

The Ecological Effects of Trait Variation in a u -Predator, v -Prey System

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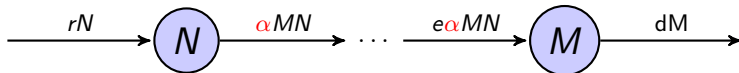
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Supported By:	Pacific Math Alliance, PUMP,	CSU Northridge

Observations

- Predator/Prey interactions are prevalent in nature
 - Crab vs. gastropod
 - Protist vs. bacteria
- There is trait variation within species
 - Thickness of plant cuticula
 - Strength of gastropod shell
- Incorporating trait variation provides richer dynamics than classical Lotka-Volterra models



Classical Lotka-Volterra Model



Variables

- $N \equiv$ Prey Density
- $M \equiv$ Predator Density

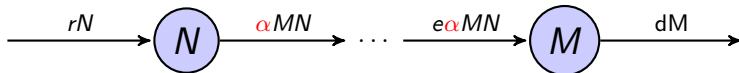
Parameters

- $\alpha \equiv$ Attack rate
- $r \equiv$ Prey birth rate
- $e \equiv$ Efficiency
- $d \equiv$ Predator death rate

$$\frac{dN}{dt} = N(r - \alpha M)$$

$$\frac{dM}{dt} = M(e\alpha N - d)$$

Classical Lotka-Volterra Model



Variables

- $N \equiv$ Prey Density
- $M \equiv$ Predator Density

Parameters

- $\alpha \equiv$ Attack rate ← No variation!
- $r \equiv$ Prey birth rate
- $e \equiv$ Efficiency
- $d \equiv$ Predator death rate

$$\frac{dN}{dt} = N(r - \alpha M)$$

$$\frac{dM}{dt} = M(e\alpha N - d)$$

Schreiber, Bürger, and Bolnick's Extension

Assume the **Predator Species** has a normally distributed trait value.

$$p(m, \bar{m}) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp \left[-\frac{(m - \bar{m})^2}{2\sigma^2} \right]$$

Attack Rate as a Function of the **Predator's Trait Value**

$$a(m) = \alpha \exp \left[-\frac{(m - \theta)^2}{2\tau^2} \right]$$

Variables

- $N \equiv$ Prey Density
- $M \equiv$ Predator Density
- (((No Prey Character)))
- $m \equiv$ **Predator Character (Trait Value)**

Parameters

- $\alpha \equiv$ Maximum attack rate
- $\theta \equiv$ **Optimal trait value**
- $\tau \equiv$ Specialization Constant
- $\sigma^2 \equiv$ Predator Trait Variance

Schreiber, Bürger, and Bolnick's Extension

Assume the **Predator Species** has a normally distributed trait value.

$$p(m, \bar{m}) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp \left[-\frac{(m - \bar{m})^2}{2\sigma^2} \right]$$

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- $M \equiv$ Predator Density
- (((No Prey Character)))
- $m \equiv$ **Predator Character (Trait Value)**

Parameters

- $\alpha \equiv$ Maximum attack rate
- $\theta \equiv$ **Optimal trait value** \leftarrow *No variation!*
- $\tau \equiv$ Specialization Constant
- $\sigma^2 \equiv$ Predator Trait Variance

Our Extension

Assume the **Predator** and **Prey** has a normally distributed trait value.

$$p(m, \bar{m}) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp \left[-\frac{(m - \bar{m})^2}{2\sigma^2} \right]$$

Attack Rate as a Function of the **Predator's Trait Value**

$$a(m) = \alpha \exp \left[-\frac{(m - \theta)^2}{2\tau^2} \right]$$

Variables

- $N \equiv$ Prey Density
- $M \equiv$ Predator Density
- $n \equiv$ Prey Character (Trait Value)
- $m \equiv$ Predator Character (Trait Value)

Parameters

- $\alpha \equiv$ Maximum attack rate
- $\theta \equiv$ Optimal trait *difference*
- $\tau \equiv$ Specialization Constant

Distribution Assumptions

- Trait values are **normally distributed** over the populations

$$p(n, \bar{n}) = \frac{1}{\sqrt{2\pi\beta^2}} \exp \left[-\frac{(n - \bar{n})^2}{2\beta^2} \right]$$

$$p(m, \bar{m}) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp \left[-\frac{(m - \bar{m})^2}{2\sigma^2} \right]$$

Variables

- $N \equiv$ Prey Density
- $\bar{n} \equiv$ Mean Prey Character
- $M \equiv$ Predator Density
- $\bar{m} \equiv$ Mean Predator Character

Parameters

- $\beta^2 \equiv$ Prey Trait Variance
- $\sigma^2 \equiv$ Predator Trait Variance

Average Attack Rate

$$\begin{aligned}\bar{a}(\bar{m}, \bar{n}) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} a(m, n) \cdot p(m, \bar{m}) \cdot p(n, \bar{n}) dm dn \\ &= \frac{\alpha \tau}{\sqrt{\sigma^2 + \beta^2 + \tau^2}} \exp \left[-\frac{(\bar{m} - \bar{n} - \theta)^2}{2(\sigma^2 + \beta^2 + \tau^2)} \right]\end{aligned}$$

Variables

- $N \equiv$ Prey Density
- $\bar{n} \equiv$ Mean Prey Character
- $M \equiv$ Predator Density
- $\bar{m} \equiv$ Mean Predator Character

Parameters

- $\beta^2 \equiv$ Prey Trait Variance
- $\sigma^2 \equiv$ Predator Trait Variance
- $\alpha \equiv$ Maximum attack rate
- $\theta \equiv$ Optimal trait difference
- $\tau \equiv$ Specialization Constant

Fitness Assumptions

- Prey experiences **logistic growth** in absence of predator
- Predator experiences **exponential decay** in absence of prey

$$Y(m, n, M, N) = r \left(1 - \frac{N}{K} \right) - Ma(m, n)$$

$$W(m, n, N) = eNa(m, n) - d$$

Variables

- $N \equiv$ Prey Density
- $n \equiv$ Prey Character
- $M \equiv$ Predator Density
- $m \equiv$ Predator Character

Parameters

- $r \equiv$ Intrinsic Prey Growth Rate
- $K \equiv$ Prey Carrying Capacity
- $d \equiv$ Predator Death Rate
- $e \equiv$ Efficiency

Average Fitness

$$\begin{aligned}\bar{Y}(\bar{m}, \bar{n}, M, N) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} Y(m, n, M, N) \cdot p(m, \bar{m}) \cdot p(n, \bar{n}) dm dn \\ &= r \left(1 - \frac{N}{K} \right) - M \bar{a}(\bar{m}, \bar{n})\end{aligned}$$

$$\begin{aligned}\bar{W}(\bar{m}, \bar{n}, N) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} W(m, n, N) \cdot p(m, \bar{m}) \cdot p(n, \bar{n}) dm dn \\ &= eN \bar{a}(\bar{m}, \bar{n}) - d\end{aligned}$$

Variables

- $N \equiv$ Prey Density
- $\bar{n} \equiv$ Mean Prey Character
- $M \equiv$ Predator Density
- $\bar{m} \equiv$ Mean Predator Character

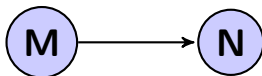
Parameters

- $r \equiv$ Intrinsic Prey Growth Rate
- $K \equiv$ Prey Carrying Capacity
- $d \equiv$ Predator Death Rate
- $e \equiv$ Efficiency

Ecological Components

$$\frac{dN}{dt} = N \cdot \overline{Y}(\overline{m}, \overline{n}, M, N) = N \left[r \left(1 - \frac{N}{K} \right) - M \overline{a}(\overline{m}, \overline{n}) \right]$$

$$\frac{dM}{dt} = M \cdot \overline{W}(\overline{m}, \overline{n}, N) = M [eN \overline{a}(\overline{m}, \overline{n}) - d]$$



Variables

- $N \equiv$ Prey Density
- $\overline{n} \equiv$ Mean Prey Character
- $M \equiv$ Predator Density
- $\overline{m} \equiv$ Mean Predator Character

Parameters

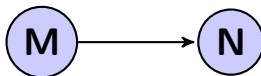
- $r \equiv$ Intrinsic Prey Growth Rate
- $K \equiv$ Prey Carrying Capacity
- $d \equiv$ Predator Death Rate
- $e \equiv$ Efficiency

Evolutionary Components

- The evolution of the mean character is always in the direction which increases the mean fitness in the population.

$$\frac{d\bar{n}}{dt} = \beta_G^2 \frac{\partial \bar{Y}}{\partial \bar{n}} = \beta_G^2 \frac{M(\theta + \bar{n} - \bar{m})}{\sigma^2 + \beta^2 + \tau^2} \bar{a}(\bar{m}, \bar{n})$$

$$\frac{d\bar{m}}{dt} = \sigma_G^2 \frac{\partial \bar{W}}{\partial \bar{m}} = \sigma_G^2 \frac{eN(\theta + \bar{n} - \bar{m})}{\sigma^2 + \beta^2 + \tau^2} \bar{a}(\bar{m}, \bar{n})$$



Variables

- $N \equiv$ Prey Density
- $\bar{n} \equiv$ Mean Prey Character
- $M \equiv$ Predator Density
- $\bar{m} \equiv$ Mean Predator Character

Parameters

- $\beta_G^2 \equiv$ Prey genetic variance
- $\sigma_G^2 \equiv$ Predator genetic variance

The Complete 1×1 Model (One Predator Species, One Prey Species)

Ecological Components

$$\frac{dN}{dt} = N \cdot \overline{Y}(\overline{m}, \overline{n}, M, N) = N \left[r \left(1 - \frac{N}{K} \right) - M \overline{a}(\overline{m}, \overline{n}) \right]$$

$$\frac{dM}{dt} = M \cdot \overline{W}(\overline{m}, \overline{n}, N) = M [eN \overline{a}(\overline{m}, \overline{n}) - d]$$

Evolutionary Components

$$\frac{d\overline{n}}{dt} = \beta_G^2 \frac{\partial \overline{Y}}{\partial \overline{n}} = \beta_G^2 \frac{M(\theta + \overline{n} - \overline{m})}{\sigma^2 + \beta^2 + \tau^2} \overline{a}(\overline{m}, \overline{n})$$

$$\frac{d\overline{m}}{dt} = \sigma_G^2 \frac{\partial \overline{W}}{\partial \overline{m}} = \sigma_G^2 \frac{eN(\theta + \overline{n} - \overline{m})}{\sigma^2 + \beta^2 + \tau^2} \overline{a}(\overline{m}, \overline{n})$$

Prey Fitness

$$Y(m, n, M, N) = r \left(1 - \frac{N}{K} \right) - Ma(m, n)$$

Predator Fitness

$$W(m, n, N) = eNa(m, n) - d$$

Prey Fitness

$$Y(m, n, M, N) = r \left(1 - \frac{N}{K} \right) - Ma(m, n)$$

↓

$$Y_j([m_i]_{i=1}^u, n_j, [M_i]_{i=1}^u, N_j) = r_j \left(1 - \frac{N_j}{K_j} \right) - \sum_{i=1}^u M_i a_{ij}(m_i, n_j)$$

Predator Fitness

$$W(m, n, N) = eNa(m, n) - d$$

Notation

$$[x_i]_{i=1}^u = x_1, \dots, x_u$$

Prey Fitness

$$Y(m, n, M, N) = r \left(1 - \frac{N}{K} \right) - Ma(m, n)$$

↓

$$Y_j([m_i]_{i=1}^u, n_j, [M_i]_{i=1}^u, N_j) = r_j \left(1 - \frac{N_j}{K_j} \right) - \sum_{i=1}^u M_i a_{ij}(m_i, n_j)$$

Predator Fitness

$$W(m, n, N) = eNa(m, n) - d$$

↓

$$W_i(m_i, [n_j]_{j=1}^v, [N_j]_{j=1}^v) = \sum_{j=1}^v \left[e_{ij} N_j a_{ij}(m_i, n_j) \right] - d_i$$

Notation

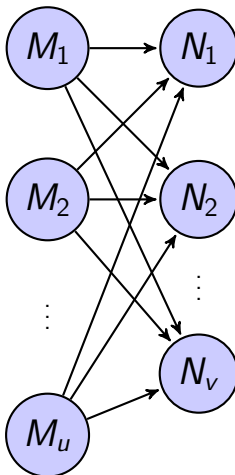
$$[x_i]_{i=1}^u = x_1, \dots, x_u$$

Average Fitness

$$\begin{aligned}
 \bar{Y}_j([\bar{m}_i]_{i=1}^u, \bar{n}_j, [M_i]_{i=1}^u, N_j) \\
 &= \int_{\mathbb{R}^{u+1}} Y_j \cdot \prod_{i=1}^u \left[p_i(m_i, \bar{m}_i) \right] \cdot p(n, \bar{n}) \prod_{i=1}^u \left[dm_i \right] dn_j \\
 &= r_j \left(1 - \frac{N_j}{K_j} \right) - \sum_{i=1}^u M_i \bar{a}_{ij}(\bar{m}_i, \bar{n}_j)
 \end{aligned}$$

$$\begin{aligned}
 \bar{W}_i(\bar{m}_i, [\bar{n}_j]_{j=1}^v, [N_j]_{j=1}^v) \\
 &= \int_{\mathbb{R}^{u+1}} W_i \cdot p_i(m_i, \bar{m}_i) \cdot \prod_{j=1}^v \left[p(n_j, \bar{n}_j) \right] dm_i \prod_{j=1}^v \left[dn_j \right] \\
 &= \sum_{j=1}^v \left[e_{ij} N_j \bar{a}_{ij}(\bar{m}_i, \bar{n}_j) \right] - d_i
 \end{aligned}$$

The Complete $u \times v$ Model (u Predator Species, v Prey Species)



The Complete $u \times v$ Model (u Predator Species, v Prey Species)

Ecological Components

$$\frac{dN_j}{dt} = N_j \bar{Y}_j = N_j \left[r_j \left(1 - \frac{N_j}{K_j} \right) - \sum_{i=1}^u M_i \bar{a}_{ij}(\bar{m}_i, \bar{n}_j) \right]$$

$$\frac{dM_i}{dt} = M_i \bar{W}_i = M_i \left[\sum_{j=1}^v \left[e_{ij} N_j \bar{a}_{ij}(\bar{m}_i, \bar{n}_j) \right] - d_i \right]$$

Evolutionary Components

$$\frac{d\bar{n}_j}{dt} = \beta_{Gj}^2 \frac{\partial \bar{Y}_j}{\partial \bar{n}_j} = \beta_{Gj}^2 \sum_{i=1}^u \left[\frac{M_i (\theta_{ij} + \bar{n}_j - \bar{m}_i)}{\sigma_i^2 + \beta_j^2 + \tau_{ij}^2} \bar{a}_{ij}(\bar{m}_i, \bar{n}_j) \right]$$

$$\frac{d\bar{m}_i}{dt} = \sigma_{Gi}^2 \frac{\partial \bar{W}_i}{\partial \bar{m}_i} = \sigma_{Gi}^2 \sum_{j=1}^v \left[\frac{e_{ij} N_j (\theta_{ij} + \bar{n}_j - \bar{m}_i)}{\sigma_i^2 + \beta_j^2 + \tau_{ij}^2} \bar{a}_{ij}(\bar{m}_i, \bar{n}_j) \right]$$

Equilibria - 1 × 1

$$\frac{dN}{dt} = N \cdot \bar{Y}(\bar{m}, \bar{n}, M, N)$$

$$\frac{d\bar{n}}{dt} = \beta_G^2 \frac{\partial \bar{Y}}{\partial \bar{n}}$$

$$\frac{dM}{dt} = M \cdot \bar{W}(\bar{m}, \bar{n}, N)$$

$$\frac{d\bar{m}}{dt} = \sigma_G^2 \frac{\partial \bar{W}}{\partial \bar{m}}$$

Extinction

$$(N^*, M^*, \bar{n}^*, \bar{m}^*) = (0, 0, _, _)$$

Exclusion

$$(N^*, M^*, \bar{n}^*, \bar{m}^*) = (K, 0, _, _)$$

Equilibria - 1 × 1

$$\frac{dN}{dt} = N \cdot \bar{Y}(\bar{m}, \bar{n}, M, N)$$

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Extinction *Unstable*

$$(N^*, M^*, \bar{n}^*, \bar{m}^*) = (0, 0, _, _)$$

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Extinction *Unstable*

$$(N^*, M^*, \bar{n}^*, \bar{m}^*) = (0, 0, _, _)$$

Exclusion *Stable under certain conditions*

$$(N^*, M^*, \bar{n}^*, \bar{m}^*) = (K, 0, _, _)$$

Equilibria - 1 × 1

$$\frac{dN}{dt} = N \cdot \bar{Y}(\bar{m}, \bar{n}, M, N)$$

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$$\frac{dM}{dt} = M \cdot \bar{W}(\bar{m}, \bar{n}, N)$$

$$\frac{d\bar{m}}{dt} = \sigma_G^2 \frac{\partial \bar{W}}{\partial \bar{m}}$$

Coexistence

$$(N^*, M^*, \bar{n}^*, \bar{m}^*) = \left(\frac{d\sqrt{A}}{e\alpha\tau}, \frac{r\sqrt{A}}{\alpha\tau} \left(1 - \frac{d\sqrt{A}}{Ke\alpha\tau} \right), \mu^*, \mu^* - \theta \right)$$

where $A = \sigma^2 + \beta^2 + \tau^2$ and μ^* is an arbitrary value.

Equilibria - 1 × 1

$$\frac{dN}{dt} = N \cdot \bar{Y}(\bar{m}, \bar{n}, M, N)$$

$$\frac{d\bar{n}}{dt} = \beta_G^2 \frac{\partial \bar{Y}}{\partial \bar{n}}$$

$$\frac{dM}{dt} = M \cdot \bar{W}(\bar{m}, \bar{n}, N)$$

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Coexistence *Stable under certain conditions*

$$(N^*, M^*, \bar{n}^*, \bar{m}^*) = \left(\frac{d\sqrt{A}}{e\alpha\tau}, \frac{r\sqrt{A}}{\alpha\tau} \left(1 - \frac{d\sqrt{A}}{Ke\alpha\tau} \right), \mu^*, \mu^* - \theta \right)$$

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Coexistence *Stable under certain conditions*

$$(N^*, M^*, \bar{n}^*, \bar{m}^*) = \left(\frac{d\sqrt{A}}{e\alpha\tau}, \frac{r\sqrt{A}}{\alpha\tau} \left(1 - \frac{d\sqrt{A}}{Ke\alpha\tau} \right), \mu^*, \mu^* - \theta \right)$$

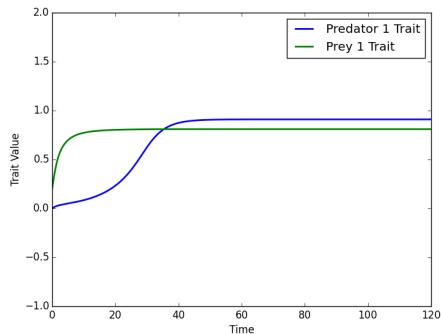
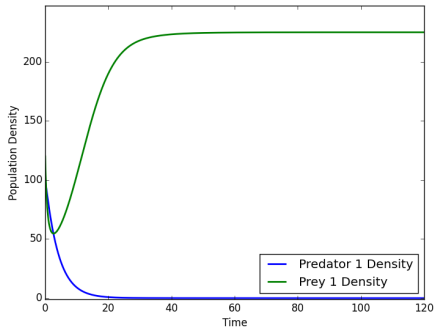
where $A = \sigma^2 + \beta^2 + \tau^2$ and μ^* is an arbitrary value.

Necessary Condition for Stable Coexistence:

- $d\sigma_G^2 > r\beta_G^2 \left(1 - \frac{d\sqrt{A}}{Ke\alpha\tau} \right)$

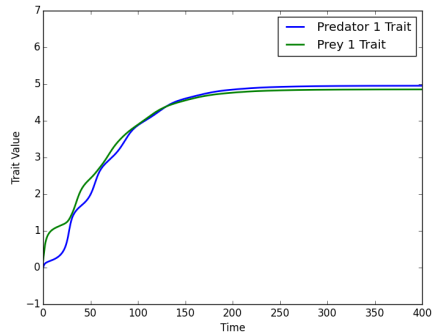
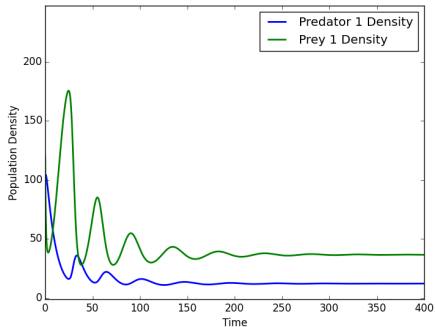
Figures - 1×1

Exclusion



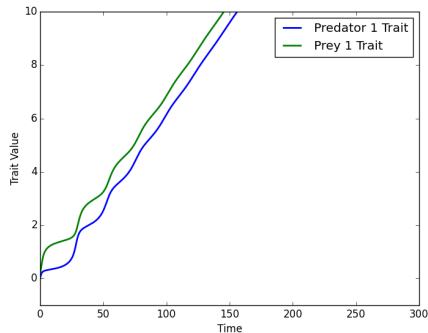
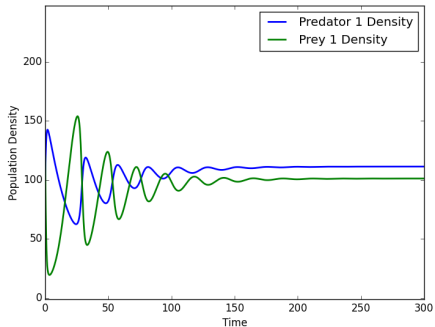
Figures - 1×1

Stable Coexistence



Figures - 1×1

Unstable Coexistence



Equilibria - 1 × 2

$$\frac{dN_1}{dt} = N_1 \cdot \bar{Y}_1(\bar{m}, \bar{n}_1, M, N_1)$$

$$\frac{dN_2}{dt} = N_2 \cdot \bar{Y}_2(\bar{m}, \bar{n}_2, M, N_2)$$

$$\frac{dM}{dt} = M \cdot \bar{W}(\bar{m}, \bar{n}_1, \bar{n}_2, N_1, N_2)$$

$$\frac{d\bar{n}_1}{dt} = \beta_{G1}^2 \frac{\partial \bar{Y}_1}{\partial \bar{n}_1}$$

$$\frac{d\bar{n}_2}{dt} = \beta_{G2}^2 \frac{\partial \bar{Y}_2}{\partial \bar{n}_2}$$

$$\frac{d\bar{m}}{dt} = \sigma_G^2 \frac{\partial \bar{W}}{\partial \bar{m}}$$

Equilibria - 1 × 2

$$\frac{dN_1}{dt} = N_1 \cdot \bar{Y}_1(\bar{m}, \bar{n}_1, M, N_1)$$

$$\frac{d\bar{n}_1}{dt} = \beta_{G1}^2 \frac{\partial \bar{Y}_1}{\partial \bar{n}_1}$$

$$\frac{dN_2}{dt} = N_2 \cdot \bar{Y}_2(\bar{m}, \bar{n}_2, M, N_2)$$

$$\frac{d\bar{n}_2}{dt} = \beta_{G2}^2 \frac{\partial \bar{Y}_2}{\partial \bar{n}_2}$$

$$\frac{dM}{dt} = M \cdot \bar{W}(\bar{m}, \bar{n}_1, \bar{n}_2, N_1, N_2)$$

$$\frac{d\bar{m}}{dt} = \sigma_G^2 \frac{\partial \bar{W}}{\partial \bar{m}}$$

Extinction

$$(N_1^*, N_2^*, M^*, \bar{n}_1^*, \bar{n}_2^*, \bar{m}^*) = (0, 0, 0, _, _, _)$$

Equilibria - 1 × 2

$$\frac{dN_1}{dt} = N_1 \cdot \bar{Y}_1(\bar{m}, \bar{n}_1, M, N_1)$$

$$\frac{d\bar{n}_1}{dt} = \beta_{G1}^2 \frac{\partial \bar{Y}_1}{\partial \bar{n}_1}$$

$$\frac{dN_2}{dt} = N_2 \cdot \bar{Y}_2(\bar{m}, \bar{n}_2, M, N_2)$$

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$$\frac{dM}{dt} = M \cdot \bar{W}(\bar{m}, \bar{n}_1, \bar{n}_2, N_1, N_2)$$

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Extinction *Unstable*

$$(N_1^*, N_2^*, M^*, \bar{n}_1^*, \bar{n}_2^*, \bar{m}^*) = (0, 0, 0, _, _, _)$$

Equilibria - 1 × 2

$$\frac{dN_1}{dt} = N_1 \cdot \bar{Y}_1(\bar{m}, \bar{n}_1, M, N_1)$$

$$\frac{d\bar{n}_1}{dt} = \beta_{G1}^2 \frac{\partial \bar{Y}_1}{\partial \bar{n}_1}$$

$$\frac{dN_2}{dt} = N_2 \cdot \bar{Y}_2(\bar{m}, \bar{n}_2, M, N_2)$$

$$\frac{d\bar{n}_2}{dt} = \beta_{G2}^2 \frac{\partial \bar{Y}_2}{\partial \bar{n}_2}$$

$$\frac{dM}{dt} = M \cdot \bar{W}(\bar{m}, \bar{n}_1, \bar{n}_2, N_1, N_2)$$

$$\frac{d\bar{m}}{dt} = \sigma_G^2 \frac{\partial \bar{W}}{\partial \bar{m}}$$

Extinction Unstable

$$(N_1^*, N_2^*, M^*, \bar{n}_1^*, \bar{n}_2^*, \bar{m}^*) = (0, 0, 0, _, _, _)$$

Exclusion

$$(N_1^*, N_2^*, M^*, \bar{n}_1^*, \bar{n}_2^*, \bar{m}^*) = (K_1, K_2, 0, _, _, _)$$

Equilibria - 1 × 2

$$\frac{dN_1}{dt} = N_1 \cdot \bar{Y}_1(\bar{m}, \bar{n}_1, M, N_1)$$

$$\frac{d\bar{n}_1}{dt} = \beta_{G1}^2 \frac{\partial \bar{Y}_1}{\partial \bar{n}_1}$$

$$\frac{dN_2}{dt} = N_2 \cdot \bar{Y}_2(\bar{m}, \bar{n}_2, M, N_2)$$

$$\frac{d\bar{n}_2}{dt} = \beta_{G2}^2 \frac{\partial \bar{Y}_2}{\partial \bar{n}_2}$$

$$\frac{dM}{dt} = M \cdot \bar{W}(\bar{m}, \bar{n}_1, \bar{n}_2, N_1, N_2)$$

$$\frac{d\bar{m}}{dt} = \sigma_G^2 \frac{\partial \bar{W}}{\partial \bar{m}}$$

Extinction *Unstable*

$$(N_1^*, N_2^*, M^*, \bar{n}_1^*, \bar{n}_2^*, \bar{m}^*) = (0, 0, 0, _, _, _)$$

Exclusion *Stable under certain conditions*

$$(N_1^*, N_2^*, M^*, \bar{n}_1^*, \bar{n}_2^*, \bar{m}^*) = (K_1, K_2, 0, _, _, _)$$

Equilibria - 1 × 2

$$\frac{dN_1}{dt} = N_1 \cdot \bar{Y}_1(\bar{m}, \bar{n}_1, M, N_1)$$

$$\frac{dN_2}{dt} = N_2 \cdot \bar{Y}_2(\bar{m}, \bar{n}_2, M, N_2)$$

$$\frac{dM}{dt} = M \cdot \bar{W}(\bar{m}, \bar{n}_1, \bar{n}_2, N_1, N_2)$$

$$\frac{d\bar{n}_1}{dt} = \beta_{G1}^2 \frac{\partial \bar{Y}_1}{\partial \bar{n}_1}$$

$$\frac{d\bar{n}_2}{dt} = \beta_{G2}^2 \frac{\partial \bar{Y}_2}{\partial \bar{n}_2}$$

$$\frac{d\bar{m}}{dt} = \sigma_G^2 \frac{\partial \bar{W}}{\partial \bar{m}}$$

Equilibria - 1 × 2

$$\frac{dN_1}{dt} = N_1 \cdot \bar{Y}_1(\bar{m}, \bar{n}_1, M, N_1)$$

$$\frac{d\bar{n}_1}{dt} = \beta_{G1}^2 \frac{\partial \bar{Y}_1}{\partial \bar{n}_1}$$

$$\frac{dN_2}{dt} = N_2 \cdot \bar{Y}_2(\bar{m}, \bar{n}_2, M, N_2)$$

$$\frac{d\bar{n}_2}{dt} = \beta_{G2}^2 \frac{\partial \bar{Y}_2}{\partial \bar{n}_2}$$

$$\frac{dM}{dt} = M \cdot \bar{W}(\bar{m}, \bar{n}_1, \bar{n}_2, N_1, N_2)$$

$$\frac{d\bar{m}}{dt} = \sigma_G^2 \frac{\partial \bar{W}}{\partial \bar{m}}$$

Generalist Becomes Specialist

$$(N_1^*, N_2^*, M^*, \bar{n}_1^*, \bar{n}_2^*, \bar{m}^*)$$

$$= \left(\frac{d\sqrt{A_1}}{e_1\alpha_1\tau_1}, K_2, \frac{r_1\sqrt{A_1}}{\alpha_1\tau_1} \left(1 - \frac{d\sqrt{A_1}}{K_1e_1\alpha_1\tau_1} \right), \mu_1^*, \mu_2^*, \mu_1^* - \theta_1 \right)$$

where $A_1 = \sigma^2 + \beta_1^2 + \tau_1^2$, μ_1^* is an arbitrary value, and μ_2^* is sufficiently far from $\mu_1^* - \theta_1$.

Equilibria - 1 × 2

$$\frac{dN_1}{dt} = N_1 \cdot \bar{Y}_1(\bar{m}, \bar{n}_1, M, N_1)$$

$$\frac{d\bar{n}_1}{dt} = \beta_{G1}^2 \frac{\partial \bar{Y}_1}{\partial \bar{n}_1}$$

$$\frac{dN_2}{dt} = N_2 \cdot \bar{Y}_2(\bar{m}, \bar{n}_2, M, N_2)$$

$$\frac{d\bar{n}_2}{dt} = \beta_{G2}^2 \frac{\partial \bar{Y}_2}{\partial \bar{n}_2}$$

$$\frac{dM}{dt} = M \cdot \bar{W}(\bar{m}, \bar{n}_1, \bar{n}_2, N_1, N_2)$$

$$\frac{d\bar{m}}{dt} = \sigma_G^2 \frac{\partial \bar{W}}{\partial \bar{m}}$$

Generalist Becomes Specialist

Stable under certain conditions???

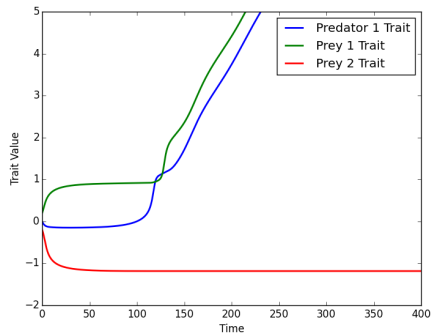
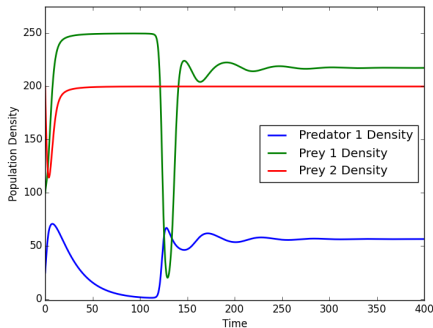
$$(N_1^*, N_2^*, M^*, \bar{n}_1^*, \bar{n}_2^*, \bar{m}^*)$$

$$= \left(\frac{d\sqrt{A_1}}{e_1\alpha_1\tau_1}, K_2, \frac{r_1\sqrt{A_1}}{\alpha_1\tau_1} \left(1 - \frac{d\sqrt{A_1}}{K_1e_1\alpha_1\tau_1} \right), \mu_1^*, \mu_2^*, \mu_1^* - \theta_1 \right)$$

where $A_1 = \sigma^2 + \beta_1^2 + \tau_1^2$, μ_1^* is an arbitrary value, and μ_2^* is sufficiently far from $\mu_1^* - \theta_1$.

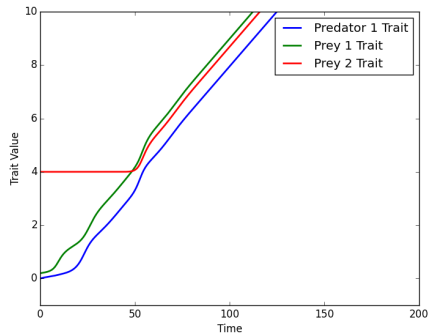
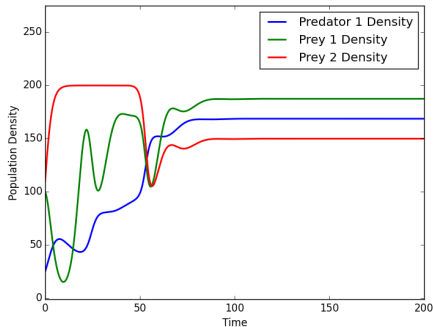
Figures - 1×2

Generalist Becomes Specialist



Figures - 1×2

Unstable Coexistence



- Two Predators competing for One Prey
- One Specialist Predator Competing with One Generalist Predator for Two Prey Species
- Two Specialist Predators Competing with One Generalist Predator for Two Prey Species
- Further Analysis of the General $u \times v$ Model
- Intra-Guild Predation
- Adding Evolutionary Cost to Prey
- Adding Evolutionary Cost to Predator

Thank You!

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Questions?