

The primary reference for this section will be the nLab page on derived functors in homological algebra ([nlab:derived'functor'in'homological'algebra]).

Recall that given abelian categories \mathcal{A} and \mathcal{B} , given an additive functor $F : \mathcal{A} \rightarrow \mathcal{B}$, if F is left exact and \mathcal{A} has enough injectives, we may form the *right derived functors* $R^n F : \mathcal{A} \rightarrow \mathcal{B}$ of F , for $n \in \mathbb{N}$. Given an object A in \mathcal{A} , we may compute $R^n F(A)$ to be the object (defined only up to isomorphism) which is obtained as follows: First, fix an injective resolution $i : A \rightarrow I^*$ of A , i.e., the data of a long exact sequence

$$0 \longrightarrow A \xrightarrow{i} I^0 \xrightarrow{d^0} I^1 \xrightarrow{d^1} I^2 \xrightarrow{d^2} I^3 \longrightarrow \dots$$

where each I^n is an injective object in \mathcal{A} . Such a sequence is guaranteed to exist since \mathcal{A} has enough injectives. Then we define $R^n F(A)$ to be the n^{th} cohomology group $H^n(F(I^*))$ of the sequence

$$0 \longrightarrow F(I^0) \xrightarrow{F(d^0)} F(I^1) \xrightarrow{F(d^1)} F(I^2) \xrightarrow{F(d^2)} F(I^3) \longrightarrow \dots$$

It is a standard result that this definition of $R^n F(A)$ does not depend on the choice of injective resolution $i : A \rightarrow I^*$.

Definition 0.1. Given an abelian category \mathcal{A} with enough injectives and an object A in \mathcal{A} , we denote the right derived functors of the left exact functor $\text{Hom}_{\mathcal{A}}(A, -) : \mathcal{A} \rightarrow \mathbf{Ab}$ by

$$\text{Ext}_{\mathcal{A}}^n(A, -) := R^n \text{Hom}_{\mathcal{A}}(A, -).$$

Remark 0.2. It is not uncommon to instead define $\text{Ext}_{\mathcal{A}}^n(-, A)$ to be the right derived functor of the functor $\text{Hom}_{\mathcal{A}}(-, A) : \mathcal{A}^{\text{op}} \rightarrow \mathbf{Ab}$, in which case we may compute $\text{Ext}_{\mathcal{A}}^n(B, A)$ by means of *projective* resolutions of A in \mathcal{A} . It is a standard result that these definitions of $\text{Ext}_{\mathcal{A}}^n(A, B)$ coincide.

Now, the first result we will state is that in order to compute the values of the right derived functors $R^n F(A)$, we do not need to consider strictly injective resolutions of A , rather, we may consider more generally “ F -acyclic resolutions”. First, we define F -acyclic objects:

Definition 0.3 ([nlab:derived'functor'in'homological'algebra]). Let $F : \mathcal{A} \rightarrow \mathcal{B}$ be a left or right exact additive functor between abelian categories, and suppose \mathcal{A} has enough injectives. An object A in \mathcal{A} is called an *F -acyclic object* if $R^n F(A) = 0$ for all $n > 0$.

Definition 0.4. Let $F : \mathcal{A} \rightarrow \mathcal{B}$ be a left exact additive functor between abelian categories, and suppose \mathcal{A} has enough injectives. Then given an object A in \mathcal{A} , an *F -acyclic resolution* $i : A \rightarrow I_F^*$ is the data of a long exact sequence in \mathcal{A}

$$0 \longrightarrow A \xrightarrow{i} I_F^0 \xrightarrow{d^0} I_F^1 \xrightarrow{d^1} I_F^2 \xrightarrow{d^2} I_F^3 \longrightarrow \dots$$

such that each I_F^n is an F -acyclic object in \mathcal{A} .

The reason that F -acyclic objects are useful is that they allow you to compute the right derived functors of F without having to use strictly injective resolutions:

Proposition 0.5 ([nlab:derived'functor'in'homological'algebra]). Let $F : \mathcal{A} \rightarrow \mathcal{B}$ be a left exact additive functor between abelian categories. Then for each object A in \mathcal{A} , given an F -acyclic resolution $i : A \rightarrow I_F^*$ of A , for each $n \in \mathbb{N}$ there is a canonical isomorphism

$$R^n F(A) \cong H^n(F(I_F^*))$$

between the n^{th} right derived functor of F evaluated on A and the cohomology of the sequence obtained by applying F to I_F^* .