Combinatorial numbers

Factorial

$n! = (n-1)(n-2)(n-3)(n-4)\ldots$

$0! = 1$

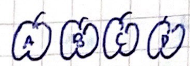
$1! = 1$

Combination

$$\binom{n}{r} = \frac{n!}{r!\,(n-r)!}$$

meaning:

Eg $\binom{4}{2}$ 4 choose 2



AB AC AD CD BC CB $=6$.

$$\frac{4!}{2!\,(4-2)!} = \frac{4!}{2!\,2!} = \frac{4\times3\times\cancel{2!}}{2!\,\cancel{2!}} = \frac{12}{2} = 6$$

Exercise example

Solve $\binom{n+2}{3} - \binom{n}{3} = n^2$

$$\frac{(n+2)!}{3!\,(n+2(-3))!} - \frac{n!}{3!\,(n-3)!} = n^2$$

$$\frac{(n+2)(n+1)\cancel{(n-1)!}}{3!\,\cancel{(n-1)!}} - \frac{n(n-1)(n-2)\cancel{(n-3)}}{3!\,\cancel{(n-3)!}} = n^2$$

$$\frac{(n+2)(n+1) - n(n-1)(n-2)}{6} = n^2$$

$$\frac{\cancel{n^3} + n^2 + 2n^2 + \cancel{2n} - \cancel{n^3} + 2n^2 + n^2 - \cancel{2n}}{6} = n^2 \Rightarrow \frac{6n^2}{6} = n^2$$

Binomial theorem

$(a+b)^0 = 1$

$(a+b)^1 = a+b$

$(a+b)^2 = a^2 + 2ab + b^2$

$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

given $(a+b)^n$

Observations:
- exponents of each term add up to n
- powers are in ascending order for b and descending for a
- always $n+1$ terms

Coefficient observations / Patterns

Pascals triangle

In order of n

1	$n=0$
1 1	$n=1$
1 2 1	$n=2$
1 3 3 1	$n=3$
1 4 6 4 1	$n=4$
1 5 10 10 5	$n=5$

* each nº is the sum of the above (except the 1)

Eg

$(a+b)^3$

$\boxed{1}\, a^3 \cancel{(b)^0} + \boxed{3}\, a^2 b + \boxed{3}\, ab^2 + \boxed{3}\, b^3 \cancel{(a)^0}$

You can also ↳ C^n_0 $\binom{n}{0}$ ↳ C^n_1 ↳ C^n_2 ...

use factorials:

Eg $(a+b)^3$ find the coeff of the term with a^2 in it

$(a+b)^3 \binom{3}{1} (a)^{\boxed{2}} (b)^{\boxed{1}}$

$\binom{n}{r} = \frac{n!}{(n-r)!\,r!}$

$\frac{3!}{1!(3-1)!} a^2 b = \frac{3 \times 2!}{1!\,2!} = 3a^2b$ ✓✓

Binomial expansion as decimals

a Find the first four terms of the expansion $(1+5x)^7$

b Hence, by setting $x = 0.01$ find an aproximation for 1.05^7 (6 s.f.)

(a $(1+5x)^7 \rightarrow 1 + C_1^7 5x + C_2^7 (5x)^2 + C_3^7 (5x)^3 + \ldots$

$= \boxed{1 + 35x + 525x^2 + 4375x^3} \ldots$

(b $x = 0.01$

$(1+5x)^7 = (1 + 5 \cdot 0.01)^7 = 1 + 35(0.01) + 525(0.01)^2 + 4375(0.01)^3 =$

≈ 1.40688 (6 s.f.)

Product of Binomial expansion

→ Find the coef of x^2y^5 in $(x-y)(x+y)^6$

$= x(x+y)^6 - y(x+y)^6$

To get x^2y^5

$x \Rightarrow x \cdot x = x^2$

$x\,(C_5^6\, x\, y^5) = 6x^2y^5$

$y \Rightarrow y \cdot y^4 = y^5$

$-y\,(C_4^6\, x^2 y^4) = 15x^2y^5$

$6x^2y^5 - 15x^2y^5 = -9x^2y^5 \rightarrow$ coef $= (-9)$

Binomial theorem extension

- $(1+x)^n = \binom{n}{0}(1)^n(x)^0 + \binom{n}{1}(1)^{n-1}(x)^1 + \binom{n}{2}(1)^{n-2}(x)^2 \ldots$

$1 + \frac{n\cancel{(n-1)!}}{1!\cancel{(n-1)!}}x + \frac{n(n-1)\cancel{(n-2)!}}{2!\cancel{(n-2)!}}x^2 + \frac{n(n-1)(n-2)\cancel{(n-3)!}}{3!\cancel{(n-3)!}}x^3$

$\approx (1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 \ldots$

* $(1+x)^{-n} = 1 + (-n)x + \frac{(-n)(-n-1)}{2!}x^2 + \frac{(-n)(-n-1)(-n-2)}{3!}x^3$

$\downarrow (-)$ $\downarrow (-)$

We could also express these ideas in terms of a and b.

$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!} \ldots$

$[a+b]^n = [(a+b)]^n = \left[a\left(1+\frac{b}{a}\right)\right]^n =$

$a^n\left(1+\frac{b}{a}\right)^n = a^n\left[1 + n\left(\frac{b}{a}\right) + \frac{n(n-1)\left(\frac{b}{a}\right)^2}{2!} + \frac{(n)(n-1)(n-2)\left(\frac{b}{a}\right)^3}{3!} \ldots\right]$

$= a^n + nba^{n-1} + \frac{n(n-1)b^2a^{n-2}}{2!} + \frac{n(n-1)(n-2)b^3a^{n-3}}{3!} \ldots$

we mult by n and bring it all up

* What we are doing with this is expressing a function as a polynomial

$(1+x)^{-n} = \frac{1}{(1+x)^n}$ [a rational f(x)]

Validity

This expression is only valid:

Standard format:

$-1 < x < 1$ → because $\frac{1}{x+2}$ eg always be smaller tan 1 or -1 $\frac{1}{4}$ $\frac{1}{-4}$

If we have xcoef ≠ x

$-\frac{1}{a} < \frac{\cancel{a}x}{\cancel{a}} < \frac{1}{a}$

Eg

$-1 < 4x < 1$ $\qquad$ $-1 < \frac{2}{3}x < 1$

$-\frac{1}{4} < x < 1$ $\qquad$ $-\frac{3}{2} < x < \frac{3}{2}$

$(1+4x)^n$ $\qquad$ $(1+\frac{2}{3}x)$

Inequalities Remember!

Mult (×) or div(÷) by neg n° → sign changes

Eg $(2<1)$ $x(-1)$

$-2 > -1$

Reciprocal → sign changes

$\frac{1}{2} < 1$ $\qquad$ $\frac{3}{2} > \frac{1}{2}$

$2 > \frac{1}{1}$ $\qquad$ $\frac{2}{3} < 1$