

AN AXIOMATIC APPROACH TO THE ADAMS SPECTRAL SEQUENCE

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1. INTRODUCTION

2. TRIANGULATED CATEGORIES WITH SUB-PICARD GRADING

2.1. Setup of \mathcal{SH} . In order to construct an abstract version of the Adams spectral sequence, we need to work in some axiomatic version of a stable homotopy category \mathcal{SH} which acts like the familiar classical stable homotopy category **hoSp** (Section 4) or the motivic stable homotopy category **SH** $_{\mathcal{S}}$ over some base scheme \mathcal{S} (Section 5).

Definition 2.1. Let \mathcal{C} be an additive category with arbitrary (small) coproducts. Then an object X in \mathcal{C} is *compact* if, for any collection of objects Y_i in \mathcal{C} indexed by some (small) set I , the canonical map

$$\bigoplus_i \mathcal{C}(X, Y_i) \rightarrow \mathcal{C}(X, \bigoplus_i Y_i)$$

is an isomorphism of abelian groups. (Explicitly, the above map takes a generator $x \in \mathcal{C}(X, Y_i)$ to the composition $X \xrightarrow{x} Y_i \hookrightarrow \bigoplus_i Y_i$.)

Definition 2.2. Given a tensor triangulated category $(\mathcal{C}, \otimes, S, \Sigma, e, \mathcal{D})$ (Definition A.13), a *sub-Picard grading* on \mathcal{C} is the following data:

- A pointed abelian group $(A, \mathbf{1})$ along with a homomorphism of pointed groups $h : (A, \mathbf{1}) \rightarrow (\text{Pic } \mathcal{C}, \Sigma S)$, where $\text{Pic } \mathcal{C}$ is the *Picard group* of isomorphism classes of invertible objects in \mathcal{C} .¹
- For each $a \in A$, a chosen representative S^a in the isomorphism class $h(a)$ such that each S^a is a compact object (Definition 2.1) and $S^0 = S$.
- For each $a, b \in A$, an isomorphism $\phi_{a,b} : S^{a+b} \rightarrow S^a \otimes S^b$. This family of isomorphisms is required to be *coherent*, in the following sense:
 - For all $a \in A$, we must have that $\phi_{a,0}$ coincides with the right unitor $S^a \xrightarrow{\cong} S^a \otimes S$ and $\phi_{0,a}$ coincides the left unitor $S^a \xrightarrow{\cong} S \otimes S^a$.
 - For all $a, b, c \in A$, the following “associativity diagram” must commute:

$$\begin{array}{ccc} S^{a+b} \otimes S^c & \xleftarrow{\phi_{a+b,c}} & S^{a+b+c} \xrightarrow{\phi_{a,b+c}} S^a \otimes S^{b+c} \\ \phi_{a,b} \otimes S^c \downarrow & & \downarrow S^a \otimes \phi_{b,c} \\ (S^a \otimes S^b) \otimes S^c & \xrightarrow{\cong} & S^a \otimes (S^b \otimes S^c) \end{array}$$

From now on we fix a monoidal closed tensor triangulated category $(\mathcal{SH}, \otimes, S, \Sigma, e, \mathcal{D})$ with arbitrary (small) (co)products and sub-Picard grading $(A, \mathbf{1}, h, \{S^a\}, \{\phi_{a,b}\})$. We also fix an isomorphism $\nu : \Sigma S \xrightarrow{\cong} S^1$ once and for all. We establish conventions. First, observe the following remark:

Remark 2.3. Note that by induction the coherence conditions for the $\phi_{a,b}$ ’s in the above definition say that given any $a_1, \dots, a_n \in A$ and $b_1, \dots, b_m \in A$ such that $a_1 + \dots + a_n = b_1 + \dots + b_m$ and any fixed parenthesizations of $X = S^{a_1} \otimes \dots \otimes S^{a_n}$ and $Y = S^{b_1} \otimes \dots \otimes S^{b_m}$, there is a *unique* isomorphism $X \rightarrow Y$ that can be obtained by forming formal compositions of products of $\phi_{a,b}$, identities, associators, unitors, and their inverses (but not symmetries).

In light of this remark, we will usually simply write ϕ or even just \cong for any isomorphism that is built by taking compositions of products of $\phi_{a,b}$ ’s, unitors, associators, identities, and their inverses. Given an object X and a natural number $n > 0$, we write

$$X^n := \overbrace{X \otimes \dots \otimes X}^{n \text{ times}} \quad \text{and} \quad X^0 := S.$$

¹Recall an object X in a symmetric monoidal category is *invertible* if there exists some object Y and an isomorphism $S \cong X \otimes Y$.

We denote the associator, symmetry, left unitor, and right unitor isomorphisms in \mathcal{SH} by

$$\begin{aligned}\alpha_{X,Y,Z} : (X \otimes Y) \otimes Z &\xrightarrow{\cong} X \otimes (Y \otimes Z) & \tau_{X,Y} : X \otimes Y &\xrightarrow{\cong} Y \otimes X \\ \lambda_X : S \otimes X &\xrightarrow{\cong} X & \rho_X : X \otimes S &\xrightarrow{\cong} X.\end{aligned}$$

Often we will drop the subscripts. Furthermore, by the coherence theorem for symmetric monoidal categories ([2]), we will often assume α , ρ , and λ are actual equalities.

Given some integer $n \in \mathbb{Z}$, we will write a bold \mathbf{n} to denote the element $n \cdot \mathbf{1}$ in A . Note that we can use the isomorphism $\nu : \Sigma S \xrightarrow{\cong} S^1$ to construct a natural isomorphism $\Sigma \cong S^1 \otimes -$:

$$\Sigma X \xrightarrow{\Sigma \lambda_X^{-1}} \Sigma(S \otimes X) \xrightarrow{e_{S,X}^{-1}} \Sigma S \otimes X \xrightarrow{\nu \otimes X} S^1 \otimes X.$$

The first two arrows are natural in X by definition. The last arrow is natural in X by functoriality of $- \otimes -$. By abuse of notation, we will also use ν to denote this natural isomorphism.

Given some $a \in A$, we define $\Sigma^a := S^a \otimes -$ and $\Omega^a := \Sigma^{-a} = S^{-a} \otimes -$. We specifically define $\Omega := \Omega^1$. We say “the a^{th} suspension of X ” to denote $\Sigma^a X$. It turns out that Σ^a is an autoequivalence of \mathcal{SH} for each $a \in A$, and furthermore, Ω^a and Σ^a form an adjoint equivalence of \mathcal{SH} for all a in A :

Proposition 2.4. *For each $a \in A$, the isomorphisms*

$$\eta_X^a : X \xrightarrow{\lambda_X^{-1}} S \otimes X \xrightarrow{\phi_{a,-a} \otimes X} (S^a \otimes S^{-a}) \otimes X \xrightarrow{\alpha} S^a \otimes (S^{-a} \otimes X) = \Sigma^a \Omega^a X$$

and

$$\varepsilon_X^a : \Omega^a \Sigma^a X = S^{-a} \otimes (S^a \otimes X) \xrightarrow{\alpha^{-1}} (S^{-a} \otimes S^a) \otimes X \xrightarrow{\phi_{-a,a}^{-1} \otimes X} S \otimes X \xrightarrow{\lambda_X} X$$

are natural in X , and furthermore, they are the unit and counit respectively of the adjoint autoequivalence $(\Omega^a, \Sigma^a, \eta^a, \varepsilon^a)$ of \mathcal{SH} . In particular, since $\Sigma \cong \Sigma^1$, $\Omega := \Omega^1$ is a left adjoint for Σ , so that $(\mathcal{SH}, \Omega, \Sigma, \eta, \varepsilon, \mathcal{D})$ is an adjointly triangulated category (Definition A.9), where η and ε are the compositions

$$\eta : \text{Id}_{\mathcal{SH}} \xrightarrow{\eta^1} \Sigma^1 \Omega \xrightarrow{\nu^{-1} \Omega} \Sigma \Omega \quad \text{and} \quad \varepsilon : \Omega \Sigma \xrightarrow{\Omega \nu} \Omega \Sigma^1 \xrightarrow{\varepsilon^1} \text{Id}_{\mathcal{SH}}.$$

Proof. In this proof, we will freely employ the coherence theorem for monoidal categories (see [2]), which essentially tells us that we may assume we are working in a strict monoidal category (i.e., that the associators and unitors are identities). Then η_X^a and ε_X^a become simply the maps

$$\eta_X^a : X \xrightarrow{\phi_{a,-a} \otimes X} S^a \otimes S^{-a} \otimes X \quad \text{and} \quad \varepsilon_X^a : S^{-a} \otimes S^a \otimes X \xrightarrow{\phi_{-a,a}^{-1} \otimes X} X.$$

That these maps are natural in X follows by functoriality of $- \otimes -$. Now, recall that in order to show that these natural isomorphisms form an *adjoint* equivalence, it suffices to show that the natural isomorphisms $\eta^a : \text{Id}_{\mathcal{SH}} \Rightarrow \Omega^a \Sigma^a$ and $\varepsilon^a : \Sigma^a \Omega^a \Rightarrow \text{Id}_{\mathcal{SH}}$ satisfy one of the two zig-zag identities:

$$\begin{array}{ccc} \Omega^a & \xrightarrow{\Omega^a \eta^a} & \Omega^a \Sigma^a \Omega^a \\ & \searrow & \downarrow \varepsilon^a \Omega^a \\ & & \Omega^a \end{array} \quad \begin{array}{ccc} \Sigma^a \Omega^a \Sigma^a & \xleftarrow{\eta^a \Sigma^a} & \Sigma^a \\ \Sigma^a \varepsilon^a \downarrow & & \nearrow \\ \Sigma^a & & \end{array}$$

(that it suffices to show only one is [5, Lemma 3.2]). We will show that the left is satisfied. Unravelling definitions, we simply wish to show that the following diagram commutes for all X

in \mathcal{SH} :

$$\begin{array}{ccc}
 S^{-a} \otimes X & \xrightarrow{S^{-a} \otimes \phi_{a,-a} \otimes X} & S^{-a} \otimes S^a \otimes S^{-a} \otimes X \\
 & \searrow & \downarrow \phi_{-a,a}^{-1} \otimes S^{-a} \otimes X \\
 & & S^{-a} \otimes X
 \end{array}$$

Yet this is simply the diagram obtained by applying $-\otimes X$ to the associativity coherence diagram for the $\phi_{a,b}$'s (since $\phi_{a,0}$ and $\phi_{0,a}$ coincide with the unitors, and here we are taking the unitors and associators to be equalities), so it does commute, as desired. \square

Given two objects X and Y in \mathcal{SH} , we will denote the hom-abelian group of morphisms from X to Y in \mathcal{SH} by $[X, Y]$, and the internal hom object by $F(X, Y)$. We can extend the abelian group $[X, Y]$ into an A -graded abelian group $[X, Y]_*$ by defining $[X, Y]_a := [S^a \otimes X, Y]$. Given an object X in \mathcal{SH} and some $a \in A$, we can define the abelian group

$$\pi_a(X) := [S^a, X],$$

which we call the a^{th} (stable) homotopy group of X . We write $\pi_*(X)$ for the A -graded abelian group $\bigoplus_{a \in A} \pi_a(X)$, so that in particular we have a canonical isomorphism

$$\pi_*(X) = [S^*, X] \cong [S, X]_*.$$

Given some other object E , we can define the A -graded abelian groups $E_*(X)$ and $E^*(X)$ by the formulas

$$E_a(X) := \pi_a(E \otimes X) = [S^a, E \otimes X] \quad \text{and} \quad E^a(X) := [X, S^a \otimes E].$$

We refer to the functor $E_*(-)$ as the *homology theory represented by E* , or just E -homology, and we refer to $E^*(-)$ as the *cohomology theory represented by E* , or just E -cohomology.

2.2. Cellular objects in \mathcal{SH} . One very important class of objects in \mathcal{SH} are the *cellular* objects. Intuitively, these are the objects that can be built out of spheres via taking coproducts and (co)fibers.

Definition 2.5. Define the class of *cellular* objects in \mathcal{SH} to be the smallest class of objects such that:

- (1) For all $a \in A$, the a -sphere S^a is cellular.
- (2) If we have a distinguished triangle

$$X \rightarrow Y \rightarrow Z \rightarrow \Sigma X$$

such that two of the three objects X , Y , and Z are cellular, then the third object is also cellular.

- (3) Given a collection of cellular objects X_i indexed by some (small) set I , the object $\bigoplus_{i \in I} X_i$ is cellular (recall we have chosen \mathcal{SH} to have arbitrary coproducts).

Lemma 2.6. *Let X and Y be two isomorphic objects in \mathcal{SH} . Then X is cellular iff Y is cellular.*

Proof. Assume we have an isomorphism $f : X \xrightarrow{\cong} Y$ and that X is cellular. Then consider the following commutative diagram

$$\begin{array}{ccccccc}
 X & \xrightarrow{f} & Y & \longrightarrow & 0 & \longrightarrow & \Sigma X \\
 \parallel & & \downarrow f^{-1} & & \parallel & & \parallel \\
 X & \xlongequal{\quad} & X & \longrightarrow & 0 & \longrightarrow & \Sigma X
 \end{array}$$

The bottom row is distinguished by axiom TR1 for a triangulated category. Hence since X is cellular, 0 is also cellular, since the class of cellular objects satisfies two-of-three for distinguished triangles. Furthermore, since the vertical arrows are all isomorphisms, the top row is distinguished as well, by axiom TR0. Thus again by two-of-three, since X and 0 are cellular, so is Y , as desired. \square

Lemma 2.7. *Let X and Y be cellular objects in \mathcal{SH} . Then $X \otimes Y$ is cellular.*

Proof. Let E be a cellular object in \mathcal{SH} , and let \mathcal{E} be the collection of objects X in \mathcal{SH} such that $E \otimes X$ is cellular. First of all, suppose we have a distinguished triangle

$$X \rightarrow Y \rightarrow Z \rightarrow \Sigma X$$

such that two of three of X , Y , and Z belong to \mathcal{E} . Then since \mathcal{SH} is tensor triangulated, we have a distinguished triangle

$$E \otimes X \rightarrow E \otimes Y \rightarrow E \otimes Z \rightarrow \Sigma(E \otimes X).$$

Per our assumptions, two of three of $E \otimes X$, $E \otimes Y$, and $E \otimes Z$ are cellular, so that the third is by definition. Thus, all three of X , Y , and Z belong to \mathcal{E} if two of them do.

Second of all, suppose we have a family X_i of objects in \mathcal{E} indexed by some (small) set I , and set $X := \bigoplus_i X_i$. Then we'd like to show X belongs to \mathcal{E} , i.e., that $E \otimes X$ is cellular. Indeed,

$$E \otimes X = E \otimes \left(\bigoplus_i X_i \right) \cong \bigoplus_i (E \otimes X_i),$$

where the isomorphism is given by the fact that \mathcal{SH} is monoidal closed, so $E \otimes -$ preserves arbitrary colimits as it is a left adjoint. Per our assumption, since each $E \otimes X_i$ is cellular, the rightmost object is cellular, since the class of cellular objects is closed under taking arbitrary coproducts, by definition. Hence $E \otimes X$ is cellular by [Lemma 2.6](#).

Finally, we would like to show that each S^a belongs to \mathcal{E} , i.e., that $S^a \otimes E$ is cellular for all $a \in A$. When $E = S^b$ for some $b \in A$, this is clearly true, since $S^b \otimes S^a \cong S^{a+b}$, which is cellular by definition, so that $S^b \otimes S^a$ is cellular by [Lemma 2.6](#). Thus by what we have shown, the class of objects X for which $S^a \otimes X$ is cellular contains every cellular object. Hence in particular $E \otimes S^a \cong S^a \otimes E$ is cellular for all $a \in A$, as desired. \square

Lemma 2.8. *Let W be a cellular object in \mathcal{SH} such that $\pi_*(W) = 0$. Then $W \cong 0$.*

Proof. Let \mathcal{E} be the collection of all X in \mathcal{SH} such that $[\Sigma^n X, W] = 0$ for all $n \in \mathbb{Z}$ (where for $n > 0$, $\Sigma^{-n} := \Omega^n = (S^{-1})^n \otimes -$). We claim \mathcal{E} contains every cellular object in \mathcal{SH} . First of all, each S^a belongs to \mathcal{E} , as

$$[\Sigma^n S^a, W] \cong [S^n \otimes S^a, W] \cong [S^{a+n}, W] \leq \pi_*(W) = 0.$$

Furthermore, suppose we are given a distinguished triangle

$$X \rightarrow Y \rightarrow Z \rightarrow \Sigma X$$

such that two of three of X , Y , and Z belong to \mathcal{E} . By [Proposition A.12](#), for all $n \in \mathbb{Z}$ we get an exact sequence

$$[\Sigma^{n+1} X, W] \rightarrow [\Sigma^n Z, W] \rightarrow [\Sigma^n Y, W] \rightarrow [\Sigma^n X, W] \rightarrow [\Sigma^{n-1} Z, W].$$

Clearly if any two of three of X , Y , and Z belong to \mathcal{E} , then by exactness of the above sequence all three of the middle terms will be zero, so that the third object will belong to \mathcal{E} as well. Finally, suppose we have a collection of objects X_i in \mathcal{E} indexed by some small set I . Then

$$\left[\Sigma^n \bigoplus_i X_i, W \right] \cong \left[\bigoplus_i \Sigma^n X_i, W \right] \cong \prod_i [\Sigma^n X_i, W] = \prod_i 0 = 0,$$

where the first isomorphism follows by the fact that Σ^n is apart of an adjoint equivalence ([Proposition 2.4](#)), so it preserves arbitrary colimits.

Thus, by definition of cellularity, \mathcal{E} contains every cellular object. In particular, \mathcal{E} contains W , so that $[W, W] = 0$, meaning in particular that $\text{id}_W = 0$, so we have a commutative diagram

$$\begin{array}{ccc} & 0 & \\ \nearrow & \xlongequal{\quad} & \searrow \\ W & \xlongequal{\quad} & W \end{array}$$

Hence the diagonals exhibit isomorphisms between 0 and W , as desired. \square

Theorem 2.9. *Let X and Y be cellular objects in \mathcal{SH} , and suppose $f : X \rightarrow Y$ is a morphism such that $f_* : \pi_*(X) \rightarrow \pi_*(Y)$ is an isomorphism. Then f is an isomorphism.*

Proof. By axiom TR2 for a triangulated category ([Definition A.1](#)), we have a distinguished triangle

$$X \xrightarrow{f} Y \xrightarrow{g} C_f \xrightarrow{h} \Sigma X.$$

First of all, note that by definition since X and Y are cellular, so is C_f . Now, we claim $\pi_*(C_f) = 0$. Indeed, given $a \in A$, by [Proposition A.12](#) we have the following exact sequence:

$$[S^a, X] \xrightarrow{f_*} [S^a, Y] \xrightarrow{g_*} [S^a, C_f] \xrightarrow{h_*} [S^a, \Sigma X] \xrightarrow{-(\Sigma f)_*} [S^a, \Sigma Y],$$

where the first arrow is an isomorphism, per our assumption that f_* is an isomorphism. To see the last arrow is an isomorphism, consider the following diagram:

$$\begin{array}{ccc} [S^a, \Sigma X] & \xrightarrow{(\Sigma f)_*} & [S^a, \Sigma Y] \\ (\nu_X)_* \downarrow & & \downarrow (\nu_Y)_* \\ [S^a, S^1 \otimes X] & \xrightarrow{(S^1 \otimes f)_*} & [S^a, S^1 \otimes Y] \\ \cong \downarrow & & \downarrow \cong \\ [S^{-1} \otimes S^a, X] & \xrightarrow{f_*} & [S^{-1} \otimes S^a, Y] \\ (\phi_{-1,a})_* \downarrow & & \downarrow (\phi_{-1,a})_* \\ [S^{a-1}, X] & \xrightarrow{f_*} & [S^{a-1}, Y] \end{array}$$

where the middle vertical arrows are the adjunction natural isomorphisms specified by [Proposition 2.4](#). The bottom arrow is an isomorphism per our assumptions, so the top arrow is likewise an isomorphism, as desired. Thus $\text{im } h_* = \ker -(\Sigma f)_* = 0$, and $\ker g_* = \text{im } f_* = [S^a, Y]$, so that $\ker h_* = \text{im } g_* = 0$. It is only possible that $\ker h_* = \text{im } g_* = 0$ if $[S^a, C_f] = 0$. Thus, we have shown $\pi_*(C_f) = 0$, and C_f is cellular, so by [Lemma 2.8](#) there is an isomorphism $C_f \cong 0$. Now

consider the following commuting diagram:

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \longrightarrow & C_f & \longrightarrow & \Sigma X \\ \downarrow f & & \parallel & & \downarrow \cong & & \downarrow \Sigma f \\ Y & \xlongequal{\quad} & Y & \longrightarrow & 0 & \longrightarrow & \Sigma Y \end{array}$$

The top row is distinguished by assumption. The bottom row is distinguished by axiom TR2. Then since the middle two vertical arrows are isomorphisms, by [Lemma A.4](#), f is an isomorphism as well, as desired. \square

Lemma 2.10. *Let $e : X \rightarrow X$ be an idempotent morphism in \mathcal{SH} , so $e \circ e = e$. Then since \mathcal{SH} is a triangulated category with arbitrary coproducts, this idempotent splits ([Proposition A.8](#)), meaning e factors as*

$$X \xrightarrow{r} Y \xrightarrow{\iota} X$$

for some object Y and morphisms r and ι with $r \circ \iota = \text{id}_Y$. Then Y is cellular if X is.

Proof. It is a general categorical fact that the splitting of an idempotent, if it exists, is unique up to unique isomorphism,² so by [Lemma 2.6](#), it suffices to show that e has some cellular splitting. In [Proposition A.8](#), it is shown that we may take Y to be the homotopy colimit ([Definition A.7](#)) of the sequence

$$X \xrightarrow{e} X \xrightarrow{e} X \xrightarrow{e} X \xrightarrow{e} \cdots,$$

so there is a distinguished triangle

$$\bigoplus_{i=0}^{\infty} X \rightarrow \bigoplus_{i=0}^{\infty} X \rightarrow Y \rightarrow \Sigma \left(\bigoplus_{i=0}^{\infty} X \right).$$

Since X is cellular, by definition $\bigoplus_{i=0}^{\infty} X$ is as well. Thus by 2-of-3 for distinguished triangles for cellular objects, Y is cellular as desired. \square

2.3. Monoid objects in \mathcal{SH} . Many of the proofs in this section are quite technical and not very euclidiating, so we direct the reader to the appendix for most proofs. To start with, recall the following definition:

Definition 2.11. Let $(\mathcal{C}, \otimes, S)$ be a symmetric monoidal category with left unitor, right unitor, associator, and symmetry isomorphisms λ , ρ , α , and τ , respectively. A *monoid object* (E, μ, e) is an object E in \mathcal{C} along with a multiplication morphism $\mu : E \otimes E \rightarrow E$ and a unit map $e : S \rightarrow E$ such that the following diagrams commute:

$$\begin{array}{ccc} E \otimes S & \xrightarrow{E \otimes e} & E \otimes E \xleftarrow{e \otimes E} S \otimes E \\ & \searrow \rho_E & \downarrow \mu \swarrow \lambda_E \\ & & E \end{array} \quad \begin{array}{ccc} (E \otimes E) \otimes E & \xrightarrow{\mu \otimes E} & E \otimes E \\ \alpha \downarrow & & \downarrow \mu \\ E \otimes (E \otimes E) & \xrightarrow{E \otimes \mu} & E \otimes E \xrightarrow{\mu} E \end{array}$$

The first diagram expresses unitality, while the second expressed associativity. If in addition the following diagram commutes,

$$\begin{array}{ccc} E \otimes E & \xrightarrow{\tau} & E \otimes E \\ & \searrow \mu & \swarrow \mu \\ & & E \end{array}$$

²In particular, given an idempotent $e : X \rightarrow X$ which splits as $X \xrightarrow{r} Y \xrightarrow{\iota} X$, r and ι are the coequalizer and equalizer, respectively, of e and id_X .

then we say (E, μ, e) is a *commutative monoid object*.

Monoid objects in \mathcal{SH} will be the focus of the rest of this paper. The most important example of a monoid object in \mathcal{SH} is the unit S , which has multiplication map $\phi_{0,0}^{-1} = \lambda_S = \rho_S : S \otimes S \rightarrow S$ and unit map $\text{id}_S : S \rightarrow S$.

Proposition 2.12 (??). *Let (E, μ, e) be a monoid object in \mathcal{SH} , then $\pi_*(E)$ is a ring under the multiplication map $\pi_*(E) \times \pi_*(E) \rightarrow \pi_*(E)$ which sends classes $x : S^a \rightarrow E$ and $y : S^b \rightarrow E$ to the composition*

$$xy : S^{a+b} \xrightarrow{\phi_{a,b}} S^a \otimes S^b \xrightarrow{x \otimes y} E \otimes E \xrightarrow{\mu} E.$$

In particular, the unit of this ring is $e \in \pi_0(E) = [S, E]$.

We call the ring $\pi_*(S)$ the *stable homotopy ring*.

Proposition 2.13 (Proposition D.13). *For all $a, b \in A$ there exists an element $\theta_{a,b} \in \pi_0(S) = [S, S]$ such that given any commutative monoid object (E, μ, e) in \mathcal{SH} , the A -graded ring structure on $\pi_*(E)$ (Proposition 2.12) has a commutativity formula given by*

$$x \cdot y = y \cdot x \cdot (e \circ \theta_{a,b})$$

for all $x \in \pi_a(E)$ and $y \in \pi_b(E)$. In particular, $\theta_{a,b} \in \text{Aut}(S)$ is the composition

$$S \xrightarrow{\cong} S^{-a-b} \otimes S^a \otimes S^b \xrightarrow{S^{-a-b} \otimes \tau} S^{-a-b} \otimes S^b \otimes S^a \xrightarrow{\cong} S,$$

where the outermost maps are the unique maps specified by Remark 2.3.

Proposition 2.14. *The $\theta_{a,b}$'s described in Proposition 2.13 satisfy the following properties for all $a, b, c \in A$:*

- (1) $\theta_{a,b} \circ \theta_{c,d} = \theta_{c,d} \cdot \theta_{a,b}$ (where \cdot denotes the product in $\pi_*(S)$ given in Proposition 2.12),
- (2) $\theta_{a,0} = \theta_{0,a} = \text{id}_S$,
- (3) $\theta_{a,b} \cdot \theta_{b,a} = \text{id}_S$,
- (4) $\theta_{a,b} \cdot \theta_{a,c} = \theta_{a,b+c}$ and $\theta_{b,a} \cdot \theta_{c,a} = \theta_{b+c,a}$,

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Proof.

□

3. THE ADAMS SPECTRAL SEQUENCE

3.1. Construction of the Adams spectral sequence. The last thing we need before we can construct the Adams spectral sequence in \mathcal{SH} is the following:

Proposition 3.1. *Suppose we are given a distinguished triangle*

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$$

and some object W in \mathcal{SH} . Then there is an infinite long exact sequence of A -graded abelian groups:

$$\cdots \rightarrow [W, Z]_{*+n+1} \xrightarrow{\partial} [W, X]_{*+n} \xrightarrow{f_*} [W, Y]_{*+n} \xrightarrow{g_*} [W, Z]_{*+n} \xrightarrow{\partial} [W, Z]_{*+n-1} \rightarrow \cdots,$$

where $\partial : [W, Z]_{+n+1} \rightarrow [W, X]_{*+n}$ sends a class $x : S^{a+n+1} \otimes W \rightarrow Z$ to the composition*

$$S^{a+n} \otimes W \xrightarrow{\phi_{-1,a+n+1}} S^{-1} \otimes S^{a+n+1} \otimes W \xrightarrow{S^{-1} \otimes x} S^{-1} \otimes Z \xrightarrow{\tilde{h}} X,$$

where here we are suppressing the associator from the notation, and $\tilde{h} : \Omega Z = S^{-1} \otimes Z \rightarrow X$ is the adjoint (Proposition 2.4) of $h : Z \rightarrow \Sigma X$.

Proof. In this proof, we will freely employ the coherence theorem for a symmetric monoidal category, which tells us we may assume associativity and unitality of $- \otimes -$ holds up to strict equality. Furthermore, we will simply write ϕ to refer to any isomorphism that can be constructed by composing copies of products of $\phi_{a,b}$'s, unitors, identities, associators, and their inverses (see [Remark 2.3](#)). Finally, given $n > 0$, we will write Σ^{-n} to denote the functor $\Omega^n = (S^{-1})^n \otimes -$.

For all $n > 0$, the $\phi_{a,b}$'s yield natural isomorphisms

$$s_X^{-n} : \Sigma^{-n} X = (S^{-1})^n \otimes X \xrightarrow{\phi \otimes X} S^{-n} \otimes X = \Omega^n X.$$

and

$$s_X^n : \Sigma^n X \xrightarrow{\nu_X^n} (S^1)^n \otimes X \xrightarrow{\phi \otimes X} S^n \otimes X = \Sigma^n X,$$

where we recursively define $\nu^1 := \nu$ and ν^{n+1} is given by the composition

$$\nu_X^{n+1} : \Sigma^{n+1} X = \Sigma^n \Sigma X \xrightarrow{\nu_{\Sigma X}^n} (S^1)^n \otimes \Sigma X \xrightarrow{(S^1)^n \otimes \nu_X} (S^1)^n \otimes S^1 \otimes X = (S^1)^{n+1} \otimes X.$$

Finally, we define s^0 to be the identity natural transformation on \mathcal{SH} . Then we get the following natural isomorphisms of A -graded abelian groups for all $n \in \mathbb{Z}$

$$\ell_V^n : [W, \Sigma^n V]_* \xrightarrow{(s_V^n)_*} [W, \Sigma^n V]_* \xrightarrow{r_{W,V}^n} [W, V]_{*-n},$$

where $r_{W,V}^n$ is the natural isomorphism given as the composition

$$[W, \Sigma^n V]_* \xrightarrow{\cong} [S^{-n} \otimes S^* \otimes W, V] \xrightarrow{(\phi \otimes W)^*} [S^{-n} \otimes W, V] = [W, V]_{*-n},$$

where the first isomorphism is the adjunction $\Omega^n \dashv \Sigma^n$ ([Proposition 2.4](#)). Now, given $n \in \mathbb{Z}$, consider the following diagram

$$(1) \quad \begin{array}{ccccccc} [W, \Sigma^{n-1} Z]_* & \xrightarrow{h_{n-1}} & [W, \Sigma^n X]_* & \xrightarrow{\Sigma^n f_*} & [W, \Sigma^n Y]_* & \xrightarrow{\Sigma^n g_*} & [W, \Sigma^n Z]_* \xrightarrow{h_n} [W, \Sigma^{n+1} X]_* \\ \ell_Z^{n-1} \downarrow & & \ell_X^n \downarrow & & \ell_Y^n \downarrow & & \ell_Z^n \downarrow \quad \ell_X^{n+1} \downarrow \\ [W, Z]_{*-n+1} & \xrightarrow{\partial} & [W, X]_{*-n} & \xrightarrow{f_*} & [W, Y]_{*-n} & \xrightarrow{g_*} & [W, Z]_{*-n} \xrightarrow{\partial} [W, X]_{*-n-1} \end{array}$$

where for $n \geq 0$, $h_n = \Sigma^n h$, and for $n > 0$, $h_{-n} = \Omega^{n-1} \tilde{h}$ (where $\tilde{h} : \Omega Z \rightarrow X$ is the adjoint of $h : Z \rightarrow \Sigma X$). We would like to show the bottom row is exact. The top row is exact since it is obtained by applying $[W, -]_*$ to a fiber sequence (see [Proposition A.12](#) for full details), and we have constructed the vertical arrows to be isomorphisms. Thus it suffices to show each square commutes. The inner two squares commute by naturality of ℓ^n . Thus, it further suffices to show the outermost squares commute. Since our choice of $n \in \mathbb{Z}$ is arbitrary, we can just show the right square commutes. First consider the case that $n \geq 0$, and consider the following diagram:

$$\begin{array}{ccccc} [W, \Sigma^n Z]_* & \xrightarrow{\Sigma^n h_*} & & & [W, \Sigma^{n+1} X]_* \\ \downarrow \ell_Z^n & & \swarrow \ell_{\Sigma X}^n & & \downarrow \ell_X^{n+1} \\ & [W, \Sigma X]_{*-n} & \xrightarrow{(\nu_X)_*} & [W, \Sigma^1 X]_{*-n} & \\ & \swarrow h_* & & \searrow r_{W,X}^1 & \\ [W, Z]_{*-n} & \xrightarrow{\partial} & [W, X]_{*-n-1} & & \end{array}$$

The leftmost region commutes by naturality of ℓ . By unravelling how $r_{W,X}^1$ and the adjoint \tilde{h} used in the definition of ∂ are defined, a simple diagram chase yields that the bottom triangle

commutes. Thus, it remains to show the rightmost triangle in the above diagram commutes. To see this, note that by unravelling how ℓ and r are defined, the rightmost square becomes

$$\begin{array}{ccccc}
[W, (S^1)^n \otimes \Sigma X] & \xleftarrow{(\nu_{\Sigma X}^n)_*} & & [W, \Sigma^{n+1} X]_* & \\
\downarrow (\phi \otimes \Sigma X)_* & \searrow ((S^1)^n \otimes \nu_X)_* & & \downarrow (\nu_X^{n+1})_* & \\
[W, \Sigma^n \Sigma X]_* & \xrightarrow{(\Sigma^n \nu_X)_*} & [W, \Sigma^n \Sigma^1 X]_* & \xrightarrow{(\phi \otimes X)_*} & [W, \Sigma^{n+1} X]_* \\
\downarrow \text{adj} & & \downarrow \text{adj} & & \downarrow \text{adj} \\
[S^{-n} \otimes S^* \otimes W, \Sigma X] & \xrightarrow{(\nu_X)_*} & [S^{-n} \otimes S^* \otimes W, S^1 \otimes X] & & [S^{-n-1} \otimes S^* \otimes W, X] \\
\downarrow (\phi \otimes W)^* & \searrow (\phi \otimes W)^* & \searrow (\phi \otimes W)^* & & \downarrow (\phi \otimes W)^* \\
[W, \Sigma X]_{*-n} & & & & [W, X]_{*-n-1} \\
\downarrow (\nu_X)_* & \swarrow & \swarrow & & \downarrow (\phi \otimes W)^* \\
[W, \Sigma^1 X]_{*-n} & \xrightarrow{\text{adj}} & [S^{-1} \otimes S^{*-n} \otimes W, X] & \xrightarrow{(\phi \otimes W)^*} & [W, X]_{*-n-1}
\end{array}$$

The top right triangle commutes by how ν^{n+1} was defined. The top left oddly-shaped region commutes by functoriality of $- \otimes -$. The middle right triangle commutes by coherence for the ϕ 's. The middle left rectangle commutes by naturality of the adjunction isomorphism. Commutativity of the bottom left triangle is clear (do a diagram chase). Commutativity of the bottom right triangle is coherence for the ϕ 's. Finally, commutativity of the remaining region is again coherence of the ϕ 's, since the adjunction isomorphism are constructed using them ([Proposition 2.4](#)).

Now we consider the negative case: Unravelling definitions, given $n > 0$, the rightmost square in diagram (1) for $-n$ becomes

$$\begin{array}{ccccc}
[W, \Omega^n Z]_* & \xrightarrow{\Omega^{n-1} \tilde{h}_*} & & [W, \Omega^{n-1} X]_* & \\
\downarrow (\phi \otimes Z)_* & \searrow (\phi \otimes \Omega Z)_* & & \downarrow (\phi \otimes X)_* & \\
[W, \Omega^n Z]_* & \xrightarrow{(\phi \otimes Z)_*} & [W, \Omega^{n-1} \Omega Z]_* & \xrightarrow{\Omega^{n-1} \tilde{h}_*} & [W, \Omega^{n-1} X]_* \\
\downarrow \text{adj} & & \downarrow \text{adj} & & \downarrow \text{adj} \\
[S^{n-1} \otimes S^* \otimes W, \Omega Z] & \xrightarrow{\tilde{h}_*} & [S^{n-1} \otimes S^* \otimes W, X] & & \\
\downarrow (\phi \otimes W)^* & & \downarrow (\phi \otimes W)^* & & \\
[W, \Omega Z]_{*+n-1} & \xrightarrow{\tilde{h}_*} & [W, X]_{*+n-1} & & \\
\downarrow \text{adj} & & \downarrow \text{adj} & & \\
[S^n \otimes S^* \otimes W, Z] & \xrightarrow{(\phi \otimes W)^*} & [S^1 \otimes S^{*+n-1} \otimes W, Z] & \xrightarrow{\tilde{h}_*} & [W, X]_{*+n-1} \\
\downarrow (\phi \otimes W)^* & \swarrow (\phi \otimes W)^* & \swarrow (\phi \otimes W)^* & & \downarrow (\phi \otimes W)^* \\
[W, Z]_{*+n} & \xrightarrow{\partial} & [W, X]_{*+n-1} & &
\end{array}$$

The top right trapezoid commutes by functoriality of $- \otimes -$. The top left triangle commutes by coherence for the ϕ 's. The middle right rectangle commutes by naturality of the adjunction. The right trapezoid below that commutes obviously. The bottom left triangle commutes by coherence of the ϕ 's. The large middle left rectangle commutes by coherence for the ϕ 's, again since the

adjunction $\Sigma^n \dashv \Omega^n$ is constructed using the ϕ 's. Finally, to see the bottom diagram commutes, we will chase some homogeneous element $f : S^{b+n-1} \otimes W \rightarrow \Omega Z$ around the region. Consider the following diagram:

$$\begin{array}{ccccccc}
S^{-1} \otimes S^{b+n} \otimes W & \xleftarrow{\phi \otimes W} & S^{b+n-1} \otimes W & & & & \\
\phi \otimes W \downarrow & & \nearrow \phi \otimes W & & \downarrow f & & \\
S^{-1} \otimes S^1 \otimes S^{b+n+1} \otimes W & & & & & & \\
S^{-1} \otimes S^1 \otimes f \downarrow & & & & & & \\
S^{-1} \otimes S^1 \otimes S^{-1} \otimes Z & \xrightarrow{\phi \otimes Z} & S^{-1} \otimes Z & \xrightarrow{S^{-1} \otimes h} & S^{-1} \otimes \Sigma X & \xrightarrow{S^{-1} \otimes \nu_X} & S^{-1} \otimes S^1 \otimes X \xrightarrow{\phi \otimes X} X
\end{array}$$

By unravelling how the adjunction and \tilde{h} are defined, the two compositions around the outside of this diagram are the two morphisms obtained by chasing f around the bottom region in diagram (2). The top left triangle commutes by coherence of the ϕ 's (Remark 2.3), while the bottom region commutes by functoriality of $-\otimes-$ and coherence of the ϕ 's. Thus we've shown diagram (1) commutes, so the bottom row is exact, as desired. \square

Remark 3.2. Expressed more compactly, the above proposition says that for each object W and distinguished triangle

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$$

in \mathcal{SH} gives rise to the following diagram of A -graded abelian groups

$$\begin{array}{ccc}
[W, X]_* & \xrightarrow{f_*} & [W, Y]_* \\
& \nwarrow \partial & \downarrow g_* \\
& & [W, Z]_*
\end{array}$$

which is exact at each vertex, and where f_* , g_* , and ∂ are A -graded homomorphisms of degree 0, 0, and -1 , respectively. Explicitly, ∂ sends a class $x : S^a \otimes W \rightarrow Z$ to the composition

$$S^{a-1} \otimes W \cong S^{-1} \otimes S^a \otimes W \xrightarrow{S^{-1} \otimes x} S^{-1} \otimes Z \xrightarrow{S^{-1} \otimes h} S^{-1} \otimes \Sigma X \xrightarrow{S^{-1} \otimes \nu_X} S^{-1} \otimes S^1 \otimes X \xrightarrow{\phi_{-1,1}^{-1} \otimes X} X.$$

In what follows, let (E, μ, e) be a commutative monoid object in \mathcal{SH} . In this section we will freely use the coherence theorem for symmetric monoidal categories without comment, in particular, we will assume unitality and associativity hold up to strict equality.

Definition 3.3. Let \overline{E} be the fiber of the unit map $e : S \rightarrow E$ (Proposition A.6). Let $Y_0 := Y$ and $W_0 := E \otimes Y$. Then for $s > 0$, define

$$Y_s := \overline{E}^s \otimes Y, \quad W_s := E \otimes Y_s = E \otimes \overline{E}^s \otimes Y,$$

where \overline{E}^s denotes the s -fold tensor product $\overline{E} \otimes \cdots \otimes \overline{E}$. Then we get fiber sequences

$$Y_{s+1} \xrightarrow{i_s} Y_s \xrightarrow{j_s} W_s \xrightarrow{k_s} \Sigma Y_{s+1}$$

obtained by applying $-\otimes Y_s$ to the fiber sequence

$$\overline{E} \rightarrow S \xrightarrow{e} E \rightarrow \Sigma \overline{E}.$$

We can splice these sequences together to get the (canonical) Adams filtration of Y :

$$\begin{array}{ccccccc}
\cdots & \longrightarrow & Y_3 & \xrightarrow{i_2} & Y_2 & \xrightarrow{i_1} & Y_1 \xrightarrow{i_0} Y_0 = Y \\
& & \downarrow j_3 & \nwarrow k_2 & \downarrow j_2 & \nwarrow k_1 & \downarrow j_1 \\
& & W_3 & & W_2 & & W_1 \\
& & & & & & \downarrow j_0 \\
& & & & & & W_0
\end{array}$$

where here each k_s is of degree -1 (in particular, the above diagram does not commute in any sense), and each i_s and j_s have degree 0. We can extend this diagram to the right by setting $Y_s = Y$, $W_s = 0$, and $i_s = \text{id}_Y$ for $s < 0$. Then we may apply the functor $[X, -]_*$, and by [Remark 3.2](#), we obtain the following A -graded unrolled exact couple ([Definition B.2](#)):

$$\begin{array}{ccccccc} \cdots & \longrightarrow & [X, Y_{s+2}]_* & \xrightarrow{i_{s+1}} & [X, Y_{s+1}]_* & \xrightarrow{i_s} & [X, Y_s]_* \xrightarrow{i_{s-1}} [X, Y_{s-1}]_* \longrightarrow \cdots \\ & & \downarrow j_{s+2} & \swarrow \partial_{s+1} & \downarrow j_{s+1} & \swarrow \partial_s & \downarrow j_s \swarrow \partial_{s-1} \downarrow j_{s-1} \\ & & [X, W_{s+2}]_* & & [X, W_{s+1}]_* & & [X, W_s]_* & & [X, W_{s-1}]_* \end{array}$$

where here we are being abusive and writing $i_s : [X, Y_{s+1}]_* \rightarrow [X, Y_s]_*$ and $j_s : [X, Y_s]_* \rightarrow [X, W_s]_*$ to denote the pushforwards of $i_s : Y_{s+1} \rightarrow Y_s$ and $j_s : Y_s \rightarrow W_s$, respectively. Each i_s , j_s , and ∂_s are A -graded homomorphisms of degrees 0, 0, and -1 , respectively.

By [Proposition B.3](#), we may associate a $\mathbb{Z} \times A$ -graded spectral sequence $r \mapsto (E_r^{*,*}(X, Y), d_r)$ to the above A -graded unrolled exact couple, where d_r has $\mathbb{Z} \times A$ -degree $(r, -1)$. We call this spectral sequence the *E-Adams spectral sequence for the computation of $[X, Y]_*$* .

3.2. The E_1 page. The goal of this subsection is to provide a nicer characterization of the E_1 page of the E -Adams spectral sequence for the computation of $[X, Y]_*$. Given a monoid object (E, μ, e) in \mathcal{SH} .

Theorem 3.4. *Let (E, μ, e) be a flat commutative monoid object in \mathcal{SH} , and let X and Y be two objects in \mathcal{SH} . Further suppose at least one of the following hold:*

- (1) *E and X are cellular objects ([Definition 2.5](#)) and $E_*(X)$ is a graded projective ([Definition C.12](#)) left $\pi_*(E)$ -module (via ??).*
- (2) *There exists a collection of $a_i \in A$ indexed by some set I such that $E \otimes X$ is a retract of $\bigoplus_i \Sigma^{a_i} E$*

Then for all $s \geq 0$ and $a \in A$, we have isomorphisms in the associated E -Adams spectral sequence

$$E_1^{s,a}(X, Y) \cong \text{Hom}_{E_*(E)}^a(E_*(X), E_*(W_s))$$

Furthermore, under these isomorphisms, the differential $d_1 : E_1^{s,a} \rightarrow E_1^{s+1,a-1}$ corresponds to the map

$$\text{Hom}_{E_*(E)}^a(E_*(X), E_*(W_s)) \rightarrow \text{Hom}_{E_*(E)}^{a-1}(E_*(X), E_*(X \otimes W_{s+1}))$$

which sends a map $f : E_(X) \rightarrow E_{*+a}(W_s)$ to the composition*

$$E_*(X) \xrightarrow{f} E_{*+a}(W_s) \xrightarrow{(X \otimes h_s)_*} E_{*+a-1}(X \otimes Y_{s+1}) \xrightarrow{(X \otimes j_{s+1})_*} E_{*+a-1}(X \otimes W_{s+1}).$$

Proof. By ??, for all $s \geq 0$ and $t, w \in \mathbb{Z}$, we have isomorphisms

$$[X, E \otimes Y_s]_{t,w} \cong \text{Hom}_{E_*(E)}^{t,w}(E_*(X), E_*(E \otimes Y_s)).$$

since $W_s = E \otimes Y_s$, we have that

$$E_1^{s,(t,w)} = [X, W_s]_{t,w} \cong \text{Hom}_{E_*(E)}^{t,w}(E_*(X), E_*(W_s)),$$

as desired. □

Definition 3.5. Let (E, μ, e) be a monoid object in \mathcal{SH} . We say E is *flat* if the canonical right $\pi_*(E)$ -module structure on $E_*(E)$ is that of a flat module.

3.3. The E_2 page.

3.4. Convergence. convergence of spectral sequences

4. THE CLASSICAL ADAMS SPECTRAL SEQUENCE

5. THE MOTIVIC ADAMS SPECTRAL SEQUENCE

APPENDIX A. TRIANGULATED CATEGORIES

We assume the reader is familiar with additive categories and (closed, symmetric) monoidal categories.

Definition A.1. A *triangulated category* $(\mathcal{C}, \Sigma, \mathcal{D})$ is the data of:

- (1) An additive category \mathcal{C} .
- (2) An additive auto-equivalence $\Sigma : \mathcal{C} \rightarrow \mathcal{C}$ called the *shift functor*.
- (3) A collection \mathcal{D} of *distinguished* triangles in \mathcal{C} , where a is a sequence of arrows of the form

$$X \rightarrow Y \rightarrow Z \rightarrow \Sigma X.$$

Distinguished triangles are also sometimes called *cofiber sequences* or *fiber sequences*.

These data must satisfy the following axioms:

TR0 Given a commutative diagram

$$\begin{array}{ccccccc} X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & \Sigma X \\ \cong \downarrow & & \cong \downarrow & & \cong \downarrow & & \cong \downarrow \\ X' & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & \Sigma X' \end{array}$$

where the vertical arrows are isomorphisms, if the top row is distinguished then so is the bottom.

TR1 For any object X in \mathcal{C} , the diagram

$$X \xrightarrow{\text{id}_X} X \rightarrow 0 \rightarrow \Sigma X$$

is a distinguished triangle.

TR2 For all $f : X \rightarrow Y$ there exists an object C_f (also sometimes denoted Y/X) called the *cofiber of f* and a distinguished triangle

$$X \xrightarrow{f} Y \rightarrow C_f \rightarrow \Sigma X.$$

TR3 Given a solid diagram

$$\begin{array}{ccccccc} X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & \Sigma X \\ f \downarrow & & \downarrow & & \downarrow & & \downarrow \Sigma f \\ X' & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & \Sigma X' \end{array}$$

such that the leftmost square commutes and both rows are distinguished, there exists a dashed arrow $Z \rightarrow Z'$ which makes the remaining two squares commute.

TR4 A triangle

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$$

is distinguished if and only if

$$Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X \xrightarrow{-\Sigma f} \Sigma Y$$

is distinguished.

TR5 (Octahedral axiom) Given three distinguished triangles

$$X \xrightarrow{f} Y \xrightarrow{h} Y/X \rightarrow \Sigma X$$

$$Y \xrightarrow{g} Z \xrightarrow{k} Z/Y \rightarrow \Sigma Y$$

$$X \xrightarrow{g \circ f} Z \xrightarrow{l} Z/X \rightarrow \Sigma X$$

there exists a distinguished triangle

$$Y/X \xrightarrow{u} Z/X \xrightarrow{v} Z/Y \xrightarrow{w} \Sigma(Y/X)$$

such that the following diagram commutes

$$\begin{array}{ccccccc}
 X & \xrightarrow{g \circ f} & Z & \xrightarrow{k} & Z/Y & \xrightarrow{w} & \Sigma(Y/X) \\
 & \searrow f & \nearrow g & \searrow l & \nearrow v & \searrow & \nearrow \Sigma h \\
 & Y & & Z/X & & \searrow & \nearrow \Sigma f \\
 & & \searrow h & \nearrow u & & \searrow & \nearrow \\
 & & Y/X & \xrightarrow{\quad} & \Sigma X & &
 \end{array}$$

It turns out that the above definition is actually redundant; TR3 and TR4 follow from the remaining axioms (see Lemmas 2.2 and 2.4 in [3]). From now on, we fix a triangulated category $(\mathcal{C}, \Sigma, \mathcal{D})$. To start, recall the following definition:

Definition A.2. A sequence

$$X_1 \rightarrow X_2 \rightarrow \cdots \rightarrow X_n$$

of arrows in \mathcal{C} is *exact* if, for any object A in \mathcal{C} , the induced sequences

$$[A, X_1] \rightarrow [A, X_2] \rightarrow \cdots \rightarrow [A, X_{n-1}] \rightarrow [A, X_n]$$

and

$$[X_n, A] \rightarrow [X_{n-1}, A] \rightarrow \cdots \rightarrow [X_2, A] \rightarrow [X_1, A]$$

are exact sequences of abelian groups.

Proposition A.3. *Every distinguished triangle is an exact sequence (in the sense of Definition A.2).*

Proof. Suppose we have some distinguished triangle

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X.$$

Then first we would like to show that given any object A in \mathcal{C} , the sequence

$$[A, X] \xrightarrow{f_*} [A, Y] \xrightarrow{g_*} [A, Z] \xrightarrow{h_*} [A, \Sigma X]$$

is exact. First we show exactness at $[A, Y]$. To see $\text{im } f_* \subseteq \ker g_*$, note it suffices to show that $g \circ f = 0$. Indeed, consider the commuting diagram

$$\begin{array}{ccccccc}
 X & \xlongequal{\quad} & X & \longrightarrow & 0 & \longrightarrow & \Sigma X \\
 \parallel & & \downarrow f & & & & \\
 X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & \Sigma X
 \end{array}$$

The top row is distinguished by axiom TR1. Thus by TR3, the following diagram commutes:

$$\begin{array}{ccccccc} X & \xlongequal{\quad} & X & \longrightarrow & 0 & \longrightarrow & \Sigma X \\ \parallel & & \downarrow f & & \downarrow & & \parallel \\ X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & \Sigma X \end{array}$$

In particular, commutativity of the second square tells us that $g \circ f = 0$, as desired. Conversely, we'd like to show that $\ker g_* \subseteq \operatorname{im} f_*$. Let $\psi : A \rightarrow Y$ be in the kernel of g_* , so that $g \circ \psi = 0$. Consider the following commutative diagram:

$$\begin{array}{ccccccc} A & \longrightarrow & 0 & \longrightarrow & \Sigma A & \xrightarrow{-\Sigma \operatorname{id}_A} & \Sigma A \\ \psi \downarrow & & \downarrow & & \downarrow & & \downarrow \\ Y & \xrightarrow{g} & Z & \xrightarrow{h} & \Sigma X & \xrightarrow{-\Sigma f} & \Sigma Y \end{array}$$

The top row is distinguished by axioms TR1 and TR4. The bottom row is distinguished by axiom TR4. Thus by axiom TR3 there exists a map $\tilde{\phi} : \Sigma A \rightarrow \Sigma X$ such that the following diagram commutes:

$$\begin{array}{ccccccc} A & \longrightarrow & 0 & \longrightarrow & \Sigma A & \xrightarrow{-\Sigma \operatorname{id}_A} & \Sigma A \\ \psi \downarrow & & \downarrow & & \tilde{\phi} \downarrow & & \Sigma \psi \downarrow \\ Y & \xrightarrow{g} & Z & \xrightarrow{h} & \Sigma X & \xrightarrow{-\Sigma f} & \Sigma Y \end{array}$$

Now, since Σ is an equivalence, it is a full functor, so that in particular there exists some $\phi : A \rightarrow X$ such that $\tilde{\phi} = \Sigma \phi$. Then by faithfulness, we may pull back the right square to get a commuting diagram

$$\begin{array}{ccc} A & \xrightarrow{-\operatorname{id}_A} & A \\ \phi \downarrow & & \downarrow \psi \\ X & \xrightarrow{-f} & Y \end{array}$$

Hence,

$$f_*(\phi) = f \circ \phi \stackrel{(*)}{=} -((-f) \circ \phi) = -(\psi \circ (-\operatorname{id}_A)) \stackrel{(*)}{=} \psi \circ \operatorname{id}_A = \psi,$$

where the equalities marked $(*)$ follow by bilinearity of composition in an additive category. Thus $\psi \in \operatorname{im} f_*$, as desired, meaning $\ker g_* \subseteq \operatorname{im} f_*$.

Now, we have shown that

$$[A, X] \xrightarrow{f_*} [A, Y] \xrightarrow{g_*} [A, Z] \xrightarrow{h_*} [A, \Sigma X]$$

is exact at $[A, Y]$. It remains to show exactness at $[A, Z]$. Yet this follows by the exact same argument given above applied to the sequence obtained from the shifted triangle (TR4)

$$Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X \xrightarrow{-\Sigma f} \Sigma Y$$

On the other hand, we would like to show that

$$[\Sigma X, A] \xrightarrow{h^*} [Z, A] \xrightarrow{g^*} [Y, A] \xrightarrow{f^*} [X, A]$$

is exact. As above, since we can shift the triangle, it suffices to show exactness at $[Z, A]$. First, since we have shown $g \circ f = 0$, we have $f^* \circ g^* = (g \circ f)^* = 0$, so that $\operatorname{im} g^* \subseteq \ker f^*$, as desired.

Conversely, in order to see $\ker f^* \subseteq \operatorname{im} g^*$, suppose $\psi : Y \rightarrow A$ is in the kernel of f^* , so that $\psi \circ f = 0$. Consider the following commuting diagram:

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & \Sigma X \\ \downarrow & & \downarrow \psi & & & & \\ 0 & \longrightarrow & A & \xlongequal{\quad} & A & \longrightarrow & 0 \end{array}$$

The top row is a distinguished triangle by assumption, and the bottom row is distinguished by axioms TR1 and TR4 for a triangulated category, along with the fact that $\Sigma 0 = 0$ since Σ is additive. Thus by axiom TR3 there exists a map $\phi : Z \rightarrow A$ such that $\phi \circ g = \psi$, i.e., $g^*(\phi) = \psi$, so that $\phi \in \operatorname{im} g^*$ as desired.

Thus, we have shown exactness of

$$[\Sigma X, A] \xrightarrow{h^*} [Z, A] \xrightarrow{g^*} [Y, A] \xrightarrow{f^*} [X, A]$$

at $[Y, A]$. To see $\ker g^* = \operatorname{im} h^*$, again the same arguments applied to the shifted triangle

$$Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X \xrightarrow{-\Sigma f} \Sigma Y$$

yield that

$$[\Sigma Y, A] \xrightarrow{(-\Sigma f)^*} [\Sigma X, A] \xrightarrow{h^*} [Z, A] \xrightarrow{g^*} [Y, A]$$

is exact at $[Z, A]$, so $\ker g^* = \operatorname{im} h^*$, as desired. \square

Lemma A.4. *Suppose we have a commutative diagram*

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & \Sigma X \\ \downarrow j & & \downarrow k & & \downarrow \ell & & \downarrow \Sigma j \\ X' & \xrightarrow{f'} & Y' & \xrightarrow{g'} & Z' & \xrightarrow{h'} & \Sigma X' \end{array}$$

with both rows distinguished. Then if any two of the maps j , k , and ℓ are isomorphisms, then so is the third.

Proof. Suppose we are given any object A in \mathcal{C} , and consider the commutative diagram

$$\begin{array}{ccccccccccc} [A, X] & \xrightarrow{f_*} & [A, Y] & \xrightarrow{g_*} & [A, Z] & \xrightarrow{k_*} & [A, \Sigma X] & \xrightarrow{-\Sigma f_*} & [A, \Sigma Y] & \xrightarrow{-\Sigma g_*} & [A, \Sigma Z] & \xrightarrow{-\Sigma h_*} & [A, \Sigma^2 X] \\ \downarrow j_* & & \downarrow k_* & & \downarrow \ell_* & & \downarrow \Sigma j_* & & \downarrow \Sigma k_* & & \downarrow \Sigma \ell_* & & \downarrow \Sigma^2 j_* \\ [A, X'] & \xrightarrow{f'_*} & [A, Y'] & \xrightarrow{g'_*} & [A, Z'] & \xrightarrow{h'_*} & [A, \Sigma X'] & \xrightarrow{-\Sigma f'_*} & [A, \Sigma Y'] & \xrightarrow{-\Sigma g'_*} & [A, \Sigma Z'] & \xrightarrow{-\Sigma h'_*} & [A, \Sigma^2 X'] \end{array}$$

The rows are exact by [Proposition A.3](#) and repeated applications of axiom TR4. It follows by the five lemma that if j and k are isomorphisms, then ℓ_* is an isomorphism. Similarly, if k and ℓ are isomorphisms then Σj_* is an isomorphism. Finally, if ℓ and j are isomorphisms, then Σk_* is an isomorphism. The desired result follows by faithfulness of Σ and the Yoneda embedding. \square

Proposition A.5. *Given a map $f : X \rightarrow Y$ in a triangulated category $(\mathcal{C}, \Sigma, \Omega, \mathcal{D})$, the cofiber sequence of f is unique up to isomorphism, in the sense that given any two distinguished triangles*

$$X \xrightarrow{f} Y \rightarrow Z \rightarrow \Sigma X \quad \text{and} \quad X \xrightarrow{f} Y \rightarrow Z' \rightarrow \Sigma X,$$

there exists an isomorphism $Z \rightarrow Z'$ which makes the following diagram commute:

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \longrightarrow & Z & \longrightarrow & \Sigma X \\ \parallel & & \parallel & & \downarrow k & & \parallel \\ X & \xrightarrow{f} & Y & \longrightarrow & Z' & \longrightarrow & \Sigma X \end{array}$$

Proof. Suppose we have two distinguished triangles

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X \quad \text{and} \quad X \xrightarrow{f} Y \xrightarrow{g'} Z' \xrightarrow{h'} \Sigma X,$$

and consider the following commutative diagram

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & \Sigma X \\ \parallel & & \parallel & & & & \\ X & \xrightarrow{f} & Y & \xrightarrow{g'} & Z' & \xrightarrow{h'} & \Sigma X \end{array}$$

By axiom TR3, there exists some map $k : Z \rightarrow Z'$ which makes the following diagram commute:

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & \Sigma X \\ \parallel & & \parallel & & \downarrow k & & \parallel \\ X & \xrightarrow{f} & Y & \xrightarrow{g'} & Z' & \xrightarrow{h'} & \Sigma X \end{array}$$

Now, by Lemma A.4, k is an isomorphism. □

Proposition A.6. *Given an arrow $f : X \rightarrow Y$ in \mathcal{C} , there exists an object F_f called the fiber of f , and a distinguished triangle*

$$F_f \rightarrow X \xrightarrow{f} Y \rightarrow \Sigma F_f (\cong C_f).$$

Proof. Since Σ is an equivalence, there exists some functor $\Omega : \mathcal{C} \rightarrow \mathcal{C}$ and natural isomorphisms $\eta : \Omega \Sigma \Rightarrow \text{Id}_{\mathcal{C}}$ and $\varepsilon : \text{Id}_{\mathcal{C}} \Rightarrow \Sigma \Omega$. By axiom TR2, we have a distinguished triangle

$$X \xrightarrow{f} Y \xrightarrow{g} C_f \xrightarrow{h} \Sigma X.$$

Now, consider the commutative diagram

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & C_f & \xrightarrow{h} & \Sigma X \\ \parallel & & \parallel & & \downarrow \eta_{C_f} & & \parallel \\ X & \xrightarrow{f} & Y & \xrightarrow{\tilde{g}} & \Sigma \Omega C_f & \xrightarrow{\tilde{h}} & \Sigma X \end{array}$$

where $\tilde{g} = \eta_{C_f} \circ g$, and $\tilde{h} = \eta_{C_f} \circ h$. Since each vertical map is an isomorphism and the top row is distinguished, the bottom row is also distinguished by axiom TR0. Now, since Σ is an equivalence of categories, it is faithful, so that in particular there exists some map $k : \Omega C_f \rightarrow X$ such that $\Sigma k = -\tilde{h} \implies -\Sigma k = \tilde{h}$. Thus, we have a distinguished triangle of the form

$$X \xrightarrow{f} Y \xrightarrow{\tilde{g}} \Sigma \Omega C_f \xrightarrow{-\Sigma k} \Sigma X.$$

Finally, by axiom TR4, we get a distinguished triangle

$$\Omega C_f \xrightarrow{k} X \xrightarrow{f} Y \xrightarrow{\tilde{g}} \Sigma \Omega C_f,$$

so we may define the fiber of f to be ΩC_f . □

Definition A.7 ([4, Definition 1.6.4]). Assume that \mathcal{C} has countable coproducts. Let

$$X_0 \xrightarrow{j_1} X_1 \xrightarrow{j_2} X_2 \xrightarrow{j_3} X_3 \rightarrow \dots$$

be a sequence of objects and morphisms in \mathcal{C} . The *homotopy colimit* of the sequence, denoted $\operatorname{holim} X_i$, is by definition given, up to non-canonical isomorphism, as the cofiber of the map

$$\coprod_{i=0}^{\infty} X_i \xrightarrow{1-\text{shift}} \coprod_{i=0}^{\infty} X_i,$$

where the shift map $\coprod_{i=0}^{\infty} X_i \xrightarrow{\text{shift}} \coprod_{i=0}^{\infty} X_i$ is understood to be the direct sum of $j_{i+1} : X_i \rightarrow X_{i+1}$. In other words, the map $1 - \text{shift}$ is the infinite matrix

$$\begin{bmatrix} \operatorname{id}_{X_0} & 0 & 0 & 0 & \cdots \\ -j_1 & \operatorname{id}_{X_1} & 0 & 0 & \cdots \\ 0 & -j_2 & \operatorname{id}_{X_2} & 0 & \cdots \\ 0 & 0 & -j_3 & \operatorname{id}_{X_3} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Proposition A.8 ([4, Proposition 1.6.8]). Suppose \mathcal{C} has countable coproducts, and suppose $e : X \rightarrow X$ is an idempotent in \mathcal{C} , so that $e \circ e = e$. Then e splits in \mathcal{C} , i.e., e factors as

$$X \xrightarrow{r} Y \xrightarrow{\iota} X$$

with $r \circ \iota = \operatorname{id}_Y$ and $\iota \circ r = e$. In particular, we may take Y to be the colimit of

$$X \xrightarrow{e} X \xrightarrow{e} X \xrightarrow{e} X \xrightarrow{e} \dots$$

write proof
down?

Proof. See [4, Proposition 1.6.8] □

For our purposes, we will always be dealing with triangulated categories with a bit of extra structure, in the following sense:

Definition A.9. An *adjointly triangulated category* $(\mathcal{C}, \Omega, \Sigma, \eta, \varepsilon, \mathcal{D})$ is the data of a triangulated category $(\mathcal{C}, \Sigma, \mathcal{D})$ along with an inverse shift functor $\Omega : \mathcal{C} \rightarrow \mathcal{C}$ and natural isomorphisms $\eta : \operatorname{Id}_{\mathcal{C}} \Rightarrow \Sigma\Omega$ and $\varepsilon : \Omega\Sigma \Rightarrow \operatorname{Id}_{\mathcal{C}}$ such that $(\Omega, \Sigma, \eta, \varepsilon)$ forms an adjoint equivalence of \mathcal{C} . In other words, η and ε are natural isomorphisms which also are the unit and counit of an adjunction $\Omega \dashv \Sigma$, so they satisfy either of the following “zig-zag identities”:

$$\begin{array}{ccc} \Omega & \xrightarrow{\Omega\eta} & \Omega\Sigma\Omega \\ & \searrow & \downarrow \varepsilon\Omega \\ & & \Omega \end{array} \quad \begin{array}{ccc} \Sigma\Omega\Sigma & \xleftarrow{\eta\Sigma} & \Sigma \\ \Sigma\varepsilon \downarrow & & \swarrow \\ \Sigma & & \Sigma \end{array}$$

(Satisfying one implies the other is automatically satisfied, see [5, Lemma 3.2]).

From now on, we will assume that \mathcal{C} is an *adjointly triangulated category* with inverse shift Ω , unit $\eta : \operatorname{Id}_{\mathcal{C}} \Rightarrow \Sigma\Omega$, and counit $\varepsilon : \Omega\Sigma \Rightarrow \operatorname{Id}_{\mathcal{C}}$.

Lemma A.10. *Given a triangle*

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X,$$

it can be shifted to the left to obtain a distinguished triangle

$$\Omega Z \xrightarrow{\widetilde{h}} X \xrightarrow{f} Y \xrightarrow{\widetilde{\Omega}g} \Sigma\Omega Z,$$

where $\tilde{h} : \Omega Z \rightarrow X$ is the adjoint of $h : Z \rightarrow \Sigma X$ and $\tilde{\Omega}g : Y \rightarrow \Sigma \Omega Z$ is the adjoint of $\Omega g : \Omega Y \rightarrow \Omega Z$.

Proof. Note that unravelling definitions, then \tilde{h} and \tilde{g} are the compositions

$$\tilde{h} : \Omega Z \xrightarrow{\Omega h} \Omega \Sigma X \xrightarrow{\varepsilon_X} X \quad \text{and} \quad \tilde{\Omega}g : Y \xrightarrow{\eta_Y} \Sigma \Omega Y \xrightarrow{\Sigma \Omega g} \Sigma \Omega Z.$$

Now consider the following diagram:

$$(3) \quad \begin{array}{ccccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & \Sigma X \\ \parallel & & \parallel & & \eta_Z \downarrow & & \parallel \\ X & \xrightarrow{f} & Y & \xrightarrow{\tilde{\Omega}g} & \Sigma \Omega Z & \xrightarrow{\Sigma \tilde{h}} & \Sigma X \end{array}$$

The left square commutes by definition. To see that the middle square commutes, expanding definitions, note it is given by the following diagram:

$$\begin{array}{ccc} Y & \xrightarrow{g} & Z \\ \parallel & & \downarrow \eta_Y \\ Y & \xrightarrow{\eta_Y} & \Sigma \Omega Y \xrightarrow{\Sigma \Omega g} \Sigma \Omega Z \end{array}$$

and this commutes by naturality of η . To see that the right square commutes, consider the following diagram:

$$\begin{array}{ccc} Z & \xrightarrow{h} & \Sigma X \\ \eta_Z \downarrow & & \swarrow \eta_{\Sigma X} \\ \Sigma \Omega Z & \xrightarrow{\Sigma \Omega h} & \Sigma \Omega \Sigma X \xrightarrow{\Sigma \varepsilon_X} \Sigma X \end{array}$$

By functoriality of Σ , the bottom composition is $\Sigma \tilde{h}$. The left region commutes by naturality of η . Commutativity of the right region is precisely one of the zig-zag identities. Hence, since diagram (3) commutes, the vertical arrows are isomorphisms, and the bottom row is distinguished, we have that the top row is distinguished as well by axiom TR0. Then by axiom TR4, since $(f, \tilde{\Omega}g, \Sigma \tilde{h})$ is distinguished, so is the triangle

$$\Omega Z \xrightarrow{\tilde{h}} X \xrightarrow{f} Y \xrightarrow{\tilde{\Omega}g} \Sigma \Omega Z. \quad \square$$

Lemma A.11. *Given a distinguished triangle*

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X,$$

for any $n > 0$, the triangle

$$\Omega^n X \xrightarrow{(-1)^n \Omega^n f} \Omega^n Y \xrightarrow{(-1)^n \Omega^n g} \Omega^n Z \xrightarrow{(-1)^n \Omega^n h} \Omega^n \Sigma X \cong \Sigma \Omega^n X,$$

is distinguished, where the final isomorphism is given by the composition

$$\Omega^n \Sigma X = \Omega^{n-1} \Omega \Sigma X \xrightarrow{\Omega^{n-1} \varepsilon_X} \Omega^{n-1} X \xrightarrow{\eta_{\Omega^{n-1} X}} \Sigma \Omega \Omega^{n-1} X = \Sigma \Omega^n X.$$

Proof. We give a proof by induction. First we show the case $n = 1$. Note by Lemma A.10, we have that given a distinguished triangle

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X,$$

we can shift it to the left to obtain a distinguished triangle

$$\Omega Z \xrightarrow{\tilde{h}} X \xrightarrow{f} Y \xrightarrow{\tilde{\Omega}g} \Sigma \Omega Z,$$

where \widetilde{h} is the adjoint of $h : Z \rightarrow \Sigma X$ and $\widetilde{\Omega}g$ is the adjoint of $\Omega g : \Omega Y \rightarrow \Omega Z$. If we apply this shifting operation again, we get the distinguished triangle

$$\Omega Y \xrightarrow{-\widetilde{\Omega}g} \Omega Z \xrightarrow{-\widetilde{h}} X \xrightarrow{\widetilde{\Omega}f} \Sigma \Omega Y,$$

where unravelling definitions, $\widetilde{\Omega}f$ is the right adjoint of $\Omega f : \Omega X \rightarrow \Omega Y$ and $\widetilde{\Omega}g$ is the right adjoint of $\widetilde{\Omega}g$, which itself is the left adjoint of Ωg , so $\widetilde{\widetilde{\Omega}g} = \Omega g$. Hence we have a distinguished triangle

$$\Omega Y \xrightarrow{-\Omega g} \Omega Z \xrightarrow{-\widetilde{h}} X \xrightarrow{\widetilde{\Omega}f} \Sigma \Omega Y.$$

We may again shift this triangle again and the above arguments yield the distinguished triangle

$$\Omega X \xrightarrow{-\Omega f} \Omega Y \xrightarrow{-\Omega g} \Omega Z \xrightarrow{\widetilde{\Omega(-\widetilde{h})}} \Sigma \Omega X,$$

where $\widetilde{\Omega(-\widetilde{h})}$ is the right adjoint of $\Omega(-\widetilde{h}) = -\Omega\widetilde{h} : \Omega \Omega Z \rightarrow \Omega X$. Explicitly unravelling definitions, $\widetilde{\Omega(-\widetilde{h})} = -\widetilde{\Omega\widetilde{h}}$ is the composition

$$\begin{aligned} [\Omega Z \xrightarrow{\eta_{\Omega Z}} \Sigma \Omega \Omega Z \xrightarrow{\Sigma(-\Omega\widetilde{h})} \Sigma \Omega X] &= -[\Omega Z \xrightarrow{\eta_{\Omega Z}} \Sigma \Omega \Omega Z \xrightarrow{\Sigma \Omega \widetilde{h}} \Sigma \Omega X] \\ &= -[\Omega Z \xrightarrow{\eta_{\Omega Z}} \Sigma \Omega \Omega Z \xrightarrow{\Sigma \Omega \Omega h} \Sigma \Omega \Omega \Sigma X \xrightarrow{\Sigma \Omega \varepsilon_X} \Sigma \Omega X] \\ &= -[\Omega Z \xrightarrow{\Omega h} \Omega \Sigma X \xrightarrow{\varepsilon_X} X \xrightarrow{\eta_X} \Sigma \Omega X], \end{aligned}$$

where the first equality follows by additivity of Σ and additivity of composition, the second follows by further unravelling how \widetilde{h} is defined, and the third follows by naturality of η , which tells us the following diagram commutes:

$$\begin{array}{ccccc} \Omega Z & \xrightarrow{\Omega h} & \Omega \Sigma X & \xrightarrow{\varepsilon_X} & X \\ \downarrow \eta_{\Omega Z} & & \downarrow \eta_{\Omega \Sigma X} & & \downarrow \eta_X \\ \Sigma \Omega \Omega Z & \xrightarrow{\Sigma \Omega \Omega h} & \Sigma \Omega \Omega \Sigma X & \xrightarrow{\Sigma \Omega \varepsilon_X} & \Sigma \Omega X \end{array}$$

Thus indeed we have a distinguished triangle

$$\Omega X \xrightarrow{-\Omega f} \Omega Y \xrightarrow{-\Omega g} \Omega Z \xrightarrow{-\Omega h} \Omega \Sigma X \cong \Sigma \Omega X,$$

where the last isomorphism is $\eta_X \circ \varepsilon_X$, as desired.

Now, we show the inductive step. Suppose we know that given a distinguished triangle

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X,$$

that for some $n > 0$ the triangle

$$\Omega^n X \xrightarrow{(-1)^n \Omega^n f} \Omega^n Y \xrightarrow{(-1)^n \Omega^n g} \Omega^n Z \xrightarrow{(-1)^n h^n} \Sigma \Omega^n X,$$

is distinguished, where $h^n : \Omega^n Z \rightarrow \Sigma \Omega^n X$ is the composition

$$\Omega^n Z \xrightarrow{\Omega^n h} \Omega^n \Sigma X \xrightarrow{\Omega^{n-1} \varepsilon_X} \Omega^{n-1} X \xrightarrow{\eta_{\Omega^{n-1} X}} \Sigma \Omega^n X.$$

Then by applying the $n = 1$ case to this triangle, we get that the following triangle is distinguished

$$\Omega^{n+1} X \xrightarrow{-\Omega((-1)^n \Omega^n f)} \Omega^{n+1} Y \xrightarrow{-\Omega((-1)^n \Omega^n g)} \Omega^{n+1} Z \xrightarrow{-\Omega((-1)^n h^n)} \Omega \Sigma \Omega^n X \cong \Sigma \Omega^{n+1} X,$$

where the final isomorphism is the composition

$$\Omega \Sigma \Omega^n X \xrightarrow{\varepsilon_{\Omega^n X}} \Omega^n X \xrightarrow{\eta_{\Omega^n X}} \Sigma \Omega \Omega^n X = \Sigma \Omega^{n+1} X.$$

We claim that this is precisely the distinguished triangle given in the statement of the lemma for $n+1$. First of all, note that $-\Omega((-1)^n \Omega^n f) = (-1)^{n+1} \Omega^{n+1} f$, $-\Omega((-1)^n \Omega^n g) = (-1)^{n+1} \Omega^{n+1} g$, and $-\Omega((-1)^n h^n) = (-1)^{n+1} \Omega h^n$ by additivity of Ω , so that the triangle becomes

$$(4) \quad \Omega^{n+1} X \xrightarrow{(-1)^{n+1} \Omega^{n+1} f} \Omega^{n+1} Y \xrightarrow{(-1)^{n+1} \Omega^{n+1} g} \Omega^{n+1} Z \xrightarrow{(-1)^{n+1} \Omega h^n} \Omega \Sigma \Omega^n X \cong \Sigma \Omega^{n+1} X.$$

Thus, in order to prove the desired characterization, it remains to show this diagram commutes:

$$\begin{array}{ccccc} \Omega^{n+1} Z & \xrightarrow{(-1)^{n+1} \Omega h^n} & \Omega \Sigma \Omega^n X & \xrightarrow{\varepsilon_{\Omega^n X}} & \Omega^n X \\ \downarrow (-1)^{n+1} \Omega^{n+1} h & & & & \downarrow \eta_{\Omega^n X} \\ \Omega^{n+1} \Sigma X & \xrightarrow{\Omega^n \varepsilon_X} & \Omega^n X & \xrightarrow{\eta_{\Omega^n X}} & \Sigma \Omega^{n+1} X \end{array}$$

(The top composition is the last two arrows in diagram (4), and the bottom composition is the last two arrows in the diagram in the statement of the lemma). Unravelling how h^n is constructed, by additivity of Ω it further suffices to show the outside of the following diagram commutes:

$$\begin{array}{ccccccc} \Omega^{n+1} Z & \xrightarrow{(-1)^{n+1} \Omega^{n+1} h} & \Omega^{n+1} \Sigma X & \xrightarrow{\Omega^n \varepsilon_X} & \Omega^n X & \xrightarrow{\Omega \eta_{\Omega^{n-1} X}} & \Omega \Sigma \Omega^n X \\ \downarrow (-1)^{n+1} \Omega^{n+1} h & & & & \parallel & & \downarrow \varepsilon_{\Omega^n X} \\ \Omega^{n+1} \Sigma X & \xrightarrow{\Omega^n \varepsilon_X} & \Omega^n X & \xrightarrow{\eta_{\Omega^n X}} & \Sigma \Omega^{n+1} X & & \downarrow \eta_{\Omega^n X} \end{array}$$

The left rectangle and bottom right triangle commute by definition. Finally, commutativity of the top right trapezoid is precisely one of the zig-zag identities applied to $\Omega^{n-1} X$. Hence, we have shown the desired result. \square

Proposition A.12. *Given a distinguished triangle*

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X,$$

let $\tilde{h} : \Omega Z \rightarrow X$ be the left adjoint of h . Then the following infinite sequence is exact:

$$\begin{array}{ccccccc} & & & & \cdots & & \\ & & & & \swarrow & & \\ \Omega^{n+1} Z & \xrightarrow[(-1)^{n+1} \Omega^n \tilde{h}]{} \Omega^n X & \xrightarrow[(-1)^n \Omega^n f]{} \Omega^n Y & \xrightarrow{(-1)^n \Omega^n g} \Omega^n Z & \xrightarrow{(-1)^n \Omega^{n-1} \tilde{h}} \Omega^{n-1} X & & \\ & & & & \swarrow & & \\ & & & & \cdots & & \\ \Omega Z & \xrightarrow[-\tilde{h}]{} X & \xrightarrow{f} Y & \xrightarrow{g} Z & \xrightarrow{h} \Sigma X & & \\ & & & & \swarrow & & \\ & & & & \cdots & & \\ \Sigma^{n-1} Z & \xrightarrow[(-1)^{n-1} \Sigma^n \tilde{h}]{} \Sigma^n X & \xrightarrow[(-1)^n \Sigma^n f]{} \Sigma^n Y & \xrightarrow{(-1)^n \Sigma^n g} \Sigma^n Z & \xrightarrow{(-1)^n \Sigma^n h} \Sigma^{n+1} X & & \\ & & & & \swarrow & & \\ & & & & \cdots & & \end{array}$$

In particular, it remains exact even if we remove the signs.

Proof. Exactness of

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X \xrightarrow{-\Sigma f} \Sigma Y$$

is [Proposition A.3](#) and axiom TR4. By induction using axiom TR4, for $n > 0$ we get that each contiguous composition of three maps below is a distinguished triangle:

$$\Sigma^n X \xrightarrow{(-1)^n \Sigma^n f} \Sigma^n Y \xrightarrow{(-1)^n \Sigma^n g} \Sigma^n Z \xrightarrow{(-1)^n \Sigma^n h} \Sigma^{n+1} X \xrightarrow{(-1)^{n+1} \Sigma^{n+1} f} \Sigma^{n+1} Y,$$

thus the sequence is exact by [Proposition A.3](#). It remains to show exactness of the LES to the left of Y . It suffices to show that the row in the following diagram is exact for all $n > 0$:

$$(5) \quad \begin{array}{ccccccc} \Omega^n X & \xrightarrow{(-1)^n \Omega^n f} & \Omega^n Y & \xrightarrow{(-1)^n \Omega^n g} & \Omega^n Z & \xrightarrow{(-1)^n \Omega^{n-1}(\varepsilon_X \circ \Omega h)} & \Omega^{n-1} X \xrightarrow{(-1)^{n-1} \Omega^{n-1} f} \Omega^{n-1} Y \\ & & & & \searrow^{(-1)^n \Omega^n h} & & \nearrow_{\Omega^{n-1} \varepsilon_X} \\ & & & & & \Omega^n \Sigma X & \end{array}$$

First of all, to see exactness at $\Omega^n Y$ and $\Omega^n Z$, consider the following commutative diagram:

$$\begin{array}{ccccccc} \Omega^n X & \xrightarrow{(-1)^n \Omega^n f} & \Omega^n Y & \xrightarrow{(-1)^n \Omega^n g} & \Omega^n Z & \xrightarrow{(-1)^n \Omega^{n-1}(\varepsilon_X \circ \Omega h)} & \Omega^{n-1} X \\ \parallel & & \parallel & & \parallel & \nearrow^{(-1)^n \Omega^n h} & \nearrow_{\Omega^{n-1} \varepsilon_X} \\ \Omega^n X & \xrightarrow{(-1)^n \Omega^n f} & \Omega^n Y & \xrightarrow{(-1)^n \Omega^n g} & \Omega^n Z & \xrightarrow{(-1)^n \Omega^n h} & \Omega^n \Sigma X \\ & & & & & \nearrow_{(-1)^n \Omega^n h} & \\ & & & & & \xrightarrow{\quad \quad \quad} & \Sigma \Omega^n X \end{array}$$

(here the dashed arrow is the morphism which makes the diagram commute).

(here the dashed arrow is the morphism which makes the diagram commute). The bottom row is distinguished by [Lemma A.11](#). Then by axiom TR0, the top row is distinguished, and thus exact by [Proposition A.3](#). Thus we have shown exactness of (5) at $\Omega^n Y$ and $\Omega^n Z$. It remains to show exactness at $\Omega^{n-1} X$. In the case $n = 1$, we want to show exactness at X in the following diagram:

$$\begin{array}{ccccc} \Omega Z & \xrightarrow{-(\varepsilon_X \circ \Omega h)} & X & \xrightarrow{f} & Y \\ & \searrow^{-\Omega h} & \nearrow_{\varepsilon_X} & & \\ & & \Omega \Sigma X & & \end{array}$$

Unravelling definitions, $\varepsilon_X \circ \Omega h$ is precisely the adjoint $\tilde{h} : \Omega Z \rightarrow X$ of $h : Z \rightarrow \Sigma X$, in which case we have that the row in the above diagram fits into a distinguished triangle by [Lemma A.10](#), and thus it is exact by [Proposition A.3](#). To see exactness at $\Omega^{n-1} X$ in diagram (5), note that if we apply [Lemma A.10](#) to the sequence [Lemma A.11](#) for $n - 1$, then we get that the following composition fits into a distinguished triangle, and is thus exact:

$$\Omega^n Z \xrightarrow{-k} \Omega^{n-1} X \xrightarrow{(-1)^{n-1} \Omega^{n-1} f} \Omega^{n-1} Y,$$

where $k : \Omega(\Omega^{n-1} Z) \rightarrow \Omega^{n-1} X$ is the adjoint of the composition

$$\Omega^{n-1} Z \xrightarrow{(-1)^{n-1} \Omega^{n-1} h} \Omega^{n-1} \Sigma X \xrightarrow{\Omega^{n-2} \varepsilon_X} \Omega^{n-2} X \xrightarrow{\eta_{\Omega^{n-2} X}} \Sigma \Omega^{n-1} X.$$

Further expanding how adjoints are constructed, k is the composition

$$\Omega^n Z \xrightarrow{(-1)^{n-1} \Omega^n h} \Omega^n \Sigma X \xrightarrow{\Omega^{n-1} \varepsilon_X} \Omega^{n-1} X \xrightarrow{\Omega \eta_{\Omega^{n-2} X}} \Omega \Sigma \Omega^{n-1} X \xrightarrow{\varepsilon_{\Omega^{n-1} X}} \Omega^{n-1} X.$$

Thus, in order to show exactness of (5) at $\Sigma^{n-1}X$, it suffices to show that $k = (-1)^{n-1}\Omega^{n-1}(\varepsilon_X \circ \Omega h)$. To that end, consider the following diagram:

$$\begin{array}{ccccc}
 \Omega^n Z & \xrightarrow{(-1)^{n-1}\Omega^n h} & \Omega^n \Sigma X & \xrightarrow{\Omega^{n-1}\varepsilon_X} & \Omega^{n-1} X & \xrightarrow{\Omega \eta_{\Omega^{n-2} X}} & \Omega \Sigma \Omega^{n-1} X \\
 \downarrow (-1)^{n-1}\Omega^n h & & & & \searrow & & \downarrow \varepsilon_{\Omega^{n-1} X} \\
 \Omega^n \Sigma X & \xrightarrow{\Omega^{n-1}\varepsilon_X} & \Omega^{n-1} X & & & &
 \end{array}$$

The top composition is k , while the bottom composition is $(-1)^{n-1}\Omega^{n-1}(\varepsilon_X \circ \Omega h)$. The left region commutes by definition, while commutativity of the right region is precisely one of the zig-zag identities applied to $\Omega^{n-2}X$. Thus, we have shown that $-k = (-1)^n\Omega^{n-1}(\varepsilon_X \circ \Omega h)$, so (5) is exact at $\Omega^{n-1}X$, as desired. \square

Also important for our work is the concept of a *tensor triangulated category*, that is, a triangulated symmetric monoidal category in which the triangulated structures are compatible, in the following sense:

Definition A.13. A *tensor triangulated category* is a triangulated symmetric monoidal category $(\mathcal{C}, \otimes, S, \Sigma, \mathcal{D})$ such that:

TT1 For all objects X and Y in \mathcal{C} , there are natural isomorphisms

$$e_{X,Y} : \Sigma X \otimes Y \xrightarrow{\cong} \Sigma(X \otimes Y).$$

TT2 For each object X in \mathcal{C} , the functor $X \otimes (-) \cong (-) \otimes X$ is an additive functor.

TT3 For each object X in \mathcal{C} , the functor $X \otimes (-) \cong (-) \otimes X$ preserves distinguished triangles, in that given a distinguished triangle/(co)fiber sequence

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A,$$

then also

$$X \otimes A \xrightarrow{X \otimes f} X \otimes B \xrightarrow{X \otimes g} X \otimes C \xrightarrow{X \otimes h} \Sigma(X \otimes A)$$

and

$$A \otimes X \xrightarrow{f \otimes X} B \otimes X \xrightarrow{g \otimes X} C \otimes X \xrightarrow{h \otimes X} \Sigma(A \otimes X)$$

are distinguished triangles, where here we are being abusive and writing $X \otimes h$ and $h \otimes X$ to denote the compositions

$$X \otimes C \xrightarrow{X \otimes h} X \otimes \Sigma A \xrightarrow{\tau} \Sigma A \otimes X \xrightarrow{e_{A,X}} \Sigma(A \otimes X) \xrightarrow{\Sigma \tau} \Sigma(X \otimes A)$$

and

$$C \otimes X \xrightarrow{h \otimes X} \Sigma A \otimes X \xrightarrow{e_{A,X}} \Sigma(A \otimes X),$$

respectively.

TT4 Given objects X , Y , and Z in \mathcal{C} , the following diagram must commute:

$$\begin{array}{ccc}
 (\Sigma X \otimes Y) \otimes Z & \xrightarrow{e_{X,Y} \otimes Z} & \Sigma(X \otimes Y) \otimes Z & \xrightarrow{e_{X \otimes Y, Z}} & \Sigma((X \otimes Y) \otimes Z) \\
 \alpha \downarrow & & & & \downarrow \Sigma \alpha \\
 \Sigma X \otimes (Y \otimes Z) & \xrightarrow{e_{X,Y \otimes Z}} & \Sigma(X \otimes (Y \otimes Z)) & &
 \end{array}$$

Usually, most tensor triangulated categories that arise in nature will satisfy additional coherence axioms (see axioms TC1–TC5 in [3]), but the above definition will suffice for our purposes. In what follows, we fix a tensor triangulated category $(\mathcal{C}, \otimes, S, \Sigma, e, \mathcal{D})$.

Definition A.14. There are natural isomorphisms

$$e'_{X,Y} : X \otimes \Sigma Y \xrightarrow{\cong} \Sigma(X \otimes Y)$$

obtained via the composition

$$X \otimes \Sigma Y \xrightarrow{\tau} \Sigma Y \otimes X \xrightarrow{e_{Y,X}} \Sigma(Y \otimes X) \xrightarrow{\Sigma\tau} \Sigma(X \otimes Y).$$

Lemma A.15. For all X and Y in \mathcal{C} , the following diagram commutes:

$$\begin{array}{ccc} \Sigma X \otimes \Sigma Y & \xrightarrow{-e_{X,\Sigma Y}} & \Sigma(X \otimes \Sigma Y) \\ e'_{\Sigma X, Y} \downarrow & & \downarrow \Sigma e'_{X,Y} \\ \Sigma(\Sigma X \otimes Y) & \xrightarrow{\Sigma e_{X,Y}} & \Sigma\Sigma(X \otimes Y) \end{array}$$

(note the sign on the top map).

Proof. Note there are natural isomorphisms

$$a_X : \Sigma X \xrightarrow{\cong} \Sigma S \otimes X$$

given by the composition

$$\Sigma X \xrightarrow{\Sigma\lambda_X} \Sigma(S \otimes X) \xrightarrow{e_{S,X}^{-1}} \Sigma S \otimes X.$$

Furthermore, under the isomorphism $a : \Sigma \cong \Sigma S \otimes -$, $e_{X,Y} : \Sigma X \otimes Y \cong \Sigma(X \otimes Y)$ corresponds to the associator $(\Sigma S \otimes X) \otimes Y \cong \Sigma S \otimes (X \otimes Y)$. Indeed, consider the following diagram:

$$\begin{array}{ccc} \Sigma X \otimes Y & \xrightarrow{e_{X,Y}} & \Sigma(X \otimes Y) \\ \Sigma\lambda_{X \otimes Y} \downarrow & \swarrow \Sigma(\lambda_X \otimes Y) & \downarrow \Sigma\lambda_{X \otimes Y} \\ \Sigma(S \otimes X) \otimes Y & \xrightarrow{e_{S \otimes X, Y}} \Sigma((S \otimes X) \otimes Y) & \xrightarrow{\Sigma\alpha} \Sigma(S \otimes (X \otimes Y)) \\ e_{S,X}^{-1} \otimes Y \downarrow & & \downarrow e_{S,X \otimes Y}^{-1} \\ (\Sigma S \otimes X) \otimes Y & \xrightarrow{\alpha} & \Sigma S \otimes (X \otimes Y) \end{array}$$

The two vertical composites are $a_X \otimes Y$ and $a_{X \otimes Y}$, respectively. The top trapezoid commutes by naturality of e . The triangle commutes by coherence for a monoidal category. Finally, commutativity of the bottom rectangle is axiom TT4 for a tensor triangulated category.

Similarly, under the isomorphism $a : \Sigma \cong \Sigma S \otimes -$, $e'_{X,Y} : X \otimes \Sigma Y \cong \Sigma(X \otimes Y)$ corresponds to the map $X \otimes \Sigma S \otimes Y \xrightarrow{\tau_{X, \Sigma S \otimes Y}} X \otimes Y \otimes \Sigma S$. To see this, consider the following diagram:

$$\begin{array}{ccccccc} X \otimes \Sigma Y & \xrightarrow{\tau_{X, \Sigma Y}} & \Sigma Y \otimes X & \xrightarrow{e_{Y,X}} & \Sigma(Y \otimes X) & \xrightarrow{\Sigma\tau_{Y,X}} & \Sigma(X \otimes Y) \\ X \otimes \Sigma\lambda_Y \downarrow & & \Sigma\lambda_Y \otimes X \downarrow & & \Sigma(\lambda_Y \otimes X) \downarrow & & \downarrow \Sigma\lambda_{X \otimes Y} \\ X \otimes \Sigma(S \otimes Y) & \xrightarrow{\tau_{X, \Sigma(S \otimes Y)}} & \Sigma(S \otimes Y) \otimes X & \xrightarrow{e_{S \otimes Y, X}} & \Sigma(S \otimes Y \otimes X) & \xrightarrow{\Sigma(S \otimes \tau_{Y,X})} & \Sigma(S \otimes X \otimes Y) \\ X \otimes e_{S,Y}^{-1} \downarrow & & e_{S,Y} \otimes X \uparrow & & e_{S,Y \otimes X} \nearrow & & \downarrow e_{S, X \otimes Y}^{-1} \\ X \otimes \Sigma S \otimes Y & \xrightarrow{\tau_{X, \Sigma S \otimes Y}} & \Sigma S \otimes Y \otimes X & \xrightarrow{\Sigma S \otimes \tau_{Y,X}} & \Sigma S \otimes X \otimes Y & & \\ & \searrow \tau_{X, \Sigma S \otimes Y} & & \searrow \tau_{X, \Sigma S \otimes Y} & & & \end{array}$$

Here we are taking the associators to be isomorphisms, by coherence for monoidal categories. The top horizontal composition is $e'_{X,Y}$, by definition. The vertical edge compositions are $X \otimes a_Y$ and $a_{X \otimes Y}$. The top left rectangle commutes by naturality of τ . The top middle rectangle commutes by naturality of e . The top right triangle commutes by naturality of τ . The bottom left trapezoid commutes by naturality of τ . The small middle triangle commutes by axiom TT4 for a tensor triangulated category. The bottom triangle commutes by coherence for a symmetric monoidal category. Finally, the remaining region on the right commutes by naturality of e . Thus, in order to show the diagram in the statement of the lemma commutes, it suffices to show the following diagram commutes:

$$\begin{array}{ccc} \Sigma S \otimes X \otimes \Sigma S \otimes Y & \xrightarrow{-\alpha} & \Sigma S \otimes X \otimes \Sigma S \otimes Y \\ \tau_{\Sigma S \otimes X, \Sigma S \otimes Y} \downarrow & & \downarrow \Sigma S \otimes \tau_{X, \Sigma S \otimes Y} \\ \Sigma S \otimes \Sigma S \otimes X \otimes Y & \xrightarrow{\alpha} & \Sigma S \otimes \Sigma S \otimes X \otimes Y \end{array}$$

To see this diagram commutes, consider the following diagram:

$$\begin{array}{ccccc} \Sigma S \otimes X \otimes \Sigma S \otimes Y & \xrightarrow{\alpha} & \Sigma S \otimes X \otimes \Sigma S \otimes Y & \xrightarrow{\Sigma S \otimes \tau_{X, \Sigma S \otimes Y}} & \Sigma S \otimes \Sigma S \otimes X \otimes Y \\ \tau_{\Sigma S \otimes X, \Sigma S \otimes Y} \downarrow & & & & \downarrow \tau_{\Sigma S, \Sigma S \otimes X \otimes Y} \\ \Sigma S \otimes \Sigma S \otimes X \otimes Y & \xrightarrow{\alpha} & \Sigma S \otimes \Sigma S \otimes X \otimes Y & & \Sigma S \otimes \Sigma S \otimes X \otimes Y \end{array}$$

The diagram commutes by coherence for a symmetric monoidal category. The desired result follows by applying axiom TT5 for a tensor triangulated category, which tells us that $\tau_{\Sigma S, \Sigma S} = -1$, and additivity of $- \otimes -$ and composition. \square

Lemma A.16. *Let $A \xrightarrow{a} B \xrightarrow{b} C \xrightarrow{c} D$ be any sequence isomorphic to a distinguished triangle. Then given any E in \mathcal{C} , the sequence*

$$E \otimes A \xrightarrow{E \otimes a} E \otimes B \xrightarrow{E \otimes b} E \otimes C \xrightarrow{E \otimes c} E \otimes D$$

is exact.

Proof. Suppose we have a commutative diagram

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & \Sigma X \\ \alpha \downarrow & & \beta \downarrow & & \gamma \downarrow & & \delta \downarrow \\ A & \xrightarrow{a} & B & \xrightarrow{b} & C & \xrightarrow{c} & D \end{array}$$

where the top row is distinguished and the vertical arrows are isomorphisms. Then the following diagram commutes by functoriality of $- \otimes -$:

$$\begin{array}{ccccccc} E \otimes X & \xrightarrow{E \otimes f} & E \otimes Y & \xrightarrow{E \otimes g} & E \otimes Z & \xrightarrow{E \otimes h} & \Sigma(E \otimes X) \\ E \otimes \alpha \downarrow & & E \otimes \beta \downarrow & & E \otimes \gamma \downarrow & & \downarrow (E \otimes \delta) \circ e'_{E, X} \\ E \otimes A & \xrightarrow{E \otimes a} & E \otimes B & \xrightarrow{E \otimes b} & E \otimes C & \xrightarrow{E \otimes c} & E \otimes D \end{array}$$

$\begin{array}{c} \nearrow E \otimes h \\ \nwarrow E \otimes \delta \end{array}$
 $\begin{array}{c} \nwarrow e'_{E, X} \\ \nearrow \end{array}$

The top triangle is distinguished by axiom TT3 for a tensor triangulated category, thus exact by [Proposition A.3](#), so that the bottom triangle is also exact since the vertical arrows are isomorphisms and each square commutes. \square

Definition A.17. An *adjointly tensor triangulated* category is a tensor triangulated category $(\mathcal{C}, \otimes, S, \Sigma, e, \mathcal{D})$ along with the structure of an adjointly triangulated category $(\mathcal{C}, \Omega, \Sigma, \eta, \varepsilon, \mathcal{D})$.

From now on, we fix an adjointly tensor triangulated category $(\mathcal{C}, \otimes, S, \Omega, \Sigma, \eta, \varepsilon, e, \mathcal{D})$.

Definition A.18. We may define natural isomorphisms $o_{X,Y} : \Omega X \otimes Y \xrightarrow{\cong} \Omega(X \otimes Y)$ and $o'_{X,Y} : X \otimes \Omega Y \xrightarrow{\cong} \Omega(X \otimes Y)$ as the compositions

$$o_{X,Y} : \Omega X \otimes Y \xrightarrow{\varepsilon_{\Omega X \otimes Y}^{-1}} \Omega \Sigma(\Omega X \otimes Y) \xrightarrow{\Omega e_{\Omega X, Y}^{-1}} \Omega(\Sigma \Omega X \otimes Y) \xrightarrow{\Omega(\eta_X^{-1} \otimes Y)} \Omega(X \otimes Y)$$

and

$$o'_{X,Y} : X \otimes \Omega Y \xrightarrow{\tau_{X, \Omega Y}} \Omega Y \otimes X \xrightarrow{o_{Y, X}} \Omega(Y \otimes X) \xrightarrow{\Omega \tau_{Y, X}} \Omega(X \otimes Y).$$

These are both clearly natural by naturality of ε , e , η , and τ .

Proposition A.19. Suppose we have a distinguished triangle

$$X \rightarrow Y \rightarrow Z \rightarrow \Sigma X$$

in \mathcal{C} . Then given any object E in \mathcal{C} , the long exact sequence given in [Proposition A.12](#) remains exact after applying $E \otimes -$.

Proof. Recall that in the proof of [Proposition A.12](#) we showed that the sequence was exact by showing that any two consecutive maps were isomorphic to a part of a distinguished triangle. Then the desired result follows from [Lemma A.16](#). \square

APPENDIX B. SPECTRAL SEQUENCES

In what follows, we fix an abelian group A . We will freely use the theory and results of [Appendix C](#)

Definition B.1. An A -graded spectral sequence $(E_r, d_r)_{r \geq r_0}$ is the data of:

- A collection of A -graded abelian groups $\{E_r^*\}_{r \geq r_0}$
- A collection of A -graded homomorphisms $d_r : E_r \rightarrow E_r$ for $r \geq r_0$ (of possibly nonzero degree) such that $d_r \circ d_r = 0$
- For each $r \geq r_0$, an A -graded isomorphism $E_{r+1} \cong \ker d_r / \text{im } d_r$ of degree 0 (where $\ker d_r$ and $\text{im } d_r$ are canonically A -graded by [Proposition C.15](#), and their quotient is canonically A -graded by [Proposition C.17](#)).

Typically we call a \mathbb{Z}^2 -graded spectral sequence a *bigraded* spectral sequence, and a \mathbb{Z}^3 -graded spectral sequence is a *trigraded* spectral sequence.

For our purposes, we will only care about spectral sequences which arise from A -graded *unrolled exact couples*. In what follows, we follow [\[1\]](#), with minor modifications for our setting, in which everything is A -graded.

Definition B.2. An A -graded *unrolled exact couple* $(D, E; i, j, k)$ is a diagram of A -graded abelian groups and A -graded homomorphisms (of possibly non-zero degree)

$$\begin{array}{ccccccc} \cdots & \longrightarrow & D^{s+2} & \xrightarrow{i} & D^{s+1} & \xrightarrow{i} & D^s & \xrightarrow{i} & D^{s-1} & \longrightarrow & \cdots \\ & & \downarrow j & \swarrow k & \downarrow j & \swarrow k & \downarrow j & \swarrow k & \downarrow j & & \\ & & E^{s+2} & & E^{s+1} & & E^s & & E^{s-1} & & \end{array}$$

in which each triangle $D^{s+1} \xrightarrow{i} D^s \xrightarrow{j} E_s \xrightarrow{k} D^{s+1}$ is an exact sequence. We require that each occurrence of i (resp. j , k) is of the same degree. In other words, an unrolled exact couple can be described as a tuple $(D, E; i, j, k)$ of $\mathbb{Z} \times A$ -graded abelian groups and $\mathbb{Z} \times A$ -graded maps $i : D \rightarrow D$, $j : D \rightarrow E$, and $k : E \rightarrow D$, such that the \mathbb{Z} -degrees of i , j , and k are -1 , 0 , and 1 , respectively. Usually i and one of j or k will be of A -degree 0 .

Given an A -graded unrolled exact couple $(D, E; i, j, k)$, we may define an associated $\mathbb{Z} \times A$ -graded spectral sequence as follows: Given some $s \in \mathbb{Z}$ and some $r \geq 1$, we first define the following subgroups of E_s :

$$Z_r^s := k^{-1}(\text{im}[i^{r-1} : D^{s+r} \rightarrow D^{s+1}]) \quad \text{and} \quad B_r^s := j(\ker[i^{r-1} : D^s \rightarrow D^{s-r+1}])$$

where we adopt the convention that i^0 is simply the identity. These are furthermore A -graded subgroups of E_s (by [Proposition C.15](#) and [Proposition C.16](#)). In this way, for each $s \in \mathbb{Z}$, we get an infinite sequence of A -graded subgroups:

$$0 = B_1^s \subseteq B_2^s \subseteq B_3^s \subseteq \cdots \subseteq \text{im } j = \ker k \subseteq \cdots \subseteq Z_3^s \subseteq Z_2^s \subseteq Z_1^s = E^s.$$

Now, for each $s \in \mathbb{Z}$ and $r \geq 1$, we define the A -graded abelian group

$$E_r^s := Z_r^s / B_r^s,$$

so that in particular $E_1^s = E^s$ for all $s \in \mathbb{Z}$, as $Z_1^s = k^{-1}(D^{s+1}) = E^s$ and $B_1^s = j(\ker \text{id}_{D^s}) = j(0) = 0$. Now we can define differentials $d_r^s : E_r^s \rightarrow E_r^{s+r}$ to be the composition

$$E_r^s = Z_r^s / B_r^s \xrightarrow{k} \text{im}[i^{r-1} : D^{s+r} \rightarrow D^{s+1}] \xrightarrow{j \circ i^{-(r-1)}} Z_r^{s+r} / B_r^{s+r} = E_r^{s+r},$$

where given some $e \in Z_r^s = k^{-1}(\text{im } i^{r-1})$, the first arrow takes a class $[e] \in E_r^s$ represented by some $e \in Z_r^s$ to the element $k(e)$, which lives in $\text{im } i^{r-1}$ by definition, and the second arrow takes $i^{r-1}(d)$ to the class $[j(d)]$. Note the first map is well-defined, as given $b \in B_r^s = j(\ker[i^{r-1}])$, we have $k(b) = 0$, as $b \in \text{im } j = \ker k$. To see the second map is well-defined, first note that given $d \in D^{s+r}$, that

$$k(j(d)) = 0 \in \text{im}[i^{r-1} : D^{s+2r} \rightarrow D^{s+r+1}],$$

so that

$$j(d) \in k^{-1}(\text{im}[i^{r-1} : D^{s+2r} \rightarrow D^{s+r+1}]) = Z_r^{s+r},$$

as desired, so that given $d \in D^{s+r}$, $j(d) \in Z_r^{s+r}$, so it makes sense to discuss the class $[j(d)] \in Z_r^{s+r} / B_r^{s+r} = E_r^{s+r}$. Secondly, if $i^{r-1}(d) = i^{r-1}(d')$ for some $d, d' \in D^{s+r}$, then

$$j(d) - j(d') = j(d - d') \in j(\ker[i^{r-1} : D^{s+r} \rightarrow D^{s+1}]) = B_r^{s+r},$$

so that $[j(d)] = [j(d')]$ in E_r^{s+r} , as desired. It is straightforward to check that these maps are also A -graded homomorphisms, so that by unravelling definitions d_r^s is an A -graded homomorphism of degree $\deg k - (r-1) \cdot \deg i + \deg j$ (so that in the standard case $\deg i = 0$, d_r^s simply has degree $\deg k + \deg j$).

These differentials square to zero, in the sense that for each $s \in \mathbb{Z}$ and $r \geq 1$ we have that $d_r^{s+r} \circ d_r^s : E_r^s \rightarrow E_r^{s+2r}$ is the zero map. Indeed, suppose we are given some class $[e] \in E_r^s$ represented by an element $e \in E^s$, so $k(e) = i^{r-1}(d)$ for some $d \in D^{s+r}$. Then

$$d_r^{s+r}(d_r^s([e])) = d_r^{s+r}([j(d)]) = [j(i^{-(r-1)}(k(j(d))))] = [j(i^{-(r-1)}(0))] = 0,$$

where the second-to-last equality follows by the fact that $k \circ j = 0$. Note that by unravelling definitions, $d_1^s = j \circ k$.

We claim that $\ker d_r^s = Z_{r+1}^s / B_r^s$. First of all, let $[e] \in E_r^s = Z_r^s / B_r^s$, so that $[e]$ is represented by some $e \in E^s$ with $k(e) = i^{r-1}(d)$ for some $d \in D^{s+r}$. Then if $[e] \in \ker d_r^s$, by definition this means $j(d) \in B_r^{s+r} = j(\ker[i^{r-1} : D^{s+r} \rightarrow D^{s+1}])$, so $j(d) = j(d')$ for some $d' \in D^{s+r}$ with

$i^{r-1}(d') = 0$. Thus $d - d' \in \ker j = \operatorname{im} i$, so there exists some $d'' \in D^{s+r+1}$ such that $i(d'') = d - d'$. Then

$$k(e) = i^{r-1}(d) = i^{r-1}(i(d'') + d') = i^r(d'') + i^{r-1}(d'),$$

but since $i^{r-1}(d') = 0$, we have $k(e) \in \operatorname{im}[i^r : D^{s+r+1} \rightarrow D^{s+1}]$, so that $e \in Z_{r+1}^s$, meaning $[e] \in Z_{r+1}^s/B_r^s$, as desired. On the other hand, suppose we are given some class $[e] \in Z_{r+1}^s/B_r^s$, represented by $e \in Z_{r+1}^s$ with $k(e) \in \operatorname{im}[i^r : D^{s+r+1} \rightarrow D^{s+1}]$. Then if we write $k(e) = i^r(d) = i^{r-1}(i(d))$, then $d_r^s([e]) = [j(i(d))] = 0$ (since $j \circ i = 0$), as asserted.

Finally, we claim that the image of $d_r^{s-r} : E_r^{s-r} \rightarrow E_r^s$ is B_{r+1}^s/B_r^s . First, let $e \in Z_r^{s-r}$, so $k(e) = i^{r-1}(d)$ for some $d \in D^s$. Then we'd like to show that $d_r^s([e]) = [j(d)]$ belongs to B_{r+1}^s/B_r^s . It suffices to show that $d \in \ker[i^r : D^s \rightarrow D^{s-r}]$. To see this, note that

$$i^r(d) = i(i^{r-1}(d)) = i(k(e)) = 0,$$

since $i \circ k = 0$. Hence we've shown $\operatorname{im} d_r^{s-r} \subseteq B_{r+1}^s/B_r^s$. Conversely, let $j(d) \in B_{r+1}^s$, so $d \in D^s$ and $i^r(d) = 0$. Then we'd like to show that $[j(d)] \in B_{r+1}^s/B_r^s$ is in the image of d_r^{s-r} . To see this, note that

$$i^r(d) = 0 \implies i^{r-1}(d) \in \ker i = \operatorname{im} k,$$

so there exists some $e \in E_r^{s-r}$ such that $k(e) = i^{r-1}(d)$, so $e \in Z_r^{s-r}$. Unravelling definitions, it follows that $d_r^{s-r}([e]) = [j(d)]$, so $[j(d)]$ is indeed in the image of d_r^{s-r} , as desired.

To recap, we have constructed for each $s \in \mathbb{Z}$ and $r \geq 1$ an A -graded abelian group E_r^s along with differentials $d_r^s : E_r^s \rightarrow E_r^{s+r}$. Furthermore, if we take homology in the middle term of the following sequence

$$E_r^{s-r} \xrightarrow{d_r^{s-r}} E_r^s \xrightarrow{d_r^s} E_r^{s+r},$$

we get

$$\ker d_r^s / \operatorname{im} d_r^{s-r} = \frac{Z_{r+1}^s/B_r^s}{B_{r+1}^s/B_r^s} \cong Z_{r+1}^s/B_{r+1}^s = E_{r+1}^s.$$

Thus, we get a spectral sequence:

Proposition B.3. *We may associate a $\mathbb{Z} \times A$ -graded spectral sequence $r \mapsto (E_r, d_r)$ to the A -graded unrolled exact couple $(D, E; i, j, k)$ by defining $E_r := \bigoplus_{s \in \mathbb{Z}} E_r^s$ and the differentials*

$$d_r : E_r \rightarrow E_r$$

are those constructed above, which have $\mathbb{Z} \times A$ -degree $(r, \deg j - (r-1) \cdot \deg i + \deg k)$.

Definition B.4 (Exact couple). An A -graded *exact couple* is a tuple $\mathcal{E} = (D, E; i, j, k)$, where D and E are A -graded abelian groups and i, j , and k are A -graded homomorphisms (of possibly nonzero degree)

$$\begin{array}{ccc} D_* & \xrightarrow{i} & D_* \\ & \swarrow k & \searrow j \\ & E_* & \end{array}$$

which form an *exact triangle*, in the sense that kernel = image at each vertex.

Definition B.5 (Derived couple). Given an exact couple $(D, E; i, j, k)$ as in the above definition, the composition $j \circ k : E \rightarrow E$ itself satisfies

$$(j \circ k) \circ (j \circ k) = j \circ (k \circ j) \circ k = j \circ 0 \circ k = 0,$$

so we may form the A -graded homology group $H(E) := \ker(j \circ k) / \operatorname{im}(j \circ k)$. Then we may form the triangle \mathcal{E}'

$$\begin{array}{ccc} i(D) & \xrightarrow{i'} & i(D) \\ & \swarrow k' \quad \searrow j' & \\ & H(E) & \end{array}$$

where i' is the restriction of i to $i(D)$, while j' and k' are given by

$$j'(i(d)) = [j(d)] \quad \text{and} \quad k'([e]) = k(e).$$

The map j' is well-defined since if $i(d) = i(d')$ then $i(d - d') = 0$, so that $d - d' \in \ker i = \operatorname{im} k$, meaning $d - d' = k(e)$ for some $e \in E$, so that

$$j(d) - j(d') = j(d - d') = j(k(e)) \in \operatorname{im}(j \circ k)$$

is a boundary, so that $[j(d)] = [j(d')]$. Similarly k' is well defined since if $[e] = [e']$ then $e - e' \in \operatorname{im}(j \circ k)$, which implies $e - e' = j(k(e''))$ for some $e'' \in E$, so that

$$k(e) - k(e') = k(e - e') = k(j(k(e''))) = 0,$$

where the last equality follows by the fact that $k \circ j = 0$. Further note that $i(D)$ and $H(E)$ are A -graded by [Proposition C.15](#) and [Proposition C.17](#), in which case by unravelling definitions, each of i' , j' , and k' are A -graded homomorphisms with

$$\deg i' = \deg i, \quad \deg j' = \deg j - \deg i, \quad \text{and} \quad \deg k' = \deg k.$$

We call \mathcal{E}' the *derived couple* of \mathcal{E} . A diagram chase (left to the reader, or see [7, Lemma 1.10]) yields that \mathcal{E}' is an exact couple.

If we iterate the process of taking the exact couple r times, the result is called the r^{th} *derived couple* \mathcal{E}_r of \mathcal{E} .

$$\begin{array}{ccc} D_r & \xrightarrow{i} & D_r \\ & \swarrow k \quad \searrow j^{(r)} & \\ & E_r & \end{array}$$

Here $D_r = i^r(D)$ is a subgroup of D , and $E_r = H(E_{r-1})$ is a subquotient of E . The maps i and k are induced from the i and k of \mathcal{E} , while $j^{(r)}$ sends $[i^r(d)]$ to $[j(d)]$. In particular, by induction it can be seen that $\deg j^{(r)} = \deg j - r \cdot \deg i$, and the degrees of i and k remain unchanged as we take successive derived couples.

Definition B.6 (The spectral sequence associated to an exact couple). An A -graded exact couple $\mathcal{E} = (D, E; i, j, k)$ gives rise to a spectral sequence $(E_r, d_r)_{r \geq 0}$, where $E_0 = E$, $d_0 = j \circ k$, and for $r > 0$, E_r is defined above and d_r is the composition $j^{(r)} \circ k$.

In practice, we will always shift everything up a degree by re-defining $E_r := E_{r-1}$ and $d_r := d_{r-1}$, so we get a spectral sequence $(E_r, d_r)_{r \geq 1}$ with $E^1 = E$ and $d^1 = j \circ k$. Then it follows that the differential $d^r = j^{(r-1)} \circ k$ has degree

$$\deg j^{(r-1)} + \deg k = \deg j - (r-1) \cdot \deg i + \deg k.$$

Remark B.7. Given an exact couple $\mathcal{E} = (D, E; i, j, k)$, we can define A -graded subgroups $Z_r = k^{-1}(i^r(D)) \subseteq E$ and $B_r = j(\ker(i^r)) \subseteq E$ for $r \geq 1$. By induction, it is straightforward to check that we have inclusions

$$B_1 \subseteq B_2 \subseteq B_3 \subseteq \cdots \subseteq \operatorname{im} j = \ker k \subseteq \cdots \subseteq Z_3 \subseteq Z_2 \subseteq Z_1$$

and that the maps

$$Z_r \rightarrow E_r$$

sending an element e to its class $[[\cdots [e] \cdots]]$ has kernel B_r , so we have identifications $E_r = Z_r/B_r$ as A -graded abelian groups. Let $e \in Z_r$, so $k(e) = i^r(d)$ for some $d \in D$. Then under this identification, it can be seen that the map $d_{r+1} : Z_r/B_r \rightarrow Z_r/B_r$ sends the coset $e + B_r \in Z_r/B_r$ to the coset $j(d) + B_r$, and that $\ker d_r = Z_r$ and $\operatorname{im} d_r = B_r$ for all $r \geq 1$.

Henceforth, we fix an A -graded exact couple $\mathcal{E} = (D, E; i, j, k)$, and we let $(E_r, d_r)_{r \geq 1}$ denote the associated A -graded spectral sequence. We make the identifications given by the above remark, so we assume E_r is the A -graded abelian group Z_r/B_r for all $r \geq 1$, so that in particular for all $a \in A$ we have identifications $E_r^a = Z_r^a/B_r^a$ (by [Proposition C.17](#)).

APPENDIX C. A -GRADED ABELIAN GROUPS, RINGS, AND MODULES

In what follows, we fix an abelian group A . We assume the reader is familiar with the basic theory of modules over not-necessarily-commutative rings.

Definition C.1. An *A -graded abelian group* is an abelian group B along with a subgroup $B_a \leq B$ for each $a \in A$ such that the canonical map

$$\bigoplus_{a \in A} B_a \rightarrow B$$

sending $(x_a)_{a \in A}$ to $\sum_{a \in A} x_a$ is an isomorphism. Given two A -graded abelian groups B and C , a homomorphism $f : B \rightarrow C$ is a *homomorphism of A -graded abelian groups*, or just an *A -graded homomorphism*, if it preserves the grading, i.e., if it restricts to a map $B_a \rightarrow C_a$ for all $a \in A$.

It is easy to see that an A -graded abelian group B is generated by its *homogeneous* elements, that is, nonzero elements $x \in B$ such that there exists some $a \in A$ with $x \in B_a$.

Remark C.2. Clearly the condition that the canonical map $\bigoplus_{a \in A} B_a \rightarrow B$ is an isomorphism requires that $B_a \cap B_b = 0$ if $a \neq b$. In particular, given a homogeneous element $x \in B$, there exists precisely one $a \in A$ such that $x \in B_a$. We call this a the *degree* of x , and we write $|x| = a$.

Definition C.3. An *A -graded ring* is a ring R such that its underlying abelian group R is A -graded and the multiplication map $R \times R \rightarrow R$ restricts to $R_a \times R_b \rightarrow R_{a+b}$ for all $a, b \in A$. A morphism of A -graded rings is a ring homomorphism whose underlying homomorphism of abelian groups is A -graded.

Explicitly, given an A -graded ring R and homogeneous elements $x, y \in R$, we must have $|xy| = |x| + |y|$. For example, given some field k , the ring $R = k[x, y]$ is \mathbb{Z}^2 -graded, where given $(n, m) \in \mathbb{Z}^2$, $R_{n,m}$ is the subgroup of those monomials of the form $ax^n y^m$ for some $a \in k$.

Definition C.4. Let R be an A -graded ring. A *left A -graded R -module* M is a left R -module M such that M is an A -graded abelian group and the action map $R \times M \rightarrow M$ restricts to a map $R_a \times M_b \rightarrow M_{a+b}$ for all $a, b \in A$. Right A -graded R -modules are defined similarly. Finally, an *A -graded R -bimodule* is an A -graded abelian group M which has the structure of both an A -graded left and right R -module such that given $r, s \in R$ and $m \in M$, $r \cdot (m \cdot s) = (r \cdot m) \cdot s$.

Morphisms between A -graded R -modules are precisely A -graded R -module homomorphisms. We write $R\text{-GrMod}$ for the category of left A -graded R -modules and $\text{GrMod-}R$ for the category of right A -graded R -modules.

Remark C.5. It is straightforward to see that an A -graded abelian group is equivalently an A -graded \mathbb{Z} -module, where here we are considering \mathbb{Z} as an A -graded ring concentrated in degree 0. Thus any result below about A -graded modules applies equally to A -graded abelian groups.

Lemma C.6. *Given an A -graded ring R and two left (resp. right) A -graded R -modules M and N , their direct sum $M \oplus N$ is naturally a left (resp. right) A -graded R -module group by defining*

$$(M \oplus N)_a := M_a \oplus N_a.$$

Proof. The canonical map $\bigoplus_{a \in A} (M_a \oplus N_a) \rightarrow M \oplus N$ factors as

$$\bigoplus_{a \in A} (M_a \oplus N_a) \xrightarrow{\cong} \bigoplus_{a \in A} M_a \oplus \bigoplus_{a \in A} N_a \xrightarrow{\cong} M \oplus N. \quad \square$$

Oftentimes when constructing A -graded rings, we do so only by defining the product of homogeneous elements, like so:

Lemma C.7. *Suppose we have an A -graded abelian group R , a distinguished element $1 \in R_0$, and \mathbb{Z} -bilinear maps $m_{a,b} : R_a \times R_b \rightarrow R_{a+b}$ for all $a, b \in A$. Further suppose that for all $x \in R_a$, $y \in R_b$, and $z \in R_c$, we have*

$$m_{a+b,c}(m_{a,b}(x, y), z) = m_{a,b+c}(x, m_{b,c}(y, z)) \quad \text{and} \quad m_{a,0}(x, 1) = m_{0,a}(1, x) = x.$$

Then there exists a unique multiplication map $m : R \times R \rightarrow R$ which endows R with the structure of an A -graded ring and restricts to $m_{a,b}$ for all $a, b \in A$.

Proof. Given $r, s \in R$, since $R \cong \bigoplus_{a \in A} R_a$, we may uniquely decompose r and s into homogeneous elements as $r = \sum_{a \in A} r_a$ and $s = \sum_{a \in A} s_a$ with each $r_a, s_a \in R_a$ such that only finitely many of the r_a 's and s_a 's are nonzero. Then in order to define a distributive product $R \times R \rightarrow R$ which restricts to $m_{a,b} : R_a \times R_b \rightarrow R_{a+b}$, note we *must* define

$$r \cdot s = \left(\sum_{a \in A} r_a \right) \cdot \left(\sum_{b \in A} s_b \right) = \sum_{a,b \in A} r_a \cdot s_b = \sum_{a,b \in A} m_{a,b}(r_a, s_b).$$

Thus, we have shown uniqueness. It remains to show this product actually gives R the structure of a ring. First we claim that the sum on the right is actually finite. Note there exists only finitely many nonzero r_a 's and s_b 's, and if $s_b = 0$ then

$$m_{a,b}(r_a, 0) = m_{a,b}(r_a, 0 + 0) \stackrel{(*)}{=} m_{a,b}(r_a, 0) + m_{a,b}(r_a, 0) \implies m_{a,b}(r_a, 0) = 0,$$

where $(*)$ follows from bilinearity of $m_{a,b}$. A similar argument yields that $m_{a,b}(0, s_b) = 0$ for all $a, b \in A$. Hence indeed $m_{a,b}(r_a, s_b)$ is zero for all but finitely many pairs $(a, b) \in A^2$, as desired. Observe that in particular

$$(r \cdot s)_a = \sum_{b+c=a} m_{b,c}(r_b, s_c) = \sum_{b \in A} m_{b,a-b}(r_b, s_{a-b}) = \sum_{c \in A} m_{a-c,c}(r_{a-c}, s_c).$$

Now we claim this multiplication is associative. Given $t = \sum_{a \in A} t_a \in R$, we have

$$\begin{aligned}
(r \cdot s) \cdot t &= \sum_{a,b \in A} m_{a,b}((r \cdot s)_a, t_b) \\
&= \sum_{a,b \in A} m_{a,b} \left(\sum_{c \in A} m_{a-c,c}(r_{a-c}, s_c), t_b \right) \\
&\stackrel{(1)}{=} \sum_{a,b,c \in A} m_{a,b}(m_{a-c,c}(r_{a-c}, s_c), t_b) \\
&\stackrel{(2)}{=} \sum_{a,b,c \in A} m_{c,a+b-c}(r_c, m_{a-c,b}(s_{a-c}, t_b)) \\
&\stackrel{(3)}{=} \sum_{a,b,c \in A} m_{a,c}(r_a, m_{b,c-b}(s_b, t_{c-b})) \\
&\stackrel{(1)}{=} \sum_{a,c \in A} m_{a,c} \left(r_a, \sum_{b \in A} m_{b,c-b}(s_b, t_{c-b}) \right) \\
&= \sum_{a,c \in A} m_{a,c}(r_a, (s \cdot t)_c) = r \cdot (s \cdot t),
\end{aligned}$$

where each occurrence of (1) follows by bilinearity of the $m_{a,b}$'s, each occurrence of (2) is associativity of the $m_{a,b}$'s, and (3) is obtained by re-indexing by re-defining $a := c$, $b := a - c$, and $c := a + b - c$. Next, we wish to show that the distinguished element $1 \in R_0$ is a unit with respect to this multiplication. Indeed, we have

$$1 \cdot r \stackrel{(1)}{=} \sum_{a \in A} m_{0,a}(1, r_a) \stackrel{(2)}{=} \sum_{a \in A} r_a = r \quad \text{and} \quad r \cdot 1 \stackrel{(1)}{=} \sum_{a \in A} m_{a,0}(r_a, 1) \stackrel{(2)}{=} \sum_{a \in A} r_a = r,$$

where (1) follows by the fact that $m_{a,b}(0, -) = m_{a,b}(-, 0) = 0$, which we have shown above, and (2) follows by unitality of the $m_{0,a}$'s and $m_{a,0}$'s, respectively. Finally, we wish to show that this product is distributive. Indeed, we have

$$\begin{aligned}
r \cdot (s + t) &= \sum_{a,b \in A} m_{a,b}(r_a, (s + t)_b) \\
&= \sum_{a,b \in A} m_{a,b}(r_a, s_b + t_b) \\
&\stackrel{(*)}{=} \sum_{a,b \in A} m_{a,b}(r_a, s_b) + \sum_{a,b \in A} m_{a,b}(r_a, t_b) = (r \cdot s) + (r \cdot t),
\end{aligned}$$

where (*) follows by bilinearity of $m_{a,b}$. An entirely analogous argument yields that $(r + s) \cdot t = (r \cdot t) + (s \cdot t)$. \square

Lemma C.8. *Let R be an A -graded ring, M an A -graded abelian group, and suppose there exists \mathbb{Z} -bilinear maps $\kappa_{a,b} : R_a \times M_b \rightarrow M_{a+b}$ for all $a, b \in A$. Further suppose that for all $r \in R_a$, $r' \in R_b$, and $m \in M_c$ that*

$$\kappa_{a+b,c}(r \cdot r', m) = \kappa_{a,b+c}(r, \kappa_{b,c}(r', m)) \quad \text{and} \quad \kappa_{0,c}(1, m) = m.$$

Then there is a unique map $\kappa : R \times M \rightarrow M$ which endows M with the structure of a left A -graded R -module and restricts to $\kappa_{a,b}$ for all $a, b \in A$.

On the other hand, suppose there exists \mathbb{Z} -bilinear maps $\kappa_{a,b} : M_a \times R_b \rightarrow M_{a+b}$ for all $a, b \in A$. Further suppose that for all $r \in R_a$, $r' \in R_b$, and $m \in M_c$ that

$$\kappa_{c,a+b}(m, r \cdot r') = \kappa_{c+a,b}(\kappa_{c,a}(m, r), r') \quad \text{and} \quad \kappa_{c,0}(m, 1) = m.$$

Then there is a unique map $\kappa : M \times R \rightarrow M$ which endows M with the structure of a right A -graded R -module and restricts to $\kappa_{a,b}$ for all $a, b \in A$.

Proof. We show the left module case, as the right module case is entirely analagous. Supposing for each $a, b \in A$ we have a map $\kappa_{a,b} : R_a \times M_b \rightarrow M_{a+b}$ satisfying the above conditions, in order to extend these to a map $R \times M \rightarrow M$, by additivity we *must* define

$$\kappa : R \times M \rightarrow M$$

to be the map sending $r = \sum_a r_a$ and $m = \sum_a m_a$ to $\sum_{a,b \in A} \kappa_{a,b}(r_a, m_b)$. Now, we need to check that for all $r, s \in R$, $x, y \in M$ that

- (1) $r \cdot (x + y) = r \cdot x + r \cdot y$
- (2) $(r + s) \cdot x = r \cdot x + s \cdot x$
- (3) $(rs) \cdot x = r \cdot (s \cdot x)$
- (4) $1 \cdot x = x$,

where above we are written $- \cdot -$ for $\kappa(-, -)$. To see the first, note

$$\begin{aligned} \kappa(r, x + y) &= \sum_{a,b \in A} \kappa_{a,b}(r_a, (x + y)_b) \\ &= \sum_{a,b \in A} \kappa_{a,b}(r_a, x_b + y_b) \\ &= \sum_{a,b \in A} (\kappa_{a,b}(r_a, x_b) + \kappa_{a,b}(r_a, y_b)) \\ &= \sum_{a,b \in A} \kappa_{a,b}(r_a, x_b) + \sum_{a,b \in A} \kappa_{a,b}(r_a, y_b) \\ &= \kappa(r, x) + \kappa(r, y). \end{aligned}$$

To see the second, note

$$\begin{aligned} \kappa(r + s, x) &= \sum_{a,b \in A} \kappa_{a,b}((r + s)_a, x_b) \\ &= \sum_{a,b \in A} \kappa_{a,b}(r_a + s_a, x_b) \\ &= \sum_{a,b \in A} (\kappa_{a,b}(r_a, x_b) + \kappa_{a,b}(s_a, x_b)) \\ &= \sum_{a,b \in A} \kappa_{a,b}(r_a, x_b) + \sum_{a,b \in A} \kappa_{a,b}(s_a, x_b) \\ &= \kappa(r, x) + \kappa(s, x). \end{aligned}$$

To see the third, note

$$\begin{aligned}
\kappa(rs, x) &= \sum_{a,b \in A} \kappa_{a,b}((rs)_a, x_b) \\
&= \sum_{a,b \in A} \kappa_{a,b} \left(\sum_{c \in A} r_c s_{a-c}, x_b \right) \\
&= \sum_{a,b,c \in A} \kappa_{a,b}(r_c s_{a-c}, x_b) \\
&= \sum_{a,b,c \in A} \kappa_{a,b}(r_c, \kappa_{a-c,b}(s_{a-c}, x_b)) \\
&=
\end{aligned}$$

FINISH 

□

When working with A -graded rings and modules, we will often freely use the above propositions without comment. In what follows, fix an A -graded ring R . We will simply say “ A -graded R -module” when we are freely considering either left or right A -graded R -modules.

Remark C.9. We often will denote an A -Graded R -module M by M_* . Given some $a \in A$, we can define the shifted A -graded abelian group M_{*+a} whose b^{th} component is M_{b+a} .

Definition C.10. More generally, given two A -graded R -modules M and N and some $d \in A$, an R -module homomorphism $f : M \rightarrow N$ is an A -graded homomorphism of degree d if it restricts to a map $M_a \rightarrow N_{a+d}$ for all $a \in A$. Thus, an A -graded homomorphism of degree d from M to N is equivalently an A -graded homomorphism $M_* \rightarrow N_{*+d}$ or an A -graded homomorphism $M_{*-d} \rightarrow N$. Given some $a \in A$ and left (resp. right) R -modules M and N , we will write

$$\text{Hom}_R^d(M, N) = \text{Hom}_R(M_*, N_{*+d}) = \text{Hom}_R(M_{*-d}, N_*)$$

to denote the set of A -graded homomorphisms of degree d from M to N , and simply

$$\text{Hom}_R(M, N)$$

to denote the set of degree-0 A -graded homomorphisms from M to N . Clearly A -graded homomorphisms may be added and subtracted, so these are further abelian groups. Thus we have an A -graded abelian group

$$\text{Hom}_R^*(M, N).$$

Unless stated otherwise, an “ A -graded homomorphism” will always refer to an A -graded homomorphism of degree 0.

Lemma C.11. Let R be an A -graded ring and M an A -graded left (resp. right) R -module. Then for all $d \in A$, the evaluation map

$$\begin{aligned}
\text{ev}_1 : \text{Hom}_R^d(R, M) &\rightarrow M_d \\
\varphi &\mapsto \varphi(1)
\end{aligned}$$

is an isomorphism of abelian groups.

Proof. We consider the case that M is a left A -graded R -module, as showing it when M is a right module is entirely analogous. First of all, this map is clearly a homomorphism, as given degree d A -graded homomorphisms $\varphi, \psi : R \rightarrow M$, we have

$$\text{ev}_1(\varphi + \psi) = (\varphi + \psi)(1) = \varphi(1) + \psi(1) = \text{ev}_1(\varphi) + \text{ev}_1(\psi).$$

Now, to see it is surjective, let $m \in M_d$, and define $\varphi_m : R \rightarrow M$ to send $r \mapsto r \cdot m$. First of all, φ_m is a module homomorphism, as given $r, s \in R$,

$$\varphi_m(r + s) = (r + s) \cdot m = r \cdot m + s \cdot m = \varphi_m(r) + \varphi_m(s) \quad \text{and} \quad \varphi_m(r \cdot s) = r \cdot s \cdot m = r \cdot \varphi_m(s).$$

Furthermore, it is clearly A -graded of degree d , as given a homogeneous element $r \in R_a$ for some $a \in A$, we have $\varphi_m(r) = r \cdot m \in R_{a+d}$, since m is homogeneous of degree d . Finally, clearly

$$\text{ev}_1(\varphi_m) = \varphi_m(1) = 1 \cdot m = m,$$

so indeed ev_1 is surjective. On the other hand, to see it is injective, suppose we are given $\varphi, \psi \in \text{Hom}_R^d(R, M)$ such that $\varphi(1) = \psi(1)$. Then given $r \in R$, we must have

$$\varphi(r) = \varphi(r \cdot 1) = r \cdot \varphi(1) = r \cdot \psi(1) = \psi(r \cdot 1) = \psi(r),$$

so φ and ψ are exactly the same map. Thus, ev_1 is injective, as desired. \square

Recall that given a ring R , a left (resp. right) module P is *projective* if, for all diagrams of R -module homomorphisms of the form

$$\begin{array}{ccc} & & M \\ & & \downarrow g \\ P & \xrightarrow{f} & N \end{array}$$

with g an epimorphism, there exists a lift $h : P \rightarrow M$ satisfying $g \circ h = f$

$$\begin{array}{ccc} & & M \\ & \nearrow h & \downarrow g \\ P & \xrightarrow{f} & N \end{array}$$

(Note h is not required to be unique.)

Definition C.12. Let R be an A -graded ring, and let P be a left (resp. right) A -graded R -module. Then P is a *graded projective* module if, for all diagrams of A -graded R -module homomorphisms of the form

$$\begin{array}{ccc} & & M \\ & & \downarrow g \\ P & \xrightarrow{f} & N \end{array}$$

with g an epimorphism, there exists an A -graded homomorphism $h : P \rightarrow M$ satisfying $g \circ h = f$.

$$\begin{array}{ccc} & & M \\ & \nearrow h & \downarrow g \\ P & \xrightarrow{f} & N \end{array}$$

(Note h is not required to be unique.)

Definition C.13. Let M be an A -graded R -module. Then an *A -graded R -submodule* is an A -graded R -module N which is a subset of M and for which the inclusion $N \hookrightarrow M$ is an A -graded homomorphism of R -modules. Equivalently, it is a submodule N one for which the canonical map

$$\bigoplus N \cap M_a \rightarrow N$$

is an isomorphism.

Lemma C.14. *Let M be an A -graded R -module. Then an R -submodule $N \leq M$ is an A -graded submodule if and only if it is generated as an R -module by homogeneous elements of M .*

Proof. If $N \leq M$ is a A -graded submodule, it is generated by the set of all its homogeneous elements, which are also homogeneous elements in M , by definition.

Conversely, suppose $N \leq M$ is a submodule which is generated by homogeneous elements of M . Then define $N_a := N \cap M_a$, and consider the canonical map

$$\Phi : \bigoplus_{a \in A} N_a \rightarrow N.$$

First of all, it is surjective, as each generator of N belongs to some N_a , by definition. To see it is injective, consider the following commutative diagram:

$$\begin{array}{ccc} \bigoplus_{a \in A} N_a & \hookrightarrow & \bigoplus_{a \in A} M_a \\ \Phi \downarrow & & \downarrow \cong \\ N & \hookrightarrow & M \end{array}$$

Since Φ composes with an injection to get an injection, clearly Φ must be injective itself. We have the desired result. \square

Proposition C.15. *Given two left (resp. right) A -graded R -modules M and N and an A -graded R -module homomorphism $\varphi : M \rightarrow N$ (of possibly nonzero degree), the kernel and images of φ are A -graded submodules of M and N , respectively.*

Proof. First recall that a degree d A -graded homomorphism $M \rightarrow N$ is simply an A -graded homomorphism $M_* \rightarrow N_{*+d}$, so it suffices to consider the case φ is of degree 0. Next, note that since the forgetful functor from R -modules to abelian groups preserves kernels and images, it suffices to consider the case that φ is a homomorphism of A -graded abelian groups. Finally, by [Lemma C.14](#), it suffices to show that $\ker \varphi$ and $\operatorname{im} \varphi$ are generated by homogeneous elements of M and N , respectively.

Note that by the universal property of the coproduct in **Ab**, the data of an A -graded homomorphism of abelian groups $\varphi : M \rightarrow N$ is precisely the data of an A -indexed collection of abelian group homomorphisms $\varphi_a : M_a \rightarrow N_a$, in which case the following diagram commutes:

$$\begin{array}{ccc} \bigoplus_a M_a & \xrightarrow{\bigoplus_a \varphi_a} & \bigoplus_a N_a \\ \cong \downarrow & & \downarrow \cong \\ M & \xrightarrow{\varphi} & N \end{array}$$

Finally, the desired result follows by the purely formal fact that taking images and kernels commutes with arbitrary direct sums. \square

Proposition C.16. *Given two left (resp. right) A -graded R -modules M and N , an A -graded submodule $K \leq N$, and an A -graded R -module homomorphism $\varphi : M \rightarrow N$ (of possibly nonzero degree), the submodule $\varphi^{-1}(K)$ of M is A -graded.*

Proof. Recall that a degree d A -graded homomorphism $M \rightarrow N$ is simply an A -graded homomorphism $M_* \rightarrow N_{*+d}$, so it suffices to consider the case φ is of degree 0. Now, let $x \in L := \varphi^{-1}(K)$. As an element of M , we may uniquely write $x = \sum_{a \in A} x_a$ where each $x_a \in M_a$. Similarly, if we set $y := \varphi(x)$, then we may uniquely write $y = \sum_{a \in A} y_a$ where each $y_a \in N_a$. Then since K is

an A -graded submodule of N and $y \in K$, by definition, we have that $y_a \in K$ for each a . Finally, note that

$$\sum_{a \in A} y_a = y = \varphi(x) = \sum_{a \in A} \varphi(x_a),$$

so that $\varphi(x_a) = y_a \in K$ for all $a \in A$, so that $x_a \in L$ for all $a \in A$. Thus we have shown that each element in L can be written as a sum of homogeneous elements in M , as desired. \square

Proposition C.17. *Given an A -graded R -module M and an A -graded subgroup $N \leq M$, the quotient M/N is canonically A -graded by defining $(M/N)_a$ to be the subgroup generated by cosets represented by homogeneous elements of degree a in M . Furthermore, the canonical maps $M_a/N_a \rightarrow (M/N)_a$ taking a coset $m + N_a$ to $m + N$ are isomorphisms.*

Proof. Consider the canonical map

$$\Phi : \bigoplus_a (M/N)_a \rightarrow M/N.$$

First of all, surjectivity of Φ follows by commutativity of the following diagram:

$$\begin{array}{ccc} \bigoplus_a M_a & \xrightarrow{\cong} & M \\ \downarrow & & \downarrow \\ \bigoplus_a (M/N)_a & \xrightarrow{\Phi} & M/N \end{array}$$

where the vertical left map sends a generator $m \in M_a$ to the coset $m + N$ in $(M/N)_a \subseteq M/N$. To see Φ is injective, suppose we are given some element $(m_a + N)_{a \in A}$ in $\bigoplus_a (M/N)_a$ such that $\sum_{a \in A} (m_a + N) = 0$ in M/N . Thus $\sum_{a \in A} m_a \in N$, and since N is A -graded this implies that each m_a belongs to $N \cap M_a = N_a$, so that in particular $m_a + N$ is zero in $(M/N)_a \subseteq M/N$, so that $(m_a + N)_{a \in A} = 0$ in $\bigoplus_a (M/N)_a$, as desired.

It remains to show that the canonical map

$$\varphi_a : M_a/N_a \rightarrow (M/N)_a$$

is an isomorphism. It is clearly surjective, as $(M/N)_a$ is generated by elements $m + N$ for $m \in M_a$, and these elements make up precisely the image of φ_a . Thus φ_a hits every generator of $(M/N)_a$, so φ_a is surjective. On the other hand, suppose we are given some $m \in M_a$ such that $\varphi(m + N_a) = m + N = 0$. Thus $m \in N$, and $m \in M_a$, so that $m \in M_a \cap N = N_a$, meaning $m + N_a = 0$ in M_a/N_a , as desired. \square

Recall that given a ring R , a left R -module M , a right R -module N , and an abelian group A , an R -balanced map $\varphi : M \times N \rightarrow B$ is one which satisfies

$$\begin{aligned} \varphi(m, n + n') &= \varphi(m, n) + \varphi(m, n') \\ \varphi(m + m', n) &= \varphi(m, n) + \varphi(m', n) \\ \varphi(m \cdot r, n) &= \varphi(m, r \cdot n). \end{aligned}$$

for all $m, m' \in M$, $n, n' \in N$, and $r \in R$. Then the tensor product $M \otimes_R N$ is the universal abelian group equipped with an R -balanced map $\otimes : M \times N \rightarrow M \otimes_R N$ such that for every abelian group B and every R -balanced map $\varphi : M \times N \rightarrow B$, there is a *unique* group homomorphism $\tilde{\varphi} : M \otimes_R N \rightarrow B$ such that $\tilde{\varphi} \circ \otimes = \varphi$. We call elements in the image of $\otimes : M \times N \rightarrow M \otimes_R N$ *pure tensors*. It is a standard fact that $M \otimes_R N$ is generated as an abelian group by its pure tensors.

Definition C.18. Suppose we have a right A -graded R -module M , a left A -graded R -module N , and an A -graded abelian group B . Then an A -graded R -balanced map $\varphi : M \times N \rightarrow B$ is an R -balanced map which restricts to $M_a \times N_b \rightarrow B_{a+b}$ for all $a, b \in A$.

Proposition C.19. Suppose we have a right A -graded R -module M and a left A -graded R -module N . Then the tensor product

$$M \otimes_R N$$

is naturally an A -graded abelian group by defining $(M \otimes_R N)_a$ to be the subgroup generated by homogeneous pure tensors $m \otimes n$ with $m \in M_b$ and $n \in N_c$ such that $b + c = a$. Furthermore, if either M (resp. N) is an A -graded bimodule, then this decomposition makes $M \otimes_R N$ into a left (resp. right) A -graded R -module. In particular, if both M and N are R -bimodules, then $M \otimes_R N$ is an R -bimodule.

Proof. By definition, since M and N are A -graded abelian groups, they are generated (as abelian groups) by their homogeneous elements. Thus it follows that $M \otimes_R N$ is generated by homogeneous pure tensors, that is, elements of the form $m \otimes n$ with $m \in M$ and $n \in N$ homogeneous. Now, given a homogeneous pure tensor $m \otimes n$, we define its *degree* by the formula $|m \otimes n| := |m| + |n|$. It follows this formula is well-defined by checking that given homogeneous elements $m \in M$, $n \in N$, and $r \in R$ that

$$|(m \cdot r) \otimes n| = |m \cdot r| + |n| = |m| + |r| + |n| = |m| + |r \cdot n| = |m \otimes (r \cdot n)|.$$

Thus, we may define $(M \otimes_R N)_a$ to be the subgroup of $M \otimes_R N$ generated by those pure homogeneous tensors of degree a . Now, consider the map

$$\Psi : M \times N \rightarrow \bigoplus_{a \in A} (M \otimes_R N)_a$$

which takes a pair $(m, n) = \sum_{a \in A} (m_a, n_a)$ to the element $\Psi(m, n)$ whose a^{th} component is

$$(\Psi(m, n))_a := \sum_{b+c=a} m_b \otimes n_c.$$

It is straightforward to see that this map is R -balanced, in the sense that it is additive in each argument and $\Psi(m \cdot r, n) = \Psi(m, r \cdot n)$ for all $m \in M$, $n \in N$, and $r \in R$. Thus by the universal property of $M \otimes_R N$, we get a homomorphism of abelian groups $\tilde{\Psi} : M \otimes_R N \rightarrow \bigoplus_{a \in A} (M \otimes_R N)_a$ lifting Ψ along the canonical map $M \times N \rightarrow M \otimes_R N$. Now, also consider the canonical map

$$\Phi : \bigoplus_{a \in A} (M \otimes_R N)_a \rightarrow M \otimes_R N.$$

We would like to show $\tilde{\Psi}$ and Φ are inverses of each other. Since $\tilde{\Psi}$ and Φ are both homomorphisms, it suffices to show this on generators. Let $m \otimes n$ be a homogeneous pure tensor with $m = m_a \in M_a$ and $n = n_b \in N_b$. Then we have

$$\Phi(\tilde{\Psi}(m \otimes n)) = \Phi\left(\bigoplus_{a \in A} \sum_{b+c=a} m_b \otimes n_c\right) \stackrel{(*)}{=} \Phi(m \otimes n) = m \otimes n,$$

and

$$\tilde{\Psi}(\Phi(m \otimes n)) = \tilde{\Psi}(m \otimes n) = \bigoplus_{a \in A} \sum_{b+c=a} m_b \otimes n_c \stackrel{(*)}{=} m \otimes n,$$

where both occurrences of $(*)$ follow by the fact that $m_b \otimes n_c = 0$ unless $b = c = a$, in which case $m_a \otimes n_a = m \otimes n$. Thus since Φ is an isomorphism, $M \otimes_R N$ is indeed an A -graded abelian group, as desired.

Now, suppose that M is an A -graded R -bimodule, so there exists left and right A -graded actions of R on M such that given $r, s \in R$ and $m \in M$ we have $r \cdot (m \cdot s) = (r \cdot m) \cdot s$. Then we would like to show that given a left A -graded R -module N that $M \otimes_R N$ is canonically a left A -graded R -module. Indeed, define the action of R on $M \otimes_R N$ on pure tensors by the formula

$$r \cdot (m \otimes n) = (r \cdot m) \otimes n.$$

First of all, clearly this map is A -graded, as if $r \in R_a$, $m \in M_b$, and $n \in N_c$ then $(r \cdot m) \otimes n$, by definition, has degree $|r \cdot m| + |n| = |r| + |m| + |n|$ (the last equality follows since the left action of R on M is A -graded). In order to show the above map defines a left module structure, it suffices to show that given pure tensors $m \otimes n, m' \otimes n' \in M \otimes_R N$ and elements $r, r' \in R$ that

- (1) $r \cdot (m \otimes n + m' \otimes n') = r \cdot (m \otimes n) + r \cdot (m' \otimes n')$,
- (2) $(r + r') \cdot (m \otimes n) = r \cdot (m \otimes n) + r' \cdot (m \otimes n)$,
- (3) $(rr') \cdot (m \otimes n) = r \cdot (r' \cdot (m \otimes n))$, and
- (4) $1 \cdot (m \otimes n) = m \otimes n$.

Axiom (1) holds by definition. To see (2), note that by the fact that R acts on M on the left that

$$(r + r') \cdot (m \otimes n) = ((r + r') \cdot m) \otimes n = (r \cdot m + r' \cdot m) \otimes n = r \cdot m \otimes n + r' \cdot m \otimes n.$$

That (3) and (4) hold follows similarly by the fact that $(rr') \cdot m = r \cdot (r' \cdot m)$ and $1 \cdot m = m$.

Conversely, if N is an A -graded R -bimodule, then showing $M \otimes_R N$ is canonically a right A -graded R -module via the rule

$$(m \otimes n) \cdot r = m \otimes (n \cdot r)$$

is entirely analagous.

Finally, if both M and N are R -bimodules, then by what we have shown, $M \otimes_R N$ is both a left and right R -module. To see these coincide to get $M \otimes_R N$ an R -bimodule structure, note that given $m \in M$, $n \in N$, and $r, r' \in R$ that

$$(r \cdot (m \otimes n)) \cdot r' = ((r \cdot m) \otimes n) \cdot r' = (r \cdot m) \otimes (n \cdot r') = r \cdot (m \otimes (n \cdot r')) = r \cdot ((m \otimes n) \cdot r'). \quad \square$$

Lemma C.20. *Let R be an A -graded ring, and suppose we have a right A -graded R -module M and a left A -graded R -module N . Then given an A -graded abelian group B and an A -graded R -balanced map*

$$\varphi : M \times N \rightarrow B,$$

the lift

$$\tilde{\varphi} : M \otimes_R N \rightarrow B$$

determined by the universal property of $M \otimes_R N$ is an A -graded homomorphism.

Proof. This simply amounts to unravelling definitions. Recall that the subgroup of homogeneous elements of degree a in $M \otimes_R N$ is that generated by pure tensors $m \otimes n$ with m and n homogeneous satisfying $|m| + |n| = a$. Thus, in order to show $\tilde{\varphi}$ is an A -graded homomorphism, it suffices to show that given homogeneous $m \in M$ and $n \in N$ that $\tilde{\varphi}(m \otimes n)$ is homogeneous and that

$$|\tilde{\varphi}(m \otimes n)| = |m \otimes n| = |m| + |n|.$$

Indeed, given two such elements m and n , consider the following diagram

$$\begin{array}{ccc} M \otimes_R N & & \\ \uparrow & \searrow \tilde{\varphi} & \\ M \times N & \xrightarrow{\varphi} & B \end{array}$$

This diagram commutes by universal property of $-\otimes_R-$. Note that the element $m \otimes n$ is mapped to by the pair (m, n) along the left vertical map. Hence by commutativity, we necessarily have

$$|\tilde{\varphi}(m \otimes n)| = |\varphi(m, n)| \stackrel{(*)}{=} |m| + |n|,$$

where $(*)$ follows by the fact that φ is an A -graded R -balanced map. \square

APPENDIX D. MONOID OBJECTS

In what follows, we fix a symmetric monoidal category $(\mathcal{C}, \otimes, S)$ with left unitor, right unitor, associator, and symmetry isomorphisms λ , ρ , α , and τ , respectively.

Definition D.1. Let (E, μ, e) be a monoid object in \mathcal{C} . Then a *left module object* (N, κ) over (E, μ, e) is the data of an object N in \mathcal{C} and a morphism $\kappa : E \otimes N \rightarrow N$ such that the following two diagrams commute in \mathcal{C} :

$$\begin{array}{ccc} S \otimes N & \xrightarrow{e \otimes N} & E \otimes N \\ & \searrow \lambda_N & \downarrow \kappa \\ & & N \end{array} \quad \begin{array}{ccc} (E \otimes E) \otimes N & \xrightarrow{\mu \otimes N} & E \otimes N \\ \alpha \downarrow & & \downarrow \kappa \\ E \otimes (E \otimes N) & \xrightarrow{E \otimes \kappa} & E \otimes N \xrightarrow{\kappa} N \end{array}$$

Definition D.2. Let (E, μ, e) be a monoid object in \mathcal{C} , and suppose we have two left module objects (N, κ) and (N', κ') over (E, μ, e) . Then a morphism $f : N \rightarrow N'$ is a *left E -module homomorphism* if the following diagram commutes in \mathcal{C} :

$$\begin{array}{ccc} E \otimes N & \xrightarrow{E \otimes f} & E \otimes N' \\ \kappa \downarrow & & \downarrow \kappa' \\ N & \xrightarrow{f} & N' \end{array}$$

Definition D.3. Given a monoid object (E, μ, e) in \mathcal{SH} , we write $E\text{-}\mathbf{Mod}$ to denote the category of left module objects over E and left E -module homomorphisms between them. We denote the homset in $E\text{-}\mathbf{Mod}$ by

$$\mathrm{Hom}_{E\text{-}\mathbf{Mod}}(M, N), \quad \text{or simply} \quad \mathrm{Hom}_E(M, N).$$

Lemma D.4. Let (E, μ, e) be a monoid object in \mathcal{C} and let (N, κ) be a left E module object. Then given some object X in \mathcal{C} and an isomorphism $\phi : N \xrightarrow{\cong} X$, X inherits the structure of a left E -module via the action map

$$\kappa_\phi : E \otimes X \xrightarrow{E \otimes \phi^{-1}} E \otimes N \xrightarrow{\kappa} N \xrightarrow{\phi} X.$$

Proof. We need to show the two coherence diagrams in [Definition D.1](#) commute. To see the former commutes, consider the following diagram:

$$\begin{array}{ccc} X & \xrightarrow{e \otimes X} & E \otimes X \\ & \searrow \phi^{-1} & \downarrow E \otimes \phi^{-1} \\ & & N \xrightarrow{e \otimes N} E \otimes N \\ & & \downarrow \kappa \\ & & N \\ & & \downarrow \phi \\ & & X \end{array}$$

The top trapezoid commutes by functoriality of $- \otimes -$. The middle small triangle commutes by unitality of κ . The remaining region commutes by definition. To see the second coherence diagram commutes, consider the following diagram:

$$\begin{array}{ccc}
 E \otimes E \otimes X & \xrightarrow{\mu \otimes X} & E \otimes X \\
 E \otimes E \otimes \phi^{-1} \downarrow & & \downarrow E \otimes \phi^{-1} \\
 E \otimes E \otimes N & \xrightarrow{\mu \otimes N} & E \otimes N \\
 E \otimes \kappa \downarrow & & \downarrow \kappa \\
 E \otimes N & \xrightarrow{\kappa} & N \\
 E \otimes \phi \downarrow & \searrow & \downarrow \phi \\
 E \otimes X & \xrightarrow{E \otimes \phi^{-1}} E \otimes N \xrightarrow{\kappa} N \xrightarrow{\phi} X
 \end{array}$$

The top rectangle commutes by functoriality of $- \otimes -$. The middle rectangle commutes by coherence for κ . The bottom two regions commute by definition. \square

Proposition D.5. *Given a monoid object (E, μ, e) in \mathcal{C} , the forgetful functor $E\text{-}\mathbf{Mod} \rightarrow \mathcal{C}$ has a left adjoint $\mathcal{C} \rightarrow E\text{-}\mathbf{Mod}$ sending an object $X \mapsto (E \otimes X, \kappa_X)$ where κ_X is the composition*

$$E \otimes (E \otimes X) \xrightarrow{\alpha^{-1}} (E \otimes E) \otimes X \xrightarrow{\mu \otimes X} E \otimes X,$$

and sending a morphism $f : X \rightarrow Y$ to $E \otimes f : E \otimes X \rightarrow E \otimes Y$.

Proof. In this proof, we work in a symmetric strict monoidal category. First, we wish to show that $E \otimes - : \mathcal{C} \rightarrow E\text{-}\mathbf{Mod}$ as constructed is well-defined. First, to see that (X, κ_X) is actually a left E -module, we need to show the two diagrams in [Definition D.1](#) commute. Indeed, consider the following diagrams:

$$\begin{array}{ccc}
 E \otimes X & \xrightarrow{e \otimes E \otimes X} & E \otimes E \otimes X \\
 & \searrow & \downarrow \mu \otimes X \\
 & & E \otimes X
 \end{array}
 \qquad
 \begin{array}{ccc}
 E \otimes E \otimes E \otimes X & \xrightarrow{\mu \otimes E \otimes X} & E \otimes E \otimes X \\
 E \otimes \mu \otimes X \downarrow & & \downarrow \mu \otimes X \\
 E \otimes E \otimes X & \xrightarrow{\mu \otimes X} & E \otimes X
 \end{array}$$

These are precisely the diagrams obtained by applying $X \otimes -$ to the coherence diagrams for μ , so that they commute as desired. Now, suppose $f : X \rightarrow Y$ is a morphism in \mathcal{C} , then we would like to show that $E \otimes f : E \otimes X \rightarrow E \otimes Y$ is a morphism of left E -module objects. Indeed, consider the following diagram:

$$\begin{array}{ccc}
 E \otimes E \otimes X & \xrightarrow{E \otimes E \otimes f} & E \otimes E \otimes Y \\
 \mu \otimes X \downarrow & & \downarrow \mu \otimes Y \\
 E \otimes X & \xrightarrow{E \otimes f} & E \otimes Y
 \end{array}$$

It commutes by functoriality of $- \otimes -$, so $E \otimes f$ is indeed a left E -module homomorphism as desired.

Now, in order to see that $E \otimes -$ is left adjoint to the forgetful functor, it suffices to construct a unit and counit for the adjunction and show they satisfy the zig-zag identities. Given X in \mathcal{C} and (N, κ) in $E\text{-}\mathbf{Mod}$, define $\eta_X := e \otimes X : X \rightarrow E \otimes X$ and $\varepsilon_{(N, \kappa)} := \kappa : E \otimes N \rightarrow N$. η_X is clearly natural in X by functoriality of $- \otimes -$, and $\varepsilon_{(N, \kappa)}$ is natural in (N, κ) by how morphisms in $E\text{-}\mathbf{Mod}$ are defined. Now, to see these are actually the unit and counit of an adjunction, we

need to show that the following diagrams commute for all X in \mathcal{C} and (N, κ) in $E\text{-}\mathbf{Mod}$:

$$\begin{array}{ccc}
 E \otimes X & \xrightarrow{E \otimes \eta_X = E \otimes e \otimes X} & E \otimes E \otimes X \\
 \searrow & & \downarrow \varepsilon_{(E \otimes X, \kappa_X)} = \mu \otimes X \\
 & & E \otimes X
 \end{array}
 \qquad
 \begin{array}{ccc}
 E \otimes N & \xleftarrow{\eta_N = e \otimes N} & N \\
 \downarrow \varepsilon_{(N, \kappa)} = \kappa & & \nearrow \\
 N & &
 \end{array}$$

Commutativity of the left diagram is unitality of μ , while commutativity of the right diagram is unitality of κ . Thus indeed $E \otimes - : \mathcal{C} \rightarrow E\text{-}\mathbf{Mod}$ is a left adjoint of the forgetful functor $E\text{-}\mathbf{Mod} \rightarrow \mathcal{C}$, as desired. \square

Definition D.6. We call the functor $E \otimes - : \mathcal{C} \rightarrow E\text{-}\mathbf{Mod}$ constructed above the *free functor*, and we call left E -modules in the image of the free functor *free modules*.

From now on we fix a monoidal closed tensor triangulated category $(\mathcal{H}, \otimes, S, \Sigma, e, \mathcal{D})$ ([Definition A.13](#)) with arbitrary (small) (co)products and sub-Picard grading $(A, \mathbf{1}, h, \{S^a\}, \{\phi_{a,b}\})$ ([Definition 2.2](#)), and we adopt the conventions outlined in [Section 2.1](#). In all proofs that follow we will freely use the coherence theorem for symmetric monoidal categories. In particular, we will assume without loss of generality that the associators and unitors in \mathcal{H} are identities.

Lemma D.7. *Let (E, μ, e) be a monoid object in \mathcal{H} , and suppose (N, κ) is a left module object over E . Then for all $a \in A$, $\Sigma^a N$ is canonically a left E -module object, with action map given by*

$$\kappa^a : E \otimes S^a \otimes N$$

$$\kappa^a : E \otimes \Sigma^a N = E \otimes S^a \otimes N \xrightarrow{\tau \otimes N} S^a \otimes E \otimes N \xrightarrow{S^a \otimes \kappa} S^a \otimes N = \Sigma^a N.$$

Proof. In this proof, we are assuming that unitality and associativity hold up to strict equality, by the coherence theorem for monoidal categories. In order to show $(\Sigma^a N, \kappa^a)$ is a left module object over E , we need to show κ^a makes the two coherence diagrams in [Definition D.1](#) commute. First, to see the first diagram commutes, consider the following diagram:

$$\begin{array}{ccc}
 S^a \otimes N & \xrightarrow{e \otimes S^a \otimes N} & E \otimes S^a \otimes N \\
 \searrow & & \downarrow \tau \otimes N \\
 & & S^a \otimes E \otimes N \\
 & & \downarrow S^a \otimes \kappa \\
 & & S^a \otimes N
 \end{array}$$

The top inner triangle commutes by coherence for a symmetric monoidal category, and the bottom inner triangle commutes by the coherence condition for κ . To see the other module condition for $\tilde{\kappa}$, consider the following diagram:

$$\begin{array}{ccccc}
 E \otimes E \otimes S^a \otimes N & \xrightarrow{\mu \otimes S^a \otimes N} & E \otimes S^a \otimes N & & \\
 \downarrow E \otimes \tau \otimes N & \searrow \tau_{E \otimes E, S^a \otimes N} & \downarrow \tau \otimes N & & \\
 E \otimes S^a \otimes E \otimes N & \xrightarrow{\tau_{E \otimes N}} & S^a \otimes E \otimes E \otimes N & \xrightarrow{S^a \otimes \mu \otimes N} & S^a \otimes E \otimes N \\
 \downarrow E \otimes S^a \otimes \kappa & & \downarrow S^a \otimes E \otimes \kappa & & \downarrow S^a \otimes \kappa \\
 E \otimes S^a \otimes N & \xrightarrow{\tau \otimes N} & S^a \otimes E \otimes N & \xrightarrow{S^a \otimes \kappa} & S^a \otimes N
 \end{array}$$

The top left triangle commutes by coherence for a symmetric monoidal category. The bottom left rectangle and top right trapezoid commute by naturality of τ . Finally, the bottom right square commutes by the coherence condition for κ . \square

Lemma D.8. *Given a monoid object (E, μ, e) in \mathcal{SH} , an object X in \mathcal{SH} , and some $a \in A$, the suspension of the free module $\Sigma^a(E \otimes X)$ is naturally isomorphic as a left E -module object to the free E -module $E \otimes \Sigma^a X$.*

Proof. It suffices to show the isomorphism $S^a \otimes E \otimes X \xrightarrow{\tau \otimes X} E \otimes S^a \otimes X$ is a homomorphism of left E -module objects. To see this, consider the following diagram:

$$\begin{array}{ccc}
 E \otimes S^a \otimes E \otimes X & \xrightarrow{E \otimes \tau \otimes X} & E \otimes E \otimes S^a \otimes X \\
 \tau \otimes E \otimes X \downarrow & \nearrow \tau_{S^a, E \otimes E \otimes X} & \downarrow \mu \otimes S^a \otimes X \\
 S^a \otimes E \otimes E \otimes X & & \\
 S^a \otimes \mu \otimes X \downarrow & & \\
 S^a \otimes E \otimes X & \xrightarrow{\tau \otimes X} & E \otimes S^a \otimes X
 \end{array}$$

The top triangle commutes by coherence for a symmetric monoidal category. The bottom trapezoid commutes by naturality of τ . \square

Definition D.9. Let (E, μ, e) be a monoid object in \mathcal{SH} , and suppose (N, κ) and (N', κ') are left E -module objects in $E\text{-Mod}$. Then the hom-sets in $E\text{-Mod}$ can be extended to A -graded abelian groups $\text{Hom}_{E\text{-Mod}}^*(N, N')$, by defining

$$\text{Hom}_{E\text{-Mod}}^a(N, N') := \text{Hom}_{E\text{-Mod}}(\Sigma^a N, N')$$

for each $a \in A$ (where $\Sigma^* N$ has the left E -module structure given by [Lemma D.7](#)).

Proposition D.10. *Let (E, μ, e) be a monoid object in \mathcal{SH} , and suppose we have a family of left E -module objects (N_i, κ_i) indexed by some small set I . Then $N := \bigoplus_{i \in I} N_i$ is canonically a left E -module, with action map given by the composition*

$$\kappa : E \otimes \bigoplus_i N_i \xrightarrow{\cong} \bigoplus_i (E \otimes N_i) \xrightarrow{\bigoplus_i \kappa_i} \bigoplus_i N_i,$$

where the first isomorphism is given by the fact that $E \otimes -$ preserves coproducts, since it is a left adjoint as \mathcal{SH} is monoidal closed. Furthermore, N is the coproduct of all the N_i 's in $E\text{-Mod}$, so that $E\text{-Mod}$ has arbitrary coproducts.

Proof. We need to show the action map κ makes the diagrams in [Definition D.1](#) commute. To see the first (unitality) diagram commutes, consider the following diagram:

$$\begin{array}{ccc}
 \bigoplus_i N_i & \xrightarrow{e \otimes \bigoplus_i N_i} & E \otimes \bigoplus_i N_i \\
 & \searrow \bigoplus_i (e \otimes N_i) & \downarrow \cong \\
 & & \bigoplus_i (E \otimes N_i) \\
 & & \downarrow \bigoplus_i \kappa_i \\
 & & \bigoplus_i N_i
 \end{array}$$

The top triangle commutes by additivity of $E \otimes -$. The bottom triangle commutes by unitality of each of the κ_i 's. To see the second coherence diagram commutes, consider the following diagram:

$$\begin{array}{ccccc}
 E \otimes E \otimes \bigoplus_i N_i & \xrightarrow{\mu \oplus \bigoplus_i N_i} & E \otimes \bigoplus_i N_i & & \\
 E \otimes \cong \downarrow & \searrow \cong & \downarrow \cong & & \\
 E \otimes \bigoplus_i (E \otimes N_i) & \xrightarrow{\cong} & \bigoplus_i (E \otimes E \otimes N_i) & \xrightarrow{\bigoplus_i (\mu \otimes N_i)} & \bigoplus_i (E \otimes N_i) \\
 E \otimes \bigoplus_i \kappa_i \downarrow & & \bigoplus_i (E \otimes \kappa_i) \downarrow & & \downarrow \bigoplus_i \kappa_i \\
 E \otimes \bigoplus_i N_i & \xrightarrow{\cong} & \bigoplus_i (E \otimes N_i) & \xrightarrow{\bigoplus_i \kappa_i} & \bigoplus_i N_i
 \end{array}$$

The bottom right square commutes by coherence for the κ_i 's. Every other region commutes by additivity of $- \otimes -$ in each variable. Thus $N = \bigoplus_i N_i$ is indeed a left E -module object, as desired.

Now, we claim that (N, κ) is the coproduct of the (N_i, κ_i) 's in $E\text{-}\mathbf{Mod}$. First, we need to show that the canonical maps $\iota_i : N_i \hookrightarrow N$ are morphisms in $E\text{-}\mathbf{Mod}$ for all $i \in I$. To see ι_i is a homomorphism of left E -module objects, consider the following diagram:

$$\begin{array}{ccc}
 E \otimes N_i & \xrightarrow{E \otimes \iota_i} & E \otimes \bigoplus_i N_i \\
 \downarrow \kappa_i & \searrow \iota_{E \otimes N_i} & \downarrow \cong \\
 & & \bigoplus_i (E \otimes N_i) \\
 & & \downarrow \bigoplus_i \kappa_i \\
 N_i & \xrightarrow{\iota_i} & \bigoplus_i N_i
 \end{array}$$

The top triangle commutes by additivity of $E \otimes -$. The bottom trapezoid commutes since, by univocal property of the coproduct, $\bigoplus_i \kappa_i$ is the unique arrow which makes the trapezoid commute for all $i \in I$. Now, it remains to show that given a left E -module object (N', κ') and homomorphisms $f_i : N_i \rightarrow N'$ of left E -module objects for all $i \in I$, that the unique arrow $f : N \rightarrow N'$ in \mathcal{SH} satisfying $f \circ \iota_i = f_i$ for all $i \in I$ is a homomorphism of left E -module objects, so that N is actually the coproduct of the N_i 's. To see this, first let $h : \bigoplus_i (E \otimes N_i) \rightarrow E \otimes N'$ be the arrow determined by the maps $E \otimes N_i \xrightarrow{E \otimes f_i} E \otimes N'$. Then consider the following diagram:

$$\begin{array}{ccc}
 E \otimes \bigoplus_i N_i & \xrightarrow{E \otimes f} & E \otimes N' \\
 \cong \downarrow & \nearrow h & \downarrow \kappa' \\
 \bigoplus_i (E \otimes N_i) & \xrightarrow{\bigoplus_i (E \otimes f_i)} & \bigoplus_i (E \otimes N') \\
 \downarrow \bigoplus_i \kappa_i & & \downarrow \bigoplus_i \kappa' \\
 \bigoplus_i N_i & \xrightarrow{f} & N' \\
 & \nearrow \bigoplus_i f_i & \searrow \nabla
 \end{array}$$

The top triangle commutes by additivity of $E \otimes -$. The triangle below that commutes by the universal property of the coproduct, since it is straightforward to check that $\nabla \circ \bigoplus_i (E \otimes f_i)$ and h both satisfy the universal property of the colimit. The left trapezoid commutes by functoriality of $- \otimes -$ and the fact that f_i is a homomorphism of left E -module objects for all $i \in I$. The right trapezoid commutes by naturality of ∇ . Finally, the bottom triangle commutes by the universal

product of the coproduct, by showing that $\nabla \circ \bigoplus_i f_i$ in place of f also satisfies the universal property of the colimit. Hence f is indeed a homomorphism of left E -module objects, as desired.

To recap, we have shown that given a set of left E -module objects $\{(N_i, \kappa_i)\}_{i \in I}$, that the inclusion maps $\iota_i : N_i \hookrightarrow \bigoplus_i N_i$ are morphisms in $E\text{-Mod}$, and that given morphism $f_i : (N_i, \kappa_i) \rightarrow (N', \kappa')$ for all $i \in I$, the unique induced map $\bigoplus_i N_i \rightarrow N'$ is a morphism in $E\text{-Mod}$. Thus, $E\text{-Mod}$ does indeed have arbitrary coproducts, and the forgetful functor $E\text{-Mod} \rightarrow \mathcal{SH}$ preserves them. \square

Proposition D.11. *Let (E, μ, e) be a monoid object in \mathcal{SH} . Then $E\text{-Mod}$ is an additive category, so that in particular the forgetful functor $E\text{-Mod} \rightarrow \mathcal{SH}$ and the free functor $\mathcal{SH} \rightarrow E\text{-Mod}$ are additive.*

Proof. It is a general fact that adjoint functors between additive categories are necessarily additive. In order to show $E\text{-Mod}$ is an additive category, it suffices to show it has finite coproducts, that $\text{Hom}_{E\text{-Mod}}(N, N')$ is an abelian group for all left E -modules N and N' , and that composition is bilinear. We know that $E\text{-Mod}$ has coproducts which are preserved by the forgetful functor $E\text{-Mod} \rightarrow \mathcal{SH}$ by Proposition D.10 (which is clearly faithful). Thus, because \mathcal{SH} is **Ab**-enriched and $\text{Hom}_{E\text{-Mod}}(N, N') \subseteq \mathcal{SH}(N, N')$, it suffices to show that $\text{Hom}_{E\text{-Mod}}(N, N')$ is closed under addition and taking inverses. To see the former, let $f, g : N \rightarrow N'$ be left E -module homomorphisms, and consider the following diagram:

$$\begin{array}{ccccc}
 E \otimes N & \xrightarrow{E \otimes \Delta_N} & E \otimes (N \oplus N) & \xrightarrow{E \otimes (f \oplus g)} & E \otimes (N' \oplus N') & \xrightarrow{E \otimes \nabla_{N'}} & E \otimes N' \\
 \downarrow \kappa & \searrow \Delta_{E \otimes N} & \downarrow \cong & & \downarrow \cong & \nearrow \nabla_{E \otimes N'} & \downarrow \kappa' \\
 & & (E \otimes N) \oplus (E \otimes N) & \xrightarrow{(E \otimes f) \oplus (E \otimes g)} & (E \otimes N') \oplus (E \otimes N') & & \\
 & & \downarrow \kappa \oplus \kappa & & \downarrow \kappa' \oplus \kappa' & & \\
 & & (E \otimes N') \otimes (E \otimes N) & \xrightarrow{(E \otimes f) \oplus (E \otimes g)} & (E \otimes N') \otimes (E \otimes N) & & \\
 & & \downarrow \kappa' \oplus \kappa & & \downarrow \kappa' \oplus \kappa & & \\
 & & N' \oplus N & \xrightarrow{f \oplus g} & N' \oplus N' & \xrightarrow{\nabla_{N'}} & N' \\
 & \nearrow f \oplus N & & & \nearrow N' \oplus g & & \\
 N & \xrightarrow{\Delta_N} & N \oplus N & \xrightarrow{f \oplus g} & N' \oplus N' & \xrightarrow{\nabla_{N'}} & N'
 \end{array}$$

The outermost trapezoids commute by naturality of Δ and ∇ . The triangles in the top corners and the top middle rectangle commute by additivity of $E \otimes -$. The middle triangle commutes by functoriality of \oplus and \otimes . The middle trapezoids commute by the fact that f and g are homomorphisms of left E -modules. Finally, the middle bottom triangle commutes by functoriality of $- \oplus -$. Commutativity of the above diagram shows that $f + g$ is a homomorphism of left E -modules as desired. Finally, to see $-f$ is a left E -module homomorphism if f is, we would like to show that $\kappa' \circ (E \otimes (-f)) = (-f) \circ \kappa$. This follows by the fact that $\kappa' \circ (E \otimes f) = f \circ \kappa$ and additivity of $- \otimes -$ and composition. \square

Proposition D.12. *Let (E, μ, e) be a monoid object in \mathcal{SH} , and consider the multiplication map $\pi_*(E) \times \pi_*(E) \rightarrow \pi_*(E)$ which sends classes $x : S^a \rightarrow E$ and $y : S^b \rightarrow E$ to the composition*

$$S^{a+b} \xrightarrow{\phi_{a,b}} S^a \otimes S^b \xrightarrow{x \otimes y} E \otimes E \xrightarrow{\mu} E.$$

Then this endows $\pi_(E)$ with the structure of an A -graded ring with unit $e \in \pi_0(E) = [S, E]$.*

Proof. Here we are using [Lemma C.7](#), so it suffices to show the given assignment is associative and unital w.r.t. homogeneous elements. Suppose we have classes x , y , and z in $\pi_a(E)$, $\pi_b(E)$, and $\pi_c(E)$, respectively. To see associativity, consider the following diagram:

$$\begin{array}{c}
 S^{a+b+c} \xrightarrow{\cong} S^a \otimes S^b \otimes S^c \xrightarrow{x \otimes y \otimes z} E \otimes E \otimes E \\
 \begin{array}{ccc}
 & \nearrow \mu \otimes E & \\
 & E \otimes E & \searrow E \otimes \mu \\
 & E & \\
 & \uparrow \mu & \\
 & E \otimes E &
 \end{array}
 \end{array}$$

(here the first arrow is the unique isomorphism obtained by composing products of $\phi_{a,b}$'s, see [Remark 2.3](#)). It commutes by associativity of μ . It follows by functoriality of $- \otimes -$ that the top composition is $(x \cdot y) \cdot z$ while the bottom is $x \cdot (y \cdot z)$, so they are equal as desired. To see that $e \in \pi_0(E)$ is a left and right unit for this multiplication, consider the following diagram

$$\begin{array}{ccccc}
 & & S^a & & \\
 & \swarrow e \otimes x & \downarrow x & \searrow x \otimes e & \\
 E \otimes E & \xleftarrow{e \otimes E} & E & \xrightarrow{E \otimes e} & E \otimes E \\
 & \searrow \mu & \parallel & \swarrow \mu & \\
 & & E & &
 \end{array}$$

Commutativity of the two top triangles is functoriality of $- \otimes -$. Commutativity of the bottom two triangles is unitality of μ . Thus the diagram commutes, so $e \cdot x = x = x \cdot e$. Finally, to see this product is bilinear (distributive). Suppose we further have some $x' \in \pi_a(E)$ and $y' \in \pi_b(E)$, and consider the following diagrams:

$$\begin{array}{c}
 \begin{array}{ccccccc}
 S^{a+b} & \xrightarrow{\phi_{a,b}} & S^a \otimes S^b & \xrightarrow{\Delta \otimes S^b} & (S^a \oplus S^a) \otimes S^b & \xrightarrow{(x \oplus x') \otimes y} & (E \oplus E) \otimes E \\
 \Delta \downarrow & & \downarrow \Delta & \swarrow \cong & & \swarrow \cong & \downarrow \nabla \otimes E \\
 S^{a+b} \oplus S^{a+b} & \xrightarrow{\phi_{a,b} \oplus \phi_{a,b}} & (S^a \otimes S^b) \oplus (S^a \otimes S^b) & \xrightarrow{(x \otimes y) \oplus (x' \otimes y)} & (E \otimes E) \oplus (E \otimes E) & \xrightarrow{\nabla} & E \otimes E \xrightarrow{\mu} E
 \end{array} \\
 \\
 \begin{array}{ccccccc}
 S^{a+b} & \xrightarrow{\phi_{a,b}} & S^a \otimes S^b & \xrightarrow{S^a \otimes \Delta} & S^b \otimes (S^b \oplus S^b) & \xrightarrow{x \otimes (y \oplus y')} & E \otimes (E \oplus E) \\
 \Delta \downarrow & & \downarrow \Delta & \swarrow \cong & & \swarrow \cong & \downarrow E \otimes \nabla \\
 S^{a+b} \oplus S^{a+b} & \xrightarrow{\phi_{a,b} \oplus \phi_{a,b}} & (S^a \otimes S^b) \oplus (S^a \otimes S^b) & \xrightarrow{(x \otimes y) \oplus (x \otimes y)} & (E \otimes E) \oplus (E \otimes E) & \xrightarrow{\nabla} & E \otimes E \xrightarrow{\mu} E
 \end{array}
 \end{array}$$

The unlabeled isomorphisms are those given by the fact that $- \otimes -$ is additive in each variable (since \mathcal{SH} is tensor triangulated). Commutativity of the left squares is naturality of $\Delta : X \rightarrow X \oplus X$ in an additive category. Commutativity of the rest of the diagram follows again from the fact that $- \otimes -$ is an additive functor in each variable. Hence, by functoriality of $- \otimes -$, these diagrams tell us that $(x + x') \cdot y = x \cdot y + x' \cdot y$ and $x \cdot (y + y') = x \cdot y + x \cdot y'$, respectively. \square

Proposition D.13. *For all $a, b \in A$ there exists an element $\theta_{a,b} \in \pi_0(S) = [S, S]$ such that given any commutative monoid object (E, μ, e) in \mathcal{SH} , the A -graded ring structure on $\pi_*(E)$ (??) has a commutativity formula given by*

$$x \cdot y = y \cdot x \cdot (e \circ \theta_{a,b})$$

for all $x \in \pi_a(E)$ and $y \in \pi_b(E)$. In particular, $\theta_{a,b} \in \text{Aut}(S)$ is the composition

$$S \xrightarrow{\cong} S^{-a-b} \otimes S^a \otimes S^b \xrightarrow{S^{-a-b} \otimes \tau} S^{-a-b} \otimes S^b \otimes S^a \xrightarrow{\cong} S,$$

where the outermost maps are the unique maps specified by [Remark 2.3](#).

Proof. Let (E, μ, e) , x , and y as in the statement of the proposition. Now consider the following diagram

$$\begin{array}{ccccc} S^{a+b} & \xrightarrow{\phi_{a,b}} & S^a \otimes S^b & \xrightarrow{x \otimes y} & E \otimes E \\ \downarrow \phi_{b,a}^{-1} \circ \tau \circ \phi_{a,b} & & \downarrow \tau & & \downarrow \tau \\ S^{a+b} & \xrightarrow{\phi_{b,a}} & S^b \otimes S^a & \xrightarrow{y \otimes x} & E \otimes E \end{array} \quad \begin{array}{c} \nearrow \mu \\ \searrow \mu \end{array} \rightarrow E$$

The left square commutes by definition. The middle square commutes by naturality of the symmetry isomorphism. Finally, the right square commutes by commutativity of E . Unravelling definitions, we have shown that under the product on $\pi_*(E)$ induced by the $\phi_{a,b}$'s,

$$x \cdot y = (y \cdot x) \circ (\phi_{b,a}^{-1} \circ \tau \circ \phi_{a,b}).$$

Thus, in order to show the desired result it further suffices to show that

$$(y \cdot x) \circ (\phi_{b,a}^{-1} \circ \tau \circ \phi_{a,b}) = y \cdot x \cdot (e \circ \theta_{a,b}).$$

Consider the following diagram:

$$\begin{array}{c} \begin{array}{ccc} S^{a+b} & \xrightarrow{\phi_{a,b}} & S^a \otimes S^b \\ \cong \downarrow & \nearrow \cong & \downarrow \tau \\ S^b \otimes S^a \otimes S^{-a-b} \otimes S^a \otimes S^b & & S^b \otimes S^a \\ S^b \otimes S^a \otimes S^{-a-b} \otimes \tau \downarrow & \nearrow \cong & \downarrow \phi_{b,a}^{-1} \\ S^b \otimes S^a \otimes S^{-a-b} \otimes S^b \otimes S^a & \xrightarrow{\cong} & S^b \otimes S^a \xrightarrow{\phi_{b,a}} S^{a+b} \end{array} \\ \begin{array}{ccc} & \searrow y \otimes x & \\ & & E \otimes E \\ y \otimes x \otimes e \nearrow & & \downarrow \mu \\ E \otimes E \otimes E & \xrightarrow{E \otimes \mu} & E \otimes E \\ \mu \otimes E \downarrow & & \downarrow \mu \\ E \otimes E & \xrightarrow{\mu} & E \end{array} \end{array}$$

Here any map simply labelled \cong is an appropriate composition of copies of $\phi_{a,b}$'s, associators, and their inverses, so that each of these maps are necessarily unique by [Remark 2.3](#). The triangles in the top large rectangle commutes by coherence for the $\phi_{a,b}$'s. The parallelogram commutes by naturality of τ and coherence of the $\phi_{a,b}$'s. The middle skewed triangle commutes by functoriality of $-\otimes -$. The triangle below that commutes by unitality of μ . Finally, the bottom rectangle commutes by associativity of μ . Hence, by unravelling definitions and applying functoriality of $-\otimes -$, we get that the right composition is $(y \cdot x) \circ (\phi_{b,a}^{-1} \circ \tau \circ \phi_{a,b})$, while the left composition is $y \cdot x \cdot (e \circ \theta_{a,b})$, so they are equal as desired. \square

Lemma D.14. Suppose we have homogeneous elements $x, y \in \pi_*(S)$ with x of degree 0, then we have $x \cdot y = y \cdot x = x \circ y$ (where the \cdot denotes the product given in [Proposition D.12](#)).

Proof. As morphisms, y is an arrow $S^a \rightarrow S$ for some a in A , and x is a morphism $S \rightarrow S$. Then consider the following diagram:

$$\begin{array}{ccccc}
 S \otimes S^a & \xleftarrow{\phi_{0,a}=\lambda_{S^a}^{-1}} & S^a & \xrightarrow{\phi_{a,0}=\rho_{S^a}^{-1}} & S^a \otimes S \\
 \downarrow y \otimes x & \swarrow S \otimes y & \downarrow y & \swarrow y \otimes S & \downarrow x \otimes y \\
 & S \otimes S & S & S \otimes S & \\
 & \xleftarrow{\lambda_S=\rho_S} & & \xleftarrow{\rho_S=\lambda_S} & \\
 & \downarrow x & & \downarrow x & \\
 S \otimes S & \xleftarrow{\phi_{0,0}^{-1}=\rho_S} & S & \xleftarrow{\phi_{0,0}^{-1}=\lambda_S} & S \otimes S
 \end{array}$$

The trapezoids commute by naturality of the unitors, and the triangles commute by functoriality of $- \otimes -$. The outside compositions are $y \cdot x$ on the left and $x \cdot y$ on the right, and the middle composition is $x \circ y$, so indeed we have $y \cdot x = x \cdot y = x \circ y$, as desired. \square

Proposition D.15. *Given $a \in A$, we have $\theta_{0,a} = \theta_{a,0} = \text{id}_S$.*

Proof. Recall $\theta_{a,0}$ is the composition

$$S \xrightarrow{\phi_{-a,a}} S^{-a} \otimes S^a \xrightarrow{S^{-a} \otimes \phi_{a,0}} S^{-a} \otimes (S^a \otimes S) \xrightarrow{S^{-a} \otimes \tau} S^{-a} \otimes (S \otimes S^a) \xrightarrow{S^{-a} \otimes \phi_{0,a}^{-1}} S^{-a} \otimes S^a \xrightarrow{\phi_{-a,a}^{-1}} S$$

By the coherence theorem for symmetric monoidal categories and the fact that $\phi_{a,0}$ and $\phi_{0,a}$ coincide with the unitors, we have that the composition

$$S^a \xrightarrow{\phi_{a,0}=\rho_{S^a}^{-1}} S^a \otimes S \xrightarrow{\tau} S \otimes S^a \xrightarrow{\phi_{0,a}^{-1}=\lambda_{S^a}} S^a$$

is precisely the identity map, so by functoriality of $- \otimes -$, we have that $\theta_{a,0}$ is the composition

$$S \xrightarrow{\phi_{-a,a}} S^{-a} \otimes S^a \xrightarrow{\cong} S^{-a} \otimes S^a \xrightarrow{\phi_{-a,a}^{-1}} S,$$

so $\theta_{a,0} = \text{id}_S$, meaning

$$x \cdot y = y \cdot x \cdot (e \circ \theta_{a,0}) = y \cdot x \cdot e = y \cdot x,$$

where the last equality follows by the fact that e is the unit for the multiplication on $\pi_*(E)$. An entirely analogous argument yields that $\theta_{0,a} = \text{id}_S$. \square

Proposition D.16. *Let $a, b \in A$. Then $\theta_{a,b} \cdot \theta_{b,a} = \text{id}_S$.*

Proof. By [Lemma D.14](#), it suffices to show that $\theta_{a,b} \circ \theta_{b,a} = \text{id}_S$. To see this, consider the following diagram:

$$\begin{array}{ccc}
 S & \xrightarrow{\phi} & S^{-a-b} \otimes S^b \otimes S^a \xrightarrow{S^{-a-b} \otimes \tau} S^{-a-b} \otimes S^a \otimes S^b \xrightarrow{\phi} S \\
 & \searrow & \downarrow \phi \\
 & & S^{-a-b} \otimes S^a \otimes S^b \\
 & \searrow & \downarrow S^{-a-b} \otimes \tau \\
 & & S^{-a-b} \otimes S^b \otimes S^a \\
 & \searrow & \downarrow \phi \\
 & & S
 \end{array}$$

Here we are suppressing associators, and any map labelled ϕ is the appropriate composition of $\phi_{a,b}$'s, unitors, associators, identities, and their inverses (see [Remark 2.3](#)). Clearly each region

commutes, the middle by the fact that $\tau^2 = 0$, and the other two regions by coherence for the ϕ 's. Thus we have shwon $\theta_{a,b} \cdot \theta_{b,a} = \theta_{a,b} \cdot \theta_{b,a} = \text{id}_S$, as desired. \square

Proposition D.17. *Let $a, b, c \in A$. Then $\theta_{a,b} \cdot \theta_{a,c} = \theta_{a,b+c}$ and $\theta_{b,a} \cdot \theta_{c,a} = \theta_{b+c,a}$.*

Proof.

$$\begin{array}{ccccc}
 S & \xrightarrow{\phi_{-a-c,a+c}} & S^{-a-c} S^{a+c} & \xrightarrow{S^{-a-c} \phi_{a,c}} & S^{-a-c} S^a S^c & \xrightarrow{S^{-a-c} \tau} & S^{-a-c} S^c S^a & \xrightarrow{S^{-a-c} \phi_{c,a}^{-1}} & S^{-a-c} S^{a+c} \\
 \downarrow \phi_{-a-b-c,a+b+c} & & \downarrow S^{-a} S^{-c} \phi_{-b,a} S^b S^c & \swarrow \phi_{-a,-c} \phi_{a-b,b} S^c & \downarrow S^{-a} S^{-c} S^c \phi_{-b,a} S^b & \swarrow \phi_{-a-c} S^c \phi_{a-b,b} & & \downarrow \phi_{-a-c}^{-1} & \downarrow S \\
 S^{-a-b-c} S^{a+b+c} & & S^{-a} S^{-c} S^{a-b} S^b S^c & \xrightarrow{S^{-a} S^{-c} \tau_{S^{a-b} S^b, S^c}} & S^{-a} S^{-c} S^c S^{a-b} S^b & & & S^{-a-b} S^{a+b} & \\
 \downarrow S^{-a-b-c} \phi_{a,b+c} & & \downarrow S^{-a} S^{-c} \phi_{-b,a} S^b S^c & & \downarrow S^{-a} S^{-c} S^c \phi_{-b,a} S^b & & & \downarrow S^{-a-b} & \\
 S^{-a-b-c} S^a S^{b+c} & \xrightarrow{\phi_{-a-c,-b} S^a} & S^{-a-c} S^{-b} S^a S^b S^c & \xrightarrow{S^{-a-c} \tau_{S^{-b} S^a S^b, S^c}} & S^{-a-c} S^c S^{-b} S^a S^b & \xleftarrow{S^{-a-c} \phi_{c,-b} S^a S^b} & S^{-a-c} S^c S^{-b} S^a S^b & \xleftarrow{S^{-a-c} \phi_{c,-b} S^a S^b} & S^{-a-b} S^a S^b \\
 \downarrow S^{-a-b-c} \tau_{a,b+c} & & \downarrow S^{-a-c} S^{-b} S^a S^b S^c & & \downarrow S^{-a-c} S^c S^{-b} \tau & & \downarrow S^{-a-c} S^c S^{-b} \tau & & \downarrow S^{-a-b} \tau \\
 S^{-a-b-c} S^{b+c} S^a & \xrightarrow{\phi_{-a-c,-b} \phi_{b,c} S^a} & S^{-a-c} S^{-b} S^b S^c S^a & \xrightarrow{S^{-a-c} \tau_{S^{-b} S^b, S^c}} & S^{-a-c} S^c S^{-b} S^b S^a & \xleftarrow{S^{-a-c} \phi_{c,-b} S^b S^a} & S^{-a-c} S^c S^{-b} S^b S^a & \xleftarrow{\phi_{-a-c,-b} S^b S^a} & S^{-a-b} S^b S^a \\
 \downarrow S^{-a-b-c} \phi_{b+c,a}^{-1} & & \downarrow S^{-a-c} \phi_{-b,b} S^c S^a & & \downarrow S^{-a-c} S^c S^c S^a & & \downarrow S^{-a-c} \phi_{c,0} S^a & & \downarrow S^{-a-b} \\
 S^{-a-b-c} S^{a+b+c} & \xrightarrow{\phi_{-a-b-c,a+b+c}^{-1}} & S & \xleftarrow{\phi_{-a-b,a+b}^{-1}} & S^{-a-b} S^{a+b} & & & & \\
 & & & & & & & &
 \end{array}$$

\square

Lemma D.18. *Let X and Y be objects in \mathcal{SH} . Then the A -graded pairing*

$$\pi_*(X) \times \pi_*(Y) \rightarrow \pi_*(X \otimes Y)$$

sending $x : S^a \rightarrow X$ and $y : S^b \rightarrow Y$ to the composition

$$S^{a+b} \xrightarrow{\phi_{a,b}} S^a \otimes S^b \xrightarrow{x \otimes y} X \otimes Y$$

is additive in each argument.

Proof. Let $a, b \in A$, and let $x_1, x_2 : S^a \rightarrow X$ and $y : S^b \rightarrow Y$. Then consider the following diagram

$$\begin{array}{ccccc}
 S^{a+b} & \xrightarrow{\cong} & S^a \otimes S^b & \xrightarrow{\Delta \otimes S^b} & (S^a \oplus S^a) \otimes S^b \\
 & & \downarrow \Delta & \swarrow \cong & \downarrow (x_1 \oplus x_2) \otimes y \\
 & & (S^a \otimes S^b) \oplus (S^a \otimes S^b) & & (X \oplus X) \otimes Y \\
 & & \downarrow (x_1 \otimes y) \oplus (x_2 \otimes y) & \swarrow \cong & \downarrow \nabla \otimes Y \\
 & & (X \otimes Y) \oplus (X \otimes Y) & \xrightarrow{\nabla} & X \otimes Y
 \end{array}$$

The isomorphisms are given by the fact that $- \otimes -$ is additive in each variable. Both triangles and the parallelogram commute since $- \otimes -$ is additive. By functoriality of $- \otimes -$, the top

composition is $(x_1 + x_2) \cdot y$ and the bottom composition is $x_1 \cdot y + x_2 \cdot y$, so they are equal, as desired. An entirely analagous argument yields that $x \cdot (y_1 + y_2) = x \cdot y_1 + x \cdot y_2$ for $x \in \pi_*(X)$ and $y_1, y_2 \in \pi_*(Y)$. \square

Lemma D.19. *Let (E, μ, e) be a monoid object. Then the assignment $\pi_* : (N, \kappa) \mapsto \pi_*(N)$ yields an additive functor from $E\text{-Mod}$ to A -graded left $\pi_*(E)$ -modules. In particular, if (N, κ) is a left E -module in $S\mathcal{H}$ then the map*

$$\pi_*(E) \times \pi_*(N) \rightarrow \pi_*(N)$$

sending a class $r : S^a \rightarrow E$ and $x : S^b \rightarrow N$ to the composition

$$r \cdot x : S^{a+b} \xrightarrow{\phi_{a,b}} S^a \otimes S^b \xrightarrow{r \otimes x} E \otimes N \xrightarrow{\kappa} N$$

endows $\pi_(N)$ with the structure of an A -graded left $\pi_*(E)$ -module.*

Proof. First let (N, κ) be an E -module object. Let $a, b, c \in A$ and $x, x' : S^a \rightarrow N$, $y : S^b \rightarrow E$, and $z, z' \in S^c \rightarrow E$. Then by [Lemma C.8](#), it suffices to show that

- (1) $y \cdot (x + x') = y \cdot x + y \cdot x'$,
- (2) $(z + z') \cdot x = z \cdot x + z' \cdot x$,
- (3) $(zy) \cdot x = z \cdot (y \cdot x)$,
- (4) $e \cdot x = x$.

The first two axioms follow by [Lemma D.18](#). To see (3), consider the diagram:

$$\begin{array}{ccccc}
 S^{a+b+c} & \xrightarrow{\cong} & S^c \otimes S^b \otimes S^a & \xrightarrow{z \otimes y \otimes x} & E \otimes E \otimes N \\
 & & & & \swarrow E \otimes \kappa \quad \searrow \mu \otimes N \\
 & & & & E \otimes N & \begin{array}{c} \downarrow \kappa \\ N \\ \uparrow \kappa \\ E \otimes N \end{array}
 \end{array}$$

It commutes by coherence for κ . By functoriality of $- \otimes -$, the two outside compositions equal $z \cdot (y \cdot x)$ on the top and $(z \cdot y) \cdot x$ on the bottom. Hence, they are equal, as desired.

Next, to see (4), consider the following diagram:

$$\begin{array}{ccc}
 S^a & \xrightarrow{x} & N \\
 & \searrow x & \nearrow \kappa \\
 & N & \\
 & \downarrow e \otimes N & \\
 & E \otimes N & \\
 & \uparrow e \otimes x & \\
 S^a & &
 \end{array}$$

The top triangle commutes by definition. The left triangle commutes by functoriality of $- \otimes -$. The right triangle commutes by unitality of κ . The top composition is x while the bottom is $e \cdot x$, thus they are necessarily equal since the diagram commutes.

Now, we'd like to show that if $f : (N, \kappa) \rightarrow (N', \kappa)$ is a homomorphism of left E -module objects, then $\pi_*(f) : \pi_*(N) \rightarrow \pi_*(N')$ is a homomorphism of left $\pi_*(E)$ -modules. To see this, let $r : S^a \rightarrow E$ in $\pi_a(E)$ and $x, x' : S^b \rightarrow N$ in $\pi_b(N)$. We'd like to show that $\pi_*(f)(x + x') = \pi_*(f)(x) + \pi_*(f)(x')$ and $\pi_*(f)(r \cdot x) = r \cdot \pi_*(f)(x)$. To see the former, consider the following

diagram:

$$\begin{array}{ccccc}
 & & & N' \oplus N' & \\
 & & f \oplus f \nearrow & \downarrow \nabla & \\
 S^a & \xrightarrow{\Delta} & S^a \oplus S^a & \xrightarrow{x \oplus x'} & N \oplus N \\
 & & \searrow \nabla & \uparrow f & \\
 & & & N' & \\
 & & & \uparrow f & \\
 & & & N &
 \end{array}$$

It commutes by naturality of ∇ in an additive category. The top composition is $\pi_*(f)(x) + \pi_*(f)(x')$, while the bottom is $\pi_*(f)(x+x')$, so they are equal as desired. To see that $\pi_*(f)(r \cdot x) = r \cdot \pi_*(f)(x)$, consider the following diagram:

$$\begin{array}{ccccc}
 & & & E \otimes N' & \\
 & & E \otimes f \nearrow & \downarrow \kappa' & \\
 S^{a+b} & \xrightarrow{\phi_{b,a}} & S^b \otimes S^a & \xrightarrow{r \otimes x} & E \otimes N \\
 & & \searrow \kappa & \uparrow f & \\
 & & & N' & \\
 & & & \uparrow f & \\
 & & & N &
 \end{array}$$

It commutes by the fact that f is a homomorphism of left E -module objects. The bottom composition is $\pi_*(f)(r \cdot x)$, while the top composition is $r \cdot \pi_*(f)(x)$, so they are equal, as desired.

It remains to show this functor is additive. It suffices to show it preserves the zero object and preserves coproducts. To see the former, note that $\pi_*(0) = [S^*, 0] = 0$ by definition, since 0 is terminal. To see the latter, we need to show that given $(N, \kappa), (N', \kappa') \in E\text{-Mod}$ that $\pi_*(N) \oplus \pi_*(N') \cong \pi_*(N \oplus N')$, and that the following diagram commutes:

$$\begin{array}{ccc}
 \pi_*(N) & & \\
 \downarrow \iota_{\pi_*(N)} & \searrow \pi_*(\iota_N) & \\
 \pi_*(N) \oplus \pi_*(N') & \xrightarrow{\cong} & \pi_*(N \oplus N')
 \end{array}$$

Since each S^a is compact, for all $a, b \in A$ we have isomorphisms

$$\pi_a(N) \oplus \pi_a(N') = [S^a, N] \oplus [S^a, N'] \cong [S^a, N \oplus N'] = \pi_a(N \oplus N'),$$

and these combine together to yield A -graded isomorphisms $\pi_*(N) \oplus \pi_*(N') \xrightarrow{\cong} \pi_*(N \oplus N')$. To see the above diagram commutes, note that since everything is an A -graded homomorphism of A -graded abelian groups, it suffices to chase homogeneous elements around to show it commutes. Indeed, it is entirely straightforward, by unravelling definitions, that both compositions around the diagram take a generator $x : S^a \rightarrow N$ in $\pi_a(N)$ to the composition

$$S^a \xrightarrow{x} N \xrightarrow{\iota_N} N \oplus N'.$$

Thus, we have shown that π_* preserves all finite coproducts, so it is additive. \square

Proposition D.20 ([6, Proposition 5.11]). *Let (E, μ, e) be a monoid object in \mathcal{SH} . Then $E_*(-)$ is a functor from \mathcal{SH} to left A -graded $\pi_*(E)$ -modules, where given some X in \mathcal{SH} , $E_*(X)$ may be endowed with the structure of a left A -graded $\pi_*(E)$ -module via the map*

$$\pi_*(E) \times E_*(X) \rightarrow E_*(X)$$

which given $a, b \in A$, sends $x : S^a \rightarrow E$ and $y : S^b \rightarrow E \otimes X$ to the composition

$$x \cdot y : S^{a+b} \cong S^a \otimes S^b \xrightarrow{x \otimes y} E \otimes (E \otimes X) \cong (E \otimes E) \otimes X \xrightarrow{\mu \otimes X} E \otimes X.$$

Similarly, the assignment $X \mapsto X_*(E)$ is a functor from \mathcal{SH} to right A -graded $\pi_*(E)$ -modules, where the structure map

$$X_*(E) \times \pi_*(E) \rightarrow X_*(E)$$

sends $x : S^a \rightarrow X \otimes E$ and $y : S^b \rightarrow E$ to the composition

$$x \cdot y : S^{a+b} \cong S^a \otimes S^b \xrightarrow{x \otimes y} (X \otimes E) \otimes E \cong X \otimes (E \otimes E) \xrightarrow{X \otimes \mu} X \otimes E.$$

Finally, $E_*(E)$ is a $\pi_*(E)$ -bimodule, in the sense that the left and right actions of $\pi_*(E)$ are compatible, so that given $y, z \in \pi_*(E)$ and $x \in E_*(E)$, $y \cdot (x \cdot z) = (y \cdot x) \cdot z$.

Proof. Note that $E_*(X) = \pi_*(E \otimes X)$, so that $E_*(-)$ is the composition of the free E -module functor (Definition D.6, which is additive by Proposition D.11) with the additive functor π_* from $E\text{-Mod}$ to left A -graded $\pi_*(E)$ -modules (Lemma D.19). Thus indeed $E_*(X)$ is a left $\pi_*(E)$ -module for all X , and given $f : X \rightarrow Y$ in \mathcal{SH} , $E_*(f)$ is a homomorphism of left $\pi_*(E)$ -modules.

Showing that $X_*(E)$ has the structure of a right $\pi_*(E)$ -module and that if $f : X \rightarrow Y$ is a morphism in \mathcal{SH} then the map

$$X_*(E) = [S^*, X \otimes E] \xrightarrow{(f \otimes E)^*} [S^*, Y \otimes E] = Y_*(E)$$

is an A -graded homomorphism of right A -graded $\pi_*(E)$ -modules is entirely analagous.

It remains to show that $E_*(E)$ is a $\pi_*(E)$ -bimodule. Let $x : S^a \rightarrow E$, $y : S^b \rightarrow E \otimes E$, and $z : S^c \rightarrow E$, and consider the following diagram:

$$\begin{array}{ccccc} & & & E \otimes E \otimes E & \\ & & \nearrow \mu \otimes E \otimes E & \downarrow E \otimes \mu & \\ S^{a+b+c} & \xrightarrow{\cong} & S^a \otimes S^b \otimes S^c & \xrightarrow{x \otimes y \otimes z} & E \otimes E \otimes E \otimes E & \xrightarrow{\mu \otimes \mu} & E \otimes E \\ & & \searrow E \otimes E \otimes \mu & \uparrow \mu \otimes E & \\ & & & E \otimes E \otimes E & \end{array}$$

Commutativity follows by functoriality of $- \otimes -$, which also tells us that the two outside compositions are $(x \cdot y) \cdot z$ (on top) and $x \cdot (y \cdot z)$ (on bottom). Hence they are equal, as desired. \square

Lemma D.21. *Let (E, μ, e) be a monoid object in \mathcal{SH} , (N, κ) a left E -module, and $a \in A$. Then the assignment*

$$\text{tw}^a : \pi_{*-a}(N) \rightarrow \pi_*(\Sigma^a N)$$

sending $x : S^{b-a} \rightarrow N$ to the composition

$$S^b \xrightarrow{\phi_{b-a,a}} S^{b-a} \otimes S^a \xrightarrow{x \otimes S^a} N \otimes S^a \xrightarrow{\tau} S^a \otimes N = \Sigma^a N$$

is an A -graded isomorphism of left A -graded $\pi_(E)$ -modules (where here $\pi_*(N)$ is a left $\pi_*(E)$ -module by Lemma D.19, and $\pi_*(\Sigma^a N)$ has the left $\pi_*(E)$ module given by Lemma D.7 and Lemma D.19).*

Proof. Unravelling definitions, the map $\text{tw}^a : \pi_{*-a}(N) \rightarrow \pi_*(\Sigma^a N)$ factors as

$$\pi_{*-a}(N) = [S^{*-a}, N] \xrightarrow{- \otimes S^a} [S^{*-a} \otimes S^a, N \otimes S^a] \xrightarrow{(\phi_{*-a,a})^*} [S^*, N \otimes S^a] \xrightarrow{\tau_*} [S^*, S^a \otimes N] = \pi_*(\Sigma^a N).$$

The arrow labeled $- \otimes S^a$ is an isomorphism of abelian groups because $- \otimes S^a \cong \Sigma^a$ is an autoequivalence of \mathcal{SH} (Proposition 2.4). Hence, we have shown the map is an isomorphism of abelian groups. Clearly the map preserves degree, so it is an A -graded homomorphism as desired. Finally, it remains to show that this map is a homomorphism of left $\pi_*(E)$ -modules, i.e., that

given $r : S^b \rightarrow E$ in $\pi_*(E)$ and $x : S^{c-a} \rightarrow N$ in $\pi_{*-a}(N)$ that $\text{tw}^a(r \cdot x) = r \cdot \text{tw}^a(x)$. Unravelling definitions, $\text{tw}^a(r \cdot x)$ is the composition

$$S^{b+c} \xrightarrow{\cong} S^b \otimes S^{c-a} \otimes S^a \xrightarrow{r \otimes x \otimes S^a} E \otimes N \otimes S^a \xrightarrow{\kappa \otimes S^a} N \otimes S^a \xrightarrow{\tau} S^a \otimes N,$$

while on the other hand $r \cdot \text{tw}^a(x)$ is the composition

$$S^{b+c} \xrightarrow{\cong} S^b \otimes S^{c-a} \otimes S^a \xrightarrow{r \otimes x \otimes S^a} E \otimes N \otimes S^a \xrightarrow{E \otimes \tau} E \otimes S^a \otimes N \xrightarrow{\tau \otimes N} S^a \otimes E \otimes N \xrightarrow{S^a \otimes \kappa} S^a \otimes N.$$

To see these are equal, consider the following diagram:

$$\begin{array}{ccccc} & & E \otimes S^a \otimes N & \xrightarrow{\tau \otimes N} & S^a \otimes E \otimes N \\ & & \uparrow E \otimes \tau & \nearrow \tau_{E \otimes N, S^a} & \downarrow S^a \otimes \kappa \\ S^{b+c} & \xrightarrow{\phi} & S^b \otimes S^{c-a} \otimes S^a & \xrightarrow{r \otimes x \otimes S^a} & E \otimes N \otimes S^a \\ & & & \searrow \kappa \otimes S^a & \uparrow \tau \\ & & & & N \otimes S^a \end{array}$$

The top triangle commutes by coherence for a symmetric monoidal category, while the right triangle commutes by naturality of τ . \square

Lemma D.22. *Let (E, μ, e) be a monoid object and (N, κ) a left E -module object in \mathcal{SH} . Then given a collection of $a_i \in A$ indexed by some set I , if (N, κ) is a retract of $\bigoplus_i (E \otimes S^{a_i})$ in $E\text{-Mod}$,³ then for all left E -module objects (N', κ') in \mathcal{SH} , the functor π_* (Lemma D.19) induces an isomorphism*

$$\pi_* : \text{Hom}_{E\text{-Mod}}(N, N') \rightarrow \text{Hom}_{\pi_*(E)}(\pi_*(N), \pi_*(N')).$$

Proof. First, we consider the case $N = \bigoplus_i (E \otimes S^{a_i})$. Consider the following diagram:

$$\begin{array}{ccc} \text{Hom}_{E\text{-Mod}}(\bigoplus_i (E \otimes S^{a_i}), N') & \xrightarrow{\pi_*} & \text{Hom}_{\pi_*(E)}(\pi_*(\bigoplus_i (E \otimes S^{a_i})), \pi_*(N')) \\ \cong \downarrow & & \downarrow \cong \\ \prod_i \text{Hom}_{E\text{-Mod}}(E \otimes S^{a_i}, N') & & \text{Hom}_{\pi_*(E)}(\bigoplus_i \pi_*(E \otimes S^{a_i}), \pi_*(N')) \\ \cong \downarrow & & \downarrow \cong \\ \prod_i [S^{a_i}, N'] & & \prod_i \text{Hom}_{\pi_*(E)}(\pi_*(E \otimes S^{a_i}), \pi_*(N')) \\ \parallel & & \downarrow \cong \\ \prod_i \pi_{a_i}(N') & \xleftarrow{\prod_i \text{ev}_1} & \prod_i \text{Hom}_{\pi_*(E)}^{a_i}(\pi_*(E), \pi_*(N')) \end{array}$$

Here the top left vertical isomorphism exhibits the universal property of the coproduct in $E\text{-Mod}$, and middle left vertical isomorphism below that is the free-forgetful adjunction for E -modules (Proposition D.5). The bottom horizontal isomorphism is the product of the evaluation-at-1 isomorphisms (Lemma C.11). On the other side, the top right vertical isomorphism is given by the fact that S^a is compact for each $a \in A$, so we have isomorphisms

$$\bigoplus_i \pi_*(E \otimes S^{a_i}) = \bigoplus_{a \in A} \bigoplus_i [S^a, E \otimes S^{a_i}] \cong \bigoplus_{a \in A} [S^a, \bigoplus_i (E \otimes S^{a_i})] = \pi_*(\bigoplus_i (E \otimes S^{a_i})),$$

³Here $\bigoplus_i (E \otimes S^{a_i})$ is a coproduct (Proposition D.10) of a bunch of left free E -module objects (Proposition D.5), so it is itself a left E -module object.

where the middle isomorphism takes a generator $x : S^a \xrightarrow{E} \otimes S^{a_i}$ to the composition $S^a \xrightarrow{x} E \otimes S^{a_i} \hookrightarrow \bigoplus_i (E \otimes S^{a_i})$. The middle right vertical isomorphism exhibits the universal property of the coproduct of modules. Finally the bottom right vertical isomorphism is given by the isomorphisms

$$\pi_{*-a_i}(E \otimes S^{a_i}) = [S^{*-a_i}, E \otimes S^{a_i}] \xrightarrow{-\otimes S^{a_i}} [S^{*-a_i} \otimes S^{a_i}, E \otimes S^{a_i}] \xrightarrow{\phi^*} [S^*, E \otimes S^{a_i}] = \pi_*(E \otimes S^{a_i}),$$

where $-\otimes S^{a_i} \cong \Sigma^{a_i}$ is an isomorphism by [Proposition 2.4](#). Now, we claim this diagram commutes. This really simply amounts to unravelling definitions, and chasing a homomorphism $f : \bigoplus_i (E \otimes S^{a_i}) \rightarrow N'$ of left E -module objects both ways around the diagram yields the composition

$$\prod_i (S^{a_i} \xrightarrow{e \otimes S^{a_i}} E \otimes S^{a_i} \hookrightarrow \bigoplus_i (E \otimes S^{a_i}) \xrightarrow{f} N').$$

Thus, since the diagram commutes, we have that

$$\pi_* : \text{Hom}_{E\text{-Mod}}(\bigoplus_i (E \otimes S^{a_i}), N') \rightarrow \text{Hom}_{\pi_*(E)}(\pi_*(\bigoplus_i (E \otimes S^{a_i})), \pi_*(N'))$$

is an isomorphism, as desired.

Now, consider the case that N is a retract of $\bigoplus_i (E \otimes S^{a_i})$ in $E\text{-Mod}$, so there exists a commuting diagram of left E -module object homomorphisms:

$$N \xrightarrow{\iota} \bigoplus_i (E \otimes S^{a_i}) \xrightarrow{r} N$$

Now consider the following diagram:

$$\begin{array}{ccccc} \text{Hom}_{E\text{-Mod}}^*(N, N') & \xrightarrow{r^*} & \text{Hom}_{E\text{-Mod}}^*(\bigoplus_i (E \otimes S^{a_i}), N') & \xrightarrow{\iota^*} & \text{Hom}_{E\text{-Mod}}^*(N, N') \\ \pi_* \downarrow & & \downarrow \pi_* & & \downarrow \pi_* \\ \text{Hom}_{\pi_*(E)}^*(\pi_*(N), \pi_*(N')) & \xrightarrow{(\pi_*(r))^*} & \text{Hom}_{\pi_*(E)}^*(\pi_*(\bigoplus_i (E \otimes S^{a_i})), \pi_*(N')) & \xrightarrow{(\pi_*(\iota))^*} & \text{Hom}_{\pi_*(E)}^*(\pi_*(N), \pi_*(N')) \end{array}$$

Each square commutes by functoriality of π_* . We have shown the middle vertical arrow is an isomorphism. Thus the outside arrows are isomorphisms as well, as a retract of an isomorphism is an isomorphism. \square

Proposition D.23. *Let (E, μ, e) be a monoid object and X an object in \mathcal{SH} . If there is a collection of $a_i \in A$ indexed by some set I such that $E \otimes X$ is a retract of $\bigoplus_i (E \otimes S^{a_i})$ in $E\text{-Mod}$,⁴ then for all left E -module objects (N, κ) the assignment*

$$\Psi : [X, N]_* \rightarrow \text{Hom}_{\pi_*(E)}^*(E_*(X), \pi_*(N))$$

sending $f : S^a \otimes X \rightarrow N$ to the map $E_{-a}(X) \rightarrow \pi_*(N)$ which sends a class $x : S^{b-a} \rightarrow E \otimes X$ to the composition*

$$\Psi(f)(x) : S^b \xrightarrow{\phi} S^{b-a} \otimes S^a \xrightarrow{x \otimes S^a} E \otimes X \otimes S^a \xrightarrow{E \otimes \tau} E \otimes S^a \otimes X \xrightarrow{E \otimes f} E \otimes N \xrightarrow{\kappa} N$$

is an A -graded isomorphism of A -graded abelian groups.

⁴Here $\bigoplus_i (E \otimes S^{a_i})$ is a coproduct ([Proposition D.10](#)) of a bunch of left free E -module objects ([Proposition D.5](#)), so it is itself a left E -module object.

Proof. Clearly as constructed, assuming $\Psi(f)$ as defined is actually a homomorphism of left $\pi_*(E)$ -modules, this map is A -graded. Thus, it suffices to show that for all $a \in A$, the restriction

$$\Psi_a : [X, N]_a \rightarrow \text{Hom}_{\pi_*(E)}^a(E_*(X), \pi_*(N))$$

is an isomorphism. First of all, note that Ψ_a factors as

$$\begin{array}{c} [X, N]_a \xlongequal{\quad} [\Sigma^a X, N] \\ \downarrow \cong \\ \text{Hom}_{E\text{-}\mathbf{Mod}}(E \otimes \Sigma^a X, N) \\ \downarrow \cong \\ \text{Hom}_{E\text{-}\mathbf{Mod}}(\Sigma^a(E \otimes X), N) \\ \downarrow \pi_* \\ \text{Hom}_{\pi_*(E)}(\pi_*(\Sigma^a(E \otimes X)), \pi_*(N)) \\ \downarrow (\text{tw}^a)^* \\ \text{Hom}_{\pi_*(E)}(E_{*-a}(X), \pi_*(N)) \xlongequal{\quad} \text{Hom}_{\pi_*(E)}^a(E_*(X), \pi_*(N)) \end{array}$$

where the first isomorphism is the free-forgetful adjunction for E -modules ([Proposition D.5](#)), the second isomorphism is given by [Lemma D.8](#), the third map is that induced by the functor π_* constructed in [Lemma D.19](#), and the final map is induced by the isomorphism $\text{tw}^a : \pi_{*-a}(E \otimes X) \xrightarrow{\cong} \pi_*(\Sigma^a(E \otimes X))$ ([Lemma D.21](#)). Unravelling definitions, this composition sends a class $f : S^a \otimes X \rightarrow N$ to the map $E_{*-a}(X) \rightarrow \pi_*(N)$ which sends a class $x : S^{b-a} \rightarrow E \otimes X$ to the composition

$$S^b \xrightarrow{\cong} S^{b-a} \otimes S^a \xrightarrow{x \otimes S^a} E \otimes X \otimes S^a \xrightarrow{\tau_{E \otimes X, S^a}} S^a \otimes E \otimes X \xrightarrow{\tau \otimes X} E \otimes S^a \otimes X \xrightarrow{E \otimes f} E \otimes N \xrightarrow{\kappa} N,$$

and clearly this equals $\Psi(f)(x)$, by coherence for the symmetries. Thus, it suffices to show that

$$\pi_* : \text{Hom}_{E\text{-}\mathbf{Mod}}(\Sigma^a(E \otimes X), N) \rightarrow \text{Hom}_{\pi_*(E)}(\pi_*(\Sigma^a(E \otimes X)), \pi_*(N))$$

is an isomorphism when $E \otimes X$ is a retract of $\bigoplus_i (E \otimes S^{a_i})$ in $E\text{-}\mathbf{Mod}$. This is precisely [Lemma D.22](#). \square

Proposition D.24. *Let (E, μ, e) be a monoid object and (N, κ) a left E -module object in \mathcal{SH} . Further suppose that E and N are cellular and that $\pi_*(N)$ is a graded projective ([Definition C.12](#)) left $\pi_*(E)$ -module ([Lemma D.19](#)). Then given some homogeneous generating set $\{x_i\}_{i \in I} \subseteq \pi_*(N)$, N is a retract of $\bigoplus_i (E \otimes S^{|x_i|})$ in $E\text{-}\mathbf{Mod}$.⁵*

Proof. Let $M := \bigoplus_i (E \otimes S^{|x_i|})$. We have a map

$$r : M \rightarrow N$$

induced by the maps

$$r_i : E \otimes S^{|x_i|} \xrightarrow{E \otimes x_i} E \otimes N \xrightarrow{\kappa} N.$$

⁵Here $\bigoplus_i (E \otimes S^{a_i})$ is a coproduct ([Proposition D.10](#)) of a bunch of left free E -module objects ([Proposition D.5](#)), so it is itself a left E -module object.

This is a homomorphism of left E -module objects:

$$\begin{array}{ccccc}
 E \otimes \bigoplus_i (E \otimes S^{|x_i|}) & \xrightarrow{E \otimes r} & E \otimes N & & \\
 \downarrow \cong & \nearrow E \otimes \bigoplus_i r_i & \nearrow E \otimes \nabla & & \\
 & E \otimes \bigoplus_i N & & & \\
 & \downarrow \cong & & & \\
 \bigoplus_i (E \otimes E \otimes S^{|x_i|}) & \xrightarrow{\bigoplus_i (E \otimes r_i)} & \bigoplus_i (E \otimes N) & & \\
 \downarrow \bigoplus_i (\mu \otimes S^{|x_i|}) & & \downarrow \bigoplus_i \kappa & & \\
 \bigoplus_i (E \otimes S^{|x_i|}) & \xrightarrow{\bigoplus_i r_i} & \bigoplus_i N & & \\
 & \searrow \nabla & & & \\
 & & N & & \\
 \downarrow \bigoplus_i (\mu \otimes S^{|x_i|}) & \nearrow \bigoplus_i r_i & \nearrow r & & \\
 \bigoplus_i (E \otimes S^{|x_i|}) & \xrightarrow{r} & N & &
 \end{array}$$

The right trapezoid commutes by naturality of ∇ . The bottom triangle commutes by the fact that $\nabla \circ \bigoplus_i r_i$ and r satisfy the same universal property for the coproduct. Every other region commutes by additivity of $E \otimes -$, except the left trapezoid: Note that by expanding out how r_i is defined, it becomes

$$\begin{array}{ccccc}
 \bigoplus_i (E \otimes E \otimes S^{|x_i|}) & \xrightarrow{\bigoplus_i (E \otimes E \otimes x_i)} & \bigoplus_i (E \otimes E \otimes N) & \xrightarrow{\bigoplus_i (E \otimes \kappa)} & \bigoplus_i (E \otimes E \otimes X) \\
 \downarrow \bigoplus_i (\mu \otimes S^{|x_i|}) & & \downarrow \bigoplus_i (\mu \otimes X) & & \downarrow \bigoplus_i \kappa \\
 \bigoplus_i (E \otimes S^{|x_i|}) & \xrightarrow{\bigoplus_i (E \otimes x_i)} & \bigoplus_i (E \otimes N) & \xrightarrow{\bigoplus_i \kappa} & \bigoplus_i (E \otimes X)
 \end{array}$$

The left square commutes by functoriality of $- \otimes -$, and the right square commutes by coherence for κ . Hence, we've shown that r is a homomorphism of left E -modules, as desired. Thus, r induces a homomorphism of left $\pi_*(E)$ -modules $\pi_*(r) \in \text{Hom}_{\pi_*(E)}(\pi_*(M), \pi_*(N))$. Further note that for all $i \in I$, x_i is in the image of $\pi_*(r)$, as by definition $\pi_*(r)$ sends the class

$$S^{|x_i|} \xrightarrow{e \otimes S^{|x_i|}} E \otimes S^{|x_i|} \hookrightarrow M$$

in $\pi_{|x_i|}(M)$ to the composition

$$S^{|x_i|} \xrightarrow{e \otimes S^{|x_i|}} E \otimes S^{|x_i|} \xrightarrow{E \otimes x_i} E \otimes N \xrightarrow{\kappa} N,$$

and by unitality of κ this composition is simply $x_i : S^{|x_i|} \rightarrow N$. Thus, we have constructed a surjective A -graded homomorphism $\pi_*(r) : \pi_*(M) \rightarrow \pi_*(N)$ of left $\pi_*(E)$ -modules, so that since $\pi_*(N)$ is projective graded module there exists an A -graded left $\pi_*(E)$ -module homomorphism $\iota : \pi_*(N) \rightarrow \pi_*(M)$ which makes the following diagram commute:

$$\begin{array}{ccc}
 & \pi_*(M) & \\
 \nearrow \iota & \downarrow \pi_*(r) & \\
 \pi_*(N) & \xlongequal{\quad} & \pi_*(N)
 \end{array}$$

which further induces the corresponding idempotent of left $\pi_*(E)$ -modules:

$$\pi_*(M) \xrightarrow{\pi_*(r)} \pi_*(N) \xrightarrow{\iota} \pi_*(M)$$

Now, by [Lemma D.22](#), since $M = \bigoplus_i (E \otimes S^{|x_i|})$, we have that this map is actually induced by some endomorphism $\ell : M \rightarrow M$ of left E -module objects. Now ℓ splits by [Proposition A.8](#),

meaning there exists a diagram in \mathcal{SH} of the form

$$\ell : M \xrightarrow{r'} X \xrightarrow{\iota'} M$$

with $r' \circ \iota' = \text{id}_X$. Note that since E and each $S^{|x_i|}$ are cellular, $E \otimes S^{|x_i|}$ is cellular for all $i \in I$ (Lemma 2.7), so that $M = \bigoplus_i (E \otimes S^{|x_i|})$ is cellular, as by definition an arbitrary coproduct of cellular objects is cellular. Thus by Lemma 2.10 X is cellular as well. Now consider the following commutative diagram

$$\begin{array}{ccccccc}
 & & \pi_*(N) & \xlongequal{\quad} & \pi_*(N) & & \\
 & \nearrow \pi_*(r) & & \searrow \iota & \nearrow \pi_*(r) & & \\
 \pi_*(N) & \xrightarrow{\iota} & \pi_*(M) & \xrightarrow{\pi_*(\ell)} & \pi_*(M) & \xrightarrow{\pi_*(\ell)} & \pi_*(M) \xrightarrow{\pi_*(r')} \pi_*(X) \\
 & \searrow \pi_*(r') & & \nearrow \pi_*(\iota') & \searrow \pi_*(r') & & \\
 & & \pi_*(X) & \xlongequal{\quad} & \pi_*(X) & &
 \end{array}$$

From this diagram we read off that the middle diagonal composition

$$\pi_*(X) \xrightarrow{\pi_*(\iota')} \pi_*(M) \xrightarrow{\pi_*(r)} \pi_*(N)$$

is an isomorphism with inverse $\pi_*(r') \circ \iota$. Now, since X and N are cellular, and $\pi_*(r \circ \iota')$ is an isomorphism, by Lemma 2.6 we have that $r \circ \iota'$ is an isomorphism, say with inverse p . Thus we have a commuting diagram

$$\begin{array}{ccccc}
 N & \xrightarrow{\quad} & M & \xrightarrow{\quad} & N \\
 & \searrow p & \nearrow \iota' & & \\
 & & X & &
 \end{array}$$

and the middle row exhibits N as a retract of $M = \bigoplus_i (E \otimes S^{|x_i|})$, as desired. \square

Corollary D.25. *Let (E, μ, e) be a monoid object and let X and Y be objects in \mathcal{SH} . Then if E and X are cellular and $E_*(X)$ is a graded projective (Definition C.12) left $\pi_*(E)$ -module (Proposition D.20), then the map*

$$\Psi_{X,Y} : [X, E \otimes Y]_* \rightarrow \text{Hom}_{\pi_*(E)}^*(E_*(X), E_*(Y))$$

sending $f : S^a \otimes X \rightarrow E \otimes Y$ to the map $E_{*-a}(X) \rightarrow E_*(Y)$ which sends a class $x : S^{b-a} \rightarrow E \otimes X$ to the composition

$$\Psi_{X,Y}(f)(x) : S^b \xrightarrow{\phi} S^{b-a} \otimes S^a \xrightarrow{x \otimes S^a} E \otimes X \otimes S^a \xrightarrow{E \otimes \tau} E \otimes S^a \otimes X \xrightarrow{E \otimes f} E \otimes E \otimes Y \xrightarrow{\mu \otimes Y} E \otimes Y$$

is an A -graded isomorphism of A -graded abelian groups.

Proof. By Proposition D.24, since $E \otimes X$ is a left E -module object (Definition D.6), $E_*(X) = \pi_*(E \otimes X)$ is a graded projective left $\pi_*(E)$ -module, and $E \otimes X$ is cellular (Lemma 2.7), it follows that $E \otimes X$ is a retract of $\bigoplus_i (E \otimes S^{a_i})$ in $E\text{-Mod}$ for some collection of $a_i \in A$ indexed by some set I . Thus the desired result follows by Proposition D.23 with $N = E \otimes Y$ (which is an E -module by Definition D.6). \square

Proposition D.26 ([7, Proposition 2.2]). *Let (E, μ, e) be a monoid object in \mathcal{SH} and let X be any object. Then the assignment*

$$E_*(E) \times E_*(X) \rightarrow E_*(E \otimes X)$$

which sends $x : S^a \rightarrow E \otimes E$ and $y : S^b \rightarrow E \otimes X$ to the composition

$$x \cdot y : S^{a+b} \cong S^a \otimes S^b \xrightarrow{x \otimes y} E \otimes E \otimes E \otimes X \xrightarrow{E \otimes \mu \otimes X} E \otimes E \otimes X$$

lifts to an A -graded homomorphism of left A -graded $\pi_*(E)$ -modules

$$\Phi_X : E_*(E) \otimes_{\pi_*(E)} E_*(X) \rightarrow E_*(E \otimes X)$$

(where here $E_*(E)$ has a $\pi_*(E)$ -bimodule structure and $E_*(X)$ has a left $\pi_*(E)$ -module structure as specified by [Proposition D.20](#), so $E_*(E) \otimes_{\pi_*(E)} E_*(X)$ is a left A -graded $\pi_*(E)$ -module by [Proposition C.19](#)). Furthermore, this homomorphism is natural in X .

Proof. First, recall by definition of the tensor product, in order to show the assignment $E_*(E) \times E_*(X) \rightarrow E_*(E \otimes X)$ induces a homomorphism $E_*(E) \otimes_{\pi_*(E)} E_*(X) \rightarrow E_*(E \otimes X)$ of A -graded abelian groups, it suffices to show that the assignment is $\pi_*(E)$ -balanced, i.e., that it is linear in each argument and satisfies $xr \cdot y = x \cdot ry$ for $x \in E_*(E)$, $y \in E_*(X)$, and $r \in \pi_*(E)$.

First, note that by the identifications $E_*(E) = \pi_*(E \otimes E)$, $E_*(X) = \pi_*(E \otimes X)$, and $E_*(E \otimes X) = \pi_*(E \otimes E \otimes X)$, and [Lemma D.18](#), it is straightforward to see that the assignment commutes with addition of maps in each argument. Now, let $a, b, c \in A$, $x : S^a \rightarrow E \otimes E$, $y : S^b \rightarrow E \otimes X$, and $z : S^c \rightarrow E$. Then we wish to show $xz \cdot y = x \cdot zy$. Consider the following diagram (where here we are passing to a permutative category):

$$\begin{array}{ccccc} & & & E \otimes E \otimes E \otimes X & \\ & & E \otimes \mu \otimes E \otimes X & \nearrow & \\ S^{a+b+c} & \xrightarrow{\cong} & S^a \otimes S^c \otimes S^b & \xrightarrow{x \otimes z \otimes y} & E \otimes E \otimes E \otimes E \otimes X \\ & & & \searrow & E \otimes E \otimes \mu \otimes X \\ & & & & E \otimes E \otimes E \otimes X \end{array}$$

It commutes by associativity of μ . By functoriality of $-\otimes-$, the top composition is given by $(xz) \cdot y$ and the bottom composition is $x \cdot (zy)$, so we have they are equal, as desired. Thus, since the map $E_*(E) \times E_*(X) \rightarrow E_*(E \otimes X)$ is $\pi_*(E)$ -balanced, we have that it induces a homomorphism of abelian groups. Furthermore, by [Lemma C.20](#) it is A -graded.

In order to see this map is a homomorphism of left $\pi_*(E)$ -modules, we must show that $z(x \cdot y) = zx \cdot y$, where x , y , and z are defined as above. Now consider the following diagram:

$$\begin{array}{ccccc} & & & E \otimes E \otimes E \otimes X & \\ & & \mu \otimes E \otimes E \otimes X & \nearrow & \\ S^{a+b+c} & \xrightarrow{\cong} & S^c \otimes S^a \otimes S^b & \xrightarrow{z \otimes x \otimes y} & E \otimes E \otimes E \otimes E \otimes X \\ & & & \searrow & \mu \otimes \mu \otimes X \\ & & & & E \otimes E \otimes E \otimes X \end{array}$$

Commutativity of the triangles is functoriality of $-\otimes-$. By functoriality of $-\otimes-$, the top composition is $zx \cdot y$, and the bottom composition is $z(x \cdot y)$. Hence they are equal, as desired, so that the map we have constructed

$$E_*(E) \otimes_{\pi_*(E)} E_*(X) \rightarrow E_*(E \otimes X)$$

is indeed an A -graded homomorphism of left A -graded $\pi_*(E)$ -modules.

Next, we would like to show that this homomorphism is natural in X . Let $f : X \rightarrow Y$ in \mathcal{SH} . Then we would like to show the following diagram commutes:

$$(6) \quad \begin{array}{ccc} E_*(E) \otimes_{\pi_*(E)} E_*(X) & \xrightarrow{\Phi_X} & E_*(E \otimes X) \\ E_*(E) \otimes_{\pi_*(E)} E_*(f) \downarrow & & \downarrow E_*(E \otimes f) \\ E_*(E) \otimes_{\pi_*(E)} E_*(Y) & \xrightarrow{\Phi_Y} & E_*(E \otimes Y) \end{array}$$

As all the maps here are homomorphisms, it suffices to chase generators around the diagram. In particular, suppose we are given $x : S^a \rightarrow E \otimes E$ and $y : S^b \rightarrow E \otimes X$, and consider the following diagram exhibiting the two possible ways to chase $x \otimes y$ around the diagram (as usual, we are passing to a symmetric strict monoidal category):

$$\begin{array}{ccccc} S^{a+b} & \xrightarrow{\phi_{a,b}} & S^a \otimes S^b & \xrightarrow{x \otimes y} & E \otimes E \otimes E \otimes X \xrightarrow{E \otimes \mu \otimes X} E \otimes E \otimes X \\ & & & & \downarrow E \otimes E \otimes E \otimes f \quad \downarrow E \otimes E \otimes f \\ & & & & E \otimes E \otimes E \otimes Y \xrightarrow{E \otimes \mu \otimes Y} E \otimes E \otimes Y \end{array}$$

This diagram commutes by functoriality of $- \otimes -$. Thus we have that diagram (6) does indeed commute, as desired. \square

Lemma D.27. *Let E and X be objects in \mathcal{SH} . Then for all $a \in A$, there is an A -graded isomorphism of A -graded abelian groups*

$$t_X^a : E_*(\Sigma^a X) \cong E_{*-a}(X)$$

which sends a class $S^b \rightarrow E \otimes \Sigma^a X = E \otimes S^a \otimes X$ to the composition

$$S^{b-a} \xrightarrow{\phi_{b,-a}} S^b \otimes S^{-a} \xrightarrow{x \otimes S^{-a}} E \otimes S^a \otimes X \otimes S^{-a} \xrightarrow{E \otimes S^a \otimes \tau_{X,S^{-a}}} E \otimes S^a \otimes S^{-a} \otimes X \xrightarrow{E \otimes \phi_{a,-a}^{-1} \otimes X} E \otimes X$$

(where here we are ignoring associators and unitors). Furthermore this isomorphism is natural in X , and if E is a monoid object in \mathcal{SH} then it is a natural isomorphism of $\pi_*(E)$ -modules.

Proof. Expressed in terms of hom-sets, t_X^a is precisely the composition

$$\begin{aligned} E_*(\Sigma^a X) &= [S^*, E \otimes S^a \otimes X] \\ &\downarrow - \otimes S^{-a} \\ [S^* \otimes S^{-a}, E \otimes S^a \otimes X \otimes S^{-a}] \\ &\downarrow (\phi_{*, -a})^* \\ [S^{*-a}, E \otimes S^a \otimes X \otimes S^{-a}] \\ &\downarrow (E \otimes S^a \otimes \tau)_* \\ [S^{*-a}, E \otimes S^a \otimes S^{-a} \otimes X] \\ &\downarrow (E \otimes \phi_{a,-a}^{-1} \otimes X)_* \\ [S^{*-a}, E \otimes X] &= E_{*-a}(E \otimes X) \end{aligned}$$

We know the first vertical arrow is an isomorphism of abelian groups as $- \otimes -$ is additive in each variable (since \mathcal{SH} is tensor triangulated) and $\Omega^a \cong - \otimes S^{-a}$ is an autoequivalence of \mathcal{SH} by [Proposition 2.4](#). The three other vertical arrows are given by composing with an isomorphism in an additive category, so they are also isomorphisms.

To see t_X^a is a homomorphism of left $\pi_*(E)$ -modules, suppose we are given classes $r : S^b \rightarrow E \pi_b(E)$ and $x : S^c \rightarrow E \otimes S^a \otimes X$ in $E_c(\Sigma^a X)$. Then we wish to show that $t_X^a(r \cdot x) = r \cdot t_X^a(x)$. Consider the following diagram:

$$\begin{array}{ccccc}
 S^{b+c-a} & & E \otimes S^a \otimes X \otimes S^{-a} & \xrightarrow{E \otimes S^a \otimes \tau_{X, S^{-a}}} & E \otimes S^a \otimes S^{-a} \otimes X \\
 \downarrow \cong & & \uparrow \mu \otimes S^a \otimes X \otimes S^{-a} & & \downarrow E \otimes \phi_{a, -a}^{-1} \otimes X \\
 S^b \otimes S^c \otimes S^{-a} & \xrightarrow{\tau \otimes x \otimes S^{-a}} & E \otimes E \otimes S^a \otimes X \otimes S^{-a} & & E \otimes X \\
 & & \downarrow E \otimes E \otimes S^a \otimes \tau_{X, S^{-a}} & \nearrow \mu \otimes S^a \otimes S^{-a} \otimes X & \uparrow \mu \otimes X \\
 & & E \otimes E \otimes S^a \otimes S^{-a} \otimes X & \xrightarrow{E \otimes E \otimes \phi_{a, -a}^{-1} \otimes X} & E \otimes E \otimes X
 \end{array}$$

Both triangles commute by functoriality of $- \otimes -$. The top composition is $t_X^a(r \cdot x)$ while the bottom is $r \cdot t_X^a(x)$, so they are equal as desired.

It remains to show t_X^a is natural in X . let $f : X \rightarrow Y$ in \mathcal{SH} , then we would like to show the following diagram commutes:

$$(7) \quad \begin{array}{ccc}
 E_*(\Sigma^a X) & \xrightarrow{t_X^a} & E_{*-a}(X) \\
 E_*(\Sigma^a f) \downarrow & & \downarrow E_{*-a}(f) \\
 E_*(\Sigma^a Y) & \xrightarrow{t_Y^a} & E_{*-a}(Y)
 \end{array}$$

We may chase a generator around the diagram since all the arrows here are homomorphisms. Let $x : S^b \rightarrow E \otimes S^a \otimes X$ in $E_*(\Sigma^a X)$. Then consider the following diagram:

$$\begin{array}{ccccccc}
 S^{b-a} & \xrightarrow{\cong} & S^b \otimes S^{-a} & \xrightarrow{x \otimes S^{-a}} & E \otimes S^a \otimes X \otimes S^{-a} & \xrightarrow{E \otimes S^a \otimes \tau} & E \otimes S^a \otimes S^{-a} \otimes X \xrightarrow{E \otimes \phi_{a, -a}^{-1} \otimes X} E \otimes X \\
 & & \downarrow E \otimes S^a \otimes f \otimes S^{-a} & & \downarrow E \otimes S^a \otimes S^{-a} \otimes f & & \downarrow E \otimes f \\
 & & E \otimes S^a \otimes Y \otimes S^{-a} & \xrightarrow{E \otimes S^a \otimes \tau} & E \otimes S^a \otimes S^{-a} \otimes Y & \xrightarrow{E \otimes \phi_{a, -a}^{-1} \otimes Y} & E \otimes Y
 \end{array}$$

The left rectangle commutes by naturality of τ , while the right rectangle commutes by functoriality of $- \otimes -$. The two outside compositions are the two ways to chase x around diagram (7), so the diagram commutes as desired. \square

Lemma D.28. *Given a monoid object (E, μ, e) in \mathcal{SH} , the maps Φ_X constructed in [Proposition D.26](#) commute with the natural isomorphisms $t_X^a : E_*(\Sigma^a X) \xrightarrow{\cong} E_{*-a}(X)$ given in [Lemma D.27](#), in the sense that the following diagram commutes for all $a \in A$ and X in \mathcal{SH} :*

$$\begin{array}{ccc}
 E_*(E) \otimes_{\pi_*(E)} E_*(\Sigma^a X) & \xrightarrow{E_*(E) \otimes t_X^a} & E_*(E) \otimes_{\pi_*(E)} E_{*-a}(X) \\
 \Phi_{\Sigma^a X} \downarrow & & \downarrow \Phi_X \\
 E_*(E \otimes \Sigma^a X) & \xrightarrow{t_X^a} & E_{*-a}(E \otimes X)
 \end{array}$$

where the top arrow is well-defined since t_X^a is a left $\pi_*(E)$ -module homomorphism by the above lemma, and we are being abusive in that the bottom arrow is given by the composition

$$E_*(E \otimes \Sigma^a X) \xrightarrow{\alpha} (E \otimes E)_*(\Sigma^a X) \xrightarrow{t_X^a} (E \otimes E)_{*-a}(X) \xrightarrow{\alpha} E_{*-a}(E \otimes X).$$

Proof. Since all the maps in the above diagram are homomorphisms, we can chase generators around to show it commutes. Let $x : S^b \rightarrow E \otimes E$ and $y : S^c \rightarrow E \otimes \Sigma^a X = E \otimes S^a \otimes X$. Then consider the following diagram:

$$\begin{array}{ccccc}
 S^{b+c-a} & & E \otimes E \otimes E \otimes S^a \otimes S^{-a} \otimes X & \xrightarrow{E \otimes E \otimes E \otimes \phi_{a,-a}^{-1} \otimes X} & E \otimes E \otimes X \\
 \downarrow \cong & & \uparrow E \otimes E \otimes E \otimes S^a \otimes \tau & & \downarrow E \otimes \mu \otimes X \\
 S^b \otimes S^c \otimes S^{-a} & \xrightarrow{x \otimes y \otimes S^{-a}} & E \otimes E \otimes E \otimes S^a \otimes X \otimes S^{-a} & \xrightarrow{E \otimes \mu \otimes S^a \otimes S^{-a} \otimes X} & E \otimes E \otimes X \\
 & & \downarrow E \otimes \mu \otimes S^a \otimes X \otimes S^{-a} & \searrow & \uparrow E \otimes E \otimes \phi_{a,-a}^{-1} \otimes X \\
 & & E \otimes E \otimes S^a \otimes X \otimes S^{-a} & \xrightarrow{E \otimes E \otimes S^a \otimes \tau} & E \otimes E \otimes S^a \otimes S^{-a} \otimes X
 \end{array}$$

Each triangle commutes by functoriality of $- \otimes -$. The two outside compositions are the two ways to chase $x \otimes y$ around the diagram in the statement of the lemma, so the diagram commutes as desired. \square

Corollary D.29. *For all X in \mathcal{SH} , we have natural isomorphisms $t_X : E_*E(\Sigma X) \xrightarrow{\cong} E_{*-1}(X)$ given by the composition*

$$E_*(\Sigma X) \xrightarrow{E_*(\nu_X)} E_*(\Sigma^1 X) \xrightarrow{t_X^1} E_{*-1}(X).$$

Furthermore, by naturality of Φ and the fact that t_X^1 commutes with Φ (in the sense of the above lemma), this isomorphism also commutes with Φ .

Proposition D.30. *Let (E, μ, e) be a flat monoid object in \mathcal{SH} (??) and let X be any cellular object in \mathcal{SH} (Definition 2.5). Then the natural homomorphism*

$$\Phi_X : E_*(E) \otimes_{\pi_*(E)} E_*(X) \rightarrow E_*(E \otimes X)$$

given in Proposition D.26 is an isomorphism of left $\pi_(E)$ -modules.*

Proof. In this proof, we will freely employ the coherence theorem for symmetric monoidal categories, and we will assume that associativity and unitality of $- \otimes -$ holds up to strict equality. To start, let \mathcal{E} be the collection of objects X in \mathcal{SH} for which this map is an isomorphism. Then in order to show \mathcal{E} contains every cellular object, it suffices to show that \mathcal{E} satisfies the three conditions given for the class of cellular objects in Definition 2.5. First, we need to show that Φ is an isomorphism when $X = S^a$ for some $a \in A$. Indeed, consider the map

$$\Psi : E_*(E \otimes S^a) \rightarrow E_*(E) \otimes_{\pi_*(E)} E_*(S^a)$$

which sends a class $x : S^b \rightarrow E \otimes E \otimes S^a$ in $E_b(E \otimes S^a)$ to the pure tensor $\tilde{x} \otimes \tilde{e}$, where $\tilde{x} \in E_{b-a}(E)$ is the composition

$$S^{b-a} \cong S^b \otimes S^{-a} \xrightarrow{x \otimes S^{-a}} E \otimes E \otimes S^a \otimes S^{-a} \xrightarrow{E \otimes E \otimes \phi_{a,-a}^{-1}} E \otimes E$$

and $\tilde{e} \in E_a(S^a)$ is the composition

$$S^a \cong S \otimes S^a \xrightarrow{e \otimes S^a} E \otimes S^a.$$

First, note Ψ is an (A -graded) homomorphism of abelian groups: Given $x, x' \in E_b(E \otimes S^a)$, we would like to show that $\tilde{x} \otimes \tilde{e} + \tilde{x}' \otimes \tilde{e} = \widetilde{x + x'} \otimes \tilde{e}$. It suffices to show that $\tilde{x} + \tilde{x}' = \widetilde{x + x'}$. To

see this, consider the following diagram (again, we are passing to a symmetric strict monoidal category):

$$\begin{array}{ccc}
S^{b-a} & \xrightarrow{\Delta} & S^{b-a} \oplus S^{b-a} \\
\phi_{b-a} \downarrow & & \downarrow \phi_{b,-a} \oplus \phi_{b,-a} \\
S^b \otimes S^{-a} & \xrightarrow{\Delta} & (S^b \otimes S^{-a}) \oplus (S^b \otimes S^{-a}) \\
\Delta \otimes S^{-a} \downarrow & \nearrow \cong & \downarrow (x \otimes S^{-a}) \oplus (x' \otimes S^{-a}) \\
(S^b \oplus S^b) \otimes S^{-a} & & (E \otimes E \otimes S^a \otimes S^{-a}) \oplus (E \otimes E \otimes S^a \otimes S^{-a}) \\
(x \oplus x') \otimes S^{-a} \downarrow & \nearrow \cong & \downarrow (E \otimes E \otimes \phi_{a,-a}^{-1}) \oplus (E \otimes E \otimes \phi_{a,-a}^{-1}) \\
((E \otimes E \otimes S^a) \oplus (E \otimes E \otimes S^a)) \otimes S^{-a} & & (E \otimes E) \oplus (E \otimes E) \\
\nabla \otimes S^{-a} \downarrow & \nwarrow \nabla & \downarrow \nabla \\
E \otimes E \otimes S^a \otimes S^{-a} & \xrightarrow{E \otimes E \otimes \phi_{a,-a}^{-1}} & E \otimes E
\end{array}$$

The top rectangle commutes by naturality of Δ in an additive category. The bottom triangle commutes by naturality of ∇ in an additive category. Finally, the remaining regions of the diagram commute by additivity of $- \otimes -$. By functoriality of $- \otimes -$, it follows that the left composition is $x + x'$ and the right composition is $\tilde{x} + \tilde{x}'$, so they are equal as desired. Thus Ψ is a homomorphism of abelian groups, as desired.

Now, we claim that Ψ is an inverse to Φ , (which is enough to show Φ is an isomorphism of left $\pi_*(E)$ -modules). Since Φ and Ψ are homomorphisms it suffices to check that they are inverses on generators. First, let $x : S^b \rightarrow E \otimes E \otimes S^a$ in $E_b(E \otimes S^a)$. We would like to show that $\Phi(\Psi(x)) = x$. Consider the following diagram, where here we are passing to a symmetric strict monoidal category:

$$\begin{array}{ccc}
S^b & \xrightarrow{\cong} & S^b \otimes S^{-a} \otimes S^a \\
x \downarrow & & \downarrow x \otimes S^{-a} \otimes e \otimes S^a \\
E \otimes E \otimes S^a & \xrightarrow{E \otimes E \otimes S^a \otimes \phi_{-a,a}} & E \otimes E \otimes S^a \otimes S^{-a} \otimes S^a \\
E \otimes \mu \otimes S^a \uparrow & & \downarrow E \otimes E \otimes \phi_{a,-a} \otimes S^a \\
E \otimes E \otimes E \otimes S^a & \xrightarrow{E \otimes E \otimes e \otimes S^a} & E \otimes E \otimes S^a \\
& & \downarrow E \otimes E \otimes e \otimes S^a \\
& & E \otimes E \otimes E \otimes S^a
\end{array}$$

The top left trapezoid commutes since the isomorphism $S^b \xrightarrow{\cong} S^b \otimes S^{-a} \otimes S^a$ may be given as $S^b \otimes \phi_{-a,a}$ (see [Remark 2.3](#)), in which case the trapezoid commutes by functoriality of $- \otimes -$. The triangle below that commutes by coherence for the $\phi_{a,b}$'s. The triangle below that commutes by definition. The bottom left triangle commutes by unitality for μ . The top right triangle commutes by functoriality of $- \otimes -$. Finally, the bottom right triangle commutes by functoriality of $- \otimes -$. It follows by unravelling definitions that the two outside compositions are x and $\Phi(\Psi(x))$, so indeed we have $\Phi(\Psi(x)) = x$ since the diagram commutes.

On the other hand, suppose we are given a homogeneous pure tensor $x \otimes y$ in $E_*(E) \otimes_{\pi_*(E)} E_*(S^a)$, so $x : S^b \rightarrow E \otimes E$ and $y : S^c \rightarrow E \otimes S^a$ for some $b, c \in A$. Then we would like to show that $\Psi(\Phi(x \otimes y)) = x \otimes y$. Unravelling definitions, $\Psi(\Phi(x \otimes y))$ is the homogeneous pure tensor $\tilde{x}\tilde{y} \otimes \tilde{e}$, where $\tilde{e} : S^a \rightarrow E \otimes S^a$ is defined above, and by functoriality of $- \otimes -$, $\tilde{x}\tilde{y} : S^{b+c-a} \rightarrow E \otimes E$

is the composition

$$\begin{aligned}
& S^{b+c-a} \\
& \downarrow \phi_{b+c,-a} \\
& S^{b+c} \otimes S^{-a} \\
& \downarrow \phi_{b,c} \otimes S^{-a} \\
& S^b \otimes S^c \otimes S^{-a} \\
& \downarrow x \otimes y \otimes S^{-a} \\
& E \otimes E \otimes E \otimes S^a \otimes S^{-a} \\
& \downarrow E \otimes \mu \otimes S^a \otimes S^{-a} \\
& E \otimes E \otimes S^a \otimes S^{-a} \\
& \downarrow E \otimes E \otimes \phi_{a,-a}^{-1} \\
& E \otimes E \otimes S \\
& \downarrow E \otimes \rho_E \\
& E \otimes E.
\end{aligned}$$

In order to see $x \otimes y = \widetilde{xy} \otimes \widetilde{e}$, it suffices to show there exists some scalar $r \in \pi_{c-a}(E)$ such that $x \cdot r = \widetilde{xy}$ and $r \cdot \widetilde{e} = y$, where here \cdot denotes the right and left action of $\pi_*(E)$ on $E_*(E)$ and $E_*(S^a)$, respectively. Now, define r to be the composition

$$S^{c-a} \cong S^c \otimes S^{-a} \xrightarrow{y \otimes S^{-a}} E \otimes S^a \otimes S^{-a} \xrightarrow{E \otimes \phi_{a,-a}^{-1}} E \otimes S \xrightarrow{\rho_E} E.$$

First, in order to see that $x \cdot r = \widetilde{xy}$, consider the following diagram, where here we are again passing to a symmetric strict monoidal category:

$$\begin{array}{c}
S^{b+c-a} \xrightarrow{\cong} S^b \otimes S^c \otimes S^{-a} \xrightarrow{x \otimes y \otimes S^{-a}} E \otimes E \otimes E \otimes S^a \otimes S^{-a} \xrightarrow{E \otimes \mu \otimes S^a \otimes S^{-a}} E \otimes E \otimes S^a \otimes S^{-a} \\
\downarrow E \otimes E \otimes E \otimes \phi_{a,-a}^{-1} \quad \searrow E \otimes \mu \otimes \phi_{a,-a}^{-1} \quad \downarrow E \otimes E \otimes \phi_{a,-a}^{-1} \\
E \otimes E \otimes E \xrightarrow{E \otimes \mu} E \otimes E
\end{array}$$

Commutativity is functoriality of $- \otimes -$, which also tells us that the two outside compositions are \widetilde{xy} (on top) and $x \cdot r$ (on the bottom), so they are equal as desired. On the other hand, in order to see that $r \cdot \widetilde{e} = y$, consider the following diagram (where here we have passed to a symmetric strict monoidal category):

$$\begin{array}{ccc}
S^c & \xrightarrow{\cong} & S^c \otimes S^{-a} \otimes S^a \\
\downarrow y & & \downarrow y \otimes S^{-a} \otimes e \otimes S^a \\
& & E \otimes S^a \otimes S^{-a} \otimes S^a \\
& \swarrow E \otimes S^a \otimes \phi_{-a,a}^{-1} & \searrow E \otimes S^a \otimes S^{-a} \otimes e \otimes S^a \\
E \otimes S^a & & E \otimes S^a \otimes S^{-a} \otimes E \otimes S^a \\
\uparrow \mu \otimes S^a & \swarrow E \otimes e \otimes S^a & \downarrow E \otimes \phi_{a,-a}^{-1} \otimes E \otimes S^a \\
E \otimes E \otimes S^a & \xrightarrow{\quad\quad\quad} & E \otimes E \otimes S^a
\end{array}$$

The top left triangle commutes since we may take the isomorphism $S^c \xrightarrow{\cong} S^c \otimes S^{-a} \otimes S^a$ to be $S^c \otimes \phi_{-a,a}$, in which case commutativity of the triangle follows by functoriality of $- \otimes -$. Commutativity of the right triangle is also functoriality of $- \otimes -$. Commutativity of the bottom

left triangle is unitality of μ . Finally, commutativity of the remaining middle 4-sided region is again functoriality of $-\otimes -$. It follows that y is equal to the outer composition, which is $r \cdot \tilde{e}$, as desired. Thus, we have shown that

$$\Psi(\Phi(x \otimes y)) = \widetilde{xy} \otimes \tilde{e} = (x \cdot r) \otimes \tilde{e} = x \otimes (r \cdot \tilde{e}) = x \otimes y,$$

as desired, so that for each $a \in A$, the object S^a belongs to the class \mathcal{E} .

Now, we would like to show that given a distinguished triangle

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X,$$

if two of three of the objects X , Y , and Z belong to \mathcal{E} , then so does the third. From now on, write $L_*^E : \mathcal{SH} \rightarrow \pi_*(E)\text{-}\mathbf{Mod}$ to denote the functor $X \mapsto E_*(E) \otimes_{\pi_*(E)} E_*(X)$, so Φ is a natural transformation $L_*^E \Rightarrow E_*(E \otimes -)$. First, note that by [Proposition A.19](#), we have the following exact sequence in \mathcal{SH} :

$$E \otimes \Omega Y \xrightarrow{E \otimes g} E \otimes \Omega Z \xrightarrow{E \otimes \tilde{h}} E \otimes X \xrightarrow{E \otimes f} E \otimes Y \xrightarrow{E \otimes g} E \otimes Z \xrightarrow{E \otimes h} E \otimes \Sigma X \xrightarrow{E \otimes \Sigma f} \Sigma Y.$$

We can then apply $[S^*, -]$ to it, which yields the following exact sequence of A -graded $\pi_*(E)$ -modules:

$$E_*(\Omega Y) \xrightarrow{E_*(\Omega g)} E_*(\Omega Z) \xrightarrow{E_*(\tilde{h})} E_*(X) \xrightarrow{E_*(f)} E_*(Y) \xrightarrow{E_*(g)} E_*(Z) \xrightarrow{E_*(h)} E_*(\Sigma X) \xrightarrow{E_*(\Sigma f)} E_*(\Sigma Y).$$

Now, we can tensor this sequence with $E_*(E)$ over $\pi_*(E)$, and since $E_*(E)$ is a flat right $\pi_*(E)$ module, we get that the top row in the following sequence is exact:

$$\begin{array}{ccccccccccc} L_*^E(\Omega Y) & \xrightarrow{L_*^E(\Omega g)} & L_*^E(\Omega Z) & \xrightarrow{L_*^E(\tilde{h})} & L_*^E(X) & \xrightarrow{L_*^E(f)} & L_*^E(Y) & \xrightarrow{L_*^E(g)} & L_*^E(Z) & \xrightarrow{L_*^E(h)} & L_*^E(\Sigma X) & \xrightarrow{L_*^E(\Sigma f)} & L_*^E(\Sigma Y) \\ \Phi_{\Omega Y} \downarrow & & \Phi_{\Omega Z} \downarrow & & \Phi_X \downarrow & & \Phi_Y \downarrow & & \Phi_Z \downarrow & & \Phi_{\Sigma X} \downarrow & & \Phi_{\Sigma Y} \downarrow \\ E_*(E \otimes \Omega Y) & \xrightarrow{E_*(E \otimes \Omega g)} & E_*(E \otimes \Omega Z) & \xrightarrow{E_*(E \otimes \tilde{h})} & E_*(E \otimes X) & \xrightarrow{E_*(E \otimes f)} & E_*(E \otimes Y) & \xrightarrow{E_*(E \otimes g)} & E_*(E \otimes Z) & \xrightarrow{E_*(E \otimes h)} & E_*(E \otimes \Sigma X) & \xrightarrow{E_*(E \otimes \Sigma f)} & E_*(E \otimes \Sigma Y) \end{array}$$

The diagram commutes since Φ is natural. The following sequence is exact in \mathcal{SH} by [Proposition A.19](#),

$$E \otimes E \otimes \Omega Y \rightarrow E \otimes E \otimes \Omega Z \rightarrow E \otimes E \otimes X \rightarrow E \otimes E \otimes Y \rightarrow E \otimes E \otimes Z \rightarrow E \otimes E \otimes \Sigma X \rightarrow E \otimes E \otimes \Sigma Y,$$

so that the bottom row in the above diagram is also exact. Now, suppose two of three of X , Y , and Z belong to \mathcal{E} . By [Lemma D.28](#), [Corollary D.29](#), if Φ_W is an isomorphism then $\Phi_{\Omega W}$ and $\Phi_{\Sigma W}$ are. Thus by the five lemma, it follows that the middle three vertical arrows in the above diagram are necessarily all isomorphisms, so we have shown that \mathcal{E} is closed under two-of-three for exact triangles, as desired.

Finally, it remains to show that \mathcal{E} is closed under arbitrary coproducts. Let $\{X_i\}_{i \in I}$ be a collection of objects in \mathcal{E} indexed by some (small) set I . Then we'd like to show that $X := \bigoplus_i X_i$ belongs to \mathcal{E} . First of all, note that $E \otimes -$ preserves arbitrary coproducts, as it has a right adjoint $F(E, -)$. Thus without loss of generality we may take $\bigoplus_i E \otimes X_i = E \otimes \bigoplus_i X_i$ (as $E \otimes \bigoplus_i X_i$ is a coproduct of all the $E \otimes X_i$'s). Now, recall that we have chosen each S^a to be a compact object ([Definition 2.1](#)), so that the canonical map

$$s : \bigoplus_i E_*(X_i) = \bigoplus_i [S^*, E \otimes X_i] \rightarrow [S^*, \bigoplus_i E \otimes X_i] = [S^*, E \otimes X] = E_*(X)$$

is an isomorphism, natural in X_i for each i . Note in particular that it is an isomorphism of left $\pi_*(E)$ -modules. To see this, first note it suffices to check that $s(r \cdot x) = r \cdot s(x)$ for some homogeneous $x \in E_*(X_i)$ for some i , as such x generate $\bigoplus_i E_*(X_i)$ by definition, and s is a

homomorphism of abelian groups. Then given $r : S^a \rightarrow E \otimes E$ and $x : S^b \rightarrow E \otimes X_i$, consider the following diagram

$$\begin{array}{c}
 S^{a+b} \xrightarrow{\phi_{a,b}} S^a \otimes S^b \xrightarrow{x \otimes y} E \otimes E \otimes E \otimes X_i \xrightarrow{E \otimes E \otimes \iota_{E \otimes X_i}} E \otimes E \otimes \bigoplus_i (E \otimes X_i) \\
 \downarrow E \otimes \mu \otimes X_i \quad \searrow E \otimes E \otimes E \otimes \iota_{X_i} \quad \parallel \\
 \quad \quad \quad E \otimes E \otimes E \otimes X \quad \downarrow E \otimes \mu \otimes X \\
 \quad \quad \quad E \otimes E \otimes X \quad \parallel \\
 E \otimes E \otimes X_i \xrightarrow{E \otimes \iota_{E \otimes X_i}} E \otimes \bigoplus_i (E \otimes X_i)
 \end{array}$$

where $\iota_{E \otimes X_i} : E \otimes X_i \hookrightarrow \bigoplus_i (E \otimes X_i)$ and $\iota_{X_i} : X_i \hookrightarrow \bigoplus_i X_i$ are the maps determined by universal property of the coproduct. Commutativity of the two triangles is again by the fact that $E \otimes -$ is colimit preserving. Commutativity of the trapezoid is functoriality of $- \otimes -$. Thus, the top arrow in the following diagram is well-defined:

$$\begin{array}{ccc}
 \bigoplus_i E_*(E) \otimes_{\pi_*(E)} E_*(X_i) & \xlongequal{\quad} & E_*(E) \otimes_{\pi_*(E)} \bigoplus_i E_*(X_i) \xrightarrow{E_*^{(E)} \otimes_{\pi_*(E)} s} E_*(E) \otimes_{\pi_*(E)} E_*(X) \\
 \downarrow \bigoplus_i \Phi_{X_i} & & \downarrow \Phi_X \\
 \bigoplus_i E_*(E \otimes X_i) & \xrightarrow{s} & E_*(\bigoplus_i E \otimes X_i) \xlongequal{\quad} E_*(E \otimes X)
 \end{array}
 \tag{8}$$

We wish to show this diagram commutes. Again, since each map here is a homomorphism, it suffices to chase generators. By definition, a generator of the top left element is a homogeneous pure tensor in $E_*(E) \otimes_{\pi_*(E)} E_*(X_i)$ for some i in I . Given classes $x : S^a \rightarrow E \otimes E$ and $y : S^b \rightarrow E \otimes X_i$, consider the following diagram:

$$\begin{array}{c}
 S^{a+b} \xrightarrow{\phi_{a,b}} S^a \otimes S^b \xrightarrow{x \otimes y} E \otimes E \otimes E \otimes X_i \xrightarrow{E \otimes E \otimes \iota_{E \otimes X_i}} E \otimes E \otimes \bigoplus_i E \otimes X_i \\
 \downarrow E \otimes \mu \otimes X_i \quad \searrow E \otimes E \otimes E \otimes \iota_{X_i} \quad \parallel \\
 E \otimes E \otimes X_i \quad \quad \quad E \otimes E \otimes E \otimes X \quad \downarrow E \otimes \mu \otimes X \\
 \downarrow \iota_{E \otimes E \otimes X_i} \quad \searrow E \otimes E \otimes \iota_{X_i} \quad \downarrow \\
 \bigoplus_i E \otimes E \otimes X_i \xlongequal{\quad} E \otimes E \otimes X
 \end{array}$$

Unravelling definitions, the two outside compositions are the two ways to chase $x \otimes y$ around diagram (8). The two triangles commute again by the fact that $- \otimes -$ preserves colimits in each argument. Commutativity of the inner parallelogram is functoriality of $- \otimes -$. Thus diagram (8) tells us Φ_X is an isomorphism, since Φ_{X_i} is an isomorphism for each i in I , meaning $\bigoplus_i \Phi_{X_i}$ is as well. \square

Proposition D.31. *Let (E, μ, e) be a ring spectrum in \mathcal{SH} , and let X and Y be two objects in \mathcal{SH} such that E and X are both cellular (Definition 2.5) and $E_*(X)$ is a projective left $\pi_*(E)$ -module (Proposition D.20). Then the map*

$$[X, E \otimes Y]_* \rightarrow \text{Hom}_{\pi_*(E)}^*(E_*(X), E_*(Y))$$

which sends a generator $f : S^a \otimes X \rightarrow E \otimes Y$ in $[X, E \otimes Y]_*$ to the assignment which sends a generator $x : S^b \rightarrow E \otimes X$ in $E_*(X)$ to the composition

$$S^{a+b} \rightarrow S^a \otimes S^b \xrightarrow{S^a \otimes x} S^a \otimes E \otimes X \xrightarrow{\tau \otimes X} E \otimes S^a \otimes X \xrightarrow{E \otimes f} E \otimes E \otimes Y \xrightarrow{\mu} E \otimes Y$$

is an A -graded isomorphism of A -graded abelian groups.

APPENDIX E. HOPF ALGEBROIDS

We will freely use the results of [Appendix C](#) in this section. In what follows, fix an A -graded ring R . Further suppose that for all $a, b \in A$, there exists units $\theta_{a,b} \in R_0$ such that:

- For all $a \in A$, $\theta_{a,0} = \theta_{0,a} = 1$,
- For all $a, b \in A$, $\theta_{a,b}^{-1} = \theta_{b,a}$,
- For all $a, b, c \in A$, $\theta_{a,b} \cdot \theta_{a,c} = \theta_{a,b+c}$ and $\theta_{b,a} \cdot \theta_{c,a} = \theta_{b+c,a}$.
- For all $x \in R_a$ and $y \in R_b$,

$$x \cdot y = y \cdot x \cdot \theta_{a,b}.$$

Definition E.1. Let $R\text{-GrCAlg}$ denote the following category:

- The objects are pairs (S, φ) where S is an A -graded ring and $\varphi : R \rightarrow S$ is an A -graded ring homomorphism such that for all $x \in S_a$ and $y \in S_b$, we have

$$x \cdot y = y \cdot x \cdot \varphi(\theta_{a,b}),$$

- The morphisms $(S, \varphi) \rightarrow (S', \varphi')$ are A -graded ring homomorphisms $f : S \rightarrow S'$ such that $f \circ \varphi = \varphi'$.

Lemma E.2. Let (S, φ) be an object in $R\text{-GrCAlg}$. Then S is canonically an A -graded R -bimodule via the action maps $(r, s) \mapsto \varphi(r) \cdot s$ and $(s, r) \mapsto s \cdot \varphi(r)$ for $r \in R$ and $s \in S$.

TODO

Proof.

□

Proposition E.3. Let (B, φ_B) and (C, φ_C) in $R\text{-GrCAlg}$. Then since B and C are A -graded R -bimodules ([Lemma E.2](#)), their R -tensor product $B \otimes_R C$ is also an A -graded R -bimodule ([Lemma C.8](#)). Then $B \otimes_R C$ is canonically an element of $R\text{-GrCAlg}$, with product

$$(B \otimes_R C) \times (B \otimes_R C) \rightarrow (B \otimes_R C)$$

sending pure homogeneous tensors $(b \otimes c, b' \otimes c')$ to the element

$$\theta_{|c|, |b'|} \cdot (bb' \otimes cc'),$$

(where here \cdot denotes the left module action of R on $B \otimes_R C$), and with structure map

$$\begin{aligned} \varphi : R &\rightarrow B \otimes_R C \\ r &\mapsto \varphi_B(r) \otimes 1_C. \end{aligned}$$

Proof. First we claim the indicated map is well-defined and actually gives $B \otimes_R C$ the structure of an A -graded ring. Here we are employing [Lemma C.7](#), so it suffices to check the map is well-defined, unital, associative, and distributive with respect to homogeneous elements. □

Definition E.4. An R -ring is a monoid object ([Definition 2.11](#)) in the category of A -graded R -bimodules. An R -coring is a monoid object in the opposite category of the category of A -graded R -bimodules.

Definition E.5. A *right R -bialgebroid* B consists of an R -bimodule B with the structure of an $R \otimes R^{\text{op}}$ -ring (B, s, t) and an R -coring (B, Δ, ε) such that:

- (i) The bimodule structure in the R -coring (B, Δ, ε) is related to the $R \otimes R^{\text{op}}$ ring (B, s, t) via

$$r \cdot b \cdot r' = b \cdot s(r') \cdot t(r), \quad \text{for } r, r' \in R, b \in B.$$

Definition E.6. A *Hopf algebroid*

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