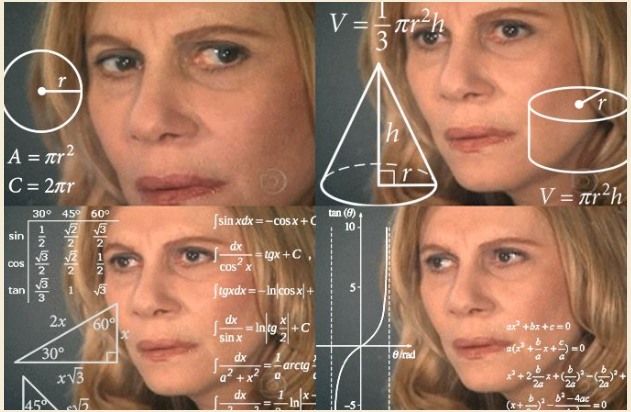
Competitive Programming

Week 10: Math and Geometry

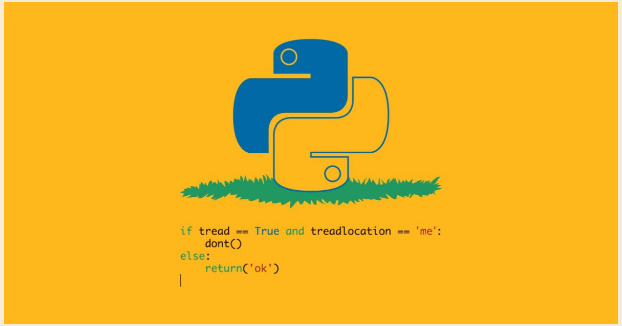
Math and Geometry

- Math is one of the best uses of a computer. Computers can do math quickly and precisely, and automate what would take a person hours of work.
- Geometry is where computers have trouble, because it usually involves floating point numbers that computers have a hard time representing accurately.



Operators

- % is the modulo operator, it returns the remainder of division of two numbers.
- += -= /= *= &= |= are equivalent to a = a + b, a = a - b, etc.
- In most programming languages, the division operator / does truncated integer division, so this will be assumed for this class unless stated otherwise.
- The ^ operator is usually an exclusive-OR operator, not an exponent.
- Python is weird.



Greatest Common Denominator (GCD)

- The GCD of two whole numbers is the largest whole number that evenly divides both. For example, gcd(8, 12) = 4.
- This is useful for adding fractions and solving spatial problems.
- Find the GCD using the Euclidean Algorithm:
  - gcd(x, 0) = x
  - gcd(a, b) = gcd(b, a % b) where a > b
  - Repeat until the base case of gcd(x, 0) is reached, and x is the gcd of a and b.
- Example:
  gcd(54, 16): 54 % 16 = 6
  gcd(16, 6): 16 % 6 = 4
  gcd(6, 4): 6 % 4 = 2
  gcd(4, 2): 4 % 2 = 0
  gcd(2, 0) = 2

Extended Euclidean Algorithm

- Find coefficients x and y such that ax + by = gcd(a, b).
- Follow a series where $x_i = x_{i-2} - (a / b) * x_{i-1}$, and $y_i = y_{i-2} - (a / b) * y_{i-1}$ and $x_0 = 1$, $x_1 = 0$, $y_0 = 0$, $y_1 = 1$ while running the Euclidean Algorithm.
- End the Euclidean Algorithm when a % b = 0, and return x and y from the previous step.
- Example:

  gcd(54, 16): 54 % 16 = 6, x = 1 - 3 * 0 = 1, y = 0 - 3 * 1 = -3

  gcd(16, 6): 16 % 6 = 4, x = 0 - 2 * 1 = -2, y = 1 - 2 * -3 = 7

  gcd(6, 4): 6 % 4 = 2, x = 1 - 1 * -2 = 3. y = -3 - 1 * 7 = -10

  gcd(4, 2): 4 % 2 = 0, x = 3, y = -10

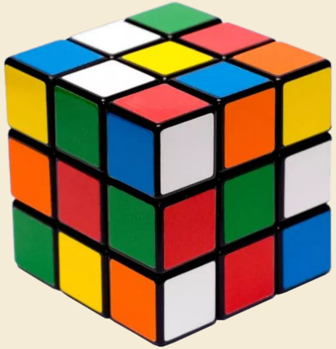
  gcd(2, 0) = 2 = 54 * 3 + 16 * -10

Least Common Multiple (LCM)

- The LCM of two numbers is the lowest positive whole number that is a multiple of both numbers.
- lcm(a, b) = (a * b) / gcd(a, b)
- LCM can be used to find when two events on different intervals occur at the same time.

Combinations and Permutations

- A combination is a grouping of items that does not consider order. C(n, k) = n! / (k!(n - k)!), where n is the number of items and k is the group size. This returns the number of possible combinations.
- A permutation is an ordering of items that considers order. P(n, k) = n! / (n - k)!, where n is the number of items and k is the group size. This returns the number of possible combinations including all orderings of each combination.
- Remember that 0! = 1.
- Rubiks cubes have permutations of where the pieces are.



Prime Numbers

- A number is prime if it is only evenly divisible by 1 and itself.
- Test for prime:
  - If n < 2, n is not prime. 1 is not considered prime.
  - If n is evenly divisible by any number from 2 to floor(sqrt(n) + 1), n is not prime.
- The Sieve of Eratosthenes can be used to quickly determine if a range of numbers are prime.
  - ```
    bool prime[n] = true;
    for (int i = 2; i < n; i++) {
        if (prime[i]) {
            for (int j = i * i; j < n; j += i) {
                Prime[j] = false;
            }
        }
    }
    ```
  - Query the resulting array for prime numbers.
    ```

Probability and Expectation

- The probability of an event can be calculated by dividing the number of outcomes in the event by the number of outcomes in the sample space.
- If you roll a die, the probability of rolling an even number is 1 / 2, because there are 3 outcomes that are even numbers out of 6 total outcomes.
- The expectation of an event is its probability multiplied by its value.
- If you win $10 by rolling a number less than or equal to 3 on the die, your expectation is $5, because the probability is 1 / 2.

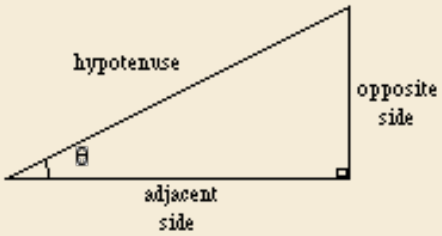
Trigonometry

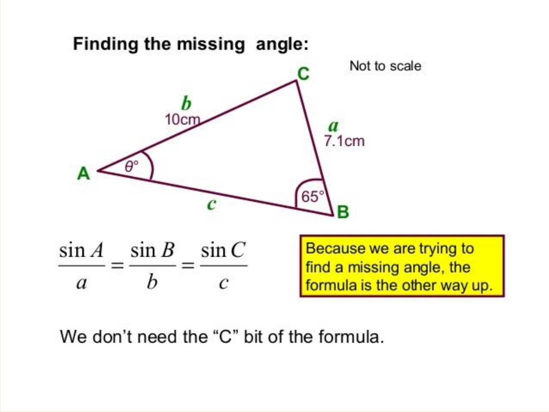
- Trigonometry is a branch of math that studies angles, usually relating to triangles.
- radians = π / (180 * degrees)

SOH $\sin\theta = \dfrac{\text{opposite}}{\text{hypotenuse}}$

CAH $\cos\theta = \dfrac{\text{adjacent}}{\text{hypotenuse}}$

TOA $\tan\theta = \dfrac{\text{opposite}}{\text{adjacent}}$



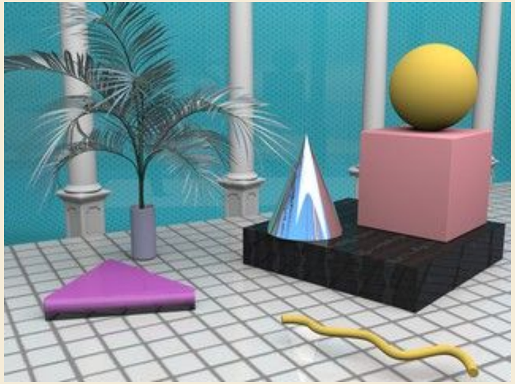


Chinese Remainder Theorem

- Determine a number given a set of congruences modulo n_i.
- All n_i must be pairwise coprime, meaning the gcd of all n_i is 1.
- Recursively solve congruences to find a single congruence that solves the set.
- Solve recurrence ax = r mod b by finding c such that ac - by = r.
- If ax = r mod b, then x = bn + c
- a = 3 mod 5, a = 2 mod 4, a = 2 mod 3
  a = 5b + 3, 5b + 3 = 2 mod 4, 5b = 3 mod 4, b = 4c + 3
  a = 20c + 18, 20c + 18 = 2 mod 3, 20c = 2 mod 3, c = 3d + 2
  a = 60d + 58
- Notice that subtracting in modulo always results in a positive value.
- 2 - 3 mod 4 = -1 mod 4 = 3 mod 4

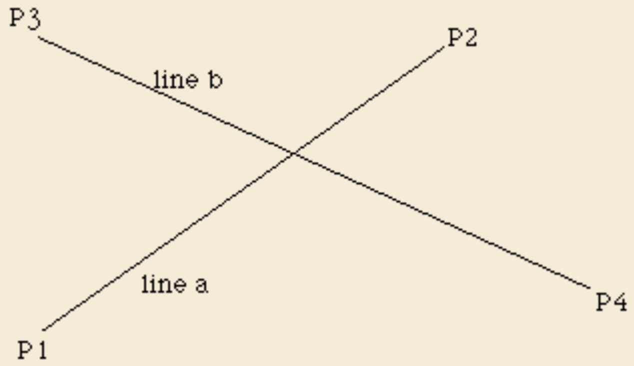
Geometry

- Avoid floating point calculations if you can.
- When comparing floating points, use a - b < ε, where ε is a fixed maximum error.
- Distance between points is sqrt((x_1 - x_2)^2 + (y_1 - y_2)^2). It is possible for this value to overflow before the computation is complete, so be careful.
- Lines can be represented as two points, a point and slope, or slope-intercept.



Intersection of Lines

- The intersection of the lines is a point where both lines share the same x and y coordinates.
- Simplify line forms into slope-intercept form: $y = m_1x + b_1$, $y = m_2x + b_2$.
- Let $m_1x + b_1 = m_2x + b_2$, so $x = (b_2 - b_1) / (m_1 - m_2)$.
- Now plug x back into one of the original equations to get y.



Example

- https://leetcode.com/problems/binary-gap/
- Input: 5, output: 2
- 5 in binary is 101