In what follows, we fix a symmetric monoidal category $(\mathcal{C}, \otimes, S)$ with left unitor, right unitor, associator, and symmetry isomorphisms λ, ρ, α, and τ, respectively.

0.1. **Monoid objects in a symmetric monoidal category.**

Definition 0.1. Let $(\mathcal{C}, \otimes, S)$ be a symmetric monoidal category with left unitor, right unitor, associator, and symmetry isomorphisms λ, ρ, α, and τ, respectively. A *monoid object* (E, μ, e) is an object E in $\mathcal{C}$ along with a multiplication morphism $\mu : E \otimes E \to E$ and a unit map $e : S \to E$ such that the following diagrams commute:

$$\begin{tikzcd} E \otimes S \arrow[r, "E\otimes e"] \arrow[dr, "\rho_E"'] & E \otimes E \arrow[d, "\mu"] & S \otimes E \arrow[l, "e\otimes E"'] \arrow[dl, "\lambda_E"] \\ & E & \end{tikzcd} \qquad \begin{tikzcd} (E \otimes E) \otimes E \arrow[rr, "\mu\otimes E"] \arrow[d, "\alpha"'] & & E \otimes E \arrow[d, "\mu"] \\ E \otimes (E \otimes E) \arrow[r, "E\otimes\mu"] & E \otimes E \arrow[r, "\mu"] & E \end{tikzcd}$$

The first diagram expresses unitality, while the second expressed associativity. If in addition the following diagram commutes,

$$\begin{tikzcd} E \otimes E \arrow[rr, "\tau"] \arrow[dr, "\mu"'] & & E \otimes E \arrow[dl, "\mu"] \\ & E & \end{tikzcd}$$

then we say (E, μ, e) is a *commutative* monoid object.

Definition 0.2. Given two monoid objects (E_1, μ_1, e_1) and (E_2, μ_2, e_2) in a symmetric monoidal category $(\mathcal{C}, \otimes, S)$, a *monoid homomorphism* from E_1 to E_2 is a morphism $f : E_1 \to E_2$ in $\mathcal{C}$ such that the following diagrams commute:

$$\begin{tikzcd} E_1 \otimes E_1 \arrow[r, "f\otimes f"] \arrow[d, "\mu_1"'] & E_2 \otimes E_2 \arrow[d, "\mu_2"] \\ E_1 \arrow[r, "f"] & E_2 \end{tikzcd} \qquad \begin{tikzcd} & S \arrow[dl, "e_1"'] \arrow[dr, "e_2"] & \\ E_1 \arrow[rr, "f"] & & E_2 \end{tikzcd}$$

It is straightforward to show that id_{E_1} is a homomorphism of monoid objects from E_1 to itself, and that the composition of monoid homomorphisms is still a monoid homomorphism. Thus, we have categories $\mathbf{Mon}_{\mathcal{C}}$ and $\mathbf{CMon}_{\mathcal{C}}$ of monoid objects and commutative monoid objects in $\mathcal{C}$, respectively, with monoid homomorphisms between them.

Lemma 0.3. *Given two monoid objects (E_1, μ_1, e_1) and (E_2, μ_2, e_2) in a symmetric monoidal category $(\mathcal{C}, \otimes, S)$, their tensor product $E_1 \otimes E_2$ canonically becomes a monoid object in $\mathcal{C}$ with unit map*

$$e : S \xrightarrow{\cong} S \otimes S \xrightarrow{e_1 \otimes e_2} E_1 \otimes E_2$$

and multiplication map

$$\mu : E_1 \otimes E_2 \otimes E_1 \otimes E_2 \xrightarrow{E_1 \otimes \tau \otimes E_2} E_1 \otimes E_1 \otimes E_2 \otimes E_2 \xrightarrow{\mu_1 \otimes \mu_2} E_1 \otimes E_2$$

(where here we are suppressing the associators from the notation). If in addition (E_1, μ_1, e_1) and (E_2, μ_2, e_2) are commutative *monoid objects, then $(E_1 \otimes E_2, \mu, e)$ is as well.*

Lemma 0.4. *Given monoid objects (E_i, μ_i, e_i) for $i = 1, 2, 3$ in a symmetric monoidal category $\mathcal{C}$, the associator $(E_1 \otimes E_2) \otimes E_3 \xrightarrow{\cong} E_1 \otimes (E_2 \otimes E_3)$ is an isomorphism of monoid objects. In other words, up to associativity, given a collection of monoid objects $E_1, \ldots, E_n$ in $\mathcal{C}$, there is no ambiguity when talking about their tensor product $E_1 \otimes \cdots \otimes E_n$ as a monoid object.*

Proof. Clearly, up to associativity, $(E_1 \otimes E_2) \otimes E_3$ and $E_1 \otimes (E_2 \otimes E_3)$ have the same unit map $S \xrightarrow{e_1 \otimes e_2 \otimes e_3} E_1 \otimes E_2 \otimes E_3$. Thus, it remains to show that they have the same product map, up to associativity. To see this, consider the following diagram, where we've passed to a symmetric strict monoidal category:

$$\begin{tikzcd}
E_1 \otimes (E_2 \otimes E_3) \otimes E_1 \otimes (E_2 \otimes E_3) \arrow[rr, equal, "\alpha"] \arrow[d, "E_1 \otimes \tau_{E_2 \otimes E_3, E_1} \otimes E_2 \otimes E_3"'] & & (E_1 \otimes E_2) \otimes E_3 \otimes (E_1 \otimes E_2) \otimes E_3 \arrow[d, "E_1 \otimes E_2 \otimes \tau_{E_3, E_1 \otimes E_2} \otimes E_3"] \\
E_1 \otimes E_1 \otimes E_2 \otimes E_3 \otimes E_2 \otimes E_3 \arrow[d, "\mu_1 \otimes E_2 \otimes \tau \otimes E_3"'] \arrow[dr, "E_1 \otimes E_1 \otimes E_2 \otimes \tau \otimes E_3"] & & E_1 \otimes E_2 \otimes E_1 \otimes E_2 \otimes E_3 \otimes E_3 \arrow[d, "E_1 \otimes \tau \otimes E_2 \otimes \mu_3"] \arrow[dl, "E_1 \otimes \tau \otimes E_2 \otimes E_3 \otimes E_3"'] \\
E_1 \otimes E_2 \otimes E_2 \otimes E_3 \otimes E_3 \arrow[d, "E_1 \otimes \mu_2 \otimes \mu_3"'] & E_1 \otimes E_1 \otimes E_2 \otimes E_2 \otimes E_3 \otimes E_3 \arrow[l, "\mu_1 \otimes E_2 \otimes E_2 \otimes E_3 \otimes E_3"] \arrow[r, "E_1 \otimes E_2 \otimes E_2 \otimes E_2 \otimes \mu_3"'] \arrow[dl, "\mu_1 \otimes \mu_2 \otimes \mu_3"] \arrow[dr, "\mu_1 \otimes \mu_2 \otimes \mu_3"'] & E_1 \otimes E_1 \otimes E_2 \otimes E_2 \otimes E_3 \arrow[d, "\mu_1 \otimes \mu_2 \otimes E_3"] \\
E_1 \otimes E_2 \otimes E_3 \arrow[rr, equal, "\alpha"] & & E_1 \otimes E_2 \otimes E_3
\end{tikzcd}$$

The top pentagonal region commutes by coherence for the τ's in a symmetric monoidal category. The bottom triangle commutes by definition. The remaining four triangles commute by functoriality of $- \otimes -$. On the left is the product for $E_1 \otimes (E_2 \otimes E_3)$, while on the right is the product for $(E_1 \otimes E_2) \otimes E_3$. Thus they are equal up to associativity, as desired. $\square$

Lemma 0.5. *Suppose we have some monoid object (E, μ, e) in $\mathcal{C}$ and some homomorphism of monoid objects $f : (E_1, \mu_1, e_1) \to (E_2, \mu_2, e_2)$ in $\mathbf{Mon}_{\mathcal{C}}$. Then $E \otimes f : E \otimes E_1 \to E \otimes E_2$ and $f \otimes E : E_1 \otimes E \to E_2 \otimes E$ are monoid homomorphisms, where here we are considering $E \otimes E_1$, $E \otimes E_2$, $E_1 \otimes E$, and $E_2 \otimes E$ to be monoid objects by Lemma 0.3.*

Proof. We will show that $E \otimes f$ is a monoid object homomorphism, as showing $f \otimes E$ is a monoid homomorphism is entirely analagous. First consider the following diagram:

$$\begin{tikzcd}
E \otimes E_1 \otimes E \otimes E_1 \arrow[rrr, "E \otimes f \otimes E \otimes f"] \arrow[d, "E \otimes \tau \otimes E_1"'] & & & E \otimes E_2 \otimes E \otimes E_2 \arrow[d, "E \otimes \tau \otimes E_2"] \\
E \otimes E \otimes E_1 \otimes E_1 \arrow[rrr, "E \otimes E \otimes f \otimes f"] \arrow[dd, "\mu \otimes \mu_1"'] \arrow[dr, "\mu \otimes E_1 \otimes E_2"] & & & E \otimes E \otimes E_2 \otimes E_2 \arrow[dd, "\mu \otimes \mu_2"] \arrow[dl, "\mu \otimes E_2 \otimes E_2"'] \\
& E \otimes E_1 \otimes E_1 \arrow[r, "E \otimes f \otimes f"] \arrow[dl, "E \otimes \mu_1"] & E \otimes E_2 \otimes E_2 \arrow[dr, "E \otimes \mu_2"'] & \\
E \otimes E_1 \arrow[rrr, "E \otimes f"] & & & E \otimes E_2
\end{tikzcd}$$

The top region commutes by naturality of τ. The bottom trapezoid commutes since f is a monoid homomorphism. The remaining three regions commute by functoriality of $- \otimes -$. Now, consider the following diagram:

$$\begin{tikzcd}
& S \arrow[d, "e"] \arrow[ddl, "e \otimes e_1"'] \arrow[ddr, "e \otimes e_2"] & \\
& E \arrow[dl, "E \otimes e_1"] \arrow[dr, "E \otimes e_2"'] & \\
E \otimes E_1 \arrow[rr, "E \otimes f"] & & E \otimes E_2
\end{tikzcd}$$

The bottom region commutes since f is a monoid homomorphism. The top two regions commute by functoriality of $- \otimes -$. Thus, we've shown $E \otimes f$ is a monoid object homomorphism, as desired. $\square$

0.2. **Modules over a monoid object.**

Definition 0.6. Let (E, μ, e) be a monoid object in $\mathcal{C}$. Then a *left module object* (N, κ) over (E, μ, e) is the data of an object N in $\mathcal{C}$ and a morphism $\kappa : E \otimes N \to N$ such that the following two diagrams commute in $\mathcal{C}$:

$$\begin{tikzcd} S \otimes N \arrow[r, "e\otimes N"] \arrow[dr, "\lambda_N"'] & E \otimes N \arrow[d, "\kappa"] \\ & N \end{tikzcd} \qquad \begin{tikzcd} (E \otimes E) \otimes N \arrow[rr, "\mu\otimes N"] \arrow[d, "\alpha"'] & & E \otimes N \arrow[d, "\kappa"] \\ E \otimes (E \otimes N) \arrow[r, "E\otimes\kappa"] & E \otimes N \arrow[r, "\kappa"] & N \end{tikzcd}$$

Definition 0.7. Let (E, μ, e) be a monoid object in $\mathcal{C}$, and suppose we have two left module objects (N, κ) and (N', κ') over (E, μ, e). Then a morphism $f : N \to N'$ is a *left E-module homomorphism* if the following diagram commutes in $\mathcal{C}$:

$$\begin{tikzcd} E \otimes N \arrow[r, "E\otimes f"] \arrow[d, "\kappa"'] & E \otimes N' \arrow[d, "\kappa'"] \\ N \arrow[r, "f"] & N' \end{tikzcd}$$

Definition 0.8. Given a monoid object (E, μ, e) in $\mathcal{C}$, we write E-**Mod** to denote the category of left module objects over E and left E-module homomorphisms between them. We denote the homset in E-**Mod** by

$$\mathrm{Hom}_{E\text{-}\mathbf{Mod}}(M, N), \qquad \text{or simply} \qquad \mathrm{Hom}_E(M, N).$$

Lemma 0.9. *Let (E, μ, e) be a monoid object in $\mathcal{C}$ and let (N, κ) be a left E module object. Then given some object X in $\mathcal{C}$ and an isomorphism $\phi : N \xrightarrow{\cong} X$, X inherits the structure of a left E-module via the action map*

$$\kappa_\phi : E \otimes X \xrightarrow{E\otimes\phi^{-1}} E \otimes N \xrightarrow{\kappa} N \xrightarrow{\phi} X.$$

Proof. We need to show the two coherence diagrams in Definition 0.6 commute. To see the former commutes, consider the following diagram:

$$\begin{tikzcd} X \arrow[rr, "e\otimes X"] \arrow[dr, "\phi^{-1}"] \arrow[ddddrr, equal] & & E \otimes X \arrow[d, "E\otimes\phi^{-1}"] \\ & N \arrow[r, "e\otimes N"] \arrow[dr, equal] & E \otimes N \arrow[d, "\kappa"] \\ & & N \arrow[d, "\phi"] \\ & & X \end{tikzcd}$$

The top trapezoid commutes by functoriality of $- \otimes -$. The middle small triangle commutes by unitality of κ. The remaining region commutes by definition. To see the second coherence diagram commutes, consider the following diagram:

$$\begin{tikzcd}
E\otimes E\otimes X \arrow[rrr, "\mu\otimes X"] \arrow[d, "E\otimes E\otimes\phi^{-1}"'] & & & E\otimes X \arrow[d, "E\otimes\phi^{-1}"] \\
E\otimes E\otimes N \arrow[rrr, "\mu\otimes N"] \arrow[d, "E\otimes\kappa"'] & & & E\otimes N \arrow[d, "\kappa"] \\
E\otimes N \arrow[rrr, "\kappa"] \arrow[d, "E\otimes\phi"'] \arrow[dr, equal] & & & N \arrow[d, "\phi"] \\
E\otimes X \arrow[r, "E\otimes\phi^{1-}"'] & E\otimes N \arrow[r, "\kappa"'] & N \arrow[r, "\phi"'] & X
\end{tikzcd}$$

The top rectangle commutes by functoriality of $-\otimes-$. The middle rectangle commutes by coherence for κ. The bottom two regions commute by definition. $\square$

Proposition 0.10. *Given a monoid object (E,μ,e) in $\mathcal{C}$, the forgetful functor E-**Mod** $\to \mathcal{C}$ has a left adjoint $\mathcal{C}\to E$-**Mod** sending an object $X\mapsto(E\otimes X,\kappa_X)$ where κ_X is the composition*

$$E\otimes(E\otimes X)\xrightarrow{\alpha^{-1}}(E\otimes E)\otimes X\xrightarrow{\mu\otimes X}E\otimes X,$$

and sending a morphism $f:X\to Y$ to $E\otimes f:E\otimes X\to E\otimes Y$.

Proof. In this proof, we work in a symmetric strict monoidal category. First, we wish to show that $E\otimes - : \mathcal{C}\to E$-**Mod** as constructed is well-defined. First, to see that (X,κ_X) is actually a left E-module, we need to show the two diagrams in Definition 0.6 commute. Indeed, consider the following diagrams:

$$\begin{tikzcd}
E\otimes X \arrow[r, "e\otimes E\otimes X"] \arrow[dr, equal] & E\otimes E\otimes X \arrow[d, "\mu\otimes X"] \\
& E\otimes X
\end{tikzcd}
\qquad
\begin{tikzcd}
E\otimes E\otimes E\otimes X \arrow[r, "\mu\otimes E\otimes X"] \arrow[d, "E\otimes\mu\otimes X"'] & E\otimes E\otimes X \arrow[d, "\mu\otimes X"] \\
E\otimes E\otimes X \arrow[r, "\mu\otimes X"'] & E\otimes X
\end{tikzcd}$$

These are precisely the diagrams obtained by applying $X\otimes-$ to the coherence diagrams for μ, so that they commute as desired. Now, suppose $f:X\to Y$ is a morphism in $\mathcal{C}$, then we would like to show that $E\otimes f:E\otimes X\to E\otimes Y$ is a morphism of left E-module objects. Indeed, consider the following diagram:

$$\begin{tikzcd}
E\otimes E\otimes X \arrow[r, "E\otimes E\otimes f"] \arrow[d, "\mu\otimes X"'] & E\otimes E\otimes Y \arrow[d, "\mu\otimes Y"] \\
E\otimes X \arrow[r, "E\otimes f"] & E\otimes Y
\end{tikzcd}$$

It commutes by functoriality of $-\otimes-$, so $E\otimes f$ is indeed a left E-module homomorphism as desired.

Now, in order to see that $E\otimes-$ is left adjoint to the forgetful functor, it suffices to construct a unit and counit for the adjunction and show they satisfy the zig-zag identities. Given X in $\mathcal{C}$ and (N,κ) in E-**Mod**, define $\eta_X := e\otimes X : X\to E\otimes X$ and $\varepsilon_{(N,\kappa)} := \kappa : E\otimes N\to N$. η_X is clearly natural in X by functoriality of $-\otimes-$, and $\varepsilon_{(N,\kappa)}$ is natural in (N,κ) by how morphisms in E-**Mod** are defined. Now, to see these are actually the unit and counit of an adjunction, we need to show that the following diagrams commute for all X in $\mathcal{C}$ and (N, κ) in E-**Mod**:

$$\begin{tikzcd} E \otimes X \arrow[r, "E\otimes\eta_X = E\otimes e\otimes X"] \arrow[dr, equal] & E \otimes E \otimes X \arrow[d, "\varepsilon_{(E\otimes X, \kappa_X)} = \mu\otimes X"] \\ & E \otimes X \end{tikzcd} \qquad \begin{tikzcd} E \otimes N \arrow[d, "\varepsilon_{(N,\kappa)} = \kappa"'] & N \arrow[l, "\eta_N = e\otimes N"'] \arrow[dl, equal] \\ N & \end{tikzcd}$$

Commutativity of the left diagram is unitality of μ, while commutativity of the right diagram is unitality of κ. Thus indeed $E \otimes - : \mathcal{C} \to E$-**Mod** is a left adjoint of the forgetful functor E-**Mod** $\to \mathcal{C}$, as desired. ☐

Definition 0.11. We call the functor $E \otimes - : \mathcal{C} \to E$-**Mod** constructed above the *free* functor, and we call left E-modules in the image of the free functor *free modules*.

From now on we fix a monoidal closed tensor triangulated category $(\mathcal{SH}, \otimes, S, \Sigma, e, \mathcal{D})$ (**??**) with arbitrary (small) (co)products and sub-Picard grading $(A, \mathbf{1}, h, \{S^a\}, \{\phi_{a,b}\})$ (**??**), and we adopt the conventions outlined in **??**. In all proofs that follow we will freely use the coherence theorem for symmetric monoidal categories. In particular, we will assume without loss of generality that the associators and unitors in $\mathcal{SH}$ are identities.

Lemma 0.12. *Let (E, μ, e) be a monoid object in $\mathcal{SH}$, and suppose (N, κ) is a left module object over E. Then for all $a \in A$, $\Sigma^a N$ is canonically a left E-module object, with action map given by*

$$\kappa^a : E \otimes S^a \otimes N$$

$$\kappa^a : E \otimes \Sigma^a N = E \otimes S^a \otimes N \xrightarrow{\tau\otimes N} S^a \otimes E \otimes N \xrightarrow{S^a\otimes\kappa} S^a \otimes N = \Sigma^a N.$$

Proof. In this proof, we are assuming that unitality and associativity hold up to strict equality, by the coherence theorem for monoidal categories. In order to show $(\Sigma^a N, \kappa^a)$ is a left module object over E, we need to show κ^a makes the two coherence diagrams in Definition 0.6 commute. First, to see the first diagram commutes, consider the following diagram:

$$\begin{tikzcd} S^a \otimes N \arrow[r, "e\otimes S^a\otimes N"] \arrow[dr, "S^a\otimes e\otimes N"] \arrow[ddr, equal] & E \otimes S^a \otimes N \arrow[d, "\tau\otimes N"] \\ & S^a \otimes E \otimes N \arrow[d, "S^a\otimes\kappa"] \\ & S^a \otimes N \end{tikzcd}$$

The top inner triangle commutes by coherence for a symmetric monoidal category, and the bottom inner triangle commutes by the coherence condition for κ. To see the other module condition for $\widetilde{\kappa}$, consider the following diagram:

$$\begin{tikzcd} E \otimes E \otimes S^a \otimes N \arrow[rr, "\mu\otimes S^a\otimes N"] \arrow[d, "E\otimes\tau\otimes N"'] \arrow[dr, "\tau_{E\otimes E, S^a}\otimes N"] & & E \otimes S^a \otimes N \arrow[d, "\tau\otimes N"] \\ E \otimes S^a \otimes E \otimes N \arrow[r, "\tau\otimes E\otimes N"'] \arrow[d, "E\otimes S^a\otimes\kappa"'] & S^a \otimes E \otimes E \otimes N \arrow[r, "S^a\otimes\mu\otimes N"] \arrow[d, dashed, "S^a\otimes E\otimes\kappa"] & S^a \otimes E \otimes N \arrow[d, "S^a\otimes\kappa"] \\ E \otimes S^a \otimes N \arrow[r, "\tau\otimes N"] & S^a \otimes E \otimes N \arrow[r, "S^a\otimes\kappa"] & S^a \otimes N \end{tikzcd}$$

The top left triangle commutes by coherence for a symmetric monoidal category. The bottom left rectangle and top right trapezoid commute by naturality of τ. Finally, the bottom right square commutes by the coherence condition for κ. □

Lemma 0.13. *Given a monoid object (E, μ, e) in $\mathcal{SH}$, an object X in $\mathcal{SH}$, and some $a \in A$, the suspension of the free module $\Sigma^a(E \otimes X)$ is naturally isomorphic as a left E-module object to the free E-module $E \otimes \Sigma^a X$.*

Proof. It suffices to show the isomorphism $S^a \otimes E \otimes X \xrightarrow{\tau \otimes X} E \otimes S^a \otimes X$ is a homomorphism of left E-module objects. To see this, consider the following diagram:

$$\begin{tikzcd}
E \otimes S^a \otimes E \otimes X \arrow[r, "E\otimes\tau\otimes X"] \arrow[d, "\tau\otimes E\otimes X"'] & E \otimes E \otimes S^a \otimes X \arrow[dd, "\mu\otimes S^a\otimes X"] \\
S^a \otimes E \otimes E \otimes X \arrow[ur, "\tau_{S^a, E\otimes E}\otimes X"'] \arrow[d, "S^a\otimes\mu\otimes X"'] & \\
S^a \otimes E \otimes X \arrow[r, "\tau\otimes X"] & E \otimes S^a \otimes X
\end{tikzcd}$$

The top triangle commutes by coherence for a symmetric monoidal category. The bottom trapezoid commutes by naturality of τ. □

Proposition 0.14. *Let (E, μ, e) be a monoid object in $\mathcal{SH}$, and suppose we have a family of left E-module objects (N_i, κ_i) indexed by some small set I. Then $N := \bigoplus_{i \in I} N_i$ is canonically a left E-module, with action map given by the composition*

$$\kappa : E \otimes \bigoplus_i N_i \xrightarrow{\cong} \bigoplus_i (E \otimes N_i) \xrightarrow{\bigoplus_i \kappa_i} \bigoplus_i N_i,$$

*where the first isomorphism is given by the fact that $E \otimes -$ preserves coproducts, since it is a left adjoint as $\mathcal{SH}$ is monoidal closed. Furthermore, N is the coproduct of all the N_i's in E-**Mod**, so that E-**Mod** has arbitrary coproducts.*

Proof. We need to show the action map κ makes the diagrams in Definition 0.6 commute. To see the first (unitality) diagram commutes, consider the following diagram:

$$\begin{tikzcd}
\bigoplus_i N_i \arrow[r, "e\otimes\bigoplus_i N_i"] \arrow[dr, "\bigoplus_i(e\otimes N_i)"'] \arrow[ddr, equal] & E \otimes \bigoplus_i N_i \arrow[d, "\cong"] \\
& \bigoplus_i (E \otimes N_i) \arrow[d, "\bigoplus_i \kappa_i"] \\
& \bigoplus_i N_i
\end{tikzcd}$$

The top triangle commutes by additivity of $E \otimes -$ The bottom triangle commutes by unitality of each of the κ_i's. To see the second coherence diagram commutes, consider the following diagram:

$$\begin{tikzcd}
E \otimes E \otimes \bigoplus_i N_i \arrow[rr, "\mu \oplus \bigoplus_i N_i"] \arrow[d, "E\otimes\cong"'] \arrow[dr, "\cong"] & & E \otimes \bigoplus_i N_i \arrow[d, "\cong"] \\
E \otimes \bigoplus_i (E \otimes N_i) \arrow[r, "\cong"] \arrow[d, "E\otimes\bigoplus_i \kappa_i"'] & \bigoplus_i (E \otimes E \otimes N_i) \arrow[r, "\bigoplus_i (\mu\otimes N_i)"] \arrow[d, "\bigoplus_i (E\otimes\kappa_i)"'] & \bigoplus_i (E \otimes N_i) \arrow[d, "\bigoplus_i \kappa_i"] \\
E \otimes \bigoplus_i N_i \arrow[r, "\cong"'] & \bigoplus_i (E \otimes N_i) \arrow[r, "\bigoplus_i \kappa_i"'] & \bigoplus_i N_i
\end{tikzcd}$$

The bottom right square commutes by coherence for the κ_i's. Every other region commutes by additivity of $- \otimes -$ in each variable. Thus $N = \bigoplus_i N_i$ is indeed a left E-module object, as desired.

Now, we claim that (N, κ) is the coproduct of the (N_i, κ_i)'s in E-**Mod**. First, we need to show that the canonical maps $\iota_i : N_i \hookrightarrow N$ are morphisms in E-**Mod** for all $i \in I$. To see ι_i is a homomorphism of left E-module objects, consider the following diagram:

$$\begin{tikzcd}
E \otimes N_i \arrow[r, "E\otimes\iota_i"] \arrow[dr, hook, "\iota_{E\otimes N_i}"] \arrow[dd, "\kappa_i"'] & E \otimes \bigoplus_i N_i \arrow[d, "\cong"] \\
 & \bigoplus_i (E \otimes N_i) \arrow[d, "\bigoplus_i \kappa_i"] \\
N_i \arrow[r, hook, "\iota_i"'] & \bigoplus_i N_i
\end{tikzcd}$$

The top triangle commutes by additivity of $E \otimes -$. The bottom trapezoid commutes since, by univeral property of the coproduct, $\bigoplus_i \kappa_i$ is the unique arrow which makes the trapezoid commute for all $i \in I$. Now, it remains to show that given a left E-module object (N', κ') and homomorphisms $f_i : N_i \to N'$ of left E-module objects for all $i \in I$, that the unique arrow $f : N \to N'$ in $\mathcal{SH}$ satisfying $f \circ \iota_i = f_i$ for all $i \in I$ is a homomorphism of left E-module objects, so that N is actually the coproduct of the N_i's. To see this, first let $h : \bigoplus_i (E \otimes N_i) \to E \otimes N'$ be the arrow determined by the maps $E \otimes N_i \xrightarrow{E\otimes f_i} E \otimes N'$. Then consider the following diagram:

$$\begin{tikzcd}
E \otimes \bigoplus_i N_i \arrow[rr, "E\otimes f"] \arrow[d, "\cong"'] & & E \otimes N' \arrow[ddd, "\kappa'"] \\
\bigoplus_i (E \otimes N_i) \arrow[urr, "h"] \arrow[r, "\bigoplus_i (E\otimes f_i)"'] \arrow[dd, "\bigoplus_i \kappa_i"'] & \bigoplus_i (E \otimes N') \arrow[ur, "\nabla"'] \arrow[d, "\bigoplus_i \kappa'"] & \\
 & \bigoplus_i N' \arrow[dr, "\nabla"] & \\
\bigoplus_i N_i \arrow[ur, "\bigoplus_i f_i"] \arrow[rr, "f"'] & & N'
\end{tikzcd}$$

The top triangle commutes by additivity of $E \otimes -$. The triangle below that commutes by the universal property of the coproduct, since it is straightforward to check that $\nabla \circ \bigoplus_i (E \otimes f_i)$ and h both satisfy the universal property of the colimit. The left trapezoid commutes by functoriality of $- \oplus -$ and the fact that f_i is a homomorphism of left E-module objects for all i in I. The right trapezoid commutes by naturality of ∇. Finally, the bottom triangle commutes by the universal product of the coproduct, by showing that $\nabla \circ \bigoplus_i f_i$ in place of f also satisfies the universal property of the colimit. Hence f is inded a homomorphism of left E-module objects, as desired.

To recap, we have shown that given a set of left E-module objects $\{(N_i, \kappa_i)\}_{i \in I}$, that the inclusion maps $\iota_i : N_i \hookrightarrow \bigoplus_i N_i$ are morphisms in E-**Mod**, and that given morphism $f_i : (N_i, \kappa_i) \to (N', \kappa')$ for all $i \in I$, the unique induced map $\bigoplus_i N_i \to N'$ is a morphism in E-**Mod**. Thus, E-**Mod** does indeed have arbitrary coproducts, and the forgetful functor $E\text{-}\mathbf{Mod} \to \mathcal{SH}$ preserves them. $\square$

Proposition 0.15. *Let (E, μ, e) be a monoid object in $\mathcal{SH}$. Then E-**Mod** is an additive category, so that in particular the forgetful functor $E\text{-}\mathbf{Mod} \to \mathcal{SH}$ and the free functor $\mathcal{SH} \to E\text{-}\mathbf{Mod}$ are additive.*

Proof. It is a general fact that adjoint functors between additive categories are necessarily additive. In order to show E-**Mod** is an additive category, it suffices to show it has finite coproducts, that $\mathrm{Hom}_{E\text{-}\mathbf{Mod}}(N, N')$ is an abelian group for all left E-modules N and N', and that composition is bilinear. We know that E-**Mod** has coproducts which are preserved by the forgetful functor $E\text{-}\mathbf{Mod} \to \mathcal{SH}$ by Proposition 0.14 (which is clearly faithful). Thus, because $\mathcal{SH}$ is **Ab**-enriched and $\mathrm{Hom}_{E\text{-}\mathbf{Mod}}(N, N') \subseteq \mathcal{SH}(N, N')$, it suffices to show that $\mathrm{Hom}_{E\text{-}\mathbf{Mod}}(N, N')$ is closed under addition and taking inverses. To see the former, let $f, g : N \to N'$ be left E-module homomorphisms, and consider the following diagram:

$$\begin{tikzcd}
E \otimes N \arrow[r, "E\otimes\Delta_N"] \arrow[dr, "\Delta_{E\otimes N}"'] \arrow[dddd, "\kappa"'] & E \otimes (N \oplus N) \arrow[rr, "E\otimes(f\oplus g)"] \arrow[d, "\cong"'] & & E \otimes (N' \oplus N') \arrow[r, "E\otimes\nabla_{N'}"] \arrow[d, "\cong"] & E \otimes N' \arrow[dddd, "\kappa'"] \\
& (E \otimes N) \oplus (E \otimes N) \arrow[rr, "(E\otimes f)\oplus(E\otimes g)"] \arrow[dr, "(E\otimes f)\oplus(E\otimes N)"'] \arrow[ddd, "\kappa\oplus\kappa"'] & & (E \otimes N') \otimes (E \otimes N') \arrow[ur, "\nabla_{E\otimes N'}"'] \arrow[ddd, "\kappa'\oplus\kappa'"] & \\
& & (E \otimes N') \otimes (E \otimes N) \arrow[ur, "(E\otimes N')\oplus(E\otimes g)"'] \arrow[d, "\kappa'\oplus\kappa"] & & \\
& & N' \oplus N \arrow[dr, "N'\oplus g"] & & \\
N \arrow[r, "\Delta_N"] & N \oplus N \arrow[rr, "f\oplus g"] \arrow[ur, "f\oplus N"] & & N' \oplus N' \arrow[r, "\nabla_{N'}"] & N'
\end{tikzcd}$$

The outermost trapezoids commute by naturality of Δ and ∇. The triangles in the top corners and the top middle rectangle commute by additivity of $E \otimes -$. The middle triangle commutes by functoriality of $\oplus$ and $\otimes$. The middle trapezoids commute by the fact that f and g are homomorphisms of left E-modules. Finally, the middle bottom triangle commutes by functoriality of $- \oplus -$. Commutativity of the above diagram shows that $f + g$ is a homomorphism of left E-modules as desired. Finally, to see $-f$ is a left E-module homomorphism if f is, we would like to show that $\kappa' \circ (E \otimes (-f)) = (-f) \circ \kappa$. This follows by the fact that $\kappa' \circ (E \otimes f) = f \circ \kappa$ and additivity of $- \otimes -$ and composition. $\square$

Definition 0.16. Let (E, μ, e) be a monoid object in $\mathcal{SH}$, and suppose (N, κ) and (N', κ') are left E-module objects in E-**Mod**. Then the hom-sets in E-**Mod** can be extended to A-graded abelian groups $\mathrm{Hom}^*_{E\text{-}\mathbf{Mod}}(N, N')$, by defining

$$\mathrm{Hom}^a_{E\text{-}\mathbf{Mod}}(N, N') := \mathrm{Hom}_{E\text{-}\mathbf{Mod}}(\Sigma^a N, N')$$

for each $a \in A$ (where $\Sigma^* N$ has the left E-module structure given by Lemma 0.12).

Proposition 0.17. *The assignment $(E, \mu, e) \mapsto \pi_*(E)$ is a functor π_* from the category $\mathbf{Mon}_{\mathcal{SH}}$ of monoid objects in $\mathcal{SH}$ (Definition 0.2) to the category of A-graded rings. In particular, given a monoid object (E, μ, e) in $\mathcal{SH}$, $\pi_*(E)$ is canonically a ring with product $\pi_*(E) \times \pi_*(E) \to \pi_*(E)$ which sends classes $x : S^a \to E$ and $y : S^b \to E$ to the composition*

$$xy : S^{a+b} \xrightarrow{\phi_{a,b}} S^a \otimes S^b \xrightarrow{x\otimes y} E \otimes E \xrightarrow{\mu} E,$$

and the unit of this ring is $e \in \pi_0(E) = [S, E]$.

Proof. First, we show that $\pi_*(E)$ is actually a ring as indicated. By **??**, in order to make the A-graded abelian group $\pi_*(E)$ into an A-graded ring, it suffices to construct an associative and unital product only with respect to homogeneous elements. Suppose we have classes x, y, and z in $\pi_a(E)$, $\pi_b(E)$, and $\pi_c(E)$, respectively. To see associativity, consider the following diagram:

$$\begin{tikzcd}
 & & & E\otimes E \arrow[d, "\mu"] \\
S^{a+b+c} \arrow[r, "\cong"] & S^a\otimes S^b\otimes S^c \arrow[r, "x\otimes y\otimes z"] & E\otimes E\otimes E \arrow[ur, "\mu\otimes E"] \arrow[dr, "E\otimes\mu"'] & E \\
 & & & E\otimes E \arrow[u, "\mu"']
\end{tikzcd}$$

(here the first arrow is the unique isomorphism obtained by composing products of $\phi_{a,b}$'s, see **??**). It commutes by associativity of μ. It follows by functoriality of $-\otimes-$ that the top composition is $(x \cdot y) \cdot z$ while the bottom is $x \cdot (y \cdot z)$, so they are equal as desired. To see that $e \in \pi_0(E)$ is a left and right unit for this multiplication, consider the following diagram

$$\begin{tikzcd}
 & S^a \arrow[dl, "e\otimes x"'] \arrow[d, "x"] \arrow[dr, "x\otimes e"] & \\
E\otimes E \arrow[dr, "\mu"'] & E \arrow[l, "e\otimes E"'] \arrow[r, "E\otimes e"] \arrow[d, equal] & E\otimes E \arrow[dl, "\mu"] \\
 & E &
\end{tikzcd}$$

Commutativity of the two top triangles is functoriality of $-\otimes-$. Commutativity of the bottom two triangles is unitality of μ. Thus the diagram commutes, so $e \cdot x = x = x \cdot e$. Finally, we wish to show this product is bilinear (distributive). Suppose we further have some $x' \in \pi_a(E)$ and $y' \in \pi_b(E)$, and consider the following diagrams:

$$\begin{tikzcd}
S^{a+b} \arrow[r, "\phi_{a,b}"] \arrow[d, "\Delta"'] & S^a\otimes S^b \arrow[r, "\Delta\otimes S^b"] \arrow[d, "\Delta"] & (S^a\oplus S^a)\otimes S^b \arrow[r, "(x\oplus x')\otimes y"] \arrow[dl, "\cong"'] & (E\oplus E)\otimes E \arrow[dl, "\cong"'] \arrow[d, "\nabla\otimes E"] & \\
S^{a+b}\oplus S^{a+b} \arrow[r, "\phi_{a,b}\oplus\phi_{a,b}"'] & (S^a\otimes S^b)\oplus(S^a\otimes S^b) \arrow[r, "(x\otimes y)\oplus(x'\otimes y)"'] & (E\otimes E)\oplus(E\otimes E) \arrow[r, "\nabla"'] & E\otimes E \arrow[r, "\mu"] & E
\end{tikzcd}$$

$$\begin{tikzcd}
S^{a+b} \arrow[r, "\phi_{a,b}"] \arrow[d, "\Delta"'] & S^a\otimes S^b \arrow[r, "S^a\otimes\Delta"] \arrow[d, "\Delta"] & S^b\otimes(S^b\oplus S^b) \arrow[r, "x\otimes(y\oplus y')"] \arrow[dl, "\cong"'] & E\otimes(E\oplus E) \arrow[dl, "\cong"'] \arrow[d, "E\otimes\nabla"] & \\
S^{a+b}\oplus S^{a+b} \arrow[r, "\phi_{a,b}\oplus\phi_{a,b}"'] & (S^a\otimes S^b)\oplus(S^a\otimes S^b) \arrow[r, "(x\otimes y)\oplus(x\otimes y')"'] & (E\otimes E)\oplus(E\otimes E) \arrow[r, "\nabla"'] & E\otimes E \arrow[r, "\mu"] & E
\end{tikzcd}$$

The unlabeled isomorphisms are those given by the fact that $-\otimes-$ is additive in each variable (since $\mathcal{SH}$ is tensor triangulated). Commutativity of the left squares is naturality of $\Delta : X \to X \oplus X$ in an additive category. Commutativity of the rest of the diagram follows again from the fact that $-\otimes-$ is an additive functor in each variable. Hence, by functoriality of $-\otimes-$, these diagrams tell us that $(x+x')\cdot y = x\cdot y + x'\cdot y$ and $x\cdot(y+y') = x\cdot y + x\cdot y'$, respectively. Thus, we have shown that if (E,μ,e) is a monoid object in $\mathcal{SH}$ then $\pi_*(E)$ is a ring, as desired.

It remains to show that given a homomorphism of monoid objects $f:(E_1,\mu_1,e_1)\to(E_2,\mu_2,e_2)$ in $\mathbf{Mon}_{\mathcal{SH}}$ that $\pi_*(f):\pi_*(E_1)\to\pi_*(E_2)$ is an A-graded ring homomorphism. First of all, we know this is an A-graded abelian group homomorphism, since $\mathcal{SH}$ is an additive category, meaning composition with f is an abelian group homomorphism. Thus, in order to show it's a ring homomorphism, it remains to show that $\pi_*(f)(e_1)=e_2$ and that for all $x,y\in\pi_*(E)$ we have $\pi_*(f)(x\cdot y)=\pi_*(f)(x)\cdot\pi_*(f)(y)$. To see the former, note that $\pi_*(f)(e_1)=f\circ e_1$, and $f\circ e_1=e_2$ since f is a monoid homomorphism in $\mathcal{SH}$. To see the latter, first note by distributivity of multiplication in $\pi_*(E_1)$ and $\pi_*(E_2)$ and the fact that $\pi_*(f)$ is a group homomorphism, it suffices to consider the case that x and y are homogeneous of the form $x:S^a\to E_1$ and $y:S^b\to E_2$. In this case, consider the following diagram:

$$\begin{tikzcd}
S^{a+b} \arrow[r, "\phi_{a,b}"] & S^a\otimes S^b \arrow[r, "x\otimes y"] & E_1\otimes E_1 \arrow[r, "f\otimes f"] \arrow[d, "\mu_1"'] & E_2\otimes E_2 \arrow[d, "\mu_2"] \\
& & E_1 \arrow[r, "f"] & E_2
\end{tikzcd}$$

The top composition is $\pi_*(f)(x)\cdot\pi_*(f)(y)$, while the bottom composition is $\pi_*(f)(x\cdot y)$. The diagram commutes since f is a monoid object homomorphism. Thus $\pi_*(f)(x\cdot y)=\pi_*(f)(x)\cdot\pi_*(f)(y)$, as desired. $\square$

Proposition 0.18. *For all $a,b\in A$ there exists an element $\theta_{a,b}\in\pi_0(S)=[S,S]$ such that given any commutative monoid object (E,μ,e) in $\mathcal{SH}$, the A-graded ring structure on $\pi_*(E)$ (**??**) has a commutativity formula given by*

$$x\cdot y = y\cdot x\cdot(e\circ\theta_{a,b})$$

for all $x\in\pi_a(E)$ and $y\in\pi_b(E)$. In particular, $\theta_{a,b}\in\mathrm{Aut}(S)$ is the composition

$$S\xrightarrow{\cong}S^{-a-b}\otimes S^a\otimes S^b\xrightarrow{S^{-a-b}\otimes\tau}S^{-a-b}\otimes S^b\otimes S^a\xrightarrow{\cong}S,$$

*where the outermost maps are the unique maps specified by **??**.*

Proof. Let (E,μ,e), x, and y as in the statement of the proposition. Now consider the following diagram

$$\begin{tikzcd}
S^{a+b} \arrow[r, "\phi_{a,b}"] \arrow[d, dashed, "\phi_{b,a}^{-1}\circ\tau\circ\phi_{a,b}"'] & S^a\otimes S^b \arrow[r, "x\otimes y"] \arrow[d, "\tau"'] & E\otimes E \arrow[d, "\tau"'] \arrow[dr, "\mu"] & \\
& & & E \\
S^{a+b} \arrow[r, "\phi_{b,a}"] & S^b\otimes S^a \arrow[r, "y\otimes x"] & E\otimes E \arrow[ur, "\mu"'] &
\end{tikzcd}$$

The left square commutes by definition. The middle square commutes by naturality of the symmetry isomorphism. Finally, the right square commutes by commutativity of E. Unravelling definitions, we have shown that under the product on $\pi_*(E)$ induced by the $\phi_{a,b}$'s,

$$x\cdot y=(y\cdot x)\circ(\phi_{b,a}^{-1}\circ\tau\circ\phi_{a,b}).$$

Thus, in order to show the desired result it further suffices to show that

$$(y\cdot x)\circ(\phi_{b,a}^{-1}\circ\tau\circ\phi_{a,b})=y\cdot x\cdot(e\circ\theta_{a,b}).$$

Consider the following diagram:

$$\begin{tikzcd}
S^{a+b} \arrow[rr, "\phi_{a,b}"] \arrow[d, "\cong"'] & & S^a \otimes S^b \arrow[d, "\tau"] \\
S^b \otimes S^a \otimes S^{-a-b} \otimes S^a \otimes S^b \arrow[urr, "\cong"'] \arrow[d, "S^b \otimes S^a \otimes S^{-a-b} \otimes \tau"'] & & S^b \otimes S^a \arrow[dl, equal] \arrow[d, "\phi^{-1}_{b,a}"] \\
S^b \otimes S^a \otimes S^{-a-b} \otimes S^b \otimes S^a \arrow[r, "\cong"] \arrow[urr, "\cong"] & S^b \otimes S^a \arrow[dr, "y \otimes x"] \arrow[ddl, "y \otimes x \otimes e"'] & S^{a+b} \arrow[l, "\phi_{b,a}"'] \\
& & E \otimes E \arrow[dll, "E \otimes E \otimes e"'] \arrow[d, equal] \\
E \otimes E \otimes E \arrow[rr, "E \otimes \mu"] \arrow[d, "\mu \otimes E"'] & & E \otimes E \arrow[d, "\mu"] \\
E \otimes E \arrow[rr, "\mu"'] & & E
\end{tikzcd}$$

Here any map simply labelled $\cong$ is an appropriate composition of copies of $\phi_{a,b}$'s, associators, and their inverses, so that each of these maps are necessarily unique by **??**. The triangles in the top large rectangle commutes by coherence for the $\phi_{a,b}$'s. The parallelogram commutes by naturality of τ and coherence of the of $\phi_{a,b}$'s. The middle skewed triangle commutes by functoriality of $- \otimes -$. The triangle below that commutes by unitality of μ. Finally, the bottom rectangle commmutes by associativity of μ. Hence, by unravelling definitions and applying functoriality of $- \otimes -$, we get that the right composition is $(y \cdot x) \circ (\phi^{-1}_{b,a} \circ \tau \circ \phi_{a,b})$, while the left composition is $y \cdot x \cdot (e \circ \theta_{a,b})$, so they are equal as desired. $\square$

Lemma 0.19. *Suppose we have homogeneous elements $x, y \in \pi_*(S)$ with x of degree 0, then we have $x \cdot y = y \cdot x = x \circ y$ (where the $\cdot$ denotes the product given in Proposition 0.17).*

Proof. As morphisms, y is an arrow $S^a \to S$ for some a in A, and x is a morphism $S \to S$. Then consider the following diagram:

$$\begin{tikzcd}
S \otimes S^a \arrow[dr, "S \otimes y"] \arrow[dd, "y \otimes x"'] & & S^a \arrow[ll, "\phi_{0,a} = \lambda^{-1}_{S^a}"'] \arrow[rr, "\phi_{a,0} = \rho^{-1}_{S^a}"] \arrow[d, "y"'] & & S^a \otimes S \arrow[dl, "y \otimes S"'] \arrow[dd, "x \otimes y"] \\
& S \otimes S \arrow[r, "\lambda_S = \rho_S"] \arrow[dl, "x \otimes S"'] & S \arrow[d, "x"'] & S \otimes S \arrow[l, "\rho_S = \lambda_S"'] \arrow[dr, "S \otimes x"] & \\
S \otimes S \arrow[rr, "\phi^{-1}_{0,0} = \rho_S"'] & & S & & S \otimes S \arrow[ll, "\phi^{-1}_{0,0} = \lambda_S"]
\end{tikzcd}$$

The trapezoids commute by naturality of the unitors, and the triangles commute by functoriality of $- \otimes -$. The outside compositions are $y \cdot x$ on the left and $x \cdot y$ on the right, and the middle composition is $x \circ y$, so indeed we have $y \cdot x = x \cdot y = x \circ y$, as desired. $\square$

Lemma 0.20. *Given $a \in A$, we have $\theta_{0,a} = \theta_{a,0} = \mathrm{id}_S$.*

Proof. Recall $\theta_{a,0}$ is the composition

$$S \xrightarrow{\phi_{-a,a}} S^{-a} \otimes S^a \xrightarrow{S^{-a} \otimes \phi_{a,0}} S^{-a} \otimes (S^a \otimes S) \xrightarrow{S^{-a} \otimes \tau} S^{-a} \otimes (S \otimes S^a) \xrightarrow{S^{-a} \otimes \phi^{-1}_{0,a}} S^{-a} \otimes S^a \xrightarrow{\phi^{-1}_{-a,a}} S$$

By the coherence theorem for symmetric monoidal categories and the fact that $\phi_{a,0}$ and $\phi_{0,a}$ coincide with the unitors, we have that the composition

$$S^a \xrightarrow{\phi_{a,0}=\rho_{S^a}^{-1}} S^a \otimes S \xrightarrow{\tau} S \otimes S^a \xrightarrow{\phi_{0,a}^{-1}=\lambda_{S^a}} S^a$$

is precisely the identity map, so by functoriality of $- \otimes -$, we have that $\theta_{a,0}$ is the composition

$$S \xrightarrow{\phi_{-a,a}} S^{-a} \otimes S^a \xrightarrow{=} S^{-a} \otimes S^a \xrightarrow{\phi_{-a,a}^{-1}} S,$$

so $\theta_{a,0} = \mathrm{id}_S$, meaning

$$x \cdot y = y \cdot x \cdot (e \circ \theta_{a,0}) = y \cdot x \cdot e = y \cdot x,$$

where the last equality follows by the fact that e is the unit for the multiplication on $\pi_*(E)$. An entirely analagous argument yields that $\theta_{0,a} = \mathrm{id}_S$. ☐

Lemma 0.21. *Let $a, b \in A$. Then $\theta_{a,b} \cdot \theta_{b,a} = \mathrm{id}_S$.*

Proof. By Lemma 0.19, it suffices to show that $\theta_{a,b} \circ \theta_{b,a} = \mathrm{id}_S$. To see this, consider the following diagram:

$$\begin{tikzcd}
S \arrow[r, "\phi"] \arrow[dddrrr, equal] & S^{-a-b} \otimes S^b \otimes S^a \arrow[r, "S^{-a-b}\otimes\tau"] \arrow[ddrr, equal] & S^{-a-b} \otimes S^a \otimes S^b \arrow[r, "\phi"] \arrow[dr, equal] & S \arrow[d, "\phi"] \\
 & & & S^{-a-b} \otimes S^a \otimes S^b \arrow[d, "S^{-a-b}\otimes\tau"] \\
 & & & S^{-a-b} \otimes S^b \otimes S^a \arrow[d, "\phi"] \\
 & & & S
\end{tikzcd}$$

Here we are suppressing associators, and any map labelled ϕ is the appropriate composition of $\phi_{a,b}$'s, unitors, associators, identities, and their inverses (see **??**). Clearly each region commutes, the middle by the fact that $\tau^2 = 0$, and the other two regions by coherence for the ϕ's. Thus we have shwon $\theta_{a,b} \cdot \theta_{b,a} = \theta_{a,b} \cdot \theta_{b,a} = \mathrm{id}_S$, as desired. ☐

Lemma 0.22. *Let $a, b, c \in A$. Then $\theta_{a,b} \cdot \theta_{a,c} = \theta_{a,b+c}$ and $\theta_{b,a} \cdot \theta_{c,a} = \theta_{b+c,a}$.*

Proof. By Lemma 0.19, it suffices to show that $\theta_{a,b} \circ \theta_{a,c} = \theta_{a,b+c}$ and $\theta_{b,a} \circ \theta_{c,a} = \theta_{b+c,a}$. First we show $\theta_{a,b} \circ \theta_{a,c} = \theta_{a,b+c}$. To see this, consider the following diagram:
(1)

$$\begin{tikzcd}
S \arrow[r, "\phi"] \arrow[d, "\phi"'] \arrow[dr, phantom, "\text{(A)}"] & S^{-a-c}S^aS^c \arrow[r, "S^{-a-c}\tau"] \arrow[d, "\phi"] \arrow[dr, phantom, "\text{(B)}"] & S^{-a-c}S^cS^a \arrow[r, "\phi"] \arrow[d, "\phi"] \arrow[dr, phantom, "\text{(C)}"] & S \arrow[d, "\phi"] \\
S^{-a-b-c}S^aS^{b+c} \arrow[r, "\phi"] \arrow[d, "S^{-a-b-c}\tau"'] \arrow[dr, phantom, "\text{(D)}"] & S^{-a-c}S^{-b}S^aS^bS^c \arrow[r, "S^{-a-c}\tau_{S^{-b}S^aS^b,S^c}"] \arrow[d, "S^{-a-c}S^{-b}\tau_{S^a,S^bS^c}"'] \arrow[dr, phantom, "\text{(E)}"] & S^{-a-c}S^cS^{-b}S^aS^b \arrow[d, "S^{-a-c}S^cS^{-b}\tau"'] \arrow[dr, phantom, "\text{(F)}"] & S^{-a-b}S^aS^b \arrow[l, "\phi"'] \arrow[d, "S^{-a-b}\tau"] \\
S^{-a-b-c}S^{b+c}S^a \arrow[r, "\phi"] \arrow[dd, "\phi"'] & S^{-a-c}S^{-b}S^bS^cS^a \arrow[r, "S^{-a-c}\tau_{S^{-b}S^b,S^c}S^a"] \arrow[dr, phantom, "\text{(G)}"] & S^{-a-c}S^cS^{-b}S^bS^a & S^{-a-b}S^bS^a \arrow[l, "\phi"'] \arrow[dd, "\phi"] \\
 & S^{-a-c}S^cS^a \arrow[u, "\phi"] \arrow[r, equal] \arrow[dr, phantom, "\text{(H)}"] & S^{-a-c}S^cS^a \arrow[u, "\phi"'] & \\
S \arrow[rrr, equal] & & & S
\end{tikzcd}$$

Here we are omitting $\otimes$ from the notation, and each occurrence of an arrow labelled ϕ indicates it is the unique arrow that can be obtained as a formal composition of tensor products of copies of $\phi_{a,b}$'s, unitors, associators, and their inverses (**??**). Clearly the composition going around the top and then the right is $\theta_{a,b} \circ \theta_{a,c}$ while the composition going left around the bottom is $\theta_{a,b+c}$. Thus, we wish to show the above diagram commutes.

Regions (A), (C), and (H) commute by coherence for the ϕ's (see previous remark). Region (E) commutes by coherence for the τ's. To see region (B) commutes, consider the following diagram, which commutes by naturality of τ:

$$\begin{tikzcd}
S^{-a-c}S^aS^c \arrow[r, "S^{-a-c}\tau"] \arrow[d, "S^{-a-c}\phi_{a-b,b}S^c"'] & S^{-a-c}S^cS^a \arrow[d, "S^{-a-c}S^c\phi_{a-b,b}"] \\
S^{-a-c}S^{a-b}S^bS^c \arrow[r, "S^{-a-c}\tau_{S^{a-b}S^b,S^c}"] \arrow[d, "S^{-a-c}\phi_{-b,a}S^bS^c"'] & S^{-a-c}S^cS^{a-b}S^b \arrow[d, "S^{-a-c}S^c\phi_{-b,a}S^b"] \\
S^{-a-c}S^{-b}S^aS^bS^c \arrow[r, "S^{-a-c}\tau_{S^{-b}S^aS^b,S^c}"] & S^{-a-c}S^cS^{-b}S^aS^b
\end{tikzcd}$$

To see region (D) commutes, note that it is simply the square

$$\begin{tikzcd}
S^{-a-b-c}S^aS^{b+c} \arrow[r, "\phi_{-a-c,-b}S^a\phi_{b,c}"] \arrow[d, "S^{-a-b-c}\tau"'] & S^{-a-c}S^{-b}S^aS^bS^c \arrow[d, "S^{-a-c}S^{-b}\tau_{S^a,S^bS^c}"] \\
S^{-a-b-c}S^{b+c}S^a \arrow[r, "\phi_{-a-c,-b}\phi_{b,c}S^a"] & S^{-a-c}S^{-b}S^bS^cS^a
\end{tikzcd}$$

This diagram commutes by naturality of τ. To see region (F) commutes, consider the following diagram, which commutes by functoriality of $- \otimes -$:

$$\begin{tikzcd}
S^{-a-c}S^cS^{-b}S^aS^b \arrow[d, "S^{-a-c}S^cS^{-b}\tau"'] & S^{-a-c}S^{c-b}S^aS^b \arrow[l, "S^{-a-c}\phi_{c,-b}S^aS^b"'] \arrow[d, "S^{-a-c}S^{c-b}\tau"] & S^{-a-b}S^aS^b \arrow[l, "\phi_{-a-c,c-b}S^aS^b"'] \arrow[d, "S^{-a-b}\tau"] \\
S^{-a-c}S^cS^{-b}S^bS^a & S^{-a-c}S^{c-b}S^bS^a \arrow[l, "S^{-a-c}\phi_{c,-b}S^bS^a"'] & S^{-a-b}S^bS^a \arrow[l, "\phi_{-a-c,c-b}S^bS^a"']
\end{tikzcd}$$

Finally, to see region (G) commutes, consider the following diagram:

$$\begin{tikzcd}
S^{-a-c}S^{-b}S^bS^cS^a \arrow[r, "S^{-a-c}\tau_{S^{-b}S^b,S^c}S^a"] & S^{-a-c}S^cS^{-b}S^bS^a \\
S^{-a-c}SS^cS^a \arrow[u, "S^{-a-c}\phi_{-b,b}S^cS^a"] \arrow[r, "S^{-a-c}\tau_{S,S^c}S^a"] & S^{-a-c}S^cSS^a \arrow[u, "S^{-a-c}S^c\phi_{-b,b}S^a"'] \\
S^{-a-c}S^cS^a \arrow[u, "S^{-a-c}\phi_{0,c}S^a=S^{-a-c}\lambda_{S^c}^{-1}S^a"] \arrow[r, equal] & S^{-a-c}S^cS^a \arrow[u, "S^{-a-c}\phi_{c,0}S^a=S^{-a-c}S\rho_{S^c}^{-1}S^a"']
\end{tikzcd}$$

The top region commutes by naturality of τ, while the bottom region commutes by coherence for a symmetric monoidal category. Thus, we have shown that diagram (1) commutes, so that $\theta_{a,b} \circ \theta_{a,c} = \theta_{a,b+c}$, as desired. Now, to see that $\theta_{b,a} \cdot \theta_{c,a} = \theta_{b+c,a}$, note that

$$\theta_{b,a} \cdot \theta_{c,a} \overset{(*)}{=} \theta_{a,b}^{-1} \cdot \theta_{a,c}^{-1} = (\theta_{a,c} \cdot \theta_{a,b})^{-1} = \theta_{a,b+c}^{-1} \overset{(*)}{=} \theta_{b+c,a},$$

where each occurrence of $(*)$ is Lemma 0.21. $\square$

Lemma 0.23. *Let X and Y be objects in $\mathcal{SH}$. Then the A-graded pairing*

$$\pi_*(X) \times \pi_*(Y) \to \pi_*(X \otimes Y)$$

sending $x : S^a \to X$ and $y : S^b \to Y$ to the composition

$$S^{a+b} \xrightarrow{\phi_{a,b}} S^a \otimes S^b \xrightarrow{x \otimes y} X \otimes Y$$

is additive in each argument.

Proof. Let $a, b \in A$, and let $x_1, x_2 : S^a \to X$ and $y : S^b \to Y$. Then consider the following diagram

$$\begin{tikzcd}
S^{a+b} \arrow[r, "\cong"] & S^a \otimes S^b \arrow[r, "\Delta \otimes S^b"] \arrow[d, "\Delta"'] & (S^a \oplus S^a) \otimes S^b \arrow[dl, "\cong"'] \arrow[d, "(x_1 \oplus x_2) \otimes y"] \\
& (S^a \otimes S^b) \oplus (S^a \otimes S^b) \arrow[d, "(x_1 \otimes y) \oplus (x_2 \otimes y)"'] & (X \oplus X) \otimes Y \arrow[dl, "\cong"'] \arrow[d, "\nabla \otimes Y"] \\
& (X \otimes Y) \oplus (X \otimes Y) \arrow[r, "\nabla"] & X \otimes Y
\end{tikzcd}$$

The isomorphisms are given by the fact that $- \otimes -$ is additive in each variable. Both triangles and the parallelogram commute since $- \otimes -$ is additive. By functoriality of $- \otimes -$, the top composition is $(x_1 + x_2) \cdot y$ and the bottom composition is $x_1 \cdot y + x_2 \cdot y$, so they are equal, as desired. An entirely analagous argument yields that $x \cdot (y_1 + y_2) = x \cdot y_1 + x \cdot y_2$ for $x \in \pi_*(X)$ and $y_1, y_2 \in \pi_*(Y)$. $\square$

Lemma 0.24. *Let (E, μ, e) be a monoid object. Then the assignment $\pi_* : (N, \kappa) \mapsto \pi_*(N)$ yields an additive functor from E-**Mod** to A-graded left $\pi_*(E)$-modules. In particular, if (N, κ) is a left E-module in $\mathcal{SH}$ then the map*

$$\pi_*(E) \times \pi_*(N) \to \pi_*(N)$$

sending a class $r : S^a \to E$ and $x : S^b \to N$ to the composition

$$r \cdot x : S^{a+b} \xrightarrow{\phi_{a,b}} S^a \otimes S^b \xrightarrow{r \otimes x} E \otimes N \xrightarrow{\kappa} N$$

endows $\pi_*(N)$ with the structure of an A-graded left $\pi_*(E)$-module.

Proof. First let (N, κ) be an E-module object. Let $a, b, c \in A$ and $x, x' : S^a \to N$, $y : S^b \to E$, and $z, z' \in S^c \to E$. Then by **??**, it suffices to show that

(1) $y \cdot (x + x') = y \cdot x + y \cdot x'$,

(2) $(z + z') \cdot x = z \cdot x + z' \cdot x$,

(3) $(zy) \cdot x = z \cdot (y \cdot x)$,

(4) $e \cdot x = x$.

The first two axioms follow by Lemma 0.23. To see (3), consider the diagram:

$$\begin{tikzcd}
 & & & E \otimes N \arrow[d, "\kappa"] \\
S^{a+b+c} \arrow[r, "\cong"] & S^c \otimes S^b \otimes S^a \arrow[r, "z\otimes y \otimes x"] & E \otimes E \otimes N \arrow[ur, "E\otimes\kappa"] \arrow[dr, "\mu\otimes N"'] & N \\
 & & & E \otimes N \arrow[u, "\kappa"']
\end{tikzcd}$$

It commutes by coherence for κ. By functoriality of $- \otimes -$, the two outside compositions equal $z \cdot (y \cdot x)$ on the top and $(z \cdot y) \cdot x$ on the bottom. Hence, they are equal, as desired.

Next, to see (4), consider the following diagram:

$$\begin{tikzcd}
S^a \arrow[rr, "x"] \arrow[dr, "x"] \arrow[ddr, bend right, "e\otimes x"'] & & N \\
 & N \arrow[ur, equal] \arrow[d, dashed, "e\otimes N"] & \\
 & E \otimes N \arrow[uur, bend right, "\kappa"'] &
\end{tikzcd}$$

The top triangle commutes by definition. The left triangle commutes by functoriality of $- \otimes -$. The right triangle commutes by unitality of κ. The top composition is x while the bottom is $e \cdot x$, thus they are necessarily equal since the diagram commutes.

Now, we'd like to show that if $f : (N, \kappa) \to (N', \kappa)$ is a homomorphism of left E-module objects, then $\pi_*(f) : \pi_*(N) \to \pi_*(N')$ is a homomorphism of left $\pi_*(E)$-modules. To see this, let $r : S^a \to E$ in $\pi_a(E)$ and $x, x : S^b \to N$ in $\pi_b(N)$. We'd like to show that $\pi_*(f)(x + x') = \pi_*(f)(x) + \pi_*(f)(x')$ and $\pi_*(f)(r \cdot x) = r \cdot \pi_*(f)(x)$. To see the former, consider the following diagram:

$$\begin{tikzcd}
 & & & N' \oplus N' \arrow[d, "\nabla"] \\
S^a \arrow[r, "\Delta"] & S^a \oplus S^a \arrow[r, "x \oplus x'"] & N \oplus N \arrow[ur, "f \oplus f"] \arrow[dr, "\nabla"'] & N' \\
 & & & N \arrow[u, "f"']
\end{tikzcd}$$

It commutes by naturality of ∇ in an additive category. The top composition is $\pi_*(f)(x) + \pi_*(f)(x')$, while the bottom is $\pi_*(f)(x+x')$, so they are equal as desired. To see that $\pi_*(f)(r \cdot x) = r \cdot \pi_*(f)(x)$, consider the following diagram:

$$\begin{tikzcd}
 & & & E \otimes N' \arrow[d, "\kappa'"] \\
S^{a+b} \arrow[r, "\phi_{b,a}"] & S^b \otimes S^a \arrow[r, "r \otimes x"] & E \otimes N \arrow[ur, "E \otimes f"] \arrow[dr, "\kappa"'] & N' \\
 & & & N \arrow[u, "f"']
\end{tikzcd}$$

It commutes by the fact that f is a homomorphism of left E-module objects. The bottom composition is $\pi_*(f)(r \cdot x)$, while the top composition is $r \cdot \pi_*(f)(x)$, so they are equal, as desired.

It remains to show this functor is additive. It suffices to show it preserves the zero object and preserves coproducts. To see the former, note that $\pi_*(0) = [S^*, 0] = 0$ by definition, since 0 is terminal. To see the latter, we need to show that given $(N, \kappa), (N', \kappa') \in E\text{-}\mathbf{Mod}$ that $\pi_*(N) \oplus \pi_*(N') \cong \pi_*(N \oplus N')$, and that the following diagram commutes:

$$\begin{tikzcd}
\pi_*(N) \arrow[d, "\iota_{\pi_*(N)}"'] \arrow[dr, "\pi_*(\iota_N)"] & \\
\pi_*(N) \oplus \pi_*(N') \arrow[r, "\cong"'] & \pi_*(N \oplus N')
\end{tikzcd}$$

Since each S^a is compact, for all $a, b \in A$ we have isomorphisms

$$\pi_a(N) \oplus \pi_a(N') = [S^a, N] \oplus [S^a, N'] \cong [S^a, N \oplus N'] = \pi_a(N, N'),$$

and these combine together to yield A-graded isomorphisms $\pi_*(N) \oplus \pi_*(N') \xrightarrow{\cong} \pi_*(N \oplus N')$. To see the above diagram commutes, note that since everything is an A-graded homomorphism of A-graded abelian groups, it suffices to chase homogeneous elements around to show it commutes. Indeed, it is entirely straightforward, by unravelling definitions, that both compositions around the diagram take a generator $x : S^a \to N$ in $\pi_a(N)$ to the composition

$$S^a \xrightarrow{x} N \xrightarrow{\iota_N} N \oplus N'.$$

Thus, we have shown that π_* preserves all finite coproducts, so it is additive. $\square$

Proposition 0.25. *Let (E, μ, e) be a monoid object in $\mathcal{SH}$. Then $E_*(-)$ is a functor from $\mathcal{SH}$ to left A-graded modules over the ring $\pi_*(E)$ (Proposition 0.17), where given some X in $\mathcal{SH}$, $E_*(X)$ may be endowed with the structure of a left A-graded $\pi_*(E)$-module via the map*

$$\pi_*(E) \times E_*(X) \to E_*(X)$$

which given $a, b \in A$, sends $x : S^a \to E$ and $y : S^b \to E \otimes X$ to the composition

$$x \cdot y : S^{a+b} \cong S^a \otimes S^b \xrightarrow{x \otimes y} E \otimes (E \otimes X) \cong (E \otimes E) \otimes X \xrightarrow{\mu \otimes X} E \otimes X.$$

Similarly, the assignment $X \mapsto X_*(E)$ is a functor from $\mathcal{SH}$ to right A-graded $\pi_*(E)$-modules, where the structure map

$$X_*(E) \times \pi_*(E) \to X_*(E)$$

sends $x : S^a \to X \otimes E$ and $y : S^b \to E$ to the composition

$$x \cdot y : S^{a+b} \cong S^a \otimes S^b \xrightarrow{x \otimes y} (X \otimes E) \otimes E \cong X \otimes (E \otimes E) \xrightarrow{X \otimes \mu} X \otimes E.$$

Finally, $E_*(E)$ is a $\pi_*(E)$-bimodule, in the sense that the left and right actions of $\pi_*(E)$ are compatible, so that given $y, z \in \pi_*(E)$ and $x \in E_*(E)$, $y \cdot (x \cdot z) = (y \cdot x) \cdot z$.

Proof. By **??**, in order to make the A-graded abelian group $E_*(X)$ into a left A-graded module over the A-graded ring $\pi_*(E)$, it suffices to define the action map $\pi_*(E) \times E_*(X) \to E_*(X)$ only for homogeneous elements, and to show that given homogeneous elements $x, x' : S^a \to E \otimes X$ in $E_a(X)$, $y : S^b \to E$ in $\pi_b(E)$, and $z, z' : S^c \to E$ in $\pi_c(E)$, that:

(1) $y \cdot (x + x') = y \cdot x + y \cdot x'$,

(2) $(z + z') \cdot x = z \cdot x + z' \cdot x$,

(3) $(zy) \cdot x = z \cdot (y \cdot x)$,

(4) $e \cdot x = x$.

Axioms (1) and (2) follow by the fact that $E_*(X) = \pi_*(E \otimes X)$ and Lemma 0.23. To see (3), consider the diagram:

$$\begin{tikzcd}
 & & & E \otimes E \otimes X \arrow[d, "\mu \otimes X"] \\
S^{a+b+c} \arrow[r, "\cong"] & S^c \otimes S^b \otimes S^a \arrow[r, "z \otimes y \otimes x"] & E \otimes E \otimes E \otimes X \arrow[ur, "E \otimes \mu \otimes X"] \arrow[dr, "\mu \otimes E \otimes X"'] & E \otimes X \\
 & & & E \otimes E \otimes X \arrow[u, "\mu \otimes X"']
\end{tikzcd}$$

It commutes by associativity of μ. By functoriality of $- \otimes -$, the two outside compositions equal $z \cdot (y \cdot x)$ on the top and $(z \cdot y) \cdot x$ on the bottom. Hence, they are equal, as desired.

Next, to see (4), consider the following diagram:

$$\begin{tikzcd}
S^a \arrow[rr, "x"] \arrow[dr, "x"'] \arrow[ddr, bend right, "e \otimes x"'] & & E \otimes X \\
 & E \otimes X \arrow[ur, equal] \arrow[d, dashed, "e \otimes E \otimes X"] & \\
 & E \otimes E \otimes X \arrow[uur, bend right, "\mu \otimes X"'] &
\end{tikzcd}$$

The top triangle commutes by definition. The left triangle commutes by functoriality of $- \otimes -$. The right triangle commutes by unitality of μ. The top composition is x while the bottom is $e \cdot x$, thus they are necessarily equal since the diagram commutes.

Thus, we have shown that the indicated map does indeed endow $E_*(X)$ with the structure of a left $\pi_*(E)$-module. It remains to show that $E_*(-)$ sends maps in $\mathcal{SH}$ to A-graded homomorphisms of left A-graded $\pi_*(E)$-modules. By definition, given $f : X \to Y$ in $\mathcal{SH}$, $E_*(f)$ is the map which takes a class $x : S^a \to E \otimes X$ to the composition

$$S^a \xrightarrow{x} E \otimes X \xrightarrow{E \otimes f} E \otimes Y.$$

To see this assignment is a homomorphism, suppose we are given some other $x' : S^a \to E \otimes X$ and some scalar $y : S^b \to E$. Then we would like to show $E_*(f)(x + x') = E_*(f)(x) + E_*(f)(x')$ and $E_*(f)(y \cdot x) = y \cdot E_*(f)(x)$. To see the former, consider the following diagram:

$$\begin{tikzcd}
 & & & (E\otimes Y)\oplus(E\otimes Y) \arrow[d, "\nabla"] \\
S^a \arrow[r, "\Delta"] & S^a \oplus S^a \arrow[r, "x\oplus x'"] & (E\otimes X)\oplus(E\otimes X) \arrow[ur, "(E\otimes f)\oplus(E\otimes f)"] \arrow[dr, "\nabla"'] & E\otimes Y \\
 & & & E\otimes X \arrow[u, "E\otimes f"']
\end{tikzcd}$$

It commutes by naturality of ∇ in an additive category. The top composition is $E_*(f)(x) + E_*(f)(x')$, while the bottom is $E_*(f)(x+x')$, so they are equal as desired. To see that $E_*(f)(y \cdot x) = y \cdot E_*(f)(x)$, consider the following diagram:

$$\begin{tikzcd}
S^{a+b} \arrow[r, "\phi_{b,a}"] & S^b \otimes S^a \arrow[r, "y\otimes x"] & E\otimes E\otimes X \arrow[r, "E\otimes E\otimes f"] \arrow[d, "\mu\otimes X"'] & E\otimes E\otimes Y \arrow[d, "\mu\otimes Y"] \\
 & & E\otimes X \arrow[r, "E\otimes f"] & E\otimes Y
\end{tikzcd}$$

It commutes by functoriality of $- \otimes -$. The top composition is $E_*(f)(y \cdot x)$, while the bottom composition is $y \cdot E_*(f)(x)$, so they are equal, as desired.

Showing that $X_*(E)$ has the structure of a right $\pi_*(E)$-module and that if $f : X \to Y$ is a morphism in $\mathcal{SH}$ then the map

$$X_*(E) = [S^*, X \otimes E] \xrightarrow{(f\otimes E)_*} [S^*, Y \otimes E] = Y_*(E)$$

is an A-graded homomorphism of right A-graded $\pi_*(E)$-modules is entirely analagous.

It remains to show that $E_*(E)$ is a $\pi_*(E)$-bimodule. Let $x : S^a \to E$, $y : S^b \to E \otimes E$, and $z : S^c \to E$, and consider the following diagram:

$$\begin{tikzcd}
 & & & E\otimes E\otimes E \arrow[d, "E\otimes\mu"] \\
S^{a+b+c} \arrow[r, "\cong"] & S^a \otimes S^b \otimes S^c \arrow[r, "x\otimes y\otimes z"] & E\otimes E\otimes E\otimes E \arrow[ur, "\mu\otimes E\otimes E"] \arrow[r, "\mu\otimes\mu"] \arrow[dr, "E\otimes E\otimes\mu"'] & E\otimes E \\
 & & & E\otimes E\otimes E \arrow[u, "\mu\otimes E"']
\end{tikzcd}$$

Commutativity follows by functoriality of $- \otimes -$, which also tells us that the two outside compositions are $(x \cdot y) \cdot z$ (on top) and $x \cdot (y \cdot z)$ (on bottom). Hence they are equal, as desired. $\square$

Lemma 0.26. *Let (E, μ, e) be a monoid object in $\mathcal{SH}$, (N, κ) a left E-module, and $a \in A$. Then the assignment*

$$\mathrm{tw}^a : \pi_{*-a}(N) \to \pi_*(\Sigma^a N)$$

sending $x : S^{b-a} \to N$ to the composition

$$S^b \xrightarrow{\phi_{b-a,a}} S^{b-a} \otimes S^a \xrightarrow{x\otimes S^a} N \otimes S^a \xrightarrow{\tau} S^a \otimes N = \Sigma^a N$$

is an A-graded isomorphism of left A-graded $\pi_*(E)$-modules (where here $\pi_*(N)$ is a left $\pi_*(E)$-module by Lemma 0.24, and $\pi_*(\Sigma^a N)$ has the left $\pi_*(E)$ module given by Lemma 0.12 and Lemma 0.24).

Proof. Unravelling definitions, the map $\mathrm{tw}^a : \pi_{*-a}(N) \to \pi_*(\Sigma^a N)$ factors as

$$\pi_{*-a}(N) = [S^{*-a}, N] \xrightarrow{-\otimes S^a} [S^{*-a} \otimes S^a, N \otimes S^a] \xrightarrow{(\phi_{*-a,a})^*} [S^*, N \otimes S^a] \xrightarrow{\tau_*} [S^*, S^a \otimes N] = \pi_*(\Sigma^a N).$$

The arrow labeled $- \otimes S^a$ is an isomorphism of abelian groups because $- \otimes S^a \cong \Sigma^a$ is an autoequivalence of $\mathcal{SH}$ (**??**). Hence, we have shown the map is an isomorphism of abelian groups. Clearly the map preserves degree, so it is an A-graded homomorphism as desired. Finally, it remains to show that this map is a homomorphism of left $\pi_*(E)$-modules, i.e., that given $r : S^b \to E$ in $\pi_*(E)$ and $x : S^{c-a} \to N$ in $\pi_{*-a}(N)$ that $\mathrm{tw}^a(r \cdot x) = r \cdot \mathrm{tw}^a(x)$. Unravelling definitions, $\mathrm{tw}^a(r \cdot x)$ is the composition

$$S^{b+c} \xrightarrow{\cong} S^b \otimes S^{c-a} \otimes S^a \xrightarrow{r \otimes x \otimes S^a} E \otimes N \otimes S^a \xrightarrow{\kappa \otimes S^a} N \otimes S^a \xrightarrow{\tau} S^a \otimes N,$$

while on the other hand $r \cdot \mathrm{tw}^a(x)$ is the composition

$$S^{b+c} \xrightarrow{\cong} S^b \otimes S^{c-a} \otimes S^a \xrightarrow{r \otimes x \otimes S^a} E \otimes N \otimes S^a \xrightarrow{E \otimes \tau} E \otimes S^a \otimes N \xrightarrow{\tau \otimes N} S^a \otimes E \otimes N \xrightarrow{S^a \otimes \kappa} S^a \otimes N.$$

To see these are equal, consider the following diagram:

$$\begin{tikzcd}
 & & E \otimes S^a \otimes N \arrow[r, "\tau \otimes N"] & S^a \otimes E \otimes N \arrow[d, "S^a \otimes \kappa"] \\
S^{b+c} \arrow[r, "\phi"] & S^b \otimes S^{c-a} \otimes S^a \arrow[r, "r \otimes x \otimes S^a"] & E \otimes N \otimes S^a \arrow[u, "E \otimes \tau"] \arrow[ur, "\tau_{E \otimes N, S^a}"] \arrow[dr, "\kappa \otimes S^a"'] & S^a \otimes N \\
 & & & N \otimes S^a \arrow[u, "\tau"']
\end{tikzcd}$$

The top triangle commutes by coherence for a symmetric monoidal category, while the right triangle commutes by naturality of τ. ☐

Lemma 0.27. *Let (E, μ, e) be a monoid object and (N, κ) a left E-module object in $\mathcal{SH}$. Then given a collection of $a_i \in A$ indexed by some set I, if (N, κ) is a retract of $\bigoplus_i (E \otimes S^{a_i})$ in E-**Mod**,[1] then for all left E-module objects (N', κ') in $\mathcal{SH}$, the functor π_* (Lemma 0.24) induces an isomorphism*

$$\pi_* : \mathrm{Hom}_{E\text{-}\mathbf{Mod}}(N, N') \to \mathrm{Hom}_{\pi_*(E)}(\pi_*(N), \pi_*(N')).$$

Proof. First, we consider the case $N = \bigoplus_i (E \otimes S^{a_i})$. Consider the following diagram:

$$\begin{tikzcd}
\mathrm{Hom}_{E\text{-}\mathbf{Mod}}(\bigoplus_i (E \otimes S^{a_i}), N') \arrow[r, "\pi_*"] \arrow[d, "\cong"'] & \mathrm{Hom}_{\pi_*(E)}(\pi_*(\bigoplus_i (E \otimes S^{a_i})), \pi_*(N')) \arrow[d, "\cong"] \\
\prod_i \mathrm{Hom}_{E\text{-}\mathbf{Mod}}(E \otimes S^{a_i}, N') \arrow[d, "\cong"'] & \mathrm{Hom}_{\pi_*(E)}(\bigoplus_i \pi_*(E \otimes S^{a_i}), \pi_*(N')) \arrow[d, "\cong"] \\
\prod_i [S^{a_i}, N'] \arrow[dd, equal] & \prod_i \mathrm{Hom}_{\pi_*(E)}(\pi_*(E \otimes S^{a_i}), \pi_*(N')) \arrow[d, "\cong"] \\
 & \prod_i \mathrm{Hom}_{\pi_*(E)}(\pi_{*-a_i}(E), \pi_*(N')) \arrow[d, equal] \\
\prod_i \pi_{a_i}(N') & \prod_i \mathrm{Hom}^{a_i}_{\pi_*(E)}(\pi_*(E), \pi_*(N')) \arrow[l, "\prod_i \mathrm{ev}_1"']
\end{tikzcd}$$

[1]Here $\bigoplus_i (E \otimes S^{a_i})$ is a coproduct (Proposition 0.14) of a bunch of left free E-module objects (Proposition 0.10), so it is itself a left E-module object.

Here the top left vertical isomorphism exibits the universal property of the coproduct in E-**Mod**, and middle left vertical isomorphism below that is the free-forgetful adjunction for E-modules (Proposition 0.10). The bottom horizontal isomorphism is the product of the evaluation-at-1 isomorphisms (**??**). On the other side, the top right vertical isomorphism is given by the fact that S^a is compact for each $a \in A$, so we have isomorphisms

$$\bigoplus_i \pi_*(E \otimes S^{a_i}) = \bigoplus_{a \in A} \bigoplus_i [S^a, E \otimes S^{a_i}] \cong \bigoplus_{a \in A} [S^a, \bigoplus_i (E \otimes S^{a_i})] = \pi_*(\bigoplus_i (E \otimes S^{a_i})),$$

where the middle isomorphism takes a generator $x : S^a \xrightarrow{E} \otimes S^{a_i}$ to the composition $S^a \xrightarrow{x} E \otimes S^{a_i} \hookrightarrow \bigoplus_i (E \otimes S^{a_i})$. The middle right vertical isomorphism exhibits the universal property of the coproduct of modules. Finally the bottom right vertical isomorphism is given by the isomorphisms

$$\pi_{*-a_i}(E \otimes S^{a_i}) = [S^{*-a_i}, E \otimes S^{a_i}] \xrightarrow{-\otimes S^{a_i}} [S^{*-a_i} \otimes S^{a_i}, E \otimes S^{a_i}] \xrightarrow{\phi^*} [S^*, E \otimes S^{a_i}] = \pi_*(E \otimes S^{a_i}),$$

where $- \otimes S^{a_i} \cong \Sigma^{a_i}$ is an isomorphism by **??**. Now, we claim this diagram commutes. This really simply amounts to unravelling definitions, and chasing a homomorphism $f : \bigoplus_i (E \otimes S^{a_i}) \to N'$ of left E-module objects both ways around the diagram yields the composition

$$\prod_i (S^{a_i} \xrightarrow{e \otimes S^{a_i}} E \otimes S^{a_i} \hookrightarrow \bigoplus_i (E \otimes S^{a_i}) \xrightarrow{f} N').$$

Thus, since the diagram commutes, we have that

$$\pi_* : \mathrm{Hom}_{E\text{-}\mathbf{Mod}}(\bigoplus_i (E \otimes S^{a_i}), N') \to \mathrm{Hom}_{\pi_*(E)}(\pi_*(\bigoplus_i (E \otimes S^{a_i})), \pi_*(N'))$$

is an isomorphism, as desired.

Now, consider the case that N is a retract of $\bigoplus_i (E \otimes S^{a_i})$ in E-**Mod**, so there exists a commuting diagram of left E-module object homomorphisms:

$$\begin{tikzcd} N \arrow[r, "\iota"'] \arrow[rr, bend left, equal] & \bigoplus_i (E \otimes S^{a_i}) \arrow[r, "r"'] & N \end{tikzcd}$$

Now consider the following diagram:

$$\begin{tikzcd} \mathrm{Hom}^*_{E\text{-}\mathbf{Mod}}(N, N') \arrow[r, "r^*"] \arrow[d, "\pi_*"'] \arrow[rr, bend left, equal] & \mathrm{Hom}^*_{E\text{-}\mathbf{Mod}}(\bigoplus_i (E \otimes S^{a_i}), N') \arrow[r, "\iota^*"] \arrow[d, "\pi_*"] & \mathrm{Hom}^*_{E\text{-}\mathbf{Mod}}(N, N') \arrow[d, "\pi_*"'] \\ \mathrm{Hom}^*_{\pi_*(E)}(\pi_*(N), \pi_*(N')) \arrow[r, "(\pi_*(r))^*"] \arrow[rr, bend right, equal] & \mathrm{Hom}^*_{\pi_*(E)}(\pi_*(\bigoplus_i (E \otimes S^{a_i})), \pi_*(N')) \arrow[r, "(\pi_*(\iota))^*"] & \mathrm{Hom}^*_{\pi_*(E)}(\pi_*(N), \pi_*(N')) \end{tikzcd}$$

Each square commutes by functoriality of π_*. We have shown the middle vertical arrow is an isomorphism. Thus the outside arrows are isomorphisms as well, as a retract of an isomorphism is an isomorphism. ☐

Proposition 0.28. *Let (E, μ, e) be a monoid object and X an object in $\mathcal{SH}$. If there is a collection of $a_i \in A$ indexed by some set I such that $E \otimes X$ is a retract of $\bigoplus_i (E \otimes S^{a_i})$ in E-**Mod**,*[2] *then for all left E-module objects (N, κ) the assignment*

$$\Psi : [X, N]_* \to \mathrm{Hom}^*_{\pi_*(E)}(E_*(X), \pi_*(N))$$

[2]Here $\bigoplus_i (E \otimes S^{a_i})$ is a coproduct (Proposition 0.14) of a bunch of left free E-module objects (Proposition 0.10), so it is itself a left E-module object.

sending $f : S^a \otimes X \to N$ *to the map* $E_{*-a}(X) \to \pi_*(N)$ *which sends a class* $x : S^{b-a} \to E \otimes X$ *to the composition*

$$\Psi(f)(x) : S^b \xrightarrow{\phi} S^{b-a} \otimes S^a \xrightarrow{x \otimes S^a} E \otimes X \otimes S^a \xrightarrow{E \otimes \tau} E \otimes S^a \otimes X \xrightarrow{E \otimes f} E \otimes N \xrightarrow{\kappa} N$$

is an A*-graded isomorphism of* A*-graded abelian groups.*

Proof. Clearly as constructed, assuming $\Psi(f)$ as defined is actually a homomorphism of left $\pi_*(E)$-modules, this map is A-graded. Thus, it suffices to show that for all $a \in A$, the restriction

$$\Psi_a : [X, N]_a \to \mathrm{Hom}^a_{\pi_*(E)}(E_*(X), \pi_*(N))$$

is an isomorphism. First of all, note that Ψ_a factors as

$$\begin{tikzcd}
{[X,N]_a} \arrow[r, equal] & {[\Sigma^a X, N]} \arrow[d, "\cong"] & \\
& \mathrm{Hom}_{E\text{-}\mathbf{Mod}}(E \otimes \Sigma^a X, N) \arrow[d, "\cong"] & \\
& \mathrm{Hom}_{E\text{-}\mathbf{Mod}}(\Sigma^a(E \otimes X), N) \arrow[d, "\pi_*"] & \\
& \mathrm{Hom}_{\pi_*(E)}(\pi_*(\Sigma^a(E \otimes X)), \pi_*(N)) \arrow[d, "(\mathrm{tw}^a)^*"] & \\
& \mathrm{Hom}_{\pi_*(E)}(E_{*-a}(X), \pi_*(N)) \arrow[r, equal] & \mathrm{Hom}^a_{\pi_*(E)}(E_*(X), \pi_*(N))
\end{tikzcd}$$

where the first isomorphism is the free-forgetful adjunction for E-modules (Proposition 0.10), the second isomorphism is given by Lemma 0.13, the third map is that induced by the functor π_* constructed in Lemma 0.24, and the final map is induced by the isomorphism $\mathrm{tw}^a : \pi_{*-a}(E \otimes X) \xrightarrow{\cong} \pi_*(\Sigma^a(E \otimes X))$ (Lemma 0.26). Unravelling definitions, this composition sends a class $f : S^a \otimes X \to N$ to the map $E_{*-a}(X) \to \pi_*(N)$ which sends a class $x : S^{b-a} \to E \otimes X$ to the composition

$$S^b \xrightarrow{\cong} S^{b-a} \otimes S^a \xrightarrow{x \otimes S^a} E \otimes X \otimes S^a \xrightarrow{\tau_{E \otimes X, S^a}} S^a \otimes E \otimes X \xrightarrow{\tau \otimes X} E \otimes S^a \otimes X \xrightarrow{E \otimes f} E \otimes N \xrightarrow{\kappa} N,$$

and clearly this equals $\Psi(f)(x)$, by coherence for the symmetries. Thus, it suffices to show that

$$\pi_* : \mathrm{Hom}_{E\text{-}\mathbf{Mod}}(\Sigma^a(E \otimes X), N) \to \mathrm{Hom}_{\pi_*(E)}(\pi_*(\Sigma^a(E \otimes X)), \pi_*(N))$$

is an isomorphism when $E \otimes X$ is a retract of $\bigoplus_i (E \otimes S^{a_i})$ in E-$\mathbf{Mod}$. This is precisely Lemma 0.27. $\square$

Proposition 0.29. *Let* (E, μ, e) *be a monoid object and* (N, κ) *a left* E*-module object in* $\mathcal{SH}$*. Further suppose that* E *and* N *are cellular and that* $\pi_*(N)$ *is a* graded projective *(**??**) left* $\pi_*(E)$*-module (Lemma 0.24). Then given some homogeneous generating set* $\{x_i\}_{i \in I} \subseteq \pi_*(N)$*,* N *is a retract of* $\bigoplus_i (E \otimes S^{|x_i|})$ *in* E-$\mathbf{Mod}$.[3]

Proof. Let $M := \bigoplus_i (E \otimes S^{|x_i|})$. We have a map

$$r : M \to N$$

[3]Here $\bigoplus_i (E \otimes S^{a_i})$ is a coproduct (Proposition 0.14) of a bunch of left free E-module objects (Proposition 0.10), so it is itself a left E-module object.

induced by the maps

$$r_i : E \otimes S^{|x_i|} \xrightarrow{E\otimes x_i} E \otimes N \xrightarrow{\kappa} N.$$

This is a homomorphism of left E-module objects:

$$\begin{tikzcd}
E \otimes \bigoplus_i (E \otimes S^{|x_i|}) \arrow[rr, "E\otimes r"] \arrow[dd, "\cong"'] \arrow[dr, "E\otimes\bigoplus_i r_i"] & & E \otimes N \arrow[dddd, "\kappa"] \\
& E \otimes \bigoplus_i N \arrow[ur, "E\otimes\nabla"] \arrow[d, "\cong"'] & \\
\bigoplus_i (E \otimes E \otimes S^{|x_i|}) \arrow[r, "\bigoplus_i (E\otimes r_i)"] \arrow[dd, "\bigoplus_i (\mu\otimes S^{|x_i|})"'] & \bigoplus_i (E \otimes N) \arrow[uur, "\nabla"'] \arrow[d, "\bigoplus_i \kappa"] & \\
& \bigoplus_i N \arrow[dr, "\nabla"] & \\
\bigoplus_i (E \otimes S^{|x_i|}) \arrow[ur, "\bigoplus_i r_i"] \arrow[rr, "r"] & & N
\end{tikzcd}$$

The right trapezoid commutes by naturality of ∇. The bottom triangle commutes by the fact that $\nabla \circ \bigoplus_i r_i$ and r satisfy the same unviersal property for the coproduct. Every other region commutes by additivity of $E \otimes -$, except the left trapezoid: Note that by expanding out how r_i is defined, it becomes

$$\begin{tikzcd}
\bigoplus_i (E \otimes E \otimes S^{|x_i|}) \arrow[r, "\bigoplus_i (E\otimes E\otimes x_i)"] \arrow[d, "\bigoplus_i (\mu\otimes S^{|x_i|})"'] & \bigoplus_i (E \otimes E \otimes N) \arrow[r, "\bigoplus_i (E\otimes\kappa)"] \arrow[d, "\bigoplus_i (\mu\otimes X)"] & \bigoplus_i (E \otimes E \otimes X) \arrow[d, "\bigoplus_i \kappa"] \\
\bigoplus_i (E \otimes S^{|x_i|}) \arrow[r, "\bigoplus_i (E\otimes x_i)"'] & \bigoplus_i (E \otimes N) \arrow[r, "\bigoplus_i \kappa"'] & \bigoplus_i (E \otimes X)
\end{tikzcd}$$

The left square commutes by functoriality of $- \otimes -$, and the right square commutes by coherence for κ. Hence, we've shown that r is a homomorphism of left E-modules, as desired. Thus, r induces a homomorphism of left $\pi_*(E)$-modules $\pi_*(r) \in \mathrm{Hom}_{\pi_*(E)}(\pi_*(M), \pi_*(N)))$ Further note that for all $i \in I$, x_i is in the image of $\pi_*(r)$, as by definition $\pi_*(r)$ sends the class

$$S^{|x_i|} \xrightarrow{e\otimes S^{|x_i|}} E \otimes S^{|x_i|} \hookrightarrow M$$

in $\pi_{|x_i|}(M)$ to the composition

$$S^{|x_i|} \xrightarrow{e\otimes S^{|x_i|}} E \otimes S^{|x_i|} \xrightarrow{E\otimes x_i} E \otimes N \xrightarrow{\kappa} N,$$

and by unitality of κ this composition is simply $x_i : S^{|x_i|} \to N$. Thus, we have constructed a surjective A-graded homomorphism $\pi_*(r) : \pi_*(M) \to \pi_*(N)$ of left $\pi_*(E)$-modules, so that since $\pi_*(N)$ is projective graded module there exists an A-graded left $\pi_*(E)$-module homomorphism $\iota : \pi_*(N) \to \pi_*(M)$ which makes the following diagram commute:

$$\begin{tikzcd}
& \pi_*(M) \arrow[d, "\pi_*(r)"] \\
\pi_*(N) \arrow[ur, "\iota"] \arrow[r, equal] & \pi_*(N)
\end{tikzcd}$$

which further induces the corresponding idempotent of left $\pi_*(E)$-modules:

$$\pi_*(M) \xrightarrow{\pi_*(r)} \pi_*(N) \xrightarrow{\iota} \pi_*(M)$$

Now, by Lemma 0.27, since $M = \bigoplus_i (E \otimes S^{|x_i|})$, we have that this map is actually induced by some endomorphism $\ell : M \to M$ of left E-module objects. Now ℓ splits by **??**, meaning there exists a diagram in $\mathcal{SH}$ of the form

$$\ell : M \xrightarrow{r'} X \xrightarrow{\iota'} M$$

with $r' \circ \iota' = \mathrm{id}_X$. Note that since E and each $S^{|x_i|}$ are cellular, $E \otimes S^{|x_i|}$ is cellular for all $i \in I$ (**??**), so that $M = \bigoplus_i (E \otimes S^{|x_i|})$ is cellular, as by definition an arbitrary coproduct of cellular objects is cellular. Thus by **??** X is cellular as well. Now consider the following commutative diagram

$$\begin{tikzcd}
 & & \pi_*(N) \arrow[rr, equal] \arrow[dr, "\iota"] & & \pi_*(N) \arrow[dr, "\iota"] & & \\
\pi_*(N) \arrow[r, "\iota"'] \arrow[urr, equal, bend left] & \pi_*(M) \arrow[ur, "\pi_*(r)"] \arrow[rr, "\pi_*(\ell)"] \arrow[dr, "\pi_*(r')"'] & & \pi_*(M) \arrow[ur, "\pi_*(r)"] \arrow[rr, "\pi_*(\ell)"] \arrow[dr, "\pi_*(r')"'] & & \pi_*(M) \arrow[r, "\pi_*(r')"] & \pi_*(X) \arrow[dll, equal, bend left] \\
 & & \pi_*(X) \arrow[rr, equal] \arrow[ur, "\pi_*(\iota')"'] & & \pi_*(X) \arrow[ur, "\pi_*(\iota')"'] & &
\end{tikzcd}$$

From this diagram we read off that the middle diagonal composition

$$\pi_*(X) \xrightarrow{\pi_*(\iota')} \pi_*(M) \xrightarrow{\pi_*(r)} \pi_*(N)$$

is an isomorphism with inverse $\pi_*(r') \circ \iota$. Now, since X and N are cellular, and $\pi_*(r \circ \iota')$ is an isomorphism, by **??** we have that $r \circ \iota'$ is an isomorphism, say with inverse p. Thus we have a commuting diagram

$$\begin{tikzcd}
N \arrow[rr, "\iota' \circ p"'] \arrow[rrrr, equal, bend left] \arrow[dr, "p"'] & & M \arrow[rr, "r"'] & & N \\
 & X \arrow[ur, "\iota'"'] & & &
\end{tikzcd}$$

and the middle row exhibits N as a retract of $M = \bigoplus_i (E \otimes S^{|x_i|})$, as desired. $\square$

Corollary 0.30. *Let (E, μ, e) be a monoid object and let X and Y be objects in $\mathcal{SH}$. Then if E and X are cellular and $E_*(X)$ is a graded projective (**??**) left $\pi_*(E)$-module (Proposition 0.25), then the map*

$$\Psi_{X,Y} : [X, E \otimes Y]_* \to \mathrm{Hom}^*_{\pi_*(E)}(E_*(X), E_*(Y))$$

sending $f : S^a \otimes X \to E \otimes Y$ to the map $E_{*-a}(X) \to E_*(Y)$ which sends a class $x : S^{b-a} \to E \otimes X$ to the composition

$$\Psi_{X,Y}(f)(x) : S^b \xrightarrow{\phi} S^{b-a} \otimes S^a \xrightarrow{x \otimes S^a} E \otimes X \otimes S^a \xrightarrow{E \otimes \tau} E \otimes S^a \otimes X \xrightarrow{E \otimes f} E \otimes E \otimes Y \xrightarrow{\mu \otimes Y} E \otimes Y$$

is an A-graded isomorphism of A-graded abelian groups.

Proof. By Proposition 0.29, since $E \otimes X$ is a left E-module object (Definition 0.11), $E_*(X) = \pi_*(E \otimes X)$ is a graded projective left $\pi_*(E)$-module, and $E \otimes X$ is cellular (**??**), it follows that $E \otimes X$ is a retract of $\bigoplus_i (E \otimes S^{a_i})$ in E-**Mod** for some collection of $a_i \in A$ indexed by some set I. Thus the desired result follows by Proposition 0.28 with $N = E \otimes Y$ (which is an E-module by Definition 0.11). $\square$

Proposition 0.31. *Let (E, μ, e) be a monoid object (Definition 0.1) and Z and W be objects in $\mathcal{SH}$. Then there is a homomorphism of abelian groups*

$$\Phi_{Z,W} : \pi_*(Z \otimes E) \otimes_{\pi_*(E)} \pi_*(E \otimes W) \to \pi_*(Z \otimes E \otimes W)$$

which given homogeneous elements $x : S^a \to Z \otimes E$ in $\pi_*(Z \otimes E)$ and $y : S^b \to E \otimes W$ in $\pi_*(E \otimes W)$, sends the homogeneous pure tensor $x \otimes y$ in $\pi_*(Z \otimes E) \otimes_{\pi_*(E)} \pi_*(E \otimes W)$ to the composition

$$S^{a+b} \xrightarrow{\phi_{a,b}} S^a \otimes S^b \xrightarrow{x \otimes y} Z \otimes E \otimes E \otimes W \xrightarrow{Z \otimes \mu \otimes W} Z \otimes E \otimes W$$

(where here we are considering the canonical A-graded right $\pi_*(E)$-module structure on $\pi_*(Z \otimes E) = Z_*(E)$ and the canonical left A-graded $\pi_*(E)$-module structure on $\pi_*(E \otimes W) = E_*(W)$ given in Proposition 0.25, so that $\pi_*(Z \otimes E) \otimes_{\pi_*(E)} \pi_*(E \otimes W)$ is a well-defined A-graded abelian group by **??***). Furthermore, this homomorphism is natural in both Z and W.*

Proof. First, recall by definition of the tensor product, in order to show the assignment $\pi_*(Z \otimes E) \times \pi_*(E \otimes W) \to \pi_*(Z \otimes E \otimes W)$ induces a homomorphism $\pi_*(Z \otimes E) \otimes_{\pi_*(E)} \pi_*(E \otimes W) \to \pi_*(Z \otimes E \otimes W)$ of A-graded abelian groups, it suffices to show that the assignment is $\pi_*(E)$-balanced, i.e., that it is linear in each argument and satisfies $zr \cdot w = z \cdot rw$ for $z \in \pi_*(Z \otimes E)$, $w \in \pi_*(E \otimes W)$, and $r \in \pi_*(E)$.

First, note that by Lemma 0.23 it is straightforward to see that the assignment commutes with addition of maps in each argument. Now, let $a, b, c \in A$, $z : S^a \to Z \otimes E$, $w : S^b \to E \otimes W$, and $r : S^c \to E$. Then we wish to show $zr \cdot w = z \cdot rw$. Consider the following diagram (where here we are passing to a symmetric strict monoidal category):

$$\begin{tikzcd}
 & & & Z \otimes E \otimes E \otimes W \arrow[d, "Z\otimes\mu\otimes W"] \\
S^{a+b+c} \arrow[r, "\cong"] & S^a \otimes S^c \otimes S^b \arrow[r, "z\otimes r\otimes w"] & Z \otimes E \otimes E \otimes E \otimes W \arrow[ur, "Z\otimes\mu\otimes E\otimes W"] \arrow[dr, "Z\otimes E\otimes\mu\otimes W"'] & Z \otimes E \otimes W \\
 & & & Z \otimes E \otimes E \otimes W \arrow[u, "Z\otimes\mu\otimes W"']
\end{tikzcd}$$

It commutes by associativity of μ. By functoriality of $- \otimes -$, the top composition is given by $(zr) \cdot w$ and the bottom composition is $z \cdot (rw)$, so we have they are equal, as desired. Thus, by **??** we get the desired A-graded homomorphism $\pi_*(Z \otimes E) \otimes_{\pi_*(E)} \pi_*(E \otimes W) \to \pi_*(Z \otimes E \otimes W)$.

Next, we would like to show that this homomorphism is natural in Z. Let $f : Z \to Z'$ in $\mathcal{SH}$. Then we would like to show the following diagram commutes:

$$\begin{tikzcd}
\pi_*(Z \otimes E) \otimes_{\pi_*(E)} \pi_*(E \otimes W) \arrow[r, "\Phi_{Z,W}"] \arrow[d, "\pi_*(f\otimes E)\otimes\pi_*(E\otimes W)"'] & \pi_*(Z \otimes E \otimes W) \arrow[d, "\pi_*(f\otimes E\otimes W)"] \\
\pi_*(Z' \otimes E) \otimes_{\pi_*(E)} \pi_*(E \otimes W) \arrow[r, "\Phi_{Z',W}"] & \pi_*(Z' \otimes E \otimes W)
\end{tikzcd} \tag{2}$$

As all the maps here are homomorphisms, in order to show it commutes, it suffices to chase generators around the diagram. In particular, suppose we are given $z : S^a \to Z \otimes E$ and $w : S^b \to E \otimes W$, and consider the following diagram exhibiting the two possible ways to chase $z \otimes w$ around the diagram (as usual, we are passing to a symmetric strict monoidal category):

$$\begin{tikzcd}
S^{a+b} \arrow[r, "\phi_{a,b}"] & S^a \otimes S^b \arrow[r, "z\otimes w"] & Z \otimes E \otimes E \otimes W \arrow[r, "Z\otimes\mu\otimes W"] \arrow[d, "f\otimes E\otimes E\otimes W"'] & Z \otimes E \otimes W \arrow[d, "f\otimes E\otimes W"] \\
 & & Z \otimes E \otimes E \otimes W \arrow[r, "Z\otimes\mu\otimes W"] & Z' \otimes E \otimes W
\end{tikzcd}$$

This diagram commutes by functoriality of $- \otimes -$. Thus we have that diagram (2) does indeed commute, so that $\Phi_{Z,W}$ is natural in Z as desired. Showing that $\Phi_{Z,W}$ is natural in W is entirely analagous. ∎

Lemma 0.32. *Let E and X be objects in $\mathcal{SH}$. Then for all $a \in A$, there is an A-graded isomorphism of A-graded abelian groups*

$$t_X^a : E_*(\Sigma^a X) \cong E_{*-a}(X)$$

which sends a class $S^b \to E \otimes \Sigma^a X = E \otimes S^a \otimes X$ to the composition

$$S^{b-a} \xrightarrow{\phi_{b,-a}} S^b \otimes S^{-a} \xrightarrow{x \otimes S^{-a}} E \otimes S^a \otimes X \otimes S^{-a} \xrightarrow{E \otimes S^a \otimes \tau_{X,S^{-a}}} E \otimes S^a \otimes S^{-a} \otimes X \xrightarrow{E \otimes \phi_{a,-a}^{-1} \otimes X} E \otimes X$$

(where here we are ignoring associators and unitors). Furthermore this isomorphism is natural in X, and if E is a monoid object in $\mathcal{SH}$ then it is a natural isomorphism of $\pi_*(E)$-modules.

Proof. Expressed in terms of hom-sets, t_X^a is precisely the composition

$$\begin{tikzcd}
E_*(\Sigma^a X) \arrow[r, equal] & {[S^*, E \otimes S^a \otimes X]} \arrow[d, "- \otimes S^{-a}"] & \\
& {[S^* \otimes S^{-a}, E \otimes S^a \otimes X \otimes S^{-a}]} \arrow[d, "(\phi_{*,-a})^*"] & \\
& {[S^{*-a}, E \otimes S^a \otimes X \otimes S^{-a}]} \arrow[d, "(E \otimes S^a \otimes \tau)_*"] & \\
& {[S^{*-a}, E \otimes S^a \otimes S^{-a} \otimes X]} \arrow[d, "(E \otimes \phi_{a,-a}^{-1} \otimes X)_*"] & \\
& {[S^{*-a}, E \otimes X]} \arrow[r, equal] & E_{*-a}(E \otimes X)
\end{tikzcd}$$

We know the first vertical arrow is an isomorphism of abelian groups as $- \otimes -$ is additive in each variable (since $\mathcal{SH}$ is tensor triangulated) and $\Omega^a \cong - \otimes S^{-a}$ is an autoequivalence of $\mathcal{SH}$ by **??**. The three other vertical arrows are given by composing with an isomorphism in an additive category, so they are also isomorphisms.

To see t_X^a is a homomorphism of left $\pi_*(E)$-modules, suppose we are given classes $r : S^b \to E$ $\pi_b(E)$ and $x : S^c \to E \otimes S^a \otimes X$ in $E_c(\Sigma^a X)$. Then we wish to show that $t_X^a(r \cdot x) = r \cdot t_X^a(x)$. Consider the following diagram:

$$\begin{tikzcd}
S^{b+c-a} \arrow[d, "\cong"] & E \otimes S^a \otimes X \otimes S^{-a} \arrow[r, "E \otimes S^a \otimes \tau_{X,S^{-a}}"] & E \otimes S^a \otimes S^{-a} \otimes X \arrow[d, "E \otimes \phi_{a,-a}^{-1} \otimes X"] \\
S^b \otimes S^c \otimes S^{-a} \arrow[r, "r \otimes x \otimes S^{-a}"] & E \otimes E \otimes S^a \otimes X \otimes S^{-a} \arrow[u, "\mu \otimes S^a \otimes X \otimes S^{-a}"'] \arrow[d, "E \otimes E \otimes S^a \otimes \tau_{X,S^{-a}}"'] & E \otimes X \\
& E \otimes E \otimes S^a \otimes S^{-a} \otimes X \arrow[r, "E \otimes E \otimes \phi_{a,-a}^{-1} \otimes X"'] \arrow[uur, "\mu \otimes S^a \otimes S^{-a} \otimes X"'] & E \otimes E \otimes X \arrow[u, "\mu \otimes X"']
\end{tikzcd}$$

Both triangles commute by functoriality of $- \otimes -$. The top composition is $t_X^a(r \cdot x)$ while the bottom is $r \cdot t_X^a(x)$, so they are equal as desired.

It remains to show t^a_X is natural in X. let $f : X \to Y$ in $\mathcal{SH}$, then we would like to show the following diagram commutes:

$$\begin{tikzcd}
E_*(\Sigma^a X) \arrow[r, "t^a_X"] \arrow[d, "E_*(\Sigma^a f)"'] & E_{*-a}(X) \arrow[d, "E_{*-a}(f)"] \\
E_*(\Sigma^a Y) \arrow[r, "t^a_Y"] & E_{*-a}(Y)
\end{tikzcd} \tag{3}$$

We may chase a generator around the diagram since all the arrows here are homomorphisms. Let $x : S^b \to E \otimes S^a \otimes X$ in $E_*(\Sigma^a X)$. Then consider the following diagram:

$$\begin{tikzcd}
S^{b-a} \arrow[r, "\cong"] & S^b \otimes S^{-a} \arrow[r, "x\otimes S^{-a}"] & E \otimes S^a \otimes X \otimes S^{-a} \arrow[r, "E\otimes S^a\otimes\tau"] \arrow[d, "E\otimes S^a\otimes f\otimes S^{-a}"'] & E \otimes S^a \otimes S^{-a} \otimes X \arrow[r, "E\otimes\phi^{-1}_{a,-a}\otimes X"] \arrow[d, "E\otimes S^a\otimes S^{-a}\otimes f"'] & E \otimes X \arrow[d, "E\otimes f"] \\
& & E \otimes S^a \otimes Y \otimes S^{-a} \arrow[r, "E\otimes S^a\otimes\tau"'] & E \otimes S^a \otimes S^{-a} \otimes Y \arrow[r, "E\otimes\phi^{-1}_{a,-a}\otimes Y"'] & E \otimes Y
\end{tikzcd}$$

The left rectangle commutes by naturality of τ, while the right rectangle commutes by functoriality of $- \otimes -$. The two outside compositions are the two ways to chase x around diagram (3), so the diagram commutes as desired. $\square$

Lemma 0.33. *Let (E, μ, e) be a monoid object and Z and W objects in $\mathcal{SH}$, and suppose the map $\Phi_{Z,W}$ constructed in Proposition 0.31 is an isomorphism. Then $\Phi_{\Sigma^a Z,W}$ and $\Phi_{Z,\Sigma^a W}$ are isomorphisms for all $a \in A$. In particular, $\Phi_{\Sigma Z,W}$ an $\Phi_{Z,\Sigma W}$ are isomorphisms.*

Proof. First to see $\Phi_{Z,\Sigma^a W}$ is an isomorphism, consider the following diagram

$$\begin{tikzcd}
Z_*(E) \otimes_{\pi_*(E)} E_*(\Sigma^a W) \arrow[r, "Z_*(E)\otimes_{\pi_*(E)} t^W_a"] \arrow[d, "\Phi_{Z,\Sigma^a W}"'] & Z_*(E) \otimes_{\pi_*(E)} E_{*-a}(W) \arrow[d, "\Phi_{Z,W}"] \\
\pi_*(Z \otimes E \otimes \Sigma^a W) = (Z \otimes E)_*(\Sigma^a W) \arrow[r, "t^W_a"] & (Z \otimes E)_{*-a}(W) = Z\pi_{*-a}(Z \otimes E \otimes W)
\end{tikzcd}$$

where the maps t_a are defined in Lemma 0.32. The top arrow is well-defined since $t^W_a : E_*(\Sigma^a W) \to E_{*-a}(W)$ is a degree $-a$ isomorphism of left $\pi_*(E)$-modules by the aforementioned lemma. In order to show the left vertical arrow is an isomorphism, it suffices to show the diagram commutes, as all the other arrows are isomorphisms. To see this, note it suffices to chase a homogeneous pure tensor around the diagram, as all the maps here are homomorphisms. So let $x : S^b \to Z \otimes E$ in $Z_*(E)$ and $y : S^c \to E \otimes S^a \otimes W$ in $E_*(\Sigma^a W)$, and consider the following diagram exhibiting the two ways to chase $x \otimes y$ around:

$$\begin{tikzcd}
S^{b+c-a} \arrow[d, "\phi"] & Z \otimes E \otimes E \otimes S^a \otimes S^{-a} \otimes W \arrow[r, "Z\otimes E\otimes E\otimes\phi^{-1}_{a,-a}\otimes W"] \arrow[ddr, "Z\otimes\mu\otimes S^a\otimes S^{-a}\otimes W"] & Z \otimes E \otimes E \otimes W \arrow[d, "Z\otimes\mu\otimes W"] \\
S^b \otimes S^c \otimes S^{-a} \arrow[r, "x\otimes y\otimes S^{-a}"] & Z \otimes E \otimes E \otimes S^a \otimes W \otimes S^{-a} \arrow[u, "Z\otimes E\otimes E\otimes S^a\otimes\tau"'] \arrow[d, "Z\otimes\mu\otimes S^a\otimes W\otimes S^{-a}"'] & Z \otimes E \otimes W \\
& Z \otimes E \otimes S^a \otimes W \otimes S^{-a} \arrow[r, "Z\otimes E\otimes S^a\otimes\tau"'] & Z \otimes E \otimes S^a \otimes S^{-a} \otimes W \arrow[u, "Z\otimes E\otimes\phi^{-1}_{a,-a}\otimes W"']
\end{tikzcd}$$

Each triangle commutes by functoriality of $- \otimes -$, so the diagram commutes as desired. Thus, we've shown $\Phi_{Z,\Sigma^a W}$ is an isomorphism for all $a \in A$.

On the other hand, in order to see $\Phi_{\Sigma^a Z,W}$ is an isomorphism, consider the following diagram:

$$\begin{tikzcd}
\pi_*(\Sigma^a Z \otimes E) \otimes_{\pi_*(E)} \pi_*(E \otimes W) \arrow[r, "\Phi_{\Sigma^a Z,W}"] \arrow[d, "\mathrm{adj} \otimes_{\pi_*(E)} \pi_*(E \otimes W)"'] & \pi_*(\Sigma^a Z \otimes E \otimes W) \arrow[d, "\mathrm{adj}"] \\
\pi_{*-a}(Z \otimes E) \otimes_{\pi_*(E)} \pi_*(E \otimes W) \arrow[r, "\Phi_{Z,W}"] & \pi_{*-a}(Z \otimes E \otimes W)
\end{tikzcd} \tag{4}$$

Here the left vertical arrow is constructed via the map $\mathrm{adj} : \pi_*(\Sigma^a Z \otimes E) \to \pi_{*-a}(Z \otimes E)$ which is given as as the composition

$$\mathrm{adj} : [S^*, S^a \otimes Z \otimes E] \xrightarrow{\cong} [S^{-a} \otimes S^*, Z \otimes E] \xrightarrow{(\phi^{-1}_{-a,*})^*} [S^{*-a}, Z \otimes E]$$

where the first arrow is the adjunction isomorphism from **??**. Explicitly, this map sends a class $x : S^b \to S^a \otimes Z \otimes E$ to the composition

$$S^{b-a} \xrightarrow{\phi_{-a,b}} S^{-a} \otimes S^b \xrightarrow{S^{-a} \otimes x} S^{-a} \otimes S^a \otimes Z \otimes E \xrightarrow{\phi^{-1}_{-a,a} \otimes Z \otimes E} Z \otimes E.$$

In order to show $\mathrm{adj} \otimes_{\pi_*(E)} \pi_*(E \otimes W)$ is well defined, it suffices to show adj is a degree $-a$ A-graded homomorphism of right A-graded $\pi_*(E)$-modules. It is clearly additive, as any adjunction between additive categories is automatically additive, as is composing with an morphism in an additive category. Thus, it remains to show adj commutes with scalar multiplication. By additivity, it suffices to consider only homogeneous elements. Let $x : S^a \to S^{\mathbf{1}} \otimes Z \otimes E$ in $\pi_*(S^{\mathbf{1}} \otimes Z \otimes E)$ and $r : S^b \to E$ in $\pi_*(E)$. Then we'd like to show that $\mathrm{adj}(x \cdot r) = \mathrm{adj}(x) \cdot r$. To see this, consider the following diagram:

$$\begin{tikzcd}
S^{a+b-\mathbf{1}} \arrow[r, "\phi"] & S^{-\mathbf{1}} \otimes S^a \otimes S^b \arrow[r, "S^{-\mathbf{1}} \otimes x \otimes y"] & S^{-\mathbf{1}} \otimes S^{\mathbf{1}} \otimes Z \otimes E \otimes E \arrow[r, "\phi^{-1}_{-\mathbf{1},\mathbf{1}} \otimes Z \otimes E \otimes E"] \arrow[d, "S^{-\mathbf{1}} \otimes S^{\mathbf{1}} \otimes Z \otimes \mu"'] & Z \otimes E \otimes E \arrow[d, "Z \otimes \mu"] \\
 & & S^{-\mathbf{1}} \otimes S^{\mathbf{1}} \otimes Z \otimes E \arrow[r, "\phi^{-1}_{-\mathbf{1},\mathbf{1}} \otimes Z \otimes E"'] & Z \otimes E
\end{tikzcd}$$

The top composition is $\mathrm{adj}(x) \cdot r$, while the bottom composition is $\mathrm{adj}(x \cdot r)$. The diagram commutes by functoriality of $- \otimes -$. Thus, it follows that $\mathrm{adj}(x) \cdot r = \mathrm{adj}(x \cdot r)$, so that adj is indeed an homomorphism of right $\pi_*(E)$-modules, in fact, an isomorphism as desired. Thus, since every arrow in diagram (4) is an isomorphism of abelian groups except the top arrow, in order to show $\Phi_{\Sigma^a Z,W}$ is an isomorphism, it suffices to show the diagram commutes. To that end, since all the arrows are homomorphisms, it suffices to chase a pure homogeneous tensor. So let $x : S^b \to \Sigma^a Z \otimes E$ and $y : S^c \to E \otimes W$, and consider the following diagram whose outside compositions exhibit the two ways to chase the pure tensor $x \otimes y$ around diagrama (4):

$$\begin{tikzcd}
S^{b+c-a} \arrow[r, "\phi"] & S^{-a} \otimes S^b \otimes S^c \arrow[r, "S^{-a} \otimes x \otimes y"] & S^{-a} \otimes S^a \otimes Z \otimes E \otimes E \otimes W \arrow[r, "S^{-a} \otimes S^a \otimes Z \otimes \mu \otimes W"] \arrow[d, "\phi^{-1}_{-a,a} \otimes Z \otimes E \otimes E \otimes W"'] & S^{-a} \otimes S^a \otimes Z \otimes E \otimes W \arrow[d, "\phi^{-1}_{-a,a} \otimes Z \otimes E \otimes W"] \\
 & & Z \otimes E \otimes E \otimes W \arrow[r, "Z \otimes \mu \otimes W"] & Z \otimes E \otimes W
\end{tikzcd}$$

The diagram clearly commutes by functoriality of $- \otimes -$, so that indeed diagram (4) commutes, so that $\Phi_{\Sigma^a Z,W}$ is indeed an isomorphism as desired.

Now, it remains to show that $\Phi_{Z,\Sigma W}$ and $\Phi_{\Sigma Z,W}$ are isomorphisms. To that end, consider the following diagram:

$$\begin{tikzcd}
\pi_*(Z\otimes E)\otimes_{\pi_*(E)}\pi_*(E\otimes \Sigma W) \arrow[r, "\Phi_{Z,\Sigma W}"] \arrow[d, "\pi_*(Z\otimes E)\otimes_{\pi_*(E)}\pi_*(E\otimes\nu_W)"'] & \pi_*(Z\otimes E\otimes \Sigma W) \arrow[d, "\pi_*(Z\otimes E\otimes\nu_W)"] \\
\pi_*(Z\otimes E)\otimes_{\pi_*(E)}\pi_*(E\otimes \Sigma^{\mathbf{1}} W) \arrow[r, "\Phi_{Z,\Sigma^{\mathbf{1}}W}"'] & \pi_*(Z\otimes E\otimes \Sigma^{\mathbf{1}} W)
\end{tikzcd}$$

It commutes by naturality of Φ. Furthermore, assuming $\Phi_{Z,W}$ is an isomorphism, by what we have shown above we know that $\Phi_{Z,\Sigma^{\mathbf{1}}W}$ is an isomorphism, and since ν_W is an isomorphism, it follows that the above diagram commutes and all arrows except $\Phi_{Z,\Sigma W}$ are isomorphisms, so that $\Phi_{Z,\Sigma W}$ must be an isomorphism itself. Finally, an entirely analagous argument using naturality of Φ with respect to ν_Z yields that $\Phi_{\Sigma Z,W}$ is an isomorphism as well. □

Proposition 0.34. *Let (E,μ,e) be a monoid object in $\mathcal{SH}$. Then if either:*

*(1) $\pi_*(Z\otimes E)=Z_*(E)$ is a flat right $\pi_*(E)$-module (via Proposition 0.25) and W is cellular (**??**), or*

*(2) $\pi_*(E\otimes W)=E_*(W)$ is a flat left $\pi_*(E)$-module (via Proposition 0.25) and Z is cellular (**??**),*

then the natural homomorphism

$$\Phi_{Z,W}:\pi_*(Z\otimes E)\otimes_{\pi_*(E)}\pi_*(E\otimes W)\to\pi_*(Z\otimes E\otimes W)$$

given in Proposition 0.31 is an isomorphism of abelian groups.

Proof. In this proof, we will freely employ the coherence theorem for symmetric monoidal categories, and we will assume that associativity and unitality of $-\otimes-$ holds up to strict equality. First we will consider the case that $\pi_*(Z\otimes E)=Z_*(E)$ is a flat right $\pi_*(E)$-module and W is cellular. To start, let $\mathcal{E}$ be the collection of objects W in $\mathcal{SH}$ for which this map is an isomorphism. Then in order to show $\mathcal{E}$ contains every cellular object, it suffices to show that $\mathcal{E}$ satisfies the three conditions given for the class of cellular objects in **??**. First, we need to show that $\Phi_{Z,W}$ is an isomorphism when $W=S^a$ for some $a\in A$. Indeed, consider the A-graded homomorphism

$$\Psi:\pi_*(Z\otimes E\otimes S^a)\to\pi_*(Z\otimes E)\otimes_{\pi_*(E)}\pi_*(E\otimes S^a)$$

which sends a class $x:S^b\to Z\otimes E\otimes S^a$ in $\pi_b(Z\otimes E\otimes S^a)$ to the pure tensor $\widetilde{x}\otimes\widetilde{e}$, where $\widetilde{x}\in\pi_{b-a}(Z\otimes E)$ is the composition

$$S^{b-a}\xrightarrow{\phi_{b,-a}}S^b\otimes S^{-a}\xrightarrow{x\otimes S^{-a}}Z\otimes E\otimes S^a\otimes S^{-a}\xrightarrow{Z\otimes E\otimes\phi^{-1}_{a,-a}}Z\otimes E$$

and $\widetilde{e}\in\pi_a(E\otimes S^a)$ is the composition

$$S^a\xrightarrow{e\otimes S^a}E\otimes S^a.$$

In order to see Ψ is an (A-graded) homomorphism of abelian groups: Given $x,x'\in\pi_b(Z\otimes E\otimes S^a)$, we would like to show that $\widetilde{x}\otimes\widetilde{e}+\widetilde{x}'\otimes\widetilde{e}=\widetilde{x+x'}\otimes\widetilde{e}$. It suffices to show that $\widetilde{x}+\widetilde{x}'=\widetilde{x+x'}$. To see this, consider the following diagram (again, we are passing to a symmetric strict monoidal category):

$$
\begin{tikzcd}
S^{b-a} \arrow[r, "\Delta"] \arrow[d, "\phi_{b-a}"'] & S^{b-a} \oplus S^{b-a} \arrow[d, "\phi_{b,-a} \oplus \phi_{b,-a}"] \\
S^b \otimes S^{-a} \arrow[r, "\Delta"] \arrow[d, "\Delta \otimes S^{-a}"'] & (S^b \otimes S^{-a}) \oplus (S^b \otimes S^{-a}) \arrow[d, "(x \otimes S^{-a}) \oplus (x' \otimes S^{-a})"] \\
(S^b \oplus S^b) \otimes S^{-a} \arrow[ur, "\cong"] \arrow[d, "(x \oplus x') \otimes S^{-a}"'] & (Z \otimes E \otimes S^a \otimes S^{-a}) \oplus (Z \otimes E \otimes S^a \otimes S^{-a}) \arrow[d, "(Z \otimes E \otimes \phi_{a,-a}^{-1}) \oplus (Z \otimes E \otimes \phi_{a,-a}^{-1})"] \arrow[ddl, "\nabla"] \\
((Z \otimes E \otimes S^a) \oplus (Z \otimes E \otimes S^a)) \otimes S^{-a} \arrow[ur, "\cong"] \arrow[d, "\nabla \otimes S^{-a}"'] & (Z \otimes E) \oplus (Z \otimes E) \arrow[d, "\nabla"] \\
Z \otimes E \otimes S^a \otimes S^{-a} \arrow[r, "Z \otimes E \otimes \phi_{a,-a}^{-1}"'] & Z \otimes E
\end{tikzcd}
$$

The top rectangle commutes by naturality of Δ in an additive category. The bottom triangle commutes by naturality of ∇ in an additive category. Finally, the remaining regions of the diagram commute by additivity of $- \otimes -$. By functoriality of $- \otimes -$, it follows that the left composition is $\widetilde{x + x'}$ and the right composition is $\widetilde{x} + \widetilde{x}'$, so they are equal as desired. Thus Ψ is a homomorphism of abelian groups, as desired.

Now, we claim that Ψ is an inverse to Φ_{Z,S^a}. Since Φ_{Z,S^a} and Ψ are homomorphisms it suffices to check that they are inverses on generators. First, let $x : S^b \to Z \otimes E \otimes S^a$ in $\pi_b(Z \otimes E \otimes S^a)$. We would like to show that $\Phi_{Z,S^a}(\Psi(x)) = x$. Consider the following diagram, where here we are passing to a symmetric strict monoidal category:

$$
\begin{tikzcd}
S^b \arrow[r, "\cong"] \arrow[dd, "x"'] & S^b \otimes S^{-a} \otimes S^a \arrow[d, dashed, "x \otimes S^{-a} \otimes S^a"] \arrow[dr, "x \otimes S^{-a} \otimes e \otimes S^a"] & \\
& Z \otimes E \otimes S^a \otimes S^{-a} \otimes S^a \arrow[r, "Z \otimes E \otimes S^a \otimes S^{-a} \otimes e \otimes S^a"'] & Z \otimes E \otimes S^a \otimes S^{-a} \otimes E \otimes S^a \arrow[ddl, "Z \otimes E \otimes \phi_{a,-a}^{-1} \otimes E \otimes S^a"] \\
Z \otimes E \otimes S^a \arrow[ur, "Z \otimes E \otimes S^a \otimes \phi_{-a,a}"] \arrow[r, equal] & Z \otimes E \otimes S^a \arrow[u, dashed, "Z \otimes E \otimes \phi_{a,-a} \otimes S^a"] \arrow[d, dashed, "Z \otimes E \otimes e \otimes S^a"] & \\
& Z \otimes E \otimes E \otimes S^a \arrow[ul, "Z \otimes \mu \otimes S^a"] &
\end{tikzcd}
$$

The top left trapezoid commutes since the isomorphism $S^b \xrightarrow{\cong} S^b \otimes S^{-a} \otimes S^a$ may be given as $S^b \otimes \phi_{-a,a}$ (see **??**), in which case the trapezoid commmutes by functoriality of $- \otimes -$. The triangle below that commutes by coherence for the $\phi_{a,b}$'s. The bottom left triangle commutes by unitality for μ. The top right triangle commutes by functoriality of $- \otimes -$. Finally, the bottom right triangle commutes by functoriality of $- \otimes -$. It follows by unravelling definitions that the two outside compositions are x and $\Phi_{Z,S^a}(\Psi(x))$, so indeed we have $\Phi_{Z,S^a}(\Psi(x)) = x$ since the diagram commutes.

On the other hand, suppose we are given a homogeneous pure tensor $x \otimes y$ in $\pi_*(Z \otimes E) \otimes_{\pi_*(E)} \pi_*(E \otimes S^a)$, so $x : S^b \to Z \otimes E$ and $y : S^c \to E \otimes S^a$ for some $b, c \in A$. Then we would like to show that $\Psi(\Phi_{Z,S^a}(x \otimes y)) = x \otimes y$. Unravelling definitions, $\Psi(\Phi_{Z,S^a}(x \otimes y))$ is the homogeneous pure tensor $\widetilde{xy} \otimes \widetilde{e}$, where $\widetilde{e}$ is the map $e \otimes S^a : S^a \to E \otimes S^a$ is defined above, and by functoriality of $- \otimes -$, $\widetilde{xy} : S^{b+c-a} \to Z \otimes E$ is the composition

$$\begin{tikzcd}
S^{b+c-a} \arrow[d, "\cong"] \\
S^b \otimes S^c \otimes S^{-a} \arrow[d, "x \otimes y \otimes S^{-a}"] \\
Z \otimes E \otimes E \otimes S^a \otimes S^{-a} \arrow[d, "Z \otimes \mu \otimes S^a \otimes S^{-a}"] \\
Z \otimes E \otimes S^a \otimes S^{-a} \arrow[d, "Z \otimes E \otimes \phi_{a,-a}^{-1}"] \\
Z \otimes E
\end{tikzcd}$$

Now, define $r \in \pi_{c-a}(E)$ to be the composition

$$S^{c-a} \cong S^c \otimes S^{-a} \xrightarrow{y \otimes S^{-a}} E \otimes S^a \otimes S^{-a} \xrightarrow{E \otimes \phi_{a,-a}^{-1}} E.$$

First, we claim that $x \cdot r = \widetilde{xy}$. To that end, consider the following diagram, where here we are again passing to a symmetric strict monoidal category:

$$\begin{tikzcd}
S^{b+c-a} \arrow[r, "\cong"] & S^b \otimes S^c \otimes S^{-a} \arrow[r, "x \otimes y \otimes S^{-a}"] & Z \otimes E \otimes E \otimes S^a \otimes S^{-a} \arrow[r, "Z \otimes \mu \otimes S^a \otimes S^{-a}"] \arrow[d, "Z \otimes E \otimes E \otimes \phi_{a,-a}^{-1}"'] & Z \otimes E \otimes S^a \otimes S^{-a} \arrow[d, "Z \otimes E \otimes \phi_{a,-a}^{-1}"] \\
& & Z \otimes E \otimes E \arrow[r, "Z \otimes \mu"'] & Z \otimes E
\end{tikzcd}$$

Commutativity is functoriality of $- \otimes -$, which also tells us that the two outside compositions are $\widetilde{xy}$ (on top) and $x \cdot r$ (on the bottom), so they are equal as desired. On the other hand, we claim that $r \cdot \widetilde{e} = y$. To see this, consider the following diagram:

$$\begin{tikzcd}
S^c \arrow[rr, "\cong"] \arrow[dd, "y"'] & & S^c \otimes S^{-a} \otimes S^a \arrow[dl, "y \otimes S^{-a} \otimes S^a"'] \arrow[d, "y \otimes S^{-a} \otimes e \otimes S^a"] \\
& E \otimes S^a \otimes S^{-a} \otimes S^a \arrow[r, "E \otimes S^a \otimes S^{-a} \otimes e \otimes S^a"'] \arrow[dl, "E \otimes S^a \otimes \phi_{-a,a}^{-1}"'] \arrow[d, "E \otimes \phi_{a,-a}^{-1} \otimes S^a"] & E \otimes S^a \otimes S^{-a} \otimes E \otimes S^a \arrow[dd, "E \otimes \phi_{a,-a}^{-1} \otimes E \otimes S^a"] \\
E \otimes S^a \arrow[r, equal] & E \otimes S^a \arrow[dr, "E \otimes e \otimes S^a"] & \\
E \otimes E \otimes S^a \arrow[u, "\mu \otimes S^a"] \arrow[rr, equal] & & E \otimes E \otimes S^a
\end{tikzcd}$$

By **??**, we may take the top arrow to be $S^c \otimes \phi_{-a,a}$, in which case the top left triangle commutes by functoriality of $- \otimes -$. The bottom trapezoid commutes by unitality of μ. Every other region commutes either by definition or by functoriality of $- \otimes -$. The top composition is $r \cdot \widetilde{e}$, so we have shown $r \cdot \widetilde{e} = y$ as desired. Thus, we have that

$$\Psi(\Phi_{Z,S^a}(x \otimes y)) = \widetilde{xy} \otimes \widetilde{e} = x \cdot r \otimes \widetilde{e} = x \otimes r \cdot \widetilde{e} = x \otimes y,$$

as desired. Hence we have shown Ψ is both a left and right inverse for Φ_{Z,S^a}, so that indeed S^a belongs to $\mathcal{E}$ as desired.

Now, we would like to show that given a distinguished triangle in $\mathcal{SH}$

$$X \xrightarrow{f} Y \xrightarrow{g} W \xrightarrow{h} \Sigma X,$$

if two of three of the objects X, Y, and W belong to $\mathcal{E}$, then so does the third. From now on, write L_*^E to denote the functor from $\mathcal{SH}$ to A-graded abelian groups sending $X \mapsto \pi_*(Z \otimes E) \otimes_{\pi_*(E)} \pi_*(E \otimes X)$. Then $\Phi_{Z,-}$ is a natural transformation $L_*^E \Rightarrow \pi_*(Z \otimes E \otimes -) = Z_*(E \otimes -)$. First, recall that it follows generally that in an adjointly tensor triangulated category (**??**), which $\mathcal{SH}$ is by **??**, given a distinguished triangle (f, g, h) we have a long exact sequence (see **??** for the definition of an exact sequence in an additive category, and see **??** for the explicit contruction of the LES associated to a distinguished triangle in an adjointly triangulated category):

$$\Omega Y \xrightarrow{\Omega g} \Omega W \xrightarrow{\widetilde{h}} X \xrightarrow{f} Y \xrightarrow{g} W \xrightarrow{h} \Sigma X \xrightarrow{\Sigma f} \Sigma Y,$$

where $\widetilde{h} : \Omega W \to X$ is the adjoint of $h : W \to \Sigma X$. Then since $\mathcal{SH}$ is further a tensor triangulated category (**??**), we have that the above sequence remains exact even after tensoring by E on the left (see **??** for details), so we have the following exact sequence in $\mathcal{SH}$:

$$E \otimes \Omega Y \xrightarrow{E \otimes \Omega g} E \otimes \Omega W \xrightarrow{E \otimes \widetilde{h}} E \otimes X \xrightarrow{E \otimes f} E \otimes Y \xrightarrow{E \otimes g} E \otimes W \xrightarrow{E \otimes h} E \otimes \Sigma X \xrightarrow{E \otimes \Sigma f} E \otimes \Sigma Y.$$

We can then apply $[S^*, -] = \pi_*(-)$ to it, which yields the following exact sequence of A-graded abelian groups:

$$E_*(\Omega Y) \xrightarrow{E_*(\Omega g)} E_*(\Omega W) \xrightarrow{E_*(\widetilde{h})} E_*(X) \xrightarrow{E_*(f)} E_*(Y) \xrightarrow{E_*(g)} E_*(W) \xrightarrow{E_*(h)} E_*(\Sigma X) \xrightarrow{E_*(f)} E_*(\Sigma Y).$$

Now, we can tensor this sequence with $\pi_*(Z \otimes E)$ on the left over $\pi_*(E)$, and since $\pi_*(Z \otimes E)$ is a flat right $\pi_*(E)$ module, we get that the top row in the following diagram is exact:

$$\begin{tikzcd}
L_*^E(\Omega Y) \arrow[r, "L_*^E(\Omega g)"] \arrow[d, "\Phi_{Z,\Omega Y}"'] & L_*^E(\Omega W) \arrow[r, "L_*^E(\widetilde{h})"] \arrow[d, "\Phi_{Z,\Omega W}"'] & L_*^E(X) \arrow[r, "L_*^E(f)"] \arrow[d, "\Phi_{Z,X}"'] & L_*^E(Y) \arrow[r, "L_*^E(g)"] \arrow[d, "\Phi_{Z,Y}"'] & L_*^E(W) \arrow[r, "L_*^E(h)"] \arrow[d, "\Phi_{Z,W}"'] & L_*^E(\Sigma X) \arrow[r, "L_*^E(\Sigma f)"] \arrow[d, "\Phi_{Z,\Sigma X}"'] & L_*^E(\Sigma Y) \arrow[d, "\Phi_{Z,\Sigma Y}"'] \\
Z_*(E \otimes \Omega Y) \arrow[r, "Z_*(E \otimes \Omega g)"'] & Z_*(E \otimes \Omega W) \arrow[r, "Z_*(E \otimes \widetilde{h})"'] & Z_*(E \otimes X) \arrow[r, "Z_*(E \otimes f)"'] & Z_*(E \otimes Y) \arrow[r, "Z_*(E \otimes g)"'] & Z_*(E \otimes W) \arrow[r, "Z_*(E \otimes h)"'] & Z_*(E \otimes \Sigma X) \arrow[r, "Z_*(E \otimes \Sigma f)"'] & Z_*(E \otimes \Sigma Y)
\end{tikzcd}$$

This diagram further commutes by naturality of $\Phi_{Z,-}$. Now, supposing that two of three of X, Y, and W belong to $\mathcal{E}$, by Lemma 0.33, if $\Phi_{Z,V}$ is an isomorphism for some object V in $\mathcal{SH}$ then $\Phi_{Z,\Omega V}$ and $\Phi_{Z,\Sigma V}$ are. Thus by the five lemma, it follows that the middle three vertical arrows in the above diagram are necessarily all isomorphisms if any two of them are, so we have shown that $\mathcal{E}$ is closed under two-of-three for exact triangles, as desired.

Finally, it remains to show that $\mathcal{E}$ is closed under arbitrary coproducts. Let $\{W_i\}_{i \in I}$ be a collection of objects in $\mathcal{E}$ indexed by some (small) set I. Then we'd like to show that $W := \bigoplus_i W_i$ belongs to $\mathcal{E}$. First of all, note that $- \otimes -$ preserves arbitrary coproducts in each argument, as it has a right adjoint $F(-,-)$. Thus without loss of generality, given any object X in $\mathcal{SH}$, we may take $\bigoplus_i X \otimes W_i = X \otimes \bigoplus_i W_i$ (as $X \otimes \bigoplus_i W_i$ *is* a coproduct of all the $X \otimes W_i$'s). Now, recall that we have chosen each S^a to be a compact object (**??**), so that given any object X and collection of objects $\{Y_i\}_{i \in I}$ in $\mathcal{SH}$, if $Y := \bigoplus_{i \in I} Y_i$, then the canonical map

$$s_{X,Y_i} : \bigoplus_i X_*(Y_i) = \bigoplus_i [S^*, X \otimes Y_i] \to [S^*, \bigoplus_i X \otimes Y_i] = [S^*, X \otimes Y] = X_*(Y)$$

is an isomorphism, natural in Y_i for each i. Note in particular that s_{E,W_i} is an isomorphism of left $\pi_*(E)$-modules. To see this, first note by additivity of s_{E,W_i}, it suffices to check that $s_{E,W_i}(r \cdot x) = r \cdot s_{E,W_i}(x)$ for each homogeneous $r \in \pi_*(E)$ and homogeneous $x \in E_*(W_i)$ for some i, as such x generate $\bigoplus_i E_*(W_i)$ by definition. Then given $r : S^a \to E$ and $x : S^b \to E \otimes W_i$, consider the following diagram

$$\begin{tikzcd}
S^{a+b} \arrow[r, "\phi_{a,b}"] & S^a \otimes S^b \arrow[r, "x\otimes y"] & E \otimes E \otimes W_i \arrow[r, "E\otimes\iota_{E\otimes W_i}"] \arrow[dddd, "\mu\otimes W_i"'] \arrow[dr, "E\otimes E\otimes\iota_{W_i}"] & E \otimes \bigoplus_i (E \otimes W_i) \arrow[d, equal] \\
 & & & E \otimes E \otimes W \arrow[d, "\mu\otimes W"] \\
 & & & E \otimes W \arrow[dd, equal] \\
 & & & \\
 & & E \otimes W_i \arrow[r, "\iota_{E\otimes W_i}"'] \arrow[uur, "E\otimes\iota_{W_i}"] & \bigoplus_i (E \otimes W_i)
\end{tikzcd}$$

where $\iota_{E\otimes W_i} : E \otimes W_i \hookrightarrow \bigoplus_i (E \otimes W_i)$ and $\iota_{W_i} : W_i \hookrightarrow \bigoplus_i W_i$ are the maps determined by the definition of the coproduct. Commutativity of the two triangles is by the fact that $E \otimes -$ is colimit preserving. Commutativity of the trapezoid is functoriality of $- \otimes -$. Thus, since s_{E,W_i} is a homomorphism of left A-graded $\pi_*(E)$-modules, the top right arrow in the following diagram is well-defined:

$$\begin{tikzcd}
\bigoplus_i Z_*(E) \otimes_{\pi_*(E)} E_*(W_i) \arrow[r, equal] \arrow[d, "\bigoplus_i \Phi_{Z,W_i}"'] & Z_*(E) \otimes_{\pi_*(E)} \bigoplus_i E_*(W_i) \arrow[r, "Z_*(E)\otimes_{\pi_*(E)} s_{E,W_i}"] & Z_*(E) \otimes_{\pi_*(E)} E_*(W) \arrow[d, "\Phi_{Z,W}"] \\
\bigoplus_i Z_*(E \otimes W_i) \arrow[r, "s_{Z,E\otimes W_i}"] & Z_*(\bigoplus_i E \otimes W_i) \arrow[r, equal] & Z_*(E \otimes W)
\end{tikzcd} \tag{5}$$

We wish to show this diagram commutes. Again, since each map here is a homomorphism, it suffices to chase generators. By definition, a generator of the top left element is a homogeneous pure tensor in $E_*(E) \otimes_{\pi_*(E)} E_*(W_i)$ for some i in I. Given classes $x : S^a \to E \otimes E$ and $y : S^b \to E \otimes W_i$, consider the following diagram:

$$\begin{tikzcd}
S^{a+b} \arrow[r, "\phi_{a,b}"] & S^a \otimes S^b \arrow[r, "x\otimes y"] & Z \otimes E \otimes E \otimes W_i \arrow[r, "Z\otimes E\otimes\iota_{E\otimes W_i}"] \arrow[d, "Z\otimes\mu\otimes W_i"'] \arrow[dr, "Z\otimes E\otimes E\otimes\iota_{W_i}"] & Z \otimes E \otimes \bigoplus_i E \otimes W_i \arrow[d, equal] \\
 & & Z \otimes E \otimes W_i \arrow[d, "\iota_{Z\otimes E\otimes W_i}"'] \arrow[dr, "Z\otimes E\otimes\iota_{W_i}"] & Z \otimes E \otimes E \otimes W \arrow[d, "Z\otimes\mu\otimes W"] \\
 & & \bigoplus_i Z \otimes E \otimes W_i \arrow[r, equal] & Z \otimes E \otimes W
\end{tikzcd}$$

Unravelling definitions, the two outside compositions are the two ways to chase $x \otimes y$ around diagram (5). The two triangles commute again by the fact that $- \otimes -$ preserves colimits in each argument. Commutativity of the inner parallelogram is functoriality of $- \otimes -$. Thus diagram (5) tells us $\Phi_{Z,W}$ is an isomorphism, since s_{E,W_i} and $s_{Z,E\otimes W_i}$ are isomorphisms, and Φ_{Z,W_i} is an isomorphism for each i in I, meaning $\bigoplus_i \Phi_{W_i}$ is as well.

Thus, we've shown the class $\mathcal{E}$ of objects W for which $\Phi_{Z,W}$ is an isomorphism contains the S^a's, is closed under two-of-three for distinguished triangles, and is closed under arbitrary coproducts. Thus, it follows that $\mathcal{E}$ contains the class of all cellular objects in $\mathcal{SH}$, as desired.

Now, suppose that $\pi_*(E \otimes W)$ is a flat left $\pi_*(E)$-module, then we'd like to show $\Phi_{Z,W}$ is an isomorphism for all cellular Z in $\mathcal{SH}$. Showing this is entirely analagous to above, so we only outline the argument. Let $\mathcal{E}$ be the class of Z in $\mathcal{SH}$ such that $\Phi_{Z,W}$ is an isomorphism. Then in order to show $\mathcal{E}$ contains every cellular object, it suffices to show it contains the S^a's, is closed under two-of-three for distinguished triangles, and is closed under arbitrary coproducts.

To see $\mathcal{E}$ contains the S^a's, consider the map

$$\Psi : \pi_*(S^a \otimes E \otimes W) \to \pi_*(S^a \otimes E) \otimes_{\pi_*(E)} \pi_*(E \otimes W)$$

sending $x : S^b \to S^a \otimes E \otimes W$ to $\widetilde{e} \otimes \widetilde{x}$, where $\widetilde{e} \in \pi_a(S^a \otimes E)$ is the map $S^a \otimes e : S^a \to S^a \otimes E$, and $\widetilde{x} \in \pi_{b-a}(E \otimes W)$ is the map

$$\widetilde{x} : S^{b-a} \xrightarrow{\phi_{-a,b}} S^{-a} \otimes S^b \xrightarrow{S^{-a} \otimes x} S^{-a} \otimes S^a \otimes E \otimes W \xrightarrow{\phi^{-1}_{-a,a} \otimes E \otimes W} E \otimes W.$$

Then checking that Ψ is a left and right inverse to $\Phi_{S^a,W}$ is entirely analagous, so that S^a belongs to $\mathcal{E}$ as desired.

To see $\mathcal{E}$ is closed under two-of-three for distinguished triangles, let

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$$

be a distinguished triangle in $\mathcal{SH}$. Then an analagous argument as above (using **??** and **??**) yields a long exact sequence of A-graded abelian groups

$$\begin{tikzcd}
& \pi_*(\Omega Y \otimes E) \arrow[r, "\pi_*(\Omega g \otimes E)"] & \pi_*(\Omega Z \otimes E) \arrow[dll, "\pi_*(\widetilde{h} \otimes E)"'] \\
\pi_*(X \otimes E) \arrow[r, "\pi_*(f \otimes E)"'] & \pi_*(Y \otimes E) \arrow[r, "\pi_*(g \otimes E)"] & \pi_*(Z \otimes E) \arrow[dll, "\pi_*(h \otimes E)"'] \\
\pi_*(\Sigma X \otimes E) \arrow[r, "\pi_*(\Sigma f \otimes E)"'] & \pi_*(\Sigma Y \otimes E) &
\end{tikzcd}$$

Then since $\pi_*(E \otimes W)$ is a flat left $\pi_*(E)$-module, we can tensor the above long exact sequence with $\pi_*(E \otimes W)$ on the right to obtain a long exact sequence which fits in the left column of the following commuting diagram:

$$\begin{tikzcd}
R^E_*(\Omega Y) \arrow[r, "\Phi_{\Omega Y,W}"] \arrow[d, "R^E_*(\Omega g)"'] & \pi_*(\Omega Y \otimes E \otimes W) \arrow[d, "\pi_*(\Omega g \otimes E \otimes W)"] \\
R^E_*(\Omega Z) \arrow[r, "\Phi_{\Omega Z,W}"] \arrow[d, "R^E_*(\widetilde{h})"'] & \pi_*(\Omega Z \otimes E \otimes W) \arrow[d, "\pi_*(\widetilde{h} \otimes E \otimes W)"] \\
R^E_*(X) \arrow[r, "\Phi_{X,W}"] \arrow[d, "R^E_*(f)"'] & \pi_*(X \otimes E \otimes W) \arrow[d, "\pi_*(f \otimes E \otimes W)"] \\
R^E_*(Y) \arrow[r, "\Phi_{Y,W}"] \arrow[d, "R^E_*(g)"'] & \pi_*(Y \otimes E \otimes W) \arrow[d, "\pi_*(g \otimes E \otimes W)"] \\
R^E_*(Z) \arrow[r, "\Phi_{Z,W}"] \arrow[d, "R^E_*(h)"'] & \pi_*(Z \otimes E \otimes W) \arrow[d, "\pi_*(h \otimes E \otimes W)"] \\
R^E_*(\Sigma X) \arrow[r, "\Phi_{\Sigma X,W}"] \arrow[d, "R^E_*(\Sigma f)"'] & \pi_*(\Sigma X \otimes E \otimes W) \arrow[d, "\pi_*(\Sigma f \otimes E \otimes W)"] \\
R^E_*(\Sigma Y) \arrow[r, "\Phi_{\Sigma Y,W}"] & \pi_*(\Sigma Y \otimes E \otimes W)
\end{tikzcd}$$

where R^E_* denotes the functor from $\mathcal{SH}$ to A-graded abelian groups sending $X \mapsto \pi_*(X \otimes E) \otimes_{\pi_*(E)} \pi_*(E \otimes W)$, so that $\Phi_{-,W}$ is a natural homomorphism $R^E_*(-) \Rightarrow \pi_*(- \otimes E \otimes W)$. Then finally by Lemma 0.33 and the five lemma, if any two of three of the middle three horizontal arrows are isomorphisms, then all three of the horizontal arrows are isomorphisms, as desired.

Finally, in order to show $\mathcal{E}$ is closed under arbitrary coproducts, suppose we have a collection of objects $\{Z_i\}_{i \in I}$ in $\mathcal{E}$ indexed by some (small) set $\mathcal{E}$. Then we'd like to show $Z := \bigoplus_{i \in I} Z_i$ also belongs to $\mathcal{E}$. First note that since the S^a's are compact, for any object Y we have isomorphisms

$$s_{Z_i,Y} : \bigoplus_i Z_{i*}(Y) = \bigoplus_i [S^*, Z_i \otimes Y] \to [S^*, \bigoplus_i (Z_i \otimes Y)] = [S^*, Z \otimes Y] = Z_*(Y).$$

It is straightforward to verify that $s_{Z_i,E} : \bigoplus_i Z_{i*}(E) \to Z_*(E)$ is not only an isomorphism of abelian groups, but an isomorphism of right A-graded $\pi_*(E)$-modules, so that the top arrow in the following diagram is well-defined:

$$\begin{tikzcd}
\bigoplus_i \left(Z_{i*}(E) \otimes_{\pi_*(E)} E_*(W)\right) \arrow[r, equal] \arrow[d, "\bigoplus_i \Phi_{Z_i,W}"'] & \bigoplus_i \left(Z_{i*}(E)\right) \otimes_{\pi_*(E)} E_*(W_i) \arrow[r, "s_{Z_i,E}"] & Z_*(E) \otimes_{\pi_*(E)} E_*(W) \arrow[d, "\Phi_{Z,W}"] \\
\bigoplus_i Z_{i*}(E \otimes W) \arrow[rr, "s_{Z_i,E\otimes W}"] & & Z_*(E \otimes W)
\end{tikzcd}$$

Then a simple diagram chase yields the diagram commutes, so that $\Phi_{Z,W}$ is an isomorphism, assuming all the $\Phi_{Z_i,W}$'s are. $\square$

Proposition 0.35. *Let (E, μ, e) be a ring spectrum in $\mathcal{SH}$, and let X and Y be two objects in $\mathcal{SH}$ such that E and X are both cellular (**??**) and $E_*(X)$ is a projective left $\pi_*(E)$-module (Proposition 0.25). Then the map*

$$[X, E \otimes Y]_* \to \mathrm{Hom}^*_{\pi_*(E)}(E_*(X), E_*(Y))$$

which sends a generator $f : S^a \otimes X \to E \otimes Y$ in $[X, E \otimes Y]_*$ to the assignment which sends a generator $x : S^b \to E \otimes X$ in $E_*(X)$ to the composition

$$S^{a+b} \to S^a \otimes S^b \xrightarrow{S^a \otimes x} S^a \otimes E \otimes X \xrightarrow{\tau \otimes X} E \otimes S^a \otimes X \xrightarrow{E \otimes f} E \otimes E \otimes Y \xrightarrow{\mu} E \otimes Y$$

is an A-graded isomorphism of A-graded abelian groups.