

0.1. **Setup of \mathcal{SH} .** In order to construct an abstract version of the Adams spectral sequence, we need to work in some axiomatic version of a stable homotopy category \mathcal{SH} which acts like the familiar classical stable homotopy category **hoSp** (??) or the motivic stable homotopy category **SH** _{\mathcal{S}} over some base scheme \mathcal{S} (??).

Definition 0.1. Let \mathcal{C} be an additive category with arbitrary (small) coproducts. Then an object X in \mathcal{C} is *compact* if, for any collection of objects Y_i in \mathcal{C} indexed by some (small) set I , the canonical map

$$\bigoplus_i \mathcal{C}(X, Y_i) \rightarrow \mathcal{C}(X, \bigoplus_i Y_i)$$

is an isomorphism of abelian groups. (Explicitly, the above map takes a generator $x \in \mathcal{C}(X, Y_i)$ to the composition $X \xrightarrow{x} Y_i \hookrightarrow \bigoplus_i Y_i$.)

Definition 0.2. Given a tensor triangulated category $(\mathcal{C}, \otimes, S, \Sigma, e, \mathcal{D})$ (??), a *sub-Picard grading* on \mathcal{C} is the following data:

- A pointed abelian group $(A, \mathbf{1})$ along with a homomorphism of pointed groups $h : (A, \mathbf{1}) \rightarrow (\text{Pic } \mathcal{C}, \Sigma S)$, where $\text{Pic } \mathcal{C}$ is the *Picard group* of isomorphism classes of invertible objects in \mathcal{C} .¹
- For each $a \in A$, a chosen representative S^a in the isomorphism class $h(a)$ such that each S^a is a compact object (Definition 0.1) and $S^0 = S$.
- For each $a, b \in A$, an isomorphism $\phi_{a,b} : S^{a+b} \rightarrow S^a \otimes S^b$. This family of isomorphisms is required to be *coherent*, in the following sense:
 - For all $a \in A$, we must have that $\phi_{a,0}$ coincides with the right unitor $S^a \xrightarrow{\cong} S^a \otimes S$ and $\phi_{0,a}$ coincides the left unitor $S^a \xrightarrow{\cong} S \otimes S^a$.
 - For all $a, b, c \in A$, the following “associativity diagram” must commute:

$$\begin{array}{ccc} S^{a+b} \otimes S^c & \xleftarrow{\phi_{a+b,c}} & S^{a+b+c} \xrightarrow{\phi_{a,b+c}} S^a \otimes S^{b+c} \\ \phi_{a,b} \otimes S^c \downarrow & & \downarrow S^a \otimes \phi_{b,c} \\ (S^a \otimes S^b) \otimes S^c & \xrightarrow{\cong} & S^a \otimes (S^b \otimes S^c) \end{array}$$

From now on we fix a monoidal closed tensor triangulated category $(\mathcal{SH}, \otimes, S, \Sigma, e, \mathcal{D})$ with arbitrary (small) (co)products and sub-Picard grading $(A, \mathbf{1}, h, \{S^a\}, \{\phi_{a,b}\})$. We also fix an isomorphism $\nu : \Sigma S \xrightarrow{\cong} S^1$ once and for all. We establish conventions. First, observe the following remark:

Remark 0.3. Note that by induction the coherence conditions for the $\phi_{a,b}$ ’s in the above definition say that given any $a_1, \dots, a_n \in A$ and $b_1, \dots, b_m \in A$ such that $a_1 + \dots + a_n = b_1 + \dots + b_m$ and any fixed parenthesizations of $X = S^{a_1} \otimes \dots \otimes S^{a_n}$ and $Y = S^{b_1} \otimes \dots \otimes S^{b_m}$, there is a *unique* isomorphism $X \rightarrow Y$ that can be obtained by forming formal compositions of products of $\phi_{a,b}$, identities, associators, unitors, and their inverses (but not symmetries).

In light of this remark, we will usually simply write ϕ or even just \cong for any isomorphism that is built by taking compositions of products of $\phi_{a,b}$ ’s, unitors, associators, identities, and their

¹Recall an object X is a symmetric monoidal category is *invertible* if there exists some object Y and an isomorphism $S \cong X \otimes Y$.

inverses. Given an object X and a natural number $n > 0$, we write

$$X^n := \overbrace{X \otimes \cdots \otimes X}^{n \text{ times}} \quad \text{and} \quad X^0 := S.$$

We denote the associator, symmetry, left unitor, and right unitor isomorphisms in \mathcal{SH} by

$$\begin{aligned} \alpha_{X,Y,Z} : (X \otimes Y) \otimes Z &\xrightarrow{\cong} X \otimes (Y \otimes Z) & \tau_{X,Y} : X \otimes Y &\xrightarrow{\cong} Y \otimes X \\ \lambda_X : S \otimes X &\xrightarrow{\cong} X & \rho_X : X \otimes S &\xrightarrow{\cong} X. \end{aligned}$$

Often we will drop the subscripts. Furthermore, by the coherence theorem for symmetric monoidal categories ([1]), we will often assume α , ρ , and λ are actual equalities.

Given some integer $n \in \mathbb{Z}$, we will write a bold \mathbf{n} to denote the element $n \cdot \mathbf{1}$ in A . Note that we can use the isomorphism $\nu : \Sigma S \xrightarrow{\cong} S^1$ to construct a natural isomorphism $\Sigma \cong S^1 \otimes -$:

$$\Sigma X \xrightarrow{\Sigma \lambda_X^{-1}} \Sigma(S \otimes X) \xrightarrow{e_{S,X}^{-1}} \Sigma S \otimes X \xrightarrow{\nu \otimes X} S^1 \otimes X.$$

The first two arrows are natural in X by definition. The last arrow is natural in X by functoriality of $- \otimes -$. By abuse of notation, we will also use ν to denote this natural isomorphism.

Given some $a \in A$, we define $\Sigma^a := S^a \otimes -$ and $\Omega^a := \Sigma^{-a} = S^{-a} \otimes -$. We specifically define $\Omega := \Omega^1$. We say “the a^{th} suspension of X ” to denote $\Sigma^a X$. It turns out that Σ^a is an autoequivalence of \mathcal{SH} for each $a \in A$, and furthermore, Ω^a and Σ^a form an adjoint equivalence of \mathcal{SH} for all a in A :

Proposition 0.4. *For each $a \in A$, the isomorphisms*

$$\eta_X^a : X \xrightarrow{\lambda_X^{-1}} S \otimes X \xrightarrow{\phi_{a,-a} \otimes X} (S^a \otimes S^{-a}) \otimes X \xrightarrow{\alpha} S^a \otimes (S^{-a} \otimes X) = \Sigma^a \Omega^a X$$

and

$$\varepsilon_X^a : \Omega^a \Sigma^a X = S^{-a} \otimes (S^a \otimes X) \xrightarrow{\alpha^{-1}} (S^{-a} \otimes S^a) \otimes X \xrightarrow{\phi_{-a,a}^{-1} \otimes X} S \otimes X \xrightarrow{\lambda_X} X$$

are natural in X , and furthermore, they are the unit and counit respectively of the adjoint autoequivalence $(\Omega^a, \Sigma^a, \eta^a, \varepsilon^a)$ of \mathcal{SH} . In particular, since $\Sigma \cong \Sigma^1$, $\Omega := \Omega^1$ is a left adjoint for Σ , so that $(\mathcal{SH}, \Omega, \Sigma, \eta, \varepsilon, \mathcal{D})$ is an adjointly triangulated category (??), where η and ε are the compositions

$$\eta : \text{Id}_{\mathcal{SH}} \xrightarrow{\eta^1} \Sigma^1 \Omega \xrightarrow{\nu^{-1} \Omega} \Sigma \Omega \quad \text{and} \quad \varepsilon : \Omega \Sigma \xrightarrow{\Omega \nu} \Omega \Sigma^1 \xrightarrow{\varepsilon^1} \text{Id}_{\mathcal{SH}}.$$

Proof. In this proof, we will freely employ the coherence theorem for monoidal categories (see [1]), which essentially tells us that we may assume we are working in a strict monoidal category (i.e., that the associators and unitors are identities). Then η_X^a and ε_X^a become simply the maps

$$\eta_X^a : X \xrightarrow{\phi_{a,-a} \otimes X} S^a \otimes S^{-a} \otimes X \quad \text{and} \quad \varepsilon_X^a : S^{-a} \otimes S^a \otimes X \xrightarrow{\phi_{-a,a}^{-1} \otimes X} X.$$

That these maps are natural in X follows by functoriality of $- \otimes -$. Now, recall that in order to show that these natural isomorphisms form an adjoint equivalence, it suffices to show that the natural isomorphisms $\eta^a : \text{Id}_{\mathcal{SH}} \Rightarrow \Omega^a \Sigma^a$ and $\varepsilon^a : \Sigma^a \Omega^a \Rightarrow \text{Id}_{\mathcal{SH}}$ satisfy one of the two zig-zag identities:

$$\begin{array}{ccc} \Omega^a & \xrightarrow{\Omega^a \eta^a} & \Omega^a \Sigma^a \Omega^a \\ & \searrow & \downarrow \varepsilon^a \Omega^a \\ & & \Omega^a \end{array} \quad \begin{array}{ccc} \Sigma^a \Omega^a \Sigma^a & \xleftarrow{\eta^a \Sigma^a} & \Sigma^a \\ \Sigma^a \varepsilon^a \downarrow & & \uparrow \\ \Sigma^a & & \end{array}$$

(that it suffices to show only one is [2, Lemma 3.2]). We will show that the left is satisfied. Unravelling definitions, we simply wish to show that the following diagram commutes for all X in \mathcal{SH} :

$$\begin{array}{ccc} S^{-a} \otimes X & \xrightarrow{S^{-a} \otimes \phi_{a,-a} \otimes X} & S^{-a} \otimes S^a \otimes S^{-a} \otimes X \\ & \searrow & \downarrow \phi_{-a,a}^{-1} \otimes S^{-a} \otimes X \\ & & S^{-a} \otimes X \end{array}$$

Yet this is simply the diagram obtained by applying $-\otimes X$ to the associativity coherence diagram for the $\phi_{a,b}$'s (since $\phi_{a,0}$ and $\phi_{0,a}$ coincide with the unitors, and here we are taking the unitors and associators to be equalities), so it does commute, as desired. \square

Given two objects X and Y in \mathcal{SH} , we will denote the hom-abelian group of morphisms from X to Y in \mathcal{SH} by $[X, Y]$, and the internal hom object by $F(X, Y)$. We can extend the abelian group $[X, Y]$ into an A -graded abelian group $[X, Y]_*$ by defining $[X, Y]_a := [S^a \otimes X, Y]$. Given an object X in \mathcal{SH} and some $a \in A$, we can define the abelian group

$$\pi_a(X) := [S^a, X],$$

which we call the a^{th} (stable) homotopy group of X . We write $\pi_*(X)$ for the A -graded abelian group $\bigoplus_{a \in A} \pi_a(X)$, so that in particular we have a canonical isomorphism

$$\pi_*(X) = [S^*, X] \cong [S, X]_*.$$

Given some other object E , we can define the A -graded abelian groups $E_*(X)$ and $E^*(X)$ by the formulas

$$E_a(X) := \pi_a(E \otimes X) = [S^a, E \otimes X] \quad \text{and} \quad E^a(X) := [X, S^a \otimes E].$$

We refer to the functor $E_*(-)$ as the *homology theory represented by E* , or just E -homology, and we refer to $E^*(-)$ as the *cohomology theory represented by E* , or just E -cohomology.

0.2. Cellular objects in \mathcal{SH} . One very important class of objects in \mathcal{SH} are the *cellular* objects. Intuitively, these are the objects that can be built out of spheres via taking coproducts and (co)fibers.

Definition 0.5. Define the class of *cellular* objects in \mathcal{SH} to be the smallest class of objects such that:

- (1) For all $a \in A$, the a -sphere S^a is cellular.
- (2) If we have a distinguished triangle

$$X \rightarrow Y \rightarrow Z \rightarrow \Sigma X$$

such that two of the three objects X , Y , and Z are cellular, then the third object is also cellular.

- (3) Given a collection of cellular objects X_i indexed by some (small) set I , the object $\bigoplus_{i \in I} X_i$ is cellular (recall we have chosen \mathcal{SH} to have arbitrary coproducts).

Lemma 0.6. *Let X and Y be two isomorphic objects in \mathcal{SH} . Then X is cellular iff Y is cellular.*

Proof. Assume we have an isomorphism $f : X \xrightarrow{\cong} Y$ and that X is cellular. Then consider the following commutative diagram

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \longrightarrow & 0 & \longrightarrow & \Sigma X \\ \parallel & & \downarrow f^{-1} & & \parallel & & \parallel \\ X & \xlongequal{\quad} & X & \longrightarrow & 0 & \longrightarrow & \Sigma X \end{array}$$

The bottom row is distinguished by axiom TR1 for a triangulated category. Hence since X is cellular, 0 is also cellular, since the class of cellular objects satisfies two-of-three for distinguished triangles. Furthermore, since the vertical arrows are all isomorphisms, the top row is distinguished as well, by axiom TR0. Thus again by two-of-three, since X and 0 are cellular, so is Y , as desired. \square

Lemma 0.7. *Let X and Y be cellular objects in \mathcal{SH} . Then $X \otimes Y$ is cellular.*

Proof. Let E be a cellular object in \mathcal{SH} , and let \mathcal{E} be the collection of objects X in \mathcal{SH} such that $E \otimes X$ is cellular. First of all, suppose we have a distinguished triangle

$$X \rightarrow Y \rightarrow Z \rightarrow \Sigma X$$

such that two of three of X , Y , and Z belong to \mathcal{E} . Then since \mathcal{SH} is tensor triangulated, we have a distinguished triangle

$$E \otimes X \rightarrow E \otimes Y \rightarrow E \otimes Z \rightarrow \Sigma(E \otimes X).$$

Per our assumptions, two of three of $E \otimes X$, $E \otimes Y$, and $E \otimes Z$ are cellular, so that the third is by definition. Thus, all three of X , Y , and Z belong to \mathcal{E} if two of them do.

Second of all, suppose we have a family X_i of objects in \mathcal{E} indexed by some (small) set I , and set $X := \bigoplus_i X_i$. Then we'd like to show X belongs to \mathcal{E} , i.e., that $E \otimes X$ is cellular. Indeed,

$$E \otimes X = E \otimes \left(\bigoplus_i X_i \right) \cong \bigoplus_i (E \otimes X_i),$$

where the isomorphism is given by the fact that \mathcal{SH} is monoidal closed, so $E \otimes -$ preserves arbitrary colimits as it is a left adjoint. Per our assumption, since each $E \otimes X_i$ is cellular, the rightmost object is cellular, since the class of cellular objects is closed under taking arbitrary coproducts, by definition. Hence $E \otimes X$ is cellular by [Lemma 0.6](#).

Finally, we would like to show that each S^a belongs to \mathcal{E} , i.e., that $S^a \otimes E$ is cellular for all $a \in A$. When $E = S^b$ for some $b \in A$, this is clearly true, since $S^b \otimes S^a \cong S^{a+b}$, which is cellular by definition, so that $S^b \otimes S^a$ is cellular by [Lemma 0.6](#). Thus by what we have shown, the class of objects X for which $S^a \otimes X$ is cellular contains every cellular object. Hence in particular $E \otimes S^a \cong S^a \otimes E$ is cellular for all $a \in A$, as desired. \square

Lemma 0.8. *Let W be a cellular object in \mathcal{SH} such that $\pi_*(W) = 0$. Then $W \cong 0$.*

Proof. Let \mathcal{E} be the collection of all X in \mathcal{SH} such that $[\Sigma^n X, W] = 0$ for all $n \in \mathbb{Z}$ (where for $n > 0$, $\Sigma^{-n} := \Omega^n = (S^{-1})^n \otimes -$). We claim \mathcal{E} contains every cellular object in \mathcal{SH} . First of all, each S^a belongs to \mathcal{E} , as

$$[\Sigma^n S^a, W] \cong [S^n \otimes S^a, W] \cong [S^{a+n}, W] \leq \pi_*(W) = 0.$$

Furthermore, suppose we are given a distinguished triangle

$$X \rightarrow Y \rightarrow Z \rightarrow \Sigma X$$

such that two of three of X , Y , and Z belong to \mathcal{E} . By ??, for all $n \in \mathbb{Z}$ we get an exact sequence

$$[\Sigma^{n+1}X, W] \rightarrow [\Sigma^n Z, W] \rightarrow [\Sigma^n Y, W] \rightarrow [\Sigma^n X, W] \rightarrow [\Sigma^{n-1}Z, W].$$

Clearly if any two of three of X , Y , and Z belong to \mathcal{E} , then by exactness of the above sequence all three of the middle terms will be zero, so that the third object will belong to \mathcal{E} as well. Finally, suppose we have a collection of objects X_i in \mathcal{E} indexed by some small set I . Then

$$\left[\Sigma^n \bigoplus_i X_i, W \right] \cong \left[\bigoplus_i \Sigma^n X_i, W \right] \cong \prod_i [\Sigma^n X_i, W] = \prod_i 0 = 0,$$

where the first isomorphism follows by the fact that Σ^n is apart of an adjoint equivalence ([Proposition 0.4](#)), so it preserves arbitrary colimits.

Thus, by definition of cellularity, \mathcal{E} contains every cellular object. In particular, \mathcal{E} contains W , so that $[W, W] = 0$, meaning in particular that $\text{id}_W = 0$, so we have a commutative diagram

$$\begin{array}{ccc} & 0 & \\ & \xlongequal{\quad} & \\ W & \xrightarrow{\quad} & 0 \\ & \searrow & \nearrow \\ & W & \\ & \xlongequal{\quad} & \end{array}$$

Hence the diagonals exhibit isomorphisms between 0 and W , as desired. \square

Theorem 0.9. *Let X and Y be cellular objects in \mathcal{SH} , and suppose $f : X \rightarrow Y$ is a morphism such that $f_* : \pi_*(X) \rightarrow \pi_*(Y)$ is an isomorphism. Then f is an isomorphism.*

Proof. By axiom TR2 for a triangulated category (??), we have a distinguished triangle

$$X \xrightarrow{f} Y \xrightarrow{g} C_f \xrightarrow{h} \Sigma X.$$

First of all, note that by definition since X and Y are cellular, so is C_f . Now, we claim $\pi_*(C_f) = 0$. Indeed, given $a \in A$, by ?? we have the following exact sequence:

$$[S^a, X] \xrightarrow{f_*} [S^a, Y] \xrightarrow{g_*} [S^a, C_f] \xrightarrow{h_*} [S^a, \Sigma X] \xrightarrow{-(\Sigma f)_*} [S^a, \Sigma Y],$$

where the first arrow is an isomorphism, per our assumption that f_* is an isomorphism. To see the last arrow is an isomorphism, consider the following diagram:

$$\begin{array}{ccc} [S^a, \Sigma X] & \xrightarrow{(\Sigma f)_*} & [S^a, \Sigma Y] \\ (\nu_X)_* \downarrow & & \downarrow (\nu_Y)_* \\ [S^a, S^1 \otimes X] & \xrightarrow{(S^1 \otimes f)_*} & [S^a, S^1 \otimes Y] \\ \cong \downarrow & & \downarrow \cong \\ [S^{-1} \otimes S^a, X] & \xrightarrow{f_*} & [S^{-1} \otimes S^a, Y] \\ (\phi_{-1,a})_* \downarrow & & \downarrow (\phi_{-1,a})_* \\ [S^{a-1}, X] & \xrightarrow{f_*} & [S^{a-1}, Y] \end{array}$$

where the middle vertical arrows are the adjunction natural isomorphisms specified by [Proposition 0.4](#). The bottom arrow is an isomorphism per our assumptions, so the top arrow is likewise an isomorphism, as desired. Thus $\text{im } h_* = \ker -(\Sigma f)_* = 0$, and $\ker g_* = \text{im } f_* = [S^a, Y]$, so that $\ker h_* = \text{im } g_* = 0$. It is only possible that $\ker h_* = \text{im } h_* = 0$ if $[S^a, C_f] = 0$. Thus, we have

shown $\pi_*(C_f) = 0$, and C_f is cellular, so by [Lemma 0.8](#) there is an isomorphism $C_f \cong 0$. Now consider the following commuting diagram:

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \longrightarrow & C_f & \longrightarrow & \Sigma X \\ \downarrow f & & \parallel & & \downarrow \cong & & \downarrow \Sigma f \\ Y & \xlongequal{\quad} & Y & \longrightarrow & 0 & \longrightarrow & \Sigma Y \end{array}$$

The top row is distinguished by assumption. The bottom row is distinguished by axiom TR2. Then since the middle two vertical arrows are isomorphisms, by [??](#), f is an isomorphism as well, as desired. \square

Lemma 0.10. *Let $e : X \rightarrow X$ be an idempotent morphism in \mathcal{SH} , so $e \circ e = e$. Then since \mathcal{SH} is a triangulated category with arbitrary coproducts, this idempotent splits ([??](#)), meaning e factors as*

$$X \xrightarrow{r} Y \xrightarrow{\iota} X$$

for some object Y and morphisms r and ι with $r \circ \iota = \text{id}_Y$. Then Y is cellular if X is.

Proof. It is a general categorical fact that the splitting of an idempotent, if it exists, is unique up to unique isomorphism,² so by [Lemma 0.6](#), it suffices to show that e has some cellular splitting. In [??](#), it is shown that we may take Y to be the homotopy colimit ([??](#)) of the sequence

$$X \xrightarrow{e} X \xrightarrow{e} X \xrightarrow{e} X \xrightarrow{e} \dots,$$

so there is a distinguished triangle

$$\bigoplus_{i=0}^{\infty} X \rightarrow \bigoplus_{i=0}^{\infty} X \rightarrow Y \rightarrow \Sigma \left(\bigoplus_{i=0}^{\infty} X \right).$$

Since X is cellular, by definition $\bigoplus_{i=0}^{\infty} X$ is as well. Thus by 2-of-3 for distinguished triangles for cellular objects, Y is cellular as desired. \square

0.3. Monoid objects in \mathcal{SH} . Many of the proofs in this section are quite technical and not very euclidiating, so we direct the reader to the appendix for most proofs. To start with, recall the following definition:

Definition 0.11. Let $(\mathcal{C}, \otimes, S)$ be a symmetric monoidal category with left unitor, right unitor, associator, and symmetry isomorphisms λ, ρ, α , and τ , respectively. A *monoid object* (E, μ, e) is an object E in \mathcal{C} along with a multiplication morphism $\mu : E \otimes E \rightarrow E$ and a unit map $e : S \rightarrow E$ such that the following diagrams commute:

$$\begin{array}{ccc} E \otimes S & \xrightarrow{E \otimes e} & E \otimes E \xleftarrow{e \otimes E} S \otimes E \\ \searrow \rho_E & & \downarrow \mu \swarrow \lambda_E \\ & E & \end{array} \quad \begin{array}{ccc} (E \otimes E) \otimes E & \xrightarrow{\mu \otimes E} & E \otimes E \\ \alpha \downarrow & & \downarrow \mu \\ E \otimes (E \otimes E) & \xrightarrow{E \otimes \mu} & E \otimes E \xrightarrow{\mu} E \end{array}$$

The first diagram expresses unitality, while the second expressed associativity. If in addition the following diagram commutes,

$$\begin{array}{ccc} E \otimes E & \xrightarrow{\tau} & E \otimes E \\ & \searrow \mu \swarrow \mu & \\ & E & \end{array}$$

²In particular, given an idempotent $e : X \rightarrow X$ which splits as $X \xrightarrow{r} Y \xrightarrow{\iota} X$, r and ι are the coequalizer and equalizer, respectively, of e and id_X .

then we say (E, μ, e) is a *commutative monoid object*.

Monoid objects in \mathcal{SH} will be the focus of the rest of this paper. The most important example of a monoid object in \mathcal{SH} is the unit S , which has multiplication map $\phi_{0,0}^{-1} = \lambda_S = \rho_S : S \otimes S \rightarrow S$ and unit map $\text{id}_S : S \rightarrow S$.

Proposition 0.12 (??). *Let (E, μ, e) be a monoid object in \mathcal{SH} , then $\pi_*(E)$ is a ring under the multiplication map $\pi_*(E) \times \pi_*(E) \rightarrow \pi_*(E)$ which sends classes $x : S^a \rightarrow E$ and $y : S^b \rightarrow E$ to the composition*

$$xy : S^{a+b} \xrightarrow{\phi_{a,b}} S^a \otimes S^b \xrightarrow{x \otimes y} E \otimes E \xrightarrow{\mu} E.$$

In particular, the unit of this ring is $e \in \pi_0(E) = [S, E]$.

We call the ring $\pi_*(S)$ the *stable homotopy ring*.

Proposition 0.13 (??). *For all $a, b \in A$ there exists an element $\theta_{a,b} \in \pi_0(S) = [S, S]$ such that given any commutative monoid object (E, μ, e) in \mathcal{SH} , the A -graded ring structure on $\pi_*(E)$ ([Proposition 0.12](#)) has a commutativity formula given by*

$$x \cdot y = y \cdot x \cdot (e \circ \theta_{a,b})$$

for all $x \in \pi_a(E)$ and $y \in \pi_b(E)$. In particular, $\theta_{a,b} \in \text{Aut}(S)$ is the composition

$$S \xrightarrow{\cong} S^{-a-b} \otimes S^a \otimes S^b \xrightarrow{S^{-a-b} \otimes \tau} S^{-a-b} \otimes S^b \otimes S^a \xrightarrow{\cong} S,$$

where the outermost maps are the unique maps specified by [Remark 0.3](#).

Proposition 0.14. *The $\theta_{a,b}$'s described in [Proposition 0.13](#) satisfy the following properties for all $a, b, c \in A$:*

- (1) $\theta_{a,b} \circ \theta_{c,d} = \theta_{c,d} \cdot \theta_{a,b}$ (where \cdot denotes the product in $\pi_*(S)$ given in [Proposition 0.12](#)),
- (2) $\theta_{a,0} = \theta_{0,a} = \text{id}_S$,
- (3) $\theta_{a,b} \cdot \theta_{b,a} = \text{id}_S$,
- (4) $\theta_{a,b} \cdot \theta_{a,c} = \theta_{a,b+c}$ and $\theta_{b,a} \cdot \theta_{c,a} = \theta_{b+c,a}$,

Proof.

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