In the first section, we fix a symmetric monoidal category $(\mathcal{C}, \otimes, S)$ withj left unitor, right unitor, associator, and symmetry isomorphism λ, ρ, α, and τ, respectively.

Definition 0.1. A *monoid object* (E, μ, e) is an object E in $\mathcal{C}$ along with a multiplication map $\mu : E \otimes E \to E$ and a unit map $e : S \to E$ such that the following diagram commutes:

$$\begin{tikzcd} E \otimes S \arrow[r, "E\otimes e"] \arrow[dr, "\rho"'] & E \otimes E \arrow[d, "\mu"] & S \otimes E \arrow[l, "e\otimes E"'] \arrow[dl, "\lambda"] \\ & E & \end{tikzcd} \qquad \begin{tikzcd} (E \otimes E) \otimes E \arrow[rr, "\mu\otimes E"] \arrow[d, "\alpha"'] & & E \otimes E \arrow[d, "\mu"] \\ E \otimes (E \otimes E) \arrow[r, "E\otimes\mu"] & E \otimes E \arrow[r, "\mu"] & E \end{tikzcd}$$

The first diagram expresses unitality, while the second expressed associativity. If in addition the following diagram commutes,

$$\begin{tikzcd} E \otimes E \arrow[rr, "\tau"] \arrow[dr, "\mu"'] & & E \otimes E \arrow[dl, "\mu"] \\ & E & \end{tikzcd}$$

then we say (E, μ, e) is a *commutative* monoid object.

From now on we fix a monoidal closed tensor triangulated category $(\mathcal{SH}, \otimes, S, \Sigma, e, \mathcal{D})$ (**??**) with arbitrary (small) (co)products and sub-Picard grading $(A, \mathbf{1}, h, \{S^a\}, \{\phi_{a,b}\})$ (**??**), and we adopt the conventions outlined in **??**. In all proofs that follow we will freely use the coherence theorem for symmetric monoidal categories. In particular, we will assume without loss of generality that the associators and unitors in $\mathcal{SH}$ are identities.

Proposition 0.2. *For each $a \in A$, the isomorphisms*

$$\eta^a_X : X \xrightarrow{\lambda_X^{-1}} S \otimes X \xrightarrow{\phi_{a,-a}\otimes X} (S^a \otimes S^{-a}) \otimes X \xrightarrow{\alpha} S^a \otimes (S^{-a} \otimes X) = \Sigma^a\Omega^a X$$

and

$$\varepsilon^a_X : \Omega^a\Sigma^a X = S^{-a} \otimes (S^a \otimes X) \xrightarrow{\alpha^{-1}} (S^{-a} \otimes S^a) \otimes X \xrightarrow{\phi^{-1}_{-a,a}\otimes X} S \otimes X \xrightarrow{\lambda_X} X$$

are natural in X, and furthermore, they are the unit and counit respectively of the adjoint autoequivalence $(\Omega^a, \Sigma^a, \eta^a, \varepsilon^a)$ of $\mathcal{SH}$. In particular, since $\Sigma \cong \Sigma^{\mathbf{1}}$, $\Omega := \Omega^{\mathbf{1}}$ is a left adjoint for Σ, so that $(\mathcal{SH}, \Omega, \Sigma, \eta, \varepsilon, \mathcal{D})$ is an adjointly *triangulated category (**??**), where η and ε are the compositions*

$$\eta : \mathrm{Id}_{\mathcal{SH}} \xRightarrow{\eta^{\mathbf{1}}} \Sigma^{\mathbf{1}}\Omega \xRightarrow{\nu^{-1}\Omega} \Sigma\Omega \qquad \textit{and} \qquad \varepsilon : \Omega\Sigma \xRightarrow{\Omega\nu} \Omega\Sigma^{\mathbf{1}} \xRightarrow{\varepsilon^{\mathbf{1}}} \mathrm{Id}_{\mathcal{SH}}.$$

Proof. In this proof, we will freely employ the coherence theorem for monoidal categories (see [1]), which essentially tells us that we may assume we are working in a strict monoidal category (i.e., that the associators and unitors and are identities). Then η^a_X and ε^a_X become simply the maps

$$\eta^a_X : X \xrightarrow{\phi_{a,-a}\otimes X} S^a \otimes S^{-a} \otimes X \qquad \text{and} \qquad \varepsilon^a_X : S^{-a} \otimes S^a \otimes X \xrightarrow{\phi^{-1}_{-a,a}\otimes X} X.$$

That these maps are natural in X follows by functoriality of $-\otimes-$. Now, recall that in order to show that these natural isomorphisms form an *adjoint* equivalence, it suffices to show that the natural isomorphisms $\eta^a : \mathrm{Id}_{\mathcal{SH}} \Rightarrow \Omega^a\Sigma^a$ and $\varepsilon^a : \Sigma^a\Omega^a \Rightarrow \mathrm{Id}_{\mathcal{SH}}$ satisfy one of the two zig-zag identities:

$$\begin{tikzcd} \Omega^a \arrow[r, "\Omega^a\eta^a"] \arrow[dr, equal] & \Omega^a\Sigma^a\Omega^a \arrow[d, "\varepsilon^a\Omega^a"] \\ & \Omega^a \end{tikzcd} \qquad \begin{tikzcd} \Sigma^a\Omega^a\Sigma^a \arrow[d, "\Sigma^a\varepsilon^a"'] & \Sigma^a \arrow[l, "\eta^a\Sigma^a"'] \arrow[dl, equal] \\ \Sigma^a & \end{tikzcd}$$

(that it suffices to show only one is [2, Lemma 3.2]). We will show that the left is satisfied. Unravelling definitions, we simply wish to show that the following diagram commutes for all X in $\mathcal{SH}$:

$$\begin{tikzcd}
S^{-a} \otimes X \arrow[r, "S^{-a}\otimes\phi_{a,-a}\otimes X"] \arrow[dr, equal] & S^{-a} \otimes S^{a} \otimes S^{-a} \otimes X \arrow[d, "\phi^{-1}_{-a,a}\otimes S^{-a}\otimes X"] \\
& S^{-a} \otimes X
\end{tikzcd}$$

Yet this is simply the diagram obtained by applying $-\otimes X$ to the associativity coherence diagram for the $\phi_{a,b}$'s (since $\phi_{a,0}$ and $\phi_{0,a}$ coincide with the unitors, and here we are taking the unitors and associators to be equalities), so it does commute, as desired. □

Proposition 0.3. *Let (E, μ, e) be a monoid object in $\mathcal{SH}$, and consider the multiplication map $\pi_*(E) \times \pi_*(E) \to \pi_*(E)$ which sends classes $x : S^a \to E$ and $y : S^b \to E$ to the composition*

$$S^{a+b} \xrightarrow{\phi_{a,b}} S^a \otimes S^b \xrightarrow{x\otimes y} E \otimes E \xrightarrow{\mu} E.$$

Then this endows $\pi_*(E)$ with the structure of an A-graded ring with unit $e \in \pi_0(E) = [S, E]$.

Proof. Here we are using **??**, so it suffices to show the given assignment is associative and unital w.r.t. homogeneous elements. Suppose we have classes x, y, and z in $\pi_a(E)$, $\pi_b(E)$, and $\pi_c(E)$, respectively. To see associativity, consider the following diagram:

$$\begin{tikzcd}
& & & E \otimes E \arrow[d, "\mu"] \\
S^{a+b+c} \arrow[r, "\cong"] & S^a \otimes S^b \otimes S^c \arrow[r, "x\otimes y\otimes z"] & E \otimes E \otimes E \arrow[ur, "\mu\otimes E"] \arrow[dr, "E\otimes\mu"'] & E \\
& & & E \otimes E \arrow[u, "\mu"']
\end{tikzcd}$$

(here the first arrow is the unique isomorphism obtained by composing products of $\phi_{a,b}$'s, see **??**). It commutes by associativity of μ. It follows by functoriality of $- \otimes -$ that the top composition is $(x \cdot y) \cdot z$ while the bottom is $x \cdot (y \cdot z)$, so they are equal as desired. To see that $e \in \pi_0(E)$ is a left and right unit for this multiplication, consider the following diagram

$$\begin{tikzcd}
& S^a \arrow[dl, "e\otimes x"'] \arrow[d, "x"] \arrow[dr, "x\otimes e"] & \\
E \otimes E \arrow[dr, "\mu"'] & E \arrow[l, "e\otimes E"'] \arrow[r, "E\otimes e"] \arrow[d, equal] & E \otimes E \arrow[dl, "\mu"] \\
& E &
\end{tikzcd}$$

Commutativity of the two top triangles is functoriality of $- \otimes -$. Commutativity of the bottom two triangles is unitality of μ. Thus the diagram commutes, so $e \cdot x = x \cdot e$. Finally, to see this product is bilinear (distributive). Suppose we further have some $x' \in \pi_a(E)$ and $y' \in \pi_b(E)$, and consider the following diagrams:

$$\begin{tikzcd}
S^{a+b} \arrow[r, "\phi_{a,b}"] \arrow[d, "\Delta"'] & S^a \otimes S^b \arrow[r, "\Delta \otimes S^b"] \arrow[d, "\Delta"] & (S^a \oplus S^a) \otimes S^b \arrow[r, "(x \oplus x') \otimes y"] \arrow[dl, "\cong"'] & (E \oplus E) \otimes E \arrow[dl, "\cong"'] \arrow[d, "\nabla \otimes E"] & \\
S^{a+b} \oplus S^{a+b} \arrow[r, "\phi_{a,b} \oplus \phi_{a,b}"'] & (S^a \otimes S^b) \oplus (S^a \otimes S^b) \arrow[r, "(x \otimes y) \oplus (x' \otimes y)"'] & (E \otimes E) \oplus (E \otimes E) \arrow[r, "\nabla"'] & E \otimes E \arrow[r, "\mu"] & E
\end{tikzcd}$$

$$\begin{tikzcd}
S^{a+b} \arrow[r, "\phi_{a,b}"] \arrow[d, "\Delta"'] & S^a \otimes S^b \arrow[r, "S^a \otimes \Delta"] \arrow[d, "\Delta"] & S^b \otimes (S^b \oplus S^b) \arrow[r, "x \otimes (y \oplus y')"] \arrow[dl, "\cong"'] & E \otimes (E \oplus E) \arrow[dl, "\cong"'] \arrow[d, "E \otimes \nabla"] & \\
S^{a+b} \oplus S^{a+b} \arrow[r, "\phi_{a,b} \oplus \phi_{a,b}"'] & (S^a \otimes S^b) \oplus (S^a \otimes S^b) \arrow[r, "(x \otimes y) \oplus (x \otimes y')"'] & (E \otimes E) \oplus (E \otimes E) \arrow[r, "\nabla"'] & E \otimes E \arrow[r, "\mu"] & E
\end{tikzcd}$$

The unlabeled isomorphisms are those given by the fact that $- \otimes -$ is additive in each variable (since $\mathcal{SH}$ is tensor triangulated). Commutativity of the left squares is naturality of $\Delta : X \to X \oplus X$ in an additive category. Commutativity of the rest of the diagram follows again from the fact that $- \otimes -$ is an additive functor in each variable. Hence, by functoriality of $- \otimes -$, these diagrams tell us that $(x + x') \cdot y = x \cdot y + x' \cdot y$ and $x \cdot (y + y') = x \cdot y + x \cdot y'$, respectively. $\square$

Proposition 0.4. *For all $a, b \in A$ there exists an element $\theta_{a,b} \in \pi_0(S) = [S, S]$ (determined by choice of coherent family $\{\phi_{a,b}\}$) such that given any commutative monoid object (E, μ, e) in $\mathcal{SH}$, the A-graded ring structure on $\pi_*(E)$ (**??**) has a commutativity formula given by*

$$x \cdot y = y \cdot x \cdot (e \circ \theta_{a,b})$$

for all $x \in \pi_a(E)$ and $y \in \pi_b(E)$. In particular, $\theta_{a,b} \in \mathrm{Aut}(S)$ is the composition

$$S \xrightarrow{\cong} S^{-a-b} \otimes S^a \otimes S^b \xrightarrow{S^{-a-b} \otimes \tau} S^{-a-b} \otimes S^b \otimes S^a \xrightarrow{\cong} S,$$

where the outermost maps are the unique maps specified by **??**.

Proof. Let (E, μ, e), x, and y as in the statement of the proposition. Now consider the following diagram

$$\begin{tikzcd}
S^{a+b} \arrow[r, "\phi_{a,b}"] \arrow[dd, dashed, "\phi_{b,a}^{-1} \circ \tau \circ \phi_{a,b}"'] & S^a \otimes S^b \arrow[r, "x \otimes y"] \arrow[dd, "\tau"'] & E \otimes E \arrow[dd, "\tau"'] \arrow[dr, "\mu"] & \\
 & & & E \\
S^{a+b} \arrow[r, "\phi_{b,a}"] & S^b \otimes S^a \arrow[r, "y \otimes x"] & E \otimes E \arrow[ur, "\mu"'] &
\end{tikzcd}$$

The left square commutes by definition. The middle square commutes by naturality of the symmetry isomorphism. Finally, the right square commutes by commutativity of E. Unravelling definitions, we have shown that under the product on $\pi_*(E)$ induced by the $\phi_{a,b}$'s,

$$x \cdot y = (y \cdot x) \circ (\phi_{b,a}^{-1} \circ \tau \circ \phi_{a,b}).$$

Thus, in order to show the desired result it further suffices to show that

$$(y \cdot x) \circ (\phi_{b,a}^{-1} \circ \tau \circ \phi_{a,b}) = y \cdot x \cdot (e \circ \theta_{a,b}).$$

Consider the following diagram:

$$\begin{tikzcd}
S^{a+b} \arrow[rr, "\phi_{a,b}"] \arrow[d, "\cong"'] & & S^a \otimes S^b \arrow[d, "\tau"] \\
S^b \otimes S^a \otimes S^{-a-b} \otimes S^a \otimes S^b \arrow[urr, "\cong"'] \arrow[d, "S^b\otimes S^a\otimes S^{-a-b}\otimes\tau"'] & & S^b \otimes S^a \arrow[d, "\phi_{b,a}^{-1}"] \arrow[dl, equal] \\
S^b \otimes S^a \otimes S^{-a-b} \otimes S^b \otimes S^a \arrow[urr, "\cong"] \arrow[r, "\cong"] & S^b \otimes S^a \arrow[dr, "y\otimes x"] \arrow[ddl, "y\otimes x\otimes e"'] & S^{a+b} \arrow[l, "\phi_{b,a}"'] \\
& & E \otimes E \arrow[dll, "E\otimes E\otimes e"'] \arrow[d, equal] \\
E \otimes E \otimes E \arrow[rr, "E\otimes\mu"] \arrow[d, "\mu\otimes E"'] & & E \otimes E \arrow[d, "\mu"] \\
E \otimes E \arrow[rr, "\mu"'] & & E
\end{tikzcd}$$

Here any map simply labelled $\cong$ is an appropriate composition of copies of $\phi_{a,b}$'s, associators, and their inverses, so that each of these maps are necessarily unique by **??**. The triangles in the top large rectangle commutes by coherence for the $\phi_{a,b}$'s. The parallelogram commutes by naturality of τ and coherence of the of $\phi_{a,b}$'s. The middle skewed triangle commutes by functoriality of $- \otimes -$. The triangle below that commutes by unitality of μ. Finally, the bottom rectangle commmutes by associativity of μ. Hence, by unravelling definitions and applying functoriality of $- \otimes -$, we get that the right composition is $(y \cdot x) \circ (\phi_{b,a}^{-1} \circ \tau \circ \phi_{a,b})$, while the left composition is $y \cdot x \cdot (e \circ \theta_{a,b})$, so they are equal as desired. $\square$

Proposition 0.5. *Given $a \in A$, we have $\theta_{0,a} = \theta_{a,0} = \mathrm{id}_S$.*

Proof. Recall $\theta_{a,0}$ is the composition

$$S \xrightarrow{\phi_{-a,a}} S^{-a} \otimes S^a \xrightarrow{S^{-a}\otimes\phi_{a,0}} S^{-a} \otimes (S^a \otimes S) \xrightarrow{S^{-a}\otimes\tau} S^{-a} \otimes (S \otimes S^a) \xrightarrow{S^{-a}\otimes\phi_{0,a}^{-1}} S^{-a} \otimes S^a \xrightarrow{\phi_{-a,a}^{-1}} S$$

By the coherence theorem for symmetric monoidal categories and the fact that $\phi_{a,0}$ and $\phi_{0,a}$ coincide with the unitors, we have that the composition

$$S^a \xrightarrow{\phi_{a,0}=\rho_{S^a}^{-1}} S^a \otimes S \xrightarrow{\tau} S \otimes S^a \xrightarrow{\phi_{0,a}^{-1}=\lambda_{S^a}} S^a$$

is precisely the identity map, so by functoriality of $- \otimes -$, we have that $\theta_{a,0}$ is the composition

$$S \xrightarrow{\phi_{-a,a}} S^{-a} \otimes S^a \xrightarrow{=} S^{-a} \otimes S^a \xrightarrow{\phi_{-a,a}^{-1}} S,$$

so $\theta_{a,0} = \mathrm{id}_S$, meaning

$$x \cdot y = y \cdot x \cdot (e \circ \theta_{a,0}) = y \cdot x \cdot e = y \cdot x,$$

where the last equality follows by the fact that e is the unit for the multiplication on $\pi_*(E)$. An entirely analagous argument yields that $\theta_{0,a} = \mathrm{id}_S$. $\square$

Proposition 0.6. *Let X and Y be objects in $\mathcal{SH}$. Then the A-graded pairing*

$$\pi_*(X) \times \pi_*(Y) \to \pi_*(X \otimes Y)$$

sending $x : S^a \to X$ and $y : S^b \to Y$ to the composition

$$S^{a+b} \xrightarrow{\phi_{a,b}} S^a \otimes S^b \xrightarrow{x\otimes y} X \otimes Y$$

is additive in each argument.

Proof. Let $a, b \in A$, and let $x_1, x_2 : S^a \to X$ and $y : S^b \to Y$. Then consider the following diagram

$$\begin{tikzcd}
S^{a+b} \arrow[r, "\cong"] & S^a \otimes S^b \arrow[r, "\Delta \otimes S^b"] \arrow[d, "\Delta"'] & (S^a \oplus S^a) \otimes S^b \arrow[dl, "\cong"'] \arrow[d, "(x_1 \oplus x_2) \otimes y"] \\
& (S^a \otimes S^b) \oplus (S^a \otimes S^b) \arrow[d, "(x_1 \otimes y) \oplus (x_2 \otimes y)"'] & (X \oplus X) \otimes Y \arrow[dl, "\cong"'] \arrow[d, "\nabla \otimes Y"] \\
& (X \otimes Y) \oplus (X \otimes Y) \arrow[r, "\nabla"] & X \otimes Y
\end{tikzcd}$$

The isomorphisms are given by the fact that $- \otimes -$ is additive in each variable. Both triangles and the parallelogram commute since $- \otimes -$ is additive. By functoriality of $- \otimes -$, the top composition is $(x_1 + x_2) \cdot y$ and the bottom composition is $x_1 \cdot y + x_2 \cdot y$, so they are equal, as desired. An entirely analagous argument yields that $x \cdot (y_1 + y_2) = x \cdot y_1 + x \cdot y_2$ for $x \in \pi_*(X)$ and $y_1, y_2 \in \pi_*(Y)$. $\square$

Proposition 0.7 ([3, Proposition 5.11]). *Let (E, μ, e) be a monoid object in $\mathcal{SH}$. Then $E_*(-)$ is a functor from $\mathcal{SH}$ to left A-graded $\pi_*(E)$-modules, where given some X in $\mathcal{SH}$, $E_*(X)$ may be endowed with the structure of a left A-graded $\pi_*(E)$-module via the map*

$$\pi_*(E) \times E_*(X) \to E_*(X)$$

which given $a, b \in A$, sends $x : S^a \to E$ and $y : S^b \to E \otimes X$ to the composition

$$x \cdot y : S^{a+b} \cong S^a \otimes S^b \xrightarrow{x \otimes y} E \otimes (E \otimes X) \cong (E \otimes E) \otimes X \xrightarrow{\mu \otimes X} E \otimes X.$$

Similarly, the assignment $X \mapsto X_*(E)$ is a functor from $\mathcal{SH}$ to right A-graded $\pi_*(E)$-modules, where the structure map

$$X_*(E) \times \pi_*(E) \to X_*(E)$$

sends $x : S^a \to X \otimes E$ and $y : S^b \to E$ to the composition

$$x \cdot y : S^{a+b} \cong S^a \otimes S^b \xrightarrow{x \otimes y} (X \otimes E) \otimes E \cong X \otimes (E \otimes E) \xrightarrow{X \otimes \mu} X \otimes E.$$

Finally, $E_*(E)$ is a $\pi_*(E)$-bimodule, in the sense that the left and right actions of $\pi_*(E)$ are compatible, so that given $y, z \in \pi_*(E)$ and $x \in E_*(E)$, $y \cdot (x \cdot z) = (y \cdot x) \cdot z$.

Proof. First we show that the map $\pi_*(E) \times E_*(X) \to E_*(X)$ endows $E_*(X)$ with the structure of a left $\pi_*(E)$-module. Let $a, b, c \in A$ and $x, x' : S^a \to E \otimes X$, $y : S^b \to E$, and $z, z' \in S^c \to E$. Then we wish to show that:

(1) $y \cdot (x + x') = y \cdot x + y \cdot x'$,

(2) $(z + z') \cdot x = z \cdot x + z' \cdot x$,

(3) $(zy) \cdot x = z \cdot (y \cdot x)$,

(4) $e \cdot x = x$.

Axioms (1) and (2) follow by the fact that $E_*(X) = \pi_*(E \otimes X)$ and Proposition 0.6. To see (3), consider the diagram:

$$\begin{tikzcd}
 & & & E \otimes E \otimes X \arrow[d, "\mu\otimes X"] \\
S^{a+b+c} \arrow[r, "\cong"] & S^c \otimes S^b \otimes S^a \arrow[r, "z\otimes y\otimes x"] & E \otimes E \otimes E \otimes X \arrow[ur, "E\otimes\mu\otimes X"] \arrow[dr, "\mu\otimes E\otimes X"'] & E \otimes X \\
 & & & E \otimes E \otimes X \arrow[u, "\mu\otimes X"']
\end{tikzcd}$$

It commutes by associativity of μ. By functoriality of $- \otimes -$, the two outside compositions equal $z \cdot (y \cdot x)$ on the top and $(z \cdot y) \cdot x$ on the bottom. Hence, they are equal, as desired.

Next, to see (4), consider the following diagram:

$$\begin{tikzcd}
S^a \arrow[rr, "x"] \arrow[dr, "x"'] \arrow[ddr, bend right, "e\otimes x"'] & & E \otimes X \\
 & E \otimes X \arrow[ur, equals] \arrow[d, "e\otimes E\otimes X"] & \\
 & E \otimes E \otimes X \arrow[uur, bend right, "\mu\otimes X"'] &
\end{tikzcd}$$

The top triangle commutes by definition. The left triangle commutes by functoriality of $- \otimes -$. The right triangle commutes by unitality of μ. The top composition is x while the bottom is $e \cdot x$, thus they are necessarily equal since the diagram commutes.

Thus, we have shown that the indicated map does indeed endow $E_*(X)$ with the structure of a left $\pi_*(E)$-module. It remains to show that $E_*(-)$ sends maps in $\mathcal{SH}$ to A-graded homomorphisms of left A-graded $\pi_*(E)$-modules. By definition, given $f : X \to Y$ in $\mathcal{SH}$, $E_*(f)$ is the map which takes a class $x : S^a \to E \otimes X$ to the composition

$$S^a \xrightarrow{x} E \otimes X \xrightarrow{E\otimes f} E \otimes Y.$$

To see this assignment is a homomorphism, suppose we are given some other $x' : S^a \to E \otimes X$ and some scalar $y : S^b \to E$. Then we would like to show $E_*(f)(x + x') = E_*(f)(x) + E_*(f)(x')$ and $E_*(f)(y \cdot x) = y \cdot E_*(f)(x)$. To see the former, consider the following diagram:

$$\begin{tikzcd}
 & & & (E \otimes Y) \oplus (E \otimes Y) \arrow[d, "\nabla"] \\
S^a \arrow[r, "\Delta"] & S^a \oplus S^a \arrow[r, "x\oplus x'"] & (E \otimes X) \oplus (E \otimes X) \arrow[ur, "(E\otimes f)\oplus(E\otimes f)"] \arrow[dr, "\nabla"'] & E \otimes Y \\
 & & & E \otimes X \arrow[u, "E\otimes f"']
\end{tikzcd}$$

It commutes by naturality of ∇ in an additive category. The top composition is $E_*(f)(x) + E_*(f)(x')$, while the bottom is $E_*(f)(x+x')$, so they are equal as desired. To see that $E_*(f)(y \cdot x) = y \cdot E_*(f)(x)$, consider the following diagram:

$$\begin{tikzcd}
S^{a+b} \arrow[r, "\phi_{b,a}"] & S^b \otimes S^a \arrow[r, "y\otimes x"] & E \otimes E \otimes X \arrow[r, "E\otimes E\otimes f"] \arrow[d, "\mu\otimes X"'] & E \otimes E \otimes Y \arrow[d, "\mu\otimes Y"] \\
 & & E \otimes X \arrow[r, "E\otimes f"] & E \otimes Y
\end{tikzcd}$$

It commutes by functoriality of $-\otimes-$. The bottom composition is $E_*(f)(y\cdot x)$, while the top composition is $y\cdot E_*(f)(x)$, so they are equal, as desired.

Showing that $X_*(E)$ has the structure of a right $\pi_*(E)$-module and that if $f : X \to Y$ is a morphism in $\mathcal{SH}$ then the map

$$X_*(E) = [S^*, X\otimes E] \xrightarrow{(f\otimes E)_*} [S^*, Y\otimes E] = Y_*(E)$$

is an A-graded homomorphism of right A-graded $\pi_*(E)$-modules is entirely analagous.

It remains to show that $E_*(E)$ is a $\pi_*(E)$-bimodule. Let $x : S^a \to E$, $y : S^b \to E\otimes E$, and $z : S^c \to E$, and consider the following diagram:

$$\begin{tikzcd}
 & & & E\otimes E\otimes E \arrow[d, "E\otimes\mu"] \\
S^{a+b+c} \arrow[r, "\cong"] & S^a\otimes S^b\otimes S^c \arrow[r, "x\otimes y\otimes z"] & E\otimes E\otimes E\otimes E \arrow[ur, "\mu\otimes E\otimes E"] \arrow[r, "\mu\otimes\mu"] \arrow[dr, "E\otimes E\otimes\mu"'] & E\otimes E \\
 & & & E\otimes E\otimes E \arrow[u, "\mu\otimes E"']
\end{tikzcd}$$

Commutativity follows by functoriality of $-\otimes-$, which also tells us that the two outside compositions are $(x\cdot y)\cdot z$ (on top) and $x\cdot(y\cdot z)$ (on bottom). Hence they are equal, as desired. □

Proposition 0.8. *Suppose we are given a distinguished triangle*

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$$

and some object W in $\mathcal{SH}$. Then there is an infinite long exact sequence

$$(1)\qquad \cdots \to [W,Z]_{*+\mathbf{n+1}} \xrightarrow{\partial} [W,X]_{*+\mathbf{n}} \xrightarrow{f_*} [W,Y]_{*+\mathbf{n}} \xrightarrow{g_*} [W,Z]_{*+\mathbf{n}} \xrightarrow{\partial} [W,Z]_{*+\mathbf{n-1}} \to \cdots,$$

where $\partial : [W,Z]_{*+\mathbf{n+1}} \to [W,X]_{*+\mathbf{n}}$ sends a class $x : S^{a+\mathbf{n+1}}\otimes W \to Z$ to the composition

$$S^{a+\mathbf{n}}\otimes W \cong S^{-\mathbf{1}}\otimes S^{a+\mathbf{n+1}}\otimes W \xrightarrow{S^{-\mathbf{1}}\otimes x} S^{-\mathbf{1}}\otimes Z \xrightarrow{S^{-\mathbf{1}}\otimes h} S^{-\mathbf{1}}\otimes\Sigma X \xrightarrow{S^{-\mathbf{1}}\otimes\nu_X} S^{-\mathbf{1}}\otimes S^{\mathbf{1}}\otimes X \xrightarrow{\phi^{-1}_{-\mathbf{1},\mathbf{1}}\otimes X} X.$$

Proof. Given $n>0$, we will write Σ^{-n} to denote the functor $\Omega^n = (S^{-\mathbf{1}})^n\otimes -$ in this proof.

For all $n>0$, the $\phi_{a,b}$'s yield natural isomorphisms

$$s^{-n} : \Omega^n X = (S^{-\mathbf{1}})^n \otimes X \xrightarrow{\phi\otimes X} S^{-\mathbf{n}}\otimes X = \Omega^{\mathbf{n}} X,$$

where here we are writing ϕ as a stand-in for the unique isomorphism $(S^{-\mathbf{1}})^n \cong S^{-\mathbf{n}}$ that can be obtained by composing copies of tensor products of $\phi_{a,b}$'s, associators, unitors, and their inverses (**??**). Similarly, we have natural isomorphisms $s^n : \Sigma^n X \cong S^{\mathbf{n}}\otimes X$ given by the composition

$$\Sigma^n X \xrightarrow{\nu_X^n} (S^{\mathbf{1}})^n\otimes X \xrightarrow{\phi\otimes X} S^{\mathbf{n}}\otimes X = \Sigma^{\mathbf{n}} X,$$

where again ϕ stands for the canonical isomorphism $(S^{\mathbf{1}})^n \cong S^{\mathbf{n}}$, and we inductively define $\nu^1 := \nu$ and ν_X^{n+1} to be the composition

$$\Sigma^{n+1}X = \Sigma^n\Sigma X \xrightarrow{\nu^n_{\Sigma X}} (S^{\mathbf{1}})^n\otimes\Sigma X \xrightarrow{(S^{\mathbf{1}})^n\otimes\nu_X} (S^{\mathbf{1}})^n\otimes S^{\mathbf{1}}\otimes X = (S^{\mathbf{1}})^{n+1}\otimes X.$$

Finally, we define s^0 to be the identity natural transformation on $\mathcal{SH}$. Then together with the natural isomorphisms $r^{\mathbf{n}}_{W,V} : [W,\Sigma^{\mathbf{n}}V]_* \cong [W,V]_{*-\mathbf{n}}$ given by **??**, we get the following natural isomorphisms of A-graded abelian groups for all $n\in\mathbb{Z}$

$$\ell^n_V : [W,\Sigma^n V]_* \xrightarrow{(s^n_V)_*} [W,\Sigma^{\mathbf{n}}V]_* \xrightarrow{r^{\mathbf{n}}_{W,V}} [W,V]_{*-\mathbf{n}}.$$

Now, given $n \in \mathbb{Z}$, consider the following diagram

$$\begin{tikzcd}
{[W, \Sigma^{n-1} Z]_*} \arrow[r, "h_{n-1}"] \arrow[d, "\ell_Z^{n-1}"'] & {[W, \Sigma^n X]_*} \arrow[r, "\Sigma^n f_*"] \arrow[d, "\ell_X^n"'] & {[W, \Sigma^n Y]_*} \arrow[r, "\Sigma^n g_*"] \arrow[d, "\ell_Y^n"'] & {[W, \Sigma^n Z]_*} \arrow[r, "h_n"] \arrow[d, "\ell_Z^n"'] & {[W, \Sigma^{n+1} X]_*} \arrow[d, "\ell_X^{n+1}"] \\
{[W, Z]_{*-\mathbf{n+1}}} \arrow[r, "\partial"'] & {[W, X]_{*-\mathbf{n}}} \arrow[r, "f_*"'] & {[W, Y]_{*-\mathbf{n}}} \arrow[r, "g_*"'] & {[W, Z]_{*-\mathbf{n}}} \arrow[r, "\partial"'] & {[W, X]_{*-\mathbf{n-1}}}
\end{tikzcd}$$

where $h_n = \Sigma^n h$ if $n \geq 0$ and $h_n = \Sigma^{n+1} \widetilde{h}$ if $n < 0$ (where $\widetilde{h} : \Omega Z \to X$ is the adjoint of $h : Z \to \Sigma X$). The top row is exact by **??**, and we have constructed the vertical arrows to be isomorphisms. The inner two squares commute by naturality of ℓ^n. Thus in order to show exactness of the bottom row, it suffices to show the outermost squares commute. Since our choice of $n \in \mathbb{Z}$ is arbitrary, it further suffices to show the right square commutes.

finish proof

$$\begin{tikzcd}
{[W, \Omega^{\mathbf{n}} Z]_*} \arrow[dddddd, "adj"'] \arrow[drr, "(\phi \otimes Z)_*"] & & {[W, \Omega^n Z]_*} \arrow[ll, "(\phi \otimes Z)_*"'] \arrow[r, "\Omega^{n-1} \widetilde{h}_*"] \arrow[d, "(\phi \otimes \Omega Z)_*"'] & {[W, \Omega^{n-1} X]_*} \arrow[d, "(\phi \otimes X)_*"] \\
& & {[W, \Omega^{\mathbf{n-1}} \Omega Z]_*} \arrow[r, "\Omega^{\mathbf{n-1}} \widetilde{h}_*"] \arrow[d, "adj"'] & {[W, \Omega^{\mathbf{n-1}} X]_*} \arrow[d, "adj"] \\
& & {[S^{\mathbf{n-1}} \otimes S^* \otimes W, \Omega Z]} \arrow[r, "\widetilde{h}_*"] \arrow[d, "(\phi \otimes W)^*"'] & {[S^{\mathbf{n-1}} \otimes S^* \otimes W, X]} \arrow[d, "(\phi \otimes W)^*"] \\
& \text{this region commutes by coherence} & {[W, \Omega Z]_{*+\mathbf{n-1}}} \arrow[r, "\widetilde{h}_*"] \arrow[d, "adj"'] & {[W, X]_{*+\mathbf{n-1}}} \\
& & {[S^{\mathbf{1}} \otimes S^{*+\mathbf{n-1}} \otimes W, Z]} \arrow[d, "(\phi \otimes W)^*"'] & \\
{[S^{\mathbf{n}} \otimes S^* \otimes W, Z]} \arrow[urr, "(\phi \otimes W)^*"] \arrow[rr, "(\phi \otimes W)^*"'] & & {[W, Z]_{*+\mathbf{n}}} \arrow[uur, "\partial"'] &
\end{tikzcd}$$

$$
\begin{tikzcd}
{[W,\Sigma^n Z]_*} \arrow[r, "\Sigma^n h_*"] \arrow[d, "\ell^n_Z"'] & {[W,\Sigma^{n+1}X]_*} \arrow[d, "\ell^n_{\Sigma X}"] \arrow[ddd, bend left, "\ell^{n+1}_X"] \\
{[W,Z]_{*-\mathbf{n}}} \arrow[r, "h_*"] \arrow[ddr, "\partial"'] & {[W,\Sigma X]_{*-\mathbf{n}}} \arrow[d, "(\nu_X)_*"] \\
 & {[W,\Sigma^{\mathbf{1}}X]_{*-\mathbf{n}}} \arrow[d, "r^{\mathbf{1}}_{W,X}"] \\
\text{seeing that this triangle commutes is easy} & {[W,X]_{*-\mathbf{n}-\mathbf{1}}}
\end{tikzcd}
$$

diagram chase around above crescent :

$$
\begin{tikzcd}
{[W,\Sigma^{n+1}X]_*} \arrow[rr, "(\nu_X^{n+1})_*"] \arrow[d, "(\nu^n_{\Sigma X})_*"'] & & {[W,(S^{\mathbf{1}})^{n+1}\otimes X]_*} \arrow[ddl, "(\phi\otimes X)"'] \arrow[dd, "(\phi\otimes X)_*"] \\
{[W,(S^{\mathbf{1}})^n\otimes \Sigma X]} \arrow[urr, "((S^{\mathbf{1}})^n\otimes\nu_X)_*"] \arrow[d, "(\phi\otimes\Sigma X)_*"'] & & \\
{[W,\Sigma^{\mathbf{n}}\Sigma X]_*} \arrow[r, "(\Sigma^{\mathbf{n}}\nu_X)_*"] \arrow[d, "adj"'] & {[W,\Sigma^{\mathbf{n}}\Sigma^{\mathbf{1}}X]_*} \arrow[r, "(\phi\otimes X)_*"] \arrow[d, "adj"] & {[W,\Sigma^{\mathbf{n+1}}X]_*} \arrow[ddddd, "adj"] \\
{[S^{-\mathbf{n}}\otimes S^*\otimes W,\Sigma X]} \arrow[r, "(\nu_X)_*"] \arrow[d, "(\phi\otimes W)^*"'] & {[S^{-\mathbf{n}}\otimes S^*\otimes W,S^{\mathbf{1}}\otimes X]} \arrow[ddl, "(\phi\otimes W)^*"] & \\
{[W,\Sigma X]_{*-\mathbf{n}}} \arrow[d, "(\nu_X)_*"'] & & \\
{[W,\Sigma^{\mathbf{1}}X]_{*-\mathbf{n}}} \arrow[d, "adj"'] & \text{this region commutes by coherence} & \\
{[S^{-\mathbf{1}}\otimes S^{*-\mathbf{n}}\otimes W,X]} \arrow[d, "(\phi\otimes W)^*"'] \arrow[drr, "(\phi\otimes W)^*"] & & \\
{[W,X]_{*-\mathbf{n}-\mathbf{1}}} & & {[S^{-\mathbf{n}-\mathbf{1}}\otimes S^*\otimes W,X]} \arrow[ll, "(\phi\otimes W)^*"]
\end{tikzcd}
$$

$\square$

Proposition 0.9 ([4, Proposition 2.2])**.** *Let (E,μ,e) be a monoid object in $\mathcal{SH}$ and let X be any object. Then the assignment*

$$E_*(E)\times E_*(X)\to E_*(E\otimes X)$$

which sends $x: S^a\to E\otimes E$ and $y: S^b\to E\otimes X$ to the composition

$$x\cdot y: S^{a+b}\cong S^a\otimes S^b\xrightarrow{x\otimes y}E\otimes E\otimes E\otimes X\xrightarrow{E\otimes\mu\otimes X}E\otimes E\otimes X$$

lifts to an A-graded homomorphism of left A-graded $\pi_*(E)$-modules

$$\Phi_X: E_*(E)\otimes_{\pi_*(E)}E_*(X)\to E_*(E\otimes X)$$

*(where here $E_*(E)$ has a $\pi_*(E)$-bimodule structure and $E_*(X)$ has a left $\pi_*(E)$-module structure as specified by Proposition 0.7, so $E_*(E)\otimes_{\pi_*(E)} E_*(X)$ is a left A-graded $\pi_*(E)$-module by **??**). Furthermore, this homomorphism is natural in X.*

Proof. First, recall by definition of the tensor product, in order to show the assignment $E_*(E)\times E_*(X)\to E_*(E\otimes X)$ induces a homomorphism $E_*(E)\otimes_{\pi_*(E)} E_*(X)\to E_*(E\otimes X)$ of A-graded abelian groups, it suffices to show that the assignment is $\pi_*(E)$-balanced, i.e., that it is linear in each argument and satisfies $xr\cdot y = x\cdot ry$ for $x\in E_*(E)$, $y\in E_*(X)$, and $r\in\pi_*(E)$.

First, note that by the identifications $E_*(E)=\pi_*(E\otimes E)$, $E_*(X)=\pi_*(E\otimes X)$, and $E_*(E\otimes X)=\pi_*(E\otimes E\otimes X)$, and Proposition 0.6, it is straightforward to see that the assignment commutes with addition of maps in each argument. Now, let $a,b,c\in A$, $x: S^a\to E\otimes E$, $y: S^b\to E\otimes X$, and $z: S^c\to E$. Then we wish to show $xz\cdot y = x\cdot zy$. Consider the following diagram (where here we are passing to a permutative category):

$$\begin{tikzcd}
 & & & E\otimes E\otimes E\otimes X \arrow[d, "E\otimes\mu\otimes X"] \\
S^{a+b+c} \arrow[r, "\cong"] & S^a\otimes S^c\otimes S^b \arrow[r, "x\otimes z\otimes y"] & E\otimes E\otimes E\otimes E\otimes X \arrow[ur, "E\otimes\mu\otimes E\otimes X"] \arrow[dr, "E\otimes E\otimes\mu\otimes X"'] & E\otimes E\otimes X \\
 & & & E\otimes E\otimes E\otimes X \arrow[u, "E\otimes\mu\otimes X"']
\end{tikzcd}$$

It commutes by associativity of μ. By functoriality of $-\otimes-$, the top composition is given by $(xz)\cdot y$ and the bottom composition is $x\cdot(zy)$, so we have they are equal, as desired. Thus, since the map $E_*(E)\times E_*(X)\to E_*(E\otimes X)$ is $\pi_*(E)$-balanced, we have that it induces a homomorphism of abelian groups. Furthermore, by **??** it is A-graded.

In order to see this map is a homomorphism of left $\pi_*(E)$-modules, we must show that $z(x\cdot y) = zx\cdot y$, where x, y, and z are defined as above. Now consider the following diagram:

$$\begin{tikzcd}
 & & & E\otimes E\otimes E\otimes X \arrow[d, "E\otimes\mu\otimes X"] \\
S^{a+b+c} \arrow[r, "\cong"] & S^c\otimes S^a\otimes S^b \arrow[r, "z\otimes x\otimes y"] & E\otimes E\otimes E\otimes E\otimes X \arrow[ur, "\mu\otimes E\otimes E\otimes X"] \arrow[dr, "E\otimes E\otimes\mu\otimes X"'] \arrow[r, "\mu\otimes\mu\otimes X"] & E\otimes E\otimes X \\
 & & & E\otimes E\otimes E\otimes X \arrow[u, "\mu\otimes E\otimes X"']
\end{tikzcd}$$

Commutativity of the triangles is functoriality of $-\otimes-$. By functoriality of $-\otimes-$, the top composition is $zx\cdot y$, and the bottom composition is $z(x\cdot y)$. Hence they are equal, as desired, so that the map we have constructed

$$E_*(E)\otimes_{\pi_*(E)} E_*(X)\to E_*(E\otimes X)$$

is indeed an A-graded homomorphism of left A-graded $\pi_*(E)$-modules.

Next, we would like to show that this homomorphism is natural in X. Let $f: X\to Y$ in $\mathcal{SH}$. Then we would like to show the following diagram commutes:

$$\begin{tikzcd}
E_*(E)\otimes_{\pi_*(E)} E_*(X) \arrow[r, "\Phi_X"] \arrow[d, "E_*(E)\otimes_{\pi_*(E)}E_*(f)"'] & E_*(E\otimes X) \arrow[d, "E_*(E\otimes f)"] \\
E_*(E)\otimes_{\pi_*(E)} E_*(Y) \arrow[r, "\Phi_Y"] & E_*(E\otimes Y)
\end{tikzcd} \tag{2}$$

As all the maps here are homomorphisms, it suffices to chase generators around the diagram. In particular, suppose we are given $x: S^a\to E\otimes E$ and $y: S^b\to E\otimes X$, and consider the following diagram exhibiting the two possible ways to chase $x \otimes y$ around the diagram (as usual, we are passing to a symmetric strict monoidal category):

$$\begin{tikzcd}
S^{a+b} \arrow[r, "\phi_{a,b}"] & S^a \otimes S^b \arrow[r, "x\otimes y"] & E \otimes E \otimes E \otimes X \arrow[r, "E\otimes\mu\otimes X"] \arrow[d, "E\otimes E\otimes E\otimes f"'] & E \otimes E \otimes X \arrow[d, "E\otimes E\otimes f"] \\
& & E \otimes E \otimes E \otimes Y \arrow[r, "E\otimes\mu\otimes Y"] & E \otimes E \otimes Y
\end{tikzcd}$$

This diagram commutes by functoriality of $- \otimes -$. Thus we have that diagram (2) does indeed commute, as desired. $\square$

Lemma 0.10. *Let E and X be objects in $\mathcal{SH}$. Then for all $a \in A$, there is an A-graded isomorphism of A-graded abelian groups*

$$t_X^a : E_*(\Sigma^a X) \cong E_{*-a}(X)$$

which sends a class $S^b \to E \otimes \Sigma^a X = E \otimes S^a \otimes X$ to the composition

$$S^{b-a} \xrightarrow{\phi_{b,-a}} S^b \otimes S^{-a} \xrightarrow{x\otimes S^{-a}} E\otimes S^a\otimes X\otimes S^{-a} \xrightarrow{E\otimes S^a\otimes \tau_{X,S^{-a}}} E\otimes S^a\otimes S^{-a}\otimes X \xrightarrow{E\otimes\phi_{a,-a}^{-1}\otimes X} E\otimes X$$

(where here we are ignoring associators and unitors). Furthermore this isomorphism is natural in X, and if E is a monoid object in $\mathcal{SH}$ then it is a natural isomorphism of $\pi_*(E)$-modules.

Proof. Expressed in terms of hom-sets, t_X^a is precisely the composition

$$\begin{tikzcd}
E_*(\Sigma^a X) \arrow[r, equal] & {[S^*, E \otimes S^a \otimes X]} \arrow[d, "-\otimes S^{-a}"] & \\
& {[S^* \otimes S^{-a}, E \otimes S^a \otimes X \otimes S^{-a}]} \arrow[d, "(\phi_{*,-a})^*"] & \\
& {[S^{*-a}, E \otimes S^a \otimes X \otimes S^{-a}]} \arrow[d, "(E\otimes S^a\otimes\tau)_*"] & \\
& {[S^{*-a}, E \otimes S^a \otimes S^{-a} \otimes X]} \arrow[d, "(E\otimes\phi_{a,-a}^{-1}\otimes X)_*"] & \\
& {[S^{*-a}, E \otimes X]} \arrow[r, equal] & E_{*-a}(E \otimes X)
\end{tikzcd}$$

We know the first vertical arrow is an isomorphism of abelian groups as $- \otimes -$ is additive in each variable (since $\mathcal{SH}$ is tensor triangulated) and $\Omega^a \cong - \otimes S^{-a}$ is an autoequivalence of $\mathcal{SH}$ by Proposition 0.2. The three other vertical arrows are given by composing with an isomorphism in an additive category, so they are also isomorphisms.

To see t_X^a is a homomorphism of left $\pi_*(E)$-modules, suppose we are given classes $r : S^b \to E$ $\pi_b(E)$ and $x : S^c \to E \otimes S^a \otimes X$ in $E_c(\Sigma^a X)$. Then we wish to show that $t_X^a(r \cdot x) = r \cdot t_X^a(x)$.

Consider the following diagram:

$$\begin{tikzcd}
S^{b+c-a} \arrow[d, "\cong"] & E \otimes S^a \otimes X \otimes S^{-a} \arrow[r, "E\otimes S^a \otimes \tau_{X,S^{-a}}"] & E \otimes S^a \otimes S^{-a} \otimes X \arrow[d, "E\otimes \phi_{a,-a}^{-1}\otimes X"] \\
S^b \otimes S^c \otimes S^{-a} \arrow[r, "r\otimes x\otimes S^{-a}"] & E \otimes E \otimes S^a \otimes X \otimes S^{-a} \arrow[u, "\mu\otimes S^a\otimes X\otimes S^{-a}"'] \arrow[d, "E\otimes E\otimes S^a\otimes \tau_{X,S^{-a}}"'] & E \otimes X \\
 & E \otimes E \otimes S^a \otimes S^{-a} \otimes X \arrow[uur, "\mu\otimes S^a\otimes S^{-a}\otimes X"'] \arrow[r, "E\otimes E\otimes \phi_{a,-a}^{-1}\otimes X"'] & E \otimes E \otimes X \arrow[u, "\mu\otimes X"']
\end{tikzcd}$$

Both triangles commute by functoriality of $-\otimes -$. The top composition is $t_X^a(r\cdot x)$ while the bottom is $r\cdot t_X^a(x)$, so they are equal as desired.

It remains to show t_X^a is natural in X. let $f : X \to Y$ in $\mathcal{SH}$, then we would like to show the following diagram commutes:

$$\begin{tikzcd}
E_*(\Sigma^a X) \arrow[r, "t_X^a"] \arrow[d, "E_*(\Sigma^a f)"'] & E_{*-a}(X) \arrow[d, "E_{*-a}(f)"] \\
E_*(\Sigma^a Y) \arrow[r, "t_Y^a"] & E_{*-a}(Y)
\end{tikzcd} \tag{3}$$

We may chase a generator around the diagram since all the arrows here are homomorphisms. Let $x : S^b \to E\otimes S^a \otimes X$ in $E_*(\Sigma^a X)$. Then consider the following diagram:

$$\begin{tikzcd}
S^{b-a} \arrow[r, "\cong"] & S^b \otimes S^{-a} \arrow[r, "x\otimes S^{-a}"] & E\otimes S^a \otimes X \otimes S^{-a} \arrow[r, "E\otimes S^a\otimes\tau"] \arrow[d, "E\otimes S^a\otimes f\otimes S^{-a}"'] & E\otimes S^a\otimes S^{-a}\otimes X \arrow[r, "E\otimes\phi_{a,-a}^{-1}\otimes X"] \arrow[d, "E\otimes S^a\otimes S^{-a}\otimes f"'] & E\otimes X \arrow[d, "E\otimes f"] \\
 & & E\otimes S^a\otimes Y\otimes S^{-a} \arrow[r, "E\otimes S^a\otimes\tau"'] & E\otimes S^a\otimes S^{-a}\otimes Y \arrow[r, "E\otimes\phi_{a,-a}^{-1}\otimes Y"'] & E\otimes Y
\end{tikzcd}$$

The left rectangle commutes by naturality of τ, while the right rectangle commutes by functoriality of $-\otimes -$. The two outside compositions are the two ways to chase x around diagram (3), so the diagram commutes as desired. $\square$

Lemma 0.11. *Given a monoid object (E, μ, e) in $\mathcal{SH}$, the maps Φ_X constructed in Proposition 0.9 commute with the natural isomorphisms $t_X^a : E_*(\Sigma^a X) \xrightarrow{\cong} E_{*-a}(X)$ given in Lemma 0.10, in the sense that the following diagram commutes for all $a \in A$ and X in $\mathcal{SH}$:*

$$\begin{tikzcd}
E_*(E)\otimes_{\pi_*(E)} E_*(\Sigma^a X) \arrow[r, "E_*(E)\otimes t_X^a"] \arrow[d, "\Phi_{\Sigma^a X}"'] & E_*(E)\otimes_{\pi_*(E)} E_{*-a}(X) \arrow[d, "\Phi_X"] \\
E_*(E\otimes \Sigma^a X) \arrow[r, "t_X^a"] & E_{*-a}(E\otimes X)
\end{tikzcd}$$

where the top arrow is well-defined since t_X^a is a left $\pi_*(E)$-modue homomorphism by the above lemma, and we are being abusive in that the bottom arrow is given by the composition

$$E_*(E\otimes\Sigma^a X) \overset{\alpha}{\cong} (E\otimes E)_*(\Sigma^a X) \xrightarrow{t_X^a} (E\otimes E)_{*-a}(X) \overset{\alpha}{\cong} E_{*-a}(E\otimes X).$$

Proof. Since all the maps in the above diagram are homomorphisms, we can chase generators around to show it commutes. Let $x : S^b \to E\otimes E$ and $y : S^c \to E\otimes \Sigma^a X = E\otimes S^a\otimes X$. Then

consider the following diagram:

$$\begin{tikzcd}
S^{b+c-a} \arrow[d, "\cong"] & E \otimes E \otimes E \otimes S^a \otimes S^{-a} \otimes X \arrow[r, "E\otimes E\otimes E\otimes \phi^{-1}_{a,-a}\otimes X"] \arrow[drr, "E\otimes\mu\otimes S^a\otimes S^{-a}\otimes X"] & E \otimes E \otimes E \otimes X \arrow[d, "E\otimes\mu\otimes X"] \\
S^b \otimes S^c \otimes S^{-a} \arrow[r, "x\otimes y\otimes S^{-a}"] & E \otimes E \otimes E \otimes S^a \otimes X \otimes S^{-a} \arrow[u, "E\otimes E\otimes E\otimes S^a\otimes\tau"] \arrow[d, "E\otimes\mu\otimes S^a\otimes X\otimes S^{-a}"] & E \otimes E \otimes X \\
& E \otimes E \otimes S^a \otimes X \otimes S^{-a} \arrow[r, "E\otimes E\otimes S^a\otimes\tau"'] & E \otimes E \otimes S^a \otimes S^{-a} \otimes X \arrow[u, "E\otimes E\otimes\phi^{-1}_{a,-a}\otimes X"']
\end{tikzcd}$$

Each triangle commutes by functoriality of $- \otimes -$. The two outside compositions are the two ways to chase $x \otimes y$ around the diagram in the statement of the lemma, so the diagram commutes as desired. ☐

Corollary 0.12. *For all X in $\mathcal{SH}$, we have natural isomorphisms $t_X : E_*E(\Sigma X) \xrightarrow{\cong} E_{*-\mathbf{1}}(X)$ given by the composition*

$$E_*(\Sigma X) \xrightarrow{E_*(\nu_X)} E_*(\Sigma^1 X) \xrightarrow{t^{\mathbf{1}}_X} E_{*-\mathbf{1}}(X).$$

Furthermore, by naturality of Φ and the fact that $t^{\mathbf{1}}_X$ commutes with Φ (in the sense of the above lemma), this isomorphism also commutes with Φ.

Proposition 0.13. *Let (E, μ, e) be a flat monoid object in $\mathcal{SH}$ (**??**) and let X be any cellular object in $\mathcal{SH}$ (**??**). Then the natural homomorphism*

$$\Phi_X : E_*(E) \otimes_{\pi_*(E)} E_*(X) \to E_*(E \otimes X)$$

given in Proposition 0.9 is an isomorphism of left $\pi_*(E)$-modules.

Proof. In this proof, we will freely employ the coherence theorem for symmetric monoidal categories, and we will assume that associativity and unitality of $-\otimes-$ holds up to strict equality. To start, let $\mathcal{E}$ be the collection of objects X in $\mathcal{SH}$ for which this map is an isomorphism. Then in order to show $\mathcal{E}$ contains every cellular object, it suffices to show that $\mathcal{E}$ satisfies the three conditions given for the class of cellular objects in **??**. First, we need to show that Φ is an isomorphism when $X = S^a$ for some $a \in A$. Indeed, consider the map

$$\Psi : E_*(E \otimes S^a) \to E_*(E) \otimes_{\pi_*(E)} E_*(S^a)$$

which sends a class $x : S^b \to E \otimes E \otimes S^a$ in $E_b(E \otimes S^a)$ to the pure tensor $\widetilde{x} \otimes \widetilde{e}$, where $\widetilde{x} \in E_{b-a}(E)$ is the composition

$$S^{b-a} \cong S^b \otimes S^{-a} \xrightarrow{x\otimes S^{-a}} E \otimes E \otimes S^a \otimes S^{-a} \xrightarrow{E\otimes E\otimes \phi^{-1}_{a,-a}} E \otimes E$$

and $\widetilde{e} \in E_a(S^a)$ is the composition

$$S^a \cong S \otimes S^a \xrightarrow{e\otimes S^a} E \otimes S^a.$$

First, note Ψ is an (A-graded) homomorphism of abelian groups: Given $x, x' \in E_b(E \otimes S^a)$, we would like to show that $\widetilde{x} \otimes \widetilde{e} + \widetilde{x}' \otimes \widetilde{e} = \widetilde{x + x'} \otimes \widetilde{e}$. It suffices to show that $\widetilde{x} + \widetilde{x}' = \widetilde{x + x'}$. To see this, consider the following diagram (again, we are passing to a symmetric strict monoidal category):

$$
\begin{tikzcd}
S^{b-a} \arrow[r, "\Delta"] \arrow[d, "\phi_{b-a}"'] & S^{b-a} \oplus S^{b-a} \arrow[d, "\phi_{b,-a} \oplus \phi_{b,-a}"] \\
S^b \otimes S^{-a} \arrow[r, "\Delta"] \arrow[d, "\Delta \otimes S^{-a}"'] & (S^b \otimes S^{-a}) \oplus (S^b \otimes S^{-a}) \arrow[d, "(x \otimes S^{-a}) \oplus (x' \otimes S^{-a})"] \\
(S^b \oplus S^b) \otimes S^{-a} \arrow[ur, "\cong"] \arrow[d, "(x \oplus x') \otimes S^{-a}"'] & (E \otimes E \otimes S^a \otimes S^{-a}) \oplus (E \otimes E \otimes S^a \otimes S^{-a}) \arrow[d, "(E \otimes E \otimes \phi_{a,-a}^{-1}) \oplus (E \otimes E \otimes \phi_{a,-a}^{-1})"] \arrow[ddl, "\nabla"] \\
((E \otimes E \otimes S^a) \oplus (E \otimes E \otimes S^a)) \otimes S^{-a} \arrow[ur, "\cong"] \arrow[d, "\nabla \otimes S^{-a}"'] & (E \otimes E) \oplus (E \otimes E) \arrow[d, "\nabla"] \\
E \otimes E \otimes S^a \otimes S^{-a} \arrow[r, "E \otimes E \otimes \phi_{a,-a}^{-1}"'] & E \otimes E
\end{tikzcd}
$$

The top rectangle commutes by naturality of Δ in an additive category. The bottom triangle commutes by naturality of ∇ in an additive category. Finally, the remaining regions of the diagram commute by additivity of $- \otimes -$. By functoriality of $- \otimes -$, it follows that the left composition is $\widetilde{x + x'}$ and the right composition is $\widetilde{x} + \widetilde{x}'$, so they are equal as desired. Thus Ψ is a homomorphism of abelian groups, as desired.

Now, we claim that Ψ is an inverse to Φ, (which is enough to show Φ is an isomorphism of left $\pi_*(E)$-modules). Since Φ and Ψ are homomorphisms it suffices to check that they are inverses on generators. First, let $x : S^b \to E \otimes E \otimes S^a$ in $E_b(E \otimes S^a)$. We would like to show that $\Phi(\Psi(x)) = x$. Consider the following diagram, where here we are passing to a symmetric strict monoidal category:

$$
\begin{tikzcd}
S^b \arrow[r, "\cong"] \arrow[dd, "x"'] & S^b \otimes S^{-a} \otimes S^a \arrow[d, dashed, "x \otimes S^{-a} \otimes S^a"] \arrow[dr, "x \otimes S^{-a} \otimes e \otimes S^a"] & \\
 & E \otimes E \otimes S^a \otimes S^{-a} \otimes S^a \arrow[r, "E \otimes E \otimes S^a \otimes S^{-a} \otimes e \otimes S^a"'] & E \otimes E \otimes S^a \otimes S^{-a} \otimes E \otimes S^a \arrow[ddl, "E \otimes E \otimes \phi_{a,-a}^{-1} \otimes E \otimes S^a"] \\
E \otimes E \otimes S^a \arrow[ur, "E \otimes E \otimes S^a \otimes \phi_{-a,a}"] \arrow[r, equal] \arrow[dr, "E \otimes E \otimes e \otimes S^a"] & E \otimes E \otimes S^a \arrow[u, dashed, "E \otimes E \otimes \phi_{a,-a} \otimes S^a"] \arrow[d, dashed, "E \otimes E \otimes e \otimes S^a"] & \\
E \otimes E \otimes E \otimes S^a \arrow[u, "E \otimes \mu \otimes S^a"] \arrow[r, equal] & E \otimes E \otimes E \otimes S^a &
\end{tikzcd}
$$

The top left trapezoid commutes since the isomorphism $S^b \xrightarrow{\cong} S^b \otimes S^{-a} \otimes S^a$ may be given as $S^b \otimes \phi_{-a,a}$ (see **??**), in which case the trapezoid commmutes by functoriality of $- \otimes -$. The triangle below that commutes by coherence for the $\phi_{a,b}$'s. The triangle below that commutes by definition. The bottom left triangle commutes by unitality for μ. The top right triangle commutes by functoriality of $- \otimes -$. Finally, the bottom right triangle commutes by functoriality of $- \otimes -$. It follows by unravelling definitions that the two outside compositions are x and $\Phi(\Psi(x))$, so indeed we have $\Phi(\Psi(x)) = x$ since the diagram commutes.

On the other hand, suppose we are given a homogeneous pure tensor $x \otimes y$ in $E_*(E) \otimes_{\pi_*(E)} E_*(S^a)$, so $x : S^b \to E \otimes E$ and $y : S^c \to E \otimes S^a$ for some $b, c \in A$. Then we would like to show that $\Psi(\Phi(x \otimes y)) = x \otimes y$. Unravelling definitions, $\Psi(\Phi(x \otimes y))$ is the homogeneous pure tensor $\widetilde{xy} \otimes \widetilde{e}$, where $\widetilde{e} : S^a \to E \otimes S^a$ is defined above, and by functoriality of $- \otimes -$, $\widetilde{xy} : S^{b+c-a} \to E \otimes E$

is the composition

$$\begin{tikzcd}
S^{b+c-a} \arrow[d, "\phi_{b+c,-a}"] \\
S^{b+c} \otimes S^{-a} \arrow[d, "\phi_{b,c}\otimes S^{-a}"] \\
S^{b} \otimes S^{c} \otimes S^{-a} \arrow[d, "x\otimes y\otimes S^{-a}"] \\
E \otimes E \otimes E \otimes S^{a} \otimes S^{-a} \arrow[d, "E\otimes\mu\otimes S^{a}\otimes S^{-a}"] \\
E \otimes E \otimes S^{a} \otimes S^{-a} \arrow[d, "E\otimes E\otimes\phi^{-1}_{a,-a}"] \\
E \otimes E \otimes S \arrow[d, "E\otimes\rho_E"] \\
E \otimes E.
\end{tikzcd}$$

In order to see $x \otimes y = \widetilde{xy} \otimes \widetilde{e}$, it suffices to show there exists some scalar $r \in \pi_{c-a}(E)$ such that $x \cdot r = \widetilde{xy}$ and $r \cdot \widetilde{e} = y$, where here $\cdot$ denotes the right and left action of $\pi_*(E)$ on $E_*(E)$ and $E_*(S^a)$, respectively. Now, define r to be the composition

$$S^{c-a} \cong S^c \otimes S^{-a} \xrightarrow{y\otimes S^{-a}} E \otimes S^a \otimes S^{-a} \xrightarrow{E\otimes\phi^{-1}_{a,-a}} E \otimes S \xrightarrow{\rho_E} E.$$

First, in order to see that $x \cdot r = \widetilde{xy}$, consider the following diagram, where here we are again passing to a symmetric strict monoidal category:

$$\begin{tikzcd}
S^{b+c-a} \arrow[r, "\cong"] & S^b \otimes S^c \otimes S^{-a} \arrow[r, "x\otimes y\otimes S^{-a}"] & E \otimes E \otimes E \otimes S^a \otimes S^{-a} \arrow[r, "E\otimes\mu\otimes S^a\otimes S^{-a}"] \arrow[d, "E\otimes E\otimes E\otimes\phi^{-1}_{a,-a}"'] \arrow[dr, "E\otimes\mu\otimes\phi^{-1}_{a,-a}"] & E \otimes E \otimes S^a \otimes S^{-a} \arrow[d, "E\otimes E\otimes\phi^{-1}_{a,-a}"] \\
& & E \otimes E \otimes E \arrow[r, "E\otimes\mu"'] & E \otimes E
\end{tikzcd}$$

Commutativity is functoriality of $-\otimes-$, which also tells us that the two outside compositions are $\widetilde{xy}$ (on top) and $x \cdot r$ (on the bottom), so they are equal as desired. On the other hand, in order to see that $r \cdot \widetilde{e} = y$, consider the following diagram (where here we have passed to a symmetric strict monoidal category):

$$\begin{tikzcd}
S^c \arrow[rr, "\cong"] \arrow[dd, "y"'] & & S^c \otimes S^{-a} \otimes S^a \arrow[dl, "y\otimes S^{-a}\otimes S^a"'] \arrow[dd, "y\otimes S^{-a}\otimes e\otimes S^a"] \\
& E \otimes S^a \otimes S^{-a} \otimes S^a \arrow[dl, "E\otimes S^a\otimes\phi^{-1}_{-a,a}"'] \arrow[dr, "E\otimes S^a\otimes S^{-a}\otimes e\otimes S^a"] & \\
E \otimes S^a \arrow[drr, "E\otimes e\otimes S^a"] & & E \otimes S^a \otimes S^{-a} \otimes E \otimes S^a \arrow[d, "E\otimes\phi^{-1}_{a,-a}\otimes E\otimes S^a"] \\
E \otimes E \otimes S^a \arrow[u, "\mu\otimes S^a"] \arrow[rr, equal] & & E \otimes E \otimes S^a
\end{tikzcd}$$

The top left triangle commutes since we may take the isomorphism $S^c \xrightarrow{\cong} S^c \otimes S^{-a} \otimes S^a$ to be $S^c \otimes \phi_{-a,a}$, in which case commutativity of the triangle follows by functoriality of $-\otimes-$. Commutativity of the right triangle is also functoriality of $-\otimes-$. Commutativity of the bottom left triangle is unitality of μ. Finally, commutativity of the remaining middle 4-sided region is again functoriality of $- \otimes -$. It follows that y is equal to the outer composition, which is $r \cdot \widetilde{e}$, as desired. Thus, we have shown that

$$\Psi(\Phi(x \otimes y)) = \widetilde{xy} \otimes \widetilde{e} = (x \cdot r) \otimes \widetilde{e} = x \otimes (r \cdot \widetilde{e}) = x \otimes y,$$

as desired, so that for each $a \in A$, the object S^a belongs to the class $\mathcal{E}$.

Now, we would like to show that given a distinguished triangle

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X,$$

if two of three of the objects X, Y, and Z belong to $\mathcal{E}$, then so does the third. From now on, write $L_*^E : \mathcal{SH} \to \pi_*(E)$-**Mod** to denote the functor $X \mapsto E_*(E) \otimes_{\pi_*(E)} E_*(X)$, so Φ is a natural transformation $L_*^E \Rightarrow E_*(E \otimes -)$. First, note that by **??**, we have the following exact sequence in $\mathcal{SH}$:

$$E \otimes \Omega Y \xrightarrow{E \otimes \Omega g} E \otimes \Omega Z \xrightarrow{E \otimes \widetilde{h}} E \otimes X \xrightarrow{E \otimes f} E \otimes Y \xrightarrow{E \otimes g} E \otimes Z \xrightarrow{E \otimes h} E \otimes \Sigma X \xrightarrow{E \otimes \Sigma f} \Sigma Y.$$

We can then apply $[S^*, -]$ to it, which yields the following exact sequence of A-graded $\pi_*(E)$-modules:

$$E_*(\Omega Y) \xrightarrow{E_*(\Omega g)} E_*(\Omega Z) \xrightarrow{E_*(\widetilde{h})} E_*(X) \xrightarrow{E_*(f)} E_*(Y) \xrightarrow{E_*(g)} E_*(Z) \xrightarrow{E_*(h)} E_*(\Sigma X) \xrightarrow{E_*(f)} E_*(\Sigma Y).$$

Now, we can tensor this sequence with $E_*(E)$ over $\pi_*(E)$, and since $E_*(E)$ is a flat right $\pi_*(E)$ module, we get that the top row in the following sequence is exact:

$$\begin{tikzcd}
L_*^E(\Omega Y) \arrow[r, "L_*^E(\Omega g)"] \arrow[d, "\Phi_{\Omega Y}"'] & L_*^E(\Omega Z) \arrow[r, "L_*^E(\widetilde{h})"] \arrow[d, "\Phi_{\Omega Z}"'] & L_*^E(X) \arrow[r, "L_*^E(f)"] \arrow[d, "\Phi_X"'] & L_*^E(Y) \arrow[r, "L_*^E(g)"] \arrow[d, "\Phi_Y"'] & L_*^E(Z) \arrow[r, "L_*^E(h)"] \arrow[d, "\Phi_Z"'] & L_*^E(\Sigma X) \arrow[r, "L_*^E(\Sigma f)"] \arrow[d, "\Phi_{\Sigma X}"'] & L_*^E(\Sigma Y) \arrow[d, "\Phi_{\Sigma Y}"'] \\
E_*(E \otimes \Omega Y) \arrow[r, "E_*(E \otimes \Omega g)"'] & E_*(E \otimes \Omega Z) \arrow[r, "E_*(E \otimes \widetilde{h})"'] & E_*(E \otimes X) \arrow[r, "E_*(E \otimes f)"'] & E_*(E \otimes Y) \arrow[r, "E_*(E \otimes g)"'] & E_*(E \otimes Z) \arrow[r, "E_*(E \otimes h)"'] & E_*(E \otimes \Sigma X) \arrow[r, "E_*(E \otimes \Sigma f)"'] & E_*(E \otimes \Sigma Y)
\end{tikzcd}$$

The diagram commutes since Φ is natural. The following sequence is exact in $\mathcal{SH}$ by **??**,

$$E \otimes E \otimes \Omega Y \to E \otimes E \otimes \Omega Z \to E \otimes E \otimes X \to E \otimes E \otimes Y \to E \otimes E \otimes Z \to E \otimes E \otimes \Sigma X \to E \otimes E \otimes \Sigma Y,$$

so that the bottom row in the above diagram is also exact. Now, suppose two of three of X, Y, and Z belong to $\mathcal{E}$. By Lemma 0.11, Corollary 0.12, if Φ_W is an isomorphism then $\Phi_{\Omega W}$ and $\Phi_{\Sigma W}$ are. Thus by the five lemma, it follows that the middle three vertical arrows in the above diagram are necessarily all isomorphisms, so we have shown that $\mathcal{E}$ is closed under two-of-three for exact triangles, as desired.

Finally, it remains to show that $\mathcal{E}$ is closed under arbitrary coproducts. Let $\{X_i\}_{i \in I}$ be a collection of objects in $\mathcal{E}$ indexed by some (small) set I. Then we'd like to show that $X := \bigoplus_i X_i$ belongs to $\mathcal{E}$. First of all, note that $E \otimes -$ preserves arbitrary coproducts, as it has a right adjoint $F(E, -)$. Thus without loss of generality we may take $\bigoplus_i E \otimes X_i = E \otimes \bigoplus_i X_i$ (as $E \otimes \bigoplus_i X_i$ *is* a coproduct of all the $E \otimes X_i$'s). Now, recall that we have chosen each S^a to be a compact object (**??**), so that the canonical map

$$s : \bigoplus_i E_*(X_i) = \bigoplus_i [S^*, E \otimes X_i] \to [S^*, \bigoplus_i E \otimes X_i] = [S^*, E \otimes X] = E_*(X)$$

is an isomorphism, natural in X_i for each i. Note in particular that it is an isomorphism of left $\pi_*(E)$-modules. To see this, first note it suffices to check that $s(r \cdot x) = r \cdot s(x)$ for some homogeneous $x \in E_*(X_i)$ for some i, as such x generate $\bigoplus_i E_*(X_i)$ by definition, and s is a homomorphism of abelian groups. Then given $r : S^a \to E \otimes E$ and $x : S^b \to E \otimes X_i$, consider the following diagram

$$\begin{tikzcd}
S^{a+b} \arrow[r, "\phi_{a,b}"] & S^a \otimes S^b \arrow[r, "x \otimes y"] & E \otimes E \otimes E \otimes X_i \arrow[r, "E\otimes E\otimes\iota_{E\otimes X_i}"] \arrow[dr, "E\otimes E\otimes E\otimes\iota_{X_i}"] \arrow[ddd, "E\otimes\mu\otimes X_i"'] & E \otimes E \otimes \bigoplus_i (E \otimes X_i) \arrow[d, equal] \\
 & & & E \otimes E \otimes E \otimes X \arrow[d, "E\otimes\mu\otimes X"] \\
 & & & E \otimes E \otimes X \arrow[d, equal] \\
 & & E \otimes E \otimes X_i \arrow[r, "E\otimes\iota_{E\otimes X_i}"'] \arrow[ur, "E\otimes E\otimes\iota_{X_i}"] & E \otimes \bigoplus_i (E \otimes X_i)
\end{tikzcd}$$

where $\iota_{E\otimes X_i} : E \otimes X_i \hookrightarrow \bigoplus_i (E \otimes X_i)$ and $\iota_{X_i} : X_i \hookrightarrow \bigoplus_i X_i$ are the maps determined by universal property of the coproduct. Commutativity of the two triangles is again by the fact that $E \otimes -$ is colimit preserving. Commutativity of the trapezoid is functoriality of $- \otimes -$. Thus, the top arrow in the following diagram is well-defined:

$$\begin{tikzcd}
\bigoplus_i E_*(E) \otimes_{\pi_*(E)} E_*(X_i) \arrow[r, equal] \arrow[d, "\bigoplus_i \Phi_{X_i}"'] & E_*(E) \otimes_{\pi_*(E)} \bigoplus_i E_*(X_i) \arrow[r, "E_*(E)\otimes_{\pi_*(E)} s"] & E_*(E) \otimes_{\pi_*(E)} E_*(X) \arrow[d, "\Phi_X"] \\
\bigoplus_i E_*(E \otimes X_i) \arrow[r, "s"] & E_*(\bigoplus_i E \otimes X_i) \arrow[r, equal] & E_*(E \otimes X)
\end{tikzcd} \tag{4}$$

We wish to show this diagram commutes. Again, since each map here is a homomorphism, it suffices to chase generators. By definition, a generator of the top left element is a homogeneous pure tensor in $E_*(E) \otimes_{\pi_*(E)} E_*(X_i)$ for some i in I. Given classes $x : S^a \to E \otimes E$ and $y : S^b \to E \otimes X_i$, consider the following diagram:

$$\begin{tikzcd}
S^{a+b} \arrow[r, "\phi_{a,b}"] & S^a \otimes S^b \arrow[r, "x \otimes y"] & E \otimes E \otimes E \otimes X_i \arrow[r, "E\otimes E\otimes\iota_{E\otimes X_i}"] \arrow[d, "E\otimes\mu\otimes X_i"'] \arrow[dr, "E\otimes E\otimes E\otimes\iota_{X_i}"] & E \otimes E \otimes \bigoplus_i E \otimes X_i \arrow[d, equal] \\
 & & E \otimes E \otimes X_i \arrow[d, "\iota_{E\otimes E\otimes X_i}"'] \arrow[dr, "E\otimes E\otimes\iota_{X_i}"] & E \otimes E \otimes E \otimes X \arrow[d, "E\otimes\mu\otimes X"] \\
 & & \bigoplus_i E \otimes E \otimes X_i \arrow[r, equal] & E \otimes E \otimes X
\end{tikzcd}$$

Unravelling definitions, the two outside compositions are the two ways to chase $x \otimes y$ around diagram (4). The two triangles commute again by the fact that $- \otimes -$ preserves colimits in each argument. Commutativity of the inner parallelogram is functoriality of $- \otimes -$. Thus diagram (4) tells us Φ_X is an isomorphism, since Φ_{X_i} is an isomorphism for each i in I, meaning $\bigoplus_i \Phi_{X_i}$ is as well. $\square$

Proposition 0.14. *Let (E, μ, e) be a ring spectrum in $\mathcal{SH}$, and let X and Y be two objects in $\mathcal{SH}$. Consider the map*

$$[X, E \otimes Y]_* \to \mathrm{Hom}^*_{\pi_*(E)}(E_*(X), E_*(Y))$$

which sends a class $f : S^a \otimes X \to E \otimes Y$ in $[X, E \otimes Y]_*$ to

Suppose there exists set I, a collection $\{a_i\}_{i\in I} \subseteq A$, and maps r and i which fit into a retract diagram:

$$\begin{tikzcd}
E \otimes X \arrow[r, "\iota"] \arrow[dr, equal] & \bigoplus_{i\in I} \Sigma^{a_i} E \arrow[d, "r"] \\
& E \otimes X
\end{tikzcd}$$

Then