0.1. **Setup of $\mathcal{SH}$.** In order to construct an abstract version of the Adams spectral sequence, we need to work in some axiomatic version of a stable homotopy category $\mathcal{SH}$ which acts like the familiar classical stable homotopy category **hoSp** (**??**) or the motivic stable homotopy category $\mathbf{SH}_{\mathscr{S}}$ over some base scheme $\mathscr{S}$ (**??**).

Definition 0.1. Let $\mathcal{C}$ be an additive category with arbitrary (small) coproducts. Then an object X in $\mathcal{C}$ is *compact* if, for any collection of objects Y_i in $\mathcal{C}$ indexed by some (small) set I, the canonical map

$$\bigoplus_i \mathcal{C}(X, Y_i) \to \mathcal{C}(X, \bigoplus_i Y_i)$$

is an isomorphism of abelian groups. (Explicitly, the above map takes a generator $x \in \mathcal{C}(X, Y_i)$ to the composition $X \xrightarrow{x} Y_i \hookrightarrow \bigoplus_i Y_i$.)

Definition 0.2. Given a tensor triangulated category $(\mathcal{C}, \otimes, S, \Sigma, e, \mathcal{D})$ (**??**), a *sub-Picard grading* on $\mathcal{C}$ is the following data:

- A pointed abelian group $(A, \mathbf{1})$ along with a homomorphism of pointed groups $h : (A, \mathbf{1}) \to (\operatorname{Pic} \mathcal{C}, \Sigma S)$, where $\operatorname{Pic} \mathcal{C}$ is the *Picard group* of isomorphism classes of invertible objects in $\mathcal{C}$.[1]
- For each $a \in A$, a chosen representative S^a in the isomorphism class $h(a)$ such that each S^a is a compact object (Definition 0.1) and $S^0 = S$.
- For each $a, b \in A$, an isomorphism $\phi_{a,b} : S^{a+b} \to S^a \otimes S^b$. This family of isomorphisms is required to be *coherent*, in the following sense:
  - For all $a \in A$, we must have that $\phi_{a,0}$ coincides with the right unitor $S^a \xrightarrow{\cong} S^a \otimes S$ and $\phi_{0,a}$ coincides the left unitor $S^a \xrightarrow{\cong} S \otimes S^a$.
  - For all $a, b, c \in A$, the following "associativity diagram" must commute:

$$\begin{tikzcd}
S^{a+b} \otimes S^c \arrow[d, "\phi_{a,b} \otimes S^c"'] & S^{a+b+c} \arrow[l, "\phi_{a+b,c}"'] \arrow[r, "\phi_{a,b+c}"] & S^a \otimes S^{b+c} \arrow[d, "S^a \otimes \phi_{b,c}"] \\
(S^a \otimes S^b) \otimes S^c \arrow[rr, "\cong"] & & S^a \otimes (S^b \otimes S^c)
\end{tikzcd}$$

Remark 0.3. Note that by induction the coherence conditions for the $\phi_{a,b}$'s in the above definition say that given any $a_1, \ldots, a_n \in A$ and $b_1, \ldots, b_m \in A$ such that $a_1 + \cdots + a_n = b_1 + \cdots + b_m$ and any fixed parenthesizations of $X = S^{a_1} \otimes \cdots \otimes S^{a_b}$ and $Y = S^{b_1} \otimes \cdots \otimes S^{b_m}$, there is a *unique* isomorphism $X \to Y$ that can be obtained by forming formal compositions of products of $\phi_{a,b}$, identities, associators, unitors, and their inverses (but not symmetries).

From now on we fix a monoidal closed tensor triangulated category $(\mathcal{SH}, \otimes, S, \Sigma, e, \mathcal{D})$ with arbitrary (small) (co)products and sub-Picard grading $(A, \mathbf{1}, h, \{S^a\}, \{\phi_{a,b}\})$. We also fix an isomorphism $\nu : \Sigma S \xrightarrow{\cong} S^{\mathbf{1}}$ once and for all. We establish some conventions. First of all, given an object X and a natural number $n > 0$, we write

$$X^n := \overbrace{X \otimes \cdots \otimes X}^{n \text{ times}} \qquad \text{and} \qquad X^0 := S.$$

[1] Recall an object X is a symmetric monoidal category is *invertible* if there exists some object Y and an isomorphism $S \cong X \otimes Y$.

We denote the associator, symmetry, left unitor, and right unitor isomorphisms in $\mathcal{SH}$ by

$$\alpha_{X,Y,Z} : (X \otimes Y) \otimes Z \xrightarrow{\cong} X \otimes (Y \otimes Z) \qquad \tau_{X,Y} : X \otimes Y \xrightarrow{\cong} Y \otimes X$$
$$\lambda_X : S \otimes X \xrightarrow{\cong} X \qquad \rho_X : X \otimes S \xrightarrow{\cong} X.$$

Often we will drop the subscripts. Furthermore, by the coherence theorem for symmetric monoidal categories, we will often assume α, ρ, and λ are actual equalities. Given some integer $n \in \mathbb{Z}$, we will write a bold $\mathbf{n}$ to denote the element $n \cdot \mathbf{1}$ in A. Note that we can use the isomorphism $\nu : \Sigma S \xrightarrow{\cong} S^{\mathbf{1}}$ to construct a natural isomorphism $\Sigma \cong S^{\mathbf{1}} \otimes -$:

$$\Sigma X \xrightarrow{\Sigma\lambda_X^{-1}} \Sigma(S \otimes X) \xrightarrow{e_{S,X}^{-1}} \Sigma S \otimes X \xrightarrow{\nu \otimes X} S^{\mathbf{1}} \otimes X.$$

The first two arrows are natural in X by definition. The last arrow is natural in X by functoriality of $- \otimes -$. By abuse of notation, we will also use ν to denote this natural isomorphism. Furthermore, under this isomorphism, $e_{X,Y} : \Sigma X \otimes Y \xrightarrow{\cong} \Sigma(X \otimes Y)$ corresponds to the associator, by commutativity of the following diagram:

is this characterization of ν and e needed?

$$\begin{tikzcd}
\Sigma X \otimes Y \arrow[r, "\Sigma\lambda_X^{-1} \otimes Y"] \arrow[dd, "e_{X,Y}"'] & \Sigma(S \otimes X) \otimes Y \arrow[r, "e_{S,X}^{-1} \otimes Y"] \arrow[d, "e_{S \otimes X, Y}"] & (\Sigma S \otimes X) \otimes Y \arrow[dd, "\alpha"] & (S^{\mathbf{1}} \otimes X) \otimes Y \arrow[l, "(\nu \otimes X) \otimes Y"'] \arrow[dd, "\alpha"] \\
 & \Sigma((S \otimes X) \otimes Y) \arrow[dl, "\Sigma(\lambda_X \otimes Y)"'] \arrow[d, "\Sigma\alpha"] & & \\
\Sigma(X \otimes Y) \arrow[r, "\Sigma\lambda_{X \otimes Y}^{-1}"'] & \Sigma(S \otimes (X \otimes Y)) \arrow[r, "e_{S,X \otimes Y}^{-1}"'] & \Sigma S \otimes (X \otimes Y) \arrow[r, "\nu \otimes (X \otimes Y)"'] & S^{\mathbf{1}} \otimes (X \otimes Y)
\end{tikzcd}$$

Commutativity of the left trapezoid is naturality of e. The bottom left triangle commutes by coherence for monoidal categories and functoriality of Σ. Commutativity of the middle square is axiom TT4 for a tensor triangulated category. Finally, the right square commutes by naturality of α.

Given some $a \in A$, we define $\Sigma^a := S^a \otimes -$ and $\Omega^a := \Sigma^{-a} = S^{-a} \otimes -$. We specifically define $\Omega := \Omega^{\mathbf{1}}$. We say "the a^{th} suspension of X" to denote $\Sigma^a X$. We will see later that for each $a \in A$, Σ^a and Ω^a form an adjoint equivalence of $\mathcal{SH}$ (Proposition 0.5), so that in particular since Ω forms an adjoint equivalence with $\Sigma^{\mathbf{1}} \cong \Sigma$, $\mathcal{SH}$ is canonically an *adjointly* triangulated category (**??**).

Given two objects X and Y in $\mathcal{SH}$, we will denote the hom-abelian group of morphisms from X to Y in $\mathcal{SH}$ by $[X, Y]$, and the internal hom object by $F(X, Y)$. We can extend the abelian group $[X, Y]$ into an A-graded abelian group $[X, Y]_*$ by defining $[X, Y]_a := [S^a \otimes X, Y]$. Given an object X in $\mathcal{SH}$ and some $a \in A$, we can define the abelian group

$$\pi_a(X) := [S^a, X],$$

which we call the a^{th} *stable homotopy group of* X. We write $\pi_*(X)$ for the A-graded abelian group $\bigoplus_{a \in A} \pi_a(X)$, so that in particular we have a canonical isomorphism

$$\pi_*(X) = [S^*, X] \cong [S, X]_*.$$

Given some other object E, we can define the A-graded abelian groups $E_*(X)$ and $E^*(X)$ by the formulas

$$E_a(X) := \pi_a(E \otimes X) = [S^a, E \otimes X] \qquad \text{and} \qquad E^a(X) := [X, S^a \otimes E].$$

We refer to the functor $E_*(-)$ as the *homology theory represented by* E, or just E-homology, and we refer to $E^*(-)$ as the *cohomology theory represented by* E, or just E-cohomology. Finally, we state the following definition in $\mathcal{SH}$:

Definition 0.4. Define the class of *cellular* objects in $\mathcal{SH}$ to be the smallest class of objects such that:

(1) For all $a \in A$, S^a is cellular.

(2) If we have a distinguished triangle
$$X \to Y \to Z \to \Sigma X$$
such that two of the three objects X, Y, and Z are cellular, than the third object is also cellular.

(3) Given a collection of cellular objects X_i indexed by some (small) set I, the object $\bigoplus_{i \in I} X_i$ is cellular (recall we have chosen $\mathcal{SH}$ to have arbitrary coproducts).

0.2. **Miscellaneous facts about $\mathcal{SH}$.**

what do I call this subsection?

Proposition 0.5. *For each $a \in A$, the functors Σ^a and Ω^a canonically form an adjoint equivalence of $\mathcal{SH}$. In particular, Ω and $\Sigma \cong \Sigma^{\mathbf{1}}$ form an adjoint equivalence, so that $\mathcal{SH}$ is an* adjointly *tensor triangulated category (**??**).*

Proof. See **??**. □

Lemma 0.6. *Let X and Y be two isomorphic objects in $\mathcal{SH}$. Then X is cellular iff Y is cellular.*

Proof. Assume we have an isomorphism $f : X \xrightarrow{\cong} Y$ and that X is cellular. Then consider the following commutative diagram

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, equal] & Y \arrow[r] \arrow[d, "f^{-1}"] & 0 \arrow[r] \arrow[d, equal] & \Sigma X \arrow[d, equal] \\
X \arrow[r, equal] & X \arrow[r] & 0 \arrow[r] & \Sigma X
\end{tikzcd}$$

The bottom row is distinguished by axiom TR1 for a triangulated category. Hence since X is cellular, 0 is also cellular, since the class of cellular objects satisfies two-of-three for distinguished triangles. Furthermore, since the vertical arrows are all isomorphisms, the top row is distinguished as well, by axiom TR0. Thus again by two-of-three, since X and 0 are cellular, so is Y, as desired. □

Lemma 0.7. *Let X and Y be cellular objects in $\mathcal{SH}$. Then $X \otimes Y$ is cellular.*

Proof. Let E be a cellular object in $\mathcal{SH}$, and let $\mathcal{E}$ be the collection of objects X in $\mathcal{SH}$ such that $E \otimes X$ is cellular. First of all, suppose we have a distinguished triangle
$$X \to Y \to Z \to \Sigma X$$
such that two of three of X, Y, and Z belong to $\mathcal{E}$. Then since $\mathcal{SH}$ is tensor triangulated, we have a distinguished triangle
$$E \otimes X \to E \otimes Y \to E \otimes Z \to \Sigma(E \otimes X).$$
Per our assumptions, two of three of $E \otimes X$, $E \otimes Y$, and $E \otimes Z$ are cellular, so that the third is by definition. Thus, all three of X, Y, and Z belong to $\mathcal{E}$ if two of them do.

Second of all, suppose we have a family X_i of objects in $\mathcal{E}$ indexed by some (small) set I, and set $X := \bigoplus_i X_i$. Then we'd like to show X belongs to $\mathcal{E}$, i.e., that $E \otimes X$ is cellular. Indeed,

$$E \otimes X = E \otimes \left(\bigoplus_i X_i \right) \cong \bigoplus_i (E \otimes X_i),$$

where the isomorphism is given by the fact that $\mathcal{SH}$ is monoidal closed, so $E \otimes -$ preserves arbitrary colimits as it is a left adjoint. Per our assumption, since each $E \otimes X_i$ is cellular, the rightmost object is cellular, since the class of cellular objects is closed under taking arbitrary coproducts, by definition. Hence $E \otimes X$ is cellular by Lemma 0.6.

Finally, we would like to show that each S^a belongs to $\mathcal{E}$, i.e., that $S^a \otimes E$ is cellular for all $a \in A$. When $E = S^b$ for some $b \in A$, this is clearly true, since $S^b \otimes S^a \cong S^{a+b}$, which is cellular by definition, so that $S^b \otimes S^a$ is cellular by Lemma 0.6. Thus by what we have shown, the class of objects X for which $S^a \otimes X$ is cellular contains every cellular object. Hence in particular $E \otimes S^a \cong S^a \otimes E$ is cellular for all $a \in A$, as desired. $\square$

Lemma 0.8. *Let W be a cellular object in $\mathcal{SH}$ such that $\pi_*(W) = 0$. Then $W \cong 0$.*

Proof. Let $\mathcal{E}$ be the collection of all X in $\mathcal{SH}$ such that $[\Sigma^n X, W] = 0$ for all $n \in \mathbb{Z}$ (where for $n > 0$, $\Sigma^{-n} := \Omega^n = (S^{-\mathbf{1}} \otimes -)^n$). We claim $\mathcal{E}$ contains every cellular object in $\mathcal{SH}$. First of all, each S^a belongs to $\mathcal{E}$, as

$$[\Sigma^n S^a, W] \cong [(\Sigma^{\mathbf{1}})^n S^a, W] \cong [S^{a+\mathbf{n}}, W] \leq \pi_*(W) = 0.$$

Furthermore, suppose we are given a distinguished triangle

$$X \to Y \to Z \to \Sigma X$$

such that two of three of X, Y, and Z belong to $\mathcal{E}$. By **??**, for all $n \in \mathbb{Z}$ we get an exact sequence

$$[\Sigma^{n+1} X, W] \to [\Sigma^n Z, W] \to [\Sigma^n Y, W] \to [\Sigma^n X, W] \to [\Sigma^{n-1} Z, W].$$

Clearly if any two of three of X, Y, and Z belong to $\mathcal{E}$, then by exactness of the above sequence all three of the middle terms will be zero, so that the third object will belong to $\mathcal{E}$ as well. Finally, suppose we have a collection of objects X_i in $\mathcal{E}$ indexed by some small set I. Then

$$\left[\Sigma^n \bigoplus_i X_i, W \right] \cong \left[\bigoplus_i \Sigma^n X_i, W \right] \cong \prod_i [\Sigma^n X_i, W] = \prod_i 0 = 0,$$

where the first isomorphism follows by the fact that Σ^n is apart of an adjoint equivalence (Proposition 0.5), so it preserves arbitrary colimits.

Thus, by definition of cellularity, $\mathcal{E}$ contains every cellular object. In particular, $\mathcal{E}$ contains W, so that $[W, W] = 0$, meaning in particular that $\mathrm{id}_W = 0$, so we have a commutative diagram

$$\begin{tikzcd} & 0 \arrow[rr, equal] \arrow[dr] & & 0 \\ W \arrow[ur] \arrow[rr, equal] & & W \arrow[ur] & \end{tikzcd}$$

Hence the diagonals exhibit isomorphisms between 0 and W, as desired. $\square$

Theorem 0.9. *Let X and Y be cellular objects in $\mathcal{SH}$, and suppose $f : X \to Y$ is a morphism such that $f_* : \pi_*(X) \to \pi_*(Y)$ is an isomorphism. Then f is an isomorphism.*

Proof. By axiom TR2 for a triangulated category (**??**), we have a distinguished triangle

$$X \xrightarrow{f} Y \xrightarrow{g} C_f \xrightarrow{h} \Sigma X.$$

First of all, note that by definition since X and Y are cellular, so is C_f. Now, we claim $\pi_*(C_f) = 0$. Indeed, given $a \in A$, by **??** we have the following exact sequence:

$$[S^a, X] \xrightarrow{f_*} [S^a, Y] \xrightarrow{g_*} [S^a, C_f] \xrightarrow{h_*} [S^a, \Sigma X] \xrightarrow{-(\Sigma f)_*} [S^a, \Sigma Y],$$

where the first arrow is an isomorphism, per our assumption that f_* is an isomorphism. To see the last arrow is an isomorphism, consider the following diagram:

$$\begin{tikzcd}
[S^a, \Sigma X] \arrow[r, "(\Sigma f)_*"] \arrow[d, "(\nu_X)_*"'] & [S^a, \Sigma Y] \arrow[d, "(\nu_Y)_*"] \\
[S^a, S^{\mathbf{1}} \otimes X] \arrow[r, "(S^{\mathbf{1}} \otimes f)_*"] \arrow[d, "\cong"'] & [S^a, S^{\mathbf{1}} \otimes Y] \arrow[d, "\cong"] \\
[S^{-\mathbf{1}} \otimes S^a, X] \arrow[r, "f_*"] \arrow[d, "(\phi_{-\mathbf{1},a})*"'] & [S^{-\mathbf{1}} \otimes S^a, Y] \arrow[d, "(\phi_{-\mathbf{1},a})*"] \\
[S^{a-\mathbf{1}}, X] \arrow[r, "f_*"] & [S^{a-\mathbf{1}}, Y]
\end{tikzcd}$$

where the middle vertical arrows are the adjunction natural isomorphisms specified by Proposition 0.5. The bottom arrow is an isomorphism per our assumptions, so the top arrow is likewise an isomorphism, as desired. Thus $\operatorname{im} h_* = \ker -(\Sigma f)_* = 0$, and $\ker g_* = \operatorname{im} f_* = [S^a, Y]$, so that $\ker h_* = \operatorname{im} g_* = 0$. It is only possible that $\ker h_* = \operatorname{im} h_* = 0$ if $[S^a, C_f] = 0$. Thus, we have shown $\pi_*(C_f) = 0$, and C_f is cellular, so by Lemma 0.8 there is an isomorphism $C_f \cong 0$. Now consider the following commuting diagram:

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "f"] & Y \arrow[r] \arrow[d, equal] & C_f \arrow[r] \arrow[d, "\cong"] & \Sigma X \arrow[d, "\Sigma f"] \\
Y \arrow[r, equal] & Y \arrow[r] & 0 \arrow[r] & \Sigma Y
\end{tikzcd}$$

The top row is distinguished by assumption. The bottom row is distinguished by axiom TR2. Then since the middle two vertical arrows are isomorphisms, by **??**, f is an isomorphism as well, as desired. ☐

Proposition 0.10. *Let $e : X \to X$ be an idempotent between cellular objects in $\mathcal{SH}$, so by* **??** *there is a diagram*

$$X \xrightarrow{r} Y \xrightarrow{\iota} X$$

with $r \circ \iota = \mathrm{id}_Y$. Then Y is cellular.

Proof. It is a general categorical fact that an idempotent splits up to unique isomorphism, so by Lemma 0.6, it suffices to show that e has some cellular splitting. In **??**, it is shown that we may take Y to be the homotopy colimit of the sequence

$$X \xrightarrow{e} X \xrightarrow{e} X \xrightarrow{e} X \xrightarrow{e} \cdots,$$

so there is a distinguished triangle

$$\bigoplus_{i=0}^{\infty} X \to \bigoplus_{i=0}^{\infty} X \to Y \to \Sigma\left(\bigoplus_{i=0}^{\infty} X\right).$$

Since X is cellular, by definition $\bigoplus_{i=0}^{\infty} X$ is as well. Thus by 2-of-3 for distinguished triangles, Y is cellular as desired. $\square$

Proposition 0.11. *Suppose we are given a distinguished triangle*

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$$

and some object W in $\mathcal{SH}$. Then there is an infinite long exact sequence of A-graded abelian groups:

$$\cdots \to [W,Z]_{*+\mathbf{n+1}} \xrightarrow{\partial} [W,X]_{*+\mathbf{n}} \xrightarrow{f_*} [W,Y]_{*+\mathbf{n}} \xrightarrow{g_*} [W,Z]_{*+\mathbf{n}} \xrightarrow{\partial} [W,Z]_{*+\mathbf{n-1}} \to \cdots,$$

where $\partial : [W,Z]_{*+\mathbf{n+1}} \to [W,X]_{*+\mathbf{n}}$ sends a class $x : S^{a+\mathbf{n+1}} \otimes W \to Z$ to the composition

$$S^{a+\mathbf{n}} \otimes W \cong S^{-\mathbf{1}} \otimes S^{a+\mathbf{n+1}} \otimes W \xrightarrow{S^{-\mathbf{1}}\otimes x} S^{-\mathbf{1}} \otimes Z \xrightarrow{S^{-\mathbf{1}}\otimes h} S^{-\mathbf{1}} \otimes \Sigma X \xrightarrow{S^{-\mathbf{1}}\otimes \nu_X} S^{-\mathbf{1}} \otimes S^{\mathbf{1}} \otimes X \xrightarrow{\phi^{-1}_{-\mathbf{1},\mathbf{1}}\otimes X} X.$$

Proof. See **??**. $\square$

Remark 0.12. Expressed more compactly, the above proposition says that each object W in $\mathcal{SH}$ and distinguished triangle

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$$

gives rise to the following diagram of A-graded abelian groups

$$\begin{tikzcd} {[W,X]_*} \arrow[r, "f_*"] & {[W,Y]_*} \arrow[d, "g_*"] \\ & {[W,Z]_*} \arrow[ul, "\partial"] \end{tikzcd}$$

which is exact at each vertex, and where f_*, g_*, and ∂ are A-graded homomorphisms of degree 0, 0, and $-\mathbf{1}$, respectively. Explicitly, ∂ sends a class $x : S^a \otimes W \to Z$ to the composition

$$S^{a-\mathbf{1}} \otimes W \cong S^{-\mathbf{1}} \otimes S^{a} \otimes W \xrightarrow{S^{-\mathbf{1}}\otimes x} S^{-\mathbf{1}} \otimes Z \xrightarrow{S^{-\mathbf{1}}\otimes h} S^{-\mathbf{1}} \otimes \Sigma X \xrightarrow{S^{-\mathbf{1}}\otimes \nu_X} S^{-\mathbf{1}} \otimes S^{\mathbf{1}} \otimes X \xrightarrow{\phi^{-1}_{-\mathbf{1},\mathbf{1}}\otimes X} X.$$

Proposition 0.13. *Let (E,μ,e) be a monoid object in $\mathcal{SH}$ (**??**). Then $\pi_*(E)$ is canonically an A-graded ring via the assignment $\pi_*(E) \times \pi_*(E) \to \pi_*(E)$ which takes classes $x : S^a \to E$ and $y : S^b \to E$ to the composition*

$$S^{a+b} \xrightarrow{\phi_{a,b}} S^a \otimes S^b \xrightarrow{x\otimes y} E \otimes E \xrightarrow{\mu} E.$$

In particular, the unit for this ring is the element $e \in [S,E] = \pi_0(E)$.

Proof. See **??**. $\square$

Proposition 0.14. *Let (E,μ,e) be a monoid object in $\mathcal{SH}$. Then $E_*(-)$ is a functor from $\mathcal{SH}$ to left A-graded $\pi_*(E)$-modules, where given some X in $\mathcal{SH}$, $E_*(X)$ may be endowed with the structure of a left A-graded $\pi_*(E)$-module via the map*

$$\pi_*(E) \times E_*(X) \to E_*(X)$$

which given $a,b \in A$, sends $x : S^a \to E$ and $y : S^b \to E \otimes X$ to the composition

$$x \cdot y : S^{a+b} \cong S^a \otimes S^b \xrightarrow{x\otimes y} E \otimes (E \otimes X) \cong (E \otimes E) \otimes X \xrightarrow{\mu\otimes X} E \otimes X.$$

Similarly, the assignment $X \mapsto X_*(E)$ is a functor from $\mathcal{SH}$ to right A-graded $\pi_*(E)$-modules, where the structure map

$$X_*(E) \times \pi_*(E) \to X_*(E)$$

sends $x : S^a \to X \otimes E$ *and* $y : S^b \to E$ *to the composition*

$$x \cdot y : S^{a+b} \cong S^a \otimes S^b \xrightarrow{x \otimes y} (X \otimes E) \otimes E \cong X \otimes (E \otimes E) \xrightarrow{X \otimes \mu} X \otimes E.$$

Finally, $E_*(E)$ *is a* $\pi_*(E)$*-bimodule, in the sense that the left and right actions of* $\pi_*(E)$ *are compatible, so that given* $y, z \in \pi_*(E)$ *and* $x \in E_*(E)$*,* $y \cdot (x \cdot z) = (y \cdot x) \cdot z$.

Proof. See **??**. □

Lemma 0.15. *Let* E *and* X *be objects in* $\mathcal{SH}$*. Then for all* $a \in A$*, there is an* A*-graded isomorphism of* A*-graded abelian groups*

$$t^a_X : E_*(\Sigma^a X) \cong E_{*-a}(X)$$

Furthermore this isomorphism is natural in X*, and if* E *is a monoid object in* $\mathcal{SH}$ *then it is a natural isomorphism of left* $\pi_*(E)$*-modules.*

Proof. See **??**. □

Definition 0.16. Given a monoid object E in $\mathcal{SH}$, we say E is *flat* if the canonical right $\pi_*(E)$-module structure on $E_*(E)$ (see the above proposition) is that of a flat module.