THE MOTIVIC ADAMS SPECTRAL SEQUENCE

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1. Introduction

One of the oldest and most fundamental problems in algebraic topology is computing the stable homotopy groups of spheres. One of the key tools used for these calculations is the Adams spectral sequence, which was introduced by Adams in 1958 ([2]). In the late 90s, Voevodsky developed the theory of $\mathbb{A}^1$-homotopy theory, also called motivic homotopy theory, which is homotopy theory done in the world of smooth schemes. It was realized that there actually exists a motivic analogue of the Adams spectral sequence, called the motivic Adams spectral sequence. Furthermore, this spectral sequence could be used to compute differentials and extensions in the classical Adams spectral sequence.

In this document, we aim to provide a construction of an Adams spectral sequence in a more general setting, which subsumes the construction of the original and the motivic Adams spectral sequences.

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2. The Adams spectral sequence

In what follows, given a monoidal category $(\mathcal{C}, \otimes, S)$ and objects $X_1, \dots, X_n$ in $\mathcal{C}$, we write $X_1 \otimes \cdots \otimes X_n$ to denote the object

$$X_1 \otimes (X_2 \otimes \cdots (X_{n-1} \otimes X_n)).$$

In particular, given an object X and some $n > 0$, we write

$$X^n := \overbrace{X \otimes \cdots \otimes X}^{n \text{ times}} \qquad \text{and} \qquad X^0 := S.$$

2.1. **Setup.** Fix a closed symmetric monoidal triangulated category $(\mathcal{C}, \otimes, S)$ with arbitrary small (co)products. Write $\alpha_{X,Y,Z} : (X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z)$, $\rho_X : X \otimes S \to X$, $\lambda_X : S \otimes X \to X$, and $\tau_{X,Y} : X \otimes Y \to Y \otimes X$ for the associator, left unitor, right unitor, and symmetry isomorphisms, respectively. Often we will supress these isomorphisms from the notation, choosing instead to simply denote them with the $\cong$ symbol. We would like the monoidal structure to be compatible with the triangulated structure, so we further require that $\mathcal{C}$ be *tensor triangulated* (see Appendix A for a review of (tensor) triangulated categories). Given two objects X and Y in $\mathcal{C}$, we denote the hom-abelian group of morphisms from X to Y in $\mathcal{C}$ by $[X, Y]$, and we denote the hom object in $\mathcal{C}$ by $F(X, Y)$.

Now, also suppose we have a fixed pointed abelian group $(A, \mathbf{1})$ (so $\mathbf{1}$ is some chosen element in A) and a homomorphism of pointed abelian groups $h : (A, \mathbf{1}) \to (\mathrm{Pic}(\mathcal{C}), \Sigma S)$, where $\mathrm{Pic}(\mathcal{C})$ is the group of isomorphism of classes of invertible objects of $\mathcal{C}$[1]. For each $a \in A$, suppose we have chosen some representative S^a belonging to the isomorphism class $h(a)$, and without loss of generality assume $S^0 = S$ and $S^{\mathbf{1}} = \Sigma S$. Given $a \in A$, define $\Sigma^a := S^a \otimes -$. Note that given any object X in $\mathcal{C}$, we have canonical natural isomorphisms

$$\Sigma^{\mathbf{1}} X = S^{\mathbf{1}} \otimes X = \Sigma S \otimes X \xrightarrow{e_{S,X}} \Sigma(S \otimes X) \xrightarrow{\Sigma \lambda_X} \Sigma X,$$

so without loss of generality we may take $\Sigma = \Sigma^{\mathbf{1}}$.

Given two objects X and Y in $\mathcal{C}$, we may extend the abelian group $[X, Y]$ to an A-graded abelian group $[X, Y]_*$ by defining $[X, Y]_a := [\Sigma^a X, Y] = [S^a \otimes X, Y]$ (see Appendix B for a discussion of A-graded abelian groups, rings, modules, etc.). Given an object X in $\mathcal{C}$ and some $a \in A$, we may define $\pi_a(X) := [S^a, X] \cong [S, X]_a$, and we write $\pi_*(X)$ for the A-graded abelian group $\bigoplus_{a \in A} \pi_a(X)$.

Now suppose we are given some monoid object E in $\mathcal{C}$, so there exists morphisms $\mu : E \wedge E \to E$ and $e : S \to E$ such that the following diagrams commute in $\mathcal{C}$:

$$\begin{tikzcd} S \otimes E \arrow[r, "e \wedge E"] \arrow[dr, "\lambda"'] & E \otimes E \arrow[d, "\mu"] & E \otimes S \arrow[l, "E \wedge e"'] \arrow[dl, "\rho"] \\ & E & \end{tikzcd} \qquad \begin{tikzcd} (E \otimes E) \otimes E \arrow[rr, "\mu \otimes E"] \arrow[d, "\alpha"'] & & E \otimes E \arrow[d, "\mu"] \\ E \otimes (E \otimes E) \arrow[r, "E \otimes \mu"] & E \otimes E \arrow[r, "\mu"] & E \end{tikzcd}$$

Then for all $a, b \in A$, we can define a "product map" $\pi_a(X) \times \pi_b(X) \to \pi_{a+b}(X)$ which sends elements $x : S^a \to X$ and $y : S^b \to X$ to the composition

$$S^{a+b} \cong S^a \otimes S^b \xrightarrow{x \otimes y} X \otimes X \xrightarrow{\mu} X.$$

Naturally, we would like to ask if this product map endows $\pi_*(X)$ with the structure of a graded ring. Note there is some ambiguity in our definition, as this map crucially relies on the isomorphism $S^{a+b} \cong S^a \otimes S^b$. This is essentially the entire discussion of [3], and as it turns out, $\pi_*(X)$

[1]Recall an object X is a symmetric monoidal category is *invertible* if there exists some Y in $\mathcal{C}$ and an isomorphism $S \cong Y \otimes X$.

is in fact a graded ring provided we can choose these morphisms to be *coherent*, in the following sense:

Definition 2.1. Suppose we have a family of isomorphisms

$$\phi_{a,b} : S^{a+b} \xrightarrow{\cong} S^a \otimes S^b$$

for all $a, b \in A$. We say this family is *coherent* if:

(1) For all $a \in A$, the isomorphisms $\phi_{a,0} : S^a \to S^a \otimes S$ and $\phi_{0,a} : S^a \to S \otimes S$ are precisely the unitor isomorphisms prescribed by the symmetric monoidal structure on $\mathcal{C}$
(2) For all $a, b, c \in A$, the following diagram commutes:

$$\begin{tikzcd}
S^{a+b} \otimes S^c \arrow[d, "\phi_{a,b} \otimes S^c"'] & S^{a+b+c} \arrow[l, "\phi_{a+b,c}"'] \arrow[r, "\phi_{a,b+c}"] & S^a \otimes S^{b+c} \arrow[d, "S^a \otimes \phi_{b,c}"] \\
(S^a \otimes S^b) \otimes S^c \arrow[rr, "\cong"] & & S^a \otimes (S^b \otimes S^c)
\end{tikzcd}$$

Remark 2.2. Note that by induction the coherence conditions say that given any $a_1, \dots, a_n \in A$ and $b_1, \dots, b_m \in A$ such that $a_1 + \cdots + a_n = b_1 + \cdots + b_m$ and any fixed parenthesizations of $X = S^{a_1} \otimes \cdots \otimes S^{a_b}$ and $Y = S^{b_1} \otimes \cdots \otimes S^{b_m}$, there is a *unique* isomorphism $X \to Y$ that can be obtained by forming compositions of tensor products of $\phi_{a,b}$, associators, and their inverses.

If we have these isomorphisms, then as it turns out $\pi_*(E)$ is indeed a graded ring if E is a monoid object:

Proposition 2.3. *Let (E, μ, e) be a monoid object in $\mathcal{C}$, and suppose we have a coherent family of isomorphisms $\phi_{a,b} : S^{a+b} \xrightarrow{\cong} S^a \otimes S^b$ (Definition 2.1). Then the $\phi_{a,b}$'s canonically endow $\pi_*(E)$ with the structure of an A-graded ring under the map*

$$\pi_*(E) \times \pi_*(E) \to \pi_*(E)$$

which takes an element $x : S^a \to E$ and $y : S^b \to E$ to the composition

$$S^{a+b} \xrightarrow{\phi_{a,b}} S^a \otimes S^b \xrightarrow{x \otimes y} E \otimes E \xrightarrow{\mu} E.$$

Proof. First we show this map is associative: Given classes x, y, and z in $\pi_a(E)$, $\pi_b(E)$, and $\pi_c(E)$, respectively, consider the following diagram:

$$\begin{tikzcd}
S^{a+b+c} \arrow[r, "\phi_{a+b,c}"] \arrow[d, "\phi_{a,b+c}"'] & S^{a+b} \otimes S^c \arrow[r, "\phi_{a,b} \otimes S^c"] & (S^a \otimes S^b) \otimes S^c \arrow[r, "(x \otimes y) \otimes z"] \arrow[dl, "\cong"] & (E \otimes E) \otimes E \arrow[r, "\mu \otimes E"] \arrow[dl, "\cong"] & E \otimes E \arrow[d, "\mu"] \\
S^a \otimes S^{b+c} \arrow[r, "S^a \otimes \phi_{b,c}"'] & S^a \otimes (S^b \otimes S^c) \arrow[r, "x \otimes (y \otimes z)"'] & E \otimes (E \otimes E) \arrow[r, "E \otimes \mu"'] & E \otimes E \arrow[r, "\mu"'] & E
\end{tikzcd}$$

Commutativity of the left pentagon is the coherence condition for the $\phi_{p,q}$'s. Commutativity of the middle parallelogram is naturality of the associator isomorphisms. Commutativity of the right pentagon is associativity of μ. The fact that the two outside compositions equal $(x \cdot y) \cdot z$ and $x \cdot (y \cdot z)$, respectively, follows by functoriality of $- \wedge -$.

Next we claim the map $e : S \to E$ is a unit for this multiplication. Given $x \in \pi_a(E)$, consider the following diagram:

$$\begin{tikzcd}
S \otimes S^a \arrow[dr, "S\otimes x"] \arrow[dd, "e\otimes x"'] & & S^a \arrow[ll, "\phi_{0,a}=\lambda_{S^a}^{-1}"'] \arrow[rr, "\phi^{-1}=\rho_{S^a}^{-1}"] \arrow[dd, "x"] & & S^a \otimes S \arrow[dl, "x\otimes S"'] \arrow[dd, "x\otimes e"] \\
& S \otimes E \arrow[dl, "e\otimes E"'] \arrow[dr, "\lambda_E"] & & E \otimes S \arrow[dl, "\rho_E"'] \arrow[dr, "E\otimes e"] & \\
E \otimes E \arrow[rr, "\mu"'] & & E & & E \wedge E \arrow[ll, "\mu"]
\end{tikzcd}$$

Commutativity of the top two large triangles is naturality of the unitor isomorphisms. Commutativity of the right and leftmost triangles functoriality of $- \wedge -$. Commutativity of the bottom triangles is unitality of μ. Hence, we have that $e \cdot x = x = x \cdot e$.

This product is also bilinear (distributive). Given $x, x' \in \pi_a(E)$ and $y, y' \in \pi_b(E)$, consider the following diagrams:

$$\begin{tikzcd}
S^{a+b} \arrow[r, "\phi_{a,b}"] \arrow[d, "\Delta"'] & S^a \otimes S^b \arrow[r, "\Delta\otimes S^b"] \arrow[d, "\Delta"] & (S^a \oplus S^a) \otimes S^b \arrow[r, "(x\oplus x')\otimes y"] \arrow[dl, "\cong"'] & (E \oplus E) \otimes E \arrow[dl, "\cong"'] \arrow[d, "\nabla\otimes E"] & \\
S^{a+b} \oplus S^{a+b} \arrow[r, "\phi_{a,b}\oplus\phi_{a,b}"'] & (S^a \otimes S^b) \oplus (S^a \otimes S^b) \arrow[r, "(x\otimes y)\oplus(x'\otimes y)"'] & (E \otimes E) \oplus (E \otimes E) \arrow[r, "\nabla"'] & E \otimes E \arrow[r, "\mu"] & E
\end{tikzcd}$$

$$\begin{tikzcd}
S^{a+b} \arrow[r, "\phi_{a,b}"] \arrow[d, "\Delta"'] & S^a \otimes S^b \arrow[r, "S^a\otimes\Delta"] \arrow[d, "\Delta"] & S^b \otimes (S^b \oplus S^b) \arrow[r, "x\otimes(y\oplus y')"] \arrow[dl, "\cong"'] & E \otimes (E \oplus E) \arrow[dl, "\cong"'] \arrow[d, "E\otimes\nabla"] & \\
S^{a+b} \oplus S^{a+b} \arrow[r, "\phi_{a,b}\oplus\phi_{a,b}"'] & (S^a \otimes S^b) \oplus (S^a \otimes S^b) \arrow[r, "(x\otimes y)\oplus(x\otimes y')"'] & (E \otimes E) \oplus (E \otimes E) \arrow[r, "\nabla"'] & E \otimes E \arrow[r, "\mu"] & E
\end{tikzcd}$$

The unlabeled isomorphisms are those given by the fact that $- \wedge -$ is additive in each variable (since $\mathcal{C}$ is tensor triangulated). Commutativity of the left squares is naturality of $\Delta : X \to X \oplus X$ in an additive category. Commutativity of the rest of the diagram follows again from the fact that $- \wedge -$ is an additive functor in each variable. Hence, by functoriality of $- \wedge -$, these diagrams tell us that $(x + x') \cdot y = x \cdot y + x' \cdot y$ and $x \cdot (y + y') = x \cdot y + x \cdot y'$, respectively. $\square$

Now, two natural questions arise:

(1) In general, can we actually find such a coherent family of $\phi_{a,b}$'s?
(2) If (E, μ, e) is a *commutative* monoid object in $\mathcal{C}$, so that the following diagram commutes,

$$\begin{tikzcd}
E \otimes E \arrow[rr, "\tau"] \arrow[dr, "\mu"'] & & E \otimes E \arrow[dl, "\mu"] \\
& E &
\end{tikzcd}$$

then in what sense, if any, is $\pi_*(E)$ a *commutative* A-graded ring?

The answer to (1) is the subject of Duggers' paper [3], and as it turns out, the answer is yes:

Theorem 2.4 ([3, Proposition 7.1]). *There exists a coherent family of isomorphisms*

$$\phi_{a,b} : S^{a+b} \xrightarrow{\cong} S^a \otimes S^b$$

in the sense of Definition 2.1, and in particular, the set of such coherent families is in bijective correspondence with the set of normalized 2*-cocycles* $Z^2(A; \mathrm{Aut}(S))_{norm}$*, i.e., the set of functions* $\alpha : A \times A \to \mathrm{Aut}(S)$ *such that* $\alpha(0,0) = \mathrm{id}_S$ *and for all* $a, b, c \in A$*,* $\alpha(a + b, c) \cdot \alpha(a, b) = \alpha(b, c) \cdot \alpha(a, b + c)$.

Henceforth, we will assume such a coherent family of $\phi_{a,b}$'s has been fixed. Question (2) still remains: given a commutative monoid object E, in what sense is $\pi_*(E)$ a *commutative* A-graded ring? The best we can do for now is the following:

Proposition 2.5 ([3, Remark 7.2])**.** *Given a commutative monoid object (E, μ, e) in $\mathcal{C}$, the A-graded ring structure on $\pi_*(E)$ has a commutativity formula given by*

$$x \cdot y = y \cdot x \cdot (e \circ \theta_{a,b})$$

for $x \in \pi_a(E)$ and $y \in \pi_b(E)$, where where $\theta_{a,b} \in \operatorname{Aut}(S)$ is the composition

$$S \xrightarrow{\cong} S^{-a-b} \otimes S^a \otimes S^b \xrightarrow{S^{-a-b}\otimes\tau} S^{-a-b} \otimes S^b \otimes S^a \xrightarrow{\cong} S,$$

where the outermost maps are the unique maps specified by Remark 2.2.

Proof. Let $\phi_{a,b}$, E, x, and y as in the statement of the proposition. Now consider the following diagram

$$\begin{tikzcd}
S^{a+b} \arrow[r, "\phi_{a,b}"] \arrow[dd, dashed, "\phi_{b,a}^{-1}\circ\tau\circ\phi_{a,b}"'] & S^a \otimes S^b \arrow[r, "x\otimes y"] \arrow[dd, "\tau"'] & E \otimes E \arrow[dr, "\mu"] \arrow[dd, "\tau"'] & \\
& & & E \\
S^{a+b} \arrow[r, "\phi_{b,a}"] & S^b \otimes S^a \arrow[r, "y\otimes x"] & E \otimes E \arrow[ur, "\mu"'] &
\end{tikzcd}$$

The left square commutes by definition. The middle square commutes by naturality of the symmetry isomorphism. Finally, the right square commutes by commutativity of E. Unravelling definitions, we have shown that under the product on $\pi_*(E)$ induced by the $\phi_{a,b}$'s,

$$x \cdot y = (y \cdot x) \circ (\phi_{b,a}^{-1} \circ \tau \circ \phi_{a,b}).$$

Thus, in order to show the desired result it further suffices to show that

$$(y \cdot x) \circ (\phi_{b,a}^{-1} \circ \tau \circ \phi_{a,b}) = y \cdot x \cdot (e \circ \theta_{a,b}).$$

Consider the following diagram:

$$\begin{tikzcd}
S^{a+b} \arrow[rr, "\phi_{a,b}"] \arrow[d, "\cong"'] & & S^a \otimes S^b \arrow[d, "\tau"] \\
S^b \otimes S^a \otimes S^{-a-b} \otimes S^a \otimes S^b \arrow[urr, "\cong"'] \arrow[d, "S^b\otimes S^a\otimes S^{-a-b}\otimes\tau"'] & & S^b \otimes S^a \arrow[d, "\phi_{b,a}^{-1}"] \\
S^b \otimes S^a \otimes S^{-a-b} \otimes S^b \otimes S^a \arrow[urr, "\cong"] \arrow[d, "\cong"'] & & S^{a+b} \arrow[d, "\phi_{b,a}"] \\
S^b \otimes S^a \otimes S \arrow[rr, "\cong"] \arrow[dr, "y\otimes x\otimes S"] \arrow[dd, "x\otimes y\otimes e"'] & & S^b \otimes S^a \arrow[dd, "y\otimes x"] \\
& E \otimes E \otimes S \arrow[dr, "\rho"] & \\
E \otimes E \otimes E \arrow[ur, "E\otimes E\otimes e"] \arrow[rr, "E\otimes\mu"] \arrow[d, "\mu\otimes E"'] & & E \otimes E \arrow[d, "\mu"] \\
E \otimes E \arrow[rr, "\mu"'] & & E
\end{tikzcd}$$

Here we are suppressing associators from the notation, and any map simply labelled $\cong$ is an appropriate composition of copies of $\phi_{a,b}$'s, associators, and their inverses, so that each of these maps are necessarily unique by Remark 2.2. The top triangle commutes by coherence for the

$\phi_{a,b}$'s. The parallelogram commutes by naturality of τ and coherence of the of $\phi_{a,b}$'s. The trapezoid commutes again by coherence for the $\phi_{a,b}$'s. The middle right large triangle commutes by naturality of the unitors (and the fact that $S^b \otimes \phi_{a,0}$ coincides with the unitor $S^b \otimes S^a \otimes S \to S^b \otimes S^a$). The middle left triangle commutes by functoriality of $- \otimes -$. The middle triangle commutes by unitality of μ. Finally, the bottom rectangle commmutes by associativity of μ. Hence, by unravelling definitions and applying functoriality of $- \otimes -$, we get that the top composition is $(y \cdot x) \circ (\phi_{b,a}^{-1} \circ \tau \circ \phi_{a,b})$, while the bottom composition is $y \cdot x \cdot (e \circ \theta_{a,b})$, so they are equal as desired. $\square$

Remark 2.6. Note that given a commutative monoid object E in $\mathcal{C}$ and elements $x \in \pi_a(E)$ and $y \in \pi_0(E)$, we have $x \cdot y = y \cdot x$, as $\theta_{a,0} = \mathrm{id}_S$. Indeed, $\theta_{a,0}$ is the composition

$$S \xrightarrow{\phi_{-a,a}} S^{-a} \otimes S^a \xrightarrow{S^{-a}\otimes\phi_{a,0}} S^{-a} \otimes (S^a \otimes S) \xrightarrow{S^{-a}\otimes\tau} S^{-a} \otimes (S \otimes S^a) \xrightarrow{S^{-a}\otimes\phi_{0,a}^{-1}} S^{-a} \otimes S^a \xrightarrow{\phi_{-a,a}^{-1}} S$$

By the coherence theorem for symmetric monoidal categories and the fact that $\phi_{a,0}$ and $\phi_{0,a}$ coincide with the unitors, we have that the composition

$$S^a \xrightarrow{\phi_{a,0}=\rho_{S^a}^{-1}} S^a \otimes S \xrightarrow{\tau} S \otimes S^a \xrightarrow{\phi_{0,a}^{-1}=\lambda_{S^a}} S^a$$

is precisely the identity map, so by functoriality of $- \otimes -$, we have that $\theta_{a,0}$ is the composition

$$S \xrightarrow{\phi_{-a,a}} S^{-a} \otimes S^a \xrightarrow{=} S^{-a} \otimes S^a \xrightarrow{\phi_{-a,a}^{-1}} S,$$

so $\theta_{a,0} = \mathrm{id}_S$, meaning

$$x \cdot y = y \cdot x \cdot (e \circ \theta_{a,0}) = y \cdot x \cdot e = y \cdot x,$$

where the last equality follows by the fact that e is the unit for the multiplication on $\pi_*(E)$.

Proposition 2.7. *Given $a \in A$, Σ^a and Σ^{-a} canonically form an adjoint equivalence of $\mathcal{C}$.*

Proof. Let $X, Y \in \mathcal{C}$. By [10, Lemma 3.2], in order to show Σ^a and Σ^{-a} are adjoint equivalences, it suffices to construct natural isomorphisms $\eta : \mathrm{Id}_\mathcal{C} \Rightarrow \Sigma^{-a} \circ \Sigma^a$ and $\varepsilon : \Sigma^a \circ \Sigma^{-a} \Rightarrow \mathrm{Id}_\mathcal{C}$ such that for all X in $\mathcal{C}$, the following diagram commutes:

(1)
$$\begin{tikzcd}
\Sigma^a X \arrow[r, "(\Sigma^a\eta)_X"] \arrow[dr] & \Sigma^a\Sigma^{-a}\Sigma^a X \arrow[d, "(\varepsilon\Sigma^a)_X"] \\
 & \Sigma^a X
\end{tikzcd}$$

Given an object X in $\mathcal{C}$, define $\eta_X : X \to \Sigma^{-a}\Sigma^a X = S^{-a} \otimes S^a \otimes X$ to be the composition

$$X \xrightarrow{\lambda_X^{-1}} S \otimes X \xrightarrow{\phi_{-a,a}\otimes X} S^{-a} \otimes S^a \otimes X.$$

Clearly this is an isomorphism. To see this is natural, let $f : X \to Y$ in $\mathcal{C}$. Then consider the following diagram:

$$\begin{tikzcd}
X \arrow[r, "\lambda_X^{-1}"] \arrow[d, "f"'] & S \otimes X \arrow[r, "\phi_{-a,a}\otimes X"] \arrow[d, "S\otimes f"] & S^{-a} \otimes S^a \otimes X \arrow[d, "S^{-a}\otimes S^a\otimes f"] \\
Y \arrow[r, "\lambda_Y^{-1}"'] & S \otimes Y \arrow[r, "\phi_{-a,a}\otimes Y"'] & S^{-a} \otimes S^a \otimes Y
\end{tikzcd}$$

The left square commutes by naturality of λ. The right square commutes by functoriality of $- \otimes -$. Hence η is indeed a natural isomorhpism.

On the other hand, given an object X in $\mathcal{C}$, define $\varepsilon_X : \Sigma^a\Sigma^{-a}X = S^a \otimes S^{-a} \otimes X \to X$ to be the composition

$$S^a \otimes S^{-a} \otimes X \xrightarrow{\phi^{-1}_{a,-a}} S \otimes X \xrightarrow{\lambda_X} X.$$

Clearly this is an isomorphism. To see it is natural, let $f : X \to Y$ in $\mathcal{C}$. Then consider the following diagram:

$$\begin{tikzcd}
S^a \otimes S^{-a} \otimes X \arrow[r, "\phi^{-1}_{a,-a}\otimes X"] \arrow[d, "S^a\otimes S^{-a}\otimes f"'] & S \otimes X \arrow[r, "\lambda_X"] \arrow[d, "S\otimes f"'] & X \arrow[d, "f"] \\
S^a \otimes S^{-a} \otimes Y \arrow[r, "\phi^{-1}_{a,-a}\otimes Y"'] & S \otimes Y \arrow[r, "\lambda_Y"'] & Y
\end{tikzcd}$$

The left square commutes by functoriality of $-\otimes-$. The right square commutes by naturality of λ. Hence, ε is natural.

Finally, let X be an object in $\mathcal{C}$. Unravelling definitions, by functoriality of $-\otimes-$, in order to show that diagram (1) commutes, it suffices to show the following diagram commutes:

$$\begin{tikzcd}
S^a \otimes X \arrow[r, "S^a\otimes\lambda_X^{-1}"] \arrow[ddrr, equal] & S^a \otimes S \otimes X \arrow[r, "S^a\otimes\phi_{-a,a}\otimes X"] & S^a \otimes S^{-a} \otimes S^a \otimes X \arrow[d, "\phi^{-1}_{a,-a}\otimes S^a\otimes X"] \\
 & & S \otimes S^a \otimes X \arrow[d, "\lambda_{S^a\otimes X}"] \\
 & & S^a \otimes X \arrow[uul, "\phi_{a,0}\otimes X"]
\end{tikzcd}$$

First, note that by the coherence theorem for monoidal categories, $\lambda_{S^a\otimes X} = \lambda_{S^a} \otimes X$[2]. And furthermore, recall $\lambda_{S^a} = \phi^{-1}_{0,a}$. Hence, the right triangle is precisely the diagram obtained by applying $-\otimes X$ to the coherence diagram for the $\phi_{a,b}$'s, so it commutes. Commutativity of the left triangle follows by the coherence theorem for monoidal categories and the fact that $\phi_{a,0} = \lambda^{-1}_{S^a}$. Hence, the diagram commutes, so (Σ^a, Σ^{-a}) forms an adjoint equivalence of $\mathcal{C}$. □

Remark 2.8. Note that because of these isomorphisms, given objects X and Y in $\mathcal{C}$, we have a canonical isomorphism of A-graded abelian groups

$$[X, \Sigma Y]_* = [S^* \otimes X, S^{\mathbf{1}} \otimes Y] \cong [S^{-\mathbf{1}} \otimes S^* \otimes X, Y] \cong [S^{*-\mathbf{1}} \otimes X, Y] = [X, Y]_{*-\mathbf{1}}$$

where the first isomorphism is obtained by the adjunction $\Sigma^{-\mathbf{1}} \dashv \Sigma^{\mathbf{1}}$ and the second isomorphism is induced by the isomorphism

$$S^{-\mathbf{1}} \otimes S^* \xrightarrow{\phi^{-1}_{-\mathbf{1},*}} S^{*-\mathbf{1}}.$$

Given objects E and X in $\mathcal{C}$, we can define an A-graded abelian group $E_*(X)$ and $E^*(X)$ by

$$E_a(X) := \pi_a(E \otimes X) = [S^a, E \otimes X] \cong [S, E \otimes X]_a$$

and

$$E^a(X) := [X, S^a \otimes E] \cong [S^{-a} \otimes X, E] = [X, E]_{-a} \cong \pi_{-a}(F(X, E)).$$

A particularly important class of elements in $\mathcal{C}$ are the *cellular* objects, which intuitively are those that can be constructed from the S^a by taking (co)fibers.

Definition 2.9. Define the class of *cellular* objects in $\mathcal{C}$ to be the smallest class of objects such that:

[2]Technically, this equality only holds up to composition with an associator, but we are ignoring such issues.

(1) For all $a \in A$, S^a is cellular.
(2) If we have a distinguished triangle
$$X \to Y \to Z \to \Sigma X (= S^{\mathbf{1}} \otimes X)$$
such that two of the three objects X, Y, and Z are cellular, than the third object is also cellular.
(3) Given a collection of cellular objects X_i indexed by some small set I, $\bigoplus_{i \in I} X_i$ is cellular.

2.2. **Construction of the Adams spectral sequence.** In what follows, let E be a commutative monoid object in $\mathcal{C}$.

Definition 2.10. Let $\overline{E}$ be the fiber of the unit map $e : S \to E$ (Proposition A.3), and for $s \geq 0$ define
$$Y_s := \overline{E}^s \otimes Y, \qquad W_s = E \otimes Y_s = E \otimes (\overline{E}^s \otimes Y),$$
where recall for $s > 0$, $\overline{E}^s$ denotes the s-fold product parenthesized as $\overline{E} \otimes (\overline{E} \otimes \cdots (\overline{E} \otimes \overline{E}))$, and $\overline{E}^0 := S$. Then we get fiber sequences
$$Y_{s+1} \xrightarrow{i_s} Y_s \xrightarrow{j_s} W_s \xrightarrow{k_s} \Sigma Y_{s+1} (= S^{\mathbf{1}} \otimes Y_{s+1})$$
obtained by applying $- \otimes Y_s$ to the sequence
$$\overline{E} \to S \xrightarrow{e} E \to \Sigma \overline{E}$$
(and applying the necessary associator isomorphisms). These sequences can be spliced together to form the *(canonical) Adams filtration of Y*:
$$\begin{tikzcd}
\cdots \arrow[r] & Y_3 \arrow[r, "i_2"] \arrow[d, "j_3"] & Y_2 \arrow[r, "i_1"] \arrow[d, "j_2"] & Y_1 \arrow[r, "i_0"] \arrow[d, "j_1"] & Y_0 = Y \arrow[d, "j_0"] \\
& W_3 & W_2 \arrow[ul, dashed, "k_2"'] & W_1 \arrow[ul, dashed, "k_1"'] & W_0 \arrow[ul, dashed, "k_0"']
\end{tikzcd}$$
where the diagonal dashed arrows are of degree $-\mathbf{1}$ (note these triangles do NOT commute in any sense). Now we may apply the functor $[X, -]_*$, and by Proposition A.4 we obtain an exact couple of $\mathbb{N} \times A$-graded abelian groups:
$$\begin{tikzcd}
{[X, Y_*]_*} \arrow[rr, "i_{**}"] & & {[X, Y_*]_*} \arrow[d, "j_{**}"] \\
& & {[X, W_*]_*} \arrow[ull, "k_{**}"]
\end{tikzcd}$$
where i_{**}, j_{**}, and k_{**} have $\mathbb{Z} \times A$-degree $(-1, 0)$, $(0, 0)$, and $(1, -\mathbf{1})$, respectively[3]. The standard argument yields a $\mathbb{N} \times A$-graded spectral sequence called from this exact couple (cf. Section 5.9 of [16]) with E_1 page given by
$$E_1^{s,a} = [X, W_s]_a$$
and r^{th} differential of $\mathbb{Z} \times A$-degree $(r, -\mathbf{1})$:
$$d_r : E_r^{s,a} \to E_r^{s+r, a-\mathbf{1}}.$$
A priori, this is all $\mathbb{N} \times A$-graded, but we regard it as being $\mathbb{Z} \times A$-graded by setting $E_r^{s,a} := 0$ for $s < 0$ and trivially extending the definition of the differentials to these zero groups. This spectral sequence is called the *E-Adams spectral sequence* for the computation of $[X, Y]_*$. The index s is called the *Adams filtration* and a is the *stem*.

[3]Explicitly, the map $k_{s,a} : [X, W_s]_a \to [X, Y_{s+1}]_{a-\mathbf{1}}$ sends a map $f : S^a \otimes X \to W_s$ to the map $S^{a-\mathbf{1}} \otimes X \to Y_{s+1}$ corresponding under the isomorphism $[X, \Sigma Y_{s+1}]_* \cong [X, Y_{s+1}]_{*-\mathbf{1}}$ to the composition $k_s \circ f : S^a \otimes X \to \Sigma Y_{s+1}$.

2.3. **The E1 page.** The goal of this subsection is to provide the following characterization for the E_1 page of the Adams spectral sequence:

Theorem 2.11. *Let E be a flat commutative ring spectrum, and let X and Y be two objects in $\mathcal{C}$ such that $E_*(X)$ is a projective module over $\pi_*(E)$. Then for all $s \geq 0$ and $a \in A$, we have isomorphisms in the associated E-Adams spectral sequence*

$$E_1^{s,a} \cong \mathrm{Hom}^a_{E_*(E)}(E_*(X), E_*(W_s))$$

Furthermore, under these isomorphisms, the differential $d_1 : E_1^{s,a} \to E_1^{s+1,a-\mathbf{1}}$ corresponds to the map

$$\mathrm{Hom}^a_{E_*(E)}(E_*(X), E_*(W_s)) \to \mathrm{Hom}^{a-\mathbf{1}}_{E_*(E)}(E_*(X), E_*(X \otimes W_{s+1}))$$

which sends a map $f : E_*(X) \to E_{*+a}(W_s)$ to the composition

$$E_*(X) \xrightarrow{f} E_{*+a}(W_s) \xrightarrow{(X\otimes h_s)_*} E_{*+a-\mathbf{1}}(X \otimes Y_{s+1}) \xrightarrow{(X\otimes j_{s+1})_*} E_{*+a-\mathbf{1}}(X \otimes W_{s+1}).$$

Proof. By Lemma B.13, for all $s \geq 0$ and $t, w \in \mathbb{Z}$, we have isomorphisms

$$[X, E \otimes Y_s]_{t,w} \cong \mathrm{Hom}^{t,w}_{E_*(E)}(E_*(X), E_*(E \otimes Y_s)).$$

since $W_s = E \otimes Y_s$, we have that

$$E_1^{s,(t,w)} = [X, W_s]_{t,w} \cong \mathrm{Hom}^{t,w}_{E_*(E)}(E_*(X), E_*(W_s)),$$

as desired. □

2.4. **The E2 page.**

2.5. **Convergence.** convergence of spectral sequences

3. The classical Adams spectral sequence

4. The motivic Adams spectral sequence

talk about motivic eilenberg-maclane spectra and cellularity

One of the key ideas in classical topology is that in order to study "nice spaces" like CW complexes or manifolds, we should work with a larger category $\mathcal{S}$ which has better formal properties, but not as nice of spaces. In topology, there are multiple candidates for this category, such as the category of simplicial sets or the category of compactly generated weak Hausdorff spaces. For our purposes, we will take $\mathcal{S} = \mathbf{Set}_{\mathbf{\Delta}}$ to be the category of simplicial sets. In this larger category, we can do homotopy theory. $\mathbb{A}^1$-homotopy theory, also called motivic homotopy theory, is motivated by applying this philosophy to the field of algebraic geometry.

4.1. **Motivic spaces.** In algebraic geometry, the key objects of study are *varieties*, i.e., smooth finite type schemes over $\operatorname{Spec} k$ for some field k. More generally, instead of considering schemes over a field k, we can consider smooth finite type schemes over some *base scheme* $\mathscr{S}$, where a "base scheme" is a Noetherian separated scheme of finite Krull dimension. We write $\mathbf{Sm}/\mathscr{S}$ to denote the category of smooth finite type schemes over $\mathscr{S}$. Often times we will write "smooth scheme over $\mathscr{S}$" or just "smooth scheme" to denote an object of $\mathbf{Sm}/\mathscr{S}$. Sadly, like the category of smooth manifolds, the category $\mathbf{Sm}/\mathscr{S}$ does not satisfy many nice formal properties, in particular, it does not have colimits, as there is no way to "glue" smooth schemes together. Taking our queue from topology, we should therefore expand the category $\mathbf{Spc}(\mathscr{S})$ to some larger category of "motivic spaces" with nice formal properties. This construction is the motivating idea behind $\mathbb{A}^1$-homotopy theory.

As it turns out, there are lots of ways to define the category of motivic spaces. We will follow the approach outlined in Section 2 of [17]. We omit many technical details, and emphasize only what we need.

Definition 4.1. A *(motivic) space* over $\mathscr{S}$ is a simplicial presheaf on $\mathbf{Sm}/\mathscr{S}$. The collection of spaces over $\mathscr{S}$ forms the category

$$\mathbf{Spc}(\mathscr{S}) := [(\mathbf{Sm}/\mathscr{S})^{\mathrm{op}}, \mathcal{S}].$$

There is already a lot we can do with this definition. Since $\mathcal{S}$ is complete and cocomplete, it follows purely formally that the category of motivic spaces is as well, so we have achieved our goal of being able to take (co)limits, which may be computed pointwise in $\mathcal{S}$. Furthermore, the requirement that objects in $\mathbf{Sm}/\mathscr{S}$ be finite type schemes over $\mathscr{S}$ ensures that $\mathbf{Sm}/\mathscr{S}$ is an essentially small category ([1]), so that $\mathbf{Spc}(\mathscr{S})$ is cartesian closed[4], and we do not have to worry about size issues (the collection of objects in $\mathbf{Spc}(\mathscr{S})$ forms a proper class).

Since $\mathcal{S} := [\mathbf{\Delta}^{\mathrm{op}}, \mathbf{Set}]$, we have an identification[5]

$$\mathbf{Spc}(\mathscr{S}) = [(\mathbf{Sm}/\mathscr{S})^{\mathrm{op}}, \mathcal{S}] = [(\mathbf{Sm}/\mathscr{S})^{\mathrm{op}}, [\mathbf{\Delta}^{\mathrm{op}}, \mathbf{Set}]] \cong [\mathbf{\Delta}^{\mathrm{op}}, [(\mathbf{Sm}/\mathscr{S})^{\mathrm{op}}, \mathbf{Set}]].$$

Hence, we may also think of motivic spaces as simplicial objects in the category of presheaves on $\mathbf{Sm}/\mathscr{S}$. In this way, by composing the Yoneda embedding with the diagonal functor, we have an embedding

$$h_{(-)} : \mathbf{Sm}/\mathscr{S} \xrightarrow{\mathcal{Y}} [(\mathbf{Sm}/\mathscr{S})^{\mathrm{op}}, \mathbf{Set}] \xrightarrow{\Delta} [\mathbf{\Delta}^{\mathrm{op}}, [(\mathbf{Sm}/\mathscr{S})^{\mathrm{op}}, \mathbf{Set}]] \cong \mathbf{Spc}(\mathscr{S})$$

taking a smooth scheme $\mathscr{X}$ to the simplicial presheaf $h_{\mathscr{X}}$ it represents. It is not hard to verify that this functor is fully faithful, since the Yoneda embedding is. Often we will not distinguish between a smooth scheme $\mathscr{X}$ and its image under this functor. We may also define based spaces:

Definition 4.2. A *based* (motivic) space is an object in the under category

$$\mathbf{Spc}_*(\mathscr{S}) := \mathbf{Spc}(S)^{\mathscr{S}/},$$

i.e., a based space is a motivic space X along with a morphism (natural transformation) $\mathscr{S} \to X$.

This definition is motivated by the observation that $\mathscr{S}$ is the terminal motivic space. Indeed, note that by definition $\mathscr{S}$ is the terminal object in $\mathbf{Sm}/\mathscr{S}$, so that given any other smooth scheme $\mathscr{U}$ over $\mathscr{S}$, there is a unique morphism $\mathscr{U} \to \mathscr{S}$, so that $h_{\mathscr{S}}(\mathscr{U}) \cong \Delta^0$. It follows purely formally that the forgetful functor $\mathbf{Spc}_*(\mathscr{S}) \to \mathbf{Spc}(S)$ has a left adjoint $(-)_+ : \mathbf{Spc}(\mathscr{S}) \to \mathbf{Spc}_*(\mathscr{S})$ taking a motivic space X to the disjoint union $X \amalg \mathscr{S}$ obtained by freely adjoining a basepoint.

We point out a couple examples of motivic spaces which will be important. In what follows, all products are taken in the category of schemes, not in $\mathbf{Sm}/\mathscr{S}$, so that in particular given any object $\mathscr{X}$ in $\mathbf{Sm}/\mathscr{S}$, there are canonical isomorphisms $\mathscr{X} \cong \mathscr{X} \times \operatorname{Spec} \mathbb{Z}$, as $\operatorname{Spec} \mathbb{Z}$ is the terminal scheme. Let $\mathbb{A}^1$ and $\mathbb{G}_m$ denote the smooth schemes $\mathscr{S} \times \operatorname{Spec} \mathbb{Z}[x]$ and $\mathscr{S} \times \operatorname{Spec} \mathbb{Z}[x, x^{-1}]$, respectively. We may consider $\mathbb{A}^1$ as canonically based via the composition

$$\mathscr{S} \cong \mathscr{S} \times \operatorname{Spec} \mathbb{Z} \to \mathscr{S} \times \operatorname{Spec} \mathbb{Z}[x] = \mathbb{A}^1,$$

where the arrow is given by $\mathscr{S} \times f$, where $f : \operatorname{Spec} \mathbb{Z} \to \operatorname{Spec} \mathbb{Z}[x]$ corresponds to the ring morphism $\mathbb{Z}[x] \to \mathbb{Z}$ sending $x \mapsto 1$. Similarly, we may view $\mathbb{G}_m$ as canonically based via the map

$$\mathscr{S} \cong \mathscr{S} \times \operatorname{Spec} \mathbb{Z} \to \mathscr{S} \times \operatorname{Spec} \mathbb{Z}[x, x^{-1}] = \mathbb{G}_m,$$

where the arrow is similarly induced by the unique ring morphism $\mathbb{Z}[x, x^{-1}] \to \mathbb{Z}$ sending $x \mapsto 1$.

Note that in the case $\mathscr{S} = \operatorname{Spec} k$ for some field k, we have identifications $\mathbb{A}^1 \cong \mathbb{A}^1_k = \operatorname{Spec} k[x]$ and $\mathbb{G}_m \cong \operatorname{Spec} k[x, x^{-1}] \cong \mathbb{A}^1_k \setminus \{0\}$, which justifies our notation. As we will see, $\mathbb{A}^1$ will play a role similar to the interval in the homotopy theory of motivic spaces. Thought of as a motivic

[4]This follows from the general categorical result that given a small category $\mathcal{C}$ and a cartesian closed complete category $\mathcal{D}$, the functor category $[\mathcal{C}, \mathcal{D}]$ is itself cartesian closed.

[5]This follows from the more general fact that given three categories $\mathcal{C}$, $\mathcal{D}$, and $\mathcal{E}$, there is a canonical isomorphism $[\mathcal{C}, [\mathcal{D}, \mathcal{E}]] \cong [\mathcal{D}, [\mathcal{C}, \mathcal{E}]]$.

space, we call $\mathbb{G}_m$ the "Tate circle". It turns out that as a motivic space, $\mathbb{G}_m$ has many of the same properties that the topological circle S^1 has in the category of topological spaces. To see this, consider the case $\mathscr{S} = \operatorname{Spec}\mathbb{C}$. We have a "realization functor" $\psi : \mathbf{Sm}/\mathbb{C} \to \mathbf{Top}$ taking a scheme $\mathscr{X}$ to its set of $\mathbb{C}$-points with the analytic topology. Under this functor, $\mathbb{A}^1$ and $\mathbb{G}_m$ are taken to the spaces $\mathbb{C}$ and $\mathbb{C} \setminus \{0\}$, respectively. Note that $\mathbb{C} \setminus \{0\}$ is homotopy equivalent to the circle, which already provides one justification for thinking of $\mathbb{G}_m$ as a circle.

We can extend the realization functor $\psi : \mathbf{Sm}/\mathbb{C} \to \mathbf{Top}$ to a functor defined on all of $\mathbf{Spc}(\mathbb{C})$:

Definition 4.3. Define the *Betti realization functor* $\psi : \mathbf{Spc}(\mathbb{C}) \to \mathbf{Top}$ to be the left Kan extension of the realization functor $\psi : \mathbf{Sm}/\mathbb{C} \to \mathbf{Top}$ along the simplicial Yoneda embedding $h_{(-)} : \mathbf{Sm}/\mathbb{C} \to \mathbf{Spc}(\mathbb{C})$:

$$\begin{tikzcd} \mathbf{Sm}/\mathbb{C} \arrow[rr, "\psi"] \arrow[dr, "h_{(-)}"'] & & \mathbf{Top} \\ & \mathbf{Spc}(\mathbb{C}) \arrow[ur, dashed, "\psi"'] & \end{tikzcd}$$

Given a space X, we often denote $\psi(X)$ by $X(\mathbb{C})$.

Since $\mathbf{Sm}/\mathbb{C}$ is (essentially) small and $\mathbf{Top}$ is (small) cocomplete, it follows that this left Kan extension does in fact exist, and we may compute it via colimits ([14, Theorem 6.2.1]).

Recall that $\mathcal{S}_*$ is a symmetric monoidal category under the smash product $- \wedge -$. The unit for this monoidal structure is given by $S^0 := \Delta^0 \amalg \Delta^0$. This induces a symmetric monoidal structure on the category $\mathbf{Spc}_*(\mathscr{S})$ of based spaces over $\mathscr{S}$:

Proposition 4.4. *Given two based motivic spaces X and Y over $\mathscr{S}$, define their* smash product *$X \wedge Y$ to be the simplicial presheaf defined by*

$$(X \wedge Y)(\mathscr{U}) := X(\mathscr{U}) \wedge Y(\mathscr{U}).$$

This smash product endows $\mathbf{Spc}_*(\mathscr{S})$ with the structure of a symmetric monoidal category, where the unit object is given by $S^{0,0} := \mathscr{S}_+ = \mathscr{S} \amalg \mathscr{S} \cong \Delta^0 \amalg \Delta^0$.

We have shown that any smooth scheme can be viewed as a motivic space, but it also turns out that any simplicial set A can viewed as a motivic space by considering the constant functor $cA : (\mathbf{Sm}/\mathscr{S})^{\mathrm{op}} \to \mathcal{S}$ on A. As we did with objects of $\mathbf{Sm}/\mathscr{S}$, we will usually simply write A to denote the corresponding motivic space cA. Per our earlier reasoning, $\mathscr{S}$ and Δ^0 are isomorphic as motivic spaces. This observation also yields a functor $\mathcal{S}_* \to \mathbf{Spc}_*(\mathscr{S})$ taking a based simplicial set $\Delta^0 \to A$ to the based motivic space $\mathscr{S} \cong c\Delta^0 \to cA$. It follows that this functor is strongly monoidal, i.e., it preserves the monoidal unit, and given any two based simplicial sets A and B we have $cA \wedge cB \cong c(A \wedge B)$ (in fact, here this isomorhpism is an equality). Furthermore, it is relatively straightforward to check that the functor $\mathbf{Spc}_*(\mathscr{S}) \to \mathcal{S}_*$ given by evaluation at $\mathscr{S}$ is a right adjoint to c.

Interestingly, this functor yields another circle in the category of pointed spaces. We can define the *simplicial circle* to be the constant simplicial presheaf S^1 pointed at its 0-simplex. As it turns out, the simplicial circle S^1 really is an entirely distinct space from the Tate circle $\mathbb{G}_m$. So which is "the" circle? As it turns out, the approach taken in motivic homotopy theory is to view them as each equally valid, but different notions, and in fact, we obtain a bigraded family of motivic spheres $S^{p,q}$ in $\mathbf{Spc}_*(\mathscr{S})$ for $p \geq q \geq 0$ by defining

$$S^{p,q} := (S^1)^{p-q} \wedge \mathbb{G}_m^q,$$

so that $S^{1,0} \cong S^1$, $S^{1,1} \cong \mathbb{G}_m$, and $S^{0,0} \cong \mathscr{S}_+ \cong S^0$ (recall $\mathscr{S}_+ = h_{\mathscr{S}} \amalg h_{\mathscr{S}}$ is the monoidal unit in $\mathbf{Spc}_*(\mathscr{S})$). The reason for this odd grading convention has to do with the theory of *motives*, which we will not explore here.

4.2. **The unstable motivic homotopy category.** So far, we have constructed motivic spaces, and given some examples of how to work with them. Yet, we still have yet to talk about how we can "do homotopy theory" in this world. To start with, we will define the *motivic model structure* on $\mathbf{Spc}_*(\mathscr{S})$. We will define this in stages, by first defining the *projective model structure* on $\mathbf{Spc}(\mathscr{S})$ and then localizing.

Proposition 4.5. *There exists a cellular, proper, simplicial monoidal model structure on* $\mathbf{Sm}/S$ *called the* projective model structure *in which*

(1) The (global) weak equivalences are those maps $f : X \to Y$ for which $f(\mathscr{U}) : X(\mathscr{U}) \to Y(\mathscr{U})$ is a weak equivalence of simplicial sets for all $\mathscr{U}$ in $\mathbf{Sm}/\mathscr{S}$,
(2) The projective fibrations are those maps $f : X \to Y$ for which $f(\mathscr{U}) : X(\mathscr{U}) \to Y(\mathscr{U})$ is a Kan fibration for all $\mathscr{U}$ in $\mathbf{Sm}/\mathscr{S}$.
(3) The projective cofibrations are those maps in $\mathbf{Spc}(\mathscr{S})$ which satisfy the left lifting property against the trivial projective fibrations.

Of course, this also endows $\mathbf{Spc}_*(\mathscr{S})$ with a model structure, which we also call the projective model structure. There exists a Grothendieck topology on $\mathbf{Spc}_*(\mathscr{S})$ called the *Nisnevich topology.*

Definition 4.6. Given a pointed space X in $\mathbf{Spc}_*(\mathscr{S})$ and some $n \geq 0$, the n^{th} simplicial homotopy sheaf $\pi_n(X)$ of X is the Nisnevich sheaffification of the presheaf $\mathscr{U} \mapsto \pi_n(X(\mathscr{U}))$. Write W_{Nis} for the class of maps $f : X \to Y$ in $\mathbf{Spc}_*(\mathscr{S})$ for which $f_* : \pi_n(X) \to \pi_n(Y)$ is an isomorphism of Nisnevich sheaves for all $n \geq 0$.

Definition 4.7. Let $W_{\mathbb{A}^1} \subseteq \text{Mor}(\mathbf{Spc}_*(\mathscr{S}))$ be the class of maps $\pi_{\mathscr{X}} : (\mathscr{X} \times \mathbb{A}^1)_+ \to \mathscr{X}_+$ for $\mathscr{X}$ in $\mathbf{Sm}/\mathscr{S}$. The *motivic model structure* on $\mathbf{Spc}_*(\mathscr{S})$ is the left Bousfield localization of the projective model structure with respect to $W_{\text{Nis}} \cup W_{\mathbb{A}^1}$. This model structure is closed symmetric monoidal, pointed simplicial, left proper, and cellular. From now on, we always write $\mathbf{Spc}_*(\mathscr{S})$ to mean the model category of pointed spaces equipped with the motivic model structure. The homotopy category of $\mathbf{Spc}_*(\mathscr{S})$ is the pointed motivic homotopy category $\mathbf{H}_*(\mathscr{S})$.

4.3. **The stable motivic homotopy category.** The canonical ring morphism $\mathbb{Z}[x] \hookrightarrow \mathbb{Z}[x, x^{-1}]$ induces a map $\mathbb{G}_m \to \mathbb{A}^1$. Let T be a cofibrant replacement of the quotient simplicial sheaf $\mathbb{A}^1/\mathbb{G}_m$ in the stable model structure on $\mathbf{Spc}_*(\mathscr{S})$. We call T the *Tate sphere*. A useful fact is that the Tate sphere is equivalent to $S^1 \wedge \mathbb{G}_m$ in the motivic model structure on $\mathbf{Spc}_*(\mathscr{S})$ ([9, Lemma 3.2.15]).

It turns out that the functor $T \wedge -$ on $\mathbf{Spc}_*(\mathscr{S})$ is a left Quillen functor, and we may invert it to create the category $\mathbf{Spt}_T(\mathscr{S})$ of T-spectra. Explicitly:

Definition 4.8. A T-spectrum X is a sequence of spaces $\{X_n\}_{n\geq 0}$ in $\mathbf{Spc}_*(\mathscr{S})$ equipped with structure maps $\sigma_n : T \wedge X_n \to X_{n+1}$. A map of T-spectra $f : X \to Y$ is a collection of maps $f_n : X_n \to Y_n$ which are compatible with the structure maps in the obvious sense. We write $\mathbf{Spt}_T(\mathscr{S})$ to denote the category of T-spectra and maps between them.

Definition 4.9. Given a based space X in $\mathbf{Spc}_*(\mathscr{S})$, we can form its *suspension spectrum* $\Sigma^\infty X$ whose n^{th} term is $X \wedge T^n$ and the structure morphisms are the canonical isomorphisms. This yields a functor $\Sigma^\infty : \mathbf{Spc}_*(\mathscr{S}) \to \mathbf{Spt}_T(\mathscr{S})$, and by composing with $(-)_+ : \mathbf{Spc}(\mathscr{S}) \to \mathbf{Spc}_*(\mathscr{S})$, we get a functor $\Sigma^\infty(-)_+ : \mathbf{Spc}(\mathscr{S}) \to \mathbf{Spt}_T(\mathscr{S})$.

Now, we would like to define the stable model structure on the category of T-specra. As we did with motivic spaces, we first start with a different model structure than the one we want, and then localize to obtain the stable model structure.

Proposition 4.10. *There exists a model structure on* $\mathbf{Spt}_T(\mathscr{S})$ *called the* level model structure *in which a map $f : X \to Y$ is a weak equivalence (resp. a fibration) if every map $f_n : X_n \to Y_n$ is a weak equivalence (resp. a fibration) in the motivic model structure on $\mathbf{Spc}_*(\mathscr{S})$. The cofibrations are determined as those with the left lifting property against the trivial level fibrations.*

Definition 4.11. Let X be a T-spectrum. For integers p and q, the $(p,q)^{\text{th}}$ stable homotopy sheaf of X, written as $\pi_{p,q}(X)$, is the Nisnevich sheafification of the presheaf

$$\mathscr{U} \mapsto \operatorname*{colim}_n \mathbf{H}_*(\mathscr{U})(S^{p+2n,q+n}, X_n|_{\mathscr{U}})$$

(note the terms in this colimit may only be defined for large enough n). A map $f : X \to Y$ is a *stable weak equivalence* if for all integers p and q the induced maps $f_* : \pi_{p,q}(X) \to \pi_{p,q}(Y)$ are isomorphisms.

Definition 4.12. The stable model structure on $\mathbf{Spt}_T(\mathscr{S})$ is the model category where the weak equivalences are the stable weak equivalences and the cofibrations are the cofibrations in the level model structure. The fibrations are those maps with the right lifting property with respect to trivial cofibrations. We write $\mathbf{SH}_{\mathscr{S}}$ for the homotopy category of $\mathbf{Spt}_T(\mathscr{S})$ with the stable model structure.

As in the case of classical spectra, we run into the unfortunate fact that the smash product does not induce a symmetric monoidal structure on on $\mathbf{Spt}_T(\mathscr{S})$. One remedy is to use the category $\mathbf{Spt}_T^{\Sigma}(S)$ of symmetric T-spectra. The construction of this category is given by Hovey in [6] and Jardine in [7], and it turns out that the smash product can be used to give $\mathbf{Spt}_T^{\Sigma}(S)$ the structure of a symmetric monoidal category, in fact, a stable symmetric monoidal model category. It is proven in [6] that there is a zig-zag of Quillen equivalences from $\mathbf{Spt}_T^{\Sigma}(S)$ to $\mathbf{Spt}_T(\mathscr{S})$, hence $\mathbf{SH}_{\mathscr{S}}$ is equivalent to the homotopy category of $\mathbf{Spt}_T^{\Sigma}(S)$ as well. In particular, the category $\mathbf{SH}_{\mathscr{S}}$ is a tensor triangulated category where the shift functor $\Sigma := \Sigma^\infty S^{1,0} \wedge -$ is given by smashing with the suspension spectrum of $S^{1,0} = S^1 \wedge \mathscr{S}_+ \cong S^1$. The monoidal product $- \wedge -$ is induced by the smash product[6], and the monoidal unit is given by the *sphere spectrum* $S := \Sigma^\infty \mathscr{S}_+ \cong \Sigma^\infty S^0$. See the appendix for a review of (tensor) triangulated categories.

Recall earlier we defined the functor $\Sigma^\infty : \mathbf{Spc}_*(\mathscr{S}) \to \mathbf{Spt}_T(\mathscr{S})$ taking a based space to its suspension spectrum. From now on, we will instead write Σ^∞ to refer to the composition

$$\mathbf{Spc}_*(\mathscr{S}) \xrightarrow{\Sigma^\infty} \mathbf{Spt}_T(\mathscr{S}) \to \mathbf{SH}_{\mathscr{S}},$$

where the second arrow is the canonical functor from a model category to its homotopy category. A useful fact is that Σ^∞ is strict monoidal, so that there are isomorphisms

$$\Sigma^\infty X \wedge \Sigma^\infty Y \cong \Sigma^\infty (X \wedge Y)$$

in $\mathbf{SH}_{\mathscr{S}}$ for all based spaces X and Y, and furthermore, this functor factors through the unstable homotopy category $\mathbf{H}_*(\mathscr{S})$. Hence since T is weakly equivalent to $S^{2,1} = S^{1,0} \wedge S^{1,1}$ in $\mathbf{Spc}_*(\mathscr{S})$, we have the following isomorphisms in $\mathbf{SH}_{\mathscr{S}}$:

$$T \cong S^{2,1} \cong S^{1,0} \wedge S^{1,1}$$

(here we are being abusive and omitting Σ^∞'s for clarity). Almost by construction, T is invertible in $\mathbf{SH}_{\mathscr{S}}$, in the sense that there exists some object T^{-1} in $\mathbf{SH}_{\mathscr{S}}$ and an isomorphism $S \cong T^{-1} \wedge T$. Now, define the spectra

$$S^{-1,0} := T^{-1} \wedge S^{1,1} \qquad \text{and} \qquad S^{-1,-1} := T^{-1} \wedge S^{1,0} (\cong \Sigma T).$$

The notation is justified by the isomorphisms

$$\xi_1 : S \cong T^{-1} \wedge T \cong T^{-1} \wedge S^{1,1} \wedge S^{1,0} = S^{-1,0} \wedge S^{1,0}$$

and

$$\xi_2 : S \cong T^{-1} \wedge T \cong T^{-1} \wedge S^{1,1} \wedge S^{1,0} \cong T^{-1} \wedge S^{1,0} \wedge S^{1,1} = S^{-1,-1} \wedge S^{1,1}.$$

[6] Sadly, explicitly constructing the monoidal product on $\mathbf{SH}_{\mathscr{S}}$ is actually quite difficult.

In this way, by abuse of notation, we may define $\mathbb{Z}\times\mathbb{Z}$–graded family of motivic sphere spectra in $\mathbf{SH}_{\mathscr{S}}$ by defining

$$S^{p,q} := (S^{1,0})^{p-q} \wedge (S^{1,1})^{q}$$

for $p, q \in \mathbb{Z}$ (recall our earlier defined conventions for powers in a monoidal category). It follows purely formally that for all $a, b \in \mathbb{Z}^2$ there exist "semi-canonical" isomorphisms[7]

$$S^{a,b} \cong S^{a} \wedge S^{b},$$

and given $p, q \in \mathbb{Z}$, the functors $S^{p,q} \wedge -$ and $S^{-p,-q} \wedge -$ form an adjoint equivalence of $\mathbf{SH}_{\mathscr{S}}$. Given a spectrum X in $\mathbf{SH}_{\mathscr{S}}$, we write $\Sigma^{p,q}$ to denote the functor defined by $\Sigma^{p,q}X := S^{p,q} \wedge X$. In particular, the shift functor [1] in the triangulated structure on $\mathbf{SH}_{\mathscr{S}}$ is given by $\Sigma^{1,0}$, and we have canonical isomorphisms $\Sigma^{p,q}S \cong S^{p,q}$. Note that since $\Sigma^\infty : \mathbf{Spc}_*(\mathscr{S}) \to \mathbf{SH}_{\mathscr{S}}$ is strict monoidal, we have isomorphisms $\Sigma^\infty S^{p,q} \cong S^{p,q}$ for all $p \geq q \geq 0$.

Given spectra X and Y, we denote the abelian group $\mathbf{SH}_{\mathscr{S}}(X, Y)$ by $[X, Y]$[8]. We may extend $[X, Y]$ to a $\mathbb{Z}^2$-graded abelian group $[X, Y]_{**}$ by defining

$$[X, Y]_{p,q} := [\Sigma^{p,q}X, Y] = [S^{p,q} \wedge X, Y].$$

We denote the category of $\mathbb{Z}\times\mathbb{Z}$-graded abelian groups by $\mathbf{Ab}^{\mathbb{Z}^2}$. Given a spectrum E, it determines functors $E^{**} : \mathbf{SH}_{\mathscr{S}}^{\mathrm{op}} \to \mathbf{Ab}^{\mathbb{Z}^2}$ and $E_{**} : \mathbf{SH}_{\mathscr{S}} \to \mathbf{Ab}^{\mathbb{Z}^2}$, by defining

$$E^{p,q}(X) := [X, S^{p,q} \wedge E] = [X, E]_{-p,-q} \qquad \text{and} \qquad E_{p,q}(X) := [S^{p,q}, E \wedge X] \cong [S, E \wedge X]_{p,q}.$$

We call the functors E^{**} and E_{**} the *cohomology* and *homology* theories represented by E, respectively. One special homology theory is that represented by the sphere spectrum S, which we denote by π_{**}:

$$\pi_{p,q}(X) := [S^{p,q}, X] \cong [S^{p,q}, S \wedge X] = S_{**}(X).$$

Given a spectrum X, we refer to the collection of $\pi_{p,q}(X)$'s as the *stable homotopy groups* of X.

Note that in what happened above, we could have actually replaced $T \simeq \mathbb{A}^1/\mathbb{G}_m$ with any compact In Section A.7 of [13], the Betti realization functor (Definition 4.3) is extended to a strong symmetric monoidal functor $\psi : \mathbf{SH}_{\mathbb{C}} \to \mathbf{hoSp}$ from the motivic stable homotopy category over $\mathbb{C}$ to the classical stable homotopy category[9]. A useful fact, one which somewhat justifies the grading for the motivic spheres, is that ψ takes the T-spectrum $S^{p,q}$ to the suspension spectrum $S^p \cong \Sigma^\infty S^p$ of the p-sphere in $\mathbf{hoSp}$.

4.4. **Grading.** First, recall the standard stable homotopy category $\mathbf{hoSp}$, obtained by formally inverting the functor $\Sigma := S^1 \wedge - : \mathcal{S}_* \to \mathcal{S}_*$. It is a tensor triangulated category, where the tensor product is denoted by $-\wedge-$ and called the *smash product*. There exists a strong monoidal functor $\Sigma^\infty : \mathcal{S}_* \to \mathbf{hoSp}$. We omit Σ^∞ from the notation, and identify a space X with its suspension spectrum $\Sigma^\infty X$. The unit for the monoidal structure on $\mathbf{hoSp}$ is given by $S := S^0$. The shift functor is given by $\Sigma := S^1 \wedge -$. In particular, since the shift functor is essentially surjective, it follows that there exists some spectrum S^{-1} in $\mathbf{hoSp}$ and an isomorphism $\xi : S \cong S^{-1} \wedge S^1$. It then follows purely formally, using only the fact that $\mathbf{hoSp}$ is a symmetric monoidal category

[7] Explicitly, these isomorphisms are obtained by forming formal compositions of unitors, associators, and the isomorphisms $\xi_1 : S \cong S^{-1,0} \wedge S^{1,0}$ and $\xi_2 : S \cong S^{-1,-1} \wedge S^{1,1}$ and their inverses as necessary.

[8] Recall that $\mathbf{SH}_{\mathscr{S}}$ is triangulated, in particular, it is an additive category.

[9] Explicitly, in [13, Theorem A.44], the category $\mathrm{Sp}^{\Sigma}(\mathbf{Top}, \mathbb{C}P^1)$ of symmetric $\mathbb{C}P^1$-spectra in $\mathbf{Top}$ is constructed, and it is shown that there is a zig-zag of Quillen equivalences between $\mathrm{Sp}^{\Sigma}(\mathbf{Top}, \mathbb{C}P^1)$ and the usual category of spectra $\mathrm{Sp}^{\Sigma}(\mathbf{Top}, S^1)$, so they have equivalent homotopy categories. Then applying the Betti realization functor levelwise yields a strict symmetric monoidal functor (Theorem A.45) from the category $\mathbf{Sp}^{\Sigma}_{\mathbb{P}^1}(\mathbb{C})$ of motivic symmetric $\mathbb{P}^1$-spectra to $\mathrm{Sp}^{\Sigma}(\mathbf{Top}, \mathbb{C}P^1)$. Finally, in the category $\mathbf{Spc}_*(\mathbb{C})$ of motivic spaces over $\mathbb{C}$, we have that T and $\mathbb{P}^1$ are equivalent, which yields a Quillen equivalence $\mathbf{Sp}^{\Sigma}_{\mathbb{P}^1}(\mathbb{C}) \simeq \mathbf{Sp}^{\Sigma}_{T}(\mathbb{C})$ (Theorem A.30). Putting all of this together yields the desired strong symmetric monoidal functor $\mathbf{SH}_{\mathbb{C}} \to \mathbf{hoSp}$.

and the isomorphism $\xi : S \cong S^{-1} \wedge S^1$, that the functors $\Sigma = S^1 \wedge -$ and $\Omega = S^{-1} \wedge -$ form an adjoint equivalence of **hoSp**. For each integer n, we may define

$$S^n := (S^1)^n.$$

In [3, Theorem 1.6], it is described how the chosen isomorphism $\xi : S \cong S^{-1} \wedge S^1$ determine canonical isomorphisms

$$\phi_{p,q} : S^{p+q} \xrightarrow{\cong} S^p \wedge S^q,$$

where $\phi_{p,q}$ is given simply by composing associators, unitors, and copies of ξ and ξ^{-1}. In particular, $\phi_{-1,1} = \xi$, and if p or q is zero then $\phi_{p,q}$ is precisely one of the unitor isomorphisms. As it turns out, these isomorphisms are very nice. For one, they are coherent, so that the obvious pentagonal diagrams commute for all $a, b, c \in \mathbb{Z}$:

$$\begin{tikzcd}
S^{a+b} \wedge S^c \arrow[d, "\phi_{a,b} \wedge S^c"'] & S^{a+b+c} \arrow[l, "\phi_{a+b,c}"'] \arrow[r, "\phi_{a,b+c}"] & S^a \wedge S^{b+c} \arrow[d, "S^a \wedge \phi_{b,c}"] \\
(S^a \wedge S^b) \wedge S^c \arrow[rr, "\cong"] & & S^a \wedge (S^b \wedge S^c)
\end{tikzcd}$$

Furthermore, these isomorphisms commute with the symmetric structure of **hoSp**, like so:

Proposition 4.13 ([5, Lemma 7.1.13] or [11, Lemma 5.9])**.** *The following diagram is commutative for arbitrary integers p and q*

$$\begin{tikzcd}
S^{p+q} \arrow[r, "\phi_{p,q}"] \arrow[d, "(-1)^{pq}"'] & S^p \wedge S^q \arrow[d, "\tau"] \\
S^{p+q} \arrow[r, "\phi_{q,p}"] & S^q \wedge S^p
\end{tikzcd}$$

where here τ is the symmetry map specified by the symmetric monoidal structure on **hoSp**, *and*

$$(-1)^{pq} = \begin{cases} \mathrm{id} & pq \equiv 0 \bmod 2 \\ -\mathrm{id} & pq \equiv 1 \bmod 2. \end{cases}$$

(Recall that **hoSp** *is a triangulated category, and in particular an additive category, so that homsets in* **hoSp** *are abelian groups.)*

Recall that a *commutative ring spectrum* is a commutative monoid object in **hoSp**, that is, a spectrum E along with maps $\mu : E \wedge E \to E$ and $e : S \to E$ such that the following diagrams commute in **hoSp**:

$$\begin{tikzcd}
 & S \wedge E \arrow[dl, "\cong"'] \arrow[d, "e \wedge E"] \\
E & E \wedge E \arrow[l, "\mu"'] \\
 & E \wedge S \arrow[ul, "\cong"] \arrow[u, "E \wedge e"']
\end{tikzcd}
\qquad
\begin{tikzcd}
(E \wedge E) \wedge E \arrow[r, "\mu \wedge E"] \arrow[d, "\cong"'] & E \wedge E \arrow[dd, "\mu"] \\
E \wedge (E \wedge E) \arrow[d, "E \wedge \mu"'] & \\
E \wedge E \arrow[r, "\mu"] & E
\end{tikzcd}
\qquad
\begin{tikzcd}
E \wedge E \arrow[dd, "\tau"'] \arrow[dr, "\mu"] & \\
 & E \\
E \wedge E \arrow[ur, "\mu"'] &
\end{tikzcd}$$

We may define the stable homotopy groups of E to be the groups

$$\pi_n(E) := [S^n, E] \cong [\Sigma^n S, E].$$

In fact, in this setting, it turns out that the graded abelian group $\pi_*(E)$ has the structure of a graded abelian group, where we may define the product

$$\pi_p(E) \times \pi_q(E) \to \pi_{p+q}(E)$$

to send a pair $(\alpha, \beta) \in \pi_p(E) \times \pi_q(E)$ to the composition

$$S^{p+q} \xrightarrow{\phi_{p,q}} S^p \wedge S^q \xrightarrow{\alpha \wedge \beta} E \wedge E \xrightarrow{\mu} E.$$

It turns out this map is associative: Given classes α, β, and γ in $\pi_a(E)$, $\pi_b(E)$, and $\pi_c(E)$, respectively, consider the following diagram:

$$\begin{tikzcd}
S^{a+b+c} \arrow[r, "\phi_{a+b,c}"] \arrow[d, "\phi_{a,b+c}"'] & S^{a+b} \wedge S^c \arrow[r, "\phi_{a,b} \wedge S^c"] & (S^a \wedge S^b) \wedge S^c \arrow[r, "(\alpha\wedge\beta)\wedge\gamma"] \arrow[dl, "\cong"] & (E \wedge E) \wedge E \arrow[r, "\mu \wedge E"] \arrow[dl, "\cong"] & E \wedge E \arrow[d, "\mu"] \\
S^a \wedge S^{b+c} \arrow[r, "S^a \wedge \phi_{b,c}"'] & S^a \wedge (S^b \wedge S^c) \arrow[r, "\alpha\wedge(\beta\wedge\gamma)"'] & E \wedge (E \wedge E) \arrow[r, "E \wedge \mu"'] & E \wedge E \arrow[r, "\mu"'] & E
\end{tikzcd}$$

Commutativity of the left pentagon is the coherence condition for the $\phi_{p,q}$'s. Commutativity of the middle parallelogram is naturality of the associator isomorphisms. Commutativity of the right pentagon is associativity of μ. The fact that the two outside compositions equal $(\alpha \cdot \beta) \cdot \gamma$ and $\alpha \cdot (\beta \cdot \gamma)$, respectively, follows by functoriality of $- \wedge -$.

It also turns out that the map $e : S \to E$ is a unit for this multiplication. Given $\alpha \in [S^p, E]$, consider the following diagram:

$$\begin{tikzcd}
S \wedge S^p \arrow[dd, "e\wedge\alpha"'] \arrow[dr, "S\wedge\alpha"] & & S^p \arrow[ll, "\phi_{0,p}^{-1}=\rho_{S^p}^{-1}"'] \arrow[rr, "\phi_{p,0}^{-1}=\lambda_{S^p}^{-1}"] \arrow[dd, "\alpha"] & & S^p \wedge S \arrow[dl, "\alpha\wedge S"'] \arrow[dd, "\alpha\wedge e"] \\
& S \wedge E \arrow[dl, "e\wedge E"'] \arrow[dr, "\rho_E"] & & E \wedge S \arrow[dl, "\lambda_E"'] \arrow[dr, "E\wedge e"] & \\
E \wedge E \arrow[rr, "\mu"'] & & E & & E \wedge E \arrow[ll, "\mu"]
\end{tikzcd}$$

Commutativity of the top two large triangles is naturality of the unitor isomorphisms. Commutativity of the right and leftmost triangles functoriality of $- \wedge -$. Commutativity of the bottom triangles is unitality of μ. Hence, we have that $e \cdot \alpha = \alpha = \alpha \cdot e$.

This composition is also bilinear. Given $\alpha, \alpha' \in \pi_p(E)$ and $\beta, \beta' \in \pi_q(E)$, consider the following diagrams:

$$\begin{tikzcd}
S^{p+q} \arrow[r, "\phi_{p,q}"] \arrow[d, "\Delta"'] & S^p \wedge S^q \arrow[r, "\Delta\wedge S^q"] \arrow[d, "\Delta"] & (S^p \oplus S^p) \wedge S^q \arrow[r, "(\alpha\oplus\alpha')\wedge\beta"] \arrow[dl, "\cong"'] & (E \oplus E) \wedge E \arrow[dl, "\cong"'] \arrow[d, "\nabla\wedge E"] & \\
S^{p+q} \oplus S^{p+q} \arrow[r, "\phi_{p,q}\oplus\phi_{p,q}"'] & (S^p \wedge S^q) \oplus (S^p \wedge S^q) \arrow[r, "(\alpha\wedge\beta)\oplus(\alpha'\wedge\beta)"'] & (E \wedge E) \oplus (E \wedge E) \arrow[r, "\nabla"'] & E \wedge E \arrow[r, "\mu"] & E
\end{tikzcd}$$

$$\begin{tikzcd}
S^{p+q} \arrow[r, "\phi_{p,q}"] \arrow[d, "\Delta"'] & S^p \wedge S^q \arrow[r, "\Delta\wedge S^q"] \arrow[d, "\Delta"] & S^p \wedge (S^q \oplus S^q) \arrow[r, "\alpha\wedge(\beta\oplus\beta')"] \arrow[dl, "\cong"'] & E \wedge (E \oplus E) \arrow[dl, "\cong"'] \arrow[d, "\nabla\wedge E"] & \\
S^{p+q} \oplus S^{p+q} \arrow[r, "\phi_{p,q}\oplus\phi_{p,q}"'] & (S^p \wedge S^q) \oplus (S^p \wedge S^q) \arrow[r, "(\alpha\wedge\beta)\oplus(\alpha\wedge\beta')"'] & (E \wedge E) \oplus (E \wedge E) \arrow[r, "\nabla"'] & E \wedge E \arrow[r, "\mu"] & E
\end{tikzcd}$$

The unlabeled isomorphisms are those given by the fact that $- \wedge -$ is additive in each variable (since **hoSp** is tensor triangulated). Commutativity of the left squares is naturality of $\Delta_X : X \to X \oplus X$ in an additive category. Commutativity of the rest of the diagram follows again from the fact that $- \wedge -$ is an additive functor in each variable. Hence, by functoriality of $- \wedge -$, these diagrams tell us that $(\alpha + \alpha') \cdot \beta = \alpha \cdot \beta + \alpha' \cdot \beta$ and $\alpha \cdot (\beta + \beta') = \alpha \cdot \beta + \alpha \cdot \beta'$, respectively.

Finally, we have that this product is graded commutative. Given $\alpha \in \pi_p(E)$ and $\beta \in \pi_q(E)$, consider the following diagram:

$$\begin{tikzcd}
S^{p+q} \arrow[r, "\phi_{p,q}"] \arrow[d, "(-1)^{pq}"'] & S^p \wedge S^q \arrow[r, "\alpha\wedge\beta"] \arrow[d, "\tau"] & E \wedge E \arrow[d, "\tau"'] \arrow[dr, "\mu"] & \\
 & & & E \\
S^{p+q} \arrow[r, "\phi_{q,p}"] & S^q \wedge S^p \arrow[r, "\beta\wedge\alpha"] & E \wedge E \arrow[ur, "\mu"'] &
\end{tikzcd}$$

Commutativity of the left square is Proposition 4.13. Commutativity of the middle square is naturality of the symmetry isomorphisms. Finally, commutativity of the right triangle is commutativity of μ. Hence by bilinearity of $- \wedge -$, it follows that $\alpha \cdot \beta = (-1)^{pq}\beta \cdot \alpha$, as desired.

To recap, we've shown that if E is a commutative ring spectrum in the stable homotopy category, then $\pi_*(E)$ is itself canonically a *graded commutative* ring.

The natural question arises: does the same thing happen in the motivic world? In other words, if we have a monoid object (E, μ, e) in the motivic stable homotopy category $\mathbf{SH}_{\mathscr{S}}$, does the $\mathbb{Z} \times \mathbb{Z}$-graded abelian group $\pi_{**}(E)$ canonically form a bigraded ring, and furthermore if E is commutative, does the $\pi_{**}(E)$ satisfy any sort of "bigraded commutativity" condition? To answer the first question, motivated by the above work in the classical stable homotopy category, we know that to make $\pi_{**}(E)$ a $\mathbb{Z}^2$-graded ring, we need a family of isomorphisms

$$\phi_{a,b} : S^{a+b} \xrightarrow{\cong} S^a \wedge S^b$$

for each $a, b \in \mathbb{Z}^2$ such that

(1) For every $a, b, c \in \mathbb{Z}^2$, the following diagram commutes:

$$\begin{tikzcd}
S^{a+b} \wedge S^c \arrow[d, "\phi_{a,b}\wedge S^c"'] & S^{a+b+c} \arrow[l, "\phi_{a+b,c}"'] \arrow[r, "\phi_{a,b+c}"] & S^a \wedge S^{b+c} \arrow[d, "S^a\wedge\phi_{b,c}"] \\
(S^a \wedge S^b) \wedge S^c \arrow[rr, "\cong"] & & S^a \wedge (S^b \wedge S^c)
\end{tikzcd}$$

(2) For every $a \in \mathbb{Z}^2$, the isomorphisms $\phi_{(0,0),a}$ and $\phi_{a,(0,0)}$ coincide with the unital isomorphisms in $\mathbf{SH}_{\mathscr{S}}$.

We call such a family of isomorphisms $\{\phi_{a,b}\}_{a,b\in\mathbb{Z}^2}$ **coherent**. Once we have a coherent family, the exact same arguments given above for monoid objects in the classical stable homotopy category endow $\pi_{**}(E)$ with the structure of a $\mathbb{Z}^2$-graded ring. So can we find such a family? Recall that we have defined the $S^{a,b}$ as wedges of the "motivic circles" $S^{1,0}$, $S^{1,1}$, and their inverses $S^{-1,0}$ and $S^{-1,-1}$. Furthermore, by [3, Theorem 1.13], we know that the isomorphisms $\xi_1 : S \cong S^{-1,0} \wedge S^{1,0}$ and $\xi_2 : S \cong S^{-1,-1} \wedge S^{1,1}$ give rise to a canonical coherent family $\{\phi_{a,b}\}_{a,b\in\mathbb{Z}^2}$ obtained by forming formal compositions of copies of associators, unitors, ξ_1 and ξ_2, and their inverses.

So, we have successfully answered our first question in the affirmative. What about the second question? As it turns out, bigraded commutativity turns out to be very subtle, but the answer is yes. First, it turns out that the functor $\mathbb{G}_m \wedge - : \mathbf{SH}_{\mathscr{S}} \to \mathbf{SH}_{\mathscr{S}}$ is an equivalence. cite In what follows, let $\epsilon \in [S, S] \cong [\mathbb{G}_m, \mathbb{G}_m]$ correspond to the endomorphism of

$$\mathbb{G}_m = \mathscr{S} \times \operatorname{Spec} \mathbb{Z}[x, x^{-1}]$$

induced by the ring morphism $\operatorname{Spec} \mathbb{Z}[x, x^{-1}] \to \operatorname{Spec} \mathbb{Z}[x, x^{-1}]$ sending $x \mapsto x^{-1}$. In particular, note that $\epsilon \circ \epsilon = \mathrm{id}_S$ in $\mathbf{SH}_{\mathscr{S}}$. Then the coherent family $\{\phi_{a,b}\}_{a,b\in\mathbb{Z}^2}$ induces the following bigraded commutativity condition:

Proposition 4.14. *Given a commutative ring spectrum E in $\mathbf{SH}_{\mathscr{S}}$ with unit $e \in [S, E] \cong \pi_{0,0}(E)$, the bigraded ring $\pi_{**}(E)$ is "bigraded commutative", in the sense that when $\alpha \in \pi_{a_1,a_2}(E)$*

and $\beta \in \pi_{b_1,b_2}(E)$, under the product determined by the coherent family $\{\phi_{a,b}\}_{a,b\in\mathbb{Z}^2}$ described above given by [3, Theorem 1.13], we have that

$$\alpha \cdot \beta = \beta \cdot \alpha \cdot (-e)^{(a_1-a_2)(b_1-b_2)} \cdot (e\epsilon)^{a_2b_2}.$$

Proof. The proof of [3, Proposition 1.18] shows this for $E = S$. The same argument works more generally. □

Sadly, as [4] describes, this product has some issues. For one, it does not agree with the graded commutativity condition described by Voevodsky for the product on the dual motivic Steenrod algebra $\mathcal{A}_{**} := M\mathbb{Z}_{**}(M\mathbb{Z}) = \pi_{**}(M\mathbb{Z} \wedge M\mathbb{Z})$ ([15, Theorem 2.2]). Furthermore, under this grading convention, given a motivic commutative ring spectrum E over $\mathscr{S} = \operatorname{Spec}\mathbb{C}$, the map

$$\pi_{*,\star}(E) \to \pi_*(E(\mathbb{C}))$$

induced by Betti realization is not a ring homomorphism—there is an annoying sign that comes up (cf. [3, Proposition 1.19]).

Can this be fixed? According to Section 7 of [3], there are in fact more coherent families of isomorphisms $\{\phi_{a,b}\}_{a,b\in\mathbb{Z}^2}$ than just the one described above, and in fact, they give rise to non-isomorphic graded rings $\pi_{**}(E)$, in general. In [4], such a family is fixed which fixes both of the above issues:

Proposition 4.15 ([4, p. 3])**.** *There exists a coherent family of isomorphisms*

$$\phi_{a,b} : S^{a+b} \xrightarrow{\cong} S^a \wedge S^b$$

for $a, b \in \mathbb{Z}^2$ such that given a motivic commutative ring spectrum E with unit $e : S \to E$, the product structure induced on the bigraded abelian group $\pi_{**}(E)$ by this family is bigraded commutative in the sense that given $\alpha \in \pi_{a_1,a_2}(E)$ and $\beta \in \pi_{b_1,b_2}(E)$, the following equation holds:[10]

$$\alpha \cdot \beta = \beta \cdot \alpha \cdot (-e)^{a_1b_1} \cdot e(-\epsilon)^{a_2b_2}.$$

Furthermore, this product is related to the product $-\star-$ given in Proposition 4.14 by the formula:

$$\alpha \cdot \beta = \alpha \star \beta \star (-e)^{a_2(b_1-b_2)}.$$

In particular, when $e \circ \epsilon = -e$ then $e \circ (-\epsilon) = e$ and thus

$$\alpha \cdot \beta = \beta \cdot \alpha \cdot (-e)^{a_1b_1}.$$

This is exactly Voevodsky's convention for commutativity in the dual Steenrod algebra ([15, Theorem 2.2]). Furthermore, this grading convention allows for the realization map

$$\pi_{*,\star}(E) \to \pi_*(E(\mathbb{C}))$$

to be a ring homomorphism for all commutative ring spectra E in $\mathbf{SH}_{\mathbb{C}}$.

Remark 4.16. For the rest of this paper, we will be using the coherent family $\{\phi_{a,b}\}_{a,b\in\mathbb{Z}^2}$ and the graded commutativity law specified by Proposition 4.15. Usually we will not label the maps, instead only writing $S^{a+b} \xrightarrow{\cong} S^a \wedge S^b$ or $S^{a+b} \cong S^a \wedge S^b$.

[10]We are fixing $u = -1$, in the notation of the Proposition in [4, p. 3].

Appendix A. Triangulated categories

Definition A.1. A *triangulated cateogory* is an additive category equipped with:

(1) An adjoint equivalence (Σ, Ω) of $\mathcal{C}$ with itself. (Σ is called the *shift functor*.)

(2) A collection of *distinguished triangles*, where a *triangle* is a diagram of the form
$$X \to Y \to Z \to \Sigma X.$$
These are also sometimes called *cofiber sequences* or *fiber sequences*.

These data must satisfy the following axioms:

TR0 Given a commutative diagram
$$\begin{tikzcd} X \arrow[r] \arrow[d, "\cong"'] & Y \arrow[r] \arrow[d, "\cong"'] & Z \arrow[r] \arrow[d, "\cong"'] & \Sigma X \arrow[d, "\cong"'] \\ X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & \Sigma X' \end{tikzcd}$$
where the top row is a distinguished triangle and the vertical arrows are isomorphisms, the bottom row is a distinguished triangle.

TR1 For any object X in $\mathcal{C}$, the diagram
$$X \xrightarrow{\mathrm{id}_X} X \to 0 \to \Sigma X$$
is a distinguished triangle.

TR2 For all $f : X \to Y$ there exists an object C_f called the *cofiber of* f and a distinguished triangle
$$X \xrightarrow{f} Y \to C_f \to \Sigma X.$$
Note that it is also common to denote C_f by X/Y.

TR3 Given two distinguished triangles
$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$$
and
$$X' \xrightarrow{f'} Y' \xrightarrow{g'} Z' \xrightarrow{h'} \Sigma X'$$
and two morphisms $\alpha : X \to X'$ and $\beta : Y \to Y'$ such that the following diagram commutes:
$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\alpha"'] & Y \arrow[d, "\beta"] \\ X' \arrow[r, "f'"] & Y' \end{tikzcd}$$
there exists a morphism $\gamma : Z \to Z'$ extending this to a morphism of distinguished triangles, in that the following diagram commutes:
$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\alpha"'] & Y \arrow[r, "g"] \arrow[d, "\beta"'] & Z \arrow[r, "h"] \arrow[d, "\gamma"'] & \Sigma X \arrow[d, "\Sigma\alpha"'] \\ X' \arrow[r, "f'"] & Y' \arrow[r, "g'"] & Z' \arrow[r, "h'"] & \Sigma X' \end{tikzcd}$$

TR4 Given a diagram
$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X,$$
it is a distinguished triangle if and only if
$$Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X \xrightarrow{-\Sigma f} \Sigma Y$$
is a distinguished triangle.

TR5 (Octahedral axiom) Given three distinguished triangles of the form

$$X \xrightarrow{f} Y \xrightarrow{h} Y/X \longrightarrow \Sigma X$$

$$Y \xrightarrow{g} Z \xrightarrow{k} Z/Y \longrightarrow \Sigma Y$$

$$X \xrightarrow{g\circ f} Z \xrightarrow{l} Z/X \longrightarrow \Sigma X$$

there exists a distinguished triangle

$$Y/X \xrightarrow{u} Z/X \xrightarrow{v} Z/Y \xrightarrow{w} \Sigma(Y/X)$$

such that the following diagram commutes

$$\begin{tikzcd}
X \arrow[rr, "g\circ f"] \arrow[dr, "f"'] & & Z \arrow[rr, "k"] \arrow[dr, "l"'] & & Z/Y \arrow[rr, "w"] \arrow[dr] & & \Sigma(Y/X) \\
& Y \arrow[ur, "g"'] \arrow[dr, "h"'] & & Z/X \arrow[ur, "v"'] \arrow[dr] & & \Sigma Y \arrow[ur, "\Sigma h"'] & \\
& & Y/X \arrow[ur, "u"'] \arrow[rr] & & \Sigma X \arrow[ur, "\Sigma f"'] & &
\end{tikzcd}$$

Note that the above definition is actually redundant; TR3 and TR4 follow from the remaining axioms (see Lemmas 2.2 and 2.4 in [8]).

Proposition A.2. *Given a map $f : X \to Y$ in $\mathcal{C}$, the cofiber sequence of f is unique up to canonical isomorphism, in the sense that given any two distinguished triangles*

$$X \xrightarrow{f} Y \to Z \to \Sigma X \qquad \text{and} \qquad X \xrightarrow{f} Y \to Z' \to \Sigma X,$$

there exists a canonical isomorphism $Z \to Z'$ (the dashed line) which makes the following diagram commute:

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, equal] & Y \arrow[r] \arrow[d, equal] & Z \arrow[r] \arrow[d, dashed] & \Sigma X \arrow[d, equal] \\
X \arrow[r, "f"] & Y \arrow[r] & Z' \arrow[r] & \Sigma X
\end{tikzcd}$$

Proposition A.3. *Given a triangulated category $\mathcal{C}$ with shift functor Σ and a morphism $f : X \to Y$, there exists an object F_f called the* fiber of f *and a distinguised triangle*

$$F_f \to X \xrightarrow{f} Y \to \Sigma F_f (\cong C_f).$$

Proposition A.4. *Let $\mathcal{C}$ be a triangluated category. Given a distinguished triangle*

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$$

and any object A in $\mathcal{C}$, there are long exact sequences of abelian groups

$$\cdots \to [\Sigma^{n+1}A, Z] \xrightarrow{h_*} [\Sigma^n A, X] \xrightarrow{f_*} [\Sigma^n A, Y] \xrightarrow{g_*} [\Sigma^n A, Z] \xrightarrow{h_*} [\Sigma^{n-1}A, X] \to \cdots$$

extending infinitely in either direction, where for $n < 0$ we set $\Sigma^{-n} := \Omega^n$.

Also important for our work is the concept of a *tensor triangulated category*, that is, a triangulated symmetric monoidal category in which the triangulated and monoidal structures play nicely with each other, in the following sense:

Definition A.5. A *tensor triangulated category* is a triangulated symmetric monoidal category $(\mathcal{C}, \otimes, S)$ with shift functor Σ such that:

(1) For all objects X, Y in $\mathcal{C}$, there are natural isomorphisms
$$e_{X,Y} : (\Sigma X) \otimes Y \xrightarrow{\cong} \Sigma(X \otimes Y).$$
(2) For each object X in $\mathcal{C}$, the functor $X \otimes (-) \cong (-) \otimes X$ is an additive functor
(3) For each object X in $\mathcal{C}$, the functor $X \otimes (-) \cong (-) \otimes X$ preserves distinguished triangles, in that given a distinguished triangle/(co)fiber sequence
$$A \xrightarrow{f} B \xrightarrow{g} B/A \xrightarrow{h} \Sigma A,$$
then also
$$X \otimes A \xrightarrow{X\otimes f} X \otimes B \xrightarrow{X\otimes g} X \otimes B/A \xrightarrow{X\otimes h} X \otimes (\Sigma A) \cong \Sigma(A \otimes X)$$
and
$$A \otimes X \xrightarrow{f\otimes X} B \otimes X \xrightarrow{g\otimes X} B/A \otimes X \xrightarrow{h\otimes X} \Sigma A \otimes X \cong \Sigma(A \otimes X)$$
are distinguished triangles.

As it turns out, pretty much any tensor triangulated category will usually satisfy additional coherence axioms (see axioms TC1–TC5 in [8]), but the above will suffice for our purposes.

Appendix B. (Co)algebra

B.1. **Grading.** First, we develop the theory of things graded by an abelian group. In what follows, we fix an abelian group A.

Definition B.1. An *A-graded abelian group* is an abelian group B along with a subgroup $B_a \leq B$ for each $a \in A$ such that the canonical map
$$\bigoplus_{a\in A} B_a \to B$$
sending $(x_a)_{a\in A}$ to $\sum_{a\in A} x_a$ is an isomorphism.

Given two A-graded abelian groups B and C, a homomorphism $f : A \to B$ is a *homomorphism of A-graded abelian groups* if it preserves the grading, i.e., if it restricts to a map $B_a \to C_a$ for all $a \in A$.

Given an A-graded abelian group B and a nonzero element $x \in B$, we say x is *homogeneous* if there exists some $a \in A$ such that $x \in B_a$.

Remark B.2. Clearly the condition that the canonical map $\bigoplus_{a\in A} B_a \to B$ is an isomorphism requires that $B_a \cap B_b = 0$ if $a \neq b$. In particular, given a homogeneous element $x \in B$, there exists precisely one $a \in A$ such that $x \in B_a$. We call this a the *degree* of x, and we denote it by $|x|$.

Definition B.3. An *A-graded ring* R is the data of a ring R such that:

(1) The underlying abelian group of R is A-graded;
(2) For all $a, b \in R$, the multiplication map $R \times R \to R$ restricts to a map
$$R_a \times R_b \to R_{a+b},$$
i.e., $|xy| = |x| + |y|$ for all nonzero $x, y \in R$.

For example, given some field k, the ring $R = k[x, y]$ is $\mathbb{Z}^2$-graded, where given $(n, m) \in \mathbb{Z}^2$, $R_{n,m}$ is the subgroup of those monomials of the form ax^ny^m for some $a \in k$. Given an A-graded ring R, we may talk about A-graded R-modules:

Definition B.4. Let R be an A-graded ring. A left A-graded R-module M is a left R-module M such that M is an A-graded abelian group, and for all $a, b \in A$, the action map $R \times M \to M$ restricts to a map $R_a \times M_b \to M_{a+b}$.

Right A-graded R-modules are defined similarly. Finally, an A-graded R-bimodule is an A-graded abelian group M along with action maps

$$\alpha_L : R \times M \to M \qquad \text{and} \qquad \alpha_R : M \times R \to M$$

which endow M with the structure of a left and right A-graded R-module, respectively, such that given $s, r \in R$ and $m \in M$, $r \cdot (m \cdot s) = (r \cdot m) \cdot s$.

Proposition B.5. *Let R be an A-graded ring, and suppose I have a right A-graded R-module M and a left A-graded R-module N. Then the tensor product*

$$M \otimes_R N$$

is naturally an A-graded abelian group, and furthermore, if either M (resp. N) is an A-graded bimodule, then it is naturally a left (resp. right) A-graded R-module

Proof. By definition, since M and N are A-graded abelian groups, they are generated (as abelian groups) by their homogeneous elements. Thus it follows that $M \otimes_R N$ is generated by *homogeneous pure tensors*, that is, elements of the form $m \otimes n$ with $m \in M$ and $n \in N$ homogeneous. Now, given a homogeneous pure tensor $m \otimes n$, we define its *degree* by the formula $|m \otimes n| := |m| + |n|$. It follows this formula is well-defined by checking that given homogeneous elements $m, m' \in M$, $n, n' \in N$, and $r \in R$ that

$$|(m \cdot r) \otimes n| = |m \otimes (r \cdot n)| = |m| + |r| + |n|.$$

Thus, we may define $(M \otimes_R N)_a$ to be the subgroup of $M \otimes_R N$ generated by those pure homogeneous tensors of degree a. Now, clearly the canonical map

$$\bigoplus_{a \in A} (M \otimes_R N)_a \to M \otimes_R N$$

finish

is surjective. To see it is injective, it suffices $\square$

Proposition B.6. *Let (E_1, μ_1, e_1) and (E_2, μ_2, e_2) be ring spectra. Then $E_1 \wedge E_2$ is canonically a ring spectrum via the maps*

$$\mu : E_1 \wedge E_2 \wedge E_1 \wedge E_2 \xrightarrow{E_1 \wedge \tau \wedge E_2} E_1 \wedge E_1 \wedge E_2 \wedge E_2 \xrightarrow{\mu_1 \wedge \mu_2} E_1 \wedge E_2$$

and

$$e : S \cong S \wedge S \xrightarrow{e_1 \wedge e_2} E_1 \wedge E_2.$$

todo

Proof. $\square$

Proposition B.7. *Let X and Y be spectra. Then the pairing*

$$\pi_{**}(X) \times \pi_{**}(Y) \to \pi_{**}(X \wedge Y)$$

sending $\alpha : S^a \to X$ and $\beta : S^b \to Y$ to the composition

$$S^{a+b} \cong S^a \wedge S^b \xrightarrow{\alpha \wedge \beta} X \wedge Y$$

is bilinear.

Proof. Let $a, b \in \mathbb{Z}^2$, and let $\alpha_1, \alpha_2 : S^a \to X$ and $\beta : S^b \to Y$. Then consider the following diagram

$$\begin{tikzcd}
S^{a+b} \arrow[r, "\cong"] & S^a \wedge S^b \arrow[r, "\Delta \wedge S^b"] \arrow[d, "\Delta"'] & (S^a \oplus S^a) \wedge S^b \arrow[d, "(\alpha_1 \oplus \alpha_2) \wedge \beta"] \arrow[dl, "\cong"'] \\
& (S^a \wedge S^b) \oplus (S^a \wedge S^b) \arrow[d, "(\alpha_1 \wedge \beta) \oplus (\alpha_2 \wedge \beta)"'] & (X \oplus X) \wedge Y \arrow[d, "\nabla \wedge Y"] \arrow[dl, "\cong"'] \\
& (X \wedge Y) \oplus (X \wedge Y) \arrow[r, "\nabla"] & X \wedge Y
\end{tikzcd}$$

The isomorphisms are given by the fact that $- \wedge -$ is additive in each variable. Both triangles and the parallelogram commute since $- \wedge -$ is additive. By functoriality of $- \wedge -$, the top composition is $(\alpha_1 + \alpha_2) \cdot \beta$ and the bottom composition is $\alpha_1 \cdot \beta + \alpha_2 \cdot \beta$, so they are equal, as desired. An entirely analagous argument yields that $\alpha \cdot (\beta_1 + \beta_2) = \alpha \cdot \beta_1 + \alpha \cdot \beta_2$ for $\alpha \in \pi_{**}(X)$ and $\beta_1, \beta_2 \in \pi_{**}(Y)$. $\square$

Proposition B.8 ([11, Proposition 5.11])**.** *Let (E, μ, e) be a ring spectrum. Then for any spectrum X, $E_{**}(X)$ canonically inherits the structure of a left graded $\pi_{**}(E)$-module via the map*

$$\pi_{**}(E) \times E_{**}(X) \to E_{**}(X)$$

which given $a, b \in \mathbb{Z}^2$, sends $\gamma : S^a \to E$ and $\alpha : S^b \to E \wedge X$ to the composition

$$\gamma\alpha : S^{a+b} \cong S^a \wedge S^b \xrightarrow{\gamma \wedge \alpha} E \wedge (E \wedge X) \cong (E \wedge E) \wedge X \xrightarrow{\mu \wedge X} E \wedge X.$$

Similarly $X_{**}(E)$ canonically inherits the structure of a right graded $\pi_{**}(E)$-module via the map

$$X_{**}(E) \times \pi_{**}(E) \to X_{**}(E)$$

which given $a, b \in \mathbb{Z}^2$, sends $\alpha : S^a \to X \wedge E$ and $\gamma : S^b \to E$ to the composition

$$\alpha\gamma : S^{a+b} \cong S^a \wedge S^b \xrightarrow{\alpha \wedge \gamma} (X \wedge E) \wedge E \cong X \wedge (E \wedge E) \xrightarrow{X \wedge \mu} X \wedge E.$$

In particular, $E_{**}(E)$ is a $\pi_{**}(E)$-bimodule, in the sense that the left and right actions of $\pi_{**}(E)$ are compatible, so that given $\beta, \gamma \in \pi_{**}(E)$ and $\alpha \in E_{**}(E)$, $\beta \cdot (\alpha \cdot \gamma) = (\beta \cdot \alpha) \cdot \gamma$.

Proof. First we show that the map $\pi_{**}(E) \times E_{**}(X) \to E_{**}(X)$ endows $E_{**}(X)$ with the structure of a left $\pi_{**}(E)$-module. Let $a, b, c \in \mathbb{Z}^2$ and $\alpha, \alpha' : S^a \to E \wedge X$, $\beta : S^b \to E$, and $\gamma, \gamma' \in S^c \to E$. Then we wish to show that:

(1) $\beta \cdot (\alpha + \alpha') = \beta \cdot \alpha + \beta \cdot \alpha'$,
(2) $(\gamma + \gamma') \cdot \alpha = \gamma \cdot \alpha + \gamma' \cdot \alpha$,
(3) $(\gamma \cdot \beta) \cdot \alpha = \gamma \cdot (\beta \cdot \alpha)$,
(4) $e \cdot \alpha = \alpha$.

Axioms (1) and (2) follow by the fact that $E_{**}(X) = \pi_{**}(E \wedge X)$ and Proposition B.7. To see (3), consider the diagram:

$$\begin{tikzcd}
S^{a+b+c} \arrow[r, "\cong"] \arrow[d, "\cong"'] & S^{c+b} \wedge S^a \arrow[dd, "\cong"] & \\
S^c \wedge S^{b+a} \arrow[d, "\cong"'] & & \\
S^c \wedge (S^b \wedge S^a) \arrow[d, "\gamma\wedge(\beta\wedge\alpha)"'] & (S^c \wedge S^b) \wedge S^a \arrow[l, "\cong"'] \arrow[d, "(\gamma\wedge\beta)\wedge\alpha"] & \\
E \wedge (E \wedge (E \wedge X)) \arrow[dd, "\cong"'] & (E \wedge E) \wedge (E \wedge X) \arrow[l, "\cong"'] \arrow[r, "\mu\wedge(E\wedge X)"] & E \wedge (E \wedge X) \arrow[d, "\cong"] \\
& ((E \wedge E) \wedge E) \wedge X \arrow[u, "\cong"'] \arrow[r, "(\mu\wedge E)\wedge X"] \arrow[d, "\cong"] & (E \wedge E) \wedge X \arrow[dd, "\mu\wedge X"] \\
E \wedge ((E \wedge E) \wedge X) \arrow[r, "\cong"] \arrow[d, "E\wedge(\mu\wedge X)"'] & (E \wedge (E \wedge E)) \wedge X \arrow[d, "(E\wedge\mu)\wedge X"] & \\
E \wedge (E \wedge X) \arrow[r, "\cong"'] & (E \wedge E) \wedge X \arrow[r, "\mu\wedge X"'] & E \wedge X
\end{tikzcd}$$

The top square commutes by coherence of the isomorphisms $S^{a+b} \cong S^a \wedge S^b$ (Proposition 4.15). The second square from the top on the left commutes by naturality of the associators. The square below that commutes by the coherence axiom for the associators in a monoidal category. The bottom left square commutes again by naturality of the associator isomorphisms. The bottom right square commutes by associativity for μ and functorialiy of $- \wedge X$. Finally, the square above that commutes again by naturality of the associator isomorphism. By functoriality of $- \wedge -$, the two outside compositions equal $(\gamma \cdot \beta) \cdot \alpha$ on the top and $\gamma \cdot (\beta \cdot \alpha)$ on the bottom. Hence, they are equal, as desired.

Next, to see (4), consider the following diagram:

$$\begin{tikzcd}
S^a \arrow[rrr, "\alpha"] \arrow[d, "\phi_{0,a}=\rho_{S^a}^{-1}"'] & & & E \wedge X \\
S \wedge S^a \arrow[r, "S\wedge\alpha"] \arrow[dd, "e\wedge\alpha"'] & S \wedge (E \wedge X) \arrow[urr, "\rho_{E\wedge X}"'] \arrow[dr, "\cong"] \arrow[ddl, "e\wedge(E\wedge X)"] & & \\
& & (S \wedge E) \wedge X \arrow[uur, "\rho_E\wedge X"] \arrow[dr, "(e\wedge E)\wedge X"] & \\
E \wedge (E \wedge X) \arrow[rrr, "\cong"] & & & (E \wedge E) \wedge X \arrow[uuu, "\mu\wedge X"']
\end{tikzcd}$$

Commutativity of the top trapezoid is naturality of the unitor. Commutativity of the left triangle is functoriality of $- \wedge -$. Commutativity of the bottom triangle is naturality of the associator isomorphisms. Commutativity of the right triangle is unitality of μ. Finally, commutativity of the remaining crooked triangle follows by coherence for monoidal categories. The two outer compositions $S^a \to E \wedge X$ are α and $e \cdot \alpha$, and by commutativity they are necessarily equal.

Thus, we have shown that the indicated map does indeed endow $E_{**}(X)$ with the structure of a left $\pi_{**}(E)$-module. Showing that $X_{**}(E)$ has the structure of a right $\pi_{**}(E)$-module is entirely analagous. $\square$

Definition B.9. Let E be a ring spectrum. We say E is *flat* if the canonical right $\pi_{**}(E)$-module structure on $E_{**}(E)$ is that of a flat module.

Proposition B.10 ([12, Proposition 2.2])**.** *Let E be a ring spectrum and let X be any spectrum. Then the assignment*

$$E_{**}(E) \times E_{**}(X) \to E_{**}(E \wedge X)$$

which sends $\alpha : S^{a,b} \to E \wedge E$ and $\beta : S^{c,d} \to E \wedge X$ to the composition

$$\alpha \cdot \beta : S^{a+c,b+d} \cong S^{a,b} \wedge S^{c,d} \xrightarrow{\alpha\wedge\beta} E \wedge E \wedge E \wedge X \xrightarrow{E\wedge\mu\wedge X} E \wedge E \wedge X$$

induces a homomorphism of bigraded abelian groups

$$E_{**}(E) \otimes_{\pi_{**}(E)} E_{**}(X) \to E_{**}(E \wedge X)$$

(where here $E_{**}(E)$ has a right $\pi_{**}(E)$-module structure and $E_{**}(X)$ has a left $\pi_{**}(E)$-module structure as specified by Proposition B.8). Furthermore, if X is cellular and E is a cellular flat commutative ring spectrum (Definition 2.9, Definition B.9), then this map is an isomorphism.

Proof. First we show that this map is $\pi_{**}(E)$-bilinear. By the identifications $E_{**}(E) = \pi_{**}(E\wedge E)$ $E_{**}(X) = \pi_{**}(E \wedge X)$, and $E_{**}(E \wedge X) = \pi_{**}(E \wedge E \wedge X)$, we know this map commutes with addition of maps in each argument by Proposition B.7. Now, let $a, b, c \in \mathbb{Z}^2$, $\alpha : S^a \to E \wedge E$, $\beta : S^b \to E \wedge X$, and $\gamma : S^c \to E$. Then we wish to show $\alpha\gamma \cdot \beta = \alpha \cdot \gamma\beta$. Consider the following diagram

$$\begin{tikzcd}
S^{a+b+c} \arrow[r, "\cong"] \arrow[d, "\cong"] & S^a \wedge S^{c+b} \arrow[r, "\cong"] & S^a \wedge S^c \wedge S^b \arrow[d, "\alpha\wedge\gamma\wedge\beta"] \\
S^{a+c} \wedge S^b \arrow[d, "\cong"'] & & E \wedge E \wedge E \wedge E \wedge X \arrow[d, "E\wedge E\wedge\mu\wedge X"] \\
S^a \wedge S^c \wedge S^b \arrow[d, "\alpha\wedge\gamma\wedge\beta"'] \arrow[uurr, "\cong"] & & E \wedge E \wedge E \wedge X \arrow[d, "E\wedge\mu\wedge X"] \\
E \wedge E \wedge E \wedge E \wedge X \arrow[r, "E\wedge\mu\wedge E\wedge X"'] \arrow[uurr, "\cong"] & E \wedge E \wedge E \wedge X \arrow[r, "E\wedge\mu\wedge X"'] & E \wedge E \wedge X
\end{tikzcd}$$

(we have suppressed the associators from the notation). The top left triangle commutes by coherence for the isomorphisms $S^{a+b} \cong S^a \wedge S^b$. The middle parallelogram commutes by naturality of the associators. Finally, the bottom right triangle is obtained by applying $E \wedge - \wedge X$ to the associativity diagram for μ, so by functoriality it commutes. Again by functoriality of $- \wedge -$, the bottom composition is given by $(\alpha\gamma) \cdot \beta$ and the top composition is $\alpha \cdot (\gamma\beta)$, so we have the desired equality.

It remains to show that if X is cellular and E is cellular flat commutative, then this map is an isomorphism. $\square$ finish or cite

Remark B.11. Let E be a ring spectrum, and N a left $\pi_{**}(E)$-module. Then

$$E_{**}(E) \otimes_{\pi_{**}(E)} N$$

is canonically a left-graded $\pi_{**}(E)$-module, as $E_{**}(E)$ is a $\pi_{**}(E)$-bimodule (Proposition B.8). In particular, the action

$$\pi_{**}(E) \times (E_{**}(E) \otimes_{\pi_{**}(E)} N) \to E_{**}(E) \otimes_{\pi_{**}(E)} N$$

sends a pair $(\gamma, \alpha \otimes n)$ to $\gamma\alpha \otimes n$, where $\gamma\alpha$ denotes the left action prescribed by Proposition B.8 of γ on α.

In the following definition, let $\varepsilon : E_{**}(E) \to \pi_{**}(E)$ be the map which sends some $\alpha : S^a \to E \wedge E$ to the composition

$$S^a \xrightarrow{\alpha} E \wedge E \xrightarrow{\mu} E.$$

Also define $\Psi : E_{**}(E) \to E_{**}(E) \otimes_{\pi_{**}(E)} E_{**}(E)$ to be the map which factors as

$$E_{**}(E) \to E_{**}(E \wedge E) \xrightarrow{\cong} E_{**}(E) \otimes_{\pi_{**}(E)} E_{**}(E)$$

where the second arrow is the isomorphism prescribed by Proposition B.10, and the first arrow sends a class $\alpha : S^a \to E \wedge E$ to the composition

$$S^a \xrightarrow{\alpha} E \wedge E \cong E \wedge S \wedge E \xrightarrow{E \wedge e \wedge E} E \wedge E \wedge E.$$

Definition B.12. Let E be a flat cellular commutative ring spectrum. Then a *left $E_{**}(E)$-comodule* is the data of

(1) A $\mathbb{Z}^2$-graded left $\pi_{**}(E)$-module M;
(2) A homomorphism of left-graded $\pi_{**}(E)$-modules:

$$\Psi_M : M \to E_{**}(E) \otimes_{\pi_{**}(E)} M.$$

(Note $E_{**}(E) \otimes_{\pi_{**}(E)} M$ is canonically a left $\pi_{**}(E)$-module by Remark B.11.)

These data must make the following diagrams commute:

(1) (Co-unitality)

$$\begin{tikzcd} M \arrow[r, "\Psi_M"] \arrow[dr, "\cong"'] & E_{**}(E) \otimes_{\pi_{**}(E)} M \arrow[d, "\varepsilon \otimes M"] \\ & \pi_{**}(E) \otimes_{\pi_{**}(E)} M \end{tikzcd}$$

(2) (Co-action property)

$$\begin{tikzcd} M \arrow[r, "\Psi_M"] \arrow[d, "\Psi_M"'] & E_{**}(E) \otimes_{\pi_{**}(E)} M \arrow[d, "\Psi \otimes M"] \\ E_{**}(E) \otimes_{\pi_{**}(E)} M \arrow[r, "E_{**}(E) \otimes \Psi_M"] & E_{**}(E) \otimes_{\pi_{**}(E)} E_{**}(E) \otimes_{\pi_{**}(E)} M \end{tikzcd}$$

Given two left $E_{**}(E)$-comodules M and N, a *homomorphism of $E_{**}(E)$-comodules* is a homomorphism $f : M \to N$ of the underlying graded left $\pi_{**}(E)$-modules such that the following diagram commutes:

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "\Psi_M"'] & N \arrow[d, "\Psi_N"] \\ E_{**}(E) \otimes_{\pi_{**}(E)} M \arrow[r, "E \otimes f"] & E_{**}(E) \otimes_{\pi_{**}(E)} N \end{tikzcd}$$

We write $E_{**}(E)$-**CoMod** for the resulting category of left $E_{**}(E)$-comodules. The notation for the hom-sets in this category is usually abbreviated to

$$\mathrm{Hom}_{E_{**}(E)}(-,-) := \mathrm{Hom}_{E_{**}(E)\text{-}\mathbf{CoMod}}(-,-).$$

Lemma B.13 ([12, Proposition 2.30, 2.33])**.** *Let E be a flat commutative ring spectrum, and let X and Y be spectra such that $E_{**}(X)$ is a projective module over $\pi_{**}(E)$. Then for all $s \geq 0$ and $t, w \in \mathbb{Z}$, there is an isomorphism*

$$\Phi : [X, E \wedge Y]_{t,w} \to \mathrm{Hom}^{t,w}_{E_{**}(E)}(E_{**}(X), E_{**}(E \wedge Y)),$$

obtained by sending a class $f : S^{t,w} \wedge X \to E \wedge Y$ *in* $[X, E \wedge Y]_{t,w}$ *to the map*

$$\Phi_f : E_{*,*}(X) \to E_{*+t,*+w}(X \wedge Y)$$

sending

$$[S^{a,b} \xrightarrow{g} E \wedge X] \mapsto [S^{a+t,b+w} \cong S^{a,b} \wedge S^{t,w} \xrightarrow{g \wedge S^{t,w}} E \wedge X \wedge S^{t,w} \cong E \wedge S^{t,w} \wedge X \xrightarrow{E \wedge f} E \wedge E \wedge Y].$$

Proof. Let $f : S^{t,w} \wedge X \to E \wedge Y$. First we want to show that Φ_f is actually an $E_{**}(E)$-comodule homomorphism. □ finish

Recall when working with the classical Adams spectral sequence, one usually develops the theory of *graded commutative Hopf algebroids*, i.e., internal groupoids in the opposite category $\mathbf{gCRing}^{\mathrm{op}}$ of $\mathbb{Z}$-graded commutative rings, regarded with its cartesian monoidal category structure. Then, one goes on to show that given a commutative ring spectrum E in **hoSp** that $E_*(E)$ is a commutative Hopf algebroid over $\pi_*(E)$.

Now, in the motivic setting, things become a bit more subtle. Namely, given a commutative ring spectrum E in $\mathbf{SH}_{\mathscr{S}}$, we would like $E_{**}(E)$ to be a "bigraded commutative Hopf algebroid over $\pi_{**}(E)$". To define such a thing, we would like to find some category $\mathcal{C}$ containing objects such as $E_{**}(E)$ and $\pi_{**}(E)$ so that the pair $(E_{**}(E), \pi_{**}(E))$ forms a groupoid object in $\mathcal{C}^{\mathrm{op}}$. The naïve answer would be to consider $E_{**}(E)$ and $\pi_{**}(E)$ as objects in some category of "bigraded commutative rings", in the same way we considered $E_*(E)$ and $\pi_*(E)$ as ojects in the category of graded commutative rings in the classical case. Yet, we run into difficulty here, as the commutative law for $E_{**}(E) = \pi_{**}(E \wedge E)$ and $\pi_{**}(E)$ (Proposition 4.15) depends on their structure as $\pi_{**}(S)$-algebras. Thus, we instead are led to the category of commutative $\pi_{**}(S)$-algebras:

Definition B.14. Let $\mathbf{CStabRing}_{\mathscr{S}}$ denote the full subcategory of $\mathbb{Z}^2$-graded algebras over $\pi_{**}^{\mathscr{S}}(S)$ (the stable homotopy groups of the sphere spectrum S in $\mathbf{SH}_{\mathscr{S}}$) containing those objects satisfying the commutativity condition given in Proposition 4.15. Explicitly, an object in $\mathbf{CStabRing}_{\mathscr{S}}$ is a $\mathbb{Z}^2$-graded ring C_{**} together with a ring morphism $e : \pi_{**}^{\mathscr{S}}(S) \to C_{**}$ such that given $x \in C_{a_1,a_2}$ and $y \in C_{b_1,b_2}$, we have

$$x \cdot y = y \cdot x \cdot (-1)^{a_1 b_1} \cdot e(-\epsilon)^{a_2 b_2}.$$

A morphism in $\mathbf{CStabRing}_{\mathscr{S}}$ is simply a morphism of $\pi_{**}^{\mathscr{S}}(S)$-algebras.

Definition B.15. For our purposes, a *bigraded commutative Hopf algebroid* (Γ, A) is an internal groupoid in the opposite category $(\mathbf{CStabRing}_{\mathscr{S}})^{\mathrm{op}}$.

Let's unravel what this definition means. First, recall the definition of a groupoid object in a category:

Definition B.16. Let $\mathcal{C}$ be a category admitting pullbacks. A *groupoid object* in $\mathcal{C}$ consists of a pair of objects (M, O) together with five morphisms

(1) *Source and target*: $s, t : M \to O$,
(2) *Identity*: $e : O \to M$,
(3) *Composition*: $c : M \times_O M \to M$,
(4) *Inverse*: $i : M \to M$

Explicitly, $M \times_O M$ fits into the following pullback diagram:

$$\begin{tikzcd} M \times_O M \arrow[r, "p_2"] \arrow[d, "p_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & M \arrow[d, "t"] \\ M \arrow[r, "s"'] & O \end{tikzcd}$$

These data must satisfy the following diagrams:

(1) Composition works correctly:

$$\begin{tikzcd} M \times_O M \arrow[r, "c"] \arrow[d, "p_1"'] & M \arrow[d, "t"] \\ M \arrow[r, "t"] & O \end{tikzcd} \qquad \begin{tikzcd} M \arrow[dr, "s"'] & O \arrow[l, "e"'] \arrow[r, "e"] \arrow[d, equal] & M \arrow[dl, "t"] \\ & O & \end{tikzcd} \qquad \begin{tikzcd} M \times_O M \arrow[r, "p_2"] \arrow[d, "c"'] & M \arrow[d, "s"] \\ M \arrow[r, "s"] & O \end{tikzcd}$$

(2) Associativity of composition:

$$\begin{tikzcd} M \times_O (M \times_O M) \arrow[rr, "\cong"] \arrow[d, "M \times c"'] & & M \times_O (M \times_O M) \arrow[d, "M \times c"] \\ M \times_O M \arrow[r, "c"] & M & M \times_O M \arrow[l, "c"'] \end{tikzcd}$$

where the top objects and the maps $M \times c$, $c \times M$ are determined like so, where both outer and inner squares in the following diagram are pullback squares:

$$\begin{tikzcd} M \times_O (M \times_O M) \arrow[rr, "p_2"] \arrow[dr, dashed, "M \times c"] \arrow[dd, "p_1"'] & & M \times_O M \arrow[d, "c"] \\ & M \times_O M \arrow[r, "p_2"] \arrow[d, "p_1"'] & M \arrow[d, "t"] \\ M \arrow[r, equal] & M \arrow[r, "s"] & O \end{tikzcd} \qquad \begin{tikzcd} (M \times_O M) \times_O M \arrow[rr, "p_2"] \arrow[dr, dashed, "c \times M"] \arrow[dd, "p_1"'] & & M \arrow[d, equal] \\ & M \times_O M \arrow[r] \arrow[d] & M \arrow[d, "t"] \\ M \times_O M \arrow[r, "c"] & M \arrow[r, "s"] & O \end{tikzcd}$$

(3) Unitality of composition: Given the maps $(\mathrm{id}_M, e \circ t), (e \circ s, \mathrm{id}_M) : M \to M \times_O M$ defined by the universal property of $M \times_O M$:

$$\begin{tikzcd} M \arrow[dr, dashed, "(\mathrm{id}_M, e\circ s)"] \arrow[drr, bend left, "e\circ s"] \arrow[ddr, bend right, equal] & & \\ & M \times_O M \arrow[r, "p_2"] \arrow[d, "p_1"'] \arrow[dr, phantom, "\lrcorner"] & M \arrow[d, "t"] \\ & M \arrow[r, "s"] & O \end{tikzcd} \qquad \begin{tikzcd} M \arrow[dr, dashed, "(e\circ t, \mathrm{id}_M)"] \arrow[drr, bend left, equal] \arrow[ddr, bend right, "e\circ t"'] & & \\ & M \times_O M \arrow[r, "p_2"] \arrow[d, "p_1"'] \arrow[dr, phantom, "\lrcorner"] & M \arrow[d, "t"] \\ & M \arrow[r, "s"] & O \end{tikzcd}$$

the following diagram commutes:

$$\begin{tikzcd} M \arrow[r, "(e\circ t, \mathrm{id}_M)"] \arrow[d, "(\mathrm{id}_M, e\circ s)"'] & M \times_O M \arrow[d, "c"] \\ M \times_O M \arrow[r, "c"'] & M \end{tikzcd}$$

(4) Inverse: The following diagrams must commute:

$$\begin{tikzcd} & M \arrow[dl, equal] \arrow[d, "i"] \\ M & M \arrow[l, "i"'] \end{tikzcd} \qquad \begin{tikzcd} M \arrow[r, "(\mathrm{id}_M, i)"] \arrow[d, "t"'] & M \times_O M \arrow[d, "c"] & M \arrow[l, "(i, \mathrm{id}_M)"'] \arrow[d, "s"] \\ O \arrow[r, "e"] & M & O \arrow[l, "e"'] \end{tikzcd} \qquad \begin{tikzcd} & M \arrow[dl, "s"'] \arrow[d, "i"] \arrow[dr, "t"] & \\ O & M \arrow[l, "t"'] \arrow[r, "s"] & O \end{tikzcd}$$

where the arrows (id_M, i) and (i, id_M) are determined by the universal property of $M \times_O M$ like so:

$$\begin{tikzcd}
M \arrow[dr, dashed, "(\mathrm{id}_M, i)"] \arrow[drr, bend left, "i"] \arrow[ddr, equal, bend right] & & \\
& M \times_O M \arrow[r, "p_2"] \arrow[d, "p_1"'] & M \arrow[d, "t"] \\
& M \arrow[r, "s"] & O
\end{tikzcd}
\qquad
\begin{tikzcd}
M \arrow[dr, dashed, "(i, \mathrm{id}_M)"] \arrow[drr, equal, bend left] \arrow[ddr, bend right, "i"'] & & \\
& M \times_O M \arrow[r, "p_2"] \arrow[d, "p_1"'] & M \arrow[d, "t"] \\
& M \arrow[r, "s"] & O
\end{tikzcd}$$

Proposition B.17 ([12, Proposition 2.3, Proposition 2.12])**.** *Let E be a flat cellular commutative ring spectrum (Definition 2.9, Definition B.9). Consider the following data:*

(1) The maps $\eta_L, \eta_R : \pi_{**}(E) \to E_{**}(E)$ which send an element $\alpha : S^a \to E$ to the compositions

$$S^a \xrightarrow{\alpha} E \xrightarrow{\cong} E \wedge S \xrightarrow{E \wedge e} E \wedge E$$

and

$$S^a \xrightarrow{\alpha} E \xrightarrow{\cong} S \wedge E \xrightarrow{e \wedge E} E \wedge E,$$

respectively.

(2) The map $\varepsilon : E_{**}(E) \to \pi_{**}(E)$ sending a class $\alpha : S^a \to E \wedge E$ to the composition

$$S^a \xrightarrow{\alpha} E \wedge E \xrightarrow{\mu} E$$

(3) The map $\Psi : E_{**}(E) \to E_{**}(E) \otimes_{\pi_{**}(E)} E_{**}(E)$ which factors as

$$E_{**}(E) \to E_{**}(E \wedge E) \xrightarrow{\cong} E_{**}(E) \otimes_{\pi_{**}(E)} E_{**}(E)$$

where the second arrow is the isomorphism prescribed by Proposition B.10, and the first arrow sends a class $\alpha : S^a \to E \wedge E$ to the composition

$$S^a \xrightarrow{\alpha} E \wedge E \cong E \wedge S \wedge E \xrightarrow{E \wedge e \wedge E} E \wedge E \wedge E.$$

(4) The map $c : E_{**}(E) \to E_{**}(E)$ sending a map $\alpha : S^a \to E \wedge E$ to the composition

$$S^a \xrightarrow{\alpha} E \wedge E \xrightarrow{\tau} E \wedge E,$$

where τ is the symmetry map prescribed by the symmetric monoidal structure on $\mathbf{SH}_{\mathscr{S}}$.

Then all of these maps are homomorphisms of $\pi_{**}(S)$-algebras, and they furthermore endow the pair $(E_{**}(E), \pi_{**}(E))$ with the structure of a bigraded commutative Hopf algebroid (Definition B.15)

Proof. □ TODO

Appendix C. Monoid objects

Appendix D. Spectral sequences

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