

# Reactive Designs in Isabelle/UTP

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## Abstract

Reactive designs combine the UTP theories of reactive processes and designs to characterise reactive programs. Whereas sequential imperative programs are expected to run until termination, reactive programs pause at instances to allow interaction with the environment using abstract events, and often do not terminate at all. Thus, whereas a design describes the precondition and postcondition for a program, to characterise initial and final states, a reactive design also has a “pericondition”, which characterises intermediate quiescent observations. This gives rise to a notion of “reactive contract”, which specifies the assumptions a program makes of its environment, and the guarantees it will make of its own behaviour in both intermediate and final observations. This Isabelle/UTP document mechanises the UTP theory of reactive designs, including its healthiness conditions, signature, and a large library of algebraic laws of reactive programming.

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## 1 Introduction

This document contains a mechanisation in Isabelle/UTP [2] of our theory of reactive designs. Reactive designs form an important semantic foundation for reactive modelling languages such as Circus [3]. For more details of this work, please see our recent paper [1].

## 2 Reactive Designs Healthiness Conditions

```
theory utp-rdes-healths
  imports UTP-Reactive.utp-reactive
begin
```

### 2.1 Preliminaries

```
named-theorems rdes and rdes-def and RD-elim
```

```
type-synonym ('s,'t) rdes = ('s,'t,unit) hrel-rsp
```

```
translations
  (type) ('s,'t) rdes <= (type) ('s,'t, unit) hrel-rsp
```

```
declare des-vars.splits [alpha-splits del]
declare rp-vars.splits [alpha-splits del]
declare rp-vars.splits [alpha-splits]
declare des-vars.splits [alpha-splits]
```

```
lemma R2-st-ex: R2 ( $\exists$  $st  $\cdot$  P) = ( $\exists$  $st  $\cdot$  R2(P))
  by (rel-auto)
```

```
lemma R2s-st'-eq-st:
  R2s($st' =u $st) = ($st' =u $st)
  by (rel-auto)
```

```
lemma R2c-st'-eq-st:
  R2c($st' =u $st) = ($st' =u $st)
  by (rel-auto)
```

```
lemma R1-des-lift-skip: R1( $\lceil$  II  $\rceil_D$ ) =  $\lceil$  II  $\rceil_D$ 
  by (rel-auto)
```

```
lemma R2-des-lift-skip:
  R2( $\lceil$  II  $\rceil_D$ ) =  $\lceil$  II  $\rceil_D$ 
  apply (rel-auto) using minus-zero-eq by blast
```

```
lemma R1-R2c-ex-st: R1 (R2c ( $\exists$  $st'  $\cdot$  Q1)) = ( $\exists$  $st'  $\cdot$  R1 (R2c Q1))
  by (rel-auto)
```

## 2.2 Identities

We define two identities for reactive designs, which correspond to the regular and state-sensitive versions of reactive designs, respectively. The former is the one used in the UTP book and related publications for CSP.

**definition**  $skip\_rea :: ('t::trace, 'α) hrel\_rp (II_C)$  **where**  
 $skip\_rea\_def \ [urel\_defs]: II_C = (II \vee (\neg \$ok \wedge \$tr \leq_u \$tr'))$

**definition**  $skip\_srea :: ('s, 't::trace, 'α) hrel\_rsp (II_R)$  **where**  
 $skip\_srea\_def \ [urel\_defs]: II_R = ((\exists \$st \cdot II_C) \triangleleft \$wait \triangleright II_C)$

**lemma**  $skip\_rea\text{-}R1\text{-lemma}: II_C = R1(\$ok \Rightarrow II)$   
**by**  $(rel\_auto)$

**lemma**  $skip\_rea\text{-}form: II_C = (II \triangleleft \$ok \triangleright R1(true))$   
**by**  $(rel\_auto)$

**lemma**  $skip\_srea\text{-}form: II_R = ((\exists \$st \cdot II) \triangleleft \$wait \triangleright II) \triangleleft \$ok \triangleright R1(true)$   
**by**  $(rel\_auto)$

**lemma**  $R1\text{-}skip\_rea: R1(II_C) = II_C$   
**by**  $(rel\_auto)$

**lemma**  $R2c\text{-}skip\_rea: R2c\ II_C = II_C$   
**by**  $(simp\ add: skip\_rea\_def\ R2c\_and\ R2c\_disj\ R2c\_skip\_r\ R2c\_not\ R2c\_ok\ R2c\_tr\_le\_tr')$

**lemma**  $R2\text{-}skip\_rea: R2(II_C) = II_C$   
**by**  $(metis\ R1\text{-}R2c\text{-}is\text{-}R2\ R1\text{-}skip\_rea\ R2c\text{-}skip\_rea)$

**lemma**  $R2c\text{-}skip\_srea: R2c(II_R) = II_R$   
**apply**  $(rel\_auto)$  **using**  $minus\_zero\_eq$  **by**  $blast+$

**lemma**  $skip\_srea\text{-}R1 \ [closure]: II_R$  *is*  $R1$   
**by**  $(rel\_auto)$

**lemma**  $skip\_srea\text{-}R2c \ [closure]: II_R$  *is*  $R2c$   
**by**  $(simp\ add: Healthy\_def\ R2c\text{-}skip\_srea)$

**lemma**  $skip\_srea\text{-}R2 \ [closure]: II_R$  *is*  $R2$   
**by**  $(metis\ Healthy\_def'\ R1\text{-}R2c\text{-}is\text{-}R2\ R2c\text{-}skip\_srea\ skip\_srea\text{-}R1)$

## 2.3 RD1: Divergence yields arbitrary traces

**definition**  $RD1 :: ('t::trace, 'α, 'β) rel\_rp \Rightarrow ('t, 'α, 'β) rel\_rp$  **where**  
 $[upred\_defs]: RD1(P) = (P \vee (\neg \$ok \wedge \$tr \leq_u \$tr'))$

$RD1$  is essentially  $H1$  from the theory of designs, but viewed through the prism of reactive processes.

**lemma**  $RD1\text{-}idem: RD1(RD1(P)) = RD1(P)$   
**by**  $(rel\_auto)$

**lemma**  $RD1\text{-}Idempotent: Idempotent\ RD1$   
**by**  $(simp\ add: Idempotent\_def\ RD1\text{-}idem)$

**lemma**  $RD1\text{-}mono: P \sqsubseteq Q \Longrightarrow RD1(P) \sqsubseteq RD1(Q)$

by (rel-auto)

**lemma** *RD1-Monotonic: Monotonic RD1*  
 using mono-def RD1-mono by blast

**lemma** *RD1-Continuous: Continuous RD1*  
 by (rel-auto)

**lemma** *R1-true-RD1-closed [closure]: R1(true) is RD1*  
 by (rel-auto)

**lemma** *RD1-wait-false [closure]: P is RD1  $\implies$  P $\llbracket$ false/\$wait $\rrbracket$  is RD1*  
 by (rel-auto)

**lemma** *RD1-wait'-false [closure]: P is RD1  $\implies$  P $\llbracket$ false/\$wait' $\rrbracket$  is RD1*  
 by (rel-auto)

**lemma** *RD1-seq: RD1(RD1(P) ;; RD1(Q)) = RD1(P) ;; RD1(Q)*  
 by (rel-auto)

**lemma** *RD1-seq-closure [closure]:  $\llbracket P \text{ is RD1}; Q \text{ is RD1} \rrbracket \implies P ;; Q \text{ is RD1}$*   
 by (metis Healthy-def' RD1-seq)

**lemma** *RD1-R1-commute: RD1(R1(P)) = R1(RD1(P))*  
 by (rel-auto)

**lemma** *RD1-R2c-commute: RD1(R2c(P)) = R2c(RD1(P))*  
 by (rel-auto)

**lemma** *RD1-via-R1: R1(H1(P)) = RD1(R1(P))*  
 by (rel-auto)

**lemma** *RD1-R1-cases: RD1(R1(P)) = (R1(P)  $\triangleleft$  \$ok  $\triangleright$  R1(true))*  
 by (rel-auto)

**lemma** *skip-rea-RD1-skip: II<sub>C</sub> = RD1(II)*  
 by (rel-auto)

**lemma** *skip-srea-RD1 [closure]: II<sub>R</sub> is RD1*  
 by (rel-auto)

**lemma** *RD1-algebraic-intro:*  
 assumes  
   *P is R1 (R1(true<sub>h</sub>) ;; P) = R1(true<sub>h</sub>) (II<sub>C</sub> ;; P) = P*  
 shows *P is RD1*  
**proof** –  
 have *P = (II<sub>C</sub> ;; P)*  
   by (simp add: assms(3))  
 also have *... = (R1(\$ok  $\Rightarrow$  II) ;; P)*  
   by (simp add: skip-rea-R1-lemma)  
 also have *... = ((( $\neg$  \$ok  $\wedge$  R1(true)) ;; P)  $\vee$  P)*  
   by (metis (no-types, lifting) R1-def seqr-left-unit seqr-or-distl skip-rea-R1-lemma skip-rea-def utp-pred-laws.inf-top-left  
 utp-pred-laws.sup-commute)  
 also have *... = ((R1( $\neg$  \$ok) ;; (R1(true<sub>h</sub>) ;; P))  $\vee$  P)*  
   using dual-order.trans by (rel-blast)

**also have** ... =  $((R1(\neg \$ok) ;; R1(true_h)) \vee P)$   
**by** (*simp add: assms(2)*)  
**also have** ... =  $(R1(\neg \$ok) \vee P)$   
**by** (*rel-auto*)  
**also have** ... =  $RD1(P)$   
**by** (*rel-auto*)  
**finally show** *?thesis*  
**by** (*simp add: Healthy-def*)  
**qed**

**theorem** *RD1-left-zero*:  
**assumes** *P is R1 P is RD1*  
**shows**  $(R1(true) ;; P) = R1(true)$   
**proof** –  
**have**  $(R1(true) ;; R1(RD1(P))) = R1(true)$   
**by** (*rel-auto*)  
**thus** *?thesis*  
**by** (*simp add: Healthy-if assms(1) assms(2)*)  
**qed**

**theorem** *RD1-left-unit*:  
**assumes** *P is R1 P is RD1*  
**shows**  $(II_C ;; P) = P$   
**proof** –  
**have**  $(II_C ;; R1(RD1(P))) = R1(RD1(P))$   
**by** (*rel-auto*)  
**thus** *?thesis*  
**by** (*simp add: Healthy-if assms(1) assms(2)*)  
**qed**

**lemma** *RD1-alt-def*:  
**assumes** *P is R1*  
**shows**  $RD1(P) = (P \triangleleft \$ok \triangleright R1(true))$   
**proof** –  
**have**  $RD1(R1(P)) = (R1(P) \triangleleft \$ok \triangleright R1(true))$   
**by** (*rel-auto*)  
**thus** *?thesis*  
**by** (*simp add: Healthy-if assms*)  
**qed**

**theorem** *RD1-algebraic*:  
**assumes** *P is R1*  
**shows**  $P \text{ is } RD1 \iff (R1(true_h) ;; P) = R1(true_h) \wedge (II_C ;; P) = P$   
**using** *RD1-algebraic-intro RD1-left-unit RD1-left-zero assms* **by** *blast*

## 2.4 R3c and R3h: Reactive design versions of R3

**definition** *R3c* ::  $(t::trace, 'a) \text{ hrel-rp} \Rightarrow (t, 'a) \text{ hrel-rp}$  **where**  
*[upred-defs]:*  $R3c(P) = (II_C \triangleleft \$wait \triangleright P)$

**definition** *R3h* ::  $(s, t::trace, 'a) \text{ hrel-rsp} \Rightarrow (s, t, 'a) \text{ hrel-rsp}$  **where**  
*R3h-def [upred-defs]:*  $R3h(P) = ((\exists \$st \cdot II_C) \triangleleft \$wait \triangleright P)$

**lemma** *R3c-idem*:  $R3c(R3c(P)) = R3c(P)$   
**by** (*rel-auto*)

**lemma** *R3c-Idempotent: Idempotent R3c*  
**by** (*simp add: Idempotent-def R3c-idem*)

**lemma** *R3c-mono:  $P \sqsubseteq Q \implies R3c(P) \sqsubseteq R3c(Q)$*   
**by** (*rel-auto*)

**lemma** *R3c-Monotonic: Monotonic R3c*  
**by** (*simp add: mono-def R3c-mono*)

**lemma** *R3c-Continuous: Continuous R3c*  
**by** (*rel-auto*)

**lemma** *R3h-idem:  $R3h(R3h(P)) = R3h(P)$*   
**by** (*rel-auto*)

**lemma** *R3h-Idempotent: Idempotent R3h*  
**by** (*simp add: Idempotent-def R3h-idem*)

**lemma** *R3h-mono:  $P \sqsubseteq Q \implies R3h(P) \sqsubseteq R3h(Q)$*   
**by** (*rel-auto*)

**lemma** *R3h-Monotonic: Monotonic R3h*  
**by** (*simp add: mono-def R3h-mono*)

**lemma** *R3h-Continuous: Continuous R3h*  
**by** (*rel-auto*)

**lemma** *R3h-inf:  $R3h(P \sqcap Q) = R3h(P) \sqcap R3h(Q)$*   
**by** (*rel-auto*)

**lemma** *R3h-UINF:*  
 $A \neq \{\} \implies R3h(\bigsqcap i \in A \cdot P(i)) = (\bigsqcap i \in A \cdot R3h(P(i)))$   
**by** (*rel-auto*)

**lemma** *R3h-cond:  $R3h(P \triangleleft b \triangleright Q) = (R3h(P) \triangleleft b \triangleright R3h(Q))$*   
**by** (*rel-auto*)

**lemma** *R3c-via-RD1-R3:  $RD1(R3(P)) = R3c(RD1(P))$*   
**by** (*rel-auto*)

**lemma** *R3c-RD1-def:  $P$  is  $RD1 \implies R3c(P) = RD1(R3(P))$*   
**by** (*simp add: Healthy-if R3c-via-RD1-R3*)

**lemma** *RD1-R3c-commute:  $RD1(R3c(P)) = R3c(RD1(P))$*   
**by** (*rel-auto*)

**lemma** *R1-R3c-commute:  $R1(R3c(P)) = R3c(R1(P))$*   
**by** (*rel-auto*)

**lemma** *R2c-R3c-commute:  $R2c(R3c(P)) = R3c(R2c(P))$*   
**apply** (*rel-auto*) **using** *minus-zero-eq* **by** *blast+*

**lemma** *R1-R3h-commute:  $R1(R3h(P)) = R3h(R1(P))$*   
**by** (*rel-auto*)

**lemma** *R2c-R3h-commute*:  $R2c(R3h(P)) = R3h(R2c(P))$   
**apply** (*rel-auto*) **using** *minus-zero-eq* **by** *blast+*

**lemma** *RD1-R3h-commute*:  $RD1(R3h(P)) = R3h(RD1(P))$   
**by** (*rel-auto*)

**lemma** *R3c-cancels-R3*:  $R3c(R3(P)) = R3c(P)$   
**by** (*rel-auto*)

**lemma** *R3-cancels-R3c*:  $R3(R3c(P)) = R3(P)$   
**by** (*rel-auto*)

**lemma** *R3h-cancels-R3c*:  $R3h(R3c(P)) = R3h(P)$   
**by** (*rel-auto*)

**lemma** *R3c-semir-form*:  
 $(R3c(P) ;; R3c(R1(Q))) = R3c(P ;; R3c(R1(Q)))$   
**by** (*rel-simp*, *safe*, *auto intro: order-trans*)

**lemma** *R3h-semir-form*:  
 $(R3h(P) ;; R3h(R1(Q))) = R3h(P ;; R3h(R1(Q)))$   
**by** (*rel-simp*, *safe*, *auto intro: order-trans*, *blast+*)

**lemma** *R3c-seq-closure*:  
**assumes**  $P$  is  $R3c$   $Q$  is  $R3c$   $Q$  is  $R1$   
**shows**  $(P ;; Q)$  is  $R3c$   
**by** (*metis Healthy-def' R3c-semir-form assms*)

**lemma** *R3h-seq-closure [closure]*:  
**assumes**  $P$  is  $R3h$   $Q$  is  $R3h$   $Q$  is  $R1$   
**shows**  $(P ;; Q)$  is  $R3h$   
**by** (*metis Healthy-def' R3h-semir-form assms*)

**lemma** *R3c-R3-left-seq-closure*:

**assumes**  $P$  is  $R3$   $Q$  is  $R3c$   
**shows**  $(P ;; Q)$  is  $R3c$

**proof** –

**have**  $(P ;; Q) = ((P ;; Q) \llbracket \text{true}/\$wait \rrbracket \triangleleft \$wait \triangleright (P ;; Q))$   
**by** (*metis cond-var-split cond-var-subst-right in-var-uvar wait-vwb-lens*)  
**also have**  $\dots = (((II \triangleleft \$wait \triangleright P) ;; Q) \llbracket \text{true}/\$wait \rrbracket \triangleleft \$wait \triangleright (P ;; Q))$   
**by** (*metis Healthy-def' R3-def assms(1)*)  
**also have**  $\dots = ((II \llbracket \text{true}/\$wait \rrbracket ;; Q) \triangleleft \$wait \triangleright (P ;; Q))$   
**by** (*subst-tac*)  
**also have**  $\dots = (((II \wedge \$wait') ;; Q) \triangleleft \$wait \triangleright (P ;; Q))$   
**by** (*metis (no-types, lifting) cond-def conj-pos-var-subst seqr-pre-var-out skip-var utp-pred-laws.inf-left-idem wait-vwb-lens*)  
**also have**  $\dots = ((II \llbracket \text{true}/\$wait' \rrbracket ;; Q \llbracket \text{true}/\$wait \rrbracket) \triangleleft \$wait \triangleright (P ;; Q))$   
**by** (*metis seqr-pre-transfer seqr-right-one-point true-alt-def uovar-convr upred-eq-true utp-rel.unrest-ouvar vwb-lens-mwb wait-vwb-lens*)  
**also have**  $\dots = ((II \llbracket \text{true}/\$wait' \rrbracket ;; (II_C \triangleleft \$wait \triangleright Q) \llbracket \text{true}/\$wait \rrbracket) \triangleleft \$wait \triangleright (P ;; Q))$   
**by** (*metis Healthy-def' R3c-def assms(2)*)  
**also have**  $\dots = ((II \llbracket \text{true}/\$wait' \rrbracket ;; II_C \llbracket \text{true}/\$wait \rrbracket) \triangleleft \$wait \triangleright (P ;; Q))$   
**by** (*subst-tac*)  
**also have**  $\dots = (((II \wedge \$wait') ;; II_C) \triangleleft \$wait \triangleright (P ;; Q))$   
**by** (*metis seqr-pre-transfer seqr-right-one-point true-alt-def uovar-convr upred-eq-true utp-rel.unrest-ouvar*)



$vwb\text{-}lens\text{-}mwb\ wait\text{-}vwb\text{-}lens)$   
**also have**  $\dots = ((II \;;\; II_C) \triangleleft \$wait \triangleright (P \;;\; Q))$   
**by** (*simp add: cond-def seqr-pre-transfer utp-rel.unrest-ouvar*)  
**also have**  $\dots = (II_C \triangleleft \$wait \triangleright (P \;;\; Q))$   
**by** *simp*  
**also have**  $\dots = R3c(P \;;\; Q)$   
**by** (*simp add: R3c-def*)  
**finally show** *?thesis*  
**by** (*simp add: Healthy-def'*)  
**qed**

**lemma** *R3c-cases*:  $R3c(P) = ((II \triangleleft \$ok \triangleright R1(true)) \triangleleft \$wait \triangleright P)$   
**by** (*rel-auto*)

**lemma** *R3h-cases*:  $R3h(P) = (((\exists \$st \cdot II) \triangleleft \$ok \triangleright R1(true)) \triangleleft \$wait \triangleright P)$   
**by** (*rel-auto*)

**lemma** *R3h-form*:  $R3h(P) = II_R \triangleleft \$wait \triangleright P$   
**by** (*rel-auto*)

**lemma** *R3c-subst-wait*:  $R3c(P) = R3c(P_f)$   
**by** (*simp add: R3c-def cond-var-subst-right*)

**lemma** *R3h-subst-wait*:  $R3h(P) = R3h(P_f)$   
**by** (*simp add: R3h-cases cond-var-subst-right*)

**lemma** *skip-srea-R3h [closure]*:  $II_R$  is *R3h*  
**by** (*rel-auto*)

**lemma** *R3h-wait-true*:  
**assumes**  $P$  is *R3h*  
**shows**  $P_t = II_R_t$   
**proof** –  
**have**  $P_t = (II_R \triangleleft \$wait \triangleright P)_t$   
**by** (*metis Healthy-if R3h-form assms*)  
**also have**  $\dots = II_{R_t}$   
**by** (*simp add: usubst*)  
**finally show** *?thesis* .  
**qed**

## 2.5 RD2: A reactive specification cannot require non-termination

**definition** *RD2* where  
 $[upred\text{-}defs]: RD2(P) = H2(P)$

*RD2* is just *H2* since the type system will automatically have J identifying the reactive variables as required.

**lemma** *RD2-idem*:  $RD2(RD2(P)) = RD2(P)$   
**by** (*simp add: H2-idem RD2-def*)

**lemma** *RD2-Idempotent*: *Idempotent* *RD2*  
**by** (*simp add: Idempotent-def RD2-idem*)

**lemma** *RD2-mono*:  $P \sqsubseteq Q \implies RD2(P) \sqsubseteq RD2(Q)$   
**by** (*simp add: H2-def RD2-def seqr-mono*)

**lemma** *RD2-Monotonic: Monotonic RD2*  
**using** *mono-def RD2-mono* **by** *blast*

**lemma** *RD2-Continuous: Continuous RD2*  
**by** *(rel-auto)*

**lemma** *RD1-RD2-commute:  $RD1(RD2(P)) = RD2(RD1(P))$*   
**by** *(rel-auto)*

**lemma** *RD2-R3c-commute:  $RD2(R3c(P)) = R3c(RD2(P))$*   
**by** *(rel-auto)*

**lemma** *RD2-R3h-commute:  $RD2(R3h(P)) = R3h(RD2(P))$*   
**by** *(rel-auto)*

## 2.6 Major healthiness conditions

**definition** *RH* ::  $(t::trace, \alpha) \text{ hrel-rp} \Rightarrow (t, \alpha) \text{ hrel-rp}$  (**R**)  
**where** *[upred-defs]:*  $RH(P) = R1(R2c(R3c(P)))$

**definition** *RHS* ::  $(s, t::trace, \alpha) \text{ hrel-rsp} \Rightarrow (s, t, \alpha) \text{ hrel-rsp}$  (**R<sub>s</sub>**)  
**where** *[upred-defs]:*  $RHS(P) = R1(R2c(R3h(P)))$

**definition** *RD* ::  $(t::trace, \alpha) \text{ hrel-rp} \Rightarrow (t, \alpha) \text{ hrel-rp}$   
**where** *[upred-defs]:*  $RD(P) = RD1(RD2(RP(P)))$

**definition** *SRD* ::  $(s, t::trace, \alpha) \text{ hrel-rsp} \Rightarrow (s, t, \alpha) \text{ hrel-rsp}$   
**where** *[upred-defs]:*  $SRD(P) = RD1(RD2(RHS(P)))$

**lemma** *RH-comp:  $RH = R1 \circ R2c \circ R3c$*   
**by** *(auto simp add: RH-def)*

**lemma** *RHS-comp:  $RHS = R1 \circ R2c \circ R3h$*   
**by** *(auto simp add: RHS-def)*

**lemma** *RD-comp:  $RD = RD1 \circ RD2 \circ RP$*   
**by** *(auto simp add: RD-def)*

**lemma** *SRD-comp:  $SRD = RD1 \circ RD2 \circ RHS$*   
**by** *(auto simp add: SRD-def)*

**lemma** *RH-idem:  $\mathbf{R}(\mathbf{R}(P)) = \mathbf{R}(P)$*   
**by** *(simp add: R1-R2c-commute R1-R3c-commute R1-idem R2c-R3c-commute R2c-idem R3c-idem RH-def)*

**lemma** *RH-Idempotent: Idempotent R*  
**by** *(simp add: Idempotent-def RH-idem)*

**lemma** *RH-Monotonic: Monotonic R*  
**by** *(metis (no-types, lifting) R1-Monotonic R2c-Monotonic R3c-mono RH-def mono-def)*

**lemma** *RH-Continuous: Continuous R*  
**by** *(simp add: Continuous-comp R1-Continuous R2c-Continuous R3c-Continuous RH-comp)*

**lemma** *RHS-idem:  $\mathbf{R}_s(\mathbf{R}_s(P)) = \mathbf{R}_s(P)$*

by (simp add: R1-R2c-is-R2 R1-R3h-commute R2-idem R2c-R3h-commute R3h-idem RHS-def)

**lemma** *RHS-Idempotent* [closure]: *Idempotent*  $\mathbf{R}_s$   
 by (simp add: Idempotent-def RHS-idem)

**lemma** *RHS-Monotonic*: *Monotonic*  $\mathbf{R}_s$   
 by (simp add: mono-def R1-R2c-is-R2 R2-mono R3h-mono RHS-def)

**lemma** *RHS-mono*:  $P \sqsubseteq Q \implies \mathbf{R}_s(P) \sqsubseteq \mathbf{R}_s(Q)$   
 using mono-def RHS-Monotonic by blast

**lemma** *RHS-Continuous* [closure]: *Continuous*  $\mathbf{R}_s$   
 by (simp add: Continuous-comp R1-Continuous R2c-Continuous R3h-Continuous RHS-comp)

**lemma** *RHS-inf*:  $\mathbf{R}_s(P \sqcap Q) = \mathbf{R}_s(P) \sqcap \mathbf{R}_s(Q)$   
 using Continuous-Disjunctous Disjunctuous-def RHS-Continuous by auto

**lemma** *RHS-INF*:  
 $A \neq \{\} \implies \mathbf{R}_s(\bigsqcap i \in A \cdot P(i)) = (\bigsqcap i \in A \cdot \mathbf{R}_s(P(i)))$   
 by (simp add: RHS-def R3h-UINF R2c-USUP R1-USUP)

**lemma** *RHS-sup*:  $\mathbf{R}_s(P \sqcup Q) = \mathbf{R}_s(P) \sqcup \mathbf{R}_s(Q)$   
 by (rel-auto)

**lemma** *RHS-SUP*:  
 $A \neq \{\} \implies \mathbf{R}_s(\bigsqcup i \in A \cdot P(i)) = (\bigsqcup i \in A \cdot \mathbf{R}_s(P(i)))$   
 by (rel-auto)

**lemma** *RHS-cond*:  $\mathbf{R}_s(P \triangleleft b \triangleright Q) = (\mathbf{R}_s(P) \triangleleft R2c\ b \triangleright \mathbf{R}_s(Q))$   
 by (simp add: RHS-def R3h-cond R2c-condr R1-cond)

**lemma** *RD-alt-def*:  $RD(P) = RD1(RD2(\mathbf{R}(P)))$   
 by (simp add: R3c-via-RD1-R3 RD1-R1-commute RD1-R2c-commute RD1-R3c-commute RD1-RD2-commute RH-def RD-def RP-def)

**lemma** *RD1-RH-commute*:  $RD1(\mathbf{R}(P)) = \mathbf{R}(RD1(P))$   
 by (simp add: RD1-R1-commute RD1-R2c-commute RD1-R3c-commute RH-def)

**lemma** *RD2-RH-commute*:  $RD2(\mathbf{R}(P)) = \mathbf{R}(RD2(P))$   
 by (metis R1-H2-commute R2c-H2-commute RD2-R3c-commute RD2-def RH-def)

**lemma** *RD-idem*:  $RD(RD(P)) = RD(P)$   
 by (simp add: RD-alt-def RD1-RH-commute RD2-RH-commute RD1-RD2-commute RD2-idem RD1-idem RH-idem)

**lemma** *RD-Monotonic*: *Monotonic*  $RD$   
 by (simp add: Monotonic-comp RD1-Monotonic RD2-Monotonic RD-comp RP-Monotonic)

**lemma** *RD-Continuous*: *Continuous*  $RD$   
 by (simp add: Continuous-comp RD1-Continuous RD2-Continuous RD-comp RP-Continuous)

**lemma** *R3-RD-RP*:  $R3(RD(P)) = RP(RD1(RD2(P)))$   
 by (metis (no-types, lifting) R1-R2c-is-R2 R2-R3-commute R3-cancels-R3c RD1-RH-commute RD2-RH-commute RD-alt-def RH-def RP-def)

**lemma** *RD1-RHS-commute*:  $RD1(\mathbf{R}_s(P)) = \mathbf{R}_s(RD1(P))$   
 by (simp add: *RD1-R1-commute RD1-R2c-commute RD1-R3h-commute RHS-def*)

**lemma** *RD2-RHS-commute*:  $RD2(\mathbf{R}_s(P)) = \mathbf{R}_s(RD2(P))$   
 by (metis *R1-H2-commute R2c-H2-commute RD2-R3h-commute RD2-def RHS-def*)

**lemma** *SRD-idem*:  $SRD(SRD(P)) = SRD(P)$   
 by (simp add: *RD1-RD2-commute RD1-RHS-commute RD1-idem RD2-RHS-commute RD2-idem RHS-idem SRD-def*)

**lemma** *SRD-Idempotent [closure]*: *Idempotent SRD*  
 by (simp add: *Idempotent-def SRD-idem*)

**lemma** *SRD-Monotonic*: *Monotonic SRD*  
 by (simp add: *Monotonic-comp RD1-Monotonic RD2-Monotonic RHS-Monotonic SRD-comp*)

**lemma** *SRD-Continuous [closure]*: *Continuous SRD*  
 by (simp add: *Continuous-comp RD1-Continuous RD2-Continuous RHS-Continuous SRD-comp*)

**lemma** *SRD-RHS-H1-H2*:  $SRD(P) = \mathbf{R}_s(\mathbf{H}(P))$   
 by (rel-auto)

**lemma** *SRD-healths [closure]*:  
 assumes *P is SRD*  
 shows *P is R1 P is R2 P is R3h P is RD1 P is RD2*  
 apply (metis *Healthy-def R1-idem RD1-RHS-commute RD2-RHS-commute RHS-def SRD-def assms*)  
 apply (metis *Healthy-def R1-R2c-is-R2 R2-idem RD1-RHS-commute RD2-RHS-commute RHS-def SRD-def assms*)  
 apply (metis *Healthy-def R1-R3h-commute R2c-R3h-commute R3h-idem RD1-R3h-commute RD2-R3h-commute RHS-def SRD-def assms*)  
 apply (metis *Healthy-def' RD1-idem SRD-def assms*)  
 apply (metis *Healthy-def' RD1-RD2-commute RD2-idem SRD-def assms*)  
 done

**lemma** *SRD-intro*:  
 assumes *P is R1 P is R2 P is R3h P is RD1 P is RD2*  
 shows *P is SRD*  
 by (metis *Healthy-def R1-R2c-is-R2 RHS-def SRD-def assms(2) assms(3) assms(4) assms(5)*)

**lemma** *SRD-ok-false [usubst]*:  $P \text{ is } SRD \implies P \llbracket \text{false}/\$ok \rrbracket = R1(true)$   
 by (metis (no-types, hide-lams) *H1-H2-eq-design Healthy-def R1-ok-false RD1-R1-commute RD1-via-R1 RD2-def SRD-def SRD-healths(1) design-ok-false*)

**lemma** *SRD-ok-true-wait-true [usubst]*:  
 assumes *P is SRD*  
 shows  $P \llbracket \text{true}, \text{true}/\$ok, \$wait \rrbracket = (\exists \$st \cdot II) \llbracket \text{true}, \text{true}/\$ok, \$wait \rrbracket$   
**proof** –  
 have  $P = (\exists \$st \cdot II) \triangleleft \$ok \triangleright R1 \text{ true} \triangleleft \$wait \triangleright P$   
 by (metis *Healthy-def R3h-cases SRD-healths(3) assms*)  
 moreover have  $((\exists \$st \cdot II) \triangleleft \$ok \triangleright R1 \text{ true} \triangleleft \$wait \triangleright P) \llbracket \text{true}, \text{true}/\$ok, \$wait \rrbracket = (\exists \$st \cdot II) \llbracket \text{true}, \text{true}/\$ok, \$wait \rrbracket$   
 by (simp add: *usubst*)  
 ultimately show *?thesis*  
 by (simp)  
**qed**

**lemma** *SRD-left-zero-1*:  $P$  is *SRD*  $\implies R1(true) ;; P = R1(true)$   
 by (*simp add: RD1-left-zero SRD-healths(1) SRD-healths(4)*)

**lemma** *SRD-left-zero-2*:

assumes  $P$  is *SRD*

shows  $(\exists \$st \cdot II)[[true, true/\$ok, \$wait]] ;; P = (\exists \$st \cdot II)[[true, true/\$ok, \$wait]]$

**proof** –

have  $(\exists \$st \cdot II)[[true, true/\$ok, \$wait]] ;; R3h(P) = (\exists \$st \cdot II)[[true, true/\$ok, \$wait]]$

by (*rel-auto*)

thus *?thesis*

by (*simp add: Healthy-if SRD-healths(3) assms*)

**qed**

## 2.7 UTP theories

We create two theory objects: one for reactive designs and one for stateful reactive designs.

**interpretation** *rdes-theory*: *utp-theory-continuous RD*

rewrites  $P \in \text{carrier } rdes\text{-theory.thy-order} \longleftrightarrow P \text{ is } RD$

and  $\text{carrier } rdes\text{-theory.thy-order} \rightarrow \text{carrier } rdes\text{-theory.thy-order} \equiv \llbracket RD \rrbracket_H \rightarrow \llbracket RD \rrbracket_H$

and  $le\ rdes\text{-theory.thy-order} = (\sqsubseteq)$

and  $eq\ rdes\text{-theory.thy-order} = (=)$

**proof** –

show *utp-theory-continuous RD*

by (*unfold-locales, simp-all add: RD-idem RD-Continuous*)

**qed** (*simp-all*)

**interpretation** *srdes-theory*: *utp-theory-continuous SRD*

rewrites  $P \in \text{carrier } srdes\text{-theory.thy-order} \longleftrightarrow P \text{ is } SRD$

and  $\text{carrier } srdes\text{-theory.thy-order} \rightarrow \text{carrier } srdes\text{-theory.thy-order} \equiv \llbracket SRD \rrbracket_H \rightarrow \llbracket SRD \rrbracket_H$

and  $le\ srdes\text{-theory.thy-order} = (\sqsubseteq)$

and  $eq\ srdes\text{-theory.thy-order} = (=)$

**proof** –

show *utp-theory-continuous SRD*

by (*unfold-locales, simp-all add: SRD-idem SRD-Continuous*)

**qed** (*simp-all*)

**interpretation** *rdes-rea-galois*:

*galois-connection*  $(RD \Leftarrow (RD1 \circ RD2, R3) \Rightarrow RP)$

**proof** (*simp add: mk-conn-def, rule galois-connectionI', simp-all add: utp-partial-order*)

show  $R3 \in \llbracket RD \rrbracket_H \rightarrow \llbracket RP \rrbracket_H$

by (*metis (no-types, lifting) Healthy-def' Pi-I R3-RD-RP RP-idem mem-Collect-eq*)

show  $RD1 \circ RD2 \in \llbracket RP \rrbracket_H \rightarrow \llbracket RD \rrbracket_H$

by (*simp add: Pi-iff Healthy-def, metis RD-def RD-idem*)

show *isotone* (*utp-order RD*) (*utp-order RP*) *R3*

by (*simp add: R3-Monotonic isotone-utp-orderI*)

show *isotone* (*utp-order RP*) (*utp-order RD*) (*RD1*  $\circ$  *RD2*)

by (*simp add: Monotonic-comp RD1-Monotonic RD2-Monotonic isotone-utp-orderI*)

fix  $P :: ('a, 'b) \text{ hrel-rp}$

assume  $P$  is *RD*

thus  $P \sqsubseteq RD1\ (RD2\ (R3\ P))$

by (*metis Healthy-if R3-RD-RP RD-def RP-idem eq-iff*)

**next**

fix  $P :: ('a, 'b) \text{ hrel-rp}$

assume  $a: P$  is *RP*

```

thus  $R3 (RD1 (RD2 P)) \sqsubseteq P$ 
proof –
  have  $R3 (RD1 (RD2 P)) = RP (RD1 (RD2(P)))$ 
    by (metis Healthy-if R3-RD-RP RD-def a)
  moreover have  $RD1(RD2(P)) \sqsubseteq P$ 
    by (rel-auto)
  ultimately show ?thesis
    by (metis Healthy-if RP-mono a)
qed
qed

```

```

interpretation rdes-rea-retract:
  retract ( $RD \Leftarrow \langle RD1 \circ RD2, R3 \rangle \Rightarrow RP$ )
  by (unfold-locales, simp-all add: mk-conn-def utp-partial-order)
    (metis Healthy-if R3-RD-RP RD-def RP-idem eq-refl)

```

```

abbreviation Chaos :: ('s, 't::trace, 'α) hrel-rsp where
Chaos  $\equiv$  srdes-theory.utp-bottom

```

```

abbreviation Miracle :: ('s, 't::trace, 'α) hrel-rsp where
Miracle  $\equiv$  srdes-theory.utp-top

```

```

thm srdes-theory.weak.bottom-lower
thm srdes-theory.weak.top-higher
thm srdes-theory.meet-bottom
thm srdes-theory.meet-top

```

```

abbreviation srd-lfp ( $\mu_R$ ) where  $\mu_R F \equiv$  srdes-theory.utp-lfp F

```

```

abbreviation srd-gfp ( $\nu_R$ ) where  $\nu_R F \equiv$  srdes-theory.utp-gfp F

```

```

syntax
  -srd-mu :: pttrn  $\Rightarrow$  logic  $\Rightarrow$  logic ( $\mu_R \cdot \cdot \cdot [0, 10] 10$ )
  -srd-nu :: pttrn  $\Rightarrow$  logic  $\Rightarrow$  logic ( $\nu_R \cdot \cdot \cdot [0, 10] 10$ )

```

```

translations
   $\mu_R X \cdot P == \mu_R (\lambda X. P)$ 
   $\nu_R X \cdot P == \mu_R (\lambda X. P)$ 

```

The reactive design weakest fixed-point can be defined in terms of relational calculus one.

```

lemma srd-mu-equiv:
  assumes Monotonic F  $F \in \llbracket SRD \rrbracket_H \rightarrow \llbracket SRD \rrbracket_H$ 
  shows  $(\mu_R X \cdot F(X)) = (\mu X \cdot F(SRD(X)))$ 
  by (metis assms srdes-theory.utp-lfp-def)

```

**end**

### 3 Reactive Design Specifications

```

theory utp-rdes-designs
  imports utp-rdes-healths
begin

```

### 3.1 Reactive design forms

**lemma** *srdes-skip-def*:  $II_R = \mathbf{R}_s(\text{true} \vdash (\$tr' =_u \$tr \wedge \neg \$wait' \wedge \lceil II \rceil_R))$   
**apply** (*rel-auto*) **using** *minus-zero-eq* **by** *blast+*

**lemma** *Chaos-def*:  $\text{Chaos} = \mathbf{R}_s(\text{false} \vdash \text{true})$   
**proof** –  
**have**  $\text{Chaos} = \text{SRD}(\text{true})$   
**by** (*metis srdes-theory.healthy-bottom*)  
**also have**  $\dots = \mathbf{R}_s(\mathbf{H}(\text{true}))$   
**by** (*simp add: SRD-RHS-H1-H2*)  
**also have**  $\dots = \mathbf{R}_s(\text{false} \vdash \text{true})$   
**by** (*metis H1-design H2-true design-false-pre*)  
**finally show** *?thesis* .  
**qed**

**lemma** *Miracle-def*:  $\text{Miracle} = \mathbf{R}_s(\text{true} \vdash \text{false})$   
**proof** –  
**have**  $\text{Miracle} = \text{SRD}(\text{false})$   
**by** (*metis srdes-theory.healthy-top*)  
**also have**  $\dots = \mathbf{R}_s(\mathbf{H}(\text{false}))$   
**by** (*simp add: SRD-RHS-H1-H2*)  
**also have**  $\dots = \mathbf{R}_s(\text{true} \vdash \text{false})$   
**by** (*metis (no-types, lifting) H1-H2-eq-design p-imp-p subst-impl subst-not utp-pred-laws.compl-bot-eq utp-pred-laws.compl-top-eq*)  
**finally show** *?thesis* .  
**qed**

**lemma** *RD1-reactive-design*:  $\text{RD1}(\mathbf{R}(P \vdash Q)) = \mathbf{R}(P \vdash Q)$   
**by** (*rel-auto*)

**lemma** *RD2-reactive-design*:  
**assumes**  $\$ok' \nmid P \ \$ok' \nmid Q$   
**shows**  $\text{RD2}(\mathbf{R}(P \vdash Q)) = \mathbf{R}(P \vdash Q)$   
**using** *assms*  
**by** (*metis H2-design RD2-RH-commute RD2-def*)

**lemma** *RD1-st-reactive-design*:  $\text{RD1}(\mathbf{R}_s(P \vdash Q)) = \mathbf{R}_s(P \vdash Q)$   
**by** (*rel-auto*)

**lemma** *RD2-st-reactive-design*:  
**assumes**  $\$ok' \nmid P \ \$ok' \nmid Q$   
**shows**  $\text{RD2}(\mathbf{R}_s(P \vdash Q)) = \mathbf{R}_s(P \vdash Q)$   
**using** *assms*  
**by** (*metis H2-design RD2-RHS-commute RD2-def*)

**lemma** *wait-false-design*:  
 $(P \vdash Q)_f = ((P_f) \vdash (Q_f))$   
**by** (*rel-auto*)

**lemma** *RD-RH-design-form*:  
 $\text{RD}(P) = \mathbf{R}((\neg P_f) \vdash P^t_f)$   
**proof** –  
**have**  $\text{RD}(P) = \text{RD1}(\text{RD2}(\text{R1}(\text{R2c}(\text{R3c}(P))))$   
**by** (*simp add: RD-alt-def RH-def*)  
**also have**  $\dots = \text{RD1}(\text{H2}(\text{R1}(\text{R2s}(\text{R3c}(P))))$

by (simp add: R1-R2s-R2c RD2-def)  
 also have ... =  $RD1(R1(H2(R2s(R3c(P)))))$   
 by (simp add: R1-H2-commute)  
 also have ... =  $R1(H1(R1(H2(R2s(R3c(P))))))$   
 by (simp add: R1-idem RD1-via-R1)  
 also have ... =  $R1(H1(H2(R2s(R3c(R1(P))))))$   
 by (simp add: R1-H2-commute R1-R2c-commute R1-R2s-R2c R1-R3c-commute RD1-via-R1)  
 also have ... =  $R1(R2s(H1(H2(R3c(R1(P))))))$   
 by (simp add: R2s-H1-commute R2s-H2-commute)  
 also have ... =  $R1(R2s(H1(R3c(H2(R1(P))))))$   
 by (metis RD2-R3c-commute RD2-def)  
 also have ... =  $R2(R1(H1(R3c(H2(R1(P))))))$   
 by (metis R1-R2-commute R1-idem R2-def)  
 also have ... =  $R2(R3c(R1(H(R1(P)))))$   
 by (simp add: R1-R3c-commute RD1-R3c-commute RD1-via-R1)  
 also have ... =  $RH(H(R1(P)))$   
 by (metis R1-R2s-R2c R1-R3c-commute R2-R1-form RH-def)  
 also have ... =  $RH(H(P))$   
 by (simp add: R1-H2-commute R1-R2c-commute R1-R3c-commute R1-idem RD1-via-R1 RH-def)  
 also have ... =  $RH((\neg P^f) \vdash P^t)$   
 by (simp add: H1-H2-eq-design)  
 also have ... =  $\mathbf{R}((\neg P^f_f) \vdash P^t_f)$   
 by (metis (no-types, lifting) R3c-subst-wait RH-def subst-not wait-false-design)  
 finally show ?thesis .  
 qed

**lemma** *RD-reactive-design*:

assumes  $P$  is *RD*  
 shows  $\mathbf{R}((\neg P^f_f) \vdash P^t_f) = P$   
 by (metis RD-RH-design-form Healthy-def' assms)

**lemma** *RD-RH-design*:

assumes  $\$ok' \# P \ \$ok' \# Q$   
 shows  $RD(\mathbf{R}(P \vdash Q)) = \mathbf{R}(P \vdash Q)$   
 by (simp add: RD1-reactive-design RD2-reactive-design RD-alt-def RH-idem assms(1) assms(2))

**lemma** *RH-design-is-RD*:

assumes  $\$ok' \# P \ \$ok' \# Q$   
 shows  $\mathbf{R}(P \vdash Q)$  is *RD*  
 by (simp add: RD-RH-design Healthy-def' assms(1) assms(2))

**lemma** *SRD-RH-design-form*:

$SRD(P) = \mathbf{R}_s((\neg P^f_f) \vdash P^t_f)$

**proof** –

have  $SRD(P) = R1(R2c(R3h(RD1(RD2(R1(P))))))$   
 by (metis (no-types, lifting) R1-H2-commute R1-R2c-commute R1-R3h-commute R1-idem R2c-H2-commute RD1-R1-commute RD1-R2c-commute RD1-R3h-commute RD2-R3h-commute RD2-def RHS-def SRD-def)  
 also have ... =  $R1(R2s(R3h(H(P))))$   
 by (metis (no-types, lifting) R1-H2-commute R1-R2c-is-R2 R1-R3h-commute R2-R1-form RD1-via-R1 RD2-def)  
 also have ... =  $\mathbf{R}_s(H(P))$   
 by (simp add: R1-R2s-R2c RHS-def)  
 also have ... =  $\mathbf{R}_s((\neg P^f) \vdash P^t)$   
 by (simp add: H1-H2-eq-design)  
 also have ... =  $\mathbf{R}_s((\neg P^f_f) \vdash P^t_f)$



by (metis (no-types, lifting) R3h-subst-wait RHS-def subst-not wait-false-design)  
 finally show ?thesis .  
 qed

**lemma** *SRD-reactive-design*:

assumes  $P$  is SRD  
 shows  $\mathbf{R}_s((\neg P^f_f) \vdash P^t_f) = P$   
 by (metis SRD-RH-design-form Healthy-def' assms)

**lemma** *SRD-RH-design*:

assumes  $\$ok' \# P \ \$ok' \# Q$   
 shows  $SRD(\mathbf{R}_s(P \vdash Q)) = \mathbf{R}_s(P \vdash Q)$   
 by (simp add: RD1-st-reactive-design RD2-st-reactive-design RHS-idem SRD-def assms(1) assms(2))

**lemma** *RHS-design-is-SRD*:

assumes  $\$ok' \# P \ \$ok' \# Q$   
 shows  $\mathbf{R}_s(P \vdash Q)$  is SRD  
 by (simp add: Healthy-def' SRD-RH-design assms(1) assms(2))

**lemma** *SRD-RHS-H1-H2*:  $SRD(P) = \mathbf{R}_s(\mathbf{H}(P))$

by (metis (no-types, lifting) H1-H2-eq-design R3h-subst-wait RHS-def SRD-RH-design-form subst-not wait-false-design)

### 3.2 Auxiliary healthiness conditions

**definition** [*upred-defs*]:  $R3c\text{-}pre(P) = (true \triangleleft \$wait \triangleright P)$

**definition** [*upred-defs*]:  $R3c\text{-}post(P) = ([II]_D \triangleleft \$wait \triangleright P)$

**definition** [*upred-defs*]:  $R3h\text{-}post(P) = ((\exists \$st \cdot [II]_D) \triangleleft \$wait \triangleright P)$

**lemma** *R3c-pre-conj*:  $R3c\text{-}pre(P \wedge Q) = (R3c\text{-}pre(P) \wedge R3c\text{-}pre(Q))$   
 by (rel-auto)

**lemma** *R3c-pre-seq*:

$(true ;; Q) = true \implies R3c\text{-}pre(P ;; Q) = (R3c\text{-}pre(P) ;; Q)$   
 by (rel-auto)

**lemma** *unrest-ok-R3c-pre* [*unrest*]:  $\$ok \# P \implies \$ok \# R3c\text{-}pre(P)$   
 by (simp add: R3c-pre-def cond-def unrest)

**lemma** *unrest-ok'-R3c-pre* [*unrest*]:  $\$ok' \# P \implies \$ok' \# R3c\text{-}pre(P)$   
 by (simp add: R3c-pre-def cond-def unrest)

**lemma** *unrest-ok-R3c-post* [*unrest*]:  $\$ok \# P \implies \$ok \# R3c\text{-}post(P)$   
 by (simp add: R3c-post-def cond-def unrest)

**lemma** *unrest-ok-R3c-post'* [*unrest*]:  $\$ok' \# P \implies \$ok' \# R3c\text{-}post(P)$   
 by (simp add: R3c-post-def cond-def unrest)

**lemma** *unrest-ok-R3h-post* [*unrest*]:  $\$ok \# P \implies \$ok \# R3h\text{-}post(P)$   
 by (simp add: R3h-post-def cond-def unrest)

**lemma** *unrest-ok-R3h-post'* [*unrest*]:  $\$ok' \# P \implies \$ok' \# R3h\text{-}post(P)$   
 by (simp add: R3h-post-def cond-def unrest)

### 3.3 Composition laws

**theorem** *R1-design-composition:*

**fixes**  $P\ Q :: ('t::trace, 'α, 'β)\ rel\text{-}rp$

**and**  $R\ S :: ('t, 'β, 'γ)\ rel\text{-}rp$

**assumes**  $\$ok' \# P\ \$ok' \# Q\ \$ok \# R\ \$ok \# S$

**shows**

$(R1(P \vdash Q) ;; R1(R \vdash S)) =$

$R1((\neg (R1(\neg P) ;; R1(true)) \wedge \neg (R1(Q) ;; R1(\neg R))) \vdash (R1(Q) ;; R1(S)))$

**proof** –

**have**  $(R1(P \vdash Q) ;; R1(R \vdash S)) = ((R1(P \vdash Q))^t ;; R1(R \vdash S)[[true/\$ok] \vee (R1(P \vdash Q))^f ;; R1(R \vdash S)[[false/\$ok]])$

**by** (*rule seqr-bool-split[of ok], simp*)

**also from** *assms* **have**  $\dots = ((R1((\$ok \wedge P) \Rightarrow (true \wedge Q)) ;; R1((true \wedge R) \Rightarrow (\$ok' \wedge S))) \vee (R1((\$ok \wedge P) \Rightarrow (false \wedge Q)) ;; R1((false \wedge R) \Rightarrow (\$ok' \wedge S))))$

**by** (*simp add: design-def usubst R1-def*)

**also from** *assms* **have**  $\dots = ((R1((\$ok \wedge P) \Rightarrow Q) ;; R1(R \Rightarrow (\$ok' \wedge S))) \vee (R1(\neg (\$ok \wedge P)) ;; R1(true)))$

**by** *simp*

**also from** *assms* **have**  $\dots = ((R1(\neg \$ok \vee \neg P \vee Q) ;; R1(\neg R \vee (\$ok' \wedge S))) \vee (R1(\neg \$ok \vee \neg P) ;; R1(true)))$

**by** (*simp add: impl-alt-def utp-pred-laws.sup.assoc*)

**also from** *assms* **have**  $\dots = (((R1(\neg \$ok \vee \neg P) \vee R1(Q)) ;; R1(\neg R \vee (\$ok' \wedge S))) \vee (R1(\neg \$ok \vee \neg P) ;; R1(true)))$

**by** (*simp add: R1-disj utp-pred-laws.disj-assoc*)

**also from** *assms* **have**  $\dots = ((R1(\neg \$ok \vee \neg P) ;; R1(\neg R \vee (\$ok' \wedge S))) \vee (R1(Q) ;; R1(\neg R \vee (\$ok' \wedge S))) \vee (R1(\neg \$ok \vee \neg P) ;; R1(true)))$

**by** (*simp add: seqr-or-distl utp-pred-laws.sup.assoc*)

**also from** *assms* **have**  $\dots = ((R1(Q) ;; R1(\neg R \vee (\$ok' \wedge S))) \vee (R1(\neg \$ok \vee \neg P) ;; R1(true)))$

**by** (*rel-blast*)

**also from** *assms* **have**  $\dots = ((R1(Q) ;; (R1(\neg R) \vee R1(S) \wedge \$ok')) \vee (R1(\neg \$ok \vee \neg P) ;; R1(true)))$

**by** (*simp add: R1-disj R1-extend-conj utp-pred-laws.inf-commute*)

**also have**  $\dots = ((R1(Q) ;; (R1(\neg R) \vee R1(S) \wedge \$ok')) \vee ((R1(\neg \$ok) :: ('t, 'α, 'β)\ rel\text{-}rp) ;; R1(true)) \vee (R1(\neg P) ;; R1(true)))$

**by** (*simp add: R1-disj seqr-or-distl*)

**also have**  $\dots = ((R1(Q) ;; (R1(\neg R) \vee R1(S) \wedge \$ok')) \vee (R1(\neg \$ok) \vee (R1(\neg P) ;; R1(true))))$

**proof** –

**have**  $((R1(\neg \$ok) :: ('t, 'α, 'β)\ rel\text{-}rp) ;; R1(true)) = (R1(\neg \$ok) :: ('t, 'α, 'γ)\ rel\text{-}rp)$

**by** (*rel-auto*)

**thus** *?thesis*

**by** *simp*

**qed**

**also have**  $\dots = ((R1(Q) ;; (R1(\neg R) \vee (R1(S \wedge \$ok')))) \vee R1(\neg \$ok) \vee (R1(\neg P) ;; R1(true)))$

**by** (*simp add: R1-extend-conj*)

**also have**  $\dots = ((R1(Q) ;; (R1(\neg R))) \vee (R1(Q) ;; (R1(S \wedge \$ok')))) \vee R1(\neg \$ok)$

$\vee (R1(\neg P) ;; R1(true))$   
 by (simp add: segr-or-distr utp-pred-laws.sup.assoc)  
 also have ... =  $R1( (R1(Q) ;; (R1(\neg R)))$   
 $\vee (R1(Q) ;; (R1(S \wedge \$ok'))$   
 $\vee (\neg \$ok)$   
 $\vee (R1(\neg P) ;; R1(true))$   
 by (simp add: R1-disj R1-segr)  
 also have ... =  $R1( (R1(Q) ;; (R1(\neg R)))$   
 $\vee ((R1(Q) ;; R1(S)) \wedge \$ok')$   
 $\vee (\neg \$ok)$   
 $\vee (R1(\neg P) ;; R1(true))$   
 by (rel-blast)  
 also have ... =  $R1(\neg(\$ok \wedge \neg (R1(\neg P) ;; R1(true)) \wedge \neg (R1(Q) ;; (R1(\neg R))))$   
 $\vee ((R1(Q) ;; R1(S)) \wedge \$ok')$   
 by (rel-blast)  
 also have ... =  $R1((\$ok \wedge \neg (R1(\neg P) ;; R1(true)) \wedge \neg (R1(Q) ;; (R1(\neg R))))$   
 $\Rightarrow (\$ok' \wedge (R1(Q) ;; R1(S))))$   
 by (simp add: impl-alt-def utp-pred-laws.inf-commute)  
 also have ... =  $R1((\neg (R1(\neg P) ;; R1(true)) \wedge \neg (R1(Q) ;; R1(\neg R))) \vdash (R1(Q) ;; R1(S)))$   
 by (simp add: design-def)  
 finally show ?thesis .  
 qed

**theorem** *R1-design-composition-RR:*

assumes *P is RR Q is RR R is RR S is RR*

shows

$(R1(P \vdash Q) ;; R1(R \vdash S)) = R1(((\neg_r P) \text{ wp}_r \text{ false} \wedge Q \text{ wp}_r R) \vdash (Q ;; S))$

apply (subst R1-design-composition)

apply (simp-all add: assms unrest wp-rea-def Healthy-if closure)

apply (rel-auto)

done

**theorem** *R1-design-composition-RC:*

assumes *P is RC Q is RR R is RR S is RR*

shows

$(R1(P \vdash Q) ;; R1(R \vdash S)) = R1((P \wedge Q \text{ wp}_r R) \vdash (Q ;; S))$

by (simp add: R1-design-composition-RR assms unrest Healthy-if closure wp)

**lemma** *R2s-design:*  $R2s(P \vdash Q) = (R2s(P) \vdash R2s(Q))$

by (simp add: R2s-def design-def usubst)

**lemma** *R2c-design:*  $R2c(P \vdash Q) = (R2c(P) \vdash R2c(Q))$

by (simp add: design-def impl-alt-def R2c-disj R2c-not R2c-ok R2c-and R2c-ok')

**lemma** *R1-R3c-design:*

$R1(R3c(P \vdash Q)) = R1(R3c\text{-pre}(P) \vdash R3c\text{-post}(Q))$

by (rel-auto)

**lemma** *R1-R3h-design:*

$R1(R3h(P \vdash Q)) = R1(R3c\text{-pre}(P) \vdash R3h\text{-post}(Q))$

by (rel-auto)

**lemma** *R3c-R1-design-composition:*

assumes  $\$ok' \# P \ \$ok' \# Q \ \$ok \# R \ \$ok \# S$

shows  $(R3c(R1(P \vdash Q)) ;; R3c(R1(R \vdash S))) =$

$$R3c(R1((\neg (R1(\neg P) ;; R1(true)) \wedge \neg ((R1(Q) \wedge \neg \$wait') ;; R1(\neg R)))$$

$$\vdash (R1(Q) ;; (\lceil II \rceil_D \triangleleft \$wait \triangleright R1(S))))$$

**proof** –

**have** 1:  $(\neg (R1(\neg R3c\text{-}pre\ P) ;; R1\ true)) = (R3c\text{-}pre(\neg (R1(\neg P) ;; R1\ true)))$   
**by** (rel-auto)  
**have** 2:  $(\neg (R1(R3c\text{-}post\ Q) ;; R1(\neg R3c\text{-}pre\ R))) = R3c\text{-}pre(\neg ((R1\ Q \wedge \neg \$wait') ;; R1(\neg R)))$   
**by** (rel-auto, blast+)  
**have** 3:  $(R1(R3c\text{-}post\ Q) ;; R1(R3c\text{-}post\ S)) = R3c\text{-}post(R1\ Q ;; (\lceil II \rceil_D \triangleleft \$wait \triangleright R1\ S))$   
**by** (rel-auto)  
**show** ?thesis  
**apply** (simp add: R3c-semir-form R1-R3c-commute[THEN sym] R1-R3c-design unrest )  
**apply** (subst R1-design-composition)  
**apply** (simp-all add: unrest assms R3c-pre-conj 1 2 3)  
**done**  
**qed**

**lemma** R3h-R1-design-composition:

**assumes**  $\$ok' \# P \ \$ok' \# Q \ \$ok \# R \ \$ok \# S$   
**shows**  $(R3h(R1(P \vdash Q)) ;; R3h(R1(R \vdash S))) =$   
 $R3h(R1((\neg (R1(\neg P) ;; R1(true)) \wedge \neg ((R1(Q) \wedge \neg \$wait') ;; R1(\neg R)))$   
 $\vdash (R1(Q) ;; ((\exists \$st \cdot \lceil II \rceil_D) \triangleleft \$wait \triangleright R1(S))))$

**proof** –

**have** 1:  $(\neg (R1(\neg R3c\text{-}pre\ P) ;; R1\ true)) = (R3c\text{-}pre(\neg (R1(\neg P) ;; R1\ true)))$   
**by** (rel-auto)  
**have** 2:  $(\neg (R1(R3h\text{-}post\ Q) ;; R1(\neg R3c\text{-}pre\ R))) = R3c\text{-}pre(\neg ((R1\ Q \wedge \neg \$wait') ;; R1(\neg R)))$   
**by** (rel-auto, blast+)  
**have** 3:  $(R1(R3h\text{-}post\ Q) ;; R1(R3h\text{-}post\ S)) = R3h\text{-}post(R1\ Q ;; ((\exists \$st \cdot \lceil II \rceil_D) \triangleleft \$wait \triangleright R1\ S))$   
**by** (rel-auto, blast+)  
**show** ?thesis  
**apply** (simp add: R3h-semir-form R1-R3h-commute[THEN sym] R1-R3h-design unrest )  
**apply** (subst R1-design-composition)  
**apply** (simp-all add: unrest assms R3c-pre-conj 1 2 3)  
**done**  
**qed**

**lemma** R2-design-composition:

**assumes**  $\$ok' \# P \ \$ok' \# Q \ \$ok \# R \ \$ok \# S$   
**shows**  $(R2(P \vdash Q) ;; R2(R \vdash S)) =$   
 $R2((\neg (R1(\neg R2c\text{-}P) ;; R1\ true) \wedge \neg (R1(R2c\text{-}Q) ;; R1(\neg R2c\text{-}R))) \vdash (R1(R2c\text{-}Q) ;; R1$   
 $(R2c\text{-}S)))$   
**apply** (simp add: R2-R2c-def R2c-design R1-design-composition assms unrest R2c-not R2c-and R2c-disj  
R1-R2c-commute[THEN sym] R2c-idem R2c-R1-seq)  
**apply** (metis (no-types, lifting) R2c-R1-seq R2c-not R2c-true)  
**done**

**lemma** RH-design-composition:

**assumes**  $\$ok' \# P \ \$ok' \# Q \ \$ok \# R \ \$ok \# S$   
**shows**  $(RH(P \vdash Q) ;; RH(R \vdash S)) =$   
 $RH((\neg (R1(\neg R2s\text{-}P) ;; R1\ true) \wedge \neg ((R1(R2s\text{-}Q) \wedge (\neg \$wait')) ;; R1(\neg R2s\text{-}R))) \vdash$   
 $(R1(R2s\text{-}Q) ;; (\lceil II \rceil_D \triangleleft \$wait \triangleright R1(R2s\text{-}S))))$

**proof** –

**have** 1:  $R2c(R1(\neg R2s\text{-}P) ;; R1\ true) = (R1(\neg R2s\text{-}P) ;; R1\ true)$   
**proof** –  
**have** 1:  $(R1(\neg R2s\text{-}P) ;; R1\ true) = (R1(R2(\neg P) ;; R2\ true))$   
**by** (rel-auto)

```

have R2c(R1(R2 (¬ P) ;; R2 true)) = R2c(R1(R2 (¬ P) ;; R2 true))
  using R2c-not by blast
also have ... = R2(R2 (¬ P) ;; R2 true)
  by (metis R1-R2c-commute R1-R2c-is-R2)
also have ... = (R2 (¬ P) ;; R2 true)
  by (simp add: R2-seqr-distribute)
also have ... = (R1 (¬ R2s P) ;; R1 true)
  by (simp add: R2-def R2s-not R2s-true)
finally show ?thesis
  by (simp add: 1)
qed

have 2:R2c ((R1 (R2s Q) ∧ ¬ $wait') ;; R1 (¬ R2s R)) = ((R1 (R2s Q) ∧ ¬ $wait') ;; R1 (¬ R2s
R))
proof -
  have ((R1 (R2s Q) ∧ ¬ $wait') ;; R1 (¬ R2s R)) = R1 (R2 (Q ∧ ¬ $wait') ;; R2 (¬ R))
    by (rel-auto)
  hence R2c ((R1 (R2s Q) ∧ ¬ $wait') ;; R1 (¬ R2s R)) = (R2 (Q ∧ ¬ $wait') ;; R2 (¬ R))
    by (metis R1-R2c-commute R1-R2c-is-R2 R2-seqr-distribute)
  also have ... = ((R1 (R2s Q) ∧ ¬ $wait') ;; R1 (¬ R2s R))
    by (rel-auto)
  finally show ?thesis .
qed

have 3:R2c((R1 (R2s Q) ;; (⌈II⌋D ◁ $wait ▷ R1 (R2s S)))) = (R1 (R2s Q) ;; (⌈II⌋D ◁ $wait ▷ R1
(R2s S)))
proof -
  have R2c(((R1 (R2s Q))⌈true/$wait'⌋ ;; (⌈II⌋D ◁ $wait ▷ R1 (R2s S))⌈true/$wait⌋))
    = ((R1 (R2s Q))⌈true/$wait'⌋ ;; (⌈II⌋D ◁ $wait ▷ R1 (R2s S))⌈true/$wait⌋)
  proof -
    have R2c(((R1 (R2s Q))⌈true/$wait'⌋ ;; (⌈II⌋D ◁ $wait ▷ R1 (R2s S))⌈true/$wait⌋)) =
      R2c(R1 (R2s (Q⌈true/$wait'⌋)) ;; ⌈II⌋D⌈true/$wait⌋)
      by (simp add: usubst cond-unit-T R1-def R2s-def)
    also have ... = R2c(R2(Q⌈true/$wait'⌋) ;; R2(⌈II⌋D⌈true/$wait⌋))
      by (metis R2-def R2-des-lift-skip R2-subst-wait-true)
    also have ... = (R2(Q⌈true/$wait'⌋) ;; R2(⌈II⌋D⌈true/$wait⌋))
      using R2c-seq by blast
    also have ... = ((R1 (R2s Q))⌈true/$wait'⌋ ;; (⌈II⌋D ◁ $wait ▷ R1 (R2s S))⌈true/$wait⌋)
      apply (simp add: usubst R2-des-lift-skip)
      apply (metis R2-def R2-des-lift-skip R2-subst-wait'-true R2-subst-wait-true)
      done
    finally show ?thesis .
  qed
  moreover have R2c(((R1 (R2s Q))⌈false/$wait'⌋ ;; (⌈II⌋D ◁ $wait ▷ R1 (R2s S))⌈false/$wait⌋))
    = ((R1 (R2s Q))⌈false/$wait'⌋ ;; (⌈II⌋D ◁ $wait ▷ R1 (R2s S))⌈false/$wait⌋)
    by (simp add: usubst cond-unit-F)
    (metis (no-types, hide-lams) R1-wait'-false R1-wait-false R2-def R2-subst-wait'-false R2-subst-wait-false
R2c-seq)
  ultimately show ?thesis
  proof -
    have ⌈II⌋D ◁ $wait ▷ R1 (R2s S) = R2 (⌈II⌋D ◁ $wait ▷ S)
      by (simp add: R1-R2c-is-R2 R1-R2s-R2c R2-condr' R2-des-lift-skip R2s-wait)
    then show ?thesis
      by (simp add: R1-R2c-is-R2 R1-R2s-R2c R2c-seq)
  qed

```

qed

have  $(R1(R2s(R3c(P \vdash Q))) \;; R1(R2s(R3c(R \vdash S)))) =$   
 $((R3c(R1(R2s(P) \vdash R2s(Q)))) \;; R3c(R1(R2s(R) \vdash R2s(S))))$   
 by *(metis (no-types, hide-lams) R1-R2s-R2c R1-R3c-commute R2c-R3c-commute R2s-design)*  
 also have  $\dots = R3c(R1((\neg(R1(\neg R2s P) \;; R1 true) \wedge \neg((R1(R2s Q) \wedge \neg \$wait') \;; R1(\neg R2s R)))) \vdash$   
 $(R1(R2s Q) \;; ([II]_D \triangleleft \$wait \triangleright R1(R2s S))))$   
 by *(simp add: R3c-R1-design-composition assms unrest)*  
 also have  $\dots = R3c(R1(R2c((\neg(R1(\neg R2s P) \;; R1 true) \wedge \neg((R1(R2s Q) \wedge \neg \$wait') \;; R1(\neg R2s R)))) \vdash$   
 $(R1(R2s Q) \;; ([II]_D \triangleleft \$wait \triangleright R1(R2s S))))$   
 by *(simp add: R2c-design R2c-and R2c-not 1 2 3)*  
 finally show *?thesis*  
 by *(simp add: R1-R2s-R2c R1-R3c-commute R2c-R3c-commute RH-def)*  
 qed

lemma *RHS-design-composition*:

assumes  $\$ok' \# P \ \$ok' \# Q \ \$ok \# R \ \$ok \# S$

shows  $(\mathbf{R}_s(P \vdash Q) \;; \mathbf{R}_s(R \vdash S)) =$

$\mathbf{R}_s((\neg(R1(\neg R2s P) \;; R1 true) \wedge \neg((R1(R2s Q) \wedge \neg \$wait') \;; R1(\neg R2s R)))) \vdash$   
 $(R1(R2s Q) \;; ((\exists \$st \cdot [II]_D \triangleleft \$wait \triangleright R1(R2s S))))$

proof –

have 1:  $R2c(R1(\neg R2s P) \;; R1 true) = (R1(\neg R2s P) \;; R1 true)$

proof –

have 1:  $(R1(\neg R2s P) \;; R1 true) = (R1(R2(\neg P) \;; R2 true))$

by *(rel-auto, blast)*

have  $R2c(R1(R2(\neg P) \;; R2 true)) = R2c(R1(R2(\neg P) \;; R2 true))$

using *R2c-not* by *blast*

also have  $\dots = R2(R2(\neg P) \;; R2 true)$

by *(metis R1-R2c-commute R1-R2c-is-R2)*

also have  $\dots = (R2(\neg P) \;; R2 true)$

by *(simp add: R2-seqr-distribute)*

also have  $\dots = (R1(\neg R2s P) \;; R1 true)$

by *(simp add: R2-def R2s-not R2s-true)*

finally show *?thesis*

by *(simp add: 1)*

qed

have 2:  $R2c((R1(R2s Q) \wedge \neg \$wait') \;; R1(\neg R2s R)) = ((R1(R2s Q) \wedge \neg \$wait') \;; R1(\neg R2s R))$

proof –

have  $((R1(R2s Q) \wedge \neg \$wait') \;; R1(\neg R2s R)) = R1(R2(Q \wedge \neg \$wait') \;; R2(\neg R))$

by *(rel-auto, blast+)*

hence  $R2c((R1(R2s Q) \wedge \neg \$wait') \;; R1(\neg R2s R)) = (R2(Q \wedge \neg \$wait') \;; R2(\neg R))$

by *(metis (no-types, lifting) R1-R2c-commute R1-R2c-is-R2 R2-seqr-distribute)*

also have  $\dots = ((R1(R2s Q) \wedge \neg \$wait') \;; R1(\neg R2s R))$

by *(rel-auto, blast+)*

finally show *?thesis* .

qed

have 3:  $R2c((R1(R2s Q) \;; ((\exists \$st \cdot [II]_D \triangleleft \$wait \triangleright R1(R2s S)))) =$   
 $(R1(R2s Q) \;; ((\exists \$st \cdot [II]_D \triangleleft \$wait \triangleright R1(R2s S))))$

proof –

have  $R2c(((R1(R2s Q))\llbracket true/\$wait' \rrbracket \;; ((\exists \$st \cdot [II]_D \triangleleft \$wait \triangleright R1(R2s S))\llbracket true/\$wait' \rrbracket))$

$$= ((R1 (R2s Q))\llbracket true/\$wait' \rrbracket ;; ((\exists \$st \cdot \lceil II \rceil_D) \triangleleft \$wait \triangleright R1 (R2s S))\llbracket true/\$wait \rrbracket)$$

**proof** –

**have**  $R2c(((R1 (R2s Q))\llbracket true/\$wait' \rrbracket ;; ((\exists \$st \cdot \lceil II \rceil_D) \triangleleft \$wait \triangleright R1 (R2s S))\llbracket true/\$wait \rrbracket)) =$   
 $R2c(R1 (R2s (Q\llbracket true/\$wait' \rrbracket)) ;; (\exists \$st \cdot \lceil II \rceil_D)\llbracket true/\$wait \rrbracket)$   
**by** (*simp add: usubst cond-unit-T R1-def R2s-def*)  
**also have**  $\dots = R2c(R2(Q\llbracket true/\$wait' \rrbracket) ;; R2((\exists \$st \cdot \lceil II \rceil_D)\llbracket true/\$wait \rrbracket))$   
**by** (*metis (no-types, lifting) R2-def R2-des-lift-skip R2-subst-wait-true R2-st-ex*)  
**also have**  $\dots = (R2(Q\llbracket true/\$wait' \rrbracket) ;; R2((\exists \$st \cdot \lceil II \rceil_D)\llbracket true/\$wait \rrbracket))$   
**using** *R2c-seq* **by** *blast*  
**also have**  $\dots = ((R1 (R2s Q))\llbracket true/\$wait' \rrbracket ;; ((\exists \$st \cdot \lceil II \rceil_D) \triangleleft \$wait \triangleright R1 (R2s S))\llbracket true/\$wait \rrbracket)$   
**apply** (*simp add: usubst R2-des-lift-skip*)  
**apply** (*metis (no-types) R2-def R2-des-lift-skip R2-st-ex R2-subst-wait'-true R2-subst-wait-true*)  
**done**  
**finally show** *?thesis* .

**qed**

**moreover have**  $R2c(((R1 (R2s Q))\llbracket false/\$wait' \rrbracket ;; ((\exists \$st \cdot \lceil II \rceil_D) \triangleleft \$wait \triangleright R1 (R2s S))\llbracket false/\$wait \rrbracket)) =$   
 $((R1 (R2s Q))\llbracket false/\$wait' \rrbracket ;; ((\exists \$st \cdot \lceil II \rceil_D) \triangleleft \$wait \triangleright R1 (R2s S))\llbracket false/\$wait \rrbracket)$   
**by** (*simp add: usubst*)  
*(metis (no-types, lifting) R1-wait'-false R1-wait-false R2-R1-form R2-subst-wait'-false R2-subst-wait-false*  
*R2c-seq)*  
**ultimately show** *?thesis*  
**by** (*smt R2-R1-form R2-condr' R2-des-lift-skip R2-st-ex R2c-seq R2s-wait*)

**qed**

**have**  $(R1(R2s(R3h(P \vdash Q))) ;; R1(R2s(R3h(R \vdash S)))) =$   
 $((R3h(R1(R2s(P) \vdash R2s(Q)))) ;; R3h(R1(R2s(R) \vdash R2s(S))))$   
**by** (*metis (no-types, hide-lams) R1-R2s-R2c R1-R3h-commute R2c-R3h-commute R2s-design*)  
**also have**  $\dots = R3h(R1((\neg (R1(\neg R2s P) ;; R1 true) \wedge \neg ((R1(R2s Q) \wedge \neg \$wait') ;; R1(\neg$   
 $R2s R))) \vdash$   
 $(R1(R2s Q) ;; ((\exists \$st \cdot \lceil II \rceil_D) \triangleleft \$wait \triangleright R1(R2s S))))$   
**by** (*simp add: R3h-R1-design-composition assms unrest*)  
**also have**  $\dots = R3h(R1(R2c((\neg (R1(\neg R2s P) ;; R1 true) \wedge \neg ((R1(R2s Q) \wedge \neg \$wait') ;; R1(\neg$   
 $R2s R))) \vdash$   
 $(R1(R2s Q) ;; ((\exists \$st \cdot \lceil II \rceil_D) \triangleleft \$wait \triangleright R1(R2s S))))$   
**by** (*simp add: R2c-design R2c-and R2c-not 1 2 3*)  
**finally show** *?thesis*  
**by** (*simp add: R1-R2s-R2c R1-R3h-commute R2c-R3h-commute RHS-def*)

**qed**

**lemma** *RHS-R2s-design-composition:*

**assumes**

$\$ok' \# P \ \$ok' \# Q \ \$ok \# R \ \$ok \# S$   
 $P \text{ is } R2s \ Q \text{ is } R2s \ R \text{ is } R2s \ S \text{ is } R2s$

**shows**  $(\mathbf{R}_s(P \vdash Q) ;; \mathbf{R}_s(R \vdash S)) =$   
 $\mathbf{R}_s((\neg (R1(\neg P) ;; R1 true) \wedge \neg ((R1 Q \wedge \neg \$wait') ;; R1(\neg R))) \vdash$   
 $(R1 Q ;; ((\exists \$st \cdot \lceil II \rceil_D) \triangleleft \$wait \triangleright R1 S)))$

**proof** –

**have**  $f1: R2s P = P$   
**by** (*meson Healthy-def assms(5)*)  
**have**  $f2: R2s Q = Q$   
**by** (*meson Healthy-def assms(6)*)  
**have**  $f3: R2s R = R$   
**by** (*meson Healthy-def assms(7)*)  
**have**  $R2s S = S$   
**by** (*meson Healthy-def assms(8)*)

**then show** *?thesis*  
**using** *f3 f2 f1* **by** (*simp add: RHS-design-composition assms(1) assms(2) assms(3) assms(4)*)  
**qed**

**lemma** *RH-design-export-R1*:  $\mathbf{R}(P \vdash Q) = \mathbf{R}(P \vdash R1(Q))$   
**by** (*rel-auto*)

**lemma** *RH-design-export-R2s*:  $\mathbf{R}(P \vdash Q) = \mathbf{R}(P \vdash R2s(Q))$   
**by** (*rel-auto*)

**lemma** *RH-design-export-R2c*:  $\mathbf{R}(P \vdash Q) = \mathbf{R}(P \vdash R2c(Q))$   
**by** (*rel-auto*)

**lemma** *RHS-design-export-R1*:  $\mathbf{R}_s(P \vdash Q) = \mathbf{R}_s(P \vdash R1(Q))$   
**by** (*rel-auto*)

**lemma** *RHS-design-export-R2s*:  $\mathbf{R}_s(P \vdash Q) = \mathbf{R}_s(P \vdash R2s(Q))$   
**by** (*rel-auto*)

**lemma** *RHS-design-export-R2c*:  $\mathbf{R}_s(P \vdash Q) = \mathbf{R}_s(P \vdash R2c(Q))$   
**by** (*rel-auto*)

**lemma** *RHS-design-export-R2*:  $\mathbf{R}_s(P \vdash Q) = \mathbf{R}_s(P \vdash R2(Q))$   
**by** (*rel-auto*)

**lemma** *R1-design-R1-pre*:  
 $\mathbf{R}_s(R1(P) \vdash Q) = \mathbf{R}_s(P \vdash Q)$   
**by** (*rel-auto*)

**lemma** *RHS-design-ok-wait*:  $\mathbf{R}_s(P \llbracket true, false / \$ok, \$wait \rrbracket \vdash Q \llbracket true, false / \$ok, \$wait \rrbracket) = \mathbf{R}_s(P \vdash Q)$   
**by** (*rel-auto*)

**lemma** *RHS-design-neg-R1-pre*:  
 $\mathbf{R}_s((\neg R1(P)) \vdash R) = \mathbf{R}_s((\neg P) \vdash R)$   
**by** (*rel-auto*)

**lemma** *RHS-design-conj-neg-R1-pre*:  
 $\mathbf{R}_s(((\neg R1(P)) \wedge Q) \vdash R) = \mathbf{R}_s(((\neg P) \wedge Q) \vdash R)$   
**by** (*rel-auto*)

**lemma** *RHS-pre-lemma*:  $(\mathbf{R}_s P)^f_f = R1(R2c(P^f_f))$   
**by** (*rel-auto*)

**lemma** *RHS-design-R2c-pre*:  
 $\mathbf{R}_s(R2c(P) \vdash Q) = \mathbf{R}_s(P \vdash Q)$   
**by** (*rel-auto*)

### 3.4 Refinement introduction laws

**lemma** *R1-design-refine*:  
**assumes**  
 $P_1 \text{ is } R1 \ P_2 \text{ is } R1 \ Q_1 \text{ is } R1 \ Q_2 \text{ is } R1$   
 $\$ok \ \# \ P_1 \ \$ok' \ \# \ P_1 \ \$ok \ \# \ P_2 \ \$ok' \ \# \ P_2$   
 $\$ok \ \# \ Q_1 \ \$ok' \ \# \ Q_1 \ \$ok \ \# \ Q_2 \ \$ok' \ \# \ Q_2$   
**shows**  $R1(P_1 \vdash P_2) \sqsubseteq R1(Q_1 \vdash Q_2) \longleftrightarrow 'P_1 \Rightarrow Q_1' \wedge 'P_1 \wedge Q_2 \Rightarrow P_2'$   
**proof** —



**have**  $R1((\exists \$ok; \$ok' \cdot P_1) \vdash (\exists \$ok; \$ok' \cdot P_2)) \sqsubseteq R1((\exists \$ok; \$ok' \cdot Q_1) \vdash (\exists \$ok; \$ok' \cdot Q_2))$   
 $\longleftrightarrow 'R1(\exists \$ok; \$ok' \cdot P_1) \Rightarrow R1(\exists \$ok; \$ok' \cdot Q_1)' \wedge 'R1(\exists \$ok; \$ok' \cdot P_1) \wedge R1(\exists \$ok; \$ok' \cdot Q_2) \Rightarrow R1(\exists \$ok; \$ok' \cdot P_2)'$   
**by** (*rel-auto, meson+*)  
**thus** *?thesis*  
**by** (*simp-all add: ex-unrest ex-plus Healthy-if assms*)  
**qed**

**lemma** *R1-design-refine-RR:*

**assumes**  $P_1$  is RR  $P_2$  is RR  $Q_1$  is RR  $Q_2$  is RR  
**shows**  $R1(P_1 \vdash P_2) \sqsubseteq R1(Q_1 \vdash Q_2) \longleftrightarrow 'P_1 \Rightarrow Q_1' \wedge 'P_1 \wedge Q_2 \Rightarrow P_2'$   
**by** (*simp add: R1-design-refine assms unrest closure*)

**lemma** *RHS-design-refine:*

**assumes**  
 $P_1$  is R1  $P_2$  is R1  $Q_1$  is R1  $Q_2$  is R1  
 $P_1$  is R2c  $P_2$  is R2c  $Q_1$  is R2c  $Q_2$  is R2c  
 $\$ok \# P_1 \$ok' \# P_1 \$ok \# P_2 \$ok' \# P_2$   
 $\$ok \# Q_1 \$ok' \# Q_1 \$ok \# Q_2 \$ok' \# Q_2$   
 $\$wait \# P_1 \$wait \# P_2 \$wait \# Q_1 \$wait \# Q_2$   
**shows**  $\mathbf{R}_s(P_1 \vdash P_2) \sqsubseteq \mathbf{R}_s(Q_1 \vdash Q_2) \longleftrightarrow 'P_1 \Rightarrow Q_1' \wedge 'P_1 \wedge Q_2 \Rightarrow P_2'$

**proof** –

**have**  $\mathbf{R}_s(P_1 \vdash P_2) \sqsubseteq \mathbf{R}_s(Q_1 \vdash Q_2) \longleftrightarrow R1(R3h(R2c(P_1 \vdash P_2))) \sqsubseteq R1(R3h(R2c(Q_1 \vdash Q_2)))$   
**by** (*simp add: R2c-R3h-commute RHS-def*)  
**also have**  $\dots \longleftrightarrow R1(R3h(P_1 \vdash P_2)) \sqsubseteq R1(R3h(Q_1 \vdash Q_2))$   
**by** (*simp add: Healthy-if R2c-design assms*)  
**also have**  $\dots \longleftrightarrow R1(R3h(P_1 \vdash P_2)) \llbracket false/\$wait \rrbracket \sqsubseteq R1(R3h(Q_1 \vdash Q_2)) \llbracket false/\$wait \rrbracket$   
**by** (*rel-auto, metis+*)  
**also have**  $\dots \longleftrightarrow R1(P_1 \vdash P_2) \llbracket false/\$wait \rrbracket \sqsubseteq R1(Q_1 \vdash Q_2) \llbracket false/\$wait \rrbracket$   
**by** (*rel-auto*)  
**also have**  $\dots \longleftrightarrow R1(P_1 \vdash P_2) \sqsubseteq R1(Q_1 \vdash Q_2)$   
**by** (*simp add: usubst assms closure unrest*)  
**also have**  $\dots \longleftrightarrow 'P_1 \Rightarrow Q_1' \wedge 'P_1 \wedge Q_2 \Rightarrow P_2'$   
**by** (*simp add: R1-design-refine assms*)  
**finally show** *?thesis* .

**qed**

**lemma** *srdes-refine-intro:*

**assumes**  $'P_1 \Rightarrow P_2' \wedge 'P_1 \wedge Q_2 \Rightarrow Q_1'$   
**shows**  $\mathbf{R}_s(P_1 \vdash Q_1) \sqsubseteq \mathbf{R}_s(P_2 \vdash Q_2)$   
**by** (*simp add: RHS-mono assms design-refine-intro*)

### 3.5 Distribution laws

**lemma** *RHS-design-choice:*  $\mathbf{R}_s(P_1 \vdash Q_1) \sqcap \mathbf{R}_s(P_2 \vdash Q_2) = \mathbf{R}_s((P_1 \wedge P_2) \vdash (Q_1 \vee Q_2))$   
**by** (*metis RHS-inf design-choice*)

**lemma** *RHS-design-sup:*  $\mathbf{R}_s(P_1 \vdash Q_1) \sqcup \mathbf{R}_s(P_2 \vdash Q_2) = \mathbf{R}_s((P_1 \vee P_2) \vdash ((P_1 \Rightarrow Q_1) \wedge (P_2 \Rightarrow Q_2)))$   
**by** (*metis RHS-sup design-inf*)

**lemma** *RHS-design-USUP:*

**assumes**  $A \neq \{\}$   
**shows**  $(\prod i \in A \cdot \mathbf{R}_s(P(i) \vdash Q(i))) = \mathbf{R}_s((\prod i \in A \cdot P(i)) \vdash (\prod i \in A \cdot Q(i)))$   
**by** (*subst RHS-INF[OF assms, THEN sym], simp add: design-UINF-mem assms*)

**end**

## 4 Reactive Design Triples

**theory** *utp-rdes-triples*  
**imports** *utp-rdes-designs*  
**begin**

### 4.1 Diamond notation

**definition** *wait'-cond* ::  
 $(t::\text{trace}, 'a, 'b) \text{ rel-rp} \Rightarrow (t, 'a, 'b) \text{ rel-rp} \Rightarrow (t, 'a, 'b) \text{ rel-rp}$  (**infixr**  $\diamond$  60) **where**  
 $[\text{upred-defs}]: P \diamond Q = (P \triangleleft \$wait' \triangleright Q)$

**utp-const** *wait'-cond*

**lemma** *wait'-cond-unrest* [*unrest*]:  
 $\llbracket \text{out-var } wait \bowtie x; x \# P; x \# Q \rrbracket \Longrightarrow x \# (P \diamond Q)$   
**by** (*simp add: wait'-cond-def unrest*)

**lemma** *wait'-cond-subst* [*usubst*]:  
 $\$wait' \#_s \sigma \Longrightarrow \sigma \dagger (P \diamond Q) = (\sigma \dagger P) \diamond (\sigma \dagger Q)$   
**by** (*simp add: wait'-cond-def usubst unrest*)

**lemma** *wait'-cond-left-false*:  $\text{false} \diamond P = (\neg \$wait' \wedge P)$   
**by** (*rel-auto*)

**lemma** *wait'-cond-seq*:  $((P \diamond Q) ;; R) = ((P ;; (\$wait' \wedge R)) \vee (Q ;; (\neg \$wait' \wedge R)))$   
**by** (*simp add: wait'-cond-def cond-def seqr-or-distl, rel-blast*)

**lemma** *wait'-cond-true*:  $(P \diamond Q \wedge \$wait') = (P \wedge \$wait')$   
**by** (*rel-auto*)

**lemma** *wait'-cond-false*:  $(P \diamond Q \wedge (\neg \$wait')) = (Q \wedge (\neg \$wait'))$   
**by** (*rel-auto*)

**lemma** *wait'-cond-idem*:  $P \diamond P = P$   
**by** (*rel-auto*)

**lemma** *wait'-cond-conj-exchange*:  
 $((P \diamond Q) \wedge (R \diamond S)) = (P \wedge R) \diamond (Q \wedge S)$   
**by** (*rel-auto*)

**lemma** *subst-wait'-cond-true* [*usubst*]:  $(P \diamond Q) \llbracket \text{true}/\$wait' \rrbracket = P \llbracket \text{true}/\$wait' \rrbracket$   
**by** (*rel-auto*)

**lemma** *subst-wait'-cond-false* [*usubst*]:  $(P \diamond Q) \llbracket \text{false}/\$wait' \rrbracket = Q \llbracket \text{false}/\$wait' \rrbracket$   
**by** (*rel-auto*)

**lemma** *subst-wait'-left-subst*:  $(P \llbracket \text{true}/\$wait' \rrbracket \diamond Q) = (P \diamond Q)$   
**by** (*rel-auto*)

**lemma** *subst-wait'-right-subst*:  $(P \diamond Q \llbracket \text{false}/\$wait' \rrbracket) = (P \diamond Q)$   
**by** (*rel-auto*)

**lemma** *wait'-cond-split*:  $P \llbracket \text{true}/\$wait' \rrbracket \diamond P \llbracket \text{false}/\$wait' \rrbracket = P$   
**by** (*simp add: wait'-cond-def cond-var-split*)

**lemma** *wait-cond'-assoc* [simp]:  $P \diamond Q \diamond R = P \diamond R$   
 by (rel-auto)

**lemma** *wait-cond'-shadow*:  $(P \diamond Q) \diamond R = P \diamond Q \diamond R$   
 by (rel-auto)

**lemma** *wait-cond'-conj* [simp]:  $P \diamond (Q \wedge (R \diamond S)) = P \diamond (Q \wedge S)$   
 by (rel-auto)

**lemma** *R1-wait'-cond*:  $R1(P \diamond Q) = R1(P) \diamond R1(Q)$   
 by (rel-auto)

**lemma** *R2s-wait'-cond*:  $R2s(P \diamond Q) = R2s(P) \diamond R2s(Q)$   
 by (simp add: wait'-cond-def R2s-def R2s-def usubst)

**lemma** *R2-wait'-cond*:  $R2(P \diamond Q) = R2(P) \diamond R2(Q)$   
 by (simp add: R2-def R2s-wait'-cond R1-wait'-cond)

**lemma** *wait'-cond-R1-closed* [closure]:  
 $\llbracket P \text{ is } R1; Q \text{ is } R1 \rrbracket \implies P \diamond Q \text{ is } R1$   
 by (simp add: Healthy-def R1-wait'-cond)

**lemma** *wait'-cond-R2c-closed* [closure]:  $\llbracket P \text{ is } R2c; Q \text{ is } R2c \rrbracket \implies P \diamond Q \text{ is } R2c$   
 by (simp add: R2c-condr wait'-cond-def Healthy-def, rel-auto)

## 4.2 Export laws

**lemma** *RH-design-peri-R1*:  $\mathbf{R}(P \vdash R1(Q) \diamond R) = \mathbf{R}(P \vdash Q \diamond R)$   
 by (metis (no-types, lifting) R1-idem R1-wait'-cond RH-design-export-R1)

**lemma** *RH-design-post-R1*:  $\mathbf{R}(P \vdash Q \diamond R1(R)) = \mathbf{R}(P \vdash Q \diamond R)$   
 by (metis R1-wait'-cond RH-design-export-R1 RH-design-peri-R1)

**lemma** *RH-design-peri-R2s*:  $\mathbf{R}(P \vdash R2s(Q) \diamond R) = \mathbf{R}(P \vdash Q \diamond R)$   
 by (metis (no-types, lifting) R2s-idem R2s-wait'-cond RH-design-export-R2s)

**lemma** *RH-design-post-R2s*:  $\mathbf{R}(P \vdash Q \diamond R2s(R)) = \mathbf{R}(P \vdash Q \diamond R)$   
 by (metis (no-types, lifting) R2s-idem R2s-wait'-cond RH-design-export-R2s)

**lemma** *RH-design-peri-R2c*:  $\mathbf{R}(P \vdash R2c(Q) \diamond R) = \mathbf{R}(P \vdash Q \diamond R)$   
 by (metis R1-R2s-R2c RH-design-peri-R1 RH-design-peri-R2s)

**lemma** *RHS-design-peri-R1*:  $\mathbf{R}_s(P \vdash R1(Q) \diamond R) = \mathbf{R}_s(P \vdash Q \diamond R)$   
 by (metis (no-types, lifting) R1-idem R1-wait'-cond RHS-design-export-R1)

**lemma** *RHS-design-post-R1*:  $\mathbf{R}_s(P \vdash Q \diamond R1(R)) = \mathbf{R}_s(P \vdash Q \diamond R)$   
 by (metis R1-wait'-cond RHS-design-export-R1 RHS-design-peri-R1)

**lemma** *RHS-design-peri-R2s*:  $\mathbf{R}_s(P \vdash R2s(Q) \diamond R) = \mathbf{R}_s(P \vdash Q \diamond R)$   
 by (metis (no-types, lifting) R2s-idem R2s-wait'-cond RHS-design-export-R2s)

**lemma** *RHS-design-post-R2s*:  $\mathbf{R}_s(P \vdash Q \diamond R2s(R)) = \mathbf{R}_s(P \vdash Q \diamond R)$   
 by (metis R2s-wait'-cond RHS-design-export-R2s RHS-design-peri-R2s)

**lemma** *RHS-design-peri-R2c*:  $\mathbf{R}_s(P \vdash R2c(Q) \diamond R) = \mathbf{R}_s(P \vdash Q \diamond R)$   
 by (metis R1-R2s-R2c RHS-design-peri-R1 RHS-design-peri-R2s)

**lemma** *RH-design-lemma1*:

$$RH(P \vdash (R1(R2c(Q)) \vee R) \diamond S) = RH(P \vdash (Q \vee R) \diamond S)$$

**by** (*metis* (*no-types*, *lifting*) *R1-R2c-is-R2* *R1-R2s-R2c* *R2-R1-form* *R2-disj* *R2c-idem* *RH-design-peri-R1* *RH-design-peri-R2s*)

**lemma** *RHS-design-lemma1*:

$$RHS(P \vdash (R1(R2c(Q)) \vee R) \diamond S) = RHS(P \vdash (Q \vee R) \diamond S)$$

**by** (*metis* (*no-types*, *lifting*) *R1-R2c-is-R2* *R1-R2s-R2c* *R2-R1-form* *R2-disj* *R2c-idem* *RHS-design-peri-R1* *RHS-design-peri-R2s*)

### 4.3 Pre-, peri-, and postconditions

#### 4.3.1 Definitions

**abbreviation**  $pre_s \equiv [\$ok \mapsto_s true, \$ok' \mapsto_s false, \$wait \mapsto_s false]$

**abbreviation**  $cmt_s \equiv [\$ok \mapsto_s true, \$ok' \mapsto_s true, \$wait \mapsto_s false]$

**abbreviation**  $peri_s \equiv [\$ok \mapsto_s true, \$ok' \mapsto_s true, \$wait \mapsto_s false, \$wait' \mapsto_s true]$

**abbreviation**  $post_s \equiv [\$ok \mapsto_s true, \$ok' \mapsto_s true, \$wait \mapsto_s false, \$wait' \mapsto_s false]$

**abbreviation**  $npre_R(P) \equiv pre_s \dagger P$

**definition** [*upred-defs*]:  $pre_R(P) = (\neg_r npre_R(P))$

**definition** [*upred-defs*]:  $cmt_R(P) = R1(cmt_s \dagger P)$

**definition** [*upred-defs*]:  $peri_R(P) = R1(peri_s \dagger P)$

**definition** [*upred-defs*]:  $post_R(P) = R1(post_s \dagger P)$

**no-utp-lift**  $pre_R cmt_R peri_R post_R npre_R$

#### 4.3.2 Unrestriction laws

**lemma** *ok-pre-unrest* [*unrest*]:  $\$ok \# pre_R P$

**by** (*simp add*: *pre\_R-def unrest usubst*)

**lemma** *ok-peri-unrest* [*unrest*]:  $\$ok \# peri_R P$

**by** (*simp add*: *peri\_R-def unrest usubst*)

**lemma** *ok-post-unrest* [*unrest*]:  $\$ok \# post_R P$

**by** (*simp add*: *post\_R-def unrest usubst*)

**lemma** *ok-cmt-unrest* [*unrest*]:  $\$ok \# cmt_R P$

**by** (*simp add*: *cmt\_R-def unrest usubst*)

**lemma** *ok'-pre-unrest* [*unrest*]:  $\$ok' \# pre_R P$

**by** (*simp add*: *pre\_R-def unrest usubst*)

**lemma** *ok'-peri-unrest* [*unrest*]:  $\$ok' \# peri_R P$

**by** (*simp add*: *peri\_R-def unrest usubst*)

**lemma** *ok'-post-unrest* [*unrest*]:  $\$ok' \# post_R P$

**by** (*simp add*: *post\_R-def unrest usubst*)

**lemma** *ok'-cmt-unrest* [*unrest*]:  $\$ok' \# cmt_R P$

**by** (*simp add*: *cmt\_R-def unrest usubst*)

**lemma** *wait-pre-unrest* [*unrest*]:  $\$wait \# pre_R P$

by (simp add: pre<sub>R</sub>-def unrest usubst)

**lemma** wait-*peri-unrest* [unrest]: \$wait # peri<sub>R</sub> P  
by (simp add: peri<sub>R</sub>-def unrest usubst)

**lemma** wait-*post-unrest* [unrest]: \$wait # post<sub>R</sub> P  
by (simp add: post<sub>R</sub>-def unrest usubst)

**lemma** wait-*cmt-unrest* [unrest]: \$wait # cmt<sub>R</sub> P  
by (simp add: cmt<sub>R</sub>-def unrest usubst)

**lemma** wait'-*peri-unrest* [unrest]: \$wait' # peri<sub>R</sub> P  
by (simp add: peri<sub>R</sub>-def unrest usubst)

**lemma** wait'-*post-unrest* [unrest]: \$wait' # post<sub>R</sub> P  
by (simp add: post<sub>R</sub>-def unrest usubst)

### 4.3.3 Substitution laws

**lemma** pre<sub>s</sub>-*design*: pre<sub>s</sub> † (P ⊢ Q) = (¬ pre<sub>s</sub> † P)  
by (simp add: design-def pre<sub>R</sub>-def usubst)

**lemma** peri<sub>s</sub>-*design*: peri<sub>s</sub> † (P ⊢ Q ◊ R) = peri<sub>s</sub> † (P ⇒ Q)  
by (simp add: design-def usubst wait'-cond-def)

**lemma** post<sub>s</sub>-*design*: post<sub>s</sub> † (P ⊢ Q ◊ R) = post<sub>s</sub> † (P ⇒ R)  
by (simp add: design-def usubst wait'-cond-def)

**lemma** cmt<sub>s</sub>-*design*: cmt<sub>s</sub> † (P ⊢ Q) = cmt<sub>s</sub> † (P ⇒ Q)  
by (simp add: design-def usubst wait'-cond-def)

**lemma** pre<sub>s</sub>-*R1* [usubst]: pre<sub>s</sub> † R1(P) = R1(pre<sub>s</sub> † P)  
by (simp add: R1-def usubst)

**lemma** pre<sub>s</sub>-*R2c* [usubst]: pre<sub>s</sub> † R2c(P) = R2c(pre<sub>s</sub> † P)  
by (simp add: R2c-def R2s-def usubst)

**lemma** peri<sub>s</sub>-*R1* [usubst]: peri<sub>s</sub> † R1(P) = R1(peri<sub>s</sub> † P)  
by (simp add: R1-def usubst)

**lemma** peri<sub>s</sub>-*R2c* [usubst]: peri<sub>s</sub> † R2c(P) = R2c(peri<sub>s</sub> † P)  
by (simp add: R2c-def R2s-def usubst)

**lemma** post<sub>s</sub>-*R1* [usubst]: post<sub>s</sub> † R1(P) = R1(post<sub>s</sub> † P)  
by (simp add: R1-def usubst)

**lemma** post<sub>s</sub>-*R2c* [usubst]: post<sub>s</sub> † R2c(P) = R2c(post<sub>s</sub> † P)  
by (simp add: R2c-def R2s-def usubst)

**lemma** cmt<sub>s</sub>-*R1* [usubst]: cmt<sub>s</sub> † R1(P) = R1(cmt<sub>s</sub> † P)  
by (simp add: R1-def usubst)

**lemma** cmt<sub>s</sub>-*R2c* [usubst]: cmt<sub>s</sub> † R2c(P) = R2c(cmt<sub>s</sub> † P)  
by (simp add: R2c-def R2s-def usubst)

**lemma** pre-*wait-false*:

$pre_R(P \llbracket false/\$wait \rrbracket) = pre_R(P)$   
**by** (*rel-auto*)

**lemma** *cmt-wait-false*:  
 $cmt_R(P \llbracket false/\$wait \rrbracket) = cmt_R(P)$   
**by** (*rel-auto*)

**lemma** *rea-pre-RHS-design*:  $pre_R(\mathbf{R}_s(P \vdash Q)) = R1(R2c(pre_s \dagger P))$   
**by** (*simp add: RHS-def usubst R3h-def pre\_R-def pre\_s-design R1-negate-R1 R2c-not rea-not-def*)

**lemma** *rea-cmt-RHS-design*:  $cmt_R(\mathbf{R}_s(P \vdash Q)) = R1(R2c(cmt_s \dagger (P \Rightarrow Q)))$   
**by** (*simp add: RHS-def usubst R3h-def cmt\_R-def cmt\_s-design R1-idem*)

**lemma** *rea-peri-RHS-design*:  $peri_R(\mathbf{R}_s(P \vdash Q \diamond R)) = R1(R2c(peri_s \dagger (P \Rightarrow_r Q)))$   
**by** (*simp add: RHS-def usubst peri\_R-def R3h-def peri\_s-design, rel-auto*)

**lemma** *rea-post-RHS-design*:  $post_R(\mathbf{R}_s(P \vdash Q \diamond R)) = R1(R2c(post_s \dagger (P \Rightarrow_r R)))$   
**by** (*simp add: RHS-def usubst post\_R-def R3h-def post\_s-design, rel-auto*)

**lemma** *peri-cmt-def*:  $peri_R(P) = (cmt_R(P)) \llbracket true/\$wait' \rrbracket$   
**by** (*rel-auto*)

**lemma** *post-cmt-def*:  $post_R(P) = (cmt_R(P)) \llbracket false/\$wait' \rrbracket$   
**by** (*rel-auto*)

**lemma** *rdes-export-cmt*:  $\mathbf{R}_s(P \vdash cmt_s \dagger Q) = \mathbf{R}_s(P \vdash Q)$   
**by** (*rel-auto*)

**lemma** *rdes-export-pre*:  $\mathbf{R}_s((P \llbracket true, false/\$ok, \$wait \rrbracket) \vdash Q) = \mathbf{R}_s(P \vdash Q)$   
**by** (*rel-auto*)

#### 4.3.4 Healthiness laws

**lemma** *wait'-unrest-pre-SRD* [*unrest*]:  
 $\$wait' \nmid pre_R(P) \implies \$wait' \nmid pre_R(SRD\ P)$   
**apply** (*rel-auto*)  
**using** *least-zero apply blast+*  
**done**

**lemma** *R1-R2s-cmt-SRD*:  
**assumes** *P is SRD*  
**shows**  $R1(R2s(cmt_R(P))) = cmt_R(P)$   
**by** (*metis (no-types, lifting) R1-R2c-commute R1-R2s-R2c R1-idem R2c-idem SRD-reactive-design assms rea-cmt-RHS-design*)

**lemma** *R1-R2s-peri-SRD*:  
**assumes** *P is SRD*  
**shows**  $R1(R2s(peri_R(P))) = peri_R(P)$   
**by** (*metis (no-types, hide-lams) Healthy-def R1-R2s-R2c R2-def R2-idem RHS-def SRD-RH-design-form assms R1-idem peri\_R-def peri\_s-R1 peri\_s-R2c*)

**lemma** *R1-peri-SRD*:  
**assumes** *P is SRD*  
**shows**  $R1(peri_R(P)) = peri_R(P)$   
**proof** –  
**have**  $R1(peri_R(P)) = R1(R1(R2s(peri_R(P))))$

by (simp add: R1-R2s-peri-SRD assms)  
 also have ... =  $\text{peri}_R(P)$   
 by (simp add: R1-idem, simp add: R1-R2s-peri-SRD assms)  
 finally show ?thesis .  
 qed

**lemma** *periR-SRD-R1 [closure]:  $P$  is SRD  $\implies \text{peri}_R(P)$  is R1*  
 by (simp add: Healthy-def' R1-peri-SRD)

**lemma** *R1-R2c-peri-RHS:*  
 assumes  $P$  is SRD  
 shows  $R1(R2c(\text{peri}_R(P))) = \text{peri}_R(P)$   
 by (metis R1-R2s-R2c R1-R2s-peri-SRD assms)

**lemma** *R1-R2s-post-SRD:*  
 assumes  $P$  is SRD  
 shows  $R1(R2s(\text{post}_R(P))) = \text{post}_R(P)$   
 by (metis (no-types, hide-lams) Healthy-def R1-R2s-R2c R1-idem R2-def R2-idem RHS-def SRD-RH-design-form  
 assms  $\text{post}_R\text{-def}$   $\text{post}_s\text{-R1}$   $\text{post}_s\text{-R2c}$ )

**lemma** *R2c-peri-SRD:*  
 assumes  $P$  is SRD  
 shows  $R2c(\text{peri}_R(P)) = \text{peri}_R(P)$   
 by (metis R1-R2c-commute R1-R2c-peri-RHS R1-peri-SRD assms)

**lemma** *R1-post-SRD:*  
 assumes  $P$  is SRD  
 shows  $R1(\text{post}_R(P)) = \text{post}_R(P)$   
**proof** –  
 have  $R1(\text{post}_R(P)) = R1(R1(R2s(\text{post}_R(P))))$   
 by (simp add: R1-R2s-post-SRD assms)  
 also have ... =  $\text{post}_R(P)$   
 by (simp add: R1-idem, simp add: R1-R2s-post-SRD assms)  
 finally show ?thesis .  
 qed

**lemma** *R2c-post-SRD:*  
 assumes  $P$  is SRD  
 shows  $R2c(\text{post}_R(P)) = \text{post}_R(P)$   
 by (metis R1-R2c-commute R1-R2s-R2c R1-R2s-post-SRD R1-post-SRD assms)

**lemma** *postR-SRD-R1 [closure]:  $P$  is SRD  $\implies \text{post}_R(P)$  is R1*  
 by (simp add: Healthy-def' R1-post-SRD)

**lemma** *R1-R2c-post-RHS:*  
 assumes  $P$  is SRD  
 shows  $R1(R2c(\text{post}_R(P))) = \text{post}_R(P)$   
 by (metis R1-R2s-R2c R1-R2s-post-SRD assms)

**lemma** *R2-cmt-conj-wait':*  
 $P$  is SRD  $\implies R2(\text{cmt}_R P \wedge \neg \$\text{wait}') = (\text{cmt}_R P \wedge \neg \$\text{wait}')$   
 by (simp add: R2-def R2s-conj R2s-not R2s-wait' R1-extend-conj R1-R2s-cmt-SRD)

**lemma** *R2c-preR:*  
 $P$  is SRD  $\implies R2c(\text{pre}_R(P)) = \text{pre}_R(P)$

by (metis (no-types, lifting) R1-R2c-commute R2c-idem SRD-reactive-design rea-pre-RHS-design)

**lemma** *preR-R2c-closed* [closure]:  $P$  is SRD  $\implies$   $\text{pre}_R(P)$  is R2c  
 by (simp add: Healthy-def' R2c-preR)

**lemma** *R2c-periR*:  
 $P$  is SRD  $\implies$   $R2c(\text{peri}_R(P)) = \text{peri}_R(P)$   
 by (metis (no-types, lifting) R1-R2c-commute R1-R2s-R2c R1-R2s-peri-SRD R2c-idem)

**lemma** *periR-R2c-closed* [closure]:  $P$  is SRD  $\implies$   $\text{peri}_R(P)$  is R2c  
 by (simp add: Healthy-def R2c-peri-SRD)

**lemma** *R2c-postR*:  
 $P$  is SRD  $\implies$   $R2c(\text{post}_R(P)) = \text{post}_R(P)$   
 by (metis (no-types, hide-lams) R1-R2c-commute R1-R2c-is-R2 R1-R2s-post-SRD R2-def R2s-idem)

**lemma** *postR-R2c-closed* [closure]:  $P$  is SRD  $\implies$   $\text{post}_R(P)$  is R2c  
 by (simp add: Healthy-def R2c-post-SRD)

**lemma** *periR-RR* [closure]:  $P$  is SRD  $\implies$   $\text{peri}_R(P)$  is RR  
 by (rule RR-intro, simp-all add: closure unrest)

**lemma** *postR-RR* [closure]:  $P$  is SRD  $\implies$   $\text{post}_R(P)$  is RR  
 by (rule RR-intro, simp-all add: closure unrest)

**lemma** *wpR-trace-ident-pre* [wp]:  
 $(\$tr' =_u \$tr \wedge \lceil II \rceil_R) \text{wp}_r \text{pre}_R P = \text{pre}_R P$   
 by (rel-auto)

**lemma** *R1-preR* [closure]:  
 $\text{pre}_R(P)$  is R1  
 by (rel-auto)

**lemma** *trace-ident-left-periR*:  
 $(\$tr' =_u \$tr \wedge \lceil II \rceil_R) ;; \text{peri}_R(P) = \text{peri}_R(P)$   
 by (rel-auto)

**lemma** *trace-ident-left-postR*:  
 $(\$tr' =_u \$tr \wedge \lceil II \rceil_R) ;; \text{post}_R(P) = \text{post}_R(P)$   
 by (rel-auto)

**lemma** *trace-ident-right-postR*:  
 $\text{post}_R(P) ;; (\$tr' =_u \$tr \wedge \lceil II \rceil_R) = \text{post}_R(P)$   
 by (rel-auto)

**lemma** *preR-R2-closed* [closure]:  $P$  is SRD  $\implies$   $\text{pre}_R(P)$  is R2  
 by (simp add: R2-comp-def Healthy-comp closure)

**lemma** *periR-R2-closed* [closure]:  $P$  is SRD  $\implies$   $\text{peri}_R(P)$  is R2  
 by (simp add: Healthy-def' R1-R2c-peri-RHS R2-R2c-def)

**lemma** *postR-R2-closed* [closure]:  $P$  is SRD  $\implies$   $\text{post}_R(P)$  is R2  
 by (simp add: Healthy-def' R1-R2c-post-RHS R2-R2c-def)



#### 4.3.5 Calculation laws

**lemma** *wait'-cond-peri-post-cmt* [rdes]:

$cmt_R P = peri_R P \diamond post_R P$   
**by** (*rel-auto*)

**lemma** *preR-rdes* [rdes]:

**assumes**  $P$  is  $RR$   
**shows**  $pre_R(\mathbf{R}_s(P \vdash Q \diamond R)) = P$   
**by** (*simp add: rea-pre-RHS-design unrest usubst assms Healthy-if RR-implies-R2c RR-implies-R1*)

**lemma** *periR-rdes* [rdes]:

**assumes**  $P$  is  $RR$   $Q$  is  $RR$   
**shows**  $peri_R(\mathbf{R}_s(P \vdash Q \diamond R)) = (P \Rightarrow_r Q)$   
**by** (*simp add: rea-peri-RHS-design unrest usubst assms Healthy-if RR-implies-R2c closure*)

**lemma** *postR-rdes* [rdes]:

**assumes**  $P$  is  $RR$   $R$  is  $RR$   
**shows**  $post_R(\mathbf{R}_s(P \vdash Q \diamond R)) = (P \Rightarrow_r R)$   
**by** (*simp add: rea-post-RHS-design unrest usubst assms Healthy-if RR-implies-R2c closure*)

**lemma** *preR-Chaos* [rdes]:  $pre_R(Chaos) = false$

**by** (*simp add: Chaos-def, rel-simp*)

**lemma** *periR-Chaos* [rdes]:  $peri_R(Chaos) = true_r$

**by** (*simp add: Chaos-def, rel-simp*)

**lemma** *postR-Chaos* [rdes]:  $post_R(Chaos) = true_r$

**by** (*simp add: Chaos-def, rel-simp*)

**lemma** *preR-Miracle* [rdes]:  $pre_R(Miracle) = true_r$

**by** (*simp add: Miracle-def, rel-auto*)

**lemma** *periR-Miracle* [rdes]:  $peri_R(Miracle) = false$

**by** (*simp add: Miracle-def, rel-auto*)

**lemma** *postR-Miracle* [rdes]:  $post_R(Miracle) = false$

**by** (*simp add: Miracle-def, rel-auto*)

**lemma** *preR-srdes-skip* [rdes]:  $pre_R(II_R) = true_r$

**by** (*rel-auto*)

**lemma** *periR-srdes-skip* [rdes]:  $peri_R(II_R) = false$

**by** (*rel-auto*)

**lemma** *postR-srdes-skip* [rdes]:  $post_R(II_R) = (\$tr' =_u \$tr \wedge [II]_R)$

**by** (*rel-auto*)

**lemma** *preR-INF* [rdes]:  $A \neq \{\} \implies pre_R(\bigcap A) = (\bigwedge P \in A \cdot pre_R(P))$

**by** (*rel-auto*)

**lemma** *periR-INF* [rdes]:  $peri_R(\bigcap A) = (\bigvee P \in A \cdot peri_R(P))$

**by** (*rel-auto*)

**lemma** *postR-INF* [rdes]:  $post_R(\bigcap A) = (\bigvee P \in A \cdot post_R(P))$

**by** (*rel-auto*)

**lemma** *preR-UINF* [rdes]:  $pre_R(\sqcap i \cdot P(i)) = (\sqcap i \cdot pre_R(P(i)))$   
**by** (*rel-auto*)

**lemma** *periR-UINF* [rdes]:  $peri_R(\sqcap i \cdot P(i)) = (\sqcap i \cdot peri_R(P(i)))$   
**by** (*rel-auto*)

**lemma** *postR-UINF* [rdes]:  $post_R(\sqcap i \cdot P(i)) = (\sqcap i \cdot post_R(P(i)))$   
**by** (*rel-auto*)

**lemma** *preR-UINF-member* [rdes]:  $A \neq \{\} \implies pre_R(\sqcap i \in A \cdot P(i)) = (\sqcap i \in A \cdot pre_R(P(i)))$   
**by** (*rel-auto*)

**lemma** *preR-UINF-member-2* [rdes]:  $A \neq \{\} \implies pre_R(\sqcap (i,j) \in A \cdot P i j) = (\sqcap (i,j) \in A \cdot pre_R(P i j))$   
**by** (*rel-auto*)

**lemma** *preR-UINF-member-3* [rdes]:  $A \neq \{\} \implies pre_R(\sqcap (i,j,k) \in A \cdot P i j k) = (\sqcap (i,j,k) \in A \cdot pre_R(P i j k))$   
**by** (*rel-auto*)

**lemma** *periR-UINF-member* [rdes]:  $peri_R(\sqcap i \in A \cdot P(i)) = (\sqcap i \in A \cdot peri_R(P(i)))$   
**by** (*rel-auto*)

**lemma** *periR-UINF-member-2* [rdes]:  $peri_R(\sqcap (i,j) \in A \cdot P i j) = (\sqcap (i,j) \in A \cdot peri_R(P i j))$   
**by** (*rel-auto*)

**lemma** *periR-UINF-member-3* [rdes]:  $peri_R(\sqcap (i,j,k) \in A \cdot P i j k) = (\sqcap (i,j,k) \in A \cdot peri_R(P i j k))$   
**by** (*rel-auto*)

**lemma** *postR-UINF-member* [rdes]:  $post_R(\sqcap i \in A \cdot P(i)) = (\sqcap i \in A \cdot post_R(P(i)))$   
**by** (*rel-auto*)

**lemma** *postR-UINF-member-2* [rdes]:  $post_R(\sqcap (i,j) \in A \cdot P i j) = (\sqcap (i,j) \in A \cdot post_R(P i j))$   
**by** (*rel-auto*)

**lemma** *postR-UINF-member-3* [rdes]:  $post_R(\sqcap (i,j,k) \in A \cdot P i j k) = (\sqcap (i,j,k) \in A \cdot post_R(P i j k))$   
**by** (*rel-auto*)

**lemma** *preR-inf* [rdes]:  $pre_R(P \sqcap Q) = (pre_R(P) \wedge pre_R(Q))$   
**by** (*rel-auto*)

**lemma** *periR-inf* [rdes]:  $peri_R(P \sqcap Q) = (peri_R(P) \vee peri_R(Q))$   
**by** (*rel-auto*)

**lemma** *postR-inf* [rdes]:  $post_R(P \sqcap Q) = (post_R(P) \vee post_R(Q))$   
**by** (*rel-auto*)

**lemma** *preR-SUP* [rdes]:  $pre_R(\sqcup A) = (\bigvee P \in A \cdot pre_R(P))$   
**by** (*rel-auto*)

**lemma** *periR-SUP* [rdes]:  $A \neq \{\} \implies peri_R(\sqcup A) = (\bigwedge P \in A \cdot peri_R(P))$   
**by** (*rel-auto*)

**lemma** *postR-SUP* [rdes]:  $A \neq \{\} \implies post_R(\sqcup A) = (\bigwedge P \in A \cdot post_R(P))$   
**by** (*rel-auto*)

#### 4.4 Formation laws

**lemma** *srdes-skip-tri-design* [*rdes-def*]:  $II_R = \mathbf{R}_s(\text{true}_r \vdash \text{false} \diamond II_r)$   
**by** (*simp add: srdes-skip-def, rel-auto*)

**lemma** *Chaos-tri-def* [*rdes-def*]:  $\text{Chaos} = \mathbf{R}_s(\text{false} \vdash \text{false} \diamond \text{false})$   
**by** (*simp add: Chaos-def design-false-pre*)

**lemma** *Miracle-tri-def* [*rdes-def*]:  $\text{Miracle} = \mathbf{R}_s(\text{true}_r \vdash \text{false} \diamond \text{false})$   
**by** (*simp add: Miracle-def R1-design-R1-pre wait'-cond-idem*)

**lemma** *RHS-tri-design-form*:

**assumes**  $P_1$  is *RR*  $P_2$  is *RR*  $P_3$  is *RR*

**shows**  $\mathbf{R}_s(P_1 \vdash P_2 \diamond P_3) = (II_R \triangleleft \$wait \triangleright ((\$ok \wedge P_1) \Rightarrow_r (\$ok' \wedge (P_2 \diamond P_3))))$

**proof** –

**have**  $\mathbf{R}_s(RR(P_1) \vdash RR(P_2) \diamond RR(P_3)) = (II_R \triangleleft \$wait \triangleright ((\$ok \wedge RR(P_1)) \Rightarrow_r (\$ok' \wedge (RR(P_2) \diamond RR(P_3))))$

**apply** (*rel-auto*) **using** *minus-zero-eq* **by** *blast*

**thus** *?thesis*

**by** (*simp add: Healthy-if assms*)

**qed**

**lemma** *RHS-design-pre-post-form*:

$\mathbf{R}_s((\neg P^f_f) \vdash P^t_f) = \mathbf{R}_s(\text{pre}_R(P) \vdash \text{cmt}_R(P))$

**proof** –

**have**  $\mathbf{R}_s((\neg P^f_f) \vdash P^t_f) = \mathbf{R}_s((\neg P^f_f) \llbracket \text{true}/\$ok \rrbracket \vdash P^t_f \llbracket \text{true}/\$ok \rrbracket)$

**by** (*simp add: design-subst-ok*)

**also have**  $\dots = \mathbf{R}_s(\text{pre}_R(P) \vdash \text{cmt}_R(P))$

**by** (*simp add: pre\_R-def cmt\_R-def usubst, rel-auto*)

**finally show** *?thesis* .

**qed**

**lemma** *SRD-as-reactive-design*:

$SRD(P) = \mathbf{R}_s(\text{pre}_R(P) \vdash \text{cmt}_R(P))$

**by** (*simp add: RHS-design-pre-post-form SRD-RH-design-form*)

**lemma** *SRD-reactive-design-alt*:

**assumes**  $P$  is *SRD*

**shows**  $\mathbf{R}_s(\text{pre}_R(P) \vdash \text{cmt}_R(P)) = P$

**proof** –

**have**  $\mathbf{R}_s(\text{pre}_R(P) \vdash \text{cmt}_R(P)) = \mathbf{R}_s((\neg P^f_f) \vdash P^t_f)$

**by** (*simp add: RHS-design-pre-post-form*)

**thus** *?thesis*

**by** (*simp add: SRD-reactive-design assms*)

**qed**

**lemma** *SRD-reactive-tri-design-lemma*:

$SRD(P) = \mathbf{R}_s((\neg P^f_f) \vdash P^t_f \llbracket \text{true}/\$wait' \rrbracket \diamond P^t_f \llbracket \text{false}/\$wait' \rrbracket)$

**by** (*simp add: SRD-RH-design-form wait'-cond-split*)

**lemma** *SRD-as-reactive-tri-design*:

$SRD(P) = \mathbf{R}_s(\text{pre}_R(P) \vdash \text{peri}_R(P) \diamond \text{post}_R(P))$

**proof** –

**have**  $SRD(P) = \mathbf{R}_s((\neg P^f_f) \vdash P^t_f \llbracket \text{true}/\$wait' \rrbracket \diamond P^t_f \llbracket \text{false}/\$wait' \rrbracket)$

**by** (*simp add: SRD-RH-design-form wait'-cond-split*)

**also have**  $\dots = \mathbf{R}_s(\text{pre}_R(P) \vdash \text{peri}_R(P) \diamond \text{post}_R(P))$

apply (simp add: usubst)  
 apply (subst design-subst-ok-ok'[THEN sym])  
 apply (simp add: pre<sub>R</sub>-def peri<sub>R</sub>-def post<sub>R</sub>-def usubst unrest)  
 apply (rel-auto)  
 done  
 finally show ?thesis .  
 qed

**lemma** *SRD-reactive-tri-design*:

assumes  $P$  is SRD  
 shows  $\mathbf{R}_s(\text{pre}_R(P) \vdash \text{peri}_R(P) \diamond \text{post}_R(P)) = P$   
 by (metis Healthy-if SRD-as-reactive-tri-design assms)

**lemma** *SRD-elim* [RD-elim]:  $\llbracket P \text{ is SRD}; Q(\mathbf{R}_s(\text{pre}_R(P) \vdash \text{peri}_R(P) \diamond \text{post}_R(P))) \rrbracket \implies Q(P)$   
 by (simp add: SRD-reactive-tri-design)

**lemma** *RHS-tri-design-is-SRD* [closure]:

assumes  $\$ok' \# P \$ok' \# Q \$ok' \# R$   
 shows  $\mathbf{R}_s(P \vdash Q \diamond R)$  is SRD  
 by (rule RHS-design-is-SRD, simp-all add: unrest assms)

**lemma** *SRD-rdes-intro* [closure]:

assumes  $P$  is RR  $Q$  is RR  $R$  is RR  
 shows  $\mathbf{R}_s(P \vdash Q \diamond R)$  is SRD  
 by (rule RHS-tri-design-is-SRD, simp-all add: unrest closure assms)

**lemma** *USUP-R1-R2s-cmt-SRD*:

assumes  $A \subseteq \llbracket \text{SRD} \rrbracket_H$   
 shows  $(\bigsqcup P \in A \cdot R1 (R2s (\text{cmt}_R P))) = (\bigsqcup P \in A \cdot \text{cmt}_R P)$   
 by (rule USUP-cong[of A], metis (mono-tags, lifting) Ball-Collect R1-R2s-cmt-SRD assms)

**lemma** *UINF-R1-R2s-cmt-SRD*:

assumes  $A \subseteq \llbracket \text{SRD} \rrbracket_H$   
 shows  $(\bigsqcap P \in A \cdot R1 (R2s (\text{cmt}_R P))) = (\bigsqcap P \in A \cdot \text{cmt}_R P)$   
 by (rule UINF-cong[of A], metis (mono-tags, lifting) Ball-Collect R1-R2s-cmt-SRD assms)

#### 4.4.1 Order laws

**lemma** *preR-antitone*:  $P \sqsubseteq Q \implies \text{pre}_R(Q) \sqsubseteq \text{pre}_R(P)$   
 by (rel-auto)

**lemma** *periR-monotone*:  $P \sqsubseteq Q \implies \text{peri}_R(P) \sqsubseteq \text{peri}_R(Q)$   
 by (rel-auto)

**lemma** *postR-monotone*:  $P \sqsubseteq Q \implies \text{post}_R(P) \sqsubseteq \text{post}_R(Q)$   
 by (rel-auto)

#### 4.5 Composition laws

**theorem** *RH-tri-design-composition*:

assumes  $\$ok' \# P \$ok' \# Q_1 \$ok' \# Q_2 \$ok \# R \$ok \# S_1 \$ok \# S_2$   
 $\$wait' \# Q_2 \$wait \# S_1 \$wait \# S_2$   
 shows  $(RH(P \vdash Q_1 \diamond Q_2) ;; RH(R \vdash S_1 \diamond S_2)) =$   
 $RH((\neg (R1 (\neg R2s P) ;; R1 \text{ true}) \wedge \neg ((R1 (R2s Q_2) \wedge \neg \$wait') ;; R1 (\neg R2s R))) \vdash$   
 $((Q_1 \vee (R1 (R2s Q_2) ;; R1 (R2s S_1))) \diamond ((R1 (R2s Q_2) ;; R1 (R2s S_2))))$

**proof** –

**have** 1:  $(\neg ((R1 (R2s (Q_1 \diamond Q_2)) \wedge \neg \$wait') ;; R1 (\neg R2s R))) =$   
 $(\neg ((R1 (R2s Q_2) \wedge \neg \$wait') ;; R1 (\neg R2s R)))$   
**by** (*metis (no-types, hide-lams) R1-extend-conj R2s-conj R2s-not R2s-wait' wait'-cond-false*)  
**have** 2:  $(R1 (R2s (Q_1 \diamond Q_2)) ;; ([II]_D \triangleleft \$wait \triangleright R1 (R2s (S_1 \diamond S_2)))) =$   
 $((R1 (R2s Q_1) \vee (R1 (R2s Q_2) ;; R1 (R2s S_1))) \diamond (R1 (R2s Q_2) ;; R1 (R2s S_2)))$   
**proof** –  
**have**  $(R1 (R2s Q_1) ;; (\$wait \wedge ([II]_D \triangleleft \$wait \triangleright R1 (R2s S_1) \diamond R1 (R2s S_2))))$   
 $= (R1 (R2s Q_1) \wedge \$wait')$   
**proof** –  
**have**  $(R1 (R2s Q_1) ;; (\$wait \wedge ([II]_D \triangleleft \$wait \triangleright R1 (R2s S_1) \diamond R1 (R2s S_2))))$   
 $= (R1 (R2s Q_1) ;; (\$wait \wedge [II]_D))$   
**by** (*rel-auto*)  
**also have** ...  $= ((R1 (R2s Q_1) ;; [II]_D) \wedge \$wait')$   
**by** (*rel-auto*)  
**also from** *assms*(2) **have** ...  $= ((R1 (R2s Q_1)) \wedge \$wait')$   
**by** (*simp add: lift-des-skip-dr-unit-unrest unrest*)  
**finally show** *?thesis* .  
**qed**

**moreover have**  $(R1 (R2s Q_2) ;; (\neg \$wait \wedge ([II]_D \triangleleft \$wait \triangleright R1 (R2s S_1) \diamond R1 (R2s S_2))))$   
 $= ((R1 (R2s Q_2)) ;; (R1 (R2s S_1) \diamond R1 (R2s S_2)))$   
**proof** –  
**have**  $(R1 (R2s Q_2) ;; (\neg \$wait \wedge ([II]_D \triangleleft \$wait \triangleright R1 (R2s S_1) \diamond R1 (R2s S_2))))$   
 $= (R1 (R2s Q_2) ;; (\neg \$wait \wedge (R1 (R2s S_1) \diamond R1 (R2s S_2))))$   
**by** (*metis (no-types, lifting) cond-def conj-disj-not-abs utp-pred-laws.double-compl utp-pred-laws.inf.left-idem*  
*utp-pred-laws.sup-assoc utp-pred-laws.sup-inf-absorb*)

**also have** ...  $= ((R1 (R2s Q_2)) \llbracket false/\$wait' \rrbracket ;; (R1 (R2s S_1) \diamond R1 (R2s S_2)) \llbracket false/\$wait' \rrbracket)$   
**by** (*metis false-alt-def seqr-right-one-point upred-eq-false wait-vwb-lens*)

**also have** ...  $= ((R1 (R2s Q_2)) ;; (R1 (R2s S_1) \diamond R1 (R2s S_2)))$   
**by** (*simp add: wait'-cond-def usubst unrest assms*)

**finally show** *?thesis* .  
**qed**

**moreover**  
**have**  $((R1 (R2s Q_1) \wedge \$wait') \vee ((R1 (R2s Q_2)) ;; (R1 (R2s S_1) \diamond R1 (R2s S_2))))$   
 $= (R1 (R2s Q_1) \vee (R1 (R2s Q_2) ;; R1 (R2s S_1))) \diamond ((R1 (R2s Q_2) ;; R1 (R2s S_2)))$   
**by** (*simp add: wait'-cond-def cond-seq-right-distr cond-and-T-integrate unrest*)

**ultimately show** *?thesis*  
**by** (*simp add: R2s-wait'-cond R1-wait'-cond wait'-cond-seq*)  
**qed**

**show** *?thesis*  
**apply** (*subst RH-design-composition*)  
**apply** (*simp-all add: assms*)  
**apply** (*simp add: assms wait'-cond-def unrest*)  
**apply** (*simp add: assms wait'-cond-def unrest*)  
**apply** (*simp add: 1 2*)  
**apply** (*simp add: R1-R2s-R2c RH-design-lemma1*)  
**done**  
**qed**

**theorem** *R1-design-composition-RR:*

**assumes** *P is RR Q is RR R is RR S is RR*

**shows**

$(R1(P \vdash Q) ;; R1(R \vdash S)) = R1(((\neg_r P) \text{ wp}_r \text{ false} \wedge Q \text{ wp}_r R) \vdash (Q ;; S))$

**apply** (*subst R1-design-composition*)

**apply** (*simp-all add: assms unrest wp-rea-def Healthy-if closure*)

**apply** (*rel-auto*)

**done**

**theorem** *R1-design-composition-RC:*

**assumes** *P is RC Q is RR R is RR S is RR*

**shows**

$(R1(P \vdash Q) ;; R1(R \vdash S)) = R1((P \wedge Q \text{ wp}_r R) \vdash (Q ;; S))$

**by** (*simp add: R1-design-composition-RR assms unrest Healthy-if closure wp*)

**theorem** *RHS-tri-design-composition:*

**assumes**  $\$ok' \# P \$ok' \# Q_1 \$ok' \# Q_2 \$ok \# R \$ok \# S_1 \$ok \# S_2$

$\$wait \# R \$wait' \# Q_2 \$wait \# S_1 \$wait \# S_2$

**shows**  $(\mathbf{R}_s(P \vdash Q_1 \diamond Q_2) ;; \mathbf{R}_s(R \vdash S_1 \diamond S_2)) =$

$\mathbf{R}_s((\neg (R1 (\neg R2s P) ;; R1 \text{ true}) \wedge \neg (R1(R2s Q_2) ;; R1 (\neg R2s R))) \vdash$   
 $((\exists \$st' \cdot Q_1) \vee (R1 (R2s Q_2) ;; R1 (R2s S_1))) \diamond ((R1 (R2s Q_2) ;; R1 (R2s S_2))))$

**proof** –

**have**  $1: (\neg ((R1 (R2s (Q_1 \diamond Q_2)) \wedge \neg \$wait') ;; R1 (\neg R2s R))) =$

$(\neg ((R1 (R2s Q_2) \wedge \neg \$wait') ;; R1 (\neg R2s R)))$

**by** (*metis (no-types, hide-lams) R1-extend-conj R2s-conj R2s-not R2s-wait' wait'-cond-false*)

**have**  $2: (R1 (R2s (Q_1 \diamond Q_2)) ;; ((\exists \$st \cdot \lceil II \rceil_D) \triangleleft \$wait \triangleright R1 (R2s (S_1 \diamond S_2)))) =$

$((\exists \$st' \cdot R1 (R2s Q_1)) \vee (R1 (R2s Q_2) ;; R1 (R2s S_1))) \diamond (R1 (R2s Q_2) ;; R1 (R2s S_2)))$

**proof** –

**have**  $(R1 (R2s Q_1) ;; (\$wait \wedge ((\exists \$st \cdot \lceil II \rceil_D) \triangleleft \$wait \triangleright R1 (R2s S_1) \diamond R1 (R2s S_2))))$

$= (\exists \$st' \cdot ((R1 (R2s Q_1)) \wedge \$wait'))$

**proof** –

**have**  $(R1 (R2s Q_1) ;; (\$wait \wedge ((\exists \$st \cdot \lceil II \rceil_D) \triangleleft \$wait \triangleright R1 (R2s S_1) \diamond R1 (R2s S_2))))$

$= (R1 (R2s Q_1) ;; (\$wait \wedge (\exists \$st \cdot \lceil II \rceil_D)))$

**by** (*rel-auto, blast+*)

**also have**  $\dots = ((R1 (R2s Q_1) ;; (\exists \$st \cdot \lceil II \rceil_D)) \wedge \$wait')$

**by** (*rel-auto*)

**also from** *assms(2)* **have**  $\dots = (\exists \$st' \cdot ((R1 (R2s Q_1)) \wedge \$wait'))$

**by** (*rel-auto, blast*)

**finally show** *?thesis* .

**qed**

**moreover have**  $(R1 (R2s Q_2) ;; (\neg \$wait \wedge ((\exists \$st \cdot \lceil II \rceil_D) \triangleleft \$wait \triangleright R1 (R2s S_1) \diamond R1 (R2s S_2))))$

$= ((R1 (R2s Q_2)) ;; (R1 (R2s S_1) \diamond R1 (R2s S_2)))$

**proof** –

**have**  $(R1 (R2s Q_2) ;; (\neg \$wait \wedge ((\exists \$st \cdot \lceil II \rceil_D) \triangleleft \$wait \triangleright R1 (R2s S_1) \diamond R1 (R2s S_2))))$

$= (R1 (R2s Q_2) ;; (\neg \$wait \wedge (R1 (R2s S_1) \diamond R1 (R2s S_2))))$

**by** (*metis (no-types, lifting) cond-def conj-disj-not-abs utp-pred-laws.double-compl utp-pred-laws.inf.left-idem utp-pred-laws.sup-assoc utp-pred-laws.sup-inf-absorb*)

**also have**  $\dots = ((R1 (R2s Q_2))\llbracket \text{false}/\$wait \rrbracket ;; (R1 (R2s S_1) \diamond R1 (R2s S_2))\llbracket \text{false}/\$wait \rrbracket)$

**by** (*metis false-alt-def seqr-right-one-point upred-eq-false wait-vwb-lens*)

**also have**  $\dots = ((R1 (R2s Q_2)) ;; (R1 (R2s S_1) \diamond R1 (R2s S_2)))$

```

    by (simp add: wait'-cond-def usubst unrest assms)

  finally show ?thesis .
qed

moreover
have ((R1 (R2s Q1) ∧ $wait') ∨ ((R1 (R2s Q2)) ;; (R1 (R2s S1) ◇ R1 (R2s S2))))
  = (R1 (R2s Q1) ∨ (R1 (R2s Q2) ;; R1 (R2s S1))) ◇ ((R1 (R2s Q2) ;; R1 (R2s S2)))
  by (simp add: wait'-cond-def cond-seq-right-distr cond-and-T-integrate unrest)

ultimately show ?thesis
  by (simp add: R2s-wait'-cond R1-wait'-cond wait'-cond-seq ex-conj-contr-right unrest)
    (simp add: cond-and-T-integrate cond-seq-right-distr unrest-var wait'-cond-def)
qed

from assms(7,8) have 3: (R1 (R2s Q2) ∧ ¬ $wait') ;; R1 (¬ R2s R) = R1 (R2s Q2) ;; R1 (¬ R2s
R)
  by (rel-auto, blast, meson)

show ?thesis
  apply (subst RHS-design-composition)
  apply (simp-all add: assms)
  apply (simp add: assms wait'-cond-def unrest)
  apply (simp add: assms wait'-cond-def unrest)
  apply (simp add: 1 2 3)
  apply (simp add: R1-R2s-R2c RHS-design-lemma1)
  apply (metis R1-R2c-ex-st RHS-design-lemma1)
done
qed

theorem RHS-tri-design-composition-wp:
  assumes $ok' # P $ok' # Q1 $ok' # Q2 $ok # R $ok # S1 $ok # S2
    $wait # R $wait' # Q2 $wait # S1 $wait # S2
    P is R2c Q1 is R1 Q1 is R2c Q2 is R1 Q2 is R2c
    R is R2c S1 is R1 S1 is R2c S2 is R1 S2 is R2c
  shows  $\mathbf{R}_s(P \vdash Q_1 \diamond Q_2) ;; \mathbf{R}_s(R \vdash S_1 \diamond S_2) =$ 
     $\mathbf{R}_s(((\neg_r P) \text{ wp}_r \text{ false} \wedge Q_2 \text{ wp}_r R) \vdash (((\exists \$st' \cdot Q_1) \sqcap (Q_2 ;; S_1)) \diamond (Q_2 ;; S_2)))$  (is ?lhs =
?rhs)
proof -
  have ?lhs =  $\mathbf{R}_s((\neg R1 (\neg P) ;; R1 \text{ true} \wedge \neg Q_2 ;; R1 (\neg R)) \vdash ((\exists \$st' \cdot Q_1) \sqcap (Q_2 ;; S_1)) \diamond (Q_2$ 
;; S2))
  by (simp add: RHS-tri-design-composition assms Healthy-if R2c-healthy-R2s disj-upred-def)
    (metis (no-types, hide-lams) R1-negate-R1 R2c-healthy-R2s assms(11,16))
  also have ... = ?rhs
  by (rel-auto)
  finally show ?thesis .
qed

theorem RHS-tri-design-composition-RR-wp:
  assumes P is RR Q1 is RR Q2 is RR
    R is RR S1 is RR S2 is RR
  shows  $\mathbf{R}_s(P \vdash Q_1 \diamond Q_2) ;; \mathbf{R}_s(R \vdash S_1 \diamond S_2) =$ 
     $\mathbf{R}_s(((\neg_r P) \text{ wp}_r \text{ false} \wedge Q_2 \text{ wp}_r R) \vdash (((\exists \$st' \cdot Q_1) \sqcap (Q_2 ;; S_1)) \diamond (Q_2 ;; S_2)))$  (is ?lhs =
?rhs)
  by (simp add: RHS-tri-design-composition-wp add: closure assms unrest RR-implies-R2c)

```

**lemma** *RHS-tri-normal-design-composition*:

**assumes**

$\$ok' \# P \$ok' \# Q_1 \$ok' \# Q_2 \$ok \# R \$ok \# S_1 \$ok \# S_2$

$\$wait \# R \$wait' \# Q_2 \$wait \# S_1 \$wait \# S_2$

$P \text{ is } R2c \ Q_1 \text{ is } R1 \ Q_1 \text{ is } R2c \ Q_2 \text{ is } R1 \ Q_2 \text{ is } R2c$

$R \text{ is } R2c \ S_1 \text{ is } R1 \ S_1 \text{ is } R2c \ S_2 \text{ is } R1 \ S_2 \text{ is } R2c$

$R1 (\neg P) ;; R1(true) = R1(\neg P) \$st' \# Q_1$

**shows**  $\mathbf{R}_s(P \vdash Q_1 \diamond Q_2) ;; \mathbf{R}_s(R \vdash S_1 \diamond S_2)$

$= \mathbf{R}_s((P \wedge Q_2 \text{ wp}_r R) \vdash (Q_1 \vee (Q_2 ;; S_1)) \diamond (Q_2 ;; S_2))$

**proof** –

**have**  $\mathbf{R}_s(P \vdash Q_1 \diamond Q_2) ;; \mathbf{R}_s(R \vdash S_1 \diamond S_2) =$

$\mathbf{R}_s((R1 (\neg P) \text{ wp}_r \text{ false} \wedge Q_2 \text{ wp}_r R) \vdash ((\exists \$st' \cdot Q_1) \sqcap (Q_2 ;; S_1)) \diamond (Q_2 ;; S_2))$

**by** (*simp-all add: RHS-tri-design-composition-wp rea-not-def assms unrest*)

**also have**  $\dots = \mathbf{R}_s((P \wedge Q_2 \text{ wp}_r R) \vdash (Q_1 \vee (Q_2 ;; S_1)) \diamond (Q_2 ;; S_2))$

**by** (*simp add: assms wp-rea-def ex-unrest, rel-auto*)

**finally show** *?thesis* .

**qed**

**lemma** *RHS-tri-normal-design-composition'* [*rdes-def*]:

**assumes**  $P \text{ is } RC \ Q_1 \text{ is } RR \ \$st' \# Q_1 \ Q_2 \text{ is } RR \ R \text{ is } RR \ S_1 \text{ is } RR \ S_2 \text{ is } RR$

**shows**  $\mathbf{R}_s(P \vdash Q_1 \diamond Q_2) ;; \mathbf{R}_s(R \vdash S_1 \diamond S_2)$

$= \mathbf{R}_s((P \wedge Q_2 \text{ wp}_r R) \vdash (Q_1 \vee (Q_2 ;; S_1)) \diamond (Q_2 ;; S_2))$

**proof** –

**have**  $R1 (\neg P) ;; R1 \text{ true} = R1(\neg P)$

**using** *RC-implies-RC1[OF assms(1)]*

**by** (*simp add: Healthy-def RC1-def rea-not-def*)

(*metis R1-negate-R1 R1-seqr utp-pred-laws.double-compl*)

**thus** *?thesis*

**by** (*simp add: RHS-tri-normal-design-composition assms closure unrest RR-implies-R2c*)

**qed**

**lemma** *RHS-tri-design-right-unit-lemma*:

**assumes**  $\$ok' \# P \$ok' \# Q \$ok' \# R \$wait' \# R$

**shows**  $\mathbf{R}_s(P \vdash Q \diamond R) ;; II_R = \mathbf{R}_s((\neg_r (\neg_r P) ;; \text{true}_r) \vdash ((\exists \$st' \cdot Q) \diamond R))$

**proof** –

**have**  $\mathbf{R}_s(P \vdash Q \diamond R) ;; II_R = \mathbf{R}_s(P \vdash Q \diamond R) ;; \mathbf{R}_s(\text{true} \vdash \text{false} \diamond (\$tr' =_u \$tr \wedge [II]_R))$

**by** (*simp add: srdes-skip-tri-design, rel-auto*)

**also have**  $\dots = \mathbf{R}_s((\neg R1 (\neg R2s P) ;; R1 \text{ true}) \vdash (\exists \$st' \cdot Q) \diamond (R1 (R2s R) ;; R1 (R2s (\$tr' =_u \$tr \wedge [II]_R))))$

**by** (*simp-all add: RHS-tri-design-composition assms unrest R2s-true R1-false R2s-false*)

**also have**  $\dots = \mathbf{R}_s((\neg R1 (\neg R2s P) ;; R1 \text{ true}) \vdash (\exists \$st' \cdot Q) \diamond R1 (R2s R))$

**proof** –

**from** *assms(3,4)* **have**  $(R1 (R2s R) ;; R1 (R2s (\$tr' =_u \$tr \wedge [II]_R))) = R1 (R2s R)$

**by** (*rel-auto, metis (no-types, lifting) minus-zero-eq, meson order-refl trace-class.diff-cancel*)

**thus** *?thesis*

**by** *simp*

**qed**

**also have**  $\dots = \mathbf{R}_s((\neg (\neg P) ;; R1 \text{ true}) \vdash ((\exists \$st' \cdot Q) \diamond R))$

**by** (*metis (no-types, lifting) R1-R2s-R1-true-lemma R1-R2s-R2c R2c-not RHS-design-R2c-pre RHS-design-neg-R1-pre RHS-design-post-R1 RHS-design-post-R2s*)

**also have**  $\dots = \mathbf{R}_s((\neg_r (\neg_r P) ;; \text{true}_r) \vdash ((\exists \$st' \cdot Q) \diamond R))$

**by** (*rel-auto*)

**finally show** *?thesis* .

**qed**



**lemma** *SRD-composition-wp*:

**assumes**  $P$  is SRD  $Q$  is SRD

**shows**  $(P ;; Q) = \mathbf{R}_s(((\neg_r \text{pre}_R P) \text{wp}_r \text{false} \wedge \text{post}_R P \text{wp}_r \text{pre}_R Q) \vdash$   
 $((\exists \$st' \cdot \text{peri}_R P) \vee (\text{post}_R P ;; \text{peri}_R Q)) \diamond (\text{post}_R P ;; \text{post}_R Q))$

**(is ?lhs = ?rhs)**

**proof** –

**have**  $(P ;; Q) = (\mathbf{R}_s(\text{pre}_R(P) \vdash \text{peri}_R(P) \diamond \text{post}_R(P)) ;; \mathbf{R}_s(\text{pre}_R(Q) \vdash \text{peri}_R(Q) \diamond \text{post}_R(Q)))$

**by** (*simp add: SRD-reactive-tri-design assms(1) assms(2)*)

**also from** *assms*

**have** ... = ?rhs

**by** (*simp add: RHS-tri-design-composition-wp disj-upred-def unrest assms closure*)

**finally show** ?thesis .

**qed**

## 4.6 Refinement introduction laws

**lemma** *RHS-tri-design-refine*:

**assumes**  $P_1$  is RR  $P_2$  is RR  $P_3$  is RR  $Q_1$  is RR  $Q_2$  is RR  $Q_3$  is RR

**shows**  $\mathbf{R}_s(P_1 \vdash P_2 \diamond P_3) \sqsubseteq \mathbf{R}_s(Q_1 \vdash Q_2 \diamond Q_3) \longleftrightarrow 'P_1 \Rightarrow Q_1' \wedge 'P_1 \wedge Q_2 \Rightarrow P_2' \wedge 'P_1 \wedge Q_3 \Rightarrow$   
 $P_3'$

**(is ?lhs = ?rhs)**

**proof** –

**have** ?lhs  $\longleftrightarrow 'P_1 \Rightarrow Q_1' \wedge 'P_1 \wedge Q_2 \diamond Q_3 \Rightarrow P_2 \diamond P_3'$

**by** (*simp add: RHS-design-refine assms closure RR-implies-R2c unrest ex-unrest*)

**also have** ...  $\longleftrightarrow 'P_1 \Rightarrow Q_1' \wedge '(P_1 \wedge Q_2) \diamond (P_1 \wedge Q_3) \Rightarrow P_2 \diamond P_3'$

**by** (*rel-auto*)

**also have** ...  $\longleftrightarrow 'P_1 \Rightarrow Q_1' \wedge '((P_1 \wedge Q_2) \diamond (P_1 \wedge Q_3) \Rightarrow P_2 \diamond P_3)[\text{true}/\$wait']' \wedge '((P_1 \wedge Q_2) \diamond$   
 $(P_1 \wedge Q_3) \Rightarrow P_2 \diamond P_3)[\text{false}/\$wait']'$

**by** (*rel-auto, metis*)

**also have** ...  $\longleftrightarrow ?rhs$

**by** (*simp add: usubst unrest assms*)

**finally show** ?thesis .

**qed**

**lemma** *RHS-tri-design-refine'*:

**assumes**  $P_1$  is RR  $P_2$  is RR  $P_3$  is RR  $Q_1$  is RR  $Q_2$  is RR  $Q_3$  is RR

**shows**  $\mathbf{R}_s(P_1 \vdash P_2 \diamond P_3) \sqsubseteq \mathbf{R}_s(Q_1 \vdash Q_2 \diamond Q_3) \longleftrightarrow (Q_1 \sqsubseteq P_1) \wedge (P_2 \sqsubseteq (P_1 \wedge Q_2)) \wedge (P_3 \sqsubseteq (P_1$   
 $\wedge Q_3))$

**by** (*simp add: RHS-tri-design-refine assms, rel-auto*)

**lemma** *srdes-tri-refine-intro*:

**assumes**  $'P_1 \Rightarrow P_2' 'P_1 \wedge Q_2 \Rightarrow Q_1' 'P_1 \wedge R_2 \Rightarrow R_1'$

**shows**  $\mathbf{R}_s(P_1 \vdash Q_1 \diamond R_1) \sqsubseteq \mathbf{R}_s(P_2 \vdash Q_2 \diamond R_2)$

**using** *assms*

**by** (*rule-tac srdes-refine-intro, simp-all, rel-auto*)

**lemma** *srdes-tri-refine-intro'*:

**assumes**  $P_2 \sqsubseteq P_1$   $Q_1 \sqsubseteq (P_1 \wedge Q_2)$   $R_1 \sqsubseteq (P_1 \wedge R_2)$

**shows**  $\mathbf{R}_s(P_1 \vdash Q_1 \diamond R_1) \sqsubseteq \mathbf{R}_s(P_2 \vdash Q_2 \diamond R_2)$

**using** *assms*

**by** (*rule-tac srdes-tri-refine-intro, simp-all add: refBy-order*)

**lemma** *SRD-peri-under-pre*:

**assumes**  $P$  is SRD  $\$wait' \# \text{pre}_R(P)$

**shows**  $(\text{pre}_R(P) \Rightarrow_r \text{peri}_R(P)) = \text{peri}_R(P)$

**proof** –

have  $peri_R(P) =$   
 $peri_R(\mathbf{R}_s(pre_R(P) \vdash peri_R(P) \diamond post_R(P)))$   
 by (simp add: SRD-reactive-tri-design assms)  
 also have  $\dots = (pre_R P \Rightarrow_r peri_R P)$   
 by (simp add: rea-pre-RHS-design rea-peri-RHS-design assms  
 unrest usubst R1-peri-SRD R2c-preR R1-rea-impl R2c-rea-impl R2c-periR)  
 finally show ?thesis ..

**qed**

**lemma** SRD-post-under-pre:

assumes  $P$  is SRD \$wait'  $\nmid pre_R(P)$   
 shows  $(pre_R(P) \Rightarrow_r post_R(P)) = post_R(P)$

**proof** –

have  $post_R(P) =$   
 $post_R(\mathbf{R}_s(pre_R(P) \vdash peri_R(P) \diamond post_R(P)))$   
 by (simp add: SRD-reactive-tri-design assms)  
 also have  $\dots = (pre_R P \Rightarrow_r post_R P)$   
 by (simp add: rea-pre-RHS-design rea-post-RHS-design assms  
 unrest usubst R1-post-SRD R2c-preR R1-rea-impl R2c-rea-impl R2c-postR)  
 finally show ?thesis ..

**qed**

**lemma** SRD-refine-intro:

assumes  
 $P$  is SRD  $Q$  is SRD  
 $'pre_R(P) \Rightarrow pre_R(Q)'$   $'pre_R(P) \wedge peri_R(Q) \Rightarrow peri_R(P)'$   $'pre_R(P) \wedge post_R(Q) \Rightarrow post_R(P)'$   
 shows  $P \sqsubseteq Q$   
 by (metis SRD-reactive-tri-design assms(1) assms(2) assms(3) assms(4) assms(5) srdes-tri-refine-intro)

**lemma** SRD-refine-intro':

assumes  
 $P$  is SRD  $Q$  is SRD  
 $'pre_R(P) \Rightarrow pre_R(Q)'$   $peri_R(P) \sqsubseteq (pre_R(P) \wedge peri_R(Q))$   $post_R(P) \sqsubseteq (pre_R(P) \wedge post_R(Q))$   
 shows  $P \sqsubseteq Q$   
 using assms by (rule-tac SRD-refine-intro, simp-all add: refBy-order)

**lemma** SRD-eq-intro:

assumes  
 $P$  is SRD  $Q$  is SRD  $pre_R(P) = pre_R(Q)$   $peri_R(P) = peri_R(Q)$   $post_R(P) = post_R(Q)$   
 shows  $P = Q$   
 by (metis SRD-reactive-tri-design assms)

**lemma** srdes-tri-eq-iff:

assumes  $P_1$  is RR  $P_2$  is RR  $P_3$  is RR  $Q_1$  is RR  $Q_2$  is RR  $Q_3$  is RR  
 shows  $\mathbf{R}_s(P_1 \vdash P_2 \diamond P_3) = \mathbf{R}_s(Q_1 \vdash Q_2 \diamond Q_3) \longleftrightarrow (P_1 = Q_1 \wedge (P_1 \wedge Q_2) = (Q_1 \wedge P_2) \wedge (P_1 \wedge Q_3) = (Q_1 \wedge P_3))$

**proof** –

have  $\mathbf{R}_s(P_1 \vdash P_2 \diamond P_3) = \mathbf{R}_s(Q_1 \vdash Q_2 \diamond Q_3) \longleftrightarrow$   
 $(\mathbf{R}_s(P_1 \vdash P_2 \diamond P_3) \sqsubseteq \mathbf{R}_s(Q_1 \vdash Q_2 \diamond Q_3) \wedge \mathbf{R}_s(Q_1 \vdash Q_2 \diamond Q_3) \sqsubseteq \mathbf{R}_s(P_1 \vdash P_2 \diamond P_3))$   
 by fastforce  
 also have  $\dots = (Q_1 \sqsubseteq P_1 \wedge P_2 \sqsubseteq (P_1 \wedge Q_2) \wedge P_3 \sqsubseteq (P_1 \wedge Q_3) \wedge P_1 \sqsubseteq Q_1 \wedge Q_2 \sqsubseteq (Q_1 \wedge P_2) \wedge Q_3 \sqsubseteq (Q_1 \wedge P_3))$   
 by (simp add: RHS-tri-design-refine' assms)  
 also have  $\dots = (P_1 = Q_1 \wedge P_2 \sqsubseteq (P_1 \wedge Q_2) \wedge P_3 \sqsubseteq (P_1 \wedge Q_3) \wedge Q_2 \sqsubseteq (Q_1 \wedge P_2) \wedge Q_3 \sqsubseteq (Q_1 \wedge P_3))$

$P_3))$   
 by *fastforce*  
 also have ... =  $(P_1 = Q_1 \wedge (P_1 \wedge Q_2) = (Q_1 \wedge P_2) \wedge (P_1 \wedge Q_3) = (Q_1 \wedge P_3))$   
 apply (*safe, simp-all*)  
 apply (*meson eq-iff utp-pred-laws.inf-greatest utp-pred-laws.inf-le1*) +  
 apply (*metis utp-pred-laws.inf-le2*) +  
 done  
 finally show *?thesis* .  
 qed

**lemma** *srdes-tri-eq-intro*:  
 assumes  $P_1 = Q_1 \ (P_1 \wedge Q_2) = (Q_1 \wedge P_2) \ (P_1 \wedge Q_3) = (Q_1 \wedge P_3)$   
 shows  $\mathbf{R}_s(P_1 \vdash P_2 \diamond P_3) = \mathbf{R}_s(Q_1 \vdash Q_2 \diamond Q_3)$   
 by (*metis (no-types, hide-lams) assms(1) assms(2) assms(3) design-export-pre wait'-cond-conj-exchange wait'-cond-idem*)

**lemma** *srdes-tri-eq-intro'*:  
 assumes  $P_1 = Q_1 \ P_2 = Q_2 \ P_3 = Q_3$   
 shows  $\mathbf{R}_s(P_1 \vdash P_2 \diamond P_3) = \mathbf{R}_s(Q_1 \vdash Q_2 \diamond Q_3)$   
 using *assms* by (*simp*)

## 4.7 Closure laws

**lemma** *SRD-srdes-skip [closure]*:  $\Pi_R$  is SRD  
 by (*simp add: srdes-skip-def RHS-design-is-SRD unrest*)

**lemma** *SRD-seqr-closure [closure]*:

assumes  $P$  is SRD  $Q$  is SRD

shows  $(P ;; Q)$  is SRD

**proof** –

have  $(P ;; Q) = \mathbf{R}_s(((\neg_r \text{pre}_R P) \text{wp}_r \text{false} \wedge \text{post}_R P \text{wp}_r \text{pre}_R Q) \vdash$   
 $((\exists \$st' \cdot \text{peri}_R P) \vee (\text{post}_R P ;; \text{peri}_R Q)) \diamond (\text{post}_R P ;; \text{post}_R Q))$

by (*simp add: SRD-composition-wp assms(1) assms(2)*)

also have ... is SRD

by (*rule RHS-design-is-SRD, simp-all add: wp-rea-def unrest*)

finally show *?thesis* .

qed

**lemma** *SRD-power-Suc [closure]*:  $P$  is SRD  $\implies P^\wedge(\text{Suc } n)$  is SRD

**proof** (*induct n*)

case 0

then show *?case*

by (*simp*)

**next**

case  $(\text{Suc } n)$

then show *?case*

using *SRD-seqr-closure* by (*simp add: SRD-seqr-closure upred-semiring.power-Suc*)

qed

**lemma** *SRD-power-comp [closure]*:  $P$  is SRD  $\implies P ;; P^\wedge n$  is SRD

by (*metis SRD-power-Suc upred-semiring.power-Suc*)

**lemma** *uplus-SRD-closed [closure]*:  $P$  is SRD  $\implies P^+$  is SRD

by (*simp add: uplus-power-def closure*)

**lemma** *SRD-Sup-closure [closure]*:

assumes  $A \subseteq \llbracket SRD \rrbracket_H$   $A \neq \{\}$   
 shows  $(\sqcap A)$  is  $SRD$   
 proof –  
 have  $SRD (\sqcap A) = (\sqcap (SRD \text{ `} A))$   
   by (*simp add: ContinuousD SRD-Continuous assms(2)*)  
 also have  $\dots = (\sqcap A)$   
   by (*simp only: Healthy-carrier-image assms*)  
 finally show ?thesis by (*simp add: Healthy-def*)  
 qed

## 4.8 Distribution laws

**lemma** *RHS-tri-design-choice* [rdes-def]:

$\mathbf{R}_s(P_1 \vdash P_2 \diamond P_3) \sqcap \mathbf{R}_s(Q_1 \vdash Q_2 \diamond Q_3) = \mathbf{R}_s((P_1 \wedge Q_1) \vdash (P_2 \vee Q_2) \diamond (P_3 \vee Q_3))$   
 apply (*simp add: RHS-design-choice*)  
 apply (*rule cong[of  $\mathbf{R}_s$   $\mathbf{R}_s$ ]*)  
 apply (*simp*)  
 apply (*rel-auto*)  
 done

**lemma** *RHS-tri-design-disj* [rdes-def]:

$(\mathbf{R}_s(P_1 \vdash P_2 \diamond P_3) \vee \mathbf{R}_s(Q_1 \vdash Q_2 \diamond Q_3)) = \mathbf{R}_s((P_1 \wedge Q_1) \vdash (P_2 \vee Q_2) \diamond (P_3 \vee Q_3))$   
 by (*simp add: RHS-tri-design-choice disj-upred-def*)

**lemma** *RHS-tri-design-sup* [rdes-def]:

$\mathbf{R}_s(P_1 \vdash P_2 \diamond P_3) \sqcup \mathbf{R}_s(Q_1 \vdash Q_2 \diamond Q_3) = \mathbf{R}_s((P_1 \vee Q_1) \vdash ((P_1 \Rightarrow_r P_2) \wedge (Q_1 \Rightarrow_r Q_2)) \diamond ((P_1 \Rightarrow_r P_3) \wedge (Q_1 \Rightarrow_r Q_3)))$   
 by (*simp add: RHS-design-sup, rel-auto*)

**lemma** *RHS-tri-design-conj* [rdes-def]:

$(\mathbf{R}_s(P_1 \vdash P_2 \diamond P_3) \wedge \mathbf{R}_s(Q_1 \vdash Q_2 \diamond Q_3)) = \mathbf{R}_s((P_1 \vee Q_1) \vdash ((P_1 \Rightarrow_r P_2) \wedge (Q_1 \Rightarrow_r Q_2)) \diamond ((P_1 \Rightarrow_r P_3) \wedge (Q_1 \Rightarrow_r Q_3)))$   
 by (*simp add: RHS-tri-design-sup conj-upred-def*)

**lemma** *SRD-UINF* [rdes-def]:

assumes  $A \neq \{\}$   $A \subseteq \llbracket SRD \rrbracket_H$   
 shows  $\sqcap A = \mathbf{R}_s((\bigwedge P \in A \cdot \text{pre}_R(P)) \vdash (\bigvee P \in A \cdot \text{peri}_R(P)) \diamond (\bigvee P \in A \cdot \text{post}_R(P)))$

proof –

have  $\sqcap A = \mathbf{R}_s(\text{pre}_R(\sqcap A) \vdash \text{peri}_R(\sqcap A) \diamond \text{post}_R(\sqcap A))$   
   by (*metis SRD-as-reactive-tri-design assms srdes-theory.healthy-inf srdes-theory.healthy-inf-def*)  
 also have  $\dots = \mathbf{R}_s((\bigwedge P \in A \cdot \text{pre}_R(P)) \vdash (\bigvee P \in A \cdot \text{peri}_R(P)) \diamond (\bigvee P \in A \cdot \text{post}_R(P)))$   
   by (*simp add: preR-INF periR-INF postR-INF assms*)  
 finally show ?thesis .

qed

**lemma** *RHS-tri-design-USUP* [rdes-def]:

assumes  $A \neq \{\}$   
 shows  $(\sqcap i \in A \cdot \mathbf{R}_s(P(i) \vdash Q(i) \diamond R(i))) = \mathbf{R}_s((\bigwedge i \in A \cdot P(i)) \vdash (\sqcap i \in A \cdot Q(i)) \diamond (\sqcap i \in A \cdot R(i)))$   
 by (*subst RHS-INF[OF assms, THEN sym], simp add: design-UINF-mem assms, rel-auto*)

**lemma** *SRD-UINF-mem*:

assumes  $A \neq \{\}$   $\bigwedge i. P\ i$  is  $SRD$   
 shows  $(\sqcap i \in A \cdot P\ i) = \mathbf{R}_s((\bigwedge i \in A \cdot \text{pre}_R(P\ i)) \vdash (\bigvee i \in A \cdot \text{peri}_R(P\ i)) \diamond (\bigvee i \in A \cdot \text{post}_R(P\ i)))$   
 (is ?lhs = ?rhs)

proof –

**have**  $?lhs = (\prod (P \text{ ' } A))$   
**by** (*rel-auto*)  
**also have**  $\dots = \mathbf{R}_s((\prod Pa \in P \text{ ' } A \cdot pre_R Pa) \vdash (\prod Pa \in P \text{ ' } A \cdot peri_R Pa) \diamond (\prod Pa \in P \text{ ' } A \cdot post_R Pa))$   
**by** (*subst rdes-def, simp-all add: assms image-subsetI*)  
**also have**  $\dots = ?rhs$   
**by** (*rel-auto*)  
**finally show**  $?thesis$  .  
**qed**

**lemma** *RHS-tri-design-UINF-ind* [*rdes-def*]:  
 $(\prod i \cdot \mathbf{R}_s(P_1(i) \vdash P_2(i) \diamond P_3(i))) = \mathbf{R}_s((\bigwedge i \cdot P_1 i) \vdash (\bigvee i \cdot P_2(i)) \diamond (\bigvee i \cdot P_3(i)))$   
**by** (*rel-auto*)

**lemma** *cond-srea-form* [*rdes-def*]:  
 $\mathbf{R}_s(P \vdash Q_1 \diamond Q_2) \triangleleft b \triangleright_R \mathbf{R}_s(R \vdash S_1 \diamond S_2) =$   
 $\mathbf{R}_s((P \triangleleft b \triangleright_R R) \vdash (Q_1 \triangleleft b \triangleright_R S_1) \diamond (Q_2 \triangleleft b \triangleright_R S_2))$   
**proof** –  
**have**  $\mathbf{R}_s(P \vdash Q_1 \diamond Q_2) \triangleleft b \triangleright_R \mathbf{R}_s(R \vdash S_1 \diamond S_2) = \mathbf{R}_s(P \vdash Q_1 \diamond Q_2) \triangleleft R2c(\lceil b \rceil_{S<}) \triangleright \mathbf{R}_s(R \vdash S_1 \diamond S_2)$   
**by** (*pred-auto*)  
**also have**  $\dots = \mathbf{R}_s(P \vdash Q_1 \diamond Q_2 \triangleleft b \triangleright_R R \vdash S_1 \diamond S_2)$   
**by** (*simp add: RHS-cond lift-cond-srea-def*)  
**also have**  $\dots = \mathbf{R}_s((P \triangleleft b \triangleright_R R) \vdash (Q_1 \diamond Q_2 \triangleleft b \triangleright_R S_1 \diamond S_2))$   
**by** (*simp add: design-condr lift-cond-srea-def*)  
**also have**  $\dots = \mathbf{R}_s((P \triangleleft b \triangleright_R R) \vdash (Q_1 \triangleleft b \triangleright_R S_1) \diamond (Q_2 \triangleleft b \triangleright_R S_2))$   
**by** (*rule cong[of  $\mathbf{R}_s$   $\mathbf{R}_s$ ], simp, rel-auto*)  
**finally show**  $?thesis$  .  
**qed**

**lemma** *SRD-cond-srea* [*closure*]:  
**assumes**  $P$  is *SRD*  $Q$  is *SRD*  
**shows**  $P \triangleleft b \triangleright_R Q$  is *SRD*  
**proof** –  
**have**  $P \triangleleft b \triangleright_R Q = \mathbf{R}_s(pre_R(P) \vdash peri_R(P) \diamond post_R(P)) \triangleleft b \triangleright_R \mathbf{R}_s(pre_R(Q) \vdash peri_R(Q) \diamond post_R(Q))$   
**by** (*simp add: SRD-reactive-tri-design assms*)  
**also have**  $\dots = \mathbf{R}_s((pre_R P \triangleleft b \triangleright_R pre_R Q) \vdash (peri_R P \triangleleft b \triangleright_R peri_R Q) \diamond (post_R P \triangleleft b \triangleright_R post_R Q))$   
**by** (*simp add: cond-srea-form*)  
**also have**  $\dots$  is *SRD*  
**by** (*simp add: RHS-tri-design-is-SRD lift-cond-srea-def unrest*)  
**finally show**  $?thesis$  .  
**qed**

## 4.9 Algebraic laws

**lemma** *SRD-left-unit*:  
**assumes**  $P$  is *SRD*  
**shows**  $\Pi_R ;; P = P$   
**by** (*simp add: SRD-composition-wp closure rdes wp C1 R1-negate-R1 R1-false rpred trace-ident-left-periR trace-ident-left-postR SRD-reactive-tri-design assms*)

**lemma** *skip-srea-self-unit* [*simp*]:  
 $\Pi_R ;; \Pi_R = \Pi_R$   
**by** (*simp add: SRD-left-unit closure*)

**lemma** *SRD-right-unit-tri-lemma*:

assumes  $P$  is *SRD*

shows  $P \;; \; II_R = \mathbf{R}_s ((\neg_r \text{pre}_R P) \text{wp}_r \text{false} \vdash (\exists \$st' \cdot \text{peri}_R P) \diamond \text{post}_R P)$

by (simp add: *SRD-composition-wp closure rdes wp rpred trace-ident-right-postR assms*)

**lemma** *Miracle-left-zero*:

assumes  $P$  is *SRD*

shows *Miracle*  $\;; \; P = \text{Miracle}$

**proof** –

have *Miracle*  $\;; \; P = \mathbf{R}_s(\text{true} \vdash \text{false}) \;; \; \mathbf{R}_s(\text{pre}_R(P) \vdash \text{cmt}_R(P))$

by (simp add: *Miracle-def SRD-reactive-design-alt assms*)

also have  $\dots = \mathbf{R}_s(\text{true} \vdash \text{false})$

by (simp add: *RHS-design-composition unrest R1-false R2s-false R2s-true*)

also have  $\dots = \text{Miracle}$

by (simp add: *Miracle-def*)

finally show *?thesis* .

**qed**

**lemma** *Chaos-left-zero*:

assumes  $P$  is *SRD*

shows (*Chaos*  $\;; \; P$ ) = *Chaos*

**proof** –

have *Chaos*  $\;; \; P = \mathbf{R}_s(\text{false} \vdash \text{true}) \;; \; \mathbf{R}_s(\text{pre}_R(P) \vdash \text{cmt}_R(P))$

by (simp add: *Chaos-def SRD-reactive-design-alt assms*)

also have  $\dots = \mathbf{R}_s ((\neg R1 \text{true} \wedge \neg (R1 \text{true} \wedge \neg \$wait')) \;; \; R1 (\neg R2s (\text{pre}_R P))) \vdash$   
 $R1 \text{true} \;; \; ((\exists \$st \cdot [II]_D) \triangleleft \$wait \triangleright R1 (R2s (\text{cmt}_R P)))$

by (simp add: *RHS-design-composition unrest R2s-false R2s-true R1-false*)

also have  $\dots = \mathbf{R}_s ((\text{false} \wedge \neg (R1 \text{true} \wedge \neg \$wait')) \;; \; R1 (\neg R2s (\text{pre}_R P))) \vdash$   
 $R1 \text{true} \;; \; ((\exists \$st \cdot [II]_D) \triangleleft \$wait \triangleright R1 (R2s (\text{cmt}_R P)))$

by (simp add: *RHS-design-conj-neg-R1-pre*)

also have  $\dots = \mathbf{R}_s(\text{true})$

by (simp add: *design-false-pre*)

also have  $\dots = \mathbf{R}_s(\text{false} \vdash \text{true})$

by (simp add: *design-def*)

also have  $\dots = \text{Chaos}$

by (simp add: *Chaos-def*)

finally show *?thesis* .

**qed**

**lemma** *SRD-right-Chaos-tri-lemma*:

assumes  $P$  is *SRD*

shows  $P \;; \; \text{Chaos} = \mathbf{R}_s (((\neg_r \text{pre}_R P) \text{wp}_r \text{false} \wedge \text{post}_R P \text{wp}_r \text{false}) \vdash (\exists \$st' \cdot \text{peri}_R P) \diamond \text{false})$

by (simp add: *SRD-composition-wp closure rdes assms wp, rel-auto*)

**lemma** *SRD-right-Miracle-tri-lemma*:

assumes  $P$  is *SRD*

shows  $P \;; \; \text{Miracle} = \mathbf{R}_s ((\neg_r \text{pre}_R P) \text{wp}_r \text{false} \vdash (\exists \$st' \cdot \text{peri}_R P) \diamond \text{false})$

by (simp add: *SRD-composition-wp closure rdes assms wp, rel-auto*)

Stateful reactive designs are left unital

**interpretation** *srdes-left-unital*: *utp-theory-left-unital SRD II<sub>R</sub>*

by (unfold-locale, simp-all add: *closure SRD-left-unit*)

## 4.10 Recursion laws

**lemma** *mono-srd-iter*:

**assumes**  $\text{mono } F \ F \in \llbracket \text{SRD} \rrbracket_H \rightarrow \llbracket \text{SRD} \rrbracket_H$   
**shows**  $\text{mono } (\lambda X. \mathbf{R}_s(\text{pre}_R(F \ X) \vdash \text{peri}_R(F \ X) \diamond \text{post}_R(F \ X)))$   
**apply** (rule monoI)  
**apply** (rule srdes-tri-refine-intro')  
**apply** (meson assms(1) monoE preR-antitone utp-pred-laws.le-infI2)  
**apply** (meson assms(1) monoE periR-monotone utp-pred-laws.le-infI2)  
**apply** (meson assms(1) monoE postR-monotone utp-pred-laws.le-infI2)  
**done**

**lemma** *mu-srd-SRD*:

**assumes**  $\text{mono } F \ F \in \llbracket \text{SRD} \rrbracket_H \rightarrow \llbracket \text{SRD} \rrbracket_H$   
**shows**  $(\mu \ X \cdot \mathbf{R}_s(\text{pre}_R(F \ X) \vdash \text{peri}_R(F \ X) \diamond \text{post}_R(F \ X))) \text{ is SRD}$   
**apply** (subst gfp-unfold)  
**apply** (simp add: mono-srd-iter assms)  
**apply** (rule RHS-tri-design-is-SRD)  
**apply** (simp-all add: unrest)  
**done**

**lemma** *mu-srd-iter*:

**assumes**  $\text{mono } F \ F \in \llbracket \text{SRD} \rrbracket_H \rightarrow \llbracket \text{SRD} \rrbracket_H$   
**shows**  $(\mu \ X \cdot \mathbf{R}_s(\text{pre}_R(F(X)) \vdash \text{peri}_R(F(X)) \diamond \text{post}_R(F(X)))) = F(\mu \ X \cdot \mathbf{R}_s(\text{pre}_R(F(X)) \vdash \text{peri}_R(F(X)) \diamond \text{post}_R(F(X))))$   
**apply** (subst gfp-unfold)  
**apply** (simp add: mono-srd-iter assms)  
**apply** (subst SRD-as-reactive-tri-design[THEN sym])  
**apply** (simp add: Healthy-apply-closed SRD-as-reactive-design SRD-reactive-design-alt assms(1) assms(2) mu-srd-SRD)  
**done**

**lemma** *mu-srd-form*:

**assumes**  $\text{mono } F \ F \in \llbracket \text{SRD} \rrbracket_H \rightarrow \llbracket \text{SRD} \rrbracket_H$   
**shows**  $\mu_R \ F = (\mu \ X \cdot \mathbf{R}_s(\text{pre}_R(F(X)) \vdash \text{peri}_R(F(X)) \diamond \text{post}_R(F(X))))$   
**proof** –  
**have** 1:  $F(\mu \ X \cdot \mathbf{R}_s(\text{pre}_R(F \ X) \vdash \text{peri}_R(F \ X) \diamond \text{post}_R(F \ X))) \text{ is SRD}$   
**by** (simp add: Healthy-apply-closed assms(1) assms(2) mu-srd-SRD)  
**have** 2:  $\text{Mono}_{\text{utp-order}} \text{SRD } F$   
**by** (simp add: assms(1) mono-Monotone-utp-order)  
**hence** 3:  $\mu_R \ F = F(\mu_R \ F)$   
**by** (simp add: srdes-theory.LFP-unfold[THEN sym] assms)  
**hence**  $\mathbf{R}_s(\text{pre}_R(F(F(\mu_R \ F))) \vdash \text{peri}_R(F(F(\mu_R \ F))) \diamond \text{post}_R(F(F(\mu_R \ F)))) = \mu_R \ F$   
**using** SRD-reactive-tri-design **by** force  
**hence**  $(\mu \ X \cdot \mathbf{R}_s(\text{pre}_R(F \ X) \vdash \text{peri}_R(F \ X) \diamond \text{post}_R(F \ X))) \sqsubseteq F(\mu_R \ F)$   
**by** (simp add: 2 srdes-theory.weak.LFP-lemma3 gfp-upperbound assms)  
**thus** ?thesis  
**using** assms 1 3 srdes-theory.weak.LFP-lowerbound eq-iff mu-srd-iter  
**by** (metis (mono-tags, lifting))  
**qed**

**lemma** *Monotonic-SRD-comp [closure]*: *Monotonic ((;;) P  $\circ$  SRD)*

**by** (simp add: mono-def R1-R2c-is-R2 R2-mono R3h-mono RD1-mono RD2-mono RHS-def SRD-def seqr-mono)

**end**

## 5 Normal Reactive Designs

```

theory utp-rdes-normal
imports
  utp-rdes-triples
  UTP-KAT.utp-kleene
begin

```

This additional healthiness condition is analogous to H3

```

definition RD3 where
[upred-defs]: RD3(P) = P ;; IIR

```

```

lemma RD3-idem: RD3(RD3(P)) = RD3(P)

```

```

proof –

```

```

  have a: IIR ;; IIR = IIR
    by (simp add: SRD-left-unit SRD-srdes-skip)
  show ?thesis
    by (simp add: RD3-def seqr-assoc a)

```

```

qed

```

```

lemma RD3-Idempotent [closure]: Idempotent RD3
  by (simp add: Idempotent-def RD3-idem)

```

```

lemma RD3-continuous: RD3( $\sqcap$  A) = ( $\sqcap$  P ∈ A. RD3(P))
  by (simp add: RD3-def seq-Sup-distr)

```

```

lemma RD3-Continuous [closure]: Continuous RD3
  by (simp add: Continuous-def RD3-continuous)

```

```

lemma RD3-right-subsumes-RD2: RD2(RD3(P)) = RD3(P)

```

```

proof –

```

```

  have a: IIR ;; J = IIR
    by (rel-auto)
  show ?thesis
    by (metis (no-types, hide-lams) H2-def RD2-def RD3-def a seqr-assoc)

```

```

qed

```

```

lemma RD3-left-subsumes-RD2: RD3(RD2(P)) = RD3(P)

```

```

proof –

```

```

  have a: J ;; IIR = IIR
    by (rel-simp, safe, blast+)
  show ?thesis
    by (metis (no-types, hide-lams) H2-def RD2-def RD3-def a seqr-assoc)

```

```

qed

```

```

lemma RD3-implies-RD2: P is RD3  $\implies$  P is RD2
  by (metis Healthy-def RD3-right-subsumes-RD2)

```

```

lemma RD3-intro-pre:

```

```

  assumes P is SRD ( $\neg_r$  preR(P)) ;; truer = ( $\neg_r$  preR(P)) $st'  $\#$  periR(P)
  shows P is RD3

```

```

proof –

```

```

  have RD3(P) = Rs (( $\neg_r$  preR P) wpr false  $\vdash$  ( $\exists$  $st' · periR P)  $\diamond$  postR P)
    by (simp add: RD3-def SRD-right-unit-tri-lemma assms)
  also have ... = Rs (( $\neg_r$  preR P) wpr false  $\vdash$  periR P  $\diamond$  postR P)
    by (simp add: assms(3) ex-unrest)

```



also have ... =  $\mathbf{R}_s((\neg_r \text{pre}_R P) \text{wp}_r \text{false} \vdash \text{cmt}_R P)$   
 by (simp add: wait'-cond-peri-post-cmt)  
 also have ... =  $\mathbf{R}_s(\text{pre}_R P \vdash \text{cmt}_R P)$   
 by (simp add: assms(2) rpred wp-rea-def R1-preR)  
 finally show ?thesis  
 by (metis Healthy-def SRD-as-reactive-design assms(1))  
 qed

**lemma** *RHS-tri-design-right-unit-lemma:*

assumes  $\$ok' \# P \$ok' \# Q \$ok' \# R \$wait' \# R$   
 shows  $\mathbf{R}_s(P \vdash Q \diamond R) ;; II_R = \mathbf{R}_s((\neg_r (\neg_r P) ;; \text{true}_r) \vdash ((\exists \$st' \cdot Q) \diamond R))$   
 proof -  
 have  $\mathbf{R}_s(P \vdash Q \diamond R) ;; II_R = \mathbf{R}_s(P \vdash Q \diamond R) ;; \mathbf{R}_s(\text{true} \vdash \text{false} \diamond (\$tr' =_u \$tr \wedge [II]_R))$   
 by (simp add: srdes-skip-tri-design, rel-auto)  
 also have ... =  $\mathbf{R}_s((\neg R1 (\neg R2s P) ;; R1 \text{true}) \vdash (\exists \$st' \cdot Q) \diamond (R1 (R2s R) ;; R1 (R2s (\$tr' =_u \$tr \wedge [II]_R))))$   
 by (simp-all add: RHS-tri-design-composition assms unrest R2s-true R1-false R2s-false)  
 also have ... =  $\mathbf{R}_s((\neg R1 (\neg R2s P) ;; R1 \text{true}) \vdash (\exists \$st' \cdot Q) \diamond R1 (R2s R))$   
 proof -  
 from assms(3,4) have  $(R1 (R2s R) ;; R1 (R2s (\$tr' =_u \$tr \wedge [II]_R))) = R1 (R2s R)$   
 by (rel-auto, metis (no-types, lifting) minus-zero-eq, meson order-refl trace-class.diff-cancel)  
 thus ?thesis  
 by simp  
 qed  
 also have ... =  $\mathbf{R}_s((\neg (\neg P) ;; R1 \text{true}) \vdash ((\exists \$st' \cdot Q) \diamond R))$   
 by (metis (no-types, lifting) R1-R2s-R1-true-lemma R1-R2s-R2c R2c-not RHS-design-R2c-pre RHS-design-neg-R1-pre RHS-design-post-R1 RHS-design-post-R2s)  
 also have ... =  $\mathbf{R}_s((\neg_r (\neg_r P) ;; \text{true}_r) \vdash ((\exists \$st' \cdot Q) \diamond R))$   
 by (rel-auto)  
 finally show ?thesis .  
 qed

**lemma** *RHS-tri-design-RD3-intro:*

assumes  
 $\$ok' \# P \$ok' \# Q \$ok' \# R \$st' \# Q \$wait' \# R$   
 $P \text{ is } R1 (\neg_r P) ;; \text{true}_r = (\neg_r P)$   
 shows  $\mathbf{R}_s(P \vdash Q \diamond R) \text{ is } RD3$   
 apply (simp add: Healthy-def RD3-def)  
 apply (subst RHS-tri-design-right-unit-lemma)  
 apply (simp-all add: assms ex-unrest rpred)  
 done

RD3 reactive designs are those whose assumption can be written as a conjunction of a precondition on (undashed) program variables, and a negated statement about the trace. The latter allows us to state that certain events must not occur in the trace – which are effectively safety properties.

**lemma** *R1-right-unit-lemma:*

$\llbracket \text{out}\alpha \# b; \text{out}\alpha \# e \rrbracket \implies (\neg_r b \vee \$tr \hat{=}^u e \leq_u \$tr') ;; R1(\text{true}) = (\neg_r b \vee \$tr \hat{=}^u e \leq_u \$tr')$   
 by (rel-auto, blast, metis (no-types, lifting) dual-order.trans)

**lemma** *RHS-tri-design-RD3-intro-form:*

assumes  
 $\text{out}\alpha \# b \text{ out}\alpha \# e \$ok' \# Q \$st' \# Q \$ok' \# R \$wait' \# R$   
 shows  $\mathbf{R}_s((b \wedge \neg_r \$tr \hat{=}^u e \leq_u \$tr') \vdash Q \diamond R) \text{ is } RD3$   
 apply (rule RHS-tri-design-RD3-intro)

apply (simp-all add: assms unrest closure rpred)  
 apply (subst R1-right-unit-lemma)  
 apply (simp-all add: assms unrest)  
 done

**definition** *NSRD* :: ('s, 't :: trace, 'α) hrel-rsp  $\Rightarrow$  ('s, 't, 'α) hrel-rsp  
**where** [upred-defs]: *NSRD* = *RD1*  $\circ$  *RD3*  $\circ$  *RHS*

**lemma** *RD1-RD3-commute*: *RD1*(*RD3*(*P*)) = *RD3*(*RD1*(*P*))  
**by** (rel-auto, blast+)

**lemma** *NSRD-is-SRD* [closure]: *P* is *NSRD*  $\implies$  *P* is *SRD*  
**by** (simp add: Healthy-def NSRD-def SRD-def, metis Healthy-def RD1-RD3-commute RD2-RHS-commute RD3-def RD3-right-subsumes-RD2 SRD-def SRD-idem SRD-seqr-closure SRD-srdes-skip)

**lemma** *NSRD-elim* [RD-elim]:  
 $\llbracket P \text{ is } NSRD; Q(\mathbf{R}_s(\text{pre}_R(P) \vdash \text{peri}_R(P) \diamond \text{post}_R(P))) \rrbracket \implies Q(P)$   
**by** (simp add: RD-elim closure)

**lemma** *NSRD-idem*: *NSRD*(*NSRD*(*P*)) = *NSRD*(*P*)  
**by** (metis (no-types, hide-lams) Healthy-def NSRD-def RD1-RD2-commute RD1-RD3-commute RD1-RHS-commute RD1-idem RD2-RHS-commute RD2-idem RD3-def RD3-idem RD3-left-subsumes-RD2 RHS-idem SRD-def comp-apply fun.map-comp srdes-left-unital.Healthy-Sequence srdes-left-unital.Healthy-Unit)

**lemma** *NSRD-Idempotent* [closure]: *Idempotent NSRD*  
**by** (simp add: Idempotent-def NSRD-idem)

**lemma** *NSRD-Continuous* [closure]: *Continuous NSRD*  
**by** (simp add: Continuous-comp NSRD-def RD1-Continuous RD3-Continuous RHS-Continuous)

**lemma** *NSRD-form*:  
 $NSRD(P) = \mathbf{R}_s((\neg_r (\neg_r \text{pre}_R(P)) ;; R1 \text{ true}) \vdash ((\exists \$st' \cdot \text{peri}_R(P)) \diamond \text{post}_R(P)))$   
**proof** –  
**have** *NSRD*(*P*) = *RD3*(*SRD*(*P*))  
**by** (metis (no-types, lifting) NSRD-def RD1-RD3-commute RD3-left-subsumes-RD2 SRD-def comp-def)  
**also have** ... = *RD3*( $\mathbf{R}_s(\text{pre}_R(P) \vdash \text{peri}_R(P) \diamond \text{post}_R(P))$ )  
**by** (simp add: SRD-as-reactive-tri-design)  
**also have** ... =  $\mathbf{R}_s(\text{pre}_R(P) \vdash \text{peri}_R(P) \diamond \text{post}_R(P)) ;; II_R$   
**by** (simp add: RD3-def)  
**also have** ... =  $\mathbf{R}_s((\neg_r (\neg_r \text{pre}_R(P)) ;; R1 \text{ true}) \vdash ((\exists \$st' \cdot \text{peri}_R(P)) \diamond \text{post}_R(P)))$   
**by** (simp add: RHS-tri-design-right-unit-lemma unrest)  
**finally show** ?thesis .  
**qed**

**lemma** *NSRD-healthy-form*:  
**assumes** *P* is *NSRD*  
**shows**  $\mathbf{R}_s((\neg_r (\neg_r \text{pre}_R(P)) ;; R1 \text{ true}) \vdash ((\exists \$st' \cdot \text{peri}_R(P)) \diamond \text{post}_R(P))) = P$   
**by** (metis Healthy-def NSRD-form assms)

**lemma** *NSRD-Sup-closure* [closure]:  
**assumes**  $A \subseteq \llbracket NSRD \rrbracket_H$   $A \neq \{\}$   
**shows**  $\sqcap A$  is *NSRD*

**proof** –  
**have** *NSRD* ( $\sqcap A$ ) = ( $\sqcap$  (*NSRD* 'A))  
**by** (simp add: ContinuousD NSRD-Continuous assms(2))

also have ... = ( $\prod$  A)  
 by (simp only: Healthy-carrier-image assms)  
 finally show ?thesis by (simp add: Healthy-def)  
 qed

**lemma** *intChoice-NSRD-closed* [closure]:  
 assumes *P is NSRD Q is NSRD*  
 shows *P  $\sqcap$  Q is NSRD*  
 using *NSRD-Sup-closure*[of {P, Q}] by (simp add: assms)

**lemma** *NSRD-SUP-closure* [closure]:  
 $\llbracket \bigwedge i. i \in A \implies P(i) \text{ is NSRD}; A \neq \{\} \rrbracket \implies (\prod_{i \in A}. P(i)) \text{ is NSRD}$   
 by (rule *NSRD-Sup-closure*, auto)

**lemma** *NSRD-neg-pre-unit*:  
 assumes *P is NSRD*  
 shows  $(\neg_r \text{pre}_R(P)) ;; \text{true}_r = (\neg_r \text{pre}_R(P))$   
**proof** –  
 have  $(\neg_r \text{pre}_R(P)) = (\neg_r \text{pre}_R(\mathbf{R}_s((\neg_r (\neg_r \text{pre}_R(P)) ;; R1 \text{true}) \vdash ((\exists \$st' \cdot \text{peri}_R(P)) \diamond \text{post}_R(P))))$   
 by (simp add: *NSRD-healthy-form* assms)  
 also have ... = *R1 (R2c (( $\neg_r \text{pre}_R P$ ) ;; R1 true))*  
 by (simp add: *rea-pre-RHS-design R1-negate-R1 R1-idem R1-rea-not' R2c-rea-not usubst rpred unrest closure*)  
 also have ... =  $(\neg_r \text{pre}_R P) ;; R1 \text{true}$   
 by (simp add: *R1-R2c-seqr-distribute closure* assms)  
 finally show ?thesis  
 by (simp add: *rea-not-def*)  
 qed

**lemma** *NSRD-neg-pre-left-zero*:  
 assumes *P is NSRD Q is R1 Q is RD1*  
 shows  $(\neg_r \text{pre}_R(P)) ;; Q = (\neg_r \text{pre}_R(P))$   
 by (metis (no-types, hide-lams) *NSRD-neg-pre-unit RD1-left-zero* assms(1) assms(2) assms(3) *seqr-assoc*)

**lemma** *NSRD-st'-unrest-peri* [unrest]:  
 assumes *P is NSRD*  
 shows  $\$st' \# \text{peri}_R(P)$   
**proof** –  
 have  $\text{peri}_R(P) = \text{peri}_R(\mathbf{R}_s((\neg_r (\neg_r \text{pre}_R(P)) ;; R1 \text{true}) \vdash ((\exists \$st' \cdot \text{peri}_R(P)) \diamond \text{post}_R(P))))$   
 by (simp add: *NSRD-healthy-form* assms)  
 also have ... = *R1 (R2c ( $\neg_r (\neg_r \text{pre}_R P)$ ) ;; R1 true  $\Rightarrow_r (\exists \$st' \cdot \text{peri}_R P)$ )*  
 by (simp add: *rea-peri-RHS-design usubst unrest*)  
 also have  $\$st' \# \dots$   
 by (simp add: *R1-def R2c-def unrest*)  
 finally show ?thesis .  
 qed

**lemma** *NSRD-wait'-unrest-pre* [unrest]:  
 assumes *P is NSRD*  
 shows  $\$wait' \# \text{pre}_R(P)$   
**proof** –  
 have  $\text{pre}_R(P) = \text{pre}_R(\mathbf{R}_s((\neg_r (\neg_r \text{pre}_R(P)) ;; R1 \text{true}) \vdash ((\exists \$st' \cdot \text{peri}_R(P)) \diamond \text{post}_R(P))))$   
 by (simp add: *NSRD-healthy-form* assms)  
 also have ... = *(R1 (R2c ( $\neg_r (\neg_r \text{pre}_R P)$ ) ;; R1 true))*  
 by (simp add: *rea-pre-RHS-design usubst unrest*)

also have  $\$st' \# \dots$   
 by (simp add: R1-def R2c-def unrest)  
 finally show ?thesis .  
 qed

lemma NSRD-st'-unrest-pre [unrest]:

assumes  $P$  is NSRD  
 shows  $\$st' \# pre_R(P)$

proof –

have  $pre_R(P) = pre_R(\mathbf{R}_s((\neg_r (\neg_r pre_R(P)) ;; R1 \text{ true}) \vdash ((\exists \$st' \cdot peri_R(P)) \diamond post_R(P))))$   
 by (simp add: NSRD-healthy-form assms)  
 also have  $\dots = R1 (R2c (\neg_r (\neg_r pre_R P) ;; R1 \text{ true}))$   
 by (simp add: rea-pre-RHS-design usubst unrest)  
 also have  $\$st' \# \dots$   
 by (simp add: R1-def R2c-def unrest)  
 finally show ?thesis .

qed

lemma NSRD-peri-under-pre [rpred]:

$P$  is NSRD  $\implies (pre_R P \Rightarrow_r peri_R P) = peri_R P$   
 by (simp add: SRD-peri-under-pre unrest closure)

lemma NSRD-post-under-pre [rpred]:

$P$  is NSRD  $\implies (pre_R P \Rightarrow_r post_R P) = post_R P$   
 by (simp add: SRD-post-under-pre unrest closure)

lemma NSRD-peri-seq-under-pre:

assumes  $P$  is NSRD  $Q$  is NSRD  
 shows  $(pre_R P \Rightarrow_r peri_R P \vee post_R P ;; peri_R Q) = (peri_R P \vee post_R P ;; peri_R Q)$   
 by (metis NSRD-peri-under-pre assms(1) rea-impl-def utp-pred-laws.disj-assoc)

lemma NSRD-postR-seq-periR-impl:

assumes  $P$  is NSRD  $Q$  is NSRD  
 shows  $(post_R P wp_r pre_R Q \Rightarrow_r (post_R P ;; peri_R Q)) = (post_R P ;; peri_R Q)$   
 by (metis NSRD-is-SRD NSRD-peri-under-pre assms postR-RR wpR-impl-post-spec)

lemma NSRD-postR-seq-postR-impl:

assumes  $P$  is NSRD  $Q$  is NSRD  
 shows  $(post_R P wp_r pre_R Q \Rightarrow_r (post_R P ;; post_R Q)) = (post_R P ;; post_R Q)$   
 by (metis NSRD-is-SRD NSRD-post-under-pre assms postR-RR wpR-impl-post-spec)

lemma NSRD-peri-under-assms:

assumes  $P$  is NSRD  $Q$  is NSRD  
 shows  $(pre_R P \wedge post_R P wp_r pre_R Q \Rightarrow_r peri_R P \vee post_R P ;; peri_R Q) = (peri_R P \vee post_R P ;; peri_R Q)$   
 by (metis (no-types, lifting) NSRD-peri-seq-under-pre assms NSRD-postR-seq-periR-impl rea-impl-conj rea-impl-disj)

lemma NSRD-peri-under-assms':

assumes  $P$  is NSRD  $Q$  is NSRD  
 shows  $(post_R P wp_r pre_R Q \Rightarrow_r peri_R P \vee post_R P ;; peri_R Q) = (peri_R P \vee post_R P ;; peri_R Q)$   
 by (simp add: NSRD-postR-seq-periR-impl assms rea-impl-disj)

lemma NSRD-post-under-assms:

assumes  $P$  is NSRD  $Q$  is NSRD

**shows**  $(pre_R P \wedge post_R P \text{ wp}_r pre_R Q \Rightarrow_r post_R P ;; post_R Q) = (pre_R P \Rightarrow_r (post_R P ;; post_R Q))$   
**by** (metis NSRD-postR-seq-postR-impl assms(1) assms(2) rea-impl-conj)

**lemma** NSRD-alt-def:  $NSRD(P) = RD3(SRD(P))$   
**by** (metis NSRD-def RD1-RD3-commute RD3-left-subsumes-RD2 SRD-def comp-eq-dest-lhs)

**lemma** preR-RR [closure]:  $P \text{ is } NSRD \implies pre_R(P) \text{ is } RR$   
**by** (rule RR-intro, simp-all add: closure unrest)

**lemma** NSRD-neg-pre-RC [closure]:  
**assumes**  $P \text{ is } NSRD$   
**shows**  $pre_R(P) \text{ is } RC$   
**by** (rule RC-intro, simp-all add: closure assms NSRD-neg-pre-unit rpred)

**lemma** NSRD-intro:  
**assumes**  $P \text{ is } SRD (\neg_r pre_R(P)) ;; true_r = (\neg_r pre_R(P)) \$st' \# peri_R(P)$   
**shows**  $P \text{ is } NSRD$

**proof** –

**have**  $NSRD(P) = \mathbf{R}_s((\neg_r (\neg_r pre_R(P)) ;; R1 true) \vdash ((\exists \$st' \cdot peri_R(P)) \diamond post_R(P)))$   
**by** (simp add: NSRD-form)  
**also have**  $\dots = \mathbf{R}_s(pre_R P \vdash peri_R P \diamond post_R P)$   
**by** (simp add: assms ex-unrest rpred closure)  
**also have**  $\dots = P$   
**by** (simp add: SRD-reactive-tri-design assms(1))  
**finally show** ?thesis  
**using** Healthy-def **by** blast

**qed**

**lemma** NSRD-intro':  
**assumes**  $P \text{ is } R2 P \text{ is } R3h P \text{ is } RD1 P \text{ is } RD3$   
**shows**  $P \text{ is } NSRD$   
**by** (metis (no-types, hide-lams) Healthy-def NSRD-def R1-R2c-is-R2 RHS-def assms comp-apply)

**lemma** NSRD-RC-intro:  
**assumes**  $P \text{ is } SRD pre_R(P) \text{ is } RC \$st' \# peri_R(P)$   
**shows**  $P \text{ is } NSRD$   
**by** (metis Healthy-def NSRD-form SRD-reactive-tri-design assms(1) assms(2) assms(3) ex-unrest rea-not-false wp-rea-RC-false wp-rea-def)

**lemma** NSRD-rdes-intro [closure]:  
**assumes**  $P \text{ is } RC Q \text{ is } RR R \text{ is } RR \$st' \# Q$   
**shows**  $\mathbf{R}_s(P \vdash Q \diamond R) \text{ is } NSRD$   
**by** (rule NSRD-RC-intro, simp-all add: rdes closure assms unrest)

**lemma** SRD-RD3-implies-NSRD:  
 $\llbracket P \text{ is } SRD; P \text{ is } RD3 \rrbracket \implies P \text{ is } NSRD$   
**by** (metis (no-types, lifting) Healthy-def NSRD-def RHS-idem SRD-healths(4) SRD-reactive-design comp-apply)

**lemma** NSRD-iff:  
 $P \text{ is } NSRD \iff ((P \text{ is } SRD) \wedge (\neg_r pre_R(P)) ;; R1(true) = (\neg_r pre_R(P)) \wedge (\$st' \# peri_R(P)))$   
**by** (meson NSRD-intro NSRD-is-SRD NSRD-neg-pre-unit NSRD-st'-unrest-peri)

**lemma** NSRD-is-RD3 [closure]:

**assumes**  $P$  is NSRD  
**shows**  $P$  is RD3  
**by** (*simp add: NSRD-is-SRD NSRD-neg-pre-unit NSRD-st'-unrest-peri RD3-intro-pre assms*)

**lemma** *NSRD-refine-elim*:

**assumes**  
 $P \sqsubseteq Q$   $P$  is NSRD  $Q$  is NSRD  
 $\llbracket 'pre_R(P) \Rightarrow pre_R(Q)'; 'pre_R(P) \wedge peri_R(Q) \Rightarrow peri_R(P)'; 'pre_R(P) \wedge post_R(Q) \Rightarrow post_R(P)'\rrbracket$   
 $\Longrightarrow R$   
**shows**  $R$   
**proof** –  
**have**  $\mathbf{R}_s(pre_R(P) \vdash peri_R(P) \diamond post_R(P)) \sqsubseteq \mathbf{R}_s(pre_R(Q) \vdash peri_R(Q) \diamond post_R(Q))$   
**by** (*simp add: NSRD-is-SRD SRD-reactive-tri-design assms(1) assms(2) assms(3)*)  
**hence** 1:  $'pre_R P \Rightarrow pre_R Q'$  **and** 2:  $'pre_R P \wedge peri_R Q \Rightarrow peri_R P'$  **and** 3:  $'pre_R P \wedge post_R Q \Rightarrow post_R P'$   
**by** (*simp-all add: RHS-tri-design-refine assms closure*)  
**with** *assms(4)* **show** *?thesis*  
**by** *simp*  
**qed**

**lemma** *NSRD-right-unit*:  $P$  is NSRD  $\Longrightarrow P ;; II_R = P$

**by** (*metis Healthy-if NSRD-is-RD3 RD3-def*)

**lemma** *NSRD-composition-wp*:

**assumes**  $P$  is NSRD  $Q$  is SRD  
**shows**  $P ;; Q =$   
 $\mathbf{R}_s((pre_R P \wedge post_R P \wp_r pre_R Q) \vdash (peri_R P \vee (post_R P ;; peri_R Q)) \diamond (post_R P ;; post_R Q))$   
**by** (*simp add: SRD-composition-wp assms NSRD-is-SRD wp-rea-def NSRD-neg-pre-unit NSRD-st'-unrest-peri R1-negate-R1 R1-preR ex-unrest rpred*)

**lemma** *preR-NSRD-seq-lemma*:

**assumes**  $P$  is NSRD  $Q$  is SRD  
**shows**  $R1(R2c(post_R P ;; (\neg_r pre_R Q))) = post_R P ;; (\neg_r pre_R Q)$   
**proof** –  
**have**  $post_R P ;; (\neg_r pre_R Q) = R1(R2c(post_R P)) ;; R1(R2c(\neg_r pre_R Q))$   
**by** (*simp add: NSRD-is-SRD R1-R2c-post-RHS R1-rea-not R2c-preR R2c-rea-not assms(1) assms(2)*)  
**also have**  $\dots = R1(R2c(post_R P ;; (\neg_r pre_R Q)))$   
**by** (*simp add: R1-seqr R2c-R1-seq calculation*)  
**finally show** *?thesis* ..  
**qed**

**lemma** *preR-NSRD-seq [rdes]*:

**assumes**  $P$  is NSRD  $Q$  is SRD  
**shows**  $pre_R(P ;; Q) = (pre_R P \wedge post_R P \wp_r pre_R Q)$   
**by** (*simp add: NSRD-composition-wp assms rea-pre-RHS-design usubst unrest wp-rea-def R2c-disj R1-disj R2c-and R2c-preR R1-R2c-commute[THEN sym] R1-extend-conj' R1-idem R2c-not closure*)  
*(metis (no-types, lifting) Healthy-def Healthy-if NSRD-is-SRD R1-R2c-commute R1-R2c-seqr-distribute R1-seqr-closure assms(1) assms(2) postR-R2c-closed postR-SRD-R1 preR-R2c-closed rea-not-R1 rea-not-R2c)*

**lemma** *periR-NSRD-seq [rdes]*:

**assumes**  $P$  is NSRD  $Q$  is NSRD  
**shows**  $peri_R(P ;; Q) = ((pre_R P \wedge post_R P \wp_r pre_R Q) \Rightarrow_r (peri_R P \vee (post_R P ;; peri_R Q)))$   
**by** (*simp add: NSRD-composition-wp assms closure rea-peri-RHS-design usubst unrest wp-rea-def*)

*R1-extend-conj' R1-disj R1-R2c-seqr-distribute R2c-disj R2c-and R2c-rea-impl R1-rea-impl'  
R2c-preR R2c-periR R1-rea-not' R2c-rea-not R1-periSRD)*

**lemma** *postR-NSRD-seq* [rdes]:

**assumes** *P is NSRD Q is NSRD*

**shows**  $\text{post}_R(P ;; Q) = ((\text{pre}_R P \wedge \text{post}_R P \text{ wp}_r \text{pre}_R Q) \Rightarrow_r (\text{post}_R P ;; \text{post}_R Q))$

**by** (*simp add: NSRD-composition-wp assms closure rea-post-RHS-design usubst unrest wp-rea-def  
R1-extend-conj' R1-disj R1-R2c-seqr-distribute R2c-disj R2c-and R2c-rea-impl R1-rea-impl'  
R2c-preR R2c-periR R1-rea-not' R2c-rea-not*)

**lemma** *NSRD-seqr-closure* [closure]:

**assumes** *P is NSRD Q is NSRD*

**shows**  $(P ;; Q) \text{ is NSRD}$

**proof** –

**have**  $(\neg_r \text{post}_R P \text{ wp}_r \text{pre}_R Q) ;; \text{true}_r = (\neg_r \text{post}_R P \text{ wp}_r \text{pre}_R Q)$

**by** (*simp add: wp-rea-def rpred assms closure seqr-assoc NSRD-neg-pre-unit*)

**moreover have**  $\$st' \# \text{pre}_R P \wedge \text{post}_R P \text{ wp}_r \text{pre}_R Q \Rightarrow_r \text{peri}_R P \vee \text{post}_R P ;; \text{peri}_R Q$

**by** (*simp add: unrest assms wp-rea-def*)

**ultimately show** *?thesis*

**by** (*rule-tac NSRD-intro, simp-all add: seqr-or-distl NSRD-neg-pre-unit assms closure rdes unrest*)

**qed**

**lemma** *RHS-tri-normal-design-composition:*

**assumes**

$\$ok' \# P \$ok' \# Q_1 \$ok' \# Q_2 \$ok \# R \$ok \# S_1 \$ok \# S_2$

$\$wait \# R \$wait' \# Q_2 \$wait \# S_1 \$wait \# S_2$

*P is R2c Q<sub>1</sub> is R1 Q<sub>1</sub> is R2c Q<sub>2</sub> is R1 Q<sub>2</sub> is R2c*

*R is R2c S<sub>1</sub> is R1 S<sub>1</sub> is R2c S<sub>2</sub> is R1 S<sub>2</sub> is R2c*

*R1 ( $\neg P$ ) ;; R1(true) = R1( $\neg P$ ) \$st' # Q<sub>1</sub>*

**shows**  $\mathbf{R}_s(P \vdash Q_1 \diamond Q_2) ;; \mathbf{R}_s(R \vdash S_1 \diamond S_2)$

$= \mathbf{R}_s((P \wedge Q_2 \text{ wp}_r R) \vdash (Q_1 \vee (Q_2 ;; S_1)) \diamond (Q_2 ;; S_2))$

**proof** –

**have**  $\mathbf{R}_s(P \vdash Q_1 \diamond Q_2) ;; \mathbf{R}_s(R \vdash S_1 \diamond S_2) =$

$\mathbf{R}_s((R1(\neg P) \text{ wp}_r \text{false} \wedge Q_2 \text{ wp}_r R) \vdash ((\exists \$st' \cdot Q_1) \sqcap (Q_2 ;; S_1)) \diamond (Q_2 ;; S_2))$

**by** (*simp-all add: RHS-tri-design-composition-wp rea-not-def assms unrest*)

**also have**  $\dots = \mathbf{R}_s((P \wedge Q_2 \text{ wp}_r R) \vdash (Q_1 \vee (Q_2 ;; S_1)) \diamond (Q_2 ;; S_2))$

**by** (*simp add: assms wp-rea-def ex-unrest, rel-auto*)

**finally show** *?thesis* .

**qed**

**lemma** *RHS-tri-normal-design-composition'* [rdes-def]:

**assumes** *P is RC Q<sub>1</sub> is RR \$st' # Q<sub>1</sub> Q<sub>2</sub> is RR R is RR S<sub>1</sub> is RR S<sub>2</sub> is RR*

**shows**  $\mathbf{R}_s(P \vdash Q_1 \diamond Q_2) ;; \mathbf{R}_s(R \vdash S_1 \diamond S_2)$

$= \mathbf{R}_s((P \wedge Q_2 \text{ wp}_r R) \vdash (Q_1 \vee (Q_2 ;; S_1)) \diamond (Q_2 ;; S_2))$

**proof** –

**have** *R1 ( $\neg P$ ) ;; R1 true = R1( $\neg P$ )*

**using** *RC-implies-RC1[OF assms(1)]*

**by** (*simp add: Healthy-def RC1-def rea-not-def*)

(*metis R1-negate-R1 R1-seqr utp-pred-laws.double-compl*)

**thus** *?thesis*

**by** (*simp add: RHS-tri-normal-design-composition assms closure unrest RR-implies-R2c*)

**qed**

If a normal reactive design has postcondition false, then it is a left zero for sequential composition.

**lemma** *NSRD-seq-post-false*:  
**assumes**  $P$  is NSRD  $Q$  is SRD  $\text{post}_R(P) = \text{false}$   
**shows**  $P \;; Q = P$   
**apply** (*simp add: NSRD-composition-wp assms wp rpred closure*)  
**using** *NSRD-is-SRD SRD-reactive-tri-design assms(1,3)* **apply** *fastforce*  
**done**

**lemma** *NSRD-srd-skip [closure]*:  $II_R$  is NSRD  
**by** (*rule NSRD-intro, simp-all add: rdes closure unrest*)

**lemma** *NSRD-Chaos [closure]*: *Chaos* is NSRD  
**by** (*rule NSRD-intro, simp-all add: closure rdes unrest*)

**lemma** *NSRD-Miracle [closure]*: *Miracle* is NSRD  
**by** (*rule NSRD-intro, simp-all add: closure rdes unrest*)

Post-composing a miracle filters out the non-terminating behaviours

**lemma** *NSRD-right-Miracle-tri-lemma*:  
**assumes**  $P$  is NSRD  
**shows**  $P \;; \text{Miracle} = \mathbf{R}_s (\text{pre}_R P \vdash \text{peri}_R P \diamond \text{false})$   
**by** (*simp add: NSRD-composition-wp closure assms rdes wp rpred*)

**lemma** *Miracle-right-anhil-iff*:  
**assumes**  $P$  is NSRD  
**shows**  $P \;; \text{Miracle} = \text{Miracle} \longleftrightarrow \text{pre}_R P = \text{true}_r \wedge (\exists \$st' \cdot \text{peri}_R P) = \text{false}$   
**by** (*simp add: SRD-right-Miracle-tri-lemma assms closure, simp add: rdes-def sdes-tri-eq-iff closure assms wp rpred, fastforce*)

The set of non-terminating behaviours is a subset

**lemma** *NSRD-right-Miracle-refines*:  
**assumes**  $P$  is NSRD  
**shows**  $P \sqsubseteq P \;; \text{Miracle}$   
**proof** –  
**have**  $\mathbf{R}_s (\text{pre}_R P \vdash \text{peri}_R P \diamond \text{post}_R P) \sqsubseteq \mathbf{R}_s (\text{pre}_R P \vdash \text{peri}_R P \diamond \text{false})$   
**by** (*rule sdes-tri-refine-intro, rel-auto+*)  
**thus** *?thesis*  
**by** (*simp add: NSRD-elim NSRD-right-Miracle-tri-lemma assms*)  
**qed**

*Chaos* is a right zero of  $P$  precisely when the conjunction of the precondition of  $P$  with the weakest precondition under which  $\text{post}_R P$  yields  $\mathbf{U}(\text{false})$  reduces to  $\mathbf{U}(\text{false})$ . This is effectively a feasibility check for reactive designs.

**lemma** *Chaos-right-anhil-iff*:  
**assumes**  $P$  is NSRD  
**shows**  $P \;; \text{Chaos} = \text{Chaos} \longleftrightarrow (\text{pre}_R P \wedge \text{post}_R P \text{ wp}_r \text{false}) = \text{false}$   
**by** (*simp add: SRD-right-Chaos-tri-lemma assms closure, simp add: rdes-def sdes-tri-eq-iff closure assms wp*)

**lemma** *upower-Suc-NSRD-closed [closure]*:  
 $P$  is NSRD  $\implies P \wedge \text{Suc } n$  is NSRD  
**proof** (*induct n*)  
**case** 0  
**then show** *?case*  
**by** (*simp*)  
**next**



```

case (Suc n)
then show ?case
  by (simp add: NSRD-seqr-closure upred-semiring.power-Suc)
qed

lemma NSRD-power-Suc [closure]:
  P is NSRD  $\implies$  P ;; P ^ n is NSRD
  by (metis upower-Suc-NSRD-closed upred-semiring.power-Suc)

lemma uplus-NSRD-closed [closure]: P is NSRD  $\implies$  P+ is NSRD
  by (simp add: uplus-power-def closure)

lemma preR-power:
  assumes P is NSRD
  shows preR(P ;; Pn) = ( $\bigsqcup$  i $\in\{0..n\}$ . (postR(P) ^ i) wpr (preR(P)))
proof (induct n)
  case 0
  then show ?case
    by (simp add: wp closure)
next
  case (Suc n) note hyp = this
  have preR (P ^ (Suc n + 1)) = preR (P ;; P ^ (n+1))
    by (simp add: upred-semiring.power-Suc)
  also have ... = (preR P  $\wedge$  postR P wpr preR (P ^ (Suc n)))
    using NSRD-iff assms preR-NSRD-seq upower-Suc-NSRD-closed by fastforce
  also have ... = (preR P  $\wedge$  postR P wpr ( $\bigsqcup$  i $\in\{0..n\}$ . postR P ^ i wpr preR P))
    by (simp add: hyp upred-semiring.power-Suc)
  also have ... = (preR P  $\wedge$  ( $\bigsqcup$  i $\in\{0..n\}$ . postR P wpr (postR P ^ i wpr preR P)))
    by (simp add: wp)
  also have ... = (preR P  $\wedge$  ( $\bigsqcup$  i $\in\{0..n\}$ . (postR P ^ (i+1) wpr preR P)))
  proof -
    have  $\bigwedge i. R1$  (postR P ^ i ;; ( $\neg_r$  preR P)) = (postR P ^ i ;; ( $\neg_r$  preR P))
      by (induct-tac i, simp-all add: closure Healthy-if assms)
    thus ?thesis
      by (simp add: wp-rea-def upred-semiring.power-Suc seqr-assoc rpred closure assms)
  qed
  also have ... = (postR P ^ 0 wpr preR P  $\wedge$  ( $\bigsqcup$  i $\in\{0..n\}$ . (postR P ^ (i+1) wpr preR P)))
    by (simp add: wp assms closure)
  also have ... = (postR P ^ 0 wpr preR P  $\wedge$  ( $\bigsqcup$  i $\in\{1..Suc\ n\}$ . (postR P ^ i wpr preR P)))
  proof -
    have ( $\bigsqcup$  i $\in\{0..n\}$ . (postR P ^ (i+1) wpr preR P)) = ( $\bigsqcup$  i $\in\{1..Suc\ n\}$ . (postR P ^ i wpr preR P))
      by (rule cong[of Inf], simp-all add: fun-eq-iff)
      (metis (no-types, lifting) image-Suc-atLeastAtMost image-cong image-image)
    thus ?thesis by simp
  qed
  also have ... = ( $\bigsqcup$  i $\in$ insert 0 {1..Suc n}. (postR P ^ i wpr preR P))
    by (simp add: conj-upred-def)
  also have ... = ( $\bigsqcup$  i $\in\{0..Suc\ n\}$ . postR P ^ i wpr preR P)
    by (simp add: atLeast0-atMost-Suc-eq-insert-0)
  finally show ?case by (simp add: upred-semiring.power-Suc)
qed

lemma preR-power' [rdes]:
  assumes P is NSRD
  shows preR(P ;; Pn) = ( $\bigsqcup$  i $\in\{0..n\}$  . (postR(P) ^ i) wpr (preR(P)))

```

```

by (simp add: preR-power assms USUP-as-Inf[THEN sym], simp add: USUP-as-Inf-image)

lemma preR-power-Suc [rdes]:
  assumes P is NSRD
  shows  $\text{pre}_R(P^\wedge(\text{Suc } n)) = (\bigsqcup_{i \in \{0..n\}} \cdot (\text{post}_R(P) \wedge i) \text{ wp}_r (\text{pre}_R(P)))$ 
  by (simp add: upred-semiring.power-Suc rdes assms)

declare upred-semiring.power-Suc [simp]

lemma periR-power:
  assumes P is NSRD
  shows  $\text{peri}_R(P ;; P^\wedge n) = (\text{pre}_R(P^\wedge(\text{Suc } n)) \Rightarrow_r (\bigsqcup_{i \in \{0..n\}} \cdot \text{post}_R(P) \wedge i) ;; \text{peri}_R(P))$ 
proof (induct n)
  case 0
  then show ?case
    by (simp add: NSRD-is-SRD NSRD-wait'-unrest-pre SRD-peri-under-pre assms)
next
  case (Suc n) note hyp = this
  have  $\text{peri}_R(P^\wedge(\text{Suc } n + 1)) = \text{peri}_R(P ;; P^\wedge(n+1))$ 
    by (simp)
  also have  $\dots = (\text{pre}_R(P^\wedge(\text{Suc } n + 1)) \Rightarrow_r (\text{peri}_R P \vee \text{post}_R P ;; \text{peri}_R(P ;; P^\wedge n)))$ 
    by (simp add: closure assms rdes)
  also have  $\dots = (\text{pre}_R(P^\wedge(\text{Suc } n + 1)) \Rightarrow_r (\text{peri}_R P \vee \text{post}_R P ;; (\text{pre}_R(P^\wedge(\text{Suc } n)) \Rightarrow_r (\bigsqcup_{i \in \{0..n\}} \cdot \text{post}_R P \wedge i) ;; \text{peri}_R P)))$ 
    by (simp only: hyp)
  also
  have  $\dots = (\text{pre}_R P \Rightarrow_r \text{peri}_R P \vee (\text{post}_R P \text{ wp}_r \text{pre}_R(P ;; P^\wedge n) \Rightarrow_r \text{post}_R P ;; (\text{pre}_R(P ;; P^\wedge n) \Rightarrow_r (\bigsqcup_{i \in \{0..n\}} \cdot \text{post}_R P \wedge i) ;; \text{peri}_R P)))$ 
    by (simp add: rdes closure assms, rel-blast)
  also
  have  $\dots = (\text{pre}_R P \Rightarrow_r \text{peri}_R P \vee (\text{post}_R P \text{ wp}_r \text{pre}_R(P ;; P^\wedge n) \Rightarrow_r \text{post}_R P ;; ((\bigsqcup_{i \in \{0..n\}} \cdot \text{post}_R P \wedge i) ;; \text{peri}_R P)))$ 
    by (simp add: Healthy-if R1-post-SRD assms closure)
  thus ?thesis
    by (simp only: wp-rea-impl-lemma, simp add: Healthy-if 1, simp add: R1-post-SRD assms closure)
qed
also
  have  $\dots = (\text{pre}_R P \wedge \text{post}_R P \text{ wp}_r \text{pre}_R(P ;; P^\wedge n) \Rightarrow_r \text{peri}_R P \vee \text{post}_R P ;; ((\bigsqcup_{i \in \{0..n\}} \cdot \text{post}_R P \wedge i) ;; \text{peri}_R P))$ 
    by (pred-auto)
  also
  have  $\dots = (\text{pre}_R P \wedge \text{post}_R P \text{ wp}_r \text{pre}_R(P ;; P^\wedge n) \Rightarrow_r \text{peri}_R P \vee ((\bigsqcup_{i \in \{0..n\}} \cdot \text{post}_R P \wedge (\text{Suc } i)) ;; \text{peri}_R P))$ 
    by (simp add: seq-Sup-distl seqr-assoc[THEN sym])

```

```

also
have ... = (preR P ∧ postR P wpr preR (P ;; P ^ n) ⇒r periR P ∨ ((⋀ i ∈ {1..Suc n}. postR P ^ i)
;; periR P))
proof -
  have (⋀ i ∈ {0..n}. postR P ^ Suc i) = (⋀ i ∈ {1..Suc n}. postR P ^ i)
  apply (rule cong[of Sup], auto)
  apply (metis atLeast0AtMost atMost-iff image-Suc-atLeastAtMost rev-image-eqI upred-semiring.power-Suc)
  using Suc-le-D apply fastforce
done
thus ?thesis by simp
qed
also
have ... = (preR P ∧ postR P wpr preR (P ;; P ^ n) ⇒r ((⋀ i ∈ {0..Suc n}. postR P ^ i) ;; periR P)
  by (simp add: SUP-atLeastAtMost-first uinf-or segr-or-distl segr-or-distr)
also
have ... = (preR(P^(Suc (Suc n)))) ⇒r ((⋀ i ∈ {0..Suc n}. postR P ^ i) ;; periR P)
  by (simp add: rdes closure assms)
finally show ?case by (simp)
qed

```

```

lemma periR-power' [rdes]:
  assumes P is NSRD
  shows periR(P ;; P ^ n) = (preR(P^(Suc n)) ⇒r (⋀ i ∈ {0..n} . postR(P) ^ i) ;; periR(P))
  by (simp add: periR-power assms UINF-as-Sup[THEN sym], simp add: UINF-as-Sup-image)

```

```

lemma periR-power-Suc [rdes]:
  assumes P is NSRD
  shows periR(P^(Suc n)) = (preR(P^(Suc n)) ⇒r (⋀ i ∈ {0..n} . postR(P) ^ i) ;; periR(P))
  by (simp add: rdes assms)

```

```

lemma postR-power [rdes]:
  assumes P is NSRD
  shows postR(P ;; P ^ n) = (preR(P^(Suc n)) ⇒r postR(P) ^ Suc n)
proof (induct n)
  case 0
  then show ?case
    by (simp add: NSRD-is-SRD NSRD-wait'-unrest-pre SRD-post-under-pre assms)
next
  case (Suc n) note hyp = this
  have postR (P ^ (Suc n + 1)) = postR (P ;; P ^ (n+1))
  by (simp)
  also have ... = (preR(P ^ (Suc n + 1)) ⇒r (postR P ;; postR (P ;; P ^ n)))
  by (simp add: closure assms rdes)
  also have ... = (preR(P ^ (Suc n + 1)) ⇒r (postR P ;; (preR (P ^ Suc n) ⇒r postR P ^ Suc n)))
  by (simp only: hyp)
  also
  have ... = (preR P ⇒r (postR P wpr preR (P ^ Suc n) ⇒r postR P ;; (preR (P ^ Suc n) ⇒r postR
P ^ Suc n)))
  by (simp add: rdes closure assms, pred-auto)
  also
  have ... = (preR P ⇒r (postR P wpr preR (P ^ Suc n) ⇒r postR P ;; postR P ^ Suc n))
  by (metis (no-types, lifting) Healthy-if NSRD-is-SRD NSRD-power-Suc R1-power assms hyp postR-SRD-R1
upred-semiring.power-Suc wp-rea-impl-lemma)
  also
  have ... = (preR P ∧ postR P wpr preR (P ^ Suc n) ⇒r postR P ^ Suc (Suc n))

```

by (*pred-auto*)  
 also have ... = ( $\text{pre}_R(P^\wedge(\text{Suc}(\text{Suc } n))) \Rightarrow_r \text{post}_R P \wedge \text{Suc}(\text{Suc } n)$ )  
 by (*simp add: rdes closure assms*)  
 finally show ?case by (*simp*)  
 qed

**lemma** *postR-power-Suc* [*rdes*]:  
 assumes *P is NSRD*  
 shows  $\text{post}_R(P^\wedge(\text{Suc } n)) = (\text{pre}_R(P^\wedge(\text{Suc } n)) \Rightarrow_r \text{post}_R(P) \wedge \text{Suc } n)$   
 by (*simp add: rdes assms*)

**lemma** *power-rdes-def* [*rdes-def*]:  
 assumes *P is RC Q is RR R is RR \$st' \# Q*  
 shows  $(\mathbf{R}_s(P \vdash Q \diamond R))^\wedge(\text{Suc } n)$   
 $= \mathbf{R}_s((\bigsqcup i \in \{0..n\} \cdot (R \wedge i) \text{ wp}_r P) \vdash ((\bigsqcap i \in \{0..n\} \cdot R \wedge i) ;; Q) \diamond (R \wedge \text{Suc } n))$

**proof** (*induct n*)

case 0

then show ?case

by (*simp add: wp assms closure*)

next

case (*Suc n*)

have 1:  $(P \wedge (\bigsqcup i \in \{0..n\} \cdot R \text{ wp}_r (R \wedge i \text{ wp}_r P))) = (\bigsqcup i \in \{0..\text{Suc } n\} \cdot R \wedge i \text{ wp}_r P)$   
 (is ?lhs = ?rhs)

**proof** –

have ?lhs =  $(P \wedge (\bigsqcup i \in \{0..n\} \cdot (R \wedge \text{Suc } i \text{ wp}_r P)))$

by (*simp add: wp closure assms*)

also have ... =  $(P \wedge (\bigsqcup i \in \{0..n\} \cdot (R \wedge \text{Suc } i \text{ wp}_r P)))$

by (*simp only: USUP-as-Inf-collect*)

also have ... =  $(P \wedge (\bigsqcup i \in \{1..\text{Suc } n\} \cdot (R \wedge i \text{ wp}_r P)))$

by (*metis (no-types, lifting) INF-cong One-nat-def image-Suc-atLeastAtMost image-image*)

also have ... =  $(\bigsqcup i \in \text{insert } 0 \{1..\text{Suc } n\} \cdot (R \wedge i \text{ wp}_r P))$

by (*simp add: wp assms closure conj-upred-def*)

also have ... =  $(\bigsqcup i \in \{0..\text{Suc } n\} \cdot (R \wedge i \text{ wp}_r P))$

by (*simp add: atLeastAtMost-insertL*)

finally show ?thesis

by (*simp add: USUP-as-Inf-collect*)

qed

have 2:  $(Q \vee R ;; (\bigsqcap i \in \{0..n\} \cdot R \wedge i) ;; Q) = (\bigsqcap i \in \{0..\text{Suc } n\} \cdot R \wedge i) ;; Q$   
 (is ?lhs = ?rhs)

**proof** –

have ?lhs =  $(Q \vee (\bigsqcap i \in \{0..n\} \cdot R \wedge \text{Suc } i) ;; Q)$

by (*simp add: seqr-assoc[THEN sym] seq-UINF-distl*)

also have ... =  $(Q \vee (\bigsqcap i \in \{0..n\} \cdot R \wedge \text{Suc } i) ;; Q)$

by (*simp only: UINF-as-Sup-collect*)

also have ... =  $(Q \vee (\bigsqcap i \in \{1..\text{Suc } n\} \cdot R \wedge i) ;; Q)$

by (*metis One-nat-def image-Suc-atLeastAtMost image-image*)

also have ... =  $((\bigsqcap i \in \text{insert } 0 \{1..\text{Suc } n\} \cdot R \wedge i) ;; Q)$

by (*simp add: disj-upred-def[THEN sym] seqr-or-distl*)

also have ... =  $((\bigsqcap i \in \{0..\text{Suc } n\} \cdot R \wedge i) ;; Q)$

by (*simp add: atLeastAtMost-insertL*)

finally show ?thesis

by (*simp add: UINF-as-Sup-collect*)

qed

```

have 3: ( $\prod i \in \{0..n\} \cdot R \wedge i$ ) ;;  $Q$  is  $RR$ 
proof -
  have ( $\prod i \in \{0..n\} \cdot R \wedge i$ ) ;;  $Q = (\prod i \in \{0..n\} \cdot R \wedge i$ ) ;;  $Q$ 
    by (simp add: UINF-as-Sup-collect)
  also have ... = ( $\prod i \in \text{insert } 0 \{1..n\} \cdot R \wedge i$ ) ;;  $Q$ 
    by (simp add: atLeastAtMost-insertL)
  also have ... = ( $Q \vee (\prod i \in \{1..n\} \cdot R \wedge i)$ ) ;;  $Q$ 
    by (metis (no-types, lifting) SUP-insert disj-upred-def segr-left-unit segr-or-distl upred-semiring.power-0)
  also have ... = ( $Q \vee (\prod i \in \{0..<n\} \cdot R \wedge \text{Suc } i)$ ) ;;  $Q$ 
    by (metis One-nat-def atLeastLessThanSuc-atLeastAtMost image-Suc-atLeastLessThan image-image)
  also have ... = ( $Q \vee (\prod i \in \{0..<n\} \cdot R \wedge \text{Suc } i)$ ) ;;  $Q$ 
    by (simp add: UINF-as-Sup-collect)
  also have ... is  $RR$ 
    by (simp-all add: closure assms)
  finally show ?thesis .
qed
from 1 2 3 Suc show ?case
  by (simp add: Suc RHS-tri-normal-design-composition' closure assms wp)
qed

```

```

declare upred-semiring.power-Suc [simp del]

```

```

theorem uplus-rdes-def [rdes-def]:

```

```

  assumes  $P$  is  $RC$   $Q$  is  $RR$   $R$  is  $RR$   $\$st' \# Q$ 
  shows  $(\mathbf{R}_s(P \vdash Q \diamond R))^+ = \mathbf{R}_s(R^{*r} \text{ wp}_r P \vdash (R^{*r} ;; Q) \diamond R^+)$ 

```

```

proof -

```

```

  have 1: ( $\prod i \cdot R \wedge i$ ) ;;  $Q = R^{*r} ;; Q$ 
    by (metis (no-types) RA1 assms(2) rea-skip-unit(2) rrel-theory.Star-def ustar-alt-def)
  show ?thesis
    by (simp add: uplus-power-def seq-UINF-distr wp closure assms rdes-def)
      (metis 1 seq-UINF-distr')

```

```

qed

```

## 5.1 UTP theory

```

lemma NSRD-false: NSRD false = Miracle

```

```

  by (metis Healthy-if NSRD-Miracle NSRD-alt-def NSRD-is-RD3 srdes-theory.healthy-top)

```

```

lemma NSRD-true: NSRD true = Chaos

```

```

  by (metis Healthy-if NSRD-Chaos NSRD-alt-def NSRD-is-RD3 srdes-theory.healthy-bottom)

```

```

interpretation nsrdes-theory: utp-theory-kleene NSRD  $II_R$ 

```

```

  rewrites  $P \in \text{carrier nsrdes-theory.thy-order} \longleftrightarrow P$  is NSRD

```

```

  and  $\text{carrier nsrdes-theory.thy-order} \rightarrow \text{carrier nsrdes-theory.thy-order} \equiv \llbracket \text{NSRD} \rrbracket_H \rightarrow \llbracket \text{NSRD} \rrbracket_H$ 

```

```

  and  $\text{le nsrdes-theory.thy-order} = (\sqsubseteq)$ 

```

```

  and  $\text{eq nsrdes-theory.thy-order} = (=)$ 

```

```

  and  $\text{nsrdes-top: nsrdes-theory.utp-top} = \text{Miracle}$ 

```

```

  and  $\text{nsrdes-bottom: nsrdes-theory.utp-bottom} = \text{Chaos}$ 

```

```

proof -

```

```

  have utp-theory-continuous NSRD

```

```

    by (unfold-locales, simp-all add: NSRD-idem NSRD-Continuous)

```

```

  then interpret utp-theory-continuous NSRD

```

```

    by simp

```

```

  show  $t$ : utp-top = Miracle and  $b$ : utp-bottom = Chaos

```

```

    by (simp-all add: healthy-top healthy-bottom NSRD-false NSRD-true)

```

**show** *utp-theory-kleene NSRD II<sub>R</sub>*  
**by** (*unfold-locales, simp-all add: closure sdes-left-unital. Unit-Left NSRD-right-unit Miracle-left-zero*  
*t*)  
**qed** (*simp-all*)

**abbreviation** *TestR (test<sub>R</sub>) where*  
*test<sub>R</sub> P ≡ nsrdes-theory.utp-test P*

**definition** *StarR :: ('s, 't::trace, 'α) hrel-rsp ⇒ ('s, 't, 'α) hrel-rsp (-<sup>\*R</sup> [999] 999) where*  
*StarR ≡ nsrdes-theory.utp-star*

We also show how to calculate the Kleene closure of a reactive design.

**thm** *rdes-def*

**lemma** *StarR-rdes-def [rdes-def]:*

**assumes** *P is RC Q is RR R is RR \$st' # Q*  
**shows**  $(\mathbf{R}_s(P \vdash Q \diamond R))^{\star R} = \mathbf{R}_s((R^{\star r} \text{wp}_r P) \vdash (R^{\star r} ;; Q) \diamond R^{\star r})$   
**by** (*simp add: StarR-def rrel-theory.Star-alt-def nsrdes-theory.Star-alt-def closure assms*)  
*(simp add: rrel-theory.Star-alt-def assms closure rdes-def unrest rpred disj-upred-def)*

**end**

## 6 Syntax for reactive design contracts

**theory** *utp-rdes-contracts*  
**imports** *utp-rdes-normal*  
**begin**

We give an experimental syntax for reactive design contracts  $[P \vdash Q|R]_R$ , where  $P$  is a precondition on undashed state variables only,  $Q$  is a pericondition that can refer to the trace and before state but not the after state, and  $R$  is a postcondition. Both  $Q$  and  $R$  can refer only to the trace contribution through a HOL variable *trace* which is bound to  $U(\&tt)$ .

**definition** *mk-RD :: 's upred ⇒ ('t::trace ⇒ 's upred) ⇒ ('t ⇒ 's hrel) ⇒ ('s, 't, 'α) hrel-rsp where*  
*mk-RD P Q R =  $\mathbf{R}_s(\lceil P \rceil_{S<} \vdash \lceil Q(x) \rceil_{S<} \llbracket x \rightarrow \&tt \rrbracket \diamond \lceil R(x) \rceil_S \llbracket x \rightarrow \&tt \rrbracket)$*

**definition** *trace-pred :: ('t::trace ⇒ 's upred) ⇒ ('s, 't, 'α) hrel-rsp where*  
*[upred-defs]: trace-pred P =  $\lceil (P \ x) \rceil_{S<} \llbracket x \rightarrow \&tt \rrbracket$*

**syntax**

*-trace-var :: logic*  
*-mk-RD :: logic ⇒ logic ⇒ logic ⇒ logic ( $\lceil \_ \rceil \vdash \_ \rceil \mid \_ \rceil_R$ )*  
*-trace-pred :: logic ⇒ logic ( $\lceil \_ \rceil_t$ )*

**parse-translation**  $\langle$

*let*  
*fun trace-var-tr [] = Syntax.free trace*  
*| trace-var-tr - = raise Match;*  
*in*  
*[ $\lceil \_ \rceil$  {syntax-const -trace-var}, K trace-var-tr]*  
*end*  
 $\rangle$

**translations**

$[P \vdash Q \mid R]_R \Rightarrow \text{CONST } \text{mk-RD } P (\lambda \text{-trace-var. } Q) (\lambda \text{-trace-var. } R)$

$[P \vdash Q \mid R]_R \leq \text{CONST } \text{mk-RD } P \ (\lambda x. Q) \ (\lambda y. R)$   
 $[P]_t \Rightarrow \text{CONST } \text{trace-pred } (\lambda \text{-trace-var. } P)$   
 $[P]_t \leq \text{CONST } \text{trace-pred } (\lambda t. P)$

**lemma** *SRD-mk-RD [closure]*:  $[P \vdash Q(\text{trace}) \mid R(\text{trace})]_R$  is *SRD*  
**by** (*simp add: mk-RD-def closure unrest*)

**lemma** *preR-mk-RD [rdes]*:  $\text{pre}_R([P \vdash Q(\text{trace}) \mid R(\text{trace})]_R) = R1([P]_{S<})$   
**by** (*simp add: mk-RD-def rea-pre-RHS-design usubst unrest R2c-not R2c-lift-state-pre*)

**lemma** *trace-pred-RR-closed [closure]*:  
 $[P \text{ trace}]_t$  is *RR*  
**by** (*rel-auto*)

**lemma** *unrest-trace-pred-st' [unrest]*:  
 $\$st' \# [P \text{ trace}]_t$   
**by** (*rel-auto*)

**lemma** *R2c-msubst-tt*:  $R2c \ (\text{msubst } (\lambda x. [Q \ x]_S) \ \&tt) = (\text{msubst } (\lambda x. [Q \ x]_S) \ \&tt)$   
**by** (*rel-auto*)

**lemma** *periR-mk-RD [rdes]*:  $\text{peri}_R([P \vdash Q(\text{trace}) \mid R(\text{trace})]_R) = ([P]_{S<} \Rightarrow_r R1((\llbracket Q(\text{trace}) \rrbracket_{S<}) \llbracket \text{trace} \rightarrow \&tt \rrbracket))$   
**by** (*simp add: mk-RD-def rea-peri-RHS-design usubst unrest R2c-not R2c-lift-state-pre R2c-disj R2c-msubst-tt R1-disj R2c-rea-impl R1-rea-impl*)

**lemma** *postR-mk-RD [rdes]*:  $\text{post}_R([P \vdash Q(\text{trace}) \mid R(\text{trace})]_R) = ([P]_{S<} \Rightarrow_r R1((\llbracket R(\text{trace}) \rrbracket_S) \llbracket \text{trace} \rightarrow \&tt \rrbracket))$   
**by** (*simp add: mk-RD-def rea-post-RHS-design usubst unrest R2c-not R2c-lift-state-pre impl-alt-def R2c-disj R2c-msubst-tt R2c-rea-impl R1-rea-impl*)

Refinement introduction law for contracts

**lemma** *RD-contract-refine*:

**assumes**

$Q$  is *SRD* ‘ $[P_1]_{S<} \Rightarrow \text{pre}_R Q$ ’  
‘ $[P_1]_{S<} \wedge \text{peri}_R Q \Rightarrow [P_2 \ x]_{S<} \llbracket x \rightarrow \&tt \rrbracket$ ’  
‘ $[P_1]_{S<} \wedge \text{post}_R Q \Rightarrow [P_3 \ x]_S \llbracket x \rightarrow \&tt \rrbracket$ ’

**shows**  $[P_1 \vdash P_2(\text{trace}) \mid P_3(\text{trace})]_R \sqsubseteq Q$

**proof** –

**have**  $[P_1 \vdash P_2(\text{trace}) \mid P_3(\text{trace})]_R \sqsubseteq \mathbf{R}_s(\text{pre}_R(Q) \vdash \text{peri}_R(Q) \diamond \text{post}_R(Q))$

**using** *assms*

**by** (*simp add: mk-RD-def, rule-tac srdes-tri-refine-intro, simp-all*)

**thus** *?thesis*

**by** (*simp add: SRD-reactive-tri-design assms(1)*)

**qed**

**end**

## 7 Reactive design tactics

**theory** *utp-rdes-tactics*

**imports** *utp-rdes-triples*

**begin**

Theorems for normalisation

**lemmas** *rdes-rel-norms* =

*prod.case-eq-if*  
*conj-assoc*  
*disj-assoc*  
*utp-pred-laws.distrib(3,4)*  
*conj-UINF-dist*  
*conj-UINF-ind-dist*  
*seqr-or-distl*  
*seqr-or-distr*  
*seq-UINF-distl*  
*seq-UINF-distl'*  
*seq-UINF-distr*  
*seq-UINF-distr'*

The following tactic can be used to simply and evaluate reactive predicates.

**method** *rpred-simp* = (*ueexpr-simp* *simps*: *rpred* *usubst* *closure* *unrest*)

Tactic to expand out healthy reactive design predicates into the syntactic triple form.

**method** *rdes-expand* **uses** *cls* = (*insert* *cls*, (*erule* *RD-elim*)*+*)

Tactic to simplify the definition of a reactive design

**method** *rdes-simp* **uses** *cls* *cong* *simps* =  
 ((*rdes-expand* *cls*: *cls*)?, (*simp* *add*: *closure*)?, (*simp* *add*: *rdes-def* *rdes-rel-norms* *rdes* *rpred* *cls* *closure*  
*alpha* *frame* *usubst* *unrest* *wp* *simps* *cong*: *cong*))

Tactic to split a refinement conjecture into three POs

**method** *rdes-refine-split* **uses** *cls* *cong* *simps* =  
 (*rdes-simp* *cls*: *cls* *cong*: *cong* *simps*: *simps*; *rule-tac* *srdes-tri-refine-intro'*)

Tactics to split an equality conjecture into three POs

**method** *rdes-eq-split* **uses** *cls* *cong* *simps* =  
 (*rdes-simp* *cls*: *cls* *cong*: *cong* *simps*: *simps*; (*rule-tac* *srdes-tri-eq-intro*))

**method** *rdes-eq-split'* **uses** *cls* *cong* *simps* =  
 (*rdes-simp* *cls*: *cls* *cong*: *cong* *simps*: *simps*; (*rule-tac* *srdes-tri-eq-intro'*))

Tactic to prove a refinement

**method** *rdes-refine* **uses** *cls* *cong* *simps* =  
 (*rdes-refine-split* *cls*: *cls* *cong*: *cong* *simps*: *simps*; (*insert* *cls*; *rel-auto*))

Tactics to prove an equality

**method** *rdes-eq* **uses** *cls* *cong* *simps* =  
 (*rdes-eq-split* *cls*: *cls* *cong*: *cong* *simps*: *simps*; *rel-auto*)

**method** *rdes-eq'* **uses** *cls* *cong* *simps* =  
 (*rdes-eq-split'* *cls*: *cls* *cong*: *cong* *simps*: *simps*; *rel-auto*)

Via antisymmetry

**method** *rdes-eq-anti* **uses** *cls* *cong* *simps* =  
 (*rdes-simp* *cls*: *cls* *cong*: *cong* *simps*: *simps*; (*rule-tac* *antisym*; (*rule-tac* *srdes-tri-refine-intro*; *rel-auto*))))

Tactic to calculate pre/peri/postconditions from reactive designs

**method** *rdes-calc* = (*simp* *add*: *rdes* *rpred* *closure* *alpha* *usubst* *unrest* *wp* *prod.case-eq-if*)



The following tactic attempts to prove a reactive design refinement by calculation of the pre-, peri-, and postconditions and then showing three implications between them using *rel-blast*.

```
method rdspl-refine =
  (rule-tac SRD-refine-intro; (simp add: closure rdes unrest usubst ; rel-blast?))
```

The following tactic combines antisymmetry with the previous tactic to prove an equality.

```
method rdspl-eq =
  (rule-tac antisym, rdes-refine, rdes-refine)
```

**end**

## 8 Reactive design parallel-by-merge

```
theory utp-rdes-parallel
imports
  utp-rdes-normal
  utp-rdes-tactics
begin
```

R3h implicitly depends on RD1, and therefore it requires that both sides be RD1. We also require that both sides are R3c, and that  $wait_m$  is a quasi-unit, and  $div_m$  yields divergence.

```
lemma st-U0-alpha:  $\exists \$st \cdot II|_0 = (\exists \$st \cdot \lceil II \rceil_0)$ 
by (rel-auto)
```

```
lemma st-U1-alpha:  $\exists \$st \cdot II|_1 = (\exists \$st \cdot \lceil II \rceil_1)$ 
by (rel-auto)
```

```
definition skip-rm :: ('s, 't::trace, 'α) rsp merge ( $II_{RM}$ ) where
  [upred-defs]:  $II_{RM} = (\exists \$<:st \cdot skip_m \vee (\neg \$<:ok \wedge \$<:tr \leq_u \$tr'))$ 
```

```
definition [upred-defs]:  $R3hm(M) = (II_{RM} \triangleleft \$<:wait \triangleright M)$ 
```

```
lemma R3hm-idem:  $R3hm(R3hm(P)) = R3hm(P)$ 
by (rel-auto)
```

```
abbreviation copytype :: 'a  $\Rightarrow$  'a  $\Rightarrow$  'a (CPTYPE'(-, -)) where CPTYPE(A, B)  $\equiv$  B
```

```
lemma R3h-par-by-merge [closure]:
  assumes P is R3h Q is R3h M is R3hm
  shows ( $P \parallel_M Q$ ) is R3h
```

**proof** –

```
  have ( $P \parallel_M Q$ ) = ((( $P \parallel_M Q$ ) $\llbracket true/\$ok \rrbracket \triangleleft \$ok \triangleright$  ( $P \parallel_M Q$ ) $\llbracket false/\$ok \rrbracket$ ) $\llbracket true/\$wait \rrbracket \triangleleft \$wait \triangleright$  ( $P \parallel_M Q$ ))
```

```
    by (simp add: cond-var-subst-left cond-var-subst-right)
```

```
  also have ... = ((( $P \parallel_M Q$ ) $\llbracket true, true/\$ok, \$wait \rrbracket \triangleleft \$ok \triangleright$  ( $P \parallel_M Q$ ) $\llbracket false, true/\$ok, \$wait \rrbracket$ )  $\triangleleft \$wait \triangleright$  ( $P \parallel_M Q$ ))
```

```
    by (rel-auto)
```

```
  also have ... = ((( $\exists \$st \cdot II$ ) $\llbracket true, true/\$ok, \$wait \rrbracket \triangleleft \$ok \triangleright$  ( $P \parallel_M Q$ ) $\llbracket false, true/\$ok, \$wait \rrbracket$ )  $\triangleleft \$wait \triangleright$  ( $P \parallel_M Q$ ))
```

**proof** –

```
  have ( $P \parallel_M Q$ ) $\llbracket true, true/\$ok, \$wait \rrbracket = ((\lceil P \rceil_0 \wedge \lceil Q \rceil_1 \wedge \$<:\mathbf{v}' =_u \$\mathbf{v}) ;; R3hm(M))\llbracket true, true/\$ok, \$wait \rrbracket$ 
    by (simp add: par-by-merge-def U0-as-alpha U1-as-alpha assms Healthy-if)
```

```
  also have ... = ((( $\lceil P \rceil_0 \wedge \lceil Q \rceil_1 \wedge \$<' =_u \$\mathbf{v}$ ) ;; ( $\exists \$<:st \cdot CPTYPE(M, \$\mathbf{v}' =_u \$<)$ )) $\llbracket true, true/\$ok, \$wait \rrbracket$ 
    by (rel-blast)
```

**also have** ... =  $((\lceil R3h(P) \rceil_0 \wedge \lceil R3h(Q) \rceil_1 \wedge \$<' =_u \$\mathbf{v}) ;; (\exists \$<:st \cdot CPTYPE(M, \$\mathbf{v}' =_u \$<))) \llbracket true, true / \$ok, \$wait \rrbracket$   
**by** (*simp add: assms Healthy-if*)  
**also have** ... =  $(\exists \$st \cdot II) \llbracket true, true / \$ok, \$wait \rrbracket$   
**by** (*rel-auto*)  
**finally show** ?thesis **by** (*simp add: closure assms unrest*)  
**qed**  
**also have** ... =  $((\exists \$st \cdot II) \llbracket true, true / \$ok, \$wait \rrbracket \triangleleft \$ok \triangleright (R1(true)) \llbracket false, true / \$ok, \$wait \rrbracket) \triangleleft \$wait$   
 $\triangleright (P \parallel_M Q)$   
**proof** –  
**have**  $(P \parallel_M Q) \llbracket false, true / \$ok, \$wait \rrbracket = ((\lceil P \rceil_0 \wedge \lceil Q \rceil_1 \wedge \$<:\mathbf{v}' =_u \$\mathbf{v}) ;; R3hm(M)) \llbracket false, true / \$ok, \$wait \rrbracket$   
**by** (*simp add: par-by-merge-def U0-as-alpha U1-as-alpha assms Healthy-if*)  
**also have** ... =  $((\lceil P \rceil_0 \wedge \lceil Q \rceil_1 \wedge \$<:\mathbf{v}' =_u \$\mathbf{v}) ;; (CPTYPE(M, \$<:tr \leq_u \$tr')) \llbracket false, true / \$ok, \$wait \rrbracket$   
**by** (*rel-blast*)  
**also have** ... =  $((\lceil R3h(P) \rceil_0 \wedge \lceil R3h(Q) \rceil_1 \wedge \$<:\mathbf{v}' =_u \$\mathbf{v}) ;; (CPTYPE(M, \$<:tr \leq_u \$tr')) \llbracket false, true / \$ok, \$wait \rrbracket$   
**by** (*simp add: assms Healthy-if*)  
**also have** ... =  $(R1(true)) \llbracket false, true / \$ok, \$wait \rrbracket$   
**by** (*rel-blast*)  
**finally show** ?thesis **by** *simp*  
**qed**  
**also have** ... =  $((\exists \$st \cdot II) \triangleleft \$ok \triangleright R1(true)) \triangleleft \$wait \triangleright (P \parallel_M Q)$   
**by** (*rel-auto*)  
**also have** ... =  $R3h(P \parallel_M Q)$   
**by** (*simp add: R3h-cases*)  
**finally show** ?thesis  
**by** (*simp add: Healthy-def*)  
**qed**

**definition** [*upred-defs*]:  $RD1m(M) = (M \vee \neg \$<:ok \wedge \$<:tr \leq_u \$tr')$

**lemma** *RD1-par-by-merge* [*closure*]:

**assumes** *P is R1 Q is R1 M is R1m P is RD1 Q is RD1 M is RD1m*

**shows**  $(P \parallel_M Q) \text{ is } RD1$

**proof** –

**have** 1:  $(RD1(R1(P)) \parallel_{RD1m(R1m(M))} RD1(R1(Q))) \llbracket false / \$ok \rrbracket = R1(true)$

**by** (*rel-blast*)

**have**  $(P \parallel_M Q) = (P \parallel_M Q) \llbracket true / \$ok \rrbracket \triangleleft \$ok \triangleright (P \parallel_M Q) \llbracket false / \$ok \rrbracket$

**by** (*simp add: cond-var-split*)

**also have** ... =  $R1(P \parallel_M Q) \triangleleft \$ok \triangleright R1(true)$

**by** (*metis 1 Healthy-if R1-par-by-merge assms calculation*)

*cond-idem cond-var-subst-right in-var-uvar ok-vwb-lens*)

**also have** ... =  $RD1(P \parallel_M Q)$

**by** (*simp add: Healthy-if R1-par-by-merge RD1-alt-def assms(3)*)

**finally show** ?thesis

**by** (*simp add: Healthy-def*)

**qed**

**lemma** *RD2-par-by-merge* [*closure*]:

**assumes** *M is RD2*

**shows**  $(P \parallel_M Q) \text{ is } RD2$

**proof** –

**have**  $(P \parallel_M Q) = ((P \parallel_s Q) ;; M)$

**by** (*simp add: par-by-merge-def*)

**also from** *assms* **have** ... =  $((P \parallel_s Q) ;; (M ;; J))$

**by** (*simp add: Healthy-def' RD2-def H2-def*)

**also from *assms* have** ... = ((( $P \parallel_s Q$ ) ;;  $M$ ) ;;  $J$ )  
**by** (*simp add: segr-assoc*)  
**also from *assms* have** ... =  $RD2(P \parallel_M Q)$   
**by** (*simp add: RD2-def H2-def par-by-merge-def*)  
**finally show** *?thesis*  
**by** (*simp add: Healthy-def'*)  
**qed**

**lemma** *SRD-par-by-merge*:

**assumes**  $P$  is *SRD*  $Q$  is *SRD*  $M$  is *R1m*  $M$  is *R2m*  $M$  is *R3hm*  $M$  is *RD1m*  $M$  is *RD2*  
**shows**  $(P \parallel_M Q)$  is *SRD*  
**by** (*rule SRD-intro, simp-all add: assms closure SRD-healths*)

**definition** *nmerge-rd0* ( $N_0$ ) **where**

*[upred-defs]*:  $N_0(M) = (\$wait' =_u (\$0:wait \vee \$1:wait) \wedge \$<:tr \leq_u \$tr' \wedge (\exists \$0:ok; \$1:ok; \$<:ok; \$ok'; \$0:wait; \$1:wait; \$<:wait; \$wait' \cdot M))$

**definition** *nmerge-rd1* ( $N_1$ ) **where**

*[upred-defs]*:  $N_1(M) = (\$ok' =_u (\$0:ok \wedge \$1:ok) \wedge N_0(M))$

**definition** *nmerge-rd* ( $N_R$ ) **where**

*[upred-defs]*:  $N_R(M) = ((\exists \$<:st \cdot \$v' =_u \$<:v) \triangleleft \$<:wait \triangleright N_1(M)) \triangleleft \$<:ok \triangleright (\$<:tr \leq_u \$tr')$

**definition** *merge-rd1* ( $M_1$ ) **where**

*[upred-defs]*:  $M_1(M) = (N_1(M) ;; II_R)$

**definition** *merge-rd* ( $M_R$ ) **where**

*[upred-defs]*:  $M_R(M) = N_R(M) ;; II_R$

**abbreviation** *rdes-par* ( $- \parallel_{R-} - [85, 0, 86]$  85) **where**

$P \parallel_{RM} Q \equiv P \parallel_{M_R(M)} Q$

Healthiness condition for reactive design merge predicates

**definition** *[upred-defs]*:  $RDM(M) = R2m(\exists \$0:ok; \$1:ok; \$<:ok; \$ok'; \$0:wait; \$1:wait; \$<:wait; \$wait' \cdot M)$

**lemma** *nmerge-rd-is-R1m [closure]*:

$N_R(M)$  is *R1m*  
**by** (*rel-blast*)

**lemma** *R2m-nmerge-rd*:  $R2m(N_R(R2m(M))) = N_R(R2m(M))$

**apply** (*rel-auto*) **using** *minus-zero-eq* **by** *blast+*

**lemma** *nmerge-rd-is-R2m [closure]*:

$M$  is *R2m*  $\implies N_R(M)$  is *R2m*  
**by** (*metis Healthy-def' R2m-nmerge-rd*)

**lemma** *nmerge-rd-is-R3hm [closure]*:  $N_R(M)$  is *R3hm*

**by** (*rel-blast*)

**lemma** *nmerge-rd-is-RD1m [closure]*:  $N_R(M)$  is *RD1m*

**by** (*rel-blast*)

**lemma** *merge-rd-is-RD3*:  $M_R(M)$  is *RD3*

**by** (*metis Healthy-Idempotent RD3-Idempotent RD3-def merge-rd-def*)

**lemma** *merge-rd-is-RD2*:  $M_R(M)$  is RD2

by (*simp add: RD3-implies-RD2 merge-rd-is-RD3*)

**lemma** *par-rdes-NSRD* [closure]:

assumes  $P$  is SRD  $Q$  is SRD  $M$  is RDM

shows  $P \parallel_{RM} Q$  is NSRD

**proof** –

have  $(P \parallel_{N_R} M \ Q \ ; \ II_R)$  is NSRD

by (*rule NSRD-intro', simp-all add: SRD-healths closure assms*)

(*metis (no-types, lifting) Healthy-def R2-par-by-merge R2-seqr-closure R2m-nmerge-rd RDM-def SRD-healths(2) assms skip-srea-R2*

,*metis Healthy-Idempotent RD3-Idempotent RD3-def*)

**thus** ?thesis

by (*simp add: merge-rd-def par-by-merge-def seqr-assoc*)

**qed**

**lemma** *RDM-intro*:

assumes  $M$  is R2m  $\$0:ok \# M \ \$1:ok \# M \ \$<:ok \# M \ \$ok' \# M$   
 $\$0:wait \# M \ \$1:wait \# M \ \$<:wait \# M \ \$wait' \# M$

shows  $M$  is RDM

using *assms*

by (*simp add: Healthy-def RDM-def ex-unrest unrest*)

**lemma** *RDM-unrests* [*unrest*]:

assumes  $M$  is RDM

shows  $\$0:ok \# M \ \$1:ok \# M \ \$<:ok \# M \ \$ok' \# M$

$\$0:wait \# M \ \$1:wait \# M \ \$<:wait \# M \ \$wait' \# M$

by (*subst Healthy-if[OF assms, THEN sym], simp-all add: RDM-def unrest, rel-auto*)+

**lemma** *RDM-R1m* [closure]:  $M$  is RDM  $\implies M$  is R1m

by (*metis (no-types, hide-lams) Healthy-def R1m-idem R2m-def RDM-def*)

**lemma** *RDM-R2m* [closure]:  $M$  is RDM  $\implies M$  is R2m

by (*metis (no-types, hide-lams) Healthy-def R2m-idem RDM-def*)

**lemma** *ex-st'-R2m-closed* [closure]:

assumes  $P$  is R2m

shows  $(\exists \ \$st' \cdot P)$  is R2m

**proof** –

have  $R2m(\exists \ \$st' \cdot R2m \ P) = (\exists \ \$st' \cdot R2m \ P)$

by (*rel-auto*)

**thus** ?thesis

by (*metis Healthy-def' assms*)

**qed**

**lemma** *parallel-RR-closed*:

assumes  $P$  is RR  $Q$  is RR  $M$  is R2m

$\$<:ok \# M \ \$<:wait \# M \ \$ok' \# M \ \$wait' \# M$

shows  $P \parallel_M Q$  is RR

by (*rule RR-R2-intro, simp-all add: unrest assms RR-implies-R2 closure*)

**lemma** *parallel-ok-cases*:

$((P \parallel_s Q) \ ; \ M) = ($

$((P^t \parallel_s Q^t) \ ; \ (M \llbracket true, true / \$0:ok, \$1:ok \rrbracket))) \vee$

$((P^f \parallel_s Q^t) ;; (M[\text{false}, \text{true}/\$0:ok, \$1:ok])) \vee$   
 $((P^t \parallel_s Q^f) ;; (M[\text{true}, \text{false}/\$0:ok, \$1:ok])) \vee$   
 $((P^f \parallel_s Q^f) ;; (M[\text{false}, \text{false}/\$0:ok, \$1:ok]))$   
**proof** –  
**have**  $((P \parallel_s Q) ;; M) = (\exists \text{ ok}_0 \cdot (P \parallel_s Q)[\llbracket \text{ok}_0 \rrbracket / \$0:ok'] ;; M[\llbracket \text{ok}_0 \rrbracket / \$0:ok])$   
**by** (*subst segr-middle[of ok ;<sub>L</sub> mrg-left], simp-all*)  
**also have**  $\dots = (\exists \text{ ok}_0 \cdot \exists \text{ ok}_1 \cdot ((P \parallel_s Q)[\llbracket \text{ok}_0 \rrbracket / \$0:ok'] [\llbracket \text{ok}_1 \rrbracket / \$1:ok']) ;; (M[\llbracket \text{ok}_0 \rrbracket / \$0:ok] [\llbracket \text{ok}_1 \rrbracket / \$1:ok]))$   
**by** (*subst segr-middle[of ok ;<sub>L</sub> mrg-right], simp-all*)  
**also have**  $\dots = (\exists \text{ ok}_0 \cdot \exists \text{ ok}_1 \cdot (P[\llbracket \text{ok}_0 \rrbracket / \$ok'] \parallel_s Q[\llbracket \text{ok}_1 \rrbracket / \$ok']) ;; (M[\llbracket \text{ok}_0 \rrbracket, \llbracket \text{ok}_1 \rrbracket / \$0:ok, \$1:ok]))$   
**by** (*rel-auto robust*)  
**also have**  $\dots =$   
 $((P^t \parallel_s Q^t) ;; (M[\text{true}, \text{true}/\$0:ok, \$1:ok])) \vee$   
 $((P^f \parallel_s Q^t) ;; (M[\text{false}, \text{true}/\$0:ok, \$1:ok])) \vee$   
 $((P^t \parallel_s Q^f) ;; (M[\text{true}, \text{false}/\$0:ok, \$1:ok])) \vee$   
 $((P^f \parallel_s Q^f) ;; (M[\text{false}, \text{false}/\$0:ok, \$1:ok]))$   
**by** (*simp add: true-alt-def[THEN sym] false-alt-def[THEN sym] disj-assoc*  
*utp-pred-laws.sup.left-commute utp-pred-laws.sup-commute usubst*)  
**finally show** *?thesis* .  
**qed**

**lemma** *skip-srea-ok-f* [*usubst*]:  
 $II_R^f = R1(\neg \$ok)$   
**by** (*rel-auto*)

**lemma** *nmerge0-rd-unrest* [*unrest*]:  
 $\$0:ok \# N_0 \ M \ \$1:ok \# N_0 \ M$   
**by** (*pred-auto*) $+$

**lemma** *parallel-assm-lemma*:

**assumes** *P* is *RD2*  
**shows**  $pre_s \dagger (P \parallel_{M_R(M)} Q) = (((pre_s \dagger P) \parallel_{N_0(M)} ;; R1(\text{true}) (cmt_s \dagger Q))$   
 $\vee ((cmt_s \dagger P) \parallel_{N_0(M)} ;; R1(\text{true}) (pre_s \dagger Q)))$

**proof** –

**have**  $pre_s \dagger (P \parallel_{M_R(M)} Q) = pre_s \dagger ((P \parallel_s Q) ;; M_R(M))$   
**by** (*simp add: par-by-merge-def*)  
**also have**  $\dots = ((P \parallel_s Q)[\text{true}, \text{false}/\$ok, \$wait] ;; N_R \ M ;; R1(\neg \$ok))$   
**by** (*simp add: merge-rd-def usubst, rel-auto*)  
**also have**  $\dots = ((P[\text{true}, \text{false}/\$ok, \$wait] \parallel_s Q[\text{true}, \text{false}/\$ok, \$wait]) ;; N_1(M) ;; R1(\neg \$ok))$   
**by** (*rel-auto robust, (metis)+*)  
**also have**  $\dots =$   
 $((P[\text{true}, \text{false}/\$ok, \$wait])^t \parallel_s (Q[\text{true}, \text{false}/\$ok, \$wait])^t) ;; ((N_1 \ M)[\text{true}, \text{true}/\$0:ok, \$1:ok] ;;$   
 $R1(\neg \$ok))) \vee$   
 $((P[\text{true}, \text{false}/\$ok, \$wait])^f \parallel_s (Q[\text{true}, \text{false}/\$ok, \$wait])^t) ;; ((N_1 \ M)[\text{false}, \text{true}/\$0:ok, \$1:ok] ;;$   
 $R1(\neg \$ok))) \vee$   
 $((P[\text{true}, \text{false}/\$ok, \$wait])^t \parallel_s (Q[\text{true}, \text{false}/\$ok, \$wait])^f) ;; ((N_1 \ M)[\text{true}, \text{false}/\$0:ok, \$1:ok] ;;$   
 $R1(\neg \$ok))) \vee$   
 $((P[\text{true}, \text{false}/\$ok, \$wait])^f \parallel_s (Q[\text{true}, \text{false}/\$ok, \$wait])^f) ;; ((N_1 \ M)[\text{false}, \text{false}/\$0:ok, \$1:ok] ;;$   
 $R1(\neg \$ok)))$   
**(is**  $\neg = (?C1 \vee_p ?C2 \vee_p ?C3 \vee_p ?C4)$ **)**  
**by** (*subst parallel-ok-cases, subst-tac*)  
**also have**  $\dots = (?C2 \vee ?C3)$   
**proof** –  
**have**  $?C1 = \text{false}$   
**by** (*rel-auto*)  
**moreover have**  $'?C4 \Rightarrow ?C3'$  **(is**  $'(?A ;; ?B) \Rightarrow (?C ;; ?D)'$ **)**

**proof** –  
**from** *assms* **have** ‘ $P^f \Rightarrow P^t$ ’  
**by** (*metis RD2-def H2-equivalence Healthy-def*)  
**hence**  $P$ : ‘ $P^f_f \Rightarrow P^t_f$ ’  
**by** (*rel-auto*)  
**have** ‘ $?A \Rightarrow ?C$ ’  
**using**  $P$  **by** (*rel-auto*)  
**moreover have** ‘ $?B \Rightarrow ?D$ ’  
**by** (*rel-auto*)  
**ultimately show** *?thesis*  
**by** (*simp add: impl-seqr-mono*)  
**qed**  
**ultimately show** *?thesis*  
**by** (*simp add: subsumption2*)  
**qed**  
**also have** ... = (  
 $((pre_s \uparrow P) \parallel_s (cmt_s \uparrow Q)) \parallel ((N_0 \ M \ ; \ R1(true))) \vee$   
 $((cmt_s \uparrow P) \parallel_s (pre_s \uparrow Q)) \parallel ((N_0 \ M \ ; \ R1(true)))$ )  
**by** (*rel-auto, metis+*)  
**also have** ... = (  
 $((pre_s \uparrow P) \parallel_{N_0 \ M \ ; \ R1(true)} (cmt_s \uparrow Q)) \vee$   
 $((cmt_s \uparrow P) \parallel_{N_0 \ M \ ; \ R1(true)} (pre_s \uparrow Q))$ )  
**by** (*simp add: par-by-merge-def*)  
**finally show** *?thesis* .  
**qed**

**lemma** *pre<sub>s</sub>-SRD*:  
**assumes**  $P$  *is SRD*  
**shows**  $pre_s \uparrow P = (\neg_r pre_R(P))$   
**proof** –  
**have**  $pre_s \uparrow P = pre_s \uparrow \mathbf{R}_s(pre_R \ P \vdash \text{peri}_R \ P \diamond \text{post}_R \ P)$   
**by** (*simp add: SRD-reactive-tri-design assms*)  
**also have** ... =  $R1(R2c(\neg pre_s \uparrow pre_R \ P))$   
**by** (*simp add: RHS-def usubst R3h-def pre<sub>s</sub>-design*)  
**also have** ... =  $R1(R2c(\neg pre_R \ P))$   
**by** (*rel-auto*)  
**also have** ... =  $(\neg_r pre_R \ P)$   
**by** (*simp add: R2c-not R2c-preR assms rea-not-def*)  
**finally show** *?thesis* .  
**qed**

**lemma** *parallel-assm*:  
**assumes**  $P$  *is SRD*  $Q$  *is SRD*  
**shows**  $pre_R(P \parallel_{M_R(M)} Q) = (\neg_r ((\neg_r pre_R(P)) \parallel_{N_0(M)} \ ; \ R1(true) \ cmt_R(Q)) \wedge$   
 $\neg_r (cmt_R(P) \parallel_{N_0(M)} \ ; \ R1(true) \ (\neg_r pre_R(Q))))$   
**(is ?lhs = ?rhs)**

**proof** –  
**have**  $pre_R(P \parallel_{M_R(M)} Q) = (\neg_r (pre_s \uparrow P) \parallel_{N_0 \ M \ ; \ R1 \ true} (cmt_s \uparrow Q) \wedge$   
 $\neg_r (cmt_s \uparrow P) \parallel_{N_0 \ M \ ; \ R1 \ true} (pre_s \uparrow Q))$   
**by** (*simp add: pre<sub>R</sub>-def parallel-assm-lemma assms SRD-healths R1-conj rea-not-def [THEN sym]*)  
**also have** ... = *?rhs*  
**by** (*simp add: pre<sub>s</sub>-SRD assms cmt<sub>R</sub>-def Healthy-if closure unrest*)  
**finally show** *?thesis* .

qed

**lemma** *parallel-assm-unrest-wait'* [unrest]:

$\llbracket P \text{ is SRD}; Q \text{ is SRD} \rrbracket \implies \$wait' \# pre_R(P \parallel_{M_R(M)} Q)$

by (simp add: parallel-assm, simp add: par-by-merge-def unrest)

**lemma** *JL1*:  $(M_1 \ M)^t \llbracket false, true / \$0:ok, \$1:ok \rrbracket = N_0(M) ;; R1(true)$

by (rel-blast)

**lemma** *JL2*:  $(M_1 \ M)^t \llbracket true, false / \$0:ok, \$1:ok \rrbracket = N_0(M) ;; R1(true)$

by (rel-blast)

**lemma** *JL3*:  $(M_1 \ M)^t \llbracket false, false / \$0:ok, \$1:ok \rrbracket = N_0(M) ;; R1(true)$

by (rel-blast)

**lemma** *JL4*:  $(M_1 \ M)^t \llbracket true, true / \$0:ok, \$1:ok \rrbracket = (\$ok' \wedge N_0 \ M) ;; II_R^t$

by (simp add: merge-rd1-def usubst nmerge-rd1-def unrest)

**lemma** *parallel-commitment-lemma-1*:

assumes *P is RD2*

shows  $cmt_s \uparrow (P \parallel_{M_R(M)} Q) = ($

$((cmt_s \uparrow P) \parallel_{(\$ok' \wedge N_0 \ M) ;; II_R^t (cmt_s \uparrow Q)}) \vee$

$((pre_s \uparrow P) \parallel_{N_0(M) ;; R1(true) (cmt_s \uparrow Q)}) \vee$

$((cmt_s \uparrow P) \parallel_{N_0(M) ;; R1(true) (pre_s \uparrow Q)}))$

**proof** –

have  $cmt_s \uparrow (P \parallel_{M_R(M)} Q) = (P \llbracket true, false / \$ok, \$wait \rrbracket \parallel_{(M_1(M))^t} Q \llbracket true, false / \$ok, \$wait \rrbracket)$

by (simp add: usubst, rel-auto)

also have  $\dots = ((P \llbracket true, false / \$ok, \$wait \rrbracket \parallel_s Q \llbracket true, false / \$ok, \$wait \rrbracket) ;; (M_1 \ M)^t)$

by (simp add: par-by-merge-def)

also have  $\dots = ($

$((cmt_s \uparrow P) \parallel_s (cmt_s \uparrow Q)) ;; ((M_1 \ M)^t \llbracket true, true / \$0:ok, \$1:ok \rrbracket)) \vee$

$((pre_s \uparrow P) \parallel_s (cmt_s \uparrow Q)) ;; ((M_1 \ M)^t \llbracket false, true / \$0:ok, \$1:ok \rrbracket)) \vee$

$((cmt_s \uparrow P) \parallel_s (pre_s \uparrow Q)) ;; ((M_1 \ M)^t \llbracket true, false / \$0:ok, \$1:ok \rrbracket)) \vee$

$((pre_s \uparrow P) \parallel_s (pre_s \uparrow Q)) ;; ((M_1 \ M)^t \llbracket false, false / \$0:ok, \$1:ok \rrbracket))$

by (subst parallel-ok-cases, subst-tac)

also have  $\dots = ($

$((cmt_s \uparrow P) \parallel_s (cmt_s \uparrow Q)) ;; ((M_1 \ M)^t \llbracket true, true / \$0:ok, \$1:ok \rrbracket)) \vee$

$((pre_s \uparrow P) \parallel_s (cmt_s \uparrow Q)) ;; (N_0(M) ;; R1(true)) \vee$

$((cmt_s \uparrow P) \parallel_s (pre_s \uparrow Q)) ;; (N_0(M) ;; R1(true)) \vee$

$((pre_s \uparrow P) \parallel_s (pre_s \uparrow Q)) ;; (N_0(M) ;; R1(true)))$

(is - =  $(?C1 \vee_p ?C2 \vee_p ?C3 \vee_p ?C4)$ )

by (simp add: JL1 JL2 JL3)

also have  $\dots = ($

$((cmt_s \uparrow P) \parallel_s (cmt_s \uparrow Q)) ;; ((M_1(M))^t \llbracket true, true / \$0:ok, \$1:ok \rrbracket)) \vee$

$((pre_s \uparrow P) \parallel_s (cmt_s \uparrow Q)) ;; (N_0(M) ;; R1(true)) \vee$

$((cmt_s \uparrow P) \parallel_s (pre_s \uparrow Q)) ;; (N_0(M) ;; R1(true)))$

**proof** –

from *assms* have  $P^f \Rightarrow P^t$

by (metis RD2-def H2-equivalence Healthy-def)

hence  $P: P^f_f \Rightarrow P^t_f$

by (rel-auto)

have  $?C4 \Rightarrow ?C3$  (is  $(?A ;; ?B) \Rightarrow (?C ;; ?D)$ )

**proof** –

have  $?A \Rightarrow ?C$

```

    using P by (rel-auto)
  thus ?thesis
    by (simp add: impl-seqr-mono)
qed
thus ?thesis
  by (simp add: subsumption2)
qed
finally show ?thesis
  by (simp add: par-by-merge-def JL4)
qed

```

**lemma parallel-commitment-lemma-2:**

```

  assumes P is RD2
  shows  $cmt_s \uparrow (P \parallel_{M_R(M)} Q) =$ 
     $((cmt_s \uparrow P) \parallel_{(\$ok' \wedge N_0 M)} ;; II_R^t (cmt_s \uparrow Q)) \vee pre_s \uparrow (P \parallel_{M_R(M)} Q)$ 
  by (simp add: parallel-commitment-lemma-1 assms parallel-assm-lemma)

```

**lemma parallel-commitment-lemma-3:**

```

  M is R1m  $\implies$   $(\$ok' \wedge N_0 M) ;; II_R^t$  is R1m
  by (rel-simp, safe, metis+)

```

**lemma parallel-commitment:**

```

  assumes P is SRD Q is SRD M is RDM
  shows  $cmt_R(P \parallel_{M_R(M)} Q) = (pre_R(P \parallel_{M_R(M)} Q) \Rightarrow_r cmt_R(P) \parallel_{(\$ok' \wedge N_0 M)} ;; II_R^t cmt_R(Q))$ 
  by (simp add: parallel-commitment-lemma-2 parallel-commitment-lemma-3 Healthy-if assms cmt_R-def
    pre_s-SRD closure rea-impl-def disj-comm unrest)

```

**theorem parallel-reactive-design:**

```

  assumes P is SRD Q is SRD M is RDM
  shows  $(P \parallel_{M_R(M)} Q) = \mathbf{R}_s($ 
     $(\neg_r ((\neg_r pre_R(P)) \parallel_{N_0(M)} ;; R1(true) cmt_R(Q)) \wedge$ 
     $\neg_r (cmt_R(P) \parallel_{N_0(M)} ;; R1(true) (\neg_r pre_R(Q)))) \vdash$ 
     $(cmt_R(P) \parallel_{(\$ok' \wedge N_0 M)} ;; II_R^t cmt_R(Q)))$  (is ?lhs = ?rhs)

```

**proof** –

```

  have  $(P \parallel_{M_R(M)} Q) = \mathbf{R}_s(pre_R(P \parallel_{M_R(M)} Q) \vdash cmt_R(P \parallel_{M_R(M)} Q))$ 
  by (metis Healthy-def NSRD-is-SRD SRD-as-reactive-design assms(1) assms(2) assms(3) par-rdes-NSRD)
  also have ... = ?rhs
  by (simp add: parallel-assm parallel-commitment design-export-spec assms, rel-auto)
  finally show ?thesis .

```

**qed**

**lemma parallel-pericondition-lemma1:**

```

   $(\$ok' \wedge P) ;; II_R \llbracket true, true / \$ok', \$wait' \rrbracket = (\exists \$st' \cdot P) \llbracket true, true / \$ok', \$wait' \rrbracket$ 
  (is ?lhs = ?rhs)

```

**proof** –

```

  have ?lhs =  $(\$ok' \wedge P) ;; (\exists \$st \cdot II) \llbracket true, true / \$ok', \$wait' \rrbracket$ 
  by (rel-blast)
  also have ... = ?rhs
  by (rel-auto)
  finally show ?thesis .

```

**qed**

**lemma parallel-pericondition-lemma2:**

```

  assumes M is RDM

```



shows  $(\exists \$st' \cdot N_0(M)) \llbracket true, true/\$ok', \$wait' \rrbracket = ((\$0:wait \vee \$1:wait) \wedge (\exists \$st' \cdot M))$   
**proof** –  
 have  $(\exists \$st' \cdot N_0(M)) \llbracket true, true/\$ok', \$wait' \rrbracket = (\exists \$st' \cdot (\$0:wait \vee \$1:wait) \wedge \$tr' \geq_u \$<:tr \wedge M)$   
 by (simp add: usubst unrest nmerge-rd0-def ex-unrest Healthy-if R1m-def assms)  
 also have ... =  $(\exists \$st' \cdot (\$0:wait \vee \$1:wait) \wedge M)$   
 by (metis (no-types, hide-lams) Healthy-if R1m-def R1m-idem R2m-def RDM-def assms utp-pred-laws.inf-commute)  
 also have ... =  $((\$0:wait \vee \$1:wait) \wedge (\exists \$st' \cdot M))$   
 by (rel-auto)  
 finally show ?thesis .  
**qed**

**lemma parallel-pericondition-lemma3:**

$((\$0:wait \vee \$1:wait) \wedge (\exists \$st' \cdot M)) = ((\$0:wait \wedge \$1:wait \wedge (\exists \$st' \cdot M)) \vee (\neg \$0:wait \wedge \$1:wait \wedge (\exists \$st' \cdot M)) \vee (\$0:wait \wedge \neg \$1:wait \wedge (\exists \$st' \cdot M)))$   
 by (rel-auto)

**lemma parallel-pericondition [rdes]:**

fixes  $M :: ('s, 't::trace, 'a) \text{rsp merge}$

assumes  $P$  is SRD  $Q$  is SRD  $M$  is RDM

shows  $peri_R(P \parallel_{M_R(M)} Q) = (pre_R(P \parallel_{M_R(M)} Q) \Rightarrow_r peri_R(P) \parallel_{\exists \$st' \cdot M} peri_R(Q) \vee post_R(P) \parallel_{\exists \$st' \cdot M} peri_R(Q) \vee peri_R(P) \parallel_{\exists \$st' \cdot M} post_R(Q))$

**proof** –

have  $peri_R(P \parallel_{M_R(M)} Q) =$   
 $(pre_R(P \parallel_{M_R(M)} Q) \Rightarrow_r cmt_R P \parallel_{(\$ok' \wedge N_0 M) ;; II \llbracket true, true/\$ok', \$wait' \rrbracket} cmt_R Q)$   
 by (simp add: peri-cmt-def parallel-commitment SRD-healths assms usubst unrest assms)  
 also have ... =  $(pre_R(P \parallel_{M_R(M)} Q) \Rightarrow_r cmt_R P \parallel_{(\exists \$st' \cdot N_0 M) \llbracket true, true/\$ok', \$wait' \rrbracket} cmt_R Q)$   
 by (simp add: parallel-pericondition-lemma1)  
 also have ... =  $(pre_R(P \parallel_{M_R(M)} Q) \Rightarrow_r cmt_R P \parallel_{(\$0:wait \vee \$1:wait) \wedge (\exists \$st' \cdot M)} cmt_R Q)$   
 by (simp add: parallel-pericondition-lemma2 assms)  
 also have ... =  $(pre_R(P \parallel_{M_R(M)} Q) \Rightarrow_r ((\lceil cmt_R P \rceil_0 \wedge \lceil cmt_R Q \rceil_1 \wedge \$<:\mathbf{v}' =_u \$\mathbf{v}) ;; (\$0:wait \wedge \$1:wait \wedge (\exists \$st' \cdot M))) \vee ((\lceil cmt_R P \rceil_0 \wedge \lceil cmt_R Q \rceil_1 \wedge \$<:\mathbf{v}' =_u \$\mathbf{v}) ;; (\neg \$0:wait \wedge \$1:wait \wedge (\exists \$st' \cdot M))) \vee ((\lceil cmt_R P \rceil_0 \wedge \lceil cmt_R Q \rceil_1 \wedge \$<:\mathbf{v}' =_u \$\mathbf{v}) ;; (\$0:wait \wedge \neg \$1:wait \wedge (\exists \$st' \cdot M))))$   
 by (simp add: par-by-merge-alt-def parallel-pericondition-lemma3 seqr-or-distr)  
 also have ... =  $(pre_R(P \parallel_{M_R(M)} Q) \Rightarrow_r ((\lceil peri_R P \rceil_0 \wedge \lceil peri_R Q \rceil_1 \wedge \$<:\mathbf{v}' =_u \$\mathbf{v}) ;; (\exists \$st' \cdot M)) \vee ((\lceil post_R P \rceil_0 \wedge \lceil peri_R Q \rceil_1 \wedge \$<:\mathbf{v}' =_u \$\mathbf{v}) ;; (\exists \$st' \cdot M)) \vee ((\lceil peri_R P \rceil_0 \wedge \lceil post_R Q \rceil_1 \wedge \$<:\mathbf{v}' =_u \$\mathbf{v}) ;; (\exists \$st' \cdot M)))$   
 by (simp add: seqr-right-one-point-true seqr-right-one-point-false cmt\_R-def post\_R-def peri\_R-def usubst unrest assms)  
 also have ... =  $(pre_R(P \parallel_{M_R(M)} Q) \Rightarrow_r peri_R(P) \parallel_{\exists \$st' \cdot M} peri_R(Q) \vee post_R(P) \parallel_{\exists \$st' \cdot M} peri_R(Q) \vee peri_R(P) \parallel_{\exists \$st' \cdot M} post_R(Q))$   
 by (simp add: par-by-merge-alt-def)  
 finally show ?thesis .  
**qed**

**lemma parallel-postcondition-lemma1:**

$(\$ok' \wedge P) ;; II \llbracket true, false/\$ok', \$wait' \rrbracket = P \llbracket true, false/\$ok', \$wait' \rrbracket$   
 (is ?lhs = ?rhs)

**proof** –

have ?lhs =  $(\$ok' \wedge P) ;; II \llbracket true, false/\$ok', \$wait' \rrbracket$

by (rel-blast)  
 also have ... = ?rhs  
 by (rel-auto)  
 finally show ?thesis .  
 qed

lemma parallel-postcondition-lemma2:

assumes  $M$  is RDM  
 shows  $(N_0(M)) \llbracket \text{true}, \text{false} / \$ok', \$wait' \rrbracket = ((\neg \$0:wait \wedge \neg \$1:wait) \wedge M)$   
 proof –  
 have  $(N_0(M)) \llbracket \text{true}, \text{false} / \$ok', \$wait' \rrbracket = ((\neg \$0:wait \wedge \neg \$1:wait) \wedge \$tr' \geq_u \$<:tr \wedge M)$   
 by (simp add: usubst unrest nmerge-rd0-def ex-unrest Healthy-if R1m-def assms)  
 also have ... =  $((\neg \$0:wait \wedge \neg \$1:wait) \wedge M)$   
 by (metis Healthy-if R1m-def RDM-R1m assms utp-pred-laws.inf-commute)  
 finally show ?thesis .  
 qed

lemma parallel-postcondition [rdes]:

fixes  $M :: ('s, 't :: \text{trace}, 'a) \text{rsp merge}$   
 assumes  $P$  is SRD  $Q$  is SRD  $M$  is RDM  
 shows  $\text{post}_R(P \parallel_{M_R(M)} Q) = (\text{pre}_R(P \parallel_{M_R(M)} Q) \Rightarrow_r \text{post}_R(P) \parallel_M \text{post}_R(Q))$   
 proof –  
 have  $\text{post}_R(P \parallel_{M_R(M)} Q) =$   
 $(\text{pre}_R(P \parallel_{M_R(M)} Q) \Rightarrow_r \text{cmt}_R P \parallel (\$ok' \wedge N_0 M) ;; H_R \llbracket \text{true}, \text{false} / \$ok', \$wait' \rrbracket \text{cmt}_R Q)$   
 by (simp add: post-cmt-def parallel-commitment assms usubst unrest SRD-healths)  
 also have ... =  $(\text{pre}_R(P \parallel_{M_R(M)} Q) \Rightarrow_r \text{cmt}_R P \parallel (\neg \$0:wait \wedge \neg \$1:wait \wedge M) \text{cmt}_R Q)$   
 by (simp add: parallel-postcondition-lemma1 parallel-postcondition-lemma2 assms,  
 simp add: utp-pred-laws.inf-commute utp-pred-laws.inf-left-commute)  
 also have ... =  $(\text{pre}_R(P \parallel_{M_R(M)} Q) \Rightarrow_r \text{post}_R P \parallel_M \text{post}_R Q)$   
 by (simp add: par-by-merge-alt-def segr-right-one-point-false usubst unrest cmt<sub>R</sub>-def post<sub>R</sub>-def assms)  
 finally show ?thesis .  
 qed

lemma parallel-precondition-lemma:

fixes  $M :: ('s, 't :: \text{trace}, 'a) \text{rsp merge}$   
 assumes  $P$  is NSRD  $Q$  is NSRD  $M$  is RDM  
 shows  $(\neg_r \text{pre}_R(P)) \parallel_{N_0(M)} ;; R1(\text{true}) \text{cmt}_R(Q) =$   
 $((\neg_r \text{pre}_R P) \parallel_M ;; R1(\text{true}) \text{peri}_R Q \vee (\neg_r \text{pre}_R P) \parallel_M ;; R1(\text{true}) \text{post}_R Q)$   
 proof –  
 have  $((\neg_r \text{pre}_R(P)) \parallel_{N_0(M)} ;; R1(\text{true}) \text{cmt}_R(Q)) =$   
 $((\neg_r \text{pre}_R(P)) \parallel_{N_0(M)} ;; R1(\text{true}) (\text{peri}_R(Q) \diamond \text{post}_R(Q)))$   
 by (simp add: wait'-cond-peri-post-cmt)  
 also have ... =  $(([\neg_r \text{pre}_R(P)]_0 \wedge [\text{peri}_R(Q) \diamond \text{post}_R(Q)]_1 \wedge \$<:\mathbf{v}' =_u \$\mathbf{v}) ;; N_0(M) ;; R1(\text{true}))$   
 by (simp add: par-by-merge-alt-def)  
 also have ... =  $(([\neg_r \text{pre}_R(P)]_0 \wedge [\text{peri}_R(Q)]_1 \triangleleft \$1:wait' \triangleright [\text{post}_R(Q)]_1 \wedge \$<:\mathbf{v}' =_u \$\mathbf{v}) ;; N_0(M) ;; R1(\text{true}))$   
 by (simp add: wait'-cond-def alpha)  
 also have ... =  $((([\neg_r \text{pre}_R(P)]_0 \wedge [\text{peri}_R(Q)]_1 \wedge \$<:\mathbf{v}' =_u \$\mathbf{v}) \triangleleft \$1:wait' \triangleright ([\neg_r \text{pre}_R(P)]_0 \wedge [\text{post}_R(Q)]_1 \wedge \$<:\mathbf{v}' =_u \$\mathbf{v})) ;; N_0(M) ;; R1(\text{true}))$   
 (is  $(?P ;; -) = (?Q ;; -)$ )  
 proof –  
 have  $?P = ?Q$   
 by (rel-auto)  
 thus ?thesis by simp

qed

also have ... =  $(([\neg_r \text{pre}_R P]_0 \wedge [\text{peri}_R Q]_1 \wedge \$<:\mathbf{v}' =_u \$\mathbf{v}) \llbracket \text{true}/\$1:\text{wait}' \rrbracket ;; (N_0 M ;; R1 \text{true}) \llbracket \text{true}/\$1:\text{wait} \rrbracket \vee$   
 $([\neg_r \text{pre}_R P]_0 \wedge [\text{post}_R Q]_1 \wedge \$<:\mathbf{v}' =_u \$\mathbf{v}) \llbracket \text{false}/\$1:\text{wait}' \rrbracket ;; (N_0 M ;; R1 \text{true}) \llbracket \text{false}/\$1:\text{wait} \rrbracket)$

by (simp add: cond-inter-var-split)

also have ... =  $(([\neg_r \text{pre}_R P]_0 \wedge [\text{peri}_R Q]_1 \wedge \$<:\mathbf{v}' =_u \$\mathbf{v}) ;; N_0 M \llbracket \text{true}/\$1:\text{wait} \rrbracket ;; R1 \text{true} \vee$   
 $([\neg_r \text{pre}_R P]_0 \wedge [\text{post}_R Q]_1 \wedge \$<:\mathbf{v}' =_u \$\mathbf{v}) ;; N_0 M \llbracket \text{false}/\$1:\text{wait} \rrbracket ;; R1 \text{true})$

by (simp add: usubst unrest)

also have ... =  $(([\neg_r \text{pre}_R P]_0 \wedge [\text{peri}_R Q]_1 \wedge \$<:\mathbf{v}' =_u \$\mathbf{v}) ;; (\$wait' \wedge M) ;; R1 \text{true} \vee$   
 $([\neg_r \text{pre}_R P]_0 \wedge [\text{post}_R Q]_1 \wedge \$<:\mathbf{v}' =_u \$\mathbf{v}) ;; (\$wait' =_u \$0:\text{wait} \wedge M) ;; R1 \text{true})$

proof –

have  $(\$tr' \geq_u \$<:tr \wedge M) = M$

using RDM-R1m[OF assms(3)]

by (simp add: Healthy-def R1m-def conj-comm)

thus ?thesis

by (simp add: nmerge-rd0-def unrest assms closure ex-unrest usubst)

qed

also have ... =  $(([\neg_r \text{pre}_R P]_0 \wedge [\text{peri}_R Q]_1 \wedge \$<:\mathbf{v}' =_u \$\mathbf{v}) ;; M ;; R1 \text{true} \vee$   
 $([\neg_r \text{pre}_R P]_0 \wedge [\text{post}_R Q]_1 \wedge \$<:\mathbf{v}' =_u \$\mathbf{v}) ;; M ;; R1 \text{true})$

(is  $(?P_1 \vee_p ?P_2) = (?Q_1 \vee ?Q_2)$ )

proof –

have  $?P_1 = ([\neg_r \text{pre}_R P]_0 \wedge [\text{peri}_R Q]_1 \wedge \$<:\mathbf{v}' =_u \$\mathbf{v}) ;; (M \wedge \$wait') ;; R1 \text{true}$

by (simp add: conj-comm)

hence 1:  $?P_1 = ?Q_1$

by (simp add: segr-left-one-point-true segr-left-one-point-false add: unrest usubst closure assms)

have  $?P_2 = (([\neg_r \text{pre}_R P]_0 \wedge [\text{post}_R Q]_1 \wedge \$<:\mathbf{v}' =_u \$\mathbf{v}) ;; (M \wedge \$wait') ;; R1 \text{true} \vee$   
 $([\neg_r \text{pre}_R P]_0 \wedge [\text{post}_R Q]_1 \wedge \$<:\mathbf{v}' =_u \$\mathbf{v}) ;; (M \wedge \neg \$wait') ;; R1 \text{true})$

by (subst segr-bool-split[of wait ;L mrg-left], simp-all add: usubst unrest assms closure conj-comm)

hence 2:  $?P_2 = ?Q_2$

by (simp add: segr-left-one-point-true segr-left-one-point-false unrest usubst closure assms)

from 1 2 show ?thesis by simp

qed

also have ... =  $((\neg_r \text{pre}_R P) \parallel_M ;; R1(\text{true}) \text{peri}_R Q \vee (\neg_r \text{pre}_R P) \parallel_M ;; R1(\text{true}) \text{post}_R Q)$

by (simp add: par-by-merge-alt-def)

finally show ?thesis .

qed

**lemma** swap-nmerge-rd0:

$\text{swap}_m ;; N_0(M) = N_0(\text{swap}_m ;; M)$

by (rel-auto, meson+)

**lemma** SymMerge-nmerge-rd0 [closure]:

$M \text{ is SymMerge} \implies N_0(M) \text{ is SymMerge}$

by (rel-auto, meson+)

**lemma** swap-merge-rd':

$\text{swap}_m ;; N_R(M) = N_R(\text{swap}_m ;; M)$

by (rel-blast)

**lemma** swap-merge-rd:

$\text{swap}_m ;; M_R(M) = M_R(\text{swap}_m ;; M)$

by (simp add: merge-rd-def segr-assoc[THEN sym] swap-merge-rd')

**lemma** SymMerge-merge-rd [closure]:

$M$  is SymMerge  $\implies M_R(M)$  is SymMerge  
 by (simp add: Healthy-def swap-merge-rd)

**lemma** *nmerge-rd1-merge3*:

**assumes**  $M$  is RDM

**shows**  $\mathbf{M3}(N_1(M)) = (\$ok' =_u (\$0:ok \wedge \$1:0:ok \wedge \$1:1:ok) \wedge$   
 $\$wait' =_u (\$0:wait \vee \$1:0:wait \vee \$1:1:wait) \wedge$   
 $\mathbf{M3}(M))$

**proof** –

**have**  $\mathbf{M3}(N_1(M)) = \mathbf{M3}(\$ok' =_u (\$0:ok \wedge \$1:ok) \wedge$   
 $\$wait' =_u (\$0:wait \vee \$1:wait) \wedge$

$\$<:tr \leq_u \$tr' \wedge$

$(\exists \{\$0:ok, \$1:ok, \$<:ok, \$ok', \$0:wait, \$1:wait, \$<:wait, \$wait'\} \cdot RDM(M)))$

**by** (simp add: nmerge-rd1-def nmerge-rd0-def assms Healthy-if)

**also have**  $\dots = \mathbf{M3}(\$ok' =_u (\$0:ok \wedge \$1:ok) \wedge \$wait' =_u (\$0:wait \vee \$1:wait) \wedge RDM(M))$

**by** (rel-blast)

**also have**  $\dots = (\$ok' =_u (\$0:ok \wedge \$1:0:ok \wedge \$1:1:ok) \wedge \$wait' =_u (\$0:wait \vee \$1:0:wait \vee \$1:1:wait)$   
 $\wedge \mathbf{M3}(RDM(M)))$

**by** (rel-blast)

**also have**  $\dots = (\$ok' =_u (\$0:ok \wedge \$1:0:ok \wedge \$1:1:ok) \wedge \$wait' =_u (\$0:wait \vee \$1:0:wait \vee \$1:1:wait)$   
 $\wedge \mathbf{M3}(M))$

**by** (simp add: assms Healthy-if)

**finally show** ?thesis .

**qed**

**lemma** *nmerge-rd-merge3*:

$\mathbf{M3}(N_R(M)) = (\exists \$<:st \cdot \$v' =_u \$<:v) \triangleleft \$<:wait \triangleright \mathbf{M3}(N_1 M) \triangleleft \$<:ok \triangleright (\$<:tr \leq_u \$tr')$

**by** (rel-blast)

**lemma** *AssocMerge-nmerge-rd*:

**assumes**  $M$  is RDM AssocMerge  $M$

**shows** AssocMerge( $N_R(M)$ )

**proof** –

**have**  $1:\mathbf{M3}(M) = rotate_m ;; \mathbf{M3}(M)$

**using** assms **by** (simp add: AssocMerge-def)

**have**  $rotate_m ;; (\mathbf{M3}(N_R(M))) =$

$rotate_m ;;$

$((\exists \$<:st \cdot \$v' =_u \$<:v) \triangleleft \$<:wait \triangleright$

$(\$ok' =_u (\$0:ok \wedge \$1:0:ok \wedge \$1:1:ok) \wedge \$wait' =_u (\$0:wait \vee \$1:0:wait \vee \$1:1:wait) \wedge$

$\mathbf{M3}(M)) \triangleleft \$<:ok \triangleright$

$(\$<:tr \leq_u \$tr'))$

**by** (simp add: AssocMerge-def nmerge-rd-merge3 nmerge-rd1-merge3 assms)

**also have**  $\dots =$

$((\exists \$<:st \cdot \$v' =_u \$<:v) \triangleleft \$<:wait \triangleright$

$(\$ok' =_u (\$0:ok \wedge \$1:0:ok \wedge \$1:1:ok) \wedge \$wait' =_u (\$0:wait \vee \$1:0:wait \vee \$1:1:wait) \wedge$

$(rotate_m ;; \mathbf{M3}(M))) \triangleleft \$<:ok \triangleright$

$(\$<:tr \leq_u \$tr'))$

**by** (rel-blast)

**also have**  $\dots =$

$((\exists \$<:st \cdot \$v' =_u \$<:v) \triangleleft \$<:wait \triangleright$

$(\$ok' =_u (\$0:ok \wedge \$1:0:ok \wedge \$1:1:ok) \wedge \$wait' =_u (\$0:wait \vee \$1:0:wait \vee \$1:1:wait) \wedge$

$\mathbf{M3}(M)) \triangleleft \$<:ok \triangleright$

$(\$<:tr \leq_u \$tr'))$

**using** 1 **by** auto

**also have**  $\dots = \mathbf{M3}(N_R(M))$

by (simp add: AssocMerge-def nmerge-rd-merge3 nmerge-rd1-merge3 assms)  
 finally show ?thesis  
 using AssocMerge-def by blast  
 qed

**lemma** swap-merge-RDM-closed [closure]:

assumes  $M$  is RDM  
 shows  $\text{swap}_m \;; M$  is RDM

**proof** –

have  $\text{RDM}(\text{swap}_m \;; \text{RDM}(M)) = (\text{swap}_m \;; \text{RDM}(M))$   
 by (rel-auto)  
 thus ?thesis  
 by (metis Healthy-def' assms)

qed

**lemma** parallel-precondition:

fixes  $M :: ('s, 't::\text{trace}, 'a) \text{rsp merge}$   
 assumes  $P$  is NSRD  $Q$  is NSRD  $M$  is RDM

shows  $\text{pre}_R(P \parallel_{M_R(M)} Q) =$   
 $(\neg_r ((\neg_r \text{pre}_R P) \parallel_M \;; R1(\text{true}) \text{peri}_R Q) \wedge$   
 $\neg_r ((\neg_r \text{pre}_R P) \parallel_M \;; R1(\text{true}) \text{post}_R Q) \wedge$   
 $\neg_r ((\neg_r \text{pre}_R Q) \parallel_{(\text{swap}_m \;; M) \;; R1(\text{true}) \text{peri}_R P} P) \wedge$   
 $\neg_r ((\neg_r \text{pre}_R Q) \parallel_{(\text{swap}_m \;; M) \;; R1(\text{true}) \text{post}_R P} P))$

**proof** –

have  $a: (\neg_r \text{pre}_R(P)) \parallel_{N_0(M)} \;; R1(\text{true}) \text{cmt}_R(Q) =$   
 $((\neg_r \text{pre}_R P) \parallel_M \;; R1(\text{true}) \text{peri}_R Q \vee (\neg_r \text{pre}_R P) \parallel_M \;; R1(\text{true}) \text{post}_R Q)$   
 by (simp add: parallel-precondition-lemma assms)

have  $b: (\neg_r \text{cmt}_R P \parallel_{N_0 M} \;; R1 \text{true} (\neg_r \text{pre}_R Q)) =$   
 $(\neg_r (\neg_r \text{pre}_R(Q)) \parallel_{N_0(\text{swap}_m \;; M) \;; R1(\text{true}) \text{cmt}_R(P)})$

by (simp add: swap-nmerge-rd0[THEN sym] seqr-assoc[THEN sym] par-by-merge-def par-sep-swap)

have  $c: (\neg_r \text{pre}_R(Q)) \parallel_{N_0(\text{swap}_m \;; M) \;; R1(\text{true}) \text{cmt}_R(P)} =$   
 $((\neg_r \text{pre}_R Q) \parallel_{(\text{swap}_m \;; M) \;; R1(\text{true}) \text{peri}_R P} P \vee (\neg_r \text{pre}_R Q) \parallel_{(\text{swap}_m \;; M) \;; R1(\text{true}) \text{post}_R P} P)$

by (simp add: parallel-precondition-lemma closure assms)

show ?thesis

by (simp add: parallel-assm closure assms a b c, rel-auto)

qed

Weakest Parallel Precondition

**definition** wrR ::

$('t::\text{trace}, 'a) \text{hrel-rp} \Rightarrow$   
 $('t :: \text{trace}, 'a) \text{rp merge} \Rightarrow$   
 $('t, 'a) \text{hrel-rp} \Rightarrow$   
 $('t, 'a) \text{hrel-rp} (- \text{wr}_R'(-) - [60,0,61] 61)$

**where** [upred-defs]:  $Q \text{wr}_R(M) P = (\neg_r ((\neg_r P) \parallel_M \;; R1(\text{true}) Q))$

**lemma** wrR-R1 [closure]:

$M \text{ is R1m} \Longrightarrow Q \text{wr}_R(M) P \text{ is R1}$   
 by (simp add: wrR-def closure)

**lemma** R2-rea-not:  $R2(\neg_r P) = (\neg_r R2(P))$

by (rel-auto)

```

lemma wrR-R2-lemma:
  assumes  $P$  is R2  $Q$  is R2  $M$  is R2m
  shows  $((\neg_r P) \parallel_M Q) ;; R1(true_h)$  is R2
proof -
  have  $(\neg_r P) \parallel_M Q$  is R2
    by (simp add: closure assms)
  thus ?thesis
    by (simp add: closure)
qed

lemma wrR-R2 [closure]:
  assumes  $P$  is R2  $Q$  is R2  $M$  is R2m
  shows  $Q wr_R(M) P$  is R2
proof -
  have  $((\neg_r P) \parallel_M Q) ;; R1(true_h)$  is R2
    by (simp add: wrR-R2-lemma assms)
  thus ?thesis
    by (simp add: wrR-def wrR-R2-lemma par-by-merge-seq-add closure)
qed

lemma wrR-RR [closure]:
  assumes  $P$  is RR  $Q$  is RR  $M$  is RDM
  shows  $Q wr_R(M) P$  is RR
  apply (rule RR-intro)
  apply (simp-all add: unrest assms closure wrR-def rpred)
  apply (metis (no-types, lifting) Healthy-def' R1-R2c-commute R1-R2c-is-R2 R1-rea-not RDM-R2m
    RR-implies-R2 assms(1) assms(2) assms(3) par-by-merge-seq-add rea-not-R2-closed
    wrR-R2-lemma)
done

lemma wrR-RC [closure]:
  assumes  $P$  is RR  $Q$  is RR  $M$  is RDM
  shows  $(Q wr_R(M) P)$  is RC
  apply (rule RC-intro)
  apply (simp add: closure assms)
  apply (simp add: wrR-def rpred closure assms)
  apply (simp add: par-by-merge-def seqr-assoc)
done

lemma wppR-choice [wp]:  $(P \vee Q) wr_R(M) R = (P wr_R(M) R \wedge Q wr_R(M) R)$ 
proof -
  have  $(P \vee Q) wr_R(M) R =$ 
     $(\neg_r ((\neg_r R) ;; U0 \wedge (P ;; U1 \vee Q ;; U1) \wedge \$<:\mathbf{v}' =_u \$\mathbf{v}) ;; M ;; true_r)$ 
    by (simp add: wrR-def par-by-merge-def seqr-or-distl)
  also have  $\dots = (\neg_r ((\neg_r R) ;; U0 \wedge P ;; U1 \wedge \$<:\mathbf{v}' =_u \$\mathbf{v} \vee (\neg_r R) ;; U0 \wedge Q ;; U1 \wedge \$<:\mathbf{v}' =_u$ 
 $\$ \mathbf{v}) ;; M ;; true_r)$ 
    by (simp add: conj-disj-distr utp-pred-laws.inf-sup-distrib2)
  also have  $\dots = (\neg_r (((\neg_r R) ;; U0 \wedge P ;; U1 \wedge \$<:\mathbf{v}' =_u \$\mathbf{v}) ;; M ;; true_r \vee$ 
 $((\neg_r R) ;; U0 \wedge Q ;; U1 \wedge \$<:\mathbf{v}' =_u \$\mathbf{v}) ;; M ;; true_r))$ 
    by (simp add: seqr-or-distl)
  also have  $\dots = (P wr_R(M) R \wedge Q wr_R(M) R)$ 
    by (simp add: wrR-def par-by-merge-def)
  finally show ?thesis .
qed

```

**lemma** *wppR-impl* [wp]:  $(P \Rightarrow_r Q) \text{ wr}_R(M) R = ((\neg_r P) \text{ wr}_R(M) R \wedge Q \text{ wr}_R(M) R)$   
 by (simp add: rea-impl-def wp)

**lemma** *wppR-miracle* [wp]:  $\text{false wr}_R(M) P = \text{true}_r$   
 by (simp add: wrR-def)

**lemma** *wppR-true* [wp]:  $P \text{ wr}_R(M) \text{true}_r = \text{true}_r$   
 by (simp add: wrR-def)

**lemma** *parallel-precondition-wr* [rdes]:  
 assumes  $P$  is NSRD  $Q$  is NSRD  $M$  is RDM  
 shows  $\text{pre}_R(P \parallel_{M_R(M)} Q) = (\text{peri}_R(Q) \text{ wr}_R(M) \text{pre}_R(P) \wedge \text{post}_R(Q) \text{ wr}_R(M) \text{pre}_R(P) \wedge \text{peri}_R(P) \text{ wr}_R(\text{swap}_m ;; M) \text{pre}_R(Q) \wedge \text{post}_R(P) \text{ wr}_R(\text{swap}_m ;; M) \text{pre}_R(Q))$   
 by (simp add: assms parallel-precondition wrR-def)

**lemma** *rea-impl-merge-left*:  $(P \Rightarrow_r Q) \parallel_M R = (((\neg_r P) \parallel_M R) \vee (Q \parallel_M R))$   
 by (simp add: rea-impl-def par-by-merge-or-left)

**lemma** *rea-impl-merge-right*:  $P \parallel_M (Q \Rightarrow_r R) = (P \parallel_M (\neg_r Q) \vee P \parallel_M R)$   
 by (simp add: rea-impl-def par-by-merge-or-right)

**lemma** *parallel-pre-lemma*:  
 $((Q_1 \Rightarrow_r Q_2) \text{ wr}_R(M) P_1 \wedge (P_1 \Rightarrow_r P_2) \parallel_{\exists \$st' . M} (Q_1 \Rightarrow_r Q_2))$   
 $= ((Q_1 \Rightarrow_r Q_2) \text{ wr}_R(M) P_1 \wedge P_2 \parallel_{\exists \$st' . M} (Q_1 \Rightarrow_r Q_2))$   
 apply (simp add: par-by-merge-alt-def)  
 apply (rel-auto)  
 apply (meson order-refl)  
 apply (meson order-refl)  
 apply blast  
 apply blast  
 apply blast  
 apply blast  
 done

**lemma** *parallel-rdes-def* [rdes-def]:  
 assumes  $P_1$  is RC  $P_2$  is RR  $P_3$  is RR  $Q_1$  is RC  $Q_2$  is RR  $Q_3$  is RR  
 $\$st' \nVdash P_2 \$st' \nVdash Q_2$   
 $M$  is RDM  
 shows  $\mathbf{R}_s(P_1 \vdash P_2 \diamond P_3) \parallel_{M_R(M)} \mathbf{R}_s(Q_1 \vdash Q_2 \diamond Q_3) =$   
 $\mathbf{R}_s(((Q_1 \Rightarrow_r Q_2) \text{ wr}_R(M) P_1 \wedge (Q_1 \Rightarrow_r Q_3) \text{ wr}_R(M) P_1 \wedge$   
 $(P_1 \Rightarrow_r P_2) \text{ wr}_R(\text{swap}_m ;; M) Q_1 \wedge (P_1 \Rightarrow_r P_3) \text{ wr}_R(\text{swap}_m ;; M) Q_1) \vdash$   
 $(P_2 \parallel_{\exists \$st' . M} Q_2 \vee$   
 $P_3 \parallel_{\exists \$st' . M} Q_2 \vee P_2 \parallel_{\exists \$st' . M} Q_3) \diamond$   
 $(P_3 \parallel_M Q_3))$  (is ?lhs = ?rhs)

**proof** –

have ?lhs =  $\mathbf{R}_s(\text{pre}_R ?lhs \vdash \text{peri}_R ?lhs \diamond \text{post}_R ?lhs)$   
 by (simp add: SRD-reactive-tri-design assms closure)

also have ... =

$\mathbf{R}_s(((Q_1 \Rightarrow_r Q_2) \text{ wr}_R(M) P_1 \wedge (Q_1 \Rightarrow_r Q_3) \text{ wr}_R(M) P_1 \wedge$   
 $(P_1 \Rightarrow_r P_2) \text{ wr}_R(\text{swap}_m ;; M) Q_1 \wedge (P_1 \Rightarrow_r P_3) \text{ wr}_R(\text{swap}_m ;; M) Q_1) \vdash$   
 $((P_1 \Rightarrow_r P_2) \parallel_{\exists \$st' . M} (Q_1 \Rightarrow_r Q_2) \vee$   
 $(P_1 \Rightarrow_r P_3) \parallel_{\exists \$st' . M} (Q_1 \Rightarrow_r Q_2) \vee (P_1 \Rightarrow_r P_2) \parallel_{\exists \$st' . M} (Q_1 \Rightarrow_r Q_3)) \diamond$   
 $((P_1 \Rightarrow_r P_3) \parallel_M (Q_1 \Rightarrow_r Q_3)))$   
 (is - =  $\mathbf{R}_s(?X$

$\vdash (?Y_1 \vee ?Y_2 \vee ?Y_3)$   
 $\diamond ?Z))$   
 by (simp add: rdes closure unrest assms, rel-auto)  
 also have ... =  $\mathbf{R}_s(?X \vdash ((?X \wedge ?Y_1) \vee (?X \wedge ?Y_2) \vee (?X \wedge ?Y_3)) \diamond (?X \wedge ?Z))$   
 by (rel-auto)  
 also have ... =  $\mathbf{R}_s(?X \vdash ((?X \wedge P_2 \parallel_{\exists} \$st' . M Q_2) \vee (?X \wedge P_3 \parallel_{\exists} \$st' . M Q_2) \vee (?X \wedge P_2 \parallel_{\exists} \$st' . M Q_3)) \diamond (?X \wedge P_3 \parallel_M Q_3))$   
 proof -  
 have 1:  $(?X \wedge ?Y_1) = (?X \wedge P_2 \parallel_{\exists} \$st' . M Q_2)$   
 by (rel-auto, meson order-refl, meson order-refl, meson order-refl, blast+)  
 have 2:  $(?X \wedge ?Y_2) = (?X \wedge P_3 \parallel_{\exists} \$st' . M Q_2)$   
 by (rel-auto, meson order-refl, meson order-refl, meson order-refl, blast+)  
 have 3:  $(?X \wedge ?Y_3) = (?X \wedge P_2 \parallel_{\exists} \$st' . M Q_3)$   
 by (rel-auto, meson order-refl, meson order-refl, meson order-refl, blast+)  
 have 4:  $(?X \wedge ?Z) = (?X \wedge P_3 \parallel_M Q_3)$   
 by (rel-auto, meson order-refl, meson order-refl, meson order-refl, blast+)  
 show ?thesis  
 by (simp add: 1 2 3 4)  
 qed  
 also have ... = ?rhs  
 by (rel-auto)  
 finally show ?thesis .  
 qed

**lemma** *Miracle-parallel-left-zero:*

assumes *P is SRD M is RDM*

shows *Miracle*  $\parallel_{RM} P = \text{Miracle}$

proof -

have  $pre_R(\text{Miracle} \parallel_{RM} P) = true_r$

by (simp add: parallel-assm wait'-cond-idem rdes closure assms)

moreover hence  $cmt_R(\text{Miracle} \parallel_{RM} P) = false$

by (simp add: rdes closure wait'-cond-idem SRD-healths assms)

ultimately have  $\text{Miracle} \parallel_{RM} P = \mathbf{R}_s(true_r \vdash false)$

by (metis NSRD-iff SRD-reactive-design-alt assms par-rdes-NSRD sdes-theory.weak.top-closed)

thus ?thesis

by (simp add: Miracle-def R1-design-R1-pre)

qed

**lemma** *Miracle-parallel-right-zero:*

assumes *P is SRD M is RDM*

shows  $P \parallel_{RM} \text{Miracle} = \text{Miracle}$

proof -

have  $pre_R(P \parallel_{RM} \text{Miracle}) = true_r$

by (simp add: wait'-cond-idem parallel-assm rdes closure assms)

moreover hence  $cmt_R(P \parallel_{RM} \text{Miracle}) = false$

by (simp add: wait'-cond-idem rdes closure SRD-healths assms)

ultimately have  $P \parallel_{RM} \text{Miracle} = \mathbf{R}_s(true_r \vdash false)$

by (metis NSRD-iff SRD-reactive-design-alt assms par-rdes-NSRD sdes-theory.weak.top-closed)

thus ?thesis

by (simp add: Miracle-def R1-design-R1-pre)

qed

## 8.1 Example basic merge

**definition** *BasicMerge* ::  $((s, 't::trace, unit) \text{ resp}) \text{ merge } (N_B)$  **where**



[upred-defs]:  $BasicMerge = (\$<:tr \leq_u \$tr' \wedge \$tr' - \$<:tr =_u \$0:tr - \$<:tr \wedge \$tr' - \$<:tr =_u \$1:tr - \$<:tr \wedge \$st' =_u \$<:st)$

**abbreviation**  $r_{basic-par} (- \parallel_B - [85,86] \ 85)$  **where**

$P \parallel_B Q \equiv P \parallel_{M_R(N_B)} Q$

**lemma**  $BasicMerge-RDM$  [closure]:  $N_B$  is RDM

**by** (rule RDM-intro, (rel-auto)+)

**lemma**  $BasicMerge-SymMerge$  [closure]:

$N_B$  is SymMerge

**by** (rel-auto)

**lemma**  $BasicMerge'-calc$ :

**assumes**  $\$ok' \# P \$wait' \# P \$ok' \# Q \$wait' \# Q$   $P$  is R2  $Q$  is R2

**shows**  $P \parallel_{N_B} Q = ((\exists \$st' \cdot P) \wedge (\exists \$st' \cdot Q) \wedge \$st' =_u \$st)$

**using** *assms*

**proof** –

**have**  $P: (\exists \{ \$ok', \$wait' \} \cdot R2(P)) = P$  (**is**  $?P' = -$ )

**by** (simp add: ex-unrest ex-plus Healthy-if *assms*)

**have**  $Q: (\exists \{ \$ok', \$wait' \} \cdot R2(Q)) = Q$  (**is**  $?Q' = -$ )

**by** (simp add: ex-unrest ex-plus Healthy-if *assms*)

**have**  $?P' \parallel_{N_B} ?Q' = ((\exists \$st' \cdot ?P') \wedge (\exists \$st' \cdot ?Q') \wedge \$st' =_u \$st)$

**by** (simp add: par-by-merge-alt-def, rel-auto, blast+)

**thus** *?thesis*

**by** (simp add:  $P \ Q$ )

qed

## 8.2 Simple parallel composition

**definition**  $rea\text{-}design\text{-}par ::$

$('s, 't::trace, 'a) \ hrel\text{-}rsp \Rightarrow ('s, 't, 'a) \ hrel\text{-}rsp \Rightarrow ('s, 't, 'a) \ hrel\text{-}rsp$  (**infixr**  $\parallel_R$  85)

**where** [upred-defs]:  $P \parallel_R Q = \mathbf{R}_s((pre_R(P) \wedge pre_R(Q)) \vdash (cmt_R(P) \wedge cmt_R(Q)))$

**lemma**  $RHS\text{-}design\text{-}par$ :

**assumes**

$\$ok' \# P_1 \$ok' \# P_2$

**shows**  $\mathbf{R}_s(P_1 \vdash Q_1) \parallel_R \mathbf{R}_s(P_2 \vdash Q_2) = \mathbf{R}_s((P_1 \wedge P_2) \vdash (Q_1 \wedge Q_2))$

**proof** –

**have**  $\mathbf{R}_s(P_1 \vdash Q_1) \parallel_R \mathbf{R}_s(P_2 \vdash Q_2) =$

$\mathbf{R}_s(P_1 \llbracket true, false / \$ok, \$wait \rrbracket \vdash Q_1 \llbracket true, false / \$ok, \$wait \rrbracket) \parallel_R \mathbf{R}_s(P_2 \llbracket true, false / \$ok, \$wait \rrbracket \vdash$

$Q_2 \llbracket true, false / \$ok, \$wait \rrbracket)$

**by** (simp add:  $RHS\text{-}design\text{-}ok\text{-}wait$ )

**also from** *assms*

**have** ... =

$\mathbf{R}_s((R1 (R2c (P_1)) \wedge R1 (R2c (P_2))) \llbracket true, false / \$ok, \$wait \rrbracket \vdash$

$(R1 (R2c (P_1 \Rightarrow Q_1)) \wedge R1 (R2c (P_2 \Rightarrow Q_2))) \llbracket true, false / \$ok, \$wait \rrbracket)$

**apply** (simp add:  $rea\text{-}design\text{-}par\text{-}def$   $rea\text{-}pre\text{-}RHS\text{-}design$   $rea\text{-}cmt\text{-}RHS\text{-}design$  *usubst unrest assms*)

**apply** (rule *cong*[of  $\mathbf{R}_s \ \mathbf{R}_s$ ], simp)

**using** *assms* **apply** (rel-auto)

**done**

**also have** ... =

$\mathbf{R}_s((R2c(P_1) \wedge R2c(P_2)) \vdash$

$(R1 (R2s (P_1 \Rightarrow Q_1)) \wedge R1 (R2s (P_2 \Rightarrow Q_2))))$

**by** (*metis* (*no-types*, *hide-lams*)  $R1\text{-}R2s\text{-}R2c$   $R1\text{-}conj$   $R1\text{-}design\text{-}R1\text{-}pre$   $RHS\text{-}design\text{-}ok\text{-}wait$ )

also have ... =  
 $\mathbf{R}_s((P_1 \wedge P_2) \vdash (R1 (R2s (P_1 \Rightarrow Q_1)) \wedge R1 (R2s (P_2 \Rightarrow Q_2))))$   
 by (*simp add: R2c-R3h-commute R2c-and R2c-design R2c-idem R2c-not RHS-def*)  
 also have ... =  $\mathbf{R}_s((P_1 \wedge P_2) \vdash ((P_1 \Rightarrow Q_1) \wedge (P_2 \Rightarrow Q_2)))$   
 by (*metis (no-types, lifting) R1-conj R2s-conj RHS-design-export-R1 RHS-design-export-R2s*)  
 also have ... =  $\mathbf{R}_s((P_1 \wedge P_2) \vdash (Q_1 \wedge Q_2))$   
 by (*rule cong[of  $\mathbf{R}_s \mathbf{R}_s$ ], simp, rel-auto*)  
 finally show ?thesis .  
 qed

**lemma** *RHS-tri-design-par*:  
 assumes  $\$ok' \# P_1 \$ok' \# P_2$   
 shows  $\mathbf{R}_s(P_1 \vdash Q_1 \diamond R_1) \parallel_R \mathbf{R}_s(P_2 \vdash Q_2 \diamond R_2) = \mathbf{R}_s((P_1 \wedge P_2) \vdash (Q_1 \wedge Q_2) \diamond (R_1 \wedge R_2))$   
 by (*simp add: RHS-design-par assms unrest wait'-cond-conj-exchange*)

**lemma** *RHS-tri-design-par-RR* [*rdes-def*]:  
 assumes  $P_1$  is RR  $P_2$  is RR  
 shows  $\mathbf{R}_s(P_1 \vdash Q_1 \diamond R_1) \parallel_R \mathbf{R}_s(P_2 \vdash Q_2 \diamond R_2) = \mathbf{R}_s((P_1 \wedge P_2) \vdash (Q_1 \wedge Q_2) \diamond (R_1 \wedge R_2))$   
 by (*simp add: RHS-tri-design-par unrest assms*)

**lemma** *RHS-comp-assoc*:  
 assumes  $P$  is NSRD  $Q$  is NSRD  $R$  is NSRD  
 shows  $(P \parallel_R Q) \parallel_R R = P \parallel_R Q \parallel_R R$   
 by (*rdes-eq cls: assms*)

end

## 9 Productive Reactive Designs

**theory** *utp-rdes-productive*  
**imports** *utp-rdes-parallel*  
**begin**

### 9.1 Healthiness condition

A reactive design is productive if it strictly increases the trace, whenever it terminates. If it does not terminate, it is also classed as productive.

**definition** *Productive* ::  $(s, t::\text{trace}, \alpha) \text{ hrel-rsp} \Rightarrow (s, t, \alpha) \text{ hrel-rsp}$  **where**  
 [*upred-defs*]:  $\text{Productive}(P) = P \parallel_R \mathbf{R}_s(\text{true} \vdash \text{true} \diamond (\$tr <_u \$tr'))$

**lemma** *Productive-RHS-design-form*:  
 assumes  $\$ok' \# P \$ok' \# Q \$ok' \# R$   
 shows  $\text{Productive}(\mathbf{R}_s(P \vdash Q \diamond R)) = \mathbf{R}_s(P \vdash Q \diamond (R \wedge \$tr <_u \$tr'))$   
 using *assms* by (*simp add: Productive-def RHS-tri-design-par unrest*)

**lemma** *Productive-form*:  
 $\text{Productive}(\text{SRD}(P)) = \mathbf{R}_s(\text{pre}_R(P) \vdash \text{peri}_R(P) \diamond (\text{post}_R(P) \wedge \$tr <_u \$tr'))$

**proof** –

have  $\text{Productive}(\text{SRD}(P)) = \mathbf{R}_s(\text{pre}_R(P) \vdash \text{peri}_R(P) \diamond \text{post}_R(P)) \parallel_R \mathbf{R}_s(\text{true} \vdash \text{true} \diamond (\$tr <_u \$tr'))$   
 by (*simp add: Productive-def SRD-as-reactive-tri-design*)  
 also have ... =  $\mathbf{R}_s(\text{pre}_R(P) \vdash \text{peri}_R(P) \diamond (\text{post}_R(P) \wedge \$tr <_u \$tr'))$   
 by (*simp add: RHS-tri-design-par unrest*)  
 finally show ?thesis .

qed

A reactive design is productive provided that the postcondition, under the precondition, strictly increases the trace.

**lemma** *Productive-intro*:

assumes  $P$  is SRD  $(\$tr <_u \$tr') \sqsubseteq (pre_R(P) \wedge post_R(P)) \$wait' \# pre_R(P)$

shows  $P$  is Productive

**proof** –

have  $P : \mathbf{R}_s(pre_R(P) \vdash peri_R(P) \diamond (post_R(P) \wedge \$tr <_u \$tr')) = P$

**proof** –

have  $\mathbf{R}_s(pre_R(P) \vdash peri_R(P) \diamond post_R(P)) = \mathbf{R}_s(pre_R(P) \vdash (pre_R(P) \wedge peri_R(P)) \diamond (pre_R(P) \wedge post_R(P)))$

by (metis (no-types, hide-lams) design-export-pre wait'-cond-conj-exchange wait'-cond-idem)

also have  $\dots = \mathbf{R}_s(pre_R(P) \vdash (pre_R(P) \wedge peri_R(P)) \diamond (pre_R(P) \wedge (post_R(P) \wedge \$tr <_u \$tr')))$

by (metis assms(2) utp-pred-laws.inf.absorb1 utp-pred-laws.inf.assoc)

also have  $\dots = \mathbf{R}_s(pre_R(P) \vdash peri_R(P) \diamond (post_R(P) \wedge \$tr <_u \$tr'))$

by (metis (no-types, hide-lams) design-export-pre wait'-cond-conj-exchange wait'-cond-idem)

finally show *?thesis*

by (simp add: SRD-reactive-tri-design assms(1))

**qed**

thus *?thesis*

by (metis Healthy-def RHS-tri-design-par Productive-def ok'-pre-unrest unrest-true utp-pred-laws.inf-right-idem utp-pred-laws.inf-top-right)

**qed**

**lemma** *Productive-post-refines-tr-increase*:

assumes  $P$  is SRD  $P$  is Productive  $\$wait' \# pre_R(P)$

shows  $(\$tr <_u \$tr') \sqsubseteq (pre_R(P) \wedge post_R(P))$

**proof** –

have  $post_R(P) = post_R(\mathbf{R}_s(pre_R(P) \vdash peri_R(P) \diamond (post_R(P) \wedge \$tr <_u \$tr')))$

by (metis Healthy-def Productive-form assms(1) assms(2))

also have  $\dots = R1(R2c(pre_R(P) \Rightarrow (post_R(P) \wedge \$tr <_u \$tr')))$

by (simp add: rea-post-RHS-design unrest usubst assms, rel-auto)

also have  $\dots = R1((pre_R(P) \Rightarrow (post_R(P) \wedge \$tr <_u \$tr')))$

by (simp add: R2c-impl R2c-preR R2c-postR R2c-and R2c-tr-less-tr' assms)

also have  $(\$tr <_u \$tr') \sqsubseteq (pre_R(P) \wedge \dots)$

by (rel-auto)

finally show *?thesis* .

**qed**

**lemma** *Continuous-Productive [closure]: Continuous Productive*

by (simp add: Continuous-def Productive-def, rel-auto)

## 9.2 Reactive design calculations

**lemma** *preR-Productive [rdes]*:

assumes  $P$  is SRD

shows  $pre_R(Productive(P)) = pre_R(P)$

**proof** –

have  $pre_R(Productive(P)) = pre_R(\mathbf{R}_s(pre_R(P) \vdash peri_R(P) \diamond (post_R(P) \wedge \$tr <_u \$tr')))$

by (metis Healthy-def Productive-form assms)

thus *?thesis*

by (simp add: rea-pre-RHS-design usubst unrest R2c-not R2c-preR R1-preR Healthy-if assms)

**qed**

**lemma** *periR-Productive [rdes]*:

assumes  $P$  is NSRD

shows  $\text{peri}_R(\text{Productive}(P)) = \text{peri}_R(P)$   
**proof** –  
 have  $\text{peri}_R(\text{Productive}(P)) = \text{peri}_R(\mathbf{R}_s(\text{pre}_R(P) \vdash \text{peri}_R(P) \diamond (\text{post}_R(P) \wedge \$tr <_u \$tr')))$   
 by (*metis Healthy-def NSRD-is-SRD Productive-form assms*)  
 also have  $\dots = R1 (R2c (\text{pre}_R P \Rightarrow_r \text{peri}_R P))$   
 by (*simp add: rea-peri-RHS-design usubst unrest R2c-not assms closure*)  
 also have  $\dots = (\text{pre}_R P \Rightarrow_r \text{peri}_R P)$   
 by (*simp add: R1-rea-impl R2c-rea-impl R2c-preR R2c-peri-SRD R1-peri-SRD assms closure R1-tr-less-tr' R2c-tr-less-tr'*)  
 finally show *?thesis*  
 by (*simp add: SRD-peri-under-pre assms unrest closure*)  
**qed**

**lemma** *postR-Productive [rdes]*:  
 assumes *P is NSRD*  
 shows  $\text{post}_R(\text{Productive}(P)) = (\text{pre}_R(P) \Rightarrow_r \text{post}_R(P) \wedge \$tr <_u \$tr')$   
**proof** –  
 have  $\text{post}_R(\text{Productive}(P)) = \text{post}_R(\mathbf{R}_s(\text{pre}_R(P) \vdash \text{peri}_R(P) \diamond (\text{post}_R(P) \wedge \$tr <_u \$tr')))$   
 by (*metis Healthy-def NSRD-is-SRD Productive-form assms*)  
 also have  $\dots = R1 (R2c (\text{pre}_R P \Rightarrow_r \text{post}_R P \wedge \$tr' >_u \$tr))$   
 by (*simp add: rea-post-RHS-design usubst unrest assms closure*)  
 also have  $\dots = (\text{pre}_R P \Rightarrow_r \text{post}_R P \wedge \$tr' >_u \$tr)$   
 by (*simp add: R1-rea-impl R2c-rea-impl R2c-preR R2c-and R1-extend-conj' R2c-post-SRD R1-post-SRD assms closure R1-tr-less-tr' R2c-tr-less-tr'*)  
 finally show *?thesis* .  
**qed**

**lemma** *preR-frame-seq-export*:  
 assumes *P is NSRD P is Productive Q is NSRD*  
 shows  $(\text{pre}_R P \wedge (\text{pre}_R P \wedge \text{post}_R P) ;; Q) = (\text{pre}_R P \wedge (\text{post}_R P ;; Q))$   
**proof** –  
 have  $(\text{pre}_R P \wedge (\text{post}_R P ;; Q)) = (\text{pre}_R P \wedge ((\text{pre}_R P \Rightarrow_r \text{post}_R P) ;; Q))$   
 by (*simp add: SRD-post-under-pre assms closure unrest*)  
 also have  $\dots = (\text{pre}_R P \wedge (((\neg_r \text{pre}_R P) ;; Q \vee (\text{pre}_R P \Rightarrow_r R1(\text{post}_R P)) ;; Q)))$   
 by (*simp add: NSRD-is-SRD R1-post-SRD assms(1) rea-impl-def seqr-or-distl R1-preR Healthy-if*)  
 also have  $\dots = (\text{pre}_R P \wedge (((\neg_r \text{pre}_R P) ;; Q \vee (\text{pre}_R P \wedge \text{post}_R P) ;; Q)))$   
**proof** –  
 have  $(\text{pre}_R P \vee \neg_r \text{pre}_R P) = R1 \text{ true}$   
 by (*simp add: R1-preR rea-not-or*)  
 then show *?thesis*  
 by (*metis (no-types, lifting) R1-def conj-comm disj-comm disj-conj-distr rea-impl-def seqr-or-distl utp-pred-laws.inf-top-left utp-pred-laws.sup.left-idem*)  
**qed**  
 also have  $\dots = (\text{pre}_R P \wedge (((\neg_r \text{pre}_R P) \vee (\text{pre}_R P \wedge \text{post}_R P) ;; Q)))$   
 by (*simp add: NSRD-neg-pre-left-zero assms closure SRD-healths*)  
 also have  $\dots = (\text{pre}_R P \wedge (\text{pre}_R P \wedge \text{post}_R P) ;; Q)$   
 by (*rel-blast*)  
 finally show *?thesis* ..  
**qed**

### 9.3 Closure laws

**lemma** *Productive-rdes-intro*:  
 assumes  $(\$tr <_u \$tr') \sqsubseteq R \$ok' \# P \$ok' \# Q \$ok' \# R \$wait \# P \$wait' \# P$   
 shows  $(\mathbf{R}_s(P \vdash Q \diamond R))$  is *Productive*  
**proof** (*rule Productive-intro*)

**show**  $\mathbf{R}_s (P \vdash Q \diamond R)$  *is SRD*  
**by** (*simp add: RHS-tri-design-is-SRD assms*)  
  
**from** *assms(1)* **show**  $(\$tr' >_u \$tr) \sqsubseteq (pre_R (\mathbf{R}_s (P \vdash Q \diamond R)) \wedge post_R (\mathbf{R}_s (P \vdash Q \diamond R)))$   
**apply** (*simp add: rea-pre-RHS-design rea-post-RHS-design usubst assms unrest*)  
**using** *assms(1)* **apply** (*rel-auto*)  
**apply** *fastforce*  
**done**  
  
**show**  $\$wait' \# pre_R (\mathbf{R}_s (P \vdash Q \diamond R))$   
**by** (*simp add: rea-pre-RHS-design rea-post-RHS-design usubst R1-def R2c-def R2s-def assms unrest*)  
**qed**

We use the  $R4$  healthiness condition to characterise that the postcondition must extend the trace for a reactive design to be productive.

**lemma** *Productive-rdes-RR-intro [closure]:*  
**assumes**  $P$  *is RR*  $Q$  *is RR*  $R$  *is RR*  $R$  *is R4*  
**shows**  $(\mathbf{R}_s(P \vdash Q \diamond R))$  *is Productive*  
**using** *assms* **by** (*simp add: Productive-rdes-intro R4-iff-refine unrest*)

**lemma** *Productive-Miracle [closure]: Miracle is Productive*  
**unfolding** *Miracle-tri-def Healthy-def*  
**by** (*subst Productive-RHS-design-form, simp-all add: unrest*)

**lemma** *Productive-Chaos [closure]: Chaos is Productive*  
**unfolding** *Chaos-tri-def Healthy-def*  
**by** (*subst Productive-RHS-design-form, simp-all add: unrest*)

**lemma** *Productive-intChoice [closure]:*  
**assumes**  $P$  *is SRD*  $P$  *is Productive*  $Q$  *is SRD*  $Q$  *is Productive*  
**shows**  $P \sqcap Q$  *is Productive*  
**proof** –  
**have**  $P \sqcap Q =$   
 $\mathbf{R}_s(pre_R(P) \vdash peri_R(P) \diamond (post_R(P) \wedge \$tr <_u \$tr')) \sqcap \mathbf{R}_s(pre_R(Q) \vdash peri_R(Q) \diamond (post_R(Q) \wedge \$tr <_u \$tr'))$   
**by** (*metis Healthy-if Productive-form assms*)  
**also have**  $\dots = \mathbf{R}_s(((pre_R P \wedge pre_R Q) \vdash (peri_R P \vee peri_R Q) \diamond ((post_R P \wedge \$tr' >_u \$tr) \vee (post_R Q \wedge \$tr' >_u \$tr))))$   
**by** (*simp add: RHS-tri-design-choice*)  
**also have**  $\dots = \mathbf{R}_s((pre_R P \wedge pre_R Q) \vdash (peri_R P \vee peri_R Q) \diamond (((post_R P) \vee (post_R Q)) \wedge \$tr' >_u \$tr))$   
**by** (*rule cong[of  $\mathbf{R}_s \mathbf{R}_s$ ], simp, rel-auto*)  
**also have**  $\dots$  *is Productive*  
**by** (*simp add: Healthy-def Productive-RHS-design-form unrest*)  
**finally show** *?thesis* .  
**qed**

**lemma** *Productive-cond-rea [closure]:*  
**assumes**  $P$  *is SRD*  $P$  *is Productive*  $Q$  *is SRD*  $Q$  *is Productive*  
**shows**  $P \triangleleft b \triangleright_R Q$  *is Productive*  
**proof** –  
**have**  $P \triangleleft b \triangleright_R Q =$   
 $\mathbf{R}_s(pre_R(P) \vdash peri_R(P) \diamond (post_R(P) \wedge \$tr <_u \$tr')) \triangleleft b \triangleright_R \mathbf{R}_s(pre_R(Q) \vdash peri_R(Q) \diamond (post_R(Q) \wedge \$tr <_u \$tr'))$   
**by** (*metis Healthy-if Productive-form assms*)

**also have** ... =  $\mathbf{R}_s ((pre_R P \triangleleft b \triangleright_R pre_R Q) \vdash (peri_R P \triangleleft b \triangleright_R peri_R Q) \diamond ((post_R P \wedge \$tr' >_u \$tr) \triangleleft b \triangleright_R (post_R Q \wedge \$tr' >_u \$tr)))$   
**by** (*simp add: cond-srea-form*)  
**also have** ... =  $\mathbf{R}_s ((pre_R P \triangleleft b \triangleright_R pre_R Q) \vdash (peri_R P \triangleleft b \triangleright_R peri_R Q) \diamond (((post_R P) \triangleleft b \triangleright_R (post_R Q)) \wedge \$tr' >_u \$tr))$   
**by** (*rule cong[of  $\mathbf{R}_s \mathbf{R}_s$ ], simp, rel-auto*)  
**also have** ... *is Productive*  
**by** (*simp add: Healthy-def Productive-RHS-design-form unrest*)  
**finally show** ?thesis .  
**qed**

**lemma** *Productive-seq-1 [closure]:*

**assumes** *P is NSRD P is Productive Q is NSRD*

**shows** *P ;; Q is Productive*

**proof** –

**have** *P ;; Q* =  $\mathbf{R}_s(pre_R(P) \vdash peri_R(P) \diamond (post_R(P) \wedge \$tr <_u \$tr')) ;; \mathbf{R}_s(pre_R(Q) \vdash peri_R(Q) \diamond (post_R(Q)))$

**by** (*metis Healthy-def NSRD-is-SRD SRD-reactive-tri-design Productive-form assms(1) assms(2) assms(3)*)

**also have** ... =  $\mathbf{R}_s ((pre_R P \wedge (post_R P \wedge \$tr' >_u \$tr) wp_r pre_R Q) \vdash (peri_R P \vee ((post_R P \wedge \$tr' >_u \$tr) ;; peri_R Q)) \diamond ((post_R P \wedge \$tr' >_u \$tr) ;; post_R Q))$

**by** (*simp add: RHS-tri-design-composition-wp rpred unrest closure assms wp NSRD-neg-pre-left-zero SRD-healths ex-unrest wp-rea-def disj-upred-def*)

**also have** ... =  $\mathbf{R}_s ((pre_R P \wedge (post_R P \wedge \$tr' >_u \$tr) wp_r pre_R Q) \vdash (peri_R P \vee ((post_R P \wedge \$tr' >_u \$tr) ;; peri_R Q)) \diamond ((post_R P \wedge \$tr' >_u \$tr) ;; post_R Q \wedge \$tr' >_u \$tr))$

**proof** –

**have**  $((post_R P \wedge \$tr' >_u \$tr) ;; R1(post_R Q)) = ((post_R P \wedge \$tr' >_u \$tr) ;; R1(post_R Q) \wedge \$tr' >_u \$tr)$

**by** (*rel-auto*)

**thus** ?thesis

**by** (*simp add: NSRD-is-SRD R1-post-SRD assms*)

**qed**

**also have** ... *is Productive*

**by** (*rule Productive-rdes-intro, simp-all add: unrest assms closure wp-rea-def*)

**finally show** ?thesis .

**qed**

**lemma** *Productive-seq-2 [closure]:*

**assumes** *P is NSRD Q is NSRD Q is Productive*

**shows** *P ;; Q is Productive*

**proof** –

**have** *P ;; Q* =  $\mathbf{R}_s(pre_R(P) \vdash peri_R(P) \diamond (post_R(P))) ;; \mathbf{R}_s(pre_R(Q) \vdash peri_R(Q) \diamond (post_R(Q) \wedge \$tr <_u \$tr'))$

**by** (*metis Healthy-def NSRD-is-SRD SRD-reactive-tri-design Productive-form assms*)

**also have** ... =  $\mathbf{R}_s ((pre_R P \wedge post_R P wp_r pre_R Q) \vdash (peri_R P \vee (post_R P ;; peri_R Q)) \diamond (post_R P ;; (post_R Q \wedge \$tr' >_u \$tr)))$

**by** (*simp add: RHS-tri-design-composition-wp rpred unrest closure assms wp NSRD-neg-pre-left-zero SRD-healths ex-unrest wp-rea-def disj-upred-def*)

**also have** ... =  $\mathbf{R}_s ((pre_R P \wedge post_R P wp_r pre_R Q) \vdash (peri_R P \vee (post_R P ;; peri_R Q)) \diamond (post_R P ;; (post_R Q \wedge \$tr' >_u \$tr) \wedge \$tr' >_u \$tr))$

**proof** –

**have**  $(R1(post_R P) ;; (post_R Q \wedge \$tr' >_u \$tr) \wedge \$tr' >_u \$tr) = (R1(post_R P) ;; (post_R Q \wedge \$tr' >_u \$tr))$

```

    by (rel-auto)
  thus ?thesis
    by (simp add: NSRD-is-SRD R1-post-SRD assms)
qed
also have ... is Productive
  by (rule Productive-rdes-intro, simp-all add: unrest assms closure wp-rea-def)
finally show ?thesis .
qed
end

```

## 10 Guarded Recursion

```

theory utp-rdes-guarded
  imports utp-rdes-productive
begin

```

### 10.1 Traces with a size measure

Guarded recursion relies on our ability to measure the trace's size, in order to see if it is decreasing on each iteration. Thus, we here equip the trace algebra with the *size* function that provides this.

```

class size-trace = trace + size +
  assumes

```

```

    size-zero: size 0 = 0 and
    size-nzero: s > 0  $\implies$  size(s) > 0 and
    size-plus: size (s + t) = size(s) + size(t)

```

— These axioms may be stronger than necessary. In particular,  $(0::'a) < ?s \implies 0 < \text{size } ?s$  requires that a non-empty trace have a positive size. But this may not be the case with all trace models and is possibly more restrictive than necessary. In future we will explore weakening.

```

begin

```

```

lemma size-mono: s ≤ t  $\implies$  size(s) ≤ size(t)
  by (metis le-add1 local.diff-add-cancel-left' local.size-plus)

```

```

lemma size-strict-mono: s < t  $\implies$  size(s) < size(t)
  by (metis cancel-ab-semigroup-add-class.add-diff-cancel-left' local.diff-add-cancel-left' local.less-iff local.minus-gr-zero-iff local.size-nzero local.size-plus zero-less-diff)

```

```

lemma trace-strict-prefixE: xs < ys  $\implies$  ( $\bigwedge$ zs.  $\llbracket$  ys = xs + zs; size(zs) > 0  $\rrbracket \implies$  thesis)  $\implies$  thesis
  by (metis local.diff-add-cancel-left' local.less-iff local.minus-gr-zero-iff local.size-nzero)

```

```

lemma size-minus-trace: y ≤ x  $\implies$  size(x - y) = size(x) - size(y)
  by (metis diff-add-inverse local.diff-add-cancel-left' local.size-plus)

```

```

end

```

Both natural numbers and lists are measurable trace algebras.

```

instance nat :: size-trace
  by (intro-classes, simp-all)

```

```

instance list :: (type) size-trace
  by (intro-classes, simp-all add: zero-list-def less-list-def' plus-list-def prefix-length-less)

```

**syntax**

$-size \quad :: \text{logic} \Rightarrow \text{logic} \ (size_u '(-))$

**translations**

$size_u(t) == \text{CONST } uop \ \text{CONST } size \ t$

## 10.2 Guardedness

**definition**  $gvert :: ((t::size\text{-}trace, 'a) \text{ } rp \times (t, 'a) \text{ } rp) \text{ } chain \text{ } \mathbf{where}$   
 $[upred\text{-}defs]: gvert(n) \equiv (\$tr \leq_u \$tr' \wedge size_u(\&tt) <_u \ll n \gg)$

**lemma**  $gvert\text{-}chain: chain \ gvert$   
**apply**  $(simp \ add: chain\text{-}def, \ safe)$   
**apply**  $(rel\text{-}simp)$   
**apply**  $(rel\text{-}simp)+$   
**done**

**lemma**  $gvert\text{-}limit: \sqcap \ (range \ gvert) = (\$tr \leq_u \$tr')$   
**by**  $(rel\text{-}auto)$

**definition**  $Guarded :: ((t::size\text{-}trace, 'a) \text{ } hrel\text{-}rp \Rightarrow (t, 'a) \text{ } hrel\text{-}rp) \Rightarrow bool \text{ } \mathbf{where}$   
 $[upred\text{-}defs]: Guarded(F) = (\forall \ X \ n. (F(X) \wedge gvert(n+1)) = (F(X \wedge gvert(n)) \wedge gvert(n+1)))$

**lemma**  $GuardedI: \ll \bigwedge \ X \ n. (F(X) \wedge gvert(n+1)) = (F(X \wedge gvert(n)) \wedge gvert(n+1)) \rr \Longrightarrow Guarded \ F$   
**by**  $(simp \ add: Guarded\text{-}def)$

Guarded reactive designs yield unique fixed-points.

**theorem**  $guarded\text{-}fp\text{-}uniq:$

**assumes**  $mono \ F \ F \in \ll id \rr_H \rightarrow \ll SRD \rr_H \ Guarded \ F$   
**shows**  $\mu \ F = \nu \ F$

**proof**  $-$

**have**  $constr \ F \ gvert$   
**using**  $assms$   
**by**  $(auto \ simp \ add: constr\text{-}def \ gvert\text{-}chain \ Guarded\text{-}def \ tcontr\text{-}alt\text{-}def')$   
**hence**  $(\$tr \leq_u \$tr' \wedge \mu \ F) = (\$tr \leq_u \$tr' \wedge \nu \ F)$   
**apply**  $(rule \ constr\text{-}fp\text{-}uniq)$   
**apply**  $(simp \ add: assms)$   
**using**  $gvert\text{-}limit$  **apply**  $blast$   
**done**

**moreover** **have**  $(\$tr \leq_u \$tr' \wedge \mu \ F) = \mu \ F$

**proof**  $-$

**have**  $\mu \ F \text{ is } R1$   
**by**  $(rule \ SRD\text{-}healths(1), \ rule \ Healthy\text{-}\mu, \ simp\text{-}all \ add: assms)$   
**thus**  $?thesis$   
**by**  $(metis \ Healthy\text{-}def \ R1\text{-}def \ conj\text{-}comm)$

**qed**

**moreover** **have**  $(\$tr \leq_u \$tr' \wedge \nu \ F) = \nu \ F$

**proof**  $-$

**have**  $\nu \ F \text{ is } R1$   
**by**  $(rule \ SRD\text{-}healths(1), \ rule \ Healthy\text{-}\nu, \ simp\text{-}all \ add: assms)$   
**thus**  $?thesis$   
**by**  $(metis \ Healthy\text{-}def \ R1\text{-}def \ conj\text{-}comm)$

**qed**

**ultimately** **show**  $?thesis$

**by**  $(simp)$

**qed**



**lemma** *Guarded-const* [closure]: *Guarded* ( $\lambda X. P$ )  
 by (*simp add: Guarded-def*)

**lemma** *UINF-Guarded* [closure]:

assumes  $\bigwedge P. P \in A \implies \text{Guarded } P$   
 shows *Guarded* ( $\lambda X. \bigwedge P \in A. P(X)$ )

**proof** (*rule GuardedI*)

fix  $X n$

have  $\bigwedge Y. ((\bigwedge P \in A. P Y) \wedge \text{gvr}(n+1)) = ((\bigwedge P \in A. (P Y \wedge \text{gvr}(n+1))) \wedge \text{gvr}(n+1))$

**proof** –

fix  $Y$

let  $?lhs = ((\bigwedge P \in A. P Y) \wedge \text{gvr}(n+1))$  and  $?rhs = ((\bigwedge P \in A. (P Y \wedge \text{gvr}(n+1))) \wedge \text{gvr}(n+1))$

have  $a: ?lhs \llbracket \text{false} / \$ok \rrbracket = ?rhs \llbracket \text{false} / \$ok \rrbracket$

by (*rel-auto*)

have  $b: ?lhs \llbracket \text{true} / \$ok \rrbracket \llbracket \text{true} / \$wait \rrbracket = ?rhs \llbracket \text{true} / \$ok \rrbracket \llbracket \text{true} / \$wait \rrbracket$

by (*rel-auto*)

have  $c: ?lhs \llbracket \text{true} / \$ok \rrbracket \llbracket \text{false} / \$wait \rrbracket = ?rhs \llbracket \text{true} / \$ok \rrbracket \llbracket \text{false} / \$wait \rrbracket$

by (*rel-auto*)

show  $?lhs = ?rhs$

using  $a b c$

by (*rule-tac bool-eq-splitI* [of *in-var ok*], *simp*, *rule-tac bool-eq-splitI* [of *in-var wait*], *simp-all*)

**qed**

moreover have  $((\bigwedge P \in A. (P X \wedge \text{gvr}(n+1))) \wedge \text{gvr}(n+1)) = ((\bigwedge P \in A. (P (X \wedge \text{gvr}(n))) \wedge \text{gvr}(n+1))) \wedge \text{gvr}(n+1))$

**proof** –

have  $(\bigwedge P \in A. (P X \wedge \text{gvr}(n+1))) = (\bigwedge P \in A. (P (X \wedge \text{gvr}(n)) \wedge \text{gvr}(n+1)))$

**proof** (*rule UINF-cong*)

fix  $P$  assume  $P \in A$

thus  $(P X \wedge \text{gvr}(n+1)) = (P (X \wedge \text{gvr}(n)) \wedge \text{gvr}(n+1))$

using *Guarded-def assms* by *blast*

**qed**

thus  $?thesis$  by *simp*

**qed**

ultimately show  $((\bigwedge P \in A. P X) \wedge \text{gvr}(n+1)) = ((\bigwedge P \in A. (P (X \wedge \text{gvr}(n)))) \wedge \text{gvr}(n+1))$

by *simp*

**qed**

**lemma** *intChoice-Guarded* [closure]:

assumes *Guarded*  $P$  *Guarded*  $Q$

shows *Guarded* ( $\lambda X. P(X) \sqcap Q(X)$ )

**proof** –

have *Guarded* ( $\lambda X. \bigwedge F \in \{P, Q\}. F(X)$ )

by (*rule UINF-Guarded*, *auto simp add: assms*)

thus  $?thesis$

by (*simp*)

**qed**

**lemma** *cond-srea-Guarded* [closure]:

assumes *Guarded*  $P$  *Guarded*  $Q$

shows *Guarded* ( $\lambda X. P(X) \triangleleft b \triangleright_R Q(X)$ )

using *assms* by (*rel-auto*)

A tail recursive reactive design with a productive body is guarded.

**lemma** *Guarded-if-Productive* [closure]:

**fixes**  $P :: ('s, 't::size-trace, 'α) \text{ hrel-rsp}$   
**assumes**  $P \text{ is NSRD } P \text{ is Productive}$   
**shows**  $\text{Guarded } (\lambda X. P ;; \text{SRD}(X))$   
**proof** (*clarsimp simp add: Guarded-def*)  
— We split the proof into three cases corresponding to valuations for ok, wait, and wait' respectively.  
**fix**  $X \ n$   
**have**  $a: (P ;; \text{SRD}(X) \wedge \text{gvt } (\text{Suc } n)) \llbracket \text{false}/\$ok \rrbracket =$   
 $(P ;; \text{SRD}(X \wedge \text{gvt } n) \wedge \text{gvt } (\text{Suc } n)) \llbracket \text{false}/\$ok \rrbracket$   
**by** (*simp add: usubst closure SRD-left-zero-1 assms*)  
**have**  $b: ((P ;; \text{SRD}(X) \wedge \text{gvt } (\text{Suc } n)) \llbracket \text{true}/\$ok \rrbracket) \llbracket \text{true}/\$wait \rrbracket =$   
 $((P ;; \text{SRD}(X \wedge \text{gvt } n) \wedge \text{gvt } (\text{Suc } n)) \llbracket \text{true}/\$ok \rrbracket) \llbracket \text{true}/\$wait \rrbracket$   
**by** (*simp add: usubst closure SRD-left-zero-2 assms*)  
**have**  $c: ((P ;; \text{SRD}(X) \wedge \text{gvt } (\text{Suc } n)) \llbracket \text{true}/\$ok \rrbracket) \llbracket \text{false}/\$wait \rrbracket =$   
 $((P ;; \text{SRD}(X \wedge \text{gvt } n) \wedge \text{gvt } (\text{Suc } n)) \llbracket \text{true}/\$ok \rrbracket) \llbracket \text{false}/\$wait \rrbracket$   
**proof** —  
**have**  $1: (P \llbracket \text{true}/\$wait' \rrbracket ;; (\text{SRD } X) \llbracket \text{true}/\$wait \rrbracket \wedge \text{gvt } (\text{Suc } n)) \llbracket \text{true}, \text{false}/\$ok, \$wait \rrbracket =$   
 $(P \llbracket \text{true}/\$wait' \rrbracket ;; (\text{SRD } (X \wedge \text{gvt } n)) \llbracket \text{true}/\$wait \rrbracket \wedge \text{gvt } (\text{Suc } n)) \llbracket \text{true}, \text{false}/\$ok, \$wait \rrbracket$   
**by** (*metis (no-types, lifting) Healthy-def R3h-wait-true SRD-healths(3) SRD-idem*)  
**have**  $2: (P \llbracket \text{false}/\$wait' \rrbracket ;; (\text{SRD } X) \llbracket \text{false}/\$wait \rrbracket \wedge \text{gvt } (\text{Suc } n)) \llbracket \text{true}, \text{false}/\$ok, \$wait \rrbracket =$   
 $(P \llbracket \text{false}/\$wait' \rrbracket ;; (\text{SRD } (X \wedge \text{gvt } n)) \llbracket \text{false}/\$wait \rrbracket \wedge \text{gvt } (\text{Suc } n)) \llbracket \text{true}, \text{false}/\$ok, \$wait \rrbracket$   
**proof** —  
**have**  $\text{exp}: \bigwedge Y :: ('s, 't, 'α) \text{ hrel-rsp}. (P \llbracket \text{false}/\$wait' \rrbracket ;; (\text{SRD } Y) \llbracket \text{false}/\$wait \rrbracket \wedge \text{gvt } (\text{Suc } n)) \llbracket \text{true}, \text{false}/\$ok, \$wait \rrbracket$   
 $=$   
 $((\neg_r \text{pre}_R P) ;; (\text{SRD}(Y)) \llbracket \text{false}/\$wait \rrbracket \vee (\text{post}_R P \wedge \$tr' >_u \$tr) ;; (\text{SRD}$   
 $Y) \llbracket \text{true}, \text{false}/\$ok, \$wait \rrbracket))$   
 $\wedge \text{gvt } (\text{Suc } n)) \llbracket \text{true}, \text{false}/\$ok, \$wait \rrbracket$   
**proof** —  
**fix**  $Y :: ('s, 't, 'α) \text{ hrel-rsp}$   
  
**have**  $(P \llbracket \text{false}/\$wait' \rrbracket ;; (\text{SRD } Y) \llbracket \text{false}/\$wait \rrbracket \wedge \text{gvt } (\text{Suc } n)) \llbracket \text{true}, \text{false}/\$ok, \$wait \rrbracket =$   
 $((\mathbf{R}_s(\text{pre}_R(P) \vdash \text{peri}_R(P) \diamond (\text{post}_R(P) \wedge \$tr <_u \$tr')) \llbracket \text{false}/\$wait' \rrbracket ;; (\text{SRD } Y) \llbracket \text{false}/\$wait \rrbracket$   
 $\wedge \text{gvt } (\text{Suc } n)) \llbracket \text{true}, \text{false}/\$ok, \$wait \rrbracket$   
**by** (*metis (no-types) Healthy-def Productive-form assms(1) assms(2) NSRD-is-SRD*)  
**also have**  $\dots =$   
 $((R1(R2c(\text{pre}_R(P) \Rightarrow (\$ok' \wedge \text{post}_R(P) \wedge \$tr <_u \$tr')))) \llbracket \text{false}/\$wait' \rrbracket ;; (\text{SRD } Y) \llbracket \text{false}/\$wait \rrbracket$   
 $\wedge \text{gvt } (\text{Suc } n)) \llbracket \text{true}, \text{false}/\$ok, \$wait \rrbracket$   
**by** (*simp add: RHS-def R1-def R2c-def R2s-def R3h-def RD1-def RD2-def usubst unrest assms*  
*closure design-def*)  
**also have**  $\dots =$   
 $((\neg_r \text{pre}_R(P) \vee (\$ok' \wedge \text{post}_R(P) \wedge \$tr <_u \$tr')) \llbracket \text{false}/\$wait' \rrbracket ;; (\text{SRD } Y) \llbracket \text{false}/\$wait \rrbracket$   
 $\wedge \text{gvt } (\text{Suc } n)) \llbracket \text{true}, \text{false}/\$ok, \$wait \rrbracket$   
**by** (*simp add: impl-alt-def R2c-disj R1-disj R2c-not assms closure R2c-and*  
*R2c-preR rea-not-def R1-extend-conj' R2c-ok' R2c-post-SRD R1-tr-less-tr' R2c-tr-less-tr'*)  
**also have**  $\dots =$   
 $((\neg_r \text{pre}_R P) ;; (\text{SRD}(Y)) \llbracket \text{false}/\$wait \rrbracket \vee (\$ok' \wedge \text{post}_R P \wedge \$tr' >_u \$tr) ;; (\text{SRD}$   
 $Y) \llbracket \text{false}/\$wait \rrbracket)) \wedge \text{gvt } (\text{Suc } n)) \llbracket \text{true}, \text{false}/\$ok, \$wait \rrbracket$   
**by** (*simp add: usubst unrest assms closure segr-or-distl NSRD-neg-pre-left-zero*)  
**also have**  $\dots =$   
 $((\neg_r \text{pre}_R P) ;; (\text{SRD}(Y)) \llbracket \text{false}/\$wait \rrbracket \vee (\text{post}_R P \wedge \$tr' >_u \$tr) ;; (\text{SRD } Y) \llbracket \text{true}, \text{false}/\$ok, \$wait \rrbracket))$   
 $\wedge \text{gvt } (\text{Suc } n)) \llbracket \text{true}, \text{false}/\$ok, \$wait \rrbracket$   
**proof** —  
**have**  $(\$ok' \wedge \text{post}_R P \wedge \$tr' >_u \$tr) ;; (\text{SRD } Y) \llbracket \text{false}/\$wait \rrbracket =$   
 $((\text{post}_R P \wedge \$tr' >_u \$tr) \wedge \$ok' =_u \text{true}) ;; (\text{SRD } Y) \llbracket \text{false}/\$wait \rrbracket$   
**by** (*rel-blast*)  
**also have**  $\dots = (\text{post}_R P \wedge \$tr' >_u \$tr) \llbracket \text{true}/\$ok' \rrbracket ;; (\text{SRD } Y) \llbracket \text{false}/\$wait \rrbracket \llbracket \text{true}/\$ok \rrbracket$

```

    using seqr-left-one-point[of ok (postR P ∧ $tr' >u $tr) True (SRD Y)⟦false/$wait⟧]
    by (simp add: true-alt-def[THEN sym])
    finally show ?thesis by (simp add: usubst unrest)
qed
finally
show (P⟦false/$wait'⟧ ;; (SRD Y)⟦false/$wait⟧ ∧ gvt (Suc n)⟦true,false/$ok,$wait⟧ =
      (((¬r preR P) ;; (SRD(Y))⟦false/$wait⟧ ∨ (postR P ∧ $tr' >u $tr) ;; (SRD
Y)⟦true,false/$ok,$wait⟧))
      ∧ gvt (Suc n)⟦true,false/$ok,$wait⟧) .
qed

have 1:((postR P ∧ $tr' >u $tr) ;; (SRD X)⟦true,false/$ok,$wait⟧ ∧ gvt (Suc n)) =
      ((postR P ∧ $tr' >u $tr) ;; (SRD (X ∧ gvt n)⟦true,false/$ok,$wait⟧ ∧ gvt (Suc n))
    apply (rel-auto)
    apply (rename-tac tr st more ok wait tr' st' more' tr0 st0 more0 ok')
    apply (rule-tac x=tr0 in exI, rule-tac x=st0 in exI, rule-tac x=more0 in exI)
    apply (simp)
    apply (erule trace-strict-prefixE)
    apply (rename-tac tr st ref ok wait tr' st' ref' tr0 st0 ref0 ok' zs)
    apply (rule-tac x=False in exI)
    apply (simp add: size-minus-trace)
    apply (subgoal-tac size(tr) < size(tr0))
    apply (simp add: less-diff-conv2 size-mono)
    using size-strict-mono apply blast
    apply (rename-tac tr st more ok wait tr' st' more' tr0 st0 more0 ok')
    apply (rule-tac x=tr0 in exI, rule-tac x=st0 in exI, rule-tac x=more0 in exI)
    apply (simp)
    apply (erule trace-strict-prefixE)
    apply (rename-tac tr st more ok wait tr' st' more' tr0 st0 more0 ok' zs)
    apply (auto simp add: size-minus-trace)
    apply (subgoal-tac size(tr) < size(tr0))
    apply (simp add: less-diff-conv2 size-mono)
    using size-strict-mono apply blast
  done
have 2:(¬r preR P) ;; (SRD X)⟦false/$wait⟧ = (¬r preR P) ;; (SRD(X ∧ gvt n)⟦false/$wait⟧)
  by (simp add: NSRD-neg-pre-left-zero closure asms SRD-healths)
show ?thesis
  by (simp add: exp 1 2 utp-pred-laws.inf-sup-distrib2)
qed

show ?thesis
proof -
  have (P ;; (SRD X) ∧ gvt (n+1))⟦true,false/$ok,$wait⟧ =
    ((P⟦true/$wait'⟧ ;; (SRD X)⟦true/$wait⟧ ∧ gvt (n+1))⟦true,false/$ok,$wait⟧ ∨
    (P⟦false/$wait'⟧ ;; (SRD X)⟦false/$wait⟧ ∧ gvt (n+1))⟦true,false/$ok,$wait⟧)
    by (subst seqr-bool-split[of wait], simp-all add: usubst utp-pred-laws.distrib(4))

  also
  have ... = ((P⟦true/$wait'⟧ ;; (SRD (X ∧ gvt n)⟦true/$wait⟧ ∧ gvt (n+1))⟦true,false/$ok,$wait⟧)
    ∨
    (P⟦false/$wait'⟧ ;; (SRD (X ∧ gvt n)⟦false/$wait⟧ ∧ gvt (n+1))⟦true,false/$ok,$wait⟧)
    by (simp add: 1 2)

  also
  have ... = ((P⟦true/$wait'⟧ ;; (SRD (X ∧ gvt n)⟦true/$wait⟧ ∨

```

```

    P[[false/$wait']] ;; (SRD (X ∧ gvirt n))[[false/$wait']] ∧ gvirt (n+1))[[true,false/$ok,$wait]]
  by (simp add: usubst utp-pred-laws.distrib(4))

  also have ... = (P ;; (SRD (X ∧ gvirt n)) ∧ gvirt (n+1))[[true,false/$ok,$wait]]
    by (subst seqr-bool-split[of wait], simp-all add: usubst)
  finally show ?thesis by (simp add: usubst)
qed

qed
show (P ;; SRD(X) ∧ gvirt (Suc n)) = (P ;; SRD(X ∧ gvirt n) ∧ gvirt (Suc n))
  apply (rule-tac bool-eq-splitI[of in-var ok])
  apply (simp-all add: a)
  apply (rule-tac bool-eq-splitI[of in-var wait])
  apply (simp-all add: b c)
done
qed



### 10.3 Tail recursive fixed-point calculations



declare upred-semiring.power-Suc [simp]

lemma mu-csp-form-1 [rdes]:
  fixes P :: ('s, 't::size-trace, 'α) hrel-rsp
  assumes P is NSRD P is Productive
  shows (μ X · P ;; SRD(X)) = (⊓ i · P ^ (i+1)) ;; Miracle
proof -
  have 1: Continuous (λX. P ;; SRD X)
    using SRD-Continuous
    by (clarsimp simp add: Continuous-def seq-SUP-distl[THEN sym], drule-tac x=A in spec, simp)
  have 2: (λX. P ;; SRD X) ∈ [[id]]H → [[SRD]]H
    by (blast intro: funcsetI closure assms)
  with 1 2 have (μ X · P ;; SRD(X)) = (ν X · P ;; SRD(X))
    by (simp add: guarded-fp-uniq Guarded-if-Productive[OF assms] funcsetI closure)
  also have ... = (⊓ i. ((λX. P ;; SRD X) ^^ i) false)
    by (simp add: sup-continuous-lfp 1 sup-continuous-Continuous false-upred-def)
  also have ... = ((λX. P ;; SRD X) ^^ 0) false ⊓ (⊓ i. ((λX. P ;; SRD X) ^^ (i+1)) false)
    by (subst Sup-power-expand, simp)
  also have ... = (⊓ i. ((λX. P ;; SRD X) ^^ (i+1)) false)
    by (simp)
  also have ... = (⊓ i. P ^ (i+1)) ;; Miracle
proof (rule SUP-cong, simp-all)
  fix i
  show P ;; SRD (((λX. P ;; SRD X) ^^ i) false) = (P ;; P ^ i) ;; Miracle
  proof (induct i)
    case 0
    then show ?case
      by (simp, metis srdes-theory.healthy-top)
  next
    case (Suc i)
    then show ?case
      by (simp add: Healthy-if NSRD-is-SRD SRD-power-comp SRD-seqr-closure assms(1) seqr-assoc[THEN sym] srdes-theory.top-closed)
  qed
qed
also have ... = (⊓ i. P ^ (i+1)) ;; Miracle
  by (simp add: seq-Sup-distr)

```

finally show ?thesis  
 by (simp add: UINF-as-Sup-collect)  
 qed

lemma mu-csp-form-NSRD [closure]:  
 fixes  $P :: ('s, 't::size-trace, 'α) hrel-rsp$   
 assumes  $P$  is NSRD  $P$  is Productive  
 shows  $(\mu X \cdot P ;; SRD(X))$  is NSRD  
 by (simp add: mu-csp-form-1 assms closure)

lemma mu-csp-form-1':  
 fixes  $P :: ('s, 't::size-trace, 'α) hrel-rsp$   
 assumes  $P$  is NSRD  $P$  is Productive  
 shows  $(\mu X \cdot P ;; SRD(X)) = (P ;; P^*) ;; Miracle$

proof –  
 have  $(\mu X \cdot P ;; SRD(X)) = (\bigcap i \in UNIV \cdot P ;; P \hat{~} i) ;; Miracle$   
 by (simp add: mu-csp-form-1 assms closure ustar-def)  
 also have  $\dots = (P ;; P^*) ;; Miracle$   
 by (simp only: seq-UINF-distl[THEN sym], simp add: ustar-def)  
 finally show ?thesis .

qed

declare upred-semiring.power-Suc [simp del]

end

## 11 Reactive Design Programs

theory utp-rdes-prog  
 imports  
 utp-rdes-normal  
 utp-rdes-tactics  
 utp-rdes-parallel  
 utp-rdes-guarded  
 UTP-KAT.utp-kleene

begin

### 11.1 State substitution

lemma srd-subst-RHS-tri-design [usubst]:  
 $\lceil \sigma \rceil_{S\sigma} \dagger \mathbf{R}_s(P \vdash Q \diamond R) = \mathbf{R}_s((\lceil \sigma \rceil_{S\sigma} \dagger P) \vdash (\lceil \sigma \rceil_{S\sigma} \dagger Q) \diamond (\lceil \sigma \rceil_{S\sigma} \dagger R))$   
 by (rel-auto)

lemma srd-subst-SRD-closed [closure]:  
 assumes  $P$  is SRD  
 shows  $\lceil \sigma \rceil_{S\sigma} \dagger P$  is SRD

proof –  
 have  $SRD(\lceil \sigma \rceil_{S\sigma} \dagger (SRD P)) = \lceil \sigma \rceil_{S\sigma} \dagger (SRD P)$   
 by (rel-auto)  
 thus ?thesis  
 by (metis Healthy-def assms)

qed

lemma preR-srd-subst [rdes]:  
 $pre_R(\lceil \sigma \rceil_{S\sigma} \dagger P) = \lceil \sigma \rceil_{S\sigma} \dagger pre_R(P)$

by (rel-auto)

**lemma** *periR-srd-subst* [rdes]:  
 $\text{peri}_R(\lceil \sigma \rceil_{S\sigma} \dagger P) = \lceil \sigma \rceil_{S\sigma} \dagger \text{peri}_R(P)$   
 by (rel-auto)

**lemma** *postR-srd-subst* [rdes]:  
 $\text{post}_R(\lceil \sigma \rceil_{S\sigma} \dagger P) = \lceil \sigma \rceil_{S\sigma} \dagger \text{post}_R(P)$   
 by (rel-auto)

**lemma** *srd-subst-NSRD-closed* [closure]:  
 assumes  $P$  is NSRD  
 shows  $\lceil \sigma \rceil_{S\sigma} \dagger P$  is NSRD  
 by (rule NSRD-RC-intro, simp-all add: closure rdes assms unrest)

## 11.2 Assignment

**definition** *assigns-srd* :: 's usubst  $\Rightarrow$  ('s, 't::trace, 'α) hrel-rsp ( $\langle \cdot \rangle_R$ ) **where**  
 $[\text{upred-defs}]: \text{assigns-srd } \sigma = \mathbf{R}_s(\text{true} \vdash (\$tr' =_u \$tr \wedge \neg \$wait' \wedge \lceil \langle \sigma \rangle_a \rceil_S \wedge \$\Sigma_S' =_u \$\Sigma_S))$

**syntax**

-assign-srd :: svids  $\Rightarrow$  uexprs  $\Rightarrow$  logic ('(-') :=<sub>R</sub> '(-'))  
 -assign-srd :: svids  $\Rightarrow$  uexprs  $\Rightarrow$  logic (**infixr** :=<sub>R</sub> 62)

**translations**

-assign-srd  $xs\ vs \Rightarrow \text{CONST assigns-srd } (-mk-usubst\ (id_s)\ xs\ vs)$   
 -assign-srd  $x\ v \leq \text{CONST assigns-srd } (\text{CONST subst-upd } (id_s)\ x\ v)$   
 -assign-srd  $x\ v \leq \text{-assign-srd } (-spvar\ x)\ v$   
 $x, y :=_R u, v \leq \text{CONST assigns-srd } (\text{CONST subst-upd } (\text{CONST subst-upd } (id_s)\ (\text{CONST pr-var } x)\ u)\ (\text{CONST pr-var } y)\ v)$

**lemma** *assigns-srd-RHS-tri-des* [rdes-def]:  
 $\langle \sigma \rangle_R = \mathbf{R}_s(\text{true}_r \vdash \text{false} \diamond \langle \sigma \rangle_r)$   
 by (rel-auto)

**lemma** *assigns-srd-NSRD-closed* [closure]:  $\langle \sigma \rangle_R$  is NSRD  
 by (simp add: rdes-def closure unrest)

**lemma** *preR-assigs-srd* [rdes]:  $\text{pre}_R(\langle \sigma \rangle_R) = \text{true}_r$   
 by (simp add: rdes-def rdes closure)

**lemma** *periR-assigs-srd* [rdes]:  $\text{peri}_R(\langle \sigma \rangle_R) = \text{false}$   
 by (simp add: rdes-def rdes closure)

**lemma** *postR-assigs-srd* [rdes]:  $\text{post}_R(\langle \sigma \rangle_R) = \langle \sigma \rangle_r$   
 by (simp add: rdes-def rdes closure rpred)

**lemma** *taut-eq-impl-property*:  
 $\llbracket \text{vwb-lens } x; \$x \# P \rrbracket \Longrightarrow '(\$x =_u \llbracket v \rrbracket \wedge Q) \Rightarrow P' = 'Q[\llbracket v \rrbracket / \$x] \Rightarrow P'$   
 by (rel-auto, meson mwb-lens-weak vwb-lens-mwb weak-lens.put-get)

**lemma** *st-subst-taut-impl*:  
 assumes  $\text{vwb-lens } x\ \$st:x \# Q\ P$  is  $RR\ Q$  is  $RR$   
 shows  $'[\&x \mapsto_s \llbracket k \rrbracket] \dagger_S P \Rightarrow Q' = '[\&x =_u \llbracket k \rrbracket]_{S<} \wedge P \Rightarrow Q'$  (is ?lhs = ?rhs)  
**proof** –  
 have ?lhs =  $'P[\llbracket k \rrbracket / \$st:x] \Rightarrow Q'$

by (simp add: usubst-st-lift-def alpha usubst)  
 also have ... = ‘ $(\$st:x =_u \ll k \gg) \wedge RR(P) \Rightarrow RR(Q)$ ’  
 by (simp add: Healthy-if assms taut-eq-impl-property)  
 also have ... = ‘ $[\&x =_u \ll k \gg]_{S<} \wedge RR(P) \Rightarrow RR(Q)$ ’  
 by (rel-blast)  
 finally show ?thesis by (simp add: assms Healthy-if)  
 qed

The following law explains how to refine a program  $Q$  when it is first initialised by an assignment. Would be good if it could be generalised to a more general precondition.

**lemma** *AssignR-init-refine-intro*:

assumes  
 $vwb\text{-}lens\ x\ \$st:x \# P_2\ \$st:x \# P_3$   
 $P_2\ is\ RR\ P_3\ is\ RR\ Q\ is\ NSRD$   
 $\mathbf{R}_s([\&x =_u \ll k \gg]_{S<} \vdash P_2 \diamond P_3) \sqsubseteq Q$   
 shows  $\mathbf{R}_s(true_r \vdash P_2 \diamond P_3) \sqsubseteq (x :=_R \ll k \gg) ;; Q$   
 proof –  
 have  $\mathbf{R}_s([\&x =_u \ll k \gg]_{S<} \vdash P_2 \diamond P_3) \sqsubseteq \mathbf{R}_s(pre_R(Q) \vdash peri_R(Q) \diamond post_R(Q))$   
 by (simp add: NSRD-is-SRD SRD-reactive-tri-design assms)  
 hence  $\mathbf{R}_s(true_r \vdash P_2 \diamond P_3) \sqsubseteq x :=_R \ll k \gg ;; \mathbf{R}_s(pre_R(Q) \vdash peri_R(Q) \diamond post_R(Q))$   
 proof (clarsimp simp add: rdes-def assms closure unrest rpred wp RHS-tri-design-refine, safe)  
 assume  $a1: [\&x =_u \ll k \gg]_{S<} \Rightarrow pre_R(Q)$  and  $a2: [\&x =_u \ll k \gg]_{S<} \wedge peri_R(Q) \Rightarrow P_2$  and  $a3: [\&x =_u \ll k \gg]_{S<} \wedge post_R(Q) \Rightarrow P_3$   
 from  $a1$  assms(1) show ‘ $R1\ true \Rightarrow [\&x \mapsto_s \ll k \gg] \upharpoonright_S pre_R(Q)$ ’  
 by (rel-simp)  
 show ‘ $[\&x \mapsto_s \ll k \gg] \upharpoonright_S peri_R(Q) \Rightarrow P_2$ ’  
 by (simp add: a2 assms st-subst-taut-impl closure)  
 show ‘ $[\&x \mapsto_s \ll k \gg] \upharpoonright_S post_R(Q) \Rightarrow P_3$ ’  
 by (simp add: a3 assms st-subst-taut-impl closure)  
 qed  
 thus ?thesis  
 by (simp add: NSRD-is-SRD SRD-reactive-tri-design assms)  
 qed

### 11.3 Conditional

**lemma** *preR-cond-srea* [rdes]:

$pre_R(P \triangleleft b \triangleright_R Q) = ([b]_{S<} \wedge pre_R(P) \vee [\neg b]_{S<} \wedge pre_R(Q))$   
 by (rel-auto)

**lemma** *periR-cond-srea* [rdes]:

assumes  $P\ is\ SRD\ Q\ is\ SRD$   
 shows  $peri_R(P \triangleleft b \triangleright_R Q) = ([b]_{S<} \wedge peri_R(P) \vee [\neg b]_{S<} \wedge peri_R(Q))$   
 proof –  
 have  $peri_R(P \triangleleft b \triangleright_R Q) = peri_R(R1(P) \triangleleft b \triangleright_R R1(Q))$   
 by (simp add: Healthy-if SRD-healths assms)  
 thus ?thesis  
 by (rel-auto)  
 qed

**lemma** *postR-cond-srea* [rdes]:

assumes  $P\ is\ SRD\ Q\ is\ SRD$   
 shows  $post_R(P \triangleleft b \triangleright_R Q) = ([b]_{S<} \wedge post_R(P) \vee [\neg b]_{S<} \wedge post_R(Q))$   
 proof –  
 have  $post_R(P \triangleleft b \triangleright_R Q) = post_R(R1(P) \triangleleft b \triangleright_R R1(Q))$

by (simp add: Healthy-if SRD-healths assms)  
 thus ?thesis  
 by (rel-auto)  
 qed

**lemma** NSRD-cond-srea [closure]:  
 assumes  $P$  is NSRD  $Q$  is NSRD  
 shows  $P \triangleleft b \triangleright_R Q$  is NSRD  
**proof** (rule NSRD-RC-intro)  
 show  $P \triangleleft b \triangleright_R Q$  is SRD  
 by (simp add: closure assms)  
 show  $\text{pre}_R (P \triangleleft b \triangleright_R Q)$  is RC  
**proof** –  
 have 1:  $(\lceil \neg b \rceil_{S<} \vee \neg_r \text{pre}_R P) ;; R1(\text{true}) = (\lceil \neg b \rceil_{S<} \vee \neg_r \text{pre}_R P)$   
 by (metis (no-types, lifting) NSRD-neg-pre-unit aext-not assms(1) seqr-or-distl st-lift-R1-true-right)  
 have 2:  $(\lceil b \rceil_{S<} \vee \neg_r \text{pre}_R Q) ;; R1(\text{true}) = (\lceil b \rceil_{S<} \vee \neg_r \text{pre}_R Q)$   
 by (simp add: NSRD-neg-pre-unit assms seqr-or-distl st-lift-R1-true-right)  
 show ?thesis  
 by (simp add: rdes closure assms)  
 qed  
 show  $\$st' \# \text{peri}_R (P \triangleleft b \triangleright_R Q)$   
 by (simp add: rdes assms closure unrest)  
 qed

## 11.4 Assumptions

**definition** AssumeR :: ' $s$  cond  $\Rightarrow$  (' $s$ , ' $t::\text{trace}$ , ' $\alpha$ )  $h\text{rel-rsp } ([\cdot]^\top_R)$  **where**  
 [upred-defs]: AssumeR  $b = II_R \triangleleft b \triangleright_R \text{Miracle}$

**lemma** AssumeR-rdes-def [rdes-def]:  
 $[b]^\top_R = \mathbf{R}_s(\text{true}_r \vdash \text{false} \diamond [b]^\top_r)$   
**unfolding** AssumeR-def **by** (rdes-eq)

**lemma** AssumeR-NSRD [closure]:  $[b]^\top_R$  is NSRD  
**by** (simp add: AssumeR-def closure)

**lemma** AssumeR-false:  $[\text{false}]^\top_R = \text{Miracle}$   
**by** (rel-auto)

**lemma** AssumeR-true:  $[\text{true}]^\top_R = II_R$   
**by** (rel-auto)

**lemma** AssumeR-comp:  $[b]^\top_R ;; [c]^\top_R = [b \wedge c]^\top_R$   
**by** (rdes-simp)

**lemma** AssumeR-choice:  $[b]^\top_R \sqcap [c]^\top_R = [b \vee c]^\top_R$   
**by** (rdes-eq)

**lemma** AssumeR-refine-skip:  $II_R \sqsubseteq [b]^\top_R$   
**by** (rdes-refine)

**lemma** AssumeR-test [closure]:  $\text{test}_R [b]^\top_R$   
**by** (simp add: AssumeR-refine-skip nsrdes-theory.utest-intro)

**lemma** Star-AssumeR:  $[b]^\top_R \star^R = II_R$   
**by** (simp add: StarR-def AssumeR-NSRD AssumeR-test nsrdes-theory.Star-test)



**lemma** *AssumeR-choice-skip*:  $II_R \sqcap [b]^\top_R = II_R$   
**by** (*rdes-eq*)

**lemma** *AssumeR-seq-refines*:  
**assumes**  $P$  is NSRD  
**shows**  $P \sqsubseteq P ;; [b]^\top_R$   
**by** (*rdes-refine cls: assms*)

**lemma** *cond-srea-AssumeR-form*:  
**assumes**  $P$  is NSRD  $Q$  is NSRD  
**shows**  $P \triangleleft b \triangleright_R Q = ([b]^\top_R ;; P) \sqcap ([\neg b]^\top_R ;; Q)$   
**by** (*rdes-eq cls: assms*)

**lemma** *cond-srea-insert-assume*:  
**assumes**  $P$  is NSRD  $Q$  is NSRD  
**shows**  $P \triangleleft b \triangleright_R Q = ([b]^\top_R ;; P \triangleleft b \triangleright_R [\neg b]^\top_R ;; Q)$   
**by** (*simp add: AssumeR-NSRD AssumeR-comp NSRD-seqr-closure RA1 assms cond-srea-AssumeR-form*)

**lemma** *AssumeR-cond-left*:  
**assumes**  $P$  is NSRD  $Q$  is NSRD  
**shows**  $[b]^\top_R ;; (P \triangleleft b \triangleright_R Q) = ([b]^\top_R ;; P)$   
**by** (*rdes-eq cls: assms*)

**lemma** *AssumeR-cond-right*:  
**assumes**  $P$  is NSRD  $Q$  is NSRD  
**shows**  $[\neg b]^\top_R ;; (P \triangleleft b \triangleright_R Q) = ([\neg b]^\top_R ;; Q)$   
**by** (*rdes-eq cls: assms*)

## 11.5 Guarded commands

**definition** *GuardedCommR* ::  $'s \text{ cond} \Rightarrow ('s, 't::\text{trace}, 'a) \text{ hrel-rsp} \Rightarrow ('s, 't, 'a) \text{ hrel-rsp} (- \rightarrow_R - [85, 86] \ 85)$  **where**  
*gcmd-def[rdes-def]*: *GuardedCommR*  $g \ A = A \triangleleft g \triangleright_R \text{Miracle}$

**lemma** *gcmd-false[simp]*:  $(\text{false} \rightarrow_R A) = \text{Miracle}$   
**unfolding** *gcmd-def* **by** (*pred-auto*)

**lemma** *gcmd-true[simp]*:  $(\text{true} \rightarrow_R A) = A$   
**unfolding** *gcmd-def* **by** (*pred-auto*)

**lemma** *gcmd-SRD*:  
**assumes**  $A$  is SRD  
**shows**  $(g \rightarrow_R A)$  is SRD  
**by** (*simp add: gcmd-def SRD-cond-srea assms srdes-theory.top-closed*)

**lemma** *gcmd-NSRD [closure]*:  
**assumes**  $A$  is NSRD  
**shows**  $(g \rightarrow_R A)$  is NSRD  
**by** (*simp add: gcmd-def NSRD-cond-srea assms NSRD-Miracle*)

**lemma** *gcmd-Productive [closure]*:  
**assumes**  $A$  is NSRD  $A$  is Productive  
**shows**  $(g \rightarrow_R A)$  is Productive  
**by** (*simp add: gcmd-def closure assms*)

**lemma** *gcmd-seq-distr*:  
**assumes** *B is NSRD*  
**shows**  $(g \rightarrow_R A) ;; B = (g \rightarrow_R (A ;; B))$   
**by** (*simp add: Miracle-left-zero NSRD-is-SRD assms cond-st-distr gcmd-def*)

**lemma** *gcmd-nondet-distr*:  
**assumes** *A is NSRD B is NSRD*  
**shows**  $(g \rightarrow_R (A \sqcap B)) = (g \rightarrow_R A) \sqcap (g \rightarrow_R B)$   
**by** (*rdes-eq cls: assms*)

**lemma** *AssumeR-as-gcmd*:  
 $[b]^\top_R = b \rightarrow_R II_R$   
**by** (*rdes-eq*)

**lemma** *AssumeR-gcomm*:  
**assumes** *P is NSRD*  
**shows**  $[b]^\top_R ;; (c \rightarrow_R P) = (b \wedge c) \rightarrow_R P$   
**by** (*rdes-eq cls: assms*)

## 11.6 Generalised Alternation

**definition** *AlternateR*  
 $:: 'a \text{ set} \Rightarrow ('a \Rightarrow 's \text{ upred}) \Rightarrow ('a \Rightarrow ('s, 't::\text{trace}, 'a) \text{ hrel-rsp}) \Rightarrow ('s, 't, 'a) \text{ hrel-rsp} \Rightarrow ('s, 't, 'a) \text{ hrel-rsp}$  **where**  
 $[upred-defs, rdes-def]: \text{AlternateR } I \ g \ A \ B = (\bigcap i \in I \cdot ((g \ i) \rightarrow_R (A \ i))) \sqcap ((\neg (\bigvee i \in I \cdot g \ i)) \rightarrow_R B)$

**definition** *AlternateR-list*  
 $:: ('s \text{ upred} \times ('s, 't::\text{trace}, 'a) \text{ hrel-rsp}) \text{ list} \Rightarrow ('s, 't, 'a) \text{ hrel-rsp} \Rightarrow ('s, 't, 'a) \text{ hrel-rsp}$  **where**  
 $[upred-defs, ndes-simp]:$   
 $\text{AlternateR-list } xs \ P = \text{AlternateR } \{0..<\text{length } xs\} \ (\lambda i. \text{map fst } xs \ ! \ i) \ (\lambda i. \text{map snd } xs \ ! \ i) \ P$

### syntax

*-altindR-els*  $:: \text{pttrn} \Rightarrow \text{logic} \Rightarrow \text{logic} \Rightarrow \text{logic} \Rightarrow \text{logic} \Rightarrow \text{logic} \ (if_R \ - \in \cdot \cdot \rightarrow \cdot \text{ else } \cdot \text{ fi})$   
*-altindR*  $:: \text{pttrn} \Rightarrow \text{logic} \Rightarrow \text{logic} \Rightarrow \text{logic} \Rightarrow \text{logic} \ (if_R \ - \in \cdot \cdot \rightarrow \cdot \text{ fi})$   
*-altgcommR-els*  $:: \text{gcomms} \Rightarrow \text{logic} \Rightarrow \text{logic} \ (if_R / \cdot \text{ else } \cdot / \text{ fi})$   
*-altgcommR*  $:: \text{gcomms} \Rightarrow \text{logic} \ (if_R / \cdot / \text{ fi})$

### translations

$if_R \ i \in I \cdot g \rightarrow A \text{ else } B \text{ fi} \rightarrow \text{CONST AlternateR } I \ (\lambda i. g) \ (\lambda i. A) \ B$   
 $if_R \ i \in I \cdot g \rightarrow A \text{ fi} \rightarrow \text{CONST AlternateR } I \ (\lambda i. g) \ (\lambda i. A) \ (\text{CONST Chaos})$   
 $if_R \ i \in I \cdot (g \ i) \rightarrow A \text{ else } B \text{ fi} \leftarrow \text{CONST AlternateR } I \ g \ (\lambda i. A) \ B$   
 $\text{-altgcommR } cs \rightarrow \text{CONST AlternateR-list } cs \ (\text{CONST Chaos})$   
 $\text{-altgcommR } (-\text{gcomm-show } cs) \leftarrow \text{CONST AlternateR-list } cs \ (\text{CONST Chaos})$   
 $\text{-altgcommR-els } cs \ P \rightarrow \text{CONST AlternateR-list } cs \ P$   
 $\text{-altgcommR-els } (-\text{gcomm-show } cs) \ P \leftarrow \text{CONST AlternateR-list } cs \ P$

**lemma** *AlternateR-NSRD-closed [closure]*:  
**assumes**  $\bigwedge i. i \in I \Rightarrow A \ i \text{ is NSRD } B \text{ is NSRD}$   
**shows**  $(if_R \ i \in I \cdot g \ i \rightarrow A \ i \text{ else } B \text{ fi}) \text{ is NSRD}$   
**proof** (*cases*  $I = \{\}$ )  
**case** *True*  
**then show** *?thesis* **by** (*simp add: AlternateR-def assms*)  
**next**  
**case** *False*

**then show** *?thesis* **by** (*simp add: AlternateR-def closure assms*)  
**qed**

**lemma** *AlternateR-empty* [*simp*]:  
 $(if_R i \in \{\} \cdot g i \rightarrow A i \text{ else } B fi) = B$   
**by** (*rdes-simp*)

**lemma** *AlternateR-Productive* [*closure*]:  
**assumes**  
 $\bigwedge i. i \in I \implies A i \text{ is NSRD } B \text{ is NSRD}$   
 $\bigwedge i. i \in I \implies A i \text{ is Productive } B \text{ is Productive}$   
**shows**  $(if_R i \in I \cdot g i \rightarrow A i \text{ else } B fi) \text{ is Productive}$   
**proof** (*cases I = \{\}*)  
**case True**  
**then show** *?thesis*  
**by** (*simp add: assms(4)*)  
**next**  
**case False**  
**then show** *?thesis*  
**by** (*simp add: AlternateR-def closure assms*)  
**qed**

**lemma** *AlternateR-singleton*:  
**assumes**  $A k \text{ is NSRD } B \text{ is NSRD}$   
**shows**  $(if_R i \in \{k\} \cdot g i \rightarrow A i \text{ else } B fi) = (A(k) \triangleleft g(k) \triangleright_R B)$   
**by** (*simp add: AlternateR-def, rdes-eq cls: assms*)

Convert an alternation over disjoint guards into a cascading if-then-else

**lemma** *AlternateR-insert-cascade*:  
**assumes**  
 $\bigwedge i. i \in I \implies A i \text{ is NSRD}$   
 $A k \text{ is NSRD } B \text{ is NSRD}$   
 $(g(k) \wedge (\bigvee i \in I \cdot g(i))) = \text{false}$   
**shows**  $(if_R i \in \text{insert } k I \cdot g i \rightarrow A i \text{ else } B fi) = (A(k) \triangleleft g(k) \triangleright_R (if_R i \in I \cdot g(i) \rightarrow A(i) \text{ else } B fi))$   
**proof** (*cases I = \{\}*)  
**case True**  
**then show** *?thesis* **by** (*simp add: AlternateR-singleton assms*)  
**next**  
**case False**  
**have**  $1: (\bigcap i \in I \cdot g i \rightarrow_R A i) = (\bigcap i \in I \cdot g i \rightarrow_R \mathbf{R}_s(\text{pre}_R(A i) \vdash \text{peri}_R(A i) \diamond \text{post}_R(A i)))$   
**by** (*simp add: NSRD-is-SRD SRD-reactive-tri-design assms(1) cong: UINF-cong*)  
**from** *assms(4)* **show** *?thesis*  
**by** (*simp add: AlternateR-def 1 False*)  
 $(\text{rdes-eq cls: assms}(1-3) \text{ False cong: UINF-cong})$   
**qed**

**lemma** *AlternateR-assume-branch*:  
**assumes**  $I \neq \{\} \bigwedge i. i \in I \implies P i \text{ is NSRD } Q \text{ is NSRD}$   
**shows**  $([\bigcap i \in I \cdot b i]^\top_R ;; \text{AlternateR } I b P Q) = (\bigcap i \in I \cdot b i \rightarrow_R P i) (\text{is } ?lhs = ?rhs)$   
**proof** –  
**have**  $?lhs = [\bigcap i \in I \cdot b i]^\top_R ;; ((\bigcap i \in I \cdot b i \rightarrow_R P i) \sqcap (\neg (\bigcap i \in I \cdot b i)) \rightarrow_R Q)$   
**by** (*simp add: AlternateR-def closure assms*)  
**also have**  $\dots = [\bigcap i \in I \cdot b i]^\top_R ;; (\bigcap i \in I \cdot b i \rightarrow_R P i) \sqcap \text{Miracle}$   
**by** (*simp add: segr-inf-distr AssumeR-gcomm closure assms*)  
**also have**  $\dots = (\bigcap i \in I \cdot ((\bigcap i \in I \cdot b i) \wedge b i) \rightarrow_R P i) \sqcap \text{Miracle}$

by (simp add: seq-UINF-distl AssumeR-gcomm closure assms cong: UINF-cong)  
 also have ... =  $(\bigcap i \in I \cdot b \ i \rightarrow_R P \ i) \sqcap \text{Miracle}$   
 proof –  
 have  $\bigwedge i. i \in I \implies ((\bigcap i \in I \cdot b \ i) \wedge b \ i) = b \ i$   
 by (rel-auto)  
 thus ?thesis  
 by (simp cong: UINF-cong)  
 qed  
 also have ... =  $(\bigcap i \in I \cdot b \ i \rightarrow_R P \ i)$   
 by (simp add: closure assms)  
 finally show ?thesis .  
 qed

## 11.7 Choose

**definition** choose-srd ::  $(s, t :: \text{trace}, \alpha) \text{ hrel-rsp } (\text{choose}_R) \text{ where}$   
 $[\text{upred-defs}, \text{rdes-def}]: \text{choose}_R = \mathbf{R}_s(\text{true}_r \vdash \text{false} \diamond \text{true}_r)$

**lemma** preR-choose [rdes]:  $\text{pre}_R(\text{choose}_R) = \text{true}_r$   
 by (rel-auto)

**lemma** periR-choose [rdes]:  $\text{peri}_R(\text{choose}_R) = \text{false}$   
 by (rel-auto)

**lemma** postR-choose [rdes]:  $\text{post}_R(\text{choose}_R) = \text{true}_r$   
 by (rel-auto)

**lemma** choose-srd-SRD [closure]:  $\text{choose}_R \text{ is SRD}$   
 by (simp add: choose-srd-def closure unrest)

**lemma** NSRD-choose-srd [closure]:  $\text{choose}_R \text{ is NSRD}$   
 by (rule NSRD-intro, simp-all add: closure unrest rdes)

## 11.8 Divergence Freedom

**definition** ndiv-srd ::  $(s, t :: \text{trace}, \alpha) \text{ hrel-rsp } (\text{ndiv}_R)$   
**where** [rdes-def]:  $\text{ndiv}_R = \mathbf{R}_s(\text{true}_r \vdash \text{true}_r \diamond \text{true}_r)$

**lemma** ndiv-NSRD [closure]:  $\text{ndiv}_R \text{ is NSRD}$   
 by (simp add: rdes-def closure unrest)

**lemma** ndiv-srd-refines-preR-true:  
 assumes  $P \text{ is SRD}$   
 shows  $\text{ndiv}_R \sqsubseteq P \longleftrightarrow \text{pre}_R(P) = \text{true}_r \text{ (is ?lhs } \longleftrightarrow \text{ ?rhs)}$   
 proof  
 assume ?lhs  
 thus ?rhs  
 by (metis R1-preR ndiv-srd-def preR-antitone preR-rdes rea-true-RR rea-true-disj(2) utp-pred-laws.sup.orderE)

next  
 assume ?rhs  
 hence  $\text{ndiv}_R \sqsubseteq \mathbf{R}_s(\text{pre}_R(P) \vdash \text{peri}_R(P) \diamond \text{post}_R(P))$   
 by (simp add: RHS-tri-design-conj assms ndiv-srd-def periR-SRD-R1 postR-SRD-R1 rea-true-conj(1)  
 rea-true-impl utp-pred-laws.inf.absorb-iff2)  
 thus ?lhs  
 by (simp add: SRD-reactive-tri-design assms)

qed

**lemma** *ndiv-srd-refines-rdes-pre-true*:  
**assumes**  $P_1$  is *RR*  $P_2$  is *RR*  $P_3$  is *RR*  
**shows**  $\text{ndiv}_R \sqsubseteq \mathbf{R}_s(P_1 \vdash P_2 \diamond P_3) \longleftrightarrow P_1 = \text{true}_r$  (**is** ?lhs  $\longleftrightarrow$  ?rhs)  
**by** (*simp add: ndiv-srd-refines-preR-true closure assms rdes unrest*)

## 11.9 State Abstraction

**definition** *state-srea* ::  
 $'s \text{ itself} \Rightarrow ('s, 't::\text{trace}, 'a, 'b) \text{ rel-rsp} \Rightarrow (\text{unit}, 't, 'a, 'b) \text{ rel-rsp}$  **where**  
*[upred-defs]*:  $\text{state-srea } t \ P = \langle \exists \ \{\$st, \$st'\} \cdot P \rangle_S$

**syntax**

*-state-srea* ::  $\text{type} \Rightarrow \text{logic} \Rightarrow \text{logic} \ (\text{state} \ - \ - \ - \ [0, 200] \ 200)$

**translations**

$\text{state } 'a \cdot P == \text{CONST state-srea TYPE('a)} \ P$

**lemma** *R1-state-srea*:  $R1(\text{state } 'a \cdot P) = (\text{state } 'a \cdot R1(P))$   
**by** (*rel-auto*)

**lemma** *R2c-state-srea*:  $R2c(\text{state } 'a \cdot P) = (\text{state } 'a \cdot R2c(P))$   
**by** (*rel-auto*)

**lemma** *R3h-state-srea*:  $R3h(\text{state } 'a \cdot P) = (\text{state } 'a \cdot R3h(P))$   
**by** (*rel-auto*)

**lemma** *RD1-state-srea*:  $RD1(\text{state } 'a \cdot P) = (\text{state } 'a \cdot RD1(P))$   
**by** (*rel-auto*)

**lemma** *RD2-state-srea*:  $RD2(\text{state } 'a \cdot P) = (\text{state } 'a \cdot RD2(P))$   
**by** (*rel-auto*)

**lemma** *RD3-state-srea*:  $RD3(\text{state } 'a \cdot P) = (\text{state } 'a \cdot RD3(P))$   
**by** (*rel-auto, blast+*)

**lemma** *SRD-state-srea* [*closure*]:  $P$  is *SRD*  $\implies \text{state } 'a \cdot P$  is *SRD*  
**by** (*simp add: Healthy-def R1-state-srea R2c-state-srea R3h-state-srea RD1-state-srea RD2-state-srea RHS-def SRD-def*)

**lemma** *NSRD-state-srea* [*closure*]:  $P$  is *NSRD*  $\implies \text{state } 'a \cdot P$  is *NSRD*  
**by** (*metis Healthy-def NSRD-is-RD3 NSRD-is-SRD RD3-state-srea SRD-RD3-implies-NSRD SRD-state-srea*)

**lemma** *preR-state-srea* [*rdes*]:  $\text{pre}_R(\text{state } 'a \cdot P) = \langle \forall \ \{\$st, \$st'\} \cdot \text{pre}_R(P) \rangle_S$   
**by** (*simp add: state-srea-def, rel-auto*)

**lemma** *periR-state-srea* [*rdes*]:  $\text{peri}_R(\text{state } 'a \cdot P) = \text{state } 'a \cdot \text{peri}_R(P)$   
**by** (*rel-auto*)

**lemma** *postR-state-srea* [*rdes*]:  $\text{post}_R(\text{state } 'a \cdot P) = \text{state } 'a \cdot \text{post}_R(P)$   
**by** (*rel-auto*)

**lemma** *state-srea-rdes-def* [*rdes-def*]:  
**assumes**  $P$  is *RC*  $Q$  is *RR*  $R$  is *RR*  
**shows**  $\text{state } 'a \cdot \mathbf{R}_s(P \vdash Q \diamond R) = \mathbf{R}_s(\langle \forall \ \{\$st, \$st'\} \cdot P \rangle_S \vdash (\text{state } 'a \cdot Q) \diamond (\text{state } 'a \cdot R))$

(is ?lhs = ?rhs)  
**proof** –  
 have ?lhs =  $\mathbf{R}_s(\text{pre}_R(?lhs) \vdash \text{peri}_R(?lhs) \diamond \text{post}_R(?lhs))$   
 by (simp add: RC-implies-RR SRD-rdes-intro SRD-reactive-tri-design SRD-state-srea assms)  
 also have ... =  $\mathbf{R}_s(\langle \forall \{ \$st, \$st' \} \cdot P \rangle_S \vdash \text{state } 'a \cdot (P \Rightarrow_r Q) \diamond \text{state } 'a \cdot (P \Rightarrow_r R))$   
 by (simp add: rdes closure assms)  
 also have ... = ?rhs  
 by (rel-auto)  
 finally show ?thesis .  
**qed**

**lemma** ext-st-rdes-dist [rdes-def]:  
 $\mathbf{R}_s(P \vdash Q \diamond R) \oplus_p \text{abs-st}_L = \mathbf{R}_s(P \oplus_p \text{abs-st}_L \vdash Q \oplus_p \text{abs-st}_L \diamond R \oplus_p \text{abs-st}_L)$   
 by (rel-auto)

**lemma** state-srea-refine:  
 $(P \oplus_p \text{abs-st}_L) \sqsubseteq Q \implies P \sqsubseteq (\text{state-srea TYPE}('s) Q)$   
 by (rel-auto)

## 11.10 Reactive Frames

**definition** rdes-frame-ext ::  $(' \alpha \implies ' \beta) \Rightarrow (' \alpha, 't::\text{trace}, 'r) \text{ hrel-rsp} \Rightarrow (' \beta, 't, 'r) \text{ hrel-rsp}$  **where**  
 [upred-defs, rdes-def]:  $\text{rdes-frame-ext } a \ P = \mathbf{R}_s(\text{rel-aext } (\text{pre}_R(P)) (\text{map-st}_L a) \vdash \text{rel-aext } (\text{peri}_R(P)) (\text{map-st}_L a) \diamond a : [\text{post}_R(P)]_r^+)$

**syntax**  
 -rdes-frame-ext ::  $\text{salpha} \Rightarrow \text{logic} \Rightarrow \text{logic } (:-)_{R^+} [99,0] 100)$

**translations**  
 -rdes-frame-ext  $x \ P \Rightarrow \text{CONST rdes-frame-ext } x \ P$   
 -rdes-frame-ext  $(\text{-salphaset } (\text{-salphamk } x)) \ P \leq \text{CONST rdes-frame-ext } x \ P$

**lemma** RC-rel-aext-st-closed [closure]:  
 assumes  $P$  is RC  
 shows  $\text{rel-aext } P (\text{map-st}_L a)$  is RC  
**proof** –  
 have  $\text{RC}(\text{rel-aext } (\text{RC}(P)) (\text{map-st}_L a)) = \text{rel-aext } (\text{RC}(P)) (\text{map-st}_L a)$   
 by (rel-auto)  
 (metis (no-types, hide-lams) diff-add-cancel-left' dual-order.trans le-add trace-class.add-diff-cancel-left trace-class.add-left-mono)  
 thus ?thesis  
 by (rule-tac Healthy-intro, simp add: assms Healthy-if)  
**qed**

**lemma** rdes-frame-ext-SRD-closed:  
 $\llbracket P \text{ is SRD}; \$wait' \# \text{pre}_R(P) \rrbracket \implies a : [P]_{R^+} \text{ is SRD}$   
 unfolding rdes-frame-ext-def  
 apply (rule SRD-rdes-intro)  
 apply (simp-all add: closure unrest )  
 apply (simp add: RR-R2-intro ok'-pre-unrest ok-pre-unrest preR-R2-closed rea-aext-RR wait-pre-unrest)  
 done

**lemma** preR-rdes-frame-ext:  
 $P \text{ is NSRD} \implies \text{pre}_R(a : [P]_{R^+}) = \text{rel-aext } (\text{pre}_R(P)) (\text{map-st}_L a)$   
 by (simp add: preR-RR preR-rdes rdes-frame-ext-def rea-aext-RR)

**lemma** *unrest-rel-aext-st'* [*unrest*]:  $\$st' \# P \implies \$st' \# \text{rel-aext } P (\text{map-st}_L a)$   
**by** (*rel-auto*)

**lemma** *rdes-frame-ext-NSRD-closed*:

*P* is NSRD  $\implies a:[P]_R^+$  is NSRD

**apply** (*rule NSRD-RC-intro*)

**apply** (*rule rdes-frame-ext-SRD-closed*)

**apply** (*simp-all add: closure unrest rdes*)

**apply** (*simp add: NSRD-neg-pre-RC RC-rel-aext-st-closed preR-RR preR-rdes rdes-frame-ext-def rea-aext-RR*)

**apply** (*simp add: rdes-frame-ext-def*)

**apply** (*simp add: rdes closure unrest*)

**done**

**lemma** *skip-srea-frame* [*frame*]:

*vwb-lens a*  $\implies a:[II_R]_R^+ = II_R$

**by** (*rdes-eq*)

**lemma** *seq-srea-frame* [*frame*]:

**assumes** *vwb-lens a P* is NSRD *Q* is NSRD

**shows**  $a:[P ;; Q]_R^+ = a:[P]_R^+ ;; a:[Q]_R^+$  (**is** *?lhs = ?rhs*)

**proof** –

**have** *?lhs* =  $\mathbf{R}_s ((pre_R P \wedge post_R P \text{wp}_r pre_R Q) \oplus_r \text{map-st}_L[a] \vdash$

$((pre_R P \wedge post_R P \text{wp}_r pre_R Q) \oplus_r \text{map-st}_L[a] \Rightarrow_r (peri_R P \vee post_R P ;; peri_R Q) \oplus_r \text{map-st}_L[a]) \diamond$

$a:[pre_R P \wedge post_R P \text{wp}_r pre_R Q \Rightarrow_r post_R P ;; post_R Q]_R^+$ )

**using** *assms(1)* **by** (*rdes-simp cls: assms(2-3)*)

**also have** ... =  $\mathbf{R}_s ((pre_R P \wedge post_R P \text{wp}_r pre_R Q) \oplus_r \text{map-st}_L[a] \vdash$

$((peri_R P \vee post_R P ;; peri_R Q) \oplus_r \text{map-st}_L[a]) \diamond$

$a:[post_R P ;; post_R Q]_R^+$ )

**by** (*rel-auto*)

**also from** *assms(1)* **have** ... = *?rhs*

**apply** (*rdes-eq-split cls: assms(2-3)*)

**apply** (*rel-auto*)

**apply** (*metis mwb-lens-def vwb-lens-mwb weak-lens.put-get*)

**apply** (*rel-auto*)

**apply** (*metis mwb-lens-def vwb-lens-mwb weak-lens.put-get*)

**apply** (*metis vwb-lens-wb wb-lens-def weak-lens.put-get*)

**apply** (*metis mwb-lens-def vwb-lens-mwb weak-lens.put-get*)

**apply** (*rel-auto*)

**apply** (*metis mwb-lens-def vwb-lens-mwb weak-lens.put-get*)

**done**

**finally show** *?thesis* .

**qed**

**lemma** *rdes-frame-ext-Productive-closed* [*closure*]:

**assumes** *P* is NSRD *P* is Productive

**shows**  $x:[P]_R^+$  is Productive

**proof** –

**have**  $x:[Productive(P)]_R^+$  is Productive

**by** (*rdes-simp cls: assms, rel-auto*)

**thus** *?thesis*

**by** (*simp add: Healthy-if assms*)

**qed**

## 11.11 While Loop

**definition**  $WhileR :: 's \text{ upred} \Rightarrow ('s, 't::\text{size-trace}, 'a) \text{ hrel-rsp} \Rightarrow ('s, 't, 'a) \text{ hrel-rsp}$  **where**  
 $WhileR \ b \ P = (\mu_R \ X \cdot (P ;; X) \triangleleft b \triangleright_R II_R)$

**syntax**  $-whileR :: \text{logic} \Rightarrow \text{logic} \Rightarrow \text{logic}$  ( $while_R$  -  $do$  -  $od$ )

**translations**  $-whileR \ b \ P == CONST \ WhileR \ b \ P$

**lemma** *Sup-power-false:*

**fixes**  $F :: 'a \text{ upred} \Rightarrow 'a \text{ upred}$

**shows**  $(\bigcap i. (F \ \hat{\wedge} \ i) \ \text{false}) = (\bigcap i. (F \ \hat{\wedge} \ (i+1)) \ \text{false})$

**proof** –

**have**  $(\bigcap i. (F \ \hat{\wedge} \ i) \ \text{false}) = (F \ \hat{\wedge} \ 0) \ \text{false} \sqcap (\bigcap i. (F \ \hat{\wedge} \ (i+1)) \ \text{false})$

**by** (*subst Sup-power-expand, simp*)

**also have**  $\dots = (\bigcap i. (F \ \hat{\wedge} \ (i+1)) \ \text{false})$

**by** (*simp*)

**finally show** *?thesis* .

**qed**

**theorem** *WhileR-unfold:*

**assumes**  $P$  *is NSRD*

**shows**  $while_R \ b \ do \ P \ od = (P ;; while_R \ b \ do \ P \ od) \triangleleft b \triangleright_R II_R$

**apply** (*simp add: WhileR-def*)

**apply** (*subst srdes-theory.LFP-unfold*)

**apply** (*simp-all add: mono-Monotone-utp-order closure assms*)

**done**

**theorem** *WhileR-iter-expand:*

**assumes**  $P$  *is NSRD*  $P$  *is Productive*

**shows**  $while_R \ b \ do \ P \ od = (\bigcap i. (P \triangleleft b \triangleright_R II_R) \ \hat{\wedge} \ i ;; (P ;; Miracle \triangleleft b \triangleright_R II_R))$  (**is** *?lhs = ?rhs*)

**proof** –

**have**  $1: \text{Continuous} \ (\lambda X. P ;; SRD \ X)$

**using** *SRD-Continuous*

**by** (*clarsimp simp add: Continuous-def seq-SUP-distl[THEN sym], drule-tac x=A in spec, simp*)

**have**  $2: \text{Continuous} \ (\lambda X. P ;; SRD \ X \triangleleft b \triangleright_R II_R)$

**by** (*simp add: 1 closure assms*)

**have**  $?lhs = (\mu_R \ X \cdot P ;; X \triangleleft b \triangleright_R II_R)$

**by** (*simp add: WhileR-def*)

**also have**  $\dots = (\mu \ X \cdot P ;; SRD(X) \triangleleft b \triangleright_R II_R)$

**by** (*auto simp add: srd-mu-equiv closure assms*)

**also have**  $\dots = (\nu \ X \cdot P ;; SRD(X) \triangleleft b \triangleright_R II_R)$

**by** (*auto simp add: guarded-fp-uniq Guarded-if-Productive[OF assms] funcsetI closure assms*)

**also have**  $\dots = (\bigcap i. ((\lambda X. P ;; SRD \ X \triangleleft b \triangleright_R II_R) \ \hat{\wedge} \ i) \ \text{false})$

**by** (*simp add: sup-continuous-lfp 2 sup-continuous-Continuous false-upred-def*)

**also have**  $\dots = (\bigcap i. ((\lambda X. P ;; SRD \ X \triangleleft b \triangleright_R II_R) \ \hat{\wedge} \ (i+1)) \ \text{false})$

**by** (*simp add: Sup-power-false*)

**also have**  $\dots = (\bigcap i. (P \triangleleft b \triangleright_R II_R) \ \hat{\wedge} \ i ;; (P ;; Miracle \triangleleft b \triangleright_R II_R))$

**proof** (*rule SUP-cong, simp*)

**fix**  $i$

**show**  $((\lambda X. P ;; SRD \ X \triangleleft b \triangleright_R II_R) \ \hat{\wedge} \ (i+1)) \ \text{false} = (P \triangleleft b \triangleright_R II_R) \ \hat{\wedge} \ i ;; (P ;; Miracle \triangleleft b \triangleright_R II_R)$

**proof** (*induct i*)

**case**  $0$

**thm** *if-eq-cancel*

**then show** *?case*

**by** (*simp, metis srdes-theory.healthy-top*)



```

next
  case (Suc i)
  show ?case
  proof -
    have  $((\lambda X. P ;; SRD\ X \triangleleft b \triangleright_R II_R) \wedge (Suc\ i + 1))\ false =$ 
       $P ;; SRD\ ((\lambda X. P ;; SRD\ X \triangleleft b \triangleright_R II_R) \wedge (i + 1))\ false) \triangleleft b \triangleright_R II_R$ 
    by simp
    also have  $\dots = P ;; SRD\ ((P \triangleleft b \triangleright_R II_R) \wedge i ;; (P ;; Miracle \triangleleft b \triangleright_R II_R)) \triangleleft b \triangleright_R II_R$ 
    using Suc.hyps by auto
    also have  $\dots = P ;; ((P \triangleleft b \triangleright_R II_R) \wedge i ;; (P ;; Miracle \triangleleft b \triangleright_R II_R)) \triangleleft b \triangleright_R II_R$ 
    by (metis (no-types, lifting) Healthy-if NSRD-cond-srea NSRD-is-SRD NSRD-power-Suc
      NSRD-srd-skip SRD-cond-srea SRD-seqr-closure assms(1) power.power-eq-if seqr-left-unit srdes-theory.top-closed)
    also have  $\dots = (P \triangleleft b \triangleright_R II_R) \wedge Suc\ i ;; (P ;; Miracle \triangleleft b \triangleright_R II_R)$ 
    proof (induct i)
      case 0
      then show ?case
      by (simp add: NSRD-is-SRD SRD-cond-srea SRD-left-unit SRD-seqr-closure SRD-srdes-skip
        assms(1) cond-L6 cond-st-distr srdes-theory.top-closed)
    next
      case (Suc i)
      have  $1: II_R ;; ((P \triangleleft b \triangleright_R II_R) ;; (P \triangleleft b \triangleright_R II_R) \wedge i) = ((P \triangleleft b \triangleright_R II_R) ;; (P \triangleleft b \triangleright_R II_R) \wedge i)$ 
      by (simp add: NSRD-is-SRD RA1 SRD-cond-srea SRD-left-unit SRD-srdes-skip assms(1))
      then show ?case
      proof -
        have  $\bigwedge u. (u ;; (P \triangleleft b \triangleright_R II_R) \wedge Suc\ i) ;; (P ;; (Miracle) \triangleleft b \triangleright_R (II_R)) \triangleleft b \triangleright_R (II_R) =$ 
           $((u \triangleleft b \triangleright_R II_R) ;; (P \triangleleft b \triangleright_R II_R) \wedge Suc\ i) ;; (P ;; (Miracle) \triangleleft b \triangleright_R (II_R))$ 
        by (metis (no-types) Suc.hyps 1 cond-L6 cond-st-distr power.power.power-Suc)
        then show ?thesis
        by (simp add: RA1 upred-semiring.power-Suc)
      qed
    qed
    qed
    finally show ?thesis .
  qed
  qed
  qed
  qed
  also have  $\dots = (\bigcap i \cdot (P \triangleleft b \triangleright_R II_R) \wedge i) ;; (P ;; Miracle \triangleleft b \triangleright_R II_R)$ 
  by (simp add: UINF-as-Sup-collect')
  finally show ?thesis .
qed

```

**theorem** *WhileR-star-expand:*

```

  assumes P is NSRD P is Productive
  shows  $while_R\ b\ do\ P\ od = (P \triangleleft b \triangleright_R II_R)^{*R} ;; (P ;; Miracle \triangleleft b \triangleright_R II_R)$  (is ?lhs = ?rhs)
proof -
  have  $?lhs = (\bigcap i \cdot (P \triangleleft b \triangleright_R II_R) \wedge i) ;; (P ;; Miracle \triangleleft b \triangleright_R II_R)$ 
  by (simp add: WhileR-iter-expand seq-UINF-distr' assms)
  also have  $\dots = (P \triangleleft b \triangleright_R II_R)^* ;; (P ;; Miracle \triangleleft b \triangleright_R II_R)$ 
  by (simp add: ustar-def)
  also have  $\dots = ((P \triangleleft b \triangleright_R II_R)^* ;; II_R) ;; (P ;; Miracle \triangleleft b \triangleright_R II_R)$ 
  by (simp add: seqr-assoc SRD-left-unit closure assms)
  also have  $\dots = (P \triangleleft b \triangleright_R II_R)^{*R} ;; (P ;; Miracle \triangleleft b \triangleright_R II_R)$ 
  by (simp add: StarR-def nsrdes-theory.Star-def)
  finally show ?thesis .
qed

```

**lemma** *WhileR-NSRD-closed* [closure]:

**assumes** *P is NSRD P is Productive*

**shows** *while<sub>R</sub> b do P od is NSRD*

**by** (*simp add: StarR-def WhileR-star-expand assms closure*)

**theorem** *WhileR-iter-form-lemma*:

**assumes** *P is NSRD*

**shows**  $(P \triangleleft b \triangleright_R II_R)^{\star R} ;; (P ;; \text{Miracle} \triangleleft b \triangleright_R II_R) = ([b]^\top_R ;; P)^{\star R} ;; [\neg b]^\top_R$

**proof** –

**have**  $(P \triangleleft b \triangleright_R II_R)^{\star R} ;; (P ;; \text{Miracle} \triangleleft b \triangleright_R II_R) = ([b]^\top_R ;; P) \sqcap [\neg b]^\top_R)^{\star R} ;; (P ;; \text{Miracle} \triangleleft b \triangleright_R II_R)$

**by** (*simp add: AssumeR-NSRD NSRD-right-unit NSRD-srd-skip assms(1) cond-srea-AssumeR-form*)

**also have**  $\dots = ([b]^\top_R ;; P)^{\star R} ;; [\neg b]^\top_R)^{\star R} ;; (P ;; \text{Miracle} \triangleleft b \triangleright_R II_R)$

**by** (*simp add: StarR-def AssumeR-NSRD NSRD-seqr-closure nsrdes-theory.Star-denest assms(1)*)

**also have**  $\dots = ([b]^\top_R ;; P)^{\star R} ;; (P ;; \text{Miracle} \triangleleft b \triangleright_R II_R)$

**by** (*metis (no-types, hide-lams) StarR-def RD3-def RD3-idem Star-AssumeR nsrdes-theory.Star-def*)

**also have**  $\dots = ([b]^\top_R ;; P)^{\star R} ;; (P ;; \text{Miracle} \triangleleft b \triangleright_R II_R)$

**by** (*simp add: StarR-def AssumeR-NSRD NSRD-seqr-closure nsrdes-theory.Star-invol assms(1)*)

**also have**  $\dots = ([b]^\top_R ;; P)^{\star R} ;; ([b]^\top_R ;; P ;; \text{Miracle}) \sqcap [\neg b]^\top_R$

**by** (*simp add: AssumeR-NSRD NSRD-Miracle NSRD-right-unit NSRD-seqr-closure NSRD-srd-skip assms(1) cond-srea-AssumeR-form*)

**also have**  $\dots = ([b]^\top_R ;; P)^{\star R} ;; [b]^\top_R ;; P ;; \text{Miracle}) \sqcap ([b]^\top_R ;; P)^{\star R} ;; [\neg b]^\top_R$

**by** (*simp add: upred-semiring.distrib-left*)

**also have**  $\dots = [b]^\top_R ;; P)^{\star R} ;; [\neg b]^\top_R$

**proof** –

**have**  $([b]^\top_R ;; P)^{\star R} ;; [\neg b]^\top_R = (II_R \sqcap ([b]^\top_R ;; P)^{\star R} ;; [b]^\top_R ;; P) ;; [\neg b]^\top_R$

**by** (*simp add: StarR-def AssumeR-NSRD NSRD-seqr-closure nsrdes-theory.Star-unfoldr-eq assms(1)*)

**also have**  $\dots = [\neg b]^\top_R \sqcap ([b]^\top_R ;; P)^{\star R} ;; [b]^\top_R ;; P ;; [\neg b]^\top_R$

**by** (*metis (no-types, lifting) StarR-def AssumeR-NSRD AssumeR-as-gcmd NSRD-srd-skip Star-AssumeR nsrdes-theory.Star-slide gcmd-seq-distr skip-srea-self-unit urel-dioid.distrib-right*)

**also have**  $\dots = [\neg b]^\top_R \sqcap ([b]^\top_R ;; P)^{\star R} ;; [b]^\top_R ;; P ;; [b \vee \neg b]^\top_R ;; [\neg b]^\top_R$

**by** (*simp add: AssumeR-true NSRD-right-unit assms(1)*)

**also have**  $\dots = [\neg b]^\top_R \sqcap ([b]^\top_R ;; P)^{\star R} ;; [b]^\top_R ;; P ;; [b]^\top_R ;; [\neg b]^\top_R$   
 $\sqcap ([b]^\top_R ;; P)^{\star R} ;; [b]^\top_R ;; P ;; [\neg b]^\top_R ;; [\neg b]^\top_R$

**by** (*metis (no-types, hide-lams) AssumeR-choice upred-semiring.add-assoc upred-semiring.distrib-left upred-semiring.distrib-right*)

**also have**  $\dots = [\neg b]^\top_R \sqcap ([b]^\top_R ;; P)^{\star R} ;; [b]^\top_R ;; P ;; ([b]^\top_R ;; [\neg b]^\top_R)$   
 $\sqcap ([b]^\top_R ;; P)^{\star R} ;; [b]^\top_R ;; P ;; ([\neg b]^\top_R ;; [\neg b]^\top_R)$

**by** (*simp add: RA1*)

**also have**  $\dots = [\neg b]^\top_R \sqcap ([b]^\top_R ;; P)^{\star R} ;; [b]^\top_R ;; P ;; \text{Miracle}$   
 $\sqcap ([b]^\top_R ;; P)^{\star R} ;; [b]^\top_R ;; P ;; [\neg b]^\top_R$

**by** (*simp add: AssumeR-comp AssumeR-false*)

**finally have**  $[b]^\top_R ;; P)^{\star R} ;; [\neg b]^\top_R \sqsubseteq ([b]^\top_R ;; P)^{\star R} ;; [b]^\top_R ;; P ;; \text{Miracle}$

**by** (*simp add: semilattice-sup-class.le-supI1*)

**thus** ?thesis

**by** (*simp add: semilattice-sup-class.le-iff-sup*)

**qed**

**finally show** ?thesis .

**qed**

**theorem** *WhileR-iter-form*:

**assumes** *P is NSRD P is Productive*

**shows** *while<sub>R</sub> b do P od = ([b]^\top\_R ;; P)^{\star R} ;; [\neg b]^\top\_R*

**by** (*simp add: WhileR-iter-form-lemma WhileR-star-expand assms*)

**theorem** *WhileR-outer-refine-intro*:

**assumes**  
 $P$  is NSRD  $P$  is Productive  
 $S \sqsubseteq ([b]^\top_R ;; P) ;; S$   $S \sqsubseteq [\neg b]^\top_R$   
**shows**  $S \sqsubseteq \text{while}_R b \text{ do } P \text{ od}$   
**apply** (*simp add: assms StarR-def WhileR-iter-form*)  
**apply** (*rule nsrdes-theory.Star-inductl*)  
**apply** (*simp-all add: closure assms*)  
**done**

**theorem** *WhileR-outer-refine-init-intro*:

**assumes**  
 $P$  is NSRD  $I$  is NSRD  $P$  is Productive  
 $S \sqsubseteq I ;; [\neg b]^\top_R$   
 $S \sqsubseteq S ;; [b]^\top_R ;; P$   
 $S \sqsubseteq I ;; [b]^\top_R ;; P$   
**shows**  $S \sqsubseteq I ;; \text{while}_R b \text{ do } P \text{ od}$   
**proof** –  
**have**  $S \sqsubseteq I ;; ([b]^\top_R ;; P) ;; ([b]^\top_R ;; P)^{\star R} ;; [\neg b]^\top_R$   
**proof** –  
**have**  $S \sqsubseteq I ;; ([b]^\top_R ;; P) ;; ([b]^\top_R ;; P)^{\star R}$   
**by** (*metis (no-types, hide-lams) StarR-def AssumeR-NSRD NSRD-seqr-closure RA1 assms(1) assms(2) assms(5) assms(6) nsrdes-theory.Star-inductr semilattice-sup-class.le-sup-iff*)  
**thus** ?thesis  
**by** (*metis (no-types, lifting) AssumeR-NSRD AssumeR-seq-refines StarR-def assms(1) dual-order.trans nsrdes-theory.Healthy-Sequence nsrdes-theory.utp-theory-kleene-axioms urel-doid.mult-isol utp-theory-kleene.Star-Healthy*)  
**qed**  
**moreover have**  $S \sqsubseteq I ;; II_R ;; [\neg b]^\top_R$   
**by** (*simp add: AssumeR-NSRD assms nsrdes-theory.Unit-Left*)  
**ultimately show** ?thesis  
**apply** (*simp add: assms WhileR-iter-form StarR-def*)  
**apply** (*subst nsrdes-theory.Star-unfoldl-eq[THEN sym]*)  
**apply** (*auto simp add: closure assms seqr-inf-distr*)  
**done**  
**qed**

**theorem** *WhileR-false*:

**assumes**  $P$  is NSRD  
**shows**  $\text{while}_R \text{false do } P \text{ od} = II_R$   
**by** (*simp add: WhileR-def rpred closure srdes-theory.LFP-const*)

**theorem** *WhileR-true*:

**assumes**  $P$  is NSRD  $P$  is Productive  
**shows**  $\text{while}_R \text{true do } P \text{ od} = P^{\star R} ;; \text{Miracle}$   
**by** (*simp add: WhileR-iter-form AssumeR-true AssumeR-false SRD-left-unit assms closure*)

**lemma** *WhileR-insert-assume*:

**assumes**  $P$  is NSRD  $P$  is Productive  
**shows**  $\text{while}_R b \text{ do } ([b]^\top_R ;; P) \text{ od} = \text{while}_R b \text{ do } P \text{ od}$   
**by** (*simp add: AssumeR-NSRD AssumeR-comp NSRD-seqr-closure Productive-seq-2 RA1 WhileR-iter-form assms*)

**theorem** *WhileR-rdes-def [rdes-def]*:

**assumes**  $P$  is RC  $Q$  is RR  $R$  is RR  $\$st' \# Q$   $R$  is R4  
**shows**  $\text{while}_R b \text{ do } \mathbf{R}_s(P \vdash Q \diamond R) \text{ od} =$

$\mathbf{R}_s (([b]^\top_r ;; R)^{*r} \text{ wp}_r ([b]_{S<} \Rightarrow_r P) \vdash (([b]^\top_r ;; R)^{*r} ;; [b]^\top_r ;; Q) \diamond (([b]^\top_r ;; R)^{*r} ;; [\neg b]^\top_r))$   
 (is ?lhs = ?rhs)  
**proof** –  
 have ?lhs =  $([b]^\top_R ;; \mathbf{R}_s (P \vdash Q \diamond R))^{*R} ;; [\neg b]^\top_R$   
 by (simp add: WhileR-iter-form Productive-rdes-RR-intro assms closure)  
 also have ... = ?rhs  
 by (simp add: rdes-def assms closure unrest rpred wp del: rea-star-wp)  
 finally show ?thesis .  
**qed**

Refinement introduction law for reactive while loops

**theorem** *WhileR-refine-intro*:

**assumes**  
 — Closure conditions  
 $Q_1$  is RC  $Q_2$  is RR  $Q_3$  is RR  $\$st' \# Q_2$   $Q_3$  is R4  
 — Refinement conditions  
 $([b]^\top_r ;; Q_3)^{*r} \text{ wp}_r ([b]_{S<} \Rightarrow_r Q_1) \sqsubseteq P_1$   
 $P_2 \sqsubseteq [b]^\top_r ;; Q_2$   
 $P_2 \sqsubseteq [b]^\top_r ;; Q_3 ;; P_2$   
 $P_3 \sqsubseteq [\neg b]^\top_r$   
 $P_3 \sqsubseteq [b]^\top_r ;; Q_3 ;; P_3$   
**shows**  $\mathbf{R}_s (P_1 \vdash P_2 \diamond P_3) \sqsubseteq \text{while}_R b \text{ do } \mathbf{R}_s (Q_1 \vdash Q_2 \diamond Q_3) \text{ od}$   
**proof** (simp add: rdes-def assms, rule srdes-tri-refine-intro')  
**show**  $([b]^\top_r ;; Q_3)^{*r} \text{ wp}_r ([b]_{S<} \Rightarrow_r Q_1) \sqsubseteq P_1$   
 by (simp add: assms)  
**show**  $P_2 \sqsubseteq (P_1 \wedge ([b]^\top_r ;; Q_3)^{*r} ;; [b]^\top_r ;; Q_2)$   
**proof** –  
 have  $P_2 \sqsubseteq ([b]^\top_r ;; Q_3)^{*r} ;; [b]^\top_r ;; Q_2$   
 by (simp add: assms rea-assume-RR rrel-theory.Star-inductl seq-RR-closed seqr-assoc)  
**thus** ?thesis  
 by (simp add: utp-pred-laws.le-infI2)  
**qed**  
**show**  $P_3 \sqsubseteq (P_1 \wedge ([b]^\top_r ;; Q_3)^{*r} ;; [\neg b]^\top_r)$   
**proof** –  
 have  $P_3 \sqsubseteq ([b]^\top_r ;; Q_3)^{*r} ;; [\neg b]^\top_r$   
 by (simp add: assms rea-assume-RR rrel-theory.Star-inductl seqr-assoc)  
**thus** ?thesis  
 by (simp add: utp-pred-laws.le-infI2)  
**qed**  
**qed**

## 11.12 Iteration Construction

**definition** *IterateR*

$:: 'a \text{ set} \Rightarrow ('a \Rightarrow 's \text{ upred}) \Rightarrow ('a \Rightarrow ('s, 't::\text{size-trace}, 'a) \text{ hrel-rsp}) \Rightarrow ('s, 't, 'a) \text{ hrel-rsp}$   
**where**  $\text{IterateR } A \ g \ P = \text{while}_R (\bigvee i \in A \cdot g(i)) \text{ do } (\text{if}_R i \in A \cdot g(i) \rightarrow P(i) \text{ fi}) \text{ od}$

**definition** *IterateR-list*

$:: ('s \text{ upred} \times ('s, 't::\text{size-trace}, 'a) \text{ hrel-rsp}) \text{ list} \Rightarrow ('s, 't, 'a) \text{ hrel-rsp}$  **where**  
 $[\text{upred-defs}, \text{ndes-simp}]$ :

$\text{IterateR-list } xs = \text{IterateR } \{0..<\text{length } xs\} (\lambda i. \text{map fst } xs ! i) (\lambda i. \text{map snd } xs ! i)$

**syntax**

$\text{-iter-srd} \quad :: \text{pttrn} \Rightarrow \text{logic} \Rightarrow \text{logic} \Rightarrow \text{logic} \Rightarrow \text{logic} \ (do_R \text{-}\in\text{-} \cdot \text{-} \rightarrow \text{-} \text{od})$   
 $\text{-iter-gcommR} \quad :: \text{gcomms} \Rightarrow \text{logic} \ (do_R / \text{-} / \text{od})$

## translations

$-iter-srd\ x\ A\ g\ P \Rightarrow CONST\ IterateR\ A\ (\lambda\ x.\ g)\ (\lambda\ x.\ P)$   
 $-iter-srd\ x\ A\ g\ P \Leftarrow CONST\ IterateR\ A\ (\lambda\ x.\ g)\ (\lambda\ x'.\ P)$   
 $-iter-gcommR\ cs \rightarrow CONST\ IterateR-list\ cs$   
 $-iter-gcommR\ (-gcomm-show\ cs) \leftarrow CONST\ IterateR-list\ cs$

**lemma** *IterateR-NSRD-closed* [closure]:

**assumes**  
 $\bigwedge i. i \in I \implies P(i)\ \text{is}\ NSRD$   
 $\bigwedge i. i \in I \implies P(i)\ \text{is}\ Productive$   
**shows**  $do_R\ i \in I \cdot g(i) \rightarrow P(i)\ \text{od}\ \text{is}\ NSRD$   
**by** (*simp add: IterateR-def closure assms*)

**lemma** *IterateR-empty*:

$do_R\ i \in \{\} \cdot g(i) \rightarrow P(i)\ \text{od} = II_R$   
**by** (*simp add: IterateR-def srd-mu-equiv closure rpred gfp-const WhileR-false*)

**lemma** *IterateR-singleton*:

**assumes**  $P\ k\ \text{is}\ NSRD\ P\ k\ \text{is}\ Productive$   
**shows**  $do_R\ i \in \{k\} \cdot g(i) \rightarrow P(i)\ \text{od} = while_R\ g(k)\ do\ P(k)\ \text{od}\ (\text{is}\ ?lhs = ?rhs)$

**proof** –

**have**  $?lhs = while_R\ g\ k\ do\ P\ k \triangleleft g\ k \triangleright_R\ Chaos\ \text{od}$   
**by** (*simp add: IterateR-def AlternateR-singleton assms closure*)  
**also have**  $\dots = while_R\ g\ k\ do\ [g\ k]^\top_R \;;\ (P\ k \triangleleft g\ k \triangleright_R\ Chaos)\ \text{od}$   
**by** (*simp add: WhileR-insert-assume closure assms*)  
**also have**  $\dots = while_R\ g\ k\ do\ P\ k\ \text{od}$   
**by** (*simp add: AssumeR-cond-left NSRD-Chaos WhileR-insert-assume assms*)  
**finally show**  $?thesis$  .

**qed**

**declare** *IterateR-list-def* [rdes-def]

**declare** *IterateR-def* [rdes-def]

**lemma** *R4-Continuous* [closure]: *Continuous R4*

**by** (*rel-auto*)

**lemma** *cond-rea-R4-closed* [closure]:

$\llbracket P\ \text{is}\ R4;\ Q\ \text{is}\ R4 \rrbracket \implies P \triangleleft b \triangleright_R Q\ \text{is}\ R4$   
**by** (*simp add: Healthy-def R4-cond*)

**lemma** *IterateR-outer-refine-intro*:

**assumes**  $I \neq \{\} \bigwedge i. i \in I \implies P\ i\ \text{is}\ NSRD \bigwedge i. i \in I \implies P\ i\ \text{is}\ Productive$   
 $\bigwedge i. i \in I \implies S \sqsubseteq (b\ i \rightarrow_R P\ i \;;\ S)$   
 $S \sqsubseteq [\neg (\bigwedge i \in I \cdot b\ i)]^\top_R$   
**shows**  $S \sqsubseteq do_R\ i \in I \cdot b(i) \rightarrow P(i)\ \text{od}$   
**apply** (*simp add: IterateR-def*)  
**apply** (*rule WhileR-outer-refine-intro*)  
**apply** (*simp-all add: assms closure AlternateR-assume-branch seq-UINF-distr UINF-refines*)  
**done**

**lemma** *IterateR-outer-refine-init-intro*:

**assumes**  
 $A \neq \{\} \bigwedge i. i \in A \implies P\ i\ \text{is}\ NSRD$   
 $\bigwedge i. i \in A \implies P\ i\ \text{is}\ Productive$   
 $I\ \text{is}\ NSRD$

$S \sqsubseteq I ;; [\neg (\bigwedge i \in A \cdot b\ i)]^\top_R$   
 $\bigwedge i. i \in A \implies S \sqsubseteq S ;; b\ i \rightarrow_R P\ i$   
 $\bigwedge i. i \in A \implies S \sqsubseteq I ;; b\ i \rightarrow_R P\ i$   
**shows**  $S \sqsubseteq I ;; do_R\ i \in A \cdot b(i) \rightarrow P(i)$  *od*  
**apply** (*simp add: IterateR-def*)  
**apply** (*rule-tac WhileR-outer-refine-init-intro*)  
**apply** (*simp-all add: assms closure AlternateR-assume-branch seq-UINF-distl UINF-refines*)  
**done**

**lemma** *IterateR-lemma1:*

$(\bigwedge i \in I \cdot b\ i)^\top_r ;; (\bigwedge i \in I \cdot P\ i \triangleleft b\ i \triangleright_R \text{false}) = (\bigwedge i \in I \cdot [b\ i]^\top_r ;; P\ i)$   
**by** (*rel-auto; fastforce*)

**lemma** *IterateR-lemma2:*

**assumes**  $I \neq \{\}$   $\bigwedge i. i \in I \implies P(i)$  *is RR*

**shows**  $(\bigwedge i \in I \cdot b\ i)_{S<} \Rightarrow_r (\bigwedge i \in I \cdot (P\ i) \triangleleft b\ i \triangleright_R R1\ \text{true}) \wedge \text{false} \triangleleft (\neg (\bigwedge i \in I \cdot b\ i)) \triangleright_R R1\ \text{true})$   
 $= (\bigwedge i \in I \cdot (P\ i) \triangleleft b\ i \triangleright_R R1\ \text{true})$

**proof** –

**from** *assms(1)*

**have**  $(\bigwedge i \in I \cdot b\ i)_{S<} \Rightarrow_r (\bigwedge i \in I \cdot RR(P\ i) \triangleleft b\ i \triangleright_R R1\ \text{true}) \wedge \text{false} \triangleleft (\neg (\bigwedge i \in I \cdot b\ i)) \triangleright_R R1\ \text{true})$   
 $= (\bigwedge i \in I \cdot RR(P\ i) \triangleleft b\ i \triangleright_R R1\ \text{true})$

**by** (*rel-auto*)

**thus** *?thesis*

**by** (*simp add: assms Healthy-if cong: USUP-cong*)

**qed**

**lemma** *IterateR-lemma3:*

**assumes**  $\bigwedge i. i \in I \implies P(i)$  *is RR*

**shows**  $(\bigwedge i \in I \cdot P\ i \triangleleft b\ i \triangleright_R R1\ \text{true}) = (\bigwedge i \in I \cdot [b\ i]_{S<} \Rightarrow_r P\ i)$

**proof** –

**have**  $(\bigwedge i \in I \cdot RR(P\ i) \triangleleft b\ i \triangleright_R R1\ \text{true}) = (\bigwedge i \in I \cdot [b\ i]_{S<} \Rightarrow_r RR(P\ i))$

**by** (*rel-auto*)

**thus** *?thesis*

**by** (*simp add: assms Healthy-if cong: USUP-cong*)

**qed**

**theorem** *IterateR-refine-intro:*

**assumes**

– Closure conditions

$\bigwedge i. i \in I \implies Q_1(i)$  *is RC*  $\bigwedge i. i \in I \implies Q_2(i)$  *is RR*  $\bigwedge i. i \in I \implies Q_3(i)$  *is RR*

$\bigwedge i. i \in I \implies \$st' \nmid Q_2(i)$   $\bigwedge i. i \in I \implies Q_3(i)$  *is R4*  $I \neq \{\}$

$(\bigwedge i \in I \cdot [b\ i]^\top_r ;; Q_3\ i)^{*r} wp_r (\bigwedge i \in I \cdot [b\ i]_{S<} \Rightarrow_r Q_1\ i) \sqsubseteq P_1$

$P_2 \sqsubseteq (\bigwedge i \in I \cdot [b\ i]^\top_r ;; Q_2\ i)$

$P_2 \sqsubseteq (\bigwedge i \in I \cdot [b\ i]^\top_r ;; Q_3\ i) ;; P_2$

$P_3 \sqsubseteq [\neg (\bigwedge i \in I \cdot b\ i)]^\top_r$

$P_3 \sqsubseteq (\bigwedge i \in I \cdot [b\ i]^\top_r ;; Q_3\ i) ;; P_3$

**shows**  $\mathbf{R}_s(P_1 \vdash P_2 \diamond P_3) \sqsubseteq do_R\ i \in I \cdot b(i) \rightarrow \mathbf{R}_s(Q_1(i) \vdash Q_2(i) \diamond Q_3(i))$  *od*

**apply** (*simp add: rdes-def closure assms unrest del: WhileR-rdes-def*)

**apply** (*rule WhileR-refine-intro*)

**apply** (*simp-all add: closure assms unrest IterateR-lemma1 IterateR-lemma2 seqr-assoc[THEN sym]*)

**apply** (*simp add: IterateR-lemma3 closure assms unrest*)

**done**

**method** *unfold-iteration* = *simp add: IterateR-list-def IterateR-def AlternateR-list-def AlternateR-def UINF-upto-expand-first*

### 11.13 Substitution Laws

**lemma** *srd-subst-Chaos* [*usubst*]:  
 $\sigma \dagger_S \text{Chaos} = \text{Chaos}$   
**by** (*rdes-simp*)

**lemma** *srd-subst-Miracle* [*usubst*]:  
 $\sigma \dagger_S \text{Miracle} = \text{Miracle}$   
**by** (*rdes-simp*)

**lemma** *srd-subst-skip* [*usubst*]:  
 $\sigma \dagger_S \Pi_R = \langle \sigma \rangle_R$   
**by** (*rdes-eq*)

**lemma** *srd-subst-assigns* [*usubst*]:  
 $\sigma \dagger_S \langle \varrho \rangle_R = \langle \varrho \circ_s \sigma \rangle_R$   
**by** (*rdes-eq*)

### 11.14 Algebraic Laws

**theorem** *assigns-srd-id*:  $\langle \text{id}_s \rangle_R = \Pi_R$   
**by** (*rdes-eq*)

**theorem** *assigns-srd-comp*:  $\langle \sigma \rangle_R ;; \langle \varrho \rangle_R = \langle \varrho \circ_s \sigma \rangle_R$   
**by** (*rdes-eq*)

**theorem** *assigns-srd-Miracle*:  $\langle \sigma \rangle_R ;; \text{Miracle} = \text{Miracle}$   
**by** (*rdes-eq*)

**theorem** *assigns-srd-Chaos*:  $\langle \sigma \rangle_R ;; \text{Chaos} = \text{Chaos}$   
**by** (*rdes-eq*)

**theorem** *assigns-srd-cond* :  $\langle \sigma \rangle_R \triangleleft b \triangleright_R \langle \varrho \rangle_R = \langle \sigma \triangleleft b \triangleright \varrho \rangle_R$   
**by** (*rdes-eq*)

**theorem** *assigns-srd-left-seq*:  
**assumes** *P is NSRD*  
**shows**  $\langle \sigma \rangle_R ;; P = \sigma \dagger_S P$   
**by** (*rdes-simp cls: assms*)

**lemma** *AlternateR-seq-distr*:  
**assumes**  $\bigwedge i. A\ i \text{ is NSRD } B \text{ is NSRD } C \text{ is NSRD}$   
**shows**  $(\text{if}_R\ i \in I \cdot g\ i \rightarrow A\ i \text{ else } B\ fi) ;; C = (\text{if}_R\ i \in I \cdot g\ i \rightarrow A\ i ;; C \text{ else } B ;; C\ fi)$   
**proof** (*cases I = {}*)  
**case** *True*  
**then show** *?thesis* **by** (*simp*)  
**next**  
**case** *False*  
**then show** *?thesis*  
**by** (*simp add: AlternateR-def upred-semiring.distrib-right seq-UINF-distr gcmd-seq-distr assms(3)*)  
**qed**

**lemma** *AlternateR-is-cond-srea*:

**assumes**  $A$  is NSRD  $B$  is NSRD  
**shows**  $(\text{if}_R i \in \{a\} \cdot g \rightarrow A \text{ else } B \text{ fi}) = (A \triangleleft g \triangleright_R B)$   
**by** (*rdes-eq cls: assms*)

**lemma** *AlternateR-Chaos*:  
 $\text{if}_R i \in A \cdot g(i) \rightarrow \text{Chaos fi} = \text{Chaos}$   
**by** (*cases*  $A = \{\}$ , *simp*, *rdes-eq*)

**lemma** *choose-srd-par*:  
 $\text{choose}_R \parallel_R \text{choose}_R = \text{choose}_R$   
**by** (*rdes-eq*)

## 11.15 Lifting designs to reactive designs

**definition** *des-rea-lift* ::  $'s \text{ hrel-des} \Rightarrow ('s, 't :: \text{trace}, 'a) \text{ hrel-rsp } (\mathbf{R}_D)$  **where**  
 $[\text{upred-defs}]: \mathbf{R}_D(P) = \mathbf{R}_s(\lceil \text{pre}_D(P) \rceil_S \vdash (\text{false} \diamond (\$tr' =_u \$tr \wedge \lceil \text{post}_D(P) \rceil_S)))$

**definition** *des-rea-drop* ::  $('s, 't :: \text{trace}, 'a) \text{ hrel-rsp} \Rightarrow 's \text{ hrel-des } (\mathbf{D}_R)$  **where**  
 $[\text{upred-defs}]: \mathbf{D}_R(P) = \lfloor (\text{pre}_R(P)) \llbracket \$tr / \$tr' \rrbracket \vdash_v \$st \rfloor_{S<} \vdash_n \lfloor (\text{post}_R(P)) \llbracket \$tr / \$tr' \rrbracket \vdash_v \{\$st, \$st'\} \rfloor_S$

**lemma** *ndesign-rea-lift-inverse*:  $\mathbf{D}_R(\mathbf{R}_D(p \vdash_n Q)) = p \vdash_n Q$   
**apply** (*simp add: des-rea-lift-def des-rea-drop-def rea-pre-RHS-design rea-post-RHS-design*)  
**apply** (*simp add: R1-def R2c-def R2s-def usubst unrest*)  
**apply** (*rel-auto*)  
**done**

**lemma** *ndesign-rea-lift-injective*:  
**assumes**  $P$  is  $\mathbf{N}$   $Q$  is  $\mathbf{N}$   $\mathbf{R}_D P = \mathbf{R}_D Q$  (**is**  $?RP(P) = ?RQ(Q)$ )  
**shows**  $P = Q$

**proof** –  
**have**  $?RP(\lfloor \text{pre}_D(P) \rfloor_{<} \vdash_n \text{post}_D(P)) = ?RQ(\lfloor \text{pre}_D(Q) \rfloor_{<} \vdash_n \text{post}_D(Q))$   
**by** (*simp add: ndesign-form assms*)  
**hence**  $\lfloor \text{pre}_D(P) \rfloor_{<} \vdash_n \text{post}_D(P) = \lfloor \text{pre}_D(Q) \rfloor_{<} \vdash_n \text{post}_D(Q)$   
**by** (*metis ndesign-rea-lift-inverse*)  
**thus** *?thesis*  
**by** (*simp add: ndesign-form assms*)  
**qed**

**lemma** *des-rea-lift-closure* [*closure*]:  $\mathbf{R}_D(P)$  is SRD  
**by** (*simp add: des-rea-lift-def RHS-design-is-SRD unrest*)

**lemma** *preR-des-rea-lift* [*rdes*]:  
 $\text{pre}_R(\mathbf{R}_D(P)) = R1(\lceil \text{pre}_D(P) \rceil_S)$   
**by** (*rel-auto*)

**lemma** *periR-des-rea-lift* [*rdes*]:  
 $\text{peri}_R(\mathbf{R}_D(P)) = (\text{false} \triangleleft \lceil \text{pre}_D(P) \rceil_S \triangleright (\$tr \leq_u \$tr'))$   
**by** (*rel-auto*)

**lemma** *postR-des-rea-lift* [*rdes*]:  
 $\text{post}_R(\mathbf{R}_D(P)) = ((\text{true} \triangleleft \lceil \text{pre}_D(P) \rceil_S \triangleright (\neg \$tr \leq_u \$tr')) \Rightarrow (\$tr' =_u \$tr \wedge \lceil \text{post}_D(P) \rceil_S))$   
**apply** (*rel-auto*) **using** *minus-zero-eq* **by** *blast*

**lemma** *ndes-rea-lift-closure* [*closure*]:  
**assumes**  $P$  is  $\mathbf{N}$



shows  $\mathbf{R}_D(P)$  is NSRD  
**proof** –  
 obtain  $p \ Q$  where  $P: P = (p \vdash_n Q)$   
 by (*metis H1-H3-commute H1-H3-is-normal-design H1-idem Healthy-def assms*)  
 show ?thesis  
 apply (*rule NSRD-intro*)  
 apply (*simp-all add: closure rdes unrest P*)  
 apply (*rel-auto*)  
 done  
**qed**

**lemma** *R-D-mono*:

assumes  $P$  is **H**  $Q$  is **H**  $P \sqsubseteq Q$   
 shows  $\mathbf{R}_D(P) \sqsubseteq \mathbf{R}_D(Q)$   
 apply (*simp add: des-rea-lift-def*)  
 apply (*rule srdes-tri-refine-intro'*)  
 apply (*meson aext-mono assms(3) design-refine-thms(1) refBy-order*)  
 apply (*rel-auto*)  
 apply (*smt aext-and aext-mono assms(1) assms(2) assms(3) rdesign-ref-monos(2) utp-pred-laws.inf.cobounded2 utp-pred-laws.inf.coboundedI2 utp-pred-laws.inf-left-commute utp-pred-laws.le-inf-iff*)  
 done

Homomorphism laws

**lemma** *R-D-Miracle*:

$\mathbf{R}_D(\top_D) = \text{Miracle}$   
 by (*simp add: Miracle-def, rel-auto*)

**lemma** *R-D-Chaos*:

$\mathbf{R}_D(\perp_D) = \text{Chaos}$

**proof** –

have  $\mathbf{R}_D(\perp_D) = \mathbf{R}_D(\text{false} \vdash_r \text{true})$   
 by (*rel-auto*)  
 also have  $\dots = \mathbf{R}_s(\text{false} \vdash \text{false} \diamond (\$tr' =_u \$tr))$   
 by (*simp add: Chaos-def des-rea-lift-def alpha*)  
 also have  $\dots = \mathbf{R}_s(\text{true})$   
 by (*rel-auto*)  
 also have  $\dots = \text{Chaos}$   
 by (*simp add: Chaos-def design-false-pre*)  
 finally show ?thesis .

**qed**

**lemma** *R-D-inf*:

$\mathbf{R}_D(P \sqcap Q) = \mathbf{R}_D(P) \sqcap \mathbf{R}_D(Q)$   
 by (*rule antisym, rel-auto+*)

**lemma** *R-D-cond*:

$\mathbf{R}_D(P \triangleleft [b]_{D<} \triangleright Q) = \mathbf{R}_D(P) \triangleleft b \triangleright_R \mathbf{R}_D(Q)$   
 by (*rule antisym, rel-auto+*)

**lemma** *R-D-seq-ndesign*:

$\mathbf{R}_D(p_1 \vdash_n Q_1) ;; \mathbf{R}_D(p_2 \vdash_n Q_2) = \mathbf{R}_D((p_1 \vdash_n Q_1) ;; (p_2 \vdash_n Q_2))$   
 apply (*rule antisym*)  
 apply (*rule SRD-refine-intro*)  
 apply (*simp-all add: closure rdes ndesign-composition-wp*)  
 using *dual-order.trans* apply (*rel-blast*)

```

using dual-order.trans apply (rel-blast)
apply (rel-auto)
apply (rule SRD-refine-intro)
  apply (simp-all add: closure rdes ndesign-composition-wp)
  apply (rel-auto)
  apply (rel-auto)
apply (rel-auto)
done

```

```

lemma R-D-seq:
  assumes  $P$  is  $\mathbf{N}$   $Q$  is  $\mathbf{N}$ 
  shows  $\mathbf{R}_D(P) ;; \mathbf{R}_D(Q) = \mathbf{R}_D(P ;; Q)$ 
  by (metis R-D-seq-ndesign assms ndesign-form)

```

These laws are applicable only when there is no further alphabet extension

```

lemma R-D-skip:
   $\mathbf{R}_D(\Pi_D) = (\Pi_R :: ('s, 't::trace, unit) hrel-rsp)$ 
  apply (rel-auto) using minus-zero-eq by blast+

```

```

lemma R-D-assigns:
   $\mathbf{R}_D(\langle \sigma \rangle_D) = (\langle \sigma \rangle_R :: ('s, 't::trace, unit) hrel-rsp)$ 
  by (simp add: assigns-d-def des-rea-lift-def alpha assigns-srd-RHS-tri-des, rel-auto)

```

## 11.16 State Invariants

**definition** *StateInvR* ::  $'s \text{ upred} \Rightarrow ('s, 't::trace, 'a) hrel-rsp \text{ } (sinv_R'(-))$  **where**  
 $[rdes-def]: sinv_R(b) = \mathbf{R}_s([b]_{S<} \vdash true_r \diamond [b]_{S>})$

```

lemma StateInvR-NSRD [closure]: sinv_R(b) is NSRD
  by (simp add: StateInvR-def closure unrest)

```

```

lemma StateInvR-srd-skip-refine:  $sinv_R(b) \sqsubseteq \Pi_R$ 
  by (rdes-refine)

```

```

lemma StateInvR-seq-idem:
   $sinv_R(b) ;; sinv_R(b) = sinv_R(b)$ 
  by (rdes-eq)

```

```

lemma StateInvR-seq-refine:
  assumes  $sinv_R(b) \sqsubseteq P$   $sinv_R(b) \sqsubseteq Q$ 
  shows  $sinv_R(b) \sqsubseteq P ;; Q$ 
  by (metis (full-types) StateInvR-seq-idem assms seqr-mono)

```

```

lemma ndiv-StateInvR:  $ndiv_R = sinv_R(true)$ 
  by (rdes-eq)

```

**end**

## 12 Instantaneous Reactive Designs

```

theory utp-rdes-instant
  imports utp-rdes-prog
begin

```

**definition** *ISR1* ::  $('s, 't::trace, 'a) hrel-rsp \Rightarrow ('s, 't, 'a) hrel-rsp$  **where**

[upred-defs]:  $ISR D1(P) = P \parallel_R \mathbf{R}_s(\text{true}_r \vdash \text{false} \diamond (\$tr' =_u \$tr))$

**definition**  $ISR D :: ('s, 't :: \text{trace}, 'a) \text{ hrel-rsp} \Rightarrow ('s, 't, 'a) \text{ hrel-rsp}$  **where**  
 [upred-defs]:  $ISR D = ISR D1 \circ NSRD$

**lemma**  $ISR D1\text{-idem}$ :  $ISR D1(ISR D1(P)) = ISR D1(P)$   
**by** (rel-auto)

**lemma**  $ISR D1\text{-monotonic}$ :  $P \sqsubseteq Q \Longrightarrow ISR D1(P) \sqsubseteq ISR D1(Q)$   
**by** (rel-auto)

**lemma**  $ISR D1\text{-RHS-design-form}$ :  
**assumes**  $\$ok' \# P \ \$ok' \# Q \ \$ok' \# R$   
**shows**  $ISR D1(\mathbf{R}_s(P \vdash Q \diamond R)) = \mathbf{R}_s(P \vdash \text{false} \diamond (R \wedge \$tr' =_u \$tr))$   
**using** *assms* **by** (simp add:  $ISR D1\text{-def}$  choose-srd-def RHS-tri-design-par unrest, rel-auto)

**lemma**  $ISR D1\text{-form}$ :  
 $ISR D1(SRD(P)) = \mathbf{R}_s(\text{pre}_R(P) \vdash \text{false} \diamond (\text{post}_R(P) \wedge \$tr' =_u \$tr))$   
**by** (simp add:  $ISR D1\text{-RHS-design-form}$  SRD-as-reactive-tri-design unrest)

**lemma**  $ISR D1\text{-rdes-def}$  [rdes-def]:  
 $\llbracket P \text{ is } RR; R \text{ is } RR \rrbracket \Longrightarrow ISR D1(\mathbf{R}_s(P \vdash Q \diamond R)) = \mathbf{R}_s(P \vdash \text{false} \diamond (R \wedge \$tr' =_u \$tr))$   
**by** (simp add:  $ISR D1\text{-def}$  rdes-def closure rpred)

**lemma**  $ISR D\text{-intro}$ :  
**assumes**  $P \text{ is } NSRD$   $\text{peri}_R(P) = (\neg_r \text{pre}_R(P)) (\$tr' =_u \$tr) \sqsubseteq \text{post}_R(P)$   
**shows**  $P \text{ is } ISR D$

**proof** –  
**have**  $\mathbf{R}_s(\text{pre}_R(P) \vdash \text{peri}_R(P) \diamond \text{post}_R(P)) \text{ is } ISR D1$   
**apply** (simp add: Healthy-def rdes-def closure assms(1–2))  
**using** *assms*(3) least-zero **apply** (rel-blast)  
**done**  
**hence**  $P \text{ is } ISR D1$   
**by** (simp add: SRD-reactive-tri-design closure assms(1))  
**thus** ?thesis  
**by** (simp add:  $ISR D\text{-def}$  Healthy-comp assms(1))  
**qed**

**lemma**  $ISR D1\text{-rdes-intro}$ :  
**assumes**  $P \text{ is } RR$   $Q \text{ is } RR$   $(\$tr' =_u \$tr) \sqsubseteq Q$   
**shows**  $\mathbf{R}_s(P \vdash \text{false} \diamond Q) \text{ is } ISR D1$   
**unfolding** Healthy-def  
**by** (simp add:  $ISR D1\text{-rdes-def}$  assms closure unrest utp-pred-laws.inf.absorb1)

**lemma**  $ISR D\text{-rdes-intro}$  [closure]:  
**assumes**  $P \text{ is } RC$   $Q \text{ is } RR$   $(\$tr' =_u \$tr) \sqsubseteq Q$   
**shows**  $\mathbf{R}_s(P \vdash \text{false} \diamond Q) \text{ is } ISR D$   
**unfolding** Healthy-def  
**by** (simp add:  $ISR D\text{-def}$  closure Healthy-if  $ISR D1\text{-rdes-def}$  assms unrest utp-pred-laws.inf.absorb1)

**lemma**  $ISR D\text{-implies-ISR D1}$ :  
**assumes**  $P \text{ is } ISR D$   
**shows**  $P \text{ is } ISR D1$   
**proof** –  
**have**  $ISR D(P) \text{ is } ISR D1$

by (simp add: ISRD-def Healthy-def ISRD1-idem)  
 thus ?thesis  
 by (simp add: assms Healthy-if)  
 qed

**lemma** *ISRD-implies-SRD*:

assumes *P* is ISRD

shows *P* is SRD

**proof** –

have 1:  $ISRD(P) = \mathbf{R}_s((\neg_r (\neg_r pre_R P) ;; R1\ true \wedge R1\ true) \vdash false \diamond (post_R P \wedge \$tr' =_u \$tr))$

by (simp add: NSRD-form ISRD1-def ISRD-def RHS-tri-design-par rdes-def unrest closure)

moreover have ... is SRD

by (simp add: closure unrest)

ultimately have  $ISRD(P)$  is SRD

by (simp)

with assms show ?thesis

by (simp add: Healthy-def)

qed

**lemma** *ISRD-implies-NSRD* [closure]:

assumes *P* is ISRD

shows *P* is NSRD

**proof** –

have 1:  $ISRD(P) = ISRD1(RD3(SRD(P)))$

by (simp add: ISRD-def NSRD-def SRD-def, metis RD1-RD3-commute RD3-left-subsumes-RD2)

also have ... =  $ISRD1(RD3(P))$

by (simp add: assms ISRD-implies-SRD Healthy-if)

also have ... =  $ISRD1(\mathbf{R}_s((\neg_r pre_R P) wp_r false_h \vdash (\exists \$st' \cdot peri_R P) \diamond post_R P))$

by (simp add: RD3-def, subst SRD-right-unit-tri-lemma, simp-all add: assms ISRD-implies-SRD)

also have ... =  $\mathbf{R}_s((\neg_r pre_R P) wp_r false_h \vdash false \diamond (post_R P \wedge \$tr' =_u \$tr))$

by (simp add: RHS-tri-design-par ISRD1-def unrest choose-srd-def rpred closure ISRD-implies-SRD

assms)

also have ... = (... ;;  $II_R$ )

by (rdes-simp, simp add: RHS-tri-normal-design-composition' closure assms unrest ISRD-implies-SRD  
wp rpred wp-rea-false-RC)

also have ... is RD3

by (simp add: Healthy-def RD3-def seqr-assoc)

finally show ?thesis

by (simp add: SRD-RD3-implies-NSRD Healthy-if assms ISRD-implies-SRD)

qed

**lemma** *ISRD-form*:

assumes *P* is ISRD

shows  $\mathbf{R}_s(pre_R(P) \vdash false \diamond (post_R(P) \wedge \$tr' =_u \$tr)) = P$

**proof** –

have  $P = ISRD1(P)$

by (simp add: ISRD-implies-ISRD1 assms Healthy-if)

also have ... =  $ISRD1(\mathbf{R}_s(pre_R(P) \vdash peri_R(P) \diamond post_R(P)))$

by (simp add: SRD-reactive-tri-design ISRD-implies-SRD assms)

also have ... =  $\mathbf{R}_s(pre_R(P) \vdash false \diamond (post_R(P) \wedge \$tr' =_u \$tr))$

by (simp add: ISRD1-rdes-def closure assms)

finally show ?thesis ..

qed

**lemma** *ISRD-elim* [RD-elim]:

$\llbracket P \text{ is ISRD}; Q(\mathbf{R}_s (pre_R(P) \vdash false \diamond (post_R(P) \wedge \$tr' =_u \$tr))) \rrbracket \implies Q(P)$   
 by (simp add: ISRD-form)

**lemma** skip-srd-ISRD [closure]:  $II_R$  is ISRD  
 by (rule ISRD-intro, simp-all add: rdes closure)

**lemma** assigns-srd-ISRD [closure]:  $\langle \sigma \rangle_R$  is ISRD  
 by (rule ISRD-intro, simp-all add: rdes closure, rel-auto)

**lemma** seq-ISRD-closed:  
 assumes  $P$  is ISRD  $Q$  is ISRD  
 shows  $P ;; Q$  is ISRD  
 apply (insert assms)  
 apply (erule ISRD-elim)+  
 apply (simp add: rdes-def closure assms unrest)  
 apply (rule ISRD-rdes-intro)  
 apply (simp-all add: rdes-def closure assms unrest)  
 apply (rel-auto)  
 done

**lemma** ISRD-Miracle-right-zero:  
 assumes  $P$  is ISRD  $pre_R(P) = true_r$   
 shows  $P ;; Miracle = Miracle$   
 by (rdes-simp cls: assms)

A recursion whose body does not extend the trace results in divergence

**lemma** ISRD-recurse-Chaos:  
 assumes  $P$  is ISRD  $post_R P ;; true_r = true_r$   
 shows  $(\mu_R X \cdot P ;; X) = Chaos$   
**proof** –  
 have 1:  $(\mu_R X \cdot P ;; X) = (\mu X \cdot P ;; SRD(X))$   
 by (auto simp add: sdes-theory.utp-lfp-def closure assms)  
 have  $(\mu X \cdot P ;; SRD(X)) \sqsubseteq Chaos$   
**proof** (rule gfp-upperbound)  
 have  $P ;; Chaos \sqsubseteq Chaos$   
 apply (rdes-refine-split cls: assms)  
 using assms(2) apply (rel-auto, metis (no-types, lifting) dual-order.antisym order-refl)  
 apply (rel-auto)+  
 done  
 thus  $P ;; SRD Chaos \sqsubseteq Chaos$   
 by (simp add: Healthy-if sdes-theory.bottom-closed)  
**qed**  
 thus ?thesis  
 by (metis 1 dual-order.antisym sdes-theory.LFP-closed sdes-theory.bottom-lower)  
**qed**

**lemma** recursive-assign-Chaos:  
 $(\mu_R X \cdot \langle \sigma \rangle_R ;; X) = Chaos$   
 by (rule ISRD-recurse-Chaos, simp-all add: closure rdes, rel-auto)

end

## 13 Meta-theory for Reactive Designs

theory utp-rea-designs

```
imports  
  utp-rdes-healths  
  utp-rdes-designs  
  utp-rdes-triples  
  utp-rdes-normal  
  utp-rdes-contracts  
  utp-rdes-tactics  
  utp-rdes-parallel  
  utp-rdes-prog  
  utp-rdes-instant  
  utp-rdes-guarded  
begin end
```

## References

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