Black-Scholes Formula Derivation and Applications

Methods in Applied Mathematics: Fourier Series and Boundary Value Problems
CU Boulder Department of Applied Mathematics

Natalie Eid - Section 001
naei4841@colorado.edu
Chloe Mulqueen - Section 003
chmu5201@colorado.edu
Isabella Bates - Section 002
isba8475@colorado.edu

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01 Abstract

In this project, the researchers will investigate the history and applications of the Black Scholes Model. They will aim to investigate under what circumstances the model is effective and what are the limitations of this model. Additionally, a simulation will be conducted to compare the actual performance of options with current market data against the forecasts made by the Black-Scholes Model. The project will also discuss other partial differential equation models that are used in Finance, in particular expansions that could be made on the Black-Scholes model.

02 Introduction

Mathematical models are fundamental to how we understand many concepts in our modern world and how we simplify different processes. An industry that saw an opportunity to improve a procedure through a complex model drives the economy: the field of finance. This led to the birth of the Black-Scholes Method, a highly regarded mathematical model that allows people to strategically price a derivative in options trading.

Options trading originated from the likes of the Ancient Greeks, as most important methodologies are. The concept is fairly simple. A man by the name of Thales of Miletus predicted a bountiful season for the olive crops and he approached the owners of the olive presses with a deal. For an agreed-upon price, Thales would pay a deposit to the owners for the right to use the olive presses when the olive harvest came. Thales was correct in his prediction and when the time came he was able to sell the right to use the olive presses at a higher price and make a profit.

The modern options trading we know today is an electronically driven version of Thales's original vision. It is important to understand the fundamentals behind options before getting into how they are traded. At its core, an option is a type of derivative which is a financial contract that is valued by an underlying asset such as stocks, bonds, or commodities. The value in owning and trading these assets is found through financial risks and how these will impact the future worth of the assets by other people participating in the market. An option is a type of derivative that allows the buyer a chance, but not an obligation, to buy or sell the underlying asset at an agreed-upon date and price. In the olive context, Thales saw an opportunity with the asset of the olive press, and he bet that the olive business would be booming and decided to buy the right to sell the olive presses at an agreed date.

The price one pays for the option is referred to as the premium, which grants the buyer the right to the derivative contract. There are two types of options, a call option which allows the buyer to buy the asset, and a put option which allows the buyer to sell the asset at the given time. The given timeframe in these contracts is widely known as the expiration date. The strike price of an option is what the contract states the asset can be bought or sold at the expiration date.

The concept of options lends itself to more complex ideas on how to price these contracts in order to trade appropriately and generate profits. Two economists by the name of Fischer Black and Merton Scholes were able to capitalize on this by creating a model to price these contracts to limit risk and heighten the probability of reward. This model is derived from a partial differential equation inspired by the famous heat equation. This concept, on the other hand, is fairly complex.

We will first show the derivation for this powerful model, starting with the Black-Scholes equation, incorporating the appropriate boundary and initial conditions. Following the derivation, we will show an application of the model using current data to further the reader's understanding on the topic.

03 Mathematical Formulation

We will derive $V(S, t)$ from the Black-Scholes equation:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0 \tag{1}$$

With boundary conditions and initial conditions:

$$V(0, t) = 0 \text{ for all } t$$
$$V(S, t) = S - K \text{ as } S \to \infty$$
$$V(S, T) = max(S - K, 0)$$

The variables for the Black-Scholes model:

- $V(S, t) = the\ price\ of\ the\ European\ option$
- $S = the\ stock\ price\ at\ the\ beginning\ of\ the\ time\ period\ of\ the\ option$
- $t = time\ in\ years,\ with\ t = 0\ generally\ representing\ the\ present\ year$
- $K = the\ strike\ price,\ or\ the\ exercise\ price$
- $(T - t) = time\ of\ option\ expiration\ minus\ t, or\ time\ until\ maturity$
- $\sigma = the\ standard\ deviation\ of\ the\ stock's\ returns,\ or\ volatility\ of\ the\ stock$
- $r = the\ risk - free\ interest\ rate$

Let $t = T - \frac{2\tau}{\sigma^2}$ and solve for τ.

$$\tau = \frac{1}{2}\sigma^2(T - t)$$

Let $S = Ke^x$ and solve for x.

$$e^x = \frac{S}{K}$$
$$x = ln(\frac{S}{K})$$

Using the above, let

$$V(S, t) = Kv(x, \tau) \quad (2)$$

Now take the first derivative of V with respect to time t.

$$\frac{\partial V}{\partial t} = K\frac{\partial v}{\partial \tau}\frac{\partial \tau}{\partial t} = K\frac{\partial v}{\partial \tau}((T - t)\frac{1}{2}\sigma^2\frac{\partial}{\partial t}) = \frac{-\sigma^2 K}{2}\frac{\partial v}{\partial \tau}$$

Take the first and second derivative of V with respect to S, the stock price.

$$\frac{\partial V}{\partial S} = K\frac{\partial v}{\partial x}\frac{\partial x}{\partial S} = K\frac{\partial v}{\partial x}(ln(\frac{S}{K})\frac{\partial}{\partial S}) = K\frac{1}{S}\frac{\partial v}{\partial x}$$

Using $\frac{\partial x}{\partial S} = \frac{1}{\frac{S}{K}} * \frac{1}{K} = \frac{1}{S}$:

$$\frac{\partial^2 V}{\partial S^2} = \frac{\partial}{\partial S}(K\frac{1}{S}\frac{\partial v}{\partial x})$$

$$= K\frac{\partial v}{\partial x}\frac{-1}{S^2} + K\frac{\partial}{\partial S}\left(\frac{\partial v}{\partial x}\right)\frac{1}{S}$$

$$= K\frac{\partial v}{\partial x}\frac{-1}{S^2} + K\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x}\right)\frac{\partial x}{\partial S}\frac{1}{S}$$

$$= K\frac{\partial v}{\partial x}\frac{-1}{S^2} + K\frac{\partial^2 v}{\partial x^2}\frac{1}{S^2}$$

Based on the above equations, we can set:

$$V(S,T) = max(S - K, 0) = max(Ke^x - K, 0)$$

Thus,

$$V(S,T) = Kv(x, 0)$$

$$v(x, 0) = max(e^x - 1, 0)$$

Take the derivatives and plug them back into equation (1).

$$(K\frac{\partial v}{\partial \tau}\frac{-\sigma^2}{2}) + \frac{\sigma^2}{2}S^2(K\frac{\partial v}{\partial x}\frac{-1}{S^2} + K\frac{\partial^2 v}{\partial x^2}\frac{1}{S^2}) + rS(K\frac{\partial v}{\partial x}\frac{1}{S}) - rKv = 0$$

Simplifying, we get

$$(\frac{\partial v}{\partial \tau}\frac{-\sigma^2}{2}) + \frac{\sigma^2}{2}S^2(\frac{\partial v}{\partial x}\frac{-1}{S^2} + \frac{\partial^2 v}{\partial x^2}\frac{1}{S^2}) + rS(\frac{\partial v}{\partial x}\frac{1}{S}) - rv = 0$$

$$(\frac{\partial v}{\partial \tau}\frac{-\sigma^2}{2}) + \frac{\sigma^2}{2}(\frac{\partial v}{\partial x} + \frac{\partial^2 v}{\partial x^2}) + r(\frac{\partial v}{\partial x}) - rv = 0$$

Factor out $\frac{\sigma^2}{2}$ and let $k = \frac{2\tau}{\sigma^2}$. Substitute and combine like terms.

$$\frac{\partial v}{\partial \tau} = (\frac{\partial^2 v}{\partial x^2} - \frac{\partial v}{\partial x}) + \frac{2\tau}{\sigma^2}(\frac{\partial v}{\partial x}) - \frac{2\tau}{\sigma^2}v$$

$$\frac{\partial v}{\partial \tau} = (\frac{\partial^2 v}{\partial x^2} - \frac{\partial v}{\partial x}) + k(\frac{\partial v}{\partial x}) - kv$$

$$\frac{\partial v}{\partial \tau} = \frac{\partial^2 v}{\partial x^2} + (k - 1)\frac{\partial v}{\partial x} - kv \tag{3}$$

Now we can let:

$$v(x, \tau) = e^{\alpha x + \beta\tau}u(x, \tau) \tag{4}$$

Thus,

$$\frac{\partial v}{\partial \tau} = \beta e^{\alpha x + \beta\tau}u + e^{\alpha x + \beta\tau}\frac{\partial u}{\partial \tau}$$

$$\frac{\partial v}{\partial x} = \alpha e^{\alpha x + \beta\tau}u + e^{\alpha x + \beta\tau}\frac{\partial u}{\partial x}$$

$$\frac{\partial^2 v}{\partial x^2} = \alpha^2 e^{\alpha x+\beta\tau} u + 2\alpha e^{\alpha x+\beta\tau}\frac{\partial u}{\partial x} + e^{\alpha x+\beta\tau}\frac{\partial^2 u}{\partial x^2}$$

Plug the above equations into equation (3):

$$\beta e^{\alpha x+\beta\tau} u + e^{\alpha x+\beta\tau}\frac{\partial u}{\partial \tau} = \alpha^2 e^{\alpha x+\beta\tau} u + 2\alpha e^{\alpha x+\beta\tau}\frac{\partial u}{\partial x} + e^{\alpha x+\beta\tau}\frac{\partial^2 u}{\partial x^2} + (k-1)(\alpha e^{\alpha x+\beta\tau} u + e^{\alpha x+\beta\tau}\frac{\partial u}{\partial x}) - k e^{\alpha x+\beta\tau} u$$

Simplifying and combining like terms,

$$\beta u + \frac{\partial u}{\partial \tau} = \alpha^2 u + 2\alpha\frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x^2} + (k-1)(\alpha u + \frac{\partial u}{\partial x}) - ku$$

$$\beta u + \frac{\partial u}{\partial \tau} = \alpha^2 u + 2\alpha\frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x^2} + k\alpha u + k\frac{\partial u}{\partial x} - \alpha u - \frac{\partial u}{\partial x} - ku$$

$$\frac{\partial u}{\partial \tau} = \alpha^2 u + 2\alpha\frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x^2} + k\alpha u + k\frac{\partial u}{\partial x} - \alpha u - \frac{\partial u}{\partial x} - ku - \beta u$$

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x}(k - 1 + 2\alpha) + u(\alpha^2 + k\alpha - \alpha - k - \beta) \qquad (5)$$

Using the conditions $u = 0$ and $\frac{\partial u}{\partial x} = 0$, we find that $\alpha = \frac{-(k-1)}{2}$ and
$\beta = \alpha^2 + (k-1)\alpha - k = \frac{-(k+1)^2}{4}$.

Plugging this into equation (5), we get:

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x}[k - 1 + 2(\frac{-(k-1)}{2})] + u[(\frac{-(k-1)}{2})^2 + k(\frac{-(k-1)}{2}) + \frac{(k-1)}{2} - k + \frac{(k+1)^2}{4}]$$

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x}[k - 1 - (k-1)] + u[\frac{k^2-2k+1}{4} - \frac{k^2-k}{2} + \frac{k-1}{2} - k + \frac{k^2-2k+1}{4}]$$

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x}[0] + u[\frac{k^2-2k+1-2k^2+2k+2k-2-4k+k^2+2k+1}{4}]$$

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x}[0] + u[0]$$

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2}$$

This leads us to the basis of the Heat Equation.
The initial condition for $u(x,\tau)$ gives:

$$u(x,0) = max(e^{\frac{(k+1)}{2}x} - e^{\frac{(k-1)}{2}x}, 0) \qquad (6)$$

We will transform the solution to the Heat equation to use for the Black-Scholes equation:

$$u(x,\tau) = \frac{1}{2\sqrt{\pi\tau}}\int_{-\infty}^{\infty} u_o(s) e^{\frac{-(x-s)^2}{4\tau}} ds$$

Let $s = z\sqrt{2\tau} + x$ and do a change of variables. Then, solve for u_0.

$$u(x,\tau) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u_o(\sqrt{2\tau} + x) e^{\frac{-z^2}{2}} dz \tag{7}$$

$$u_0 = e^{\frac{k+1}{2}(z\sqrt{2\tau}+x)} - e^{\frac{k-1}{2}(z\sqrt{2\tau}+x)} \tag{8}$$

$u_0 > 0$, since the time cannot be less than 0. Thus, $x > \frac{-x}{\sqrt{2\tau}}$.

Plugging this into equation (7),

$$u(x,\tau) = \frac{1}{\sqrt{2\pi}} \int_{-\frac{x}{\sqrt{2\tau}}}^{\infty} (e^{\frac{k+1}{2}(z\sqrt{2\tau}+x)} - e^{\frac{k-1}{2}(z\sqrt{2\tau}+x)}) e^{\frac{-z^2}{2}} dz$$

$$u(x,\tau) = \frac{1}{\sqrt{2\pi}} \int_{-\frac{x}{\sqrt{2\tau}}}^{\infty} e^{\frac{k+1}{2}(z\sqrt{2\tau}+x)} e^{\frac{-z^2}{2}} dz - \frac{1}{\sqrt{2\pi}} \int_{-\frac{x}{\sqrt{2\tau}}}^{\infty} e^{\frac{k-1}{2}(z\sqrt{2\tau}+x)} e^{\frac{-z^2}{2}} dz$$

$$u(x,\tau) = \frac{1}{\sqrt{2\pi}} \int_{-\frac{x}{\sqrt{2\tau}}}^{\infty} e^{\frac{k+1}{2}(z\sqrt{2\tau}+x)-\frac{z^2}{2}} dz - \frac{1}{\sqrt{2\pi}} \int_{-\frac{x}{\sqrt{2\tau}}}^{\infty} e^{\frac{k-1}{2}(z\sqrt{2\tau}+x)-\frac{z^2}{2}} dz \tag{9}$$

Completing the square,

$$\begin{aligned}
\frac{k+1}{2}(z\sqrt{2\tau} + x) - \frac{z^2}{2} &=- \frac{1}{2}(z^2 - z\sqrt{2\tau}(k+1)) + \frac{x(k+1)}{2} \\
&=- \frac{1}{2}\left[z^2 - z\sqrt{2\tau}(k+1) + \frac{\tau}{2}(k+1)^2\right] + \frac{x(k+1)}{2} - (-\frac{\tau(k+1)^2}{4}) \\
&=- \frac{1}{2}\left[z - z\sqrt{\frac{\tau}{2}}(k+1)\right]^2 + \frac{x(k+1)}{2} + \frac{\tau(k+1)^2}{4}
\end{aligned}$$

Plugging the above into equation (9),

$$u(x,\tau) = \frac{1}{\sqrt{2\pi}} e^{\frac{x(k+1)}{2}+\frac{\tau(k+1)^2}{4}} \int_{-\frac{x}{\sqrt{2\tau}}}^{\infty} e^{-\frac{1}{2}(z-\sqrt{\frac{\tau}{2}}(k+1))^2} dz - \frac{1}{\sqrt{2\pi}} e^{\frac{x(k-1)}{2}+\frac{\tau(k+1)^2}{4}} \int_{-\frac{x}{\sqrt{2\tau}}}^{\infty} e^{-\frac{1}{2}(z-\sqrt{\frac{\tau}{2}}(k+1))^2} dz$$

Let $y = z - \sqrt{\frac{\tau}{2}}(k+1)$, $dy = dz$ and $z = \frac{-x}{\sqrt{2\tau}}$.

So,

$$u(x,\tau) = \frac{1}{\sqrt{2\pi}} e^{\frac{x(k+1)}{2}+\frac{\tau(k+1)^2}{4}} \int_{-\frac{x}{\sqrt{2\tau}}-\sqrt{\frac{\tau}{2}}(k+1)}^{\infty} e^{-\frac{y^2}{2}} dy - \frac{1}{\sqrt{2\pi}} e^{\frac{x(k-1)}{2}+\frac{\tau(k+1)^2}{4}} \int_{-\frac{x}{\sqrt{2\tau}}-\sqrt{\frac{\tau}{2}}(k+1)}^{\infty} e^{-\frac{y^2}{2}} dy \tag{10}$$

The area $(-\infty, d)$ under the curve given by the cdf of the Normal distribution is:

$$N(d) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d} e^{-\frac{y^2}{2}} dy$$

$$d = \frac{x}{\sqrt{2\tau}} + \sqrt{\frac{\tau}{2}}(k + 1)$$

Since the Normal curve is symmetrical, the area $(-\infty, d)$ is the same as $(-d, \infty)$. Thus, d_1 is the same as d_2, except it has $(k - 1)$ instead of $(k + 1)$.

$$d_1 = \frac{x}{\sqrt{2\tau}} + \sqrt{\frac{\tau}{2}}(k + 1)$$

$$d_2 = \frac{x}{\sqrt{2\tau}} + \sqrt{\frac{\tau}{2}}(k - 1)$$

Plugging N into equation (10),

$$u(x,\tau) = e^{\frac{x(k+1)}{2} + \frac{\tau(k+1)^2}{4}} N(d_1) - e^{\frac{x(k-1)}{2} + \frac{\tau(k+1)^2}{4}} N(d_2) \tag{11}$$

Plugging α, β, and equation (11) into equation (4),

$$v(x,\tau) = e^{\frac{x(k+1)}{2} - \frac{\tau(k+1)^2}{4}} u(x,\tau)$$

$$v(x,\tau) = e^{\frac{x(k+1)}{2} - \frac{\tau(k+1)^2}{4}} \left[e^{\frac{x(k+1)}{2} + \frac{\tau(k+1)^2}{4}} N(d_1) - e^{\frac{x(k-1)}{2} + \frac{\tau(k+1)^2}{4}} N(d_2) \right]$$

Simplifying,

$$v(x,\tau) = e^{x} N(d_1) - e^{-k\tau} N(d_2)$$

Plugging in $\tau = \frac{1}{2}\sigma^2(T - t)$ and $x = ln(\frac{S}{K})$ from the beginning, we get

$$v(x,\tau) = e^{ln(\frac{S}{K})} N(d_1) - e^{-k\frac{1}{2}\sigma^2(T-t)} N(d_2)$$

$$v(x,\tau) = \frac{S}{K} N(d_1) - e^{-k\frac{1}{2}\sigma^2(T-t)} N(d_2) \tag{12}$$

Plugging in $\tau = \frac{1}{2}\sigma^2(T - t)$ and $x = ln(\frac{S}{K})$ into d_1 and d_2,

$$d_1 = \frac{ln(\frac{S}{K})}{\sqrt{2(\frac{1}{2}\sigma^2(T-t)}} + \sqrt{\frac{\frac{1}{2}\sigma^2(T-t)}{2}}(k + 1)$$

$$= \frac{ln(\frac{S}{K})}{\sigma\sqrt{T-t}} + \frac{\sigma}{2}\sqrt{T - t}(k + 1)$$

$$= \frac{ln(\frac{S}{K})\left(\frac{\sigma^2}{2}k+\frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}$$

$$d_2 = \frac{ln(\frac{S}{K})\left(\frac{\sigma^2}{2}k-\frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}$$

The risk-free interest rate $r = \frac{k}{2}\sigma^2$. Plugging r into equation (12) and d_1,

$$v(x, \tau) = \frac{S}{K}N(d_1) - e^{-r(T-t)}N(d_2) \quad (13)$$

$$d_1 = \frac{ln(\frac{S}{K})+(r+\frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}$$

Plugging equation (13) into equation (2), we get

$$\mathrm{V}(S, t) = K\frac{S}{K}N(d_1) - Ke^{-r(T-t)}N(d_2)$$

$$= SN(d_1) - Ke^{-r(T-t)}N(d_2)$$

Thus, the solution to the Black-Scholes equation is

$$V(S, t) = SN(d_1) - Ke^{-r(T-t)}N(d_2) \quad (14)$$

With

$$d_1 = \frac{ln(\frac{S}{K})+(r+\frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}} \quad (15)$$

$$d_2 = \frac{ln(\frac{S}{K})+(r-\frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}} \quad (16)$$

This final solution lends itself nicely for financial professionals to use with ease on a regular basis. Through deriving the equation, a few assumptions must be made. The one that affects the output price the most is that the option is exercised on the expiration date. This is only true for European call options, as the United States call options can be acted on at any time up to the date of maturity. This means that buyers or sellers are not fully informed of a convenient price at any time throughout the duration of their option, only when it reaches its maturity date. The model also assumes a constant volatility and risk-free rate of return, when in reality neither of these are constant. The use of the normal distribution is prominent throughout the derivation, but stock prices do not always follow a lognormal pattern. Many times, stock prices can experience large fluctuations that are not foreseen. These assumptions can lead to inaccuracies in the resulting call price.

04 Numerical Example

Let us examine the Black Scholes model in practice with some examples. As we know the Black Scholes model takes the input of the current stock price, the exercise price, the risk-free interest rate, the time to expiration, and the standard deviation of log returns (aka the expected volatility of a stock) and gives what a buyer would value the option at. However, this does not guarantee wins for every option, as the volatility of a stock is perpetually changing and is not constant. Particularly over periods volatility can fluctuate. This is where some predictions must be made and influence the choice to buy and sell options.

While we may not be able to predict the volatility of a stock in the future, we may use foresight to decide what risks to take when investing in options. We can use an option's market price to find a stock's implied volatility, working backward from the Black-Scholes model. Say for example, that an investor predicts that a future event will cause the price of a stock to increase drastically, meaning that the market participant expects the volatility to be greater than the historical volatility. If this assumption proves to be accurate, investing in an option at market price would lead to the exercise price being lower than the current market price, meaning gained assets. However, the seller of the option is anticipating the opposite- that the volatility of the stock is less than the implied volatility, or that the option will decrease in price. This means that the seller will earn the option's premium. Recall that while a call buyer's earnings are unlimited with unlimited growth of the asset, their losses are limited to the premium.

Moreover, since the Black-Scholes model considers only European options (that can only be exercised at the end of the option period), there is much more room for fluctuation in the American method of trading options, since the option can be exercised at any time (meaning that the implied volatility does not necessarily need to be consistent over the term for the buyer to make a profit, just so long as the option exceeds the price of the option plus the premium once during the period).

To investigate this, we will examine how well the Black-Scholes model would have performed for several different implied volatilities, over 10 days for several major stocks. We collected information about stock prices and some data on typical options for that stock at that time that expired recently (with the initial bid on November 29th and ending on December 8th, 2023 so that the time to expiration was 10 days, note that on this day the risk-free interest rate was 4.44%). Moreover, data from Yahoo Finance was used to conduct this investigation. Next, we calculated the implied volatility of these options traded on this day and compared the options that resulted in the investor buying the put that gained them money versus those that did not. Implied versus realized volatility will then be analyzed to show that what the market predicts is only sometimes in line with what the market does.

First, we investigated Apple Inc. (AAPL) which had an opening price of $189.84 on November 29, 2023, and a closing price of $195.71 on December 8th. Since this stock is increasing in price, the buyer of the option gains if the final price is less than the strike price, plus the premium. AAPL has a historical volatility of 14.4%, meaning that we might predict the price of the stock to rise to a bit less than $194, or in the next 10 days (using Apple's average yearly dividend yield of 0.5%. So in this scenario, we would gain money, but only a little since, we would pay $1 less for the asset than what it is valued at at the end of the period.

Stock Price	Strike price	Volatility	Risk free interest rate	Time to ex		Dividend Yield	D1	D2	D	D	Call Price
$189.84	$193.00	14.40%	4.44%	1		0.50%					**$0.00755**
$189.84	$193.00	14.40%	4.44%	2		0.50%					**$0.05601**
$189.84	$193.00	14.40%	4.44%	3		0.50%					**$0.12912**
$189.84	$193.00	14.40%	4.44%	4		0.50%					**$0.21142**
$189.84	$193.00	14.40%	4.44%	5		0.50%					**$0.29665**
$189.84	$193.00	14.40%	4.44%	6		0.50%					**$0.38217**
$189.84	$193.00	14.40%	4.44%	7		0.50%					**$0.46680**
$189.84	$193.00	14.40%	4.44%	8		0.50%					**$0.55002**
$189.84	$193.00	14.40%	4.44%	9		0.50%					**$0.63161**
$189.84	$193.00	14.40%	4.44%	10		0.50%					**$0.71150**
$189.84	$193.00	14.40%	4.44%	11		0.50%					**$0.78970**
$189.84	$193.00	14.40%	4.44%	12		0.50%					**$0.86627**
$189.84	$193.00	14.40%	4.44%	13		0.50%					**$0.94126**

However, say, for example, we are conservative with our estimation of variance and guess that Apple will have lower volatility than what is historically true, then we would value the option for a lower value, and may not buy the option at all if we believe that the price will not move. This would result in the buyer gaining more than simply relying on the historical data.

Stock Price	Strike price	Volatility	Risk free interest rate	Time to e		Dividend Yield	D1	D2	D	D	Call Price
$189.84	$193.00	5.00%	4.44%	1		0.50%					**$0.00000**
$189.84	$193.00	5.00%	4.44%	2		0.50%					**$0.00000**
$189.84	$193.00	5.00%	4.44%	3		0.50%					**$0.00004**
$189.84	$193.00	5.00%	4.44%	4		0.50%					**$0.00030**
$189.84	$193.00	5.00%	4.44%	5		0.50%					**$0.00108**
$189.84	$193.00	5.00%	4.44%	6		0.50%					**$0.00266**
$189.84	$193.00	5.00%	4.44%	7		0.50%					**$0.00522**
$189.84	$193.00	5.00%	4.44%	8		0.50%					**$0.00882**
$189.84	$193.00	5.00%	4.44%	9		0.50%					**$0.01349**
$189.84	$193.00	5.00%	4.44%	10		0.50%					**$0.01919**
$189.84	$193.00	5.00%	4.44%	11		0.50%					**$0.02586**
$189.84	$193.00	5.00%	4.44%	12		0.50%					**$0.03344**

Alternatively, say we were aggressive with our estimation of variance and believed that the variance would be significantly higher than the actual volatility of the stock over the period from purchase to expiration, we would overvalue the stock.

Stock Price	Strike price	Volatility	Risk free interest rate	Time to e		Dividend Yield	D1	D2	D	D	Call Price
$189.84	$193.00	35.00%	4.44%	1		0.50%					**$0.35523**
$189.84	$193.00	35.00%	4.44%	2		0.50%					**$0.79776**
$189.84	$193.00	35.00%	4.44%	3		0.50%					**$1.18230**
$189.84	$193.00	35.00%	4.44%	4		0.50%					**$1.52382**
$189.84	$193.00	35.00%	4.44%	5		0.50%					**$1.83375**
$189.84	$193.00	35.00%	4.44%	6		0.50%					**$2.11947**
$189.84	$193.00	35.00%	4.44%	7		0.50%					**$2.38596**
$189.84	$193.00	35.00%	4.44%	8		0.50%					**$2.63669**
$189.84	$193.00	35.00%	4.44%	9		0.50%					**$2.87424**
$189.84	$193.00	35.00%	4.44%	10		0.50%					**$3.10053**
$189.84	$193.00	35.00%	4.44%	11		0.50%					**$3.31708**

Looking at the information above we notice that since we are paying the strike price and the call price, we would be losing a little money, note however the high for the call over this period was 195.99, meaning that for a European call option, we would lose a little, but for an American option since we can trade the option at any time, if we took advantage of the high and exercised at that time and were able to sell while the price of the option still exceeded the strike and the premium combined, we would have gained from buying this option. So when looking at the historical data from this period, many of the option buyers gained assets as they predicted volatility within the range of the historical data or slightly above or below it, those that betted on the margins typically lost assets.

Let's now look at a stock that decreased in value over the past 10 days Tesla Inc. (TSLA). Since the Black Scholes model only works for stocks increasing in price, volatility is seen in what we expect the strike price to be. The lower the strike price, the higher the implied volatility. In this case, just as with the stock that gains value, to gain assets, the strike price must be less than the initial stock price plus the call price. The price of TSLA opened at $249.14 on

November 29th and closed at $243.84 on December 8th, and has a high historical volatility of 49%.

Strike Price	Stock price	Volatility	Risk free interest rate	Time to ex		Dividend Yield	D1	D2	D	D	Call Price	Put Price
$240.00	$249.14	49.00%	4.44%	1		0.00%					**$0.20460**	**$9.31429**
$240.00	$249.14	49.00%	4.44%	2		0.00%					**$0.70598**	**$9.78537**
$240.00	$249.14	49.00%	4.44%	3		0.00%					**$1.23080**	**$10.27990**
$240.00	$249.14	49.00%	4.44%	4		0.00%					**$1.73399**	**$10.75279**
$240.00	$249.14	49.00%	4.44%	5		0.00%					**$2.21004**	**$11.19856**
$240.00	$249.14	49.00%	4.44%	6		0.00%					**$2.66052**	**$11.61874**
$240.00	$249.14	49.00%	4.44%	7		0.00%					**$3.08821**	**$12.01616**
$240.00	$249.14	49.00%	4.44%	8		0.00%					**$3.49587**	**$12.39354**
$240.00	$249.14	49.00%	4.44%	9		0.00%					**$3.88587**	**$12.75326**
$240.00	$249.14	49.00%	4.44%	10		0.00%					**$4.26024**	**$13.09737**
$240.00	$249.14	49.00%	4.44%	11		0.00%					**$4.62069**	**$13.42754**
$240.00	$249.14	49.00%	4.44%	12		0.00%					**$4.96863**	**$13.74522**

In this case, while the stock price goes down, it decreases more than what the historical volatility implies it would, so guessing solely with historical data causes us to gain assets in this case. However, exercising the option before the time of expiration could result in the gaining of assets if timed correctly.

Strike Price	Stock price	Volatility	Risk free interest rate	Time to ex		Dividend Yield	D1	D2	D	D	Call Price
$235.00	$249.14	49.00%	4.44%	1		0.00%					**$0.02455**
$235.00	$249.14	49.00%	4.44%	2		0.00%					**$0.20193**
$235.00	$249.14	49.00%	4.44%	3		0.00%					**$0.48189**
$235.00	$249.14	49.00%	4.44%	4		0.00%					**$0.80267**
$235.00	$249.14	49.00%	4.44%	5		0.00%					**$1.13754**
$235.00	$249.14	49.00%	4.44%	6		0.00%					**$1.47480**
$235.00	$249.14	49.00%	4.44%	7		0.00%					**$1.80910**
$235.00	$249.14	49.00%	4.44%	8		0.00%					**$2.13791**
$235.00	$249.14	49.00%	4.44%	9		0.00%					**$2.46014**
$235.00	$249.14	49.00%	4.44%	10		0.00%					**$2.77537**
$235.00	$249.14	49.00%	4.44%	11		0.00%					**$3.08355**

When we are less conservative with our estimate of market volatility we win in this case. So when observing historical options data, unlike the last example, it is better not to be conservative, since with decreasing stock price it is more likely that the stock will go out of bounds of profitability. When evaluating the historical options of TSLA, not many of the options traded were profitable because there is not much incentive for call sellers to sell these options, as they are selling for a significantly lower price than what the option is currently worth.

In conclusion, this trial of the Black Scholes model using Apple and Tesla reveals several insights into implied and realized volatility. Firstly, the Black-Scholes forces the assumption of constant volatility and forces investors to predict future volatility over a short period. While predicting constant volatility tends to not cause issues with call options that can only be exercised on the expiration date, it can cause varying results for calls that can be exercised any time between the purchase of the option and the expiration date as stocks can have big fluctuations, particularly over extended periods. Traders must also rely on historical data and predictions about news to make decisions to decide if the stock will maintain its historical volatility or if end-of-period announcements or other news can wildly affect the volatility of a stock. Moreover, the direction in which we predict that the price of the stock will move governs the choice to be conservative or aggressive with stock volatility. While predicting that the stock value will increase favors a more conservative estimation of volatility, falling stock prices favor aggressive tactics. It is clear that in the practical application of the Black-Scholes Model, a nuanced understanding of market dynamics is invaluable to managing risk and accurately assessing trends.

04 Summary and Conclusion

Through this project, we were able to deepen our understanding of how partial differential equations can be applied to real-world applications and solved for important results. We researched the history of options trading which helped us understand where there was a need for a model such as the one created by Fischer Black, Merton Scholes, and Robert Merton. We used their techniques to derive this formula and found an interesting application by analyzing the Tesla and Apple stocks.

To derive the formula, we wrote the time variable t as a function of the time to expiration, (T-t) and σ^2, which represents the volatility of the returns. We then used the changing stock price at the beginning of the time period to plug into the desired equation. Through taking derivatives with respect to the changing variables in the function, we eventually were left with a very familiar equation: the Heat Equation. Using different methods discussed in our partial differential equation course, we were able to solve this equation with the given initial conditions. Balancing the different variables and how they worked together proved tedious, but we arrived at the solution that we were able to incorporate into our numerical example.

This formula is fundamental to the market efficiency we see today, but it does not come without limitations. The derivation of the formula showed restrictions with the need to keep certain variables held constant. We saw that the difference between European and American options affects the outcome of the fair call price, as the Black-Scholes formula only considers the calls made on the expiration date.

Testing the formula with current data in the Apple and Tesla stocks helped us better understand the impact of volatility on the expected outcome. We combatted this issue by being less conservative with our estimate of market volatility when choosing which figures to analyze. There is a delicate balance for predicting future volatility and the impact it has on option pricing.

If given more time for our project, we would have liked to show an example of where the Black-Scholes model “fails” to give us an accurate call price and show other methods that could be used in its substitute. We would have liked to research the Binomial Option Pricing method and analyze how this model compares to the likes of Black-Scholes. We also learned through our research that there are other, more difficult methods to deriving and solving the Black-Scholes model. One that Fischer Black and Merton Scholes discuss in their original paper is using the capital asset pricing model, widely referred to as CAPM, to derive the equation. This method is more strenuous in which it requires one to use a discount rate that depends on both time and price of the given stock.

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