In this appendix, we fix a symmetric monoidal category  $(\mathcal{C}, \otimes, S)$  with left unitor, right unitor, associator, and symmetry isomorphisms  $\lambda$ ,  $\rho$ ,  $\alpha$ , and  $\tau$ , respectively.

## 0.1. Monoid objects in a symmetric monoidal category.

**Definition 0.1.** A monoid object  $(E, \mu, e)$  is an object E in  $\mathfrak{C}$  along with a multiplication morphism  $\mu: E \otimes E \to E$  and a unit map  $e: S \to E$  such that the following diagrams commute:

$$E \otimes S \xrightarrow{E \otimes e} E \otimes E \xleftarrow{e \otimes E} S \otimes E \qquad (E \otimes E) \otimes E \xrightarrow{\mu \otimes E} E \otimes E$$

$$\downarrow^{\mu} \qquad \qquad \downarrow^{\mu} \qquad \qquad \downarrow^{\mu}$$

The first diagram expresses unitality, while the second expressed associativity. If in addition the following diagram commutes,

$$E \otimes E \xrightarrow{\tau} E \otimes E$$

then we say  $(E, \mu, e)$  is a *commutative* monoid object.

**Example 0.2.** The object S is a monoid object, with multiplication map  $\rho_S = \lambda_S : S \otimes S \to S$  and unit  $\mathrm{id}_S : S \to S$ .

**Definition 0.3.** Given two monoid objects  $(E_1, \mu_1, e_1)$  and  $(E_2, \mu_2, e_2)$  in a symmetric monoidal category  $(\mathcal{C}, \otimes, S)$ , a monoid homomorphism from  $E_1$  to  $E_2$  is a morphism  $f: E_1 \to E_2$  in  $\mathcal{C}$  such that the following diagrams commute:

$$E_{1} \otimes E_{1} \xrightarrow{f \otimes f} E_{2} \otimes E_{2} \qquad S$$

$$\downarrow^{\mu_{1}} \qquad \downarrow^{\mu_{2}} \qquad E_{1} \xrightarrow{f} E_{2} \qquad E_{1} \xrightarrow{f} E_{2}$$

It is straightforward to show that  $\mathrm{id}_{E_1}$  is a homomorphism of monoid objects from  $E_1$  to itself, and that the composition of monoid homomorphisms is still a monoid homomorphism. Thus, we have categories  $\mathbf{Mon}_{\mathbb{C}}$  and  $\mathbf{CMon}_{\mathbb{C}}$  of monoid objects and commutative monoid objects in  $\mathbb{C}$ , respectively, with monoid homomorphisms between them.

**Lemma 0.4.** Given two monoid objects  $(E_1, \mu_1, e_1)$  and  $(E_2, \mu_2, e_2)$  in a symmetric monoidal category  $(\mathfrak{C}, \otimes, S)$ , their tensor product  $E_1 \otimes E_2$  canonically becomes a monoid object in  $\mathfrak{C}$  with unit map

$$e: S \xrightarrow{\cong} S \otimes S \xrightarrow{e_1 \otimes e_2} E_1 \otimes E_2$$

and multiplication map

$$\mu: E_1 \otimes E_2 \otimes E_1 \otimes E_2 \xrightarrow{E_1 \otimes \tau \otimes E_2} E_1 \otimes E_1 \otimes E_2 \otimes E_2 \xrightarrow{\mu_1 \otimes \mu_2} E_1 \otimes E_2$$

(where here we are suppressing the associators from the notation). If in addition  $(E_1, \mu_1, e_1)$  and  $(E_2, \mu_2, e_2)$  are commutative monoid objects, then  $(E_1 \otimes E_2, \mu, e)$  is as well.

 **Lemma 0.5.** Let  $(E, \mu, e)$  be a monoid object in SH. Then the maps  $e: S \to E$  and  $\mu: E \otimes E \to E$ are monoid object homomorphisms (where here S and  $E \otimes E$  are considered to be monoid objects by Example 0.2 and Lemma 0.4, respectively).

todo Proof.

> **Lemma 0.6.** Given monoid objects  $(E_i, \mu_i, e_i)$  for i = 1, 2, 3 in a symmetric monoidal category  $\mathbb{C}$ , the associator  $(E_1 \otimes E_2) \otimes E_3 \xrightarrow{\cong} E_1 \otimes (E_2 \otimes E_3)$  is an isomorphism of monoid objects. In other words, up to associativity, given a collection of monoid objects  $E_1, \ldots, E_n$  in  $\mathbb{C}$ , there is no ambiguity when talking about their tensor product  $E_1 \otimes \cdots \otimes E_n$  as a monoid object.

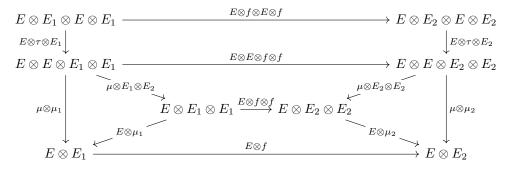
> *Proof.* Clearly, up to associativity,  $(E_1 \otimes E_2) \otimes E_3$  and  $E_1 \otimes (E_2 \otimes E_3)$  have the same unit map  $S \xrightarrow{e_1 \otimes e_2 \otimes e_3} E_1 \otimes E_2 \otimes E_3$ . Thus, it remains to show that they have the same product map, up to associativity. To see this, consider the following diagram, where we've passed to a symmetric strict monoidal category:

$$E_1 \otimes (E_2 \otimes E_3) \otimes E_1 \otimes (E_2 \otimes E_3) = \frac{\alpha}{(E_1 \otimes E_2) \otimes E_3 \otimes (E_1 \otimes E_2) \otimes E_3} \\ E_1 \otimes \tau_{E_2 \otimes E_3, E_1} \otimes E_2 \otimes E_3 \\ E_1 \otimes E_1 \otimes E_2 \otimes E_3 \otimes E_2 \otimes E_3 \\ E_1 \otimes E_2 \otimes \tau_{E_3, E_1} \otimes E_2 \otimes E_3 \otimes E_2 \otimes E_3 \\ E_1 \otimes E_2 \otimes E_3 \otimes E_2 \otimes E_3 \otimes E_3 \otimes E_2 \otimes E_3 \otimes E_3 \\ E_1 \otimes E_2 \otimes E_2 \otimes E_3 \otimes E_3 \otimes E_2 \otimes E_3 \otimes E_$$

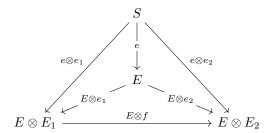
The top pentagonal region commutes by coherence for the  $\tau$ 's in a symmetric monoidal category. The bottom triangle commutes by definition. The remaining four triangles commute by functoriality of  $-\otimes$  -. On the left is the product for  $E_1\otimes (E_2\otimes E_3)$ , while on the right is the product for  $(E_1 \otimes E_2) \otimes E_3$ . Thus they are equal up to associativity, as desired.

**Lemma 0.7.** Suppose we have some monoid object  $(E, \mu, e)$  in  $\mathfrak C$  and some homomorphism of monoid objects  $f:(E_1,\mu_1,e_1)\to (E_2,\mu_2,e_2)$  in  $\mathbf{Mon}_{\mathfrak{C}}$ . Then  $E\otimes f:E\otimes E_1\to E\otimes E_2$  and  $f \otimes E : E_1 \otimes E \to E_2 \otimes E$  are monoid homomorphisms, where here we are considering  $E \otimes E_1$ ,  $E \otimes E_2$ ,  $E_1 \otimes E$ , and  $E_2 \otimes E$  to be monoid objects by Lemma 0.4.

*Proof.* We will show that  $E \otimes f$  is a monoid object homomorphism, as showing  $f \otimes E$  is a monoid homomorphism is entirely analogous. First consider the following diagram:



The top region commutes by naturality of  $\tau$ . The bottom trapezoid commutes since f is a monoid homomorphism. The remaining three regions commute by functoriality of  $-\otimes -$ . Now, consider the following diagram:



The bottom region commutes since f is a monoid homomorphism. The top two regions commute by functoriality of  $-\otimes -$ . Thus, we've shown  $E\otimes f$  is a monoid object homomorphism, as desired.

## 0.2. Modules over monoid objects in a symmetric monoidal category.

**Definition 0.8.** Let  $(E, \mu, e)$  be a monoid object in  $\mathcal{C}$ . Then a (left) module object  $(N, \kappa)$  over  $(E, \mu, e)$  is the data of an object N in  $\mathcal{C}$  and a morphism  $\kappa : E \otimes N \to N$  such that the following two diagrams commute in  $\mathcal{C}$ :

$$S \otimes N \xrightarrow{e \otimes N} E \otimes N \qquad (E \otimes E) \otimes N \xrightarrow{\mu \otimes N} E \otimes N$$

$$\downarrow^{\kappa} \qquad \qquad \downarrow^{\kappa} \qquad \qquad \downarrow^{\kappa}$$

$$E \otimes (E \otimes N) \xrightarrow{E \otimes \kappa} E \otimes N \xrightarrow{\kappa} N$$

**Definition 0.9.** Let  $(E, \mu, e)$  be a monoid object in  $\mathcal{C}$ , and suppose we have two (left) module objects  $(N, \kappa)$  and  $(N', \kappa')$  over  $(E, \mu, e)$ . Then a morphism  $f: N \to N'$  is a (left) E-module homomorphism if the following diagram commutes in  $\mathcal{C}$ :

$$E \otimes N \xrightarrow{E \otimes f} E \otimes N'$$

$$\downarrow \kappa \downarrow \qquad \qquad \downarrow \kappa'$$

$$N \xrightarrow{f} N'$$

**Definition 0.10.** Given a monoid object  $(E, \mu, e)$  in  $\mathcal{C}$ , we write E-**Mod** to denote the category of (left) module objects over E and E-module homomorphisms between them. We denote the homset in E-**Mod** by

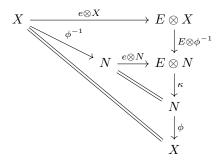
$$\operatorname{Hom}_{E\operatorname{-Mod}}(M,N)$$
, or simply  $\operatorname{Hom}_E(M,N)$ .

For our purposes, we will only consider left module objects, so we will usually drop the quanitier "left" and just refer to them as "module objects".

**Lemma 0.11.** Let  $(E, \mu, e)$  be a monoid object in  $\mathfrak{C}$  and let  $(N, \kappa)$  be an E module object. Then given some object X in  $\mathfrak{C}$  and an isomorphism  $\phi: N \xrightarrow{\cong} X$ , X inherits the structure of an E-module via the action map

$$\kappa_{\phi}: E \otimes X \xrightarrow{E \otimes \phi^{-1}} E \otimes N \xrightarrow{\kappa} N \xrightarrow{\phi} X.$$

*Proof.* We need to show the two coherence diagrams in Definition 0.8 commute. To see the former commutes, consider the following diagram:



The top trapezoid commutes by functoriality of  $-\otimes -$ . The middle small triangle commutes by unitality of  $\kappa$ . The remaining region commutes by definition. To see the second coherence diagram commutes, consider the following diagram:

The top rectangle commutes by functoriality of  $-\otimes -$ . The middle rectangle commutes by coherence for  $\kappa$ . The bottom two regions commute by definition.

**Proposition 0.12.** Given a monoid object  $(E, \mu, e)$  in  $\mathbb{C}$ , the forgetful functor E-**Mod**  $\to \mathbb{C}$  has a left adjoint  $\mathbb{C} \to E$ -**Mod** sending an object X in  $\mathbb{C}$  to  $(E \otimes X, \kappa_X)$  where  $\kappa_X$  is the composition

$$E \otimes (E \otimes X) \xrightarrow{\alpha^{-1}} (E \otimes E) \otimes X \xrightarrow{\mu \otimes X} E \otimes X,$$

and sending a morphism  $f: X \to Y$  to  $E \otimes f: E \otimes X \to E \otimes Y$ .

We call this functor  $E \otimes -: \mathcal{C} \to E\text{-}\mathbf{Mod}$  the free functor, and we call E-modules in the image of the free functor free modules.

*Proof.* In this proof, we work in a symmetric strict monoidal category. First, we wish to show that  $E \otimes -: \mathcal{C} \to E\text{-}\mathbf{Mod}$  as constructed is well-defined. First, to see that  $(X, \kappa_X)$  is actually a  $E\text{-}\mathrm{module}$ , we need to show the two diagrams in Definition 0.8 commute. Indeed, consider the following diagrams:

$$E \otimes X \xrightarrow{e \otimes E \otimes X} E \otimes E \otimes X \qquad E \otimes E \otimes E \otimes X \xrightarrow{\mu \otimes E \otimes X} E \otimes E \otimes X$$

$$\downarrow^{\mu \otimes X} \qquad E \otimes \mu \otimes X \downarrow \qquad \downarrow^{\mu \otimes X} \qquad \downarrow^{\mu \otimes X} \qquad E \otimes E \otimes X \xrightarrow{\mu \otimes X} E \otimes X$$

These are precisely the diagrams obtained by applying  $X \otimes -$  to the coherence diagrams for  $\mu$ , so that they commute as desired. Now, suppose  $f: X \to Y$  is a morphism in  $\mathcal{C}$ , then we would

like to show that  $E \otimes f : E \otimes X \to E \otimes Y$  is a morphism of E-module objects. Indeed, consider the following diagram:

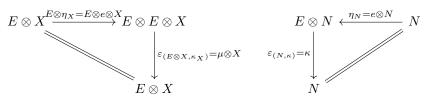
$$E \otimes E \otimes X \xrightarrow{E \otimes E \otimes f} E \otimes E \otimes Y$$

$$\downarrow^{\mu \otimes X} \qquad \qquad \downarrow^{\mu \otimes Y}$$

$$E \otimes X \xrightarrow{E \otimes f} E \otimes Y$$

It commutes by functoriality of  $-\otimes -$ , so  $E\otimes f$  is indeed an E-module homomorphism as desired.

Now, in order to see that  $E \otimes -$  is left adjoint to the forgetful functor, it suffices to construct a unit and counit for the adjunction and show they satisfy the zig-zag identities. Given X in  $\mathfrak{C}$  and  $(N,\kappa)$  in E-Mod, define  $\eta_X := e \otimes X : X \to E \otimes X$  and  $\varepsilon_{(N,\kappa)} := \kappa : E \otimes N \to N$ .  $\eta_X$  is clearly natural in X by functoriality of  $-\otimes -$ , and  $\varepsilon_{(N,\kappa)}$  is natural in  $(N,\kappa)$  by how morphisms in E-Mod are defined. Now, to see these are actually the unit and counit of an adjunction, we need to show that the following diagrams commute for all X in  $\mathfrak{C}$  and  $(N,\kappa)$  in E-Mod:



Commutativity of the left diagram is unitality of  $\mu$ , while commutativity of the right diagram is unitality of  $\kappa$ . Thus indeed  $E \otimes - : \mathcal{C} \to E\text{-}\mathbf{Mod}$  is a left adjoint of the forgetful functor  $E\text{-}\mathbf{Mod} \to \mathcal{C}$ , as desired.

**Lemma 0.13.** Let  $(E, \mu, e)$  be a monoid object in  $\mathbb{C}$ . Further suppose we have some object X in  $\mathbb{C}$  and an E-module object  $(N, \kappa)$ , along with a commuting diagram in  $\mathbb{C}$ 

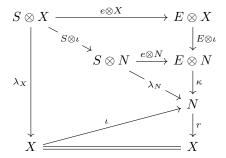
$$X \xrightarrow{\iota} N \xrightarrow{r} X$$

Then if  $\ell := \iota \circ r : N \to N$  is an E-module homomorphism, then X is canonically an E-module object with structure map

$$\kappa_X : E \otimes X \xrightarrow{E \otimes \iota} E \otimes N \xrightarrow{\kappa} N \xrightarrow{r} X,$$

and furthermore, the maps  $\iota: X \to N$  and  $r: N \to X$  are E-module homomorphisms.

*Proof.* First, in order to show  $(X, \kappa_X)$  is an *E*-module, we need to show the two diagrams in Definition 0.8 commute. To see the unitality diagram holds, consider the following diagram:



The large left triangle commutes by naturality of  $\lambda$ . The top trapezoid commutes by functoriality of  $-\otimes$ . The small middle right triangle commutes by unitality of  $\kappa$ . Finally, the bottom triangle

commutes by definition, since we are assuming  $r \circ \iota = \mathrm{id}_X$ . Now the right composition is  $\kappa_X$ , so we have shown  $\kappa_X \circ (e \otimes X) = \lambda_X$ , as desired. Now, consider the following diagram:

$$E \otimes E \otimes X \xrightarrow{\mu \otimes X} E \otimes X$$

$$E \otimes E \otimes \iota \downarrow \qquad \downarrow E \otimes X$$

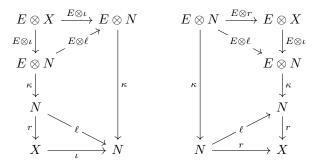
$$E \otimes E \otimes N \xrightarrow{E \otimes E \otimes \ell} E \otimes E \otimes N \xrightarrow{\mu \otimes N} E \otimes N$$

$$E \otimes \iota \downarrow \qquad \downarrow \kappa$$

$$E \otimes \iota \downarrow$$

The top trapezoid commutes by functoriality of  $-\otimes -$ . The top left triangle commutes by functoriality of  $-\otimes -$  and the fact that  $\ell \circ \iota = \iota \circ r \circ \iota = \iota \circ \operatorname{id}_X = \iota$ . The middle left trapezoid commutes by since  $\ell$  is an E-module homomorphism, by assumption. The bottom left triangle commutes by functoriality of  $-\otimes -$  and the fact that  $\iota \circ r = \ell$ . Thus, we have shown that  $(X, \kappa_X)$  is an E-module object, as desired.

Now, it remains to show that  $\iota: X \to N$  and  $r: N \to X$  are E-module homomorphisms. To that end, consider the following two diagrams:



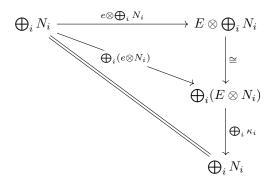
The trapezoids in each diagram commute since we are assuming  $\ell$  is a E-module homomorphism. The four triangles commute since  $\ell \circ \iota = \iota$  and  $r \circ \ell = r$ . Thus, we have shown that  $\kappa_X \circ (E \otimes r) = r \circ \kappa$  and  $\kappa \circ (E \otimes \iota) = \iota \circ \kappa_X$ , so we indeed have that  $\iota$  and r are E-module homomorphisms, as desired.

**Proposition 0.14.** Suppose that  $\mathfrak{C}$  is further an additive symmetric monoidal closed category. Let  $(E, \mu, e)$  be a monoid object in  $\mathfrak{C}$ , and suppose we have a family of E-module objects  $(N_i, \kappa_i)$  indexed by some small set I. Then  $N := \bigoplus_{i \in I} N_i$  is canonically an E-module, with action map given by the composition

$$\kappa : E \otimes \bigoplus_{i} N_i \xrightarrow{\cong} \bigoplus_{i} (E \otimes N_i) \xrightarrow{\bigoplus_{i} \kappa_i} \bigoplus_{i} N_i,$$

where the first isomorphism is given by the fact that  $E \otimes -$  preserves coproducts, since it is a left adjoint. Furthermore, N is the coproduct of all the  $N_i$ 's in E-Mod, so that E-Mod has arbitrary coproducts.

*Proof.* We need to show the action map  $\kappa$  makes the diagrams in Definition 0.8 commute. To see the first (unitality) diagram commutes, consider the following diagram:



The top triangle commutes since  $E \otimes -$  preserves coproducts, as it is a left adjoint. The bottom triangle commutes by unitality of each of the  $\kappa_i$ 's. To see the second coherence diagram commutes, consider the following diagram:

$$E \otimes E \otimes \bigoplus_{i} N_{i} \xrightarrow{\mu \oplus \bigoplus_{i} N_{i}} E \otimes \bigoplus_{i} N_{i}$$

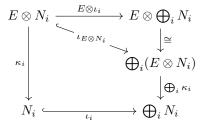
$$E \otimes \bigoplus_{i} (E \otimes N_{i}) \xrightarrow{\cong} \bigoplus_{i} (E \otimes E \otimes N_{i}) \xrightarrow{\bigoplus_{i} (\mu \otimes N_{i})} \bigoplus_{i} (E \otimes N_{i})$$

$$E \otimes \bigoplus_{i} \kappa_{i} \downarrow \qquad \bigoplus_{i} (E \otimes \kappa_{i}) \downarrow \qquad \bigoplus_{i} \kappa_{i}$$

$$E \otimes \bigoplus_{i} N_{i} \xrightarrow{\cong} \bigoplus_{i} (E \otimes N_{i}) \xrightarrow{\bigoplus_{i} \kappa_{i}} \bigoplus_{i} N_{i}$$

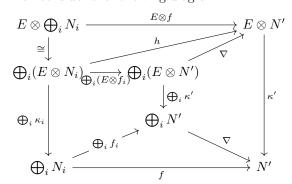
The bottom right square commutes by coherence for the  $\kappa_i$ 's. Every other region commutes since  $-\otimes -$  preserves colimits in each variable. Thus  $N = \bigoplus_i N_i$  is indeed an E-module object, as desired.

Now, we claim that  $(N, \kappa)$  is the coproduct of the  $(N_i, \kappa_i)$ 's in E-Mod. First, we need to show that the canonical maps  $\iota_i : N_i \hookrightarrow N$  are morphisms in E-Mod for all  $i \in I$ . To see  $\iota_i$  is a homomorphism of E-module objects, consider the following diagram:



The top triangle commutes by additivity of  $E \otimes -$ . The bottom trapezoid commutes since, by univeral property of the coproduct,  $\bigoplus_i \kappa_i$  is the unique arrow which makes the trapezoid commute for all  $i \in I$ . Now, it remains to show that given an E-module object  $(N', \kappa')$  and homomorphisms  $f_i : N_i \to N'$  of E-module objects for all  $i \in I$ , that the unique arrow  $f : N \to N'$  in  $S\mathcal{H}$  satisfying  $f \circ \iota_i = f_i$  for all  $i \in I$  is a homomorphism of E-module objects, so that N is actually the coproduct of the  $N_i$ 's. To see this, first let  $h : \bigoplus_i (E \otimes N_i) \to E \otimes N'$  be the arrow determined by the maps

 $E \otimes N_i \xrightarrow{E \otimes f_i} E \otimes N'$ . Then consider the following diagram:



The top triangle commutes by additivity of  $E \otimes -$ . The triangle below that commutes by the universal property of the coproduct, since it is straightforward to check that  $\nabla \circ \bigoplus_i (E \otimes f_i)$  and h both satisfy the universal property of the colimit. The left trapezoid commutes by functoriality of  $-\oplus$  and the fact that  $f_i$  is a homomorphism of E-module objects for all i in I. The right trapezoid commutes by naturality of  $\nabla$ . Finally, the bottom triangle commutes by the universal product of the coproduct, by showing that  $\nabla \circ \bigoplus_i f_i$  in place of f also satisfies the universal property of the colimit. Hence f is indeed a homomorphism of E-module objects, as desired.

To recap, we have shown that given a set of E-module objects  $\{(N_i, \kappa_i)\}_{i \in I}$ , the inclusion maps  $\iota_i : N_i \hookrightarrow \bigoplus_i N_i$  are morphisms in E-**Mod**, and that given morphisms  $f_i : (N_i, \kappa_i) \to (N', \kappa')$  for all  $i \in I$ , the unique induced map  $\bigoplus_i N_i \to N'$  is a morphism in E-**Mod**. Thus, E-**Mod** does indeed have arbitrary coproducts, and the forgetful functor E-**Mod**  $\to \mathcal{SH}$  preserves them.  $\square$ 

**Proposition 0.15.** Suppose that C is further an additive closed symmetric monoidal category, and let  $(E, \mu, e)$  be a monoid object in C. Then E-Mod is itself an additive category, so that in particular the forgetful functor E-Mod  $\to C$  and the free functor  $C \to E$ -Mod (Proposition 0.12) are additive.

*Proof.* It is a general fact that adjoint functors between additive categories are necessarily additive. In order to show E- $\mathbf{Mod}$  is an additive category, it suffices to show it has finite coproducts, that  $\mathrm{Hom}_{E\mathbf{-Mod}}(N,N')$  is an abelian group for all E-modules N and N', and that composition is bilinear. We know that  $E\mathbf{-Mod}$  has coproducts which are preserved by the forgetful functor  $E\mathbf{-Mod} \to \mathcal{C}$  by Proposition 0.14 (which is clearly faithful). Thus, because  $\mathcal{C}$  is  $\mathbf{Ab}$ -enriched and  $\mathrm{Hom}_{E\mathbf{-Mod}}(N,N') \subseteq \mathcal{C}(N,N')$ , it suffices to show that  $\mathrm{Hom}_{E\mathbf{-Mod}}(N,N')$  is closed under addition and taking inverses. To see the former, let  $f,g:N\to N'$  be  $E\mathbf{-module}$  homomorphisms, and consider the following diagram:

The outermost trapezoids commute by naturality of  $\Delta$  and  $\nabla$ . The triangles in the top corners and the top middle rectangle commute by additivity of  $E \otimes -$ . Finally, the middle bottom rectangle commutes by functoriality of  $-\oplus -$  and  $-\otimes -$ , and the fact that f and g are E-module homomorphisms. Commutativity of the above diagram shows that f + g is a homomorphism of

E-modules as desired. Finally, to see -f is a E-module homomorphism if f is, we would like to show that  $\kappa' \circ (E \otimes (-f)) = (-f) \circ \kappa$ . This follows by the fact that  $\kappa' \circ (E \otimes f) = f \circ \kappa$  and additivity of  $-\otimes$  – and composition.