

In this appendix, we will define the notion of *A-graded anticommutative Hopf algebroids* (Definition 0.2) over an *A-graded anticommutative ring* R (??), and left comodules over them (Definition 0.6).

0.1. A-graded anticommutative Hopf algebroids over R . Given an *A-graded anticommutative ring* R , we will define an *A-graded anticommutative Hopf algebroid* over R to be a groupoid object in $R\text{-}\mathbf{GCA}^A$, i.e., a groupoid object in $(R\text{-}\mathbf{GCA}^A)^{\text{op}}$. First, recall the definition of a *groupoid object* in a category with pullbacks:

Definition 0.1. Let \mathcal{C} be a category with pullbacks. A *groupoid object* in \mathcal{C} consists of a pair of objects (M, O) together with five morphisms

- (1) *Source and target*: $s, t : M \rightarrow O$,
- (2) *Identity*: $e : O \rightarrow M$,
- (3) *Composition*: $c : M \times_O M \rightarrow M$,
- (4) *Inverse*: $i : M \rightarrow M$

Where $M \times_O M$ will always refer to the object which into the following pullback diagram in \mathcal{C} :

$$\begin{array}{ccc} M \times_O M & \xrightarrow{p_2} & M \\ p_1 \downarrow & \lrcorner & \downarrow t \\ M & \xrightarrow{s} & O \end{array}$$

For example, if we're working in $\mathcal{C} = \mathbf{Set}$, we should think of M as a set of morphisms, and O as a set of objects. The functions s and t take a morphism to their domain and codomain, respectively, and $M \times_O M$ is the collection of pairs of morphisms $(g, f) \in M \times M$ such that $t(f) = s(g)$, and the composition map $c : M \times_O M \rightarrow M$ takes such a pair to the element $g \circ f \in M$. We think of the identity $e : O \rightarrow M$ as taking some object $x \in O$ to the identity morphism $e(x) = \text{id}_x \in M$ on x , and the inverse map $i : M \rightarrow M$ takes a morphism f to its inverse f^{-1} . These data are required to make the following diagrams commute:

- (1) Composition works correctly:

$$\begin{array}{ccc} M \times_O M & \xrightarrow{c} & M \\ p_1 \downarrow & & \downarrow t \\ M & \xrightarrow{t} & O \end{array} \quad \begin{array}{ccc} M & \xleftarrow{e} & O \\ s \searrow & \parallel & \swarrow t \\ & O & \end{array} \quad \begin{array}{ccc} M \times_O M & \xrightarrow{p_2} & M \\ c \downarrow & & \downarrow s \\ M & \xrightarrow{s} & O \end{array}$$

Expressed in terms of sets, the first diagram says that the target of $g \circ f$ is the target of g . The second diagram says that the domain and codomain of the identity on some object x is x . The third diagram says that the domain of $g \circ f$ is the domain of f .

- (2) Associativity of composition: Write $M \times_O (M \times_O M)$ and $(M \times_O M) \times_O M$ for the pullbacks of $(s, t \circ c)$ and $(s \circ c, t)$, respectively, so we have commuting diagrams

$$\begin{array}{ccc} (M \times_O M) \times_O M & \xrightarrow{p'_2} & M \\ p'_1 \downarrow & \searrow c \times M & \parallel \\ M \times_O M & \xrightarrow{p_2} & M \\ p_1 \downarrow & & \downarrow t \\ M \times_O M & \xrightarrow{c} & M \xrightarrow{s} O \end{array} \quad \begin{array}{ccc} M \times_O (M \times_O M) & \xrightarrow{p''_2} & M \times_O M \\ p''_1 \downarrow & \searrow M \times c & \downarrow c \\ M \times_O M & \xrightarrow{p_2} & M \\ p_1 \downarrow & & \downarrow t \\ M & \xrightarrow{s} & O \end{array}$$

the following diagram commutes:

$$\begin{array}{ccc}
 M & \xrightarrow{(eot, \text{id}_M)} & M \times_O M \\
 (\text{id}_M, eos) \downarrow & \searrow & \downarrow c \\
 M \times_O M & \xrightarrow{c} & M
 \end{array}$$

Expressed in terms of sets, this diagram says that given $f \in M$ with $s(f) = x$ and $t(f) = y$, that $f \circ \text{id}_x = f$ and $\text{id}_y \circ f = f$.

(4) Inverse: The following diagrams must commute:

$$\begin{array}{ccccc}
 M & & M & \xrightarrow{(\text{id}_M, i)} & M \times_O M & \xleftarrow{(i, \text{id}_M)} & M \\
 \swarrow i & \downarrow i & t \downarrow & & \downarrow c & & \downarrow s \\
 M & \xleftarrow{i} & M & & O & \xleftarrow{e} & M & \xleftarrow{e} & O
 \end{array}
 \quad
 \begin{array}{ccc}
 M & & M \\
 s \swarrow & \downarrow i & \searrow t \\
 O & \xleftarrow{t} & M & \xrightarrow{s} & O
 \end{array}$$

where the arrows (id_M, i) and (i, id_M) are determined by the universal property of $M \times_O M$ like so:

$$\begin{array}{ccc}
 M & \xrightarrow{i} & M \\
 (\text{id}_M, i) \searrow & & \downarrow p_1 \\
 M \times_O M & \xrightarrow{p_2} & M \\
 p_1 \downarrow & & \downarrow t \\
 M & \xrightarrow{s} & O
 \end{array}
 \quad
 \begin{array}{ccc}
 M & \xrightarrow{(i, \text{id}_M)} & M \\
 i \searrow & & \downarrow p_1 \\
 M \times_O M & \xrightarrow{p_2} & M \\
 p_1 \downarrow & & \downarrow t \\
 M & \xrightarrow{s} & O
 \end{array}$$

Expressed in terms of sets, given $f \in M$ with $s(f) = x$ and $t(f) = y$, the first diagram says that $(f^{-1})^{-1} = f$, the second says that $f \circ f^{-1} = \text{id}_y$ and $f^{-1} \circ f = \text{id}_x$, and the last diagram says that the domain and codomain of f^{-1} are y and x , respectively.

It can be seen that groupoid objects in $\mathcal{C} = \mathbf{Set}$ are precisely (small) groupoids. Now, we can state and unravel the definition of a Hopf algebroid:

Definition 0.2. Given an A -graded anticommutative ring R (??), an A -graded anticommutative Hopf algebroid over R is a co-groupoid object in $R\text{-GCA}^A$, i.e., a groupoid object in $(R\text{-AGrCAlg})^{\text{op}}$. Explicitly, an A -graded anticommutative Hopf algebroid over E is a pair (Γ, B) of objects in $R\text{-AGrCAlg}$ along with morphisms

- (1) *left unit*: $\eta_L : B \rightarrow \Gamma$ (corresponding to t),
- (2) *right unit*: $\eta_R : B \rightarrow \Gamma$ (corresponding to s),
- (3) *comultiplication*: $\Psi : \Gamma \rightarrow \Gamma \otimes_B \Gamma$ (corresponding to c),
- (4) *counit*: $\epsilon : \Gamma \rightarrow B$ (corresponding to e),
- (5) *conjugation*: $c : \Gamma \rightarrow \Gamma$ (corresponding to i),

where here Γ may be viewed as a B -bimodule with left B -module structure induced by η_L and right B -module structure induced by η_R , so we may form the tensor product of bimodules $\Gamma \otimes_B \Gamma$, which further may be given the structure of an A -graded anticommutative R -algebra (by ??), and

fits into the following pushout diagram in $R\text{-}\mathbf{GCA}^A g$ (??):

$$\begin{array}{ccc} B & \xrightarrow{\eta_L} & \Gamma \\ \eta_R \downarrow & & \downarrow g \mapsto 1 \otimes g \\ \Gamma & \xrightarrow{g \mapsto g \otimes 1} & \Gamma \otimes_B \Gamma \end{array}$$

These data must make the following diagrams commute:

(1) (Composition works correctly)

$$\begin{array}{ccc} B & \xrightarrow{\eta_L} & \Gamma \\ \eta_L \downarrow & & \downarrow \Psi \\ \Gamma & \xrightarrow{g \mapsto g \otimes 1} & \Gamma \otimes_B \Gamma \end{array} \quad \begin{array}{ccc} & B & \\ \eta_R \swarrow & \parallel & \searrow \eta_L \\ \Gamma & \xrightarrow{\epsilon} B \xleftarrow{\epsilon} & \Gamma \end{array} \quad \begin{array}{ccc} B & \xrightarrow{\eta_R} & \Gamma \\ \eta_R \downarrow & & \downarrow g \mapsto 1 \otimes g \\ \Gamma & \xrightarrow{\Psi} & \Gamma \otimes_B \Gamma \end{array}$$

(2) (Coassociativity) The following diagram must commute

$$\begin{array}{ccccc} \Gamma \otimes_B \Gamma & \xleftarrow{\Psi} & \Gamma & \xrightarrow{\Psi} & \Gamma \otimes_B \Gamma \\ \Psi \otimes_B \Gamma \downarrow & & & & \downarrow \Gamma \otimes_B \Psi \\ (\Gamma \otimes_B \Gamma) \otimes_B \Gamma & \xrightarrow{\cong} & & & \Gamma \otimes_B (\Gamma \otimes_B \Gamma) \end{array}$$

where $(\Gamma \otimes_B \Gamma) \otimes_B \Gamma$ and $\Gamma \otimes_B (\Gamma \otimes_B \Gamma)$ denote the rings which fit into the following pushout diagrams in $R\text{-}\mathbf{GCA}^A$:

$$\begin{array}{ccc} B & \xrightarrow{\eta_L} & \Gamma \\ \eta_R \downarrow & & \downarrow g \mapsto (1 \otimes 1) \otimes g \\ \Gamma & & \downarrow \Psi \\ \Gamma \otimes_B \Gamma & \xrightarrow{g \otimes g' \mapsto (g \otimes g') \otimes 1} & (\Gamma \otimes_B \Gamma) \otimes_B \Gamma \end{array} \quad \begin{array}{ccc} B & \xrightarrow{\eta_L} \Gamma & \xrightarrow{\Psi} \Gamma \otimes_B \Gamma \\ \eta_R \downarrow & & \downarrow (g \otimes g') \mapsto 1 \otimes (g \otimes g') \\ \Gamma & \xrightarrow{g \mapsto g \otimes (1 \otimes 1)} & \Gamma \otimes_B (\Gamma \otimes_B \Gamma) \end{array}$$

and the isomorphism $(\Gamma \otimes_B \Gamma) \otimes_B \Gamma \rightarrow \Gamma \otimes_B (\Gamma \otimes_B \Gamma)$ sends $(g \otimes g') \otimes g''$ to $g \otimes (g' \otimes g'')$, the left vertical arrow $\Psi \otimes \Gamma$ sends $g \otimes g'$ to $\Psi(g) \otimes g$, and the right vertical arrow $\Gamma \otimes \Psi$ sends $g \otimes g'$ to $g \otimes \Psi(g')$.

(3) (Co-unitality):

$$\begin{array}{ccc} \Gamma & \xrightarrow{\Psi} & \Gamma \otimes_B \Gamma \\ \Psi \downarrow & \searrow & \downarrow (\eta_L \circ \epsilon) \cdot \text{id}_\Gamma \\ \Gamma \otimes_B \Gamma & \xrightarrow{\text{id}_\Gamma \cdot (\eta_R \circ \epsilon)} & \Gamma \end{array}$$

where the right vertical arrow sends $g \otimes g'$ to $\eta_L(\epsilon(g))g'$ and the bottom horizontal arrow sends $g \otimes g'$ to $g\eta_R(\epsilon(g'))$.

(4) (Convolution):

$$\begin{array}{ccc} & \Gamma & \\ & \parallel & \downarrow c \\ \Gamma & \xleftarrow{c} & \Gamma \end{array} \quad \begin{array}{ccccc} B & \xleftarrow{\epsilon} & \Gamma & \xrightarrow{\epsilon} & B \\ \eta_L \downarrow & & \downarrow i & & \downarrow \eta_R \\ \Gamma & \xleftarrow{\text{id}_\Gamma \cdot c} & \Gamma \otimes_B \Gamma & \xrightarrow{c \cdot \text{id}_\Gamma} & \Gamma \end{array} \quad \begin{array}{ccccc} B & \xrightarrow{\eta_L} & \Gamma & \xleftarrow{\eta_R} & B \\ & \searrow \eta_R & \downarrow c & \swarrow \eta_L & \\ & & \Gamma & & \end{array}$$

where the bottom left arrow in the middle diagram sends $g \otimes g'$ to $gc(g')$ and the bottom right arrow in the middle diagram sends $g \otimes g'$ to $c(g)g'$.

The remainder of this subsection is devoted to proving some technical lemmas about A -graded anticommutative Hopf algebroids.

Proposition 0.3. *Suppose we have an A -graded anticommutative Hopf algebroid (Γ, B) over (R, θ) with structure maps η_L , η_R , Ψ , ϵ , and c (Definition 0.2). Recall in the definition, we considered $\Gamma \otimes_B \Gamma$ to be the A -graded R -commutative ring whose underlying abelian group was given by the tensor product of B -bimodules, where Γ has left B -module structure induced by η_L and right B -module structure induced by η_R . Thus $\Gamma \otimes_B \Gamma$ is canonically a B -bimodule, as it is a tensor product of B -bimodules. Then the canonical left (resp. right) B -module structure on $\Gamma \otimes_B \Gamma$ coincides with that induced by the ring homomorphism $\Psi \circ \eta_L$ (resp. $\Psi \circ \eta_R$).*

Proof. First we show the left module structures coincide. By additivity, in order to show the module structures coincide, it suffices to show that given a homogeneous pure tensor $g \otimes g'$ in $\Gamma \otimes_B \Gamma$ and some $b \in B$ that $\Psi(\eta_L(b)) \cdot (g \otimes g') = (\eta_L(b) \cdot g) \otimes g'$, where \cdot on the left denotes the product in $\Gamma \otimes_B \Gamma$ and the \cdot on the right denotes the product in Γ . By the axioms for a Hopf algebroid, we have that $\Psi(\eta_L(b)) = \eta_L(b) \otimes 1$. Thus by how the product in $\Gamma \otimes_B \Gamma$ is defined (??), we have that

$$\Psi(\eta_L(b)) \cdot (g \otimes g') = (\eta_L(b) \otimes 1) \cdot (g \otimes g') = (\varphi_\Gamma(\theta_{0,|g|}) \cdot \eta_L(b) \cdot g) \otimes (g' \cdot 1) = (\eta_L(b) \cdot g) \otimes g',$$

where $\varphi_\Gamma : R \rightarrow \Gamma$ is the structure map, and the last equality follows by the fact that $\theta_{0,|g|} = 1$. An entirely analogous argument yields that the canonical right module structure on $\Gamma \otimes_B \Gamma$ coincides with that induced by $\Psi \circ \eta_R$, since $\Psi \circ \eta_R = 1 \otimes \eta_R$. \square

Remark 0.4. By the above proposition, given an A -graded commutative Hopf algebroid (Γ, B) over R , there is no ambiguity when discussing the objects $\Gamma \otimes_B (\Gamma \otimes_B \Gamma)$ and $(\Gamma \otimes_B \Gamma) \otimes_B \Gamma$ — they may both be considered as the threefold tensor product of the B -bimodule Γ with itself. In particular, we have a canonical isomorphism of B -bimodules

$$(\Gamma \otimes_B \Gamma) \otimes_B \Gamma \rightarrow \Gamma \otimes_B (\Gamma \otimes_B \Gamma)$$

sending $(g \otimes g') \otimes g''$ to $g \otimes (g' \otimes g'')$, and this is precisely the isomorphism in the coassociativity diagram in the definition of a Hopf algebroid (Definition 0.2).

Proposition 0.5. *Suppose we have an A -graded commutative Hopf algebroid (Γ, B) over R with structure maps η_L , η_R , Ψ , ϵ , and c . Then $\eta_L : B \rightarrow \Gamma$ is a homomorphism of left B -modules, $\eta_R : B \rightarrow \Gamma$ is a homomorphism of right B -modules, and $\Psi : \Gamma \rightarrow \Gamma \otimes_B \Gamma$ and $\epsilon : \Gamma \rightarrow B$ are homomorphisms of B -bimodules.*

Proof. Since the left (resp. right) B -module structure on Γ is induced by η_L (resp. η_R), the map η_L (resp. η_R) is a homomorphism of left (resp. right) B -modules by definition.

Next, we want to show Ψ is a homomorphism of B -bimodules. The left (resp. right) B -module structure on Γ is that induced by η_L (resp. η_R), and in Proposition 0.3, we showed that the left (resp. right) B -module structure on $\Gamma \otimes_B \Gamma$ is that induced by $\Psi \circ \eta_L$ (resp. $\Psi \circ \eta_R$), so that by definition $\Psi : \Gamma \rightarrow \Gamma \otimes_B \Gamma$ is a homomorphism of left (resp. right) B -modules.

Lastly, we claim that $\epsilon : \Gamma \rightarrow B$ is a homomorphism of B -bimodules. We need to show that given $g \in \Gamma$ and $b, b' \in B$ that $\epsilon(\eta_L(b)g\eta_R(g')) = b\epsilon(g)b'$. This follows from the fact that ϵ is a ring homomorphism satisfying $\epsilon \circ \eta_L = \epsilon \circ \eta_R = \text{id}_B$. \square

0.2. Comodules over a Hopf algebroid. In what follows, fix an A -graded anticommutative ring (R, θ) and an A -graded anticommutative Hopf algebroid (Γ, B) over R with structure maps $\eta_L, \eta_R, \Psi, \epsilon$, and c . We will always view Γ with its *canonical* B -bimodule structure, with left B -module structure induced by η_L , and right B -module structure induced by η_R . In particular, any tensor product over B involving Γ will always refer to Γ with this bimodule structure.

Definition 0.6. A *left comodule over Γ* is a pair (N, Ψ_N) , where N is a left A -graded B -module and $\Psi_N : N \rightarrow \Gamma \otimes_B N$ is an A -graded homomorphism of left A -graded B -modules. These data are required to make the following diagrams commute

$$\begin{array}{ccc} N & \xrightarrow{\Psi_N} & \Gamma \otimes_B N \\ & \searrow \cong & \downarrow \epsilon \otimes N \\ & & B \otimes_B N \end{array} \quad \begin{array}{ccccc} \Gamma \otimes_B N & \xleftarrow{\Psi_N} & N & \xrightarrow{\Psi_N} & \Gamma \otimes_B N \\ \Psi \otimes N \downarrow & & & & \downarrow \Gamma \otimes \Psi_N \\ (\Gamma \otimes_B \Gamma) \otimes_B N & \xrightarrow{\cong} & & & \Gamma \otimes_B (\Gamma \otimes_B N) \end{array}$$

The maps $\epsilon \otimes N$ and $\Psi \otimes N$ are well-defined by [Proposition 0.5](#), and the bottom isomorphism in the right diagram is the canonical one sending $(g \otimes g') \otimes n \mapsto g \otimes (g' \otimes n)$.

Given two left A -graded Γ -comodules (N_1, Ψ_{N_1}) and (N_2, Ψ_{N_2}) , a homomorphism of left A -graded comodules $f : N_1 \rightarrow N_2$ is an A -graded homomorphism of the underlying left B -modules such that the following diagram commutes:

$$\begin{array}{ccc} N_1 & \xrightarrow{f} & N_2 \\ \Psi_{N_1} \downarrow & & \downarrow \Psi_{N_2} \\ \Gamma \otimes_B N_1 & \xrightarrow{\Gamma \otimes f} & \Gamma \otimes_B N_2 \end{array}$$

We write $\Gamma\text{-CoMod}^A$ for the resulting category of left A -graded comodules over Γ . In the above definition, we required A -graded left Γ -comodule homomorphisms to strictly preserve the grading, but we could have instead considered left Γ -comodule homomorphisms which are of degree d for some $d \in A$, or equivalently, the set of degree zero A -graded Γ -comodule homomorphisms from N_1 to the shifted comodule $(N_2)_{*+d}$. We denote the hom-set of degree- d A -graded left Γ -comodule homomorphisms from (N_1, Ψ_{N_1}) to (N_2, Ψ_{N_2}) by

$$\text{Hom}_{\Gamma\text{-CoMod}^A}^d(N_1, N_2) \quad \text{or usually just} \quad \text{Hom}_{\Gamma}^d(N_1, N_2).$$

In particular, we simply write $\text{Hom}_{\Gamma\text{-CoMod}^A}(N_1, N_2)$ or $\text{Hom}_{\Gamma}(N_1, N_2)$ for the set of strictly degree preserving (degree 0) A -graded left Γ -comodule homomorphisms from (N_1, Ψ_{N_1}) to (N_2, Ψ_{N_2}) .

Proposition 0.7. *The category $\Gamma\text{-CoMod}^A$ is an additive category.*

Proof. First, we show the category is **Ab**-enriched. Since the forgetful functor $\Gamma\text{-CoMod}^A \rightarrow B\text{-Mod}^A$ is clearly faithful, we may view hom-sets in $\Gamma\text{-CoMod}^A$ as subsets of hom-groups in $B\text{-Mod}^A$, so that in order to show $\Gamma\text{-CoMod}^A$ is **Ab**-enriched, it suffices to show that hom-sets in $\Gamma\text{-CoMod}^A$ are closed under addition of module homomorphisms and taking inverses. To that end, suppose we have two A -graded left Γ -comodule homomorphisms $f, g : (N_1, \Psi_{N_1}) \rightarrow (N_2, \Psi_{N_2})$, then we have

$$\begin{aligned} \Psi_{N_2} \circ (f + g) &= (\Psi_{N_2} \circ f) + (\Psi_{N_2} \circ g) \\ &= ((\Gamma \otimes_B f) \circ \Psi_{N_1}) + ((\Gamma \otimes_B g) \circ \Psi_{N_1}) \\ &= ((\Gamma \otimes_B f) + (\Gamma \otimes_B g)) \circ \Psi_{N_1} \\ &= (\Gamma \otimes_B (f + g)) \circ \Psi_{N_1}, \end{aligned}$$

where the first equality follows since Ψ_{N_2} is a homomorphism, the second follows since f and g are left Γ -comodule homomorphisms, the third follows since Ψ_{N_1} is a homomorphism, and the last equality follows by definition of the tensor product of modules. Hence $f + g$ is indeed an A -graded left Γ -comodule homomorphism, as desired. Now, we also claim $-f$ is an A -graded left Γ -comodule homomorphism. To that end, note that

$$\Psi_{N_2} \circ (-f) = -\Psi_{N_2} \circ f = -(\Gamma \otimes_B f) \circ \Psi_{N_1} = (\Gamma \otimes_B (-f)) \circ \Psi_{N_1},$$

where the first equality follows since Ψ_{N_2} is a homomorphism, the second follows since f is an A -graded left Γ -comodule homomorphism, and the third equality follows by definition of the tensor product.

Thus, we've shown that the hom-sets in $\Gamma\text{-CoMod}^A$ are abelian groups, and composition is clearly bilinear, so that $\Gamma\text{-CoMod}^A$ is indeed **Ab**-enriched.

Now, in order to show $\Gamma\text{-CoMod}^A$ is additive, it suffices to show that it contains a zero object and has binary coproducts. First of all, it is straightforward to check that the zero left B -module is clearly an A -graded left Γ -comodule with structure map the unique map $0 \rightarrow \Gamma \otimes_B 0 \cong 0$, and that given any other A -graded left Γ -comodule (N, Ψ_N) , the unique homomorphisms of left B -modules $0 \rightarrow N$ and $N \rightarrow 0$ are left comodule homomorphisms.

Now, suppose we have two A -graded left Γ -comodules (N_1, Ψ_{N_1}) and (N_2, Ψ_{N_2}) . First, we claim their direct sum as left B -modules $N_1 \oplus N_2$ is canonically an A -graded left Γ -comodule. We know that $N_1 \oplus N_2$ is an A -graded left B -module by ??, and we can define the structure map

$$\Psi_{N_1 \oplus N_2} : N_1 \oplus N_2 \xrightarrow{\Psi_{N_1} \oplus \Psi_{N_2}} (\Gamma \otimes_B N_1) \oplus (\Gamma \otimes_B N_2) \cong \Gamma \otimes_B (N_1 \oplus N_2),$$

where the final isomorphism is the canonical one sending $(g_1 \otimes n_1) \oplus (g_2 \otimes n_2)$ to $(g_1 \otimes n_1) + (g_2 \otimes n_2)$. Then □ finish

Proposition 0.8. *The forgetful functor $\Gamma\text{-CoMod}^A \rightarrow B\text{-Mod}^A$ (where here $B\text{-Mod}^A$ is the category of A -graded left B -modules and degree-preserving module homomorphisms between them) has a right adjoint $\Gamma \otimes_B - : B\text{-Mod}^A \rightarrow \Gamma\text{-CoMod}^A$ called the co-free construction, where the co-free left A -graded Γ -comodule on a left A -graded B -module M is the B -module $\Gamma \otimes_B M$ equipped with the coaction*

$$\Psi_{\Gamma \otimes_B M} : \Gamma \otimes_B M \xrightarrow{\Psi \otimes_B M} (\Gamma \otimes_B \Gamma) \otimes_B M \xrightarrow{\cong} \Gamma \otimes_B (\Gamma \otimes_B M).$$

Explicitly, given some (N, Ψ_N) in $\Gamma\text{-CoMod}$ and some M in $B\text{-Mod}^A$, the counit and unit of this adjunction are given by

$$\eta_{(N, \Psi_N)} : N \xrightarrow{\Psi_N} \Gamma \otimes_B N$$

and

$$\varepsilon_M : \Gamma \otimes_B M \xrightarrow{\varepsilon \otimes_B M} B \otimes_B M \xrightarrow{\cong} M.$$

Proof. □ todo

Proposition 0.9. *Suppose that Γ is flat as a right B -module, i.e., suppose $\eta_R : B \rightarrow \Gamma$ is a flat ring homomorphism. Then the category $\Gamma\text{-CoMod}^A$ is an abelian category.*

Proof. □ finish