Attempted solution – Exercise sheet II (Combinatorics I)

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Question 1

Prove the following supersaturation theorem for cliques:

Theorem 1. *A graph with* $o(n^r)$ *copies of* K_r *has at most* $\mathrm{ex}(n, K_r) + o(n^2)$ *edges.*

Hint: apply Turán's theorem in a random subset of the vertices of constant size.

Solution:

Question 2

Show that, for every graph H, there exists $\delta > 0$ such that the following holds for all sufficiently large $n \in \mathbb{N}$. If G is a graph on n vertices with

$$e(G) > (1 - \delta)\binom{n}{2},$$

then in every $r-$colouring of $E(G)$ there are at least $\delta n^{v(H)}$ monochromatic copies of H.

Solution:

Question 3

Say that a $k-$uniform hypergraph G is said to be $2-$colourable if there exists a partition $V(G) = A \cup B$ with no edges entirely contained in either A or B. Let $b(k)$ denote the minimum number of edges in a $k-$uniform hypergraph that is not $2-$colourable.

(a) By considering a random colouring, show that $b(k) \geqslant 2k - 1$.

(b) By considering a random hypergraph, prove an upper bound for $b(k)$.

Solution:

Question 4

Show that any finite set A of integers contains a sum-free subset of size at least $|A|/3$.

Solution:

Question 5

Prove that there exists a tournament of order n containing at least $2^{-n}(n-1)!$ directed Hamiltonian cycles. (An ordering $(v_1, \ldots, v_n)$ of the vertices is a directed Hamiltonian cycle if v_i beat v_{i+1} for every $i \in \mathbb{Z}_n$.)

Solution:

Question 6

Let $(e_1, \dots, e_m)$ be an arbitrary ordering of the edges of a graph G on n vertices. Show that there exists an increasing walk (in this ordering) of length at least $d = 2m/n$.

Solution:

Question 7

Let G be a (not necessarily planar) graph with $|G| = n$ and $e(G) = m$. Suppose that G is drawn in the plane, but with edges allowed to cross. Let t be the number of pairs of edges which cross. Show that $t \geqslant m - 3n + 6$.

Suppose now $m \geqslant 4n$. By considering a random set $W \subset V(G)$ containing each vertex of G independently with probability $4n/m$, show that in fact $t \geqslant m^3/64n^2$.

Solution:

Question 8

Use the Janson inequalities and Harris' lemma to determine, for each constant $c \in (0, 1)$, a function $p = p(n)$ such that

$$\mathbb{P}\left(K_r \subset G(n, p)\right) \to c$$

as $n \to \infty$.

Solution:

Question 9

Use the Janson inequalities to prove Bollobás' theorem, that

$$\chi\left(G(n, 1/2)\right) = \left(\frac{1}{2} + o(1)\right)\frac{n}{\log_2 n}.$$

Solution: