The Use of Random-Fuzzy Variables for the Implementation of Decision Rules in the Presence of Measurement Uncertainty

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***Abstract*—The practical, everyday final applications of measurement processes are mostly aimed at making a decision, on the basis of a comparison between the measured value and a reference value. If uncertainty in measurement is considered, this comparison must be performed between an interval of confidence (the measurement result) and a scalar quantity (the reference value). The result of such a comparison is quite often not univocal, so that making a decision may become quite troublesome. This paper shows how the use of the random-fuzzy variables in the expression of uncertainty in measurement allows the implementation of simple decision rules capable of taking into account the measurement uncertainty correctly. The proposed decision rules are applied to measurement procedures based on measurement algorithms that contain *if … then … else* structures where the *if* condition is applied to intermediate measurement results. An example of implementation of these decision rules is reported and discussed.**

***Index Terms*—Decision rules, digital signal-processing (DSP)-based measurement, fuzzy variables, measurement characterization, uncertainty.**

I. Introduction

THE final aim of almost any measurement process is a decision, based on the result of the measurement process itself. Several examples can be taken from very different fields of activity, ranging from the industrial applications to the everyday life. Dimensional measurements of geometric properties are used to assess whether a workpiece stays within the required tolerance so that it can be accepted or must be rejected; again, dimensional measurements are used to classify eggs into the different sizes. Time measurements in a GPS system are used to assess whether a person, or an object, is in the right place or not. Electromagnetic field measurements are used to assess whether living in a region of space is dangerous or not for human beings.

All the above examples come to the right decision by comparing the measurement result with a given threshold. This is a quite immediate operation when both quantities to be compared are expressed by scalar values. Unfortunately, at least one of the above quantities, the measurement result, cannot be expressed by a single scalar value, since the presence of the measurement uncertainty forces the expression of it in terms of an interval of confidence. In this case the comparison is not univocal, and taking a decision may become quite troublesome.

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Furthermore, very few indications are given by the standard organizations on this point. To the authors' knowledge, only [1] provides some indications on how to compare the result of geometrical measurements with a specific threshold in order to prove conformance or nonconformance with specification.

These recommendations are in agreement with the statistical approach suggested by the IEC-ISO *Guide to the Expression of Uncertainty in Measurement* [2] and, therefore, appear to be a good starting point, at least when industrial applications are concerned. However, they may lead to unsatisfactory decisions in more critical situations, for instance, when the nonconformance with specification may threaten the human health, as in the case of exposure to electromagnetic fields.

The problem becomes even more complex when the decision is part of the measurement process, and not only the final step of the process itself. This is typical of the complex measurement systems, based on digital signal-processing (DSP) techniques, when the measurement algorithms contain *if…then…else* structures where the *if* condition is applied to intermediate measurement results.

In this case, the problem is not only how the uncertainty associated with the intermediate measurement result affects the decision but also how it propagates through the *then…else* cases and affects the uncertainty of the final measurement result.

The guide [2] is practically useless in this case, since its recommended procedure to obtain the combined standard uncertainty requires the relationship between the direct measurement results and the final result to be mathematically derivable. This is not the case of an algorithm with *if…then…else* structures embedded.

This paper proposes an approach based on the random-fuzzy variables (RFVs). Previous papers have shown how these variables can be effectively employed to express uncertainty in measurement, especially when complex measurement algorithms are implemented [3]–[6].

After recalling the fundamentals of the RFVs, this paper shows how decision rules can be implemented on RFVs in order to compare a measurement result, expressed in terms of RFV, with a threshold. An example will be discussed.

II. Proposed Approach

A. Random-Fuzzy Variable Background

Random-fuzzy variables represent an extension of the fuzzy variables defined by Zadeh [7]–[9]. While a fuzzy variable is defined by a membership function over a reference set X, an

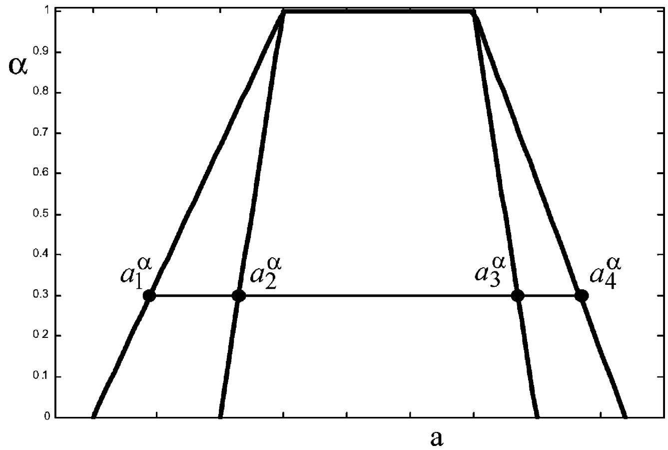

Fig. 1. Example of random-fuzzy variable.

RFV is completely defined by two membership functions [6], as shown in Fig. 1. By definition, the membership functions take only values between zero and one.

When the RFV is employed to represent a measurement result, the uncertainty associated to the measurement itself is also given [5], [6], [10]. Moreover, the two membership functions have a different meaning from the metrological point of view [5], [6], [10]. The internal membership function is obtained by taking into account only the unknown systematic contributions to the measurement uncertainty, while the external membership function is obtained by taking into account all contributions, so that random contributions are taken into account by that part of the RFV among the two membership functions.

According to the mathematical definitions provided in [6], an RFV A also can be defined in terms of α-cuts. The α-cut is the closed interval $[a_1^\alpha, a_4^\alpha]$ defined by the external membership function of the RFV at level α, with $0 \leq \alpha \leq 1$.

It has been proven [6] that, when an RFV is used to represent the result of a measurement, each α-cut represents a possible confidence interval, within which the measurement result is supposed to lie with a level of confidence $1-\alpha$.

As shown in Fig. 1, the α-cuts of an RFV are all nested. Furthermore, each α-cut is represented by four numbers

$$A_\alpha = \{a_1^\alpha, a_2^\alpha, a_3^\alpha, a_4^\alpha\} \tag{1}$$

where $a_1^\alpha \leq a_2^\alpha \leq a_3^\alpha \leq a_4^\alpha$, $\forall\alpha$, which define three closed intervals: $[a_1^\alpha, a_2^\alpha]$, $[a_2^\alpha, a_3^\alpha]$, and $[a_3^\alpha, a_4^\alpha]$. The central interval $[a_2^\alpha, a_3^\alpha]$ is intercepted on the internal membership function, while the side intervals $[a_1^\alpha, a_2^\alpha]$ and $[a_3^\alpha, a_4^\alpha]$ represent the distance between the internal and the external membership functions.

According to the way the membership functions are obtained [6], [10], it can be stated that, for each level α, the central interval considers all unknown systematic contributions to the measurement uncertainty, while the side intervals consider all random contributions [6].

Moreover, a suitable mathematics has been defined for the RFVs [6], so that a measurement algorithm can be executed directly on the RFVs, thus providing the measurement result and its associated uncertainty in terms of an RFV [10].

B. Implementation of the Single Decision Rule

The decision rule that is proposed here comes from the comparison between the result of a measurement represented in terms of RFV and a fixed threshold, represented by a scalar value.

When an RFV A is compared with a scalar value, this comparison yields three cases.

1) The upper limit a_4^0 of the α-cut with $\alpha = 0$ is lower than the threshold. This means that all elements in all α-cuts of A are lower than the threshold. This case leads to the decision that the measurement result is lower than the threshold. Since this decision can be taken with full certainty, a credibility factor $\lambda = 1$ can be associated to it. On the other side, it can be stated that A is greater than the threshold with a credibility factor equal to zero.
2) The lower limit a_1^0 of the α-cut with $\alpha = 0$ is greater than the threshold. This means that all elements in all α-cuts of A are greater than the threshold. This case leads to the decision that the measurement result is greater than the threshold. Since this decision can be taken with full certainty, a credibility factor $\lambda = 1$ can be associated to it. On the other side, it can be stated that A is lower than the threshold with a credibility factor equal to zero.
3) The threshold value falls inside the α-cut with $\alpha = 0$. More precisely, if h is the threshold value, this means that $a_1^0 \leq h \leq a_4^0$. In this case, the measurement result can be considered both lower and greater than the threshold and taking a decision is quite difficult. Therefore, it becomes very important to associate the correct credibility factors to both "pending" decisions. Obviously, since the two "pending" decisions are opposite, the sum of the two credibility factors must be one. These credibility factors can be directly obtained from the external membership function of A. In particular, the credibility factor related to the statement that A is lower than the threshold can be defined as the relative area subtended by the external membership function of A in the interval $a_1^0 \leq a < h$. Similarly, the credibility factor related to the statement that A is greater than the threshold can be defined as the relative area subtended by the external membership function of A in the interval $h \leq a < a_4^0$, as shown in Fig. 2. When this case occurs, the decision that must be taken is the one to which the greater credibility factor corresponds, that is, the one with a credibility factor $\lambda > 0.5$.

According to the above considerations, a credibility factor is always associated to the result of a comparison, in order to assess the "certainty level" of the result itself. Moreover, when multiple comparisons are needed, the values of the credibility factors of each single comparison must be processed together, in order to provide a single final credibility factor, associated to the final result.

C. Multiple Decision Rules

When the measurement algorithm contains multiple comparisons, the proposed decision rule must be applied a certain number of times, thus providing different credibility factors. In this section, it is shown how the different credibility factors can

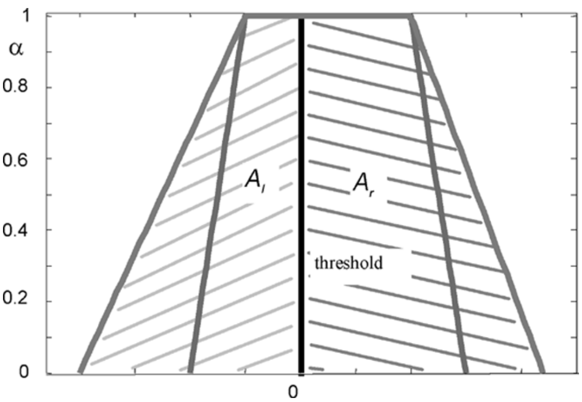

Fig. 2. Example of implementation of the decision rules on a random-fuzzy variable. The random-fuzzy variable is lower than the threshold with a credibility factor A_l/A and greater with a credibility factor equal to A_r/A. A is the total area.

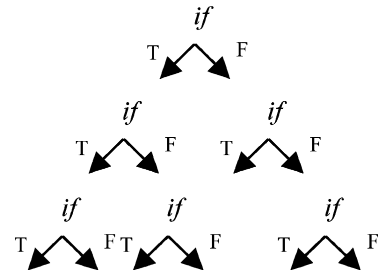

Fig. 3. Exemplification of in-series decision rules.

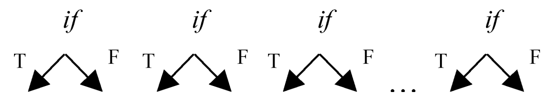

Fig. 4. Exemplification of in-parallel decision rules.

be combined together, in order to associate a unique credibility factor to the final measurement result. This combination generally depends on the considered measurement algorithm, and an "optimal" way to compose the single credibility factors must be found for each kind of measurement algorithm.

Though the standard fuzzy inference rules based, for instance, on the MIN-MAX composition might be considered, it must be taken into account that the random-fuzzy technique applied to the expression of uncertainty requires only to combine RFVs, and it is not based on standard fuzzy inference. Therefore, simpler rules, specifically tailored on the typical problems commonly found in measurement applications, are considered here in order to obtain the final credibility factor in a straightforward way.

The multiple comparisons can be divided into two main classes.

1) In-Series Decision Rules: This situation is shown in Fig. 3. Different conditions must be checked, as a consequence of the previously checked condition, and the *if…then…else* cases are in a cascaded structure. From a theoretical point of view, if n is the number of the comparisons, 2^n results are possible. If the simple decision rule described in the previous section is applied to each comparison, only one result is finally obtained. It seems reasonable to associate the following credibility factor to it:

$$\lambda = \prod_{k=1}^{n} \lambda_k \tag{2}$$

where λ_k is the credibility factor associated to the result of the kth comparison. Of course, while it is always $\lambda_k > 0.5$, the final credibility factor could be lower than 0.5.

2) In-Parallel Decision Rules: This situation is shown in Fig. 4. Different conditions are checked independently from each other. The result of the n comparisons is a combination of *true/false* results. For instance, if $n = 2$, the possible combinations are: TF, FT, FF, TT.

The selection of the "best" combination depends on the single credibility factors. In fact, for each comparison, the result with the credibility factor greater than 0.5 must be chosen.

In this case, it seems reasonable to associate to the final *true/false* combination the following credibility factor:

$$\lambda = \frac{1}{n} \cdot \sum_{k=1}^{n} \lambda_k \tag{3}$$

where $\lambda_k > 0.5$ is the credibility factor associated to the result of the kth comparison.

Since (3) is the mean of the credibility factors associated to each comparison result, which are all greater than 0.5, the credibility factor associated to the final combination is surely greater than 0.5.

The combination rules given in (2) and (3), which allow the evaluation of a final credibility factor, do not refer to the final measurement result but only to the final chosen combination. This means that, if a measurement algorithm is executed either between two successive comparisons, in the case of in-series structure, or at the end of the multiple comparisons, a credibility factor must be evaluated to be associated to the result of the measurement algorithm. In simple cases, the credibility factor that must be associated to the final measurement result is equal to the credibility factor associated to the *true/false* final combination. In most complicated cases, a different credibility factor, still based on the credibility factor associated to the *true/false* final combination, could be associated to the final measurement result.

III. Example

A common measurement procedure that requires the implementation of decision rules is the classification of a measured quantity according to the result of the comparison of this same quantity, or another measured quantity, with a given threshold [11], [12].

In the electric field, an example is given by the classification of the current harmonic components into "linear" and "nonlinear" in order to assign a "nonlinearity" index to the considered device. A current harmonic component is considered linear if the correspondent voltage harmonic component is present;

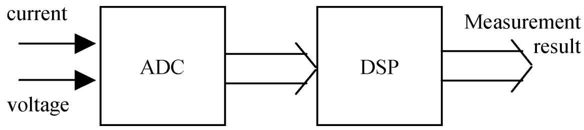

Fig. 5. DSP-based instrument.

otherwise, it is considered nonlinear. After this simple classification, a nonlinearity index can be defined as the square sum of all nonlinear current harmonic components

$$I_{nl}^2 = \sum_k I_k^2, \qquad \forall k \,|V_k = 0 \,. \tag{4}$$

Similarly, a linearity index can be defined as the square sum of all linear current harmonic components

$$I_l^2 = \sum_k I_k^2, \qquad \forall k \,|V_k > 0 \,. \tag{5}$$

According to (4) and (5), the amplitude of each voltage harmonic component must be compared with a zero threshold. In practice, such a direct comparison leads to meaningless results, because of the presence of noise, which forces all measured values of the voltage harmonic components to be greater than zero. On the contrary, meaningful results can be obtained if the measured values are compared with the instruments resolution. However, since the measured values are characterized by their uncertainties, the comparisons must consider also the uncertainty, as discussed in the previous section. Therefore, the instrument's features must be taken into account.

The measurement system considered in this example is schematically shown in Fig. 5. The instrument's resolution is the resolution of the ADC. Moreover, the system has been experimentally characterized [10], [13] in order to associate an RFV to each introduced uncertainty contribution. Therefore, each acquired sample of voltage and current can be corrupted by the uncertainty contributions and transformed into an RFV [10], thus taking into account its associated uncertainty.

In this example, a load is supposed to be supplied by a periodic distorted voltage with a 50 Hz fundamental frequency and a third, fifth, and seventh harmonic component. These components are set so that, after considering the uncertainty contributions

1) the RFVs representing the first, third, and fifth harmonic components are completely above the threshold;
2) the RFV representing the seventh harmonic component is across the threshold.

Obviously, all other harmonic components, evaluated by means of a frequency-domain analysis, are below the threshold.

Two examples are considered: a linear load and a nonlinear load. In the first case, the current presents the same harmonic components of the applied voltage. In the second case, the presence of a nonlinear load has been simulated by simply adding to the line current the 11th, 13th, 17th, and 21st harmonic components. Figs. 6 and 7 show the voltage and current signals in case of linear and nonlinear load, respectively.

Since they have the same applied voltage, in both cases the multiple parallel comparisons lead to the following conclusions.

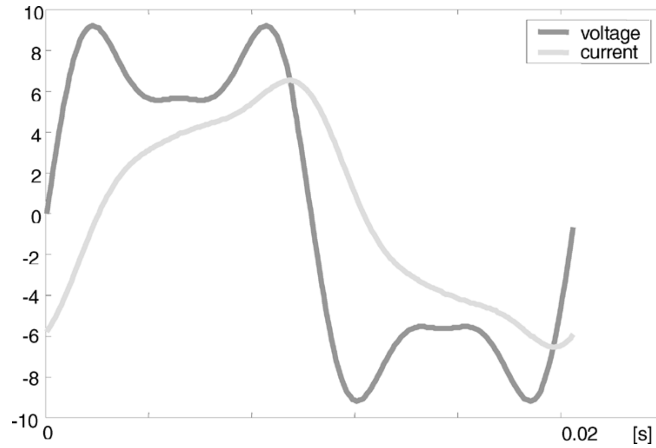

Fig. 6. Simulated voltage and current signals in case of linear load.

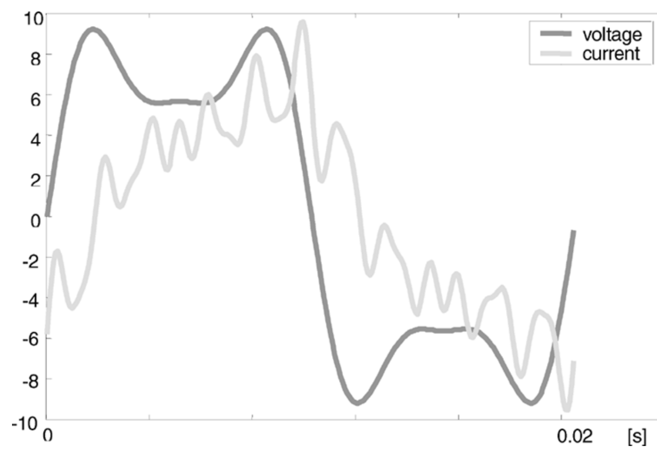

Fig. 7. Simulated voltage and current signals in case of nonlinear load.

- The first, third, and fifth voltage harmonic components are totally above the threshold; therefore the correspondent current harmonic components are "linear" with a credibility factor $\lambda = 1$.
- The seventh voltage harmonic component is greater than the threshold with a credibility factor $\lambda = 0.69$; therefore the seventh current harmonic component is "linear" with a credibility factor $\lambda = 0.69$.
- All other voltage harmonic components are totally below the threshold; therefore the correspondent current harmonic components are "nonlinear" with a credibility factor $\lambda = 1$.

The credibility factor associated with the *true/false* final combination is given by (3). Since the linearity and nonlinearity indices are built as the square sum of current harmonic components, the credibility that a given component should be considered in the sum is the same as the credibility of the result of the considered comparison. Therefore, it seems reasonable to associate, to the measured indices, the same credibility factor as that of the *true/false* combination considered in their evaluation.

In the case of a linear load, the results are given in Fig. 8. The credibility factors, associated to the obtained RFVs by following the proposed approach, are

- $\lambda = 1$ for the "nonlinearity" index;
- $\lambda = 0.92$ for the "linearity" index.

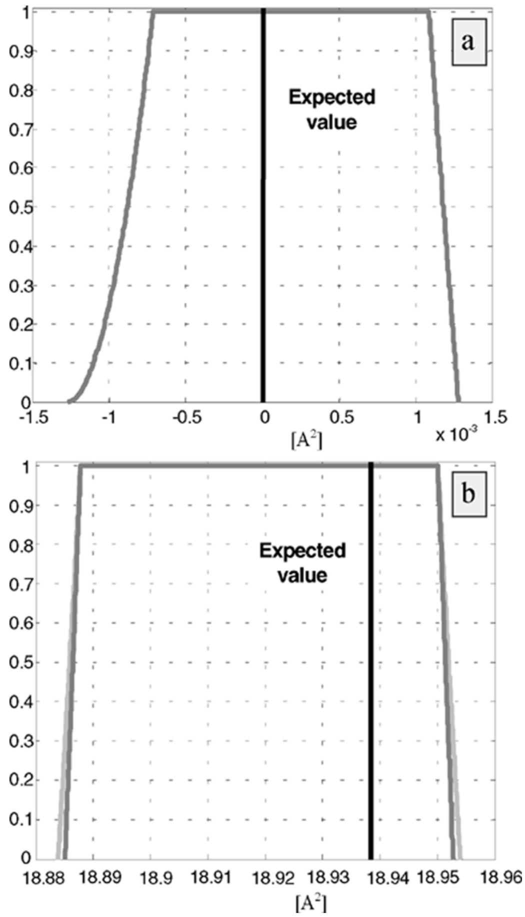

Fig. 8. (a) Nonlinearity and (b) linearity indices, in case of linear load, in terms of random fuzzy variables and expected values.

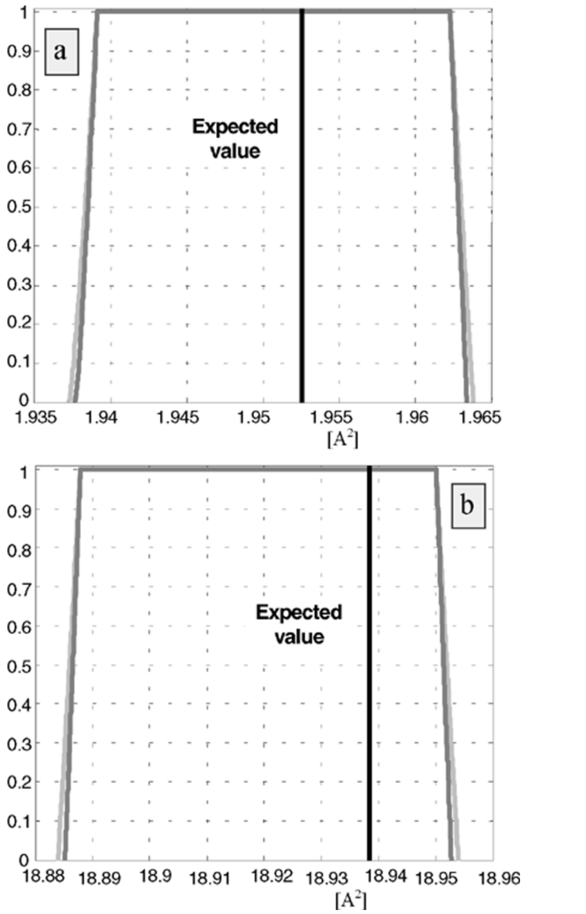

Fig. 9. (a) Nonlinearity and (b) linearity indices, in case of nonlinear load, in terms of random fuzzy variables and expected values.

In case of a nonlinear load, the results are given in Fig. 9. Since the *true/false* combination is the same as in the previous case, the credibility factors are the same, for both the linearity and nonlinearity indices.

In both figures, the expected values for the linearity and nonlinearity indices, evaluated in the absence of the measurement uncertainty, are also reported. The expected values are perfectly included within the obtained RFVs, thus proving the correctness of the proposed method.

The "nonlinearity" index in Fig. 8(a) requires further considerations. Since the "nonlinearity" index is given, by definition, by the square sum of current harmonic components, one should expect to obtain a nonnegative value. Despite this, the "nonlinearity index" in Fig. 8(a) is an RFV across zero. This is perfectly normal when the RFVs and the associated mathematics are considered. In fact, when an RFV is associated to a small measured quantity, it may happen that it falls across the zero value. Furthermore, due to the way the product between two RFVs is defined [6], [14], the square of an RFV across zero is still an RFV across zero. This explains why the amplitude of a small harmonic component can be represented by an RFV across zero. Since, in the considered example of a linear load, all the current harmonic components that contribute to the "nonlinearity" index are noise (they are theoretically zero), all of them are represented by RFV across the zero value. Therefore, it is quite normal to obtain a result, for the "nonlinearity" index, across the zero value.

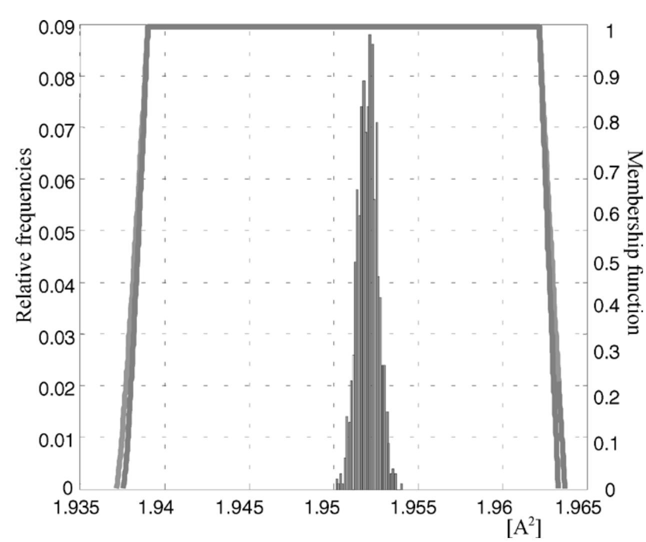

Fig. 10. Nonlinearity index, in case of nonlinear load, in terms of relative frequency histogram and RFV.

It is also interesting to compare these results, obtained with the RFV approach, with the results obtained with with other available methods.

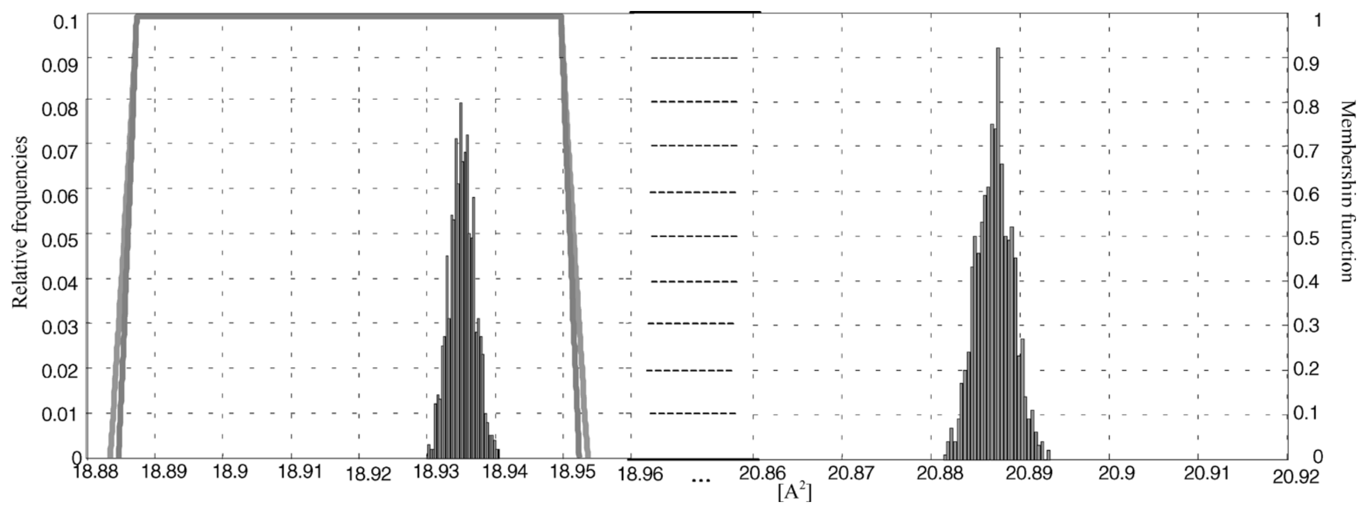

Fig. 11. Linearity index, in case of nonlinear load, in terms of relative frequency histogram and RFV.

The IEC-ISO guide [2] is obviously useless, since the considered structure is not derivable. The Monte Carlo method [15] is expected to solve this problem, but it too is not totally effective.

Let us consider the measurement system in Fig. 5 and the example of a nonlinear load already considered. If the uncertainty contributions introduced by the considered measurement system are expressed in terms of probability density functions, the Monte Carlo method can be employed in order to obtain the probability distributions of the results [13].

Figs. 10 and 11 show the "nonlinearity" and "linearity" indices, respectively, in terms of relative frequency histograms, obtained by means of the Monte Carlo method: Only 1000 iterations have been considered, since no significant improvements have been noticed with more iterations. In the same figures the previously obtained results in terms of RFVs are also reported, in order to easily compare the results obtained with the two different approaches. Fig. 10 shows that the relative frequency histogram is perfectly included into the RFV. On the other side, Fig. 11 shows, as far as the Monte Carlo approach is concerned, a division of the relative frequency histogram into two subhistograms. This division is due to the way the Monte Carlo method works: in fact, when the seventh voltage harmonic component is considered, because of the associated measurement uncertainty, in the different iterations, the extracted value falls sometimes below and sometimes above the threshold, so that the correspondent current is classified sometimes as "nonlinear" and sometimes as "linear." Therefore, a unique decision is not taken, thus giving rise to the double peak of the distribution of the linearity index, where one distribution comes from the correct decision and the other one comes from the wrong decision. Of course, the correct result is the single distribution under the obtained RFV, thus proving that the approach in terms of RFVs is more effective.

IV. Conclusion

In this paper, the decision rules have been defined in a measurement process, based on the representation of the measurement results in terms of RFVs. This approach allows the estimation of the measurement uncertainty also when the measurement process contains *if…then…else* structures.

The proposed method provides good results, generally better than those obtained by using other approaches, such as the Monte Carlo method, as shown by the considered example.

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