Reactive Designs in Isabelle/UTP

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Abstract

Reactive designs combine the UTP theories of reactive processes and designs to characterise reactive programs. Whereas sequential imperative programs are expected to run until termination, reactive programs pause at instances to allow interaction with the environment using abstract events, and often do not terminate at all. Thus, whereas a design describes the precondition and postcondition for a program, to characterise initial and final states, a reactive design also has a "pericondition", which characterises intermediate quiescent observations. This gives rise to a notion of "reactive contract", which specifies the assumptions a program makes of its environment, and the guarantees it will make of its own behaviour in both intermediate and final observations. This Isabelle/UTP document mechanises the UTP theory of reactive designs, including its healthiness conditions, signature, and a large library of algebraic laws of reactive programming.

Contents

1 Introduction

This document contains a mechanisation in Isabelle/UTP [2] of our theory of reactive designs. Reactive designs form an important semantic foundation for reactive modelling languages such as Circus [3]. For more details of this work, please see our recent paper [1].

2 Reactive Designs Healthiness Conditions

theory *utp-rdes-healths*
imports *UTP−Reactive.utp-reactive*
begin

2.1 Preliminaries

named-theorems *rdes* **and** *rdes-def* **and** *RD-elim*

type-synonym $('s,'t)$ *rdes* = $('s,'t,unit)$ *hrel-rsp*

translations
$(type)$ $('s,'t)$ *rdes* <= $(type)$ $('s, 't, unit)$ *hrel-rsp*

declare *des-vars.splits* [*alpha-splits del*]
declare *rp-vars.splits* [*alpha-splits del*]
declare *rp-vars.splits* [*alpha-splits*]
declare *des-vars.splits* [*alpha-splits*]

lemma *R2-st-ex*: *R2* $(\exists\ \$st \cdot P) = (\exists\ \$st \cdot R2(P))$
by (*rel-auto*)

lemma *R2s-st'-eq-st*:
$R2s(\$st' =_u \$st) = (\$st' =_u \$st)$
by (*rel-auto*)

lemma *R2c-st'-eq-st*:
$R2c(\$st' =_u \$st) = (\$st' =_u \$st)$
by (*rel-auto*)

lemma *R1-des-lift-skip*: $R1(\lceil II \rceil_D) = \lceil II \rceil_D$

by (*rel-auto*)

lemma *R2-des-lift-skip*:
$R2(\lceil II \rceil_D) = \lceil II \rceil_D$
apply (*rel-auto*) **using** *minus-zero-eq* **by** *blast*

lemma *R1-R2c-ex-st*: $R1\ (R2c\ (\exists\ \$st' \cdot Q_1)) = (\exists\ \$st' \cdot R1\ (R2c\ Q_1))$
by (*rel-auto*)

2.2 Identities

We define two identities fro reactive designs, which correspond to the regular and state-sensitive versions of reactive designs, respectively. The former is the one used in the UTP book and related publications for CSP.

definition *skip-rea* :: ('*t*::*trace*, 'α) *hrel-rp* (II_C) **where**
skip-rea-def [*urel-defs*]: $II_C = (II \vee (\neg\ \$ok \wedge \$tr \leq_u \$tr'))$

definition *skip-srea* :: ('*s*, '*t*::*trace*, 'α) *hrel-rsp* (II_R) **where**
skip-srea-def [*urel-defs*]: $II_R = ((\exists\ \$st \cdot II_C) \lhd \$wait \rhd II_C)$

lemma *skip-rea-R1-lemma*: $II_C = R1(\$ok \Rightarrow II)$
by (*rel-auto*)

lemma *skip-rea-form*: $II_C = (II \lhd \$ok \rhd R1(true))$
by (*rel-auto*)

lemma *skip-srea-form*: $II_R = ((\exists\ \$st \cdot II) \lhd \$wait \rhd II) \lhd \$ok \rhd R1(true)$
by (*rel-auto*)

lemma *R1-skip-rea*: $R1(II_C) = II_C$
by (*rel-auto*)

lemma *R2c-skip-rea*: $R2c\ II_C = II_C$
by (*simp add*: *skip-rea-def R2c-and R2c-disj R2c-skip-r R2c-not R2c-ok R2c-tr-le-tr'*)

lemma *R2-skip-rea*: $R2(II_C) = II_C$
by (*metis R1-R2c-is-R2 R1-skip-rea R2c-skip-rea*)

lemma *R2c-skip-srea*: $R2c(II_R) = II_R$
apply (*rel-auto*) **using** *minus-zero-eq* **by** *blast+*

lemma *skip-srea-R1* [*closure*]: II_R *is R1*
by (*rel-auto*)

lemma *skip-srea-R2c* [*closure*]: II_R *is R2c*
by (*simp add*: *Healthy-def R2c-skip-srea*)

lemma *skip-srea-R2* [*closure*]: II_R *is R2*
by (*metis Healthy-def' R1-R2c-is-R2 R2c-skip-srea skip-srea-R1*)

2.3 RD1: Divergence yields arbitrary traces

definition *RD1* :: ('*t*::*trace*,'α,'β) *rel-rp* $\Rightarrow$ ('*t*,'α,'β) *rel-rp* **where**
[*upred-defs*]: $RD1(P) = (P \vee (\neg\ \$ok \wedge \$tr \leq_u \$tr'))$

$RD1$ is essentially $H1$ from the theory of designs, but viewed through the prism of reactive processes.

lemma *RD1-idem*: $RD1(RD1(P)) = RD1(P)$
by (*rel-auto*)

lemma *RD1-Idempotent*: *Idempotent RD1*
by (*simp add*: *Idempotent-def RD1-idem*)

lemma *RD1-mono*: $P \sqsubseteq Q \Longrightarrow RD1(P) \sqsubseteq RD1(Q)$
by (*rel-auto*)

lemma *RD1-Monotonic*: *Monotonic RD1*
using *mono-def RD1-mono* **by** *blast*

lemma *RD1-Continuous*: *Continuous RD1*
by (*rel-auto*)

lemma *R1-true-RD1-closed* [*closure*]: *R1*(*true*) *is RD1*
by (*rel-auto*)

lemma *RD1-wait-false* [*closure*]: *P is RD1* $\Longrightarrow$ *P*⟦*false*/$*wait*⟧ *is RD1*
by (*rel-auto*)

lemma *RD1-wait'-false* [*closure*]: *P is RD1* $\Longrightarrow$ *P*⟦*false*/$*wait´*⟧ *is RD1*
by (*rel-auto*)

lemma *RD1-seq*: $RD1(RD1(P) ;; RD1(Q)) = RD1(P) ;; RD1(Q)$
by (*rel-auto*)

lemma *RD1-seq-closure* [*closure*]: ⟦ *P is RD1*; *Q is RD1* ⟧ $\Longrightarrow$ *P* ;; *Q is RD1*
by (*metis Healthy-def' RD1-seq*)

lemma *RD1-R1-commute*: $RD1(R1(P)) = R1(RD1(P))$
by (*rel-auto*)

lemma *RD1-R2c-commute*: $RD1(R2c(P)) = R2c(RD1(P))$
by (*rel-auto*)

lemma *RD1-via-R1*: $R1(H1(P)) = RD1(R1(P))$
by (*rel-auto*)

lemma *RD1-R1-cases*: $RD1(R1(P)) = (R1(P) \lhd \$ok \rhd R1(true))$
by (*rel-auto*)

lemma *skip-rea-RD1-skip*: $II_C = RD1(II)$
by (*rel-auto*)

lemma *skip-srea-RD1* [*closure*]: II_R *is RD1*
by (*rel-auto*)

lemma *RD1-algebraic-intro*:
assumes
P *is R1* $(R1(true_h) ;; P) = R1(true_h)$ $(II_C ;; P) = P$
shows *P is RD1*
proof −

have $P = (II_C \mathbin{;;} P)$
by (*simp add*: *assms*(*3*))
also have ... $= (R1(\$ok \Rightarrow II) \mathbin{;;} P)$
by (*simp add*: *skip-rea-R1-lemma*)
also have ... $= (((\neg\ \$ok \wedge R1(true)) \mathbin{;;} P) \vee P)$
by (*metis* (*no-types*, *lifting*) *R1-def seqr-left-unit seqr-or-distl skip-rea-R1-lemma skip-rea-def utp-pred-laws.inf-top-left utp-pred-laws.sup-commute*)
also have ... $= ((R1(\neg\ \$ok) \mathbin{;;} (R1(true_h) \mathbin{;;} P)) \vee P)$
using *dual-order.trans* **by** (*rel-blast*)
also have ... $= ((R1(\neg\ \$ok) \mathbin{;;} R1(true_h)) \vee P)$
by (*simp add*: *assms*(*2*))
also have ... $= (R1(\neg\ \$ok) \vee P)$
by (*rel-auto*)
also have ... $= RD1(P)$
by (*rel-auto*)
finally show *?thesis*
by (*simp add*: *Healthy-def*)
qed

theorem *RD1-left-zero*:
assumes *P is R1 P is RD1*
shows $(R1(true) \mathbin{;;} P) = R1(true)$
proof −
have $(R1(true) \mathbin{;;} R1(RD1(P))) = R1(true)$
by (*rel-auto*)
thus *?thesis*
by (*simp add*: *Healthy-if assms*(*1*) *assms*(*2*))
qed

theorem *RD1-left-unit*:
assumes *P is R1 P is RD1*
shows $(II_C \mathbin{;;} P) = P$
proof −
have $(II_C \mathbin{;;} R1(RD1(P))) = R1(RD1(P))$
by (*rel-auto*)
thus *?thesis*
by (*simp add*: *Healthy-if assms*(*1*) *assms*(*2*))
qed

lemma *RD1-alt-def*:
assumes *P is R1*
shows $RD1(P) = (P \lhd \$ok \rhd R1(true))$
proof −
have $RD1(R1(P)) = (R1(P) \lhd \$ok \rhd R1(true))$
by (*rel-auto*)
thus *?thesis*
by (*simp add*: *Healthy-if assms*)
qed

theorem *RD1-algebraic*:
assumes *P is R1*
shows *P is RD1* $\longleftrightarrow (R1(true_h) \mathbin{;;} P) = R1(true_h) \wedge (II_C \mathbin{;;} P) = P$
using *RD1-algebraic-intro RD1-left-unit RD1-left-zero assms* **by** *blast*

2.4 R3c and R3h: Reactive design versions of R3

definition *R3c* :: ('*t*::*trace*, 'α) *hrel-rp* ⇒ ('*t*, 'α) *hrel-rp* **where**
[*upred-defs*]: *R3c*(*P*) = (*II*$_C$ ◁ $*wait* ▷ *P*)

definition *R3h* :: ('*s*, '*t*::*trace*, 'α) *hrel-rsp* ⇒ ('*s*, '*t*, 'α) *hrel-rsp* **where**
R3h-def [*upred-defs*]: *R3h*(*P*) = ((∃ $*st* · *II*$_C$) ◁ $*wait* ▷ *P*)

lemma *R3c-idem*: *R3c*(*R3c*(*P*)) = *R3c*(*P*)
by (*rel-auto*)

lemma *R3c-Idempotent*: *Idempotent R3c*
by (*simp add*: *Idempotent-def R3c-idem*)

lemma *R3c-mono*: *P* ⊑ *Q* ⟹ *R3c*(*P*) ⊑ *R3c*(*Q*)
by (*rel-auto*)

lemma *R3c-Monotonic*: *Monotonic R3c*
by (*simp add*: *mono-def R3c-mono*)

lemma *R3c-Continuous*: *Continuous R3c*
by (*rel-auto*)

lemma *R3h-idem*: *R3h*(*R3h*(*P*)) = *R3h*(*P*)
by (*rel-auto*)

lemma *R3h-Idempotent*: *Idempotent R3h*
by (*simp add*: *Idempotent-def R3h-idem*)

lemma *R3h-mono*: *P* ⊑ *Q* ⟹ *R3h*(*P*) ⊑ *R3h*(*Q*)
by (*rel-auto*)

lemma *R3h-Monotonic*: *Monotonic R3h*
by (*simp add*: *mono-def R3h-mono*)

lemma *R3h-Continuous*: *Continuous R3h*
by (*rel-auto*)

lemma *R3h-inf*: *R3h*(*P* ⊓ *Q*) = *R3h*(*P*) ⊓ *R3h*(*Q*)
by (*rel-auto*)

lemma *R3h-UINF*:
A ≠ {} ⟹ *R3h*(⨅ *i* ∈ *A* · *P*(*i*)) = (⨅ *i* ∈ *A* · *R3h*(*P*(*i*)))
by (*rel-auto*)

lemma *R3h-cond*: *R3h*(*P* ◁ *b* ▷ *Q*) = (*R3h*(*P*) ◁ *b* ▷ *R3h*(*Q*))
by (*rel-auto*)

lemma *R3c-via-RD1-R3*: *RD1*(*R3*(*P*)) = *R3c*(*RD1*(*P*))
by (*rel-auto*)

lemma *R3c-RD1-def*: *P is RD1* ⟹ *R3c*(*P*) = *RD1*(*R3*(*P*))
by (*simp add*: *Healthy-if R3c-via-RD1-R3*)

lemma *RD1-R3c-commute*: *RD1*(*R3c*(*P*)) = *R3c*(*RD1*(*P*))
by (*rel-auto*)

lemma *R1-R3c-commute*: $R1(R3c(P)) = R3c(R1(P))$
by (*rel-auto*)

lemma *R2c-R3c-commute*: $R2c(R3c(P)) = R3c(R2c(P))$
apply (*rel-auto*) **using** *minus-zero-eq* **by** *blast+*

lemma *R1-R3h-commute*: $R1(R3h(P)) = R3h(R1(P))$
by (*rel-auto*)

lemma *R2c-R3h-commute*: $R2c(R3h(P)) = R3h(R2c(P))$
apply (*rel-auto*) **using** *minus-zero-eq* **by** *blast+*

lemma *RD1-R3h-commute*: $RD1(R3h(P)) = R3h(RD1(P))$
by (*rel-auto*)

lemma *R3c-cancels-R3*: $R3c(R3(P)) = R3c(P)$
by (*rel-auto*)

lemma *R3-cancels-R3c*: $R3(R3c(P)) = R3(P)$
by (*rel-auto*)

lemma *R3h-cancels-R3c*: $R3h(R3c(P)) = R3h(P)$
by (*rel-auto*)

lemma *R3c-semir-form*:
$(R3c(P) ;; R3c(R1(Q))) = R3c(P ;; R3c(R1(Q)))$
by (*rel-simp*, *safe*, *auto intro*: *order-trans*)

lemma *R3h-semir-form*:
$(R3h(P) ;; R3h(R1(Q))) = R3h(P ;; R3h(R1(Q)))$
by (*rel-simp*, *safe*, *auto intro*: *order-trans*, *blast+*)

lemma *R3c-seq-closure*:
assumes *P is R3c Q is R3c Q is R1*
shows $(P ;; Q)$ *is R3c*
by (*metis Healthy-def' R3c-semir-form assms*)

lemma *R3h-seq-closure* [*closure*]:
assumes *P is R3h Q is R3h Q is R1*
shows $(P ;; Q)$ *is R3h*
by (*metis Healthy-def' R3h-semir-form assms*)

lemma *R3c-R3-left-seq-closure*:
assumes *P is R3 Q is R3c*
shows $(P ;; Q)$ *is R3c*
proof −
have $(P ;; Q) = ((P ;; Q)[\![true/\$wait]\!] \triangleleft \$wait \triangleright (P ;; Q))$
by (*metis cond-var-split cond-var-subst-right in-var-uvar wait-vwb-lens*)
also have ... $= (((II \triangleleft \$wait \triangleright P) ;; Q)[\![true/\$wait]\!] \triangleleft \$wait \triangleright (P ;; Q))$
by (*metis Healthy-def' R3-def assms*(*1*))
also have ... $= ((II[\![true/\$wait]\!] ;; Q) \triangleleft \$wait \triangleright (P ;; Q))$
by (*subst-tac*)
also have ... $= (((II \wedge \$wait') ;; Q) \triangleleft \$wait \triangleright (P ;; Q))$
by (*metis* (*no-types*, *lifting*) *cond-def conj-pos-var-subst seqr-pre-var-out skip-var utp-pred-laws.inf-left-idem*

wait-vwb-lens)
also have ... = $((II[\![true/\$wait\acute{}]\!] ;; Q[\![true/\$wait]\!]) \lhd \$wait \rhd (P ;; Q))$
by (*metis seqr-pre-transfer seqr-right-one-point true-alt-def uovar-convr upred-eq-true utp-rel.unrest-ouvar vwb-lens-mwb wait-vwb-lens*)
also have ... = $((II[\![true/\$wait\acute{}]\!] ;; (II_C \lhd \$wait \rhd Q)[\![true/\$wait]\!]) \lhd \$wait \rhd (P ;; Q))$
by (*metis Healthy-def' R3c-def assms*(2))
also have ... = $((II[\![true/\$wait\acute{}]\!] ;; II_C[\![true/\$wait]\!]) \lhd \$wait \rhd (P ;; Q))$
by (*subst-tac*)
also have ... = $(((II \wedge \$wait\acute{}) ;; II_C) \lhd \$wait \rhd (P ;; Q))$
by (*metis seqr-pre-transfer seqr-right-one-point true-alt-def uovar-convr upred-eq-true utp-rel.unrest-ouvar vwb-lens-mwb wait-vwb-lens*)
also have ... = $((II ;; II_C) \lhd \$wait \rhd (P ;; Q))$
by (*simp add*: *cond-def seqr-pre-transfer utp-rel.unrest-ouvar*)
also have ... = $(II_C \lhd \$wait \rhd (P ;; Q))$
by *simp*
also have ... = $R3c(P ;; Q)$
by (*simp add*: *R3c-def*)
finally show *?thesis*
by (*simp add*: *Healthy-def'*)
qed

lemma *R3c-cases*: $R3c(P) = ((II \lhd \$ok \rhd R1(true)) \lhd \$wait \rhd P)$
by (*rel-auto*)

lemma *R3h-cases*: $R3h(P) = (((\exists\ \$st \cdot II) \lhd \$ok \rhd R1(true)) \lhd \$wait \rhd P)$
by (*rel-auto*)

lemma *R3h-form*: $R3h(P) = II_R \lhd \$wait \rhd P$
by (*rel-auto*)

lemma *R3c-subst-wait*: $R3c(P) = R3c(P\ _f)$
by (*simp add*: *R3c-def cond-var-subst-right*)

lemma *R3h-subst-wait*: $R3h(P) = R3h(P\ _f)$
by (*simp add*: *R3h-cases cond-var-subst-right*)

lemma *skip-srea-R3h* [*closure*]: II_R *is R3h*
by (*rel-auto*)

lemma *R3h-wait-true*:
assumes *P is R3h*
shows $P\ _t = II_R\ _t$
proof −
have $P\ _t = (II_R \lhd \$wait \rhd P)\ _t$
by (*metis Healthy-if R3h-form assms*)
also have ... = $II_R\ _t$
by (*simp add*: *usubst*)
finally show *?thesis* **.**
qed

lemma *R3c-refines-R3h*: $R3h(P) \sqsubseteq R3c(P)$
by (*rel-auto*)

2.5 RD2: A reactive specification cannot require non-termination

definition *RD2* **where**

[*upred-defs*]: $RD2(P) = H2(P)$

$RD2$ is just $H2$ since the type system will automatically have J identifying the reactive variables as required.

lemma *RD2-idem*: $RD2(RD2(P)) = RD2(P)$
by (*simp add*: *H2-idem RD2-def*)

lemma *RD2-Idempotent*: *Idempotent RD2*
by (*simp add*: *Idempotent-def RD2-idem*)

lemma *RD2-mono*: $P \sqsubseteq Q \Longrightarrow RD2(P) \sqsubseteq RD2(Q)$
by (*simp add*: *H2-def RD2-def seqr-mono*)

lemma *RD2-Monotonic*: *Monotonic RD2*
using *mono-def RD2-mono* **by** *blast*

lemma *RD2-Continuous*: *Continuous RD2*
by (*rel-auto*)

lemma *RD1-RD2-commute*: $RD1(RD2(P)) = RD2(RD1(P))$
by (*rel-auto*)

lemma *RD2-R3c-commute*: $RD2(R3c(P)) = R3c(RD2(P))$
by (*rel-auto*)

lemma *RD2-R3h-commute*: $RD2(R3h(P)) = R3h(RD2(P))$
by (*rel-auto*)

2.6 Major healthiness conditions

definition $RH :: ('t::trace,'\alpha)\ hrel\text{-}rp \Rightarrow ('t,'\alpha)\ hrel\text{-}rp\ (\mathbf{R})$
where [*upred-defs*]: $RH(P) = R1(R2c(R3c(P)))$

definition $RHS :: ('s,'t::trace,'\alpha)\ hrel\text{-}rsp \Rightarrow ('s,'t,'\alpha)\ hrel\text{-}rsp\ (\mathbf{R}_s)$
where [*upred-defs*]: $RHS(P) = R1(R2c(R3h(P)))$

definition $RD :: ('t::trace,'\alpha)\ hrel\text{-}rp \Rightarrow ('t,'\alpha)\ hrel\text{-}rp$
where [*upred-defs*]: $RD(P) = RD1(RD2(RP(P)))$

definition $SRD :: ('s,'t::trace,'\alpha)\ hrel\text{-}rsp \Rightarrow ('s,'t,'\alpha)\ hrel\text{-}rsp$
where [*upred-defs*]: $SRD(P) = RD1(RD2(RHS(P)))$

lemma *RH-comp*: $RH = R1 \circ R2c \circ R3c$
by (*auto simp add*: *RH-def*)

lemma *RHS-comp*: $RHS = R1 \circ R2c \circ R3h$
by (*auto simp add*: *RHS-def*)

lemma *RD-comp*: $RD = RD1 \circ RD2 \circ RP$
by (*auto simp add*: *RD-def*)

lemma *SRD-comp*: $SRD = RD1 \circ RD2 \circ RHS$
by (*auto simp add*: *SRD-def*)

lemma *RH-idem*: $\mathbf{R}(\mathbf{R}(P)) = \mathbf{R}(P)$

by (*simp add*: *R1-R2c-commute R1-R3c-commute R1-idem R2c-R3c-commute R2c-idem R3c-idem RH-def*)

lemma *RH-Idempotent*: *Idempotent* $\mathbf{R}$
by (*simp add*: *Idempotent-def RH-idem*)

lemma *RH-Monotonic*: *Monotonic* $\mathbf{R}$
by (*metis* (*no-types*, *lifting*) *R1-Monotonic R2c-Monotonic R3c-mono RH-def mono-def*)

lemma *RH-mono*: $P \sqsubseteq Q \Longrightarrow \mathbf{R}(P) \sqsubseteq \mathbf{R}(Q)$
using *mono-def RH-Monotonic* **by** *blast*

lemma *RH-Continuous*: *Continuous* $\mathbf{R}$
by (*simp add*: *Continuous-comp R1-Continuous R2c-Continuous R3c-Continuous RH-comp*)

lemma *RHS-idem*: $\mathbf{R}_s(\mathbf{R}_s(P)) = \mathbf{R}_s(P)$
by (*simp add*: *R1-R2c-is-R2 R1-R3h-commute R2-idem R2c-R3h-commute R3h-idem RHS-def*)

lemma *RHS-Idempotent* [*closure*]: *Idempotent* $\mathbf{R}_s$
by (*simp add*: *Idempotent-def RHS-idem*)

lemma *RHS-Monotonic*: *Monotonic* $\mathbf{R}_s$
by (*simp add*: *mono-def R1-R2c-is-R2 R2-mono R3h-mono RHS-def*)

lemma *RHS-mono*: $P \sqsubseteq Q \Longrightarrow \mathbf{R}_s(P) \sqsubseteq \mathbf{R}_s(Q)$
using *mono-def RHS-Monotonic* **by** *blast*

lemma *RHS-Continuous* [*closure*]: *Continuous* $\mathbf{R}_s$
by (*simp add*: *Continuous-comp R1-Continuous R2c-Continuous R3h-Continuous RHS-comp*)

lemma *RHS-inf*: $\mathbf{R}_s(P \sqcap Q) = \mathbf{R}_s(P) \sqcap \mathbf{R}_s(Q)$
using *Continuous-Disjunctous Disjunctuous-def RHS-Continuous* **by** *auto*

lemma *RHS-INF*:
$A \neq \{\} \Longrightarrow \mathbf{R}_s(\bigsqcap\ i \in A \cdot P(i)) = (\bigsqcap\ i \in A \cdot \mathbf{R}_s(P(i)))$
by (*simp add*: *RHS-def R3h-UINF R2c-USUP R1-USUP*)

lemma *RHS-sup*: $\mathbf{R}_s(P \sqcup Q) = \mathbf{R}_s(P) \sqcup \mathbf{R}_s(Q)$
by (*rel-auto*)

lemma *RHS-SUP*:
$A \neq \{\} \Longrightarrow \mathbf{R}_s(\bigsqcup\ i \in A \cdot P(i)) = (\bigsqcup\ i \in A \cdot \mathbf{R}_s(P(i)))$
by (*rel-auto*)

lemma *RHS-cond*: $\mathbf{R}_s(P \lhd b \rhd Q) = (\mathbf{R}_s(P) \lhd R2c\ b \rhd \mathbf{R}_s(Q))$
by (*simp add*: *RHS-def R3h-cond R2c-condr R1-cond*)

lemma *RD-alt-def*: $RD(P) = RD1(RD2(\mathbf{R}(P)))$
by (*simp add*: *R3c-via-RD1-R3 RD1-R1-commute RD1-R2c-commute RD1-R3c-commute RD1-RD2-commute RH-def RD-def RP-def*)

lemma *RD1-RH-commute*: $RD1(\mathbf{R}(P)) = \mathbf{R}(RD1(P))$
by (*simp add*: *RD1-R1-commute RD1-R2c-commute RD1-R3c-commute RH-def*)

lemma *RD2-RH-commute*: $RD2(\mathbf{R}(P)) = \mathbf{R}(RD2(P))$

by (*metis R1-H2-commute R2c-H2-commute RD2-R3c-commute RD2-def RH-def*)

lemma *RD-idem*: $RD(RD(P)) = RD(P)$
by (*simp add*: *RD-alt-def RD1-RH-commute RD2-RH-commute RD1-RD2-commute RD2-idem RD1-idem RH-idem*)

lemma *RD-Monotonic*: *Monotonic RD*
by (*simp add*: *Monotonic-comp RD1-Monotonic RD2-Monotonic RD-comp RP-Monotonic*)

lemma *RD-Continuous*: *Continuous RD*
by (*simp add*: *Continuous-comp RD1-Continuous RD2-Continuous RD-comp RP-Continuous*)

lemma *R3-RD-RP*: $R3(RD(P)) = RP(RD1(RD2(P)))$
by (*metis* (*no-types*, *lifting*) *R1-R2c-is-R2 R2-R3-commute R3-cancels-R3c RD1-RH-commute RD2-RH-commute RD-alt-def RH-def RP-def*)

lemma *RD-healths* [*closure*]:
assumes *P is RD*
shows *P is R1 P is R2 P is R3c P is RD1 P is RD2*
apply (*metis Healthy-def R1-idem RD1-RH-commute RD2-RH-commute RH-def RD-alt-def assms*)
apply (*metis Healthy-def R1-R2c-is-R2 R2-idem RD1-RH-commute RD2-RH-commute RH-def RD-alt-def assms*)
apply (*metis Healthy-def R3-RD-RP R3c-via-RD1-R3 RD1-RD2-commute RD1-RH-commute RD2-R3c-commute RD2-RH-commute RD-alt-def RD-def assms*)
apply (*metis Healthy-Idempotent Healthy-def RD1-Idempotent RD-alt-def assms*)
apply (*metis H2-idem Healthy-def RD1-RD2-commute RD2-def RD-def assms*)
done

lemma *RD1-RHS-commute*: $RD1(\mathbf{R}_s(P)) = \mathbf{R}_s(RD1(P))$
by (*simp add*: *RD1-R1-commute RD1-R2c-commute RD1-R3h-commute RHS-def*)

lemma *RD2-RHS-commute*: $RD2(\mathbf{R}_s(P)) = \mathbf{R}_s(RD2(P))$
by (*metis R1-H2-commute R2c-H2-commute RD2-R3h-commute RD2-def RHS-def*)

lemma *SRD-idem*: $SRD(SRD(P)) = SRD(P)$
by (*simp add*: *RD1-RD2-commute RD1-RHS-commute RD1-idem RD2-RHS-commute RD2-idem RHS-idem SRD-def*)

lemma *SRD-Idempotent* [*closure*]: *Idempotent SRD*
by (*simp add*: *Idempotent-def SRD-idem*)

lemma *SRD-Monotonic*: *Monotonic SRD*
by (*simp add*: *Monotonic-comp RD1-Monotonic RD2-Monotonic RHS-Monotonic SRD-comp*)

lemma *SRD-Continuous* [*closure*]: *Continuous SRD*
by (*simp add*: *Continuous-comp RD1-Continuous RD2-Continuous RHS-Continuous SRD-comp*)

lemma *SRD-RHS-H1-H2*: $SRD(P) = \mathbf{R}_s(\mathbf{H}(P))$
by (*rel-auto*)

lemma *SRD-healths* [*closure*]:
assumes *P is SRD*
shows *P is R1 P is R2 P is R3h P is RD1 P is RD2*
apply (*metis Healthy-def R1-idem RD1-RHS-commute RD2-RHS-commute RHS-def SRD-def assms*)
apply (*metis Healthy-def R1-R2c-is-R2 R2-idem RD1-RHS-commute RD2-RHS-commute RHS-def*

SRD-def assms)
apply (*metis Healthy-def R1-R3h-commute R2c-R3h-commute R3h-idem RD1-R3h-commute RD2-R3h-commute RHS-def SRD-def assms*)
apply (*metis Healthy-def' RD1-idem SRD-def assms*)
apply (*metis Healthy-def' RD1-RD2-commute RD2-idem SRD-def assms*)
done

lemma *SRD-intro*:
assumes *P is R1 P is R2 P is R3h P is RD1 P is RD2*
shows *P is SRD*
by (*metis Healthy-def R1-R2c-is-R2 RHS-def SRD-def assms*(*2*) *assms*(*3*) *assms*(*4*) *assms*(*5*))

lemma *SRD-ok-false* [*usubst*]: *P is SRD* $\Longrightarrow$ $P[\![false/\$ok]\!] = R1(true)$
by (*metis* (*no-types*, *hide-lams*) *H1-H2-eq-design Healthy-def R1-ok-false RD1-R1-commute RD1-via-R1 RD2-def SRD-def SRD-healths*(*1*) *design-ok-false*)

lemma *SRD-ok-true-wait-true* [*usubst*]:
assumes *P is SRD*
shows $P[\![true,true/\$ok,\$wait]\!] = (\exists\ \$st \cdot II)[\![true,true/\$ok,\$wait]\!]$
proof −
have $P = (\exists\ \$st \cdot II) \lhd \$ok \rhd R1\ true \lhd \$wait \rhd P$
by (*metis Healthy-def R3h-cases SRD-healths*(*3*) *assms*)
moreover have $((\exists\ \$st \cdot II) \lhd \$ok \rhd R1\ true \lhd \$wait \rhd P)[\![true,true/\$ok,\$wait]\!] = (\exists\ \$st \cdot II)[\![true,true/\$ok,\$wait]\!]$
by (*simp add*: *usubst*)
ultimately show *?thesis*
by (*simp*)
qed

lemma *SRD-left-zero-1*: *P is SRD* $\Longrightarrow R1(true) ;; P = R1(true)$
by (*simp add*: *RD1-left-zero SRD-healths*(*1*) *SRD-healths*(*4*))

lemma *SRD-left-zero-2*:
assumes *P is SRD*
shows $(\exists\ \$st \cdot II)[\![true,true/\$ok,\$wait]\!] ;; P = (\exists\ \$st \cdot II)[\![true,true/\$ok,\$wait]\!]$
proof −
have $(\exists\ \$st \cdot II)[\![true,true/\$ok,\$wait]\!] ;; R3h(P) = (\exists\ \$st \cdot II)[\![true,true/\$ok,\$wait]\!]$
by (*rel-auto*)
thus *?thesis*
by (*simp add*: *Healthy-if SRD-healths*(*3*) *assms*)
qed

2.7 UTP theories

We create two theory objects: one for reactive designs and one for stateful reactive designs.

interpretation *rdes-theory*: *utp-theory-continuous RD*
rewrites $P \in carrier\ rdes\text{-}theory.thy\text{-}order \longleftrightarrow P\ is\ RD$
and $carrier\ rdes\text{-}theory.thy\text{-}order \to carrier\ rdes\text{-}theory.thy\text{-}order \equiv [\![RD]\!]_H \to [\![RD]\!]_H$
and $le\ rdes\text{-}theory.thy\text{-}order = (\sqsubseteq)$
and $eq\ rdes\text{-}theory.thy\text{-}order = (=)$
proof −
show *utp-theory-continuous RD*
by (*unfold-locales*, *simp-all add*: *RD-idem RD-Continuous*)
qed (*simp-all*)

interpretation *srdes-theory*: *utp-theory-continuous SRD*
  rewrites $P \in$ *carrier srdes-theory.thy-order* $\longleftrightarrow$ *P is SRD*
  and *carrier srdes-theory.thy-order* $\rightarrow$ *carrier srdes-theory.thy-order* $\equiv [\![SRD]\!]_H \rightarrow [\![SRD]\!]_H$
  and *le srdes-theory.thy-order* $= (\sqsubseteq)$
  and *eq srdes-theory.thy-order* $= (=)$
proof −
  show *utp-theory-continuous SRD*
    by (*unfold-locales*, *simp-all add*: *SRD-idem SRD-Continuous*)
qed (*simp-all*)

interpretation *rdes-rea-galois*:
  galois-connection $(RD \Leftarrow\langle RD1 \circ RD2, R3\rangle\Rightarrow RP)$
proof (*simp add*: *mk-conn-def*, *rule galois-connectionI'*, *simp-all add*: *utp-partial-order*)
  show $R3 \in [\![RD]\!]_H \rightarrow [\![RP]\!]_H$
    by (*metis* (*no-types*, *lifting*) *Healthy-def' Pi-I R3-RD-RP RP-idem mem-Collect-eq*)
  show $RD1 \circ RD2 \in [\![RP]\!]_H \rightarrow [\![RD]\!]_H$
    by (*simp add*: *Pi-iff Healthy-def*, *metis RD-def RD-idem*)
  show *isotone* (*utp-order RD*) (*utp-order RP*) *R3*
    by (*simp add*: *R3-Monotonic isotone-utp-orderI*)
  show *isotone* (*utp-order RP*) (*utp-order RD*) $(RD1 \circ RD2)$
    by (*simp add*: *Monotonic-comp RD1-Monotonic RD2-Monotonic isotone-utp-orderI*)
  fix $P :: ('a, 'b)$ *hrel-rp*
  assume *P is RD*
  thus $P \sqsubseteq RD1\ (RD2\ (R3\ P))$
    by (*metis Healthy-if R3-RD-RP RD-def RP-idem eq-iff*)
next
  fix $P :: ('a, 'b)$ *hrel-rp*
  assume *a*: *P is RP*
  thus $R3\ (RD1\ (RD2\ P)) \sqsubseteq P$
  proof −
    have $R3\ (RD1\ (RD2\ P)) = RP\ (RD1\ (RD2(P)))$
      by (*metis Healthy-if R3-RD-RP RD-def a*)
    moreover have $RD1(RD2(P)) \sqsubseteq P$
      by (*rel-auto*)
    ultimately show *?thesis*
      by (*metis Healthy-if RP-mono a*)
  qed
qed

interpretation *rdes-rea-retract*:
  retract $(RD \Leftarrow\langle RD1 \circ RD2, R3\rangle\Rightarrow RP)$
  by (*unfold-locales*, *simp-all add*: *mk-conn-def utp-partial-order*)
    (*metis Healthy-if R3-RD-RP RD-def RP-idem eq-refl*)

abbreviation *Chaos* :: $('s, 't::trace, 'α)$ *hrel-rsp* **where**
Chaos ≡ *srdes-theory.utp-bottom*

abbreviation *Miracle* :: $('s, 't::trace, 'α)$ *hrel-rsp* **where**
Miracle ≡ *srdes-theory.utp-top*

thm *srdes-theory.weak.bottom-lower*
thm *srdes-theory.weak.top-higher*
thm *srdes-theory.meet-bottom*
thm *srdes-theory.meet-top*

abbreviation *srd-lfp* (μ_R) **where** $\mu_R\ F \equiv$ *srdes-theory.utp-lfp F*

abbreviation *srd-gfp* (ν_R) **where** $\nu_R\ F \equiv$ *srdes-theory.utp-gfp F*

syntax
-srd-mu :: *pttrn* ⇒ *logic* ⇒ *logic* (μ_R - ⋅ - [0, 10] 10)
-srd-nu :: *pttrn* ⇒ *logic* ⇒ *logic* (ν_R - ⋅ - [0, 10] 10)

translations
$\mu_R\ X \cdot P == \mu_R\ (\lambda\ X.\ P)$
$\nu_R\ X \cdot P == \mu_R\ (\lambda\ X.\ P)$

The reactive design weakest fixed-point can be defined in terms of relational calculus one.

lemma *srd-mu-equiv*:
assumes *Monotonic F F* ∈ $[\![SRD]\!]_H \to [\![SRD]\!]_H$
shows $(\mu_R\ X \cdot F(X)) = (\mu\ X \cdot F(SRD(X)))$
by (*metis assms srdes-theory.utp-lfp-def*)

end

3 Reactive Design Specifications

theory *utp-rdes-designs*
imports *utp-rdes-healths*
begin

3.1 Reactive design forms

lemma *rdes-skip-def*: $II_C = \mathbf{R}(true \vdash (\$tr' =_u \$tr \wedge \neg\ \$wait' \wedge \lceil II \rceil_R))$
apply (*rel-auto*) **using** *minus-zero-eq* **by** *blast+*

lemma *srdes-skip-def*: $II_R = \mathbf{R}_s(true \vdash (\$tr' =_u \$tr \wedge \neg\ \$wait' \wedge \lceil II \rceil_R))$
apply (*rel-auto*) **using** *minus-zero-eq* **by** *blast+*

lemma *Chaos-def*: $Chaos = \mathbf{R}_s(false \vdash true)$
proof −
have $Chaos = SRD(true)$
by (*metis srdes-theory.healthy-bottom*)
also have ... = $\mathbf{R}_s(\mathbf{H}(true))$
by (*simp add*: *SRD-RHS-H1-H2*)
also have ... = $\mathbf{R}_s(false \vdash true)$
by (*metis H1-design H2-true design-false-pre*)
finally show *?thesis* **.**
qed

lemma *Miracle-def*: $Miracle = \mathbf{R}_s(true \vdash false)$
proof −
have $Miracle = SRD(false)$
by (*metis srdes-theory.healthy-top*)
also have ... = $\mathbf{R}_s(\mathbf{H}(false))$
by (*simp add*: *SRD-RHS-H1-H2*)
also have ... = $\mathbf{R}_s(true \vdash false)$
by (*metis* (*no-types*, *lifting*) *H1-H2-eq-design p-imp-p subst-impl subst-not utp-pred-laws.compl-bot-eq utp-pred-laws.compl-top-eq*)
finally show *?thesis* **.**

qed

lemma *RD1-reactive-design*: $RD1(\mathbf{R}(P \vdash Q)) = \mathbf{R}(P \vdash Q)$
by (*rel-auto*)

lemma *RD2-reactive-design*:
assumes $\$ok' \sharp P$ $\$ok' \sharp Q$
shows $RD2(\mathbf{R}(P \vdash Q)) = \mathbf{R}(P \vdash Q)$
using *assms*
by (*metis H2-design RD2-RH-commute RD2-def*)

lemma *RD1-st-reactive-design*: $RD1(\mathbf{R}_s(P \vdash Q)) = \mathbf{R}_s(P \vdash Q)$
by (*rel-auto*)

lemma *RD2-st-reactive-design*:
assumes $\$ok' \sharp P$ $\$ok' \sharp Q$
shows $RD2(\mathbf{R}_s(P \vdash Q)) = \mathbf{R}_s(P \vdash Q)$
using *assms*
by (*metis H2-design RD2-RHS-commute RD2-def*)

lemma *wait-false-design*:
$(P \vdash Q)_f = ((P_f) \vdash (Q_f))$
by (*rel-auto*)

lemma *RD-RH-design-form*:
$RD(P) = \mathbf{R}((\neg P^f{}_f) \vdash P^t{}_f)$
proof −
have $RD(P) = RD1(RD2(R1(R2c(R3c(P)))))$
by (*simp add*: *RD-alt-def RH-def*)
also have ... $= RD1(H2(R1(R2s(R3c(P)))))$
by (*simp add*: *R1-R2s-R2c RD2-def*)
also have ... $= RD1(R1(H2(R2s(R3c(P)))))$
by (*simp add*: *R1-H2-commute*)
also have ... $= R1(H1(R1(H2(R2s(R3c(P))))))$
by (*simp add*: *R1-idem RD1-via-R1*)
also have ... $= R1(H1(H2(R2s(R3c(R1(P))))))$
by (*simp add*: *R1-H2-commute R1-R2c-commute R1-R2s-R2c R1-R3c-commute RD1-via-R1*)
also have ... $= R1(R2s(H1(H2(R3c(R1(P))))))$
by (*simp add*: *R2s-H1-commute R2s-H2-commute*)
also have ... $= R1(R2s(H1(R3c(H2(R1(P))))))$
by (*metis RD2-R3c-commute RD2-def*)
also have ... $= R2(R1(H1(R3c(H2(R1(P))))))$
by (*metis R1-R2-commute R1-idem R2-def*)
also have ... $= R2(R3c(R1(\mathbf{H}(R1(P)))))$
by (*simp add*: *R1-R3c-commute RD1-R3c-commute RD1-via-R1*)
also have ... $= RH(\mathbf{H}(R1(P)))$
by (*metis R1-R2s-R2c R1-R3c-commute R2-R1-form RH-def*)
also have ... $= RH(\mathbf{H}(P))$
by (*simp add*: *R1-H2-commute R1-R2c-commute R1-R3c-commute R1-idem RD1-via-R1 RH-def*)
also have ... $= RH((\neg P^f) \vdash P^t)$
by (*simp add*: *H1-H2-eq-design*)
also have ... $= \mathbf{R}((\neg P^f{}_f) \vdash P^t{}_f)$
by (*metis* (*no-types*, *lifting*) *R3c-subst-wait RH-def subst-not wait-false-design*)
finally show *?thesis* **.**
qed

lemma *RD-reactive-design*:
assumes *P is RD*
shows $\mathbf{R}((\neg\ P^f{}_f) \vdash P^t{}_f) = P$
by (*metis RD-RH-design-form Healthy-def' assms*)

lemma *RD-RH-design*:
assumes $\$ok' \sharp P$ $\$ok' \sharp Q$
shows $RD(\mathbf{R}(P \vdash Q)) = \mathbf{R}(P \vdash Q)$
by (*simp add*: *RD1-reactive-design RD2-reactive-design RD-alt-def RH-idem assms*(*1*) *assms*(*2*))

lemma *RH-design-is-RD*:
assumes $\$ok' \sharp P$ $\$ok' \sharp Q$
shows $\mathbf{R}(P \vdash Q)$ *is RD*
by (*simp add*: *RD-RH-design Healthy-def' assms*(*1*) *assms*(*2*))

lemma *SRD-RH-design-form*:
$SRD(P) = \mathbf{R}_s((\neg\ P^f{}_f) \vdash P^t{}_f)$
proof −
have $SRD(P) = R1(R2c(R3h(RD1(RD2(R1(P))))))$
by (*metis* (*no-types*, *lifting*) *R1-H2-commute R1-R2c-commute R1-R3h-commute R1-idem R2c-H2-commute RD1-R1-commute RD1-R2c-commute RD1-R3h-commute RD2-R3h-commute RD2-def RHS-def SRD-def*)
also have ... $= R1(R2s(R3h(\mathbf{H}(P))))$
by (*metis* (*no-types*, *lifting*) *R1-H2-commute R1-R2c-is-R2 R1-R3h-commute R2-R1-form RD1-via-R1 RD2-def*)
also have ... $= \mathbf{R}_s(\mathbf{H}(P))$
by (*simp add*: *R1-R2s-R2c RHS-def*)
also have ... $= \mathbf{R}_s((\neg\ P^f) \vdash P^t)$
by (*simp add*: *H1-H2-eq-design*)
also have ... $= \mathbf{R}_s((\neg\ P^f{}_f) \vdash P^t{}_f)$
by (*metis* (*no-types*, *lifting*) *R3h-subst-wait RHS-def subst-not wait-false-design*)
finally show *?thesis* **.**
qed

lemma *SRD-reactive-design*:
assumes *P is SRD*
shows $\mathbf{R}_s((\neg\ P^f{}_f) \vdash P^t{}_f) = P$
by (*metis SRD-RH-design-form Healthy-def' assms*)

lemma *SRD-RH-design*:
assumes $\$ok' \sharp P$ $\$ok' \sharp Q$
shows $SRD(\mathbf{R}_s(P \vdash Q)) = \mathbf{R}_s(P \vdash Q)$
by (*simp add*: *RD1-st-reactive-design RD2-st-reactive-design RHS-idem SRD-def assms*(*1*) *assms*(*2*))

lemma *RHS-design-is-SRD*:
assumes $\$ok' \sharp P$ $\$ok' \sharp Q$
shows $\mathbf{R}_s(P \vdash Q)$ *is SRD*
by (*simp add*: *Healthy-def' SRD-RH-design assms*(*1*) *assms*(*2*))

lemma *SRD-RHS-H1-H2*: $SRD(P) = \mathbf{R}_s(\mathbf{H}(P))$
by (*metis* (*no-types*, *lifting*) *H1-H2-eq-design R3h-subst-wait RHS-def SRD-RH-design-form subst-not wait-false-design*)

3.2 Auxiliary healthiness conditions

definition [*upred-defs*]: $R3c\text{-}pre(P) = (true \lhd \$wait \rhd P)$

definition [*upred-defs*]: $R3c\text{-}post(P) = (\lceil II \rceil_D \triangleleft \$wait \triangleright P)$

definition [*upred-defs*]: $R3h\text{-}post(P) = ((\exists\ \$st \cdot \lceil II \rceil_D) \triangleleft \$wait \triangleright P)$

lemma *R3c-pre-conj*: $R3c\text{-}pre(P \wedge Q) = (R3c\text{-}pre(P) \wedge R3c\text{-}pre(Q))$
by (*rel-auto*)

lemma *R3c-pre-seq*:
$(true ;; Q) = true \Longrightarrow R3c\text{-}pre(P ;; Q) = (R3c\text{-}pre(P) ;; Q)$
by (*rel-auto*)

lemma *unrest-ok-R3c-pre* [*unrest*]: $\$ok \sharp P \Longrightarrow \$ok \sharp R3c\text{-}pre(P)$
by (*simp add*: *R3c-pre-def cond-def unrest*)

lemma *unrest-ok'-R3c-pre* [*unrest*]: $\$ok' \sharp P \Longrightarrow \$ok' \sharp R3c\text{-}pre(P)$
by (*simp add*: *R3c-pre-def cond-def unrest*)

lemma *unrest-ok-R3c-post* [*unrest*]: $\$ok \sharp P \Longrightarrow \$ok \sharp R3c\text{-}post(P)$
by (*simp add*: *R3c-post-def cond-def unrest*)

lemma *unrest-ok-R3c-post'* [*unrest*]: $\$ok' \sharp P \Longrightarrow \$ok' \sharp R3c\text{-}post(P)$
by (*simp add*: *R3c-post-def cond-def unrest*)

lemma *unrest-ok-R3h-post* [*unrest*]: $\$ok \sharp P \Longrightarrow \$ok \sharp R3h\text{-}post(P)$
by (*simp add*: *R3h-post-def cond-def unrest*)

lemma *unrest-ok-R3h-post'* [*unrest*]: $\$ok' \sharp P \Longrightarrow \$ok' \sharp R3h\text{-}post(P)$
by (*simp add*: *R3h-post-def cond-def unrest*)

3.3 Composition laws

theorem *R1-design-composition*:
fixes $P\ Q :: ('t::trace, 'α, 'β)\ rel\text{-}rp$
and $R\ S :: ('t, 'β, 'γ)\ rel\text{-}rp$
assumes $\$ok' \sharp P\ \$ok' \sharp Q\ \$ok \sharp R\ \$ok \sharp S$
shows
$(R1(P \vdash Q) ;; R1(R \vdash S)) =$
$R1((\neg\ (R1(\neg\ P) ;; R1(true)) \wedge \neg\ (R1(Q) ;; R1(\neg\ R))) \vdash (R1(Q) ;; R1(S)))$
proof −
have $(R1(P \vdash Q) ;; R1(R \vdash S)) = ((R1\ (P \vdash Q))^t ;; R1\ (R \vdash S)[\![true/\$ok]\!] \vee (R1\ (P \vdash Q))^f ;; R1\ (R \vdash S)[\![false/\$ok]\!])$
by (*rule seqr-bool-split*[*of ok*], *simp*)
also from *assms* **have** ... $= ((R1((\$ok \wedge P) \Rightarrow (true \wedge Q)) ;; R1((true \wedge R) \Rightarrow (\$ok' \wedge S))) \vee (R1((\$ok \wedge P) \Rightarrow (false \wedge Q)) ;; R1((false \wedge R) \Rightarrow (\$ok' \wedge S))))$
by (*simp add*: *design-def usubst R1-def*)
also from *assms* **have** ... $= ((R1((\$ok \wedge P) \Rightarrow Q) ;; R1(R \Rightarrow (\$ok' \wedge S))) \vee (R1(\neg\ (\$ok \wedge P)) ;; R1(true)))$
by *simp*
also from *assms* **have** ... $= ((R1(\neg\ \$ok \vee \neg\ P \vee Q) ;; R1(\neg\ R \vee (\$ok' \wedge S))) \vee (R1(\neg\ \$ok \vee \neg\ P) ;; R1(true)))$
by (*simp add*: *impl-alt-def utp-pred-laws.sup.assoc*)
also from *assms* **have** ... $= (((R1(\neg\ \$ok \vee \neg\ P) \vee R1(Q)) ;; R1(\neg\ R \vee (\$ok' \wedge S))) \vee (R1(\neg\ \$ok \vee \neg\ P) ;; R1(true)))$
by (*simp add*: *R1-disj utp-pred-laws.disj-assoc*)
also from *assms* **have** ... $= ((R1(\neg\ \$ok \vee \neg\ P) ;; R1(\neg\ R \vee (\$ok' \wedge S)))$

$\vee$ ($R1(Q)$;; $R1(\neg\ R \vee (\$ok' \wedge S))$)
$\vee$ ($R1(\neg\ \$ok \vee \neg\ P)$;; $R1(true)$))
by (*simp add*: *seqr-or-distl utp-pred-laws.sup.assoc*)
also from *assms* **have** ... = (($R1(Q)$;; $R1(\neg\ R \vee (\$ok' \wedge S))$)
$\vee$ ($R1(\neg\ \$ok \vee \neg\ P)$;; $R1(true)$))
by (*rel-blast*)
also from *assms* **have** ... = (($R1(Q)$;; ($R1(\neg\ R) \vee R1(S) \wedge \$ok'$))
$\vee$ ($R1(\neg\ \$ok \vee \neg\ P)$;; $R1(true)$))
by (*simp add*: *R1-disj R1-extend-conj utp-pred-laws.inf-commute*)
also have ... = (($R1(Q)$;; ($R1(\neg\ R) \vee R1(S) \wedge \$ok'$))
$\vee$ (($R1(\neg\ \$ok)$ :: $('t,'\alpha,'\beta)$ *rel-rp*) ;; $R1(true)$)
$\vee$ ($R1(\neg\ P)$;; $R1(true)$))
by (*simp add*: *R1-disj seqr-or-distl*)
also have ... = (($R1(Q)$;; ($R1(\neg\ R) \vee R1(S) \wedge \$ok'$))
$\vee$ ($R1(\neg\ \$ok)$)
$\vee$ ($R1(\neg\ P)$;; $R1(true)$))
proof −
have (($R1(\neg\ \$ok)$ :: $('t,'\alpha,'\beta)$ *rel-rp*) ;; $R1(true)$) =
($R1(\neg\ \$ok)$ :: $('t,'\alpha,'\gamma)$ *rel-rp*)
by (*rel-auto*)
thus *?thesis*
by *simp*
qed
also have ... = (($R1(Q)$;; ($R1(\neg\ R) \vee (R1(S \wedge \$ok'))$))
$\vee$ $R1(\neg\ \$ok)$
$\vee$ ($R1(\neg\ P)$;; $R1(true)$))
by (*simp add*: *R1-extend-conj*)
also have ... = (($R1(Q)$;; ($R1\ (\neg\ R)$))
$\vee$ ($R1(Q)$;; ($R1(S \wedge \$ok')$))
$\vee$ $R1(\neg\ \$ok)$
$\vee$ ($R1(\neg\ P)$;; $R1(true)$))
by (*simp add*: *seqr-or-distr utp-pred-laws.sup.assoc*)
also have ... = $R1$(($R1(Q)$;; ($R1\ (\neg\ R)$))
$\vee$ ($R1(Q)$;; ($R1(S \wedge \$ok')$))
$\vee$ ($\neg\ \$ok$)
$\vee$ ($R1(\neg\ P)$;; $R1(true)$))
by (*simp add*: *R1-disj R1-seqr*)
also have ... = $R1$(($R1(Q)$;; ($R1\ (\neg\ R)$))
$\vee$ (($R1(Q)$;; $R1(S)$) $\wedge$ $\$ok'$)
$\vee$ ($\neg\ \$ok$)
$\vee$ ($R1(\neg\ P)$;; $R1(true)$))
by (*rel-blast*)
also have ... = $R1(\neg(\$ok \wedge \neg\ (R1(\neg\ P)$;; $R1(true)) \wedge \neg\ (R1(Q)$;; $(R1\ (\neg\ R))))$
$\vee$ (($R1(Q)$;; $R1(S)$) $\wedge$ $\$ok'$))
by (*rel-blast*)
also have ... = $R1((\$ok \wedge \neg\ (R1(\neg\ P)$;; $R1(true)) \wedge \neg\ (R1(Q)$;; $(R1\ (\neg\ R))))$
$\Rightarrow$ ($\$ok' \wedge (R1(Q)$;; $R1(S))))$
by (*simp add*: *impl-alt-def utp-pred-laws.inf-commute*)
also have ... = $R1((\neg\ (R1(\neg\ P)$;; $R1(true)) \wedge \neg\ (R1(Q)$;; $R1(\neg\ R))) \vdash (R1(Q)$;; $R1(S)))$
by (*simp add*: *design-def*)
finally show *?thesis* **.**
qed

theorem *R1-design-composition-RR*:
assumes *P is RR Q is RR R is RR S is RR*

shows
$(R1(P \vdash Q) ;; R1(R \vdash S)) = R1(((\neg_r\ P)\ wp_r\ false \wedge Q\ wp_r\ R) \vdash (Q ;; S))$
apply (*subst R1-design-composition*)
apply (*simp-all add*: *assms unrest wp-rea-def Healthy-if closure*)
apply (*rel-auto*)
done

theorem *R1-design-composition-RC*:
assumes *P is RC Q is RR R is RR S is RR*
shows
$(R1(P \vdash Q) ;; R1(R \vdash S)) = R1((P \wedge Q\ wp_r\ R) \vdash (Q ;; S))$
by (*simp add*: *R1-design-composition-RR assms unrest Healthy-if closure wp*)

lemma *R2s-design*: $R2s(P \vdash Q) = (R2s(P) \vdash R2s(Q))$
by (*simp add*: *R2s-def design-def usubst*)

lemma *R2c-design*: $R2c(P \vdash Q) = (R2c(P) \vdash R2c(Q))$
by (*simp add*: *design-def impl-alt-def R2c-disj R2c-not R2c-ok R2c-and R2c-ok'*)

lemma *R1-R3c-design*:
$R1(R3c(P \vdash Q)) = R1(R3c\text{-}pre(P) \vdash R3c\text{-}post(Q))$
by (*rel-auto*)

lemma *R1-R3h-design*:
$R1(R3h(P \vdash Q)) = R1(R3c\text{-}pre(P) \vdash R3h\text{-}post(Q))$
by (*rel-auto*)

lemma *R3c-R1-design-composition*:
assumes $\$ok' \sharp P\ \$ok' \sharp Q\ \$ok \sharp R\ \$ok \sharp S$
shows $(R3c(R1(P \vdash Q)) ;; R3c(R1(R \vdash S))) =$
$R3c(R1((\neg\ (R1(\neg\ P) ;; R1(true)) \wedge \neg\ ((R1(Q) \wedge \neg\ \$wait') ;; R1(\neg\ R)))$
$\vdash (R1(Q) ;; (\lceil II \rceil_D \triangleleft \$wait \triangleright R1(S)))))$
proof −
have *1*:$(\neg\ (R1\ (\neg\ R3c\text{-}pre\ P) ;; R1\ true)) = (R3c\text{-}pre\ (\neg\ (R1\ (\neg\ P) ;; R1\ true)))$
by (*rel-auto*)
have *2*:$(\neg\ (R1\ (R3c\text{-}post\ Q) ;; R1\ (\neg\ R3c\text{-}pre\ R))) = R3c\text{-}pre(\neg\ ((R1\ Q \wedge \neg\ \$wait')\ ;;\ R1\ (\neg\ R)))$
by (*rel-auto*, *blast+*)
have *3*:$(R1\ (R3c\text{-}post\ Q) ;; R1\ (R3c\text{-}post\ S)) = R3c\text{-}post\ (R1\ Q ;; (\lceil II \rceil_D \triangleleft \$wait \triangleright R1\ S))$
by (*rel-auto*)
show *?thesis*
apply (*simp add*: *R3c-semir-form R1-R3c-commute*[*THEN sym*] *R1-R3c-design unrest*)
apply (*subst R1-design-composition*)
apply (*simp-all add*: *unrest assms R3c-pre-conj 1 2 3*)
done
qed

lemma *R3h-R1-design-composition*:
assumes $\$ok' \sharp P\ \$ok' \sharp Q\ \$ok \sharp R\ \$ok \sharp S$
shows $(R3h(R1(P \vdash Q)) ;; R3h(R1(R \vdash S))) =$
$R3h(R1((\neg\ (R1(\neg\ P) ;; R1(true)) \wedge \neg\ ((R1(Q) \wedge \neg\ \$wait') ;; R1(\neg\ R)))$
$\vdash (R1(Q) ;; ((\exists\ \$st \cdot \lceil II \rceil_D) \triangleleft \$wait \triangleright R1(S)))))$
proof −
have *1*:$(\neg\ (R1\ (\neg\ R3c\text{-}pre\ P) ;; R1\ true)) = (R3c\text{-}pre\ (\neg\ (R1\ (\neg\ P) ;; R1\ true)))$
by (*rel-auto*)
have *2*:$(\neg\ (R1\ (R3h\text{-}post\ Q) ;; R1\ (\neg\ R3c\text{-}pre\ R))) = R3c\text{-}pre(\neg\ ((R1\ Q \wedge \neg\ \$wait')\ ;;\ R1\ (\neg\ R)))$

```
    by (rel-auto, blast+)
  have 3:(R1 (R3h-post Q) ;; R1 (R3h-post S)) = R3h-post (R1 Q ;; ((∃ $st · ⌈II⌉_D) ◁ $wait ▷ R1 S))
    by (rel-auto, blast+)
  show ?thesis
    apply (simp add: R3h-semir-form R1-R3h-commute[THEN sym] R1-R3h-design unrest )
    apply (subst R1-design-composition)
    apply (simp-all add: unrest assms R3c-pre-conj 1 2 3)
  done
qed

lemma R2-design-composition:
  assumes $ok´ ♯ P $ok´ ♯ Q $ok ♯ R $ok ♯ S
  shows (R2(P ⊢ Q) ;; R2(R ⊢ S)) =
        R2((¬ (R1 (¬ R2c P) ;; R1 true) ∧ ¬ (R1 (R2c Q) ;; R1 (¬ R2c R))) ⊢ (R1 (R2c Q) ;; R1
(R2c S)))
 apply (simp add: R2-R2c-def R2c-design R1-design-composition assms unrest R2c-not R2c-and R2c-disj
R1-R2c-commute[THEN sym] R2c-idem R2c-R1-seq)
  apply (metis (no-types, lifting) R2c-R1-seq R2c-not R2c-true)
done

lemma RH-design-composition:
  assumes $ok´ ♯ P $ok´ ♯ Q $ok ♯ R $ok ♯ S
  shows (RH(P ⊢ Q) ;; RH(R ⊢ S)) =
       RH((¬ (R1 (¬ R2s P) ;; R1 true) ∧ ¬ ((R1 (R2s Q) ∧ (¬ $wait´)) ;; R1 (¬ R2s R))) ⊢
                  (R1 (R2s Q) ;; (⌈II⌉_D ◁ $wait ▷ R1 (R2s S))))
proof −
  have 1: R2c (R1 (¬ R2s P) ;; R1 true) = (R1 (¬ R2s P) ;; R1 true)
  proof −
    have 1:(R1 (¬ R2s P) ;; R1 true) = (R1(R2 (¬ P) ;; R2 true))
      by (rel-auto)
    have R2c(R1(R2 (¬ P) ;; R2 true)) = R2c(R1(R2 (¬ P) ;; R2 true))
      using R2c-not by blast
    also have ... = R2(R2 (¬ P) ;; R2 true)
      by (metis R1-R2c-commute R1-R2c-is-R2)
    also have ... = (R2 (¬ P) ;; R2 true)
      by (simp add: R2-seqr-distribute)
    also have ... = (R1 (¬ R2s P) ;; R1 true)
      by (simp add: R2-def R2s-not R2s-true)
    finally show ?thesis
      by (simp add: 1)
  qed

  have 2:R2c ((R1 (R2s Q) ∧ ¬ $wait´) ;; R1 (¬ R2s R)) = ((R1 (R2s Q) ∧ ¬ $wait´) ;; R1 (¬ R2s
R))
  proof −
    have ((R1 (R2s Q) ∧ ¬ $wait´) ;; R1 (¬ R2s R)) = R1 (R2 (Q ∧ ¬ $wait´) ;; R2 (¬ R))
      by (rel-auto)
    hence R2c ((R1 (R2s Q) ∧ ¬ $wait´) ;; R1 (¬ R2s R)) = (R2 (Q ∧ ¬ $wait´) ;; R2 (¬ R))
      by (metis R1-R2c-commute R1-R2c-is-R2 R2-seqr-distribute)
    also have ... = ((R1 (R2s Q) ∧ ¬ $wait´) ;; R1 (¬ R2s R))
      by (rel-auto)
    finally show ?thesis .
  qed

  have 3:R2c((R1 (R2s Q) ;; (⌈II⌉_D ◁ $wait ▷ R1 (R2s S)))) = (R1 (R2s Q) ;; (⌈II⌉_D ◁ $wait ▷ R1
```

$(R2s\ S)))$
proof −
have $R2c(((R1\ (R2s\ Q))[\![true/\$wait´]\!] ;; (\lceil II \rceil_D \lhd \$wait \rhd R1\ (R2s\ S))[\![true/\$wait]\!]))$
$= ((R1\ (R2s\ Q))[\![true/\$wait´]\!] ;; (\lceil II \rceil_D \lhd \$wait \rhd R1\ (R2s\ S))[\![true/\$wait]\!])$
proof −
have $R2c(((R1\ (R2s\ Q))[\![true/\$wait´]\!] ;; (\lceil II \rceil_D \lhd \$wait \rhd R1\ (R2s\ S))[\![true/\$wait]\!])) =$
$R2c(R1\ (R2s\ (Q[\![true/\$wait´]\!])) ;; \lceil II \rceil_D[\![true/\$wait]\!])$
by (*simp add*: *usubst cond-unit-T R1-def R2s-def*)
also have $... = R2c(R2(Q[\![true/\$wait´]\!]) ;; R2(\lceil II \rceil_D[\![true/\$wait]\!]))$
by (*metis R2-def R2-des-lift-skip R2-subst-wait-true*)
also have $... = (R2(Q[\![true/\$wait´]\!]) ;; R2(\lceil II \rceil_D[\![true/\$wait]\!]))$
using *R2c-seq* **by** *blast*
also have $... = ((R1\ (R2s\ Q))[\![true/\$wait´]\!] ;; (\lceil II \rceil_D \lhd \$wait \rhd R1\ (R2s\ S))[\![true/\$wait]\!])$
apply (*simp add*: *usubst R2-des-lift-skip*)
apply (*metis R2-def R2-des-lift-skip R2-subst-wait'-true R2-subst-wait-true*)
done
finally show *?thesis* **.**
qed
moreover have $R2c(((R1\ (R2s\ Q))[\![false/\$wait´]\!] ;; (\lceil II \rceil_D \lhd \$wait \rhd R1\ (R2s\ S))[\![false/\$wait]\!]))$
$= ((R1\ (R2s\ Q))[\![false/\$wait´]\!] ;; (\lceil II \rceil_D \lhd \$wait \rhd R1\ (R2s\ S))[\![false/\$wait]\!])$
by (*simp add*: *usubst cond-unit-F*)
(*metis* (*no-types*, *hide-lams*) *R1-wait'-false R1-wait-false R2-def R2-subst-wait'-false R2-subst-wait-false R2c-seq*)
ultimately show *?thesis*
proof −
have $\lceil II \rceil_D \lhd \$wait \rhd R1\ (R2s\ S) = R2\ (\lceil II \rceil_D \lhd \$wait \rhd S)$
by (*simp add*: *R1-R2c-is-R2 R1-R2s-R2c R2-condr' R2-des-lift-skip R2s-wait*)
then show *?thesis*
by (*simp add*: *R1-R2c-is-R2 R1-R2s-R2c R2c-seq*)
qed
qed

have $(R1(R2s(R3c(P \vdash Q))) ;; R1(R2s(R3c(R \vdash S)))) =$
$((R3c(R1(R2s(P) \vdash R2s(Q)))) ;; R3c(R1(R2s(R) \vdash R2s(S))))$
by (*metis* (*no-types*, *hide-lams*) *R1-R2s-R2c R1-R3c-commute R2c-R3c-commute R2s-design*)
also have $... = R3c\ (R1\ ((\neg\ (R1\ (\neg\ R2s\ P) ;;\ R1\ true) \wedge \neg\ ((R1\ (R2s\ Q) \wedge \neg\ \$wait´) ;;\ R1\ (\neg\ R2s\ R))) \vdash$
$(R1\ (R2s\ Q) ;; (\lceil II \rceil_D \lhd \$wait \rhd R1\ (R2s\ S)))))$
by (*simp add*: *R3c-R1-design-composition assms unrest*)
also have $... = R3c(R1(R2c((\neg\ (R1\ (\neg\ R2s\ P) ;;\ R1\ true) \wedge \neg\ ((R1\ (R2s\ Q) \wedge \neg\ \$wait´) ;;\ R1\ (\neg\ R2s\ R))) \vdash$
$(R1\ (R2s\ Q) ;; (\lceil II \rceil_D \lhd \$wait \rhd R1\ (R2s\ S))))))$
by (*simp add*: *R2c-design R2c-and R2c-not 1 2 3*)
finally show *?thesis*
by (*simp add*: *R1-R2s-R2c R1-R3c-commute R2c-R3c-commute RH-def*)
qed

lemma *RHS-design-composition*:
assumes $\$ok´ \sharp P\ \$ok´ \sharp Q\ \$ok \sharp R\ \$ok \sharp S$
shows $(\mathbf{R}_s(P \vdash Q) ;; \mathbf{R}_s(R \vdash S)) =$
$\mathbf{R}_s((\neg\ (R1\ (\neg\ R2s\ P) ;;\ R1\ true) \wedge \neg\ ((R1\ (R2s\ Q) \wedge (\neg\ \$wait´)) ;;\ R1\ (\neg\ R2s\ R))) \vdash$
$(R1\ (R2s\ Q) ;; ((\exists\ \$st \cdot \lceil II \rceil_D) \lhd \$wait \rhd R1\ (R2s\ S))))$
proof −
have *1*: $R2c\ (R1\ (\neg\ R2s\ P) ;;\ R1\ true) = (R1\ (\neg\ R2s\ P) ;;\ R1\ true)$
proof −

have *1*:$(R1\ (\neg\ R2s\ P)\ ;;\ R1\ true) = (R1(R2\ (\neg\ P)\ ;;\ R2\ true))$
by (*rel-auto*, *blast*)
have $R2c(R1(R2\ (\neg\ P)\ ;;\ R2\ true)) = R2c(R1(R2\ (\neg\ P)\ ;;\ R2\ true))$
using *R2c-not* **by** *blast*
also have $... = R2(R2\ (\neg\ P)\ ;;\ R2\ true)$
by (*metis R1-R2c-commute R1-R2c-is-R2*)
also have $... = (R2\ (\neg\ P)\ ;;\ R2\ true)$
by (*simp add*: *R2-seqr-distribute*)
also have $... = (R1\ (\neg\ R2s\ P)\ ;;\ R1\ true)$
by (*simp add*: *R2-def R2s-not R2s-true*)
finally show *?thesis*
by (*simp add*: *1*)
qed

have *2*:$R2c\ ((R1\ (R2s\ Q) \wedge \neg\ \$wait´)\ ;;\ R1\ (\neg\ R2s\ R)) = ((R1\ (R2s\ Q) \wedge \neg\ \$wait´)\ ;;\ R1\ (\neg\ R2s\ R))$
proof −
have $((R1\ (R2s\ Q) \wedge \neg\ \$wait´)\ ;;\ R1\ (\neg\ R2s\ R)) = R1\ (R2\ (Q \wedge \neg\ \$wait´)\ ;;\ R2\ (\neg\ R))$
by (*rel-auto*, *blast+*)
hence $R2c\ ((R1\ (R2s\ Q) \wedge \neg\ \$wait´)\ ;;\ R1\ (\neg\ R2s\ R)) = (R2\ (Q \wedge \neg\ \$wait´)\ ;;\ R2\ (\neg\ R))$
by (*metis* (*no-types*, *lifting*) *R1-R2c-commute R1-R2c-is-R2 R2-seqr-distribute*)
also have $... = ((R1\ (R2s\ Q) \wedge \neg\ \$wait´)\ ;;\ R1\ (\neg\ R2s\ R))$
by (*rel-auto*, *blast+*)
finally show *?thesis* **.**
qed

have *3*:$R2c((R1\ (R2s\ Q)\ ;;\ ((\exists\ \$st \cdot \lceil II \rceil_D) \triangleleft \$wait \triangleright R1\ (R2s\ S)))) =$
$(R1\ (R2s\ Q)\ ;;\ ((\exists\ \$st \cdot \lceil II \rceil_D) \triangleleft \$wait \triangleright R1\ (R2s\ S)))$
proof −
have $R2c(((R1\ (R2s\ Q))[\![true/\$wait´]\!]\ ;;\ ((\exists\ \$st \cdot \lceil II \rceil_D) \triangleleft \$wait \triangleright R1\ (R2s\ S))[\![true/\$wait]\!]))$
$= ((R1\ (R2s\ Q))[\![true/\$wait´]\!]\ ;;\ ((\exists\ \$st \cdot \lceil II \rceil_D) \triangleleft \$wait \triangleright R1\ (R2s\ S))[\![true/\$wait]\!])$
proof −
have $R2c(((R1\ (R2s\ Q))[\![true/\$wait´]\!]\ ;;\ ((\exists\ \$st \cdot \lceil II \rceil_D) \triangleleft \$wait \triangleright R1\ (R2s\ S))[\![true/\$wait]\!])) =$
$R2c(R1\ (R2s\ (Q[\![true/\$wait´]\!]))\ ;;\ (\exists\ \$st \cdot \lceil II \rceil_D)[\![true/\$wait]\!])$
by (*simp add*: *usubst cond-unit-T R1-def R2s-def*)
also have $... = R2c(R2(Q[\![true/\$wait´]\!])\ ;;\ R2((\exists\ \$st \cdot \lceil II \rceil_D)[\![true/\$wait]\!]))$
by (*metis* (*no-types*, *lifting*) *R2-def R2-des-lift-skip R2-subst-wait-true R2-st-ex*)
also have $... = (R2(Q[\![true/\$wait´]\!])\ ;;\ R2((\exists\ \$st \cdot \lceil II \rceil_D)[\![true/\$wait]\!]))$
using *R2c-seq* **by** *blast*
also have $... = ((R1\ (R2s\ Q))[\![true/\$wait´]\!]\ ;;\ ((\exists\ \$st \cdot \lceil II \rceil_D) \triangleleft \$wait \triangleright R1\ (R2s\ S))[\![true/\$wait]\!])$
apply (*simp add*: *usubst R2-des-lift-skip*)
apply (*metis* (*no-types*) *R2-def R2-des-lift-skip R2-st-ex R2-subst-wait'-true R2-subst-wait-true*)
done
finally show *?thesis* **.**
qed
moreover have $R2c(((R1\ (R2s\ Q))[\![false/\$wait´]\!]\ ;;\ ((\exists\ \$st \cdot \lceil II \rceil_D) \triangleleft \$wait \triangleright R1\ (R2s\ S))[\![false/\$wait]\!]))$
$= ((R1\ (R2s\ Q))[\![false/\$wait´]\!]\ ;;\ ((\exists\ \$st \cdot \lceil II \rceil_D) \triangleleft \$wait \triangleright R1\ (R2s\ S))[\![false/\$wait]\!])$
by (*simp add*: *usubst*)
(*metis* (*no-types*, *lifting*) *R1-wait'-false R1-wait-false R2-R1-form R2-subst-wait'-false R2-subst-wait-false R2c-seq*)
ultimately show *?thesis*
by (*smt R2-R1-form R2-condr' R2-des-lift-skip R2-st-ex R2c-seq R2s-wait*)
qed

have $(R1(R2s(R3h(P \vdash Q)))\ ;;\ R1(R2s(R3h(R \vdash S)))) =$

$((R3h(R1(R2s(P) \vdash R2s(Q)))) ;; R3h(R1(R2s(R) \vdash R2s(S))))$
by (*metis* (*no-types*, *hide-lams*) *R1-R2s-R2c R1-R3h-commute R2c-R3h-commute R2s-design*)
also have ... = $R3h\ (R1\ ((\neg\ (R1\ (\neg\ R2s\ P)\ ;;\ R1\ true)\ \wedge\ \neg\ ((R1\ (R2s\ Q)\ \wedge\ \neg\ \$wait´)\ ;;\ R1\ (\neg\ R2s\ R)))\ \vdash$
$(R1\ (R2s\ Q)\ ;;\ ((\exists\ \$st \cdot \lceil II \rceil_D) \triangleleft \$wait \triangleright R1\ (R2s\ S)))))$
by (*simp add*: *R3h-R1-design-composition assms unrest*)
also have ... = $R3h(R1(R2c((\neg\ (R1\ (\neg\ R2s\ P)\ ;;\ R1\ true)\ \wedge\ \neg\ ((R1\ (R2s\ Q)\ \wedge\ \neg\ \$wait´)\ ;;\ R1\ (\neg\ R2s\ R)))\ \vdash$
$(R1\ (R2s\ Q)\ ;;\ ((\exists\ \$st \cdot \lceil II \rceil_D) \triangleleft \$wait \triangleright R1\ (R2s\ S))))))$
by (*simp add*: *R2c-design R2c-and R2c-not 1 2 3*)
finally show *?thesis*
by (*simp add*: *R1-R2s-R2c R1-R3h-commute R2c-R3h-commute RHS-def*)
qed

lemma *RHS-R2s-design-composition*:
assumes
$\$ok´ \sharp P$ $\$ok´ \sharp Q$ $\$ok \sharp R$ $\$ok \sharp S$
P is R2s Q is R2s R is R2s S is R2s
shows $(\mathbf{R}_s(P \vdash Q)\ ;;\ \mathbf{R}_s(R \vdash S)) =$
$\mathbf{R}_s((\neg\ (R1\ (\neg\ P)\ ;;\ R1\ true)\ \wedge\ \neg\ ((R1\ Q\ \wedge\ \neg\ \$wait´)\ ;;\ R1\ (\neg\ R)))\ \vdash$
$(R1\ Q\ ;;\ ((\exists\ \$st \cdot \lceil II \rceil_D) \triangleleft \$wait \triangleright R1\ S)))$
proof −
have *f1*: $R2s\ P = P$
by (*meson Healthy-def assms*(*5*))
have *f2*: $R2s\ Q = Q$
by (*meson Healthy-def assms*(*6*))
have *f3*: $R2s\ R = R$
by (*meson Healthy-def assms*(*7*))
have $R2s\ S = S$
by (*meson Healthy-def assms*(*8*))
then show *?thesis*
using *f3 f2 f1* **by** (*simp add*: *RHS-design-composition assms*(*1*) *assms*(*2*) *assms*(*3*) *assms*(*4*))
qed

lemma *RH-design-export-R1*: $\mathbf{R}(P \vdash Q) = \mathbf{R}(P \vdash R1(Q))$
by (*rel-auto*)

lemma *RH-design-export-R2s*: $\mathbf{R}(P \vdash Q) = \mathbf{R}(P \vdash R2s(Q))$
by (*rel-auto*)

lemma *RH-design-export-R2c*: $\mathbf{R}(P \vdash Q) = \mathbf{R}(P \vdash R2c(Q))$
by (*rel-auto*)

lemma *RHS-design-export-R1*: $\mathbf{R}_s(P \vdash Q) = \mathbf{R}_s(P \vdash R1(Q))$
by (*rel-auto*)

lemma *RHS-design-export-R2s*: $\mathbf{R}_s(P \vdash Q) = \mathbf{R}_s(P \vdash R2s(Q))$
by (*rel-auto*)

lemma *RHS-design-export-R2c*: $\mathbf{R}_s(P \vdash Q) = \mathbf{R}_s(P \vdash R2c(Q))$
by (*rel-auto*)

lemma *RHS-design-export-R2*: $\mathbf{R}_s(P \vdash Q) = \mathbf{R}_s(P \vdash R2(Q))$
by (*rel-auto*)

lemma *R1-design-R1-pre*:
  $\mathbf{R}_s(R1(P) \vdash Q) = \mathbf{R}_s(P \vdash Q)$
  by (*rel-auto*)

lemma *RHS-design-ok-wait*: $\mathbf{R}_s(P[\![true,false/\$ok,\$wait]\!] \vdash Q[\![true,false/\$ok,\$wait]\!]) = \mathbf{R}_s(P \vdash Q)$
  by (*rel-auto*)

lemma *RH-design-neg-R1-pre*:
  $\mathbf{R}\ ((\neg\ R1\ P) \vdash R) = \mathbf{R}\ ((\neg\ P) \vdash R)$
  by (*rel-auto*)

lemma *RHS-design-neg-R1-pre*:
  $\mathbf{R}_s\ ((\neg\ R1\ P) \vdash R) = \mathbf{R}_s\ ((\neg\ P) \vdash R)$
  by (*rel-auto*)

lemma *RHS-design-conj-neg-R1-pre*:
  $\mathbf{R}_s\ (((\neg\ R1\ P) \wedge Q) \vdash R) = \mathbf{R}_s\ (((\neg\ P) \wedge Q) \vdash R)$
  by (*rel-auto*)

lemma *RHS-pre-lemma*: $(\mathbf{R}_s\ P)^f{}_f = R1(R2c(P^f{}_f))$
  by (*rel-auto*)

lemma *RH-design-R2c-pre*:
  $\mathbf{R}(R2c(P) \vdash Q) = \mathbf{R}(P \vdash Q)$
  by (*rel-auto*)

lemma *RHS-design-R2c-pre*:
  $\mathbf{R}_s(R2c(P) \vdash Q) = \mathbf{R}_s(P \vdash Q)$
  by (*rel-auto*)

3.4 Refinement introduction laws

lemma *R1-design-refine*:
  assumes
    P_1 *is R1* P_2 *is R1* Q_1 *is R1* Q_2 *is R1*
    $\$ok \sharp P_1\ \$ok' \sharp P_1\ \$ok \sharp P_2\ \$ok' \sharp P_2$
    $\$ok \sharp Q_1\ \$ok' \sharp Q_1\ \$ok \sharp Q_2\ \$ok' \sharp Q_2$
  shows $R1(P_1 \vdash P_2) \sqsubseteq R1(Q_1 \vdash Q_2) \longleftrightarrow$ ‘$P_1 \Rightarrow Q_1$‘ $\wedge$ ‘$P_1 \wedge Q_2 \Rightarrow P_2$‘
proof −
  have $R1((\exists\ \$ok;\$ok' \cdot P_1) \vdash (\exists\ \$ok;\$ok' \cdot P_2)) \sqsubseteq R1((\exists\ \$ok;\$ok' \cdot Q_1) \vdash (\exists\ \$ok;\$ok' \cdot Q_2))$
    $\longleftrightarrow$ ‘$R1(\exists\ \$ok;\$ok' \cdot P_1) \Rightarrow R1(\exists\ \$ok;\$ok' \cdot Q_1)$‘ $\wedge$ ‘$R1(\exists\ \$ok;\$ok' \cdot P_1) \wedge R1(\exists\ \$ok;\$ok' \cdot Q_2) \Rightarrow R1(\exists\ \$ok;\$ok' \cdot P_2)$‘
    by (*rel-auto*, *meson+*)
  thus *?thesis*
    by (*simp-all add*: *ex-unrest ex-plus Healthy-if assms*)
qed

lemma *R1-design-refine-RR*:
  assumes P_1 *is RR* P_2 *is RR* Q_1 *is RR* Q_2 *is RR*
  shows $R1(P_1 \vdash P_2) \sqsubseteq R1(Q_1 \vdash Q_2) \longleftrightarrow$ ‘$P_1 \Rightarrow Q_1$‘ $\wedge$ ‘$P_1 \wedge Q_2 \Rightarrow P_2$‘
  by (*simp add*: *R1-design-refine assms unrest closure*)

lemma *RH-design-refine*:
  assumes
    P_1 *is R1* P_2 *is R1* Q_1 *is R1* Q_2 *is R1*
    P_1 *is R2c* P_2 *is R2c* Q_1 *is R2c* Q_2 *is R2c*

$ok ♯ P_1 $ok´ ♯ P_1 $ok ♯ P_2 $ok´ ♯ P_2
$ok ♯ Q_1 $ok´ ♯ Q_1 $ok ♯ Q_2 $ok´ ♯ Q_2
$wait ♯ P_1 $wait ♯ P_2 $wait ♯ Q_1 $wait ♯ Q_2
shows $\mathbf{R}(P_1 \vdash P_2) \sqsubseteq \mathbf{R}(Q_1 \vdash Q_2) \longleftrightarrow \text{`}P_1 \Rightarrow Q_1\text{`} \wedge \text{`}P_1 \wedge Q_2 \Rightarrow P_2\text{`}$
proof −
have $\mathbf{R}(P_1 \vdash P_2) \sqsubseteq \mathbf{R}(Q_1 \vdash Q_2) \longleftrightarrow R1(R3c(R2c(P_1 \vdash P_2))) \sqsubseteq R1(R3c(R2c(Q_1 \vdash Q_2)))$
by (*simp add*: *R2c-R3c-commute RH-def*)
also have ... $\longleftrightarrow R1(R3c(P_1 \vdash P_2)) \sqsubseteq R1(R3c(Q_1 \vdash Q_2))$
by (*simp add*: *Healthy-if R2c-design assms*)
also have ... $\longleftrightarrow R1(R3c(P_1 \vdash P_2))[\![false/\$wait]\!] \sqsubseteq R1(R3c(Q_1 \vdash Q_2))[\![false/\$wait]\!]$
by (*rel-auto*, *metis+*)
also have ... $\longleftrightarrow R1(P_1 \vdash P_2)[\![false/\$wait]\!] \sqsubseteq R1(Q_1 \vdash Q_2)[\![false/\$wait]\!]$
by (*rel-auto*)
also have ... $\longleftrightarrow R1(P_1 \vdash P_2) \sqsubseteq R1(Q_1 \vdash Q_2)$
by (*simp add*: *usubst assms closure unrest*)
also have ... $\longleftrightarrow \text{`}P_1 \Rightarrow Q_1\text{`} \wedge \text{`}P_1 \wedge Q_2 \Rightarrow P_2\text{`}$
by (*simp add*: *R1-design-refine assms*)
finally show *?thesis* **.**
qed

lemma *RHS-design-refine*:
assumes
P_1 *is R1* P_2 *is R1* Q_1 *is R1* Q_2 *is R1*
P_1 *is R2c* P_2 *is R2c* Q_1 *is R2c* Q_2 *is R2c*
$ok ♯ P_1 $ok´ ♯ P_1 $ok ♯ P_2 $ok´ ♯ P_2
$ok ♯ Q_1 $ok´ ♯ Q_1 $ok ♯ Q_2 $ok´ ♯ Q_2
$wait ♯ P_1 $wait ♯ P_2 $wait ♯ Q_1 $wait ♯ Q_2
shows $\mathbf{R}_s(P_1 \vdash P_2) \sqsubseteq \mathbf{R}_s(Q_1 \vdash Q_2) \longleftrightarrow \text{`}P_1 \Rightarrow Q_1\text{`} \wedge \text{`}P_1 \wedge Q_2 \Rightarrow P_2\text{`}$
proof −
have $\mathbf{R}_s(P_1 \vdash P_2) \sqsubseteq \mathbf{R}_s(Q_1 \vdash Q_2) \longleftrightarrow R1(R3h(R2c(P_1 \vdash P_2))) \sqsubseteq R1(R3h(R2c(Q_1 \vdash Q_2)))$
by (*simp add*: *R2c-R3h-commute RHS-def*)
also have ... $\longleftrightarrow R1(R3h(P_1 \vdash P_2)) \sqsubseteq R1(R3h(Q_1 \vdash Q_2))$
by (*simp add*: *Healthy-if R2c-design assms*)
also have ... $\longleftrightarrow R1(R3h(P_1 \vdash P_2))[\![false/\$wait]\!] \sqsubseteq R1(R3h(Q_1 \vdash Q_2))[\![false/\$wait]\!]$
by (*rel-auto*, *metis+*)
also have ... $\longleftrightarrow R1(P_1 \vdash P_2)[\![false/\$wait]\!] \sqsubseteq R1(Q_1 \vdash Q_2)[\![false/\$wait]\!]$
by (*rel-auto*)
also have ... $\longleftrightarrow R1(P_1 \vdash P_2) \sqsubseteq R1(Q_1 \vdash Q_2)$
by (*simp add*: *usubst assms closure unrest*)
also have ... $\longleftrightarrow \text{`}P_1 \Rightarrow Q_1\text{`} \wedge \text{`}P_1 \wedge Q_2 \Rightarrow P_2\text{`}$
by (*simp add*: *R1-design-refine assms*)
finally show *?thesis* **.**
qed

lemma *rdes-refine-intro*:
assumes `}P_1 \Rightarrow P_2\text{` `}P_1 \wedge Q_2 \Rightarrow Q_1\text{`
shows $\mathbf{R}(P_1 \vdash Q_1) \sqsubseteq \mathbf{R}(P_2 \vdash Q_2)$
by (*simp add*: *RH-mono assms design-refine-intro*)

lemma *srdes-refine-intro*:
assumes `}P_1 \Rightarrow P_2\text{` `}P_1 \wedge Q_2 \Rightarrow Q_1\text{`
shows $\mathbf{R}_s(P_1 \vdash Q_1) \sqsubseteq \mathbf{R}_s(P_2 \vdash Q_2)$
by (*simp add*: *RHS-mono assms design-refine-intro*)

3.5 Distribution laws

lemma *RHS-design-choice*: $\mathbf{R}_s(P_1 \vdash Q_1) \sqcap \mathbf{R}_s(P_2 \vdash Q_2) = \mathbf{R}_s((P_1 \wedge P_2) \vdash (Q_1 \vee Q_2))$
by (*metis RHS-inf design-choice*)

lemma *RHS-design-sup*: $\mathbf{R}_s(P_1 \vdash Q_1) \sqcup \mathbf{R}_s(P_2 \vdash Q_2) = \mathbf{R}_s((P_1 \vee P_2) \vdash ((P_1 \Rightarrow Q_1) \wedge (P_2 \Rightarrow Q_2)))$
by (*metis RHS-sup design-inf*)

lemma *RHS-design-USUP*:
assumes $A \neq \{\}$
shows $(\bigsqcap\ i \in A \cdot \mathbf{R}_s(P(i) \vdash Q(i))) = \mathbf{R}_s((\bigsqcup\ i \in A \cdot P(i)) \vdash (\bigsqcap\ i \in A \cdot Q(i)))$
by (*subst RHS-INF*[*OF assms*, *THEN sym*], *simp add*: *design-UINF-mem assms*)

end

4 Reactive Design Triples

theory *utp-rdes-triples*
imports *utp-rdes-designs*
begin

4.1 Diamond notation

definition *wait'-cond* ::
($'t$::*trace*,$'\alpha$,$'\beta$) *rel-rp* $\Rightarrow$ ($'t$,$'\alpha$,$'\beta$) *rel-rp* $\Rightarrow$ ($'t$,$'\alpha$,$'\beta$) *rel-rp* (**infixr** $\diamond$ *60*) **where**
[*upred-defs*]: $P \diamond Q = (P \lhd \$wait' \rhd Q)$

utp-const *wait'-cond*

lemma *wait'-cond-unrest* [*unrest*]:
$[\![$ *out-var wait* $\bowtie x$; $x \sharp P$; $x \sharp Q$ $]\!] \Longrightarrow x \sharp (P \diamond Q)$
by (*simp add*: *wait'-cond-def unrest*)

lemma *wait'-cond-subst* [*usubst*]:
$\$wait' \sharp_s \sigma \Longrightarrow \sigma \dagger (P \diamond Q) = (\sigma \dagger P) \diamond (\sigma \dagger Q)$
by (*simp add*: *wait'-cond-def usubst unrest usubst-apply-unrest*)

lemma *wait'-cond-left-false*: $false \diamond P = (\neg\ \$wait' \wedge P)$
by (*rel-auto*)

lemma *wait'-cond-seq*: $((P \diamond Q)\ ;;\ R) = ((P\ ;;\ (\$wait \wedge R)) \vee (Q\ ;;\ (\neg \$wait \wedge R)))$
by (*simp add*: *wait'-cond-def cond-def seqr-or-distl*, *rel-blast*)

lemma *wait'-cond-true*: $(P \diamond Q \wedge \$wait') = (P \wedge \$wait')$
by (*rel-auto*)

lemma *wait'-cond-false*: $(P \diamond Q \wedge (\neg \$wait')) = (Q \wedge (\neg \$wait'))$
by (*rel-auto*)

lemma *wait'-cond-idem*: $P \diamond P = P$
by (*rel-auto*)

lemma *wait'-cond-conj-exchange*:
$((P \diamond Q) \wedge (R \diamond S)) = (P \wedge R) \diamond (Q \wedge S)$
by (*rel-auto*)

lemma *subst-wait′-cond-true* [*usubst*]: $(P \diamond Q)[\![true/\$wait\acute{}]\!] = P[\![true/\$wait\acute{}]\!]$
by (*rel-auto*)

lemma *subst-wait′-cond-false* [*usubst*]: $(P \diamond Q)[\![false/\$wait\acute{}]\!] = Q[\![false/\$wait\acute{}]\!]$
by (*rel-auto*)

lemma *subst-wait′-left-subst*: $(P[\![true/\$wait\acute{}]\!] \diamond Q) = (P \diamond Q)$
by (*rel-auto*)

lemma *subst-wait′-right-subst*: $(P \diamond Q[\![false/\$wait\acute{}]\!]) = (P \diamond Q)$
by (*rel-auto*)

lemma *wait′-cond-split*: $P[\![true/\$wait\acute{}]\!] \diamond P[\![false/\$wait\acute{}]\!] = P$
by (*simp add*: *wait′-cond-def cond-var-split*)

lemma *wait-cond′-assoc* [*simp*]: $P \diamond Q \diamond R = P \diamond R$
by (*rel-auto*)

lemma *wait-cond′-shadow*: $(P \diamond Q) \diamond R = P \diamond Q \diamond R$
by (*rel-auto*)

lemma *wait-cond′-conj* [*simp*]: $P \diamond (Q \wedge (R \diamond S)) = P \diamond (Q \wedge S)$
by (*rel-auto*)

lemma *R1-wait′-cond*: $R1(P \diamond Q) = R1(P) \diamond R1(Q)$
by (*rel-auto*)

lemma *R2s-wait′-cond*: $R2s(P \diamond Q) = R2s(P) \diamond R2s(Q)$
by (*simp add*: *wait′-cond-def R2s-def R2s-def usubst*)

lemma *R2-wait′-cond*: $R2(P \diamond Q) = R2(P) \diamond R2(Q)$
by (*simp add*: *R2-def R2s-wait′-cond R1-wait′-cond*)

lemma *wait′-cond-R1-closed* [*closure*]:
$[\![P \text{ is } R1;\ Q \text{ is } R1]\!] \Longrightarrow P \diamond Q \text{ is } R1$
by (*simp add*: *Healthy-def R1-wait′-cond*)

lemma *wait′-cond-R2c-closed* [*closure*]: $[\![P \text{ is } R2c;\ Q \text{ is } R2c]\!] \Longrightarrow P \diamond Q \text{ is } R2c$
by (*simp add*: *R2c-condr wait′-cond-def Healthy-def*, *rel-auto*)

4.2 Export laws

lemma *RH-design-peri-R1*: $\mathbf{R}(P \vdash R1(Q) \diamond R) = \mathbf{R}(P \vdash Q \diamond R)$
by (*metis* (*no-types*, *lifting*) *R1-idem R1-wait′-cond RH-design-export-R1*)

lemma *RH-design-post-R1*: $\mathbf{R}(P \vdash Q \diamond R1(R)) = \mathbf{R}(P \vdash Q \diamond R)$
by (*metis R1-wait′-cond RH-design-export-R1 RH-design-peri-R1*)

lemma *RH-design-peri-R2s*: $\mathbf{R}(P \vdash R2s(Q) \diamond R) = \mathbf{R}(P \vdash Q \diamond R)$
by (*metis* (*no-types*, *lifting*) *R2s-idem R2s-wait′-cond RH-design-export-R2s*)

lemma *RH-design-post-R2s*: $\mathbf{R}(P \vdash Q \diamond R2s(R)) = \mathbf{R}(P \vdash Q \diamond R)$
by (*metis* (*no-types*, *lifting*) *R2s-idem R2s-wait′-cond RH-design-export-R2s*)

lemma *RH-design-peri-R2c*: $\mathbf{R}(P \vdash R2c(Q) \diamond R) = \mathbf{R}(P \vdash Q \diamond R)$
by (*metis R1-R2s-R2c RH-design-peri-R1 RH-design-peri-R2s*)

lemma *RHS-design-peri-R1*: $\mathbf{R}_s(P \vdash R1(Q) \diamond R) = \mathbf{R}_s(P \vdash Q \diamond R)$
by (*metis* (*no-types*, *lifting*) *R1-idem R1-wait'-cond RHS-design-export-R1*)

lemma *RHS-design-post-R1*: $\mathbf{R}_s(P \vdash Q \diamond R1(R)) = \mathbf{R}_s(P \vdash Q \diamond R)$
by (*metis R1-wait'-cond RHS-design-export-R1 RHS-design-peri-R1*)

lemma *RHS-design-peri-R2s*: $\mathbf{R}_s(P \vdash R2s(Q) \diamond R) = \mathbf{R}_s(P \vdash Q \diamond R)$
by (*metis* (*no-types*, *lifting*) *R2s-idem R2s-wait'-cond RHS-design-export-R2s*)

lemma *RHS-design-post-R2s*: $\mathbf{R}_s(P \vdash Q \diamond R2s(R)) = \mathbf{R}_s(P \vdash Q \diamond R)$
by (*metis R2s-wait'-cond RHS-design-export-R2s RHS-design-peri-R2s*)

lemma *RHS-design-peri-R2c*: $\mathbf{R}_s(P \vdash R2c(Q) \diamond R) = \mathbf{R}_s(P \vdash Q \diamond R)$
by (*metis R1-R2s-R2c RHS-design-peri-R1 RHS-design-peri-R2s*)

lemma *RH-design-lemma1*:
$RH(P \vdash (R1(R2c(Q)) \vee R) \diamond S) = RH(P \vdash (Q \vee R) \diamond S)$
by (*metis* (*no-types*, *lifting*) *R1-R2c-is-R2 R1-R2s-R2c R2-R1-form R2-disj R2c-idem RH-design-peri-R1 RH-design-peri-R2s*)

lemma *RHS-design-lemma1*:
$RHS(P \vdash (R1(R2c(Q)) \vee R) \diamond S) = RHS(P \vdash (Q \vee R) \diamond S)$
by (*metis* (*no-types*, *lifting*) *R1-R2c-is-R2 R1-R2s-R2c R2-R1-form R2-disj R2c-idem RHS-design-peri-R1 RHS-design-peri-R2s*)

4.3 Pre-, peri-, and postconditions

4.3.1 Definitions

abbreviation $pre_s \equiv [\$ok \mapsto_s true, \$ok' \mapsto_s false, \$wait \mapsto_s false]$
abbreviation $cmt_s \equiv [\$ok \mapsto_s true, \$ok' \mapsto_s true, \$wait \mapsto_s false]$
abbreviation $peri_s \equiv [\$ok \mapsto_s true, \$ok' \mapsto_s true, \$wait \mapsto_s false, \$wait' \mapsto_s true]$
abbreviation $post_s \equiv [\$ok \mapsto_s true, \$ok' \mapsto_s true, \$wait \mapsto_s false, \$wait' \mapsto_s false]$

abbreviation $npre_R(P) \equiv pre_s \dagger P$

definition [*upred-defs*]: $pre_R(P) = (\neg_r\ npre_R(P))$
definition [*upred-defs*]: $cmt_R(P) = R1(cmt_s \dagger P)$
definition [*upred-defs*]: $peri_R(P) = R1(peri_s \dagger P)$
definition [*upred-defs*]: $post_R(P) = R1(post_s \dagger P)$

no-utp-lift $pre_R\ cmt_R\ peri_R\ post_R\ npre_R$

4.3.2 Unrestriction laws

lemma *ok-pre-unrest* [*unrest*]: $\$ok \sharp pre_R\ P$
by (*simp add*: pre_R*-def unrest usubst*)

lemma *ok-peri-unrest* [*unrest*]: $\$ok \sharp peri_R\ P$
by (*simp add*: $peri_R$*-def unrest usubst*)

lemma *ok-post-unrest* [*unrest*]: $\$ok \sharp post_R\ P$
by (*simp add*: $post_R$*-def unrest usubst*)

lemma *ok-cmt-unrest* [*unrest*]: $\$ok \sharp cmt_R\ P$

by (*simp add*: cmt_R-*def unrest usubst*)

lemma *ok′-pre-unrest* [*unrest*]: $\$ok' \sharp pre_R\ P$
by (*simp add*: pre_R-*def unrest usubst*)

lemma *ok′-peri-unrest* [*unrest*]: $\$ok' \sharp peri_R\ P$
by (*simp add*: $peri_R$-*def unrest usubst*)

lemma *ok′-post-unrest* [*unrest*]: $\$ok' \sharp post_R\ P$
by (*simp add*: $post_R$-*def unrest usubst*)

lemma *ok′-cmt-unrest* [*unrest*]: $\$ok' \sharp cmt_R\ P$
by (*simp add*: cmt_R-*def unrest usubst*)

lemma *wait-pre-unrest* [*unrest*]: $\$wait \sharp pre_R\ P$
by (*simp add*: pre_R-*def unrest usubst*)

lemma *wait-peri-unrest* [*unrest*]: $\$wait \sharp peri_R\ P$
by (*simp add*: $peri_R$-*def unrest usubst*)

lemma *wait-post-unrest* [*unrest*]: $\$wait \sharp post_R\ P$
by (*simp add*: $post_R$-*def unrest usubst*)

lemma *wait-cmt-unrest* [*unrest*]: $\$wait \sharp cmt_R\ P$
by (*simp add*: cmt_R-*def unrest usubst*)

lemma *wait′-peri-unrest* [*unrest*]: $\$wait' \sharp peri_R\ P$
by (*simp add*: $peri_R$-*def unrest usubst*)

lemma *wait′-post-unrest* [*unrest*]: $\$wait' \sharp post_R\ P$
by (*simp add*: $post_R$-*def unrest usubst*)

4.3.3 Substitution laws

lemma pre_s-*design*: $pre_s\ \dagger\ (P \vdash Q) = (\neg\ pre_s\ \dagger\ P)$
by (*simp add*: *design-def* pre_R-*def usubst*)

lemma $peri_s$-*design*: $peri_s\ \dagger\ (P \vdash Q \diamond R) = peri_s\ \dagger\ (P \Rightarrow Q)$
by (*simp add*: *design-def usubst wait′-cond-def*)

lemma $post_s$-*design*: $post_s\ \dagger\ (P \vdash Q \diamond R) = post_s\ \dagger\ (P \Rightarrow R)$
by (*simp add*: *design-def usubst wait′-cond-def*)

lemma cmt_s-*design*: $cmt_s\ \dagger\ (P \vdash Q) = cmt_s\ \dagger\ (P \Rightarrow Q)$
by (*simp add*: *design-def usubst wait′-cond-def*)

lemma pre_s-*R1* [*usubst*]: $pre_s\ \dagger\ R1(P) = R1(pre_s\ \dagger\ P)$
by (*simp add*: *R1-def usubst*)

lemma pre_s-*R2c* [*usubst*]: $pre_s\ \dagger\ R2c(P) = R2c(pre_s\ \dagger\ P)$
by (*simp add*: *R2c-def R2s-def usubst*)

lemma $peri_s$-*R1* [*usubst*]: $peri_s\ \dagger\ R1(P) = R1(peri_s\ \dagger\ P)$
by (*simp add*: *R1-def usubst*)

lemma $peri_s$-*R2c* [*usubst*]: $peri_s\ \dagger\ R2c(P) = R2c(peri_s\ \dagger\ P)$

by (*simp add*: *R2c-def R2s-def usubst*)

lemma $post_s$*-R1* [*usubst*]: $post_s \dagger R1(P) = R1(post_s \dagger P)$
by (*simp add*: *R1-def usubst*)

lemma $post_s$*-R2c* [*usubst*]: $post_s \dagger R2c(P) = R2c(post_s \dagger P)$
by (*simp add*: *R2c-def R2s-def usubst*)

lemma cmt_s*-R1* [*usubst*]: $cmt_s \dagger R1(P) = R1(cmt_s \dagger P)$
by (*simp add*: *R1-def usubst*)

lemma cmt_s*-R2c* [*usubst*]: $cmt_s \dagger R2c(P) = R2c(cmt_s \dagger P)$
by (*simp add*: *R2c-def R2s-def usubst*)

lemma *pre-wait-false*:
$pre_R(P[\![false/\$wait]\!]) = pre_R(P)$
by (*rel-auto*)

lemma *cmt-wait-false*:
$cmt_R(P[\![false/\$wait]\!]) = cmt_R(P)$
by (*rel-auto*)

lemma *rea-pre-RH-design*: $pre_R(\mathbf{R}(P \vdash Q)) = R1(R2c(pre_s \dagger P))$
by (*simp add*: *RH-def usubst R3c-def* pre_R*-def* pre_s*-design R1-negate-R1 R2c-not rea-not-def*)

lemma *rea-pre-RHS-design*: $pre_R(\mathbf{R}_s(P \vdash Q)) = R1(R2c(pre_s \dagger P))$
by (*simp add*: *RHS-def usubst R3h-def* pre_R*-def* pre_s*-design R1-negate-R1 R2c-not rea-not-def*)

lemma *rea-cmt-RH-design*: $cmt_R(\mathbf{R}(P \vdash Q)) = R1(R2c(cmt_s \dagger (P \Rightarrow Q)))$
by (*simp add*: *RH-def usubst R3c-def* cmt_R*-def* cmt_s*-design R1-idem*)

lemma *rea-cmt-RHS-design*: $cmt_R(\mathbf{R}_s(P \vdash Q)) = R1(R2c(cmt_s \dagger (P \Rightarrow Q)))$
by (*simp add*: *RHS-def usubst R3h-def* cmt_R*-def* cmt_s*-design R1-idem*)

lemma *rea-peri-RH-design*: $peri_R(\mathbf{R}(P \vdash Q \diamond R)) = R1(R2c(peri_s \dagger (P \Rightarrow_r Q)))$
by *rel-auto*

lemma *rea-peri-RHS-design*: $peri_R(\mathbf{R}_s(P \vdash Q \diamond R)) = R1(R2c(peri_s \dagger (P \Rightarrow_r Q)))$
by (*simp add*:*RHS-def usubst* $peri_R$*-def R3h-def* $peri_s$*-design*, *rel-auto*)

lemma *rea-post-RH-design*: $post_R(\mathbf{R}(P \vdash Q \diamond R)) = R1(R2c(post_s \dagger (P \Rightarrow_r R)))$
by *rel-auto*

lemma *rea-post-RHS-design*: $post_R(\mathbf{R}_s(P \vdash Q \diamond R)) = R1(R2c(post_s \dagger (P \Rightarrow_r R)))$
by (*simp add*:*RHS-def usubst* $post_R$*-def R3h-def* $post_s$*-design*, *rel-auto*)

lemma *peri-cmt-def*: $peri_R(P) = (cmt_R(P))[\![true/\$wait\,']\!]$
by (*rel-auto*)

lemma *post-cmt-def*: $post_R(P) = (cmt_R(P))[\![false/\$wait\,']\!]$
by (*rel-auto*)

lemma *rdes-export-cmt*: $\mathbf{R}_s(P \vdash cmt_s \dagger Q) = \mathbf{R}_s(P \vdash Q)$
by (*rel-auto*)

lemma *rdes-export-pre*: $\mathbf{R}_s((P[\![true,false/\$ok,\$wait]\!]) \vdash Q) = \mathbf{R}_s(P \vdash Q)$
by (*rel-auto*)

4.3.4 Healthiness laws

lemma *wait′-unrest-pre-SRD* [*unrest*]:
$\$wait' \sharp pre_R(P) \Longrightarrow \$wait' \sharp pre_R\ (SRD\ P)$
apply (*rel-auto*)
using *least-zero* **apply** *blast+*
done

lemma *R1-R2s-cmt-SRD*:
assumes *P is SRD*
shows $R1(R2s(cmt_R(P))) = cmt_R(P)$
by (*metis* (*no-types*, *lifting*) *R1-R2c-commute R1-R2s-R2c R1-idem R2c-idem SRD-reactive-design assms rea-cmt-RHS-design*)

lemma *R1-R2s-peri-SRD*:
assumes *P is SRD*
shows $R1(R2s(peri_R(P))) = peri_R(P)$
by (*metis* (*no-types*, *hide-lams*) *Healthy-def R1-R2s-R2c R2-def R2-idem RHS-def SRD-RH-design-form assms R1-idem* $peri_R$*-def* $peri_s$*-R1* $peri_s$*-R2c*)

lemma *R1-peri-SRD*:
assumes *P is SRD*
shows $R1(peri_R(P)) = peri_R(P)$
proof −
have $R1(peri_R(P)) = R1(R1(R2s(peri_R(P))))$
by (*simp add*: *R1-R2s-peri-SRD assms*)
also have ... $= peri_R(P)$
by (*simp add*: *R1-idem*, *simp add*: *R1-R2s-peri-SRD assms*)
finally show *?thesis* **.**
qed

lemma *R1-R2c-peri-RHS*:
assumes *P is SRD*
shows $R1(R2c(peri_R(P))) = peri_R(P)$
by (*metis R1-R2s-R2c R1-R2s-peri-SRD assms*)

lemma *R1-R2s-post-SRD*:
assumes *P is SRD*
shows $R1(R2s(post_R(P))) = post_R(P)$
by (*metis* (*no-types*, *hide-lams*) *Healthy-def R1-R2s-R2c R1-idem R2-def R2-idem RHS-def SRD-RH-design-form assms* $post_R$*-def* $post_s$*-R1* $post_s$*-R2c*)

lemma *R2c-peri-SRD*:
assumes *P is SRD*
shows $R2c(peri_R(P)) = peri_R(P)$
by (*metis R1-R2c-commute R1-R2c-peri-RHS R1-peri-SRD assms*)

lemma *R1-post-SRD*:
assumes *P is SRD*
shows $R1(post_R(P)) = post_R(P)$
proof −
have $R1(post_R(P)) = R1(R1(R2s(post_R(P))))$
by (*simp add*: *R1-R2s-post-SRD assms*)

also have ... = $post_R(P)$
by (*simp add*: *R1-idem*, *simp add*: *R1-R2s-post-SRD assms*)
finally show *?thesis* **.**
qed

lemma *R2c-post-SRD*:
assumes *P is SRD*
shows $R2c(post_R(P)) = post_R(P)$
by (*metis R1-R2c-commute R1-R2s-R2c R1-R2s-post-SRD R1-post-SRD assms*)

lemma *R1-R2c-post-RHS*:
assumes *P is SRD*
shows $R1(R2c(post_R(P))) = post_R(P)$
by (*metis R1-R2s-R2c R1-R2s-post-SRD assms*)

lemma *R2-cmt-conj-wait'*:
P *is SRD* $\Longrightarrow R2(cmt_R\ P \wedge \neg\ \$wait') = (cmt_R\ P \wedge \neg\ \$wait')$
by (*simp add*: *R2-def R2s-conj R2s-not R2s-wait' R1-extend-conj R1-R2s-cmt-SRD*)

lemma *R2c-preR*:
P *is SRD* $\Longrightarrow R2c(pre_R(P)) = pre_R(P)$
by (*metis* (*no-types*, *lifting*) *R1-R2c-commute R2c-idem SRD-reactive-design rea-pre-RHS-design*)

lemma *preR-R2-closed* [*closure*]:
assumes *P is R2*
shows $pre_R\ P$ *is R2*
proof −
have $R2(pre_R(R2(P))) = pre_R(R2(P))$
by (*rel-auto*)
thus *?thesis*
by (*metis Healthy-def assms*)
qed

lemma *periR-R2-closed* [*closure*]:
assumes *P is R2*
shows $peri_R\ P$ *is R2*
proof −
have $R2(peri_R(R2(P))) = peri_R(R2(P))$
by (*rel-auto*)
thus *?thesis*
by (*metis Healthy-def assms*)
qed

lemma *postR-R2-closed* [*closure*]:
assumes *P is R2*
shows $post_R\ P$ *is R2*
proof −
have $R2(post_R(R2(P))) = post_R(R2(P))$
by (*rel-auto*)
thus *?thesis*
by (*metis Healthy-def assms*)
qed

lemma *postR-SRD-R1* [*closure*]: P *is SRD* $\Longrightarrow post_R(P)$ *is R1*
by (*simp add*: *Healthy-def' R1-post-SRD*)

lemma *R2c-periR*:
P is SRD $\Longrightarrow$ $R2c(peri_R(P)) = peri_R(P)$
by (*metis* (*no-types*, *lifting*) *R1-R2c-commute R1-R2s-R2c R1-R2s-peri-SRD R2c-idem*)

lemma *R2c-postR*:
P is SRD $\Longrightarrow$ $R2c(post_R(P)) = post_R(P)$
by (*metis* (*no-types*, *hide-lams*) *R1-R2c-commute R1-R2c-is-R2 R1-R2s-post-SRD R2-def R2s-idem*)

lemma *periR-RR* [*closure*]: *P is R2* $\Longrightarrow$ $peri_R(P)$ *is RR*
by (*rule RR-intro*, *simp-all add*: *closure unrest*)

lemma *postR-RR* [*closure*]: *P is R2* $\Longrightarrow$ $post_R(P)$ *is RR*
by (*rule RR-intro*, *simp-all add*: *closure unrest*)

lemma *wpR-trace-ident-pre* [*wp*]:
$(\$tr' =_u \$tr \wedge \lceil II \rceil_R)\ wp_r\ pre_R\ P = pre_R\ P$
by (*rel-auto*)

lemma *R1-preR* [*closure*]:
$pre_R(P)$ *is R1*
by (*rel-auto*)

lemma *trace-ident-left-periR*:
$(\$tr' =_u \$tr \wedge \lceil II \rceil_R)\ ;;\ peri_R(P) = peri_R(P)$
by (*rel-auto*)

lemma *trace-ident-left-postR*:
$(\$tr' =_u \$tr \wedge \lceil II \rceil_R)\ ;;\ post_R(P) = post_R(P)$
by (*rel-auto*)

lemma *trace-ident-right-postR*:
$post_R(P)\ ;;\ (\$tr' =_u \$tr \wedge \lceil II \rceil_R) = post_R(P)$
by (*rel-auto*)

4.3.5 Calculation laws

lemma *wait'-cond-peri-post-cmt* [*rdes*]:
$cmt_R\ P = peri_R\ P \diamond post_R\ P$
by (*rel-auto*)

lemma *preR-rdes* [*rdes*]:
assumes *P is RR*
shows $pre_R(\mathbf{R}(P \vdash Q \diamond R)) = P$
by (*simp add*: *rea-pre-RH-design unrest usubst assms Healthy-if RR-implies-R2c RR-implies-R1*)

lemma *preR-srdes* [*rdes*]:
assumes *P is RR*
shows $pre_R(\mathbf{R}_s(P \vdash Q \diamond R)) = P$
by (*simp add*: *rea-pre-RHS-design unrest usubst assms Healthy-if RR-implies-R2c RR-implies-R1*)

lemma *periR-rdes* [*rdes*]:
assumes *P is RR Q is RR*
shows $peri_R(\mathbf{R}(P \vdash Q \diamond R)) = (P \Rightarrow_r Q)$
by (*simp add*: *rea-peri-RH-design unrest usubst assms Healthy-if RR-implies-R2c closure*)

lemma *periR-srdes* [*rdes*]:
assumes *P is RR Q is RR*
shows $peri_R(\mathbf{R}_s(P \vdash Q \diamond R)) = (P \Rightarrow_r Q)$
by (*simp add*: *rea-peri-RHS-design unrest usubst assms Healthy-if RR-implies-R2c closure*)

lemma *postR-rdes* [*rdes*]:
assumes *P is RR R is RR*
shows $post_R(\mathbf{R}(P \vdash Q \diamond R)) = (P \Rightarrow_r R)$
by (*simp add*: *rea-post-RH-design unrest usubst assms Healthy-if RR-implies-R2c closure*)

lemma *postR-srdes* [*rdes*]:
assumes *P is RR R is RR*
shows $post_R(\mathbf{R}_s(P \vdash Q \diamond R)) = (P \Rightarrow_r R)$
by (*simp add*: *rea-post-RHS-design unrest usubst assms Healthy-if RR-implies-R2c closure*)

lemma *preR-Chaos* [*rdes*]: $pre_R(Chaos) = false$
by (*simp add*: *Chaos-def*, *rel-simp*)

lemma *periR-Chaos* [*rdes*]: $peri_R(Chaos) = true_r$
by (*simp add*: *Chaos-def*, *rel-simp*)

lemma *postR-Chaos* [*rdes*]: $post_R(Chaos) = true_r$
by (*simp add*: *Chaos-def*, *rel-simp*)

lemma *preR-Miracle* [*rdes*]: $pre_R(Miracle) = true_r$
by (*simp add*: *Miracle-def*, *rel-auto*)

lemma *periR-Miracle* [*rdes*]: $peri_R(Miracle) = false$
by (*simp add*: *Miracle-def*, *rel-auto*)

lemma *postR-Miracle* [*rdes*]: $post_R(Miracle) = false$
by (*simp add*: *Miracle-def*, *rel-auto*)

lemma *preR-srdes-skip* [*rdes*]: $pre_R(II_R) = true_r$
by (*rel-auto*)

lemma *periR-srdes-skip* [*rdes*]: $peri_R(II_R) = false$
by (*rel-auto*)

lemma *postR-srdes-skip* [*rdes*]: $post_R(II_R) = (\$tr\,' =_u \$tr \wedge \lceil II \rceil_R)$
by (*rel-auto*)

lemma *preR-INF* [*rdes*]: $A \neq \{\} \Longrightarrow pre_R(\bigsqcap A) = (\bigwedge P{\in}A \cdot pre_R(P))$
by (*rel-auto*)

lemma *periR-INF* [*rdes*]: $peri_R(\bigsqcap A) = (\bigvee P{\in}A \cdot peri_R(P))$
by (*rel-auto*)

lemma *postR-INF* [*rdes*]: $post_R(\bigsqcap A) = (\bigvee P{\in}A \cdot post_R(P))$
by (*rel-auto*)

lemma *preR-UINF* [*rdes*]: $pre_R(\bigsqcap i \cdot P(i)) = (\bigsqcup i \cdot pre_R(P(i)))$
by (*rel-auto*)

lemma *periR-UINF* [*rdes*]: $peri_R(\bigsqcap i \cdot P(i)) = (\bigsqcap i \cdot peri_R(P(i)))$

by (*rel-auto*)

lemma *postR-UINF* [*rdes*]: $post_R(\bigsqcap\ i \cdot P(i)) = (\bigsqcap\ i \cdot post_R(P(i)))$
by (*rel-auto*)

lemma *preR-UINF-member* [*rdes*]: $A \neq \{\} \Longrightarrow pre_R(\bigsqcap\ i{\in}A \cdot P(i)) = (\bigsqcup\ i{\in}A \cdot pre_R(P(i)))$
by (*rel-auto*)

lemma *preR-UINF-member-2* [*rdes*]: $A \neq \{\} \Longrightarrow pre_R(\bigsqcap\ (i,j){\in}A \cdot P\ i\ j) = (\bigsqcup\ (i,j){\in}A \cdot pre_R(P\ i\ j))$
by (*rel-auto*)

lemma *preR-UINF-member-3* [*rdes*]: $A \neq \{\} \Longrightarrow pre_R(\bigsqcap\ (i,j,k){\in}A \cdot P\ i\ j\ k) = (\bigsqcup\ (i,j,k){\in}A \cdot pre_R(P\ i\ j\ k))$
by (*rel-auto*)

lemma *periR-UINF-member* [*rdes*]: $peri_R(\bigsqcap\ i{\in}A \cdot P(i)) = (\bigsqcap\ i{\in}A \cdot peri_R(P(i)))$
by (*rel-auto*)

lemma *periR-UINF-member-2* [*rdes*]: $peri_R(\bigsqcap\ (i,j){\in}A \cdot P\ i\ j) = (\bigsqcap\ (i,j){\in}A \cdot peri_R(P\ i\ j))$
by (*rel-auto*)

lemma *periR-UINF-member-3* [*rdes*]: $peri_R(\bigsqcap\ (i,j,k){\in}A \cdot P\ i\ j\ k) = (\bigsqcap\ (i,j,k){\in}A \cdot peri_R(P\ i\ j\ k))$
by (*rel-auto*)

lemma *postR-UINF-member* [*rdes*]: $post_R(\bigsqcap\ i{\in}A \cdot P(i)) = (\bigsqcap\ i{\in}A \cdot post_R(P(i)))$
by (*rel-auto*)

lemma *postR-UINF-member-2* [*rdes*]: $post_R(\bigsqcap\ (i,j){\in}A \cdot P\ i\ j) = (\bigsqcap\ (i,j){\in}A \cdot post_R(P\ i\ j))$
by (*rel-auto*)

lemma *postR-UINF-member-3* [*rdes*]: $post_R(\bigsqcap\ (i,j,k){\in}A \cdot P\ i\ j\ k) = (\bigsqcap\ (i,j,k){\in}A \cdot post_R(P\ i\ j\ k))$
by (*rel-auto*)

lemma *preR-inf* [*rdes*]: $pre_R(P \sqcap Q) = (pre_R(P) \wedge pre_R(Q))$
by (*rel-auto*)

lemma *periR-inf* [*rdes*]: $peri_R(P \sqcap Q) = (peri_R(P) \vee peri_R(Q))$
by (*rel-auto*)

lemma *postR-inf* [*rdes*]: $post_R(P \sqcap Q) = (post_R(P) \vee post_R(Q))$
by (*rel-auto*)

lemma *preR-SUP* [*rdes*]: $pre_R(\bigsqcup\ A) = (\bigvee\ P{\in}A \cdot pre_R(P))$
by (*rel-auto*)

lemma *periR-SUP* [*rdes*]: $A \neq \{\} \Longrightarrow peri_R(\bigsqcup\ A) = (\bigwedge\ P{\in}A \cdot peri_R(P))$
by (*rel-auto*)

lemma *postR-SUP* [*rdes*]: $A \neq \{\} \Longrightarrow post_R(\bigsqcup\ A) = (\bigwedge\ P{\in}A \cdot post_R(P))$
by (*rel-auto*)

4.4 Formation laws

4.4.1 Regular

lemma *rdes-skip-tri-design* [*rdes-def*]: $II_C = \mathbf{R}(true_r \vdash false \diamond II_r)$

apply (*simp add*: *skip-rea-def*, *rel-auto*)
using *minus-zero-eq* **apply** *blast+*
done

lemma *RH-tri-design-form*:
assumes P_1 *is RR* P_2 *is RR* P_3 *is RR*
shows $\mathbf{R}(P_1 \vdash P_2 \diamond P_3) = (II_C \lhd \$wait \rhd ((\$ok \wedge P_1) \Rightarrow_r (\$ok' \wedge (P_2 \diamond P_3))))$
proof −
have $\mathbf{R}(RR(P_1) \vdash RR(P_2) \diamond RR(P_3)) = (II_C \lhd \$wait \rhd ((\$ok \wedge RR(P_1)) \Rightarrow_r (\$ok' \wedge (RR(P_2) \diamond RR(P_3)))))$
apply (*rel-auto*) **using** *minus-zero-eq* **by** *blast*
thus *?thesis*
by (*simp add*: *Healthy-if assms*)
qed

lemma *RH-design-pre-post-form*:
$\mathbf{R}((\neg\ {P^f}_f) \vdash {P^t}_f) = \mathbf{R}(pre_R(P) \vdash cmt_R(P))$
proof −
have $\mathbf{R}((\neg\ {P^f}_f) \vdash {P^t}_f) = \mathbf{R}((\neg\ {P^f}_f)[\![true/\$ok]\!] \vdash {P^t}_f[\![true/\$ok]\!])$
by (*simp add*: *design-subst-ok*)
also have ... $= \mathbf{R}(pre_R(P) \vdash cmt_R(P))$
by (*simp add*: pre_R*-def* cmt_R*-def usubst*, *rel-auto*)
finally show *?thesis* **.**
qed

lemma *RD-as-reactive-design*:
$RD(P) = \mathbf{R}(pre_R(P) \vdash cmt_R(P))$
by (*simp add*: *RH-design-pre-post-form RD-RH-design-form*)

lemma *RD-reactive-design-alt*:
assumes P *is RD*
shows $\mathbf{R}(pre_R(P) \vdash cmt_R(P)) = P$
proof −
have $\mathbf{R}(pre_R(P) \vdash cmt_R(P)) = \mathbf{R}((\neg\ {P^f}_f) \vdash {P^t}_f)$
by (*simp add*: *RH-design-pre-post-form*)
thus *?thesis*
by (*simp add*: *RD-reactive-design assms*)
qed

lemma *RD-reactive-tri-design-lemma*:
$RD(P) = \mathbf{R}((\neg\ {P^f}_f) \vdash {P^t}_f[\![true/\$wait']\!] \diamond {P^t}_f[\![false/\$wait']\!])$
by (*simp add*: *RD-RH-design-form wait'-cond-split*)

lemma *RD-as-reactive-tri-design*:
$RD(P) = \mathbf{R}(pre_R(P) \vdash peri_R(P) \diamond post_R(P))$
proof −
have $RD(P) = \mathbf{R}((\neg\ {P^f}_f) \vdash {P^t}_f[\![true/\$wait']\!] \diamond {P^t}_f[\![false/\$wait']\!])$
by (*simp add*: *RD-RH-design-form wait'-cond-split*)
also have ... $= \mathbf{R}(pre_R(P) \vdash peri_R(P) \diamond post_R(P))$
by (*rel-auto*)
finally show *?thesis* **.**
qed

lemma *RD-reactive-tri-design*:
assumes P *is RD*

shows $\mathbf{R}(pre_R(P) \vdash peri_R(P) \diamond post_R(P)) = P$
by (*metis Healthy-if RD-as-reactive-tri-design assms*)

lemma *RD-elimination* [*RD-elim*]: $[\![P \text{ is } RD;\ Q(\mathbf{R}(pre_R(P) \vdash peri_R(P) \diamond post_R(P)))]\!] \Longrightarrow Q(P)$
by (*simp add*: *RD-reactive-tri-design*)

lemma *RH-tri-design-is-RD* [*closure*]:
assumes $\$ok' \sharp P$ $\$ok' \sharp Q$ $\$ok' \sharp R$
shows $\mathbf{R}(P \vdash Q \diamond R)$ *is RD*
by (*rule RH-design-is-RD*, *simp-all add*: *unrest assms*)

lemma *RD-rdes-intro* [*closure*]:
assumes *P is RR Q is RR R is RR*
shows $\mathbf{R}(P \vdash Q \diamond R)$ *is RD*
by (*rule RH-tri-design-is-RD*, *simp-all add*: *unrest closure assms*)

4.4.2 Stateful

lemma *srdes-skip-tri-design* [*rdes-def*]: $II_R = \mathbf{R}_s(true_r \vdash false \diamond II_r)$
by (*simp add*: *srdes-skip-def*, *rel-auto*)

lemma *Chaos-tri-def* [*rdes-def*]: $Chaos = \mathbf{R}_s(false \vdash false \diamond false)$
by (*simp add*: *Chaos-def design-false-pre*)

lemma *Miracle-tri-def* [*rdes-def*]: $Miracle = \mathbf{R}_s(true_r \vdash false \diamond false)$
by (*simp add*: *Miracle-def R1-design-R1-pre wait'-cond-idem*)

lemma *RHS-tri-design-form*:
assumes *P₁ is RR P₂ is RR P₃ is RR*
shows $\mathbf{R}_s(P_1 \vdash P_2 \diamond P_3) = (II_R \lhd \$wait \rhd ((\$ok \wedge P_1) \Rightarrow_r (\$ok' \wedge (P_2 \diamond P_3))))$
proof −
have $\mathbf{R}_s(RR(P_1) \vdash RR(P_2) \diamond RR(P_3)) = (II_R \lhd \$wait \rhd ((\$ok \wedge RR(P_1)) \Rightarrow_r (\$ok' \wedge (RR(P_2) \diamond RR(P_3)))))$
apply (*rel-auto*) **using** *minus-zero-eq* **by** *blast*
thus *?thesis*
by (*simp add*: *Healthy-if assms*)
qed

lemma *RHS-design-pre-post-form*:
$\mathbf{R}_s((\neg\ {P^f}_f) \vdash {P^t}_f) = \mathbf{R}_s(pre_R(P) \vdash cmt_R(P))$
proof −
have $\mathbf{R}_s((\neg\ {P^f}_f) \vdash {P^t}_f) = \mathbf{R}_s((\neg\ {P^f}_f)[\![true/\$ok]\!] \vdash {P^t}_f[\![true/\$ok]\!])$
by (*simp add*: *design-subst-ok*)
also have ... $= \mathbf{R}_s(pre_R(P) \vdash cmt_R(P))$
by (*simp add*: pre_R*-def* cmt_R*-def usubst*, *rel-auto*)
finally show *?thesis* **.**
qed

lemma *SRD-as-reactive-design*:
$SRD(P) = \mathbf{R}_s(pre_R(P) \vdash cmt_R(P))$
by (*simp add*: *RHS-design-pre-post-form SRD-RH-design-form*)

lemma *SRD-reactive-design-alt*:
assumes *P is SRD*
shows $\mathbf{R}_s(pre_R(P) \vdash cmt_R(P)) = P$
proof −

have $\mathbf{R}_s(pre_R(P) \vdash cmt_R(P)) = \mathbf{R}_s((\neg\ {P^f}_f) \vdash {P^t}_f)$
by (*simp add*: *RHS-design-pre-post-form*)
thus *?thesis*
by (*simp add*: *SRD-reactive-design assms*)
qed

lemma *SRD-reactive-tri-design-lemma*:
$SRD(P) = \mathbf{R}_s((\neg\ {P^f}_f) \vdash {P^t}_f[\![true/\$wait\acute{}]\!] \diamond {P^t}_f[\![false/\$wait\acute{}]\!])$
by (*simp add*: *SRD-RH-design-form wait'-cond-split*)

lemma *SRD-as-reactive-tri-design*:
$SRD(P) = \mathbf{R}_s(pre_R(P) \vdash peri_R(P) \diamond post_R(P))$
proof −
have $SRD(P) = \mathbf{R}_s((\neg\ {P^f}_f) \vdash {P^t}_f[\![true/\$wait\acute{}]\!] \diamond {P^t}_f[\![false/\$wait\acute{}]\!])$
by (*simp add*: *SRD-RH-design-form wait'-cond-split*)
also have ... $= \mathbf{R}_s(pre_R(P) \vdash peri_R(P) \diamond post_R(P))$
apply (*simp add*: *usubst*)
apply (*subst design-subst-ok-ok'*[*THEN sym*])
apply (*simp add*: pre_R*-def* $peri_R$*-def* $post_R$*-def usubst unrest*)
apply (*rel-auto*)
done
finally show *?thesis* **.**
qed

lemma *SRD-reactive-tri-design*:
assumes *P is SRD*
shows $\mathbf{R}_s(pre_R(P) \vdash peri_R(P) \diamond post_R(P)) = P$
by (*metis Healthy-if SRD-as-reactive-tri-design assms*)

lemma *SRD-elim* [*RD-elim*]: $[\![\ P \text{ is } SRD;\ Q(\mathbf{R}_s(pre_R(P) \vdash peri_R(P) \diamond post_R(P)))\ ]\!] \Longrightarrow Q(P)$
by (*simp add*: *SRD-reactive-tri-design*)

lemma *RHS-tri-design-is-SRD* [*closure*]:
assumes $\$ok\acute{}\ \sharp\ P\ \$ok\acute{}\ \sharp\ Q\ \$ok\acute{}\ \sharp\ R$
shows $\mathbf{R}_s(P \vdash Q \diamond R)$ *is SRD*
by (*rule RHS-design-is-SRD*, *simp-all add*: *unrest assms*)

lemma *SRD-rdes-intro* [*closure*]:
assumes *P is RR Q is RR R is RR*
shows $\mathbf{R}_s(P \vdash Q \diamond R)$ *is SRD*
by (*rule RHS-tri-design-is-SRD*, *simp-all add*: *unrest closure assms*)

lemma *USUP-R1-R2s-cmt-SRD*:
assumes $A \subseteq [\![SRD]\!]_H$
shows $(\bigsqcup\ P \in A \cdot R1\ (R2s\ (cmt_R\ P))) = (\bigsqcup\ P \in A \cdot cmt_R\ P)$
by (*rule USUP-cong*[*of A*], *metis* (*mono-tags*, *lifting*) *Ball-Collect R1-R2s-cmt-SRD assms*)

lemma *UINF-R1-R2s-cmt-SRD*:
assumes $A \subseteq [\![SRD]\!]_H$
shows $(\bigsqcap\ P \in A \cdot R1\ (R2s\ (cmt_R\ P))) = (\bigsqcap\ P \in A \cdot cmt_R\ P)$
by (*rule UINF-cong*[*of A*], *metis* (*mono-tags*, *lifting*) *Ball-Collect R1-R2s-cmt-SRD assms*)

4.4.3 Order laws

lemma *preR-antitone*: $P \sqsubseteq Q \Longrightarrow pre_R(Q) \sqsubseteq pre_R(P)$
by (*rel-auto*)

lemma *periR-monotone*: $P \sqsubseteq Q \Longrightarrow peri_R(P) \sqsubseteq peri_R(Q)$
by (*rel-auto*)

lemma *postR-monotone*: $P \sqsubseteq Q \Longrightarrow post_R(P) \sqsubseteq post_R(Q)$
by (*rel-auto*)

4.5 Composition laws

theorem *R1-design-composition-RR*:
assumes *P is RR Q is RR R is RR S is RR*
shows
$(R1(P \vdash Q) \mathbin{;;} R1(R \vdash S)) = R1(((\neg_r\ P)\ wp_r\ false \wedge Q\ wp_r\ R) \vdash (Q \mathbin{;;} S))$
apply (*subst R1-design-composition*)
apply (*simp-all add*: *assms unrest wp-rea-def Healthy-if closure*)
apply (*rel-auto*)
done

theorem *R1-design-composition-RC*:
assumes *P is RC Q is RR R is RR S is RR*
shows
$(R1(P \vdash Q) \mathbin{;;} R1(R \vdash S)) = R1((P \wedge Q\ wp_r\ R) \vdash (Q \mathbin{;;} S))$
by (*simp add*: *R1-design-composition-RR assms unrest Healthy-if closure wp*)

4.5.1 Regular

theorem *RH-tri-design-composition*:
assumes $\$ok' \sharp P\ \$ok' \sharp Q_1\ \$ok' \sharp Q_2\ \$ok \sharp R\ \$ok \sharp S_1\ \$ok \sharp S_2$
$\$wait \sharp R\ \$wait' \sharp Q_2\ \$wait \sharp S_1\ \$wait \sharp S_2$
shows $(\mathbf{R}(P \vdash Q_1 \diamond Q_2) \mathbin{;;} \mathbf{R}(R \vdash S_1 \diamond S_2)) =$
$\mathbf{R}((\neg\ (R1\ (\neg\ R2s\ P) \mathbin{;;} R1\ true) \wedge \neg\ (R1(R2s\ Q_2) \mathbin{;;} R1\ (\neg\ R2s\ R))) \vdash$
$((Q_1 \vee (R1\ (R2s\ Q_2) \mathbin{;;} R1\ (R2s\ S_1))) \diamond ((R1\ (R2s\ Q_2) \mathbin{;;} R1\ (R2s\ S_2)))))$
proof −
have *1*:$(\neg\ ((R1\ (R2s\ (Q_1 \diamond Q_2)) \wedge \neg\ \$wait') \mathbin{;;} R1\ (\neg\ R2s\ R))) =$
$(\neg\ ((R1\ (R2s\ Q_2) \wedge \neg\ \$wait') \mathbin{;;} R1\ (\neg\ R2s\ R)))$
by (*metis* (*no-types*, *hide-lams*) *R1-extend-conj R2s-conj R2s-not R2s-wait' wait'-cond-false*)
have *2*: $(R1\ (R2s\ (Q_1 \diamond Q_2)) \mathbin{;;} (\lceil II \rceil_D \triangleleft \$wait \triangleright R1\ (R2s\ (S_1 \diamond S_2)))) =$
$(((R1\ (R2s\ Q_1)) \vee (R1\ (R2s\ Q_2) \mathbin{;;} R1\ (R2s\ S_1))) \diamond (R1\ (R2s\ Q_2) \mathbin{;;} R1\ (R2s\ S_2)))$
proof −
have $(R1\ (R2s\ Q_1) \mathbin{;;} (\$wait \wedge (\lceil II \rceil_D \triangleleft \$wait \triangleright R1\ (R2s\ S_1) \diamond R1\ (R2s\ S_2))))$
$= (((R1\ (R2s\ Q_1)) \wedge \$wait'))$
proof −
have $(R1\ (R2s\ Q_1) \mathbin{;;} (\$wait \wedge ((\lceil II \rceil_D) \triangleleft \$wait \triangleright R1\ (R2s\ S_1) \diamond R1\ (R2s\ S_2))))$
$= (R1\ (R2s\ Q_1) \mathbin{;;} (\$wait \wedge (\lceil II \rceil_D)))$
by (*rel-auto*)
also have ... $= ((R1\ (R2s\ Q_1) \mathbin{;;} \lceil II \rceil_D) \wedge \$wait')$
by (*rel-auto*)
also from *assms*(*2*) **have** ... $= ((R1\ (R2s\ Q_1)) \wedge \$wait')$
by (*rel-auto*, *blast*)
finally show *?thesis* **.**
qed

moreover have $(R1\ (R2s\ Q_2) \mathbin{;;} (\neg\ \$wait \wedge ((\lceil II \rceil_D) \triangleleft \$wait \triangleright R1\ (R2s\ S_1) \diamond R1\ (R2s\ S_2))))$
$= ((R1\ (R2s\ Q_2)) \mathbin{;;} (R1\ (R2s\ S_1) \diamond R1\ (R2s\ S_2)))$
proof −
have $(R1\ (R2s\ Q_2) \mathbin{;;} (\neg\ \$wait \wedge (\lceil II \rceil_D \triangleleft \$wait \triangleright R1\ (R2s\ S_1) \diamond R1\ (R2s\ S_2))))$

$= (R1\ (R2s\ Q_2)\ ;;\ (\neg\ \$wait \wedge (R1\ (R2s\ S_1) \diamond R1\ (R2s\ S_2))))$
by (*metis* (*no-types*, *lifting*) *cond-def conj-disj-not-abs utp-pred-laws.double-compl utp-pred-laws.inf.left-idem utp-pred-laws.sup-assoc utp-pred-laws.sup-inf-absorb*)

also have ... $= ((R1\ (R2s\ Q_2))[\![false/\$wait´]\!]\ ;;\ (R1\ (R2s\ S_1) \diamond R1\ (R2s\ S_2))[\![false/\$wait]\!])$
by (*metis false-alt-def seqr-right-one-point upred-eq-false wait-vwb-lens*)

also have ... $= ((R1\ (R2s\ Q_2))\ ;;\ (R1\ (R2s\ S_1) \diamond R1\ (R2s\ S_2)))$
by (*simp add*: *wait'-cond-def usubst unrest assms*)

finally show *?thesis* **.**
qed

moreover
have $((R1\ (R2s\ Q_1) \wedge \$wait´) \vee ((R1\ (R2s\ Q_2))\ ;;\ (R1\ (R2s\ S_1) \diamond R1\ (R2s\ S_2))))$
$= (R1\ (R2s\ Q_1) \vee (R1\ (R2s\ Q_2)\ ;;\ R1\ (R2s\ S_1))) \diamond ((R1\ (R2s\ Q_2)\ ;;\ R1\ (R2s\ S_2)))$
by (*simp add*: *wait'-cond-def cond-seq-right-distr cond-and-T-integrate unrest*)

ultimately show *?thesis*
by (*simp add*: *R2s-wait'-cond R1-wait'-cond wait'-cond-seq ex-conj-contr-right unrest*)
qed

from *assms*(*7*,*8*) **have** *3*: $(R1\ (R2s\ Q_2) \wedge \neg\ \$wait´)\ ;;\ R1\ (\neg\ R2s\ R) = R1\ (R2s\ Q_2)\ ;;\ R1\ (\neg\ R2s\ R)$
by (*rel-auto*, *meson*)

show *?thesis*
by (*simp add*: *RH-design-composition unrest assms 1 2 3*, *simp add*: *R1-R2s-R2c RH-design-lemma1*)
qed

theorem *RH-tri-design-composition-wp*:
assumes $\$ok´ \sharp P\ \$ok´ \sharp Q_1\ \$ok´ \sharp Q_2\ \$ok \sharp R\ \$ok \sharp S_1\ \$ok \sharp S_2$
$\$wait \sharp R\ \$wait´ \sharp Q_2\ \$wait \sharp S_1\ \$wait \sharp S_2$
P is R2c Q_1 *is R1* Q_1 *is R2c* Q_2 *is R1* Q_2 *is R2c*
R is R2c S_1 *is R1* S_1 *is R2c* S_2 *is R1* S_2 *is R2c*
shows $\mathbf{R}(P \vdash Q_1 \diamond Q_2)\ ;;\ \mathbf{R}(R \vdash S_1 \diamond S_2) =$
$\mathbf{R}(((\neg_r\ P)\ wp_r\ false \wedge Q_2\ wp_r\ R) \vdash ((Q_1 \sqcap (Q_2\ ;;\ S_1)) \diamond (Q_2\ ;;\ S_2)))$ (**is** *?lhs* = *?rhs*)
proof −
have *?lhs* $= \mathbf{R}\ ((\neg\ R1\ (\neg\ P)\ ;;\ R1\ true \wedge \neg\ Q_2\ ;;\ R1\ (\neg\ R)) \vdash (Q_1 \sqcap (Q_2\ ;;\ S_1)) \diamond (Q_2\ ;;\ S_2))$
by (*simp add*: *RH-tri-design-composition assms Healthy-if R2c-healthy-R2s disj-upred-def*)
(*metis* (*no-types*, *hide-lams*) *R1-negate-R1 R2c-healthy-R2s assms*(*11*,*16*))
also have ... = *?rhs*
by (*rel-auto*)
finally show *?thesis* **.**
qed

theorem *RH-tri-design-composition-RR-wp*:
assumes *P is RR* Q_1 *is RR* Q_2 *is RR*
R is RR S_1 *is RR* S_2 *is RR*
shows $\mathbf{R}(P \vdash Q_1 \diamond Q_2)\ ;;\ \mathbf{R}(R \vdash S_1 \diamond S_2) =$
$\mathbf{R}(((\neg_r\ P)\ wp_r\ false \wedge Q_2\ wp_r\ R) \vdash ((Q_1 \sqcap (Q_2\ ;;\ S_1)) \diamond (Q_2\ ;;\ S_2)))$ (**is** *?lhs* = *?rhs*)
by (*simp add*: *RH-tri-design-composition-wp add*: *closure assms unrest RR-implies-R2c*)

lemma *RH-tri-normal-design-composition*:
assumes

$ok´ \sharp P$ $ok´ \sharp Q_1$ $ok´ \sharp Q_2$ $ok \sharp R$ $ok \sharp S_1$ $ok \sharp S_2$
$wait \sharp R$ $wait´ \sharp Q_2$ $wait \sharp S_1$ $wait \sharp S_2$
P *is R2c* Q_1 *is R1* Q_1 *is R2c* Q_2 *is R1* Q_2 *is R2c*
R *is R2c* S_1 *is R1* S_1 *is R2c* S_2 *is R1* S_2 *is R2c*
R1 $(\neg P)$;; *R1*(*true*) = *R1*$(\neg P)$
shows $\mathbf{R}(P \vdash Q_1 \diamond Q_2)$;; $\mathbf{R}(R \vdash S_1 \diamond S_2)$
$= \mathbf{R}((P \wedge Q_2\ wp_r\ R) \vdash (Q_1 \vee (Q_2 ;; S_1)) \diamond (Q_2 ;; S_2))$
proof −
have $\mathbf{R}(P \vdash Q_1 \diamond Q_2)$;; $\mathbf{R}(R \vdash S_1 \diamond S_2) =$
$\mathbf{R}((R1\ (\neg P)\ wp_r\ false \wedge Q_2\ wp_r\ R) \vdash (Q_1 \sqcap (Q_2 ;; S_1)) \diamond (Q_2 ;; S_2))$
by (*simp-all add*: *RH-tri-design-composition-wp rea-not-def assms unrest*)
also have ... = $\mathbf{R}((P \wedge Q_2\ wp_r\ R) \vdash (Q_1 \vee (Q_2 ;; S_1)) \diamond (Q_2 ;; S_2))$
by (*simp add*: *assms wp-rea-def ex-unrest*, *rel-auto*)
finally show *?thesis* **.**
qed

lemma *RH-tri-normal-design-composition'* [*rdes-def*]:
assumes P *is RC* Q_1 *is RR* Q_2 *is RR* R *is RR* S_1 *is RR* S_2 *is RR*
shows $\mathbf{R}(P \vdash Q_1 \diamond Q_2)$;; $\mathbf{R}(R \vdash S_1 \diamond S_2)$
$= \mathbf{R}((P \wedge Q_2\ wp_r\ R) \vdash (Q_1 \vee (Q_2 ;; S_1)) \diamond (Q_2 ;; S_2))$
proof −
have *R1* $(\neg P)$;; *R1 true* = *R1*$(\neg P)$
using *RC-implies-RC1*[*OF assms*(*1*)]
by (*simp add*: *Healthy-def RC1-def rea-not-def*)
(*metis R1-negate-R1 R1-seqr utp-pred-laws.double-compl*)
thus *?thesis*
by (*simp add*: *RH-tri-normal-design-composition assms closure unrest RR-implies-R2c*)
qed

lemma *RH-tri-design-right-unit-lemma*:
assumes $ok´ \sharp P$ $ok´ \sharp Q$ $ok´ \sharp R$ $wait´ \sharp R$
shows $\mathbf{R}(P \vdash Q \diamond R)$;; $II_C = \mathbf{R}((\neg_r (\neg_r P) ;; true_r) \vdash (Q \diamond R))$
proof −
have $\mathbf{R}(P \vdash Q \diamond R)$;; $II_C = \mathbf{R}(P \vdash Q \diamond R)$;; $\mathbf{R}(true \vdash false \diamond (\$tr´ =_u \$tr \wedge \lceil II \rceil_R))$
by (*simp add*: *rdes-skip-tri-design*, *rel-auto*)
also have ... = $\mathbf{R}$ $((\neg\ R1\ (\neg\ R2s\ P) ;; R1\ true) \vdash Q \diamond (R1\ (R2s\ R) ;; R1\ (R2s\ (\$tr´ =_u \$tr \wedge \lceil II \rceil_R))))$
by (*simp-all add*: *RH-tri-design-composition assms unrest R2s-true R1-false R2s-false*)
also have ... = $\mathbf{R}$ $((\neg\ R1\ (\neg\ R2s\ P) ;; R1\ true) \vdash Q \diamond R1\ (R2s\ R))$
proof −
from *assms*(*3*,*4*) **have** $(R1\ (R2s\ R) ;; R1\ (R2s\ (\$tr´ =_u \$tr \wedge \lceil II \rceil_R))) = R1\ (R2s\ R)$
by (*rel-auto*, *metis* (*no-types*, *lifting*) *minus-zero-eq*, *meson order-refl trace-class.diff-cancel*)
thus *?thesis*
by *simp*
qed
also have ... = $\mathbf{R}((\neg\ (\neg\ P) ;; R1\ true) \vdash (Q \diamond R))$
by (*metis* (*no-types*, *lifting*) *R1-R2s-R1-true-lemma R1-R2s-R2c R2c-not RH-design-R2c-pre RH-design-neg-R1-pre RH-design-post-R1 RH-design-post-R2s*)
also have ... = $\mathbf{R}((\neg_r (\neg_r P) ;; true_r) \vdash Q \diamond R)$
by (*rel-auto*)
finally show *?thesis* **.**
qed

4.5.2 Stateful

theorem *RHS-tri-design-composition*:

assumes $ok´ ♯ P $ok´ ♯ Q_1 $ok´ ♯ Q_2 $ok ♯ R $ok ♯ S_1 $ok ♯ S_2
$wait ♯ R $wait´ ♯ Q_2 $wait ♯ S_1 $wait ♯ S_2
shows (**R**_s(P ⊢ Q_1 ⋄ Q_2) ;; **R**_s(R ⊢ S_1 ⋄ S_2)) =
R_s((¬ (R1 (¬ R2s P) ;; R1 true) ∧ ¬ (R1(R2s Q_2) ;; R1 (¬ R2s R))) ⊢
(((∃ $st´ · Q_1) ∨ (R1 (R2s Q_2) ;; R1 (R2s S_1))) ⋄ ((R1 (R2s Q_2) ;; R1 (R2s S_2)))))
proof −
have *1*:(¬ ((R1 (R2s (Q_1 ⋄ Q_2)) ∧ ¬ $wait´) ;; R1 (¬ R2s R))) =
(¬ ((R1 (R2s Q_2) ∧ ¬ $wait´) ;; R1 (¬ R2s R)))
by (*metis* (*no-types*, *hide-lams*) *R1-extend-conj R2s-conj R2s-not R2s-wait' wait'-cond-false*)
have *2*: (R1 (R2s (Q_1 ⋄ Q_2)) ;; ((∃ $st · ⌈II⌉_D) ◁ $wait ▷ R1 (R2s (S_1 ⋄ S_2)))) =
(((∃ $st´ · R1 (R2s Q_1)) ∨ (R1 (R2s Q_2) ;; R1 (R2s S_1))) ⋄ (R1 (R2s Q_2) ;; R1 (R2s S_2)))
proof −
have (R1 (R2s Q_1) ;; ($wait ∧ ((∃ $st · ⌈II⌉_D) ◁ $wait ▷ R1 (R2s S_1) ⋄ R1 (R2s S_2))))
= (∃ $st´ · ((R1 (R2s Q_1)) ∧ $wait´))
proof −
have (R1 (R2s Q_1) ;; ($wait ∧ ((∃ $st · ⌈II⌉_D) ◁ $wait ▷ R1 (R2s S_1) ⋄ R1 (R2s S_2))))
= (R1 (R2s Q_1) ;; ($wait ∧ (∃ $st · ⌈II⌉_D)))
by (*rel-auto*, *blast+*)
also have ... = ((R1 (R2s Q_1) ;; (∃ $st · ⌈II⌉_D)) ∧ $wait´)
by (*rel-auto*)
also from *assms(2)* **have** ... = (∃ $st´ · ((R1 (R2s Q_1)) ∧ $wait´))
by (*rel-auto*, *blast*)
finally show *?thesis* **.**
qed

moreover have (R1 (R2s Q_2) ;; (¬ $wait ∧ ((∃ $st · ⌈II⌉_D) ◁ $wait ▷ R1 (R2s S_1) ⋄ R1 (R2s S_2))))
= ((R1 (R2s Q_2)) ;; (R1 (R2s S_1) ⋄ R1 (R2s S_2)))
proof −
have (R1 (R2s Q_2) ;; (¬ $wait ∧ ((∃ $st · ⌈II⌉_D) ◁ $wait ▷ R1 (R2s S_1) ⋄ R1 (R2s S_2))))
= (R1 (R2s Q_2) ;; (¬ $wait ∧ (R1 (R2s S_1) ⋄ R1 (R2s S_2))))
by (*metis* (*no-types*, *lifting*) *cond-def conj-disj-not-abs utp-pred-laws.double-compl utp-pred-laws.inf.left-idem utp-pred-laws.sup-assoc utp-pred-laws.sup-inf-absorb*)

also have ... = ((R1 (R2s Q_2))⟦*false*/$wait´⟧ ;; (R1 (R2s S_1) ⋄ R1 (R2s S_2))⟦*false*/$wait⟧)
by (*metis false-alt-def seqr-right-one-point upred-eq-false wait-vwb-lens*)

also have ... = ((R1 (R2s Q_2)) ;; (R1 (R2s S_1) ⋄ R1 (R2s S_2)))
by (*simp add*: *wait'-cond-def usubst unrest assms*)

finally show *?thesis* **.**
qed

moreover
have ((R1 (R2s Q_1) ∧ $wait´) ∨ ((R1 (R2s Q_2)) ;; (R1 (R2s S_1) ⋄ R1 (R2s S_2))))
= (R1 (R2s Q_1) ∨ (R1 (R2s Q_2) ;; R1 (R2s S_1))) ⋄ ((R1 (R2s Q_2) ;; R1 (R2s S_2)))
by (*simp add*: *wait'-cond-def cond-seq-right-distr cond-and-T-integrate unrest*)

ultimately show *?thesis*
by (*simp add*: *R2s-wait'-cond R1-wait'-cond wait'-cond-seq ex-conj-contr-right unrest*)
(*simp add*: *cond-and-T-integrate cond-seq-right-distr unrest-var wait'-cond-def*)
qed

from *assms(7,8)* **have** *3*: (R1 (R2s Q_2) ∧ ¬ $wait´) ;; R1 (¬ R2s R) = R1 (R2s Q_2) ;; R1 (¬ R2s

R)
by (*rel-auto*, *blast*, *meson*)

show *?thesis*
apply (*subst RHS-design-composition*)
apply (*simp-all add*: *assms*)
apply (*simp add*: *assms wait′-cond-def unrest*)
apply (*simp add*: *assms wait′-cond-def unrest*)
apply (*simp add*: *1 2 3*)
apply (*simp add*: *R1-R2s-R2c RHS-design-lemma1*)
apply (*metis R1-R2c-ex-st RHS-design-lemma1*)
done
qed

theorem *RHS-tri-design-composition-wp*:
assumes $ok´ ♯ P $ok´ ♯ Q_1 $ok´ ♯ Q_2 $ok ♯ R $ok ♯ S_1 $ok ♯ S_2
$wait ♯ R $wait´ ♯ Q_2 $wait ♯ S_1 $wait ♯ S_2
P is R2c Q_1 *is R1* Q_1 *is R2c* Q_2 *is R1* Q_2 *is R2c*
R is R2c S_1 *is R1* S_1 *is R2c* S_2 *is R1* S_2 *is R2c*
shows $\mathbf{R}_s(P \vdash Q_1 \diamond Q_2)$;; $\mathbf{R}_s(R \vdash S_1 \diamond S_2) =$
$\mathbf{R}_s(((\neg_r\ P)\ wp_r\ false \wedge Q_2\ wp_r\ R) \vdash (((\exists\ \$st´ \cdot Q_1) \sqcap (Q_2 ;; S_1)) \diamond (Q_2 ;; S_2)))$ (**is** *?lhs* = *?rhs*)
proof −
have *?lhs* = $\mathbf{R}_s$ $((\neg\ R1\ (\neg\ P) ;; R1\ true \wedge \neg\ Q_2 ;; R1\ (\neg\ R)) \vdash ((\exists\ \$st´ \cdot Q_1) \sqcap (Q_2 ;; S_1)) \diamond (Q_2 ;; S_2))$
by (*simp add*: *RHS-tri-design-composition assms Healthy-if R2c-healthy-R2s disj-upred-def*)
(*metis* (*no-types*, *hide-lams*) *R1-negate-R1 R2c-healthy-R2s assms*(*11*,*16*))
also have ... = *?rhs*
by (*rel-auto*)
finally show *?thesis* **.**
qed

theorem *RHS-tri-design-composition-RR-wp*:
assumes *P is RR* Q_1 *is RR* Q_2 *is RR*
R is RR S_1 *is RR* S_2 *is RR*
shows $\mathbf{R}_s(P \vdash Q_1 \diamond Q_2)$;; $\mathbf{R}_s(R \vdash S_1 \diamond S_2) =$
$\mathbf{R}_s(((\neg_r\ P)\ wp_r\ false \wedge Q_2\ wp_r\ R) \vdash (((\exists\ \$st´ \cdot Q_1) \sqcap (Q_2 ;; S_1)) \diamond (Q_2 ;; S_2)))$ (**is** *?lhs* = *?rhs*)
by (*simp add*: *RHS-tri-design-composition-wp add*: *closure assms unrest RR-implies-R2c*)

lemma *RHS-tri-normal-design-composition*:
assumes
$ok´ ♯ P $ok´ ♯ Q_1 $ok´ ♯ Q_2 $ok ♯ R $ok ♯ S_1 $ok ♯ S_2
$wait ♯ R $wait´ ♯ Q_2 $wait ♯ S_1 $wait ♯ S_2
P is R2c Q_1 *is R1* Q_1 *is R2c* Q_2 *is R1* Q_2 *is R2c*
R is R2c S_1 *is R1* S_1 *is R2c* S_2 *is R1* S_2 *is R2c*
R1 $(\neg\ P) ;; R1(true) = R1(\neg\ P)$ $st´ ♯ Q_1
shows $\mathbf{R}_s(P \vdash Q_1 \diamond Q_2)$;; $\mathbf{R}_s(R \vdash S_1 \diamond S_2)$
$= \mathbf{R}_s((P \wedge Q_2\ wp_r\ R) \vdash (Q_1 \vee (Q_2 ;; S_1)) \diamond (Q_2 ;; S_2))$
proof −
have $\mathbf{R}_s(P \vdash Q_1 \diamond Q_2)$;; $\mathbf{R}_s(R \vdash S_1 \diamond S_2) =$
$\mathbf{R}_s$ $((R1\ (\neg\ P)\ wp_r\ false \wedge Q_2\ wp_r\ R) \vdash ((\exists\ \$st´ \cdot Q_1) \sqcap (Q_2 ;; S_1)) \diamond (Q_2 ;; S_2))$
by (*simp-all add*: *RHS-tri-design-composition-wp rea-not-def assms unrest*)
also have ... = $\mathbf{R}_s((P \wedge Q_2\ wp_r\ R) \vdash (Q_1 \vee (Q_2 ;; S_1)) \diamond (Q_2 ;; S_2))$
by (*simp add*: *assms wp-rea-def ex-unrest*, *rel-auto*)

finally show *?thesis* **.**
qed

lemma *RHS-tri-normal-design-composition'* [*rdes-def*]:
assumes *P is RC* Q_1 *is RR* $\$st' \sharp Q_1$ Q_2 *is RR R is RR* S_1 *is RR* S_2 *is RR*
shows $\mathbf{R}_s(P \vdash Q_1 \diamond Q_2) ;; \mathbf{R}_s(R \vdash S_1 \diamond S_2)$
$= \mathbf{R}_s((P \wedge Q_2\ wp_r\ R) \vdash (Q_1 \vee (Q_2 ;; S_1)) \diamond (Q_2 ;; S_2))$
proof −
have *R1* $(\neg P)$;; *R1 true* = *R1*$(\neg P)$
using *RC-implies-RC1*[*OF assms*(*1*)]
by (*simp add*: *Healthy-def RC1-def rea-not-def*)
(*metis R1-negate-R1 R1-seqr utp-pred-laws.double-compl*)
thus *?thesis*
by (*simp add*: *RHS-tri-normal-design-composition assms closure unrest RR-implies-R2c*)
qed

lemma *RHS-tri-design-right-unit-lemma*:
assumes $\$ok' \sharp P$ $\$ok' \sharp Q$ $\$ok' \sharp R$ $\$wait' \sharp R$
shows $\mathbf{R}_s(P \vdash Q \diamond R) ;; II_R = \mathbf{R}_s((\neg_r (\neg_r P) ;; true_r) \vdash ((\exists\ \$st' \cdot Q) \diamond R))$
proof −
have $\mathbf{R}_s(P \vdash Q \diamond R) ;; II_R = \mathbf{R}_s(P \vdash Q \diamond R) ;; \mathbf{R}_s(true \vdash false \diamond (\$tr' =_u \$tr \wedge \lceil II \rceil_R))$
by (*simp add*: *srdes-skip-tri-design*, *rel-auto*)
also have ... $= \mathbf{R}_s ((\neg\ R1\ (\neg\ R2s\ P) ;; R1\ true) \vdash (\exists\ \$st' \cdot Q) \diamond (R1\ (R2s\ R) ;; R1\ (R2s\ (\$tr' =_u$
$\$tr \wedge \lceil II \rceil_R))))$
by (*simp-all add*: *RHS-tri-design-composition assms unrest R2s-true R1-false R2s-false*)
also have ... $= \mathbf{R}_s ((\neg\ R1\ (\neg\ R2s\ P) ;; R1\ true) \vdash (\exists\ \$st' \cdot Q) \diamond R1\ (R2s\ R))$
proof −
from *assms*(*3*,*4*) **have** $(R1\ (R2s\ R) ;; R1\ (R2s\ (\$tr' =_u \$tr \wedge \lceil II \rceil_R))) = R1\ (R2s\ R)$
by (*rel-auto*, *metis* (*no-types*, *lifting*) *minus-zero-eq*, *meson order-refl trace-class.diff-cancel*)
thus *?thesis*
by *simp*
qed
also have ... $= \mathbf{R}_s((\neg\ (\neg\ P) ;; R1\ true) \vdash ((\exists\ \$st' \cdot Q) \diamond R))$
by (*metis* (*no-types*, *lifting*) *R1-R2s-R1-true-lemma R1-R2s-R2c R2c-not RHS-design-R2c-pre RHS-design-neg-R1-pre*
RHS-design-post-R1 RHS-design-post-R2s)
also have ... $= \mathbf{R}_s((\neg_r (\neg_r P) ;; true_r) \vdash ((\exists\ \$st' \cdot Q) \diamond R))$
by (*rel-auto*)
finally show *?thesis* **.**
qed

lemma *RD-composition-wp*:
assumes *P is RD Q is RD*
shows $(P ;; Q) = \mathbf{R}\ (((\neg_r\ pre_R\ P)\ wp_r\ false \wedge post_R\ P\ wp_r\ pre_R\ Q) \vdash$
$(peri_R\ P \vee (post_R\ P ;; peri_R\ Q)) \diamond (post_R\ P ;; post_R\ Q))$
(**is** *?lhs* = *?rhs*)
proof −
have $(P ;; Q) = (\mathbf{R}(pre_R(P) \vdash peri_R(P) \diamond post_R(P)) ;; \mathbf{R}(pre_R(Q) \vdash peri_R(Q) \diamond post_R(Q)))$
by (*simp add*: *RD-reactive-tri-design assms*(*1*) *assms*(*2*))
also from *assms*
have ... = *?rhs*
by (*simp add*: *RH-tri-design-composition-wp unrest closure disj-upred-def*)
finally show *?thesis* **.**
qed

lemma *SRD-composition-wp*:

assumes *P is SRD Q is SRD*
shows $(P \;;;\; Q) = \mathbf{R}_s\ (((\neg_r\ pre_R\ P)\ wp_r\ false \wedge post_R\ P\ wp_r\ pre_R\ Q) \vdash$
$((\exists\ \$st' \cdot peri_R\ P) \vee (post_R\ P \;;;\; peri_R\ Q)) \diamond (post_R\ P \;;;\; post_R\ Q))$
(**is** *?lhs = ?rhs*)
proof −
have $(P \;;;\; Q) = (\mathbf{R}_s(pre_R(P) \vdash peri_R(P) \diamond post_R(P)) \;;;\; \mathbf{R}_s(pre_R(Q) \vdash peri_R(Q) \diamond post_R(Q)))$
by (*simp add*: *SRD-reactive-tri-design assms*(*1*) *assms*(*2*))
also from *assms*
have ... = *?rhs*
by (*simp add*: *RHS-tri-design-composition-wp disj-upred-def unrest assms closure*)
finally show *?thesis* **.**
qed

4.6 Refinement introduction laws

4.6.1 Regular

lemma *RH-tri-design-refine*:
assumes P_1 *is RR* P_2 *is RR* P_3 *is RR* Q_1 *is RR* Q_2 *is RR* Q_3 *is RR*
shows $\mathbf{R}(P_1 \vdash P_2 \diamond P_3) \sqsubseteq \mathbf{R}(Q_1 \vdash Q_2 \diamond Q_3) \longleftrightarrow$ ‘$P_1 \Rightarrow Q_1$‘ ∧ ‘$P_1 \wedge Q_2 \Rightarrow P_2$‘ ∧ ‘$P_1 \wedge Q_3 \Rightarrow P_3$‘
(**is** *?lhs = ?rhs*)
proof −
have *?lhs* $\longleftrightarrow$ ‘$P_1 \Rightarrow Q_1$‘ ∧ ‘$P_1 \wedge Q_2 \diamond Q_3 \Rightarrow P_2 \diamond P_3$‘
by (*simp add*: *RH-design-refine assms closure RR-implies-R2c unrest ex-unrest*)
also have ... $\longleftrightarrow$ ‘$P_1 \Rightarrow Q_1$‘ ∧ ‘$(P_1 \wedge Q_2) \diamond (P_1 \wedge Q_3) \Rightarrow P_2 \diamond P_3$‘
by (*rel-auto*)
also have ... $\longleftrightarrow$ ‘$P_1 \Rightarrow Q_1$‘ ∧ ‘$((P_1 \wedge Q_2) \diamond (P_1 \wedge Q_3) \Rightarrow P_2 \diamond P_3)[\![true/\$wait']\!]$‘ ∧ ‘$((P_1 \wedge Q_2) \diamond (P_1 \wedge Q_3) \Rightarrow P_2 \diamond P_3)[\![false/\$wait']\!]$‘
by (*rel-auto*, *metis*)
also have ... $\longleftrightarrow$ *?rhs*
by (*simp add*: *usubst unrest assms*)
finally show *?thesis* **.**
qed

lemma *RH-tri-design-refine′*:
assumes P_1 *is RR* P_2 *is RR* P_3 *is RR* Q_1 *is RR* Q_2 *is RR* Q_3 *is RR*
shows $\mathbf{R}(P_1 \vdash P_2 \diamond P_3) \sqsubseteq \mathbf{R}(Q_1 \vdash Q_2 \diamond Q_3) \longleftrightarrow (Q_1 \sqsubseteq P_1) \wedge (P_2 \sqsubseteq (P_1 \wedge Q_2)) \wedge (P_3 \sqsubseteq (P_1 \wedge Q_3))$
by (*simp add*: *RH-tri-design-refine assms*, *rel-auto*)

lemma *rdes-tri-refine-intro*:
assumes ‘$P_1 \Rightarrow P_2$‘ ‘$P_1 \wedge Q_2 \Rightarrow Q_1$‘ ‘$P_1 \wedge R_2 \Rightarrow R_1$‘
shows $\mathbf{R}(P_1 \vdash Q_1 \diamond R_1) \sqsubseteq \mathbf{R}(P_2 \vdash Q_2 \diamond R_2)$
using *assms*
by (*rule-tac rdes-refine-intro*, *simp-all*, *rel-auto*)

lemma *rdes-tri-refine-intro′*:
assumes $P_2 \sqsubseteq P_1$ $Q_1 \sqsubseteq (P_1 \wedge Q_2)$ $R_1 \sqsubseteq (P_1 \wedge R_2)$
shows $\mathbf{R}(P_1 \vdash Q_1 \diamond R_1) \sqsubseteq \mathbf{R}(P_2 \vdash Q_2 \diamond R_2)$
using *assms*
by (*rule-tac rdes-tri-refine-intro*, *simp-all add*: *refBy-order*)

4.6.2 Stateful

lemma *RHS-tri-design-refine*:

assumes P_1 *is RR* P_2 *is RR* P_3 *is RR* Q_1 *is RR* Q_2 *is RR* Q_3 *is RR*
shows $\mathbf{R}_s(P_1 \vdash P_2 \diamond P_3) \sqsubseteq \mathbf{R}_s(Q_1 \vdash Q_2 \diamond Q_3) \longleftrightarrow$ ‘$P_1 \Rightarrow Q_1$‘ $\wedge$ ‘$P_1 \wedge Q_2 \Rightarrow P_2$‘ $\wedge$ ‘$P_1 \wedge Q_3 \Rightarrow P_3$‘
(**is** *?lhs* = *?rhs*)
proof −
have *?lhs* $\longleftrightarrow$ ‘$P_1 \Rightarrow Q_1$‘ $\wedge$ ‘$P_1 \wedge Q_2 \diamond Q_3 \Rightarrow P_2 \diamond P_3$‘
by (*simp add*: *RHS-design-refine assms closure RR-implies-R2c unrest ex-unrest*)
also have ... $\longleftrightarrow$ ‘$P_1 \Rightarrow Q_1$‘ $\wedge$ ‘$(P_1 \wedge Q_2) \diamond (P_1 \wedge Q_3) \Rightarrow P_2 \diamond P_3$‘
by (*rel-auto*)
also have ... $\longleftrightarrow$ ‘$P_1 \Rightarrow Q_1$‘ $\wedge$ ‘$((P_1 \wedge Q_2) \diamond (P_1 \wedge Q_3) \Rightarrow P_2 \diamond P_3)[\![true/\$wait´]\!]$‘ $\wedge$ ‘$((P_1 \wedge Q_2) \diamond (P_1 \wedge Q_3) \Rightarrow P_2 \diamond P_3)[\![false/\$wait´]\!]$‘
by (*rel-auto*, *metis*)
also have ... $\longleftrightarrow$ *?rhs*
by (*simp add*: *usubst unrest assms*)
finally show *?thesis* **.**
qed

lemma *RHS-tri-design-refine′*:
assumes P_1 *is RR* P_2 *is RR* P_3 *is RR* Q_1 *is RR* Q_2 *is RR* Q_3 *is RR*
shows $\mathbf{R}_s(P_1 \vdash P_2 \diamond P_3) \sqsubseteq \mathbf{R}_s(Q_1 \vdash Q_2 \diamond Q_3) \longleftrightarrow (Q_1 \sqsubseteq P_1) \wedge (P_2 \sqsubseteq (P_1 \wedge Q_2)) \wedge (P_3 \sqsubseteq (P_1 \wedge Q_3))$
by (*simp add*: *RHS-tri-design-refine assms*, *rel-auto*)

lemma *srdes-tri-refine-intro*:
assumes ‘$P_1 \Rightarrow P_2$‘ ‘$P_1 \wedge Q_2 \Rightarrow Q_1$‘ ‘$P_1 \wedge R_2 \Rightarrow R_1$‘
shows $\mathbf{R}_s(P_1 \vdash Q_1 \diamond R_1) \sqsubseteq \mathbf{R}_s(P_2 \vdash Q_2 \diamond R_2)$
using *assms*
by (*rule-tac srdes-refine-intro*, *simp-all*, *rel-auto*)

lemma *srdes-tri-refine-intro′*:
assumes $P_2 \sqsubseteq P_1$ $Q_1 \sqsubseteq (P_1 \wedge Q_2)$ $R_1 \sqsubseteq (P_1 \wedge R_2)$
shows $\mathbf{R}_s(P_1 \vdash Q_1 \diamond R_1) \sqsubseteq \mathbf{R}_s(P_2 \vdash Q_2 \diamond R_2)$
using *assms*
by (*rule-tac srdes-tri-refine-intro*, *simp-all add*: *refBy-order*)

lemma *SRD-peri-under-pre*:
assumes *P is SRD* $\$wait´ \sharp pre_R(P)$
shows $(pre_R(P) \Rightarrow_r peri_R(P)) = peri_R(P)$
proof −
have $peri_R(P) =$
$peri_R(\mathbf{R}_s(pre_R(P) \vdash peri_R(P) \diamond post_R(P)))$
by (*simp add*: *SRD-reactive-tri-design assms*)
also have ... $= (pre_R\ P \Rightarrow_r peri_R\ P)$
by (*simp add*: *rea-pre-RHS-design rea-peri-RHS-design assms*
unrest usubst R1-peri-SRD R2c-preR R1-rea-impl R2c-rea-impl R2c-periR)
finally show *?thesis* **..**
qed

lemma *SRD-post-under-pre*:
assumes *P is SRD* $\$wait´ \sharp pre_R(P)$
shows $(pre_R(P) \Rightarrow_r post_R(P)) = post_R(P)$
proof −
have $post_R(P) =$
$post_R(\mathbf{R}_s(pre_R(P) \vdash peri_R(P) \diamond post_R(P)))$
by (*simp add*: *SRD-reactive-tri-design assms*)

also have ... = ($pre_R\ P \Rightarrow_r post_R\ P$)
by (*simp add*: *rea-pre-RHS-design rea-post-RHS-design assms unrest usubst R1-post-SRD R2c-preR R1-rea-impl R2c-rea-impl R2c-postR*)
finally show *?thesis* **..**
qed

lemma *SRD-refine-intro*:
assumes
P is SRD Q is SRD
‘$pre_R(P) \Rightarrow pre_R(Q)$‘ ‘$pre_R(P) \wedge peri_R(Q) \Rightarrow peri_R(P)$‘ ‘$pre_R(P) \wedge post_R(Q) \Rightarrow post_R(P)$‘
shows $P \sqsubseteq Q$
by (*metis SRD-reactive-tri-design assms*(*1*) *assms*(*2*) *assms*(*3*) *assms*(*4*) *assms*(*5*) *srdes-tri-refine-intro*)

lemma *SRD-refine-intro′*:
assumes
P is SRD Q is SRD
‘$pre_R(P) \Rightarrow pre_R(Q)$‘ $peri_R(P) \sqsubseteq (pre_R(P) \wedge peri_R(Q))$ $post_R(P) \sqsubseteq (pre_R(P) \wedge post_R(Q))$
shows $P \sqsubseteq Q$
using *assms* **by** (*rule-tac SRD-refine-intro*, *simp-all add*: *refBy-order*)

lemma *SRD-eq-intro*:
assumes
P is SRD Q is SRD $pre_R(P) = pre_R(Q)$ $peri_R(P) = peri_R(Q)$ $post_R(P) = post_R(Q)$
shows $P = Q$
by (*metis SRD-reactive-tri-design assms*)

lemma *srdes-tri-eq-iff*:
assumes P_1 *is RR* P_2 *is RR* P_3 *is RR* Q_1 *is RR* Q_2 *is RR* Q_3 *is RR*
shows $\mathbf{R}_s(P_1 \vdash P_2 \diamond P_3) = \mathbf{R}_s(Q_1 \vdash Q_2 \diamond Q_3) \longleftrightarrow (P_1 = Q_1 \wedge (P_1 \wedge Q_2) = (Q_1 \wedge P_2) \wedge (P_1 \wedge Q_3) = (Q_1 \wedge P_3))$
proof −
have $\mathbf{R}_s(P_1 \vdash P_2 \diamond P_3) = \mathbf{R}_s(Q_1 \vdash Q_2 \diamond Q_3) \longleftrightarrow$
$(\mathbf{R}_s(P_1 \vdash P_2 \diamond P_3) \sqsubseteq \mathbf{R}_s(Q_1 \vdash Q_2 \diamond Q_3) \wedge \mathbf{R}_s(Q_1 \vdash Q_2 \diamond Q_3) \sqsubseteq \mathbf{R}_s(P_1 \vdash P_2 \diamond P_3))$
by *fastforce*
also have ... = $(Q_1 \sqsubseteq P_1 \wedge P_2 \sqsubseteq (P_1 \wedge Q_2) \wedge P_3 \sqsubseteq (P_1 \wedge Q_3) \wedge P_1 \sqsubseteq Q_1 \wedge Q_2 \sqsubseteq (Q_1 \wedge P_2) \wedge Q_3 \sqsubseteq (Q_1 \wedge P_3))$
by (*simp add*: *RHS-tri-design-refine′ assms*)
also have ... = $(P_1 = Q_1 \wedge P_2 \sqsubseteq (P_1 \wedge Q_2) \wedge P_3 \sqsubseteq (P_1 \wedge Q_3) \wedge Q_2 \sqsubseteq (Q_1 \wedge P_2) \wedge Q_3 \sqsubseteq (Q_1 \wedge P_3))$
by *fastforce*
also have ... = $(P_1 = Q_1 \wedge (P_1 \wedge Q_2) = (Q_1 \wedge P_2) \wedge (P_1 \wedge Q_3) = (Q_1 \wedge P_3))$
apply (*safe*, *simp-all*)
apply (*meson eq-iff utp-pred-laws.inf-greatest utp-pred-laws.inf-le1*)+
apply (*metis utp-pred-laws.inf-le2*)+
done
finally show *?thesis* **.**
qed

lemma *rdes-tri-eq-intro*:
assumes $P_1 = Q_1$ $(P_1 \wedge Q_2) = (Q_1 \wedge P_2)$ $(P_1 \wedge Q_3) = (Q_1 \wedge P_3)$
shows $\mathbf{R}(P_1 \vdash P_2 \diamond P_3) = \mathbf{R}(Q_1 \vdash Q_2 \diamond Q_3)$
by (*metis* (*no-types*, *hide-lams*) *assms*(*1*) *assms*(*2*) *assms*(*3*) *design-export-pre wait′-cond-conj-exchange wait′-cond-idem*)

lemma *srdes-tri-eq-intro*:

assumes $P_1 = Q_1$ $(P_1 \wedge Q_2) = (Q_1 \wedge P_2)$ $(P_1 \wedge Q_3) = (Q_1 \wedge P_3)$
shows $\mathbf{R}_s(P_1 \vdash P_2 \diamond P_3) = \mathbf{R}_s(Q_1 \vdash Q_2 \diamond Q_3)$
by (*metis* (*no-types*, *hide-lams*) *assms*(*1*) *assms*(*2*) *assms*(*3*) *design-export-pre wait'-cond-conj-exchange wait'-cond-idem*)

lemma *rdes-tri-eq-intro'*:
assumes $P_1 = Q_1$ $P_2 = Q_2$ $P_3 = Q_3$
shows $\mathbf{R}(P_1 \vdash P_2 \diamond P_3) = \mathbf{R}(Q_1 \vdash Q_2 \diamond Q_3)$
using *assms* **by** (*simp*)

lemma *srdes-tri-eq-intro'*:
assumes $P_1 = Q_1$ $P_2 = Q_2$ $P_3 = Q_3$
shows $\mathbf{R}_s(P_1 \vdash P_2 \diamond P_3) = \mathbf{R}_s(Q_1 \vdash Q_2 \diamond Q_3)$
using *assms* **by** (*simp*)

4.7 Closure laws

4.7.1 Regular

lemma *RD-srdes-skip* [*closure*]: II_C *is RD*
by (*simp add*: *rdes-skip-def RH-design-is-RD unrest*)

lemma *RD-seqr-closure* [*closure*]:
assumes *P is RD Q is RD*
shows (*P* ;; *Q*) *is RD*
proof −
have $(P ;; Q) = \mathbf{R}\ (((\neg_r\ pre_R\ P)\ wp_r\ false \wedge post_R\ P\ wp_r\ pre_R\ Q) \vdash$
$(peri_R\ P \vee (post_R\ P ;; peri_R\ Q)) \diamond (post_R\ P ;; post_R\ Q))$
by (*simp add*: *RD-composition-wp assms*(*1*) *assms*(*2*))
also have *... is RD*
by (*rule RH-design-is-RD*, *simp-all add*: *wp-rea-def unrest*)
finally show *?thesis* **.**
qed

lemma *RD-power-Suc* [*closure*]: *P is RD* $\Longrightarrow$ *P^(Suc n) is RD*
proof (*induct n*)
case *0*
then show *?case*
by (*simp*)
next
case (*Suc n*)
then show *?case*
using *RD-seqr-closure* **by** (*simp add*: *RD-seqr-closure upred-semiring.power-Suc*)
qed

lemma *RD-power-comp* [*closure*]: *P is RD* $\Longrightarrow$ *P ;; P^n is RD*
by (*metis RD-power-Suc upred-semiring.power-Suc*)

lemma *uplus-RD-closed* [*closure*]: *P is RD* $\Longrightarrow$ P^+ *is RD*
by (*simp add*: *uplus-power-def closure*)

4.7.2 Stateful

lemma *SRD-srdes-skip* [*closure*]: II_R *is SRD*
by (*simp add*: *srdes-skip-def RHS-design-is-SRD unrest*)

lemma *SRD-seqr-closure* [*closure*]:
assumes *P is SRD Q is SRD*
shows (*P* ;; *Q*) *is SRD*
proof −
have (*P* ;; *Q*) = $\mathbf{R}_s$ (((¬$_r$ *pre*$_R$ *P*) *wp*$_r$ *false* ∧ *post*$_R$ *P wp*$_r$ *pre*$_R$ *Q*) ⊢
((∃ $*st*′ · *peri*$_R$ *P*) ∨ (*post*$_R$ *P* ;; *peri*$_R$ *Q*)) ⋄ (*post*$_R$ *P* ;; *post*$_R$ *Q*))
by (*simp add*: *SRD-composition-wp assms*(*1*) *assms*(*2*))
also have ... *is SRD*
by (*rule RHS-design-is-SRD*, *simp-all add*: *wp-rea-def unrest*)
finally show *?thesis* **.**
qed

lemma *SRD-power-Suc* [*closure*]: *P is SRD* ⟹ *P*^(*Suc n*) *is SRD*
proof (*induct n*)
case *0*
then show *?case*
by (*simp*)
next
case (*Suc n*)
then show *?case*
using *SRD-seqr-closure* **by** (*simp add*: *SRD-seqr-closure upred-semiring.power-Suc*)
qed

lemma *SRD-power-comp* [*closure*]: *P is SRD* ⟹ *P* ;; *P*^*n is SRD*
by (*metis SRD-power-Suc upred-semiring.power-Suc*)

lemma *uplus-SRD-closed* [*closure*]: *P is SRD* ⟹ *P*$^+$ *is SRD*
by (*simp add*: *uplus-power-def closure*)

lemma *SRD-Sup-closure* [*closure*]:
assumes *A* ⊆ ⟦*SRD*⟧$_H$ *A* ≠ {}
shows (⨅ *A*) *is SRD*
proof −
have *SRD* (⨅ *A*) = (⨅ (*SRD* ‘*A*))
by (*simp add*: *ContinuousD SRD-Continuous assms*(*2*))
also have ... = (⨅ *A*)
by (*simp only*: *Healthy-carrier-image assms*)
finally show *?thesis* **by** (*simp add*: *Healthy-def*)
qed

4.8 Distribution laws

lemma *RHS-tri-design-choice* [*rdes-def*]:
$\mathbf{R}_s(P_1 \vdash P_2 \diamond P_3) \sqcap \mathbf{R}_s(Q_1 \vdash Q_2 \diamond Q_3) = \mathbf{R}_s((P_1 \wedge Q_1) \vdash (P_2 \vee Q_2) \diamond (P_3 \vee Q_3))$
apply (*simp add*: *RHS-design-choice*)
apply (*rule cong*[*of* $\mathbf{R}_s$ $\mathbf{R}_s$])
apply (*simp*)
apply (*rel-auto*)
done

lemma *RHS-tri-design-disj* [*rdes-def*]:
$(\mathbf{R}_s(P_1 \vdash P_2 \diamond P_3) \vee \mathbf{R}_s(Q_1 \vdash Q_2 \diamond Q_3)) = \mathbf{R}_s((P_1 \wedge Q_1) \vdash (P_2 \vee Q_2) \diamond (P_3 \vee Q_3))$
by (*simp add*: *RHS-tri-design-choice disj-upred-def*)

lemma *RHS-tri-design-sup* [*rdes-def*]:
$\mathbf{R}_s(P_1 \vdash P_2 \diamond P_3) \sqcup \mathbf{R}_s(Q_1 \vdash Q_2 \diamond Q_3) = \mathbf{R}_s((P_1 \vee Q_1) \vdash ((P_1 \Rightarrow_r P_2) \wedge (Q_1 \Rightarrow_r Q_2)) \diamond ((P_1$

$\Rightarrow_r$ P_3) $\wedge$ (Q_1 $\Rightarrow_r$ Q_3)))
by (*simp add*: *RHS-design-sup*, *rel-auto*)

lemma *RHS-tri-design-conj* [*rdes-def*]:
($\mathbf{R}_s(P_1 \vdash P_2 \diamond P_3) \wedge \mathbf{R}_s(Q_1 \vdash Q_2 \diamond Q_3)) = \mathbf{R}_s((P_1 \vee Q_1) \vdash ((P_1 \Rightarrow_r P_2) \wedge (Q_1 \Rightarrow_r Q_2)) \diamond ((P_1$ $\Rightarrow_r$ P_3) $\wedge$ (Q_1 $\Rightarrow_r$ Q_3)))
by (*simp add*: *RHS-tri-design-sup conj-upred-def*)

lemma *SRD-UINF* [*rdes-def*]:
assumes $A \neq \{\}$ $A \subseteq [\![SRD]\!]_H$
shows $\bigsqcap A = \mathbf{R}_s((\bigwedge P{\in}A \cdot pre_R(P)) \vdash (\bigvee P{\in}A \cdot peri_R(P)) \diamond (\bigvee P{\in}A \cdot post_R(P)))$
proof −
have $\bigsqcap A = \mathbf{R}_s(pre_R(\bigsqcap A) \vdash peri_R(\bigsqcap A) \diamond post_R(\bigsqcap A))$
by (*metis SRD-as-reactive-tri-design assms srdes-theory.healthy-inf srdes-theory.healthy-inf-def*)
also have ... $= \mathbf{R}_s((\bigwedge P{\in}A \cdot pre_R(P)) \vdash (\bigvee P{\in}A \cdot peri_R(P)) \diamond (\bigvee P{\in}A \cdot post_R(P)))$
by (*simp add*: *preR-INF periR-INF postR-INF assms*)
finally show *?thesis* **.**
qed

lemma *RHS-tri-design-USUP* [*rdes-def*]:
assumes $A \neq \{\}$
shows ($\bigsqcap i \in A \cdot \mathbf{R}_s(P(i) \vdash Q(i) \diamond R(i))) = \mathbf{R}_s((\bigsqcup i \in A \cdot P(i)) \vdash (\bigsqcap i \in A \cdot Q(i)) \diamond (\bigsqcap i \in A$ $\cdot$ $R(i)))$
by (*subst RHS-INF*[*OF assms*, *THEN sym*], *simp add*: *design-UINF-mem assms*, *rel-auto*)

lemma *SRD-UINF-mem*:
assumes $A \neq \{\}$ $\bigwedge i.$ $P\ i$ *is SRD*
shows ($\bigsqcap i{\in}A \cdot P\ i) = \mathbf{R}_s((\bigwedge i{\in}A \cdot pre_R(P\ i)) \vdash (\bigvee i{\in}A \cdot peri_R(P\ i)) \diamond (\bigvee i{\in}A \cdot post_R(P\ i)))$
(**is** *?lhs* = *?rhs*)
proof −
have *?lhs* = ($\bigsqcap$ (P ‘ A))
by (*rel-auto*)
also have ... = $\mathbf{R}_s$ (($\bigsqcup Pa \in P$ ‘ $A \cdot pre_R\ Pa) \vdash (\bigsqcap Pa \in P$ ‘ $A \cdot peri_R\ Pa) \diamond (\bigsqcap Pa \in P$ ‘ $A \cdot$ $post_R\ Pa))$
by (*subst rdes-def*, *simp-all add*: *assms image-subsetI*)
also have ... = *?rhs*
by (*rel-auto*)
finally show *?thesis* **.**
qed

lemma *RHS-tri-design-UINF-ind* [*rdes-def*]:
($\bigsqcap i \cdot \mathbf{R}_s(P_1(i) \vdash P_2(i) \diamond P_3(i))) = \mathbf{R}_s((\bigwedge i \cdot P_1\ i) \vdash (\bigvee i \cdot P_2(i)) \diamond (\bigvee i \cdot P_3(i)))$
by (*rel-auto*)

lemma *cond-srea-form* [*rdes-def*]:
$\mathbf{R}_s(P \vdash Q_1 \diamond Q_2) \lhd b \rhd_R \mathbf{R}_s(R \vdash S_1 \diamond S_2) =$
$\mathbf{R}_s((P \lhd b \rhd_R R) \vdash (Q_1 \lhd b \rhd_R S_1) \diamond (Q_2 \lhd b \rhd_R S_2))$
proof −
have $\mathbf{R}_s(P \vdash Q_1 \diamond Q_2) \lhd b \rhd_R \mathbf{R}_s(R \vdash S_1 \diamond S_2) = \mathbf{R}_s(P \vdash Q_1 \diamond Q_2) \lhd R2c(\lceil b \rceil_{S<}) \rhd \mathbf{R}_s(R \vdash S_1 \diamond$ $S_2)$
by (*pred-auto*)
also have ... $= \mathbf{R}_s$ ($P \vdash Q_1 \diamond Q_2 \lhd b \rhd_R R \vdash S_1 \diamond S_2$)
by (*simp add*: *RHS-cond lift-cond-srea-def*)
also have ... $= \mathbf{R}_s$ (($P \lhd b \rhd_R R) \vdash (Q_1 \diamond Q_2 \lhd b \rhd_R S_1 \diamond S_2)$)
by (*simp add*: *design-condr lift-cond-srea-def*)

also have ... = $\mathbf{R}_s((P \triangleleft b \triangleright_R R) \vdash (Q_1 \triangleleft b \triangleright_R S_1) \diamond (Q_2 \triangleleft b \triangleright_R S_2))$
by (*rule cong*[*of* $\mathbf{R}_s$ $\mathbf{R}_s$], *simp*, *rel-auto*)
finally show *?thesis* **.**
qed

lemma *SRD-cond-srea* [*closure*]:
assumes *P is SRD Q is SRD*
shows $P \triangleleft b \triangleright_R Q$ *is SRD*
proof −
have $P \triangleleft b \triangleright_R Q = \mathbf{R}_s(pre_R(P) \vdash peri_R(P) \diamond post_R(P)) \triangleleft b \triangleright_R \mathbf{R}_s(pre_R(Q) \vdash peri_R(Q) \diamond post_R(Q))$
by (*simp add*: *SRD-reactive-tri-design assms*)
also have ... = $\mathbf{R}_s$ $((pre_R\ P \triangleleft b \triangleright_R pre_R\ Q) \vdash (peri_R\ P \triangleleft b \triangleright_R peri_R\ Q) \diamond (post_R\ P \triangleleft b \triangleright_R post_R\ Q))$
by (*simp add*: *cond-srea-form*)
also have ... *is SRD*
by (*simp add*: *RHS-tri-design-is-SRD lift-cond-srea-def unrest*)
finally show *?thesis* **.**
qed

4.9 Algebraic laws

lemma *RD-left-unit*:
assumes *P is RD*
shows II_C ;; $P = P$
by (*simp add*: *RD1-left-unit RD-healths*(*1*) *RD-healths*(*4*) *assms*)

lemma *skip-rdes-self-unit* [*simp*]:
II_C ;; $II_C = II_C$
by (*simp add*: *RD-left-unit closure*)

lemma *SRD-left-unit*:
assumes *P is SRD*
shows II_R ;; $P = P$
by (*simp add*: *SRD-composition-wp closure rdes wp C1 R1-negate-R1 R1-false rpred trace-ident-left-periR trace-ident-left-postR SRD-reactive-tri-design assms*)

lemma *skip-srea-self-unit* [*simp*]:
II_R ;; $II_R = II_R$
by (*simp add*: *SRD-left-unit closure*)

lemma *SRD-right-unit-tri-lemma*:
assumes *P is SRD*
shows P ;; $II_R = \mathbf{R}_s$ $((\neg_r\ pre_R\ P)\ wp_r\ false \vdash (\exists\ \$st' \cdot peri_R\ P) \diamond post_R\ P)$
by (*simp add*: *SRD-composition-wp closure rdes wp rpred trace-ident-right-postR assms*)

lemma *Miracle-left-zero*:
assumes *P is SRD*
shows *Miracle* ;; P = *Miracle*
proof −
have *Miracle* ;; $P = \mathbf{R}_s(true \vdash false)$;; $\mathbf{R}_s(pre_R(P) \vdash cmt_R(P))$
by (*simp add*: *Miracle-def SRD-reactive-design-alt assms*)
also have ... = $\mathbf{R}_s(true \vdash false)$
by (*simp add*: *RHS-design-composition unrest R1-false R2s-false R2s-true*)
also have ... = *Miracle*
by (*simp add*: *Miracle-def*)
finally show *?thesis* **.**

qed

lemma *Chaos-left-zero*:
assumes *P is SRD*
shows (*Chaos* ;; *P*) = *Chaos*
proof −
have *Chaos* ;; *P* = $\mathbf{R}_s$(*false* ⊢ *true*) ;; $\mathbf{R}_s$(*pre*$_R$(*P*) ⊢ *cmt*$_R$(*P*))
by (*simp add*: *Chaos-def SRD-reactive-design-alt assms*)
also have ... = $\mathbf{R}_s$ ((¬ *R1 true* ∧ ¬ (*R1 true* ∧ ¬ $*wait*´) ;; *R1* (¬ *R2s* (*pre*$_R$ *P*))) ⊢
R1 true ;; ((∃ $*st* · ⌈*II*⌉$_D$) ◁ $*wait* ▷ *R1* (*R2s* (*cmt*$_R$ *P*))))
by (*simp add*: *RHS-design-composition unrest R2s-false R2s-true R1-false*)
also have ... = $\mathbf{R}_s$ ((*false* ∧ ¬ (*R1 true* ∧ ¬ $*wait*´) ;; *R1* (¬ *R2s* (*pre*$_R$ *P*))) ⊢
R1 true ;; ((∃ $*st* · ⌈*II*⌉$_D$) ◁ $*wait* ▷ *R1* (*R2s* (*cmt*$_R$ *P*))))
by (*simp add*: *RHS-design-conj-neg-R1-pre*)
also have ... = $\mathbf{R}_s$(*true*)
by (*simp add*: *design-false-pre*)
also have ... = $\mathbf{R}_s$(*false* ⊢ *true*)
by (*simp add*: *design-def*)
also have ... = *Chaos*
by (*simp add*: *Chaos-def*)
finally show *?thesis* **.**
qed

lemma *SRD-right-Chaos-tri-lemma*:
assumes *P is SRD*
shows *P* ;; *Chaos* = $\mathbf{R}_s$ (((¬$_r$ *pre*$_R$ *P*) *wp*$_r$ *false* ∧ *post*$_R$ *P wp*$_r$ *false*) ⊢ (∃ $*st*´ · *peri*$_R$ *P*) ⋄ *false*)
by (*simp add*: *SRD-composition-wp closure rdes assms wp*, *rel-auto*)

lemma *SRD-right-Miracle-tri-lemma*:
assumes *P is SRD*
shows *P* ;; *Miracle* = $\mathbf{R}_s$ ((¬$_r$ *pre*$_R$ *P*) *wp*$_r$ *false* ⊢ (∃ $*st*´ · *peri*$_R$ *P*) ⋄ *false*)
by (*simp add*: *SRD-composition-wp closure rdes assms wp*, *rel-auto*)

Reactive designs are left unital

interpretation *rdes-left-unital*: *utp-theory-left-unital RD II*$_C$
by (*unfold-locales*, *simp-all add*: *closure RD-left-unit*)

Stateful reactive designs are left unital

interpretation *srdes-left-unital*: *utp-theory-left-unital SRD II*$_R$
by (*unfold-locales*, *simp-all add*: *closure SRD-left-unit*)

4.10 Recursion laws

lemma *mono-srd-iter*:
assumes *mono F* $F \in [\![SRD]\!]_H \to [\![SRD]\!]_H$
shows *mono* (λ*X*. $\mathbf{R}_s$(*pre*$_R$(*F X*) ⊢ *peri*$_R$(*F X*) ⋄ *post*$_R$ (*F X*)))
apply (*rule monoI*)
apply (*rule srdes-tri-refine-intro′*)
apply (*meson assms*(*1*) *monoE preR-antitone utp-pred-laws.le-infI2*)
apply (*meson assms*(*1*) *monoE periR-monotone utp-pred-laws.le-infI2*)
apply (*meson assms*(*1*) *monoE postR-monotone utp-pred-laws.le-infI2*)
done

lemma *mu-srd-SRD*:
assumes *mono F* $F \in [\![SRD]\!]_H \to [\![SRD]\!]_H$

shows ($\mu\ X \cdot \mathbf{R}_s\ (pre_R\ (F\ X) \vdash peri_R\ (F\ X) \diamond post_R\ (F\ X))$) *is SRD*
apply (*subst gfp-unfold*)
apply (*simp add*: *mono-srd-iter assms*)
apply (*rule RHS-tri-design-is-SRD*)
apply (*simp-all add*: *unrest*)
done

lemma *mu-srd-iter*:
assumes *mono F F* $\in [\![SRD]\!]_H \to [\![SRD]\!]_H$
shows ($\mu\ X \cdot \mathbf{R}_s(pre_R(F(X)) \vdash peri_R(F(X)) \diamond post_R(F(X)))) = F(\mu\ X \cdot \mathbf{R}_s(pre_R(F(X)) \vdash peri_R(F(X)) \diamond post_R(F(X))))$
apply (*subst gfp-unfold*)
apply (*simp add*: *mono-srd-iter assms*)
apply (*subst SRD-as-reactive-tri-design*[*THEN sym*])
apply (*simp add*: *Healthy-apply-closed SRD-as-reactive-design SRD-reactive-design-alt assms*(*1*) *assms*(*2*) *mu-srd-SRD*)
done

lemma *mu-srd-form*:
assumes *mono F F* $\in [\![SRD]\!]_H \to [\![SRD]\!]_H$
shows $\mu_R\ F = (\mu\ X \cdot \mathbf{R}_s(pre_R(F(X)) \vdash peri_R(F(X)) \diamond post_R(F(X))))$
proof −
have *1*: $F\ (\mu\ X \cdot \mathbf{R}_s(pre_R\ (F\ X) \vdash peri_R(F\ X) \diamond post_R\ (F\ X)))$ *is SRD*
by (*simp add*: *Healthy-apply-closed assms*(*1*) *assms*(*2*) *mu-srd-SRD*)
have *2*:$Mono_{utp\text{-}order\ SRD}\ F$
by (*simp add*: *assms*(*1*) *mono-Monotone-utp-order*)
hence *3*:$\mu_R\ F = F\ (\mu_R\ F)$
by (*simp add*: *srdes-theory.LFP-unfold*[*THEN sym*] *assms*)
hence $\mathbf{R}_s(pre_R\ (F\ (F\ (\mu_R\ F))) \vdash peri_R\ (F\ (F\ (\mu_R\ F))) \diamond post_R\ (F\ (F\ (\mu_R\ F)))) = \mu_R\ F$
using *SRD-reactive-tri-design* **by** *force*
hence $(\mu\ X \cdot \mathbf{R}_s(pre_R\ (F\ X) \vdash peri_R(F\ X) \diamond post_R\ (F\ X))) \sqsubseteq F\ (\mu_R\ F)$
by (*simp add*: *2 srdes-theory.weak.LFP-lemma3 gfp-upperbound assms*)
thus *?thesis*
using *assms 1 3 srdes-theory.weak.LFP-lowerbound eq-iff mu-srd-iter*
by (*metis* (*mono-tags*, *lifting*))
qed

lemma *Monotonic-SRD-comp* [*closure*]: *Monotonic* ((;;) $P \circ$ *SRD*)
by (*simp add*: *mono-def R1-R2c-is-R2 R2-mono R3h-mono RD1-mono RD2-mono RHS-def SRD-def seqr-mono*)

end

5 Normal Reactive Designs

theory *utp-rdes-normal*
imports
utp-rdes-triples
UTP−KAT.utp-kleene
begin

These additional healthiness conditions are analogous to H3

definition *RD3* **where**
[*upred-defs*]: $RD3(P) = P\ ;;\ II_R$

definition *RD3c* **where**
[*upred-defs*]: $RD3c(P) = P \mathbin{;;} II_C$

lemma *RD3-idem*: $RD3(RD3(P)) = RD3(P)$
proof −
  have *a*: $II_R \mathbin{;;} II_R = II_R$
    by (*simp add*: *SRD-left-unit SRD-srdes-skip*)
  show *?thesis*
    by (*simp add*: *RD3-def seqr-assoc a*)
qed

lemma *RD3-Idempotent* [*closure*]: *Idempotent RD3*
  by (*simp add*: *Idempotent-def RD3-idem*)

lemma *RD3-continuous*: $RD3(\bigsqcap A) = (\bigsqcap P{\in}A.\ RD3(P))$
  by (*simp add*: *RD3-def seq-Sup-distr*)

lemma *RD3-Continuous* [*closure*]: *Continuous RD3*
  by (*simp add*: *Continuous-def RD3-continuous*)

lemma *RD3-right-subsumes-RD2*: $RD2(RD3(P)) = RD3(P)$
proof −
  have *a*:$II_R \mathbin{;;} J = II_R$
    by (*rel-auto*)
  show *?thesis*
    by (*metis* (*no-types*, *hide-lams*) *H2-def RD2-def RD3-def a seqr-assoc*)
qed

lemma *RD3c-idem*: $RD3c(RD3c(P)) = RD3c(P)$
proof −
  have *a*: $II_C \mathbin{;;} II_C = II_C$
    by *simp*
  show *?thesis*
    by (*simp add*: *RD3c-def seqr-assoc a*)
qed

lemma *RD3c-Idempotent* [*closure*]: *Idempotent RD3c*
  by (*simp add*: *Idempotent-def RD3c-idem*)

lemma *RD3c-continuous*: $RD3c(\bigsqcap A) = (\bigsqcap P{\in}A.\ RD3c(P))$
  by (*simp add*: *RD3c-def seq-Sup-distr*)

lemma *RD3c-Continuous* [*closure*]: *Continuous RD3c*
  by (*simp add*: *Continuous-def RD3c-continuous*)

lemma *RD3c-right-subsumes-RD2*: $RD2\ (RD3c\ P) = RD3c\ P$
proof −
  have *a*:$II_C \mathbin{;;} J = II_C$
    by (*rel-auto*)
  thus *?thesis*
    by (*simp add*: *a H2-def RD2-def RD3c-def seqr-assoc*)
qed

lemma *RD3-left-subsumes-RD2*: $RD3(RD2(P)) = RD3(P)$
proof −

have $a{:}J$;; $II_R = II_R$
by (*rel-simp*, *safe*, *blast+*)
show *?thesis*
by (*metis* (*no-types*, *hide-lams*) *H2-def RD2-def RD3-def a seqr-assoc*)
qed

lemma *RD3c-left-subsumes-RD2*: $RD3c(RD2(P)) = RD3c(P)$
proof −
have $a{:}J$;; $II_C = II_C$
by (*rel-auto*)
show *?thesis*
by (*simp add*: *H2-def RD2-def RD3c-def a seqr-assoc*)
qed

lemma *RD3-implies-RD2*: *P is RD3* $\Longrightarrow$ *P is RD2*
by (*metis Healthy-def RD3-right-subsumes-RD2*)

lemma *RD3-intro-pre*:
assumes *P is SRD* $(\neg_r\ pre_R(P))$;; $true_r = (\neg_r\ pre_R(P))$ $\$st' \sharp peri_R(P)$
shows *P is RD3*
proof −
have $RD3(P) = \mathbf{R}_s\ ((\neg_r\ pre_R\ P)\ wp_r\ false \vdash (\exists\ \$st' \cdot peri_R\ P) \diamond post_R\ P)$
by (*simp add*: *RD3-def SRD-right-unit-tri-lemma assms*)
also have ... $= \mathbf{R}_s\ ((\neg_r\ pre_R\ P)\ wp_r\ false \vdash peri_R\ P \diamond post_R\ P)$
by (*simp add*: *assms*(*3*) *ex-unrest*)
also have ... $= \mathbf{R}_s\ ((\neg_r\ pre_R\ P)\ wp_r\ false \vdash cmt_R\ P)$
by (*simp add*: *wait'-cond-peri-post-cmt*)
also have ... $= \mathbf{R}_s\ (pre_R\ P \vdash cmt_R\ P)$
by (*simp add*: *assms*(*2*) *rpred wp-rea-def R1-preR*)
finally show *?thesis*
by (*metis Healthy-def SRD-as-reactive-design assms*(*1*))
qed

lemma *RHS-tri-design-RD3-intro*:
assumes
$\$ok' \sharp P\ \$ok' \sharp Q\ \$ok' \sharp R\ \$st' \sharp Q\ \ \$wait' \sharp R$
P is R1 $(\neg_r\ P)$;; $true_r = (\neg_r\ P)$
shows $\mathbf{R}_s(P \vdash Q \diamond R)$ *is RD3*
apply (*simp add*: *Healthy-def RD3-def*)
apply (*subst RHS-tri-design-right-unit-lemma*)
apply (*simp-all add*:*assms ex-unrest rpred*)
done

RD3 reactive designs are those whose assumption can be written as a conjunction of a precondition on (undashed) program variables, and a negated statement about the trace. The latter allows us to state that certain events must not occur in the trace – which are effectively safety properties.

lemma *R1-right-unit-lemma*:
$[\![\ out\alpha \sharp b;\ out\alpha \sharp e\ ]\!] \Longrightarrow (\neg_r\ b \vee \$tr\ ^\frown_u\ e \leq_u \$tr') ;;\ R1(true) = (\neg_r\ b \vee \$tr\ ^\frown_u\ e \leq_u \$tr')$
by (*rel-auto*, *blast*, *metis* (*no-types*, *lifting*) *dual-order.trans*)

lemma *RHS-tri-design-RD3-intro-form*:
assumes
$out\alpha \sharp b\ out\alpha \sharp e\ \$ok' \sharp Q\ \$st' \sharp Q\ \$ok' \sharp R\ \$wait' \sharp R$
shows $\mathbf{R}_s((b \wedge \neg_r\ \$tr\ ^\frown_u\ e \leq_u \$tr') \vdash Q \diamond R)$ *is RD3*

apply (*rule RHS-tri-design-RD3-intro*)
apply (*simp-all add*: *assms unrest closure rpred*)
apply (*subst R1-right-unit-lemma*)
apply (*simp-all add*: *assms unrest*)
done

lemma *RD1-RD3-commute*: $RD1(RD3(P)) = RD3(RD1(P))$
by (*rel-auto*, *blast+*)

lemma *RD1-RD3c-commute*: $RD1(RD3c(P)) = RD3c(RD1(P))$
by (*rel-auto*)

definition *NSRD* :: $('s,'t::trace,'\alpha)$ *hrel-rsp* $\Rightarrow$ $('s,'t,'\alpha)$ *hrel-rsp*
where [*upred-defs*]: $NSRD = RD1 \circ RD3 \circ RHS$

definition *NRD* :: $('t::trace,'\alpha)$ *hrel-rp* $\Rightarrow$ $('t,'\alpha)$ *hrel-rp*
where [*upred-defs*]: $NRD = RD1 \circ RD3c \circ RH$

lemma *NRD-is-RD* [*closure*]: *P is NRD* $\Longrightarrow$ *P is RD*
by (*simp add*: *Healthy-def NRD-def RD-def*)
(*metis* (*no-types*, *lifting*) *R1-R3c-commute R1-seqr R1-skip-rea R2c-R1-seq R2c-R3c-commute R2c-skip-rea R3-skipr R3c-semir-form R3c-via-RD1-R3 RD1-R1-commute RD1-R2c-commute RD1-R3c-commute RD1-RD2-commute RD1-idem RD3c-def RD3c-right-subsumes-RD2 RH-def RP-def RP-idem skip-rea-RD1-skip*)

lemma *NRD-elim* [*RD-elim*]:
$[\![$ *P is NRD*; $Q(\mathbf{R}(pre_R(P) \vdash peri_R(P) \diamond post_R(P)))$ $]\!] \Longrightarrow Q(P)$
by (*simp add*: *RD-elim closure*)

lemma *NRD-idem*: $NRD(NRD(P)) = NRD(P)$
by (*metis* (*no-types*, *hide-lams*) *Healthy-Idempotent Healthy-def Idempotent-def NRD-def R1-R3c-commute R1-skip-rea R2-R2c-def R2-idem R2-seqr-closure R2c-R3c-commute R2c-skip-rea R3c-Idempotent R3c-def R3c-seq-closure R3c-via-RD1-R3 RD1-RD3c-commute RD1-RH-commute RD1-idem RD3c-def RD3c-idem RD-healths*(*3*) *RH-def comp-assoc comp-eq-dest-lhs rdes-left-unital.Healthy-Unit rea-lift-R1 rea-lift-def*)

lemma *NRD-Idempotent* [*closure*]: *Idempotent NRD*
by (*simp add*: *Idempotent-def NRD-idem*)

lemma *NRD-Continuous* [*closure*]: *Continuous NRD*
by (*simp add*: *Continuous-comp NRD-def RD1-Continuous RD3c-Continuous RH-Continuous*)

lemma *NRD-form*:
$NRD(P) = \mathbf{R}((\neg_r\ (\neg_r\ pre_R(P))\ ;;\ R1\ true) \vdash (peri_R(P) \diamond post_R(P)))$
proof −
have $NRD(P) = RD3c(RD(P))$
by (*metis NRD-def RD1-RD3c-commute RD3c-left-subsumes-RD2 RD-alt-def comp-eq-dest-lhs*)
also have $... = RD3c(\mathbf{R}(pre_R(P) \vdash peri_R(P) \diamond post_R(P)))$
by (*simp add*: *RD-as-reactive-tri-design*)
also have $... = \mathbf{R}(pre_R(P) \vdash peri_R(P) \diamond post_R(P))\ ;;\ II_C$
by (*simp add*: *RD3c-def*)
also have $... = \mathbf{R}((\neg_r\ (\neg_r\ pre_R(P))\ ;;\ R1\ true) \vdash (peri_R(P) \diamond post_R(P)))$
by (*simp add*: *RH-tri-design-right-unit-lemma unrest*)
finally show *?thesis* **.**
qed

lemma *NRD-healthy-form*:

assumes *P is NRD*
shows $\mathbf{R}((\neg_r\ (\neg_r\ pre_R(P))\ ;;\ R1\ true) \vdash (peri_R(P) \diamond post_R(P))) = P$
by (*metis Healthy-def NRD-form assms*)

lemma *NRD-neg-pre-unit*:
assumes *P is NRD*
shows $(\neg_r\ pre_R(P))\ ;;\ true_r = (\neg_r\ pre_R(P))$
proof −
have $(\neg_r\ pre_R(P)) = (\neg_r\ pre_R(\mathbf{R}((\neg_r\ (\neg_r\ pre_R(P)))\ ;;\ R1\ true) \vdash (peri_R(P) \diamond post_R(P)))))$
by (*simp add*: *NRD-healthy-form assms*)
also have $... = R1\ (R2c\ ((\neg_r\ pre_R\ P)\ ;;\ R1\ true))$
by (*simp add*: *rea-pre-RH-design R1-negate-R1 R1-idem R1-rea-not′ R2c-rea-not usubst rpred unrest closure*)
also have $... = (\neg_r\ pre_R\ P)\ ;;\ R1\ true$
by (*simp add*: *R1-R2c-seqr-distribute closure assms*)
finally show *?thesis*
by (*simp add*: *rea-not-def*)
qed

lemma *NRD-wait′-unrest-pre* [*unrest*]:
assumes *P is NRD*
shows $\$wait´ \sharp pre_R(P)$
proof −
have $pre_R(P) = pre_R(\mathbf{R}((\neg_r\ (\neg_r\ pre_R(P)))\ ;;\ R1\ true) \vdash (peri_R(P) \diamond post_R(P))))$
by (*simp add*: *NRD-healthy-form assms*)
also have $... = (R1\ (R2c\ (\neg_r\ (\neg_r\ pre_R\ P)\ ;;\ R1\ true)))$
by (*simp add*: *rea-pre-RH-design usubst unrest*)
also have $\$wait´ \sharp ...$
by (*simp add*: *R1-def R2c-def unrest*)
finally show *?thesis* **.**
qed

lemma *preR-NRD-RR* [*closure*]: *P is NRD* $\Longrightarrow$ $pre_R(P)$ *is RR*
by (*rule RR-intro*, *simp-all add*: *closure unrest*)

lemma *NRD-neg-pre-RC* [*closure*]:
assumes *P is NRD*
shows $pre_R(P)$ *is RC*
by (*rule RC-intro*, *simp-all add*: *closure assms NRD-neg-pre-unit rpred*)

lemma *NRD-intro*:
assumes *P is RD* $(\neg_r\ pre_R(P))\ ;;\ true_r = (\neg_r\ pre_R(P))$
shows *P is NRD*
proof −
have $NRD(P) = \mathbf{R}((\neg_r\ (\neg_r\ pre_R(P)))\ ;;\ R1\ true) \vdash (peri_R(P) \diamond post_R(P)))$
by (*simp add*: *NRD-form*)
also have $... = \mathbf{R}(pre_R\ P \vdash peri_R\ P \diamond post_R\ P)$
by (*simp add*: *assms rpred closure*)
also have $... = P$
by (*simp add*: *RD-reactive-tri-design assms*(*1*))
finally show *?thesis*
using *Healthy-def* **by** *blast*
qed

lemma *NRD-intro′*:

assumes *P is R2 P is R3c P is RD1 P is RD3c*
shows *P is NRD*
by (*metis* (*no-types*, *hide-lams*) *Healthy-def NRD-def R1-R2c-is-R2 RH-def assms comp-apply*)

lemma *NRD-RC-intro*:
assumes *P is RD* $pre_R(P)$ *is RC*
shows *P is NRD*
by (*metis Healthy-def NRD-form RD-reactive-tri-design assms*(*1*) *assms*(*2*)
rea-not-false wp-rea-RC-false wp-rea-def)

lemma *NRD-rdes-intro* [*closure*]:
assumes *P is RC Q is RR R is RR*
shows $\mathbf{R}(P \vdash Q \diamond R)$ *is NRD*
by (*rule NRD-RC-intro*, *simp-all add*: *rdes closure assms unrest*)

lemma *NRD-composition-wp*:
assumes *P is NRD Q is RD*
shows $P \mathbin{;;} Q =$
$\mathbf{R}\ ((pre_R\ P \wedge post_R\ P\ wp_r\ pre_R\ Q) \vdash (peri_R\ P \vee (post_R\ P \mathbin{;;} peri_R\ Q)) \diamond (post_R\ P \mathbin{;;} post_R\ Q))$
by (*simp add*: *RD-composition-wp assms NRD-is-RD wp-rea-def NRD-neg-pre-unit R1-negate-R1 R1-preR ex-unrest rpred*)

lemma *NSRD-is-SRD* [*closure*]: *P is NSRD* $\Longrightarrow$ *P is SRD*
by (*simp add*: *Healthy-def NSRD-def SRD-def*, *metis Healthy-def RD1-RD3-commute RD2-RHS-commute RD3-def RD3-right-subsumes-RD2 SRD-def SRD-idem SRD-seqr-closure SRD-srdes-skip*)

lemma *NSRD-elim* [*RD-elim*]:
$[\![\ P \text{ is NSRD};\ Q(\mathbf{R}_s(pre_R(P) \vdash peri_R(P) \diamond post_R(P)))\ ]\!] \Longrightarrow Q(P)$
by (*simp add*: *RD-elim closure*)

lemma *NSRD-idem*: $NSRD(NSRD(P)) = NSRD(P)$
by (*metis* (*no-types*, *hide-lams*) *Healthy-def NSRD-def RD1-RD2-commute RD1-RD3-commute RD1-RHS-commute RD1-idem RD2-RHS-commute RD2-idem RD3-def RD3-idem RD3-left-subsumes-RD2 RHS-idem SRD-def comp-apply fun.map-comp srdes-left-unital.Healthy-Sequence srdes-left-unital.Healthy-Unit*)

lemma *NSRD-Idempotent* [*closure*]: *Idempotent NSRD*
by (*simp add*: *Idempotent-def NSRD-idem*)

lemma *NSRD-Continuous* [*closure*]: *Continuous NSRD*
by (*simp add*: *Continuous-comp NSRD-def RD1-Continuous RD3-Continuous RHS-Continuous*)

lemma *NSRD-form*:
$NSRD(P) = \mathbf{R}_s((\neg_r\ (\neg_r\ pre_R(P)) \mathbin{;;} R1\ true) \vdash ((\exists\ \$st' \cdot peri_R(P)) \diamond post_R(P)))$
proof −
have $NSRD(P) = RD3(SRD(P))$
by (*metis* (*no-types*, *lifting*) *NSRD-def RD1-RD3-commute RD3-left-subsumes-RD2 SRD-def comp-def*)
also have ... $= RD3(\mathbf{R}_s(pre_R(P) \vdash peri_R(P) \diamond post_R(P)))$
by (*simp add*: *SRD-as-reactive-tri-design*)
also have ... $= \mathbf{R}_s(pre_R(P) \vdash peri_R(P) \diamond post_R(P)) \mathbin{;;} II_R$
by (*simp add*: *RD3-def*)
also have ... $= \mathbf{R}_s((\neg_r\ (\neg_r\ pre_R(P)) \mathbin{;;} R1\ true) \vdash ((\exists\ \$st' \cdot peri_R(P)) \diamond post_R(P)))$
by (*simp add*: *RHS-tri-design-right-unit-lemma unrest*)
finally show *?thesis* **.**
qed

lemma *NSRD-healthy-form*:
assumes *P is NSRD*
shows $\mathbf{R}_s((\neg_r\ (\neg_r\ pre_R(P))\ ;;\ R1\ true) \vdash ((\exists\ \$st' \cdot peri_R(P)) \diamond post_R(P))) = P$
by (*metis Healthy-def NSRD-form assms*)

lemma *NSRD-Sup-closure* [*closure*]:
assumes $A \subseteq [\![NSRD]\!]_H$ $A \neq \{\}$
shows $\bigsqcap A$ *is NSRD*
proof −
have *NSRD* $(\bigsqcap A) = (\bigsqcap (NSRD\ `A))$
by (*simp add*: *ContinuousD NSRD-Continuous assms*(*2*))
also have ... $= (\bigsqcap A)$
by (*simp only*: *Healthy-carrier-image assms*)
finally show *?thesis* **by** (*simp add*: *Healthy-def*)
qed

lemma *intChoice-NSRD-closed* [*closure*]:
assumes *P is NSRD Q is NSRD*
shows $P \sqcap Q$ *is NSRD*
using *NSRD-Sup-closure*[*of* $\{P, Q\}$] **by** (*simp add*: *assms*)

lemma *NRSD-SUP-closure* [*closure*]:
$[\![\bigwedge i.\ i \in A \Longrightarrow P(i)\ is\ NSRD;\ A \neq \{\}\ ]\!] \Longrightarrow (\bigsqcap i{\in}A.\ P(i))\ is\ NSRD$
by (*rule NSRD-Sup-closure*, *auto*)

lemma *NSRD-neg-pre-unit*:
assumes *P is NSRD*
shows $(\neg_r\ pre_R(P))\ ;;\ true_r = (\neg_r\ pre_R(P))$
proof −
have $(\neg_r\ pre_R(P)) = (\neg_r\ pre_R(\mathbf{R}_s((\neg_r\ (\neg_r\ pre_R(P))\ ;;\ R1\ true) \vdash ((\exists\ \$st' \cdot peri_R(P)) \diamond post_R(P)))))$
by (*simp add*: *NSRD-healthy-form assms*)
also have ... $= R1\ (R2c\ ((\neg_r\ pre_R\ P)\ ;;\ R1\ true))$
by (*simp add*: *rea-pre-RHS-design R1-negate-R1 R1-idem R1-rea-not' R2c-rea-not usubst rpred unrest closure*)
also have ... $= (\neg_r\ pre_R\ P)\ ;;\ R1\ true$
by (*simp add*: *R1-R2c-seqr-distribute closure assms*)
finally show *?thesis*
by (*simp add*: *rea-not-def*)
qed

lemma *NSRD-neg-pre-left-zero*:
assumes *P is NSRD Q is R1 Q is RD1*
shows $(\neg_r\ pre_R(P))\ ;;\ Q = (\neg_r\ pre_R(P))$
by (*metis* (*no-types*, *hide-lams*) *NSRD-neg-pre-unit RD1-left-zero assms*(*1*) *assms*(*2*) *assms*(*3*) *seqr-assoc*)

lemma *NSRD-st'-unrest-peri* [*unrest*]:
assumes *P is NSRD*
shows $\$st' \sharp peri_R(P)$
proof −
have $peri_R(P) = peri_R(\mathbf{R}_s((\neg_r\ (\neg_r\ pre_R(P))\ ;;\ R1\ true) \vdash ((\exists\ \$st' \cdot peri_R(P)) \diamond post_R(P))))$
by (*simp add*: *NSRD-healthy-form assms*)
also have ... $= R1\ (R2c\ (\neg_r\ (\neg_r\ pre_R\ P)\ ;;\ R1\ true \Rightarrow_r (\exists\ \$st' \cdot peri_R\ P)))$
by (*simp add*: *rea-peri-RHS-design usubst unrest*)
also have $\$st' \sharp$ ...

by (*simp add*: *R1-def R2c-def unrest*)
finally show *?thesis* **.**
qed

lemma *NSRD-wait′-unrest-pre* [*unrest*]:
assumes *P is NSRD*
shows $\$wait\acute{} \sharp pre_R(P)$
proof −
have $pre_R(P) = pre_R(\mathbf{R}_s((\neg_r\ (\neg_r\ pre_R(P))\ ;;\ R1\ true) \vdash ((\exists\ \$st\acute{}\ \cdot\ peri_R(P)) \diamond post_R(P))))$
by (*simp add*: *NSRD-healthy-form assms*)
also have $... = (R1\ (R2c\ (\neg_r\ (\neg_r\ pre_R\ P)\ ;;\ R1\ true)))$
by (*simp add*: *rea-pre-RHS-design usubst unrest*)
also have $\$wait\acute{} \sharp ...$
by (*simp add*: *R1-def R2c-def unrest*)
finally show *?thesis* **.**
qed

lemma *NSRD-st′-unrest-pre* [*unrest*]:
assumes *P is NSRD*
shows $\$st\acute{} \sharp pre_R(P)$
proof −
have $pre_R(P) = pre_R(\mathbf{R}_s((\neg_r\ (\neg_r\ pre_R(P))\ ;;\ R1\ true) \vdash ((\exists\ \$st\acute{}\ \cdot\ peri_R(P)) \diamond post_R(P))))$
by (*simp add*: *NSRD-healthy-form assms*)
also have $... = R1\ (R2c\ (\neg_r\ (\neg_r\ pre_R\ P)\ ;;\ R1\ true))$
by (*simp add*: *rea-pre-RHS-design usubst unrest*)
also have $\$st\acute{} \sharp ...$
by (*simp add*: *R1-def R2c-def unrest*)
finally show *?thesis* **.**
qed

lemma *NSRD-peri-under-pre* [*rpred*]:
$P\ is\ NSRD \Longrightarrow (pre_R\ P \Rightarrow_r peri_R\ P) = peri_R\ P$
by (*simp add*: *SRD-peri-under-pre unrest closure*)

lemma *NSRD-post-under-pre* [*rpred*]:
$P\ is\ NSRD \Longrightarrow (pre_R\ P \Rightarrow_r post_R\ P) = post_R\ P$
by (*simp add*: *SRD-post-under-pre unrest closure*)

lemma *NSRD-peri-seq-under-pre*:
assumes *P is NSRD Q is NSRD*
shows $(pre_R\ P \Rightarrow_r peri_R\ P \vee post_R\ P\ ;;\ peri_R\ Q) = (peri_R\ P \vee post_R\ P\ ;;\ peri_R\ Q)$
by (*metis NSRD-peri-under-pre assms*(*1*) *rea-impl-def utp-pred-laws.disj-assoc*)

lemma *NSRD-postR-seq-periR-impl*:
assumes *P is NSRD Q is NSRD*
shows $(post_R\ P\ wp_r\ pre_R\ Q \Rightarrow_r (post_R\ P\ ;;\ peri_R\ Q)) = (post_R\ P\ ;;\ peri_R\ Q)$
by (*metis NSRD-is-SRD NSRD-peri-under-pre SRD-healths*(*2*) *assms postR-RR wpR-impl-post-spec*)

lemma *NSRD-postR-seq-postR-impl*:
assumes *P is NSRD Q is NSRD*
shows $(post_R\ P\ wp_r\ pre_R\ Q \Rightarrow_r (post_R\ P\ ;;\ post_R\ Q)) = (post_R\ P\ ;;\ post_R\ Q)$
by (*metis NSRD-is-SRD NSRD-post-under-pre SRD-healths*(*2*) *assms postR-RR wpR-impl-post-spec*)

lemma *NSRD-peri-under-assms*:
assumes *P is NSRD Q is NSRD*

shows ($pre_R\ P \wedge post_R\ P\ wp_r\ pre_R\ Q \Rightarrow_r peri_R\ P \vee post_R\ P\ ;;\ peri_R\ Q) = (peri_R\ P \vee post_R\ P\ ;;\ peri_R\ Q$)
by (*metis* (*no-types*, *lifting*) *NSRD-peri-seq-under-pre assms NSRD-postR-seq-periR-impl rea-impl-conj rea-impl-disj*)

lemma *NSRD-peri-under-assms′*:
assumes *P is NSRD Q is NSRD*
shows ($post_R\ P\ wp_r\ pre_R\ Q \Rightarrow_r peri_R\ P \vee post_R\ P\ ;;\ peri_R\ Q) = (peri_R\ P \vee post_R\ P\ ;;\ peri_R\ Q$)
by (*simp add*: *NSRD-postR-seq-periR-impl assms rea-impl-disj*)

lemma *NSRD-post-under-assms*:
assumes *P is NSRD Q is NSRD*
shows ($pre_R\ P \wedge post_R\ P\ wp_r\ pre_R\ Q \Rightarrow_r post_R\ P\ ;;\ post_R\ Q) = (pre_R\ P \Rightarrow_r (post_R\ P\ ;;\ post_R\ Q)$)
by (*metis NSRD-postR-seq-postR-impl assms*(*1*) *assms*(*2*) *rea-impl-conj*)

lemma *NSRD-alt-def*: $NSRD(P) = RD3(SRD(P))$
by (*metis NSRD-def RD1-RD3-commute RD3-left-subsumes-RD2 SRD-def comp-eq-dest-lhs*)

lemma *preR-RR* [*closure*]: *P is NSRD* $\Longrightarrow$ $pre_R(P)$ *is RR*
by (*rule RR-intro*, *simp-all add*: *closure unrest*)

lemma *NSRD-neg-pre-RC* [*closure*]:
assumes *P is NSRD*
shows $pre_R(P)$ *is RC*
by (*rule RC-intro*, *simp-all add*: *closure assms NSRD-neg-pre-unit rpred*)

lemma *NSRD-intro*:
assumes *P is SRD* $(\neg_r\ pre_R(P))\ ;;\ true_r = (\neg_r\ pre_R(P))\ \$st' \sharp peri_R(P)$
shows *P is NSRD*
proof −
have $NSRD(P) = \mathbf{R}_s((\neg_r\ (\neg_r\ pre_R(P))\ ;;\ R1\ true) \vdash ((\exists\ \$st' \cdot peri_R(P)) \diamond post_R(P)))$
by (*simp add*: *NSRD-form*)
also have $... = \mathbf{R}_s(pre_R\ P \vdash peri_R\ P \diamond post_R\ P)$
by (*simp add*: *assms ex-unrest rpred closure*)
also have $... = P$
by (*simp add*: *SRD-reactive-tri-design assms*(*1*))
finally show *?thesis*
using *Healthy-def* **by** *blast*
qed

lemma *NSRD-intro′*:
assumes *P is R2 P is R3h P is RD1 P is RD3*
shows *P is NSRD*
by (*metis* (*no-types*, *hide-lams*) *Healthy-def NSRD-def R1-R2c-is-R2 RHS-def assms comp-apply*)

lemma *NSRD-RC-intro*:
assumes *P is SRD* $pre_R(P)$ *is RC* $\$st' \sharp peri_R(P)$
shows *P is NSRD*
by (*metis Healthy-def NSRD-form SRD-reactive-tri-design assms*(*1*) *assms*(*2*) *assms*(*3*)
ex-unrest rea-not-false wp-rea-RC-false wp-rea-def)

lemma *NSRD-rdes-intro* [*closure*]:
assumes *P is RC Q is RR R is RR* $\$st' \sharp Q$
shows $\mathbf{R}_s(P \vdash Q \diamond R)$ *is NSRD*
by (*rule NSRD-RC-intro*, *simp-all add*: *rdes closure assms unrest*)

lemma *SRD-RD3-implies-NSRD*:
⟦ *P is SRD*; *P is RD3* ⟧ ⟹ *P is NSRD*
by (*metis* (*no-types*, *lifting*) *Healthy-def NSRD-def RHS-idem SRD-healths*(*4*) *SRD-reactive-design comp-apply*)

lemma *NSRD-iff*:
P is NSRD ⟷ ((*P is SRD*) ∧ (¬$_r$ *pre*$_R$(*P*)) ;; *R1*(*true*) = (¬$_r$ *pre*$_R$(*P*)) ∧ ($*st*′ ♯ *peri*$_R$(*P*)))
by (*meson NSRD-intro NSRD-is-SRD NSRD-neg-pre-unit NSRD-st'-unrest-peri*)

lemma *NSRD-is-RD3* [*closure*]:
assumes *P is NSRD*
shows *P is RD3*
by (*simp add*: *NSRD-is-SRD NSRD-neg-pre-unit NSRD-st'-unrest-peri RD3-intro-pre assms*)

lemma *NSRD-refine-elim*:
assumes
P ⊑ *Q P is NSRD Q is NSRD*
⟦ ‘*pre*$_R$(*P*) ⇒ *pre*$_R$(*Q*)‘; ‘*pre*$_R$(*P*) ∧ *peri*$_R$(*Q*) ⇒ *peri*$_R$(*P*)‘; ‘*pre*$_R$(*P*) ∧ *post*$_R$(*Q*) ⇒ *post*$_R$(*P*)‘ ⟧ ⟹ *R*
shows *R*
proof −
have **R**$_s$(*pre*$_R$(*P*) ⊢ *peri*$_R$(*P*) ⋄ *post*$_R$(*P*)) ⊑ **R**$_s$(*pre*$_R$(*Q*) ⊢ *peri*$_R$(*Q*) ⋄ *post*$_R$(*Q*))
by (*simp add*: *NSRD-is-SRD SRD-reactive-tri-design assms*(*1*) *assms*(*2*) *assms*(*3*))
hence *1*:‘*pre*$_R$ *P* ⇒ *pre*$_R$ *Q*‘ **and** *2*:‘*pre*$_R$ *P* ∧ *peri*$_R$ *Q* ⇒ *peri*$_R$ *P*‘ **and** *3*:‘*pre*$_R$ *P* ∧ *post*$_R$ *Q* ⇒ *post*$_R$ *P*‘
by (*simp-all add*: *RHS-tri-design-refine assms closure*)
with *assms*(*4*) **show** *?thesis*
by *simp*
qed

lemma *NSRD-right-unit*: *P is NSRD* ⟹ *P* ;; *II*$_R$ = *P*
by (*metis Healthy-if NSRD-is-RD3 RD3-def*)

lemma *NSRD-composition-wp*:
assumes *P is NSRD Q is SRD*
shows *P* ;; *Q* =
R$_s$ ((*pre*$_R$ *P* ∧ *post*$_R$ *P* *wp*$_r$ *pre*$_R$ *Q*) ⊢ (*peri*$_R$ *P* ∨ (*post*$_R$ *P* ;; *peri*$_R$ *Q*)) ⋄ (*post*$_R$ *P* ;; *post*$_R$ *Q*))
by (*simp add*: *SRD-composition-wp assms NSRD-is-SRD wp-rea-def NSRD-neg-pre-unit NSRD-st'-unrest-peri R1-negate-R1 R1-preR ex-unrest rpred*)

lemma *preR-NSRD-seq-lemma*:
assumes *P is NSRD Q is SRD*
shows *R1* (*R2c* (*post*$_R$ *P* ;; (¬$_r$ *pre*$_R$ *Q*))) = *post*$_R$ *P* ;; (¬$_r$ *pre*$_R$ *Q*)
proof −
have *post*$_R$ *P* ;; (¬$_r$ *pre*$_R$ *Q*) = *R1*(*R2c*(*post*$_R$ *P*)) ;; *R1*(*R2c*(¬$_r$ *pre*$_R$ *Q*))
by (*simp add*: *NSRD-is-SRD R1-R2c-post-RHS R1-rea-not R2c-preR R2c-rea-not assms*(*1*) *assms*(*2*))
also have ... = *R1* (*R2c* (*post*$_R$ *P* ;; (¬$_r$ *pre*$_R$ *Q*)))
by (*simp add*: *R1-seqr R2c-R1-seq calculation*)
finally show *?thesis* **..**
qed

lemma *preR-NRD-seq* [*rdes*]:

assumes *P is NRD Q is RD*
shows $pre_R(P \;;; Q) = (pre_R\ P \wedge post_R\ P\ wp_r\ pre_R\ Q)$
by (*simp add*: *NRD-composition-wp assms rea-pre-RH-design usubst unrest wp-rea-def R2c-disj R1-disj R2c-and R2c-preR R1-R2c-commute*[*THEN sym*] *R1-extend-conj' R1-idem R2c-not closure R1-rea-not' R2c-rea-not preR-NSRD-seq-lemma Healthy-if*)

lemma *preR-NSRD-seq* [*rdes*]:
assumes *P is NSRD Q is SRD*
shows $pre_R(P \;;; Q) = (pre_R\ P \wedge post_R\ P\ wp_r\ pre_R\ Q)$
by (*simp add*: *NSRD-composition-wp assms rea-pre-RHS-design usubst unrest wp-rea-def R2c-disj R1-disj R2c-and R2c-preR R1-R2c-commute*[*THEN sym*] *R1-extend-conj' R1-idem R2c-not closure R1-rea-not' R2c-rea-not preR-NSRD-seq-lemma*)

lemma *periR-NSRD-seq* [*rdes*]:
assumes *P is NSRD Q is NSRD*
shows $peri_R(P \;;; Q) = ((pre_R\ P \wedge post_R\ P\ wp_r\ pre_R\ Q) \Rightarrow_r (peri_R\ P \vee (post_R\ P \;;; peri_R\ Q)))$
by (*simp add*: *NSRD-composition-wp assms closure rea-peri-RHS-design usubst unrest wp-rea-def R1-extend-conj' R1-disj R1-R2c-seqr-distribute R2c-disj R2c-and R2c-rea-impl R1-rea-impl' R2c-preR R2c-periR R1-rea-not' R2c-rea-not R1-peri-SRD*)

lemma *postR-NSRD-seq* [*rdes*]:
assumes *P is NSRD Q is NSRD*
shows $post_R(P \;;; Q) = ((pre_R\ P \wedge post_R\ P\ wp_r\ pre_R\ Q) \Rightarrow_r (post_R\ P \;;; post_R\ Q))$
by (*simp add*: *NSRD-composition-wp assms closure rea-post-RHS-design usubst unrest wp-rea-def R1-extend-conj' R1-disj R1-R2c-seqr-distribute R2c-disj R2c-and R2c-rea-impl R1-rea-impl' R2c-preR R2c-periR R1-rea-not' R2c-rea-not*)

lemma *NRD-seqr-closure* [*closure*]:
assumes *P is NRD Q is NRD*
shows $(P \;;; Q)$ *is NRD*
proof −
have $(\neg_r\ post_R\ P\ wp_r\ pre_R\ Q) \;;; true_r = (\neg_r\ post_R\ P\ wp_r\ pre_R\ Q)$
by (*simp add*: *wp-rea-def rpred assms closure seqr-assoc NRD-neg-pre-unit*)
thus *?thesis*
by (*rule-tac NRD-intro*, *simp-all add*: *seqr-or-distl NRD-neg-pre-unit assms closure rdes unrest*)
qed

lemma *NSRD-seqr-closure* [*closure*]:
assumes *P is NSRD Q is NSRD*
shows $(P \;;; Q)$ *is NSRD*
proof −
have $(\neg_r\ post_R\ P\ wp_r\ pre_R\ Q) \;;; true_r = (\neg_r\ post_R\ P\ wp_r\ pre_R\ Q)$
by (*simp add*: *wp-rea-def rpred assms closure seqr-assoc NSRD-neg-pre-unit*)
moreover have $\$st' \sharp pre_R\ P \wedge post_R\ P\ wp_r\ pre_R\ Q \Rightarrow_r peri_R\ P \vee post_R\ P \;;; peri_R\ Q$
by (*simp add*: *unrest assms wp-rea-def*)
ultimately show *?thesis*
by (*rule-tac NSRD-intro*, *simp-all add*: *seqr-or-distl NSRD-neg-pre-unit assms closure rdes unrest*)
qed

lemma *RHS-tri-normal-design-composition*:
assumes
$\$ok' \sharp P\ \$ok' \sharp Q_1\ \$ok' \sharp Q_2\ \$ok \sharp R\ \$ok \sharp S_1\ \$ok \sharp S_2$
$\$wait \sharp R\ \$wait' \sharp Q_2\ \$wait \sharp S_1\ \$wait \sharp S_2$

P is R2c Q₁ is R1 Q₁ is R2c Q₂ is R1 Q₂ is R2c
R is R2c S₁ is R1 S₁ is R2c S₂ is R1 S₂ is R2c
R1 (¬ P) ;; R1(true) = R1(¬ P) $st´ ♯ Q₁
shows $\mathbf{R}_s(P \vdash Q_1 \diamond Q_2) ;; \mathbf{R}_s(R \vdash S_1 \diamond S_2)$
$= \mathbf{R}_s((P \wedge Q_2\ wp_r\ R) \vdash (Q_1 \vee (Q_2 ;; S_1)) \diamond (Q_2 ;; S_2))$
proof −
have $\mathbf{R}_s(P \vdash Q_1 \diamond Q_2) ;; \mathbf{R}_s(R \vdash S_1 \diamond S_2) =$
$\mathbf{R}_s\ ((R1\ (\neg\ P)\ wp_r\ false \wedge Q_2\ wp_r\ R) \vdash ((\exists\ \$st´ \cdot Q_1) \sqcap (Q_2 ;; S_1)) \diamond (Q_2 ;; S_2))$
by (*simp-all add*: *RHS-tri-design-composition-wp rea-not-def assms unrest*)
also have ... $= \mathbf{R}_s((P \wedge Q_2\ wp_r\ R) \vdash (Q_1 \vee (Q_2 ;; S_1)) \diamond (Q_2 ;; S_2))$
by (*simp add*: *assms wp-rea-def ex-unrest*, *rel-auto*)
finally show *?thesis* **.**
qed

lemma *RHS-tri-normal-design-composition'* [*rdes-def*]:
assumes *P is RC Q₁ is RR $st´ ♯ Q₁ Q₂ is RR R is RR S₁ is RR S₂ is RR*
shows $\mathbf{R}_s(P \vdash Q_1 \diamond Q_2) ;; \mathbf{R}_s(R \vdash S_1 \diamond S_2)$
$= \mathbf{R}_s((P \wedge Q_2\ wp_r\ R) \vdash (Q_1 \vee (Q_2 ;; S_1)) \diamond (Q_2 ;; S_2))$
proof −
have *R1 (¬ P) ;; R1 true = R1(¬ P)*
using *RC-implies-RC1*[*OF assms*(*1*)]
by (*simp add*: *Healthy-def RC1-def rea-not-def*)
(*metis R1-negate-R1 R1-seqr utp-pred-laws.double-compl*)
thus *?thesis*
by (*simp add*: *RHS-tri-normal-design-composition assms closure unrest RR-implies-R2c*)
qed

If a normal reactive design has postcondition false, then it is a left zero for sequential composition.

lemma *NSRD-seq-post-false*:
assumes *P is NSRD Q is SRD* $post_R(P) = false$
shows *P ;; Q = P*
apply (*simp add*: *NSRD-composition-wp assms wp rpred closure*)
using *NSRD-is-SRD SRD-reactive-tri-design assms*(*1*,*3*) **apply** *fastforce*
done

lemma *NSRD-srd-skip* [*closure*]: II_R *is NSRD*
by (*rule NSRD-intro*, *simp-all add*: *rdes closure unrest*)

lemma *NSRD-Chaos* [*closure*]: *Chaos is NSRD*
by (*rule NSRD-intro*, *simp-all add*: *closure rdes unrest*)

lemma *NSRD-Miracle* [*closure*]: *Miracle is NSRD*
by (*rule NSRD-intro*, *simp-all add*: *closure rdes unrest*)

Post-composing a miracle filters out the non-terminating behaviours

lemma *NSRD-right-Miracle-tri-lemma*:
assumes *P is NSRD*
shows $P ;; Miracle = \mathbf{R}_s\ (pre_R\ P \vdash peri_R\ P \diamond false)$
by (*simp add*: *NSRD-composition-wp closure assms rdes wp rpred*)

lemma *Miracle-right-anhil-iff*:
assumes *P is NSRD*
shows $P ;; Miracle = Miracle \longleftrightarrow pre_R\ P = true_r \wedge (\exists\ \$st´ \cdot peri_R\ P) = false$
by (*simp add*: *SRD-right-Miracle-tri-lemma assms closure*, *simp add*: *rdes-def srdes-tri-eq-iff closure*

assms wp rpred, *fastforce*)

The set of non-terminating behaviours is a subset

lemma *NSRD-right-Miracle-refines*:
assumes *P is NSRD*
shows $P \sqsubseteq P$;; *Miracle*
proof −
have $\mathbf{R}_s$ (pre_R $P \vdash peri_R$ $P \diamond post_R$ P) $\sqsubseteq \mathbf{R}_s$ (pre_R $P \vdash peri_R$ $P \diamond$ *false*)
by (*rule srdes-tri-refine-intro*, *rel-auto+*)
thus *?thesis*
by (*simp add*: *NSRD-elim NSRD-right-Miracle-tri-lemma assms*)
qed

Chaos is a right zero of P precisely when the conjunction of the precondition of P with the weakest precondition under which $post_R$ P yields $\boldsymbol{U}(false)$ reduces to $\boldsymbol{U}(false)$. This is effectively a feasibility check for reactive designs.

lemma *Chaos-right-anhil-iff*:
assumes *P is NSRD*
shows P ;; *Chaos* = *Chaos* $\longleftrightarrow$ (pre_R $P \wedge post_R$ P wp_r *false*) = *false*
by (*simp add*: *SRD-right-Chaos-tri-lemma assms closure*, *simp add*: *rdes-def srdes-tri-eq-iff closure assms wp*)

lemma *upower-Suc-NSRD-closed* [*closure*]:
P is NSRD $\Longrightarrow$ P ^ *Suc n is NSRD*
proof (*induct n*)
case *0*
then show *?case*
by (*simp*)
next
case (*Suc n*)
then show *?case*
by (*simp add*: *NSRD-seqr-closure upred-semiring.power-Suc*)
qed

lemma *NSRD-power-Suc* [*closure*]:
P is NSRD $\Longrightarrow$ P ;; P ^ *n is NSRD*
by (*metis upower-Suc-NSRD-closed upred-semiring.power-Suc*)

lemma *uplus-NSRD-closed* [*closure*]: *P is NSRD* $\Longrightarrow$ P^+ *is NSRD*
by (*simp add*: *uplus-power-def closure*)

lemma *preR-power*:
assumes *P is NSRD*
shows $pre_R(P$;; $P\hat{}n) = (\bigsqcup\ i{\in}\{0..n\}.\ (post_R(P)\ \hat{}\ i)\ wp_r\ (pre_R(P)))$
proof (*induct n*)
case *0*
then show *?case*
by (*simp add*: *wp closure*)
next
case (*Suc n*) **note** *hyp* = *this*
have pre_R (P ^ (*Suc n* + *1*)) = pre_R (P ;; P ^ (*n*+*1*))
by (*simp add*: *upred-semiring.power-Suc*)
also have ... = (pre_R $P \wedge post_R$ P wp_r pre_R (P ^ (*Suc n*)))
using *NSRD-iff assms preR-NSRD-seq upower-Suc-NSRD-closed* **by** *fastforce*
also have ... = (pre_R $P \wedge post_R$ P wp_r ($\bigsqcup i{\in}\{0..n\}.\ post_R$ P ^ i wp_r pre_R P))

by (*simp add*: *hyp upred-semiring.power-Suc*)
also have ... = ($pre_R\ P \wedge (\bigsqcup i{\in}\{0..n\}.\ post_R\ P\ wp_r\ (post_R\ P\ \hat{}\ i\ wp_r\ pre_R\ P))$)
by (*simp add*: *wp*)
also have ... = ($pre_R\ P \wedge (\bigsqcup i{\in}\{0..n\}.\ (post_R\ P\ \hat{}\ (i{+}1)\ wp_r\ pre_R\ P))$)
proof −
have $\bigwedge i.\ R1\ (post_R\ P\ \hat{}\ i\ ;;\ (\neg_r\ pre_R\ P)) = (post_R\ P\ \hat{}\ i\ ;;\ (\neg_r\ pre_R\ P))$
by (*induct-tac i*, *simp-all add*: *closure Healthy-if assms*)
thus *?thesis*
by (*simp add*: *wp-rea-def upred-semiring.power-Suc seqr-assoc rpred closure assms*)
qed
also have ... = ($post_R\ P\ \hat{}\ 0\ wp_r\ pre_R\ P \wedge (\bigsqcup i{\in}\{0..n\}.\ (post_R\ P\ \hat{}\ (i{+}1)\ wp_r\ pre_R\ P))$)
by (*simp add*: *wp assms closure*)
also have ... = ($post_R\ P\ \hat{}\ 0\ wp_r\ pre_R\ P \wedge (\bigsqcup i{\in}\{1..Suc\ n\}.\ (post_R\ P\ \hat{}\ i\ wp_r\ pre_R\ P))$)
proof −
have $(\bigsqcup i{\in}\{0..n\}.\ (post_R\ P\ \hat{}\ (i{+}1)\ wp_r\ pre_R\ P)) = (\bigsqcup i{\in}\{1..Suc\ n\}.\ (post_R\ P\ \hat{}\ i\ wp_r\ pre_R\ P))$
by (*rule cong*[*of Inf*], *simp-all add*: *fun-eq-iff*)
(*metis* (*no-types*, *lifting*) *image-Suc-atLeastAtMost image-cong image-image*)
thus *?thesis* **by** *simp*
qed
also have ... = ($\bigsqcup i{\in}insert\ 0\ \{1..Suc\ n\}.\ (post_R\ P\ \hat{}\ i\ wp_r\ pre_R\ P)$)
by (*simp add*: *conj-upred-def*)
also have ... = ($\bigsqcup i{\in}\{0..Suc\ n\}.\ post_R\ P\ \hat{}\ i\ wp_r\ pre_R\ P$)
by (*simp add*: *atLeast0-atMost-Suc-eq-insert-0*)
finally show *?case* **by** (*simp add*: *upred-semiring.power-Suc*)
qed

lemma *preR-power'* [*rdes*]:
assumes *P is NSRD*
shows $pre_R(P\ ;;\ P\hat{}n) = (\bigsqcup\ i{\in}\{0..n\} \cdot (post_R(P)\ \hat{}\ i)\ wp_r\ (pre_R(P)))$
by (*simp add*: *preR-power assms USUP-as-Inf*[*THEN sym*], *simp add*: *USUP-as-Inf-image*)

lemma *preR-power-Suc* [*rdes*]:
assumes *P is NSRD*
shows $pre_R(P\hat{}(Suc\ n)) = (\bigsqcup\ i{\in}\{0..n\} \cdot (post_R(P)\ \hat{}\ i)\ wp_r\ (pre_R(P)))$
by (*simp add*: *upred-semiring.power-Suc rdes assms*)

declare *upred-semiring.power-Suc* [*simp*]

lemma *periR-power*:
assumes *P is NSRD*
shows $peri_R(P\ ;;\ P\hat{}n) = (pre_R(P\hat{}(Suc\ n)) \Rightarrow_r (\bigsqcap\ i{\in}\{0..n\}.\ post_R(P)\ \hat{}\ i)\ ;;\ peri_R(P))$
proof (*induct n*)
case *0*
then show *?case*
by (*simp add*: *NSRD-is-SRD NSRD-wait'-unrest-pre SRD-peri-under-pre assms*)
next
case (*Suc n*) **note** *hyp* = *this*
have $peri_R\ (P\ \hat{}\ (Suc\ n + 1)) = peri_R\ (P\ ;;\ P\ \hat{}\ (n{+}1))$
by (*simp*)
also have ... = ($pre_R(P\ \hat{}\ (Suc\ n + 1)) \Rightarrow_r (peri_R\ P \vee post_R\ P\ ;;\ peri_R\ (P\ ;;\ P\ \hat{}\ n))$)
by (*simp add*: *closure assms rdes*)
also have ... = ($pre_R(P\ \hat{}\ (Suc\ n + 1)) \Rightarrow_r (peri_R\ P \vee post_R\ P\ ;;\ (pre_R\ (P\ \hat{}\ (Suc\ n)) \Rightarrow_r (\bigsqcap i{\in}\{0..n\}.$
$post_R\ P\ \hat{}\ i)\ ;;\ peri_R\ P))$)
by (*simp only*: *hyp*)
also

have ... = ($pre_R\ P \Rightarrow_r peri_R\ P \vee (post_R\ P\ wp_r\ pre_R\ (P\ ;;\ P\ \hat{}\ n) \Rightarrow_r post_R\ P\ ;;\ (pre_R\ (P\ ;;\ P\ \hat{}\ n) \Rightarrow_r (\bigsqcap i{\in}\{0..n\}.\ post_R\ P\ \hat{}\ i)\ ;;\ peri_R\ P)))$
by (*simp add*: *rdes closure assms*, *rel-blast*)
also
have ... = ($pre_R\ P \Rightarrow_r peri_R\ P \vee (post_R\ P\ wp_r\ pre_R\ (P\ ;;\ P\ \hat{}\ n) \Rightarrow_r post_R\ P\ ;;\ ((\bigsqcap i{\in}\{0..n\}.\ post_R\ P\ \hat{}\ i)\ ;;\ peri_R\ P)))$
proof −
have $(\bigsqcap i{\in}\{0..n\}.\ post_R\ P\ \hat{}\ i)$ *is R1*
by (*simp add*: *R1-Continuous assms closure*)
hence *1*:$((\bigsqcap i{\in}\{0..n\}.\ post_R\ P\ \hat{}\ i)\ ;;\ peri_R\ P)$ *is R1*
by (*simp add*: *closure assms*)
hence $(pre_R\ (P\ ;;\ P\ \hat{}\ n) \Rightarrow_r (\bigsqcap i{\in}\{0..n\}.\ post_R\ P\ \hat{}\ i)\ ;;\ peri_R\ P)$ *is R1*
by (*simp add*: *closure*)
hence $(post_R\ P\ wp_r\ pre_R\ (P\ ;;\ P\ \hat{}\ n) \Rightarrow_r post_R\ P\ ;;\ (pre_R\ (P\ ;;\ P\ \hat{}\ n) \Rightarrow_r (\bigsqcap i{\in}\{0..n\}.\ post_R\ P\ \hat{}\ i)\ ;;\ peri_R\ P))$
$= (post_R\ P\ wp_r\ pre_R\ (P\ ;;\ P\ \hat{}\ n) \Rightarrow_r R1(post_R\ P)\ ;;\ R1(pre_R\ (P\ ;;\ P\ \hat{}\ n) \Rightarrow_r (\bigsqcap i{\in}\{0..n\}.\ post_R\ P\ \hat{}\ i)\ ;;\ peri_R\ P))$
by (*simp add*: *Healthy-if R1-post-SRD assms closure*)
thus *?thesis*
by (*simp only*: *wp-rea-impl-lemma*, *simp add*: *Healthy-if 1*, *simp add*: *R1-post-SRD assms closure*)
qed
also
have ... = ($pre_R\ P \wedge post_R\ P\ wp_r\ pre_R\ (P\ ;;\ P\ \hat{}\ n) \Rightarrow_r peri_R\ P \vee post_R\ P\ ;;\ ((\bigsqcap i{\in}\{0..n\}.\ post_R\ P\ \hat{}\ i)\ ;;\ peri_R\ P)$)
by (*pred-auto*)
also
have ... = ($pre_R\ P \wedge post_R\ P\ wp_r\ pre_R\ (P\ ;;\ P\ \hat{}\ n) \Rightarrow_r peri_R\ P \vee ((\bigsqcap i{\in}\{0..n\}.\ post_R\ P\ \hat{}\ (Suc\ i))\ ;;\ peri_R\ P)$)
by (*simp add*: *seq-Sup-distl seqr-assoc*[*THEN sym*])
also
have ... = ($pre_R\ P \wedge post_R\ P\ wp_r\ pre_R\ (P\ ;;\ P\ \hat{}\ n) \Rightarrow_r peri_R\ P \vee ((\bigsqcap i{\in}\{1..Suc\ n\}.\ post_R\ P\ \hat{}\ i)\ ;;\ peri_R\ P)$)
proof −
have $(\bigsqcap i{\in}\{0..n\}.\ post_R\ P\ \hat{}\ Suc\ i) = (\bigsqcap i{\in}\{1..Suc\ n\}.\ post_R\ P\ \hat{}\ i)$
apply (*rule cong*[*of Sup*], *auto*)
apply (*metis atLeast0AtMost atMost-iff image-Suc-atLeastAtMost rev-image-eqI upred-semiring.power-Suc*)
using *Suc-le-D* **apply** *fastforce*
done
thus *?thesis* **by** *simp*
qed
also
have ... = ($pre_R\ P \wedge post_R\ P\ wp_r\ pre_R\ (P\ ;;\ P\ \hat{}\ n) \Rightarrow_r ((\bigsqcap i{\in}\{0..Suc\ n\}.\ post_R\ P\ \hat{}\ i))\ ;;\ peri_R\ P$)
by (*simp add*: *SUP-atLeastAtMost-first uinf-or seqr-or-distl seqr-or-distr*)
also
have ... = ($pre_R(P\hat{}(Suc\ (Suc\ n))) \Rightarrow_r ((\bigsqcap i{\in}\{0..Suc\ n\}.\ post_R\ P\ \hat{}\ i)\ ;;\ peri_R\ P)$)
by (*simp add*: *rdes closure assms*)
finally show *?case* **by** (*simp*)
qed

lemma *periR-power′* [*rdes*]:
assumes *P is NSRD*
shows $peri_R(P\ ;;\ P\hat{}n) = (pre_R(P\hat{}(Suc\ n)) \Rightarrow_r (\bigsqcap\ i{\in}\{0..n\} \cdot post_R(P)\ \hat{}\ i)\ ;;\ peri_R(P))$
by (*simp add*: *periR-power assms UINF-as-Sup*[*THEN sym*], *simp add*: *UINF-as-Sup-image*)

lemma *periR-power-Suc* [*rdes*]:

assumes *P is NSRD*
shows $peri_R(P\hat{}(Suc\ n)) = (pre_R(P\hat{}(Suc\ n)) \Rightarrow_r (\bigsqcap\ i{\in}\{0..n\} \cdot post_R(P)\ \hat{}\ i)\ ;;\ peri_R(P))$
by (*simp add*: *rdes assms*)

lemma *postR-power* [*rdes*]:
assumes *P is NSRD*
shows $post_R(P\ ;;\ P\hat{}n) = (pre_R(P\hat{}(Suc\ n)) \Rightarrow_r post_R(P)\ \hat{}\ Suc\ n)$
proof (*induct n*)
case *0*
then show *?case*
by (*simp add*: *NSRD-is-SRD NSRD-wait'-unrest-pre SRD-post-under-pre assms*)
next
case (*Suc n*) **note** *hyp* = *this*
have $post_R\ (P\ \hat{}\ (Suc\ n + 1)) = post_R\ (P\ ;;\ P\ \hat{}\ (n+1))$
by (*simp*)
also have ... = $(pre_R(P\ \hat{}\ (Suc\ n + 1)) \Rightarrow_r (post_R\ P\ ;;\ post_R\ (P\ ;;\ P\ \hat{}\ n)))$
by (*simp add*: *closure assms rdes*)
also have ... = $(pre_R(P\ \hat{}\ (Suc\ n + 1)) \Rightarrow_r (post_R\ P\ ;;\ (pre_R\ (P\ \hat{}\ Suc\ n) \Rightarrow_r post_R\ P\ \hat{}\ Suc\ n)))$
by (*simp only*: *hyp*)
also
have ... = $(pre_R\ P \Rightarrow_r (post_R\ P\ wp_r\ pre_R\ (P\ \hat{}\ Suc\ n) \Rightarrow_r post_R\ P\ ;;\ (pre_R\ (P\ \hat{}\ Suc\ n) \Rightarrow_r post_R\ P\ \hat{}\ Suc\ n)))$
by (*simp add*: *rdes closure assms*, *pred-auto*)
also
have ... = $(pre_R\ P \Rightarrow_r (post_R\ P\ wp_r\ pre_R\ (P\ \hat{}\ Suc\ n) \Rightarrow_r post_R\ P\ ;;\ post_R\ P\ \hat{}\ Suc\ n))$
by (*metis* (*no-types*, *lifting*) *Healthy-if NSRD-is-SRD NSRD-power-Suc R1-power assms hyp postR-SRD-R1 upred-semiring.power-Suc wp-rea-impl-lemma*)
also
have ... = $(pre_R\ P \wedge post_R\ P\ wp_r\ pre_R\ (P\ \hat{}\ Suc\ n) \Rightarrow_r post_R\ P\ \hat{}\ Suc\ (Suc\ n))$
by (*pred-auto*)
also have ... = $(pre_R(P\hat{}(Suc\ (Suc\ n))) \Rightarrow_r post_R\ P\ \hat{}\ Suc\ (Suc\ n))$
by (*simp add*: *rdes closure assms*)
finally show *?case* **by** (*simp*)
qed

lemma *postR-power-Suc* [*rdes*]:
assumes *P is NSRD*
shows $post_R(P\hat{}(Suc\ n)) = (pre_R(P\hat{}(Suc\ n)) \Rightarrow_r post_R(P)\ \hat{}\ Suc\ n)$
by (*simp add*: *rdes assms*)

lemma *power-rdes-def* [*rdes-def*]:
assumes *P is RC Q is RR R is RR* $\$st' \sharp Q$
shows $(\mathbf{R}_s(P \vdash Q \diamond R))\hat{}(Suc\ n)$
$= \mathbf{R}_s((\bigsqcup\ i{\in}\{0..n\} \cdot (R\ \hat{}\ i)\ wp_r\ P) \vdash ((\bigsqcap\ i{\in}\{0..n\} \cdot R\ \hat{}\ i)\ ;;\ Q) \diamond (R\ \hat{}\ Suc\ n))$
proof (*induct n*)
case *0*
then show *?case*
by (*simp add*: *wp assms closure*)
next
case (*Suc n*)

have *1*: $(P \wedge (\bigsqcup\ i \in \{0..n\} \cdot R\ wp_r\ (R\ \hat{}\ i\ wp_r\ P))) = (\bigsqcup\ i \in \{0..Suc\ n\} \cdot R\ \hat{}\ i\ wp_r\ P)$
(**is** *?lhs* = *?rhs*)
proof −
have $?lhs = (P \wedge (\bigsqcup\ i \in \{0..n\} \cdot (R\ \hat{}\ Suc\ i\ wp_r\ P)))$

by (*simp add*: *wp closure assms*)
also have ... = $(P \wedge (\bigsqcup\ i \in \{0..n\}.\ (R\ ^\frown\ Suc\ i\ wp_r\ P)))$
by (*simp only*: *USUP-as-Inf-collect*)
also have ... = $(P \wedge (\bigsqcup\ i \in \{1..Suc\ n\}.\ (R\ ^\frown\ i\ wp_r\ P)))$
by (*metis* (*no-types*, *lifting*) *INF-cong One-nat-def image-Suc-atLeastAtMost image-image*)
also have ... = $(\bigsqcup\ i \in insert\ 0\ \{1..Suc\ n\}.\ (R\ ^\frown\ i\ wp_r\ P))$
by (*simp add*: *wp assms closure conj-upred-def*)
also have ... = $(\bigsqcup\ i \in \{0..Suc\ n\}.\ (R\ ^\frown\ i\ wp_r\ P))$
by (*simp add*: *atLeastAtMost-insertL*)
finally show *?thesis*
by (*simp add*: *USUP-as-Inf-collect*)
qed

have *2*: $(Q \vee R\ ;;\ (\bigsqcap\ i \in \{0..n\} \cdot R\ ^\frown\ i)\ ;;\ Q) = (\bigsqcap\ i \in \{0..Suc\ n\} \cdot R\ ^\frown\ i)\ ;;\ Q$
(**is** *?lhs* = *?rhs*)
proof −
have *?lhs* = $(Q \vee (\bigsqcap\ i \in \{0..n\} \cdot R\ ^\frown\ Suc\ i)\ ;;\ Q)$
by (*simp add*: *seqr-assoc*[*THEN sym*] *seq-UINF-distl*)
also have ... = $(Q \vee (\bigsqcap\ i \in \{0..n\}.\ R\ ^\frown\ Suc\ i)\ ;;\ Q)$
by (*simp only*: *UINF-as-Sup-collect*)
also have ... = $(Q \vee (\bigsqcap\ i \in \{1..Suc\ n\}.\ R\ ^\frown\ i)\ ;;\ Q)$
by (*metis One-nat-def image-Suc-atLeastAtMost image-image*)
also have ... = $((\bigsqcap\ i \in insert\ 0\ \{1..Suc\ n\}.\ R\ ^\frown\ i)\ ;;\ Q)$
by (*simp add*: *disj-upred-def*[*THEN sym*] *seqr-or-distl*)
also have ... = $((\bigsqcap\ i \in \{0..Suc\ n\}.\ R\ ^\frown\ i)\ ;;\ Q)$
by (*simp add*: *atLeastAtMost-insertL*)
finally show *?thesis*
by (*simp add*: *UINF-as-Sup-collect*)
qed

have *3*: $(\bigsqcap\ i \in \{0..n\} \cdot R\ ^\frown\ i)\ ;;\ Q$ *is RR*
proof −
have $(\bigsqcap\ i \in \{0..n\} \cdot R\ ^\frown\ i)\ ;;\ Q = (\bigsqcap\ i \in \{0..n\}.\ R\ ^\frown\ i)\ ;;\ Q$
by (*simp add*: *UINF-as-Sup-collect*)
also have ... = $(\bigsqcap\ i \in insert\ 0\ \{1..n\}.\ R\ ^\frown\ i)\ ;;\ Q$
by (*simp add*: *atLeastAtMost-insertL*)
also have ... = $(Q \vee (\bigsqcap\ i \in \{1..n\}.\ R\ ^\frown\ i)\ ;;\ Q)$
by (*metis* (*no-types*, *lifting*) *SUP-insert disj-upred-def seqr-left-unit seqr-or-distl upred-semiring.power-0*)
also have ... = $(Q \vee (\bigsqcap\ i \in \{0..<n\}.\ R\ ^\frown\ Suc\ i)\ ;;\ Q)$
by (*metis One-nat-def atLeastLessThanSuc-atLeastAtMost image-Suc-atLeastLessThan image-image*)
also have ... = $(Q \vee (\bigsqcap\ i \in \{0..<n\} \cdot R\ ^\frown\ Suc\ i)\ ;;\ Q)$
by (*simp add*: *UINF-as-Sup-collect*)
also have ... *is RR*
by (*simp-all add*: *closure assms*)
finally show *?thesis* **.**
qed
from *1 2 3 Suc* **show** *?case*
by (*simp add*: *Suc RHS-tri-normal-design-composition′ closure assms wp*)
qed

declare *upred-semiring.power-Suc* [*simp del*]

theorem *uplus-rdes-def* [*rdes-def*]:
assumes *P is RC Q is RR R is RR* $st′ \sharp Q$
shows $(\mathbf{R}_s(P \vdash Q \diamond R))^+ = \mathbf{R}_s(R^{\star r}\ wp_r\ P \vdash (R^{\star r}\ ;;\ Q) \diamond R^+)$

proof −
have *1*:($\bigsqcap$ i ∈ R ^ i) ;; Q = $R^{\star r}$;; Q
by (*metis* (*no-types*) *RA1 assms*(*2*) *rea-skip-unit*(*2*) *rrel-theory.Star-def ustar-alt-def*)
show *?thesis*
by (*simp add*: *uplus-power-def seq-UINF-distr wp closure assms rdes-def*)
(*metis 1 seq-UINF-distr'*)
qed

5.1 UTP theory

lemma *NSRD-false*: *NSRD false* = *Miracle*
by (*metis Healthy-if NSRD-Miracle NSRD-alt-def NSRD-is-RD3 srdes-theory.healthy-top*)

lemma *NSRD-true*: *NSRD true* = *Chaos*
by (*metis Healthy-if NSRD-Chaos NSRD-alt-def NSRD-is-RD3 srdes-theory.healthy-bottom*)

interpretation *nsrdes-theory*: *utp-theory-kleene NSRD* II_R
rewrites $P \in$ *carrier nsrdes-theory.thy-order* $\longleftrightarrow$ P *is NSRD*
and *carrier nsrdes-theory.thy-order* → *carrier nsrdes-theory.thy-order* ≡ $[\![NSRD]\!]_H$ → $[\![NSRD]\!]_H$
and *le nsrdes-theory.thy-order* = (⊑)
and *eq nsrdes-theory.thy-order* = (=)
and *nsrdes-top*: *nsrdes-theory.utp-top* = *Miracle*
and *nsrdes-bottom*: *nsrdes-theory.utp-bottom* = *Chaos*
proof −
have *utp-theory-continuous NSRD*
by (*unfold-locales*, *simp-all add*: *NSRD-idem NSRD-Continuous*)
then interpret *utp-theory-continuous NSRD*
by *simp*
show *t*: *utp-top* = *Miracle* **and** *b*:*utp-bottom* = *Chaos*
by (*simp-all add*: *healthy-top healthy-bottom NSRD-false NSRD-true*)
show *utp-theory-kleene NSRD* II_R
by (*unfold-locales*, *simp-all add*: *closure srdes-left-unital.Unit-Left NSRD-right-unit Miracle-left-zero t*)
qed (*simp-all*)

abbreviation *TestR* ($test_R$) **where**
$test_R$ P ≡ *nsrdes-theory.utp-test* P

definition *StarR* :: ('*s*, '*t*::*trace*, 'α) *hrel-rsp* ⇒ ('*s*, '*t*, 'α) *hrel-rsp* (-$^{\star R}$ [*999*] *999*) **where**
StarR ≡ *nsrdes-theory.utp-star*

We also show how to calculate the Kleene closure of a reactive design.

thm *rdes-def*

lemma *StarR-rdes-def* [*rdes-def*]:
assumes P *is RC* Q *is RR* R *is RR* $st' ♯ Q
shows ($\mathbf{R}_s(P \vdash Q \diamond R)$)$^{\star R}$ = $\mathbf{R}_s((R^{\star r}\ wp_r\ P) \vdash (R^{\star r}\ ;;\ Q) \diamond R^{\star r})$
by (*simp add*: *StarR-def rrel-theory.Star-alt-def nsrdes-theory.Star-alt-def closure assms*)
(*simp add*: *rrel-theory.Star-alt-def assms closure rdes-def unrest rpred disj-upred-def*)

end

6 Syntax for reactive design contracts

theory *utp-rdes-contracts*

imports *utp-rdes-normal*
begin

We give an experimental syntax for reactive design contracts $[P \vdash Q|R]_R$, where P is a precondition on undashed state variables only, Q is a pericondition that can refer to the trace and before state but not the after state, and R is a postcondition. Both Q and R can refer only to the trace contribution through a HOL variable *trace* which is bound to $\boldsymbol{U}(\&tt)$.

definition *mk-RD* :: *'s upred* ⇒ (*'t*::*trace* ⇒ *'s upred*) ⇒ (*'t* ⇒ *'s hrel*) ⇒ (*'s*, *'t*, *'a*) *hrel-rsp* **where**
mk-RD P Q R = $\mathbf{R}_s(\lceil P \rceil_{S<} \vdash \lceil Q(x) \rceil_{S<}[\![x \rightarrow \&tt]\!] \diamond \lceil R(x) \rceil_S[\![x \rightarrow \&tt]\!])$

definition *trace-pred* :: (*'t*::*trace* ⇒ *'s upred*) ⇒ (*'s*, *'t*, *'α*) *hrel-rsp* **where**
[*upred-defs*]: *trace-pred P* = $[(P\ x)]_{S<}[\![x \rightarrow \&tt]\!]$

syntax
-trace-var :: *logic*
-mk-RD :: *logic* ⇒ *logic* ⇒ *logic* ⇒ *logic* ($[-/ \vdash -/ \mid -]_R$)
-trace-pred :: *logic* ⇒ *logic* ($[-]_t$)

parse-translation ‹
let
fun trace-var-tr [] = *Syntax.free trace*
| *trace-var-tr* - = *raise Match*;
in
[(@{*syntax-const -trace-var*}, *K trace-var-tr*)]
end
›

translations
$[P \vdash Q \mid R]_R$ => *CONST mk-RD P* (λ *-trace-var*. *Q*) (λ *-trace-var*. *R*)
$[P \vdash Q \mid R]_R$ <= *CONST mk-RD P* (λ *x*. *Q*) (λ *y*. *R*)
$[P]_t$ => *CONST trace-pred* (λ *-trace-var*. *P*)
$[P]_t$ <= *CONST trace-pred* (λ *t*. *P*)

lemma *SRD-mk-RD* [*closure*]: $[P \vdash Q(trace) \mid R(trace)]_R$ *is SRD*
by (*simp add*: *mk-RD-def closure unrest*)

lemma *preR-mk-RD* [*rdes*]: $pre_R([P \vdash Q(trace) \mid R(trace)\ ]_R) = R1(\lceil P \rceil_{S<})$
by (*simp add*: *mk-RD-def rea-pre-RHS-design usubst unrest R2c-not R2c-lift-state-pre*)

lemma *trace-pred-RR-closed* [*closure*]:
$[P\ trace]_t$ *is RR*
by (*rel-auto*)

lemma *unrest-trace-pred-st'* [*unrest*]:
$\$st' \sharp [P\ trace]_t$
by (*rel-auto*)

lemma *R2c-msubst-tt*: *R2c* (*msubst* (λ*x*. $\lceil Q\ x \rceil_S$) *&tt*) = (*msubst* (λ*x*. $\lceil Q\ x \rceil_S$) *&tt*)
by (*rel-auto*)

lemma *periR-mk-RD* [*rdes*]: $peri_R([P \vdash Q(trace) \mid R(trace)\ ]_R) = (\lceil P \rceil_{S<} \Rightarrow_r R1((\lceil Q(trace) \rceil_{S<})[\![trace \rightarrow \&tt]\!]))$
by (*simp add*: *mk-RD-def rea-peri-RHS-design usubst unrest R2c-not R2c-lift-state-pre*
R2c-disj R2c-msubst-tt R1-disj R2c-rea-impl R1-rea-impl)

lemma *postR-mk-RD* [*rdes*]: $post_R([P \vdash Q(trace) \mid R(trace)\ ]_R) = (\lceil P \rceil_{S<} \Rightarrow_r R1((\lceil R(trace) \rceil_S)[\![trace \rightarrow \&tt]\!]))$
by (*simp add*: *mk-RD-def rea-post-RHS-design usubst unrest R2c-not R2c-lift-state-pre impl-alt-def R2c-disj R2c-msubst-tt R2c-rea-impl R1-rea-impl*)

Refinement introduction law for contracts

lemma *RD-contract-refine*:
assumes
Q *is SRD* ‘$\lceil P_1 \rceil_{S<} \Rightarrow pre_R\ Q$‘
‘$\lceil P_1 \rceil_{S<} \wedge peri_R\ Q \Rightarrow \lceil P_2\ x \rceil_{S<}[\![x \rightarrow \&tt]\!]$‘
‘$\lceil P_1 \rceil_{S<} \wedge post_R\ Q \Rightarrow \lceil P_3\ x \rceil_S[\![x \rightarrow \&tt]\!]$‘
shows $[P_1 \vdash P_2(trace) \mid P_3(trace)]_R \sqsubseteq Q$
proof −
have $[P_1 \vdash P_2(trace) \mid P_3(trace)]_R \sqsubseteq \mathbf{R}_s(pre_R(Q) \vdash peri_R(Q) \diamond post_R(Q))$
using *assms*
by (*simp add*: *mk-RD-def*, *rule-tac srdes-tri-refine-intro*, *simp-all*)
thus *?thesis*
by (*simp add*: *SRD-reactive-tri-design assms*(*1*))
qed

end

7 Reactive design tactics

theory *utp-rdes-tactics*
imports *utp-rdes-triples*
begin

Theorems for normalisation

lemmas *rdes-rel-norms* =
prod.case-eq-if
conj-assoc
disj-assoc
utp-pred-laws.distrib(*3*,*4*)
conj-UINF-dist
conj-UINF-ind-dist
seqr-or-distl
seqr-or-distr
seq-UINF-distl
seq-UINF-distl′
seq-UINF-distr
seq-UINF-distr′

The following tactic can be used to simply and evaluate reactive predicates.

method *rpred-simp* = (*uexpr-simp simps*: *rpred usubst closure unrest*)

Tactic to expand out healthy reactive design predicates into the syntactic triple form.

method *rdes-expand* **uses** *cls* = (*insert cls*, (*erule RD-elim*)+)

Tactic to simplify the definition of a reactive design

method *rdes-simp* **uses** *cls cong simps* =
((*rdes-expand cls*: *cls*)*?*, (*simp add*: *closure*)*?*, (*simp add*: *rdes-def rdes-rel-norms rdes rpred cls closure alpha frame usubst unrest wp simps cong*: *cong*))

Tactic to split a refinement conjecture into three POs

method *rdes-refine-split* **uses** *cls cong simps* =
(*rdes-simp cls*: *cls cong*: *cong simps*: *simps*; *rule-tac rdes-tri-refine-intro′ srdes-tri-refine-intro′*)

Tactics to split an equality conjecture into three POs

method *rdes-eq-split* **uses** *cls cong simps* =
(*rdes-simp cls*: *cls cong*: *cong simps*: *simps*; (*rule-tac rdes-tri-eq-intro srdes-tri-eq-intro*))

method *rdes-eq-split′* **uses** *cls cong simps* =
(*rdes-simp cls*: *cls cong*: *cong simps*: *simps*; (*rule-tac rdes-tri-eq-intro′ srdes-tri-eq-intro′*))

Tactic to prove a refinement

method *rdes-refine* **uses** *cls cong simps* =
(*rdes-refine-split cls*: *cls cong*: *cong simps*: *simps*; (*insert cls*; *rel-auto*))

Tactics to prove an equality

method *rdes-eq* **uses** *cls cong simps* =
(*rdes-eq-split cls*: *cls cong*: *cong simps*: *simps*; *rel-auto*)

method *rdes-eq′* **uses** *cls cong simps* =
(*rdes-eq-split′ cls*: *cls cong*: *cong simps*: *simps*; *rel-auto*)

Via antisymmetry

method *rdes-eq-anti* **uses** *cls cong simps* =
(*rdes-simp cls*: *cls cong*: *cong simps*: *simps*; (*rule-tac antisym*; (*rule-tac rdes-tri-refine-intro srdes-tri-refine-intro*; *rel-auto*)))

Tactic to calculate pre/peri/postconditions from reactive designs

method *rdes-calc* = (*simp add*: *rdes rpred closure alpha usubst unrest wp prod.case-eq-if*)

The following tactic attempts to prove a reactive design refinement by calculation of the pre-, peri-, and postconditions and then showing three implications between them using rel-blast.

method *rdspl-refine* =
(*rule-tac SRD-refine-intro*; (*simp add*: *closure rdes unrest usubst* ; *rel-blast?*))

The following tactic combines antisymmetry with the previous tactic to prove an equality.

method *rdspl-eq* =
(*rule-tac antisym*, *rdes-refine*, *rdes-refine*)

end

8 Reactive design parallel-by-merge

theory *utp-rdes-parallel*
imports
utp-rdes-normal
utp-rdes-tactics
begin

R3h implicitly depends on RD1, and therefore it requires that both sides be RD1. We also require that both sides are R3c, and that $wait_m$ is a quasi-unit, and div_m yields divergence.

lemma *st-U0-alpha*: $\lceil \exists\ \$st \cdot II \rceil_0 = (\exists\ \$st \cdot \lceil II \rceil_0)$
by (*rel-auto*)

lemma *st-U1-alpha*: $\lceil \exists\ \$st \cdot II \rceil_1 = (\exists\ \$st \cdot \lceil II \rceil_1)$
by (*rel-auto*)

definition *skip-rm* :: ('*s*,'*t*::*trace*,'α) *rsp merge* (II_{RM}) **where**
[*upred-defs*]: $II_{RM} = (\exists\ \$<:st \cdot skip_m \vee (\neg\ \$<:ok \wedge \$<:tr \leq_u \$tr\,'))$

definition [*upred-defs*]: $R3hm(M) = (II_{RM} \lhd \$<:wait \rhd M)$

lemma *R3hm-idem*: $R3hm(R3hm(P)) = R3hm(P)$
by (*rel-auto*)

abbreviation *copytype* :: '*a* ⇒ '*a* ⇒ '*a* (*CPTYPE'*(-, -')) **where** $CPTYPE(A,\ B) \equiv B$

lemma *R3h-par-by-merge* [*closure*]:
assumes *P is R3h Q is R3h M is R3hm*
shows ($P \parallel_M Q$) *is R3h*
proof −
have $(P \parallel_M Q) = (((P \parallel_M Q)[\![true/\$ok]\!] \lhd \$ok \rhd (P \parallel_M Q)[\![false/\$ok]\!])[\![true/\$wait]\!] \lhd \$wait \rhd (P \parallel_M Q))$
by (*simp add*: *cond-var-subst-left cond-var-subst-right*)
also have ... $= (((P \parallel_M Q)[\![true,true/\$ok,\$wait]\!] \lhd \$ok \rhd (P \parallel_M Q)[\![false,true/\$ok,\$wait]\!]) \lhd \$wait \rhd (P \parallel_M Q))$
by (*rel-auto*)
also have ... $= (((\exists\ \$st \cdot II)[\![true,true/\$ok,\$wait]\!] \lhd \$ok \rhd (P \parallel_M Q)[\![false,true/\$ok,\$wait]\!]) \lhd \$wait \rhd (P \parallel_M Q))$
proof −
have $(P \parallel_M Q)[\![true,true/\$ok,\$wait]\!] = ((\lceil P \rceil_0 \wedge \lceil Q \rceil_1 \wedge \$<:\mathbf{v}\,' =_u \$\mathbf{v})\ ;;\ R3hm(M))[\![true,true/\$ok,\$wait]\!]$
by (*simp add*: *par-by-merge-def U0-as-alpha U1-as-alpha assms Healthy-if*)
also have ... $= ((\lceil P \rceil_0 \wedge \lceil Q \rceil_1 \wedge \$<' =_u \$\mathbf{v})\ ;;\ (\exists\ \$<:st \cdot CPTYPE(M, \$\mathbf{v}\,' =_u \$<)))[\![true,true/\$ok,\$wait]\!]$
by (*rel-blast*)
also have ... $= ((\lceil R3h(P) \rceil_0 \wedge \lceil R3h(Q) \rceil_1 \wedge \$<' =_u \$\mathbf{v})\ ;;\ (\exists\ \$<:st \cdot CPTYPE(M,\ \$\mathbf{v}\,' =_u \$<)))[\![true,true/\$ok,\$wait]\!]$
by (*simp add*: *assms Healthy-if*)
also have ... $= (\exists\ \$st \cdot II)[\![true,true/\$ok,\$wait]\!]$
by (*rel-auto*)
finally show *?thesis* **by** (*simp add*: *closure assms unrest*)
qed
also have ... $= (((\exists\ \$st \cdot II)[\![true,true/\$ok,\$wait]\!] \lhd \$ok \rhd (R1(true))[\![false,true/\$ok,\$wait]\!]) \lhd \$wait \rhd (P \parallel_M Q))$
proof −
have $(P \parallel_M Q)[\![false,true/\$ok,\$wait]\!] = ((\lceil P \rceil_0 \wedge \lceil Q \rceil_1 \wedge \$<:\mathbf{v}\,' =_u \$\mathbf{v})\ ;;\ R3hm(M))[\![false,true/\$ok,\$wait]\!]$
by (*simp add*: *par-by-merge-def U0-as-alpha U1-as-alpha assms Healthy-if*)
also have ... $= ((\lceil P \rceil_0 \wedge \lceil Q \rceil_1 \wedge \$<:\mathbf{v}\,' =_u \$\mathbf{v})\ ;;\ (CPTYPE(M, \$<:tr \leq_u \$tr\,')))[\![false,true/\$ok,\$wait]\!]$
by (*rel-blast*)
also have ... $= ((\lceil R3h(P) \rceil_0 \wedge \lceil R3h(Q) \rceil_1 \wedge \$<:\mathbf{v}\,' =_u \$\mathbf{v})\ ;;\ (CPTYPE(M, \$<:tr \leq_u \$tr\,')))[\![false,true/\$ok,\$wait]\!]$
by (*simp add*: *assms Healthy-if*)
also have ... $= (R1(true))[\![false,true/\$ok,\$wait]\!]$
by (*rel-blast*)
finally show *?thesis* **by** *simp*
qed
also have ... $= (((\exists\ \$st \cdot II) \lhd \$ok \rhd R1(true)) \lhd \$wait \rhd (P \parallel_M Q))$
by (*rel-auto*)
also have ... $= R3h(P \parallel_M Q)$
by (*simp add*: *R3h-cases*)

finally show *?thesis*
by (*simp add*: *Healthy-def*)
qed

definition [*upred-defs*]: $RD1m(M) = (M \vee \neg \${<}{:}ok \wedge \${<}{:}tr \leq_u \$tr')$

lemma *RD1-par-by-merge* [*closure*]:
assumes *P is R1 Q is R1 M is R1m P is RD1 Q is RD1 M is RD1m*
shows $(P \parallel_M Q)$ *is RD1*
proof −
have *1*: $(RD1(R1(P)) \parallel_{RD1m(R1m(M))} RD1(R1(Q)))[\![false/\$ok]\!] = R1(true)$
by (*rel-blast*)
have $(P \parallel_M Q) = (P \parallel_M Q)[\![true/\$ok]\!] \lhd \$ok \rhd (P \parallel_M Q)[\![false/\$ok]\!]$
by (*simp add*: *cond-var-split*)
also have $... = R1(P \parallel_M Q) \lhd \$ok \rhd R1(true)$
by (*metis 1 Healthy-if R1-par-by-merge assms calculation*
cond-idem cond-var-subst-right in-var-uvar ok-vwb-lens)
also have $... = RD1(P \parallel_M Q)$
by (*simp add*: *Healthy-if R1-par-by-merge RD1-alt-def assms*(*3*))
finally show *?thesis*
by (*simp add*: *Healthy-def*)
qed

lemma *RD2-par-by-merge* [*closure*]:
assumes *M is RD2*
shows $(P \parallel_M Q)$ *is RD2*
proof −
have $(P \parallel_M Q) = ((P \parallel_s Q) ;; M)$
by (*simp add*: *par-by-merge-def*)
also from *assms* **have** $... = ((P \parallel_s Q) ;; (M ;; J))$
by (*simp add*: *Healthy-def′ RD2-def H2-def*)
also from *assms* **have** $... = (((P \parallel_s Q) ;; M) ;; J)$
by (*simp add*: *seqr-assoc*)
also from *assms* **have** $... = RD2(P \parallel_M Q)$
by (*simp add*: *RD2-def H2-def par-by-merge-def*)
finally show *?thesis*
by (*simp add*: *Healthy-def′*)
qed

lemma *SRD-par-by-merge*:
assumes *P is SRD Q is SRD M is R1m M is R2m M is R3hm M is RD1m M is RD2*
shows $(P \parallel_M Q)$ *is SRD*
by (*rule SRD-intro*, *simp-all add*: *assms closure SRD-healths*)

definition *nmerge-rd0* (N_0) **where**
[*upred-defs*]: $N_0(M) = (\$wait' =_u (\$0{:}wait \vee \$1{:}wait) \wedge \${<}{:}tr \leq_u \$tr'$
$\wedge\ (\exists\ \$0{:}ok;\$1{:}ok;\${<}{:}ok;\$ok';\$0{:}wait;\$1{:}wait;\${<}{:}wait;\$wait' \bullet M))$

definition *nmerge-rd1* (N_1) **where**
[*upred-defs*]: $N_1(M) = (\$ok' =_u (\$0{:}ok \wedge \$1{:}ok) \wedge N_0(M))$

definition *nmerge-rd* (N_R) **where**
[*upred-defs*]: $N_R(M) = ((\exists\ \${<}{:}st \bullet \$\mathbf{v}' =_u \${<}{:}\mathbf{v}) \lhd \${<}{:}wait \rhd N_1(M)) \lhd \${<}{:}ok \rhd (\${<}{:}tr \leq_u \$tr')$

definition *merge-rd1* (M_1) **where**

[*upred-defs*]: $M_1(M) = (N_1(M) \mathbin{;;} II_R)$

definition *merge-rd* (M_R) **where**
[*upred-defs*]: $M_R(M) = N_R(M) \mathbin{;;} II_R$

abbreviation *rdes-par* (- $\|_{R}$ - [*85*,*0*,*86*] *85*) **where**
$P \parallel_{RM} Q \equiv P \parallel_{M_R(M)} Q$

Healthiness condition for reactive design merge predicates

definition [*upred-defs*]: *RDM*(M) = *R2m*(∃ $*0*:*ok*;$*1*:*ok*;$<:*ok*;$*ok´*;$*0*:*wait*;$*1*:*wait*;$<:*wait*;$*wait´* · *M*)

lemma *nmerge-rd-is-R1m* [*closure*]:
N$_R(M)$ *is R1m*
by (*rel-blast*)

lemma *R2m-nmerge-rd*: *R2m*$(N_R(R2m(M))) = N_R(R2m(M))$
apply (*rel-auto*) **using** *minus-zero-eq* **by** *blast+*

lemma *nmerge-rd-is-R2m* [*closure*]:
M is R2m $\Longrightarrow$ *N*$_R(M)$ *is R2m*
by (*metis Healthy-def´ R2m-nmerge-rd*)

lemma *nmerge-rd-is-R3hm* [*closure*]: *N*$_R(M)$ *is R3hm*
by (*rel-blast*)

lemma *nmerge-rd-is-RD1m* [*closure*]: *N*$_R(M)$ *is RD1m*
by (*rel-blast*)

lemma *merge-rd-is-RD3*: *M*$_R(M)$ *is RD3*
by (*metis Healthy-Idempotent RD3-Idempotent RD3-def merge-rd-def*)

lemma *merge-rd-is-RD2*: *M*$_R(M)$ *is RD2*
by (*simp add*: *RD3-implies-RD2 merge-rd-is-RD3*)

lemma *par-rdes-NSRD* [*closure*]:
assumes *P is SRD Q is SRD M is RDM*
shows *P* $\|_{RM}$ *Q is NSRD*
proof −
have (*P* $\|_{N_R\ M}$ *Q* ;; *II*$_R$) *is NSRD*
by (*rule NSRD-intro´*, *simp-all add*: *SRD-healths closure assms*)
(*metis* (*no-types*, *lifting*) *Healthy-def R2-par-by-merge R2-seqr-closure R2m-nmerge-rd RDM-def SRD-healths*(*2*) *assms skip-srea-R2*
,*metis Healthy-Idempotent RD3-Idempotent RD3-def*)
thus *?thesis*
by (*simp add*: *merge-rd-def par-by-merge-def seqr-assoc*)
qed

lemma *RDM-intro*:
assumes *M is R2m* $*0*:*ok* ♯ *M* $*1*:*ok* ♯ *M* $<:*ok* ♯ *M* $*ok´* ♯ *M*
$*0*:*wait* ♯ *M* $*1*:*wait* ♯ *M* $<:*wait* ♯ *M* $*wait´* ♯ *M*
shows *M is RDM*
using *assms*
by (*simp add*: *Healthy-def RDM-def ex-unrest unrest*)

lemma *RDM-unrests* [*unrest*]:
assumes *M is RDM*
shows $0:ok ♯ M $1:ok ♯ M $<:ok ♯ M $ok´ ♯ M
$0:wait ♯ M $1:wait ♯ M $<:wait ♯ M $wait´ ♯ M
by (*subst Healthy-if*[*OF assms*, *THEN sym*], *simp-all add*: *RDM-def unrest*, *rel-auto*)+

lemma *RDM-R1m* [*closure*]: *M is RDM* ⟹ *M is R1m*
by (*metis* (*no-types*, *hide-lams*) *Healthy-def R1m-idem R2m-def RDM-def*)

lemma *RDM-R2m* [*closure*]: *M is RDM* ⟹ *M is R2m*
by (*metis* (*no-types*, *hide-lams*) *Healthy-def R2m-idem RDM-def*)

lemma *ex-st'-R2m-closed* [*closure*]:
assumes *P is R2m*
shows (∃ $st´ · *P*) *is R2m*
proof −
have *R2m*(∃ $st´ · *R2m P*) = (∃ $st´ · *R2m P*)
by (*rel-auto*)
thus *?thesis*
by (*metis Healthy-def' assms*)
qed

lemma *parallel-RR-closed*:
assumes *P is RR Q is RR M is R2m*
$<:ok ♯ M $<:wait ♯ M $ok´ ♯ M $wait´ ♯ M
shows *P* ‖$_M$ *Q is RR*
by (*rule RR-R2-intro*, *simp-all add*: *unrest assms RR-implies-R2 closure*)

lemma *parallel-ok-cases*:
((*P* ‖$_s$ *Q*) ;; *M*) = (
((*P*^{t} ‖$_s$ *Q*^{t}) ;; (*M*⟦*true*,*true*/$0:ok,$1:ok⟧)) ∨
((*P*^{f} ‖$_s$ *Q*^{t}) ;; (*M*⟦*false*,*true*/$0:ok,$1:ok⟧)) ∨
((*P*^{t} ‖$_s$ *Q*^{f}) ;; (*M*⟦*true*,*false*/$0:ok,$1:ok⟧)) ∨
((*P*^{f} ‖$_s$ *Q*^{f}) ;; (*M*⟦*false*,*false*/$0:ok,$1:ok⟧)))
proof −
have ((*P* ‖$_s$ *Q*) ;; *M*) = (∃ *ok*$_0$ · (*P* ‖$_s$ *Q*)⟦«*ok*$_0$»/$0:ok´⟧ ;; *M*⟦«*ok*$_0$»/$0:ok⟧)
by (*subst seqr-middle*[*of ok* ;$_L$ *mrg-left*], *simp-all*)
also have ... = (∃ *ok*$_0$ · ∃ *ok*$_1$ · ((*P* ‖$_s$ *Q*)⟦«*ok*$_0$»/$0:ok´⟧⟦«*ok*$_1$»/$1:ok´⟧) ;; (*M*⟦«*ok*$_0$»/$0:ok⟧⟦«*ok*$_1$»/$1:ok⟧))
by (*subst seqr-middle*[*of ok* ;$_L$ *mrg-right*], *simp-all*)
also have ... = (∃ *ok*$_0$ · ∃ *ok*$_1$ · (*P*⟦«*ok*$_0$»/$ok´⟧ ‖$_s$ *Q*⟦«*ok*$_1$»/$ok´⟧) ;; (*M*⟦«*ok*$_0$»,«*ok*$_1$»/$0:ok,$1:ok⟧))
by (*rel-auto robust*)
also have ... = (
((*P*^{t} ‖$_s$ *Q*^{t}) ;; (*M*⟦*true*,*true*/$0:ok,$1:ok⟧)) ∨
((*P*^{f} ‖$_s$ *Q*^{t}) ;; (*M*⟦*false*,*true*/$0:ok,$1:ok⟧)) ∨
((*P*^{t} ‖$_s$ *Q*^{f}) ;; (*M*⟦*true*,*false*/$0:ok,$1:ok⟧)) ∨
((*P*^{f} ‖$_s$ *Q*^{f}) ;; (*M*⟦*false*,*false*/$0:ok,$1:ok⟧)))
by (*simp add*: *true-alt-def*[*THEN sym*] *false-alt-def*[*THEN sym*] *disj-assoc*
utp-pred-laws.sup.left-commute utp-pred-laws.sup-commute usubst)
finally show *?thesis* **.**
qed

lemma *skip-srea-ok-f* [*usubst*]:
II$_R{}^f$ = *R1*(¬$ok)
by (*rel-auto*)

lemma *nmerge0-rd-unrest* [*unrest*]:
  $\$0{:}ok \sharp N_0\ M\ \$1{:}ok \sharp N_0\ M$
  by (*pred-auto*)+

lemma *parallel-assm-lemma*:
  assumes *P is RD2*
  shows $pre_s \dagger (P \parallel_{M_R(M)} Q) = (((pre_s \dagger P) \parallel_{N_0(M)} ;;\ R1(true)\ (cmt_s \dagger Q))$
    $\vee\ ((cmt_s \dagger P) \parallel_{N_0(M)} ;;\ R1(true)\ (pre_s \dagger Q)))$
proof −
  have $pre_s \dagger (P \parallel_{M_R(M)} Q) = pre_s \dagger ((P \parallel_s Q) ;; M_R(M))$
    by (*simp add*: *par-by-merge-def*)
  also have ... $= ((P \parallel_s Q)[\![true,false/\$ok,\$wait]\!] ;; N_R\ M ;; R1(\neg\ \$ok))$
    by (*simp add*: *merge-rd-def usubst*, *rel-auto*)
  also have ... $= ((P[\![true,false/\$ok,\$wait]\!] \parallel_s Q[\![true,false/\$ok,\$wait]\!]) ;; N_1(M) ;; R1(\neg\ \$ok))$
    by (*rel-auto robust*, (*metis*)+)
  also have ... $= ($(
      $(((P[\![true,false/\$ok,\$wait]\!])^t \parallel_s (Q[\![true,false/\$ok,\$wait]\!])^t) ;; ((N_1\ M)[\![true,true/\$0{:}ok,\$1{:}ok]\!] ;;$
$R1(\neg\ \$ok))) \vee$
      $(((P[\![true,false/\$ok,\$wait]\!])^f \parallel_s (Q[\![true,false/\$ok,\$wait]\!])^t) ;; ((N_1\ M)[\![false,true/\$0{:}ok,\$1{:}ok]\!] ;;$
$R1(\neg\ \$ok))) \vee$
      $(((P[\![true,false/\$ok,\$wait]\!])^t \parallel_s (Q[\![true,false/\$ok,\$wait]\!])^f) ;; ((N_1\ M)[\![true,false/\$0{:}ok,\$1{:}ok]\!] ;;$
$R1(\neg\ \$ok))) \vee$
      $(((P[\![true,false/\$ok,\$wait]\!])^f \parallel_s (Q[\![true,false/\$ok,\$wait]\!])^f) ;; ((N_1\ M)[\![false,false/\$0{:}ok,\$1{:}ok]\!] ;;$
$R1(\neg\ \$ok))))\ )$
    (**is** - $= (?C1 \vee_p ?C2 \vee_p ?C3 \vee_p ?C4)$)
    by (*subst parallel-ok-cases*, *subst-tac*)
  also have ... $= (?C2 \vee ?C3)$
  proof −
    have $?C1 = false$
      by (*rel-auto*)
    moreover have ‘$?C4 \Rightarrow ?C3$‘ (**is** ‘$(?A ;; ?B) \Rightarrow (?C ;; ?D)$‘)
    proof −
      from *assms* **have** ‘$P^f \Rightarrow P^t$‘
        by (*metis RD2-def H2-equivalence Healthy-def'*)
      hence *P*: ‘${P^f}_f \Rightarrow {P^t}_f$‘
        by (*rel-auto*)
      have ‘$?A \Rightarrow ?C$‘
        using *P* **by** (*rel-auto*)
      moreover have ‘$?B \Rightarrow ?D$‘
        by (*rel-auto*)
      ultimately show *?thesis*
        by (*simp add*: *impl-seqr-mono*)
    qed
    ultimately show *?thesis*
      by (*simp add*: *subsumption2*)
  qed
  also have ... $= ($
      $(((pre_s \dagger P) \parallel_s (cmt_s \dagger Q)) ;; ((N_0\ M ;; R1(true)))) \vee$
      $(((cmt_s \dagger P) \parallel_s (pre_s \dagger Q)) ;; ((N_0\ M ;; R1(true)))))$
    by (*rel-auto*, *metis*+)
  also have ... $= ($
      $((pre_s \dagger P) \parallel_{N_0\ M ;;\ R1(true)} (cmt_s \dagger Q)) \vee$
      $((cmt_s \dagger P) \parallel_{N_0\ M ;;\ R1(true)} (pre_s \dagger Q)))$
    by (*simp add*: *par-by-merge-def*)
  finally show *?thesis* **.**

qed

lemma *pre$_s$-SRD*:
assumes *P is SRD*
shows *pre$_s$ † P = (¬$_r$ pre$_R$(P))*
proof −
have *pre$_s$ † P = pre$_s$ † **R**$_s$(pre$_R$ P ⊢ peri$_R$ P ⋄ post$_R$ P)*
by (*simp add*: *SRD-reactive-tri-design assms*)
also have *... = R1(R2c(¬ pre$_s$ † pre$_R$ P))*
by (*simp add*: *RHS-def usubst R3h-def pre$_s$-design*)
also have *... = R1(R2c(¬ pre$_R$ P))*
by (*rel-auto*)
also have *... = (¬$_r$ pre$_R$ P)*
by (*simp add*: *R2c-not R2c-preR assms rea-not-def*)
finally show *?thesis* **.**
qed

lemma *parallel-assm*:
assumes *P is SRD Q is SRD*
shows *pre$_R$(P ‖$_{M_R(M)}$ Q) = (¬$_r$ ((¬$_r$ pre$_R$(P)) ‖$_{N_0(M)}$;; R1(true) cmt$_R$(Q)) ∧*
¬$_r$ (cmt$_R$(P) ‖$_{N_0(M)}$;; R1(true) (¬$_r$ pre$_R$(Q))))
(**is** *?lhs = ?rhs*)
proof −
have *pre$_R$(P ‖$_{M_R(M)}$ Q) = (¬$_r$ (pre$_s$ † P) ‖$_{N_0\ M}$;; R1 true (cmt$_s$ † Q) ∧*
¬$_r$ (cmt$_s$ † P) ‖$_{N_0\ M}$;; R1 true (pre$_s$ † Q))
by (*simp add*: *pre$_R$-def parallel-assm-lemma assms SRD-healths R1-conj rea-not-def*[*THEN sym*])
also have *... = ?rhs*
by (*simp add*: *pre$_s$-SRD assms cmt$_R$-def Healthy-if closure unrest*)
finally show *?thesis* **.**
qed

lemma *parallel-assm-unrest-wait′* [*unrest*]:
⟦ *P is SRD*; *Q is SRD* ⟧ ⟹ *$wait´ ♯ pre$_R$(P ‖$_{M_R(M)}$ Q)*
by (*simp add*: *parallel-assm*, *simp add*: *par-by-merge-def unrest*)

lemma *JL1*: *(M$_1$ M)^t⟦false,true/$0:ok,$1:ok⟧ = N$_0$(M) ;; R1(true)*
by (*rel-blast*)

lemma *JL2*: *(M$_1$ M)^t⟦true,false/$0:ok,$1:ok⟧ = N$_0$(M) ;; R1(true)*
by (*rel-blast*)

lemma *JL3*: *(M$_1$ M)^t⟦false,false/$0:ok,$1:ok⟧ = N$_0$(M) ;; R1(true)*
by (*rel-blast*)

lemma *JL4*: *(M$_1$ M)^t⟦true,true/$0:ok,$1:ok⟧ = ($ok´ ∧ N$_0$ M) ;; II$_R{}^t$*
by (*simp add*: *merge-rd1-def usubst nmerge-rd1-def unrest*)

lemma *parallel-commitment-lemma-1*:
assumes *P is RD2*
shows *cmt$_s$ † (P ‖$_{M_R(M)}$ Q) = (*
((cmt$_s$ † P) ‖$_{(\$ok´\ \wedge\ N_0\ M)}$;; II$_R{}^t$ (cmt$_s$ † Q)) ∨
((pre$_s$ † P) ‖$_{N_0(M)}$;; R1(true) (cmt$_s$ † Q)) ∨

```
  ((cmt_s † P) ‖_{N_0(M)} ;; R1(true) (pre_s † Q)))
proof −
  have cmt_s † (P ‖_{M_R(M)} Q) = (P⟦true,false/$ok,$wait⟧ ‖_{(M_1(M))^t} Q⟦true,false/$ok,$wait⟧)
    by (simp add: usubst, rel-auto)
  also have ... = ((P⟦true,false/$ok,$wait⟧ ‖_s Q⟦true,false/$ok,$wait⟧) ;; (M_1 M)^t)
    by (simp add: par-by-merge-def)
  also have ... = (
      (((cmt_s † P) ‖_s (cmt_s † Q)) ;; ((M_1 M)^t⟦true,true/$0:ok,$1:ok⟧)) ∨
      (((pre_s † P) ‖_s (cmt_s † Q)) ;; ((M_1 M)^t⟦false,true/$0:ok,$1:ok⟧)) ∨
      (((cmt_s † P) ‖_s (pre_s † Q)) ;; ((M_1 M)^t⟦true,false/$0:ok,$1:ok⟧)) ∨
      (((pre_s † P) ‖_s (pre_s † Q)) ;; ((M_1 M)^t⟦false,false/$0:ok,$1:ok⟧)))
    by (subst parallel-ok-cases, subst-tac)
  also have ... = (
      (((cmt_s † P) ‖_s (cmt_s † Q)) ;; ((M_1 M)^t⟦true,true/$0:ok,$1:ok⟧)) ∨
      (((pre_s † P) ‖_s (cmt_s † Q)) ;; (N_0(M) ;; R1(true))) ∨
      (((cmt_s † P) ‖_s (pre_s † Q)) ;; (N_0(M) ;; R1(true))) ∨
      (((pre_s † P) ‖_s (pre_s † Q)) ;; (N_0(M) ;; R1(true))))
      (is - = (?C1 ∨_p ?C2 ∨_p ?C3 ∨_p ?C4))
    by (simp add: JL1 JL2 JL3)
  also have ... = (
      (((cmt_s † P) ‖_s (cmt_s † Q)) ;; ((M_1(M))^t⟦true,true/$0:ok,$1:ok⟧)) ∨
      (((pre_s † P) ‖_s (cmt_s † Q)) ;; (N_0(M) ;; R1(true))) ∨
      (((cmt_s † P) ‖_s (pre_s † Q)) ;; (N_0(M) ;; R1(true))))
  proof −
    from assms have ‘P^f ⇒ P^t‘
      by (metis RD2-def H2-equivalence Healthy-def′)
    hence P:‘P^f_f ⇒ P^t_f‘
      by (rel-auto)
    have ‘?C4 ⇒ ?C3‘ (is ‘(?A ;; ?B) ⇒ (?C ;; ?D)‘)
    proof −
      have ‘?A ⇒ ?C‘
        using P by (rel-auto)
      thus ?thesis
        by (simp add: impl-seqr-mono)
    qed
    thus ?thesis
      by (simp add: subsumption2)
  qed
  finally show ?thesis
    by (simp add: par-by-merge-def JL4)
qed

lemma parallel-commitment-lemma-2:
  assumes P is RD2
  shows cmt_s † (P ‖_{M_R(M)} Q) =
        (((cmt_s † P) ‖_{($ok´ ∧ N_0 M) ;; II_R^t} (cmt_s † Q)) ∨ pre_s † (P ‖_{M_R(M)} Q))
  by (simp add: parallel-commitment-lemma-1 assms parallel-assm-lemma)

lemma parallel-commitment-lemma-3:
  M is R1m ⟹ ($ok´ ∧ N_0 M) ;; II_R^t is R1m
  by (rel-simp, safe, metis+)

lemma parallel-commitment:
  assumes P is SRD Q is SRD M is RDM
  shows cmt_R(P ‖_{M_R(M)} Q) = (pre_R(P ‖_{M_R(M)} Q) ⇒_r cmt_R(P) ‖_{($ok´ ∧ N_0 M) ;; II_R^t} cmt_R(Q))
```

by (*simp add*: *parallel-commitment-lemma-2 parallel-commitment-lemma-3 Healthy-if assms* cmt_R*-def* pre_s*-SRD closure rea-impl-def disj-comm unrest*)

theorem *parallel-reactive-design*:
assumes *P is SRD Q is SRD M is RDM*
shows $(P \parallel_{M_R(M)} Q) = \mathbf{R}_s($
$(\neg_r\ ((\neg_r\ pre_R(P))\ \parallel_{N_0(M)}\ ;;\ R1(true)\ cmt_R(Q)) \wedge$
$\neg_r\ (cmt_R(P)\ \parallel_{N_0(M)}\ ;;\ R1(true)\ (\neg_r\ pre_R(Q)))) \vdash$
$(cmt_R(P)\ \parallel_{(\$ok' \wedge N_0\ M)}\ ;;\ II_R{}^t\ cmt_R(Q)))$ (**is** *?lhs = ?rhs*)
proof −
have $(P \parallel_{M_R(M)} Q) = \mathbf{R}_s(pre_R(P \parallel_{M_R(M)} Q) \vdash cmt_R(P \parallel_{M_R(M)} Q))$
by (*metis Healthy-def NSRD-is-SRD SRD-as-reactive-design assms*(*1*) *assms*(*2*) *assms*(*3*) *par-rdes-NSRD*)
also have *... = ?rhs*
by (*simp add*: *parallel-assm parallel-commitment design-export-spec assms*, *rel-auto*)
finally show *?thesis* **.**
qed

lemma *parallel-pericondition-lemma1*:
$(\$ok' \wedge P)\ ;;\ II_R[\![true,true/\$ok', \$wait']\!] = (\exists\ \$st' \cdot P)[\![true,true/\$ok',\$wait']\!]$
(**is** *?lhs = ?rhs*)
proof −
have *?lhs* $= (\$ok' \wedge P)\ ;;\ (\exists\ \$st \cdot II)[\![true,true/\$ok', \$wait']\!]$
by (*rel-blast*)
also have *... = ?rhs*
by (*rel-auto*)
finally show *?thesis* **.**
qed

lemma *parallel-pericondition-lemma2*:
assumes *M is RDM*
shows $(\exists\ \$st' \cdot N_0(M))[\![true,true/\$ok', \$wait']\!] = ((\$0{:}wait \vee \$1{:}wait) \wedge (\exists\ \$st' \cdot M))$
proof −
have $(\exists\ \$st' \cdot N_0(M))[\![true,true/\$ok', \$wait']\!] = (\exists\ \$st' \cdot (\$0{:}wait \vee \$1{:}wait) \wedge \$tr' \geq_u \${<}{:}tr \wedge M)$
by (*simp add*: *usubst unrest nmerge-rd0-def ex-unrest Healthy-if R1m-def assms*)
also have *... =* $(\exists\ \$st' \cdot (\$0{:}wait \vee \$1{:}wait) \wedge M)$
by (*metis* (*no-types*, *hide-lams*) *Healthy-if R1m-def R1m-idem R2m-def RDM-def assms utp-pred-laws.inf-commute*)
also have *...* $= ((\$0{:}wait \vee \$1{:}wait) \wedge (\exists\ \$st' \cdot M))$
by (*rel-auto*)
finally show *?thesis* **.**
qed

lemma *parallel-pericondition-lemma3*:
$((\$0{:}wait \vee \$1{:}wait) \wedge (\exists\ \$st' \cdot M)) = ((\$0{:}wait \wedge \$1{:}wait \wedge (\exists\ \$st' \cdot M)) \vee (\neg\ \$0{:}wait \wedge \$1{:}wait \wedge (\exists\ \$st' \cdot M)) \vee (\$0{:}wait \wedge \neg\ \$1{:}wait \wedge (\exists\ \$st' \cdot M)))$
by (*rel-auto*)

lemma *parallel-pericondition* [*rdes*]:
fixes $M :: ('s,'t::trace,'\alpha)$ *rsp merge*
assumes *P is SRD Q is SRD M is RDM*
shows $peri_R(P \parallel_{M_R(M)} Q) = (pre_R\ (P \parallel_{M_R\ M} Q) \Rightarrow_r peri_R(P) \parallel_{\exists\ \$st' \cdot M} peri_R(Q)$
$\vee\ post_R(P) \parallel_{\exists\ \$st' \cdot M} peri_R(Q)$
$\vee\ peri_R(P) \parallel_{\exists\ \$st' \cdot M} post_R(Q))$
proof −

have $peri_R(P \parallel_{M_R(M)} Q) =$
$(pre_R\ (P \parallel_{M_R\ M} Q) \Rightarrow_r cmt_R\ P \parallel_{(\$ok' \wedge N_0\ M)} ;;\ II_R[\![true,true/\$ok', \$wait']\!]\ cmt_R\ Q)$
by (*simp add*: *peri-cmt-def parallel-commitment SRD-healths assms usubst unrest assms*)
also have ... $= (pre_R\ (P \parallel_{M_R\ M} Q) \Rightarrow_r cmt_R\ P \parallel_{(\exists\ \$st' \cdot N_0\ M)[\![true,true/\$ok', \$wait']\!]} cmt_R\ Q)$
by (*simp add*: *parallel-pericondition-lemma1*)
also have ... $= (pre_R\ (P \parallel_{M_R\ M} Q) \Rightarrow_r cmt_R\ P \parallel_{(\$0{:}wait \vee \$1{:}wait) \wedge (\exists\ \$st' \cdot M)} cmt_R\ Q)$
by (*simp add*: *parallel-pericondition-lemma2 assms*)
also have ... $= (pre_R\ (P \parallel_{M_R\ M} Q) \Rightarrow_r ((\lceil cmt_R\ P \rceil_0 \wedge \lceil cmt_R\ Q \rceil_1 \wedge \${<}{:}\mathbf{v}' =_u \$\mathbf{v});; (\$0{:}wait \wedge \$1{:}wait \wedge (\exists\ \$st' \cdot M))$
$\vee (\lceil cmt_R\ P \rceil_0 \wedge \lceil cmt_R\ Q \rceil_1 \wedge \${<}{:}\mathbf{v}' =_u \$\mathbf{v});; (\neg\ \$0{:}wait \wedge \$1{:}wait \wedge (\exists\ \$st' \cdot M))$
$\vee (\lceil cmt_R\ P \rceil_0 \wedge \lceil cmt_R\ Q \rceil_1 \wedge \${<}{:}\mathbf{v}' =_u \$\mathbf{v});; (\$0{:}wait \wedge \neg\ \$1{:}wait \wedge (\exists\ \$st' \cdot M))))$
by (*simp add*: *par-by-merge-alt-def parallel-pericondition-lemma3 seqr-or-distr*)
also have ... $= (pre_R\ (P \parallel_{M_R\ M} Q) \Rightarrow_r ((\lceil peri_R\ P \rceil_0 \wedge \lceil peri_R\ Q \rceil_1 \wedge \$<' =_u \$\mathbf{v});; (\exists\ \$st' \cdot M)$
$\vee (\lceil post_R\ P \rceil_0 \wedge \lceil peri_R\ Q \rceil_1 \wedge \$<' =_u \$\mathbf{v});; (\exists\ \$st' \cdot M)$
$\vee (\lceil peri_R\ P \rceil_0 \wedge \lceil post_R\ Q \rceil_1 \wedge \$<' =_u \$\mathbf{v});; (\exists\ \$st' \cdot M)))$
by (*simp add*: *seqr-right-one-point-true seqr-right-one-point-false* cmt_R*-def* $post_R$*-def* $peri_R$*-def usubst unrest assms*)
also have ... $= (pre_R\ (P \parallel_{M_R\ M} Q) \Rightarrow_r peri_R(P) \parallel_{\exists\ \$st' \cdot M} peri_R(Q)$
$\vee\ post_R(P) \parallel_{\exists\ \$st' \cdot M} peri_R(Q)$
$\vee\ peri_R(P) \parallel_{\exists\ \$st' \cdot M} post_R(Q))$
by (*simp add*: *par-by-merge-alt-def*)
finally show *?thesis* **.**
qed

lemma *parallel-postcondition-lemma1*:
$(\$ok' \wedge P);;\ II_R[\![true,false/\$ok',\$wait']\!] = P[\![true,false/\$ok',\$wait']\!]$
(**is** *?lhs* = *?rhs*)
proof −
have *?lhs* $= (\$ok' \wedge P);;\ II[\![true,false/\$ok', \$wait']\!]$
by (*rel-blast*)
also have ... = *?rhs*
by (*rel-auto*)
finally show *?thesis* **.**
qed

lemma *parallel-postcondition-lemma2*:
assumes *M is RDM*
shows $(N_0(M))[\![true,false/\$ok',\$wait']\!] = ((\neg\ \$0{:}wait \wedge \neg\ \$1{:}wait) \wedge M)$
proof −
have $(N_0(M))[\![true,false/\$ok',\$wait']\!] = ((\neg\ \$0{:}wait \wedge \neg\ \$1{:}wait) \wedge \$tr' \geq_u \${<}{:}tr \wedge M)$
by (*simp add*: *usubst unrest nmerge-rd0-def ex-unrest Healthy-if R1m-def assms*)
also have ... $= ((\neg\ \$0{:}wait \wedge \neg\ \$1{:}wait) \wedge M)$
by (*metis Healthy-if R1m-def RDM-R1m assms utp-pred-laws.inf-commute*)
finally show *?thesis* **.**
qed

lemma *parallel-postcondition* [*rdes*]:
fixes $M :: ('s,'t::trace,'\alpha)$ *rsp merge*
assumes *P is SRD Q is SRD M is RDM*
shows $post_R(P \parallel_{M_R(M)} Q) = (pre_R\ (P \parallel_{M_R\ M} Q) \Rightarrow_r post_R(P) \parallel_M post_R(Q))$
proof −
have $post_R(P \parallel_{M_R(M)} Q) =$

$(pre_R\ (P \parallel_{M_R\ M}\ Q) \Rightarrow_r\ cmt_R\ P \parallel_{(\$ok' \wedge N_0\ M)}\ ;;\ II_R[\![true,false/\$ok', \$wait']\!]\ cmt_R\ Q)$
by (*simp add*: *post-cmt-def parallel-commitment assms usubst unrest SRD-healths*)
also have ... = $(pre_R\ (P \parallel_{M_R\ M}\ Q) \Rightarrow_r\ cmt_R\ P \parallel_{(\neg\ \$0{:}wait \wedge \neg\ \$1{:}wait \wedge M)}\ cmt_R\ Q)$
by (*simp add*: *parallel-postcondition-lemma1 parallel-postcondition-lemma2 assms*, *simp add*: *utp-pred-laws.inf-commute utp-pred-laws.inf-left-commute*)
also have ... = $(pre_R\ (P \parallel_{M_R\ M}\ Q) \Rightarrow_r\ post_R\ P \parallel_M\ post_R\ Q)$
by (*simp add*: *par-by-merge-alt-def seqr-right-one-point-false usubst unrest* cmt_R*-def* $post_R$*-def assms*)
finally show *?thesis* **.**
qed

lemma *parallel-precondition-lemma*:
fixes $M :: ('s,'t::trace,'\alpha)\ rsp\ merge$
assumes *P is NSRD Q is NSRD M is RDM*
shows $(\neg_r\ pre_R(P)) \parallel_{N_0(M)}\ ;;\ R1(true)\ cmt_R(Q) =$
$((\neg_r\ pre_R\ P) \parallel_M\ ;;\ R1(true)\ peri_R\ Q \vee (\neg_r\ pre_R\ P) \parallel_M\ ;;\ R1(true)\ post_R\ Q)$
proof −
have $((\neg_r\ pre_R(P)) \parallel_{N_0(M)}\ ;;\ R1(true)\ cmt_R(Q)) =$
$((\neg_r\ pre_R(P)) \parallel_{N_0(M)}\ ;;\ R1(true)\ (peri_R(Q) \diamond post_R(Q)))$
by (*simp add*: *wait'-cond-peri-post-cmt*)
also have ... = $((\lceil\neg_r\ pre_R(P)\rceil_0 \wedge \lceil peri_R(Q) \diamond post_R(Q)\rceil_1 \wedge \$<{:}\mathbf{v}' =_u \$\mathbf{v})\ ;;\ N_0(M)\ ;;\ R1(true))$
by (*simp add*: *par-by-merge-alt-def*)
also have ... = $((\lceil\neg_r\ pre_R(P)\rceil_0 \wedge \lceil peri_R(Q)\rceil_1 \lhd \$1{:}wait' \rhd \lceil post_R(Q)\rceil_1 \wedge \$<{:}\mathbf{v}' =_u \$\mathbf{v})\ ;;\ N_0(M)$;; $R1(true))$
by (*simp add*: *wait'-cond-def alpha*)
also have ... = $(((\lceil\neg_r\ pre_R(P)\rceil_0 \wedge \lceil peri_R(Q)\rceil_1 \wedge \$<{:}\mathbf{v}' =_u \$\mathbf{v}) \lhd \$1{:}wait' \rhd (\lceil\neg_r\ pre_R(P)\rceil_0 \wedge \lceil post_R(Q)\rceil_1 \wedge \$<{:}\mathbf{v}' =_u \$\mathbf{v}))\ ;;\ N_0(M)\ ;;\ R1(true))$
(**is** $(?P\ ;;\ \text{-}) = (?Q\ ;;\ \text{-})$)
proof −
have $?P = ?Q$
by (*rel-auto*)
thus *?thesis* **by** *simp*
qed
also have ... = $((\lceil\neg_r\ pre_R\ P\rceil_0 \wedge \lceil peri_R\ Q\rceil_1 \wedge \$<{:}\mathbf{v}' =_u \$\mathbf{v})[\![true/\$1{:}wait']\!]\ ;;\ (N_0\ M\ ;;\ R1\ true)[\![true/\$1{:}wait]\!] \vee$
$(\lceil\neg_r\ pre_R\ P\rceil_0 \wedge \lceil post_R\ Q\rceil_1 \wedge \$<{:}\mathbf{v}' =_u \$\mathbf{v})[\![false/\$1{:}wait']\!]\ ;;\ (N_0\ M\ ;;\ R1\ true)[\![false/\$1{:}wait]\!])$
by (*simp add*: *cond-inter-var-split*)
also have ... = $((\lceil\neg_r\ pre_R\ P\rceil_0 \wedge \lceil peri_R\ Q\rceil_1 \wedge \$<{:}\mathbf{v}' =_u \$\mathbf{v})\ ;;\ N_0\ M[\![true/\$1{:}wait]\!]\ ;;\ R1\ true \vee$
$(\lceil\neg_r\ pre_R\ P\rceil_0 \wedge \lceil post_R\ Q\rceil_1 \wedge \$<{:}\mathbf{v}' =_u \$\mathbf{v})\ ;;\ N_0\ M[\![false/\$1{:}wait]\!]\ ;;\ R1\ true)$
by (*simp add*: *usubst unrest*)
also have ... = $((\lceil\neg_r\ pre_R\ P\rceil_0 \wedge \lceil peri_R\ Q\rceil_1 \wedge \$<{:}\mathbf{v}' =_u \$\mathbf{v})\ ;;\ (\$wait' \wedge M)\ ;;\ R1\ true \vee$
$(\lceil\neg_r\ pre_R\ P\rceil_0 \wedge \lceil post_R\ Q\rceil_1 \wedge \$<{:}\mathbf{v}' =_u \$\mathbf{v})\ ;;\ (\$wait' =_u \$0{:}wait \wedge M)\ ;;\ R1\ true)$
proof −
have $(\$tr' \geq_u \$<{:}tr \wedge M) = M$
using *RDM-R1m*[*OF assms*(*3*)]
by (*simp add*: *Healthy-def R1m-def conj-comm*)
thus *?thesis*
by (*simp add*: *nmerge-rd0-def unrest assms closure ex-unrest usubst*)
qed
also have ... = $((\lceil\neg_r\ pre_R\ P\rceil_0 \wedge \lceil peri_R\ Q\rceil_1 \wedge \$<{:}\mathbf{v}' =_u \$\mathbf{v})\ ;;\ M\ ;;\ R1\ true \vee$
$(\lceil\neg_r\ pre_R\ P\rceil_0 \wedge \lceil post_R\ Q\rceil_1 \wedge \$<{:}\mathbf{v}' =_u \$\mathbf{v})\ ;;\ M\ ;;\ R1\ true)$
(**is** $(?P_1 \vee_p\ ?P_2) = (?Q_1 \vee\ ?Q_2)$)
proof −
have $?P_1 = (\lceil\neg_r\ pre_R\ P\rceil_0 \wedge \lceil peri_R\ Q\rceil_1 \wedge \$<{:}\mathbf{v}' =_u \$\mathbf{v})\ ;;\ (M \wedge \$wait')\ ;;\ R1\ true$

by (*simp add*: *conj-comm*)
hence *1*: $?P_1 = ?Q_1$
by (*simp add*: *seqr-left-one-point-true seqr-left-one-point-false add*: *unrest usubst closure assms*)
have $?P_2 = ((\lceil \neg_r\ pre_R\ P \rceil_0 \wedge \lceil post_R\ Q \rceil_1 \wedge \$<:\mathbf{v}' =_u \$\mathbf{v})\ ;;\ (M \wedge \$wait')\ ;;\ R1\ true \vee$
$(\lceil \neg_r\ pre_R\ P \rceil_0 \wedge \lceil post_R\ Q \rceil_1 \wedge \$<:\mathbf{v}' =_u \$\mathbf{v})\ ;;\ (M \wedge \neg\ \$wait')\ ;;\ R1\ true)$
by (*subst seqr-bool-split*[*of wait* $;_L$ *mrg-left*], *simp-all add*: *usubst unrest assms closure conj-comm*)
hence *2*: $?P_2 = ?Q_2$
by (*simp add*: *seqr-left-one-point-true seqr-left-one-point-false unrest usubst closure assms*)
from *1 2* **show** *?thesis* **by** *simp*
qed
also have ... $= ((\neg_r\ pre_R\ P) \parallel_{M\ ;;\ R1(true)} peri_R\ Q \vee (\neg_r\ pre_R\ P) \parallel_{M\ ;;\ R1(true)} post_R\ Q)$
by (*simp add*: *par-by-merge-alt-def*)
finally show *?thesis* **.**
qed

lemma *swap-nmerge-rd0*:
$swap_m\ ;;\ N_0(M) = N_0(swap_m\ ;;\ M)$
by (*rel-auto*, *meson*+)

lemma *SymMerge-nmerge-rd0* [*closure*]:
M is SymMerge $\Longrightarrow$ $N_0(M)$ *is SymMerge*
by (*rel-auto*, *meson*+)

lemma *swap-merge-rd'*:
$swap_m\ ;;\ N_R(M) = N_R(swap_m\ ;;\ M)$
by (*rel-blast*)

lemma *swap-merge-rd*:
$swap_m\ ;;\ M_R(M) = M_R(swap_m\ ;;\ M)$
by (*simp add*: *merge-rd-def seqr-assoc*[*THEN sym*] *swap-merge-rd'*)

lemma *SymMerge-merge-rd* [*closure*]:
M is SymMerge $\Longrightarrow$ $M_R(M)$ *is SymMerge*
by (*simp add*: *Healthy-def swap-merge-rd*)

lemma *nmerge-rd1-merge3*:
assumes *M is RDM*
shows $\mathbf{M}3(N_1(M)) = (\$ok' =_u (\$0{:}ok \wedge \$1{:}0{:}ok \wedge \$1{:}1{:}ok) \wedge$
$\$wait' =_u (\$0{:}wait \vee \$1{:}0{:}wait \vee \$1{:}1{:}wait) \wedge$
$\mathbf{M}3(M))$
proof −
have $\mathbf{M}3(N_1(M)) = \mathbf{M}3(\$ok' =_u (\$0{:}ok \wedge \$1{:}ok) \wedge$
$\$wait' =_u (\$0{:}wait \vee \$1{:}wait) \wedge$
$\$<:tr \leq_u \$tr' \wedge$
$(\exists\ \{\$0{:}ok, \$1{:}ok, \$<:ok, \$ok', \$0{:}wait, \$1{:}wait, \$<:wait, \$wait'\} \cdot RDM(M)))$
by (*simp add*: *nmerge-rd1-def nmerge-rd0-def assms Healthy-if*)
also have ... $= \mathbf{M}3(\$ok' =_u (\$0{:}ok \wedge \$1{:}ok) \wedge \$wait' =_u (\$0{:}wait \vee \$1{:}wait) \wedge RDM(M))$
by (*rel-blast*)
also have ... $= (\$ok' =_u (\$0{:}ok \wedge \$1{:}0{:}ok \wedge \$1{:}1{:}ok) \wedge \$wait' =_u (\$0{:}wait \vee \$1{:}0{:}wait \vee \$1{:}1{:}wait)$
$\wedge\ \mathbf{M}3(RDM(M)))$
by (*rel-blast*)
also have ... $= (\$ok' =_u (\$0{:}ok \wedge \$1{:}0{:}ok \wedge \$1{:}1{:}ok) \wedge \$wait' =_u (\$0{:}wait \vee \$1{:}0{:}wait \vee \$1{:}1{:}wait)$
$\wedge\ \mathbf{M}3(M))$
by (*simp add*: *assms Healthy-if*)
finally show *?thesis* **.**

qed

lemma *nmerge-rd-merge3*:
M3($N_R(M)$) = ($\exists$ $<:st · $**v**´ =$_u$ $<:**v**) ◁ $<:wait ▷ **M**3($N_1$ M) ◁ $<:ok ▷ ($<:tr ≤$_u$ $tr´)
by (*rel-blast*)

lemma *AssocMerge-nmerge-rd*:
assumes *M is RDM AssocMerge M*
shows *AssocMerge*($N_R(M)$)
proof −
have *1*:**M**3(M) = *rotate*$_m$;; **M**3(M)
using *assms* **by** (*simp add*: *AssocMerge-def*)
have *rotate*$_m$;; (**M**3($N_R(M)$)) =
rotate$_m$;;
(($\exists$ $<:st · $**v**´ =$_u$ $<:**v**) ◁ $<:wait ▷
($ok´ =$_u$ ($0:ok ∧ $1:0:ok ∧ $1:1:ok) ∧ $wait´ =$_u$ ($0:wait ∨ $1:0:wait ∨ $1:1:wait) ∧
M3(M)) ◁ $<:ok ▷
($<:tr ≤$_u$ $tr´))
by (*simp add*: *AssocMerge-def nmerge-rd-merge3 nmerge-rd1-merge3 assms*)
also have ... =
(($\exists$ $<:st · $**v**´ =$_u$ $<:**v**) ◁ $<:wait ▷
($ok´ =$_u$ ($0:ok ∧ $1:0:ok ∧ $1:1:ok) ∧ $wait´ =$_u$ ($0:wait ∨ $1:0:wait ∨ $1:1:wait) ∧
(*rotate*$_m$;; **M**3(M))) ◁ $<:ok ▷
($<:tr ≤$_u$ $tr´))
by (*rel-blast*)
also have ... =
(($\exists$ $<:st · $**v**´ =$_u$ $<:**v**) ◁ $<:wait ▷
($ok´ =$_u$ ($0:ok ∧ $1:0:ok ∧ $1:1:ok) ∧ $wait´ =$_u$ ($0:wait ∨ $1:0:wait ∨ $1:1:wait) ∧
M3(M)) ◁ $<:ok ▷
($<:tr ≤$_u$ $tr´))
using *1* **by** *auto*
also have ... = **M**3($N_R(M)$)
by (*simp add*: *AssocMerge-def nmerge-rd-merge3 nmerge-rd1-merge3 assms*)
finally show *?thesis*
using *AssocMerge-def* **by** *blast*
qed

lemma *swap-merge-RDM-closed* [*closure*]:
assumes *M is RDM*
shows *swap*$_m$;; *M is RDM*
proof −
have *RDM*(*swap*$_m$;; *RDM*(M)) = (*swap*$_m$;; *RDM*(M))
by (*rel-auto*)
thus *?thesis*
by (*metis Healthy-def′ assms*)
qed

lemma *parallel-precondition*:
fixes M :: ('*s*,'*t*::*trace*,'α) *rsp merge*
assumes *P is NSRD Q is NSRD M is RDM*
shows *pre*$_R$(P ‖$_{M_R(M)}$ Q) =
(¬$_r$ ((¬$_r$ *pre*$_R$ P) ‖$_{M\ ;;\ R1(true)}$ *peri*$_R$ Q) ∧
¬$_r$ ((¬$_r$ *pre*$_R$ P) ‖$_{M\ ;;\ R1(true)}$ *post*$_R$ Q) ∧
¬$_r$ ((¬$_r$ *pre*$_R$ Q) ‖$_{(swap_m\ ;;\ M)\ ;;\ R1(true)}$ *peri*$_R$ P) ∧
¬$_r$ ((¬$_r$ *pre*$_R$ Q) ‖$_{(swap_m\ ;;\ M)\ ;;\ R1(true)}$ *post*$_R$ P))

proof −
have *a*: $(\neg_r\ pre_R(P))\ \|_{N_0(M)}\ ;;\ R1(true)\ cmt_R(Q) =$
$((\neg_r\ pre_R\ P)\ \|_M\ ;;\ R1(true)\ peri_R\ Q \vee (\neg_r\ pre_R\ P)\ \|_M\ ;;\ R1(true)\ post_R\ Q)$
by (*simp add*: *parallel-precondition-lemma assms*)

have *b*: $(\neg_r\ cmt_R\ P\ \|_{N_0}\ M\ ;;\ R1\ true\ (\neg_r\ pre_R\ Q)) =$
$(\neg_r\ (\neg_r\ pre_R(Q))\ \|_{N_0(swap_m\ ;;\ M)}\ ;;\ R1(true)\ cmt_R(P))$
by (*simp add*: *swap-nmerge-rd0*[*THEN sym*] *seqr-assoc*[*THEN sym*] *par-by-merge-def par-sep-swap*)
have *c*: $(\neg_r\ pre_R(Q))\ \|_{N_0(swap_m\ ;;\ M)}\ ;;\ R1(true)\ cmt_R(P) =$
$((\neg_r\ pre_R\ Q)\ \|_{(swap_m\ ;;\ M)}\ ;;\ R1(true)\ peri_R\ P \vee (\neg_r\ pre_R\ Q)\ \|_{(swap_m\ ;;\ M)}\ ;;\ R1(true)\ post_R\ P)$
by (*simp add*: *parallel-precondition-lemma closure assms*)

show *?thesis*
by (*simp add*: *parallel-assm closure assms a b c*, *rel-auto*)
qed

Weakest Parallel Precondition

definition *wrR* ::
('*t*::*trace*, 'α) *hrel-rp* ⇒
('*t* :: *trace*, 'α) *rp merge* ⇒
('*t*, 'α) *hrel-rp* ⇒
('*t*, 'α) *hrel-rp* (- wr_R'(-') - [*60*,*0*,*61*] *61*)
where [*upred-defs*]: $Q\ wr_R(M)\ P = (\neg_r\ ((\neg_r\ P)\ \|_{M\ ;;\ R1(true)}\ Q))$

lemma *wrR-R1* [*closure*]:
M is R1m ⟹ $Q\ wr_R(M)\ P$ *is R1*
by (*simp add*: *wrR-def closure*)

lemma *R2-rea-not*: $R2(\neg_r\ P) = (\neg_r\ R2(P))$
by (*rel-auto*)

lemma *wrR-R2-lemma*:
assumes *P is R2 Q is R2 M is R2m*
shows $((\neg_r\ P)\ \|_M\ Q)$;; $R1(true_h)$ *is R2*
proof −
have $(\neg_r\ P)\ \|_M\ Q$ *is R2*
by (*simp add*: *closure assms*)
thus *?thesis*
by (*simp add*: *closure*)
qed

lemma *wrR-R2* [*closure*]:
assumes *P is R2 Q is R2 M is R2m*
shows $Q\ wr_R(M)\ P$ *is R2*
proof −
have $((\neg_r\ P)\ \|_M\ Q)$;; $R1(true_h)$ *is R2*
by (*simp add*: *wrR-R2-lemma assms*)
thus *?thesis*
by (*simp add*: *wrR-def wrR-R2-lemma par-by-merge-seq-add closure*)
qed

lemma *wrR-RR* [*closure*]:
assumes *P is RR Q is RR M is RDM*
shows $Q\ wr_R(M)\ P$ *is RR*

apply (*rule RR-intro*)
apply (*simp-all add*: *unrest assms closure wrR-def rpred*)
apply (*metis* (*no-types*, *lifting*) *Healthy-def' R1-R2c-commute R1-R2c-is-R2 R1-rea-not RDM-R2m RR-implies-R2 assms*(*1*) *assms*(*2*) *assms*(*3*) *par-by-merge-seq-add rea-not-R2-closed wrR-R2-lemma*)
done

lemma *wrR-RC* [*closure*]:
assumes *P is RR Q is RR M is RDM*
shows (*Q* $wr_R(M)$ *P*) *is RC*
apply (*rule RC-intro*)
apply (*simp add*: *closure assms*)
apply (*simp add*: *wrR-def rpred closure assms*)
apply (*simp add*: *par-by-merge-def seqr-assoc*)
done

lemma *wppR-choice* [*wp*]: (*P* ∨ *Q*) $wr_R(M)$ *R* = (*P* $wr_R(M)$ *R* ∧ *Q* $wr_R(M)$ *R*)
proof −
have (*P* ∨ *Q*) $wr_R(M)$ *R* =
(¬$_r$ ((¬$_r$ *R*) ;; *U0* ∧ (*P* ;; *U1* ∨ *Q* ;; *U1*) ∧ $<:**v**´ =$_u$ $**v**) ;; *M* ;; *true*$_r$)
by (*simp add*: *wrR-def par-by-merge-def seqr-or-distl*)
also have ... = (¬$_r$ ((¬$_r$ *R*) ;; *U0* ∧ *P* ;; *U1* ∧ $<:**v**´ =$_u$ $**v** ∨ (¬$_r$ *R*) ;; *U0* ∧ *Q* ;; *U1* ∧ $<:**v**´ =$_u$ $**v**) ;; *M* ;; *true*$_r$)
by (*simp add*: *conj-disj-distr utp-pred-laws.inf-sup-distrib2*)
also have ... = (¬$_r$ (((¬$_r$ *R*) ;; *U0* ∧ *P* ;; *U1* ∧ $<:**v**´ =$_u$ $**v**) ;; *M* ;; *true*$_r$ ∨
((¬$_r$ *R*) ;; *U0* ∧ *Q* ;; *U1* ∧ $<:**v**´ =$_u$ $**v**) ;; *M* ;; *true*$_r$))
by (*simp add*: *seqr-or-distl*)
also have ... = (*P* $wr_R(M)$ *R* ∧ *Q* $wr_R(M)$ *R*)
by (*simp add*: *wrR-def par-by-merge-def*)
finally show *?thesis* **.**
qed

lemma *wppR-impl* [*wp*]: (*P* ⇒$_r$ *Q*) $wr_R(M)$ *R* = ((¬$_r$ *P*) $wr_R(M)$ *R* ∧ *Q* $wr_R(M)$ *R*)
by (*simp add*: *rea-impl-def wp*)

lemma *wppR-miracle* [*wp*]: *false* $wr_R(M)$ *P* = *true*$_r$
by (*simp add*: *wrR-def*)

lemma *wppR-true* [*wp*]: *P* $wr_R(M)$ *true*$_r$ = *true*$_r$
by (*simp add*: *wrR-def*)

lemma *parallel-precondition-wr* [*rdes*]:
assumes *P is NSRD Q is NSRD M is RDM*
shows $pre_R(P \parallel_{M_R(M)} Q)$ = ($peri_R(Q)$ $wr_R(M)$ $pre_R(P)$ ∧ $post_R(Q)$ $wr_R(M)$ $pre_R(P)$ ∧
$peri_R(P)$ $wr_R(swap_m$;; *M*) $pre_R(Q)$ ∧ $post_R(P)$ $wr_R(swap_m$;; *M*) $pre_R(Q)$)
by (*simp add*: *assms parallel-precondition wrR-def*)

lemma *rea-impl-merge-left*: (*P* ⇒$_r$ *Q*) ‖$_M$ *R* = (((¬$_r$ *P*) ‖$_M$ *R*) ∨ (*Q* ‖$_M$ *R*))
by (*simp add*: *rea-impl-def par-by-merge-or-left*)

lemma *rea-impl-merge-right*: *P* ‖$_M$ (*Q* ⇒$_r$ *R*) = (*P* ‖$_M$ (¬$_r$ *Q*) ∨ *P* ‖$_M$ *R*)
by (*simp add*: *rea-impl-def par-by-merge-or-right*)

lemma *parallel-pre-lemma*:
((*Q*$_1$ ⇒$_r$ *Q*$_2$) $wr_R(M)$ *P*$_1$ ∧ (*P*$_1$ ⇒$_r$ *P*$_2$) ‖$_{∃\ \$st´\ .\ M}$ (*Q*$_1$ ⇒$_r$ *Q*$_2$))

$= ((Q_1 \Rightarrow_r Q_2)\ wr_R(M)\ P_1 \wedge P_2 \parallel_{\exists\ \$st'\ .\ M} (Q_1 \Rightarrow_r Q_2))$
apply (*simp add*: *par-by-merge-alt-def*)
apply (*rel-auto*)
apply (*meson order-refl*)
apply (*meson order-refl*)
apply *blast*
apply *blast*
apply *blast*
apply *blast*
done

lemma *parallel-rdes-def* [*rdes-def*]:
assumes P_1 *is RC* P_2 *is RR* P_3 *is RR* Q_1 *is RC* Q_2 *is RR* Q_3 *is RR*
$\$st' \sharp P_2$ $\$st' \sharp Q_2$
M *is RDM*
shows $\mathbf{R}_s(P_1 \vdash P_2 \diamond P_3) \parallel_{M_R(M)} \mathbf{R}_s(Q_1 \vdash Q_2 \diamond Q_3) =$
$\mathbf{R}_s(((Q_1 \Rightarrow_r Q_2)\ wr_R(M)\ P_1 \wedge (Q_1 \Rightarrow_r Q_3)\ wr_R(M)\ P_1 \wedge$
$(P_1 \Rightarrow_r P_2)\ wr_R(swap_m ;; M)\ Q_1 \wedge (P_1 \Rightarrow_r P_3)\ wr_R(swap_m ;; M)\ Q_1) \vdash$
$(P_2 \parallel_{\exists\ \$st'\ .\ M} Q_2 \vee$
$P_3 \parallel_{\exists\ \$st'\ .\ M} Q_2 \vee P_2 \parallel_{\exists\ \$st'\ .\ M} Q_3) \diamond$
$(P_3 \parallel_M Q_3))$ (**is** *?lhs* = *?rhs*)
proof −
have *?lhs* = $\mathbf{R}_s$ (pre_R *?lhs* ⊢ $peri_R$ *?lhs* ◇ $post_R$ *?lhs*)
by (*simp add*: *SRD-reactive-tri-design assms closure*)
also have ... =
$\mathbf{R}_s(((Q_1 \Rightarrow_r Q_2)\ wr_R(M)\ P_1 \wedge (Q_1 \Rightarrow_r Q_3)\ wr_R(M)\ P_1 \wedge$
$(P_1 \Rightarrow_r P_2)\ wr_R(swap_m ;; M)\ Q_1 \wedge (P_1 \Rightarrow_r P_3)\ wr_R(swap_m ;; M)\ Q_1) \vdash$
$((P_1 \Rightarrow_r P_2) \parallel_{\exists\ \$st'\ .\ M} (Q_1 \Rightarrow_r Q_2) \vee$
$(P_1 \Rightarrow_r P_3) \parallel_{\exists\ \$st'\ .\ M} (Q_1 \Rightarrow_r Q_2) \vee (P_1 \Rightarrow_r P_2) \parallel_{\exists\ \$st'\ .\ M} (Q_1 \Rightarrow_r Q_3)) \diamond$
$((P_1 \Rightarrow_r P_3) \parallel_M (Q_1 \Rightarrow_r Q_3)))$
(**is** - = $\mathbf{R}_s$(*?X*
⊢ (*?*Y_1 ∨ *?*Y_2 ∨ *?*Y_3)
◇ *?Z*))
by (*simp add*: *rdes closure unrest assms*, *rel-auto*)
also have ... = $\mathbf{R}_s$(*?X* ⊢ ((*?X* ∧ *?*Y_1) ∨ (*?X* ∧ *?*Y_2) ∨ (*?X* ∧ *?*Y_3)) ◇ (*?X* ∧ *?Z*))
by (*rel-auto*)
also have ... = $\mathbf{R}_s$(*?X* ⊢ ((*?X* ∧ $P_2 \parallel_{\exists\ \$st'\ .\ M} Q_2$) ∨ (*?X* ∧ $P_3 \parallel_{\exists\ \$st'\ .\ M} Q_2$) ∨ (*?X* ∧ $P_2 \parallel_{\exists\ \$st'\ .\ M} Q_3$)) ◇ (*?X* ∧ $P_3 \parallel_M Q_3$))
proof −
have *1*:(*?X* ∧ *?*Y_1) = (*?X* ∧ $P_2 \parallel_{\exists\ \$st'\ .\ M} Q_2$)
by (*rel-auto*, *meson order-refl*, *meson order-refl*, *meson order-refl*, *blast*+)
have *2*:(*?X* ∧ *?*Y_2) = (*?X* ∧ $P_3 \parallel_{\exists\ \$st'\ .\ M} Q_2$)
by (*rel-auto*, *meson order-refl*, *meson order-refl*, *meson order-refl*, *blast*+)
have *3*:(*?X* ∧ *?*Y_3) = (*?X* ∧ $P_2 \parallel_{\exists\ \$st'\ .\ M} Q_3$)
by (*rel-auto*, *meson order-refl*, *meson order-refl*, *meson order-refl*, *blast*+)
have *4*:(*?X* ∧ *?Z*) = (*?X* ∧ $P_3 \parallel_M Q_3$)
by (*rel-auto*, *meson order-refl*, *meson order-refl*, *meson order-refl*, *blast*+)
show *?thesis*
by (*simp add*: *1 2 3 4*)
qed
also have ... = *?rhs*
by (*rel-auto*)
finally show *?thesis* .
qed

lemma *Miracle-parallel-left-zero*:
assumes *P is SRD M is RDM*
shows *Miracle* $\parallel_{RM}$ *P* = *Miracle*
proof −
have pre_R(*Miracle* $\parallel_{RM}$ *P*) = $true_r$
by (*simp add*: *parallel-assm wait'-cond-idem rdes closure assms*)
moreover hence cmt_R(*Miracle* $\parallel_{RM}$ *P*) = *false*
by (*simp add*: *rdes closure wait'-cond-idem SRD-healths assms*)
ultimately have *Miracle* $\parallel_{RM}$ *P* = $\mathbf{R}_s$($true_r$ ⊢ *false*)
by (*metis NSRD-iff SRD-reactive-design-alt assms par-rdes-NSRD srdes-theory.weak.top-closed*)
thus *?thesis*
by (*simp add*: *Miracle-def R1-design-R1-pre*)
qed

lemma *Miracle-parallel-right-zero*:
assumes *P is SRD M is RDM*
shows *P* $\parallel_{RM}$ *Miracle* = *Miracle*
proof −
have pre_R(*P* $\parallel_{RM}$ *Miracle*) = $true_r$
by (*simp add*: *wait'-cond-idem parallel-assm rdes closure assms*)
moreover hence cmt_R(*P* $\parallel_{RM}$ *Miracle*) = *false*
by (*simp add*: *wait'-cond-idem rdes closure SRD-healths assms*)
ultimately have *P* $\parallel_{RM}$ *Miracle* = $\mathbf{R}_s$($true_r$ ⊢ *false*)
by (*metis NSRD-iff SRD-reactive-design-alt assms par-rdes-NSRD srdes-theory.weak.top-closed*)
thus *?thesis*
by (*simp add*: *Miracle-def R1-design-R1-pre*)
qed

8.1 Example basic merge

definition *BasicMerge* :: (('*s*, '*t*::*trace*, *unit*) *rsp*) *merge* (N_B) **where**
[*upred-defs*]: *BasicMerge* = ($<:*tr* ≤$_u$ $*tr*´ ∧ $*tr*´ − $<:*tr* =$_u$ $*0*:*tr* − $<:*tr* ∧ $*tr*´ − $<:*tr* =$_u$ $*1*:*tr* − $<:*tr* ∧ $*st*´ =$_u$ $<:*st*)

abbreviation *rbasic-par* (- $\parallel_B$ - [*85*,*86*] *85*) **where**
P $\parallel_B$ *Q* ≡ *P* $\parallel_{M_R(N_B)}$ *Q*

lemma *BasicMerge-RDM* [*closure*]: N_B *is RDM*
by (*rule RDM-intro*, (*rel-auto*)+)

lemma *BasicMerge-SymMerge* [*closure*]:
N_B *is SymMerge*
by (*rel-auto*)

lemma *BasicMerge'-calc*:
assumes $*ok*´ ♯ *P* $*wait*´ ♯ *P* $*ok*´ ♯ *Q* $*wait*´ ♯ *Q P is R2 Q is R2*
shows *P* $\parallel_{N_B}$ *Q* = ((∃ $*st*´ • *P*) ∧ (∃ $*st*´ • *Q*) ∧ $*st*´ =$_u$ $*st*)
using *assms*
proof −
have *P*:(∃ {$*ok*´,$*wait*´} • *R2*(*P*)) = *P* (**is** *?P'* = -)
by (*simp add*: *ex-unrest ex-plus Healthy-if assms*)
have *Q*:(∃ {$*ok*´,$*wait*´} • *R2*(*Q*)) = *Q* (**is** *?Q'* = -)
by (*simp add*: *ex-unrest ex-plus Healthy-if assms*)
have *?P'* $\parallel_{N_B}$ *?Q'* = ((∃ $*st*´ • *?P'*) ∧ (∃ $*st*´ • *?Q'*) ∧ $*st*´ =$_u$ $*st*)
by (*simp add*: *par-by-merge-alt-def*, *rel-auto*, *blast*+)

thus *?thesis*
by (*simp add*: *P Q*)
qed

8.2 Simple parallel composition

definition *rea-design-par* ::
('*s*, '*t*::*trace*, 'α) *hrel-rsp* ⇒ ('*s*, '*t*, 'α) *hrel-rsp* ⇒ ('*s*, '*t*, 'α) *hrel-rsp* (**infixr** $\|_R$ *85*)
where [*upred-defs*]: $P \parallel_R Q = \mathbf{R}_s((pre_R(P) \ \wedge pre_R(Q)) \vdash (cmt_R(P) \wedge cmt_R(Q)))$

lemma *RHS-design-par*:
assumes
$\$ok' \sharp P_1$ $\$ok' \sharp P_2$
shows $\mathbf{R}_s(P_1 \vdash Q_1) \parallel_R \mathbf{R}_s(P_2 \vdash Q_2) = \mathbf{R}_s((P_1 \wedge P_2) \vdash (Q_1 \wedge Q_2))$
proof −
have $\mathbf{R}_s(P_1 \vdash Q_1) \parallel_R \mathbf{R}_s(P_2 \vdash Q_2) =$
$\mathbf{R}_s(P_1[\![true,false/\$ok,\$wait]\!] \vdash Q_1[\![true,false/\$ok,\$wait]\!]) \parallel_R \mathbf{R}_s(P_2[\![true,false/\$ok,\$wait]\!] \vdash Q_2[\![true,false/\$ok,\$wait]\!])$
by (*simp add*: *RHS-design-ok-wait*)

also from *assms*
have ... =
$\mathbf{R}_s((R1\ (R2c\ (P_1)) \wedge R1\ (R2c\ (P_2)))[\![true,false/\$ok,\$wait]\!] \vdash$
$(R1\ (R2c\ (P_1 \Rightarrow Q_1)) \wedge R1\ (R2c\ (P_2 \Rightarrow Q_2)))[\![true,false/\$ok,\$wait]\!])$
apply (*simp add*: *rea-design-par-def rea-pre-RHS-design rea-cmt-RHS-design usubst unrest assms*)
apply (*rule cong*[*of* $\mathbf{R}_s$ $\mathbf{R}_s$], *simp*)
using *assms* **apply** (*rel-auto*)
done
also have ... =
$\mathbf{R}_s((R2c(P_1) \wedge R2c(P_2)) \vdash$
$(R1\ (R2s\ (P_1 \Rightarrow Q_1)) \wedge R1\ (R2s\ (P_2 \Rightarrow Q_2))))$
by (*metis* (*no-types*, *hide-lams*) *R1-R2s-R2c R1-conj R1-design-R1-pre RHS-design-ok-wait*)
also have ... =
$\mathbf{R}_s((P_1 \wedge P_2) \vdash (R1\ (R2s\ (P_1 \Rightarrow Q_1)) \wedge R1\ (R2s\ (P_2 \Rightarrow Q_2))))$
by (*simp add*: *R2c-R3h-commute R2c-and R2c-design R2c-idem R2c-not RHS-def*)
also have ... = $\mathbf{R}_s((P_1 \wedge P_2) \vdash ((P_1 \Rightarrow Q_1) \wedge (P_2 \Rightarrow Q_2)))$
by (*metis* (*no-types*, *lifting*) *R1-conj R2s-conj RHS-design-export-R1 RHS-design-export-R2s*)
also have ... = $\mathbf{R}_s((P_1 \wedge P_2) \vdash (Q_1 \wedge Q_2))$
by (*rule cong*[*of* $\mathbf{R}_s$ $\mathbf{R}_s$], *simp*, *rel-auto*)
finally show *?thesis* **.**
qed

lemma *RHS-tri-design-par*:
assumes $\$ok' \sharp P_1$ $\$ok' \sharp P_2$
shows $\mathbf{R}_s(P_1 \vdash Q_1 \diamond R_1) \parallel_R \mathbf{R}_s(P_2 \vdash Q_2 \diamond R_2) = \mathbf{R}_s((P_1 \wedge P_2) \vdash (Q_1 \wedge Q_2) \diamond (R_1 \wedge R_2))$
by (*simp add*: *RHS-design-par assms unrest wait'-cond-conj-exchange*)

lemma *RHS-tri-design-par-RR* [*rdes-def*]:
assumes P_1 *is RR* P_2 *is RR*
shows $\mathbf{R}_s(P_1 \vdash Q_1 \diamond R_1) \parallel_R \mathbf{R}_s(P_2 \vdash Q_2 \diamond R_2) = \mathbf{R}_s((P_1 \wedge P_2) \vdash (Q_1 \wedge Q_2) \diamond (R_1 \wedge R_2))$
by (*simp add*: *RHS-tri-design-par unrest assms*)

lemma *RHS-comp-assoc*:
assumes *P is NSRD Q is NSRD R is NSRD*
shows $(P \parallel_R Q) \parallel_R R = P \parallel_R Q \parallel_R R$
by (*rdes-eq cls*: *assms*)

end

9 Productive Reactive Designs

theory *utp-rdes-productive*
imports *utp-rdes-parallel*
begin

9.1 Healthiness condition

A reactive design is productive if it strictly increases the trace, whenever it terminates. If it does not terminate, it is also classed as productive.

definition *Productive* :: ('*s*, '*t*::*trace*, 'α) *hrel-rsp* ⇒ ('*s*, '*t*, 'α) *hrel-rsp* **where**
[*upred-defs*]: *Productive*(*P*) = *P* $\parallel_R$ $\mathbf{R}_s$(*true* ⊢ *true* ⋄ ($tr <$_u$ $tr´$))

lemma *Productive-RHS-design-form*:
assumes $ok´ ♯ *P* $ok´ ♯ *Q* $ok´ ♯ *R*
shows *Productive*($\mathbf{R}_s$(*P* ⊢ *Q* ⋄ *R*)) = $\mathbf{R}_s$(*P* ⊢ *Q* ⋄ (*R* ∧ $tr <$_u$ $tr´$))
using *assms* **by** (*simp add*: *Productive-def RHS-tri-design-par unrest*)

lemma *Productive-form*:
Productive(*SRD*(*P*)) = $\mathbf{R}_s$(*pre*$_R$(*P*) ⊢ *peri*$_R$(*P*) ⋄ (*post*$_R$(*P*) ∧ $tr <$_u$ $tr´$))
proof −
have *Productive*(*SRD*(*P*)) = $\mathbf{R}_s$(*pre*$_R$(*P*) ⊢ *peri*$_R$(*P*) ⋄ *post*$_R$(*P*)) $\parallel_R$ $\mathbf{R}_s$(*true* ⊢ *true* ⋄ ($tr <$_u$ $tr´$))
by (*simp add*: *Productive-def SRD-as-reactive-tri-design*)
also have ... = $\mathbf{R}_s$(*pre*$_R$(*P*) ⊢ *peri*$_R$(*P*) ⋄ (*post*$_R$(*P*) ∧ $tr <$_u$ $tr´$))
by (*simp add*: *RHS-tri-design-par unrest*)
finally show *?thesis* **.**
qed

A reactive design is productive provided that the postcondition, under the precondition, strictly increases the trace.

lemma *Productive-intro*:
assumes *P is SRD* ($tr <$_u$ $tr´$) ⊑ (*pre*$_R$(*P*) ∧ *post*$_R$(*P*)) $wait´ ♯ *pre*$_R$(*P*)
shows *P is Productive*
proof −
have *P*:$\mathbf{R}_s$(*pre*$_R$(*P*) ⊢ *peri*$_R$(*P*) ⋄ (*post*$_R$(*P*) ∧ $tr <$_u$ $tr´$)) = *P*
proof −
have $\mathbf{R}_s$(*pre*$_R$(*P*) ⊢ *peri*$_R$(*P*) ⋄ *post*$_R$(*P*)) = $\mathbf{R}_s$(*pre*$_R$(*P*) ⊢ (*pre*$_R$(*P*) ∧ *peri*$_R$(*P*)) ⋄ (*pre*$_R$(*P*) ∧ *post*$_R$(*P*)))
by (*metis* (*no-types*, *hide-lams*) *design-export-pre wait'-cond-conj-exchange wait'-cond-idem*)
also have ... = $\mathbf{R}_s$(*pre*$_R$(*P*) ⊢ (*pre*$_R$(*P*) ∧ *peri*$_R$(*P*)) ⋄ (*pre*$_R$(*P*) ∧ (*post*$_R$(*P*) ∧ $tr <$_u$ $tr´$)))
by (*metis assms*(*2*) *utp-pred-laws.inf.absorb1 utp-pred-laws.inf.assoc*)
also have ... = $\mathbf{R}_s$(*pre*$_R$(*P*) ⊢ *peri*$_R$(*P*) ⋄ (*post*$_R$(*P*) ∧ $tr <$_u$ $tr´$))
by (*metis* (*no-types*, *hide-lams*) *design-export-pre wait'-cond-conj-exchange wait'-cond-idem*)
finally show *?thesis*
by (*simp add*: *SRD-reactive-tri-design assms*(*1*))
qed
thus *?thesis*
by (*metis Healthy-def RHS-tri-design-par Productive-def ok'-pre-unrest unrest-true utp-pred-laws.inf-right-idem utp-pred-laws.inf-top-right*)
qed

lemma *Productive-post-refines-tr-increase*:
assumes *P is SRD P is Productive* $wait' ♯ $pre_R(P)$
shows $(\$tr <_u \$tr') \sqsubseteq (pre_R(P) \wedge post_R(P))$
proof −
have $post_R(P) = post_R(\mathbf{R}_s(pre_R(P) \vdash peri_R(P) \diamond (post_R(P) \wedge \$tr <_u \$tr')))$
by (*metis Healthy-def Productive-form assms*(1) *assms*(2))
also have ... $= R1(R2c(pre_R(P) \Rightarrow (post_R(P) \wedge \$tr <_u \$tr')))$
by (*simp add*: *rea-post-RHS-design unrest usubst assms*, *rel-auto*)
also have ... $= R1((pre_R(P) \Rightarrow (post_R(P) \wedge \$tr <_u \$tr')))$
by (*simp add*: *R2c-impl R2c-preR R2c-postR R2c-and R2c-tr-less-tr' assms*)
also have $(\$tr <_u \$tr') \sqsubseteq (pre_R(P) \wedge ...)$
by (*rel-auto*)
finally show *?thesis* **.**
qed

lemma *Continuous-Productive* [*closure*]: *Continuous Productive*
by (*simp add*: *Continuous-def Productive-def*, *rel-auto*)

9.2 Reactive design calculations

lemma *preR-Productive* [*rdes*]:
assumes *P is SRD*
shows $pre_R(Productive(P)) = pre_R(P)$
proof −
have $pre_R(Productive(P)) = pre_R(\mathbf{R}_s(pre_R(P) \vdash peri_R(P) \diamond (post_R(P) \wedge \$tr <_u \$tr')))$
by (*metis Healthy-def Productive-form assms*)
thus *?thesis*
by (*simp add*: *rea-pre-RHS-design usubst unrest R2c-not R2c-preR R1-preR Healthy-if assms*)
qed

lemma *periR-Productive* [*rdes*]:
assumes *P is NSRD*
shows $peri_R(Productive(P)) = peri_R(P)$
proof −
have $peri_R(Productive(P)) = peri_R(\mathbf{R}_s(pre_R(P) \vdash peri_R(P) \diamond (post_R(P) \wedge \$tr <_u \$tr')))$
by (*metis Healthy-def NSRD-is-SRD Productive-form assms*)
also have ... $= R1\ (R2c\ (pre_R\ P \Rightarrow_r peri_R\ P))$
by (*simp add*: *rea-peri-RHS-design usubst unrest R2c-not assms closure*)
also have ... $= (pre_R\ P \Rightarrow_r peri_R\ P)$
by (*simp add*: *R1-rea-impl R2c-rea-impl R2c-preR R2c-peri-SRD*
R1-peri-SRD assms closure R1-tr-less-tr' R2c-tr-less-tr')
finally show *?thesis*
by (*simp add*: *SRD-peri-under-pre assms unrest closure*)
qed

lemma *postR-Productive* [*rdes*]:
assumes *P is NSRD*
shows $post_R(Productive(P)) = (pre_R(P) \Rightarrow_r post_R(P) \wedge \$tr <_u \$tr')$
proof −
have $post_R(Productive(P)) = post_R(\mathbf{R}_s(pre_R(P) \vdash peri_R(P) \diamond (post_R(P) \wedge \$tr <_u \$tr')))$
by (*metis Healthy-def NSRD-is-SRD Productive-form assms*)
also have ... $= R1\ (R2c\ (pre_R\ P \Rightarrow_r post_R\ P \wedge \$tr' >_u \$tr))$
by (*simp add*: *rea-post-RHS-design usubst unrest assms closure*)
also have ... $= (pre_R\ P \Rightarrow_r post_R\ P \wedge \$tr' >_u \$tr)$
by (*simp add*: *R1-rea-impl R2c-rea-impl R2c-preR R2c-and R1-extend-conj' R2c-post-SRD*
R1-post-SRD assms closure R1-tr-less-tr' R2c-tr-less-tr')

finally show *?thesis* **.**
qed

lemma *preR-frame-seq-export*:
assumes *P is NSRD P is Productive Q is NSRD*
shows $(pre_R\ P \wedge (pre_R\ P \wedge post_R\ P)\ ;;\ Q) = (pre_R\ P \wedge (post_R\ P\ ;;\ Q))$
proof −
have $(pre_R\ P \wedge (post_R\ P\ ;;\ Q)) = (pre_R\ P \wedge ((pre_R\ P \Rightarrow_r post_R\ P)\ ;;\ Q))$
by (*simp add*: *SRD-post-under-pre assms closure unrest*)
also have ... $= (pre_R\ P \wedge (((\neg_r\ pre_R\ P)\ ;;\ Q \vee (pre_R\ P \Rightarrow_r R1(post_R\ P))\ ;;\ Q)))$
by (*simp add*: *NSRD-is-SRD R1-post-SRD assms*(*1*) *rea-impl-def seqr-or-distl R1-preR Healthy-if*)
also have ... $= (pre_R\ P \wedge (((\neg_r\ pre_R\ P)\ ;;\ Q \vee (pre_R\ P \wedge post_R\ P)\ ;;\ Q)))$
proof −
have $(pre_R\ P \vee \neg_r\ pre_R\ P) = R1\ true$
by (*simp add*: *R1-preR rea-not-or*)
then show *?thesis*
by (*metis* (*no-types*, *lifting*) *R1-def conj-comm disj-comm disj-conj-distr rea-impl-def seqr-or-distl utp-pred-laws.inf-top-left utp-pred-laws.sup.left-idem*)
qed
also have ... $= (pre_R\ P \wedge (((\neg_r\ pre_R\ P) \vee (pre_R\ P \wedge post_R\ P)\ ;;\ Q)))$
by (*simp add*: *NSRD-neg-pre-left-zero assms closure SRD-healths*)
also have ... $= (pre_R\ P \wedge (pre_R\ P \wedge post_R\ P)\ ;;\ Q)$
by (*rel-blast*)
finally show *?thesis* **..**
qed

9.3 Closure laws

lemma *Productive-rdes-intro*:
assumes $(\$tr <_u \$tr') \sqsubseteq R\ \$ok'\ \sharp\ P\ \$ok'\ \sharp\ Q\ \$ok'\ \sharp\ R\ \$wait\ \sharp\ P\ \$wait'\ \sharp\ P$
shows $(\mathbf{R}_s(P \vdash Q \diamond R))$ *is Productive*
proof (*rule Productive-intro*)
show $\mathbf{R}_s\ (P \vdash Q \diamond R)$ *is SRD*
by (*simp add*: *RHS-tri-design-is-SRD assms*)

from *assms*(*1*) **show** $(\$tr' >_u \$tr) \sqsubseteq (pre_R\ (\mathbf{R}_s\ (P \vdash Q \diamond R)) \wedge post_R\ (\mathbf{R}_s\ (P \vdash Q \diamond R)))$
apply (*simp add*: *rea-pre-RHS-design rea-post-RHS-design usubst assms unrest*)
using *assms*(*1*) **apply** (*rel-auto*)
apply *fastforce*
done

show $\$wait' \sharp pre_R\ (\mathbf{R}_s\ (P \vdash Q \diamond R))$
by (*simp add*: *rea-pre-RHS-design rea-post-RHS-design usubst R1-def R2c-def R2s-def assms unrest*)
qed

We use the *R4* healthiness condition to characterise that the postcondition must extend the trace for a reactive design to be productive.

lemma *Productive-rdes-RR-intro* [*closure*]:
assumes *P is RR Q is RR R is RR R is R4*
shows $(\mathbf{R}_s(P \vdash Q \diamond R))$ *is Productive*
using *assms* **by** (*simp add*: *Productive-rdes-intro R4-iff-refine unrest*)

lemma *Productive-Miracle* [*closure*]: *Miracle is Productive*
unfolding *Miracle-tri-def Healthy-def*
by (*subst Productive-RHS-design-form*, *simp-all add*: *unrest*)

lemma *Productive-Chaos* [*closure*]: *Chaos is Productive*
unfolding *Chaos-tri-def Healthy-def*
by (*subst Productive-RHS-design-form*, *simp-all add*: *unrest*)

lemma *Productive-intChoice* [*closure*]:
assumes *P is SRD P is Productive Q is SRD Q is Productive*
shows *P* ⊓ *Q is Productive*
proof −
have *P* ⊓ *Q* =
$\mathbf{R}_s(pre_R(P) \vdash peri_R(P) \diamond (post_R(P) \wedge \$tr <_u \$tr') \sqcap \mathbf{R}_s(pre_R(Q) \vdash peri_R(Q) \diamond (post_R(Q) \wedge \$tr <_u \$tr'))$
by (*metis Healthy-if Productive-form assms*)
also have ... = $\mathbf{R}_s((pre_R\ P \wedge pre_R\ Q) \vdash (peri_R\ P \vee peri_R\ Q) \diamond ((post_R\ P \wedge \$tr' >_u \$tr) \vee (post_R\ Q \wedge \$tr' >_u \$tr)))$
by (*simp add*: *RHS-tri-design-choice*)
also have ... = $\mathbf{R}_s((pre_R\ P \wedge pre_R\ Q) \vdash (peri_R\ P \vee peri_R\ Q) \diamond (((post_R\ P) \vee (post_R\ Q)) \wedge \$tr' >_u \$tr))$
by (*rule cong*[*of* $\mathbf{R}_s$ $\mathbf{R}_s$], *simp*, *rel-auto*)
also have ... *is Productive*
by (*simp add*: *Healthy-def Productive-RHS-design-form unrest*)
finally show *?thesis* **.**
qed

lemma *Productive-cond-rea* [*closure*]:
assumes *P is SRD P is Productive Q is SRD Q is Productive*
shows *P* ◁ *b* ▷$_R$ *Q is Productive*
proof −
have *P* ◁ *b* ▷$_R$ *Q* =
$\mathbf{R}_s(pre_R(P) \vdash peri_R(P) \diamond (post_R(P) \wedge \$tr <_u \$tr')) \lhd b \rhd_R \mathbf{R}_s(pre_R(Q) \vdash peri_R(Q) \diamond (post_R(Q) \wedge \$tr <_u \$tr'))$
by (*metis Healthy-if Productive-form assms*)
also have ... = $\mathbf{R}_s((pre_R\ P \lhd b \rhd_R pre_R\ Q) \vdash (peri_R\ P \lhd b \rhd_R peri_R\ Q) \diamond ((post_R\ P \wedge \$tr' >_u \$tr) \lhd b \rhd_R (post_R\ Q \wedge \$tr' >_u \$tr)))$
by (*simp add*: *cond-srea-form*)
also have ... = $\mathbf{R}_s((pre_R\ P \lhd b \rhd_R pre_R\ Q) \vdash (peri_R\ P \lhd b \rhd_R peri_R\ Q) \diamond (((post_R\ P) \lhd b \rhd_R (post_R\ Q)) \wedge \$tr' >_u \$tr))$
by (*rule cong*[*of* $\mathbf{R}_s$ $\mathbf{R}_s$], *simp*, *rel-auto*)
also have ... *is Productive*
by (*simp add*: *Healthy-def Productive-RHS-design-form unrest*)
finally show *?thesis* **.**
qed

lemma *Productive-seq-1* [*closure*]:
assumes *P is NSRD P is Productive Q is NSRD*
shows *P* ;; *Q is Productive*
proof −
have *P* ;; *Q* = $\mathbf{R}_s(pre_R(P) \vdash peri_R(P) \diamond (post_R(P) \wedge \$tr <_u \$tr')) ;; \mathbf{R}_s(pre_R(Q) \vdash peri_R(Q) \diamond (post_R(Q)))$
by (*metis Healthy-def NSRD-is-SRD SRD-reactive-tri-design Productive-form assms*(*1*) *assms*(*2*) *assms*(*3*))
also have ... = $\mathbf{R}_s((pre_R\ P \wedge (post_R\ P \wedge \$tr' >_u \$tr)\ wp_r\ pre_R\ Q) \vdash$
$(peri_R\ P \vee ((post_R\ P \wedge \$tr' >_u \$tr) ;; peri_R\ Q)) \diamond ((post_R\ P \wedge \$tr' >_u \$tr) ;; post_R\ Q))$
by (*simp add*: *RHS-tri-design-composition-wp rpred unrest closure assms wp NSRD-neg-pre-left-zero*

SRD-healths ex-unrest wp-rea-def disj-upred-def)
also have ... = $\mathbf{R}_s$ ((pre_R P ∧ ($post_R$ P ∧ $tr´ >_u $tr) wp_r pre_R Q) ⊢
($peri_R$ P ∨ (($post_R$ P ∧ $tr´ >_u $tr) ;; $peri_R$ Q)) ⋄ (($post_R$ P ∧ $tr´ >_u $tr) ;; $post_R$ Q ∧ $tr´ >_u $tr))
proof −
have (($post_R$ P ∧ $tr´ >_u $tr) ;; $R1$($post_R$ Q)) = (($post_R$ P ∧ $tr´ >_u $tr) ;; $R1$($post_R$ Q) ∧ $tr´ >_u $tr)
by (*rel-auto*)
thus *?thesis*
by (*simp add*: *NSRD-is-SRD R1-post-SRD assms*)
qed
also have ... *is Productive*
by (*rule Productive-rdes-intro*, *simp-all add*: *unrest assms closure wp-rea-def*)
finally show *?thesis* **.**
qed

lemma *Productive-seq-2* [*closure*]:
assumes *P is NSRD Q is NSRD Q is Productive*
shows *P* ;; *Q is Productive*
proof −
have P ;; Q = $\mathbf{R}_s$(pre_R(P) ⊢ $peri_R$(P) ⋄ ($post_R$(P))) ;; $\mathbf{R}_s$(pre_R(Q) ⊢ $peri_R$(Q) ⋄ ($post_R$(Q) ∧ $tr <_u $tr´))
by (*metis Healthy-def NSRD-is-SRD SRD-reactive-tri-design Productive-form assms*)
also have ... = $\mathbf{R}_s$ ((pre_R P ∧ $post_R$ P wp_r pre_R Q) ⊢ ($peri_R$ P ∨ ($post_R$ P ;; $peri_R$ Q)) ⋄ ($post_R$ P ;; ($post_R$ Q ∧ $tr´ >_u $tr)))
by (*simp add*: *RHS-tri-design-composition-wp rpred unrest closure assms wp NSRD-neg-pre-left-zero SRD-healths ex-unrest wp-rea-def disj-upred-def*)
also have ... = $\mathbf{R}_s$ ((pre_R P ∧ $post_R$ P wp_r pre_R Q) ⊢ ($peri_R$ P ∨ ($post_R$ P ;; $peri_R$ Q)) ⋄ ($post_R$ P ;; ($post_R$ Q ∧ $tr´ >_u $tr) ∧ $tr´ >_u $tr))
proof −
have ($R1$($post_R$ P) ;; ($post_R$ Q ∧ $tr´ >_u $tr) ∧ $tr´ >_u $tr) = ($R1$($post_R$ P) ;; ($post_R$ Q ∧ $tr´ >_u $tr))
by (*rel-auto*)
thus *?thesis*
by (*simp add*: *NSRD-is-SRD R1-post-SRD assms*)
qed
also have ... *is Productive*
by (*rule Productive-rdes-intro*, *simp-all add*: *unrest assms closure wp-rea-def*)
finally show *?thesis* **.**
qed

end

10 Guarded Recursion

theory *utp-rdes-guarded*
imports *utp-rdes-productive*
begin

10.1 Traces with a size measure

Guarded recursion relies on our ability to measure the trace's size, in order to see if it is decreasing on each iteration. Thus, we here equip the trace algebra with the *size* function that provides this.

class *size-trace* = *trace* + *size* +
  assumes
    size-zero: *size 0* = *0* **and**
    size-nzero: $s > 0 \Longrightarrow size(s) > 0$ **and**
    size-plus: *size* $(s + t) = size(s) + size(t)$
  — These axioms may be stronger than necessary. In particular, $(0::'a) < ?s \Longrightarrow 0 < size\ ?s$ requires that a non-empty trace have a positive size. But this may not be the case with all trace models and is possibly more restrictive than necessary. In future we will explore weakening.
begin

lemma *size-mono*: $s \leq t \Longrightarrow size(s) \leq size(t)$
  by (*metis le-add1 local.diff-add-cancel-left' local.size-plus*)

lemma *size-strict-mono*: $s < t \Longrightarrow size(s) < size(t)$
  by (*metis cancel-ab-semigroup-add-class.add-diff-cancel-left' local.diff-add-cancel-left' local.less-iff local.minus-gr-zero-iff local.size-nzero local.size-plus zero-less-diff*)

lemma *trace-strict-prefixE*: $xs < ys \Longrightarrow (\bigwedge zs.\ [\![\ ys = xs + zs;\ size(zs) > 0\ ]\!] \Longrightarrow thesis) \Longrightarrow thesis$
  by (*metis local.diff-add-cancel-left' local.less-iff local.minus-gr-zero-iff local.size-nzero*)

lemma *size-minus-trace*: $y \leq x \Longrightarrow size(x - y) = size(x) - size(y)$
  by (*metis diff-add-inverse local.diff-add-cancel-left' local.size-plus*)

end

Both natural numbers and lists are measurable trace algebras.

instance *nat* :: *size-trace*
  by (*intro-classes*, *simp-all*)

instance *list* :: (*type*) *size-trace*
  by (*intro-classes*, *simp-all add*: *zero-list-def less-list-def' plus-list-def prefix-length-less*)

syntax
  -usize :: *logic* ⇒ *logic* ($size_u$'(-'))

translations
  $size_u(t)$ == *CONST uop CONST size t*

10.2 Guardedness

definition *gvrt* :: (('*t*::*size-trace*,'α) *rp* × ('*t*,'α) *rp*) *chain* **where**
[*upred-defs*]: $gvrt(n) \equiv (\$tr \leq_u \$tr' \wedge size_u(\&tt) <_u \ll n \gg)$

lemma *gvrt-chain*: *chain gvrt*
  apply (*simp add*: *chain-def*, *safe*)
  apply (*rel-simp*)
  apply (*rel-simp*)+
done

lemma *gvrt-limit*: $\bigsqcap (range\ gvrt) = (\$tr \leq_u \$tr')$
  by (*rel-auto*)

definition *Guarded* :: (('*t*::*size-trace*,'α) *hrel-rp* ⇒ ('*t*,'α) *hrel-rp*) ⇒ *bool* **where**
[*upred-defs*]: $Guarded(F) = (\forall\ X\ n.\ (F(X) \wedge gvrt(n+1)) = (F(X \wedge gvrt(n)) \wedge gvrt(n+1)))$

lemma *GuardedI*: $[\![\ \bigwedge X\ n.\ (F(X) \wedge gvrt(n+1)) = (F(X \wedge gvrt(n)) \wedge gvrt(n+1))\ ]\!] \Longrightarrow Guarded\ F$

by (*simp add*: *Guarded-def*)

Guarded reactive designs yield unique fixed-points.

theorem *guarded-fp-uniq*:
assumes *mono F* $F \in [\![id]\!]_H \to [\![SRD]\!]_H$ *Guarded F*
shows $\mu\ F = \nu\ F$
proof −
have *constr F gvrt*
using *assms*
by (*auto simp add*: *constr-def gvrt-chain Guarded-def tcontr-alt-def'*)
hence ($tr $\leq_u$ $tr´ $\wedge$ $\mu\ F$) = ($tr $\leq_u$ $tr´ $\wedge$ $\nu\ F$)
apply (*rule constr-fp-uniq*)
apply (*simp add*: *assms*)
using *gvrt-limit* **apply** *blast*
done
moreover have ($tr $\leq_u$ $tr´ $\wedge$ $\mu\ F$) = $\mu\ F$
proof −
have $\mu\ F$ *is R1*
by (*rule SRD-healths*(*1*), *rule Healthy-mu*, *simp-all add*: *assms*)
thus *?thesis*
by (*metis Healthy-def R1-def conj-comm*)
qed
moreover have ($tr $\leq_u$ $tr´ $\wedge$ $\nu\ F$) = $\nu\ F$
proof −
have $\nu\ F$ *is R1*
by (*rule SRD-healths*(*1*), *rule Healthy-nu*, *simp-all add*: *assms*)
thus *?thesis*
by (*metis Healthy-def R1-def conj-comm*)
qed
ultimately show *?thesis*
by (*simp*)
qed

lemma *Guarded-const* [*closure*]: *Guarded* ($\lambda\ X.\ P$)
by (*simp add*: *Guarded-def*)

lemma *UINF-Guarded* [*closure*]:
assumes $\bigwedge P.\ P \in A \Longrightarrow$ *Guarded P*
shows *Guarded* ($\lambda\ X.\ \prod P{\in}A \cdot P(X)$)
proof (*rule GuardedI*)
fix *X n*
have $\bigwedge\ Y.\ ((\prod P{\in}A \cdot P\ Y) \wedge gvrt(n{+}1)) = ((\prod P{\in}A \cdot (P\ Y \wedge gvrt(n{+}1))) \wedge gvrt(n{+}1))$
proof −
fix *Y*
let *?lhs* = $((\prod P{\in}A \cdot P\ Y) \wedge gvrt(n{+}1))$ **and** *?rhs* = $((\prod P{\in}A \cdot (P\ Y \wedge gvrt(n{+}1))) \wedge gvrt(n{+}1))$
have *a*:*?lhs*$[\![false/\$ok]\!]$ = *?rhs*$[\![false/\$ok]\!]$
by (*rel-auto*)
have *b*:*?lhs*$[\![true/\$ok]\!][\![true/\$wait]\!]$ = *?rhs*$[\![true/\$ok]\!][\![true/\$wait]\!]$
by (*rel-auto*)
have *c*:*?lhs*$[\![true/\$ok]\!][\![false/\$wait]\!]$ = *?rhs*$[\![true/\$ok]\!][\![false/\$wait]\!]$
by (*rel-auto*)
show *?lhs* = *?rhs*
using *a b c*
by (*rule-tac bool-eq-splitI*[*of in-var ok*], *simp*, *rule-tac bool-eq-splitI*[*of in-var wait*], *simp-all*)
qed

moreover have $((\bigsqcap P{\in}A \cdot (P\ X \wedge gvrt(n{+}1))) \wedge gvrt(n{+}1)) = ((\bigsqcap P{\in}A \cdot (P\ (X \wedge gvrt(n)) \wedge gvrt(n{+}1))) \wedge gvrt(n{+}1))$
proof −
have $(\bigsqcap P{\in}A \cdot (P\ X \wedge gvrt(n{+}1))) = (\bigsqcap P{\in}A \cdot (P\ (X \wedge gvrt(n)) \wedge gvrt(n{+}1)))$
proof (*rule UINF-cong*)
fix P **assume** $P \in A$
thus $(P\ X \wedge gvrt(n{+}1)) = (P\ (X \wedge gvrt(n)) \wedge gvrt(n{+}1))$
using *Guarded-def assms* **by** *blast*
qed
thus *?thesis* **by** *simp*
qed
ultimately show $((\bigsqcap P{\in}A \cdot P\ X) \wedge gvrt(n{+}1)) = ((\bigsqcap P{\in}A \cdot (P\ (X \wedge gvrt(n)))) \wedge gvrt(n{+}1))$
by *simp*
qed

lemma *intChoice-Guarded* [*closure*]:
assumes *Guarded P Guarded Q*
shows *Guarded* $(\lambda\ X.\ P(X) \sqcap Q(X))$
proof −
have *Guarded* $(\lambda\ X.\ \bigsqcap F{\in}\{P,Q\} \cdot F(X))$
by (*rule UINF-Guarded*, *auto simp add*: *assms*)
thus *?thesis*
by (*simp*)
qed

lemma *cond-srea-Guarded* [*closure*]:
assumes *Guarded P Guarded Q*
shows *Guarded* $(\lambda\ X.\ P(X) \lhd b \rhd_R Q(X))$
using *assms* **by** (*rel-auto*)

A tail recursive reactive design with a productive body is guarded.

lemma *Guarded-if-Productive* [*closure*]:
fixes $P :: ('s, 't{::}size\text{-}trace, 'a)\ hrel\text{-}rsp$
assumes *P is NSRD P is Productive*
shows *Guarded* $(\lambda\ X.\ P ;; SRD(X))$
proof (*clarsimp simp add*: *Guarded-def*)
— We split the proof into three cases corresponding to valuations for ok, wait, and wait' respectively.
fix $X\ n$
have *a*:$(P ;; SRD(X) \wedge gvrt\ (Suc\ n))[\![false/\$ok]\!] =$
$(P ;; SRD(X \wedge gvrt\ n) \wedge gvrt\ (Suc\ n))[\![false/\$ok]\!]$
by (*simp add*: *usubst closure SRD-left-zero-1 assms*)
have *b*:$((P ;; SRD(X) \wedge gvrt\ (Suc\ n))[\![true/\$ok]\!])[\![true/\$wait]\!] =$
$((P ;; SRD(X \wedge gvrt\ n) \wedge gvrt\ (Suc\ n))[\![true/\$ok]\!])[\![true/\$wait]\!]$
by (*simp add*: *usubst closure SRD-left-zero-2 assms*)
have *c*:$((P ;; SRD(X) \wedge gvrt\ (Suc\ n))[\![true/\$ok]\!])[\![false/\$wait]\!] =$
$((P ;; SRD(X \wedge gvrt\ n) \wedge gvrt\ (Suc\ n))[\![true/\$ok]\!])[\![false/\$wait]\!]$
proof −
have *1*:$(P[\![true/\$wait´]\!] ;; (SRD\ X)[\![true/\$wait]\!] \wedge gvrt\ (Suc\ n))[\![true,false/\$ok,\$wait]\!] =$
$(P[\![true/\$wait´]\!] ;; (SRD\ (X \wedge gvrt\ n))[\![true/\$wait]\!] \wedge gvrt\ (Suc\ n))[\![true,false/\$ok,\$wait]\!]$
by (*metis* (*no-types*, *lifting*) *Healthy-def R3h-wait-true SRD-healths*(*3*) *SRD-idem*)
have *2*:$(P[\![false/\$wait´]\!] ;; (SRD\ X)[\![false/\$wait]\!] \wedge gvrt\ (Suc\ n))[\![true,false/\$ok,\$wait]\!] =$
$(P[\![false/\$wait´]\!] ;; (SRD\ (X \wedge gvrt\ n))[\![false/\$wait]\!] \wedge gvrt\ (Suc\ n))[\![true,false/\$ok,\$wait]\!]$
proof −
have *exp*:$\bigwedge Y{::}('s, 't, 'a)\ hrel\text{-}rsp.\ (P[\![false/\$wait´]\!] ;; (SRD\ Y)[\![false/\$wait]\!] \wedge gvrt\ (Suc\ n))[\![true,false/\$ok,\$wait]\!]$
=

$$((((\neg_r\ pre_R\ P)\ ;;\ (SRD(Y))[\![false/\$wait]\!] \vee (post_R\ P \wedge \$tr' >_u \$tr)\ ;;\ (SRD\ Y)[\![true,false/\$ok,\$wait]\!]))$$
$$\wedge\ gvrt\ (Suc\ n))[\![true,false/\$ok,\$wait]\!]$$

proof −
fix $Y :: ('s, 't, 'α)$ *hrel-rsp*

have $(P[\![false/\$wait']\!]\ ;;\ (SRD\ Y)[\![false/\$wait]\!] \wedge gvrt\ (Suc\ n))[\![true,false/\$ok,\$wait]\!] =$
$((\mathbf{R}_s(pre_R(P) \vdash peri_R(P) \diamond (post_R(P) \wedge \$tr <_u \$tr')))[\![false/\$wait']\!]\ ;;\ (SRD\ Y)[\![false/\$wait]\!]$
$\wedge\ gvrt\ (Suc\ n))[\![true,false/\$ok,\$wait]\!]$
by (*metis* (*no-types*) *Healthy-def Productive-form assms*(1) *assms*(2) *NSRD-is-SRD*)
also have ... =
$((R1(R2c(pre_R(P) \Rightarrow (\$ok' \wedge post_R(P) \wedge \$tr <_u \$tr'))))[\![false/\$wait']\!]\ ;;\ (SRD\ Y)[\![false/\$wait]\!]$
$\wedge\ gvrt\ (Suc\ n))[\![true,false/\$ok,\$wait]\!]$
by (*simp add*: *RHS-def R1-def R2c-def R2s-def R3h-def RD1-def RD2-def usubst unrest assms closure design-def*)
also have ... =
$(((\neg_r\ pre_R(P) \vee (\$ok' \wedge post_R(P) \wedge \$tr <_u \$tr')))[\![false/\$wait']\!]\ ;;\ (SRD\ Y)[\![false/\$wait]\!]$
$\wedge\ gvrt\ (Suc\ n))[\![true,false/\$ok,\$wait]\!]$
by (*simp add*: *impl-alt-def R2c-disj R1-disj R2c-not assms closure R2c-and*
R2c-preR rea-not-def R1-extend-conj' R2c-ok' R2c-post-SRD R1-tr-less-tr' R2c-tr-less-tr')
also have ... =
$((((\neg_r\ pre_R\ P)\ ;;\ (SRD(Y))[\![false/\$wait]\!] \vee (\$ok' \wedge post_R\ P \wedge \$tr' >_u \$tr)\ ;;\ (SRD\ Y)[\![false/\$wait]\!])) \wedge gvrt\ (Suc\ n))[\![true,false/\$ok,\$wait]\!]$
by (*simp add*: *usubst unrest assms closure seqr-or-distl NSRD-neg-pre-left-zero*)
also have ... =
$((((\neg_r\ pre_R\ P)\ ;;\ (SRD(Y))[\![false/\$wait]\!] \vee (post_R\ P \wedge \$tr' >_u \$tr)\ ;;\ (SRD\ Y)[\![true,false/\$ok,\$wait]\!]))$
$\wedge\ gvrt\ (Suc\ n))[\![true,false/\$ok,\$wait]\!]$
proof −
have $(\$ok' \wedge post_R\ P \wedge \$tr' >_u \$tr)\ ;;\ (SRD\ Y)[\![false/\$wait]\!] =$
$((post_R\ P \wedge \$tr' >_u \$tr) \wedge \$ok' =_u true)\ ;;\ (SRD\ Y)[\![false/\$wait]\!]$
by (*rel-blast*)
also have ... $= (post_R\ P \wedge \$tr' >_u \$tr)[\![true/\$ok']\!]\ ;;\ (SRD\ Y)[\![false/\$wait]\!][\![true/\$ok]\!]$
using *seqr-left-one-point*[*of ok* $(post_R\ P \wedge \$tr' >_u \$tr)$ *True* $(SRD\ Y)[\![false/\$wait]\!]$]
by (*simp add*: *true-alt-def*[*THEN sym*])
finally show *?thesis* **by** (*simp add*: *usubst unrest*)
qed
finally
show $(P[\![false/\$wait']\!]\ ;;\ (SRD\ Y)[\![false/\$wait]\!] \wedge gvrt\ (Suc\ n))[\![true,false/\$ok,\$wait]\!] =$
$((((\neg_r\ pre_R\ P)\ ;;\ (SRD(Y))[\![false/\$wait]\!] \vee (post_R\ P \wedge \$tr' >_u \$tr)\ ;;\ (SRD\ Y)[\![true,false/\$ok,\$wait]\!]))$
$\wedge\ gvrt\ (Suc\ n))[\![true,false/\$ok,\$wait]\!]$.
qed

have *1*:$((post_R\ P \wedge \$tr' >_u \$tr)\ ;;\ (SRD\ X)[\![true,false/\$ok,\$wait]\!] \wedge gvrt\ (Suc\ n)) =$
$((post_R\ P \wedge \$tr' >_u \$tr)\ ;;\ (SRD\ (X \wedge gvrt\ n))[\![true,false/\$ok,\$wait]\!] \wedge gvrt\ (Suc\ n))$
apply (*rel-auto*)
apply (*rename-tac tr st more ok wait tr' st' more' tr$_0$ st$_0$ more$_0$ ok'*)
apply (*rule-tac x=tr$_0$* **in** *exI*, *rule-tac x=st$_0$* **in** *exI*, *rule-tac x=more$_0$* **in** *exI*)
apply (*simp*)
apply (*erule trace-strict-prefixE*)
apply (*rename-tac tr st ref ok wait tr' st' ref' tr$_0$ st$_0$ ref$_0$ ok' zs*)
apply (*rule-tac x=False* **in** *exI*)
apply (*simp add*: *size-minus-trace*)
apply (*subgoal-tac size*$(tr) <$ *size*(tr_0))
apply (*simp add*: *less-diff-conv2 size-mono*)

using *size-strict-mono* **apply** *blast*
apply (*rename-tac tr st more ok wait tr′ st′ more′* tr_0 st_0 $more_0$ *ok′*)
apply (*rule-tac x=*tr_0 **in** *exI*, *rule-tac x=*st_0 **in** *exI*, *rule-tac x=*$more_0$ **in** *exI*)
apply (*simp*)
apply (*erule trace-strict-prefixE*)
apply (*rename-tac tr st more ok wait tr′ st′ more′* tr_0 st_0 $more_0$ *ok′ zs*)
apply (*auto simp add*: *size-minus-trace*)
apply (*subgoal-tac size*(*tr*) < *size*(tr_0))
apply (*simp add*: *less-diff-conv2 size-mono*)
using *size-strict-mono* **apply** *blast*
done
have *2*:($\neg_r$ pre_R *P*) ;; (*SRD X*)⟦*false*/$*wait*⟧ = ($\neg_r$ pre_R *P*) ;; (*SRD*(*X* ∧ *gvrt n*))⟦*false*/$*wait*⟧
by (*simp add*: *NSRD-neg-pre-left-zero closure assms SRD-healths*)
show *?thesis*
by (*simp add*: *exp 1 2 utp-pred-laws.inf-sup-distrib2*)
qed

show *?thesis*
proof −
have (*P* ;; (*SRD X*) ∧ *gvrt* (*n*+*1*))⟦*true*,*false*/$*ok*,$*wait*⟧ =
((*P*⟦*true*/$*wait´*⟧ ;; (*SRD X*)⟦*true*/$*wait*⟧ ∧ *gvrt* (*n*+*1*))⟦*true*,*false*/$*ok*,$*wait*⟧ ∨
(*P*⟦*false*/$*wait´*⟧ ;; (*SRD X*)⟦*false*/$*wait*⟧ ∧ *gvrt* (*n*+*1*))⟦*true*,*false*/$*ok*,$*wait*⟧)
by (*subst seqr-bool-split*[*of wait*], *simp-all add*: *usubst utp-pred-laws.distrib*(*4*))

also
have ... = ((*P*⟦*true*/$*wait´*⟧ ;; (*SRD* (*X* ∧ *gvrt n*))⟦*true*/$*wait*⟧ ∧ *gvrt* (*n*+*1*))⟦*true*,*false*/$*ok*,$*wait*⟧
∨
(*P*⟦*false*/$*wait´*⟧ ;; (*SRD* (*X* ∧ *gvrt n*))⟦*false*/$*wait*⟧ ∧ *gvrt* (*n*+*1*))⟦*true*,*false*/$*ok*,$*wait*⟧)
by (*simp add*: *1 2*)

also
have ... = ((*P*⟦*true*/$*wait´*⟧ ;; (*SRD* (*X* ∧ *gvrt n*))⟦*true*/$*wait*⟧ ∨
P⟦*false*/$*wait´*⟧ ;; (*SRD* (*X* ∧ *gvrt n*))⟦*false*/$*wait*⟧) ∧ *gvrt* (*n*+*1*))⟦*true*,*false*/$*ok*,$*wait*⟧
by (*simp add*: *usubst utp-pred-laws.distrib*(*4*))

also have ... = (*P* ;; (*SRD* (*X* ∧ *gvrt n*)) ∧ *gvrt* (*n*+*1*))⟦*true*,*false*/$*ok*,$*wait*⟧
by (*subst seqr-bool-split*[*of wait*], *simp-all add*: *usubst*)
finally show *?thesis* **by** (*simp add*: *usubst*)
qed

qed
show (*P* ;; *SRD*(*X*) ∧ *gvrt* (*Suc n*)) = (*P* ;; *SRD*(*X* ∧ *gvrt n*) ∧ *gvrt* (*Suc n*))
apply (*rule-tac bool-eq-splitI*[*of in-var ok*])
apply (*simp-all add*: *a*)
apply (*rule-tac bool-eq-splitI*[*of in-var wait*])
apply (*simp-all add*: *b c*)
done
qed

10.3 Tail recursive fixed-point calculations

declare *upred-semiring.power-Suc* [*simp*]

lemma *mu-csp-form-1* [*rdes*]:
fixes *P* :: ('*s*, '*t*::*size-trace*,'α) *hrel-rsp*
assumes *P is NSRD P is Productive*

shows ($\mu\ X \cdot P\ ;;\ SRD(X)) = (\bigsqcap i \cdot P\ \hat{}\ (i{+}1))\ ;;\ Miracle$
proof −
have *1*:*Continuous* ($\lambda X.\ P\ ;;\ SRD\ X$)
using *SRD-Continuous*
by (*clarsimp simp add*: *Continuous-def seq-SUP-distl*[*THEN sym*], *drule-tac x*=*A* **in** *spec*, *simp*)
have *2*: ($\lambda X.\ P\ ;;\ SRD\ X$) $\in [\![id]\!]_H \rightarrow [\![SRD]\!]_H$
by (*blast intro*: *funcsetI closure assms*)
with *1 2* **have** ($\mu\ X \cdot P\ ;;\ SRD(X)) = (\nu\ X \cdot P\ ;;\ SRD(X)$)
by (*simp add*: *guarded-fp-uniq Guarded-if-Productive*[*OF assms*] *funcsetI closure*)
also have ... = ($\bigsqcap i.\ ((\lambda X.\ P\ ;;\ SRD\ X)\ \hat{}\hat{}\ i)\ false$)
by (*simp add*: *sup-continuous-lfp 1 sup-continuous-Continuous false-upred-def*)
also have ... = ($(\lambda X.\ P\ ;;\ SRD\ X)\ \hat{}\hat{}\ 0)\ false \sqcap (\bigsqcap i.\ ((\lambda X.\ P\ ;;\ SRD\ X)\ \hat{}\hat{}\ (i{+}1))\ false$)
by (*subst Sup-power-expand*, *simp*)
also have ... = ($\bigsqcap i.\ ((\lambda X.\ P\ ;;\ SRD\ X)\ \hat{}\hat{}\ (i{+}1))\ false$)
by (*simp*)
also have ... = ($\bigsqcap i.\ P\ \hat{}\ (i{+}1)\ ;;\ Miracle$)
proof (*rule SUP-cong*, *simp-all*)
fix *i*
show *P* ;; *SRD* ((($\lambda X.\ P\ ;;\ SRD\ X)\ \hat{}\hat{}\ i)\ false) = (P\ ;;\ P\ \hat{}\ i)\ ;;\ Miracle$
proof (*induct i*)
case *0*
then show *?case*
by (*simp*, *metis srdes-theory.healthy-top*)
next
case (*Suc i*)
then show *?case*
by (*simp add*: *Healthy-if NSRD-is-SRD SRD-power-comp SRD-seqr-closure assms*(*1*) *seqr-assoc*[*THEN sym*] *srdes-theory.top-closed*)
qed
qed
also have ... = ($\bigsqcap i.\ P\ \hat{}\ (i{+}1))\ ;;\ Miracle$
by (*simp add*: *seq-Sup-distr*)
finally show *?thesis*
by (*simp add*: *UINF-as-Sup-collect*)
qed

lemma *mu-csp-form-NSRD* [*closure*]:
fixes *P* :: ('*s*, '*t*::*size-trace*,'α) *hrel-rsp*
assumes *P is NSRD P is Productive*
shows ($\mu\ X \cdot P\ ;;\ SRD(X)$) *is NSRD*
by (*simp add*: *mu-csp-form-1 assms closure*)

lemma *mu-csp-form-1'*:
fixes *P* :: ('*s*, '*t*::*size-trace*,'α) *hrel-rsp*
assumes *P is NSRD P is Productive*
shows ($\mu\ X \cdot P\ ;;\ SRD(X)) = (P\ ;;\ P^\star)\ ;;\ Miracle$
proof −
have ($\mu\ X \cdot P\ ;;\ SRD(X)) = (\bigsqcap\ i{\in}UNIV \cdot P\ ;;\ P\ \hat{}\ i)\ ;;\ Miracle$
by (*simp add*: *mu-csp-form-1 assms closure ustar-def*)
also have ... = ($P\ ;;\ P^\star)\ ;;\ Miracle$
by (*simp only*: *seq-UINF-distl*[*THEN sym*], *simp add*: *ustar-def*)
finally show *?thesis* **.**
qed

declare *upred-semiring.power-Suc* [*simp del*]

end

11 Reactive Design Programs

theory *utp-rdes-prog*
imports
utp-rdes-normal
utp-rdes-tactics
utp-rdes-parallel
utp-rdes-guarded
UTP−KAT.utp-kleene
begin

11.1 State substitution

lemma *srd-subst-RHS-tri-design* [*usubst*]:
$\lceil\sigma\rceil_{S\sigma} \dagger \mathbf{R}_s(P \vdash Q \diamond R) = \mathbf{R}_s((\lceil\sigma\rceil_{S\sigma} \dagger P) \vdash (\lceil\sigma\rceil_{S\sigma} \dagger Q) \diamond (\lceil\sigma\rceil_{S\sigma} \dagger R))$
by (*rel-auto*)

lemma *srd-subst-SRD-closed* [*closure*]:
assumes *P is SRD*
shows $\lceil\sigma\rceil_{S\sigma} \dagger P$ *is SRD*
proof −
have $SRD(\lceil\sigma\rceil_{S\sigma} \dagger (SRD\ P)) = \lceil\sigma\rceil_{S\sigma} \dagger (SRD\ P)$
by (*rel-auto*)
thus *?thesis*
by (*metis Healthy-def assms*)
qed

lemma *preR-srd-subst* [*rdes*]:
$pre_R(\lceil\sigma\rceil_{S\sigma} \dagger P) = \lceil\sigma\rceil_{S\sigma} \dagger pre_R(P)$
by (*rel-auto*)

lemma *periR-srd-subst* [*rdes*]:
$peri_R(\lceil\sigma\rceil_{S\sigma} \dagger P) = \lceil\sigma\rceil_{S\sigma} \dagger peri_R(P)$
by (*rel-auto*)

lemma *postR-srd-subst* [*rdes*]:
$post_R(\lceil\sigma\rceil_{S\sigma} \dagger P) = \lceil\sigma\rceil_{S\sigma} \dagger post_R(P)$
by (*rel-auto*)

lemma *srd-subst-NSRD-closed* [*closure*]:
assumes *P is NSRD*
shows $\lceil\sigma\rceil_{S\sigma} \dagger P$ *is NSRD*
by (*rule NSRD-RC-intro*, *simp-all add*: *closure rdes assms unrest*)

11.2 Assignment

definition *assigns-srd* :: *'s usubst* ⇒ (*'s*, *'t*::*trace*, *'α*) *hrel-rsp* ($\langle\text{-}\rangle_R$) **where**
[*upred-defs*]: $assigns\text{-}srd\ \sigma = \mathbf{R}_s(true \vdash (\$tr' =_u \$tr \wedge \neg \$wait' \wedge \lceil\langle\sigma\rangle_a\rceil_S \wedge \$\Sigma_S' =_u \$\Sigma_S))$

syntax
-assign-srd :: *svids* ⇒ *uexprs* ⇒ *logic* ($'(\text{-}') :=_R '(\text{-}')$)
-assign-srd :: *svids* ⇒ *uexprs* ⇒ *logic* (**infixr** $:=_R$ *62*)

translations
-assign-srd xs vs => CONST assigns-srd (-mk-usubst (id_s) *xs vs)*
-assign-srd x v <= CONST assigns-srd (CONST subst-upd (id_s) *x v)*
-assign-srd x v <= -assign-srd (-spvar x) v
$x,y :=_R u,v$ *<= CONST assigns-srd (CONST subst-upd (CONST subst-upd* (id_s) *(CONST pr-var x) u) (CONST pr-var y) v)*

lemma *assigns-srd-RHS-tri-des* [*rdes-def*]:
$\langle\sigma\rangle_R = \mathbf{R}_s(true_r \vdash false \diamond \langle\sigma\rangle_r)$
by (*rel-auto*)

lemma *assigns-srd-NSRD-closed* [*closure*]: $\langle\sigma\rangle_R$ *is NSRD*
by (*simp add*: *rdes-def closure unrest*)

lemma *preR-assigns-srd* [*rdes*]: $pre_R(\langle\sigma\rangle_R) = true_r$
by (*simp add*: *rdes-def rdes closure*)

lemma *periR-assigns-srd* [*rdes*]: $peri_R(\langle\sigma\rangle_R) = false$
by (*simp add*: *rdes-def rdes closure*)

lemma *postR-assigns-srd* [*rdes*]: $post_R(\langle\sigma\rangle_R) = \langle\sigma\rangle_r$
by (*simp add*: *rdes-def rdes closure rpred*)

lemma *taut-eq-impl-property*:
$[\![$ *vwb-lens x*; $\$x \sharp P$ $]\!] \Longrightarrow$ ‘$(\$x =_u \ll v\gg \wedge Q) \Rightarrow P$‘ = ‘$Q[\![\ll v\gg/\$x]\!] \Rightarrow P$‘
by (*rel-auto*, *meson mwb-lens-weak vwb-lens-mwb weak-lens.put-get*)

lemma *st-subst-taut-impl*:
assumes *vwb-lens x* $\$st{:}x \sharp Q$ *P is RR Q is RR*
shows ‘$[\&x \mapsto_s \ll k\gg]$ $\dagger_S$ $P \Rightarrow Q$‘ = ‘$[\&x =_u \ll k\gg]_{S<} \wedge P \Rightarrow Q$‘ (**is** *?lhs = ?rhs*)
proof −
have *?lhs* = ‘$P[\![\ll k\gg/\$st{:}x]\!] \Rightarrow Q$‘
by (*simp add*: *usubst-st-lift-def alpha usubst*)
also have ... = ‘$(\$st{:}x =_u \ll k\gg) \wedge RR(P) \Rightarrow RR(Q)$‘
by (*simp add*: *Healthy-if assms taut-eq-impl-property*)
also have ... = ‘$[\&x =_u \ll k\gg]_{S<} \wedge RR(P) \Rightarrow RR(Q)$‘
by (*rel-blast*)
finally show *?thesis* **by** (*simp add*: *assms Healthy-if*)
qed

The following law explains how to refine a program Q when it is first initialised by an assignment. Would be good if it could be generalised to a more general precondition.

lemma *AssignR-init-refine-intro*:
assumes
vwb-lens x $\$st{:}x \sharp P_2$ $\$st{:}x \sharp P_3$
P_2 *is RR* P_3 *is RR Q is NSRD*
$\mathbf{R}_s([\&x =_u \ll k\gg]_{S<} \vdash P_2 \diamond P_3) \sqsubseteq Q$
shows $\mathbf{R}_s(true_r \vdash P_2 \diamond P_3) \sqsubseteq (x :=_R \ll k\gg)$;; Q
proof −
have $\mathbf{R}_s([\&x =_u \ll k\gg]_{S<} \vdash P_2 \diamond P_3) \sqsubseteq \mathbf{R}_s(pre_R(Q) \vdash peri_R(Q) \diamond post_R(Q))$
by (*simp add*: *NSRD-is-SRD SRD-reactive-tri-design assms*)
hence $\mathbf{R}_s(true_r \vdash P_2 \diamond P_3) \sqsubseteq x :=_R \ll k\gg$;; $\mathbf{R}_s(pre_R(Q) \vdash peri_R(Q) \diamond post_R(Q))$
proof (*clarsimp simp add*: *rdes-def assms closure unrest rpred wp RHS-tri-design-refine*, *safe*)
assume *a1*:‘$[\&x =_u \ll k\gg]_{S<} \Rightarrow pre_R(Q)$‘ **and** *a2*:‘$[\&x =_u \ll k\gg]_{S<} \wedge peri_R(Q) \Rightarrow P_2$‘ **and** *a3*:‘$[\&x$

$=_u \ll k \gg]_{S<} \wedge post_R(Q) \Rightarrow P_3$‘
from *a1 assms*(*1*) **show** ‘*R1 true* $\Rightarrow$ [$\&x \mapsto_s \ll k \gg$] $\dagger_S$ $pre_R(Q)$‘
by (*rel-simp*)
show ‘[$\&x \mapsto_s \ll k \gg$] $\dagger_S$ $peri_R(Q) \Rightarrow P_2$‘
by (*simp add*: *a2 assms st-subst-taut-impl closure*)
show ‘[$\&x \mapsto_s \ll k \gg$] $\dagger_S$ $post_R(Q) \Rightarrow P_3$‘
by (*simp add*: *a3 assms st-subst-taut-impl closure*)
qed
thus *?thesis*
by (*simp add*: *NSRD-is-SRD SRD-reactive-tri-design assms*)
qed

11.3 Conditional

lemma *preR-cond-srea* [*rdes*]:
$pre_R(P \lhd b \rhd_R Q) = ([b]_{S<} \wedge pre_R(P) \vee [\neg b]_{S<} \wedge pre_R(Q))$
by (*rel-auto*)

lemma *periR-cond-srea* [*rdes*]:
assumes *P is SRD Q is SRD*
shows $peri_R(P \lhd b \rhd_R Q) = ([b]_{S<} \wedge peri_R(P) \vee [\neg b]_{S<} \wedge peri_R(Q))$
proof −
have $peri_R(P \lhd b \rhd_R Q) = peri_R(R1(P) \lhd b \rhd_R R1(Q))$
by (*simp add*: *Healthy-if SRD-healths assms*)
thus *?thesis*
by (*rel-auto*)
qed

lemma *postR-cond-srea* [*rdes*]:
assumes *P is SRD Q is SRD*
shows $post_R(P \lhd b \rhd_R Q) = ([b]_{S<} \wedge post_R(P) \vee [\neg b]_{S<} \wedge post_R(Q))$
proof −
have $post_R(P \lhd b \rhd_R Q) = post_R(R1(P) \lhd b \rhd_R R1(Q))$
by (*simp add*: *Healthy-if SRD-healths assms*)
thus *?thesis*
by (*rel-auto*)
qed

lemma *NSRD-cond-srea* [*closure*]:
assumes *P is NSRD Q is NSRD*
shows $P \lhd b \rhd_R Q$ *is NSRD*
proof (*rule NSRD-RC-intro*)
show $P \lhd b \rhd_R Q$ *is SRD*
by (*simp add*: *closure assms*)
show $pre_R\ (P \lhd b \rhd_R Q)$ *is RC*
proof −
have *1*:$(\lceil \neg\ b \rceil_{S<} \vee \neg_r\ pre_R\ P)$;; $R1(true) = (\lceil \neg\ b \rceil_{S<} \vee \neg_r\ pre_R\ P)$
by (*metis* (*no-types*, *lifting*) *NSRD-neg-pre-unit aext-not assms*(*1*) *seqr-or-distl st-lift-R1-true-right*)
have *2*:$(\lceil b \rceil_{S<} \vee \neg_r\ pre_R\ Q)$;; $R1(true) = (\lceil b \rceil_{S<} \vee \neg_r\ pre_R\ Q)$
by (*simp add*: *NSRD-neg-pre-unit assms seqr-or-distl st-lift-R1-true-right*)
show *?thesis*
by (*simp add*: *rdes closure assms*)
qed
show $\$st´ \sharp peri_R\ (P \lhd b \rhd_R Q)$
by (*simp add*: *rdes assms closure unrest*)
qed

11.4 Assumptions

definition *AssumeR* :: *'s cond* ⇒ (*'s*, *'t*::*trace*, *'α*) *hrel-rsp* ([-]$^\top{}_R$) **where**
[*upred-defs*]: *AssumeR b* = $II_R \lhd b \rhd_R$ *Miracle*

lemma *AssumeR-rdes-def* [*rdes-def*]:
$[b]^\top{}_R = \mathbf{R}_s(true_r \vdash false \diamond [b]^\top{}_r)$
unfolding *AssumeR-def* **by** (*rdes-eq*)

lemma *AssumeR-NSRD* [*closure*]: $[b]^\top{}_R$ *is NSRD*
by (*simp add*: *AssumeR-def closure*)

lemma *AssumeR-false*: $[false]^\top{}_R$ = *Miracle*
by (*rel-auto*)

lemma *AssumeR-true*: $[true]^\top{}_R = II_R$
by (*rel-auto*)

lemma *AssumeR-comp*: $[b]^\top{}_R$;; $[c]^\top{}_R = [b \wedge c]^\top{}_R$
by (*rdes-simp*)

lemma *AssumeR-choice*: $[b]^\top{}_R \sqcap [c]^\top{}_R = [b \vee c]^\top{}_R$
by (*rdes-eq*)

lemma *AssumeR-refine-skip*: $II_R \sqsubseteq [b]^\top{}_R$
by (*rdes-refine*)

lemma *AssumeR-test* [*closure*]: $test_R$ $[b]^\top{}_R$
by (*simp add*: *AssumeR-refine-skip nsrdes-theory.utest-intro*)

lemma *Star-AssumeR*: $[b]^\top{}_R{}^{\star R} = II_R$
by (*simp add*: *StarR-def AssumeR-NSRD AssumeR-test nsrdes-theory.Star-test*)

lemma *AssumeR-choice-skip*: $II_R \sqcap [b]^\top{}_R = II_R$
by (*rdes-eq*)

lemma *AssumeR-seq-refines*:
assumes *P is NSRD*
shows $P \sqsubseteq P$;; $[b]^\top{}_R$
by (*rdes-refine cls*: *assms*)

lemma *cond-srea-AssumeR-form*:
assumes *P is NSRD Q is NSRD*
shows $P \lhd b \rhd_R Q = ([b]^\top{}_R$;; $P) \sqcap ([\neg b]^\top{}_R$;; $Q)$
by (*rdes-eq cls*: *assms*)

lemma *cond-srea-insert-assume*:
assumes *P is NSRD Q is NSRD*
shows $P \lhd b \rhd_R Q = ([b]^\top{}_R$;; $P \lhd b \rhd_R [\neg b]^\top{}_R$;; $Q)$
by (*simp add*: *AssumeR-NSRD AssumeR-comp NSRD-seqr-closure RA1 assms cond-srea-AssumeR-form*)

lemma *AssumeR-cond-left*:
assumes *P is NSRD Q is NSRD*
shows $[b]^\top{}_R$;; $(P \lhd b \rhd_R Q) = ([b]^\top{}_R$;; $P)$
by (*rdes-eq cls*: *assms*)

lemma *AssumeR-cond-right*:
assumes *P is NSRD Q is NSRD*
shows $[\neg b]^\top{}_R \mathbin{;;} (P \mathrel{\triangleleft} b \mathrel{\triangleright}_R Q) = ([\neg b]^\top{}_R \mathbin{;;} Q)$
by (*rdes-eq cls*: *assms*)

11.5 Guarded commands

definition *GuardedCommR* :: *'s cond* ⇒ (*'s*, *'t*::*trace*, *'α*) *hrel-rsp* ⇒ (*'s*, *'t*, *'α*) *hrel-rsp* (- →$_R$ - [*85*, *86*] *85*) **where**
gcmd-def[*rdes-def*]: *GuardedCommR g A* = *A* ◁ *g* ▷$_R$ *Miracle*

lemma *gcmd-false*[*simp*]: (*false* →$_R$ *A*) = *Miracle*
unfolding *gcmd-def* **by** (*pred-auto*)

lemma *gcmd-true*[*simp*]: (*true* →$_R$ *A*) = *A*
unfolding *gcmd-def* **by** (*pred-auto*)

lemma *gcmd-SRD*:
assumes *A is SRD*
shows (*g* →$_R$ *A*) *is SRD*
by (*simp add*: *gcmd-def SRD-cond-srea assms srdes-theory.top-closed*)

lemma *gcmd-NSRD* [*closure*]:
assumes *A is NSRD*
shows (*g* →$_R$ *A*) *is NSRD*
by (*simp add*: *gcmd-def NSRD-cond-srea assms NSRD-Miracle*)

lemma *gcmd-Productive* [*closure*]:
assumes *A is NSRD A is Productive*
shows (*g* →$_R$ *A*) *is Productive*
by (*simp add*: *gcmd-def closure assms*)

lemma *gcmd-seq-distr*:
assumes *B is NSRD*
shows (*g* →$_R$ *A*) ;; *B* = (*g* →$_R$ (*A* ;; *B*))
by (*simp add*: *Miracle-left-zero NSRD-is-SRD assms cond-st-distr gcmd-def*)

lemma *gcmd-nondet-distr*:
assumes *A is NSRD B is NSRD*
shows (*g* →$_R$ (*A* ⊓ *B*)) = (*g* →$_R$ *A*) ⊓ (*g* →$_R$ *B*)
by (*rdes-eq cls*: *assms*)

lemma *AssumeR-as-gcmd*:
$[b]^\top{}_R$ = *b* →$_R$ *II*$_R$
by (*rdes-eq*)

lemma *AssumeR-gcomm*:
assumes *P is NSRD*
shows $[b]^\top{}_R \mathbin{;;} (c \to_R P) = (b \wedge c) \to_R P$
by (*rdes-eq cls*: *assms*)

11.6 Generalised Alternation

definition *AlternateR*
:: *'a set* ⇒ (*'a* ⇒ *'s upred*) ⇒ (*'a* ⇒ (*'s*, *'t*::*trace*, *'α*) *hrel-rsp*) ⇒ (*'s*, *'t*, *'α*) *hrel-rsp* ⇒ (*'s*, *'t*, *'α*) *hrel-rsp* **where**

[*upred-defs*, *rdes-def*]: *AlternateR I g A B* = $(\bigsqcap\ i \in I \cdot ((g\ i) \rightarrow_R (A\ i))) \sqcap ((\neg\ (\bigsqcup\ i \in I \cdot g\ i)) \rightarrow_R B)$

definition *AlternateR-list*
:: ('*s upred* × ('*s*, '*t*::*trace*, 'α) *hrel-rsp*) *list* ⇒ ('*s*, '*t*, 'α) *hrel-rsp* ⇒ ('*s*, '*t*, 'α) *hrel-rsp* **where**
[*upred-defs*, *ndes-simp*]:
AlternateR-list xs P = *AlternateR* {*0*..<*length xs*} (λ *i*. *map fst xs* ! *i*) (λ *i*. *map snd xs* ! *i*) *P*

syntax
-altindR-els :: *pttrn* ⇒ *logic* ⇒ *logic* ⇒ *logic* ⇒ *logic* ⇒ *logic* (*if*$_R$ -∈- • - → - *else* - *fi*)
-altindR :: *pttrn* ⇒ *logic* ⇒ *logic* ⇒ *logic* ⇒ *logic* (*if*$_R$ -∈- • - → - *fi*)

-altgcommR-els :: *gcomms* ⇒ *logic* ⇒ *logic* (*if*$_R$/ - *else* - /*fi*)
-altgcommR :: *gcomms* ⇒ *logic* (*if*$_R$/ - /*fi*)

translations
if$_R$ *i*∈*I* • *g* → *A else B fi* ⇀ *CONST AlternateR I* (λ*i*. *g*) (λ*i*. *A*) *B*
if$_R$ *i*∈*I* • *g* → *A fi* ⇀ *CONST AlternateR I* (λ*i*. *g*) (λ*i*. *A*) (*CONST Chaos*)
if$_R$ *i*∈*I* • (*g i*) → *A else B fi* ↼ *CONST AlternateR I g* (λ*i*. *A*) *B*
-altgcommR cs ⇀ *CONST AlternateR-list cs* (*CONST Chaos*)
-altgcommR (*-gcomm-show cs*) ↼ *CONST AlternateR-list cs* (*CONST Chaos*)
-altgcommR-els cs P ⇀ *CONST AlternateR-list cs P*
-altgcommR-els (*-gcomm-show cs*) *P* ↼ *CONST AlternateR-list cs P*

lemma *AlternateR-NSRD-closed* [*closure*]:
assumes ⋀ *i*. *i* ∈ *I* ⟹ *A i is NSRD B is NSRD*
shows (*if*$_R$ *i*∈*I* • *g i* → *A i else B fi*) *is NSRD*
proof (*cases I* = {})
case *True*
then show *?thesis* **by** (*simp add*: *AlternateR-def assms*)
next
case *False*
then show *?thesis* **by** (*simp add*: *AlternateR-def closure assms*)
qed

lemma *AlternateR-empty* [*simp*]:
(*if*$_R$ *i* ∈ {} • *g i* → *A i else B fi*) = *B*
by (*rdes-simp*)

lemma *AlternateR-Productive* [*closure*]:
assumes
⋀ *i*. *i* ∈ *I* ⟹ *A i is NSRD B is NSRD*
⋀ *i*. *i* ∈ *I* ⟹ *A i is Productive B is Productive*
shows (*if*$_R$ *i*∈*I* • *g i* → *A i else B fi*) *is Productive*
proof (*cases I* = {})
case *True*
then show *?thesis*
by (*simp add*: *assms*(*4*))
next
case *False*
then show *?thesis*
by (*simp add*: *AlternateR-def closure assms*)
qed

lemma *AlternateR-singleton*:

assumes *A k is NSRD B is NSRD*
shows $(if_R\ i \in \{k\} \cdot g\ i \to A\ i\ else\ B\ fi) = (A(k) \lhd g(k) \rhd_R B)$
by (*simp add*: *AlternateR-def*, *rdes-eq cls*: *assms*)

Convert an alternation over disjoint guards into a cascading if-then-else

lemma *AlternateR-insert-cascade*:
assumes
$\bigwedge i.\ i \in I \Longrightarrow A\ i\ is\ NSRD$
A k is NSRD B is NSRD
$(g(k) \wedge (\bigvee i{\in}I \cdot g(i))) = false$
shows $(if_R\ i \in insert\ k\ I \cdot g\ i \to A\ i\ else\ B\ fi) = (A(k) \lhd g(k) \rhd_R (if_R\ i{\in}I \cdot g(i) \to A(i)\ else\ B\ fi))$
proof (*cases* $I = \{\}$)
case *True*
then show *?thesis* **by** (*simp add*: *AlternateR-singleton assms*)
next
case *False*
have *1*: $(\prod\ i \in I \cdot g\ i \to_R A\ i) = (\prod\ i \in I \cdot g\ i \to_R \mathbf{R}_s(pre_R(A\ i) \vdash peri_R(A\ i) \diamond post_R(A\ i)))$
by (*simp add*: *NSRD-is-SRD SRD-reactive-tri-design assms*(*1*) *cong*: *UINF-cong*)
from *assms*(*4*) **show** *?thesis*
by (*simp add*: *AlternateR-def 1 False*)
(*rdes-eq cls*: *assms*(*1−3*) *False cong*: *UINF-cong*)
qed

lemma *AlternateR-assume-branch*:
assumes $I \neq \{\} \bigwedge i.\ i \in I \Longrightarrow P\ i\ is\ NSRD\ Q\ is\ NSRD$
shows $([\prod\ i \in I \cdot b\ i]^\top_R ;;\ AlternateR\ I\ b\ P\ Q) = (\prod\ i \in I \cdot b\ i \to_R P\ i)$ (**is** *?lhs* = *?rhs*)
proof −
have $?lhs = [\prod\ i \in I \cdot b\ i]^\top_R ;; ((\prod\ i \in I \cdot b\ i \to_R P\ i) \sqcap (\neg\ (\prod\ i \in I \cdot b\ i)) \to_R Q)$
by (*simp add*: *AlternateR-def closure assms*)
also have ... $= [\prod\ i \in I \cdot b\ i]^\top_R ;; (\prod\ i \in I \cdot b\ i \to_R P\ i) \sqcap Miracle$
by (*simp add*: *seqr-inf-distr AssumeR-gcomm closure assms*)
also have ... $= (\prod\ i \in I \cdot ((\prod\ i \in I \cdot b\ i) \wedge b\ i) \to_R P\ i) \sqcap Miracle$
by (*simp add*: *seq-UINF-distl AssumeR-gcomm closure assms cong*: *UINF-cong*)
also have ... $= (\prod\ i \in I \cdot b\ i \to_R P\ i) \sqcap Miracle$
proof −
have $\bigwedge i.\ i \in I \Longrightarrow ((\prod\ i \in I \cdot b\ i) \wedge b\ i) = b\ i$
by (*rel-auto*)
thus *?thesis*
by (*simp cong*: *UINF-cong*)
qed
also have ... $= (\prod\ i \in I \cdot b\ i \to_R P\ i)$
by (*simp add*: *closure assms*)
finally show *?thesis* **.**
qed

11.7 Choose

definition *choose-srd* :: ('*s*,'*t*::*trace*,'α) *hrel-rsp* ($choose_R$) **where**
[*upred-defs*, *rdes-def*]: $choose_R = \mathbf{R}_s(true_r \vdash false \diamond true_r)$

lemma *preR-choose* [*rdes*]: $pre_R(choose_R) = true_r$
by (*rel-auto*)

lemma *periR-choose* [*rdes*]: $peri_R(choose_R) = false$
by (*rel-auto*)

lemma *postR-choose* [*rdes*]: $post_R(choose_R) = true_r$
by (*rel-auto*)

lemma *choose-srd-SRD* [*closure*]: $choose_R$ *is SRD*
by (*simp add*: *choose-srd-def closure unrest*)

lemma *NSRD-choose-srd* [*closure*]: $choose_R$ *is NSRD*
by (*rule NSRD-intro*, *simp-all add*: *closure unrest rdes*)

11.8 Divergence Freedom

definition *ndiv-srd* :: ('*s*,'*t*::*trace*,'α) *hrel-rsp* ($ndiv_R$)
where [*rdes-def*]: *ndiv-srd* = $\mathbf{R}_s(true_r \vdash true_r \diamond true_r)$

lemma *ndiv-NSRD* [*closure*]: $ndiv_R$ *is NSRD*
by (*simp add*: *rdes-def closure unrest*)

lemma *ndiv-srd-refines-preR-true*:
assumes *P is SRD*
shows $ndiv_R \sqsubseteq P \longleftrightarrow pre_R(P) = true_r$ (**is** *?lhs* $\longleftrightarrow$ *?rhs*)
proof
assume *?lhs*
thus *?rhs*
by (*metis R1-preR ndiv-srd-def preR-antitone preR-srdes rea-true-RR rea-true-disj*(*2*) *utp-pred-laws.sup.orderE*)

next
assume *?rhs*
hence $ndiv_R \sqsubseteq \mathbf{R}_s(pre_R(P) \vdash peri_R(P) \diamond post_R(P))$
by (*simp add*: *RHS-tri-design-conj assms ndiv-srd-def closure rea-true-conj*(*1*) *rea-true-impl utp-pred-laws.inf.absorb-if*
thus *?lhs*
by (*simp add*: *SRD-reactive-tri-design assms*)
qed

lemma *ndiv-srd-refines-rdes-pre-true*:
assumes P_1 *is RR* P_2 *is RR* P_3 *is RR*
shows $ndiv_R \sqsubseteq \mathbf{R}_s(P_1 \vdash P_2 \diamond P_3) \longleftrightarrow P_1 = true_r$ (**is** *?lhs* $\longleftrightarrow$ *?rhs*)
by (*simp add*: *ndiv-srd-refines-preR-true closure assms rdes unrest*)

11.9 State Abstraction

definition *state-srea* ::
'*s itself* $\Rightarrow$ ('*s*,'*t*::*trace*,'α,'β) *rel-rsp* $\Rightarrow$ (*unit*,'*t*,'α,'β) *rel-rsp* **where**
[*upred-defs*]: *state-srea t P* = $\langle \exists$ {$st,$st´} $\cdot$ *P*$\rangle_S$

syntax
-state-srea :: *type* $\Rightarrow$ *logic* $\Rightarrow$ *logic* (*state* - $\cdot$ - [*0*,*200*] *200*)

translations
state '*a* $\cdot$ *P* == *CONST state-srea TYPE*('*a*) *P*

lemma *R1-state-srea*: *R1*(*state* '*a* $\cdot$ *P*) = (*state* '*a* $\cdot$ *R1*(*P*))
by (*rel-auto*)

lemma *R2c-state-srea*: *R2c*(*state* '*a* $\cdot$ *P*) = (*state* '*a* $\cdot$ *R2c*(*P*))
by (*rel-auto*)

lemma *R3h-state-srea*: $R3h(state\ 'a \cdot P) = (state\ 'a \cdot R3h(P))$
by (*rel-auto*)

lemma *RD1-state-srea*: $RD1(state\ 'a \cdot P) = (state\ 'a \cdot RD1(P))$
by (*rel-auto*)

lemma *RD2-state-srea*: $RD2(state\ 'a \cdot P) = (state\ 'a \cdot RD2(P))$
by (*rel-auto*)

lemma *RD3-state-srea*: $RD3(state\ 'a \cdot P) = (state\ 'a \cdot RD3(P))$
by (*rel-auto*, *blast+*)

lemma *SRD-state-srea* [*closure*]: *P is SRD* $\Longrightarrow$ *state 'a · P is SRD*
by (*simp add*: *Healthy-def R1-state-srea R2c-state-srea R3h-state-srea RD1-state-srea RD2-state-srea RHS-def SRD-def*)

lemma *NSRD-state-srea* [*closure*]: *P is NSRD* $\Longrightarrow$ *state 'a · P is NSRD*
by (*metis Healthy-def NSRD-is-RD3 NSRD-is-SRD RD3-state-srea SRD-RD3-implies-NSRD SRD-state-srea*)

lemma *preR-state-srea* [*rdes*]: $pre_R(state\ 'a \cdot P) = \langle \forall\ \{\$st,\$st\acute{}\} \cdot pre_R(P)\rangle_S$
by (*simp add*: *state-srea-def*, *rel-auto*)

lemma *periR-state-srea* [*rdes*]: $peri_R(state\ 'a \cdot P) = state\ 'a \cdot peri_R(P)$
by (*rel-auto*)

lemma *postR-state-srea* [*rdes*]: $post_R(state\ 'a \cdot P) = state\ 'a \cdot post_R(P)$
by (*rel-auto*)

lemma *state-srea-rdes-def* [*rdes-def*]:
assumes *P is RC Q is RR R is RR*
shows $state\ 'a \cdot \mathbf{R}_s(P \vdash Q \diamond R) = \mathbf{R}_s(\langle \forall\ \{\$st,\$st\acute{}\} \cdot P\rangle_S \vdash (state\ 'a \cdot Q) \diamond (state\ 'a \cdot R))$
(**is** *?lhs = ?rhs*)
proof −
have $?lhs = \mathbf{R}_s(pre_R(?lhs) \vdash peri_R(?lhs) \diamond post_R(?lhs))$
by (*simp add*: *RC-implies-RR SRD-rdes-intro SRD-reactive-tri-design SRD-state-srea assms*)
also have ... = $\mathbf{R}_s\ (\langle \forall\ \{\$st, \$st\acute{}\} \cdot P\rangle_S \vdash state\ 'a \cdot (P \Rightarrow_r Q) \diamond state\ 'a \cdot (P \Rightarrow_r R))$
by (*simp add*: *rdes closure assms*)
also have ... = *?rhs*
by (*rel-auto*)
finally show *?thesis* **.**
qed

lemma *ext-st-rdes-dist* [*rdes-def*]:
$\mathbf{R}_s(P \vdash Q \diamond R) \oplus_p abs\text{-}st_L = \mathbf{R}_s(P \oplus_p abs\text{-}st_L \vdash Q \oplus_p abs\text{-}st_L \diamond R \oplus_p abs\text{-}st_L)$
by (*rel-auto*)

lemma *state-srea-refine*:
$(P \oplus_p abs\text{-}st_L) \sqsubseteq Q \Longrightarrow P \sqsubseteq (state\text{-}srea\ TYPE('s)\ Q)$
by (*rel-auto*)

11.10 Reactive Frames

definition *rdes-frame-ext* :: $('\alpha \Longrightarrow '\beta) \Rightarrow ('\alpha, 't::trace, 'r)\ hrel\text{-}rsp \Rightarrow ('\beta, 't, 'r)\ hrel\text{-}rsp$ **where**
[*upred-defs*, *rdes-def*]: *rdes-frame-ext a P* = $\mathbf{R}_s(rel\text{-}aext\ (pre_R(P))\ (map\text{-}st_L\ a) \vdash rel\text{-}aext\ (peri_R(P))\ (map\text{-}st_L\ a) \diamond a{:}[post_R(P)]_r{}^+)$

syntax
-rdes-frame-ext :: salpha ⇒ logic ⇒ logic (-:[-]$_R{}^+$ [99,0] 100)

translations
-rdes-frame-ext x P => CONST rdes-frame-ext x P
-rdes-frame-ext (-salphaset (-salphamk x)) P <= CONST rdes-frame-ext x P

lemma *RC-rel-aext-st-closed* [*closure*]:
assumes *P is RC*
shows *rel-aext P (map-st$_L$ a) is RC*
proof −
have *RC(rel-aext (RC(P)) (map-st$_L$ a)) = rel-aext (RC(P)) (map-st$_L$ a)*
by (*rel-auto*)
(*metis* (*no-types*, *hide-lams*) *diff-add-cancel-left' dual-order.trans le-add trace-class.add-diff-cancel-left trace-class.add-left-mono*)
thus *?thesis*
by (*rule-tac Healthy-intro*, *simp add*: *assms Healthy-if*)
qed

lemma *rdes-frame-ext-SRD-closed*:
⟦ *P is SRD*; *$wait´ ♯ pre$_R$(P)* ⟧ ⟹ *a:[P]$_R{}^+$ is SRD*
unfolding *rdes-frame-ext-def*
apply (*rule SRD-rdes-intro*)
apply (*simp-all add*: *closure unrest*)
apply (*simp add*: *RR-R2-intro ok'-pre-unrest ok-pre-unrest wait-pre-unrest closure*)
done

lemma *preR-rdes-frame-ext*:
P is NSRD ⟹ pre$_R$(a:[P]$_R{}^+$) = rel-aext (pre$_R$(P)) (map-st$_L$ a)
by (*simp add*: *preR-RR preR-srdes rdes-frame-ext-def rea-aext-RR*)

lemma *unrest-rel-aext-st'* [*unrest*]: *$st´ ♯ P ⟹ $st´ ♯ rel-aext P (map-st$_L$ a)*
by (*rel-auto*)

lemma *rdes-frame-ext-NSRD-closed*:
P is NSRD ⟹ a:[P]$_R{}^+$ is NSRD
apply (*rule NSRD-RC-intro*)
apply (*rule rdes-frame-ext-SRD-closed*)
apply (*simp-all add*: *closure unrest rdes*)
apply (*simp add*: *NSRD-neg-pre-RC RC-rel-aext-st-closed preR-RR preR-srdes rdes-frame-ext-def rea-aext-RR*)
apply (*simp add*: *rdes-frame-ext-def*)
apply (*simp add*: *rdes closure unrest*)
done

lemma *skip-srea-frame* [*frame*]:
vwb-lens a ⟹ a:[II$_R$]$_R{}^+$ = II$_R$
by (*rdes-eq*)

lemma *seq-srea-frame* [*frame*]:
assumes *vwb-lens a P is NSRD Q is NSRD*
shows *a:[P ;; Q]$_R{}^+$ = a:[P]$_R{}^+$;; a:[Q]$_R{}^+$* (**is** *?lhs = ?rhs*)
proof −
have *?lhs = **R**$_s$ ((pre$_R$ P ∧ post$_R$ P wp$_r$ pre$_R$ Q) ⊕$_r$ map-st$_L$[a] ⊢*
((pre$_R$ P ∧ post$_R$ P wp$_r$ pre$_R$ Q) ⊕$_r$ map-st$_L$[a] ⇒$_r$ (peri$_R$ P ∨ post$_R$ P ;; peri$_R$ Q)

$\oplus_r$ *map-st*$_L[a]$) $\diamond$
$a{:}[pre_R\ P \wedge post_R\ P\ wp_r\ pre_R\ Q \Rightarrow_r post_R\ P\ ;;\ post_R\ Q]_r{}^+$)
using *assms*(*1*) **by** (*rdes-simp cls*: *assms*(*2*−*3*))
also have ... = $\mathbf{R}_s$ ((*pre*$_R$ *P* $\wedge$ *post*$_R$ *P wp*$_r$ *pre*$_R$ *Q*) $\oplus_r$ *map-st*$_L[a] \vdash$
((*peri*$_R$ *P* $\vee$ *post*$_R$ *P* ;; *peri*$_R$ *Q*) $\oplus_r$ *map-st*$_L[a]$) $\diamond$
$a{:}[post_R\ P\ ;;\ post_R\ Q]_r{}^+$)
by (*rel-auto*)
also from *assms*(*1*) **have** ... = *?rhs*
apply (*rdes-eq-split cls*: *assms*(*2*−*3*))
apply (*rel-auto*)
apply (*metis mwb-lens-def vwb-lens-mwb weak-lens.put-get*)
apply (*rel-auto*)
apply (*metis mwb-lens-def vwb-lens-mwb weak-lens.put-get*)
apply (*metis vwb-lens-wb wb-lens-def weak-lens.put-get*)
apply (*metis mwb-lens-def vwb-lens-mwb weak-lens.put-get*)
apply (*rel-auto*)
apply (*metis mwb-lens-def vwb-lens-mwb weak-lens.put-get*)
done
finally show *?thesis* **.**
qed

lemma *rdes-frame-ext-Productive-closed* [*closure*]:
assumes *P is NSRD P is Productive*
shows $x{:}[P]_R{}^+$ *is Productive*
proof −
have $x{:}[Productive(P)]_R{}^+$ *is Productive*
by (*rdes-simp cls*: *assms*, *rel-auto*)
thus *?thesis*
by (*simp add*: *Healthy-if assms*)
qed

11.11 While Loop

definition *WhileR* :: *'s upred* ⇒ (*'s*, *'t*::*size-trace*, *'α*) *hrel-rsp* ⇒ (*'s*, *'t*, *'α*) *hrel-rsp* **where**
WhileR b P = (μ_R *X* · (*P* ;; *X*) ◁ *b* ▷$_R$ *II*$_R$)

syntax *-whileR* :: *logic* ⇒ *logic* ⇒ *logic* (*while*$_R$ *- do - od*)
translations *-whileR b P* == *CONST WhileR b P*

lemma *Sup-power-false*:
fixes *F* :: *'α upred* ⇒ *'α upred*
shows ($\bigsqcap i.$ (*F* ^^ *i*) *false*) = ($\bigsqcap i.$ (*F* ^^ (*i*+*1*)) *false*)
proof −
have ($\bigsqcap i.$ (*F* ^^ *i*) *false*) = (*F* ^^ *0*) *false* ⊓ ($\bigsqcap i.$ (*F* ^^ (*i*+*1*)) *false*)
by (*subst Sup-power-expand*, *simp*)
also have ... = ($\bigsqcap i.$ (*F* ^^ (*i*+*1*)) *false*)
by (*simp*)
finally show *?thesis* **.**
qed

theorem *WhileR-unfold*:
assumes *P is NSRD*
shows *while*$_R$ *b do P od* = (*P* ;; *while*$_R$ *b do P od*) ◁ *b* ▷$_R$ *II*$_R$
apply (*simp add*: *WhileR-def*)
apply (*subst srdes-theory.LFP-unfold*)
apply (*simp-all add*: *mono-Monotone-utp-order closure assms*)

done

theorem *WhileR-iter-expand*:
assumes *P is NSRD P is Productive*
shows $while_R\ b\ do\ P\ od = (\bigsqcap i \cdot (P \lhd b \rhd_R II_R)\ \hat{}\ i\ ;;\ (P\ ;;\ Miracle \lhd b \rhd_R II_R))$ (**is** *?lhs = ?rhs*)
proof −
have *1*:*Continuous* $(\lambda X.\ P\ ;;\ SRD\ X)$
using *SRD-Continuous*
by (*clarsimp simp add*: *Continuous-def seq-SUP-distl*[*THEN sym*], *drule-tac x*=*A* **in** *spec*, *simp*)
have *2*: *Continuous* $(\lambda X.\ P\ ;;\ SRD\ X \lhd b \rhd_R II_R)$
by (*simp add*: *1 closure assms*)
have *?lhs* $= (\mu_R\ X \cdot P\ ;;\ X \lhd b \rhd_R II_R)$
by (*simp add*: *WhileR-def*)
also have ... $= (\mu\ X \cdot P\ ;;\ SRD(X) \lhd b \rhd_R II_R)$
by (*auto simp add*: *srd-mu-equiv closure assms*)
also have ... $= (\nu\ X \cdot P\ ;;\ SRD(X) \lhd b \rhd_R II_R)$
by (*auto simp add*: *guarded-fp-uniq Guarded-if-Productive*[*OF assms*] *funcsetI closure assms*)
also have ... $= (\bigsqcap i.\ ((\lambda X.\ P\ ;;\ SRD\ X \lhd b \rhd_R II_R)\ \hat{\ }\hat{\ }\ i)\ false)$
by (*simp add*: *sup-continuous-lfp 2 sup-continuous-Continuous false-upred-def*)
also have ... $= (\bigsqcap i.\ ((\lambda X.\ P\ ;;\ SRD\ X \lhd b \rhd_R II_R)\ \hat{\ }\hat{\ }\ (i{+}1))\ false)$
by (*simp add*: *Sup-power-false*)
also have ... $= (\bigsqcap i.\ (P \lhd b \rhd_R II_R)\hat{}i\ ;;\ (P\ ;;\ Miracle \lhd b \rhd_R II_R))$
proof (*rule SUP-cong*, *simp*)
fix *i*
show $((\lambda X.\ P\ ;;\ SRD\ X \lhd b \rhd_R II_R)\ \hat{\ }\hat{\ }\ (i+1))\ false = (P \lhd b \rhd_R II_R)\ \hat{}\ i\ ;;\ (P\ ;;\ Miracle \lhd b \rhd_R II_R)$
proof (*induct i*)
case *0*
thm *if-eq-cancel*
then show *?case*
by (*simp*, *metis srdes-theory.healthy-top*)
next
case (*Suc i*)
show *?case*
proof −
have $((\lambda X.\ P\ ;;\ SRD\ X \lhd b \rhd_R II_R)\ \hat{\ }\hat{\ }\ (Suc\ i + 1))\ false =$
$P\ ;;\ SRD\ (((\lambda X.\ P\ ;;\ SRD\ X \lhd b \rhd_R II_R)\ \hat{\ }\hat{\ }\ (i+1))\ false) \lhd b \rhd_R II_R$
by *simp*
also have ... $= P\ ;;\ SRD\ ((P \lhd b \rhd_R II_R)\ \hat{}\ i\ ;;\ (P\ ;;\ Miracle \lhd b \rhd_R II_R)) \lhd b \rhd_R II_R$
using *Suc.hyps* **by** *auto*
also have ... $= P\ ;;\ ((P \lhd b \rhd_R II_R)\ \hat{}\ i\ ;;\ (P\ ;;\ Miracle \lhd b \rhd_R II_R)) \lhd b \rhd_R II_R$
by (*metis* (*no-types*, *lifting*) *Healthy-if NSRD-cond-srea NSRD-is-SRD NSRD-power-Suc NSRD-srd-skip SRD-cond-srea SRD-seqr-closure assms*(*1*) *power.power-eq-if seqr-left-unit srdes-theory.top-closed*)
also have ... $= (P \lhd b \rhd_R II_R)\ \hat{}\ Suc\ i\ ;;\ (P\ ;;\ Miracle \lhd b \rhd_R II_R)$
proof (*induct i*)
case *0*
then show *?case*
by (*simp add*: *NSRD-is-SRD SRD-cond-srea SRD-left-unit SRD-seqr-closure SRD-srdes-skip assms*(*1*) *cond-L6 cond-st-distr srdes-theory.top-closed*)
next
case (*Suc i*)
have *1*: $II_R\ ;;\ ((P \lhd b \rhd_R II_R)\ ;;\ (P \lhd b \rhd_R II_R)\ \hat{}\ i) = ((P \lhd b \rhd_R II_R)\ ;;\ (P \lhd b \rhd_R II_R)\ \hat{}\ i)$
by (*simp add*: *NSRD-is-SRD RA1 SRD-cond-srea SRD-left-unit SRD-srdes-skip assms*(*1*))
then show *?case*
proof −

have $\bigwedge u.\ (u\ ;;\ (P \lhd b \rhd_R II_R)\ \hat{}\ Suc\ i)\ ;;\ (P\ ;;\ (Miracle) \lhd b \rhd_R (II_R)) \lhd b \rhd_R (II_R) =$
$((u \lhd b \rhd_R II_R)\ ;;\ (P \lhd b \rhd_R II_R)\ \hat{}\ Suc\ i)\ ;;\ (P\ ;;\ (Miracle) \lhd b \rhd_R (II_R))$
by (*metis* (*no-types*) *Suc.hyps 1 cond-L6 cond-st-distr power.power.power-Suc*)
then show *?thesis*
by (*simp add*: *RA1 upred-semiring.power-Suc*)
qed
qed
finally show *?thesis* **.**
qed
qed
qed
also have $... = (\bigsqcap i \cdot (P \lhd b \rhd_R II_R)\hat{}i\ ;;\ (P\ ;;\ Miracle \lhd b \rhd_R II_R))$
by (*simp add*: *UINF-as-Sup-collect'*)
finally show *?thesis* **.**
qed

theorem *WhileR-star-expand*:
assumes *P is NSRD P is Productive*
shows $while_R\ b\ do\ P\ od = (P \lhd b \rhd_R II_R)^{\star R}\ ;;\ (P\ ;;\ Miracle \lhd b \rhd_R II_R)$ (**is** *?lhs* = *?rhs*)
proof −
have $?lhs = (\bigsqcap i \cdot (P \lhd b \rhd_R II_R)\ \hat{}\ i)\ ;;\ (P\ ;;\ Miracle \lhd b \rhd_R II_R)$
by (*simp add*: *WhileR-iter-expand seq-UINF-distr' assms*)
also have $... = (P \lhd b \rhd_R II_R)^{\star}\ ;;\ (P\ ;;\ Miracle \lhd b \rhd_R II_R)$
by (*simp add*: *ustar-def*)
also have $... = ((P \lhd b \rhd_R II_R)^{\star}\ ;;\ II_R)\ ;;\ (P\ ;;\ Miracle \lhd b \rhd_R II_R)$
by (*simp add*: *seqr-assoc SRD-left-unit closure assms*)
also have $... = (P \lhd b \rhd_R II_R)^{\star R}\ ;;\ (P\ ;;\ Miracle \lhd b \rhd_R II_R)$
by (*simp add*: *StarR-def nsrdes-theory.Star-def*)
finally show *?thesis* **.**
qed

lemma *WhileR-NSRD-closed* [*closure*]:
assumes *P is NSRD P is Productive*
shows $while_R\ b\ do\ P\ od$ *is NSRD*
by (*simp add*: *StarR-def WhileR-star-expand assms closure*)

theorem *WhileR-iter-form-lemma*:
assumes *P is NSRD*
shows $(P \lhd b \rhd_R II_R)^{\star R}\ ;;\ (P\ ;;\ Miracle \lhd b \rhd_R II_R) = ([b]^{\top}{}_R\ ;;\ P)^{\star R}\ ;;\ [\neg\ b]^{\top}{}_R$
proof −
have $(P \lhd b \rhd_R II_R)^{\star R}\ ;;\ (P\ ;;\ Miracle \lhd b \rhd_R II_R) = (([b]^{\top}{}_R\ ;;\ P) \sqcap [\neg b]^{\top}{}_R)^{\star R}\ ;;\ (P\ ;;\ Miracle \lhd b \rhd_R II_R)$
by (*simp add*: *AssumeR-NSRD NSRD-right-unit NSRD-srd-skip assms*(*1*) *cond-srea-AssumeR-form*)
also have $... = (([b]^{\top}{}_R\ ;;\ P)^{\star R}\ ;;\ [\neg\ b]^{\top}{}_R{}^{\star R})^{\star R}\ ;;\ (P\ ;;\ Miracle \lhd b \rhd_R II_R)$
by (*simp add*: *StarR-def AssumeR-NSRD NSRD-seqr-closure nsrdes-theory.Star-denest assms*(*1*))
also have $... = (([b]^{\top}{}_R\ ;;\ P)^{\star R})^{\star R}\ ;;\ (P\ ;;\ Miracle \lhd b \rhd_R II_R)$
by (*metis* (*no-types*, *hide-lams*) *StarR-def RD3-def RD3-idem Star-AssumeR nsrdes-theory.Star-def*)
also have $... = (([b]^{\top}{}_R\ ;;\ P)^{\star R})\ ;;\ (P\ ;;\ Miracle \lhd b \rhd_R II_R)$
by (*simp add*: *StarR-def AssumeR-NSRD NSRD-seqr-closure nsrdes-theory.Star-invol assms*(*1*))
also have $... = (([b]^{\top}{}_R\ ;;\ P)^{\star R})\ ;;\ (([b]^{\top}{}_R\ ;;\ P\ ;;\ Miracle) \sqcap [\neg b]^{\top}{}_R)$
by (*simp add*: *AssumeR-NSRD NSRD-Miracle NSRD-right-unit NSRD-seqr-closure NSRD-srd-skip assms*(*1*) *cond-srea-AssumeR-form*)
also have $... = ((([b]^{\top}{}_R\ ;;\ P)^{\star R})\ ;;\ [b]^{\top}{}_R\ ;;\ P\ ;;\ Miracle) \sqcap (([b]^{\top}{}_R\ ;;\ P)^{\star R}\ ;;\ [\neg b]^{\top}{}_R)$
by (*simp add*: *upred-semiring.distrib-left*)
also have $... = ([b]^{\top}{}_R\ ;;\ P)^{\star R}\ ;;\ [\neg\ b]^{\top}{}_R$

proof −
have $((([b]^\top{}_R ;; P)^{\star R}) ;; [\neg b]^\top{}_R = (II_R \sqcap ((([b]^\top{}_R ;; P)^{\star R} ;; [b]^\top{}_R ;; P)) ;; [\neg\ b]^\top{}_R$
by (*simp add*: *StarR-def AssumeR-NSRD NSRD-seqr-closure nsrdes-theory.Star-unfoldr-eq assms(1)*)
also have ... $= [\neg\ b]^\top{}_R \sqcap ((([b]^\top{}_R ;; P)^{\star R} ;; [b]^\top{}_R ;; P) ;; [\neg\ b]^\top{}_R)$
by (*metis* (*no-types*, *lifting*) *StarR-def AssumeR-NSRD AssumeR-as-gcmd NSRD-srd-skip Star-AssumeR nsrdes-theory.Star-slide gcmd-seq-distr skip-srea-self-unit urel-dioid.distrib-right'*)
also have ... $= [\neg\ b]^\top{}_R \sqcap ((([b]^\top{}_R ;; P)^{\star R} ;; [b]^\top{}_R ;; P ;; [b \vee \neg\ b]^\top{}_R) ;; [\neg\ b]^\top{}_R)$
by (*simp add*: *AssumeR-true NSRD-right-unit assms(1)*)
also have ... $= [\neg\ b]^\top{}_R \sqcap ((([b]^\top{}_R ;; P)^{\star R} ;; [b]^\top{}_R ;; P ;; [b]^\top{}_R) ;; [\neg\ b]^\top{}_R)$
$\sqcap ((([b]^\top{}_R ;; P)^{\star R} ;; [b]^\top{}_R ;; P ;; [\neg\ b]^\top{}_R) ;; [\neg\ b]^\top{}_R)$
by (*metis* (*no-types*, *hide-lams*) *AssumeR-choice upred-semiring.add-assoc upred-semiring.distrib-left upred-semiring.distrib-right*)
also have ... $= [\neg\ b]^\top{}_R \sqcap ((([b]^\top{}_R ;; P)^{\star R} ;; [b]^\top{}_R ;; P ;; ([b]^\top{}_R ;; [\neg\ b]^\top{}_R))$
$\sqcap ((([b]^\top{}_R ;; P)^{\star R} ;; [b]^\top{}_R ;; P ;; ([\neg\ b]^\top{}_R ;; [\neg\ b]^\top{}_R))$
by (*simp add*: *RA1*)
also have ... $= [\neg\ b]^\top{}_R \sqcap ((([b]^\top{}_R ;; P)^{\star R} ;; [b]^\top{}_R ;; P ;; Miracle)$
$\sqcap ((([b]^\top{}_R ;; P)^{\star R} ;; [b]^\top{}_R ;; P ;; [\neg\ b]^\top{}_R)$
by (*simp add*: *AssumeR-comp AssumeR-false*)
finally have $([b]^\top{}_R ;; P)^{\star R} ;; [\neg\ b]^\top{}_R \sqsubseteq ((([b]^\top{}_R ;; P)^{\star R}) ;; [b]^\top{}_R ;; P ;; Miracle$
by (*simp add*: *semilattice-sup-class.le-supI1*)
thus *?thesis*
by (*simp add*: *semilattice-sup-class.le-iff-sup*)
qed
finally show *?thesis* **.**
qed

theorem *WhileR-iter-form*:
assumes *P is NSRD P is Productive*
shows $while_R\ b\ do\ P\ od = ([b]^\top{}_R ;; P)^{\star R} ;; [\neg\ b]^\top{}_R$
by (*simp add*: *WhileR-iter-form-lemma WhileR-star-expand assms*)

theorem *WhileR-outer-refine-intro*:
assumes
P is NSRD P is Productive
$S \sqsubseteq ([b]^\top{}_R ;; P) ;; S\ S \sqsubseteq [\neg\ b]^\top{}_R$
shows $S \sqsubseteq while_R\ b\ do\ P\ od$
apply (*simp add*: *assms StarR-def WhileR-iter-form*)
apply (*rule nsrdes-theory.Star-inductl*)
apply (*simp-all add*: *closure assms*)
done

theorem *WhileR-outer-refine-init-intro*:
assumes
P is NSRD I is NSRD P is Productive
$S \sqsubseteq I ;; [\neg\ b]^\top{}_R$
$S \sqsubseteq S ;; [b]^\top{}_R ;; P$
$S \sqsubseteq I ;; [b]^\top{}_R ;; P$
shows $S \sqsubseteq I ;; while_R\ b\ do\ P\ od$
proof −
have $S \sqsubseteq I ;; ((([b]^\top{}_R ;; P) ;; ([b]^\top{}_R ;; P)^{\star R}) ;; [\neg\ b]^\top{}_R$
proof −
have $S \sqsubseteq I ;; ([b]^\top{}_R ;; P) ;; ([b]^\top{}_R ;; P)^{\star R}$
by (*metis* (*no-types*, *hide-lams*) *StarR-def AssumeR-NSRD NSRD-seqr-closure RA1 assms(1) assms(2) assms(5) assms(6) nsrdes-theory.Star-inductr semilattice-sup-class.le-sup-iff*)
thus *?thesis*

by (*metis* (*no-types*, *lifting*) *AssumeR-NSRD AssumeR-seq-refines StarR-def assms*(*1*) *dual-order.trans nsrdes-theory.Healthy-Sequence nsrdes-theory.utp-theory-kleene-axioms urel-dioid.mult-isol utp-theory-kleene.Star-Healthy*
qed
moreover have $S \sqsubseteq I \mathbin{;;} II_R \mathbin{;;} [\neg\ b]^\top{}_R$
by (*simp add*: *AssumeR-NSRD assms nsrdes-theory.Unit-Left*)
ultimately show *?thesis*
apply (*simp add*: *assms WhileR-iter-form StarR-def*)
apply (*subst nsrdes-theory.Star-unfoldl-eq*[*THEN sym*])
apply (*auto simp add*: *closure assms seqr-inf-distr*)
done
qed

theorem *WhileR-false*:
assumes *P is NSRD*
shows $while_R\ false\ do\ P\ od = II_R$
by (*simp add*: *WhileR-def rpred closure srdes-theory.LFP-const*)

theorem *WhileR-true*:
assumes *P is NSRD P is Productive*
shows $while_R\ true\ do\ P\ od = P^{\star R} \mathbin{;;} Miracle$
by (*simp add*: *WhileR-iter-form AssumeR-true AssumeR-false SRD-left-unit assms closure*)

lemma *WhileR-insert-assume*:
assumes *P is NSRD P is Productive*
shows $while_R\ b\ do\ ([b]^\top{}_R \mathbin{;;} P)\ od = while_R\ b\ do\ P\ od$
by (*simp add*: *AssumeR-NSRD AssumeR-comp NSRD-seqr-closure Productive-seq-2 RA1 WhileR-iter-form assms*)

theorem *WhileR-rdes-def* [*rdes-def*]:
assumes *P is RC Q is RR R is RR* $st' \sharp Q$ *R is R4*
shows $while_R\ b\ do\ \mathbf{R}_s(P \vdash Q \diamond R)\ od =$
$\mathbf{R}_s\ (([b]^\top{}_r \mathbin{;;} R)^{\star r}\ wp_r\ ([b]_{S<} \Rightarrow_r P) \vdash (([b]^\top{}_r \mathbin{;;} R)^{\star r} \mathbin{;;} [b]^\top{}_r \mathbin{;;} Q) \diamond (([b]^\top{}_r \mathbin{;;} R)^{\star r} \mathbin{;;} [\neg\ b]^\top{}_r))$
(**is** *?lhs* = *?rhs*)
proof −
have $?lhs = ([b]^\top{}_R \mathbin{;;} \mathbf{R}_s\ (P \vdash Q \diamond R))^{\star R} \mathbin{;;} [\neg\ b]^\top{}_R$
by (*simp add*: *WhileR-iter-form Productive-rdes-RR-intro assms closure*)
also have ... = *?rhs*
by (*simp add*: *rdes-def assms closure unrest rpred wp del*: *rea-star-wp*)
finally show *?thesis* **.**
qed

Refinement introduction law for reactive while loops

theorem *WhileR-refine-intro*:
assumes
— Closure conditions
Q_1 *is RC* Q_2 *is RR* Q_3 *is RR* $st' \sharp Q_2\ Q_3$ *is R4*
— Refinement conditions
$([b]^\top{}_r \mathbin{;;} Q_3)^{\star r}\ wp_r\ ([b]_{S<} \Rightarrow_r Q_1) \sqsubseteq P_1$
$P_2 \sqsubseteq [b]^\top{}_r \mathbin{;;} Q_2$
$P_2 \sqsubseteq [b]^\top{}_r \mathbin{;;} Q_3 \mathbin{;;} P_2$
$P_3 \sqsubseteq [\neg b]^\top{}_r$
$P_3 \sqsubseteq [b]^\top{}_r \mathbin{;;} Q_3 \mathbin{;;} P_3$
shows $\mathbf{R}_s(P_1 \vdash P_2 \diamond P_3) \sqsubseteq while_R\ b\ do\ \mathbf{R}_s(Q_1 \vdash Q_2 \diamond Q_3)\ od$
proof (*simp add*: *rdes-def assms*, *rule srdes-tri-refine-intro′*)
show $([b]^\top{}_r \mathbin{;;} Q_3)^{\star r}\ wp_r\ ([b]_{S<} \Rightarrow_r Q_1) \sqsubseteq P_1$

by (*simp add*: *assms*)
show $P_2 \sqsubseteq (P_1 \wedge ([b]^\top{}_r \;;;\; Q_3)^{\star r} \;;;\; [b]^\top{}_r \;;;\; Q_2)$
proof −
have $P_2 \sqsubseteq ([b]^\top{}_r \;;;\; Q_3)^{\star r} \;;;\; [b]^\top{}_r \;;;\; Q_2$
by (*simp add*: *assms rea-assume-RR rrel-theory.Star-inductl seq-RR-closed seqr-assoc*)
thus *?thesis*
by (*simp add*: *utp-pred-laws.le-infI2*)
qed
show $P_3 \sqsubseteq (P_1 \wedge ([b]^\top{}_r \;;;\; Q_3)^{\star r} \;;;\; [\neg\ b]^\top{}_r)$
proof −
have $P_3 \sqsubseteq ([b]^\top{}_r \;;;\; Q_3)^{\star r} \;;;\; [\neg\ b]^\top{}_r$
by (*simp add*: *assms rea-assume-RR rrel-theory.Star-inductl seqr-assoc*)
thus *?thesis*
by (*simp add*: *utp-pred-laws.le-infI2*)
qed
qed

lemma *WhileR-post-lemma*:
assumes
P_1 *is RC* P_2 *is RR* P_3 *is RR*
Q_1 *is RC* Q_2 *is RR* Q_3 *is RR* $\$st' \sharp Q_2$ Q_3 *is R4*
$\mathbf{R}_s(P_1 \vdash P_2 \diamond P_3) \sqsubseteq \mathit{while}_R\ b\ \mathit{do}\ \mathbf{R}_s(Q_1 \vdash Q_2 \diamond Q_3)\ \mathit{od}$
shows $P_3 \sqsubseteq (P_1 \wedge [\neg b]^\top{}_r)$
proof −
from *assms* **have** $P_3 \sqsubseteq (P_1 \wedge ([b]^\top{}_r \;;;\; Q_3)^{\star r} \;;;\; [\neg\ b]^\top{}_r)$
by (*simp add*: *rdes-def assms RHS-tri-design-refine' closure*)
moreover have $(P_1 \wedge ([b]^\top{}_r \;;;\; Q_3)^{\star r} \;;;\; [\neg\ b]^\top{}_r) \sqsubseteq (P_1 \wedge [\neg\ b]^\top{}_r)$
by (*metis* (*no-types*, *hide-lams*) *order-refl rea-assume-RR rea-skip-unit*(*2*) *rrel-theory.Star-unfoldl semilattice-sup-class.le-sup-iff urel-dioid.mult-isor utp-pred-laws.inf-mono*)
thus *?thesis*
by (*meson calculation order.trans*)
qed

11.12 Iteration Construction

definition *IterateR*
:: *'a set* ⇒ (*'a* ⇒ *'s upred*) ⇒ (*'a* ⇒ (*'s*, *'t*::*size-trace*, *'α*) *hrel-rsp*) ⇒ (*'s*, *'t*, *'α*) *hrel-rsp*
where $\mathit{IterateR}\ A\ g\ P = \mathit{while}_R\ (\bigvee i{\in}A \cdot g(i))\ \mathit{do}\ (\mathit{if}_R\ i{\in}A \cdot g(i) \rightarrow P(i)\ \mathit{fi})\ \mathit{od}$

definition *IterateR-list*
:: (*'s upred* × (*'s*, *'t*::*size-trace*, *'α*) *hrel-rsp*) *list* ⇒ (*'s*, *'t*, *'α*) *hrel-rsp* **where**
[*upred-defs*, *ndes-simp*]:
$\mathit{IterateR\text{-}list}\ xs = \mathit{IterateR}\ \{0..<\mathit{length}\ xs\}\ (\lambda\ i.\ \mathit{map}\ \mathit{fst}\ xs\ !\ i)\ (\lambda\ i.\ \mathit{map}\ \mathit{snd}\ xs\ !\ i)$

syntax
-iter-srd :: *pttrn* ⇒ *logic* ⇒ *logic* ⇒ *logic* ⇒ *logic* (do_R -∈- • - → - *od*)
-iter-gcommR :: *gcomms* ⇒ *logic* (do_R/ - /*od*)

translations
-iter-srd x A g P => *CONST IterateR A* (λ *x*. *g*) (λ *x*. *P*)
-iter-srd x A g P <= *CONST IterateR A* (λ *x*. *g*) (λ *x'*. *P*)
-iter-gcommR cs ⇀ *CONST IterateR-list cs*
-iter-gcommR (*-gcomm-show cs*) ↽ *CONST IterateR-list cs*

lemma *IterateR-NSRD-closed* [*closure*]:
assumes

$\bigwedge i.\ i \in I \Longrightarrow P(i)$ *is NSRD*
$\bigwedge i.\ i \in I \Longrightarrow P(i)$ *is Productive*
shows $do_R\ i{\in}I \cdot g(i) \rightarrow P(i)\ od$ *is NSRD*
by (*simp add*: *IterateR-def closure assms*)

lemma *IterateR-empty*:
$do_R\ i{\in}\{\} \cdot g(i) \rightarrow P(i)\ od = II_R$
by (*simp add*: *IterateR-def srd-mu-equiv closure rpred gfp-const WhileR-false*)

lemma *IterateR-singleton*:
assumes *P k is NSRD P k is Productive*
shows $do_R\ i{\in}\{k\} \cdot g(i) \rightarrow P(i)\ od = while_R\ g(k)\ do\ P(k)\ od$ (**is** *?lhs = ?rhs*)
proof −
have $?lhs = while_R\ g\ k\ do\ P\ k \lhd g\ k \rhd_R Chaos\ od$
by (*simp add*: *IterateR-def AlternateR-singleton assms closure*)
also have $... = while_R\ g\ k\ do\ [g\ k]^\top_R ;; (P\ k \lhd g\ k \rhd_R Chaos)\ od$
by (*simp add*: *WhileR-insert-assume closure assms*)
also have $... = while_R\ g\ k\ do\ P\ k\ od$
by (*simp add*: *AssumeR-cond-left NSRD-Chaos WhileR-insert-assume assms*)
finally show *?thesis* **.**
qed

declare *IterateR-list-def* [*rdes-def*]
declare *IterateR-def* [*rdes-def*]

lemma *R4-Continuous* [*closure*]: *Continuous R4*
by (*rel-auto*)

lemma *cond-rea-R4-closed* [*closure*]:
$[\![\ P\ is\ R4;\ Q\ is\ R4\ ]\!] \Longrightarrow P \lhd b \rhd_R Q\ is\ R4$
by (*simp add*: *Healthy-def R4-cond*)

lemma *IterateR-outer-refine-intro*:
assumes $I \neq \{\}$ $\bigwedge i.\ i \in I \Longrightarrow P\ i\ is\ NSRD$ $\bigwedge i.\ i \in I \Longrightarrow P\ i\ is\ Productive$
$\bigwedge i.\ i \in I \Longrightarrow S \sqsubseteq (b\ i \rightarrow_R P\ i ;; S)$
$S \sqsubseteq [\neg (\bigsqcap\ i \in I \cdot b\ i)]^\top_R$
shows $S \sqsubseteq do_R\ i{\in}I \cdot b(i) \rightarrow P(i)\ od$
apply (*simp add*: *IterateR-def*)
apply (*rule WhileR-outer-refine-intro*)
apply (*simp-all add*: *assms closure AlternateR-assume-branch seq-UINF-distr UINF-refines*)
done

lemma *IterateR-outer-refine-init-intro*:
assumes
$A \neq \{\}$ $\bigwedge i.\ i \in A \Longrightarrow P\ i\ is\ NSRD$
$\bigwedge i.\ i \in A \Longrightarrow P\ i\ is\ Productive$
$I\ is\ NSRD$
$S \sqsubseteq I ;; [\neg (\bigsqcap\ i \in A \cdot b\ i)]^\top_R$
$\bigwedge i.\ i \in A \Longrightarrow S \sqsubseteq S ;; b\ i \rightarrow_R P\ i$
$\bigwedge i.\ i \in A \Longrightarrow S \sqsubseteq I ;; b\ i \rightarrow_R P\ i$
shows $S \sqsubseteq I ;; do_R\ i{\in}A \cdot b(i) \rightarrow P(i)\ od$
apply (*simp add*: *IterateR-def*)
apply (*rule-tac WhileR-outer-refine-init-intro*)
apply (*simp-all add*: *assms closure AlternateR-assume-branch seq-UINF-distl UINF-refines*)
done

lemma *IterateR-lemma1*:
$[\bigsqcap\ i \in I \cdot b\ i]^\top{}_r\ ;;\ (\bigsqcap\ i \in I \cdot P\ i \lhd b\ i \rhd_R\ false) = (\bigsqcap\ i \in I \cdot [b\ i]^\top{}_r\ ;;\ P\ i)$
by (*rel-auto*; *fastforce*)

lemma *IterateR-lemma2*:
assumes $I \neq \{\}$ $\bigwedge\ i.\ i{\in}I \Longrightarrow P(i)$ *is RR*
shows $([\bigsqcap\ i \in I \cdot b\ i]_{S<} \Rightarrow_r (\bigsqcup\ i \in I \cdot (P\ i) \lhd b\ i \rhd_R\ R1\ true) \wedge false \lhd (\neg\ (\bigsqcap\ i \in I \cdot b\ i)) \rhd_R R1\ true)$
$= (\bigsqcup\ i \in I \cdot (P\ i) \lhd b\ i \rhd_R\ R1\ true)$
proof −
from *assms(1)*
have $([\bigsqcap\ i \in I \cdot b\ i]_{S<} \Rightarrow_r (\bigsqcup\ i \in I \cdot RR(P\ i) \lhd b\ i \rhd_R\ R1\ true) \wedge false \lhd (\neg\ (\bigsqcap\ i \in I \cdot b\ i)) \rhd_R R1\ true)$
$= (\bigsqcup\ i \in I \cdot RR(P\ i) \lhd b\ i \rhd_R\ R1\ true)$
by (*rel-auto*)
thus *?thesis*
by (*simp add*: *assms Healthy-if cong*: *USUP-cong*)
qed

lemma *IterateR-lemma3*:
assumes $\bigwedge\ i.\ i{\in}I \Longrightarrow P(i)$ *is RR*
shows $(\bigsqcup\ i \in I \cdot P\ i \lhd b\ i \rhd_R\ R1\ true) = (\bigsqcup\ i \in I \cdot [b\ i]_{S<} \Rightarrow_r\ P\ i)$
proof −
have $(\bigsqcup\ i \in I \cdot RR(P\ i) \lhd b\ i \rhd_R\ R1\ true) = (\bigsqcup\ i \in I \cdot [b\ i]_{S<} \Rightarrow_r\ RR(P\ i))$
by (*rel-auto*)
thus *?thesis*
by (*simp add*: *assms Healthy-if cong*: *USUP-cong*)
qed

theorem *IterateR-refine-intro*:
assumes
— Closure conditions
$\bigwedge\ i.\ i{\in}I \Longrightarrow Q_1(i)$ *is RC* $\bigwedge\ i.\ i{\in}I \Longrightarrow Q_2(i)$ *is RR* $\bigwedge\ i.\ i{\in}I \Longrightarrow Q_3(i)$ *is RR*
$\bigwedge\ i.\ i{\in}I \Longrightarrow \$st' \sharp Q_2(i)$ $\bigwedge\ i.\ i{\in}I \Longrightarrow Q_3(i)$ *is R4* $I \neq \{\}$
$(\bigsqcap\ i \in I \cdot [b\ i]^\top{}_r\ ;;\ Q_3\ i)^{\star r}\ wp_r\ (\bigsqcup\ i \in I \cdot [b\ i]_{S<} \Rightarrow_r\ Q_1\ i) \sqsubseteq P_1$
$P_2 \sqsubseteq (\bigsqcap\ i \in I \cdot [b\ i]^\top{}_r\ ;;\ Q_2\ i)$
$P_2 \sqsubseteq (\bigsqcap\ i \in I \cdot [b\ i]^\top{}_r\ ;;\ Q_3\ i)\ ;;\ P_2$
$P_3 \sqsubseteq [\neg\ (\bigsqcap\ i \in I \cdot b\ i)]^\top{}_r$
$P_3 \sqsubseteq (\bigsqcap\ i \in I \cdot [b\ i]^\top{}_r\ ;;\ Q_3\ i)\ ;;\ P_3$
shows $\mathbf{R}_s(P_1 \vdash P_2 \diamond P_3) \sqsubseteq do_R\ i{\in}I \cdot b(i) \rightarrow \mathbf{R}_s(Q_1(i) \vdash Q_2(i) \diamond Q_3(i))\ od$
apply (*simp add*: *rdes-def closure assms unrest del*: *WhileR-rdes-def*)
apply (*rule WhileR-refine-intro*)
apply (*simp-all add*: *closure assms unrest IterateR-lemma1 IterateR-lemma2 seqr-assoc*[*THEN sym*])
apply (*simp add*: *IterateR-lemma3 closure assms unrest*)
done

method *unfold-iteration* = *simp add*: *IterateR-list-def IterateR-def AlternateR-list-def AlternateR-def UINF-upto-expand-first*

11.13 Substitution Laws

lemma *srd-subst-Chaos* [*usubst*]:
$\sigma \dagger_S Chaos = Chaos$
by (*rdes-simp*)

lemma *srd-subst-Miracle* [*usubst*]:
$\sigma \dagger_S Miracle = Miracle$
by (*rdes-simp*)

lemma *srd-subst-skip* [*usubst*]:
$\sigma \dagger_S II_R = \langle \sigma \rangle_R$
by (*rdes-eq*)

lemma *srd-subst-assigns* [*usubst*]:
$\sigma \dagger_S \langle \varrho \rangle_R = \langle \varrho \circ_s \sigma \rangle_R$
by (*rdes-eq*)

11.14 Algebraic Laws

theorem *assigns-srd-id*: $\langle id_s \rangle_R = II_R$
by (*rdes-eq*)

theorem *assigns-srd-comp*: $\langle \sigma \rangle_R ;; \langle \varrho \rangle_R = \langle \varrho \circ_s \sigma \rangle_R$
by (*rdes-eq*)

theorem *assigns-srd-Miracle*: $\langle \sigma \rangle_R ;; Miracle = Miracle$
by (*rdes-eq*)

theorem *assigns-srd-Chaos*: $\langle \sigma \rangle_R ;; Chaos = Chaos$
by (*rdes-eq*)

theorem *assigns-srd-cond* : $\langle \sigma \rangle_R \lhd b \rhd_R \langle \varrho \rangle_R = \langle \sigma \lhd b \rhd \varrho \rangle_R$
by (*rdes-eq*)

theorem *assigns-srd-left-seq*:
assumes *P is NSRD*
shows $\langle \sigma \rangle_R ;; P = \sigma \dagger_S P$
by (*rdes-simp cls*: *assms*)

lemma *AlternateR-seq-distr*:
assumes $\bigwedge i.$ *A i is NSRD B is NSRD C is NSRD*
shows $(if_R\ i \in I \cdot g\ i \to A\ i\ else\ B\ fi) ;; C = (if_R\ i \in I \cdot g\ i \to A\ i ;; C\ else\ B ;; C\ fi)$
proof (*cases* $I = \{\}$)
case *True*
then show *?thesis* **by** (*simp*)
next
case *False*
then show *?thesis*
by (*simp add*: *AlternateR-def upred-semiring.distrib-right seq-UINF-distr gcmd-seq-distr assms*(*3*))
qed

lemma *AlternateR-is-cond-srea*:
assumes *A is NSRD B is NSRD*
shows $(if_R\ i \in \{a\} \cdot g \to A\ else\ B\ fi) = (A \lhd g \rhd_R B)$
by (*rdes-eq cls*: *assms*)

lemma *AlternateR-Chaos*:
$if_R\ i{\in}A \cdot g(i) \to Chaos\ fi = Chaos$
by (*cases* $A = \{\}$, *simp*, *rdes-eq*)

lemma *choose-srd-par*:

$choose_R \parallel_R choose_R = choose_R$
by (*rdes-eq*)

11.15 Lifting designs to reactive designs

definition *des-rea-lift* :: *'s hrel-des* ⇒ (*'s*,*'t*::*trace*,*'α*) *hrel-rsp* ($\mathbf{R}_D$) **where**
[*upred-defs*]: $\mathbf{R}_D(P) = \mathbf{R}_s(\lceil pre_D(P) \rceil_S \vdash (false \diamond (\$tr\,' =_u \$tr \wedge \lceil post_D(P) \rceil_S)))$

definition *des-rea-drop* :: (*'s*,*'t*::*trace*,*'α*) *hrel-rsp* ⇒ *'s hrel-des* ($\mathbf{D}_R$) **where**
[*upred-defs*]: $\mathbf{D}_R(P) = \lfloor (pre_R(P))[\![\$tr/\$tr\,']\!] \upharpoonright_v \$st \rfloor_{S<}$
$\vdash_n \lfloor (post_R(P))[\![\$tr/\$tr\,']\!] \upharpoonright_v \{\$st,\$st\,'\} \rfloor_S$

lemma *ndesign-rea-lift-inverse*: $\mathbf{D}_R(\mathbf{R}_D(p \vdash_n Q)) = p \vdash_n Q$
apply (*simp add*: *des-rea-lift-def des-rea-drop-def rea-pre-RHS-design rea-post-RHS-design*)
apply (*simp add*: *R1-def R2c-def R2s-def usubst unrest*)
apply (*rel-auto*)
done

lemma *ndesign-rea-lift-injective*:
assumes *P is* **N** *Q is* **N** $\mathbf{R}_D\ P = \mathbf{R}_D\ Q$ (**is** *?RP*(*P*) = *?RQ*(*Q*))
shows *P* = *Q*
proof −
have *?RP*($\lfloor pre_D(P) \rfloor_<$ ⊢$_n$ $post_D(P)$) = *?RQ*($\lfloor pre_D(Q) \rfloor_<$ ⊢$_n$ $post_D(Q)$)
by (*simp add*: *ndesign-form assms*)
hence $\lfloor pre_D(P) \rfloor_<$ ⊢$_n$ $post_D(P) = \lfloor pre_D(Q) \rfloor_<$ ⊢$_n$ $post_D(Q)$
by (*metis ndesign-rea-lift-inverse*)
thus *?thesis*
by (*simp add*: *ndesign-form assms*)
qed

lemma *des-rea-lift-closure* [*closure*]: $\mathbf{R}_D(P)$ *is SRD*
by (*simp add*: *des-rea-lift-def RHS-design-is-SRD unrest*)

lemma *preR-des-rea-lift* [*rdes*]:
$pre_R(\mathbf{R}_D(P)) = R1(\lceil pre_D(P) \rceil_S)$
by (*rel-auto*)

lemma *periR-des-rea-lift* [*rdes*]:
$peri_R(\mathbf{R}_D(P)) = (false \lhd \lceil pre_D(P) \rceil_S \rhd (\$tr \leq_u \$tr\,'))$
by (*rel-auto*)

lemma *postR-des-rea-lift* [*rdes*]:
$post_R(\mathbf{R}_D(P)) = ((true \lhd \lceil pre_D(P) \rceil_S \rhd (\neg\ \$tr \leq_u \$tr\,')) \Rightarrow (\$tr\,' =_u \$tr \wedge \lceil post_D(P) \rceil_S))$
apply (*rel-auto*) **using** *minus-zero-eq* **by** *blast*

lemma *ndes-rea-lift-closure* [*closure*]:
assumes *P is* **N**
shows $\mathbf{R}_D(P)$ *is NSRD*
proof −
obtain *p Q* **where** *P*: *P* = (*p* ⊢$_n$ *Q*)
by (*metis H1-H3-commute H1-H3-is-normal-design H1-idem Healthy-def assms*)
show *?thesis*
apply (*rule NSRD-intro*)
apply (*simp-all add*: *closure rdes unrest P*)
apply (*rel-auto*)
done

qed

lemma *R-D-mono*:
assumes *P is* **H** *Q is* **H** $P \sqsubseteq Q$
shows $\mathbf{R}_D(P) \sqsubseteq \mathbf{R}_D(Q)$
apply (*simp add*: *des-rea-lift-def*)
apply (*rule srdes-tri-refine-intro'*)
apply (*meson aext-mono assms*(*3*) *design-refine-thms*(*1*) *refBy-order*)
apply (*rel-auto*)
apply (*smt aext-and aext-mono assms*(*1*) *assms*(*2*) *assms*(*3*) *rdesign-ref-monos*(*2*) *utp-pred-laws.inf.cobounded2 utp-pred-laws.inf.coboundedI2 utp-pred-laws.inf-left-commute utp-pred-laws.le-inf-iff*)
done

Homomorphism laws

lemma *R-D-Miracle*:
$\mathbf{R}_D(\top_D) = Miracle$
by (*simp add*: *Miracle-def*, *rel-auto*)

lemma *R-D-Chaos*:
$\mathbf{R}_D(\bot_D) = Chaos$
proof −
have $\mathbf{R}_D(\bot_D) = \mathbf{R}_D(false \vdash_r true)$
by (*rel-auto*)
also have ... $= \mathbf{R}_s\ (false \vdash false \diamond (\$tr\acute{} =_u \$tr))$
by (*simp add*: *Chaos-def des-rea-lift-def alpha*)
also have ... $= \mathbf{R}_s\ (true)$
by (*rel-auto*)
also have ... $= Chaos$
by (*simp add*: *Chaos-def design-false-pre*)
finally show *?thesis* **.**
qed

lemma *R-D-inf*:
$\mathbf{R}_D(P \sqcap Q) = \mathbf{R}_D(P) \sqcap \mathbf{R}_D(Q)$
by (*rule antisym*, *rel-auto+*)

lemma *R-D-cond*:
$\mathbf{R}_D(P \lhd \lceil b \rceil_{D<} \rhd Q) = \mathbf{R}_D(P) \lhd b \rhd_R \mathbf{R}_D(Q)$
by (*rule antisym*, *rel-auto+*)

lemma *R-D-seq-ndesign*:
$\mathbf{R}_D(p_1 \vdash_n Q_1)\ ;;\ \mathbf{R}_D(p_2 \vdash_n Q_2) = \mathbf{R}_D((p_1 \vdash_n Q_1)\ ;;\ (p_2 \vdash_n Q_2))$
apply (*rule antisym*)
apply (*rule SRD-refine-intro*)
apply (*simp-all add*: *closure rdes ndesign-composition-wp*)
using *dual-order.trans* **apply** (*rel-blast*)
using *dual-order.trans* **apply** (*rel-blast*)
apply (*rel-auto*)
apply (*rule SRD-refine-intro*)
apply (*simp-all add*: *closure rdes ndesign-composition-wp*)
apply (*rel-auto*)
apply (*rel-auto*)
apply (*rel-auto*)
done

lemma *R-D-seq*:
assumes P *is* **N** Q *is* **N**
shows $\mathbf{R}_D(P)$;; $\mathbf{R}_D(Q) = \mathbf{R}_D(P$;; $Q)$
by (*metis R-D-seq-ndesign assms ndesign-form*)

These laws are applicable only when there is no further alphabet extension

lemma *R-D-skip*:
$\mathbf{R}_D(II_D) = (II_R :: ('s,'t::trace,unit)$ *hrel-rsp*)
apply (*rel-auto*) **using** *minus-zero-eq* **by** *blast+*

lemma *R-D-assigns*:
$\mathbf{R}_D(\langle\sigma\rangle_D) = (\langle\sigma\rangle_R :: ('s,'t::trace,unit)$ *hrel-rsp*)
by (*simp add*: *assigns-d-def des-rea-lift-def alpha assigns-srd-RHS-tri-des*, *rel-auto*)

11.16 State Invariants

definition *StateInvR* :: *'s upred* ⇒ (*'s*, *'t::trace*, *'α*) *hrel-rsp* (*sinv*$_R$*'*(*-'*)) **where**
[*rdes-def*]: *sinv*$_R(b) = \mathbf{R}_s([b]_{S<} \vdash true_r \diamond [b]_{S>})$

lemma *StateInvR-NSRD* [*closure*]: *sinv*$_R(b)$ *is NSRD*
by (*simp add*: *StateInvR-def closure unrest*)

lemma *StateInvR-srd-skip-refine*: *sinv*$_R(b) \sqsubseteq II_R$
by (*rdes-refine*)

lemma *StateInvR-seq-idem*:
sinv$_R(b)$;; *sinv*$_R(b)$ = *sinv*$_R(b)$
by (*rdes-eq*)

lemma *StateInvR-seq-refine*:
assumes *sinv*$_R(b) \sqsubseteq P$ *sinv*$_R(b) \sqsubseteq Q$
shows *sinv*$_R(b) \sqsubseteq P$;; Q
by (*metis* (*full-types*) *StateInvR-seq-idem assms seqr-mono*)

lemma *ndiv-StateInvR*: *ndiv*$_R$ = *sinv*$_R$(*true*)
by (*rdes-eq*)

end

12 Instantaneous Reactive Designs

theory *utp-rdes-instant*
imports *utp-rdes-prog*
begin

definition *ISRD1* :: (*'s*,*'t::trace*,*'α*) *hrel-rsp* ⇒ (*'s*,*'t*,*'α*) *hrel-rsp* **where**
[*upred-defs*]: *ISRD1*$(P) = P \parallel_R \mathbf{R}_s(true_r \vdash false \diamond (\$tr\,' =_u \$tr))$

definition *ISRD* :: (*'s*,*'t::trace*,*'α*) *hrel-rsp* ⇒ (*'s*,*'t*,*'α*) *hrel-rsp* **where**
[*upred-defs*]: *ISRD* = *ISRD1* ∘ *NSRD*

lemma *ISRD1-idem*: *ISRD1*(*ISRD1*(P)) = *ISRD1*(P)
by (*rel-auto*)

lemma *ISRD1-monotonic*: $P \sqsubseteq Q \Longrightarrow$ *ISRD1*$(P) \sqsubseteq$ *ISRD1*(Q)

by (*rel-auto*)

lemma *ISRD1-RHS-design-form*:
assumes $ok´ ♯ P $ok´ ♯ Q $ok´ ♯ R
shows *ISRD1*($\mathbf{R}_s(P \vdash Q \diamond R)$) = $\mathbf{R}_s(P \vdash false \diamond (R \wedge \$tr´ =_u \$tr))$
using *assms* **by** (*simp add*: *ISRD1-def choose-srd-def RHS-tri-design-par unrest*, *rel-auto*)

lemma *ISRD1-form*:
ISRD1(*SRD*(*P*)) = $\mathbf{R}_s(pre_R(P) \vdash false \diamond (post_R(P) \wedge \$tr´ =_u \$tr))$
by (*simp add*: *ISRD1-RHS-design-form SRD-as-reactive-tri-design unrest*)

lemma *ISRD1-rdes-def* [*rdes-def*]:
⟦ *P is RR*; *R is RR* ⟧ ⟹ *ISRD1*($\mathbf{R}_s(P \vdash Q \diamond R)$) = $\mathbf{R}_s(P \vdash false \diamond (R \wedge \$tr´ =_u \$tr))$
by (*simp add*: *ISRD1-def rdes-def closure rpred*)

lemma *ISRD-intro*:
assumes *P is NSRD* $peri_R(P) = (\neg_r\ pre_R(P))\ (\$tr´ =_u \$tr) \sqsubseteq post_R(P)$
shows *P is ISRD*
proof −
have $\mathbf{R}_s(pre_R(P) \vdash peri_R(P) \diamond post_R(P))$ *is ISRD1*
apply (*simp add*: *Healthy-def rdes-def closure assms*(*1*−*2*))
using *assms*(*3*) *least-zero* **apply** (*rel-blast*)
done
hence *P is ISRD1*
by (*simp add*: *SRD-reactive-tri-design closure assms*(*1*))
thus *?thesis*
by (*simp add*: *ISRD-def Healthy-comp assms*(*1*))
qed

lemma *ISRD1-rdes-intro*:
assumes *P is RR Q is RR* ($\$tr´ =_u \$tr) \sqsubseteq Q$
shows $\mathbf{R}_s(P \vdash false \diamond Q)$ *is ISRD1*
unfolding *Healthy-def*
by (*simp add*: *ISRD1-rdes-def assms closure unrest utp-pred-laws.inf.absorb1*)

lemma *ISRD-rdes-intro* [*closure*]:
assumes *P is RC Q is RR* ($\$tr´ =_u \$tr) \sqsubseteq Q$
shows $\mathbf{R}_s(P \vdash false \diamond Q)$ *is ISRD*
unfolding *Healthy-def*
by (*simp add*: *ISRD-def closure Healthy-if ISRD1-rdes-def assms unrest utp-pred-laws.inf.absorb1*)

lemma *ISRD-implies-ISRD1*:
assumes *P is ISRD*
shows *P is ISRD1*
proof −
have *ISRD*(*P*) *is ISRD1*
by (*simp add*: *ISRD-def Healthy-def ISRD1-idem*)
thus *?thesis*
by (*simp add*: *assms Healthy-if*)
qed

lemma *ISRD-implies-SRD*:
assumes *P is ISRD*
shows *P is SRD*
proof −

have *1*:$ISRD(P) = \mathbf{R}_s((\neg_r\ (\neg_r\ pre_R\ P)\ ;;\ R1\ true \wedge R1\ true) \vdash false \diamond (post_R\ P \wedge \$tr' =_u \$tr))$
by (*simp add*: *NSRD-form ISRD1-def ISRD-def RHS-tri-design-par rdes-def unrest closure*)
moreover have ... *is SRD*
by (*simp add*: *closure unrest*)
ultimately have $ISRD(P)$ *is SRD*
by (*simp*)
with *assms* **show** *?thesis*
by (*simp add*: *Healthy-def*)
qed

lemma *ISRD-implies-NSRD* [*closure*]:
assumes P *is ISRD*
shows P *is NSRD*
proof −
have *1*:$ISRD(P) = ISRD1(RD3(SRD(P)))$
by (*simp add*: *ISRD-def NSRD-def SRD-def*, *metis RD1-RD3-commute RD3-left-subsumes-RD2*)
also have ... $= ISRD1(RD3(P))$
by (*simp add*: *assms ISRD-implies-SRD Healthy-if*)
also have ... $= ISRD1\ (\mathbf{R}_s\ ((\neg_r\ pre_R\ P)\ wp_r\ false_h \vdash (\exists\ \$st' \cdot peri_R\ P) \diamond post_R\ P))$
by (*simp add*: *RD3-def*, *subst SRD-right-unit-tri-lemma*, *simp-all add*: *assms ISRD-implies-SRD*)
also have ... $= \mathbf{R}_s\ ((\neg_r\ pre_R\ P)\ wp_r\ false_h \vdash false \diamond (post_R\ P \wedge \$tr' =_u \$tr))$
by (*simp add*: *RHS-tri-design-par ISRD1-def unrest choose-srd-def rpred closure ISRD-implies-SRD assms*)
also have ... $= (... ;;\ II_R)$
by (*rdes-simp*, *simp add*: *RHS-tri-normal-design-composition′ closure assms unrest ISRD-implies-SRD wp rpred wp-rea-false-RC*)
also have ... *is RD3*
by (*simp add*: *Healthy-def RD3-def seqr-assoc*)
finally show *?thesis*
by (*simp add*: *SRD-RD3-implies-NSRD Healthy-if assms ISRD-implies-SRD*)
qed

lemma *ISRD-form*:
assumes P *is ISRD*
shows $\mathbf{R}_s(pre_R(P) \vdash false \diamond (post_R(P) \wedge \$tr' =_u \$tr)) = P$
proof −
have $P = ISRD1(P)$
by (*simp add*: *ISRD-implies-ISRD1 assms Healthy-if*)
also have ... $= ISRD1(\mathbf{R}_s(pre_R(P) \vdash peri_R(P) \diamond post_R(P)))$
by (*simp add*: *SRD-reactive-tri-design ISRD-implies-SRD assms*)
also have ... $= \mathbf{R}_s(pre_R(P) \vdash false \diamond (post_R(P) \wedge \$tr' =_u \$tr))$
by (*simp add*: *ISRD1-rdes-def closure assms*)
finally show *?thesis* **..**
qed

lemma *ISRD-elim* [*RD-elim*]:
$[\![\ P\ is\ ISRD;\ Q(\mathbf{R}_s\ (pre_R(P) \vdash false \diamond (post_R(P) \wedge \$tr' =_u \$tr)))\ ]\!] \Longrightarrow Q(P)$
by (*simp add*: *ISRD-form*)

lemma *skip-srd-ISRD* [*closure*]: II_R *is ISRD*
by (*rule ISRD-intro*, *simp-all add*: *rdes closure*)

lemma *assigns-srd-ISRD* [*closure*]: $\langle\sigma\rangle_R$ *is ISRD*
by (*rule ISRD-intro*, *simp-all add*: *rdes closure*, *rel-auto*)

lemma *seq-ISRD-closed*:
assumes *P is ISRD Q is ISRD*
shows *P ;; Q is ISRD*
apply (*insert assms*)
apply (*erule ISRD-elim*)+
apply (*simp add*: *rdes-def closure assms unrest*)
apply (*rule ISRD-rdes-intro*)
apply (*simp-all add*: *rdes-def closure assms unrest*)
apply (*rel-auto*)
done

lemma *ISRD-Miracle-right-zero*:
assumes *P is ISRD* $pre_R(P) = true_r$
shows *P ;; Miracle = Miracle*
by (*rdes-simp cls*: *assms*)

A recursion whose body does not extend the trace results in divergence

lemma *ISRD-recurse-Chaos*:
assumes *P is ISRD* $post_R\ P\ ;;\ true_r = true_r$
shows $(\mu_R\ X \cdot P\ ;;\ X) = Chaos$
proof −
have *1*: $(\mu_R\ X \cdot P\ ;;\ X) = (\mu\ X \cdot P\ ;;\ SRD(X))$
by (*auto simp add*: *srdes-theory.utp-lfp-def closure assms*)
have $(\mu\ X \cdot P\ ;;\ SRD(X)) \sqsubseteq Chaos$
proof (*rule gfp-upperbound*)
have $P\ ;;\ Chaos \sqsubseteq Chaos$
apply (*rdes-refine-split cls*: *assms*)
using *assms*(*2*) **apply** (*rel-auto*, *metis* (*no-types*, *lifting*) *dual-order.antisym order-refl*)
apply (*rel-auto*)+
done
thus $P\ ;;\ SRD\ Chaos \sqsubseteq Chaos$
by (*simp add*: *Healthy-if srdes-theory.bottom-closed*)
qed
thus *?thesis*
by (*metis 1 dual-order.antisym srdes-theory.LFP-closed srdes-theory.bottom-lower*)
qed

lemma *recursive-assign-Chaos*:
$(\mu_R\ X \cdot \langle\sigma\rangle_R\ ;;\ X) = Chaos$
by (*rule ISRD-recurse-Chaos*, *simp-all add*: *closure rdes*, *rel-auto*)

end

13 Meta-theory for Reactive Designs

theory *utp-rea-designs*
imports
utp-rdes-healths
utp-rdes-designs
utp-rdes-triples
utp-rdes-normal
utp-rdes-contracts
utp-rdes-tactics
utp-rdes-parallel
utp-rdes-prog

utp-rdes-instant
utp-rdes-guarded
begin end

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