Section 1

Exercise 1.1.i

(i) Let $f : x \to y$ be a morphism and $g, g' : y \to x$ be inverses. Then $fg = \mathrm{id}_y$ and $gf = \mathrm{id}_x$ (resp g'). Hence $gf = g'f$ so

$$\begin{aligned} gfg &= g'fg \\ g &= g' \end{aligned}$$

Therefore any morphism has at most one inverse. ///

(ii) Suppose there are morphisms $g, h : y \rightrightarrows x$ such that $fg = \mathrm{id}_y$ and $hf = \mathrm{id}_x$. Then

$$\begin{aligned} hfg &= h \\ g &= h \end{aligned}$$

Hence $fg = \mathrm{id}_y$ and $gf = \mathrm{id}_x$ so f is an isomorphism. ///

Exercise 1.1.ii Consider a category C and the subset of isomorphisms it contains. The subset contains the identity morphisms for each object because the identity is an isomorphism. The subset inherits associativity from C. Now pick two isomorphisms $f : x \to y$ and $g : y \to z$ and consider gf. We have $(f^{-1}g^{-1})gf = f^{-1}g^{-1}gf = f^{-1}f = \mathrm{id}_x$ and $gf(f^{-1}g^{-1}) = \mathrm{id}_y$ so gf is also an isomorphism. Hence C contains a maximal groupoid. ///

Exercise 1.1.iii

(i) Observe that for any object $f : c \to x$ in c/C, the morphism $\mathrm{id}_x : x \to x$ is the identity morphism for f. Now pick two morphisms in c/C, $h : x \to y$ and $k : y \to z$ and consider kh.

$$\begin{tikzcd} & c \arrow[dl, "f"'] \arrow[d, dashed, "g"] \arrow[dr, "b"] & \\ x \arrow[r, "h"] & y \arrow[r, "k"] & z \end{tikzcd}$$

Because $hf = g$ and $kg = b$ we have $khf = b$ so kh is a morphism in c/C which is enough to show it is a category. ///

(ii) A similar argument as above shows that C/c is a category. ///

Section 2

Exercise 1.2.ii

(i) ($\Longrightarrow$) Suppose $f : x \to y$ is a split epimorphism. Now pick a morphism h in $C(c, y)$. We may write

Exercise 1.2.iii Let $f : x \hookrightarrow y$ and $g : y \hookrightarrow z$ be monomorphisms and $h, k : w \to x$ be morphisms such that $gfh = gfk$. Because g is monic, it follows that $fh = fk$. Similarly, because f is monic we have $h = k$. Hence gf is monic.

Now suppose f and g are simply morphisms and that gf is monic. BWOC suppose that f is not monic. Then there exist morphisms h, k such that $fh = fk$ but $h \neq k$. Hence we have $gfh = gfk$ this is a contradiction because gf being monic implies that $h = k$. Hence f is monic.

Now observe that the monomorphisms in a category form a subcategory. Every object has an identity morphism because the identity is monic. Associativity is inherited from the larger category and composition is preserved because the composition of two monomorphisms is also a monomorphisms as shown above.

By duality the epimorphisms also form a subcategory. ///

Exercise 1.2.v In **Ring**, consider the inclusion map $i : \mathbb{Z} \hookrightarrow \mathbb{Q}$. Now pick two morphisms, $f, g : R \rightrightarrows \mathbb{Z}$ such that $if = ig$ where R is some ring. Now BWOC suppose that $f \neq g$. Then there exists an $r \in R$ such that $f(r) \neq g(r)$. Hence $i(f(r)) \neq i(g(r))$ so $if \neq ig$ which is a contradiction. Hence the above implies that $f = g$ so i is monic.

Now pick morphisms $h, k : \mathbb{Q} \rightrightarrows R$ such that $hi = ki$. This means that for all integers z, $h(z) = k(z)$. Each rational q may be written canonically as nd^{-1} for $n, d \in \mathbb{Z}$. Hence $h(q) = h(nd^{-1}) = h(n)h(d^{-1})$. Further, because every element in $\mathbb{Q}$ is a unit, $h(n)h(d^{-1}) = h(n)h(d)^{-1}$. Because h and k agree on the integers, we have $h(n)h(d)^{-1} = k(n)k(d)^{-1}$. Hence $h = k$ and i is an epimorphism.

$\mathbb{Q}$ and $\mathbb{Z}$ are not isomorphic in **Ring** because $\mathbb{Q}$ is a field while $\mathbb{Z}$ is not. Hence in **Ring**, monic and epic does not imply isomorphism. ///