THE MOTIVIC ADAMS SPECTRAL SEQUENCE

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2.1. **Setup.** In order to construct an abstract version of the Adams spectral sequence, we need to work in some "axiomatic stable homotopy category" $\mathcal{SH}$ which acts like the familiar stable homotopy category **hoSp** (Section 3) or the motivic stable homotopy category $\mathbf{SH}_{\mathscr{S}}$ over some base scheme $\mathscr{S}$ (Section 4). As it turns out, practically all the data we need is the following:

Definition 2.1. A *stable homotopy category* is the following data:

- A closed tensor triangulated category $(\mathcal{SH}, \otimes, S, \Sigma, \Omega)$ with arbitrary small (co)products.
- A pointed abelian group $(A, \mathbf{1})$ and a homomorphism $h : (A, \mathbf{1}) \to (\mathrm{Pic}(\mathcal{SH}), \Sigma S)$ of pointed groups (i.e., $\mathbf{1}$ is sent to the isomorphism class of ΣS), where $\mathrm{Pic}(\mathcal{SH})$ is the group of isomorphism classes of invertible objects in $\mathcal{SH}$[1].
- For each $a \in A$, a chosen object S^a in the isomorphism class $h(a)$.

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[1]Recall an object X in a symmetric monoidal category is *invertible* if there exists some object Y in $\mathcal{SH}$ and an isomorphism $S \cong Y \otimes X$. To see ΣS is invertible, note that we have isomorphisms

$$\Sigma S \otimes \Omega S \cong \Sigma(S \otimes \Omega S) \cong \Sigma(\Omega S \otimes S) \cong \Sigma\Omega S \otimes S \cong S \otimes S \cong S,$$

where the first isomorphism is axiom TT1 for a tensor triangulated category (Definition A.5), the second isomorphism is given by the symmetry in $\mathcal{SH}$, the third isomorphism is again axiom TT1, the fourth isomorphism is the fact that Σ and Ω for an adjoint equivalence, and finally the last isomorphism follows by the fact that S is the monoidal unit in $\mathcal{SH}$.

Given an abstract stable homotopy category as above, we will always assume without loss of generality that $S^0 = S$ and $\Sigma = S^{\mathbf{1}} \otimes -$ (by Proposition A.7). we establish the following conventions:

- Given objects $X_1, \dots, X_n$ in $\mathcal{SH}$, we write $X_1 \otimes \cdots \otimes X_n$ to denote the object
$$X_1 \otimes (X_2 \otimes \cdots (X_{n-1} \otimes X_n)).$$
In particular, given an object X and a natural number $n > 0$, we write
$$X^n := \overbrace{X \otimes \cdots \otimes X}^{n \text{ times}} \qquad \text{and} \qquad X^0 := S.$$
- We denote the associator, symmetry, left unitor, and right unitor isomorphisms in $\mathcal{SH}$ by
$$\alpha_{X,Y,Z} : (X \otimes Y) \otimes Z \xrightarrow{\cong} X \otimes (Y \otimes Z) \qquad \tau_{X,Y} : X \otimes Y \xrightarrow{\cong} Y \otimes X$$
$$\lambda_X : S \otimes X \xrightarrow{\cong} X \qquad \rho_X : X \otimes S \xrightarrow{\cong} X.$$
Often we will suppress these isomorphisms from the notation (particularly the associators), choosing instead to denote them without their subscripts or simply with the symbol $\cong$.
- Given some $a \in A$, we define the functor $\Sigma^a := S^a \otimes -$, so that in particular $\Sigma^{\mathbf{1}} = \Sigma$.
- Given two objects X and Y, we denote the hom-abelian group of morphisms from X to Y in $\mathcal{SH}$ by $[X, Y]$, and we denote the internal hom object by $F(X, Y)$. We will often refer to morphisms in $\mathcal{SH}$ as *classes*, as we will think of them as representing homotopy classes of maps.
- Given two objects X and Y in $\mathcal{SH}$, we may extend the abelian group $[X, Y]$ to an A-graded abelian group $[X, Y]_*$ defined by
$$[X, Y]_a := [\Sigma^a X, Y] = [S^a \otimes X, Y].$$
(See Appendix C for a review of the theory of A-graded abelian groups, rings, modules, etc.)
- Given an object X in $\mathcal{SH}$ and some $a \in A$, define the abelian group
$$\pi_a(X) := [S^a, X],$$
and write $\pi_*(X)$ for the associated A-graded abelian group $\bigoplus_{a \in A} \pi_a(X)$. We call $\pi_a(X)$ the a^{th} *stable homotopy group of* X.
- Given two objects E and X in $\mathcal{SH}$, we define the A-graded abelian groups $E_*(X)$ and $E^*(X)$ by
$$E_a(X) := \pi_a(E \otimes X) = [S^a, E \otimes X] \qquad \text{and} \qquad E^a(X) := [X, S^a \otimes E].$$
We refer to the functor $E_*(-)$ as the *homology theory represented by* E, or just E-homology, and we refer to $E^*(-)$ as the *cohomology theory represented by* E, or just E-cohomology.

From now on, we fix the data of a stable homotopy category $\mathcal{SH}$ given above once and for all. Observe that for all $a, b \in A$, the objects S^{a+b} and $S^a \otimes S^b$ are isomorphic, since $h : A \to \operatorname{Pic}(\mathcal{SH})$ is a group homomorphism. Hence given a monoid object (E, μ, e) in $\mathcal{SH}$ (Definition D.1), supposing we had fixed isomorphisms $S^{a+b} \cong S^a \otimes S^b$ for all $a, b \in A$, we get a multiplication map $\pi_*(E) \times \pi_*(E) \to \pi_*(E)$ which sends classes $x : S^a \to E$ and $y : S^b \to E$ to the product
$$S^{a+b} \cong S^a \otimes S^b \xrightarrow{x \otimes y} E \otimes E \xrightarrow{\mu} E.$$
Naturally, we would like this product to make $\pi_*(E)$ into an A-graded ring (with unit $e \in \pi_0(E) = [S, E]$), rather than just an A-graded abelian group. This is essentially the entire discussion of

Dugger's paper [1], and as it turns out, $\pi_*(E)$ is in fact a graded ring provided we can choose these morphisms to be *coherent*, in the following sense:

Definition 2.2. Suppose we have a family of isomorphisms

$$\phi_{a,b} : S^{a+b} \xrightarrow{\cong} S^a \otimes S^b$$

for all $a, b \in A$. We say this family is *coherent* if:

(1) For all $a \in A$, we have equalities $\phi_{a,0} = \rho_{S^a}^{-1} : S^a \to S^a \otimes S$ and $\phi_{0,a} = \lambda_{S^a}^{-1} : S^a \to S \otimes S^a$.
(2) For all $a, b, c \in A$, the following diagram commutes:

$$\begin{tikzcd} S^{a+b} \otimes S^c \arrow[d, "\phi_{a,b}\otimes S^c"'] & S^{a+b+c} \arrow[l, "\phi_{a+b,c}"'] \arrow[r, "\phi_{a,b+c}"] & S^a \otimes S^{b+c} \arrow[d, "S^a\otimes\phi_{b,c}"] \\ (S^a \otimes S^b) \otimes S^c \arrow[rr, "\cong"] & & S^a \otimes (S^b \otimes S^c) \end{tikzcd}$$

Furthermore, Dugger gaurantees that we can always find such a coherent family:

Theorem 2.3 ([1, Proposition 7.1])**.** *There exists a coherent family of isomorphisms*

$$\phi_{a,b} : S^{a+b} \xrightarrow{\cong} S^a \otimes S^b$$

in the sense of Definition 2.2, and in particular, the set of such coherent families is in bijective correspondence with the set of normalized 2*-cocycles* $Z^2(A; \mathrm{Aut}(S))_{norm}$*, i.e., the set of functions* $\alpha : A \times A \to \mathrm{Aut}(S)$ *such that* $\alpha(0,0) = \mathrm{id}_S$ *and for all* $a, b, c \in A$*,* $\alpha(a+b, c) \cdot \alpha(a, b) = \alpha(b, c) \cdot \alpha(a, b+c)$*.*

Thus, from now on we will suppose once and for all we have fixed a coherent family $\{\phi_{a,b}\}_{a,b\in A}$. Such a coherent family has very nice properties, in particular:

Remark 2.4. Note that by induction the coherence conditions say that given any $a_1, \dots, a_n \in A$ and $b_1, \dots, b_m \in A$ such that $a_1 + \cdots + a_n = b_1 + \cdots + b_m$ and any fixed parenthesizations of $X = S^{a_1} \otimes \cdots \otimes S^{a_b}$ and $Y = S^{b_1} \otimes \cdots \otimes S^{b_m}$, there is a *unique* isomorphism $X \to Y$ that can be obtained by forming formal compositions of tensor products of $\phi_{a,b}$, associators, and their inverses.

Of course, we get our desired result: $\pi_*(E)$ is indeed an A-graded ring if E is a monoid object.

Proposition 2.5. *Let* (E, μ, e) *be a commutative monoid object in* $\mathcal{SH}$*, and consider the multiplication map* $\pi_*(E) \times \pi_*(E) \to \pi_*(E)$ *which sends classes* $x : S^a \to E$ *and* $y : S^b \to E$ *to the composition*

$$S^{a+b} \xrightarrow{\phi_{a,b}} S^a \otimes S^b \xrightarrow{x\otimes y} E \otimes E \xrightarrow{\mu} E.$$

Then this endows $\pi_*(E)$ *with the structure of an* A*-graded ring with unit* $e \in \pi_0(E) = [S, E]$*.*

Proof. See Proposition D.3. □

Furthermore, it turns out that if E is a *commutative* monoid object in $\mathcal{SH}$, then $\pi_*(E)$ is "A-graded commutative," in the following sense:

Proposition 2.6. *For all* $a, b \in A$ *there exists an element* $\theta_{a,b} \in \pi_0(S) = [S, S]$ *(determined by choice of coherent family* $\{\phi_{a,b}\}$*) such that given any commutative monoid object* (E, μ, e) *in* $\mathcal{SH}$*, the* A*-graded ring structure on* $\pi_*(E)$ *(Proposition 2.5) has a commutativity formula given by*

$$x \cdot y = y \cdot x \cdot (e \circ \theta_{a,b})$$

for all $x \in \pi_a(E)$ *and* $y \in \pi_b(E)$*.*

Furthermore, $\theta_{0,a} = \theta_{a,0} = \mathrm{id}_S$ *for all* $a \in A$*, so that if either* x *or* y *has degree* 0*,* $x \cdot y = y \cdot x$*.*

Proof. See Proposition D.4 and Proposition D.5. □

We also have the following result:

Proposition 2.7. *Given some $a \in A$, the functors Σ^a and Σ^{-a} canonically form an adjoint equivalence of $\mathcal{SH}$.*

Proof. See Proposition D.6. $\square$

In particular, note that this tells us that given objects E and X in $\mathcal{SH}$, we have isomorphisms

$$E^*(X) = [X, S^* \otimes X] \cong [S^{-*} \otimes X, E] \cong [S^{-*}, F(X, E)] = \pi_{-*}(F(X, E)).$$

Similarly, given any objects X and Y in $\mathcal{SH}$, we have isomorphisms of A-graded abelian groups

$$[X, \Sigma Y]_* = [S^* \otimes X, S^{\mathbf{1}} \otimes Y] \cong [S^{-\mathbf{1}} \otimes S^* \otimes X, Y] \cong [S^{*-\mathbf{1}} \otimes X, Y] = [X, Y]_{*-\mathbf{1}},$$

where the first isomorphism is the adjunction specified by the above proposition, and the second isomorphism is induced by the isomorphism

$$S^{*-\mathbf{1}} \otimes X \xrightarrow{\phi_{-\mathbf{1},*} \otimes X} S^{-\mathbf{1}} \otimes S^* \otimes X.$$

The last ingredient in order to develop the Adams spectral sequence abstractly is a notion of *cellularity* in $\mathcal{SH}$:

Definition 2.8. Define the class of *cellular* objects in $\mathcal{SH}$ to be the smallest class of objects such that:

(1) For all $a \in A$, S^a is cellular.
(2) If we have a distinguished triangle
$$X \to Y \to Z \to \Sigma X (= S^{\mathbf{1}} \otimes X)$$
such that two of the three objects X, Y, and Z are cellular, than the third object is also cellular.
(3) Given a collection of cellular objects X_i indexed by some small set I, $\bigoplus_{i \in I} X_i$ is cellular.

2.2. **Construction of the Adams spectral sequence.** In what follows, let E be a commutative monoid object in $\mathcal{SH}$.

Definition 2.9. Let $\overline{E}$ be the fiber of the unit map $e : S \to E$ (**??**), and for $s \geq 0$ define

$$Y_s := \overline{E}^s \otimes Y, \qquad W_s = E \otimes Y_s = E \otimes (\overline{E}^s \otimes Y),$$

where recall for $s > 0$, $\overline{E}^s$ denotes the s-fold product parenthesized as $\overline{E} \otimes (\overline{E} \otimes \cdots (\overline{E} \otimes \overline{E}))$, and $\overline{E}^0 := S$. Then we get fiber sequences

$$Y_{s+1} \xrightarrow{i_s} Y_s \xrightarrow{j_s} W_s \xrightarrow{k_s} \Sigma Y_{s+1} (= S^{\mathbf{1}} \otimes Y_{s+1})$$

obtained by applying $- \otimes Y_s$ to the sequence

$$\overline{E} \to S \xrightarrow{e} E \to \Sigma \overline{E}$$

(and applying the necessary associator isomorphisms). These sequences can be spliced together to form the *(canonical) Adams filtration of Y*:

$$\begin{tikzcd}
\cdots \arrow[r] & Y_3 \arrow[r, "i_2"] \arrow[d, "j_3"] & Y_2 \arrow[r, "i_1"] \arrow[d, "j_2"] & Y_1 \arrow[r, "i_0"] \arrow[d, "j_1"] & Y_0 = Y \arrow[d, "j_0"] \\
& W_3 & W_2 \arrow[ul, dashed, "k_2"'] & W_1 \arrow[ul, dashed, "k_1"'] & W_0 \arrow[ul, dashed, "k_0"']
\end{tikzcd}$$

where the diagonal dashed arrows are of degree $-\mathbf{1}$ (note these triangles do NOT commute in any sense). Now we may apply the functor $[X,-]_*$, and by **??** we obtain an exact couple of $\mathbb{N}\times A$-graded abelian groups:

$$\begin{tikzcd} {[X,Y_*]_*} \arrow[r, "i_{**}"] & {[X,Y_*]_*} \arrow[d, "j_{**}"] \\ & {[X,W_*]_*} \arrow[ul, "k_{**}"] \end{tikzcd}$$

where i_{**}, j_{**}, and k_{**} have $\mathbb{Z}\times A$-degree $(-1,0)$, $(0,0)$, and $(1,-\mathbf{1})$, respectively[2]. The standard argument yields a $\mathbb{N}\times A$-graded spectral sequence called from this exact couple (cf. Section 5.9 of [6]) with E_1 page given by

$$E_1^{s,a} = [X,W_s]_a$$

and r^{th} differential of $\mathbb{Z}\times A$-degree $(r,-\mathbf{1})$:

$$d_r : E_r^{s,a} \to E_r^{s+r,a-\mathbf{1}}.$$

A priori, this is all $\mathbb{N}\times A$-graded, but we regard it as being $\mathbb{Z}\times A$-graded by setting $E_r^{s,a} := 0$ for $s<0$ and trivially extending the definition of the differentials to these zero groups. This spectral sequence is called the *E-Adams spectral sequence* for the computation of $[X,Y]_*$. The index s is called the *Adams filtration* and a is the *stem*.

2.3. **The E_1 page.** The goal of this subsection is to provide the following characterization for the E_1 page of the Adams spectral sequence:

Theorem 2.10. *Let E be a flat commutative ring spectrum, and let X and Y be two objects in $\mathcal{SH}$ such that $E_*(X)$ is a projective module over $\pi_*(E)$. Then for all $s\geq 0$ and $a\in A$, we have isomorphisms in the associated E-Adams spectral sequence*

$$E_1^{s,a} \cong \operatorname{Hom}^a_{E_*(E)}(E_*(X),E_*(W_s))$$

Furthermore, under these isomorphisms, the differential $d_1 : E_1^{s,a}\to E_1^{s+1,a-\mathbf{1}}$ corresponds to the map

$$\operatorname{Hom}^a_{E_*(E)}(E_*(X),E_*(W_s)) \to \operatorname{Hom}^{a-\mathbf{1}}_{E_*(E)}(E_*(X),E_*(X\otimes W_{s+1}))$$

which sends a map $f : E_*(X)\to E_{*+a}(W_s)$ to the composition

$$E_*(X) \xrightarrow{f} E_{*+a}(W_s) \xrightarrow{(X\otimes h_s)_*} E_{*+a-\mathbf{1}}(X\otimes Y_{s+1}) \xrightarrow{(X\otimes j_{s+1})_*} E_{*+a-\mathbf{1}}(X\otimes W_{s+1}).$$

Proof. By Lemma C.8, for all $s\geq 0$ and $t,w\in\mathbb{Z}$, we have isomorphisms

$$[X,E\otimes Y_s]_{t,w} \cong \operatorname{Hom}^{t,w}_{E_*(E)}(E_*(X),E_*(E\otimes Y_s)).$$

since $W_s = E\otimes Y_s$, we have that

$$E_1^{s,(t,w)} = [X,W_s]_{t,w} \cong \operatorname{Hom}^{t,w}_{E_*(E)}(E_*(X),E_*(W_s)),$$

as desired. □

2.4. **The E_2 page.**

2.5. **Convergence.** convergence of spectral sequences

[2]Explicitly, the map $k_{s,a} : [X,W_s]_a \to [X,Y_{s+1}]_{a-\mathbf{1}}$ sends a map $f : S^a\otimes X\to W_s$ to the map $S^{a-\mathbf{1}}\otimes X\to Y_{s+1}$ corresponding under the isomorphism $[X,\Sigma Y_{s+1}]_* \cong [X,Y_{s+1}]_{*-\mathbf{1}}$ to the composition $k_s\circ f : S^a\otimes X\to \Sigma Y_{s+1}$.

3. The classical Adams spectral sequence

4. The motivic Adams spectral sequence

Appendix A. Triangulated categories

We assume the reader is familiar with additive categories and (closed, symmetric) monoidal categories.

Definition A.1. A *triangulated category* is a tuple $(\mathcal{C}, \Sigma, \Omega, \mathcal{D})$ such that

(1) $\mathcal{C}$ is an additive category.
(2) $\Sigma, \Omega : \mathcal{C} \to \mathcal{C}$ form an adjoint equivalence of $\mathcal{C}$ with itself. (Σ is calld the *shift functor*.)
(3) $\mathcal{D}$ is a collection of *distinguished triangles*, where a *triangle* is a diagram of the form

$$X \to Y \to Z \to \Sigma X.$$

These are also sometimes called *cofiber sequences* or *fiber sequences.*

These data must satisfy the following axioms:

TR0 Given a commutative diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d, "\cong"'] & Y \arrow[r] \arrow[d, "\cong"'] & Z \arrow[r] \arrow[d, "\cong"'] & \Sigma X \arrow[d, "\cong"'] \\ X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & \Sigma X' \end{tikzcd}$$

where the vertical arrows are isomorphisms, if the top row is distinguished then so is the bottom.

TR1 For any object X in $\mathcal{C}$, the diagram

$$X \xrightarrow{\mathrm{id}_X} X \to 0 \to \Sigma X$$

is a distinguished triangle.

TR2 For all $f : X \to Y$ there exists an object C_f (also sometimes denoted Y/X) called the *cofiber of f* and a distinguished triangle

$$X \xrightarrow{f} Y \to C_f \to \Sigma X.$$

TR3 Given a solid diagram with both rows commutative

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] & Y \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d, dashed] & \Sigma X \arrow[d, "\Sigma f"] \\ X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & \Sigma X' \end{tikzcd}$$

such that the leftmost square commmutes and both rows are distinguished, there exists a dashed arrow $Z \to Z'$ which makes the remaining two squares commute.

TR4 A triangle

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{\Sigma} X$$

is distinguished if and only if

$$Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X \xrightarrow{-\Sigma f} \Sigma Y$$

is distinguished.

TR5 (Octahedral axiom) Given three distinguished triangles

$$X \xrightarrow{f} Y \xrightarrow{h} Y/X \to \Sigma X$$

$$Y \xrightarrow{g} \xrightarrow{k} Z/Y \to \Sigma Y$$

$$X \xrightarrow{g \circ f} Z \xrightarrow{l} Z/X \to \Sigma X$$

there exists a distinguished triangle

$$Y/X \xrightarrow{u} Z/X \xrightarrow{v} Z/Y \xrightarrow{w} \Sigma(Y/X)$$

such that the following diagram commutes

$$\begin{tikzcd}
X \arrow[rr, "g\circ f"] \arrow[dr, "f"'] & & Z \arrow[rr, "k"] \arrow[dr, "l"'] & & Z/Y \arrow[rr, "w"] \arrow[dr] & & \Sigma(Y/X) \\
& Y \arrow[ur, "g"'] \arrow[dr, "h"'] & & Z/X \arrow[ur, "v"'] \arrow[dr] & & \Sigma Y \arrow[ur, "\Sigma h"'] & \\
& & Y/X \arrow[ur, "u"'] \arrow[rr] & & \Sigma X \arrow[ur, "\Sigma f"'] & &
\end{tikzcd}$$

It turns out that the above definition is actually redundant; TR3 and TR4 follow from the remaining axioms (see Lemmas 2.2 and 2.4 in [2]).

We now recall several important propositions for triangulated categories:

Proposition A.2. *Given a map $f : X \to Y$ in a triangulated category $(\mathcal{C}, \Sigma, \Omega, \mathcal{D})$, the cofiber sequence of f is unique up to isomorphism, in the sense that given any two distinguished triangles*

$$X \xrightarrow{f} Y \to Z \to \Sigma X \qquad \textit{and} \qquad X \xrightarrow{f} Y \to Z' \to \Sigma X,$$

there exists an isomorphism $Z \to Z'$ which makes the following diagram commute:

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, equal] & Y \arrow[r] \arrow[d, equal] & Z \arrow[r] \arrow[d, dashed, "k"] & \Sigma X \arrow[d, equal] \\
X \arrow[r, "f"] & Y \arrow[r] & Z' \arrow[r] & \Sigma X
\end{tikzcd}$$

Proposition A.3. *Given an arrow $f : X \to Y$ in a triangulated category $(\mathcal{C}, \Sigma, \Omega, \mathcal{D})$, there exists an object F_f called the* fiber *of f, and a distinguished triangle*

$$F_f \to X \xrightarrow{f} Y \to \Sigma F_f (\cong C_f).$$

Proposition A.4. *Let $(\mathcal{C}, \Sigma, \Omega, \mathcal{D})$ be a triangulated category. Given a distinguished triangle*

$$X \xrightarrow{f} Y \xrightarrow{g} \xrightarrow{h} \Sigma X$$

and any object A in $\mathcal{C}$, there is a long exact sequence of abelian groups

$$\cdots \to [\Sigma^{n+1} A, Z] \xrightarrow{h_*} [\Sigma^n X, X] \xrightarrow{f_*} [\Sigma^n A, Y] \xrightarrow{g_*} [\Sigma^n A, Z] \xrightarrow{h_*} [\Sigma^{n-1} A, X] \to \cdots$$

extending infinitely in either direction, where for $n < 0$ we define $\Sigma^{-n} := \Omega^n$.

Also important for our work is the concept of a *tensor triangulated category*, that is, a triangulated symmetric monoidal category in which the triangulated structures are compatible, in the following sense:

Definition A.5. A *tensor triangulated category* is a triangulated symmetric monoidal category $(\mathcal{C}, \otimes, S, \Sigma, \Omega, \mathcal{D})$ such that:

TT1 For all objects X and Y in $\mathcal{C}$, there are natural isomorphisms

$$e_{X,Y} : (\Sigma X) \otimes Y \xrightarrow{\cong} \Sigma(X \otimes Y).$$

TT2 For each object X in $\mathcal{C}$, the functor $X \otimes (-) \cong (-) \otimes X$ is an additive functor.

TT3 For each object X in $\mathcal{C}$, the functor $X \otimes (-) \cong (-) \otimes X$ preserves distinguished triangles, in that given a distinguished triangle/(co)fiber sequence

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{\Sigma} A,$$

then also

$$X \otimes A \xrightarrow{X \otimes f} X \otimes B \xrightarrow{X \otimes g} X \otimes C \xrightarrow{\Sigma(X \otimes h)} \Sigma(X \otimes A)$$

and

$$A \otimes X \xrightarrow{f \otimes X} B \otimes X \xrightarrow{g \otimes X} C \otimes X \xrightarrow{\Sigma(h \otimes X)} \Sigma(A \otimes X)$$

are distinguished triangles.

Usually, most tensor triangulated categories that arise in nature will satisfy additional coherence axioms (see axioms TC1–TC5 in [2]), but the above definition will suffice for our purposes. To avoid the awkwardness of saying "a tensor triangulated category which is also a closed symmetric monoidal category," we introduce the following (nonstandard) terminology:

Definition A.6. We say a tensor triangulated category $(\mathcal{C}, \otimes, S, \Sigma, \Omega)$ is *closed* if $\mathcal{C}$ is a closed symmetric monoidal category, in the sense that for each object $X \in \mathcal{C}$, the functor $- \otimes X$ has a right adjoint $F(X, -)$.

Note that given a tensor triangulated category, we have the following characterization of the shift functor:

Proposition A.7. *Given a tensor triangulated category* $(\mathcal{C}, \otimes, S, \Sigma, \Omega)$*, there is a canonical natural isomorphism* $\Sigma S \otimes - \cong \Sigma$.

Proof. Given an object X in $\mathcal{C}$, we have natural isomorphisms

$$\Sigma S \otimes X \xrightarrow{e_{S,X}} \Sigma(S \otimes X) \xrightarrow{\Sigma \lambda_X} \Sigma X,$$

where λ_X is the left unitor specified by the monoidal structure on $\mathcal{C}$. $\square$

Appendix B. Spectral sequences

Appendix C. (Co)algebra

C.1. **Grading.** First, we develop the theory of things graded by an abelian group. In what follows, we fix an abelian group A.

Definition C.1. An *A-graded abelian group* is an abelian group B along with a subgroup $B_a \leq B$ for each $a \in A$ such that the canonical map

$$\bigoplus_{a \in A} B_a \to B$$

sending $(x_a)_{a \in A}$ to $\sum_{a \in A} x_a$ is an isomorphism. Given two A-graded abelian groups B and C, a homomorphism $f : B \to C$ is a *homomorphism of A-graded abelian groups* if it preserves the grading, i.e., if it restricts to a map $B_a \to C_a$ for all $a \in A$.

It is easy to see that an A-graded abelian group B is generated by its *homogeneous* elements, that is, nonzero elements $x \in B$ such that there exists some $a \in A$ with $x \in B_a$.

Remark C.2. Clearly the condition that the canonical map $\bigoplus_{a \in A} B_a \to B$ is an isomorphism requires that $B_a \cap B_b = 0$ if $a \neq b$. In particular, given a homogeneous element $x \in B$, there exists precisely one $a \in A$ such that $x \in B_a$. We call this a the *degree* of x, and we write $|x| = a$.

Definition C.3. An *A-graded ring* R is the data of a ring R such that:

(1) The underlying abelian group of R is A-graded;

(2) For all $a, b \in A$, the multiplication map $R \times R \to R$ restricts to a map

$$R_a \times R_b \to R_{a+b},$$

i.e., $|x \cdot y| = |x| + |y|$ for all nonzero $x, y \in R$.

For example, given some field k, the ring $R = k[x, y]$ is $\mathbb{Z}^2$-graded, where given $(n, m) \in \mathbb{Z}^2$, $R_{n,m}$ is the subgroup of those monomials of the form ax^ny^m for some $a \in k$. Oftentimes we constructing A-graded rings, we do so only by defining the product of homogeneous elements, like so:

Proposition C.4. *Given an A-graded abelian group R, a distinguished element $1 \in R_0$, and $\mathbb{Z}$-bilinear maps $m_{a,b} : R_a \times R_b \to R_{a+b}$ for all $a, b \in A$ such that given $x \in R_a$, $y \in R_b$, and $z \in R_c$,*

$$m_{a+b,c}(m_{a,b}(x, y), z) = m_{a,b+c}(x, m_{b,c}(y, z)) \qquad \textit{and} \qquad m_{a,0}(x, 1) = m_{0,a}(1, x) = x,$$

there exists a unique multiplication map $m : R \times R \to R$ which endows R with the structure of an A-graded ring and restricts to $m_{a,b}$ for all $a, b \in A$.

Proof. Given $r, s \in R$, since $R \cong \bigoplus_{a \in A} R_a$, we may uniquely decompose r and s into homogeneous elements as $r = \sum_{a \in A} r_a$ and $s = \sum_{a \in A} s_a$ with each $r_a, s_a \in R_a$ such that only finitely many of the r_a's and s_a's are nonzero. Then in order to define a distributive product $R \times R \to R$ which restricts to $m_{a,b} : R_a \times R_b \to R_{a+b}$, note we *must* define

$$r \cdot = \left(\sum_{a \in A} r_a\right) \cdot \left(\sum_{b \in A} s_b\right) = \sum_{a,b \in A} r_a \cdot s_b = \sum_{a,b \in A} m_{a,b}(r_a, s_b).$$

Thus, we have shown uniqueness. It remains to show this product actually gives R the structure of a ring. First we claim that the sum on the right is actually finite. Note there exists only finitely many nonzero r_a's and s_b's, and if $s_b = 0$ then

$$m_{a,b}(r_a, 0) = m_{a,b}(r_a, 0 + 0) \overset{(*)}{=} m_{a,b}(r_a, 0) + m_{a,b}(r_a, 0) \implies m_{a,b}(r_a, 0) = 0,$$

where $(*)$ follows from bilinearity of $m_{a,b}$. A similar argument yields that $m_{a,b}(0, r_b) = 0$ for all $a, b \in A$. Hence indeed $m_{a,b}(r_a, s_b)$ is zero for all but finitely many pairs $(a, b) \in A^2$, as desired. Observe that in particular

$$(r \cdot s)_a = \sum_{b+c=a} m_{b,c}(r_b, s_c) = \sum_{b \in A} m_{b,a-b}(r_b, s_{a-b}) = \sum_{c \in A} m_{a-c,c}(r_{a-c}, s_c).$$

Now we claim this multiplication is associative. Given $t = \sum_{a\in A} t_a \in R$, we have

$$\begin{aligned}
(r\cdot s)\cdot t &= \sum_{a,b\in A} m_{a,b}((r\cdot s)_a, t_b)\\
&= \sum_{a,b\in A} m_{a,b}\left(\sum_{c\in A} m_{a-c,c}(r_{a-c}, s_c), t_b\right)\\
&\overset{(1)}{=} \sum_{a,b,c\in A} m_{a,b}(m_{a-c,c}(r_{a-c}, s_c), t_b)\\
&\overset{(2)}{=} \sum_{a,b,c\in A} m_{c,a+b-c}(r_c, m_{a-c,b}(s_{a-c}, t_b))\\
&\overset{(3)}{=} \sum_{a,b,c\in A} m_{a,c}(r_a, m_{b,c-b}(s_b, t_{c-b}))\\
&\overset{(1)}{=} \sum_{a,c\in A} m_{a,c}\left(r_a, \sum_{b\in A} m_{b,c-b}(s_b, t_{c-b})\right)\\
&= \sum_{a,c\in A} m_{a,c}(r_a, (s\cdot t)_c) = r\cdot(s\cdot t),
\end{aligned}$$

where each occurrence of (1) follows by bilinearity of the $m_{a,b}$'s, each occurrence of (2) is associativity of the $m_{a,b}$'s, and (3) is obtained by re-indexing by re-defining $a := c$, $b := a-c$, and $c := a+b-c$. Next, we wish to show that the distinguished element $1 \in R_0$ is a unit with respect to this multiplication. Indeed, we have

$$1\cdot r \overset{(1)}{=} \sum_{a\in A} m_{0,a}(1, r_a) \overset{(2)}{=} \sum_{a\in A} r_a = r$$

and

$$r\cdot 1 \overset{(1)}{=} \sum_{a\in A} m_{a,0}(r_a, 1) \overset{(2)}{=} \sum_{a\in A} r_a = r,$$

where (1) follows by the fact that $m_{a,b}(0,-) = m_{a,b}(-,0) = 0$, which we have shown above, and (2) follows by unitality of the $m_{0,a}$'s and $m_{0,a}$'s, respectively. Finally, we wish to show that this product is distributive. Indeed, we have

$$\begin{aligned}
r\cdot(s+t) &= \sum_{a,b\in A} m_{a,b}(r_a, (s+t)_b)\\
&= \sum_{a,b\in A} m_{a,b}(r_a, s_b + t_b)\\
&\overset{(*)}{=} \sum_{a,b\in A} m_{a,b}(r_a, s_b) + \sum_{a,b\in A} m_{a,b}(r_a, t_b) = (r\cdot s) + (r\cdot t),
\end{aligned}$$

where $(*)$ follows by bilinearity of $m_{a,b}$. An entirely analagous argument yields that $(r+s)\cdot t = (r\cdot t) + (s\cdot t)$. $\square$

When working with A-graded abelian groups, we will freely use the above proposition without comment. Given an A-graded ring R, we may talk about A-graded R-modules:

Definition C.5. Let R be an A-graded ring. A left A-graded R-module M is a left R-module M such that M is an A-graded abelian group, and for all $a, b \in A$, the action map $R \times M \to M$ restricts to a map $R_a \times M_b \to M_{a+b}$.

Right A-graded R-modules are defined similarly. Finally, an A-graded R-bimodule is an A-graded abelian group M along with action maps

$$\alpha_L : R \times M \to M \qquad \text{and} \qquad \alpha_R : M \times R \to M$$

which endow M with the structure of a left and right A-graded R-module, respectively, such that given $s, r \in R$ and $m \in M$, $r \cdot (m \cdot s) = (r \cdot m) \cdot s$.

Proposition C.6. *Let R be an A-graded ring, and suppose I have a right A-graded R-module M and a left A-graded R-module N. Then the tensor product*

$$M \otimes_R N$$

is naturally an A-graded abelian group, and furthermore, if either M (resp. N) is an A-graded bimodule, then it is naturally a left (resp. right) A-graded R-module

Proof. By definition, since M and N are A-graded abelian groups, they are generated (as abelian groups) by their homogeneous elements. Thus it follows that $M \otimes_R N$ is generated by *homogeneous pure tensors*, that is, elements of the form $m \otimes n$ with $m \in M$ and $n \in N$ homogeneous. Now, given a homogeneous pure tensor $m \otimes n$, we define its *degree* by the formula $|m \otimes n| := |m| + |n|$. It follows this formula is well-defined by checking that given homogeneous elements $m, m' \in M$, $n, n' \in N$, and $r \in R$ that

$$|(m \cdot r) \otimes n| = |m \otimes (r \cdot n)| = |m| + |r| + |n|.$$

Thus, we may define $(M \otimes_R N)_a$ to be the subgroup of $M \otimes_R N$ generated by those pure homogeneous tensors of degree a. Now, clearly the canonical map

$$\bigoplus_{a \in A} (M \otimes_R N)_a \to M \otimes_R N$$

is surjective. To see it is injective, it suffices ∎ finish

Definition C.7. Let E be a flat cellular commutative ring spectrum. Then a *left $E_{**}(E)$-comodule* is the data of

(1) A $\mathbb{Z}^2$-graded left $\pi_{**}(E)$-module M;
(2) A homomorphism of left-graded $\pi_{**}(E)$-modules:

$$\Psi_M : M \to E_{**}(E) \otimes_{\pi_{**}(E)} M.$$

(Note $E_{**}(E) \otimes_{\pi_{**}(E)} M$ is canonically a left $\pi_{**}(E)$-module by Remark D.11.)

These data must make the following diagrams commute:

(1) (Co-unitality)

$$\begin{tikzcd} M \arrow[r, "\Psi_M"] \arrow[dr, "\cong"'] & E_{**}(E) \otimes_{\pi_{**}(E)} M \arrow[d, "\varepsilon \otimes M"] \\ & \pi_{**}(E) \otimes_{\pi_{**}(E)} M \end{tikzcd}$$

(2) (Co-action property)

$$\begin{tikzcd} M \arrow[rr, "\Psi_M"] \arrow[d, "\Psi_M"'] & & E_{**}(E) \otimes_{\pi_{**}(E)} M \arrow[d, "\Psi \otimes M"] \\ E_{**}(E) \otimes_{\pi_{**}(E)} M \arrow[rr, "E_{**}(E) \otimes \Psi_M"] & & E_{**}(E) \otimes_{\pi_{**}(E)} E_{**}(E) \otimes_{\pi_{**}(E)} M \end{tikzcd}$$

Given two left $E_{**}(E)$-comodules M and N, a *homomorphism of $E_{**}(E)$-comodules* is a homomorphism $f : M \to N$ of the underlying graded left $\pi_{**}(E)$-modules such that the following diagram commutes:

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "\Psi_M"'] & N \arrow[d, "\Psi_N"] \\ E_{**}(E) \otimes_{\pi_{**}(E)} M \arrow[r, "E \otimes f"] & E_{**}(E) \otimes_{\pi_{**}(E)} N \end{tikzcd}$$

We write $E_{**}(E)$-**CoMod** for the resulting category of left $E_{**}(E)$-comodules. The notation for the hom-sets in this category is usually abbreviated to

$$\mathrm{Hom}_{E_{**}(E)}(-,-) := \mathrm{Hom}_{E_{**}(E)\text{-}\mathbf{CoMod}}(-,-).$$

Lemma C.8 ([5, Proposition 2.30, 2.33])**.** *Let E be a flat commutative ring spectrum, and let X and Y be spectra such that $E_{**}(X)$ is a projective module over $\pi_{**}(E)$. Then for all $s \geq 0$ and $t, w \in \mathbb{Z}$, there is an isomorphism*

$$\Phi : [X, E \wedge Y]_{t,w} \to \mathrm{Hom}^{t,w}_{E_{**}(E)}(E_{**}(X), E_{**}(E \wedge Y)),$$

obtained by sending a class $f : S^{t,w} \wedge X \to E \wedge Y$ in $[X, E \wedge Y]_{t,w}$ to the map

$$\Phi_f : E_{*,*}(X) \to E_{*+t,*+w}(X \wedge Y)$$

sending

$$[S^{a,b} \xrightarrow{g} E \wedge X] \mapsto [S^{a+t,b+w} \cong S^{a,b} \wedge S^{t,w} \xrightarrow{g \wedge S^{t,w}} E \wedge X \wedge S^{t,w} \cong E \wedge S^{t,w} \wedge X \xrightarrow{E \wedge f} E \wedge E \wedge Y].$$

Proof. Let $f : S^{t,w} \wedge X \to E \wedge Y$. First we want to show that Φ_f is actually an $E_{**}(E)$-comodule
finish homomorphism. □

Recall when working with the classical Adams spectral sequence, one usually develops the theory of *graded commutative Hopf algebroids*, i.e., internal groupoids in the opposite category $\mathbf{gCRing}^{\mathrm{op}}$ of $\mathbb{Z}$-graded commutative rings, regarded with its cartesian monoidal category structure. Then, one goes on to show that given a commutative ring spectrum E in **hoSp** that $E_*(E)$ is a commutative Hopf algebroid over $\pi_*(E)$.

Now, in the motivic setting, things become a bit more subtle. Namely, given a commutative ring spectrum E in $\mathbf{SH}_{\mathscr{S}}$, we would like $E_{**}(E)$ to be a "bigraded commutative Hopf algebroid over $\pi_{**}(E)$". To define such a thing, we would like to find some category $\mathcal{C}$ containing objects such as $E_{**}(E)$ and $\pi_{**}(E)$ so that the pair $(E_{**}(E), \pi_{**}(E))$ forms a groupoid object in $\mathcal{C}^{\mathrm{op}}$. The naïve answer would be to consider $E_{**}(E)$ and $\pi_{**}(E)$ as objects in some category of "bigraded commutative rings", in the same way we considered $E_*(E)$ and $\pi_*(E)$ as ojects in the category of graded commutative rings in the classical case. Yet, we run into difficulty here, as the commutative law for $E_{**}(E) = \pi_{**}(E \wedge E)$ and $\pi_{**}(E)$ (**??**) depends on their structure as $\pi_{**}(S)$-algebras. Thus, we instead are led to the category of commutative $\pi_{**}(S)$-algebras:

Definition C.9. Let $\mathbf{CStabRing}_{\mathscr{S}}$ denote the full subcategory of $\mathbb{Z}^2$-graded algebras over $\pi^{\mathscr{S}}_{**}(S)$ (the stable homotopy groups of the sphere spectrum S in $\mathbf{SH}_{\mathscr{S}}$) containing those objects satisfying the commutativity condition given in **??**. Explicitly, an object in $\mathbf{CStabRing}_{\mathscr{S}}$ is a $\mathbb{Z}^2$-graded ring C_{**} together with a ring morphism $e : \pi^{\mathscr{S}}_{**}(S) \to C_{**}$ such that given $x \in C_{a_1,a_2}$ and $y \in C_{b_1,b_2}$, we have

$$x \cdot y = y \cdot x \cdot (-1)^{a_1 b_1} \cdot e(-\epsilon)^{a_2 b_2}.$$

A morphism in $\mathbf{CStabRing}_{\mathscr{S}}$ is simply a morphism of $\pi^{\mathscr{S}}_{**}(S)$-algebras.

Definition C.10. For our purposes, a *bigraded commutative Hopf algebroid* (Γ, A) is an internal groupoid in the opposite category $(\mathbf{CStabRing}_{\mathscr{S}})^{\mathrm{op}}$.

Let's unravel what this definition means. First, recall the definition of a groupoid object in a category:

Definition C.11. Let $\mathcal{C}$ be a category admitting pullbacks. A *groupoid object* in $\mathcal{C}$ consists of a pair of objects (M, O) together with five morphisms

(1) *Source and target*: $s, t : M \to O$,
(2) *Identity*: $e : O \to M$,
(3) *Composition*: $c : M \times_O M \to M$,
(4) *Inverse*: $i : M \to M$

Explicitly, $M \times_O M$ fits into the following pullback diagram:

$$\begin{tikzcd}
M \times_O M \arrow[r, "p_2"] \arrow[d, "p_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & M \arrow[d, "t"] \\
M \arrow[r, "s"'] & O
\end{tikzcd}$$

These data must satisfy the following diagrams:

(1) Composition works correctly:

$$\begin{tikzcd}
M \times_O M \arrow[r, "c"] \arrow[d, "p_1"'] & M \arrow[d, "t"] \\
M \arrow[r, "t"] & O
\end{tikzcd}
\qquad
\begin{tikzcd}
M \arrow[dr, "s"'] & O \arrow[l, "e"'] \arrow[r, "e"] \arrow[d, equal] & M \arrow[dl, "t"] \\
 & O &
\end{tikzcd}
\qquad
\begin{tikzcd}
M \times_O M \arrow[r, "p_2"] \arrow[d, "c"'] & M \arrow[d, "s"] \\
M \arrow[r, "s"] & O
\end{tikzcd}$$

(2) Associativity of composition:

$$\begin{tikzcd}
M \times_O (M \times_O M) \arrow[rr, "\cong"] \arrow[d, "M\times c"'] & & M \times_O (M \times_O M) \arrow[d, "M\times c"] \\
M \times_O M \arrow[r, "c"] & M & M \times_O M \arrow[l, "c"']
\end{tikzcd}$$

where the top objects and the maps $M \times c$, $c \times M$ are determined like so, where both outer and inner squares in the following diagram are pullback squares:

$$\begin{tikzcd}
M \times_O (M \times_O M) \arrow[rr, "p_2"] \arrow[dr, dashed, "M\times c"] \arrow[dd, "p_1"'] & & M \times_O M \arrow[d, "c"] \\
 & M \times_O M \arrow[r, "p_2"] \arrow[d, "p_1"'] & M \arrow[d, "t"] \\
M \arrow[r, equal] & M \arrow[r, "s"] & O
\end{tikzcd}
\qquad
\begin{tikzcd}
(M \times_O M) \times_O M \arrow[rr, "p_2"] \arrow[dr, dashed, "c\times M"] \arrow[d, "p_1"'] & & M \arrow[d, equal] \\
 & M \times_O M \arrow[r] \arrow[d] & M \arrow[d, "t"] \\
M \times_O M \arrow[r, "c"] & M \arrow[r, "s"] & O
\end{tikzcd}$$

(3) Unitality of composition: Given the maps $(\mathrm{id}_M, e\circ t), (e \circ s, \mathrm{id}_M) : M \to M \times_O M$ defined by the universal property of $M \times_O M$:

$$\begin{tikzcd}
M \arrow[dr, dashed, "(\mathrm{id}_M{,}e\circ s)"] \arrow[drr, bend left, "e\circ s"] \arrow[ddr, bend right, equal] & & \\
 & M \times_O M \arrow[r, "p_2"] \arrow[d, "p_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & M \arrow[d, "t"] \\
 & M \arrow[r, "s"] & O
\end{tikzcd}
\qquad
\begin{tikzcd}
M \arrow[dr, dashed, "(e\circ t{,}\mathrm{id}_M)"] \arrow[drr, bend left, equal] \arrow[ddr, bend right, "e\circ t"'] & & \\
 & M \times_O M \arrow[r, "p_2"] \arrow[d, "p_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & M \arrow[d, "t"] \\
 & M \arrow[r, "s"] & O
\end{tikzcd}$$

the following diagram commutes:

$$\begin{tikzcd} M \arrow[r, "(e\circ t, \mathrm{id}_M)"] \arrow[d, "(\mathrm{id}_M, e\circ s)"'] & M \times_O M \arrow[d, "c"] \\ M \times_O M \arrow[r, "c"'] & M \end{tikzcd}$$

(4) Inverse: The following diagrams must commute:

$$\begin{tikzcd} & M \arrow[dl, equal] \arrow[d, "i"] \\ M & M \arrow[l, "i"'] \end{tikzcd} \qquad \begin{tikzcd} M \arrow[r, "(\mathrm{id}_M, i)"] \arrow[d, "t"'] & M \times_O M \arrow[d, "c"] & M \arrow[l, "(i, \mathrm{id}_M)"'] \arrow[d, "s"] \\ O \arrow[r, "e"'] & M & O \arrow[l, "e"] \end{tikzcd} \qquad \begin{tikzcd} & M \arrow[dl, "s"'] \arrow[d, "i"] \arrow[dr, "t"] & \\ O & M \arrow[l, "t"] \arrow[r, "s"'] & O \end{tikzcd}$$

where the arrows (id_M, i) and (i, id_M) are determined by the universal property of $M \times_O M$ like so:

$$\begin{tikzcd} M \arrow[dr, dashed, "(\mathrm{id}_M, i)"] \arrow[drr, bend left, "i"] \arrow[ddr, bend right, equal] & & \\ & M \times_O M \arrow[r, "p_2"] \arrow[d, "p_1"'] & M \arrow[d, "t"] \\ & M \arrow[r, "s"'] & O \end{tikzcd} \qquad \begin{tikzcd} M \arrow[dr, dashed, "(i, \mathrm{id}_M)"] \arrow[drr, bend left, equal] \arrow[ddr, bend right, "i"'] & & \\ & M \times_O M \arrow[r, "p_2"] \arrow[d, "p_1"'] & M \arrow[d, "t"] \\ & M \arrow[r, "s"'] & O \end{tikzcd}$$

Proposition C.12 ([5, Proposition 2.3, Proposition 2.12])**.** *Let E be a flat cellular commutative ring spectrum (Definition 2.8, Definition D.9). Consider the following data:*

(1) The maps $\eta_L, \eta_R : \pi_{**}(E) \to E_{**}(E)$ which send an element $\alpha : S^a \to E$ to the compositions

$$S^a \xrightarrow{\alpha} E \xrightarrow{\cong} E \wedge S \xrightarrow{E\wedge e} E \wedge E$$

and

$$S^a \xrightarrow{\alpha} E \xrightarrow{\cong} S \wedge E \xrightarrow{e\wedge E} E \wedge E,$$

respectively.

(2) The map $\varepsilon : E_{**}(E) \to \pi_{**}(E)$ sending a class $\alpha : S^a \to E \wedge E$ to the composition

$$S^a \xrightarrow{\alpha} E \wedge E \xrightarrow{\mu} E$$

(3) The map $\Psi : E_{**}(E) \to E_{**}(E) \otimes_{\pi_{**}(E)} E_{**}(E)$ which factors as

$$E_{**}(E) \to E_{**}(E \wedge E) \xrightarrow{\cong} E_{**}(E) \otimes_{\pi_{**}(E)} E_{**}(E)$$

where the second arrow is the isomorphism prescribed by Proposition D.10, and the first arrow sends a class $\alpha : S^a \to E \wedge E$ to the composition

$$S^a \xrightarrow{\alpha} E \wedge E \cong E \wedge S \wedge E \xrightarrow{E\wedge e\wedge E} E \wedge E \wedge E.$$

(4) The map $c : E_{**}(E) \to E_{**}(E)$ sending a map $\alpha : S^a \to E \wedge E$ to the composition

$$S^a \xrightarrow{\alpha} E \wedge E \xrightarrow{\tau} E \wedge E,$$

where τ is the symmetry map prescribed by the symmetric monoidal structure on $\mathbf{SH}_{\mathscr{S}}$.

Then all of these maps are homomorphisms of $\pi_{**}(S)$-algebras, and they furthermore endow the pair $(E_{**}(E), \pi_{**}(E))$ with the structure of a bigraded commutative Hopf algebroid (Definition C.10)

TODO

Proof. □

Appendix D. Monoid objects in a stable homotopy category

Definition D.1. Let $(\mathcal{C}, \otimes, S)$ be a symmetric monoidal category with left unitor, right unitor, and associator, and symmetry isomorphism λ, ρ, α, and τ, respectively. Then a *monoid object* (E, μ, e) is an object E in $\mathcal{C}$ along with a multiplication map $\mu : E \otimes E \to E$ and a unit map $e : S \to E$ such that the following diagram commutes:

$$\begin{tikzcd} E \otimes S \arrow[r, "E\otimes e"] \arrow[dr, "\rho"'] & E \otimes E \arrow[d, "\mu"] & S \otimes E \arrow[l, "e\otimes E"'] \arrow[dl, "\lambda"] \\ & E & \end{tikzcd} \qquad \begin{tikzcd} (E\otimes E)\otimes E \arrow[rr, "\mu\otimes E"] \arrow[d, "\alpha"'] & & E \otimes E \arrow[d, "\mu"] \\ E\otimes(E\otimes E) \arrow[r, "E\otimes\mu"] & E\otimes E \arrow[r, "\mu"] & E \end{tikzcd}$$

The first diagram expresses unitality, while the second expressed associativity. If in addition the following diagram commutes,

$$\begin{tikzcd} E \otimes E \arrow[rr, "\tau"] \arrow[dr, "\mu"'] & & E \otimes E \arrow[dl, "\mu"] \\ & E & \end{tikzcd}$$

then we say (E, μ, e) is a *commutative* monoid object.

Proposition D.2. *Let (E_1, μ_1, e_1) and (E_2, μ_2, e_2) be monoid objects in a symmetric monoidal category $(\mathcal{C}, \otimes, S)$. Then $E_1 \otimes E_2$ is canonically a ring spectrum via the maps*

$$\mu : E_1 \otimes E_2 \otimes E_1 \otimes E_2 \xrightarrow{E_1\otimes\tau\otimes E_2} E_1 \otimes E_1 \otimes E_2 \otimes E_2 \xrightarrow{\mu_1\otimes\mu_2} E_1 \otimes E_2$$

and

$$e : S \cong S \otimes S \xrightarrow{e_1\otimes e_2} E_1 \otimes E_2.$$

Proof. □ todo

In what follows, fix a stable homotopy category $\mathcal{SH}$ (Definition 2.1) along with the additional data therewithin, and adopt the conventions outlined in Section 2.1. Further suppose we have fixed a coherent family of isomorphisms

$$\phi_{a,b} : S^{a+b} \xrightarrow{\cong} S^a \otimes S^b,$$

in the sense of Definition 2.2 (the existence of such a coherent family is guaranteed by Theorem 2.3).

Proposition D.3. *Let (E, μ, e) be a commutative monoid object in $\mathcal{SH}$, and consider the multiplication map $\pi_*(E) \times \pi_*(E) \to \pi_*(E)$ which sends classes $x : S^a \to E$ and $y : S^b \to E$ to the composition*

$$S^{a+b} \xrightarrow{\phi_{a,b}} S^a \otimes S^b \xrightarrow{x\otimes y} E \otimes E \xrightarrow{\mu} E.$$

Then this endows $\pi_*(E)$ with the structure of an A-graded ring with unit $e \in \pi_0(E) = [S, E]$.

Proof. First we show this map is associative: Given classes x, y, and z in $\pi_a(E)$, $\pi_b(E)$, and $\pi_c(E)$, respectively, consider the following diagram:

$$\begin{tikzcd} S^{a+b+c} \arrow[r, "\phi_{a+b,c}"] \arrow[d, "\phi_{a,b+c}"'] & S^{a+b}\otimes S^c \arrow[r, "\phi_{a,b}\otimes S^c"] & (S^a\otimes S^b)\otimes S^c \arrow[r, "(x\otimes y)\otimes z"] \arrow[dl, "\cong"] & (E\otimes E)\otimes E \arrow[r, "\mu\otimes E"] \arrow[dl, "\cong"] & E\otimes E \arrow[d, "\mu"] \\ S^a\otimes S^{b+c} \arrow[r, "S^a\otimes\phi_{b,c}"'] & S^a\otimes(S^b\otimes S^c) \arrow[r, "x\otimes(y\otimes z)"'] & E\otimes(E\otimes E) \arrow[r, "E\otimes\mu"'] & E\otimes E \arrow[r, "\mu"'] & E \end{tikzcd}$$

Commutativity of the left pentagon is the coherence condition for the $\phi_{a,b}$'s. Commutativity of the middle parallelogram is naturality of the associator isomorphisms. Commutativity of the right pentagon is associativity of μ. The fact that the two outside compositions equal $(x \cdot y) \cdot z$ and $x \cdot (y \cdot z)$, respectively, follows by functoriality of $- \otimes -$.

Next we claim the map $e : S \to E$ is a unit for this multiplication. Given $x \in \pi_a(E)$, consider the following diagram:

$$\begin{tikzcd}
S\otimes S^a \arrow[dd, "e\otimes x"'] \arrow[dr, "S\otimes x"] & & S^a \arrow[ll, "\phi_{0,a}=\lambda^{-1}_{S^a}"'] \arrow[rr, "\phi_{a,0}=\rho^{-1}_{S^a}"] \arrow[dd, "x"] & & S^a\otimes S \arrow[dl, "x\otimes S"'] \arrow[dd, "x\otimes e"] \\
& S\otimes E \arrow[dl, "e\otimes E"'] \arrow[dr, "\lambda_E"] & & E\otimes S \arrow[dl, "\rho_E"'] \arrow[dr, "E\otimes e"] & \\
E\otimes E \arrow[rr, "\mu"'] & & E & & E\otimes E \arrow[ll, "\mu"]
\end{tikzcd}$$

Commutativity of the top two large triangles is naturality of the unitor isomorphisms. Commutativity of the right and leftmost triangles is functoriality of $-\otimes-$. Commutativity of the bottom triangles is unitality of μ. Hence, we have that $e \cdot x = x = x \cdot e$.

This product is also bilinear (distributive). Given $x, x' \in \pi_a(E)$ and $y, y' \in \pi_b(E)$, consider the following diagrams:

$$\begin{tikzcd}
S^{a+b} \arrow[r, "\phi_{a,b}"] \arrow[d, "\Delta"'] & S^a\otimes S^b \arrow[r, "\Delta\otimes S^b"] \arrow[d, "\Delta"] & (S^a\oplus S^a)\otimes S^b \arrow[r, "(x\oplus x')\otimes y"] \arrow[dl, "\cong"'] & (E\oplus E)\otimes E \arrow[dl, "\cong"'] \arrow[d, "\nabla\otimes E"] & \\
S^{a+b}\oplus S^{a+b} \arrow[r, "\phi_{a,b}\oplus\phi_{a,b}"'] & (S^a\otimes S^b)\oplus(S^a\otimes S^b) \arrow[r, "(x\otimes y)\oplus(x'\otimes y)"'] & (E\otimes E)\oplus(E\otimes E) \arrow[r, "\nabla"'] & E\otimes E \arrow[r, "\mu"] & E
\end{tikzcd}$$

$$\begin{tikzcd}
S^{a+b} \arrow[r, "\phi_{a,b}"] \arrow[d, "\Delta"'] & S^a\otimes S^b \arrow[r, "S^a\otimes\Delta"] \arrow[d, "\Delta"] & S^b\otimes(S^b\oplus S^b) \arrow[r, "x\otimes(y\oplus y')"] \arrow[dl, "\cong"'] & E\otimes(E\oplus E) \arrow[dl, "\cong"'] \arrow[d, "E\otimes\nabla"] & \\
S^{a+b}\oplus S^{a+b} \arrow[r, "\phi_{a,b}\oplus\phi_{a,b}"'] & (S^a\otimes S^b)\oplus(S^a\otimes S^b) \arrow[r, "(x\otimes y)\oplus(x\otimes y')"'] & (E\otimes E)\oplus(E\otimes E) \arrow[r, "\nabla"'] & E\otimes E \arrow[r, "\mu"] & E
\end{tikzcd}$$

The unlabeled isomorphisms are those given by the fact that $- \otimes -$ is additive in each variable (since $\mathcal{SH}$ is tensor triangulated). Commutativity of the left squares is naturality of $\Delta : X \to X \oplus X$ in an additive category. Commutativity of the rest of the diagram follows again from the fact that $- \otimes -$ is an additive functor in each variable. Hence, by functoriality of $- \otimes -$, these diagrams tell us that $(x + x') \cdot y = x \cdot y + x' \cdot y$ and $x \cdot (y + y') = x \cdot y + x \cdot y'$, respectively. □

Proposition D.4. *For all $a, b \in A$ there exists an element $\theta_{a,b} \in \pi_0(S) = [S, S]$ (determined by choice of coherent family $\{\phi_{a,b}\}$) such that given any commutative monoid object (E, μ, e) in $\mathcal{SH}$, the A-graded ring structure on $\pi_*(E)$ (Proposition 2.5) has a commutativity formula given by*

$$x \cdot y = y \cdot x \cdot (e \circ \theta_{a,b})$$

for all $x \in \pi_a(E)$ and $y \in \pi_b(E)$. In particular, $\theta_{a,b} \in \mathrm{Aut}(S)$ is the composition

$$S \xrightarrow{\cong} S^{-a-b} \otimes S^a \otimes S^b \xrightarrow{S^{-a-b}\otimes\tau} S^{-a-b} \otimes S^b \otimes S^a \xrightarrow{\cong} S,$$

where the outermost maps are the unique maps specified by Remark 2.4.

Proof. Let $\phi_{a,b}$, E, x, and y as in the statement of the proposition. Now consider the following diagram

$$\begin{tikzcd}
S^{a+b} \arrow[r, "\phi_{a,b}"] \arrow[dd, dashed, "\phi_{b,a}^{-1}\circ\tau\circ\phi_{a,b}"'] & S^a \otimes S^b \arrow[r, "x\otimes y"] \arrow[dd, "\tau"'] & E \otimes E \arrow[dd, "\tau"'] \arrow[dr, "\mu"] & \\
 & & & E \\
S^{a+b} \arrow[r, "\phi_{b,a}"] & S^b \otimes S^a \arrow[r, "y\otimes x"] & E \otimes E \arrow[ur, "\mu"'] &
\end{tikzcd}$$

The left square commutes by definition. The middle square commutes by naturality of the symmetry isomorphism. Finally, the right square commutes by commutativity of E. Unravelling definitions, we have shown that under the product on $\pi_*(E)$ induced by the $\phi_{a,b}$'s,

$$x \cdot y = (y \cdot x) \circ (\phi_{b,a}^{-1} \circ \tau \circ \phi_{a,b}).$$

Thus, in order to show the desired result it further suffices to show that

$$(y \cdot x) \circ (\phi_{b,a}^{-1} \circ \tau \circ \phi_{a,b}) = y \cdot x \cdot (e \circ \theta_{a,b}).$$

Consider the following diagram:

$$\begin{tikzcd}
S^{a+b} \arrow[rr, "\phi_{a,b}"] \arrow[d, "\cong"'] & & S^a \otimes S^b \arrow[d, "\tau"] \\
S^b \otimes S^a \otimes S^{-a-b} \otimes S^a \otimes S^b \arrow[urr, "\cong"'] \arrow[d, "S^b\otimes S^a\otimes S^{-a-b}\otimes\tau"'] & & S^b \otimes S^a \arrow[d, "\phi_{b,a}^{-1}"] \\
S^b \otimes S^a \otimes S^{-a-b} \otimes S^b \otimes S^a \arrow[urr, "\cong"] \arrow[d, "\cong"'] & & S^{a+b} \arrow[d, "\phi_{b,a}"] \\
S^b \otimes S^a \otimes S \arrow[rr, "\cong"] \arrow[dr, "y\otimes x\otimes S"] \arrow[dd, "x\otimes y\otimes e"'] & & S^b \otimes S^a \arrow[dd, "y\otimes x"] \\
 & E \otimes E \otimes S \arrow[dr, "\rho"] & \\
E \otimes E \otimes E \arrow[ur, "E\otimes E\otimes e"] \arrow[rr, "E\otimes\mu"] \arrow[d, "\mu\otimes E"'] & & E \otimes E \arrow[d, "\mu"] \\
E \otimes E \arrow[rr, "\mu"'] & & E
\end{tikzcd}$$

Here we are suppressing associators from the notation, and any map simply labelled $\cong$ is an appropriate composition of copies of $\phi_{a,b}$'s, associators, and their inverses, so that each of these maps are necessarily unique by Remark 2.4. The top triangle commutes by coherence for the $\phi_{a,b}$'s. The parallelogram commutes by naturality of τ and coherence of the of $\phi_{a,b}$'s. The trapezoid commutes again by coherence for the $\phi_{a,b}$'s. The middle right large triangle commutes by naturality of the unitors (and the fact that $S^b \otimes \phi_{a,0}$ coincides with the unitor $S^b \otimes S^a \otimes S \to S^b \otimes S^a$). The middle left triangle commutes by functoriality of $- \otimes -$. The middle triangle commutes by unitality of μ. Finally, the bottom rectangle commmutes by associativity of μ. Hence, by unravelling definitions and applying functoriality of $- \otimes -$, we get that the top composition is $(y \cdot x) \circ (\phi_{b,a}^{-1} \circ \tau \circ \phi_{a,b})$, while the bottom composition is $y \cdot x \cdot (e \circ \theta_{a,b})$, so they are equal as desired. $\square$

Proposition D.5. *Given $a \in A$, we have $\theta_{0,a} = \theta_{a,0} = \mathrm{id}_S$.*

Proof. Recall $\theta_{a,0}$ is the composition

$$S \xrightarrow{\phi_{-a,a}} S^{-a} \otimes S^{a} \xrightarrow{S^{-a}\otimes\phi_{a,0}} S^{-a} \otimes (S^{a} \otimes S) \xrightarrow{S^{-a}\otimes\tau} S^{-a} \otimes (S \otimes S^{a}) \xrightarrow{S^{-a}\otimes\phi_{0,a}^{-1}} S^{-a} \otimes S^{a} \xrightarrow{\phi_{-a,a}^{-1}} S$$

By the coherence theorem for symmetric monoidal categories and the fact that $\phi_{a,0}$ and $\phi_{0,a}$ coincide with the unitors, we have that the composition

$$S^{a} \xrightarrow{\phi_{a,0}=\rho_{S^a}^{-1}} S^{a} \otimes S \xrightarrow{\tau} S \otimes S^{a} \xrightarrow{\phi_{0,a}^{-1}=\lambda_{S^a}} S^{a}$$

is precisely the identity map, so by functoriality of $- \otimes -$, we have that $\theta_{a,0}$ is the composition

$$S \xrightarrow{\phi_{-a,a}} S^{-a} \otimes S^{a} \xrightarrow{=} S^{-a} \otimes S^{a} \xrightarrow{\phi_{-a,a}^{-1}} S,$$

so $\theta_{a,0} = \mathrm{id}_S$, meaning

$$x \cdot y = y \cdot x \cdot (e \circ \theta_{a,0}) = y \cdot x \cdot e = y \cdot x,$$

where the last equality follows by the fact that e is the unit for the multiplication on $\pi_*(E)$. An entirely analagous argument yields that $\theta_{0,a} = \mathrm{id}_S$. $\square$

Proposition D.6. *Given some $a \in A$, the functors Σ^a and Σ^{-a} canonically form an adjoint equivalence of $\mathcal{SH}$.*

Proof. Let $X, Y \in \mathcal{SH}$. By [3, Lemma 3.2], in order to show Σ^a and Σ^{-a} are adjoint equivalences, it suffices to construct natural isomorphisms $\eta : \mathrm{Id}_{\mathcal{SH}} \Rightarrow \Sigma^{-a} \circ \Sigma^{a}$ and $\varepsilon : \Sigma^{a} \circ \Sigma^{-a} \Rightarrow \mathrm{Id}_{\mathcal{SH}}$ such that for all X in $\mathcal{SH}$, the following diagram commutes:

(1)
$$\begin{tikzcd} \Sigma^a X \arrow[r, "(\Sigma^a \eta)_X"] \arrow[dr] & \Sigma^a \Sigma^{-a} \Sigma^a X \arrow[d, "(\varepsilon \Sigma^a)_X"] \\ & \Sigma^a X \end{tikzcd}$$

Given an object X in $\mathcal{SH}$, define $\eta_X : X \to \Sigma^{-a}\Sigma^{a} X = S^{-a} \otimes S^{a} \otimes X$ to be the composition

$$X \xrightarrow{\lambda_X^{-1}} S \otimes X \xrightarrow{\phi_{-a,a}\otimes X} S^{-a} \otimes S^{a} \otimes X.$$

Clearly this is an isomorphism. To see this is natural, let $f : X \to Y$ in $\mathcal{SH}$. Then consider the following diagram:

$$\begin{tikzcd} X \arrow[r, "\lambda_X^{-1}"] \arrow[d, "f"'] & S \otimes X \arrow[r, "\phi_{-a,a}\otimes X"] \arrow[d, "S\otimes f"] & S^{-a} \otimes S^{a} \otimes X \arrow[d, "S^{-a}\otimes S^{a}\otimes f"] \\ Y \arrow[r, "\lambda_Y^{-1}"'] & S \otimes Y \arrow[r, "\phi_{-a,a}\otimes Y"'] & S^{-a} \otimes S^{a} \otimes Y \end{tikzcd}$$

The left square commutes by naturality of λ. The right square commutes by functoriality of $- \otimes -$. Hence η is indeed a natural isomorhpism.

On the other hand, given an object X in $\mathcal{SH}$, define $\varepsilon_X : \Sigma^{a}\Sigma^{-a} X = S^{a} \otimes S^{-a} \otimes X \to X$ to be the composition

$$S^{a} \otimes S^{-a} \otimes X \xrightarrow{\phi_{a,-a}^{-1}} S \otimes X \xrightarrow{\lambda_X} X.$$

Clearly this is an isomorphism. To see it is natural, let $f : X \to Y$ in $\mathcal{SH}$. Then consider the following diagram:

$$\begin{tikzcd}
S^a \otimes S^{-a} \otimes X \arrow[r, "\phi_{a,-a}^{-1}\otimes X"] \arrow[d, "S^a\otimes S^{-a}\otimes f"'] & S \otimes X \arrow[r, "\lambda_X"] \arrow[d, "S\otimes f"'] & X \arrow[d, "f"] \\
S^a \otimes S^{-a} \otimes Y \arrow[r, "\phi_{a,-a}^{-1}\otimes Y"'] & S \otimes Y \arrow[r, "\lambda_Y"'] & Y
\end{tikzcd}$$

The left square commutes by functoriality of $-\otimes-$. The right square commutes by naturality of λ. Hence, ε is natural.

Finally, let X be an object in $\mathcal{SH}$. Unravelling definitions, by functoriality of $-\otimes-$, in order to show that diagram (1) commutes, it suffices to show the following diagram commutes:

$$\begin{tikzcd}
S^a \otimes X \arrow[r, "S^a\otimes\lambda_X^{-1}"] \arrow[ddrr, equal] & S^a \otimes S \otimes X \arrow[r, "S^a\otimes\phi_{-a,a}\otimes X"] & S^a \otimes S^{-a} \otimes S^a \otimes X \arrow[d, "\phi_{a,-a}^{-1}\otimes S^a\otimes X"] \\
 & & S \otimes S^a \otimes X \arrow[d, "\lambda_{S^a\otimes X}"] \\
 & & S^a \otimes X \arrow[uul, "\phi_{a,0}\otimes X"']
\end{tikzcd}$$

First, note that by the coherence theorem for monoidal categories, $\lambda_{S^a\otimes X} = \lambda_{S^a} \otimes X$[3]. And furthermore, recall $\lambda_{S^a} = \phi_{0,a}^{-1}$. Hence, the right triangle is precisely the diagram obtained by applying $-\otimes X$ to the coherence diagram for the $\phi_{a,b}$'s, so it commutes. Commutativity of the left triangle follows by the coherence theorem for monoidal categories and the fact that $\phi_{a,0} = \lambda_{S^a}^{-1}$. Hence, the diagram commutes, so (Σ^a, Σ^{-a}) forms an adjoint equivalence of $\mathcal{SH}$. □

Proposition D.7. *Let X and Y be objects in $\mathcal{SH}$. Then the pairing*

$$\pi_*(X) \times \pi_*(Y) \to \pi_*(X \otimes Y)$$

sending $x : S^a \to X$ and $y : S^b \to Y$ to the composition

$$S^{a+b} \xrightarrow{\phi_{a,b}} S^a \otimes S^b \xrightarrow{x\otimes y} X \otimes Y$$

is bilinear.

Proof. Let $a, b \in A$, and let $x_1, x_2 : S^a \to X$ and $y : S^b \to Y$. Then consider the following diagram

$$\begin{tikzcd}
S^{a+b} \arrow[r, "\cong"] & S^a \otimes S^b \arrow[r, "\Delta\otimes S^b"] \arrow[d, "\Delta"'] & (S^a \oplus S^a) \otimes S^b \arrow[dl, "\cong"'] \arrow[d, "(x_1\oplus x_2)\otimes y"] \\
 & (S^a \otimes S^b) \oplus (S^a \otimes S^b) \arrow[d, "(x_1\otimes y)\oplus(x_2\otimes y)"'] & (X \oplus X) \otimes Y \arrow[dl, "\cong"'] \arrow[d, "\nabla\otimes Y"] \\
 & (X \otimes Y) \oplus (X \otimes Y) \arrow[r, "\nabla"] & X \otimes Y
\end{tikzcd}$$

The isomorphisms are given by the fact that $-\otimes-$ is additive in each variable. Both triangles and the parallelogram commute since $-\otimes-$ is additive. By functoriality of $-\otimes-$, the top composition is $(x_1 + x_2) \cdot y$ and the bottom composition is $x_1 \cdot y + x_2 \cdot y$, so they are equal, as

[3]Technically, this equality only holds up to composition with an associator, but we are ignoring such issues.

desired. An entirely analagous argument yields that $x \cdot (y_1 + y_2) = x \cdot y_1 + x \cdot y_2$ for $x \in \pi_*(X)$ and $y_1, y_2 \in \pi_*(Y)$. ☐

Proposition D.8 ([4, Proposition 5.11])**.** *Let (E, μ, e) be a ring spectrum. Then for any spectrum X, $E_*(X)$ canonically inherits the structure of a left graded $\pi_*(E)$-module via the map*

$$\pi_*(E) \times E_*(X) \to E_*(X)$$

which given $a, b \in A$, sends $x : S^a \to E$ and $y : S^b \to E \otimes X$ to the composition

$$x \cdot y : S^{a+b} \cong S^a \otimes S^b \xrightarrow{x \otimes y} E \otimes (E \otimes X) \cong (E \otimes E) \otimes X \xrightarrow{\mu \otimes X} E \otimes X.$$

Similarly $X_*(E)$ canonically inherits the structure of a right graded $\pi_*(E)$-module via the map

$$X_*(E) \times \pi_*(E) \to X_*(E)$$

which given $a, b \in A$, sends $x : S^a \to X \otimes E$ and $y : S^b \to E$ to the composition

$$x \cdot y : S^{a+b} \cong S^a \otimes S^b \xrightarrow{x \otimes y} (X \otimes E) \otimes E \cong X \otimes (E \otimes E) \xrightarrow{X \otimes \mu} X \otimes E.$$

In particular, $E_*(E)$ is a $\pi_*(E)$-bimodule, in the sense that the left and right actions of $\pi_*(E)$ are compatible, so that given $y, z \in \pi_*(E)$ and $x \in E_*(E)$, $y \cdot (x \cdot z) = (y \cdot x) \cdot z$.

Proof. First we show that the map $\pi_*(E) \times E_*(X) \to E_*(X)$ endows $E_*(X)$ with the structure of a left $\pi_*(E)$-module. Let $a, b, c \in A$ and $x, x' : S^a \to E \otimes X$, $y : S^b \to E$, and $z, z' \in S^c \to E$. Then we wish to show that:

(1) $y \cdot (x + x') = y \cdot x + y \cdot x'$,
(2) $(z + z') \cdot x = z \cdot x + z' \cdot x$,
(3) $(zy) \cdot x = z \cdot (y \cdot x)$,
(4) $e \cdot x = x$.

Axioms (1) and (2) follow by the fact that $E_*(X) = \pi_*(E \otimes X)$ and Proposition D.7. To see (3), consider the diagram:

$$\begin{tikzcd}
S^{a+b+c} \arrow[r, "\cong"] \arrow[d, "\cong"'] & S^{c+b} \otimes S^a \arrow[dd, "\cong"] & \\
S^c \otimes S^{b+a} \arrow[d, "\cong"'] & & \\
S^c \otimes (S^b \otimes S^a) \arrow[d, "z\otimes(y\otimes x)"'] & (S^c \otimes S^b) \otimes S^a \arrow[l, "\cong"'] \arrow[d, "(z\otimes y)\otimes x"] & \\
E \otimes (E \otimes (E \otimes X)) \arrow[dd, "\cong"'] & (E \otimes E) \otimes (E \otimes X) \arrow[l, "\cong"'] \arrow[r, "\mu\otimes(E\otimes X)"] & E \otimes (E \otimes X) \arrow[d, "\cong"] \\
& ((E \otimes E) \otimes E) \otimes X \arrow[u, "\cong"'] \arrow[r, "(\mu\otimes E)\otimes X"] \arrow[d, "\cong"] & (E \otimes E) \otimes X \arrow[dd, "\mu\otimes X"] \\
E \otimes ((E \otimes E) \otimes X) \arrow[r, "\cong"] \arrow[d, "E\otimes(\mu\otimes X)"'] & (E \otimes (E \otimes E)) \otimes X \arrow[d, "(E\otimes\mu)\otimes X"] & \\
E \otimes (E \otimes X) \arrow[r, "\cong"'] & (E \otimes E) \otimes X \arrow[r, "\mu\otimes X"'] & E \otimes X
\end{tikzcd}$$

The top square commutes by coherence of the isomorphisms $S^{a+b} \cong S^a \otimes S^b$ (**??**). The second square from the top on the left commutes by naturality of the associators. The square below that commutes by the coherence axiom for the associators in a monoidal category. The bottom left square commutes again by naturality of the associator isomorphisms. The bottom right square commutes by associativity for μ and functorialiy of $- \otimes X$. Finally, the square above that commutes again by naturality of the associator isomorphism. By functoriality of $- \otimes -$, the two outside compositions equal $(z \cdot y) \cdot x$ on the top and $z \cdot (y \cdot x)$ on the bottom. Hence, they are equal, as desired.

Next, to see (4), consider the following diagram:

$$\begin{tikzcd}
S^a \arrow[rrr, "x"] \arrow[d, "\phi_{0,a}=\lambda_{S^a}^{-1}"'] & & & E \otimes X \\
S \otimes S^a \arrow[r, "S\otimes x"] \arrow[dd, "e\otimes x"'] & S \otimes (E \otimes X) \arrow[urr, "\lambda_{E\otimes X}"'] \arrow[dr, "\cong"] \arrow[ddl, "e\otimes(E\otimes X)"] & & \\
 & & (S \otimes E) \otimes X \arrow[uur, "\lambda_E \otimes X"'] \arrow[dr, "(e\otimes E)\otimes X"] & \\
E \otimes (E \otimes X) \arrow[rrr, "\cong"] & & & (E \otimes E) \otimes X \arrow[uuu, "\mu\otimes X"']
\end{tikzcd}$$

Commutativity of the top trapezoid is naturality of the unitor. Commutativity of the left triangle is functoriality of $- \otimes -$. Commutativity of the bottom triangle is naturality of the associator isomorphisms. Commutativity of the right triangle is unitality of μ and functoriality of $- \otimes X$. Finally, commutativity of the remaining crooked triangle follows by coherence for monoidal categories. The two outer compositions $S^a \to E \otimes X$ are x and $e \cdot x$, and by commutativity they are necessarily equal.

Thus, we have shown that the indicated map does indeed endow $E_*(X)$ with the structure of a left $\pi_*(E)$-module. Showing that $X_*(E)$ has the structure of a right $\pi_*(E)$-module is entirely analagous.

It remains to show that $E_*(E)$ is a bimodule. □ finish

Definition D.9. Let E be a ring spectrum. We say E is *flat* if the canonical right $\pi_*(E)$-module structure on $E_*(E)$ is that of a flat module.

Proposition D.10 ([5, Proposition 2.2])**.** *Let E be a ring spectrum and let X be any spectrum. Then the assignment*

$$E_*(E) \times E_*(X) \to E_*(E \otimes X)$$

which sends $\alpha : S^{a,b} \to E \otimes E$ and $\beta : S^{c,d} \to E \otimes X$ to the composition

$$\alpha \cdot \beta : S^{a+c,b+d} \cong S^{a,b} \otimes S^{c,d} \xrightarrow{\alpha\otimes\beta} E \otimes E \otimes E \otimes X \xrightarrow{E\otimes\mu\otimes X} E \otimes E \otimes X$$

induces a homomorphism of bigraded abelian groups

$$E_*(E) \otimes_{\pi_*(E)} E_*(X) \to E_*(E \otimes X)$$

(where here $E_*(E)$ has a right $\pi_*(E)$-module structure and $E_*(X)$ has a left $\pi_*(E)$-module structure as specified by Proposition D.8). Furthermore, if X is cellular and E is a cellular flat commutative ring spectrum (Definition 2.8, Definition D.9), then this map is an isomorphism.

Proof. First we show that this map is $\pi_*(E)$-bilinear. By the identifications $E_*(E) = \pi_*(E \otimes E)$ $E_*(X) = \pi_*(E \otimes X)$, and $E_*(E \otimes X) = \pi_*(E \otimes E \otimes X)$, we know this map commutes with addition of maps in each argument by Proposition D.7. Now, let $a, b, c \in \mathbb{Z}^2$, $\alpha : S^a \to E \otimes E$, $\beta : S^b \to E \otimes X$, and $\gamma : S^c \to E$. Then we wish to show $\alpha\gamma \cdot \beta = \alpha \cdot \gamma\beta$. Consider the following diagram

$$
\begin{tikzcd}
S^{a+b+c} \arrow[r, "\cong"] \arrow[d, "\cong"] & S^a \otimes S^{c+b} \arrow[r, "\cong"] & S^a \otimes S^c \otimes S^b \arrow[d, "\alpha\otimes\gamma\otimes\beta"] \\
S^{a+c} \otimes S^b \arrow[d, "\cong"'] & & E \otimes E \otimes E \otimes E \otimes X \arrow[d, "E\otimes E\otimes\mu\otimes X"] \\
S^a \otimes S^c \otimes S^b \arrow[d, "\alpha\otimes\gamma\otimes\beta"'] \arrow[uurr, "\cong"] & & E \otimes E \otimes E \otimes X \arrow[d, "E\otimes\mu\otimes X"] \\
E \otimes E \otimes E \otimes E \otimes X \arrow[r, "E\otimes\mu\otimes E\otimes X"'] \arrow[uurr, "\cong"] & E \otimes E \otimes E \otimes X \arrow[r, "E\otimes\mu\otimes X"'] & E \otimes E \otimes X
\end{tikzcd}
$$

(we have suppressed the associators from the notation). The top left triangle commutes by coherence for the isomorphisms $S^{a+b} \cong S^a \otimes S^b$. The middle parallelogram commutes by naturality of the associators. Finally, the bottom right triangle is obtained by applying $E \otimes - \otimes X$ to the associativity diagram for μ, so by functoriality it commutes. Again by functoriality of $- \otimes -$, the bottom composition is given by $(\alpha\gamma) \cdot \beta$ and the top composition is $\alpha \cdot (\gamma\beta)$, so we have the desired equality.

It remains to show that if X is cellular and E is cellular flat commutative, then this map is an isomorphism. □

finish or cite

Remark D.11. Let E be a ring spectrum, and N a left $\pi_*(E)$-module. Then

$$E_*(E) \otimes_{\pi_*(E)} N$$

is canonically a left-graded $\pi_*(E)$-module, as $E_*(E)$ is a $\pi_*(E)$-bimodule (Proposition D.8). In particular, the action

$$\pi_*(E) \times (E_*(E) \otimes_{\pi_*(E)} N) \to E_*(E) \otimes_{\pi_*(E)} N$$

sends a pair $(\gamma, \alpha \otimes n)$ to $\gamma\alpha \otimes n$, where $\gamma\alpha$ denotes the left action prescribed by Proposition D.8 of γ on α.

In the following definition, let $\varepsilon : E_*(E) \to \pi_*(E)$ be the map which sends some $\alpha : S^a \to E \otimes E$ to the composition

$$S^a \xrightarrow{\alpha} E \otimes E \xrightarrow{\mu} E.$$

Also define $\Psi : E_*(E) \to E_*(E) \otimes_{\pi_*(E)} E_*(E)$ to be the map which factors as

$$E_*(E) \to E_*(E \otimes E) \xrightarrow{\cong} E_*(E) \otimes_{\pi_*(E)} E_*(E)$$

where the second arrow is the isomorphism prescribed by Proposition D.10, and the first arrow sends a class $\alpha : S^a \to E \otimes E$ to the composition

$$S^a \xrightarrow{\alpha} E \otimes E \cong E \otimes S \otimes E \xrightarrow{E\otimes e\otimes E} E \otimes E \otimes E.$$

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