Section 6

Exercise 1.6.i Let i and t be initial and terminal in C respectively and suppose there is a morphism $f : t \to i$. Because i and t are terminal and initial, there is a unique morphism $g : i \to t$. Now observe that $fg = \mathrm{id}_i$ and $gf = \mathrm{id}_t$ because there is a unique morphism $i \to i$ and a unique morphism $t \to t$ which must then each be the identity. Thus $t \cong i$. ///

Exercise 1.6.ii Let p and q be terminal objects. Then there are unique morphism $p \to q$ and $q \to p$. Because the only morphisms $p \to p$ and $q \to q$ are the identities, pq and qp are the identities so $p \cong q$ uniquely. ///

Exercise 1.6.iv The forgetful functor **Ring** $\to$ **Set** is faithful but does not preserve epimorphisms because the inclusion $\mathbb{Z} \hookrightarrow \mathbb{Q}$ is epic but not a surjection.

Argue by duality...

Exercise 1.6.v Consider the concrete category $(\mathbf{2}, U)$ where $\mathbf{2}$ is the category containing a single morphism $0 \to 1$ and where $U : \mathbf{2} \to \mathbf{Set}$ takes 0 to a singleton set, 1 to a doubleton set, and the morphism to some morphism $U0 \to U1$. Observe that the morphism in $\mathbf{2}$ is a monomorphism but its underlying function is not injective because there are no injections between finite sets of different cardinalities.

As we saw in a previous exercise, the inclusion $\mathbb{Z} \hookrightarrow \mathbb{Q}$ in the category of Rings is an epimorphism but the underlying function is not a surjection. ///