Mathematics: Analysis and Interpretations IA

OPTIMIZING AN INVESTMENT PORTFOLIO AND MODELLING ITS EXPECTED RETURNS

Introduction:

I have always been very interested in finance and wanted to learn how to invest. Recently, I took an online course on financial markets by Yale University and was introduced to the Modern Portfolio Theory (MPT). The MPT was introduced by Henry Markowitz in 1952, and it is an investment strategy that uses the correlation between stocks to help investors maximize return for a given risk, under the assumptions that you cannot view assets in a portfolio in isolation and that it is very difficult to forecast future investment returns. Hence, investors must look at long-term historical returns to approximate future returns and analyse how the potential returns and risk level of each stock relate to each other[1].

In this investigation, I will apply the MPT to build a 1-million-dollar investment portfolio and model its future returns.

Key concepts:

The expected return on an investment is *the profit or loss that an investor anticipates on an investment based on historical rates of return*[2]. The risk of a stock is *the chance that an investment's actual return will differ from what is expected*[3]. This can be determined using the standard deviation of the stock, which is the *average amount by which individual data points differ from the mean*[4]. The risk-return trade-off is the idea *that investors can only achieve higher*

[1] https://www.forbes.com/advisor/investing/modern-portfolio-theory/
[2] https://www.investopedia.com/terms/e/expectedreturn.asp
[3] https://modelinvesting.com/articles/the-risk-return-trade-off/
[4] https://www.investopedia.com/ask/answers/021915/how-standard-deviation-used-determine-risk.asp

returns if they accept higher levels of risk[5]. Covariance is *a measure of the relationship between two random variables, evaluating how much the variables change together*[6].

Selecting stocks for the analysis:

The MPT highlights diversification to reduce risk, selecting stocks with low or negative correlations (covariances). Thus, I chose one stock from each of the 11 stock market sectors because it is likely they would be affected differently by economic conditions.

ENERGY: Exxon Mobil Corporation (XOM), MATERIALS: Southern Copper Corp (SCCO), INDUSTRIALS: Caterpillar Inc. (CAT), UTILITIES: Southern Co (SO), HEALTHCARE: Johnson & Johnson (JNJ). FINANCIALS: JPMorgan Chase & Co (JPM), CONSUMER DISCRETIONARY: Amazon.com Inc (AMZN), CONSUMER STAPLES: Coca-Cola Co (KO), INFORMATION TECHNOLOGY: Microsoft Corp (MSFT), COMMUNICATION SERVICES: Alphabet Inc (GOOG), REAL ESTATE: Welltower Inc (WELL).

Data collection and calculations:

I collected daily returns of the 11 selected stocks over the last 5 years (July 2019 to July 2024) from Yahoo finance. This made for 1257 data points, which formed a statistically significant sample.

To find the optimal portfolio, one must consider both the returns and variances of the portfolio when attributing the weights to each asset. Harry Markowitz coined the Sharpe ratio in the MPT, which is the ratio between the expected excess return of the portfolio and the standard deviation of the portfolio. Therefore, when the Sharpe ratio is maximized, we have maximized the return

[5] https://www.sciencedirect.com/topics/economics-econometrics-and-finance/risk-return-tradeoff
https://www.investopedia.com/terms/c/covariance.asp#:~:text=Covariance%20evaluates%20how%20the%20mean,said%20to%20have%20positive%20covariance.

[6] https://corporatefinanceinstitute.com/resources/data-science/covariance/

per unit of risk[7]. I will attempt to find the weights for each stock in the portfolio that maximizes the Sharpe ratio.

$$Sharpe\ Ratio = \frac{\bar{R}_p - R_f}{\sigma_p}$$

Where: $\bar{R}_p$= expected return of the portfolio; R_f= risk-free rate, which is the rate of return of an investment with zero risk; $(\bar{R}_p - R_f)$ = expected excess return of the portfolio; σ_p= standard deviation of the portfolio

The expected return of the entire portfolio is the weighted average of the expected returns of the individual stocks in the portfolio. For an 11-asset portfolio, this is represented by the formula:[8]

$$\bar{R}_p = \sum_{i=1}^{11} (w_i \bar{R}_i)$$

Where: i= asset; w_i= weight of asset i; $\bar{R}_i$= expected return of asset i

I began by calculating the expected return of each asset. Assuming that, because of the large sample size, the data collected accounts for the variations I can expect in future returns of the stocks, I used the average return as a measure of the expected return of the stocks. The formula for the average percentage return of a stock, $\bar{R}_i$, is:

$$\bar{R}_i = \frac{1}{n} \sum_{t=1}^{n} R_t$$

Where: R_t= daily percentage return, $R_t = \frac{P_t - P_{t-1}}{P_{t-1}} \times 100$

Where: R_t= return at day t; P_t= price at day t; P_{t-1}= price at the previous period

[7] https://www.investopedia.com/terms/s/sharperatio.asp
[8] https://www.investopedia.com/ask/answers/042815/what-difference-between-expected-return-and-standard-deviation-portfolio.asp#:~:text=The%20expected%20return%20is%20calculated,x%20Expected%20Return)...

I used excel to calculate the daily percentage return and the average return due to the large number of data points, but, manually, this would look like:

Calculating the periodic return for t=2 for Microsoft:

Table 1: Sample Results Table for Daily Percentage Return of Microsoft Stock

Date	**Adj Close for MSFT**	**Daily Percentage Return (%)**
7/2/19	130.255753	
7/3/19	131.095032	$\left(\frac{131.10}{130.26} - 1\right) \times 100 = 0.644$

Calculating the expected return for Microsoft:

$$\bar{R}_{MSFT} = \frac{1}{1257}(0.644 - 0.291 - 0.0729 + \cdots - 1.30 + 2.19 + 0.558 = 0.117\%$$

See expected returns for all stocks, in decimal form, in the appendix (page 24).

The Sharpe ratio also uses the standard deviation of the portfolio. The standard deviation of an entire portfolio depends on the individual standard deviations of each stock and the correlation between stocks. This is because when stocks move opposite each other, they offset each other's risk, decreasing the overall risk of the portfolio.

The formula for calculating the variance of a 2-asset portfolio is:[9]

$${\sigma_p}^2 = {w_1}^2{\sigma_1}^2 + {w_2}^2{\sigma_2}^2 + 2w_1w_2Cov_{1,2}\sigma_1\sigma_2$$

Where: w_1= weight of asset 1; w_2= weight of asset 2; $Cov_{1,2}$= covariance between assets 1 and 2; σ_1= standard deviation of asset 1, σ_2= standard deviation of asset 2

The standard deviation would be the square root of this.

[9] https://www.investopedia.com/ask/answers/071515/how-can-i-measure-portfolio-variance.asp#:~:text=Portfolio%20variance%20is%20a%20measure,between%20securities%20in%20the%20portfolio.

This involves calculating the covariance between all pairs of stocks as well as their individual standard deviations. However, the variance of the portfolio can also be expressed in matrix form without having to calculate standard deviation for each asset, streamlining the process:

$$\sigma_p{}^2 = w^T \Omega \mathrm{w}$$

Where, for an 11-asset portfolio:

$$\text{Weights Vector, } w = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \\ w_5 \\ w_6 \\ w_7 \\ w_8 \\ w_9 \\ w_{10} \\ w_{11} \end{pmatrix}$$

Covariance Matrix, Ω =

$$\begin{pmatrix}
Cov_{1,1} & Cov_{1,2} & Cov_{1,3} & Cov_{1,4} & Cov_{1,5} & Cov_{1,6} & Cov_{1,7} & Cov_{1,8} & Cov_{1,9} & Cov_{1,10} & Cov_{1,11} \\
Cov_{2,1} & Cov_{2,2} & Cov_{2,3} & Cov_{2,4} & Cov_{2,5} & Cov_{2,6} & Cov_{2,7} & Cov_{2,8} & Cov_{2,9} & Cov_{2,10} & Cov_{2,11} \\
Cov_{3,1} & Cov_{3,2} & Cov_{3,3} & Cov_{3,4} & Cov_{3,5} & Cov_{3,6} & Cov_{3,7} & Cov_{3,8} & Cov_{3,9} & Cov_{3,10} & Cov_{3,11} \\
Cov_{4,1} & Cov_{4,2} & Cov_{4,3} & Cov_{4,4} & Cov_{4,5} & Cov_{4,6} & Cov_{4,7} & Cov_{4,8} & Cov_{4,9} & Cov_{4,10} & Cov_{4,11} \\
Cov_{5,1} & Cov_{5,2} & Cov_{5,3} & Cov_{5,4} & Cov_{5,5} & Cov_{5,6} & Cov_{5,7} & Cov_{5,8} & Cov_{5,9} & Cov_{5,10} & Cov_{5,11} \\
Cov_{6,1} & Cov_{6,2} & Cov_{6,3} & Cov_{6,4} & Cov_{6,5} & Cov_{6,6} & Cov_{6,7} & Cov_{6,8} & Cov_{6,9} & Cov_{6,10} & Cov_{6,11} \\
Cov_{7,1} & Cov_{7,2} & Cov_{7,3} & Cov_{7,4} & Cov_{7,5} & Cov_{7,6} & Cov_{7,7} & Cov_{7,8} & Cov_{7,9} & Cov_{7,10} & Cov_{7,11} \\
Cov_{8,1} & Cov_{8,2} & Cov_{8,3} & Cov_{8,4} & Cov_{8,5} & Cov_{8,6} & Cov_{8,7} & Cov_{8,8} & Cov_{8,9} & Cov_{8,10} & Cov_{8,11} \\
Cov_{9,1} & Cov_{9,2} & Cov_{9,3} & Cov_{9,4} & Cov_{9,5} & Cov_{9,6} & Cov_{9,7} & Cov_{9,8} & Cov_{9,9} & Cov_{9,10} & Cov_{9,11} \\
Cov_{10,1} & Cov_{10,2} & Cov_{10,3} & Cov_{10,4} & Cov_{10,5} & Cov_{10,6} & Cov_{10,7} & Cov_{10,8} & Cov_{10,9} & Cov_{10,10} & Cov_{10,11} \\
Cov_{11,1} & Cov_{11,2} & Cov_{11,3} & Cov_{11,4} & Cov_{11,5} & Cov_{11,6} & Cov_{11,7} & Cov_{11,8} & Cov_{11,9} & Cov_{11,10} & Cov_{11,11}
\end{pmatrix}$$

Transposed weights vector, $w^T =$

$$\begin{pmatrix} w_1 & w_2 & w_3 & w_4 & w_5 & w_6 & w_7 & w_8 & w_9 & w_{10} & w_{11} \end{pmatrix}$$

Which gives:

$$\sigma_p{}^2 = \sum_{i=1}^{11} w_1 Cov_{i,1} \times w_1 + \sum_{i=1}^{11} w_2 Cov_{i,2} \times w_2 + \sum_{i=1}^{11} w_3 Cov_{i,3} \times w_3 + \sum_{i=1}^{11} w_4 Cov_{i,4} \times w_4 + \sum_{i=1}^{11} w_5 Cov_{i,5} \times w_5 + \sum_{i=1}^{11} w_6 Cov_{i,6} \times w_6 + \sum_{i=1}^{11} w_7 Cov_{i,7} \times w_7 + \sum_{i=1}^{11} w_8 Cov_{i,8} \times w_8 + \sum_{i=1}^{11} w_9 Cov_{i,9} \times w_9 + \sum_{i=1}^{11} w_{10} Cov_{i,10} \times w_{10} + \sum_{i=1}^{11} w_{11} Cov_{i,11} \times w_{11}$$

See appendix pages 26 to 29 for full derivation of the formula.

The next step was to calculate the covariance of each stock pair. A positive covariance indicates the variables move in the same direction, and the greater the value, the stronger the positive relationship. Conversely, a negative covariance indicates the variables move in opposite directions, and the more negative the value, the stronger the inverse relationship. The formula for the population covariance between two variables X and Y with n data points (days) is:[10]

$$Cov(X, Y) = \frac{1}{n}\sum_{t=1}^{n}(X_t - \bar{X})(Y_t - \bar{Y})$$

This formula applies for when you have data for the entire population. However, I collected data from a specific number of days, which represents just a sample of the data for the returns of the stocks. Hence, the mean I calculate is a sample mean, which is only an estimate of the population mean. Bessel's correction changes the denominator of the formula to $n - 1$ to account for this bias[11].

$$Cov(X, Y) = \frac{1}{1257 - 1}\sum_{t=1}^{1257}(X_t - \bar{X})(Y_t - \bar{Y}) = \frac{1}{1256}\sum_{t=1}^{1257}(X_t - \bar{X})(Y_t - \bar{Y})$$

I used excel to calculate this for all combination of stocks. As an example, a calculation using the stocks MSFT and XOM at t=1 and t=1257 would be as follows:

[10] https://www.investopedia.com/terms/c/covariance.asp
[11] https://towardsdatascience.com/the-reasoning-behind-bessels-correction-n-1-eeea25ec9bc9

$$Cov(MSFT, XOM) = \frac{1}{2-1}\sum_{t=1}^{1257}(MSFT_t - \overline{MSFT})(XOM_t - \overline{XOM})$$

When t=1:

$$(MSFT_1 - \overline{MSFT})(XOM_1 - \overline{XOM})$$

$$(0.06633 - 0.1169)(-0.01097 - 0.07578) = 0.004387$$

When t=1257:

$$(MSFT_{1257} - \overline{MSFT})(XOM_{1257} - \overline{XOM})$$

$$(-1.3029 - 0.1169)(0.1915 - 0.07578) = -0.16430$$

Thus, the covariance of the stocks considering only these two data points would be:

$$Cov(MSFT, XOM) = 0.004387 \times -0.1643 = -0.00072078$$

See covariance matrix in appendix page 29.

I inputted the covariance values into the covariance matrix (see page x), where: 1= Microsoft; 2 = Exxon Mobile; 3 = Southern Copper; 4 = Caterpillar Inc; 5 = Southern Co; 6 = Johnson & Johnson; 7 = JPMorgan Chase & Co; 8 = Amazon; 9 = Coca-Cola; 10 = Google; 11 = Welltower Inc. The variance formula would thus be:

$$\sigma_p{}^2 = (w_1 \quad w_2 \quad w_3 \quad w_4 \quad w_5 \quad w_6 \quad w_7 \quad w_8 \quad w_9 \quad w_{10} \quad w_{11}) \times$$

$$\begin{vmatrix} 0.0003639 & 0.0001092 & 0.0001722 & 0.0001376 & 0.0001209 & 0.0000959 & 0.0001667 & 0.0002867 & 0.0001129 & 0.0002880 & 0.0001659 \\ 0.0001092 & 0.0004675 & 0.0002739 & 0.0002655 & 0.0001356 & 0.0000803 & 0.0002550 & 0.0000734 & 0.0001207 & 0.0001237 & 0.0002362 \\ 0.0001722 & 0.0002739 & 0.0006025 & 0.0002961 & 0.0001092 & 0.0000806 & 0.0002549 & 0.0001605 & 0.0001105 & 0.0001763 & 0.0002211 \\ 0.0001376 & 0.0002655 & 0.0002961 & 0.0004132 & 0.0001204 & 0.0000899 & 0.0002596 & 0.0001088 & 0.0001169 & 0.0001453 & 0.0002155 \\ 0.0001209 & 0.0001356 & 0.0001092 & 0.0001204 & 0.0002781 & 0.0001192 & 0.0001538 & 0.0000753 & 0.0001430 & 0.0001071 & 0.0001931 \\ 0.0000959 & 0.0000803 & 0.0000806 & 0.0000899 & 0.0001192 & 0.0001571 & 0.0001047 & 0.0000585 & 0.0000919 & 0.0000816 & 0.0000822 \\ 0.0001667 & 0.0002550 & 0.0002549 & 0.0002596 & 0.0001538 & 0.0001047 & 0.0004036 & 0.0001188 & 0.0001459 & 0.0001702 & 0.0002790 \\ 0.0002867 & 0.0000734 & 0.0001605 & 0.0001088 & 0.0000753 & 0.0000585 & 0.0001188 & 0.0004888 & 0.0000707 & 0.0002890 & 0.0001147 \\ 0.0001129 & 0.0001207 & 0.0001105 & 0.0001169 & 0.0001430 & 0.0000919 & 0.0001459 & 0.0000707 & 0.0001738 & 0.0001055 & 0.0001614 \\ 0.0002880 & 0.0001237 & 0.0001763 & 0.0001453 & 0.0001071 & 0.0000816 & 0.0001702 & 0.0002890 & 0.0001055 & 0.0004064 & 0.0001705 \\ 0.0001659 & 0.0002362 & 0.0002211 & 0.0002155 & 0.0001931 & 0.0000822 & 0.0002790 & 0.0001147 & 0.0001614 & 0.0001705 & 0.0006396 \end{vmatrix} \times \begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \\ w_5 \\ w_6 \\ w_7 \\ w_8 \\ w_9 \\ w_{10} \\ w_{11} \end{pmatrix}$$

With this, I can begin to try to optimize the portfolio so it would maximize the Sharpe ratio.

Portfolio Optimization:

The risk-free rate, R_f, is part of the Sharpe ratio formula. It represents the theoretical rate of return of an investment with zero risk[12]. This can be estimated by the rate of return of government bonds, which are very low-risk investments, and adjusting the value to the rate of inflation[13]. For this investigation, I used the average daily return on the 5-year US Treasury Bonds as I am using daily statistics over the span of 5 years. I will be considering the average yearly US inflation rate from 2019 to 2023, as there is still not finished data for 2024. The annual return on a 5-year Treasury Bond was averaged at around 4.86% in 2023[14], when the average inflation rate was 3.97%,[15] however, I had to convert this into daily returns to use it together with my collected data.

$$Risk\ Free\ Rate = \frac{1 + Annual\ Returns\ of\ US\ Treasury\ Bond}{1 + Annual\ Inflation\ Rate)} - 1$$

$$Daily\ Risk\ Free\ Rate = (\frac{1 + Annual\ Returns\ of\ US\ Treasury\ Bond}{1 + Annual\ Inflation\ Rate)})^{\frac{1}{365}} - 1$$

$$Daily\ Risk\ Free\ Rate = (\frac{1 + 4.86}{1 + 3.97})^{\frac{1}{365}} - 1 = 0.000451\%$$

Before using the solver tool in excel to optimize the portfolio, I calculated the Sharpe ratio if I allocated the same weight to all stocks:

Weight of each stock, $w = \frac{1}{11} = 0.0909$

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[12] https://www.wallstreetprep.com/knowledge/risk-free-rate/#:~:text=The%20Risk%20Free%20Rate%20(rf)%20is%20the%20theoretical%20rate%20of,of%20the%20projected%20cash%20flows.

[13] https://www.wallstreetmojo.com/risk-free-rate-formula/

[14] https://finance.yahoo.com/quote/%5EFVX/

[15] https://www.macrotrends.net/global-metrics/countries/usa/united-states/inflation-rate-cpi

Return of the portfolio, $R_p = \sum_{i=1}^{11}(w_i \bar{R}_i) = (0.0909 \times 0.1169) + (0.0909 \times 0.0758) + (0.0909 \times 0.1305) + (0.0909 \times 0.1010) + (0.0909 \times 0.05756) + (0.0909 \times 0.02257) + (0.0909 \times 0.07776) + (0.0909 \times 0.08000) + (0.0909 \times 0.03780) + (0.0909 \times 0.11164) + (0.0909 \times 0.06526) = 0.07970\%$

Standard deviation of the portfolio, $\sigma_p =$

$$\left(\begin{pmatrix} 0.0909 & 0.0909 & 0.0909 & 0.0909 & 0.0909 & 0.0909 & 0.0909 & 0.0909 & 0.0909 & 0.0909 & 0.0909 \end{pmatrix} \times \begin{vmatrix} 0.0003639 & 0.0001092 & 0.0001722 & 0.0001376 & 0.0001209 & 0.0000959 & 0.0001667 & 0.0002867 & 0.0001129 & 0.0002880 & 0.0001659 \\ 0.0001092 & 0.0004675 & 0.0002739 & 0.0002655 & 0.0001356 & 0.0000803 & 0.0002550 & 0.0000734 & 0.0001207 & 0.0001237 & 0.0002362 \\ 0.0001722 & 0.0002739 & 0.0006025 & 0.0002961 & 0.0001092 & 0.0000806 & 0.0002549 & 0.0001605 & 0.0001105 & 0.0001763 & 0.0002211 \\ 0.0001376 & 0.0002655 & 0.0002961 & 0.0004132 & 0.0001204 & 0.0000899 & 0.0002596 & 0.0001088 & 0.0001169 & 0.0001453 & 0.0002155 \\ 0.0001209 & 0.0001356 & 0.0001092 & 0.0001204 & 0.0002781 & 0.0001192 & 0.0001538 & 0.0000753 & 0.0001430 & 0.0001071 & 0.0001931 \\ 0.0000959 & 0.0000803 & 0.0000806 & 0.0000899 & 0.0001192 & 0.0001571 & 0.0001047 & 0.0000585 & 0.0000919 & 0.0000816 & 0.0000822 \\ 0.0001667 & 0.0002550 & 0.0002549 & 0.0002596 & 0.0001538 & 0.0001047 & 0.0004036 & 0.0001188 & 0.0001459 & 0.0001702 & 0.0002790 \\ 0.0002867 & 0.0000734 & 0.0001605 & 0.0001088 & 0.0000753 & 0.0000585 & 0.0001188 & 0.0004888 & 0.0000707 & 0.0002890 & 0.0001147 \\ 0.0001129 & 0.0001207 & 0.0001105 & 0.0001169 & 0.0001430 & 0.0000919 & 0.0001459 & 0.0000707 & 0.0001738 & 0.0001055 & 0.0001614 \\ 0.0002880 & 0.0001237 & 0.0001763 & 0.0001453 & 0.0001071 & 0.0000816 & 0.0001702 & 0.0002890 & 0.0001055 & 0.0004064 & 0.0001705 \\ 0.0001659 & 0.0002362 & 0.0002211 & 0.0002155 & 0.0001931 & 0.0000822 & 0.0002790 & 0.0001147 & 0.0001614 & 0.0001705 & 0.0006396 \end{vmatrix} \times \begin{pmatrix} 0.0909 \\ 0.0909 \\ 0.0909 \\ 0.0909 \\ 0.0909 \\ 0.0909 \\ 0.0909 \\ 0.0909 \\ 0.0909 \\ 0.0909 \\ 0.0909 \end{pmatrix}\right)^{0.5}) = 0.0137$$

See full working out for standard deviation in appendix page 30.

Hence, the Sharpe Ratio is:

$$Sharpe\ Ratio = \frac{0.0797 - 0.000451}{0.0137} = 5.78$$

Using whole values on excel, I obtained 5.99 for the Sharpe Ratio, but this discrepancy can be attributed to the rounding I did to facilitate the manual calculation.

To find the maximum Sharpe ratio using the solver tool in excel, I assembled the covariance matrix, inputted the variance formula for each asset under each covariance column, and inserted the sum formula for the weights column. See the results and the excel formulas in appendix pages 31 to 33.

The portfolio variance for the given weights would be the sum of the bottom row, for example:

$$\sigma_p{}^2 = 0.00001669 + 0.00001770 + 0.00002031 + 0.00001796 + 0.00001286 + 0.000008611 + 0.00001911 + 0.00001525 + 0.00001118 + 0.00001705 + 0.00002049 = 0.000177184$$

I inputted the formulas for variance, standard deviation, average return, and Sharpe ratio into the excel sheet to make it easier to optimize the portfolio (see excel formulas in appendix, pages 32 and 33). To find the weight distribution that would maximize my Sharpe ratio, I used the "solver" tool in excel. I set my goal to maximizing the cell containing the Sharpe ratio formula by changing the number on the column containing the weights of each stock. I had to input a constraint of maintaining the cell that calculates the sum of the weights equal to one.

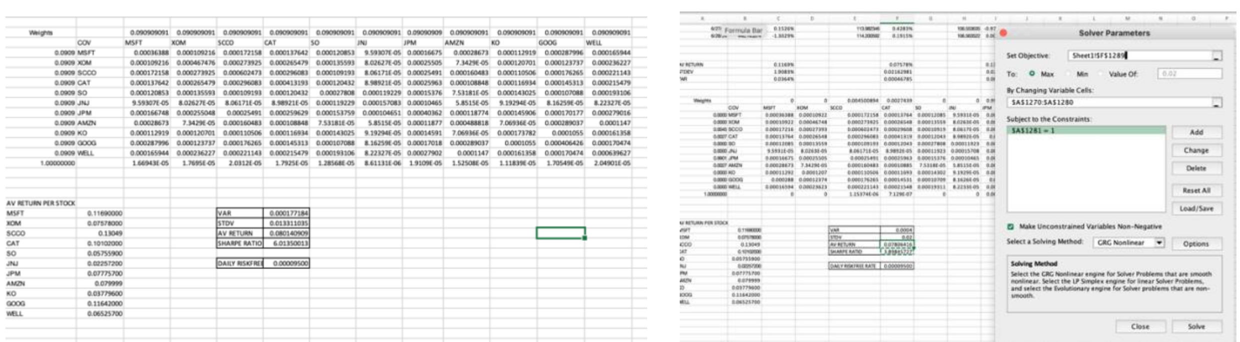

Figure 1: Screenshots of Excel Sheet and Excel Solver

After clicking solve, I obtained the following weights:

Table 2: Weights of each Stock in MPT Portfolio

	MSFT	XOM	SCCO	CAT	SO	JNJ	JPM	AMZN	KO	GOOG	WELL
Weight	0.350	0.000	0.216	0.170	0.0601	0.000	0.000	0.000	0.000	0.204	0.000

This means that, according to the Modern Portfolio Theory, my investment portfolio should be made up of 34.96% MSFT, 21.63% SCCO, 17.03% CAT, 6.01% SO, and 20.37% GOOG.

For these optimum weights, the results for the variance, standard deviation, return and Sharpe ratio were the following:

Table 3: Results for Variance, Standard Deviation, Average Return, and Sharpe Ratio

VAR	STDV	AV RETURN	SHARPE RATIO
0.000246399	0.015697113	0.113473303	7.22287596

Modeling the Optimum Portfolio:

To predict the future value of the optimal portfolio if I started investing 1000000 dollars now, I first simplified the returns using the expected return. The value of the portfolio at day t would thus be:

$$Value\ of\ investment\ in\ day\ t = initial\ investment\ \times \left(\frac{daily\ average\ percentage\ return}{100}\right)^{t-1}$$

$$Value\ of\ the\ investment\ in\ day\ t = 1000000 \times \left(1 + \frac{0.113}{100}\right)^{t-1}$$

The standard deviation calculated, which was 0.0159%, shows that the formula above can be either an overestimation or an underestimation of how the stock actually performs. See appendix page 34 for calculations accounting for the standard deviation.

This power model uses a compound interest approach, assuming a constant daily percentage return, resulting in an exponential growth curve. To see the suitability of this model, I graphed it with the actual return of the portfolio considering that I had started investing 1 million dollars in July 2019. I calculated the actual return of the portfolio at each day by multiplying the actual return of each stock with their weight in the portfolio and adding them together, as I did for the stocks in pages 10 and 11. I did this using excel, but an example calculation for $\bar{R}_t$ at t=1 is:

Table 4: Return of each Stock and their Weights at t=1

	MSFT	SCCO	CAT	SO	GOOG
R_i at t=1	0.00663	-0.0114	-0.00549	0.0186	0.0121
w_i	0.345	0.216	0.17	0.0601	0.204

$$\bar{R}_i = (0.00663 \times 0.345) + (-0.0114 \times 0.216) + (-0.00549 \times 0.10) + (0.0186 \times 0.0601) \times (0.0121 \times 0.204) = 0.00250 = 0.250\%$$

Then, to calculate the value of the portfolio at each day, I used the formula:

$$Value\ of\ the\ Investment\ at\ day\ t = value\ of\ the\ investment\ on\ the\ previous\ day\ \times (1 + \frac{\%\ return\ at\ day\ t}{100})$$

Example calculation at t=1:

$$Value\ of\ the\ Investment\ at\ day\ 1 = 1000000\ \times \left(1 + \frac{0.250}{100}\right) = 1002500$$

Then, I plotted a graph of the expected return, maximum return, minimum return, and actual return against time:

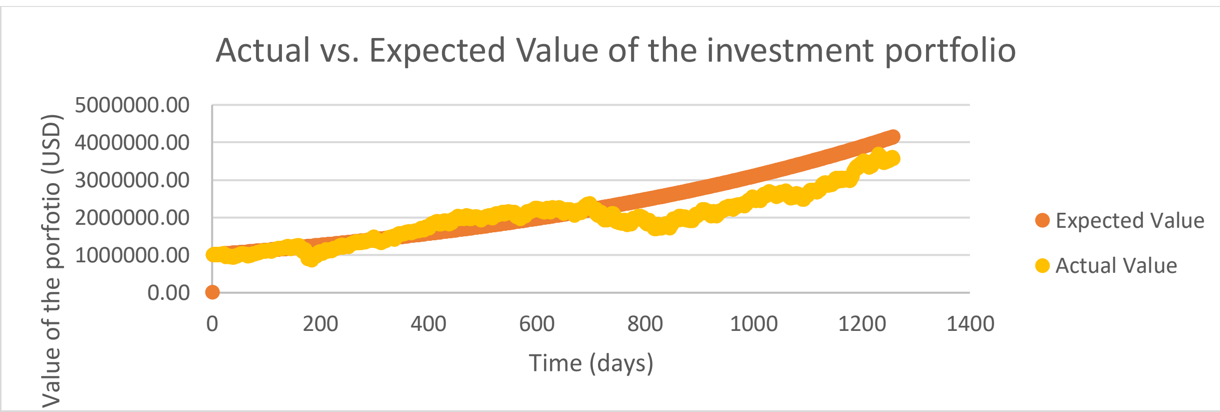

Figure 2: Graph of Actual and Expected Value of the MPT Portfolio

The graph shows the actual value of the portfolio doesn't fully follow the model for the expected value, dipping below the expected value curve between February 2020 and October 2020, then going above the curve until June 2022, and finishing below the curve until the end of the time period considered. To quantify the suitability of the model, calculated the R-Squared value by using the expected and actual values at t=1, t=300, t=600, t=900 and t=1200:

$$R^2 = 1 - \frac{\sum_{i=1}^{5}(y_i - \hat{y}_i)^2}{\sum_{i=1}^{5}(y_i - \bar{y}_i)^2}$$

Where:

y_i = observed value; $\hat{y}_i$ = predicted value from the model; $\bar{y}_i$= mean of observed values

Table 5: Sample R-Squared Calculation for Model 1

y_i	$\hat{y}_i$	$(y_i - \hat{y}_i)^2$	$(y_i - \bar{y}_i)^2$
1000000	$1000000 \times (1 + \frac{0.113}{100})^{1-1} = 1000000$	$(1000000 - 1000000)^2 = 0$	$\bar{y}_i = \frac{1000000+1406729.65+2215280.16+2087477.42+3426718.54}{5}$ $\bar{y}_i = 2027241.15$ $(1000000 - 2027241.15)^2 = 1.06 \times 10^{12}$

See full table in appendix page 35.

$$R^2 = 1 - \frac{0 + 25389707 + 6.17 \times 10^{10} + 4.53 \times 10^{11} + 4.53 \times 10^{11} + 1.99 \times 10^{11}}{1.06 \times 10^{12} + 3.85 \times 10^{11} + 3.63 \times 10^{9} + 1.96 \times 10^{12}} = 0.34$$

This means that 34% of the variation in the actual value of the portfolio can be predicted using the model, hence proving this model is unsuitable for predicting returns of investment portfolios. To find a better way to model future returns, I compared the portfolio's returns with the return of the market to see if there is a relationship. I used daily data for the S&P 500 index for the comparison, as it models 500 of the largest businesses publicly traded in the US[16], thus closely following the market trend. I calculated the return (see sample results table in appendix page 35), and used the same formula for the *Value of Investment at day t* to plot the graph:

[16] https://www.forbes.com/advisor/investing/what-is-sp-500/#:~:text=The%20S%26P%20500%20is%20a,the%20performance%20of%20U.S.%20stocks.

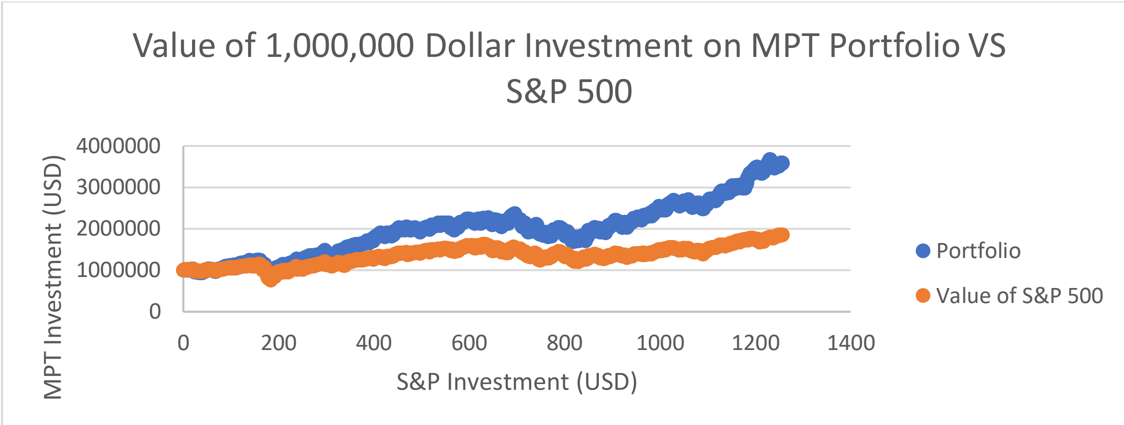

Figure 3: MPT Portfolio and S&P 500 Investment Against Time, 2019-2024

The shapes of the curves look similar, hence there seems to be a relationship between the movements of the market and the portfolio. To better visualize the relationship, I plotted the value of the portfolio against the value of S&P 500:

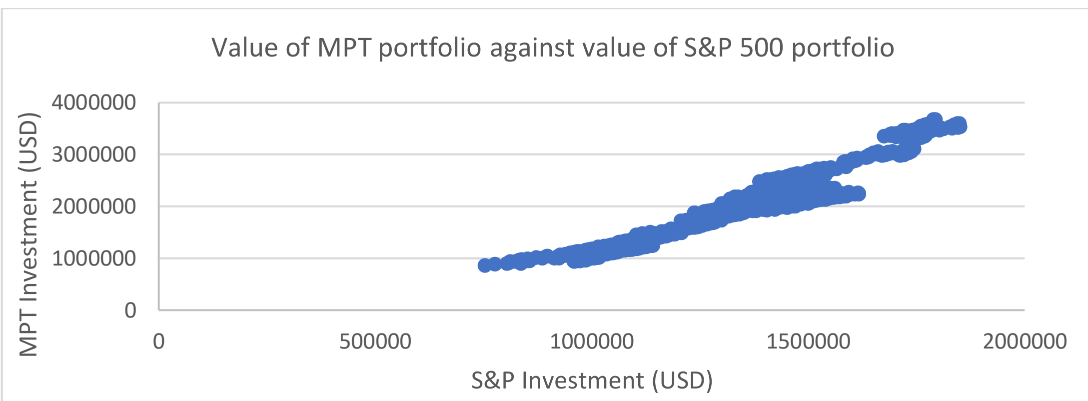

Figure 4: MPT Portfolio against S&P 500 Investment

I added an excel trend line with the R^2 value to see which model would fit best (see results in appendix pages 36 and 37). The R^2 value for the polynomial model is the greatest, suggesting the relationship approximates a segment of a quadratic model when the y-value is increasing. This indicates that, as time progresses, returns increase, suggesting exponential growth. The quadratic model equation is:

$$y = ax^2 + bx + c$$

In this case, the y-value is the value of the MPT portfolio, and the x-value is the value of the S&P 500 investment. To derive the equation, I selected three points in Figure 6, one in the beginning (t=1), middle (t=600), and end (t=1200) of the curve and built the system of equations:

$$1000000 = a(1000000)^2 + 1000000b + c$$

$$2215280.15 = a(1568405)^2 + 1568405b + c$$

$$3562132.22 = a(1842062.12)^2 + 1842062.12b + c$$

Using matrices to solve the equation (see appendix pages 37 and 38 for full calculation):

$$\begin{pmatrix} 1000000^2 & 1000000 & 1 \\ 1568405^2 & 1568405 & 1 \\ 1842062.12^2 & 1842062.12 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 1000000 \\ 2215280.15 \\ 3562132.22 \end{pmatrix}$$

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 1000000^2 & 1000000 & 1 \\ 1568405^2 & 1568405 & 1 \\ 1842062.12^2 & 1842062.12 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1000000 \\ 2215280.15 \\ 3562132.22 \end{pmatrix} = \begin{pmatrix} 3.31 \times 10^{-6} \\ -6.35 \\ 4046659.38 \end{pmatrix}$$

$$y = 3.31 \times 10^{-6} x^2 - 6.35x + 4046659.38$$

I substituted in the values of the S&P 500 Investment to calculate the predicted value of the MPT portfolio over the 5 years and plotted the actual value against it:

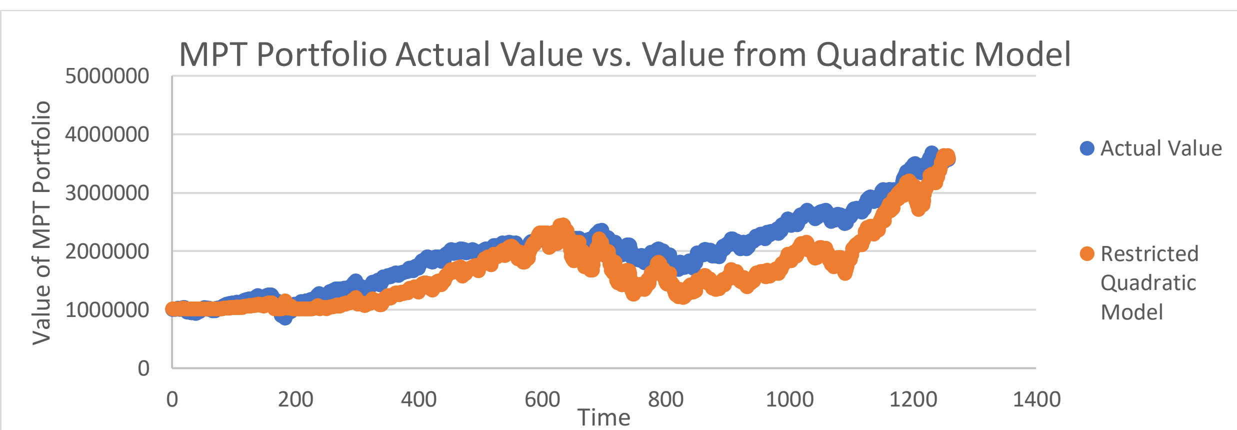

Figure 5: Actual Value and Quadratic Model Against Time, 2019-2024

This model seems to predict the return of the portfolio very well, following the fluctuations over time. I can assume that the R^2 is close to the one calculated by excel when I added the trendline, 0.923, which is much higher than for the first model I built.

The relationship between the return of the portfolio and the market is very important for investors, who usually consider the beta index when making investment decisions. The beta index in finance denotes the volatility or systematic risk of a security or portfolio compared to the market[17]. The S&P 500 has a beta of 1.0, meaning that it moves exactly with the market.

Even though I already created a good model, I saw the opportunity to build the Capital Asset Pricing Model (CAPM), which is a common strategy used by investors to predict future returns. The equation is[18]:

$$R_{pt} = R_f + \beta_p(R_{mt} - R_f)$$

Where: R_{pt}= expected return at time t; R_f= risk-free rate; β_p= beta index; R_{mt}= return of the market at time t

With the regression between the return of the market and the portfolio calculated before, I estimated the beta index using the slope of the graph using differentiation:

$$y = -3.31 \times 10^{-6}x^2 - 6.35x + 4046659.38$$

$$\frac{dy}{dx} = 2(3.31 \times 10^{-6})x - 6.35$$

I inserted this formula in excel for all x-values, with results ranging from -0.00632 to 5.89 as the relationship between the variables is not linear. The average was 2.55. Hence, we could interpret

[17] https://www.investopedia.com/terms/b/beta.asp

[18] https://www.investopedia.com/terms/c/capm.asp#:~:text=The%20capital%20asset%20pricing%20model%2C%20or%20CAPM%2C%20is%20a%20financial,to%20the%20market%20(beta).

that the beta index is close to 2.55, meaning that the portfolio tends to move 2.55 times the movement of the market.

Thus, the CAPM equation is:

$$R_p = 0.000000451 + 2.55(R_m - 0.000000451)$$

The formula for the future value of the portfolio according to the CAPM is:

$$Value\ of\ the\ Investment\ at\ day\ t = value\ of\ the\ investment\ on\ the\ previous\ day \times$$

$$(1 + \left(0.000000451 + 2.55(R_m - 0.000000451)\right))$$

Plotting the returns against time, I generated the following graph:

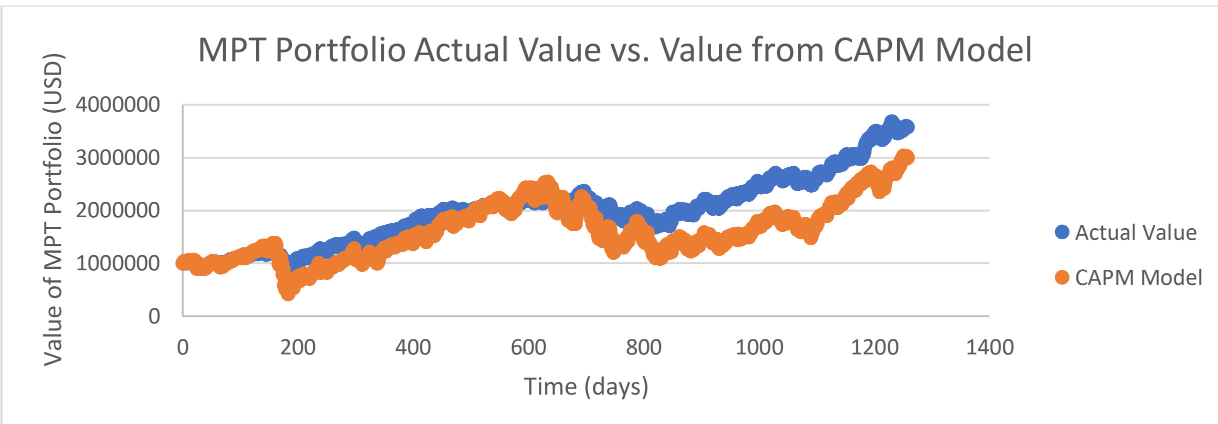

Figure 6: Actual Value and CAPM Against Time, 2019-2024

This seems to be less accurate than the quadratic model, especially when predicting the value of the portfolio after around day 710.

To quantify how well the CAPM predicts the actual return of the portfolio, I calculated the R^2 value for it using the same 5 points as I did for the first model. The result was 0.66 (see working out in appendix, page 38), meaning that 66% of the variation in the actual value of the portfolio can be predicted using the CAPM model. This is almost double the value of the power model, but less than the quadratic model.

<u>Conclusion and Evaluation:</u>

To conclude, the MPT enabled me to construct an investment portfolio that maximized return for a given risk, and it helped me better understand the balance between risk and return and the importance of considering how stocks are related. This theory is very useful for beginner investors like me who face challenges in stock selection and fund distribution.

Nonetheless, the optimization part of my investigation had some limitations. Firstly, I chose well-known stocks from large companies, and these are often more sensitive to market sentiments and economic factors. The portfolio showed a positive correlation with the market as seen in the modelling part, which shows it is more vulnerable to market-wide downturns. To improve, I could have selected lesser-known stocks and expanded my selection to a broader geographical area instead of mostly US companies, which would improve the diversification of the portfolio. Secondly, by using historical returns to calculate expected returns, I assumed that past trends would continue in the future, which might not necessarily be true. It also doesn’t account for long-term economic cycles or extreme events, which could impact the performance of the portfolio. To improve this, I could have considered historical returns over a longer period, such as 20 years, or conducted a stress test to measure how the portfolio would perform under extreme events. Thirdly, after modelling, I saw that the portfolio is still very volatile, which might not fit all investors’ profiles. To improve this, I could have plotted the efficient frontier, which is a component of the MPT that graphically represents a set of optimal portfolios with the highest expected return for a given level of risk. It is constructed by plotting the standard deviation of the portfolio against the excess return of the portfolio for a range of standard deviation values[19], so it would be useful to visualize the risk-return tradeoff and select which combination of standard deviation and expected return best fit certain investment objectives. In the modelling part, I first attempted to use the expected return, then quadratic regression between the return of the market and of the portfolio, and finally the CAPM model. After finding the R^2 value for all the models, the best model proved to be the quadratic regression, although it also didn't fully follow the curve of the actual returns. This was unexpected as I thought that the CAPM would be the best fit, given that it is a model used specifically for investments. Nonetheless, there were some limitations in the modelling part. By considering the relationship between the market and the portfolio as quadratic, I limit my analysis to a certain domain after the turning point, so I wouldn't be able to use it to model past trends, and in the long run, the model might not be accurate. To improve this, I could have performed a power regression, which had the second-highest R^2 value according to excel and would probably be better for modelling in the long run. Moreover, when estimating the beta index for the CAPM equation, I used the regression between the return of the S&P 500 and the MPT portfolio. However, the formula for calculating the beta of the stock is dividing the covariance between the return of the market and the portfolio by the variance of the return of the market[20]. Thus, the beta value I got is probably not fully accurate, and to improve the investigation, I could have calculated the beta using the formula. Furthermore, the R^2 values I calculated for the first and third model manually was not fully accurate as I decided to choose 5 data points to simplify the calculations, meaning it could be an over or underestimation of the actual R^2, but because the values for the three models were

[19] https://www.investopedia.com/terms/e/efficientfrontier.asp#:~:text=The%20efficient%20frontier%20is%20the,for%20the%20level%20of%20risk.

[20] https://www.google.com/url?sa=i&url=https%3A%2F%2Fwww.wallstreetmojo.com%2Fbeta-formula%2F&psig=AOvVaw0GBkVmJKR_o_Nb0O5i6PWU&ust=1730200746441000&source=images&cd=vfe&opi=89978449&ved=0CBcQjhxqFwoTCICwvJj6sIkDFQAAAAAdAAAAABAE

significantly different, I was able to assume they were suitable enough to compare the models and determine which one works best, even considering approximation errors. Moreover, the CAPM and the quadratic regression predicts the value of the portfolio when we know the return of the market, which is not suitable for future predictions if we don't know the return of the market. To improve, I could have calculated the average return of the market and use the CAPM model as a base to build a linear model that could estimate the future returns. I could also have attempted other ways of modelling, such as how the portfolio moves with changes in certain macroeconomic factors such as GDP, interest rates, and inflation rates, which would give me a better idea of how the stock would perform under different settings.

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Appendix:

Average return, standard deviation and variance of selected stocks:

	Average Return	Standard Deviation	Variance
MSFT	0.00117	0.0191	0.000364
XOM	0.000758	0.0216	0.000468
SCCO	0.00130	0.0246	0.000603
CAT	0.00101	0.0203	0.000414
SO	0.000575	0.0167	0.000278
JNJ	0.000226	0.0125	0.000157
JPM	0.000778	0.0201	0.000404
AMZN	0.000800	0.0221	0.000489
KO	0.000378	0.0132	0.000174
GOOG	0.00116	0.0202	0.000407
WELL	0.000653	0.0253	0.000640

Explaining the variance formula for an 11-asset portfolio:

$$\sigma_p{}^2 = w_1{}^2\sigma_1{}^2 + w_2{}^2\sigma_2{}^2 + w_3{}^2\sigma_3{}^2 + w_4{}^2\sigma_4{}^2 + w_5{}^2\sigma_5{}^2 + w_6{}^2\sigma_6{}^2 + w_7{}^2\sigma_7{}^2 + w_8{}^2\sigma_8{}^2$$

$$+ w_9{}^2\sigma_9{}^2 + w_{10}{}^2\sigma_{10}{}^2 + w_{11}{}^2\sigma_{11}{}^2 + 2w_1w_2\rho_{1,2} + 2w_1w_3\rho_{1,3} + 2w_1w_4Cov_{1,4}$$

$$+ 2w_1w_5Cov_{1,5} + 2w_1w_6Cov_{1,6} + 2w_1w_7Cov_{1,7} + 2w_1w_8Cov_{1,8}$$

$$+ 2w_1w_9Cov_{1,9} + 2w_1w_{10}Cov_{1,10} + 2w_1w_{11}Cov_{1,11} + 2w_2w_3Cov_{2,3}$$

$$+ 2w_2w_4Cov_{2,4} + 2w_2w_5Cov_{2,5} + 2w_2w_6Cov_{2,6} + 2w_2w_7Cov_{2,7}$$

$$+ 2w_2w_8Cov_{2,8} + 2w_2w_9Cov_{2,9} + 2w_2w_{10}Cov_{2,10} + 2w_2w_{11}Cov_{2,11}$$

$$+ 2w_3w_4Cov_{3,4} + 2w_3w_5Cov_{3,5} + 2w_3w_6Cov_{3,6} + 2w_3w_7Cov_{3,7}$$

$$+ 2w_3w_8Cov_{3,8} + 2w_3w_9Cov_{3,9} + 2w_3w_{10}Cov_{3,10} + 2w_3w_{11}Cov_{3,11}$$

$$+ 2w_4w_5Cov_{4,5} + 2w_4w_6Cov_{4,6} + 2w_4w_7Cov_{4,7} + 2w_4w_8Cov_{4,8}$$

$$+ 2w_4w_9Cov_{4,9} + 2w_4w_{10}Cov_{4,10} + 2w_4w_{11}Cov_{4,11} + 2w_5w_6Cov_{5,6}$$

$$+ 2w_5w_7Cov_{5,7} + 2w_5w_7Cov_{5,7} + 2w_5w_8Cov_{5,8} + 2w_5w_9Cov_{5,9}$$

$$+ 2w_5w_{10}Cov_{5,10} + 2w_5w_{11}Cov_{5,11} + 2w_6w_7Cov_{6,7} + 2w_6w_8Cov_{6,8}$$

$$+ 2w_6w_9Cov_{6,9} + 2w_6w_{10}Cov_{6,10} + 2w_7w_8Cov_{7,8} + 2w_7w_9Cov_{7,9}$$

$$+ 2w_7w_{10}Cov_{7,10} + 2w_7w_{11}Cov_{7,11} + 2w_8w_9Cov_{8,9} + 2w_8w_{10}Cov_{8,10}$$

$$+ 2w_8w_{11}Cov_{8,11} + 2w_9w_{10}Cov_{9,10} + 2w_9w_{11}Cov_{9,11} + 2w_{10}w_{11}Cov_{10,11}$$

Simplifying this would give:

$$\sigma_p{}^2 = \sum_{i=1}^{11} w_i{}^2\sigma_i{}^2 + \sum_{i=1}^{11}\sum_{j=1, j\neq i}^{11} w_iw_j\, Cov_{i,j}\sigma_i\sigma_j$$

Or, further:

$$\sigma_p{}^2 = \sum_{i=1}^{11}\sum_{j=1, j\neq i}^{11} w_iw_j\, Cov_{i,j}\sigma_i\sigma_j$$

Since:

$$w_i w_i Cov_{i,i} \sigma_i \sigma_i = {w_i}^2 {\sigma_i}^2$$

Hence, the standard deviation formula is:

$$\sigma = \sqrt{\sum_{i=1}^{n} \sum_{j=1}^{n} w_i \, w_j Cov_{ij} \sigma_i \sigma_j}$$

Explaining the variance formula for an 11-asset portfolio in matrix form:

The variance of the portfolio can also be represented in matrix form[21]:

$${\sigma_p}^2 = w^T \Omega \mathrm{w}$$

Where:

Weights Vector, w:

$$w = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \\ w_5 \\ w_6 \\ w_7 \\ w_8 \\ w_9 \\ w_{10} \\ w_{11} \end{pmatrix}$$

Covariance Matrix, Ω:

[21] https://www.google.com/url?sa=i&url=https%3A%2F%2Fmedium.com%2Fapplied-finance%2Fmatrices-and-portfolio-variance-bf90408295d7&psig=AOvVaw1Ob5eDvpNUHNJ3_GDtdk8X&ust=1725056080412000&source=images&cd=vfe&opi=89978449&ved=0CBcQjhxqFwoTCNi3o-mcm4gDFQAAAAAdAAAAABAE

$$\Omega = \begin{pmatrix} Cov_{1,1} & Cov_{1,2} & Cov_{1,3} & Cov_{1,4} & Cov_{1,5} & Cov_{1,6} & Cov_{1,7} & Cov_{1,8} & Cov_{1,9} & Cov_{1,10} & Cov_{1,11} \\ Cov_{2,1} & Cov_{2,2} & Cov_{2,3} & Cov_{2,4} & Cov_{2,5} & Cov_{2,6} & Cov_{2,7} & Cov_{2,8} & Cov_{2,9} & Cov_{2,10} & Cov_{2,11} \\ Cov_{3,1} & Cov_{3,2} & Cov_{3,3} & Cov_{3,4} & Cov_{3,5} & Cov_{3,6} & Cov_{3,7} & Cov_{3,8} & Cov_{3,9} & Cov_{3,10} & Cov_{3,11} \\ Cov_{4,1} & Cov_{4,2} & Cov_{4,3} & Cov_{4,4} & Cov_{4,5} & Cov_{4,6} & Cov_{4,7} & Cov_{4,8} & Cov_{4,9} & Cov_{4,10} & Cov_{4,11} \\ Cov_{5,1} & Cov_{5,2} & Cov_{5,3} & Cov_{5,4} & Cov_{5,5} & Cov_{5,6} & Cov_{5,7} & Cov_{5,8} & Cov_{5,9} & Cov_{5,10} & Cov_{5,11} \\ Cov_{6,1} & Cov_{6,2} & Cov_{6,3} & Cov_{6,4} & Cov_{6,5} & Cov_{6,6} & Cov_{6,7} & Cov_{6,8} & Cov_{6,9} & Cov_{6,10} & Cov_{6,11} \\ Cov_{7,1} & Cov_{7,2} & Cov_{7,3} & Cov_{7,4} & Cov_{7,5} & Cov_{7,6} & Cov_{7,7} & Cov_{7,8} & Cov_{7,9} & Cov_{7,10} & Cov_{7,11} \\ Cov_{8,1} & Cov_{8,2} & Cov_{8,3} & Cov_{8,4} & Cov_{8,5} & Cov_{8,6} & Cov_{8,7} & Cov_{8,8} & Cov_{8,9} & Cov_{8,10} & Cov_{8,11} \\ Cov_{9,1} & Cov_{9,2} & Cov_{9,3} & Cov_{9,4} & Cov_{9,5} & Cov_{9,6} & Cov_{9,7} & Cov_{9,8} & Cov_{9,9} & Cov_{9,10} & Cov_{9,11} \\ Cov_{10,1} & Cov_{10,2} & Cov_{10,3} & Cov_{10,4} & Cov_{10,5} & Cov_{10,6} & Cov_{10,7} & Cov_{10,8} & Cov_{10,9} & Cov_{10,10} & Cov_{10,11} \\ Cov_{11,1} & Cov_{11,2} & Cov_{11,3} & Cov_{11,4} & Cov_{11,5} & Cov_{11,6} & Cov_{11,7} & Cov_{11,8} & Cov_{11,9} & Cov_{11,10} & Cov_{11,11} \end{pmatrix}$$

Transposed weights vector, w^T:

$$w^T = (w_1 \quad w_2 \quad w_3 \quad w_4 \quad w_5 \quad w_6 \quad w_7 \quad w_8 \quad w_9 \quad w_{10} \quad w_{11})$$

$$\sigma_p{}^2$$

$$= (w_1 \quad w_2 \quad w_3 \quad w_4 \quad w_5 \quad w_6 \quad w_7 \quad w_8 \quad w_9 \quad w_{10} \quad w_{11})$$

$$\times \begin{pmatrix} Cov_{1,1} & Cov_{1,2} & Cov_{1,3} & Cov_{1,4} & Cov_{1,5} & Cov_{1,6} & Cov_{1,7} & Cov_{1,8} & Cov_{1,9} & Cov_{1,10} & Cov_{1,11} \\ Cov_{2,1} & Cov_{2,2} & Cov_{2,3} & Cov_{2,4} & Cov_{2,5} & Cov_{2,6} & Cov_{2,7} & Cov_{2,8} & Cov_{2,9} & Cov_{2,10} & Cov_{2,11} \\ Cov_{3,1} & Cov_{3,2} & Cov_{3,3} & Cov_{3,4} & Cov_{3,5} & Cov_{3,6} & Cov_{3,7} & Cov_{3,8} & Cov_{3,9} & Cov_{3,10} & Cov_{3,11} \\ Cov_{4,1} & Cov_{4,2} & Cov_{4,3} & Cov_{4,4} & Cov_{4,5} & Cov_{4,6} & Cov_{4,7} & Cov_{4,8} & Cov_{4,9} & Cov_{4,10} & Cov_{4,11} \\ Cov_{5,1} & Cov_{5,2} & Cov_{5,3} & Cov_{5,4} & Cov_{5,5} & Cov_{5,6} & Cov_{5,7} & Cov_{5,8} & Cov_{5,9} & Cov_{5,10} & Cov_{5,11} \\ Cov_{6,1} & Cov_{6,2} & Cov_{6,3} & Cov_{6,4} & Cov_{6,5} & Cov_{6,6} & Cov_{6,7} & Cov_{6,8} & Cov_{6,9} & Cov_{6,10} & Cov_{6,11} \\ Cov_{7,1} & Cov_{7,2} & Cov_{7,3} & Cov_{7,4} & Cov_{7,5} & Cov_{7,6} & Cov_{7,7} & Cov_{7,8} & Cov_{7,9} & Cov_{7,10} & Cov_{7,11} \\ Cov_{8,1} & Cov_{8,2} & Cov_{8,3} & Cov_{8,4} & Cov_{8,5} & Cov_{8,6} & Cov_{8,7} & Cov_{8,8} & Cov_{8,9} & Cov_{8,10} & Cov_{8,11} \\ Cov_{9,1} & Cov_{9,2} & Cov_{9,3} & Cov_{9,4} & Cov_{9,5} & Cov_{9,6} & Cov_{9,7} & Cov_{9,8} & Cov_{9,9} & Cov_{9,10} & Cov_{9,11} \\ Cov_{10,1} & Cov_{10,2} & Cov_{10,3} & Cov_{10,4} & Cov_{10,5} & Cov_{10,6} & Cov_{10,7} & Cov_{10,8} & Cov_{10,9} & Cov_{10,10} & Cov_{10,11} \\ Cov_{11,1} & Cov_{11,2} & Cov_{11,3} & Cov_{11,4} & Cov_{11,5} & Cov_{11,6} & Cov_{11,7} & Cov_{11,8} & Cov_{11,9} & Cov_{11,10} & Cov_{11,11} \end{pmatrix}$$

$$\times \begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \\ w_5 \\ w_6 \\ w_7 \\ w_8 \\ w_9 \\ w_{10} \\ w_{11} \end{pmatrix}$$

$$\sigma_p{}^2 = \left(\sum_{i=1}^{11} w_1 Cov_{i,1} \quad \sum_{i=1}^{11} w_2 Cov_{i,2} \quad \sum_{i=1}^{11} w_3 Cov_{i,3} \quad \sum_{i=1}^{11} w_4 Cov_{i,4} \quad \sum_{i=1}^{11} w_5 Cov_{i,5} \quad \sum_{i=1}^{11} w_6 Cov_{i,6} \quad \sum_{i=1}^{11} w_7 Cov_{i,7} \quad \sum_{i=1}^{11} w_8 Cov_{i,8} \quad \sum_{i=1}^{11} w_9 Cov_{i,9} \quad \sum_{i=1}^{11} w_{10} Cov_{i,10} \quad \sum_{i=1}^{11} w_{11} Cov_{i,11} \right) \times \begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \\ w_5 \\ w_6 \\ w_7 \\ w_8 \\ w_9 \\ w_{10} \\ w_{11} \end{pmatrix}$$

$$\begin{aligned} \sigma_p{}^2 = {} & \sum_{i=1}^{11} w_1 Cov_{i,1} \times w_1 + \sum_{i=1}^{11} w_2 Cov_{i,2} \times w_2 + \sum_{i=1}^{11} w_3 Cov_{i,3} \times w_3 + \sum_{i=1}^{11} w_4 Cov_{i,4} \times w_4 \\ & + \sum_{i=1}^{11} w_5 Cov_{i,5} \times w_5 + \sum_{i=1}^{11} w_6 Cov_{i,6} \times w_6 + \sum_{i=1}^{11} w_7 Cov_{i,7} \times w_7 \\ & + \sum_{i=1}^{11} w_8 Cov_{i,8} \times w_8 + \sum_{i=1}^{11} w_9 Cov_{i,9} \times w_9 + \sum_{i=1}^{11} w_{10} Cov_{i,10} \times w_{10} \\ & + \sum_{i=1}^{11} w_{11} Cov_{i,11} \times w_{11} \end{aligned}$$

Which gives:

$$\sigma_p{}^2 = \sum_{i=1}^{11} w_1 Cov_{i,1} \times w_1 + \sum_{i=1}^{11} w_2 Cov_{i,2} \times w_2 + \sum_{i=1}^{11} w_3 Cov_{i,3} \times w_3 + \sum_{i=1}^{11} w_4 Cov_{i,4} \times w_4 + \sum_{i=1}^{11} w_5 Cov_{i,5} \times w_5 + \sum_{i=1}^{11} w_6 Cov_{i,6} \times w_6 + \sum_{i=1}^{11} w_7 Cov_{i,7} \times w_7 + \sum_{i=1}^{11} w_8 Cov_{i,8} \times w_8 + \sum_{i=1}^{11} w_9 Cov_{i,9} \times w_9 + \sum_{i=1}^{11} w_{10} Cov_{i,10} \times w_{10} + \sum_{i=1}^{11} w_{11} Cov_{i,11} \times w_{11}$$

The standard deviation would be the square root of the formula above.

Covariance Table for the 11 stocks:

COV	MSFT	XOM	SCCO	CAT	SO	JNJ	JPM	AMZN	KO	GOOG	WELL
MSFT	0.00036388	0.000109216	0.000172158	0.000137642	0.000120853	9.59307E-05	0.00016675	0.00028673	0.000112919	0.000287996	0.000165944
XOM	0.000109216	0.000467476	0.000273925	0.000265479	0.000135593	8.02627E-05	0.00025505	7.3429E-05	0.000120701	0.000123737	0.000236227
SCCO	0.000172158	0.000273925	0.000602473	0.000296083	0.000109193	8.06171E-05	0.00025491	0.000160483	0.000110506	0.000176265	0.000221143
CAT	0.000137642	0.000265479	0.000296083	0.000413193	0.000120432	8.98921E-05	0.00025963	0.000108848	0.000116934	0.000145313	0.000215479
SO	0.000120853	0.000135593	0.000109193	0.000120432	0.00027808	0.000119229	0.00015376	7.53181E-05	0.000143025	0.000107088	0.000193106
JNJ	9.59307E-05	8.02627E-05	8.06171E-05	8.98921E-05	0.000119229	0.000157083	0.00010465	5.8515E-05	9.19294E-05	8.16259E-05	8.22327E-05
JPM	0.000166748	0.000255048	0.00025491	0.000259629	0.000153759	0.000104651	0.00040362	0.000118774	0.000145906	0.000170177	0.000279016
AMZN	0.00028673	7.3429E-05	0.000160483	0.000108848	7.53181E-05	5.8515E-05	0.00011877	0.000488818	7.06936E-05	0.000289037	0.0001147
KO	0.000112919	0.000120701	0.000110506	0.000116934	0.000143025	9.19294E-05	0.00014591	7.06936E-05	0.000173782	0.0001055	0.000161358
GOOG	0.000287996	0.000123737	0.000176265	0.000145313	0.000107088	8.16259E-05	0.00017018	0.000289037	0.0001055	0.000406426	0.000170474
WELL	0.000165944	0.000236227	0.000221143	0.000215479	0.000193106	8.22327E-05	0.00027902	0.0001147	0.000161358	0.000170474	0.000639627

Standard deviation for equal weighted portfolio:

Standard deviation of the portfolio, $\sigma_p =$

$((0.0909 \quad 0.0909 \quad 0.0909 \quad 0.0909 \quad 0.0909 \quad 0.0909 \quad 0.0909 \quad 0.0909 \quad 0.0909 \quad 0.0909 \quad 0.0909) \times$

$$
\begin{vmatrix}
0.0003639 & 0.0001092 & 0.0001722 & 0.0001376 & 0.0001209 & 0.0000959 & 0.0001667 & 0.0002867 & 0.0001129 & 0.0002880 & 0.0001659 \\
0.0001092 & 0.0004675 & 0.0002739 & 0.0002655 & 0.0001356 & 0.0000803 & 0.0002550 & 0.0000734 & 0.0001207 & 0.0001237 & 0.0002362 \\
0.0001722 & 0.0002739 & 0.0006025 & 0.0002961 & 0.0001092 & 0.0000806 & 0.0002549 & 0.0001605 & 0.0001105 & 0.0001763 & 0.0002211 \\
0.0001376 & 0.0002655 & 0.0002961 & 0.0004132 & 0.0001204 & 0.0000899 & 0.0002596 & 0.0001088 & 0.0001169 & 0.0001453 & 0.0002155 \\
0.0001209 & 0.0001356 & 0.0001092 & 0.0001204 & 0.0002781 & 0.0001192 & 0.0001538 & 0.0000753 & 0.0001430 & 0.0001071 & 0.0001931 \\
0.0000959 & 0.0000803 & 0.0000806 & 0.0000899 & 0.0001192 & 0.0001571 & 0.0001047 & 0.0000585 & 0.0000919 & 0.0000816 & 0.0000822 \\
0.0001667 & 0.0002550 & 0.0002549 & 0.0002596 & 0.0001538 & 0.0001047 & 0.0004036 & 0.0001188 & 0.0001459 & 0.0001702 & 0.0002790 \\
0.0002867 & 0.0000734 & 0.0001605 & 0.0001088 & 0.0000753 & 0.0000585 & 0.0001188 & 0.0004888 & 0.0000707 & 0.0002890 & 0.0001147 \\
0.0001129 & 0.0001207 & 0.0001105 & 0.0001169 & 0.0001430 & 0.0000919 & 0.0001459 & 0.0000707 & 0.0001738 & 0.0001055 & 0.0001614 \\
0.0002880 & 0.0001237 & 0.0001763 & 0.0001453 & 0.0001071 & 0.0000816 & 0.0001702 & 0.0002890 & 0.0001055 & 0.0004064 & 0.0001705 \\
0.0001659 & 0.0002362 & 0.0002211 & 0.0002155 & 0.0001931 & 0.0000822 & 0.0002790 & 0.0001147 & 0.0001614 & 0.0001705 & 0.0006396
\end{vmatrix}
$$

$$
\times \begin{pmatrix} 0.0909 \\ 0.0909 \\ 0.0909 \\ 0.0909 \\ 0.0909 \\ 0.0909 \\ 0.0909 \\ 0.0909 \\ 0.0909 \\ 0.0909 \\ 0.0909 \end{pmatrix})^{0.5})
$$

$$
= (0.0909^2(0.01909^2 + 0.02163^2 + 0.02456^2 + 0.02034^2 + 0.01668^2 + 0.01254^2 + 0.02010^2 + 0.02212^2 + 0.01319^2 + 0.02017^2 + 0.02530^2) + (2 \times 0.0909^2)(0.00011 + 0.00017 + 0.00012 + 0.00096 + 0.00017 + 0.00029 + 0.00011 + 0.00029 + 0.00017 + 0.00027 + 0.00027 + 0.00014 + 0.000080 + 0.00026 + 0.000073 + 0.00012 + 0.00012 + 0.00014 + 0.00030 + 0.00011 + 0.000081 + 0.00025 + 0.00016 + 0.00011 + 0.00018 + 0.00022 + 0.00012 + 0.000090 + 0.00026 + 0.00011 + 0.00012 + 0.00015 + 0.00022 + 0.00012 + 0.00015 + 0.000075 + 0.00014 + 0.00011 + 0.00019 + 0.00010 + 0.000059 + 0.000092 + 0.000082 + 0.000082 + 0.00012 + 0.00015 + 0.00017 + 0.00028 + 0.000071 + 0.00029 + 0.00011 + 0.00011 + 0.00016 + 0.00017))^{0.5} = 0.0137
$$

Excel return formula:

Date	MSFT Adj Close	Return(%)
7/1/19	129.164612	
7/2/19	130.021423	=(B3/B2)-1

Excel average return formula:

AV RETURN		=AVERAGE(C3:C1259)

Portfolio variance calculation using excel:

Matrix Table:

Weights		0.0909	0.0909	0.0909	0.0909	0.0909	0.0909	0.0909	0.0909	0.09091	0.0909	0.0909
	COV	**MSFT**	**XOM**	**SCCO**	**CAT**	**SO**	**JNJ**	**JPM**	**AMZN**	**KO**	**GOOG**	**WELL**
0.0909	**MSFT**	0.00036388	0.000109216	0.000172158	0.000137642	0.000120853	9.59307E-05	0.00016675	0.00028673	0.000112919	0.000287996	0.000165944
0.0909	**XOM**	0.000109216	0.000467476	0.000273925	0.000265479	0.000135593	8.02627E-05	0.00025505	7.3429E-05	0.000120701	0.000123737	0.000236227
0.0909	**SCCO**	0.000172158	0.000273925	0.000602473	0.000296083	0.000109193	8.06171E-05	0.00025491	0.000160483	0.000110506	0.000176265	0.000221143
0.0909	**CAT**	0.000137642	0.000265479	0.000296083	0.000413193	0.000120432	8.98921E-05	0.00025963	0.000108848	0.000116934	0.000145313	0.000215479
0.0909	**SO**	0.000120853	0.000135593	0.000109193	0.000120432	0.00027808	0.000119229	0.00015376	7.53181E-05	0.000143025	0.000107088	0.000193106
0.0909	**JNJ**	9.59307E-05	8.02627E-05	8.06171E-05	8.98921E-05	0.000119229	0.000157083	0.00010465	5.8515E-05	9.19294E-05	8.16259E-05	8.22327E-05
0.0909	**JPM**	0.000166748	0.000255048	0.00025491	0.000259629	0.000153759	0.000104651	0.00040362	0.000118774	0.000145906	0.000170177	0.000279016
0.0909	**AMZN**	0.00028673	7.3429E-05	0.000160483	0.000108848	7.53181E-05	5.8515E-05	0.00011877	0.000488818	7.06936E-05	0.000289037	0.0001147
0.0909	**KO**	0.000112919	0.000120701	0.000110506	0.000116934	0.000143025	9.19294E-05	0.00014591	7.06936E-05	0.000173782	0.0001055	0.000161358
0.0909	**GOOG**	0.000287996	0.000123737	0.000176265	0.000145313	0.000107088	8.16259E-05	0.00017018	0.000289037	0.0001055	0.000406426	0.000170474
0.0909	**WELL**	0.000165944	0.000236227	0.000221143	0.000215479	0.000193106	8.22327E-05	0.00027902	0.0001147	0.000161358	0.000170474	0.000639627
1.00000000		1.66943E-05	1.7695E-05	2.0312E-05	1.7925E-05	1.28568E-05	8.61131E-06	1.9109E-05	1.52508E-05	1.11839E-05	1.70549E-05	2.04901E-05

I used the COVAR function for each possible combination of stocks:

Weights		0.0909	0.09090909	0.09
	COV	MSFT	XOM	SCCO
0.0909	MSFT	=COVAR(C3:C1259,C3:C1259)		
0.0909	XOM	0.00010922	0.00046748	0.00
0.0909	SCCO	0.00017216	0.00027393	0.00
0.0909	CAT	0.00013764	0.00026548	0.00

I used the first column as the weights vector and the first row as the transposed weights vector:

Weights		=A1270	0.090909091	0.090909091	0.0
	COV	MSFT	XOM	SCCO	CAT
0.0909	MSFT	0.00036388	0.000109216	0.000172158	0.0
0.0909	XOM	0.000109216	0.000467476	0.000273925	0.0
0.0909	SCCO	0.000172158	0.000273925	0.000602473	0.0
0.0909	CAT	0.000137642	0.000265479	0.000296083	0.0
0.0909	SO	0.000120853	0.000135593	0.000109193	0.0

I used the sum-product function in the bottom row for the matrix multiplication:

Weights		0.0909	0.090909091	0.090909091
	COV	MSFT	XOM	SCCO
0.0909	MSFT	0.00036388	0.000109216	0.000172158
0.0909	XOM	0.000109216	0.000467476	0.000273925
0.0909	SCCO	0.000172158	0.000273925	0.000602473
0.0909	CAT	0.000137642	0.000265479	0.000296083
0.0909	SO	0.000120853	0.000135593	0.000109193
0.0909	JNJ	9.59307E-05	8.02627E-05	8.06171E-05
0.0909	JPM	0.000166748	0.000255048	0.00025491
0.0909	AMZN	0.00028673	7.3429E-05	0.000160483
0.0909	KO	0.000112919	0.000120701	0.000110506
0.0909	GOOG	0.000287996	0.000123737	0.000176265
0.0909	WELL	0.000165944	0.000236227	0.000221143
1.00000000		=C1268*SUMPRODUCT(A1270:A1280,C1270:C1280)		

I added the values in the bottom row for the variance:

0.000165944	0.000236227	0.000221143	0.000215479	0.000193106	8.22327E-05	0.000279016	0.0001147	0.000161358	0.000170474	0.000639627
1.66943E-05	1.7695E-05	2.0312E-05	1.7925E-05	1.28568E-05	8.61131E-06	1.91094E-05	1.52508E-05	1.11839E-05	1.70549E-05	2.04901E-05
0.01926		VAR	=SUM(C1281:M1281)							

Calculating the standard deviation:

I square-rooted the variance:

VAR	0.00017718
STDV	=F1286^0.5
AV RETURN	0.08014091

Calculating the average return of the portfolio:

I used the sum-product function for the average returns of each stock and their relative weights in the portfolio:

Weights							
		0.0909	0.09090909	0.090909091	0.09090909	0.090909091	0.090
	COV	MSFT	XOM	SCCO	CAT	SO	JNJ
0.0909	MSFT	0.00036388	0.00010922	0.000172158	0.00013764	0.000120853	9.59307E-0
0.0909	XOM	0.00010922	0.00046748	0.000273925	0.00026548	0.000135593	8.02627E-0
0.0909	SCCO	0.00017216	0.00027393	0.000602473	0.00029608	0.000109193	8.06171E-0
0.0909	CAT	0.00013764	0.00026548	0.000296083	0.00041319	0.000120432	8.98921E-0
0.0909	SO	0.00012085	0.00013559	0.000109193	0.00012043	0.00027808	0.00011922
0.0909	JNJ	9.5931E-05	8.0263E-05	8.06171E-05	8.9892E-05	0.000119229	0.00015708
0.0909	JPM	0.00016675	0.00025505	0.00025491	0.00025963	0.000153759	0.00010465
0.0909	AMZN	0.00028673	7.3429E-05	0.000160483	0.00010885	7.53181E-05	5.8515E-0
0.0909	KO	0.00011292	0.0001207	0.000110506	0.00011693	0.000143025	9.19294E-0
0.0909	GOOG	0.000288	0.00012374	0.000176265	0.00014531	0.000107088	8.16259E-0
0.0909	WELL	0.00016594	0.00023623	0.000221143	0.00021548	0.000193106	8.22327E-0
1.00000000		1.6694E-05	1.7695E-05	2.0312E-05	1.7925E-05	1.28568E-05	8.61131E-0

AV RETURN PER STOCK		
MSFT	0.11690000	0.01926
XOM	0.07578000	0.02163
SCCO	0.13049	0.02456
CAT	0.10102000	0.02034
SO	0.05755900	0.01668
JNJ	0.02257200	0.01254
JPM	0.07775700	0.02010
AMZN	0.079999	0.02212
KO	0.03779600	0.01319
GOOG	0.11642000	0.02017
WELL	0.06525700	0.02530

VAR	0.00017718
STDV	0.01331103
AV RETURN	=SUMPRODUCT(A1270:A1280,B1286:B1296
SHARPE RATIO	5.9867554
DAILY RISKFREE RATE	0.00045100

Calculating the Sharpe Ratio:

VAR	0.00017718
STDV	0.01331103
AV RETURN	0.08014091
SHARPE RATIO	=(F1288-F1291)/F1287
DAILY RISKFREE RATE	0.00045100

Finding the return I would have gotten if I started investing in the optimal portfolio in 2019:

Calculating daily return:

AF	AG	AH	AI	AJ	AK	AL	AM	AN	AO	AP	AQ
							MSFT	SCCO	CAT	SO	GOOG
MSFT	SCCO	CAT	SO	GOOG	rp	weights	34.96%	21.63%	17.03%	6.01%	20.37%
0.6633%	-1.1432%	-0.5491%	1.8600%	1.2114%	=SUMPRODUCT(AM2:AQ2,AF3:AJ3)			21.63%	17.03%	6.01%	20.37%
0.6443%	1.1564%	-0.1914%	0.8056%	0.9296%	0.006806		34.96%	21.63%	17.03%	6.01%	20.37%
-0.2910%	-0.8574%	-0.0737%	0.0888%	0.8925%	-0.001126		34.96%	21.63%	17.03%	6.01%	20.37%

Calculating the value of the portfolio:

actual retur	1000000
0.002497	=AC2*(1+(AB3))
0.006806	1009319.262

Calculating the value of the S&P portfolio:

S&P return	1000000	a
0.00293	=AA2*(1+(Z3))	
0.00767	1010622.97	

Models:

Future value of the MPT Portfolio considering the standard deviation:

$$\begin{aligned} &Value\ of\ the\ portfolio\ in\ day\ t \\ &\qquad = initial\ investment \\ &\qquad \times \left(1 + \frac{daily\ average\ percentage\ return - standard\ deviation}{100}\right)^{t-1} \end{aligned}$$

$$Minimum\ value\ of\ the\ portfolio\ in\ day\ t = 1000000 \times \left(1 + \frac{0.113 - 0.0159}{100}\right)^{t-1}$$

$$Minimum\ value\ of\ the\ portfolio\ in\ day\ t = 1000000 \times \left(1 + \frac{0.0971}{100}\right)^{t-1}$$

$$Maximum\ value\ of\ the\ portfolio\ in\ day\ t = 1000000 \times \left(1 + \frac{0.113 + 0.0159}{100}\right)^{t-1}$$

R-Squared for Model 1:

R-Squared Calculation for Model 1

y_i	$\hat{y}_i$	$(y_i - \hat{y}_i)^2$	$(y_i - \bar{y}_i)^2$
1000000	$1000000 \times (1 + \frac{0.113}{100})^{1-1} = 1000000$	$(1000000 - 1000000)^2 = 0$	$\bar{y}_i = \frac{1000000+1406729.65+2215280.16+2087477.42+3426718.54}{5}$ $\bar{y}_i = 2027241.15$ $(1000000 - 2027241.15)^2 = 1.06 \times 10^{12}$
1406729.65	1401690.83	25389707	3.85×10^{11}
2215280.16	1966957.35	6.17×10^{10}	3.54×10^{10}
2087477.42	2760181.57	4.53×10^{11}	3.63×10^{9}
3426718.54	3873293.10	1.99×10^{11}	1.96×10^{12}

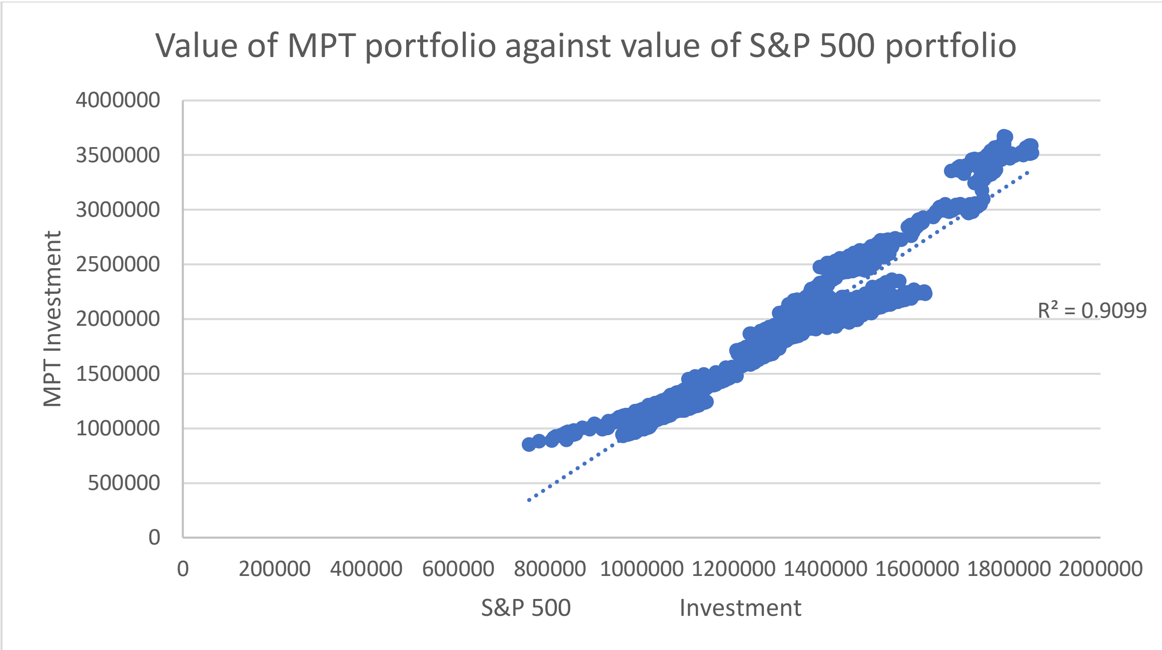
Relationship between MPT and S&P 500:

Sample table with S&P 500 returns:

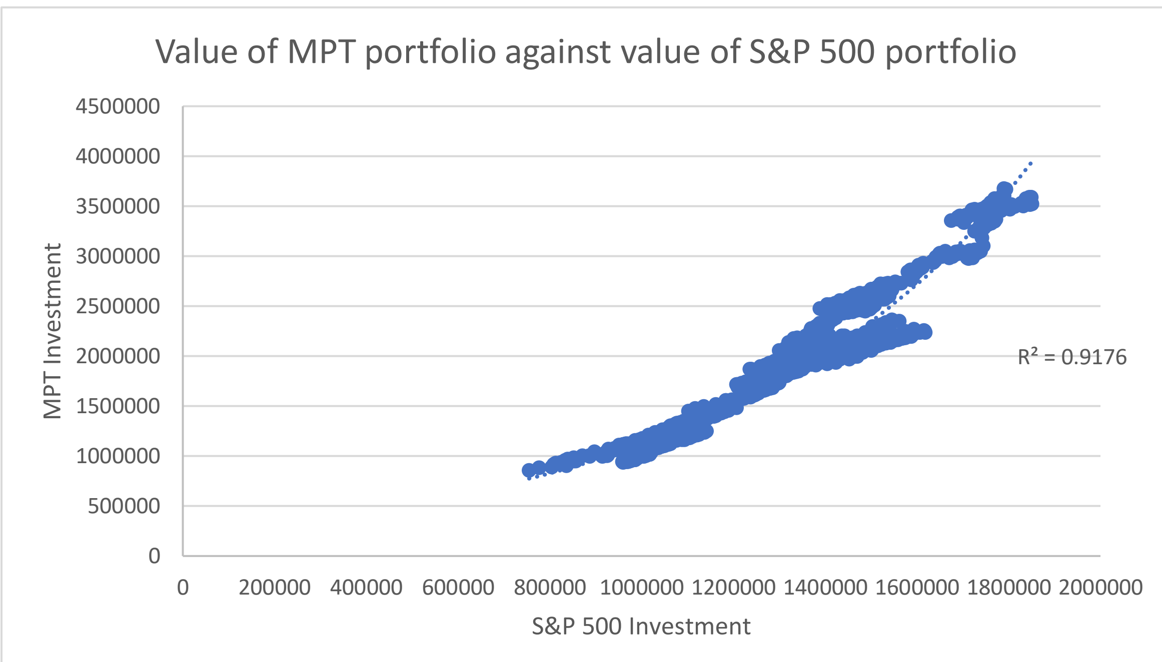
Day	S&P return
1	0.00293
2	0.00767
3	-0.00181
4	-0.00484
5	0.00124
6	0.00451
7	0.00229
8	0.00462
9	0.00018
10	-0.00340

Models for the relationship between the S&P 500 and the MPT Portfolio:

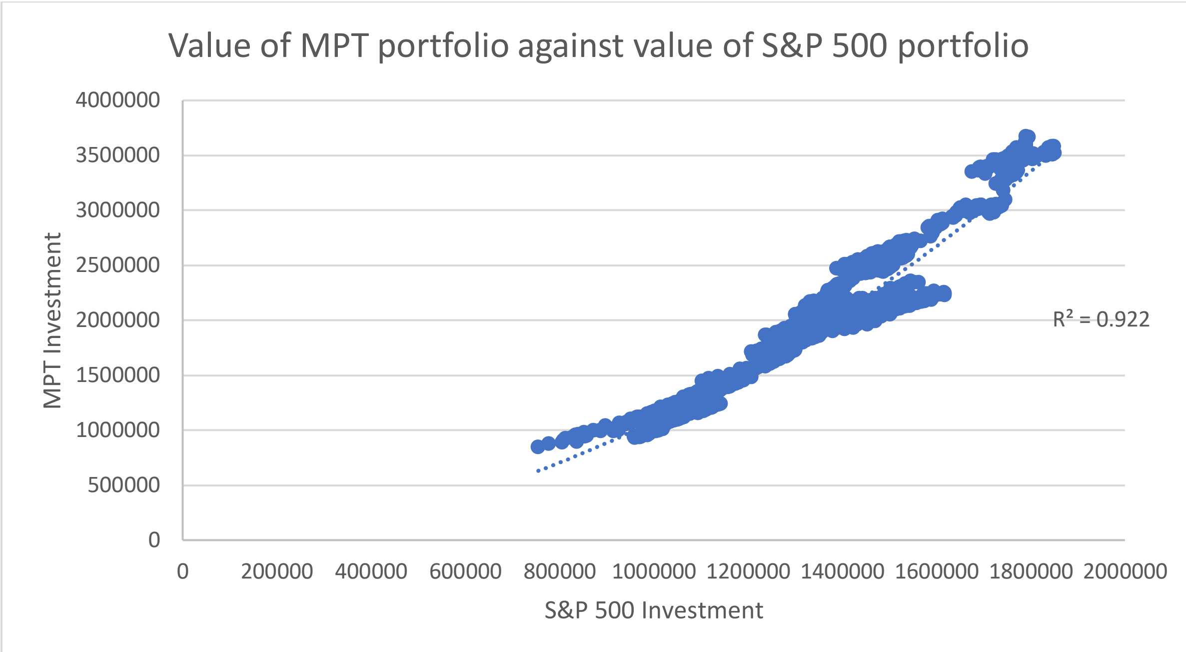
Linear model:



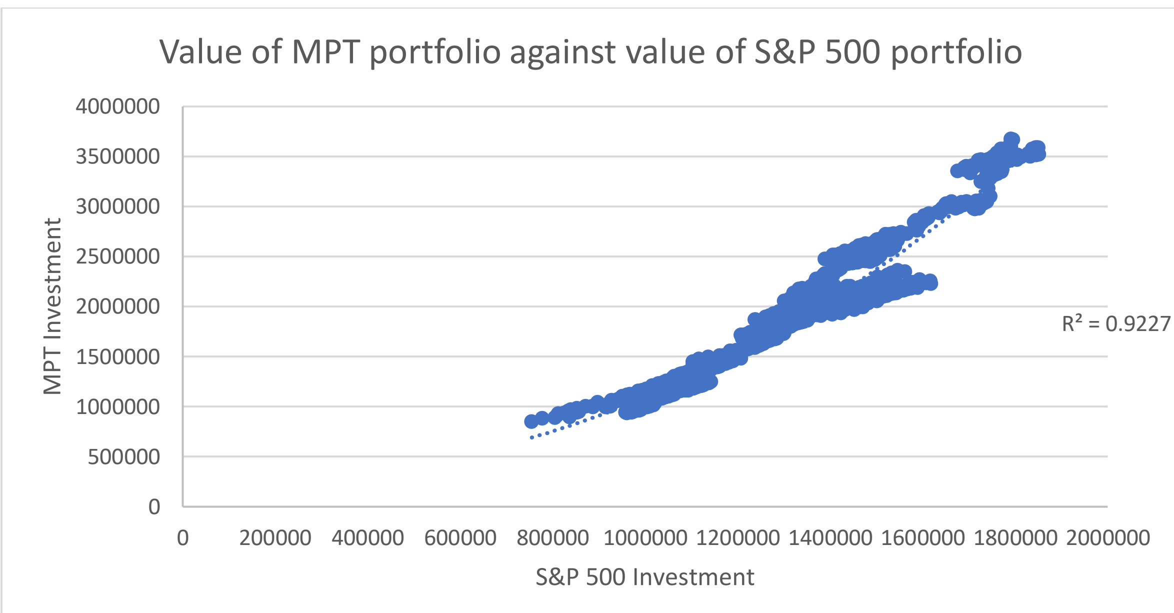
Exponential model:



Power model:



Polynomial model:



Quadratic regression using matrices:

$$\begin{pmatrix} 1000000^2 & 1000000 & 1 \\ 1568405^2 & 1568405 & 1 \\ 1842062.12^2 & 1842062.12 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 1000000 \\ 2215280.15 \\ 3562132.22 \end{pmatrix}$$

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 1000000^2 & 1000000 & 1 \\ 1568405^2 & 1568405 & 1 \\ 1842062.12^2 & 1842062.12 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1000000 \\ 2215280.15 \\ 3562132.22 \end{pmatrix}$$

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 2.09 \times 10^{-12} & -6.43 \times 10^{-12} & 4.34 \times 10^{-12} \\ -7.13 \times 10^{-6} & 1.83 \times 10^{-5} & -1.110 \times 10^{-5} \\ 6.04 & -11.8 & 6.81 \end{pmatrix} \begin{pmatrix} 1000000 \\ 2215280.15 \\ 3562132.22 \end{pmatrix}$$

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} (2.09 \times 10^{-12} \times 1000000) + (-6.43 \times 10^{-12} \times 2215280.15) + (4.34 \times 10^{-12} \times 3562132.22) \\ (-7.13 \times 10^{-6} \times 1000000) + (1.83 \times 10^{-5} \times 2215280.15) + (-1.110 \times 10^{-5} \times 3562132.22) \\ (6.04 \times 1000000) + (-11.84 \times 2215280.15) + (6.81 \times 3562132.22) \end{pmatrix}$$

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 3.31 \times 10^{-6} \\ -6.35 \\ 4046659.38 \end{pmatrix}$$

Calculating R-Squared for CAPM:

R-Squared Calculation for CAPM

y_i	$\hat{y}_i$	$(y_i - \hat{y}_i)^2$	$(y_i - \bar{y}_i)^2$
1000000	$1000000 \times (1 + \big(0.000451 + 2.55(0 - 0.000000451)\big) = 1000449.85$	$(1000000 - 999300.95 = 499560.9$	$\bar{y}_i = 2027241.15$ $(1000000 - 2027241.15)^2 = 1.06 \times 10^{12}$
1406729.65	1134094.15	7.43×10^{10}	3.85×10^{11}
2215280.16	2393077.06	3.16×10^{10}	3.54×10^{10}
2087477.42	1422751.18	4.42×10^{11}	3.63×10^{9}
3426718.54	2649888.44	6.06×10^{11}	1.96×10^{12}

$$R^2 = 1 - \frac{0 + 499560.9 + 7.43 \times 10^{10} + 3.16 \times 10^{10} + 4.42 \times 10^{11} + 6.06 \times 10^{11}}{1.06 \times 10^{12} + 3.85 \times 10^{11} + 3.63 \times 10^{9} + 1.96 \times 10^{12}}$$

Calculating the VaR:

Because of the inability of the standard deviation to account for all fluctuations in the value of the portfolio, I decided to calculate the value at risk (VaR) of the portfolio, which is a way to calculate the maximum loss expected for an investment portfolio with a significance level of α%.

Because of the central limit theorem, which suggests that the average of a large number of independent random variables follow a normal distribution, it could be assumed that the historical returns of the assets follow a normal distribution. The calculation of the VaR uses this assumption, however, there are limitations to this approach, as financial returns are fat-tailed, meaning that more extreme events happen than expected in a normal distribution.

Assuming that daily returns are normally distributed, we can calculate the VaR using the formula:[22]

$$VaR = \bar{R}_p - \sigma_p \times z_a$$

Where:

z_a= critical value from the normal distribution corresponding to the confidence level α

For this investigation, I will take the significance level as 95%. Looking at the standard normal distribution, the z-value for a confidence interval of 95% is 1.96[23]. With that, I can substitute the values into the formula:

$$VaR = 0.113473303 - 0.015697113 \times 1.96 = 0.08270$$

[22] https://www.investopedia.com/terms/v/var.asp

[23] https://sphweb.bumc.bu.edu/otlt/MPH-Modules/BS/BS704_Confidence_Intervals/BS704_Confidence_Intervals_print.html#:~:text=The%20Z%20value%20for%2095%25%20confidence%20is%20Z%3D1.96.

This means that there is a 5% chance that the portfolio could incur a daily loss greater than 0.0827%. This is a significant number as the expected return calculated was 0.113%, thus showing that even though the MFT tries to minimize risk for a given return, investments in stocks are unpredictable due to their high volatility, which makes them really difficult to model.