Assignment 3: Going Viral

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MATH-466

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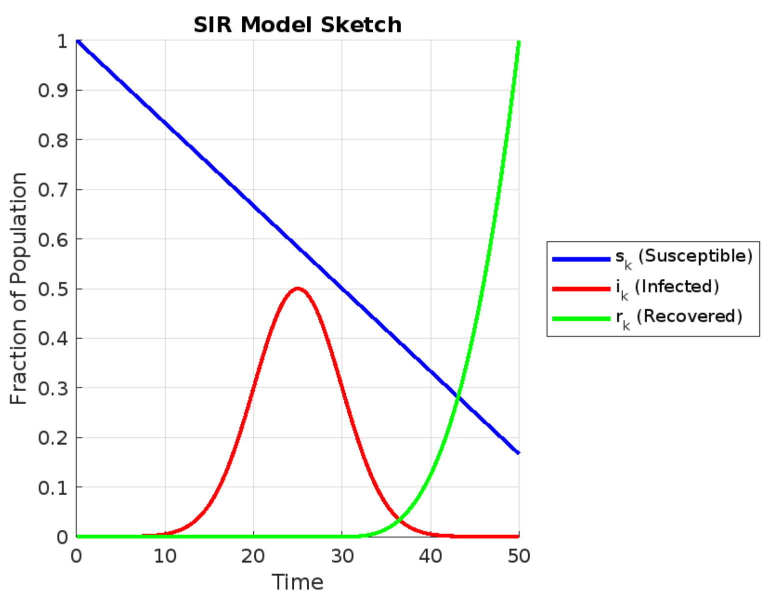
1. Estimated Behavior of Variables

- The susceptible fraction, s_k, gradually decreases over time as individuals become infected.
- The infected fraction, i_k, initially increases as new infections occur, reaches a peak, and decreases as people recover.
- The recovered fraction, r_k, is increasing over time as infected individuals recover.

2. Sketch of the Graphs



3. Conservation of Population

At each sampling time k, the total number of individuals is conserved:

$$S_k + I_k + R_k = N,$$

and when expressed as fractions,

$$s_k + i_k + r_k = 1,$$

since $s_k = \frac{S_k}{N}$, $i_k = \frac{I_k}{N}$, and $r_k = \frac{R_k}{N}$.

4. Difference Equation for the Susceptible Compartment

Based on the assumption that each infected individual makes b contacts per day and that the fraction of these contacts with susceptibles is s_{k-1}, the number of new infections during one sampling interval is:

$$\text{New infections} = b\, s_{k-1}\, I_{k-1}.$$

The change in the susceptible compartment satisfies:

$$S_k - S_{k-1} = -b\, s_{k-1}\, I_{k-1}.$$

5. Difference Equation for the Infected Compartment

The infected category increases by the number of new infections and decreases by the number of recoveries. With a fraction a of infected individuals recovering per day, the recoveries per time interval are:

$$\text{Recoveries} = a\, I_{k-1}.$$

The change in the infected compartment satisfies:

$$I_k - I_{k-1} = b\, s_{k-1}\, I_{k-1} - a\, I_{k-1}.$$

6. Difference Equation for the Recovered Compartment

Since every individual leaving the infected group recovers, the recovered compartment increases by:

$$R_k - R_{k-1} = a\, I_{k-1}.$$

7. Converting to Fractions of the Total Population

Divide each of the equations by N:

$$\begin{aligned} s_k - s_{k-1} &= -b\, s_{k-1}\, i_{k-1}, \\ i_k - i_{k-1} &= b\, s_{k-1}\, i_{k-1} - a\, i_{k-1}, \\ r_k - r_{k-1} &= a\, i_{k-1}. \end{aligned}$$

8. Initial Conditions

Given the population data for New York City during the late 1960's:

$$S_0 = 7{,}899{,}990, \quad I_0 = 10, \quad R_0 = 0, \quad N = 7{,}900{,}000,$$

the initial fractions are:

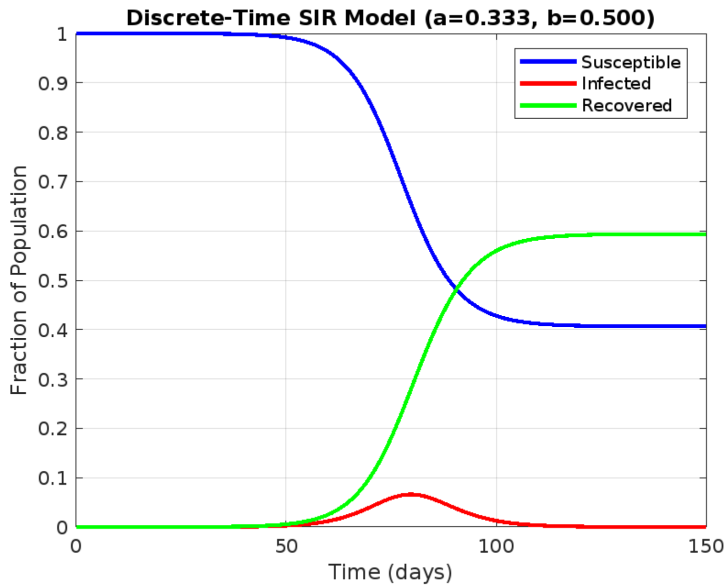
$$\begin{aligned} s_0 &= \frac{S_0}{N} = \frac{7{,}899{,}990}{7{,}900{,}000} \approx 0.9999987, \\ i_0 &= \frac{I_0}{N} = \frac{10}{7{,}900{,}000} \approx 1.2658 \times 10^{-6}, \\ r_0 &= \frac{R_0}{N} = 0. \end{aligned}$$

9. Complete SIR Model

The complete model is given by:

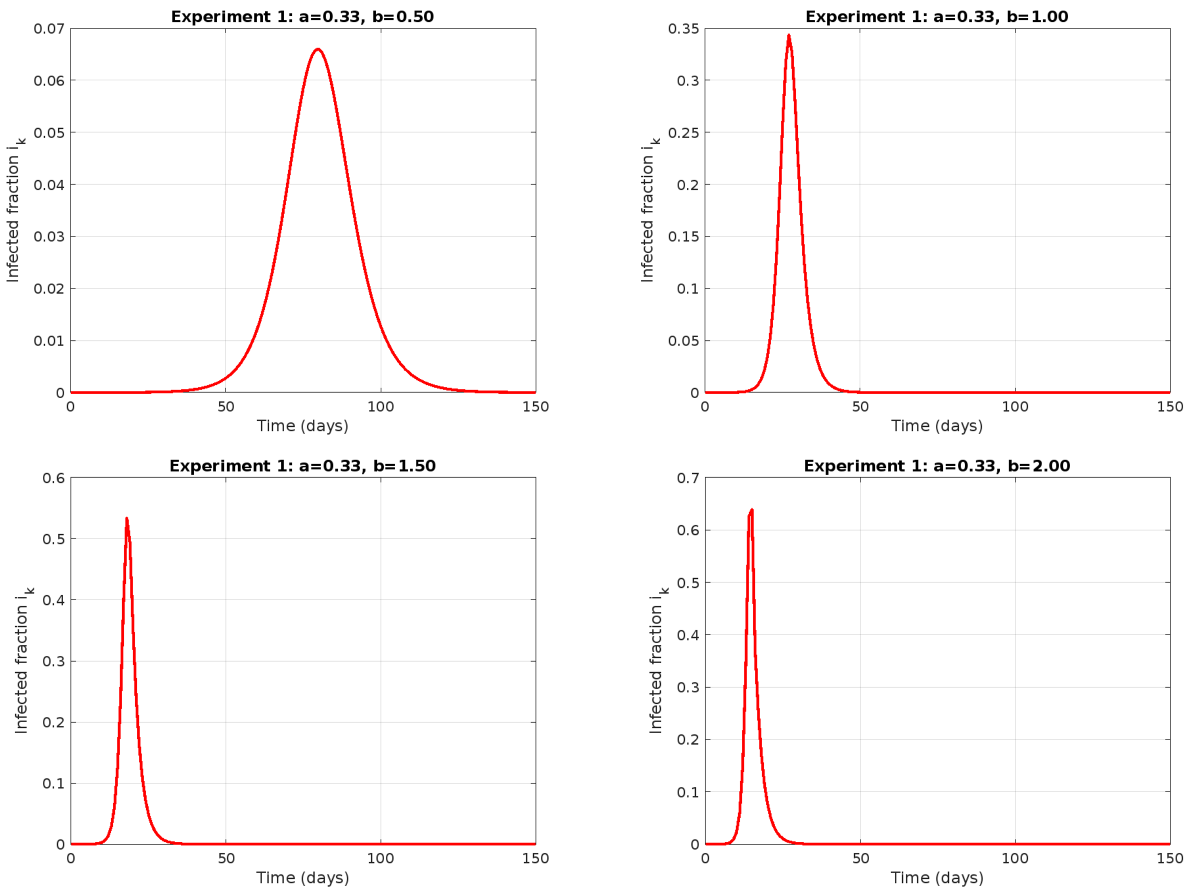
$$\begin{aligned} s_k - s_{k-1} &= -b\, s_{k-1}\, i_{k-1}, & s_0 &= \frac{7{,}899{,}990}{7{,}900{,}000}, \\ i_k - i_{k-1} &= b\, s_{k-1}\, i_{k-1} - a\, i_{k-1}, & i_0 &= \frac{10}{7{,}900{,}000}, \\ r_k - r_{k-1} &= a\, i_{k-1}, & r_0 &= 0. \end{aligned}$$

10. Run Simulation with a=1/3 and b=1/2



11. Experiment with b

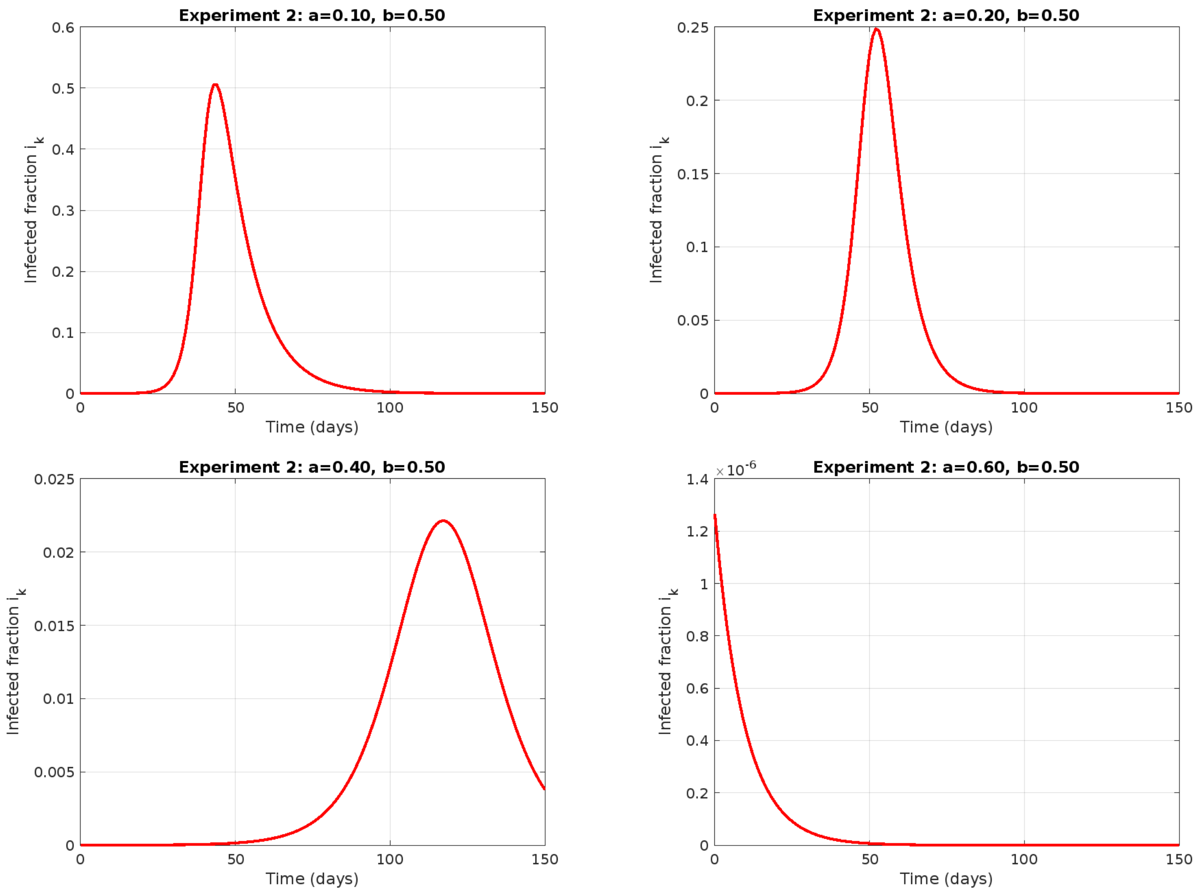
Below, i_k is graphed with $a = \{1/3\}$ and varying values of $b = \{0.5, 1, 1.5, 2\}$. As b increases, the peak happens earlier in time, and it becomes much higher. and the maximum proportion of individuals infected becomes higher.



12. Explain Changes in i_k from b

This makes sense because b represents how fast susceptible individuals become infected. It is the infection rate per contact. As this value increases, the spread of disease becomes faster.

13. Experiment with a



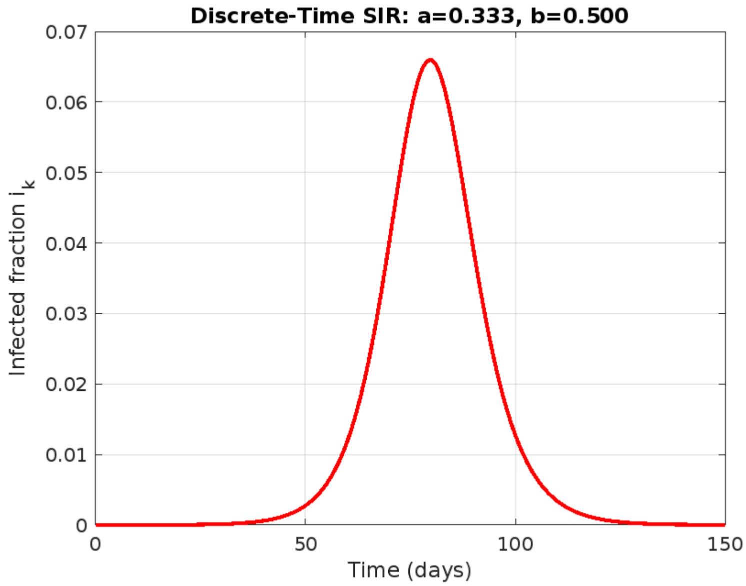
14. Explain the Changes in i_k from a

The variable a represents the recovery rate. The simulation outcomes for varying values of a make sense. When recovery is slow, the infection builds up to a high peak and lasts longer. When recovery is fast, fewer people are infected at once, and the epidemic is contained quickly.

15. Change in Graph Character

Between $a = 0.4$ and $a = 0.6$, the graph loses its typical bell curve. With this higher recovery rate, people recover before they have the chance to infect others, so the disease doesn't continue to spread to more of the population.

16. Compare Model With Data



The observed death data, when graphed, show a single wave over about 13 weeks with a peak around weeks 6–7, and the SIR model with $a = 1/3$ and $b = 0.5$ produces a similarly shaped single wave of infection that starts near zero, rises to a peak, and then declines back toward zero. Flu deaths typically lag behind infections. The model has a peak in infected individuals around day 75, which is around weeks 10-11. It makes sense that there would be a lag of about 4 weeks between peak infections and peak deaths.

17. Compare with Expectations

My expectation for i_k was pretty accurate. I think this is because it's easy to assume a disease outbreak will have a peak and then settle down. I wasn't as accurate in my initial sketch for s_k and r_k. In my sketch, they both went to the opposite extreme of the graph, instead of leveling out.

a. What do you think about the relatively low level of infection at the peak of the epidemic?

- As soon as infections begin to rise, the susceptible population starts shrinking. Fewer susceptible people in the population means fewer opportunities for the disease to spread, which slows down the infection rate

b. Can you see how a low peak level of infection can nevertheless lead to more than half the population getting sick? Explain.

- The cumulative number of infections will be much larger than the peak infection level at any moment because infections are spread out over time rather than all happening at once.

18. Preventing an Epidemic

$i_k > i_{k-1}$ means the epidemic is growing at step k. Conversely, if $i_k - i_{k-1}$ remains negative (or zero) from $k = 0$ onward, then i_k never exceeds its previous value, and no epidemic can get started. Preventing the outbreak is equivalent to ensuring

$$i_k - i_{k-1} \leq 0 \quad \text{for all } k \geq 0.$$

19. Infected-Fraction Equation for $i_k - i_{k-1}$ Sign

The infected-fraction difference equation is

$$i_k - i_{k-1} \;=\; b\,s_{k-1}\,i_{k-1} \;-\; a\,i_{k-1}.$$

Factor out i_{k-1}:

$$i_k - i_{k-1} \;=\; i_{k-1}\big(b\,s_{k-1} \;-\; a\big).$$

The factor i_{k-1} is always nonnegative. The other factor, $(b\,s_{k-1} - a)$, can be positive or negative depending on s_{k-1}. If $b\,s_{k-1} > a$, then $i_k - i_{k-1} > 0$, and the infection grows. If $b\,s_{k-1} < a$, then $i_k - i_{k-1} < 0$, and the infection declines.

20. s_k is a Decreasing Function

The susceptible update equation is

$$s_k \;=\; s_{k-1} \;-\; b\,s_{k-1}\,i_{k-1}.$$

Therefore,

$$s_k - s_{k-1} \;=\; -\,b\,s_{k-1}\,i_{k-1} \;\le\; 0,$$

meaning $s_k \le s_{k-1}$. Since s_k is decreasing in k, so its largest value is s_0. Because the infected difference equation can be written as

$$i_k - i_{k-1} \;=\; i_{k-1}\,\big(b\,s_{k-1} - a\big),$$

if $b\,s_0 < a$ at $k = 0$, then $i_1 - i_0 < 0$. For following times, $s_{k-1} \;\le\; s_0$, so $b\,s_{k-1} \;\le\; b\,s_0 < a$, and $i_k - i_{k-1}$ remains negative. Thus the infection never grows.

21. Show that $i_1 - i_0 \;=\; (b\,s_0 - a)\,i_0\,\tau$ and explain why, if s_0 is less than $1/c$, then no epidemic can develop.

Assume each time step has length τ. The infected-fraction difference equation is

$$i_{k+1} - i_k \;=\; \big(b\,s_k\,i_k - a\,i_k\big)\,\tau \;=\; i_k\,\big(b\,s_k - a\big)\,\tau.$$

At $k = 0$, we get

$$i_1 - i_0 \;=\; i_0\,\big(b\,s_0 - a\big)\,\tau.$$

Define the contact number as $c = \frac{b}{a}$, so $b = c\,a$. If $s_0 < \frac{1}{c}$, then

$$b\,s_0 - a \;=\; a\,\big(c\,s_0 - 1\big) \;<\; 0,$$

hence $i_1 - i_0 < 0$. As s_k is monotonically decreasing, $b\,s_{k-1} \le b\,s_0 < a$ for all $k \ge 1$, so $i_k - i_{k-1}$ remains negative. Therefore, no epidemic can develop.

22. Measles Vaccine Effectiveness

From 1912 to 1928, the contact number for measles in the U.S. was observed to be

$$c = 12.8.$$

We assume it remains the same today. The fraction of the population that must be vaccinated to prevent an epidemic can be found by requiring the initial susceptible fraction s_0 to be below $1/c$, so $s_0 < 1/c$. If the vaccine is 100% effective, vaccinated individuals move directly into the recovered category, reducing the susceptible population.

Because $s_0 + v_0 = 1$, where v_0 is the fraction vaccinated, we need $s_0 = 1 - v_0 < \frac{1}{c}$. Hence,

$$1 - v_0 \;<\; \frac{1}{12.8}$$

$$v_0 \;>\; 1 - \frac{1}{12.8} \;\approx\; 0.9219.$$

At least 92.19% of the population must be vaccinated to prevent a measles epidemic.

23. Vaccine 95% Effective

If the vaccine is only 95% effective, then a fraction v_0 of the population is vaccinated, but only $0.95\,v_0$ of the population actually becomes immune. The fraction of the population that remains susceptible is

$$s_0 \;=\; 1 \;-\; 0.95\,v_0.$$

$s_0 < \frac{1}{c}$ to avoid an epidemic, so

$$1 - 0.95\,v_0 < \frac{1}{12.8}$$
$$0.95\,v_0 > 1 - \frac{1}{12.8}$$
$$v_0 > \frac{1}{0.95}\Big(1 - \frac{1}{12.8}\Big) \;\approx\; 0.97.$$

At least 97% of the population must receive the vaccine to ensure herd immunity against measles.

24. Modeling Steps for SIR Model

- **Identify Population Categories:** Partition the population into Susceptible, Infected, and Recovered based on the initial conditions.
- **Define Variables:** Let S_k, I_k, R_k be the number of individuals in each category at time t. Make these values proportional to the population. $s_k = \frac{S_k}{N}$, $i_k = \frac{I_k}{N}$, and $r_k = \frac{R_k}{N}$. Set a as the recovery rate and set b as the infection rate.
- **Set Assumptions:** (a) total population is constant, (b) infection arises through contact between susceptibles and infecteds at a certain rate, (c) infected individuals recover at a fixed rate and become immune.
- **Formulate Equations:** Translate these assumptions into either continuous ODEs or difference equations capturing how S, I, R change over time.

25. Outcomes of Trace Infection

According to the SIR model, when a trace of infection is introduced:

- **Either** the infection dies out quickly without growing,
- **Or** it grows into an epidemic wave, infecting a significant fraction of the population before subsiding.

We can see which scenario occurs with the infection fraction i_k: either initially increases or remains stable/declines.

26. Epidemic: Nearly Everyone Infected?

An epidemic doesn't mean almost everyone gets the disease. Sometimes only a moderate fraction of the population becomes infected before the epidemic dies out. Once some fraction of individuals have recovered (and hence are immune), the number of susceptible individuals drops low enough that transmission slows. Eventually, $S(t)$ can become small enough that $b\,S(t)$ drops below the recovery rate a, causing $I(t)$ to decline again.

27. Peak vs Cumulative Infections

It depends on the different growth factors. Here, we used a and b. During experimentation done with these factors, when b is relatively lower and a is higher, the i_k curve was flattened. It reduced the amount of individuals infected at one time, at the maximum. However, it was more spread out.

28. Solutions Without Explicit Formulas

In the SIR model, explicit solution formulas for $S(t)$, $R(t)$, and $I(t)$ can be more complicated. Instead, the key idea is to:

- Express the infection fraction $I(t)$ in terms of the susceptible fraction $S(t)$ (or vice versa), often integrating a relationship like $\frac{dI}{dS}$
- Use the initial conditions to find how variables change in relation to eachother

29. Contact Number

The contact number, $c = \frac{b}{a}$, is the average number of "close contacts" (sufficient to spread the disease) that each infected individual makes over the infectious period. It measures how *contagious* a disease is in a given population.

30. Herd Immunity

Definition: Herd immunity occurs when a large enough fraction of the population is immune that an outbreak cannot sustain itself, i.e. the disease cannot find enough new susceptible individuals to create an epidemic.
Vaccination Program: By vaccinating a certain fraction of the population (so that $s_0 < 1/c$), we reduce the susceptible fraction so that i_k never increases. Even if a trace infection enters the population, it won't start an epidemic.

31. Polio Eradication vs. Measles

The contact number for poliomyelitis in the U.S. in 1955 was 4.9. For measles it was 12.8. We have eradicated poliomyelitis but not measles. Because c was smaller for polio, the fraction of the population needing vaccinations to achieve herd immunity was lower than for measles. With poliomyelitis, the $1/c \approx 1/4.9$ was easier to achieve.