AN AXIOMATIC APPROACH TO THE ADAMS SPECTRAL SEQUENCE

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Contents

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1. Introduction

Homotopy theory is the study of spaces, where we consider two spaces to be "the same" if one can be stretched or deformed into the other. More specifically, given two continuous based maps of based spaces $f, g : X \to Y$, we say these maps are *homotopic* if there exists a family of based maps $h_t : X \to Y$ indexed by $t \in [0, 1]$ such that $h_0 = f$, $h_1 = g$, and the assignment

$$X \times I \to Y, \qquad (x, t) \mapsto h_t(x)$$

is continuous. Two spaces X and Y are *homotopy equivalent* if there exists maps $f : X \to Y$ and $g : Y \to X$ such that $f \circ g$ is homotopic to id_Y and $g \circ f$ is homotopic to id_X. Intuitively, two based spaces are homotopy equivalent if we can contiously squish and deform one into the other without moving the basepoint. Thus, one of the most important problems in algebraic topology is computing the homotopy classes of based maps $[X, Y]$ between two based spaces. Sadly, this is an extremely difficult problem, and in general there is no way to characterize these sets. Yet, there is some hope, if we focus our attention to nicer spaces. For one, topologists are mainly interested in *CW complexes*, which roughly are spaces that can be built inductively by gluing spheres and disks of increasingly higher dimension together. Thus, the hope is that if we can understand the sets $\pi_n(S^m) := [S^n, S^m]$ of based homotopy classes of maps of spheres, then we can understand all the ways to build CW complexes and construct (nice) maps between them. But how would we even begin to understand these sets? First of all, for $n > 0$, these sets become groups, and for $n > 1$ they are in fact abelian groups. There are also some computations that can be made. For $n > 0$, it is known that $[S^n, S^n] \cong \mathbb{Z}$. Furthermore, if $n < m$, then $[S^n, S^m] = 0$. We also have the following famous theorem of Freudenthal:

Theorem 1.1. *Given $k \geq 0$, the homotopy groups $\pi_{n+k}(S^n) = [S^{n+k}, S^n]$ are independent of n for $n > k + 1$.*

In other words, in order to compute the groups $[S^n, S^m]$, if n and m are large enough, it suffices only to know the difference between n and m. Thus, we may consider the *stable homotopy groups of spheres* π_k^S, which, up to isomorphism, are the abelian groups $[S^{n+k}, S^k]$ for $n > k + 1$. This theorem is actually a corollary of a more general result. Given a based space X, we may take its *suspension* ΣX. If X is a CW complex, taking suspension has the effect of shifting all the cells of X up a dimension. In particular, for all $n \geq 0$, there are homeomorphisms $\Sigma S^n \cong S^{n+1}$.

Theorem 1.2. *For X a based CW complex with no cells of dimension $\geq 2n$ and Y a based CW complex with $[S^k, Y] = *$ for $k \leq n$, there are isomorphisms*

$$[X, Y] \xrightarrow{\cong} [\Sigma Y, \Sigma X] \xrightarrow{\cong} [\Sigma^2 Y, \Sigma^2 X],$$

and $[X, Y]$ is canonically an abelian group.

Outline: justify construction of stable homotopy category, talk about how goal of ASS is to compute $[X, Y]$ via E-homology of X and Y. "The majority of this paper is dedicated to indentifying as much structure as possible on the groups $E_*(X)$ for E a multiplicative (co)homology theory, so that we can successively approximate $[X, Y]$ by means of sequences of maps from $E_*(X)$ to $E_*(Y)$, via the ASS"

This paper strives to achieve several goals, in particular, we aim to:

(1) Provide an axiomatic generalization of the Adams spectral sequence which applies equally to the classical, motivic, and equivariant stable homotopy categories.

(2) Provide a reference for the full and explicit details of the construction of the classical, motivic, and equivariant E-Adams spectral sequences, the characterization of their E_2 pages, and the proofs of their convergence.

This project originally aimed to achieve the second goal, specifically for the motivic Adams spectral sequence. Along the way, the idea for the generalization came to the author after reading the pair of papers [2] and [3]. We provide several innovations:

(1) We provide a general construction of the Adams spectral sequence which equally applies to the classical, motivic, and equivariant stable homotopy categories. In particular, in the equivariant case, we construct a spectral sequence which intrisically keeps track of the $RO(G)$ grading, or, if one likes, could be constructed to be graded by the entirety of the Picard group of the equivariant stable homotopy category.

(2) We develop the notion of a "tensor-triangulated category with sub-Picard grading," which roughly is a category which is graded by some abelian group, symmetric monoidal, and triangulated, all in a compatible way. This provides a surprisingly powerful axiomatization of the (classical, motivic, equivariant) stable homotopy category, and a shockingly large amount of the theory therewithin can be carried out entirely in this framework.

(3) We provide an encompassing notion of "cellularity" in a tensor triangulated category with sub-Picard grading, which parallels the same notion in the motivic stable homotopy category.

(4) We work out some of the graded-commutativity properties of $\pi_*(E)$ for a commutative monoid object (E, μ, e) in a tensor triangulated category with sub-Picard grading.

(5) We develop much of the theory of A-graded abelian groups, rings, modules, submodules, quotient modules, tensor products, homomorpisms, etc.

(6) We suggest a definition for the correct notion of an "anticommutative A-graded ring" for a general abelian group A. In particular, we suggest a new candidate for the category in which the motivic Steenrod algebra is a Hopf algebroid.

(7) add more

In Section 2, we will develop the notion of tensor triangulated categories with sub-Picard grading, which may be thought of as categories which are symmetric monoidal, triangulated, and graded by an abelian group A, all in a compatible way. We will then fix such a category $\mathcal{SH}$ (with a few extra categorical conditions), which acts as an axiomatic model for the classical, motivic, and equivariant stable homotopy categories. In this category, we will be able to develop much of the theory of stable homotopy theory, in particular, we will be able to formulate the notion of A-graded stable homotopy groups $\pi_*(X)$ of objects X in $\mathcal{SH}$, as well as homology, and cohomology represented by objects in this category. We will show that (co)fiber sequences (distinguished triangles) in $\mathcal{SH}$ give rise to long exact sequences of homotopy groups, and that $\mathcal{SH}$ is equipped with an A-indexed family of suspension and loop autoequivalences.

After just this first section, we will actually have all the data needed to construct the Adams spectral sequence, yet we will not actually do so until Section 7. The goal of this spectral sequence will be to compute the A-graded abelian groups of stable homotopy classes of maps $[X, Y]_*$ between objects X and Y in $\mathcal{SH}$, by means of algebraic information about the E-homology of X and Y. Yet, looking at just the definition of the E-Adams spectral sequence, it will not be immediately clear how exactly it achieves this goal in any sense. Thus, before constructing the sequence, Sections 2–6 will be devoted to formulating suitable conditions on E, X, and Y under which the E-homology groups $E_*(X)$ and $E_*(Y)$ have enough structure that algebraic information about maps between them gives suitable information about the groups $[X, Y]_*$.

In Section 3, we will formulate the notion of *cellular* objects in $\mathcal{SH}$. Intuitively, these are the objects in $\mathcal{SH}$ which may be constructed by gluing together copies of spheres. In the case $\mathcal{SH}$ is the motivic stable homotopy category, these objects will correspond to the standard notion of cellular motivic spaces. In the case $\mathcal{SH}$ is the classical stable homotopy category, every object will turn out to be cellular, as a consequence of the fact that every space is weakly equivalent to a CW complex. The class of cellular objects in $\mathcal{SH}$ will satisfy many very important properties, for example, given cellular objects X and Y in $\mathcal{SH}$, a map $f : X \to Y$ will be an isomorphism iff it induces an isomorphism on stable homotopy groups $\pi_*(f) : \pi_*(X) \to \pi_*(Y)$. Many of the important theorems and propositions presented in this paper will require some sort of cellularity condition.

In Section 4, we will discuss the theory of monoid objects in $\mathcal{SH}$, which correspond to ring spectra in stable homotopy theory. We will show that given a monoid object E in $\mathcal{SH}$, its stable homotopy groups $\pi_*(E)$ naturally form an A-graded ring, and furthermore, E-homology $E_*(-)$ will yield a functor from $\mathcal{SH}$ to the category of A-graded left modules over $\pi_*(E)$. A great deal of effort will be put into formulating the exact sense in which the rings $\pi_*(E)$ are *A-graded anticommutative* when E is a commutative monoid object in $\mathcal{SH}$. In particular, we will show that $\pi_*(E)$ is an *A-graded anticommutative algebra* over the A-graded anticommutative stable homotopy ring $\pi_*(S)$ (where S is the monoidal unit in $\mathcal{SH}$), in a suitable sense.

In Section 5, we will prove analogues of important theorems for homology in $\mathcal{SH}$. First of all, we will prove that for E a commutative monoid object and objects X and Y in $\mathcal{SH}$, under suitable conditions we have a *Künneth isomorphism*

$$Z_*(E) \otimes_{\pi_*(E)} E_*(W) \to \pi_*(Z \otimes E \otimes W)$$

relating the Z-homology of E and the E-homology of W to the stable homotopy groups of $Z \otimes E \otimes W$. We will then take a bit to develop the theory of module objects over monoid objects in $\mathcal{SH}$, with which we will prove a generalization of the universal coefficient theorem, which will tell us that under suitable conditions, for a monoid object E in $\mathcal{SH}$ and an object X, the cohomology $E^*(X)$ of X is the dual of the homology $E_*(X)$ as a $\pi_*(E)$-module. These two theorems will be very important for our later work.

In Section 6, we will show that for nice enough commutative monoid objects E in $\mathcal{SH}$, that the E-self homology $E_*(E)$, along with the ring $\pi_*(E)$, forms an *A-graded anticommutative Hopf algebroid*, which we define to be a co-groupoid object in the category $\pi_*(S)$-$\mathbf{GCA}^A$ of A-graded anticommutative $\pi_*(S)$-algebras. This pair $(E_*(E), \pi_*(E))$ with its additional structure as a Hopf algebroid is called the *dual E-Steenrod algebra*, over which the A-graded E-homology group $E_*(X)$ of X is canonically an A-graded left comodule for each X in $\mathcal{SH}$.

Finally, in Section 7, we will construct the $\mathbb{Z} \times A$-graded E-Adams spectral sequence for the computation of X and Y, and we will show that under suitable conditions,

$$E_2^{*,*}(X, Y) \cong \mathrm{Ext}^{*,*}(E_*(X), E_*(Y)).$$

Furthermore, we will show this spectral sequence converges to the groups $[X, Y_E]_*$, where Y_E is the E-completion of Y in some sense. Thus, we will have developed a tool to compute information about the groups $[X, Y]_*$ from data about the E-homology groups of X and Y.

fix above (ugh), add blurbs about sections 8 and 9, as well as appendices

2. Preliminaries

2.1. **Triangulated categories with sub-Picard grading.** In order to construct an abstract version of the Adams spectral sequence, we need to work in some axiomatic version of a stable homotopy category $\mathcal{SH}$ which acts like the familiar classical stable homotopy category **hoSp** (**??**) or the motivic stable homotopy category $\mathbf{SH}_{\mathscr{S}}$ over some base scheme $\mathscr{S}$ (**??**). This setting will be a category which is:

(1) Symmetric monoidal closed
(2) Triangulated
(3) Graded by (some collection of) its invertible objects

These data are furthermore required to be compatible.

Definition 2.1. Let $\mathcal{C}$ be an additive category with arbitrary (small) coproducts. Then an object X in $\mathcal{C}$ is *compact* if, for any collection of objects Y_i in $\mathcal{C}$ indexed by some (small) set I, the canonical map

$$\bigoplus_i \mathcal{C}(X, Y_i) \to \mathcal{C}(X, \bigoplus_i Y_i)$$

is an isomorphism of abelian groups. (Explicitly, the above map takes a generator $x \in \mathcal{C}(X, Y_i)$ to the composition $X \xrightarrow{x} Y_i \hookrightarrow \bigoplus_i Y_i$.)

Definition 2.2. Given a tensor triangulated category $(\mathcal{C}, \otimes, S, \Sigma, e, \mathcal{D})$ (Definition A.12), a *sub-Picard grading* on $\mathcal{C}$ is the following data:

- A pointed abelian group $(A, \mathbf{1})$ along with a homomorphism of pointed groups $h : (A, \mathbf{1}) \to (\operatorname{Pic}\mathcal{C}, \Sigma S)$, where $\operatorname{Pic}\mathcal{C}$ is the *Picard group* of isomorphism classes of invertible objects in $\mathcal{C}$.[1]
- For each $a \in A$, a chosen representative S^a called the *a-sphere* in the isomorphism class $h(a)$ such that each S^a is a compact object (Definition 2.1) and $S^0 = S$.
- For each $a, b \in A$, an isomorphism $\phi_{a,b} : S^{a+b} \to S^a \otimes S^b$. This family of isomorphisms is required to be *coherent*, in the following sense:
  - For all $a \in A$, we must have that $\phi_{a,0}$ coincides with the right unitor $\rho^{-1}_{S^a} : S^a \xrightarrow{\cong} S^a \otimes S$ and $\phi_{0,a}$ coincides the left unitor $\lambda^{-1}_{S^a} : S^a \xrightarrow{\cong} S \otimes S^a$.
  - For all $a, b, c \in A$, the following "associativity diagram" must commute:

$$\begin{tikzcd}
S^{a+b} \otimes S^c \arrow[d, "\phi_{a,b} \otimes S^c"'] & S^{a+b+c} \arrow[l, "\phi_{a+b,c}"'] \arrow[r, "\phi_{a,b+c}"] & S^a \otimes S^{b+c} \arrow[d, "S^a \otimes \phi_{b,c}"] \\
(S^a \otimes S^b) \otimes S^c \arrow[rr, "\cong"] & & S^a \otimes (S^b \otimes S^c)
\end{tikzcd}$$

From now on we fix a monoidal closed tensor triangulated category $(\mathcal{SH}, \otimes, S, \Sigma, e, \mathcal{D})$ with arbitrary (small) (co)products and sub-Picard grading $(A, \mathbf{1}, h, \{S^a\}, \{\phi_{a,b}\})$. We also fix an

[1] Recall an object X is a symmetric monoidal category is *invertible* if there exists some object Y and an isomorphism $S \cong X \otimes Y$.

isomorphism $\nu : \Sigma S \xrightarrow{\cong} S^{\mathbf{1}}$ once and for all. We establish conventions. First, observe the following remark:

Remark 2.3. Note that by induction the coherence conditions for the $\phi_{a,b}$'s in the above definition say that given any $a_1, \dots, a_n \in A$ and $b_1, \dots, b_m \in A$ such that $a_1 + \cdots + a_n = b_1 + \cdots + b_m$ and any fixed parenthesizations of $X = S^{a_1} \otimes \cdots \otimes S^{a_b}$ and $Y = S^{b_1} \otimes \cdots \otimes S^{b_m}$, there is a *unique* isomorphism $X \to Y$ that can be obtained by forming formal compositions of products of $\phi_{a,b}$, identities, associators, unitors, and their inverses (but not symmetries).

In light of this remark, we will usually simply write ϕ or even just $\cong$ for any isomorphism that is built by taking compositions of products of $\phi_{a,b}$'s, unitors, associators, identities, and their inverses. Given an object X and a natural number $n > 0$, we write

$$X^n := \overbrace{X \otimes \cdots \otimes X}^{n \text{ times}} \qquad \text{and} \qquad X^0 := S.$$

We denote the associator, symmetry, left unitor, and right unitor isomorphisms in $\mathcal{SH}$ by

$$\alpha_{X,Y,Z} : (X \otimes Y) \otimes Z \xrightarrow{\cong} X \otimes (Y \otimes Z) \qquad\qquad \tau_{X,Y} : X \otimes Y \xrightarrow{\cong} Y \otimes X$$
$$\lambda_X : S \otimes X \xrightarrow{\cong} X \qquad\qquad \rho_X : X \otimes S \xrightarrow{\cong} X.$$

Often we will drop the subscripts. Furthermore, by the coherence theorem for symmetric monoidal categories ([4]), we will often assume α, ρ, and λ are actual equalities.

Given some integer $n \in \mathbb{Z}$, we will write a bold $\mathbf{n}$ to denote the element $n \cdot \mathbf{1}$ in A. Note that we can use the isomorphism $\nu : \Sigma S \xrightarrow{\cong} S^{\mathbf{1}}$ to construct a natural isomorphism $\Sigma \cong S^{\mathbf{1}} \otimes -$:

$$\Sigma X \xrightarrow{\Sigma \lambda_X^{-1}} \Sigma(S \otimes X) \xrightarrow{e_{S,X}^{-1}} \Sigma S \otimes X \xrightarrow{\nu \otimes X} S^{\mathbf{1}} \otimes X,$$

where $e_{X,Y} : \Sigma X \otimes Y \to \Sigma(X \otimes Y)$ is the isomorphism specified by the fact that $\mathcal{SH}$ is tensor-triangulated. The first two arrows are natural in X by definition. The last arrow is natural in X by functoriality of $- \otimes -$. By abuse of notation, we will always use ν to denote this natural isomorphism, rather than the isomorphism $\Sigma S \xrightarrow{\cong} S^{\mathbf{1}}$.

Given some $a \in A$, we define $\Sigma^a := S^a \otimes -$ and $\Omega^a := \Sigma^{-a} = S^{-a} \otimes -$. We specifically define $\Omega := \Omega^{\mathbf{1}}$. We say "the a^{th} suspension of X" to denote $\Sigma^a X$. It turns out that Σ^a is an autoequivalence of $\mathcal{SH}$ for each $a \in A$, and furthermore, Ω^a and Σ^a form an adjoint equivalence of $\mathcal{SH}$ for all a in A:

Proposition 2.4. *For each $a \in A$, the isomorphisms*

$$\eta_X^a : X \xrightarrow{\lambda_X^{-1}} S \otimes X \xrightarrow{\phi_{a,-a} \otimes X} (S^a \otimes S^{-a}) \otimes X \xrightarrow{\alpha} S^a \otimes (S^{-a} \otimes X) = \Sigma^a \Omega^a X$$

and

$$\varepsilon_X^a : \Omega^a \Sigma^a X = S^{-a} \otimes (S^a \otimes X) \xrightarrow{\alpha^{-1}} (S^{-a} \otimes S^a) \otimes X \xrightarrow{\phi_{-a,a}^{-1} \otimes X} S \otimes X \xrightarrow{\lambda_X} X$$

are natural in X, and furthermore, they are the unit and counit respectively of the adjoint autoequivalence $(\Omega^a, \Sigma^a, \eta^a, \varepsilon^a)$ of $\mathcal{SH}$. In particular, since $\Sigma \cong \Sigma^{\mathbf{1}}$, $\Omega := \Omega^{\mathbf{1}}$ is a left adjoint for Σ, so that $(\mathcal{SH}, \Omega, \Sigma, \eta, \varepsilon, \mathcal{D})$ is an adjointly *triangulated category (Definition A.8), where η and ε are the compositions*

$$\eta : \mathrm{Id}_{\mathcal{SH}} \xRightarrow{\eta^{\mathbf{1}}} \Sigma^{\mathbf{1}} \Omega \xRightarrow{\nu^{-1}\Omega} \Sigma\Omega \qquad \textit{and} \qquad \varepsilon : \Omega\Sigma \xRightarrow{\Omega\nu} \Omega\Sigma^{\mathbf{1}} \xRightarrow{\varepsilon^{\mathbf{1}}} \mathrm{Id}_{\mathcal{SH}}.$$

Proof. In this proof, we will freely employ the coherence theorem for monoidal categories (see [4]), which essentially tells us that we may assume we are working in a strict monoidal category (i.e., that the associators and unitors and are identities). Then η^a_X and ε^a_X become simply the maps

$$\eta^a_X : X \xrightarrow{\phi_{a,-a}\otimes X} S^a \otimes S^{-a} \otimes X \qquad \text{and} \qquad \varepsilon^a_X : S^{-a} \otimes S^a \otimes X \xrightarrow{\phi^{-1}_{-a,a}\otimes X} X.$$

That these maps are natural in X follows by functoriality of $-\otimes-$. Now, recall that in order to show that these natural isomorphisms form an *adjoint* equivalence, it suffices to show that the natural isomorphisms $\eta^a : \mathrm{Id}_{\mathcal{SH}} \Rightarrow \Omega^a\Sigma^a$ and $\varepsilon^a : \Sigma^a\Omega^a \Rightarrow \mathrm{Id}_{\mathcal{SH}}$ satisfy one of the two zig-zag identities:

$$\begin{tikzcd} \Omega^a \arrow[r, "\Omega^a\eta^a"] \arrow[dr, equal] & \Omega^a\Sigma^a\Omega^a \arrow[d, "\varepsilon^a\Omega^a"] \\ & \Omega^a \end{tikzcd} \qquad \begin{tikzcd} \Sigma^a\Omega^a\Sigma^a \arrow[d, "\Sigma^a\varepsilon^a"'] & \Sigma^a \arrow[l, "\eta^a\Sigma^a"'] \arrow[dl, equal] \\ \Sigma^a & \end{tikzcd}$$

(that it suffices to show only one is [7, Lemma 3.2]). We will show that the left is satisfied. Unravelling definitions, we simply wish to show that the following diagram commutes for all X in $\mathcal{SH}$:

$$\begin{tikzcd} S^{-a}\otimes X \arrow[r, "S^{-a}\otimes\phi_{a,-a}\otimes X"] \arrow[dr, equal] & S^{-a}\otimes S^a\otimes S^{-a}\otimes X \arrow[d, "\phi^{-1}_{-a,a}\otimes S^{-a}\otimes X"] \\ & S^{-a}\otimes X \end{tikzcd}$$

Yet this is simply the diagram obtained by applying $-\otimes X$ to the associativity coherence diagram for the $\phi_{a,b}$'s (since $\phi_{a,0}$ and $\phi_{0,a}$ coincide with the unitors, and here we are taking the unitors and associators to be equalities), so it does commute, as desired. □

Given two objects X and Y in $\mathcal{SH}$, we will denote the hom-abelian group of morphisms from X to Y in $\mathcal{SH}$ by $[X,Y]$, and the internal hom object by $F(X,Y)$. We can extend the abelian group $[X,Y]$ into an A-graded abelian group $[X,Y]_*$ by defining $[X,Y]_a := [S^a\otimes X, Y]$.

Given an object X in $\mathcal{SH}$ and some $a\in A$, we can define the abelian group

$$\pi_a(X) := [S^a, X],$$

which we call the a^{th} *(stable) homotopy group of* X. We write $\pi_*(X)$ for the A-graded abelian group $\bigoplus_{a\in A}\pi_a(X)$, so that in particular we have a canonical isomorphism

$$\pi_*(X) = [S^*, X] \cong [S, X]_*.$$

Given some other object E, we can define the A-graded abelian groups $E_*(X)$ and $E^*(X)$ by the formulas

$$E_a(X) := \pi_a(E\otimes X) = [S^a, E\otimes X] \qquad \text{and} \qquad E^a(X) := [X, S^a\otimes E].$$

We refer to the functor $E_*(-)$ as the *homology theory represented by* E, or just E-homology, and we refer to $E^*(-)$ as the *cohomology theory represented by* E, or just E-cohomology.

A nice result is that in $\mathcal{SH}$, (co)fiber sequences (distinguished triangles) give rise to homotopy long exact sequences. To see this, we first need the following lemma:

Definition 2.5. For all X, Y in $\mathcal{SH}$ and $a\in A$, there are A-graded isomorphisms

$$s^a_{X,Y} : [X, \Sigma^a Y]_* \to [X,Y]_{*-a}$$

sending $x : S^b\otimes X \to S^a\otimes Y$ in $[X,\Sigma^a Y]_*$ to the composition

$$S^{b-a}\otimes X \xrightarrow{\phi_{-a,b}\otimes X} S^{-a}\otimes S^b\otimes X \xrightarrow{S^{-a}\otimes x} S^{-a}\otimes S^a\otimes Y \xrightarrow{\phi^{-1}_{-a,a}\otimes Y} Y.$$

Furthermore, these isomorphisms are natural in both X and Y.

In particular, for each $a \in A$ and object X in $\mathcal{SH}$, we have natural isomorphisms

$$s^a_X : \pi_*(\Sigma^a X) = [S^*, \Sigma^a X] \xrightarrow{\cong} [S, \Sigma^a X]_* \xrightarrow{s^a_{S,X}} [S, X]_{*-a} \xrightarrow{\cong} \pi_{*-a}(X)$$

sending $x : S^b \to S^a \otimes X$ in $\pi_*(\Sigma^a X)$ to the composition

$$S^{b-a} \xrightarrow{\phi_{-a,b}} S^{-a} \otimes S^b \xrightarrow{S^{-a}\otimes x} S^{-a} \otimes S^a \otimes X \xrightarrow{\phi^{-1}_{-a,a}\otimes X} X.$$

Proof. First, by unravelling definitions, note that $s^a_{X,Y}$ is precisely the composition

$$[X, \Sigma^a Y]_* = [S^* \otimes X, S^a \otimes Y] \xrightarrow{\text{adj}} [S^{-a} \otimes S^* \otimes X, Y] \xrightarrow{(\phi_{-a,*}\otimes X)^*} [S^{*-a} \otimes X, Y] = [X, Y]_{*-a},$$

where the adjunction is that from Proposition 2.4. The adjunction is natural in $S^* \otimes X$ and Y by definition, so that in particular it is natural in X and Y. It is furthermore straightforward to see by functoriality of $- \otimes -$ that the second arrow is natural in both X and Y. Thus $s^a_{X,Y}$ is natural in X and Y, as desired. ∎

Proposition 2.6. *Suppose we are given a distinguished triangle*

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$$

and an object W in $\mathcal{SH}$. Then there exists a "connecting homomorphism" of degree $-\mathbf{1}$

$$\partial : [W, Z]_* \to [W, X]_{*-\mathbf{1}}$$

such that the following triangle is exact at each vertex:

$$\begin{tikzcd} {[W,X]_*} \arrow[r, "f_*"] & {[W,Y]_*} \arrow[d, "g_*"] \\ & {[W,Z]_*} \arrow[ul, "\partial"] \end{tikzcd}$$

Proof. By axiom TR4 for a triangulated category and the fact that distinguished triangles are exact (Proposition A.3), we have the following exact sequence in $\mathcal{SH}$

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X \xrightarrow{\Sigma f} \Sigma Y.$$

Thus, we may apply $[W, -]_*$ to get an exact sequence of A-graded abelian groups which fits into the top row in the following diagram:

$$\begin{tikzcd} {[W,X]_*} \arrow[r, "f_*"] \arrow[dd, equal] & {[W,Y]_*} \arrow[r, "g_*"] \arrow[dd, equal] & {[W,Z]_*} \arrow[r, "h_*"] \arrow[dd, equal] & {[W,\Sigma X]_*} \arrow[r, "\Sigma f_*"] \arrow[d, "(\nu_X)_*"] & {[W,\Sigma Y]_*} \arrow[d, "(\nu_Y)_*"] \\ & & & {[W,\Sigma^{\mathbf{1}} X]_*} \arrow[r, "\Sigma^{\mathbf{1}} f_*"] \arrow[d, "s^{\mathbf{1}}_{W,X}"] & {[W,\Sigma^{\mathbf{1}} Y]_*} \arrow[d, "s^{\mathbf{1}}_{W,Y}"] \\ {[W,X]_*} \arrow[r, "f_*"] & {[W,Y]_*} \arrow[r, "g_*"] & {[W,Z]_*} \arrow[r, dashed, "\partial"] & {[W,X]_{*-\mathbf{1}}} \arrow[r, "f_*"] & {[W,Y]_{*-\mathbf{1}}} \end{tikzcd}$$

where here we define $\partial : [W, Z]_* \to [W, X]_{*-\mathbf{1}}$ to be the composition which makes the diagram commute. The diagram commutes by naturality of ν and $s^{\mathbf{1}}$, so that the bottom row is exact since the top row is exact and the vertical arrows are isomorphisms. Hence may roll the bottom row up to get the desired exact triangle:

$$\begin{tikzcd} {[W,X]_*} \arrow[r, "f_*"] & {[W,Y]_*} \arrow[d, "g_*"] \\ & {[W,Z]_*} \arrow[ul, "\partial"] \end{tikzcd}$$

□

3. Cellular objects in $\mathcal{SH}$

One very important class of objects in $\mathcal{SH}$ are the *cellular* objects. Intuitively, these are the objects that can be built out of the S^a's via taking coproducts and (co)fibers.

Definition 3.1. Define the class of *cellular* objects in $\mathcal{SH}$ to be the smallest class of objects such that:

(1) For all $a \in A$, the a-sphere S^a is cellular.

(2) If we have a distinguished triangle

$$X \to Y \to Z \to \Sigma X$$

such that two of the three objects X, Y, and Z are cellular, than the third object is also cellular.

(3) Given a collection of cellular objects X_i indexed by some (small) set I, the object $\bigoplus_{i\in I} X_i$ is cellular (recall we have chosen $\mathcal{SH}$ to have arbitrary coproducts).

We write $\mathcal{SH}$-**Cell** to denote the full subcategory of $\mathcal{SH}$ on the cellular objects.

Lemma 3.2. *Let X and Y be two isomorphic objects in $\mathcal{SH}$. Then X is cellular iff Y is cellular.*

Proof. Assume we have an isomorphism $f : X \xrightarrow{\cong} Y$ and that X is cellular. Then consider the following commutative diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, equal] & Y \arrow[r] \arrow[d, "f^{-1}"] & 0 \arrow[r] \arrow[d, equal] & \Sigma X \arrow[d, equal] \\ X \arrow[r, equal] & X \arrow[r] & 0 \arrow[r] & \Sigma X \end{tikzcd}$$

The bottom row is distinguished by axiom TR1 for a triangulated category. Hence since X is cellular, 0 is also cellular, since the class of cellular objects satisfies two-of-three for distinguished triangles. Furthermore, since the vertical arrows are all isomorphisms, the top row is distinguished as well, by axiom TR0. Thus again by two-of-three, since X and 0 are cellular, so is Y, as desired. □

Lemma 3.3. *Let X and Y be cellular objects in $\mathcal{SH}$. Then $X \otimes Y$ is cellular.*

Proof. Let E be a cellular object in $\mathcal{SH}$, and let $\mathcal{E}$ be the collection of objects X in $\mathcal{SH}$ such that $E \otimes X$ is cellular. First of all, suppose we have a distinguished triangle

$$X \to Y \to Z \to \Sigma X$$

such that two of three of X, Y, and Z belong to $\mathcal{E}$. Then since $\mathcal{SH}$ is tensor triangulated, we have a distinguished triangle

$$E \otimes X \to E \otimes Y \to E \otimes Z \to \Sigma(E \otimes X).$$

Per our assumptions, two of three of $E \otimes X$, $E \otimes Y$, and $E \otimes Z$ are cellular, so that the third is by definition. Thus, all three of X, Y, and Z belong to $\mathcal{E}$ if two of them do.

Second of all, suppose we have a family X_i of objects in $\mathcal{E}$ indexed by some (small) set I, and set $X := \bigoplus_i X_i$. Then we'd like to show X belongs to $\mathcal{E}$, i.e., that $E \otimes X$ is cellular. Indeed,

$$E \otimes X = E \otimes \left(\bigoplus_i X_i \right) \cong \bigoplus_i (E \otimes X_i),$$

where the isomorphism is given by the fact that $\mathcal{SH}$ is monoidal closed, so $E \otimes -$ preserves arbitrary colimits as it is a left adjoint. Per our assumption, since each $E \otimes X_i$ is cellular, the rightmost object is cellular, since the class of cellular objects is closed under taking arbitrary coproducts, by definition. Hence $E \otimes X$ is cellular by Lemma 3.2.

Finally, we would like to show that each S^a belongs to $\mathcal{E}$, i.e., that $S^a \otimes E$ is cellular for all $a \in A$. When $E = S^b$ for some $b \in A$, this is clearly true, since $S^b \otimes S^a \cong S^{a+b}$, which is cellular by definition, so that $S^b \otimes S^a$ is cellular by Lemma 3.2. Thus by what we have shown, the class of objects X for which $S^a \otimes X$ is cellular contains every cellular object. Hence in particular $E \otimes S^a \cong S^a \otimes E$ is cellular for all $a \in A$, as desired. $\square$

Lemma 3.4. *Let W be a cellular object in $\mathcal{SH}$ such that $\pi_*(W) = 0$. Then $W \cong 0$.*

Proof. Let $\mathcal{E}$ be the collection of all X in $\mathcal{SH}$ such that and $[\Sigma^n X, W] = 0$ for all $n \in \mathbb{Z}$ (where for $n < 0$ we define $\Sigma^n := \Omega^{-n} = (S^{-\mathbf{1}} \otimes -)^n$). We claim $\mathcal{E}$ contains every cellular object in $\mathcal{SH}$. First of all, each S^a belongs to $\mathcal{E}$, as

$$[\Sigma^n S^a, W] \cong [S^{\mathbf{n}} \otimes S^a, W] \cong [S^{a+\mathbf{n}}, W] \leq \pi_*(W) = 0.$$

Furthermore, suppose we are given a distinguished triangle

$$X \to Y \to Z \to \Sigma X$$

such that two of three of X, Y, and Z belong to $\mathcal{E}$. By Proposition A.11, for all $n \in \mathbb{Z}$ we get an exact sequence of abelian groups

$$[\Sigma^{n+1} X, W] \to [\Sigma^n Z, W] \to [\Sigma^n Y, W] \to [\Sigma^n X, W] \to [\Sigma^{n-1} Z, W].$$

Clearly if any two of three of X, Y, and Z belong to $\mathcal{E}$, then by exactness of the above sequence all three of the middle terms will be zero, so that the third object will belong to $\mathcal{E}$ as well. Finally, suppose we have a collection of objects X_i in $\mathcal{E}$ indexed by some small set I. Then

$$\left[\Sigma^n \bigoplus_i X_i, W \right] \cong \left[\bigoplus_i \Sigma^n X_i, W \right] \cong \prod_i [\Sigma^n X_i, W] = \prod_i 0 = 0,$$

where the first isomorphism follows by the fact that Σ^n is apart of an adjoint equivalence (Proposition 2.4), so it preserves arbitrary colimits.

Thus, by definition of cellularity, $\mathcal{E}$ contains every cellular object. In particular, $\mathcal{E}$ contains W, so that $[W, W] = 0$, meaning in particular that $\mathrm{id}_W = 0$, so we have a commutative diagram

$$\begin{tikzcd} & 0 \arrow[r, equal] \arrow[dr] & 0 \\ W \arrow[ur] \arrow[rr, equal] & & W \arrow[u] \end{tikzcd}$$

Hence the diagonals exhibit isomorphisms between 0 and W, as desired. $\square$

Theorem 3.5. *Let X and Y be cellular objects in $\mathcal{SH}$, and suppose $f : X \to Y$ is a morphism such that $f_* : \pi_*(X) \to \pi_*(Y)$ is an isomorphism. Then f is an isomorphism.*

Proof. By axiom TR2 for a triangulated category, we have a distinguished triangle

$$X \xrightarrow{f} Y \xrightarrow{g} C_f \xrightarrow{h} \Sigma X.$$

First of all, note that by definition since X and Y are cellular, so is C_f. We claim $\pi_*(C_f) = 0$. Indeed, given $a \in A$, by axiom TR4 for a triangulated category and the fact that distinguished triangles are exact, the following sequence of abelian groups is exact:

$$[S^a, X] \xrightarrow{f_*} [S^a, Y] \xrightarrow{g_*} [S^a, C_f] \xrightarrow{h_*} [S^a, \Sigma X] \xrightarrow{\Sigma f_*} [S^a, \Sigma Y].$$

where the first arrow is and last arrows are isomorphisms, per our assumption that f is an isomorphism. Then by exactness we have $\operatorname{im} h_* = \ker(\Sigma f_*) = 0$. Yet we also have $\ker g_* = \operatorname{im} f_* = [S^a, Y]$, so that $\ker h_* = \operatorname{im} g_* = 0$. It is only possible that $\ker h_* = \operatorname{im} h_* = 0$ if $[S^a, C_f] = 0$. Thus, we have shown $\pi_*(C_f) = 0$, and C_f is cellular, so by Lemma 3.4 there is an isomorphism $C_f \cong 0$. Now consider the following diagram:

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "f"] & Y \arrow[r] \arrow[d, equal] & C_f \arrow[r] \arrow[d, "\cong"] & \Sigma X \arrow[d, "\Sigma f"] \\
Y \arrow[r, equal] & Y \arrow[r] & 0 \arrow[r] & \Sigma Y
\end{tikzcd}$$

The middle square commutes since 0 is terminal, while the right square commutes since $C_f \cong 0$ is initial. The top row is distinguished by assumption. The bottom row is distinguished by axiom TR2. Then since the middle two vertical arrows are isomorphisms, by Lemma A.4, f is an isomorphism as well, as desired. □

Lemma 3.6. *Let $e : X \to X$ be an idempotent morphism in $\mathcal{SH}$, so $e \circ e = e$. Then since $\mathcal{SH}$ is a triangulated category with arbitrary coproducts, this idempotent splits (Proposition A.7), meaning e factors as*

$$X \xrightarrow{r} Y \xrightarrow{\iota} X$$

for some object Y and morphisms r and ι with $r \circ \iota = \mathrm{id}_Y$. Then Y is cellular if X is.

Proof. It is a general categorical fact that the splitting of an idempotent, if it exists, is unique up to unique isomorphism,[2] so by Lemma 3.2, it suffices to show that e has some cellular splitting. In Proposition A.7, it is shown that we may take Y to be the homotopy colimit (Definition A.6) of the sequence

$$X \xrightarrow{e} X \xrightarrow{e} X \xrightarrow{e} X \xrightarrow{e} \cdots,$$

so there is a distinguished triangle

$$\bigoplus_{i=0}^{\infty} X \to \bigoplus_{i=0}^{\infty} X \to Y \to \Sigma\left(\bigoplus_{i=0}^{\infty} X\right).$$

Since X is cellular, by definition $\bigoplus_{i=0}^{\infty} X$ is as well. Thus by 2-of-3 for distinguished triangles for cellular objects, Y is cellular as desired. □

4. Monoid objects in $\mathcal{SH}$

4.1. **Monoid objects in $\mathcal{SH}$ and their associated rings.** For a review of monoid objects in a symmetric monoidal category, see **??**. The most important example of a monoid object in $\mathcal{SH}$ is the unit S, which has multiplication map $\phi_{0,0}^{-1} = \lambda_S = \rho_S : S \otimes S \to S$ and unit map $\mathrm{id}_S : S \to S$. fix ref

[2] In particular, given an idempotent $e : X \to X$ which splits as $X \xrightarrow{r} Y \xrightarrow{\iota} X$, r and ι are the coequalizer and equalizer, respectively, of e and id_X.

Proposition 4.1. *The assignment $(E, \mu, e) \mapsto \pi_*(E)$ is a functor π_* from the category $\mathbf{Mon}_{\mathcal{SH}}$ of monoid objects in $\mathcal{SH}$ (Definition C.3) to the category of A-graded rings. In particular, given a monoid object (E, μ, e) in $\mathcal{SH}$, $\pi_*(E)$ is canonically a ring with unit $e \in \pi_0(e) = [S, E]$ and product $\pi_*(E) \times \pi_*(E) \to \pi_*(E)$ which sends classes $x : S^a \to E$ and $y : S^b \to E$ to the composition*

$$xy : S^{a+b} \xrightarrow{\phi_{a,b}} S^a \otimes S^b \xrightarrow{x \otimes y} E \otimes E \xrightarrow{\mu} E.$$

Proof. First, we show that $\pi_*(E)$ is actually a ring as indicated. By Lemma B.8, in order to make the A-graded abelian group $\pi_*(E)$ into an A-graded ring, it suffices to construct an associative, unital, and bilinear (distributive) product only with respect to homogeneous elements. Suppose we have classes x, y, and z in $\pi_a(E)$, $\pi_b(E)$, and $\pi_c(E)$, respectively. To see associativity, consider the following diagram:

$$\begin{tikzcd}
 & & & E \otimes E \arrow[d, "\mu"] \\
S^{a+b+c} \arrow[r, "\cong"] & S^a \otimes S^b \otimes S^c \arrow[r, "x \otimes y \otimes z"] & E \otimes E \otimes E \arrow[ur, "\mu \otimes E"] \arrow[dr, "E \otimes \mu"'] & E \\
 & & & E \otimes E \arrow[u, "\mu"']
\end{tikzcd}$$

(here the first arrow is the unique isomorphism obtained by composing products of $\phi_{a,b}$'s, see Remark 2.3). It commutes by associativity of μ. It follows by functoriality of $- \otimes -$ that the top composition is $(x \cdot y) \cdot z$ while the bottom is $x \cdot (y \cdot z)$, so they are equal as desired. To see that $e \in \pi_0(E)$ is a left and right unit for this multiplication, consider the following diagram

$$\begin{tikzcd}
 & S^a \arrow[dl, "e \otimes x"'] \arrow[d, "x"] \arrow[dr, "x \otimes e"] & \\
E \otimes E \arrow[dr, "\mu"'] & E \arrow[l, "e \otimes E"'] \arrow[r, "E \otimes e"] \arrow[d, equal] & E \otimes E \arrow[dl, "\mu"] \\
 & E &
\end{tikzcd}$$

Commutativity of the two top triangles is functoriality of $- \otimes -$. Commutativity of the bottom two triangles is unitality of μ. Thus the diagram commutes, so $e \cdot x = x = x \cdot e$. Finally, we wish to show this product is bilinear (distributive). Suppose we further have some $x' \in \pi_a(E)$ and $y' \in \pi_b(E)$, and consider the following diagrams:

$$\begin{tikzcd}
S^{a+b} \arrow[r, "\phi_{a,b}"] \arrow[d, "\Delta"'] & S^a \otimes S^b \arrow[r, "\Delta \otimes S^b"] \arrow[d, "\Delta"] & (S^a \oplus S^a) \otimes S^b \arrow[r, "(x \oplus x') \otimes y"] \arrow[dl, "\cong"'] & (E \oplus E) \otimes E \arrow[dl, "\cong"'] \arrow[d, "\nabla \otimes E"] & \\
S^{a+b} \oplus S^{a+b} \arrow[r, "\phi_{a,b} \oplus \phi_{a,b}"'] & (S^a \otimes S^b) \oplus (S^a \otimes S^b) \arrow[r, "(x \otimes y) \oplus (x' \otimes y)"'] & (E \otimes E) \oplus (E \otimes E) \arrow[r, "\nabla"'] & E \otimes E \arrow[r, "\mu"] & E
\end{tikzcd}$$

$$\begin{tikzcd}
S^{a+b} \arrow[r, "\phi_{a,b}"] \arrow[d, "\Delta"'] & S^a \otimes S^b \arrow[r, "S^a \otimes \Delta"] \arrow[d, "\Delta"] & S^b \otimes (S^b \oplus S^b) \arrow[r, "x \otimes (y \oplus y')"] \arrow[dl, "\cong"'] & E \otimes (E \oplus E) \arrow[dl, "\cong"'] \arrow[d, "E \otimes \nabla"] & \\
S^{a+b} \oplus S^{a+b} \arrow[r, "\phi_{a,b} \oplus \phi_{a,b}"'] & (S^a \otimes S^b) \oplus (S^a \otimes S^b) \arrow[r, "(x \otimes y) \oplus (x \otimes y')"'] & (E \otimes E) \oplus (E \otimes E) \arrow[r, "\nabla"'] & E \otimes E \arrow[r, "\mu"] & E
\end{tikzcd}$$

The unlabeled isomorphisms are those given by the fact that $- \otimes -$ is additive in each variable (since $\mathcal{SH}$ is tensor triangulated). Commutativity of the left squares is naturality of $\Delta : X \to X \oplus X$ in an additive category. Commutativity of the rest of the diagram follows again from the fact that $- \otimes -$ is an additive functor in each variable. Hence, by functoriality of $- \otimes -$, these diagrams tell us that $(x+x')\cdot y = x\cdot y + x'\cdot y$ and $x\cdot(y+y') = x\cdot y + x\cdot y'$, respectively. Thus, we have shown that if (E,μ,e) is a monoid object in $\mathcal{SH}$ then $\pi_*(E)$ is a ring, as desired.

It remains to show that given a homomorphism of monoid objects $f : (E_1,\mu_1,e_1) \to (E_2,\mu_2,e_2)$ in $\mathbf{Mon}_{\mathcal{SH}}$ that $\pi_*(f) : \pi_*(E_1) \to \pi_*(E_2)$ is an A-graded ring homomorphism. First of all, we know this is an A-graded abelian group homomorphism, since $\mathcal{SH}$ is an additive category, meaning composition with f is an abelian group homomorphism. Thus, in order to show it's a ring homomorphism, it remains to show that $\pi_*(f)(e_1) = e_2$ and that for all $x, y \in \pi_*(E)$ we have $\pi_*(f)(x\cdot y) = \pi_*(f)(x)\cdot\pi_*(f)(y)$. The former follows since $\pi_*(f)(e_1) = f\circ e_1 = e_2$, since f is a monoid homomorphism in $\mathcal{SH}$. To see the latter, first note by distributivity of multiplication in $\pi_*(E_1)$ and $\pi_*(E_2)$ and the fact that $\pi_*(f)$ is a group homomorphism, it suffices to consider the case that x and y are homogeneous of the form $x : S^a \to E_1$ and $y : S^b \to E_2$. In this case, consider the following diagram:

$$\begin{tikzcd}
S^{a+b} \arrow[r, "\phi_{a,b}"] & S^a \otimes S^b \arrow[r, "x\otimes y"] & E_1 \otimes E_1 \arrow[r, "f\otimes f"] \arrow[d, "\mu_1"'] & E_2 \otimes E_2 \arrow[d, "\mu_2"] \\
& & E_1 \arrow[r, "f"] & E_2
\end{tikzcd}$$

The top composition is $\pi_*(f)(x)\cdot\pi_*(f)(y)$, while the bottom composition is $\pi_*(f)(x\cdot y)$. The diagram commutes since f is a monoid object homomorphism. Thus $\pi_*(f)(x\cdot y) = \pi_*(f)(x)\cdot\pi_*(f)(y)$, as desired. $\square$

We call the ring $\pi_*(S)$ the *stable homotopy ring.* We have shown that π_* takes monoids to rings. Given a monoid object (E,μ,e) in $\mathcal{SH}$, we further have that E_* sends objects to $\pi_*(E)$-modules. First, we prove the following lemma:

Lemma 4.2. *Let X and Y be objects in $\mathcal{SH}$. Then the A-graded pairing*

$$\pi_*(X)\times\pi_*(Y) \to \pi_*(X\otimes Y)$$

sending $x : S^a \to X$ and $y : S^b \to Y$ to the composition

$$S^{a+b} \xrightarrow{\phi_{a,b}} S^a \otimes S^b \xrightarrow{x\otimes y} X\otimes Y$$

is additive in each argument.

Proof. Let $a, b \in A$, and let $x_1, x_2 : S^a \to X$ and $y : S^b \to Y$. Then consider the following diagram

$$\begin{tikzcd}
S^{a+b} \arrow[r, "\cong"] & S^a \otimes S^b \arrow[r, "\Delta\otimes S^b"] \arrow[d, "\Delta"'] & (S^a \oplus S^a)\otimes S^b \arrow[dl, "\cong"'] \arrow[d, "(x_1\oplus x_2)\otimes y"] \\
& (S^a\otimes S^b)\oplus(S^a\otimes S^b) \arrow[d, "(x_1\otimes y)\oplus(x_2\otimes y)"'] & (X\oplus X)\otimes Y \arrow[dl, "\cong"'] \arrow[d, "\nabla\otimes Y"] \\
& (X\otimes Y)\oplus(X\otimes Y) \arrow[r, "\nabla"] & X\otimes Y
\end{tikzcd}$$

The isomorphisms are given by the fact that $-\otimes-$ is additive in each variable. Both triangles and the parallelogram commute since $-\otimes-$ is additive. By functoriality of $-\otimes-$, the top composition is $(x_1+x_2)\cdot y$ and the bottom composition is $x_1\cdot y + x_2\cdot y$, so they are equal, as desired. An entirely analagous argument yields that $x\cdot(y_1+y_2) = x\cdot y_1 + x\cdot y_2$ for $x \in \pi_*(X)$ and $y_1, y_2 \in \pi_*(Y)$. $\square$

Proposition 4.3. *Let (E, μ, e) be a monoid object in $\mathcal{SH}$. Then $E_*(-)$ is an additive functor from $\mathcal{SH}$ to the category $\pi_*(E)$-$\mathbf{Mod}(A)$ of left A-graded modules over the ring $\pi_*(E)$ (Proposition 4.1) and degree-preserving homomorphisms between them, where given some X in $\mathcal{SH}$, $E_*(X)$ may be endowed with its* canonical *structure as a left A-graded $\pi_*(E)$-module via the map*

$$\pi_*(E) \times E_*(X) \to E_*(X)$$

which given $a, b \in A$, sends $x : S^a \to E$ and $y : S^b \to E \otimes X$ to the composition

$$x \cdot y : S^{a+b} \cong S^a \otimes S^b \xrightarrow{x \otimes y} E \otimes (E \otimes X) \cong (E \otimes E) \otimes X \xrightarrow{\mu \otimes X} E \otimes X.$$

Similarly, the assignment $X \mapsto X_*(E)$ is a functor from $\mathcal{SH}$ to right A-graded $\pi_*(E)$-modules, where the structure map

$$X_*(E) \times \pi_*(E) \to X_*(E)$$

sends $x : S^a \to X \otimes E$ and $y : S^b \to E$ to the composition

$$x \cdot y : S^{a+b} \cong S^a \otimes S^b \xrightarrow{x \otimes y} (X \otimes E) \otimes E \cong X \otimes (E \otimes E) \xrightarrow{X \otimes \mu} X \otimes E.$$

Finally, $E_*(E)$ is a $\pi_*(E)$-bimodule, in the sense that the left and right actions of $\pi_*(E)$ are compatible, so that given $y, z \in \pi_*(E)$ and $x \in E_*(E)$, $y \cdot (x \cdot z) = (y \cdot x) \cdot z$.

Proof. By Lemma B.9, in order to make the A-graded abelian group $E_*(X)$ into a left A-graded module over the A-graded ring $\pi_*(E)$, it suffices to define the action map $\pi_*(E) \times E_*(X) \to E_*(X)$ only for homogeneous elements, and to show that given homogeneous elements $x, x' : S^a \to E \otimes X$ in $E_a(X)$, $y : S^b \to E$ in $\pi_b(E)$, and $z, z' : S^c \to E$ in $\pi_c(E)$, that:

(1) $y \cdot (x + x') = y \cdot x + y \cdot x'$,

(2) $(z + z') \cdot x = z \cdot x + z' \cdot x$,

(3) $(zy) \cdot x = z \cdot (y \cdot x)$,

(4) $e \cdot x = x$.

Axioms (1) and (2) follow by the fact that $E_*(X) = \pi_*(E \otimes X)$ and Lemma 4.2. To see (3), consider the diagram:

$$\begin{tikzcd}
& & & E \otimes E \otimes X \arrow[d, "\mu \otimes X"] \\
S^{a+b+c} \arrow[r, "\cong"] & S^c \otimes S^b \otimes S^a \arrow[r, "z \otimes y \otimes x"] & E \otimes E \otimes E \otimes X \arrow[ur, "E \otimes \mu \otimes X"] \arrow[dr, "\mu \otimes E \otimes X"'] & E \otimes X \\
& & & E \otimes E \otimes X \arrow[u, "\mu \otimes X"']
\end{tikzcd}$$

It commutes by associativity of μ. By functoriality of $- \otimes -$, the two outside compositions equal $z \cdot (y \cdot x)$ on the top and $(z \cdot y) \cdot x$ on the bottom. Hence, they are equal, as desired.

Next, to see (4), consider the following diagram:

$$\begin{tikzcd}
S^a \arrow[rr, "x"] \arrow[dr, "x"] \arrow[ddr, bend right, "e \otimes x"'] & & E \otimes X \\
& E \otimes X \arrow[ur, equal] \arrow[d, dashed, "e \otimes E \otimes X"] & \\
& E \otimes E \otimes X \arrow[uur, bend right, "\mu \otimes X"'] &
\end{tikzcd}$$

The top triangle commutes by definition. The left triangle commutes by functoriality of $- \otimes -$. The right triangle commutes by unitality of μ. The top composition is x while the bottom is $e \cdot x$, thus they are necessarily equal since the diagram commutes.

Thus, we have shown that the indicated map does indeed endow $E_*(X)$ with the structure of a left $\pi_*(E)$-module. Next we would like to show that $E_*(-)$ sends maps in $\mathcal{SH}$ to A-graded homomorphisms of left A-graded $\pi_*(E)$-modules. By definition, given $f : X \to Y$ in $\mathcal{SH}$, $E_*(f)$ is the map which takes a class $x : S^a \to E \otimes X$ to the composition

$$S^a \xrightarrow{x} E \otimes X \xrightarrow{E \otimes f} E \otimes Y.$$

To see this assignment is a homomorphism, suppose we are given some other $x' : S^a \to E \otimes X$ and some scalar $y : S^b \to E$. Then we would like to show $E_*(f)(x + x') = E_*(f)(x) + E_*(f)(x')$ and $E_*(f)(y \cdot x) = y \cdot E_*(f)(x)$. To see the former, consider the following diagram:

$$\begin{tikzcd}
 & & & (E \otimes Y) \oplus (E \otimes Y) \arrow[d, "\nabla"] \\
S^a \arrow[r, "\Delta"] & S^a \oplus S^a \arrow[r, "x \oplus x'"] & (E \otimes X) \oplus (E \otimes X) \arrow[ur, "(E \otimes f) \oplus (E \otimes f)"] \arrow[dr, "\nabla"'] & E \otimes Y \\
 & & & E \otimes X \arrow[u, "E \otimes f"']
\end{tikzcd}$$

It commutes by naturality of ∇ in an additive category. The top composition is $E_*(f)(x) + E_*(f)(x')$, while the bottom is $E_*(f)(x+x')$, so they are equal as desired. To see that $E_*(f)(y \cdot x) = y \cdot E_*(f)(x)$, consider the following diagram:

$$\begin{tikzcd}
S^{a+b} \arrow[r, "\phi_{b,a}"] & S^b \otimes S^a \arrow[r, "y \otimes x"] & E \otimes E \otimes X \arrow[r, "E \otimes E \otimes f"] \arrow[d, "\mu \otimes X"'] & E \otimes E \otimes Y \arrow[d, "\mu \otimes Y"] \\
 & & E \otimes X \arrow[r, "E \otimes f"] & E \otimes Y
\end{tikzcd}$$

It commutes by functoriality of $- \otimes -$. The top composition is $E_*(f)(y \cdot x)$, while the bottom composition is $y \cdot E_*(f)(x)$, so they are equal, as desired.

Thus, we've shown $E_*(-)$ yields a functor $\mathcal{SH} \to \pi_*(E)$-$\mathbf{Mod}(A)$; it remains to show this functor is additive, equivalently, $\mathbf{Ab}$-enriched. This is clear, as given $f, g : X \to Y$ in $\mathcal{SH}$, we have

$$E_*(f + g) = [S^*, E \otimes (f + g)] = [S^*, (E \otimes f) + (E \otimes g)] = E_*(f) + E_*(g),$$

where the second equality follows since $- \otimes -$ is additive in each variable.

Showing that $X_*(E)$ has the structure of a right $\pi_*(E)$-module and that if $f : X \to Y$ is a morphism in $\mathcal{SH}$ then the map

$$X_*(E) = [S^*, X \otimes E] \xrightarrow{(f \otimes E)_*} [S^*, Y \otimes E] = Y_*(E)$$

is an A-graded homomorphism of right A-graded $\pi_*(E)$-modules is entirely analagous.

It remains to show that $E_*(E)$ is a $\pi_*(E)$-bimodule. Let $x : S^a \to E$, $y : S^b \to E \otimes E$, and $z : S^c \to E$, and consider the following diagram:

$$\begin{tikzcd}
 & & & E \otimes E \otimes E \arrow[d, "E\otimes\mu"] \\
S^{a+b+c} \arrow[r, "\cong"] & S^a \otimes S^b \otimes S^c \arrow[r, "x\otimes y\otimes z"] & E \otimes E \otimes E \otimes E \arrow[ur, "\mu\otimes E\otimes E"] \arrow[r, "\mu\otimes\mu"] \arrow[dr, "E\otimes E\otimes\mu"'] & E \otimes E \\
 & & & E \otimes E \otimes E \arrow[u, "\mu\otimes E"']
\end{tikzcd}$$

Commutativity follows by functoriality of $- \otimes -$, which also tells us that the two outside compositions are $(x \cdot y) \cdot z$ (on top) and $x \cdot (y \cdot z)$ (on bottom). Hence they are equal, as desired. $\square$

Lemma 4.4. *Let E and X be objects in $\mathcal{SH}$. Then for all $a \in A$, there is an A-graded isomorphism of A-graded abelian groups*

$$t^a_X : E_*(\Sigma^a X) \cong E_{*-a}(X)$$

which sends a class $x : S^b \to E \otimes \Sigma^a X = E \otimes S^a \otimes X$ to the composition

$$S^{b-a} \xrightarrow{\phi_{b,-a}} S^b \otimes S^{-a} \xrightarrow{x\otimes S^{-a}} E \otimes S^a \otimes X \otimes S^{-a} \xrightarrow{E\otimes\tau\otimes S^{-a}} E \otimes X \otimes S^a \otimes S^{-a} \xrightarrow{E\otimes X\otimes\phi^{-1}_{a,-a}} E \otimes X$$

with inverse $(t^a_X)^{-1} : E_{*-a}(X) \to E_*(\Sigma^a X)$ sending a class $x : S^{b-a} \to E \otimes X$ to the composition

$$S^b \xrightarrow{\phi_{b-a,a}} S^{b-a} \otimes S^a \xrightarrow{x\otimes S^a} E \otimes X \otimes S^a \xrightarrow{E\otimes\tau} E \otimes S^a \otimes X$$

(where here we are suppressing associators and unitors from the notation). Furthermore this isomorphism is natural in X, and if E is a monoid object in $\mathcal{SH}$ then it is an isomorphism of left $\pi_*(E)$-modules.

Proof. Expressed in terms of hom-sets, t^a_X is precisely the composition

$$\begin{tikzcd}
E_*(\Sigma^a X) \arrow[r, equal] & {[S^*, E \otimes S^a \otimes X]} \arrow[d, "(E\otimes\tau)_*"] & \\
 & {[S^*, E \otimes X \otimes S^a]} \arrow[d, "-\otimes S^{-a}"] & \\
 & {[S^* \otimes S^{-a}, E \otimes X \otimes S^a \otimes S^{-a}]} \arrow[d, "(E\otimes X\otimes\phi^{-1}_{a,-a})_*"] & \\
 & {[S^* \otimes S^{-a}, E \otimes X]} \arrow[d, "(\phi_{*,-a})^*"] & \\
 & {[S^{*-a}, E \otimes X]} \arrow[r, equal] & E_{*-a}(E \otimes X)
\end{tikzcd}$$

We know the second vertical arrow is an isomorphism of abelian groups as $- \otimes -$ is additive in each variable (since $\mathcal{SH}$ is tensor triangulated) and $\Omega^a \cong - \otimes S^{-a}$ is an autoequivalence of $\mathcal{SH}$ by Proposition 2.4. The three other vertical arrows are given by composing with an isomorphism in an additive category, so they are also isomorphisms. Now, note the proposed inverse constructed above can be factored into the following composition:

$$\begin{tikzcd}
E_{*-a}(E \otimes X) \arrow[r, equal] & {[S^{*-a}, E \otimes X]} \arrow[d, "-\otimes S^a"] & \\
& {[S^{*-a} \otimes S^a, E \otimes X \otimes S^a]} \arrow[d, "(\phi_{*-a,a})^*"] & \\
& {[S^*, E \otimes X \otimes S^a]} \arrow[d, "(E\otimes\tau)_*"] & \\
& {[S^*, E \otimes S^a \otimes X]} \arrow[r, equal] & E_*(\Sigma^a X)
\end{tikzcd}$$

It is entirely straightforward to check that this is an inverse to t_X^a, and we leave it to the reader to check this. (Since we already know t_X^a is an isomorphism, it suffices to show this composition is either a left or right inverse.)

Now, to see t_X^a is a homomorphism of left $\pi_*(E)$-modules, suppose we are given classes $r : S^b \to E$ in $\pi_b(E)$ and $x : S^c \to E \otimes S^a \otimes X$ in $E_c(\Sigma^a X)$. Then we wish to show that $t_X^a(r \cdot x) = r \cdot t_X^a(x)$. To that end, consider the following diagram:

$$\begin{tikzcd}
S^{b+c-a} \arrow[d, "\cong"] & E \otimes S^a \otimes X \otimes S^{-a} \arrow[r, "E\otimes\tau\otimes S^{-a}"] & E \otimes X \otimes S^a \otimes S^{-a} \arrow[d, "E\otimes X\otimes\phi^{-1}_{a,-a}"] \\
S^b \otimes S^c \otimes S^{-a} \arrow[r, "r\otimes x\otimes S^{-a}"] & E \otimes E \otimes S^a \otimes X \otimes S^{-a} \arrow[u, "\mu\otimes S^a\otimes X\otimes S^{-a}"] \arrow[d, "E\otimes E\otimes\tau\otimes S^{-a}"'] & E \otimes X \\
& E \otimes E \otimes X \otimes S^a \otimes S^{-a} \arrow[r, "E\otimes E\otimes X\otimes\phi^{-1}_{a,-a}"'] \arrow[ur, "\mu\otimes X\otimes S^a\otimes S^{-a}"] & E \otimes E \otimes X \arrow[u, "\mu\otimes X"']
\end{tikzcd}$$

Both triangles commute by functoriality of $- \otimes -$. The top composition is $t_X^a(r \cdot x)$ while the bottom is $r \cdot t_X^a(x)$, so they are equal as desired.

It remains to show t_X^a is natural in X. let $f : X \to Y$ in $\mathcal{SH}$, then we would like to show the following diagram commutes:

$$\begin{tikzcd}
E_*(\Sigma^a X) \arrow[r, "t_X^a"] \arrow[d, "E_*(\Sigma^a f)"'] & E_{*-a}(X) \arrow[d, "E_{*-a}(f)"] \\
E_*(\Sigma^a Y) \arrow[r, "t_Y^a"] & E_{*-a}(Y)
\end{tikzcd} \tag{1}$$

We may chase a generator around the diagram since all the arrows here are homomorphisms. Let $x : S^b \to E \otimes S^a \otimes X$ in $E_*(\Sigma^a X)$. Then consider the following diagram:

$$\begin{tikzcd}
S^{b-a} \arrow[r, "\cong"] & S^b \otimes S^{-a} \arrow[r, "x\otimes S^{-a}"] & E \otimes S^a \otimes X \otimes S^{-a} \arrow[r, "E\otimes\tau\otimes S^{-a}"] \arrow[d, "E\otimes S^a\otimes f\otimes S^{-a}"'] & E \otimes X \otimes S^a \otimes S^{-a} \arrow[r, "E\otimes X\otimes\phi^{-1}_{a,-a}"] \arrow[d, dashed, "E\otimes f\otimes S^a\otimes S^{-a}"'] & E \otimes X \arrow[d, "E\otimes f"] \\
& & E \otimes S^a \otimes Y \otimes S^{-a} \arrow[r, "E\otimes\tau\otimes S^{-a}"'] & E \otimes Y \otimes S^a \otimes S^{-a} \arrow[r, "E\otimes Y\otimes\phi^{-1}_{a,-a}"'] & E \otimes Y
\end{tikzcd}$$

The left rectangle commutes by naturality of τ, while the right rectangle commutes by functoriality of $- \otimes -$. The two outside compositions are the two ways to chase x around diagram (1), so the diagram commutes as desired. $\square$

4.2. **Commutative monoid objects in $\mathcal{SH}$ and their associated rings.** A natural question that arises is: In what sense is $\pi_*(E)$ "graded commutative" if (E, μ, e) is a commutative monoid object in $\mathcal{SH}$? It turns out that it satisfies a rather strong commutativity condition. In this subsection, we will show that $\pi_*(E)$ is an *A-graded anticommutative ring*, in the following sense:

Definition 4.5. An *A-graded anticommutative ring* is an A-graded ring R along with an assignment $\theta : A^2 \to R_0^\times$ sending $(a, b) \mapsto \theta_{a,b}$ such that for all $a, b, c \in A$,

- $\theta_{a,0} = \theta_{0,a} = 1$,
- $\theta_{a,b}^{-1} = \theta_{b,a}$,
- $\theta_{a,b} \cdot \theta_{a,c} = \theta_{a,b+c}$ and $\theta_{b,a} \cdot \theta_{c,a} = \theta_{b+c,a}$, and
- for all homogeneous x and y in R,
$$x \cdot y = y \cdot x \cdot \theta_{|x|,|y|}.$$

Given two A-graded anticommutative rings (R, θ) and (R', θ'), an A-graded ring homomorphism $f : R \to R'$ is a homomorphism of A-graded anticommutative rings if it satisfies $f \circ \theta = \theta'$. We write $\mathbf{GrCRing}^A$ for the resulting category.

In fact, the above definition will be entirely motivated by the work we do here. An interesting fact is that the initial object in the category $\mathbf{GrCRing}^A$ is the group algebra $\mathbb{Z}[A \wedge A]$ on the exterior square $A \wedge A$, viewed as an A-graded ring concentrated in degree 0, and where $\theta_{a,b}$ corresponds to the element $a \wedge b$.

is this actually true?

In fact, we will show that not only is $\pi_*(E)$ an A-graded anticommutative ring, but it is an A-graded anticommutative algebra over the stable homotopy ring $\pi_*(S)$, defined as follows:

Definition 4.6. Given an A-graded anticommutative ring (R, θ) (Definition 4.5), we write R-$\mathbf{GCA}^A$ to denote the slice category $\mathbf{GrCRing}^A/(R, \theta)$. Explicitly:

- The objects are pairs (S, φ) called *A-graded anticommutative R-algebras*, where S is an A-graded ring and $\varphi : R \to S$ is an A-graded ring homomorphism such that for all $x \in S_a$ and $y \in S_b$, we have
$$x \cdot y = y \cdot x \cdot \varphi(\theta_{a,b}),$$
- The morphisms $(S, \varphi) \to (S', \varphi')$ are A-graded ring homomorphisms $f : S \to S'$ such that $f \circ \varphi = \varphi'$.

Note that our notation for the category R-$\mathbf{GCA}^A$ is somewhat deficient, as there may be multiple choices of families of units $\theta_{a,b} \in R_0$ satisfying the required properties which give rise to strictly different categories, as the following example illustrates. The following example exhibits this issue.

Example 4.7. Consider $R = \mathbb{Z}$ as a ring graded over $A = \mathbb{Z}$ concentrated in degree 0, and let $\theta_{n,m} := (-1)^{n \cdot m}$ for all $n, m \in \mathbb{Z}$, then R-$\mathbf{GCA}^A$ is simply the standard category of graded anticommutative rings, i.e., $\mathbb{Z}$-graded rings R such that for all homogeneous $x, y \in R$, $x \cdot y = y \cdot x \cdot (-1)^{|x||y|}$. On the other hand, if we instead define $\theta_{n,m} = 1$ for all $n, m \in \mathbb{Z}$, then the resulting category R-$\mathbf{GCA}^A$ becomes the category of strictly commutative $\mathbb{Z}$-graded rings.

Like the standard category of $\mathbb{Z}$-graded anticommutative rings, it turns out that the category R-$\mathbf{GCA}^A$ has many nice properties, some of which are detailed in Appendix B.4. In particular, we show that R-$\mathbf{GCA}^A$ has finite coproducts and pushouts, and as in the standard category of (graded anti)commutative rings, they are formed by taking the underlying tensor product of bimodules and endowing it with a (graded anti)commutative multiplication.

The rest of this subsection will be devoted to proving that for each commutative monoid object (E, μ, e) in $\mathcal{SH}$, $\pi_*(E)$ is an A-graded anticommutative algebra over the A-graded anticommutative ring $\pi_*(S)$.

Proposition 4.8. *For all $a, b \in A$ there exists an element $\theta_{a,b} \in \pi_0(S) = [S, S]$ such that given any commutative monoid object (E, μ, e) in $\mathcal{SH}$, the A-graded ring structure on $\pi_*(E)$ (Proposition 4.1) has a commutativity formula given by*

$$x \cdot y = y \cdot x \cdot (e \circ \theta_{a,b})$$

for all $x \in \pi_a(E)$ and $y \in \pi_b(E)$.

Proof. Given $a, b \in A$, define $\theta_{a,b} \in \mathrm{Aut}(S)$ to be the composition

$$S \xrightarrow{\cong} S^{-a-b} \otimes S^a \otimes S^b \xrightarrow{S^{-a-b} \otimes \tau} S^{-a-b} \otimes S^b \otimes S^a \xrightarrow{\cong} S,$$

where the outermost maps are the unique maps specified by Remark 2.3. Now let (E, μ, e), x, and y as in the statement of the proposition, and consider the following diagram

$$\begin{tikzcd}
S^{a+b} \arrow[r, "\phi_{a,b}"] \arrow[d, dashed, "\phi_{b,a}^{-1} \circ \tau \circ \phi_{a,b}"'] & S^a \otimes S^b \arrow[r, "x \otimes y"] \arrow[d, "\tau"'] & E \otimes E \arrow[d, "\tau"'] \arrow[dr, "\mu"] & \\
 & & & E \\
S^{a+b} \arrow[r, "\phi_{b,a}"] & S^b \otimes S^a \arrow[r, "y \otimes x"] & E \otimes E \arrow[ur, "\mu"'] &
\end{tikzcd}$$

The left square commutes by definition. The middle square commutes by naturality of the symmetry isomorphism. Finally, the right square commutes by commutativity of E. Unravelling definitions, we have shown that under the product on $\pi_*(E)$ induced by the $\phi_{a,b}$'s,

$$x \cdot y = (y \cdot x) \circ (\phi_{b,a}^{-1} \circ \tau \circ \phi_{a,b}).$$

Thus, in order to show the desired result it further suffices to show that

$$(y \cdot x) \circ (\phi_{b,a}^{-1} \circ \tau \circ \phi_{a,b}) = y \cdot x \cdot (e \circ \theta_{a,b}).$$

Consider the following diagram:

$$\begin{tikzcd}
S^{a+b} \arrow[rr, "\phi_{a,b}"] \arrow[d, "\cong"'] & & S^a \otimes S^b \arrow[d, "\tau"] \\
S^b \otimes S^a \otimes S^{-a-b} \otimes S^a \otimes S^b \arrow[urr, "\cong"'] \arrow[d, "S^b \otimes S^a \otimes S^{-a-b} \otimes \tau"'] & & S^b \otimes S^a \arrow[d, "\phi_{b,a}^{-1}"] \arrow[dl, equal] \\
S^b \otimes S^a \otimes S^{-a-b} \otimes S^b \otimes S^a \arrow[r, "\cong"] \arrow[urr, "\cong"] & S^b \otimes S^a \arrow[dr, "y \otimes x"] \arrow[ddl, "y \otimes x \otimes e"'] & S^{a+b} \arrow[l, "\phi_{b,a}"'] \\
& & E \otimes E \arrow[d, equal] \arrow[dll, "E \otimes E \otimes e"'] \\
E \otimes E \otimes E \arrow[rr, "E \otimes \mu"] \arrow[d, "\mu \otimes E"'] & & E \otimes E \arrow[d, "\mu"] \\
E \otimes E \arrow[rr, "\mu"'] & & E
\end{tikzcd}$$

Here any map simply labelled $\cong$ is an appropriate composition of copies of $\phi_{a,b}$'s, associators, and their inverses, so that each of these maps are necessarily unique by Remark 2.3. The triangles in the top large rectangle commutes by coherence for the $\phi_{a,b}$'s. The parallelogram commutes by naturality of τ and coherence of the of $\phi_{a,b}$'s. The middle skewed triangle commutes by functoriality of $-\otimes-$. The triangle below that commutes by unitality of μ. Finally, the bottom rectangle commmutes by associativity of μ. Hence, by unravelling definitions and applying functoriality of $-\otimes-$, we get that the right composition is $(y \cdot x) \circ (\phi_{b,a}^{-1} \circ \tau \circ \phi_{a,b})$, while the left composition is $y \cdot x \cdot (e \circ \theta_{a,b})$, so they are equal as desired. $\square$

Lemma 4.9. *Suppose we have homogeneous elements $x, y \in \pi_*(S)$ with x of degree 0, then we have $x \cdot y = y \cdot x = x \circ y$ (where the $\cdot$ denotes the product given in Proposition 4.1).*

Proof. As morphisms, y is an arrow $S^a \to S$ for some a in A, and x is a morphism $S \to S$. Then consider the following diagram:

$$\begin{tikzcd}
S \otimes S^a \arrow[dd, "y \otimes x"'] \arrow[dr, "S \otimes y"] & & S^a \arrow[ll, "\phi_{0,a} = \lambda_{S^a}^{-1}"'] \arrow[rr, "\phi_{a,0} = \rho_{S^a}^{-1}"] \arrow[d, "y"'] & & S^a \otimes S \arrow[dd, "x \otimes y"] \arrow[dl, "y \otimes S"'] \\
& S \otimes S \arrow[r, "\lambda_S = \rho_S"] \arrow[dl, "x \otimes S"'] & S \arrow[d, "x"'] & S \otimes S \arrow[l, "\rho_S = \lambda_S"'] \arrow[dr, "S \otimes x"] & \\
S \otimes S \arrow[rr, "\phi_{0,0}^{-1} = \rho_S"'] & & S & & S \otimes S \arrow[ll, "\phi_{0,0}^{-1} = \lambda_S"]
\end{tikzcd}$$

The trapezoids commute by naturality of the unitors, and the triangles commute by functoriality of $-\otimes-$. The outside compositions are $y \cdot x$ on the left and $x \cdot y$ on the right, and the middle composition is $x \circ y$, so indeed we have $y \cdot x = x \cdot y = x \circ y$, as desired. $\square$

Lemma 4.10. *Given $a \in A$, we have $\theta_{0,a} = \theta_{a,0} = \mathrm{id}_S$.*

Proof. Recall $\theta_{a,0}$ is the composition

$$S \xrightarrow{\phi_{-a,a}} S^{-a} \otimes S^a \xrightarrow{S^{-a} \otimes \phi_{a,0}} S^{-a} \otimes (S^a \otimes S) \xrightarrow{S^{-a} \otimes \tau} S^{-a} \otimes (S \otimes S^a) \xrightarrow{S^{-a} \otimes \phi_{0,a}^{-1}} S^{-a} \otimes S^a \xrightarrow{\phi_{-a,a}^{-1}} S$$

By the coherence theorem for symmetric monoidal categories and the fact that $\phi_{a,0}$ and $\phi_{0,a}$ coincide with the unitors, we have that the composition

$$S^a \xrightarrow{\phi_{a,0}=\rho_{S^a}^{-1}} S^a \otimes S \xrightarrow{\tau} S \otimes S^a \xrightarrow{\phi_{0,a}^{-1}=\lambda_{S^a}} S^a$$

is precisely the identity map, so by functoriality of $- \otimes -$, we have that $\theta_{a,0}$ is the composition

$$S \xrightarrow{\phi_{-a,a}} S^{-a} \otimes S^a \xrightarrow{=} S^{-a} \otimes S^a \xrightarrow{\phi_{-a,a}^{-1}} S.$$

Hence $\theta_{a,0} = \mathrm{id}_S$, as desired. An entirely analagous argument yields that $\theta_{0,a} = \mathrm{id}_S$. $\square$

Lemma 4.11. *Let $a, b \in A$. Then $\theta_{a,b} \cdot \theta_{b,a} = \mathrm{id}_S$.*

Proof. By Lemma 4.9, it suffices to show that $\theta_{a,b} \circ \theta_{b,a} = \mathrm{id}_S$. To see this, consider the following diagram:

$$\begin{tikzcd}
S \arrow[r, "\phi"] \arrow[dddrrr, equal] & S^{-a-b} \otimes S^b \otimes S^a \arrow[r, "S^{-a-b}\otimes\tau"] \arrow[ddrr, equal] & S^{-a-b} \otimes S^a \otimes S^b \arrow[r, "\phi"] \arrow[dr, equal] & S \arrow[d, "\phi"] \\
 & & & S^{-a-b} \otimes S^a \otimes S^b \arrow[d, "S^{-a-b}\otimes\tau"] \\
 & & & S^{-a-b} \otimes S^b \otimes S^a \arrow[d, "\phi"] \\
 & & & S
\end{tikzcd}$$

Here we are suppressing associators, and any map labelled ϕ is the appropriate composition of $\phi_{a,b}$'s, unitors, associators, identities, and their inverses (see Remark 2.3). Clearly each region commutes, the middle by the fact that $\tau^2 = 0$, and the other two regions by coherence for the ϕ's. Thus we have shown $\theta_{a,b} \cdot \theta_{b,a} = \theta_{a,b} \cdot \theta_{b,a} = \mathrm{id}_S$, as desired. $\square$

Lemma 4.12. *Let $a, b, c \in A$. Then $\theta_{a,b} \cdot \theta_{a,c} = \theta_{a,b+c}$ and $\theta_{b,a} \cdot \theta_{c,a} = \theta_{b+c,a}$.*

Proof. By Lemma 4.9, it suffices to show that $\theta_{a,b} \circ \theta_{a,c} = \theta_{a,b+c}$ and $\theta_{b,a} \circ \theta_{c,a} = \theta_{b+c,a}$. First we show $\theta_{a,b} \circ \theta_{a,c} = \theta_{a,b+c}$. To see this, consider the following diagram:
(2)

$$\begin{tikzcd}
S \arrow[rr, "\phi"] \arrow[dd, "\phi"'] & & S^{-a-c}S^aS^c \arrow[rr, "S^{-a-c}\tau"] \arrow[dd, "\phi"'] & & S^{-a-c}S^cS^a \arrow[rr, "\phi"] \arrow[dd, "\phi"] & & S \arrow[dd, "\phi"] \\
 & \text{(A)} & & \text{(B)} & & \text{(C)} & \\
S^{-a-b-c}S^aS^{b+c} \arrow[rr, "\phi"] \arrow[dd, "S^{-a-b-c}\tau"'] & & S^{-a-c}S^{-b}S^aS^bS^c \arrow[rr, "S^{-a-c}\tau_{S^{-b}S^aS^b,S^c}"] \arrow[dd, "S^{-a-c}S^{-b}\tau_{S^a,S^bS^c}"'] & & S^{-a-c}S^cS^{-b}S^aS^b \arrow[dd, "S^{-a-c}S^cS^{-b}\tau"'] & & S^{-a-b}S^aS^b \arrow[ll, "\phi"'] \arrow[dd, "S^{-a-b}\tau"] \\
 & \text{(D)} & & \text{(E)} & & \text{(F)} & \\
S^{-a-b-c}S^{b+c}S^a \arrow[rr, "\phi"] \arrow[dddd, "\phi"'] & & S^{-a-c}S^{-b}S^bS^cS^a \arrow[rr, "S^{-a-c}\tau_{S^{-b}S^b,S^cS^a}"] & & S^{-a-c}S^cS^{-b}S^bS^a & & S^{-a-b}S^bS^a \arrow[ll, "\phi"'] \arrow[dddd, "\phi"] \\
 & & & \text{(G)} & & & \\
 & & S^{-a-c}S^cS^a \arrow[uu, "\phi"] \arrow[rr, equal] & & S^{-a-c}S^cS^a \arrow[uu, "\phi"'] & & \\
 & & & \text{(H)} & & & \\
S \arrow[rrrrrr, equal] & & & & & & S
\end{tikzcd}$$

Here we are omitting $\otimes$ from the notation, and each occurrence of an arrow labelled ϕ indicates it is the unique arrow that can be obtained as a formal composition of tensor products of copies of $\phi_{a,b}$'s, unitors, associators, and their inverses (Remark 2.3). Clearly the composition going around the top and then the right is $\theta_{a,b} \circ \theta_{a,c}$ while the composition going left around the bottom is $\theta_{a,b+c}$. Thus, we wish to show the above diagram commutes.

Regions (A), (C), and (H) commute by coherence for the ϕ's (see previous remark). Region (E) commutes by coherence for the τ's. To see region (B) commutes, consider the following diagram, which commutes by naturality of τ:

$$\begin{tikzcd}
S^{-a-c}S^aS^c \arrow[r, "S^{-a-c}\tau"] \arrow[d, "S^{-a-c}\phi_{a-b,b}S^c"'] & S^{-a-c}S^cS^a \arrow[d, "S^{-a-c}S^c\phi_{a-b,b}"] \\
S^{-a-c}S^{a-b}S^bS^c \arrow[r, "S^{-a-c}\tau_{S^{a-b}S^b,S^c}"] \arrow[d, "S^{-a-c}\phi_{-b,a}S^bS^c"'] & S^{-a-c}S^cS^{a-b}S^b \arrow[d, "S^{-a-c}S^c\phi_{-b,a}S^b"] \\
S^{-a-c}S^{-b}S^aS^bS^c \arrow[r, "S^{-a-c}\tau_{S^{-b}S^aS^b,S^c}"] & S^{-a-c}S^cS^{-b}S^aS^b
\end{tikzcd}$$

To see region (D) commutes, note that it is simply the square

$$\begin{tikzcd}
S^{-a-b-c}S^aS^{b+c} \arrow[r, "\phi_{-a-c,-b}S^a\phi_{b,c}"] \arrow[d, "S^{-a-b-c}\tau"'] & S^{-a-c}S^{-b}S^aS^bS^c \arrow[d, "S^{-a-c}S^{-b}\tau_{S^a,S^bS^c}"] \\
S^{-a-b-c}S^{b+c}S^a \arrow[r, "\phi_{-a-c,-b}\phi_{b,c}S^a"] & S^{-a-c}S^{-b}S^bS^cS^a
\end{tikzcd}$$

This diagram commutes by naturality of τ. To see region (F) commutes, consider the following diagram, which commutes by functoriality of $-\otimes-$:

$$\begin{tikzcd}
S^{-a-c}S^cS^{-b}S^aS^b \arrow[d, "S^{-a-c}S^cS^{-b}\tau"'] & S^{-a-c}S^{c-b}S^aS^b \arrow[l, "S^{-a-c}\phi_{c,-b}S^aS^b"'] \arrow[d, "S^{-a-c}S^{c-b}\tau"] & S^{-a-b}S^aS^b \arrow[l, "\phi_{-a-c,c-b}S^aS^b"'] \arrow[d, "S^{-a-b}\tau"] \\
S^{-a-c}S^cS^{-b}S^bS^a & S^{-a-c}S^{c-b}S^bS^a \arrow[l, "S^{-a-c}\phi_{c,-b}S^bS^a"'] & S^{-a-b}S^bS^a \arrow[l, "\phi_{-a-c,c-b}S^bS^a"']
\end{tikzcd}$$

Finally, to see region (G) commutes, consider the following diagram:

$$\begin{tikzcd}
S^{-a-c}S^{-b}S^bS^cS^a \arrow[r, "S^{-a-c}\tau_{S^{-b}S^b,S^c}S^a"] & S^{-a-c}S^cS^{-b}S^bS^a \\
S^{-a-c}SS^cS^a \arrow[u, "S^{-a-c}\phi_{-b,b}S^cS^a"] \arrow[r, "S^{-a-c}\tau_{S,S^c}S^a"] & S^{-a-c}S^cSS^a \arrow[u, "S^{-a-c}S^c\phi_{-b,b}S^a"'] \\
S^{-a-c}S^cS^a \arrow[u, "S^{-a-c}\phi_{0,c}S^a=S^{-a-c}\lambda^{-1}_{S^c}S^a"] \arrow[r, equal] & S^{-a-c}S^cS^a \arrow[u, "S^{-a-c}\phi_{c,0}S^a=S^{-a-c}S\rho^{-1}_{S^c}S^a"']
\end{tikzcd}$$

The top region commutes by naturality of τ, while the bottom region commutes by coherence for a symmetric monoidal category. Thus, we have shown that diagram (2) commutes, so that $\theta_{a,b} \circ \theta_{a,c} = \theta_{a,b+c}$, as desired. Now, to see that $\theta_{b,a} \cdot \theta_{c,a} = \theta_{b+c,a}$, note that

$$\theta_{b,a} \cdot \theta_{c,a} \overset{(*)}{=} \theta_{a,b}^{-1} \cdot \theta_{a,c}^{-1} = (\theta_{a,c} \cdot \theta_{a,b})^{-1} = \theta_{a,b+c}^{-1} \overset{(*)}{=} \theta_{b+c,a},$$

where each occurrence of $(*)$ is Lemma 4.11. $\square$

To recap, we have shown that the assignment $\theta : A^2 \to \pi_0(S)^\times$ satisfies the following for all $a, b, c \in A$:

- $\theta_{a,0} = \theta_{0,a} = 1$,
- $\theta_{a,b}^{-1} = \theta_{b,a}$,
- $\theta_{a,b} \cdot \theta_{a,c} = \theta_{a,b+c}$ and $\theta_{b,a} \cdot \theta_{c,a} = \theta_{b+c,a}$, and
- for all homogeneous x and y in $\pi_*(S)$,
$$x \cdot y = y \cdot x \cdot \theta_{|x|,|y|}.$$

Thus, the stable homotopy ring $\pi_*(S)$ is an A-graded anticommutative ring, as desired.

Proposition 4.13. *The assignment $(E, \mu, e) \mapsto (\pi_*(E), \pi_*(e))$ yields a functor*
$$\pi_* : \mathbf{CMon}_{\mathcal{SH}} \to \pi_*(S)\text{-}\mathbf{GCA}^A$$
from the category of commutative monoid objects in $\mathcal{SH}$ (Definition C.3) to the category of A-graded anticommutative $\pi_*(S)$-algebras (Definition 4.6).

Proof. By Proposition 4.1, we know that π_* yields a functor from $\mathbf{CMon}_{\mathcal{SH}}$ to A-graded rings. Furthermore, by Proposition 4.8, we know that for all homogeneous $x, y \in \pi_*(E)$ that
$$x \cdot y = y \cdot x \cdot (e \circ \theta_{|x|,|y|}) = y \cdot x \cdot \pi_*(e)(\theta_{|x|,|y|}),$$
as desired. Thus, it remains to show that $\pi_*(e) : \pi_*(S) \to \pi_*(E)$ is an A-graded ring homomorphism for any (commutative) monoid object (E, μ, e) in $\mathcal{SH}$, and that given a monoid homomorphism $f : (E_1, \mu_1, e_1) \to (E_2, \mu_2, e_2)$ in $\mathbf{CMon}_{\mathcal{SH}}$, that $\pi_*(f)$ satisfies $\pi_*(f) \circ \pi_*(e_1) = \pi_*(e_2)$. The latter clearly holds, as since f is a monoid homomorphism, we have $f \circ e_1 = e_2$, so that
$$\pi_*(f) \circ \pi_*(e_1) = \pi_*(f \circ e_1) = \pi_*(e_2).$$
Furthermore, since $e : S \to E$ is a monoid object homomorphism (Lemma C.6), we know that $\pi_*(e) : \pi_*(S) \to \pi_*(E)$ is an A-graded ring homomorphism by Proposition 4.1. ☐

5. Some important theorems in $\mathcal{SH}$

So far, we have already identified a good amount of structure on the objects $\pi_*(E)$, $E_*(E) = \pi_*(E \otimes E)$, and $E_*(X)$ for E a (commutative) monoid object and X an object in $\mathcal{SH}$. Namely, we have shown that $\pi_*(E)$ and $E_*(E)$ are canonically A-graded anticommutative algebras over the stable homotopy ring (Proposition 4.13), and that $E_*(X)$ is canonically an A-graded left $\pi_*(E)$-module (Proposition 4.3). We would like to identify even more structure on these objects, namely, in Section 6, we will show that the pair $(E_*(E), \pi_*(E))$ is an *A-graded anticommutative Hopf algebroid*, over which $E_*(X)$ is an A-graded left comodule. To that end, we need two important theorems, namely, we need analogs of the Künneth isomorphism and the universal coefficient theorem from algebraic topology. This section is dedicated to formulating and proving these theorems. The proofs of these theorems are arguably the most technical and difficult in this paper, so we will be especially careful to give them in their full and explicit detail.

5.1. **A Künneth isomorphism.**

The goal of this subsection will be to prove the following theorem, which, given a monoid object (E, μ, e) and objects Z and W in $\mathcal{SH}$, relates the Z-homology of E with the E-homology of W to $\pi_*(Z \otimes E \otimes W)$:

Theorem 5.1 (The Künneth isomorphism)**.** *Let (E, μ, e) be a monoid object and Z and W objects in $\mathcal{SH}$. Then if*

- $Z_*(E)$ *is a flat right* $\pi_*(E)$*-module (via Proposition 4.3) and* W *is cellular (Definition 3.1),* or
- $E_*(W)$ *is a flat left* $\pi_*(E)$*-module (via Proposition 4.3) and* Z *is cellular,*

then there is a natural A*-graded isomorphism of* A*-graded abelian groups, called the* Künneth isomorphism*:*

$$\Phi_{Z,W} : Z_*(E) \otimes_{\pi_*(E)} E_*(W) \xrightarrow{\cong} \pi_*(Z \otimes E \otimes W).$$

There is much work to be done. First, we construct the map and show it is natural:

Proposition 5.2. *Let* (E, μ, e) *be a monoid object and* Z *and* W *be objects in* $\mathcal{SH}$*. Then there is an* A*-graded homomorphism of abelian groups*

$$\Phi_{Z,W} : Z_*(E) \otimes_{\pi_*(E)} E_*(W) \to \pi_*(Z \otimes E \otimes W)$$

which given homogeneous elements $x : S^a \to Z \otimes E$ *in* $Z_*(E) = \pi_*(Z \otimes E)$ *and* $y : S^b \to E \otimes W$ *in* $E_*(W) = \pi_*(E \otimes W)$*, sends the homogeneous pure tensor* $x \otimes y$ *in* $Z_*(E) \otimes_{\pi_*(E)} E_*(W)$ *to the composition*

$$\Phi_{Z,W}(x \otimes y) : S^{a+b} \xrightarrow{\phi_{a,b}} S^a \otimes S^b \xrightarrow{x \otimes y} Z \otimes E \otimes E \otimes W \xrightarrow{Z \otimes \mu \otimes W} Z \otimes E \otimes W$$

Furthermore, this homomorphism is natural in both Z *and* W*.*

Proof. By Lemma B.14, in order to get an A-graded homomorphism

$$\Phi_{Z,W} : Z_*(E) \otimes_{\pi_*(E)} E_*(W) \to \pi_*(Z \otimes E \otimes W),$$

it suffices to define an assignment $P : Z_*(E) \times E_*(W) \to \pi_*(Z \otimes E \otimes W)$ on homogeneous elements (which we have), and show that it is additive in each argument for homogeneous elements of the same degree, and that for all homogeneous $z \in Z_*(E)$, $r \in \pi_*(E)$, and $w \in E_*(W)$ that $P(zr, w) = P(z, rw)$, where concatenation denotes the module action.

First, note that by Lemma 4.2 it is straightforward to see that the assignment commutes with addition of maps of the same degree in each argument. Now, let $a, b, c \in A$, $z : S^a \to Z \otimes E$, $w : S^b \to E \otimes W$, and $r : S^c \to E$. Then we wish to show $P(zr, w) = P(z, rw)$. Consider the following diagram (where here we are passing to a symmetric strict monoidal category):

$$\begin{tikzcd}
 & & & Z \otimes E \otimes E \otimes W \arrow[d, "Z\otimes\mu\otimes W"] \\
S^{a+b+c} \arrow[r, "\cong"] & S^a \otimes S^c \otimes S^b \arrow[r, "z\otimes r\otimes w"] & Z \otimes E \otimes E \otimes E \otimes W \arrow[ur, "Z\otimes\mu\otimes E\otimes W"] \arrow[dr, "Z\otimes E\otimes\mu\otimes W"'] & Z \otimes E \otimes W \\
 & & & Z \otimes E \otimes E \otimes W \arrow[u, "Z\otimes\mu\otimes W"']
\end{tikzcd}$$

It commutes by associativity of μ. By functoriality of $- \otimes -$, the top composition is given by $P(zr, w)$ and the bottom composition is $P(z, rw)$, so they are equal as desired. Thus, by Lemma B.14 we get the desired A-graded homomorphism $\pi_*(Z \otimes E) \otimes_{\pi_*(E)} \pi_*(E \otimes W) \to \pi_*(Z \otimes E \otimes W)$.

Next, we would like to show that this homomorphism is natural in Z. Let $f : Z \to Z'$ in $\mathcal{SH}$. Then we would like to show the following diagram commutes:

$$\begin{tikzcd}
\pi_*(Z \otimes E) \otimes_{\pi_*(E)} \pi_*(E \otimes W) \arrow[r, "\Phi_{Z,W}"] \arrow[d, "\pi_*(f\otimes E)\otimes\pi_*(E\otimes W)"'] & \pi_*(Z \otimes E \otimes W) \arrow[d, "\pi_*(f\otimes E\otimes W)"] \\
\pi_*(Z' \otimes E) \otimes_{\pi_*(E)} \pi_*(E \otimes W) \arrow[r, "\Phi_{Z',W}"] & \pi_*(Z' \otimes E \otimes W)
\end{tikzcd} \tag{3}$$

As all the maps here are homomorphisms, in order to show it commutes, it suffices to chase generators around the diagram. In particular, suppose we are given $z : S^a \to Z \otimes E$ and $w : S^b \to E \otimes W$, and consider the following diagram exhibiting the two possible ways to chase $z \otimes w$ around the diagram (as usual, we are passing to a symmetric strict monoidal category):

$$\begin{tikzcd}
S^{a+b} \arrow[r, "\phi_{a,b}"] & S^a \otimes S^b \arrow[r, "z\otimes w"] & Z \otimes E \otimes E \otimes W \arrow[r, "Z\otimes\mu\otimes W"] \arrow[d, "f\otimes E\otimes E\otimes W"'] & Z \otimes E \otimes W \arrow[d, "f\otimes E\otimes W"] \\
 & & Z \otimes E \otimes E \otimes W \arrow[r, "Z\otimes\mu\otimes W"] & Z' \otimes E \otimes W
\end{tikzcd}$$

This diagram commutes by functoriality of $- \otimes -$. Thus we have that diagram (3) does indeed commute, so that $\Phi_{Z,W}$ is natural in Z as desired. Showing that $\Phi_{Z,W}$ is natural in W is entirely analagous. $\square$

Now, before proving the Künneth map is an isomorphism under the conditions given in Theorem 5.1, we prove the following lemmas:

Lemma 5.3. *Let (E, μ, e) be a monoid object and Z and W be objects in $\mathcal{SH}$. Then for all $a \in A$, the following diagram commutes*

$$\begin{tikzcd}
Z_*(E) \otimes_{\pi_*(E)} E_*(\Sigma^a W) \arrow[r, "Z_*(E)\otimes_{\pi_*(E)} t_a^W"] \arrow[d, "\Phi_{Z,\Sigma^a W}"'] & Z_*(E) \otimes_{\pi_*(E)} E_{*-a}(W) \arrow[d, "\Phi_{Z,W}"] \\
\pi_*(Z \otimes E \otimes \Sigma^a W) \arrow[d, equal] & \pi_{*-a}(Z \otimes E \otimes W) \arrow[d, equal] \\
(Z \otimes E)_*(\Sigma^a W) \arrow[r, "t_a^W"] & (Z \otimes E)_{*-a}(W)
\end{tikzcd}$$

where the maps t_a are constructed and proven to be A-graded isomorphisms of abelian groups in **??**.

Proof. Note that in **??**, it is shown that $t_a^W : E_*(\Sigma^a W) \to E_{*-a}(W)$ is not just an A-graded isomorphism of abelian groups, but it is furthermore a left $\pi_*(E)$-module isomorphism. Thus, the top arrow in the above diagram is well-defined. Since all the arrows involved are A-graded homomorphisms, in order to show the diagram commutes it suffices to chase a pure homogeneous tensor around, as they generate the top left object. To that end, let $x : S^b \to Z \otimes E$ in $Z_*(E)$ and $y : S^c \to E \otimes S^a \otimes W$ in $E_*(\Sigma^a W)$, and consider the following diagram exhibiting the two ways to chase $x \otimes y$ around:

$$\begin{tikzcd}
S^{b+c-a} \arrow[d, "\phi"] & Z \otimes E \otimes E \otimes W \otimes S^a \otimes S^{-a} \arrow[r, "Z\otimes E\otimes E\otimes W\otimes\phi^{-1}_{a,-a}"] \arrow[ddr, "Z\otimes\mu\otimes W\otimes S^a\otimes S^{-a}"] & Z \otimes E \otimes E \otimes W \arrow[d, "Z\otimes\mu\otimes W"] \\
S^b \otimes S^c \otimes S^{-a} \arrow[r, "x\otimes y\otimes S^{-a}"] & Z \otimes E \otimes E \otimes S^a \otimes W \otimes S^{-a} \arrow[u, "Z\otimes E\otimes E\otimes\tau\otimes S^{-a}"] \arrow[d, "Z\otimes\mu\otimes S^a\otimes W\otimes S^{-a}"'] & Z \otimes E \otimes W \\
 & Z \otimes E \otimes S^a \otimes W \otimes S^{-a} \arrow[r, "Z\otimes E\otimes\tau\otimes S^{-a}"'] & Z \otimes E \otimes W \otimes S^a \otimes S^{-a} \arrow[u, "Z\otimes E\otimes W\otimes\phi^{-1}_{a,-a}"']
\end{tikzcd}$$

Each triangle commutes by functoriality of $- \otimes -$, so the diagram commutes as desired. $\square$

Lemma 5.4. *Given a monoid object (E, μ, e) and an object X in $\mathcal{SH}$, for all $a \in A$ the A-graded isomorphisms*

$$s^a_{X\otimes E} : \pi_*(\Sigma^a X \otimes E) \to \pi_{*-a}(X \otimes E)$$

from Definition 2.5 are isomorphisms of right $\pi_*(E)$-modules, where here $\pi_*(\Sigma^a X \otimes E)$ and $\pi_*(X \otimes E) = X_*(E)$ are considered with their canonical right $\pi_*(E)$-module structure given in Proposition 4.3.

Proof. By additivity, in order to show $s^a_{X\otimes E}$ is a homomorphism of right $\pi_*(E)$-modules, it suffices to show that for all homogeneous $x : S^b \to S^a \otimes X \otimes E$ in $\pi_*(\Sigma^a X \otimes E)$ and $r : S^b \to E$ in $\pi_*(E)$ that $s^a_{X\otimes E}(x \cdot r) = s^a_{X\otimes E}(x) \cdot r$. To that end, consider the following diagram:

$$\begin{tikzcd}
S^{b+c-a} \arrow[r, "\phi"] & S^{-a} \otimes S^b \otimes S^c \arrow[r, "S^{-a}\otimes x\otimes r"] & S^{-a} \otimes S^a \otimes X \otimes E \otimes E \arrow[r, "S^{-a}\otimes S^a\otimes X\otimes\mu"] \arrow[d, "\phi^{-1}_{-a,a}\otimes X\otimes E\otimes E"'] & S^{-a} \otimes S^a \otimes X \otimes E \arrow[d, "\phi^{-1}_{-a,a}\otimes X\otimes E"] \\
& & X \otimes E \otimes E \arrow[r, "X\otimes\mu"] & X \otimes E
\end{tikzcd}$$

The top composition is $s^a_{X\otimes E}(x \cdot r)$, while the bottom composition is $s^a_{X\otimes E}(x) \cdot r$. The diagram commutes by functoriality of $- \otimes -$, so that $s^a_{X\otimes E}(x \cdot r) = s^a_{X\otimes E}(x) \cdot r$ as desired, meaning $s^a_{X\otimes E}$ is indeed a right $\pi_*(E)$-module homomorphism. □

Lemma 5.5. *Let (E, μ, e) be a monoid object and Z and W objects in $\mathcal{SH}$, and suppose the Künneth map $\Phi_{Z,W}$ is an isomorphism. Then $\Phi_{\Sigma^a Z,W}$ and $\Phi_{Z,\Sigma^a W}$ are isomorphisms for all $a \in A$, and so are $\Phi_{\Sigma Z,W}$ an $\Phi_{Z,\Sigma W}$.*

Proof. If $\Phi_{Z,W}$ is an isomorphism, it follows that $\Phi_{Z,\Sigma^a W}$ is an isomorphism by Lemma 5.3. On the other hand, in order to see $\Phi_{\Sigma^a Z,W}$ is an isomorphism, consider the following diagram:

$$\begin{tikzcd}
\pi_*(\Sigma^a Z \otimes E) \otimes_{\pi_*(E)} \pi_*(E \otimes W) \arrow[r, "\Phi_{\Sigma^a Z,W}"] \arrow[d, "s^a_{Z\otimes E}\otimes_{\pi_*(E)}\pi_*(E\otimes W)"'] & \pi_*(\Sigma^a Z \otimes E \otimes W) \arrow[d, "s^a_{Z\otimes E\otimes W}"] \\
\pi_{*-a}(Z \otimes E) \otimes_{\pi_*(E)} \pi_*(E \otimes W) \arrow[r, "\Phi_{Z,W}"] & \pi_{*-a}(Z \otimes E \otimes W)
\end{tikzcd} \tag{4}$$

Here the vertical arrows are induced via the isomorphisms constructed in Definition 2.5, and the left vertical arrow is well-defined since $s^a_{Z\otimes E}$ is a right $\pi_*(E)$-module homomorphism by Lemma 5.4. Since every arrow in diagram (4) is an isomorphism of abelian groups except the top arrow, in order to show $\Phi_{\Sigma^a Z,W}$ is an isomorphism, it suffices to show the diagram commutes. To that end, since all the arrows are homomorphisms, it suffices to chase a pure homogeneous tensor around. So let $x : S^b \to \Sigma^a Z \otimes E$ and $y : S^c \to E \otimes W$, and consider the following diagram whose outside compositions exhibit the two ways to chase the pure tensor $x \otimes y$ around diagrama (4):

$$\begin{tikzcd}
S^{b+c-a} \arrow[r, "\phi"] & S^{-a} \otimes S^b \otimes S^c \arrow[r, "S^{-a}\otimes x\otimes y"] & S^{-a} \otimes S^a \otimes Z \otimes E \otimes E \otimes W \arrow[r, "S^{-a}\otimes S^a\otimes Z\otimes\mu\otimes W"] \arrow[d, "\phi^{-1}_{-a,a}\otimes Z\otimes E\otimes E\otimes W"'] & S^{-a} \otimes S^a \otimes Z \otimes E \otimes W \arrow[d, "\phi^{-1}_{-a,a}\otimes Z\otimes E\otimes W"] \\
& & Z \otimes E \otimes E \otimes W \arrow[r, "Z\otimes\mu\otimes W"] & Z \otimes E \otimes W
\end{tikzcd}$$

The diagram clearly commutes by functoriality of $- \otimes -$, so that indeed diagram (4) commutes, so that $\Phi_{\Sigma^a Z,W}$ is indeed an isomorphism as desired.

Now, it remains to show that $\Phi_{Z,\Sigma W}$ and $\Phi_{\Sigma Z,W}$ are isomorphisms. To that end, consider the following diagram:

$$\begin{tikzcd}
\pi_*(Z\otimes E)\otimes_{\pi_*(E)}\pi_*(E\otimes\Sigma W) \arrow[r, "\Phi_{Z,\Sigma W}"] \arrow[d, "\pi_*(Z\otimes E)\otimes_{\pi_*(E)}\pi_*(E\otimes\nu_W)"'] & \pi_*(Z\otimes E\otimes\Sigma W) \arrow[d, "\pi_*(Z\otimes E\otimes\nu_W)"] \\
\pi_*(Z\otimes E)\otimes_{\pi_*(E)}\pi_*(E\otimes\Sigma^{\mathbf{1}}W) \arrow[r, "\Phi_{Z,\Sigma^{\mathbf{1}}W}"'] & \pi_*(Z\otimes E\otimes\Sigma^{\mathbf{1}}W)
\end{tikzcd}$$

It commutes by naturality of Φ. Furthermore, assuming $\Phi_{Z,W}$ is an isomorphism, by what we have shown above we know that $\Phi_{Z,\Sigma^{\mathbf{1}}W}$ is an isomorphism, and since ν_W is an isomorphism, it follows that the above diagram commutes and all arrows except $\Phi_{Z,\Sigma W}$ are isomorphisms, so that $\Phi_{Z,\Sigma W}$ must be an isomorphism itself. Finally, an entirely analagous argument using naturality of Φ with respect to ν_Z yields that $\Phi_{\Sigma Z,W}$ is an isomorphism as well. $\square$

Now, we can finally prove the desired theorem:

Proposition 5.6. *Let (E,μ,e) be a monoid object and Z and W objects in $\mathcal{SH}$. Then if either:*

(1) $Z_*(E)$ is a flat right $\pi_*(E)$-module (via Proposition 4.3) and W is cellular (Definition 3.1), or

(2) $E_*(W)$ is a flat left $\pi_*(E)$-module (via Proposition 4.3) and Z is cellular,

then the natural homomorphism

$$\Phi_{Z,W}: Z_*(E)\otimes_{\pi_*(E)} E_*(W)\to\pi_*(Z\otimes E\otimes W)$$

given in Proposition 5.2 is an isomorphism of abelian groups.

Proof. In this proof, we will freely employ the coherence theorem for symmetric monoidal categories, and we will assume that associativity and unitality of $-\otimes-$ holds up to strict equality. First we will consider the case that $\pi_*(Z\otimes E)=Z_*(E)$ is a flat right $\pi_*(E)$-module and W is cellular. To start, let $\mathcal{E}$ be the collection of objects W in $\mathcal{SH}$ for which $\Phi_{Z,W}$ is an isomorphism. Then in order to show $\mathcal{E}$ contains every cellular object, it suffices to show that $\mathcal{E}$ satisfies the three conditions given for the class of cellular objects in Definition 3.1. First, we need to show that $\Phi_{Z,W}$ is an isomorphism when $W=S^a$ for some $a\in A$. Indeed, consider the A-graded homomorphism

$$\Psi:\pi_*(Z\otimes E\otimes S^a)\to\pi_*(Z\otimes E)\otimes_{\pi_*(E)}\pi_*(E\otimes S^a)$$

which sends a class $x:S^b\to Z\otimes E\otimes S^a$ in $\pi_b(Z\otimes E\otimes S^a)$ to the pure tensor $\widetilde{x}\otimes\widetilde{e}$, where $\widetilde{x}\in\pi_{b-a}(Z\otimes E)$ is the composition

$$S^{b-a}\xrightarrow{\phi_{b,-a}}S^b\otimes S^{-a}\xrightarrow{x\otimes S^{-a}}Z\otimes E\otimes S^a\otimes S^{-a}\xrightarrow{Z\otimes E\otimes\phi_{a,-a}^{-1}}Z\otimes E$$

and $\widetilde{e}\in\pi_a(E\otimes S^a)$ is the composition

$$S^a\xrightarrow{e\otimes S^a}E\otimes S^a.$$

In order to see Ψ is an (A-graded) homomorphism of abelian groups: Given $x,x'\in\pi_b(Z\otimes E\otimes S^a)$, we would like to show that $\widetilde{x}\otimes\widetilde{e}+\widetilde{x}'\otimes\widetilde{e}=\widetilde{x+x'}\otimes\widetilde{e}$. It suffices to show that $\widetilde{x}+\widetilde{x}'=\widetilde{x+x'}$.

To see this, consider the following diagram (again, we are passing to a symmetric strict monoidal category):

$$\begin{tikzcd}
S^{b-a} \arrow[r, "\Delta"] \arrow[d, "\phi_{b-a}"'] & S^{b-a} \oplus S^{b-a} \arrow[d, "\phi_{b,-a} \oplus \phi_{b,-a}"] \\
S^b \otimes S^{-a} \arrow[r, "\Delta"] \arrow[d, "\Delta \otimes S^{-a}"'] & (S^b \otimes S^{-a}) \oplus (S^b \otimes S^{-a}) \arrow[d, "(x \otimes S^{-a}) \oplus (x' \otimes S^{-a})"] \\
(S^b \oplus S^b) \otimes S^{-a} \arrow[ur, "\cong"] \arrow[d, "(x \oplus x') \otimes S^{-a}"'] & (Z \otimes E \otimes S^a \otimes S^{-a}) \oplus (Z \otimes E \otimes S^a \otimes S^{-a}) \arrow[d, "(Z \otimes E \otimes \phi^{-1}_{a,-a}) \oplus (Z \otimes E \otimes \phi^{-1}_{a,-a})"] \arrow[ddl, "\nabla"] \\
((Z \otimes E \otimes S^a) \oplus (Z \otimes E \otimes S^a)) \otimes S^{-a} \arrow[ur, "\cong"] \arrow[d, "\nabla \otimes S^{-a}"'] & (Z \otimes E) \oplus (Z \otimes E) \arrow[d, "\nabla"] \\
Z \otimes E \otimes S^a \otimes S^{-a} \arrow[r, "Z \otimes E \otimes \phi^{-1}_{a,-a}"'] & Z \otimes E
\end{tikzcd}$$

The top rectangle commutes by naturality of Δ in an additive category. The bottom triangle commutes by naturality of ∇ in an additive category. Finally, the remaining regions of the diagram commute by additivity of $- \otimes -$. By functoriality of $- \otimes -$, it follows that the left composition is $\widetilde{x + x'}$ and the right composition is $\widetilde{x} + \widetilde{x}'$, so they are equal as desired. Thus Ψ is a homomorphism of abelian groups, as desired.

Now, we claim that Ψ is an inverse to Φ_{Z,S^a}. Since Φ_{Z,S^a} and Ψ are homomorphisms it suffices to check that they are inverses on generators. First, let $x : S^b \to Z \otimes E \otimes S^a$ in $\pi_b(Z \otimes E \otimes S^a)$. We would like to show that $\Phi_{Z,S^a}(\Psi(x)) = x$. Consider the following diagram, where here we are passing to a symmetric strict monoidal category:

$$\begin{tikzcd}
S^b \arrow[r, "\cong"] \arrow[dd, "x"'] & S^b \otimes S^{-a} \otimes S^a \arrow[d, dashed, "x \otimes S^{-a} \otimes S^a"] \arrow[dr, "x \otimes S^{-a} \otimes e \otimes S^a"] & \\
 & Z \otimes E \otimes S^a \otimes S^{-a} \otimes S^a \arrow[r, "Z \otimes E \otimes S^a \otimes S^{-a} \otimes e \otimes S^a"'] & Z \otimes E \otimes S^a \otimes S^{-a} \otimes E \otimes S^a \arrow[ddl, "Z \otimes E \otimes \phi^{-1}_{a,-a} \otimes E \otimes S^a"] \\
Z \otimes E \otimes S^a \arrow[ur, "Z \otimes E \otimes S^a \otimes \phi_{-a,a}"] \arrow[r, equal] & Z \otimes E \otimes S^a \arrow[u, dashed, "Z \otimes E \otimes \phi_{a,-a} \otimes S^a"'] \arrow[d, dashed, "Z \otimes E \otimes e \otimes S^a"] & \\
 & Z \otimes E \otimes E \otimes S^a \arrow[ul, "Z \otimes \mu \otimes S^a"] &
\end{tikzcd}$$

The top left trapezoid commutes since the isomorphism $S^b \xrightarrow{\cong} S^b \otimes S^{-a} \otimes S^a$ may be given as $S^b \otimes \phi_{-a,a}$ (see Remark 2.3), in which case the trapezoid commmutes by functoriality of $- \otimes -$. The triangle below that commutes by coherence for the $\phi_{a,b}$'s. The bottom left triangle commutes by unitality for μ. The top right triangle commutes by functoriality of $- \otimes -$. Finally, the bottom right triangle commutes by functoriality of $- \otimes -$. It follows by unravelling definitions that the two outside compositions are x and $\Phi_{Z,S^a}(\Psi(x))$, so indeed we have $\Phi_{Z,S^a}(\Psi(x)) = x$ since the diagram commutes.

On the other hand, suppose we are given a homogeneous pure tensor $x \otimes y$ in $\pi_*(Z \otimes E) \otimes_{\pi_*(E)} \pi_*(E \otimes S^a)$, so $x : S^b \to Z \otimes E$ and $y : S^c \to E \otimes S^a$ for some $b, c \in A$. Then we would like to show that $\Psi(\Phi_{Z,S^a}(x \otimes y)) = x \otimes y$. Unravelling definitions, $\Psi(\Phi_{Z,S^a}(x \otimes y))$ is the homogeneous pure tensor $\widetilde{xy} \otimes \widetilde{e}$, where $\widetilde{e}$ is the map $e \otimes S^a : S^a \to E \otimes S^a$ is defined above, and by functoriality of $-\otimes -$, $\widetilde{xy} : S^{b+c-a} \to Z \otimes E$ is the composition

$$\begin{tikzcd}
S^{b+c-a} \arrow[d, "\cong"] \\
S^b \otimes S^c \otimes S^{-a} \arrow[d, "x\otimes y\otimes S^{-a}"] \\
Z \otimes E \otimes E \otimes S^a \otimes S^{-a} \arrow[d, "Z\otimes\mu\otimes S^a\otimes S^{-a}"] \\
Z \otimes E \otimes S^a \otimes S^{-a} \arrow[d, "Z\otimes E\otimes\phi^{-1}_{a,-a}"] \\
Z \otimes E
\end{tikzcd}$$

Now, define $r \in \pi_{c-a}(E)$ to be the composition

$$S^{c-a} \cong S^c \otimes S^{-a} \xrightarrow{y\otimes S^{-a}} E \otimes S^a \otimes S^{-a} \xrightarrow{E\otimes\phi^{-1}_{a,-a}} E.$$

First, we claim that $x \cdot r = \widetilde{xy}$. To that end, consider the following diagram, where here we are again passing to a symmetric strict monoidal category:

$$\begin{tikzcd}
S^{b+c-a} \arrow[r, "\cong"] & S^b \otimes S^c \otimes S^{-a} \arrow[r, "x\otimes y\otimes S^{-a}"] & Z \otimes E \otimes E \otimes S^a \otimes S^{-a} \arrow[r, "Z\otimes\mu\otimes S^a\otimes S^{-a}"] \arrow[d, "Z\otimes E\otimes E\otimes\phi^{-1}_{a,-a}"'] & Z \otimes E \otimes S^a \otimes S^{-a} \arrow[d, "Z\otimes E\otimes\phi^{-1}_{a,-a}"] \\
& & Z \otimes E \otimes E \arrow[r, "Z\otimes\mu"'] & Z \otimes E
\end{tikzcd}$$

Commutativity is functoriality of $-\otimes-$, which also tells us that the two outside compositions are $\widetilde{xy}$ (on top) and $x \cdot r$ (on the bottom), so they are equal as desired. On the other hand, we claim that $r \cdot \widetilde{e} = y$. To see this, consider the following diagram:

$$\begin{tikzcd}
S^c \arrow[rr, "\cong"] \arrow[dd, "y"'] & & S^c \otimes S^{-a} \otimes S^a \arrow[dl, "y\otimes S^{-a}\otimes S^a"'] \arrow[d, "y\otimes S^{-a}\otimes e\otimes S^a"] \\
& E \otimes S^a \otimes S^{-a} \otimes S^a \arrow[r, "E\otimes S^a\otimes S^{-a}\otimes e\otimes S^a"'] \arrow[dl, "E\otimes S^a\otimes\phi^{-1}_{-a,a}"'] \arrow[d, "E\otimes\phi^{-1}_{a,-a}\otimes S^a"] & E \otimes S^a \otimes S^{-a} \otimes E \otimes S^a \arrow[dd, "E\otimes\phi^{-1}_{a,-a}\otimes E\otimes S^a"] \\
E \otimes S^a \arrow[r, equal] & E \otimes S^a \arrow[dr, "E\otimes e\otimes S^a"] & \\
E \otimes E \otimes S^a \arrow[u, "\mu\otimes S^a"] \arrow[rr, equal] & & E \otimes E \otimes S^a
\end{tikzcd}$$

By Remark 2.3, we may take the top arrow to be $S^c \otimes \phi_{-a,a}$, in which case the top left triangle commutes by functoriality of $-\otimes-$. The bottom trapezoid commutes by unitality of μ. Every other region commutes either by definition or by functoriality of $-\otimes-$. The top composition is $r \cdot \widetilde{e}$, so we have shown $r \cdot \widetilde{e} = y$ as desired. Thus, we have that

$$\Psi(\Phi_{Z,S^a}(x \otimes y)) = \widetilde{xy} \otimes \widetilde{e} = x \cdot r \otimes \widetilde{e} = x \otimes r \cdot \widetilde{e} = x \otimes y,$$

as desired. Hence we have shown Ψ is both a left and right inverse for Φ_{Z,S^a}, so that indeed S^a belongs to $\mathcal{E}$ as desired.

Now, we would like to show that given a distinguished triangle in $\mathcal{SH}$

$$X \xrightarrow{f} Y \xrightarrow{g} W \xrightarrow{h} \Sigma X,$$

if two of three of the objects X, Y, and W belong to $\mathcal{E}$, then so does the third. From now on, write L_*^E to denote the functor from $\mathcal{SH}$ to A-graded abelian groups sending $X \mapsto \pi_*(Z \otimes E) \otimes_{\pi_*(E)} \pi_*(E \otimes X)$. Then $\Phi_{Z,-}$ is a natural transformation $L_*^E \Rightarrow \pi_*(Z \otimes E \otimes -) = Z_*(E \otimes -)$. First, recall that it follows generally that in an adjointly triangulated category (Definition A.8), which $\mathcal{SH}$ is by Proposition 2.4, given a distinguished triangle (f, g, h) we have a long exact sequence (see Definition A.2 for the definition of an exact sequence in an additive category, and see Proposition A.11 for the explicit contruction of the LES associated to a distinguished triangle in an adjointly triangulated category):

$$\Omega Y \xrightarrow{\Omega g} \Omega W \xrightarrow{\widetilde{h}} X \xrightarrow{f} Y \xrightarrow{g} W \xrightarrow{h} \Sigma X \xrightarrow{\Sigma f} \Sigma Y,$$

where $\widetilde{h} : \Omega W \to X$ is the adjoint of $h : W \to \Sigma X$. Then since $\mathcal{SH}$ is further a tensor triangulated category (Definition A.12), we have that the above sequence remains exact even after tensoring by E on the left (see Proposition A.15 for details), so we have the following exact sequence in $\mathcal{SH}$:

$$E \otimes \Omega Y \xrightarrow{E\otimes\Omega g} E \otimes \Omega W \xrightarrow{E\otimes\widetilde{h}} E \otimes X \xrightarrow{E\otimes f} E \otimes Y \xrightarrow{E\otimes g} E \otimes W \xrightarrow{E\otimes h} E \otimes \Sigma X \xrightarrow{E\otimes\Sigma f} E \otimes \Sigma Y.$$

We can then apply $[S^*, -] = \pi_*(-)$ to it, which yields the following exact sequence of A-graded abelian groups:

$$E_*(\Omega Y) \xrightarrow{E_*(\Omega g)} E_*(\Omega W) \xrightarrow{E_*(\widetilde{h})} E_*(X) \xrightarrow{E_*(f)} E_*(Y) \xrightarrow{E_*(g)} E_*(W) \xrightarrow{E_*(h)} E_*(\Sigma X) \xrightarrow{E_*(f)} E_*(\Sigma Y).$$

Now, we can tensor this sequence with $\pi_*(Z \otimes E)$ on the left over $\pi_*(E)$, and since $\pi_*(Z \otimes E)$ is a flat right $\pi_*(E)$ module, we get that the top row in the following diagram is exact:

$$\begin{tikzcd}
L_*^E(\Omega Y) \arrow[r, "L_*^E(\Omega g)"] \arrow[d, "\Phi_{Z,\Omega Y}"'] & L_*^E(\Omega W) \arrow[r, "L_*^E(\widetilde{h})"] \arrow[d, "\Phi_{Z,\Omega W}"'] & L_*^E(X) \arrow[r, "L_*^E(f)"] \arrow[d, "\Phi_{Z,X}"'] & L_*^E(Y) \arrow[r, "L_*^E(g)"] \arrow[d, "\Phi_{Z,Y}"'] & L_*^E(W) \arrow[r, "L_*^E(h)"] \arrow[d, "\Phi_{Z,W}"'] & L_*^E(\Sigma X) \arrow[r, "L_*^E(\Sigma f)"] \arrow[d, "\Phi_{Z,\Sigma X}"'] & L_*^E(\Sigma Y) \arrow[d, "\Phi_{Z,\Sigma Y}"'] \\
Z_*(E \otimes \Omega Y) \arrow[r, "Z_*(E\otimes\Omega g)"'] & Z_*(E \otimes \Omega W) \arrow[r, "Z_*(E\otimes\widetilde{h})"'] & Z_*(E \otimes X) \arrow[r, "Z_*(E\otimes f)"'] & Z_*(E \otimes Y) \arrow[r, "Z_*(E\otimes g)"'] & Z_*(E \otimes W) \arrow[r, "Z_*(E\otimes h)"'] & Z_*(E \otimes \Sigma X) \arrow[r, "Z_*(E\otimes\Sigma f)"'] & Z_*(E \otimes \Sigma Y)
\end{tikzcd}$$

This diagram further commutes by naturality of $\Phi_{Z,-}$. Now, supposing that two of three of X, Y, and W belong to $\mathcal{E}$, by Lemma 5.5, if $\Phi_{Z,V}$ is an isomorphism for some object V in $\mathcal{SH}$ then $\Phi_{Z,\Omega V}$ and $\Phi_{Z,\Sigma V}$ are. Thus by the five lemma, it follows that the middle three vertical arrows in the above diagram are necessarily all isomorphisms if any two of them are, so we have shown that $\mathcal{E}$ is closed under two-of-three for exact triangles, as desired.

Finally, it remains to show that $\mathcal{E}$ is closed under arbitrary coproducts. Let $\{W_i\}_{i\in I}$ be a collection of objects in $\mathcal{E}$ indexed by some (small) set I. Then we'd like to show that $W := \bigoplus_i W_i$ belongs to $\mathcal{E}$. First of all, note that $- \otimes -$ preserves arbitrary coproducts in each argument, as it has a right adjoint $F(-, -)$. Thus without loss of generality, given any object X in $\mathcal{SH}$, we may take $\bigoplus_i X \otimes W_i = X \otimes \bigoplus_i W_i$ (as $X \otimes \bigoplus_i W_i$ *is* a coproduct of all the $X \otimes W_i$'s). Now, recall that we have chosen each S^a to be a compact object (Definition 2.1), so that given any object X and collection of objects $\{Y_i\}_{i\in I}$ in $\mathcal{SH}$, if $Y := \bigoplus_{i\in I} Y_i$, then the canonical map

$$q_{X,Y_i} : \bigoplus_i X_*(Y_i) = \bigoplus_i [S^*, X \otimes Y_i] \to [S^*, \bigoplus_i X \otimes Y_i] = [S^*, X \otimes Y] = X_*(Y)$$

is an isomorphism, natural in Y_i for each i. Note in particular that q_{E,W_i} is an isomorphism of left $\pi_*(E)$-modules. To see this, first note by additivity of q_{E,W_i}, it suffices to check that $q_{E,W_i}(r \cdot x) = r \cdot q_{E,W_i}(x)$ for each homogeneous $r \in \pi_*(E)$ and homogeneous $x \in E_*(W_i)$ for some i, as such x generate $\bigoplus_i E_*(W_i)$ by definition. Then given $r : S^a \to E$ and $x : S^b \to E \otimes W_i$, consider the following diagram

$$\begin{tikzcd}
S^{a+b} \arrow[r, "\phi_{a,b}"] & S^a \otimes S^b \arrow[r, "x\otimes y"] & E \otimes E \otimes W_i \arrow[r, "E\otimes\iota_{E\otimes W_i}"] \arrow[dddd, "\mu\otimes W_i"'] \arrow[dr, "E\otimes E\otimes\iota_{W_i}"] & E \otimes \bigoplus_i (E \otimes W_i) \arrow[d, equal] \\
 & & & E \otimes E \otimes W \arrow[d, "\mu\otimes W"] \\
 & & & E \otimes W \arrow[dd, equal] \\
 & & & \\
 & & E \otimes W_i \arrow[r, "\iota_{E\otimes W_i}"'] \arrow[uur, "E\otimes\iota_{W_i}"] & \bigoplus_i (E \otimes W_i)
\end{tikzcd}$$

where $\iota_{E\otimes W_i} : E \otimes W_i \hookrightarrow \bigoplus_i (E \otimes W_i)$ and $\iota_{W_i} : W_i \hookrightarrow \bigoplus_i W_i$ are the maps determined by the definition of the coproduct. Commutativity of the two triangles is by the fact that $E \otimes -$ is colimit preserving. Commutativity of the trapezoid is functoriality of $- \otimes -$. Thus, since q_{E,W_i} is a homomorphism of left A-graded $\pi_*(E)$-modules, the top right arrow in the following diagram is well-defined:

$$\begin{tikzcd}
\bigoplus_i Z_*(E) \otimes_{\pi_*(E)} E_*(W_i) \arrow[r, equal] \arrow[d, "\bigoplus_i \Phi_{Z,W_i}"'] & Z_*(E) \otimes_{\pi_*(E)} \bigoplus_i E_*(W_i) \arrow[r, "Z_*(E)\otimes_{\pi_*(E)} q_{E,W_i}"] & Z_*(E) \otimes_{\pi_*(E)} E_*(W) \arrow[d, "\Phi_{Z,W}"] \\
\bigoplus_i Z_*(E \otimes W_i) \arrow[r, "q_{Z,E\otimes W_i}"] & Z_*(\bigoplus_i E \otimes W_i) \arrow[r, equal] & Z_*(E \otimes W)
\end{tikzcd} \tag{5}$$

We wish to show this diagram commutes. Again, since each map here is a homomorphism, it suffices to chase generators. By definition, a generator of the top left element is a homogeneous pure tensor in $E_*(E) \otimes_{\pi_*(E)} E_*(W_i)$ for some i in I. Given classes $x : S^a \to Z \otimes E$ in $Z_*(E)$ and $y : S^b \to E \otimes W_i$ in $E_*(W_i)$, consider the following diagram:

$$\begin{tikzcd}
S^{a+b} \arrow[r, "\phi_{a,b}"] & S^a \otimes S^b \arrow[r, "x\otimes y"] & Z \otimes E \otimes E \otimes W_i \arrow[r, "Z\otimes E\otimes\iota_{E\otimes W_i}"] \arrow[d, "Z\otimes\mu\otimes W_i"'] \arrow[dr, "Z\otimes E\otimes E\otimes\iota_{W_i}"] & Z \otimes E \otimes \bigoplus_i E \otimes W_i \arrow[d, equal] \\
 & & Z \otimes E \otimes W_i \arrow[d, "\iota_{Z\otimes E\otimes W_i}"'] \arrow[dr, "Z\otimes E\otimes\iota_{W_i}"] & Z \otimes E \otimes E \otimes W \arrow[d, "Z\otimes\mu\otimes W"] \\
 & & \bigoplus_i Z \otimes E \otimes W_i \arrow[r, equal] & Z \otimes E \otimes W
\end{tikzcd}$$

Unravelling definitions, the two outside compositions are the two ways to chase $x \otimes y$ around diagram (5). The two triangles commute again by the fact that $- \otimes -$ preserves colimits in each argument. Commutativity of the inner parallelogram is functoriality of $- \otimes -$. Thus diagram (5) tells us $\Phi_{Z,W}$ is an isomorphism, since q_{E,W_i} and $q_{Z,E\otimes W_i}$ are isomorphisms, and Φ_{Z,W_i} is an isomorphism for each i in I, meaning $\bigoplus_i \Phi_{W_i}$ is as well.

Thus, we've shown the class $\mathcal{E}$ of objects W for which $\Phi_{Z,W}$ is an isomorphism contains the S^a's, is closed under two-of-three for distinguished triangles, and is closed under arbitrary coproducts. Thus, it follows that $\mathcal{E}$ contains the class of all cellular objects in $\mathcal{SH}$, as desired.

Now, suppose that $\pi_*(E \otimes W)$ is a flat left $\pi_*(E)$-module, then we'd like to show $\Phi_{Z,W}$ is an isomorphism for all cellular Z in $\mathcal{SH}$. Showing this is entirely analagous to above, so we only outline the argument. Let $\mathcal{E}$ be the class of Z in $\mathcal{SH}$ such that $\Phi_{Z,W}$ is an isomorphism. Then in order to show $\mathcal{E}$ contains every cellular object, it suffices to show it contains the S^a's, is closed under two-of-three for distinguished triangles, and is closed under arbitrary coproducts.

To see $\mathcal{E}$ contains the S^a's, consider the map

$$\Psi : \pi_*(S^a \otimes E \otimes W) \to \pi_*(S^a \otimes E) \otimes_{\pi_*(E)} \pi_*(E \otimes W)$$

sending $x : S^b \to S^a \otimes E \otimes W$ to $\widetilde{e} \otimes \widetilde{x}$, where $\widetilde{e} \in \pi_a(S^a \otimes E)$ is the map $S^a \otimes e : S^a \to S^a \otimes E$, and $\widetilde{x} \in \pi_{b-a}(E \otimes W)$ is the map

$$\widetilde{x} : S^{b-a} \xrightarrow{\phi_{-a,b}} S^{-a} \otimes S^b \xrightarrow{S^{-a} \otimes x} S^{-a} \otimes S^a \otimes E \otimes W \xrightarrow{\phi^{-1}_{-a,a} \otimes E \otimes W} E \otimes W.$$

Then checking that Ψ is a left and right inverse to $\Phi_{S^a,W}$ is entirely analagous, so that S^a belongs to $\mathcal{E}$ as desired.

To see $\mathcal{E}$ is closed under two-of-three for distinguished triangles, let

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$$

be a distinguished triangle in $\mathcal{SH}$. Then an analagous argument as above (using Proposition A.11 and Proposition A.15) yields a long exact sequence of A-graded abelian groups

$$\begin{tikzcd}
 & \pi_*(\Omega Y \otimes E) \arrow[r, "\pi_*(\Omega g \otimes E)"] & \pi_*(\Omega Z \otimes E) \arrow[dll, "\pi_*(\widetilde{h} \otimes E)"'] \\
\pi_*(X \otimes E) \arrow[r, "\pi_*(f \otimes E)"'] & \pi_*(Y \otimes E) \arrow[r, "\pi_*(g \otimes E)"] & \pi_*(Z \otimes E) \arrow[dll, "\pi_*(h \otimes E)"'] \\
\pi_*(\Sigma X \otimes E) \arrow[r, "\pi_*(\Sigma f \otimes E)"'] & \pi_*(\Sigma Y \otimes E) &
\end{tikzcd}$$

Then since $\pi_*(E \otimes W)$ is a flat left $\pi_*(E)$-module, we can tensor the above long exact sequence with $\pi_*(E \otimes W)$ on the right to obtain a long exact sequence which fits in the left column of the following commuting diagram:

$$\begin{tikzcd}
R^E_*(\Omega Y) \arrow[r, "\Phi_{\Omega Y,W}"] \arrow[d, "R^E_*(\Omega g)"'] & \pi_*(\Omega Y \otimes E \otimes W) \arrow[d, "\pi_*(\Omega g \otimes E \otimes W)"] \\
R^E_*(\Omega Z) \arrow[r, "\Phi_{\Omega Z,W}"] \arrow[d, "R^E_*(\widetilde{h})"'] & \pi_*(\Omega Z \otimes E \otimes W) \arrow[d, "\pi_*(\widetilde{h} \otimes E \otimes W)"] \\
R^E_*(X) \arrow[r, "\Phi_{X,W}"] \arrow[d, "R^E_*(f)"'] & \pi_*(X \otimes E \otimes W) \arrow[d, "\pi_*(f \otimes E \otimes W)"] \\
R^E_*(Y) \arrow[r, "\Phi_{Y,W}"] \arrow[d, "R^E_*(g)"'] & \pi_*(Y \otimes E \otimes W) \arrow[d, "\pi_*(g \otimes E \otimes W)"] \\
R^E_*(Z) \arrow[r, "\Phi_{Z,W}"] \arrow[d, "R^E_*(h)"'] & \pi_*(Z \otimes E \otimes W) \arrow[d, "\pi_*(h \otimes E \otimes W)"] \\
R^E_*(\Sigma X) \arrow[r, "\Phi_{\Sigma X,W}"] \arrow[d, "R^E_*(\Sigma f)"'] & \pi_*(\Sigma X \otimes E \otimes W) \arrow[d, "\pi_*(\Sigma f \otimes E \otimes W)"] \\
R^E_*(\Sigma Y) \arrow[r, "\Phi_{\Sigma Y,W}"] & \pi_*(\Sigma Y \otimes E \otimes W)
\end{tikzcd}$$

where R^E_* denotes the functor from $\mathcal{SH}$ to A-graded abelian groups sending $X \mapsto \pi_*(X \otimes E) \otimes_{\pi_*(E)} \pi_*(E \otimes W)$, so that $\Phi_{-,W}$ is a natural homomorphism $R^E_*(-) \Rightarrow \pi_*(- \otimes E \otimes W)$. Then finally by Lemma 5.5 and the five lemma, if any two of three of the middle three horizontal arrows are isomorphisms, then all three of the horizontal arrows are isomorphisms, as desired.

Finally, in order to show $\mathcal{E}$ is closed under arbitrary coproducts, suppose we have a collection of objects $\{Z_i\}_{i \in I}$ in $\mathcal{E}$ indexed by some (small) set $\mathcal{E}$. Then we'd like to show $Z := \bigoplus_{i \in I} Z_i$ also belongs to $\mathcal{E}$. First note that since the S^a's are compact, for any object Y we have isomorphisms

$$q_{Z_i,Y} : \bigoplus_i Z_{i*}(Y) = \bigoplus_i [S^*, Z_i \otimes Y] \to [S^*, \bigoplus_i (Z_i \otimes Y)] = [S^*, Z \otimes Y] = Z_*(Y).$$

It is straightforward to verify that $q_{Z_i,E} : \bigoplus_i Z_{i*}(E) \to Z_*(E)$ is not only an isomorphism of abelian groups, but an isomorphism of right A-graded $\pi_*(E)$-modules, so that the top arrow in the following diagram is well-defined:

$$\begin{tikzcd}
\bigoplus_i \left(Z_{i*}(E) \otimes_{\pi_*(E)} E_*(W)\right) \arrow[r, equal] \arrow[d, "\bigoplus_i \Phi_{Z_i,W}"'] & \bigoplus_i \left(Z_{i*}(E)\right) \otimes_{\pi_*(E)} E_*(W) \arrow[r, "q_{Z_i,E} \otimes E_*(W_i)"] & Z_*(E) \otimes_{\pi_*(E)} E_*(W) \arrow[d, "\Phi_{Z,W}"] \\
\bigoplus_i Z_{i*}(E \otimes W) \arrow[rr, "q_{Z_i,E\otimes W}"] & & Z_*(E \otimes W)
\end{tikzcd}$$

Then a simple diagram chase yields the diagram commutes, so that $\Phi_{Z,W}$ is an isomorphism, assuming all the $\Phi_{Z_i,W}$'s are. $\square$

5.2. **Modules over monoid objects in $\mathcal{SH}$.** Now, before we prove our next theorem (an analog of the universal coefficient theorem in $\mathcal{SH}$), we need to develop some of the theory of (left) module objects over monoid objects in $\mathcal{SH}$. For a review of the basic definitions and properties of module objects over monoid objects in symmetric monoidal categories, see Appendix C.2. Recall specifically that given a monoid object (E, μ, e) in $\mathcal{SH}$, the category E-**Mod** of (left) E-module objects is additive (Proposition C.15), and the forgetful functor E-**Mod** $\to \mathcal{SH}$ preserves arbitrary coproducts and has a right adjoint $\mathcal{SH} \to E$-**Mod** taking an object X in $\mathcal{SH}$ to the *free E-module* $E \otimes X$ (Proposition C.12).

Proposition 5.7. *Let (E, μ, e) be a monoid object in $\mathcal{SH}$. Then the assignment $\pi_* : (N, \kappa) \mapsto \pi_*(N)$ yields an additive functor from E-**Mod** to the category $\pi_*(E)$-**Mod**^{A} of A-graded left $\pi_*(E)$-modules and degree-preserving homomorphisms between them. In particular, if (N, κ) is an E-module object in $\mathcal{SH}$, then we view it with its* canonical *A-graded left $\pi_*(E)$-module structure given by the graded map*

$$\pi_*(E) \times \pi_*(N) \to \pi_*(N)$$

sending a class $r : S^a \to E$ and $x : S^b \to N$ to the composition

$$r \cdot x : S^{a+b} \xrightarrow{\phi_{a,b}} S^a \otimes S^b \xrightarrow{r \otimes x} E \otimes N \xrightarrow{\kappa} N.$$

Proof. First let (N, κ) be an E-module object. Let $a, b, c \in A$ and $x, x' : S^a \to N$, $y : S^b \to E$, and $z, z' \in S^c \to E$. Then by Lemma B.9, it suffices to show that

(1) $y \cdot (x + x') = y \cdot x + y \cdot x'$,

(2) $(z + z') \cdot x = z \cdot x + z' \cdot x$,

(3) $(zy) \cdot x = z \cdot (y \cdot x)$,

(4) $e \cdot x = x$.

The first two axioms follow by Lemma 4.2. To see (3), consider the diagram:

$$
\begin{tikzcd}
 & & & E\otimes N \arrow[d, "\kappa"] \\
S^{a+b+c} \arrow[r, "\cong"] & S^c\otimes S^b\otimes S^a \arrow[r, "z\otimes y\otimes x"] & E\otimes E\otimes N \arrow[ur, "E\otimes\kappa"] \arrow[dr, "\mu\otimes N"'] & N \\
 & & & E\otimes N \arrow[u, "\kappa"']
\end{tikzcd}
$$

It commutes by coherence for κ. By functoriality of $-\otimes-$, the two outside compositions equal $z\cdot(y\cdot x)$ on the top and $(z\cdot y)\cdot x$ on the bottom. Hence, they are equal, as desired.

Next, to see (4), consider the following diagram:

$$
\begin{tikzcd}
S^a \arrow[rr, "x"] \arrow[dr, "x"'] \arrow[ddr, bend right, "e\otimes x"'] & & N \\
 & N \arrow[ur, equal] \arrow[d, dashed, "e\otimes N"] & \\
 & E\otimes N \arrow[uur, bend right, "\kappa"'] &
\end{tikzcd}
$$

The top triangle commutes by definition. The left triangle commutes by functoriality of $-\otimes-$. The right triangle commutes by unitality of κ. The top composition is x while the bottom is $e\cdot x$, thus they are necessarily equal since the diagram commutes.

Now, we'd like to show that if $f:(N,\kappa)\to(N',\kappa)$ is a homomorphism of E-module objects, then $\pi_*(f):\pi_*(N)\to\pi_*(N')$ is a homomorphism of left $\pi_*(E)$-modules. To see this, let $r: S^a\to E$ in $\pi_a(E)$ and $x,x: S^b\to N$ in $\pi_b(N)$. We'd like to show that $\pi_*(f)(x+x')=\pi_*(f)(x)+\pi_*(f)(x')$ and $\pi_*(f)(r\cdot x)=r\cdot\pi_*(f)(x)$. To see the former, consider the following diagram:

$$
\begin{tikzcd}
 & & & N'\oplus N' \arrow[d, "\nabla"] \\
S^a \arrow[r, "\Delta"] & S^a\oplus S^a \arrow[r, "x\oplus x'"] & N\oplus N \arrow[ur, "f\oplus f"] \arrow[dr, "\nabla"'] & N' \\
 & & & N \arrow[u, "f"']
\end{tikzcd}
$$

It commutes by naturality of ∇ in an additive category. The top composition is $\pi_*(f)(x)+\pi_*(f)(x')$, while the bottom is $\pi_*(f)(x+x')$, so they are equal as desired. To see that $\pi_*(f)(r\cdot x)=r\cdot\pi_*(f)(x)$, consider the following diagram:

$$
\begin{tikzcd}
 & & & E\otimes N' \arrow[d, "\kappa'"] \\
S^{a+b} \arrow[r, "\phi_{b,a}"] & S^b\otimes S^a \arrow[r, "r\otimes x"] & E\otimes N \arrow[ur, "E\otimes f"] \arrow[dr, "\kappa"'] & N' \\
 & & & N \arrow[u, "f"']
\end{tikzcd}
$$

It commutes by the fact that f is a homomorphism of E-module objects. The bottom composition is $\pi_*(f)(r\cdot x)$, while the top composition is $r\cdot\pi_*(f)(x)$, so they are equal, as desired.

Next we claim this functor is additive. It suffices to show it preserves the zero object and preserves coproducts. To see the former, note that $\pi_*(0)=[S^*,0]=0$ by definition, since

0 is terminal. To see the latter, we need to show that given $(N, \kappa), (N', \kappa') \in E$-**Mod** that $\pi_*(N) \oplus \pi_*(N') \cong \pi_*(N \oplus N')$, and that the following diagram commutes:

$$\begin{tikzcd}
\pi_*(N) \arrow[d, "\iota_{\pi_*(N)}"'] \arrow[dr, "\pi_*(\iota_N)"] & \\
\pi_*(N) \oplus \pi_*(N') \arrow[r, "\cong"'] & \pi_*(N \oplus N')
\end{tikzcd}$$

Since each S^a is compact, for all $a, b \in A$ we have isomorphisms

$$\pi_a(N) \oplus \pi_a(N') = [S^a, N] \oplus [S^a, N'] \cong [S^a, N \oplus N'] = \pi_a(N \oplus N'),$$

and these combine together to yield A-graded isomorphisms $\pi_*(N) \oplus \pi_*(N') \xrightarrow{\cong} \pi_*(N \oplus N')$. To see the above diagram commutes, note that since everything is an A-graded homomorphism of A-graded abelian groups, it suffices to chase homogeneous elements around to show it commutes. Indeed, it is entirely straightforward, by unravelling definitions, that both compositions around the diagram take a generator $x : S^a \to N$ in $\pi_a(N)$ to the composition

$$S^a \xrightarrow{x} N \xrightarrow{\iota_N} N \oplus N'.$$

Thus, we have shown that π_* preserves all finite coproducts, so it is additive. ☐

Remark 5.8. In the above proposition, we have shown that given an E-module object (N, κ) in $\mathcal{SH}$, $\pi_*(N)$ is canonically an A-graded left $\pi_*(E)$-module. In particular, we may apply this proposition to the free E-module $E \otimes X$ (Proposition C.12). It is straightforward to see, and we leave it to the reader to check, that the A-graded left $\pi_*(E)$-module structure on $E_*(X) = \pi_*(E \otimes X)$ induced by the above proposition is precisely the canonical module structure from Proposition 4.3. In fact, the above proposition entirely subsumes the first half of Proposition 4.3 (although we give the two separate statements for the sake of clarity). Thus, there continues to be no ambiguity when talking about the left $\pi_*(E)$-module structure on $E_*(X)$.

Lemma 5.9. *Let (E, μ, e) be a monoid object in $\mathcal{SH}$, and suppose (N, κ) is a module object over E (Definition C.8). Then for all $a \in A$, the a^{th} suspension $\Sigma^a N$ of N is canonically an E-module object, with action map given by*

$$\kappa^a : E \otimes \Sigma^a N = E \otimes S^a \otimes N \xrightarrow{\tau \otimes N} S^a \otimes E \otimes N \xrightarrow{S^a \otimes \kappa} S^a \otimes N = \Sigma^a N.$$

Furthermore, given an E-module homomorphism $f : (N, \kappa) \to (N', \kappa')$, $\Sigma^a f : \Sigma^a N \to \Sigma^a N'$ is likewise an E-module homomorphism.

Proof. In this proof, we are assuming that unitality and associativity hold up to strict equality, by the coherence theorem for monoidal categories. In order to show $(\Sigma^a N, \kappa^a)$ is a module object over E, we need to show κ^a makes the two coherence diagrams in Definition C.8 commute. First, to see the first diagram commutes, consider the following diagram:

$$\begin{tikzcd}
S^a \otimes N \arrow[r, "e \otimes S^a \otimes N"] \arrow[dr, "S^a \otimes e \otimes N"'] \arrow[ddr, equal] & E \otimes S^a \otimes N \arrow[d, "\tau \otimes N"] \\
& S^a \otimes E \otimes N \arrow[d, "S^a \otimes \kappa"] \\
& S^a \otimes N
\end{tikzcd}$$

The top inner triangle commutes by coherence for a symmetric monoidal category, and the bottom inner triangle commutes by the coherence condition for κ. To see the other module condition for

$\widetilde{\kappa}$, consider the following diagram:

$$\begin{tikzcd}
E \otimes E \otimes S^a \otimes N \arrow[rr, "\mu \otimes S^a \otimes N"] \arrow[d, "E \otimes \tau \otimes N"'] \arrow[dr, "\tau_{E \otimes E, S^a} \otimes N"] & & E \otimes S^a \otimes N \arrow[d, "\tau \otimes N"] \\
E \otimes S^a \otimes E \otimes N \arrow[r, "\tau \otimes E \otimes N"'] \arrow[d, "E \otimes S^a \otimes \kappa"'] & S^a \otimes E \otimes E \otimes N \arrow[r, "S^a \otimes \mu \otimes N"] \arrow[d, "S^a \otimes E \otimes \kappa"] & S^a \otimes E \otimes N \arrow[d, "S^a \otimes \kappa"] \\
E \otimes S^a \otimes N \arrow[r, "\tau \otimes N"] & S^a \otimes E \otimes N \arrow[r, "S^a \otimes \kappa"] & S^a \otimes N
\end{tikzcd}$$

The top left triangle commutes by coherence for a symmetric monoidal category. The bottom left rectangle and top right trapezoid commute by naturality of τ. Finally, the bottom right square commutes by the coherence condition for κ.

Thus, we have shown that $\Sigma^a N$ is indeed an object in E-**Mod**, as desired. Now let $f : (N, \kappa) \to (N', \kappa')$ be a morphism in E-**Mod**, we would like to show $\Sigma^a f : \Sigma^a N \to \Sigma^a N'$ is also a homomorphism of E-modules. To that end, consider the following diagram:

$$\begin{tikzcd}
E \otimes S^a \otimes N \arrow[r, "E \otimes S^a \otimes f"] \arrow[d, "\tau \otimes N"'] & E \otimes S^a \otimes N' \arrow[d, "\tau \otimes N'"] \\
S^a \otimes E \otimes N \arrow[r, "S^a \otimes E \otimes f"] \arrow[d, "S^a \otimes \kappa"'] & S^a \otimes E \otimes N' \arrow[d, "S^a \otimes \kappa'"] \\
S^a \otimes N \arrow[r, "S^a \otimes f"'] & S^a \otimes N'
\end{tikzcd}$$

The top rectangle commutes by functoriality of $- \otimes -$, while the bottom commutes since f is an E-module homomorphism. Thus, $S^a \otimes f = \Sigma^a f$ is an E-module homomorphism, as desired. $\square$

Definition 5.10. We can extend the hom-groups in E-**Mod** (which is additive by Proposition C.15) to A-graded abelian groups $\mathrm{Hom}^*_{E\text{-}\mathbf{Mod}}(N, N')$ defined by

$$\mathrm{Hom}^a_{E\text{-}\mathbf{Mod}}(N, N') := \mathrm{Hom}_E(\Sigma^a N, N'),$$

where $\Sigma^a N$ is considered as an E-module object by the above lemma.

Lemma 5.11. *Given a monoid object (E, μ, e) in $\mathcal{SH}$, an object X in $\mathcal{SH}$, and some $a \in A$, the suspension of the free module $\Sigma^a(E \otimes X)$ is naturally isomorphic as an E-module object to the free E-module $E \otimes \Sigma^a X$.*

Proof. It suffices to show the map $S^a \otimes E \otimes X \xrightarrow{\tau \otimes X} E \otimes S^a \otimes X$ is a homomorphism of E-module objects, as we know it is an isomorphism and natural in X. To that end, consider the following diagram:

$$\begin{tikzcd}
E \otimes S^a \otimes E \otimes X \arrow[r, "E \otimes \tau \otimes X"] \arrow[d, "\tau \otimes E \otimes X"'] & E \otimes E \otimes S^a \otimes X \arrow[dd, "\mu \otimes S^a \otimes X"] \\
S^a \otimes E \otimes E \otimes X \arrow[ur, "\tau_{S^a, E \otimes E} \otimes X"'] \arrow[d, "S^a \otimes \mu \otimes X"'] & \\
S^a \otimes E \otimes X \arrow[r, "\tau \otimes X"] & E \otimes S^a \otimes X
\end{tikzcd}$$

The top triangle commutes by coherence for a symmetric monoidal category. The bottom trapezoid commutes by naturality of τ. $\square$

Lemma 5.12. *Let (E, μ, e) be a monoid object in $\mathcal{SH}$, and suppose we have a collection of objects (N_i, κ_i) in E-**Mod**. Then for all $a \in A$, since Σ^a has a right adjoint Σ^{-a} (Proposition 2.4), it preserves coproducts in $\mathcal{SH}$, (which are coproducts in E-**Mod** by Proposition C.14), so we have an isomorphism*

$$\Sigma^a \bigoplus_i N_i \cong \bigoplus_i \Sigma^a N_i.$$

Then this isomorphism is an E-module homomorphism.

Proof. □ TODO

5.3. **A universal coefficient theorem.** Finally, we have the ingredients required to state and prove the following universal coefficient theorem:

Theorem 5.13. *Let (E, μ, e) be a monoid object and let X and Y be objects in $\mathcal{SH}$. Then if E and X are cellular and $E_*(X)$ is a graded projective (Definition B.15) left $\pi_*(E)$-module (via Proposition 4.3), then the map*

$$[X, E \otimes Y] \to \mathrm{Hom}_{\pi_*(E)}(E_*(X), E_*(Y)), \qquad [X \xrightarrow{f} E \otimes Y] \mapsto [\pi_*(\mu \otimes Y) \circ \pi_*(E \otimes f)]$$

is an isomorphism, and extends to an A-graded isomorphism

$$[X, E \otimes Y]_* \to \mathrm{Hom}^*_{\pi_*(E)}(E_*(X), E_*(Y)).$$

Proof. Since: (1) $E \otimes X$ is a free E-module object (Proposition C.12), (2) $E_*(X) = \pi_*(E \otimes X)$ is a graded projective left $\pi_*(E)$-module, and (3) E and $E \otimes X$ are cellular (by Lemma 3.3), by Proposition 5.16 below it follows that $E \otimes X$ is a retract of $\bigoplus_i (E \otimes S^{a_i})$ in E-**Mod** for some collection of $a_i \in A$ indexed by some set I. Thus the desired result follows by Proposition 5.15 below with $N = E \otimes Y$ (which is considered as a free E-module by Proposition C.12). □

In the case that $Y = S$, this theorem becomes the more familiar statement:

$$E^*(X) \cong [X, E]_{-*} \cong [X, E \otimes S]_{-*} \cong \mathrm{Hom}^{-*}_{\pi_*(E)}(E_*(X), \pi_*(E)),$$

i.e., the E-cohomology of X is isomorphic to the dual of the E-homology of X when $E_*(X)$ is a graded projective module. Hence why we call it the universal coefficient theorem.

Proposition 5.14. *Let (E, μ, e) be a monoid object and (N, κ) an E-module object in $\mathcal{SH}$. Then given a collection of $a_i \in A$ indexed by some set I, if (N, κ) is a retract of $\bigoplus_i (E \otimes S^{a_i})$ in E-**Mod**,*[3] *then for all E-module objects (N', κ'), the functor $\pi_* : E\text{-}\mathbf{Mod} \to \pi_*(E)\text{-}\mathbf{Mod}(A)$ (Proposition 5.7) induces an isomorphism of abelian groups*

$$\pi_* : \mathrm{Hom}_{E\text{-}\mathbf{Mod}}(N, N') \to \mathrm{Hom}_{\pi_*(E)}(\pi_*(N), \pi_*(N')).$$

[3]Here $\bigoplus_i (E \otimes S^{a_i})$ is a coproduct (Proposition C.14) of a bunch of free E-module objects (Proposition C.12), so it is itself an E-module object.

Proof. To start, we consider the case $N = \bigoplus_i (E \otimes S^{a_i})$. Consider the following diagram:

$$\begin{tikzcd}
\mathrm{Hom}_E(\bigoplus_i (E \otimes S^{a_i}), N') \arrow[r, "\pi_*"] \arrow[d, "\cong"'] & \mathrm{Hom}_{\pi_*(E)}(\pi_*(\bigoplus_i (E \otimes S^{a_i})), \pi_*(N')) \arrow[d, "\cong"] \\
\prod_i \mathrm{Hom}_E(E \otimes S^{a_i}, N') \arrow[d, "\cong"'] & \mathrm{Hom}_{\pi_*(E)}(\bigoplus_i \pi_*(E \otimes S^{a_i}), \pi_*(N')) \arrow[d, "\cong"] \\
\prod_i [S^{a_i}, N'] \arrow[dd, equal] & \prod_i \mathrm{Hom}_{\pi_*(E)}(\pi_*(E \otimes S^{a_i}), \pi_*(N')) \arrow[d, "\cong"] \\
 & \prod_i \mathrm{Hom}_{\pi_*(E)}(\pi_{*-a_i}(E), \pi_*(N')) \arrow[d, equal] \\
\prod_i \pi_{a_i}(N') & \prod_i \mathrm{Hom}^{a_i}_{\pi_*(E)}(\pi_*(E), \pi_*(N')) \arrow[l, "\prod_i \mathrm{ev}_1"']
\end{tikzcd}$$

Here the top left vertical isomorphism exibits the universal property of the coproduct in E-**Mod**, and middle left vertical isomorphism below that is the free-forgetful adjunction for E-modules (Proposition C.12). The bottom horizontal isomorphism is the product of the evaluation-at-1 isomorphisms (Lemma B.10). On the other side, the top right vertical isomorphism is given by the fact that S^a is compact for each $a \in A$, so we have isomorphisms

$$\bigoplus_i \pi_*(E \otimes S^{a_i}) = \bigoplus_{a \in A} \bigoplus_i [S^a, E \otimes S^{a_i}] \cong \bigoplus_{a \in A} [S^a, \bigoplus_i (E \otimes S^{a_i})] = \pi_*(\bigoplus_i (E \otimes S^{a_i})),$$

where the middle isomorphism takes a generator $x : S^a \xrightarrow{E} \otimes S^{a_i}$ to the composition $S^a \xrightarrow{x} E \otimes S^{a_i} \hookrightarrow \bigoplus_i (E \otimes S^{a_i})$. The middle right vertical isomorphism exhibits the universal property of the coproduct of modules. Finally the bottom right vertical isomorphism is given by the isomorphisms

$$\pi_{*-a_i}(E \otimes S^{a_i}) = [S^{*-a_i}, E \otimes S^{a_i}] \xrightarrow{-\otimes S^{a_i}} [S^{*-a_i} \otimes S^{a_i}, E \otimes S^{a_i}] \xrightarrow{\phi^*} [S^*, E \otimes S^{a_i}] = \pi_*(E \otimes S^{a_i}),$$

where $-\otimes S^{a_i} \cong \Sigma^{a_i}$ is an isomorphism by Proposition 2.4. Now, we claim this diagram commutes. This really simply amounts to unravelling definitions, and chasing a homomorphism $f : \bigoplus_i (E \otimes S^{a_i}) \to N'$ of E-module objects both ways around the diagram yields the composition

$$\prod_i (S^{a_i} \xrightarrow{e \otimes S^{a_i}} E \otimes S^{a_i} \hookrightarrow \bigoplus_i (E \otimes S^{a_i}) \xrightarrow{f} N').$$

Thus, since the diagram commutes, we have that

$$\pi_* : \mathrm{Hom}_E(\bigoplus_i (E \otimes S^{a_i}), N') \to \mathrm{Hom}_{\pi_*(E)}(\pi_*(\bigoplus_i (E \otimes S^{a_i})), \pi_*(N'))$$

is an isomorphism, as desired.

Now, consider the case that N is a retract of $\bigoplus_i (E \otimes S^{a_i})$ in E-**Mod**, so there exists a commuting diagram of E-module object homomorphisms:

$$\begin{tikzcd}
N \arrow[r, "\iota"'] \arrow[rr, bend left, equal] & \bigoplus_i (E \otimes S^{a_i}) \arrow[r, "r"'] & N
\end{tikzcd}$$

Now consider the following diagram:

$$\begin{tikzcd}
\mathrm{Hom}_E(N,N') \arrow[r, "r^*"] \arrow[d, "\pi_*"'] \arrow[rr, bend left, equal] & \mathrm{Hom}_E(\bigoplus_i (E\otimes S^{a_i}), N') \arrow[r, "\iota^*"] \arrow[d, "\pi_*"] & \mathrm{Hom}_E(N,N') \arrow[d, "\pi_*"'] \\
\mathrm{Hom}_{\pi_*(E)}(\pi_*(N),\pi_*(N')) \arrow[r, "(\pi_*(r))^*"] \arrow[rr, bend right, equal] & \mathrm{Hom}_{\pi_*(E)}(\pi_*(\bigoplus_i (E\otimes S^{a_i})),\pi_*(N')) \arrow[r, "(\pi_*(\iota))^*"] & \mathrm{Hom}_{\pi_*(E)}(\pi_*(N),\pi_*(N'))
\end{tikzcd}$$

Each square commutes by functoriality of π_*. We have shown the middle vertical arrow is an isomorphism. Thus the outside arrows are isomorphisms as well, as a retract of an isomorphism is an isomorphism. □

Proposition 5.15. *Let (E,μ,e) be a monoid object and X an object in $\mathcal{SH}$. If there is a collection of $a_i \in A$ indexed by some set I such that $E\otimes X$ is a retract of $\bigoplus_i(E\otimes S^{a_i})$ in E-**Mod**,[4] then for all E-module objects (N,κ), the assignment*

$$[X,N] \to \mathrm{Hom}_{\pi_*(E)}(E_*(X),\pi_*(N)), \qquad [X \xrightarrow{f} N] \mapsto [\pi_*(\kappa)\circ\pi_*(E\otimes f)]$$

is an isomorphism, and further extends to an A-graded isomorphism of A-graded abelian groups

$$[X,N]_* \to \mathrm{Hom}^*_{\pi_*(E)}(E_*(X),\pi_*(N)).$$

Proof. For each $a\in A$, define

$$U_a : [X,N]_a \to \mathrm{Hom}^a_{\pi_*(E)}(E_*(X),\pi_*(N))$$

to be the composition

$$\begin{tikzcd}
{[X,N]_a} \arrow[r, equal] & {[\Sigma^a X, N]} \arrow[d, "\mathrm{adj}"] & \\
 & \mathrm{Hom}_{E\text{-}\mathbf{Mod}}(E\otimes\Sigma^a X, N) \arrow[d, "\pi_*(-)"] & \\
 & \mathrm{Hom}_{\pi_*(E)}(E_*(\Sigma^a X),\pi_*(N) \arrow[d, "((t^a_X)^{-1})^*"] & \\
 & \mathrm{Hom}_{\pi_*(E)}(E_{*-a}(X),\pi_*(N)) \arrow[r, equal] & \mathrm{Hom}^a_{\pi_*(E)}(E_*(X),\pi_*(N))
\end{tikzcd}$$

where the first isomorphism is the free-forgetful adjunction for E-modules (Proposition C.12), the second map is that induced by the functor π_* constructed in Proposition 5.7, and the third map is induced by the A-graded isomorphism of left $\pi_*(E)$-modules $(t^a_X)^{-1} : E_{*-a}(X) \to E_*(\Sigma^a X)$ from **??**. By unravelling definitions, is straightforward to see that under the identification $[X,N] \cong [X,N]_0$, the map $U_0 : [X,N]_0 \to \mathrm{Hom}^0_{\pi_*(E)}(E_*(X),\pi_*(N))$ coincides with the assignment

$$[X,N] \to \mathrm{Hom}_{\pi_*(E)}(E_*(X),\pi_*(N)) \qquad [X \xrightarrow{f} N] \mapsto [\pi_*(\kappa)\circ\pi_*(E\otimes f)].$$

Furthermore, note we have isomorphisms in E-**Mod**

$$E\otimes\Sigma^a X = E\otimes S^a\otimes X \cong S^a\otimes E\otimes X$$

[4]Here $\bigoplus_i(E\otimes S^{a_i})$ is a coproduct (Proposition C.14) of a bunch of free E-module objects (Proposition C.12), so it is itself a E-module object.

(by Lemma 5.11) and

$$S^a \otimes \bigoplus_i (E \otimes S^{a_i}) \cong \bigoplus_i (S^a \otimes E \otimes S^{a_i}) \cong \bigoplus_i (E \otimes S^a \otimes S^{a_i}) \cong \bigoplus_i (E \otimes S^{a+a_i}),$$

where the first isomorphism is in E-**Mod** by Lemma 5.12, the second is in E-**Mod** by Lemma 5.11, and the last is a coproduct of homomorphisms of free E-modules (Proposition C.12), so it is also an E-module homomorphism. Hence we have that $E \otimes \Sigma^a X \cong S^a \otimes E \otimes X$ is a retract of $\bigoplus_i (E \otimes S^{a+a_i}) \cong S^a \otimes \bigoplus_i (E \otimes S^{a_i})$ in E-**Mod**, as $E \otimes X$ is a retract of $\bigoplus_i (E \otimes S^{a_i})$ in E-**Mod**, so that by Proposition 5.14, the map

$$\pi_* : \mathrm{Hom}_{E\text{-}\mathbf{Mod}}(E \otimes \Sigma^a X, N) \to \mathrm{Hom}_{\pi_*(E)}(E_*(\Sigma^a X), \pi_*(N))$$

is an isomorphism. Thus, we have constructed a bunch of isomorphisms

$$U_a : [X, N]_a \to \mathrm{Hom}^a_{\pi_*(E)}(E_*(X), \pi_*(N)),$$

so that by the universal property of the coproduct of abelian groups, there is a unique A-graded isomorphism

$$[X, N]_* \to \mathrm{Hom}^*_{\pi_*(E)}(E_*(X), \pi_*(N))$$

extending these maps, as desired. □

Proposition 5.16. *Let (E, μ, e) be a monoid object and (N, κ) an E-module object in $\mathcal{SH}$. Further suppose that E and N are cellular and that $\pi_*(N)$ is a* graded projective *(Definition B.15) left $\pi_*(E)$-module (via Proposition 5.7). Then given some homogeneous generating set $\{x_i\}_{i \in I} \subseteq \pi_*(N)$, N is a retract of $\bigoplus_i (E \otimes S^{|x_i|})$ in E-**Mod**.*[5]

Proof. Let $M := \bigoplus_i (E \otimes S^{|x_i|})$. We have a map

$$r : M \to N$$

induced by the maps

$$r_i : E \otimes S^{|x_i|} \xrightarrow{E \otimes x_i} E \otimes N \xrightarrow{\kappa} N.$$

This is a homomorphism of E-module objects:

$$\begin{tikzcd}
E \otimes \bigoplus_i (E \otimes S^{|x_i|}) \arrow[rr, "E\otimes r"] \arrow[dr, "E\otimes\bigoplus_i r_i"] \arrow[dd, "\cong"'] & & E \otimes N \arrow[dddd, "\kappa"] \\
& E \otimes \bigoplus_i N \arrow[ur, "E\otimes\nabla"'] \arrow[d, "\cong"'] & \\
\bigoplus_i (E \otimes E \otimes S^{|x_i|}) \arrow[r, "\bigoplus_i(E\otimes r_i)"] \arrow[dd, "\bigoplus_i(\mu\otimes S^{|x_i|})"'] & \bigoplus_i (E \otimes N) \arrow[uur, "\nabla"'] \arrow[d, "\bigoplus_i \kappa"] & \\
& \bigoplus_i N \arrow[dr, "\nabla"] & \\
\bigoplus_i (E \otimes S^{|x_i|}) \arrow[ur, "\bigoplus_i r_i"] \arrow[rr, "r"] & & N
\end{tikzcd}$$

The right trapezoid commutes by naturality of ∇. The bottom triangle commutes by the fact that $\nabla \circ \bigoplus_i r_i$ and r satisfy the same universal property for the coproduct. Every other region

[5]Here $\bigoplus_i (E \otimes S^{a_i})$ is a coproduct (Proposition C.14) of a bunch of free E-module objects (Proposition C.12), so it is itself an E-module object.

commutes by additivity of $E \otimes -$, except the left trapezoid: Note that by expanding out how r_i is defined, it becomes

$$\begin{tikzcd}
\bigoplus_i (E \otimes E \otimes S^{|x_i|}) \arrow[r, "\bigoplus_i (E\otimes E\otimes x_i)"] \arrow[d, "\bigoplus_i (\mu \otimes S^{|x_i|})"'] & \bigoplus_i (E \otimes E \otimes N) \arrow[r, "\bigoplus_i (E\otimes \kappa)"] \arrow[d, "\bigoplus_i (\mu\otimes X)"] & \bigoplus_i (E \otimes E \otimes X) \arrow[d, "\bigoplus_i \kappa"] \\
\bigoplus_i (E \otimes S^{|x_i|}) \arrow[r, "\bigoplus_i (E\otimes x_i)"'] & \bigoplus_i (E \otimes N) \arrow[r, "\bigoplus_i \kappa"'] & \bigoplus_i (E \otimes X)
\end{tikzcd}$$

The left square commutes by functoriality of $- \otimes -$, and the right square commutes by coherence for κ. Hence, we've shown that r is a homomorphism of E-modules, as desired. Thus, r induces a homomorphism of left $\pi_*(E)$-modules $\pi_*(r) \in \mathrm{Hom}_{\pi_*(E)}(\pi_*(M), \pi_*(N))$. Further note that for all $i \in I$, x_i is in the image of $\pi_*(r)$, as by definition $\pi_*(r)$ sends the class

$$S^{|x_i|} \xrightarrow{e\otimes S^{|x_i|}} E \otimes S^{|x_i|} \hookrightarrow M$$

in $\pi_{|x_i|}(M)$ to the composition

$$S^{|x_i|} \xrightarrow{e\otimes S^{|x_i|}} E \otimes S^{|x_i|} \xrightarrow{E\otimes x_i} E \otimes N \xrightarrow{\kappa} N,$$

and by unitality of κ this composition is simply $x_i : S^{|x_i|} \to N$. Thus, we have constructed a surjective A-graded homomorphism $\pi_*(r) : \pi_*(M) \to \pi_*(N)$ of left $\pi_*(E)$-modules, so that since $\pi_*(N)$ is projective graded module there exists an A-graded left $\pi_*(E)$-module homomorphism $\iota : \pi_*(N) \to \pi_*(M)$ which makes the following diagram commute:

$$\begin{tikzcd}
 & \pi_*(M) \arrow[d, "\pi_*(r)"] \\
\pi_*(N) \arrow[ur, "\iota"] \arrow[r, equal] & \pi_*(N)
\end{tikzcd}$$

Thus we have an idempotent of left A-graded $\pi_*(E)$-modules:

$$\pi_*(M) \xrightarrow{\pi_*(r)} \pi_*(N) \xrightarrow{\iota} \pi_*(M)$$

Now, by Proposition 5.14, since $M = \bigoplus_i (E \otimes S^{|x_i|})$, we have that the map

$$\pi_* : \mathrm{Hom}_{E\text{-}\mathbf{Mod}}(M, M) \to \mathrm{Hom}_{\pi_*(E)\text{-}\mathbf{Mod}}(\pi_*(M), \pi_*(M))$$

is an isomorphism of abelian groups, so that the above idempotent is induced by some endomorphism $\ell : M \to M$ of E-module objects. Further note that by functoriality of π_*,

$$\pi_*(\ell \circ \ell) = \pi_*(\ell) \circ \pi_*(\ell) = \pi_*(\ell),$$

and again since π_* is an isomorphism here, we have that $\ell \circ \ell = \ell$, so that ℓ is an idempotent in $\mathcal{SH}$. By Proposition A.7, every idempotent in $\mathcal{SH}$ splits, meaning ℓ factors in $\mathcal{SH}$ as

$$\ell : M \xrightarrow{r'} X \xrightarrow{\iota'} M$$

with $r' \circ \iota' = \mathrm{id}_X$. Since X is a retract of an E-module object, and the corresponding idempotent is an E-module homomorphism, it follows purely formally that X may be canonically viewed as an E-module object, and that $r' : M \to X$ and $\iota' : X \to M$ are homomorphisms of E-module objects (see Lemma C.13 for details). Note that since E and each $S^{|x_i|}$ are cellular, $E \otimes S^{|x_i|}$ is cellular for all $i \in I$ (by Lemma 3.3), so that $M = \bigoplus_i (E \otimes S^{|x_i|})$ is cellular, as by definition an arbitrary coproduct of cellular objects is cellular. Thus by Lemma 3.6, X is cellular as well. Now consider the following commutative diagram

$$\begin{tikzcd}
 & & \pi_*(N) \arrow[rr, equal] \arrow[dr, "\iota"] & & \pi_*(N) \arrow[dr, "\iota"] & & \\
\pi_*(N) \arrow[r, "\iota"'] \arrow[urr, bend left, equal] & \pi_*(M) \arrow[ur, "\pi_*(r)"] \arrow[rr, "\pi_*(\ell)"] \arrow[dr, "\pi_*(r')"'] & & \pi_*(M) \arrow[ur, "\pi_*(r)"] \arrow[rr, "\pi_*(\ell)"] \arrow[dr, "\pi_*(r')"] & & \pi_*(M) \arrow[r, "\pi_*(r')"] & \pi_*(X) \\
 & & \pi_*(X) \arrow[ur, "\pi_*(\iota')"'] \arrow[rr, equal] & & \pi_*(X) \arrow[ur, "\pi_*(\iota')"'] \arrow[urr, bend right, equal] & &
\end{tikzcd}$$

From this diagram we read off that the middle diagonal composition

$$\pi_*(X) \xrightarrow{\pi_*(\iota')} \pi_*(M) \xrightarrow{\pi_*(r)} \pi_*(N)$$

is an isomorphism with inverse $\pi_*(r') \circ \iota$. Now, since X and N are cellular, and $\pi_*(r \circ \iota')$ is an isomorphism, by Lemma 3.2 we have that $r \circ \iota'$ is an isomorphism, say with inverse p. Thus we have a commuting diagram

$$\begin{tikzcd}
N \arrow[rr, "\iota' \circ p"'] \arrow[rrr, bend left, equal] \arrow[dr, "p"'] & & M \arrow[r, "r"'] & N \\
 & X \arrow[ur, "\iota'"'] & &
\end{tikzcd}$$

and the middle row exhibits N as a retract of $M = \bigoplus_i (E \otimes S^{|x_i|})$, as desired. It remains to show this is a retract in E-**Mod**, i.e., that r and $\iota' \circ p$ are homomorphisms of E-module objects. Above we constructed r to be a homomorphism of E-modules. We also know that X is an E-module object and that ι' is an E-module homomorphism. Thus, it remains to show that $p : N \to X$ is an E-module homomorphism. But we know that p is the inverse of $r \circ \iota'$ in $\mathcal{SH}$, and we know r and ι' are morphisms in E-**Mod**, so that p is the inverse of $r \circ \iota'$ in E-**Mod**, meaning p is indeed an E-module homomorphism as desired. $\square$

6. The dual E-Steenrod algebra

In Section 4.1, we showed that given a monoid object (E, μ, e) in $\mathcal{SH}$, that $E_*(E)$ is canonically an A-graded bimodule over the ring $\pi_*(E)$. In this subsection, we will outline some additional structure carried by the pair $(E_*(E), \pi_*(E))$. In particular, we will show that if (E, μ, e) is a flat (Definition 6.5) commutative monoid object, then this pair, called the *dual E-Steenrod algebra*, is canonically an *A-graded anticommutative Hopf algebroid* over the stable homotopy ring $\pi_*(S)$ (Definition D.2). To start with, we outline some structure maps relating $E_*(E)$ and $\pi_*(E)$.

First, recall that given a monoid object (E, μ, e) in $\mathcal{SH}$, $\pi_*(E)$ is canonically an A-graded ring by Proposition 4.1, and so is $E_*(E) = \pi_*(E \otimes E)$ and $E_*(E \otimes E) = \pi_*(E \otimes E \otimes E)$, since the tensor product of monoid objects in a symmetric monoidal category is again a monoid object (Lemma C.4).

Proposition 6.1. *Let (E, μ, e) be a commutative monoid object in $\mathcal{SH}$. Then the maps*

(1) $E \xrightarrow{\cong} E \otimes S \xrightarrow{E \otimes e} E \otimes E$,

(2) $E \xrightarrow{\cong} S \otimes E \xrightarrow{e \otimes E} E \otimes E$,

(3) $E \otimes E \xrightarrow{\cong} E \otimes S \otimes E \xrightarrow{E \otimes e \otimes E} E \otimes E \otimes E$,

(4) $E \otimes E \xrightarrow{\mu} E$, *and*

(5) $E \otimes E \xrightarrow{\tau_{E,E}} E \otimes E$

are homomorphisms of monoid objects in $\mathcal{SH}$ *(where here* $E \otimes E$ *and* $E \otimes E \otimes E$ *are considered as monoid objects in* $\mathcal{SH}$ *by Lemma C.4 and Lemma C.5, respectively), so that by* **??**, *under* π_* *they induce morphisms in* $\pi_*(S)$-$\mathbf{GCA}^A$*:*

(1) $\eta_L : \pi_*(E) \to E_*(E)$,

(2) $\eta_R : \pi_*(E) \to E_*(E)$,

(3) $h : E_*(E) \to E_*(E \otimes E)$,

(4) $\epsilon : E_*(E) \to \pi_*(E)$, *and*

(5) $c : E_*(E) \to E_*(E)$.

Proof. It is a general fact that the unit and multiplication maps $e : S \to E$ and $\mu : E \otimes E \to E$ for a monoid are monoid homomorphisms (Lemma C.6), so that furthermore the maps $E \otimes e$, and $e \otimes E$ from E to $E \otimes E$ are monoid homomorphisms, by Lemma C.7. Similarly, $E \otimes e \otimes E : E \otimes E \to E \otimes E \otimes E$ is a monoid homomorphism. Thus, it remains to show that $\tau_{E,E} : E \otimes E \to E \otimes E$ is a monoid homomorphism. First, consider the following diagram:

$$\begin{tikzcd}
E_1 \otimes E_2 \otimes E_3 \otimes E_4 \arrow[r, "\tau\otimes\tau"] \arrow[d, "E\otimes\tau\otimes E"'] & E_2 \otimes E_1 \otimes E_4 \otimes E_3 \arrow[d, "E\otimes\tau\otimes E"] \\
E_1 \otimes E_3 \otimes E_2 \otimes E_4 \arrow[r, "\tau_{E\otimes E, E\otimes E}"] \arrow[d, "\mu\otimes\mu"'] & E_2 \otimes E_4 \otimes E_1 \otimes E_3 \arrow[d, "\mu\otimes\mu"] \\
E_{1,3} \otimes E_{2,4} \arrow[r, "\tau"'] & E_{2,4} \otimes E_{1,3}
\end{tikzcd}$$

(Here we've labelled the E's to make the action of the braidings clearer). The top region commutes by coherence for the symmetries in a symmetric monoidal category, while the bottom region commutes by naturality of τ. Now, consider the following diagram:

$$\begin{tikzcd}
 & & S \arrow[dl, "\cong"'] \arrow[dr, "\cong"] & & \\
 & S \otimes S \arrow[rr, "\tau"] \arrow[dl, "e\otimes e"'] & & S \otimes S \arrow[dr, "e\otimes e"] & \\
E \otimes E \arrow[rrrr, "\tau"] & & & & E \otimes E
\end{tikzcd}$$

The top triangle commutes by coherence for a symmetric monoidal category, while the bottom region commutes by naturality of τ. Thus, we have shown $\tau_{E,E}$ is a homomorphism of monoid objects, as desired. $\square$

Recall a that given a homomorphism of rings $f : R \to R'$, R' canonically becomes an R-bimodule with left action $r \cdot x := f(r)x$ and right action $x \cdot r := xf(r)$. In particular, the ring homomorphisms $\eta_L : \pi_*(E) \to E_*(E)$ and $\eta_R : \pi_*(E) \to E_*(E)$ endow $E_*(E)$ with the structure of a bimodule over $\pi_*(E)$. Naturally, one may ask in what sense these bimodule structures coincide with the canonical one (from Proposition 4.3). The following lemma tells us that the canonical $\pi_*(E)$-bimodule structure on $E_*(E)$ is that with left action induced by η_L and right action induced by η_R:

Lemma 6.2. *Let* (E, μ, e) *be a commutative monoid object in* $\mathcal{SH}$*. Then the left (resp. right)* $\pi_*(E)$*-module structure induced on* $E_*(E)$ *by the ring homomorphism* η_L *(resp.* η_R*) coincides with the canonical left (resp. right)* $\pi_*(E)$*-module structure on* $E_*(E)$ *given in* **??**.

Proof. What's going on here is a bit subtle, so we're going to be really explicit. In Proposition 4.3, it was shown that $E_*(E)$ is a left $\pi_*(E)$-module via the assignment

$$\pi_*(E) \times E_*(E) \to E_*(E)$$

which sends homogeneous elements $r : S^a \to E$ and $x : S^b \to E \otimes E$ to the composition

$$S^{a+b} \xrightarrow{\cong} S^a \otimes S^b \xrightarrow{r \otimes x} E \otimes E \otimes E \xrightarrow{\mu \otimes E} E \otimes E.$$

We'd like to show that this is the same thing as the assignment $\pi_*(E) \times E_*(E) \to E_*(E)$ sending $(r, x) \mapsto \eta_L(r)x$, where $\eta_L(r)x$ denotes the product of $\eta_L(r)$ and x taken in the ring $E_*(E)$. Explicitly, the product structure on $E_*(E) = \pi_*(E \otimes E)$ is that induced by the fact that $E \otimes E$ is a monoid object in $\mathcal{SH}$ (by Lemma C.4), with product

$$E \otimes E \otimes E \otimes E \xrightarrow{E \otimes \tau \otimes E} E \otimes E \otimes E \otimes E \xrightarrow{\mu \otimes \mu} E \otimes E$$

(note the middle two factors are swapped). By linearity of module actions, in order to show the canonical left $\pi_*(E)$-module structure on $E_*(E)$ agrees with that induced by η_L, it suffices to show the module actions agree on homogeneous elements. Now, suppose we have homogeneous elements $r : S^a \to E$ in $\pi_*(E)$ and $x : S^b \to E \otimes E$ in $E_*(E)$, and consider the following diagram, where we've passed to a symmetric strict monoidal category:

$$\begin{tikzcd}
S^{a+b} \arrow[d, "\phi_{a,b}"] \\
S^a \otimes S^b \arrow[d, "r \otimes x"] \\
E_1 \otimes E_2 \otimes E_3 \arrow[rrrr, "\mu \otimes E"] \arrow[dr, equal] \arrow[dd, "E \otimes e \otimes E"'] & & & & E_{1,2} \otimes E_3 \arrow[dd, equal] \\
& E_1 \otimes E_2 \otimes E_3 \arrow[r, equal] & E_1 \otimes E_2 \otimes E_3 \arrow[r, equal] \arrow[d, "E \otimes E \otimes e \otimes E"] & E_1 \otimes E_2 \otimes E_3 \arrow[dr, "\mu \otimes E"] & \\
E_1 \otimes E \otimes E_2 \otimes E_3 \arrow[ur, "E \otimes \mu \otimes E"] \arrow[rr, "E \otimes \tau \otimes E"'] & & E_1 \otimes E_2 \otimes E \otimes E_3 \arrow[ul, "E \otimes \mu \otimes E"] \arrow[ur, "E \otimes E \otimes \mu"'] \arrow[rr, "\mu \otimes \mu"'] & & E_{1,2} \otimes E_3
\end{tikzcd}$$

Here we've numbered the E's to make it clear what's going on. The bottom composition is $\eta_L(r)x$, while the top composition is the canonical left action of r on x given in **??**. The leftmost triangle commutes by unitality of μ. The triangle to the right of that commutes by commutativity of μ. The triangle to the right of that commutes by unitality of μ, as does the next triangle. The remaining triangle on the right commutes by functoriality of $- \otimes -$. Finally, the top region commutes by definition. Thus, we've shown that the left $\pi_*(E)$-module structure induced on $E_*(E)$ by η_L is in fact the canonical one. On the other hand, showing that the right $\pi_*(E)$-module structure induced on $E_*(E)$ by η_R is the canonical one is entirely analagous, and we leave it as an exercise for the reader. $\square$

Recall (Proposition B.22) that the pushout of two morphisms $f : B \to C$ and $g : B \to D$ in R-$\mathbf{GCA}^A$ is obtained by taking the tensor product of B-modules $C \otimes_B D$, where C has right B-module action induced by f, and D has left B-module action induced by g, and giving it an anticommutative product which makes $C \otimes_B D$ a ring. Thus, by the above lemma, we may view the tensor product of bimodules $E_*(E) \otimes_{\pi_*(E)} E_*(E)$ (where $E_*(E)$ is considered with its canonical $\pi_*(E)$-bimodule structure from Proposition 4.3) as not just an A-graded abelian group or a $\pi_*(E)$-bimodule, but as an A-graded anticommutative $\pi_*(S)$-algebra:

Corollary 6.3. *Given a commutative monoid object (E, μ, e) in $\mathcal{SH}$, the domain of the homomorphism*

$$\Phi_{E,E} : E_*(E) \otimes_{\pi_*(E)} E_*(E) \to E_*(E \otimes E)$$

constructed in **??** *is canonically an A-graded $\pi_*(S)$-ring, and sits in the following pushout diagram in $\pi_*(S)$-$\mathbf{GCA}^A$:*

$$\begin{tikzcd}
\pi_*(E) \arrow[r, "\eta_L"] \arrow[d, "\eta_R"'] & E_*(E) \arrow[d, "x \mapsto 1 \otimes x"] \\
E_*(E) \arrow[r, "x \mapsto x \otimes 1"'] & E_*(E) \otimes_{\pi_*(E)} E_*(E)
\end{tikzcd}$$

Furthermore, with respect to this ring structure, $\Phi_{E,E}$ is a homomorphism of rings:

Lemma 6.4. *Let (E, μ, e) be a commutative monoid object in $\mathcal{SH}$. Then the homomorphism*

$$\Phi_{E,E} : E_*(E) \otimes_{\pi_*(E)} E_*(E) \to E_*(E \otimes E)$$

constructed in Proposition 5.2 is a homomorphism of A-graded anticommutative $\pi_*(S)$-algebras.

Proof. Consider the maps

$$f : E \otimes E \xrightarrow{e \otimes E \otimes E} E \otimes E \otimes E$$

and

$$g : E \otimes E \xrightarrow{E \otimes E \otimes e} E \otimes E \otimes E.$$

We know that the maps

$$E \xrightarrow{e \otimes E} E \otimes E \qquad \text{and} \qquad E \xrightarrow{E \otimes e} E \otimes E$$

are monoid homomorphisms by Proposition 6.1, so that f and g are monoid homomorphisms by Lemma C.7. Furthermore, by Lemma C.5, they are monoid homomorphisms between the same monoid objects in $\mathcal{SH}$ (up to associativity). Finally, note that we have the following commutative diagram

$$\begin{tikzcd}
E \arrow[r, "E \otimes e"] \arrow[d, "e \otimes E"'] \arrow[dr, "e \otimes E \otimes e"] & E \otimes E \arrow[d, "e \otimes E \otimes E"] \\
E \otimes E \arrow[r, "E \otimes E \otimes e"'] & E \otimes E \otimes E
\end{tikzcd}$$

where the outer arrows are monoid object homomorphisms, thus, we may apply π_*, which yields the following commutative diagram in $\pi_*(S)$-$\mathbf{GCA}^A$ (**??**):

$$\begin{tikzcd}
\pi_*(E) \arrow[r, "\eta_L"] \arrow[d, "\eta_R"'] & E_*(E) \arrow[d, "\pi_*(f)"] \\
E_*(E) \arrow[r, "\pi_*(g)"'] & E_*(E \otimes E)
\end{tikzcd}$$

Hence by Lemma 6.4 and the universal property of the pushout, there exists some unique morphism $\ell : E_*(E) \otimes_{\pi_*(E)} E_*(E) \to E_*(E \otimes E)$ in $\pi_*(S)$-$\mathbf{GCA}^A$ which makes the following diagram commute:

$$\begin{tikzcd}
\pi_*(E) \arrow[r, "\eta_L"] \arrow[d, "\eta_R"'] & E_*(E) \arrow[d, "x \mapsto 1 \otimes x"'] \arrow[ddr, bend left, "\pi_*(f)"] & \\
E_*(E) \arrow[r, "x \mapsto x \otimes 1"] \arrow[drr, bend right, "\pi_*(g)"'] & E_*(E) \otimes_{\pi_*(E)} E_*(E) \arrow[dr, dashed, "\ell"] & \\
 & & E_*(E \otimes E)
\end{tikzcd}$$

Thus in order to show Φ_E is a morphism in $\pi_*(S)$-$\mathbf{GCA}^A$, it suffices to show that Φ_E and ℓ are the same map, since we know ℓ is a homomorphism of A-graded $\pi_*(S)$-commutative rings. Since Φ_E and ℓ are both abelian group homomorphisms, it further suffices to show they agree on homogeneous pure tensors, which generate $E_*(E) \otimes_{\pi_*(E)} E_*(E)$ as an abelian group. Given homogeneous elements $x : S^a \to E \otimes E$ and $y : S^b \to E \otimes E$ in $E_*(E)$, unravelling how pushouts in $\pi_*(S)$-$\mathbf{GCA}^A$ are defined (Proposition B.22), ℓ sends the pure homogeneous tensor $x \otimes y$ to the element $\pi_*(g)(x) \cdot \pi_*(f)(y)$, where here $\cdot$ denotes the product taken in $E_*(E \otimes E) = \pi_*(E \otimes E \otimes E)$. Now, consider the following diagram:

$$
\begin{tikzcd}
S^{a+b} \arrow[d, "\phi_{a,b}"] & & \\
S^a \otimes S^b \arrow[d, "x \otimes y"] & & \\
E_1 \otimes E_2 \otimes E_3 \otimes E_4 \arrow[rr, "g \otimes f = E \otimes E \otimes e \otimes e \otimes E \otimes E"] \arrow[drr, "E \otimes e \otimes E \otimes e \otimes E \otimes E"] \arrow[ddr, equal] \arrow[ddd, "E \otimes \mu \otimes E"'] & & E_1 \otimes E_2 \otimes E_a \otimes E_b \otimes E_3 \otimes E_4 \arrow[d, "E \otimes \tau_{E \otimes E, E} \otimes E \otimes E"] \\
& & E_1 \otimes E_b \otimes E_2 \otimes E_a \otimes E_3 \otimes E_4 \arrow[d, "\mu \otimes E \otimes \tau \otimes E"] \\
& E_1 \otimes E_2 \otimes E_3 \otimes E_4 \arrow[ur, "E \otimes e \otimes E \otimes e \otimes E \otimes E"] \arrow[r, "E \otimes E \otimes E \otimes e \otimes E"'] \arrow[dl, "E \otimes \mu \otimes E"'] & E_1 \otimes E_2 \otimes E_3 \otimes E_a \otimes E_4 \arrow[d, "E \otimes \mu \otimes \mu"] \\
E_1 \otimes E_{2,3} \otimes E_4 \arrow[rr, equal] & & E_1 \otimes E_{2,3} \otimes E_4
\end{tikzcd}
$$

Here we have labelled the E's to make things clearer. The left outside composition is $\Phi_E(x \otimes y)$, while the right composition is $\pi_*(g)(x) \cdot \pi_*(f)(y)$. The top right triangle commutes by coherence for a symmetric monoidal category. The middle tright triangle commutes by unitality of μ and coherence for a symmetric monoidal category. The bottom trapezoid commutes by unitality of μ. The rest of the diagram commutes by definition. Thus we have $\Phi_E(x \otimes y) = \pi_*(g)(x) \cdot \pi_*(f)(y)$, so that $\Phi_E = \ell$ is not just an isomorphism of left $\pi_*(E)$-modules, but an isomorphism of A-graded anticommutative $\pi_*(S)$-algebras, as desired. □

For the sake of conciseness, we make the following definition:

Definition 6.5. We say that a monoid object (E, μ, e) in $\mathcal{SH}$ is *flat* if the canonical right $\pi_*(E)$-module structure on $E_*(E)$ from Proposition 4.3 is that of a flat module, or equivalently by Lemma 6.2, if the the map $\eta_R : \pi_*(E) \to E_*(E)$ constructed in Proposition 6.1 is a flat ring homomorphism.

Finally, we can package all of this information into an object called the *dual E-Steenrod algebra*:

Definition 6.6. Let (E, μ, e) be a *commutative* monoid object in $\mathcal{SH}$ which is flat (Definition 6.5) and cellular (Definition 3.1). Then the *dual E-Steenrod algebra* is the pair of A-graded abelian groups $(E_*(E), \pi_*(E))$ equipped with the following structure:

1. The A-graded $\pi_*(S)$-commutative ring structure on $\pi_*(E)$ induced from E being a commutative monoid object in $\mathcal{SH}$ (**??**).
2. The A-graded $\pi_*(S)$-commutative ring structure on $E_*(E)$ induced from the fact that $E \otimes E$ is canonically a commutative monoid object in $\mathcal{SH}$ (Lemma C.4), so that also $E_*(E) = \pi_*(E \otimes E)$ is an A-graded $\pi_*(S)$-commutative ring (**??**).

3. The homomorphisms of A-graded $\pi_*(S)$-commutative rings

$$\eta_L : \pi_*(E) \to E_*(E)$$

and

$$\eta_R : \pi_*(E) \to E_*(E)$$

induced under π_* by the monoid object homomorphisms

$$E \xrightarrow{\cong} E \otimes S \xrightarrow{E \otimes e} E \otimes E$$

and

$$E \xrightarrow{\cong} S \otimes E \xrightarrow{e \otimes E} E \otimes E.$$

4. The homomorphism of A-graded $\pi_*(S)$-commutative rings

$$\Psi_E : E_*(E) \to E_*(E) \otimes_{\pi_*(E)} E_*(E)$$

given by the composition

$$E_*(E) \xrightarrow{h} E_*(E \otimes E) \xrightarrow{\Phi_{E,E}^{-1}} E_*(E) \otimes_{\pi_*(E)} E_*(E),$$

where h is a homomorphism of A-graded $\pi_*(S)$-commutative rings induced under π_* by the monoid object homomorphism

$$E \otimes E \xrightarrow{\cong} E \otimes S \otimes E \xrightarrow{E \otimes e \otimes E} E \otimes E \otimes E,$$

and $\Phi_{E,E}$ is morphism constructed in Proposition 5.2, which is proven to be an isomorphism in Proposition 5.6, and furthermore an isomorphism in $\pi_*(S)$-$\mathbf{GCA}^A$ by **??**.

5. The homomorphism of A-graded $\pi_*(S)$-commutative rings

$$\epsilon : E_*(E) \to \pi_*(E)$$

induced under π_* by the monoid object homomorphism

$$E \otimes E \xrightarrow{\mu} E.$$

6. The homomorphism of A-graded $\pi_*(S)$-commutative rings

$$c : E_*(E) \to E_*(E)$$

induced under π_* from the monoid object homomorphism

$$E \otimes E \xrightarrow{\tau} E \otimes E.$$

The curious reader may wonder why we call $(E_*(E), \pi_*(E))$ the *dual* E-Steenrod algebra. The "dual" is there because the E-Steenrod algebra refers instead to the E-self cohomology $E^*(E) \cong [E, E]_{-*}$. Clasically, the Adams spectral sequence was originally constructed in such a way that the E_1 and E_2 pages could be characterized in terms of cohomology and the E-Steenrod algebra, but it turns out that our approach using homology and the dual E-Steenrod algebra is somewhat better behaved, at least when E is flat in the sense of Definition 6.5.

6.1. **The dual E-Steenrod algebra is a Hopf algebroid.** Above, given a flat and cellular commutative monoid object (E, μ, e) in $\mathcal{SH}$, we constructed an algebraic gadget $(E_*(E), \pi_*(E))$ in the category $\pi_*(S)$-$\mathbf{GCA}^A$ of A-graded anticommutative $\pi_*(S)$-algebras called the *dual E-Steenrod algebra*. In this subsection, we will show this object is an example of the general notion of an *A-graded anticommutative Hopf algebroid*:

Proposition 6.7. *Let (E, μ, e) be a commutative monoid object in $\mathcal{SH}$ which is flat (Definition 6.5) and cellular (Definition 3.1). Then the dual E-Steenrod algebra $(E_*(E), \pi_*(E))$ with the structure maps $(\eta_L, \eta_R, \Psi, \epsilon, c)$ from Definition 6.6 is an A-graded anticommutative Hopf algebroid over $\pi_*(S)$ (Definition D.2), i.e., a co-groupoid object in the category $\pi_*(S)$-$\mathbf{GCA}^A$.*

Proof. We need to show all the diagrams in Definition D.2 commute. Since we are dealing with A-graded homomorphisms, when showing these diagrams commute, it always suffices to chase homogeneous elements around. To that end, we fix homogeneous elements $x : S^a \to E$ in $\pi_*(E)$ and $y : S^b \to E \otimes E$ in $E_*(E \otimes E)$ now.

First, we wish to show the outside of the following diagram commutes:

$$\begin{tikzcd}
\pi_*(E) \arrow[rr, "\eta_R"] \arrow[dd, "\eta_R"'] & & E_*(E) \arrow[dl, "\pi_*(E\otimes e\otimes E)"'] \arrow[dd, "\Psi"] \\
& E_*(E\otimes E) & \\
E_*(E) \arrow[rr, "x\mapsto 1\otimes x"'] & & E_*(E)\otimes_{\pi_*(E)} E_*(E) \arrow[ul, "\Phi_{E,E}"]
\end{tikzcd}$$

The right region commutes by how Ψ is defined (**??**), and $\Phi_{E,E}$ is an isomorphism, so it suffices to show the left region commutes. To that end, consider the following diagram:

$$\begin{tikzcd}
S^a \arrow[r, "x"] \arrow[d, equal, "\phi_{0,a}=\lambda_{S^a}^{-1}"'] \arrow[ddddr, "e\otimes e\otimes x"] & E \arrow[r, "e\otimes E"] & E\otimes E \arrow[dddd, "E\otimes e\otimes E"] \\
S\otimes S^a \arrow[d, "e\otimes e\otimes x"'] & & \\
E\otimes E\otimes E \arrow[d, "E\otimes E\otimes e\otimes E"'] \arrow[ddrr, equal] & & \\
E\otimes E\otimes E\otimes E \arrow[rr, "E\otimes\mu\otimes E"'] & & E\otimes E\otimes E
\end{tikzcd}$$

The top composition is $\pi_*(E \otimes e \otimes E)(\eta_R(x))$, while the bottom composition is $\Phi_{E,E}(1 \otimes \eta_R(x))$. The top right region commutes by functoriality of $- \otimes -$. The bottom left triangle commutes by unitality of μ. Finally, the middle triangle commutes by definition.

Now, we wish to show the following diagram commutes

$$\begin{tikzcd}
E_*(E) \arrow[dr, "\epsilon"'] & \pi_*(E) \arrow[l, "\eta_L"'] \arrow[r, "\eta_R"] \arrow[d, equal] & E_*(E) \arrow[dl, "\epsilon"] \\
& \pi_*(E) &
\end{tikzcd}$$

Unravelling how η_L, η_R, and ϵ are defined, this is the diagram obtained by applying π_* to the following diagram:

$$\begin{tikzcd} E \otimes E \arrow[dr, "\mu"'] & E \arrow[l, "E\otimes e"'] \arrow[r, "e\otimes E"] \arrow[d, equal] & E \otimes E \arrow[dl, "\mu"] \\ & E & \end{tikzcd}$$

This commutes by unitality of μ.

Showing that the third diagram in item (1) in Definition D.2 is entirely analagous to how we showed the first diagram commutes.

Now, we'd like to show the following diagram commutes: □

finish proof, add reference to that old Adams book

6.2. Comodules over the dual E-Steenrod algebra.

Proposition 6.8. *Let (E, μ, e) be a flat (Definition 6.5) and cellular (Definition 3.1) commutative monoid object in $\mathcal{SH}$. Then $E_*(-)$ is a functor from $\mathcal{SH}$ to the category $E_*(E)$-***CoMod** *of left A-graded comodules (Definition D.6) over the dual E-Steenrod algebra, which is an A-graded commutative Hopf algebroid over $\pi_*(S)$, by Proposition 6.7.*

*In particular, given an object X in $\mathcal{SH}$, we are viewing $E_*(X)$ with its canonical left $\pi_*(E)$-module structure (**??**), and the action map*

$$\Psi_X : E_*(X) \to E_*(E) \otimes_{\pi_*(E)} E_*(X)$$

is given by the composition

$$\Psi_X : E_*(X) \xrightarrow{E_*(e\otimes X)} E_*(E \otimes X) \xrightarrow{\Phi^{-1}_{E,X}} E_*(E) \otimes_{\pi_*(E)} E_*(X).$$

Proof. □

TODO

Proposition 6.9. *Let (E, μ, e) be a flat (Definition 6.5) and cellular (Definition 3.1) commutative monoid object in $\mathcal{SH}$. Then given an object X in $\mathcal{SH}$, the map*

$$\Phi_{E,X} : E_*(E) \otimes_{\pi_*(E)} E_*(X) \to E_*(E \otimes X)$$

*constructed in **??** is a homomorphism of A-graded left Γ-comodules, where here by Proposition D.8 we are viewing $E_*(E) \otimes_{\pi_*(E)} E_*(X)$ as the co-free $E_*(E)$-comodule on $E_*(X)$ with its canonical A-graded left $\pi_*(E)$-module structure (from **??**).*

Proof. □

TODO

7. The Adams spectral sequence

7.1. Construction of the spectral sequence. In the sections that follow, let (E, μ, e) be a monoid object and X and Y be objects in $\mathcal{SH}$.

Definition 7.1 (The Adams Spectral Sequence)**.** Let $\overline{E}$ be the fiber of the unit map $e : S \to E$ (Proposition A.5). Let $Y_0 := Y$ and $W_0 := E \otimes Y$. Then for $s > 0$, define

$$Y_s := \overline{E}^s \otimes Y, \qquad W_s := E \otimes Y_s = E \otimes \overline{E}^s \otimes Y,$$

where $\overline{E}^s$ denotes the s-fold tensor product $\overline{E} \otimes \cdots \otimes \overline{E}$. Then we get fiber sequences

$$Y_{s+1} \xrightarrow{i_s} Y_s \xrightarrow{j_s} W_s \xrightarrow{k_s} \Sigma Y_{s+1}$$

obtained by applying $-\otimes Y_s$ to the fiber sequence

$$\overline{E} \to S \xrightarrow{e} E \to \Sigma\overline{E}.$$

We can splice these sequences together to get the *(canonical) Adams filtration of* Y:

$$\begin{tikzcd}
\cdots \arrow[r] & Y_3 \arrow[r, "i_2"] \arrow[d, "j_3"] & Y_2 \arrow[r, "i_1"] \arrow[d, "j_2"] & Y_1 \arrow[r, "i_0"] \arrow[d, "j_1"] & Y_0 = Y \arrow[d, "j_0"] \\
& W_3 & W_2 \arrow[ul, dashed, "k_2"] & W_1 \arrow[ul, dashed, "k_1"] & W_0 \arrow[ul, dashed, "k_0"]
\end{tikzcd}$$

where here each k_s is of degree $-\mathbf{1}$ (in particular, the above diagram does not commute in any sense), and each i_s and j_s have degree 0. We can extend this diagram to the right by setting $Y_s = Y$, $W_s = 0$, and $i_s = \mathrm{id}_Y$ for $s < 0$. Then we may apply the functor $[X, -]_*$, and by Proposition 2.6, we obtain the following A-graded unrolled exact couple (Definition E.2):

$$\begin{tikzcd}
\cdots \arrow[r] & {[X, Y_{s+2}]_*} \arrow[r, "i_{s+1}"] \arrow[d, "j_{s+2}"] & {[X, Y_{s+1}]_*} \arrow[r, "i_s"] \arrow[d, "j_{s+1}"] & {[X, Y_s]_*} \arrow[r, "i_{s-1}"] \arrow[d, "j_s"] & {[X, Y_{s-1}]_*} \arrow[r] \arrow[d, "j_{s-1}"] & \cdots \\
& {[X, W_{s+2}]_*} & {[X, W_{s+1}]_*} \arrow[ul, "\partial_{s+1}"] & {[X, W_s]_*} \arrow[ul, "\partial_s"] & {[X, W_{s-1}]_*} \arrow[ul, "\partial_{s-1}"] &
\end{tikzcd}$$

where here we are being abusive and writing $i_s : [X, Y_{s+1}]_* \to [X, Y_s]_*$ and $j_s : [X, Y_s]_* \to [X, W_s]_*$ to denote the pushforward maps induced by $i_s : Y_{s+1} \to Y_s$ and $j_s : Y_s \to W_s$, respectively. Each i_s, j_s, and ∂_s are A-graded homomorphisms of degrees 0, 0, and $-\mathbf{1}$, respectively.

By Proposition E.3, we may associate a $\mathbb{Z} \times A$-graded spectral sequence $r \mapsto (E_r^{*,*}(X, Y), d_r)$ to the above A-graded unrolled exact couple, where d_r has $\mathbb{Z} \times A$-degree $(r, -\mathbf{1})$. We call this spectral sequence the *E-Adams spectral sequence for the computation of* $[X, Y]_*$.

For those who would rather not lose themselves in the appendix, we give a brief unravelling of how Proposition E.3 applies to the present situation. Given some $s \in \mathbb{Z}$ and some $r \geq 1$, we may define the following A-graded subgroups of $[X, W_s]$:

$$Z_r^s := \partial_s^{-1}(\mathrm{im}[i^{(r-1)} : [X, Y_{s+r}]_* \to [X, Y_{s+1}]_*])$$

and

$$B_r^s := j_s(\ker[i^{(r-1)} : [X, Y_s]_* \to [X, Y_{s-r+1}]_*]),$$

where we adopt the convention that $i^{(0)}$ is simply the identity. This yields an infinite sequence of inclusions

$$0 = B_1^s \subseteq B_2^s \subseteq B_3^s \subseteq \cdots \subseteq \mathrm{im}\, j_s = \ker \partial_s \subseteq \cdots \subseteq Z_3^s \subseteq Z_2^s \subseteq Z_1^s = [X, W_s]_*.$$

Then for $r \geq 1$, we define E_r^s to be the A-graded quotient group

$$E_r^s := Z_r^s / B_r^s.$$

Thus taking the direct sum of all the E_r^s's yields the r^{th} page of the spectral sequence

$$E_r := \bigoplus_{s \in \mathbb{Z}} E_r^s,$$

which is a $\mathbb{Z} \times A$-graded abelian group.

The differential $d_r : E_r \to E_r$ is a map of $\mathbb{Z} \times A$-degree $(r, \mathbf{1})$, and is constructed as follows: an element of $E_r^s = Z_r^s / B_r^s$ is a coset represented by some $x \in Z_r^s$, so that $\partial_s(x) = i^{(r-1)}(y)$ for some $y \in [X, Y_{s+r}]_*$. Then we define $d_r([x])$ to be the coset $[j_{s+r}(y)]$ in Z_r^{s+r}/B_r^{s+r}.

In the case $r = 1$, since $B_1^s = 0$ and $Z_1^s = [X, W_s]_*$, we have that $E_1^s = [X, W_s]_*$, and given some $x \in E_1^s = [X, W_s]_*$, the differential d_1 is given by $d_1(x) = j_{s+1}(\partial_s(x))$, so that $d_1 = j \circ \partial$.

In **??**, it is shown in explicit detail that all of these definitions make sense and are well-defined. In particular, it is shown that the differentials are well-defined A-graded homomorphisms, that $d_r \circ d_r = 0$, and that

$$\ker d_r^s / \operatorname{im} d_r^s = \frac{Z_{r+1}^s / B_r^s}{B_{r+1}^s / B_r^s} \cong Z_{r+1}^s / B_{r+1}^s = E_{r+1}^s.$$

7.2. The E_1 page.

8. The classical Adams spectral sequence

intro

9. The motivic Adams spectral sequence

intro

Appendix A. Triangulated categories

A.1. Triangulated categories and their basic properties.

Definition A.1. A *triangulated category* $(\mathcal{C}, \Sigma, \mathcal{D})$ is the data of:

(1) An additive category $\mathcal{C}$.

(2) An additive auto-equivalence $\Sigma : \mathcal{C} \to \mathcal{C}$ called the *shift functor*.

(3) A collection $\mathcal{D}$ of *distinguished* triangles in $\mathcal{C}$, where a *triangle* is a sequence of arrows of the form

$$X \to Y \to Z \to \Sigma X.$$

Distinguished triangles are also sometimes called *cofiber sequences* or *fiber sequences*.

These data must satisfy the following axioms:

TR0 Given a commutative diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d, "\cong"'] & Y \arrow[r] \arrow[d, "\cong"'] & Z \arrow[r] \arrow[d, "\cong"'] & \Sigma X \arrow[d, "\cong"'] \\ X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & \Sigma X' \end{tikzcd}$$

where the vertical arrows are isomorphisms, if the top row is distinguished then so is the bottom.

TR1 For any object X in $\mathcal{C}$, the diagram

$$X \xrightarrow{\mathrm{id}_X} X \to 0 \to \Sigma X$$

is a distinguished triangle.

TR2 For all $f : X \to Y$ there exists an object C_f (also sometimes denoted Y/X) called the *cofiber of f* and a distinguished triangle

$$X \xrightarrow{f} Y \to C_f \to \Sigma X.$$

TR3 Given a solid diagram

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "f"'] & Y \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d, dashed] & \Sigma X \arrow[d, "\Sigma f"] \\
X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & \Sigma X'
\end{tikzcd}$$

such that the leftmost square commmutes and both rows are distinguished, there exists a dashed arrow $Z \to Z'$ which makes the remaining two squares commute.

TR4 A triangle

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$$

is distinguished if and only if

$$Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X \xrightarrow{-\Sigma f} \Sigma Y$$

is distinguished.

TR5 (Octahedral axiom) Given three distinguished triangles

$$X \xrightarrow{f} Y \xrightarrow{h} Y/X \to \Sigma X$$

$$Y \xrightarrow{g} Z \xrightarrow{k} Z/Y \to \Sigma Y$$

$$X \xrightarrow{g \circ f} Z \xrightarrow{l} Z/X \to \Sigma X$$

there exists a distinguished triangle

$$Y/X \xrightarrow{u} Z/X \xrightarrow{v} Z/Y \xrightarrow{w} \Sigma(Y/X)$$

such that the following diagram commutes

$$\begin{tikzcd}
X \arrow[rr, "g \circ f"] \arrow[dr, "f"'] & & Z \arrow[rr, "k"] \arrow[dr, "l"'] & & Z/Y \arrow[rr, "w"] \arrow[dr] & & \Sigma(Y/X) \\
& Y \arrow[ur, "g"'] \arrow[dr, "h"'] & & Z/X \arrow[ur, "v"'] \arrow[dr] & & \Sigma Y \arrow[ur, "\Sigma h"'] & \\
& & Y/X \arrow[ur, "u"'] \arrow[rr] & & \Sigma X \arrow[ur, "\Sigma f"'] & &
\end{tikzcd}$$

It turns out that the above definition is actually redundant; TR3 and TR4 follow from the remaining axioms (see Lemmas 2.2 and 2.4 in [5]). From now on, we fix a triangulated category $(\mathcal{C}, \Sigma, \mathcal{D})$. To start, recall the following definition:

Definition A.2. A sequence

$$X_1 \to X_2 \to \cdots \to X_n$$

of arrows in $\mathcal{C}$ is *exact* if, for any object A in $\mathcal{C}$, the induced sequences

$$[A, X_1] \to [A, X_2] \to \cdots \to [A, X_{n-1}] \to [A, X_n]$$

and

$$[X_n, A] \to [X_{n-1}, A] \to \cdots \to [X_2, A] \to [X_1, A]$$

are exact sequences of abelian groups.

Proposition A.3. *Every distinguished triangle is an exact sequence (in the sense of Definition A.2).*

Proof. Suppose we have some distinguished triangle

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X.$$

Then first we would like to show that given any object A in $\mathcal{C}$, the sequence

$$[A, X] \xrightarrow{f_*} [A, Y] \xrightarrow{g_*} [A, Z] \xrightarrow{h_*} [A, \Sigma X]$$

is exact. First we show exactness at $[A, Y]$. To see $\operatorname{im} f_* \subseteq \ker g_*$, note it suffices to show that $g \circ f = 0$. Indeed, consider the commuting diagram

$$\begin{tikzcd} X \arrow[r, equal] \arrow[d, equal] & X \arrow[r] \arrow[d, "f"] & 0 \arrow[r] & \Sigma X \\ X \arrow[r, "f"] & Y \arrow[r, "g"] & Z \arrow[r, "h"] & \Sigma X \end{tikzcd}$$

The top row is distinguished by axiom TR1. Thus by TR3, the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, equal] \arrow[d, equal] & X \arrow[r] \arrow[d, "f"] & 0 \arrow[r] \arrow[d] & \Sigma X \arrow[d, equal] \\ X \arrow[r, "f"] & Y \arrow[r, "g"] & Z \arrow[r, "h"] & \Sigma X \end{tikzcd}$$

In particular, commutativity of the second square tells us that $g \circ f = 0$, as desired. Conversely, we'd like to show that $\ker g_* \subseteq \operatorname{im} f_*$. Let $\psi : A \to Y$ be in the kernel of g_*, so that $g \circ \psi = 0$. Consider the following commutative diagram:

$$\begin{tikzcd} A \arrow[r] \arrow[d, "\psi"'] & 0 \arrow[r] \arrow[d] & \Sigma A \arrow[r, "-\Sigma \mathrm{id}_A"] & \Sigma A \\ Y \arrow[r, "g"] & Z \arrow[r, "h"] & \Sigma X \arrow[r, "-\Sigma f"] & \Sigma Y \end{tikzcd}$$

The top row is distinguished by axioms TR1 and TR4. The bottom row is distinguished by axiom TR4. Thus by axiom TR3 there exists a map $\widetilde{\phi} : \Sigma A \to \Sigma X$ such that the following diagram commutes:

$$\begin{tikzcd} A \arrow[r] \arrow[d, "\psi"'] & 0 \arrow[r] \arrow[d] & \Sigma A \arrow[r, "-\Sigma \mathrm{id}_A"] \arrow[d, "\widetilde{\phi}"'] & \Sigma A \arrow[d, "\Sigma\psi"'] \\ Y \arrow[r, "g"] & Z \arrow[r, "h"] & \Sigma X \arrow[r, "-\Sigma f"] & \Sigma Y \end{tikzcd}$$

Now, since Σ is an equivalence, it is a full functor, so that in particular there exists some $\phi : A \to X$ such that $\widetilde{\phi} = \Sigma\phi$. Then by faithfullness, we may pull back the right square to get a commuting diagram

$$\begin{tikzcd} A \arrow[r, "-\mathrm{id}_A"] \arrow[d, "\phi"'] & A \arrow[d, "\psi"] \\ X \arrow[r, "-f"'] & Y \end{tikzcd}$$

Hence,

$$f_*(\phi) = f \circ \phi \overset{(*)}{=} -((-f) \circ \phi) = -(\psi \circ (-\mathrm{id}_A)) \overset{(*)}{=} \psi \circ \mathrm{id}_A = \psi,$$

where the equalities marked $(*)$ follow by bilinearity of composition in an additive category. Thus $\psi \in \operatorname{im} f_*$, as desired, meaning $\ker g_* \subseteq \operatorname{im} f_*$.

Now, we have shown that

$$[A, X] \xrightarrow{f_*} [A, Y] \xrightarrow{g_*} [A, Z] \xrightarrow{h_*} [A, \Sigma X]$$

is exact at $[A, Y]$. It remains to show exactness at $[A, Z]$. Yet this follows by the exact same argument given above applied to the sequence obtained from the shifted triangle (TR4)

$$Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X \xrightarrow{-\Sigma f} \Sigma Y$$

On the other hand, we would like to show that

$$[\Sigma X, A] \xrightarrow{h^*} [Z, A] \xrightarrow{g^*} [Y, A] \xrightarrow{f^*} [X, A]$$

is exact. As above, since we can shift the triangle, it suffices to show exactness at $[Z, A]$. First, since we have shown $g \circ f = 0$, we have $f^* \circ g^* = (g \circ f)^* = 0$, so that $\operatorname{im} g^* \subseteq \ker f^*$, as desired. Conversely, in order to see $\ker f^* \subseteq \operatorname{im} g^*$, suppose $\psi : Y \to A$ is in the kernel of f^*, so that $\psi \circ f = 0$. Consider the following commuting diagram:

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d] & Y \arrow[r, "g"] \arrow[d, "\psi"] & Z \arrow[r, "h"] & \Sigma X \\
0 \arrow[r] & A \arrow[r, equal] & A \arrow[r] & 0
\end{tikzcd}$$

The top row is a distinguished triangle by assumption, and the bottom row is distinguished by axioms TR1 and TR4 for a triangulated category, along with the fact that $\Sigma 0 = 0$ since Σ is additive. Thus by axiom TR3 there exists a map $\phi : Z \to A$ such that $\phi \circ g = \psi$, i.e., $g^*(\phi) = \psi$, so that $\phi \in \operatorname{im} g^*$ as desired. $\square$

Lemma A.4. *Suppose we have a commutative diagram*

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "j"] & Y \arrow[r, "g"] \arrow[d, "k"] & Z \arrow[r, "h"] \arrow[d, "\ell"] & \Sigma X \arrow[d, "\Sigma j"] \\
X' \arrow[r, "f'"] & Y' \arrow[r, "g'"] & Z' \arrow[r, "h'"] & \Sigma X'
\end{tikzcd}$$

with both rows distinguished. Then if any two of the maps j, k, and ℓ are isomorphisms, then so is the third.

Proof. Suppose we are given any object W in $\mathcal{C}$, and consider the commutative diagram

$$\begin{tikzcd}
{[W, X]} \arrow[r, "f_*"] \arrow[d, "j_*"] & {[W, Y]} \arrow[r, "g_*"] \arrow[d, "k_*"] & {[W, Z]} \arrow[r, "k_*"] \arrow[d, "\ell_*"] & {[W, \Sigma X]} \arrow[r, "-\Sigma f_*"] \arrow[d, "\Sigma j_*"] & {[W, \Sigma Y]} \arrow[r, "-\Sigma g_*"] \arrow[d, "\Sigma k_*"] & {[W, \Sigma Z]} \arrow[r, "-\Sigma h_*"] \arrow[d, "\Sigma \ell_*"] & {[W, \Sigma^2 X]} \arrow[d, "\Sigma^2 j_*"] \\
{[W, X']} \arrow[r, "f'_*"] & {[W, Y']} \arrow[r, "g'_*"] & {[W, Z']} \arrow[r, "h'_*"] & {[W, \Sigma X']} \arrow[r, "-\Sigma f'_*"] & {[W, \Sigma Y']} \arrow[r, "-\Sigma g'_*"] & {[W, \Sigma Z']} \arrow[r, "-\Sigma h'_*"] & {[W, \Sigma^2 X']}
\end{tikzcd}$$

The rows are exact by Proposition A.3 and repeated applications of axiom TR4. It follows by the five lemma and faithfulness of Σ that if j and k are isomorphisms, then ℓ_* is an isomorphism. Similarly, if k and ℓ are isomorphisms then Σj_* is an isomorphism. Finally, if ℓ and j are isomorphisms, then Σk_* is an isomorphism. The desired result follows by faithfullness of Σ and the Yoneda embedding. $\square$

Proposition A.5. *Given an arrow $f : X \to Y$ in $\mathcal{C}$, there exists an object F_f called the* fiber *of f, and a distinguished triangle*

$$F_f \to X \xrightarrow{f} Y \to \Sigma F_f (\cong C_f).$$

Proof. Since Σ is an equivalence, there exists some functor $\Omega : \mathcal{C} \to \mathcal{C}$ and natural isomorphisms $\varepsilon : \Omega\Sigma \Rightarrow \mathrm{Id}_{\mathcal{C}}$ and $\eta : \mathrm{Id}_{\mathcal{C}} \Rightarrow \Sigma\Omega$. By axiom TR2, we have a distinguished triangle

$$X \xrightarrow{f} Y \xrightarrow{g} C_f \xrightarrow{h} \Sigma X.$$

Now, consider the commutative diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, equal] & Y \arrow[r, "g"] \arrow[d, equal] & C_f \arrow[r, "h"] \arrow[d, "\eta_{C_f}"] & \Sigma X \arrow[d, equal] \\ X \arrow[r, "f"] & Y \arrow[r, "\widetilde{g}"] & \Sigma\Omega C_f \arrow[r, "\widetilde{h}"] & \Sigma X \end{tikzcd}$$

where $\widetilde{g} = \eta_{C_f} \circ g$, and $\widetilde{h} = h \circ \eta_{C_f}^{-1}$. Since each vertical map is an isomorphism and the top row is distinguished, the bottom row is also distinguished by axiom TR0. Now, since Σ is an equivalence of categories, it is faithful, so that in particular there exists some map $k : \Omega C_f \to X$ such that $\Sigma k = -\widetilde{h} \implies -\Sigma k = \widetilde{h}$. Thus, we have a distinguished triangle of the form

$$X \xrightarrow{f} Y \xrightarrow{\widetilde{g}} \Sigma\Omega C_f \xrightarrow{-\Sigma k} \Sigma X.$$

Finally, by axiom TR4, we get a distinguished triangle

$$\Omega C_f \xrightarrow{k} X \xrightarrow{f} Y \xrightarrow{\widetilde{g}} \Sigma\Omega C_f,$$

so we may define the fiber of f to be ΩC_f. □

A.2. **Homotopy (co)limits in a triangulated category.**

Definition A.6 ([6, Definition 1.6.4])**.** Assume that $\mathcal{C}$ has countable coproducts. Let

$$X_0 \xrightarrow{j_1} X_1 \xrightarrow{j_2} X_2 \xrightarrow{j_3} X_3 \to \cdots$$

be a sequence of objects and morphisms in $\mathcal{C}$. The *homotopy colimit* of the sequence, denoted $\operatorname{holim} X_i$, is by definition given, up to non-canonical isomorphism, as the cofiber of the map

$$\coprod_{i=0}^{\infty} X_i \xrightarrow{1-\text{shift}} \coprod_{i=0}^{\infty} X_i,$$

where the shift map $\coprod_{i=0}^{\infty} X_i \xrightarrow{\text{shift}} \coprod_{i=0}^{\infty} X_i$ is understood to be the direct sum of $j_{i+1} : X_i \to X_{i+1}$. In other words, the map $1 - \text{shift}$ is the infinite matrix

$$\begin{bmatrix} \mathrm{id}_{X_0} & 0 & 0 & 0 & \cdots \\ -j_1 & \mathrm{id}_{X_1} & 0 & 0 & \cdots \\ 0 & -j_2 & \mathrm{id}_{X_2} & 0 & \cdots \\ 0 & 0 & -j_3 & \mathrm{id}_{X_3} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Proposition A.7 ([6, Proposition 1.6.8])**.** *Suppose $\mathcal{C}$ has countable coproducts, and suppose $e : X \to X$ is an idempotent in $\mathcal{C}$, so that $e \circ e = e$. Then e splits in $\mathcal{C}$, i.e., e factors as*

$$X \xrightarrow{r} Y \xrightarrow{\iota} X$$

with $r \circ \iota = \mathrm{id}_Y$ and $\iota \circ r = e$. In particular, we may take Y to be the colimit of

$$X \xrightarrow{e} X \xrightarrow{e} X \xrightarrow{e} X \xrightarrow{e} \cdots .$$

Proof. See [6, Proposition 1.6.8]. □ write proof down?

A.3. **Adjointly triangulated categories.** For our purposes, we will always be dealing with triangulated categories with a bit of extra structure, in the following sense:

Definition A.8. An *adjointly triangulated category* $(\mathcal{C}, \Omega, \Sigma, \eta, \varepsilon, \mathcal{D})$ is the data of a triangulated category $(\mathcal{C}, \Sigma, \mathcal{D})$ along with an inverse shift functor $\Omega : \mathcal{C} \to \mathcal{C}$ and natural isomorphisms $\eta : \mathrm{Id}_{\mathcal{C}} \Rightarrow \Sigma\Omega$ and $\varepsilon : \Omega\Sigma \Rightarrow \mathrm{Id}_{\mathcal{C}}$ such that $(\Omega, \Sigma, \eta, \varepsilon)$ forms an adjoint equivalence of $\mathcal{C}$. In other words, η and ε are natural isomorphisms which also are the unit and counit of an adjunction $\Omega \dashv \Sigma$, so they satisfy either of the following "zig-zag identities":

$$\begin{tikzcd} \Omega \arrow[r, "\Omega\eta"] \arrow[dr, equal] & \Omega\Sigma\Omega \arrow[d, "\varepsilon\Omega"] \\ & \Omega \end{tikzcd} \qquad \begin{tikzcd} \Sigma\Omega\Sigma \arrow[d, "\Sigma\varepsilon"'] & \Sigma \arrow[l, "\eta\Sigma"'] \arrow[dl, equal] \\ \Sigma & \end{tikzcd}$$

(Satisfying one implies the other is automatically satisfied, see [7, Lemma 3.2]).

From now on, we will assume that $\mathcal{C}$ is an *adjointly* triangulated category with inverse shift Ω, unit $\eta : \mathrm{Id}_{\mathcal{C}} \Rightarrow \Sigma\Omega$, and counit $\varepsilon : \Omega\Sigma \Rightarrow \mathrm{Id}_{\mathcal{C}}$.

Lemma A.9. *Given a triangle*

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X,$$

it can be shifted to the left to obtain a distinguished triangle

$$\Omega Z \xrightarrow{-\widetilde{h}} X \xrightarrow{f} Y \xrightarrow{\widetilde{\Omega g}} \Sigma\Omega Z,$$

where $\widetilde{h} : \Omega Z \to X$ is the adjoint of $h : Z \to \Sigma X$ and $\widetilde{\Omega g} : Y \to \Sigma\Omega Z$ is the adjoint of $\Omega g : \Omega Y \to \Omega Z$.

Proof. Note that unravelling definitions, $\widetilde{h}$ and $\widetilde{g}$ are the compositions

$$\widetilde{h} : \Omega Z \xrightarrow{\Omega h} \Omega\Sigma X \xrightarrow{\varepsilon_X} X \qquad \text{and} \qquad \widetilde{\Omega g} : Y \xrightarrow{\eta_Y} \Sigma\Omega Y \xrightarrow{\Sigma\Omega g} \Sigma\Omega Z.$$

Now consider the following diagram:

(6)
$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, equal] & Y \arrow[r, "g"] \arrow[d, equal] & Z \arrow[r, "h"] \arrow[d, "\eta_Z"'] & \Sigma X \arrow[d, equal] \\ X \arrow[r, "f"] & Y \arrow[r, "\widetilde{\Omega g}"] & \Sigma\Omega Z \arrow[r, "\Sigma\widetilde{h}"] & \Sigma X \end{tikzcd}$$

The left square commutes by definition. To see that the middle square commutes, expanding definitions, note it is given by the following diagram:

$$\begin{tikzcd} Y \arrow[rr, "g"] \arrow[d, equal] & & Z \arrow[d, "\eta_Y"] \\ Y \arrow[r, "\eta_Y"] & \Sigma\Omega Y \arrow[r, "\Sigma\Omega g"] & \Sigma\Omega Z \end{tikzcd}$$

and this commutes by naturality of η. To see that the right square commutes, consider the following diagram:

$$\begin{tikzcd} Z \arrow[rr, "h"] \arrow[d, "\eta_Z"'] & & \Sigma X \arrow[d, equal] \arrow[dl, "\eta_{\Sigma X}"'] \\ \Sigma\Omega Z \arrow[r, "\Sigma\Omega h"'] & \Sigma\Omega\Sigma X \arrow[r, "\Sigma\varepsilon_X"'] & \Sigma X \end{tikzcd}$$

By functoriality of Σ, the bottom composition is $\Sigma\widetilde{h}$. The left region commutes by naturality of η. Commutativity of the right region is precisely one of the the zig-zag identities. Hence, since diagram (6) commutes, the vertical arrows are isomorphisms, and the top row is distinguished, we have that the bottom row is distinguished as well by axiom TR0. Then by axiom TR4, since $(f, \widetilde{\Omega g}, \Sigma\widetilde{h})$ is distinguished, so is the triangle

$$\Omega Z \xrightarrow{-\widetilde{h}} X \xrightarrow{f} Y \xrightarrow{\widetilde{\Omega g}} \Sigma\Omega Z. \qquad \square$$

Lemma A.10. *Given a distinguished triangle*

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X,$$

for any $n > 0$, the triangle

$$\Omega^n X \xrightarrow{(-1)^n\Omega^n f} \Omega^n Y \xrightarrow{(-1)^n\Omega^n g} \Omega^n Z \xrightarrow{(-1)^n\Omega^n h} \Omega^n\Sigma X \cong \Sigma\Omega^n X,$$

is distinguished, where the final isomorphism is given by the composition

$$\Omega^n\Sigma X = \Omega^{n-1}\Omega\Sigma X \xrightarrow{\Omega^{n-1}\varepsilon_X} \Omega^{n-1}X \xrightarrow{\eta_{\Omega^{n-1}X}} \Sigma\Omega\Omega^{n-1}X = \Sigma\Omega^n X.$$

Proof. We give a proof by induction. First we show the case $n = 1$. Note by Lemma A.9, we have that given a distinguished triangle

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X,$$

we can shift it to the left to obtain a distinguished triangle

$$\Omega Z \xrightarrow{-\widetilde{h}} X \xrightarrow{f} Y \xrightarrow{\widetilde{\Omega g}} \Sigma\Omega Z,$$

where $\widetilde{h}$ is the adjoint of $h : Z \to \Sigma X$ and $\widetilde{\Omega g}$ is the adjoint of $\Omega g : \Omega Y \to \Omega Z$. If we apply this shifting operation again, we get the distinguished triangle

$$\Omega Y \xrightarrow{-\widetilde{\widetilde{\Omega g}}} \Omega Z \xrightarrow{-\widetilde{h}} X \xrightarrow{\widetilde{\Omega f}} \Sigma\Omega Y,$$

where unravelling definitions, $\widetilde{\Omega f}$ is the right adjoint of $\Omega f : \Omega X \to \Omega Y$ and $\widetilde{\widetilde{\Omega g}}$ is the right adjoint of $\widetilde{\Omega g}$, which itself is the left adjoint of Ωg, so $\widetilde{\widetilde{\Omega g}} = \Omega g$. Hence we have a distinguished triangle

$$\Omega Y \xrightarrow{-\Omega g} \Omega Z \xrightarrow{-\widetilde{h}} X \xrightarrow{\widetilde{\Omega f}} \Sigma\Omega Y.$$

We may again shift this triangle again and the above arguments yield the distinguished triangle

$$\Omega X \xrightarrow{-\Omega f} \Omega Y \xrightarrow{-\Omega g} \Omega Z \xrightarrow{\widetilde{\Omega(-\widetilde{h})}} \Sigma\Omega X,$$

where $\widetilde{\Omega(-\widetilde{h})}$ is the right adjoint of $\Omega(-\widetilde{h}) = -\Omega\widetilde{h} : \Omega\Omega Z \to \Omega X$. Explicitly unravelling definitions, $\widetilde{\Omega(-\widetilde{h})} = -\widetilde{\Omega\widetilde{h}}$ is the composition

$$\begin{aligned}
[\Omega Z \xrightarrow{\eta_{\Omega Z}} \Sigma\Omega\Omega Z \xrightarrow{\Sigma(-\Omega\widetilde{h})} \Sigma\Omega X] &= -[\Omega Z \xrightarrow{\eta_{\Omega Z}} \Sigma\Omega\Omega Z \xrightarrow{\Sigma\Omega\widetilde{h}} \Sigma\Omega X] \\
&= -[\Omega Z \xrightarrow{\eta_{\Omega Z}} \Sigma\Omega\Omega Z \xrightarrow{\Sigma\Omega\Omega h} \Sigma\Omega\Omega\Sigma X \xrightarrow{\Sigma\Omega\varepsilon_X} \Sigma\Omega X] \\
&= -[\Omega Z \xrightarrow{\Omega h} \Omega\Sigma X \xrightarrow{\varepsilon_X} X \xrightarrow{\eta_X} \Sigma\Omega X],
\end{aligned}$$

where the first equality follows by additivity of Σ and additivity of composition, the second follows by further unravelling how $\widetilde{h}$ is defined, and the third follows by naturality of η, which tells us the following diagram commutes:

$$\begin{tikzcd}
\Omega Z \arrow[r, "\Omega h"] \arrow[d, "\eta_{\Omega Z}"] & \Omega\Sigma X \arrow[r, "\varepsilon_X"] \arrow[d, "\eta_{\Omega\Sigma X}"] & X \arrow[d, "\eta_X"] \\
\Sigma\Omega\Omega Z \arrow[r, "\Sigma\Omega\Omega h"'] & \Sigma\Omega\Omega\Sigma X \arrow[r, "\Sigma\Omega\varepsilon_X"'] & \Sigma\Omega X
\end{tikzcd}$$

Thus indeed we have a distinguished triangle

$$\Omega X \xrightarrow{-\Omega f} \Omega Y \xrightarrow{-\Omega g} \Omega Z \xrightarrow{-\Omega h} \Omega\Sigma X \cong \Sigma\Omega X,$$

where the last isomorphism is $\eta_X \circ \varepsilon_X$, as desired.

Now, we show the inductive step. Suppose we know that given a distinguished triangle

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X,$$

that for some $n > 0$ the triangle

$$\Omega^n X \xrightarrow{(-1)^n\Omega^n f} \Omega^n Y \xrightarrow{(-1)^n\Omega^n g} \Omega^n Z \xrightarrow{(-1)^n h^n} \Sigma\Omega^n X,$$

is distinguished, where $h^n : \Omega^n Z \to \Sigma\Omega^n X$ is the composition

$$\Omega^n Z \xrightarrow{\Omega^n h} \Omega^n\Sigma X \xrightarrow{\Omega^{n-1}\varepsilon_X} \Omega^{n-1} X \xrightarrow{\eta_{\Omega^{n-1}X}} \Sigma\Omega^n X.$$

Then by applying the $n = 1$ case to this triangle, we get that the following triangle is distinguished

$$\Omega^{n+1} X \xrightarrow{-\Omega((-1)^n\Omega^n f)} \Omega^{n+1} Y \xrightarrow{-\Omega((-1)^n\Omega^n g)} \Omega^{n+1} Z \xrightarrow{-\Omega((-1)^n h^n)} \Omega\Sigma\Omega^n X \cong \Sigma\Omega^{n+1} X,$$

where the final isomorphism is the composition

$$\Omega\Sigma\Omega^n X \xrightarrow{\varepsilon_{\Omega^n X}} \Omega^n X \xrightarrow{\eta_{\Omega^n X}} \Sigma\Omega\Omega^n X = \Sigma\Omega^{n+1} X.$$

We claim that this is precisely the distinguished triangle given in the statement of the lemma for $n+1$. First of all, note that $-\Omega((-1)^n\Omega^n f) = (-1)^{n+1}\Omega^{n+1} f$, $-\Omega((-1)^n\Omega^n g) = (-1)^{n+1}\Omega^{n+1} g$, and $-\Omega((-1)^n h^n) = (-1)^{n+1}\Omega h^n$ by additivity of Ω, so that the triangle becomes

$$\Omega^{n+1} X \xrightarrow{(-1)^{n+1}\Omega^{n+1} f} \Omega^{n+1} Y \xrightarrow{(-1)^{n+1}\Omega^{n+1} g} \Omega^{n+1} Z \xrightarrow{(-1)^{n+1}\Omega h^n} \Omega\Sigma\Omega^n X \cong \Sigma\Omega^{n+1} X. \tag{7}$$

Thus, in order to prove the desired characterization, it remains to show this diagram commutes:

$$\begin{tikzcd}
\Omega^{n+1} Z \arrow[r, "(-1)^{n+1}\Omega h^n"] \arrow[d, "(-1)^{n+1}\Omega^{n+1} h"'] & \Omega\Sigma\Omega^n X \arrow[r, "\varepsilon_{\Omega^n X}"] & \Omega^n X \arrow[d, "\eta_{\Omega^n X}"] \\
\Omega^{n+1}\Sigma X \arrow[r, "\Omega^n\varepsilon_X"'] & \Omega^n X \arrow[r, "\eta_{\Omega^n X}"'] & \Sigma\Omega^{n+1} X
\end{tikzcd}$$

(The top composition is the last two arrows in diagram (7), and the bottom composition is the last two arrows in the diagram in the statement of the lemma). Unravelling how h^n is constructed, by additivity of Ω it further suffices to show the outside of the following diagram commutes:

$$\begin{tikzcd}
\Omega^{n+1} Z \arrow[r, "(-1)^{n+1}\Omega^{n+1} h"] \arrow[dd, "(-1)^{n+1}\Omega^{n+1} h"'] & \Omega^{n+1}\Sigma X \arrow[r, "\Omega^n\varepsilon_X"] & \Omega^n X \arrow[r, "\Omega\eta_{\Omega^{n-1}X}"] \arrow[dd, equal] & \Omega\Sigma\Omega^n X \arrow[d, "\varepsilon_{\Omega^n X}"] \\
 & & & \Omega^n X \arrow[d, "\eta_{\Omega^n X}"] \arrow[dl, equal] \\
\Omega^{n+1}\Sigma X \arrow[rr, "\Omega^n\varepsilon_X"'] & & \Omega^n X \arrow[r, "\eta_{\Omega^n X}"'] & \Sigma\Omega^{n+1} X
\end{tikzcd}$$

The left rectangle and bottom right triangle commute by definition. Finally, commutativity of the top right trapezoid is precisely one of the zig-zag identities applied to $\Omega^{n-1}X$. Hence, we have shown the desired result. $\square$

Proposition A.11. *Given a distinguished triangle*

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X,$$

let $\widetilde{h} : \Omega Z \to X$ be the left adjoint of h. Then the following infinite sequence is exact:

$$\begin{tikzcd}
 & & \cdots \arrow[dll] & & \\
\Omega^{n+1}Z \arrow[r, "(-1)^{n+1}\Omega^n\widetilde{h}"'] & \Omega^n X \arrow[r, "(-1)^n\Omega^n f"'] & \Omega^n Y \arrow[r, "(-1)^n\Omega^n g"] & \Omega^n Z \arrow[r, "(-1)^n\Omega^{n-1}\widetilde{h}"] & \Omega^{n-1}X \arrow[dll] \\
 & & \cdots \arrow[dll] & & \\
\Omega Z \arrow[r, "-\widetilde{h}"'] & X \arrow[r, "f"'] & Y \arrow[r, "g"] & Z \arrow[r, "h"] & \Sigma X \arrow[dll] \\
 & & \cdots \arrow[dll] & & \\
\Sigma^{n-1}Z \arrow[r, "(-1)^{n-1}\Sigma^n h"'] & \Sigma^n X \arrow[r, "(-1)^n\Sigma^n f"'] & \Sigma^n Y \arrow[r, "(-1)^n\Sigma^n g"] & \Sigma^n Z \arrow[r, "(-1)^n\Sigma^n h"] & \Sigma^{n+1}X \arrow[dll] \\
 & & \cdots & &
\end{tikzcd}$$

In particular, it remains exact even if we remove the signs.

Proof. Exactness of

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X \xrightarrow{-\Sigma f} \Sigma Y$$

is Proposition A.3 and axiom TR4. By induction using axiom TR4, for $n > 0$ we get that each contiguous composition of three maps below is a distinguished triangle:

$$\Sigma^n X \xrightarrow{(-1)^n\Sigma^n f} \Sigma^n Y \xrightarrow{(-1)^n\Sigma^n g} \Sigma^n Z \xrightarrow{(-1)^n\Sigma^n h} \Sigma^{n+1}X \xrightarrow{(-1)^{n+1}\Sigma^{n+1}f} \Sigma^{n+1}Y,$$

thus the sequence is exact by Proposition A.3. It remains to show exactness of the LES to the left of Y. It suffices to show that the row in the following diagram is exact for all $n > 0$:

$$\begin{tikzcd}
\Omega^n X \arrow[r, "(-1)^n\Omega^n f"] & \Omega^n Y \arrow[r, "(-1)^n\Omega^n g"] & \Omega^n Z \arrow[rr, "(-1)^n\Omega^{n-1}(\varepsilon_X \circ \Omega h)"] \arrow[dr, "(-1)^n\Omega^n h"'] & & \Omega^{n-1}X \arrow[r, "(-1)^{n-1}\Omega^{n-1}f"] & \Omega^{n-1}Y \\
 & & & \Omega^n\Sigma X \arrow[ur, "\Omega^{n-1}\varepsilon_X"'] & &
\end{tikzcd} \tag{8}$$

First of all, to see exactness at $\Omega^n Y$ and $\Omega^n Z$, consider the following commutative diagram:

$$\begin{tikzcd}
\Omega^n X \arrow[r, "(-1)^n\Omega^n f"] \arrow[dd, equal] & \Omega^n Y \arrow[r, "(-1)^n\Omega^n g"] \arrow[dd, equal] & \Omega^n Z \arrow[rr, "(-1)^n\Omega^{n-1}(\varepsilon_X \circ \Omega h)"] \arrow[dd, equal] \arrow[dr, dashed, "(-1)^n\Omega^n h"'] & & \Omega^{n-1}X \arrow[dd, "\eta_{\Omega^{n-1}X}"] \\
 & & & \Omega^n\Sigma X \arrow[ur, dashed, "\Omega^{n-1}\varepsilon_X"'] & \\
\Omega^n X \arrow[r, "(-1)^n\Omega^n f"] & \Omega^n Y \arrow[r, "(-1)^n\Omega^n g"] & \Omega^n Z \arrow[rr, dashed] \arrow[ur, dashed, "(-1)^n\Omega^n h"] & & \Sigma\Omega^n X
\end{tikzcd}$$

(here the dashed arrow is the morphism which makes the diagram commute). The bottom row is distinguished by Lemma A.10. Then by axiom TR0, the top row is distinguished, and thus exact by Proposition A.3. Thus we have shown exactness of (8) at $\Omega^n Y$ and $\Omega^n Z$. It remains to show exactness at $\Omega^{n-1}X$. In the case $n = 1$, we want to show exactness at X in the following diagram:

$$\begin{tikzcd} \Omega Z \arrow[rr, "-(\varepsilon_X \circ \Omega h)"] \arrow[dr, "-\Omega h"'] & & X \arrow[r, "f"] & Y \\ & \Omega\Sigma X \arrow[ur, "\varepsilon_X"'] & & \end{tikzcd}$$

Unravelling definitions, $\varepsilon_X \circ \Omega h$ is precisely the adjoint $\widetilde{h} : \Omega Z \to X$ of $h : Z \to \Sigma X$, in which case we have that the row in the above diagram fits into a distinguished triangle by Lemma A.9, and thus it is exact by Proposition A.3. To see exactness at $\Omega^{n-1}X$ in diagram (8), note that if we apply Lemma A.9 to the sequence Lemma A.10 for $n-1$, then we get that the following composition fits into a distinguished triangle, and is thus exact:

$$\Omega^n Z \xrightarrow{-k} \Omega^{n-1}X \xrightarrow{(-1)^{n-1}\Omega^{n-1}f} \Omega^{n-1}Y,$$

where $k : \Omega(\Omega^{n-1}Z) \to \Omega^{n-1}X$ is the adjoint of the composition

$$\Omega^{n-1}Z \xrightarrow{(-1)^{n-1}\Omega^{n-1}h} \Omega^{n-1}\Sigma X \xrightarrow{\Omega^{n-2}\varepsilon_X} \Omega^{n-2}X \xrightarrow{\eta_{\Omega^{n-2}X}} \Sigma\Omega^{n-1}X.$$

Further expanding how adjoints are constructed, k is the composition

$$\Omega^n Z \xrightarrow{(-1)^{n-1}\Omega^n h} \Omega^n\Sigma X \xrightarrow{\Omega^{n-1}\varepsilon_X} \Omega^{n-1}X \xrightarrow{\Omega\eta_{\Omega^{n-2}X}} \Omega\Sigma\Omega^{n-1}X \xrightarrow{\varepsilon_{\Omega^{n-1}X}} \Omega^{n-1}X.$$

Thus, in order to show exactness of (8) at $\Sigma^{n-1}X$, it suffices to show that $k = (-1)^{n-1}\Omega^{n-1}(\varepsilon_X \circ \Omega h)$. To that end, consider the following diagram:

$$\begin{tikzcd} \Omega^n Z \arrow[r, "(-1)^{n-1}\Omega^n h"] \arrow[d, "(-1)^{n-1}\Omega^n h"'] & \Omega^n\Sigma X \arrow[r, "\Omega^{n-1}\varepsilon_X"] & \Omega^{n-1}X \arrow[r, "\Omega\eta_{\Omega^{n-2}X}"] \arrow[dr, equal] & \Omega\Sigma\Omega^{n-1}X \arrow[d, "\varepsilon_{\Omega^{n-1}X}"] \\ \Omega^n\Sigma X \arrow[rrr, "\Omega^{n-1}\varepsilon_X"'] & & & \Omega^{n-1}X \end{tikzcd}$$

The top composition is k, while the bottom composition is $(-1)^{n-1}\Omega^{n-1}(\varepsilon_X \circ \Omega h)$. The left region commutes by definition, while commutativity of the right region is precisely one of the zig-zag identities applied to $\Omega^{n-2}X$. Thus, we have shown that $-k = (-1)^n\Omega^{n-1}(\varepsilon_X \circ \Omega h)$, so (8) is exact at $\Omega^{n-1}X$, as desired. $\square$

A.4. **Tensor triangulated categories.** Also important for our work is the concept of a *tensor triangulated category*, that is, a triangulated symmetric monoidal category in which the triangulated structures are compatible, in the following sense:

Definition A.12. A *tensor triangulated category* is a triangulated symmetric monoidal category $(\mathcal{C}, \otimes, S, \Sigma, \mathcal{D})$ such that:

TT1 For all objects X and Y in $\mathcal{C}$, there are natural isomorphisms

$$e_{X,Y} : \Sigma X \otimes Y \xrightarrow{\cong} \Sigma(X \otimes Y).$$

TT2 For each object X in $\mathcal{C}$, the functor $X \otimes (-) \cong (-) \otimes X$ is an additive functor.

TT3 For each object X in $\mathcal{C}$, the functor $X \otimes (-) \cong (-) \otimes X$ preserves distinguished triangles, in that given a distinguished triangle/(co)fiber sequence

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A,$$

then also

$$X \otimes A \xrightarrow{X \otimes f} X \otimes B \xrightarrow{X \otimes g} X \otimes C \xrightarrow{X \otimes' h} \Sigma(X \otimes A)$$

and

$$A \otimes X \xrightarrow{f \otimes X} B \otimes X \xrightarrow{g \otimes X} C \otimes X \xrightarrow{h \otimes' X} \Sigma(A \otimes X)$$

are distinguished triangles, where here we writing $X \otimes' h$ and $h \otimes' X$ to denote the compositions

$$X \otimes C \xrightarrow{X \otimes h} X \otimes \Sigma A \xrightarrow{\tau} \Sigma A \otimes X \xrightarrow{e_{A,X}} \Sigma(A \otimes X) \xrightarrow{\Sigma\tau} \Sigma(X \otimes A)$$

and

$$C \otimes X \xrightarrow{h \otimes X} \Sigma A \otimes X \xrightarrow{e_{A,X}} \Sigma(A \otimes X),$$

respectively.

TT4 Given objects X, Y, and Z in $\mathcal{C}$, the following diagram must commute:

$$\begin{tikzcd}
(\Sigma X \otimes Y) \otimes Z \arrow[r, "e_{X,Y} \otimes Z"] \arrow[d, "\alpha"'] & \Sigma(X \otimes Y) \otimes Z \arrow[r, "e_{X \otimes Y, Z}"] & \Sigma((X \otimes Y) \otimes Z) \arrow[d, "\Sigma\alpha"] \\
\Sigma X \otimes (Y \otimes Z) \arrow[rr, "e_{X, Y \otimes Z}"'] & & \Sigma(X \otimes (Y \otimes Z))
\end{tikzcd}$$

Usually, most tensor triangulated categories that arise in nature will satisfy additional coherence axioms (see axioms TC1–TC5 in [5]), but the above definition will suffice for our purposes. In what follows, we fix a tensor triangulated category $(\mathcal{C}, \otimes, S, \Sigma, e, \mathcal{D})$.

Definition A.13. There are natural isomorphisms

$$e'_{X,Y} : X \otimes \Sigma Y \xrightarrow{\cong} \Sigma(X \otimes Y)$$

obtained via the composition

$$X \otimes \Sigma Y \xrightarrow{\tau} \Sigma Y \otimes X \xrightarrow{e_{Y,X}} \Sigma(Y \otimes X) \xrightarrow{\Sigma\tau} \Sigma(X \otimes Y).$$

Lemma A.14. *Let $A \xrightarrow{a} B \xrightarrow{b} C \xrightarrow{c} D$ be any sequence isomorphic to a distinguished triangle. Then given any E in $\mathcal{C}$, the sequences*

$$E \otimes A \xrightarrow{E \otimes a} E \otimes B \xrightarrow{E \otimes b} E \otimes C \xrightarrow{E \otimes c} E \otimes D$$

and

$$A \otimes E \xrightarrow{a \otimes E} B \otimes E \xrightarrow{b \otimes E} C \otimes E \xrightarrow{c \otimes E} D \otimes E$$

are exact.

Proof. Since (a, b, c) is isomorphic to a distinguished triangle, there exists a commuting diagram in $\mathcal{SH}$

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\alpha"'] & Y \arrow[r, "g"] \arrow[d, "\beta"'] & Z \arrow[r, "h"] \arrow[d, "\gamma"'] & \Sigma X \arrow[d, "\delta"'] \\
A \arrow[r, "a"'] & B \arrow[r, "b"'] & C \arrow[r, "c"'] & D
\end{tikzcd}$$

where the top row is distinguished and the vertical arrows are isomorphisms. Then the following diagram commutes by functoriality of $-\otimes-$:

$$\begin{tikzcd}
E\otimes X \arrow[r, "E\otimes f"] \arrow[d, "E\otimes\alpha"'] & E\otimes Y \arrow[r, "E\otimes g"] \arrow[d, "E\otimes\beta"'] & E\otimes Z \arrow[rr, "E\otimes' h"] \arrow[d, "E\otimes\gamma"'] \arrow[dr, dashed, "E\otimes h"] & & \Sigma(E\otimes X) \arrow[d, dashed, "(E\otimes\delta)\circ(e'_{E,X})^{-1}"] \\
& & & E\otimes\Sigma X \arrow[ur, dashed, "e'_{E,X}"] \arrow[dr, dashed, "E\otimes\delta"] & \\
E\otimes A \arrow[r, "E\otimes a"'] & E\otimes B \arrow[r, "E\otimes b"'] & E\otimes C \arrow[rr, "E\otimes c"'] & & E\otimes D
\end{tikzcd}$$

The top triangle is distinguished by axiom TT3 for a tensor triangulated category, thus exact by Proposition A.3, so that the bottom triangle is also exact since the vertical arrows are isomorphisms and each square commutes. Similarly, the following diagram also commutes by functoriality of $-\otimes-$:

$$\begin{tikzcd}
X\otimes E \arrow[r, "f\otimes E"] \arrow[d, "\alpha\otimes E"'] & Y\otimes E \arrow[r, "g\otimes E"] \arrow[d, "\beta\otimes E"'] & Z\otimes E \arrow[rr, "h\otimes' E"] \arrow[d, "\gamma\otimes E"'] \arrow[dr, dashed, "h\otimes E"] & & \Sigma(X\otimes E) \arrow[d, "(\delta\otimes E)\circ e_{X,E}^{-1}"] \\
& & & \Sigma X\otimes E \arrow[ur, dashed, "e_{X,E}"] \arrow[dr, dashed, "\delta\otimes E"] & \\
A\otimes E \arrow[r, "a\otimes E"'] & B\otimes E \arrow[r, "b\otimes E"'] & C\otimes E \arrow[rr, "c\otimes E"'] & & D\otimes E
\end{tikzcd}$$

The top row is distinguished by axiom TT3 for a tensor triangulated category, thus exact by Proposition A.3, so that the bottom triangle is also exact since the vertical arrows are isomorphisms and each square commutes. ☐

Proposition A.15. *Suppose we have a distinguished triangle*

$$X \to Y \to Z \to \Sigma X$$

in $\mathcal{C}$. Then given any object E in $\mathcal{C}$, the long exact sequence given in Proposition A.11 remains exact after applying $E\otimes -$ or $-\otimes E$.

Proof. Recall that in the proof of Proposition A.11 we showed that the sequence was exact by showing that any two consecutive maps were isomorphic to a part of a distinguished triangle. Then the desired result follows from Lemma A.14. ☐

Definition A.16. An *adjointly tensor triangulated* category is a tensor triangulated category $(\mathcal{C}, \otimes, S, \Sigma, e, \mathcal{D})$ along with the structure of an adjointly triangulated category $(\mathcal{C}, \Omega, \Sigma, \eta, \varepsilon, \mathcal{D})$.

From now on, we fix an adjointly tensor triangulated category $(\mathcal{C}, \otimes, S, \Omega, \Sigma, \eta, \varepsilon, e, \mathcal{D})$.

Definition A.17. We may define natural isomorphisms $o_{X,Y} : \Omega X \otimes Y \xrightarrow{\cong} \Omega(X\otimes Y)$ and $o'_{X,Y} : X\otimes \Omega Y \xrightarrow{\cong} \Omega(X\otimes Y)$ as the compositions

$$o_{X,Y} : \Omega X\otimes Y \xrightarrow{\varepsilon^{-1}_{\Omega X\otimes Y}} \Omega\Sigma(\Omega X\otimes Y) \xrightarrow{\Omega e^{-1}_{\Omega X,Y}} \Omega(\Sigma\Omega X\otimes Y) \xrightarrow{\Omega(\eta_X^{-1}\otimes Y)} \Omega(X\otimes Y)$$

and

$$o'_{X,Y} : X\otimes\Omega Y \xrightarrow{\tau_{X,\Omega Y}} \Omega Y\otimes X \xrightarrow{o_{Y,X}} \Omega(Y\otimes X) \xrightarrow{\Omega\tau_{Y,X}} \Omega(X\otimes Y).$$

These are both clearly natural by naturality of ε, e, η, and τ.

Appendix B. A-graded objects

In this appendix, we fix an abelian group A once and for all. We assume the reader is familiar with the basic theory of (non-commutative, unital) rings and modules over them.

B.1. A-graded abelian groups, rings, and modules.

Definition B.1. An A-*graded abelian group* is an abelian group B along with a subgroup $B_a \leq B$ for each $a \in A$ such that the canonical map

$$\bigoplus_{a\in A} B_a \to B$$

sending $(x_a)_{a\in A}$ to $\sum_{a\in A} x_a$ is an isomorphism. Given two A-graded abelian groups B and C, a homomorphism $f : B \to C$ is a *homomorphism of A-graded abelian groups*, or just an A-*graded homomorphism*, if it preserves the grading, i.e., if it restricts to a map $B_a \to C_a$ for all $a \in A$.

We denote the category of A-graded abelian groups and A-graded homomorphisms between them by $\mathbf{Ab}^A$

It is easy to see that an A-graded abelian group B is generated by its *homogeneous* elements, that is, nonzero elements $x \in B$ such that there exists some $a \in A$ with $x \in B_a$.

Remark B.2. Clearly the condition that the canonical map $\bigoplus_{a\in A} B_a \to B$ is an isomorphism requires that $B_a \cap B_b = 0$ if $a \neq b$. In particular, given a homogeneous element $x \in B$, there exists precisely one $a \in A$ such that $x \in B_a$. We call this a the *degree* of x, and we write $|x| = a$.

Definition B.3. An A-*graded ring* is a ring R such that its underlying abelian group R is A-graded and the multiplication map $R \times R \to R$ restricts to $R_a \times R_b \to R_{a+b}$ for all $a, b \in A$. A morphism of A-graded rings is a ring homomorphism whose underlying homomorphism of abelian groups is A-graded.

Explicitly, given an A-graded ring R and homogeneous elements $x, y \in R$, we must have $|xy| = |x| + |y|$. For example, given some field k, the ring $R = k[x, y]$ is $\mathbb{Z}^2$-graded, where given $(n, m) \in \mathbb{Z}^2$, $R_{n,m}$ is the subgroup of those monomials of the form ax^ny^m for some $a \in k$.

Definition B.4. Let R be an A-graded ring. A *left A-graded R-module* M is a left R-module M such that M is an A-graded abelian group and the action map $R \times M \to M$ restricts to a map $R_a \times M_b \to M_{a+b}$ for all $a, b \in A$. Right A-graded R-modules are defined similarly. Finally, an A-graded R-bimodule is an A-graded abelian group M which has the structure of both an A-graded left and right R-module such that given $r, s \in R$ and $m \in M$, $r \cdot (m \cdot s) = (r \cdot m) \cdot s$.

Morphisms between A-graded R-modules are precisely A-graded R-module homomorphisms. We write R-$\mathbf{Mod}^A$ for the category of left A-graded R-modules and $\mathbf{Mod}^A$-R for the category of right A-graded R-modules.

Remark B.5. It is straightforward to see that an A-graded abelian group is equivalently an A-graded $\mathbb{Z}$-module, where here we are considering $\mathbb{Z}$ as an A-graded ring concentrated in degree 0. Thus any result below about A-graded modules applies equally to A-graded abelian groups.

Remark B.6. We often will denote an A-graded R-module M by M_*. Given some $a \in A$, we can define the shifted A-graded abelian group M_{*+a} whose b^{th} component is M_{b+a}. We will also sometimes write $\Sigma^a M$ to denote the shifted module M_{*-a}.

Definition B.7. More generally, given two A-graded R-modules M and N and some $d \in A$, an R-module homomorphism $f : M \to N$ is *an A-graded homomorphism of degree d* if it restricts to a map $M_a \to N_{a+d}$ for all $a \in A$. Thus, an A-graded homomorphism of degree d from M to N is equivalently an A-graded homomorphism $M_* \to N_{*+d}$ or an A-graded homomorphism $M_{*-d} \to N$. Given some $a \in A$ and left (resp. right) R-modules M and N, we will write

$$\mathrm{Hom}_R^d(M, N) = \mathrm{Hom}_R(M_*, N_{*+d}) = \mathrm{Hom}_R(M_{*-d}, N_*)$$

to denote the set of A-graded homomorphisms of degree d from M to N, and simply

$$\mathrm{Hom}_R(M, N)$$

to denote the set of degree-0 A-graded homomorphisms from M to N. Clearly A-graded homomorphisms may be added and subtracted, so these are further abelian groups. Thus we have an A-graded abelian group

$$\mathrm{Hom}_R^*(M, N).$$

Unless stated otherwise, an "A-graded homomorphism" will always refer to an A-graded homomorphism of degree 0.

Oftentimes when constructing A-graded rings, we do so only by defining the product of homogeneous elements, like so:

Lemma B.8. *Suppose we have an A-graded abelian group R, a distinguished element $1 \in R_0$, and $\mathbb{Z}$-bilinear maps $m_{a,b} : R_a \times R_b \to R_{a+b}$ for all $a, b \in A$. Further suppose that for all $x \in R_a$, $y \in R_b$, and $z \in R_c$, we have*

$$m_{a+b,c}(m_{a,b}(x, y), z) = m_{a,b+c}(x, m_{b,c}(y, z)) \qquad \textit{and} \qquad m_{a,0}(x, 1) = m_{0,a}(1, x) = x.$$

Then there exists a unique multiplication map $m : R \times R \to R$ which endows R with the structure of an A-graded ring and restricts to $m_{a,b}$ for all $a, b \in A$.

Proof. Given $r, s \in R$, since $R \cong \bigoplus_{a \in A} R_a$, we may uniquely decompose r and s into homogeneous elements as $r = \sum_{a \in A} r_a$ and $s = \sum_{a \in A} s_a$ with each $r_a, s_a \in R_a$ such that only finitely many of the r_a's and s_a's are nonzero. Then in order to define a distributive product $R \times R \to R$ which restricts to $m_{a,b} : R_a \times R_b \to R_{a+b}$, note we *must* define

$$r \cdot s = \left(\sum_{a \in A} r_a\right) \cdot \left(\sum_{b \in A} s_b\right) = \sum_{a,b \in A} r_a \cdot s_b = \sum_{a,b \in A} m_{a,b}(r_a, s_b).$$

Thus, we have shown uniqueness. It remains to show this product actually gives R the structure of a ring. First we claim that the sum on the right is actually finite. Note there exists only finitely many nonzero r_a's and s_b's, and if $s_b = 0$ then

$$m_{a,b}(r_a, 0) = m_{a,b}(r_a, 0 + 0) \overset{(*)}{=} m_{a,b}(r_a, 0) + m_{a,b}(r_a, 0) \implies m_{a,b}(r_a, 0) = 0,$$

where $(*)$ follows from bilinearity of $m_{a,b}$. A similar argument yields that $m_{a,b}(0, s_b) = 0$ for all $a, b \in A$. Hence indeed $m_{a,b}(r_a, s_b)$ is zero for all but finitely many pairs $(a, b) \in A^2$, as desired. Observe that in particular

$$(r \cdot s)_a = \sum_{b+c=a} m_{b,c}(r_b, s_c) = \sum_{b \in A} m_{b,a-b}(r_b, s_{a-b}) = \sum_{c \in A} m_{a-c,c}(r_{a-c}, s_c).$$

Now we claim this multiplication is associative. Given $t = \sum_{a\in A} t_a \in R$, we have

$$\begin{aligned}
(r\cdot s)\cdot t &= \sum_{a,b\in A} m_{a,b}((r\cdot s)_a, t_b)\\
&= \sum_{a,b\in A} m_{a,b}\left(\sum_{c\in A} m_{a-c,c}(r_{a-c}, s_c), t_b\right)\\
&\overset{(1)}{=} \sum_{a,b,c\in A} m_{a,b}(m_{a-c,c}(r_{a-c}, s_c), t_b)\\
&\overset{(2)}{=} \sum_{a,b,c\in A} m_{c,a+b-c}(r_c, m_{a-c,b}(s_{a-c}, t_b))\\
&\overset{(3)}{=} \sum_{a,b,c\in A} m_{a,c}(r_a, m_{b,c-b}(s_b, t_{c-b}))\\
&\overset{(1)}{=} \sum_{a,c\in A} m_{a,c}\left(r_a, \sum_{b\in A} m_{b,c-b}(s_b, t_{c-b})\right)\\
&= \sum_{a,c\in A} m_{a,c}(r_a, (s\cdot t)_c) = r\cdot(s\cdot t),
\end{aligned}$$

where each occurrence of (1) follows by bilinearity of the $m_{a,b}$'s, each occurrence of (2) is associativity of the $m_{a,b}$'s, and (3) is obtained by re-indexing by re-defining $a := c$, $b := a-c$, and $c := a+b-c$. Next, we wish to show that the distinguished element $1 \in R_0$ is a unit with respect to this multiplication. Indeed, we have

$$1\cdot r \overset{(1)}{=} \sum_{a\in A} m_{0,a}(1, r_a) \overset{(2)}{=} \sum_{a\in A} r_a = r \qquad \text{and} \qquad r\cdot 1 \overset{(1)}{=} \sum_{a\in A} m_{a,0}(r_a, 1) \overset{(2)}{=} \sum_{a\in A} r_a = r,$$

where (1) follows by the fact that $m_{a,b}(0,-) = m_{a,b}(-,0) = 0$, which we have shown above, and (2) follows by unitality of the $m_{0,a}$'s and $m_{0,a}$'s, respectively. Finally, we wish to show that this product is distributive. Indeed, we have

$$\begin{aligned}
r\cdot(s+t) &= \sum_{a,b\in A} m_{a,b}(r_a, (s+t)_b)\\
&= \sum_{a,b\in A} m_{a,b}(r_a, s_b + t_b)\\
&\overset{(*)}{=} \sum_{a,b\in A} m_{a,b}(r_a, s_b) + \sum_{a,b\in A} m_{a,b}(r_a, t_b) = (r\cdot s) + (r\cdot t),
\end{aligned}$$

where $(*)$ follows by bilinearity of $m_{a,b}$. An entirely analagous argument yields that $(r+s)\cdot t = (r\cdot t) + (s\cdot t)$. ☐

Similarly, when defining A-graded modules, we will only define the action maps for homogeneous elements:

Lemma B.9. *Let R be an A-graded ring, M an A-graded abelian group, and suppose there exists $\mathbb{Z}$-bilinear maps $\kappa_{a,b} : R_a \times M_b \to M_{a+b}$ for all $a, b \in A$. Further suppose that for all $r \in R_a$, $r' \in R_b$, and $m \in M_c$ that*

$$\kappa_{a+b,c}(r\cdot r', m) = \kappa_{a,b+c}(r, \kappa_{b,c}(r', m)) \qquad \text{and} \qquad \kappa_{0,c}(1, m) = m.$$

Then there is a unique map $\kappa : R\times M \to M$ which endows M with the structure of a left A-graded R-module and restricts to $\kappa_{a,b}$ for all $a, b \in A$.

On the other hand, suppose there exists $\mathbb{Z}$-bilinear maps $\kappa_{a,b} : M_a \times R_b \to M_{a+b}$ for all $a, b \in A$. Further suppose that for all $r \in R_a$, $r' \in R_b$, and $m \in M_c$ that

$$\kappa_{c,a+b}(m, r \cdot r') = \kappa_{c+a,b}(\kappa_{c,a}(m, r), r') \qquad \text{and} \qquad \kappa_{c,0}(m, 1) = m.$$

Then there is a unique map $\kappa : M \times R \to M$ which endows M with the structure of a right A-graded R-module and restricts to $\kappa_{a,b}$ for all $a, b \in A$.

Finally, if we have maps $\lambda_{a,b} : R_a \times M_b \to M_{a+b}$ and $\rho_{a,b} : M_a \times R_b \to M_{a+b}$ satisfying all of the above conditions, and if we further have that

$$\lambda_{a,b+c}(r, \rho_{b,c}(x, s)) = \rho_{a+b,c}(\lambda_{a,b}(r, x), s)$$

for all $r \in R_a$, $x \in M_b$, and $s \in R_c$, then the left and right A-graded R-module structures induced on M by the λ's and ρ's give M the structure of an A-graded R-bimodule.

Proof. We show the left module case, as the right module case is entirely analagous. Supposing for each $a, b \in A$ we have a map $\kappa_{a,b} : R_a \times M_b \to M_{a+b}$ satisfying the above conditions, in order to extend these to a map $R \times M \to M$, by additivity we *must* define

$$\kappa : R \times M \to M$$

to be the map sending $r = \sum_a r_a$ and $m = \sum_a m_a$ to $\sum_{a,b \in A} \kappa_{a,b}(r_a, m_b)$. Now, we need to check that for all $r, s \in R$, $x, y \in M$ that

(1) $r \cdot (x + y) = r \cdot x + r \cdot y$

(2) $(r + s) \cdot x = r \cdot x + s \cdot x$

(3) $(rs) \cdot x = r \cdot (s \cdot x)$

(4) $1 \cdot x = x$,

where above we have written $- \cdot -$ for $\kappa(-, -)$. To see the first, note

$$\begin{aligned}
\kappa(r, x + y) &= \sum_{a,b \in A} \kappa_{a,b}(r_a, (x + y)_b) \\
&= \sum_{a,b \in A} \kappa_{a,b}(r_a, x_b + y_b) \\
&= \sum_{a,b \in A} (\kappa_{a,b}(r_a, x_b) + \kappa_{a,b}(r_a, y_b)) \\
&= \sum_{a,b \in A} \kappa_{a,b}(r_a, x_b) + \sum_{a,b \in A} \kappa_{a,b}(r_a, y_b) \\
&= \kappa(r, x) + \kappa(r, y).
\end{aligned}$$

To see the second, note

$$\begin{aligned}
\kappa(r + s, x) &= \sum_{a,b \in A} \kappa_{a,b}((r + s)_a, x_b) \\
&= \sum_{a,b \in A} \kappa_{a,b}(r_a + s_a, x_b) \\
&= \sum_{a,b \in A} (\kappa_{a,b}(r_a, x_b) + \kappa_{a,b}(s_a, x_b)) \\
&= \sum_{a,b \in A} \kappa_{a,b}(r_a, x_b) + \sum_{a,b \in A} \kappa_{a,b}(s_a, x_b) \\
&= \kappa(r, x) + \kappa(s, x).
\end{aligned}$$

To see the third, note

$$\begin{aligned}
\kappa(rs, x) &= \sum_{a,b\in A} \kappa_{a,b}((rs)_a, x_b) \\
&= \sum_{a,b\in A} \kappa_{a,b}\left(\sum_{c\in A} r_c s_{a-c}, x_b\right) \\
&= \sum_{a,b,c\in A} \kappa_{a,b}(r_c s_{a-c}, x_b) \\
&= \sum_{a,b,c\in A} \kappa_{a,b}(r_c, \kappa_{a-c,b}(s_{a-c}, x_b)) \\
&=
\end{aligned}$$

□ FINISH

When working with A-graded rings and modules, we will often freely use the above propositions without comment.

Lemma B.10. *Let R be an A-graded ring, and let M be an A-graded left (resp. right) R-module. Then for all $d \in A$, the evaluation map*

$$\begin{aligned}
\mathrm{ev}_1 : \mathrm{Hom}_R^d(R, M) &\to M_d \\
\varphi &\mapsto \varphi(1)
\end{aligned}$$

is an isomorphism of abelian groups.

Proof. We consider the case that M is a left A-graded R-module, as showing it when M is a right module is entirely analagous. First of all, this map is clearly a homomorphism, as given degree d A-graded homomorphisms $\varphi, \psi : R \to M$, we have

$$\mathrm{ev}_1(\varphi + \psi) = (\varphi + \psi)(1) = \varphi(1) + \psi(1) = \mathrm{ev}_1(\varphi) + \mathrm{ev}_1(\psi).$$

Now, to see it is surjective, let $m \in M_d$, and define $\varphi_m : R \to M$ to send $r \mapsto r \cdot m$. First of all, φ_m is a module homomorphism, as given $r, s \in R$,

$$\varphi_m(r + s) = (r + s) \cdot m = r \cdot m + s \cdot m = \varphi_m(r) + \varphi_m(s) \quad \text{and} \quad \varphi_m(r \cdot s) = r \cdot s \cdot m = r \cdot \varphi_m(s).$$

Furthermore, it is clearly A-graded of degree d, as given a homogeneous element $r \in R_a$ for some $a \in A$, we have $\varphi_m(r) = r \cdot m \in R_{a+d}$, since m is homogeneous of degree d. Finally, clearly

$$\mathrm{ev}_1(\varphi_m) = \varphi_m(1) = 1 \cdot m = m,$$

so indeed ev_1 is surjective. On the other hand, to see it is injective, suppose we are given $\varphi, \psi \in \mathrm{Hom}_R^d(R, M)$ such that $\varphi(1) = \psi(1)$. Then given $r \in R$, we must have

$$\varphi(r) = \varphi(r \cdot 1) = r \cdot \varphi(1) = r \cdot \psi(1) = \psi(r \cdot 1) = \psi(r),$$

so φ and ψ are exactly the same map. Thus, ev_1 is injective, as desired. □

B.2. **Tensor products of A-graded modules.**

Lemma B.11. *Given an A-graded ring R and two left (resp. right) A-graded R-modules M and N, their direct sum $M \oplus N$ is naturally a left (resp. right) A-graded R-module by defining*

$$(M \oplus N)_a := M_a \oplus N_a.$$

Proof. The canonical map $\bigoplus_{a\in A}(M_a \oplus N_a) \to M \oplus N$ factors as

$$\bigoplus_{a\in A}(M_a \oplus N_a) \xrightarrow{\cong} \bigoplus_{a\in A} M_a \oplus \bigoplus_{a\in A} N_a \xrightarrow{\cong} M \oplus N. \qquad \square$$

Recall that given a ring R, a left R-module M, a right R-module N, and an abelian group A, an *R-balanced map* $\varphi : M \times N \to B$ is one which satisifies

$$\begin{aligned}
\varphi(m, n + n') &= \varphi(m, n) + \varphi(m, n') \\
\varphi(m + m', n) &= \varphi(m, n) + \varphi(m', n) \\
\varphi(m \cdot r, n) &= \varphi(m, r \cdot n).
\end{aligned}$$

for all $m, m' \in M$, $n, n' \in N$, and $r \in R$. Then the tensor product $M \otimes_R N$ is the universal abelian group equipped with an R-balanced map $\otimes : M \times N \to M \otimes_R N$ such that for every abelian group B and every R-balanced map $\varphi : M \times N \to B$, there is a *unique* group homomorphism $\widetilde{\varphi} : M \otimes_R N \to B$ such that $\widetilde{f} \circ \otimes = f$. We call elements in the image of $\otimes : M \times N \to M \otimes_R N$ *pure tensors*. It is a standard fact that $M \otimes_R N$ is generated as an abelian group by its pure tensors.

Definition B.12. Suppose we have a right A-graded R-module M, a left A-graded R-module N, and an A-graded abelian group B. Then an *A-graded R-balanced map* $\varphi : M \times N \to B$ is an R-balanced map which restricts to $M_a \times N_b \to B_{a+b}$ for all $a, b \in A$.

Proposition B.13. *Suppose we have a right A-graded R-module M and a left A-graded R-module N. Then the tensor product*

$$M \otimes_R N$$

is naturally an A-graded abelian group by defining $(M \otimes_R N)_a$ to be the subgroup generated by homogeneous *pure tensors $m \otimes n$ with $m \in M_b$ and $n \in N_c$ such that $b + c = a$. Furthermore, if either M (resp. N) is an A-graded bimodule, then this decomposition makes $M \otimes_R N$ into a left (resp. right) A-graded R-module. In particular, if both M and N are R-bimodules, then $M \otimes_R N$ is an R-bimodule.*

Proof. By definition, since M and N are A-graded abelian groups, they are generated (as abelian groups) by their homogeneous elements. Thus it follows that $M \otimes_R N$ is generated by *homogeneous pure tensors*, that is, elements of the form $m \otimes n$ with $m \in M$ and $n \in N$ homogeneous. Now, given a homogeneous pure tensor $m \otimes n$, we define its *degree* by the formula $|m \otimes n| := |m| + |n|$. It follows this formula is well-defined by checking that given homogeneous elements $m \in M$, $n \in N$, and $r \in R$ that

$$|(m \cdot r) \otimes n| = |m \cdot r| + |n| = |m| + |r| + |n| = |m| + |r \cdot n| = |m \otimes (r \cdot n)|.$$

Thus, we may define $(M \otimes_R N)_a$ to be the subgroup of $M \otimes_R N$ generated by those pure homogeneous tensors of degree a. Now, consider the map

$$\Psi : M \times N \to \bigoplus_{a\in A}(M \otimes_R N)_a$$

which takes a pair $(m, n) = \sum_{a\in A}(m_a, n_a)$ to the element $\Psi(m, n)$ whose a^{th} component is

$$(\Psi(m, n))_a := \sum_{b+c=a} m_b \otimes n_c.$$

It is straightforward to see that this map is R-balanced, in the sense that it is additive in each argument and $\Psi(m \cdot r, n) = \Psi(m, r \cdot n)$ for all $m \in M$, $n \in N$, and $r \in R$. Thus by the universal property of $M \otimes_R N$, we get a homomorphism of abelian groups $\widetilde{\Psi} : M \otimes_R N \to \bigoplus_{a \in A}(M \otimes_R N)_a$ lifting Ψ along the canonical map $M \times N \to M \otimes_R N$. Now, also consider the canonical map

$$\Phi : \bigoplus_{a \in A}(M \otimes_R N)_a \to M \otimes_R N.$$

We would like to show $\widetilde{\Psi}$ and Φ are inverses of eah other. Since $\widetilde{\Psi}$ and Φ are both homomorphisms, it suffices to show this on generators. Let $m \otimes n$ be a homogeneous pure tensor with $m = m_a \in M_a$ and $n = n_b \in N_b$. Then we have

$$\Phi(\widetilde{\Psi}(m \otimes n)) = \Phi\left(\bigoplus_{a \in A} \sum_{b+c=a} m_b \otimes n_c\right) \overset{(*)}{=} \Phi(m \otimes n) = m \otimes n,$$

and

$$\widetilde{\Psi}(\Phi(m \otimes n)) = \widetilde{\Psi}(m \otimes n) = \bigoplus_{a \in A} \sum_{b+c=a} m_b \otimes n_c \overset{(*)}{=} m \otimes n,$$

where both occurrences of $(*)$ follow by the fact that $m_b \otimes n_c = 0$ unless $b = c = a$, in which case $m_a \otimes n_a = m \otimes n$. Thus since Φ is an isomorphism, $M \otimes_R N$ is indeed an A-graded abelian group, as desired.

Now, suppose that M is an A-graded R-bimodule, so there exists left and right A-graded actions of R on M such that given $r, s \in R$ and $m \in M$ we have $r \cdot (m \cdot s) = (r \cdot m) \cdot s$. Then we would like to show that given a left A-graded R-module N that $M \otimes_R N$ is canonically a left A-graded R-module. Indeed, define the action of R on $M \otimes_R N$ on pure tensors by the formula

$$r \cdot (m \otimes n) = (r \cdot m) \otimes n.$$

First of all, clearly this map is A-graded, as if $r \in R_a$, $m \in M_b$, and $n \in N_c$ then $(r \cdot m) \otimes n$, by definition, has degree $|r \cdot m| + |n| = |r| + |m| + |n|$ (the last equality follows since the left action of R on M is A-graded). In order to show the above map defines a left module structure, it suffices to show that given pure tensors $m \otimes n, m' \otimes n' \in M \otimes_R N$ and elements $r, r' \in R$ that

(1) $r \cdot (m \otimes n + m' \otimes n') = r \cdot (m \otimes n) + r \cdot (m' \otimes n')$,

(2) $(r + r') \cdot (m \otimes n) = r \cdot (m \otimes n) + r' \cdot (m' \otimes n')$,

(3) $(rr') \cdot (m \otimes n) = r \cdot (r' \cdot (m \otimes n))$, and

(4) $1 \cdot (m \otimes n) = m \otimes n$.

Axiom (1) holds by definition. To see (2), note that by the fact that R acts on M on the left that

$$(r + r') \cdot (m \otimes n) = ((r + r') \cdot m) \otimes n = (r \cdot m + r' \cdot m) \otimes n = r \cdot m \otimes n + r' \cdot m \otimes n.$$

That (3) and (4) hold follows similarly by the fact that $(rr') \cdot m = r \cdot (r' \cdot m)$ and $1 \cdot m = m$.

Conversely, if N is an A-graded R-bimodule, then showing $M \otimes_R N$ is canonically a right A-graded R-module via the rule

$$(m \otimes n) \cdot r = m \otimes (n \cdot r)$$

is entirely analagous.

Finally, if both M and N are R-bimodules, then by what we have shown, $M \otimes_R N$ is both a left and right R-module. To see these coincide to give $M \otimes_R N$ an R-bimodule structure, note that given $m \in M$, $n \in N$, and $r, r' \in R$ that

$$(r \cdot (m \otimes n)) \cdot r' = ((r \cdot m) \otimes n) \cdot r' = (r \cdot m) \otimes (n \cdot r') = r \cdot (m \otimes (n \cdot r')) = r \cdot ((m \otimes n) \cdot r'). \quad \square$$

Lemma B.14. *Let R be an A-graded ring, B an A-graded abelian group, M a right A-graded R-module, and N a left A-graded R-module. Further suppose we are given a map $\varphi_{a,b} : M_a \times N_b \to B_{a+b}$ for all $a, b \in A$ which commutes with addition in each argument, and such that for all $m \in M_a$, $n \in N_b$, and $r \in R_c$ that*

$$\varphi_{a+b,c}(m \cdot r, n) = \varphi_{a,b+c}(m, r \cdot n).$$

Then there is a unique A-graded R-balanced map $\varphi : M \times N \to B$ which restricts to $\varphi_{a,b}$ for all $a, b \in A$, and furthermore, the induced homorphism $\widetilde{\varphi} : M \otimes_R N \to B$ is an A-graded homomorphism of abelian groups.

TODO

Proof. ☐

B.3. **A-graded submodules and quotient modules.** In what follows, fix an A-graded ring R. We will simply say "A-graded R-module" when we are freely considering either left or right A-graded R-modules. Recall that a left (resp. right) module P is *projective* if, for all diagrams of R-module homomorphisms of the form

$$\begin{tikzcd} & M \arrow[d, two heads, "g"] \\ P \arrow[r, "f"'] & N \end{tikzcd}$$

with g an epimorphism, there exists a lift $h : P \to M$ satisfying $g \circ h = f$

$$\begin{tikzcd} & M \arrow[d, two heads, "g"] \\ P \arrow[r, "f"'] \arrow[ur, dashed, "h"] & N \end{tikzcd}$$

(Note h is not required to be unique.)

Definition B.15. Let R be an A-graded ring, and let P be a left (resp. right) A-graded R-module. Then P is a *graded projective* module if, for all diagrams of A-graded R-module homomorphisms of the form

$$\begin{tikzcd} & M \arrow[d, two heads, "g"] \\ P \arrow[r, "f"'] & N \end{tikzcd}$$

with g an epimorphism, there exists an A-graded homomorphism $h : P \to M$ satisfying $g \circ h = f$.

$$\begin{tikzcd} & M \arrow[d, two heads, "g"] \\ P \arrow[r, "f"'] \arrow[ur, dashed, "h"] & N \end{tikzcd}$$

(Note h is not required to be unique.)

Definition B.16. Let M be an A-graded R-module. Then an *A-graded R-submodule* is an A-graded R-module N which is a subset of M and for which the inclusion $N \hookrightarrow M$ is an A-graded homomorphism of R-modules. Equivalently, it is a submodule N for which the canonical map

$$\bigoplus_{a \in A} N \cap M_a \to N$$

is an isomorphism.

Lemma B.17. *Let M be an A-graded R-module. Then an R-submodule $N \leq M$ is an A-graded submodule if and only if it is generated as an R-module by homogeneous elements of M.*

Proof. If $N \leq M$ is a A-graded submodule, it is generated by the set of all its homogeneous elements, which are also homogeneous elements in M, by definition.

Conversely, suppose $N \leq M$ is a submodule which is generated by homogeneous elements of M. Then define $N_a := N \cap M_a$, and consider the canonical map

$$\Phi : \bigoplus_{a \in A} N_a \to N.$$

First of all, it is surjective, as each generator of N belongs to some N_a, by definition. To see it is injective, consider the following commutative diagram:

$$\begin{tikzcd} \bigoplus_{a\in A} N_a \arrow[r, hook] \arrow[d, "\Phi"'] & \bigoplus_{a\in A} M_a \arrow[d, "\cong"] \\ N \arrow[r, hook] & M \end{tikzcd}$$

Since Φ composes with an injection to get an injection, clearly Φ must be injective itself. We have the desired result. $\square$

Proposition B.18. *Given two left (resp. right) A-graded R-modules M and N and an A-graded R-module homomorphism $\varphi : M \to N$ (of possibly nonzero degree), the kernel and images of φ are A-graded submodules of M and N, respectively.*

Proof. First recall that a degree d A-graded homomorphism $M \to N$ is simply an A-graded homomorphism $M_* \to N_{*+d}$, so it suffices to consider the case φ is of degree 0. Next, note that since the forgetful functor from R-modules to abelian groups preserves kernels and images, it suffices to consider the case that φ is a homomorphism of A-graded abelian groups. Finally, by Lemma B.17, it suffices to show that $\ker\varphi$ and $\operatorname{im}\varphi$ are generated by homogeneous elements of M and N, respectively.

Note that by the universal property of the coproduct in **Ab**, the data of an A-graded homomorphism of abelian groups $\varphi : M \to N$ is precisely the data of an A-indexed collection of abelian group homomorphisms $\varphi_a : M_a \to N_a$, in which case the following diagram commutes:

$$\begin{tikzcd} \bigoplus_a M_a \arrow[r, "\bigoplus_a \varphi_a"] \arrow[d, "\cong"'] & \bigoplus_a N_a \arrow[d, "\cong"] \\ M \arrow[r, "\varphi"] & N \end{tikzcd}$$

Finally, the desired result follows by the purely formal fact that taking images and kernels commutes with arbitrary direct sums. $\square$

Proposition B.19. *Given two left (resp. right) A-graded R-modules M and N, an A-graded submodule $K \leq N$, and an A-graded R-module homomorphism $\varphi : M \to N$ (of possibly nonzero degree), the submodule $\varphi^{-1}(K)$ of M is A-graded.*

Proof. Recall that a degree d A-graded homomorphism $M \to N$ is simply an A-graded homomorphism $M_* \to N_{*+d}$, so it suffices to consider the case φ is of degree 0. Now, let $x \in L := \varphi^{-1}(K)$. As an element of M, we may uniquely write $x = \sum_{a\in A} x_a$ where each $x_a \in M_a$. Similarly, if we set $y := \varphi(x)$, then we may uniquely write $y = \sum_{a\in A} y_a$ where each $y_a \in N_a$. Then since K is an A-graded submodule of N and $y \in K$, by definition, we have that $y_a \in K$ for each a. Finally, note that

$$\sum_{a \in A} y_a = y = \varphi(x) = \sum_{a \in A} \varphi(x_a),$$

so that $\varphi(x_a) = y_a \in K$ for all $a \in A$, so that $x_a \in L$ for all $a \in A$. Thus we have shown that each element in L can be written as a sum of homogeneous elements in M, as desired. □

Proposition B.20. *Given an A-graded R-module M and an A-graded subgroup $N \leq M$, the quotient M/N is canonically A-graded by defining $(M/N)_a$ to be the subgroup generated by cosets represented by homogeneous elements of degree a in M. Furthermore, the canonical maps $M_a/N_a \to (M/N)_a$ taking a coset $m + N_a$ to $m + N$ are isomorphisms.*

Proof. Consider the canonical map

$$\Phi : \bigoplus_a (M/N)_a \to M/N.$$

First of all, surjectivity of Φ follows by commutativity of the following diagram:

$$\begin{tikzcd}
\bigoplus_a M_a \arrow[r, "\cong"] \arrow[d, two heads] & M \arrow[d, two heads] \\
\bigoplus_a (M/N)_a \arrow[r, "\Phi"] & M/N
\end{tikzcd}$$

where the vertical left map sends a generator $m \in M_a$ to the coset $m + N$ in $(M/N)_a \subseteq M/N$. To see Φ is injective, suppose we are given some element $(m_a + N)_{a \in A}$ in $\bigoplus_a (M/C)_a$ such that $\sum_{a \in A}(m_a + N) = 0$ in M/N. Thus $\sum_{a \in A} m_a \in N$, and since N is A-graded this implies that each m_a belongs to $N \cap M_a = N_a$, so that in particular $m_a + N$ is zero in $(M/N)_a \subseteq M/N$, so that $(m_a + N)_{a \in A} = 0$ in $\bigoplus_a (M/N)_a$, as desired.

It remains to show that the canonical map

$$\varphi_a : M_a/N_a \to (M/N)_a$$

is an isomorphism. It is clearly surjective, as $(M/N)_a$ is generated by elements $m + N$ for $m \in M_a$, and these elements make up precisely the image of φ_a. Thus φ_a hits every generator of $(M/N)_a$, so φ_a is surjective. On the other hand, suppose we are given some $m \in M_a$ such that $\varphi(m + N_a) = m + N = 0$. Thus $m \in N$, and $m \in M_a$, so that $m \in M_a \cap N = N_a$, meaning $m + N_a = 0$ in M_a/N_a, as desired. □

B.4. **Pushouts of A-graded anticommutative rings.** The key definitions for this section are Definition 4.5 and Definition 4.6. The goal of this section is to show that given an A-graded anticommutative ring R that the category R-$\mathbf{GrCAlg}(A)$ of A-graded anticommutative R-algebras has pushouts and binary coproducts, which are formed by taking the tensor product of the underlying A-graded modules and endowing it with an anticommutative product. The proofs here are entirely analagous to showing that the standard category of anticommutative $\mathbb{Z}$-graded rings has pushouts, so rather than giving complete proofs in this section we simply outline what needs to be shown, and leave it to the reader to fill in the details.

Proposition B.21. *Suppose we have an A-graded anticommutative ring R (Definition 4.5) and two morphisms $f : (B, \varphi_B) \to (C, \varphi_C)$ and $g : (B, \varphi_B) \to (D, \varphi_D)$ in R-$\mathbf{GrCAlg}(A)$ (Definition 4.6). Then f and g make C and D both B-bimodules, respectively,[6] so we may form their tensor product $C \otimes_B D$, which is itself an A-graded B-bimodule (Proposition B.13). Then $C \otimes_B D$ canonically inherits the structure of an A-graded R-commutative ring with unit $1_C \otimes 1_D$ via a product*

$$(C \otimes_B D) \times (C \otimes_B D) \to C \otimes_B D$$

which sends a pair $(x \otimes y, x' \otimes y')$ of homogeneous pure tensors to the element

$$\varphi_B(\theta_{|x|,|y'|}) \cdot (xx' \otimes yy') = \varphi_C(\theta_{|x|,|y'|})xx' \otimes yy',$$

(where here $\cdot$ denotes the left module action of B on $C \otimes_B D$), and with structure map

$$\begin{aligned} \varphi : R &\to C \otimes_B D \\ r &\mapsto \varphi_B(r) \cdot (1_C \otimes 1_D) = (\varphi_C(r) \otimes 1_D) = (1_C \otimes \varphi_D(r)). \end{aligned}$$

Proof sketch. We simply lay out everything that needs to be shown, and we leave it to the reader to fill in the details. First to show that the indicated product is actually well-defined and distributive, by Lemma B.14 it suffices to show that for all homogeneous $c, c', c'' \in C$, $d, d', d'' \in D$, and $b \in B$ with $|c'| = |c''|$ and $|d'| = |d''|$, that

$$\begin{aligned}
\varphi_B(\theta_{|d|,|c'+c''|}) \cdot (c(c' + c'') \otimes dd') &= \varphi_B(\theta_{|d|,|c'|}) \cdot (cc' \otimes dd') + \varphi_B(\theta_{|d|,|c''|}) \cdot (cc'' \otimes dd') \\
\varphi_B(\theta_{|d|,|c'|}) \cdot (cc' \otimes d(d' + d'')) &= \varphi_B(\theta_{|d|,|c'|}) \cdot (cc' \otimes dd') + \varphi_B(\theta_{|d|,|c'|}) \cdot (cc' \otimes dd'') \\
\varphi_B(\theta_{|d|,|c' \cdot b|}) \cdot (c(c' \cdot b) \otimes dd') &= \varphi_B(\theta_{|d|,|c'|}) \cdot (cc' \otimes d(b \cdot d')) \\
\varphi_B(\theta_{|d'|,|c|}) \cdot ((c' + c'')c \otimes d'd) &= \varphi_B(\theta_{|d'|,|c|}) \cdot (c'c \otimes d'd) + \varphi_B(\theta_{|d'|,|c|}) \cdot (c''c \otimes d'd) \\
\varphi_B(\theta_{|d'+d''|,|c|}) \cdot (c'c \otimes (d' + d'')d) &= \varphi_B(\theta_{|d'|,|c|}) \cdot (c'c \otimes d'd) + \varphi_B(\theta_{|d''|,|c|}) \cdot (c'c \otimes d''d) \\
\varphi_B(\theta_{|d'|,|c|})((c' \cdot b)c \otimes d'd) &= \varphi_B(\theta_{|c|,|b \cdot d'|}) \cdot (c'c \otimes (b \cdot d')d),
\end{aligned}$$

where each occurrence of $\cdot$ denotes the left or right module action of B. These tell us that for all $x \in C \otimes_B D$ that the maps $C \otimes_B D \to C \otimes_B D$ sending $y \mapsto xy$ and $y \mapsto yx$ are well-defined A-graded homomorphisms of abelian groups, so we have a distributive product $(x, y) \mapsto xy$. Then to show that this product makes $C \otimes_B D$ an A-graded ring, we need to show it is associative and unital. By Lemma B.8, it suffices to show that for all *homogeneous* $x, y, z \in C \otimes_B D$ that $(xy)z = x(yz)$ and $x(1_C \otimes 1_D) = x = (1_C \otimes 1_D)x$. By distributivity, it further suffices to consider the case that x, y, and z are homogeneous *pure tensors* in $C \otimes_B D$, i.e., it suffices to show that for all homogeneous $c, c', c'' \in C$ and $d, d', d'' \in D$ that

$$((c \otimes d)(c' \otimes d'))(c'' \otimes d'') = (c \otimes d)((c' \otimes d')(c'' \otimes d''))$$

and

$$(c \otimes d)(1_C \otimes 1_D) = (c \otimes d) = (1_C \otimes 1_D)(c \otimes d).$$

Thus, proving these hold will show $C \otimes_B D$ has the structure of an A-graded ring, as desired. Now, we wish to show that the given map $\varphi : R \to C \otimes_B D$ is a ring homomorphism. Clearly it sends 1 to $1_C \otimes 1_D$, and again by linearity, it suffices to show that given *homogeneous* $r, s \in R$ that

$$\varphi(r + s) = \varphi_B(r + s)(1_C \otimes 1_D) = \varphi_B(r)(1_C \otimes 1_D) + \varphi_B(s)(1_C \otimes 1_D) = \varphi(r) + \varphi(s)$$

and

$$\varphi(rs) = \varphi_B(rs)(1_C \otimes 1_D) = (\varphi_B(r)(1_C \otimes 1_D))(\varphi_B(s)(1_C \otimes 1_D)) = \varphi(r)\varphi(s).$$

[6] Explicitly, it is a standard fact that given a ring homomorphism $\varphi : R \to S$ that S canonically becomes an R-bimodule with left action $r \cdot s := \varphi(r)s$ and right action $s \cdot r := s\varphi(r)$, so that in particular if φ is an A-graded homomorphism of A-graded rings, then φ makes S an A-graded R-bimodule.

Finally, we need to show that $C \otimes_B D$ satisfies the graded commutativity condition, for which again by linearity it suffices to show that given homogeneous $c, c' \in C$ and $d, d' \in D$ that

$$(c \otimes d)(c' \otimes d') = \varphi(\theta_{|c\otimes d|,|c'\otimes d'|})(c' \otimes d')(c \otimes d) = \varphi(\theta_{|c|+|d|,|c'|+|d'|})(c' \otimes d')(c \otimes d).$$

Showing all of these is relatively straightforward. □

Proposition B.22. *Given an A-graded anticommutative ring (R, θ), the category R-**GrCAlg**(A) has pushouts, where given $f : (B, \varphi_B) \to (C, \varphi_C)$ and $g : (B, \varphi_B) \to (D, \varphi_D)$, their pushout is the object $(C \otimes_B D, \varphi)$ constructed in Proposition B.21, along with the canonical maps $(C, \varphi_C) \to (C \otimes_B D, \varphi)$ sending $c \mapsto c \otimes 1_D$ and $(D, \varphi_D) \to (C \otimes_B D, \varphi)$ sending $d \mapsto 1_C \otimes d$. In particular, since (R, id_R) is initial, R-**GrCAlg**(A) has binary coproducts.*

Proof sketch. First, we need to show that the given maps $i_C : (C, \varphi_C) \to (C \otimes_B D, \varphi)$ and $i_D : (D, \varphi_D) \to (C \otimes_B D, \varphi)$ are actually morphisms in R-**GrCAlg**(A), i.e., that they are ring homomorphisms and that the following diagram commutes:

$$\begin{tikzcd} & R \arrow[dl, "\varphi_C"'] \arrow[d, dashed, "\varphi"] \arrow[dr, "\varphi_D"] & \\ C \arrow[r, "i_C"'] & C \otimes_B D & D \arrow[l, "i_D"] \end{tikzcd}$$

Showing this is entirely straightforward. Furthermore, i_C and i_D clearly make the following diagram commute:

$$\begin{tikzcd} B \arrow[r, "g"] \arrow[d, "f"'] & D \arrow[d, "i_D"] \\ C \arrow[r, "i_C"'] & C \otimes_B D \end{tikzcd}$$

It remains to show that i_C and i_D are the universal such arrows. Suppose we have some object (E, φ_E) in R-**GrCAlg**(A) and a commuting diagram

$$\begin{tikzcd} B \arrow[r, "g"] \arrow[d, "f"'] & D \arrow[d, "k"] \\ C \arrow[r, "h"] & E \end{tikzcd}$$

of morphisms in R-**GrCAlg**(A). Then we'd like to show there exists a unique morphism $\ell : C \otimes_B D \to E$ in R-**GrCAlg**(A) which makes the following diagram commute:

$$\begin{tikzcd} B \arrow[r, "g"] \arrow[d, "f"'] & D \arrow[d, "i_D"'] \arrow[ddr, bend left, "k"] & \\ C \arrow[r, "i_C"] \arrow[drr, bend right, "h"'] & C \otimes_B D \arrow[dr, dashed, "\ell"] & \\ & & E \end{tikzcd}$$

First we show uniqueness. Supposing such an arrow ℓ existed, given elements $c \in C$ and $d \in D$, we must have

$$\ell(c \otimes d) = \ell((c \otimes 1_D)(1_C \otimes d)) = \ell(c \otimes 1_D)\ell(1_C \otimes d) = \ell(i_C(c))\ell(i_D(d)) = h(c)k(d).$$

Since pure tensors generate $C \otimes_B D$, we have uniquely determined ℓ, and clearly it makes the above diagram commute. Now, it remains to show that as defined ℓ is a morphism in R-**GrCAlg**(A), i.e., that it is an A-graded ring homomorphism and that the following diagram commutes:

$$\begin{tikzcd} & R \arrow[dl, "\varphi"'] \arrow[dr, "\varphi_E"] & \\ C \otimes_B D \arrow[rr, "\ell"] & & E \end{tikzcd}$$

This is all entirely straightforward to show. □

Appendix C. Monoid objects

In this appendix, we fix a symmetric monoidal category $(\mathcal{C}, \otimes, S)$ with left unitor, right unitor, associator, and symmetry isomorphisms λ, ρ, α, and τ, respectively.

C.1. **Monoid objects in a symmetric monoidal category.**

Definition C.1. A *monoid object* (E, μ, e) is an object E in $\mathcal{C}$ along with a multiplication morphism $\mu : E \otimes E \to E$ and a unit map $e : S \to E$ such that the following diagrams commute:

$$\begin{tikzcd} E \otimes S \arrow[r, "E\otimes e"] \arrow[dr, "\rho_E"'] & E \otimes E \arrow[d, "\mu"] & S \otimes E \arrow[l, "e\otimes E"'] \arrow[dl, "\lambda_E"] \\ & E & \end{tikzcd} \qquad \begin{tikzcd} (E \otimes E) \otimes E \arrow[rr, "\mu\otimes E"] \arrow[d, "\alpha"'] & & E \otimes E \arrow[d, "\mu"] \\ E \otimes (E \otimes E) \arrow[r, "E\otimes\mu"] & E \otimes E \arrow[r, "\mu"] & E \end{tikzcd}$$

The first diagram expresses unitality, while the second expressed associativity. If in addition the following diagram commutes,

$$\begin{tikzcd} E \otimes E \arrow[rr, "\tau"] \arrow[dr, "\mu"'] & & E \otimes E \arrow[dl, "\mu"] \\ & E & \end{tikzcd}$$

then we say (E, μ, e) is a *commutative* monoid object.

Example C.2. The object S is a monoid object, with multiplication map $\rho_S = \lambda_S : S \otimes S \to S$ and unit $\mathrm{id}_S : S \to S$.

Definition C.3. Given two monoid objects (E_1, μ_1, e_1) and (E_2, μ_2, e_2) in a symmetric monoidal category $(\mathcal{C}, \otimes, S)$, a *monoid homomorphism* from E_1 to E_2 is a morphism $f : E_1 \to E_2$ in $\mathcal{C}$ such that the following diagrams commute:

$$\begin{tikzcd} E_1 \otimes E_1 \arrow[r, "f\otimes f"] \arrow[d, "\mu_1"'] & E_2 \otimes E_2 \arrow[d, "\mu_2"] \\ E_1 \arrow[r, "f"] & E_2 \end{tikzcd} \qquad \begin{tikzcd} & S \arrow[dl, "e_1"'] \arrow[dr, "e_2"] & \\ E_1 \arrow[rr, "f"] & & E_2 \end{tikzcd}$$

It is straightforward to show that id_{E_1} is a homomorphism of monoid objects from E_1 to itself, and that the composition of monoid homomorphisms is still a monoid homomorphism. Thus, we have categories $\mathbf{Mon}_{\mathcal{C}}$ and $\mathbf{CMon}_{\mathcal{C}}$ of monoid objects and commutative monoid objects in $\mathcal{C}$, respectively, with monoid homomorphisms between them.

Lemma C.4. *Given two monoid objects (E_1, μ_1, e_1) and (E_2, μ_2, e_2) in a symmetric monoidal category $(\mathcal{C}, \otimes, S)$, their tensor product $E_1 \otimes E_2$ canonically becomes a monoid object in $\mathcal{C}$ with unit map*

$$e : S \xrightarrow{\cong} S \otimes S \xrightarrow{e_1 \otimes e_2} E_1 \otimes E_2$$

and multiplication map

$$\mu : E_1 \otimes E_2 \otimes E_1 \otimes E_2 \xrightarrow{E_1 \otimes \tau \otimes E_2} E_1 \otimes E_1 \otimes E_2 \otimes E_2 \xrightarrow{\mu_1 \otimes \mu_2} E_1 \otimes E_2$$

(where here we are suppressing the associators from the notation). If in addition (E_1, μ_1, e_1) and (E_2, μ_2, e_2) are commutative *monoid objects, then $(E_1 \otimes E_2, \mu, e)$ is as well.*

todo

Proof. ☐

Lemma C.5. *Given monoid objects (E_i, μ_i, e_i) for $i = 1, 2, 3$ in a symmetric monoidal category $\mathcal{C}$, the associator $(E_1 \otimes E_2) \otimes E_3 \xrightarrow{\cong} E_1 \otimes (E_2 \otimes E_3)$ is an isomorphism of monoid objects. In other words, up to associativity, given a collection of monoid objects $E_1, \ldots, E_n$ in $\mathcal{C}$, there is no ambiguity when talking about their tensor product $E_1 \otimes \cdots \otimes E_n$ as a monoid object.*

Proof. Clearly, up to associativity, $(E_1 \otimes E_2) \otimes E_3$ and $E_1 \otimes (E_2 \otimes E_3)$ have the same unit map $S \xrightarrow{e_1 \otimes e_2 \otimes e_3} E_1 \otimes E_2 \otimes E_3$. Thus, it remains to show that they have the same product map, up to associativity. To see this, consider the following diagram, where we've passed to a symmetric strict monoidal category:

$$\begin{tikzcd}
E_1 \otimes (E_2 \otimes E_3) \otimes E_1 \otimes (E_2 \otimes E_3) \arrow[rrrr, equal, "\alpha"] \arrow[d, "E_1 \otimes \tau_{E_2 \otimes E_3, E_1} \otimes E_2 \otimes E_3"'] & & & & (E_1 \otimes E_2) \otimes E_3 \otimes (E_1 \otimes E_2) \otimes E_3 \arrow[d, "E_1 \otimes E_2 \otimes \tau_{E_3, E_1 \otimes E_2} \otimes E_3"] \\
E_1 \otimes E_1 \otimes E_2 \otimes E_3 \otimes E_2 \otimes E_3 \arrow[d, "\mu_1 \otimes E_2 \otimes \tau \otimes E_3"'] \arrow[drr, "E_1 \otimes E_1 \otimes E_2 \otimes \tau \otimes E_3"] & & & & E_1 \otimes E_2 \otimes E_1 \otimes E_2 \otimes E_3 \otimes E_3 \arrow[d, "E_1 \otimes \tau \otimes E_2 \otimes \mu_3"] \arrow[dll, "E_1 \otimes \tau \otimes E_2 \otimes E_3 \otimes E_3"'] \\
E_1 \otimes E_2 \otimes E_2 \otimes E_3 \otimes E_3 \arrow[d, "E_1 \otimes \mu_2 \otimes \mu_3"'] & & E_1 \otimes E_1 \otimes E_2 \otimes E_2 \otimes E_3 \otimes E_3 \arrow[ll, "\mu_1 \otimes E_2 \otimes E_2 \otimes E_3 \otimes E_3"] \arrow[rr, "E_1 \otimes E_2 \otimes E_2 \otimes E_2 \otimes \mu_3"'] \arrow[dll, "\mu_1 \otimes \mu_2 \otimes \mu_3"] \arrow[drr, "\mu_1 \otimes \mu_2 \otimes \mu_3"'] & & E_1 \otimes E_1 \otimes E_2 \otimes E_2 \otimes E_3 \arrow[d, "\mu_1 \otimes \mu_2 \otimes E_3"] \\
E_1 \otimes E_2 \otimes E_3 \arrow[rrrr, equal, "\alpha"] & & & & E_1 \otimes E_2 \otimes E_3
\end{tikzcd}$$

The top pentagonal region commutes by coherence for the τ's in a symmetric monoidal category. The bottom triangle commutes by definition. The remaining four triangles commute by functoriality of $- \otimes -$. On the left is the product for $E_1 \otimes (E_2 \otimes E_3)$, while on the right is the product for $(E_1 \otimes E_2) \otimes E_3$. Thus they are equal up to associativity, as desired. ☐

Lemma C.6. *Let (E, μ, e) be a monoid object in $\mathcal{SH}$. Then the maps $e : S \to E$ and $\mu : E \otimes E \to E$ are monoid object homomorphisms (where here S and $E \otimes E$ are considered to be monoid objects by Example C.2 and Lemma C.4, respectively).*

todo

Proof. ☐

Lemma C.7. *Suppose we have some monoid object (E, μ, e) in $\mathcal{C}$ and some homomorphism of monoid objects $f : (E_1, \mu_1, e_1) \to (E_2, \mu_2, e_2)$ in $\mathbf{Mon}_{\mathcal{C}}$. Then $E \otimes f : E \otimes E_1 \to E \otimes E_2$ and $f \otimes E : E_1 \otimes E \to E_2 \otimes E$ are monoid homomorphisms, where here we are considering $E \otimes E_1$, $E \otimes E_2$, $E_1 \otimes E$, and $E_2 \otimes E$ to be monoid objects by Lemma C.4.*

Proof. We will show that $E \otimes f$ is a monoid object homomorphism, as showing $f \otimes E$ is a monoid homomorphism is entirely analagous. First consider the following diagram:

$$\begin{tikzcd}
E \otimes E_1 \otimes E \otimes E_1 \arrow[rrr, "E\otimes f\otimes E\otimes f"] \arrow[d, "E\otimes\tau\otimes E_1"'] & & & E \otimes E_2 \otimes E \otimes E_2 \arrow[d, "E\otimes\tau\otimes E_2"] \\
E \otimes E \otimes E_1 \otimes E_1 \arrow[rrr, "E\otimes E\otimes f\otimes f"] \arrow[ddd, "\mu\otimes\mu_1"'] \arrow[dr, "\mu\otimes E_1\otimes E_2"] & & & E \otimes E \otimes E_2 \otimes E_2 \arrow[ddd, "\mu\otimes\mu_2"] \arrow[dl, "\mu\otimes E_2\otimes E_2"'] \\
& E \otimes E_1 \otimes E_1 \arrow[r, "E\otimes f\otimes f"] \arrow[ddl, "E\otimes\mu_1"'] & E \otimes E_2 \otimes E_2 \arrow[ddr, "E\otimes\mu_2"] & \\
& & & \\
E \otimes E_1 \arrow[rrr, "E\otimes f"] & & & E \otimes E_2
\end{tikzcd}$$

The top region commutes by naturality of τ. The bottom trapezoid commutes since f is a monoid homomorphism. The remaining three regions commute by functoriality of $- \otimes -$. Now, consider the following diagram:

$$\begin{tikzcd}
& S \arrow[ddl, "e\otimes e_1"'] \arrow[d, "e"] \arrow[ddr, "e\otimes e_2"] & \\
& E \arrow[dl, "E\otimes e_1"'] \arrow[dr, "E\otimes e_2"] & \\
E \otimes E_1 \arrow[rr, "E\otimes f"] & & E \otimes E_2
\end{tikzcd}$$

The bottom region commutes since f is a monoid homomorphism. The top two regions commute by functoriality of $- \otimes -$. Thus, we've shown $E \otimes f$ is a monoid object homomorphism, as desired. $\square$

C.2. **Modules over monoid objects in a symmetric monoidal category.**

Definition C.8. Let (E, μ, e) be a monoid object in $\mathcal{C}$. Then a *(left) module object* (N, κ) over (E, μ, e) is the data of an object N in $\mathcal{C}$ and a morphism $\kappa : E \otimes N \to N$ such that the following two diagrams commute in $\mathcal{C}$:

$$\begin{tikzcd}
S \otimes N \arrow[r, "e\otimes N"] \arrow[dr, "\lambda_N"'] & E \otimes N \arrow[d, "\kappa"] \\
& N
\end{tikzcd}
\qquad
\begin{tikzcd}
(E \otimes E) \otimes N \arrow[rr, "\mu\otimes N"] \arrow[d, "\alpha"'] & & E \otimes N \arrow[d, "\kappa"] \\
E \otimes (E \otimes N) \arrow[r, "E\otimes\kappa"] & E \otimes N \arrow[r, "\kappa"] & N
\end{tikzcd}$$

Definition C.9. Let (E, μ, e) be a monoid object in $\mathcal{C}$, and suppose we have two (left) module objects (N, κ) and (N', κ') over (E, μ, e). Then a morphism $f : N \to N'$ is a *(left) E-module homomorphism* if the following diagram commutes in $\mathcal{C}$:

$$\begin{tikzcd}
E \otimes N \arrow[r, "E\otimes f"] \arrow[d, "\kappa"'] & E \otimes N' \arrow[d, "\kappa'"] \\
N \arrow[r, "f"] & N'
\end{tikzcd}$$

Definition C.10. Given a monoid object (E, μ, e) in $\mathcal{C}$, we write E-**Mod** to denote the category of (left) module objects over E and E-module homomorphisms between them. We denote the homset in E-**Mod** by

$$\mathrm{Hom}_{E\text{-}\mathbf{Mod}}(M,N), \qquad \text{or simply} \qquad \mathrm{Hom}_E(M,N).$$

For our purposes, we will only consider left module objects, so we will usually drop the quanitfier "left" and just refer to them as "module objects".

Lemma C.11. *Let (E,μ,e) be a monoid object in $\mathcal{C}$ and let (N,κ) be an E module object. Then given some object X in $\mathcal{C}$ and an isomorphism $\phi : N \xrightarrow{\cong} X$, X inherits the structure of an E-module via the action map*

$$\kappa_\phi : E\otimes X \xrightarrow{E\otimes\phi^{-1}} E\otimes N \xrightarrow{\kappa} N \xrightarrow{\phi} X.$$

Proof. We need to show the two coherence diagrams in Definition C.8 commute. To see the former commutes, consider the following diagram:

$$\begin{tikzcd}
X \arrow[rr, "e\otimes X"] \arrow[dr, "\phi^{-1}"] \arrow[drrr, equal] & & E\otimes X \arrow[d, "E\otimes\phi^{-1}"] \\
& N \arrow[r, "e\otimes N"] \arrow[dr, equal] & E\otimes N \arrow[d, "\kappa"] \\
& & N \arrow[d, "\phi"] \\
& & X
\end{tikzcd}$$

The top trapezoid commutes by functoriality of $-\otimes-$. The middle small triangle commutes by unitality of κ. The remaining region commutes by definition. To see the second coherence diagram commutes, consider the following diagram:

$$\begin{tikzcd}
E\otimes E\otimes X \arrow[rrr, "\mu\otimes X"] \arrow[d, "E\otimes E\otimes\phi^{-1}"'] & & & E\otimes X \arrow[d, "E\otimes\phi^{-1}"] \\
E\otimes E\otimes N \arrow[rrr, "\mu\otimes N"] \arrow[d, "E\otimes\kappa"'] & & & E\otimes N \arrow[d, "\kappa"] \\
E\otimes N \arrow[rrr, "\kappa"] \arrow[d, "E\otimes\phi"'] \arrow[dr, equal] & & & N \arrow[d, "\phi"] \\
E\otimes X \arrow[r, "E\otimes\phi^{1-}"'] & E\otimes N \arrow[r, "\kappa"'] & N \arrow[r, "\phi"'] & X
\end{tikzcd}$$

The top rectangle commutes by functoriality of $-\otimes-$. The middle rectangle commutes by coherence for κ. The bottom two regions commute by definition. $\square$

Proposition C.12. *Given a monoid object (E,μ,e) in $\mathcal{C}$, the forgetful functor E-**Mod** $\to \mathcal{C}$ has a left adjoint $\mathcal{C} \to E$-**Mod** sending an object X in $\mathcal{C}$ to $(E\otimes X, \kappa_X)$ where κ_X is the composition*

$$E\otimes(E\otimes X) \xrightarrow{\alpha^{-1}} (E\otimes E)\otimes X \xrightarrow{\mu\otimes X} E\otimes X,$$

and sending a morphism $f : X \to Y$ to $E\otimes f : E\otimes X \to E\otimes Y$.

*We call this functor $E\otimes - : \mathcal{C} \to E$-**Mod** the* free *functor, and we call E-modules in the image of the free functor* free modules.

Proof. In this proof, we work in a symmetric strict monoidal category. First, we wish to show that $E \otimes - : \mathcal{C} \to E\text{-}\mathbf{Mod}$ as constructed is well-defined. First, to see that (X, κ_X) is actually a E-module, we need to show the two diagrams in Definition C.8 commute. Indeed, consider the following diagrams:

$$\begin{tikzcd} E \otimes X \arrow[r, "e\otimes E\otimes X"] \arrow[dr, equal] & E \otimes E \otimes X \arrow[d, "\mu\otimes X"] \\ & E \otimes X \end{tikzcd} \qquad \begin{tikzcd} E \otimes E \otimes E \otimes X \arrow[r, "\mu\otimes E\otimes X"] \arrow[d, "E\otimes\mu\otimes X"'] & E \otimes E \otimes X \arrow[d, "\mu\otimes X"] \\ E \otimes E \otimes X \arrow[r, "\mu\otimes X"'] & E \otimes X \end{tikzcd}$$

These are precisely the diagrams obtained by applying $X \otimes -$ to the coherence diagrams for μ, so that they commute as desired. Now, suppose $f : X \to Y$ is a morphism in $\mathcal{C}$, then we would like to show that $E \otimes f : E \otimes X \to E \otimes Y$ is a morphism of E-module objects. Indeed, consider the following diagram:

$$\begin{tikzcd} E \otimes E \otimes X \arrow[r, "E\otimes E\otimes f"] \arrow[d, "\mu\otimes X"'] & E \otimes E \otimes Y \arrow[d, "\mu\otimes Y"] \\ E \otimes X \arrow[r, "E\otimes f"] & E \otimes Y \end{tikzcd}$$

It commutes by functoriality of $-\otimes-$, so $E\otimes f$ is indeed an E-module homomorphism as desired.

Now, in order to see that $E \otimes -$ is left adjoint to the forgetful functor, it suffices to construct a unit and counit for the adjunction and show they satisfy the zig-zag identities. Given X in $\mathcal{C}$ and (N, κ) in E-**Mod**, define $\eta_X := e \otimes X : X \to E \otimes X$ and $\varepsilon_{(N,\kappa)} := \kappa : E \otimes N \to N$. η_X is clearly natural in X by functoriality of $- \otimes -$, and $\varepsilon_{(N,\kappa)}$ is natural in (N, κ) by how morphisms in E-**Mod** are defined. Now, to see these are actually the unit and counit of an adjunction, we need to show that the following diagrams commute for all X in $\mathcal{C}$ and (N, κ) in E-**Mod**:

$$\begin{tikzcd} E \otimes X \arrow[r, "E\otimes\eta_X=E\otimes e\otimes X"] \arrow[dr, equal] & E \otimes E \otimes X \arrow[d, "\varepsilon_{(E\otimes X,\kappa_X)}=\mu\otimes X"] \\ & E \otimes X \end{tikzcd} \qquad \begin{tikzcd} E \otimes N \arrow[d, "\varepsilon_{(N,\kappa)}=\kappa"'] & N \arrow[l, "\eta_N=e\otimes N"'] \arrow[dl, equal] \\ N & \end{tikzcd}$$

Commutativity of the left diagram is unitality of μ, while commutativity of the right diagram is unitality of κ. Thus indeed $E \otimes - : \mathcal{C} \to E\text{-}\mathbf{Mod}$ is a left adjoint of the forgetful functor $E\text{-}\mathbf{Mod} \to \mathcal{C}$, as desired. ☐

Lemma C.13. *Let (E, μ, e) be a monoid object in $\mathcal{C}$. Further suppose we have some object X in $\mathcal{C}$ and an E-module object (N, κ), along with a commuting diagram in $\mathcal{C}$*

$$\begin{tikzcd} X \arrow[r, "\iota"'] \arrow[rr, bend left, equal] & N \arrow[r, "r"'] & X \end{tikzcd}$$

Then if $\ell := \iota \circ r : N \to N$ is an E-module homomorphism, then X is canonically an E-module object with structure map

$$\kappa_X : E \otimes X \xrightarrow{E\otimes\iota} E \otimes N \xrightarrow{\kappa} N \xrightarrow{r} X,$$

and furthermore, the maps $\iota : X \to N$ and $r : N \to X$ are E-module homomorphisms.

Proof. First, in order to show (X, κ_X) is an E-module, we need to show the two diagrams in Definition C.8 commute. To see the unitality diagram holds, consider the following diagram:

$$\begin{tikzcd}
S\otimes X \arrow[rr, "e\otimes X"] \arrow[dr, "S\otimes\iota"] \arrow[ddd, "\lambda_X"'] & & E\otimes X \arrow[d, "E\otimes\iota"] \\
& S\otimes N \arrow[r, "e\otimes N"] \arrow[dr, "\lambda_N"] & E\otimes N \arrow[d, "\kappa"] \\
& & N \arrow[d, "r"] \\
X \arrow[urr, "\iota"] \arrow[rr, equal] & & X
\end{tikzcd}$$

The large left triangle commutes by naturality of λ. The top trapezoid commutes by functoriality of $-\otimes-$. The small middle right triangle commutes by unitality of κ. Finally, the bottom triangle commutes by definition, since we are assuming $r \circ \iota = \mathrm{id}_X$. Now the right composition is κ_X, so we have shown $\kappa_X \circ (e \otimes X) = \lambda_X$, as desired. Now, consider the following diagram:

$$\begin{tikzcd}
E\otimes E\otimes X \arrow[rrr, "\mu\otimes X"] \arrow[d, "E\otimes E\otimes\iota"'] \arrow[dr, "E\otimes E\otimes\iota"] & & & E\otimes X \arrow[d, "E\otimes\iota"] \\
E\otimes E\otimes N \arrow[r, "E\otimes E\otimes\ell"'] \arrow[d, "E\otimes\kappa"'] & E\otimes E\otimes N \arrow[rr, "\mu\otimes N"] \arrow[dd, "E\otimes\kappa"] & & E\otimes N \arrow[d, "\kappa"] \\
E\otimes N \arrow[d, "E\otimes r"'] \arrow[dr, "E\otimes\ell"] & & & N \arrow[d, "r"] \arrow[dl, equal] \\
E\otimes X \arrow[r, "E\otimes\iota"'] & E\otimes N \arrow[r, "\kappa"'] & N \arrow[r, "r"'] & X
\end{tikzcd}$$

The top trapezoid commutes by funtoriality of $-\otimes-$. The top left triangle commutes by functoriality of $-\otimes-$ and the fact that $\ell\circ\iota = \iota\circ r\circ\iota = \iota\circ \mathrm{id}_X = \iota$. The middle left trapezoid commutes by since ℓ is an E-module homomorphism, by assumption. The bottom left triangle commutes by functoriality of $-\otimes-$ and the fact that $\iota\circ r = \ell$. Thus, we have shown that (X, κ_X) is an E-module object, as desired.

Now, it remains to show that $\iota : X \to N$ and $r : N \to X$ are E-module homomorphisms. To that end, consider the following two diagrams:

$$\begin{tikzcd}
E\otimes X \arrow[r, "E\otimes\iota"] \arrow[d, "E\otimes\iota"'] & E\otimes N \arrow[ddd, "\kappa"] \\
E\otimes N \arrow[ur, "E\otimes\ell"] \arrow[d, "\kappa"'] & \\
N \arrow[d, "r"'] \arrow[dr, "\ell"] & \\
X \arrow[r, "\iota"'] & N
\end{tikzcd}
\qquad
\begin{tikzcd}
E\otimes N \arrow[r, "E\otimes r"] \arrow[dr, "E\otimes\ell"] \arrow[ddd, "\kappa"'] & E\otimes X \arrow[d, "E\otimes\iota"] \\
& E\otimes N \arrow[d, "\kappa"] \\
& N \arrow[d, "r"] \\
N \arrow[ur, "\ell"] \arrow[r, "r"] & X
\end{tikzcd}$$

The trapezoids in each diagram commute since we are assuming ℓ is a E-module homomorphism. The four triangles commute since $\ell\circ\iota = \iota$ and $r\circ\ell = r$. Thus, we have shown that $\kappa_X\circ(E\otimes r) = r\circ\kappa$ and $\kappa\circ(E\otimes\iota) = \iota\circ\kappa_X$, so we indeed have that ι and r are E-module homomorphisms, as desired. $\square$

Proposition C.14. *Suppose that $\mathcal{C}$ is an additive symmetric monoidal closed category. Let (E, μ, e) be a monoid object in $\mathcal{C}$, and suppose we have a family of E-module objects (N_i, κ_i) indexed by some small set I. Then $N := \bigoplus_{i \in I} N_i$ is canonically an E-module, with action map given by the composition*

$$\kappa : E \otimes \bigoplus_i N_i \xrightarrow{\cong} \bigoplus_i (E \otimes N_i) \xrightarrow{\bigoplus_i \kappa_i} \bigoplus_i N_i,$$

*where the first isomorphism is given by the fact that $E \otimes -$ preserves coproducts, since it is a left adjoint. Furthermore, N is the coproduct of all the N_i's in E-**Mod**, so that E-**Mod** has arbitrary coproducts.*

Proof. We need to show the action map κ makes the diagrams in Definition C.8 commute. To see the first (unitality) diagram commutes, consider the following diagram:

$$\begin{tikzcd}
\bigoplus_i N_i \arrow[r, "e \otimes \bigoplus_i N_i"] \arrow[dr, "\bigoplus_i (e \otimes N_i)"] \arrow[ddr, equal] & E \otimes \bigoplus_i N_i \arrow[d, "\cong"] \\
& \bigoplus_i (E \otimes N_i) \arrow[d, "\bigoplus_i \kappa_i"] \\
& \bigoplus_i N_i
\end{tikzcd}$$

The top triangle commutes since $E \otimes -$ preserves coproducts, as it is a left adjoint. The bottom triangle commutes by unitality of each of the κ_i's. To see the second coherence diagram commutes, consider the following diagram:

$$\begin{tikzcd}
E \otimes E \otimes \bigoplus_i N_i \arrow[rr, "\mu \oplus \bigoplus_i N_i"] \arrow[d, "E \otimes \cong"'] \arrow[dr, "\cong"] & & E \otimes \bigoplus_i N_i \arrow[d, "\cong"] \\
E \otimes \bigoplus_i (E \otimes N_i) \arrow[r, "\cong"] \arrow[d, "E \otimes \bigoplus_i \kappa_i"'] & \bigoplus_i (E \otimes E \otimes N_i) \arrow[r, "\bigoplus_i (\mu \otimes N_i)"] \arrow[d, "\bigoplus_i (E \otimes \kappa_i)"'] & \bigoplus_i (E \otimes N_i) \arrow[d, "\bigoplus_i \kappa_i"] \\
E \otimes \bigoplus_i N_i \arrow[r, "\cong"'] & \bigoplus_i (E \otimes N_i) \arrow[r, "\bigoplus_i \kappa_i"'] & \bigoplus_i N_i
\end{tikzcd}$$

The bottom right square commutes by coherence for the κ_i's. Every other region commutes since $- \otimes -$ preserves colimits in each variable. Thus $N = \bigoplus_i N_i$ is indeed an E-module object, as desired.

Now, we claim that (N, κ) is the coproduct of the (N_i, κ_i)'s in E-**Mod**. First, we need to show that the canonical maps $\iota_i : N_i \hookrightarrow N$ are morphisms in E-**Mod** for all $i \in I$. To see ι_i is a homomorphism of E-module objects, consider the following diagram:

$$\begin{tikzcd}
E \otimes N_i \arrow[r, "E \otimes \iota_i"] \arrow[dr, hook, "\iota_{E \otimes N_i}"] \arrow[dd, "\kappa_i"'] & E \otimes \bigoplus_i N_i \arrow[d, "\cong"] \\
& \bigoplus_i (E \otimes N_i) \arrow[d, "\bigoplus_i \kappa_i"] \\
N_i \arrow[r, hook, "\iota_i"'] & \bigoplus_i N_i
\end{tikzcd}$$

The top triangle commutes by additivity of $E \otimes -$. The bottom trapezoid commutes since, by univeral property of the coproduct, $\bigoplus_i \kappa_i$ is the unique arrow which makes the trapezoid commute for all $i \in I$. Now, it remains to show that given an E-module object (N', κ') and homomorphisms $f_i : N_i \to N'$ of E-module objects for all $i \in I$, that the unique arrow $f : N \to N'$ in $\mathcal{SH}$ satisfying $f \circ \iota_i = f_i$ for all $i \in I$ is a homomorphism of E-module objects, so that N is actually the coproduct of the N_i's. To see this, first let $h : \bigoplus_i (E \otimes N_i) \to E \otimes N'$ be the arrow determined by the maps $E \otimes N_i \xrightarrow{E \otimes f_i} E \otimes N'$. Then consider the following diagram:

$$\begin{tikzcd}
E \otimes \bigoplus_i N_i \arrow[rr, "E \otimes f"] \arrow[d, "\cong"'] & & E \otimes N' \arrow[ddd, "\kappa'"] \\
\bigoplus_i (E \otimes N_i) \arrow[r, "\bigoplus_i (E \otimes f_i)"'] \arrow[urr, "h"] \arrow[dd, "\bigoplus_i \kappa_i"'] & \bigoplus_i (E \otimes N') \arrow[ur, "\nabla"'] \arrow[d, "\bigoplus_i \kappa'"] & \\
& \bigoplus_i N' \arrow[dr, "\nabla"] & \\
\bigoplus_i N_i \arrow[ur, "\bigoplus_i f_i"] \arrow[rr, "f"'] & & N'
\end{tikzcd}$$

The top triangle commutes by additivity of $E \otimes -$. The triangle below that commutes by the universal property of the coproduct, since it is straightforward to check that $\nabla \circ \bigoplus_i (E \otimes f_i)$ and h both satisfy the universal property of the colimit. The left trapezoid commutes by functoriality of $- \oplus -$ and the fact that f_i is a homomorphism of E-module objects for all i in I. The right trapezoid commutes by naturality of ∇. Finally, the bottom triangle commutes by the universal product of the coproduct, by showing that $\nabla \circ \bigoplus_i f_i$ in place of f also satisfies the universal property of the colimit. Hence f is inded a homomorphism of E-module objects, as desired.

To recap, we have shown that given a set of E-module objects $\{(N_i, \kappa_i)\}_{i \in I}$, the inclusion maps $\iota_i : N_i \hookrightarrow \bigoplus_i N_i$ are morphisms in E-**Mod**, and that given morphisms $f_i : (N_i, \kappa_i) \to (N', \kappa')$ for all $i \in I$, the unique induced map $\bigoplus_i N_i \to N'$ is a morphism in E-**Mod**. Thus, E-**Mod** does indeed have arbitrary coproducts, and the forgetful functor E-**Mod** $\to \mathcal{SH}$ preserves them. $\square$

Proposition C.15. *Suppose that $\mathcal{C}$ is an additive closed symmetric monoidal category, and let (E, μ, e) be a monoid object in $\mathcal{C}$. Then E-**Mod** is itself an additive category, so that in particular the forgetful functor E-**Mod** $\to \mathcal{C}$ and the free functor $\mathcal{C} \to E$-**Mod** (Proposition C.12) are additive.*

Proof. It is a general fact that adjoint functors between additive categories are necessarily additive. In order to show E-**Mod** is an additive category, it suffices to show it has finite coproducts, that $\mathrm{Hom}_{E\text{-}\mathbf{Mod}}(N, N')$ is an abelian group for all E-modules N and N', and that composition is bilinear. We know that E-**Mod** has coproducts which are preserved by the forgetful functor E-**Mod** $\to \mathcal{C}$ by Proposition C.14 (which is clearly faithful). Thus, because $\mathcal{C}$ is **Ab**-enriched and $\mathrm{Hom}_{E\text{-}\mathbf{Mod}}(N, N') \subseteq \mathcal{C}(N, N')$, it suffices to show that $\mathrm{Hom}_{E\text{-}\mathbf{Mod}}(N, N')$ is closed under addition and taking inverses. To see the former, let $f, g : N \to N'$ be E-module homomorphisms, and consider the following diagram:

$$\begin{tikzcd}
E \otimes N \arrow[r, "E\otimes\Delta_N"] \arrow[dd, "\kappa"'] \arrow[dr, "\Delta_{E\otimes N}"'] & E \otimes (N \oplus N) \arrow[r, "E\otimes(f\oplus g)"] \arrow[d, "\cong"'] & E \otimes (N' \oplus N') \arrow[r, "E\otimes\nabla_{N'}"] \arrow[d, "\cong"] & E \otimes N' \arrow[dd, "\kappa'"] \\
 & (E \otimes N) \oplus (E \otimes N) \arrow[r, "(E\otimes f)\oplus(E\otimes g)"] \arrow[d, "\kappa\oplus\kappa"'] & (E \otimes N') \otimes (E \otimes N') \arrow[d, "\kappa'\oplus\kappa'"] \arrow[ur, "\nabla_{E\otimes N'}"'] & \\
N \arrow[r, "\Delta_N"] & N \oplus N \arrow[r, "f\oplus g"] & N' \oplus N' \arrow[r, "\nabla_{N'}"] & N'
\end{tikzcd}$$

The outermost trapezoids commute by naturality of Δ and ∇. The triangles in the top corners and the top middle rectangle commute by additivity of $E \otimes -$. Finally, the middle bottom rectangle commutes by functoriality of $- \oplus -$ and $- \otimes -$, and the fact that f and g are E-module homomorphisms. Commutativity of the above diagram shows that $f + g$ is a homomorphism of E-modules as desired. Finally, to see $-f$ is a E-module homomorphism if f is, we would like to show that $\kappa' \circ (E \otimes (-f)) = (-f) \circ \kappa$. This follows by the fact that $\kappa' \circ (E \otimes f) = f \circ \kappa$ and additivity of $- \otimes -$ and composition. ☐

Appendix D. Hopf algebroids

In this appendix, we will define the notion of *A-graded anticommutative Hopf algebroids* (Definition D.2) over an A-graded anticommutative ring R (Definition 4.5), and left comodules over them (Definition D.6).

D.1. **A-graded anticommutative Hopf algebroids over R.** Given an A-graded anticommutative ring R, we will define an A-graded anticommutative Hopf algebroid over R to be a cogroupoid object in R-$\mathbf{GCA}^A$, i.e., a groupoid object in $(R\text{-}\mathbf{GCA}^A)^{\mathrm{op}}$. First, recall the definition of a *groupoid object* in a category with pullbacks:

Definition D.1. Let $\mathcal{C}$ be a category with pullbacks. A *groupoid object* in $\mathcal{C}$ consists of a pair of objects (M, O) together with five morphisms

(1) *Source and target*: $s, t : M \to O$,
(2) *Identity*: $e : O \to M$,
(3) *Composition*: $c : M \times_O M \to M$,
(4) *Inverse*: $i : M \to M$

Where $M \times_O M$ will always refer to the object which into the following pullback diagram in $\mathcal{C}$:

$$\begin{tikzcd}
M \times_O M \arrow[r, "p_2"] \arrow[d, "p_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & M \arrow[d, "t"] \\
M \arrow[r, "s"'] & O
\end{tikzcd}$$

For example, if we're working in $\mathcal{C} = \mathbf{Set}$, we should think of M as a set of morphisms, and O as a set of objects. The functions s and t take a morphism to their domain and codomain, respectively, and $M \times_O M$ is the collection of pairs of morphisms $(g, f) \in M \times M$ such that $t(f) = s(g)$, and the composition map $c : M \times_O M \to M$ takes such a pair to the element $g \circ f \in M$. We think of the identity $e : O \to M$ as taking some object $x \in O$ to the identity morphism $e(x) = \mathrm{id}_x \in M$ on x, and the inverse map $i : M \to M$ takes a morphism f to its inverse f^{-1}. These data are required to make the following diagrams commute:

(1) Composition works correctly:

$$
\begin{tikzcd}
M \times_O M \arrow[r, "c"] \arrow[d, "p_1"'] & M \arrow[d, "t"] \\
M \arrow[r, "t"] & O
\end{tikzcd}
\qquad
\begin{tikzcd}
M \arrow[dr, "s"'] & O \arrow[l, "e"'] \arrow[r, "e"] \arrow[d, equal] & M \arrow[dl, "t"] \\
 & O &
\end{tikzcd}
\qquad
\begin{tikzcd}
M \times_O M \arrow[r, "p_2"] \arrow[d, "c"'] & M \arrow[d, "s"] \\
M \arrow[r, "s"] & O
\end{tikzcd}
$$

Expressed in terms of sets, the first diagram says that the target of $g \circ f$ is the target of g. The second diagram says that the domain and codomain of the identity on some object x is x. The third diagram says that the domain of $g \circ f$ is the domain of f.

(2) Associativity of composition: Write $M \times_O (M \times_O M)$ and $(M \times_O M) \times_O M$ for the pullbacks of $(s, t \circ c)$ and $(s \circ c, t)$, respectively, so we have commuting diagrams

$$
\begin{tikzcd}
(M \times_O M) \times_O M \arrow[rr, "p_2'"] \arrow[dr, dashed, "c \times M"'] \arrow[dd, "p_1'"'] & & M \arrow[d, equal] \\
 & M \times_O M \arrow[r, "p_2"] \arrow[d, "p_1"'] & M \arrow[d, "t"] \\
M \times_O M \arrow[r, "c"] & M \arrow[r, "s"] & O
\end{tikzcd}
\qquad
\begin{tikzcd}
M \times_O (M \times_O M) \arrow[rr, "p_2''"] \arrow[dr, dashed, "M \times c"'] \arrow[dd, "p_1''"'] & & M \times_O M \arrow[d, "c"] \\
 & M \times_O M \arrow[r, "p_2"] \arrow[d, "p_1"'] & M \arrow[d, "t"] \\
M \arrow[r, equal] & M \arrow[r, "s"] & O
\end{tikzcd}
$$

where the inner and outer squares in both diagrams are pullback squares. Furthermore, assuming the diagrams in condition (1) above are satisfied, we have that $t \circ p_1 \circ p_2'' = t \circ c \circ p_2'' = s \circ p_1''$, so that by the universal property of the pullback we have a map $M \times p_1 : M \times_O (M \times_O M) \to M \times_O M$ like so:

$$
\begin{tikzcd}
M \times_O (M \times_O M) \arrow[dr, dashed, "M \times p_1"] \arrow[drr, bend left, "p_1 \circ p_2''"] \arrow[ddr, bend right, "p_1''"'] & & \\
 & M \times_O M \arrow[r, "p_2"] \arrow[d, "p_1"'] & M \arrow[d, "t"] \\
 & M \arrow[r, "s"] & O
\end{tikzcd}
$$

Now note that again assuming the diagrams above in (1) commute, we have $s \circ c = s \circ p_2$, so that

$$
s \circ c \circ (M \times p_1) = s \circ p_2 \circ (M \times p_1) = s \circ p_1 \circ p_2'' = t \circ p_2 \circ p_2''.
$$

Then by the unviersal property of the pullback we get a map $a : M \times_O (M \times_O M) \to (M \times_O M) \times_O M$ like so:

$$
\begin{tikzcd}
M \times_O (M \times_O M) \arrow[dr, dashed, "a"] \arrow[drrr, bend left, "p_2 \circ p_2''"] \arrow[dddr, bend right, "M \times p_1"'] & & & \\
 & (M \times_O M) \times_O M \arrow[rr, "p_2'"] \arrow[dd, "p_1'"'] & & M \arrow[d, equal] \\
 & & & M \arrow[d, "t"] \\
 & M \times_O M \arrow[r, "c"] & M \arrow[r, "s"] & O
\end{tikzcd}
$$

Exercise: Show that this map a is an isomorphism. Then we require that the following diagram commutes:

$$\begin{tikzcd}
M \times_O (M \times_O M) \arrow[rr, "a"] \arrow[d, "M\times c"'] & & (M \times_O M) \times_O M \arrow[d, "c\times M"] \\
M \times_O M \arrow[r, "c"] & M & M \times_O M \arrow[l, "c"']
\end{tikzcd}$$

Expressed in terms of sets, this diagram says $h \circ (g \circ f) = (h \circ g) \circ f$.

(3) Unitality of composition: Given the maps $(\mathrm{id}_M, e \circ t), (e \circ s, \mathrm{id}_M) : M \to M \times_O M$ defined by the universal property of $M \times_O M$:

$$\begin{tikzcd}
M \arrow[dr, dashed, "(\mathrm{id}_M{,}e\circ s)"] \arrow[drr, bend left, "e\circ s"] \arrow[ddr, bend right, equal] & & \\
& M \times_O M \arrow[r, "p_2"] \arrow[d, "p_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & M \arrow[d, "t"] \\
& M \arrow[r, "s"] & O
\end{tikzcd}
\qquad
\begin{tikzcd}
M \arrow[dr, dashed, "(e\circ t{,}\mathrm{id}_M)"] \arrow[drr, bend left, equal] \arrow[ddr, bend right, "e\circ t"'] & & \\
& M \times_O M \arrow[r, "p_2"] \arrow[d, "p_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & M \arrow[d, "t"] \\
& M \arrow[r, "s"] & O
\end{tikzcd}$$

the following diagram commutes:

$$\begin{tikzcd}
M \arrow[r, "(e\circ t{,}\mathrm{id}_M)"] \arrow[d, "(\mathrm{id}_M{,}e\circ s)"'] \arrow[dr, equal] & M \times_O M \arrow[d, "c"] \\
M \times_O M \arrow[r, "c"'] & M
\end{tikzcd}$$

Expressed in terms of sets, this diagram says that given $f \in M$ with $s(f) = x$ and $t(f) = y$, that $f \circ \mathrm{id}_x = f$ and $\mathrm{id}_y \circ f = f$.

(4) Inverse: The following diagrams must commute:

$$\begin{tikzcd}
& M \arrow[dl, equal] \arrow[d, "i"] \\
M & M \arrow[l, "i"']
\end{tikzcd}
\qquad
\begin{tikzcd}
M \arrow[r, "(\mathrm{id}_M{,}i)"] \arrow[d, "t"'] & M \times_O M \arrow[d, "c"] & M \arrow[l, "(i{,}\mathrm{id}_M)"'] \arrow[d, "s"] \\
O \arrow[r, "e"] & M & O \arrow[l, "e"']
\end{tikzcd}
\qquad
\begin{tikzcd}
& M \arrow[dl, "s"'] \arrow[d, "i"] \arrow[dr, "t"] & \\
O & M \arrow[l, "t"'] \arrow[r, "s"] & O
\end{tikzcd}$$

where the arrows (id_M, i) and (i, id_M) are determined by the universal property of $M \times_O M$ like so:

$$\begin{tikzcd}
M \arrow[dr, dashed, "(\mathrm{id}_M{,}i)"] \arrow[drr, bend left, "i"] \arrow[ddr, bend right, equal] & & \\
& M \times_O M \arrow[r, "p_2"] \arrow[d, "p_1"'] & M \arrow[d, "t"] \\
& M \arrow[r, "s"] & O
\end{tikzcd}
\qquad
\begin{tikzcd}
M \arrow[dr, dashed, "(i{,}\mathrm{id}_M)"] \arrow[drr, bend left, equal] \arrow[ddr, bend right, "i"'] & & \\
& M \times_O M \arrow[r, "p_2"] \arrow[d, "p_1"'] & M \arrow[d, "t"] \\
& M \arrow[r, "s"] & O
\end{tikzcd}$$

Expressed in terms of sets, given $f \in M$ with $s(f) = x$ and $t(f) = y$, the first diagram says that $(f^{-1})^{-1} = f$, the second says that $f \circ f^{-1} = \mathrm{id}_y$ and $f^{-1} \circ f = \mathrm{id}_x$, and the last diagram says that the domain and codomain of f^{-1} are y and x, respectively.

It can be seen that groupoid objects in $\mathcal{C} = \mathbf{Set}$ are precisely (small) groupoids. Now, we can state and unravel the definition of a Hopf algebroid:

Definition D.2. Given an A-graded anticommutative ring R (Definition 4.5), an *A-graded anticommutative Hopf algebroid over R* is a co-groupoid object in R-$\mathbf{GCA}^A$, i.e., a groupoid object in $(R\text{-}A\mathbf{GrCAlg})^{\mathrm{op}}$. Explicitly, an A-graded anticommutative Hopf algebroid over E is a pair (Γ, B) of objects in R-$A\mathbf{GrCAlg}$ along with morphisms

(1) *left unit*: $\eta_L : B \to \Gamma$ (corresponding to t),
(2) *right unit*: $\eta_R : B \to \Gamma$ (corresponding to s),
(3) *comultiplication*: $\Psi : \Gamma \to \Gamma \otimes_B \Gamma$ (corresponding to c),
(4) *counit*: $\epsilon : \Gamma \to B$ (corresponding to e),
(5) *conjugation*: $c : \Gamma \to \Gamma$ (corresponding to i),

where here Γ may be viewed as a B-bimodule with left B-module structure induced by η_L and right B-module structure induced by η_R, so we may form the tensor product of bimodules $\Gamma \otimes_B \Gamma$, which further may be given the structure of an A-graded anticommutative R-algebra (by Proposition B.21), and fits into the following pushout diagram in R-$\mathbf{GCA}^A g$ (Proposition B.22):

$$\begin{tikzcd} B \arrow[r, "\eta_L"] \arrow[d, "\eta_R"'] & \Gamma \arrow[d, "g \mapsto 1 \otimes g"] \\ \Gamma \arrow[r, "g \mapsto g \otimes 1"'] & \Gamma \otimes_B \Gamma \end{tikzcd}$$

These data must make the following diagrams commute:

(1) (Composition works correctly)

$$\begin{tikzcd} B \arrow[r, "\eta_L"] \arrow[d, "\eta_L"'] & \Gamma \arrow[d, "\Psi"] \\ \Gamma \arrow[r, "g \mapsto g \otimes 1"'] & \Gamma \otimes_B \Gamma \end{tikzcd} \qquad \begin{tikzcd} & B \arrow[dl, "\eta_R"'] \arrow[d, equal] \arrow[dr, "\eta_L"] & \\ \Gamma \arrow[r, "\epsilon"'] & B & \Gamma \arrow[l, "\epsilon"] \end{tikzcd} \qquad \begin{tikzcd} B \arrow[r, "\eta_R"] \arrow[d, "\eta_R"'] & \Gamma \arrow[d, "g \mapsto 1 \otimes g"] \\ \Gamma \arrow[r, "\Psi"] & \Gamma \otimes_B \Gamma \end{tikzcd}$$

(2) (Coassociativity) The following diagram must commute

$$\begin{tikzcd} \Gamma \otimes_B \Gamma \arrow[d, "\Psi \otimes_B \Gamma"'] & \Gamma \arrow[l, "\Psi"'] \arrow[r, "\Psi"] & \Gamma \otimes_B \Gamma \arrow[d, "\Gamma \otimes_B \Psi"] \\ (\Gamma \otimes_B \Gamma) \otimes_B \Gamma \arrow[rr, "\cong"] & & \Gamma \otimes_B (\Gamma \otimes_B \Gamma) \end{tikzcd}$$

where $(\Gamma \otimes_B \Gamma) \otimes_B \Gamma$ and $\Gamma \otimes_B (\Gamma \otimes_B \Gamma)$ denote the rings which fit into the following pushout diagrams in R-$\mathbf{GCA}^A$:

$$\begin{tikzcd} B \arrow[r, "\eta_L"] \arrow[d, "\eta_R"'] & \Gamma \arrow[dd, "g \mapsto (1 \otimes 1) \otimes g"] \\ \Gamma \arrow[d, "\Psi"'] & \\ \Gamma \otimes_B \Gamma \arrow[r, "g \otimes g' \mapsto (g \otimes g') \otimes 1"'] & (\Gamma \otimes_B \Gamma) \otimes_B \Gamma \end{tikzcd} \qquad \begin{tikzcd} B \arrow[r, "\eta_L"] \arrow[d, "\eta_R"'] & \Gamma \arrow[r, "\Psi"] & \Gamma \otimes_B \Gamma \arrow[d, "(g \otimes g') \mapsto 1 \otimes (g \otimes g')"] \\ \Gamma \arrow[rr, "g \mapsto g \otimes (1 \otimes 1)"'] & & \Gamma \otimes_B (\Gamma \otimes_B \Gamma) \end{tikzcd}$$

and the isomorphism $(\Gamma \otimes_B \Gamma) \otimes_B \Gamma \to \Gamma \otimes_B (\Gamma \otimes_B \Gamma)$ sends $(g \otimes g') \otimes g''$ to $g \otimes (g' \otimes g'')$, the left vertical arrow $\Psi \otimes \Gamma$ sends $g \otimes g'$ to $\Psi(g) \otimes g$, and the right vertical arrow $\Gamma \otimes \Psi$ sends $g \otimes g'$ to $g \otimes \Psi(g')$.

(3) (Co-unitality):

$$\begin{tikzcd} \Gamma \arrow[r, "\Psi"] \arrow[d, "\Psi"'] \arrow[dr, equal] & \Gamma \otimes_B \Gamma \arrow[d, "(\eta_L \circ \epsilon)\cdot \mathrm{id}_\Gamma"] \\ \Gamma \otimes_B \Gamma \arrow[r, "\mathrm{id}_\Gamma \cdot (\eta_R \circ \epsilon)"'] & \Gamma \end{tikzcd}$$

where the right vertical arrow sends $g \otimes g'$ to $\eta_L(\epsilon(g))g'$ and the bottom horizontal arrow sends $g \otimes g'$ to $g\eta_R(\epsilon(g'))$.

(4) (Convolution):

$$\begin{tikzcd} & \Gamma \arrow[d, "c"] \arrow[dl, equal] \\ \Gamma & \Gamma \arrow[l, "c"] \end{tikzcd} \qquad \begin{tikzcd} B \arrow[d, "\eta_L"'] & \Gamma \arrow[l, "\epsilon"'] \arrow[r, "\epsilon"] \arrow[d, "i"'] & B \arrow[d, "\eta_R"] \\ \Gamma & \Gamma \otimes_B \Gamma \arrow[l, "\mathrm{id}_\Gamma \cdot c"] \arrow[r, "c \cdot \mathrm{id}_\Gamma"'] & \Gamma \end{tikzcd} \qquad \begin{tikzcd} B \arrow[r, "\eta_L"] \arrow[dr, "\eta_R"'] & \Gamma \arrow[d, "c"'] & B \arrow[l, "\eta_R"'] \arrow[dl, "\eta_L"] \\ & \Gamma & \end{tikzcd}$$

where the bottom left arrow in the middle diagram sends $g \otimes g'$ to $gc(g')$ and the bottom right arrow in the middle diagram sends $g \otimes g'$ to $c(g)g'$.

The remainder of this subsection is devoted to proving some technical lemmas about A-graded anticommutative Hopf algebroids.

Proposition D.3. *Suppose we have an A-graded anticommutative Hopf algebroid (Γ, B) over (R, θ) with structure maps η_L, η_R, Ψ, ϵ, and c (Definition D.2). Recall in the definition, we considered $\Gamma \otimes_B \Gamma$ to be the A-graded R-commutative ring whose underlying abelian group was given by the tensor product of B-bimodules, where Γ has left B-module structure induced by η_L and right B-module structure induced by η_R. Thus $\Gamma \otimes_B \Gamma$ is canonically a B-bimodule, as it is a tensor product of B-bimodules. Then the canonical left (resp. right) B-module structure on $\Gamma \otimes_B \Gamma$ coincides with that induced by the ring homomorphism $\Psi \circ \eta_L$ (resp. $\Psi \circ \eta_R$).*

Proof. First we show the left module structures coincide. By additivity, in order to show the module structures coincide, it suffices to show that given a homogeneous pure tensor $g \otimes g'$ in $\Gamma \otimes_B \Gamma$ and some $b \in B$ that $\Psi(\eta_L(b)) \cdot (g \otimes g') = (\eta_L(b) \cdot g) \otimes g'$, where $\cdot$ on the left denotes the product in $\Gamma \otimes_B \Gamma$ and the $\cdot$ on the right denotes the product in Γ. By the axioms for a Hopf algebroid, we have that $\Psi(\eta_L(b)) = \eta_L(b) \otimes 1$. Thus by how the product in $\Gamma \otimes_B \Gamma$ is defined (Proposition B.21), we have that

$$\Psi(\eta_L(b)) \cdot (g \otimes g') = (\eta_L(b) \otimes 1) \cdot (g \otimes g') = (\varphi_\Gamma(\theta_{0,|g|}) \cdot \eta_L(b) \cdot g) \otimes (g' \cdot 1) = (\eta_L(b) \cdot g) \otimes g',$$

where $\varphi_\Gamma : R \to \Gamma$ is the structure map, and the last equality follows by the fact that $\theta_{0,|g|} = 1$. An entirely analagous argument yields that the canonical right module structure on $\Gamma \otimes_B \Gamma$ coincides with that induced by $\Psi \circ \eta_R$, since $\Psi \circ \eta_R = 1 \otimes \eta_R$. □

Remark D.4. By the above proposition, given an A-graded commutative Hopf algebroid (Γ, B) over R, there is no ambiguity when discussing the objects $\Gamma \otimes_B (\Gamma \otimes_B \Gamma)$ and $(\Gamma \otimes_B \Gamma) \otimes_B \Gamma$ — they may both be considered as the threefold tensor product of the B-bimodule Γ with itself. In particular, we have a canonical isomorphism of B-bimodules

$$(\Gamma \otimes_B \Gamma) \otimes_B \Gamma \to \Gamma \otimes_B (\Gamma \otimes_B \Gamma)$$

sending $(g \otimes g') \otimes g''$ to $g \otimes (g' \otimes g'')$, and this is precisely the isomorphism in the coassociativity diagram in the definition of a Hopf algebroid (Definition D.2).

Proposition D.5. *Suppose we have an A-graded commutative Hopf algebroid (Γ, B) over R with structure maps η_L, η_R, Ψ, ϵ, and c. Then $\eta_L : B \to \Gamma$ is a homomorphism of left B-modules, $\eta_R : B \to \Gamma$ is a homomorphism of right B-modules, and $\Psi : \Gamma \to \Gamma \otimes_B \Gamma$ and $\epsilon : \Gamma \to B$ are homomorphisms of B-bimodules.*

Proof. Since the left (resp. right) B-module structure on Γ is induced by η_L (resp. η_R), the map η_L (resp. η_R) is a homomorphism of left (resp. right) B-modules by definition.

Next, we want to show Ψ is a homomorphism of B-bimodules. The left (resp. right) B-module structure on Γ is that induced by η_L (resp. η_R), and in Proposition D.3, we showed that the left (resp. right) B-module structure on $\Gamma \otimes_B \Gamma$ is that induced by $\Psi \circ \eta_L$ (resp. $\Psi \circ \eta_R$), so that by definition $\Psi : \Gamma \to \Gamma \otimes_B \Gamma$ is a homomorphism of left (resp. right) B-modules.

Lastly, we claim that $\epsilon : \Gamma \to B$ is a homomorphism of B-bimodules. We need to show that given $g \in \Gamma$ and $b, b' \in B$ that $\epsilon(\eta_L(b) g \eta_R(g')) = b\epsilon(g)b'$. This follows from the fact that ϵ is a ring homomorphism satisfying $\epsilon \circ \eta_L = \epsilon \circ \eta_R = \mathrm{id}_B$. ☐

D.2. **Comodules over a Hopf algebroid.** In what follows, fix an A-graded anticommutative ring (R, θ) and an A-graded anticommutative Hopf algebroid (Γ, B) over R with structure maps η_L, η_R, Ψ, ϵ, and c. We will always view Γ with its *canonical* B-bimodule structure, with left B-module structure induced by η_L, and right B-module structure induced by η_R. In particular, any tensor product over B involving Γ will always refer to Γ with this bimodule structure.

Definition D.6. A *left comodule over* Γ is a pair (N, Ψ_N), where N is a left A-graded B-module and $\Psi_N : N \to \Gamma \otimes_B N$ is an A-graded homomorphism of left A-graded B-modules. These data are required to make the following diagrams commute

$$\begin{tikzcd} N \arrow[r, "\Psi_N"] \arrow[dr, "\cong"'] & \Gamma \otimes_B N \arrow[d, "\epsilon \otimes N"] \\ & B \otimes_B N \end{tikzcd} \qquad \begin{tikzcd} \Gamma \otimes_B N \arrow[d, "\Psi \otimes N"'] & N \arrow[l, "\Psi_N"'] \arrow[r, "\Psi_N"] & \Gamma \otimes_B N \arrow[d, "\Gamma \otimes \Psi_N"] \\ (\Gamma \otimes_B \Gamma) \otimes_B N \arrow[rr, "\cong"] & & \Gamma \otimes_B (\Gamma \otimes_B N) \end{tikzcd}$$

The maps $\epsilon \otimes N$ and $\Psi \otimes N$ are well-defined by Proposition D.5, and the bottom isomorphism in the right diagram is the canonical one sending $(g \otimes g') \otimes n \mapsto g \otimes (g' \otimes n)$.

Given two left A-graded Γ-comodules (N_1, Ψ_{N_1}) and (N_2, Ψ_{N_2}), a homomorphism of left A-graded comodules $f : N_1 \to N_2$ is an A-graded homomorphism of the underlying left B-modules such that the following diagram commutes:

$$\begin{tikzcd} N_1 \arrow[r, "f"] \arrow[d, "\Psi_{N_1}"'] & N_2 \arrow[d, "\Psi_{N_2}"] \\ \Gamma \otimes_B N_1 \arrow[r, "\Gamma \otimes f"] & \Gamma \otimes_B N_2 \end{tikzcd}$$

We write Γ-$\mathbf{CoMod}^A$ for the resulting category of left A-graded comodules over Γ. In the above definition, we required A-graded left Γ-comodule homomorphisms to strictly preserve the grading, but we could have instead considered left Γ-comodule homomorphisms which are of degree d for some $d \in A$, or equivalently, the set of degree zero A-graded Γ-comodule homomorphisms from N_1 to the shifted comodule $(N_2)_{*+d}$. We denote the hom-set of degree-d A-graded left Γ-comodule homomorphisms from (N_1, Ψ_{N_1}) to (N_2, Ψ_{N_2}) by

$$\mathrm{Hom}^d_{\Gamma\text{-}\mathbf{CoMod}^A}(N_1, N_2) \qquad \text{or usually just} \qquad \mathrm{Hom}^d_{\Gamma}(N_1, N_2).$$

In particular, we simply write $\mathrm{Hom}_{\Gamma\text{-}\mathbf{CoMod}^A}(N_1, N_2)$ or $\mathrm{Hom}_{\Gamma}(N_1, N_2)$ for the set of strictly degree preserving (degree 0) A-graded left Γ-comodule homomorphisms from (N_1, Ψ_{N_1}) to (N_2, Ψ_{N_2}).

Proposition D.7. *The category* Γ-$\mathbf{CoMod}^A$ *is an additive category.*

Proof. First, we show the category is $\mathbf{Ab}$-enriched. Since the forgetful functor Γ-$\mathbf{CoMod}^A \to B$-$\mathbf{Mod}^A$ is clearly faithful, we may view hom-sets in Γ-$\mathbf{CoMod}^A$ as subsets of hom-groups in B-$\mathbf{Mod}^A$, so that in order to show Γ-$\mathbf{CoMod}^A$ is $\mathbf{Ab}$-enriched, it suffices to show that hom-sets in Γ-$\mathbf{CoMod}^A$ are closed under addition of module homomorphisms and taking inverses. To that end, suppose we have two A-graded left Γ-comodule homomorphisms $f, g : (N_1, \Psi_{N_1}) \to (N_2, \Psi_{N_2})$, then we have

$$\begin{aligned}\Psi_{N_2} \circ (f+g) &= (\Psi_{N_2} \circ f) + (\Psi_{N_2} \circ g) \\ &= ((\Gamma \otimes_B f) \circ \Psi_{N_1}) + ((\Gamma \otimes_B g) \circ \Psi_{N_1}) \\ &= ((\Gamma \otimes_B f) + (\Gamma \otimes_B g)) \circ \Psi_{N_1} \\ &= (\Gamma \otimes_B (f+g)) \circ \Psi_{N_1},\end{aligned}$$

where the first equality follows since Ψ_{N_2} is a homomorphism, the second follows since f and g are left Γ-comodule homomorphisms, the third follows since Ψ_{N_1} is a homomorphism, and the last equality follows by definition of the tensor product of modules. Hence $f+g$ is indeed an A-graded left Γ-comodule homomorphism, as desired. Now, we also claim $-f$ is an A-graded left Γ-comodule homomorphism. To that end, note that

$$\Psi_{N_2} \circ (-f) = -\Psi_{N_2} \circ f = -(\Gamma \otimes_B f) \circ \Psi_{N_1} = (\Gamma \otimes_B (-f)) \circ \Psi_{N_1},$$

where the first equality follows since Ψ_{N_2} is a homomorphism, the second follows since f is an A-graded left Γ-comodule homomorphism, and the third equality follows by definition of the tensor product.

Thus, we've shown that the hom-sets in Γ-$\mathbf{CoMod}^A$ are abelian groups, and composition is clearly bilinear, so that Γ-$\mathbf{CoMod}^A$ is indeed $\mathbf{Ab}$-enriched.

Now, in order to show Γ-$\mathbf{CoMod}^A$ is additive, it suffices to show that it contains a zero object and has binary coproducts. First of all, it is straightforward to check that the zero left B-module is clearly an A-graded left Γ-comodule with structure map the unique map $0 \to \Gamma \otimes_B 0 \cong 0$, and that given any other A-graded left Γ-comodule (N, Ψ_N), the unique homomorphisms of left B-modules $0 \to N$ and $N \to 0$ are left comodule homomorphisms.

Now, suppose we have two A-graded left Γ-comodules (N_1, Ψ_{N_1}) and (N_2, Ψ_{N_2}). First, we claim their direct sum as left B-modules $N_1 \oplus N_2$ is canonically an A-graded left Γ-comodule. We know that $N_1 \oplus N_2$ is an A-graded left B-module by Lemma B.11, and we can define the structure map

$$\Psi_{N_1 \oplus N_2} : N_1 \oplus N_2 \xrightarrow{\Psi_{N_1} \oplus \Psi_{N_2}} (\Gamma \otimes_B N_1) \oplus (\Gamma \otimes_B N_2) \cong \Gamma \otimes_B (N_1 \oplus N_2),$$

where the final isomorphism is the canonical one sending $(g_1 \otimes n_1) \oplus (g_2 \otimes n_2)$ to $(g_1 \otimes n_1) + (g_2 \otimes n_2)$. Then □ finish

Proposition D.8. *The forgetful functor* Γ-$\mathbf{CoMod}^A \to B$-$\mathbf{Mod}^A$ *(where here* B-$\mathbf{Mod}^A$ *is the category of* A*-graded left* B*-modules and degree-preserving module homomorphisms between them) has a right adjoint* $\Gamma \otimes_B - : B$-$\mathbf{Mod}^A \to \Gamma$-$\mathbf{CoMod}^A$ *called the* co-free construction*, where the co-free left* A*-graded* Γ*-comodule on a left* A*-graded* B*-module* M *is the* B*-module* $\Gamma \otimes_B M$ *equipped with the coaction*

$$\Psi_{\Gamma \otimes_B M} : \Gamma \otimes_B M \xrightarrow{\Psi \otimes_B M} (\Gamma \otimes_B \Gamma) \otimes_B M \xrightarrow{\cong} \Gamma \otimes_B (\Gamma \otimes_B M).$$

Explicitly, given some (N, Ψ_N) *in* Γ-**CoMod** *and some* M *in* B-$\mathbf{Mod}^A$*, the counit and unit of this adjunction are given by*

$$\eta_{(N,\Psi_N)} : N \xrightarrow{\Psi_N} \Gamma \otimes_B N$$

and

$$\varepsilon_M : \Gamma \otimes_B M \xrightarrow{\epsilon \otimes_B M} B \otimes_B M \xrightarrow{\cong} M.$$

todo

Proof. □

Proposition D.9. *Suppose that* Γ *is flat as a right* B*-module, i.e., suppose* $\eta_R : B \to \Gamma$ *is a flat ring homomorphism. Then the category* Γ-$\mathbf{CoMod}^A$ *is an abelian category.*

finish

Proof. □

Appendix E. Spectral sequences

In what follows, we fix an abelian group A. We will freely use the theory and results of **??**

Definition E.1. An *A-graded spectral sequence* $(E_r, d_r)_{r \geq r_0}$ is the data of:

- A collection of A-graded abelian groups $\{E_r^*\}_{r \geq r_0}$
- A collection of A-graded homomorphisms $d_r : E_r \to E_r$ for $r \geq r_0$ (of possibly nonzero degree) such that $d_r \circ d_r = 0$
- For each $r \geq r_0$, an A-graded isomorphism $E_{r+1} \cong \ker d_r / \operatorname{im} d_r$ of degree 0 (where $\ker d_r$ and $\operatorname{im} d_r$ are canonically A-graded by Proposition B.18, and their quotient is canonically A-graded by Proposition B.20).

Typically we call a $\mathbb{Z}^2$-graded spectral sequence a *bigraded* spectral sequence, and a $\mathbb{Z}^3$-graded spectral sequence is a *trigraded* spectral sequence.

E.1. Unrolled exact couples and their associated spectral sequences.

For our purposes, we will only care about spectral sequences which arise from *A-graded unrolled exact couples*. In what follows, we follow [1], with minor modifications for our setting, in which everything is A-graded.

Definition E.2. An A-graded *unrolled exact couple* $(D, E; i, j, k)$ is a diagram of A-graded abelian groups and A-graded homomorphisms (of possibly non-zero degree)

$$\begin{tikzcd}
\cdots \arrow[r] & D^{s+2} \arrow[r, "i"] \arrow[d, "j"] & D^{s+1} \arrow[r, "i"] \arrow[d, "j"] & D^s \arrow[r, "i"] \arrow[d, "j"] & D^{s-1} \arrow[r] \arrow[d, "j"] & \cdots \\
& E^{s+2} & E^{s+1} \arrow[ul, "k"] & E^s \arrow[ul, "k"] & E^{s-1} \arrow[ul, "k"] &
\end{tikzcd}$$

in which each triangle $D^{s+1} \xrightarrow{i} D^s \xrightarrow{j} E_s \xrightarrow{k} D^{s+1}$ is an exact sequence. We require that each occurrence of i (resp. j, k) is of the same degree. In other words, an unrolled exact couple can be described as a tuple $(D, E; i, j, k)$ of $\mathbb{Z} \times A$-graded abelian groups and $\mathbb{Z} \times A$-graded maps $i : D \to D$, $j : D \to E$, and $k : E \to D$, such that the $\mathbb{Z}$-degrees of i, j, and k are -1, 0, and 1, respectively. Usually i and one of j or k will be of A-degree 0.

Given an A-graded unrolled exact couple $(D, E; i, j, k)$, we may define an associated $\mathbb{Z} \times A$-graded spectral sequence as follows: Given some $s \in \mathbb{Z}$ and some $r \geq 1$, we first define the following subgroups of E_s:

$$Z_r^s := k^{-1}(\operatorname{im}[i^{r-1} : D^{s+r} \to D^{s+1}]) \qquad \text{and} \qquad B_r^s := j(\ker[i^{r-1} : D^s \to D^{s-r+1}])$$

where we adopt the convention that i^0 is simply the identity. These are furthermore A-graded subgrous of E_s (by Proposition B.18 and Proposition B.19). In this way, for each $s \in \mathbb{Z}$, we get an infinite sequence of A-graded subgroups:

$$0 = B_1^s \subseteq B_2^s \subseteq B_3^s \subseteq \cdots \subseteq \operatorname{im} j = \ker k \subseteq \cdots \subseteq Z_3^s \subseteq Z_2^s \subseteq Z_1^s = E^s.$$

Now, for each $s \in \mathbb{Z}$ and $r \geq 1$, we define the A-graded abelian group

$$E_r^s := Z_r^s / B_r^s,$$

so that in particular $E_1^s = E^s$ for all $s \in \mathbb{Z}$, as $Z_1^s = k^{-1}(D^{s+1}) = E^s$ and $B_1^s = j(\ker \mathrm{id}_{D^s}) = j(0) = 0$. Now we can define differentials $d_r^s : E_r^s \to E_r^{s+r}$ to be the composition

$$E_r^s = Z_r^s / B_r^s \xrightarrow{k} \operatorname{im}[i^{r-1} : D^{s+r} \to D^{s+1}] \xrightarrow{j \circ i^{-(r-1)}} Z_r^{s+r} / B_r^{s+r} = E_r^{s+r},$$

where given some $e \in Z_r^s = k^{-1}(\operatorname{im} i^{r-1})$, the first arrow takes a class $[e] \in E_r^s$ represented by some $e \in Z_r^s$ to the element $k(e)$, which lives in $\operatorname{im} i^{r-1}$ by definition, and the second arrow takes $i^{r-1}(d)$ to the class $[j(d)]$. Note the first map is well-defined, as given $b \in B_r^s = j(\ker[i^{r-1}])$, we have $k(b) = 0$, as $b \in \operatorname{im} j = \ker k$. To see the second map is well-defined, first note that given $d \in D^{s+r}$, that

$$k(j(d)) = 0 \in \operatorname{im}[i^{r-1} : D^{s+2r} \to D^{s+r+1}],$$

so that

$$j(d) \in k^{-1}(\operatorname{im}[i^{r-1} : D^{s+2r} \to D^{s+r+1}]) = Z_r^{s+r},$$

as desired, so that given $d \in D^{s+r}$, $j(d) in Z_r^{s+r}$, so it makes sense to discuss the class $[j(d)] \in Z_r^{s+r} / B_r^{s+r} = E_r^{s+r}$. Secondly, if $i^{r-1}(d) = i^{r-1}(d')$ for some $d, d' \in D^{s+r}$, then

$$j(d) - j(d') = j(d - d') \in j(\ker[i^{r-1} : D^{s+r} \to D^{s+1}]) = B_r^{s+r},$$

so that $[j(d)] = [j(d')]$ in E_r^{s+r}, as desired. It is straightforward to check that these maps are also A-graded homomorphisms, so that by unravelling definitions d_r^s is an A-graded homomorphism of degree $\deg k - (r-1) \cdot \deg i + \deg j$ (so that in the standard case $\deg i = 0$, d_r^s simply has degree $\deg k + \deg j$).

These differentials square to zero, in the sense that for each $s \in \mathbb{Z}$ and $r \geq 1$ we have that $d_r^{s+r} \circ d_r^s : E_r^s \to E_r^{s+2r}$ is the zero map. Indeed, suppose we are given some class $[e] \in E_r^s$ represented by an element $e \in E^s$, so $k(e) = i^{r-1}(d)$ for some $d \in D^{s+r}$. Then

$$d_r^{s+r}(d_r^s([e])) = d_r^{s+r}([j(d)]) = [j(i^{-(r-1)}(k(j(d))))] = [j(i^{-(r-1)}(0))] = 0,$$

where the second-to-last equality follows by the fact that $k \circ j = 0$. Note that by unravelling definitions, $d_1^s = j \circ k$.

We claim that $\ker d_r^s = Z_{r+1}^s / B_r^s$. First of all, let $[e] \in E_r^s = Z_r^s / B_r^s$, so that $[e]$ is represented by some $e \in E^s$ with $k(e) = i^{r-1}(d)$ for some $d \in D^{s+r}$. Then if $[e] \in \ker d_r^s$, by definition this means $j(d) \in B_r^{s+r} = j(\ker[i^{r-1} : D^{s+r} \to D^{s+1}])$, so $j(d) = j(d')$ for some $d' \in D^{s+r}$ with $i^{r-1}(d') = 0$. Thus $d - d' \in \ker j = \operatorname{im} i$, so there exists some $d'' \in D^{s+r+1}$ such that $i(d'') = d - d'$. Then

$$k(e) = i^{r-1}(d) = i^{r-1}(i(d'') + d') = i^r(d'') + i^{r-1}(d'),$$

but since $i^{r-1}(d') = 0$, we have $k(e) \in \operatorname{im}[i^r : D^{s+r+1} \to D^{s+1}]$, so that $e \in Z_{r+1}^s$, meaning $[e] \in Z_{r+1}^s / B_r^s$, as desired. On the other hand, suppose we are given some class $[e] \in Z_{r+1}^s / B_r^s$, represented by $e \in Z_{r+1}^s$ with $k(e) \in \operatorname{im}[i^r : D^{s+r+1} \to D^{s+1}]$. Then if we write $k(e) = i^r(d) = i^{r-1}(i(d))$, then $d_r^s([e]) = [j(i(d))] = 0$ (since $j \circ i = 0$), as asserted.

Finally, we claim that the image of $d_r^{s-r} : E_r^{s-r} \to E_r^s$ is B_{r+1}^s / B_r^s. First, let $e \in Z_r^{s-r}$, so $k(e) = i^{r-1}(d)$ for some $d \in D^s$. Then we'd like to show that $d_r^s([e]) = [j(d)]$ belongs to B_{r+1}^s / B_r^s. It suffices to show that $d \in \ker[i^r : D^s \to D^{s-r}]$. To see this, note that

$$i^r(d) = i(i^{r-1}(d)) = i(k(e)) = 0,$$

since $i \circ k = 0$. Hence we've shown $\operatorname{im} d_r^{s-r} \subseteq B_{r+1}^s / B_r^s$. Conversely, let $j(d) \in B_{r+1}^s$, so $d \in D^s$ and $i^r(d) = 0$. Then we'd like to show that $[j(d)] \in B_{r+1}^s / B_r^s$ is in the image of d_r^{s-r}. To see this, note that

$$i^r(d) = 0 \implies i^{r-1}(d) \in \ker i = \operatorname{im} k,$$

so there exists some $e \in E^{s-r}$ such that $k(e) = i^{r-1}(d)$, so $e \in Z_r^{s-r}$. Unravelling definitions, it follows that $d_r^{s-r}([e]) = [j(d)]$, so $[j(d)]$ is indeed in the image of d_r^{s-r}, as desired.

To recap, we have constructed for each $s \in \mathbb{Z}$ and $r \geq 1$ an A-graded abelian group E_r^s along with differentials $d_r^s : E_r^s \to E_r^{s+r}$. Furthermore, if we take homology in the middle term of the following sequence

$$E_r^{s-r} \xrightarrow{d_r^{s-r}} E_r^s \xrightarrow{d_r^s} E_r^{s+r},$$

we get

$$\ker d_r^s / \operatorname{im} d_r^{s-r} = \frac{Z_{r+1}^s / B_r^s}{B_{r+1}^s / B_r^s} \cong Z_{r+1}^s / B_{r+1}^s = E_{r+1}^s.$$

Thus, we get a spectral sequence:

Proposition E.3. *We may associate a $\mathbb{Z} \times A$-graded spectral sequence $r \mapsto (E_r, d_r)$ to the A-graded unrolled exact couple $(D, E; i, j, k)$ by defining $E_r := \bigoplus_{s \in \mathbb{Z}} E_r^s$ and the differentials*

$$d_r : E_r \to E_r$$

are those constructed above, which have $\mathbb{Z} \times A$-degree $(r, \deg j - (r-1) \cdot \deg i + \deg k)$.

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