

0.1. Triangulated categories and their basic properties. In this appendix, we fix a triangulated category $(\mathcal{C}, \Sigma, \mathcal{D})$ (??). We will denote the hom-group $\mathcal{C}(X, Y)$ by $[X, Y]$. To start, recall the following definition:

Definition 0.1. A sequence

$$X_1 \rightarrow X_2 \rightarrow \cdots \rightarrow X_n$$

of arrows in \mathcal{C} is *exact* if, for any object A in \mathcal{C} , the induced sequences

$$[A, X_1] \rightarrow [A, X_2] \rightarrow \cdots \rightarrow [A, X_{n-1}] \rightarrow [A, X_n]$$

and

$$[X_n, A] \rightarrow [X_{n-1}, A] \rightarrow \cdots \rightarrow [X_2, A] \rightarrow [X_1, A]$$

are exact sequences of abelian groups.

It is straightforward to verify that if we have an exact sequence in \mathcal{C}

$$X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \cdots \rightarrow X_n,$$

then the sequence remains exact if we change the signs of any of the maps involved. We will use this fact often without comment.

Proposition 0.2. *Every distinguished triangle is an exact sequence (in the sense of Definition 0.1).*

Proof. Suppose we have some distinguished triangle

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X.$$

Then first we would like to show that given any object A in \mathcal{C} , the sequence

$$[A, X] \xrightarrow{f_*} [A, Y] \xrightarrow{g_*} [A, Z] \xrightarrow{h_*} [A, \Sigma X]$$

is exact. First we show exactness at $[A, Y]$. To see $\text{im } f_* \subseteq \ker g_*$, note it suffices to show that $g \circ f = 0$. Indeed, consider the commuting diagram

$$\begin{array}{ccccccc} X & \xlongequal{\quad} & X & \longrightarrow & 0 & \longrightarrow & \Sigma X \\ \parallel & & \downarrow f & & & & \\ X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & \Sigma X \end{array}$$

The top row is distinguished by axiom TR1. Thus by TR3, the following diagram commutes:

$$\begin{array}{ccccccc} X & \xlongequal{\quad} & X & \longrightarrow & 0 & \longrightarrow & \Sigma X \\ \parallel & & \downarrow f & & \downarrow & & \parallel \\ X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & \Sigma X \end{array}$$

In particular, commutativity of the second square tells us that $g \circ f = 0$, as desired. Conversely, we'd like to show that $\ker g_* \subseteq \text{im } f_*$. Let $\psi : A \rightarrow Y$ be in the kernel of g_* , so that $g \circ \psi = 0$. Consider the following commutative diagram:

$$\begin{array}{ccccccc} A & \longrightarrow & 0 & \longrightarrow & \Sigma A & \xrightarrow{-\Sigma \text{id}_A} & \Sigma A \\ \psi \downarrow & & \downarrow & & & & \\ Y & \xrightarrow{g} & Z & \xrightarrow{h} & \Sigma X & \xrightarrow{-\Sigma f} & \Sigma Y \end{array}$$

The top row is distinguished by axioms TR1 and TR4. The bottom row is distinguished by axiom TR4. Thus by axiom TR3 there exists a map $\tilde{\phi} : \Sigma A \rightarrow \Sigma X$ such that the following diagram commutes:

$$\begin{array}{ccccccc} A & \longrightarrow & 0 & \longrightarrow & \Sigma A & \xrightarrow{-\Sigma \text{id}_A} & \Sigma A \\ \psi \downarrow & & \downarrow & & \tilde{\phi} \downarrow & & \Sigma \psi \downarrow \\ Y & \xrightarrow{g} & Z & \xrightarrow{h} & \Sigma X & \xrightarrow{-\Sigma f} & \Sigma Y \end{array}$$

Now, since Σ is an equivalence, it is a full functor, so that in particular there exists some $\phi : A \rightarrow X$ such that $\tilde{\phi} = \Sigma \phi$. Then by faithfulness, we may pull back the right square to get a commuting diagram

$$\begin{array}{ccc} A & \xrightarrow{-\text{id}_A} & A \\ \phi \downarrow & & \downarrow \psi \\ X & \xrightarrow{-f} & Y \end{array}$$

Hence,

$$f_*(\phi) = f \circ \phi \stackrel{(*)}{=} -((-f) \circ \phi) = -(\psi \circ (-\text{id}_A)) \stackrel{(*)}{=} \psi \circ \text{id}_A = \psi,$$

where the equalities marked $(*)$ follow by bilinearity of composition in an additive category. Thus $\psi \in \text{im } f_*$, as desired, meaning $\ker g_* \subseteq \text{im } f_*$.

Now, we have shown that

$$[A, X] \xrightarrow{f_*} [A, Y] \xrightarrow{g_*} [A, Z] \xrightarrow{h_*} [A, \Sigma X]$$

is exact at $[A, Y]$. It remains to show exactness at $[A, Z]$. Yet this follows by the exact same argument given above applied to the sequence obtained from the shifted triangle (TR4)

$$Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X \xrightarrow{-\Sigma f} \Sigma Y$$

On the other hand, we would like to show that

$$[\Sigma X, A] \xrightarrow{h^*} [Z, A] \xrightarrow{g^*} [Y, A] \xrightarrow{f^*} [X, A]$$

is exact. As above, since we can shift the triangle, it suffices to show exactness at $[Z, A]$. First, since we have shown $g \circ f = 0$, we have $f^* \circ g^* = (g \circ f)^* = 0$, so that $\text{im } g^* \subseteq \ker f^*$, as desired. Conversely, in order to see $\ker f^* \subseteq \text{im } g^*$, suppose $\psi : Y \rightarrow A$ is in the kernel of f^* , so that $\psi \circ f = 0$. Consider the following commuting diagram:

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & \Sigma X \\ \downarrow & & \downarrow \psi & & & & \\ 0 & \longrightarrow & A & \xlongequal{\quad} & A & \longrightarrow & 0 \end{array}$$

The top row is a distinguished triangle by assumption, and the bottom row is distinguished by axioms TR1 and TR4 for a triangulated category, along with the fact that $\Sigma 0 = 0$ since Σ is additive. Thus by axiom TR3 there exists a map $\phi : Z \rightarrow A$ such that $\phi \circ g = \psi$, i.e., $g^*(\phi) = \psi$, so that $\phi \in \text{im } g^*$ as desired. \square

Lemma 0.3. *Suppose we have a commutative diagram*

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & \Sigma X \\ \downarrow j & & \downarrow k & & \downarrow \ell & & \downarrow \Sigma j \\ X' & \xrightarrow{f'} & Y' & \xrightarrow{g'} & Z' & \xrightarrow{h'} & \Sigma X' \end{array}$$

with both rows distinguished. Then if any two of the maps j , k , and ℓ are isomorphisms, then so is the third.

Proof. Suppose we are given any object W in \mathcal{C} , and consider the commutative diagram

$$\begin{array}{ccccccccccc} [W, X] & \xrightarrow{f_*} & [W, Y] & \xrightarrow{g_*} & [W, Z] & \xrightarrow{k_*} & [W, \Sigma X] & \xrightarrow{\Sigma f_*} & [W, \Sigma Y] & \xrightarrow{\Sigma g_*} & [W, \Sigma Z] & \xrightarrow{\Sigma h_*} & [W, \Sigma^2 X] \\ \downarrow j_* & & \downarrow k_* & & \downarrow \ell_* & & \downarrow \Sigma j_* & & \downarrow \Sigma k_* & & \downarrow \Sigma \ell_* & & \downarrow \Sigma^2 j_* \\ [W, X'] & \xrightarrow{f'_*} & [W, Y'] & \xrightarrow{g'_*} & [W, Z'] & \xrightarrow{h'_*} & [W, \Sigma X'] & \xrightarrow{\Sigma f'_*} & [W, \Sigma Y'] & \xrightarrow{\Sigma g'_*} & [W, \Sigma Z'] & \xrightarrow{\Sigma h'_*} & [W, \Sigma^2 X'] \end{array}$$

The rows are exact by **Proposition 0.2** and repeated applications of axiom TR4. It follows by the five lemma and faithfulness of Σ that if j and k are isomorphisms, then ℓ_* is an isomorphism. Similarly, if k and ℓ are isomorphisms then Σj_* is an isomorphism. Finally, if ℓ and j are isomorphisms, then Σk_* is an isomorphism. The desired result follows by faithfulness of Σ and the Yoneda embedding. \square

Proposition 0.4. *Given an arrow $f : X \rightarrow Y$ in \mathcal{C} , there exists an object F_f called the fiber of f , and a distinguished triangle*

$$F_f \rightarrow X \xrightarrow{f} Y \rightarrow \Sigma F_f (\cong C_f).$$

Proof. Since Σ is an equivalence, there exists some functor $\Omega : \mathcal{C} \rightarrow \mathcal{C}$ and natural isomorphisms $\varepsilon : \Omega \Sigma \Rightarrow \text{Id}_{\mathcal{C}}$ and $\eta : \text{Id}_{\mathcal{C}} \Rightarrow \Sigma \Omega$. By axiom TR2, we have a distinguished triangle

$$X \xrightarrow{f} Y \xrightarrow{g} C_f \xrightarrow{h} \Sigma X.$$

Now, consider the commutative diagram

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & C_f & \xrightarrow{h} & \Sigma X \\ \parallel & & \parallel & & \downarrow \eta_{C_f} & & \parallel \\ X & \xrightarrow{f} & Y & \xrightarrow{\tilde{g}} & \Sigma \Omega C_f & \xrightarrow{\tilde{h}} & \Sigma X \end{array}$$

where $\tilde{g} = \eta_{C_f} \circ g$, and $\tilde{h} = h \circ \eta_{C_f}^{-1}$. Since each vertical map is an isomorphism and the top row is distinguished, the bottom row is also distinguished by axiom TR0. Now, since Σ is an equivalence of categories, it is faithful, so that in particular there exists some map $k : \Omega C_f \rightarrow X$ such that $\Sigma k = -\tilde{h} \implies -\Sigma k = \tilde{h}$. Thus, we have a distinguished triangle of the form

$$X \xrightarrow{f} Y \xrightarrow{\tilde{g}} \Sigma \Omega C_f \xrightarrow{-\Sigma k} \Sigma X.$$

Finally, by axiom TR4, we get a distinguished triangle

$$\Omega C_f \xrightarrow{k} X \xrightarrow{f} Y \xrightarrow{\tilde{g}} \Sigma \Omega C_f,$$

so we may define the fiber of f to be ΩC_f . \square

0.2. Homotopy (co)limits in a triangulated category. In this subsection, we will assume \mathcal{C} has countable products and coproducts.

Definition 0.5 ([1, Definition 1.6.4]). Let

$$X_0 \xrightarrow{j_1} X_1 \xrightarrow{j_2} X_2 \xrightarrow{j_3} X_3 \rightarrow \dots$$

be a sequence of objects and morphisms in \mathcal{C} . The *homotopy colimit* of the sequence, denoted $\text{hocolim } X_i$, is given (up to non-canonical isomorphism) as the cofiber of the map

$$\bigoplus_{i=0}^{\infty} X_i \xrightarrow{1-\text{shift}} \bigoplus_{i=0}^{\infty} X_i,$$

where the shift map $\bigoplus_{i=0}^{\infty} X_i \xrightarrow{\text{shift}} \bigoplus_{i=0}^{\infty} X_i$ is understood to be the direct sum of $j_{i+1} : X_i \rightarrow X_{i+1}$, i.e., by the universal property of the coproduct, it is induced by the maps

$$X_s \xrightarrow{j_{s+1}} X_{s+1} \hookrightarrow \bigoplus_{i=0}^{\infty} X_i.$$

Definition 0.6. Assume that \mathcal{C} has countable products, and let

$$\cdots \rightarrow X_3 \xrightarrow{j_3} X_2 \xrightarrow{j_2} X_1 \xrightarrow{j_1} X_0$$

be a sequence of objects and morphisms in \mathcal{C} . The *homotopy limit* of the sequence, denoted $\text{holim } X_i$, is given (up to non-canonical isomorphism) as the fiber ([Proposition 0.4](#)) of the map

$$\prod_{i=0}^{\infty} X_i \xrightarrow{1-\text{shift}} \prod_{i=0}^{\infty} X_i,$$

where the shift map $\prod_{i=0}^{\infty} X_i \xrightarrow{\text{shift}} \prod_{i=0}^{\infty} X_i$ is understood to be the product of $j_i : X_i \rightarrow X_{i-1}$, i.e., by the universal property of the product, it is induced by the maps

$$\prod_{i=0}^{\infty} X_i \rightarrow X_{s+1} \xrightarrow{j_{s+1}} X_s.$$

0.3. Adjointly triangulated categories. From now on, we will assume that \mathcal{C} is an *adjointly triangulated category* (??) with inverse shift Ω , unit $\eta : \text{Id}_{\mathcal{C}} \Rightarrow \Sigma\Omega$, and counit $\varepsilon : \Omega\Sigma \Rightarrow \text{Id}_{\mathcal{C}}$.

Lemma 0.7. *Given a triangle*

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X,$$

it can be shifted to the left to obtain a distinguished triangle

$$\Omega Z \xrightarrow{\tilde{h}} X \xrightarrow{f} Y \xrightarrow{\widetilde{\Omega g}} \Sigma\Omega Z,$$

where $\tilde{h} : \Omega Z \rightarrow X$ is the adjoint of $h : Z \rightarrow \Sigma X$ and $\widetilde{\Omega g} : Y \rightarrow \Sigma\Omega Z$ is the adjoint of $\Omega g : \Omega Y \rightarrow \Omega Z$.

Proof. Note that unravelling definitions, \tilde{h} and $\widetilde{\Omega g}$ are the compositions

$$\tilde{h} : \Omega Z \xrightarrow{\Omega h} \Omega\Sigma X \xrightarrow{\varepsilon_X} X \quad \text{and} \quad \widetilde{\Omega g} : Y \xrightarrow{\eta_Y} \Sigma\Omega Y \xrightarrow{\Sigma\Omega g} \Sigma\Omega Z.$$

Now consider the following diagram:

$$(1) \quad \begin{array}{ccccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & \Sigma X \\ \parallel & & \parallel & & \eta_Z \downarrow & & \parallel \\ X & \xrightarrow{f} & Y & \xrightarrow{\widetilde{\Omega g}} & \Sigma\Omega Z & \xrightarrow{\Sigma\tilde{h}} & \Sigma X \end{array}$$

The left square commutes by definition. To see that the middle square commutes, expanding definitions, note it is given by the following diagram:

$$\begin{array}{ccccc} Y & \xrightarrow{g} & Z & & \\ \parallel & & \downarrow \eta_Y & & \\ Y & \xrightarrow{\eta_Y} & \Sigma\Omega Y & \xrightarrow{\Sigma\Omega g} & \Sigma\Omega Z \end{array}$$

and this commutes by naturality of η . To see that the right square commutes, consider the following diagram:

$$\begin{array}{ccccc} Z & \xrightarrow{h} & \Sigma X & & \\ \eta_Z \downarrow & & \nwarrow \eta_{\Sigma X} & & \parallel \\ \Sigma\Omega Z & \xrightarrow{\Sigma\Omega h} & \Sigma\Omega\Sigma X & \xrightarrow{\Sigma\varepsilon_X} & \Sigma X \end{array}$$

By functoriality of Σ , the bottom composition is $\Sigma\tilde{h}$. The left region commutes by naturality of η . Commutativity of the right region is precisely one of the zig-zag identities. Hence, since diagram (1) commutes, the vertical arrows are isomorphisms, and the top row is distinguished, we have that the bottom row is distinguished as well by axiom TR0. Then by axiom TR4, since $(f, \tilde{\Omega}g, \Sigma\tilde{h})$ is distinguished, so is the triangle

$$\Omega Z \xrightarrow{\tilde{h}} X \xrightarrow{f} Y \xrightarrow{\tilde{\Omega}g} \Sigma\Omega Z. \quad \square$$

Lemma 0.8. *Given a distinguished triangle*

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X,$$

for any $n > 0$, the triangle

$$\Omega^n X \xrightarrow{(-1)^n \Omega^n f} \Omega^n Y \xrightarrow{(-1)^n \Omega^n g} \Omega^n Z \xrightarrow{(-1)^n \Omega^n h} \Omega^n \Sigma X \cong \Sigma \Omega^n X,$$

is distinguished, where the final isomorphism is given by the composition

$$\Omega^n \Sigma X = \Omega^{n-1} \Omega \Sigma X \xrightarrow{\Omega^{n-1} \varepsilon_X} \Omega^{n-1} X \xrightarrow{\eta_{\Omega^{n-1} X}} \Sigma \Omega \Omega^{n-1} X = \Sigma \Omega^n X.$$

Proof. We give a proof by induction. First we show the case $n = 1$. Note by Lemma 0.7, we have that given a distinguished triangle

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X,$$

we can shift it to the left to obtain a distinguished triangle

$$\Omega Z \xrightarrow{\tilde{h}} X \xrightarrow{f} Y \xrightarrow{\tilde{\Omega}g} \Sigma\Omega Z,$$

where \tilde{h} is the adjoint of $h : Z \rightarrow \Sigma X$ and $\tilde{\Omega}g$ is the adjoint of $\Omega g : \Omega Y \rightarrow \Omega Z$. If we apply this shifting operation again, we get the distinguished triangle

$$\Omega Y \xrightarrow{\widetilde{\tilde{\Omega}g}} \Omega Z \xrightarrow{\tilde{h}} X \xrightarrow{\widetilde{\Omega f}} \Sigma\Omega Y,$$

where unravelling definitions, $\widetilde{\Omega f}$ is the right adjoint of $\Omega f : \Omega X \rightarrow \Omega Y$ and $\widetilde{\tilde{\Omega}g}$ is the right adjoint of $\tilde{\Omega}g$, which itself is the left adjoint of Ωg , so $\widetilde{\tilde{\Omega}g} = \Omega g$. Hence we have a distinguished triangle

$$\Omega Y \xrightarrow{\Omega g} \Omega Z \xrightarrow{\tilde{h}} X \xrightarrow{\widetilde{\Omega f}} \Sigma\Omega Y.$$

We may again shift this triangle again and the above arguments yield the distinguished triangle

$$\Omega X \xrightarrow{-\Omega f} \Omega Y \xrightarrow{-\Omega g} \Omega Z \xrightarrow{\widetilde{\Omega(-\tilde{h})}} \Sigma \Omega X,$$

where $\widetilde{\Omega(-\tilde{h})}$ is the right adjoint of $\Omega(-\tilde{h}) = -\Omega\tilde{h} : \Omega\Omega Z \rightarrow \Omega X$. Explicitly unravelling definitions, $\widetilde{\Omega(-\tilde{h})} = -\widetilde{\Omega\tilde{h}}$ is the composition

$$\begin{aligned} [\Omega Z \xrightarrow{\eta_{\Omega Z}} \Sigma \Omega \Omega Z \xrightarrow{\Sigma(-\Omega\tilde{h})} \Sigma \Omega X] &= -[\Omega Z \xrightarrow{\eta_{\Omega Z}} \Sigma \Omega \Omega Z \xrightarrow{\Sigma \Omega \tilde{h}} \Sigma \Omega X] \\ &= -[\Omega Z \xrightarrow{\eta_{\Omega Z}} \Sigma \Omega \Omega Z \xrightarrow{\Sigma \Omega \Omega h} \Sigma \Omega \Omega \Sigma X \xrightarrow{\Sigma \Omega \varepsilon_X} \Sigma \Omega X] \\ &= -[\Omega Z \xrightarrow{\Omega h} \Omega \Sigma X \xrightarrow{\varepsilon_X} X \xrightarrow{\eta_X} \Sigma \Omega X], \end{aligned}$$

where the first equality follows by additivity of Σ and additivity of composition, the second follows by further unravelling how \tilde{h} is defined, and the third follows by naturality of η , which tells us the following diagram commutes:

$$\begin{array}{ccccc} \Omega Z & \xrightarrow{\Omega h} & \Omega \Sigma X & \xrightarrow{\varepsilon_X} & X \\ \downarrow \eta_{\Omega Z} & & \downarrow \eta_{\Omega \Sigma X} & & \downarrow \eta_X \\ \Sigma \Omega \Omega Z & \xrightarrow{\Sigma \Omega \Omega h} & \Sigma \Omega \Omega \Sigma X & \xrightarrow{\Sigma \Omega \varepsilon_X} & \Sigma \Omega X \end{array}$$

Thus indeed we have a distinguished triangle

$$\Omega X \xrightarrow{-\Omega f} \Omega Y \xrightarrow{-\Omega g} \Omega Z \xrightarrow{-\Omega h} \Omega \Sigma X \cong \Sigma \Omega X,$$

where the last isomorphism is $\eta_X \circ \varepsilon_X$, as desired.

Now, we show the inductive step. Suppose we know that given a distinguished triangle

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X,$$

that for some $n > 0$ the triangle

$$\Omega^n X \xrightarrow{(-1)^n \Omega^n f} \Omega^n Y \xrightarrow{(-1)^n \Omega^n g} \Omega^n Z \xrightarrow{(-1)^n h^n} \Sigma \Omega^n X,$$

is distinguished, where $h^n : \Omega^n Z \rightarrow \Sigma \Omega^n X$ is the composition

$$\Omega^n Z \xrightarrow{\Omega^n h} \Omega^n \Sigma X \xrightarrow{\Omega^{n-1} \varepsilon_X} \Omega^{n-1} X \xrightarrow{\eta_{\Omega^{n-1} X}} \Sigma \Omega^n X.$$

Then by applying the $n = 1$ case to this triangle, we get that the following triangle is distinguished

$$\Omega^{n+1} X \xrightarrow{-\Omega((-1)^n \Omega^n f)} \Omega^{n+1} Y \xrightarrow{-\Omega((-1)^n \Omega^n g)} \Omega^{n+1} Z \xrightarrow{-\Omega((-1)^n h^n)} \Omega \Sigma \Omega^n X \cong \Sigma \Omega^{n+1} X,$$

where the final isomorphism is the composition

$$\Omega \Sigma \Omega^n X \xrightarrow{\varepsilon_{\Omega^n X}} \Omega^n X \xrightarrow{\eta_{\Omega^n X}} \Sigma \Omega \Omega^n X = \Sigma \Omega^{n+1} X.$$

We claim that this is precisely the distinguished triangle given in the statement of the lemma for $n + 1$. First of all, note that $-\Omega((-1)^n \Omega^n f) = (-1)^{n+1} \Omega^{n+1} f$, $-\Omega((-1)^n \Omega^n g) = (-1)^{n+1} \Omega^{n+1} g$, and $-\Omega((-1)^n h^n) = (-1)^{n+1} \Omega h^n$ by additivity of Ω , so that the triangle becomes

$$(2) \quad \Omega^{n+1} X \xrightarrow{(-1)^{n+1} \Omega^{n+1} f} \Omega^{n+1} Y \xrightarrow{(-1)^{n+1} \Omega^{n+1} g} \Omega^{n+1} Z \xrightarrow{(-1)^{n+1} \Omega h^n} \Omega \Sigma \Omega^n X \cong \Sigma \Omega^{n+1} X.$$

Thus, in order to prove the desired characterization, it remains to show this diagram commutes:

$$\begin{array}{ccccc} \Omega^{n+1} Z & \xrightarrow{(-1)^{n+1} \Omega h^n} & \Omega \Sigma \Omega^n X & \xrightarrow{\varepsilon_{\Omega^n X}} & \Omega^n X \\ (-1)^{n+1} \Omega^{n+1} h \downarrow & & & & \downarrow \eta_{\Omega^n X} \\ \Omega^{n+1} \Sigma X & \xrightarrow{\Omega^n \varepsilon_X} & \Omega^n X & \xrightarrow{\eta_{\Omega^n X}} & \Sigma \Omega^{n+1} X \end{array}$$

(The top composition is the last two arrows in diagram (2), and the bottom composition is the last two arrows in the diagram in the statement of the lemma). Unravelling how h^n is constructed, by additivity of Ω it further suffices to show the outside of the following diagram commutes:

$$\begin{array}{ccccccc}
 \Omega^{n+1}Z & \xrightarrow{(-1)^{n+1}\Omega^{n+1}h} & \Omega^{n+1}\Sigma X & \xrightarrow{\Omega^n \varepsilon_X} & \Omega^n X & \xrightarrow{\Omega \eta_{\Omega^{n-1}X}} & \Omega \Sigma \Omega^n X \\
 \downarrow (-1)^{n+1}\Omega^{n+1}h & & & & \parallel & & \downarrow \varepsilon_{\Omega^n X} \\
 \Omega^{n+1}\Sigma X & \xrightarrow{\Omega^n \varepsilon_X} & \Omega^n X & \xrightarrow{\eta_{\Omega^n X}} & \Sigma \Omega^{n+1}X & & \Omega^n X \\
 & & & & & & \downarrow \eta_{\Omega^n X} \\
 & & & & & & \Sigma \Omega^{n+1}X
 \end{array}$$

The left rectangle and bottom right triangle commute by definition. Finally, commutativity of the top right trapezoid is precisely one of the zig-zag identities applied to $\Omega^{n-1}X$. Hence, we have shown the desired result. \square

Proposition 0.9. *Given a distinguished triangle*

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X,$$

let $\tilde{h} : \Omega Z \rightarrow X$ be the left adjoint of h . Then the following infinite sequence is exact:

$$\begin{array}{ccccccc}
 & & & & \dots & & \\
 & & & \swarrow & & & \\
 \Omega^{n+1}Z & \xleftarrow[(-1)^{n+1}\Omega^n \tilde{h}]{} \Omega^n X & \xrightarrow[(-1)^n \Omega^n f]{} \Omega^n Y & \xrightarrow{(-1)^n \Omega^n g} \Omega^n Z & \xrightarrow{(-1)^n \Omega^{n-1} \tilde{h}} \Omega^{n-1}X & & \\
 & & & \swarrow & & & \\
 & & \dots & \swarrow & & & \\
 \Omega Z & \xleftarrow[-\tilde{h}]{} X & \xrightarrow{f} Y & \xrightarrow{g} Z & \xrightarrow{h} \Sigma X & & \\
 & & & \swarrow & & & \\
 & & \dots & \swarrow & & & \\
 \Sigma^{n-1}Z & \xleftarrow[(-1)^{n-1}\Sigma^n \tilde{h}]{} \Sigma^n X & \xrightarrow[(-1)^n \Sigma^n f]{} \Sigma^n Y & \xrightarrow{(-1)^n \Sigma^n g} \Sigma^n Z & \xrightarrow{(-1)^n \Sigma^n h} \Sigma^{n+1}X & & \\
 & & & \swarrow & & & \\
 & & \dots & \swarrow & & &
 \end{array}$$

In particular, it remains exact even if we remove the signs.

Proof. Exactness of

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X \xrightarrow{-\Sigma f} \Sigma Y$$

is **Proposition 0.2** and axiom TR4. By induction using axiom TR4, for $n > 0$ we get that each contiguous composition of three maps below is a distinguished triangle:

$$\Sigma^n X \xrightarrow{(-1)^n \Sigma^n f} \Sigma^n Y \xrightarrow{(-1)^n \Sigma^n g} \Sigma^n Z \xrightarrow{(-1)^n \Sigma^n h} \Sigma^{n+1} X \xrightarrow{(-1)^{n+1} \Sigma^{n+1} f} \Sigma^{n+1} Y,$$

thus the sequence is exact by [Proposition 0.2](#). It remains to show exactness of the LES to the left of Y . It suffices to show that the row in the following diagram is exact for all $n > 0$:

$$(3) \quad \begin{array}{ccccccc} \Omega^n X & \xrightarrow{(-1)^n \Omega^n f} & \Omega^n Y & \xrightarrow{(-1)^n \Omega^n g} & \Omega^n Z & \xrightarrow{(-1)^n \Omega^{n-1}(\varepsilon_X \circ \Omega h)} & \Omega^{n-1} X \xrightarrow{(-1)^{n-1} \Omega^{n-1} f} \Omega^{n-1} Y \\ & & & & \searrow & & \nearrow \\ & & & & \Omega^n \Sigma X & & \Omega^{n-1} \varepsilon_X \end{array}$$

First of all, to see exactness at $\Omega^n Y$ and $\Omega^n Z$, consider the following commutative diagram:

$$\begin{array}{ccccccc} \Omega^n X & \xrightarrow{(-1)^n \Omega^n f} & \Omega^n Y & \xrightarrow{(-1)^n \Omega^n g} & \Omega^n Z & \xrightarrow{(-1)^n \Omega^{n-1}(\varepsilon_X \circ \Omega h)} & \Omega^{n-1} X \\ \parallel & & \parallel & & \parallel & \nearrow \scriptstyle (-1)^n \Omega^n h & \nearrow \scriptstyle \Omega^{n-1} \varepsilon_X \\ & & & & \Omega^n \Sigma X & & \downarrow \scriptstyle \eta_{\Omega^{n-1} X} \\ \Omega^n X & \xrightarrow{(-1)^n \Omega^n f} & \Omega^n Y & \xrightarrow{(-1)^n \Omega^n g} & \Omega^n Z & \dashrightarrow & \Sigma \Omega^n X \end{array}$$

(here the dashed arrow is the morphism which makes the diagram commute). The bottom row is distinguished by [Lemma 0.8](#). Then by axiom TR0, the top row is distinguished, and thus exact by [Proposition 0.2](#). Thus we have shown exactness of (3) at $\Omega^n Y$ and $\Omega^n Z$. It remains to show exactness at $\Omega^{n-1} X$. In the case $n = 1$, we want to show exactness at X in the following diagram:

$$\begin{array}{ccccc} \Omega Z & \xrightarrow{-(\varepsilon_X \circ \Omega h)} & X & \xrightarrow{f} & Y \\ & \searrow \scriptstyle -\Omega h & \nearrow \scriptstyle \varepsilon_X & & \\ & \Omega \Sigma X & & & \end{array}$$

Unravelling definitions, $\varepsilon_X \circ \Omega h$ is precisely the adjoint $\tilde{h} : \Omega Z \rightarrow X$ of $h : Z \rightarrow \Sigma X$, in which case we have that the row in the above diagram fits into a distinguished triangle by [Lemma 0.7](#), and thus it is exact by [Proposition 0.2](#). To see exactness at $\Omega^{n-1} X$ in diagram (3), note that if we apply [Lemma 0.7](#) to the sequence [Lemma 0.8](#) for $n - 1$, then we get that the following composition fits into a distinguished triangle, and is thus exact:

$$\Omega^n Z \xrightarrow{-k} \Omega^{n-1} X \xrightarrow{(-1)^{n-1} \Omega^{n-1} f} \Omega^{n-1} Y,$$

where $k : \Omega(\Omega^{n-1} Z) \rightarrow \Omega^{n-1} X$ is the adjoint of the composition

$$\Omega^{n-1} Z \xrightarrow{(-1)^{n-1} \Omega^{n-1} h} \Omega^{n-1} \Sigma X \xrightarrow{\Omega^{n-2} \varepsilon_X} \Omega^{n-2} X \xrightarrow{\eta_{\Omega^{n-2} X}} \Sigma \Omega^{n-1} X.$$

Further expanding how adjoints are constructed, k is the composition

$$\Omega^n Z \xrightarrow{(-1)^{n-1} \Omega^n h} \Omega^n \Sigma X \xrightarrow{\Omega^{n-1} \varepsilon_X} \Omega^{n-1} X \xrightarrow{\Omega \eta_{\Omega^{n-2} X}} \Omega \Sigma \Omega^{n-1} X \xrightarrow{\varepsilon_{\Omega^{n-1} X}} \Omega^{n-1} X.$$

Thus, in order to show exactness of (3) at $\Omega^{n-1} X$, it suffices to show that $k = (-1)^{n-1} \Omega^{n-1}(\varepsilon_X \circ \Omega h)$. To that end, consider the following diagram:

$$\begin{array}{ccccc} \Omega^n Z & \xrightarrow{(-1)^{n-1} \Omega^n h} & \Omega^n \Sigma X & \xrightarrow{\Omega^{n-1} \varepsilon_X} & \Omega^{n-1} X \xrightarrow{\Omega \eta_{\Omega^{n-2} X}} \Omega \Sigma \Omega^{n-1} X \\ \downarrow \scriptstyle (-1)^{n-1} \Omega^n h & & & & \searrow \scriptstyle \varepsilon_{\Omega^{n-1} X} \\ \Omega^n \Sigma X & \xrightarrow{\hspace{10em}} & \Omega^{n-1} \varepsilon_X & & \Omega^{n-1} X \end{array}$$

The top composition is k , while the bottom composition is $(-1)^{n-1} \Omega^{n-1}(\varepsilon_X \circ \Omega h)$. The left region commutes by definition, while commutativity of the right region is precisely one of the

zig-zag identities applied to $\Omega^{n-2}X$. Thus, we have shown that $-k = (-1)^n \Omega^{n-1}(\varepsilon_X \circ \Omega h)$, so (3) is exact at $\Omega^{n-1}X$, as desired. \square

0.4. Tensor triangulated categories. In what follows, we fix a tensor triangulated category $(\mathcal{C}, \otimes, S, \Sigma, e, \mathcal{D})$ (??).

Lemma 0.10. *Let $A \xrightarrow{a} B \xrightarrow{b} C \xrightarrow{c} D$ be any sequence isomorphic to a distinguished triangle. Then given any E in \mathcal{C} , the sequences*

$$E \otimes A \xrightarrow{E \otimes a} E \otimes B \xrightarrow{E \otimes b} E \otimes C \xrightarrow{E \otimes c} E \otimes D$$

and

$$A \otimes E \xrightarrow{a \otimes E} B \otimes E \xrightarrow{b \otimes E} C \otimes E \xrightarrow{c \otimes E} D \otimes E$$

are exact.

Proof. Since (a, b, c) is isomorphic to a distinguished triangle, there exists a commuting diagram in \mathcal{SH}

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & \Sigma X \\ \alpha \downarrow & & \beta \downarrow & & \gamma \downarrow & & \delta \downarrow \\ A & \xrightarrow{a} & B & \xrightarrow{b} & C & \xrightarrow{c} & D \end{array}$$

where the top row is distinguished and the vertical arrows are isomorphisms. Then the following diagram commutes by functoriality of $- \otimes -$:

$$\begin{array}{ccccccc} E \otimes X & \xrightarrow{E \otimes f} & E \otimes Y & \xrightarrow{E \otimes g} & E \otimes Z & \xrightarrow{E \otimes h} & \Sigma(E \otimes X) \\ \downarrow E \otimes \alpha & & \downarrow E \otimes \beta & & \downarrow E \otimes \gamma & & \downarrow (E \otimes \delta) \circ (e'_{E,X})^{-1} \\ E \otimes A & \xrightarrow{E \otimes a} & E \otimes B & \xrightarrow{E \otimes b} & E \otimes C & \xrightarrow{E \otimes c} & E \otimes D \end{array}$$

(Note: The diagram above is a simplified representation of the complex commutative diagram in the image, which includes additional arrows and labels like $E \otimes h$, $e'_{E,X}$, and $E \otimes \delta$ connecting the top and bottom rows.)

The top triangle is distinguished by axiom TT3 for a tensor triangulated category, thus exact by [Proposition 0.2](#), so that the bottom triangle is also exact since the vertical arrows are isomorphisms and each square commutes. Similarly, the following diagram also commutes by functoriality of $- \otimes -$:

$$\begin{array}{ccccccc} X \otimes E & \xrightarrow{f \otimes E} & Y \otimes E & \xrightarrow{g \otimes E} & Z \otimes E & \xrightarrow{h \otimes E} & \Sigma(X \otimes E) \\ \downarrow \alpha \otimes E & & \downarrow \beta \otimes E & & \downarrow \gamma \otimes E & & \downarrow (\delta \otimes E) \circ e_{X,E}^{-1} \\ A \otimes E & \xrightarrow{a \otimes E} & B \otimes E & \xrightarrow{b \otimes E} & C \otimes E & \xrightarrow{c \otimes E} & D \otimes E \end{array}$$

(Note: The diagram above is a simplified representation of the complex commutative diagram in the image, which includes additional arrows and labels like $h \otimes E$, $e_{X,E}$, and $\delta \otimes E$ connecting the top and bottom rows.)

The top row is distinguished by axiom TT3 for a tensor triangulated category, thus exact by [Proposition 0.2](#), so that the bottom triangle is also exact since the vertical arrows are isomorphisms and each square commutes. \square

Proposition 0.11. *Suppose we have a distinguished triangle*

$$X \rightarrow Y \rightarrow Z \rightarrow \Sigma X$$

in \mathcal{C} . Then given any object E in \mathcal{C} , the long exact sequence given in [Proposition 0.9](#) remains exact after applying $E \otimes -$ or $- \otimes E$.

Proof. Recall that in the proof of [Proposition 0.9](#) we showed that the sequence was exact by showing that any two consecutive maps were isomorphic to a part of a distinguished triangle. Then the desired result follows from [Lemma 0.10](#). \square