# 理论力学 CAI

## 刚体平面运动学

- 前言
- 刚体的连体基 刚体位形的描述
- 刚体的平面运动
- 刚体的姿态及其变化
- 基点的位置、速度与加速度
- 刚体上给定点的位置、速度与加速度
- 相对刚体运动的任意点的位置、速度与加速度

## 基点的位置 速度与加速度






## 基点的位置、速度与加速度

- 直角坐标
- 极坐标

# 直角坐标

- 基点的位置

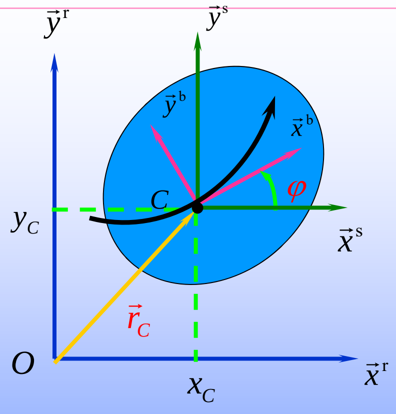
$\vec{r}_C \xrightarrow{\vec{e}^{\,\mathrm{r}}} \boldsymbol{r}_C = (x_C \quad y_C)^{\mathrm{T}}$

- 基点的绝对轨迹

$$\boldsymbol{r}_C(t) = (x_C(t) \quad y_C(t))^{\mathrm{T}}$$

$$\begin{cases} x = x_C(t) \\ y = y_C(t) \end{cases}$$

$\vec{r}_C$ 的矢端轨迹方程




质点运动学问题





- 基点的速度

$$\vec{v}_C \overset{\text{def}}{=} \frac{{}^{\mathrm{r}}\mathrm{d}}{\mathrm{d}t}\vec{r}_C = \dot{\vec{r}}_C$$

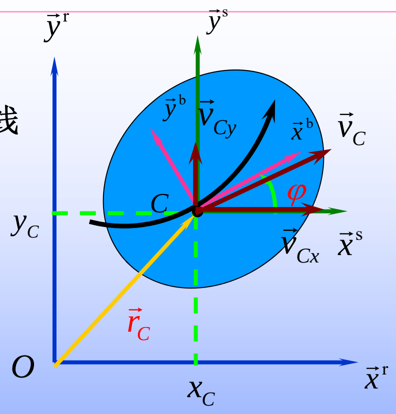
绝对轨迹的切线

$\vec{e}^{\,\mathrm{r}} \Downarrow$

$$\vec{v}_C = \boldsymbol{v}_C^{\mathrm{T}} \vec{\boldsymbol{e}}$$

$$\boldsymbol{v}_C = \dot{\boldsymbol{r}}_C \qquad \boldsymbol{v}_C = (v_{Cx} \quad v_{Cy})^{\mathrm{T}}$$

$$\boldsymbol{r}_C = (x_C \quad y_C)^{\mathrm{T}}$$




$$v_{Cx} = \dot{x}_C$$

$$v_{Cy} = \dot{y}_C$$

- 基点的加速度

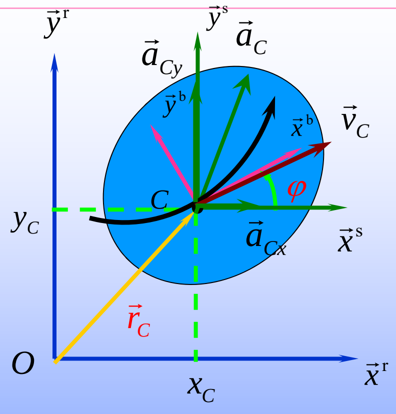
$$\vec{a}_C \overset{\text{def}}{=} \frac{{}^{\mathrm{r}}\mathrm{d}}{\mathrm{d}t}\vec{v}_C = \dot{\vec{v}}_C = \ddot{\vec{r}}_C$$

$\vec{\boldsymbol{e}}^{\mathrm{r}}$ ⬇ $\qquad \vec{a}_C = \boldsymbol{a}_C^{\mathrm{T}}\vec{\boldsymbol{e}}$

$$\boldsymbol{a}_C = \dot{\boldsymbol{v}}_C = \ddot{\boldsymbol{r}}_C \qquad \boldsymbol{a}_C = \begin{pmatrix} a_{Cx} & a_{Cy} \end{pmatrix}^{\mathrm{T}}$$

$$\boldsymbol{v}_C = \begin{pmatrix} v_{Cx} & v_{Cy} \end{pmatrix}^{\mathrm{T}}$$

$$\boldsymbol{r}_C = \begin{pmatrix} x_C & y_C \end{pmatrix}^{\mathrm{T}}$$




$$a_{Cx} = \dot{v}_{Cx} = \ddot{x}_C$$

$$a_{Cy} = \dot{v}_{Cy} = \ddot{y}_C$$





# 基点的位置、速度与加速度

- 直角坐标
- 极坐标

# 极坐标

- 基点的位置（直角坐标）

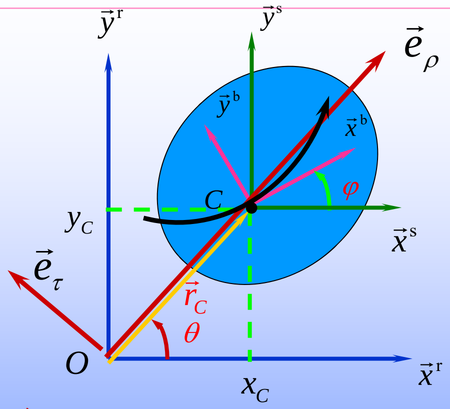
$\vec{r}_C \xrightarrow{\vec{\boldsymbol{e}}^{\mathrm{r}}} \boldsymbol{r}_C = (x_C \quad y_C)^{\mathrm{T}}$

- 动参考基

$\vec{\boldsymbol{e}}^{\mathrm{p}} = (\vec{e}_\rho \quad \vec{e}_\tau)^{\mathrm{T}} \qquad \vec{e}_\rho \,//\, \vec{r}_C$

姿态角$\boldsymbol{\theta}$　　　不是刚体的姿态角$\boldsymbol{\varphi}$

$$\boldsymbol{A}^{\mathrm{rP}} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$







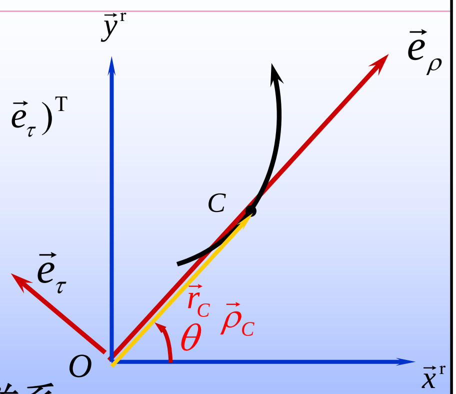

- 基点相对动基 $\vec{\boldsymbol{e}}^{\mathrm{p}}$的位置

$\vec{\rho}_C = \rho_C \vec{e}_\rho = \boldsymbol{\rho}_C^{\mathrm{PT}} \vec{\boldsymbol{e}}^{\mathrm{P}} \qquad \vec{\boldsymbol{e}}^{\mathrm{p}} = (\vec{e}_\rho \quad \vec{e}_\tau)^{\mathrm{T}}$

$\vec{\rho}_C \xrightarrow{\vec{\boldsymbol{e}}^{\mathrm{p}}} \underline{\boldsymbol{\rho}_C^{\mathrm{P}} = (\rho_C \quad 0)^{\mathrm{T}}}$

- 基点的极坐标

$\boldsymbol{q}_C = (\rho_C \quad \theta)^{\mathrm{T}}$




- 基点的直角坐标与极坐标关系

$\vec{r}_C = \vec{\rho}_C$

$\boldsymbol{r}_C = \boldsymbol{\rho}_C = \boldsymbol{A}^{\mathrm{rP}} \boldsymbol{\rho}_C^{\mathrm{P}}$

$$\boldsymbol{A}^{\mathrm{rP}} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$$\begin{pmatrix} x_C \\ y_C \end{pmatrix} = \rho_C \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$$

- 基点速度 $\vec{e}^{\mathrm{P}} = (\vec{e}_\rho \quad \vec{e}_\tau)^{\mathrm{T}}$

$$\vec{v}_C = \frac{{}^{\mathrm{r}}\mathrm{d}}{\mathrm{d}t}\vec{r}_C = \frac{{}^{\mathrm{r}}\mathrm{d}}{\mathrm{d}t}\vec{\rho}_C = \frac{{}^{\mathrm{P}}\mathrm{d}}{\mathrm{d}t}\vec{\rho}_C + \vec{\omega}^{\mathrm{P}} \times \vec{\rho}_C$$

$$\frac{{}^{\mathrm{P}}\mathrm{d}}{\mathrm{d}t}\vec{\rho}_C = \dot{\rho}_C \vec{e}_\rho$$

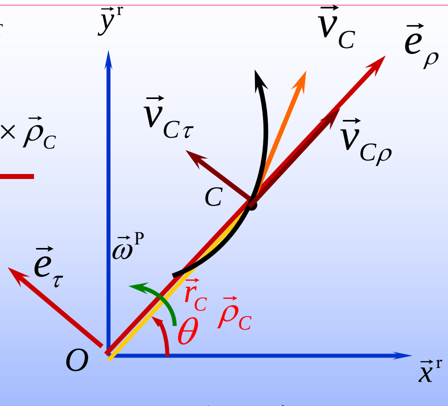
$$\vec{\omega}^{\mathrm{P}} \times \vec{\rho}_C = \dot{\theta}\vec{z} \times \vec{\rho}_C = \dot{\theta}\rho_C \vec{e}_\tau$$

$$\vec{v}_C = \boldsymbol{v}_C^{\mathrm{PT}} \vec{\boldsymbol{e}}^{\mathrm{P}}$$

$$\boldsymbol{v}_C^{\mathrm{P}} = \begin{pmatrix} v_{C\rho} & v_{C\tau} \end{pmatrix}^{\mathrm{T}}$$

$$v_{C\rho} = \dot{\rho}_C \qquad v_{C\tau} = \dot{\theta}\rho_C$$

基点*C*的径向速度 基点*C*的切向速度




$$\vec{r}_C = \vec{\rho}_C$$

$$\vec{\omega}^{\mathrm{P}} = \dot{\theta}\vec{z}$$

$\vec{\boldsymbol{e}}^{\mathrm{P}}$ 的角速度

不是刚体的角速度





- 基点速度 $\vec{e}^{\mathrm{P}} = (\vec{e}_\rho \quad \vec{e}_\tau)^{\mathrm{T}}$

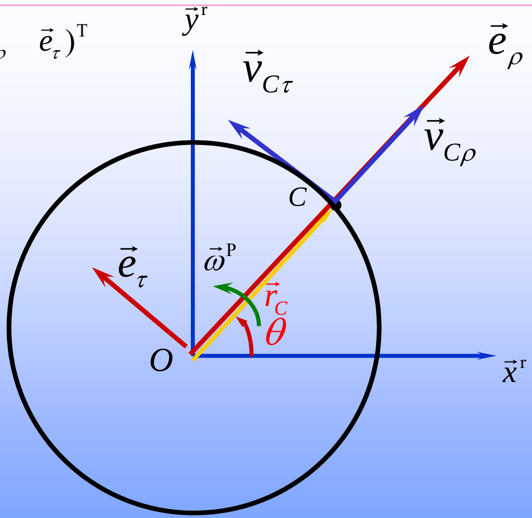
$\rho_C$为常数

基点*C*的径向速度

$$v_{C\rho} = \dot{\rho}_C = 0$$

基点*C*的切向速度

$$v_{C\tau} = \dot{\theta}\rho_C$$



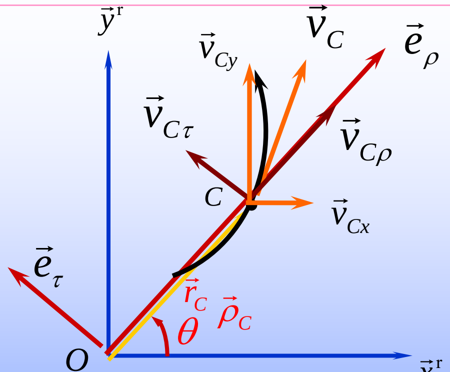

圆周运动

## 基点速度

$\vec v_C \xrightarrow{\vec{\boldsymbol e}^{\mathrm P}} \boldsymbol v_C^{\mathrm P} = (v_{C\rho}\quad v_{C\tau})^{\mathrm T}$

$v_{C\rho}=\dot\rho_C$ 基点*C*的径向速度

$v_{C\tau}=\dot\theta\rho_C$ 基点*C*的切向速度




## 基点速度两坐标阵的关系

$\vec v_C \xrightarrow{\vec{\boldsymbol e}^{\mathrm r}} \boldsymbol v_C = (v_{Cx}\quad v_{Cy})^{\mathrm T}$

$$\boldsymbol A^{\mathrm{rP}} = \begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}$$

$$\boldsymbol v_C = \boldsymbol A^{\mathrm{rP}}\boldsymbol v_C^{\mathrm P}$$

$$\begin{pmatrix}v_{Cx}\\ v_{Cy}\end{pmatrix} = \dot\rho_C\begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix} + \rho_C\dot\theta\begin{pmatrix}-\sin\theta\\ \cos\theta\end{pmatrix}$$

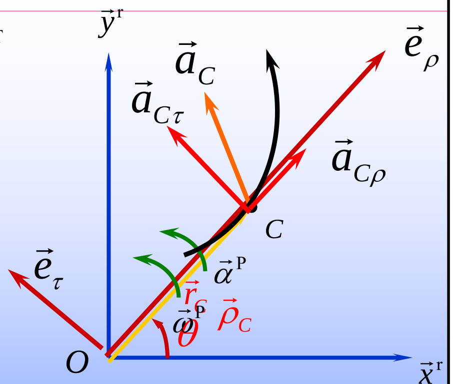




- ## 基点加速度 $\vec{\boldsymbol e}^{\mathrm P} = (\vec e_\rho\quad \vec e_\tau)^{\mathrm T}$

$$\vec a_C = \frac{^{\mathrm P}\mathrm d^2}{\mathrm dt^2}\vec\rho_C + \vec\alpha^{\mathrm P}\times\vec\rho_C + 2\vec\omega^{\mathrm P}\times\frac{^{\mathrm P}\mathrm d}{\mathrm dt}\vec\rho_C + \vec\omega^{\mathrm P}\times(\vec\omega^{\mathrm P}\times\vec\rho_C)$$

$$\vec a_C = \boldsymbol a_C^{\mathrm{P\,T}}\vec{\boldsymbol e}^{\mathrm P}$$




$$\frac{^{\mathrm P}\mathrm d^2}{\mathrm dt^2}\vec\rho_C = \underline{\ddot\rho_C\vec e_\rho}$$

$$\vec\alpha^{\mathrm P}\times\vec\rho_C = \ddot\theta\vec z\times\rho_C = \underline{\ddot\theta\rho_C\vec e_\tau},$$

$$2\vec\omega^{\mathrm P}\times\frac{^{\mathrm P}\mathrm d}{\mathrm dt}\vec\rho_C = 2\dot\theta\vec z\times\dot\rho_C\vec e_\rho = \underline{2\dot\theta\dot\rho_C\vec e_\tau}$$

$$\vec\omega^{\mathrm P}\times(\vec\omega^{\mathrm P}\times\vec\rho_C) = \dot\theta\vec z\times(\dot\theta\vec z\times\rho_C\vec e_\rho) = \underline{-\dot\theta^2\rho_C\vec e_\rho}$$

$$\boldsymbol a_C^{\mathrm P} = (a_{C\rho}\quad a_{C\tau})^{\mathrm T}$$

$$a_{C\rho} = \ddot\rho_C - \dot\theta^2\rho_C$$

$$a_{C\tau} = \ddot\theta\rho_C + 2\dot\theta\dot\rho_C$$

## 基点加速度

$\rho_C$为常数

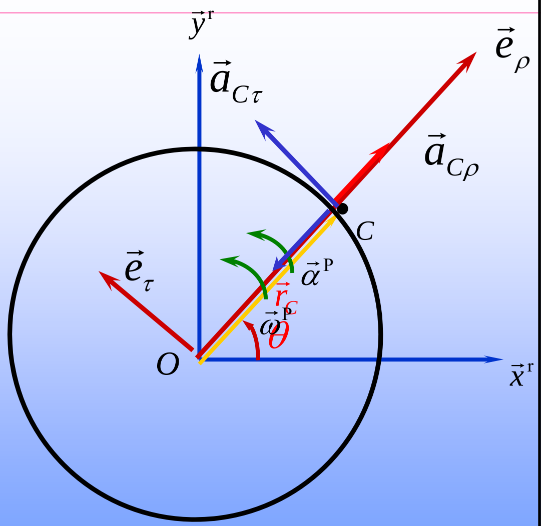
基点$C$的径向加速度

$$a_{C\rho} = \ddot{\rho}_C - \dot{\theta}^2 \rho_C$$

$$= -\dot{\theta}^2 \rho_C \quad = -v_C^2 / \rho_C$$

基点$C$的切向加速度

$$a_{C\tau} = \ddot{\theta}\rho_C + 2\dot{\theta}\dot{\rho}_C \quad = \ddot{\theta}\rho_C$$




圆周运动



刚体平面运动学

## 基点加速度

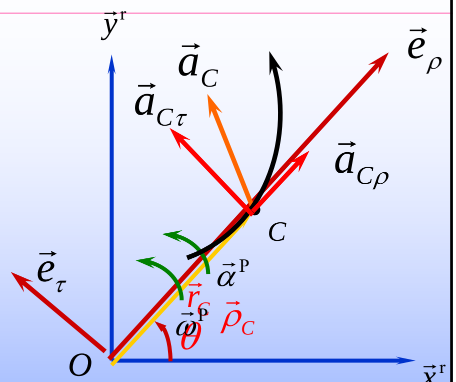
$$\vec{a}_C \xrightarrow{\boldsymbol{e}^{\mathrm{P}}} \boldsymbol{a}_C^{\mathrm{P}} = \begin{pmatrix} a_{C\rho} & a_{C\tau} \end{pmatrix}^{\mathrm{T}}$$

$$a_{C\rho} = \ddot{\rho}_C - \dot{\theta}^2 \rho_C$$

基点$C$的径向加速度

$$a_{C\tau} = \ddot{\theta}\rho_C + 2\dot{\theta}\dot{\rho}_C$$

基点$C$的切向加速度




## 基点加速度两坐标阵的关系

$$\boldsymbol{A}^{\mathrm{rP}} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$$\vec{a}_C \xrightarrow{\vec{e}^{\mathrm{r}}} \boldsymbol{a}_C = \begin{pmatrix} a_{Cx} & a_{Cy} \end{pmatrix}^{\mathrm{T}} \quad \boldsymbol{a}_C = \boldsymbol{A}^{\mathrm{rP}} \boldsymbol{a}_C^{\mathrm{P}}$$

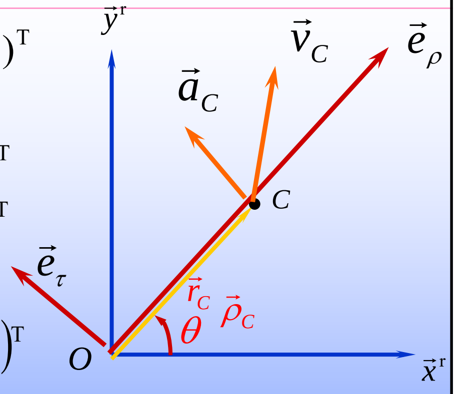
$$\begin{pmatrix} a_{Cx} \\ a_{Cy} \end{pmatrix} = \left(\rho_C - \rho_C\dot{\theta}^2\right)\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} + \left(\rho_C\ddot{\theta} + 2\dot{\rho}_C\dot{\theta}\right)\begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$$

- 小结 $\vec{e}^{\mathrm{p}} = (\vec{e}_\rho \quad \vec{e}_\tau)^{\mathrm{T}}$

基点位置 $\vec{r}_C$ $\vec{\rho}_C$

$$\boldsymbol{r}_C = (x_C \quad y_C)^{\mathrm{T}} \qquad \boldsymbol{\rho}_C^{\mathrm{P}} = (\rho_C \quad 0)^{\mathrm{T}}$$

$$\boldsymbol{q}_C = (\rho_C \quad \theta)^{\mathrm{T}}$$




基点速度 $\vec{v}_C$

$$\boldsymbol{v}_C = (v_{Cx} \quad v_{Cy})^{\mathrm{T}} \qquad \boldsymbol{v}_C^{\mathrm{P}} = (v_{C\rho} \quad v_{C\tau})^{\mathrm{T}}$$

基点加速度 $\vec{a}_C$

$$\boldsymbol{a}_C = (a_{Cx} \quad a_{Cy})^{\mathrm{T}} \qquad \boldsymbol{a}_C^{\mathrm{P}} = (a_{C\rho} \quad a_{C\tau})^{\mathrm{T}}$$

$$\boldsymbol{r}_C = \boldsymbol{A}^{\mathrm{rP}} \boldsymbol{\rho}_C^{\mathrm{P}}$$

$$\boldsymbol{v}_C = \boldsymbol{A}^{\mathrm{rP}} \boldsymbol{v}_C^{\mathrm{P}}$$

$$\boldsymbol{a}_C = \boldsymbol{A}^{\mathrm{rP}} \boldsymbol{a}_C^{\mathrm{P}}$$

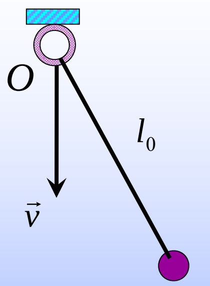




**[例]**

变摆长的摆套在环上，摆绳的变化规律

$$l_0 + \sin bt$$

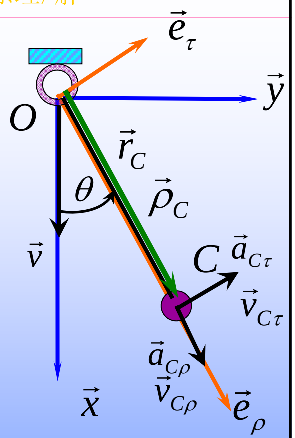



对于摆的速度、加速度矢量

**(1)**写出它们的极坐标表达式

**(2)**写出它们的直角坐标表达式

**[解1]** 利用极坐标推导的公式




直角基 $O-\vec{\boldsymbol{e}}$ 极坐标基 $O-\vec{e}^{\mathrm{P}}$

$$r_C=\rho_C=l_0+\sin bt$$

速度

$$v_{C\rho}=\dot{\rho}_C \qquad v_{C\rho}=b\cos bt$$

$$v_{C\tau}=\dot{\theta}\rho_C \qquad v_{C\tau}=\dot{\theta}\left(l_0+\sin bt\right)$$

加速度

$$a_{C\rho}=\ddot{\rho}_C-\dot{\theta}^2\rho_C$$

$$a_{C\rho}=-b^2\sin bt-\dot{\theta}^2\left(l_0+\sin bt\right)$$

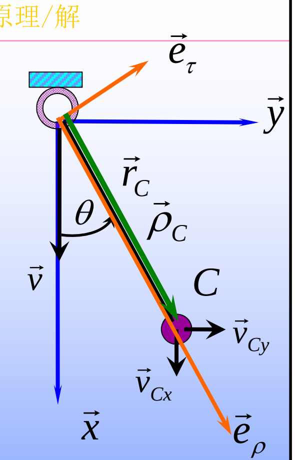
$$a_{C\tau}=\ddot{\theta}\rho_C+2\dot{\theta}\dot{\rho}_C$$

$$a_{C\tau}=\ddot{\theta}\left(l_0+\sin bt\right)+2\dot{\theta}b\cos bt$$





**[解2]** 摆长与摆角为点*C*的极坐标




速度 $\boldsymbol{v}_C=\boldsymbol{A}^{\mathrm{rP}}\boldsymbol{v}_C^{\mathrm{P}}$

$$\begin{pmatrix} v_{Cx} \\ v_{Cy} \end{pmatrix}=\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} v_{C\rho} \\ v_{C\tau} \end{pmatrix}$$

$$v_{C\rho}=b\cos bt \qquad v_{C\tau}=\dot{\theta}\left(l_0+\sin bt\right)$$

$$\begin{pmatrix} v_{Cx} \\ v_{Cy} \end{pmatrix}=v_{C\rho}\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}+v_{C\tau}\begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$$

$$\boldsymbol{A}^{\mathrm{rP}}=\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$$=b\cos bt\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}+\dot{\theta}(l_0+\sin bt)\begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$$

加速度 $\boldsymbol{a}_C = \boldsymbol{A}^{\mathrm{rP}}\boldsymbol{a}_C^{\mathrm{P}}$ $\boldsymbol{A}^{\mathrm{rP}} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

$$\begin{pmatrix} a_{Cx} \\ a_{Cy} \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} a_{C\rho} \\ a_{C\tau} \end{pmatrix}$$

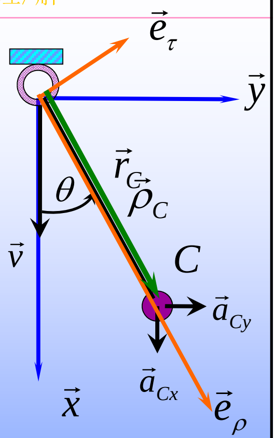
$$a_{C\rho} = -b^2 \sin bt - \dot{\theta}^2 (l_0 + \sin bt)$$

$$a_{C\tau} = \ddot{\theta}(l_0 + \sin bt) + 2\dot{\theta}b\cos bt$$

$$\begin{pmatrix} a_{Cx} \\ a_{Cy} \end{pmatrix} = a_{C\rho}\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} + a_{C\tau}\begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$$

$$= \left(-b^2 \sin bt - \dot{\theta}^2 (l_0 + \sin bt)\right)\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$$

$$+ \left(\ddot{\theta}(l_0 + \sin bt) + 2\dot{\theta}b\cos bt\right)\begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$$








比较 $\vec{v}_C$，$\vec{a}_C$

$$v_{C\rho} = b\cos bt \qquad v_{C\tau} = \dot{\theta}(l_0 + \sin bt)$$

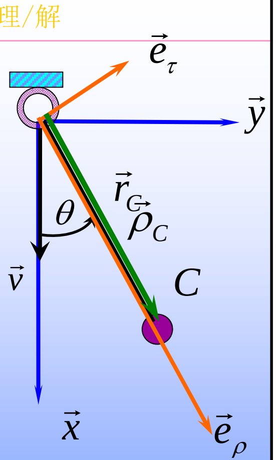
$$a_{C\rho} = -b^2 \sin bt - \dot{\theta}^2 (l_0 + \sin bt)$$

$$a_{C\tau} = \ddot{\theta}(l_0 + \sin bt) + 2\dot{\theta}b\cos bt$$

$$\begin{pmatrix} v_{Cx} \\ v_{Cy} \end{pmatrix} = b\cos bt\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} + \dot{\theta}(l_0 + \sin bt)\begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$$

$$\begin{pmatrix} a_{Cx} \\ a_{Cy} \end{pmatrix} = \left(-b^2 \sin bt - \dot{\theta}^2 (l_0 + \sin bt)\right)\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$$

$$+ \left(\ddot{\theta}(l_0 + \sin bt) + 2\dot{\theta}b\cos bt\right)\begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$$




有时直角坐标表达式不一定简单,注意利用极坐标表达式