  $gf$  is monic. BWO suppose that  $f$  is not monic. Then there exist morphisms  $h, k$  such that  $fh = fk$  but  $h \neq k$ . Hence

we have  $gfh = gfk$  this is a contradiction because  $gf$  being monic implies that  $h = k$ . Hence  $f$  is monic.

Now observe that the monomorphisms in a category form a subcategory. Every object has an identity morphism because the identity is monic. Associativity is inherited from the larger category and composition is preserved because the composition of two monomorphisms is also a monomorphism as shown above.

By duality the epimorphisms also form a subcategory. ///

**Exercise 1.2.iv** Recall that the only ideals in a field  $F$  are  $\{0\}$  and  $F$ . Pick a morphism  $f : X \rightarrow Y$  in **Field**. The kernel of  $f$  is an ideal so it is either  $\{0\}$  or  $X$ . By the first isomorphism theorem,  $f$  is either 1-1 or the 0 map. Now pick  $g, h$  such that  $fg = fh$ . Then because  $f$  is 1-1, then it maps every element of its domain to a unique element in the codomain hence  $h = g$ . Therefore  $f$  is monic. Hence the monomorphisms in **Field** are the non-zero maps. ///

\***Exercise 1.2.v** In **Ring**, consider the inclusion map  $i : \mathbb{Z} \hookrightarrow \mathbb{Q}$ . Now pick two morphisms,  $f, g : R \rightrightarrows \mathbb{Z}$  such that  $if = ig$  where  $R$  is some ring. Now BWOC suppose that  $f \neq g$ . Then there exists an  $r \in R$  such that  $f(r) \neq g(r)$ . Hence  $i(f(r)) \neq i(g(r))$  so  $if \neq ig$  which is a contradiction. Hence the above implies that  $f = g$  so  $i$  is monic.

Now pick morphisms  $h, k : \mathbb{Q} \rightrightarrows R$  such that  $hi = ki$ . This means that for all integers  $z$ ,  $h(z) = k(z)$ . Each rational  $q$  may be written canonically as  $nd^{-1}$  for  $n, d \in \mathbb{Z}$ . Hence  $h(q) = h(nd^{-1}) = h(n)h(d^{-1})$ . Further, because every element in  $\mathbb{Q}$  is a unit,  $h(n)h(d^{-1}) = h(n)h(d)^{-1}$ . Because  $h$  and  $k$  agree on the integers, we have  $h(n)h(d)^{-1} = k(n)k(d)^{-1}$ . Hence  $h = k$  and  $i$  is an epimorphism.

$\mathbb{Q}$  and  $\mathbb{Z}$  are not isomorphic in **Ring** because  $\mathbb{Q}$  is a field while  $\mathbb{Z}$  is not. Hence in **Ring**, monic and epic does not imply isomorphism. ///

**Exercise 1.2.vii** For a given subset of objects  $A \subseteq P$ , consider  $M_A = \{x \in \text{obj}P \mid \forall a \in A, \exists f \in P(a, x)\}$ , i.e. the set of all elements in  $P$  targeted by a morphism from every element in  $A$ . If it exists, the supremum of  $A$  is the least element in  $M_A$  w.r.t.  $\leq$ .

We define the infimum by duality as the greatest element in  $m_A = \{x \in \text{obj}P \mid \forall a \in A, \exists f \in P(x, a)\}$ .

Now suppose we have  $p, q$  both suprema of  $A$ . Then by definition  $p \leq q$  and  $q \leq p$  because each is a member of  $A$ . Hence by the antireflexivity of  $\leq$ , we have  $p = q$ . The dual proof shows uniqueness for the infimum. ///

## Section 3

**Exercise 1.3.i** Given two groups  $G$  and  $H$ , a functor  $BG \rightarrow BH$  is a group homomorphism. Given such a functor  $F$ , and elements  $x, y \in G$ , that is morphisms in  $BG$ , we have  $F(x)F(y) = F(xy)$  which is exactly a group homomorphism. ///

**Exercise 1.3.iii** Pick a category  $C$  where  $\text{obj}C = \{a, b, b', c\}$  and  $|\text{obj}C| = 4$  such that the only morphisms included beyond the identities are  $f : a \rightarrow b$  and  $g : b' \rightarrow c$ .

Now define a functor  $F : C \rightarrow D$  for some category  $D$  where  $\text{obj}D = \{Fa, Fb = Fb', Fc\}$  and  $|\text{obj}D| = 3$ . Let  $F$  preserve identities and  $Ff : Fa \rightarrow Fb$  and  $Fg : Fb \rightarrow Fc$ . Observe that this is in fact a functor because it preserves identities and there are no nontrivial compositions in  $C$ .

Because  $Fg \circ Ff$  is a morphism in  $D$ , but is not in the image of  $F$ , the image of  $F$  is not a sub category. ///

**Exercise 1.3.iv** First observe that  $C(c, -)$  and  $C(-, c)$  both preserve identities. Consider the action of  $C(c, -)$  on  $\text{id}_x : x \rightarrow x$ .  $\text{id}_x$  maps to post composition by the identity on  $C(c, x)$  which is the identity morphism for  $C(c, x)$ . Similar reason applies for  $C(-, c)$ .

Now pick two morphisms  $f : x \rightarrow y$  and  $g : y \rightarrow z$  in  $C$ . Then  $C(c, -)g \cdot C(c, -)f = g_*f_* = (gf)_* = C(c, -)(gf)$ . Hence  $C(c, -)$  is in fact a functor. The same holds for  $C(-, c)$  by duality. ///

**Exercise 1.3.viii** Consider the trivial Functor  $\mathbf{0} : \text{Set} \rightarrow \text{Set}$  that maps every set to  $\emptyset$  and every morphism to  $\text{id}_\emptyset$ . Observe that this is indeed a functor because  $\mathbf{0}(\text{id}_x) = \text{id}_{\mathbf{0}x} = \text{id}_\emptyset$  and  $\mathbf{0}(gf) = \mathbf{0}(g)\mathbf{0}(f) = \text{id}_\emptyset$ . Now pick any morphism in  $\text{Set}$  that is not an isomorphism. It is mapped to  $\text{id}_\emptyset$  which is an isomorphism. Hence functors do not reflect isomorphisms. ///

**Exercise 1.3.vi** Consider the map  $\text{dom} : F \downarrow G \rightarrow D$  that takes an object  $(d, e, f) \mapsto d$  and a morphism  $(h, k) \mapsto h$ . Now pick an identity morphism  $(\text{id}_d, \text{id}_e)$ ; clearly this functor preserves identity. Now pick a composable pair of morphisms  $(h, k)$  and  $(h', k')$ . We have  $\text{dom}((h', k')(h, k)) = \text{dom}(h'h, k'k) = h'h = \text{dom}(h', k')\text{dom}(h, k)$ . Hence  $\text{dom}$  is a functor. Define  $\text{cod}$  analogously. ///

**Exercise 1.3.x** Let  $\text{cl}(g)$  be the conjugacy class of  $g$  and  $\text{Cl}(G)$  be the set of all conjugacy classes of  $G$  (I don't know what the standard notation for this is). Consider the following mapping:

$$\begin{aligned} \text{Conj} : \text{Group} &\longrightarrow \text{Set} \\ G &\longmapsto \text{Cl}(G) \\ (f : G \rightarrow H) &\longmapsto (f' : \text{Cl}(G) \rightarrow \text{Cl}(H)) \end{aligned}$$

Where  $f'$  maps  $\text{cl}(x)$  to  $\text{cl}(f(x))$ . Note that this is indeed a well defined map: pick an element  $b$  in  $G$ . Then  $f'(b) = \text{cl}(f(b))$ . For any other element  $a \in \text{cl}(b)$ , we have  $f'(a) = \text{cl}(f(a)) = \text{cl}(f(g)f(b)f(g)^{-1}) = \text{cl}(f(b))$ .

Observe that  $\text{Conj}$  preserves identities. Now pick  $h : G \rightarrow H$  and  $k : H \rightarrow I$ . Then  $\text{Conj}(kh) = f' : \text{Cl}(G) \rightarrow \text{Cl}(I) = \text{Conj}(k)\text{Conj}(h)$ . Hence  $\text{Conj}$  is a functor.

Now pick a pair of groups such that the cardinalities of their sets of conjugacy classes differ. If there was an isomorphism between the two groups, say  $k : G \rightarrow H$ , then

$\text{Conj}k : \text{Cl}(G) \rightarrow \text{Cl}(H)$  would be an isomorphism as well. But this cannot be because there are no isomorphisms between sets with different cardinalities. ///

## Section 4

**Exercise 1.4.i** Consider the inverse of a component of the natural isomorphism  $\alpha : F \Rightarrow G$  such as  $\alpha_c^{-1} : Gc \rightarrow Fc$ . It remains to show that the following diagram commutes:

$$\begin{array}{ccc} Gc & \xrightarrow{\alpha_c^{-1}} & Fc \\ G\phi \downarrow & & \downarrow F\phi \\ Gc' & \xrightarrow{\alpha_{c'}^{-1}} & Fc' \end{array}$$

By the naturality of  $\alpha$  we have that  $G\phi \cdot \alpha_c = \alpha_{c'} \cdot F\phi$ . Hence by composing each side with  $\alpha_c^{-1}$  and  $\alpha_{c'}^{-1}$  we get  $\alpha_{c'}^{-1} \cdot G\phi = F\phi \cdot \alpha_c^{-1}$ . Hence  $\alpha^{-1} : G \Rightarrow F$  is a natural isomorphism. ///

**Exercise 1.4.ii** Pick two groups  $H, K$  and two functors  $F, G : BH \Rightarrow BK$  such that there exists a natural transformation  $\alpha : F \Rightarrow G$ . By exercise 1.3.i,  $F$  and  $G$  are group homomorphisms. Now note that because each category contains only one element, there is exactly one component of  $\alpha$ , say  $k$ . Now pick a morphism (group element) in  $BH$ , say  $h$  and consider the corresponding naturality square. Examining it shows that  $kF(h) = G(h)k$  so  $F(h) = k^{-1}G(h)k$ . Hence we may say that  $\alpha$  is an element  $k \in K$  such that for every  $h \in H$ ,  $F(h) = k^{-1}G(h)k$ . ///

**Exercise 1.4.iv** Observe that  $f$  and  $g$  define the same natural transformation exactly when their components are equal. The naturality condition for  $f_*$  is that  $h^*f_* = f_*h^*$  and for  $g_*$  it is  $h^*g_* = g_*h^*$ . Hence because  $f$  and  $g$  are distinct morphisms, we do not have  $h^*f_* = h^*g_*$ , for example if  $h^*$  is the identity. ///

**Exercise 1.4.vi** The components of a natural transformation are defined to be morphisms in the shared target category of two functors. Hence the definition of natural (or extranatural) transformation does not allow for different target categories because the same notion of morphism does not apply. ///

## Section 5

**Exercise 1.5.i** Let  $\phi : 0 \rightarrow 1$  be the single nonidentity morphism in **2**. Then consider the category  $C \times \mathbf{2}$  and pick a morphism  $f : x \rightarrow y$  in  $C$ . We have

$$\begin{array}{ccc} (x, 0) & \xrightarrow{(\text{id}_x, \phi)} & (x, 1) \\ (f, \text{id}_0) \downarrow & \searrow (f, \phi) & \downarrow (f, \text{id}_1) \\ (y, 0) & \xrightarrow{(\text{id}_y, \phi)} & (y, 1) \end{array}$$

A quick check shows that this diagram commutes. Now apply  $H$  to  $C \times \mathbf{2}$ . Observe that because  $F = H\text{id}_0$ , we have  $H(f, \text{id}_0) = Ff$ . Similarly for  $G$ , we have  $H(f, \text{id}_1) = Gf$ .

Now because  $H$  is itself a single functor, it preserves compositions of morphisms. Hence the above square is lifted to  $D$  where it forms a naturality square for  $f$  for some natural transformation where  $H(\text{id}_x, \phi)$  and  $H(\text{id}_y, \phi)$  are the components.

Let  $\mathcal{H}$  be the set of all functors  $C \times \mathbf{2} \rightarrow D$  that restrict along  $i_0$  and  $i_1$  to  $F$  and  $G$  and  $\mathcal{A}$  be the set of natural transformations from  $F$  to  $G$ . Observe then that applying  $H$  defines a function  $\mathcal{F} : \mathcal{H} \rightarrow \mathcal{A}$ .

We may now define  $\mathcal{F}^{-1}$ . For any  $\alpha : F \Rightarrow G$ , map the component  $\alpha_x$  to  $(\text{id}_x, \phi)$ . This is enough to define a particular functor in  $\mathcal{H}$  as the other mappings are constrained by  $F$  and  $G$ .

Hence  $\mathcal{H} \cong \mathcal{A}$ . ///

**Exercise 1.5.iii** Let the two isomorphisms be  $g : a \rightarrow a'$  and  $h : b \rightarrow b'$  and let the leftmost diagram define  $f'$ . Then  $f' = hfg^{-1}$ . Hence the second diagram commutes because  $f'g = hf$ , the third because  $h^{-1}f' = fg^{-1}$ , and the fourth because  $h^{-1}f'g = f$ . ///

**Exercise 1.5.iv**

- (i) Because  $F$  is fully faithful,  $C(y, x) \cong D(Fy, Fx)$ . Hence  $(Ff)^{-1}$  has a preimage in  $C$ , say  $g$ . Thus  $Ff \circ Fg = \text{id}_y = F(fg) = F\text{id}_y$ . Hence  $fg = \text{id}_y$ . Similar reasoning shows that  $gf = \text{id}_x$ . Therefore  $f$  is an isomorphism. ///
- (ii) Because  $F$  is fully faithful, we know that for any  $x, y \in C$ ,  $C(x, y) \cong D(Fx, Fy)$ . There is at least one morphism in  $D(Fx, Fy)$ , namely the isomorphism  $Ff$ . Thus there is a morphism  $f \in C(x, y)$ . Now apply (i) to see that  $f$  is an isomorphism so  $x \cong y$ . ///

**Exercise 1.5.ix** Theorem 1.5.9 says that a functor defining an equivalence of categories is fully faithful. Hence, for a category to be equivalent to a locally small category, the hom sets of each category must be isomorphic. Sets are only isomorphic to other sets; therefore the equivalent category is also locally small. ///

**Exercise 1.5.xi** Consider the inclusion  $\text{Ab} \rightarrow \text{Group}$ . It is fully faithful because the hom sets between abelian groups are the same in each category. It is not essentially surjective because nonabelian groups are not isomorphic to abelian groups. ///

## Section 6

**Exercise 1.6.i** Let  $i$  and  $t$  be initial and terminal in  $C$  respectively and suppose there is a morphism  $f : t \rightarrow i$ . Because  $i$  and  $t$  are terminal and initial, there is a unique morphism  $g : i \rightarrow t$ . Now observe that  $fg = \text{id}_i$  and  $gf = \text{id}_t$  because there is a unique morphism  $i \rightarrow i$  and a unique morphism  $t \rightarrow t$  which must then each be the identity. Thus  $t \cong i$ . ///

**Exercise 1.6.ii** Let  $p$  and  $q$  be terminal objects. Then there are unique morphism  $p \rightarrow q$  and  $q \rightarrow p$ . Because the only morphisms  $p \rightarrow p$  and  $q \rightarrow q$  are the identities,  $pq$

and  $qp$  are the identities so  $p \cong q$  uniquely. ///

**Exercise 1.6.iv** The forgetful functor  $\mathbf{Ring} \rightarrow \mathbf{Set}$  is faithful but does not preserve epimorphisms because the inclusion  $\mathbb{Z} \hookrightarrow \mathbb{Q}$  is epic but not a surjection.

Argue by duality...

**Exercise 1.6.v** Consider the concrete category  $(\mathbf{2}, U)$  where  $\mathbf{2}$  is the category containing a single morphism  $0 \rightarrow 1$  and where  $U : \mathbf{2} \rightarrow \mathbf{Set}$  takes  $0$  to a singleton set,  $1$  to a doubleton set, and the morphism to some morphism  $U0 \rightarrow U1$ . Observe that the morphism in  $\mathbf{2}$  is a monomorphism but its underlying function is not injective because there are no injections between finite sets of different cardinalities.

As we saw in a previous exercise, the inclusion  $\mathbb{Z} \hookrightarrow \mathbb{Q}$  in the category of Rings is an epimorphism but the underlying function is not a surjection. ///

## Section 7

**Exercise 1.7.i** Let  $F, G : C \rightrightarrows D$  be functors. Observe that for any  $c \in C$ ,  $D(Fc, Gc)$  is a set because  $D$  is locally small. Now consider

$$A = \bigcup_{c \in C} D(Fc, Gc)$$

$A$  is a set because it is a union of sets indexed by another set.

Now, every natural transformation is uniquely defined by its components. Each component is a map  $\alpha_c : Fc \rightarrow Gc$  for some  $c$ . Hence, each natural transformation corresponds to an element in  $\mathcal{P}(A)$  that contains all of its components.

Thus the collection of natural transformations between  $F, G$  is a set because it is in bijection with a subset of  $\mathcal{P}(A)$  which is a set.

Hence There are set-many morphisms between any two objects in  $D^C$  so it is locally small. ///

**Exercise 1.7.ii** Observe that the following diagram commutes as it is the diagram defining horizontal composition:

$$H(Fc) \quad K(Fc)$$

Hence by functoriality of  $L$ ,

$$\begin{aligned} L(\beta_{Fc'} \circ H F f) &= L(K F f \circ \beta_{Fc}) \\ L(\beta_{Fc'}) L(H F f) &= L(K F f) L(\beta_{Fc}) \end{aligned}$$

so the diagram commutes and  $L\beta F$  is natural.

**Exercise 1.7.v** Define the monoid operation to be vertical composition of natural transformations  $\mathbf{1}_C \Rightarrow \mathbf{1}_C$ . The identity natural transformation serves as the identity. Further, this operation is associative because the composition of components is associative.

**Exercise 1.7.vii** For each  $c \in C$ ,  $F(c, -)$  inherits functoriality from  $F$ , hence  $F$  determines a functor for each  $c$ .  $F$  also determines a natural transformation  $F(f, -) : F(c, -) \Rightarrow F(c', -)$  as follows. For some morphism  $g : x \rightarrow y$  in  $D$ , we have

$$\begin{array}{ccc} F(c, x) & \xrightarrow{F(f, x)} & F(c', x) \\ \downarrow F(c, g) & & \downarrow F(c', g) \\ F(c, y) & \xrightarrow{F(f, y)} & F(c', y) \end{array}$$

The square commutes because  $f$  and  $g$  each act independently on the left and right components respectively.

Now, for each  $c \in C$ , pick a functor  $K_c : D \rightarrow E$  and for each morphism  $f : c \rightarrow c'$ , pick a natural transformation  $\alpha : K_c \Rightarrow K_{c'}$ . Then we may define  $F : C \times D \rightarrow E$  for objects as  $(c, d) \mapsto K_c(d)$ . For a morphism  $(f, g) : (c, d) \rightarrow (c', d')$ , we need to construct a mapping  $F(f, g) : F(c, d) \rightarrow F(c', d')$ . By the definition of  $F$  on objects, this is  $F(f, g) : K_c(d) \rightarrow K_{c'}(d)$ . Hence define  $F(f, g) = K_{c'}(g) \circ \alpha_d$ .

Show that  $F$  is a functor...

Hence there is a bijection between  $C \times D \rightarrow E$  and  $C \rightarrow E^D$ .