0.1. **Construction of the Adams spectral sequence.** The last thing we need before we can construct the Adams spectral sequence in $\mathcal{SH}$ is the following:

Proposition 0.1. *Suppose we are given a distinguished triangle*

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$$

and some object W in $\mathcal{SH}$. Then there is an infinite long exact sequence of A-graded abelian groups:

$$\cdots \to [W, Z]_{*+\mathbf{n+1}} \xrightarrow{\partial} [W, X]_{*+\mathbf{n}} \xrightarrow{f_*} [W, Y]_{*+\mathbf{n}} \xrightarrow{g_*} [W, Z]_{*+\mathbf{n}} \xrightarrow{\partial} [W, Z]_{*+\mathbf{n-1}} \to \cdots,$$

where $\partial : [W, Z]_{*+\mathbf{n+1}} \to [W, X]_{*+\mathbf{n}}$ sends a class $x : S^{a+\mathbf{n+1}} \otimes W \to Z$ to the composition

$$S^{a+\mathbf{n}} \otimes W \xrightarrow{\phi_{-\mathbf{1},a+\mathbf{n+1}}} S^{-\mathbf{1}} \otimes S^{a+\mathbf{n+1}} \otimes W \xrightarrow{S^{-\mathbf{1}}\otimes x} S^{-\mathbf{1}} \otimes Z \xrightarrow{\widetilde{h}} X,$$

*where here we are suppressing the associator from the notation, and $\widetilde{h} : \Omega Z = S^{-\mathbf{1}} \otimes Z \to X$ is the adjoint (**??**) of $h : Z \to \Sigma X$.*

Proof. In this proof, we will freely employ the coherence theorem for a symmetric monoidal category, which tells us we may assume associativity and unitality of $- \otimes -$ holds up to strict equality. Furthermore, we will simply write ϕ to refer to any isomorphism that can be constructed by composing copies of products of $\phi_{a,b}$'s, unitors, identities, associators, and their inverses (see **??**). Finally, given $n > 0$, we will write Σ^{-n} to denote the functor $\Omega^n = (S^{-\mathbf{1}})^n \otimes -$.

For all $n > 0$, the $\phi_{a,b}$'s yield natural isomorphisms

$$s_X^{-n} : \Sigma^{-n} X = (S^{-\mathbf{1}})^n \otimes X \xrightarrow{\phi\otimes X} S^{-\mathbf{n}} \otimes X = \Omega^{\mathbf{n}} X.$$

and

$$s_X^n : \Sigma^n X \xrightarrow{\nu_X^n} (S^{\mathbf{1}})^n \otimes X \xrightarrow{\phi\otimes X} S^{\mathbf{n}} \otimes X = \Sigma^{\mathbf{n}} X,$$

where we recursively define $\nu^1 := \nu$ and ν^{n+1} is given by the composition

$$\nu_X^{n+1} : \Sigma^{n+1} X = \Sigma^n \Sigma X \xrightarrow{\nu^n_{\Sigma X}} (S^{\mathbf{1}})^n \otimes \Sigma X \xrightarrow{(S^{\mathbf{1}})^n \otimes \nu_X} (S^{\mathbf{1}})^n \otimes S^{\mathbf{1}} \otimes X = (S^{\mathbf{1}})^{n+1} \otimes X.$$

Finally, we define s^0 to be the identity natural transformation on $\mathcal{SH}$. Then we get the following natural isomorphisms of A-graded abelian groups for all $n \in \mathbb{Z}$

$$\ell_V^n : [W, \Sigma^n V]_* \xrightarrow{(s_V^n)_*} [W, \Sigma^{\mathbf{n}} V]_* \xrightarrow{r^{\mathbf{n}}_{W,V}} [W, V]_{*-\mathbf{n}},$$

where $r^{\mathbf{n}}_{W,V}$ is the natural isomorphism given as the composition

$$[W, \Sigma^{\mathbf{n}} V]_* \xrightarrow{\cong} [S^{-\mathbf{n}} \otimes S^* \otimes W, V] \xrightarrow{(\phi\otimes W)^*} [S^{*-\mathbf{n}} \otimes W, V] = [W, V]_{*-\mathbf{n}},$$

where the first isomorphism is the adjunction $\Omega^{\mathbf{n}} \dashv \Sigma^{\mathbf{n}}$ (**??**). Now, given $n \in \mathbb{Z}$, consider the following diagram

$$\begin{tikzcd}
{[W, \Sigma^{n-1} Z]_*} \arrow[r, "h_{n-1}"] \arrow[d, "\ell_Z^{n-1}"'] & {[W, \Sigma^n X]_*} \arrow[r, "\Sigma^n f_*"] \arrow[d, "\ell_X^n"'] & {[W, \Sigma^n Y]_*} \arrow[r, "\Sigma^n g_*"] \arrow[d, "\ell_Y^n"'] & {[W, \Sigma^n Z]_*} \arrow[r, "h_n"] \arrow[d, "\ell_Z^n"'] & {[W, \Sigma^{n+1} X]_*} \arrow[d, "\ell_X^{n+1}"] \\
{[W, Z]_{*-\mathbf{n+1}}} \arrow[r, "\partial"'] & {[W, X]_{*-\mathbf{n}}} \arrow[r, "f_*"'] & {[W, Y]_{*-\mathbf{n}}} \arrow[r, "g_*"'] & {[W, Z]_{*-\mathbf{n}}} \arrow[r, "\partial"'] & {[W, X]_{*-\mathbf{n-1}}}
\end{tikzcd} \tag{1}$$

where for $n \geq 0$, $h_n = \Sigma^n h$, and for $n > 0$, $h_{-n} = \Omega^{n-1} \widetilde{h}$ (where $\widetilde{h} : \Omega Z \to X$ is the adjoint of $h : Z \to \Sigma X$). We would like to show the bottom row is exact. The top row is exact since it is obtained by applying $[W, -]_*$ to a fiber sequence (see **??** for full details), and we have constructed the vertical arrows to be isomorphisms. Thus it suffices to show each square commutes. The inner two squares commute by naturality of ℓ^n. Thus, it further suffices to show the outermost squares commute. Since our choice of $n \in \mathbb{Z}$ is arbitrary, we can just show the right square commutes. First consider the case that $n \geq 0$, and consider the following diagram:

$$\begin{tikzcd}
{[W, \Sigma^n Z]_*} \arrow[rrr, "\Sigma^n h_*"] \arrow[ddd, "\ell^n_Z"'] & & & {[W, \Sigma^{n+1} X]_*} \arrow[dll, "\ell^n_{\Sigma X}"] \arrow[ddd, "\ell^{n+1}_X"] \\
& {[W, \Sigma X]_{*-\mathbf{n}}} \arrow[dr, "(\nu_X)_*"] & & \\
& & {[W, \Sigma^{\mathbf{1}} X]_{*-\mathbf{n}}} \arrow[dr, "r^{\mathbf{1}}_{W,X}"] & \\
{[W, Z]_{*-\mathbf{n}}} \arrow[uur, "h_*"] \arrow[rrr, "\partial"'] & & & {[W, X]_{*-\mathbf{n}-\mathbf{1}}}
\end{tikzcd}$$

The leftmost region commutes by naturality of ℓ. By unravelling how $r^{\mathbf{1}}_{W,X}$ and the adjoint $\widetilde{h}$ used in the definition of ∂ are defined, a simple diagram chase yields that the bottom triangle commutes. Thus, it remains to show the rightmost triangle in the above diagram commutes. To see this, note that by unravelling how ℓ and r are defined, the rightmost square becomes

$$\begin{tikzcd}
{[W, (S^{\mathbf{1}})^n \otimes \Sigma X]} \arrow[ddr, "((S^{\mathbf{1}})^n \otimes \nu_X)_*" near start, shift left=0] \arrow[dd, "(\phi \otimes \Sigma X)_*"'] & & {[W, \Sigma^{n+1} X]_*} \arrow[ll, "(\nu^n_{\Sigma X})_*"'] \arrow[d, "(\nu^{n+1}_X)_*"] \\
& & {[W, (S^{\mathbf{1}})^{n+1} \otimes X]_*} \arrow[dl, "(\phi \otimes X)_*"'] \arrow[d, "(\phi \otimes X)_*"] \\
{[W, \Sigma^{\mathbf{n}} \Sigma X]_*} \arrow[r, "(\Sigma^{\mathbf{n}} \nu_X)_*"] \arrow[d, "\mathrm{adj}"'] & {[W, \Sigma^{\mathbf{n}} \Sigma^{\mathbf{1}} X]_*} \arrow[r, "(\phi \otimes X)_*"'] \arrow[d, "\mathrm{adj}"] & {[W, \Sigma^{\mathbf{n+1}} X]_*} \arrow[d, "\mathrm{adj}"] \\
{[S^{-\mathbf{n}} \otimes S^* \otimes W, \Sigma X]} \arrow[r, "(\nu_X)_*"] \arrow[d, "(\phi \otimes W)^*"'] & {[S^{-\mathbf{n}} \otimes S^* \otimes W, S^{\mathbf{1}} \otimes X]} \arrow[ddl, "(\phi \otimes W)^*"] & {[S^{-\mathbf{n}-\mathbf{1}} \otimes S^* \otimes W, X]} \arrow[dd, "(\phi \otimes W)^*"] \\
{[W, \Sigma X]_{*-\mathbf{n}}} \arrow[d, "(\nu_X)_*"'] & & \\
{[W, \Sigma^{\mathbf{1}} X]_{*-\mathbf{n}}} \arrow[r, "\mathrm{adj}"'] & {[S^{-\mathbf{1}} \otimes S^{*-\mathbf{n}} \otimes W, X]} \arrow[uur, "(\phi \otimes W)^*"] \arrow[r, "(\phi \otimes W)^*"'] & {[W, X]_{*-\mathbf{n}-\mathbf{1}}}
\end{tikzcd}$$

The top right triangle commutes by how ν^{n+1} was defined. The top left oddly-shaped region commutes by functoriality of $- \otimes -$. The middle right triangle commutes by coherence for the ϕ's. The middle left rectangle commutes by naturality of the adjunction isomorphism. Commutativity of the bottom left triangle is clear (do a diagram chase). Commutativity of the bottom right triangle is coherence for the ϕ's. Finally, commutativity of the remaining region is again coherence of the ϕ's, since the adjunction isomorphism are constructed using them (**??**).

Now we consider the negative case: Unravelling definitions, given $n > 0$, the rightmost square in diagram (1) for $-n$ becomes

$$\begin{tikzcd}
{[W,\Omega^{n}Z]_*} \arrow[rr, "\Omega^{n-1}\widetilde{h}_*"] \arrow[d, "(\phi\otimes Z)_*"'] \arrow[dr, "(\phi\otimes\Omega Z)_*"] & & {[W,\Omega^{n-1}X]_*} \arrow[d, "(\phi\otimes X)_*"] \\
{[W,\Omega^{\mathbf{n}}Z]_*} \arrow[r, "(\phi\otimes Z)_*"'] \arrow[dddd, "\mathrm{adj}"'] & {[W,\Omega^{\mathbf{n-1}}\Omega Z]_*} \arrow[r, "\Omega^{\mathbf{n-1}}\widetilde{h}_*"] \arrow[d, "\mathrm{adj}"'] & {[W,\Omega^{\mathbf{n-1}}X]_*} \arrow[d, "\mathrm{adj}"] \\
 & {[S^{\mathbf{n-1}}\otimes S^*\otimes W,\Omega Z]} \arrow[r, "\widetilde{h}_*"] \arrow[d, "(\phi\otimes W)^*"'] & {[S^{\mathbf{n-1}}\otimes S^*\otimes W,X]} \arrow[ddd, "(\phi\otimes W)^*"] \\
 & {[W,\Omega Z]_{*+\mathbf{n-1}}} \arrow[d, "\mathrm{adj}"'] \arrow[ddr, "\widetilde{h}_*"] & \\
{[S^{\mathbf{n}}\otimes S^*\otimes W,Z]} \arrow[r, "(\phi\otimes W)^*"] \arrow[d, "(\phi\otimes W)^*"'] & {[S^{\mathbf{1}}\otimes S^{*+\mathbf{n-1}}\otimes W,Z]} \arrow[dl, "(\phi\otimes W)^*"] & \\
{[W,Z]_{*+\mathbf{n}}} \arrow[rr, "\partial"'] & & {[W,X]_{*+\mathbf{n-1}}}
\end{tikzcd} \tag{2}$$

The top right trapezoid commutes by functoriality of $-\otimes-$. The top left triangle commutes by coherence for the ϕ's. The middle right rectangle commutes by naturality of the adjunction. The right trapezoid below that commutes obviously. The bottom left triangle commutes by coherence of the ϕ's. The large middle left rectangle commutes by coherence for the ϕ's, again since the adjunction $\Sigma^{\mathbf{n}} \dashv \Omega^{\mathbf{n}}$ is constructed using the ϕ's. Finally, to see the bottom diagram commutes, we will chase some homogeneous element $f : S^{b+\mathbf{n-1}} \otimes W \to \Omega Z$ around the region. Consider the following diagram:

$$\begin{tikzcd}
S^{\mathbf{-1}}\otimes S^{b+\mathbf{n}}\otimes W \arrow[d, "\phi\otimes W"'] & S^{b+\mathbf{n-1}}\otimes W \arrow[l, "\phi\otimes W"'] \arrow[dd, "f"] & & & \\
S^{\mathbf{-1}}\otimes S^{\mathbf{1}}\otimes S^{b+\mathbf{n+1}}\otimes W \arrow[ur, "\phi\otimes W"] \arrow[d, "S^{\mathbf{-1}}\otimes S^{\mathbf{1}}\otimes f"'] & & & & \\
S^{\mathbf{-1}}\otimes S^{\mathbf{1}}\otimes S^{\mathbf{-1}}\otimes Z \arrow[r, "\phi\otimes Z"'] & S^{-1}\otimes Z \arrow[r, "S^{\mathbf{-1}}\otimes h"] & S^{\mathbf{-1}}\otimes \Sigma X \arrow[r, "S^{\mathbf{-1}}\otimes\nu_X"] & S^{\mathbf{-1}}\otimes S^{\mathbf{1}}\otimes X \arrow[r, "\phi\otimes X"] & X
\end{tikzcd}$$

By unravelling how the adjunction and $\widetilde{h}$ are defined, the two compositions around the outside of this diagram are the two morphisms obtained by chasing f around the bottom region in diagram (2). The top left triangle commutes by coherence of the ϕ's (**??**), while the bottom region commutes by functoriality of $-\otimes-$ and coherence of the ϕ's. Thus we've shown diagram (1) commutes, so the bottom row is exact, as desired. $\square$

Remark 0.2. Expressed more compactly, the above proposition says that for each object W and distinguished triangle

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$$

in $\mathcal{SH}$ gives rise to the following diagram of A-graded abelian groups

$$\begin{tikzcd}
{[W,X]_*} \arrow[r, "f_*"] & {[W,Y]_*} \arrow[d, "g_*"] \\
 & {[W,Z]_*} \arrow[ul, "\partial"]
\end{tikzcd}$$

which is exact at each vertex, and where f_*, g_*, and ∂ are A-graded homomorphisms of degree 0, 0, and $-\mathbf{1}$, respectively. Explicitly, ∂ sends a class $x : S^a \otimes W \to Z$ to the composition

$$S^{a-\mathbf{1}} \otimes W \cong S^{-\mathbf{1}} \otimes S^a \otimes W \xrightarrow{S^{-\mathbf{1}}\otimes x} S^{-\mathbf{1}} \otimes Z \xrightarrow{S^{-\mathbf{1}}\otimes h} S^{-\mathbf{1}} \otimes \Sigma X \xrightarrow{S^{-\mathbf{1}}\otimes \nu_X} S^{-\mathbf{1}} \otimes S^{\mathbf{1}} \otimes X \xrightarrow{\phi^{-1}_{-\mathbf{1},\mathbf{1}}\otimes X} X.$$

In what follows, let (E, μ, e) be a commutative monoid object in $\mathcal{SH}$. In this section we will freely use the coherence theorem for symmetric monoidal categories without comment, in particular, we will assume unitality and associativity hold up to strict equality.

Definition 0.3. Let $\overline{E}$ be the fiber of the unit map $e : S \to E$ (**??**). Let $Y_0 := Y$ and $W_0 := E \otimes Y$. Then for $s > 0$, define

$$Y_s := \overline{E}^s \otimes Y, \qquad W_s := E \otimes Y_s = E \otimes \overline{E}^s \otimes Y,$$

where $\overline{E}^s$ denotes the s-fold tensor product $\overline{E} \otimes \cdots \otimes \overline{E}$. Then we get fiber sequences

$$Y_{s+1} \xrightarrow{i_s} Y_s \xrightarrow{j_s} W_s \xrightarrow{k_s} \Sigma Y_{s+1}$$

obtained by applying $- \otimes Y_s$ to the fiber sequence

$$\overline{E} \to S \xrightarrow{e} E \to \Sigma \overline{E}.$$

We can splice these sequences together to get the *(canonical) Adams filtration of* Y:

$$\begin{tikzcd}
\cdots \arrow[r] & Y_3 \arrow[r, "i_2"] \arrow[d, "j_3"] & Y_2 \arrow[r, "i_1"] \arrow[d, "j_2"] & Y_1 \arrow[r, "i_0"] \arrow[d, "j_1"] & Y_0 = Y \arrow[d, "j_0"] \\
& W_3 & W_2 \arrow[ul, dashed, "k_2"] & W_1 \arrow[ul, dashed, "k_1"] & W_0 \arrow[ul, dashed, "k_0"]
\end{tikzcd}$$

where here each k_s is of degree $-\mathbf{1}$ (in particular, the above diagram does not commute in any sense), and each i_s and j_s have degree 0. We can extend this diagram to the right by setting $Y_s = Y$, $W_s = 0$, and $i_s = \mathrm{id}_Y$ for $s < 0$. Then we may apply the functor $[X, -]_*$, and by Remark 0.2, we obtain the following A-graded unrolled exact couple (**??**):

$$\begin{tikzcd}
\cdots \arrow[r] & {[X, Y_{s+2}]_*} \arrow[r, "i_{s+1}"] \arrow[d, "j_{s+2}"] & {[X, Y_{s+1}]_*} \arrow[r, "i_s"] \arrow[d, "j_{s+1}"] & {[X, Y_s]_*} \arrow[r, "i_{s-1}"] \arrow[d, "j_s"] & {[X, Y_{s-1}]_*} \arrow[r] \arrow[d, "j_{s-1}"] & \cdots \\
& {[X, W_{s+2}]_*} & {[X, W_{s+1}]_*} \arrow[ul, "\partial_{s+1}"] & {[X, W_s]_*} \arrow[ul, "\partial_s"] & {[X, W_{s-1}]_*} \arrow[ul, "\partial_{s-1}"] &
\end{tikzcd}$$

where here we are being abusive and writing $i_s : [X, Y_{s+1}]_* \to [X, Y_s]_*$ and $j_s : [X, Y_s]_* \to [X, W_s]_*$ to denote the pushforwards of $i_s : Y_{s+1} \to Y_s$ and $j_s : Y_s \to W_s$, respectively. Each i_s, j_s, and ∂_s are A-graded homomorphisms of degrees 0, 0, and $-\mathbf{1}$, respectively.

By **??**, we may associate a $\mathbb{Z} \times A$-graded spectral sequence $r \mapsto (E_r^{*,*}(X, Y), d_r)$ to the above A-graded unrolled exact couple, where d_r has $\mathbb{Z} \times A$-degree $(r, -\mathbf{1})$. We call this spectral sequence the *E-Adams spectral sequence for the computation of* $[X, Y]_*$.

0.2. The E_1 page.

The goal of this subsection is to provide a nicer characterization of the E_1 page of the E-Adams spectral sequence for the computation of $[X, Y]_*$. Given a monoid object (E, μ, e) in $\mathcal{SH}$.

Theorem 0.4. *Let (E, μ, e) be a flat commutative monoid object in $\mathcal{SH}$, and let X and Y be two objects in $\mathcal{SH}$. Further suppose at least one of the following hold:*

*(1) E and X are cellular objects (**??**) and $E_*(X)$ is a graded projective (**??**) left $\pi_*(E)$-module (via **??**).*

(2) There exists a collection of $a_i \in A$ indexed by some set I such that $E \otimes X$ is a retract of $\bigoplus_i \Sigma^{a_i} E$

Then for all $s \geq 0$ and $a \in A$, we have isomorphisms in the associated E-Adams spectral sequence

$$E_1^{s,a}(X, Y) \cong \mathrm{Hom}^a_{E_*(E)}(E_*(X), E_*(W_s))$$

Furthermore, under these isomorphisms, the differential $d_1 : E_1^{s,a} \to E_1^{s+1,a-\mathbf{1}}$ corresponds to the map

$$\mathrm{Hom}^a_{E_*(E)}(E_*(X), E_*(W_s)) \to \mathrm{Hom}^{a-\mathbf{1}}_{E_*(E)}(E_*(X), E_*(X \otimes W_{s+1}))$$

which sends a map $f : E_*(X) \to E_{*+a}(W_s)$ to the composition

$$E_*(X) \xrightarrow{f} E_{*+a}(W_s) \xrightarrow{(X \otimes h_s)_*} E_{*+a-\mathbf{1}}(X \otimes Y_{s+1}) \xrightarrow{(X \otimes j_{s+1})_*} E_{*+a-\mathbf{1}}(X \otimes W_{s+1}).$$

Proof. By **??**, for all $s \geq 0$ and $t, w \in \mathbb{Z}$, we have isomorphisms

$$[X, E \otimes Y_s]_{t,w} \cong \mathrm{Hom}^{t,w}_{E_*(E)}(E_*(X), E_*(E \otimes Y_s)).$$

since $W_s = E \otimes Y_s$, we have that

$$E_1^{s,(t,w)} = [X, W_s]_{t,w} \cong \mathrm{Hom}^{t,w}_{E_*(E)}(E_*(X), E_*(W_s)),$$

as desired. □

Definition 0.5. Let (E, μ, e) be a monoid object in $\mathcal{SH}$. We say E is *flat* if the canonical right $\pi_*(E)$-module structure on $E_*(E)$ is that of a flat module.

0.3. The E_2 page.

0.4. Convergence.

convergence of spectral sequences