Unscented Transform: A Powerful Tool for Measurement Uncertainty Evaluation

Leopoldo Angrisani, Massimo D'Apuzzo, and Rosario Schiano Lo Moriello

***Abstract*—An original approach for uncertainty evaluation in indirect measurements is presented hereinafter. The approach applies the unscented transform to the measurement model (i.e., the functional relationship between output and input quantities) in order to gain a reliable estimate of output expectation and standard deviation (measurement uncertainty). Thanks to some useful properties of the transform, notable limits of the current GUM recommendations can be overcome. In particular, reliable estimates are also granted in the presence of nonlinear and/or nonanalytical measurement models or complex digital signal processing algorithms. A number of numerical tests are conducted on simulated and actual measurement data. Remarkable concurrence between obtained estimates and those granted by Monte Carlo simulations confirms the efficacy of the proposed approach.**

***Index Terms*—GUM, indirect measurements, measurement uncertainty, Monte Carlo simulation, unscented transform.**

I. Introduction

In RECENT years, measurement uncertainty evaluation has shown itself as a fundamental practice for both scientific and industrial fields. To this aim, the expression of the measurement result in terms of nominal value and error has slowly been replaced by a proper set of values that could reasonably be associated with the quantity under measurement (i.e., the measurand) within a specified confidence level (CL). This concept was formalized in the early 1990s by the IEC-ISO recommendation "Guide to the expression of uncertainty in measurement" (GUM) [1]. GUM clearly asserts that measurement uncertainty is "a parameter associated with the result of a measurement that characterizes the dispersion of the values that could reasonably be attributed to the measurand."

When the desired measurand Y cannot be measured directly, its value is derived from the values of N input quantities $X_1, X_2, \ldots, X_N$ modeled as random variates and functionally related to Y as

$$Y = f(X_1, X_2, \ldots, X_N). \tag{1}$$

It is worth noting that the function f could model not only a physical law but also a complex measurement process, and, in particular, it should contain all quantities that can contribute a significant uncertainty to the measurement result.

According to the GUM, an estimate y of the measurand Y is the value that the function f assumes in correspondence of the best estimates $x_1, x_2, \ldots, x_N$ of the expectation of input random variates. Moreover, an estimate of the standard uncertainty associated with y, expressed as $u_c(y)$ and representing the output standard deviation, is the positive square root of the estimated variance $u_c^2(y)$ obtained from

$$u_c^2(y) = \sum_{i=1}^{N} \left(\frac{\partial f}{\partial x_i}\right)^2 u^2(x_i) + 2\sum_{i=1}^{N-1}\sum_{j=i+1}^{N} \frac{\partial f}{\partial x_i}\frac{\partial f}{\partial x_j} u(x_i, x_j) \tag{2}$$

where $u(x_i)$ is either the standard uncertainty on x_i or an estimate of it, and the covariance $u(x_i, x_j)$ is given by $r(x_i, x_j) \cdot u(x_i) \cdot u(x_j)$, where $r(x_i, x_j)$ stands for the correlation coefficient associated with x_i and x_j. Equation (2) is usually referred to as the law of propagation of uncertainty (LPU) [1] and represents a first-order Taylor series approximation of the output actual variance. In addition, GUM establishes that the measurement result has to be provided as an interval expressed by $Y = y \pm k u_c(y)$, where the coverage factor k is determined according to both the output probability density function (pdf; Gaussian or t-student) assumed and "CL" required.

As confirmed in [2] and [3], LPU proves efficient only if the following conditions are satisfied: 1) negligible nonlinearity of f; 2) availability of the analytical expression of f; 3) absence of discontinuities in f; and 4) small values of all contributory uncertainties $u(x_i)$.

After the publication of the GUM, several approaches have been proposed with the aim of assuring reliable uncertainty estimates also when the aforementioned conditions are not met. Some of them exploit useful properties of fuzzy or random-fuzzy variables and related algebra [4]–[6]; others use higher order Taylor series approximations [7].

The authors present an original agile approach based on the unscented transform (UT) [8] for overcoming the key limitations of the GUM in the estimation of both output expectation and standard deviation. UT has originally been adopted to improve estimates provided by the extended Kalman filter and moves from the assumption that "it is easier to approximate a probability distribution than it is to approximate an arbitrary nonlinear function or transformation" [8]. To this aim, a set of points (referred to as sigma points) is chosen, in the N-dimensional input domain of the function f, in such a

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L. Angrisani is with the Dipartimento di Informatica e Sistemistica, Università di Napoli Federico II, Napoli 80125, Italy (e-mail: angrisan@unina.it).

M. D'Apuzzo and R. S. Lo Moriello are with the Dipartimento di Ingegneria Elettrica, Università di Napoli Federico II, Napoli 80125, Italy (e-mail: dapuzzo@unina.it; rschiano@unina.it).

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way that their moments are equal to those characterizing input random variates. The function f is then applied to each sigma point, in turn, to yield a cloud of transformed points. Major statistics of transformed points can then be calculated to form an estimate of output expectation and standard deviation.

In the following, the fundamental steps of the proposed approach are first described in detail and then clarified through an application example. Results obtained from the application of the proposed approach to linear and nonlinear measurement processes as well as a typical digital signal processing (DSP) algorithm are presented. According to the recent trend [3], [9], [10], the obtained results are compared to those provided by K Monte Carlo (MC) simulations, taken as reference. A value of $K = 10^6$ can, in fact, be expected to deliver a 95% coverage interval for the output quantity value, whatever the "shape" of its pdf [3]. The theoretical background underlying the proposed approach and its suitability for uncertainty evaluation in indirect measurements are finally outlined in a separate Appendix.

II. Proposed Approach

A. Operative Procedure

Let us consider a vector $\mathbf{X} = [X_1, X_2, \ldots, X_N]$ of uncorrelated random variates, each of which is characterized by its own pdf and models an input quantity of an indirect measurement $Y = f(X_1, X_2, \ldots, X_N)$.

For each $X_i (i = 1, \ldots, N)$, its expectation and a suitable collection of central moments have to be estimated first. This task can be fulfilled through either repeated measurements or already available information and/or user knowledge and experience.

According to UT theory, matrix $\underline{\underline{\boldsymbol{\chi}}}$, the columns of which are the so-called sigma points, has then to be arranged as

$$\underline{\underline{\boldsymbol{\chi}}} = [\underline{\underline{\mathbf{X}}} + \underline{\underline{\Sigma}}_1 \; \underline{\underline{\mathbf{X}}} + \underline{\underline{\Sigma}}_2 \; \ldots \; \underline{\underline{\mathbf{X}}} + \underline{\underline{\Sigma}}_G \; \overline{\mathbf{x}}]. \tag{3}$$

$\underline{\underline{\boldsymbol{\chi}}}$ has N rows and $G \cdot N + 1$ columns, where G is the number of considered central moments. Vector $\overline{\mathbf{x}}$, the last column of $\underline{\underline{\boldsymbol{\chi}}}$, contains the best estimates (x_i, $i = 1, \ldots, N$) of the expectation of variates $X_i (i = 1, \ldots, N)$. Moreover, $\underline{\underline{\mathbf{X}}}$ is an N-dimensional square matrix given by

$$\underline{\underline{\mathbf{X}}} = \begin{bmatrix} x_1 & \cdots & x_1 \\ \vdots & \ddots & \vdots \\ x_N & \cdots & x_N \end{bmatrix} \tag{4}$$

and $\underline{\underline{\boldsymbol{\Sigma}}}_i$, $i = 1, \ldots, G$ are $N \times N$ diagonal matrices

$$\underline{\underline{\boldsymbol{\Sigma}}}_i = \begin{bmatrix} s_{i1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & s_{iN} \end{bmatrix} \tag{5}$$

obtained by solving the nonlinear equation system

$$\begin{cases} 1 = W_0 + N(W_1 + W_2 + \cdots + W_G) \\ \underline{\underline{0}} = W_1 \underline{\underline{\Sigma}}_1 + W_2 \underline{\underline{\Sigma}}_2 + \cdots + W_G \underline{\underline{\Sigma}}_G \\ \underline{\underline{\mu^2}} = W_1 \underline{\underline{\Sigma}}_1^2 + W_2 \underline{\underline{\Sigma}}_2^2 + \cdots + W_G \underline{\underline{\Sigma}}_G^2 \\ \vdots \\ \underline{\underline{\mu^G}} = W_1 \underline{\underline{\Sigma}}_1^G + W_2 \underline{\underline{\Sigma}}_2^G + \cdots + W_G \underline{\underline{\Sigma}}_G^G \end{cases} \tag{6}$$

where $\underline{\underline{\mu^k}}$ are $N \times N$ diagonal matrices, the generic element of which μ_{ii}^k is equal to the kth central moment of the ith input random variate. The value of each weight W_i can be fixed arbitrarily; any choice grants the same final results of the whole procedure. In practice, the choice of the value of each W_i should be helpful in solving equation system (6) easily. Statistics of the resulting sigma points are inherently equal to those characterizing vector X.

Function f is then applied to each sigma point

$$\psi_j = f(\underline{\underline{\boldsymbol{\chi}}}|_j), \qquad j = 1, \ldots, GN + 1 \tag{7}$$

where $\underline{\underline{\boldsymbol{\chi}}}|_j$ stands for the jth column of the matrix $\underline{\underline{\boldsymbol{\chi}}}$. The obtained values ψ_j are processed according to the expressions

$$y = \sum_{j=1}^{N} W_1 \psi_j + \sum_{j=N+1}^{2N} W_2 \psi_j + \cdots + \sum_{j=(G-1)N+1}^{GN} W_G \psi_j + W_0 \psi_{GN+1} \tag{8}$$

$$\mu_y^i = \sum_{j=1}^{N} W_1 (\psi_j - \overline{y})^i + \sum_{j=N+1}^{2N} W_2 (\psi_j - \overline{y})^i + \sum_{j=(G-1)N+1}^{GN} W_G (\psi_j - \overline{y})^i + W_0 (\psi_{GN+1} - \overline{y})^i \tag{9}$$

in order to gain the desired estimates of the output expectation as well as the central moments (see Appendix for further details).

In particular, for $i = 2$, (9) provides the estimate of the output variance.

In the presence of input quantities modeled by random variates characterized by symmetric pdfs, the solution of equation system (6) can be optimized through the choices

$$\underline{\underline{\Sigma}}_{2k} = -\underline{\underline{\Sigma}}_{2k-1}, \quad W_{2k} = -W_{2k-1}, \quad k = 1, \ldots, \frac{G}{2}. \tag{10}$$

This way, only $G/2$ equations along with related weights W_i and unknown matrices $\underline{\underline{\boldsymbol{\Sigma}}}_i$ are necessary to assure that 1) even central moments, up to the Gth order, of the resulting sigma points are equal to the corresponding ones of input variates, and 2) odd central moments are null according to the assumption of input random variates characterized by symmetric pdfs. Thanks to a proper selection of the values of W_i ($W_1 = W_3$ and $W_2 = W_4 = -W_1$), the authors are able to solve the equation

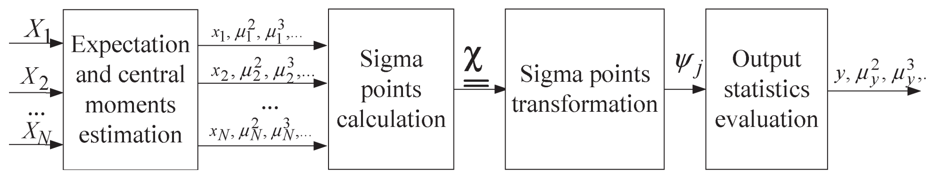

Fig. 1. Operating steps of the proposed approach.

TABLE I
HIGHER ORDER MOMENTS CHARACTERIZING THE INPUT RANDOM VARIATES IN THE CONSIDERED APPLICATION EXAMPLE

	m	φ
μ^4	$3\ \Omega^4$	$3(\pi/40)^4$ rad^4
μ^6	$15\ \Omega^6$	$15(\pi/40)^6$ rad^6
μ^8	$105\ \Omega^8$	$105(\pi/40)^8$ rad^8

system (6) in explicit form for central moments up to the eighth order.

It is worth highlighting that the higher the number of considered input central moments, the more accurate the estimates of output expectation and standard deviation [8]. Moreover, results similar to those granted by higher order Taylor series approximations can be assured with no need of any derivative of f (see the Appendix). This is a very attractive feature whenever the analytical form of f is not available, as in the presence of complex DSP algorithms [11].

B. Application Example

The operating steps of the proposed approach, shown in Fig. 1, are clarified with reference to an application example. Let us suppose to be interested in gaining the real and imaginary parts of a capacitive impedance $\dot{Z}$ from available measures of both its modulus and phase. To this aim, well-known conversion formulas from polar to Cartesian coordinates can be used as

$$\begin{cases} \mathrm{Re}\,(\dot{Z}) = m\,\cos\,(\varphi) \\ \mathrm{Im}\,(\dot{Z}) = m\,\sin\,(\varphi) \end{cases} \tag{11}$$

where m and φ, respectively, stands for impedance modulus and phase; each equation of system (11) can be regarded as a nonlinear measurement model.

According to that stated above, the first step of the proposed approach is the estimation of expectation and central moments of all random variates modeling the considered input quantities. For the sake of simplicity, let us suppose that both modulus and phase of the impedance $\dot{Z}$ are modeled as normal distributed (Gaussian) variates. As for the modulus, expectation and standard deviation have been set equal, respectively, to 1000 and 1 Ω; concerning the phase, they have been chosen equal, respectively, to $\pi/2$ and $\pi/40$ rad. Higher order moments can easily be evaluated; the corresponding values are shown in Table I. If no information about the pdf of the input random variates is available, estimates of their expectation and central moments can be gathered from the values that well-known statistical estimators exhibit when applied to the results obtained through independent repeated measurements.

Sigma points are then established. Since the considered pdfs are symmetric, the optimized solution of the equation system

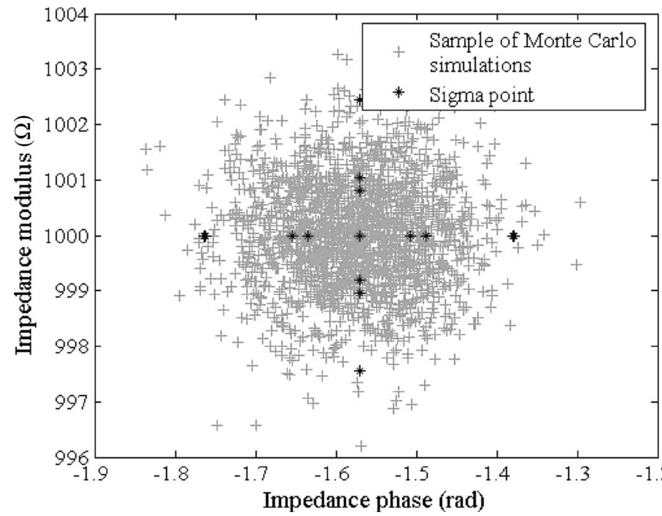

Fig. 2. Location of the resulting sigma points in the phase–modulus plane. Some samples provided by MC simulations are also given.

(6) can be pursued; a set of sigma points capable of propagating moments up to the eighth order can thus be gained. For the considered example, four matrices $\underline{\underline{\boldsymbol{\Sigma}}}_i$, $i = 1, \ldots, 4$, are determined; each matrix has the form

$$\underline{\underline{\boldsymbol{\Sigma}}}_i = \begin{bmatrix} m_i & 0 \\ 0 & \varphi_i \end{bmatrix} \tag{12}$$

while matrix $\underline{\underline{X}}$ can be written as

$$\underline{\underline{\mathbf{X}}} = \begin{bmatrix} m & m \\ \varphi & \varphi \end{bmatrix}. \tag{13}$$

Elements of the matrices $\underline{\underline{\boldsymbol{\Sigma}}}_i$ are, in particular, calculated by solving the equation systems

$$\begin{cases} \sigma_m^2 = 2W_1 m_1^2 - 2W_1 m_2^2 + 2W_1 m_3^2 - 2W_1 m_4^2 \\ 3\sigma_m^4 = 2W_1 m_1^4 - 2W_1 m_2^4 + 2W_1 m_3^4 - 2W_1 m_4^4 \\ 15\sigma_m^6 = 2W_1 m_1^6 - 2W_1 m_2^6 + 2W_1 m_3^6 - 2W_1 m_4^6 \\ 105\sigma_m^8 = 2W_1 m_1^8 - 2W_1 m_2^8 + 2W_1 m_3^8 - 2W_1 m_4^8 \end{cases} \tag{14}$$

$$\begin{cases} \sigma_\varphi^2 = 2W_1 \varphi_1^2 - 2W_1 \varphi_2^2 + 2W_1 \varphi_3^2 - 2W_1 \varphi_4^2 \\ 3\sigma_\varphi^4 = 2W_1 \varphi_1^4 - 2W_1 \varphi_2^4 + 2W_1 \varphi_3^4 - 2W_1 \varphi_4^4 \\ 15\sigma_\varphi^6 = 2W_1 \varphi_1^6 - 2W_1 \varphi_2^6 + 2W_1 \varphi_3^6 - 2W_1 \varphi_4^6 \\ 105\sigma_\varphi^8 = 2W_1 \varphi_1^8 - 2W_1 \varphi_2^8 + 2W_1 \varphi_3^8 - 2W_1 \varphi_4^8. \end{cases} \tag{15}$$

Letting W_1 be equal to 1, the resulting sigma points on the phase modulus plane are given in Fig. 2. The same figure also shows some samples of input random variates attained through MC simulations.

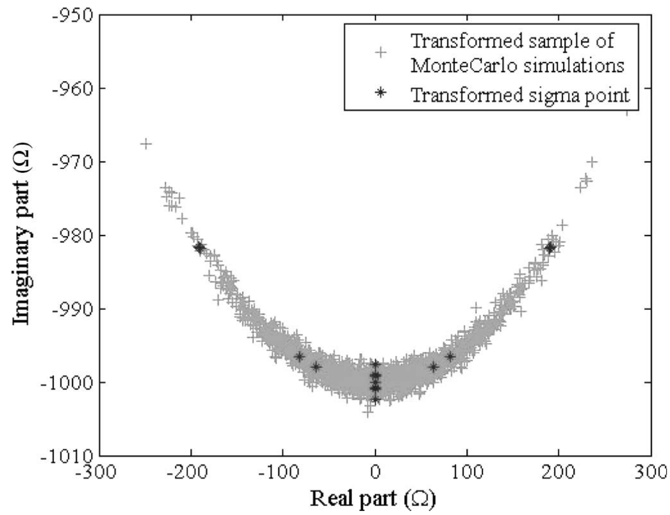

Fig. 3. Location on the Nyquist plane of the transformed sigma points. Results obtained from the application of the measurement model to MC samples are also presented.

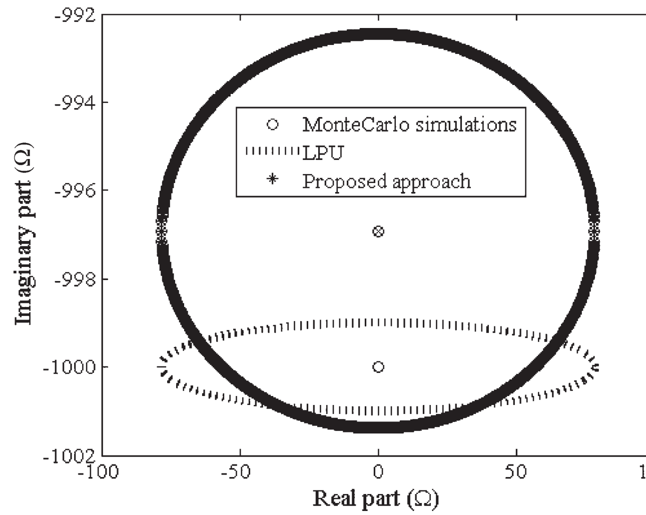

Fig. 4. Comparison of the 1σ contour characterizing the proposed approach to those peculiar to MC simulations and LPU.

TABLE II
RESULTS PROVIDED BY THE PROPOSED APPROACH

	$\mathrm{Re}(\dot{z})$ (Ω)		$\mathrm{Im}(\dot{z})$(Ω)	
	y_{Re}	u_{Re}	y_{Im}	u_{Im}
UT	0.000	78.29	-996.920	4.46
MC	-0.03	78.22	-996.927	4.44

The third step of the proposed approach is the application of the measurement model (11) to the resulting sigma points. The transformed sigma points are reported in Fig. 3 along with the results obtained from the application of the measurement model to the aforementioned MC samples.

Estimates of expectation (y_{Re} and y_{Im}) and standard uncertainty (u_{Re} and u_{Im}) of both $\mathrm{Re}\,(\dot{Z})$ and $\mathrm{Im}\,(\dot{Z})$ are finally attained through (8) and (9). The obtained results are given in Table II and compared to those provided by 10^6 MC simulations. In particular, rows referred to as UT and MC report estimates provided, respectively, by the proposed approach and MC simulations for both $\mathrm{Re}(\dot{Z})$ and $\mathrm{Im}(\dot{Z})$. A remarkable concurrence can be noticed.

Fig. 4 gives an interesting representation of the performance granted by the proposed approach. In particular, its 1σ contour, i.e., the ellipse centered in $(y_{\mathrm{Re}}, y_{\mathrm{Im}})$ and whose semimajor and semiminor axes are equal, respectively, to u_{Re} and u_{Im}, has been sketched and compared to those related to MC simulations and LPU. It is worth noting how the 1σ contour characterizing the proposed approach fits well that granted by MC simulations. The same consideration does not hold for the LPU 1σ contour; biased and inconsistent estimates are, in fact, experienced.

III. EXPERIMENTS

A number of tests have been conducted on simulated and actual measurement data; different measurement models have been considered. For each measurement model, estimates of output expectation y and standard uncertainty u_c have been determined both through the proposed approach and LPU. The achieved performance is expressed as differences Δy and Δu_c between the obtained estimates and those granted by 10^6 MC simulations. In the following, the symbol UT8 refers to the capability of the proposed approach in capturing central moments up to the eighth order in the presence of input random variates characterized by symmetric pdfs. In a similar way, the symbol UTa4 refers to the capability of the proposed approach of capturing central moments up to the fourth order in the presence of input random variates characterized by asymmetric pdfs.

TABLE III
RESULTS OBTAINED IN TESTS ON THE CONSIDERED LINEAR MEASUREMENT MODEL

Test	Δy		Δu_c	
	LPU	UT8	LPU	UT8
L1	0.00034	0.00034	$2.2\cdot10^{-4}$	$2.2\cdot10^{-4}$
L2	0.0065	0.0065	$-5.0\cdot10^{-4}$	$-5.0\cdot10^{-4}$
L3	0.0077	0.0077	$-2.2\cdot10^{-4}$	$-2.2\cdot10^{-4}$
L4	-0.0084	-0.0084	$-4.5\cdot10^{-4}$	$-4.5\cdot10^{-4}$

A. Linear Measurement Model

According to that reported in [3], the considered measurement model accounts for the addition of four input quantities

$$Y = X_1 + X_3 + X_3 + X_4. \tag{16}$$

Four different test conditions have been analyzed. In the test referred to as L1, all input quantities have been modeled as standard Gaussian variates ($X_i = \mathcal{N}(0, 1),\ i = 1, \ldots, 4$). As for the test L2, zero-mean rectangular distributed input random variates, characterized by the same width ($X_i = \mathcal{U}(0, \sqrt{3})$, $i = 1, \ldots, 4$), have been considered. Concerning the test L3, all input quantities have been modeled as rectangular distributed variates; X_1, X_2, and X_3 have the same features ($X_i = \mathcal{U}(0, \sqrt{3})$, $i = 1, \ldots, 3$), while X_4 has a higher variance ($X_4 = \mathcal{U}(0, \sqrt{30})$). In the last test referred to as L4, the pdfs of the input random variates have been different from one another [$X_1 = \mathcal{N}(0, 1),\ X_2 = \mathcal{N}(1, 2),\ X_3 = \mathcal{U}(0, \sqrt{3})$, and $X_4 = \mathcal{U}(2, 1)$]. Obtained results are summarized in Table III.

TABLE IV
RESULTS OBTAINED IN TESTS ON THE CONSIDERED NONLINEAR MEASUREMENT MODEL WITH X_1 AND X_2 MODELED AS SYMMETRICALLY DISTRIBUTED VARIATES. Δy AND Δu_c ARE EXPRESSED IN PERCENTAGE RELATIVE TERMS

Test	σ	Δ	Δy (%)		Δu_c (%)		C.L. (%)	
			LPU	UT8	LPU	UT8	LPU	UT8
S1	$0.0001\cdot\pi$	0.00001	$5.1\cdot10^{-6}$	$1.8\cdot10^{-7}$	0.047	0.034	100	100
S2	$0.001\cdot\pi$	0.0001	$4.9\cdot10^{-4}$	$6.0\cdot10^{-8}$	-0.63	0.086	99.6	99.8
S3	$0.01\cdot\pi$	0.001	0.049	$-2.2\cdot10^{-4}$	-36	0.10	82.5	96.1
S4	$0.05\cdot\pi$	0.005	1.2	0.023	-83	-0.13	48.3	94.9
S5	$0.1\cdot\pi$	0.05	5.1	-0.044	-91	0.15	35.7	94.8

TABLE V
RESULTS OBTAINED IN TESTS ON THE CONSIDERED NONLINEAR MEASUREMENT MODEL WITH X_1 AND X_2 MODELED RESPECTIVELY AS SYMMETRICALLY AND ASYMMETRICALLY DISTRIBUTED VARIATES. Δy AND Δu_c ARE EXPRESSED IN PERCENTAGE RELATIVE TERMS

Test	σ	Δ	Δy (%)		Δu_c (%)		C.L. (%)	
			LPU	UTa4	LPU	UTa4	LPU	UTa4
As1	$0.0001\cdot\pi$	0.00001	-0.016	$-3.5\cdot10^{-5}$	97	0.056	19.3	96.2
As2	$0.001\cdot\pi$	0.0001	-0.16	$-2\cdot10^{-4}$	98	0.25	19.5	96.2
As3	$0.01\cdot\pi$	0.001	-1.6	$1.7\cdot10^{-4}$	97	-0.7	19.3	96.3

As expected, both the proposed approach and the LPU have shown similar performances.

B. Nonlinear Measurement Model

The nonlinear measurement model $Y = X_1 \cos(X_2)$ has been taken into account. Input quantities X_1 and X_2 have been modeled, respectively, as a rectangular distributed variate $(X_1 = \mathcal{U}(1, \Delta))$ and as a Gaussian random variate $(X_2 = \mathcal{N}(\pi, \sigma))$. Increasing values of standard deviation σ and width Δ have been considered; they are summarized in Table IV. For each couple of values of σ and Δ, estimates of output expectation and standard uncertainty have been gained both through the proposed approach and LPU. 10^6 MC simulations have also been carried out to estimate the output cumulative distribution function. This way, it has been possible to evaluate the CL associated to the interval $[y - 2u_c, y + 2u_c]$, for both the proposed approach and the LPU. Results presented in Table IV clearly prove the superior performance of the proposed approach with respect to LPU, especially in critical conditions (high values of σ and Δ).

Further examples have concerned asymmetrically distributed input variates in the same nonlinear measurement model; X_2 has, in particular, been modeled as a Rayleigh distribution with parameter σ. The obtained results are given in Table V; considerations very similar to those stated above can be drawn.

C. DSP Algorithm

The proposed approach works well also in the presence of DSP algorithms as measurement models. The given example refers to the application of the fast Fourier transform (FFT) to a portion (16 samples) of an actual 100-kHz sinusoidal signal in order to gain its amplitude spectrum. Samples have been acquired at a rate of 1 MS/s and quantized through an 8-bit analog-to-digital converter. The same rectangular pdf has been assumed for each sample. For each spectral line (bin), differences (Δy and Δu_c) between expectation and standard deviation estimates provided by the proposed approach and those granted by 10^6 MC simulations have been calculated. Their values, in percentage relative terms, have never been greater than 0.2% (Fig. 5). Moreover, the time taken by the proposed approach has been much lower (31 ms) than that required by MC simulations (about 23 s), hardware and software resources being equal.

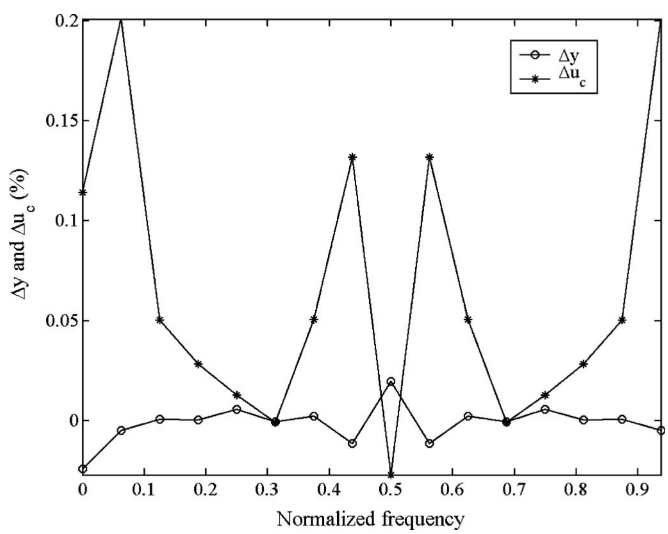

Fig. 5. Differences, for each spectral line, between expectation and standard deviation estimates provided by the proposed approach and those granted by 10^6 MC simulations.

IV. CONCLUSION

An original approach for estimating output expectation y and standard uncertainty u_c in indirect measurements has been presented. It proposes the use of the UT to overcome typical problems affecting current GUM recommendations. To assess its performance, several tests have been conducted on a linear measurement model, a nonlinear measurement model, and an FFT-based algorithm. Differences between obtained estimates and those granted by 10^6 MC simulations have, in particular, been evaluated; values always lower than 0.7% have been achieved. Moreover, the effective CL associated with the interval $[y - 2u_c, y + 2u_c]$ has also been derived; CLs higher than 90% have been experienced.

Future activity is mainly oriented to 1) make the proposed approach effective in the presence of correlated input variates (preliminary results have already been presented in [12]) and in a real-time context, 2) evaluate uncertainty components due to the analog-to-digital conversion on the output of DSP algorithms, and 3) find out how to derive the CL on the measurement result from moments of the transformed sigma points [13].

APPENDIX

Let us consider expression (8); it can be written as

$$\begin{aligned} y &= \sum_{j=1}^{N} W_1 f(\overline{\mathbf{x}} + \mathbf{s}_{1j}) + \sum_{j=N+1}^{2N} W_2 f(\overline{\mathbf{x}} + \mathbf{s}_{2j}) \\ &\quad + \sum_{j=(G-1)N+1}^{GN} W_G f(\overline{\mathbf{x}} + \mathbf{s}_{Gj}) + W_0 f(\overline{\mathbf{x}}) \\ &= \sum_{i=1}^{G}\sum_{j=1}^{N} W_i f(\overline{\mathbf{x}} + \mathbf{s}_{ij}) + W_0 f(\overline{\mathbf{x}}) \end{aligned} \tag{17}$$

where $\mathbf{s}_{ij}$ stands for the jth column of matrix $\underline{\underline{\boldsymbol{\Sigma}}}_i$. Considering multidimensional Taylor series expansion, the generic transformed sigma point can be expressed as

$$f(\overline{\mathbf{x}} + \mathbf{s}_{ij}) = f(\overline{\mathbf{x}}) + \nabla f \mathbf{s}_{ij} + \frac{\nabla^2 f \mathbf{s}_{ij}^2}{2!} + \frac{\nabla^3 f \mathbf{s}_{ij}^3}{3!} + \cdots \tag{18}$$

where the ∇f operator evaluates the total differential of $f(\cdot)$ when perturbed by $\mathbf{s}_{ij}$ around a nominal value $\overline{\mathbf{x}}$. The kth term in the Taylor series is

$$\frac{\nabla^k f \mathbf{s}_{ij}^k}{k!} = \frac{1}{k!}\left(\sum_{m=1}^{N} s_{imj}\frac{\partial}{\partial x_m}\right)^k f(\cdot)\big|_{\mathbf{x}=\overline{\mathbf{x}}} \tag{19}$$

where s_{imj} is the element of matrix $\underline{\underline{\Sigma}}_i$ characterized by row and column indexes equal, respectively, to m and j. Since matrices $\underline{\underline{\Sigma}}_i$ are diagonal, it results in $s_{imj} = 0$ for each $m \neq j$; this way, it is possible to verify that

$$\frac{\nabla^k f \mathbf{s}_{ij}^k}{k!} = \frac{1}{k!}\, s_{ijj}^k \frac{\partial^k f}{\partial x_j^k}\bigg|_{\mathbf{x}=\overline{\mathbf{x}}}. \tag{20}$$

The last term of (17) can thus be written as

$$\begin{aligned} y &= W_0 f(\overline{\mathbf{x}}) + \sum_{j=1}^{N}\sum_{i=1}^{G} W_i \\ &\quad \times \left(f(\overline{\mathbf{x}}) + s_{ijj}\frac{\partial f}{\partial x_j}\bigg|_{\mathbf{x}=\overline{\mathbf{x}}} + \frac{1}{2!}\, s_{ijj}^2 \frac{\partial^2 f}{\partial x_j^2}\bigg|_{\mathbf{x}=\overline{x}} + \cdots \right) \end{aligned} \tag{21}$$

where the higher order terms of the Taylor series have been dropped.

Let us analyze each term on the right side of (21). Thanks to the properties of the determined sigma points (6), it can be verified that

$$\begin{aligned} W_0 f(\overline{\mathbf{x}}) + \sum_{j=1}^{N}\sum_{i=1}^{G} W_i f(\overline{\mathbf{x}}) &= f(\overline{\mathbf{x}})\left(W_0 + \sum_{i=1}^{G} N W_i\right) \\ &= f(\overline{\mathbf{x}}) \end{aligned} \tag{22}$$

$$\begin{aligned} \sum_{j=1}^{N}\sum_{i=1}^{G} W_i\, s_{ijj}\frac{\partial f}{\partial x_j}\bigg|_{\mathbf{x}=\overline{\mathbf{x}}} &= \sum_{j=1}^{N}\frac{\partial f}{\partial x_j}\bigg|_{\mathbf{x}=\overline{\mathbf{x}}} \sum_{i=1}^{G} W_i s_{ijj} \\ &= 0 \end{aligned} \tag{23}$$

$$\begin{aligned} \sum_{j=1}^{N}\sum_{i=1}^{G} W_i\, s_{ijj}^2\frac{\partial^2 f}{\partial x_j^2}\bigg|_{\mathbf{x}=\overline{\mathbf{x}}} &= \sum_{j=1}^{N}\frac{\partial^2 f}{\partial x_j^2}\bigg|_{\mathbf{x}=\overline{\mathbf{x}}} \sum_{i=1}^{G} W_i s_{ijj}^2 \\ &= \sum_{j=1}^{N}\mu_j^2\frac{\partial^2 f}{\partial x_j^2}\bigg|_{\mathbf{x}=\overline{\mathbf{x}}} \end{aligned} \tag{24}$$

$$\begin{aligned} \sum_{j=1}^{N}\sum_{i=1}^{G} W_i\, s_{ijj}^k\frac{\partial^k f}{\partial x_j^k}\bigg|_{\mathbf{x}=\overline{\mathbf{x}}} &= \sum_{j=1}^{N}\frac{\partial^k f}{\partial x_j^k}\bigg|_{\mathbf{x}=\overline{\mathbf{x}}} \sum_{i=1}^{G} W_i s_{ijj}^k \\ &= \sum_{j=1}^{N}\mu_j^k\frac{\partial^k f}{\partial x_j^k}\bigg|_{\mathbf{x}=\overline{\mathbf{x}}} \end{aligned} \tag{25}$$

Thus, the output expectation estimate becomes

$$y = f(\overline{\mathbf{x}}) + \frac{1}{2!}\sum_{j=1}^{N}\mu_j^2\frac{\partial^2 f}{\partial x_j^2}\bigg|_{\mathbf{x}=\overline{\mathbf{x}}} + \cdots + \frac{1}{k!}\sum_{j=1}^{N}\mu_j^k\frac{\partial^k f}{\partial x_j^k}\bigg|_{\mathbf{x}=\overline{\mathbf{x}}} + \cdots. \tag{26}$$

Comparing (26) to the Gth-order Taylor series approximation of the measurement model in (1), the following considerations arise.

- In the presence of uncorrelated input random variates, (26) includes all terms of the considered Taylor series approximation up to the second order.
- According to equation system (6), a number of the kth-order $(3 < k < G)$ terms of the Taylor series approximation are present in (26), thus enhancing the quality of the output expectation estimate. Only the kth-order joint moments are, in particular, missed out. The effect of their absence is, however, mitigated by the fact that the terms in the Taylor series have inverse factorial weights.

Similar considerations also hold for the output standard deviation estimate.

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Leopoldo Angrisani was born in Nocera Superiore, SA, Italy, on April 16, 1969. He received the M.S. degree (*cum laude*) in electronic engineering from the University of Salerno, Salerno, Italy, in 1993 and the Ph.D. degree in electrical engineering from the Università di Napoli Federico II, Napoli, Italy, in 1997.

Since 2002, he has been an Associate Professor at the Dipartimento di Informatica e Sistemistica, Università di Napoli Federico II. He is currently involved in research into new digital signal processing methods for detecting, measuring, and classifying transient signals, estimating the instantaneous frequency of monocomponent and multicomponent signals, assessing the performance of RF equipment in wireless and/or mobile communication systems, and testing communication networks.

Massimo D'Apuzzo was born in Napoli, Italy, in 1947. He received the M.S. degree in chemical engineering and electrical engineering from the Università di Napoli Federico II, Napoli, Italy, in 1971 and 1973, respectively.

In 1974, he was an Assistant Professor of electrical measurements at the Institute of Electrical Engineering, Università di Napoli Federico II. In 1980, he became an Associate Professor of electrical measurements. He was a Full Professor of electrical measurements at the University of L'Aquila, L'Aquila, Italy, from 1986 to 1988 and at the University of Salerno, Salerno, Italy, from 1989 to 1991. Since 1991, he has been a Full Professor of electrical measurements at the Università di Napoli Federico II. His current research interests are new approaches for measurement uncertainty evaluation, use of wavelet transform for measurement applications, and new methods based on time–frequency transforms for testing RF equipment for mobile communications.

Rosario Schiano Lo Moriello was born in Castellammare di Stabia (NA), Italy, in 1975. He received the M.S. degree (*cum laude*) in materials engineering and the Ph.D. degree in electrical engineering from the Università di Napoli Federico II, Napoli, Italy, in 2001 and 2004, respectively.

His current research interests are concerned with distance smart sensors based on ultrasonics, uncertainty evaluation in indirect measurements, new methods based on wavelet transforms for instantaneous frequency estimation, and real-time apparatuses for phase noise measurement.