Mathematical Toolkit for Isabelle/UTP

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Abstract

This document describes our mathematical toolkit for Isabelle/UTP, which provides a foundational collection of definition, theorems, and proof facilities. This includes extensions to existing HOL libraries, such as for list and partial functions, and also new type definitions, theorems, and Isabelle/HOL commands.

Contents

1 Introduction

This document contains the description of our mathematical toolkit for Isabelle/UTP [2, 3, 4, 7], a mechanisation of Hoare and He's *Unifying Theories of Programming* [5, 1]. The toolkit provides a foundational collection of additional HOL theorems, new abstract types, and proof facilities, upon which Isabelle/UTP depends. In brief, the toolkit contains the following principal items:

- additional laws and functions for the list, map (partial functions), countable set, and finite set types;

- type definitions for partial and finite functions, together with additional functions and laws derived from the Z mathematical toolkit [6];

- positive subtypes of existing types;

- infinite sequences;

- the "total recall" package, which allows us to precisely control overriding of existing syntax annotations.

A few other theories exist that add smaller utilities and additional laws.

2 Lists: extra functions and properties

theory *List-Extra*
  imports
  HOL−Library.Sublist
  HOL−Library.Monad-Syntax
  HOL−Library.Prefix-Order
  Optics.Lens-Instances
begin

2.1 Useful Abbreviations

abbreviation *list-sum xs ≡ foldr (+) xs 0*

2.2 Folds

context *abel-semigroup*
begin

  lemma *foldr-snoc*: *foldr (∗) (xs @ [x]) k = (foldr (∗) xs k) ∗ x*
    by *(induct xs, simp-all add: commute left-commute)*

end

2.3 List Lookup

The following variant of the standard *nth* function returns ⊥ if the index is out of range.

primrec
  nth-el :: *'a list ⇒ nat ⇒ 'a option (-⟨-⟩$_l$ [90, 0] 91)*
where
  []⟨i⟩$_l$ = None
| (x # xs)⟨i⟩$_l$ = (case i of 0 ⇒ Some x | Suc j ⇒ xs ⟨j⟩$_l$)

lemma *nth-el-appendl*[*simp*]: *i < length xs ⟹ (xs @ ys)⟨i⟩$_l$ = xs⟨i⟩$_l$*
  apply *(induct xs arbitrary: i)*
   apply *simp*
  apply *(case-tac i)*
   apply *simp-all*
  done

lemma *nth-el-appendr*[*simp*]: *length xs ≤ i ⟹ (xs @ ys)⟨i⟩$_l$ = ys⟨i − length xs⟩$_l$*
  apply *(induct xs arbitrary: i)*
   apply *simp*
  apply *(case-tac i)*
   apply *simp-all*
  done

2.4 Extra List Theorems

2.4.1 Map

lemma *map-nth-Cons-atLeastLessThan*:
  map (nth (x # xs)) [Suc m..<n] = map (nth xs) [m..<n − 1]
proof −
  have *nth xs = nth (x # xs) ∘ Suc*
    by *auto*
  hence *map (nth xs) [m..<n − 1] = map (nth (x # xs) ∘ Suc) [m..<n − 1]*
    by *simp*
  also have *... = map (nth (x # xs)) (map Suc [m..<n − 1])*
    by *simp*
  also have *... = map (nth (x # xs)) [Suc m..<n]*
    by *(metis Suc-diff-1 le-0-eq length-upt list.simps(8) list.size(3) map-Suc-upt not-less upt-0)*
  finally show *?thesis* **..**
qed

2.4.2 Sorted Lists

lemma *sorted-last* [*simp*]: ⟦ $x \in set\ xs$; *sorted xs* ⟧ $\Longrightarrow x \le last\ xs$
  by (*induct xs, auto*)

lemma *sorted-prefix*:
  assumes $xs \le ys$ *sorted ys*
  shows *sorted xs*
proof −
  obtain *zs* **where** $ys = xs\ @\ zs$
    using *Prefix-Order.prefixE assms(1)* **by** *auto*
  thus *?thesis*
    using *assms(2) sorted-append* **by** *blast*
qed

lemma *sorted-map*: ⟦ *sorted xs*; *mono f* ⟧ $\Longrightarrow$ *sorted* (*map f xs*)
  by (*simp add*: *monoD sorted-iff-nth-mono*)

lemma *sorted-distinct* [*intro*]: ⟦ *sorted* (*xs*); *distinct*(*xs*) ⟧ $\Longrightarrow$ ($\forall$ *i*<*length xs* − *1*. *xs*!*i* < *xs*!(*i* + *1*))
  apply (*induct xs*)
   apply (*auto*)
  apply (*metis* (*no-types, hide-lams*) *Suc-leI Suc-less-eq Suc-pred gr0-conv-Suc not-le not-less-iff-gr-or-eq nth-Cons-Suc nth-mem nth-non-equal-first-eq*)
  done

The concatenation of two lists is sorted provided (1) both the lists are sorted, and (2) the final and first elements are ordered.

lemma *sorted-append-middle*:
  $sorted(xs@ys) = (sorted\ xs \land sorted\ ys \land (xs \ne [] \land ys \ne [] \longrightarrow xs!(length\ xs - 1) \le ys!0))$
proof −
  have $\bigwedge x\ y.$ ⟦ *sorted xs*; *sorted ys*; $xs\ !\ (length\ xs - Suc\ 0) \le ys\ !\ 0$ ⟧ $\Longrightarrow x \in set\ xs \Longrightarrow y \in set\ ys \Longrightarrow x \le y$
  proof −
   fix $x\ y$
   assume *sorted xs sorted ys* $xs\ !\ (length\ xs - Suc\ 0) \le ys\ !\ 0$ $x \in set\ xs$ $y \in set\ ys$
   moreover then obtain $i\ j$ **where** *i*: $x = xs!i\ i < length\ xs$ **and** *j*: $y = ys!j\ j < length\ ys$
    by (*auto simp add*: *in-set-conv-nth*)
   moreover have $xs\ !\ i \le xs!(length\ xs - 1)$
    by (*metis One-nat-def Suc-diff-Suc Suc-leI Suc-le-mono* ‹*i < length xs*› ‹*sorted xs*› *diff-less diff-zero gr-implies-not-zero nat.simps(3) sorted-iff-nth-mono zero-less-iff-neq-zero*)
   moreover have $ys!0 \le ys\ !\ j$
    by (*simp add*: *calculation(2) calculation(9) sorted-nth-mono*)
   ultimately have $xs\ !\ i \le ys\ !\ j$
    by (*metis One-nat-def dual-order.trans*)
   thus $x \le y$
    by (*simp add*: *i j*)
  qed
  thus *?thesis*
   by (*auto simp add*: *sorted-append*)
qed

Is the given list a permutation of the given set?

definition *is-sorted-list-of-set* :: ($'a$::*ord*) *set* $\Rightarrow$ $'a$ *list* $\Rightarrow$ *bool* **where**
is-sorted-list-of-set A xs = (($\forall$ *i*<*length*(*xs*) − *1*. *xs*!*i* < *xs*!(*i* + *1*)) $\land$ *set*(*xs*) = *A*)

lemma *sorted-is-sorted-list-of-set*:

  assumes *is-sorted-list-of-set A xs*
  shows *sorted(xs)* **and** *distinct(xs)*
using *assms* **proof** (*induct xs arbitrary: A*)
  show *sorted* []
    by *auto*
next
  show *distinct* []
    by *auto*
next
  fix *A* :: *'a set*
  case (*Cons x xs*) **note** *hyps = this*
  assume *isl*: *is-sorted-list-of-set A (x # xs)*
  hence *srt*: ($\forall i{<}length\ xs - Suc\ 0.\ xs\ !\ i < xs\ !\ Suc\ i$)
    using *less-diff-conv* **by** (*auto simp add: is-sorted-list-of-set-def*)
  with *hyps(1)* **have** *srtd*: *sorted xs*
    by (*simp add: is-sorted-list-of-set-def*)
  with *isl* **show** *sorted (x # xs)*
    apply (*auto simp add: is-sorted-list-of-set-def*)
    apply (*metis* (*mono-tags, lifting*) *all-nth-imp-all-set less-le-trans linorder-not-less not-less-iff-gr-or-eq*
nth-Cons-0 sorted-iff-nth-mono zero-order(3))
    done
  from *srt hyps(2)* **have** *distinct xs*
    by (*simp add: is-sorted-list-of-set-def*)
  with *isl* **show** *distinct (x # xs)*
  proof −
    have ($\forall n.\ \neg\ n < length\ (x\ \#\ xs) - 1 \lor (x\ \#\ xs)\ !\ n < (x\ \#\ xs)\ !\ (n + 1)$) $\land$ *set (x # xs) = A*
      by (*meson ‹is-sorted-list-of-set A (x # xs)› is-sorted-list-of-set-def*)
    then show *?thesis*
      by (*metis ‹distinct xs› add.commute add-diff-cancel-left' distinct.simps(2) leD length-Cons length-greater-0-conv*
length-pos-if-in-set less-le nth-Cons-0 nth-Cons-Suc plus-1-eq-Suc set-ConsD sorted.elims(2) srtd)
  qed
qed

lemma *is-sorted-list-of-set-alt-def*:
  is-sorted-list-of-set A xs $\longleftrightarrow$ *sorted (xs)* $\land$ *distinct (xs)* $\land$ *set(xs) = A*
  apply (*auto intro: sorted-is-sorted-list-of-set*)
    apply (*auto simp add: is-sorted-list-of-set-def*)
  apply (*metis Nat.add-0-right One-nat-def add-Suc-right sorted-distinct*)
  done

definition *sorted-list-of-set-alt* :: (*'a::ord*) *set* $\Rightarrow$ *'a list* **where**
sorted-list-of-set-alt A =
  (*if (A = {}) then* [] *else (THE xs. is-sorted-list-of-set A xs)*)

lemma *is-sorted-list-of-set*:
  finite A $\implies$ *is-sorted-list-of-set A (sorted-list-of-set A)*
  apply (*simp add: is-sorted-list-of-set-def*)
  apply (*metis One-nat-def add.right-neutral add-Suc-right sorted-distinct sorted-list-of-set*)
  done

lemma *sorted-list-of-set-other-def*:
  finite A $\implies$ *sorted-list-of-set(A) = (THE xs. sorted(xs)* $\land$ *distinct(xs)* $\land$ *set xs = A)*
  apply (*rule sym*)
  apply (*rule the-equality*)
    apply (*auto*)

apply (*simp add*: *sorted-distinct-set-unique*)
done

lemma *sorted-list-of-set-alt* [*simp*]:
  finite A $\implies$ *sorted-list-of-set-alt*(*A*) = *sorted-list-of-set*(*A*)
  apply (*rule sym*)
  apply (*auto simp add*: *sorted-list-of-set-alt-def is-sorted-list-of-set-alt-def sorted-list-of-set-other-def*)
  done

Sorting lists according to a relation

definition *is-sorted-list-of-set-by* :: $'a\ rel \Rightarrow\ 'a\ set \Rightarrow\ 'a\ list \Rightarrow\ bool$ **where**
is-sorted-list-of-set-by R A xs = (($\forall\ i{<}length(xs) - 1.\ (xs!i,\ xs!(i + 1)) \in R$) $\wedge$ *set*(*xs*) = *A*)

definition *sorted-list-of-set-by* :: $'a\ rel \Rightarrow\ 'a\ set \Rightarrow\ 'a\ list$ **where**
sorted-list-of-set-by R A = (*THE xs. is-sorted-list-of-set-by R A xs*)

definition *fin-set-lexord* :: $'a\ rel \Rightarrow\ 'a\ set\ rel$ **where**
fin-set-lexord R = {(*A, B*). *finite A* $\wedge$ *finite B* $\wedge$
  ($\exists$ *xs ys. is-sorted-list-of-set-by R A xs* $\wedge$ *is-sorted-list-of-set-by R B ys*
  $\wedge$ (*xs, ys*) $\in$ *lexord R*)}

lemma *is-sorted-list-of-set-by-mono*:
  $[\![$ $R \subseteq S$; *is-sorted-list-of-set-by R A xs* $]\!]$ $\implies$ *is-sorted-list-of-set-by S A xs*
  by (*auto simp add*: *is-sorted-list-of-set-by-def*)

lemma *lexord-mono'*:
  $[\![$ ($\bigwedge$ *x y. f x y* $\longrightarrow$ *g x y*); (*xs, ys*) $\in$ *lexord* {(*x, y*). *f x y*} $]\!]$ $\implies$ (*xs, ys*) $\in$ *lexord* {(*x, y*). *g x y*}
  by (*metis case-prodD case-prodI lexord-take-index-conv mem-Collect-eq*)

lemma *fin-set-lexord-mono* [*mono*]:
  ($\bigwedge$ *x y. f x y* $\longrightarrow$ *g x y*) $\implies$ (*xs, ys*) $\in$ *fin-set-lexord* {(*x, y*). *f x y*} $\longrightarrow$ (*xs, ys*) $\in$ *fin-set-lexord* {(*x, y*). *g x y*}
proof
  assume
    fin: (*xs, ys*) $\in$ *fin-set-lexord* {(*x, y*). *f x y*} **and**
    hyp: ($\bigwedge$ *x y. f x y* $\longrightarrow$ *g x y*)

  from *fin* **have** *finite xs finite ys*
    using *fin-set-lexord-def* **by** *fastforce+*

  with *fin hyp* **show** (*xs, ys*) $\in$ *fin-set-lexord* {(*x, y*). *g x y*}
    apply (*auto simp add*: *fin-set-lexord-def*)
    apply (*rename-tac xs' ys'*)
    apply (*rule-tac x=xs' in exI*)
    apply (*auto*)
     apply (*metis case-prodD case-prodI is-sorted-list-of-set-by-def mem-Collect-eq*)
    apply (*metis case-prodD case-prodI is-sorted-list-of-set-by-def lexord-mono' mem-Collect-eq*)
    done
qed

definition *distincts* :: $'a\ set \Rightarrow\ 'a\ list\ set$ **where**
distincts A = {*xs* $\in$ *lists A. distinct*(*xs*)}

lemma *tl-element*:
  $[\![$ *x* $\in$ *set xs; x* $\neq$ *hd*(*xs*) $]\!]$ $\implies$ *x* $\in$ *set*(*tl*(*xs*))

by (*metis in-set-insert insert-Nil list.collapse list.distinct(2) set-ConsD*)

2.4.3 List Update

lemma *listsum-update*:
  fixes *xs* :: *'a::ring list*
  assumes *i < length xs*
  shows *list-sum* (*xs*[*i* := *v*]) = *list-sum xs* − *xs* ! *i* + *v*
using *assms* **proof** (*induct xs arbitrary*: *i*)
  case *Nil*
  then show *?case* **by** (*simp*)
next
  case (*Cons a xs*)
  then show *?case*
  proof (*cases i*)
    case *0*
    thus *?thesis*
      by (*simp add*: *add.commute*)
  next
    case (*Suc i'*)
    with *Cons* **show** *?thesis*
      by (*auto*)
  qed
qed

2.4.4 Drop While and Take While

lemma *dropWhile-sorted-le-above*:
  ⟦ *sorted xs*; *x* ∈ *set* (*dropWhile* (λ *x*. *x* ≤ *n*) *xs*) ⟧ ⟹ *x* > *n*
  apply (*induct xs*)
   apply (*auto*)
  apply (*rename-tac a xs*)
  apply (*case-tac a* ≤ *n*)
   apply (*auto*)
done

lemma *set-dropWhile-le*:
  sorted xs ⟹ *set* (*dropWhile* (λ *x*. *x* ≤ *n*) *xs*) = {*x*∈*set xs*. *x* > *n*}
  apply (*induct xs*)
   apply (*simp*)
  apply (*rename-tac x xs*)
  apply (*subgoal-tac sorted xs*)
   apply (*simp*)
   apply (*safe*)
    apply (*auto*)
done

lemma *set-takeWhile-less-sorted*:
  ⟦ *sorted I*; *x* ∈ *set I*; *x* < *n* ⟧ ⟹ *x* ∈ *set* (*takeWhile* (λ*x*. *x* < *n*) *I*)
proof (*induct I arbitrary*: *x*)
  case *Nil* **thus** *?case*
    by (*simp*)
next
  case (*Cons a I*) **thus** *?case*
    by *auto*
qed

lemma *nth-le-takeWhile-ord*: ⟦ *sorted xs*; $i \geq$ *length* (*takeWhile* (λ *x*. *x* $\leq$ *n*) *xs*); $i <$ *length xs* ⟧ $\Longrightarrow$
n $\leq$ *xs* ! *i*
  apply (*induct xs arbitrary*: *i*, *auto*)
  apply (*rename-tac x xs i*)
  apply (*case-tac x* $\leq$ *n*)
   apply (*auto*)
   apply (*metis One-nat-def Suc-eq-plus1 le-less-linear le-less-trans less-imp-le list.size(4) nth-mem*
set-ConsD)
  done

lemma *length-takeWhile-less*:
  ⟦ *a* $\in$ *set xs*; ¬ *P a* ⟧ $\Longrightarrow$ *length* (*takeWhile P xs*) $<$ *length xs*
  by (*metis in-set-conv-nth length-takeWhile-le nat-neq-iff not-less set-takeWhileD takeWhile-nth*)

lemma *nth-length-takeWhile-less*:
  ⟦ *sorted xs*; *distinct xs*; ($\exists$ *a* $\in$ *set xs*. *a* $\geq$ *n*) ⟧ $\Longrightarrow$ *xs* ! *length* (*takeWhile* (λx. *x* $<$ *n*) *xs*) $\geq$ *n*
  by (*induct xs*, *auto*)

2.4.5 Last and But Last

lemma *length-gt-zero-butlast-concat*:
  assumes *length ys* $>$ *0*
  shows *butlast* (*xs* @ *ys*) = *xs* @ (*butlast ys*)
  using *assms* **by** (*metis butlast-append length-greater-0-conv*)

lemma *length-eq-zero-butlast-concat*:
  assumes *length ys* = *0*
  shows *butlast* (*xs* @ *ys*) = *butlast xs*
  using *assms* **by** (*metis append-Nil2 length-0-conv*)

lemma *butlast-single-element*:
  shows *butlast* [*e*] = []
  by (*metis butlast.simps(2)*)

lemma *last-single-element*:
  shows *last* [*e*] = *e*
  by (*metis last.simps*)

lemma *length-zero-last-concat*:
  assumes *length t* = *0*
  shows *last* (*s* @ *t*) = *last s*
  by (*metis append-Nil2 assms length-0-conv*)

lemma *length-gt-zero-last-concat*:
  assumes *length t* $>$ *0*
  shows *last* (*s* @ *t*) = *last t*
  by (*metis assms last-append length-greater-0-conv*)

2.4.6 Prefixes and Strict Prefixes

lemma *prefix-length-eq*:
  ⟦ *length xs* = *length ys*; *prefix xs ys* ⟧ $\Longrightarrow$ *xs* = *ys*
  by (*metis not-equal-is-parallel parallel-def*)

lemma *prefix-Cons-elim* [*elim*]:

assumes *prefix (x # xs) ys*
obtains *ys′* **where** *ys = x # ys′ prefix xs ys′*
using *assms*
by (*metis append-Cons prefix-def*)

lemma *prefix-map-inj*:
  ⟦ *inj-on f (set xs ∪ set ys); prefix (map f xs) (map f ys)* ⟧ ⟹
  prefix xs ys
  apply (*induct xs arbitrary:ys*)
   apply (*simp-all*)
  apply (*erule prefix-Cons-elim*)
  apply (*auto*)
  apply (*metis image-insert insertI1 insert-Diff-if singletonE*)
  done

lemma *prefix-map-inj-eq* [*simp*]:
  inj-on f (set xs ∪ set ys) ⟹
  prefix (map f xs) (map f ys) ⟷ *prefix xs ys*
  using *map-mono-prefix prefix-map-inj* **by** *blast*

lemma *strict-prefix-Cons-elim* [*elim*]:
  assumes *strict-prefix (x # xs) ys*
  obtains *ys′* **where** *ys = x # ys′ strict-prefix xs ys′*
  using *assms*
  by (*metis Sublist.strict-prefixE′ Sublist.strict-prefixI′ append-Cons*)

lemma *strict-prefix-map-inj*:
  ⟦ *inj-on f (set xs ∪ set ys); strict-prefix (map f xs) (map f ys)* ⟧ ⟹
  strict-prefix xs ys
  apply (*induct xs arbitrary:ys*)
   apply (*auto*)
  using *prefix-bot.bot.not-eq-extremum* **apply** *fastforce*
  apply (*erule strict-prefix-Cons-elim*)
  apply (*auto*)
  apply (*metis (hide-lams, full-types) image-insert insertI1 insert-Diff-if singletonE*)
  done

lemma *strict-prefix-map-inj-eq* [*simp*]:
  inj-on f (set xs ∪ set ys) ⟹
  strict-prefix (map f xs) (map f ys) ⟷ *strict-prefix xs ys*
  by (*simp add: inj-on-map-eq-map strict-prefix-def*)

lemma *prefix-drop*:
  ⟦ *drop (length xs) ys = zs; prefix xs ys* ⟧
  ⟹ *ys = xs @ zs*
  by (*metis append-eq-conv-conj prefix-def*)

lemma *list-append-prefixD* [*dest*]: *x @ y ≤ z ⟹ x ≤ z*
  using *append-prefixD less-eq-list-def* **by** *blast*

lemma *prefix-not-empty*:
  assumes *strict-prefix xs ys* **and** *xs ≠ []*
  shows *ys ≠ []*
  using *Sublist.strict-prefix-simps(1) assms(1)* **by** *blast*

lemma *prefix-not-empty-length-gt-zero*:
  assumes *strict-prefix xs ys* **and** *xs ≠ []*
  shows *length ys > 0*
  using *assms prefix-not-empty* **by** *auto*

lemma *butlast-prefix-suffix-not-empty*:
  assumes *strict-prefix (butlast xs) ys*
  shows *ys ≠ []*
  using *assms prefix-not-empty-length-gt-zero* **by** *fastforce*

lemma *prefix-and-concat-prefix-is-concat-prefix*:
  assumes *prefix s t prefix (e @ t) u*
  shows *prefix (e @ s) u*
  using *Sublist.same-prefix-prefix assms(1) assms(2) prefix-order.dual-order.trans* **by** *blast*

lemma *prefix-eq-exists*:
  prefix s t ⟷ (∃ xs . s @ xs = t)
  using *prefix-def* **by** *auto*

lemma *strict-prefix-eq-exists*:
  strict-prefix s t ⟷ (∃ xs . s @ xs = t ∧ (length xs) > 0)
  using *prefix-def strict-prefix-def* **by** *auto*

lemma *butlast-strict-prefix-eq-butlast*:
  assumes *length s = length t* **and** *strict-prefix (butlast s) t*
  shows *strict-prefix (butlast s) t ⟷ (butlast s) = (butlast t)*
  by (*metis append-butlast-last-id append-eq-append-conv assms(1) assms(2) length-0-conv length-butlast strict-prefix-eq-exists*)

lemma *butlast-eq-if-eq-length-and-prefix*:
  assumes *length s > 0 length z > 0*
       length s = length z strict-prefix (butlast s) t strict-prefix (butlast z) t
  shows *(butlast s) = (butlast z)*
  using *assms* **by** (*auto simp add:strict-prefix-eq-exists*)

lemma *prefix-imp-length-lteq*:
  assumes *prefix s t*
  shows *length s ≤ length t*
  using *assms* **by** (*simp add: Sublist.prefix-length-le*)

lemma *prefix-imp-length-not-gt*:
  assumes *prefix s t*
  shows *¬ length t < length s*
  using *assms* **by** (*simp add: Sublist.prefix-length-le leD*)

lemma *prefix-and-eq-length-imp-eq-list*:
  assumes *prefix s t* **and** *length t = length s*
  shows *s=t*
  using *assms* **by** (*simp add: prefix-length-eq*)

lemma *butlast-prefix-imp-length-not-gt*:
  assumes *length s > 0 strict-prefix (butlast s) t*
  shows *¬ (length t < length s)*
  using *assms prefix-length-less* **by** *fastforce*

lemma *length-not-gt-iff-eq-length*:
  assumes *length s > 0* **and** *strict-prefix* (*butlast s*) *t*
  shows (¬ (*length s < length t*)) = (*length s = length t*)
proof −
  have (¬ (*length s < length t*)) = ((*length t < length s*) ∨ (*length s = length t*))
    by (*metis not-less-iff-gr-or-eq*)
  also have ... = (*length s = length t*)
    using *assms*
    by (*simp add:butlast-prefix-imp-length-not-gt*)

  finally show *?thesis* **.**
qed

Greatest common prefix

fun *gcp* :: ′*a list* ⇒ ′*a list* ⇒ ′*a list* **where**
gcp [] *ys* = [] |
gcp (*x # xs*) (*y # ys*) = (*if* (*x = y*) *then x # gcp xs ys else* []) |
gcp - - = []

lemma *gcp-right* [*simp*]: *gcp xs* [] = []
  by (*induct xs, auto*)

lemma *gcp-append* [*simp*]: *gcp* (*xs @ ys*) (*xs @ zs*) = *xs @ gcp ys zs*
  by (*induct xs, auto*)

lemma *gcp-lb1*: *prefix* (*gcp xs ys*) *xs*
  apply (*induct xs arbitrary: ys, auto*)
  apply (*case-tac ys, auto*)
  done

lemma *gcp-lb2*: *prefix* (*gcp xs ys*) *ys*
  apply (*induct ys arbitrary: xs, auto*)
  apply (*case-tac xs, auto*)
  done

interpretation *prefix-semilattice*: *semilattice-inf gcp prefix strict-prefix*
proof
  fix *xs ys* :: ′*a list*
  show *prefix* (*gcp xs ys*) *xs*
    by (*induct xs arbitrary: ys, auto, case-tac ys, auto*)
  show *prefix* (*gcp xs ys*) *ys*
    by (*induct ys arbitrary: xs, auto, case-tac xs, auto*)
next
  fix *xs ys zs* :: ′*a list*
  assume *prefix xs ys prefix xs zs*
  thus *prefix xs* (*gcp ys zs*)
    by (*simp add: prefix-def , auto*)
qed

2.4.7   Lexicographic Order

lemma *lexord-append*:
  assumes (*xs*$_1$ @ *ys*$_1$, *xs*$_2$ @ *ys*$_2$) ∈ *lexord R length*(*xs*$_1$) = *length*(*xs*$_2$)
  shows (*xs*$_1$, *xs*$_2$) ∈ *lexord R* ∨ (*xs*$_1$ = *xs*$_2$ ∧ (*ys*$_1$, *ys*$_2$) ∈ *lexord R*)
using *assms*
proof (*induct xs*$_2$ *arbitrary*: *xs*$_1$)

case (*Cons x₂ xs₂′*) **note** *hyps* = *this*
from *hyps(3)* **obtain** x_1 $xs_1′$ **where** xs_1: $xs_1 = x_1 \# xs_1′$ *length*($xs_1′$) = *length*($xs_2′$)
  by (*auto*, *metis Suc-length-conv*)
with *hyps(2)* **have** *xcases*: $(x_1, x_2) \in R \vee (xs_1′ @ ys_1, xs_2′ @ ys_2) \in lexord\ R$
  by (*auto*)
show *?case*
proof (*cases* $(x_1, x_2) \in R$)
  case *True* **with** xs_1 **show** *?thesis*
    by (*auto*)
next
  case *False*
  with *xcases* **have** $(xs_1′ @ ys_1, xs_2′ @ ys_2) \in lexord\ R$
    by (*auto*)
  with *hyps(1)* xs_1 **have** *dichot*: $(xs_1′, xs_2′) \in lexord\ R \vee (xs_1′ = xs_2′ \wedge (ys_1, ys_2) \in lexord\ R)$
    by (*auto*)
  have $x_1 = x_2$
    using *False hyps(2)* $xs_1(1)$ **by** *auto*
  with *dichot* xs_1 **show** *?thesis*
    by (*simp*)
qed
next
  case *Nil* **thus** *?case*
    by *auto*
qed

lemma *strict-prefix-lexord-rel*:
  strict-prefix xs ys $\implies$ $(xs, ys) \in lexord\ R$
  by (*metis Sublist.strict-prefixE′ lexord-append-rightI*)

lemma *strict-prefix-lexord-left*:
  assumes *trans R* $(xs, ys) \in lexord\ R$ *strict-prefix xs′ xs*
  shows $(xs′, ys) \in lexord\ R$
  by (*metis assms lexord-trans strict-prefix-lexord-rel*)

lemma *prefix-lexord-right*:
  assumes *trans R* $(xs, ys) \in lexord\ R$ *strict-prefix ys ys′*
  shows $(xs, ys′) \in lexord\ R$
  by (*metis assms lexord-trans strict-prefix-lexord-rel*)

lemma *lexord-eq-length*:
  assumes $(xs, ys) \in lexord\ R$ *length xs* = *length ys*
  shows $\exists\ i.\ (xs!i, ys!i) \in R \wedge i < length\ xs \wedge (\forall\ j{<}i.\ xs!j = ys!j)$
using *assms* **proof** (*induct xs arbitrary*: *ys*)
  case (*Cons x xs*) **note** *hyps* = *this*
  then obtain y $ys′$ **where** *ys*: *ys* = $y \# ys′$ *length ys′* = *length xs*
    by (*metis Suc-length-conv*)
  show *?case*
  proof (*cases* $(x, y) \in R$)
    case *True* **with** *ys* **show** *?thesis*
      by (*rule-tac x=0* **in** *exI*, *simp*)
  next
    case *False*
    with *ys hyps(2)* **have** *xy*: $x = y$ $(xs, ys′) \in lexord\ R$
      by *auto*
    with *hyps(1,3)* *ys* **obtain** i **where** $(xs!i, ys′!i) \in R$ $i < length\ xs$ $(\forall\ j{<}i.\ xs!j = ys′!j)$

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by *force*
  with *xy ys* **show** *?thesis*
    apply (*rule-tac x=Suc i* **in** *exI*)
    apply (*auto simp add*: *less-Suc-eq-0-disj*)
  done
  qed
next
  case *Nil* **thus** *?case* **by** (*auto*)
qed

lemma *lexord-intro-elems*:
  assumes *length xs > i length ys > i (xs!i, ys!i)* ∈ *R* ∀ *j<i. xs!j = ys!j*
  shows *(xs, ys)* ∈ *lexord R*
using *assms* **proof** (*induct i arbitrary*: *xs ys*)
  case *0* **thus** *?case*
    by (*auto, metis lexord-cons-cons list.exhaust nth-Cons-0*)
next
  case (*Suc i*) **note** *hyps = this*
  then obtain *x′ y′ xs′ ys′* **where** *xs = x′ # xs′ ys = y′ # ys′*
    by (*metis Suc-length-conv Suc-lessE*)
  moreover with *hyps(5)* **have** ∀ *j<i. xs′ ! j = ys′ ! j*
    by (*auto*)
  ultimately show *?case* **using** *hyps*
    by (*auto*)
qed

2.5 Distributed Concatenation

definition *uncurry* :: $(′a \Rightarrow ′b \Rightarrow ′c) \Rightarrow (′a \times ′b \Rightarrow ′c)$ **where**
[*simp*]: *uncurry f = (λ(x, y). f x y)*

definition *dist-concat* ::
  $′a\ list\ set \Rightarrow ′a\ list\ set \Rightarrow ′a\ list\ set$ (**infixr** ⌢ *100*) **where**
dist-concat ls1 ls2 = (uncurry (@) ' (ls1 × ls2))

lemma *dist-concat-left-empty* [*simp*]:
  {} ⌢ *ys* = {}
  by (*simp add*: *dist-concat-def*)

lemma *dist-concat-right-empty* [*simp*]:
  xs ⌢ {} = {}
  by (*simp add*: *dist-concat-def*)

lemma *dist-concat-insert* [*simp*]:
insert l ls1 ⌢ *ls2 = ((@) l ' (ls2))* ∪ *(ls1* ⌢ *ls2)*
  by (*auto simp add*: *dist-concat-def*)

2.6 List Domain and Range

abbreviation *seq-dom* :: $′a\ list \Rightarrow nat\ set$ (dom_l) **where**
seq-dom xs ≡ {*0..<length xs*}

abbreviation *seq-ran* :: $′a\ list \Rightarrow ′a\ set$ (ran_l) **where**
seq-ran xs ≡ *set xs*

2.7 Extracting List Elements

definition *seq-extract :: nat set ⇒ 'a list ⇒ 'a list* (**infix** $\uparrow_l$ *80*) **where**
seq-extract A xs = nths xs A

lemma *seq-extract-Nil* [*simp*]: *A* $\uparrow_l$ [] = []
  by (*simp add*: *seq-extract-def*)

lemma *seq-extract-Cons*:
  A $\uparrow_l$ (*x # xs*) = (*if 0 ∈ A then* [*x*] *else* []) @ {*j. Suc j ∈ A*} $\uparrow_l$ *xs*
  by (*simp add*: *seq-extract-def nths-Cons*)

lemma *seq-extract-empty* [*simp*]: {} $\uparrow_l$ *xs* = []
  by (*simp add*: *seq-extract-def*)

lemma *seq-extract-ident* [*simp*]: {*0..<length xs*} $\uparrow_l$ *xs* = *xs*
  unfolding *list-eq-iff-nth-eq*
  by (*auto simp add*: *seq-extract-def length-nths atLeast0LessThan*)

lemma *seq-extract-split*:
  assumes *i ≤ length xs*
  shows {*0..<i*} $\uparrow_l$ *xs* @ {*i..<length xs*} $\uparrow_l$ *xs* = *xs*
using *assms*
proof (*induct xs arbitrary*: *i*)
  case *Nil* **thus** *?case* **by** (*simp add*: *seq-extract-def*)
next
  case (*Cons x xs*) **note** *hyp* = *this*
  have {*j. Suc j < i*} = {*0..<i − 1*}
    by (*auto*)
  moreover have {*j. i ≤ Suc j ∧ j < length xs*} = {*i − 1..<length xs*}
    by (*auto*)
  ultimately show *?case*
    using *hyp* **by** (*force simp add*: *seq-extract-def nths-Cons*)
qed

lemma *seq-extract-append*:
  A $\uparrow_l$ (*xs* @ *ys*) = (*A* $\uparrow_l$ *xs*) @ ({*j. j + length xs ∈ A*} $\uparrow_l$ *ys*)
  by (*simp add*: *seq-extract-def nths-append*)

lemma *seq-extract-range*: *A* $\uparrow_l$ *xs* = (*A ∩ dom$_l$(xs)*) $\uparrow_l$ *xs*
  apply (*auto simp add*: *seq-extract-def nths-def*)
  apply (*metis* (*no-types, lifting*) *atLeastLessThan-iff filter-cong in-set-zip nth-mem set-upt*)
done

lemma *seq-extract-out-of-range*:
  A ∩ dom$_l$(xs) = {} ⟹ *A* $\uparrow_l$ *xs* = []
  by (*metis seq-extract-def seq-extract-range nths-empty*)

lemma *seq-extract-length* [*simp*]:
  length (*A* $\uparrow_l$ *xs*) = *card* (*A ∩ dom$_l$(xs)*)
proof −
  have {*i. i < length(xs) ∧ i ∈ A*} = (*A ∩ {0..<length(xs)}*)
    by (*auto*)
  thus *?thesis*
    by (*simp add*: *seq-extract-def length-nths*)
qed

lemma *seq-extract-Cons-atLeastLessThan*:
  assumes $m < n$
  shows $\{m..<n\} \upharpoonright_l (x \# xs) = (if \ (m = 0) \ then \ x \# (\{0..<n-1\} \upharpoonright_l xs) \ else \ \{m-1..<n-1\} \upharpoonright_l xs)$
proof $-$
  have $\{j.\ Suc\ j < n\} = \{0..<n - Suc\ 0\}$
    by (*auto*)
  moreover have $\{j.\ m \leq Suc\ j \wedge Suc\ j < n\} = \{m - Suc\ 0..<n - Suc\ 0\}$
    by (*auto*)

  ultimately show *?thesis* **using** *assms*
    by (*auto simp add*: *seq-extract-Cons*)
qed

lemma *seq-extract-singleton*:
  assumes $i < length\ xs$
  shows $\{i\} \upharpoonright_l xs = [xs\ !\ i]$
  using *assms*
  apply (*induct xs arbitrary*: i)
  apply (*auto simp add*: *seq-extract-Cons*)
  apply (*rename-tac xs i*)
  apply (*subgoal-tac* $\{j.\ Suc\ j = i\} = \{i - 1\}$)
  apply (*auto*)
done

lemma *seq-extract-as-map*:
  assumes $m < n\ \ n \leq length\ xs$
  shows $\{m..<n\} \upharpoonright_l xs = map\ (nth\ xs)\ [m..<n]$
using *assms* **proof** (*induct xs arbitrary*: $m\ n$)
  case *Nil* **thus** *?case* **by** *simp*
next
  case (*Cons x xs*)
  have $[m..<n] = m \# [m+1..<n]$
    using *Cons.prems(1) upt-eq-Cons-conv* **by** *blast*
  moreover have $map\ (nth\ (x \# xs))\ [Suc\ m..<n] = map\ (nth\ xs)\ [m..<n-1]$
    by (*simp add*: *map-nth-Cons-atLeastLessThan*)
  ultimately show *?case*
    using *Cons upt-rec*
    by (*auto simp add*: *seq-extract-Cons-atLeastLessThan*)
qed

lemma *seq-append-as-extract*:
  $xs = ys\ @\ zs \longleftrightarrow (\exists\ i \leq length(xs).\ ys = \{0..<i\} \upharpoonright_l xs \wedge zs = \{i..<length(xs)\} \upharpoonright_l xs)$
proof
  assume *xs*: $xs = ys\ @\ zs$
  moreover have $ys = \{0..<length\ ys\} \upharpoonright_l (ys\ @\ zs)$
    by (*simp add*: *seq-extract-append*)
  moreover have $zs = \{length\ ys..<length\ ys + length\ zs\} \upharpoonright_l (ys\ @\ zs)$
  proof $-$
    have $\{length\ ys..<length\ ys + length\ zs\} \cap \{0..<length\ ys\} = \{\}$
      by *auto*
    moreover have *s1*: $\{j.\ j < length\ zs\} = \{0..<length\ zs\}$
      by *auto*
    ultimately show *?thesis*
      by (*simp add*: *seq-extract-append seq-extract-out-of-range*)

qed
  ultimately show $(\exists\ i \leq length(xs).\ ys = \{0..<i\}\ \restriction_l\ xs\ \wedge\ zs = \{i..<length(xs)\}\ \restriction_l\ xs)$
    by *(rule-tac x=length ys* **in** *exI, auto)*
next
  assume $\exists\, i \leq length\ xs.\ ys = \{0..<i\}\ \restriction_l\ xs\ \wedge\ zs = \{i..<length\ xs\}\ \restriction_l\ xs$
  thus $xs = ys\ @\ zs$
    by *(auto simp add: seq-extract-split)*
qed

2.8 Filtering a list according to a set

definition *seq-filter* :: $'a\ list \Rightarrow\ 'a\ set \Rightarrow\ 'a\ list$ (**infix** $\restriction_l$ *80*) **where**
seq-filter xs A = filter $(\lambda\ x.\ x \in A)\ xs$

lemma *seq-filter-Cons-in* [*simp*]:
  $x \in cs \Longrightarrow (x\ \#\ xs)\ \restriction_l\ cs = x\ \#\ (xs\ \restriction_l\ cs)$
  by *(simp add: seq-filter-def)*

lemma *seq-filter-Cons-out* [*simp*]:
  $x \notin cs \Longrightarrow (x\ \#\ xs)\ \restriction_l\ cs = (xs\ \restriction_l\ cs)$
  by *(simp add: seq-filter-def)*

lemma *seq-filter-Nil* [*simp*]: $[]\ \restriction_l\ A = []$
  by *(simp add: seq-filter-def)*

lemma *seq-filter-empty* [*simp*]: $xs\ \restriction_l\ \{\} = []$
  by *(simp add: seq-filter-def)*

lemma *seq-filter-append*: $(xs\ @\ ys)\ \restriction_l\ A = (xs\ \restriction_l\ A)\ @\ (ys\ \restriction_l\ A)$
  by *(simp add: seq-filter-def)*

lemma *seq-filter-UNIV* [*simp*]: $xs\ \restriction_l\ UNIV = xs$
  by *(simp add: seq-filter-def)*

lemma *seq-filter-twice* [*simp*]: $(xs\ \restriction_l\ A)\ \restriction_l\ B = xs\ \restriction_l\ (A \cap B)$
  by *(simp add: seq-filter-def)*

2.9 Minus on lists

instantiation *list* :: *(type) minus*
begin

We define list minus so that if the second list is not a prefix of the first, then an arbitrary list longer than the combined length is produced. Thus we can always determined from the output whether the minus is defined or not.

definition $xs - ys = (if\ (prefix\ ys\ xs)\ then\ drop\ (length\ ys)\ xs\ else\ [])$

instance ..
end

lemma *minus-cancel* [*simp*]: $xs - xs = []$
  by *(simp add: minus-list-def)*

lemma *append-minus* [*simp*]: $(xs\ @\ ys) - xs = ys$
  by *(simp add: minus-list-def)*

lemma *minus-right-nil* [*simp*]: $xs - [] = xs$
  by (*simp add*: *minus-list-def*)

lemma *list-concat-minus-list-concat*: $(s @ t) - (s @ z) = t - z$
  by (*simp add*: *minus-list-def*)

lemma *length-minus-list*: $y \leq x \implies length(x - y) = length(x) - length(y)$
  by (*simp add*: *less-eq-list-def minus-list-def*)

lemma *map-list-minus*:
  $xs \leq ys \implies map\ f\ (ys - xs) = map\ f\ ys - map\ f\ xs$
  by (*simp add*: *drop-map less-eq-list-def map-mono-prefix minus-list-def*)

lemma *list-minus-first-tl* [*simp*]:
  $[x] \leq xs \implies (xs - [x]) = tl\ xs$
  by (*metis Prefix-Order.prefixE append.left-neutral append-minus list.sel(3) not-Cons-self2 tl-append2*)

Extra lemmas about *prefix* and *strict-prefix*

lemma *prefix-concat-minus*:
  assumes *prefix xs ys*
  shows $xs @ (ys - xs) = ys$
  using *assms* **by** (*metis minus-list-def prefix-drop*)

lemma *prefix-minus-concat*:
  assumes *prefix s t*
  shows $(t - s) @ z = (t @ z) - s$
  using *assms* **by** (*simp add*: *Sublist.prefix-length-le minus-list-def*)

lemma *strict-prefix-minus-not-empty*:
  assumes *strict-prefix xs ys*
  shows $ys - xs \neq []$
  using *assms* **by** (*metis append-Nil2 prefix-concat-minus strict-prefix-def*)

lemma *strict-prefix-diff-minus*:
  assumes *prefix xs ys* **and** $xs \neq ys$
  shows $(ys - xs) \neq []$
  using *assms* **by** (*simp add*: *strict-prefix-minus-not-empty*)

lemma *length-tl-list-minus-butlast-gt-zero*:
  assumes *length s < length t* **and** *strict-prefix (butlast s) t* **and** *length s > 0*
  shows *length (tl (t - (butlast s))) > 0*
  using *assms*
  by (*metis Nitpick.size-list-simp(2) butlast-snoc hd-Cons-tl length-butlast length-greater-0-conv length-tl*
less-trans nat-neq-iff strict-prefix-minus-not-empty prefix-order.dual-order.strict-implies-order prefix-concat-minus)

lemma *list-minus-butlast-eq-butlast-list*:
  assumes *length t = length s* **and** *strict-prefix (butlast s) t*
  shows $t - (butlast\ s) = [last\ t]$
  using *assms*
  by (*metis append-butlast-last-id append-eq-append-conv butlast.simps(1) length-butlast less-numeral-extra(3)*
list.size(3) prefix-order.dual-order.strict-implies-order prefix-concat-minus prefix-length-less)

lemma *butlast-strict-prefix-length-lt-imp-last-tl-minus-butlast-eq-last*:
  assumes *length s > 0 strict-prefix (butlast s) t length s < length t*

shows *last* (*tl* (*t* − (*butlast s*))) = (*last t*)
  using *assms* **by** (*metis last-append last-tl length-tl-list-minus-butlast-gt-zero less-numeral-extra(3)*
list.size(3) append-minus strict-prefix-eq-exists)

lemma *tl-list-minus-butlast-not-empty*:
  assumes *strict-prefix* (*butlast s*) *t* **and** *length s > 0* **and** *length t > length s*
  shows *tl* (*t* − (*butlast s*)) ≠ []
  using *assms length-tl-list-minus-butlast-gt-zero* **by** *fastforce*

lemma *tl-list-minus-butlast-empty*:
  assumes *strict-prefix* (*butlast s*) *t* **and** *length s > 0* **and** *length t = length s*
  shows *tl* (*t* − (*butlast s*)) = []
  using *assms* **by** (*simp add: list-minus-butlast-eq-butlast-list*)

lemma *concat-minus-list-concat-butlast-eq-list-minus-butlast*:
  assumes *prefix* (*butlast u*) *s*
  shows (*t @ s*) − (*t @ (butlast u*)) = *s* − (*butlast u*)
  using *assms* **by** (*metis append-assoc prefix-concat-minus append-minus*)

lemma *tl-list-minus-butlast-eq-empty*:
  assumes *strict-prefix* (*butlast s*) *t* **and** *length s = length t*
  shows *tl* (*t* − (*butlast s*)) = []
  using *assms* **by** (*metis list.sel(3) list-minus-butlast-eq-butlast-list*)

lemma *prefix-length-tl-minus*:
  assumes *strict-prefix s t*
  shows *length* (*tl* (*t*−*s*)) = (*length* (*t*−*s*)) − *1*
  by (*auto*)

lemma *length-list-minus*:
  assumes *strict-prefix s t*
  shows *length*(*t* − *s*) = *length*(*t*) − *length*(*s*)
  using *assms* **by** (*simp add: minus-list-def prefix-order.dual-order.strict-implies-order*)

2.10   Laws on *take*, *drop*, and *nths*

lemma *take-prefix*: *m ≤ n* ⟹ *take m xs ≤ take n xs*
  by (*metis Prefix-Order.prefixI append-take-drop-id min-absorb2 take-append take-take*)

lemma *nths-atLeastAtMost-0-take*: *nths xs {0..m} = take* (*Suc m*) *xs*
  by (*metis atLeast0AtMost lessThan-Suc-atMost nths-upt-eq-take*)

lemma *nths-atLeastLessThan-0-take*: *nths xs {0..<m} = take m xs*
  by (*simp add: atLeast0LessThan*)

lemma *nths-atLeastAtMost-prefix*: *m ≤ n* ⟹ *nths xs {0..m} ≤ nths xs {0..n}*
  by (*simp add: nths-atLeastAtMost-0-take take-prefix*)

lemma *sorted-nths-atLeastAtMost-0*: ⟦ *m ≤ n; sorted* (*nths xs {0..n}*) ⟧ ⟹ *sorted* (*nths xs {0..m}*)
  using *nths-atLeastAtMost-prefix sorted-prefix* **by** *blast*

lemma *sorted-nths-atLeastLessThan-0*: ⟦ *m ≤ n; sorted* (*nths xs {0..<n}*) ⟧ ⟹ *sorted* (*nths xs {0..<m}*)
  by (*metis atLeast0LessThan nths-upt-eq-take sorted-prefix take-prefix*)

lemma *list-augment-as-update*:

```
      k < length xs ⟹ list-augment xs k x = list-update xs k x
      by (metis list-augment-def list-augment-idem list-update-overwrite)


lemma nths-list-update-out: k ∉ A ⟹ nths (list-update xs k x) A = nths xs A
  apply (induct xs arbitrary: k x A)
   apply (auto)
  apply (rename-tac a xs k x A)
  apply (case-tac k)
   apply (auto simp add: nths-Cons)
  done


lemma nths-list-augment-out: ⟦ k < length xs; k ∉ A ⟧ ⟹ nths (list-augment xs k x) A = nths xs A
  by (simp add: list-augment-as-update nths-list-update-out)


lemma nths-single: n < length xs ⟹ nths xs {n} = [xs ! n]
proof (induct xs arbitrary: n)
  case Nil
  then show ?case by (simp)
next
  case (Cons a xs)
  have ⋀ n. n > 0 ⟹ {j. Suc j = n} = {n−1} by auto
  with Cons show ?case by (auto simp add: nths-Cons)
qed


lemma nths-uptoLessThan:
  ⟦ m ≤ n; n < length xs ⟧ ⟹ nths xs {m..n} = xs ! m # nths xs {Suc m..n}
proof (induct xs arbitrary: m n)
case Nil
  then show ?case by (simp)
next
  case (Cons a xs)
  have l1: ⋀ m n. ⟦ 0 < m; m ≤ n ⟧ ⟹ {j. m ≤ Suc j ∧ Suc j ≤ n} = {m−1..n−1}
    by (auto)
  have l2: ⋀ m n. ⟦ 0 < m; m ≤ n ⟧ ⟹ {j. m ≤ j ∧ Suc j ≤ n} = {m..n−1}
    by (auto)
  from Cons show ?case by (auto simp add: nths-Cons l1 l2)
qed


lemma nths-upt-nth: ⟦ j < i; i < length xs ⟧ ⟹ (nths xs {0..<i}) ! j = xs ! j
  by (metis lessThan-atLeast0 nth-take nths-upt-eq-take)


lemma nths-upt-length: ⟦ m ≤ n; n ≤ length xs ⟧ ⟹ length (nths xs {m..<n}) = n−m
  by (metis atLeastLessThan-empty diff-is-0-eq length-map length-upt list.size(3) not-less nths-empty
seq-extract-as-map seq-extract-def)


lemma nths-upt-le-length:
  ⟦ m ≤ n; Suc n ≤ length xs ⟧ ⟹ length (nths xs {m..n}) = Suc n − m
  by (metis atLeastLessThanSuc-atLeastAtMost le-SucI nths-upt-length)


lemma sl1: n > 0 ⟹ {j. Suc j ≤ n} = {0..n−1}
  by (auto)


lemma sl2: ⟦ 0 < m; m ≤ n ⟧ ⟹ {j. m ≤ Suc j ∧ Suc j ≤ n} = {m−1..n−1}
  by auto
```

lemma *nths-upt-le-nth*: $\llbracket\ m \le n;\ Suc\ n \le length\ xs;\ i < Suc\ n - m\ \rrbracket$
$\implies (nths\ xs\ \{m..n\})\ !\ i = xs\ !\ (i + m)$
proof (*induct xs arbitrary: m n i*)
  case *Nil*
  then show *?case* **by** (*simp*)
next
  case (*Cons a xs*)
  then show *?case*
  proof (*cases i = 0*)
    case *True*
    with *Cons* **show** *?thesis* **by** (*auto simp add*: *nths-Cons sl2*)
  next
    case *False*
    with *Cons* **show** *?thesis* **by** (*auto simp add*: *nths-Cons sl1 sl2*)
  qed
qed

lemma *nths-upt-le-append-split*:
  $\llbracket\ j \le i;\ i < length\ xs\ \rrbracket \implies nths\ xs\ \{0..<j\}\ @\ nths\ xs\ \{j..i\} = nths\ xs\ \{0..i\}$
  by (*auto simp add*: *list-eq-iff-nth-eq nths-upt-length nths-upt-le-length nths-upt-le-nth nths-upt-nth nth-append*)

2.11 List power

overloading
  $listpow \equiv compow :: nat \Rightarrow {}'a\ list \Rightarrow {}'a\ list$
begin

fun $listpow :: nat \Rightarrow {}'a\ list \Rightarrow {}'a\ list$ **where**
  $listpow\ 0\ xs = []$
| $listpow\ (Suc\ n)\ xs = xs\ @\ listpow\ n\ xs$

end

lemma *listpow-Nil* [*simp*]: $[]\ \hat{}\ \hat{}\ n = []$
  by (*induct n*) *simp-all*

lemma *listpow-Suc-right*: $xs\ \hat{}\ \hat{}\ Suc\ n = xs\ \hat{}\ \hat{}\ n\ @\ xs$
  by (*induct n*) *simp-all*

lemma *listpow-add*: $xs\ \hat{}\ \hat{}\ (m + n) = xs\ \hat{}\ \hat{}\ m\ @\ xs\ \hat{}\ \hat{}\ n$
  by (*induct m*) *simp-all*

end

3   Infinite Sequences

theory *Sequence*
imports
  HOL.Real
  List-Extra
  HOL−Library.Sublist
  HOL−Library.Nat-Bijection
begin

typedef $'a\ seq = UNIV :: (nat \Rightarrow 'a)\ set$
  by (*auto*)

setup-lifting *type-definition-seq*

definition *ssubstr* $:: nat \Rightarrow nat \Rightarrow 'a\ seq \Rightarrow 'a\ list$ **where**
ssubstr $i\ j\ xs = map\ (Rep\text{-}seq\ xs)\ [i\ ..<j]$

lift-definition *nth-seq* $:: 'a\ seq \Rightarrow nat \Rightarrow 'a$ (**infixl** $!_s$ *100*)
is $\lambda\ f\ i.\ f\ i$.

abbreviation *sinit* $:: nat \Rightarrow 'a\ seq \Rightarrow 'a\ list$ **where**
sinit $i\ xs \equiv ssubstr\ 0\ i\ xs$

lemma *sinit-len* [*simp*]:
  $length\ (sinit\ i\ xs) = i$
  by (*simp add*: *ssubstr-def*)

lemma *sinit-0* [*simp*]: $sinit\ 0\ xs = [\,]$
  by (*simp add*: *ssubstr-def*)

lemma *prefix-upt-0* [*intro*]:
  $i \le j \Longrightarrow prefix\ [0..<i]\ [0..<j]$
 by (*induct i, auto, metis append-prefixD le0 prefix-order.lift-Suc-mono-le prefix-order.order-refl upt-Suc*)

lemma *sinit-prefix*:
  $i \le j \Longrightarrow prefix\ (sinit\ i\ xs)\ (sinit\ j\ xs)$
  by (*simp add*: *map-mono-prefix prefix-upt-0 ssubstr-def*)

lemma *sinit-strict-prefix*:
  $i < j \Longrightarrow strict\text{-}prefix\ (sinit\ i\ xs)\ (sinit\ j\ xs)$
  by (*metis sinit-len sinit-prefix le-less nat-neq-iff prefix-order.dual-order.strict-iff-order*)

lemma *nth-sinit*:
  $i < n \Longrightarrow sinit\ n\ xs\ !\ i = xs\ !_s\ i$
  apply (*auto simp add*: *ssubstr-def*)
  apply (*transfer, auto*)
  done

lemma *sinit-append-split*:
  assumes $i < j$
  shows $sinit\ j\ xs = sinit\ i\ xs\ @\ ssubstr\ i\ j\ xs$
proof −
  have $[0..<i]\ @\ [i..<j] = [0..<j]$
    by (*metis assms le0 le-add-diff-inverse le-less upt-add-eq-append*)
  thus *?thesis*
    by (*auto simp add*: *ssubstr-def, transfer, simp add*: *map-append*[*THEN sym*])
qed

lemma *sinit-linear-asym-lemma1*:
  assumes *asym R* $i < j$ $(sinit\ i\ xs, sinit\ i\ ys) \in lexord\ R$ $(sinit\ j\ ys, sinit\ j\ xs) \in lexord\ R$
  shows *False*
proof −
  have *sinit-xs*: $sinit\ j\ xs = sinit\ i\ xs\ @\ ssubstr\ i\ j\ xs$

    by (*metis assms(2) sinit-append-split*)
  have *sinit-ys*: *sinit j ys = sinit i ys @ ssubstr i j ys*
    by (*metis assms(2) sinit-append-split*)
  from *sinit-xs sinit-ys assms(4)*
  have (*sinit i ys, sinit i xs*) $\in$ *lexord R* $\vee$ (*sinit i ys = sinit i xs* $\wedge$ (*ssubstr i j ys, ssubstr i j xs*) $\in$ *lexord R*)
    by (*auto dest*: *lexord-append*)
  with *assms lexord-asymmetric* **show** *False*
    by (*force*)
qed

lemma *sinit-linear-asym-lemma2*:
  assumes *asym R* (*sinit i xs, sinit i ys*) $\in$ *lexord R* (*sinit j ys, sinit j xs*) $\in$ *lexord R*
  shows *False*
proof (*cases i j rule*: *linorder-cases*)
  case *less* **with** *assms* **show** *?thesis*
    by (*auto dest*: *sinit-linear-asym-lemma1*)
next
  case *equal* **with** *assms* **show** *?thesis*
    by (*simp add*: *lexord-asymmetric*)
next
  case *greater* **with** *assms* **show** *?thesis*
    by (*auto dest*: *sinit-linear-asym-lemma1*)
qed

lemma *range-ext*:
  assumes $\forall\, i :: nat.\ \forall\, x \in \{0..<i\}.\ f(x) = g(x)$
  shows $f = g$
proof (*rule ext*)
  fix *x* :: *nat*
  obtain *i* :: *nat* **where** $i > x$
    by (*metis lessI*)
  with *assms* **show** $f(x) = g(x)$
    by (*auto*)
qed

lemma *sinit-ext*:
  ($\forall$ *i. sinit i xs = sinit i ys*) $\Longrightarrow$ *xs = ys*
  by (*simp add*: *ssubstr-def*, *transfer*, *auto intro*: *range-ext*)

definition *seq-lexord* :: $'a\ rel \Rightarrow ('a\ seq)\ rel$ **where**
seq-lexord R = {(*xs, ys*). ($\exists$ *i*. (*sinit i xs, sinit i ys*) $\in$ *lexord R*)}

lemma *seq-lexord-irreflexive*:
  $\forall\, x.$ (*x, x*) $\notin$ *R* $\Longrightarrow$ (*xs, xs*) $\notin$ *seq-lexord R*
  by (*auto dest*: *lexord-irreflexive simp add*: *irrefl-def seq-lexord-def*)

lemma *seq-lexord-irrefl*:
  irrefl R $\Longrightarrow$ *irrefl* (*seq-lexord R*)
  by (*simp add*: *irrefl-def seq-lexord-irreflexive*)

lemma *seq-lexord-transitive*:
  assumes *trans R*
  shows *trans* (*seq-lexord R*)
unfolding *seq-lexord-def*

proof (*rule transI, clarify*)
  fix *xs ys zs* :: *'a seq* **and** *m n* :: *nat*
  assume *las*: (*sinit m xs, sinit m ys*) ∈ *lexord R* (*sinit n ys, sinit n zs*) ∈ *lexord R*
  hence *inz*: *m > 0*
    using *gr0I* **by** *force*
  from *las(1)* **obtain** *i* **where** *sinitm*: (*sinit m xs!i, sinit m ys!i*) ∈ *R i < m* ∀ *j<i. sinit m xs!j =*
sinit m ys!j
    using *lexord-eq-length* **by** *force*
  from *las(2)* **obtain** *j* **where** *sinitn*: (*sinit n ys!j, sinit n zs!j*) ∈ *R j < n* ∀ *k<j. sinit n ys!k = sinit*
n zs!k
    using *lexord-eq-length* **by** *force*
  show ∃ *i.* (*sinit i xs, sinit i zs*) ∈ *lexord R*
  proof (*cases i ≤ j*)
    case *True* **note** *lt = this*
    with *sinitm sinitn* **have** (*sinit n xs!i, sinit n zs!i*) ∈ *R*
      by (*metis assms le-eq-less-or-eq le-less-trans nth-sinit transD*)
    moreover from *lt sinitm sinitn* **have** ∀ *j<i. sinit m xs!j = sinit m zs!j*
      by (*metis less-le-trans less-trans nth-sinit*)
    ultimately have (*sinit n xs, sinit n zs*) ∈ *lexord R* **using** *sinitm(2) sinitn(2) lt*
      apply (*rule-tac lexord-intro-elems*)
        apply (*auto*)
      apply (*metis less-le-trans less-trans nth-sinit*)
      done
    thus *?thesis* **by** *auto*
  next
    case *False*
    then have *ge*: *i > j* **by** *auto*
    with *assms sinitm sinitn* **have** (*sinit n xs!j, sinit n zs!j*) ∈ *R*
      by (*metis less-trans nth-sinit*)
    moreover from *ge sinitm sinitn* **have** ∀ *k<j. sinit m xs!k = sinit m zs!k*
      by (*metis dual-order.strict-trans nth-sinit*)
    ultimately have (*sinit n xs, sinit n zs*) ∈ *lexord R* **using** *sinitm(2) sinitn(2) ge*
      apply (*rule-tac lexord-intro-elems*)
        apply (*auto*)
      apply (*metis less-trans nth-sinit*)
      done
    thus *?thesis* **by** *auto*
  qed
qed

lemma *seq-lexord-trans*:
  ⟦ (*xs, ys*) ∈ *seq-lexord R*; (*ys, zs*) ∈ *seq-lexord R*; *trans R* ⟧ ⟹ (*xs, zs*) ∈ *seq-lexord R*
  by (*meson seq-lexord-transitive transE*)

lemma *seq-lexord-antisym*:
  ⟦ *asym R*; (*a, b*) ∈ *seq-lexord R* ⟧ ⟹ (*b, a*) ∉ *seq-lexord R*
  by (*auto dest*: *sinit-linear-asym-lemma2 simp add*: *seq-lexord-def*)

lemma *seq-lexord-asym*:
  assumes *asym R*
  shows *asym* (*seq-lexord R*)
  by (*meson assms asym.simps seq-lexord-antisym seq-lexord-irrefl*)

lemma *seq-lexord-total*:
  assumes *total R*

shows *total* (*seq-lexord R*)
  using *assms* **by** (*auto simp add: total-on-def seq-lexord-def*, *meson lexord-linear sinit-ext*)

lemma *seq-lexord-strict-linear-order*:
  assumes *strict-linear-order R*
  shows *strict-linear-order* (*seq-lexord R*)
  using *assms*
  by (*auto simp add: strict-linear-order-on-def partial-order-on-def preorder-on-def*
        intro: *seq-lexord-transitive seq-lexord-irrefl seq-lexord-total*)

lemma *seq-lexord-linear*:
  assumes (∀ *a b.* (*a,b*)∈ *R* ∨ *a* = *b* ∨ (*b,a*) ∈ *R*)
  shows (*x,y*) ∈ *seq-lexord R* ∨ *x* = *y* ∨ (*y,x*) ∈ *seq-lexord R*
proof −
  have *total R*
    using *assms total-on-def* **by** *blast*
  hence *total* (*seq-lexord R*)
    using *seq-lexord-total* **by** *blast*
  thus *?thesis*
    by (*auto simp add: total-on-def*)
qed

instantiation *seq* :: (*ord*) *ord*
begin

definition *less-seq* :: ′*a seq* ⇒ ′*a seq* ⇒ *bool* **where**
less-seq xs ys ⟷ (*xs, ys*) ∈ *seq-lexord* {(*xs, ys*). *xs* < *ys*}

definition *less-eq-seq* :: ′*a seq* ⇒ ′*a seq* ⇒ *bool* **where**
less-eq-seq xs ys = (*xs* = *ys* ∨ *xs* < *ys*)

instance ..

end

instance *seq* :: (*order*) *order*
proof
  fix *xs* :: ′*a seq*
  show *xs* ≤ *xs* **by** (*simp add: less-eq-seq-def*)
next
  fix *xs ys zs* :: ′*a seq*
  assume *xs* ≤ *ys* **and** *ys* ≤ *zs*
  then show *xs* ≤ *zs*
    by (*force dest: seq-lexord-trans simp add: less-eq-seq-def less-seq-def trans-def*)
next
  fix *xs ys* :: ′*a seq*
  assume *xs* ≤ *ys* **and** *ys* ≤ *xs*
  then show *xs* = *ys*
    apply (*auto simp add: less-eq-seq-def less-seq-def*)
    apply (*rule seq-lexord-irreflexive* [*THEN notE*])
     defer
     apply (*rule seq-lexord-trans*)
       apply (*auto intro: transI*)
    done
next

fix *xs ys* :: *'a seq*
show *xs* < *ys* ⟷ *xs* ≤ *ys* ∧ ¬ *ys* ≤ *xs*
  apply (*auto simp add*: *less-seq-def less-eq-seq-def*)
   defer
   apply (*rule seq-lexord-irreflexive* [*THEN notE*])
    apply *auto*
   apply (*rule seq-lexord-irreflexive* [*THEN notE*])
    defer
   apply (*rule seq-lexord-trans*)
    apply (*auto intro*: *transI*)
  apply (*simp add*: *seq-lexord-irreflexive*)
  done
qed

instance *seq* :: (*linorder*) *linorder*
proof
  fix *xs ys* :: *'a seq*
  have (*xs, ys*) ∈ *seq-lexord* {(*u, v*). *u* < *v*} ∨ *xs* = *ys* ∨ (*ys, xs*) ∈ *seq-lexord* {(*u, v*). *u* < *v*}
   by (*rule seq-lexord-linear*) *auto*
  then show *xs* ≤ *ys* ∨ *ys* ≤ *xs*
   by (*auto simp add*: *less-eq-seq-def less-seq-def*)
qed

lemma *seq-lexord-mono* [*mono*]:
  (⋀ *x y*. *f x y* ⟶ *g x y*) ⟹ (*xs, ys*) ∈ *seq-lexord* {(*x, y*). *f x y*} ⟶ (*xs, ys*) ∈ *seq-lexord* {(*x, y*). *g x y*}
  apply (*auto simp add*: *seq-lexord-def*)
  apply (*metis case-prodD case-prodI lexord-take-index-conv mem-Collect-eq*)
done

fun *insort-rel* :: *'a rel* ⇒ *'a* ⇒ *'a list* ⇒ *'a list* **where**
insort-rel R x [] = [*x*] |
insort-rel R x (*y* # *ys*) = (*if* (*x* = *y* ∨ (*x,y*) ∈ *R*) *then x* # *y* # *ys else y* # *insort-rel R x ys*)

inductive *sorted-rel* :: *'a rel* ⇒ *'a list* ⇒ *bool* **where**
Nil-rel [*iff*]: *sorted-rel R* [] |
Cons-rel: ∀ *y* ∈ *set xs*. (*x* = *y* ∨ (*x, y*) ∈ *R*) ⟹ *sorted-rel R xs* ⟹ *sorted-rel R* (*x* # *xs*)

definition *list-of-set* :: *'a rel* ⇒ *'a set* ⇒ *'a list* **where**
list-of-set R = *folding.F* (*insort-rel R*) []

lift-definition *seq-inj* :: *'a seq seq* ⇒ *'a seq* **is**
λ *f i*. *f* (*fst* (*prod-decode i*)) (*snd* (*prod-decode i*)) .

lift-definition *seq-proj* :: *'a seq* ⇒ *'a seq seq* **is**
λ *f i j*. *f* (*prod-encode* (*i, j*)) .

lemma *seq-inj-inverse*: *seq-proj* (*seq-inj x*) = *x*
  by (*transfer*, *simp*)

lemma *seq-proj-inverse*: *seq-inj* (*seq-proj x*) = *x*
  by (*transfer*, *simp*)

lemma *seq-inj*: *inj seq-inj*
  by (*metis injI seq-inj-inverse*)

```
lemma seq-inj-surj: bij seq-inj
  apply (rule bijI)
   apply (auto simp add: seq-inj)
  apply (metis rangeI seq-proj-inverse)
  done
end
```

4   Finite Sets: extra functions and properties

theory *FSet-Extra*
imports
  HOL−Library.FSet
  HOL−Library.Countable-Set-Type
begin

setup-lifting *type-definition-fset*

notation *fempty* (⦃⦄)
notation *fset* (⟨-⟩$_f$)
notation *fminus* (**infixl** $-_f$ *65*)

syntax
  -FinFset :: args => 'a fset   (⦃(-)⦄)

translations
  ⦃*x, xs*⦄ == *CONST finsert x* ⦃*xs*⦄
  ⦃*x*⦄ == *CONST finsert x* ⦃⦄

term *fBall*

syntax
  -fBall :: pttrn => 'a fset => bool => bool ((3∀ -|∈|-./ -) [0, 0, 10] 10)
  -fBex :: pttrn => 'a fset => bool => bool ((3∃ -|∈|-./ -) [0, 0, 10] 10)

translations
  ∀ *x*|∈|*A. P* == *CONST fBall A* (%*x. P*)
  ∃ *x*|∈|*A. P* == *CONST fBex A* (%*x. P*)

definition *FUnion :: 'a fset fset ⇒ 'a fset* (⋃$_f$- [90] 90) **where**
FUnion xs = Abs-fset (⋃*x*∈⟨*xs*⟩$_f$. ⟨*x*⟩$_f$)

definition *FInter :: 'a fset fset ⇒ 'a fset* (⋂$_f$- [90] 90) **where**
FInter xs = Abs-fset (⋂*x*∈⟨*xs*⟩$_f$. ⟨*x*⟩$_f$)

Finite power set

definition *FinPow :: 'a fset ⇒ 'a fset fset* **where**
FinPow xs = Abs-fset (*Abs-fset* ' *Pow* ⟨*xs*⟩$_f$)

Set of all finite subsets of a set

definition *Fow :: 'a set ⇒ 'a fset set* **where**
Fow A = {x. ⟨*x*⟩$_f$ ⊆ *A*}

declare *Abs-fset-inverse* [*simp*]

```

**lemma** *fset-intro*:
  *fset x = fset y* $\Longrightarrow$ *x = y*
  **by** (*simp add:fset-inject*)

**lemma** *fset-elim*:
  $\llbracket$ *x = y*; *fset x = fset y* $\Longrightarrow$ *P* $\rrbracket$ $\Longrightarrow$ *P*
  **by** (*auto*)

**lemma** *fmember-intro*:
  $\llbracket$ *x $\in$ fset(xs)* $\rrbracket$ $\Longrightarrow$ *x* $|\in|$ *xs*
  **by** (*metis fmember.rep-eq*)

**lemma** *fmember-elim*:
  $\llbracket$ *x* $|\in|$ *xs*; *x $\in$ fset(xs)* $\Longrightarrow$ *P* $\rrbracket$ $\Longrightarrow$ *P*
  **by** (*metis fmember.rep-eq*)

**lemma** *fnmember-intro* [*intro*]:
  $\llbracket$ *x $\notin$ fset(xs)* $\rrbracket$ $\Longrightarrow$ *x* $|\notin|$ *xs*
  **by** (*metis fmember.rep-eq*)

**lemma** *fnmember-elim* [*elim*]:
  $\llbracket$ *x* $|\notin|$ *xs*; *x $\notin$ fset(xs)* $\Longrightarrow$ *P* $\rrbracket$ $\Longrightarrow$ *P*
  **by** (*metis fmember.rep-eq*)

**lemma** *fsubset-intro* [*intro*]:
  $\langle xs \rangle_f \subseteq \langle ys \rangle_f$ $\Longrightarrow$ *xs* $|\subseteq|$ *ys*
  **by** (*metis less-eq-fset.rep-eq*)

**lemma** *fsubset-elim* [*elim*]:
  $\llbracket$ *xs* $|\subseteq|$ *ys*; $\langle xs \rangle_f \subseteq \langle ys \rangle_f$ $\Longrightarrow$ *P* $\rrbracket$ $\Longrightarrow$ *P*
  **by** (*metis less-eq-fset.rep-eq*)

**lemma** *fBall-intro* [*intro*]:
  *Ball* $\langle A \rangle_f$ *P* $\Longrightarrow$ *fBall A P*
  **by** (*metis* (*poly-guards-query*) *fBallI fmember.rep-eq*)

**lemma** *fBall-elim* [*elim*]:
  $\llbracket$ *fBall A P*; *Ball* $\langle A \rangle_f$ *P* $\Longrightarrow$ *Q* $\rrbracket$ $\Longrightarrow$ *Q*
  **by** (*metis fBallE fmember.rep-eq*)

**lift-definition** *finset* :: $'a$ *list* $\Rightarrow$ $'a$ *fset* **is** *set* **..**

**context** *linorder*
**begin**

**lemma** *sorted-list-of-set-inj*:
  $\llbracket$ *finite xs*; *finite ys*; *sorted-list-of-set xs = sorted-list-of-set ys* $\rrbracket$
  $\Longrightarrow$ *xs = ys*
  **apply** (*simp add:sorted-list-of-set-def*)
  **apply** (*induct xs rule:finite-induct*)
   **apply** (*induct ys rule:finite-induct*)
    **apply** (*simp-all*)
  **apply** (*metis finite.insertI insert-not-empty sorted-list-of-set-def sorted-list-of-set-empty sorted-list-of-set-eq-Nil-iff*)
  **apply** (*metis finite.insertI finite-list set-remdups set-sort sorted-list-of-set-def sorted-list-of-set-sort-remdups*)
  **done**

**definition** *flist* :: *'a fset ⇒ 'a list* **where**
*flist xs = sorted-list-of-set (fset xs)*

**lemma** *flist-inj*: *inj flist*
  **apply** (*simp add:flist-def inj-on-def*)
  **apply** (*clarify*)
  **apply** (*rename-tac x y*)
  **apply** (*subgoal-tac fset x = fset y*)
   **apply** (*simp add:fset-inject*)
  **apply** (*rule sorted-list-of-set-inj*, *simp-all*)
  **done**

**lemma** *flist-props* [*simp*]:
  *sorted* (*flist xs*)
  *distinct* (*flist xs*)
  **by** (*simp-all add:flist-def*)

**lemma** *flist-empty* [*simp*]:
  *flist {|\|} = []*
  **by** (*simp add:flist-def*)

**lemma** *flist-inv* [*simp*]: *finset* (*flist xs*) = *xs*
  **by** (*simp add:finset-def flist-def fset-inverse*)

**lemma** *flist-set* [*simp*]: *set* (*flist xs*) = *fset xs*
  **by** (*simp add:finset-def flist-def fset-inverse*)

**lemma** *fset-inv* [*simp*]: ⟦ *sorted xs*; *distinct xs* ⟧ ⟹ *flist* (*finset xs*) = *xs*
  **apply** (*simp add:finset-def flist-def fset-inverse*)
  **apply** (*metis local.sorted-list-of-set-sort-remdups local.sorted-sort-id remdups-id-iff-distinct*)
  **done**

**lemma** *fcard-flist*:
  *fcard xs = length* (*flist xs*)
  **apply** (*simp add:fcard-def*)
  **apply** (*fold flist-set*)
  **apply** (*unfold distinct-card*[*OF flist-props(2)*])
  **apply** (*rule refl*)
  **done**

**lemma** *flist-nth*:
  *i < fcard vs* ⟹ *flist vs ! i* |∈| *vs*
  **apply** (*simp add*: *fmember-def flist-def fcard-def*)
  **apply** (*metis fcard.rep-eq fcard-flist finset.rep-eq flist-def flist-inv nth-mem*)
  **done**

**definition** *fmax* :: *'a fset ⇒ 'a* **where**
*fmax xs = (if (xs = {|\|}) then undefined else last (flist xs))*

**end**

**definition** *flists* :: *'a fset ⇒ 'a list set* **where**
*flists A = {xs. distinct xs ∧ finset xs = A}*

**lemma** *flists-nonempty*: $\exists$ *xs. xs* $\in$ *flists A*
  **apply** (*simp add*: *flists-def*)
  **apply** (*metis Abs-fset-cases Abs-fset-inverse finite-distinct-list finite-fset finset.rep-eq*)
  **done**

**lemma** *flists-elem-uniq*: $\llbracket$ *x* $\in$ *flists A*; *x* $\in$ *flists B* $\rrbracket$ $\Longrightarrow$ *A = B*
  **by** (*simp add*: *flists-def*)

**definition** *flist-arb* :: ′*a fset* $\Rightarrow$ ′*a list* **where**
*flist-arb A = (SOME xs. xs* $\in$ *flists A)*

**lemma** *flist-arb-distinct* [*simp*]: *distinct* (*flist-arb A*)
  **by** (*metis* (*mono-tags*) *flist-arb-def flists-def flists-nonempty mem-Collect-eq someI-ex*)

**lemma** *flist-arb-inv* [*simp*]: *finset* (*flist-arb A*) = *A*
  **by** (*metis* (*mono-tags*) *flist-arb-def flists-def flists-nonempty mem-Collect-eq someI-ex*)

**lemma** *flist-arb-inj*:
  *inj flist-arb*
  **by** (*metis flist-arb-inv injI*)

**lemma** *flist-arb-lists*: *flist-arb* ‘ *Fow A* $\subseteq$ *lists A*
  **apply** (*auto*)
  **using** *Fow-def finset.rep-eq* **apply** *fastforce*
  **done**

**lemma** *countable-Fow*:
  **fixes** *A* :: ′*a set*
  **assumes** *countable A*
  **shows** *countable* (*Fow A*)
**proof** −
  **from** *assms* **obtain** *to-nat-list* :: ′*a list* $\Rightarrow$ *nat* **where** *inj-on to-nat-list* (*lists A*)
    **by** *blast*
  **thus** *?thesis*
    **apply** (*simp add*: *countable-def*)
    **apply** (*rule-tac x=to-nat-list* $\circ$ *flist-arb* **in** *exI*)
    **apply** (*rule comp-inj-on*)
     **apply** (*metis flist-arb-inv inj-on-def*)
    **apply** (*simp add*: *flist-arb-lists subset-inj-on*)
    **done**
**qed**

**definition** *flub* :: ′*a fset set* $\Rightarrow$ ′*a fset* $\Rightarrow$ ′*a fset* **where**
*flub A  t = (if* ($\forall$ *a*$\in$*A. a* $|\subseteq|$ *t) then Abs-fset* ($\bigcup$ *x*$\in$*A.* $\langle x \rangle_f$) *else t)*

**lemma** *finite-Union-subsets*:
  $\llbracket$ $\forall$ *a* $\in$ *A. a* $\subseteq$ *b; finite b* $\rrbracket$ $\Longrightarrow$ *finite* ($\bigcup A$)
  **by** (*metis Sup-le-iff finite-subset*)

**lemma** *finite-UN-subsets*:
  $\llbracket$ $\forall$ *a* $\in$ *A. B a* $\subseteq$ *b; finite b* $\rrbracket$ $\Longrightarrow$ *finite* ($\bigcup$ *a*$\in$*A. B a*)
  **by** (*metis UN-subset-iff finite-subset*)

**lemma** *flub-rep-eq*:
  $\langle$*flub A t*$\rangle_f$ = (*if* ($\forall$ *a*$\in$*A. a* $|\subseteq|$ *t) then* ($\bigcup$ *x*$\in$*A.* $\langle x \rangle_f$) *else* $\langle t \rangle_f$)

**apply** (*subgoal-tac* (*if* ($\forall$ $a \in A$. $a$ $|\subseteq|$ $t$) *then* ($\bigcup x \in A$. $\langle x \rangle_f$) *else* $\langle t \rangle_f$) $\in \{x.\ finite\ x\}$)
  **apply** (*auto simp add:flub-def*)
 **apply** (*rule finite-UN-subsets[of - - $\langle t \rangle_f$]*)
  **apply** (*auto*)
 **done**

**definition** *fglb* :: *$'a$ fset set $\Rightarrow$ $'a$ fset $\Rightarrow$ $'a$ fset* **where**
*fglb A t = (if (A = {}) then t else Abs-fset ($\bigcap x \in A$. $\langle x \rangle_f$))*

**lemma** *fglb-rep-eq*:
  $\langle fglb\ A\ t \rangle_f$ = (*if* ($A$ = {}) *then* $\langle t \rangle_f$ *else* ($\bigcap x \in A$. $\langle x \rangle_f$))
  **apply** (*subgoal-tac* (*if* ($A$ = {}) *then* $\langle t \rangle_f$ *else* ($\bigcap x \in A$. $\langle x \rangle_f$)) $\in \{x.\ finite\ x\}$)
   **apply** (*metis Abs-fset-inverse fglb-def*)
  **apply** (*auto*)
  **apply** (*metis finite-INT finite-fset*)
  **done**

**lemma** *FinPow-rep-eq* [*simp*]:
  *fset (FinPow xs) = {ys. ys $|\subseteq|$ xs}*
  **apply** (*subgoal-tac finite (Abs-fset ' Pow $\langle xs \rangle_f$)*)
   **apply** (*auto simp add: fmember-def FinPow-def*)
   **apply** (*rename-tac $x'$ $y'$*)
   **apply** (*subgoal-tac finite $x'$*)
    **apply** (*auto*)
   **apply** (*metis finite-fset finite-subset*)
  **apply** (*metis (full-types) Pow-iff fset-inverse imageI less-eq-fset.rep-eq*)
  **done**

**lemma** *FUnion-rep-eq* [*simp*]:
  $\langle \bigcup_f xs \rangle_f$ = ($\bigcup x \in \langle xs \rangle_f$. $\langle x \rangle_f$)
  **by** (*simp add:FUnion-def*)

**lemma** *FInter-rep-eq* [*simp*]:
  $xs \neq \{\!\!\{\}\!\!\}$ $\Longrightarrow$ $\langle \bigcap_f xs \rangle_f$ = ($\bigcap x \in \langle xs \rangle_f$. $\langle x \rangle_f$)
  **apply** (*simp add:FInter-def*)
  **apply** (*subgoal-tac finite ($\bigcap x \in \langle xs \rangle_f$. $\langle x \rangle_f$)*)
   **apply** (*simp*)
  **apply** (*metis (poly-guards-query) bot-fset.rep-eq fglb-rep-eq finite-fset fset-inverse*)
  **done**

**lemma** *FUnion-empty* [*simp*]:
  $\bigcup_f \{\!\!\{\}\!\!\} = \{\!\!\{\}\!\!\}$
  **by** (*auto simp add:FUnion-def fmember-def*)

**lemma** *FinPow-member* [*simp*]:
  *xs $|\in|$ FinPow xs*
  **by** (*auto simp add:fmember-def*)

**lemma** *FUnion-FinPow* [*simp*]:
  $\bigcup_f (FinPow\ x) = x$
  **by** (*auto simp add:fmember-def less-eq-fset-def*)

**lemma** *Fow-mem* [*iff*]: *$x \in Fow\ A \longleftrightarrow \langle x \rangle_f \subseteq A$*
  **by** (*auto simp add:Fow-def*)

**lemma** *Fow-UNIV* [*simp*]: *Fow UNIV = UNIV*
  **by** (*simp add:Fow-def*)


**lift-definition** *FMax* :: (*'a::linorder*) *fset* ⇒ *'a* **is** *Max* **.**


**end**


# 5   Countable Sets: Extra functions and properties

**theory** *Countable-Set-Extra*
**imports**
  *HOL−Library.Countable-Set-Type*
  *Sequence*
  *FSet-Extra*
**begin**


## 5.1   Extra syntax

**notation** *cempty* ({}$_c$)
**notation** *cin* (**infix** ∈$_c$ *50*)
**notation** *cUn* (**infixl** ∪$_c$ *65*)
**notation** *cInt* (**infixl** ∩$_c$ *70*)
**notation** *cDiff* (**infixl** −$_c$ *65*)
**notation** *cUnion* (⋃$_c$- [*900*] *900*)
**notation** *cimage* (**infixr** '$_c$ *90*)


**abbreviation** *csubseteq* :: *'a cset* ⇒ *'a cset* ⇒ *bool* ((-/ ⊆$_c$ -) [*51*, *51*] *50*)
**where** *A* ⊆$_c$ *B* ≡ *A* ≤ *B*


**abbreviation** *csubset* :: *'a cset* ⇒ *'a cset* ⇒ *bool* ((-/ ⊂$_c$ -) [*51*, *51*] *50*)
**where** *A* ⊂$_c$ *B* ≡ *A* < *B*


## 5.2   Countable set functions

**setup-lifting** *type-definition-cset*


**lift-definition** *cnin* :: *'a* ⇒ *'a cset* ⇒ *bool* (**infix** ∉$_c$ *50*) **is** (∉) **.**


**definition** *cBall* :: *'a cset* ⇒ (*'a* ⇒ *bool*) ⇒ *bool* **where**
*cBall A P* = (∀ *x. x* ∈$_c$ *A* ⟶ *P x*)


**definition** *cBex* :: *'a cset* ⇒ (*'a* ⇒ *bool*) ⇒ *bool* **where**
*cBex A P* = (∃ *x. x* ∈$_c$ *A* ⟶ *P x*)


**declare** *cBall-def* [*mono,simp*]
**declare** *cBex-def* [*mono,simp*]


**syntax**
  *-cBall* :: *pttrn* => *'a cset* => *bool* => *bool* ((*3*∀ -∈$_c$-./ -) [*0*, *0*, *10*] *10*)
  *-cBex* :: *pttrn* => *'a cset* => *bool* => *bool* ((*3*∃ -∈$_c$-./ -) [*0*, *0*, *10*] *10*)


**translations**
  ∀ *x*∈$_c$*A. P* == *CONST cBall A* (%*x. P*)
  ∃ *x*∈$_c$*A. P* == *CONST cBex  A* (%*x. P*)

**definition** *cset-Collect* :: $(\,'a \Rightarrow bool) \Rightarrow\, 'a \ cset$ **where**
*cset-Collect* = (*acset o Collect*)

**lift-definition** *cset-Coll* :: $'a \ cset \Rightarrow (\,'a \Rightarrow bool) \Rightarrow\, 'a \ cset$ **is** $\lambda \ A \ P.\ \{x \in A.\ P\ x\}$
  **by** (*auto*)

**lemma** *cset-Coll-equiv*: *cset-Coll A P = cset-Collect* $(\lambda \ x.\ x \in_c A \wedge P\ x)$
  **by** (*simp add:cset-Collect-def cset-Coll-def cin-def*)

**declare** *cset-Collect-def* [*simp*]

**syntax**
  *-cColl* :: *pttrn => bool =>* $'a \ cset$ $((1\{-./\ -\}_c))$

**translations**
  $\{x\ .\ P\}_c \rightleftharpoons (CONST\ cset\text{-}Collect)\ (\lambda \ x\ .\ P)$

**syntax** (*xsymbols*)
  *-cCollect* :: *pttrn =>* $'a \ cset$ *=> bool =>* $'a \ cset$    $((1\{-\ \in_c/\ -./\ -\}_c))$
**translations**
  $\{x \in_c A.\ P\}_c => CONST\ cset\text{-}Coll\ A\ (\lambda \ x.\ P)$

**lemma** *cset-CollectI*: $P\ (a :: 'a\text{::}countable) \Longrightarrow a \in_c \{x.\ P\ x\}_c$
  **by** (*simp add: cin-def*)

**lemma** *cset-CollI*: $⟦\ a \in_c A;\ P\ a\ ⟧ \Longrightarrow a \in_c \{x \in_c A.\ P\ x\}_c$
  **by** (*simp add: cin.rep-eq cset-Coll.rep-eq*)

**lemma** *cset-CollectD*: $(a :: 'a\text{::}countable) \in_c \{x.\ P\ x\}_c \Longrightarrow P\ a$
  **by** (*simp add: cin-def*)

**lemma** *cset-Collect-cong*: $(\bigwedge x.\ P\ x = Q\ x) ==> \{x.\ P\ x\}_c = \{x.\ Q\ x\}_c$
  **by** *simp*

Avoid eta-contraction for robust pretty-printing.

**print-translation** ‹
[*Syntax-Trans.preserve-binder-abs-tr′*
  @{*const-syntax cset-Collect*} @{*syntax-const -cColl*}]
›

**lift-definition** *cset-set* :: $'a \ list \Rightarrow 'a \ cset$ **is** *set*
  **using** *countable-finite* **by** *blast*

**lemma** *countable-finite-power*:
  $countable(A) \Longrightarrow countable\ \{B.\ B \subseteq A \wedge finite(B)\}$
  **by** (*metis Collect-conj-eq Int-commute countable-Collect-finite-subset*)

**lift-definition** *cInter* :: $'a \ cset\ cset \Rightarrow 'a \ cset$  $(\bigcap_c\text{-}\ [900]\ 900)$
  **is** $\lambda A.\ if\ A = \{\}\ then\ \{\}\ else\ \bigcap\ A$
  **using** *countable-INT* [*of - - id*] **by** *auto*

**abbreviation** (*input*) *cINTER* :: $'a \ cset \Rightarrow (\,'a \Rightarrow 'b\ cset) \Rightarrow 'b\ cset$
  **where** *cINTER A f* $\equiv$ *cInter* (*cimage f A*)

**lift-definition** *cfinite* :: $'a \ cset \Rightarrow bool$ **is** *finite* .

**lift-definition** *cInfinite* :: *'a cset ⇒ bool* **is** *infinite* .
**lift-definition** *clist* :: *'a::linorder cset ⇒ 'a list* **is** *sorted-list-of-set* .
**lift-definition** *ccard* :: *'a cset ⇒ nat* **is** *card* .
**lift-definition** *cPow* :: *'a cset ⇒ 'a cset cset* **is** $\lambda$ *A.* $\{B.\ B \subseteq_c A \wedge cfinite(B)\}$
**proof** −
  **fix** *A*
  **have** $\{B :: 'a\ cset.\ B \subseteq_c A \wedge cfinite\ B\} = acset\ `\ \{B :: 'a\ set.\ B \subseteq rcset\ A \wedge finite\ B\}$
    **apply** (*auto simp add: cfinite.rep-eq cin-def less-eq-cset-def countable-finite*)
    **using** *image-iff* **apply** *fastforce*
    **done**

  **moreover have** *countable* $\{B :: 'a\ set.\ B \subseteq rcset\ A \wedge finite\ B\}$
    **by** (*auto intro: countable-finite-power*)

  **ultimately show** *countable* $\{B.\ B \subseteq_c A \wedge cfinite\ B\}$
    **by** *simp*
**qed**

**definition** *CCollect* :: *('a ⇒ bool option) ⇒ 'a cset option* **where**
*CCollect p = (if (None ∉ range p) then Some (cset-Collect (the ∘ p)) else None)*

**definition** *cset-mapM* :: *'a option cset ⇒ 'a cset option* **where**
*cset-mapM A = (if (None* $\in_c$ *A) then None else Some (the* $`_c$ *A))*

**lemma** *cset-mapM-Some-image* [*simp*]:
  *cset-mapM (cimage Some A) = Some A*
  **apply** (*auto simp add: cset-mapM-def*)
  **apply** (*metis cimage-cinsert cinsertI1 option.sel set-cinsert*)
  **done**

**definition** *CCollect-ext* :: *('a ⇒ 'b option) ⇒ ('a ⇒ bool option) ⇒ 'b cset option* **where**
*CCollect-ext f p = do { xs ← CCollect p; cset-mapM (f* $`_c$ *xs) }*

**lemma** *the-Some-image* [*simp*]:
  *the ` Some ` xs = xs*
  **by** (*auto simp add:image-iff*)

**lemma** *CCollect-ext-Some* [*simp*]:
  *CCollect-ext Some xs = CCollect xs*
  **apply** (*case-tac CCollect xs*)
   **apply** (*auto simp add:CCollect-ext-def*)
  **done**

**lift-definition** *list-of-cset* :: *'a :: linorder cset ⇒ 'a list* **is** *sorted-list-of-set* .

**lift-definition** *fset-cset* :: *'a fset ⇒ 'a cset* **is** *id*
  **using** *uncountable-infinite* **by** *auto*

**definition** *cset-count* :: *'a cset ⇒ 'a ⇒ nat* **where**
*cset-count A =*
  *(if (finite (rcset A))*
  *then (SOME f::'a⇒nat. inj-on f (rcset A))*
  *else (SOME f::'a⇒nat. bij-betw f (rcset A) UNIV))*

**lemma** *cset-count-inj-seq*:

*inj-on* (*cset-count A*) (*rcset A*)
**proof** (*cases finite* (*rcset A*))
  **case** *True* **note** *fin = this*
  **obtain** *count ::* $'a \Rightarrow nat$ **where** *count-inj: inj-on count* (*rcset A*)
    **by** (*metis countable-def mem-Collect-eq rcset*)
  **with** *fin* **show** *?thesis*
    **by** (*metis* (*poly-guards-query*) *cset-count-def someI-ex*)
**next**
  **case** *False* **note** *inf = this*
  **obtain** *count ::* $'a \Rightarrow nat$ **where** *count-bij: bij-betw count* (*rcset A*) *UNIV*
    **by** (*metis countableE-infinite inf mem-Collect-eq rcset*)
  **with** *inf* **have** *bij-betw* (*cset-count A*) (*rcset A*) *UNIV*
    **by** (*metis* (*poly-guards-query*) *cset-count-def someI-ex*)
  **thus** *?thesis*
    **by** (*metis bij-betw-imp-inj-on*)
**qed**

**lemma** *cset-count-infinite-bij*:
  **assumes** *infinite* (*rcset A*)
  **shows** *bij-betw* (*cset-count A*) (*rcset A*) *UNIV*
**proof** −
  **from** *assms* **obtain** *count ::* $'a \Rightarrow nat$ **where** *count-bij: bij-betw count* (*rcset A*) *UNIV*
    **by** (*metis countableE-infinite mem-Collect-eq rcset*)
  **with** *assms* **show** *?thesis*
    **by** (*metis* (*poly-guards-query*) *cset-count-def someI-ex*)
**qed**

**definition** *cset-seq ::* $'a\ cset \Rightarrow (nat \rightharpoonup 'a)$ **where**
*cset-seq A i* = (*if* (*i* ∈ *range* (*cset-count A*) ∧ *inv-into* (*rcset A*) (*cset-count A*) *i* ∈$_c$ *A*)
        *then Some* (*inv-into* (*rcset A*) (*cset-count A*) *i*)
        *else None*)

**lemma** *cset-seq-ran: ran* (*cset-seq A*) = *rcset*(*A*)
  **apply** (*auto simp add: ran-def cset-seq-def cin.rep-eq*)
  **apply** (*metis cset-count-inj-seq inv-into-f-f rangeI*)
  **done**

**lemma** *cset-seq-inj: inj cset-seq*
**proof** (*rule injI*)
  **fix** *A B ::* $'a\ cset$
  **assume** *cset-seq A = cset-seq B*
  **thus** *A = B*
    **by** (*metis cset-seq-ran rcset-inverse*)
**qed**

**lift-definition** *cset2seq ::* $'a\ cset \Rightarrow 'a\ seq$
**is** ($\lambda$ *A i. if* (*i* ∈ *cset-count A* ' *rcset A*) *then inv-into* (*rcset A*) (*cset-count A*) *i else* (*SOME x. x* ∈$_c$
*A*)) **.**

**lemma** *range-cset2seq*:
  *A* ≠ {}$_c$ ⟹ *range* (*Rep-seq* (*cset2seq A*)) = *rcset A*
  **by** (*force intro: someI2 simp add: cset2seq.rep-eq cset-count-inj-seq bot-cset.rep-eq cin.rep-eq*)

**lemma** *infinite-cset-count-surj: infinite* (*rcset A*) ⟹ *surj* (*cset-count A*)
  **using** *bij-betw-imp-surj cset-count-infinite-bij* **by** *auto*

**lemma** *cset2seq-inj*:
  *inj-on cset2seq* {*A. A* ≠ {}$_c$}
  **apply** (*rule inj-onI*)
  **apply** (*simp*)
  **apply** (*metis range-cset2seq rcset-inject*)
  **done**

**lift-definition** *nat-seq2set* :: *nat seq* ⇒ *nat set* **is**
λ *f. prod-encode* ' {(*x, f x*) | *x. True*} **.**

**lemma** *inj-nat-seq2set*: *inj nat-seq2set*
**proof** (*rule injI, transfer*)
  **fix** *f g*
  **assume** *prod-encode* ' {(*x, f x*) |*x. True*} = *prod-encode* ' {(*x, g x*) |*x. True*}
  **hence** {(*x, f x*) |*x. True*} = {(*x, g x*) |*x. True*}
    **by** (*simp add: inj-image-eq-iff* [*OF inj-prod-encode*])
  **thus** *f* = *g*
    **by** (*auto simp add: set-eq-iff*)
**qed**

**lift-definition** *bit-seq-of-nat-set* :: *nat set* ⇒ *bool seq*
**is** λ *A i. i* ∈ *A* **.**

**lemma** *bit-seq-of-nat-set-inj*: *inj bit-seq-of-nat-set*
  **apply** (*rule injI*)
  **apply** *transfer*
  **apply** (*auto simp add: fun-eq-iff*)
  **done**

**lemma** *bit-seq-of-nat-cset-bij*: *bij bit-seq-of-nat-set*
  **apply** (*rule bijI*)
   **apply** (*fact bit-seq-of-nat-set-inj*)
  **apply** *transfer*
  **apply** (*rule surjI*)
  **apply** *auto*
  **done**

This function is a partial injection from countable sets of natural sets to natural sets. When used with the Schroeder-Bernstein theorem, it can be used to conjure a total bijection between these two types.

**definition** *nat-set-cset-collapse* :: *nat set cset* ⇒ *nat set* **where**
*nat-set-cset-collapse* = *inv bit-seq-of-nat-set* ∘ *seq-inj* ∘ *cset2seq* ∘ (λ *A.* (*bit-seq-of-nat-set* '$_c$ *A*))

**lemma** *nat-set-cset-collapse-inj*: *inj-on nat-set-cset-collapse* {*A. A* ≠ {}$_c$}
**proof** −
  **have** ('$_c$) *bit-seq-of-nat-set* ' {*A. A* ≠ {}$_c$} ⊆ {*A. A* ≠ {}$_c$}
    **by** (*auto simp add:cimage.rep-eq*)
  **thus** *?thesis*
    **apply** (*simp add: nat-set-cset-collapse-def*)
    **apply** (*rule comp-inj-on*)
     **apply** (*meson bit-seq-of-nat-set-inj cset.inj-map injD inj-onI*)
    **apply** (*rule comp-inj-on*)
     **apply** (*metis cset2seq-inj subset-inj-on*)
    **apply** (*rule comp-inj-on*)

  **apply** (*rule subset-inj-on*)
   **apply** (*rule seq-inj*)
   **apply** (*simp*)
   **apply** (*meson UNIV-I bij-imp-bij-inv bij-is-inj bit-seq-of-nat-cset-bij subsetI subset-inj-on*)
   **done**
**qed**

**lemma** *inj-csingle*:
 *inj csingle*
 **by** (*auto intro*: *injI simp add*: *cinsert-def bot-cset.rep-eq*)

**lemma** *range-csingle*:
 *range csingle* $\subseteq \{A.\ A \neq \{\}_c\}$
 **by** (*auto*)

**lift-definition** *csets* :: $'a\ set \Rightarrow 'a\ cset\ set$ **is**
$\lambda\ A.\ \{B.\ B \subseteq A \wedge countable\ B\}$ **by** *auto*

**lemma** *csets-finite*: *finite A* $\Longrightarrow$ *finite* (*csets A*)
 **by** (*auto simp add*: *csets-def*)

**lemma** *csets-infinite*: *infinite A* $\Longrightarrow$ *infinite* (*csets A*)
 **by** (*auto simp add*: *csets-def*, *metis csets.abs-eq csets.rep-eq finite-countable-subset finite-imageI*)

**lemma** *csets-UNIV*:
 *csets* (*UNIV* :: $'a\ set$) = (*UNIV* :: $'a\ cset\ set$)
 **by** (*auto simp add*: *csets-def*, *metis image-iff rcset rcset-inverse*)

**lemma** *infinite-nempty-cset*:
 **assumes** *infinite* (*UNIV* :: $'a\ set$)
 **shows** *infinite* ($\{A.\ A \neq \{\}_c\}$ :: $'a\ cset\ set$)
**proof** $-$
 **have** *infinite* (*UNIV* :: $'a\ cset\ set$)
  **by** (*metis assms csets-UNIV csets-infinite*)
 **hence** *infinite* ((*UNIV* :: $'a\ cset\ set$) $- \{\{\}_c\}$)
  **by** (*rule infinite-remove*)
 **thus** *?thesis*
  **by** (*auto*)
**qed**

**lemma** *nat-set-cset-partial-bij*:
 **obtains** $f$ :: *nat set cset* $\Rightarrow$ *nat set* **where** *bij-betw f* $\{A.\ A \neq \{\}_c\}$ *UNIV*
  **using** *Schroeder-Bernstein*[*OF nat-set-cset-collapse-inj, of UNIV csingle, simplified, OF inj-csingle range-csingle*]
 **by** (*auto*)

**lemma** *nat-set-cset-bij*:
 **obtains** $f$ :: *nat set cset* $\Rightarrow$ *nat set* **where** *bij f*
**proof** $-$
 **obtain** $g$ :: *nat set cset* $\Rightarrow$ *nat set* **where** *bij-betw g* $\{A.\ A \neq \{\}_c\}$ *UNIV*
  **using** *nat-set-cset-partial-bij* **by** *blast*
 **moreover obtain** $h$ :: *nat set cset* $\Rightarrow$ *nat set cset* **where** *bij-betw h UNIV* $\{A.\ A \neq \{\}_c\}$
 **proof** $-$
  **have** *infinite* (*UNIV* :: *nat set cset set*)
   **by** (*metis Finite-Set.finite-set csets-UNIV csets-infinite infinite-UNIV-char-0*)

**then obtain** $h'$ :: *nat set cset* $\Rightarrow$ *nat set cset* **where** *bij-betw h' UNIV* (*UNIV* $-$ {{}$_c$})
  **using** *infinite-imp-bij-betw*[*of UNIV* :: *nat set cset set* {}$_c$] **by** *auto*
 **moreover have** (*UNIV* :: *nat set cset set*) $-$ {{}$_c$} = {$A$. $A \neq$ {}$_c$}
  **by** (*auto*)
 **ultimately show** *?thesis*
  **using** *that* **by** (*auto*)
**qed**
**ultimately have** *bij* ($g \circ h$)
 **using** *bij-betw-trans* **by** *blast*
**with** *that* **show** *?thesis*
 **by** (*auto*)
**qed**

**definition** *nat-set-cset-bij* = (*SOME f* :: *nat set cset* $\Rightarrow$ *nat set. bij f*)

**lemma** *bij-nat-set-cset-bij*:
 *bij nat-set-cset-bij*
 **by** (*metis nat-set-cset-bij nat-set-cset-bij-def someI-ex*)

**lemma** *inj-on-image-csets*:
 *inj-on f A* $\Longrightarrow$ *inj-on* (('$_c$) *f*) (*csets A*)
 **by** (*fastforce simp add*: *inj-on-def cimage-def cin-def csets-def*)

**lemma** *image-csets-surj*:
 ⟦ *inj-on f A*; *f ' A* = *B* ⟧ $\Longrightarrow$ ('$_c$) *f ' csets A* = *csets B*
 **apply** (*auto simp add*: *cimage-def csets-def image-mono map-fun-def*)
 **apply** (*simp add*: *image-comp*)
 **apply** (*auto simp add*: *image-Collect*)
 **apply** (*erule subset-imageE*)
 **using** *countable-image-inj-on subset-inj-on* **by** *blast*

**lemma** *bij-betw-image-csets*:
 *bij-betw f A B* $\Longrightarrow$ *bij-betw* (('$_c$) *f*) (*csets A*) (*csets B*)
 **by** (*simp add*: *bij-betw-def inj-on-image-csets image-csets-surj*)
**end**


# 6    Extra Relational Definitions and Theorems

**theory** *Relation-Extra*
 **imports** *HOL*−*Library.FuncSet*
**begin**

We set up some nice syntax for heterogeneous relations at the type level

**syntax**
 *-rel-type* :: *type* $\Rightarrow$ *type* $\Rightarrow$ *type* (**infixr** $\leftrightarrow$ *0*)

**translations**
 (*type*) $'a \leftrightarrow {}'b$ == (*type*) ($'a \times {}'b$) *set*


## 6.1    Relational Function Operations

**definition** *rel-apply* :: ($'a \leftrightarrow {}'b$) $\Rightarrow$ $'a$ $\Rightarrow$ $'b$ (*-'(-')$_r$* [*999,0*] *999*) **where**
*rel-apply R x* = (*if x* $\in$ *Domain*(*R*) *then THE y*. (*x, y*) $\in$ *R else undefined*)

**definition** *rel-domres* :: $'a$ *set* $\Rightarrow$ ($'a \leftrightarrow {}'b$) $\Rightarrow$ $'a \leftrightarrow {}'b$ (**infixr** $\lhd_r$ *85*) **where**

*rel-domres A R = {(k, v) ∈ R. k ∈ A}*

**definition** *rel-override* :: $('a \leftrightarrow 'b) \Rightarrow ('a \leftrightarrow 'b) \Rightarrow 'a \leftrightarrow 'b$ (**infixl** $+_r$ *65*) **where**
*rel-override R S = (− Domain S)* $\lhd_r$ *R ∪ S*

**definition** *rel-update* :: $('a \leftrightarrow 'b) \Rightarrow 'a \Rightarrow 'b \Rightarrow 'a \leftrightarrow 'b$ **where**
*rel-update R k v = rel-override R {(k, v)}*

## 6.2   Domain Restriction

**lemma** *Domain-rel-domres* [*simp*]: *Domain* $(A \lhd_r R) = A \cap Domain(R)$
  **by** (*auto simp add*: *rel-domres-def*)

**lemma** *rel-domres-empty* [*simp*]: {} $\lhd_r$ *R = {}*
  **by** (*simp add*: *rel-domres-def*)

**lemma** *rel-domres-UNIV* [*simp*]: *UNIV* $\lhd_r$ *R = R*
  **by** (*simp add*: *rel-domres-def*)

**lemma** *rel-domres-nil* [*simp*]: *A* $\lhd_r$ *{} = {}*
  **by** (*simp add*: *rel-domres-def*)

**lemma** *rel-domres-inter* [*simp*]: *A* $\lhd_r$ *B* $\lhd_r$ *R = (A ∩ B)* $\lhd_r$ *R*
  **by** (*auto simp add*: *rel-domres-def*)

## 6.3   Relational Override

**interpretation** *rel-override-monoid*: *monoid-add* $(+_r)$ *{}*
  **by** (*unfold-locales, simp-all add*: *rel-override-def*, *auto simp add*: *rel-domres-def*)

**lemma** *Domain-rel-override* [*simp*]: *Domain* $(R +_r S) = Domain(R) ∪ Domain(S)$
  **by** (*auto simp add*: *rel-override-def Domain-Un-eq*)

**lemma** *Range-rel-override*: *Range*$(R +_r S) \subseteq Range(R) ∪ Range(S)$
  **by** (*auto simp add*: *rel-override-def rel-domres-def*)

## 6.4   Functional Relations

**definition** *functional* :: $('a \leftrightarrow 'b) \Rightarrow bool$ **where**
*functional g = inj-on fst g*

**lemma** *functional-algebraic*: *functional R* $\longleftrightarrow$ $R^{-1}$ *O R* $\subseteq$ *Id*
  **apply** (*auto simp add*: *functional-def subset-iff relcomp-unfold*)
  **using** *inj-on-eq-iff* **apply** *fastforce*
  **apply** (*metis inj-onI surjective-pairing*)
  **done**

**lemma** *functional-determine*: ⟦ *functional R*; $(x, y) ∈ R$; $(x, z) ∈ R$ ⟧ $\Longrightarrow$ *y = z*
  **by** (*auto simp add*: *functional-algebraic subset-iff relcomp-unfold*)

**lemma** *functional-apply*:
  **assumes** *functional R* $(x, y) ∈ R$
  **shows** $R(x)_r = y$
  **by** (*metis* (*no-types, lifting*) *Domain.intros DomainE assms(1) assms(2) functional-determine rel-apply-def*
*theI-unique*)

**lemma** *functional-elem*:
  **assumes** *functional R* $x \in Domain(R)$
  **shows** $(x, R(x)_r) \in R$
  **using** *assms(1) assms(2) functional-apply* **by** *fastforce*

**lemma** *functional-empty* [*simp*]: *functional* {}
  **by** (*simp add*: *functional-def*)

**lemma** *functional-override* [*intro*]: ⟦ *functional R*; *functional S* ⟧ $\Longrightarrow$ *functional* $(R +_r S)$
  **by** (*auto simp add*: *functional-algebraic rel-override-def rel-domres-def*)

**definition** *functional-list* :: $('a \times 'b)$ *list* $\Rightarrow$ *bool* **where**
*functional-list xs* = ($\forall$ *x y z. ListMem* $(x,y)$ *xs* $\land$ *ListMem* $(x,z)$ *xs* $\longrightarrow$ $y = z$)

**lemma** *functional-insert* [*simp*]: *functional* (*insert* $(x,y)$ *g*) $\longleftrightarrow$ ($g$''$\{x\} \subseteq \{y\} \land$ *functional g*)
  **by** (*auto simp add:functional-def inj-on-def image-def*)

**lemma** *functional-list-nil*[*simp*]: *functional-list* []
  **by** (*simp add:functional-list-def ListMem-iff*)

**lemma** *functional-list*: *functional-list xs* $\longleftrightarrow$ *functional* (*set xs*)
  **apply** (*induct xs*)
   **apply** (*simp add:functional-def*)
  **apply** (*simp add:functional-def functional-list-def ListMem-iff*)
  **apply** (*safe*)
        **apply** (*force*)
       **apply** (*force*)
      **apply** (*force*)
     **apply** (*force*)
    **apply** (*force*)
   **apply** (*force*)
   **apply** (*force*)
  **apply** (*force*)
  **done**

**definition** *fun-rel* :: $('a \Rightarrow 'b) \Rightarrow ('a \leftrightarrow 'b)$ **where**
*fun-rel f* = $\{(x, y).\ y = f\ x\}$

**lemma** *functional-fun-rel*: *functional* (*fun-rel f*)
  **by** (*simp add*: *fun-rel-def functional-def*)
    (*metis* (*mono-tags, lifting*) *Product-Type.Collect-case-prodD inj-onI prod.expand*)

**lemma** *rel-apply-fun* [*simp*]: (*fun-rel f*)$(x)_r$ = $f\ x$
  **by** (*simp add*: *fun-rel-def rel-apply-def*)

## 6.5   Left-Total Relations

**definition** *left-totalr-on* :: $'a\ set \Rightarrow ('a \leftrightarrow 'b) \Rightarrow bool$ **where**
*left-totalr-on A R* $\longleftrightarrow$ ($\forall x \in A.\ \exists y.\ (x, y) \in R$)

**abbreviation** *left-totalr R* $\equiv$ *left-totalr-on UNIV R*

**lemma** *left-totalr-algebraic*: *left-totalr R* $\longleftrightarrow$ $Id \subseteq R\ O\ R^{-1}$
  **by** (*auto simp add*: *left-totalr-on-def*)

**lemma** *left-totalr-fun-rel*: *left-totalr* (*fun-rel f*)

**by** (*simp add*: *left-totalr-on-def fun-rel-def*)

## 6.6 Relation Sets

**definition** *rel-typed* :: $'a\ set \Rightarrow 'b\ set \Rightarrow ('a \leftrightarrow 'b)\ set$ (**infixr** $\leftrightarrow_r$ *55*) **where**
*rel-typed A B* = $\{R.\ Domain(R) \subseteq A \land Range(R) \subseteq B\}$

**lemma** *rel-typed-intro*: $[\![\ Domain(R) \subseteq A;\ Range(R) \subseteq B\ ]\!] \Longrightarrow R \in A \leftrightarrow_r B$
  **by** (*simp add*: *rel-typed-def*)

**definition** *rel-pfun* :: $'a\ set \Rightarrow 'b\ set \Rightarrow ('a \leftrightarrow 'b)\ set$ (**infixr** $\rightharpoonup_r$ *55*) **where**
*rel-pfun A B* = $\{R.\ R \in A \leftrightarrow_r B \land functional\ R\}$

**lemma** *rel-pfun-intro*: $[\![\ R \in A \leftrightarrow_r B;\ functional\ R\ ]\!] \Longrightarrow R \in A \rightharpoonup_r B$
  **by** (*simp add*: *rel-pfun-def*)

**definition** *rel-tfun* :: $'a\ set \Rightarrow 'b\ set \Rightarrow ('a \leftrightarrow 'b)\ set$ (**infixr** $\rightarrow_r$ *55*) **where**
*rel-tfun A B* = $\{R.\ R \in A \rightharpoonup_r B \land left\text{-}totalr\ R\}$

**definition** *rel-ffun* :: $'a\ set \Rightarrow 'b\ set \Rightarrow ('a \leftrightarrow 'b)\ set$ **where**
*rel-ffun A B* = $\{R.\ R \in A \rightharpoonup_r B \land finite(Domain\ R)\}$

## 6.7 Closure Properties

These can be seen as typing rules for relational functions

**named-theorems** *rclos*

**lemma** *rel-ffun-is-pfun* [*rclos*]: $R \in rel\text{-}ffun\ A\ B \Longrightarrow R \in A \rightharpoonup_r B$
  **by** (*simp add*: *rel-ffun-def*)

**lemma** *rel-tfun-is-pfun* [*rclos*]: $R \in A \rightarrow_r B \Longrightarrow R \in A \rightharpoonup_r B$
  **by** (*simp add*: *rel-tfun-def*)

**lemma** *rel-pfun-is-typed* [*rclos*]: $R \in A \rightharpoonup_r B \Longrightarrow R \in A \leftrightarrow_r B$
  **by** (*simp add*: *rel-pfun-def*)

**lemma** *rel-ffun-empty* [*rclos*]: $\{\} \in rel\text{-}ffun\ A\ B$
  **by** (*simp add*: *rel-ffun-def rel-pfun-def rel-typed-def*)

**lemma** *rel-pfun-apply* [*rclos*]: $[\![\ x \in Domain(R);\ R \in A \rightharpoonup_r B\ ]\!] \Longrightarrow R(x)_r \in B$
  **using** *functional-apply* **by** (*fastforce simp add*: *rel-pfun-def rel-typed-def*)

**lemma** *rel-tfun-apply* [*rclos*]: $[\![\ x \in A;\ R \in A \rightarrow_r B\ ]\!] \Longrightarrow R(x)_r \in B$
  **by** (*metis* (*no-types, lifting*) *Domain-iff iso-tuple-UNIV-I left-totalr-on-def mem-Collect-eq rel-pfun-apply rel-tfun-def*)

**lemma** *rel-typed-insert* [*rclos*]: $[\![\ R \in A \leftrightarrow_r B;\ x \in A;\ y \in B\ ]\!] \Longrightarrow insert\ (x,\ y)\ R \in A \leftrightarrow_r B$
  **by** (*simp add*: *rel-typed-def*)

**lemma** *rel-pfun-insert* [*rclos*]: $[\![\ R \in A \rightharpoonup_r B;\ x \in A;\ y \in B;\ x \notin Domain(R)\ ]\!] \Longrightarrow insert\ (x,\ y)\ R \in A \rightharpoonup_r B$
  **by** (*auto intro*: *rclos simp add*: *rel-pfun-def*)

**lemma** *rel-pfun-override* [*rclos*]: $[\![\ R \in A \rightharpoonup_r B;\ S \in A \rightharpoonup_r B\ ]\!] \Longrightarrow (R +_r S) \in A \rightharpoonup_r B$
  **apply** (*rule rel-pfun-intro*)

**apply** (*rule rel-typed-intro*)
 **apply** (*auto simp add: rel-pfun-def rel-typed-def*)
 **apply** (*metis* (*no-types, hide-lams*) *Range.simps Range-Un-eq Range-rel-override Un-iff rev-subsetD*)
 **done**

**end**

# 7  Map Type: extra functions and properties

**theory** *Map-Extra*
 **imports**
 *Relation-Extra*
 *HOL−Library.Countable-Set*
 *HOL−Library.Monad-Syntax*
**begin**

## 7.1  Graphing Maps

**definition** *map-graph* :: $('a \rightharpoonup 'b) \Rightarrow ('a \leftrightarrow 'b)$ **where**
*map-graph f* = $\{(x,y) \mid x\ y.\ f\ x = Some\ y\}$

**definition** *graph-map* :: $('a \leftrightarrow 'b) \Rightarrow ('a \rightharpoonup 'b)$ **where**
*graph-map g* = $(\lambda\ x.\ if\ (x \in fst\ `\ g)\ then\ Some\ (SOME\ y.\ (x,y) \in g)\ else\ None)$

**definition** *graph-map′* :: $('a \leftrightarrow 'b) \rightharpoonup ('a \rightharpoonup 'b)$ **where**
*graph-map′ R* = (*if* (*functional R*) *then Some* (*graph-map R*) *else None*)

**lemma** *map-graph-mem-equiv*: $(x, y) \in map\text{-}graph\ f \longleftrightarrow f(x) = Some\ y$
 **by** (*simp add: map-graph-def*)

**lemma** *map-graph-functional*[*simp*]: *functional* (*map-graph f*)
 **by** (*simp add:functional-def map-graph-def inj-on-def*)

**lemma** *map-graph-countable* [*simp*]: *countable* (*dom f*) $\implies$ *countable* (*map-graph f*)
 **apply** (*auto simp add:map-graph-def countable-def*)
 **apply** (*rename-tac f′*)
 **apply** (*rule-tac x=f′ ∘ fst* **in** *exI*)
 **apply** (*auto simp add:inj-on-def dom-def*)
 **apply** *fastforce*
 **done**

**lemma** *map-graph-inv* [*simp*]: *graph-map* (*map-graph f*) = *f*
 **by** (*auto intro!:ext simp add:map-graph-def graph-map-def image-def*)

**lemma** *graph-map-empty*[*simp*]: *graph-map* {} = *Map.empty*
 **by** (*simp add:graph-map-def*)

**lemma** *graph-map-insert* [*simp*]: ⟦*functional g*; *g``*{*x*} ⊆ {*y*}⟧ $\implies$ *graph-map* (*insert* (*x,y*) *g*) = (*graph-map g*)(*x* ↦ *y*)
 **by** (*rule ext*, *auto simp add:graph-map-def*)

**lemma** *dom-map-graph*: *dom f* = *Domain*(*map-graph f*)
 **by** (*simp add: map-graph-def dom-def image-def*)

**lemma** *ran-map-graph*: *ran f* = *Range*(*map-graph f*)

**by** (*simp add*: *map-graph-def ran-def image-def*)

**lemma** *rel-apply-map-graph*:
  $x \in dom(f) \implies (map\text{-}graph\ f)(x)_r = the\ (f\ x)$
  **by** (*auto simp add*: *rel-apply-def map-graph-def*)

**lemma** *ran-map-add-subset*:
  $ran\ (x \mathbin{++} y) \subseteq (ran\ x) \cup (ran\ y)$
  **by** (*auto simp add*:*ran-def*)

**lemma** *finite-dom-graph*: $finite\ (dom\ f) \implies finite\ (map\text{-}graph\ f)$
  **by** (*metis dom-map-graph finite-imageD fst-eq-Domain functional-def map-graph-functional*)

**lemma** *finite-dom-ran* [*simp*]: $finite\ (dom\ f) \implies finite\ (ran\ f)$
  **by** (*metis finite-Range finite-dom-graph ran-map-graph*)

**lemma** *graph-map-inv* [*simp*]: $functional\ g \implies map\text{-}graph\ (graph\text{-}map\ g) = g$
  **apply** (*auto simp add*:*map-graph-def graph-map-def functional-def*)
    **apply** (*metis* (*lifting, no-types*) *image-iff option.distinct*(*1*) *option.inject someI surjective-pairing*)
   **apply** (*simp add*:*inj-on-def*)
   **apply** (*metis fst-conv snd-conv some-equality*)
   **apply** (*metis* (*lifting*) *fst-conv image-iff*)
  **done**

**lemma** *graph-map-dom*: $dom\ (graph\text{-}map\ R) = fst\ `\ R$
  **by** (*simp add*: *graph-map-def dom-def*)

**lemma** *graph-map-countable-dom*: $countable\ R \implies countable\ (dom\ (graph\text{-}map\ R))$
  **by** (*simp add*: *graph-map-dom*)

**lemma** *countable-ran*:
  **assumes** $countable\ (dom\ f)$
  **shows** $countable\ (ran\ f)$
**proof** −
  **have** $countable\ (map\text{-}graph\ f)$
    **by** (*simp add*: *assms*)
  **then have** $countable\ (Range(map\text{-}graph\ f))$
    **by** (*simp add*: *Range-snd*)
  **thus** *?thesis*
    **by** (*simp add*: *ran-map-graph*)
**qed**

**lemma** *map-graph-inv′* [*simp*]:
  $graph\text{-}map′\ (map\text{-}graph\ f) = Some\ f$
  **by** (*simp add*: *graph-map′-def*)

**lemma** *map-graph-inj*:
  *inj map-graph*
  **by** (*metis injI map-graph-inv*)

**lemma** *map-eq-graph*: $f = g \longleftrightarrow map\text{-}graph\ f = map\text{-}graph\ g$
  **by** (*auto simp add*: *inj-eq map-graph-inj*)

**lemma** *map-le-graph*: $f \subseteq_m g \longleftrightarrow map\text{-}graph\ f \subseteq map\text{-}graph\ g$
  **by** (*force simp add*: *map-le-def map-graph-def*)

**lemma** *map-graph-comp*: *map-graph* ($g \circ_m f$) = (*map-graph f*) $O$ (*map-graph g*)
  **apply** (*auto simp add*: *map-comp-def map-graph-def relcomp-unfold*)
  **apply** (*rename-tac a b*)
  **apply** (*case-tac f a, auto*)
  **done**

## 7.2   Map Application

**definition** *map-apply* :: ($'a \rightharpoonup 'b$) $\Rightarrow$ $'a$ $\Rightarrow$ $'b$ (-'(-')$_m$ [999,0] 999) **where**
*map-apply* = ($\lambda$ *f x*. *the* (*f x*))

## 7.3   Map Membership

**fun** *map-member* :: $'a \times 'b \Rightarrow$ ($'a \rightharpoonup 'b$) $\Rightarrow$ *bool* (**infix** $\in_m$ 50) **where**
($k$, $v$) $\in_m$ $m$ $\longleftrightarrow$ $m(k)$ = *Some*($v$)

**lemma** *map-ext*:
  ⟦ $\bigwedge$ $x$ $y$. ($x$, $y$) $\in_m$ $A$ $\longleftrightarrow$ ($x$, $y$) $\in_m$ $B$ ⟧ $\Longrightarrow$ $A$ = $B$
  **by** (*rule ext, auto, metis not-Some-eq*)

**lemma** *map-member-alt-def*:
  ($x$, $y$) $\in_m$ $A$ $\longleftrightarrow$ ($x$ $\in$ *dom A* $\wedge$ $A(x)_m$ = $y$)
  **by** (*auto simp add*: *map-apply-def*)

**lemma** *map-le-member*:
  $f \subseteq_m g$ $\longleftrightarrow$ ($\forall$ $x$ $y$. ($x$,$y$) $\in_m$ $f$ $\longrightarrow$ ($x$,$y$) $\in_m$ $g$)
  **by** (*force simp add*: *map-le-def*)

## 7.4   Preimage

**definition** *preimage* :: ($'a \rightharpoonup 'b$) $\Rightarrow$ $'b$ *set* $\Rightarrow$ $'a$ *set* **where**
*preimage f B* = \{$x$ $\in$ *dom*($f$). *the*($f(x)$) $\in$ $B$\}

**lemma** *preimage-range*: *preimage f* (*ran f*) = *dom f*
  **by** (*auto simp add*: *preimage-def ran-def*)

**lemma** *dom-preimage*: *dom* ($m \circ_m f$) = *preimage f* (*dom m*)
  **apply** (*auto simp add*: *dom-def preimage-def*)
   **apply** (*meson map-comp-Some-iff*)
  **apply** (*metis map-comp-def option.case-eq-if option.distinct(1)*)
  **done**

**lemma** *countable-preimage*:
  ⟦ *countable A*; *inj-on f* (*preimage f A*) ⟧ $\Longrightarrow$ *countable* (*preimage f A*)
  **apply** (*auto simp add*: *countable-def*)
  **apply** (*rename-tac g*)
  **apply** (*rule-tac x=g $\circ$ the $\circ$ f* **in** *exI*)
  **apply** (*rule inj-onI*)
  **apply** (*drule inj-onD*)
    **apply** (*auto simp add*: *preimage-def inj-onD*)
  **done**

## 7.5   Minus operation for maps

**definition** *map-minus* :: ($'a \rightharpoonup 'b$) $\Rightarrow$ ($'a \rightharpoonup 'b$) $\Rightarrow$ ($'a \rightharpoonup 'b$) (**infixl** $--$ 100)

**where** *map-minus f g = ($\lambda$ x. if (f x = g x) then None else f x)*

**lemma** *map-minus-apply* [*simp*]: $y \in dom(f\ {-}{-}\ g) \Longrightarrow (f\ {-}{-}\ g)(y)_m = f(y)_m$
  **by** (*auto simp add: map-minus-def dom-def map-apply-def*)


**lemma** *map-member-plus*:
  $(x,\ y) \in_m f\ {+}{+}\ g \longleftrightarrow ((x \notin dom(g) \wedge (x,\ y) \in_m f) \vee (x,\ y) \in_m g)$
  **by** (*auto simp add: map-add-Some-iff*)


**lemma** *map-member-minus*:
  $(x,\ y) \in_m f\ {-}{-}\ g \longleftrightarrow (x,\ y) \in_m f \wedge (\neg\ (x,\ y) \in_m g)$
  **by** (*auto simp add: map-minus-def*)


**lemma** *map-minus-plus-commute*:
  $dom(g) \cap dom(h) = \{\} \Longrightarrow (f\ {-}{-}\ g)\ {+}{+}\ h = (f\ {+}{+}\ h)\ {-}{-}\ g$
  **apply** (*rule map-ext*)
  **apply** (*auto simp add: map-member-plus map-member-minus simp del: map-member.simps*)
  **apply** (*auto simp add: map-member-alt-def*)
  **done**


**lemma** *map-graph-minus*: *map-graph (f $-\!-$ g) = map-graph f $-$ map-graph g*
  **by** (*auto simp add: map-minus-def map-graph-def, (meson option.distinct(1))+*)


**lemma** *map-minus-common-subset*:
  $[\![\ h \subseteq_m f;\ h \subseteq_m g\ ]\!] \Longrightarrow (f\ {-}{-}\ h = g\ {-}{-}\ h) = (f = g)$
  **by** (*auto simp add: map-eq-graph map-graph-minus map-le-graph*)


## 7.6 Map Bind

Create some extra intro/elim rules to help dealing with proof about option bind.

**lemma** *option-bindSomeE* [*elim!*]:
  $[\![\ X >\!\!>\!\!= F = Some(v); \bigwedge x.\ [\![\ X = Some(x);\ F(x) = Some(v)\ ]\!] \Longrightarrow P\ ]\!] \Longrightarrow P$
  **by** (*case-tac X, auto*)


**lemma** *option-bindSomeI* [*intro*]:
  $[\![\ X = Some(x);\ F(x) = Some(y)\ ]\!] \Longrightarrow X >\!\!>\!\!= F = Some(y)$
  **by** (*simp*)


**lemma** *ifSomeE* [*elim*]: $[\![\ (if\ c\ then\ Some(x)\ else\ None) = Some(y);\ [\![\ c;\ x = y\ ]\!] \Longrightarrow P\ ]\!] \Longrightarrow P$
  **by** (*case-tac c, auto*)


## 7.7 Range Restriction

A range restriction operator; only domain restriction is provided in HOL.

**definition** *ran-restrict-map* :: $('a \rightharpoonup 'b) \Rightarrow 'b\ set \Rightarrow 'a \rightharpoonup 'b$ (-$\restriction$- [*111,110*] *110*) **where**
*ran-restrict-map f B = ($\lambda$x. do { v <$-$ f(x); if (v $\in$ B) then Some(v) else None })*


**lemma** *ran-restrict-empty* [*simp*]: $f\restriction_{\{\}} = Map.empty$
  **by** (*simp add:ran-restrict-map-def*)


**lemma** *ran-restrict-ran* [*simp*]: $f\restriction_{ran(f)} = f$
  **apply** (*auto simp add:ran-restrict-map-def ran-def*)
  **apply** (*rule ext*)
  **apply** (*case-tac f(x), auto*)

**done**

**lemma** *ran-ran-restrict* [*simp*]: $ran(f{\restriction}_B) = ran(f) \cap B$
  **by** (*auto intro*!:*option-bindSomeI simp add*:*ran-restrict-map-def ran-def*)

**lemma** *dom-ran-restrict*: $dom(f{\restriction}_B) \subseteq dom(f)$
  **by** (*auto simp add*:*ran-restrict-map-def dom-def*)

**lemma** *ran-restrict-finite-dom* [*intro*]:
  $finite(dom(f)) \implies finite(dom(f{\restriction}_B))$
  **by** (*metis finite-subset dom-ran-restrict*)

**lemma** *dom-Some* [*simp*]: $dom\ (Some \circ f) = UNIV$
  **by** (*auto*)

**lemma** *dom-left-map-add* [*simp*]: $x \in dom\ g \implies (f \mathbin{++} g)\ x = g\ x$
  **by** (*auto simp add*:*map-add-def dom-def*)

**lemma** *dom-right-map-add* [*simp*]: $⟦\ x \notin dom\ g;\ x \in dom\ f\ ⟧ \implies (f \mathbin{++} g)\ x = f\ x$
  **by** (*auto simp add*:*map-add-def dom-def*)

**lemma** *map-add-restrict*:
  $f \mathbin{++} g = (f\ |`\ (-\ dom\ g)) \mathbin{++} g$
  **by** (*rule ext, auto simp add*: *map-add-def restrict-map-def*)

## 7.8 Map Inverse and Identity

**definition** *map-inv* :: $('a \rightharpoonup 'b) \Rightarrow ('b \rightharpoonup 'a)$ **where**
*map-inv* $f \equiv \lambda\ y.\ if\ (y \in ran\ f)\ then\ Some\ (SOME\ x.\ f\ x = Some\ y)\ else\ None$

**definition** *map-id-on* :: $'a\ set \Rightarrow ('a \rightharpoonup 'a)$ **where**
*map-id-on* $xs \equiv \lambda\ x.\ if\ (x \in xs)\ then\ Some\ x\ else\ None$

**lemma** *map-id-on-in* [*simp*]:
  $x \in xs \implies map\text{-}id\text{-}on\ xs\ x = Some\ x$
  **by** (*simp add*:*map-id-on-def*)

**lemma** *map-id-on-out* [*simp*]:
  $x \notin xs \implies map\text{-}id\text{-}on\ xs\ x = None$
  **by** (*simp add*:*map-id-on-def*)

**lemma** *map-id-dom* [*simp*]: $dom\ (map\text{-}id\text{-}on\ xs) = xs$
  **by** (*simp add*:*dom-def map-id-on-def*)

**lemma** *map-id-ran* [*simp*]: $ran\ (map\text{-}id\text{-}on\ xs) = xs$
  **by** (*force simp add*:*ran-def map-id-on-def*)

**lemma** *map-id-on-UNIV* [*simp*]: $map\text{-}id\text{-}on\ UNIV = Some$
  **by** (*simp add*:*map-id-on-def*)

**lemma** *map-id-on-inj* [*simp*]:
  $inj\text{-}on\ (map\text{-}id\text{-}on\ xs)\ xs$
  **by** (*simp add*:*inj-on-def*)

**lemma** *map-inv-empty* [*simp*]: $map\text{-}inv\ Map.empty = Map.empty$
  **by** (*simp add*:*map-inv-def*)

**lemma** *map-inv-id* [*simp*]:
  *map-inv* (*map-id-on xs*) = *map-id-on xs*
  **by** (*force simp add:map-inv-def map-id-on-def ran-def*)

**lemma** *map-inv-Some* [*simp*]: *map-inv Some* = *Some*
  **by** (*simp add:map-inv-def ran-def*)

**lemma** *map-inv-f-f* [*simp*]:
  ⟦ *inj-on f* (*dom f*); *f x* = *Some y* ⟧ ⟹ *map-inv f y* = *Some x*
  **apply** (*auto simp add: map-inv-def*)
   **apply** (*rule some-equality*)
    **apply** (*auto simp add:inj-on-def dom-def ran-def*)
  **done**

**lemma** *dom-map-inv* [*simp*]:
  *dom* (*map-inv f*) = *ran f*
  **by** (*auto simp add:map-inv-def*)

**lemma** *ran-map-inv* [*simp*]:
  *inj-on f* (*dom f*) ⟹ *ran* (*map-inv f*) = *dom f*
  **apply** (*auto simp add:map-inv-def ran-def*)
   **apply** (*rename-tac a b*)
   **apply** (*rule-tac x=a* **in** *exI*)
   **apply** (*force intro:someI*)
  **apply** (*rename-tac x y*)
  **apply** (*rule-tac x=y* **in** *exI*)
  **apply** (*auto*)
  **apply** (*rule some-equality, simp-all*)
  **apply** (*auto simp add:inj-on-def dom-def*)
  **done**

**lemma** *dom-image-ran*: *f* ' *dom f* = *Some* ' *ran f*
  **by** (*auto simp add:dom-def ran-def image-def*)

**lemma** *inj-map-inv* [*intro*]:
  *inj-on f* (*dom f*) ⟹ *inj-on* (*map-inv f*) (*ran f*)
  **apply** (*auto simp add:map-inv-def inj-on-def dom-def ran-def*)
  **apply** (*rename-tac x y u v*)
  **apply** (*frule-tac P=λ xa. f xa* = *Some x* **in** *some-equality*)
   **apply** (*auto*)
  **apply** (*metis* (*mono-tags*) *option.sel someI*)
  **done**

**lemma** *inj-map-bij*: *inj-on f* (*dom f*) ⟹ *bij-betw f* (*dom f*) (*Some* ' *ran f*)
  **by** (*auto simp add:inj-on-def dom-def ran-def image-def bij-betw-def*)

**lemma** *map-inv-map-inv* [*simp*]:
  **assumes** *inj-on f* (*dom f*)
  **shows** *map-inv* (*map-inv f*) = *f*
**proof** −

  **from** *assms* **have** *inj-on* (*map-inv f*) (*ran f*)
    **by** *auto*

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**thus** *?thesis*
  **apply** (*rule-tac ext*)
  **apply** (*rename-tac x*)
  **apply** (*case-tac ∃ y. map-inv f y = Some x*)
   **apply** (*auto*)[*1*]
   **apply** (*simp add:map-inv-def*)
   **apply** (*rename-tac x y*)
   **apply** (*case-tac y ∈ ran f, simp-all*)
   **apply** (*auto*)
   **apply** (*rule someI2-ex*)
    **apply** (*simp add:ran-def*)
   **apply** (*simp*)
  **apply** (*metis assms dom-image-ran dom-map-inv image-iff map-add-dom-app-simps(2) map-add-dom-app-simps(3) ran-map-inv*)
   **done**
**qed**

**lemma** *map-self-adjoin-complete* [*intro*]:
  **assumes** *dom f ∩ ran f = {} inj-on f (dom f)*
  **shows** *inj-on (map-inv f ++ f) (dom f ∪ ran f)*
  **apply** (*rule inj-onI*)
  **apply** (*insert assms*)
  **apply** (*rename-tac x y*)
  **apply** (*case-tac x ∈ dom f*)
   **apply** (*simp*)
   **apply** (*case-tac y ∈ dom f*)
    **apply** (*simp add:inj-on-def*)
   **apply** (*case-tac y ∈ ran f*)
    **apply** (*subgoal-tac y ∈ dom (map-inv f)*)
     **apply** (*simp*)
     **apply** (*metis Int-iff domD empty-iff ranI ran-map-inv*)
    **apply** (*simp*)
   **apply** (*simp*)
  **apply** (*simp*)
  **apply** (*case-tac y ∈ dom f*)
   **apply** (*simp*)
   **apply** (*case-tac y ∈ ran f*)
    **apply** (*subgoal-tac y ∈ dom (map-inv f)*)
     **apply** (*simp*)
     **apply** (*metis Int-iff empty-iff*)
    **apply** (*simp*)
   **apply** (*metis Int-iff domD empty-iff ranI ran-map-inv*)
  **apply** (*simp*)
  **apply** (*metis (lifting) inj-map-inv inj-on-contraD*)
  **done**

**lemma** *inj-completed-map* [*intro*]:
  ⟦ *dom f = ran f; inj-on f (dom f)* ⟧ ⟹ *inj (Some ++ f)*
  **apply** (*drule inj-map-bij*)
  **apply** (*auto simp add:bij-betw-def*)
  **apply** (*auto simp add:inj-on-def*)[*1*]
  **apply** (*rename-tac x y*)
  **apply** (*case-tac x ∈ dom f*)
   **apply** (*simp*)
   **apply** (*case-tac y ∈ dom f*)

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**apply** (*simp*)
 **apply** (*simp add:ran-def*)
 **apply** (*case-tac y ∈ dom f*)
 **apply** (*auto intro:ranI*)
 **done**

**lemma** *bij-completed-map* [*intro*]:
  ⟦ *dom f = ran f*; *inj-on f* (*dom f*) ⟧ ⟹
  *bij-betw* (*Some ++ f*) *UNIV* (*range Some*)
  **apply** (*auto simp add:bij-betw-def*)
   **apply** (*rename-tac x*)
   **apply** (*case-tac x ∈ dom f*)
    **apply** (*simp*)
    **apply** (*metis domD rangeI*)
   **apply** (*simp*)
  **apply** (*simp add:image-def*)
  **apply** (*metis* (*full-types*) *dom-image-ran dom-left-map-add image-iff map-add-dom-app-simps(3)*)
  **done**

**lemma** *bij-map-Some*:
  *bij-betw f a* (*Some ' b*) ⟹ *bij-betw* (*the ∘ f*) *a b*
  **apply** (*simp add:bij-betw-def*)
  **apply** (*safe*)
    **apply** (*metis* (*hide-lams, no-types*) *comp-inj-on-iff f-the-inv-into-f inj-on-inverseI option.sel*)
   **apply** (*metis* (*hide-lams, no-types*) *image-iff option.sel*)
  **apply** (*metis Option.these-def Some-image-these-eq image-image these-image-Some-eq*)
  **done**

**lemma** *ran-map-add* [*simp*]:
  *m'*(*dom m ∩ dom n*) = *n'*(*dom m ∩ dom n*) ⟹
  *ran*(*m++n*) = *ran n ∪ ran m*
  **apply** (*auto simp add:ran-def*)
   **apply** (*metis map-add-find-right*)
  **apply** (*rename-tac x a*)
  **apply** (*case-tac a ∈ dom n*)
   **apply** (*subgoal-tac ∃ b. n b = Some x*)
    **apply** (*auto*)
    **apply** (*rename-tac x a b y*)
    **apply** (*rule-tac x=b* **in** *exI*)
    **apply** (*simp*)
   **apply** (*metis* (*hide-lams, no-types*) *IntI domI image-iff*)
 **apply** (*metis* (*full-types*) *map-add-None map-add-dom-app-simps(1) map-add-dom-app-simps(3) not-None-eq*)
  **done**

**lemma** *ran-maplets* [*simp*]:
  ⟦ *length xs = length ys*; *distinct xs* ⟧ ⟹ *ran* [*xs* [↦] *ys*] = *set ys*
    **by** (*induct rule:list-induct2, simp-all*)

**lemma** *inj-map-add*:
  ⟦ *inj-on f* (*dom f*); *inj-on g* (*dom g*); *ran f ∩ ran g = {}* ⟧ ⟹
  *inj-on* (*f ++ g*) (*dom f ∪ dom g*)
  **apply** (*auto simp add:inj-on-def*)
     **apply** (*metis* (*full-types*) *disjoint-iff-not-equal domI dom-left-map-add map-add-dom-app-simps(3)
ranI*)
     **apply** (*metis domI*)

```
      apply (metis disjoint-iff-not-equal ranI)
       apply (metis disjoint-iff-not-equal domIff map-add-Some-iff ranI)
      apply (metis domI)
      done

lemma map-inv-add [simp]:
  assumes inj-on f (dom f) inj-on g (dom g)
          dom f ∩ dom g = {} ran f ∩ ran g = {}
  shows map-inv (f ++ g) = map-inv f ++ map-inv g
proof (rule ext)

  from assms have minj: inj-on (f ++ g) (dom (f ++ g))
    by (simp, metis inj-map-add sup-commute)

  fix x
  have x ∈ ran g ⟹ map-inv (f ++ g) x = (map-inv f ++ map-inv g) x
  proof −
    assume ran:x ∈ ran g
    then obtain y where dom:g y = Some x y ∈ dom g
      by (auto simp add:ran-def)

    hence (f ++ g) y = Some x
      by simp

    with assms minj ran dom show map-inv (f ++ g) x = (map-inv f ++ map-inv g) x
      by simp
  qed

  moreover have ⟦ x ∉ ran g; x ∈ ran f ⟧ ⟹ map-inv (f ++ g) x = (map-inv f ++ map-inv g) x
  proof −
    assume ran:x ∉ ran g x ∈ ran f
    with assms obtain y where dom:f y = Some x y ∈ dom f y ∉ dom g
      by (auto simp add:ran-def)

    with ran have (f ++ g) y = Some x
      by (simp)

    with assms minj ran dom show map-inv (f ++ g) x = (map-inv f ++ map-inv g) x
      by simp
  qed

  moreover from assms minj have ⟦ x ∉ ran g; x ∉ ran f ⟧ ⟹ map-inv (f ++ g) x = (map-inv f
++ map-inv g) x
    apply (auto simp add:map-inv-def ran-def map-add-def)
    apply (metis dom-left-map-add map-add-def map-add-dom-app-simps(3))
    done

  ultimately show map-inv (f ++ g) x = (map-inv f ++ map-inv g) x
    apply (case-tac x ∈ ran g)
     apply (simp)
    apply (case-tac x ∈ ran f)
     apply (simp-all)
    done
qed
```

**lemma** *map-add-lookup* [*simp*]:
  $x \notin dom\ f \implies ([x \mapsto y] \mathbin{++} f)\ x = Some\ y$
  **by** (*simp add:map-add-def dom-def*)

**lemma** *map-add-Some*: $Some \mathbin{++} f = map\text{-}id\text{-}on\ (-\ dom\ f) \mathbin{++} f$
  **apply** (*rule ext*)
  **apply** (*rename-tac x*)
  **apply** (*case-tac* $x \in dom\ f$)
   **apply** (*simp-all*)
  **done**

**lemma** *distinct-map-dom*:
  $x \notin set\ xs \implies x \notin dom\ [xs\ [\mapsto]\ ys]$
  **by** (*simp add:dom-def*)

**lemma** *distinct-map-ran*:
  $\llbracket\ distinct\ xs;\ y \notin set\ ys;\ length\ xs = length\ ys\ \rrbracket \implies$
  $y \notin ran\ ([xs\ [\mapsto]\ ys])$
  **apply** (*simp add:map-upds-def*)
  **apply** (*subgoal-tac distinct* (*map fst* (*rev* (*zip xs ys*))))
  **apply** (*simp add:ran-distinct*)
  **apply** (*metis* (*hide-lams, no-types*) *image-iff set-zip-rightD surjective-pairing*)
  **apply** (*simp add:zip-rev*[*THEN sym*])
**done**

**lemma** *maplets-lookup*[*rule-format,dest*]:
  $\llbracket\ length\ xs = length\ ys;\ distinct\ xs\ \rrbracket \implies$
    $\forall\ y.\ [xs\ [\mapsto]\ ys]\ x = Some\ y \longrightarrow y \in set\ ys$
  **by** (*induct rule:list-induct2, auto*)

**lemma** *maplets-distinct-inj* [*intro*]:
  $\llbracket\ length\ xs = length\ ys;\ distinct\ xs;\ distinct\ ys;\ set\ xs \cap set\ ys = \{\}\ \rrbracket \implies$
  $inj\text{-}on\ [xs\ [\mapsto]\ ys]\ (set\ xs)$
  **apply** (*induct rule:list-induct2*)
   **apply** (*simp-all*)
  **apply** (*rule conjI*)
   **apply** (*rule inj-onI*)
   **apply** (*rename-tac x xs y ys xa ya*)
   **apply** (*case-tac* $xa = x$)
    **apply** (*simp*)
   **apply** (*case-tac* $xa = y$)
    **apply** (*simp*)
   **apply** (*simp*)
   **apply** (*case-tac* $ya = x$)
    **apply** (*simp*)
   **apply** (*simp add:inj-on-def*)
  **apply** (*auto*)
  **apply** (*rename-tac x xs y ys xa*)
  **apply** (*case-tac* $xa = y$)
   **apply** (*simp*)
  **apply** (*metis maplets-lookup*)
  **done**

**lemma** *map-inv-maplet*[*simp*]: $map\text{-}inv\ [x \mapsto y] = [y \mapsto x]$
  **by** (*auto simp add:map-inv-def*)

**lemma** *map-inv-maplets* [*simp*]:
 ⟦ *length xs = length ys*; *distinct xs*; *distinct ys*; *set xs ∩ set ys = {}* ⟧ ⟹
 *map-inv* [*xs* [↦] *ys*] = [*ys* [↦] *xs*]
 **apply** (*induct rule*:*list-induct2*)
  **apply** (*simp-all*)
 **apply** (*rename-tac x xs y ys*)
 **apply** (*subgoal-tac map-inv* ([*xs* [↦] *ys*] ++ [*x* ↦ *y*]) = *map-inv* [*xs* [↦] *ys*] ++ *map-inv* [*x* ↦ *y*])
  **apply** (*simp*)
 **apply** (*rule map-inv-add*)
   **apply** (*auto*)
  **done**

**lemma** *maplets-lookup-nth* [*rule-format*,*simp*]:
 ⟦ *length xs = length ys*; *distinct xs* ⟧ ⟹
 ∀ *i < length ys*. [*xs* [↦] *ys*] (*xs* ! *i*) = *Some* (*ys* ! *i*)
 **apply** (*induct rule*:*list-induct2*)
  **apply** (*auto*)
  **apply** (*rename-tac x xs y ys i*)
  **apply** (*case-tac i*)
   **apply** (*simp-all*)
  **apply** (*metis nth-mem*)
 **apply** (*rename-tac x xs y ys i*)
 **apply** (*case-tac i*)
  **apply** (*auto*)
 **done**

**theorem** *inv-map-inv*:
 ⟦ *inj-on f* (*dom f*); *ran f = dom f* ⟧
 ⟹ *inv* (*the ∘* (*Some* ++ *f*)) = *the ∘ map-inv* (*Some* ++ *f*)
 **apply** (*rule ext*)
 **apply** (*simp add*:*map-add-Some*)
 **apply** (*simp add*:*inv-def*)
 **apply** (*rename-tac x*)
 **apply** (*case-tac* ∃ *y. f y = Some x*)
  **apply** (*erule exE*)
  **apply** (*rename-tac x y*)
  **apply** (*subgoal-tac x ∈ ran f*)
   **apply** (*subgoal-tac y ∈ dom f*)
    **apply** (*simp*)
    **apply** (*rule some-equality*)
     **apply** (*simp*)
    **apply** (*metis* (*hide-lams, mono-tags*) *domD domI dom-left-map-add inj-on-contraD map-add-Some map-add-dom-app-simps*(*3*) *option.sel*)
   **apply** (*simp add*:*dom-def*)
  **apply** (*metis ranI*)
 **apply** (*simp*)
 **apply** (*rename-tac x*)
 **apply** (*subgoal-tac x ∉ ran f*)
  **apply** (*simp*)
  **apply** (*rule some-equality*)
   **apply** (*simp*)
  **apply** (*metis domD dom-left-map-add map-add-Some map-add-dom-app-simps*(*3*) *option.sel*)
 **apply** (*metis dom-image-ran image-iff*)
 **done**

**lemma** *map-comp-dom*: *dom* $(g \circ_m f) \subseteq dom\ f$
  **by** (*metis* (*lifting*, *full-types*) *Collect-mono dom-def map-comp-simps*(*1*))

**lemma** *map-comp-assoc*: $f \circ_m (g \circ_m h) = f \circ_m g \circ_m h$
**proof**
  **fix** $x$
  **show** $(f \circ_m (g \circ_m h))\ x = (f \circ_m g \circ_m h)\ x$
  **proof** (*cases h x*)
    **case** *None* **thus** *?thesis*
      **by** (*auto simp add*: *map-comp-def*)
  **next**
    **case** (*Some y*) **thus** *?thesis*
      **by** (*auto simp add*: *map-comp-def*)
  **qed**
**qed**

**lemma** *map-comp-runit* [*simp*]: $f \circ_m Some = f$
  **by** (*simp add*: *map-comp-def*)

**lemma** *map-comp-lunit* [*simp*]: $Some \circ_m f = f$
**proof**
  **fix** $x$
  **show** $(Some \circ_m f)\ x = f\ x$
  **proof** (*cases f x*)
    **case** *None* **thus** *?thesis*
      **by** (*simp add*: *map-comp-def*)
  **next**
    **case** (*Some y*) **thus** *?thesis*
      **by** (*simp add*: *map-comp-def*)
  **qed**
**qed**

**lemma** *map-comp-apply* [*simp*]: $(f \circ_m g)\ x = g(x) >>= f$
  **by** (*auto simp add*: *map-comp-def option.case-eq-if*)

## 7.9 Merging of compatible maps

**definition** *comp-map* :: $('a \rightharpoonup 'b) \Rightarrow ('a \rightharpoonup 'b) \Rightarrow bool$ (**infixl** $\|_m$ *60*) **where**
$comp\text{-}map\ f\ g = (\forall\ x \in dom(f) \cap dom(g).\ the(f(x)) = the(g(x)))$

**lemma** *comp-map-unit*: $Map.empty \|_m f$
  **by** (*simp add*: *comp-map-def*)

**lemma** *comp-map-refl*: $f \|_m f$
  **by** (*simp add*: *comp-map-def*)

**lemma** *comp-map-sym*: $f \|_m g \Longrightarrow g \|_m f$
  **by** (*simp add*: *comp-map-def*)

**definition** *merge* :: $('a \rightharpoonup 'b)\ set \Rightarrow 'a \rightharpoonup 'b$ **where**
$merge\ fs =$
  $(\lambda\ x.\ if\ (\exists\ f \in fs.\ x \in dom(f))\ then\ (THE\ y.\ \forall\ f \in fs.\ x \in dom(f) \longrightarrow f(x) = y)\ else\ None)$

**lemma** *merge-empty*: $merge\ \{\} = Map.empty$
  **by** (*simp add*: *merge-def*)

**lemma** *merge-singleton*: *merge* {*f*} = *f*
  **apply** (*auto intro*!: *ext simp add*: *merge-def*)
  **using** *option.collapse* **apply** *fastforce*
  **done**

## 7.10   Conversion between lists and maps

**definition** *map-of-list* :: *'a list* ⇒ (*nat* ⇀ *'a*) **where**
*map-of-list xs* = (λ *i*. *if* (*i* < *length xs*) *then Some* (*xs*!*i*) *else None*)

**lemma** *map-of-list-nil* [*simp*]: *map-of-list* [] = *Map.empty*
  **by** (*simp add*: *map-of-list-def*)

**lemma** *dom-map-of-list* [*simp*]: *dom* (*map-of-list xs*) = {*0..<length xs*}
  **by** (*auto simp add*: *map-of-list-def dom-def*)

**lemma** *ran-map-of-list* [*simp*]: *ran* (*map-of-list xs*) = *set xs*
  **apply** (*simp add*: *ran-def map-of-list-def*)
  **apply** (*safe*)
   **apply** (*force*)
  **apply** (*meson in-set-conv-nth*)
  **done**

**definition** *list-of-map* :: (*nat* ⇀ *'a*) ⇒ *'a list* **where**
*list-of-map f* = (*if* (*f* = *Map.empty*) *then* [] *else map* (*the* ∘ *f*) [*0 ..< Suc*(*GREATEST x. x* ∈ *dom f*)])

**lemma** *list-of-map-empty* [*simp*]: *list-of-map Map.empty* = []
  **by** (*simp add*: *list-of-map-def*)

**definition** *list-of-map'* :: (*nat* ⇀ *'a*) ⇀ *'a list* **where**
*list-of-map' f* = (*if* (∃ *n*. *dom f* = {*0..<n*}) *then Some* (*list-of-map f*) *else None*)

**lemma** *map-of-list-inv* [*simp*]: *list-of-map* (*map-of-list xs*) = *xs*
**proof** (*cases xs* = [])
  **case** *True* **thus** *?thesis* **by** (*simp*)
**next**
  **case** *False*
  **moreover hence** (*GREATEST x. x* ∈ *dom* (*map-of-list xs*)) = *length xs* − 1
    **by** (*auto intro*: *Greatest-equality*)
  **moreover from** *False* **have** *map-of-list xs* ≠ *Map.empty*
    **by** (*metis ran-empty ran-map-of-list set-empty*)
  **ultimately show** *?thesis*
    **by** (*auto intro*!:*nth-equalityI simp add*: *list-of-map-def map-of-list-def  fun-eq-iff*)
**qed**

## 7.11   Map Comprehension

Map comprehension simply converts a relation built through set comprehension into a map.

**syntax**
  *-Mapcompr* :: *'a* ⇒ *'b* ⇒ *idts* ⇒ *bool* ⇒ *'a* ⇀ *'b*   ((*1*[- ↦ - |/-./ -]))

**translations**
  *-Mapcompr F G xs P* == *CONST graph-map* {(*F*, *G*) | *xs*. *P*}

**lemma** *map-compr-eta*:
$[x \mapsto y \mid x\ y.\ (x,\ y) \in_m f] = f$
  **apply** (*rule ext*)
  **apply** (*auto simp add*: *graph-map-def*)
  **apply** (*metis* (*mono-tags*, *lifting*) *Domain.DomainI fst-eq-Domain mem-Collect-eq old.prod.case option.distinct*(*1*) *option.expand option.sel*)
  **done**

**lemma** *map-compr-simple*:
$[x \mapsto F\ x\ y \mid x\ y.\ (x,\ y) \in_m f] = (\lambda\ x.\ do\ \{\ y \leftarrow f(x);\ Some(F\ x\ y)\ \})$
  **apply** (*rule ext*)
  **apply** (*auto simp add*: *graph-map-def image-Collect*)
  **done**

**lemma** *map-compr-dom-simple* [*simp*]:
  $dom\ [x \mapsto f\ x \mid x.\ P\ x] = \{x.\ P\ x\}$
  **by** (*force simp add*: *graph-map-dom image-Collect*)

**lemma** *map-compr-ran-simple* [*simp*]:
 $ran\ [x \mapsto f\ x \mid x.\ P\ x] = \{f\ x \mid x.\ P\ x\}$
  **apply** (*auto simp add*: *graph-map-def ran-def*)
  **apply** (*metis* (*mono-tags*, *lifting*) *fst-eqD image-eqI mem-Collect-eq someI*)
  **done**

**lemma** *map-compr-eval-simple* [*simp*]:
  $[x \mapsto f\ x \mid x.\ P\ x]\ x = (if\ (P\ x)\ then\ Some\ (f\ x)\ else\ None)$
  **by** (*auto simp add*: *graph-map-def image-Collect*)

## 7.12 Sorted lists from maps

**definition** *sorted-list-of-map* :: $('a::linorder \rightharpoonup 'b) \Rightarrow ('a \times 'b)\ list$ **where**
*sorted-list-of-map* $f = map\ (\lambda\ k.\ (k,\ the\ (f\ k)))\ (sorted$-$list$-$of$-$set(dom(f)))$

**lemma** *sorted-list-of-map-empty* [*simp*]:
 *sorted-list-of-map Map.empty* $= []$
  **by** (*simp add*: *sorted-list-of-map-def*)

**lemma** *sorted-list-of-map-inv*:
  **assumes** *finite*(*dom*(*f*))
  **shows** *map-of* (*sorted-list-of-map f*) $= f$
**proof** −
  **obtain** *A* **where** *finite A A = dom*(*f*)
    **by** (*simp add*: *assms*)
  **thus** *?thesis*
  **proof** (*induct A rule*: *finite-induct*)
    **case** *empty* **thus** *?thesis*
      **by** (*simp add*: *sorted-list-of-map-def*, *metis dom-empty empty-iff map-le-antisym map-le-def*)
  **next**
    **case** (*insert x A*) **thus** *?thesis*
      **by** (*simp add*: *assms sorted-list-of-map-def map-of-map-keys*)
  **qed**
**qed**

**declare** *map-member.simps* [*simp del*]

## 7.13   Extra map lemmas

**lemma** *map-eqI*:
  ⟦ *dom f = dom g*; ∀ *x*∈*dom(f)*. *the(f x) = the(g x)* ⟧ ⟹ *f = g*
  **by** (*metis domIff map-le-antisym map-le-def option.expand*)

**lemma** *map-restrict-dom* [*simp*]: *f* |' *dom f = f*
  **by** (*simp add*: *map-eqI*)

**lemma** *map-restrict-dom-compl*: *f* |' (− *dom f*) = *Map.empty*
  **by** (*metis dom-eq-empty-conv dom-restrict inf-compl-bot*)

**lemma** *restrict-map-neg-disj*:
  *dom(f)* ∩ *A* = {} ⟹ *f* |' (− *A*) = *f*
  **by** (*auto simp add*: *restrict-map-def*, *rule ext*, *auto*, *metis disjoint-iff-not-equal domIff*)

**lemma** *map-plus-restrict-dist*: (*f* ++ *g*) |' *A* = (*f* |' *A*) ++ (*g* |' *A*)
  **by** (*auto simp add*: *restrict-map-def map-add-def*)

**lemma** *map-plus-eq-left*:
  **assumes** *f* ++ *h* = *g* ++ *h*
  **shows** (*f* |' (− *dom h*)) = (*g* |' (− *dom h*))
**proof** −
  **have** *h* |' (− *dom h*) = *Map.empty*
    **by** (*metis Compl-disjoint dom-eq-empty-conv dom-restrict*)
  **then have** *f2*: *f* |' (− *dom h*) = (*f* ++ *h*) |' (− *dom h*)
    **by** (*simp add*: *map-plus-restrict-dist*)
  **have** *h* |' (− *dom h*) = *Map.empty*
    **by** (*metis* (*no-types*) *Compl-disjoint dom-eq-empty-conv dom-restrict*)
  **then show** *?thesis*
    **using** *f2 assms* **by** (*simp add*: *map-plus-restrict-dist*)
**qed**

**lemma** *map-add-split*:
  *dom(f)* = *A* ∪ *B* ⟹ (*f* |' *A*) ++ (*f* |' *B*) = *f*
  **by** (*rule ext*, *auto simp add*: *map-add-def restrict-map-def option.case-eq-if*)

**lemma** *map-le-via-restrict*:
  *f* ⊆_m *g* ⟷ *g* |' *dom(f)* = *f*
  **by** (*auto simp add*: *map-le-def restrict-map-def dom-def fun-eq-iff*)

**lemma** *map-add-cancel*:
  *f* ⊆_m *g* ⟹ *f* ++ (*g* −− *f*) = *g*
  **by** (*auto simp add*: *map-le-def map-add-def map-minus-def fun-eq-iff option.case-eq-if*)
    (*metis domIff*)

**lemma** *map-le-iff-add*: *f* ⊆_m *g* ⟷ (∃ *h*. *dom(f)* ∩ *dom(h)* = {} ∧ *f* ++ *h* = *g*)
  **apply** (*auto*)
  **apply** (*rule-tac x=g* −− *f* **in** *exI*)
  **apply** (*metis* (*no-types*, *lifting*) *Int-emptyI domIff map-add-cancel map-le-def map-minus-def*)
  **apply** (*simp add*: *map-add-comm*)
  **done**

**lemma** *map-add-comm-weak*: (∀ *k* ∈ *dom m1* ∩ *dom m2*. *m1(k)* = *m2(k)*) ⟹ *m1* ++ *m2* = *m2* ++ *m1*
  **by** (*auto simp add*: *map-add-def option.case-eq-if fun-eq-iff*)

(*metis IntI domI option.inject*)

**end**

# 8 Alternative List Lexicographic Order

**theory** *List-Lexord-Alt*
  **imports** *Main*
**begin**

Since we can't instantiate the order class twice for lists, and we want prefix as the default order for the UTP we here add syntax for the lexicographic order relation.

**definition** *list-lex-less* :: $'a$::*linorder list* $\Rightarrow$ $'a$ *list* $\Rightarrow$ *bool* (**infix** $<_l$ *50*)
**where** $xs <_l ys \longleftrightarrow (xs, ys) \in lexord\ \{(u, v).\ u < v\}$

**lemma** *list-lex-less-neq* [*simp*]: $x <_l y \Longrightarrow x \neq y$
  **apply** (*simp add*: *list-lex-less-def*)
  **apply** (*meson case-prodD less-irrefl lexord-irreflexive mem-Collect-eq*)
**done**

**lemma** *not-less-Nil* [*simp*]: $\neg\ x <_l\ []$
  **by** (*simp add*: *list-lex-less-def*)

**lemma** *Nil-less-Cons* [*simp*]: $[] <_l a \# x$
  **by** (*simp add*: *list-lex-less-def*)

**lemma** *Cons-less-Cons* [*simp*]: $a \# x <_l b \# y \longleftrightarrow a < b \lor a = b \land x <_l y$
  **by** (*simp add*: *list-lex-less-def*)
**end**

# 9 Partial Functions

**theory** *Partial-Fun*
**imports** *Optics.Lenses Map-Extra*
**begin**

I'm not completely satisfied with partial functions as provided by Map.thy, since they don't have a unique type and so we can't instantiate classes, make use of adhoc-overloading etc. Consequently I've created a new type and derived the laws.

## 9.1 Partial function type and operations

**typedef** $('a, 'b)$ *pfun* = *UNIV* :: $('a \rightharpoonup 'b)$ *set* **..**

**setup-lifting** *type-definition-pfun*

**lift-definition** *pfun-app* :: $('a, 'b)$ *pfun* $\Rightarrow$ $'a$ $\Rightarrow$ $'b$ ($-'(-')_p$ [*999,0*] *999*) **is**
$\lambda\ f\ x.\ if\ (x \in dom\ f)\ then\ the\ (f\ x)\ else\ undefined$ .

**lift-definition** *pfun-upd* :: $('a, 'b)$ *pfun* $\Rightarrow$ $'a$ $\Rightarrow$ $'b$ $\Rightarrow$ $('a, 'b)$ *pfun*
**is** $\lambda\ f\ k\ v.\ f(k := Some\ v)$ .

**lift-definition** *pdom* :: $('a, 'b)$ *pfun* $\Rightarrow$ $'a$ *set* **is** *dom* .

**lift-definition** *pran* :: $('a, 'b)$ *pfun* $\Rightarrow$ $'b$ *set* **is** *ran* .

**lift-definition** *pfun-comp* :: $('b, 'c)$ *pfun* $\Rightarrow$ $('a, 'b)$ *pfun* $\Rightarrow$ $('a, 'c)$ *pfun* (**infixl** $\circ_p$ *55*) **is** *map-comp* .

**lift-definition** *pfun-member* :: $'a \times 'b \Rightarrow ('a, 'b)$ *pfun* $\Rightarrow$ *bool* (**infix** $\in_p$ *50*) **is** $(\in_m)$ .

**lift-definition** *pId-on* :: $'a$ *set* $\Rightarrow$ $('a, 'a)$ *pfun* **is** $\lambda A\ x.\ if\ (x \in A)\ then\ Some\ x\ else\ None$ .

**abbreviation** *pId* :: $('a, 'a)$ *pfun* **where**
$pId \equiv pId\text{-}on\ UNIV$

**lift-definition** *plambda* :: $('a \Rightarrow bool) \Rightarrow ('a \Rightarrow 'b) \Rightarrow ('a, 'b)$ *pfun*
**is** $\lambda P\ f\ x.\ if\ (P\ x)\ then\ Some\ (f\ x)\ else\ None$ .

**lift-definition** *pdom-res* :: $'a$ *set* $\Rightarrow$ $('a, 'b)$ *pfun* $\Rightarrow$ $('a, 'b)$ *pfun* (**infixr** $\lhd_p$ *85*)
**is** $\lambda A\ f.\ restrict\text{-}map\ f\ A$ .

**lift-definition** *pran-res* :: $('a, 'b)$ *pfun* $\Rightarrow$ $'b$ *set* $\Rightarrow$ $('a, 'b)$ *pfun* (**infixl** $\rhd_p$ *85*)
**is** *ran-restrict-map* .

**lift-definition** *pfun-graph* :: $('a, 'b)$ *pfun* $\Rightarrow$ $('a \times 'b)$ *set* **is** *map-graph* .

**lift-definition** *graph-pfun* :: $('a \times 'b)$ *set* $\Rightarrow$ $('a, 'b)$ *pfun* **is** *graph-map* .

**lift-definition** *pfun-entries* :: $'k$ *set* $\Rightarrow$ $('k \Rightarrow 'v) \Rightarrow ('k, 'v)$ *pfun* **is**
$\lambda d\ f\ x.\ if\ (x \in d)\ then\ Some\ (f\ x)\ else\ None$ .

**definition** *pcard* :: $('a, 'b)$ *pfun* $\Rightarrow$ *nat*
**where** $pcard\ f = card\ (pdom\ f)$

**instantiation** *pfun* :: $(type, type)$ *zero*
**begin**
**lift-definition** *zero-pfun* :: $('a, 'b)$ *pfun* **is** *Map.empty* .
**instance** ..
**end**

**abbreviation** *pempty* :: $('a, 'b)$ *pfun* $(\{\}_p)$
**where** $pempty \equiv 0$

**instantiation** *pfun* :: $(type, type)$ *plus*
**begin**
**lift-definition** *plus-pfun* :: $('a, 'b)$ *pfun* $\Rightarrow$ $('a, 'b)$ *pfun* $\Rightarrow$ $('a, 'b)$ *pfun* **is** $(++)$ .
**instance** ..
**end**

**instantiation** *pfun* :: $(type, type)$ *minus*
**begin**
**lift-definition** *minus-pfun* :: $('a, 'b)$ *pfun* $\Rightarrow$ $('a, 'b)$ *pfun* $\Rightarrow$ $('a, 'b)$ *pfun* **is** $(--)$ .
**instance** ..
**end**

**instance** *pfun* :: $(type, type)$ *monoid-add*
  **by** $(intro\text{-}classes, (transfer, auto)+)$

**instantiation** *pfun* :: $(type, type)$ *inf*

**begin**
**lift-definition** *inf-pfun* :: $('a, 'b)$ *pfun* $\Rightarrow$ $('a, 'b)$ *pfun* $\Rightarrow$ $('a, 'b)$ *pfun* **is**
$\lambda\ f\ g\ x.\ if\ (x \in dom(f) \cap dom(g) \wedge f(x) = g(x))\ then\ f(x)\ else\ None$ **.**
**instance ..**
**end**

**abbreviation** *pfun-inter* :: $('a, 'b)$ *pfun* $\Rightarrow$ $('a, 'b)$ *pfun* $\Rightarrow$ $('a, 'b)$ *pfun* (**infixl** $\cap_p$ *80*)
**where** *pfun-inter* $\equiv$ *inf*

**instantiation** *pfun* :: $(type, type)$ *order*
**begin**
  **lift-definition** *less-eq-pfun* :: $('a, 'b)$ *pfun* $\Rightarrow$ $('a, 'b)$ *pfun* $\Rightarrow$ *bool* **is**
  $\lambda\ f\ g.\ f \subseteq_m g$ **.**
  **lift-definition** *less-pfun* :: $('a, 'b)$ *pfun* $\Rightarrow$ $('a, 'b)$ *pfun* $\Rightarrow$ *bool* **is**
  $\lambda\ f\ g.\ f \subseteq_m g \wedge f \neq g$ **.**
**instance**
  **by** (*intro-classes*, (*transfer*, *auto intro*: *map-le-trans simp add*: *map-le-antisym*)+)
**end**

**abbreviation** *pfun-subset* :: $('a, 'b)$ *pfun* $\Rightarrow$ $('a, 'b)$ *pfun* $\Rightarrow$ *bool* (**infix** $\subset_p$ *50*)
**where** *pfun-subset* $\equiv$ *less*

**abbreviation** *pfun-subset-eq* :: $('a, 'b)$ *pfun* $\Rightarrow$ $('a, 'b)$ *pfun* $\Rightarrow$ *bool* (**infix** $\subseteq_p$ *50*)
**where** *pfun-subset-eq* $\equiv$ *less-eq*

**instance** *pfun* :: $(type, type)$ *semilattice-inf*
  **by** (*intro-classes*, (*transfer*, *auto simp add*: *map-le-def dom-def*)+)

**lemma** *pfun-subset-eq-least* [*simp*]:
  $\{\}_p \subseteq_p f$
  **by** (*transfer*, *auto*)

**syntax**
  *-PfunUpd* :: $[('a, 'b)\ pfun, maplets] \Rightarrow ('a, 'b)\ pfun$ (*-'(-')$_p$* [*900,0*]*900*)
  *-Pfun*    :: $maplets \Rightarrow ('a, 'b)\ pfun$          (($1\{-\}_p$))
  *-plam*    :: $pttrn \Rightarrow logic \Rightarrow logic \Rightarrow logic$ ($\lambda$ - | - . - [*0,0,10*] *10*)

**translations**
  *-PfunUpd m* (*-Maplets xy ms*)  == *-PfunUpd* (*-PfunUpd m xy*) *ms*
  *-PfunUpd m* (*-maplet  x y*)    == *CONST pfun-upd m x y*
  *-Pfun ms*                 => *-PfunUpd* (*CONST pempty*) *ms*
  *-Pfun* (*-Maplets ms1 ms2*)     <= *-PfunUpd* (*-Pfun ms1*) *ms2*
  *-Pfun ms*                 <= *-PfunUpd* (*CONST pempty*) *ms*
  $\lambda\ x\ |\ P\ .\ e$           => *CONST plambda* ($\lambda\ x.\ P$) ($\lambda\ x.\ e$)
  $\lambda\ x\ |\ P\ .\ e$           <= *CONST plambda* ($\lambda\ x.\ P$) ($\lambda\ y.\ e$)
  $\lambda\ y\ |\ P\ .\ e$           <= *CONST plambda* ($\lambda\ x.\ P$) ($\lambda\ y.\ e$)
  $\lambda\ y\ |\ f\ v\ y\ .\ e$         <= *CONST plambda* ($f\ v$) ($\lambda\ y.\ e$)

## 9.2 Algebraic laws

**lemma** *pfun-comp-assoc*: $f \circ_p (g \circ_p h) = (f \circ_p g) \circ_p h$
  **by** (*transfer*, *simp add*: *map-comp-assoc*)

**lemma** *pfun-comp-left-id* [*simp*]: $pId \circ_p f = f$
  **by** (*transfer*, *auto*)

**lemma** *pfun-comp-right-id* [*simp*]: $f \circ_p pId = f$
  **by** (*transfer*, *auto*)

**lemma** *pfun-override-dist-comp*:
  $(f + g) \circ_p h = (f \circ_p h) + (g \circ_p h)$
  **apply** (*transfer*)
  **apply** (*rule ext*)
  **apply** (*auto simp add*: *map-add-def*)
  **apply** (*rename-tac f g h x*)
  **apply** (*case-tac h x*)
   **apply** (*auto*)
  **apply** (*rename-tac f g h x y*)
  **apply** (*case-tac g y*)
   **apply** (*auto*)
  **done**

**lemma** *pfun-minus-unit* [*simp*]:
  **fixes** $f :: ('a, 'b) \ pfun$
  **shows** $f - 0 = f$
  **by** (*transfer*, *simp add*: *map-minus-def*)

**lemma** *pfun-minus-zero* [*simp*]:
  **fixes** $f :: ('a, 'b) \ pfun$
  **shows** $0 - f = 0$
  **by** (*transfer*, *simp add*: *map-minus-def*)

**lemma** *pfun-minus-self* [*simp*]:
  **fixes** $f :: ('a, 'b) \ pfun$
  **shows** $f - f = 0$
  **by** (*transfer*, *simp add*: *map-minus-def*)

**lemma** *pfun-plus-commute*:
  $pdom(f) \cap pdom(g) = \{\} \Longrightarrow f + g = g + f$
  **by** (*transfer*, *metis map-add-comm*)

**lemma** *pfun-plus-commute-weak*:
  $(\forall \ k \in pdom(f) \cap pdom(g). \ f(k)_p = g(k)_p) \Longrightarrow f + g = g + f$
  **by** (*transfer*, *simp*, *metis IntD1 IntD2 domD map-add-comm-weak option.sel*)

**lemma** *pfun-minus-plus-commute*:
  $pdom(g) \cap pdom(h) = \{\} \Longrightarrow (f - g) + h = (f + h) - g$
  **by** (*transfer*, *simp add*: *map-minus-plus-commute*)

**lemma** *pfun-plus-minus*:
  $f \subseteq_p g \Longrightarrow (g - f) + f = g$
  **by** (*transfer*, *rule ext*, *auto simp add*: *map-le-def map-minus-def map-add-def option.case-eq-if*)

**lemma** *pfun-minus-common-subset*:
  $[\![ \ h \subseteq_p f; \ h \subseteq_p g \ ]\!] \Longrightarrow (f - h = g - h) = (f = g)$
  **by** (*transfer*, *simp add*: *map-minus-common-subset*)

**lemma** *pfun-minus-plus*:
  $pdom(f) \cap pdom(g) = \{\} \Longrightarrow (f + g) - g = f$
  **by** (*transfer*, *simp add*: *map-add-def map-minus-def option.case-eq-if*, *rule ext*, *auto*)
    (*metis Int-commute domIff insert-disjoint(1) insert-dom*)

**lemma** *pfun-plus-pos*: $x + y = \{\}_p \implies x = \{\}_p$
  **by** (*transfer*, *simp*)

**lemma** *pfun-le-plus*: *pdom* $x \cap$ *pdom* $y = \{\} \implies x \leq x + y$
  **by** (*transfer*, *auto simp add*: *map-le-iff-add*)

## 9.3  Lambda abstraction

**lemma** *plambda-app* [*simp*]: $(\lambda\ x\ |\ P\ x\ .\ f\ x)(v)_p = (if\ (P\ v)\ then\ (f\ v)\ else\ undefined)$
  **by** (*transfer*, *auto*)

**lemma** *plambda-eta* [*simp*]: $(\lambda\ x\ |\ x \in pdom(f).\ f(x)_p) = f$
  **by** (*transfer*; *auto simp add*: *domIff*)

**lemma** *plambda-id* [*simp*]: $(\lambda\ x\ |\ P\ x\ .\ x) = pId\text{-}on\ \{x.\ P\ x\}$
  **by** (*transfer*, *simp*)

## 9.4  Membership, application, and update

**lemma** *pfun-ext*: $\llbracket \bigwedge x\ y.\ (x,\ y) \in_p f \longleftrightarrow (x,\ y) \in_p g \rrbracket \implies f = g$
  **by** (*transfer*, *simp add*: *map-ext*)

**lemma** *pfun-member-alt-def*:
  $(x,\ y) \in_p f \longleftrightarrow (x \in pdom\ f \wedge f(x)_p = y)$
  **by** (*transfer*, *auto simp add*: *map-member-alt-def map-apply-def*)

**lemma** *pfun-member-plus*:
  $(x,\ y) \in_p f + g \longleftrightarrow ((x \notin pdom(g) \wedge (x,\ y) \in_p f) \vee (x,\ y) \in_p g)$
  **by** (*transfer*, *simp add*: *map-member-plus*)

**lemma** *pfun-member-minus*:
  $(x,\ y) \in_p f - g \longleftrightarrow (x,\ y) \in_p f \wedge (\neg\ (x,\ y) \in_p g)$
  **by** (*transfer*, *simp add*: *map-member-minus*)

**lemma** *pfun-app-upd-1* [*simp*]: $x = y \implies (f(x \mapsto v)_p)(y)_p = v$
  **by** (*transfer*, *simp*)

**lemma** *pfun-app-upd-2* [*simp*]: $x \neq y \implies (f(x \mapsto v)_p)(y)_p = f(y)_p$
  **by** (*transfer*, *simp*)

**lemma** *pfun-graph-apply* [*simp*]: *rel-apply* (*pfun-graph* $f$) $x = f(x)_p$
  **by** (*transfer*, *auto simp add*: *rel-apply-def map-graph-def*)

**lemma** *pfun-upd-ext* [*simp*]: $x \in pdom(f) \implies f(x \mapsto f(x)_p)_p = f$
  **by** (*transfer*, *simp add*: *domIff*)

**lemma** *pfun-app-add* [*simp*]: $x \in pdom(g) \implies (f + g)(x)_p = g(x)_p$
  **by** (*transfer*, *auto*)

**lemma** *pfun-upd-add* [*simp*]: $f + g(x \mapsto v)_p = (f + g)(x \mapsto v)_p$
  **by** (*transfer*, *simp*)

**lemma** *pfun-upd-twice* [*simp*]: $f(x \mapsto u,\ x \mapsto v)_p = f(x \mapsto v)_p$
  **by** (*transfer*, *simp*)

**lemma** *pfun-upd-comm*:
  **assumes** $x \neq y$
  **shows** $f(y \mapsto u, \ x \mapsto v)_p = f(x \mapsto v, \ y \mapsto u)_p$
  **using** *assms* **by** (*transfer*, *auto*)

**lemma** *pfun-upd-comm-linorder* [*simp*]:
  **fixes** $x \ y :: \ 'a :: linorder$
  **assumes** $x < y$
  **shows** $f(y \mapsto u, \ x \mapsto v)_p = f(x \mapsto v, \ y \mapsto u)_p$
  **using** *assms* **by** (*transfer*, *auto*)

**lemma** *pfun-app-minus* [*simp*]: $x \notin pdom \ g \implies (f - g)(x)_p = f(x)_p$
  **by** (*transfer*, *auto simp add*: *map-minus-def*)

**lemma** *pfun-app-empty* [*simp*]: $\{\}_p(x)_p = undefined$
  **by** (*transfer*, *simp*)

**lemma** *pfun-app-not-in-dom*:
  $x \notin pdom(f) \implies f(x)_p = undefined$
  **by** (*transfer*, *simp*)

**lemma** *pfun-upd-minus* [*simp*]:
  $x \notin pdom \ g \implies (f - g)(x \mapsto v)_p = (f(x \mapsto v)_p - g)$
  **by** (*transfer*, *auto simp add*: *map-minus-def*)

**lemma** *pdom-member-minus-iff* [*simp*]:
  $x \notin pdom \ g \implies x \in pdom(f - g) \longleftrightarrow x \in pdom(f)$
  **by** (*transfer*, *simp add*: *domIff map-minus-def*)

**lemma** *psubseteq-pfun-upd1* [*intro*]:
  $[\![ \ f \subseteq_p g; \ x \notin pdom(g) \ ]\!] \implies f \subseteq_p g(x \mapsto v)_p$
  **by** (*transfer*, *auto simp add*: *map-le-def dom-def*)

**lemma** *psubseteq-pfun-upd2* [*intro*]:
  $[\![ \ f \subseteq_p g; \ x \notin pdom(f) \ ]\!] \implies f \subseteq_p g(x \mapsto v)_p$
  **by** (*transfer*, *auto simp add*: *map-le-def dom-def*)

**lemma** *psubseteq-pfun-upd3* [*intro*]:
  $[\![ \ f \subseteq_p g; \ g(x)_p = v \ ]\!] \implies f \subseteq_p g(x \mapsto v)_p$
  **by** (*transfer*, *auto simp add*: *map-le-def dom-def*)

**lemma** *psubseteq-dom-subset*:
  $f \subseteq_p g \implies pdom(f) \subseteq pdom(g)$
  **by** (*transfer*, *auto simp add*: *map-le-def dom-def*)

**lemma** *psubseteq-ran-subset*:
  $f \subseteq_p g \implies pran(f) \subseteq pran(g)$
  **by** (*transfer*, *auto simp add*: *map-le-def dom-def ran-def*, *fastforce*)

## 9.5   Domain laws

**lemma** *pdom-zero* [*simp*]: $pdom \ 0 = \{\}$
  **by** (*transfer*, *simp*)

**lemma** *pdom-pId-on* [*simp*]: $pdom \ (pId\text{-}on \ A) = A$
  **by** (*transfer*, *auto*)

**lemma** *pdom-plus* [*simp*]: *pdom* $(f + g) = pdom\ f \cup pdom\ g$
  **by** (*transfer*, *auto*)

**lemma** *pdom-minus* [*simp*]: $g \leq f \implies pdom\ (f - g) = pdom\ f - pdom\ g$
  **apply** (*transfer*, *auto simp add*: *map-minus-def*)
  **apply** (*meson option.distinct(1)*)
  **apply** (*metis domIff map-le-def option.simps(3)*)
  **done**

**lemma** *pdom-inter*: *pdom* $(f \cap_p g) \subseteq pdom\ f \cap pdom\ g$
  **by** (*transfer*, *auto simp add*: *dom-def*)

**lemma** *pdom-comp* [*simp*]: *pdom* $(g \circ_p f) = pdom\ (f \rhd_p pdom\ g)$
  **by** (*transfer*, *auto simp add*: *ran-restrict-map-def*)

**lemma** *pdom-upd* [*simp*]: *pdom* $(f(k \mapsto v)_p) = insert\ k\ (pdom\ f)$
  **by** (*transfer*, *simp*)

**lemma** *pdom-plamda* [*simp*]: *pdom* $(\lambda\ x \mid P\ x\ .\ f\ x) = \{x.\ P\ x\}$
  **by** (*transfer*, *auto*)

**lemma** *pdom-pdom-res* [*simp*]: *pdom* $(A \lhd_p f) = A \cap pdom(f)$
  **by** (*transfer*, *auto*)

**lemma** *pdom-graph-pfun* [*simp*]: *pdom* $(graph\text{-}pfun\ R) = Domain\ R$
  **by** (*transfer*, *simp add*: *Domain-fst graph-map-dom*)

**lemma** *pdom-pran-res-finite* [*simp*]:
  *finite* $(pdom\ f) \implies finite\ (pdom\ (f \rhd_p A))$
  **by** (*transfer*, *auto*)

**lemma** *pdom-pfun-graph-finite* [*simp*]:
  *finite* $(pdom\ f) \implies finite\ (pfun\text{-}graph\ f)$
  **by** (*transfer*, *simp add*: *finite-dom-graph*)

## 9.6 Range laws

**lemma** *pran-zero* [*simp*]: *pran* $0 = \{\}$
  **by** (*transfer*, *simp*)

**lemma** *pran-pId-on* [*simp*]: *pran* $(pId\text{-}on\ A) = A$
  **by** (*transfer*, *auto simp add*: *ran-def*)

**lemma** *pran-upd* [*simp*]: *pran* $(f(k \mapsto v)_p) = insert\ v\ (pran\ ((-\ \{k\}) \lhd_p f))$
  **by** (*transfer*, *auto simp add*: *ran-def restrict-map-def*)

**lemma** *pran-plamda* [*simp*]: *pran* $(\lambda\ x \mid P\ x\ .\ f\ x) = \{f\ x \mid x.\ P\ x\}$
  **by** (*transfer*, *auto simp add*: *ran-def*)

**lemma** *pran-pran-res* [*simp*]: *pran* $(f \rhd_p A) = pran(f) \cap A$
  **by** (*transfer*, *auto*)

**lemma** *pran-comp* [*simp*]: *pran* $(g \circ_p f) = pran\ (pran\ f \lhd_p g)$
  **by** (*transfer*, *auto simp add*: *ran-def restrict-map-def*)

**lemma** *pran-finite* [*simp*]: *finite* (*pdom f*) $\Longrightarrow$ *finite* (*pran f*)
  **by** (*transfer*, *auto*)

## 9.7  Domain restriction laws

**lemma** *pdom-res-zero* [*simp*]: $A \lhd_p \{\}_p = \{\}_p$
  **by** (*transfer*, *auto*)

**lemma** *pdom-res-empty* [*simp*]:
  $(\{\} \lhd_p f) = \{\}_p$
  **by** (*transfer*, *auto*)

**lemma** *pdom-res-pdom* [*simp*]:
  $pdom(f) \lhd_p f = f$
  **by** (*transfer*, *auto*)

**lemma** *pdom-res-UNIV* [*simp*]: $UNIV \lhd_p f = f$
  **by** (*transfer*, *auto*)

**lemma** *pdom-res-alt-def*: $A \lhd_p f = f \circ_p pId\text{-}on\ A$
  **by** (*transfer*, *rule ext*, *auto simp add*: *restrict-map-def*)

**lemma** *pdom-res-upd-in* [*simp*]:
  $k \in A \Longrightarrow A \lhd_p f(k \mapsto v)_p = (A \lhd_p f)(k \mapsto v)_p$
  **by** (*transfer*, *auto*)

**lemma** *pdom-res-upd-out* [*simp*]:
  $k \notin A \Longrightarrow A \lhd_p f(k \mapsto v)_p = A \lhd_p f$
  **by** (*transfer*, *auto*)

**lemma** *pfun-pdom-antires-upd* [*simp*]:
  $k \in A \Longrightarrow ((- A) \lhd_p m)(k \mapsto v)_p = ((- (A - \{k\})) \lhd_p m)(k \mapsto v)_p$
  **by** (*transfer*, *simp*)

**lemma** *pdom-antires-insert-notin* [*simp*]:
  $k \notin pdom(f) \Longrightarrow (- insert\ k\ A) \lhd_p f = (- A) \lhd_p f$
  **by** (*transfer*, *auto simp add*: *restrict-map-def*)

**lemma** *pdom-res-override* [*simp*]: $A \lhd_p (f + g) = (A \lhd_p f) + (A \lhd_p g)$
  **by** (*simp add*: *pdom-res-alt-def pfun-override-dist-comp*)

**lemma** *pdom-res-minus* [*simp*]: $A \lhd_p (f - g) = (A \lhd_p f) - g$
  **by** (*transfer*, *auto simp add*: *map-minus-def restrict-map-def*)

**lemma** *pdom-res-swap*: $(A \lhd_p f) \rhd_p B = A \lhd_p (f \rhd_p B)$
  **by** (*transfer*, *auto simp add*: *restrict-map-def ran-restrict-map-def*)

**lemma** *pdom-res-twice* [*simp*]: $A \lhd_p (B \lhd_p f) = (A \cap B) \lhd_p f$
  **by** (*transfer*, *auto simp add*: *Int-commute*)

**lemma** *pdom-res-comp* [*simp*]: $A \lhd_p (g \circ_p f) = g \circ_p (A \lhd_p f)$
  **by** (*simp add*: *pdom-res-alt-def pfun-comp-assoc*)

**lemma** *pdom-res-apply* [*simp*]:
  $x \in A \Longrightarrow (A \lhd_p f)(x)_p = f(x)_p$
  **by** (*transfer*, *auto*)

## 9.8   Range restriction laws

**lemma** *pran-res-zero* [*simp*]: $\{\}_p \rhd_p A = \{\}_p$
  **by** (*transfer*, *auto simp add*: *ran-restrict-map-def*)

**lemma** *pran-res-upd-1* [*simp*]: $v \in A \Longrightarrow f(x \mapsto v)_p \rhd_p A = (f \rhd_p A)(x \mapsto v)_p$
  **by** (*transfer*, *auto simp add*: *ran-restrict-map-def*)

**lemma** *pran-res-upd-2* [*simp*]: $v \notin A \Longrightarrow f(x \mapsto v)_p \rhd_p A = ((- \{x\}) \lhd_p f) \rhd_p A$
  **by** (*transfer*, *auto simp add*: *ran-restrict-map-def*)

**lemma** *pran-res-alt-def*: $f \rhd_p A = pId\text{-}on\ A \circ_p f$
  **by** (*transfer*, *rule ext*, *auto simp add*: *ran-restrict-map-def*)

**lemma** *pran-res-override*: $(f + g) \rhd_p A \subseteq_p (f \rhd_p A) + (g \rhd_p A)$
  **apply** (*transfer*, *auto simp add*: *map-add-def ran-restrict-map-def map-le-def*)
  **apply** (*rename-tac f g A a y x*)
  **apply** (*case-tac g a*)
   **apply** (*auto*)
  **done**

## 9.9   Graph laws

**lemma** *pfun-graph-inv*: *graph-pfun* (*pfun-graph f*) = *f*
  **by** (*transfer*, *simp*)

**lemma** *pfun-graph-zero*: *pfun-graph* $0 = \{\}$
  **by** (*transfer*, *simp add*: *map-graph-def*)

**lemma** *pfun-graph-pId-on*: *pfun-graph* (*pId-on A*) = *Id-on A*
  **by** (*transfer*, *auto simp add*: *map-graph-def*)

**lemma** *pfun-graph-minus*: *pfun-graph* $(f - g) = pfun\text{-}graph\ f - pfun\text{-}graph\ g$
  **by** (*transfer*, *simp add*: *map-graph-minus*)

**lemma** *pfun-graph-inter*: *pfun-graph* $(f \cap_p g) = pfun\text{-}graph\ f \cap pfun\text{-}graph\ g$
  **apply** (*transfer*, *auto simp add*: *map-graph-def*)
   **apply** (*metis option.discI*)+
  **done**

## 9.10   Entries

**lemma** *pfun-entries-empty* [*simp*]: *pfun-entries* $\{\}\ f = \{\}_p$
  **by** (*transfer*, *simp*)

**lemma** *pfun-entries-apply-1* [*simp*]:
  $x \in d \Longrightarrow (pfun\text{-}entries\ d\ f)(x)_p = f\ x$
  **by** (*transfer*, *auto*)

**lemma** *pfun-entries-apply-2* [*simp*]:
  $x \notin d \Longrightarrow (pfun\text{-}entries\ d\ f)(x)_p = undefined$
  **by** (*transfer*, *auto*)

## 9.11   Summation

**definition** *pfun-sum* :: $('k,\ 'v::comm\text{-}monoid\text{-}add)\ pfun \Rightarrow 'v$ **where**

*pfun-sum f = sum (pfun-app f) (pdom f)*

**lemma** *pfun-sum-empty* [*simp*]: *pfun-sum* $\{\}_p$ = *0*
  **by** (*simp add*: *pfun-sum-def*)

**lemma** *pfun-sum-upd-1*:
  **assumes** *finite(pdom(m))* $k \notin pdom(m)$
  **shows** *pfun-sum* $(m(k \mapsto v)_p)$ = *pfun-sum m* + *v*
  **by** (*simp-all add*: *pfun-sum-def assms*, *metis add.commute assms(2) pfun-app-upd-2 sum.cong*)

**lemma** *pfun-sums-upd-2*:
  **assumes** *finite(pdom(m))*
  **shows** *pfun-sum* $(m(k \mapsto v)_p)$ = *pfun-sum* $((- \{k\}) \lhd_p m)$ + *v*
**proof** (*cases* $k \notin pdom(m)$)
  **case** *True*
  **then show** *?thesis*
    **by** (*simp add*: *pfun-sum-upd-1 assms*)
**next**
  **case** *False*
  **then show** *?thesis*
    **using** *assms pfun-sum-upd-1* [*of* $((- \{k\}) \lhd_p m)$ *k v*]
    **by** (*simp add*: *pfun-sum-upd-1*)
**qed**

**lemma** *pfun-sum-dom-res-insert* [*simp*]:
  **assumes** $x \in pdom f$ $x \notin A$ *finite A*
  **shows** *pfun-sum* $((insert\ x\ A) \lhd_p f)$ = $f(x)_p$ + *pfun-sum* $(A \lhd_p f)$
  **using** *assms* **by** (*simp add*: *pfun-sum-def*)

**lemma** *pfun-sum-pdom-res*:
  **fixes** *f* :: ($'a,'b$::*ab-group-add*) *pfun*
  **assumes** *finite(pdom f)*
  **shows** *pfun-sum* $(A \lhd_p f)$ = *pfun-sum f* − $(pfun\text{-}sum\ ((- A) \lhd_p f))$
**proof** −
  **have** *1*:$A \cap pdom(f)$ = $pdom(f)$ − $(pdom(f) - A)$
    **by** (*auto*)
  **show** *?thesis*
    **apply** (*simp add*: *pfun-sum-def*)
    **apply** (*subst 1*)
    **apply** (*subst sum-diff*)
      **apply** (*auto simp add*: *sum-diff Diff-subset Int-commute boolean-algebra-class.diff-eq assms*)
    **done**
**qed**

**lemma** *pfun-sum-pdom-antires* [*simp*]:
  **fixes** *f* :: ($'a,'b$::*ab-group-add*) *pfun*
  **assumes** *finite(pdom f)*
  **shows** *pfun-sum* $((- A) \lhd_p f)$ = *pfun-sum f* − *pfun-sum* $(A \lhd_p f)$
  **by** (*subst pfun-sum-pdom-res*, *simp-all add*: *assms*)

## 9.12   Partial Function Lens

**definition** *pfun-lens* :: $'a \Rightarrow ('b \Longrightarrow ('a, 'b)\ pfun)$ **where**
[*lens-defs*]: *pfun-lens i* = ⦇ *lens-get* = $\lambda\ s.\ s(i)_p$, *lens-put* = $\lambda\ s\ v.\ s(i \mapsto v)_p$ ⦈

**lemma** *pfun-lens-mwb* [*simp*]: *mwb-lens* (*pfun-lens i*)

**by** (*unfold-locales*, *simp-all add*: *pfun-lens-def*)

**lemma** *pfun-lens-src*: $\mathcal{S}_{pfun\text{-}lens\ i} = \{f.\ i \in pdom(f)\}$
  **by** (*auto simp add*: *lens-defs lens-source-def*, *transfer*, *force*)

Hide implementation details for partial functions

**lifting-update** *pfun.lifting*
**lifting-forget** *pfun.lifting*

**end**

# 10 Finite Functions

**theory** *Finite-Fun*
**imports** *Map-Extra Partial-Fun FSet-Extra*
**begin**

## 10.1 Finite function type and operations

**typedef** $('a,\ 'b)\ ffun = \{f :: ('a,\ 'b)\ pfun.\ finite(pdom(f))\}$
  **morphisms** *pfun-of Abs-pfun*
  **by** (*rule-tac* $x=\{\}_p$ **in** *exI*, *auto*)

**setup-lifting** *type-definition-ffun*

**lift-definition** *ffun-app* :: $('a,\ 'b)\ ffun \Rightarrow\ 'a \Rightarrow\ 'b\ (\text{-}'(\text{-}')_f\ [999,0]\ 999)$ **is** *pfun-app* .

**lift-definition** *ffun-upd* :: $('a,\ 'b)\ ffun \Rightarrow\ 'a \Rightarrow\ 'b \Rightarrow\ ('a,\ 'b)\ ffun$ **is** *pfun-upd* **by** *simp*

**lift-definition** *fdom* :: $('a,\ 'b)\ ffun \Rightarrow\ 'a\ set$ **is** *pdom* .

**lift-definition** *fran* :: $('a,\ 'b)\ ffun \Rightarrow\ 'b\ set$ **is** *pran* .

**lift-definition** *ffun-comp* :: $('b,\ 'c)\ ffun \Rightarrow ('a,\ 'b)\ ffun \Rightarrow ('a,\ 'c)\ ffun$ (**infixl** $\circ_f$ *55*) **is** *pfun-comp* **by** *auto*

**lift-definition** *ffun-member* :: $'a \times\ 'b \Rightarrow ('a,\ 'b)\ ffun \Rightarrow\ bool$ (**infix** $\in_f$ *50*) **is** $(\in_p)$ .

**lift-definition** *fdom-res* :: $'a\ set \Rightarrow ('a,\ 'b)\ ffun \Rightarrow ('a,\ 'b)\ ffun$ (**infixl** $\lhd_f$ *85*)
**is** *pdom-res* **by** *simp*

**lift-definition** *fran-res* :: $('a,\ 'b)\ ffun \Rightarrow\ 'b\ set \Rightarrow ('a,\ 'b)\ ffun$ (**infixl** $\rhd_f$ *85*)
**is** *pran-res* **by** *simp*

**lift-definition** *ffun-graph* :: $('a,\ 'b)\ ffun \Rightarrow ('a \times\ 'b)\ set$ **is** *pfun-graph* .

**lift-definition** *graph-ffun* :: $('a \times\ 'b)\ set \Rightarrow ('a,\ 'b)\ ffun$ **is**
  $\lambda\ R.\ if\ (finite\ (Domain\ R))\ then\ graph\text{-}pfun\ R\ else\ pempty$
  **by** (*simp add*: *finite-Domain*)

**instantiation** *ffun* :: (*type*, *type*) *zero*
**begin**
**lift-definition** *zero-ffun* :: $('a,\ 'b)\ ffun$ **is** *0* **by** *simp*
**instance** **..**
**end**

**abbreviation** *fempty* :: $('a, 'b)$ *ffun* $(\{\}_f)$
**where** *fempty* $\equiv$ *0*

**instantiation** *ffun* :: (*type*, *type*) *plus*
**begin**
**lift-definition** *plus-ffun* :: $('a, 'b)$ *ffun* $\Rightarrow$ $('a, 'b)$ *ffun* $\Rightarrow$ $('a, 'b)$ *ffun* **is** $(+)$ **by** *simp*
**instance ..**
**end**

**instantiation** *ffun* :: (*type*, *type*) *minus*
**begin**
**lift-definition** *minus-ffun* :: $('a, 'b)$ *ffun* $\Rightarrow$ $('a, 'b)$ *ffun* $\Rightarrow$ $('a, 'b)$ *ffun* **is** $(-)$
 **by** (*metis finite-Diff finite-Domain pdom-graph-pfun pdom-pfun-graph-finite pfun-graph-inv pfun-graph-minus*)
**instance ..**
**end**

**instance** *ffun* :: (*type*, *type*) *monoid-add*
  **by** (*intro-classes*, (*transfer*, *simp add*: *add.assoc*)+)

**instantiation** *ffun* :: (*type*, *type*) *inf*
**begin**
**lift-definition** *inf-ffun* :: $('a, 'b)$ *ffun* $\Rightarrow$ $('a, 'b)$ *ffun* $\Rightarrow$ $('a, 'b)$ *ffun* **is** *inf*
  **by** (*meson finite-Int infinite-super pdom-inter*)
**instance ..**
**end**

**abbreviation** *ffun-inter* :: $('a, 'b)$ *ffun* $\Rightarrow$ $('a, 'b)$ *ffun* $\Rightarrow$ $('a, 'b)$ *ffun* (**infixl** $\cap_f$ *80*)
**where** *ffun-inter* $\equiv$ *inf*

**instantiation** *ffun* :: (*type*, *type*) *order*
**begin**
  **lift-definition** *less-eq-ffun* :: $('a, 'b)$ *ffun* $\Rightarrow$ $('a, 'b)$ *ffun* $\Rightarrow$ *bool* **is**
  $\lambda\, f\, g.\ f \subseteq_p g$ .
  **lift-definition** *less-ffun* :: $('a, 'b)$ *ffun* $\Rightarrow$ $('a, 'b)$ *ffun* $\Rightarrow$ *bool* **is**
  $\lambda\, f\, g.\ f < g$ .
**instance**
  **by** (*intro-classes*, (*transfer*, *auto*)+)
**end**

**abbreviation** *ffun-subset* :: $('a, 'b)$ *ffun* $\Rightarrow$ $('a, 'b)$ *ffun* $\Rightarrow$ *bool* (**infix** $\subset_f$ *50*)
**where** *ffun-subset* $\equiv$ *less*

**abbreviation** *ffun-subset-eq* :: $('a, 'b)$ *ffun* $\Rightarrow$ $('a, 'b)$ *ffun* $\Rightarrow$ *bool* (**infix** $\subseteq_f$ *50*)
**where** *ffun-subset-eq* $\equiv$ *less-eq*

**instance** *ffun* :: (*type*, *type*) *semilattice-inf*
  **by** (*intro-classes*, (*transfer*, *auto*)+)

**lemma** *ffun-subset-eq-least* [*simp*]:
  $\{\}_f \subseteq_f f$
  **by** (*transfer*, *auto*)

**syntax**
  *-FfunUpd* :: [$('a, 'b)$ *ffun*, *maplets*] => $('a, 'b)$ *ffun* ($-'(-')_f$ [*900,0*]*900*)

*-Ffun*  $::$ *maplets* $=>$ $('a,\ 'b)\ ffun$  $((1\{-\}_f))$

**translations**
 *-FfunUpd m (-Maplets xy ms)*  $==$ *-FfunUpd (-FfunUpd m xy) ms*
 *-FfunUpd m (-maplet  x y)*  $==$ *CONST ffun-upd m x y*
 *-Ffun ms*  $=>$ *-FfunUpd (CONST fempty) ms*
 *-Ffun (-Maplets ms1 ms2)*  $<=$ *-FfunUpd (-Ffun ms1) ms2*
 *-Ffun ms*  $<=$ *-FfunUpd (CONST fempty) ms*

## 10.2  Algebraic laws

**lemma** *ffun-comp-assoc*: $f \circ_f (g \circ_f h) = (f \circ_f g) \circ_f h$
 **by** (*transfer*, *simp add*: *pfun-comp-assoc*)

**lemma** *pfun-override-dist-comp*:
 $(f + g) \circ_f h = (f \circ_f h) + (g \circ_f h)$
 **by** (*transfer*, *simp add*: *pfun-override-dist-comp*)

**lemma** *ffun-minus-unit* [*simp*]:
 **fixes** $f :: ('a,\ 'b)\ ffun$
 **shows** $f - 0 = f$
 **by** (*transfer*, *simp*)

**lemma** *ffun-minus-zero* [*simp*]:
 **fixes** $f :: ('a,\ 'b)\ ffun$
 **shows** $0 - f = 0$
 **by** (*transfer*, *simp*)

**lemma** *ffun-minus-self* [*simp*]:
 **fixes** $f :: ('a,\ 'b)\ ffun$
 **shows** $f - f = 0$
 **by** (*transfer*, *simp*)

**lemma** *ffun-plus-commute*:
 $fdom(f) \cap fdom(g) = \{\} \implies f + g = g + f$
 **by** (*transfer*, *metis pfun-plus-commute*)

**lemma** *ffun-minus-plus-commute*:
 $fdom(g) \cap fdom(h) = \{\} \implies (f - g) + h = (f + h) - g$
 **by** (*transfer*, *simp add*: *pfun-minus-plus-commute*)

**lemma** *ffun-plus-minus*:
 $f \subseteq_f g \implies (g - f) + f = g$
 **by** (*transfer*, *simp add*: *pfun-plus-minus*)

**lemma** *ffun-minus-common-subset*:
 $[\![\ h \subseteq_f f;\ h \subseteq_f g\ ]\!] \implies (f - h = g - h) = (f = g)$
 **by** (*transfer*, *simp add*: *pfun-minus-common-subset*)

**lemma** *ffun-minus-plus*:
 $fdom(f) \cap fdom(g) = \{\} \implies (f + g) - g = f$
 **by** (*transfer*, *simp add*: *pfun-minus-plus*)

**lemma** *ffun-plus-pos*: $x + y = \{\}_f \implies x = \{\}_f$
 **by** (*transfer*, *simp add*: *pfun-plus-pos*)

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**lemma** *ffun-le-plus*: $fdom\ x \cap fdom\ y = \{\} \implies x \le x + y$
  **by** (*transfer*, *simp add*: *pfun-le-plus*)

## 10.3 Membership, application, and update

**lemma** *ffun-ext*: $[\![ \bigwedge x\ y.\ (x,\ y) \in_f f \longleftrightarrow (x,\ y) \in_f g ]\!] \implies f = g$
  **by** (*transfer*, *simp add*: *pfun-ext*)

**lemma** *ffun-member-alt-def*:
  $(x,\ y) \in_f f \longleftrightarrow (x \in fdom\ f \wedge f(x)_f = y)$
  **by** (*transfer*, *simp add*: *pfun-member-alt-def*)

**lemma** *ffun-member-plus*:
  $(x,\ y) \in_f f + g \longleftrightarrow ((x \notin fdom(g) \wedge (x,\ y) \in_f f) \vee (x,\ y) \in_f g)$
  **by** (*transfer*, *simp add*: *pfun-member-plus*)

**lemma** *ffun-member-minus*:
  $(x,\ y) \in_f f - g \longleftrightarrow (x,\ y) \in_f f \wedge (\neg\ (x,\ y) \in_f g)$
  **by** (*transfer*, *simp add*: *pfun-member-minus*)

**lemma** *ffun-app-upd-1* [*simp*]: $x = y \implies (f(x \mapsto v)_f)(y)_f = v$
  **by** (*transfer*, *simp*)

**lemma** *ffun-app-upd-2* [*simp*]: $x \ne y \implies (f(x \mapsto v)_f)(y)_f = f(y)_f$
  **by** (*transfer*, *simp*)

**lemma** *ffun-upd-ext* [*simp*]: $x \in fdom(f) \implies f(x \mapsto f(x)_f)_f = f$
  **by** (*transfer*, *simp*)

**lemma** *ffun-app-add* [*simp*]: $x \in fdom(g) \implies (f + g)(x)_f = g(x)_f$
  **by** (*transfer*, *simp*)

**lemma** *ffun-upd-add* [*simp*]: $f + g(x \mapsto v)_f = (f + g)(x \mapsto v)_f$
  **by** (*transfer*, *simp*)

**lemma** *ffun-upd-twice* [*simp*]: $f(x \mapsto u,\ x \mapsto v)_f = f(x \mapsto v)_f$
  **by** (*transfer*, *simp*)

**lemma** *ffun-upd-comm*:
  **assumes** $x \ne y$
  **shows** $f(y \mapsto u,\ x \mapsto v)_f = f(x \mapsto v,\ y \mapsto u)_f$
  **using** *assms* **by** (*transfer*, *simp add*: *pfun-upd-comm*)

**lemma** *ffun-upd-comm-linorder* [*simp*]:
  **fixes** $x\ y :: 'a :: linorder$
  **assumes** $x < y$
  **shows** $f(y \mapsto u,\ x \mapsto v)_f = f(x \mapsto v,\ y \mapsto u)_f$
  **using** *assms* **by** (*transfer*, *auto*)

**lemma** *ffun-app-minus* [*simp*]: $x \notin fdom\ g \implies (f - g)(x)_f = f(x)_f$
  **by** (*transfer*, *auto*)

**lemma** *ffun-upd-minus* [*simp*]:
  $x \notin fdom\ g \implies (f - g)(x \mapsto v)_f = (f(x \mapsto v)_f - g)$
  **by** (*transfer*, *auto*)

**lemma** *fdom-member-minus-iff* [*simp*]:
$x \notin fdom\ g \Longrightarrow x \in fdom(f - g) \longleftrightarrow x \in fdom(f)$
**by** (*transfer*, *simp*)

**lemma** *fsubseteq-ffun-upd1* [*intro*]:
$[\![ f \subseteq_f g;\ x \notin fdom(g) ]\!] \Longrightarrow f \subseteq_f g(x \mapsto v)_f$
**by** (*transfer*, *auto*)

**lemma** *fsubseteq-ffun-upd2* [*intro*]:
$[\![ f \subseteq_f g;\ x \notin fdom(f) ]\!] \Longrightarrow f \subseteq_f g(x \mapsto v)_f$
**by** (*transfer*, *auto*)

**lemma** *psubseteq-pfun-upd3* [*intro*]:
$[\![ f \subseteq_f g;\ g(x)_f = v ]\!] \Longrightarrow f \subseteq_f g(x \mapsto v)_f$
**by** (*transfer*, *auto*)

**lemma** *fsubseteq-dom-subset*:
$f \subseteq_f g \Longrightarrow fdom(f) \subseteq fdom(g)$
**by** (*transfer*, *auto simp add*: *psubseteq-dom-subset*)

**lemma** *fsubseteq-ran-subset*:
$f \subseteq_f g \Longrightarrow fran(f) \subseteq fran(g)$
**by** (*transfer*, *simp add*: *psubseteq-ran-subset*)

## 10.4 Domain laws

**lemma** *fdom-finite* [*simp*]: $finite(fdom(f))$
**by** (*transfer*, *simp*)

**lemma** *fdom-zero* [*simp*]: $fdom\ 0 = \{\}$
**by** (*transfer*, *simp*)

**lemma** *fdom-plus* [*simp*]: $fdom\ (f + g) = fdom\ f \cup fdom\ g$
**by** (*transfer*, *auto*)

**lemma** *fdom-inter*: $fdom\ (f \cap_f g) \subseteq fdom\ f \cap fdom\ g$
**by** (*transfer*, *meson pdom-inter*)

**lemma** *fdom-comp* [*simp*]: $fdom\ (g \circ_f f) = fdom\ (f \rhd_f fdom\ g)$
**by** (*transfer*, *auto*)

**lemma** *fdom-upd* [*simp*]: $fdom\ (f(k \mapsto v)_f) = insert\ k\ (fdom\ f)$
**by** (*transfer*, *simp*)

**lemma** *fdom-fdom-res* [*simp*]: $fdom\ (A \lhd_f f) = A \cap fdom(f)$
**by** (*transfer*, *auto*)

**lemma** *fdom-graph-ffun* [*simp*]:
$finite\ (Domain\ R) \Longrightarrow fdom\ (graph\text{-}ffun\ R) = Domain\ R$
**by** (*transfer*, *simp add*: *Domain-fst graph-map-dom*)

## 10.5 Range laws

**lemma** *fran-zero* [*simp*]: $fran\ 0 = \{\}$
**by** (*transfer*, *simp*)

**lemma** *fran-upd* [*simp*]: *fran* $(f(k \mapsto v)_f)$ = *insert v* $(fran ((- \{k\}) \lhd_f f))$
  **by** (*transfer*, *auto*)

**lemma** *fran-fran-res* [*simp*]: *fran* $(f \rhd_f A)$ = $fran(f) \cap A$
  **by** (*transfer*, *auto*)

**lemma** *fran-comp* [*simp*]: *fran* $(g \circ_f f)$ = *fran* $(fran f \lhd_f g)$
  **by** (*transfer*, *auto*)

## 10.6   Domain restriction laws

**lemma** *fdom-res-zero* [*simp*]: $A \lhd_f \{\}_f = \{\}_f$
  **by** (*transfer*, *auto*)

**lemma** *fdom-res-empty* [*simp*]:
  $(\{\} \lhd_f f) = \{\}_f$
  **by** (*transfer*, *auto*)

**lemma** *fdom-res-fdom* [*simp*]:
  $fdom(f) \lhd_f f = f$
  **by** (*transfer*, *auto*)

**lemma** *pdom-res-upd-in* [*simp*]:
  $k \in A \implies A \lhd_f f(k \mapsto v)_f = (A \lhd_f f)(k \mapsto v)_f$
  **by** (*transfer*, *auto*)

**lemma** *pdom-res-upd-out* [*simp*]:
  $k \notin A \implies A \lhd_f f(k \mapsto v)_f = A \lhd_f f$
  **by** (*transfer*, *auto*)

**lemma** *fdom-res-override* [*simp*]: $A \lhd_f (f + g) = (A \lhd_f f) + (A \lhd_f g)$
  **by** (*metis fdom-res.rep-eq pdom-res-override pfun-of-inject plus-ffun.rep-eq*)

**lemma** *fdom-res-minus* [*simp*]: $A \lhd_f (f - g) = (A \lhd_f f) - g$
  **by** (*transfer*, *auto*)

**lemma** *fdom-res-swap*: $(A \lhd_f f) \rhd_f B = A \lhd_f (f \rhd_f B)$
  **by** (*transfer*, *simp add*: *pdom-res-swap*)

**lemma** *fdom-res-twice* [*simp*]: $A \lhd_f (B \lhd_f f) = (A \cap B) \lhd_f f$
  **by** (*transfer*, *auto*)

**lemma** *fdom-res-comp* [*simp*]: $A \lhd_f (g \circ_f f) = g \circ_f (A \lhd_f f)$
  **by** (*transfer*, *simp*)

## 10.7   Range restriction laws

**lemma** *fran-res-zero* [*simp*]: $\{\}_f \rhd_f A = \{\}_f$
  **by** (*transfer*, *auto*)

**lemma** *fran-res-upd-1* [*simp*]: $v \in A \implies f(x \mapsto v)_f \rhd_f A = (f \rhd_f A)(x \mapsto v)_f$
  **by** (*transfer*, *auto*)

**lemma** *fran-res-upd-2* [*simp*]: $v \notin A \implies f(x \mapsto v)_f \rhd_f A = ((- \{x\}) \lhd_f f) \rhd_f A$
  **by** (*transfer*, *auto*)

**lemma** *fran-res-override*: $(f + g) \rhd_f A \subseteq_f (f \rhd_f A) + (g \rhd_f A)$
  **by** (*transfer*, *simp add*: *pran-res-override*)

## 10.8 Graph laws

**lemma** *ffun-graph-inv*: *graph-ffun* (*ffun-graph f*) = *f*
  **by** (*transfer*, *auto simp add*: *pfun-graph-inv finite-Domain*)

**lemma** *ffun-graph-zero*: *ffun-graph 0* = {}
  **by** (*transfer*, *simp add*: *pfun-graph-zero*)

**lemma** *ffun-graph-minus*: *ffun-graph* $(f - g)$ = *ffun-graph f* − *ffun-graph g*
  **by** (*transfer*, *simp add*: *pfun-graph-minus*)

**lemma** *ffun-graph-inter*: *ffun-graph* $(f \cap_f g)$ = *ffun-graph f* ∩ *ffun-graph g*
  **by** (*transfer*, *simp add*: *pfun-graph-inter*)

## 10.9 Partial Function Lens

**definition** *ffun-lens* :: $'a \Rightarrow ('b \Longrightarrow ('a, 'b)\ ffun)$ **where**
[*lens-defs*]: *ffun-lens i* = (| *lens-get* = $\lambda\ s.\ s(i)_f$, *lens-put* = $\lambda\ s\ v.\ s(i \mapsto v)_f$ |)

**lemma** *ffun-lens-mwb* [*simp*]: *mwb-lens* (*ffun-lens i*)
  **by** (*unfold-locales*, *simp-all add*: *ffun-lens-def*)

**lemma** *ffun-lens-src*: $\mathcal{S}_{\textit{ffun-lens } i}$ = {*f*. $i \in fdom(f)$}
  **by** (*auto simp add*: *lens-defs lens-source-def*, *metis ffun-upd-ext*)

Hide implementation details for finite functions

**lifting-update** *ffun.lifting*
**lifting-forget** *ffun.lifting*

**end**

# 11 Infinity Supplement

**theory** *Infinity*
**imports** *HOL.Real*
  *HOL−Library.Infinite-Set*
  *Optics.Two*
**begin**

This theory introduces a type class *infinite* that guarantees that the underlying universe of the type is infinite. It also provides useful theorems to prove infinity of the universes for various HOL types.

## 11.1 Type class *infinite*

The type class postulates that the universe (carrier) of a type is infinite.

**class** *infinite* =
  **assumes** *infinite-UNIV* [*simp*]: *infinite* (*UNIV* :: $'a\ set$)

## 11.2 Infinity Theorems

Useful theorems to prove that a type's *UNIV* is infinite.

Note that *infinite-UNIV-nat* is already a simplification rule by default.

**lemmas** *infinite-UNIV-int* [*simp*]

**theorem** *infinite-UNIV-real* [*simp*]:
*infinite* (*UNIV* :: *real set*)
  **by** (*rule infinite-UNIV-char-0*)

**theorem** *infinite-UNIV-fun1* [*simp*]:
*infinite* (*UNIV* :: $'a$ *set*) $\implies$
*card* (*UNIV* :: $'b$ *set*) $\neq$ *Suc 0* $\implies$
*infinite* (*UNIV* :: ($'a \Rightarrow 'b$) *set*)
  **apply** (*erule contrapos-nn*)
  **apply** (*erule finite-fun-UNIVD1*)
  **apply** (*assumption*)
  **done**

**theorem** *infinite-UNIV-fun2* [*simp*]:
*infinite* (*UNIV* :: $'b$ *set*) $\implies$
*infinite* (*UNIV* :: ($'a \Rightarrow 'b$) *set*)
  **apply** (*erule contrapos-nn*)
  **apply** (*erule finite-fun-UNIVD2*)
  **done**

**theorem** *infinite-UNIV-set* [*simp*]:
*infinite* (*UNIV* :: $'a$ *set*) $\implies$
*infinite* (*UNIV* :: $'a$ *set set*)
  **apply** (*erule contrapos-nn*)
  **apply** (*simp add*: *Finite-Set.finite-set*)
  **done**

**theorem** *infinite-UNIV-prod1* [*simp*]:
*infinite* (*UNIV* :: $'a$ *set*) $\implies$
*infinite* (*UNIV* :: ($'a \times 'b$) *set*)
  **apply** (*erule contrapos-nn*)
  **apply** (*simp add*: *finite-prod*)
  **done**

**theorem** *infinite-UNIV-prod2* [*simp*]:
*infinite* (*UNIV* :: $'b$ *set*) $\implies$
*infinite* (*UNIV* :: ($'a \times 'b$) *set*)
  **apply** (*erule contrapos-nn*)
  **apply** (*simp add*: *finite-prod*)
  **done**

**theorem** *infinite-UNIV-sum1* [*simp*]:
*infinite* (*UNIV* :: $'a$ *set*) $\implies$
*infinite* (*UNIV* :: ($'a + 'b$) *set*)
  **apply** (*erule contrapos-nn*)
  **apply** (*simp*)
  **done**

**theorem** *infinite-UNIV-sum2* [*simp*]:

*infinite* (*UNIV* :: ′*b set*) $\Longrightarrow$
  *infinite* (*UNIV* :: (′*a* + ′*b*) *set*)
  **apply** (*erule contrapos-nn*)
  **apply** (*simp*)
  **done**

**theorem** *infinite-UNIV-list* [*simp*]:
*infinite* (*UNIV* :: ′*a list set*)
  **apply** (*rule infinite-UNIV-listI*)
  **done**

**theorem** *infinite-UNIV-option* [*simp*]:
*infinite* (*UNIV* :: ′*a set*) $\Longrightarrow$
  *infinite* (*UNIV* :: ′*a option set*)
  **apply** (*erule contrapos-nn*)
  **apply** (*simp*)
  **done**

**theorem** *infinite-image* [*intro*]:
*infinite A* $\Longrightarrow$ *inj-on f A* $\Longrightarrow$ *infinite* (*f ` A*)
  **apply** (*metis finite-imageD*)
  **done**

**theorem** *infinite-transfer* :
*infinite B* $\Longrightarrow$ *B* $\subseteq$ *f ` A* $\Longrightarrow$ *infinite A*
  **using** *infinite-super*
  **apply** (*blast*)
  **done**

## 11.3   Instantiations

The instantiations for product and sum types have stronger caveats than in principle needed. Namely, it would be sufficient for one type of a product or sum to be infinite. A corresponding rule, however, cannot be formulated using type classes. Generally, classes are not entirely adequate for the purpose of deriving the infinity of HOL types, which is perhaps why a class such as *infinite* was omitted from the Isabelle/HOL library.

**instance** *nat* :: *infinite* **by** (*intro-classes*, *simp*)
**instance** *int* :: *infinite* **by** (*intro-classes*, *simp*)
**instance** *real* :: *infinite* **by** (*intro-classes*, *simp*)
**instance** *fun* :: (*type*, *infinite*) *infinite* **by** (*intro-classes*, *simp*)
**instance** *set* :: (*infinite*) *infinite* **by** (*intro-classes*, *simp*)
**instance** *prod* :: (*infinite*, *infinite*) *infinite* **by** (*intro-classes*, *simp*)
**instance** *sum* :: (*infinite*, *infinite*) *infinite* **by** (*intro-classes*, *simp*)
**instance** *list* :: (*type*) *infinite* **by** (*intro-classes*, *simp*)
**instance** *option* :: (*infinite*) *infinite* **by** (*intro-classes*, *simp*)

**subclass** (**in** *infinite*) *two*  **by** (*intro-classes*, *auto*)

**end**

# 12   Positive Subtypes

**theory** *Positive*
**imports**

*Infinity*
*HOL−Library.Countable*
**begin**

## 12.1 Type Definition

**typedef** (**overloaded**) $'a$::{*zero*, *linorder*} *pos* = {$x$::$'a$. $x \geq 0$}
  **apply** (*rule-tac x = 0* **in** *exI*)
  **apply** (*clarsimp*)
  **done**

**syntax**
  *-type-pos* :: *type* $\Rightarrow$ *type* (*-$^+$ [999] 999*)

**translations**
  (*type*) $'a^+$ == (*type*) $'a$ *pos*

**setup-lifting** *type-definition-pos*

**type-synonym** *preal* = *real pos*

## 12.2 Operators

**lift-definition** *mk-pos* :: $'a$::{*zero*, *linorder*} $\Rightarrow$ $'a$ *pos* **is**
$\lambda$ *n. if* ($n \geq 0$) *then n else 0* **by** *auto*

**lift-definition** *real-of-pos* :: *real pos* $\Rightarrow$ *real* **is** *id* **.**

**declare** [[*coercion real-of-pos*]]

## 12.3 Instantiations

**instantiation** *pos* :: ({*zero*, *linorder*}) *zero*
**begin**
  **lift-definition** *zero-pos* :: $'a$ *pos*
    **is** *0* :: $'a$ **..**
  **instance ..**
**end**

**instantiation** *pos* :: ({*zero*, *linorder*}) *linorder*
**begin**
  **lift-definition** *less-eq-pos* :: $'a$ *pos* $\Rightarrow$ $'a$ *pos* $\Rightarrow$ *bool*
    **is** ($\leq$) :: $'a$ $\Rightarrow$ $'a$ $\Rightarrow$ *bool* **.**
  **lift-definition** *less-pos* :: $'a$ *pos* $\Rightarrow$ $'a$ *pos* $\Rightarrow$ *bool*
    **is** ($<$) :: $'a$ $\Rightarrow$ $'a$ $\Rightarrow$ *bool* **.**
  **instance**
    **apply** (*intro-classes*; *transfer*)
      **apply** (*auto*)
  **done**
**end**

**instance** *pos* :: ({*zero*, *linorder*, *no-top*}) *no-top*
  **apply** (*intro-classes*)
  **apply** (*transfer*)
  **apply** (*clarsimp*)
  **apply** (*meson gt-ex less-imp-le order.strict-trans1*)

**done**

**instance** *pos* :: ({*zero*, *linorder*, *no-top*}) *infinite*
  **apply** (*intro-classes*)
  **apply** (*rule notI*)
  **apply** (*subgoal-tac* $\forall$ *x*::$'a$ *pos*. *x* $\leq$ *Max UNIV*)
  **using** *gt-ex leD* **apply** (*blast*)
  **apply** (*simp*)
  **done**

**instantiation** *pos* :: (*linordered-semidom*) *linordered-semidom*
**begin**
  **lift-definition** *one-pos* :: $'a$ *pos*
    **is** *1* :: $'a$ **by** (*simp*)
  **lift-definition** *plus-pos* :: $'a$ *pos* $\Rightarrow$ $'a$ *pos* $\Rightarrow$ $'a$ *pos*
    **is** (+) **by** (*simp*)
  **lift-definition** *minus-pos* :: $'a$ *pos* $\Rightarrow$ $'a$ *pos* $\Rightarrow$ $'a$ *pos*
    **is** $\lambda x\ y.$ *if* $y \leq x$ *then* $x - y$ *else 0*
    **by** (*simp add: add-le-imp-le-diff*)
  **lift-definition** *times-pos* :: $'a$ *pos* $\Rightarrow$ $'a$ *pos* $\Rightarrow$ $'a$ *pos*
    **is** *times* **by** (*simp*)
  **instance**
    **apply** (*intro-classes*; *transfer*; *simp?*)
       **apply** (*simp add: add.assoc*)
      **apply** (*simp add: add.commute*)
     **apply** (*safe*; *clarsimp?*) [1]
      **apply** (*simp add: diff-diff-add*)
      **apply** (*metis add-le-cancel-left le-add-diff-inverse*)
      **apply** (*simp add: add.commute add-le-imp-le-diff*)
      **apply** (*metis add-increasing2 antisym linear*)
     **apply** (*simp add: mult.assoc*)
    **apply** (*simp add: mult.commute*)
    **apply** (*simp add: comm-semiring-class.distrib*)
    **apply** (*simp add: mult-strict-left-mono*)
   **apply** (*safe*; *clarsimp?*) [1]
    **apply** (*simp add: right-diff-distrib'*)
    **apply** (*simp add: mult-left-mono*)
   **using** *mult-left-le-imp-le* **apply** (*fastforce*)
   **apply** (*simp add: distrib-left*)
   **done**
**end**

**instantiation** *pos* :: (*linordered-field*) *semidom-divide*
**begin**
  **lift-definition** *divide-pos* :: $'a$ *pos* $\Rightarrow$ $'a$ *pos* $\Rightarrow$ $'a$ *pos*
    **is** *divide* **by** (*simp*)
  **instance**
    **apply** (*intro-classes*; *transfer*)
    **apply** (*simp-all*)
    **done**
**end**

**instantiation** *pos* :: (*linordered-field*) *inverse*
**begin**
  **lift-definition** *inverse-pos* :: $'a$ *pos* $\Rightarrow$ $'a$ *pos*

**is** *inverse* **by** (*simp*)
  **instance ..**
**end**

**lemma** *pos-positive* [*simp*]: $0 \leq (x::'a::\{zero,linorder\}\ pos)$
  **by** (*transfer*, *simp*)

## 12.4    Theorems

**lemma** *mk-pos-zero* [*simp*]: *mk-pos 0 = 0*
  **by** (*transfer*, *simp*)

**lemma** *mk-pos-one* [*simp*]: *mk-pos 1 = 1*
  **by** (*transfer*, *simp*)

**lemma** *mk-pos-leq*:
  $[\![\ 0 \leq x;\ x \leq y\ ]\!] \implies$ *mk-pos x* $\leq$ *mk-pos y*
  **by** (*transfer*, *auto*)

**lemma** *mk-pos-less*:
  $[\![\ 0 \leq x;\ x < y\ ]\!] \implies$ *mk-pos x* $<$ *mk-pos y*
  **by** (*transfer*, *auto*)

**lemma** *real-of-pos* [*simp*]: $x \geq 0 \implies$ *real-of-pos (mk-pos x) = x*
  **by** (*transfer*, *simp*)

**lemma** *mk-pos-real-of-pos* [*simp*]: *mk-pos (real-of-pos x) = x*
  **by** (*transfer*, *simp*)

## 12.5    Transfer to Reals

**named-theorems** *pos-transfer*

**lemma** *real-of-pos-0* [*pos-transfer*]:
  *real-of-pos 0 = 0*
  **by** (*transfer*, *auto*)

**lemma** *real-of-pos-1* [*pos-transfer*]:
  *real-of-pos 1 = 1*
  **by** (*transfer*, *auto*)

**lemma** *real-op-pos-plus* [*pos-transfer*]:
  *real-of-pos (x + y) = real-of-pos x + real-of-pos y*
  **by** (*transfer*, *simp*)

**lemma** *real-op-pos-minus* [*pos-transfer*]:
  $x \geq y \implies$ *real-of-pos (x − y) = real-of-pos x − real-of-pos y*
  **by** (*transfer*, *simp*)

**lemma** *real-op-pos-mult* [*pos-transfer*]:
  *real-of-pos (x ∗ y) = real-of-pos x ∗ real-of-pos y*
  **by** (*transfer*, *simp*)

**lemma** *real-op-pos-div* [*pos-transfer*]:
  *real-of-pos (x / y) = real-of-pos x / real-of-pos y*
  **by** (*transfer*, *simp*)

**lemma** *real-of-pos-numeral* [*pos-transfer*]:
  *real-of-pos* (*numeral n*) = *numeral n*
  **by** (*induct n, simp-all only*: *numeral.simps pos-transfer*)

**lemma** *real-of-pos-eq-transfer* [*pos-transfer*]:
  $x = y \longleftrightarrow$ *real-of-pos* $x =$ *real-of-pos* $y$
  **by** (*transfer, auto*)

**lemma** *real-of-pos-less-eq-transfer* [*pos-transfer*]:
  $x \leq y \longleftrightarrow$ *real-of-pos* $x \leq$ *real-of-pos* $y$
  **by** (*transfer, auto*)

**lemma** *real-of-pos-less-transfer* [*pos-transfer*]:
  $x < y \longleftrightarrow$ *real-of-pos* $x <$ *real-of-pos* $y$
  **by** (*transfer, auto*)

**end**

# 13 Recall Undeclarations

**theory** *Total-Recall*
**imports** *Main*
**keywords**
  *purge-syntax* :: *thy-decl* **and**
  *purge-notation* :: *thy-decl* **and**
  *recall-syntax* :: *thy-decl*
**begin**

## 13.1 ML File Import

**ML-file** *Total-Recall.ML*

## 13.2 Outer Commands

**ML** ‹
  *val - =*
    *Outer-Syntax.command* @{*command-keyword purge-syntax*}
      *purge raw syntax clauses*
      ((*Parse.syntax-mode* −− *Scan.repeat1 Parse.const-decl*) >>
        (*Toplevel.theory o* (*fn* (*mode, args*) =>
          (*TotalRecall.record-no-syntax mode args*) *o*
          (*Sign.del-syntax-cmd mode args*))));

  *val - =*
    *Outer-Syntax.local-theory* @{*command-keyword purge-notation*}
      *purge concrete syntax for constants / fixed variables*
      ((*Parse.syntax-mode* −− *Parse.and-list1* (*Parse.const* −− *Parse.mixfix*)) >>
        (*fn* (*mode, args*) =>
          (*Local-Theory.background-theory*
            (*TotalRecall.record-no-notation mode args*)) *o*
          (*Specification.notation-cmd false mode args*)));

  *val - =*
    *Outer-Syntax.command* @{*command-keyword recall-syntax*}

*recall undecarations of all purged items*
*(Scan.succeed (Toplevel.theory TotalRecall.execute-all))*
⟩
**end**

# 14  Meta-theory for UTP Toolkit

**theory** *utp-toolkit*
  **imports**
  *HOL.Deriv*
  *HOL−Library.Adhoc-Overloading*
  *HOL−Library.Char-ord*
  *HOL−Library.Countable-Set*
  *HOL−Library.FSet*
  *HOL−Library.Monad-Syntax*
  *HOL−Library.Countable*
  *HOL−Library.Order-Continuity*
  *HOL−Library.Prefix-Order*
  *HOL−Library.Product-Order*
  *HOL−Library.Sublist*
  *HOL−Algebra.Complete-Lattice*
  *HOL−Algebra.Galois-Connection*
  *HOL−Eisbach.Eisbach*
  *Optics.Optics*
  *Countable-Set-Extra*
  *FSet-Extra*
  *Relation-Extra*
  *Map-Extra*
  *List-Extra*
  *List-Lexord-Alt*
  *Partial-Fun*
  *Finite-Fun*
  *Infinity*
  *Positive*
  *Total-Recall*
**begin end**

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