Measurement Uncertainty in a Multivariate Model: A Novel Approach

Gaetano Iuculano, Andrea Zanobini, Annarita Lazzari, and Gabriella Pellegrini Gualtieri

***Abstract*—The confidence region and the related confidence level are the bases for the uncertainty expression in a measurement process. In this work, a multivariate model is considered, and different situations with respect to the parameters constituting the model are examined. Computational results are reported and analyzed to assess the validity of the proposed approach.**

***Index Terms*—Uncertain systems, uncertainty-confidence region.**

I. Introduction

THE ISO/BIPM *Guide to the Expression of Uncertainty in Measurement* [1], usually referred to as the GUM or the Guide, and the associated NIST adaptation [2] have described a unified convention for expressing measurement uncertainty. Application of the Guide has extended beyond calibration and research laboratories and into the industrial domain of manufactured products.

According to the GUM [1], the measurement uncertainty defines the range of values that could reasonably be attributed to the measured quantity, and it can be appropriately expressed by a statistical coverage interval. This interval, with acceptable probability (i.e., confidence level), can include a relevant share of the above-mentioned values population. The method that the Guide implicitly suggests for evaluating uncertainty is essentially limited to the so-called Gaussian "propagation law" for uncertainty, while the calculation of the probability intervals is based on the approximation method.

This paper aims to give the uncertainty measurement expression in a multivariate model that considers a vectorial measurand, not explicitly treated in the Guide. This model is essential in the case of a set of measurands simultaneously determined in the same measurement process or when it wants to specify the compatibility of multiple measurements that represent the same measurand in different conditions.

From the operative point of view, in this work, the confidence levels of the probability regions are evaluated in order to describe the uncertainty expression in different situations of practical interest.

Manuscript received May 26, 2002; revised June 18, 2003.

G. Iuculano and A. Zanobini are with the Department of Electronics and Telecommunications, University of Florence, Florence, Italy (e-mail: willis@ingfi1.ing.unifi.it).

A. Lazzari is with the Mitutoyo Institute of Metrology, Mitutoyo Italiana S.r.l., Milano, Italy (e-mail: annarita.lazzari@mitutoyo.it).

G. P. Gualtieri is with the Department of Applied Mathematics, Engineering Faculty, University of Florence, Florence, Italy (e-mail: Gualtieri@dma.unifi.it).

Digital Object Identifier 10.1109/TIM.2003.817935

II. Measurement and the Uncertainty Expression: The Method Presented in GUM

The aim and objective of a measurement are to attribute a value to a measurand Y. From the operational point of view, this is possible by comparing the measurand Y with a known quantity or with a quantity formed by other quantities $\{\underline{X} = X_1, X_2, \ldots, X_n\}$, which are easy to determine. For this, a model of the evaluation is needed which expresses this assignment of a value to the measurand in mathematical terms (i.e., a functional relationship or equation)

$$Y = f(X_1, X_2, \ldots, X_n) = f(\underline{X}). \tag{1}$$

In particular, the function f includes corrections for systematic effects, accounts for sources of variability, such as those due to different instruments, observers, laboratories. and so on. Thus, the general functional relationship represents not only a physical law but also a measurement process.

The result of such process, that is the output quantity Y, is obtained by (1) using input estimates $\underline{x} = (x_1, x_2, \ldots, x_n)$, usually coinciding with the expected values, for the values of the n input quantities $X_1, X_2, \ldots, X_n$

$$y = f(x_1, x_2, \ldots, x_n) = f(\underline{x}) \tag{2}$$

consequently reported as $Y = y \pm U$ where y is the estimate (or expectation) of Y and U is the expanded uncertainty defined by $U = ku_c(y)$. Here, $u_c(y)$ is the combined standard uncertainty representing the estimated standard deviation of the result Y, and k is a coverage factor chosen to produce an interval having a level of confidence close to the expectation value.

The standard uncertainties of measurement to be attributed to the input values are obtained as square roots from the variances of the distributions: $u(x_i) = \sqrt{\mathrm{Var}[X_i]}, \mathrm{i} = 1, \ldots \mathrm{n}$. The combined standard uncertainty of the measurement result Y is the positive square root of the estimated variance $u_c^2(y)$ obtained from

$$u_c^2(y) = \sum_{i=1}^{n} A_i + 2\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} B_{ij} \tag{3}$$

where the terms $A_i = (\partial f/\partial x_i)^2 u^2(x_i)$ are weighted variances associated with x_i $\mathrm{i} = 1, 2, \ldots, \mathrm{n}$, and $B_{ij} = (\partial f/\partial x_i)(\partial f/\partial x_j) u(x_i, x_j)$ are weighted covariances associated with x_i, x_j $\mathrm{i}, \mathrm{j} = 1, 2, \ldots, \mathrm{n}$, respectively. The terms A_{i}, B_{ij} depend on the probability distributions that characterize one's knowledge of the input quantities. The weight coefficients $((\partial f(\underline{X}))/\partial x_i)$, referred to as sensitivity coefficients, are equal to $((\partial f(\underline{X}))/\partial X_i)$ evaluated at $\underline{X} = \underline{x}$.

Equation (3) derives from the following considerations: if the function $f(\underline{X})$ is supposed to be a continuous function of the input quantity $X_1, X_2, \ldots, X_n$, it can be approximated using a second-order Taylor's series expansion about the means

$$Y = f(\underline{x}) + \sum_{i=1}^{n} \left(\frac{\partial f(\underline{X})}{\partial x_i} \right)_{\underline{X}=\underline{x}} (X_i - x_i) + W. \tag{4}$$

W is the remainder expressed by

$$W = \frac{1}{2!} \left[\sum_{i=1}^{n} \left(\frac{\partial^2 f}{\partial X_i^2} \right)_{\underline{X}=\underline{x}+\theta(\underline{X}-\underline{x})} (X_i - x_i)^2 + 2 \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \times \left(\frac{\partial^2 f(\underline{X})}{\partial X_i \partial X_j} \right)_{\underline{X}=\underline{x}+\theta(\underline{X}-\underline{x})} (X_i - x_i)(X_j - x_j) \right] \tag{5}$$

with $0 < \theta < 1$. When $\underline{X} \to \underline{x}$, the remainder approaches zero more quickly than the first-order terms in (4), so W and all the higher terms are normally neglected, provided that the uncertainties in $X_1, X_2, \ldots, X_n$ are small and the vector $\underline{X} = (X_1, X_2, \ldots, X_n)$ is close to $\underline{x} = (x_1, x_2, \cdots, x_n)$. For a linear model W, is zero.

Equation (3) is commonly referred to as the law of propagation of uncertainty, and it is based on the first-order Taylor series approximation of (1) assuming $W = 0$ in (4)

$$u(y) \cong \sqrt{\sum_{i=1}^{n} \frac{\partial f(\underline{X})}{\partial x_i} u^2(x_i) + 2 \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \frac{\partial f}{\partial x_i} \frac{\partial f}{\partial x_j} u(x_i, x_j)} \tag{6}$$

where $u(x_i, x_j) = \rho_{ij} u(x_i) u(x_j)$, and ρ_{ij} the correlation coefficients between $x_i x_j$, $i \neq j$.

The correlation coefficients are the measure in which potential dependencies of the knowledge of the input quantities for the evaluation are expressed. Their value lies in the range: $|\rho_{ij}| \leq 1$. In the most frequent case that the knowledge of the input quantities can be considered noncorrelated, the correlation coefficients have the values

$$\rho_{i,j} = \begin{cases} 0, & \text{if } i \neq j \\ 1, & \text{if } i = j \end{cases}. \tag{7}$$

III. "Measurement" and its Confidence Region: A Novel Approach

The "measurement" that is related to n different measurands in the same measurement process, or to n reproductions of the same measurand, is considered a random multivariate variable represented through the vector $\underline{M} = (M_1, \ldots, M_{\mathrm{n}})$. It belongs to a probability n-dimensional space $S^{(n)}$, which represents the set of arrays, each of them of n values jointly attributable within the limits of the measurements done.

Let us denote by $C^{(n)}$ a limited domain of $S^{(n)}$ a variability region of $\underline{M}$. This region, according to the mono-dimensional case of the GUM, can be called the statistical confidence region, and the related confidence level, denoted with p, is defined by

$$p = P\{\underline{M} \in C^{(n)}\} = \int \underset{C^{(n)}}{\cdots} \int f_{\underline{\mathrm{M}}}(\underline{m}) dm_1 dm_2 \ldots dm_n \tag{8}$$

where $\underline{m} = (m_1, m_2, \ldots, m_n)$ and $f_{\underline{\mathrm{M}}}(\underline{m})$ is the nth varied joint probability density function

$$f_{\underline{\mathrm{M}}}(\underline{m}) = \lim_{\underline{h} \to \underline{0}} \frac{P\{\underline{m} \leq M \leq \underline{m} + \underline{h}\}}{\underline{h}} \tag{9}$$

with $\underline{h} = (h_1, h_2, \ldots, h_n)$ and $\underline{0}$ is zero vector. The confidence region $C^{(n)}$ and the related confidence level p are the bases for the uncertainty expression in the multivariate model.

IV. Applications and Results of the Proposed Approach

In this section, the proposed approach is applied considering the multivariate normal distribution and multivariate uniform distribution. In both situations, square and circular domains are taken into account.

A. Two-Variate Distribution

First, we considered the two-varied model ($n = 2$). Let two measurements M_1, M_2 with mean values $\mu_1 = E\{M_1\}$, $\mu_2 = E\{M_2\}$ have uncertainties $u(M_1) = \sqrt{\mathrm{Var}\{M_1\}}$ and $u(M_2) = \sqrt{\mathrm{Var}\{M_2\}}$, respectively.

Consider the measurements in the reduced (or standardized) form $M_1^* = (M_1 - \mu_1)/(\sigma_1)$, $M_2^* = (M_2 - \mu_2)/(\sigma_2)$.

The confidence level is

$$p = P\left\{M_1^*, M_2^* \in C^{(2)}\right\} = \iint_{\mathrm{C}^{(2)}} f_{\mathrm{M}_1^*\mathrm{M}_2^*}(m_1, m_2) dm_1 dm_2. \tag{10}$$

Obviously, p depends on $C^{(2)}$, where $C^{(2)}$ is the "two-varied confidence region" [5], [7], and it depends also on the correlation coefficient $\rho_{12} = E\{M_1^* M_2^*\}$.

We have considered two cases:

- *Square domain*: $C^{(2)} \equiv \{(m_1, m_2) \in R^2 : -(h+B) \leq m_1 \leq h - B, -(h+B) \leq m_2 \leq h - B\}$, where B is a convenient parameter

$$p = \int_{-(h+B)}^{h-B} \int_{-(h+B)}^{h-B} f_{\mathrm{M}_1^*\mathrm{M}_2^*}(m_1, m_2) dm_1 dm_2. \tag{11}$$

- *Circular domain*: $C^{(2)} \equiv \{(m_1, m_2) \in R^2 : m_1^2 + m_2^2 \leq a^2\}$, where a is the radius and

$$p = \iint_{m_1^2 + m_2^2 \leq a^2} f_{\mathrm{M}_1^*\mathrm{M}_2^*}(m_1, m_2) dm_1 dm_2. \tag{12}$$

For each of the two above-mentioned "confidence regions, " we considered two distributions:

a) *Normal two-variate distribution* ($-\infty < m_i < +\infty$, $\mathrm{i} = 1{,}2$)

$$\begin{aligned} p &= \iint_{C^{(2)}} f_{M_1^*, M_2^*}(m_1, m_2) dm_1 dm_2 \\ &= \frac{1}{2\pi\sqrt{1 - \rho_{12}^2}} \iint_{C^{(2)}} \times \exp\left\{ -\frac{1}{2\left(1 - \rho_{12}^2\right)} \times \left[m_1^2 - 2\rho_{12} m_1 m_2 + m_2^2\right] \right\} dm_1 dm_2. \end{aligned} \tag{13}$$

TABLE I
DIFFERENT VALUES OF THE CONFIDENCE LEVEL p

	B	**Square domain (parameter h)**					
		p=0.70	**p=0.75**	**p=0.80**	**p=0.85**	**p=0.90**	**p=0.95**
ρ_{12} = -0.90	0	1.449	1.500	1.549	1.597	1.643	1.688
	0.02	1.449	1.500	1.549	1.597	1.643	1.688
	0.06	1.451	1.502	1.552	1.599	1.646	1.691
	0.1	1.456	1.507	1.556	1.604	1.651	1.696
ρ_{12} = -0.50	0	1.449	1.500	1.549	1.597	1.643	1.688
	0.02	1.449	1.500	1.549	1.597	1.643	1.688
	0.06	1.450	1.501	1.551	1.598	1.645	1.690
	0.1	1.453	1.504	1.553	1.601	1.647	1.692
ρ_{12} = 0	0	1.449	1.500	1.549	1.597	1.643	1.688
	0.02	1.449	1.500	1.549	1.597	1.643	1.688
	0.06	1.449	1.500	1.549	1.597	1.643	1.688
	0.1	1.449	1.500	1.549	1.597	1.643	1.688
ρ_{12} = 0.50	0	1.449	1.500	1.549	1.597	1.643	1.688
	0.02	1.449	1.500	1.549	1.597	1.643	1.688
	0.06	1.448	1.499	1.548	1.595	1.642	1.687
	0.1	1.446	1.496	1.545	1.593	1.639	1.684
ρ_{12} = 0.90	0	1.449	1.500	1.549	1.597	1.643	1.688
	0.02	1.449	1.500	1.549	1.597	1.643	1.688
	0.06	1.447	1.498	1.547	1.594	1.641	1.685
	0.1	1.443	1.493	1.542	1.590	1.636	1.681

b) *Uniform two-variate distribution*

By introducing the marginal distributions for the uniform random variables expressed in reduced form, with probability density equal to

$$f_{M_1^*}(m_i)\begin{cases}\frac{1}{2\sqrt{3}}, & -\sqrt{3}\le m_i\le\sqrt{3}\quad i=1,2\\ 0, & \text{elsewhere}\end{cases}\tag{14}$$

the two-varied joint probability density function is [7]

$$\begin{aligned}f_{M_1^*M_2^*}(m_1,m_2)&=f_{M_1^*}(m_1)f_{M_2^*}(m_2)\\&\quad\times\left\{1+3\rho_{12}\left[1-2F_{M_1^*}(m_1)\right]\left[1-2F_{M_2^*}(m_2)\right]\right\}\end{aligned}\tag{15}$$

where the distribution functions $F_{M_1^*}(m_1)$ and $F_{M_2^*}(m_2)$ are

$$F_{M_1^*}(m_i)=\int_{-\sqrt{3}}^{m_i}f_{M_1^*}\,(m_i')\,dm_i'=\frac{1}{2}\left[1+\frac{m_i}{\sqrt{3}}\right],\quad i=1,2.\tag{16}$$

By substituting (14) and (16) in (15), we obtain

$$\begin{aligned}f_{M_1^*M_2^*}(m_1,m_2)&=\frac{1}{12}\left\{1+3\rho_{12}\left[-\frac{m_1}{\sqrt{3}}\right]\left[-\frac{m_2}{\sqrt{3}}\right]\right\}\\&=\frac{1}{12}\{1+\rho_{12}m_1m_2\}\\&\quad-\sqrt{3}\le m_1,m_2\le\sqrt{3}.\end{aligned}\tag{17}$$

In particular, we verified that

$$\frac{1}{12}\int_{-\sqrt{3}}^{\sqrt{3}}\int_{-\sqrt{3}}^{\sqrt{3}}\{1+\rho_{12}m_1m_2\}dm_1dm_2=1\quad\text{(normalization condition)}\tag{18}$$

and that the correlation coefficient is

$$\frac{1}{12}\int_{-\sqrt{3}}^{\sqrt{3}}\int_{-\sqrt{3}}^{\sqrt{3}}m_1m_2\{1+\rho_{12}m_1m_2\}dm_1dm_2=\rho_{12}.\tag{19}$$

The confidence level p is expressed as follows:

$$p=\frac{1}{12}\iint_{C^{(2)}}\{1+\rho_{12}m_1m_2\}dm_1dm_2.\tag{20}$$

Numerical results of these two models are reported in the following practical examples.

We considered the uniform two-variate distribution with square domain

$$\begin{aligned}C^{(2)}\equiv\{&-(h+B)\le m_1\le h-B,\\&-(h+B)\le m_2\le h-B\}.\end{aligned}\tag{21}$$

TABLE II
DIFFERENT VALUES OF THE CONFIDENCE LEVEL p

	B	***Circular domain* (radius *a*)**					
		p=0.70	***p=0.75***	***p=0.80***	***p=0.85***	***p=0.90***	***p=0.95***
ρ_{12} = -0.90	0	1.635	1.693	1.748	1.802	1.854	1.905
	0.02	1.635	1.693	1.748	1.802	1.854	1.905
	0.06	1.637	1.695	1.751	1.804	1.857	1.908
	0.1	1.643	1.700	1.756	1.810	1.863	1.914
ρ_{12} = -0.50	0	1.635	1.693	1.748	1.802	1.854	1.905
	0.02	1.635	1.693	1.748	1.802	1.854	1.905
	0.06	1.636	1.694	1.750	1.803	1.856	1.907
	0.1	1.640	1.697	1.752	1.807	1.858	1.909
ρ_{12} = 0	0	1.635	1.693	1.748	1.802	1.854	1.905
	0.02	1.635	1.693	1.748	1.802	1.854	1.905
	0.06	1.635	1.693	1.748	1.802	1.854	1.905
	0.1	1.635	1.693	1.748	1.802	1.854	1.905
ρ_{12} = 0.50	0	1.635	1.693	1.748	1.802	1.854	1.905
	0.02	1.635	1.693	1.748	1.802	1.854	1.905
	0.06	1.634	1.691	1.747	1.800	1.853	1.904
	0.1	1.632	1.688	1.743	1.798	1.849	1.900
ρ_{12} = 0.90	0	1.635	1.693	1.748	1.802	1.854	1.905
	0.02	1.635	1.693	1.748	1.802	1.854	1.905
	0.06	1.633	1.690	1.746	1.799	1.852	1.901
	0.1	1.628	1.685	1.740	1.794	1.846	1.897

In this case, the domain must be asymmetrical $(B \neq 0)$ so that the confidence level remains dependent of the correlation coefficient. In fact, for $B = 0$, $p = h^2/3$.

For the circular domain :

$$C^{(2)} \equiv \left\{(m_1 + B)^2 + (m_2 + B)^2 \leq a^2\right\}. \tag{22}$$

The equation becomes after an appropriate transformation in polar coordinates [see (23) at bottom of the page].

In conclusion, the confidence level is

$$\begin{aligned} p = &\int_0^{2\pi}\int_0^1 \frac{\mathrm{ra}^2}{12}\left\{1 + \rho_{12}\left[\frac{r^2a^2}{2}\sin 2\theta + B^2\right.\right.\\ &\left.\left.-\mathrm{Bra}(\cos\theta + \sin\theta)\right]\right\} dr\, d\theta\\ = &\frac{a^2\pi}{12}(1 + \rho_{12} B^2) \end{aligned} \tag{24}$$

In Tables I and II, we show for different values of the confidence level p, the related values of the parameter h, in the square domain, and of the radius a, in the circular domain, when ρ_{12} and B vary.

B. Three-Variate Distribution

We now consider the normal three-variate distribution. To this aim, three random variables $M_{\mathrm{i}} \approx \mathrm{N}(\mu_{\mathrm{i}}, \sigma_i^2)$ are taken into account, with normal distribution and $\sigma_i = \sqrt{\mathrm{Var}\{M_i\}}$ $\mathrm{i} = 1$, 2, 3.

Three random variables M_{i} $\mathrm{i} = 1, 2, 3$ with normal distribution $M_{\mathrm{i}} = \mathrm{N}(\mu_{\mathrm{i}}, \sigma_i^2)$, with $\sigma_i = \sqrt{\mathrm{Var}\{M_i\}}$ $\mathrm{i} = 1, 2, 3$, are considered. The expanded uncertainty, U_i of the measure M_{i}, as specified in GUM [1], is $\mu_i \pm u_i = \mu_i \pm k_i\sigma_i$ where k_i is the coverage factor. The correlation coefficients in the reduced form are

$$\rho_{ij} = E\left({M^*}_i {M^*}_j\right) = \frac{E\{M_iM_j\} - \mu_i\mu_j}{\sigma_i\sigma_j}, \quad i,j = 1,2,3 \tag{25}$$

with $|\rho_{ij}| \leq 1$. The variability of the vector $\underline{M} = (M_1, M_2, M_3)^T$ may be summarized into a probability region $C^{(3)} \subset R^3$, with an assigned coverage probability p that is

$$p = P\{\underline{M} \in C^{(3)}\} = \iiint_{C^3} f_{\underline{M}}(\underline{m})\, dm_1\, dm_2\, dm_3 \tag{26}$$

$$\begin{cases} m_1 + B = \mathrm{ra}\cos\theta\\ m_2 + B = \mathrm{ra}\sin\theta \end{cases} \quad \begin{cases} f_{M_1^*,M_1^*}(r,\theta) = \frac{\mathrm{ra}^2}{12}\left\{1 + \rho_{12}\left[\frac{r^2a^2}{2}\sin 2\theta + B^2 - \mathrm{Bra}(\cos\theta + \sin\theta)\right]\right\}\\ \overline{C}^{(2)} \equiv \left\{(r,\theta) \in \Re^2 : \quad 0 \leq r \leq 1, \quad 0 \leq \theta \leq 2\pi\right\} \end{cases} \tag{23}$$

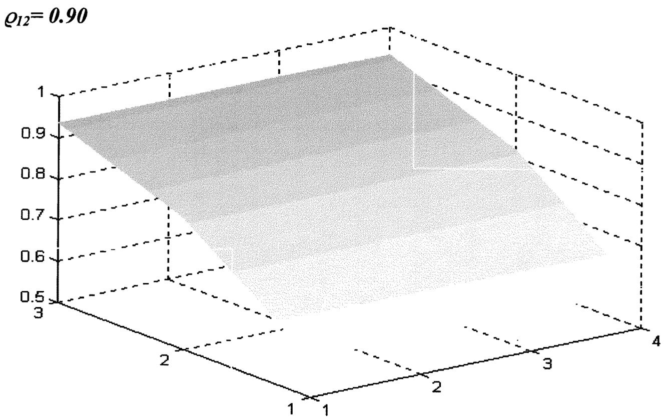

Fig. 1. Values in the X and Y axes represent, respectively, the row and the column numbers of the matrix for $\rho_{12} = 0.90$.

with

$$f_{\underline{M}}(\underline{m}) = (2\pi)^{-\frac{3}{2}}(\det \underline{\underline{D}})^{-\frac{1}{2}} \times \exp\left[-\frac{1}{2}(\underline{m}-\underline{\mu})\underline{\underline{D}}^{-1}(\underline{m}-\underline{\mu})^T\right] \quad (27)$$

where

$$\underline{\underline{D}} = \begin{bmatrix} \sigma_1^2 & \rho_{12}\sigma_1\sigma_2 & \rho_{13}\sigma_1\sigma_3 \\ \rho_{12}\sigma_1\sigma_2 & \sigma_2^2 & \rho_{23}\sigma_2\sigma_3 \\ \rho_{13}\sigma_1\sigma_3 & \rho_{23}\sigma_2\sigma_3 & \sigma_3^2 \end{bmatrix} \quad (28)$$

is the symmetric dispersion matrix.

Assuming $C^{(3)} = I_1 \times I_2 \times I_3$ where $I_i = [\mu_i \pm k_i\sigma_i]$ for i = 1, 2, 3, the confidence level p of the region $C^{(3)}$ is

$$p = (\det \underline{\underline{D}})^{-\frac{1}{2}} |\det \underline{\underline{Q}}| \prod_{i=1}^{3} \frac{1}{\sqrt{\lambda_i}} erf\left(\sqrt{\frac{\lambda_i}{2}} k_i^*\right) \quad (29)$$

where λ_i are the eigenvalues of the matrix $\underline{\underline{D}}$, and $\underline{\underline{Q}} = (q_{ij})$ for (i, j = 1, 2, 3) is the unitary matrix such that $\underline{\underline{Q}}^T \underline{\underline{D}}\,\underline{\underline{Q}} = \underline{\underline{\Lambda}}$, $\underline{\underline{\Lambda}}$ being the diagonal matrix with the nonnull elements equal to λ_i. In (29) $k_i^* = \sum_{j=1}^{3} q_{ij}k_j$, and erf(-) is the error function.

V. Experimental Results

The proposed approach has been proved in both the two-variate and three-variate distributions.

A. Two-Variate Distribution

We have evaluated the confidence levels related to different values of the correlation coefficient

$$\rho_{12} = 0, \pm 0.5, \pm 0.9.$$

For the normal, two-variate distribution, we first considered the square-domain

$$C^{(2)} \equiv \{(m_1, m_2) \in R^2 : -(h+B) \le m_1 \le h - B \quad -(h+B) \le m_2 \le h - B\} \quad (30)$$

with h values: 1, 1.5, 2 and B values: 0, 0.02, 0.06, 0.1.

For each couple of h and B values, represented for sake of simplicity as seen below, the corresponding confidence level p is evaluated according to (29). The results are summarized in the matrices for different values of the correlation coefficient ρ_{12}.

h/B	1	2	3	4
1	1, 0	1, 0.02	1, 0.06	1, 0.1
2	1.5, 0	1.5, 0.02	1.5, 0.06	1.5, 0.1
3	2, 0	2, 0.02	2, 0.06	2, 0.1

$\rho_{12} = 0$

.4661	.4659	.4649	.4628
.7507	.7505	.7494	.7473
.9111	.9110	.9103	.9090

$\rho_{12} = 0.50$

.4980	.4979	.4971	.4954
.7698	.7697	.7688	.7670
.9171	.9171	.9165	.9154

$\rho_{12} = 0.90$

.5964	.5963	.5955	.5939
.8207	.8206	.8199	.8184
.9357	.9357	.9352	.9343

A plot of data for this last case is shown in Fig. 1. The corresponding values related to the negative correlation coefficient are as follows.

$\rho_{12} = -0.50$

.4980	.4978	.4961	.4926
.7698	.7696	.7682	.7655
.9171	.9170	.9163	.9148

$\rho_{12} = -0.90$

.5964	.5959	.5924	.5855
.8207	.8204	.8183	.8140
.9357	.9356	.9346	.9326

A plot of the data for this last matrix is shown in Fig. 2.

In the case of circular domain, see (12), we considered an area equal to the square domain with the radius $a = (2)/(\sqrt{\pi})h$ and h values 1, 1.5, 2. In this case, we have

$$\begin{cases} m_1 = \frac{1}{\sqrt{2}}\overline{m}_1 + \frac{1}{\sqrt{2}}\overline{m}_2 \\ m_2 = -\frac{1}{\sqrt{2}}\overline{m}_1 + \frac{1}{\sqrt{2}}\overline{m}_2 \end{cases} \Rightarrow \begin{cases} f_{\mathrm{M}_1^*,\mathrm{M}_1^*}(\overline{m}_1, \overline{m}_2) = \frac{1}{2\pi\sqrt{1-\rho_{12}^2}} \times \exp\left\{-\frac{1}{2}\left[\frac{\overline{m}_1^2}{1-\rho_{12}} + \frac{\overline{m}_2^2}{1+\rho_{12}}\right]\right\} \\ \overline{C}^{(2)} \equiv \left\{(\overline{m}_1, \overline{m}_2) \in R^2 : \quad \overline{m}_1^2 + \overline{m}_2^2 \le a^2\right\} \end{cases} . \quad (31)$$

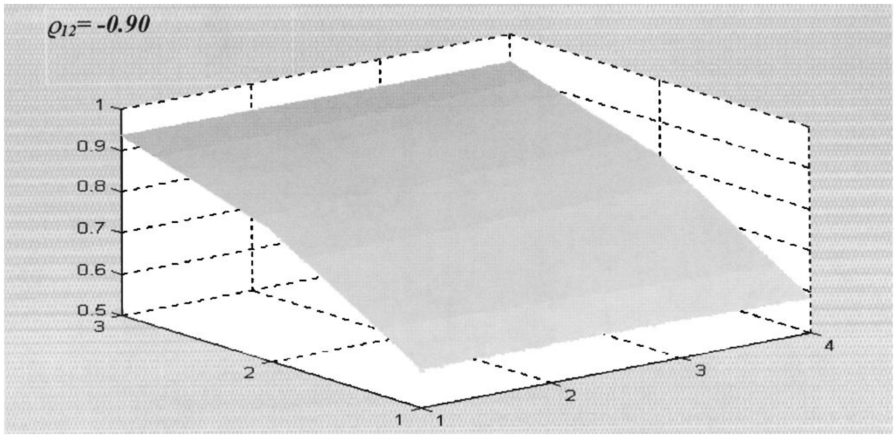

Fig. 2. Values in the X and Y axes represent, respectively, the row and the column numbers of the matrix for $\rho_{12} = -0.90$.

ρ_{12}/a	1	2	3
1	$0, \frac{2}{\sqrt{\pi}}$	$0, \frac{3}{\sqrt{\pi}}$	$0, \frac{4}{\sqrt{\pi}}$
2	$\pm 0.5, \frac{2}{\sqrt{\pi}}$	$\pm 0.5, \frac{3}{\sqrt{\pi}}$	$\pm 0.5, \frac{4}{\sqrt{\pi}}$
3	$\pm 0.9, \frac{2}{\sqrt{\pi}}$	$\pm 0.9, \frac{3}{\sqrt{\pi}}$	$\pm 0.9, \frac{4}{\sqrt{\pi}}$

$\longrightarrow$

.4792	.7613	.9216
.5014	.7711	.9138
.5663	.7719	.8949

Fig. 3. Results concerning the different confidence levels p.

Using polar coordinates r and a, (31) becomes

$$\begin{cases} \overline{m}_1 = ra\cos\theta \\ \overline{m}_2 = ra\sin\theta \end{cases} \Rightarrow$$

$$\begin{cases} f_{M_1^*,M_2^*}(r,\theta) = \frac{ra^2}{2\pi\sqrt{1-\rho_{12}^2}} \\ \qquad\qquad \times \exp\left\{-\frac{r^2a^2}{2\left(1-\rho_{12}^2\right)}\left[1+\rho_{12}\cos 2\theta\right]\right\}. \\ \overline{\overline{C}}^{(2)} \equiv \left\{(r,\theta)\in\Re^2 : \quad 0\le r\le 1, \quad 0\le\theta\le 2\pi\right\} \end{cases} \tag{32}$$

In conclusion, taking into account the Jacobian

$$\begin{aligned} p = &\int_0^{2\pi}\int_0^1 \frac{ra^2}{2\pi\sqrt{1-\rho_{12}^2}} \\ &\times \exp\left\{-\frac{r^2a^2}{2\left(1-\rho_{12}^2\right)}\left[1+\rho_{12}\cos 2\theta\right]\right\} \mathrm{d}r\mathrm{d}\theta \\ = &\frac{\left(1-\rho_{12}^2\right)^{\frac{1}{2}}}{2\pi}\int_0^{2\pi}\frac{1}{1+\rho_{12}\cos 2\theta} \\ &\times \left[1-\exp\left\{-a^2\frac{1+\rho_{12}\cos 2\theta}{2\left(1-\rho_{12}^2\right)}\right\}\right] d\theta \end{aligned} \tag{33}$$

In this circular domain case, we verify that the confidence level is an even function of the correlation coefficient ρ_{12}.

The results concerning the different confidence levels p when ρ_{12} and a assume the following values

$$\rho_{12} = 0, \pm 0.5, \pm 0.9; \quad a = \frac{2}{\sqrt{\pi}}, \frac{2.6}{\sqrt{\pi}}, \frac{3.2}{\sqrt{\pi}}$$

are summarized as seen in Fig. 3.

A plot of the data for the matrix in Fig. 3 is shown in Fig. 4.

B. Three-Variate Distribution

In order to prove the three-variate model, a circuit configuration in which the resistance R and its reactance X can be determined (following Ohm's laws) by simultaneous measurements is considered:

1) The amplitude V of a sinusoidal voltage across its terminals.
2) The amplitude I of the alternating current passing through it.
3) The phase-shift angle ϕ of the alternating potential difference relative to the alternating current.

The characteristic parameters of the experimental model are reported in Table III, and the correlation coefficients ρ_{ij} $(\mathrm{i,j} = 1,2,3)$ are subjected to the following constraints:

$$1 + 2\rho_{12}\rho_{13}\rho_{23} \ge \rho_{12}^2 + \rho_{13}^2 + \rho_{23}^2 \tag{34}$$

with the matrix $\underline{\underline{D}}$ being positive semi-definite.

The level of confidence is evaluated for different values of the coverage factors assuming that $k_i = k$, i $=$ 1, 2, 3. The computational results are reported in Table IV.

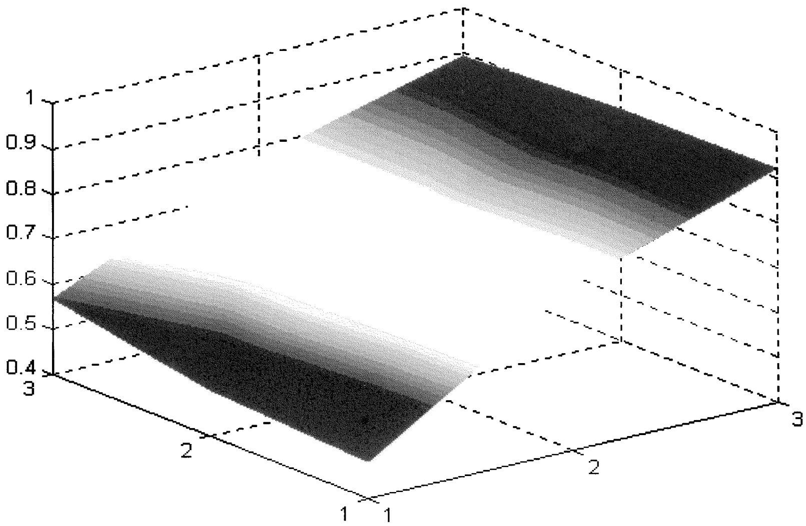

Fig. 4. Values in the X and Y axes represent, respectively, the row and the column numbers of the matrix.

TABLE III
CHARACTERISTIC PARAMETERS OF THE MODEL

μ_1=4.999 57 V	μ_2=19.661 01 mA	μ_3 =1.044 46 rad
σ_1=0.003 2 V	σ_2=0.009 5 mA	σ_3=0.000 75 rad
ρ_{12}=-0.36	ρ_{13}=0.86	ρ_{23}=-0.65

TABLE IV
COMPUTATIONAL RESULTS

k	p
2	0.9110
3	0.9892
4	0.9993

p = the level of confidence

k= the coverage factor

TABLE V
p_a = ANALYTICAL VALUE; p_c = MONTE CARLO VALUE

k	p_a	p_c
2	0.9110	0.9325
3	0.9892	0.9901
4	0.9993	0.9995

Finally, we want to underline how computer-intensive techniques such as Monte Carlo (MC) methods can be utilized to evaluate the level of confidence p of the probability region $C^{(n)}$ in an nth dimensional model.

Reported in the following is the basic idea associated with the sample-mean MC algorithm.

The multiple integral $p = \int_{\mathrm{C}^n} \cdot \cdot \int f_{\underline{M}}(\underline{m}) dm_1 \cdot \cdot dm_{\mathrm{n}}$ where the domain of integration $C^n \subset R^n$ is assumed to be bounded and to have a finite measure $V = \int_{C^n} \cdot \cdot \int dx_1 \cdot \cdot dx_{\mathrm{n}}$, may be represented as an expected value of some random variable. Indeed, rewriting the integral as $p = \int_{C^n} \cdot \cdot \int ((f_{\underline{\mathrm{M}}}(\underline{\mathrm{x}}))/(f_{\underline{\mathrm{X}}}(\underline{\mathrm{x}}))) f_{\underline{\mathrm{X}}}(\underline{\mathrm{x}}) dx_1 \cdot \cdot dx_{\mathrm{n}}$ and assuming that $f_{\underline{\mathrm{x}}}(\underline{x})$ is any probability density function such that $f_{\underline{\mathrm{X}}}(\underline{x}) > 0$ when $f_{\underline{\mathrm{X}}}(\underline{x}) \neq 0$, then $p = E[(f_{\underline{\mathrm{M}}}(\underline{X}))/(f_{\underline{\mathrm{X}}}(\underline{X}))]$. The random vector X, constituted by the n auxiliary random variables $(X_1, \ldots, X_n)$, is distributed according to the n-dimensional joint probability density $f_{\underline{\mathrm{X}}}(\underline{x})$, by defining $f_{\underline{\mathrm{X}}}(\underline{x})$ as

$$f_{\underline{\mathrm{X}}}(\underline{x}) = -\begin{cases} \frac{1}{V}, & \text{if } \underline{x} \in C^n \\ 0, & \text{elsewhere} \end{cases} \quad \text{then} \quad p = V\, E\left[f_{\underline{\mathrm{M}}}(\underline{X})\right]. \tag{35}$$

An unbiased estimator of p is its sample mean $\tilde{p} = (V)/(N) \sum_{j=1}^{N} f_{\underline{M}}(\underline{X}_j)$, where $(\underline{X}_1, \ldots, \underline{X}_N)$ is a random sample, that is, a set of independent random vectors each with the same n-dimensional distribution of $\underline{X}$.

Reported in TableV are the MC computational results expressing the level of confidence p of the experimental model, in comparison with the analytical ones (Table IV). The MC numerical approximation has been carried out with $N = 10^6$ random points.

VI. CONCLUSION

The present research has to be considered as an approach to understand the behavior of the uncertainty expression, in a measurement process, concerning multivariate models. To check the potentiality of the proposed method, different situations have been examined, and the level of confidence of the probability regions has been evaluated. The results have been reported and compared, in a three-dimensional experimental model, also with MC numerical approximation. The results show that the method could deserve further attention inside more complicated models.

REFERENCES

[1] *Guide to the Expression of Uncertainty in Measurement,* 1st ed. Geneva, Switzerland: IOS, 1993.

[2] B. N. Taylor and C. E. Kuyatt, *Guidelines for Evaluating and Expressing the Uncertainty of NIST Measurement Results*. Gaithersburg, MD: NIST, 1994.

[3] *International Standard ISO 3534-1 Statistics—Vocabulary and Symbols—Part I: Probability and General Statistical Terms*, 1st ed. Geneva, Switzerland: IOS, 1993.

[4] G. Iuculano, S. D'Emilio, and G. Pellegrini, "Uncertainty interpretation by defining a probability interval with a high level of confidence," in *IMEKO Tech. Comm. 8: Traceability in Metrology*, Torino, Italy, Sept. 1998.

[5] G. Iuculano, E. Arri, and G. Pellegrini, "Measurement compatibility: a novel approach based on the definition of probability intervals," in *Proc. IEEE Instrumentation and Measurement Technology Conf.*, Venice, Italy, May 1999.

[6] G. Iuculano, S. D'Emilio, and G. Pellegrini, "Using probability interval for measurement uncertainty evaluation," in *IMEKO-XV*, Osaka, Japan, June 1999.

[7] A. Zanobini, G. Pellegrini, and G. Iuculano, "The estimation of a confidence belt for the uncertainty expression in a measurement process by computationally intensive methods," in *EuroConf. Advanced Mathematical and Computational Tools in Metrology*, Caparica, Portugal, May 10–13, 2000.

[8] M. Kendall, A. Stuart, and J. K. Ord, *Kendall's Advanced Theory of Statistics*, 5th ed. London, U.K.: Charles Griffin, 1987.

Gaetano Iuculano received the degree in electronic engineering from University of Bologna, Bologna, Italy.

He is currently a Professor of electrical measurements and metrology with the Department of Electronics and Telecommunications, University of Florence, Florence, Italy. He has experience in calibration applications and planning experiments, reliability analysis and life testing for electronic devices and systems, and considerable expertise in practical statistical analysis for electrical engineering. He has authored and coauthored more than two hundred technical papers in his current research interests.

Andrea Zanobini was born in Florence, Italy, in 1962. He received the M.S degree in electronic engineering from the University of Florence in 1989.

He joined S.M.I. Europa Metalli in 1990, where he worked in the Quality Control and Assurance Group. Since 1991, he has been with the Department of Electronics and Telecommunications of the University of Florence and his research interests include robot sensors, software in measurements, statistics, and reliability analysis for electronic devices and systems.

Annarita Lazzari received the degree in electronic engineering from the Engineering Faculty of the University of Bologna, Bologna, Italy, in 1999. She is currently pursuing the Doctorate of research in electrotechnical engineering, for which she has developed research activity in the field of the bases of measurement, statistic process control, reliability and quality control, statistic methods for industrial applications, analysis of multivariate models and evaluation of the uncertainties, and application of computer-intensive techniques.

Currently, she is the individual responsible for the Mitutoyo Italiana and Educational Technical Support of the Mitutoyo Institute of Metrology (MIM), Milano, Italy.

Gabriella Pellegrini Gualtieri received the degree from the University of Rome, Rome, Italy, in 1965.

She is currently an Associate Professor of mathematical analysis at the University of Florence, Florence, Italy. Her research interests include numerical methods of nonlinear dynamic systems and uncertainty estimation in metrology by statistical techniques and probability theory.