

# The Effects of Intraspecific Genetic Variation on the Dynamics of Predator-Prey Ecological Communities

Sam Fleischer, Pablo Chavarria

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## Preparing Undergraduates through Mentoring for PhDs (PUMP) Research Symposium

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Supported By: Pacific Math Alliance, PUMP, CSU Northridge

# Overview

- Motivation
  - Observations in Nature
  - Previous Models
- Our Expansions
  - Gaussian Attack Rate under Coevolution
  - Introduce Stabilizing Selection
  - General Ditrophic Expansion
- Discussion

# Observations in Nature

- Predator/Prey interactions are prevalent in nature
  - Crab vs. gastropod [Saloniemi, 1993]
  - Classical Lotka-Volterra model
    - Genetic adaptation is insignificant

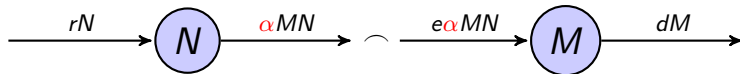


# Observations in Nature

- Predator/Prey interactions are prevalent in nature
  - Crab vs. gastropod [Saloniemi, 1993]
  - Classical Lotka-Volterra model
    - Genetic adaptation is insignificant
- There is trait variation within species, which causes **variation in fundamental model parameters**
  - Relative strength of crab claw vs. gastropod shell [Saloniemi, 1993]
  - Incorporating trait variation provides **richer dynamics** than classical Lotka-Volterra models



# Classical Lotka-Volterra Model



$$\frac{dN}{dt} = N(r - \alpha M)$$

$$\frac{dM}{dt} = M(e\alpha N - d)$$

- Prey Exponential Growth
- Predator Exponential Decay
- Linear Functional Response

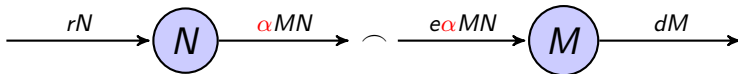
## Variables

- $N \equiv$  Prey Density
- $M \equiv$  Predator Density

## Parameters

- $\alpha \equiv$  Attack rate
- $r \equiv$  Prey Growth rate
- $e \equiv$  Efficiency
- $d \equiv$  Predator death rate

# Classical Lotka-Volterra Model



$$\frac{dN}{dt} = N(r - \alpha M)$$

$$\frac{dM}{dt} = M(e\alpha N - d)$$

## Variables

- $N \equiv$  Prey Density
- $M \equiv$  Predator Density

## Parameters

- $\alpha \equiv$  Attack rate ← No variation!
- $r \equiv$  Prey Growth rate
- $e \equiv$  Efficiency
- $d \equiv$  Predator death rate

- Prey Exponential Growth
- Predator Exponential Decay
- Linear Functional Response

# Schreiber, Bürger, and Bolnick's Expansion

Assume the **Predator Species** has a normally distributed trait value.

$$p(\textcolor{red}{m}, \bar{m}) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp \left[ -\frac{(\textcolor{red}{m} - \bar{m})^2}{2\sigma^2} \right]$$

## Parameters

- $\sigma^2 \equiv$  Predator Trait Variance

## Variables

- $m \equiv$  Predator Trait Value

# Schreiber, Bürger, and Bolnick's Expansion

Assume the **Predator Species** has a normally distributed trait value.

$$p(m, \bar{m}) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp \left[ -\frac{(m - \bar{m})^2}{2\sigma^2} \right]$$

Attack Rate is a Function of the **Predator's Trait Value**

$$a(m) = \alpha \exp \left[ -\frac{(m - \theta)^2}{2\tau^2} \right]$$

## Parameters

### Variables

- $m \equiv$  Predator Trait Value

- $\sigma^2 \equiv$  Predator Trait Variance
- $\alpha \equiv$  Maximum attack rate
- $\tau \equiv$  Specialization Constant
- $\theta \equiv$  Optimal trait value



# Schreiber, Bürger, and Bolnick's Expansion

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Attack Rate is a Function of the **Predator's Trait Value**

$$a(m) = \alpha \exp \left[ -\frac{(m - \theta)^2}{2\tau^2} \right]$$

## Parameters

### Variables

- $m \equiv$  Predator Trait Value
- (((No Prey Trait Value)))

- $\sigma^2 \equiv$  Predator Trait Variance
- $\alpha \equiv$  Maximum attack rate
- $\tau \equiv$  Specialization Constant
- $\theta \equiv$  Optimal trait value  
↑ **No variation!**

# Normally Distributed Trait Values

Assume **Prey** and **Predator** have normally distributed trait values.

$$p(n, \bar{n}) = \frac{1}{\sqrt{2\pi\beta^2}} \exp \left[ -\frac{(n - \bar{n})^2}{2\beta^2} \right]$$

$$p(m, \bar{m}) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp \left[ -\frac{(m - \bar{m})^2}{2\sigma^2} \right]$$

## Variables

- $n \equiv$  Prey Trait Value
- $\bar{n} \equiv$  **Average** Prey Trait Value
- $m \equiv$  Predator Trait Value
- $\bar{m} \equiv$  **Average** Predator Trait Value

## Parameters

- $\beta^2 \equiv$  Prey Trait Variance
- $\sigma^2 \equiv$  Predator Trait Variance

# Attack Rate

Attack Rate is a Gaussian Function of the **Prey's Trait Value** and the **Predator's Trait Value**

$$a(n, m) = \alpha \exp \left[ -\frac{((m - n) - \theta)^2}{2\tau^2} \right]$$

## Variables

- $n \equiv$  Prey Trait Value
- $\bar{n} \equiv$  **Average** Prey Trait Value
- $m \equiv$  Predator Trait Value
- $\bar{m} \equiv$  **Average** Predator Trait Value

## Parameters

- $\alpha \equiv$  Maximum attack rate
- $\theta \equiv$  **Optimal trait difference**
- $\tau^2 \equiv$  Specialization Constant

# Attack Rate

Attack Rate is a Gaussian Function of the **Prey's Trait Value** and the **Predator's Trait Value**

$$a(n, m) = \alpha \exp \left[ -\frac{((m - n) - \theta)^2}{2\tau^2} \right]$$

## Average Attack Rate

$$\begin{aligned} \bar{a}(\bar{n}, \bar{m}) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} a(n, m) \cdot p(n, \bar{n}) \cdot p(m, \bar{m}) \, dn \, dm \\ &= \frac{\alpha\tau}{\sqrt{\sigma^2 + \beta^2 + \tau^2}} \exp \left[ -\frac{((\bar{m} - \bar{n}) - \theta)^2}{2(\sigma^2 + \beta^2 + \tau^2)} \right] \end{aligned}$$

## Variables

- $n \equiv$  Prey Trait Value
- $\bar{n} \equiv$  Average Prey Trait Value
- $m \equiv$  Predator Trait Value
- $\bar{m} \equiv$  Average Predator Trait Value

## Parameters

- $\alpha \equiv$  Maximum attack rate
- $\theta \equiv$  Optimal trait difference
- $\tau^2 \equiv$  Specialization Constant
- $\beta^2 \equiv$  Prey Trait Variance
- $\sigma^2 \equiv$  Predator Trait Variance

# Fitness Assumptions

- Prey experiences **logistic growth** in absence of predator
- Predator experiences **exponential decay** in absence of prey

$$Y(N, n, M, m) = r \left( 1 - \frac{N}{K} \right) - Ma(n, m)$$

$$W(N, n, M, m) = eNa(n, m) - d$$

## Variables

- $N \equiv$  Prey Density
- $n \equiv$  Prey Trait Value
- $M \equiv$  Predator Density
- $m \equiv$  Predator Trait Value

## Parameters

- $r \equiv$  Intrinsic Prey Growth Rate
- $K \equiv$  Prey Carrying Capacity
- $d \equiv$  Predator Death Rate
- $e \equiv$  Efficiency

# Average Fitness

$$\begin{aligned}\overline{Y}(N, \bar{n}, M, \bar{m}) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} Y(N, n, M, m) \cdot p(m, \bar{m}) \cdot p(n, \bar{n}) \, dmdn \\ &= r \left( 1 - \frac{N}{K} \right) - M\bar{a}(\bar{n}, \bar{m})\end{aligned}$$

$$\begin{aligned}\overline{W}(N, \bar{n}, M, \bar{m}) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} W(N, n, M, m) \cdot p(m, \bar{m}) \cdot p(n, \bar{n}) \, dmdn \\ &= eN\bar{a}(\bar{n}, \bar{m}) - d\end{aligned}$$

## Variables

- $N \equiv$  Prey Density
- $\bar{n} \equiv$  **Average** Prey Trait Value
- $M \equiv$  Predator Density
- $\bar{m} \equiv$  **Average** Predator Trait Value

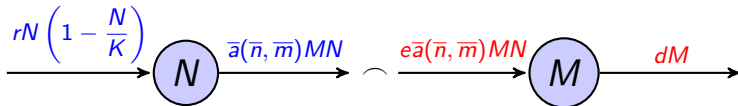
## Parameters

- $r \equiv$  Intrinsic Prey Growth Rate
- $K \equiv$  Prey Carrying Capacity
- $d \equiv$  Predator Death Rate
- $e \equiv$  Efficiency

# Ecological Components

$$\frac{dN}{dt} = N \cdot \overline{Y}(N, \bar{n}, M, \bar{m}) = N \left[ r \left( 1 - \frac{N}{K} \right) - M \bar{a}(\bar{n}, \bar{m}) \right]$$

$$\frac{dM}{dt} = M \cdot \overline{W}(N, \bar{n}, M, \bar{m}) = M [e N \bar{a}(\bar{n}, \bar{m}) - d]$$



## Variables

- $N \equiv$  Prey Density
- $\bar{n} \equiv$  **Average** Prey Trait Value
- $M \equiv$  **Predator** Density
- $\bar{m} \equiv$  **Average** Predator Trait Value

## Parameters

- $r \equiv$  Intrinsic Prey Growth Rate
- $K \equiv$  Prey Carrying Capacity
- $d \equiv$  Predator Death Rate
- $e \equiv$  Efficiency

# Evolutionary Components

- The evolution of the mean trait value is always in the direction which increases the mean fitness in the population. [Lande, 1976]

$$\frac{d\bar{n}}{dt} = \beta_G^2 \frac{\partial \bar{Y}}{\partial \bar{n}} = \beta_G^2 \frac{M(\theta - (\bar{m} - \bar{n}))}{\sigma^2 + \beta^2 + \tau^2} \bar{a}(\bar{n}, \bar{m})$$

$$\frac{d\bar{m}}{dt} = \sigma_G^2 \frac{\partial \bar{W}}{\partial \bar{m}} = \sigma_G^2 \frac{eN(\theta - (\bar{m} - \bar{n}))}{\sigma^2 + \beta^2 + \tau^2} \bar{a}(\bar{n}, \bar{m})$$

## Variables

- $N \equiv$  Prey Density
- $\bar{n} \equiv$  Mean Prey Character
- $M \equiv$  Predator Density
- $\bar{m} \equiv$  Mean Predator Character

## Parameters

- $\beta_G^2 \equiv$  Prey genetic variance
- $\sigma_G^2 \equiv$  Predator genetic variance



# The Complete $1 \times 1$ Model (One Prey Species, One Predator Species)

## Ecological Components

$$\frac{dN}{dt} = N \cdot \overline{Y}(N, \bar{n}, M, \bar{m}) = N \left[ r \left( 1 - \frac{N}{K} \right) - M \bar{a}(\bar{m}, \bar{n}) \right]$$

$$\frac{dM}{dt} = M \cdot \overline{W}(N, \bar{n}, M, \bar{m}) = M [eN \bar{a}(\bar{m}, \bar{n}) - d]$$

## Evolutionary Components

$$\frac{d\bar{n}}{dt} = \beta_G^2 \frac{\partial \overline{Y}}{\partial \bar{n}} = \beta_G^2 \frac{M(\theta - (\bar{m} - \bar{n}))}{\sigma^2 + \beta^2 + \tau^2} \bar{a}(\bar{n}, \bar{m})$$

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# Equilibria - $1 \times 1$

## Extinction

$$(N^*, M^*, \bar{n}^*, \bar{m}^*) = (0, 0, \_, \_)$$

# Equilibria - $1 \times 1$

Extinction *Unstable*

$$(N^*, M^*, \bar{n}^*, \bar{m}^*) = (0, 0, \_, \_)$$

# Equilibria - $1 \times 1$

Extinction *Unstable*

$$(N^*, M^*, \bar{n}^*, \bar{m}^*) = (0, 0, \_, \_)$$

Exclusion

$$(N^*, M^*, \bar{n}^*, \bar{m}^*) = (K, 0, \mu^*, \mu^* + \theta)$$

where  $\mu^*$  is arbitrary



# Equilibria - $1 \times 1$

## Coexistence

$$(N^*, M^*, \bar{n}^*, \bar{m}^*) = \left( \frac{d\sqrt{A}}{e\alpha\tau}, \frac{r\sqrt{A}}{\alpha\tau} \left( 1 - \frac{d\sqrt{A}}{Ke\alpha\tau} \right), \mu^*, \mu^* + \theta \right)$$

# Equilibria - $1 \times 1$

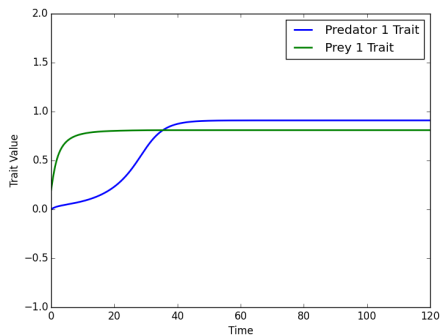
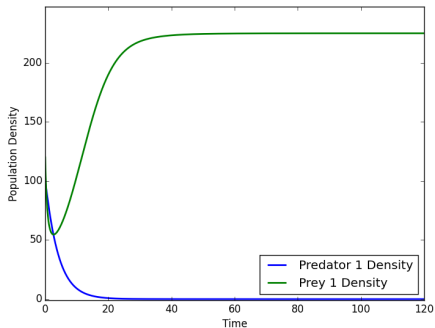
## Coexistence

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Necessary Condition for Asymptotically Stable Coexistence:

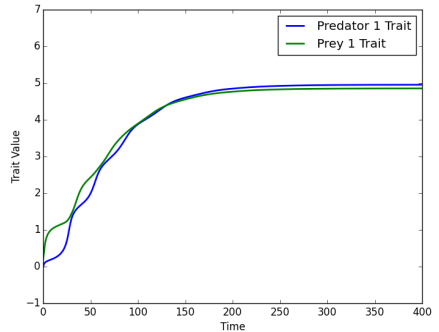
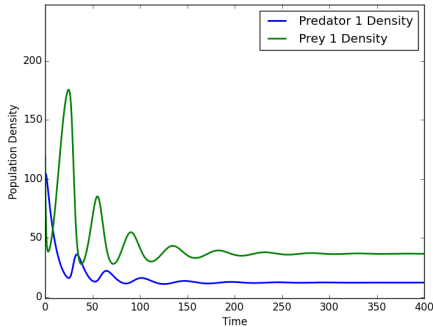
$$\frac{\sigma_G^2}{\beta_G^2} > \frac{r}{d} \left( 1 - \frac{d\sqrt{A}}{Ke\alpha\tau} \right)$$

# Figures - $1 \times 1$ - Stable Exclusion

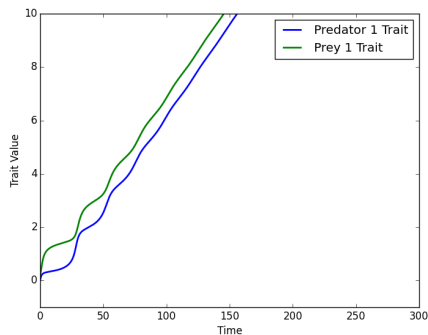
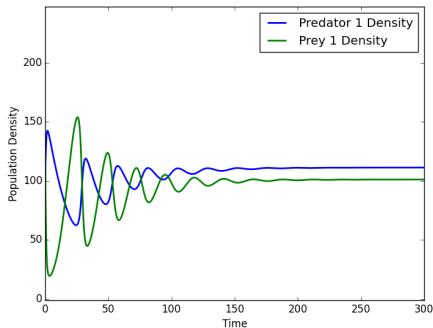




# Figures - $1 \times 1$ - Stable Coexistence



# Figures - $1 \times 1$ - “Arms Race” Coexistence



# Avoiding an “Arms Race” with Stabilizing Selection

Assume Prey Growth Rate is a Function of the **Prey's Trait Value**

$$r(n) = \rho \exp \left[ -\frac{(n - \phi)^2}{2\gamma^2} \right]$$

## Variables

- $n \equiv$  Prey Trait Value
- $\bar{n} \equiv$  **Average Prey Trait Value**

## Parameters

- $\rho \equiv$  Maximum Growth Rate
- $\phi \equiv$  **Prey Optimum Trait Value**
- $\gamma^2 \equiv$  Stabilizing Selection Constant

# Avoiding an “Arms Race” with Stabilizing Selection

Assume Prey Growth Rate is a Function of the **Prey's Trait Value**

$$r(n) = \rho \exp \left[ -\frac{(n - \phi)^2}{2\gamma^2} \right]$$

Average Growth Rate

$$\begin{aligned} \bar{r}(\bar{n}) &= \int_{-\infty}^{\infty} r(n) \cdot p(n, \bar{n}) dn \\ &= \frac{\rho\gamma}{\sqrt{\beta^2 + \gamma^2}} \exp \left[ -\frac{(n - \phi)^2}{2\gamma^2} \right] \end{aligned}$$

## Variables

- $n \equiv$  Prey Trait Value
- $\bar{n} \equiv$  Average Prey Trait Value

## Parameters

- $\rho \equiv$  Maximum Growth Rate
- $\phi \equiv$  Prey Optimum Trait Value
- $\gamma^2 \equiv$  Stabilizing Selection Constant
- $\beta^2 \equiv$  Prey Trait Variance

# Fitness Assumptions

- Prey experiences **logistic growth** in absence of predator
- Predator experiences **exponential decay** in absence of prey

$$Y(N, n, M, m) = r(n) \left(1 - \frac{N}{K}\right) - Ma(n, m)$$

$$W(N, n, M, m) = eNa(n, m) - d$$

## Variables

- $N \equiv$  Prey Density
- $n \equiv$  Prey Trait Value
- $M \equiv$  Predator Density
- $m \equiv$  Predator Trait Value

## Parameters

- $r \equiv$  Intrinsic Prey Growth Rate Function
- $K \equiv$  Prey Carrying Capacity
- $d \equiv$  Predator Death Rate
- $e \equiv$  Efficiency

# The Complete $1 \times 1$ Model (One Prey Species, One Predator Species)

## Ecological Components

$$\frac{dN}{dt} = N \cdot \bar{Y}(N, \bar{n}, M, \bar{m}) = N \left[ \bar{r}(\bar{n}) \left( 1 - \frac{N}{K} \right) - M \bar{a}(\bar{n}, \bar{m}) \right]$$

$$\frac{dM}{dt} = M \cdot \bar{W}(N, \bar{n}, M, \bar{m}) = M [eN \bar{a}(\bar{n}, \bar{m}) - d]$$

## Evolutionary Components

$$\frac{d\bar{n}}{dt} = \beta_G^2 \frac{\partial \bar{Y}}{\partial \bar{n}} = \beta_G^2 \left[ \bar{r}(\bar{n}) \left( 1 - \frac{N}{K} \right) \frac{(\phi - \bar{n})}{\beta^2 + \gamma^2} + \frac{M(\theta - (\bar{m} - \bar{n}))}{\sigma^2 + \beta^2 + \tau^2} \bar{a}(\bar{m}, \bar{n}) \right]$$

$$\frac{d\bar{m}}{dt} = \sigma_G^2 \frac{\partial \bar{W}}{\partial \bar{m}} = \sigma_G^2 \frac{eN(\theta - (\bar{m} - \bar{n}))}{\sigma^2 + \beta^2 + \tau^2} \bar{a}(\bar{n}, \bar{m})$$

# Equilibria - $1 \times 1$

## Extinction

$$(N^*, M^*, \bar{n}^*, \bar{m}^*) = (0, 0, \_, \_)$$

# Equilibria - $1 \times 1$

Extinction *Unstable*

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# Equilibria - $1 \times 1$

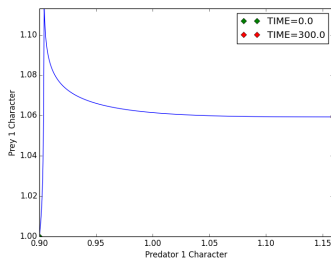
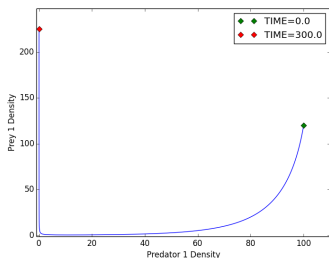
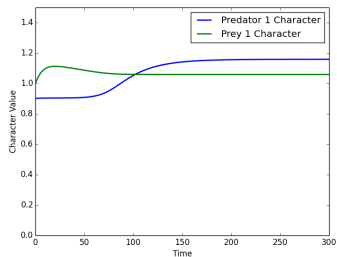
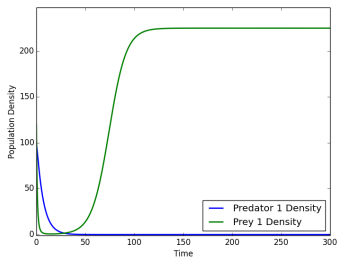
## Coexistence

$$(N^*, M^*, \bar{n}^*, \bar{m}^*) = \left( \frac{d\sqrt{A}}{e\alpha\tau}, \frac{\rho\gamma\sqrt{A}}{\alpha\tau\sqrt{B}} \left( 1 - \frac{d\sqrt{A}}{Ke\alpha\tau} \right), \phi, \theta + \phi \right)$$

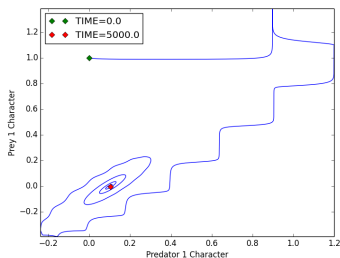
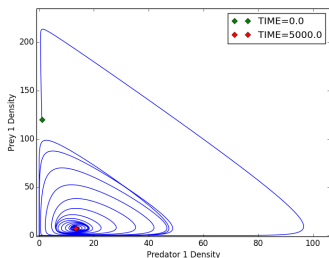
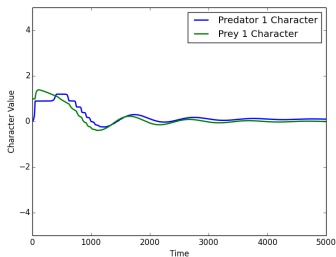
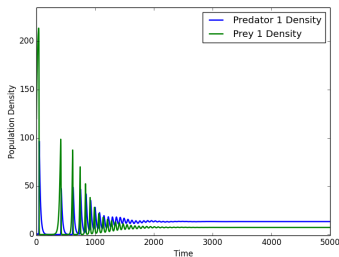
where  $A = \sigma^2 + \beta^2 + \tau^2$  and  $B = \beta^2 + \gamma^2$



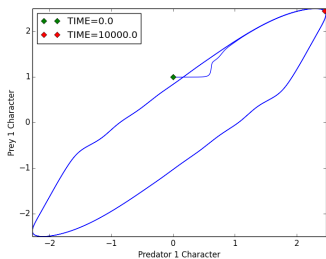
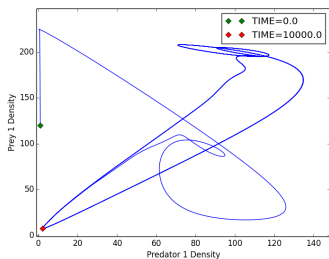
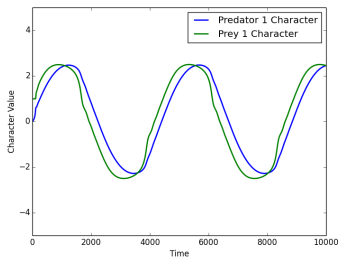
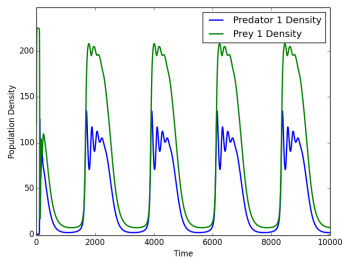
# Figures - $1 \times 1$ - Stable Exclusion



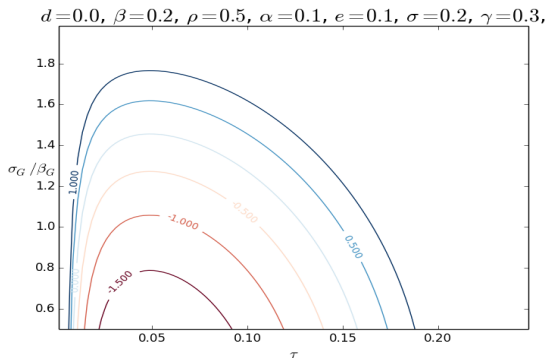
# Figures - $1 \times 1$ - Stable Coexistence



# Figures - $1 \times 1$ - Stable Cycles (Red Queen Dynamics) [Kindrik, Kondrashov, 1994]



# Contour Plot - Coexistence Asymptotic Stability Criterion



$$f_{\text{stable}}(\text{system parameters}) = \frac{\sigma_G^2}{\beta_G^2} - \frac{\rho\gamma}{d\sqrt{B}} \left( 1 - \frac{d\sqrt{A}}{Ke\alpha\tau} \right) \left( 1 - \frac{A}{B} \right)$$

$f_{\text{stable}} > 0 \implies$  Coexistence is *stable*

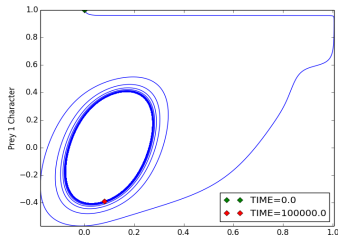
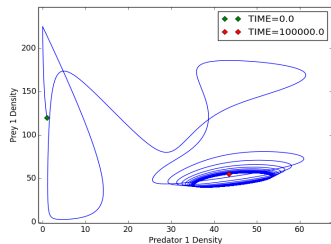
$f_{\text{stable}} < 0 \implies$  Coexistence is *unstable*

$f_{\text{stable}} = 0 \implies$  Hopf Bifurcation



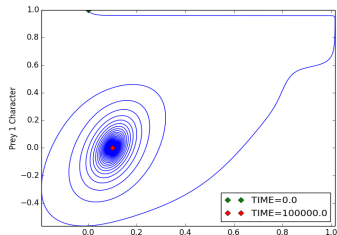
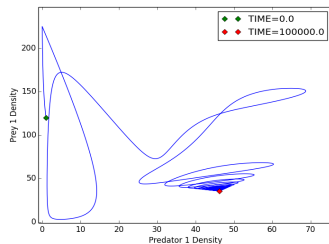
# $\tau = 0.05$ : Limit Cycle vs. Node

$$\frac{\sigma_G}{\beta_G} = 1.3 \Rightarrow f_{\text{stable}} < 0$$



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$$\frac{\sigma_G}{\beta_G} = 1.5 \Rightarrow f_{\text{stable}} > 0$$



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# Summary of the $1 \times 1$ Model

- 4-dimensional system of ODEs
  - 2 ODEs describing the change in **population size** over time
  - 2 ODEs describing the change in **trait value** over time

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- 3 equilibrium points
  - Extinction (Unstable)
  - Exclusion (Stable under certain conditions)
  - Coexistence (Stable under certain conditions)

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- 4-dimensional system of ODEs
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- 3 equilibrium points
  - Extinction (Unstable)
  - Exclusion (Stable under certain conditions)
  - Coexistence (Stable under certain conditions)
- Non-Equilibrium Dynamics
  - Constant Growth Rate
    - “Arms Race” Coexistence
  - Gaussian Growth Rate Function
    - Stable Limit Cycles

# Expansion of Fitness Functions

## Prey Fitness

$$Y(N, n, M, m) = r(n) \left( 1 - \frac{N}{K} \right) - Ma(n, m)$$

## Predator Fitness

$$W(N, n, M, m) = eNa(n, m) - d$$

# Expansion of Fitness Functions

## Prey Fitness

$$Y(N, n, M, m) = r(n) \left( 1 - \frac{N}{K} \right) - Ma(n, m)$$

↓

$$Y_j(N_j, n_j, [M_i]_{i=1}^u, [m_i]_{i=1}^u) = r_j(n_j) \left( 1 - \frac{N_j}{K_j} \right) - \sum_{i=1}^u M_i a_{ij}(n_j, m_i)$$

## Predator Fitness

$$W(N, n, M, m) = eNa(n, m) - d$$

**Notation:**  $[x_i]_{i=1}^u = x_1, \dots, x_u$

# Expansion of Fitness Functions

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## Predator Fitness

$$W(N, n, M, m) = eNa(n, m) - d$$

↓

$$W_i([N_j]_{j=1}^v, [n_j]_{j=1}^v, M_i, m_i) = \sum_{j=1}^v \left[ e_{ij} N_j a_{ij}(n_j, m_i) \right] - d_i$$

**Notation:**  $[x_i]_{i=1}^u = x_1, \dots, x_u$

# Average Fitness Calculation

$$\begin{aligned}
 \overline{Y}_j(N_j, \overline{n}_j, [M_i]_{i=1}^u, [\overline{m}_i]_{i=1}^u) \\
 &= \int_{\mathbb{R}^{u+1}} Y_j \cdot \prod_{i=1}^u \left[ p_i(m_i, \overline{m}_i) \right] \cdot p(n, \overline{n}) \prod_{i=1}^u [dm_i] dn_j \\
 &= \overline{r}_j(\overline{n}_j) \left( 1 - \frac{N_j}{K_j} \right) - \sum_{i=1}^u M_i \overline{a}_{ij}(\overline{n}_j, \overline{m}_i)
 \end{aligned}$$

$$\begin{aligned}
 \overline{W}_i(N_j, \overline{n}_j, [M_i]_{i=1}^u, [\overline{m}_i]_{i=1}^u) \\
 &= \int_{\mathbb{R}^{u+1}} W_i \cdot p_i(m_i, \overline{m}_i) \cdot \prod_{j=1}^v \left[ p(n_j, \overline{n}_j) \right] dm_i \prod_{j=1}^v [dn_j] \\
 &= \sum_{j=1}^v \left[ e_{ij} N_j \overline{a}_{ij}(\overline{n}_j, \overline{m}_i) \right] - d_i
 \end{aligned}$$



# The Complete $v \times u$ Model - ( $v$ Prey Species, $u$ Predator Species)

## Ecological Components

$$\frac{dN_j}{dt} = N_j \overline{Y_j} = N_j \left[ \overline{r_j}(\overline{n_j}) \left( 1 - \frac{N_j}{K_j} \right) - \sum_{i=1}^u M_i \overline{a_{ij}}(\overline{n_j}, \overline{m_i}) \right]$$

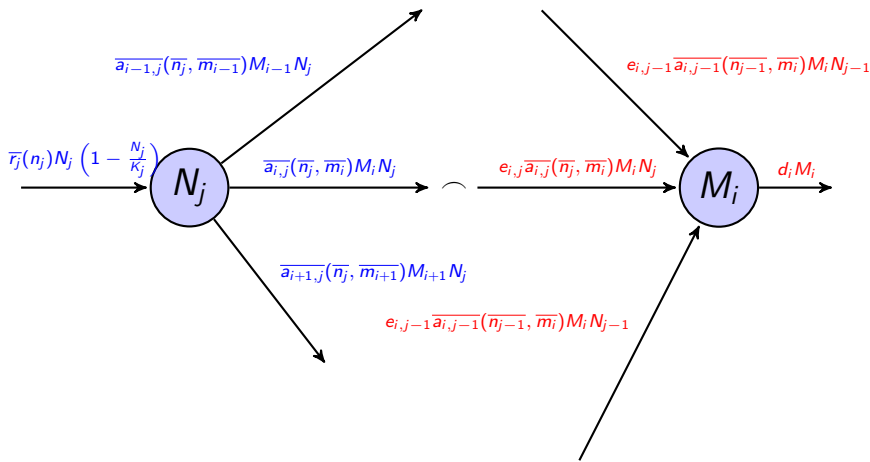
$$\frac{dM_i}{dt} = M_i \overline{W_i} = M_i \left[ \sum_{j=1}^v \left[ e_{ij} N_j \overline{a_{ij}}(\overline{n_j}, \overline{m_i}) \right] - d_i \right]$$

## Evolutionary Components

$$\begin{aligned} \frac{d\overline{n_j}}{dt} = \beta_{Gj}^2 \frac{\partial \overline{Y_j}}{\partial \overline{n_j}} = \beta_{Gj}^2 \left[ \overline{r_j}(\overline{n_j}) \left( 1 - \frac{N_j}{K_j} \right) \frac{(\phi_j - \overline{n_j})}{\beta_j^2 + \gamma_j^2} \right. \\ \left. + \sum_{i=1}^u \left[ \frac{M_i(\theta_{ij} - (\overline{m_i} - \overline{n_j}))}{\sigma_i^2 + \beta_j^2 + \tau_{ij}^2} \overline{a_{ij}}(\overline{n_j}, \overline{m_i}) \right] \right] \end{aligned}$$

$$\frac{d\overline{m_i}}{dt} = \sigma_{Gi}^2 \frac{\partial \overline{W_i}}{\partial \overline{m_i}} = \sigma_{Gi}^2 \sum_{j=1}^v \left[ \frac{e_{ij} N_j (\theta_{ij} - (\overline{m_i} - \overline{n_j}))}{\sigma_i^2 + \beta_j^2 + \tau_{ij}^2} \overline{a_{ij}}(\overline{n_j}, \overline{m_i}) \right]$$

# The Complete $v \times u$ Model - ( $v$ Prey Species, $u$ Predator Species)



# The Complete $2 \times 1$ Model - (2 Prey Species, 1 Predator Species)

## Ecological Components

$$\frac{dN_1}{dt} = N_1 \overline{Y_1} = N_1 \left[ \overline{r_1}(\overline{n_1}) \left( 1 - \frac{N_1}{K_1} \right) - M \overline{a_1}(\overline{n_1}, \overline{m}) \right]$$

$$\frac{dN_2}{dt} = N_2 \overline{Y_2} = N_2 \left[ \overline{r_2}(\overline{n_2}) \left( 1 - \frac{N_2}{K_2} \right) - M \overline{a_2}(\overline{n_2}, \overline{m}) \right]$$

$$\frac{dM}{dt} = M \overline{W} = M \left[ \sum_{j=1}^2 \left[ e_j N_j \overline{a_j}(\overline{n_j}, \overline{m}) \right] - d \right]$$

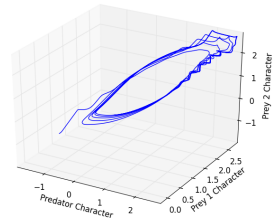
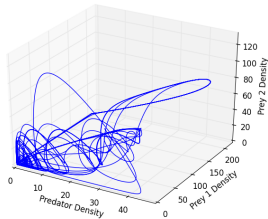
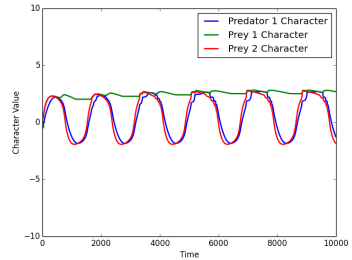
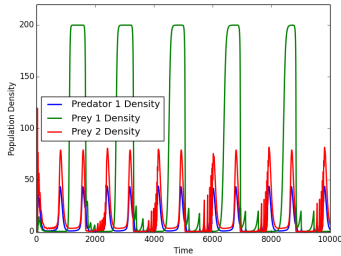
## Evolutionary Components

$$\frac{d\overline{n_1}}{dt} = \beta_{G1}^2 \frac{\partial \overline{Y_1}}{\partial \overline{n_1}} = \beta_{G1}^2 \left[ \overline{r_1}(\overline{n_1}) \left( 1 - \frac{N_1}{K_1} \right) \frac{(\phi_1 - \overline{n_1})}{\beta_1^2 + \gamma_1^2} + \frac{M(\theta_1 - (\overline{m} - \overline{n_1}))}{\sigma^2 + \beta_1^2 + \tau_1^2} \overline{a_1}(\overline{n_1}, \overline{m}) \right]$$

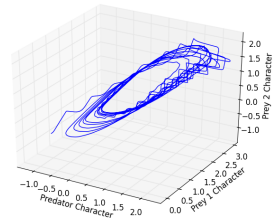
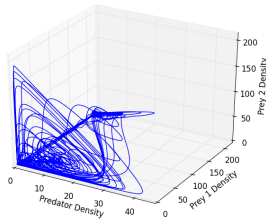
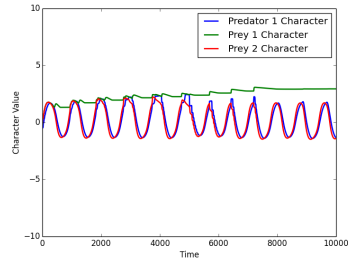
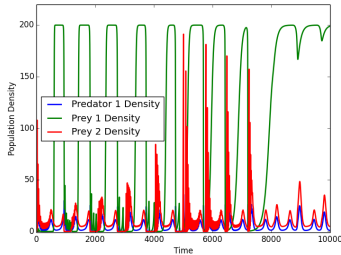
$$\frac{d\overline{n_2}}{dt} = \beta_{G1}^2 \frac{\partial \overline{Y_2}}{\partial \overline{n_2}} = \beta_{G2}^2 \left[ \overline{r_2}(\overline{n_2}) \left( 1 - \frac{N_2}{K_2} \right) \frac{(\phi_2 - \overline{n_2})}{\beta_2^2 + \gamma_2^2} + \frac{M(\theta_2 - (\overline{m} - \overline{n_2}))}{\sigma^2 + \beta_2^2 + \tau_2^2} \overline{a_2}(\overline{n_2}, \overline{m}) \right]$$

$$\frac{d\overline{m}}{dt} = \sigma_G^2 \frac{\partial \overline{W}}{\partial \overline{m}} = \sigma_G^2 \sum_{j=1}^2 \left[ \frac{e_j N_j (\theta_j - (\overline{m} - \overline{n_j}))}{\sigma^2 + \beta_j^2 + \tau_j^2} \overline{a_j}(\overline{n_j}, \overline{m}) \right]$$

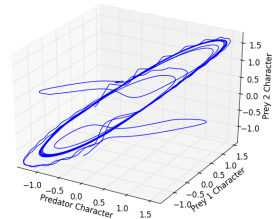
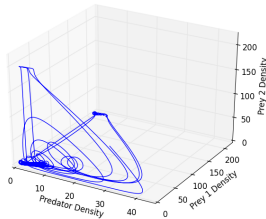
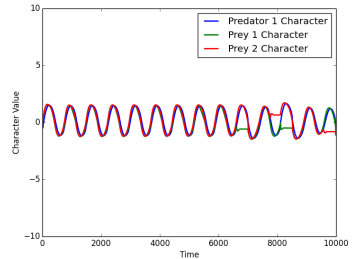
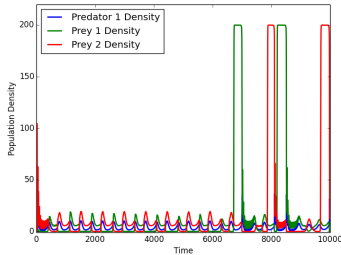
# Figures - $2 \times 1$



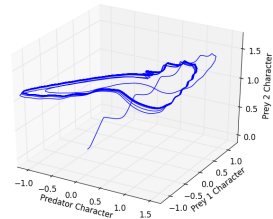
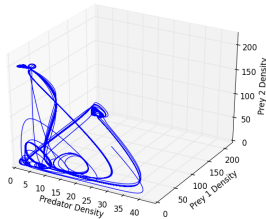
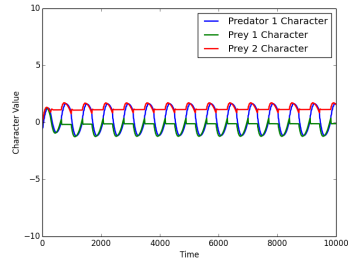
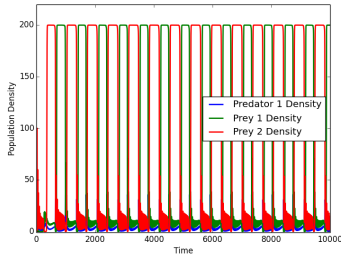
# Figures - $2 \times 1$



# Figures - $2 \times 1$



# Figures - $2 \times 1$



# The Complete $1 \times 2$ Model - (1 Prey Species, 2 Predator Species)

## Ecological Components

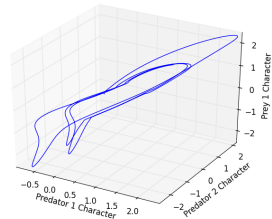
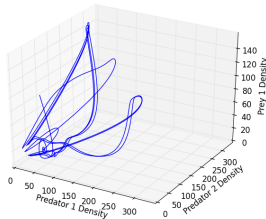
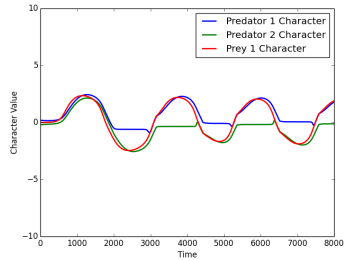
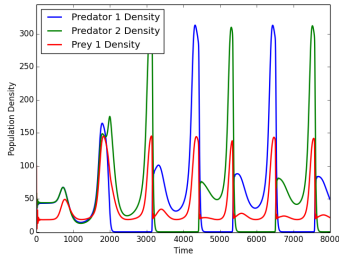
$$\begin{aligned}\frac{dN}{dt} &= N\overline{Y} = N \left[ \overline{r}(\overline{n}) \left( 1 - \frac{N}{K} \right) - \sum_{i=1}^2 M_i \overline{a}_i(\overline{n}, \overline{m}_i) \right] \\ \frac{dM_1}{dt} &= M_1 \overline{W} = M_1 \left[ e_1 N \overline{a}_1(\overline{n}, \overline{m}_1) - d_1 \right] \\ \frac{dM_2}{dt} &= M_2 \overline{W} = M_2 \left[ e_2 N \overline{a}_2(\overline{n}, \overline{m}_2) - d_2 \right]\end{aligned}$$

## Evolutionary Components

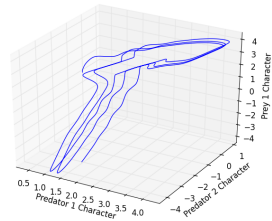
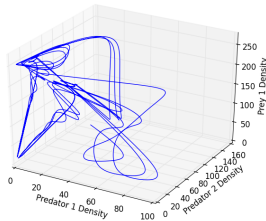
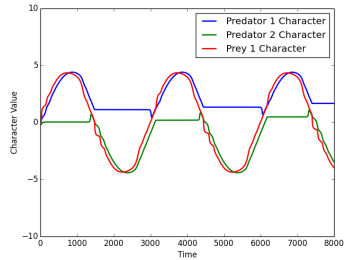
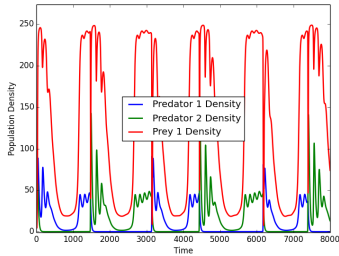
$$\begin{aligned}\frac{d\overline{n}}{dt} &= \beta_G^2 \frac{\partial \overline{Y}}{\partial \overline{n}} = \beta_G^2 \left[ \overline{r}(\overline{n}) \left( 1 - \frac{N}{K} \right) \frac{(\phi - \overline{n})}{\beta^2 + \gamma^2} + \sum_{i=1}^2 \left[ \frac{M_i (\theta_i - (\overline{m}_i - \overline{n}))}{\sigma_i^2 + \beta^2 + \tau_i^2} \overline{a}_i(\overline{n}, \overline{m}_i) \right] \right] \\ \frac{d\overline{m}_1}{dt} &= \sigma_{G1}^2 \frac{\partial \overline{W}_1}{\partial \overline{m}_1} = \sigma_{G1}^2 \left[ \frac{e_1 N (\theta_1 - (\overline{m}_1 - \overline{n}))}{\sigma_1^2 + \beta^2 + \tau_1^2} \overline{a}_1(\overline{n}, \overline{m}_1) \right] \\ \frac{d\overline{m}_2}{dt} &= \sigma_{G1}^2 \frac{\partial \overline{W}_2}{\partial \overline{m}_2} = \sigma_{G1}^2 \left[ \frac{e_2 N (\theta_2 - (\overline{m}_2 - \overline{n}))}{\sigma_2^2 + \beta^2 + \tau_2^2} \overline{a}_2(\overline{n}, \overline{m}_2) \right]\end{aligned}$$



# Figures - $1 \times 2$



# Figures - $1 \times 2$



## Future Work

- $(1 \times 2)$  Two predator species in competition for one prey
- $(2 \times 1)$  Two prey species in apparent competition via one generalist predator
- $(2 \times 2)$  One specialist predator competing with one generalist predator for two prey
- $(2 \times 3)$  Two specialist predators competing with one generalist predator for two prey species
- $(u \times v)$  The General Ditrophic Expansion
- Intraguild Predation and General Multitrophic Expansion

# Thank You!

- PUMP (Preparing Undergraduates through Mentoring towards PhDs)
- The Pacific Math Alliance
- The National Math Alliance
- Dr. Helena Noronha, Dr. Ramin Vakilian, and all other PUMP organizers
- National Science Foundation
- California State University, Northridge
- Dr. Jing Li and Dr. Casey terHorst

## Questions?