理论力学 CAI

矢量动力学基础

动量定理

- 前言
- 惯量
- 动量定理
- 动量矩定理
- 动能定理

动量定理

- 动量
- 动量定理与质心运动定理
- 变质量质心运动定理

动量定理

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- 动量定理与质心运动定理
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动量

- 质点的动量
- 质点系的动量

- 质点的动量

建惯性基 $O-\vec{e}$ 静止

质点P_k的动量 $\vec{p}_k \overset{\text{def}}{=} m_k \vec{v}_k = m_k \dfrac{\mathrm{d}}{\mathrm{d}t}\vec{r}_k$

绝对速度　绝对导数

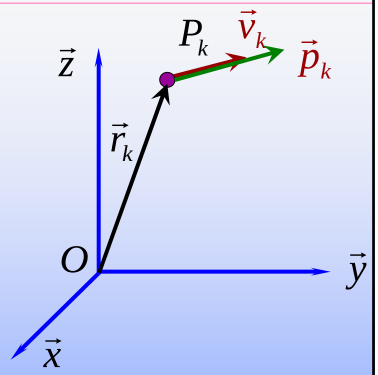
$$\vec{p}_k \overset{\text{def}}{=} m_k \vec{v}_k = m_k \dot{\vec{r}}_k$$

$$\vec{e}: \quad \boldsymbol{p}_k = m_k \boldsymbol{v}_k = m_k \dot{\boldsymbol{r}}_k$$

$p_{kx} = m_k v_{kx} = m_k \dot{x}_k$ x向动量

$p_{ky} = m_k v_{ky} = m_k \dot{y}_k$ y向动量

$p_{kz} = m_k v_{kz} = m_k \dot{z}_k$ z向动量



$$\boldsymbol{p}_k = \begin{pmatrix} p_{kx} & p_{ky} & p_{kz} \end{pmatrix}^{\mathrm{T}}$$

$$\boldsymbol{v}_k = \begin{pmatrix} v_{kx} & v_{ky} & v_{kz} \end{pmatrix}^{\mathrm{T}}$$

$$\boldsymbol{r}_k = \begin{pmatrix} x_k & y_k & z_k \end{pmatrix}^{\mathrm{T}}$$

- 质点系的动量

质点系 $(P_1, P_2, \cdots, P_n)$

质点P_k的动量

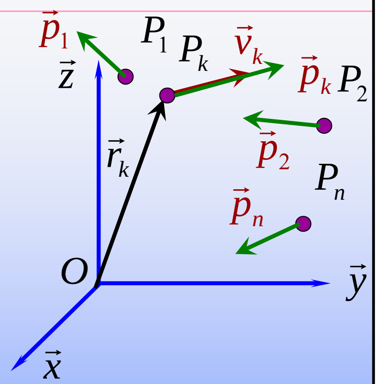
$$\vec{p}_k = m_k \vec{v}_k = m_k \dot{\vec{r}}_k \qquad (k = 1, 2, \cdots, n)$$

质点系的动量

$$\vec{p} \overset{\text{def}}{=} \sum_{k=1}^{n} \vec{p}_k$$

质点系的动量是各质点动量的矢量和

$$\vec{p} = \sum_{k=1}^{n} m_k \vec{v}_k = \sum_{k=1}^{n} m_k \dot{\vec{r}}_k$$



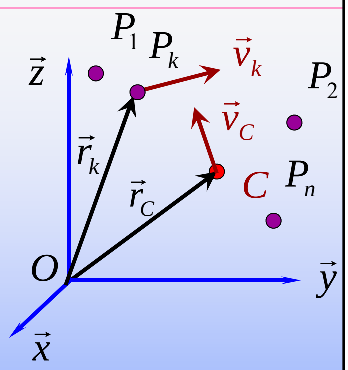
质点系 $(P_1, P_2, \cdots, P_n)$

质点系的动量 $\vec{p} = \sum_{k=1}^{n} m_k \vec{v}_k = \sum_{k=1}^{n} m_k \dot{\vec{r}}_k$

令质点系的质心 C $\vec{r}_C$

$$\sum_{k=1}^{n} m_k \vec{r}_k = m\vec{r}_C \qquad \sum_{k=1}^{n} m_k \dot{\vec{r}}_k = m\dot{\vec{r}}_C = m\vec{v}_C$$

$$\vec{p} = m\vec{v}_C$$



质点系的动量为系统的质量与系统质心速度乘积

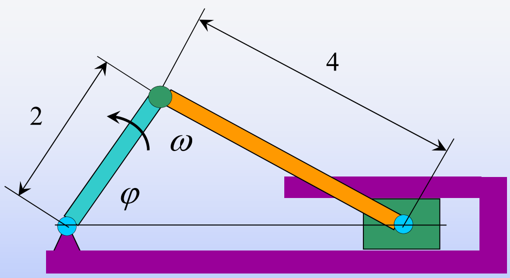
$$\underline{e}: \quad \underline{p} = m\underline{v_C}$$

$$p_x = mv_{Cx} = m\dot{x}_C$$
$$p_y = mv_{Cy} = m\dot{y}_C$$
$$p_z = mv_{Cz} = m\dot{z}_C$$

$$\underline{p} = \begin{pmatrix} p_x & p_y & p_z \end{pmatrix}^{\mathrm{T}}$$
$$\underline{v_C} = \begin{pmatrix} v_{Cx} & v_{Cy} & v_{Cz} \end{pmatrix}^{\mathrm{T}}$$
$$\underline{r_C} = \begin{pmatrix} x_C & y_C & z_C \end{pmatrix}^{\mathrm{T}}$$

[例]

写出此瞬时系统的动量



[解] 每一个刚体的质心 $\boldsymbol{C_i}$

计算每一个刚体质心 速度

$\vec{v}_1 \quad \vec{v}_2 \quad \vec{v}_3$

$\omega_1 \to v_1 \to v_A \to s \to \omega_2 \to v_2 \to v_3$

写出每一个刚体的动量

$$\vec{p}_i = m_i \vec{v}_i$$

$$\vec{p} = \sum \vec{p}_i$$

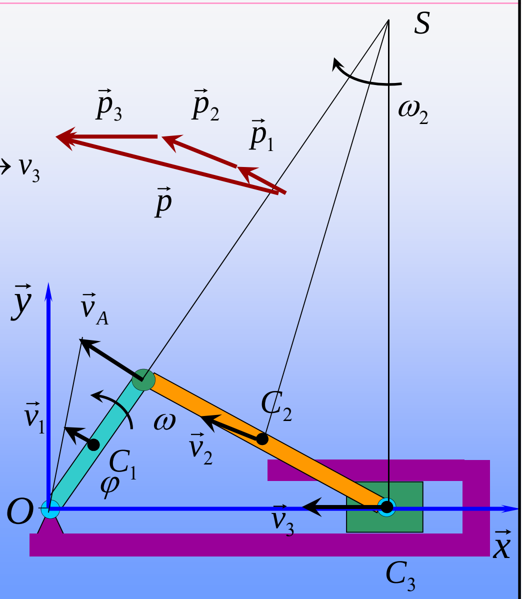
解析表达 $\quad O - \vec{\boldsymbol{e}}$

建惯性基

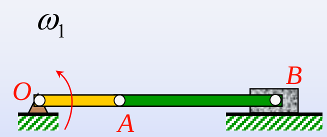
$$\boldsymbol{p}_i = m_i \boldsymbol{v}_i \qquad \boldsymbol{p} = \sum \boldsymbol{p}_i$$

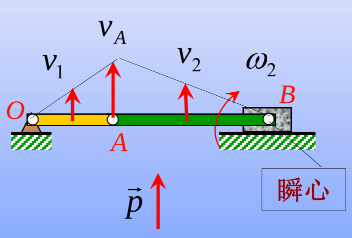
解题的关键：

正确分析刚体质心的速度



求系统动量 $\quad OA = r, AB = 2r$





$$v_1 = \frac{\omega_1 r}{2} \qquad p_1 = \frac{m_1 \omega_1 r}{2}$$

$$v_A = \omega_1 r$$

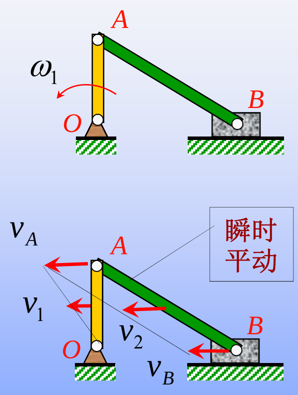
$$\omega_2 = \frac{\omega_1}{2}$$

$$v_2 = \frac{\omega_1 r}{2} \qquad p_2 = \frac{m_2 \omega_1 r}{2}$$

$$v_B = 0 \qquad p_3 = 0$$

$$p = p_1 + p_2 + p_3 = \frac{(m_1 + m_2)}{2} \omega_1 r$$

求系统动量



$\leftarrow \quad \vec{p}$

$$v_1 = \frac{\omega_1 r}{2} \qquad p_1 = \frac{m_1\omega_1 r}{2}$$

$$v_A = \omega_1 r$$

$$\omega_2 = 0$$

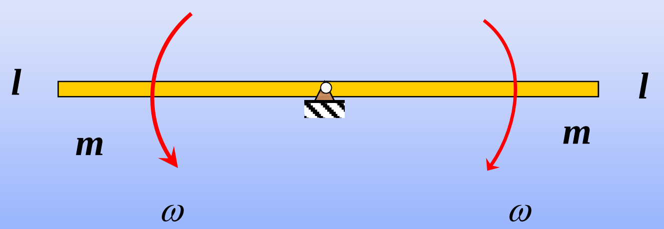
$$v_2 = v_B = v_A = \omega_1 r$$

$$p_2 = m_2\omega_1 r \qquad p_3 = m_3\omega_1 r$$

$$p = \left(\frac{m_1}{2} + m_2 + m_3\right)\omega_1 r$$

求刚体系动量

长度质量相等的两根杆子

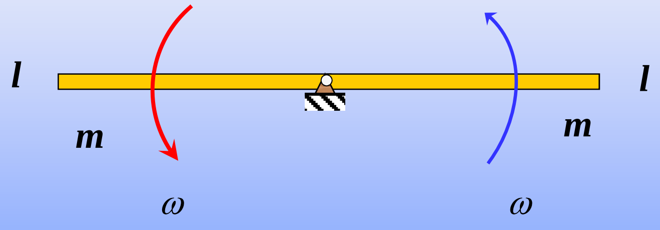


刚体系质心速度=0?

$$p = m\omega l\left(\downarrow\right)$$

求刚体系动量

长度质量相等的两根杆子

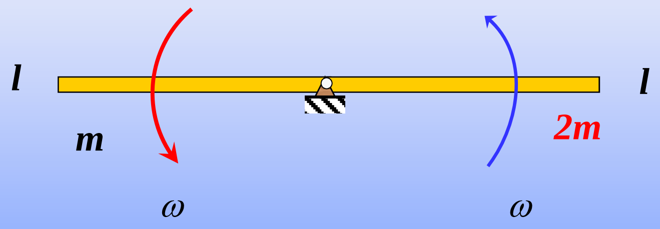


刚体系质心速度=0

$$p = 0$$

求刚体系动量

长度质量相等的两根杆子

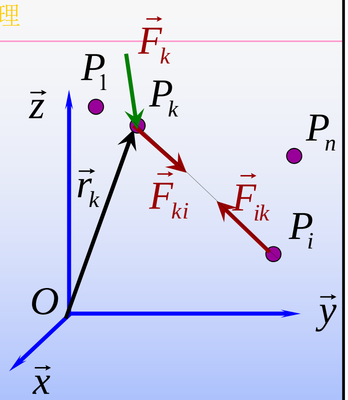


$$p = ?$$

动量定理与质心运动定理

- 质点系动量定理
- 动量定理的积分形式
- 质心运动定理
- 动量守恒定律

质点系动量定理



质点系 $(P_1, P_2, \cdots, P_n)$

质点P_k $(k=1,2,\cdots,n)$ 所受的力

外力 $\vec{F}_k$

内力 质点P_i对P_k的作用力

$\vec{F}_{ki}$ $(i=1,2,\cdots,n; \neq k)$

牛顿定律

$$m_k \ddot{\vec{r}}_k = \vec{F}_k + \sum_{i=1,\neq k}^{n} \vec{F}_{ki} \quad (k=1,2,\cdots,n)$$

n个方程相加

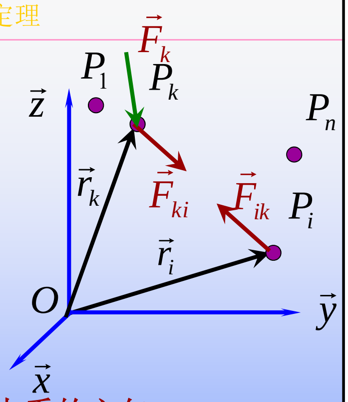
$$\sum_{k=1}^{n} m_k \ddot{\vec{r}}_k = \sum_{k=1}^{n} \vec{F}_k + \cancel{\sum_{k=1}^{n} \sum_{i=1,\neq k}^{n} \vec{F}_{ki}}$$

$$\vec{F}_{ki} = -\vec{F}_{ik}$$

$(i,k=1,2,\cdots,n)$

$$\sum_{k=1}^{n} m_k \ddot{\vec{r}}_k = \sum_{k=1}^{n} \vec{F}_k$$

$$\sum_{k=1}^{n} m_k \ddot{\vec{r}}_k = \sum_{k=1}^{n} \vec{F}_k$$



$\vec{p} = \sum_{k=1}^{n} m_k \dot{\vec{r}}_k$ $\quad \dot{\vec{p}} = \sum_{k=1}^{n} m_k \ddot{\vec{r}}_k$ 质点质量不变

$\sum_{k=1}^{n} \vec{F}_k = \vec{F}_{\mathrm{R}}$ 作用于质点系外力系的主矢

$$\dot{\vec{p}} = \vec{F}_{\mathrm{R}}$$

质点系的动量对时间的绝对导数等于外力系的主矢

$\vec{e}:$ $\quad \dot{\boldsymbol{p}} = \boldsymbol{F}_{\mathrm{R}}$ $\qquad \boldsymbol{p} = (p_x \quad p_y \quad p_z)^{\mathrm{T}}$

$\dot{p}_x = F_{\mathrm{R}x} \quad \dot{p}_y = F_{\mathrm{R}y} \quad \dot{p}_z = F_{\mathrm{R}z}$ $\qquad \boldsymbol{F}_{\mathrm{R}} = (F_{\mathrm{R}x} \quad F_{\mathrm{R}y} \quad F_{\mathrm{R}z})^{\mathrm{T}}$

动量定理的积分形式

质点系动量定理 $\quad \dot{\boldsymbol{p}} = \boldsymbol{F}_{\mathrm{R}} \quad \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{p} = \boldsymbol{F}_{\mathrm{R}} \quad \mathrm{d}\boldsymbol{p} = \boldsymbol{F}_{\mathrm{R}}\,\mathrm{d}t$

积分形式

$$\boldsymbol{p} - \boldsymbol{p}_0 = \int_{t_0}^{t} \boldsymbol{F}_{\mathrm{R}}\,\mathrm{d}t$$

时刻 t 质点系的动量

时刻 t_0 质点系的动量 $\qquad t_0$ 到 t 间隔内外力系主矢的冲量

质点系动量在时间间隔内的变化等于外力系的主矢在同一时间间隔内的冲量

$$p_x - p_{x0} = \int_{t_0}^{t} F_{\mathrm{R}x}\,\mathrm{d}t$$

$$p_y - p_{y0} = \int_{t_0}^{t} F_{\mathrm{R}y}\,\mathrm{d}t$$

$$p_z - p_{z0} = \int_{t_0}^{t} F_{\mathrm{R}z}\,\mathrm{d}t,$$

$\boldsymbol{p} = (p_x \quad p_y \quad p_z)^{\mathrm{T}}$

$\boldsymbol{F}_{\mathrm{R}} = (F_{\mathrm{R}x} \quad F_{\mathrm{R}y} \quad F_{\mathrm{R}z})^{\mathrm{T}}$

质心运动定理

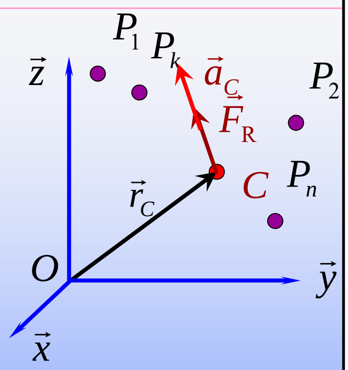
- 定理描述

质点系动量定理 $\dot{\vec{p}} = \vec{F}_{\mathrm{R}}$

$$\vec{p} = m\vec{v}_C = m\dot{\vec{r}}_C$$

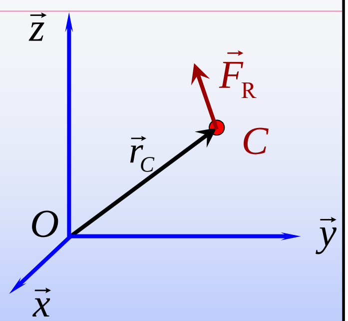
$$m\vec{a}_C = \vec{F}_{\mathrm{R}} \qquad m\ddot{\vec{r}}_C = \vec{F}_{\mathrm{R}}$$

质点系质量与其质心加速度矢量的乘积等于外力系的主矢



- 某瞬时
  - 质心速度与外力系主矢的方向一致
  - 大小成正比，系统质量为比例系数

$$m\vec{a}_C = \vec{F}_{\mathrm{R}} \qquad m\ddot{\vec{r}}_C = \vec{F}_{\mathrm{R}}$$



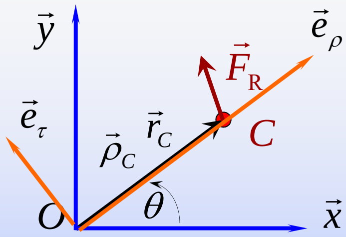
- 直角坐标

$\vec{\boldsymbol{e}}$: $m\boldsymbol{a}_C = \boldsymbol{F}_{\mathrm{R}}$ $m\ddot{\boldsymbol{r}}_C = \boldsymbol{F}_{\mathrm{R}}$

牛顿方程

$$m a_x = F_{\mathrm{R}x} \qquad m\ddot{x}_C = F_{\mathrm{R}x}$$
$$m a_y = F_{\mathrm{R}y} \qquad m\ddot{y}_C = F_{\mathrm{R}y}$$
$$m a_z = F_{\mathrm{R}z} \qquad m\ddot{z}_C = F_{\mathrm{R}z}$$

$m\vec{a}_C = \vec{F}_{\rm R}$　　$m\ddot{\vec{r}}_C = \vec{F}_{\rm R}$



- 极坐标(平面)

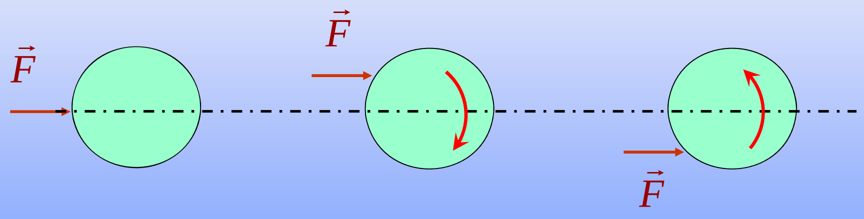
$\vec{e}^{\rm P}$:

$$ma_\rho = F_{{\rm R}\,\rho}$$
$$ma_\tau = F_{{\rm R}\,\tau}$$

$$\boldsymbol{a}_C = \begin{pmatrix} a_{C\rho} & a_{C\tau} \end{pmatrix}^{\rm T}$$
$$\boldsymbol{F}_{\rm R} = \begin{pmatrix} F_{{\rm R}\,\rho} & F_{{\rm R}\,\tau} \end{pmatrix}^{\rm T}$$

$$m\left(\ddot{\rho}_C - \dot{\theta}^2\rho_C\right) = F_{{\rm R}\,\rho}$$
$$m\left(\ddot{\theta}\rho_C + 2\dot{\theta}\dot{\rho}_C\right) = F_{{\rm R}\,\tau}$$

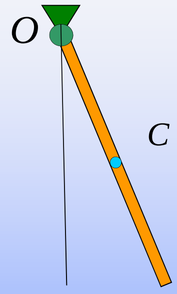
1.三个圆盘的运动是否一样?



2.三个圆盘质心的运动是否一样?

[例]

单摆质量为**m**,长为**2*l*.**



1. 写出单摆质心运动方程

2. 如果单摆作匀速圆周运动，角速度为ω,
求圆柱铰O的约束反力

[解1] 直角坐标

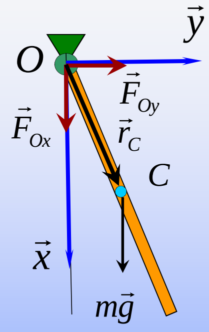
建惯性基 $O-\vec{e}$

受力分析：

主动力： $m\vec{g}$ 已知

系统外力

铰**O**的理想约束力： $\vec{F}_{Ox}$ $\vec{F}_{Oy}$ 设定正向



牛顿方程 $m\ddot{\boldsymbol{r}}_C=\boldsymbol{F}_{\mathrm{R}}$

$$m\ddot{x}_C=F_{\mathrm{R}x}$$

$$m\ddot{x}_C=F_{Ox}+mg$$

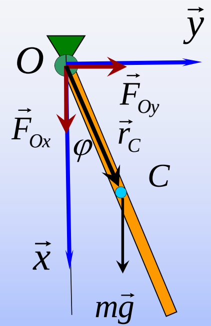
$$m\ddot{y}_C=F_{\mathrm{R}y}$$

$$m\ddot{y}_C=F_{Oy}$$

$$m\ddot{x}_C = F_{Ox} + mg$$

$$m\ddot{y}_C = F_{Oy}$$

单摆质心运动方程



如果单摆作匀速圆周运动，角速度为ω
求约束反力

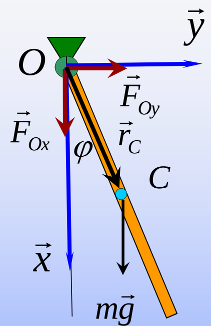
已知 φ —方程？→ 求 F_{Ox} F_{Oy}

↓ 运动学问题

$\ddot{x}_C$ $\ddot{y}_C$ —方程→ 求 F_{Ox} F_{Oy}

$$m\ddot{x}_C = F_{Ox} + mg \qquad m\ddot{y}_C = F_{Oy}$$

运动学问题：单摆作匀速圆周运动，求质心加速度



约束方程 $x_C = l\cos\varphi$ $y_C = l\sin\varphi$

加速度约束方程

$$\ddot{x}_C = -l\dot{\varphi}^2\cos\varphi - l\ddot{\varphi}\sin\varphi = -l\omega^2\cos\varphi$$

$$\ddot{y}_C = -l\dot{\varphi}^2\sin\varphi + l\ddot{\varphi}\cos\varphi = -l\omega^2\sin\varphi$$

动力学问题：由方程求理想约束力

$$F_{Ox} = -mg + m\ddot{x}_C = -mg - \underline{ml\omega^2\cos\varphi}$$

$$F_{Oy} = m\ddot{y}_C = \underline{-ml\omega^2\sin\varphi}$$

$$\dot{\varphi} = \omega$$

$$\ddot{\varphi} = 0$$

动理想约束力 交变

$$\varphi = \varphi_0 + \omega t$$

[解2] 惯性基 $O-\vec{e}$ 极坐标 $\boldsymbol{e}^{\mathrm{P}}$

受力分析：

主动力： $m\vec{g}$ 已知

理想约束力： $\vec{F}_{O\rho}$ $\vec{F}_{O\tau}$ 设定正向

系统外力

牛顿方程 $m\ddot{\boldsymbol{r}}_C=\boldsymbol{F}_{\mathrm{R}}$

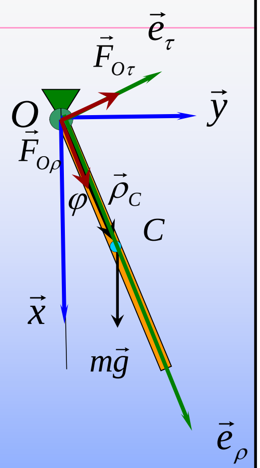
$$m\left(\ddot{\rho}_C-\dot{\varphi}^2\rho_C\right)=mg\cos\varphi+F_{O\rho}$$

$$m\left(\ddot{\varphi}\rho_C+2\dot{\varphi}\dot{\rho}_C\right)=-mg\sin\varphi+F_{O\tau}$$

对于摆 $\rho_C=l,\quad \dot{\rho}_C=\ddot{\rho}_C=0$

$$-m\dot{\varphi}^2 l=mg\cos\varphi+F_{O\rho}$$

$$m\ddot{\varphi}l=-mg\sin\varphi+F_{O\tau}$$



$$-m\dot{\varphi}^2 l=mg\cos\varphi+F_{O\rho}$$

$$m\ddot{\varphi}l=-mg\sin\varphi+F_{O\tau}$$

单摆质心运动方程

解约束反力

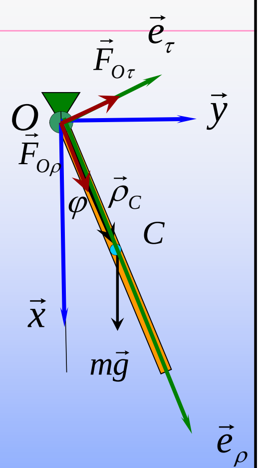
$$\dot{\varphi}=\omega,\quad \ddot{\varphi}=0$$

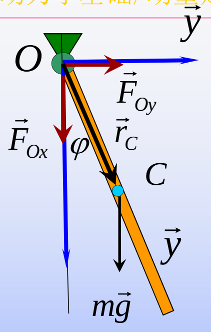
理想约束力：

$$F_{O\rho}=-mg\cos\varphi-m\omega^2 l$$

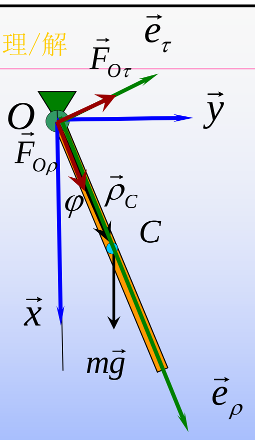
$$F_{O\tau}=mg\sin\varphi$$

动理想约束力 不变? $\varphi=\varphi_0+\omega t$





$\varphi = \varphi_0 + \omega t$



理想约束力：

$F_{Ox} = -mg \boxed{-ml\omega^2 \cos\varphi}$ $\qquad$ $F_{O\rho} = -mg\cos\varphi - \boxed{m\omega^2 l}$

$F_{Oy} = \boxed{-ml\omega^2 \sin\varphi}$ $\qquad$ $F_{O\tau} = mg\sin\varphi$

交变 动理想约束力 不变

相对惯性基 相对连体基

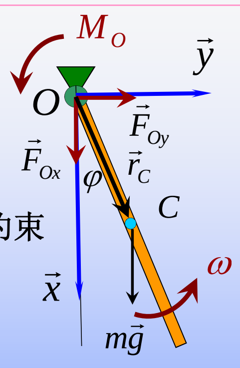
讨论 如果单摆作匀速圆周运动，角速度为ω

较O理想约束反力 F_{Ox} F_{Oy}

有其他理想约束否？

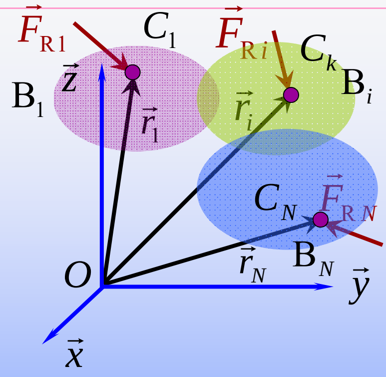
强制单摆作匀速圆周运动也是一种理想约束

这种理想约束力为一约束力偶



重力矩

- 刚体系的质心运动



$$\sum_{k=1}^{n} m_k \ddot{\vec{r}}_k = \sum_{k=1}^{n} \vec{F}_k$$

刚体系 $(B_1, B_2, \cdots, B_N)$

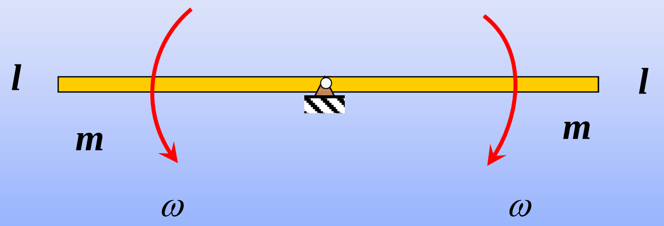
$$\sum_{i=1}^{N} m_i \ddot{\vec{r}}_i = \sum_{i=1}^{N} \vec{F}_{Ri}$$

$$\sum_{i=1}^{N} m_i \vec{a}_i = \sum_{i=1}^{N} \vec{F}_{Ri} = m\vec{a}_C = m\ddot{\vec{r}}_C$$

	m_i	$\vec{r}_i$	$\vec{a}_i$	$\vec{F}_{Ri}$
刚体$\mathbf{B}_i$	质量	质心矢径	质心加速度	外力主矢
	m	$\vec{r}_C$	$\vec{a}_C$	
刚体系	总质量	总质心矢径	总质心加速度	

刚体系质心加速度

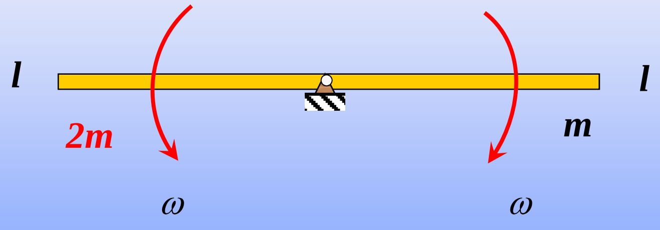
两根杆子



刚体系质心加速度？ 等于0

求刚体系质心加速度

两根杆子



刚体系质心加速度？

$$a_C = \frac{\omega^2 l}{6}(\rightarrow)$$

矢量分正负

动量守恒定律

$$\boldsymbol{p} - \boldsymbol{p}_0 = \int_{t_0}^{t} \boldsymbol{F}_\mathrm{R}\, \mathrm{d}t$$

$$\boldsymbol{F}_\mathrm{R} = \boldsymbol{0}$$

$$\boldsymbol{p} - \boldsymbol{p}_0 = \boldsymbol{0} \qquad m\boldsymbol{v}_C - m\boldsymbol{v}_{C0} = \boldsymbol{0}$$

当作用于质点系外力的主矢为零时质点系的动量保持不变

$$p_x - p_{x0} = \int_{t_0}^{t} F_{\mathrm{R}x}\, \mathrm{d}t \qquad F_{\mathrm{R}x} = 0 \quad p_x - p_{x0} = 0$$

$$m\dot{x}_{Cx} - m\dot{x}_{C0} = mv_{Cx} - mv_{Cx0} = 0$$

$$p_y - p_{y0} = \int_{t_0}^{t} F_{\mathrm{R}y}\, \mathrm{d}t \qquad F_{\mathrm{R}y} = 0 \quad p_y - p_{y0} = 0$$

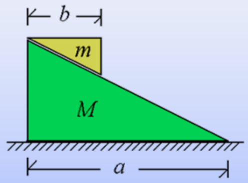
$$m\dot{x}_{Cy} - m\dot{x}_{Cy0} = mv_{Cy} - mv_{Cy0} = 0$$

$$p_z - p_{z0} = \int_{t_0}^{t} F_{\mathrm{R}z}\, \mathrm{d}t \qquad F_{\mathrm{R}z} = 0 \quad p_z - p_{z0} = 0$$

$$m\dot{x}_{Cz} - m\dot{x}_{Cz0} = mv_{Cz} - mv_{Cz0} = 0$$

【例】质量为M的大三角形柱体，放于光滑水平面上，斜面上另放一质量为m的小三角形柱体,求小三角形柱体滑到底时,大三角形柱体的位移。

解：选两物体组成的系统为研究对象。



受力分析：

$F_x^e = 0$ 水平方向动量守恒

初始静止，质心速度为零

质心的x坐标不变

$$X = \frac{m}{M+m}(a-b)$$