# THE MOTIVIC ADAMS SPECTRAL SEQUENCE

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## 1. Introduction

## 2. Triangulated categories with sub-Picard grading

2.1. **Setup.** In order to construct an abstract version of the Adams spectral sequence, we need to work in some axiomatic version of a stable homotopy category $\mathcal{SH}$ which acts like the familiar classical stable homotopy category **hoSp** (Section 4) or the motivic stable homotopy category $\mathbf{SH}_{\mathscr{S}}$ over some base scheme $\mathscr{S}$ (Section 5).

**Definition 2.1.** Let $\mathcal{C}$ be an additive category with arbitrary (small) coproducts. Then an object $X$ in $\mathcal{C}$ is *compact* if, for any collection of objects $Y_i$ in $\mathcal{C}$ indexed by some (small) set $I$, the canonical map

$$\coprod_i \mathcal{C}(X, Y_i) \to \mathcal{C}(X, \coprod_i Y_i)$$

is an isomorphism of abelian groups. (Explicitly, the above map takes a generator $x \in \mathcal{C}(X, Y_i)$ to the composition $X \xrightarrow{x} Y_i \hookrightarrow \coprod_i Y_i$.)

**Definition 2.2.** Given a tensor triangulated category $(\mathcal{C}, \otimes, S, \Sigma, e, \mathcal{D})$ (Definition A.11), a *sub-Picard grading* on $\mathcal{C}$ is the following data:

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- A pointed abelian group $(A, \mathbf{1})$ along with a homomorphism of pointed groups $h : (A, \mathbf{1}) \to (\operatorname{Pic} \mathcal{C}, \Sigma S)$, where $\operatorname{Pic} \mathcal{C}$ is the *Picard group* of isomorphism classes of invertible objects in $\mathcal{C}$.[1]
- For each $a \in A$, a chosen representative $S^a$ in the isomorphism class $h(a)$ such that each $S^a$ is a compact object (Definition 2.1) and $S^0 = S$.
- A chosen isomorphism $\nu : \Sigma S \xrightarrow{\cong} S^{\mathbf{1}}$.
- For each $a, b \in A$, an isomorphism $\phi_{a,b} : S^{a+b} \to S^a \otimes S^b$. This family of isomorphisms is required to be *coherent*, in the following sense:
  - For all $a \in A$, we must have that $\phi_{a,0}$ coincides with the right unitor $S^a \xrightarrow{\cong} S^a \otimes S$ and $\phi_{0,a}$ coincides the left unitor $S^a \xrightarrow{\cong} S \otimes S^a$.
  - For all $a, b, c \in A$, the following "associativity diagram" must commute:

$$\begin{tikzcd}
S^{a+b} \otimes S^c \arrow[d, "\phi_{a,b} \otimes S^c"'] & S^{a+b+c} \arrow[l, "\phi_{a+b,c}"'] \arrow[r, "\phi_{a,b+c}"] & S^a \otimes S^{b+c} \arrow[d, "S^a \otimes \phi_{b,c}"] \\
(S^a \otimes S^b) \otimes S^c \arrow[rr, "\cong"] & & S^a \otimes (S^b \otimes S^c)
\end{tikzcd}$$

**Remark 2.3.** Note that by induction the coherence conditions for the $\phi_{a,b}$'s in the above definition say that given any $a_1, \ldots, a_n \in A$ and $b_1, \ldots, b_m \in A$ such that $a_1 + \cdots + a_n = b_1 + \cdots + b_m$ and any fixed parenthesizations of $X = S^{a_1} \otimes \cdots \otimes S^{a_b}$ and $Y = S^{b_1} \otimes \cdots \otimes S^{b_m}$, there is a *unique* isomorphism $X \to Y$ that can be obtained by forming formal compositions of products of $\phi_{a,b}$, identities, associators, unitors, and their inverses.

From now on we fix a monoidal closed tensor triangulated category $(\mathcal{SH}, \otimes, S, \Sigma, e, \mathcal{D})$ with arbitrary (small) (co)products and sub-Picard grading $(A, \mathbf{1}, h, \{S^a\}, \nu, \{\phi_{a,b}\})$. We establish some conventions. First of all, given an object $X$ and a natural number $n > 0$, we write

$$X^n := \overbrace{X \otimes \cdots \otimes X}^{n \text{ times}} \qquad \text{and} \qquad X^0 := S.$$

We denote the associator, symmetry, left unitor, and right unitor isomorphisms in $\mathcal{SH}$ by

$$\alpha_{X,Y,Z} : (X \otimes Y) \otimes Z \xrightarrow{\cong} X \otimes (Y \otimes Z) \qquad \tau_{X,Y} : X \otimes Y \xrightarrow{\cong} Y \otimes X$$
$$\lambda_X : S \otimes X \xrightarrow{\cong} X \qquad \rho_X : X \otimes S \xrightarrow{\cong} X.$$

Often we will drop the subscripts. Furthermore, by the coherence theorem for symmetric monoidal categories, we will often assume $\alpha$, $\rho$, and $\lambda$ are actual equalities. Given some inte, there exists some isomorphismger $n \in \mathbb{Z}$, we will write a bold $\mathbf{n}$ to denote the element $n \cdot \mathbf{1}$ in $A$. Note that we can use the isomorphism $\nu : S^{\mathbf{1}} \otimes - \cong \Sigma$ to construct a natural isomorphism $S^{\mathbf{1}} \otimes - \cong \Sigma$:

$$S^{\mathbf{1}} \otimes X \xrightarrow{\nu \otimes X} \Sigma S \otimes X \xrightarrow{e_{S,X}} \Sigma(S \otimes X) \xrightarrow{\Sigma \lambda_X} \Sigma X.$$

The last two arrows are natural in $X$ by definition. The first arrow is natural in $X$ by functoriality of $- \otimes -$. By abuse of notation, we will also use $\nu$ to denote this natural isomorphism. Furthermore, under this isomorphism, $e_{X,Y} : \Sigma X \otimes Y \xrightarrow{\cong} \Sigma(X \otimes Y)$ corresponds to the associator,

[1]Recall an object $X$ is a symmetric monoidal category is *invertible* if there exists some object $Y$ and an isomorphism $S \cong X \otimes Y$.

by commutativity of the following diagram:

$$\begin{tikzcd}
(S^{\mathbf{1}} \otimes X) \otimes Y \arrow[r, "(\nu\otimes X)\otimes Y"] \arrow[dd, "\alpha"'] & (\Sigma S \otimes X) \otimes Y \arrow[r, "e_{S,X}\otimes Y"] \arrow[dd, "\alpha"] & \Sigma(S \otimes X) \otimes Y \arrow[r, "\Sigma\lambda_X \otimes Y"] \arrow[d, "e_{S\otimes X,Y}"'] & \Sigma X \otimes Y \arrow[dd, "e_{X,Y}"] \\
 & & \Sigma((S \otimes X) \otimes Y) \arrow[d, "\Sigma\alpha"'] \arrow[dr, "\Sigma(\lambda_X \otimes Y)"] & \\
S^{\mathbf{1}} \otimes (X \otimes Y) \arrow[r, "\nu\otimes(X\otimes Y)"'] & \Sigma S \otimes (X \otimes Y) \arrow[r, "e_{S,X\otimes Y}"'] & \Sigma(S \otimes (X \otimes Y)) \arrow[r, "\Sigma\lambda_{X\otimes Y}"'] & \Sigma(X \otimes Y)
\end{tikzcd}$$

The left square commutes by naturality of $\alpha$. Commutativity of the middle square is axiom TT4 for a tensor triangulated category. Commutativity of the right trapezoid is naturality of $e$. Finally the bottom triangle commutes by coherence for monoidal categories and functoriality of $\Sigma$.

Given some $a \in A$, we define $\Sigma^a := S^a \otimes -$ and $\Omega^a := \Sigma^{-a} = S^{-a} \otimes -$. We specifically define $\Omega := \Omega^{\mathbf{1}}$. We will see later that for each $a \in A$, $\Sigma^a$ and $\Omega^a$ form an adjoint equivalence of $\mathcal{SH}$ (Proposition 2.5), so that in particular since $\Omega$ forms an adjoint equivalence with $\Sigma^{\mathbf{1}} \cong \Sigma$, $\mathcal{SH}$ is canonically an *adjointly* triangulated category (Definition A.7).

Given two objects $X$ and $Y$ in $\mathcal{SH}$, we will denote the hom-abelian group of morphisms from $X$ to $Y$ in $\mathcal{SH}$ by $[X, Y]$, and the internal hom object by $F(X, Y)$. We can extend the abelian group $[X, Y]$ into an $A$-graded abelian group $[X, Y]_*$ by defining $[X, Y]_a := [S^a \otimes X, Y]$. Given an object $X$ in $\mathcal{SH}$ and some $a \in A$, we can define the abelian group

$$\pi_a(X) := [S^a, X],$$

which we call the $a^{th}$ *stable homotopy group of* $X$. We write $\pi_*(X)$ for the $A$-graded abelian group $\bigoplus_{a\in A} \pi_a(X)$, so that in particular we have a canonical isomorphism

$$\pi_*(X) = [S^*, X] \cong [S, X]_*.$$

Given some other object $E$, we can define the $A$-graded abelian groups $E_*(X)$ and $E^*(X)$ by the formulas

$$E_a(X) := \pi_a(E \otimes X) = [S^a, E \otimes X] \qquad \text{and} \qquad E^a(X) := [X, S^a \otimes E].$$

We refer to the functor $E_*(-)$ as the *homology theory represented by* $E$, or just $E$-homology, and we refer to $E^*(-)$ as the *cohomology theory represented by* $E$, or just $E$-cohomology. Finally, we state the following definition in $\mathcal{SH}$:

**Definition 2.4.** Define the class of *cellular* objects in $\mathcal{SH}$ to be the smallest class of objects such that:

(1) For all $a \in A$, $S^a$ is cellular.
(2) If we have a distinguished triangle
$$X \to Y \to Z \to \Sigma X$$
such that two of the three objects $X$, $Y$, and $Z$ are cellular, than the third object is also cellular.
(3) Given a collection of cellular objects $X_i$ indexed by some (small) set $I$, the object $\coprod_{i\in I} X_i$ is cellular (recall we have chosen $\mathcal{SH}$ to have arbitrary (co)products).

## 2.2. Miscellaneous facts about $\mathcal{SH}$.

what do I call this subsection?

**Proposition 2.5.** *For each $a \in A$, the isomorphisms*

$$\eta^a_X : X \xrightarrow{\lambda_X^{-1}} S \otimes X \xrightarrow{\Phi_{a,-a}\otimes X} (S^a \otimes S^{-a}) \otimes X \xrightarrow{\alpha} S^a \otimes (S^{-a} \otimes X) = \Sigma^a\Omega^a X$$

*and*

$$\varepsilon^a_X : \Omega^a\Sigma^a X = S^{-a} \otimes (S^a \otimes X) \xrightarrow{\alpha^{-1}} (S^{-a} \otimes S^a) \otimes X \xrightarrow{\phi^{-1}_{-a,a}\otimes X} S \otimes X \xrightarrow{\lambda_X} X$$

*are natural in $X$, and furthermore, they are the unit and counit respectively of the adjoint autoequivalence $(\Omega^a, \Sigma^a, \eta^a, \varepsilon^a)$ of $\mathcal{SH}$. In particular, since $\Sigma \cong \Sigma^{\mathbf{1}}$, $\Omega := \Omega^{\mathbf{1}}$ is a left adjoint for $\Sigma$, so that $(\mathcal{SH}, \Omega, \Sigma, \eta, \varepsilon, \mathcal{D})$ is an* adjointly *triangulated category (Definition A.7), where $\eta$ and $\varepsilon$ are the compositions*

$$\eta : \mathrm{Id}_{\mathcal{SH}} \xRightarrow{\eta^{\mathbf{1}}} \Sigma^{\mathbf{1}}\Omega \xRightarrow{\nu\Omega} \Sigma\Omega \qquad \textit{and} \qquad \varepsilon : \Omega\Sigma \xRightarrow{\Omega\nu^{-1}} \Omega\Sigma^{\mathbf{1}} \xRightarrow{\varepsilon^{\mathbf{1}}} \mathrm{Id}_{\mathcal{SH}}.$$

*Proof.* In this proof, we will freely employ the coherence theorem for monoidal categories (see [1]), which essentially tells us that we may assume we are working in a strict monoidal category (i.e., that the associators and unitors and are identities). Then $\eta^a_X$ and $\varepsilon^a_X$ become simply the maps

$$\eta^a_X : X \xrightarrow{\phi_{a,-a}\otimes X} S^a \otimes S^{-a} \otimes X \qquad \text{and} \qquad \varepsilon^a_X : S^{-a} \otimes S^a \otimes X \xrightarrow{\phi^{-1}_{-a,a}\otimes X} X.$$

That these maps are natural in $X$ follows by functoriality of $-\otimes-$. Now, recall that in order to show that these natural isomorphisms form an *adjoint* equivalence, it suffices to show that the natural isomorphisms $\eta^a : \mathrm{Id}_{\mathcal{SH}} \Rightarrow \Omega^a\Sigma^a$ and $\varepsilon^a : \Sigma^a\Omega^a \Rightarrow \mathrm{Id}_{\mathcal{SH}}$ satisfy one of the two zig-zag identities:

$$\begin{tikzcd} \Omega^a \arrow[r, "\Omega^a\eta^a"] \arrow[dr, equal] & \Omega^a\Sigma^a\Omega^a \arrow[d, "\varepsilon^a\Omega^a"] \\ & \Omega^a \end{tikzcd} \qquad \begin{tikzcd} \Sigma^a\Omega^a\Sigma^a \arrow[d, "\Sigma^a\varepsilon^a"'] & \Sigma^a \arrow[l, "\eta^a\Sigma^a"'] \arrow[dl, equal] \\ \Sigma^a & \end{tikzcd}$$

(that it suffices to show only one is [3, Lemma 3.2]). We will show that the left is satisfied. Unravelling definitions, we simply wish to show that the following diagram commutes for all $X$ in $\mathcal{SH}$:

$$\begin{tikzcd} S^{-a}\otimes X \arrow[r, "S^{-a}\otimes\phi_{a,-a}\otimes X"] \arrow[dr, equal] & S^{-a}\otimes S^a\otimes S^{-a}\otimes X \arrow[d, "\phi^{-1}_{-a,a}\otimes S^{-a}\otimes X"] \\ & S^{-a}\otimes X \end{tikzcd}$$

Yet this is simply the diagram obtained by applying $-\otimes X$ to the associativity coherence diagram for the $\phi_{a,b}$'s (since $\phi_{a,0}$ and $\phi_{0,a}$ coincide with the unitors, and here we are taking the unitors and associators to be equalities), so it does commute, as desired. $\square$

### 2.3. Monoid objects in $\mathcal{SH}$.

**Proposition 2.6.** *Let $(E, \mu, e)$ be a monoid object in $\mathcal{SH}$ (Definition D.1). Then $\pi_*(E)$ is canonically an $A$-graded ring via the assignment $\pi_*(E) \times \pi_*(E) \to \pi_*(E)$ which takes classes $x : S^a \to E$ and $y : S^b \to E$ to the composition*

$$S^{a+b} \xrightarrow{\phi}_{a,b} S^a \otimes S^b \xrightarrow{x\otimes y} E \otimes E \xrightarrow{\mu} E.$$

*In particular, the unit for this ring is the element $e \in [S, E] = \pi_0(E)$.*

*Proof.* See Proposition D.2. $\square$

**Proposition 2.7.** *Let $(E, \mu, e)$ be a monoid object in $\mathcal{SH}$. Then $E_*(-)$ is a functor from $\mathcal{SH}$ to left $A$-graded $\pi_*(E)$-modules, where given some $X$ in $\mathcal{SH}$, $E_*(X)$ may be endowed with the structure of a left $A$-graded $\pi_*(E)$-module via the map*

$$\pi_*(E) \times E_*(X) \to E_*(X)$$

*which given $a, b \in A$, sends $x : S^a \to E$ and $y : S^b \to E \otimes X$ to the composition*

$$x \cdot y : S^{a+b} \cong S^a \otimes S^b \xrightarrow{x\otimes y} E \otimes (E \otimes X) \cong (E \otimes E) \otimes X \xrightarrow{\mu\otimes X} E \otimes X.$$

*Similarly, the assignment $X \mapsto X_*(E)$ is a functor from $\mathcal{SH}$ to right $A$-graded $\pi_*(E)$-modules, where the structure map*

$$X_*(E) \times \pi_*(E) \to X_*(E)$$

*sends $x : S^a \to X \otimes E$ and $y : S^b \to E$ to the composition*

$$x \cdot y : S^{a+b} \cong S^a \otimes S^b \xrightarrow{x \otimes y} (X \otimes E) \otimes E \cong X \otimes (E \otimes E) \xrightarrow{X \otimes \mu} X \otimes E.$$

*Finally, $E_*(E)$ is a $\pi_*(E)$-bimodule, in the sense that the left and right actions of $\pi_*(E)$ are compatible, so that given $y, z \in \pi_*(E)$ and $x \in E_*(E)$, $y \cdot (x \cdot z) = (y \cdot x) \cdot z$.*

*Proof.* See Proposition D.6. □

**Definition 2.8.** Given a monoid object $E$ in $\mathcal{SH}$, we say $E$ is *flat* if the canonical right $\pi_*(E)$-module structure on $E_*(E)$ (see the above proposition) is that of a flat module.

## 3. The Adams spectral sequence

3.1. **Construction of the Adams spectral sequence.** In what follows, let $E$ be a commutative monoid object in $\mathcal{SH}$.

**Definition 3.1.** Let $\overline{E}$ be the fiber of the unit map $e : S \to E$ (Proposition A.6), and for $s \geq 0$ define

$$Y_s := \overline{E}^s \otimes Y, \qquad W_s = E \otimes Y_s = E \otimes (\overline{E}^s \otimes Y),$$

where recall for $s > 0$, $\overline{E}^s$ denotes the $s$-fold product parenthesized as $\overline{E} \otimes (\overline{E} \otimes \cdots (\overline{E} \otimes \overline{E}))$, and $\overline{E}^0 := S$. Then we get fiber sequences

$$Y_{s+1} \xrightarrow{i_s} Y_s \xrightarrow{j_s} W_s \xrightarrow{k_s} \Sigma Y_{s+1} (= S^{\mathbf{1}} \otimes Y_{s+1})$$

obtained by applying $- \otimes Y_s$ to the sequence

$$\overline{E} \to S \xrightarrow{e} E \to \Sigma \overline{E}$$

(and applying the necessary associator and unitor isomorphisms). These sequences can be spliced together to form the *(canonical) Adams filtration of $Y$*:

$$\begin{tikzcd}
\cdots \arrow[r] & Y_3 \arrow[r, "i_2"] \arrow[d, "j_3"] & Y_2 \arrow[r, "i_1"] \arrow[d, "j_2"] & Y_1 \arrow[r, "i_0"] \arrow[d, "j_1"] & Y_0 = Y \arrow[d, "j_0"] \\
& W_3 & W_2 \arrow[ul, dashed, "k_2"] & W_1 \arrow[ul, dashed, "k_1"] & W_0 \arrow[ul, dashed, "k_0"]
\end{tikzcd}$$

where the diagonal dashed arrows are of degree $-\mathbf{1}$ (note these triangles do NOT commute in any sense). Now we may apply the functor $[X, -]_*$, and by Proposition A.10 we obtain an exact couple of $\mathbb{N} \times A$-graded abelian groups:

$$\begin{tikzcd}
{[X, Y_*]_*} \arrow[rr, "i_{**}"] & & {[X, Y_*]_*} \arrow[d, "j_{**}"] \\
& & {[X, W_*]_*} \arrow[ull, "k_{**}"]
\end{tikzcd}$$

where $i_{**}$, $j_{**}$, and $k_{**}$ have $\mathbb{Z} \times A$-degree $(-1, 0)$, $(0, 0)$, and $(1, -\mathbf{1})$, respectively[2]. The standard argument yields an $\mathbb{N} \times A$-graded spectral sequence called from this exact couple (cf. Section 5.9 of [6]) with $E_1$ page given by

$$E_1^{s,a} = [X, W_s]_a$$

[2]Explicitly, the map $k_{s,a} : [X, W_s]_a \to [X, Y_{s+1}]_{a-\mathbf{1}}$ sends a map $f : S^a \otimes X \to W_s$ to the map $S^{a-\mathbf{1}} \otimes X \to Y_{s+1}$ corresponding under the isomorphism $[X, \Sigma Y_{s+1}]_* \cong [X, Y_{s+1}]_{*-\mathbf{1}}$ to the composition $k_s \circ f : S^a \otimes X \to \Sigma Y_{s+1}$.

and $r^{\text{th}}$ differential of $\mathbb{Z} \times A$-degree $(r, -\mathbf{1})$:

$$d_r : E_r^{s,a} \to E_r^{s+r,a-\mathbf{1}}.$$

A priori, this is all $\mathbb{N} \times A$-graded, but we regard it as being $\mathbb{Z} \times A$-graded by setting $E_r^{s,a} := 0$ for $s < 0$ and trivially extending the definition of the differentials to these zero groups. This spectral sequence is called the *$E$-Adams spectral sequence* for the computation of $[X, Y]_*$. The index $s$ is called the *Adams filtration* and $a$ is the *stem*.

3.2. **The $E_1$ page.** The goal of this subsection is to provide the following characterization for the $E_1$ page of the Adams spectral sequence:

**Theorem 3.2.** *Let $E$ be a flat commutative monoid object in $\mathcal{SH}$, and let $X$ and $Y$ be two objects in $\mathcal{SH}$ such that $E_*(X)$ is a projective module over $\pi_*(E)$. Then for all $s \geq 0$ and $a \in A$, we have isomorphisms in the associated $E$-Adams spectral sequence*

$$E_1^{s,a} \cong \operatorname{Hom}^a_{E_*(E)}(E_*(X), E_*(W_s))$$

*Furthermore, under these isomorphisms, the differential $d_1 : E_1^{s,a} \to E_1^{s+1,a-\mathbf{1}}$ corresponds to the map*

$$\operatorname{Hom}^a_{E_*(E)}(E_*(X), E_*(W_s)) \to \operatorname{Hom}^{a-\mathbf{1}}_{E_*(E)}(E_*(X), E_*(X \otimes W_{s+1}))$$

*which sends a map $f : E_*(X) \to E_{*+a}(W_s)$ to the composition*

$$E_*(X) \xrightarrow{f} E_{*+a}(W_s) \xrightarrow{(X \otimes h_s)_*} E_{*+a-\mathbf{1}}(X \otimes Y_{s+1}) \xrightarrow{(X \otimes j_{s+1})_*} E_{*+a-\mathbf{1}}(X \otimes W_{s+1}).$$

*Proof.* By **??**, for all $s \geq 0$ and $t, w \in \mathbb{Z}$, we have isomorphisms

$$[X, E \otimes Y_s]_{t,w} \cong \operatorname{Hom}^{t,w}_{E_*(E)}(E_*(X), E_*(E \otimes Y_s)).$$

since $W_s = E \otimes Y_s$, we have that

$$E_1^{s,(t,w)} = [X, W_s]_{t,w} \cong \operatorname{Hom}^{t,w}_{E_*(E)}(E_*(X), E_*(W_s)),$$

as desired. $\square$

**Definition 3.3.** Let $(E, \mu, e)$ be a monoid object in $\mathcal{SH}$. We say $E$ is *flat* if the canonical right $\pi_*(E)$-module structure on $E_*(E)$ is that of a flat module.

3.3. **The $E_2$ page.**

3.4. **Convergence.** convergence of spectral sequences

## 4. The classical Adams spectral sequence

## 5. The motivic Adams spectral sequence

## Appendix A. Triangulated categories

We assume the reader is familiar with additive categories and (closed, symmetric) monoidal categories.

**Definition A.1.** A *triangulated category* $(\mathcal{C}, \Sigma, \mathcal{D})$ is the data of:

(1) An additive category $\mathcal{C}$.
(2) An additive auto-equivalence $\Sigma : \mathcal{C} \to \mathcal{C}$ called the *shift functor*.
(3) A collection $\mathcal{D}$ of *distinguished* triangles in $\mathcal{C}$, where a is a sequence of arrows of the form

$$X \to Y \to Z \to \Sigma X.$$

Distinguished triangles are also sometimes called *cofiber sequences* or *fiber sequences*.

These data must satisfy the following axioms:

**TR0** Given a commutative diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d, "\cong"'] & Y \arrow[r] \arrow[d, "\cong"'] & Z \arrow[r] \arrow[d, "\cong"'] & \Sigma X \arrow[d, "\cong"'] \\ X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & \Sigma X' \end{tikzcd}$$

where the vertical arrows are isomorphisms, if the top row is distinguished then so is the bottom.

**TR1** For any object $X$ in $\mathcal{C}$, the diagram

$$X \xrightarrow{\mathrm{id}_X} X \to 0 \to \Sigma X$$

is a distinguished triangle.

**TR2** For all $f : X \to Y$ there exists an object $C_f$ (also sometimes denoted $Y/X$) called the *cofiber of* $f$ and a distinguished triangle

$$X \xrightarrow{f} Y \to C_f \to \Sigma X.$$

**TR3** Given a solid diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] & Y \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d, dashed] & \Sigma X \arrow[d, "\Sigma f"] \\ X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & \Sigma X' \end{tikzcd}$$

such that the leftmost square commmutes and both rows are distinguished, there exists a dashed arrow $Z \to Z'$ which makes the remaining two squares commute.

**TR4** A triangle

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{\Sigma} X$$

is distinguished if and only if

$$Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X \xrightarrow{-\Sigma f} \Sigma Y$$

is distinguished.

**TR5** (Octahedral axiom) Given three distinguished triangles

$$X \xrightarrow{f} Y \xrightarrow{h} Y/X \to \Sigma X$$

$$Y \xrightarrow{g} Z \xrightarrow{k} Z/Y \to \Sigma Y$$

$$X \xrightarrow{g \circ f} Z \xrightarrow{l} Z/X \to \Sigma X$$

there exists a distinguished triangle

$$Y/X \xrightarrow{u} Z/X \xrightarrow{v} Z/Y \xrightarrow{w} \Sigma(Y/X)$$

such that the following diagram commutes

$$\begin{tikzcd} X \arrow[rr, "g \circ f"] \arrow[dr, "f"'] & & Z \arrow[rr, "k"] \arrow[dr, "l"'] & & Z/Y \arrow[rr, "w"] \arrow[dr] & & \Sigma(Y/X) \\ & Y \arrow[ur, "g"'] \arrow[dr, "h"'] & & Z/X \arrow[ur, "v"'] \arrow[dr] & & \Sigma Y \arrow[ur, "\Sigma h"'] & \\ & & Y/X \arrow[ur, "u"'] \arrow[rr] & & \Sigma X \arrow[ur, "\Sigma f"'] & & \end{tikzcd}$$

It turns out that the above definition is actually redundant; TR3 and TR4 follow from the remaining axioms (see Lemmas 2.2 and 2.4 in [2]). From now on, we fix a triangulated category $(\mathcal{C}, \Sigma, \mathcal{D})$. To start, recall the following definition:

**Definition A.2.** A sequence
$$X_1 \to X_2 \to \cdots \to X_n$$
of arrows in $\mathcal{C}$ is *exact* if, for any object $A$ in $\mathcal{C}$, the induced sequence
$$[A, X_1] \to [A, X_2] \to \cdots \to [A, X_n]$$
is an exact sequence of abelian groups.

**Proposition A.3.** *Any distinguished triangle is an exact sequence (in the sense of Definition A.2).*

*Proof.* Suppose we have some distinguished triangle
$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X.$$
Then we would like to show that given any object $A$ in $\mathcal{C}$, the sequence
$$[A, X] \xrightarrow{f_*} [A, Y] \xrightarrow{g_*} [A, Z] \xrightarrow{h_*} [A, \Sigma X]$$
is exact. First we show exactness at $[A, Y]$. To see $\operatorname{im} f_* \subseteq \ker g_*$, note it suffices to show that $g \circ f = 0$. Indeed, consider the commuting diagram
$$\begin{tikzcd} X \arrow[r, equal] \arrow[d, equal] & X \arrow[r] \arrow[d, "f"] & 0 \arrow[r] & \Sigma X \\ X \arrow[r, "f"] & Y \arrow[r, "g"] & Z \arrow[r, "h"] & \Sigma X \end{tikzcd}$$
The top row is distinguished by axiom TR1. Thus by TR3, the following diagram commutes:
$$\begin{tikzcd} X \arrow[r, equal] \arrow[d, equal] & X \arrow[r] \arrow[d, "f"] & 0 \arrow[r] \arrow[d] & \Sigma X \arrow[d, equal] \\ X \arrow[r, "f"] & Y \arrow[r, "g"] & Z \arrow[r, "h"] & \Sigma X \end{tikzcd}$$
In particular, commutativity of the second square tells us that $g \circ f = 0$, as desired. Conversely, we'd like to show that $\ker g_* \subseteq \operatorname{im} f_*$. Let $\psi : A \to Y$ be in the kernel of $g_*$, so that $g \circ \psi = 0$. Consider the following commutative diagram:
$$\begin{tikzcd} A \arrow[r] \arrow[d, "\psi"'] & 0 \arrow[r] \arrow[d] & \Sigma A \arrow[r, "-\Sigma \mathrm{id}_A"] & \Sigma A \\ Y \arrow[r, "g"] & Z \arrow[r, "h"] & \Sigma X \arrow[r, "-\Sigma f"] & \Sigma Y \end{tikzcd}$$
The top row is distinguished by axioms TR1 and TR4. The bottom row is distinguished by axiom TR4. Thus by axiom TR3 there exists a map $\widetilde{\phi} : \Sigma A \to \Sigma X$ such that the following diagram commutes:
$$\begin{tikzcd} A \arrow[r] \arrow[d, "\psi"'] & 0 \arrow[r] \arrow[d] & \Sigma A \arrow[r, "-\Sigma \mathrm{id}_A"] \arrow[d, "\widetilde{\phi}"'] & \Sigma A \arrow[d, "\Sigma\psi"'] \\ Y \arrow[r, "g"] & Z \arrow[r, "h"] & \Sigma X \arrow[r, "-\Sigma f"] & \Sigma Y \end{tikzcd}$$
Now, since $\Sigma$ is an equivalence, it is a full functor, so that in particular there exists some $\phi : A \to X$ such that $\widetilde{\phi} = \Sigma\phi$. Then by faithfullness, we may pull back the right square to get a commuting diagram
$$\begin{tikzcd} A \arrow[r, "-\mathrm{id}_A"] \arrow[d, "\phi"'] & A \arrow[d, "\psi"] \\ X \arrow[r, "-f"'] & Y \end{tikzcd}$$

Hence,

$$f_*(\phi) = f \circ \phi \overset{(*)}{=} -((-f) \circ \phi) = -(\psi \circ (-\mathrm{id}_A)) \overset{(*)}{=} \psi \circ \mathrm{id}_A = \psi,$$

where the equalities marked $(*)$ follow by bilinearity of composition in an additive category. Thus $\psi \in \operatorname{im} f_*$, as desired, meaning $\ker g_* \subseteq \operatorname{im} f_*$.

Now, we have shown that

$$[A, X] \xrightarrow{f_*} [A, Y] \xrightarrow{g_*} [A, Z] \xrightarrow{h_*} [A, \Sigma X]$$

is exact at $[A, Y]$. It remains to show exactness at $[A, Z]$. Yet this follows by the exact same argument given above applied to the sequence obtained from the shifted triangle (TR4)

$$Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X \xrightarrow{-\Sigma f} \Sigma Y. \qquad \square$$

**Lemma A.4.** *Suppose we have a commutative diagram*

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "j"] & Y \arrow[r, "g"] \arrow[d, "k"] & Z \arrow[r, "h"] \arrow[d, "\ell"] & \Sigma X \arrow[d, "\Sigma j"] \\
X' \arrow[r, "f'"] & Y' \arrow[r, "g'"] & Z' \arrow[r, "h'"] & \Sigma X'
\end{tikzcd}$$

*with both rows distinguished. Then if any two of the maps $j$, $k$, and $\ell$ are isomorphisms, then so is the third.*

*Proof.* Suppose we are given any object $A$ in $\mathcal{C}$, and consider the commutative diagram

$$\begin{tikzcd}
{[A,X]} \arrow[r, "f_*"] \arrow[d, "j_*"] & {[A,Y]} \arrow[r, "g_*"] \arrow[d, "k_*"] & {[A,Z]} \arrow[r, "k_*"] \arrow[d, "\ell_*"] & {[A,\Sigma X]} \arrow[r, "-\Sigma f_*"] \arrow[d, "\Sigma j_*"] & {[A,\Sigma Y]} \arrow[r, "-\Sigma g_*"] \arrow[d, "\Sigma k_*"] & {[A,\Sigma Z]} \arrow[r, "-\Sigma h_*"] \arrow[d, "\Sigma \ell_*"] & {[A,\Sigma^2 X]} \arrow[d, "\Sigma^2 j_*"] \\
{[A,X']} \arrow[r, "f'_*"] & {[A,Y']} \arrow[r, "g'_*"] & {[A,Z']} \arrow[r, "h'_*"] & {[A,\Sigma X']} \arrow[r, "-\Sigma f'_*"] & {[A,\Sigma Y']} \arrow[r, "-\Sigma g'_*"] & {[A,\Sigma Z']} \arrow[r, "-\Sigma h'_*"] & {[A,\Sigma^2 X']}
\end{tikzcd}$$

The rows are exact by Proposition A.3 and repeated applications of axiom TR4. It follows by the five lemma that if $j$ and $k$ are isomorphisms, then $\ell_*$ is an isomorphism. Similarly, if $k$ and $\ell$ are isomorphisms then $\Sigma j_*$ is an isomorphism. Finally, if $\ell$ and $j$ are isomorphisms, then $\Sigma k_*$ is an isomorphism. The desired result follows by faithfullness of $\Sigma$ and the Yoneda embedding. $\square$

**Proposition A.5.** *Given a map $f : X \to Y$ in a triangulated category $(\mathcal{C}, \Sigma, \Omega, \mathcal{D})$, the cofiber sequence of $f$ is unique up to isomorphism, in the sense that given any two distinguished triangles*

$$X \xrightarrow{f} Y \to Z \to \Sigma X \qquad \textit{and} \qquad X \xrightarrow{f} Y \to Z' \to \Sigma X,$$

*there exists an isomorphism $Z \to Z'$ which makes the following diagram commute:*

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, equal] & Y \arrow[r] \arrow[d, equal] & Z \arrow[r] \arrow[d, dashed, "k"] & \Sigma X \arrow[d, equal] \\
X \arrow[r, "f"] & Y \arrow[r] & Z' \arrow[r] & \Sigma X
\end{tikzcd}$$

*Proof.* Suppose we have two distinguished triangles

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X \qquad \text{and} \qquad X \xrightarrow{f} Y \xrightarrow{g'} Z' \xrightarrow{h'} \Sigma X,$$

and consider the following commutative diagram

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, equal] & Y \arrow[r, "g"] \arrow[d, equal] & Z \arrow[r, "h"] & \Sigma X \\
X \arrow[r, "f"] & Y \arrow[r, "g'"] & Z' \arrow[r, "h'"] & \Sigma X
\end{tikzcd}$$

By axiom TR3, there exists some map $k : Z \to Z'$ which makes the following diagram commute:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, equal] & Y \arrow[r, "g"] \arrow[d, equal] & Z \arrow[r, "h"] \arrow[d, "k"] & \Sigma X \arrow[d, equal] \\ X \arrow[r, "f"] & Y \arrow[r, "g'"] & Z' \arrow[r, "h'"] & \Sigma X \end{tikzcd}$$

Now, by Lemma A.4, $k$ is an isomorphism. □

**Proposition A.6.** *Given an arrow $f : X \to Y$ in $\mathcal{C}$, there exists an object $F_f$ called the* fiber *of $f$, and a distinguished triangle*

$$F_f \to X \xrightarrow{f} Y \to \Sigma F_f (\cong C_f).$$

*Proof.* Since $\Sigma$ is an equivalence, there exists some functor $\Omega : \mathcal{C} \to \mathcal{C}$ and natural isomorphisms $\eta : \Omega\Sigma \Rightarrow \mathrm{Id}_{\mathcal{C}}$ and $\varepsilon : \mathrm{Id}_{\mathcal{C}} \Rightarrow \Sigma\Omega$. By axiom TR2, we have a distinguished triangle

$$X \xrightarrow{f} Y \xrightarrow{g} C_f \xrightarrow{h} \Sigma X.$$

Now, consider the commutative diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, equal] & Y \arrow[r, "g"] \arrow[d, equal] & C_f \arrow[r, "h"] \arrow[d, "\eta_{C_f}"] & \Sigma X \arrow[d, equal] \\ X \arrow[r, "f"] & Y \arrow[r, "\widetilde{g}"] & \Sigma\Omega C_f \arrow[r, "\widetilde{h}"] & \Sigma X \end{tikzcd}$$

where $\widetilde{g} = \eta_{C_f} \circ g$, and $\widetilde{h} = \eta_{C_f} \circ h$. Since each vertical map is an isomorphism and the top row is distinguished, the bottom row is also distinguished by axiom TR0. Now, since $\Sigma$ is an equivalence of categories, it is faithful, so that in particular there exists some map $k : \Omega C_f \to X$ such that $\Sigma k = -\widetilde{h} \implies -\Sigma k = \widetilde{h}$. Thus, we have a distinguished triangle of the form

$$X \xrightarrow{f} Y \xrightarrow{\widetilde{g}} \Sigma\Omega C_f \xrightarrow{-\Sigma k} \Sigma X.$$

Finally, by axiom TR4, we get a distinguished triangle

$$\Omega C_f \xrightarrow{k} X \xrightarrow{f} Y \xrightarrow{\widetilde{g}} \Sigma\Omega C_f,$$

so we may define the fiber of $f$ to be $\Omega C_f$. □

For our purposes, we will always be dealing with triangulated categories with a bit of extra structure, in the following sense:

**Definition A.7.** An *adjointly triangulated category* $(\mathcal{C}, \Omega, \Sigma, \eta, \varepsilon, \mathcal{D})$ is the data of a triangulated category $(\mathcal{C}, \Sigma, \mathcal{D})$ along with a functor $\Omega : \mathcal{C} \to \mathcal{C}$ and natural isomorphisms $\eta : \mathrm{Id}_{\mathcal{C}} \Rightarrow \Sigma\Omega$ and $\varepsilon : \Omega\Sigma \Rightarrow \mathrm{Id}_{\mathcal{C}}$ such that $(\Omega, \Sigma, \eta, \varepsilon)$ forms an adjoint equivalence of $\mathcal{C}$. In other words, $\eta$ and $\varepsilon$ are natural isomorphisms which also are the unit and counit of an adjunction $\Omega \dashv \Sigma$, so they satisfy either of the following "zig-zag identities":

$$\begin{tikzcd} \Omega \arrow[r, "\Omega\eta"] \arrow[dr, equal] & \Omega\Sigma\Omega \arrow[d, "\varepsilon\Omega"] \\ & \Omega \end{tikzcd} \qquad \begin{tikzcd} \Sigma\Omega\Sigma \arrow[d, "\Sigma\varepsilon"'] & \Sigma \arrow[l, "\eta\Sigma"'] \arrow[dl, equal] \\ \Sigma & \end{tikzcd}$$

(Satisfying one implies the other is automatically satisfied, see [3, Lemma 3.2]).

From now on, we will assume that $\mathcal{C}$ is an *adjointly* triangulated category with inverse shift $\Omega$, unit $\eta : \mathrm{Id}_{\mathcal{C}} \Rightarrow \Sigma\Omega$, and counit $\varepsilon : \Omega\Sigma \Rightarrow \mathrm{Id}_{\mathcal{C}}$.

**Lemma A.8.** *Given a triangle*

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X,$$

*it can be shifted to the left to obtain a distinguished triangle*

$$\Omega Z \xrightarrow{-\widetilde{h}} X \xrightarrow{f} Y \xrightarrow{\widetilde{\Omega g}} \Sigma\Omega Z,$$

*where* $\widetilde{h} : \Omega Z \to X$ *is the adjoint of* $h : Z \to \Sigma X$ *and* $\widetilde{\Omega g} : Y \to \Sigma\Omega Z$ *is the adjoint of* $\Omega g : \Omega Y \to \Omega Z$*.*

*Proof.* Note that unravelling definitions, then $\widetilde{h}$ and $\widetilde{g}$ are the compositions

$$\widetilde{h} : \Omega Z \xrightarrow{\Omega h} \Omega\Sigma X \xrightarrow{\varepsilon_X} X \qquad \text{and} \qquad \widetilde{\Omega g} : Y \xrightarrow{\eta_Y} \Sigma\Omega Y \xrightarrow{\Sigma\Omega g} \Sigma\Omega Z.$$

Now consider the following diagram:

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, equal] & Y \arrow[r, "g"] \arrow[d, equal] & Z \arrow[r, "h"] \arrow[d, "\eta_Z"'] & \Sigma X \arrow[d, equal] \\
X \arrow[r, "f"] & Y \arrow[r, "\widetilde{\Omega g}"] & \Sigma\Omega Z \arrow[r, "\Sigma\widetilde{h}"] & \Sigma X
\end{tikzcd} \tag{1}$$

The left square commutes by definition. To see that the middle square commutes, expanding definitions, note it is given by the following diagram:

$$\begin{tikzcd}
Y \arrow[rr, "g"] \arrow[d, equal] & & Z \arrow[d, "\eta_Y"] \\
Y \arrow[r, "\eta_Y"] & \Sigma\Omega Y \arrow[r, "\Sigma\Omega g"] & \Sigma\Omega Z
\end{tikzcd}$$

and this commutes by naturality of $\eta$. To see that the right square commutes, consider the following diagram:

$$\begin{tikzcd}
Z \arrow[rr, "h"] \arrow[d, "\eta_Z"'] & & \Sigma X \arrow[d, equal] \arrow[dl, "\eta_{\Sigma X}"'] \\
\Sigma\Omega Z \arrow[r, "\Sigma\Omega h"'] & \Sigma\Omega\Sigma X \arrow[r, "\Sigma\varepsilon_X"'] & \Sigma X
\end{tikzcd}$$

By functoriality of $\Sigma$, the bottom composition is $\Sigma\widetilde{h}$. The left region commutes by naturality of $\eta$. Commutativity of the right region is precisely one of the the zig-zag identities. Hence, since diagram (1) commutes, the vertical arrows are isomorphisms, and the bottom row is distinguished, we have that the top row is distinguished as well by axiom TR0. Then by axiom TR4, since $(f, \widetilde{\Omega g}, \Sigma\widetilde{h})$ is distinguished, so is the triangle

$$\Omega Z \xrightarrow{-\widetilde{h}} X \xrightarrow{f} Y \xrightarrow{\widetilde{\Omega g}} \Sigma\Omega Z. \qquad \square$$

**Lemma A.9.** *Given a distinguished triangle*

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X,$$

*for any* $n > 0$*, the triangle*

$$\Omega^n X \xrightarrow{(-1)^n\Omega^n f} \Omega^n Y \xrightarrow{(-1)^n\Omega^n g} \Omega^n Z \xrightarrow{(-1)^n\Omega^n h} \Omega^n\Sigma X \cong \Sigma\Omega^n X,$$

*is distinguished, where the final isomorphism is given by the composition*

$$\Omega^n\Sigma X = \Omega^{n-1}\Omega\Sigma X \xrightarrow{\Omega^{n-1}\varepsilon_X} \Omega^{n-1} X \xrightarrow{\eta_{\Omega^{n-1}X}} \Sigma\Omega\Omega^{n-1} X = \Sigma\Omega^n X.$$

*Proof.* We give a proof by induction. First we show the case $n = 1$. Note by Lemma A.8, we have that given a distinguished triangle

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X,$$

we can shift it to the left to obtain a distinguished triangle

$$\Omega Z \xrightarrow{-\widetilde{h}} X \xrightarrow{f} Y \xrightarrow{\widetilde{\Omega g}} \Sigma\Omega Z,$$

where $\widetilde{h}$ is the adjoint of $h : Z \to \Sigma X$ and $\widetilde{\Omega g}$ is the adjoint of $\Omega g : \Omega Y \to \Omega Z$. If we apply this shifting operation again, we get the distinguished triangle

$$\Omega Y \xrightarrow{-\widetilde{\widetilde{\Omega g}}} \Omega Z \xrightarrow{-\widetilde{h}} X \xrightarrow{\widetilde{\Omega f}} \Sigma\Omega Y,$$

where unravelling definitions, $\widetilde{\Omega f}$ is the right adjoint of $\Omega f : \Omega X \to \Omega Y$ and $\widetilde{\widetilde{\Omega g}}$ is the right adjoint of $\widetilde{\Omega g}$, which itself is the left adjoint of $\Omega g$, so $\widetilde{\widetilde{\Omega g}} = \Omega g$. Hence we have a distinguished triangle

$$\Omega Y \xrightarrow{-\Omega g} \Omega Z \xrightarrow{-\widetilde{h}} X \xrightarrow{\widetilde{\Omega f}} \Sigma\Omega Y.$$

We may again shift this triangle again and the above arguments yield the distinguished triangle

$$\Omega X \xrightarrow{-\Omega f} \Omega Y \xrightarrow{-\Omega g} \Omega Z \xrightarrow{\widetilde{\Omega(-\widetilde{h})}} \Sigma\Omega X,$$

where $\widetilde{\Omega(-\widetilde{h})}$ is the right adjoint of $\Omega(-\widetilde{h}) = -\Omega\widetilde{h} : \Omega\Omega Z \to \Omega X$. Explicitly unravelling definitions, $\widetilde{\Omega(-\widetilde{h})} = \widetilde{-\Omega\widetilde{h}}$ is the composition

$$\begin{aligned}
[\Omega Z \xrightarrow{\eta_{\Omega Z}} \Sigma\Omega\Omega Z \xrightarrow{\Sigma(-\Omega\widetilde{h})} \Sigma\Omega X] &= -[\Omega Z \xrightarrow{\eta_{\Omega Z}} \Sigma\Omega\Omega Z \xrightarrow{\Sigma\Omega\widetilde{h}} \Sigma\Omega X] \\
&= -[\Omega Z \xrightarrow{\eta_{\Omega Z}} \Sigma\Omega\Omega Z \xrightarrow{\Sigma\Omega\Omega h} \Sigma\Omega\Omega\Sigma X \xrightarrow{\Sigma\Omega\varepsilon_X} \Sigma\Omega X] \\
&= -[\Omega Z \xrightarrow{\Omega h} \Omega\Sigma X \xrightarrow{\varepsilon_X} X \xrightarrow{\eta_X} \Sigma\Omega X],
\end{aligned}$$

where the first equality follows by additivity of $\Sigma$ and additivity of composition, the second follows by further unravelling how $\widetilde{h}$ is defined, and the third follows by naturality of $\eta$, which tells us the following diagram commutes:

$$\begin{tikzcd}
\Omega Z \arrow[r, "\Omega h"] \arrow[d, "\eta_{\Omega Z}"] & \Omega\Sigma X \arrow[r, "\varepsilon_X"] \arrow[d, "\eta_{\Omega\Sigma X}"] & X \arrow[d, "\eta_X"] \\
\Sigma\Omega\Omega Z \arrow[r, "\Sigma\Omega\Omega h"'] & \Sigma\Omega\Omega\Sigma X \arrow[r, "\Sigma\Omega\varepsilon_X"'] & \Sigma\Omega X
\end{tikzcd}$$

Thus indeed we have a distinguished triangle

$$\Omega X \xrightarrow{-\Omega f} \Omega Y \xrightarrow{-\Omega g} \Omega Z \xrightarrow{-\Omega h} \Omega\Sigma X \cong \Sigma\Omega X,$$

where the last isomorphism is $\eta_X \circ \varepsilon_X$, as desired.

Now, we show the inductive step. Suppose we know that given a distinguished triangle

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X,$$

that for some $n > 0$ the triangle

$$\Omega^n X \xrightarrow{(-1)^n\Omega^n f} \Omega^n Y \xrightarrow{(-1)^n\Omega^n g} \Omega^n Z \xrightarrow{(-1)^n h^n} \Sigma\Omega^n X,$$

is distinguished, where $h^n : \Omega^n Z \to \Sigma\Omega^n X$ is the composition

$$\Omega^n Z \xrightarrow{\Omega^n h} \Omega^n\Sigma X \xrightarrow{\Omega^{n-1}\varepsilon_X} \Omega^{n-1} X \xrightarrow{\eta_{\Omega^{n-1}X}} \Sigma\Omega^n X.$$

Then by applying the $n = 1$ case to this triangle, we get that the following triangle is distinguished

$$\Omega^{n+1}X \xrightarrow{-\Omega((-1)^n\Omega^n f)} \Omega^{n+1}Y \xrightarrow{-\Omega((-1)^n\Omega^n g)} \Omega^{n+1}Z \xrightarrow{-\Omega((-1)^n h^n)} \Omega\Sigma\Omega^n X \cong \Sigma\Omega^{n+1}X,$$

where the final isomorphism is the composition

$$\Omega\Sigma\Omega^n X \xrightarrow{\varepsilon_{\Omega^n X}} \Omega^n X \xrightarrow{\eta_{\Omega^n X}} \Sigma\Omega\Omega^n X = \Sigma\Omega^{n+1}X.$$

We claim that this is precisely the distinguished triangle given in the statement of the lemma for $n+1$. First of all, note that $-\Omega((-1)^n\Omega^n f) = (-1)^{n+1}\Omega^{n+1}f$, $-\Omega((-1)^n\Omega^n g) = (-1)^{n+1}\Omega^{n+1}g$, and $-\Omega((-1)^n h^n) = (-1)^{n+1}\Omega h^n$ by additivity of $\Omega$, so that the triangle becomes

$$\Omega^{n+1}X \xrightarrow{(-1)^{n+1}\Omega^{n+1}f} \Omega^{n+1}Y \xrightarrow{(-1)^{n+1}\Omega^{n+1}g} \Omega^{n+1}Z \xrightarrow{(-1)^{n+1}\Omega h^n} \Omega\Sigma\Omega^n X \cong \Sigma\Omega^{n+1}X. \tag{2}$$

Thus, in order to prove the desired characterization, it remains to show this diagram commutes:

$$\begin{tikzcd}
\Omega^{n+1}Z \arrow[r, "(-1)^{n+1}\Omega h^n"] \arrow[d, "(-1)^{n+1}\Omega^{n+1}h"'] & \Omega\Sigma\Omega^n X \arrow[r, "\varepsilon_{\Omega^n X}"] & \Omega^n X \arrow[d, "\eta_{\Omega^n X}"] \\
\Omega^{n+1}\Sigma X \arrow[r, "\Omega^n\varepsilon_X"'] & \Omega^n X \arrow[r, "\eta_{\Omega^n X}"'] & \Sigma\Omega^{n+1}X
\end{tikzcd}$$

(The top composition is the last two arrows in diagram (2), and the bottom composition is the last two arrows in the diagram in the statement of the lemma). Unravelling how $h^n$ is constructed, by additivity of $\Omega$ it further suffices to show the outside of the following diagram commutes:

$$\begin{tikzcd}
\Omega^{n+1}Z \arrow[r, "(-1)^{n+1}\Omega^{n+1}h"] \arrow[dd, "(-1)^{n+1}\Omega^{n+1}h"'] & \Omega^{n+1}\Sigma X \arrow[r, "\Omega^n\varepsilon_X"] & \Omega^n X \arrow[r, "\Omega\eta_{\Omega^{n-1}X}"] \arrow[dd, equal] & \Omega\Sigma\Omega^n X \arrow[d, "\varepsilon_{\Omega^n X}"] \\
& & & \Omega^n X \arrow[d, "\eta_{\Omega^n X}"] \arrow[dl, equal] \\
\Omega^{n+1}\Sigma X \arrow[rr, "\Omega^n\varepsilon_X"'] & & \Omega^n X \arrow[r, "\eta_{\Omega^n X}"'] & \Sigma\Omega^{n+1}X
\end{tikzcd}$$

The left rectangle and bottom right triangle commute by definition. Finally, commutativity of the top right trapezoid is precisely one of the zig-zag identities applied to $\Omega^{n-1}X$. Hence, we have shown the desired result. $\square$

**Proposition A.10.** *Given a distinguished triangle*

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X,$$

*let $\widetilde{h} : \Omega Z \to X$ be the left adjoint of $h$. Then the following infinite sequence is exact:*

$$\begin{tikzcd}
 & & \cdots \arrow[dll] & & \\
\Omega^{n+1}Z \arrow[r, "(-1)^{n+1}\Omega^n\widetilde{h}"'] & \Omega^n X \arrow[r, "(-1)^n\Omega^n f"'] & \Omega^n Y \arrow[r, "(-1)^n\Omega^n g"] & \Omega^n Z \arrow[r, "(-1)^n\Omega^{n-1}\widetilde{h}"] & \Omega^{n-1}X \arrow[dll] \\
 & & \cdots \arrow[dll] & & \\
\Omega Z \arrow[r, "-\widetilde{h}"'] & X \arrow[r, "f"'] & Y \arrow[r, "g"] & Z \arrow[r, "h"] & \Sigma X \arrow[dll] \\
 & & \cdots \arrow[dll] & & \\
\Sigma^{n-1}Z \arrow[r, "(-1)^{n-1}\Sigma^n h"'] & \Sigma^n X \arrow[r, "(-1)^n\Sigma^n f"'] & \Sigma^n Y \arrow[r, "(-1)^n\Sigma^n g"] & \Sigma^n Z \arrow[r, "(-1)^n\Sigma^n h"] & \Sigma^{n+1}X \arrow[dll] \\
 & & \cdots & &
\end{tikzcd}$$

*In particular, it remains exact even if we remove the signs.*

*Proof.* Exactness of

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X \xrightarrow{-\Sigma f} \Sigma Y$$

is Proposition A.3 and axiom TR4. By induction using axiom TR4, for $n > 0$ we get that each contiguous composition of three maps below is a distinguished triangle:

$$\Sigma^n X \xrightarrow{(-1)^n\Sigma^n f} \Sigma^n Y \xrightarrow{(-1)^n\Sigma^n g} \Sigma^n Z \xrightarrow{(-1)^n\Sigma^n h} \Sigma^{n+1}X \xrightarrow{(-1)^{n+1}\Sigma^{n+1} f} \Sigma^{n+1}Y,$$

thus the sequence is exact by Proposition A.3. It remains to show exactness of the LES to the left of $Y$. It suffices to show that the row in the following diagram is exact for all $n > 0$:

$$\begin{tikzcd}
\Omega^n X \arrow[r, "(-1)^n\Omega^n f"] & \Omega^n Y \arrow[r, "(-1)^n\Omega^n g"] & \Omega^n Z \arrow[rr, "(-1)^n\Omega^{n-1}(\varepsilon_X\circ\Omega h)"] \arrow[dr, "(-1)^n\Omega^n h"'] & & \Omega^{n-1}X \arrow[r, "(-1)^{n-1}\Omega^{n-1}f"] & \Omega^{n-1}Y \\
 & & & \Omega^n\Sigma X \arrow[ur, "\Omega^{n-1}\varepsilon_X"'] & &
\end{tikzcd} \tag{3}$$

First of all, to see exactness at $\Omega^n Y$ and $\Omega^n Z$, consider the following commutative diagram:

$$\begin{tikzcd}
\Omega^n X \arrow[r, "(-1)^n\Omega^n f"] \arrow[dd, equal] & \Omega^n Y \arrow[r, "(-1)^n\Omega^n g"] \arrow[dd, equal] & \Omega^n Z \arrow[rr, "(-1)^n\Omega^{n-1}(\varepsilon_X\circ\Omega h)"] \arrow[dd, equal] \arrow[dr, "(-1)^n\Omega^n h"'] & & \Omega^{n-1}X \arrow[dd, "\eta_{\Omega^{n-1}X}"] \\
 & & & \Omega^n\Sigma X \arrow[ur, "\Omega^{n-1}\varepsilon_X"'] & \\
\Omega^n X \arrow[r, "(-1)^n\Omega^n f"] & \Omega^n Y \arrow[r, "(-1)^n\Omega^n g"] & \Omega^n Z \arrow[rr, dashed] \arrow[ur, "(-1)^n\Omega^n h"] & & \Sigma\Omega^n X
\end{tikzcd}$$

(here the dashed arrow is the morphism which makes the diagram commute). The bottom row is distinguished by Lemma A.9. Then by axiom TR0, the top row is distinguished, and thus exact by Proposition A.3. Thus we have shown exactness of (3) at $\Omega^n Y$ and $\Omega^n Z$. It remains to show exactness at $\Omega^{n-1}X$. In the case $n = 1$, we want to show exactness at $X$ in the following diagram:

$$\begin{tikzcd}
\Omega Z \arrow[rr, "-(\varepsilon_X\circ\Omega h)"] \arrow[dr, "-\Omega h"'] & & X \arrow[r, "f"] & Y \\
 & \Omega\Sigma X \arrow[ur, "\varepsilon_X"'] & &
\end{tikzcd}$$

Unravelling definitions, $\varepsilon_X \circ \Omega h$ is precisely the adjoint $\widetilde{h} : \Omega Z \to X$ of $h : Z \to \Sigma X$, in which case we have that the row in the above diagram fits into a distinguished triangle by Lemma A.8, and thus it is exact by Proposition A.3. To see exactness at $\Omega^{n-1}X$ in diagram (3), note that if we apply Lemma A.8 to the sequence Lemma A.9 for $n-1$, then we get that the following composition fits into a distinguished triangle, and is thus exact:

$$\Omega^n Z \xrightarrow{-k} \Omega^{n-1} X \xrightarrow{(-1)^{n-1}\Omega^{n-1} f} \Omega^{n-1} Y,$$

where $k : \Omega(\Omega^{n-1}Z) \to \Omega^{n-1}X$ is the adjoint of the composition

$$\Omega^{n-1} Z \xrightarrow{(-1)^{n-1}\Omega^{n-1} h} \Omega^{n-1}\Sigma X \xrightarrow{\Omega^{n-2}\varepsilon_X} \Omega^{n-2} X \xrightarrow{\eta_{\Omega^{n-2}X}} \Sigma\Omega^{n-1} X.$$

Further expanding how adjoints are constructed, $k$ is the composition

$$\Omega^n Z \xrightarrow{(-1)^{n-1}\Omega^n h} \Omega^n \Sigma X \xrightarrow{\Omega^{n-1}\varepsilon_X} \Omega^{n-1} X \xrightarrow{\Omega\eta_{\Omega^{n-2}X}} \Omega\Sigma\Omega^{n-1} X \xrightarrow{\varepsilon_{\Omega^{n-1}X}} \Omega^{n-1} X.$$

Thus, in order to show exactness of (3) at $\Sigma^{n-1}X$, it suffices to show that $k = (-1)^{n-1}\Omega^{n-1}(\varepsilon_X \circ \Omega h)$. To that end, consider the following diagram:

$$\begin{tikzcd}
\Omega^n Z \arrow[r, "(-1)^{n-1}\Omega^n h"] \arrow[d, "(-1)^{n-1}\Omega^n h"'] & \Omega^n \Sigma X \arrow[r, "\Omega^{n-1}\varepsilon_X"] & \Omega^{n-1} X \arrow[r, "\Omega\eta_{\Omega^{n-2}X}"] \arrow[dr, equal] & \Omega\Sigma\Omega^{n-1} X \arrow[d, "\varepsilon_{\Omega^{n-1}X}"] \\
\Omega^n \Sigma X \arrow[rrr, "\Omega^{n-1}\varepsilon_X"'] & & & \Omega^{n-1} X
\end{tikzcd}$$

The top composition is $k$, while the bottom composition is $(-1)^{n-1}\Omega^{n-1}(\varepsilon_X \circ \Omega h)$. The left region commutes by definition, while commutativity of the right region is precisely one of the zig-zag identities applied to $\Omega^{n-2}X$. Thus, we have shown that $-k = (-1)^n\Omega^{n-1}(\varepsilon_X \circ \Omega h)$, so (3) is exact at $\Omega^{n-1}X$, as desired. $\square$

Also important for our work is the concept of a *tensor triangulated category*, that is, a triangulated symmetric monoidal category in which the triangulated structures are compatible, in the following sense:

**Definition A.11.** A *tensor triangulated category* is a triangulated symmetric monoidal category $(\mathcal{C}, \otimes, S, \Sigma, \mathcal{D})$ such that:

**TT1** For all objects $X$ and $Y$ in $\mathcal{C}$, there are natural isomorphisms

$$e_{X,Y} : \Sigma X \otimes Y \xrightarrow{\cong} \Sigma(X \otimes Y).$$

**TT2** For each object $X$ in $\mathcal{C}$, the functor $X \otimes (-) \cong (-) \otimes X$ is an additive functor.

**TT3** For each object $X$ in $\mathcal{C}$, the functor $X \otimes (-) \cong (-) \otimes X$ preserves distinguished triangles, in that given a distinguished triangle/(co)fiber sequence

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A,$$

then also

$$X \otimes A \xrightarrow{X\otimes f} X \otimes B \xrightarrow{X \otimes g} X \otimes C \xrightarrow{X\otimes h} \Sigma(X \otimes A)$$

and

$$A \otimes X \xrightarrow{f\otimes X} B \otimes X \xrightarrow{g \otimes X} C \otimes X \xrightarrow{h\otimes X} \Sigma(A \otimes X)$$

are distinguished triangles, where here we are being abusive and writing $X \otimes h$ and $h \otimes X$ to denote the compositions

$$X \otimes C \xrightarrow{X\otimes h} X \otimes \Sigma A \xrightarrow{\tau} \Sigma A \otimes X \xrightarrow{e_{A,X}} \Sigma(A \otimes X) \xrightarrow{\Sigma\tau} \Sigma(X \otimes A)$$

and

$$C \otimes X \xrightarrow{h \otimes X} \Sigma A \otimes X \xrightarrow{e_{A,X}} \Sigma(A \otimes X),$$

respectively.

**TT4** Given objects $X$, $Y$, and $Z$ in $\mathcal{C}$, the following diagram must commute:

$$\begin{tikzcd}
(\Sigma X \otimes Y) \otimes Z \arrow[r, "e_{X,Y} \otimes Z"] \arrow[d, "\alpha"'] & \Sigma(X \otimes Y) \otimes Z \arrow[r, "e_{X \otimes Y, Z}"] & \Sigma((X \otimes Y) \otimes Z) \arrow[d, "\Sigma\alpha"] \\
\Sigma X \otimes (Y \otimes Z) \arrow[rr, "e_{X, Y \otimes Z}"'] & & \Sigma(X \otimes (Y \otimes Z))
\end{tikzcd}$$

**TT5** The following diagram must commute

$$\begin{tikzcd}
\Sigma S \otimes \Sigma S \arrow[r, "e_{S,S}"] \arrow[d, "\tau"'] & \Sigma(S \otimes \Sigma S) \arrow[r, "\Sigma\lambda_{\Sigma S}"] & \Sigma^2 S \arrow[d, "-1"] \\
\Sigma S \otimes \Sigma S \arrow[r, "e_{S,S}"] & \Sigma(S \otimes \Sigma S) \arrow[r, "\Sigma\lambda_{\Sigma S}"] & \Sigma^2 S
\end{tikzcd}$$

In other words, $\tau_{\Sigma S, \Sigma S} = -1$.

Usually, most tensor triangulated categories that arise in nature will satisfy additional coherence axioms (see axioms TC1–TC5 in [2]), but the above definition will suffice for our purposes. In what follows, we fix a tensor triangulated category $(\mathcal{C}, \otimes, S, \Sigma, e, \mathcal{D})$.

**Definition A.12.** There are natural isomorphisms

$$e'_{X,Y} : X \otimes \Sigma Y \xrightarrow{\cong} \Sigma(X \otimes Y)$$

obtained via the composition

$$X \otimes \Sigma Y \xrightarrow{\tau} \Sigma Y \otimes X \xrightarrow{e_{Y,X}} \Sigma(Y \otimes X) \xrightarrow{\Sigma\tau} \Sigma(X \otimes Y).$$

**Lemma A.13.** *For all $X$ and $Y$ in $\mathcal{C}$, the following diagram commutes:*

$$\begin{tikzcd}
\Sigma X \otimes \Sigma Y \arrow[r, "-e_{X,\Sigma Y}"] \arrow[d, "e'_{\Sigma X, Y}"'] & \Sigma(X \otimes \Sigma Y) \arrow[d, "\Sigma e'_{X,Y}"] \\
\Sigma(\Sigma X \otimes Y) \arrow[r, "\Sigma e_{X,Y}"'] & \Sigma\Sigma(X \otimes Y)
\end{tikzcd}$$

*(note the sign on the top map).*

*Proof.* Note there are natural isomorphisms

$$a_X : \Sigma X \xrightarrow{\cong} \Sigma S \otimes X$$

given by the composition

$$\Sigma X \xrightarrow{\Sigma\lambda_X} \Sigma(S \otimes X) \xrightarrow{e^{-1}_{S,X}} \Sigma S \otimes X.$$

Furthermore, under the isomorphism $a : \Sigma \cong \Sigma S \otimes -$, $e_{X,Y} : \Sigma X \otimes Y \cong \Sigma(X \otimes Y)$ corresponds to the associator $(\Sigma S \otimes X) \otimes Y \cong \Sigma S \otimes (X \otimes Y)$. Indeed, consider the following diagram:

$$\begin{tikzcd}
\Sigma X \otimes Y \arrow[rr, "e_{X,Y}"] \arrow[d, "\Sigma\lambda_X \otimes Y"'] & & \Sigma(X \otimes Y) \arrow[d, "\Sigma\lambda_{X \otimes Y}"] \arrow[dl, "\Sigma(\lambda_X \otimes Y)"'] \\
\Sigma(S \otimes X) \otimes Y \arrow[r, "e_{S \otimes X, Y}"'] \arrow[d, "e^{-1}_{S,X} \otimes Y"'] & \Sigma((S \otimes X) \otimes Y) \arrow[r, "\Sigma\alpha"'] & \Sigma(S \otimes (X \otimes Y)) \arrow[d, "e^{-1}_{S, X \otimes Y}"] \\
(\Sigma S \otimes X) \otimes Y \arrow[rr, "\alpha"'] & & \Sigma S \otimes (X \otimes Y)
\end{tikzcd}$$

The two vertical composites are $a_X \otimes Y$ and $a_{X\otimes Y}$, respectively. The top trapezoid commutes by naturality of $e$. The triangle commutes by coherence for a monoidal category. Finally, commutativity of the bottom rectangle is axiom TT4 for a tensor triangulated category.

Similarly, under the isomorphism $a : \Sigma \cong \Sigma S \otimes -$, $e'_{X,Y} : X \otimes \Sigma Y \cong \Sigma(X \otimes Y)$ corresponds to the map $X \otimes \Sigma S \otimes Y \xrightarrow{\tau_{X,\Sigma S \otimes Y}} X \otimes Y \otimes \Sigma S$. To see this, consider the following diagram:

$$\begin{tikzcd}
X \otimes \Sigma Y \arrow[r, "\tau_{X,\Sigma Y}"] \arrow[d, "X\otimes\Sigma\lambda_Y"'] & \Sigma Y \otimes X \arrow[r, "e_{Y,X}"] \arrow[d, "\Sigma\lambda_Y\otimes X"'] & \Sigma(Y \otimes X) \arrow[r, "\Sigma\tau_{Y,X}"] \arrow[d, "\Sigma(\lambda_Y\otimes X)"'] & \Sigma(X \otimes Y) \arrow[d, "\Sigma\lambda_{X\otimes Y}"] \\
X \otimes \Sigma(S \otimes Y) \arrow[r, "\tau_{X,\Sigma(S\otimes Y)}"] \arrow[dd, "X\otimes e^{-1}_{S,Y}"'] & \Sigma(S \otimes Y) \otimes X \arrow[r, "e_{S\otimes Y,X}"] & \Sigma(S \otimes Y \otimes X) \arrow[r, "\Sigma(S\otimes\tau_{Y,X})"] & \Sigma(S \otimes X \otimes Y) \arrow[dd, "e^{-1}_{S,X\otimes Y}"] \\
 & \Sigma S \otimes Y \otimes X \arrow[u, "e_{S,Y}\otimes X"] \arrow[ur, "e_{S,Y\otimes X}"'] \arrow[drr, "\Sigma S\otimes\tau_{Y,X}"] & & \\
X \otimes \Sigma S \otimes Y \arrow[ur, "\tau_{X,\Sigma S\otimes Y}"] \arrow[rrr, "\tau_{X,\Sigma S\otimes Y}"] & & & \Sigma S \otimes X \otimes Y
\end{tikzcd}$$

Here we are taking the associators to be isomorphisms, by coherence for monoidal categories. The top horizontal composition is $e'_{X,Y}$, by definition. The vertical edge compositions are $X \otimes a_Y$ and $a_{X\otimes Y}$. The top left rectangle commutes by naturality of $\tau$. The top middle rectangle commutes by naturality of $e$. The top right triangle commutes by naturality of $\tau$. The bottom left trapezoid commutes by naturality of $\tau$. The small middle triangle commutes by axiom TT4 for a tensor triangulated category. The bottom triangle commutes by coherence for a symmetric monoidal category. Finally, the remaining region on the right commutes by naturality of $e$. Thus, in order to show the diagram in the statement of the lemma commutes, it suffices to show the following diagram commutes:

$$\begin{tikzcd}
\Sigma S \otimes X \otimes \Sigma S \otimes Y \arrow[r, "-\alpha"] \arrow[d, "\tau_{\Sigma S\otimes X,\Sigma S\otimes Y}"'] & \Sigma S \otimes X \otimes \Sigma S \otimes Y \arrow[d, "\Sigma S\otimes\tau_{X,\Sigma S\otimes Y}"] \\
\Sigma S \otimes \Sigma S \otimes X \otimes Y \arrow[r, "\alpha"'] & \Sigma S \otimes \Sigma S \otimes X \otimes Y
\end{tikzcd}$$

To see this diagram commutes, consider the following diagram:

$$\begin{tikzcd}
\Sigma S \otimes X \otimes \Sigma S \otimes Y \arrow[r, "\alpha"] \arrow[d, "\tau_{\Sigma S\otimes X,\Sigma S\otimes Y}"'] & \Sigma S \otimes X \otimes \Sigma S \otimes Y \arrow[r, "\Sigma S\otimes\tau_{X,\Sigma S\otimes Y}"] & \Sigma S \otimes \Sigma S \otimes X \otimes Y \arrow[d, "\tau_{\Sigma S,\Sigma S}\otimes X\otimes Y"] \\
\Sigma S \otimes \Sigma S \otimes X \otimes Y \arrow[rr, "\alpha"'] & & \Sigma S \otimes \Sigma S \otimes X \otimes Y
\end{tikzcd}$$

The diagram commutes by coherence for a symmetric monoidal category. The desired result follows by applying axiom TT5 for a tensor triangulated category, which tells us that $\tau_{\Sigma S,\Sigma S} = -1$, and additivity of $- \otimes -$ and composition. $\square$

**Lemma A.14.** *Let $A \xrightarrow{a} B \xrightarrow{b} C \xrightarrow{c} D$ be any sequence isomorphic to a distinguished triangle. Then given any $E$ in $\mathcal{C}$, the sequence*

$$E \otimes A \xrightarrow{E\otimes a} E \otimes B \xrightarrow{E\otimes b} E \otimes C \xrightarrow{E\otimes c} E \otimes D$$

*is exact.*

*Proof.* Suppose we have a commutative diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\alpha"'] & Y \arrow[r, "g"] \arrow[d, "\beta"'] & Z \arrow[r, "h"] \arrow[d, "\gamma"'] & \Sigma X \arrow[d, "\delta"'] \\ A \arrow[r, "a"'] & B \arrow[r, "b"'] & C \arrow[r, "c"'] & D \end{tikzcd}$$

where the top row is distinguished and the vertical arrows are isomorphisms. Then the following diagram commutes by functoriality of $- \otimes -$:

$$\begin{tikzcd} E\otimes X \arrow[r, "E\otimes f"] \arrow[dd, "E\otimes\alpha"'] & E\otimes Y \arrow[r, "E\otimes g"] \arrow[dd, "E\otimes\beta"'] & E\otimes Z \arrow[rr, "E\otimes' h"] \arrow[dd, "E\otimes\gamma"'] \arrow[dr, "E\otimes h"] & & \Sigma(E\otimes X) \arrow[dl, "e'_{E,X}"'] \arrow[dd, dashed, "(E\otimes\delta)\circ e'_{E,X}"] \\ & & & E\otimes\Sigma X \arrow[dr, "E\otimes\delta"] & \\ E\otimes A \arrow[r, "E\otimes a"'] & E\otimes B \arrow[r, "E\otimes b"'] & E\otimes C \arrow[rr, "E\otimes c"'] & & E\otimes D \end{tikzcd}$$

The top triangle is distinguished by axiom TT3 for a tensor triangulated category, thus exact by Proposition A.3, so that the bottom triangle is also exact since the vertical arrows are isomorphisms and each square commutes. □

**Definition A.15.** An *adjointly tensor triangulated* category is a tensor triangulated category $(\mathcal{C}, \otimes, S, \Sigma, e, \mathcal{D})$ along with the structure of an adjointly triangulated category $(\mathcal{C}, \Omega, \Sigma, \eta, \varepsilon, \mathcal{D})$.

From now on, we fix an adjointly tensor triangulated category $(\mathcal{C}, \otimes, S, \Omega, \Sigma, \eta, \varepsilon, e, \mathcal{D})$. First of all, note we may define natural isomorphisms $o_{X,Y} : \Omega X \otimes Y \xrightarrow{\cong} \Omega(X \otimes Y)$ and $o'_{X,Y} : X \otimes \Omega Y \xrightarrow{\cong} \Omega(X \otimes Y)$ as the compositions

$$o_{X,Y} : \Omega X \otimes Y \xrightarrow{\varepsilon^{-1}_{\Omega X \otimes Y}} \Omega\Sigma(\Omega X \otimes Y) \xrightarrow{\Omega e^{-1}_{\Omega X, Y}} \Omega(\Sigma\Omega X \otimes Y) \xrightarrow{\Omega(\eta_X^{-1} \otimes Y)} \Omega(X \otimes Y)$$

and

$$o'_{X,Y} : X \otimes \Omega Y \xrightarrow{\tau_{X,\Omega Y}} \Omega Y \otimes X \xrightarrow{o_{Y,X}} \Omega(Y \otimes X) \xrightarrow{\Omega\tau_{Y,X}} \Omega(X \otimes Y).$$

These are both clearly natural by naturality of $\varepsilon$, $e$, $\eta$, and $\tau$.

**Proposition A.16.** *Suppose we have a distinguished triangle*

$$X \to Y \to Z \to \Sigma X$$

*in $\mathcal{C}$. Then given any object $E$ in $\mathcal{C}$, the long exact sequence given in Proposition A.10 remains exact after applying $E \otimes -$.*

*Proof.* Recall that in the proof of Proposition A.10 we showed that the sequence was exact by showing that any two consecutive maps were isomorphic to a part of a distinguished triangle. Then the desired result follows from Lemma A.14. □

## Appendix B. Spectral sequences

In what follows, we fix an abelian group $A$. We will freely use the theory and results of Appendix C

**Definition B.1.** An *$A$-graded spectral sequence* is the data of a collection of $A$-graded abelian groups $\{E_r^*\}_{r \geq r_0}$ along with homomorphisms of $A$-graded abelian groups $d_r : E_r \to E_r$ (usually of nonzero degree) such that $d_r \circ d_r = 0$ and $E_{r+1} = \ker d_r / \operatorname{im} d_r$.

## Appendix C. (Co)algebra

C.1. **Grading.** First, we develop the theory of things graded by an abelian group. In what follows, we fix an abelian group $A$. We assume the reader is familiar with the basic theory of modules over non-commutative rings.

**Definition C.1.** An *$A$-graded abelian group* is an abelian group $B$ along with a subgroup $B_a \leq B$ for each $a \in A$ such that the canonical map

$$\bigoplus_{a\in A} B_a \to B$$

sending $(x_a)_{a\in A}$ to $\sum_{a\in A} x_a$ is an isomorphism. Given two $A$-graded abelian groups $B$ and $C$, a homomorphism $f : B \to C$ is a *homomorphism of $A$-graded abelian groups* if it preserves the grading, i.e., if it restricts to a map $B_a \to C_a$ for all $a \in A$.

**Remark C.2.** We often will denote an $A$-graded abelian group $B$ by $B_*$. Given some $a \in A$, we can define the shifted $A$-graded abelian group $B_{*+a}$ whose $b^{\text{th}}$ component is $B_{b+a}$.

**Remark C.3.** By the universal property of the coproduct in **Ab**, an $A$-graded homomorphism $\varphi : B \to C$ of $A$-graded abelian groups is precisely the data of an arbitrary collection of homomorphisms $\varphi_a : B_a \to C_a$ indexed by $a \in A$. We will nearly always use this fact without comment.

**Definition C.4.** More generally, given two $A$-graded abelian groups $B$ and $C$ and some $d \in A$, a group homomorphism $f : B \to C$ is *an $A$-graded homomorphism of degree $d$* if it restricts to a map $B_a \to C_{a+d}$ for all $a \in A$. Thus, an $A$-graded homomorphism of degree $d$ from $B$ to $C$ is equivalently an $A$-graded homomorphism $B_* \to C_{*+d}$.

Unless stated otherwise, an "$A$-graded homomorphism" will always refer to an $A$-graded homomorphism of degree 0. It is easy to see that an $A$-graded abelian group $B$ is generated by its *homogeneous* elements, that is, nonzero elements $x \in B$ such that there exists some $a \in A$ with $x \in B_a$.

**Remark C.5.** Clearly the condition that the canonical map $\bigoplus_{a\in A} B_a \to B$ is an isomorphism requires that $B_a \cap B_b = 0$ if $a \neq b$. In particular, given a homogeneous element $x \in B$, there exists precisely one $a \in A$ such that $x \in B_a$. We call this $a$ the *degree* of $x$, and we write $|x| = a$.

**Lemma C.6.** *Given two $A$-graded abelian groups $B$ and $C$, their product $B \oplus C$ is naturally an $A$-graded abelian group by defining*

$$(B \oplus C)_a := \bigoplus_{b+c=a} B_b \oplus C_c.$$

*Proof.* This is entirely straightforward, as

$$B \oplus C \cong \left(\bigoplus_{b\in A} B_b\right) \oplus \left(\bigoplus_{c\in A} C_c\right) \cong \bigoplus_{b,c\in A} B_b \oplus C_c \cong \bigoplus_{a\in A}\bigoplus_{b\in A} B_b \oplus C_{a-b} \cong \bigoplus_{a\in A}\left(\bigoplus_{b+c=a} B_b \oplus C_c\right).$$

□

**Definition C.7.** An $A$-graded ring $R$ is a ring such that is underlying abelian group is $A$-graded, and the multiplication map $R \times R \to R$ is a (degree 0) homomorphism of $A$-graded abelian groups (here $R \times R$ has the structure of an $A$-graded abelian group by Lemma C.6).

**Definition C.8.** An $A$-graded map of $A$-graded rings (resp. left/right $A$-graded $R$-modules) is a homomorphism of rings (resp. left/right $R$-modules) such that the underlying homomorphism of abelian groups is $A$-graded.

Explicitly, given an $A$-graded ring $R$ and homogeneous elements $x, y \in R$, we must have $|xy| = |x| + |y|$. For example, given some field $k$, the ring $R = k[x, y]$ is $\mathbb{Z}^2$-graded, where given $(n, m) \in \mathbb{Z}^2$, $R_{n,m}$ is the subgroup of those monomials of the form $ax^n y^m$ for some $a \in k$. Oftentimes when constructing $A$-graded rings, we do so only by defining the product of homogeneous elements, like so:

**Proposition C.9.** *Given an $A$-graded abelian group $R$, a distinguished element $1 \in R_0$, and $\mathbb{Z}$-bilinear maps $m_{a,b} : R_a \times R_b \to R_{a+b}$ for all $a, b \in A$ such that given $x \in R_a$, $y \in R_b$, and $z \in R_c$,*

$$m_{a+b,c}(m_{a,b}(x, y), z) = m_{a,b+c}(x, m_{b,c}(y, z)) \qquad \text{and} \qquad m_{a,0}(x, 1) = m_{0,a}(1, x) = x,$$

*there exists a unique multiplication map $m : R \times R \to R$ which endows $R$ with the structure of an $A$-graded ring and restricts to $m_{a,b}$ for all $a, b \in A$.*

*Proof.* Given $r, s \in R$, since $R \cong \bigoplus_{a \in A} R_a$, we may uniquely decompose $r$ and $s$ into homogeneous elements as $r = \sum_{a \in A} r_a$ and $s = \sum_{a \in A} s_a$ with each $r_a, s_a \in R_a$ such that only finitely many of the $r_a$'s and $s_a$'s are nonzero. Then in order to define a distributive product $R \times R \to R$ which restricts to $m_{a,b} : R_a \times R_b \to R_{a+b}$, note we *must* define

$$r \cdot s = \left(\sum_{a \in A} r_a\right) \cdot \left(\sum_{b \in A} s_b\right) = \sum_{a,b \in A} r_a \cdot s_b = \sum_{a,b \in A} m_{a,b}(r_a, s_b).$$

Thus, we have shown uniqueness. It remains to show this product actually gives $R$ the structure of a ring. First we claim that the sum on the right is actually finite. Note there exists only finitely many nonzero $r_a$'s and $s_b$'s, and if $s_b = 0$ then

$$m_{a,b}(r_a, 0) = m_{a,b}(r_a, 0 + 0) \overset{(*)}{=} m_{a,b}(r_a, 0) + m_{a,b}(r_a, 0) \implies m_{a,b}(r_a, 0) = 0,$$

where $(*)$ follows from bilinearity of $m_{a,b}$. A similar argument yields that $m_{a,b}(0, s_b) = 0$ for all $a, b \in A$. Hence indeed $m_{a,b}(r_a, s_b)$ is zero for all but finitely many pairs $(a, b) \in A^2$, as desired. Observe that in particular

$$(r \cdot s)_a = \sum_{b+c=a} m_{b,c}(r_b, s_c) = \sum_{b \in A} m_{b,a-b}(r_b, s_{a-b}) = \sum_{c \in A} m_{a-c,c}(r_{a-c}, s_c).$$

Now we claim this multiplication is associative. Given $t = \sum_{a \in A} t_a \in R$, we have

$$\begin{aligned}
(r \cdot s) \cdot t &= \sum_{a,b \in A} m_{a,b}((r \cdot s)_a, t_b) \\
&= \sum_{a,b \in A} m_{a,b}\left(\sum_{c \in A} m_{a-c,c}(r_{a-c}, s_c), t_b\right) \\
&\overset{(1)}{=} \sum_{a,b,c \in A} m_{a,b}(m_{a-c,c}(r_{a-c}, s_c), t_b) \\
&\overset{(2)}{=} \sum_{a,b,c \in A} m_{c,a+b-c}(r_c, m_{a-c,b}(s_{a-c}, t_b)) \\
&\overset{(3)}{=} \sum_{a,b,c \in A} m_{a,c}(r_a, m_{b,c-b}(s_b, t_{c-b})) \\
&\overset{(1)}{=} \sum_{a,c \in A} m_{a,c}\left(r_a, \sum_{b \in A} m_{b,c-b}(s_b, t_{c-b})\right) \\
&= \sum_{a,c \in A} m_{a,c}(r_a, (s \cdot t)_c) = r \cdot (s \cdot t),
\end{aligned}$$

where each occurrence of (1) follows by bilinearity of the $m_{a,b}$'s, each occurrence of (2) is associativity of the $m_{a,b}$'s, and (3) is obtained by re-indexing by re-defining $a := c$, $b := a - c$, and $c := a + b - c$. Next, we wish to show that the distinguished element $1 \in R_0$ is a unit with respect to this multiplication. Indeed, we have

$$1 \cdot r \overset{(1)}{=} \sum_{a \in A} m_{0,a}(1, r_a) \overset{(2)}{=} \sum_{a \in A} r_a = r \qquad \text{and} \qquad r \cdot 1 \overset{(1)}{=} \sum_{a \in A} m_{a,0}(r_a, 1) \overset{(2)}{=} \sum_{a \in A} r_a = r,$$

where (1) follows by the fact that $m_{a,b}(0, -) = m_{a,b}(-, 0) = 0$, which we have shown above, and (2) follows by unitality of the $m_{0,a}$'s and $m_{0,a}$'s, respectively. Finally, we wish to show that this product is distributive. Indeed, we have

$$\begin{aligned} r \cdot (s + t) &= \sum_{a,b \in A} m_{a,b}(r_a, (s + t)_b) \\ &= \sum_{a,b \in A} m_{a,b}(r_a, s_b + t_b) \\ &\overset{(*)}{=} \sum_{a,b \in A} m_{a,b}(r_a, s_b) + \sum_{a,b \in A} m_{a,b}(r_a, t_b) = (r \cdot s) + (r \cdot t), \end{aligned}$$

where $(*)$ follows by bilinearity of $m_{a,b}$. An entirely analagous argument yields that $(r + s) \cdot t = (r \cdot t) + (s \cdot t)$. ☐

When working with $A$-graded rings, we will often freely use the above proposition without comment.

**Definition C.10.** Let $R$ be an $A$-graded ring. A left $A$-graded $R$-module $M$ is a left $R$-module $M$ such that $M$ is an $A$-graded abelian group, and the multiplication map $R \times M \to M$ is a homomorphism of $A$-graded abelian groups (i.e., for all $a, b \in A$ this map must restrict to $R_a \times M_b \to M_{a+b}$). Right $A$-graded $R$-modules are defined similarly. Finally, an $A$-graded $R$-bimodule is an $A$-graded abelian group $M$ along with $A$-graded action maps

$$R \times M \to M \qquad \text{and} \qquad M \times R \to M$$

which endow $M$ with the structure of a left and right $A$-graded $R$-module, respectively, such that given $r, s \in R$ and $m \in M$, $r \cdot (m \cdot s) = (r \cdot m) \cdot s$. Morphisms between $A$-graded $R$-(bi)modules are precisely $A$-graded $R$-(bi)module homomorphisms.

**Proposition C.11.** *Let $R$ be an $A$-graded ring, and suppose we have a right $A$-graded $R$-module $M$ and a left $A$-graded $R$-module $N$. Then the tensor product*

$$M \otimes_R N$$

*is naturally an $A$-graded abelian group by defining $(M \otimes_R N)_a$ to be the subgroup generated by homogeneous pure tensors $m \otimes n$ with $m \in M_b$ and $n \in N_c$ such that $b + c = a$. Furthermore, if either $M$ (resp. $N$) is an $A$-graded bimodule, then $M \otimes_R N$ is naturally a left (resp. right) $A$-graded $R$-module*

*Proof.* By definition, since $M$ and $N$ are $A$-graded abelian groups, they are generated (as abelian groups) by their homogeneous elements. Thus it follows that $M \otimes_R N$ is generated by *homogeneous pure tensors*, that is, elements of the form $m \otimes n$ with $m \in M$ and $n \in N$ homogeneous. Now, given a homogeneous pure tensor $m \otimes n$, we define its *degree* by the formula $|m \otimes n| := |m| + |n|$. It follows this formula is well-defined by checking that given homogeneous elements $m \in M$, $n \in N$, and $r \in R$ that

$$|(m \cdot r) \otimes n| = |m \cdot r| + |n| = |m| + |r| + |n| = |m| + |r \cdot n| = |m \otimes (r \cdot n)|.$$

Thus, we may define $(M \otimes_R N)_a$ to be the subgroup of $M \otimes_R N$ generated by those pure homogeneous tensors of degree $a$. Now, consider the map

$$\Phi : M \times N \to \bigoplus_{a \in A} (M \otimes_R N)_a$$

which takes a pair $(m, n) = \sum_{a \in A} (m_a, n_a)$ to the element $\Phi(m, n)$ whose $a^{\text{th}}$ component is

$$(\Phi(m, n))_a := \sum_{b+c=a} m_b \otimes n_c.$$

It is straightforward to see that this map is $R$-balanced, in the sense that it is additive in each argument and $\Phi(m \cdot r, n) = \Phi(m, r \cdot n)$ for all $m \in M$, $n \in N$, and $r \in R$. Thus by the universal property of $M \otimes_R N$, we get a homomorphism of abelian groups $\widetilde{\Phi} : M \otimes_R N \to \bigoplus_{a \in A} (M \otimes_R N)_a$ lifting $\Phi$ along the canonical map $M \times N \to M \otimes_R N$. Now, also consider the canonical map

$$\Psi : \bigoplus_{a \in A} (M \otimes_R N)_a \to M \otimes_R N.$$

We would like to show $\widetilde{\Phi}$ and $\Psi$ are inverses of eah other. Since $\widetilde{\Phi}$ and $\Psi$ are both homomorphisms, it suffices to show this on generators. Let $m \otimes n$ be a homogeneous pure tensor with $m = m_a \in M_a$ and $n = n_b \in N_b$. Then we have

$$\Psi(\widetilde{\Phi}(m \otimes n)) = \Psi\left(\bigoplus_{a \in A} \sum_{b+c=a} m_b \otimes n_c\right) \overset{(*)}{=} \Psi(m \otimes n) = m \otimes n,$$

and

$$\widetilde{\Phi}(\Psi(m \otimes n)) = \widetilde{\Phi}(m \otimes n) = \bigoplus_{a \in A} \sum_{b+c=a} m_b \otimes n_c \overset{(*)}{=} m \otimes n,$$

where both occurrences of $(*)$ follow by the fact that $m_b \otimes n_c = 0$ unless $b = c = a$, in which case $m_a \otimes n_a = m \otimes n$. Thus since $\Psi$ is an isomorphism, $M \otimes_R N$ is indeed an $A$-graded abelian group, as desired.

Now, suppose that $M$ is an $A$-graded $R$-bimodule, so there exists left and right $A$-graded actions of $R$ on $M$ such that given $r, s \in R$ and $m \in M$ we have $r \cdot (m \cdot s) = (r \cdot m) \cdot s$. Then we would like to show that given a left $A$-graded $R$-module $N$ that $M \otimes_R N$ is canonically a left $A$-graded $R$-module. Indeed, define the action of $R$ on $M \otimes_R N$ on pure tensors by the formula

$$r \cdot (m \otimes n) = (r \cdot m) \otimes n.$$

First of all, clearly this map is $A$-graded, as if $r \in R_a$, $m \in M_b$, and $n \in N_c$ then $(r \cdot m) \otimes n$, by definition, has degree $|r \cdot m| + |n| = |r| + |m| + |n|$ (the last equality follows since the left action of $R$ on $M$ is $A$-graded). In order to show the above map defines a left module structure, it suffices to show that given pure tensors $m \otimes n, m' \otimes n' \in M \otimes_R N$ and elements $r, r' \in R$ that

(1) $r \cdot (m \otimes n + m' \otimes n') = r \cdot (m \otimes n) + r \cdot (m' \otimes n')$,
(2) $(r + r') \cdot (m \otimes n) = r \cdot (m \otimes n) + r' \cdot (m' \otimes n')$,
(3) $(rr') \cdot (m \otimes n) = r \cdot (r' \cdot (m \otimes n))$, and
(4) $1 \cdot (m \otimes n) = m \otimes n$.

Axiom (1) holds by definition. To see (2), note that by the fact that $R$ acts on $M$ on the left that

$$(r + r') \cdot (m \otimes n) = ((r + r') \cdot m) \otimes n = (r \cdot m + r' \cdot m) \otimes n = r \cdot m \otimes n + r' \cdot m \otimes n.$$

That (3) and (4) hold follows similarly by the fact that $(rr') \cdot m = r \cdot (r' \cdot m)$ and $1 \cdot m = m$.

Conversely, if $N$ is an $A$-graded $R$-bimodule, then showing $M \otimes_R N$ is canonically a right $A$-graded $R$-module via the rule

$$(m \otimes n) \cdot r = m \otimes (n \cdot r)$$

is entirely analagous. $\square$

**Lemma C.12.** *Let $R$ be an $A$-graded ring, and suppose we have a right $A$-graded $R$-module $M$ and a left $A$-graded $R$-module $N$. Then given an $A$-graded abelian group $B$ and an $A$-graded $R$-balanced map*

$$\varphi : M \times N \to B$$

*(here $M \times N$ is regarded as an $A$-graded abelian group by Lemma C.6), the lift*

$$\widetilde{\varphi} : M \otimes_R N \to B$$

*determined by the universal property of $M \otimes_R N$ is an $A$-graded map.*

*Proof.* This simply amounts to unravelling definitions. Recall that the subgroup of homogeneous elements of degree $a$ in $M \otimes_R N$ is that generated by pure tensors $m \otimes n$ with $m$ and $n$ homogeneous satisfying $|m| + |n| = a$. Thus, in order to show $\widetilde{\varphi}$ is an $A$-graded homomorphism, it suffices to show that given homogeneous $m \in M$ and $n \in N$, we have

$$|\widetilde{\varphi}(m \otimes n)| = |m \otimes n| = |m| + |n|.$$

Indeed, given two such elements, consider the following diagram

$$\begin{tikzcd} M \otimes_R N \arrow[dr, "\widetilde{\varphi}"] & \\ M \times N \arrow[u] \arrow[r, "\varphi"'] & B \end{tikzcd}$$

This diagram commutes by universal property of $- \otimes_R -$. Note that the element $m \otimes n$ is mapped to by the pair $(m, n)$ along the left vertical map. Hence by commutativity, we necessarily have

$$|\widetilde{\varphi}(m \otimes n)| = |\varphi(m, n)| \overset{(*)}{=} |(m, n)| = |m| + |n|,$$

where $(*)$ follows by the fact that $\varphi$ is an $A$-graded map. ∎

## Appendix D. Monoid objects

**Definition D.1.** Let $(\mathcal{C}, \otimes, S)$ be a symmetric monoidal category with left unitor, right unitor, and associator, and symmetry isomorphism $\lambda$, $\rho$, $\alpha$, and $\tau$, respectively. Then a *monoid object* $(E, \mu, e)$ is an object $E$ in $\mathcal{C}$ along with a multiplication map $\mu : E \otimes E \to E$ and a unit map $e : S \to E$ such that the following diagram commutes:

$$\begin{tikzcd} E \otimes S \arrow[r, "E \otimes e"] \arrow[dr, "\rho"'] & E \otimes E \arrow[d, "\mu"] & S \otimes E \arrow[l, "e \otimes E"'] \arrow[dl, "\lambda"] \\ & E & \end{tikzcd} \qquad \begin{tikzcd} (E \otimes E) \otimes E \arrow[rr, "\mu \otimes E"] \arrow[d, "\alpha"'] & & E \otimes E \arrow[d, "\mu"] \\ E \otimes (E \otimes E) \arrow[r, "E \otimes \mu"] & E \otimes E \arrow[r, "\mu"] & E \end{tikzcd}$$

The first diagram expresses unitality, while the second expressed associativity. If in addition the following diagram commutes,

$$\begin{tikzcd} E \otimes E \arrow[rr, "\tau"] \arrow[dr, "\mu"'] & & E \otimes E \arrow[dl, "\mu"] \\ & E & \end{tikzcd}$$

then we say $(E, \mu, e)$ is a *commutative* monoid object.

From now on we fix a monoidal closed tensor triangulated category $(\mathcal{SH}, \otimes, S, \Sigma, e, \mathcal{D})$ (Definition A.11) with arbitrary (small) (co)products and sub-Picard grading $(A, \mathbf{1}, h, \{S^a\}, \nu, \{\phi_{a,b}\})$ (Definition 2.2), and we adopt the conventions outlined in Section 2.1. In all proofs that follow we will freely use the coherence theorem for symmetric monoidal categories. In particular, we will assume without loss of generality that the associators and unitors in $\mathcal{SH}$ are identities.

**Proposition D.2.** *Let $(E, \mu, e)$ be a commutative monoid object in $\mathcal{SH}$, and consider the multiplication map $\pi_*(E) \times \pi_*(E) \to \pi_*(E)$ which sends classes $x : S^a \to E$ and $y : S^b \to E$ to the composition*

$$S^{a+b} \xrightarrow{\phi_{a,b}} S^a \otimes S^b \xrightarrow{x \otimes y} E \otimes E \xrightarrow{\mu} E.$$

*Then this endows $\pi_*(E)$ with the structure of an $A$-graded ring with unit $e \in \pi_0(E) = [S, E]$.*

*Proof.* Here we are using Proposition C.9, so it suffices to show the given assignment is associative and unital w.r.t. homogeneous elements. Suppose we have classes $x$, $y$, and $z$ in $\pi_a(E)$, $\pi_b(E)$, and $\pi_c(E)$, respectively. To see associativity, consider the following diagram:

$$\begin{tikzcd}
 & & & E \otimes E \arrow[d, "\mu"] \\
S^{a+b+c} \arrow[r, "\cong"] & S^a \otimes S^b \otimes S^c \arrow[r, "x \otimes y \otimes z"] & E \otimes E \otimes E \arrow[ur, "\mu \otimes E"] \arrow[dr, "E \otimes \mu"'] & E \\
 & & & E \otimes E \arrow[u, "\mu"']
\end{tikzcd}$$

(here the first arrow is the unique isomorphism obtained by composing products of $\phi_{a,b}$'s, see Remark 2.3). It commutes by associativity of $\mu$. It follows by functoriality of $- \otimes -$ that the top composition is $(x \cdot y) \cdot z$ while the bottom is $x \cdot (y \cdot z)$, so they are equal as desired. To see that $e \in \pi_0(E)$ is a left and right unit for this multiplication, consider the following diagram

$$\begin{tikzcd}
 & S^a \arrow[dl, "e \otimes x"'] \arrow[d, "x"] \arrow[dr, "x \otimes e"] & \\
E \otimes E \arrow[dr, "\mu"'] & E \arrow[l, "e \otimes E"'] \arrow[r, "E \otimes e"] \arrow[d, equal] & E \otimes E \arrow[dl, "\mu"] \\
 & E &
\end{tikzcd}$$

Commutativity of the two top triangles is functoriality of $- \otimes -$. Commutativity of the bottom two triangles is unitality of $\mu$. Thus the diagram commutes, so $e \cdot x = x \cdot e$. Finally, to see this product is bilinear (distributive). Suppose we further have some $x' \in \pi_a(E)$ and $y' \in \pi_b(E)$, and consider the following diagrams:

$$\begin{tikzcd}
S^{a+b} \arrow[r, "\phi_{a,b}"] \arrow[d, "\Delta"'] & S^a \otimes S^b \arrow[r, "\Delta \otimes S^b"] \arrow[d, "\Delta"] & (S^a \oplus S^a) \otimes S^b \arrow[r, "(x \oplus x') \otimes y"] \arrow[dl, "\cong"'] & (E \oplus E) \otimes E \arrow[d, "\nabla \otimes E"] \arrow[dl, "\cong"'] & \\
S^{a+b} \oplus S^{a+b} \arrow[r, "\phi_{a,b} \oplus \phi_{a,b}"'] & (S^a \otimes S^b) \oplus (S^a \otimes S^b) \arrow[r, "(x \otimes y) \oplus (x' \otimes y)"'] & (E \otimes E) \oplus (E \otimes E) \arrow[r, "\nabla"'] & E \otimes E \arrow[r, "\mu"] & E
\end{tikzcd}$$

$$\begin{tikzcd}
S^{a+b} \arrow[r, "\phi_{a,b}"] \arrow[d, "\Delta"'] & S^a \otimes S^b \arrow[r, "S^a \otimes \Delta"] \arrow[d, "\Delta"] & S^b \otimes (S^b \oplus S^b) \arrow[r, "x \otimes (y \oplus y')"] \arrow[dl, "\cong"'] & E \otimes (E \oplus E) \arrow[d, "E \otimes \nabla"] \arrow[dl, "\cong"'] & \\
S^{a+b} \oplus S^{a+b} \arrow[r, "\phi_{a,b} \oplus \phi_{a,b}"'] & (S^a \otimes S^b) \oplus (S^a \otimes S^b) \arrow[r, "(x \otimes y) \oplus (x \otimes y')"'] & (E \otimes E) \oplus (E \otimes E) \arrow[r, "\nabla"'] & E \otimes E \arrow[r, "\mu"] & E
\end{tikzcd}$$

The unlabeled isomorphisms are those given by the fact that $- \otimes -$ is additive in each variable (since $\mathcal{SH}$ is tensor triangulated). Commutativity of the left squares is naturality of $\Delta : X \to X \oplus X$ in an additive category. Commutativity of the rest of the diagram follows again from the fact that $- \otimes -$ is an additive functor in each variable. Hence, by functoriality of $- \otimes -$, these diagrams tell us that $(x + x') \cdot y = x \cdot y + x' \cdot y$ and $x \cdot (y + y') = x \cdot y + x \cdot y'$, respectively. $\square$

**Proposition D.3.** *For all $a, b \in A$ there exists an element $\theta_{a,b} \in \pi_0(S) = [S, S]$ (determined by choice of coherent family $\{\phi_{a,b}\}$) such that given any commutative monoid object $(E, \mu, e)$ in $\mathcal{SH}$, the $A$-graded ring structure on $\pi_*(E)$ (**??**) has a commutativity formula given by*

$$x \cdot y = y \cdot x \cdot (e \circ \theta_{a,b})$$

*for all $x \in \pi_a(E)$ and $y \in \pi_b(E)$. In particular, $\theta_{a,b} \in \mathrm{Aut}(S)$ is the composition*

$$S \xrightarrow{\cong} S^{-a-b} \otimes S^a \otimes S^b \xrightarrow{S^{-a-b}\otimes\tau} S^{-a-b} \otimes S^b \otimes S^a \xrightarrow{\cong} S,$$

*where the outermost maps are the unique maps specified by Remark 2.3.*

*Proof.* Let $(E, \mu, e)$, $x$, and $y$ as in the statement of the proposition. Now consider the following diagram

$$\begin{tikzcd}
S^{a+b} \arrow[r, "\phi_{a,b}"] \arrow[d, dashed, "\phi_{b,a}^{-1}\circ\tau\circ\phi_{a,b}"'] & S^a \otimes S^b \arrow[r, "x\otimes y"] \arrow[d, "\tau"'] & E \otimes E \arrow[d, "\tau"'] \arrow[dr, "\mu"] & \\
S^{a+b} \arrow[r, "\phi_{b,a}"] & S^b \otimes S^a \arrow[r, "y\otimes x"] & E \otimes E \arrow[r, "\mu"'] & E
\end{tikzcd}$$

The left square commutes by definition. The middle square commutes by naturality of the symmetry isomorphism. Finally, the right square commutes by commutativity of $E$. Unravelling definitions, we have shown that under the product on $\pi_*(E)$ induced by the $\phi_{a,b}$'s,

$$x \cdot y = (y \cdot x) \circ (\phi_{b,a}^{-1} \circ \tau \circ \phi_{a,b}).$$

Thus, in order to show the desired result it further suffices to show that

$$(y \cdot x) \circ (\phi_{b,a}^{-1} \circ \tau \circ \phi_{a,b}) = y \cdot x \cdot (e \circ \theta_{a,b}).$$

Consider the following diagram:

$$\begin{tikzcd}
S^{a+b} \arrow[rr, "\phi_{a,b}"] \arrow[d, "\cong"'] & & S^a \otimes S^b \arrow[d, "\tau"] \\
S^b \otimes S^a \otimes S^{-a-b} \otimes S^a \otimes S^b \arrow[urr, "\cong"'] \arrow[d, "S^b\otimes S^a\otimes S^{-a-b}\otimes\tau"'] & & S^b \otimes S^a \arrow[d, "\phi_{b,a}^{-1}"] \arrow[dl, equal] \\
S^b \otimes S^a \otimes S^{-a-b} \otimes S^b \otimes S^a \arrow[r, "\cong"] \arrow[urr, "\cong"] & S^b \otimes S^a \arrow[dr, "y\otimes x"] \arrow[ddl, "y\otimes x\otimes e"'] & S^{a+b} \arrow[l, "\phi_{b,a}"'] \\
 & & E \otimes E \arrow[dll, "E\otimes E\otimes e"'] \arrow[d, equal] \\
E \otimes E \otimes E \arrow[rr, "E\otimes\mu"] \arrow[d, "\mu\otimes E"'] & & E \otimes E \arrow[d, "\mu"] \\
E \otimes E \arrow[rr, "\mu"'] & & E
\end{tikzcd}$$

Here any map simply labelled $\cong$ is an appropriate composition of copies of $\phi_{a,b}$'s, associators, and their inverses, so that each of these maps are necessarily unique by Remark 2.3. The triangles in the top large rectangle commutes by coherence for the $\phi_{a,b}$'s. The parallelogram commutes by naturality of $\tau$ and coherence of the of $\phi_{a,b}$'s. The middle skewed triangle commutes by functoriality of $-\otimes-$. The triangle below that commutes by unitality of $\mu$. Finally, the bottom rectangle commmutes by associativity of $\mu$. Hence, by unravelling definitions and applying functoriality of $-\otimes-$, we get that the right composition is $(y \cdot x) \circ (\phi_{b,a}^{-1} \circ \tau \circ \phi_{a,b})$, while the left composition is $y \cdot x \cdot (e \circ \theta_{a,b})$, so they are equal as desired. $\square$

**Proposition D.4.** *Given $a \in A$, we have $\theta_{0,a} = \theta_{a,0} = \mathrm{id}_S$.*

*Proof.* Recall $\theta_{a,0}$ is the composition

$$S \xrightarrow{\phi_{-a,a}} S^{-a} \otimes S^a \xrightarrow{S^{-a}\otimes\phi_{a,0}} S^{-a} \otimes (S^a \otimes S) \xrightarrow{S^{-a}\otimes\tau} S^{-a} \otimes (S \otimes S^a) \xrightarrow{S^{-a}\otimes\phi_{0,a}^{-1}} S^{-a} \otimes S^a \xrightarrow{\phi_{-a,a}^{-1}} S$$

By the coherence theorem for symmetric monoidal categories and the fact that $\phi_{a,0}$ and $\phi_{0,a}$ coincide with the unitors, we have that the composition

$$S^a \xrightarrow{\phi_{a,0}=\rho_{S^a}^{-1}} S^a \otimes S \xrightarrow{\tau} S \otimes S^a \xrightarrow{\phi_{0,a}^{-1}=\lambda_{S^a}} S^a$$

is precisely the identity map, so by functoriality of $- \otimes -$, we have that $\theta_{a,0}$ is the composition

$$S \xrightarrow{\phi_{-a,a}} S^{-a} \otimes S^a \xrightarrow{=} S^{-a} \otimes S^a \xrightarrow{\phi_{-a,a}^{-1}} S,$$

so $\theta_{a,0} = \mathrm{id}_S$, meaning

$$x \cdot y = y \cdot x \cdot (e \circ \theta_{a,0}) = y \cdot x \cdot e = y \cdot x,$$

where the last equality follows by the fact that $e$ is the unit for the multiplication on $\pi_*(E)$. An entirely analagous argument yields that $\theta_{0,a} = \mathrm{id}_S$. $\square$

**Proposition D.5.** *Let $X$ and $Y$ be objects in $\mathcal{SH}$. Then the $A$-graded pairing*

$$\pi_*(X) \times \pi_*(Y) \to \pi_*(X \otimes Y)$$

*sending $x : S^a \to X$ and $y : S^b \to Y$ to the composition*

$$S^{a+b} \xrightarrow{\phi_{a,b}} S^a \otimes S^b \xrightarrow{x\otimes y} X \otimes Y$$

*is additive in each argument.*

*Proof.* Let $a, b \in A$, and let $x_1, x_2 : S^a \to X$ and $y : S^b \to Y$. Then consider the following diagram

$$\begin{tikzcd}
S^{a+b} \arrow[r, "\cong"] & S^a \otimes S^b \arrow[r, "\Delta\otimes S^b"] \arrow[d, "\Delta"'] & (S^a \oplus S^a) \otimes S^b \arrow[d, "(x_1\oplus x_2)\otimes y"] \arrow[dl, "\cong"'] \\
& (S^a \otimes S^b) \oplus (S^a \otimes S^b) \arrow[d, "(x_1\otimes y)\oplus(x_2\otimes y)"'] & (X \oplus X) \otimes Y \arrow[d, "\nabla\otimes Y"] \arrow[dl, "\cong"'] \\
& (X \otimes Y) \oplus (X \otimes Y) \arrow[r, "\nabla"] & X \otimes Y
\end{tikzcd}$$

The isomorphisms are given by the fact that $- \otimes -$ is additive in each variable. Both triangles and the parallelogram commute since $- \otimes -$ is additive. By functoriality of $- \otimes -$, the top composition is $(x_1 + x_2) \cdot y$ and the bottom composition is $x_1 \cdot y + x_2 \cdot y$, so they are equal, as desired. An entirely analagous argument yields that $x \cdot (y_1 + y_2) = x \cdot y_1 + x \cdot y_2$ for $x \in \pi_*(X)$ and $y_1, y_2 \in \pi_*(Y)$. $\square$

**Proposition D.6** ([4, Proposition 5.11])**.** *Let $(E, \mu, e)$ be a monoid object in $\mathcal{SH}$. Then $E_*(-)$ is a functor from $\mathcal{SH}$ to left $A$-graded $\pi_*(E)$-modules, where given some $X$ in $\mathcal{SH}$, $E_*(X)$ may be endowed with the structure of a left $A$-graded $\pi_*(E)$-module via the map*

$$\pi_*(E) \times E_*(X) \to E_*(X)$$

*which given $a, b \in A$, sends $x : S^a \to E$ and $y : S^b \to E \otimes X$ to the composition*

$$x \cdot y : S^{a+b} \cong S^a \otimes S^b \xrightarrow{x\otimes y} E \otimes (E \otimes X) \cong (E \otimes E) \otimes X \xrightarrow{\mu\otimes X} E \otimes X.$$

*Similarly, the assignment $X \mapsto X_*(E)$ is a functor from $\mathcal{SH}$ to right $A$-graded $\pi_*(E)$-modules, where the structure map*

$$X_*(E) \times \pi_*(E) \to X_*(E)$$

*sends $x : S^a \to X \otimes E$ and $y : S^b \to E$ to the composition*

$$x \cdot y : S^{a+b} \cong S^a \otimes S^b \xrightarrow{x \otimes y} (X \otimes E) \otimes E \cong X \otimes (E \otimes E) \xrightarrow{X \otimes \mu} X \otimes E.$$

*Finally, $E_*(E)$ is a $\pi_*(E)$-bimodule, in the sense that the left and right actions of $\pi_*(E)$ are compatible, so that given $y, z \in \pi_*(E)$ and $x \in E_*(E)$, $y \cdot (x \cdot z) = (y \cdot x) \cdot z$.*

*Proof.* First we show that the map $\pi_*(E) \times E_*(X) \to E_*(X)$ endows $E_*(X)$ with the structure of a left $\pi_*(E)$-module. Let $a, b, c \in A$ and $x, x' : S^a \to E \otimes X$, $y : S^b \to E$, and $z, z' \in S^c \to E$. Then we wish to show that:

(1) $y \cdot (x + x') = y \cdot x + y \cdot x'$,
(2) $(z + z') \cdot x = z \cdot x + z' \cdot x$,
(3) $(zy) \cdot x = z \cdot (y \cdot x)$,
(4) $e \cdot x = x$.

Axioms (1) and (2) follow by the fact that $E_*(X) = \pi_*(E \otimes X)$ and Proposition D.5. To see (3), consider the diagram:

$$\begin{tikzcd}
 & & & & E \otimes E \otimes X \arrow[d, "\mu \otimes X"] \\
S^{a+b+c} \arrow[r, "\cong"] & S^c \otimes S^b \otimes S^a \arrow[r, "z \otimes y \otimes x"] & E \otimes E \otimes E \otimes X \arrow[urr, "E \otimes \mu \otimes X"] \arrow[drr, "\mu \otimes E \otimes X"'] & & E \otimes X \\
 & & & & E \otimes E \otimes X \arrow[u, "\mu \otimes X"']
\end{tikzcd}$$

It commutes by associativity of $\mu$. By functoriality of $- \otimes -$, the two outside compositions equal $z \cdot (y \cdot x)$ on the top and $(z \cdot y) \cdot x$ on the bottom. Hence, they are equal, as desired.

Next, to see (4), consider the following diagram:

$$\begin{tikzcd}
S^a \arrow[rr, "x"] \arrow[dr, "x"'] \arrow[ddr, bend right, "e \otimes x"'] & & E \otimes X \\
 & E \otimes X \arrow[ur, equal] \arrow[d, "e \otimes E \otimes X"] & \\
 & E \otimes E \otimes X \arrow[uur, bend right, "\mu \otimes X"'] &
\end{tikzcd}$$

The top triangle commutes by definition. The left triangle commutes by functoriality of $- \otimes -$. The right triangle commutes by unitality of $\mu$. The top composition is $x$ while the bottom is $e \cdot x$, thus they are necessarily equal since the diagram commutes.

Thus, we have shown that the indicated map does indeed endow $E_*(X)$ with the structure of a left $\pi_*(E)$-module. It remains to show that $E_*(-)$ sends maps in $\mathcal{SH}$ to $A$-graded homomorphisms of left $A$-graded $\pi_*(E)$-modules. By definition, given $f : X \to Y$ in $\mathcal{SH}$, $E_*(f)$ is the map which takes a class $x : S^a \to E \otimes X$ to the composition

$$S^a \xrightarrow{x} E \otimes X \xrightarrow{E \otimes f} E \otimes Y.$$

To see this assignment is a homomorphism, suppose we are given some other $x' : S^a \to E \otimes X$ and some scalar $y : S^b \to E$. Then we would like to show $E_*(f)(x + x') = E_*(f)(x) + E_*(f)(x')$

and $E_*(f)(y \cdot x) = y \cdot E_*(f)(x)$. To see the former, consider the following diagram:

$$\begin{tikzcd}
 & & & (E\otimes Y)\oplus(E\otimes Y) \arrow[d, "\nabla"] \\
S^a \arrow[r, "\Delta"] & S^a \oplus S^a \arrow[r, "x\oplus x'"] & (E\otimes X)\oplus(E\otimes X) \arrow[ur, "(E\otimes f)\oplus(E\otimes f)"] \arrow[dr, "\nabla"'] & E\otimes Y \\
 & & & E\otimes X \arrow[u, "E\otimes f"']
\end{tikzcd}$$

It commutes by naturality of $\nabla$ in an additive category. The top composition is $E_*(f)(x) + E_*(f)(x')$, while the bottom is $E_*(f)(x+x')$, so they are equal as desired. To see that $E_*(f)(y \cdot x) = y \cdot E_*(f)(x)$, consider the following diagram:

$$\begin{tikzcd}
S^{a+b} \arrow[r, "\phi_{b,a}"] & S^b \otimes S^a \arrow[r, "y\otimes x"] & E\otimes E\otimes X \arrow[r, "E\otimes E\otimes f"] \arrow[d, "\mu\otimes X"'] & E\otimes E\otimes Y \arrow[d, "\mu\otimes Y"] \\
 & & E\otimes X \arrow[r, "E\otimes f"] & E\otimes Y
\end{tikzcd}$$

It commutes by functoriality of $- \otimes -$. The bottom composition is $E_*(f)(y \cdot x)$, while the top composition is $y \cdot E_*(f)(x)$, so they are equal, as desired.

Showing that $X_*(E)$ has the structure of a right $\pi_*(E)$-module and that if $f : X \to Y$ is a morphism in $\mathcal{SH}$ then the map

$$X_*(E) = [S^*, X \otimes E] \xrightarrow{(f\otimes E)_*} [S^*, Y \otimes E] = Y_*(E)$$

is an $A$-graded homomorphism of right $A$-graded $\pi_*(E)$-modules is entirely analagous.

It remains to show that $E_*(E)$ is a $\pi_*(E)$-bimodule. Let $x : S^a \to E$, $y : S^b \to E \otimes E$, and $z : S^c \to E$, and consider the following diagram:

$$\begin{tikzcd}
 & & & E\otimes E\otimes E \arrow[d, "E\otimes\mu"] \\
S^{a+b+c} \arrow[r, "\cong"] & S^a\otimes S^b\otimes S^c \arrow[r, "x\otimes y\otimes z"] & E\otimes E\otimes E\otimes E \arrow[ur, "\mu\otimes E\otimes E"] \arrow[r, "\mu\otimes\mu"] \arrow[dr, "E\otimes E\otimes\mu"'] & E\otimes E \\
 & & & E\otimes E\otimes E \arrow[u, "\mu\otimes E"']
\end{tikzcd}$$

Commutativity follows by functoriality of $- \otimes -$, which also tells us that the two outside compositions are $(x \cdot y) \cdot z$ (on top) and $x \cdot (y \cdot z)$ (on bottom). Hence they are equal, as desired. $\square$

**Proposition D.7** ([5, Proposition 2.2]). *Let $(E, \mu, e)$ be a monoid object in $\mathcal{SH}$ and let $X$ be any object. Then the assignment*

$$E_*(E) \times E_*(X) \to E_*(E \otimes X)$$

*which sends $x : S^a \to E \otimes E$ and $y : S^b \to E \otimes X$ to the composition*

$$x \cdot y : S^{a+b} \cong S^a \otimes S^b \xrightarrow{x\otimes y} E \otimes E \otimes E \otimes X \xrightarrow{E\otimes\mu\otimes X} E \otimes E \otimes X$$

*lifts to an $A$-graded homomorphism of left $A$-graded $\pi_*(E)$-modules*

$$\Phi_X : E_*(E) \otimes_{\pi_*(E)} E_*(X) \to E_*(E \otimes X)$$

*(where here $E_*(E)$ has a $\pi_*(E)$-bimodule structure and $E_*(X)$ has a left $\pi_*(E)$-module structure as specified by Proposition D.6, so $E_*(E) \otimes_{\pi_*(E)} E_*(X)$ is a left $A$-graded $\pi_*(E)$-module by Proposition C.11). Furthermore, this homomorphism is natural in $X$.*

*Proof.* First, recall by definition of the tensor product, in order to show the assignment $E_*(E) \times E_*(X) \to E_*(E \otimes X)$ induces a homomorphism $E_*(E) \otimes_{\pi_*(E)} E_*(X) \to E_*(E \otimes X)$ of $A$-graded abelian groups, it suffices to show that the assignment is $\pi_*(E)$-balanced, i.e., that it is linear in each argument and satisfies $xr \cdot y = x \cdot ry$ for $x \in E_*(E)$, $y \in E_*(X)$, and $r \in \pi_*(E)$.

First, note that by the identifications $E_*(E) = \pi_*(E \otimes E)$, $E_*(X) = \pi_*(E \otimes X)$, and $E_*(E \otimes X) = \pi_*(E \otimes E \otimes X)$, and Proposition D.5, it is straightforward to see that the assignment commutes with addition of maps in each argument. Now, let $a, b, c \in A$, $x : S^a \to E \otimes E$, $y : S^b \to E \otimes X$, and $z : S^c \to E$. Then we wish to show $xz \cdot y = x \cdot zy$. Consider the following diagram (where here we are passing to a permutative category):

$$\begin{tikzcd}
 & & & E \otimes E \otimes E \otimes X \arrow[d, "E\otimes\mu\otimes X"] \\
S^{a+b+c} \arrow[r, "\cong"] & S^a \otimes S^c \otimes S^b \arrow[r, "x\otimes z\otimes y"] & E \otimes E \otimes E \otimes E \otimes X \arrow[ur, "E\otimes\mu\otimes E\otimes X"] \arrow[dr, "E\otimes E\otimes\mu\otimes X"'] & E \otimes E \otimes X \\
 & & & E \otimes E \otimes E \otimes X \arrow[u, "E\otimes\mu\otimes X"']
\end{tikzcd}$$

It commutes by associativity of $\mu$. By functoriality of $-\otimes-$, the top composition is given by $(xz)\cdot y$ and the bottom composition is $x \cdot (zy)$, so we have they are equal, as desired. Thus, since the map $E_*(E) \times E_*(X) \to E_*(E \otimes X)$ is $\pi_*(E)$-balanced, we have that it induces a homomorphism of abelian groups. Furthermore, by Lemma C.12 it is $A$-graded.

In order to see this map is a homomorphism of left $\pi_*(E)$-modules, we must show that $z(x\cdot y) = zx \cdot y$, where $x$, $y$, and $z$ are defined as above. Now consider the following diagram:

$$\begin{tikzcd}
 & & & E \otimes E \otimes E \otimes X \arrow[d, "E\otimes\mu\otimes X"] \\
S^{a+b+c} \arrow[r, "\cong"] & S^c \otimes S^a \otimes S^b \arrow[r, "z\otimes x\otimes y"] & E \otimes E \otimes E \otimes E \otimes X \arrow[ur, "\mu\otimes E\otimes E\otimes X"] \arrow[dr, "E\otimes E\otimes\mu\otimes X"'] \arrow[r, "\mu\otimes\mu\otimes X"] & E \otimes E \otimes X \\
 & & & E \otimes E \otimes E \otimes X \arrow[u, "\mu\otimes E\otimes X"']
\end{tikzcd}$$

Commutativity of the triangles is functoriality of $- \otimes -$. By functoriality of $- \otimes -$, the top composition is $zx \cdot y$, and the bottom composition is $z(x \cdot y)$. Hence they are equal, as desired, so that the map we have constructed

$$E_*(E) \otimes_{\pi_*(E)} E_*(X) \to E_*(E \otimes X)$$

is indeed an $A$-graded homomorphism of left $A$-graded $\pi_*(E)$-modules.

Next, we would like to show that this homomorphism is natural in $X$. Let $f : X \to Y$ in $\mathcal{SH}$. Then we would like to show the following diagram commutes:

$$\begin{tikzcd}
E_*(E) \otimes_{\pi_*(E)} E_*(X) \arrow[r, "\Phi_X"] \arrow[d, "E_*(E)\otimes_{\pi_*(E)}E_*(f)"'] & E_*(E \otimes X) \arrow[d, "E_*(E\otimes f)"] \\
E_*(E) \otimes_{\pi_*(E)} E_*(Y) \arrow[r, "\Phi_Y"] & E_*(E \otimes Y)
\end{tikzcd} \tag{4}$$

As all the maps here are homomorphisms, it suffices to chase generators around the diagram. In particular, suppose we are given $x : S^a \to E \otimes E$ and $y : S^b \to E \otimes X$, and consider the following diagram exhibiting the two possible ways to chase $x \otimes y$ around the diagram (as usual, we are passing to a symmetric strict monoidal category):

$$\begin{tikzcd}
S^{a+b} \arrow[r, "\phi_{a,b}"] & S^a \otimes S^b \arrow[r, "x\otimes y"] & E \otimes E \otimes E \otimes X \arrow[r, "E\otimes\mu\otimes X"] \arrow[d, "E\otimes E\otimes E\otimes f"'] & E \otimes E \otimes X \arrow[d, "E\otimes E\otimes f"] \\
 & & E \otimes E \otimes E \otimes Y \arrow[r, "E\otimes\mu\otimes Y"] & E \otimes E \otimes Y
\end{tikzcd}$$

This diagram commutes by functoriality of $- \otimes -$. Thus we have that diagram (4) does indeed commute, as desired. $\square$

**Lemma D.8.** *Let $E$ and $X$ be objects in $\mathcal{SH}$. Then for all $a \in A$, there is an $A$-graded isomorphism of left $\pi_*(E)$-modules*

$$t^a_X : E_*(\Sigma^a X) \cong E_{*-a}(X)$$

*which sends a class $S^b \to E \otimes \Sigma^a X = E \otimes S^a \otimes X$ to the composition*

$$S^{b-a} \xrightarrow{\phi_{b,-a}} S^b \otimes S^{-a} \xrightarrow{x\otimes S^{-a}} E\otimes S^a\otimes X\otimes S^{-a} \xrightarrow{E\otimes S^a\otimes\tau_{X,S^{-a}}} E\otimes S^a\otimes S^{-a}\otimes X \xrightarrow{E\otimes\phi^{-1}_{a,-a}\otimes X} E\otimes X$$

*(where here we are ignoring associators and unitors). Furthermore this isomorphism is natural in $X$.*

*Proof.* Expressed in terms of hom-sets, $t^a_X$ is precisely the composition

$$\begin{tikzcd}
E_*(\Sigma^a X) \arrow[r, equal] & {[S^*, E \otimes S^a \otimes X]} \arrow[d, "-\otimes S^{-a}"] & \\
 & {[S^* \otimes S^{-a}, E \otimes S^a \otimes X \otimes S^{-a}]} \arrow[d, "(\phi_{*,-a})^*"] & \\
 & {[S^{*-a}, E \otimes S^a \otimes X \otimes S^{-a}]} \arrow[d, "(E\otimes S^a\otimes\tau)_*"] & \\
 & {[S^{*-a}, E \otimes S^a \otimes S^{-a} \otimes X]} \arrow[d, "(E\otimes\phi^{-1}_{a,-a}\otimes X)_*"] & \\
 & {[S^{*-a}, E \otimes X]} \arrow[r, equal] & E_{*-a}(E \otimes X)
\end{tikzcd}$$

We know the first vertical arrow is an isomorphism of abelian groups as $- \otimes -$ is additive in each variable (since $\mathcal{SH}$ is tensor triangulated) and $\Omega^a \cong - \otimes S^{-a}$ is an autoequivalence of $\mathcal{SH}$ by Proposition 2.5. The three other vertical arrows are given by composing with an isomorphism in an additive category, so they are also isomorphisms.

To see $t^a_X$ is a homomorphism of left $\pi_*(E)$-modules, suppose we are given classes $r : S^b \to E$ $\pi_b(E)$ and $x : S^c \to E \otimes S^a \otimes X$ in $E_c(\Sigma^a X)$. Then we wish to show that $t^a_X(r \cdot x) = r \cdot t^a_X(x)$.

Consider the following diagram:

$$\begin{tikzcd}
S^{b+c-a} \arrow[d, "\cong"] & E \otimes S^a \otimes X \otimes S^{-a} \arrow[r, "E\otimes S^a\otimes\tau_{X,S^{-a}}"] & E \otimes S^a \otimes S^{-a} \otimes X \arrow[d, "E\otimes\phi_{a,-a}^{-1}\otimes X"] \\
S^b \otimes S^c \otimes S^{-a} \arrow[r, "r\otimes x\otimes S^{-a}"] & E \otimes E \otimes S^a \otimes X \otimes S^{-a} \arrow[u, "\mu\otimes S^a\otimes X\otimes S^{-a}"] \arrow[d, "E\otimes E\otimes S^a\otimes\tau_{X,S^{-a}}"'] & E \otimes X \\
& E \otimes E \otimes S^a \otimes S^{-a} \otimes X \arrow[r, "E\otimes E\otimes\phi_{a,-a}^{-1}\otimes X"'] \arrow[uur, "\mu\otimes S^a\otimes S^{-a}\otimes X"'] & E \otimes E \otimes X \arrow[u, "\mu\otimes X"']
\end{tikzcd}$$

Both triangles commute by functoriality of $-\otimes-$. The top composition is $t_X^a(r\cdot x)$ while the bottom is $r\cdot t_X^a(x)$, so they are equal as desired.

It remains to show $t_X^a$ is natural in $X$. let $f : X \to Y$ in $\mathcal{SH}$, then we would like to show the following diagram commutes:

$$\begin{tikzcd}
E_*(\Sigma^a X) \arrow[r, "t_X^a"] \arrow[d, "E_*(\Sigma^a f)"'] & E_{*-a}(X) \arrow[d, "E_{*-a}(f)"] \\
E_*(\Sigma^a Y) \arrow[r, "t_Y^a"] & E_{*-a}(Y)
\end{tikzcd} \tag{5}$$

We may chase a generator around the diagram since all the arrows here are homomorphisms. Let $x : S^b \to E \otimes S^a \otimes X$ in $E_*(\Sigma^a X)$. Then consider the following diagram:

$$\begin{tikzcd}
S^{b-a} \arrow[r, "\cong"] & S^b \otimes S^{-a} \arrow[r, "x\otimes S^{-a}"] & E \otimes S^a \otimes X \otimes S^{-a} \arrow[r, "E\otimes S^a\otimes\tau"] \arrow[d, "E\otimes S^a\otimes f\otimes S^{-a}"'] & E \otimes S^a \otimes S^{-a} \otimes X \arrow[r, "E\otimes\phi_{a,-a}^{-1}\otimes X"] \arrow[d, "E\otimes S^a\otimes S^{-a}\otimes f"'] & E \otimes X \arrow[d, "E\otimes f"] \\
& & E \otimes S^a \otimes Y \otimes S^{-a} \arrow[r, "E\otimes S^a\otimes\tau"'] & E \otimes S^a \otimes S^{-a} \otimes Y \arrow[r, "E\otimes\phi_{a,-a}^{-1}\otimes Y"'] & E \otimes Y
\end{tikzcd}$$

The left rectangle commutes by naturality of $\tau$, while the right rectangle commutes by functoriality of $-\otimes-$. The two outside compositions are the two ways to chase $x$ around diagram (5), so the diagram commutes as desired. $\square$

**Lemma D.9.** *Given a monoid object $(E, \mu, e)$ in $\mathcal{SH}$, the maps $\Phi_X$ constructed in Proposition D.7 commute with the natural isomorphisms $t_X^a$, in the sense that the following diagram commutes for all $a \in A$ and $X$ in $\mathcal{SH}$:*

$$\begin{tikzcd}
E_*(E) \otimes_{\pi_*(E)} E_*(\Sigma^a X) \arrow[r, "E_*(E)\otimes t_X^a"] \arrow[d, "\Phi_{\Sigma^a X}"'] & E_*(E) \otimes_{\pi_*(E)} E_{*-a}(X) \arrow[d, "\Phi_X"] \\
E_*(E \otimes \Sigma^a X) \arrow[r, "t_X^a"] & E_{*-a}(E \otimes X)
\end{tikzcd}$$

*where the top arrow is well-defined since $t_X^a$ is a left $\pi_*(E)$-modue homomorphism by the above lemma, and we are being abusive in that the bottom arrow is given by the composition*

$$E_*(E \otimes \Sigma^a X) \overset{\alpha}{\cong} (E \otimes E)_*(\Sigma^a X) \xrightarrow{t_X^a} (E \otimes E)_{*-a}(X) \overset{\alpha}{\cong} E_{*-a}(E \otimes X).$$

*Proof.* Since all the maps in the above diagram are homomorphisms, we can chase generators around to show it commutes. Let $x : S^b \to E \otimes E$ and $y : S^c \to E \otimes \Sigma^a X = E \otimes S^a \otimes X$. Then consider the following diagram:

$$\begin{tikzcd}
S^{b+c-a} \arrow[d, "\cong"] & E\otimes E\otimes E\otimes S^a\otimes S^{-a}\otimes X \arrow[r, "E\otimes E\otimes E\otimes \phi^{-1}_{a,-a}\otimes X"] \arrow[dr, "E\otimes\mu\otimes S^a\otimes S^{-a}\otimes X"] \arrow[ddr] & E\otimes E\otimes E\otimes X \arrow[d, "E\otimes\mu\otimes X"] \\
S^b\otimes S^c\otimes S^{-a} \arrow[r, "x\otimes y\otimes S^{-a}"] & E\otimes E\otimes E\otimes S^a\otimes X\otimes S^{-a} \arrow[u, "E\otimes E\otimes E\otimes S^a\otimes\tau"] \arrow[d, "E\otimes\mu\otimes S^a\otimes X\otimes S^{-a}"] & E\otimes E\otimes X \\
 & E\otimes E\otimes S^a\otimes X\otimes S^{-a} \arrow[r, "E\otimes E\otimes S^a\otimes\tau"'] & E\otimes E\otimes S^a\otimes S^{-a}\otimes X \arrow[u, "E\otimes E\otimes \phi^{-1}_{a,-a}\otimes X"']
\end{tikzcd}$$

Each triangle commutes by functoriality of $-\otimes -$. The two outside compositions are the two ways to chase $x\otimes y$ around the diagram in the statement of the lemma, so the diagram commutes as desired. ☐

**Corollary D.10.** *For all $X$ in $\mathcal{SH}$, we have natural isomorphisms $t_X : E_*E(\Sigma X) \xrightarrow{\cong} E_{*-\mathbf{1}}(X)$ given by the composition*

$$E_*(\Sigma X) \xrightarrow{E_*(\nu_X^{-1})} E_*(\Sigma^1 X) \xrightarrow{t_X^{\mathbf{1}}} E_{*-\mathbf{1}}(X).$$

*Furthermore, by naturality of $\Phi$ and the fact that $t_X^{\mathbf{1}}$ commutes with $\Phi$ (in the sense of the above lemma), this isomorphism also commutes with $\Phi$.*

**Proposition D.11.** *Let $(E,\mu,e)$ be a flat monoid object in $\mathcal{SH}$ (Definition 2.8) and let $X$ be any cellular object in $\mathcal{SH}$ (Definition 2.4). Then the natural homomorphism*

$$\Phi_X : E_*(E)\otimes_{\pi_*(E)} E_*(X) \to E_*(E\otimes X)$$

*given in Proposition D.7 is an isomorphism of left $\pi_*(E)$-modules.*

*Proof.* In this proof, we will freely employ the coherence theorem for symmetric monoidal categories, and we will assume that associativity and unitality of $-\otimes-$ holds up to strict equality. To start, let $\mathcal{E}$ be the collection of objects $X$ in $\mathcal{SH}$ for which this map is an isomorphism. Then in order to show $\mathcal{E}$ contains every cellular object, it suffices to show that $\mathcal{E}$ satisfies the three conditions given for the class of cellular objects in Definition 2.4. First, we need to show that $\Phi$ is an isomorphism when $X = S^a$ for some $a\in A$. Indeed, consider the map

$$\Psi : E_*(E\otimes S^a) \to E_*(E)\otimes_{\pi_*(E)} E_*(S^a)$$

which sends a class $x : S^b \to E\otimes E\otimes S^a$ in $E_b(E\otimes S^a)$ to the pure tensor $\widetilde{x}\otimes\widetilde{e}$, where $\widetilde{x}\in E_{b-a}(E)$ is the composition

$$S^{b-a} \cong S^b\otimes S^{-a} \xrightarrow{x\otimes S^{-a}} E\otimes E\otimes S^a\otimes S^{-a} \xrightarrow{E\otimes E\otimes\phi^{-1}_{a,-a}} E\otimes E$$

and $\widetilde{e}\in E_a(S^a)$ is the composition

$$S^a \cong S\otimes S^a \xrightarrow{e\otimes S^a} E\otimes S^a.$$

First, note $\Psi$ is an ($A$-graded) homomorphism of abelian groups: Given $x,x'\in E_b(E\otimes S^a)$, we would like to show that $\widetilde{x}\otimes\widetilde{e} + \widetilde{x}'\otimes\widetilde{e} = \widetilde{x+x'}\otimes\widetilde{e}$. It suffices to show that $\widetilde{x}+\widetilde{x}' = \widetilde{x+x'}$. To see this, consider the following diagram (again, we are passing to a symmetric strict monoidal category):

$$\begin{tikzcd}
S^{b-a} \arrow[r, "\Delta"] \arrow[d, "\phi_{b-a}"'] & S^{b-a} \oplus S^{b-a} \arrow[d, "\phi_{b,-a} \oplus \phi_{b,-a}"] \\
S^b \otimes S^{-a} \arrow[r, "\Delta"] \arrow[d, "\Delta \otimes S^{-a}"'] & (S^b \otimes S^{-a}) \oplus (S^b \otimes S^{-a}) \arrow[d, "(x \otimes S^{-a}) \oplus (x' \otimes S^{-a})"] \\
(S^b \oplus S^b) \otimes S^{-a} \arrow[ur, "\cong"] \arrow[d, "(x \oplus x') \otimes S^{-a}"'] & (E \otimes E \otimes S^a \otimes S^{-a}) \oplus (E \otimes E \otimes S^a \otimes S^{-a}) \arrow[d, "(E \otimes E \otimes \phi_{a,-a}^{-1}) \oplus (E \otimes E \otimes \phi_{a,-a}^{-1})"] \arrow[ddl, "\nabla"] \\
((E \otimes E \otimes S^a) \oplus (E \otimes E \otimes S^a)) \otimes S^{-a} \arrow[ur, "\cong"] \arrow[d, "\nabla \otimes S^{-a}"'] & (E \otimes E) \oplus (E \otimes E) \arrow[d, "\nabla"] \\
E \otimes E \otimes S^a \otimes S^{-a} \arrow[r, "E \otimes E \otimes \phi_{a,-a}^{-1}"'] & E \otimes E
\end{tikzcd}$$

The top rectangle commutes by naturality of $\Delta$ in an additive category. The bottom triangle commutes by naturality of $\nabla$ in an additive category. Finally, the remaining regions of the diagram commute by additivity of $- \otimes -$. By functoriality of $- \otimes -$, it follows that the left composition is $\widetilde{x + x'}$ and the right composition is $\widetilde{x} + \widetilde{x}'$, so they are equal as desired. Thus $\Psi$ is a homomorphism of abelian groups, as desired.

Now, we claim that $\Psi$ is an inverse to $\Phi$, (which is enough to show $\Phi$ is an isomorphism of left $\pi_*(E)$-modules). Since $\Phi$ and $\Psi$ are homomorphisms it suffices to check that they are inverses on generators. First, let $x : S^b \to E \otimes E \otimes S^a$ in $E_b(E \otimes S^a)$. We would like to show that $\Phi(\Psi(x)) = x$. Consider the following diagram, where here we are passing to a symmetric strict monoidal category:

$$\begin{tikzcd}
S^b \arrow[r, "\cong"] \arrow[dd, "x"'] & S^b \otimes S^{-a} \otimes S^a \arrow[d, dashed, "x \otimes S^{-a} \otimes S^a"] \arrow[dr, "x \otimes S^{-a} \otimes e \otimes S^a"] & \\
 & E \otimes E \otimes S^a \otimes S^{-a} \otimes S^a \arrow[r, "E \otimes E \otimes S^a \otimes S^{-a} \otimes e \otimes S^a"'] & E \otimes E \otimes S^a \otimes S^{-a} \otimes E \otimes S^a \arrow[ddl, "E \otimes E \otimes \phi_{a,-a}^{-1} \otimes E \otimes S^a"] \\
E \otimes E \otimes S^a \arrow[ur, "E \otimes E \otimes S^a \otimes \phi_{-a,a}"] \arrow[r, equal] \arrow[dr, "E \otimes E \otimes e \otimes S^a"] & E \otimes E \otimes S^a \arrow[u, dashed, "E \otimes E \otimes \phi_{a,-a} \otimes S^a"] \arrow[d, dashed, "E \otimes E \otimes e \otimes S^a"] & \\
E \otimes E \otimes E \otimes S^a \arrow[u, "E \otimes \mu \otimes S^a"] \arrow[r, equal] & E \otimes E \otimes E \otimes S^a &
\end{tikzcd}$$

The top left trapezoid commutes since the isomorphism $S^b \xrightarrow{\cong} S^b \otimes S^{-a} \otimes S^a$ may be given as $S^b \otimes \phi_{-a,a}$ (see Remark 2.3), in which case the trapezoid commmutes by functoriality of $- \otimes -$. The triangle below that commutes by coherence for the $\phi_{a,b}$'s. The triangle below that commutes by definition. The bottom left triangle commutes by unitality for $\mu$. The top right triangle commutes by functoriality of $- \otimes -$. Finally, the bottom right triangle commutes by functoriality of $- \otimes -$. It follows by unravelling definitions that the two outside compositions are $x$ and $\Phi(\Psi(x))$, so indeed we have $\Phi(\Psi(x)) = x$ since the diagram commutes.

On the other hand, suppose we are given a homogeneous pure tensor $x \otimes y$ in $E_*(E) \otimes_{\pi_*(E)} E_*(S^a)$, so $x : S^b \to E \otimes E$ and $y : S^c \to E \otimes S^a$ for some $b, c \in A$. Then we would like to show that $\Psi(\Phi(x \otimes y)) = x \otimes y$. Unravelling definitions, $\Psi(\Phi(x \otimes y))$ is the homogeneous pure tensor $\widetilde{xy} \otimes \widetilde{e}$, where $\widetilde{e} : S^a \to E \otimes S^a$ is defined above, and by functoriality of $- \otimes -$, $\widetilde{xy} : S^{b+c-a} \to E \otimes E$

is the composition

$$\begin{tikzcd}
S^{b+c-a} \arrow[d, "\phi_{b+c,-a}"] \\
S^{b+c} \otimes S^{-a} \arrow[d, "\phi_{b,c} \otimes S^{-a}"] \\
S^{b} \otimes S^{c} \otimes S^{-a} \arrow[d, "x \otimes y \otimes S^{-a}"] \\
E \otimes E \otimes E \otimes S^{a} \otimes S^{-a} \arrow[d, "E \otimes \mu \otimes S^{a} \otimes S^{-a}"] \\
E \otimes E \otimes S^{a} \otimes S^{-a} \arrow[d, "E \otimes E \otimes \phi_{a,-a}^{-1}"] \\
E \otimes E \otimes S \arrow[d, "E \otimes \rho_E"] \\
E \otimes E.
\end{tikzcd}$$

In order to see $x \otimes y = \widetilde{xy} \otimes \widetilde{e}$, it suffices to show there exists some scalar $r \in \pi_{c-a}(E)$ such that $x \cdot r = \widetilde{xy}$ and $r \cdot \widetilde{e} = y$, where here $\cdot$ denotes the right and left action of $\pi_*(E)$ on $E_*(E)$ and $E_*(S^a)$, respectively. Now, define $r$ to be the composition

$$S^{c-a} \cong S^c \otimes S^{-a} \xrightarrow{y \otimes S^{-a}} E \otimes S^a \otimes S^{-a} \xrightarrow{E \otimes \phi_{a,-a}^{-1}} E \otimes S \xrightarrow{\rho_E} E.$$

First, in order to see that $x \cdot r = \widetilde{xy}$, consider the following diagram, where here we are again passing to a symmetric strict monoidal category:

$$\begin{tikzcd}
S^{b+c-a} \arrow[r, "\cong"] & S^b \otimes S^c \otimes S^{-a} \arrow[r, "x \otimes y \otimes S^{-a}"] & E \otimes E \otimes E \otimes S^a \otimes S^{-a} \arrow[r, "E \otimes \mu \otimes S^a \otimes S^{-a}"] \arrow[d, "E \otimes E \otimes E \otimes \phi_{a,-a}^{-1}"'] \arrow[dr, "E \otimes \mu \otimes \phi_{a,-a}^{-1}"] & E \otimes E \otimes S^a \otimes S^{-a} \arrow[d, "E \otimes E \otimes \phi_{a,-a}^{-1}"] \\
 & & E \otimes E \otimes E \arrow[r, "E \otimes \mu"'] & E \otimes E
\end{tikzcd}$$

Commutativity is functoriality of $- \otimes -$, which also tells us that the two outside compositions are $\widetilde{xy}$ (on top) and $x \cdot r$ (on the bottom), so they are equal as desired. On the other hand, in order to see that $r \cdot \widetilde{e} = y$, consider the following diagram (where here we have passed to a symmetric strict monoidal category):

$$\begin{tikzcd}
S^c \arrow[rr, "\cong"] \arrow[dd, "y"'] & & S^c \otimes S^{-a} \otimes S^a \arrow[dl, "y \otimes S^{-a} \otimes S^a"'] \arrow[dd, "y \otimes S^{-a} \otimes e \otimes S^a"] \\
 & E \otimes S^a \otimes S^{-a} \otimes S^a \arrow[dl, "E \otimes S^a \otimes \phi_{-a,a}^{-1}"'] \arrow[dr, "E \otimes S^a \otimes S^{-a} \otimes e \otimes S^a"] & \\
E \otimes S^a \arrow[drr, "E \otimes e \otimes S^a"] & & E \otimes S^a \otimes S^{-a} \otimes E \otimes S^a \arrow[d, "E \otimes \phi_{a,-a}^{-1} \otimes E \otimes S^a"] \\
E \otimes E \otimes S^a \arrow[u, "\mu \otimes S^a"] \arrow[rr, equal] & & E \otimes E \otimes S^a
\end{tikzcd}$$

The top left triangle commutes since we may take the isomorphism $S^c \xrightarrow{\cong} S^c \otimes S^{-a} \otimes S^a$ to be $S^c \otimes \phi_{-a,a}$, in which case commutativity of the triangle follows by functoriality of $- \otimes -$. Commutativity of the right triangle is also functoriality of $- \otimes -$. Commutativity of the bottom left triangle is unitality of $\mu$. Finally, commutativity of the remaining middle 4-sided region is again functoriality of $- \otimes -$. It follows that $y$ is equal to the outer composition, which is $r \cdot \widetilde{e}$, as desired. Thus, we have shown that

$$\Psi(\Phi(x \otimes y)) = \widetilde{xy} \otimes \widetilde{e} = (x \cdot r) \otimes \widetilde{e} = x \otimes (r \cdot \widetilde{e}) = x \otimes y,$$

as desired, so that for each $a \in A$, the object $S^a$ belongs to the class $\mathcal{E}$.

Now, we would like to show that given a distinguished triangle

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X,$$

if two of three of the objects $X$, $Y$, and $Z$ belong to $\mathcal{E}$, then so does the third. From now on, write $L_*^E : \mathcal{SH} \to \pi_*(E)$-**Mod** to denote the functor $X \mapsto E_*(E) \otimes_{\pi_*(E)} E_*(X)$, so $\Phi$ is a natural transformation $L_*^E \Rightarrow E_*(E \otimes -)$. First, note that by Proposition A.16, we have the following exact sequence in $\mathcal{SH}$:

$$E \otimes \Omega Y \xrightarrow{E \otimes \Omega g} E \otimes \Omega Z \xrightarrow{E \otimes \widetilde{h}} E \otimes X \xrightarrow{E \otimes f} E \otimes Y \xrightarrow{E \otimes g} E \otimes Z \xrightarrow{E \otimes h} E \otimes \Sigma X \xrightarrow{E \otimes \Sigma f} \Sigma Y.$$

We can then apply $[S^*, -]$ to it, which yields the following exact sequence of $A$-graded $\pi_*(E)$-modules:

$$E_*(\Omega Y) \xrightarrow{E_*(\Omega g)} E_*(\Omega Z) \xrightarrow{E_*(\widetilde{h})} E_*(X) \xrightarrow{E_*(f)} E_*(Y) \xrightarrow{E_*(g)} E_*(Z) \xrightarrow{E_*(h)} E_*(\Sigma X) \xrightarrow{E_*(f)} E_*(\Sigma Y).$$

Now, we can tensor this sequence with $E_*(E)$ over $\pi_*(E)$, and since $E_*(E)$ is a flat right $\pi_*(E)$ module, we get that the top row in the following sequence is exact:

$$\begin{tikzcd}
L_*^E(\Omega Y) \arrow[r, "L_*^E(\Omega g)"] \arrow[d, "\Phi_{\Omega Y}"'] & L_*^E(\Omega Z) \arrow[r, "L_*^E(\widetilde{h})"] \arrow[d, "\Phi_{\Omega Z}"'] & L_*^E(X) \arrow[r, "L_*^E(f)"] \arrow[d, "\Phi_X"'] & L_*^E(Y) \arrow[r, "L_*^E(g)"] \arrow[d, "\Phi_Y"'] & L_*^E(Z) \arrow[r, "L_*^E(h)"] \arrow[d, "\Phi_Z"'] & L_*^E(\Sigma X) \arrow[r, "L_*^E(\Sigma f)"] \arrow[d, "\Phi_{\Sigma X}"'] & L_*^E(\Sigma Y) \arrow[d, "\Phi_{\Sigma Y}"'] \\
E_*(E \otimes \Omega Y) \arrow[r, "E_*(E \otimes \Omega g)"'] & E_*(E \otimes \Omega Z) \arrow[r, "E_*(E \otimes h)"'] & E_*(E \otimes X) \arrow[r, "E_*(E \otimes f)"'] & E_*(E \otimes Y) \arrow[r, "E_*(E \otimes g)"'] & E_*(E \otimes Z) \arrow[r, "E_*(E \otimes h)"'] & E_*(E \otimes \Sigma X) \arrow[r, "E_*(E \otimes \Sigma f)"'] & E_*(E \otimes \Sigma Y)
\end{tikzcd}$$

The diagram commutes since $\Phi$ is natural. The following sequence is exact in $\mathcal{SH}$ by Proposition A.16,

$$E \otimes E \otimes \Omega Y \to E \otimes E \otimes \Omega Z \to E \otimes E \otimes X \to E \otimes E \otimes Y \to E \otimes E \otimes Z \to E \otimes E \otimes \Sigma X \to E \otimes E \otimes \Sigma Y,$$

so that the bottom row in the above diagram is also exact. Now, suppose two of three of $X$, $Y$, and $Z$ belong to $\mathcal{E}$. By Lemma D.9, Corollary D.10, if $\Phi_W$ is an isomorphism then $\Phi_{\Omega W}$ and $\Phi_{\Sigma W}$ are. Thus by the five lemma, it follows that the middle three vertical arrows in the above diagram are necessarily all isomorphisms, so we have shown that $\mathcal{E}$ is closed under two-of-three for exact triangles, as desired.

Finally, it remains to show that $\mathcal{E}$ is closed under arbitrary products. Let $\{X_i\}_{i \in I}$ be a collection of objects in $\mathcal{E}$ indexed by some (small) set $I$. Then we'd like to show that $X := \coprod_i X_i$ belongs to $\mathcal{E}$. First of all, note that $E \otimes -$ preserves arbitrary coproducts, as it has a right adjoint $F(E, -)$. Thus without loss of generality we may take $\coprod_i E \otimes X_i = E \otimes \coprod_i X_i$ (as $E \otimes \coprod_i X_i$ *is* a coproduct of all the $E \otimes X_i$'s). Now, recall that we have chosen each $S^a$ to be a compact object (Definition 2.1), so that the canonical map

$$s : \coprod_i E_*(X_i) = \coprod_i [S^*, E \otimes X_i] \to [S^*, \coprod_i E \otimes X_i] = [S^*, E \otimes X] = E_*(X)$$

is an isomorphism, natural in $X_i$ for each $i$. Note in particular that it is an isomorphism of left $\pi_*(E)$-modules. To see this, first note it suffices to check that $s(r \cdot x) = r \cdot s(x)$ for some homogeneous $x \in E_*(X_i)$ for some $i$, as such $x$ generate $\coprod_i E_*(X_i)$ by definition, and $s$ is a homomorphism of abelian groups. Then given $r : S^a \to E \otimes E$ and $x : S^b \to E \otimes X_i$, consider the following diagram

$$\begin{tikzcd}
S^{a+b} \arrow[r, "\phi_{a,b}"] & S^a \otimes S^b \arrow[r, "x\otimes y"] & E \otimes E \otimes E \otimes X_i \arrow[r, "E\otimes E\otimes \iota_{E\otimes X_i}"] \arrow[dr, "E\otimes E\otimes E\otimes \iota_{X_i}"] \arrow[ddd, "E\otimes \mu \otimes X_i"'] & E \otimes E \otimes \coprod_i (E \otimes X_i) \arrow[d, equal] \\
& & & E \otimes E \otimes E \otimes X \arrow[d, "E\otimes \mu\otimes X"] \\
& & & E \otimes E \otimes X \arrow[d, equal] \\
& & E \otimes E \otimes X_i \arrow[r, "E\otimes \iota_{E\otimes X_i}"'] \arrow[ur, "E\otimes E\otimes \iota_{X_i}"] & E \otimes \coprod_i (E \otimes X_i)
\end{tikzcd}$$

where $\iota_{E\otimes X_i} : E \otimes X_i \hookrightarrow \coprod_i (E \otimes X_i)$ and $\iota_{X_i} : X_i \hookrightarrow \coprod_i X_i$ are the maps determined by universal property of the coproduct. Commutativity of the two triangles is again by the fact that $E \otimes -$ is colimit preserving. Commutativity of the trapezoid is functoriality of $- \otimes -$. Thus, the top arrow in the following diagram is well-defined:

$$\begin{tikzcd}
\coprod_i E_*(E) \otimes_{\pi_*(E)} E_*(X_i) \arrow[r, equal] \arrow[d, "\coprod_i \Phi_{X_i}"'] & E_*(E) \otimes_{\pi_*(E)} \coprod_i E_*(X_i) \arrow[r, "E_*(E)\otimes_{\pi_*(E)} s"] & E_*(E) \otimes_{\pi_*(E)} E_*(X) \arrow[d, "\Phi_X"] \\
\coprod_i E_*(E \otimes X_i) \arrow[r, "s"] & E_*(\coprod_i E \otimes X_i) \arrow[r, equal] & E_*(E \otimes X)
\end{tikzcd} \tag{6}$$

We wish to show this diagram commutes. Again, since each map here is a homomorphism, it suffices to chase generators. By definition, a generator of the top left element is a homogeneous pure tensor in $E_*(E) \otimes_{\pi_*(E)} E_*(X_i)$ for some $i$ in $I$. Given classes $x : S^a \to E \otimes E$ and $y : S^b \to E \otimes X_i$, consider the following diagram:

$$\begin{tikzcd}
S^{a+b} \arrow[r, "\phi_{a,b}"] & S^a \otimes S^b \arrow[r, "x\otimes y"] & E \otimes E \otimes E \otimes X_i \arrow[r, "E\otimes E\otimes \iota_{E\otimes X_i}"] \arrow[d, "E\otimes \mu\otimes X_i"'] \arrow[dr, "E\otimes E\otimes E\otimes \iota_{X_i}"] & E \otimes E \otimes \coprod_i E \otimes X_i \arrow[d, equal] \\
& & E \otimes E \otimes X_i \arrow[d, "\iota_{E\otimes E\otimes X_i}"'] \arrow[dr, "E\otimes E\otimes \iota_{X_i}"] & E \otimes E \otimes E \otimes X \arrow[d, "E\otimes \mu \otimes X"] \\
& & \coprod_i E \otimes E \otimes X_i \arrow[r, equal] & E \otimes E \otimes X
\end{tikzcd}$$

Unravelling definitions, the two outside compositions are the two ways to chase $x \otimes y$ around diagram (6). The two triangles commute again by the fact that $- \otimes -$ preserves colimits in each argument. Commutativity of the inner parallelogram is functoriality of $- \otimes -$. Thus diagram (6) tells us $\Phi_X$ is an isomorphism, since $\Phi_{X_i}$ is an isomorphism for each $i$ in $I$, meaning $\coprod_i \Phi_{X_i}$ is as well. □

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