CSCI 3022

intro to data science with probability & statistics

Lecture 16
March 12, 2018

Introduction to Hypothesis Testing

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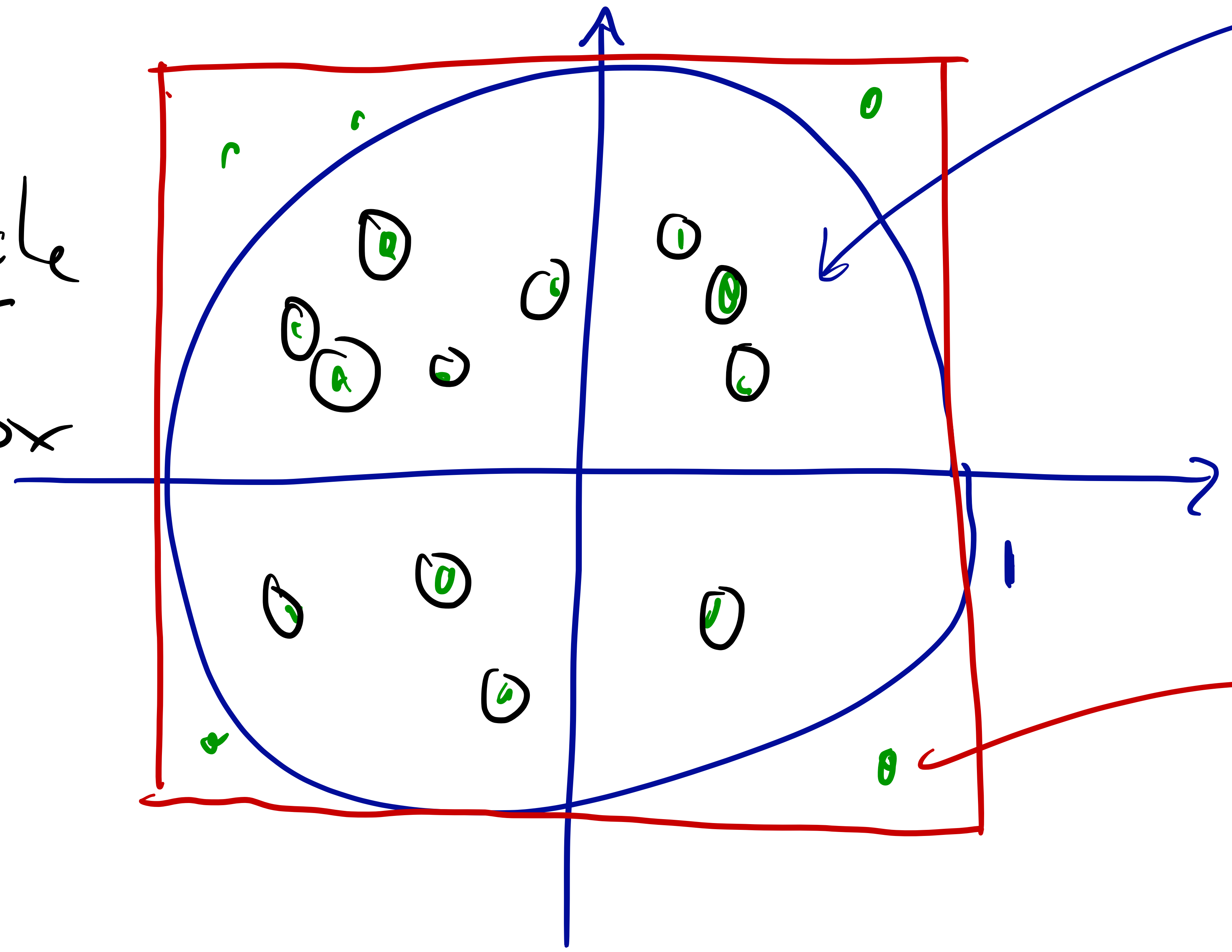
Stuff & Things

- **HW4** due on Friday.

Monte Carlo to be posted on Piazza.

$$P(\text{in circle}) = \frac{\text{Area of circle}}{\text{Area of box}} = \frac{\pi}{4}$$

Sample a jillion times to estimate this



Area $= \pi r^2 = \pi 1^2 = \pi$

Area $= 2 \times 2 = 4$

A thought experiment 🤔

- **Example**: After the introduction of the Euro, Polish mathematicians claimed that the new Belgian 1 Euro coin is not a fair coin. Suppose I hand you a Belgian 1 Euro coin. How could you decide whether or not it is fair?

Flip it a jillion times
[n]

→ record # H, # flips

Conclusion: if $\frac{\# H}{\# \text{flips}} \neq 0.5$ then not a fair coin.

Goal: anchor this test with stats!

Statistical Hypotheses

- **Definition**: A *statistical hypothesis* is a claim about the value of a parameter of a population characteristic.
- **Examples**:
  - Suppose the recovery time of a person suffering from disease D be normally distributed with mean μ_1 and standard deviation σ_1. **Hypothesis**: $\mu_1 > 10$ days.
  - Suppose μ_2 is the recovery time of a person suffering from disease D and given treatment for D. **Hypothesis**: $\mu_2 < \mu_1$
  - Suppose μ_1 is the mean internet speed for Comcast and μ_2 is the mean internet speed for Century Link. **Hypothesis**: $\mu_1 \neq \mu_2$

Null vs Alternative Hypotheses

- In any hypothesis testing problem, there are always two competing hypotheses that we consider:

1. Null Hypothesis H_0 — status quo, default, e.g. coin is fair, $p = 0.5$
2. Alternative Hypothesis H_1 — "research" hypothesis, what we want to test — e.g. coin is biased, $p \neq 0.5$

- The **objective** of hypothesis testing is to decide, based on the data that we've sampled, *whether the alternative hypothesis is actually supported by the data.*

The classic jury analogy

- Think about a jury in a criminal trial.

H_0

- When a defendant is accused of a crime, the jury is supposed to presume that the defendant is not guilty. **“Not guilty” is the null hypothesis**.

H_1

- The jury is then presented with **evidence** (data). If the evidence seems implausible under the assumption of not-guilty, they may **reject** the “not guilty” status, and claim that the defendant is likely guilty.

Null vs Alternative Hypotheses

alternative Hypothesis

- **Is there strong evidence for the alternative**?
- The burden of proof is placed on those that believe the alternative claim, just like in a jury.
- The initially favored claim, written as H_0, will not be rejected in favor of the alternative claim, written as H_1, unless the sample evidence provides a lot of support for the alternative.
- Two possible conclusions:

1. Reject H_0 in favor of H_1
2. Fail to reject H_0

Null vs Alternative Hypotheses

- **Why assume the Null Hypothesis**?
  - Sometimes we don't want to accept a particular assertion unless/until data can be shown to strongly support it.
  - Reluctance (measured in cost or time) to change.
- **Example**: A company is considering hiring a new advertising company to help generate traffic to their website. Under their current advertising they get, on average, 200K hits per day. With μ denoting the true average number of hits they'd get per day under the new company's advertising, they would not want to switch companies (because it would be costly) unless evidence strongly suggested that μ exceeds 200K.

Null vs Alternative Hypotheses

- **Example**: A company is considering hiring a new advertising company to help generate traffic to their website. Under their current advertising they get, on average, 200K hits per day. With μ denoting the true average number of hits they'd get per day under the new company's advertising, they would not want to switch companies (because it would be costly) unless evidence strongly suggested that μ exceeds 200K.
- An appropriate problem formulation would involve testing:

H_0: $\mu = 200{,}000$ (status quo) H_1: $\mu > 200{,}000$ (alternative)

- The conclusion that change is justified is identified with the alternative hypothesis and it would take conclusive evidence to justify rejecting H_0 and switching to the new company "show me enough data to convince me."

Null vs Alternative Hypotheses

- The alternative to the Null Hypothesis $H_0 : \theta = \theta_0$ will look like one of the following assertions (or hypotheses):

1. $\theta > \theta_0$ $\quad$ $p > 0.5$
2. $\theta < \theta_0$ $\quad$ $p < 0.5$
3. $\theta \neq \theta_0$ $\quad$ $p \neq 0.5$

examples

- The equals sign is **always** the Null Hypothesis $\theta = \theta_0$
- The alternative hypothesis is the one for which we are seeking statistical evidence.

Test statistics and evidence

- **Def**: A test statistic is a quantity derived from the sample data and calculated assuming that the Null hypothesis is true. It is used in the decision about whether or not to reject the Null hypothesis.
- **Intuition**:
  - We can think of the test statistics as our evidence about the competing hypotheses.
  - We consider the test statistic under the assumption that H_0 is true by asking: **How likely would we obtain this evidence if the Null were true?**
- **Example**: To determine if the Belgian 1 Euro coin is fair you flip it 100 times and record the number of Heads. What is the test statistic? What are the Null and alternative hypotheses?

Test statistics and evidence

- **Example**: To determine if the Belgian 1 Euro coin is fair you flip it n times and record the number of Heads. What is the test statistic? What are the Null and alternative hypotheses?

Test statistic: $\hat{p} = \frac{\#\text{Heads}}{100}$ proportion of heads in our data.

$H_0: p = 0.5$ Under the null, $\hat{p} = \frac{X}{n}$ $X \sim \text{Bin}(n=100, p=0.5)$

null!

$H_1: p \neq 0.5$

$\frac{p(1-p)}{n}$

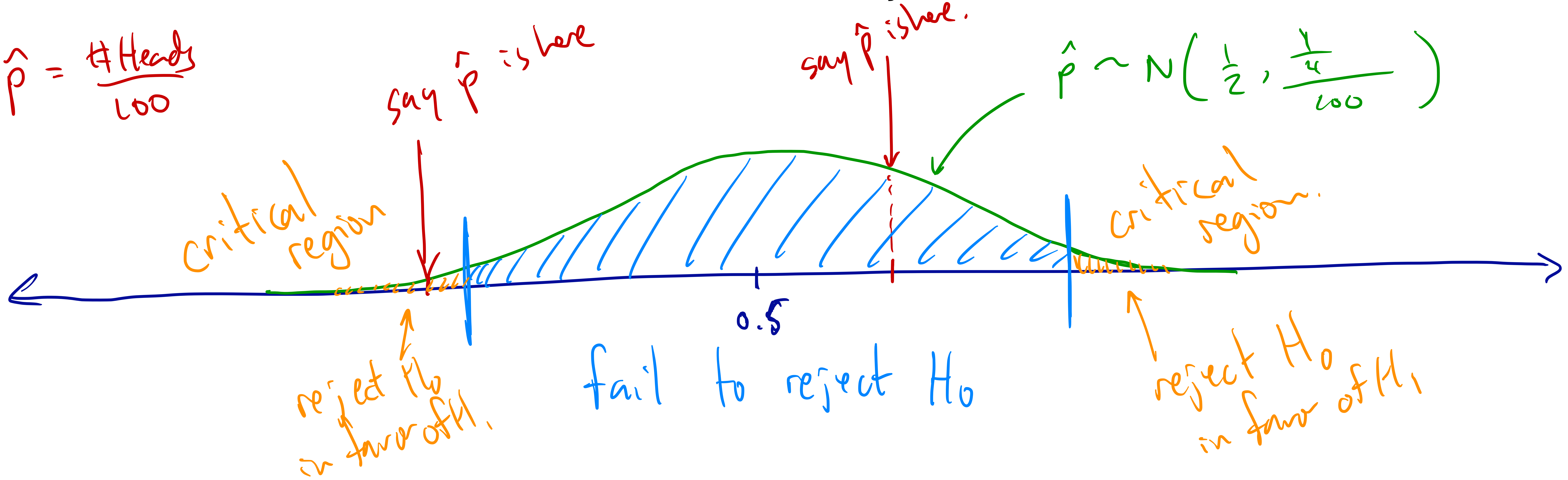
Under the null, $\hat{p} \sim N\left(0.5, \frac{0.5(1-0.5)}{100}\right)$

How likely is it that our ~~actual~~ observed $\hat{p}$ occurs under $N\left(0.5, \frac{0.5(1-0.5)}{100}\right)$

Test statistics and evidence

- **Example**: To determine if the Belgian 1 Euro coin is fair you flip it n times and record the number of Heads. What is the test statistic? What are the Null and alternative hypotheses?

- **Question**: What would it take to convince you that the coin is not fair?



Test statistics and evidence

- **Example**: To determine if the Belgian 1 Euro coin is fair you flip it n times and record the number of Heads. What is the test statistic? What are the Null and alternative hypotheses?

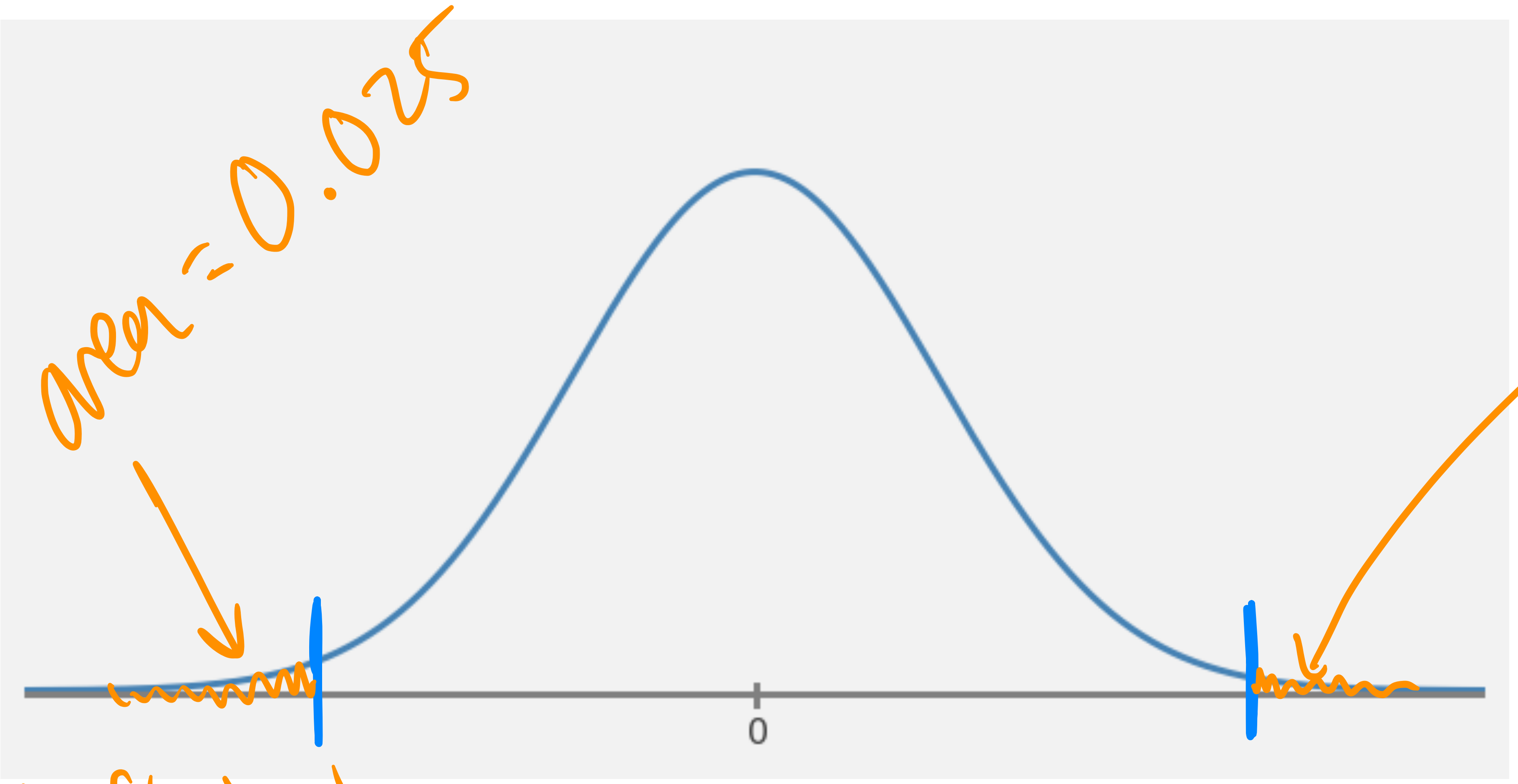
- **Question**: What would it take to convince you that the coin is not fair?

Convert to a std. normal Z. $\alpha = 0.05$

two-tailed test

area = 0.025

area 0.025



left tail

right tail

Rejection regions and significance level

- **Example**: To determine if the Belgian 1 Euro coin is fair you flip it n times and record the number of Heads.
- **Def**: The **rejection region** is a range of values of the test statistic that would lead you to **reject** the Null hypothesis.
- **Def**: The **significance level** α indicates the largest probability of the test statistic occurring under the Null hypothesis that would lead you to reject the Null hypothesis.

$\alpha = 0.10$

really intuitive!

see next slides

two-tailed test.

rejection region

critical values

Detecting Biased Coins

- **Example**: To test if the Belgian 1 Euro coin is fair you flip it 100 times and get 38 Heads. Do you reject the Null at the .05 significance level or not?

$H_0: p = 0.5$

$H_1: p \neq 0.5$

$\alpha = 0.05$

$z_{\alpha/2} = 1.96$

$-z_{\alpha/2} = -1.96$

$$\hat{p} \sim N\left(\frac{1}{2}, \frac{\frac{1}{2}\left(1-\frac{1}{2}\right)}{100}\right)$$

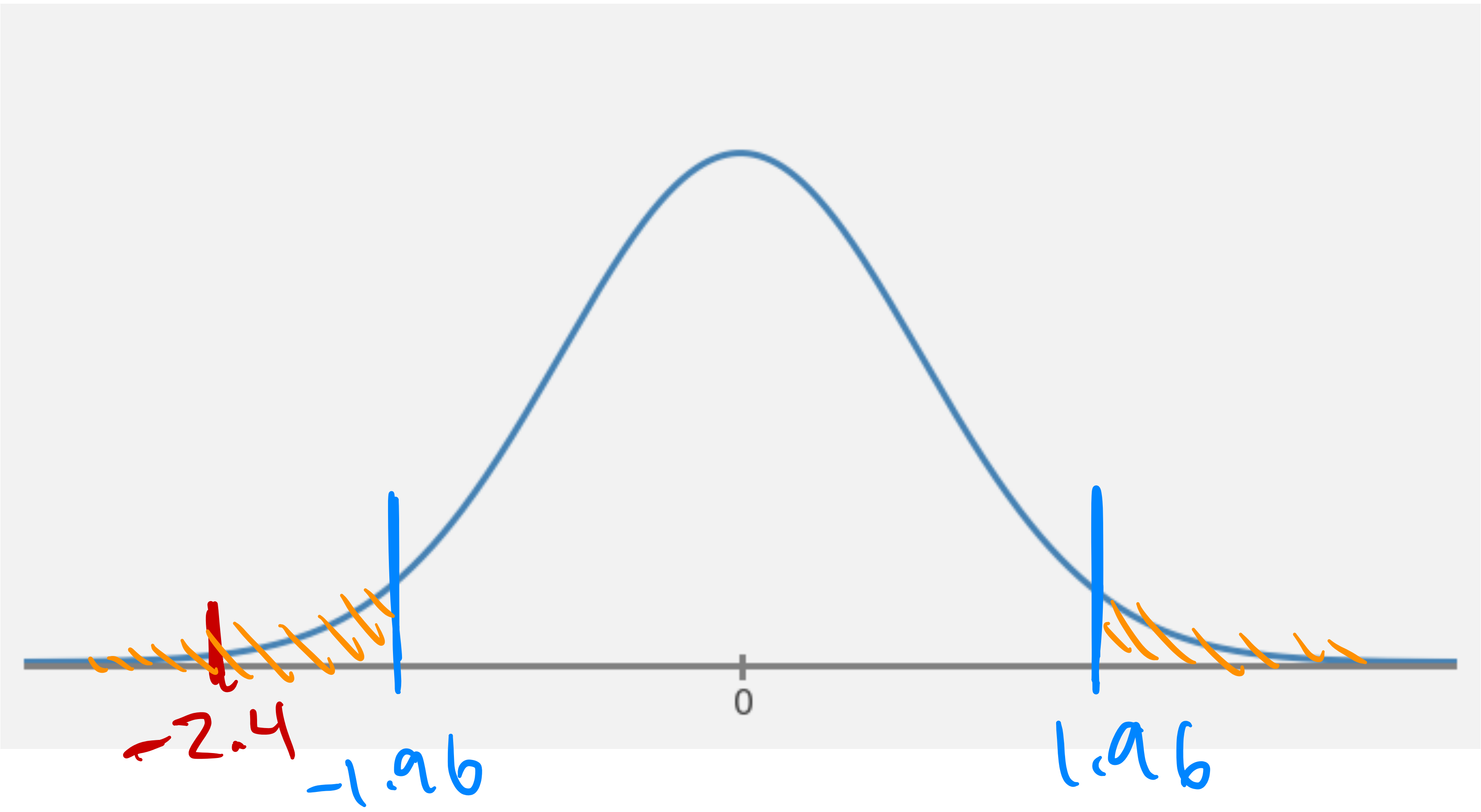
CLT

$$\sigma^2 = \frac{\frac{1}{2}\cdot\frac{1}{2}}{100}$$

$$\sigma = \frac{\frac{1}{2}}{10} = \frac{1}{20}$$

$$Z = \frac{X-\mu}{\sigma}$$

$$= \frac{0.38 - 0.5}{\frac{1}{20}} = -2.4$$



Detecting Biased Coins

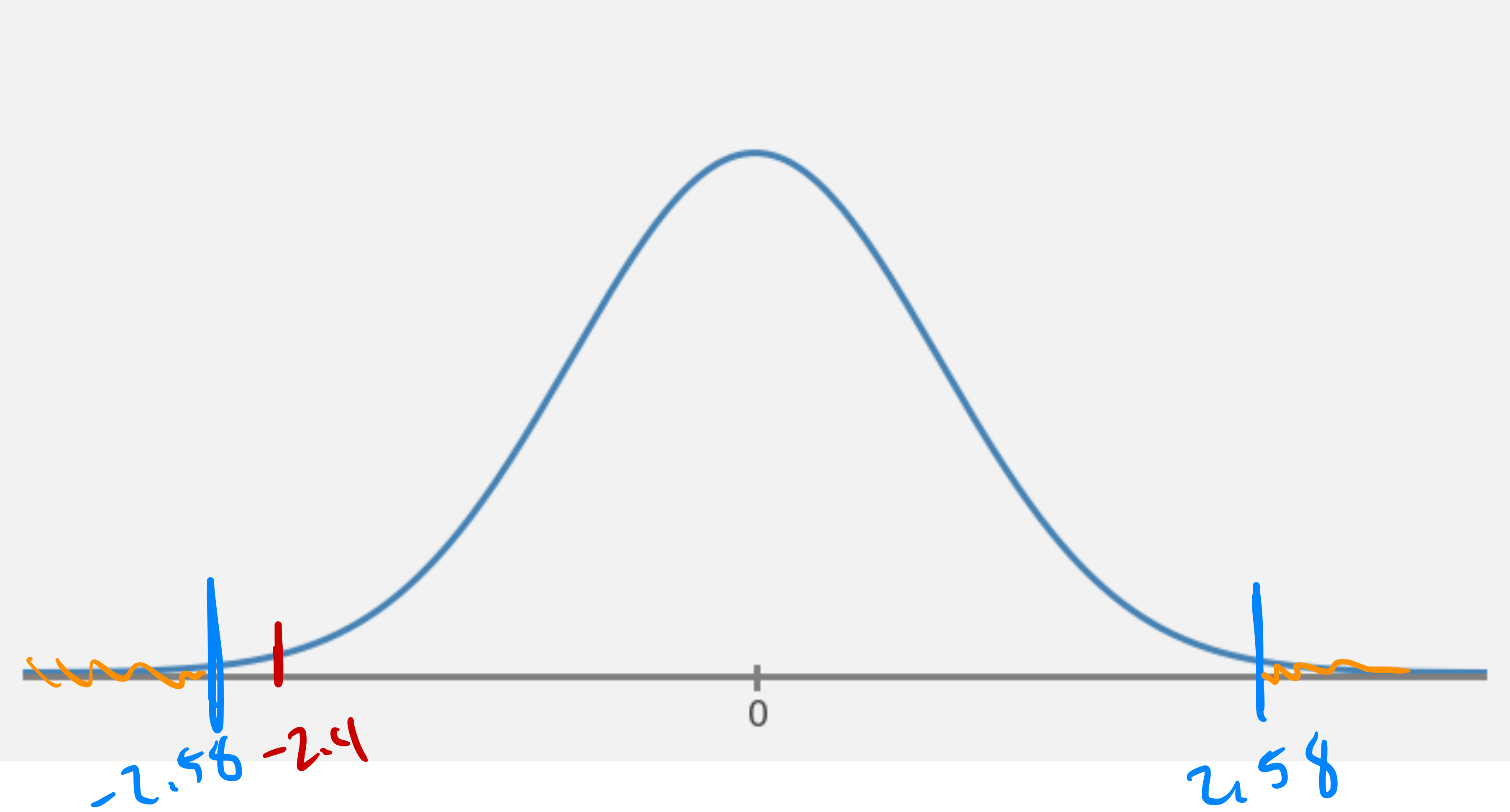
- **Example**: To test if the Belgian 1 Euro coin is fair you flip it 100 times and get 38 Heads. Do you reject the Null at the .01 significance level or not?

$\alpha = 0.01$ $\quad z_{\alpha/2} = z_{0.005} = 2.58$

$-z_{0.005} = -2.58$

prev slide:

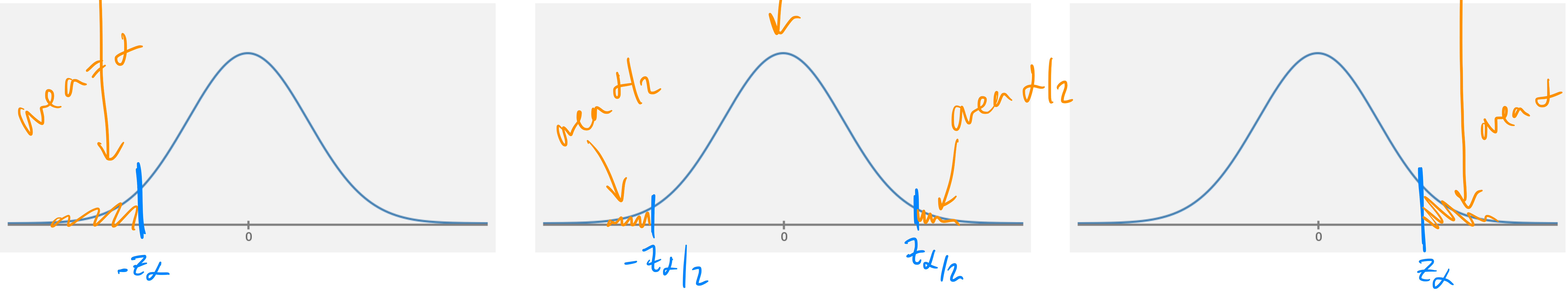
test statistic = -2.4



Different tests for different hypotheses

- The coin example was an exampled of a **two-tailed hypothesis test**, because we would have rejected the Null hypothesis had the coin been been biased towards heads OR tails.

Alternative Hypothesis	Rejection Region
$H_1 : \theta > \theta_0$	$z \geq z_\alpha$
$H_1 : \theta < \theta_0$	$z \leq -z_\alpha$
$H_1 : \theta \neq \theta_0$	$z \leq -z_{\alpha/2}$ or $z \geq z_{\alpha/2}$



Switching advertising strategies

$H_0: \mu = 200$ $\qquad$ $H_1: \mu > 200$

- **Example**: Suppose a company is considering hiring a new outside advertising company to help generate traffic to their website. Under their current advertising they get, on average, 200 thousand hits per day with a standard deviation of 50 thousand hits per day. You decide to hire the new ad company for a 30 day trial. During those 30 days, your website gets 210 thousand hits per day. Perform a hypothesis test to determine if the new ad campaign outperforms the old one at the .05 significance level.

$z_\alpha = z_{.05} = 1.645$

CLT $N\left(\mu, \frac{\sigma^2}{n}\right)$

If null H_0 were true

$$\bar{X} \sim N\left(200, \frac{50^2}{30}\right)$$

$$Z = \frac{\bar{X} - \mu}{\sigma} \sim N(0,1)$$

$$\frac{210 - 200}{50/\sqrt{30}} = 1.1$$