Section 1

Exercise 1

(i) Let $f : x \to y$ be a morphism and $g, g' : y \to x$ be inverses. Then $fg = \mathrm{id}_y$ and $gf = \mathrm{id}_x$ (resp g'). Hence $gf = g'f$ so

$$\begin{aligned} gfg &= g'fg \\ g &= g' \end{aligned}$$

Therefore any morphism has at most one inverse. ///

(ii) Suppose there are morphisms $g, h : y \rightrightarrows x$ such that $fg = \mathrm{id}_y$ and $hf = \mathrm{id}_x$. Then

$$\begin{aligned} hfg &= h \\ g &= h \end{aligned}$$

Hence $fg = \mathrm{id}_y$ and $gf = \mathrm{id}_x$ so f is an isomorphism. ///

Exercise 2 Consider a category C and the subset of isomorphisms it contains. The subset contains the identity morphisms for each object because the identity is an isomorphism. The subset inherits associativity from C. Now pick two isomorphisms $f : x \to y$ and $g : y \to z$ and consider gf. We have $(f^{-1}g^{-1})gf = f^{-1}g^{-1}gf = f^{-1}f = \mathrm{id}_x$ and $gf(f^{-1}g^{-1}) = \mathrm{id}_y$ so gf is also an isomorphism. Hence C contains a maximal groupoid. ///

Exercise 3

(i) Observe that for any object $f : c \to x$ in c/C, the morphism $\mathrm{id}_x : x \to x$ is the identity morphism for f. Now pick two morphisms in c/C, $h : x \to y$ and $k : y \to z$ and consider kh.

$$\begin{tikzcd} & c \arrow[dl, "f"'] \arrow[d, dashed, "g"] \arrow[dr, "b"] & \\ x \arrow[r, "h"] & y \arrow[r, "k"] & z \end{tikzcd}$$

Because $hf = g$ and $kg = b$ we have $khf = b$ so kh is a morphism in c/C which is enough to show it is a category. ///

(ii) A similar argument as above shows that C/c is a category. ///