

Section 1

Exercise 1.1.i

- (i) Let $f : x \rightarrow y$ be a morphism and $g, g' : y \rightarrow x$ be inverses. Then $fg = \text{id}_y$ and $gf = \text{id}_x$ (resp g'). Hence $gf = g'f$ so

$$\begin{aligned} gfg &= g'fg \\ g &= g' \end{aligned}$$

Therefore any morphism has at most one inverse. ///

- (ii) Suppose there are morphisms $g, h : y \rightrightarrows x$ such that $fg = \text{id}_y$ and $hf = \text{id}_x$. Then

$$\begin{aligned} hfg &= h \\ g &= h \end{aligned}$$

Hence $fg = \text{id}_y$ and $gf = \text{id}_x$ so f is an isomorphism. ///

Exercise 1.1.ii

Consider a category C and the subset of isomorphisms it contains. The subset contains the identity morphisms for each object because the identity is an isomorphism. The subset inherits associativity from C . Now pick two isomorphisms $f : x \rightarrow y$ and $g : y \rightarrow z$ and consider gf . We have $(f^{-1}g^{-1})gf = f^{-1}g^{-1}gf = f^{-1}f = \text{id}_x$ and $gf(f^{-1}g^{-1}) = \text{id}_y$ so gf is also an isomorphism. Hence C contains a maximal groupoid. ///

Exercise 1.1.iii

- (i) Observe that for any object $f : c \rightarrow x$ in c/C , the morphism $\text{id}_x : x \rightarrow x$ is the identity morphism for f . Now pick two morphisms in c/C , $h : x \rightarrow y$ and $k : y \rightarrow z$ and consider kh .

$$\begin{array}{ccccc} & & c & & \\ & f \swarrow & \downarrow g & \searrow b & \\ x & \xrightarrow{h} & y & \xrightarrow{k} & z \end{array}$$

Because $hf = g$ and $kg = b$ we have $khf = b$ so kh is a morphism in c/C which is enough to show it is a category. ///

- (ii) A similar argument as above shows that C/c is a category. ///

Section 2

Exercise 1.2.iii

Let $f : x \hookrightarrow y$ and $g : y \hookrightarrow z$ be monomorphisms and $h, k : w \rightarrow x$ be morphisms such that $gh = gfk$. Because g is monic, it follows that $fh = fk$. Similarly, because f is monic we have $h = k$. Hence gf is monic.

Now suppose f and g are simply morphisms and that gf is monic. BWO suppose that f is not monic. Then there exist morphisms h, k such that $fh = fk$ but $h \neq k$. Hence

we have $gfh = gfk$ this is a contradiction because gf being monic implies that $h = k$. Hence f is monic.

Now observe that the monomorphisms in a category form a subcategory. Every object has an identity morphism because the identity is monic. Associativity is inherited from the larger category and composition is preserved because the composition of two monomorphisms is also a monomorphism as shown above.

By duality the epimorphisms also form a subcategory. ///

Exercise 1.2.iv Q: What are the monomorphisms in **Field**. Haven't been able to crack this one yet.

Exercise 1.2.v In **Ring**, consider the inclusion map $i : \mathbb{Z} \hookrightarrow \mathbb{Q}$. Now pick two morphisms, $f, g : R \rightrightarrows \mathbb{Z}$ such that $if = ig$ where R is some ring. Now BWOC suppose that $f \neq g$. Then there exists an $r \in R$ such that $f(r) \neq g(r)$. Hence $i(f(r)) \neq i(g(r))$ so $if \neq ig$ which is a contradiction. Hence the above implies that $f = g$ so i is monic.

Now pick morphisms $h, k : \mathbb{Q} \rightrightarrows R$ such that $hi = ki$. This means that for all integers z , $h(z) = k(z)$. Each rational q may be written canonically as nd^{-1} for $n, d \in \mathbb{Z}$. Hence $h(q) = h(nd^{-1}) = h(n)h(d^{-1})$. Further, because every element in \mathbb{Q} is a unit, $h(n)h(d^{-1}) = h(n)h(d)^{-1}$. Because h and k agree on the integers, we have $h(n)h(d)^{-1} = k(n)k(d)^{-1}$. Hence $h = k$ and i is an epimorphism.

\mathbb{Q} and \mathbb{Z} are not isomorphic in **Ring** because \mathbb{Q} is a field while \mathbb{Z} is not. Hence in **Ring**, monic and epic does not imply isomorphism. ///

Exercise 1.2.vii For a given subset of objects $A \subseteq P$, consider $M_A = \{x \in \text{obj}P \mid \forall a \in A, \exists f \in P(a, x)\}$, i.e. the set of all elements in P targeted by a morphism from every element in A . If it exists (when could it not exist?), the supremum of A is the least element in M_A w.r.t. \leq .

By duality...