In **??**, we showed that given a monoid object $(E, \mu, e)$ in $\mathcal{SH}$, that $E_*(E) = \pi_*(E \otimes E)$ is both a ring (since $E \otimes E$ is a monoid object if $E$ is), and an $A$-graded bimodule over the ring $\pi_*(E)$. In this subsection, we will outline some additional structure carried by the pair $(E_*(E), \pi_*(E))$. Namely, we will show that if $(E, \mu, e)$ is a flat (Definition 0.5) commutative monoid object, then this pair, called the *dual $E$-Steenrod algebra*, is canonically an *$A$-graded anticommutative Hopf algebroid* over the stable homotopy ring $\pi_*(S)$ (**??**). To start with, we outline some structure maps relating $E_*(E)$ and $\pi_*(E)$.

First, recall that given a monoid object $(E, \mu, e)$ in $\mathcal{SH}$, $\pi_*(E)$ is canonically an $A$-graded ring by **??**, and so is $E_*(E) = \pi_*(E \otimes E)$ and $E_*(E \otimes E) = \pi_*(E \otimes E \otimes E)$, since the tensor product of monoid objects in a symmetric monoidal category is again a monoid object (**??**).

**Lemma 0.1.** *Let $(E, \mu, e)$ be a commutative monoid object in $\mathcal{SH}$. Then the maps*

(1) $E \xrightarrow{\cong} E \otimes S \xrightarrow{E \otimes e} E \otimes E$,

(2) $E \xrightarrow{\cong} S \otimes E \xrightarrow{e \otimes E} E \otimes E$,

(3) $E \otimes E \xrightarrow{\cong} E \otimes S \otimes E \xrightarrow{E \otimes e \otimes E} E \otimes E \otimes E$,

(4) $E \otimes E \xrightarrow{\mu} E$, and

(5) $E \otimes E \xrightarrow{\tau_{E,E}} E \otimes E$

*are homomorphisms of monoid objects in $\mathcal{SH}$ (where here $E \otimes E$ and $E \otimes E \otimes E$ are considered as monoid objects in $\mathcal{SH}$ by **??** and **??**, respectively), so that by **??**, under $\pi_*$ they induce morphisms in $\pi_*(S)$-$\mathbf{GCA}^A$:*

(1) $\eta_L : \pi_*(E) \to E_*(E)$,

(2) $\eta_R : \pi_*(E) \to E_*(E)$,

(3) $h : E_*(E) \to E_*(E \otimes E)$,

(4) $\epsilon : E_*(E) \to \pi_*(E)$, and

(5) $c : E_*(E) \to E_*(E)$.

*Proof.* It is a general fact that the unit and multiplication maps $e : S \to E$ and $\mu : E \otimes E \to E$ for a monoid are monoid homomorphisms when $(E, \mu, e)$ is a commutative monoid object (**??**), so that the maps $E \otimes e$, and $e \otimes E$ from $E$ to $E \otimes E$ are monoid homomorphisms, by **??**. Similarly, $E \otimes e \otimes E : E \otimes E \to E \otimes E \otimes E$ is a monoid homomorphism. Thus, it remains to show that $\tau_{E,E} : E \otimes E \to E \otimes E$ is a monoid homomorphism. First, consider the following diagram:

$$
\begin{array}{ccc}
E_1 \otimes E_2 \otimes E_3 \otimes E_4 & \xrightarrow{\tau \otimes \tau} & E_2 \otimes E_1 \otimes E_4 \otimes E_3 \\
{\scriptstyle E \otimes \tau \otimes E} \downarrow & & \downarrow {\scriptstyle E \otimes \tau \otimes E} \\
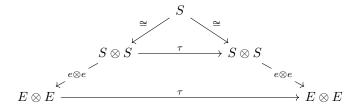E_1 \otimes E_3 \otimes E_2 \otimes E_4 & \xrightarrow{\tau_{E \otimes E, E \otimes E}} & E_2 \otimes E_4 \otimes E_1 \otimes E_3 \\
{\scriptstyle \mu \otimes \mu} \downarrow & & \downarrow {\scriptstyle \mu \otimes \mu} \\
E_{1,3} \otimes E_{2,4} & \xrightarrow{\tau} & E_{2,4} \otimes E_{1,3}
\end{array}
$$

(Here we've labelled the $E$'s to make the action of the braidings clearer). The top region commutes by coherence for the symmetries in a symmetric monoidal category, while the bottom region

commutes by naturality of $\tau$. Now, consider the following diagram:

$$
\begin{array}{ccc}
& S & \\
{\scriptstyle \cong} \swarrow & & \searrow {\scriptstyle \cong} \\
S \otimes S & \xrightarrow{\ \tau\ } & S \otimes S \\
{\scriptstyle e \otimes e} \swarrow & & \searrow {\scriptstyle e \otimes e} \\
E \otimes E & \xrightarrow{\ \ \ \ \tau\ \ \ \ } & E \otimes E
\end{array}
$$

The top triangle commutes by coherence for a symmetric monoidal category, while the bottom region commutes by naturality of $\tau$. Thus, we have shown $\tau_{E,E}$ is a homomorphism of monoid objects, as desired. $\qquad\square$

Recall a that given a homomorphism of rings $f : R \to R'$, the ring $R'$ canonically becomes an $R$-bimodule with left action $r \cdot x := f(r)x$ and right action $x \cdot r := xf(r)$. In particular, the ring homomorphisms $\eta_L : \pi_*(E) \to E_*(E)$ and $\eta_R : \pi_*(E) \to E_*(E)$ endow $E_*(E)$ with the structure of a bimodule over $\pi_*(E)$. Naturally, one may ask in what sense these bimodule structures coincide with the canonical one (from **??**). The following lemma tells us that the canonical $\pi_*(E)$-bimodule structure on $E_*(E)$ is that with left action induced by $\eta_L$ and right action induced by $\eta_R$:

**Lemma 0.2.** *Let $(E, \mu, e)$ be a commutative monoid object in $\mathcal{SH}$. Then the left (resp. right) $\pi_*(E)$-module structure induced on $E_*(E)$ by the ring homomorphism $\eta_L$ (resp. $\eta_R$) coincides with the canonical left (resp. right) $\pi_*(E)$-module structure on $E_*(E)$ given in **??**.*

*Proof.* What's going on here is a bit subtle, so we're going to be really explicit. In **??**, it was shown that $E_*(E)$ is a left $\pi_*(E)$-module via the assignment

$$
\pi_*(E) \times E_*(E) \to E_*(E)
$$

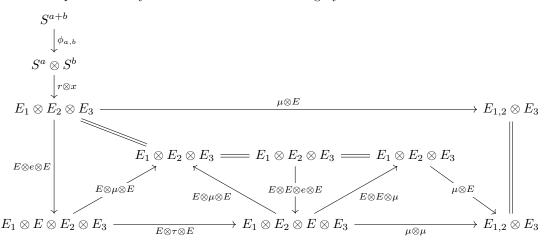which sends homogeneous elements $r : S^a \to E$ and $x : S^b \to E \otimes E$ to the composition

$$
S^{a+b} \xrightarrow{\cong} S^a \otimes S^b \xrightarrow{r \otimes x} E \otimes E \otimes E \xrightarrow{\mu \otimes E} E \otimes E.
$$

We'd like to show that this is the same thing as the assignment $\pi_*(E) \times E_*(E) \to E_*(E)$ sending $(r, x) \mapsto \eta_L(r)x$, where $\eta_L(r)x$ denotes the product of $\eta_L(r)$ and $x$ taken in the ring $E_*(E)$. Explicitly, the product structure on $E_*(E) = \pi_*(E \otimes E)$ is that induced by the fact that $E \otimes E$ is a monoid object in $\mathcal{SH}$ (by **??**), with product

$$
E \otimes E \otimes E \otimes E \xrightarrow{E \otimes \tau \otimes E} E \otimes E \otimes E \otimes E \xrightarrow{\mu \otimes \mu} E \otimes E
$$

(note the middle two factors are swapped). By linearity of module actions, in order to show the canonical left $\pi_*(E)$-module structure on $E_*(E)$ agrees with that induced by $\eta_L$, it suffices to show the module actions agree on homogeneous elements. Now, suppose we have homogeneous elements $r : S^a \to E$ in $\pi_*(E)$ and $x : S^b \to E \otimes E$ in $E_*(E)$, and consider the following diagram,

where we've passed to a symmetric strict monoidal category:



Here we've numbered the $E$'s to make it clear what's going on. The bottom composition is $\eta_L(r)x$, while the top composition is the canonical left action of $r$ on $x$ given in **??**. The leftmost triangle commutes by unitality of $\mu$. The triangle to the right of that commutes by commutativity of $\mu$. The triangle to the right of that commutes by unitality of $\mu$, as does the next triangle. The remaining triangle on the right commutes by functoriality of $-\otimes-$. Finally, the top region commutes by definition. Thus, we've shown that the left $\pi_*(E)$-module structure induced on $E_*(E)$ by $\eta_L$ is in fact the canonical one. On the other hand, showing that the right $\pi_*(E)$-module structure induced on $E_*(E)$ by $\eta_R$ is the canonical one is entirely analagous, and we leave it as an exercise for the reader. $\qquad\square$

Recall (**??**) that the pushout of two morphisms $f : B \to C$ and $g : B \to D$ in $R\text{-}\mathbf{GCA}^A$ is obtained by taking the tensor product of $B$-modules $C \otimes_B D$, where $C$ has right $B$-module action induced by $f$, and $D$ has left $B$-module action induced by $g$, and giving it an anticommutative product which makes $C \otimes_B D$ a ring. Thus, by the above lemma, we may view the tensor product of bimodules $E_*(E) \otimes_{\pi_*(E)} E_*(E)$ (where $E_*(E)$ is considered with its canonical $\pi_*(E)$-bimodule structure from **??**) as not just an $A$-graded abelian group or a $\pi_*(E)$-bimodule, but as an $A$-graded anticommutative $\pi_*(S)$-algebra:

**Corollary 0.3.** *Given a commutative monoid object $(E, \mu, e)$ in $\mathcal{SH}$, the domain of the homomorphism*

$$\Phi_{E,E} : E_*(E) \otimes_{\pi_*(E)} E_*(E) \to E_*(E \otimes E)$$

*constructed in* **??** *is canonically an $A$-graded anticommutative $\pi_*(S)$-algebra, and sits in the following pushout diagram in $\pi_*(S)\text{-}\mathbf{GCA}^A$:*

$$
\begin{array}{ccc}
\pi_*(E) & \xrightarrow{\;\eta_L\;} & E_*(E) \\
{\scriptstyle \eta_R}\downarrow & & \downarrow{\scriptstyle x\mapsto 1\otimes x} \\
E_*(E) & \xrightarrow[x\mapsto x\otimes 1]{} & E_*(E) \otimes_{\pi_*(E)} E_*(E)
\end{array}
$$

Furthermore, with respect to this ring structure, $\Phi_{E,E}$ is a homomorphism of rings:

**Lemma 0.4.** *Let $(E, \mu, e)$ be a commutative monoid object in $\mathcal{SH}$. Then the homomorphism*

$$\Phi_{E,E} : E_*(E) \otimes_{\pi_*(E)} E_*(E) \to E_*(E \otimes E)$$

*constructed in* **??** *is a homomorphism of $A$-graded anticommutative $\pi_*(S)$-algebras.*

*Proof.* Consider the maps

$$f : E \otimes E \xrightarrow{e \otimes E \otimes E} E \otimes E \otimes E$$

and

$$g : E \otimes E \xrightarrow{E \otimes E \otimes e} E \otimes E \otimes E.$$

We know that the maps

$$E \xrightarrow{e \otimes E} E \otimes E \qquad \text{and} \qquad E \xrightarrow{E \otimes e} E \otimes E$$

are monoid homomorphisms by Lemma 0.1, so that $f$ and $g$ are monoid homomorphisms by **??**. Furthermore, by **??**, they are monoid homomorphisms between the same monoid objects in $S\mathcal{H}$ (when we assume that strict associativity holds). Finally, note that we have the following commutative diagram

$$
\begin{array}{ccc}
E & \xrightarrow{\ E \otimes e\ } & E \otimes E \\
{\scriptstyle e \otimes E}\big\downarrow & {\scriptstyle e \otimes E \otimes e} \searrow & \big\downarrow {\scriptstyle e \otimes E \otimes E} \\
E \otimes E & \xrightarrow[E \otimes E \otimes e]{} & E \otimes E \otimes E
\end{array}
$$

where the outer arrows are monoid object homomorphisms, thus, we may apply $\pi_*$, which yields the following commutative diagram in $\pi_*(S)\text{-}\mathbf{GCA}^A$ (**??**):
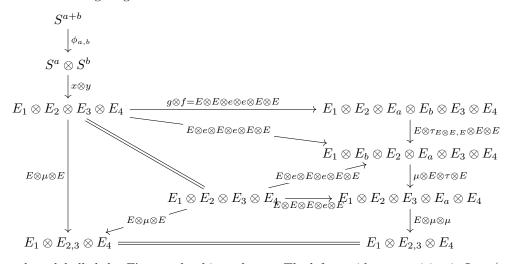
$$
\begin{array}{ccc}
\pi_*(E) & \xrightarrow{\ \eta_L\ } & E_*(E) \\
{\scriptstyle \eta_R}\big\downarrow & & \big\downarrow {\scriptstyle \pi_*(f)} \\
E_*(E) & \xrightarrow[\pi_*(g)]{} & E_*(E \otimes E)
\end{array}
$$

Hence by Lemma 0.4 and the universal property of the pushout, there exists some unique morphism $\ell : E_*(E) \otimes_{\pi_*(E)} E_*(E) \to E_*(E \otimes E)$ in $\pi_*(S)\text{-}\mathbf{GCA}^A$ which makes the following diagram commute:

$$
\begin{array}{ccc}
\pi_*(E) & \xrightarrow{\ \eta_L\ } & E_*(E) \\
{\scriptstyle \eta_R}\big\downarrow & {\scriptstyle x \mapsto 1 \otimes x}\big\downarrow & \\
E_*(E) & \xrightarrow{x \mapsto x \otimes 1} E_*(E) \otimes_{\pi_*(E)} E_*(E) & \\
 & \xrightarrow[\pi_*(g)]{} & E_*(E \otimes E)
\end{array}
$$

with $\pi_*(f)$ and $\ell$ arrows to $E_*(E \otimes E)$.

Thus in order to show $\Phi_{E,E}$ is a morphism in $\pi_*(S)\text{-}\mathbf{GCA}^A$, it suffices to show that $\Phi_{E,E}$ and $\ell$ are the same map, since we know $\ell$ is a homomorphism of $A$-graded anticommutative $\pi_*(S)$-algebras. Since $\Phi_{E,E}$ and $\ell$ are both abelian group homomorphisms, it further suffices to show they agree on homogeneous pure tensors, which generate $E_*(E) \otimes_{\pi_*(E)} E_*(E)$ as an abelian group. Given homogeneous elements $x : S^a \to E \otimes E$ and $y : S^b \to E \otimes E$ in $E_*(E)$, unravelling how pushouts in $\pi_*(S)\text{-}\mathbf{GCA}^A$ are defined (**??**), $\ell$ sends the pure homogeneous tensor $x \otimes y$ to the element $\pi_*(g)(x) \cdot \pi_*(f)(y)$, where here $\cdot$ denotes the product taken in $E_*(E \otimes E) = \pi_*(E \otimes E \otimes E)$. Now,

consider the following diagram:



Here we have labelled the $E$'s to make things clearer. The left outside composition is $\Phi_{E,E}(x \otimes y)$, while the right composition is $\pi_*(g)(x) \cdot \pi_*(f)(y)$. The top right triangle commutes by coherence for a symmetric monoidal category. The middle tright triangle commutes by unitality of $\mu$ and coherence for a symmetric monoidal category. The bottom trapezoid commutes by unitality of $\mu$. The rest of the diagram commutes by definition. Thus we have $\Phi_{E,E}(x \otimes y) = \pi_*(g)(x) \cdot \pi_*(f)(y)$, so that $\Phi_{E,E} = \ell$ is not just an isomorphism of left $\pi_*(E)$-modules, but an isomorphism of $A$-graded anticommutative $\pi_*(S)$-algebras, as desired. $\qquad\square$

For the sake of conciseness, we make the following definition:

**Definition 0.5.** We say that a monoid object $(E, \mu, e)$ in $\mathcal{SH}$ is *flat* if the canonical right $\pi_*(E)$-module structure on $E_*(E)$ from **??** is that of a flat module, or equivalently by Lemma 0.2, if the the map $\eta_R : \pi_*(E) \to E_*(E)$ constructed in Lemma 0.1 is a flat ring homomorphism.

Finally, we can package all of this information into an object called the *dual E-Steenrod algebra*:

**Definition 0.6.** Let $(E, \mu, e)$ be a *commutative* monoid object in $\mathcal{SH}$ which is flat (Definition 0.5) and cellular (**??**). Then the *dual E-Steenrod algebra* is the pair of $A$-graded abelian groups $(E_*(E), \pi_*(E))$ equipped with the following structure:

1. The $A$-graded anticommutative $\pi_*(S)$-algebra structure on $\pi_*(E)$ induced from $E$ being a commutative monoid object in $\mathcal{SH}$ (**??**).

2. The $A$-graded anticommutative $\pi_*(S)$-algebra structure on $E_*(E)$ induced from the fact that $E \otimes E$ is canonically a commutative monoid object in $\mathcal{SH}$ (**??**), so that also $E_*(E) = \pi_*(E \otimes E)$ is an $A$-graded anticommutative $\pi_*(S)$-algebra (**??**).

3. The homomorphisms of $A$-graded anticommutative $\pi_*(S)$-algebras

$$\eta_L : \pi_*(E) \to E_*(E)$$

and

$$\eta_R : \pi_*(E) \to E_*(E)$$

induced under $\pi_*$ by the monoid object homomorphisms

$$E \xrightarrow{\cong} E \otimes S \xrightarrow{E \otimes e} E \otimes E$$

and
$$E \xrightarrow{\cong} S \otimes E \xrightarrow{e \otimes E} E \otimes E.$$

4. The homomorphism of $A$-graded anticommutative $\pi_*(S)$-algebras
$$\Psi_E : E_*(E) \to E_*(E) \otimes_{\pi_*(E)} E_*(E)$$
given by the composition
$$E_*(E) \xrightarrow{h} E_*(E \otimes E) \xrightarrow{\Phi_{E,E}^{-1}} E_*(E) \otimes_{\pi_*(E)} E_*(E),$$
where $h$ is a homomorphism of $A$-graded anticommutative $\pi_*(S)$-algebras induced under $\pi_*$ by the monoid object homomorphism
$$E \otimes E \xrightarrow{\cong} E \otimes S \otimes E \xrightarrow{E \otimes e \otimes E} E \otimes E \otimes E,$$
and $\Phi_{E,E}$ is morphism constructed in **??**, which is proven to be an isomorphism in **??** (since $E$ is flat and cellular), and furthermore an isomorphism in $\pi_*(S)$-$\mathbf{GCA}^A$ by Lemma 0.4.

5. The homomorphism of $A$-graded anticommutative $\pi_*(S)$-algebras
$$\epsilon : E_*(E) \to \pi_*(E)$$
induced under $\pi_*$ by the monoid object homomorphism
$$E \otimes E \xrightarrow{\mu} E.$$

6. The homomorphism of $A$-graded anticommutative $\pi_*(S)$-algebras
$$c : E_*(E) \to E_*(E)$$
induced under $\pi_*$ from the monoid object homomorphism
$$E \otimes E \xrightarrow{\tau} E \otimes E.$$

The curious reader may wonder why we call $(E_*(E), \pi_*(E))$ the *dual $E$-Steenrod algebra*. The "dual" is there because the $E$-Steenrod algebra refers instead to the $E$-self cohomology $E^*(E) \cong [E, E]_{-*}$. Clasically, the Adams spectral sequence was originally constructed in such a way that the $E_1$ and $E_2$ pages could be characterized in terms of cohomology groups as modules over the $E$-Steenrod algebra, but it turns out that our approach using homology groups as comodules over the dual $E$-Steenrod algebra is somewhat better behaved in practice.

0.1. **The dual $E$-Steenrod algebra is a Hopf algebroid.** Above, given a flat and cellular commutative monoid object $(E, \mu, e)$ in $\mathcal{SH}$, we constructed an algebraic gadget $(E_*(E), \pi_*(E))$ in the category $\pi_*(S)$-$\mathbf{GCA}^A$ of $A$-graded anticommutative $\pi_*(S)$-algebras called the *dual $E$-Steenrod algebra*. In this subsection, we will show this object is an example of the general notion of an *$A$-graded anticommutative Hopf algebroid*:

**Proposition 0.7.** *Let $(E, \mu, e)$ be a commutative monoid object in $\mathcal{SH}$ which is flat (Definition 0.5) and cellular (**??**). Then the dual $E$-Steenrod algebra $(E_*(E), \pi_*(E))$ with the structure maps $(\eta_L, \eta_R, \Psi, \epsilon, c)$ from Definition 0.6 is an $A$-graded anticommutative Hopf algebroid over $\pi_*(S)$ (**??**), i.e., a co-groupoid object in the category $\pi_*(S)$-$\mathbf{GCA}^A$.*

*Proof.* All that needs to be done is to show all the diagrams in **??** commute. This is nearly all entirely straightforward, the only real difficulty that arises is showing the co-associativity diagram holds. The argument is sketched in the case $\mathcal{SH}$ is the classical stable homotopy category in sufficent detail in Lecture 3 of the article [1] by Adams. The argument given there works essentially the exact same way here in our more general setting. $\square$

0.2. **Comodules over the dual $E$-Steenrod algebra.** Finally, we can identify some additional structure on $E$-homology groups of (cellular) objects in $\mathcal{SH}$ in terms of the Hopf algebroid structure on the dual $E$-Steenrod algebra.

**Proposition 0.8.** *Let $(E, \mu, e)$ be a flat (Definition 0.5) and cellular (??) commutative monoid object in $\mathcal{SH}$. Then $E_*(-)$ is an additive functor from the full subcategory $\mathcal{SH}$-**Cell** of cellular objects in $\mathcal{SH}$ to the category $E_*(E)$-**CoMod**$^A$ of left $A$-graded comodules (??) over the dual $E$-Steenrod algebra, which is an $A$-graded commutative Hopf algebroid over $\pi_*(S)$, by Proposition 0.7.*

*In particular, given an object $X$ in $\mathcal{SH}$-**Cell**, we are viewing $E_*(X)$ with its canonical left $\pi_*(E)$-module structure (??), and the action map is given by the composition*

$$\Psi_X : E_*(X) \xrightarrow{E_*(e \otimes X)} E_*(E \otimes X) \xrightarrow{\Phi_{E,X}^{-1}} E_*(E) \otimes_{\pi_*(E)} E_*(X).$$

*Proof.* Again, we refer the reader to Lecture 3 in [1], where this is shown in the classical stable homotopy category (although the proof carries over basically verbatim to our setting). $\square$
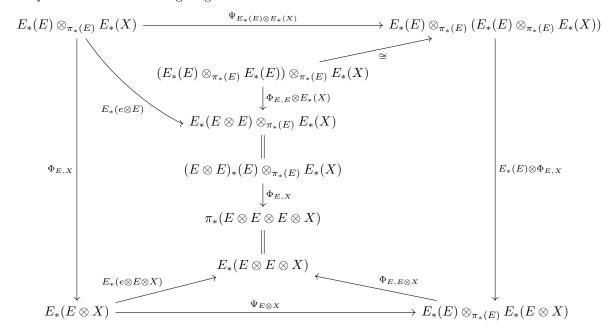
Now, we can use this structure in order to identify the group of maps $X \to E \otimes Y$ with graded $E_*(E)$-comodule homomorphisms from $E_*(X)$ to $E_*(Y)$. First, we need the following two technical lemmas:

**Lemma 0.9.** *Let $(E, \mu, e)$ be a flat (Definition 0.5) and cellular (??) commutative monoid object in $\mathcal{SH}$. Then given an object $X$ in $\mathcal{SH}$, the map*

$$\Phi_{E,X} : E_*(E) \otimes_{\pi_*(E)} E_*(X) \to E_*(E \otimes X)$$

*constructed in ?? is a homomorphism of $A$-graded left $\Gamma$-comodules, where here by ?? we are viewing $E_*(E) \otimes_{\pi_*(E)} E_*(X)$ as the co-free $E_*(E)$-comodule on $E_*(X)$ with its canonical $A$-graded left $\pi_*(E)$-module structure (from ??), and $E_*(E \otimes X)$ with its canonical left $E_*(E)$-comodule structure from Proposition 0.8.*

*Proof.* Consider the following diagram:

The top and bottom regions commute by definition. The left region commutes by naturality of $\Phi_{E,X}$. Thus, it remains to show the rightmost region commutes. To that end, since all the arrows involved are homomorphisms, it suffices to chase a homogeneous pure tensor around. Let $x : S^a \to E \otimes E$, $y : S^b \to E \otimes E$, and $z : S^c \to E \otimes X$, and consider the following diagram:

$$
\begin{array}{c}
S^{a+b+c} \\
\downarrow \phi \\
S^a \otimes S^b \otimes S^c \\
\downarrow x \otimes y \otimes z \\
E \otimes E \otimes E \otimes E \otimes E \otimes X \xrightarrow{E \otimes \mu \otimes E \otimes E \otimes X} E \otimes E \otimes E \otimes E \otimes X \\
\downarrow{\scriptstyle E \otimes E \otimes E \otimes \mu \otimes X} \qquad\qquad\qquad\qquad \downarrow{\scriptstyle E \otimes E \otimes \mu \otimes X} \\
E \otimes E \otimes E \otimes E \otimes X \xrightarrow{E \otimes \mu \otimes E \otimes X} E \otimes E \otimes E \otimes X
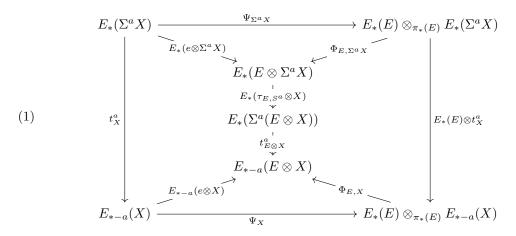\end{array}
$$

The two compositions are the two results of chasing $(x \otimes y) \otimes z$ around the rightmost region in the above diagram. It clearly commutes by functoriality of $- \otimes -$. Hence, indeed we have that $\Phi_{E,X}$ is a homomorphism of left $E_*(E)$-comodules, as desired. $\qquad\square$

**Lemma 0.10.** *Let $(E, \mu, e)$ be a flat (Definition 0.5) and cellular (??) commutative monoid object in $\mathcal{SH}$. Then the isomorphism*

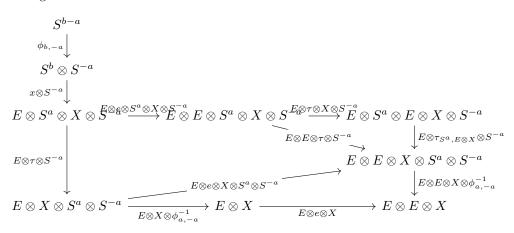$$t^a_X : E_*(\Sigma^a X) \to E_{*-a}(X)$$

*from ?? is an $A$-graded isomorphism of left $E_*(E)$-comodules.*

*Proof.* We know that $t^a_X : E_*(\Sigma^a X) \to E_{*-a}(X)$ is already an $A$-graded isomorphism of left $\pi_*(E)$-modules, so clearly it simply suffices to show that $t^a_X$ is a homomorphism of left $E_*(E)$-comodules. To that end, consider the following diagram:

(1)

$$
\begin{array}{ccc}
E_*(\Sigma^a X) & \xrightarrow{\Psi_{\Sigma^a X}} & E_*(E) \otimes_{\pi_*(E)} E_*(\Sigma^a X) \\
& {\scriptstyle E_*(e \otimes \Sigma^a X)} \searrow \quad \nwarrow {\scriptstyle \Phi_{E, \Sigma^a X}} & \\
& E_*(E \otimes \Sigma^a X) & \\
& \Big\downarrow{\scriptstyle E_*(\tau_{E,S^a} \otimes X)} & \\
t^a_X \Big\downarrow & E_*(\Sigma^a(E \otimes X)) & \Big\downarrow {\scriptstyle E_*(E) \otimes t^a_X} \\
& \Big\downarrow{\scriptstyle t^a_{E \otimes X}} & \\
& E_{*-a}(E \otimes X) & \\
& {\scriptstyle E_{*-a}(e \otimes X)} \nearrow \quad \swarrow {\scriptstyle \Phi_{E,X}} & \\
E_{*-a}(X) & \xrightarrow{\Psi_X} & E_*(E) \otimes_{\pi_*(E)} E_{*-a}(X)
\end{array}
$$

The top and bottom regions commute by definition. To see the left and right regions commute, we'll do a diagram chase of homogeneous elements. First of all, let $x : S^b \to E \otimes S^a \otimes X$ in $E_*(\Sigma^a X)$, and consider the following diagram exhibiting the two ways to chase $x$ around the

leftmost region:

$$
\begin{array}{c}
S^{b-a} \\
\big\downarrow{\scriptstyle\phi_{b,-a}} \\
S^b \otimes S^{-a} \\
\big\downarrow{\scriptstyle x\otimes S^{-a}}
\end{array}
$$

$$
E \otimes S^a \otimes X \otimes S^{-a} \xrightarrow{E\otimes e \otimes S^a \otimes X \otimes S^{-a}} E \otimes E \otimes S^a \otimes X \otimes S^{-a} \xrightarrow{E\otimes \tau \otimes X \otimes S^{-a}} E \otimes S^a \otimes E \otimes X \otimes S^{-a}
$$

$$
\xrightarrow{E\otimes E\otimes \tau \otimes S^{-a}} E \otimes E \otimes X \otimes S^a \otimes S^{-a}
$$

$$
\big\downarrow{\scriptstyle E\otimes \tau_{S^a, E\otimes X}\otimes S^{-a}}
$$

$$
{\scriptstyle E\otimes \tau \otimes S^{-a}}\big\downarrow \qquad\qquad \big\downarrow{\scriptstyle E\otimes E\otimes X \otimes \phi_{a,-a}^{-1}}
$$

$$
E \otimes X \otimes S^a \otimes S^{-a} \xrightarrow[E\otimes X\otimes \phi_{a,-a}^{-1}]{E\otimes e\otimes X\otimes S^a \otimes S^{-a}} E \otimes X \xrightarrow{E\otimes e\otimes X} E \otimes E \otimes X
$$

The top right region commutes by coherence for the symmetries, while the other two regions commute by functoriality of $-\otimes-$. Thus, it remains to show the rightmost region in diagram (1) commutes. To that end, let $x : S^b \to E \otimes E$ in $E_*(E)$ and $y : S^c \to E \otimes S^a \otimes X$ in $E_*(\Sigma^a X)$, and consider the following diagram, which exhibits the two ways to chase $x \otimes y$ around the rightmost region of diagram (1):

$$
\begin{array}{c}
S^{b+c-a} \\
\big\downarrow{\scriptstyle\phi} \\
S^b \otimes S^c \otimes S^{-a} \\
\big\downarrow{\scriptstyle x\otimes y\otimes S^{-a}}
\end{array}
$$

$$
E \otimes E \otimes E \otimes S^a \otimes X \otimes S^{-a} \xrightarrow{E\otimes \mu \otimes S^a \otimes X \otimes S^{-a}} E \otimes E \otimes S^a \otimes X \otimes S^{-a} \xrightarrow{E\otimes \tau \otimes X \otimes S^{-a}} E \otimes S^a \otimes E \otimes X \otimes S^{-a}
$$

$$
{\scriptstyle E\otimes E\otimes E\otimes \tau \otimes S^{-a}}\big\downarrow \qquad {\scriptstyle E\otimes E\otimes \tau \otimes S^{-a}}\big\downarrow \qquad \big\downarrow{\scriptstyle E\otimes \tau_{S^a, E\otimes X} S^{-a}}
$$

$$
E \otimes E \otimes E \otimes X \otimes S^a \otimes S^{-a} \xrightarrow{E\otimes \mu \otimes X\otimes S^a \otimes S^{-a}} E \otimes E \otimes X \otimes S^a \otimes S^{-a}
$$

$$
{\scriptstyle E\otimes E\otimes E\otimes X \otimes \phi_{a,-a}^{-1}}\big\downarrow \qquad\qquad \big\downarrow{\scriptstyle E\otimes E\otimes X \otimes \phi_{a,-a}^{-1}}
$$

$$
E \otimes E \otimes E \otimes X \xrightarrow{E\otimes \mu \otimes X} E \otimes E \otimes X
$$

The top right region commutes by coherence for the symmetries. The remaining two regions commute by functoriality of $-\otimes-$. Thus, indeed we have that diagram (1) commutes, so $t_X^a$ is a homomorphism of left $E_*(E)$-comodules, as desired. $\qquad\square$

Now we may prove the theorem.

**Theorem 0.11.** *Let $(E, \mu, e)$ be a commutative monoid object, and $X$ and $Y$ objects in $\mathcal{SH}$. Suppose further that:*

- *$E$ is flat (Definition 0.5) and cellular (??),*
- *$X$ is cellular and $E_*(X)$ is a graded projective left $\pi_*(E)$-module (via ??), and*
- *$Y$ is cellular.*

*Then the assignment*

$$
E_*(-) : [X, E \otimes Y] \to \operatorname{Hom}_{E_*(E)}(E_*(X), E_*(E \otimes Y)), \qquad f \mapsto E_*(f)
$$

*induced by the functor* $E_*(-) : \mathcal{SH}\text{-}\mathbf{Cell} \to E_*(E)\text{-}\mathbf{CoMod}^A$ *(Proposition 0.8) is an isomorphism of abelian groups.*

*Proof.* Since $X$ is cellular, by Proposition 0.8 we have that $E_*(X)$ is canonically an $A$-graded left $E_*(E)$-comodule. Similarly, since $E$ and $Y$ are cellular, we know that $E \otimes Y$ is cellular, so that $E_*(E \otimes Y)$ is also canonically an $E_*(E)$-comodule. Thus, we have a well-defined assignment

$$[X, E \otimes Y] \xrightarrow{E_*(-)} \mathrm{Hom}_{E_*(E)}(E_*(X), E_*(E \otimes Y)).$$

To see this arrow is an isomorphism, consider the following diagram:

$$
\begin{array}{ccc}
[X, E \otimes Y] & \xrightarrow{\quad E_*(-) \quad} & \mathrm{Hom}_{E_*(E)}(E_*(X), E_*(E \otimes Y)) \\
{\scriptstyle \pi_*(\mu \otimes Y) \circ E_*(-)} \downarrow & {\scriptstyle \pi_*(\mu \otimes Y) \circ (-)} \nearrow & \uparrow {\scriptstyle (\Phi_{E,Y})_*} \\
\mathrm{Hom}_{\pi_*(E)}(E_*(X), E_*(Y)) & \xleftarrow{\quad \mathrm{adj} \quad} & \mathrm{Hom}_{E_*(E)}(E_*(X), E_*(E) \otimes_{\pi_*(E)} E_*(Y))
\end{array}
$$

We know the left vertical map is an isomorphism by **??**, and the bottom horizontal isomorphism is the forgetful-cofree adjunction (**??**) for $A$-graded left comodules over the dual $E$-Steenrod algebra. The right vertical arrow is a well-defined isomorphism, as $\Phi_{E,Y}$ is a homomorphism of $A$-graded left $E_*(E)$-comodules (Lemma 0.9), and in fact it is an isomorphism by **??**, since $E_*(E)$ is flat and $Y$ is cellular. Thus in order to see the top arrow is an isomorphism, it suffices to show that the diagram commutes. The left triangle clearly commutes; to see the right triangle commutes, recall that by how the how forgetful-cofree adjunction for left comodules over a Hopf algebroid is defined, that the bottom vertical arrow sends an $A$-graded homomorphism of left $E_*(E)$-comodules $\psi : E_*(X) \to E_*(E) \otimes_{\pi_*(E)} E_*(Y)$ to the composition

$$E_*(X) \xrightarrow{\psi} E_*(E) \otimes_{\pi_*(E)} E_*(Y) \xrightarrow{\pi_*(\mu) \otimes E_*(Y)} \pi_*(E) \otimes_{\pi_*(E)} E_*(Y) \xrightarrow{\cong} E_*(Y).$$

Thus, in order to show that this composition equals $\pi_*(\mu \otimes Y) \circ \Phi_{E,Y} \circ \psi$, it suffices to show the following diagram commutes:

$$
\begin{array}{ccc}
E_*(E) \otimes_{\pi_*(E)} E_*(Y) & \xrightarrow{\pi_*(\mu) \otimes E_*(Y)} & \pi_*(E) \otimes_{\pi_*(E)} E_*(Y) \\
{\scriptstyle \Phi_{E,Y}} \downarrow & & \downarrow {\scriptstyle \cong} \\
E_*(E \otimes Y) & \xrightarrow{\quad \pi_*(\mu \otimes Y) \quad} & E_*(Y)
\end{array}
$$

Since all the arrows here are homomorphisms of abelian groups, in order to show the diagram commutes, it suffices to chase pure homogeneous tensors around. To that end, let $x : S^a \to E \otimes E$ and $y : S^b \to E \otimes Y$, and consider the following diagram exhibiting the two ways to chase $x \otimes y$ around:

$$
\begin{array}{ccccccc}
S^{a+b} & \xrightarrow{\phi_{a,b}} & S^a \otimes S^b & \xrightarrow{x \otimes y} & E \otimes E \otimes E \otimes Y & \xrightarrow{\mu \otimes E \otimes Y} & E \otimes E \otimes Y \\
& & & & {\scriptstyle E \otimes \mu \otimes Y} \downarrow & & \downarrow {\scriptstyle \mu \otimes Y} \\
& & & & E \otimes E \otimes Y & \xrightarrow{\quad \mu \otimes Y \quad} & E \otimes Y
\end{array}
$$

The diagram commutes by associtiavity of $\mu$. Thus, we have indeed shown that

$$E_*(-) : [X, E \otimes Y] \to \mathrm{Hom}_{E_*(E)}(E_*(X), E_*(E \otimes Y))$$

is an isomorphism of abelian groups, as desired. $\qquad\square$