Attempted solution – Exam II (Combinatorics I)

March 8, 2021 / Isabella B. Amaral

Note: Once again — I didn't really have the background to take on this course, so most solutions are **really** quite handwavy and mostly inspired by what was given in class.

Question 1

Show that every graph G has a bipartite subgraph with at least $e(G)/2$ edges.

Solution:

Define an $n-$vertex random graph G such that $\mathbb{P}(e \in E(G)) = p$ (independent for each vertex). Define $\mathbb{I}_k$ as the indicator function for the $k-$th edge and thus, by linearity of expectation we have

$$e(G) = \mathbb{E}\left[\#\text{ of edges}\right] = \sum_{k=1}^{\binom{n}{2}} \mathbb{E}\left[\mathbb{I}_k\right] = \sum_{k=1}^{\binom{n}{2}} \mathbb{P}\left(k\text{-th edge is present}\right) = \binom{n}{2} p.$$

By Mantel's theorem (thm. 1) the least edges a $s-$vertex graph needs before it isn't bipartite anymore is $\lfloor s^2/4 \rfloor$, thus if there exists an $s < n$ such that $e(G) \geqslant \lfloor s^2/4 \rfloor > e(G)/2$ we are sure to have at least a complete $s-$vertex bipartite graph. Thus

$$e(G) \geqslant \frac{s^2}{4} > \frac{e(G)}{2} = \binom{n}{2} p \frac{1}{2} \geqslant \left(\frac{n}{2}\right)^2 \frac{p}{2} \implies (\mathrm{e}\, n)^2 \frac{p}{2} \geqslant s^2 \geqslant \frac{p}{2} n^2 \implies e\sqrt{\frac{p}{2}} \geqslant \frac{s}{n} \geqslant \sqrt{\frac{p}{2}}.$$

As $n \geqslant s > 0$ and $1 > p > 0$ we have that

$$e\sqrt{\frac{1}{2}} > 1 > e\sqrt{\frac{p}{2}} \geqslant \frac{s}{n} \geqslant \sqrt{\frac{p}{2}} > 0$$

which is always true.

appendix

Theorem 1. $\mathrm{ex}(n, K_3) = \lfloor n^2/4 \rfloor$

Proof. Let G be the largest K_3-free $n-$vertex graph.

So, suppose we have $e(G) \leqslant \lfloor n^2/4 \rfloor$, then, if $K_3 \nsubseteq G$ we have

- for $n = 1$:
$$e(G) = 0 \leqslant \lfloor 1^2/4 \rfloor = 0.$$
- for $n = 2$:
$$e(G) = 1 \leqslant \lfloor 2^2/4 \rfloor = 1.$$
- for $n = 3$:
$$e(G) = 2 \leqslant \lfloor 3^2/4 \rfloor = 2.$$

This hypothesis is equivalent to taking a complete bipartite graph, so we can make a clever construction to get a bound on the number of edges:

We can grab one of its edges $e \in E(G)$, so that each of e's vertices determines a disjoint subset, for if they had any vertex in common, there would be a copy of K_3 contained in G.

Let G' be this new graph we get by selecting an edge $e \in E(G)$ and removing it from G.

If G doesn't contain any copies of K_3 we can have at most $n-2$ vertices between e and G' (because there can be no more than one edge between the vertices in e and G' without forming a triangle), plus the edge e we've selected. Thus, by applying an induction on n we have:

$$\begin{aligned} e(G) &\leqslant e(G') + n - 2 + 1 \\ &\overset{IH}{\leqslant} \left\lfloor \frac{(n-2)^2}{4} \right\rfloor + n - 1 = \left\lfloor \frac{n^2 - 4n + 4 + 4n - 4}{4} \right\rfloor = \left\lfloor \frac{n^2}{4} \right\rfloor. \end{aligned}$$

$\square$