École Centrale de Nantes

Humanoid Robots

Laboratory Report

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1 Introduction

In this first laboratory we were asked to define three Matlab functions that will be used in the next assignment:

- function_dyn, describing the dynamic model of the biped in single support
- function_reactionforce, computing the reaction force
- function_impact, evaluating the impact model

The main goal of these functions is to describe the dynamic modeling of a compass-walking robot, defined in this figure 1

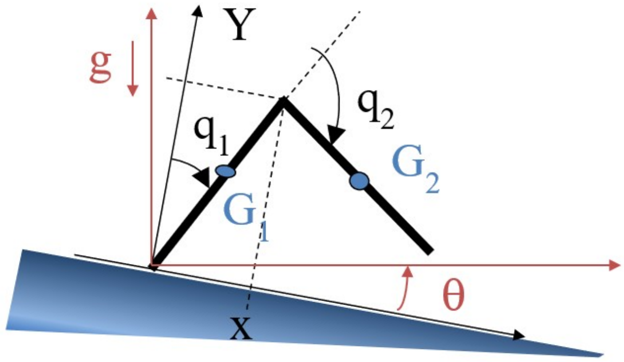


Figure 1: Compass-walking robot

As it is possible too see from the image, the robot is considered in single support and it is composed by two identical legs of length l and mass m, having an inertia I and a distance between the center of mass G and the hip of s. Furthermore the robot is walking along a slope, having θ angle; it is possible in fact to observe two frames. The black one is the reference frame R_0, which is attached to the ground with slope. In order to check the correctness of the three required Matlab functions, is necessary to run the **main.m** file; the resultant matrices will be printed on the command window section and can be compared with the required results. The known quantities that have been used to test the code are the following:

- l = 0.8 m;
- m = 2 Kg;
- I = 0.1 Kg m^2;
- S = 0.5 m;
- g = 9.8 Kg s^{-2};
- $\theta = 2\pi/180$;
- q1 = 1 , q2 = -0.5;
- $\dot{q}_1 = 0.1$, $\dot{q}_2 = 0.2$;

- $\ddot{q}_1 = 0.2$, $\ddot{q}_2 = 0.3$;

In this report, the mainly mathematical steps to achieve our objectives are described: further and detailed calculation were derived by hand and can be checked in the *Hand computations* section at the end of this PDF.

2 Dynamic model of the biped in single support

In this first section, the goal was to write a function that allows to find the matrices A and H to build the model under the form

$$A \begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix} + H + Q = B\Gamma \tag{1}$$

This model has been define with the *Lagrange approach.*

2.1 Positions of G1 and G2

The centers of mass *G1* and *G2* were found in function of *q1* and *q2*, deriving the following equations in vectorial form from the drawing:

$$\overrightarrow{OG_1} = (l-s) \begin{bmatrix} -sin(q1) \\ cos(q1) \end{bmatrix} \tag{2}$$

$$\overrightarrow{OG_2} = l \begin{bmatrix} -sin(q1) \\ cos(q1) \end{bmatrix} + s \begin{bmatrix} -sin(q1+q2) \\ cos(q1+q2) \end{bmatrix} \tag{3}$$

2.2 Linear velocities

From the previous two equations, I am now able to find *G1* and *G2* velocities deriving their positions, and obtaining:

$$\dot{\overrightarrow{OG_1}} = (l-s) \begin{bmatrix} -cos(q1) \\ -sin(q1) \end{bmatrix} \dot{q}_1 = \overrightarrow{v_{G_1}} \tag{4}$$

$$\dot{\overrightarrow{OG_2}} = l \begin{bmatrix} -cos(q1) \\ -sin(q1) \end{bmatrix} \dot{q}_1 + s \begin{bmatrix} -cos(q1+q2) \\ -sin(q1+q2) \end{bmatrix} (\dot{q}_1 + \dot{q}_2) = \overrightarrow{v_{G_2}} \tag{5}$$

3 Angular velocities

Considering a positive sign for a counterclockwise rotation, the angular velocities of each leg are obtained:

$$\overrightarrow{w_{G_1}} = -\dot{q}_1 \overrightarrow{z_0} \tag{6}$$

$$\overrightarrow{w_{G_2}} = -(\dot{q}_1 + \dot{q}_2) \overrightarrow{z_0} \tag{7}$$

where $\overrightarrow{z_0}$ is the z versor around which the rotation takes place.

3.1 Kinetic energies

At this point I am able to evaluate the kinetic energies of each link, using the formula:

$$E_{C_i} = \frac{1}{2}(m{V_{G_i}}^T V_{G_i} + {w_i}^T I w_i) \tag{8}$$

Applying the previous equation on the two link of my robot I can obtain:

$$E_1 = \frac{1}{2}m(l-s)^2 \dot{q_1}^2 + \frac{1}{2}I\dot{q_1}^2 \tag{9}$$

$$E_2 = \frac{1}{2}ml^2\dot{q_1}^2 + \frac{1}{2}ms^2(\dot{q_1}+\dot{q_2})^2 + mls\dot{q_1}(\dot{q_1}+\dot{q_2})cos(q_2) + \frac{1}{2}I(\dot{q_1}+\dot{q_2})^2 \tag{10}$$

and the total Kinetic energy, which is the sum of the two:

$$E_{tot} = E_1 + E_2 \tag{11}$$

3.2 Inertia matrix

The inertia matrix is now deduced using the shape of this model:

$$E_{tot} = E_1 + E_2 = \frac{1}{2}\begin{bmatrix}\dot{q_1} & \dot{q_2}\end{bmatrix} \mathbf{A} \begin{bmatrix}\dot{q_1} \\ \dot{q_2}\end{bmatrix} \tag{12}$$

All the mathematical steps are derived in the *Hand computations* section at the end of this PDF. Finally the 2x2 matrix A is extracted:

$$A = \begin{bmatrix} m(l-s)^2 + ms^2 + ml^2 + 2I + 2mlscos(q_2) & ms^2 + 2I + 2mlscos(q_2) \\ ms^2 + I + 2mlscos(q_2) & ms^2 + I \end{bmatrix} \tag{13}$$

4 Potential energy

In order to obtain the potential energy, the following formula is used:

$$U_i = -mg^T\overrightarrow{OG_i} \tag{14}$$

Applying this formula is important to notice that the gravity vector $\overrightarrow{g}$ has to be written with respect to the R_0 reference frame, so we need to perform a rotation around Z axis of θ, obtaining:

$$\overrightarrow{g} = \begin{bmatrix} -gsin(\theta) \\ gcos(\theta) \\ 0 \end{bmatrix} \tag{15}$$

The potential energy formula is now applicable and I can retrieve:

$$G_1 = -mg(l-s)cos(\theta - q_1) \tag{16}$$

$$G_2 = -mg[lcos(\theta - q_1) + scos(\theta - (q_1 + q_2))] \tag{17}$$

The total potential energy is nothing but the sum of G_1 and G_2:

$$U_{tot} = G_1 + G_2 \tag{18}$$

5 Gravity effect

The gravity effects are found by partially deriving equation 18 by q_1 to find Q_1 and by q_2 to find Q_2.

$$Q_1 = mg(l-s)sin(\theta - q_1) + mglsin(\theta - q_1) + mgssin(\theta - (q_1 + q_2)) \tag{19}$$

$$Q_2 = mgssin(\theta - (q_1 + q_2)) \tag{20}$$

5.1 H vector

Finally we can find the H vector using the formula

$$\overrightarrow{H} = B\dot{q_1}\dot{q_2} + C \begin{bmatrix} \dot{q_1}^2 \\ \dot{q_2}^2 \end{bmatrix} \tag{21}$$

I have followed the steps described in the assignment text and deriving the calculations I have found the matrices B and C:

$$B = \begin{bmatrix} -2mlssin(q_2) \\ 0 \end{bmatrix} \tag{22}$$

$$C = \begin{bmatrix} 0 & -mlssin(q_2) \\ mlssin(q_2) & 0 \end{bmatrix} \tag{23}$$

6 Reaction force

The objective of this section is to check the condition of contact in support in order to then be able to calculate the reaction force during support on leg 1. The contact focused on a point, thus only the reaction force exist, without momentum.

6.1 Position of robot center of mass

At this point it is possible to find the center of mass of the robot X_G as function of q_1 and q_2. I have easily found it using:

$$X_G = \frac{G_1 + G_2}{2} \tag{24}$$

The resultant matrix is the following:

$$G = \begin{bmatrix} -lsin(q_1) + \frac{1}{2}ssin(q_1) - \frac{1}{2}ssin(q_1 + q_2) \\ lcos(q_1) - \frac{1}{2}scos(q_1) + \frac{1}{2}scos(q_1 + q_2) \end{bmatrix} \tag{25}$$

6.2 Velocity and acceleration of robot center of mass

Deriving the position of the robot's center of mass with respect to time, I can find its velocity:

$$\begin{bmatrix} -lcos(q_1)\dot{q_1} + \frac{1}{2}scos(q_1)\dot{q_1} - \frac{1}{2}scos(q_1 + q_2)(\dot{q_1} + \dot{q_2}) \\ -lsin(q_1)\dot{q_1} + \frac{1}{2}ssin(q_1)\dot{q_1} - \frac{1}{2}ssin(q_1 + q_2)(\dot{q_1} + \dot{q_2}) \end{bmatrix} \tag{26}$$

Now deriving with respect to time the previous equation I can get the robot's center of mass acceleration.

$$\begin{bmatrix} lsin(q_1)\dot{q_1}^2 - lcos(q_1)\ddot{q_1} - \frac{1}{s}ssin(q_1)\dot{q_1}^2 + \frac{1}{2}scos(q_1)\ddot{q_1} + \frac{1}{2}ssin(q_1+q_2)(\dot{q_1}+\dot{q_2})^2 - \frac{1}{2}scos(q_1+q_2)(\ddot{q_1}+\ddot{q_2}) \\ -lcos(q_1)\dot{q_1}^2 - lsin(q_1)\ddot{q_1} + \frac{1}{s}scos(q_1)\dot{q_1}^2 + \frac{1}{2}ssin(q_1)\ddot{q_1} - \frac{1}{2}scos(q_1+q_2)(\dot{q_1}+\dot{q_2})^2 - \frac{1}{2}ssin(q_1+q_2)(\ddot{q_1}+\ddot{q_2}) \end{bmatrix} \tag{27}$$

6.3 Reaction force

Now I can finally find the reaction force using the previous evaluations and the formula:

$$F = M\ddot{x_G} - M\overrightarrow{g} \tag{28}$$

Letting M be the total mass of the robot, corresponding to:

$$M = 2m \tag{29}$$

7 Impact model

In this last section the impact model is evaluated: in fact at this instant the second leg is touching the ground whereas the first one takes off. The model that describes this situation is the following:

$$\begin{bmatrix} A_1 & -{J_{R2}}^T \\ J_{R2} & 0_{2x2} \end{bmatrix} \begin{bmatrix} \dot{x}^+ \\ \dot{y}^+ \\ \dot{q_1}^+ \\ \dot{q_2}^+ \\ I_{R_2x} \\ I_{R_2y} \end{bmatrix} = \begin{bmatrix} A_1 \\ 0_{2x4} \end{bmatrix} \begin{bmatrix} \dot{x}^- \\ \dot{y}^- \\ \dot{q_1}^- \\ \dot{q_2}^- \end{bmatrix} \tag{30}$$

In this model I can consider as known the x and y coordinates and I am able to compute in a different way $\overrightarrow{OG_1}$ $\overrightarrow{OG_2}$ and their respective velocities and accelerations, leading to a different kinetic matrix A that can be found using the decomposed model:

$$E_{tot} = E_1 + E_2 = \frac{1}{2} \begin{bmatrix} \dot{x} & \dot{y} & \dot{q_1} & \dot{q_2} \end{bmatrix} \mathbf{A} \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{q_1} \\ \dot{q_2} \end{bmatrix} \tag{31}$$

So we can now extract:

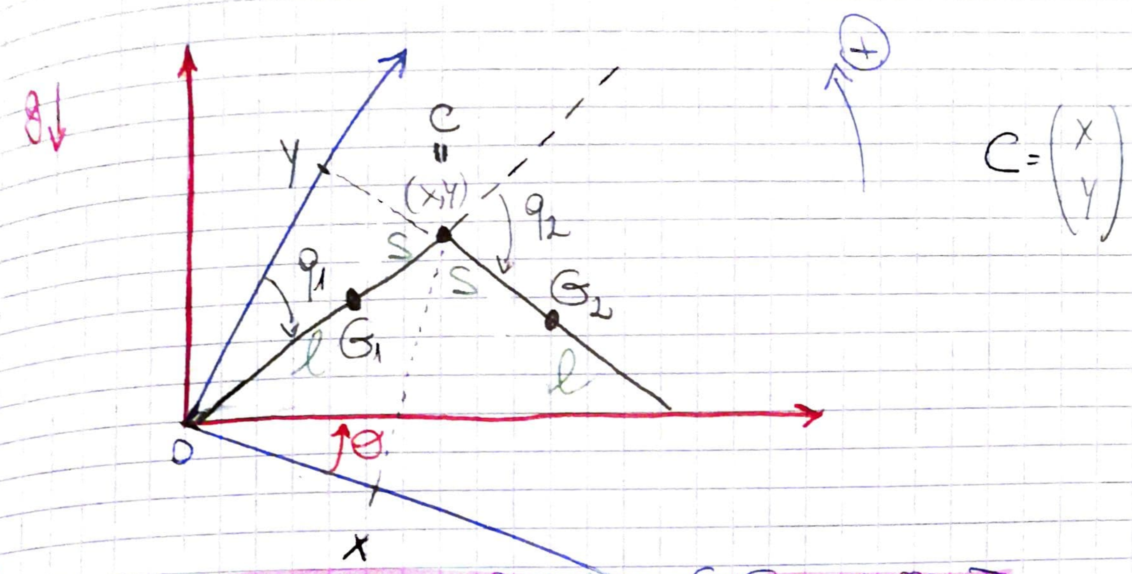
$$A = \begin{bmatrix} 2m & 0 & mscos(q_1){-}mscos(q_1{+}q_2) & -mscos(q_1{+}q_2) \\ 0 & 2m & mscos(q_1){-}mssin(q_1{+}q_2) & -mssin(q_1{+}q_2) \\ mscos(q_1){-}mscos(q_1{+}q_2) & mscos(q_1){-}mssin(q_1{+}q_2) & 2ms^2{+}2I & ms^2{+}I \\ -mscos(q_1{+}q_2) & -mssin(q_1{+}q_2) & ms^2{+}I & ms^2{+}I \end{bmatrix} \tag{32}$$

Finally considering the assumptions that the first leg was in support, the second leg arrives in impact, the first leg takes off whereas the second one stays on the ground without sliding. Knowing this and deriving the contact condition, I could find the matrix J_{R2} in order to find the relation which links the instants before and after the impact:

$$J_R = \begin{bmatrix} 1 & 0 & -lcos(q_1+q_2) & -lcos(q_1+q_2) \\ 0 & 1 & -lsin(q_1+q_2) & -lsin(q_1+q_2) \end{bmatrix} \tag{33}$$

Hand computations

HUMRO LAB 1



1) DYNAMIC MODEL OF THE BIPED IN SINGLE SUPPORT.

$[A,H] = \text{function\_dyn}(q_1, q_2, \dot{q}_1, \dot{q}_2, \theta)$

Model to find $\rightarrow$ $A\begin{bmatrix}\ddot{q}_1\\ \ddot{q}_2\end{bmatrix} + H + Q = B\tau$

Lagrange approach (FROM AHDRO LECTURES)

① POSITION OF G_1 & G_2 AS FUNCTION OF q_1 & q_2.

$$\vec{OG_1} = (l-S)\begin{bmatrix}\cos(90^\circ+q_1)\\ \sin(90^\circ+q_1)\end{bmatrix} = (l-S)\begin{bmatrix}-\sin q_1\\ \cos q_1\end{bmatrix}$$

$$\boxed{\vec{OG_1}} = (l-S)\begin{bmatrix}-\sin q_1\\ \cos q_1\end{bmatrix}$$

$$\boxed{\vec{OG_2}} = \vec{OC} + \vec{CG_2} = l\begin{bmatrix}-\sin q_1\\ \cos q_1\end{bmatrix} + S\begin{bmatrix}-\sin(q_1+q_2)\\ \cos(q_1+q_2)\end{bmatrix}$$

So, now I have found the position of G_1 & G_2 as function of q_1 & q_2.

② VELOCITY OF CENTER OF MASS

$$\vec{OG_1} = (l-s)\begin{bmatrix}-\sin q_1 \\ \cos q_1\end{bmatrix} \qquad \vec{OG_2} = l\begin{bmatrix}-\sin q_1 \\ \cos q_1\end{bmatrix} + s\begin{bmatrix}-\sin(q_1+q_2) \\ \cos(q_1+q_2)\end{bmatrix}$$

$$\dot{\vec{OG_1}} = (l-s)\begin{bmatrix}-\cos q_1 \\ -\sin q_1\end{bmatrix}\dot{q}_1 = \boxed{\vec{v}_{G1}}$$

$$\dot{\vec{OG_2}} = l\begin{bmatrix}-\cos q_1 \\ -\sin q_1\end{bmatrix}\dot{q}_1 + s\begin{bmatrix}-\cos(q_1+q_2) \\ -\sin(q_1+q_2)\end{bmatrix}(\dot{q}_1+\dot{q}_2) = \boxed{\vec{v}_{G2}}$$

③ ANGULAR VELOCITY OF EACH LEG.

LEG 1 $\boxed{\vec{\omega}_1} = -\dot{q}_1\vec{z}_0$

LEG 2 $\boxed{\vec{\omega}_2} = -(\dot{q}_1+\dot{q}_2)\vec{z}_0$

④ KINETIC ENERGY OF EACH LINK

$$\boxed{E_1} = \frac{1}{2}\left[m(l-s)^2\dot{q}_1^2 + \dot{q}_1^2 I\right] = \frac{1}{2}m(l-s)^2\dot{q}_1^2 + \frac{1}{2}\dot{q}_1^2 I$$

$$E_2 = \frac{1}{2}\left[m(l\dot{q}_1 + s(\dot{q}_1+\dot{q}_2))^2 + (\dot{q}_1+\dot{q}_2)^2 I\right] =$$

$$= \frac{1}{2}ml^2\dot{q}_1^2 + \frac{1}{2}ms^2(\dot{q}_1+\dot{q}_2)^2 + \frac{1}{2}(\dot{q}_1+\dot{q}_2)^2 I$$

$$+\frac{1}{2}m\,2ls\,\dot{q}_1(\dot{q}_1+\dot{q}_2)\cos q_2$$

$$\boxed{E_2} = \frac{1}{2}ml^2\dot{q}_1^2 + \frac{1}{2}ms^2(\dot{q}_1+\dot{q}_2)^2 + mls\,\dot{q}_1(\dot{q}_1+\dot{q}_2)\cos q_2 + \frac{1}{2}(\dot{q}_1+\dot{q}$$

$+ q_2)$

$+ q_2)]$

5 INERZIA MATRIX

$$E_c = E_{c1} + E_{c2} = \frac{1}{2}\begin{bmatrix}\dot q_1 & \dot q_2\end{bmatrix} A \begin{bmatrix}\dot q_1 \\ \dot q_2\end{bmatrix}$$

$$E_{c1} + E_{c2} = \frac{1}{2} m (l-S)^2 \dot q_1^2 + \frac{1}{2}\dot q_1^2 I + \frac{1}{2} m l^2 \dot q_1^2 + \frac{1}{2} m S^2 (\dot q_1 + \dot q_2)^2$$

$$+ m l S \dot q_1 (\dot q_1 + \dot q_2)\cos q_2 + \frac{1}{2}(\dot q_1 + \dot q_2)^2 I$$

$$E_c = \frac{1}{2}\Big[m(l-S)^2 \dot q_1^2 + \dot q_1^2 I + m l^2 \dot q_1^2 + m S^2 \overbrace{(\dot q_1 + \dot q_2)^2}^{\dot q_1^2 + \dot q_2^2 + 2\dot q_1 \dot q_2} +$$

$$+ 2 m l S \dot q_1 (\dot q_1 + \dot q_2)\cos q_2 + \underbrace{(\dot q_1 + \dot q_2)^2}_{\dot q_1^2 + \dot q_2^2 + 2\dot q_1 \dot q_2} I\Big] = (*)$$

$$\Rightarrow \frac{1}{2}\underset{1\times 2}{\begin{bmatrix}\dot q_1 & \dot q_2\end{bmatrix}} \underset{2\times 2}{\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22}\end{bmatrix}} \underset{2\times 1}{\begin{bmatrix}\dot q_1 \\ \dot q_2\end{bmatrix}} =$$

$$\frac{1}{2}\underset{1\times 2}{\begin{bmatrix} A_{11}\dot q_1 + A_{21}\dot q_2 & A_{12}\dot q_1 + A_{22}\dot q_2\end{bmatrix}} \begin{bmatrix}\dot q_1 \\ \dot q_2\end{bmatrix} =$$

$$= \Big[A_{11}\dot q_1^2 + A_{21}\dot q_1 \dot q_2 + A_{12}\dot q_1 \dot q_2 + A_{22}\dot q_2^2\Big]\frac{1}{2}$$

(*)

$(\dot q_1 + \dot q_2)^2 I$

$$\Rightarrow E_c = \frac{1}{2}\Big[m(l-S)^2 \dot q_1^2 + \dot q_1^2 I + m l^2 \dot q_1^2 + m S^2 \dot q_1^2 + m S^2 \dot q_2^2 +$$

$$+ 2 m S^2 \dot q_1 \dot q_2 + 2 m l S \dot q_1^2 \cos q_2 + 2 m l S \dot q_1 \dot q_2 \cos q_2 + \dot q_1^2 I + \dot q_2^2 I +$$

$$+ 2 \dot q_1 \dot q_2 I\Big] =$$

INERTIA MATRIX

$$\boxed{A} = \begin{bmatrix} m(l-s)^2 + 2I + ml^2 + ms^2 + 2mls\cos q_2 & ms^2 + I + mls\cos q_2 \\ ms^2 + I + mls\cos q_2 & ms^2 + I \end{bmatrix}$$

6) POTENTIAL ENERGIES $\longrightarrow$ $\vec{g} = \begin{bmatrix} -g\sin\theta \\ g\cos\theta \\ 0 \end{bmatrix}$

$$\boxed{U_1} = -m({}^0g^T)\,\overrightarrow{OG_1} =$$

$$= -mg(l-s)\begin{bmatrix} -\sin q_1 \\ \cos q_1 \end{bmatrix}\begin{bmatrix} -\sin\theta & \cos\theta \end{bmatrix} =$$

$$= -mg(l-s)\underbrace{(\sin\theta_1 \sin q_1 + \cos\theta_1 \cos q_1)}_{\cos(\theta - q_1)} = -mg(l-s)\cos(\theta - q_1)$$

$$\boxed{U_2} = -mg\left[l\begin{bmatrix} -\sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} -\sin q_1 \\ \cos q_1 \end{bmatrix} + s\begin{bmatrix} -\sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} -\sin(q_1+q_2) \\ \cos(q_1+q_2) \end{bmatrix}\right] =$$

$$= -mg\left[l\left[\sin\theta\sin q_1 + \cos\theta\cos q_1\right] + s\left[\sin\theta\sin(q_1+q_2) + \cos\theta\cos(q_1+q_2)\right]\right]$$

$$= -mg\left[l\cos(\theta - q_1) + s\cos(\theta - (q_1+q_2))\right]$$

7) GRAVITY EFFECT

$$Q_i = \frac{\partial U}{\partial q_i}$$

$$U = U_1 + U_2 = + mg(l-S)\cos(\theta - q_1) + mg\left[l\cos(\theta - q_1) + S\cos(\theta - (q_1 + q_2))\right]$$

$$Q_1 = \frac{\partial U}{\partial q_1} = + mg(l-S)\operatorname{sen}(\theta - q_1) + mg\,l\operatorname{sen}(\theta - q_1) + \, + mgS\operatorname{sen}(\theta - (q_1 + q_2))$$

$$Q_2 = \frac{\partial U}{\partial q_2} = + mg\,S\operatorname{sen}(\theta - (q_1 + q_2))$$

$$Q = \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix}$$

8) VECTOR H

$$H = B\dot{q}\dot{q} + C\dot{q}^2$$

$$\dot{q}\dot{q} = [\dot{q}_1\dot{q}_2, \ldots \dot{q}_1\dot{q}_n, \dot{q}_2\dot{q}_3, \ldots, \dot{q}_{n-1}\dot{q}_n]^T \text{ and } \dot{q}^2 = [\dot{q}_1^2 \ldots \dot{q}_n^2]^T$$

CHECK THE TEST

$$\vec{H} = B\dot{q}_1\dot{q}_2 + C\begin{bmatrix} \dot{q}_1^2 \\ \dot{q}_2^2 \end{bmatrix}$$

$$B_{1,1} = B_{11} = \frac{\partial A_{11}}{\partial q_2} + \frac{\partial A_{12}}{\partial q_1} - \frac{\partial A_{12}}{\partial q_1} = -2 m l S \operatorname{sen} q_2$$

$i = 1$

$j = 1$

$k = 2$

$\boxed{n = 2}$

$i = n = 2 \quad j = n - 1 = 2 - 1 = 1 \quad k = 2$

$$B_{21} = \frac{\partial A_{21}}{\partial q_2} + \frac{\partial A_{12}}{\partial q_1} - \frac{\partial A_{12}}{\partial q_2} = 0$$

$$= 0$$

same

$$B = \begin{bmatrix} -2 m l S \operatorname{sen} q_2 \\ 0 \end{bmatrix}$$

$i = 1 \quad j = 1$

$$C_{11} = \frac{\partial A_{11}}{\partial q_1} - \frac{1}{2} \frac{\partial A_{11}}{\partial q_1} = 0$$

$i = 1 \quad j = 2$

$$C_{12} = \frac{\partial A_{12}}{\partial q_2} - \frac{1}{2} \cancel{\frac{\partial A_{22}}{\partial q_1}} = - m l S \operatorname{sen} q_2$$

$i = 2 \quad j = 1$

$$C_{21} = \cancel{\frac{\partial A_{21}}{\partial q_1}} - \frac{1}{2} \frac{\partial A_{11}}{\partial q_2} = + \frac{1}{2} \cdot 2 m l S \operatorname{sen} q_2 = m l S \operatorname{sen} q_2$$

$i = 2 \quad j = 2$

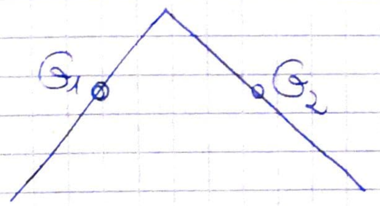
$$C_{22} = \frac{\partial A_{22}}{\partial q_2} - \frac{1}{2} \frac{\partial A_{22}}{\partial q_2} = 0$$

$$C = \begin{bmatrix} 0 & -m l S \,\mathrm{sen}(q_2) \\ m l S \,\mathrm{sen}\, q_2 & 0 \end{bmatrix}$$

2) Reaction force

$[F] =$ function_reaction force $(q_1, q_2, \dot{q}_1, \dot{q}_2, \ddot{q}_1, \ddot{q}_2)$

1 POSITION OF THE MASS CENTER XG OF THE ROBOT AS FUNCTION OF q_1 AND q_2. + 2 + 3



$$G = \frac{G_1 + G_2}{2} \Rightarrow = \begin{bmatrix} \dfrac{(l-S)(-\mathrm{sen}\, q_1) + l(-\mathrm{sen}\, q_1) + S(-\mathrm{sen}(q_1+q_2))}{2} \\[2ex] \dfrac{(l-S)\cos q_1 + l\cos q_1 + S\cos(q_1+q_2)}{2} \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{\overbrace{-l\,\mathrm{sen}\, q_1 + S\,\mathrm{sen}\, q_1 - l\,\mathrm{sen}\, q_1}^{-2l\,\mathrm{sen}\, q_1} - S\,\mathrm{sen}(q_1+q_2)}{2} \\[2ex] \dfrac{\overbrace{l\cos q_1 - S\cos q_1 + l\cos q_1}^{2l\cos q_1} + S\cos(q_1+q_2)}{2} \end{bmatrix} =$$

$$G = \begin{bmatrix} -l\,\text{sen}\,q_1 + \frac{1}{2} s\,\text{sen}\,q_1 - \frac{1}{2} s\,\text{sen}(q_1+q_2) \\ l\cos q_1 - \frac{1}{2} s\cos q_1 + \frac{1}{2} s\cos(q_1+q_2) \end{bmatrix}$$

$$\dot{G} = \begin{bmatrix} -l\cos q_1\,\dot{q}_1 + \frac{1}{2} s\cos q_1\,\dot{q}_1 - \frac{1}{2} s\cos(q_1+q_2)(\dot{q}_1+\dot{q}_2) \\ -l\,\text{sen}\,q_1\,\dot{q}_1 + \frac{1}{2} s\,\text{sen}\,q_1\,\dot{q}_1 - \frac{1}{2} s\,\text{sen}(q_1+q_2)(\dot{q}_1+\dot{q}_2) \end{bmatrix}$$

$$\ddot{G} = \begin{bmatrix} l\,\text{sen}\,q_1\,\dot{q}_1^2 - l\cos q_1\,\ddot{q}_1 - \frac{1}{2} s\,\text{sen}\,q_1\,\dot{q}_1^2 + \frac{1}{2} s\cos q_1\,\ddot{q}_1 + \\ + \frac{1}{2} s\,\text{sen}(q_1+q_2)(\dot{q}_1+\dot{q}_2)^2 - \frac{1}{2} s\cos(q_1+q_2)(\ddot{q}_1+\ddot{q}_2) \\ -l\cos q_1\,\dot{q}_1^2 - l\,\text{sen}\,q_1\,\ddot{q}_1 + \frac{1}{2} s\cos q_1\,\dot{q}_1^2 + \frac{1}{2} s\,\text{sen}\,q_1\,\ddot{q}_1 + \\ - \frac{1}{2} s\cos(q_1+q_2)(\dot{q}_1+\dot{q}_2)^2 - \frac{1}{2} s\,\text{sen}(q_1+q_2)(\ddot{q}_1+\ddot{q}_2) \end{bmatrix}$$

$$\vec{g} = \begin{bmatrix} -g\,\text{sen}\,\theta \\ g\cos\theta \\ 0 \end{bmatrix}$$

$$\vec{F} = 2m\ddot{G} - 2m\vec{g}$$

3) Impact model

① A_1 matrix

$$E_c = E_{c1} + E_{c2} = \frac{1}{2}\begin{bmatrix}\dot{x} & \dot{y} & \dot{q}_1 & \dot{q}_2\end{bmatrix} A_1 \begin{bmatrix}\dot{x}\\ \dot{y}\\ \dot{q}_1\\ \dot{q}_2\end{bmatrix}$$

So now I have to take into account also $\dot{x}$ and $\dot{y}$ to evaluate $\vec{OG_1}$ & $\vec{OG_2}$

so

$$\vec{OG_1} = \begin{bmatrix} x - S\sin q_1 \\ y - S\cos q_1 \end{bmatrix} \qquad \vec{OG_2} = \begin{bmatrix} x - S\sin(q_1+q_2) \\ y + S\cos(q_1+q_2)\end{bmatrix}$$

$$\dot{\vec{OG_1}} = \begin{bmatrix} \dot{x} - S\cos q_1 \dot{q}_1 \\ \dot{y} + S\sin q_1 \dot{q}_1 \end{bmatrix} \qquad \dot{\vec{OG_2}} = \begin{bmatrix} \dot{x} - S\cos(q_1+q_2)(\dot{q}_1+\dot{q}_2) \\ \dot{y} - S\sin(q_1+q_2)(\dot{q}_1+\dot{q}_2)\end{bmatrix}$$

$$\dot{\omega}_1 = \dot{q}_1 \vec{z_0} \quad \dot{\omega}_2 = -(\dot{q}_1+\dot{q}_2)\vec{z_0}$$

$$\boxed{E_1} = \frac{1}{2}\left[m \begin{bmatrix} \dot{x} - S\cos q_1 \dot{q}_1 \\ \dot{y} + S\sin q_1 \dot{q}_1 \end{bmatrix}^T \begin{bmatrix} \dot{x} - S\cos q_1 \dot{q}_1 \\ \dot{y} + S\sin q_1 \dot{q}_1 \end{bmatrix} + \dot{q}_1^2 I \right] =$$

$$= \frac{1}{2}\Big[m \big(\dot{x}^2 \underbrace{- \dot{x} S\cos q_1 \dot{q}_1 - \dot{x} S\cos q_1 \dot{q}_1}_{-2\dot{x}S\cos q_1 \dot{q}_1} + S^2 \cos^2 q_1 \dot{q}_1^2 +$$

$$+ \dot{y}^2 + 2\dot{y} S\sin q_1 \dot{q}_1 + S^2 \sin^2 q_1 \dot{q}_1^2 \big) + \dot{q}_1^2 I \Big] =$$

$$= \frac{1}{2}\left[m\left(\dot{x}^2 - 2\dot{x}S\cos q_1 \dot{q}_1 + S^2 \dot{q}_1^2 + \dot{y}^2 + 2\dot{y}S\sin q_1 \dot{q}_1\right) + \dot{q}_1^2 I\right]$$

$$\boxed{E_2} = \frac{1}{2}\left[m \begin{bmatrix} \dot{x} - s\cos(q_1+q_2)(\dot{q}_1+\dot{q}_2) \\ \dot{y} - s\,\mathrm{sen}(q_1+q_2)(\dot{q}_1+\dot{q}_2) \end{bmatrix}^T \begin{bmatrix} \dot{x} - s\cos(q_1+q_2)(\dot{q}_1+\dot{q}_2) \\ \dot{y} - s\,\mathrm{sen}(q_1+q_2)(\dot{q}_1+\dot{q}_2) \end{bmatrix} + I(\dot{q}_1+\dot{q}_2)^2 \right]$$

$$= \frac{1}{2}\Big[m\Big(\dot{x}^2 + s^2\cos^2(q_1+q_2)(\dot{q}_1+\dot{q}_2)^2 - 2\dot{x}\,s\cos(q_1+q_2)(\dot{q}_1+\dot{q}_2) + \dot{y}^2 + s^2\,\mathrm{sen}^2(q_1+q_2)(\dot{q}_1+\dot{q}_2)^2 - 2\dot{y}\,s\,\mathrm{sen}(q_1+q_2)(\dot{q}_1+\dot{q}_2) \Big) + I(\dot{q}_1+\dot{q}_2)^2 \Big] =$$

$$= \frac{1}{2}\Big[m\Big(\dot{x}^2 + s^2(\dot{q}_1+\dot{q}_2)^2 - 2\dot{x}\,s\cos(q_1+q_2)(\dot{q}_1+\dot{q}_2) + \dot{y}^2 - 2\dot{y}\,s\,\mathrm{sen}(q_1+q_2)(\dot{q}_1+\dot{q}_2) \Big) + I(\dot{q}_1+\dot{q}_2)^2 \Big]$$

$$E_{tot} = E_1 + E_2 = \frac{1}{2} m\dot{x}^2 + \frac{1}{2} m\, 2s\dot{q}_1(-\dot{x}\cos q_1 + \dot{y}\,\mathrm{sen}\, q_1) + \frac{1}{2} m\dot{y}^2 + \frac{1}{2} m s^2 \dot{q}_1^2 + \frac{1}{2}\dot{q}_1^2 I + \frac{1}{2} m\dot{x}^2 + \frac{1}{2} m s^2(\dot{q}_1+\dot{q}_2)^2 - \frac{1}{2} m\, 2s(\dot{q}_1+\dot{q}_2)(\dot{x}\cos(q_1+q_2) + \dot{y}\,\mathrm{sen}(q_1+q_2)) + \frac{1}{2} m\dot{y}^2 + \frac{1}{2}(\dot{q}_1+\dot{q}_2)^2 I$$

$$E_{tot} = m\dot{x}^2 + m s\dot{q}_1(-\dot{x}\cos q_1 + \dot{y}\,\mathrm{sen}\, q_1) + m\dot{y}^2 + \frac{1}{2} m s^2\dot{q}_1^2 + \frac{1}{2}\dot{q}_1^2 I + \frac{1}{2} m s^2(\dot{q}_1+\dot{q}_2)^2 - m s(\dot{q}_1+\dot{q}_2)(\dot{x}\cos(q_1+q_2) + \dot{y}\,\mathrm{sen}(q_1+q_2)) + \frac{1}{2}(\dot{q}_1+\dot{q}_2)^2 I$$

Knowing that

$$E_c = \frac{1}{2}\underbrace{\begin{bmatrix} \dot{x} & \dot{y} & \dot{q}_1 & \dot{q}_2 \end{bmatrix}}_{1\times4} \underbrace{A_1}_{4\times4} \underbrace{\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{q}_1 \\ \dot{q}_2 \end{bmatrix}}_{4\times1}$$

$$E_C = \begin{bmatrix} \dot{x} & \dot{y} & \dot{q}_1 & \dot{q}_2 \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{22} & A_{23} & A_{24} \\ A_{31} & A_{32} & A_{33} & A_{34} \\ A_{41} & A_{42} & A_{43} & A_{44} \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{q}_1 \\ \dot{q}_2 \end{bmatrix}$$

1×4

$$= \begin{bmatrix} \dot{x}A_{11} + \dot{y}A_{21} + \dot{q}_1 A_{31} + \dot{q}_2 A_{41} & \dot{x}A_{21} + \dot{y}A_{22} + \dot{q}_1 A_{23} + \dot{q}_2 A_{24} & \cdots \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} =$$

$$= \dot{x}^2 A_{11} + \dot{y}\dot{x}A_{21} + \dot{q}_1 \dot{x} A_{31} + \dot{q}_2 \dot{x} A_{41} + \dot{x}\dot{y}A_{21} + \dot{y}^2 A_{22} + \dot{q}_1 \dot{y} A_{23} + \cdots$$

$$\Rightarrow A_1 = \begin{bmatrix} 2m & 0 & ms\cos q_1 - ms\cos(q_1+q_2) & -ms\cos(q_1+q_2) \\ 0 & 2m & ms\cos q_1 - ms\,\mathrm{sen}(q_1+q_2) & -ms\,\mathrm{sen}(q_1+q_2) \\ ms\cos(q_1) - ms\cos(q_1+q_2) & ms\cos(q_1) - ms\,\mathrm{sen}(q_1+q_2) & 2ms^2 + 2I & ms^2 + I \\ -ms\cos(q_1+q_2) & -ms\,\mathrm{sen}(q_1+q_2) & ms^2 + I & ms^2 + I \end{bmatrix}$$

then B can easily find

$$\dot{x} = J_R \dot{q} \quad \Rightarrow \quad J_R = \begin{bmatrix} 1 & 0 & -l\cos(q_1+q_2) & -l\cos(q_1+q_2) \\ 0 & 1 & -l\,\mathrm{sen}(q_1+q_2) & -l\,\mathrm{sen}(q_1+q_2) \end{bmatrix}$$