Holography Duality (8.821) Fall 2024
Assignment 4

Oct. 16th, 2024 **Due Oct. 30th, 2024**

- Please remember to put **your name** at the top of your paper.

Note:

For reviews of the large N expansion of various types of quantum field theories, see e.g.

- S. Coleman, "$1/N$", in *Aspect of Symmetry*, Cambridge Univ. Press, (1985).

  Highly recommend you to read p.368-p.378. Coleman's discussion is very elegant and transparent.

- Moshe Moshe and Jean Zinn-Justin, "Quantum Field Theory in the large N limit: a review", arXiv:hep-th/0306133.

- A. Manohar, "Large N QCD", arXiv:hep-ph/9802419.

  The discussion by Manohar contains pedagogic introduction to all the basic ingredients and also extensive discussion of applications to QCD.

Problem Set 4

1. **Reparameterization invariance (10 points)**

   The Nambu-Goto action for a relativistic string (moving in Minkowski spacetime) is given by

$$S_{\mathrm{NG}} = -\frac{T_0}{c} \int d^2\xi \, \sqrt{-\det \gamma} \tag{1}$$

   where T_0 is a constant and

$$\gamma_{\alpha\beta} = \partial_\alpha X^\mu \partial_\beta X_\mu, \quad \alpha, \beta = 0, 1 \tag{2}$$

   is the induced metric on the worldsheet. Show that the action is reparameterization invariant.

2. **Non-relativistic limit (10 points)**

   Consider the Nambu-Goto action (1) with $\xi^\alpha = (\tau, \sigma)$ for an open string stretched from $x^1 = 0$ and $x^1 = a$. Use the reparameterization invariance we can set $X^0 \equiv t = \tau$, and $X^1 \equiv x = \frac{a\sigma}{\ell}$ with $\sigma \in [0, \ell]$.

   The other embedding coordinates $X^i(\tau, \sigma)$, with $i = 2, 3, \cdots d$ can then be viewed as functions $X^i(t, x)$ of parameters t and x. They will be collectively denoted as $\vec{y}\,(t, x)$. Consider the nonrelativistic approximation based on

   $$|\partial_x \vec{y}\,| \ll 1 \quad \text{and} \quad \Big|\frac{1}{c}\partial_t\,\vec{y}\,\Big| \ll 1\ . \tag{3}$$

   Show that the Nambu-Goto action reduces to, up to an additive constant, to the action for a non-relativistic string performing small transverse oscillations. What are the corresponding tension and mass per unit length of the string? Also work out explicitly the additive constant and explain its physical meaning.

3. **Relation between string couplings (15 points)**

   The coupling for closed $\to$ open is denoted as g_o'. The coupling for open $\to$ open + open is given by g_o. The coupling for closed $\to$ closed + closed is g_s. These are basic interactions of string theory. In this problem we will show that these couplings are related.

   (a) Draw the simplest worldsheet (i.e. lowest order diagram in coupling expansion) for a single closed string transitioning to two open strings, and use it to show that the corresponding coupling g_s' is given by

   $$g_s' = g_o g_o'\ . \tag{4}$$

   (b) Consider the simplest worldsheet (i.e. lowest order diagram in coupling expansion) for open + open $\to$ closed + closed (i.e. two initial open strings going to two final closed strings). The same diagram viewed sideway can be considered as open + closed $\to$ open + closed. Show that the equivalence of two perspectives (channels) gives

   $$g_s' = g_s\ . \tag{5}$$

   (c) Draw the diagram for open + open $\to$ closed $\to$ open + open, where two open strings merge into a closed string which then splits again into two open strings. Now view the same diagram sideway and argue it can also be interpreted as open + open $\to$ open + open $\to$ open + open. Use the equivalence of the two perspectives (channels) to show that

   $$g_s'^2 = g_o^4\ . \tag{6}$$

   Combining (4)–(6), we thus conclude that

   $$g_o = g_o', \qquad g_s' = g_s = g_o^2\ . \tag{7}$$

4. **Random matrix integrals (30 points)**

Consider a $N \times N$ hermitian matrix Φ with a probability distribution

$$P(\Phi) = \frac{1}{Z} e^{-S} \tag{8}$$

where

$$S = \mathrm{Tr}\left(\frac{a}{2}\Phi^2 + \lambda\Phi^3\right) \ . \tag{9}$$

a is a constant and the coupling λ is taken to be small. Z is given by

$$Z = \int d\Phi\, e^{-S} \tag{10}$$

so that the total probability is one. (Since we will always treat $\lambda\,\mathrm{Tr}\,\Phi^3$ term perturbatively, we will not be concerned that the integral over $P(\Phi)$ is not defined non-perturbatively in λ.)

(a) Draw the Feynman rules for (9) in both single-line and double line notations. You need to be careful about indices and coupling dependence, but no need to be careful about numerical factors or other constants.

(b) Draw the diagrams (both single-line and double line) for the computation of Z to the lowest nonzero order. You should give N and λ dependence for each diagram, but no need to worry about other numerical factors.

(c) Draw the diagrams (both single-line and double line) for the computation of

$$\langle \Phi^a{}_b \Phi^c{}_d \rangle \equiv \int d\Phi\, P(\Phi)\, \Phi^a{}_b \Phi^c{}_d \tag{11}$$

to order λ^2. Again you should give N and λ dependence for each diagram, but no need to worry about other numerical factors.

5. $U(N)$ **Yang-Mills theory (25 points)**

Consider Yang-Mills theory with gauge group $U(N)$ and coupling constant g

$$\mathcal{L} = -\frac{c}{g^2}\,\mathrm{Tr}\, F_{\mu\nu}F^{\mu\nu} \tag{12}$$

with

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu] \ . \tag{13}$$

$A_\mu = A^a_\mu T^a$ with T^a $U(N)$ generators in the fundamental representation. c is a constant which makes A_μ canonical normalized. You can forget about ghosts and gauge fixing.

(a) Give the Feynman rules in both single-line and double line notations. You need to be careful about indices and coupling dependence, but no need to be careful about numerical factors or other constants.

(b) Draw the diagrams in both single-line and double line notations for the self-energy of A_μ to two loops. Give the coupling and N dependence for each diagram.

(c) Write down the 1-loop β-function for 't Hooft coupling $\lambda = g^2 N$ at leading order in large N. (You can directly use the standard Yang-Mills β-function as your starting point.)

6. **Wilson loop in the large N limit (10 points)**

Consider a Wilson loop in an $SU(N)$ gauge theory along some closed loop C,

$$W_R(C) = \left\langle \operatorname{Tr} P \exp\left(ig \oint_C A_\mu dx^\mu \right) \right\rangle \tag{14}$$

where $A_\mu = A_\mu^a T_R^a$ with T_R^a $SU(N)$ generators in a representation R and $\langle \cdots \rangle$ denotes the expectation value in some state (say vacuum or thermal state).

(a) In the large N limit, find the relation between $W_F(C)$ and $W_A(C)$ where subscript F and A denote the fundamental and adjoint representations respectively.

(b) Now consider a confining theory. Remind yourself (or find out) how a Wilson loop should behave in such a theory. What does (a) imply about the string tension for a fundamental and an adjoint quark? Try to give an intuitive explanation for the result.