Beating the Taxman Asymptotically

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Abstract

I describe, with proof, a strategy for the taxman game that wins for all sufficiently large pot sizes. The Taxman Game is a number-theoretic game in which the player tries to collect from a pot natural numbers totalling more than the numbers that his automated opponent collects.

1 Introduction

Robert Moniot describes the taxman game [1]:

"Here's how to play: you start with a pot consisting of all the positive integers up to [and including] some chosen limit, N. You take one, and the taxman gets all the others that divide it evenly. The number you picked is added to your score, and the numbers chosen by the taxman are added to his score. This process is repeated until there's nothing left in the pot. What makes the game interesting is that the taxman has to get something on every turn, so you can't pick a number that has no divisors left. And when the numbers remaining in the pot have no divisors left, the taxman gets them all!"

Moniot specifies a strategy that he has verified by computer to win for $N \leq 1000$ (except for the cases $N = 1$ and $N = 3$). He conjectures that the player has an advantage for large N since the prime numbers, which the player can only collect one of, become sparser as they get larger. I will present a strategy that always wins for N sufficiently large.

2 An asymptotically winnning strategy

Let $N \in \mathbb{N}$. For $0 \leq n \leq N/2$, let a_n be the largest multiple of n such that $N/2 < a_n \leq N$ and such that a_n has no proper divisors greater than n. If no number meets these conditions, let $a_n = 0$. Another way to look at this is that we attempt to set $a_n = p_n \cdot n$, where p_n is the largest prime such that (1) no prime less than p_n divides n and (2) $N/2 < p_n \cdot n \leq N$. If conditions (1) and (2) cannot be satisfied, then $a_n = 0$ instead.

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Let $\{b_m\}$ be the subsequence of nonzero values of a_n, and let k denote the length of $\{b_m\}$.

Theorem 1 *For N sufficiently large, the following is a winning strategy in the taxman game: On the mth turn, $(m < k)$, pick b_m. After k turns, even if the game has not yet ended we already have enough points to win, so pick any legal number each turn after the kth.*

Let us verify that the strategy of theorem 1 is legal in the taxman game. Each a_n is divisible by n and has no proper divisor greater than n. Therefore, the taxman will always get something, namely n, since all the numbers the taxman took before the player chose a_n are less than n, and the player has not taken n since $n \leq N/2$, but the player only takes numbers greater than $N/2$. Furthermore, the taxman will not take a number that the player intends to take later; this is again because the taxman always takes numbers less than or equal to $N/2$, whereas the player always takes numbers greater than $N/2$.

Finally, we must show that the player never attempts to take the same number twice. We have established that when $a_n \neq 0$, $a_n = p_n \cdot n$ for some prime p_n, and p_n is the least prime that divides a_n. Suppose $n \neq m$ and $a_n, a_m \neq 0$. Without loss of generality, let $n > m$. If $p_n = p_m$, then $a_n > a_m$, so $a_n \neq a_m$. If $p_n \neq p_m$, then $a_n \neq a_m$ because p_n is the least prime that divides a_n, whereas p_m is the least prime that divides a_m.

To show that our strategy wins for N sufficiently large, it is equivalent to show that, asymptotically speaking, the player will take more than half the total pot. Stated more formally, we must show that for all N sufficiently large,

$$\frac{\sum_{1 \leq n \leq \frac{N}{2}} a_n}{\sum_{n=1}^{N} n} > \frac{1}{2}$$

where a_n is implicitly a function of both N and n. For convenience, let S denote the total pot:

$$S = \sum_{n=1}^{N} n = \frac{N(N+1)}{2}$$

Before we finish proving theorem 1, we must prove the following lemma.

Lemma 1 *Let X be a subset of $\mathbb{Z}_j$, the integers modulo j, for some natural number j. Let $0 < a < b < 1$. If we hold a, b, and j constant, then*

$$\lim_{N \to \infty} \frac{1}{S} \sum_{\substack{aN < i \leq bN \\ i \bmod j \in X}} i = (b^2 - a^2) \frac{|X|}{j}$$

Proof:

Begin by breaking down the limit into two factors.

$$\frac{1}{S} \sum_{\substack{aN < i \leq bN \\ i \bmod j \in X}} i = \left(\frac{1}{S} \sum_{aN < i \leq bN} i \right) \times \left(\left(\sum_{\substack{aN < i \leq bN \\ i \bmod j \in X}} i \right) \left(\sum_{aN < i \leq bN} i \right)^{-1} \right)$$

The sum of all the integers from 1 to bN is approximately (this is not quite precise since aN and bN may not be integers, but close enough for an asymptotic result) $\frac{bN(bN+1)}{2}$, and the sum of all the integers from 1 to aN is approximately

$$\frac{aN(aN+1)}{2},$$

so the sum of all the integers from aN to bN is

$$\frac{bN(bN+1)}{2} - \frac{aN(aN+1)}{2} \approx \frac{\left(b^2-a^2\right)N^2}{2}, \tag{1}$$

where the approximation above follows from looking at only the N^2 term. Therefore,

$$\lim_{N\to\infty}\left(\frac{1}{S}\sum_{aN<i\leq bN} i\right) = b^2 - a^2.$$

To complete the proof, we must show that

$$\lim_{N\to\infty}\left(\sum_{\substack{aN<i\leq bN\\ i \bmod j\in X}} i\right)\left(\sum_{aN<i\leq bN} i\right)^{-1} = \frac{|X|}{j} \tag{2}$$

For any integer i, let M_i denote the least multiple of j greater than i, and let m_i denote the greatest multiple of j less than or equal to i. The definitions of m_i and M_i imply that

$$\left(\sum_{\substack{aN<i\leq bN\\ i \bmod j\in X}} m_i\right)\left(\sum_{aN<i\leq bN} i\right)^{-1} \leq \left(\sum_{\substack{aN<i\leq bN\\ i \bmod j\in X}} i\right)\left(\sum_{aN<i\leq bN} i\right)^{-1} \tag{3}$$

and

$$\left(\sum_{\substack{aN<i\leq bN\\ i \bmod j\in X}} i\right)\left(\sum_{aN<i\leq bN} i\right)^{-1} \leq \left(\sum_{\substack{aN<i\leq bN\\ i \bmod j\in X}} M_i\right)\left(\sum_{aN<i\leq bN} i\right)^{-1}. \tag{4}$$

Furthermore, for N sufficiently large, it is clear that

$$\left(\sum_{\substack{aN<i\leq bN\\ i \bmod j\in X}} m_i\right)\left(\sum_{aN<i\leq bN} i\right)^{-1} < \frac{|X|}{j} < \left(\sum_{\substack{aN<i\leq bN\\ i \bmod j\in X}} M_i\right)\left(\sum_{aN<i\leq bN} i\right)^{-1}. \tag{5}$$

However,

$$\lim_{N\to\infty}\left(\sum_{\substack{aN<i\le bN\\ i \bmod j\in X}} M_i\right)\left(\sum_{aN<i\le bN} i\right)^{-1} - \left(\sum_{\substack{aN<i\le bN\\ i \bmod j\in X}} m_i\right)\left(\sum_{aN<i\le bN} i\right)^{-1} = 0 \tag{6}$$

because

$$\sum_{\substack{aN<i\le bN\\ i \bmod j\in X}} M_i - \sum_{\substack{aN<i\le bN\\ i \bmod j\in X}} m_i \le Nj$$

since each of the no more than N pairs of corresponding terms in the two sums differ by j, whereas

$$\left(\sum_{aN<i\le bN} i\right) \approx \frac{(b^2-a^2)N^2}{2},$$

as noted in equation 1.

Equations 5 and 6 together show that

$$\lim_{N\to\infty}\left(\sum_{\substack{aN<i\le bN\\ i \bmod j\in X}} m_i\right)\left(\sum_{aN<i\le bN} i\right)^{-1} = \frac{|X|}{j} = \left(\sum_{\substack{aN<i\le bN\\ i \bmod j\in X}} M_i\right)\left(\sum_{aN<i\le bN} i\right)^{-1}.$$

Therefore, by equations 3 and 4 and the squeeze theorem, we have proven equation 2, so the lemma is proven. QED

Now we are ready to finish the proof of theorem 1. It will suffice to restrict our analysis to the subset of $\{a_n\}$ where $n > N/14$. Suppose $N/14 < n \le N/11$. For each n in this range, a_n will be nonzero iff n is not divisible by $2, 3$, or 5, in which case $a_n = 7n$ if n is divisible by 7, and $a_n = 11n$ otherwise. By the Chinese Remainder Theorem and the multiplication principle, the system

$$\begin{aligned} n &\equiv 1, 2, 3, 4, 5, \text{ or } 6 \bmod 7\\ n &\equiv 1, 2, 3, \text{ or } 4 \bmod 5\\ n &\equiv 1 \text{ or } 2 \bmod 3\\ n &\equiv 1 \bmod 2 \end{aligned}$$

has $6\cdot 4\cdot 2\cdot 1 = 48$ solutions modulo $7\cdot 5\cdot 3\cdot 2 = 210$. Each n satisfying these properties will be multiplied by 11 to get a_n, so the corresponding a_n values will be the set of all numbers such that $11N/14 < a_n < N$ and a_n is congruent to one of precisely 48 values modulo $210 \cdot 11 = 2310$. (We multiplied 210 by 11 because multiplying the values of n by 11 makes the corresponding values of a_n 11 times as sparse – a_n must satisfy all the conditions on n above for congruence modulo $2, 3, 5$, and 7, as well as the condition of being congruent to

0 modulo 11.) By lemma 1, the sum of these a_n values in proportion to S will be asymptotically equal to

$$\frac{48}{2310}\left(1-\left(\frac{11}{14}\right)^2\right) = .0079,$$

truncated four places past the decimal. Similarly, the asymptotic proportional sum of the numbers between $N/14$ and $N/11$ that get multiplied by 7 will be

$$\left(\frac{1}{2}\cdot\frac{2}{3}\cdot\frac{4}{5}\cdot\frac{1}{7}\right)\cdot\frac{1}{7}\cdot\left(\left(\frac{7}{11}\right)^2-\left(\frac{1}{2}\right)^2\right) > .0008$$

So the sum of all the a_n corresponding to this range of n is greater than $0.0079 + 0.0008 = 0087$.

I will now use the technique outlined above to compute the asymptotic proportional sum of a_n for all the ranges in question. The results are tabulated below. I have given the totals, in decimal form, for each range of n values. The equals signs in the table should actually be interpreted as $>$, as I have truncated all decimals given at three places past the decimal point rather than rounding. The rightmost column of the table gives the prime, p_n, that n is multiplied by to get a_n in each case. In each of the products of fractions below, all the factors but the last are counting the number of n that meet the divisibility requirements to be multiplied by the given p_n value, and the last factor is used because when n is multiplied by p_n, the distribution of the a_n values is p_n times as sparse as the distribution of the corresponding n values. We have some wiggle room here since we could have carried our analysis down to a lower value of n. In fact, when I carried out my analysis, I started out looking at n greater than $N/4$ and lowered the lower bound on n until I had added up enough terms to exceed $1/2$.

Note that these computations make use of the lemma because in each case the a_n values we are adding up range from aN to bN for some a, b, and take on fixed congruence classes.

$$
\begin{array}{llll}
\frac{N}{14} < n \le \frac{N}{11} & \frac{1\cdot 2\cdot 4\cdot 6\cdot 1}{2\cdot 3\cdot 5\cdot 7\cdot 11}\cdot\left(1-\frac{121}{196}\right) & & \text{when } n \text{ is multiplied by } 11 \\
 & +\frac{1\cdot 2\cdot 4\cdot 1\cdot 1}{2\cdot 3\cdot 5\cdot 7\cdot 7}\cdot\left(\frac{49}{121}-\frac{1}{4}\right) & = .008 & \text{when } n \text{ is multiplied by } 7 \\
\frac{N}{11} < n \le \frac{N}{10} & \frac{1\cdot 2\cdot 4\cdot 1}{2\cdot 3\cdot 5\cdot 7}\cdot\left(\frac{49}{100}-\frac{49}{121}\right) & = .003 & \text{when } n \text{ is multiplied by } 7 \\
\frac{N}{10} < n \le \frac{N}{7} & \frac{1\cdot 2\cdot 4\cdot 1}{2\cdot 3\cdot 5\cdot 7}\cdot\left(1-\frac{49}{100}\right) & & \text{when } n \text{ is multiplied by } 7 \\
 & +\frac{1\cdot 2\cdot 1\cdot 1}{2\cdot 3\cdot 5\cdot 5}\cdot\left(\frac{25}{49}-\frac{1}{4}\right) & = .022 & \text{when } n \text{ is multiplied by } 5 \\
\frac{N}{7} < n \le \frac{N}{6} & \frac{1\cdot 2\cdot 1}{2\cdot 3\cdot 5}\cdot\left(\frac{25}{36}-\frac{25}{49}\right) & = .012 & \text{when } n \text{ is multiplied by } 5 \\
\frac{N}{6} < n \le \frac{N}{5} & \frac{1\cdot 2\cdot 1}{2\cdot 3\cdot 5}\cdot\left(1-\frac{25}{36}\right) & & \text{when } n \text{ is multiplied by } 5 \\
 & \frac{1\cdot 1\cdot 1}{2\cdot 3\cdot 3}\cdot\left(\frac{9}{25}-\frac{1}{4}\right) & = .026 & \text{when } n \text{ is multiplied by } 3 \\
\frac{N}{5} < n \le \frac{N}{4} & \frac{1\cdot 1}{2\cdot 3}\cdot\left(\frac{9}{16}-\frac{9}{25}\right) & = .033 & \text{when } n \text{ is multiplied by } 3 \\
\frac{N}{4} < n \le \frac{N}{3} & \frac{1\cdot 1}{2\cdot 3}\cdot\left(1-\frac{9}{16}\right) & & \text{when } n \text{ is multiplied by } 3 \\
 & +\frac{1\cdot 1}{2\cdot 2}\cdot\left(\frac{4}{9}-\frac{1}{4}\right) & = .121 & \text{when } n \text{ is multiplied by } 2 \\
\frac{N}{3} < n \le \frac{N}{2} & \frac{1}{2}\cdot\left(1-\frac{4}{9}\right) & = .277 & \text{when } n \text{ is multiplied by } 2
\end{array}
$$

The grand total of the proportional asymptotic sums is greater than $.502 > 1/2$. Since the number of j values that we plugged into the lemma is finite, it is safe to add these asymptotic sums. Therefore, the proof is complete.

3 Open questions

3.1 Does the strategy win for all $N > 19$?

Can we compute a specific lower bound for N, above which the specified strategy always wins? If so, and if the lower bound is reasonably small, we could use a computer to check all the remaining cases. Based on a computer simulation testing I wrote to test many values, I conjecture that my strategy wins for all N other than $1, 3$, and 19.

3.2 What is the closed form of the asymptotic sum?

Does the following limit exist, and if so, can it be evaluated in closed form?

$$\lim_{N\to\infty} \frac{\sum_{1\le n\le \frac{N}{2}} a_n}{\sum_{n=1}^{N} n} \tag{7}$$

An extrapolation of the approach taken in this paper shows that

$$\lim_{N\to\infty} \frac{\sum_{N/k\le n\le \frac{N}{2}} a_n}{\sum_{n=1}^{N} n}$$

exists for any k, but not that the limit of line 7 necessarily exists. My computer simulation suggests that the sum converges to about 0.53, staying within .01 of 0.53 value for $N > 300$.

3.3 Taxman as a two player game

Suppose Taxman is made into a two player game. The players take turns choosing numbers that have proper divisors remaining in the pot and each player's opponent gets the divisors of the number the player chooses. The game ends when no numbers have divisors remaining in the pot. The player with the higher sum wins, or if the players' sums are equal, the game is a tie. Does the first player have a winning strategy for N sufficiently large? Suppose we allow the players the option of passing and the game ends when no numbers have proper divisors remaining in the pot or when both players pass in succession. Does this change which player, if either, has a winning strategy? The passing rule certainly prevents the second player from having a winning strategy, because if he did, then both players would pass their first moves, resulting in a draw.

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References

[1] Moniot, R., "The Taxman Game", Math Horizons, Feb. 2007, pp.18-20.

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Norman Perlmutter was born and raised in Toledo, OH. He graduated from Grinnell College with honors in mathematics in May 2007. In September 2007, he received the Pamela Ferguson Prize, which is awarded to up to two outstanding senior Grinnell mathematics student each year. This paper is the result of

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