Attempted solution – Exercise sheet I (Combinatorics I)

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Question 1

By defining, and calculating the expectation of, a suitable random variable, show that every graph G has a bipartite subgraph with at least $e(G)/2$ edges.

Solution:

Define an $n-$vertex random graph G such that $\mathbb{P}(e \in E(G)) = p$ (independent for each vertex), and thus has an expected average degree of $\mathbb{E}\left[\bar{d}(G)\right] = n\,p$ as it is a binomial distribution.

As we're dealing with bounded n, we assume it's non-zero. By definition of an **average** we get that:

$$\overbrace{\left(\sum_{v \in V(G)} d(v)\right)}^{=2e(G)} /n = \bar{d}(G)$$

If we let $\bar{d}(G) = n\,p$, we then have

$$n\,p = \frac{2e(G)}{n} \implies e(G) = \frac{n^2\,p}{2}$$

If we let s be the least set size that fills our requirement, we must then have that the number of edges of a $K_{s,s}$ be $e(G)/2$, and thus:

$$\frac{s^2}{4} = \frac{e(G)}{2} = \left(\frac{n^2}{2}\right)\frac{1}{2} \implies s^2 = n^2\,p.$$

By Kővári–Sós–Turán[1] we have that a bipartite graph composed of two $s-$vertex sets is bound to occur for

$$e(G) > C\,n^{2-1/s}$$

so, replacing $e(G)$ and s by our previous equations we have that, for

$$\frac{n^2\,p}{2} > C\,n^{2-1/(n\sqrt{p})} \implies n\sqrt{p}\log_n(p/C) + 1 > 0 \implies \frac{p}{C} \geqslant 1 \iff p \geqslant C$$

a complete bipartite graph of with set size s must exist.

Question 2

Show that $R(3,4) \leqslant 9$, $R(4,4) \leqslant 18$ and $R(3,3,3) \leqslant 17$.

Solution:

Here we adopt the lemma (proven in class[2]) that

$$R(s,t) \leqslant R(s-1,t) + R(s,t-1)$$

and, thus, we have $R(3,4) \leqslant R(2,4) + R(3,3)$. We can clearly see that $R(2,4)$ must be 4, as for $n \geqslant 4$ it's possible to form a K_4, and if we don't any other edge we draw is gonna form a K_2 of the other

[1]Proof in the end of the document 1.

colour (the same thinking applies to any other $R(2,n)$, i.e. $R(2,n) = R(n,2) = n$).

As for $R(3,3)$ we already know the result[3], which is 6, thus:

$$R(3,4) \leqslant 4 + 6 = 10,$$

so, we must be able to construct a K_9 without a K_3 or a K_4, but notice that, by the same construction method as the proof of lemma 3, we must be able to select any vertex $v \in V(G)$ and get a subset A which must have less than 6 vertices (as $K_{3,3} = 6$). But if we repeat this procedure for each vertex, we would have a total of $9 \cdot 5 = 45$ blue edges. How is this possible as, if we're counting each edge twice, we should have an even number? Contradiction!

Which implies $R(3,4) \leqslant 9$.

By the same lemma one notices that

$$R(4,4) \leqslant R(3,4) + R(4,3) = 9 + 9 = 18.$$

And then, as we can guarantee that we have at least a monochromatic copy of a K_4 in a 18−vertex graph, we know that $K_3 \subset K_4$, so $R(3,3,3) \leqslant 17$.

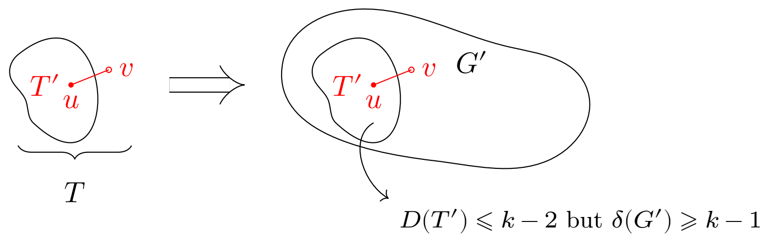
Question 3

Show that every graph of average degree d contains a subgraph of minimum degree at least $d/2$. Deduce that $\mathrm{ex}(n,T) \leqslant (k-1)n$ for every tree T with k vertices.

Solution:

As was shown in class:

Lemma 1. *Let G be a graph with average degree d then $\exists G' \in G$ with a minimum degree (δ) $x = x(d) \geqslant d/2$*



Proof.

Dispose $V(G)$ in a row, if every vertex has less than x edges going forwards ($\rightarrow$) than we must have at most $n\,d/2 = e(G) \leqslant x\,n \implies x > d/2$. □

Lemma 2. *Let G be a graph with minimum degree $\delta(G) \geqslant k-1$. Then $T \subset G\,\forall$ trees T with k vertices.*

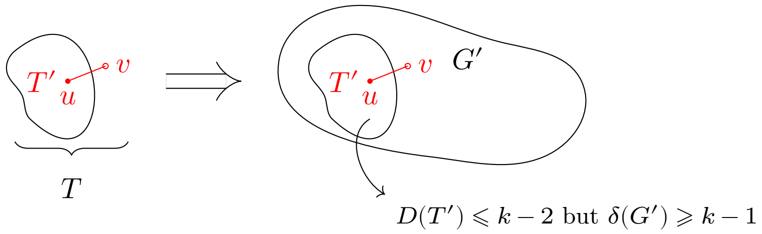
Proof. By induction on k we have that:

For $k = 1, 2$ it's quite trivial, as for $k = 1$ we have a mere vertex, and for $k = 2$ we have $\delta(G) \geqslant 1$, and thus we have such a tree.

If T has k vertices, take out a leaf:

[2] Proof in the end of the document 3.
[3] Proof in the end of the document 2

Then suppose it's valid for k, thus we have $\delta(G') \geqslant k > k-1$. So, there exists a tree $T' \subset T$, where T' is T without a leaf v, neighbour of $u \in V(T)$.



So, it follows that $d(u) \geqslant \delta(G') = k-1$. □

Let T be a tree with a fixed number of vertices k. If we wish to bound $\text{ex}(n, T)$ we then have

$$e(G) = C\,n = \bar{d}(G) = 2C \overset{\exists G' \subset G}{\overset{(1)}{\Longrightarrow}} \delta(G') \geqslant C \overset{\exists T \subset G'}{\overset{(2)}{\Longrightarrow}} (k-2)\frac{n}{2} \leqslant \text{ex}(n, T) \leqslant (k-1)n$$

□

appendix

Theorem 1 (Kővári–Sós–Turán 1950's)**.** $\text{ex}(n, H) = o(n^2)$

Proof. We can start by noticing that, given a bipartite graph H it must fit into another, larger, bipartite graph with disjoint sets of sizes s and t, denoted $K_{s,t}$. So we can write the inequality

$$\text{ex}(n, H) \leqslant \text{ex}(n, K_{s,t})$$

so now we can focus on bounding $\text{ex}(n, K_{s,t})$ from below, which can be done by generalizing the idea from the previous proof (for C_4), only this time we'll be counting s—cherries:

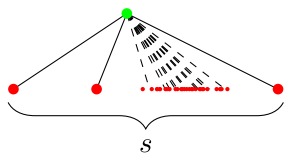


Figure 1: s—cherry.

$$\left(\frac{n}{s}\right)^s \geqslant (t-1)\binom{n}{s} \geqslant \#\text{ cherries} = \sum_{v \in V(G)} \binom{d(v)}{s}$$

again, by Jensen's inequality

$$\begin{aligned}
\#\text{ cherries} &\geqslant n\binom{\sum d(v)/n}{s} = n\binom{2e(G)/n}{s} \\
&= \frac{n}{s!}\left(\frac{2e(G)}{n}\right)\left(\frac{2e(G)}{n} - 1\right)\cdots\left(\frac{2e(G)}{n} - s + 1\right) \\
\left(\frac{n}{s}\right)^s \geqslant \#\text{ cherries} &\geqslant \frac{e(G)^s}{s^s\, n^{s-1}} \\
&\Longrightarrow e(G) \lesssim C\, n^{2-1/s}.
\end{aligned}$$

So that $\text{ex}(n, H) \lesssim C\, n^{2-1/s}$. □

Lemma 3. $R(s,t) \leqslant R(s-1,t) + R(s,t-1)$

Proof. Let G be a graph with $n = R(s,t) - 1$ which we will colour with $c : E(K_n) \longrightarrow \{R, B\}$, such that it doesn't have a blue K_s or a red K_t. Take $v \in V(K_n)$ and separate its neighbours into two groups: those which have a blue edge connecting v to them (say, group A) and the same for those with a red edge (group B). Then we cannot have K_{s-1} or K_t inside A.

So we must *always* have $|A| \leqslant R(s-1,t) - 1$ and $|B| \leqslant R(s,t-1) - 1$ for any step of the recursion (say, take a vertex $v' \in A$ and apply the same logic for some A' and B').

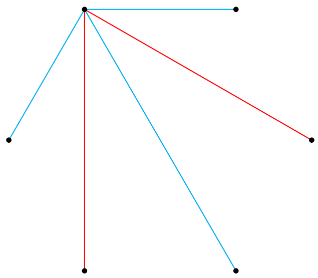
Thus, we have

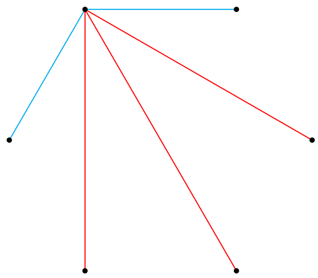
$$|V| = |A| + |B| + 1 \implies R(s,t) - 1 = n = |A| + |B| + 1 \leqslant R(s-1,t) + R(s,t-1) - 1.$$

□

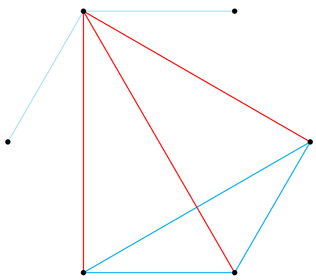
Theorem 2. $K(3,3) = 6$

Proof. Draw 6 vertices and notice that, by selecting any of them, we colour at least two vertices with the same colour:

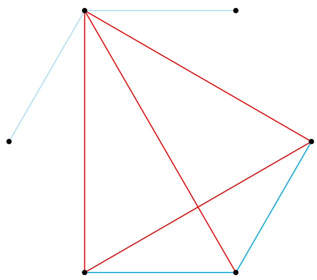




Let's select any 3 edges which are monochromatic say, the middle ones in the rightmost drawing, and then we can see that:



In case we draw everything in the other colour, we end up forming a triangle.



|If we try to avoid that by changing any one of the |edges we form a triangle anyway.

□