The Effects of Intraspecific Genetic Variation on the Dynamic of Predator-Prey Ecological Communities

Sam Fleischer, Pablo Chavarria

May 3, 2015

Preparing Undergraduates through Mentoring for PhDs (PUMP) Research Symposium

Advisor:	Dr. Jing Li, Mathematics,	CSU Northridge
Consultant:	Dr. Casey terHorst, Biology,	CSU Northridge

Supported By:	Pacific Math Alliance, PUMP,	CSU Northridge

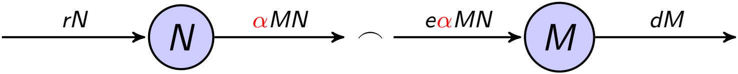
Overview

- Motivation
  - Observations in Nature
  - Previous Models
- Our Expansions
  - Gaussian Attack Rate under Coevolution
  - Introduce Stabilizing Selection
  - General Ditrophic Expansion
- Discussion

Observations in Nature

- Predator/Prey interactions are prevalent in nature
    - Crab vs. gastropod [Saloniemi, 1993]
    - Protist vs. bacteria [terHorst]
- There is trait variation within species
    - Thickness of plant cuticula [Saloniemi, 1993]
    - Strength of gastropod shell [Saloniemi, 1993]
- Incorporating trait variation provides **richer dynamics** than classical Lotka-Volterra models

Classical Lotka-Volterra Model



$$\frac{dN}{dt} = N(r - \alpha M)$$

$$\frac{dM}{dt} = M(e\alpha N - d)$$

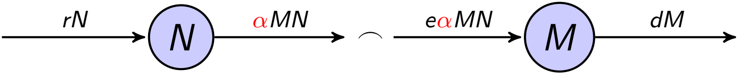
Variables

- $N \equiv$ Prey Density
- $M \equiv$ Predator Density

Parameters

- $\alpha \equiv$ Attack rate
- $r \equiv$ Prey birth rate
- $e \equiv$ Efficiency
- $d \equiv$ Predator death rate

Classical Lotka-Volterra Model



$$\frac{dN}{dt} = N(r - \alpha M)$$

$$\frac{dM}{dt} = M(e\alpha N - d)$$

Variables

- $N \equiv$ Prey Density
- $M \equiv$ Predator Density

Parameters

- $\alpha \equiv$ Attack rate $\longleftarrow$ **No variation!**
- $r \equiv$ Prey birth rate
- $e \equiv$ Efficiency
- $d \equiv$ Predator death rate

Schreiber, Bürger, and Bolnick's Expansion

Assume the Predator Species has a normally distributed trait value.

$$p(m, \overline{m}) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{(m-\overline{m})^2}{2\sigma^2}\right]$$

Parameters

- $\sigma^2 \equiv$ Predator Trait Variance

Variables

- $m \equiv$ Predator Trait Value

Schreiber, Bürger, and Bolnick's Expansion

Assume the Predator Species has a normally distributed trait value.

$$p(m, \overline{m}) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{(m-\overline{m})^2}{2\sigma^2}\right]$$

Attack Rate is a Function of the Predator's Trait Value

$$a(m) = \alpha \exp\left[-\frac{(m-\theta)^2}{2\tau^2}\right]$$

Variables

- $m \equiv$ Predator Trait Value

Parameters

- $\sigma^2 \equiv$ Predator Trait Variance
- $\alpha \equiv$ Maximum attack rate
- $\tau \equiv$ Specialization Constant
- $\theta \equiv$ Optimal trait value

Schreiber, Bürger, and Bolnick's Expansion

Assume the Predator Species has a normally distributed trait value.

$$p(m, \overline{m}) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{(m - \overline{m})^2}{2\sigma^2}\right]$$

Attack Rate is a Function of the Predator's Trait Value

$$a(m) = \alpha \exp\left[-\frac{(m - \theta)^2}{2\tau^2}\right]$$

Variables

- $m \equiv$ Predator Trait Value
- (((No Prey Trait Value)))

Parameters

- $\sigma^2 \equiv$ Predator Trait Variance
- $\alpha \equiv$ Maximum attack rate
- $\tau \equiv$ Specialization Constant
- $\theta \equiv$ Optimal trait value

  $\uparrow$ **No variation!**

Normally Distributed Trait Values

Assume Prey and Predator have normally distributed trait values.

$$p(n,\overline{n}) = \frac{1}{\sqrt{2\pi\beta^2}} \exp\left[-\frac{(n-\overline{n})^2}{2\beta^2}\right]$$

$$p(m,\overline{m}) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{(m-\overline{m})^2}{2\sigma^2}\right]$$

Variables

- $n \equiv$ Prey Trait Value
- $\overline{n} \equiv$ **Average** Prey Trait Value
- $m \equiv$ Predator Trait Value
- $\overline{m} \equiv$ **Average** Predator Trait Value

Parameters

- $\beta^2 \equiv$ Prey Trait Variance
- $\sigma^2 \equiv$ Predator Trait Variance

Attack Rate

Attack Rate is a Gaussian Function of the Prey's Trait Value and the Predator's Trait Value

$$a(n, m) = \alpha \exp\left[-\frac{((m-n)-\theta)^2}{2\tau^2}\right]$$

Variables

- $n \equiv$ Prey Trait Value
- $\overline{n} \equiv$ **Average** Prey Trait Value
- $m \equiv$ Predator Trait Value
- $\overline{m} \equiv$ **Average** Predator Trait Value

Parameters

- $\alpha \equiv$ Maximum attack rate
- $\theta \equiv$ Optimal trait **difference**
- $\tau^2 \equiv$ Specialization Constant

Attack Rate

Attack Rate is a Gaussian Function of the Prey's Trait Value and the Predator's Trait Value

$$a(n, m) = \alpha \exp\left[-\frac{((m-n)-\theta)^2}{2\tau^2}\right]$$

Average Attack Rate

$$\bar{a}(\bar{n}, \bar{m}) = \int_{-\infty}^{\infty}\int_{\infty}^{\infty} a(n, m)\cdot p(n, \bar{n})\cdot p(m, \bar{m})\; dn\,dm$$

$$= \frac{\alpha\tau}{\sqrt{\sigma^2+\beta^2+\tau^2}} \exp\left[-\frac{((\bar{m}-\bar{n})-\theta)^2}{2(\sigma^2+\beta^2+\tau^2)}\right]$$

Variables

- $n \equiv$ Prey Trait Value
- $\bar{n} \equiv$ **Average** Prey Trait Value
- $m \equiv$ Predator Trait Value
- $\bar{m} \equiv$ **Average** Predator Trait Value

Parameters

- $\alpha \equiv$ Maximum attack rate
- $\theta \equiv$ Optimal trait **difference**
- $\tau^2 \equiv$ Specialization Constant
- $\beta^2 \equiv$ Prey Trait Variance
- $\sigma^2 \equiv$ Predator Trait Variance

Fitness Assumptions

- Prey experiences logistic growth in absence of predator
- Predator experiences exponential decay in absence of prey

$$Y(N, n, M, m) = r\left(1 - \frac{N}{K}\right) - Ma(n, m)$$

$$W(N, n, M, m) = eNa(n, m) - d$$

Variables

- $N \equiv$ Prey Density
- $n \equiv$ Prey Trait Value
- $M \equiv$ Predator Density
- $m \equiv$ Predator Trait Value

Parameters

- $r \equiv$ Intrinsic Prey Growth Rate
- $K \equiv$ Prey Carrying Capacity
- $d \equiv$ Predator Death Rate
- $e \equiv$ Efficiency

Average Fitness

$$\overline{Y}(N,\overline{n},M,\overline{m}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} Y(N,n,M,m)\cdot p(m,\overline{m})\cdot p(n,\overline{n})\ dmdn$$
$$= r\left(1-\frac{N}{K}\right) - M\overline{a}(\overline{n},\overline{m})$$

$$\overline{W}(N,\overline{n},M,\overline{m}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} W(N,n,M,m)\cdot p(m,\overline{m})\cdot p(n,\overline{n})\ dmdn$$
$$= eN\overline{a}(\overline{n},\overline{m}) - d$$

Variables

- $N \equiv$ Prey Density
- $\overline{n} \equiv$ **Average** Prey Trait Value
- $M \equiv$ Predator Density
- $\overline{m} \equiv$ **Average** Predator Trait Value

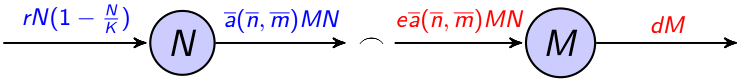
Parameters

- $\overline{r} \equiv$ Intrinsic Prey Growth Rate
- $K \equiv$ Prey Carrying Capacity
- $d \equiv$ Predator Death Rate
- $e \equiv$ Efficiency

Ecological Components

$$\frac{dN}{dt} = N \cdot \overline{Y}(N, \overline{n}, M, \overline{m}) = N\left[r\left(1 - \frac{N}{K}\right) - M\overline{a}(\overline{n}, \overline{m})\right]$$

$$\frac{dM}{dt} = M \cdot \overline{W}(N, \overline{n}, M, \overline{m}) = M[eN\overline{a}(\overline{n}, \overline{m}) - d]$$



Variables

- $N \equiv$ Prey Density
- $\overline{n} \equiv$ **Average** Prey Trait Value
- $M \equiv$ Predator Density
- $\overline{m} \equiv$ **Average** Predator Trait Value

Parameters

- $\overline{r} \equiv$ Intrinsic Prey Growth Rate
- $K \equiv$ Prey Carrying Capacity
- $d \equiv$ Predator Death Rate
- $e \equiv$ Efficiency

Evolutionary Components

- The evolution of the mean trait value is always in the direction which increases the mean fitness in the population. [Lande, 1976]

$$\frac{d\overline{n}}{dt} = \beta_G^2 \frac{\partial \overline{Y}}{\partial \overline{n}} = \beta_G^2 \frac{M(\theta - (\overline{m} - \overline{n}))}{\sigma^2 + \beta^2 + \tau^2} \overline{a}(\overline{m}, \overline{n})$$

$$\frac{d\overline{m}}{dt} = \sigma_G^2 \frac{\partial \overline{W}}{\partial \overline{m}} = \sigma_G^2 \frac{eN(\theta - (\overline{m} - \overline{n}))}{\sigma^2 + \beta^2 + \tau^2} \overline{a}(\overline{m}, \overline{n})$$

Variables

- $N \equiv$ Prey Density
- $\overline{n} \equiv$ Mean Prey Character
- $M \equiv$ Predator Density
- $\overline{m} \equiv$ Mean Predator Character

Parameters

- $\beta_G^2 \equiv$ Prey genetic variance
- $\sigma_G^2 \equiv$ Predator genetic variance

The Complete 1×1 Model (One Predator Species, One Prey Species)

Ecological Components

$$\frac{dN}{dt} = N \cdot \overline{Y}(\overline{m}, \overline{n}, M, N) = N\left[r\left(1 - \frac{N}{K}\right) - M\overline{a}(\overline{m}, \overline{n})\right]$$

$$\frac{dM}{dt} = M \cdot \overline{W}(\overline{m}, \overline{n}, N) = M[eN\overline{a}(\overline{m}, \overline{n}) - d]$$

Evolutionary Components

$$\frac{d\overline{n}}{dt} = \beta_G^2 \frac{\partial \overline{Y}}{\partial \overline{n}} = \beta_G^2 \frac{M(\theta + \overline{n} - \overline{m})}{\sigma^2 + \beta^2 + \tau^2} \overline{a}(\overline{m}, \overline{n})$$

$$\frac{d\overline{m}}{dt} = \sigma_G^2 \frac{\partial \overline{W}}{\partial \overline{m}} = \sigma_G^2 \frac{eN(\theta + \overline{n} - \overline{m})}{\sigma^2 + \beta^2 + \tau^2} \overline{a}(\overline{m}, \overline{n})$$

Equilibria - 1×1

$$\frac{dN}{dt} = N \cdot \overline{Y}(\overline{m}, \overline{n}, M, N) \qquad \frac{d\overline{n}}{dt} = \beta_G^2 \frac{\partial \overline{Y}}{\partial \overline{n}}$$

$$\frac{dM}{dt} = M \cdot \overline{W}(\overline{m}, \overline{n}, N) \qquad \frac{d\overline{m}}{dt} = \sigma_G^2 \frac{\partial \overline{W}}{\partial \overline{m}}$$

Extinction

$$(N^*, M^*, \overline{n}^*, \overline{m}^*) = (0, 0, \_, \_)$$

Exclusion

$$(N^*, M^*, \overline{n}^*, \overline{m}^*) = (K, 0, \mu^*, \mu^* + \theta) \qquad \text{where } \mu^* \text{ is arbitrary}$$

Equilibria - 1×1

$$\frac{dN}{dt} = N \cdot \overline{Y}(\overline{m}, \overline{n}, M, N) \qquad \frac{d\overline{n}}{dt} = \beta_G^2 \frac{\partial \overline{Y}}{\partial \overline{n}}$$

$$\frac{dM}{dt} = M \cdot \overline{W}(\overline{m}, \overline{n}, N) \qquad \frac{d\overline{m}}{dt} = \sigma_G^2 \frac{\partial \overline{W}}{\partial \overline{m}}$$

Extinction $\boxed{\textit{Unstable}}$

$$(N^*, M^*, \overline{n}^*, \overline{m}^*) = (0, 0, \_, \_)$$

Exclusion

$$(N^*, M^*, \overline{n}^*, \overline{m}^*) = (K, 0, \mu^*, \mu^* + \theta) \qquad \text{where } \mu^* \text{ is arbitrary}$$

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Extinction $\boxed{\textit{Unstable}}$

$$(N^*, M^*, \overline{n}^*, \overline{m}^*) = (0, 0, \_, \_)$$

Exclusion $\boxed{\textit{Stable under certain conditions}}$

$$(N^*, M^*, \overline{n}^*, \overline{m}^*) = (K, 0, \mu^*, \mu^* + \theta) \qquad \text{where } \mu^* \text{ is arbitrary}$$

Equilibria - 1×1

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$$\frac{dM}{dt} = M \cdot \overline{W}(\overline{m}, \overline{n}, N) \qquad \frac{d\overline{m}}{dt} = \sigma_G^2 \frac{\partial \overline{W}}{\partial \overline{m}}$$

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$$(N^*, M^*, \overline{n}^*, \overline{m}^*) = (K, 0, \mu^*, \mu^* + \theta) \qquad \text{where } \mu^* \text{ is arbitrary}$$

Necessary Condition for Stable Exclusion:

- $d > \frac{Ke\alpha\tau}{\sqrt{A}}$ where $A = \sigma^2 + \beta^2 + \tau^2$.

Equilibria - 1×1

$$\frac{dN}{dt} = N \cdot \overline{Y}(\overline{m}, \overline{n}, M, N) \qquad \frac{d\overline{n}}{dt} = \beta_G^2 \frac{\partial \overline{Y}}{\partial \overline{n}}$$

$$\frac{dM}{dt} = M \cdot \overline{W}(\overline{m}, \overline{n}, N) \qquad \frac{d\overline{m}}{dt} = \sigma_G^2 \frac{\partial \overline{W}}{\partial \overline{m}}$$

Coexistence

$$(N^*, M^*, \overline{n}^*, \overline{m}^*) = (\frac{d\sqrt{A}}{e\alpha\tau}\ ,\ \frac{r\sqrt{A}}{\alpha\tau}\left(1 - \frac{d\sqrt{A}}{Ke\alpha\tau}\right)\ ,\ \mu^*\ ,\ \mu^* + \theta)$$

where $A = \sigma^2 + \beta^2 + \tau^2$ and μ^* is an arbitrary value.

Equilibria - 1×1

$$\frac{dN}{dt} = N \cdot \overline{Y}(\overline{m}, \overline{n}, M, N) \qquad \frac{d\overline{n}}{dt} = \beta_G^2 \frac{\partial \overline{Y}}{\partial \overline{n}}$$

$$\frac{dM}{dt} = M \cdot \overline{W}(\overline{m}, \overline{n}, N) \qquad \frac{d\overline{m}}{dt} = \sigma_G^2 \frac{\partial \overline{W}}{\partial \overline{m}}$$

Coexistence $\boxed{\textit{Stable under certain conditions}}$

$$(N^*, M^*, \overline{n}^*, \overline{m}^*) = \left(\frac{d\sqrt{A}}{e\alpha\tau}\ ,\ \frac{r\sqrt{A}}{\alpha\tau}\left(1 - \frac{d\sqrt{A}}{Ke\alpha\tau}\right)\ ,\ \mu^*\ ,\ \mu^* + \theta\right)$$

where $A = \sigma^2 + \beta^2 + \tau^2$ and μ^* is an arbitrary value.

Equilibria - 1 × 1

$$\frac{dN}{dt} = N \cdot \overline{Y}(\overline{m}, \overline{n}, M, N) \qquad \frac{d\overline{n}}{dt} = \beta_G^2 \frac{\partial \overline{Y}}{\partial \overline{n}}$$

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Coexistence $\boxed{\textit{Stable under certain conditions}}$

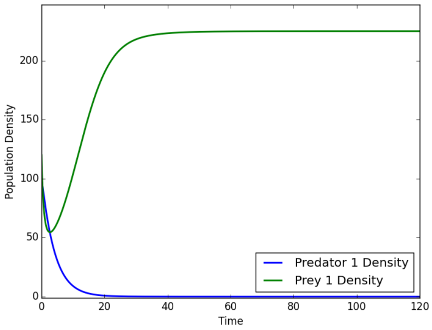
$$(N^*, M^*, \overline{n}^*, \overline{m}^*) = \left(\frac{d\sqrt{A}}{e\alpha\tau}\ ,\ \frac{r\sqrt{A}}{\alpha\tau}\left(1 - \frac{d\sqrt{A}}{Ke\alpha\tau}\right)\ ,\ \mu^*\ ,\ \mu^* + \theta\right)$$

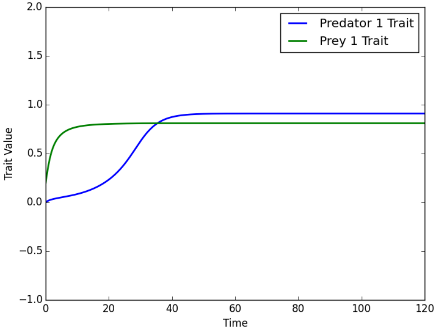
where $A = \sigma^2 + \beta^2 + \tau^2$ and μ^* is an arbitrary value.

Necessary Condition for Stable Coexistence:

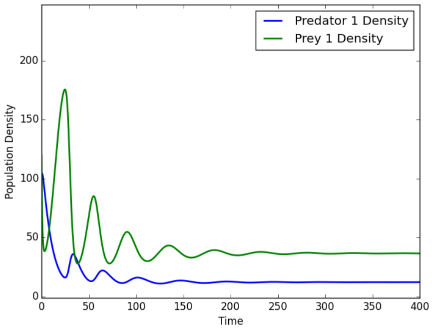
- $\frac{\sigma_G^2}{\beta_G^2} > \frac{r}{d}\left(1 - \frac{d\sqrt{A}}{Ke\alpha\tau}\right)$

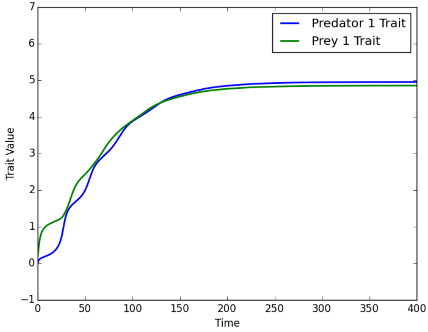
Figures - 1×1 - Stable Exclusion



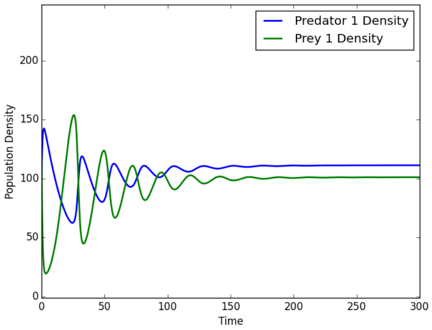


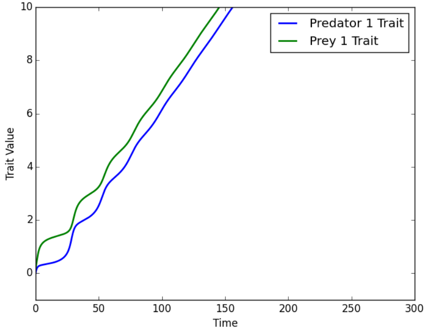
Figures - 1 $\times$ 1 - Stable Coexistence





Figures - 1 $\times$ 1 - "Arms Race" Coexistence





Avoiding an "Arms Race" with Stabilizing Selection

Assume Prey Growth Rate is a Function of the Prey's Trait Value

$$r(n) = \rho \exp\left[-\frac{(n-\phi)^2}{2\gamma^2}\right]$$

Variables

- $n \equiv$ Prey Trait Value
- $\bar{n} \equiv$ **Average** Prey Trait Value

Parameters

- $\rho \equiv$ Maximum Growth Rate
- $\phi \equiv$ Prey Optimum Trait Value
- $\gamma^2 \equiv$ Stabilizing Selection Constant

Avoiding an "Arms Race" with Stabilizing Selection

Assume Prey Growth Rate is a Function of the Prey's Trait Value

$$r(n) = \rho \exp\left[-\frac{(n-\phi)^2}{2\gamma^2}\right]$$

Averge Growth Rate

$$\begin{aligned}\bar{r}(\bar{n}) &= \int_{-\infty}^{\infty} r(n)\cdot p(n,\bar{n})dn \\ &= \frac{\rho\gamma}{\sqrt{\beta^2+\gamma^2}}\exp\left[-\frac{(n-\phi)^2}{2\gamma^2}\right]\end{aligned}$$

Variables

- $n \equiv$ Prey Trait Value
- $\bar{n} \equiv$ **Average** Prey Trait Value

Parameters

- $\rho \equiv$ Maximum Growth Rate
- $\phi \equiv$ Prey Optimum Trait Value
- $\gamma^2 \equiv$ Stabilizing Selection Constant
- $\beta^2 \equiv$ Prey Trait Variance

Fitness Assumptions

- Prey experiences logistic growth in absence of predator
- Predator experiences exponential decay in absence of prey

$$Y(N, n, M, m) = r(n)\left(1 - \frac{N}{K}\right) - Ma(n, m)$$

$$W(N, n, M, m) = eNa(n, m) - d$$

Variables

- $N \equiv$ Prey Density
- $n \equiv$ Prey Trait Value
- $M \equiv$ Predator Density
- $m \equiv$ Predator Trait Value

Parameters

- $r \equiv$ Intrinsic Prey Growth Rate Function
- $K \equiv$ Prey Carrying Capacity
- $d \equiv$ Predator Death Rate
- $e \equiv$ Efficiency

The Complete 1×1 Model (One Predator Species, One Prey Species)

Ecological Components

$$\frac{dN}{dt} = N \cdot \overline{Y}(N, \overline{n}, M, \overline{m}) \; = N\left[\overline{r}(\overline{n})\left(1 - \frac{N}{K}\right) - M\overline{a}(\overline{n}, \overline{m})\right]$$

$$\frac{dM}{dt} = M \cdot \overline{W}(N, \overline{n}, M, \overline{m}) = M[eN\overline{a}(\overline{n}, \overline{m}) - d]$$

Evolutionary Components

$$\frac{d\overline{n}}{dt} = \beta_G^2 \frac{\partial \overline{Y}}{\partial \overline{n}} = \beta_G^2 \left[\overline{r}(\overline{n})\left(1 - \frac{N}{K}\right)\frac{(\phi - \overline{n})}{\beta^2 + \gamma^2} + \frac{M(\theta - (\overline{m} - \overline{n}))}{\sigma^2 + \beta^2 + \tau^2}\overline{a}(\overline{m}, \overline{n})\right]$$

$$\frac{d\overline{m}}{dt} = \sigma_G^2 \frac{\partial \overline{W}}{\partial \overline{m}} = \sigma_G^2 \frac{eN(\theta - (\overline{m} - \overline{n}))}{\sigma^2 + \beta^2 + \tau^2}\overline{a}(\overline{m}, \overline{n})$$

Equilibria - 1×1

$$\frac{dN}{dt} = N \cdot \overline{Y}(N, \overline{n}, M, \overline{m}) \qquad \frac{d\overline{n}}{dt} = \beta_G^2 \frac{\partial \overline{Y}}{\partial \overline{n}}$$

$$\frac{dM}{dt} = M \cdot \overline{W}(N, \overline{n}, M, \overline{m}) \qquad \frac{d\overline{m}}{dt} = \sigma_G^2 \frac{\partial \overline{W}}{\partial \overline{m}}$$

Extinction

$$(N^*, M^*, \overline{n}^*, \overline{m}^*) = (0, 0, \_, \_)$$

Exclusion

$$(N^*, M^*, \overline{n}^*, \overline{m}^*) = (K, 0, \mu^*, \mu^* + \theta) \qquad \text{where } \mu^* \text{ is arbitrary}$$

Equilibria - 1×1

$$\frac{dN}{dt} = N \cdot \overline{Y}(N, \overline{n}, M, \overline{m}) \qquad \frac{d\overline{n}}{dt} = \beta_G^2 \frac{\partial \overline{Y}}{\partial \overline{n}}$$

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Extinction $\boxed{\textit{Unstable}}$

$$(N^*, M^*, \overline{n}^*, \overline{m}^*) = (0, 0, \_, \_)$$

Exclusion

$$(N^*, M^*, \overline{n}^*, \overline{m}^*) = (K, 0, \mu^*, \mu^* + \theta) \qquad \text{where } \mu^* \text{ is arbitrary}$$

Equilibria - 1×1

$$\frac{dN}{dt} = N \cdot \overline{Y}(N, \overline{n}, M, \overline{m}) \qquad \frac{d\overline{n}}{dt} = \beta_G^2 \frac{\partial \overline{Y}}{\partial \overline{n}}$$

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Extinction $\boxed{\textit{Unstable}}$

$$(N^*, M^*, \overline{n}^*, \overline{m}^*) = (0, 0, \_, \_)$$

Exclusion $\boxed{\textit{Locally stable under certain conditions}}$

$$(N^*, M^*, \overline{n}^*, \overline{m}^*) = (K, 0, \mu^*, \mu^* + \theta) \qquad \text{where } \mu^* \text{ is arbitrary}$$

Equilibria - 1×1

$$\frac{dN}{dt} = N \cdot \overline{Y}(N, \overline{n}, M, \overline{m}) \qquad \frac{d\overline{n}}{dt} = \beta_G^2 \frac{\partial \overline{Y}}{\partial \overline{n}}$$

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$$(N^*, M^*, \overline{n}^*, \overline{m}^*) = (K, 0, \mu^*, \mu^* + \theta) \qquad \text{where } \mu^* \text{ is arbitrary}$$

Necessary Conditions for Locally Stable Exclusion: Necessary Condition for Stable Exclusion:

- $d > \dfrac{Ke\alpha\tau}{\sqrt{A}}$ where $A = \sigma^2 + \beta^2 + \tau^2$.

Equilibria - 1×1

$$\frac{dN}{dt} = N \cdot \overline{Y}(N, \overline{n}, M, \overline{m}) \qquad \frac{d\overline{n}}{dt} = \beta_G^2 \frac{\partial \overline{Y}}{\partial \overline{n}}$$

$$\frac{dM}{dt} = M \cdot \overline{W}(N, \overline{n}, M, \overline{m}) \qquad \frac{d\overline{m}}{dt} = \sigma_G^2 \frac{\partial \overline{W}}{\partial \overline{m}}$$

Coexistence

$$(N^*, M^*, \overline{n}^*, \overline{m}^*) = \left(\frac{d\sqrt{A}}{e\alpha\tau}\ ,\ \frac{\rho\gamma\sqrt{A}}{\alpha\tau\sqrt{B}}\left(1 - \frac{d\sqrt{A}}{Ke\alpha\tau}\right)\ ,\ \phi\ ,\ \theta + \phi\right)$$

where $A = \sigma^2 + \beta^2 + \tau^2$ and $B = \beta^2 + \gamma^2$

Equilibria - 1×1

$$\frac{dN}{dt} = N \cdot \overline{Y}(N, \overline{n}, M, \overline{m}) \qquad \frac{d\overline{n}}{dt} = \beta_G^2 \frac{\partial \overline{Y}}{\partial \overline{n}}$$

$$\frac{dM}{dt} = M \cdot \overline{W}(N, \overline{n}, M, \overline{m}) \qquad \frac{d\overline{m}}{dt} = \sigma_G^2 \frac{\partial \overline{W}}{\partial \overline{m}}$$

Coexistence $\boxed{\textit{Locally stable under certain conditions}}$

$$(N^*, M^*, \overline{n}^*, \overline{m}^*) = \left(\frac{d\sqrt{A}}{e\alpha\tau}\ ,\ \frac{\rho\gamma\sqrt{A}}{\alpha\tau\sqrt{B}}\left(1 - \frac{d\sqrt{A}}{Ke\alpha\tau}\right)\ ,\ \phi\ ,\ \theta + \phi\right)$$

where $A = \sigma^2 + \beta^2 + \tau^2$ and $B = \beta^2 + \gamma^2$

Equilibria - 1×1

$$\frac{dN}{dt} = N \cdot \overline{Y}(N, \overline{n}, M, \overline{m}) \qquad \frac{d\overline{n}}{dt} = \beta_G^2 \frac{\partial \overline{Y}}{\partial \overline{n}}$$

$$\frac{dM}{dt} = M \cdot \overline{W}(N, \overline{n}, M, \overline{m}) \qquad \frac{d\overline{m}}{dt} = \sigma_G^2 \frac{\partial \overline{W}}{\partial \overline{m}}$$

Coexistence *Locally stable under certain conditions*

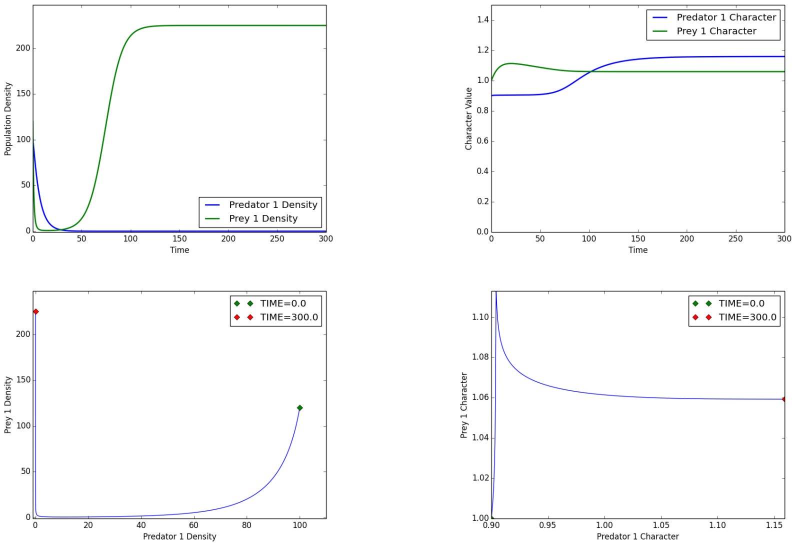
$$(N^*, M^*, \overline{n}^*, \overline{m}^*) = \left(\frac{d\sqrt{A}}{e\alpha\tau} \,,\, \frac{\rho\gamma\sqrt{A}}{\alpha\tau\sqrt{B}} \left(1 - \frac{d\sqrt{A}}{Ke\alpha\tau}\right) \,,\, \phi \,,\, \theta + \phi \right)$$

where $A = \sigma^2 + \beta^2 + \tau^2$ and $B = \beta^2 + \gamma^2$

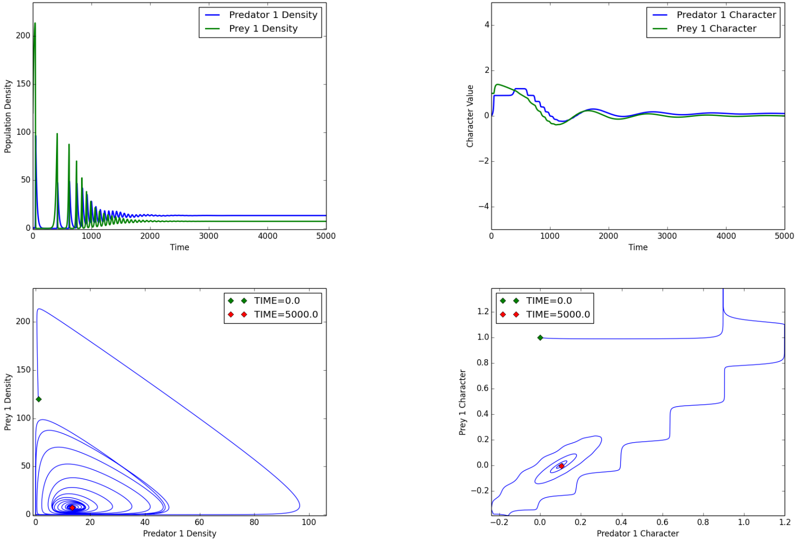
Necessary Condition for Locally Stable Coexistence:

- $\dfrac{\sigma_G^2}{\beta_G^2} > \dfrac{\rho\gamma}{d\sqrt{B}} \left(1 - \dfrac{d\sqrt{A}}{Ke\alpha\tau}\right)\left(1 - \dfrac{A}{B}\right)$

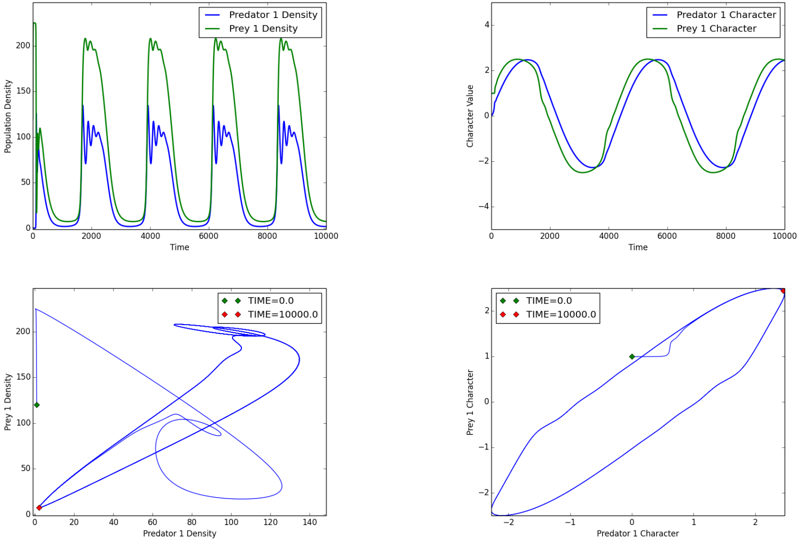
Figures - 1×1 - Stable Exclusion



Figures - 1×1 - Stable Coexistence



Figures - 1×1 - Stable Cycles (Red Queen Dynamics)[Kindrik, Kondrashov, 1994]



Question

What happens when the exclusion **AND** coexistence stability criterion are **NOT** met?

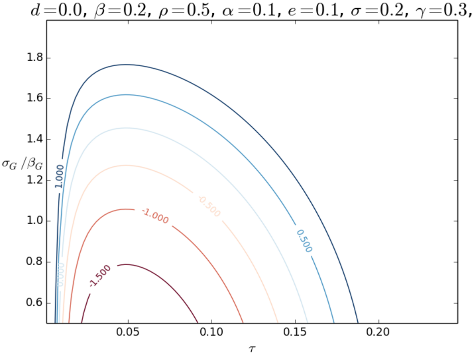
Question

What happens when the exclusion **AND** coexistence stability criterion are **NOT** met?

Answer

Stable Limit Cycles

Contour Plot

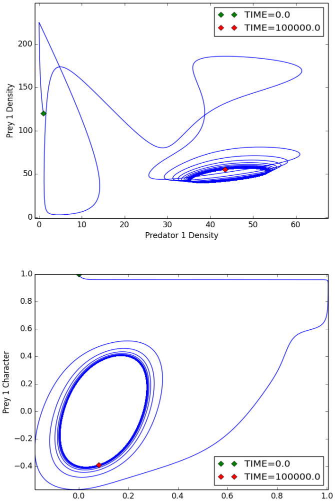


$$f_{\text{stable}}(\textit{system parameters}) = \frac{\sigma_G^2}{\beta_G^2} - \frac{\rho\gamma}{d\sqrt{B}}\left(1 - \frac{d\sqrt{A}}{Ke\alpha\tau}\right)\left(1 - \frac{A}{B}\right)$$

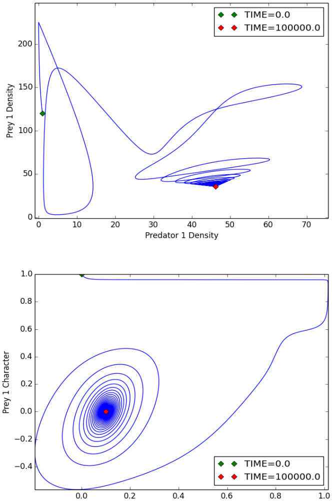
$f_{\text{stable}} > 0 \implies$ Coexistence is *stable*
$f_{\text{stable}} < 0 \implies$ Coexistence is *unstable*
$f_{\text{stable}} = 0 \implies$ Hopf Bifurcation

$\tau = 0.05$: Limit Cycle vs. Node

$$\frac{\sigma_G}{\beta_G} = 1.3$$



$$\frac{\sigma_G}{\beta_G} = 1.5$$



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Expansion of Fitness Functions

Prey Fitness

$$Y(N, n, M, m) = r(n)\left(1 - \frac{N}{K}\right) - Ma(n, m)$$

Predator Fitness

$$W(N, n, M, m) = eNa(n, m) - d$$

Expansion of Fitness Functions

Prey Fitness

$$Y(N, n, M, m) = r(n)\left(1 - \frac{N}{K}\right) - Ma(n, m)$$

$$\downarrow$$

$$Y_j(N_j, n_j, [M_i]_{i=1}^{u}, [m_i]_{i=1}^{u}) = r_j(n_j)\left(1 - \frac{N_j}{K_j}\right) - \sum_{i=1}^{u} M_i a_{ij}(n_j, m_i)$$

Predator Fitness

$$W(N, n, M, m) = eNa(n, m) - d$$

Notation: $[x_i]_{i=1}^{u} = x_1, \ldots, x_u$

Expansion of Fitness Functions

Prey Fitness

$$Y(N, n, M, m) = r(n)\left(1 - \frac{N}{K}\right) - Ma(n, m)$$

$$\downarrow$$

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Predator Fitness

$$W(N, n, M, m) = eNa(n, m) - d$$

$$\downarrow$$

$$W_i([N_j]_{j=1}^{v}, [n_j]_{j=1}^{v}, M_i, m_i) = \sum_{j=1}^{v}\left[e_{ij} N_j a_{ij}(n_j, m_i)\right] - d_i$$

Notation: $[x_i]_{i=1}^{u} = x_1, \ldots, x_u$

Average Fitness Calculation

$$\overline{Y}_j(N_j, \overline{n_j}, [M_i]_{i=1}^u, [\overline{m_i}]_{i=1}^u)$$

$$= \int_{\mathbb{R}^{u+1}} Y_j \cdot \prod_{i=1}^{u} \left[p_i(m_i, \overline{m_i}) \right] \cdot p(n, \overline{n}) \prod_{i=1}^{u} \left[dm_i \right] dn_j$$

$$= \overline{r_j}(\overline{n_j}) \left(1 - \frac{N_j}{K_j} \right) - \sum_{i=1}^{u} M_i \overline{a}_{ij}(\overline{n}_j, \overline{m}_i)$$

$$\overline{W}_i(N_j, \overline{n_j}, [M_i]_{i=1}^u, [\overline{m_i}]_{i=1}^u)$$

$$= \int_{\mathbb{R}^{u+1}} W_i \cdot p_i(m_i, \overline{m_i}) \cdot \prod_{j=1}^{v} \left[p(n_j, \overline{n}_j) \right] dm_i \prod_{j=1}^{v} \left[dn_j \right]$$

$$= \sum_{j=1}^{v} \left[e_{ij} N_j \overline{a}_{ij}(\overline{n}_j, \overline{m}_i) \right] - d_i$$

The Complete $u \times v$ Model - (u Predator Species, v Prey Species)

Ecological Components

$$\frac{dN_j}{dt} = N_j \overline{Y_j} = N_j \left[\overline{r_j}(\overline{n_j}) \left(1 - \frac{N_j}{K_j}\right) - \sum_{i=1}^{u} M_i \overline{a}_{ij}(\overline{n}_j, \overline{m}_i) \right]$$

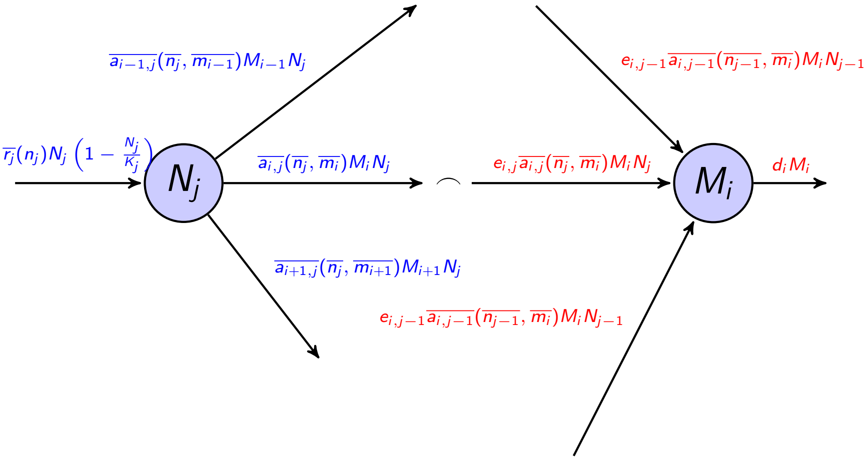
$$\frac{dM_i}{dt} = M_i \overline{W_i} = M_i \left[\sum_{j=1}^{v} \left[e_{ij} N_j \overline{a}_{ij}(\overline{m}_i, \overline{n}_j) \right] - d_i \right]$$

Evolutionary Components

$$\frac{d\overline{n}_j}{dt} = \beta_{Gj}^2 \frac{\partial \overline{Y_j}}{\partial \overline{n_j}} = \beta_{Gj}^2 \left[\overline{r_j}(\overline{n_j}) \left(1 - \frac{N_j}{K_j}\right) \frac{(\phi_j - \overline{n_j})}{\beta_j^2 + \gamma_j^2} + \sum_{i=1}^{u} \left[\frac{M_i(\theta_{ij} - (\overline{m_i} - \overline{n_j}))}{\sigma_i^2 + \beta_j^2 + \tau_{ij}^2} \overline{a}_{ij}(\overline{m_i}, \overline{n_j}) \right] \right]$$

$$\frac{d\overline{m}_i}{dt} = \sigma_{Gi}^2 \sum_{j=1}^{v} \left[\frac{e_{ij} N_j(\theta_{ij} - (\overline{m_i} - \overline{n_j}))}{\sigma_i^2 + \beta_j^2 + \tau_{ij}^2} \overline{a}_{ij}(\overline{m_i}, \overline{n_j}) \right]$$

The Complete $u \times v$ Model - (u Predator Species, v Prey Species)



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Future Work

- (1×2) Two predator species in competition for one prey
- (2×1) Two prey species in apparent competition via one generalist predator
- (2×2) One specialist predator competing with one generalist predator for two prey
- (2×3) Two specialist predators competing with one generalist predator for two prey species
- ($u \times v$) The General Ditrophic Expansion
- Intraguild Predation and General Multitrophic Expansion

Thank You!

- PUMP (Preparing Undergraduates through Mentoring towards PhDs)
- The Pacific Math Alliance
- The National Math Alliance
- Dr. Helena Noronha, Dr. Ramin Vakilian, and all other PUMP organizers
- National Science Foundation
- California State University, Northridge
- Dr. Jing Li and Dr. Casey terHorst

Questions?