理论力学 CAI

矢量动力学基础

- 前言
- 惯量
- 动量定理
- 动量矩定理
- 动能定理

惯 量

惯量

- 转动惯量
- 转动惯量的平行轴定理

惯量

- 转动惯量
- 转动惯量的平行轴定理

转动惯量

- 质量
- 转动惯量

质量

- 质点的质量
  - 质点惯性的度量

$$\vec{F}=m\vec{a}$$

- 某瞬时
  - 力与加速度方向一致
  - 力与加速度大小成正比
  - 质量是它们的比例系数

$$m=F/a \quad \text{一维的情况下}$$

- 质点系的质量

质点系 $(P_1,P_2,\cdots,P_N)$

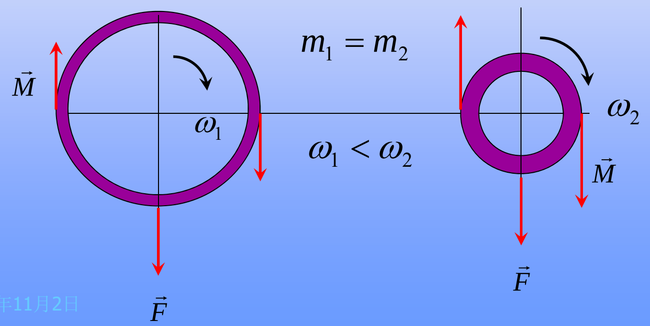
各个质点的质量

$$m_i\ (i=1,2,\cdots,N)$$

$$m=\sum_{i=1}^{n}m_i$$

质量　转动惯量

- 质量是质点或质点系平移惯性的度量
- 转动惯量是质点系（刚体）转动惯性的度量



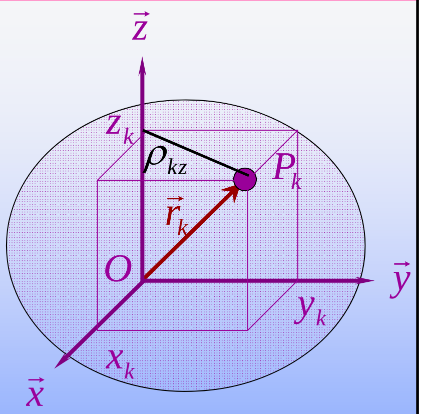
- 刚体转动惯量的定义

在刚体的给定点O
建立连体基　　$O-\vec{e}$

刚体任意质点P_k的质量 m_k

质点P_k的矢径　$\vec{r}_k$

$O-\vec{e}$: $\boldsymbol{r}_k = (x_k \quad y_k \quad z_k)^{\mathrm{T}}$　常值阵



$$J_{Oz} \overset{\text{def}}{=} \sum_k m_k \left(x_k^2 + y_k^2\right) = \sum_k m_k \rho_{kz}^2$$

刚体关于Oz轴的转动惯量

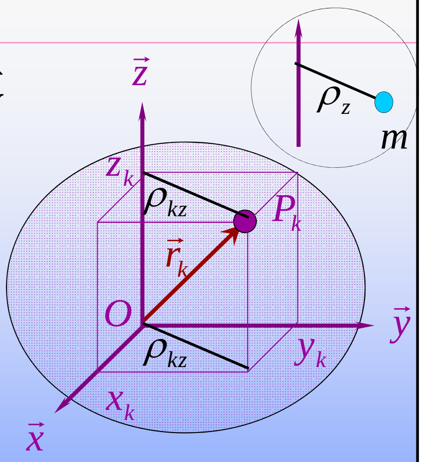
$$J_{Ox} \overset{\text{def}}{=} \sum_k m_k \left(y_k^2 + z_k^2\right)$$　刚体关于Ox轴的转动惯量

$$J_{Oy} \overset{\text{def}}{=} \sum_k m_k \left(z_k^2 + x_k^2\right)$$　刚体关于Oy轴的转动惯量

转动惯量恒正　　单位　$\mathrm{kg \cdot m^2}$

- 刚体转动惯量的另一种表达

$$J_{Oz}=\sum_k m_k\left(x_k^2+y_k^2\right)=\sum_k m_k\rho_{kz}^2$$



刚体关于**Oz**轴的转动惯量等于所有质点的质量与其到**Oz**轴距离的平方乘积的代数和

令 $J_{Oz}\overset{\text{def}}{=}m\rho_z^2$ **m**为刚体的质量

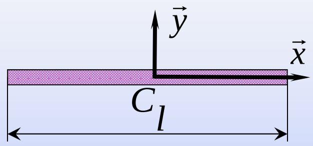
ρ_z 刚体对**Oz**轴的回转半径

$$m\rho_z^2=\sum_k m_k\rho_{kz}^2$$

$$J_{Ox}=\sum_k m_k\left(y_k^2+z_k^2\right)=\sum_k m_k\rho_{kx}^2\overset{\text{def}}{=}m\rho_x^2$$ ρ_x 刚体对**Ox**轴的回转半径

$$J_{Oy}=\sum_k m_k\left(z_k^2+x_k^2\right)=\sum_k m_k\rho_{ky}^2\overset{\text{def}}{=}m\rho_y^2$$ ρ_y 刚体对**Oy**轴的回转半径

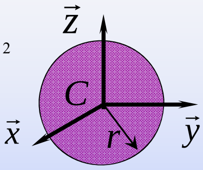
- 常见的均质几何体的转动惯量



细直杆

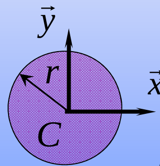
$$J_{Cy}=J_{Cz}=\frac{1}{12}ml^2$$

$$J_{Cx}=0$$



球体

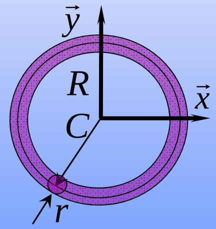
$$J_{Cx}=J_{Cy}=J_{Cz}=\frac{2}{5}mr^2$$



圆形薄板

$$J_{Cz}=\frac{1}{2}mr^2$$

$$J_{Cx}=J_{Cy}=\frac{1}{4}mr^2$$



圆环

$$J_{Cz}=m(R^2+\cancel{\tfrac{3}{4}r^2})$$

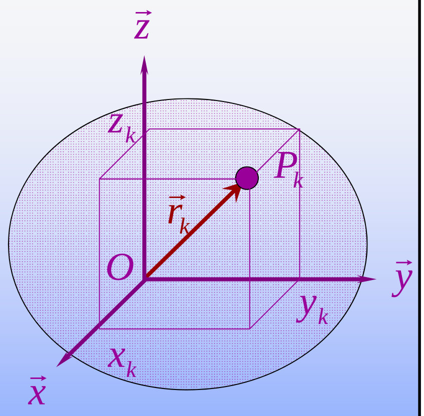
$$J_{Cx}=J_{Cy}=\frac{1}{2}m(R^2+\cancel{\tfrac{5}{4}r^2})$$

$r<<R$

$$J_{Cz}=mR^2$$

$$J_{Cx}=J_{Cy}=\frac{1}{2}mR^2$$

- 刚体惯性积的定义



在刚体的给定点***O***
建立连体基 $O-\vec{e}$

刚体任意质点$\boldsymbol{P_k}$的质量 $\boldsymbol{m_k}$

质点$\boldsymbol{P_k}$的矢径 $\vec{r}_k$

$$O-\vec{e}:\ \boldsymbol{r}_k=\begin{pmatrix}x_k & y_k & z_k\end{pmatrix}^{\mathrm{T}}$$ 常值阵

$$J_{Oxy}=J_{Oyx}\overset{\mathrm{def}}{=}\sum_k m_k x_k y_k$$

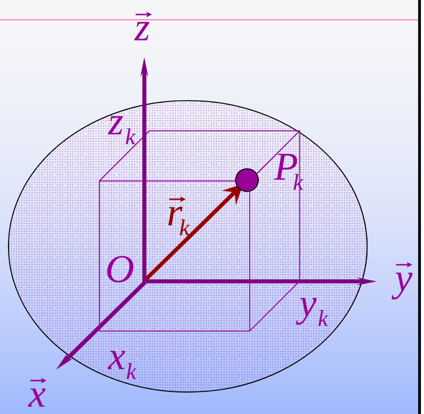
刚体关于***Oxy***平面的惯性积

$$J_{Oyz}=J_{Ozy}\overset{\mathrm{def}}{=}\sum_k m_k y_k z_k$$ 刚体关于***Oyz***平面的惯性积

$$J_{Ozx}=J_{Oxz}\overset{\mathrm{def}}{=}\sum_k m_k z_k x_k$$ 刚体关于***Ozx***平面的惯性积

惯性积可正可负 单位 $\mathrm{kg\cdot m^2}$

- 刚体的惯量主轴



$J_{Oyz}=0$
$J_{Oxz}=0$ 称***Oz***轴为刚体的惯量主轴

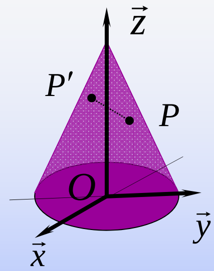
$J_{Oxz}=0$
$J_{Oxy}=0$ 称***Ox***轴为刚体的惯量主轴

$J_{Oxy}=0$
$J_{Oyz}=0$ 称***Oy***轴为刚体的惯量主轴

Oz轴为刚体的旋转对称轴

$J_{Oxz} = 0 \quad J_{Oyz} = 0$

Oz轴为刚体的惯量主轴



$P(x_k, y_k, z_k)$

$P'(-x_k, -y_k, z_k)$

$m_k x_k z_k + m_k(-x_k)z_k = 0$

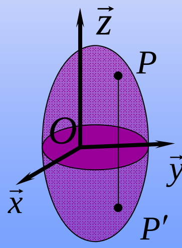
$m_k y_k z_k + m_k(-y_k)z_k = 0$

Oxy为刚体的对称面

Oz轴为垂直于该面的任意轴

$J_{Oxz} = 0 \quad J_{Oyz} = 0$

Oz轴为刚体的惯量主轴



$P(x_k, y_k, z_k)$

$P'(x_k, y_k, -z_k)$

$m_k x_k z_k + m_k x_k(-z_k) = 0$

$m_k y_k z_k + m_k y_k(-z_k) = 0$

- 主轴连体基

转动惯量和惯性积与连体基的基点位置和连体基基矢量的方位有关

同一个基点**O**，不同连体基的刚体转动惯量与惯性积不同，但存在关系

$O - \vec{\boldsymbol{e}}^i \qquad J_{Ox^i}, J_{Oy^i}, J_{Oz^i}$

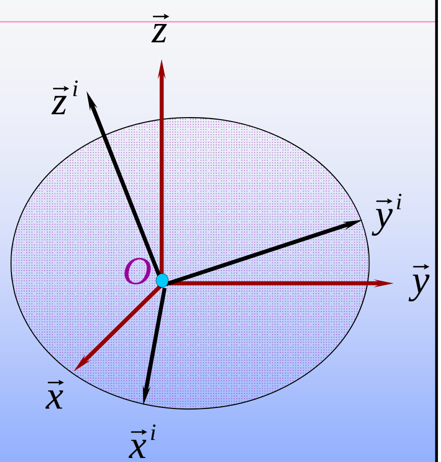
$J_{Ox^iy^i}, J_{Oy^iz^i}, J_{Oz^ix^i}$

过点**O**至少存在一个连体基，该基的三根轴同时为刚体的主轴

$O - \vec{\boldsymbol{e}} \qquad J_{Ox}, J_{Oy}, J_{Oz}$

$J_{Oxy} = J_{Oyz} = J_{Ozx} = 0$

$O - \vec{\boldsymbol{e}}$ 刚体关于点**O**的主轴连体基



- 中心主轴连体基

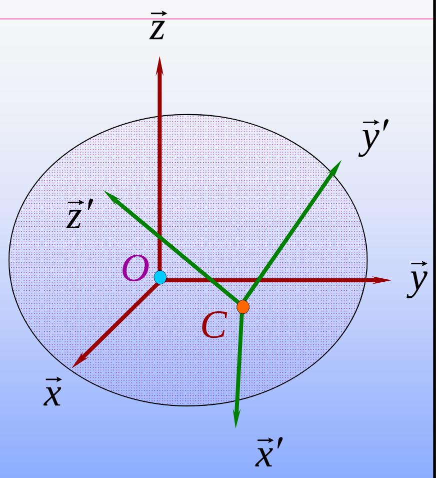
刚体关于质心C的惯量主轴连体基
称为中心惯量主轴连体基

$$C-\vec{\boldsymbol{e}}' \quad J_{Cx'}, J_{Cy'}, J_{Cz'}$$

刚体的中心主转动惯量

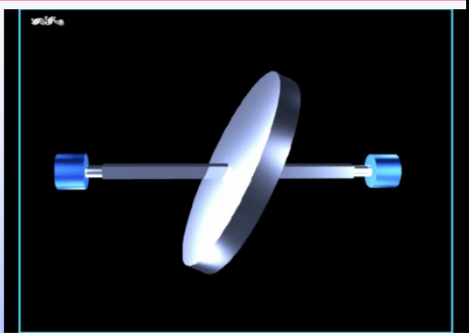
$$J_{Cx'y'} = J_{Cy'z'} = J_{Cz'x'} = 0$$

刚体对不同基点的连体基的转动惯量
与惯性积不同，但存在关系



[例]

均质圆盘的转子，质心为C。转子的转轴$C\vec{z}$与圆盘中心轴$C\vec{z}'$有如图所示一小偏角θ



试计算惯性积J_{Czx}

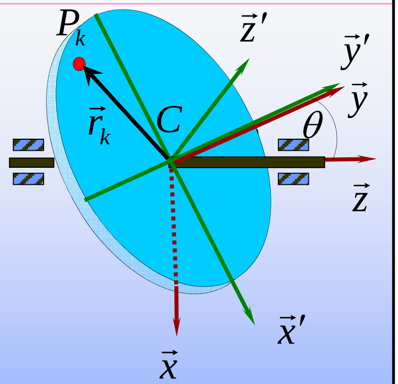
[解]

圆盘上质点$\boldsymbol{P}_k$的矢径 $\vec{r}_k$

$C-\vec{\boldsymbol{e}}$ $\quad \boldsymbol{r}_k = (x_k \quad y_k \quad z_k)^{\mathrm{T}}$

$C-\vec{\boldsymbol{e}}'$ $\quad \boldsymbol{r}'_k = (x'_k \quad y'_k \quad z'_k)^{\mathrm{T}}$

$$\boldsymbol{r}_k = \boldsymbol{A}'\boldsymbol{r}'_k \qquad \begin{aligned} x_k &= x'_k\cos\theta - z'_k\sin\theta \\ y_k &= y'_k \\ z_k &= x'_k\sin\theta + z'_k\cos\theta \end{aligned}$$



$$J_{Cxz} = \sum_k m_k x_k z_k$$

$$= \sum_k m_k(x'^2_k - z'^2_k)\sin\theta\cos\theta + \sum_k m_k x'_k z'_k(\cos^2\theta - \sin^2\theta)$$

$$+y'^2_k - y'^2_k$$

$$= (J_{Cz'} - J_{Cx'})\sin\theta\cos\theta + J_{Cx'z'}(\cos^2\theta - \sin^2\theta)$$

$\vec{\boldsymbol{e}}'$相对于$\vec{\boldsymbol{e}}$ 方向余弦阵

$$\boldsymbol{A}' = \begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix}$$

$$J_{Cxz} = (J_{Cz'} - J_{Cx'})\sin\theta\cos\theta + J_{Cx'z'}(\cos^2\theta - \sin^2\theta)$$

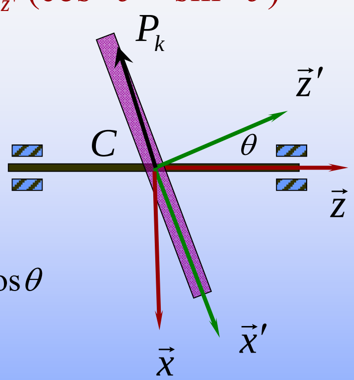
对于均质圆盘的转子

$C-\vec{\boldsymbol{e}}'$ 中心惯量主轴连体基

$$J_{Cz'} = \frac{1}{2}mr^2 \qquad J_{Cx'} = \frac{1}{4}mr^2 \qquad J_{Cx'z'} = 0$$

$$J_{Cxz} = (J_{Cz'} - J_{Cx'})\sin\theta\cos\theta = \frac{mr^2}{4}\sin\theta\cos\theta$$

显然 $C-\vec{\boldsymbol{e}}$ 非中心惯量主轴连体基

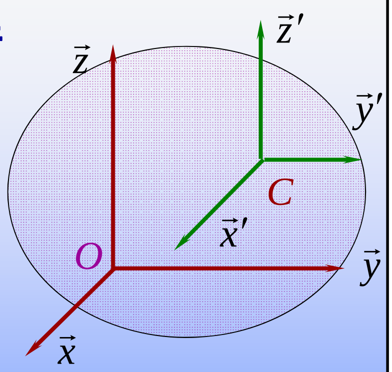


惯量

- 转动惯量
- 转动惯量的平行轴定理

转动惯量的平行轴定理

- 刚体对不同基点连体基的转动惯量与惯量积不同，但存在关系
- 讨论刚体对不同基点相互平行两连体基的惯量特性间的关系



- **定理描述**

同一刚体两个连体基相互平行

$O-\vec{e}$　　$C-\vec{e}'$

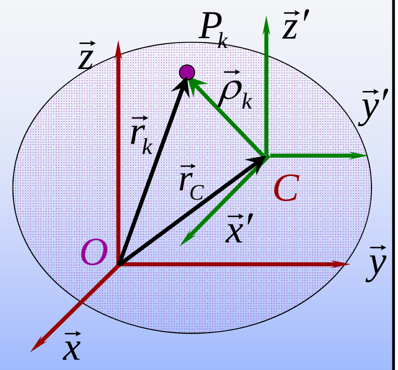
C为质心　　$\vec{r}_C$

质点P_k的矢径　$\vec{r}_k$　$\vec{\rho}_k$

$$\vec{r}_k = \vec{r}_C + \vec{\rho}_k$$

$$\vec{e}:\quad \boldsymbol{r}_k = \boldsymbol{r}_C + \boldsymbol{A}\boldsymbol{\rho}'_k = \boldsymbol{r}_C + \boldsymbol{\rho}'_k$$

$$\begin{pmatrix} x_k \\ y_k \\ z_k \end{pmatrix} = \begin{pmatrix} x_C \\ y_C \\ z_C \end{pmatrix} + \begin{pmatrix} x'_k \\ y'_k \\ z'_k \end{pmatrix}$$



$\vec{e}'$相对于$\vec{e}$　方向余弦阵

$$\boldsymbol{A}' = \boldsymbol{I}$$

$$J_{Oz} = \sum_k m_k\left(x_k^2 + y_k^2\right) = \sum_k m_k\left[(x_C + x'_k)^2 + (y_C + y'_k)^2\right]$$

$$= \sum_k m_k(x_k'^2 + y_k'^2) + (x_C^2 + y_C^2)\sum_k m_k + 2x_C\sum_k m_k y'_k + 2y_C\sum_k m_k x'_k$$

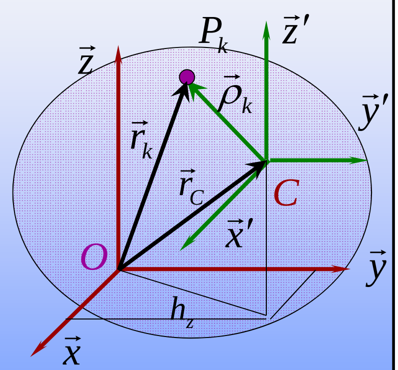
$$J_{Oz} = \underbrace{\sum_k m_k(x_k'^2 + y_k'^2)}_{J_{Cz'}} + \underbrace{(x_C^2 + y_C^2)}_{h_z^2}\underbrace{\sum_k m_k}_{m} + 2x_C\sum_k \cancel{m_k y'_k} + 2y_C\sum_k \cancel{m_k x'_k}$$

$$\sum_k m_k\vec{\rho}_k = m\vec{\rho}_C = \vec{0}$$　　**C为质心**

$$\sum_k m_k x'_k = 0 \qquad \sum_k m_k y'_k = 0 \qquad \sum_k m_k z'_k = 0$$

$$J_{Oz} = J_{Cz} + mh_z^2$$

刚体对任意轴的转动惯量等于它对过质心的平行轴转动惯量加上刚体的质量与两轴垂直距离平方的乘积



$$J_{Ox} = J_{Cx} + mh_x^2 \qquad J_{Oy} = J_{Cy} + mh_y^2$$

刚体对所有平行轴的转动惯量中过质心的最小

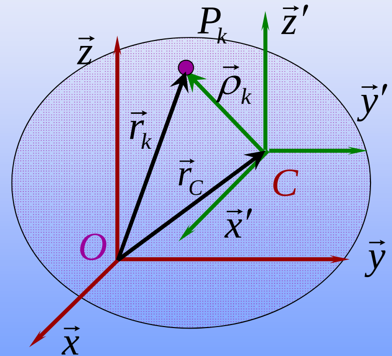
$$J_{Oxy} = \sum_k m_k x_k y_k = \sum_k m_k (x_C + x'_k)(y_C + y'_k)$$

$$= \underbrace{\sum_k m_k x'_k y'_k}_{J_{Cx'y'}} + x_C y_C \underbrace{\sum_k m_k}_{m} + x_C \cancel{\sum_k m_k y'_k} + y_C \cancel{\sum_k m_k x'_k}$$

$$J_{Oxy} = J_{Cxy} + m x_C y_C$$

$$J_{Oyz} = J_{Cyz} + m y_C z_C$$

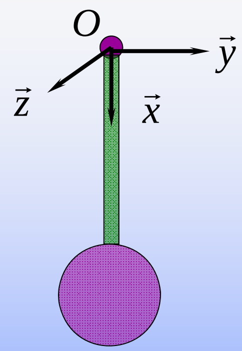
$$J_{Ozx} = J_{Czx} + m z_C x_C$$



$$\begin{pmatrix} x_k \\ y_k \\ z_k \end{pmatrix} = \begin{pmatrix} x_C \\ y_C \\ z_C \end{pmatrix} + \begin{pmatrix} x'_k \\ y'_k \\ z'_k \end{pmatrix}$$

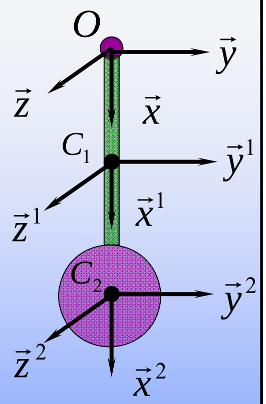
[例]

图示一摆由长为 l 均质杆与一半径为 r 的均质圆球刚连而成。质量分别为 m_1 与 m_2



试计算该摆对过O且垂直杆的z轴的转动惯量

[解]



过均质杆质心C_1建连体基 $C_1-\vec{e}^1$ 平行于 $\vec{e}$

过均质球质心C_2建连体基 $C_2-\vec{e}^2$ 平行于 $\vec{e}$

均质杆 $J_{1C_1z}=\frac{1}{12}m_1 l^2$

$$J_{1Oz}=J_{1C_1z}+m_1\left(\frac{l}{2}\right)^2=\frac{1}{3}m_1 l^2$$

均质球 $J_{2C_2z}=\frac{2}{5}m_2 r^2$

$$J_{2Oz}=J_{2C_2z}+m_2(l+r)^2=\frac{2}{5}m_2 r^2+m_2(l+r)^2$$

系统

$$J_{Oz}=J_{1Oz}+J_{2Oz}=\frac{1}{3}m_1 l^2+\frac{2}{5}m_2 r^2+m_2(l+r)^2$$

小结

$$J_{Ox}\overset{\text{def}}{=}\sum_k m_k\left(y_k^2+z_k^2\right)\overset{\text{def}}{=}m\rho_x^2$$

$$J_{Oyx}=J_{Oxy}\overset{\text{def}}{=}\sum_k m_k y_k x_k$$

$$J_{Ozx}=J_{Oxz}\overset{\text{def}}{=}\sum_k m_k z_k x_k$$

- 转动惯量，惯性积
  - 质量分布
  - 主轴
- 转动惯量的平行轴定理

$$J_{Ox}=J_{Cx}+mh_x^2$$

$$J_{Oxy}=J_{Cxy}+mx_C y_C$$