

Section 1

Exercise 1.1.i

- (i) Let $f : x \rightarrow y$ be a morphism and $g, g' : y \rightarrow x$ be inverses. Then $fg = \text{id}_y$ and $gf = \text{id}_x$ (resp g'). Hence $gf = g'f$ so

$$\begin{aligned} gfg &= g'fg \\ g &= g' \end{aligned}$$

Therefore any morphism has at most one inverse. ///

- (ii) Suppose there are morphisms $g, h : y \rightrightarrows x$ such that $fg = \text{id}_y$ and $hf = \text{id}_x$. Then

$$\begin{aligned} hfg &= h \\ g &= h \end{aligned}$$

Hence $fg = \text{id}_y$ and $gf = \text{id}_x$ so f is an isomorphism. ///

Exercise 1.1.ii Consider a category C and the subset of isomorphisms it contains. The subset contains the identity morphisms for each object because the identity is an isomorphism. The subset inherits associativity from C . Now pick two isomorphisms $f : x \rightarrow y$ and $g : y \rightarrow z$ and consider gf . We have $(f^{-1}g^{-1})gf = f^{-1}g^{-1}gf = f^{-1}f = \text{id}_x$ and $gf(f^{-1}g^{-1}) = \text{id}_y$ so gf is also an isomorphism. Hence C contains a maximal groupoid. ///

Exercise 1.1.iii

- (i) Observe that for any object $f : c \rightarrow x$ in c/C , the morphism $\text{id}_x : x \rightarrow x$ is the identity morphism for f . Now pick two morphisms in c/C , $h : x \rightarrow y$ and $k : y \rightarrow z$ and consider kh .

$$\begin{array}{ccccc} & & c & & \\ & f \swarrow & \downarrow g & \searrow b & \\ x & \xrightarrow{h} & y & \xrightarrow{k} & z \end{array}$$

Because $hf = g$ and $kg = b$ we have $khf = b$ so kh is a morphism in c/C which is enough to show it is a category. ///

- (ii) A similar argument as above shows that C/c is a category. ///

Section 2

Exercise 1.2.iii Let $f : x \hookrightarrow y$ and $g : y \hookrightarrow z$ be monomorphisms and $h, k : w \rightarrow x$ be morphisms such that $gh = gfk$. Because g is monic, it follows that $fh = fk$. Similarly, because f is monic we have $h = k$. Hence gf is monic.

Now suppose f and g are simply morphisms and that gf is monic. BWO suppose that f is not monic. Then there exist morphisms h, k such that $fh = fk$ but $h \neq k$. Hence

we have $gfh = gfk$ this is a contradiction because gf being monic implies that $h = k$. Hence f is monic.

Now observe that the monomorphisms in a category form a subcategory. Every object has an identity morphism because the identity is monic. Associativity is inherited from the larger category and composition is preserved because the composition of two monomorphisms is also a monomorphism as shown above.

By duality the epimorphisms also form a subcategory. ///

Exercise 1.2.iv Question: What are the monomorphisms in Field?

Exercise 1.2.v In **Ring**, consider the inclusion map $i : \mathbb{Z} \hookrightarrow \mathbb{Q}$. Now pick two morphisms, $f, g : R \rightrightarrows \mathbb{Z}$ such that $if = ig$ where R is some ring. Now BWOC suppose that $f \neq g$. Then there exists an $r \in R$ such that $f(r) \neq g(r)$. Hence $i(f(r)) \neq i(g(r))$ so $if \neq ig$ which is a contradiction. Hence the above implies that $f = g$ so i is monic.

Now pick morphisms $h, k : \mathbb{Q} \rightrightarrows R$ such that $hi = ki$. This means that for all integers z , $h(z) = k(z)$. Each rational q may be written canonically as nd^{-1} for $n, d \in \mathbb{Z}$. Hence $h(q) = h(nd^{-1}) = h(n)h(d^{-1})$. Further, because every element in \mathbb{Q} is a unit, $h(n)h(d^{-1}) = h(n)h(d)^{-1}$. Because h and k agree on the integers, we have $h(n)h(d)^{-1} = k(n)k(d)^{-1}$. Hence $h = k$ and i is an epimorphism.

\mathbb{Q} and \mathbb{Z} are not isomorphic in **Ring** because \mathbb{Q} is a field while \mathbb{Z} is not. Hence in **Ring**, monic and epic does not imply isomorphism. ///

Exercise 1.2.vii For a given subset of objects $A \subseteq P$, consider $M_A = \{x \in \text{obj}P \mid \forall a \in A, \exists f \in P(a, x)\}$, i.e. the set of all elements in P targeted by a morphism from every element in A . If it exists (when could the supremum not exist?), the supremum of A is the least element in M_A w.r.t. \leq .

By duality...

Section 3

Exercise 1.3.i Given two groups G and H , a functor $BG \rightarrow BH$ is a group homomorphism. Given such a functor F , and elements $x, y \in G$, that is morphisms in BG , we have $F(x)F(y) = F(xy)$ which is exactly a group homomorphism. ///

Exercise 1.3.iii Question: Give an example of a functor such that its image is not a subcategory of the codomain.

Exercise 1.3.iv First observe that $C(c, -)$ and $C(-, c)$ both preserve identities. Consider the action of $C(c, -)$ on $\text{id}_x : x \rightarrow x$. id_x maps to post composition by the identity on $C(c, x)$ which is the identity morphism for $C(c, x)$. Similar reason applies for $C(-, c)$.

Now pick two morphisms $f : x \rightarrow y$ and $g : y \rightarrow z$ in C . Then $C(c, -)g \cdot C(c, -)f = g_*f_* = (gf)_* = C(c, -)(gf)$. Hence $C(c, -)$ is in fact a functor. A similar argument shows the same holds for $C(-, c)$.

Is this by duality? ///

Exercise 1.3.v I'm not sure I understand the question but I think the pairs of functors are duals: A functor $F : C^{\text{op}} \rightarrow D$ sends a morphism $f : x \rightarrow y$ to $Ff : Fy \rightarrow Fx$. We may construct a new functor $F' : C \rightarrow D^{\text{op}}$ such that $f : y \rightarrow x \mapsto F'f : F'x \rightarrow F'y$.

Exercise 1.3.viii Consider the trivial Functor $\mathbf{0} : \text{Set} \rightarrow \text{Set}$ that maps every set to \emptyset and every morphism to id_{\emptyset} . Observe that this is indeed a functor because $\mathbf{0}(\text{id}_x) = \text{id}_{\mathbf{0}x} = \text{id}_{\emptyset}$ and $\mathbf{0}(gf) = \mathbf{0}(g)\mathbf{0}(f) = \text{id}_{\emptyset}$. Now pick any morphism in Set that is not an isomorphism. It is mapped to id_{\emptyset} which is an isomorphism. Hence functors do not reflect isomorphisms. ///

Exercise 1.3.vi Consider the map $\text{dom} : F \downarrow G \rightarrow D$ that takes an object $(d, e, f) \mapsto d$ and a morphism $(h, k) \mapsto h$. Now pick an identity morphism $(\text{id}_d, \text{id}_e)$; clearly this functor preserves identity. Now pick a composable pair of morphisms (h, k) and (h', k') . We have $\text{dom}((h', k')(h, k)) = \text{dom}(h'h, k'k) = h'h = \text{dom}(h', k')\text{dom}(h, k)$. Hence dom is a functor. Define cod analogously. ///

Exercise 1.3.x Let $\text{cl}(g)$ be the conjugacy class of g and $\text{Cl}(G)$ be the set of all conjugacy classes of G (I don't know what the standard notation for this is). Consider the following mapping:

$$\begin{aligned} \text{Conj} : \text{Group} &\longrightarrow \text{Set} \\ G &\longmapsto \text{Cl}(G) \\ (f : G \rightarrow H) &\longmapsto (f' : \text{Cl}(G) \rightarrow \text{Cl}(H)) \end{aligned}$$

Where f' maps $\text{cl}(x)$ to $\text{cl}(f(x))$. Note that this is indeed a well defined map: pick an element b in G . Then $f'(b) = \text{cl}(f(b))$. For any other element $a \in \text{cl}(b)$, we have $f'(a) = \text{cl}(f(a)) = \text{cl}(f(g)f(b)f(g)^{-1}) = \text{cl}(f(b))$.

Observe that Conj preserves identities. Now pick $h : G \rightarrow H$ and $k : H \rightarrow I$. Then $\text{Conj}(kh) = f' : \text{Cl}(G) \rightarrow \text{Cl}(I) = \text{Conj}(k) \text{Conj}(h)$. Hence Conj is a functor.

Now pick a pair of groups such that the cardinalities of their sets of conjugacy classes differ. (How do I conclude that they are not isomorphic?).

Section 1.4

Exercise 1.4.i Consider the inverse of a component of the natural isomorphism $\alpha : F \Rightarrow G$ such as $\alpha_c^{-1} : Gc \rightarrow Fc$. It remains to show that the following diagram commutes:

$$\begin{array}{ccc} Gc & \xrightarrow{\alpha_c^{-1}} & Fc \\ G\phi \downarrow & & \downarrow F\phi \\ Gc' & \xrightarrow{\alpha_{c'}^{-1}} & Fc' \end{array}$$

By the naturality of α we have that $G\phi \cdot \alpha_c = \alpha_{c'} \cdot F\phi$. Hence by composing each side with α_c^{-1} and $\alpha_{c'}^{-1}$ we get $\alpha_{c'}^{-1} \cdot G\phi = F\phi \cdot \alpha_c^{-1}$. Hence $\alpha^{-1} : G \Rightarrow F$ is a natural isomorphism. ///