The Random-Fuzzy Variables: A New Approach to the Expression of Uncertainty in Measurement

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***Abstract*—The good measurement practice requires that the measurement uncertainty is estimated and provided together with the measurement result. The practice today, which is reflected in the reference standard provided by the IEC-ISO "Guide to the expression of uncertainty in measurement," adopts a statistical approach for the expression and estimation of the uncertainty, since the probability theory is the most known and used mathematical tool to deal with distributions of values. However, the probability theory is not the only tool to deal with distributions of values and is not the most suitable one when the values do not distribute in a totally random way. In this case, a more general theory, the theory of the evidence, should be considered. This paper recalls the fundamentals of the theory of the evidence and frames the random-fuzzy variables within this theory, showing how they can usefully be employed to represent the result of a measurement together with its associated uncertainty. The mathematics is defined on the random-fuzzy variables, so that the uncertainty can be processed, and simple examples are given.**

***Index Terms*—Fuzzy variables, metrology, uncertainty expression.**

I. Introduction

THE correct estimation of the measurement uncertainty has represented a true challenge to the measurement experts since the early beginning of the measurement science and has steadily evolved as the state of the art of the measurement practice has developed.

Nowadays, state of the art reflects the scientific discussion that took place in the 1970s and 1980s and is perfectly expressed in the present reference Standard, that is the IEC-ISO "Guide to the expression of uncertainty in measurement" [1]. The definition of uncertainty of a measurement result provided by the Guide and the guidelines to estimate, express and process it, are not only a standard reference, but do represent concepts that are widely accepted by the scientific and technical community.

For this reason, the definitions given by the Guide [1] will be shortly recalled here, since they appear to be the best definitions presently available, in order to frame the uncertainty concept in a broad context and to outline the limits of the present practice of uncertainty estimation.

The measurement uncertainty is defined by the Guide as "a parameter, associated with the result of a measurement, that characterizes the dispersion of the values that could reasonably be attributed to the measurand" [1]. The Guide recognizes that "the word 'uncertainty' means 'doubt,' and, thus, in its broadest sense, 'uncertainty of measurement' means doubt about the exactness or accuracy of the result of a measurement" [1]. It also states that "the uncertainty of the result of a measurement reflects the lack of exact knowledge of the value of the measurand. The result of a measurement after correction for recognized systematic effects is still only an *estimate* of the value of the measurand because of the uncertainty arising from random effects and from imperfect correction of the result for systematic effects" [1].

The guide takes also into account the concepts of 'level of confidence' and 'confidence interval.' In fact, it states that "in many industrial and commercial applications, as well as in the areas of health and safety, it is often necessary to provide an interval about the measurement result within which the values that could reasonably be attributed to the quantity subject to measurement may be expected to lie with a high level of confidence. Thus, the ideal method for evaluating and expressing measurement uncertainty should be capable of readily providing such a *confidence interval*, in particular, one that corresponds in a realistic way with the required level of confidence" [1].

The definitions and concepts above shortly recalled give clear evidence that a measurement result can no longer be expressed by a single scalar value, but it must be expressed by a distribution of values over an interval within which the measurement result is expected to lie with a given level of confidence.

One of the most known and used mathematical tools to deal with distributions of values is the well known probability theory, to which the present measurement practice and the IEC-ISO Guide refer almost totally. The followed probabilistic approach is somehow justified by the consideration that the result of a measurement after correction of recognized systematic effects is still affected by an uncertainty due to the imperfect correction of the systematic effects and to random effects. If the random effects are supposed to be the prevailing ones, then the probabilistic approach appears to be quite natural, since the probability theory is the most suitable mathematical tool to deal with random phenomena.

In recent years, however, the limitations of this probabilistic approach have been outlined, and the criticism has focused mainly on the following points.

- In many practical applications, the random effects do not prevail over the systematic ones, especially when these latter ones are unknown, or the applied corrections are not totally effective. A probabilistic processing of non-negligible systematic effects may yield a wrong evaluation of the measurement uncertainty.

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- The evaluation of the combined uncertainty may become a difficult task, especially when the number of factors involved is high. In this case the accurate estimate of the correlation factors may become quite troublesome.
- The evaluation of the combined uncertainty becomes impossible when the functional relationship between the measurand and the measured quantities can no longer be described by a continuous function, since the sensitivity coefficients [1] cannot be evaluated.

Statistical methods have been proposed to overcome the last two problems [2]–[4], but their computational burden is generally high [4], so that their efficiency is limited. Moreover, since they are still based on a probabilistic and statistical approach, they do not overcome the first problem listed previously.

The probability theory is not, however, the only theory for handling distributions of values and non deterministic information. This theory can be seen as a particular case of the more general "theory of the evidence" [5], which encompasses also the possibility theory that, though less known than the probability theory, appears suitable for expressing uncertainty in measurement too.

Some attempts have already been proposed, to express the measurement uncertainty within the possibility theory [6], [7], though, at the authors' knowledge, they still lack in generality in the way they express and combine all possible contributions to the measurement uncertainty.

This paper is aimed at proposing a more general approach, framed within the theory of the evidence, able to profit by both the possibilistic and the probabilistic approaches, thus overcoming the limitations of these approaches when individually followed.

The core of this proposal is an original definition of the random-fuzzy variables, tailored to effectively express uncertainty in measurement. A suitable mathematics is also defined on these variables, in order to allow an effective combination of all uncertainty contributions.

II. Possibility Theory and the Fuzzy Variables

The theory of the evidence is based on two concepts, the *belief measure* and the *plausibility measure*, where the term "measure" must be interpreted in its strict mathematical meaning.

Let us consider set $A, A \in \wp(X)$, where $\wp(X)$ is the set of all crisp subsets of the universal set X. From a strict mathematical point of view [8], given a finite universal set X, a *belief measure* is a function Bel: $\wp(X) \rightarrow [0,1]$ such that $\mathrm{Bel}(\oslash) = 0, \mathrm{Bel}(X) = 1$, and, for all possible families of subsets of X:

$$\mathrm{Bel}(A_1 \cup A_2 \cup \cdots \cup A_n) \geq \sum_j \mathrm{Bel}(A_j) - \sum_{j<k} \mathrm{Bel}(A_j \cap A_k) + \cdots + (-1)^{n+1}\mathrm{Bel}(A_1 \cap A_2 \cap \cdots \cap A_n).$$

The belief measure $\mathrm{Bel}(A)$ is interpreted as the degree of belief (based on available evidence) that a given element of X belongs to set A. Associated with each belief measure is a plausibility measure, $\mathrm{Pl}(A)$, defined, for all $A \in \wp(X)$, by the equation [8]

$$\mathrm{Pl}(A) = 1 - \mathrm{Bel}(\bar{A}) \tag{1}$$

where $\bar{A}$ is the complement set of A. The plausibility measure can be interpreted as the degree of plausibility (based on available evidence) that a given element of X belongs to set A [8].

Both the belief and plausibility measures can be characterized in terms of a function m, called *basic probability assignment*

$$m: \wp(X) \rightarrow [0,1]$$

such that $m(\oslash) = 0$, and

$$\sum_{A \in \wp(X)} m(A) = 1. \tag{2}$$

For each set $A, A \in \wp(X), m(A)$ represents the extent to which the available evidence supports the statement that an element in X belongs to set A [8].

It can be proved [8] that, for any set $A \in \wp(X)$, the belief and plausibility measures can be expressed as

$$\mathrm{Bel}(A) = \sum_{B \mid B \subseteq A} m(B) \tag{3}$$

and

$$\mathrm{Pl}(A) = \sum_{B \mid A \cap B \neq \phi} m(B). \tag{4}$$

The relationship between $m(A)$ and $\mathrm{Bel}(A)$ has the following meaning: While $m(A)$ characterizes the degree of evidence or belief that the element in question belongs exactly to set $A, \mathrm{Bel}(A)$ represents the total evidence or belief that the element belongs to A as well as to the various special subset of A. On the other hand, $\mathrm{Pl}(A)$ represents not only the total evidence or belief that the element in question belongs to set A or to any of its subsets, but also the additional evidence or belief associated with the sets that overlap with A. Therefore, $\mathrm{Pl}(A) \geq \mathrm{Bel}(A)$ for all $A \in \wp(X)$.

Sets $A \in \wp(X)$ for which $m(A) > 0$ are called focal elements of m. If a set $\Im$ of focal elements is considered together with the associated basic assignments m, set $\langle \Im, m \rangle$ is called a *body of evidence*.

A particular branch of the theory of the evidence, called *possibility theory*, deals with bodies of evidence whose focal elements are nested. When the bodies of evidence are nested, the belief and plausibility measures satisfy the following more restrictive conditions [8]:

$$\begin{aligned} \mathrm{Bel}(A \cap B) &= \mathbf{min}[\mathrm{Bel}(A), \mathrm{Bel}(B)] \\ \mathrm{Pl}(A \cup B) &= \mathbf{max}[\mathrm{Pl}(A), \mathrm{Pl}(B)] \end{aligned} \tag{5}$$

for all A and $B \in \wp(X)$.

When these conditions are satisfied, the belief measure is called *necessity measure* (Nec) and the plausibility measure is called *possibility measure* (Pos). Therefore, (5) becomes

$$\begin{aligned} \mathrm{Nec}(A \cap B) &= \mathbf{min}[\mathrm{Nec}(A), \mathrm{Nec}(B)] \\ \mathrm{Pos}(A \cup B) &= \mathbf{max}[\mathrm{Pos}(A), \mathrm{Pos}(B)] \end{aligned}$$

for every A and $B \in \wp(X)$.

In addition, possibility measures and necessity measures constrain each other as expressed by the following conditions [8]:

$$\begin{aligned} \mathrm{Nec}(A) > 0 &\Rightarrow \mathrm{Pos}(A) = 1 \\ \mathrm{Pos}(A) < 1 &\Rightarrow \mathrm{Nec}(A) = 0 \end{aligned}$$

for every $A \in \wp(X)$.

Moreover, (1) brings to

$$\mathrm{Nec}(A) = 1 - \mathrm{Pos}(\bar{A}) \tag{6}$$

and

$$\mathrm{Nec}(\bar{A}) = 1 - \mathrm{Pos}(A). \tag{7}$$

These considerations give the possibility of framing the fuzzy set theory within the possibility theory.

The concept of fuzzy variables and fuzzy sets has been introduced by Zadeh [9]–[11], who extended the traditional concept of membership of a variable a to a set A, that in the crisp set theory could only take values 0 ($a \notin A$) or 1 ($a \in A$), by considering membership functions $\mu_{\mathrm{A}}(a)$, where $0 \leq \mu_{\mathrm{A}}(a) \leq 1, \mu_{\mathrm{A}}(a)$ is convex and normal (which means that its maximum value must be always 1).

If a number $\alpha, 0 \leq \alpha \leq 1$, is considered, the α-cut of the fuzzy variable A can be defined as

$$A_\alpha = \{a \,|\, \mu_{\mathrm{A}}(a) \geq \alpha\} \tag{8}$$

which defines an interval $[a_1^\alpha, a_2^\alpha]$, where $a_1^\alpha \leq a_2^\alpha$.

The whole set of α-cuts can be seen as a body of nested sets; in fact it can be readily proven that $A_{\alpha_1} \subset A_{\alpha_2}$ for every $\alpha_1 > \alpha_2$. In particular, the α-cut with $\alpha = 0$ nests all other α-cuts, while the α-cut with $\alpha = 1$ is nested by all others. In this respect, the α-cuts of A play the same role as the focal elements in a possibilistic body of evidence formulated within the theory of the evidence.

According to the definition of a fuzzy variable [11], [12], each α-cut A_α can be seen as a confidence interval within which variable a is supposed to lie with a level of confidence $1-\alpha$, which corresponds, from a stricter mathematical point of view [8], to the value assumed by the necessity measure Nec associated with the α-cut itself:

$$\mathrm{Nec}(A_\alpha) = 1 - \alpha. \tag{9}$$

This statement can be proved by considering the following.

Let us now consider a fuzzy variable X and its α-cuts X_α. Let us consider, for the possible values of α, the increasing order $\alpha_1 < \alpha_2 < \cdots < \alpha_j < \cdots < \alpha_N$, where $\alpha_1 = 0$ and $\alpha_N = 1$. Then, the following applies, for the corresponding α-cuts: $X_{\alpha_N} \subset X_{\alpha_{N-1}} \subset \cdots \subset X_{\alpha_2} \subset X_{\alpha_1}$. The set inclusion is strict, since a fuzzy variable is, by definition, convex.

Under these considerations, (3) and (4) can be rewritten as

$$\begin{aligned}\mathrm{Nec}\big(X_{\alpha_j}\big) &= \sum_{X_\alpha \,|\, X_\alpha \subseteq X_{\alpha_j}} m(X_\alpha) \\ &= \sum_{\alpha=\alpha_j}^{1} m(X_\alpha) = \sum_{i=j}^{N} m\left(X_{\alpha_i}\right) \end{aligned} \tag{10}$$

$$\begin{aligned}\mathrm{Pos}\big(X_{\alpha_j}\big) &= \sum_{X_\alpha \,|\, X_\alpha \cap X_{\alpha_j} \neq \emptyset} m(X_\alpha) \\ &= \sum_{\alpha=0}^{1} m(X_\alpha) = \sum_{i=1}^{N} m\left(X_{\alpha_i}\right) \equiv 1. \end{aligned} \tag{11}$$

Moreover, if the complement sets $\overline{X_{\alpha_j}}$ are considered, together with (6) and (7), it follows:

$$\mathrm{Nec}\left(\overline{X_{\alpha_j}}\right) \equiv 0 \tag{12}$$

$$\mathrm{Pos}\left(\overline{X_{\alpha_j}}\right) = \sum_{\alpha=0}^{\alpha_{j-1}} m(X_\alpha) = \sum_{i=1}^{j-1} m\left(X_{\alpha_i}\right). \tag{13}$$

Furthermore, it is possible to establish a relationship between the different values of $\mathrm{Pos}(\overline{X_{\alpha_j}})$ and those of levels α_j.

Let us consider level $\alpha = 0$, to which the α-cut $X_{\alpha=0}$ is associated. This α-cut is, among all different α-cuts, the greatest (see Fig. 1). If the negation of $X_{\alpha=0}, \overline{X_{\alpha=0}}$, is considered, it is obvious that the possibility that an element belongs to that set is zero. In fact, the membership function is null outside the interval $X_{\alpha=0}$.

Let us now consider increasing values of α. As α increases, the interval defined by the α-cut X_α decreases, while the interval $\overline{X_\alpha}$ increases (Fig. 1). Therefore, it is obvious that, as α increases, the credibility that a generic element belongs to interval X_α decreases, while the credibility that the same element belongs to interval $\overline{X_\alpha}$ increases, and attains its maximum value when α is maximum, that is, when $\alpha = 1$. Therefore, as also shown by (13), $\mathrm{Pos}(\overline{X_{\alpha_j}})$ is an increasing measure as α increases, that is, as index j increases. Since a possibility measure is defined between 0 and 1, it must be $\mathrm{Pos}(\overline{X_{\alpha_j}}) = 1$ when $\alpha_j = 1 (j = N)$.

Therefore, since

- $0 \leq \mathrm{Pos}(\overline{X_\alpha}) \leq 1$;
- $0 \leq \alpha \leq 1$;
- $\mathrm{Pos}(\overline{X_{\alpha=0}}) = 0$;
- $\mathrm{Pos}(\overline{X_{\alpha=1}}) = 1$;

it can be concluded that, for every α_j:

$$\mathrm{Pos}\left(\overline{X_{\alpha_j}}\right) = \alpha_j. \tag{14}$$

Finally, from (6) and (14), it follows that

$$\mathrm{Nec}\left(X_{\alpha_j}\right) = 1 - \alpha_j \tag{15}$$

and (9) is thus proved.

Moreover

$$\alpha_j = \sum_{i=1}^{j-1} m\left(X_{\alpha_i}\right) \tag{16}$$

and

$$m\left(X_{\alpha_j}\right) = \alpha_{j+1} - \alpha_j \tag{17}$$

where $\alpha_{N+1} = 1$ by definition.

Equation (15) proves, in a strict mathematical way, that each α-cut of a fuzzy variable represents a confidence interval with a confidence level equal to $1-\alpha$.

In conclusion of this short theoretical survey, it is worth while analysing briefly the main difference between the possibility theory and the probability theory, since they represent two particular branches of the theory of the evidence. The fundamental difference between the probability and possibility theories lies in the different structure of the respective bodies of evidence. Probabilistic bodies of evidence consist of singletons, while

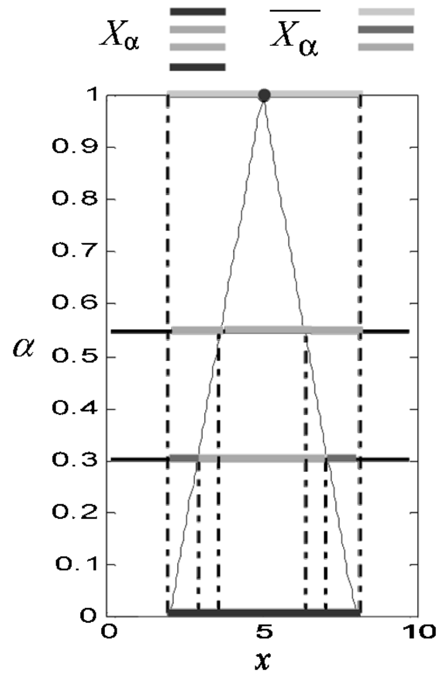

Fig. 1. Generic α-cuts X_α and their negations $\overline{X_\alpha}$.

possibilistic bodies of evidence are nested sets. In the probability theory, a probability density function is defined over an interval (the interval can be both finite and infinite). Therefore, for each singleton, it is possible to define a probability measure, that is the value assumed by the probability density function in the particular point. On the other hand, for each confidence interval (α-cut) in the possibility theory, it is possible to define a possibility measure, but no further information is given about the single values within the interval.

III. Possibility Theory and the Expression of Uncertainty in Measurement

As it has been shortly recalled in Section I, a widely accepted definition of uncertainty of a measurement result is "a parameter . . . that characterizes the dispersion of the values that could reasonably be attributed to the measurand" [1]. This dispersion can be represented by a confidence interval within which the measurand is expected to lie with a given level of confidence. This concept can be reformulated, referring to the possibility theory, by stating that the uncertainty of the measurement result, when associated with the result itself, identifies a set of values, whose associated necessity measure represents the degree of belief (based on available evidence) that the measurand belongs to that set.

The "available evidence" can be based on experimental data or by "judgment using all relevant information on the possible variability" [1] of the measurement result.

According to the conclusion of the last section and the above reformulation of the definition of uncertainty, the result of a measurement, together with the associated uncertainty, can be represented by a fuzzy variable: each α-cut represents a confidence interval and the degree of belief (or level of confidence) that the measurand lies within the α-cut is given by (15).

The use of fuzzy variables for expressing a measurement result has recently been proposed in the literature [6], [7], [13], though it has not yet been framed into the possibility theory.

The main advantage of referring to this theory is that it is possible to handle confidence intervals without making any assumption about how the elements belonging to the interval are distributed over the interval itself. No assumption about the probability distribution over the interval is required, and all effects that contribute to the measurement uncertainty should no longer be considered as random ones. Non-random effects can hence be treated in a correct mathematical way.

As a first, simple basic example, let us consider the dc resistance measurement of a resistor, obtained as the ratio of the measured values of the applied voltage and the current flowing through the resistor. Let us also suppose that two DMMs are used to measure voltage and current and that the only information available about the DMMs accuracy is that provided by the manufacturer. In this case, the manufacturer is likely to provide the maximum error interval about the measured voltage of the whole family of those DMMs. It is reasonable to suppose that the values measured by the employed DMMs lie somewhere within the provided interval and that all values measured by these instruments fall inside an interval that is much narrower than the one provided by the manufacturer and is totally included into this last one. Under this assumption, the measurement result is affected by prevailing systematic effects and there is no available evidence that the systematic effects acting on the measured voltage value may compensate those acting on the measured current value.

The possible measured voltage and current values can be therefore represented by the two fuzzy variables V and I shown in Fig. 2(a) and (b). Having defined the mathematical operations between the fuzzy variables [12], [13], the measured resistance value, together with its associated uncertainty, can be represented by the fuzzy variable

$$R = \frac{V}{I} \tag{18}$$

shown in Fig. 2(c). Fig. 2(d) shows also the fuzzy variable associated with the measured dc power value $P = V \cdot I$.

The obtained values are compatible with the uncertainty values evaluated according to the indications given by the IEC-ISO Guide when total correlation is considered between the measured values of voltage and current. However, though this situation is compatible with a total correlation between these two instruments, it represents also a more general situation reflecting a total ignorance on how the measurement results are distributed over the confidence interval. This situation is better represented by a fuzzy variable than by a probability distribution.

On the other hand, when random effects are actually present, the fuzzy variables cannot take into account the probabilistic compensation that, in this case, may affect the measurement process [13]. In other words, the fuzzy variables are a very effective tool in expressing and processing uncertainty in measurement when the prevailing sources of uncertainty show a systematic behavior. When also random sources are present, they cannot be expressed and processed in terms of simple

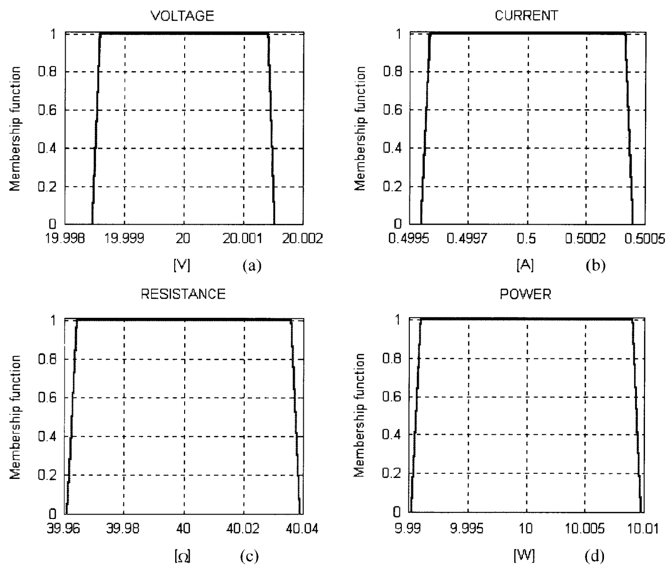

Fig. 2. Expression of the measurement results by means of fuzzy variables. Figs. 2(a) and (b) show the distribution of the possible measured values of voltage and current, while Figs. 2(c) and (d) show the distribution of the possible measured values of dc resistance and power obtained from the measured values of voltage and current.

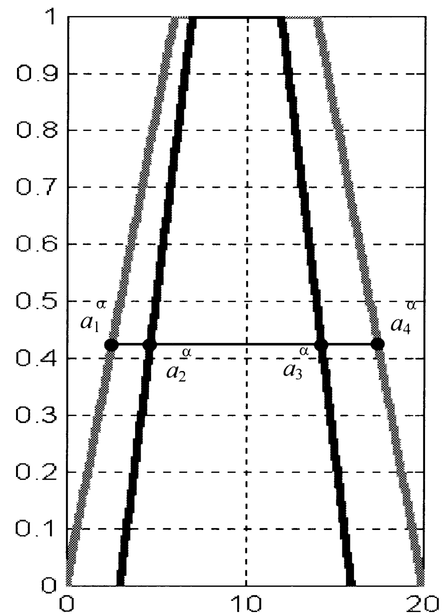

Fig. 3. Example of membership function of an RFV.

fuzzy variables any longer, since random phenomena are handled by the probability theory, which is a branch of the theory of the evidence different from the possibility theory. Attempts have been proposed to consider also random phenomena in terms of fuzzy variables [7], but they do not refer directly to the probability theory, which remains the main tool for handling random phenomena.

IV. Random-Fuzzy Variables and Associated Mathematics

In order to overcome the problem outlined previously, and represent all kinds of possible effects inside the unifying framework of the theory of the evidence, a more general variable must be defined, showing the basic properties of both the possibility and probability theories.

This section proposes an original definition for the RFVs [12] and the associated mathematics.

The membership function of a RFV A, defined on the reference set X, can be still defined in terms of α-cuts, where an α-cut is now represented by four numbers

$$A_\alpha = [a_1^\alpha, a_2^\alpha, a_3^\alpha, a_4^\alpha] \tag{19}$$

where $a_1^\alpha \leq a_2^\alpha \leq a_3^\alpha \leq a_4^\alpha \; \forall\alpha$. It can be readily recognized that all these α-cuts still represent nested focal elements of a body of evidence.

According to the meaning assigned to an α-cut, interval $[a_1^\alpha, a_4^\alpha]$ represents a confidence interval with the level of confidence $1 - \alpha$. As shown in Fig. 3, within this interval, three subintervals can be recognized, which differ from each other from the way the possible values are distributed over the intervals themselves.

As far as the inner interval $[a_2^\alpha, a_3^\alpha]$ is concerned, nothing can be said about the way the possible values $a, a_2^\alpha \leq a < a_3^\alpha$, distribute.

As far as intervals $[a_1^\alpha, a_2^\alpha]$ and $[a_3^\alpha, a_4^\alpha]$ are concerned, the possible values $a, a_1^\alpha \leq a < a_2^\alpha$ and $a_3^\alpha < a \leq a_4^\alpha$, are supposed to be randomly distributed, according to a normal probability density function.

This last assumption comes directly from the need of describing purely random effects affecting the measurement process: These effects can be effectively represented by a normal probability distribution. Under this assumption, the distribution over interval $[a_1^\alpha, a_2^\alpha]$ is, by definition, the left half side of the normal distribution, with its peak value located at a_2^α and the value taken at 3σ (σ being the standard deviation of the normal distribution) located at a_1^α. Similarly, the right half side of the normal distribution is distributed over interval $[a_3^\alpha, a_4^\alpha]$, with its peak value located at a_3^α and the value taken at 3σ located at a_4^α, by definition.

It can be readily recognized that if $a_1^\alpha = a_2^\alpha$ and $a_3^\alpha = a_4^\alpha$ the random effects are supposed to be absent and the RFV becomes a simple fuzzy variable; if $a_2^\alpha = a_3^\alpha$ only random effects are represented by the RFV.

In the following, the mathematical operations for the RFVs are defined, so that the result of a measurement together with the associated uncertainty can be processed when both random and nonrandom effects are present.

Since the RFVs lie in the theory of the evidence, of which the possibility theory and the probability theory are two distinct particular cases, the mathematics of the RFVs can be defined by referring to two distinct mathematics: the mathematics of the intervals, defined in the possibility theory, and the well-known statistics, defined in the probability theory. For each value of α, which determines (19), the mathematics of the intervals will be used for the internal interval $[a_2^\alpha, a_3^\alpha]$, while statistics will be used for the external intervals $[a_1^\alpha, a_2^\alpha]$ and $[a_3^\alpha, a_4^\alpha]$.

Let us consider two RFVs A and B. A and B are represented, for each value of α, by the α-cuts $A_\alpha = [a_1^\alpha, a_2^\alpha, a_3^\alpha, a_4^\alpha]$, and $B_\alpha = [b_1^\alpha, b_2^\alpha, b_3^\alpha, b_4^\alpha]$. Let us now consider the RFV C, defined as $C = f(A, B)$. For each value of α, C will be represented by the α-cut $C_\alpha = [c_1^\alpha, c_2^\alpha, c_3^\alpha, c_4^\alpha]$. The internal interval $[c_2^\alpha, c_3^\alpha]$ is defined according to the mathematics of the intervals. If the four algebraic operations are considered, the following expressions are derived [13]:

$$\begin{aligned} c_2^\alpha &= a_2^\alpha + b_2^\alpha \\ c_3^\alpha &= a_3^\alpha + b_3^\alpha \end{aligned} \tag{20}$$

if function f is a sum

$$\begin{aligned} c_2^\alpha &= a_2^\alpha - b_3^\alpha \\ c_3^\alpha &= a_3^\alpha - b_2^\alpha \end{aligned} \tag{21}$$

if function f is a difference

$$\begin{aligned} c_2^\alpha &= \mathbf{min}(a_2^\alpha \cdot b_3^\alpha, a_2^\alpha \cdot b_2^\alpha, a_3^\alpha \cdot b_3^\alpha, a_3^\alpha \cdot b_2^\alpha) \\ c_3^\alpha &= \mathbf{max}(a_2^\alpha \cdot b_3^\alpha, a_2^\alpha \cdot b_2^\alpha, a_3^\alpha \cdot b_3^\alpha, a_3^\alpha \cdot b_2^\alpha) \end{aligned} \tag{22}$$

if function f is a product

$$\begin{aligned} c_2^\alpha &= \mathbf{min}(\mathrm{a}_2^\alpha/\mathrm{b}_2^\alpha, \mathrm{a}_3^\alpha/\mathrm{b}_3^\alpha, \mathrm{a}_2^\alpha/\mathrm{b}_3^\alpha, \mathrm{a}_3^\alpha/\mathrm{b}_2^\alpha) \\ c_3^\alpha &= \mathbf{max}(\mathrm{a}_2^\alpha/\mathrm{b}_2^\alpha, \mathrm{a}_3^\alpha/\mathrm{b}_3^\alpha, \mathrm{a}_2^\alpha/\mathrm{b}_3^\alpha, \mathrm{a}_3^\alpha/\mathrm{b}_2^\alpha) \end{aligned} \tag{23}$$

if function f is a division, provided that $0 \notin [b_1^\alpha, b_4^\alpha]$.

The composition of the external intervals requires to refer to statistics. Within these intervals, values are distributed, by definition, in a random way, with a normal probability density function. A normal distribution is completely described by its mean value and its standard deviation, and these values can be obtained from the extreme values of the external intervals, according to the given definitions for these extreme values. In particular

$$\begin{aligned} \sigma_{\mathrm{A}i}^\alpha &= (a_2^\alpha - a_1^\alpha)/3 \\ \sigma_{\mathrm{Ar}}^\alpha &= (a_4^\alpha - a_3^\alpha)/3 \end{aligned} \tag{24}$$

and

$$\begin{aligned} \sigma_{\mathrm{B}i}^\alpha &= (b_2^\alpha - b_1^\alpha)/3 \\ \sigma_{\mathrm{Br}}^\alpha &= (b_4^\alpha - b_3^\alpha)/3 \end{aligned} \tag{25}$$

where $\sigma_{\mathrm{Ar}}^\alpha, \sigma_{\mathrm{Ai}}^\alpha, \sigma_{\mathrm{Br}}^\alpha, \sigma_{\mathrm{Bi}}^\alpha$ represent the standard deviations of the right and left half-normal distributions defined over the external intervals of the RFVs A and B, respectively. Therefore, the standard deviation of the RFV $C = f(A, B)$ can be found by simply composing the standard deviations, as shown in (26a) and (26b) at the bottom of the page, where $r(A, B)$ is the correlation factor between the two considered normal distributions.

Finally, the external values c_1^α and c_4^α can be derived as

$$\begin{aligned} c_1^\alpha &= c_2^\alpha - 3 \cdot \sigma_{\mathrm{Cl}}^\alpha \\ c_4^\alpha &= c_3^\alpha + 3 \cdot \sigma_{\mathrm{Cr}}^\alpha. \end{aligned} \tag{27}$$

Of course, the derivatives in (26) depend on relation f. If the four algebraic operations are considered

$$\left(\frac{\partial f}{\partial A}\right)^2 = \left(\frac{\partial f}{\partial B}\right)^2 = 1 \tag{28}$$

if function f is a sum or a difference

$$\left(\frac{\partial f}{\partial A}\right)^2 = B_c^2; \quad \left(\frac{\partial f}{\partial B}\right)^2 = A_c^2 \tag{29}$$

if function f is a product, where A_c and B_c represent the central values of the α-cuts with $\alpha = 1$ of the RFVs A and B, respectively

$$\left(\frac{\partial f}{\partial A}\right) = \frac{1}{B_c} \quad \left(\frac{\partial f}{\partial B}\right) = -\frac{A_c}{B_c^2} \tag{30}$$

if function f is a division.

V. Expression of Uncertainty

According to the definitions given in the previous sections, the RFVs extend the concepts described in Section III when both random and nonrandom contributions are affecting the measurement process. In order to illustrate this concept, let us consider the same simple example considered in the previous section, concerning the measurement of dc resistance and power. In this case, however, let us suppose that the employed DMMs have been calibrated, and the resulting correction for the systematic effects applied, so that only the residual systematic effects due to the uncertainty on the calibration are still present, together with the random effects. In this case, the random effects are supposed to be comparable with the residual systematic effects.

Let us suppose that the measured values of voltage and current, together with their associated uncertainties, can be represented by the RFVs shown in Fig. 4(a) and (b), and that voltage and current measurements are independent, so that no correlation is considered. By processing (18), where now R, V, and I are RFVs, the measured value for the resistance is obtained, as shown in Fig. 4(c). Similarly, the measured power value P is obtained as an RFV, as shown in Fig. 4(d). The possible compensation of the random effects is clearly shown by the different

$$\sigma_{Ci}^\alpha = \sqrt{\left(\frac{\partial f}{\partial A}\right)^2 \cdot (\sigma_{\mathrm{Al}}^\alpha)^2 + \left(\frac{\partial f}{\partial B}\right)^2 \cdot (\sigma_{\mathrm{Bl}}^\alpha)^2 + 2 \cdot \frac{\partial f}{\partial A} \cdot \frac{\partial f}{\partial B} \cdot \sigma_{\mathrm{Al}}^\alpha \cdot \sigma_{\mathrm{Bl}}^\alpha \cdot r(A, B)} \tag{26a}$$

$$\sigma_{Cr}^\alpha = \sqrt{\left(\frac{\partial f}{\partial A}\right)^2 \cdot (\sigma_{\mathrm{Ar}}^\alpha)^2 + \left(\frac{\partial f}{\partial B}\right)^2 \cdot (\sigma_{\mathrm{Br}}^\alpha)^2 + 2 \cdot \frac{\partial f}{\partial A} \cdot \frac{\partial f}{\partial B} \cdot \sigma_{\mathrm{Ar}}^\alpha \cdot \sigma_{\mathrm{Br}}^\alpha \cdot r(A, B)} \tag{26b}$$

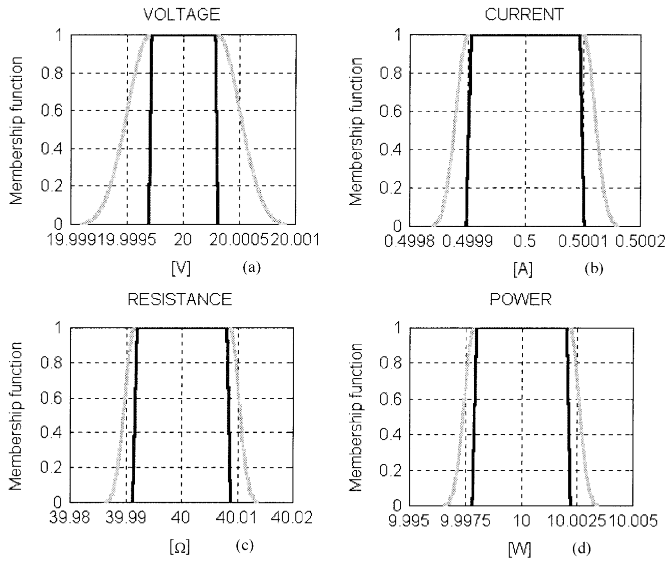

Fig. 4. Expression of the measurement results by means of RFVs. Figs. 4(a) and (b) show the distribution of the possible measured values of voltage and current due to systematic (black line) and random effects (gray line). Figs. 4(c) and (d) show the distribution of the measured values of dc resistance and power obtained from the measured values of voltage and current, giving evidence again of the systematic and random effects.

width of the random parts of the RFVs in Fig. 4, while the permanence of the residual systematic effects is shown by the unchanged shape of the pure fuzzy part of the RFVs.

The measurement results in Fig. 4 shows that the result of a measurement can be effectively expressed in terms of RFVs, and all possible contributions to the uncertainty can be considered.

This simple example shows how the proposed method can be generalized. Let us suppose that a measurand y is determined by the functional relationship on scalar terms

$$y = f(x_1, x_2, \dots x_n) \tag{31}$$

where $x_1, x_2, \dots, x_n$ are measured values.

If $x_1, x_2, \dots, x_n$, together with their associated uncertainty, are expressed in terms of the RFVs $X_1, X_2, \dots X_n$, relationship (31) is changed into the same functional relationship on RFVs

$$Y = f(X_1, X_2, \dots X_n) \tag{32}$$

which provides the measured value in terms of an RFV.

An experimental example of how this method can be applied to practical cases, more significant than the basic example here considered, can be found in [14].

VI. Conclusion

The concept of uncertainty in measurement has been analyzed in terms of the more general theory of the evidence, instead of the probability theory, usually referred to in the measurement practice [1]. It has been shown that the theory of the evidence allows representing the dispersion of the values that could reasonably be attributed to the measurand in a more suitable way than the probability theory, especially when nonnegligible nonrandom effects are present.

The fuzzy variables have been proved to play, in the possibility theory, the same role as the random variables in the probability theory and are hence suitable to represent a measurement result together with its associated uncertainty. In order to take into account and process all kinds of effects, the random-fuzzy variables have been considered and suitably defined. A suitable mathematics has also been defined, in order to process the uncertainty for the estimation of the combined uncertainty in indirect measurements. According to the theoretical considerations developed in the paper, the RFVs can be considered effective in expressing uncertainty in measurement in a more general way than that followed by the present measurement practice, reflected in the recommendations of the IEC-ISO Guide [1]. For instance, nonnegligible systematic effects, which very often affect the measurement process in industrial applications, can be taken into account properly in the uncertainty evaluation, as shown in the reported simple basic example.

Although this paper is focused mainly on theoretical considerations, evidence is available in the literature that the proposed approach can be applied to the practical estimation of the measurement uncertainty of DSP-based instruments [14], in a quite straightforward way.

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