(1) Let $f : A \to B$ be a ring homomorphism and let $S \subseteq A$ be a multiplicative subset. Define $T = f(S)$. Let M be a B-module. Construct an $S^{-1}A$ linear isomorphism between $S^{-1}M$ (where M is considered an A-module via f) and $T^{-1}B$ (considered an $S^{-1}A$-module via the map $S^{-1}A \to T^{-1}B$ given by $a/s \mapsto f(a)/f(s)$)

Solution. Define $\Phi : S^{-1}M \to T^{-1}M$ by $m/s \mapsto m/f(s)$.

First, we show this map is well-defined. Suppose m/s and m'/s' are equivalent elements of $S^{-1}M$, so that there exists some $x \in S$ such that

$$x \cdot (s' \cdot m - s \cdot m') = f(x)(f(s')m - f(s)m') = 0.$$

The right-hand equality gives that the fractions $\Phi(m/s) = m/f(s)$ and $\Phi(m'/s') = m'/f(s')$ are equivalent in $T^{-1}M$ via the element $f(x) \in T$.

Next, we show that Φ is $S^{-1}A$-linear. Indeed, given $m/s, m'/s' \in S^{-1}M$ and $a/x \in S^{-1}A$, we have that:

$$\begin{aligned}
\Phi(m/s + m'/s') &= \Phi\left(\frac{s' \cdot m + s \cdot m'}{ss'}\right) \\
&= \Phi\left(\frac{f(s')m + f(s)m'}{ss'}\right) \\
&= \frac{f(s')m + f(s)m'}{f(ss')} \\
&= \frac{f(s')m + f(s)m'}{f(s)f(s')} \\
&= \frac{m}{f(s)} + \frac{m'}{f(s')} \\
&= \Phi(m/s) + \Phi(m'/s'),
\end{aligned}$$

and

$$\Phi\left(\frac{a}{x} \cdot \frac{m}{s}\right) = \Phi\left(\frac{f(a)m}{xs}\right) = \frac{f(a)m}{f(xs)} = \frac{f(a)m}{f(x)f(s)} = \frac{a}{x} \cdot \frac{m}{f(s)} = \frac{a}{x} \cdot \Phi\left(\frac{m}{s}\right).$$

Next, we show that Φ is injective. Suppose $\Phi(m/s) = \Phi(m'/s')$. Then there exists $f(x) \in T$ such that

$$x \cdot (f(s')m - f(s)m') = f(x)(f(s')m - f(s)m') = x \cdot (s' \cdot m - s \cdot m') = 0,$$

so that s/m must have been equal to s'/m' in the first place.

Finally, we show that Φ is surjective. Let $m/f(s)$ in $T^{-1}M$. Then clearly we have that $\Phi(m/s) = m/f(s)$.

(2) Let M be an A-module. Define the **support** of M to be

$$\operatorname{Supp} M := \{\mathfrak{p} \in \operatorname{Spec} A \mid M_{\mathfrak{p}} \neq 0\}.$$

(a) Show that $\operatorname{Supp} M \subseteq V(\operatorname{Ann} M)$ (recall that $\operatorname{Ann} M := \{x \in A \mid xm = 0 \text{ for all } m \in M\}$ and here we use the notation from §1.4), and that equality holds if M is finitely generated.

Proof. In order to show $\operatorname{Supp} M \subseteq V(\operatorname{Ann} M)$, it suffices to show that for any prime ideal $\mathfrak{p} \in \operatorname{Spec} A$ such that $M_\mathfrak{p} \neq 0$, $\operatorname{Ann} M \subseteq \mathfrak{p}$. It further suffices to show the contrapositive, namely, that if $\operatorname{Ann} M \not\subseteq \mathfrak{p}$, then $M_\mathfrak{p} = 0$. Indeed, given such a $\mathfrak{p}$, we have that there exists $x \in \operatorname{Ann} M \cap (A - \mathfrak{p})$, so that given any m/s in $M_\mathfrak{p}$, we have that $m/s = 0/1$, as

$$x(1m - s0) = xm = 0,$$

so that indeed $M_\mathfrak{p} = 0$.

Now, we assume that M is finitely generated and show the converse, namely, that if $M_\mathfrak{p} = 0$ then $\operatorname{Ann} M \not\subseteq \mathfrak{p}$. Let $m_1, \ldots, m_n$ generate M. Then since $M_\mathfrak{p} = 0$, $m_i/1 = 0/1$ for each $i = 1, \ldots, n$, so that for each i there exists $x_i \in A - \mathfrak{p}$ such that

$$x_i(1m_i - 1 \cdot 0) = x_i m_i = 0.$$

Then since $A - \mathfrak{p}$ is multiplicatively closed, we have that $x := x_1 \cdots x_n \in A - \mathfrak{p}$. Clearly $xm_i = 0$ for all i. Furthermore, $x \in \operatorname{Ann} M$, as given any $m \in M$, we have that

$$m = \sum_{i=1}^{n} a_i m_i$$

for some $a_1, \ldots, a_n \in A$, so that

$$xm = \sum_{i-1}^{n} a_i x m_i = \sum_{i=1}^{n} a_i 0 = 0.$$ □

(b) Let N be another A-module. Show that $\operatorname{Supp}(M \otimes_A N) \subseteq \operatorname{Supp} M \cap \operatorname{Supp} N$, and that equality holds if M and N are finitely generated (Exercise 2.3 of Atiyah-Macdonald may be helpful here).

Proof. In order to show $\operatorname{Supp}(M \otimes_A N) \subseteq \operatorname{Supp} M \cap \operatorname{Supp} N$, it suffices to show that if $\mathfrak{p} \in \operatorname{Spec} A$ then $(M \otimes_A N)_\mathfrak{p} \neq 0 \implies M_\mathfrak{p} \neq 0$ and $N_\mathfrak{p} \neq 0$. It further suffices to show the contrapositive, namely, that $M_\mathfrak{p} = 0$ or $N_\mathfrak{p} = 0 \implies (M \otimes_A N)_\mathfrak{p} = 0$. Indeed, this follows by Proposition 2.2.5, which gives that $(M \otimes_A N)_\mathfrak{p} = M_\mathfrak{p} \otimes_{A_\mathfrak{p}} N_\mathfrak{p}$.

Now, suppose that M and N are finitely generated. We wish to show that if $(M \otimes_A N)_\mathfrak{p} = M_\mathfrak{p} \otimes_{A_\mathfrak{p}} N_\mathfrak{p}$ is zero, then at least one of $M_\mathfrak{p}$ or $N_\mathfrak{p}$ are. This follows by A&M Exercise 2.3, as $A_\mathfrak{p}$ is a local ring. □

(3) Let M be an A-module and let $S \subset A$ be a multiplicative set. Show that the natural $M \to S^{-1}M$ given by $m \mapsto m/1$ is a bijection if and only if, for all $x \in S$, the multiplication map $m \mapsto xm$ is an isomorphism of M.

Proof. Let $\varphi : M \to S^{-1}M$ denote the canonical map and let $t_x : M \to M$ denote the multiplication-by-x map.

First, suppose that φ is a bijection. Let $x \in S$. Then first we claim t_x is injective. It suffices to show that $\ker t_x = 0$. Suppose for the sake of a contradiction that there existed some nonzero $m \in M$ such that $xm = 0$. Then we have

$$x(1m - 1 \cdot 0) = 0,$$

so that $\varphi(m) = m/1 = 0/1 = \varphi(0)$ in $S^{-1}M$, a contradiction of the fact that φ is a bijection.

Next, we claim that t_x is surjective. Let $m \in M$. Since φ is surjective, there exists n in M such that $m/x = \varphi(n) = n/1$. In particular, this means there exists some $y \in S$ such that

$$y(1m - xn) = t_y(1m - xn) = 0 \implies 1m - xn = 0 \implies m = xn = t_x(n),$$

where the first implication follows by the fact that t_y is injective. Therefore m is indeed in the image of t_x, t_x is surjective.

Conversely, suppose that t_x is an isomorphism for all $x \in S$. First, we show φ is surjective. Let m/s in $S^{-1}M$. Since t_s is a bijection, there exists $n \in M$ such that $t_s(n) = sn = m$. Then we have $m/s = n/1 = \varphi(n)$, as

$$1(m - sn) = 1 \cdot 0 = 0.$$

Hence, indeed φ is surjective.

Finally, we show that φ is injective. Let $n, m \in M$ such that $\varphi(n) = \varphi(m)$, so that there exists $x \in S$ such that

$$x(n - m) = 0 \implies xn = xm \implies t_x(n) = t_x(m) \implies n = m,$$

where the last implication follows as t_x is injective. □

(4) Let A be a ring and let $0 \to X \to Y \to Z \to 0$ be a chain complex of A-modules. Show that the following are equivalent:

(a) $0 \to X \to Y \to Z \to 0$ is exact.

(b) $0 \to X_\mathfrak{p} \to Y_\mathfrak{p} \to Z_\mathfrak{p} \to 0$ is exact for all prime ideals $\mathfrak{p} \leq A$.

(c) $0 \to X_\mathfrak{m} \to Y_\mathfrak{m} \to Z_\mathfrak{m} \to 0$ is exact for all maximal ideals $\mathfrak{m} \leq A$.

Proof. (a) implies (b) by Proposition 2.2.1. Clearly (b) implies (c), as every maximal ideal is prime. It therefore suffices to show that (c) implies (a).

First, we show that given a A-linear map $f : M \to N$ between A-modules, that $\ker f_\mathfrak{p} = (\ker f)_\mathfrak{p}$ and $\operatorname{im} f_\mathfrak{p} = (\operatorname{im} f)_\mathfrak{p}$ for all prime ideals $\mathfrak{p} \leq A$. Indeed, we have that

$$0 \to \ker f \hookrightarrow M \xrightarrow{f} N$$

is an exact sequence, so that

$$0 \to (\ker f)_\mathfrak{p} \hookrightarrow M_\mathfrak{p} \xrightarrow{f_\mathfrak{p}} N_\mathfrak{p}$$

is likewise exact, giving that $\ker f_\mathfrak{p} = (\ker f)_\mathfrak{p}$. Similarly, exactness of

$$M \xrightarrow{f} N \xrightarrow{\pi} \operatorname{coker} f \to 0$$

(so that $\ker \pi = \operatorname{im} f$) gives that

$$M_\mathfrak{p} \xrightarrow{f_\mathfrak{p}} N_\mathfrak{p} \xrightarrow{\pi_\mathfrak{p}} (\operatorname{coker} f)_\mathfrak{p} \to 0$$

is exact, so that $\operatorname{im} f_\mathfrak{p} = \ker \pi_\mathfrak{p} = (\ker \pi)_\mathfrak{p} = (\operatorname{im} f)_\mathfrak{p}$.

Now, suppose that that we have a sequence

$$X \xrightarrow{g} Y \xrightarrow{f} Z \tag{1}$$

such that
$$X_{\mathfrak{m}} \xrightarrow{g_{\mathfrak{m}}} Y_{\mathfrak{m}} \xrightarrow{f_{\mathfrak{m}}} Z_{\mathfrak{m}}$$
is exact for all maximal ideals $\mathfrak{m} \leq A$. Then we have
$$\frac{\ker f_{\mathfrak{m}}}{\operatorname{im} g_{\mathfrak{m}}} = \frac{(\ker f)_{\mathfrak{m}}}{(\operatorname{im} g)_{\mathfrak{m}}} \overset{(*)}{\cong} \left(\frac{\ker f}{\operatorname{im} g}\right)_{\mathfrak{m}} = 0$$
for all maximal ideals $\mathfrak{m} \leq A$, where $(*)$ follows by Corollary 2.2.2(3). Thus by Proposition 2.3.1(3), we have that $\ker f / \operatorname{im} g = 0$, so that indeed the sequence in Equation 1 is exact. □

(5) Let M be an A-module. Suppose that for each maximal ideal $\mathfrak{m}$ of A, there exists $f \notin \mathfrak{m}$ such that M_f is a finitely generated A_f-module. Pick one such element with this property and call it $f_{\mathfrak{m}}$.

(a) Show that there is a finite subset $\{f_1, \ldots, f_r\}$ of $\{f_{\mathfrak{m}}\}$ that generates the unit ideal.

Proof. Let I denote the ideal generated by the $f_{\mathfrak{m}}$ for all $\mathfrak{m} \in \operatorname{MaxSpec} A$. Then I is not contained in any maximal ideal $\mathfrak{m}$, as $f_{\mathfrak{m}} \in I - \mathfrak{m}$. Hence, I is necessarily the unit ideal, so that $1 \in I$. The desired result follows. □

(b) Use the generators for $M_{f_1}, \ldots, M_{f_r}$ to get a finite generating set for M.

Proof. Let $m_{i,1}/f_i^{e_{i,1}}, m_{i,2}/f_i^{e_{i,2}}, \ldots, m_{i,n}/f_i^{e_{i,n}}$ generate M_i. We claim
$$\{m_{i,j} \mid 1 \leq i \leq r, 1 \leq j \leq n\}$$
generates M.
Let $x \in M$. For $i = 1, \ldots, r$, since M_{f_i} is finitely generated as an A_{f_i}-module, there exists $a_{i,j}/f_i^{k_{i,j}} \in A_{f_i}$ for $j = 1, \ldots, n$ such that
$$\frac{x}{1} = \sum_{j=1}^{n} \frac{a_{i,j}}{f_i^{k_{i,j}}} \cdot \frac{m_{i,j}}{f_i^{e_{i,j}}}$$
so that for $i = 1, \ldots, r$ there exists $\ell_i \in \mathbb{N}$ and $A_{i,j} \in A$ for $j = 1, \ldots, n$ such that
$$f_i^{\ell_i} x = \sum_{j=1}^{n} A_{i,j} m_{i,j}.$$
Then because there exists $b_1, \ldots, b_r \in A$ such that
$$\sum_{i=1}^{r} b_i f_i = 1,$$
so we have that
$$x = x\left(\sum_{i=1}^{r} b_i f_i\right)^{\ell_1 + \cdots + \ell_r}$$
can be written as an A-linear combination of $m_{i,j}$'s. □

(6) Let A be a ring with multiplicative set $S \subset A$. Let M, N be A-modules and assume that M is finitely presented. Construct an $S^{-1}A$-linear isomorphism between $\operatorname{Hom}_{S^{-1}A}(S^{-1}M, S^{-1}N)$ and $S^{-1}\operatorname{Hom}_A(M, N)$.

Proof. Fix an A-module N. Given an A-module M, define

$$\begin{aligned}\Phi_M : S^{-1}\mathrm{Hom}_A(M,N) &\to \mathrm{Hom}_{S^{-1}A}(S^{-1}M, S^{-1}N)\\ \frac{f}{s} &\mapsto \left(\frac{m}{t} \mapsto \frac{f(m)}{st}\right).\end{aligned}$$

First, we show that given $f/s \in S^{-1}\mathrm{Hom}_A(M,N)$ that $\Phi_M(f/s)$ is a well-defined $S^{-1}A$-linear map. First, well-definedness. Suppose $m/t = m'/t'$, so that there exists $x \in S$ with

$$xt'm = xtm'.$$

Then we have by A-linearity of f that

$$x(stf(m') - st'f(m)) = sf(xtm' - xt'm) = sf(0) = 0,$$

so that indeed

$$\Phi_M\left(\frac{f}{s}\right)\left(\frac{m}{t}\right) = \frac{f(m)}{st} = \frac{f(m')}{st'} = \Phi_M\left(\frac{f}{s}\right)\left(\frac{m'}{t'}\right).$$

Next, we show that $\Phi_M(f/s)$ is $S^{-1}A$-linear. Let $m, m' \in M$, $s', t, t' \in S$, and $a \in A$. Then

$$\begin{aligned}\Phi_M\left(\frac{f}{s}\right)\left(\frac{m}{t} + \frac{a}{s'}\cdot\frac{m'}{t'}\right) &= \Phi_M\left(\frac{f}{s}\right)\left(\frac{s't'm + atm'}{tt's'}\right)\\ &= \frac{f(s't'm + atm')}{tt'ss'}\\ &= \frac{s't'f(m)}{tt'ss'} + \frac{atf(m')}{tt'ss'}\\ &= \frac{f(m)}{st} + \frac{a}{s'}\cdot\frac{f(m')}{st'}\\ &= \Phi_M\left(\frac{f}{s}\right)\left(\frac{m}{t}\right) + \frac{a}{s'}\cdot\Phi_M\left(\frac{f}{s}\right)\left(\frac{m'}{t'}\right).\end{aligned}$$

Next, we show that Φ_M itself is $S^{-1}A$-linear. Let $f, g \in \mathrm{Hom}_A(M,N)$, $s, s', t, t' \in S$, and $a \in A$. Then given any $m/t' \in S^{-1}M$, we have:

$$\begin{aligned}\Phi_M\left(\frac{f}{s} + \frac{a}{t}\cdot\frac{g}{s'}\right)\left(\frac{m}{t'}\right) &= \Phi_M\left(\frac{s'tf + asg}{ss't}\right)\left(\frac{m}{t'}\right)\\ &= \frac{s'tf(m) + asg(m)}{ss'tt'}\\ &= \frac{f(m)}{st'} + \frac{a}{t}\cdot\frac{g(m)}{s't'}\\ &= \Phi_M\left(\frac{f}{s}\right)\left(\frac{m}{t'}\right) + \frac{a}{t}\cdot\Phi_M\left(\frac{g}{s'}\right)\left(\frac{m}{t'}\right).\end{aligned}$$

Therefore, indeed Φ_M is a well-defined $S^{-1}A$-linear map. Finally, we claim that Φ_M is natural in M. Let $\varphi : L \to M$ be an A-linear map. Then we claim the following diagram commutes:

$$\begin{tikzcd}
S^{-1}\mathrm{Hom}_A(M,N) \arrow[r, "S^{-1}\mathrm{Hom}(\varphi,N)"] \arrow[d, "\Phi_M"'] & S^{-1}\mathrm{Hom}_A(L,N) \arrow[d, "\Phi_L"] \\
\mathrm{Hom}_{S^{-1}A}(S^{-1}M, S^{-1}N) \arrow[r, "\mathrm{Hom}_{S^{-1}A}(S^{-1}\varphi, S^{-1}N)"'] & \mathrm{Hom}_{S^{-1}A}(S^{-1}L, S^{-1}N)
\end{tikzcd}$$

Indeed, chasing an element f/s around the diagram yields

$$\begin{tikzcd}
\frac{f}{s} \arrow[r, mapsto] \arrow[d, mapsto] & \frac{f\circ\varphi}{s} \arrow[d, mapsto] \\
\left(\frac{m}{t} \mapsto \frac{f(m)}{st}\right) \arrow[r, mapsto] & \left(\frac{\ell}{t} \mapsto \frac{f(\varphi(m))}{st}\right).
\end{tikzcd}$$

It remains to show that if M is finitely presented then Φ_M is an isomorphism. If M is finitely presented, there exists a finite free presentation

$$F \xrightarrow{\alpha} G \xrightarrow{\beta} M \longrightarrow 0.$$

We can apply the natural transformation Φ to this diagram, and since $S^{-1}\mathrm{Hom}_A(-,N)$ and $\mathrm{Hom}_{S^{-1}A}(S^{-1}-,N)$ are left-exact contravariant functors (since $S^{-1}-$ and $\mathrm{Hom}(-,N)$ are left-exact), each row in the following diagram is exact.

$$\begin{tikzcd}
0 \arrow[r] & S^{-1}\mathrm{Hom}_A(M,N) \arrow[r] \arrow[d, "\Phi_M"] & S^{-1}\mathrm{Hom}_A(G,N) \arrow[r] \arrow[d, "\Phi_G"] & S^{-1}\mathrm{Hom}_A(F,N) \arrow[d, "\Phi_F"] \\
0 \arrow[r] & \mathrm{Hom}_{S^{-1}A}(S^{-1}M, S^{-1}N) \arrow[r] & \mathrm{Hom}_{S^{-1}A}(S^{-1}G, S^{-1}N) \arrow[r] & \mathrm{Hom}_{S^{-1}A}(S^{-1}F, S^{-1}N)
\end{tikzcd}$$

By naturality of Φ, this diagram commutes. Hence, in order to show that Φ_M is an isomorpism, by the five lemma it suffices to show that Φ_G and Φ_F are. This follows by the fact that F and G are free A-modules and the fact that the following diagram commutes for all $n \in \mathbb{N}$ (I have ommited the check that this diagram commutes, as well as the explicit descriptions of the isomorphisms involved, it is straightforward to check).

$$\begin{tikzcd}
S^{-1}\mathrm{Hom}_A(A^{\oplus n}, N) \arrow[r, "\Phi_{A^{\oplus n}}"] \arrow[d, "\cong"'] & \mathrm{Hom}_{S^{-1}A}(S^{-1}A^{\oplus n}, S^{-1}N) \arrow[d, "\cong"] \\
S^{-1}\left((\mathrm{Hom}_A(A,N))^{\oplus n}\right) \arrow[d, "\cong"'] & \mathrm{Hom}_{S^{-1}A}((S^{-1}A)^{\oplus n}, S^{-1}N) \arrow[d, "\cong"] \\
S^{-1}(N^{\oplus n}) \arrow[d, "\cong"'] & \left(\mathrm{Hom}_{S^{-1}A}(S^{-1}A, S^{-1}N)\right)^{\oplus n} \arrow[d, "\cong"] \\
(S^{-1}N)^{\oplus n} \arrow[r, equal] & (S^{-1}N)^{\oplus n}
\end{tikzcd}$$

Hence, Φ_G and Φ_F must be isomorphisms, so that Φ_M is as well. □

(7) (Atiyah-Macdonald, Exercise 3.12) Let A be an integral domain and M an A-module. An element $x \in M$ is a *torsion element* of M if $\mathrm{Ann}(x) \neq 0$, that is if x is killed by some nonzero element of A. Show that the torsion elements of M form a submodule of M.

Proof. Let $m, m' \in M$ be torsion elements of M annihilated by $x, x' \in A$, respectively. Then

$$xx'(m+m') = x'(xm) + x(x'm') = x'0 + x0 = 0.$$

Similarly, given $a \in A$, we have

$$x(am) = a(xm) = a0 = 0.$$ □

This submodule is called the *torsion submodule* of M and is denoted by $T(M)$. If $T(M) = 0$, the module M is said to be torsion-free. Show that

i) If M is any A-module, then $M/T(M)$ is torsion-free.

Proof. Let $\overline{m} \in M/T(M)$ such that there exists some nonzero $a \in \operatorname{Ann}(\overline{m})$. Then

$$\begin{aligned} a\overline{m} = \overline{0} &\implies \overline{am} = \overline{0} \\ &\implies am \in T(M) \\ &\implies \exists b \in A \mid bam = 0 \\ &\implies m \in T(M) \\ &\implies \overline{m} = \overline{0}, \end{aligned}$$

so that indeed $T(M/T(M)) = \overline{0}$. □

ii) If $f : M \to N$ is a module homomorphism, then $f(T(M)) \subseteq T(N)$.

Proof. Let $m \in T(M)$, so that there exists nonzero $a \in A$ with $am = 0$. Then

$$af(m) = f(am) = f(0) = 0,$$

so that $f(m) \in T(N)$. □

iii) If $0 \to M' \to M \to M''$ is an exact sequence, then the sequence $0 \to T(M') \to T(M) \to T(M'')$ is exact.

Proof. Label the sequence like so

$$0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M''.$$

We wish to show the following sequence is exact

$$0 \longrightarrow T(M') \xrightarrow{T(f)} T(M) \xrightarrow{T(g)} T(M'').$$

Before showing this sequence is exact, note the following useful properties: Let $\varphi : P \to Q$ be an A-linear map. Then $\ker T(\varphi) = \ker \varphi \cap T(P)$ and $\operatorname{im} T(\varphi) = \varphi(T(P))$. Now, we show the above sequence is exact. First, we know it is exact at $T(M')$ as

$$\ker T(f) = \ker f \cap T(M') = 0 \cap T(M') = 0,$$

so that $T(f)$ is indeed a monomorphism. Since $g \circ f = 0$, clearly $T(g) \circ T(f) = 0$, as $T(g)(T(f)(x)) = g(f(x))$ for all $x \in T(M')$, so that $\operatorname{im} T(f) \subseteq \ker T(g)$. Finally, we have that:

$$\ker T(g) = \ker g \cap T(M) = \operatorname{im} f \cap T(M) \supseteq \operatorname{im} T(f).$$ □

iv) If M is any A-module, then $T(M)$ is the kernel of the mapping $x \mapsto 1 \otimes x$ of M into $K \otimes_A M$, where K is the field of fractions of A.

Proof. Let $S := A - \{0\}$, then $K = S^{-1}A$ so that $K \otimes_A M \cong S^{-1}M$ via the map $a/s \otimes m \mapsto am/s$ (Proposition 2.2.3). It follows that it suffices to show that $T(M)$ is the kernel of the canonical map $\varphi : M \to S^{-1}M$. Indeed, we have that

$$\begin{aligned} m \in T(M) &\iff \exists a \in \operatorname{Ann}(M) - 0 \\ &\iff \exists a \in S \text{ s.t. } am = 0 \\ &\iff \exists a \in S \text{ s.t. } a(1 \cdot m - 1 \cdot 0) = 0 \\ &\iff \frac{m}{1} = \frac{0}{1} \\ &\iff m \in \ker \varphi, \end{aligned}$$

so indeed $\ker \varphi = T(M)$. $\square$

(8) (Atiyah-Macdonald, Exercise 3.13) Let S be a multiplicatively closed subset of an integral domain A. In the notation of Exercise 12, show that $T(S^{-1}M) = S^{-1}(TM)$.

Proof. First, suppose that $m/s \in T(S^{-1}M)$, so that $s \in S$ and there exists some nonzero $a \in A$ such that $a(m/s) = 0$. In particular, this means that there exists some $t \in S$ with $t(1 \cdot am - s \cdot 0) = 0$, i.e. $(ta)m = 0$, so that $m \in T(M)$, giving that $m/s \in S^{-1}(TM)$.

Conversely, suppose that $m/s \in S^{-1}(TM)$, so that $m \in T(M)$ and $s \in S$. Then because $m \in T(M)$, there exists some nonzero $a \in A$ with $am = 0$. In particular, this gives that $1(am - 0s) = 0$, and $1 \in S$, so that necessarily $a(m/s) = 0$, meaning $m/s \in T(S^{-1}M)$. $\square$

Deduce that the following are equivalent

i) M is torsion-free.

ii) $M_{\mathfrak{p}}$ is torsion-free for all prime ideals $\mathfrak{p}$.

iii) $M_{\mathfrak{m}}$ is torsion-free for all maximal ideals $\mathfrak{m}$.

First, we show (i) implies (ii). Given a prime ideal $\mathfrak{p} \leq A$, since M is torsion free, we have that $0 = 0_{\mathfrak{p}} = (TM)_{\mathfrak{p}} = T(M_{\mathfrak{p}})$ for all prime ideals, as localization preserves torsion.

Clearly (ii) implies (iii).

It remains to show that (iii) implies (i). By Proposition 2.3.1, it suffices to show that $(TM)_{\mathfrak{m}} = 0$ for all maximal ideals $\mathfrak{m} \leq A$. This follows by the fact that localization (faithfully) preserves torsion, as $T(M_{\mathfrak{m}}) = 0$ for all maximal ideals $\mathfrak{m} \leq A$.