Section 1

Exercise 2.1.i The question asks us to the consider natural transformations between $\mathbf{Cat}(\mathbf{1}, -)$ and $\mathbf{Cat}(\mathbf{2}, -)$ induced by functors between $\mathbf{1}$ and $\mathbf{2}$.

For the collapsing functor $! : \mathbf{2} \to \mathbf{1}$, we have the following naturality square:

$$\begin{tikzcd}
\mathbf{Cat}(\mathbf{1}, x) \arrow[r, "!^*"] \arrow[d, "f_*"] & \mathbf{Cat}(\mathbf{2}, x) \arrow[d, "f_*"] \\
\mathbf{Cat}(\mathbf{1}, y) \arrow[r, "!^*"] & \mathbf{Cat}(\mathbf{2}, y)
\end{tikzcd}$$

This diagram commutes because for any suitable input x, $f_*!^*(x) = !xf = !^*f_*(x)$.

We achieve similar results for 0 and 2 by drawing a similar diagram for natural transformations from $\mathbf{Cat}(\mathbf{2}, -) \to \mathbf{Cat}(\mathbf{1}, -)$.

what else should be said about these natural transformations?

Exercise 2.1.ii If F is representable then we have a natural isomorphism α and the following commutative square:

$$\begin{tikzcd}
Fx \arrow[r, leftrightarrow, "\alpha_x"] \arrow[d, "Ff"'] & C(c, x) \arrow[d, "f_*"] \\
Fy \arrow[r, leftrightarrow, "\alpha_y"] & C(c, y)
\end{tikzcd}$$

Now suppose f is monic and pick two morphisms $g, h : W \rightrightarrows C(c, x)$ for some set W, such that $f_*g = f_*h$. This is equivalent to saying $\forall w \in W, f_*(g(w)) = f_*(h(w)) \implies f \circ (g(w)) = f \circ (h(w)) \implies g(w) = h(w) \implies g = h$ by the fact that f is monic. Hence f_* is monic.

Now, by the square, we have $\alpha_y Ff = f_* \alpha_x \implies Ff = \alpha_y^{-1} f_* \alpha_x$. Hence Ff is a composition of three injective maps which together are also injective. Therefore F preserves monomorphisms.

As for part two, pick a functor $\mathbf{2} \to \mathbf{Set}$ that takes the morphism to a non-injective function. By the contrapositive of the above statement, the functor is not representable because it does not preserve monomorphisms. ///