We assume the reader is familiar with additive categories and (closed, symmetric) monoidal categories.

Definition 0.1. Let $\mathcal{C}$ be an additive category. Given a sequence

$$X_1 \to \cdots \to X_n$$

of morphisms in $\mathcal{C}$, we say this sequence is *exact* if, for any object A in $\mathcal{C}$, the induced sequence

$$\mathcal{C}(A, X_1) \to \mathcal{C}(A, X_n) \to \cdots \to \mathcal{C}(A, X_n)$$

is an exact sequence of abelian groups.

Definition 0.2. A *triangulated category* is a tuple $(\mathcal{C}, \Sigma, \Omega, \eta, \varepsilon, \mathcal{D})$ such that

(1) $\mathcal{C}$ is an additive category.
(2) An adjoint pair of additive functors $\Sigma, \Omega : \mathcal{C} \to \mathcal{C}$ such that the unit $\eta : \mathrm{Id}_{\mathcal{C}} \Rightarrow \Omega\Sigma$ and counit $\varepsilon : \Sigma\Omega \Rightarrow \mathrm{Id}_{\mathcal{C}}$ are natural isomorphisms, i.e., the tuple $(\Sigma, \Omega, \eta, \varepsilon)$ forms an adjoint autoequivalence of $\mathcal{C}$. Usually Σ is called the *shift functor*.
(3) $\mathcal{D}$ is a collection of *distinguished triangles*, where a *triangle* is a diagram of the form

$$X \to Y \to Z \to \Sigma X.$$

These are also sometimes called *cofiber sequences* or *fiber sequences*.

These data must satisfy the following axioms:

TR0 Given a commutative diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d, "\cong"'] & Y \arrow[r] \arrow[d, "\cong"'] & Z \arrow[r] \arrow[d, "\cong"'] & \Sigma X \arrow[d, "\cong"'] \\ X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & \Sigma X' \end{tikzcd}$$

where the vertical arrows are isomorphisms, if the top row is distinguished then so is the bottom.

TR1 For any object X in $\mathcal{C}$, the diagram

$$X \xrightarrow{\mathrm{id}_X} X \to 0 \to \Sigma X$$

is a distinguished triangle.

TR2 For all $f : X \to Y$ there exists an object C_f (also sometimes denoted Y/X) called the *cofiber of* f and a distinguished triangle

$$X \xrightarrow{f} Y \to C_f \to \Sigma X.$$

TR3 Given a solid diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] & Y \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d, dashed] & \Sigma X \arrow[d, "\Sigma f"] \\ X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & \Sigma X' \end{tikzcd}$$

such that the leftmost square commmutes and both rows are distinguished, there exists a dashed arrow $Z \to Z'$ which makes the remaining two squares commute.

TR4 A triangle

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{\Sigma} X$$

is distinguished if and only if

$$Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X \xrightarrow{-\Sigma f} \Sigma Y$$

is distinguished.

TR5 (Octahedral axiom) Given three distinguished triangles

$$X \xrightarrow{f} Y \xrightarrow{h} Y/X \to \Sigma X$$

$$Y \xrightarrow{g} Z \xrightarrow{k} Z/Y \to \Sigma Y$$

$$X \xrightarrow{g\circ f} Z \xrightarrow{l} Z/X \to \Sigma X$$

there exists a distinguished triangle

$$Y/X \xrightarrow{u} Z/X \xrightarrow{v} Z/Y \xrightarrow{w} \Sigma(Y/X)$$

such that the following diagram commutes

$$\begin{tikzcd}
X \arrow[rr, "g\circ f"] \arrow[dr, "f"'] & & Z \arrow[rr, "k"] \arrow[dr, "l"'] & & Z/Y \arrow[rr, "w"] \arrow[dr] & & \Sigma(Y/X) \\
& Y \arrow[ur, "g"'] \arrow[dr, "h"'] & & Z/X \arrow[ur, "v"'] \arrow[dr] & & \Sigma Y \arrow[ur, "\Sigma h"'] & \\
& & Y/X \arrow[ur, "u"'] \arrow[rr] & & \Sigma X \arrow[ur, "\Sigma f"'] & &
\end{tikzcd}$$

It turns out that the above definition is actually redundant; TR3 and TR4 follow from the remaining axioms (see Lemmas 2.2 and 2.4 in [1]).

In this section, we fix a triangulated category $\mathcal{C}$, and we will always use brackets $[-,-]$ to denote the homset in $\mathcal{C}$. Note our definition of a triangulated category is slightly nonstandard, namely, we require that the shift functor gives an *adjoint* autoequivalence of $\mathcal{C}$, not just an equivalence. In other words, we require that the natural isomorphisms $\eta : \mathrm{Id}_{\mathcal{C}} \Rightarrow \Sigma\Omega$ and $\varepsilon : \Sigma\Omega \Rightarrow \mathrm{Id}_{\mathcal{C}}$ satisfy either of the following zig-zag identities:

$$\begin{tikzcd}
\Sigma \arrow[r, "\Sigma\eta"] \arrow[dr, equal] & \Sigma\Omega\Sigma \arrow[d, "\varepsilon\Sigma"] \\
& \Sigma
\end{tikzcd}
\qquad
\begin{tikzcd}
\Omega\Sigma\Omega \arrow[d, "\Omega\varepsilon"'] & \Omega \arrow[l, "\eta\Omega"'] \arrow[dl, equal] \\
\Omega &
\end{tikzcd}$$

(Satisfying one implies the other is automatically satisfied, see [2, Lemma 3.2]). We have also fixed the data of natural isomorphisms η and ε exhibiting the adjoint equivalence $(\Sigma, \Omega, \eta, \varepsilon)$ in our definition. We very well could have only required that Σ is *an* adjoint equivalence, and everything we will do will go through all the same.

Proposition 0.3. *Let $(\mathcal{C}, \Sigma)$ be a triangulated category. Then any distinguished triangle in $\mathcal{C}$ is an exact sequence (in the sense of Definition 0.1).*

Proof. Suppose we have some distinguished triangle

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X.$$

Then we would like to show that given any object A in $\mathcal{C}$, the sequence

$$[A, X] \xrightarrow{f_*} [A, Y] \xrightarrow{g_*} [A, Z] \xrightarrow{h_*} [A, \Sigma X]$$

is exact. First we show exactness at $[A, Y]$. To see $\operatorname{im} f_* \subseteq \ker g_*$, note it suffices to show that $g \circ f = 0$. Indeed, consider the commuting diagram

$$\begin{tikzcd}
X \arrow[r, equal] \arrow[d, equal] & X \arrow[r] \arrow[d, "f"] & 0 \arrow[r] & \Sigma X \\
X \arrow[r, "f"] & Y \arrow[r, "g"] & Z \arrow[r, "h"] & \Sigma X
\end{tikzcd}$$

The top row is distinguished by axiom TR1. Thus by TR3, the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, equal] \arrow[d, equal] & X \arrow[r] \arrow[d, "f"] & 0 \arrow[r] \arrow[d] & \Sigma X \arrow[d, equal] \\ X \arrow[r, "f"] & Y \arrow[r, "g"] & Z \arrow[r, "h"] & \Sigma X \end{tikzcd}$$

In particular, commutativity of the second square tells us that $g \circ f = 0$, as desired. Conversely, we'd liek to show that $\ker g_* \subseteq \operatorname{im} f_*$. Let $\psi : A \to Y$ be in the kernel of g_*, so that $g \circ \psi = 0$. Consider the following commutative diagram:

$$\begin{tikzcd} A \arrow[r] \arrow[d, "\psi"'] & 0 \arrow[r] \arrow[d] & \Sigma A \arrow[r, "-\Sigma \mathrm{id}_A"] & \Sigma A \\ Y \arrow[r, "g"] & Z \arrow[r, "h"] & \Sigma X \arrow[r, "-\Sigma f"] & \Sigma Y \end{tikzcd}$$

The top row is distinguished by axioms TR1 and TR4. The bottom row is distinguished by axiom TR4. Thus by axiom TR3 there exists a map $\widetilde{\phi} : \Sigma A \to \Sigma X$ such that the following diagram commutes:

$$\begin{tikzcd} A \arrow[r] \arrow[d, "\psi"'] & 0 \arrow[r] \arrow[d] & \Sigma A \arrow[r, "-\Sigma \mathrm{id}_A"] \arrow[d, "\widetilde{\phi}"'] & \Sigma A \arrow[d, "\Sigma\psi"'] \\ Y \arrow[r, "g"] & Z \arrow[r, "h"] & \Sigma X \arrow[r, "-\Sigma f"] & \Sigma Y \end{tikzcd}$$

Now, since Σ is an equivalence, it is a full functor, so that in particular there exists some $\phi : A \to X$ such that $\widetilde{\phi} = \Sigma\phi$. Then by faithfullness, we may pull back the right square to get a commuting diagram

$$\begin{tikzcd} A \arrow[r, "-\mathrm{id}_A"] \arrow[d, "\phi"'] & A \arrow[d, "\psi"] \\ X \arrow[r, "-f"'] & Y \end{tikzcd}$$

Hence,

$$f_*(\phi) = f \circ \phi \overset{(*)}{=} -((-f) \circ \phi) = -(\psi \circ (-\mathrm{id}_A)) \overset{(*)}{=} \psi \circ \mathrm{id}_A = \psi,$$

where the equalities marked $(*)$ follow by bilinearity of composition in an additive category. Thus $\psi \in \operatorname{im} f_*$, as desired, meaning $\ker g_* \subseteq \operatorname{im} f_*$.

Now, we have shown that

$$[A, X] \xrightarrow{f_*} [A, Y] \xrightarrow{g_*} [A, Z] \xrightarrow{h_*} [A, \Sigma X]$$

is exact at $[A, Y]$. It remains to show exactness at $[A, Z]$. Yet this follows by the exact same argument given above applied to the sequence obtained from the shifted triangle (TR4)

$$Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X \xrightarrow{-\Sigma f} \Sigma Y.$$

□

Lemma 0.4. *Suppose we have a commutative diagram*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "j"] & Y \arrow[r, "g"] \arrow[d, "k"] & Z \arrow[r, "h"] \arrow[d, "\ell"] & \Sigma X \arrow[d, "\Sigma j"] \\ X' \arrow[r, "f'"] & Y' \arrow[r, "g'"] & Z' \arrow[r, "h'"] & \Sigma X' \end{tikzcd}$$

with both rows distinguished. Then if any two of the maps j, k, and ℓ are isomorphisms, then so is the third.

Proof. Suppose we are given any object A in $\mathcal{C}$, and consider the commutative diagram

$$\begin{tikzcd}
{[A,X]} \arrow[r, "f_*"] \arrow[d, "j_*"] & {[A,Y]} \arrow[r, "g_*"] \arrow[d, "k_*"] & {[A,Z]} \arrow[r, "k_*"] \arrow[d, "\ell_*"] & {[A,\Sigma X]} \arrow[r, "-\Sigma f_*"] \arrow[d, "\Sigma j_*"] & {[A,\Sigma Y]} \arrow[r, "-\Sigma g_*"] \arrow[d, "\Sigma k_*"] & {[A,\Sigma Z]} \arrow[r, "-\Sigma h_*"] \arrow[d, "\Sigma \ell_*"] & {[A,\Sigma^2 X]} \arrow[d, "\Sigma^2 j_*"] \\
{[A,X']} \arrow[r, "f'_*"] & {[A,Y']} \arrow[r, "g'_*"] & {[A,Z']} \arrow[r, "h'_*"] & {[A,\Sigma X']} \arrow[r, "-\Sigma f'_*"] & {[A,\Sigma Y']} \arrow[r, "-\Sigma g'_*"] & {[A,\Sigma Z']} \arrow[r, "-\Sigma h'_*"] & {[A,\Sigma^2 X']}
\end{tikzcd}$$

The rows are exact by Proposition 0.3 and repeated applications of axiom TR4. It follows by the five lemma that if j and k are isomorphisms, then ℓ_* is an isomorphism. Similarly, if k and ℓ are isomorphisms then Σj_* is an isomorphism. Finally, if ℓ and j are isomorphisms, then Σk_* is an isomorphism. The desired result follows by faithfullness of Σ and the Yoneda embedding. □

Proposition 0.5. *Given a map $f : X \to Y$ in a triangulated category $(\mathcal{C}, \Sigma, \Omega, \mathcal{D})$, the cofiber sequence of f is unique up to isomorphism, in the sense that given any two distinguished triangles*

$$X \xrightarrow{f} Y \to Z \to \Sigma X \qquad \textit{and} \qquad X \xrightarrow{f} Y \to Z' \to \Sigma X,$$

there exists an isomorphism $Z \to Z'$ which makes the following diagram commute:

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, equal] & Y \arrow[r] \arrow[d, equal] & Z \arrow[r] \arrow[d, dashed, "k"] & \Sigma X \arrow[d, equal] \\
X \arrow[r, "f"] & Y \arrow[r] & Z' \arrow[r] & \Sigma X
\end{tikzcd}$$

Proof. Suppose we have two distinguished triangles

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X \qquad \text{and} \qquad X \xrightarrow{f} Y \xrightarrow{g'} Z' \xrightarrow{h'} \Sigma X,$$

and consider the following commutative diagram

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, equal] & Y \arrow[r, "g"] \arrow[d, equal] & Z \arrow[r, "h"] & \Sigma X \\
X \arrow[r, "f"] & Y \arrow[r, "g'"] & Z' \arrow[r, "h'"] & \Sigma X
\end{tikzcd}$$

By axiom TR3, there exists some map $k : Z \to Z'$ which makes the following diagram commute:

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, equal] & Y \arrow[r, "g"] \arrow[d, equal] & Z \arrow[r, "h"] \arrow[d, "k"] & \Sigma X \arrow[d, equal] \\
X \arrow[r, "f"] & Y \arrow[r, "g'"] & Z' \arrow[r, "h'"] & \Sigma X
\end{tikzcd}$$

Now, by Lemma 0.4, k is an isomorphism. □

Proposition 0.6. *Given an arrow $f : X \to Y$ in a triangulated category $(\mathcal{C}, \Sigma, \Omega, \mathcal{D})$, there exists an object F_f called the* fiber *of f, and a distinguished triangle*

$$F_f \to X \xrightarrow{f} Y \to \Sigma F_f (\cong C_f).$$

Proof. By axiom TR2, we have a distinguished triangle

$$X \xrightarrow{f} Y \xrightarrow{g} C_f \xrightarrow{h} \Sigma X.$$

Now, consider the commutative diagram

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, equal] & Y \arrow[r, "\widetilde{g}"] \arrow[d, equal] & \Sigma\Omega C_f \arrow[r, "\widetilde{h}"] \arrow[d, "\varepsilon_{C_f}"'] & \Sigma X \arrow[d, equal] \\
X \arrow[r, "f"] & Y \arrow[r, "g"] & C_f \arrow[r, "h"] & \Sigma X
\end{tikzcd}$$

where $\varepsilon : \Sigma\Omega \Rightarrow \mathrm{Id}_{\mathcal{C}}$ is the counit of the adjunction $\Sigma \dashv \Omega$, $\widetilde{g} = \varepsilon_{C_f}^{-1} \circ g$, and $\widetilde{h} = \varepsilon_{C_f}^{-1} \circ h$. Since each vertical map is an isomorphism and the bottom row is distinguished, the top row is also distinguished by axiom TR0. Now, since Σ is an equivalence of categories, it is faithful, so that in particular there exists some map $k : \Omega C_f \to X$ such that $\Sigma k = -\widetilde{h} \implies -\Sigma k = \widetilde{h}$. Thus, we have a distinguished triangle of the form

$$X \xrightarrow{f} Y \xrightarrow{\widetilde{g}} \Sigma\Omega C_f \xrightarrow{-\Sigma k} \Sigma X.$$

Finally, by axiom TR4, we get a distinguished triangle

$$\Omega C_f \xrightarrow{k} X \xrightarrow{f} Y \xrightarrow{\widetilde{g}} \Sigma\Omega C_f,$$

so we may define the fiber of f to be ΩC_f. $\square$

Lemma 0.7. *Given a triangle*

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X,$$

it can be shifted to the left to obtain a distinguished triangle

$$\Omega Z \xrightarrow{-\widetilde{h}} X \xrightarrow{f} Y \xrightarrow{\widetilde{\Omega g}} \Sigma\Omega Z,$$

where $\widetilde{h} : \Omega Z \to X$ is the adjoint of $h : Z \to \Sigma X$ and $\widetilde{\Omega g} : Y \to \Sigma\Omega Z$ is the adjoint of $\Omega g : \Omega Y \to \Omega Z$.

Proof. Note that unravelling definitions, if ε and η are the counit and unit of the adjoint equivalence $\Sigma \dashv \Omega$, respectively (so η^{-1} and ε^{-1} are the counit and unit of the adjunction $\Omega \dashv \Sigma$), then $\widetilde{h}$ and $\widetilde{g}$ are the compositions

$$\widetilde{h} : \Omega Z \xrightarrow{\Omega h} \Omega\Sigma X \xrightarrow{\eta_X^{-1}} X \qquad \text{and} \qquad \widetilde{\Omega g} : Y \xrightarrow{\varepsilon_Y^{-1}} \Sigma\Omega Y \xrightarrow{\Sigma\Omega g} \Sigma\Omega Z.$$

Now consider the following diagram:

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, equal] & Y \arrow[r, "\widetilde{\Omega g}"] \arrow[d, equal] & \Sigma\Omega Z \arrow[r, "\Sigma\widetilde{h}"] \arrow[d, "\varepsilon_Z"] & \Sigma X \arrow[d, equal] \\
X \arrow[r, "f"] & Y \arrow[r, "g"] & Z \arrow[r, "h"] & \Sigma X
\end{tikzcd}$$

The left square commutes by definition. To see that the middle square commutes, expanding definitions, note it is given by the following diagram:

$$\begin{tikzcd}
Y \arrow[r, "\varepsilon_Y^{-1}"] \arrow[d, equal] & \Sigma\Omega Y \arrow[r, "\Sigma\Omega g"] & \Sigma\Omega Z \arrow[d, "\varepsilon_Y"] \\
Y \arrow[rr, "g"] & & Z
\end{tikzcd}$$

and this commutes by naturality of ε. To see that the right square commutes, consider the following diagram:

$$\begin{tikzcd}
\Sigma\Omega Z \arrow[r, "\Sigma\Omega h"] \arrow[d, "\varepsilon_Z"'] & \Sigma\Omega\Sigma X \arrow[r, "\Sigma\eta_X^{-1}"] \arrow[dr, "\varepsilon_{\Sigma X}"] & \Sigma X \arrow[d, equal] \\
Z \arrow[rr, "h"'] & & \Sigma X
\end{tikzcd}$$

By functoriality of Σ, the top composition is $\Sigma\widetilde{h}$. The left region commutes by naturality of ε. Commutativity of the right region is precisely one of the the zig-zag identities for the unit and counit of an adjunction. Hence, since diagram (**??**) commutes, the vertical arrows are isomorphisms, and the bottom row is distinguished, we have that the top row is distinguished as well by axiom TR0. Then by axiom TR4, since $(f, \widetilde{\Omega g}, \Sigma\widetilde{h})$ is distinguished, so is the triangle

$$\Omega Z \xrightarrow{-\widetilde{h}} X \xrightarrow{f} Y \xrightarrow{\widetilde{\Omega g}} \Sigma\Omega Z. \qquad \square$$

Lemma 0.8. *Given a distinguished triangle*

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X,$$

for any $n > 0$, the triangle

$$\Omega^n X \xrightarrow{(-1)^n\Omega^n f} \Omega^n Y \xrightarrow{(-1)^n\Omega^n g} \Omega^n Z \xrightarrow{(-1)^n\Omega^n h} \Omega^n\Sigma X \cong \Sigma\Omega^n X,$$

is distinguished, where the final isomorphism is given by the composition

$$\Omega^n\Sigma X = \Omega^{n-1}\Omega\Sigma X \xrightarrow{\Omega^{n-1}\eta_X^{-1}} \Omega^{n-1}X \xrightarrow{\varepsilon^{-1}_{\Omega^{n-1}X}} \Sigma\Omega\Omega^{n-1}X = \Sigma\Omega^n X,$$

where ε and η are the counit and unit of the adjunction $\Sigma \dashv \Omega$, respectively.

Proof. We give a proof by induction. First we show the case $n = 1$. Note by Lemma 0.7, we have that given a distinguished triangle

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X,$$

we can shift it to the left to obtain a distinguished triangle

$$\Omega Z \xrightarrow{-\widetilde{h}} X \xrightarrow{f} Y \xrightarrow{\widetilde{\Omega g}} \Sigma\Omega Z,$$

where $\widetilde{h} : \Omega Z \to X$ is the adjoint of $h : Z \to \Sigma X$ and $\widetilde{\Omega g}$ is the adjoint of $\Omega g : \Omega Y \to \Omega Z$. If we apply this shifting operation again, we get the distinguished triangle

$$\Omega Y \xrightarrow{-\widetilde{\widetilde{\Omega g}}} \Omega Z \xrightarrow{-\widetilde{h}} X \xrightarrow{\widetilde{\Omega f}} \Sigma\Omega Y,$$

where unravelling definitions, $\widetilde{\Omega f}$ is the right adjoint of $\Omega f : \Omega X \to \Omega Y$ and $\widetilde{\widetilde{\Omega g}}$ is the right adjoint of $\widetilde{\Omega g}$, which itself is the left adjoint of Ωg, so $\widetilde{\widetilde{\Omega g}} = \Omega g$. Hence we have a distinguished triangle

$$\Omega Y \xrightarrow{-\Omega g} \Omega Z \xrightarrow{-\widetilde{h}} X \xrightarrow{\widetilde{\Omega f}} \Sigma\Omega Y.$$

We may again shift this triangle again and the above arguments yield the distinguished triangle

$$\Omega X \xrightarrow{-\Omega f} \Omega Y \xrightarrow{-\Omega g} \Omega Z \xrightarrow{\widetilde{\Omega(-\widetilde{h})}} \Sigma\Omega X,$$

where $\widetilde{\Omega(-\widetilde{h})}$ is the right adjoint of $\Omega(-\widetilde{h}) = -\Omega\widetilde{h} : \Omega\Omega Z \to \Omega X$. Explicitly unravelling definitions, $\widetilde{\Omega(-\widetilde{h})} = -\widetilde{\Omega\widetilde{h}}$ is the composition

$$\begin{aligned}
[\Omega Z \xrightarrow{\varepsilon^{-1}_{\Omega Z}} \Sigma\Omega\Omega Z \xrightarrow{\Sigma(-\Omega\widetilde{h})} \Sigma\Omega X] &= -[\Omega Z \xrightarrow{\varepsilon^{-1}_{\Omega Z}} \Sigma\Omega\Omega Z \xrightarrow{\Sigma\Omega\widetilde{h}} \Sigma\Omega X] \\
&= -[\Omega Z \xrightarrow{\varepsilon^{-1}_{\Omega Z}} \Sigma\Omega\Omega Z \xrightarrow{\Sigma\Omega\Omega h} \Sigma\Omega\Omega\Sigma X \xrightarrow{\Sigma\Omega\eta_X^{-1}} \Sigma\Omega X] \\
&= -[\Omega Z \xrightarrow{\Omega h} \Omega\Sigma X \xrightarrow{\eta_X^{-1}} X \xrightarrow{\varepsilon_X^{-1}} \Sigma\Omega X],
\end{aligned}$$

where the first equality follows by additivity of Σ and additivity of composition, the second follows by further unravelling how $\widetilde{h}$ is defined, and the third follows by naturality of ε, which tells us the following diagram commutes:

$$\begin{tikzcd}
\Omega Z \arrow[r, "\Omega h"] \arrow[d, "\varepsilon^{-1}_{\Omega Z}"] & \Omega\Sigma X \arrow[r, "\eta_X^{-1}"] \arrow[d, "\varepsilon^{-1}_{\Omega\Sigma X}"] & X \arrow[d, "\varepsilon_X^{-1}"] \\
\Sigma\Omega\Omega Z \arrow[r, "\Sigma\Omega\Omega h"'] & \Sigma\Omega\Omega\Sigma X \arrow[r, "\Sigma\Omega\eta_X^{-1}"'] & \Sigma\Omega X
\end{tikzcd}$$

Thus indeed we have a distinguished triangle

$$\Omega X \xrightarrow{-\Omega f} \Omega Y \xrightarrow{-\Omega g} \Omega Z \xrightarrow{-\Omega h} \Omega\Sigma X \cong \Sigma\Omega X,$$

where the last isomorphism is $\varepsilon_X^{-1} \circ \eta_X^{-1}$, as desired.

Now, we show the inductive step. Suppose we know that given a distinguished triangle

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X,$$

that for some $n > 0$ the triangle

$$\Omega^n X \xrightarrow{(-1)^n\Omega^n f} \Omega^n Y \xrightarrow{(-1)^n\Omega^n g} \Omega^n Z \xrightarrow{(-1)^n h^n} \Sigma\Omega^n X,$$

is distinguished, where $h^n : \Omega^n Z \to \Sigma\Omega^n X$ is the composition

$$\Omega^n Z \xrightarrow{\Omega^n h} \Omega^n\Sigma X \xrightarrow{\Omega^{n-1}\eta_X^{-1}} \Omega^{n-1} X \xrightarrow{\varepsilon^{-1}_{\Omega^{n-1}X}} \Sigma\Omega^n X.$$

Then by applying the $n = 1$ case to this triangle, we get that the following triangle is distinguished

$$\Omega^{n+1} X \xrightarrow{-\Omega((-1)^n\Omega^n f)} \Omega^{n+1} Y \xrightarrow{-\Omega((-1)^n\Omega^n g)} \Omega^{n+1} Z \xrightarrow{-\Omega((-1)^n h^n)} \Omega\Sigma\Omega^n X \cong \Sigma\Omega^{n+1} X,$$

where the final isomorphism is the composition

$$\Omega\Sigma\Omega^n X \xrightarrow{\eta^{-1}_{\Omega^n X}} \Omega^n X \xrightarrow{\varepsilon^{-1}_{\Omega^n X}} \Sigma\Omega\Omega^n X = \Sigma\Omega^{n+1} X.$$

We claim that this is precisely the distinguished triangle given in the statement of the lemma for $n+1$. First of all, note that $-\Omega((-1)^n\Omega^n f) = (-1)^{n+1}\Omega^{n+1} f$, $-\Omega((-1)^n\Omega^n g) = (-1)^{n+1}\Omega^{n+1} g$, and $-\Omega((-1)^n h^n) = (-1)^{n+1}\Omega h^n$ by additivity of Ω, so that the triangle becomes

$$\Omega^{n+1} X \xrightarrow{(-1)^{n+1}\Omega^{n+1} f} \Omega^{n+1} Y \xrightarrow{(-1)^{n+1}\Omega^{n+1} g} \Omega^{n+1} Z \xrightarrow{(-1)^{n+1}\Omega h^n} \Omega\Sigma\Omega^n X \cong \Sigma\Omega^{n+1} X. \tag{1}$$

Thus, in order to prove the desired characterization, it remains to show this diagram commutes:

$$\begin{tikzcd}
\Omega^{n+1} Z \arrow[r, "(-1)^{n+1}\Omega h^n"] \arrow[d, "(-1)^{n+1}\Omega^{n+1} h"'] & \Omega\Sigma\Omega^n X \arrow[r, "\eta^{-1}_{\Omega^n X}"] & \Omega^n X \arrow[d, "\varepsilon^{-1}_{\Omega^n X}"] \\
\Omega^{n+1}\Sigma X \arrow[r, "\Omega^n\eta_X^{-1}"'] & \Omega^n X \arrow[r, "\varepsilon^{-1}_{\Omega^n X}"'] & \Sigma\Omega^{n+1} X
\end{tikzcd}$$

(The top composition is the last two arrows in diagram (1), and the bottom composition is the last two arrows in the diagram). Unravelling how h^n is constructed, by additivity of Ω it further suffices to show the outside of the following diagram commutes:

$$\begin{tikzcd}
\Omega^{n+1}Z \arrow[r, "(-1)^{n+1}\Omega^{n+1}h"] \arrow[dd, "(-1)^{n+1}\Omega^{n+1}h"'] & \Omega^{n+1}\Sigma X \arrow[r, "\Omega^n\eta_X^{-1}"] & \Omega^n X \arrow[r, "\Omega\varepsilon^{-1}_{\Omega^{n-1}X}"] \arrow[dd, equal] & \Omega\Sigma\Omega^n X \arrow[d, "\eta^{-1}_{\Omega^n X}"] \\
& & & \Omega^n X \arrow[d, "\varepsilon^{-1}_{\Omega^n X}"] \\
\Omega^{n+1}\Sigma X \arrow[rr, "\Omega^n\eta_X^{-1}"'] & & \Omega^n X \arrow[r, "\varepsilon^{-1}_{\Omega^n X}"'] \arrow[ur, equal] & \Sigma\Omega^{n+1}X
\end{tikzcd}$$

The left rectangle and bottom right triangle commute by definition. Finally, commutativity of the top right trapezoid is precisely one of the zig-zag identities for the adjunction $\Omega \dashv \Sigma$ applied to $\Omega^{n-1}X$. Hence, we have shown the desired result. $\square$

Lemma 0.9. *Suppose we have a distinguished triangle*

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X.$$

Then for any integer $n \geq 0$, the sequence

$$\Sigma^n X \xrightarrow{\Sigma^n f} \Sigma^n Y \xrightarrow{\Sigma^n g} \Sigma^n Z \xrightarrow{\Sigma^n h} \Sigma^{n+1} X$$

is exact. Similarly, for any integer $n > 0$, the sequence

$$\Omega^n X \xrightarrow{\Omega^n f} \Omega^n Y \xrightarrow{\Omega^n g} \Omega^n Z \xrightarrow{\Omega^n h} \Omega^n \Sigma X \cong \Omega^{n-1} X$$

are exact (Definition 0.1), where the isomorphism is given by $\Omega^{n-1}\eta_X^{-1}$, where $\eta : \mathrm{Id}_{\mathcal{C}} \to \Omega\Sigma$ is the unit of the adjunction $\Sigma \dashv \Omega$.

Proof. By Proposition 0.3, the first statement holds when $n = 0$. Now suppose we are given some $n > 0$. Using axiom TR4, by induction we have that the triangle

$$\Sigma^n X \xrightarrow{(-1)^n\Sigma^n f} \Sigma^n Y \xrightarrow{(-1)^n\Sigma^n g} \Sigma^n Z \xrightarrow{(-1)^n\Sigma^n h} \Sigma^{n+1} X$$

is also distinguished. Thus, again by Proposition 0.3, given any object A in $\mathcal{C}$, the sequence of abelian groups

$$[A, \Sigma^n X] \xrightarrow{(-1)^n\Sigma^n f_*} [A, \Sigma^n Y] \xrightarrow{(-1)^n\Sigma^n g_*} [A, \Sigma^n Z] \xrightarrow{(-1)^n\Sigma^n h_*} [A, \Sigma^{n+1} X]$$

is exact. A simple diagram chase yields that we can remove the signs and the sequence is still exact, so we have shown the desired statement for Σ. Now, we would like to show for $n > 0$ that the sequence

$$\Omega^n X \xrightarrow{\Omega^n f} \Omega^n Y \xrightarrow{\Omega^n g} \Omega^n Z \xrightarrow{\Omega^n h} \Omega^n \Sigma X \cong \Omega^{n-1} X$$

is exact, where the final isomorphism is $\Omega^{n-1}\eta_X^{-1}$. Now, consider the following diagram:

$$\begin{tikzcd}
\Omega^n X \arrow[r, "(-1)^n\Omega^n f"] \arrow[dd, equal] & \Omega^n Y \arrow[r, "(-1)^n\Omega^n g"] \arrow[dd, equal] & \Omega^n Z \arrow[rr, dashed] \arrow[dd, equal] \arrow[dr, dashed, "(-1)^n\Omega^n h"] & & \Sigma\Omega^n X \\
& & & \Omega^n\Sigma X \arrow[dr, dashed, "\Omega^{n-1}\eta_X^{-1}"] & \\
\Omega^n X \arrow[r, "(-1)^n\Omega^n f"'] & \Omega^n Y \arrow[r, "(-1)^n\Omega^n g"'] & \Omega^n Z \arrow[rr, dashed] \arrow[ur, dashed, "(-1)^n\Omega^n h"] & & \Omega^{n-1} X \arrow[uu, "\varepsilon^{-1}_{\Omega^{n-1}X}"']
\end{tikzcd}$$

where the dashed arrows are the necessary compositions which makes this diagram commute. By Lemma 0.8, the top row is distinguished, thus exact (by Proposition 0.3). Thus, since the diagram commutes and the vertical arrows are isomorphisms, it follows that the bottom row is exact. Again, a simple diagram chase yields that we can forget the signs and the sequence is still exact, so we get precisely the desired result. $\square$

Proposition 0.10. *Given a distinguished triangle*

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X,$$

the sequence

$$\cdots \to \Sigma^{n-1}Z \xrightarrow{\Sigma^{n-1}h} \Sigma^n X \xrightarrow{\Sigma^n f} \Sigma^n Y \xrightarrow{\Sigma^n g} \Sigma^n Z \xrightarrow{\Sigma^n h} \Sigma^{n+1}X \to \cdots$$

extending infinitely in either direction is exact (in the sense of Definition 0.1), where for $n > 0$ we are defining $\Sigma^{-n} := \Omega^n$, and by abusive of notation we are writing $\Sigma^{-n}h$ to mean the composition

$$\Sigma^{-n}Z = \Omega^n Z \xrightarrow{\Omega^n h} \Omega^n \Sigma X \xrightarrow{\Omega^{n-1}\eta_X^{-1}} \Omega^{n-1}X = \Sigma^{-n+1}X,$$

where $\eta : \mathrm{Id}_{\mathcal{C}} \Rightarrow \Omega\Sigma$ is the unit of the adjunction $\Sigma \dashv \Omega$.

Proof. Exactness of

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X \xrightarrow{-\Sigma f} \Sigma Y$$

is Proposition 0.3 and axiom TR4. A simple diagram chase yields that the sequence is still exact after removing the sign (change $-\Sigma f$ to Σf). Similarly, for $n > 0$, exactness of

$$\Sigma^n X \xrightarrow{\Sigma^n f} \Sigma^n Y \xrightarrow{\Sigma^n g} \Sigma^n Z \xrightarrow{\Sigma^n h} \Sigma^n X \xrightarrow{-\Sigma^n f} \Sigma^n Y$$

is Lemma 0.9 combined with axiom TR4. Again, we can remove the sign and it is still exact. It remains to show exactness to the left. It suffices to show that the row in the following diagram is exact for all $n > 0$:

$$\begin{tikzcd}
\Omega^n X \arrow[r, "\Omega^n f"] & \Omega^n Y \arrow[r, "\Omega^n g"] & \Omega^n Z \arrow[rr, "\Omega^{n-1}(\eta_X^{-1}\circ\Omega h)"] \arrow[dr, "\Omega^n h"'] & & \Omega^{n-1}X \arrow[r, "\Omega^{n-1}f"] & \Omega^{n-1}Y \\
& & & \Omega^n\Sigma X \arrow[ur, "\Omega^{n-1}\eta_X^{-1}"'] & &
\end{tikzcd}$$

Exactness of the first three arrows is Lemma 0.9. Thus, we would like to show that for all $n > 0$ that the row in the following diagram is exact:

$$\begin{tikzcd}
\Omega^n Z \arrow[rr, "\Omega^{n-1}(\eta_X^{-1}\circ\Omega h)"] \arrow[dr, "\Omega^n h"'] & & \Omega^{n-1}X \arrow[r, "\Omega^{n-1}f"] & \Omega^{n-1}Y \\
& \Omega^n\Sigma X \arrow[ur, "\Omega^{n-1}\eta_X^{-1}"'] & &
\end{tikzcd} \tag{2}$$

By Lemma 0.7, the row in the following commutative diagram is a distinguished triangle:

$$\begin{tikzcd}
\Omega Z \arrow[rr, "-\widetilde{h}"] \arrow[dr, "-\Omega h"'] & & X \arrow[r, "f"] & Y \arrow[r, "\widetilde{\Omega g}"] & \Sigma\Omega Z \\
& \Omega\Sigma X \arrow[ur, "\eta_X^{-1}"'] & & &
\end{tikzcd}$$

Thus the row is exact by Proposition 0.3, and again we can remove the sign to get that the row in (2) is exact when $n = 1$. Finally, we can apply Lemma 0.9 to the distinguished triangle in the above diagram to get exactness of (2) when $n > 1$. □

Also important for our work is the concept of a *tensor triangulated category*, that is, a triangulated symmetric monoidal category in which the triangulated structures are compatible, in the following sense:

Definition 0.11. A *tensor triangulated category* is a triangulated symmetric monoidal category $(\mathcal{C}, \otimes, S, \Sigma, \Omega, \mathcal{D})$ such that:

TT1 For all objects X and Y in $\mathcal{C}$, there are natural isomorphisms

$$e_{X,Y} : \Sigma X \otimes Y \xrightarrow{\cong} \Sigma(X \otimes Y).$$

TT2 For each object X in $\mathcal{C}$, the functor $X \otimes (-) \cong (-) \otimes X$ is an additive functor.

TT3 For each object X in $\mathcal{C}$, the functor $X \otimes (-) \cong (-) \otimes X$ preserves distinguished triangles, in that given a distinguished triangle/(co)fiber sequence

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A,$$

then also

$$X \otimes A \xrightarrow{X\otimes f} X \otimes B \xrightarrow{X\otimes g} X \otimes C \xrightarrow{X\otimes h} \Sigma(X \otimes A)$$

and

$$A \otimes X \xrightarrow{f\otimes X} B \otimes X \xrightarrow{g\otimes X} C \otimes X \xrightarrow{h\otimes X} \Sigma(A \otimes X)$$

are distinguished triangles, where here we are being abusive and writing $X \otimes h$ and $h \otimes X$ to denote the compositions

$$X \otimes C \xrightarrow{X\otimes h} X \otimes \Sigma A \xrightarrow{\tau} \Sigma A \otimes X \xrightarrow{e_{A,X}} \Sigma(A \otimes X) \xrightarrow{\Sigma\tau} \Sigma(X \otimes A)$$

and

$$C \otimes X \xrightarrow{h\otimes X} \Sigma A \otimes X \xrightarrow{e_{A,X}} \Sigma(A \otimes X),$$

respectively.

TT4 Given objects X, Y, and Z in $\mathcal{C}$, the following diagram must commute:

$$\begin{tikzcd}
(\Sigma X \otimes Y) \otimes Z \arrow[r, "e_{X,Y\otimes Z}"] \arrow[d, "\alpha"'] & \Sigma(X \otimes Y) \otimes Z \arrow[r, "e_{X\otimes Y,Z}"] & \Sigma((X \otimes Y) \otimes Z) \arrow[d, "\alpha"] \\
\Sigma X \otimes (Y \otimes Z) \arrow[rr, "e_{X,Y\otimes Z}"'] & & \Sigma(X \otimes (Y \otimes Z))
\end{tikzcd}$$

TT5 The following diagram must commute for all $n, m \in \mathbb{Z}$:

$$\begin{tikzcd}
\Sigma S \otimes \Sigma S \arrow[r, "e_{S,S}"] \arrow[d, "\tau"'] & \Sigma(S \otimes \Sigma S) \arrow[r, "\Sigma\lambda_{\Sigma S}"] & \Sigma^2 S \arrow[d, "-1"] \\
\Sigma S \otimes \Sigma S \arrow[r, "e_{S,S}"] & \Sigma(S \otimes \Sigma S) \arrow[r, "\Sigma\lambda_{\Sigma S}"] & \Sigma^2 S
\end{tikzcd}$$

Usually, most tensor triangulated categories that arise in nature will satisfy additional coherence axioms (see axioms TC1–TC5 in [1]), but the above definition will suffice for our purposes. Note that in the definition of the tensor triangulated category, we chose isomorphisms

$$e_{X,Y} : \Sigma X \otimes Y \xrightarrow{\cong} \Sigma(X \otimes Y),$$

but we just as well could have chosen isomorphisms

$$e'_{X,Y} : X \otimes \Sigma Y \xrightarrow{\cong} \Sigma(X \otimes Y).$$

Definition 0.12. Given a tensor triangulated category $(\mathcal{C}, \otimes, S, \Sigma, \Omega, e)$, there are natural isomorphisms

$$e'_{X,Y} : X \otimes \Sigma Y \xrightarrow{\cong} \Sigma(X \otimes Y)$$

obtained via the composition

$$X \otimes \Sigma Y \xrightarrow{\tau} \Sigma Y \otimes X \xrightarrow{e_{Y,X}} \Sigma(Y \otimes X) \xrightarrow{\Sigma\tau} \Sigma(X \otimes Y).$$

Proposition 0.13. *The isomorphisms $e'_{X,Y} : X \otimes \Sigma Y \to \Sigma(X \otimes Y)$ defined above satisfy the following coherence condition for any objects X, Y, and Z:*

$$\begin{tikzcd}
(X \otimes Y) \otimes \Sigma Z \arrow[rr, "e'_{X\otimes Y,\Sigma Z}"] \arrow[d, "\alpha"'] & & \Sigma((X \otimes Y) \otimes Z) \arrow[d, "\Sigma\alpha"] \\
X \otimes (Y \otimes \Sigma Z) \arrow[r, "X\otimes e'_{Y,Z}"] & X \otimes \Sigma(Y \otimes Z) \arrow[r, "e'_{X,Y\otimes Z}"] & \Sigma(X \otimes (Y \otimes Z))
\end{tikzcd}$$

Proof. By the coherence theorem for monoidal categories, we may assume associativity holds up to strict equality, in which case we simply wish to show that the following diagram commutes:

$$\begin{tikzcd}
X \otimes Y \otimes \Sigma Z \arrow[r, "X \otimes e'_{Y,Z}"] \arrow[dr, "e'_{X \otimes Y,Z}"'] & X \otimes \Sigma(Y \otimes Z) \arrow[d, "e'_{X,Y \otimes Z}"] \\
 & \Sigma(X \otimes Y \otimes Z)
\end{tikzcd}$$

Now consider the following diagram:

$$\begin{tikzcd}
X \otimes Y \otimes \Sigma Z \arrow[r, "X \otimes \tau_{Y,\Sigma Z}"] \arrow[d, "\tau_{X \otimes Y,\Sigma Z}"'] & X \otimes \Sigma Z \otimes Y \arrow[r, "X \otimes e_{Z,Y}"] \arrow[d, "\tau_{X,\Sigma Z \otimes Y}"] & X \otimes \Sigma(Z \otimes Y) \arrow[r, "X \otimes \Sigma \tau_{Z,Y}"] & X \otimes \Sigma(Y \otimes Z) \arrow[d, "\tau_{X,\Sigma(Y \otimes Z)}"] \\
\Sigma Z \otimes X \otimes Y \arrow[r, "\Sigma Z \otimes \tau_{X,Y}"] \arrow[dd, "e_{Z,X \otimes Y}"'] & \Sigma Z \otimes Y \otimes X \arrow[r, "e_{Z,Y} \otimes X"] \arrow[d, "e_{Z,Y \otimes X}"'] & \Sigma(Z \otimes Y) \otimes X \arrow[r, "\Sigma \tau_{Z,Y} \otimes X"] \arrow[dl, "e_{Z \otimes Y,X}"] & \Sigma(Y \otimes Z) \otimes X \arrow[d, "e_{Y \otimes Z,X}"] \\
 & \Sigma(Z \otimes Y \otimes X) \arrow[rr, "\Sigma(\tau_{Z,Y} \otimes X)"] \arrow[d, "\Sigma \tau_{Z \otimes Y,X}"] & & \Sigma(Y \otimes Z \otimes X) \arrow[d, "\Sigma \tau_{Y \otimes Z,X}"] \\
\Sigma(Z \otimes X \otimes Y) \arrow[ur, "\Sigma(Z \otimes \tau_{X,Y})"] \arrow[r, "\Sigma(\tau_{Z,X} \otimes Y)"] \arrow[rrr, bend right, "\Sigma \tau_{Z,X \otimes Y}"'] & \Sigma(X \otimes Z \otimes Y) \arrow[rr, "\Sigma(X \otimes \tau_{Z,Y})"] & & \Sigma(X \otimes Y \otimes Z)
\end{tikzcd}$$

Unravelling definitions, the top composition is $e'_{X,Y \otimes Z} \circ X \otimes e'_{Y,Z}$ and the bottom composition is $e'_{X \otimes Y,Z}$, so it suffices to show this diagram commutes. The top left square commutes by coherence for symmetric monoidal categories. The trapezoid below that on the left commutes by naturality of e. The triangle below that commutes by coherence for symmetric monoidal categories. The top right rectangle commutes by functoriality of $- \otimes -$ and naturality of τ. The small triangle below that in the middle of the diagram commutes by axiom TT4 for a tensor triangulated category. Commutativity of the trapezoid on the middle right is naturality of e. Finally, the remaining two regions on the bottom commutes by coherence for symmetric monoidal categories. $\square$