Section 1

Exercise 1.1.i

(i) Let $f : x \to y$ be a morphism and $g, g' : y \to x$ be inverses. Then $fg = \mathrm{id}_y$ and $gf = \mathrm{id}_x$ (resp g'). Hence $gf = g'f$ so

$$\begin{aligned} gfg &= g'fg \\ g &= g' \end{aligned}$$

Therefore any morphism has at most one inverse. ///

(ii) Suppose there are morphisms $g, h : y \rightrightarrows x$ such that $fg = \mathrm{id}_y$ and $hf = \mathrm{id}_x$. Then

$$\begin{aligned} hfg &= h \\ g &= h \end{aligned}$$

Hence $fg = \mathrm{id}_y$ and $gf = \mathrm{id}_x$ so f is an isomorphism. ///

Exercise 1.1.ii Consider a category C and the subset of isomorphisms it contains. The subset contains the identity morphisms for each object because the identity is an isomorphism. The subset inherits associativity from C. Now pick two isomorphisms $f : x \to y$ and $g : y \to z$ and consider gf. We have $(f^{-1}g^{-1})gf = f^{-1}g^{-1}gf = f^{-1}f = \mathrm{id}_x$ and $gf(f^{-1}g^{-1}) = \mathrm{id}_y$ so gf is also an isomorphism. Hence C contains a maximal groupoid. ///

Exercise 1.1.iii

(i) Observe that for any object $f : c \to x$ in c/C, the morphism $\mathrm{id}_x : x \to x$ is the identity morphism for f. Now pick two morphisms in c/C, $h : x \to y$ and $k : y \to z$ and consider kh.

$$\begin{tikzcd}
 & c \arrow[dl, "f"'] \arrow[d, dashed, "g"] \arrow[dr, "b"] & \\
x \arrow[r, "h"] & y \arrow[r, "k"] & z
\end{tikzcd}$$

Because $hf = g$ and $kg = b$ we have $khf = b$ so kh is a morphism in c/C which is enough to show it is a category. ///

(ii) A similar argument as above shows that C/c is a category. ///

Section 2

Exercise 1.2.iii Let $f : x \hookrightarrow y$ and $g : y \hookrightarrow z$ be monomorphisms and $h, k : w \to x$ be morphisms such that $gfh = gfk$. Because g is monic, it follows that $fh = fk$. Similarly, because f is monic we have $h = k$. Hence gf is monic.

Now suppose f and g are simply morphisms and that gf is monic. BWOC suppose that f is not monic. Then there exist morphisms h, k such that $fh = fk$ but $h \neq k$. Hence we have $gfh = gfk$ this is a contradiction because gf being monic implies that $h = k$. Hence f is monic.

Now observe that the monomorphisms in a category form a subcategory. Every object has an identity morphism because the identity is monic. Associativity is inherited from the larger category and composition is preserved because the composition of two monomorphisms is also a monomorphisms as shown above.

By duality the epimorphisms also form a subcategory. ///

Exercise 1.2.iv Question: What are the monomorphisms in Field?

Exercise 1.2.v In Ring, consider the inclusion map $i : \mathbb{Z} \hookrightarrow \mathbb{Q}$. Now pick two morphisms, $f, g : R \rightrightarrows \mathbb{Z}$ such that $if = ig$ where R is some ring. Now BWOC suppose that $f \neq g$. Then there exists an $r \in R$ such that $f(r) \neq g(r)$. Hence $i(f(r)) \neq i(g(r))$ so $if \neq ig$ which is a contradiction. Hence the above implies that $f = g$ so i is monic.

Now pick morphisms $h, k : \mathbb{Q} \rightrightarrows R$ such that $hi = ki$. This means that for all integers z, $h(z) = k(z)$. Each rational q may be written canonically as nd^{-1} for $n, d \in \mathbb{Z}$. Hence $h(q) = h(nd^{-1}) = h(n)h(d^{-1})$. Further, because every element in $\mathbb{Q}$ is a unit, $h(n)h(d^{-1}) = h(n)h(d)^{-1}$. Because h and k agree on the integers, we have $h(n)h(d)^{-1} = k(n)k(d)^{-1}$. Hence $h = k$ and i is an epimorphism.

$\mathbb{Q}$ and $\mathbb{Z}$ are not isomorphic in Ring because $\mathbb{Q}$ is a field while $\mathbb{Z}$ is not. Hence in Ring, monic and epic does not imply isomorphism. ///

Exercise 1.2.vii For a given subset of objects $A \subseteq P$, consider $M_A = \{x \in \text{obj}P | \forall a \in A, \exists f \in P(a, x)\}$, i.e. the set of all elements in P targeted by a morphism from every element in A. If it exists (when could the supremum not exist?), the supremum of A is the least element in M_A w.r.t. $\leq$.

By duality...

Section 3

Exercise 1.3.i Given two groups G and H, a functor $\mathrm{B}G \to \mathrm{B}H$ is a group homomorphism. Given such a functor F, and elements $x, y \in G$, that is morphisms in $\mathrm{B}G$, we have $F(x)F(y) = F(xy)$ which is exactly a group homomorphism. ///

Exercise 1.3.iii Question: Give an example of a functor such that its image is not a subcategory of the codomain.

Exercise 1.3.iv First observe that $C(c, -)$ and $C(-, c)$ both preserve identities. Consider the action of $C(c, -)$ on $\text{id}_x : x \to x$. id_x maps to post composition by the identity on $C(c, x)$ which is the identity morphism for $C(c, x)$. Similar reason applies for $C(-, c)$.

Now pick two morphisms $f : x \to y$ and $g : y \to z$ in C. Then $C(c, -)g \cdot C(c, -)f = g_* f_* = (gf)_* = C(c, -)(gf)$. Hence $C(c, -)$ is in fact a functor. The same holds for $C(-, c)$ by duality. ///

Exercise 1.3.viii Consider the trivial Functor $\mathbf{0} : \mathrm{Set} \to \mathrm{Set}$ that maps every set to $\emptyset$ and every morphism to $\mathrm{id}_\emptyset$. Observe that this is indeed a functor because $\mathbf{0}(\mathrm{id}_x) = \mathrm{id}_{\mathbf{0}x} = \mathrm{id}_\emptyset$ and $\mathbf{0}(gf) = \mathbf{0}(g)\mathbf{0}(f) = \mathrm{id}_\emptyset$. Now pick any morphism in Set that is not an isomorphism. It is mapped to $\mathrm{id}_\emptyset$ which is an isomorphism. Hence functors do not reflect isomorphisms. ///

Exercise 1.3.vi Consider the map $\mathrm{dom} : F{\downarrow}G \to D$ that takes an object $(d, e, f) \mapsto d$ and a morphism $(h, k) \mapsto h$. Now pick an identity morphism $(\mathrm{id}_d, \mathrm{id}_e)$; clearly this functor preserves identity. Now pick a composable pair of morphisms (h, k) and (h', k'). We have $\mathrm{dom}((h', k')(h, k)) = \mathrm{dom}(h'h, k'k) = h'h = \mathrm{dom}(h', k')\mathrm{dom}(h, k)$. Hence dom is a functor. Define cod analagously. ///

Exercise 1.3.x Let $\mathrm{cl}(g)$ be the conjugacy class of g and $\mathrm{Cl}(G)$ be the set of all conjugacy classes of G (I don't know what the standard notation for this is). Consider the following mapping:

$$\begin{aligned} \mathrm{Conj} : \mathrm{Group} &\longrightarrow \mathrm{Set} \\ G &\longmapsto \mathrm{Cl}(G) \\ (f : G \to H) &\longmapsto (f' : \mathrm{Cl}(G) \to \mathrm{Cl}(H)) \end{aligned}$$

Where f' maps $\mathrm{cl}(x)$ to $\mathrm{cl}(f(x))$. Note that this is indeed a well defined map: pick an element b in G. Then $f'(b) = \mathrm{cl}(f(b))$. For any other element $a \in \mathrm{cl}(b)$, we have $f'(a) = \mathrm{cl}(f(a)) = \mathrm{cl}(f(g)f(b)f(g)^{-1}) = \mathrm{cl}(f(b))$.

Observe that Conj preserves identities. Now pick $h : G \to H$ and $k : H \to I$. Then $\mathrm{Conj}(kh) = f' : \mathrm{Cl}(G) \to \mathrm{Cl}(I) = \mathrm{Conj}(k)\,\mathrm{Conj}(h)$. Hence Conj is a functor.

Now pick a pair of groups such that the cardinalities of their sets of conjugacy classes differ. (How do I conclude that they are not isomorphic?).

Section 1.4

Exercise 1.4.i Consider the inverse of a component of the natural isomorphism $\alpha : F \Rightarrow G$ such as $\alpha_c^{-1} : Gc \to Fc$. It remains to show that the following diagram commutes:

$$\begin{tikzcd} Gc \arrow[r, "\alpha_c^{-1}"] \arrow[d, "G\phi"'] & Fc \arrow[d, "F\phi"] \\ Gc' \arrow[r, "\alpha_{c'}^{-1}"] & Fc' \end{tikzcd}$$

By the naturality of α we have that $G\phi \cdot \alpha_c = \alpha_{c'} \cdot F\phi$. Hence by composing each side with α_c^{-1} and $\alpha_{c'}^{-1}$ we get $\alpha_{c'}^{-1} \cdot G\phi = F\phi \cdot \alpha_c^{-1}$. Hence $\alpha^{-1} : G \Rightarrow F$ is a natural isomorphism. ///

Exercise 1.4.ii Pick two groups H, K and two functors $F, G : \mathrm{B}H \rightrightarrows \mathrm{B}K$. By exercise 1.3.i, F and G are simply group homomorphisms. Now pick a natural transformation $\alpha : F \Rightarrow G$.

$\boxed{\textbf{Exercise 1.4.vi}}$ The target category of both of the functors must be the same...

Question: Why is A included in the definition? Why not just B and B^{op}?