Modeling and Processing Measurement Uncertainty Within the Theory of Evidence: Mathematics of Random–Fuzzy Variables

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***Abstract*—Random–fuzzy variables (RFVs) are mathematical variables defined within the theory of evidence. Their importance in measurement activities is due to the fact that they can be employed for the representation of measurement results, together with the associated uncertainty, whether its nature is random effects, systematic effects, or unknown effects. Of course, their importance and usability also depend on the fact that they can be employed for processing measurement results. This paper proposes suitable mathematics and related calculus for processing RFVs, which consider the different nature and the different behavior of the uncertainty effects. The proposed approach yields to process measurement algorithms directly in terms of RFVs so that the final measurement result (and all associated available information) is provided as an RFV.**

***Index Terms*—Fuzzy averaging operators, measurement uncertainty, random contributions, random–fuzzy variables (RFVs), systematic contributions, theory of evidence, uncertainty processing, unknown contributions.**

I. Introduction

IT HAS always been recognized in measurement science that the result of a measurement provides incomplete knowledge about the measurand. One of the challenges posed by measurement science is the estimation of how incomplete (or, equivalently, how complete) this knowledge is.

Different solutions have been proposed over the years, which start from the classical theory of errors and the related mathematics of the intervals. This approach has been strongly criticized because it refers to the "true value" concept: both philosophical and scientific considerations proved that the true value is not only unknown but is also unknowable, even if its existence can be assumed.

A more recent approach is the one encompassed by the Guide to the Expression of Uncertainty in Measurement (GUM) [1], [2], based on the uncertainty concept, which does not require knowledge of the true value and is totally framed within the probability theory. In this framework, provided that all systematic contributions to uncertainty are recognized and compensated for, as required by [1], the measurement results can be mathematically represented by random variables. Unfortunately, the compensation of all systematic effects can almost never be attained in practice, especially in industrial applications. Two main reasons can be given: the generally high cost of compensations (when possible) and the fact that systematic effects are generally known in terms of a closed interval within which the value of the systematic effect itself lays, but this value is often, and generally, unknown.

In order to overcome the limitations of the probability theory, which handles, with its random variables, only that particular class of incomplete information due to random effects, some contributions have been published in recent years, which propose the use of possibility theory and fuzzy variables (FVs) [3]–[5]. However, the use of FVs as an alternative to random variables for the mathematical representation of measurement results shows opposite problems: they can suitably handle incomplete information due to unknown effects and systematic effects, but they cannot handle incomplete information due to random effects. In this respect, some attempts, which are based on the use of t-norm operators, have been proposed in the literature [5], [6] to make the FVs provide, in the presence of random effects, similar results to those provided by random variables. However, the mathematical foundations of this approach seem to be not totally appropriate, and its effectiveness has not been fully proved in practical applications.

More recently, random–fuzzy variables (RFVs) have been proposed to mathematically represent and propagate measurement results in the most general situation, i.e., when both random and nonrandom effects are present [7]–[12]. To prove the effectiveness of the RFV representation of measurement results, this paper, after briefly recalling the fundamentals of RFVs, defines the mathematics and related calculus for RFVs that are capable of correctly taking into account the different kinds of effects that contribute to measurement uncertainty.

II. RFVs and the Expression of Uncertainty

RFVs are defined within the mathematical theory of evidence [13], which encompasses, as particular cases, both the probability theory (within which random variables are defined) and the possibility theory (within which FVs are defined). In this respect, RFVs can be considered as a generalization of both random variables and FVs and are, hence, expected to be more effective in representing measurement results, whatever the associated uncertainty contribution is: random, systematic, or unknown. This statement can be proved if the definition of RFV as a set of suitable confidence intervals is briefly recalled [12].

Manuscript received June 15, 2006; revised October 19, 2006.

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Digital Object Identifier 10.1109/TIM.2007.894907

It is widely known that a confidence interval is a closed interval in the set of real numbers $\Re$ within which the possible values of an uncertain result can be located, i.e.,

$$A = [a_1, a_2] \tag{1}$$

where $a_1 \leq a_2$.

This definition is very simple, from a mathematical point of view, and defines the so-called intervals of type 1. However, when measurement results are concerned, confidence intervals can take different meanings. Let us consider a confidence interval no matter which is the way it has been obtained (type A or type B evaluation of uncertainty [1]). Given this confidence interval, it is known (with the corresponding level of confidence) that the result will surely fall in one point of this interval, even if this point is unknown. Moreover, different interpretations can be given according to the available information.

- Nothing else is known. In this case, no assumptions can be done about the values obtained from different measurements (under the same conditions). The effect of this kind of contribution on the possible distribution of measurement results within the given interval is unknown. This interval must be considered as a whole, and the mathematics of the intervals must be applied when dealing with it [12].
- It is known that the given interval is associated with a systematic contribution. This situation can be found, for instance, when the confidence interval is retrieved from a data sheet of a given instrument. It is known that when different measurements are performed (under the same conditions), the same result is always provided. Of course, the systematic contribution is not precisely known, and the entire interval must be considered and processed, according to the mathematics of the intervals, unless compensation algorithms are applied (a typical example is given by the double weight on a weight bridge) [12].
- It is known that the given interval is associated with a random contribution. This situation can be found, for instance, when the confidence interval is obtained from the possible values provided by repeated measurements. In this case, it is known that when different measurements are performed (under the same conditions), the value of the result varies randomly within the interval. Also in this case, the entire interval must be considered and processed but, this time, according to a different mathematics [12]. If the random contribution is the only one affecting the measurement result, the probability theory can be employed.

It can be readily understood that when different kinds of contributions to uncertainty are simultaneously present (as in the most general case), the confidence interval mathematically defined in (1) is no longer capable of fully describing the possible distribution of the measurement results. This full description can be achieved if a more general definition than (1) is given for confidence intervals. This more general definition is that of the so-called intervals of type 2 [14] and can be obtained under the following assumptions.

Let us suppose that the lower and upper bounds of the closed interval (1) are themselves uncertain. In other words, instead of being ordinary numbers, they are closed intervals in $\Re$, i.e.,

$$B = [[b_1, b_2], [b_3, b_4]] \tag{2}$$

where $b_1 \leq b_2 \leq b_3 \leq b_4$.

This means that the left and right bounds of the closed interval (2) can vary in $[b_1, b_2]$ and $[b_3, b_4]$, respectively. The way the bounds vary within these intervals can be determined starting from the available information. When measurement results are concerned, it is convenient to associate this variation with the effect of random contributions [12].

Therefore, under this assumption, it can be stated that the internal interval $[b_2, b_3]$ (which is still a confidence interval of type 1) represents the effect of all systematic and unknown contributions to uncertainty, while the external intervals $[b_1, b_2]$ and $[b_3, b_4]$ represent the effect of all random contributions. Hence, the mathematical confidence intervals of type 2 can be suitably employed to describe the possible distribution of measurement results due to the effects of both random and nonrandom contributions.

Confidence intervals of types 1 and 2 are used to define FVs of types 1 and 2, respectively [12].

An FV (of type 1) can be defined as a set of confidence intervals of type 1 $A_\alpha = [a_1^\alpha, a_2^\alpha]$, $\alpha \in [0, 1]$ that obey the following constraints [12]:

- $a_1^\alpha \leq a_2^\alpha \ \forall \alpha$;
- $\forall \alpha, \alpha'$ in the range [0,1]: $\alpha' > \alpha \Rightarrow A_{\alpha'} \subset A_\alpha$.

Similarly, an FV of type 2 can be defined by suitable sets of confidence intervals of type 2 [14]. FVs of type 2 are a large class of mathematical objects. As far as the representation of measurement results is concerned, a subclass of FVs of type 2 can be effectively employed, as proved in [12]: the RFVs.

RFVs can be defined as a set of confidence intervals $B_\alpha = [[b_1^\alpha, b_2^\alpha], [b_3^\alpha, b_4^\alpha]]$, $\alpha \in [0, 1]$ that obey the following constraints [12]:

- $b_1^\alpha \leq b_2^\alpha \leq b_3^\alpha \leq b_4^\alpha \ \forall \alpha$;
- the sequences of intervals of confidence of type 1 $[b_1^\alpha, b_4^\alpha]$ and $[b_2^\alpha, b_3^\alpha]$ generate two membership functions (MFs) that are normal and convex;
- $\forall \alpha, \alpha'$ in the range [0,1]:

$$\alpha' > \alpha \Rightarrow \begin{cases} \left[b_1^{\alpha'}, b_3^{\alpha'}\right] \subset [b_1^\alpha, b_3^\alpha] \\ \left[b_2^{\alpha'}, b_4^{\alpha'}\right] \subset [b_2^\alpha, b_4^\alpha] \end{cases}$$

- $[b_2^{\alpha=1}, b_3^{\alpha=1}] \equiv [b_1^{\alpha=1}, b_4^{\alpha=1}]$.

An example of RFV is given in Fig. 1.

Each confidence interval at level α, which defines the FV or RFV, is called α-cut [12], and the level of confidence that the result of a measurement lies within this specific interval is $1 - \alpha$ [12].

As also shown in Fig. 1, the bounds of the wider and inner intervals $[b_1^\alpha, b_4^\alpha]$ and $[b_2^\alpha, b_3^\alpha]$ define two functions, which are generally called MFs [12], [15], that represent, in the theory of evidence, two distributions of possibility [7], [12].

Within this same theory, the distribution of possibility is a mathematical tool for representing incomplete information.

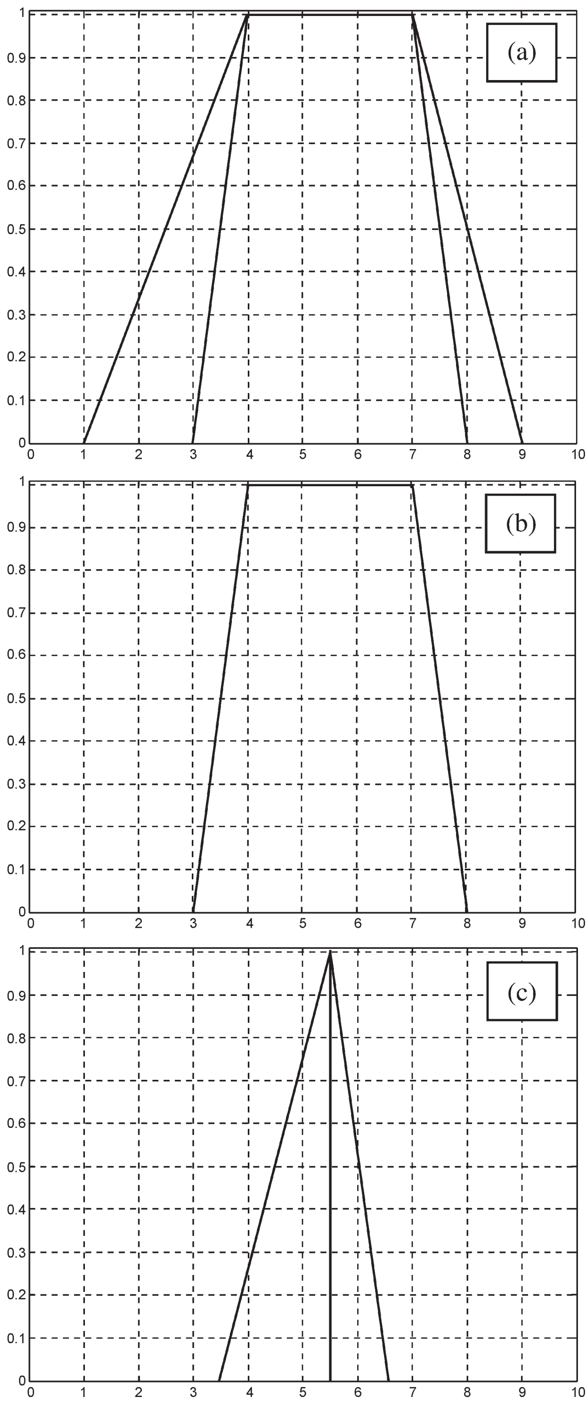

Fig. 1. Example of RFVs. (a) Generic RFV: both random and nonrandom contributions to uncertainty are present. (b) "Pure fuzzy" RFV: only nonrandom contributions are present. (c) "Pure random" RFV: only random contributions are present.

In this respect, it has similar meaning as the distribution of probability, although this last function represents a different kind of incomplete information [12]. In particular, a distribution of probability is best suited for representing incomplete information due to random effects, while a distribution of possibility is best suited for representing incomplete information due to nonrandom effects [12].

They are both suited for defining intervals of confidence at a given level of confidence p, although a distribution of possibility yields these intervals in a more straightforward way than a distribution of probability, since it is quite simply provided by the α cut at level $\alpha = 1 - p$ [12].

Let us now consider that the GUM states that "in many industrial and commercial applications, as well as in the areas of health and safety, it is often necessary to provide an interval about the measurement result within which the values that can reasonably be attributed to the quantity subject to measurement may be expected to lie with a high level of confidence. Thus, the ideal method for evaluating and expressing measurement uncertainty should be capable of readily providing such a confidence interval, in particular, one that corresponds in a realistic way with the required level of confidence." In this respect, a distribution of possibility appears to be the most useful tool for representing such confidence intervals.

This statement is reinforced by the availability of probability–possibility transformations that change a given probability distribution into a possibility distribution capable of providing the same information in terms of confidence intervals [12]. This means that for each given level of confidence, the same intervals of confidence are provided by the two distributions.

Therefore, according to both the available evidence and the result obtained by suitably applying the aforementioned probability–possibility transformations [12], it is possible to build an RFV in such a way that its inner MF represents the effects of the unknown or uncompensated systematic contributions, of total ignorance and partial ignorance, and its external MF also includes the effects of random contributions [12].

Of course, if only nonrandom contributions are affecting the measurement result, the external MF will coincide with the inner one, and the resulting RFV is called "pure fuzzy" RFV [Fig. 1(b)] [12]. On the other hand, if only random contributions are affecting the measurement result, all intervals associated with the inner MF will degenerate into zero-width intervals, as shown in Fig. 1(c) [12]. In this case, the RFV is called "pure random" RFV.

III. RFVs AND THE PROPAGATION OF UNCERTAINTY

As required by [1], any method used to evaluate measurement uncertainty must also provide a way to combine it in order to also evaluate the uncertainty associated with combined measurement results. Therefore, the RFV approach is truly consistent with the GUM if RFVs can be employed not only to represent uncertainty but also to propagate it. In other words, a suitable mathematics must be defined so that if (direct) measurement results are represented as RFVs, these same results can be suitably combined in order to provide the result of any indirect measurement, together with the associated uncertainty, directly in terms of an RFV.

As shown in [7] and [12] and according to the considerations reported in the previous section, the two MFs of an RFV are built, starting from the available information, in two different ways. It follows almost immediately that in order to correctly take into account the behavior of the different uncertainty contributions when they combine with each other, the two MFs should also be processed in two different mathematical ways.

In the previous papers [7]–[11], a hybrid way has been proposed to compose RFVs, i.e., partly framed within the possibility theory and partly within the probability theory. However, that approach showed to be effective only in particular situations, i.e., when the random contributions are represented by Gaussian probability density functions (pdfs). Examples of the effectiveness of this particular solution are reported in [16] and [17].

This paper is aimed to show a different approach, whose peculiarities, with respect to the previous one, are as follows.

- It is more general since it does not need to refer to Gaussian pdfs.
- It is fully consistent with the definition of RFVs. In fact, since both MFs of an RFV are possibility distribution functions, they both should be processed according to some particular fuzzy operators [12].

Several fuzzy operators are defined in the literature. The simplest one makes use of the widely known mathematics of the intervals. According to this fuzzy operator, when two MFs must be combined together, the mathematics of the interval is applied to each α-cut at different levels α.

However, it has already been proven [7]–[12] that the mathematics of the intervals models only the combination of systematic effects and ignorance (total or partial). Therefore, this is the suitable mathematics for the combination of the internal MFs of RFVs. The problem of the combination of the external MFs, which should also consider random effects, must be solved.

The following sections are focused on a possible effective solution to this problem.

IV. Preliminary Considerations

In this section, different steps in the combination of RFVs are considered in order to provide the required background to the definitions given in Section V.

Let us consider two generic RFVs A and B, as in the example in Fig. 2,[1] and let $A_\alpha = [[a_1^\alpha, a_2^\alpha], [a_3^\alpha, a_4^\alpha]]$ and $B_\alpha = [[b_1^\alpha, b_2^\alpha], [b_3^\alpha, b_4^\alpha]]$ be their α cuts.

Let us start by considering their sum.

In our first step, we suppose to combine both the internal and the external MFs of the given RFVs with the mathematics of the intervals. The RFV shown in Fig. 3 is obtained. It can be proved that the result is an RFV again, and therefore, it satisfies to the constraints given in Section II.

The result provided by the first step is generally not the final one, but useful considerations can be drawn.

Let $C_\alpha = [[c_1^\alpha, c_2^\alpha], [c_3^\alpha, c_4^\alpha]]$ and $D_\alpha = [[d_1^\alpha, d_2^\alpha], [d_3^\alpha, d_4^\alpha]]$ be, respectively, the α cuts of the final result (still unknown) and the α cuts of the result obtained in the first step (Fig. 3). First of all, it can be stated that the final internal intervals $[c_2^\alpha, c_3^\alpha]$ at different levels α are exactly the internal intervals $[d_2^\alpha, d_3^\alpha]$ already obtained (Fig. 3). In fact, as already stated, the internal MFs, taking into account only ignorance and uncompensated systematic effects, can be suitably combined by applying the

[1]The simple MFs in Fig. 2 are considered here for the sake of clarity. Of course, the following mathematical derivations are valid for any possible shape of the MFs, provided that they satisfy the constraints given in Section II.

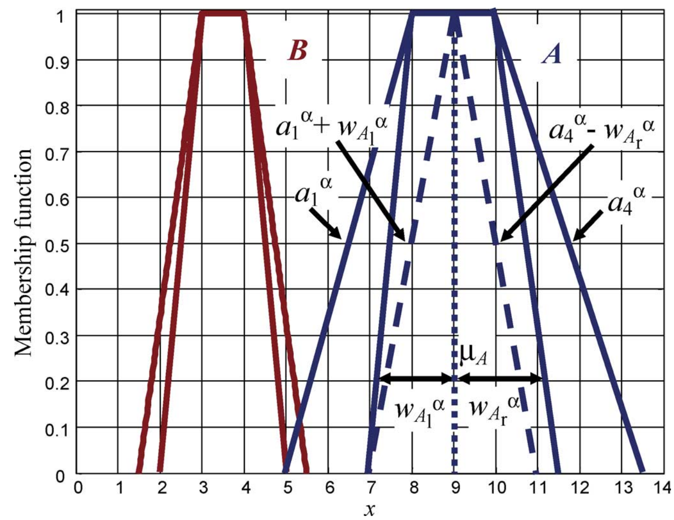

Fig. 2. Example of two RFVs. The way to determine the pure random part is also shown.

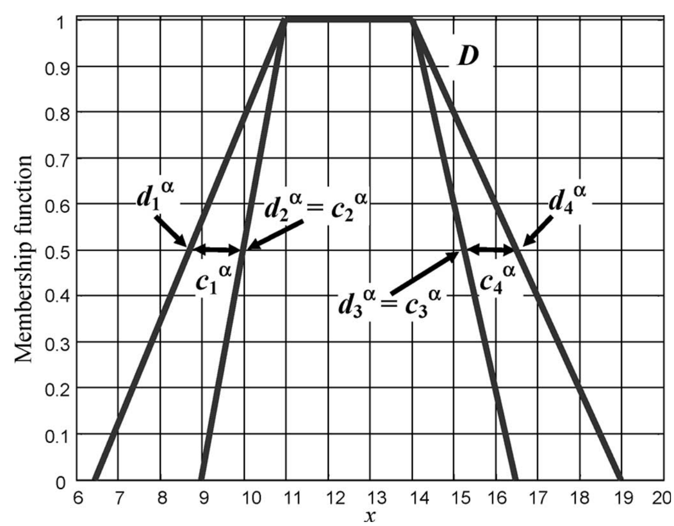

Fig. 3. Sum of the two RFVs in Fig. 2 according to the mathematics of the intervals for both MFs.

mathematics of the intervals. On the other hand, bounds of the final external intervals $[c_1^\alpha, c_4^\alpha]$ satisfy the following constraints (Fig. 3):

$$d_1^\alpha \le c_1^\alpha \le d_2^\alpha \tag{3}$$
$$d_3^\alpha \le c_4^\alpha \le d_4^\alpha. \tag{4}$$

The two inequalities can be understood by considering that α cuts $[d_1^\alpha, d_4^\alpha]$ and $[d_2^\alpha, d_3^\alpha]$ represent, respectively, the maximum and minimum intervals within which the possible final results could lie [12]. In fact, the final external intervals $[c_1^\alpha, c_4^\alpha]$ will coincide with the intervals $[d_1^\alpha, d_4^\alpha]$ when the random contributions combine with each other in such a way that they do not compensate. On the other hand, the final external intervals $[c_1^\alpha, c_4^\alpha]$ will coincide with the intervals $[d_2^\alpha, d_3^\alpha]$ when the random contributions compensate each other totally.

Hence, this first step yields the first requirement for the determination of the final RFV C: it is necessary to find a fuzzy operator capable of providing, given two MFs, a third

MF that lies in between. Therefore, the second step consists of the definition of a suitable fuzzy operator.

The most widely known fuzzy operators are t-norms and t-conorms, which, however, are not useful for this purpose. In fact, if two MFs, where one is included into the other one[2] (Fig. 3), are given, every kind of t-norm provides an MF that is not greater than the smaller one, and every kind of t-conorm provides an MF that is not smaller than the greater one [12], [14], [15]. In other words, no t-norms or t-conorms that provide an MF included between the two initial ones exist. On the other hand, different fuzzy operators available in the literature [14], [15], which are called averaging operators, are capable of providing MFs that lie in between the two initial ones [12], [14], [15].

Many kinds of averaging operators are available, but the more appropriate one is the ordered weighed averaging (OWA) operator [12], [15]. Considering two MFs, where one is included into the other one (Fig. 3), whose α cuts are, respectively, $[d_1^\alpha, d_4^\alpha]$ and $[d_2^\alpha, d_3^\alpha]$, an OWA operator provides a new MF, which is included between the two given ones, whose α-cuts are [12]

$$[w_1 \cdot ex\,(d_1^\alpha, d_2^\alpha) + w_2 \cdot in\,(d_1^\alpha, d_2^\alpha)\ , \\ w_1 \cdot ex\,(d_3^\alpha, d_4^\alpha) + w_2 \cdot in\,(d_3^\alpha, d_4^\alpha)] \quad (5)$$

where w_1 and w_2 are numerical weights that satisfy to

$$w_i \in [0, 1] \quad \text{and} \quad w_1 + w_2 = 1$$

and in and ex are functions that provide the value belonging, respectively, to the internal and external MF [12]. Equation (5) shows that the same OWA operator is applied twice for each level α, i.e., for the determination of the left and right bounds of the final α-cut, respectively. Hence, the second step is concluded: the OWA operator has been chosen among all possible fuzzy operators. The following step is, of course, the definition of the most suitable OWA operator. Let us note that its definition can be obtained after defining

- the values of the two weights;
- the proper MFs that have to be averaged.

The answer to the above two points is not immediate. Let us be reminded that the desired MF is the external MF of an RFV. Therefore, it must take into account the effects of the combination of random contributions. The requested OWA operator can hence be obtained only after the behavior of random effects, when they are combined, has been deeply investigated. This requires the study of the combination of random variables under the hypothesis of different correlation coefficients.

This investigation involves the determination of both the support and the shape of the distribution of the result, as shown in Section IV-A and B.

A. Support of the result

Let us consider two random variables that must be combined. From a mathematical point of view, the two random variables can be represented by two pdfs. In this section, we are not interested in the determination of the pdf of the result of the combination but only in its support. Let us consider, in this respect, that the support of the result is indeed the most important information regarding the result itself. In fact, the support of the final random variable corresponds exactly to the interval of confidence at the level of confidence 1.

[2]As in the case when RFVs are considered.

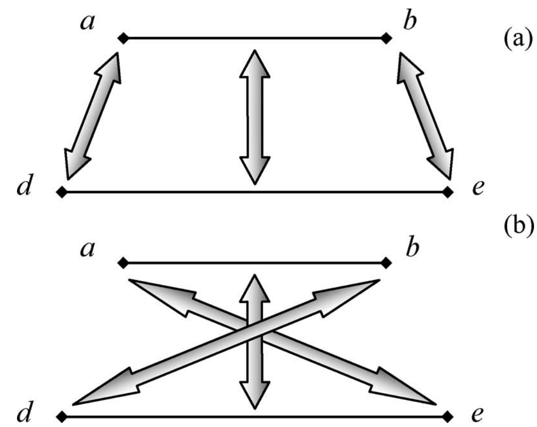

Fig. 4. Effect of (a) a total positive correlation and (b) a total negative correlation.

Given two random variables, the support of the result of their combination depends on [12] the following:

- the supports of the initial pdfs;
- the considered mathematical operation;
- the correlation [1] between the two random variables.

It can also be proved that the support does not depend on the shape of the pdfs.

Let the closed intervals $[a, b]$ and $[d, e]$ be the supports of the initial pdfs, and let us consider the four arithmetic operations. As far as the correlation between the initial random variables is concerned, let us first consider the three correlation coefficients $\rho = 0, +1, -1$, which represent the three most frequently met situations. It can be stated that when a zero correlation coefficient is taken into account, all values in the two considered intervals can combine randomly. On the other hand, the total correlation (both positive and negative) acts in the following way: when a random extraction from the first interval is taken, then the extraction in the second interval is fully determined. This means that the values in the two intervals cannot randomly combine, and only some couples of values can be extracted, as schematically shown in Fig. 4.

In the following, the support of the result is given for the four arithmetic operations and for these three correlation coefficients [12].

Sum

- $\rho = 0$:

$$[a + d, b + e] \quad (6)$$

- $\rho = +1$:

$$[a + d, b + e] \quad (7)$$

- $\rho = -1$:

$$[\min\{a + e; b + d\}, \max\{a + e; b + d\}] \,. \quad (8)$$

Difference

- $\rho = 0$:

$$[a-e, b-d] \tag{9}$$

- $\rho = +1$:

$$[\min\{a-d; b-e\}, \max\{a-d; b-e\}] \tag{10}$$

- $\rho = -1$:

$$[a-e, b-d]. \tag{11}$$

Product

Let us now denote with $[f, g]$ the support of the result:

- $\rho = 0$:

$$[\min\{ad, ae, bd, be\}, \max\{ad, ae, bd, be\}]\,. \tag{12}$$

- $\rho = +1$:

$$f = \begin{cases} x_m y_m, & \text{if } a < x_m < b \\ \min\{ad, be\}, & \text{otherwise} \end{cases} \tag{13}$$

$$g = \max\{ad, be\} \tag{14}$$

where

$$x_m = \frac{\mu_1 - \mu_2 \cdot r}{2}, \quad y_m = \mu_2 + \frac{x_m - \mu_1}{r}$$
$$\mu_1 = \frac{a+b}{2}, \quad \mu_2 = \frac{d+e}{2}, \quad r = \frac{b-a}{e-d}.$$

- $\rho = -1$:

$$f = \min\{ae, bd\} \tag{15}$$

$$g = \begin{cases} x_M y_M, & \text{if } a < x_M < b \\ \max\{ae, bd\}, & \text{otherwise} \end{cases} \tag{16}$$

where

$$x_M = \frac{\mu_1 + \mu_2 \cdot r}{2}$$
$$y_M = \mu_2 - \frac{x_M - \mu_1}{r}.$$

Division

Provided that $0 \notin [d, e]$

- $\rho = 0$:

$$[\min\{a/d, a/e, b/d, b/e\}, \max\{a/d, a/e, b/d, b/e\}] \tag{17}$$

- $\rho = +1$:

$$[\min\{a/d; b/e\}, \max\{a/d; b/e\}] \tag{18}$$

- $\rho = -1$:

$$[\min\{a/e; b/d\}, \max\{a/e; b/d\}] \tag{19}$$

The above equations still define the mathematics of the intervals since only the given intervals are involved, although it is different from the classical one, which can be applied only to the case $\rho = 0$.

When correlation coefficients $0 < \rho < 1$ are considered, it can be stated that the final support is between the one evaluated for $\rho = 0$ and the one evaluated for $\rho = 1$. Similarly, when correlation coefficients $-1 < \rho < 0$ are considered, it can be stated that the final support is between the one evaluated for $\rho = 0$ and the one evaluated for $\rho = -1$.[3]

As stated above, the defined supports of the final pdf yield the interval of confidence at level of confidence 1. By definition, this same interval of confidence is the α-cut at level $\alpha = 0$ of the corresponding "pure random" RFV, i.e., the RFV containing the same information as the pdf and obtained by applying the probability–possibility transformation defined in [7] and [12].

Hence, the support obtained by combining, with the above equations, the supports of the two initial pdfs is the same as that obtained by combining the supports of the two corresponding "pure random" RFVs. This means that, when two "pure random" RFVs have to be combined, the above definitions must be applied to their α-cuts at level $\alpha = 0$.

Since every α-cut of an RFV is a confidence interval, the above equations could also be applied to every other level α. However, this always leads to acceptable results, i.e., to MFs, where the α cuts for higher levels of α are included in those for lower levels, only as far as (6), (7), (9), and (11) are concerned.

On the other hand, (8), (10), and (12)–(19) lead to acceptable results only in the particular situation where the external MFs of A and B have the same shape. Therefore, in more general cases, it must be checked that the obtained intervals are nested, and if necessary, suitable adjustments must be applied [12].

As far as sum and difference are concerned, a very simple solution can be applied.

- Sum. When $\rho = -1$, (8) is applied only to determine the support of the result (α-cut at level $\alpha = 0$), and the same shape of the MF obtained for $\rho = +1$ (obtained by applying (7) for every level α) is taken. This follows by considering that the shape of the result of the sum of two pdfs is the same for $\rho = 1$ and $\rho = -1$ [12].
- Difference. When $\rho = +1$, (10) is applied only to determine the support of the result (α-cut at level $\alpha = 0$), and the same shape of the MF obtained for $\rho = -1$ (obtained by applying (11) for every level α) is taken. This follows by considering that the shape of the result of the difference of two pdfs is the same for $\rho = 1$ and $\rho = -1$ [12].

On the other hand, as far as product and difference are concerned, the following can be applied: α-cuts at levels $\alpha > 0$ are forced not to jut out the α-cut at level $\alpha = 0$. It can be noted that the applied adjustment is consistent with the fact that all confidence intervals at levels of confidence lower than 1 must be included into the confidence interval at level of confidence 1 (support).

[3]In general, a strict close-form solution is seldom available for ρ values different from $\rho = 0, \pm 1$. However, these values are seldom met in practice and are also difficult to be accurately estimated. Therefore, approximated solutions are generally acceptable.

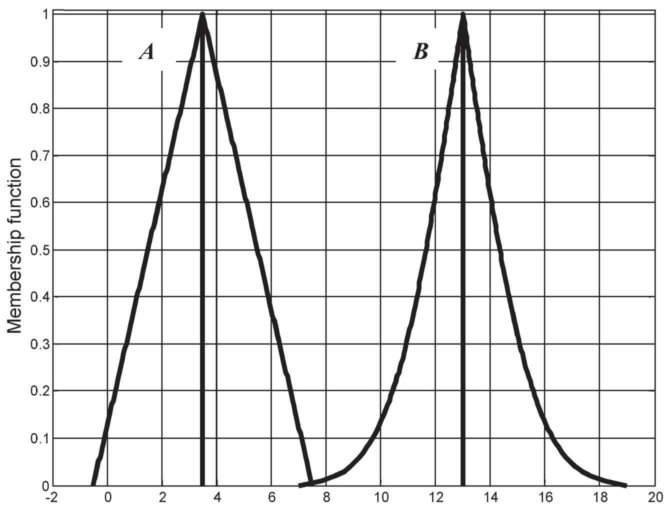

Fig. 5. Example of two pure random RFVs with different shapes.

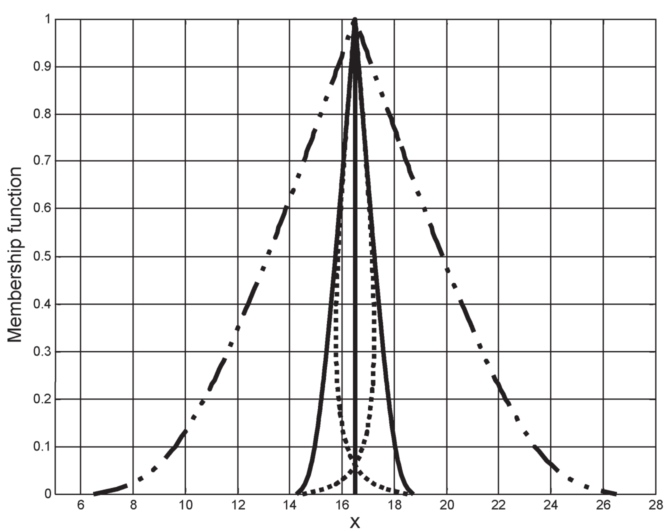

Fig. 6. Sum for $\rho = -1$ of the pure random RFVs in Fig. 5. The dotted line is the result obtained by applying (8) to all α cuts. The dashed-dotted line is the result for $\rho = 1$. The solid line is the final result.

Examples are shown in Figs. 5–7. Fig. 5 shows two pure random RFVs with different shapes, Fig. 6 shows their sum for $\rho = -1$, and Fig. 7 shows their product for the same value of the correlation coefficient. Besides the final result (shown in solid line), these figures also show the improper results obtained by the strict application of (8), (15), and (16), respectively.

When values of the correlation factors different from $\rho = 0, \pm 1$ are given, the external MF of the final RFV is expected to satisfy to the following conditions.

- If $0 < \rho < 1$, then the external MF of the final RFV lies in between the external MF obtained for $\rho = 0$ and the one obtained for $\rho = 1$.
- If $-1 < \rho < 0$, then the external MF of the final RFV lies in between the external MF obtained for $\rho = 0$ and the one obtained for $\rho = -1$.

This means that a suitable OWA operator must be applied to obtain the required final result.[3] As stated above, once the two MFs to be averaged are known, the OWA operator is fully defined by its two weights. If (5) is considered, it follows,

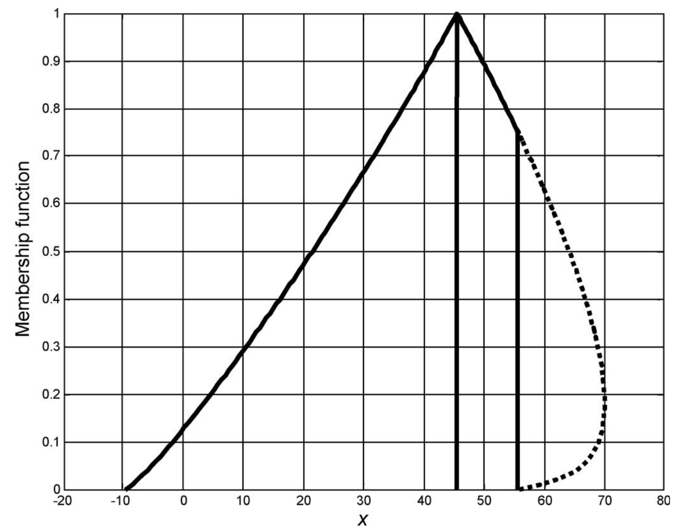

Fig. 7. Product for $\rho = -1$ of the pure random RFVs in Fig. 5. The dotted line is the result obtained by simply applying (15) and (16) to all α cuts. The solid line is the final result.

for $0 < \rho < 1$, that the two MFs to be averaged are the ones obtained for $\rho = 0$ and $\rho = 1$, and the two weights

$$w_1 = 1 - \rho \quad \text{and} \quad w_2 = \rho$$

can be defined in order to obtain an effective, although approximated, solution.

Similarly, when $-1 < \rho < 0$, the two MFs to be averaged are the ones obtained for $\rho = 0$ and $\rho = -1$, and the two weights are

$$w_1 = 1 + \rho \quad \text{and} \quad w_2 = -\rho.$$

B. Shape of the result

According to the considerations in Section I, given two generic RFVs, the internal MFs are combined according to the classical mathematics of the intervals, and the external MFs are combined according to the new mathematics of the intervals defined in Section I. However, if this procedure is followed, the following is obtained.

- When the sum is taken into account, the same MFs are provided for both $\rho = 0$ and $\rho = +1$ and, consequently, for all positive values of ρ.
- When the difference is taken into account, the same MFs are provided for both $\rho = 0$ and $\rho = -1$ and, consequently, for all negative values of ρ.

These results are of course not correct, since the different correlation coefficients should somehow influence the shape of the final result. This again can be easily proved if the behavior of random variables is considered. Fig. 8 shows, as an example, the sum of two uniform pdfs, which show the same width, for values of the correlation factors 0, $+1$, and -1. Let us consider the two cases $\rho = 0$ and $\rho = +1$: the support of the result is the same, but the shape of the two pdfs is completely different. In particular, when uncorrelated uniform pdfs (having the same width of the supports) are summed up, a triangular pdf

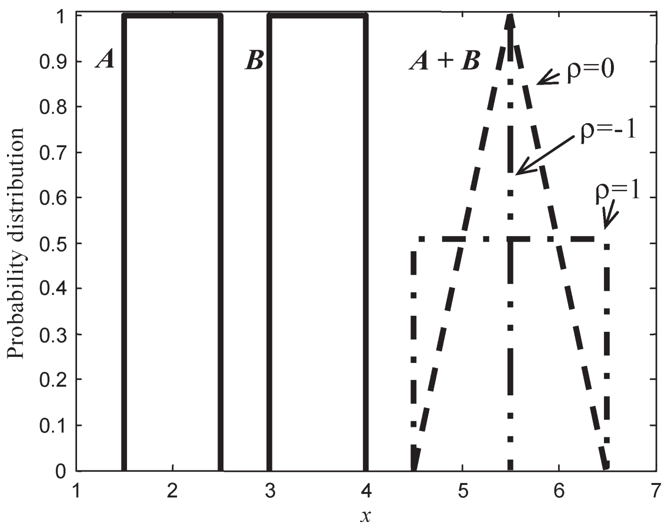

Fig. 8. Sum of the effects of two random contributions at different values of the correlation factor.

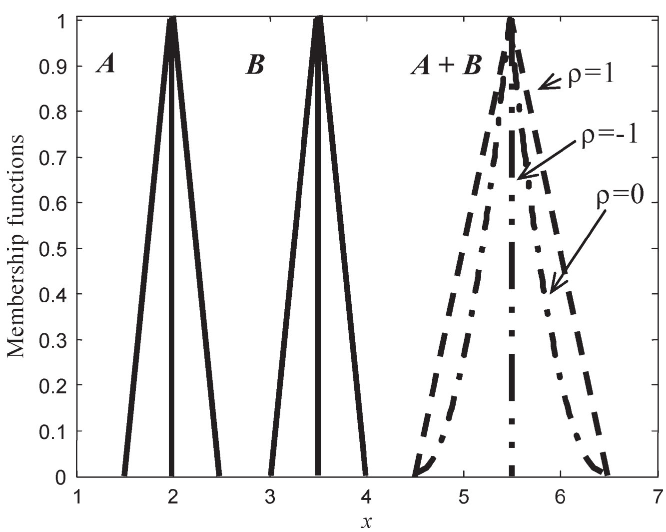

Fig. 9. Corresponding RFVs of the pdfs in Fig. 8.

is obtained; on the other hand, if the pdfs are totally correlated, the shape of the pdfs is maintained.

If the same situation is represented in terms of RFVs, different MFs should be similarly obtained. In fact, if the probability–possibility transformation [7]–[12] is applied to each pdf in Fig. 8, the RFVs shown in Fig. 9 are obtained, which are the correct ones. On the other hand, if the procedure defined in Section IV-A were applied, the following results would have been obtained:

- for $\rho = +1$, the same result shown in Fig. 9 for $\rho = +1$;
- for $\rho = -1$, the same result shown in Fig. 9 for $\rho = -1$;
- for $\rho = 0$, an RFV equal to that obtained for $\rho = +1$ that is a different result from the correct one shown in Fig. 9.

Similar considerations can also be drawn when the difference is concerned.

This leads to the conclusion that the formulas given in Section I for sum and difference, when the correlation coefficient is equal to 0, are not correct, except for the support (or, equivalently, the α-cut at level $\alpha = 0$).

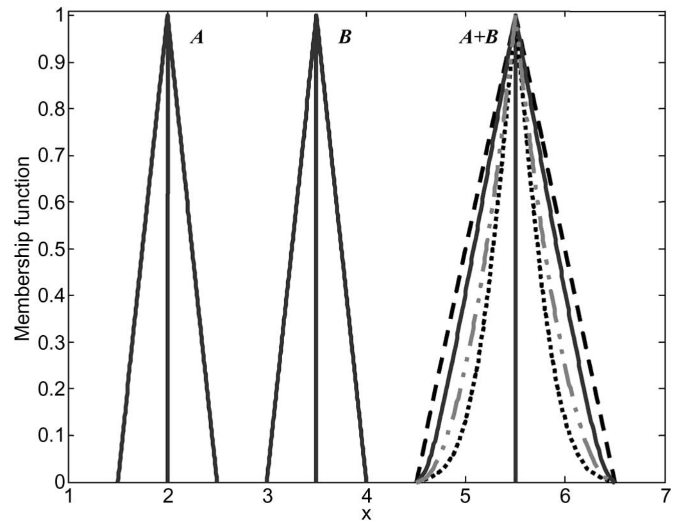

Fig. 10. Construction of the sum of two pure random RFVs for $\rho = 0$. Dashed line: step 1; dotted line: step 2; solid line: step 3; dashed-dotted line: strictly correct result.

Once again, the analysis of the way random variables combine with each other is useful. In fact, the Central Limit Theorem states that the sum of uncorrelated random variables tends to a normal distribution. If this property is "transformed" for RFVs, it follows that the sum of uncorrelated pure random RFVs should tend to a normal possibility distribution.[4]

Therefore, when $\rho = 0$ is considered, the result of the sum (or difference) of two pure random RFVs can be obtained by applying the following steps, as graphically shown in Fig. 10.

1) An MF is evaluated by applying (6) for the sum and (9) for the difference to every level α.
2) The normal possibility distribution having the same mean and support[5] as the MF obtained in step 1 is evaluated.
3) A suitable OWA operator is applied between the two MFs evaluated in steps 1 and 2.

The two weights of this last OWA operator must be chosen in such a way that the greater the number of RFVs to be summed up, the better the result approximates a normal possibility distribution. The following values can be proved [12] to satisfy this requirement:

$$w_1 = 1/\sqrt{2} \quad \text{and} \quad w_2 = 1 - 1/\sqrt{2}.$$

As shown by the solid line in Fig. 10, the obtained result is a good approximation of the strictly correct result shown by the dashed-dotted line in the same figure.

[4]The normal possibility distribution is the possibility distribution obtained by applying the probability–possibility transformation defined in [7] and [12] to the normal probability distribution. The normal pdf is supposed to be limited in the $\pm 3\sigma$ interval; this same interval is therefore the support (α-cut at level $\alpha = 0$) of the corresponding RFV.

[5]This applies only if the α-cut at level $\alpha = 0$ of the pure random RFV obtained in step 1 is symmetric with respect to the mean value of the same RFV. Otherwise, the support of the normal possibility distribution is twice the width of the narrower interval among the two external intervals of the α-cut at level $\alpha = 0$.

V. Mathematics of the RFVs

In the previous section, the mathematics for the combination of pure random RFVs has been proposed. Starting from these considerations, the mathematics can be readily developed for "complete" RFVs.

As already stated, the random and nonrandom parts of the RFVs must be combined in two different ways, according to the different nature of the effects they represent. Therefore, it is necessary to recognize the "pure random" and "pure fuzzy" parts of the two initial RFVs. As already stated, the "pure random" parts will be combined according to the new mathematics of the intervals and the three OWA operators defined in Section IV, and the "pure fuzzy" parts will be combined according to the classical mathematics of the intervals.

As shown in the example in Fig. 2 for RFV A, its "pure random" part is an MF whose α-cuts are [12]

$$[a_1^\alpha + w_{Al}^\alpha ; a_4^\alpha - w_{Ar}^\alpha]$$

where $\mu_A = (a_2^{\alpha=1} + a_3^{\alpha=1})/2$ is the mean of the RFV [12], and

$$w_{Al}^\alpha = \mu_A - a_2^\alpha \quad \text{and} \quad w_{Ar}^\alpha = a_3^\alpha - \mu_A$$

are the widths of the left and right sides of the α-cuts of the internal MF (see Fig. 2).

Let us now denote by $[a_{r1}^\alpha ; a_{r4}^\alpha]$ and $[b_{r1}^\alpha ; b_{r4}^\alpha]$ the α-cuts of the "pure random" MFs obtained, respectively, for RFVs A and B. Hence, the following definitions are given for the four arithmetic operations. $[c_1^\alpha ; c_2^\alpha ; c_3^\alpha ; c_4^\alpha]$ are the α-cuts of the result.

Sum

$$c_2^\alpha = a_2^\alpha + b_2^\alpha$$
$$c_3^\alpha = a_3^\alpha + b_3^\alpha$$

while c_1^α and c_4^α depend on the considered value for ρ.

- $\rho = 0$:

$$c_1^\alpha(\rho=0) = c_2^\alpha - \mu_c + k \cdot ex\left(a_{r1}^\alpha + b_{r1}^\alpha, g_1^\alpha\right) + (1-k) \cdot in\left(a_{r1}^\alpha + b_{r1}^\alpha, g_1^\alpha\right)$$
$$c_4^\alpha(\rho=0) = c_3^\alpha - \mu_c + k \cdot ex\left(a_{r4}^\alpha + b_{r4}^\alpha, g_4^\alpha\right) + (1-k) \cdot in\left(a_{r4}^\alpha + b_{r4}^\alpha, g_4^\alpha\right)$$

  where k is a constant $(k = 1/\sqrt{2})$, μ_c is the mean value of the sum $(\mu_c = (c_2^{\alpha=1} + c_3^{\alpha=1})/2)$, and $[g_1^\alpha, g_4^\alpha]$ is the generic α-cut of the normal possibility distribution having mean value μ_c and standard deviation

$$\sigma = \frac{1}{3} \min\left\{\mu_c - a_{r1}^{\alpha=0} - b_{r1}^{\alpha=0} ; a_{r4}^{\alpha=0} + b_{r4}^{\alpha=0} - \mu_c\right\}.$$

- $\rho = +1$:

$$c_1^\alpha(\rho=1) = a_1^\alpha + b_1^\alpha$$
$$c_4^\alpha(\rho=1) = a_4^\alpha + b_4^\alpha.$$

- $\rho = -1$:

$$c_1^\alpha(\rho=-1) = c_2^\alpha + s_1 \cdot \left(a_{r1}^\alpha + b_{r1}^\alpha - \mu_c\right)$$
$$c_4^\alpha(\rho=-1) = c_3^\alpha + s_4 \cdot \left(a_{r4}^\alpha + b_{r4}^\alpha - \mu_c\right)$$

  where

$$s_1 = \frac{\mu_c - \min\left\{a_{r1}^{\alpha=0} + b_{r4}^{\alpha=0} ; a_{r4}^{\alpha=0} + b_{r1}^{\alpha=0}\right\}}{\mu_c - \left(a_{r1}^{\alpha=0} + b_{r1}^{\alpha=0}\right)}$$
$$s_4 = \frac{\max\left\{a_{r1}^{\alpha=0} + b_{r4}^{\alpha=0} ; a_{r4}^{\alpha=0} + b_{r1}^{\alpha=0}\right\} - \mu_c}{\left(a_{r4}^{\alpha=0} + b_{r4}^{\alpha=0}\right) - \mu_c}.$$

Difference

$$c_2^\alpha = a_2^\alpha - b_3^\alpha$$
$$c_3^\alpha = a_3^\alpha - b_2^\alpha$$

while c_1^α and c_4^α depend on the considered value for ρ.

- $\rho = 0$:

$$c_1^\alpha(\rho=0) = c_2^\alpha - \mu_c + k \cdot ex\left(a_{r1}^\alpha - b_{r4}^\alpha, g_1^\alpha\right) + (1-k) \cdot in\left(a_{r1}^\alpha - b_{r4}^\alpha, g_1^\alpha\right)$$
$$c_4^\alpha(\rho=0) = c_3^\alpha - \mu_c + k \cdot ex\left(a_{r4}^\alpha - b_{r1}^\alpha, g_4^\alpha\right) + (1-k) \cdot in\left(a_{r4}^\alpha - b_{r1}^\alpha, g_4^\alpha\right)$$

  where k is a constant $(k = 1/\sqrt{2})$, μ_c is the mean value of the difference $(\mu_c = (c_2^{\alpha=1} + c_3^{\alpha=1})/2)$, and $[g_1^\alpha, g_4^\alpha]$ is the generic α-cut of the normal possibility distribution having mean value μ_c and standard deviation

$$\sigma = \frac{1}{3} \min\left\{\mu_c - a_{r1}^{\alpha=0} + b_{r4}^{\alpha=0} ; a_{r4}^{\alpha=0} - b_{r1}^{\alpha=0} - \mu_c\right\}.$$

- $\rho = +1$:

$$c_1^\alpha(\rho=1) = c_2^\alpha + s_1 \cdot \left(a_{r1}^\alpha - b_{r4}^\alpha - \mu_c\right)$$
$$c_4^\alpha(\rho=1) = c_3^\alpha + s_4 \cdot \left(a_{r4}^\alpha - b_{r1}^\alpha - \mu_c\right)$$

  where

$$s_1 = \frac{\mu_c - \min\left\{a_{r1}^{\alpha=0} - b_{r1}^{\alpha=0} ; a_{r4}^{\alpha=0} - b_{r4}^{\alpha=0}\right\}}{\mu_c - \left(a_{r1}^{\alpha=0} - b_{r4}^{\alpha=0}\right)}$$
$$s_4 = \frac{\max\left\{a_{r1}^{\alpha=0} - b_{r1}^{\alpha=0} ; a_{r4}^{\alpha=0} - b_{r4}^{\alpha=0}\right\} - \mu_c}{\left(a_{r4}^{\alpha=0} - b_{r1}^{\alpha=0}\right) - \mu_c}.$$

- $\rho = -1$:

$$c_1^\alpha(\rho=-1) = a_1^\alpha - b_4^\alpha$$
$$c_4^\alpha(\rho=-1) = a_4^\alpha - b_1^\alpha.$$

Product

$$c_1^\alpha = c_2^\alpha - \mu_r + c_{r1}^\alpha$$
$$c_2^\alpha = \min\left\{a_2^\alpha b_2^\alpha ; a_2^\alpha b_3^\alpha ; a_3^\alpha b_2^\alpha ; a_3^\alpha b_3^\alpha\right\}$$
$$c_3^\alpha = \max\left\{a_2^\alpha b_2^\alpha ; a_2^\alpha b_3^\alpha ; a_3^\alpha b_2^\alpha ; a_3^\alpha b_3^\alpha\right\}$$
$$c_4^\alpha = c_3^\alpha - \mu_r + c_{r4}^\alpha$$

where $\mu_r = (c_{r1}^{\alpha=1} + c_{r4}^{\alpha=1})/2$, and c_{r1}^{α} and c_{r4}^{α} depend on the considered value for ρ.

- $\rho = 0$:

$$c_{r1}^{\alpha}(\rho = 0) = \min\left\{a_{r1}^{\alpha}b_{r1}^{\alpha}; a_{r1}^{\alpha}b_{r4}^{\alpha}; a_{r4}^{\alpha}b_{r1}^{\alpha}; a_{r4}^{\alpha}b_{r4}^{\alpha}\right\}$$
$$c_{r4}^{\alpha}(\rho = 0) = \max\left\{a_{r1}^{\alpha}b_{r1}^{\alpha}; a_{r1}^{\alpha}b_{r4}^{\alpha}; a_{r4}^{\alpha}b_{r1}^{\alpha}; a_{r4}^{\alpha}b_{r4}^{\alpha}\right\}.$$

- $\rho = +1$:

$$c_{r1}^{\alpha}(\rho = 1) = \begin{cases} x_m^{\alpha}y_m^{\alpha}, & \text{if } a_{r1}^{\alpha} < x_m^{\alpha} < a_{r4}^{\alpha} \\ \min\left\{a_{r1}^{\alpha}b_{r1}^{\alpha}, a_{r4}^{\alpha}b_{r4}^{\alpha}\right\}, & \text{otherwise} \end{cases}$$
$$c_{r4}^{\alpha}(\rho = 1) = \max\left\{a_{r1}^{\alpha}b_{r1}^{\alpha}, a_{r4}^{\alpha}b_{r4}^{\alpha}\right\}$$

where according to the above definitions

$$x_m^{\alpha} = \frac{a_{r1}^{\alpha} + a_{r4}^{\alpha}}{4} - \frac{r^{\alpha}}{4}\left(b_{r1}^{\alpha} + b_{r4}^{\alpha}\right)$$
$$y_m^{\alpha} = \frac{b_{r1}^{\alpha} + b_{r4}^{\alpha}}{4} - \frac{a_{r1}^{\alpha} + a_{r4}^{\alpha}}{4r^{\alpha}}$$
$$r^{\alpha} = \frac{a_{r4}^{\alpha} - a_{r1}^{\alpha}}{b_{r4}^{\alpha} - b_{r1}^{\alpha}}.$$

In case the values c_{r1}^{α} and c_{r4}^{α} provide a multivalued function, then a correction must be applied, as shown in Section IV-A and Fig. 7.

- $\rho = -1$:

$$c_{r1}^{\alpha}(\rho = -1) = \min\left\{a_{r1}^{\alpha}b_{r4}^{\alpha}, a_{r4}^{\alpha}b_{r1}^{\alpha}\right\}$$
$$c_{r4}^{\alpha}(\rho = -1) = \begin{cases} x_M^{\alpha}y_M^{\alpha}, & \text{if } a_{r1}^{\alpha} < x_M^{\alpha} < a_{r4}^{\alpha} \\ \max\left\{a_{r1}^{\alpha}b_{r4}^{\alpha}, a_{r4}^{\alpha}b_{r1}^{\alpha}\right\}, & \text{otherwise} \end{cases}$$

where r^{α} is defined above, and

$$x_M^{\alpha} = \frac{a_{r1}^{\alpha} + a_{r4}^{\alpha}}{4} + \frac{r^{\alpha}}{4}\left(b_{r1}^{\alpha} + b_{r4}^{\alpha}\right)$$
$$y_M^{\alpha} = \frac{b_{r1}^{\alpha} + b_{r4}^{\alpha}}{4} + \frac{a_{r1}^{\alpha} + a_{r4}^{\alpha}}{4r^{\alpha}}.$$

In case the values c_{r1}^{α} and c_{r4}^{α} provide a multivalued function, then a correction must be applied, as shown in Section IV-A and Fig. 7.

Division

Provided that $0 \notin [b_1^{\alpha}, b_4^{\alpha}]$

$$c_1^{\alpha} = c_2^{\alpha} - \mu_r + c_{r1}^{\alpha}$$
$$c_2^{\alpha} = \min\left\{a_2^{\alpha}/b_2^{\alpha}; a_2^{\alpha}/b_3^{\alpha}; a_3^{\alpha}/b_2^{\alpha}; a_3^{\alpha}/b_3^{\alpha}\right\}$$
$$c_3^{\alpha} = \max\left\{a_2^{\alpha}/b_2^{\alpha}; a_2^{\alpha}/b_3^{\alpha}; a_3^{\alpha}/b_2^{\alpha}; a_3^{\alpha}/b_3^{\alpha}\right\}$$
$$c_4^{\alpha} = c_3^{\alpha} - \mu_r + c_{r4}^{\alpha}$$

where $\mu_r = (c_{r1}^{\alpha=1} + c_{r4}^{\alpha=1})/2$, and c_{r1}^{α} and c_{r4}^{α} depend on the considered value for ρ.

- $\rho = 0$:

$$c_{r1}^{\alpha}(\rho = 0) = \min\left\{a_{r1}^{\alpha}/b_{r1}^{\alpha}; a_{r1}^{\alpha}/b_{r4}^{\alpha}; a_{r4}^{\alpha}/b_{r1}^{\alpha}; a_{r4}^{\alpha}/b_{r4}^{\alpha}\right\}$$
$$c_{r4}^{\alpha}(\rho = 0) = \max\left\{a_{r1}^{\alpha}/b_{r1}^{\alpha}; a_{r1}^{\alpha}/b_{r4}^{\alpha}; a_{r4}^{\alpha}/b_{r1}^{\alpha}; a_{r4}^{\alpha}/b_{r4}^{\alpha}\right\}.$$

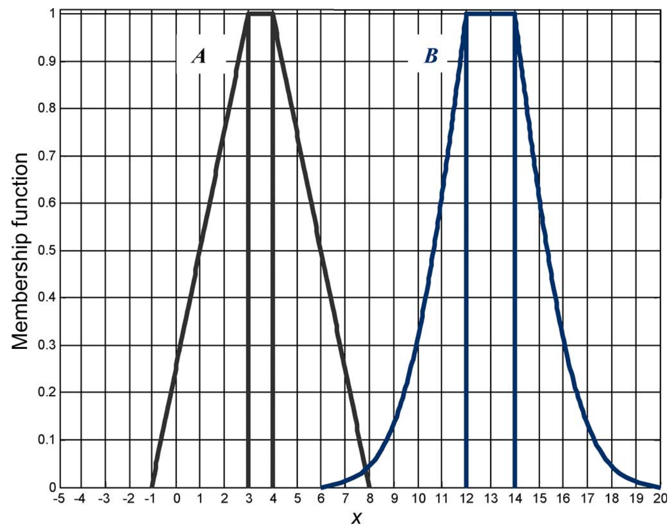

Fig. 11. RFVs used in the following examples.

- $\rho = +1$:

$$c_{r1}^{\alpha}(\rho = 1) = \min\left\{a_{r1}^{\alpha}/b_{r1}^{\alpha}, a_{r4}^{\alpha}/b_{r4}^{\alpha}\right\}$$
$$c_{r4}^{\alpha}(\rho = 1) = \max\left\{a_{r1}^{\alpha}/b_{r1}^{\alpha}, a_{r4}^{\alpha}/b_{r4}^{\alpha}\right\}.$$

In case the values c_{r1}^{α} and c_{r4}^{α} provide a multivalued function, then a correction must be applied, as shown in Section IV-A and Fig. 7.

- $\rho = -1$:

$$c_{r1}^{\alpha}(\rho = -1) = \min\left\{a_{r1}^{\alpha}/b_{r4}^{\alpha}, a_{r4}^{\alpha}/b_{r1}^{\alpha}\right\}$$
$$c_{r4}^{\alpha}(\rho = -1) = \max\left\{a_{r1}^{\alpha}/b_{r4}^{\alpha}, a_{r4}^{\alpha}/b_{r1}^{\alpha}\right\}.$$

In case the values c_{r1}^{α} and c_{r4}^{α} provide a multivalued function, then a correction must be applied, as shown in Section IV-A and Fig. 7.

Moreover, for all the considered operations, for values of ρ different from 0, +1, and −1, according to the OWA operator defined in Section IV-A, the following applies.

- if $0 < \rho < 1$:

$$c_1^{\alpha}(\rho > 0) = (1 - \rho) \cdot c_1^{\alpha}(\rho = 0) + \rho \cdot c_1^{\alpha}(\rho = 1)$$
$$c_4^{\alpha}(\rho > 0) = (1 - \rho) \cdot c_4^{\alpha}(\rho = 0) + \rho \cdot c_4^{\alpha}(\rho = 1)$$

- if $-1 < \rho < 0$:

$$c_1^{\alpha}(\rho < 0) = (1 + \rho) \cdot c_1^{\alpha}(\rho = 0) - \rho \cdot c_1^{\alpha}(\rho = -1)$$
$$c_4^{\alpha}(\rho < 0) = (1 + \rho) \cdot c_4^{\alpha}(\rho = 0) - \rho \cdot c_4^{\alpha}(\rho = -1).$$

Figs. 11–15 show some examples for the four operations and different values of the correlation coefficient ρ.

VI. Conclusion

Previous works have shown that RFVs, which are defined within the theory of evidence, are suitable variables for representing measurement results and the associated uncertainty. Their main advantage is the capability of representing all possible contributions to uncertainty in the correct mathematical way.

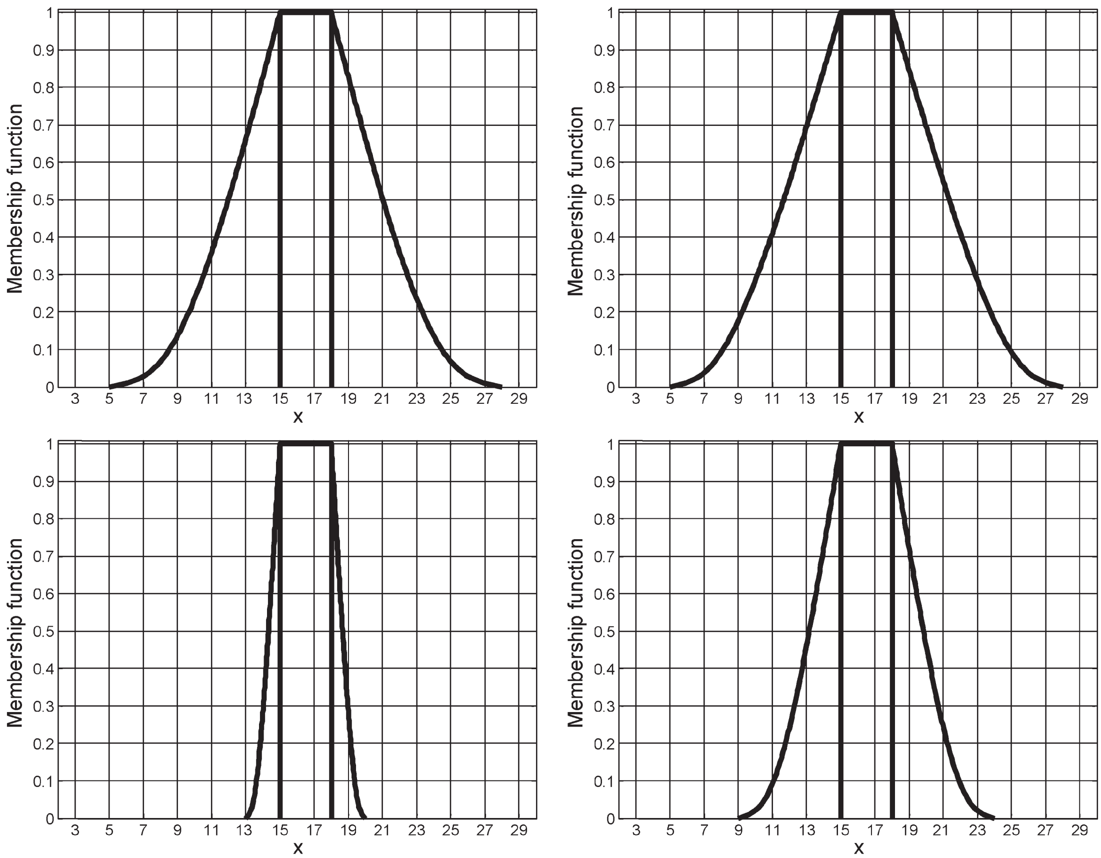

Fig. 12. Sum of the RFVs in Fig. 11 for $\rho = 0$ (upper left), $\rho = 1$ (upper right), $\rho = -1$ (lower left), and $\rho = -0.5$ (lower right).

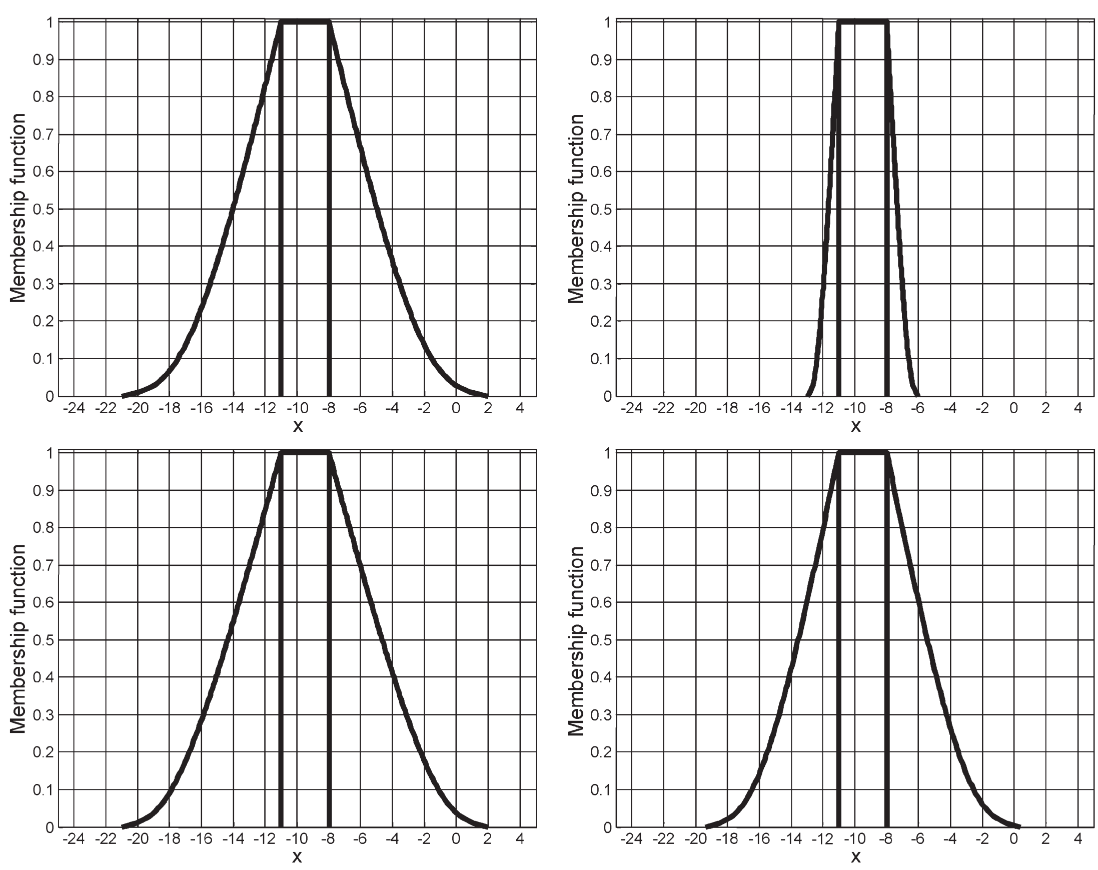

Fig. 13. Difference of the RFVs in Fig. 11 for $\rho = 0$ (upper left), $\rho = 1$ (upper right), $\rho = -1$ (lower left), $\rho = 0.2$ (lower right).

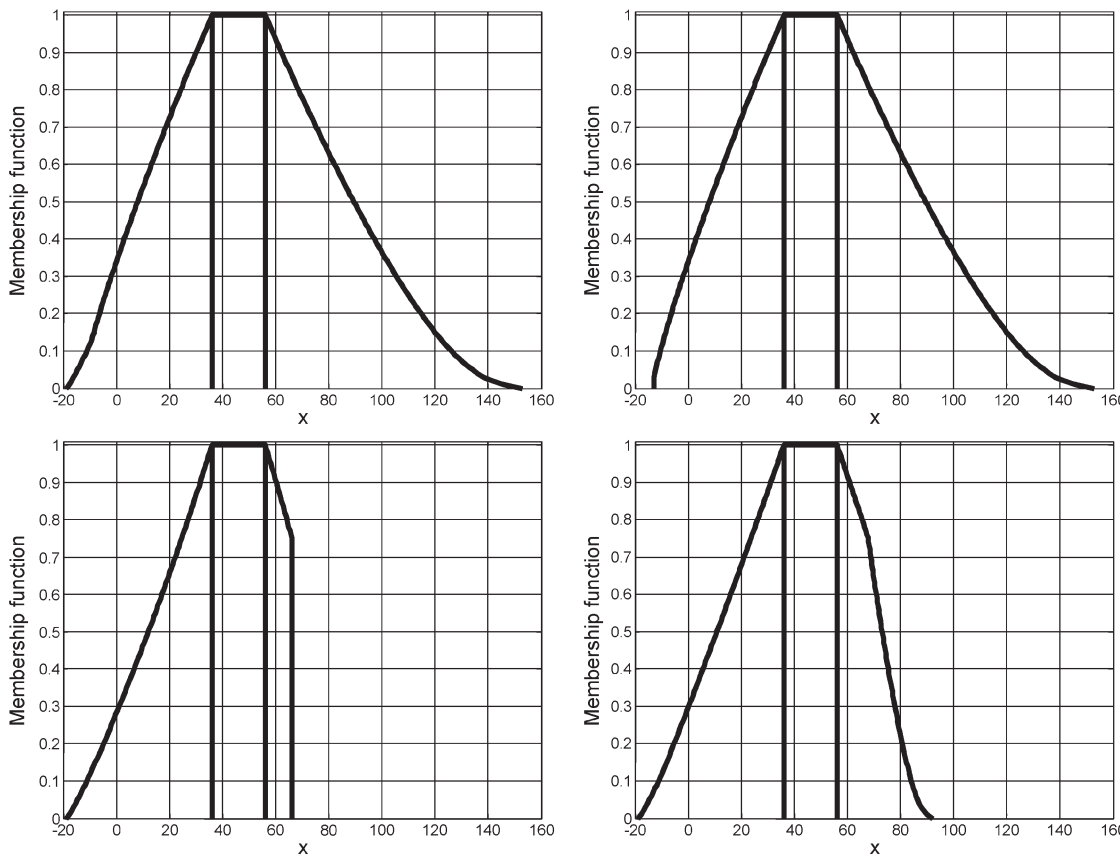

Fig. 14. Product of the RFVs in Fig. 11 for (upper left) $\rho = 0$, (upper right) $\rho = 1$, (lower left) $\rho = -1$, and (lower right) $\rho = -0.7$.

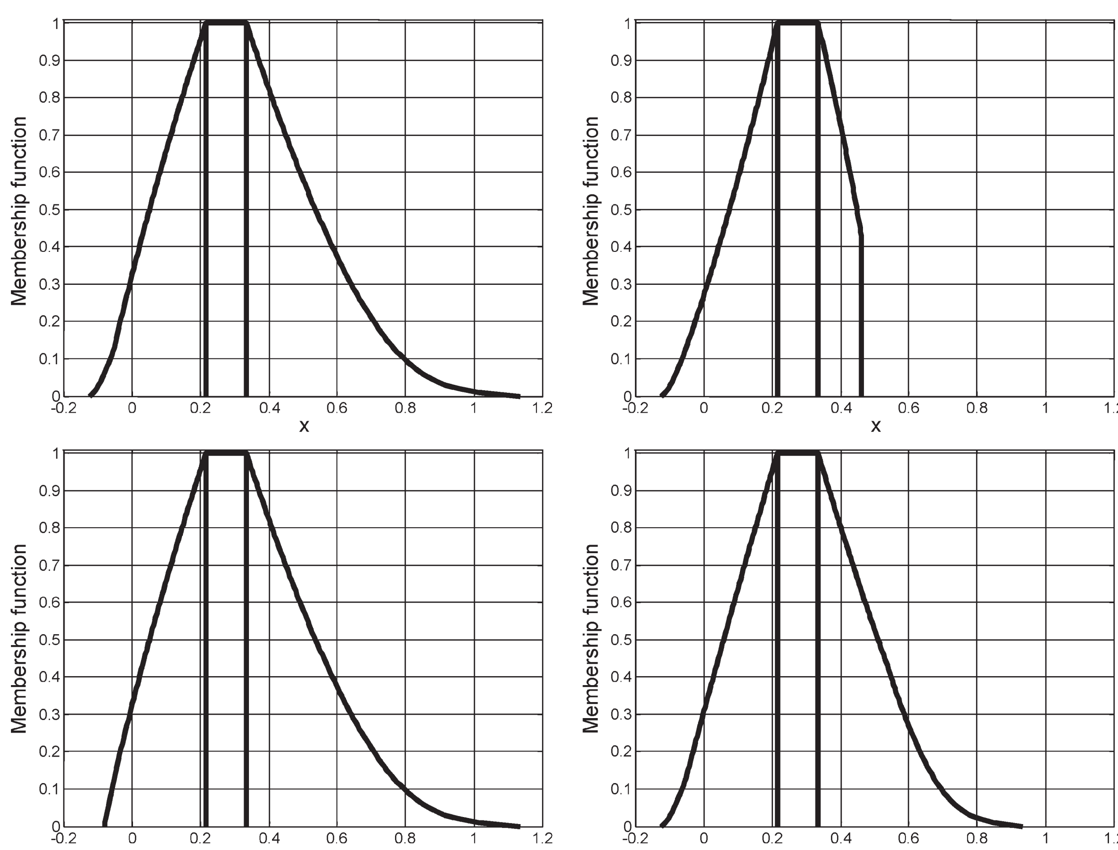

Fig. 15. Division of the RFVs in Fig. 11 for (upper left) $\rho = 0$, (upper right) $\rho = 1$, (lower left) $\rho = -1$, and (lower right) $\rho = 0.3$.

This paper proposes an original mathematics for combining RFVs so that measurement results can be combined in such a way that the uncertainty of the final result is directly obtained.

The mathematics has been defined by considering the different ways random, unknown, and systematic contributions propagate through the measurement process; the mathematics of the intervals and the probability theory have been considered, and a unified approach has been followed, according to the theory of evidence.

The availability of such mathematics allows uncertainty to be processed directly within the measurement procedure, together with the measurement data, so that the different uncertainty components can be tracked when they propagate throughout the measurement procedure and contribute to the uncertainty of the final result, which is directly provided in terms of an RFV.

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