理论力学 CAI 数学基础

For 计算机辅助分析

数学基础
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矩阵对变量的导数

矩阵对时间的导数

- 元素为时间 t 的函数，记为$A_{ij}(t)$，该矩阵记为$\boldsymbol{A}(t)$

$$\boldsymbol{A}(t)=\left(A_{ij}(t)\right)_{m\times n}$$

例

$$\boldsymbol{A}=\begin{pmatrix}\cos t & \sin t\\ -\sin t & \cos t\end{pmatrix}\qquad \boldsymbol{B}=\begin{pmatrix}t & 0\\ 0 & t^2\end{pmatrix}$$

- 矩阵对时间导数的定义
  - 矩阵对时间的导数为一同阶矩阵
  - 其各元素为原矩阵的元素$A_{ij}(t)$对时间的导数

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{A}\overset{\text{def}}{=}\left(\frac{\mathrm{d}A_{ij}}{\mathrm{d}t}\right)_{m\times n}\qquad \dot{\boldsymbol{A}}=\left(\dot{A}_{ij}\right)_{m\times n}$$

例

$$\boldsymbol{A}=\begin{pmatrix}\cos t & \sin t\\ -\sin t & \cos t\end{pmatrix}$$

$$\dot{\boldsymbol{A}}=\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\cos t & \sin t\\ -\sin t & \cos t\end{pmatrix}=\begin{pmatrix}\frac{\mathrm{d}}{\mathrm{d}t}\cos t & \frac{\mathrm{d}}{\mathrm{d}t}\sin t\\ \frac{\mathrm{d}}{\mathrm{d}t}(-\sin t) & \frac{\mathrm{d}}{\mathrm{d}t}\cos t\end{pmatrix}=\begin{pmatrix}-\sin t & \cos t\\ -\cos t & -\sin t\end{pmatrix}$$

矩阵对变量的导数

- 标量函数对变量的偏导数
- 标量函数阵对变量的偏导数

标量函数对变量的偏导数

- 多变量的标量函数(多元函数)

如果有一个标量a，它是$\boldsymbol{n}$个变量的函数，记为

$$a = a(q_1, q_2, \cdots, q_n)$$

对于这组（$\boldsymbol{n}$个）变量，通常引入一 $\boldsymbol{n}$ 阶列矩阵表示这组变量，即

$$\boldsymbol{q} = \begin{pmatrix} q_1 & q_2 & \cdots & q_n \end{pmatrix}^{\mathrm{T}}$$

标量a，可记为

$$a = a(q_1, q_2, \cdots, q_n) = a(\boldsymbol{q})$$

[例] 对于二元函数 $a=\sin\theta_1\cos 2\theta_2$

标量a是2个变量$\boldsymbol{\theta}_1$， $\boldsymbol{\theta}_2$的函数

引入二阶变量阵

$$\boldsymbol{q}=\begin{pmatrix}\theta_1 & \theta_2\end{pmatrix}^{\mathrm{T}}$$

标量函数a可表为

$$a=a(\boldsymbol{q})\ =\sin\theta_1\cos 2\theta_2$$

$$a=a(q_1,q_2,\cdots,q_n)=a(\mathbf{q})$$

- 标量函数对变量的偏导数
  - 多变量函数 a 对n 阶变量阵$\boldsymbol{q}$ 的偏导数可构成一 n 阶行阵
  - 其元素分别为该标量函数对各自变量 q_j 的偏导数

$$a_{\mathbf{q}}\overset{\text{def}}{=}\frac{\partial a}{\partial \mathbf{q}}\overset{\text{def}}{=}\begin{pmatrix}\dfrac{\partial a}{\partial q_1} & \dfrac{\partial a}{\partial q_2} & \cdots & \dfrac{\partial a}{\partial q_n}\end{pmatrix}_{1\times n}=\begin{pmatrix}\dfrac{\partial a}{\partial q_j}\end{pmatrix}_{1\times n}$$

$$a = a(q_1, q_2, \cdots, q_n) = a(\mathbf{q})$$

$$a_{\mathbf{q}} \overset{\text{def}}{=} \frac{\partial a}{\partial \mathbf{q}} \overset{\text{def}}{=} \begin{pmatrix} \frac{\partial a}{\partial q_1} & \frac{\partial a}{\partial q_2} & \cdots & \frac{\partial a}{\partial q_n} \end{pmatrix}_{1\times n} = \begin{pmatrix} \frac{\partial a}{\partial q_j} \end{pmatrix}_{1\times n}$$

[例]对于二阶变量阵 $\underset{2\times 1}{\mathbf{q} = (\theta_1 \quad \theta_2)^{\mathrm{T}}}$ 的标量函数 $a = a(\mathbf{q}) = \sin\theta_1 \cos 2\theta_2$

$$a_{\mathbf{q}} = \underset{1\times 2}{\begin{pmatrix} \frac{\partial a}{\partial \theta_1} & \frac{\partial a}{\partial \theta_2} \end{pmatrix}} = \begin{pmatrix} \cos\theta_1 \cos 2\theta_2 & -2\sin\theta_1 \sin 2\theta_2 \end{pmatrix}$$

标量函数阵对变量的偏导数

- 多变量的标量函数阵

对于有n个变量，通常引入一 n 阶列矩阵表示这组变量，即

$$\mathbf{q} = (q_1 \quad q_2 \quad \cdots \quad q_n)^{\mathrm{T}}$$

如果有一个m 阶标量函数列阵它的元素是上述变量的函数，记为

$$\boldsymbol{\Phi} = \begin{pmatrix} \Phi_1 \\ \Phi_2 \\ \vdots \\ \Phi_m \end{pmatrix} = \begin{pmatrix} \Phi_1(\mathbf{q}) \\ \Phi_2(\mathbf{q}) \\ \vdots \\ \Phi_m(\mathbf{q}) \end{pmatrix}$$

$$\boldsymbol{\Phi} = (\Phi_1 \quad \Phi_2 \quad \cdots \quad \Phi_m)^{\mathrm{T}} = (\Phi_1(\mathbf{q}) \quad \Phi_2(\mathbf{q}) \quad \cdots \quad \Phi_m(\mathbf{q}))^{\mathrm{T}}$$

- 标量函数阵对变量的偏导数
  - 多变量的标量函数阵

$$\boldsymbol{\Phi}=\begin{pmatrix}\Phi_1\\ \Phi_2\\ \vdots\\ \Phi_m\end{pmatrix}=\begin{pmatrix}\Phi_1(\boldsymbol{q})\\ \Phi_2(\boldsymbol{q})\\ \vdots\\ \Phi_m(\boldsymbol{q})\end{pmatrix}=\begin{pmatrix}\Phi_1 & \Phi_2 & \cdots & \Phi_m\end{pmatrix}^{\mathrm{T}}$$

[例]对于二阶变量阵 $\boldsymbol{q}=\begin{pmatrix}\theta_1 & \theta_2\end{pmatrix}^{\mathrm{T}}$有如下的标量函数阵

2×1

$$\boldsymbol{\Phi}=\begin{pmatrix}\sin(\theta_1+2\theta_2)\\ \cos(2\theta_1-\theta_2)\\ \cos(2\theta_1-2\theta_2)\end{pmatrix}=\left(\sin(\theta_1+2\theta_2)\quad\cos(2\theta_1-\theta_2)\quad\cos(2\theta_1-2\theta_2)\right)^{\mathrm{T}}$$

3×1

$$\boldsymbol{\Phi}=\begin{pmatrix}\Phi_1\\ \Phi_2\\ \vdots\\ \Phi_m\end{pmatrix}=\begin{pmatrix}\Phi_1(\boldsymbol{q})\\ \Phi_2(\boldsymbol{q})\\ \vdots\\ \Phi_m(\boldsymbol{q})\end{pmatrix}$$

$$\begin{aligned}\boldsymbol{\Phi}_{1\mathbf{q}}&=\begin{pmatrix}\frac{\partial\Phi_1}{\partial q_1} & \frac{\partial\Phi_1}{\partial q_2} & \cdots & \frac{\partial\Phi_1}{\partial q_n}\end{pmatrix}\\ \boldsymbol{\Phi}_{2\mathbf{q}}&=\begin{pmatrix}\frac{\partial\Phi_2}{\partial q_1} & \frac{\partial\Phi_2}{\partial q_2} & \cdots & \frac{\partial\Phi_2}{\partial q_n}\end{pmatrix}\\ &\vdots\\ \boldsymbol{\Phi}_{m\mathbf{q}}&=\begin{pmatrix}\frac{\partial\Phi_m}{\partial q_1} & \frac{\partial\Phi_m}{\partial q_2} & \cdots & \frac{\partial\Phi_m}{\partial q_n}\end{pmatrix}\end{aligned}$$

标量函数阵对变量的偏导数

$$\boldsymbol{\Phi}_{\mathbf{q}}\overset{\text{def}}{=}\frac{\partial\boldsymbol{\Phi}}{\partial\mathbf{q}}\overset{\text{def}}{=}\begin{pmatrix}\boldsymbol{\Phi}_{1\mathbf{q}}\\ \boldsymbol{\Phi}_{2\mathbf{q}}\\ \vdots\\ \boldsymbol{\Phi}_{m\mathbf{q}}\end{pmatrix}=\begin{pmatrix}\frac{\partial\Phi_1}{\partial q_1} & \frac{\partial\Phi_1}{\partial q_2} & \cdots & \frac{\partial\Phi_1}{\partial q_n}\\ \frac{\partial\Phi_2}{\partial q_1} & \frac{\partial\Phi_2}{\partial q_2} & \cdots & \frac{\partial\Phi_2}{\partial q_n}\\ \vdots & \vdots & & \vdots\\ \frac{\partial\Phi_m}{\partial q_1} & \frac{\partial\Phi_m}{\partial q_2} & \cdots & \frac{\partial\Phi_m}{\partial q_n}\end{pmatrix}=\left(\frac{\partial\Phi_i}{\partial q_j}\right)_{m\times n}$$

- 标量函数阵对变量的偏导数
  - m 阶标量函数列阵对n 阶变量阵$\boldsymbol{q}$ 的偏导数定义为一 $m\times n$ 阶矩阵
  - 且

$$\boldsymbol{\Phi}=\begin{pmatrix}\Phi_1\\ \Phi_2\\ \vdots\\ \Phi_m\end{pmatrix}=\begin{pmatrix}\Phi_1(\boldsymbol{q})\\ \Phi_2(\boldsymbol{q})\\ \vdots\\ \Phi_m(\boldsymbol{q})\end{pmatrix}$$

$$\boldsymbol{\Phi}_{\boldsymbol{q}}\overset{\text{def}}{=}\frac{\partial\boldsymbol{\Phi}}{\partial\boldsymbol{q}}\overset{\text{def}}{=}\left(\frac{\partial\Phi_i}{\partial q_j}\right)_{m\times n}=\begin{array}{c|c} & \begin{matrix}q_1 & \quad q_2 & \quad & \quad q_n\end{matrix}\\ \hline \begin{matrix}\Phi_1\\[2ex] \Phi_2\\[2ex] \\ \Phi_m\end{matrix} & \begin{pmatrix}\frac{\partial\Phi_1}{\partial q_1} & \frac{\partial\Phi_1}{\partial q_2} & \cdots & \frac{\partial\Phi_1}{\partial q_n}\\ \frac{\partial\Phi_2}{\partial q_1} & \frac{\partial\Phi_2}{\partial q_2} & \cdots & \frac{\partial\Phi_2}{\partial q_n}\\ \vdots & \vdots & & \vdots\\ \frac{\partial\Phi_m}{\partial q_1} & \frac{\partial\Phi_m}{\partial q_2} & \cdots & \frac{\partial\Phi_m}{\partial q_n}\end{pmatrix}\end{array}$$

[例]对于二阶变量阵 $\underset{2\times 1}{\boldsymbol{q}=(\theta_1\quad\theta_2)^{\mathrm{T}}}$ 有如下的标量函数

$$\boldsymbol{\Phi}=\underset{3\times 1}{\begin{pmatrix}\sin(\theta_1+2\theta_2)\\ \cos(2\theta_1-\theta_2)\\ \cos(2\theta_1-2\theta_2)\end{pmatrix}}$$

$$\boldsymbol{\Phi}_{\boldsymbol{q}}\overset{\text{def}}{=}\frac{\partial\boldsymbol{\Phi}}{\partial\boldsymbol{q}}\overset{\text{def}}{=}\left(\frac{\partial\Phi_i}{\partial q_j}\right)_{m\times n}=\begin{pmatrix}\frac{\partial\Phi_1}{\partial q_1} & \frac{\partial\Phi_1}{\partial q_2} & \cdots & \frac{\partial\Phi_1}{\partial q_n}\\ \frac{\partial\Phi_2}{\partial q_1} & \frac{\partial\Phi_2}{\partial q_2} & \cdots & \frac{\partial\Phi_2}{\partial q_n}\\ \vdots & \vdots & & \vdots\\ \frac{\partial\Phi_m}{\partial q_1} & \frac{\partial\Phi_m}{\partial q_2} & \cdots & \frac{\partial\Phi_m}{\partial q_n}\end{pmatrix}$$

$$\boldsymbol{\Phi}_{\boldsymbol{q}}=\underset{3\times 2}{\begin{pmatrix}\frac{\partial\Phi_1}{\partial q_1} & \frac{\partial\Phi_1}{\partial q_2}\\ \frac{\partial\Phi_2}{\partial q_1} & \frac{\partial\Phi_2}{\partial q_2}\\ \frac{\partial\Phi_3}{\partial q_1} & \frac{\partial\Phi_3}{\partial q_2}\end{pmatrix}}=\begin{pmatrix}\cos(\theta_1+2\theta_2) & 2\cos(\theta_1+2\theta_2)\\ -2\sin(2\theta_1-\theta_2) & \sin(2\theta_1-\theta_2)\\ -2\sin(2\theta_1-2\theta_2) & 2\sin(2\theta_1-2\theta_2)\end{pmatrix}$$

小结

- 矩阵导数公式

$$a_{\mathbf{q}} \overset{\text{def}}{=} \frac{\partial a}{\partial \mathbf{q}} \overset{\text{def}}{=} \begin{pmatrix} \frac{\partial a}{\partial q_1} & \frac{\partial a}{\partial q_2} & \cdots & \frac{\partial a}{\partial q_n} \end{pmatrix} = \begin{pmatrix} \frac{\partial a}{\partial q_j} \end{pmatrix}_{1\times n}$$

$$\boldsymbol{\Phi}_{\mathbf{q}} \overset{\text{def}}{=} \frac{\partial \boldsymbol{\Phi}}{\partial \mathbf{q}} \overset{\text{def}}{=} \begin{pmatrix} \frac{\partial \Phi_i}{\partial q_j} \end{pmatrix}_{m\times n} = \begin{pmatrix} \frac{\partial \Phi_1}{\partial q_1} & \frac{\partial \Phi_1}{\partial q_2} & \cdots & \frac{\partial \Phi_1}{\partial q_n} \\ \frac{\partial \Phi_2}{\partial q_1} & \frac{\partial \Phi_2}{\partial q_2} & \cdots & \frac{\partial \Phi_2}{\partial q_n} \\ \vdots & \vdots & & \vdots \\ \frac{\partial \Phi_m}{\partial q_1} & \frac{\partial \Phi_m}{\partial q_2} & \cdots & \frac{\partial \Phi_m}{\partial q_n} \end{pmatrix}$$