AN AXIOMATIC APPROACH TO THE ADAMS SPECTRAL SEQUENCE

ISAIAH DAILEY

Contents

1. Introduction 2
2. Triangulated categories with sub-Picard grading 2
   2.1. Setup of $\mathcal{SH}$ 2
   2.2. The homotopy long exact sequence 5
   2.3. Cellular objects in $\mathcal{SH}$ 8
   2.4. Monoid objects in $\mathcal{SH}$ 11
3. The Adams spectral sequence 12
4. The E_1 page 14
   4.1. Flat monoid objects in $\mathcal{SH}$ 14
   4.2. The dual E-Steenrod algebra 15
   4.3. Comodules over the dual E-Steenrod algebra 17
5. The E_2 page 17
6. Convergence of the Adams spectral sequence 17
7. The classical Adams spectral sequence 17
8. The motivic Adams spectral sequence 17
   8.1. Motivic spaces 18
   8.2. The unstable motivic homotopy category 20
   8.3. The stable motivic homotopy category 21
   8.4. Grading 23
Appendix A. Triangulated categories 27
   A.1. Triangulated categories and their properties 27
   A.2. Homotopy (co)limits in a triangulated category 32
   A.3. Adjointly triangulated categories 33
   A.4. Tensor triangulated categories 37
Appendix B. Spectral sequences 40
   B.1. Unrolled exact couples and their associated spectral sequences 40
Appendix C. A-graded abelian groups, rings, and modules 42

Date: September 13, 2023.

C.1. A-graded submodules and quotient modules 47
C.2. Tensor product of A-graded modules 49
Appendix D. Monoid objects 51
D.1. Monoid objects in a symmetric monoidal category 52
D.2. Modules over a monoid object 54
Appendix E. Hopf algebroids 85
E.1. The category R-**GrCAlg** of A-graded R-commutative rings 85
E.2. A-graded commutative Hopf algebroids over R 88
E.3. Comodules over a Hopf algebroid 92
E.4. The dual E-Steenrod algebra is a Hopf algebroid 94
References 102

1. Introduction

2. Triangulated categories with sub-Picard grading

2.1. **Setup of $\mathcal{SH}$.** In order to construct an abstract version of the Adams spectral sequence, we need to work in some axiomatic version of a stable homotopy category $\mathcal{SH}$ which acts like the familiar classical stable homotopy category **hoSp** (Section 7) or the motivic stable homotopy category $\mathbf{SH}_{\mathscr{S}}$ over some base scheme $\mathscr{S}$ (Section 8).

Definition 2.1. Let $\mathcal{C}$ be an additive category with arbitrary (small) coproducts. Then an object X in $\mathcal{C}$ is *compact* if, for any collection of objects Y_i in $\mathcal{C}$ indexed by some (small) set I, the canonical map

$$\bigoplus_i \mathcal{C}(X, Y_i) \to \mathcal{C}(X, \bigoplus_i Y_i)$$

is an isomorphism of abelian groups. (Explicitly, the above map takes a generator $x \in \mathcal{C}(X, Y_i)$ to the composition $X \xrightarrow{x} Y_i \hookrightarrow \bigoplus_i Y_i$.)

Definition 2.2. Given a tensor triangulated category $(\mathcal{C}, \otimes, S, \Sigma, e, \mathcal{D})$ (Definition A.13), a *sub-Picard grading* on $\mathcal{C}$ is the following data:

- A pointed abelian group $(A, \mathbf{1})$ along with a homomorphism of pointed groups $h : (A, \mathbf{1}) \to (\operatorname{Pic} \mathcal{C}, \Sigma S)$, where $\operatorname{Pic} \mathcal{C}$ is the *Picard group* of isomorphism classes of invertible objects in $\mathcal{C}$.[1]
- For each $a \in A$, a chosen representative S^a called the *a-sphere* in the isomorphism class $h(a)$ such that each S^a is a com pact object (Definition 2.1) and $S^0 = S$.
- For each $a, b \in A$, an isomorphism $\phi_{a,b} : S^{a+b} \to S^a \otimes S^b$. This family of isomorphisms is required to be *coherent*, in the following sense:
  - For all $a \in A$, we must have that $\phi_{a,0}$ coincides with the right unitor $\rho_{S^a}^{-1} : S^a \xrightarrow{\cong} S^a \otimes S$ and $\phi_{0,a}$ coincides the left unitor $\lambda_{S^a}^{-1} : S^a \xrightarrow{\cong} S \otimes S^a$.

[1] Recall an object X is a symmetric monoidal category is *invertible* if there exists some object Y and an isomorphism $S \cong X \otimes Y$.

– For all $a, b, c \in A$, the following "associativity diagram" must commute:

$$\begin{tikzcd} S^{a+b} \otimes S^c \arrow[d, "\phi_{a,b} \otimes S^c"'] & S^{a+b+c} \arrow[l, "\phi_{a+b,c}"'] \arrow[r, "\phi_{a,b+c}"] & S^a \otimes S^{b+c} \arrow[d, "S^a \otimes \phi_{b,c}"] \\ (S^a \otimes S^b) \otimes S^c \arrow[rr, "\cong"] & & S^a \otimes (S^b \otimes S^c) \end{tikzcd}$$

From now on we fix a monoidal closed tensor triangulated category $(\mathcal{SH}, \otimes, S, \Sigma, e, \mathcal{D})$ with arbitrary (small) (co)products and sub-Picard grading $(A, \mathbf{1}, h, \{S^a\}, \{\phi_{a,b}\})$. We also fix an isomorphism $\nu : \Sigma S \xrightarrow{\cong} S^{\mathbf{1}}$ once and for all. We establish conventions. First, observe the following remark:

Remark 2.3. Note that by induction the coherence conditions for the $\phi_{a,b}$'s in the above definition say that given any $a_1, \ldots, a_n \in A$ and $b_1, \ldots, b_m \in A$ such that $a_1 + \cdots + a_n = b_1 + \cdots + b_m$ and any fixed parenthesizations of $X = S^{a_1} \otimes \cdots \otimes S^{a_b}$ and $Y = S^{b_1} \otimes \cdots \otimes S^{b_m}$, there is a *unique* isomorphism $X \to Y$ that can be obtained by forming formal compositions of products of $\phi_{a,b}$, identities, associators, unitors, and their inverses (but not symmetries).

In light of this remark, we will usually simply write ϕ or even just $\cong$ for any isomorphism that is built by taking compositions of products of $\phi_{a,b}$'s, unitors, associators, identities, and their inverses. Given an object X and a natural number $n > 0$, we write

$$X^n := \overbrace{X \otimes \cdots \otimes X}^{n \text{ times}} \qquad \text{and} \qquad X^0 := S.$$

We denote the associator, symmetry, left unitor, and right unitor isomorphisms in $\mathcal{SH}$ by

$$\alpha_{X,Y,Z} : (X \otimes Y) \otimes Z \xrightarrow{\cong} X \otimes (Y \otimes Z) \qquad \tau_{X,Y} : X \otimes Y \xrightarrow{\cong} Y \otimes X$$
$$\lambda_X : S \otimes X \xrightarrow{\cong} X \qquad \rho_X : X \otimes S \xrightarrow{\cong} X.$$

Often we will drop the subscripts. Furthermore, by the coherence theorem for symmetric monoidal categories ([8]), we will often assume α, ρ, and λ are actual equalities.

Given some integer $n \in \mathbb{Z}$, we will write a bold $\mathbf{n}$ to denote the element $n \cdot \mathbf{1}$ in A. Note that we can use the isomorphism $\nu : \Sigma S \xrightarrow{\cong} S^{\mathbf{1}}$ to construct a natural isomorphism $\Sigma \cong S^{\mathbf{1}} \otimes -$:

$$\Sigma X \xrightarrow{\Sigma\lambda_X^{-1}} \Sigma(S \otimes X) \xrightarrow{e_{S,X}^{-1}} \Sigma S \otimes X \xrightarrow{\nu \otimes X} S^{\mathbf{1}} \otimes X.$$

The first two arrows are natural in X by definition. The last arrow is natural in X by functoriality of $- \otimes -$. By abuse of notation, we will also use ν to denote this natural isomorphism.

Given some $a \in A$, we define $\Sigma^a := S^a \otimes -$ and $\Omega^a := \Sigma^{-a} = S^{-a} \otimes -$. We specifically define $\Omega := \Omega^{\mathbf{1}}$. We say "the a^{th} suspension of X" to denote $\Sigma^a X$. It turns out that Σ^a is an autoequivalence of $\mathcal{SH}$ for each $a \in A$, and furthermore, Ω^a and Σ^a form an adjoint equivalence of $\mathcal{SH}$ for all a in A:

Proposition 2.4. *For each $a \in A$, the isomorphisms*

$$\eta_X^a : X \xrightarrow{\lambda_X^{-1}} S \otimes X \xrightarrow{\phi_{a,-a} \otimes X} (S^a \otimes S^{-a}) \otimes X \xrightarrow{\alpha} S^a \otimes (S^{-a} \otimes X) = \Sigma^a \Omega^a X$$

and

$$\varepsilon_X^a : \Omega^a \Sigma^a X = S^{-a} \otimes (S^a \otimes X) \xrightarrow{\alpha^{-1}} (S^{-a} \otimes S^a) \otimes X \xrightarrow{\phi_{-a,a}^{-1} \otimes X} S \otimes X \xrightarrow{\lambda_X} X$$

are natural in X, and furthermore, they are the unit and counit respectively of the adjoint autoequivalence $(\Omega^a, \Sigma^a, \eta^a, \varepsilon^a)$ of $\mathcal{SH}$. In particular, since $\Sigma \cong \Sigma^{\mathbf{1}}$, $\Omega := \Omega^{\mathbf{1}}$ is a left adjoint for Σ, so *that* $(\mathcal{SH}, \Omega, \Sigma, \eta, \varepsilon, \mathcal{D})$ *is an* adjointly *triangulated category (Definition A.9), where* η *and* ε *are the compositions*

$$\eta : \mathrm{Id}_{\mathcal{SH}} \xRightarrow{\eta^{\mathbf{1}}} \Sigma^{\mathbf{1}}\Omega \xRightarrow{\nu^{-1}\Omega} \Sigma\Omega \qquad \textit{and} \qquad \varepsilon : \Omega\Sigma \xRightarrow{\Omega\nu} \Omega\Sigma^{\mathbf{1}} \xRightarrow{\varepsilon^{\mathbf{1}}} \mathrm{Id}_{\mathcal{SH}}.$$

Proof. In this proof, we will freely employ the coherence theorem for monoidal categories (see [8]), which essentially tells us that we may assume we are working in a strict monoidal category (i.e., that the associators and unitors and are identities). Then η^a_X and ε^a_X become simply the maps

$$\eta^a_X : X \xrightarrow{\phi_{a,-a} \otimes X} S^a \otimes S^{-a} \otimes X \qquad \text{and} \qquad \varepsilon^a_X : S^{-a} \otimes S^a \otimes X \xrightarrow{\phi^{-1}_{-a,a} \otimes X} X.$$

That these maps are natural in X follows by functoriality of $- \otimes -$. Now, recall that in order to show that these natural isomorphisms form an *adjoint* equivalence, it suffices to show that the natural isomorphisms $\eta^a : \mathrm{Id}_{\mathcal{SH}} \Rightarrow \Omega^a\Sigma^a$ and $\varepsilon^a : \Sigma^a\Omega^a \Rightarrow \mathrm{Id}_{\mathcal{SH}}$ satisfy one of the two zig-zag identities:

$$\begin{tikzcd} \Omega^a \arrow[r, "\Omega^a\eta^a"] \arrow[dr, equal] & \Omega^a\Sigma^a\Omega^a \arrow[d, "\varepsilon^a\Omega^a"] \\ & \Omega^a \end{tikzcd} \qquad \begin{tikzcd} \Sigma^a\Omega^a\Sigma^a \arrow[d, "\Sigma^a\varepsilon^a"'] & \Sigma^a \arrow[l, "\eta^a\Sigma^a"'] \arrow[dl, equal] \\ \Sigma^a \end{tikzcd}$$

(that it suffices to show only one is [12, Lemma 3.2]). We will show that the left is satisfied. Unravelling definitions, we simply wish to show that the following diagram commutes for all X in $\mathcal{SH}$:

$$\begin{tikzcd} S^{-a} \otimes X \arrow[r, "S^{-a}\otimes\phi_{a,-a}\otimes X"] \arrow[dr, equal] & S^{-a} \otimes S^a \otimes S^{-a} \otimes X \arrow[d, "\phi^{-1}_{-a,a}\otimes S^{-a}\otimes X"] \\ & S^{-a} \otimes X \end{tikzcd}$$

Yet this is simply the diagram obtained by applying $- \otimes X$ to the associativity coherence diagram for the $\phi_{a,b}$'s (since $\phi_{a,0}$ and $\phi_{0,a}$ coincide with the unitors, and here we are taking the unitors and associators to be equalities), so it does commute, as desired. □

Given two objects X and Y in $\mathcal{SH}$, we will denote the hom-abelian group of morphisms from X to Y in $\mathcal{SH}$ by $[X, Y]$, and the internal hom object by $F(X, Y)$. We can extend the abelian group $[X, Y]$ into an A-graded abelian group $[X, Y]_*$ by defining $[X, Y]_a := [S^a \otimes X, Y]$. Given an object X in $\mathcal{SH}$ and some $a \in A$, we can define the abelian group

$$\pi_a(X) := [S^a, X],$$

which we call the a^{th} *(stable) homotopy group of* X. We write $\pi_*(X)$ for the A-graded abelian group $\bigoplus_{a\in A} \pi_a(X)$, so that in particular we have a canonical isomorphism

$$\pi_*(X) = [S^*, X] \cong [S, X]_*.$$

Given some other object E, we can define the A-graded abelian groups $E_*(X)$ and $E^*(X)$ by the formulas

$$E_a(X) := \pi_a(E \otimes X) = [S^a, E \otimes X] \qquad \text{and} \qquad E^a(X) := [X, S^a \otimes E].$$

We refer to the functor $E_*(-)$ as the *homology theory represented by* E, or just E-homology, and we refer to $E^*(-)$ as the *cohomology theory represented by* E, or just E-cohomology.

2.2. The homotopy long exact sequence.

Proposition 2.5. *Suppose we are given a distinguished triangle*

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$$

and some object W in $\mathcal{SH}$. Then there is an infinite long exact sequence of A-graded abelian groups:

$$\cdots \to [W,Z]_{*+\mathbf{n+1}} \xrightarrow{\partial} [W,X]_{*+\mathbf{n}} \xrightarrow{f_*} [W,Y]_{*+\mathbf{n}} \xrightarrow{g_*} [W,Z]_{*+\mathbf{n}} \xrightarrow{\partial} [W,Z]_{*+\mathbf{n-1}} \to \cdots,$$

where $\partial : [W,Z]_{*+\mathbf{n+1}} \to [W,X]_{*+\mathbf{n}}$ sends a class $x : S^{a+\mathbf{n+1}} \otimes W \to Z$ to the composition

$$S^{a+\mathbf{n}} \otimes W \xrightarrow{\phi_{-\mathbf{1},a+\mathbf{n+1}}} S^{-\mathbf{1}} \otimes S^{a+\mathbf{n+1}} \otimes W \xrightarrow{S^{-\mathbf{1}}\otimes x} S^{-\mathbf{1}} \otimes Z \xrightarrow{\widetilde{h}} X,$$

where here we are suppressing the associator from the notation, and $\widetilde{h} : \Omega Z = S^{-\mathbf{1}} \otimes Z \to X$ is the adjoint (Proposition 2.4) of $h : Z \to \Sigma X$.

Proof. In this proof, we will freely employ the coherence theorem for a symmetric monoidal category, which tells us we may assume associativity and unitality of $-\otimes-$ holds up to strict equality. Furthermore, we will simply write ϕ to refer to any isomorphism that can be constructed by composing copies of products of $\phi_{a,b}$'s, unitors, identities, associators, and their inverses (see Remark 2.3). Finally, given $n > 0$, we will write Σ^{-n} to denote the functor $\Omega^n = (S^{-\mathbf{1}})^n \otimes -$.

For all $n > 0$, the $\phi_{a,b}$'s yield natural isomorphisms

$$s_X^{-n} : \Sigma^{-n} X = (S^{-\mathbf{1}})^n \otimes X \xrightarrow{\phi\otimes X} S^{-\mathbf{n}} \otimes X = \Omega^{\mathbf{n}} X.$$

and

$$s_X^n : \Sigma^n X \xrightarrow{\nu_X^n} (S^{\mathbf{1}})^n \otimes X \xrightarrow{\phi\otimes X} S^{\mathbf{n}} \otimes X = \Sigma^{\mathbf{n}} X,$$

where we recursively define $\nu^1 := \nu$ and ν^{n+1} is given by the composition

$$\nu_X^{n+1} : \Sigma^{n+1} X = \Sigma^n \Sigma X \xrightarrow{\nu_{\Sigma X}^n} (S^{\mathbf{1}})^n \otimes \Sigma X \xrightarrow{(S^{\mathbf{1}})^n\otimes \nu_X} (S^{\mathbf{1}})^n \otimes S^{\mathbf{1}} \otimes X = (S^{\mathbf{1}})^{n+1} \otimes X.$$

Finally, we define s^0 to be the identity natural transformation on $\mathcal{SH}$. Then we get the following natural isomorphisms of A-graded abelian groups for all $n \in \mathbb{Z}$

$$\ell_V^n : [W, \Sigma^n V]_* \xrightarrow{(s_V^n)_*} [W, \Sigma^{\mathbf{n}} V]_* \xrightarrow{r^{\mathbf{n}}_{W,V}} [W,V]_{*-\mathbf{n}},$$

where $r^{\mathbf{n}}_{W,V}$ is the natural isomorphism given as the composition

$$[W, \Sigma^{\mathbf{n}} V]_* \xrightarrow{\cong} [S^{-\mathbf{n}} \otimes S^* \otimes W, V] \xrightarrow{(\phi\otimes W)^*} [S^{*-\mathbf{n}} \otimes W, V] = [W,V]_{*-\mathbf{n}},$$

where the first isomorphism is the adjunction $\Omega^{\mathbf{n}} \dashv \Sigma^{\mathbf{n}}$ (Proposition 2.4). Now, given $n \in \mathbb{Z}$, consider the following diagram

$$\begin{tikzcd}
{[W,\Sigma^{n-1}Z]_*} \arrow[r, "h_{n-1}"] \arrow[d, "\ell_Z^{n-1}"'] & {[W,\Sigma^n X]_*} \arrow[r, "\Sigma^n f_*"] \arrow[d, "\ell_X^n"'] & {[W,\Sigma^n Y]_*} \arrow[r, "\Sigma^n g_*"] \arrow[d, "\ell_Y^n"'] & {[W,\Sigma^n Z]_*} \arrow[r, "h_n"] \arrow[d, "\ell_Z^n"'] & {[W,\Sigma^{n+1} X]_*} \arrow[d, "\ell_X^{n+1}"] \\
{[W,Z]_{*-\mathbf{n+1}}} \arrow[r, "\partial"'] & {[W,X]_{*-\mathbf{n}}} \arrow[r, "f_*"'] & {[W,Y]_{*-\mathbf{n}}} \arrow[r, "g_*"'] & {[W,Z]_{*-\mathbf{n}}} \arrow[r, "\partial"'] & {[W,X]_{*-\mathbf{n-1}}}
\end{tikzcd} \tag{1}$$

where for $n \geq 0$, $h_n = \Sigma^n h$, and for $n > 0$, $h_{-n} = \Omega^{n-1}\widetilde{h}$ (where $\widetilde{h} : \Omega Z \to X$ is the adjoint of $h : Z \to \Sigma X$). We would like to show the bottom row is exact. The top row is exact since it is obtained by applying $[W,-]_*$ to a fiber sequence (see Proposition A.12 for full details), and we have constructed the vertical arrows to be isomorphisms. Thus it suffices to show each square commutes. The inner two squares commute by naturality of ℓ^n. Thus, it further suffices to show the outermost squares commute. Since our choice of $n \in \mathbb{Z}$ is arbitrary, we can just show the right square commutes. First consider the case that $n \geq 0$, and consider the following diagram:

$$
\begin{tikzcd}
{[W, \Sigma^n Z]_*} \arrow[rrr, "\Sigma^n h_*"] \arrow[ddd, "\ell^n_Z"'] & & & {[W, \Sigma^{n+1} X]_*} \arrow[dll, "\ell^n_{\Sigma X}"] \arrow[ddd, "\ell^{n+1}_X"] \\
& {[W, \Sigma X]_{*-\mathbf{n}}} \arrow[dr, "(\nu_X)_*"] & & \\
& & {[W, \Sigma^{\mathbf{1}} X]_{*-\mathbf{n}}} \arrow[dr, "r^{\mathbf{1}}_{W,X}"] & \\
{[W, Z]_{*-\mathbf{n}}} \arrow[uur, "h_*"] \arrow[rrr, "\partial"'] & & & {[W, X]_{*-\mathbf{n}-\mathbf{1}}}
\end{tikzcd}
$$

The leftmost region commutes by naturality of ℓ. By unravelling how $r^{\mathbf{1}}_{W,X}$ and the adjoint $\widetilde{h}$ used in the definition of ∂ are defined, a simple diagram chase yields that the bottom triangle commutes. Thus, it remains to show the rightmost triangle in the above diagram commutes. To see this, note that by unravelling how ℓ and r are defined, this triangle becomes

$$
\begin{tikzcd}
{[W, (S^{\mathbf{1}})^n \otimes \Sigma X]} \arrow[ddr, "((S^{\mathbf{1}})^n \otimes \nu_X)_*"] \arrow[dd, "(\phi \otimes \Sigma X)_*"'] & & {[W, \Sigma^{n+1} X]_*} \arrow[ll, "(\nu^n_{\Sigma X})_*"'] \arrow[d, "(\nu^{n+1}_X)_*"] \\
& & {[W, (S^{\mathbf{1}})^{n+1} \otimes X]_*} \arrow[dl, "(\phi \otimes X)_*"'] \arrow[d, "(\phi \otimes X)_*"] \\
{[W, \Sigma^{\mathbf{n}} \Sigma X]_*} \arrow[r, "(\Sigma^{\mathbf{n}} \nu_X)_*"] \arrow[d, "\mathrm{adj}"'] & {[W, \Sigma^{\mathbf{n}} \Sigma^{\mathbf{1}} X]_*} \arrow[r, "(\phi \otimes X)_*"'] \arrow[d, "\mathrm{adj}"] & {[W, \Sigma^{\mathbf{n}+\mathbf{1}} X]_*} \arrow[d, "\mathrm{adj}"] \\
{[S^{-\mathbf{n}} \otimes S^* \otimes W, \Sigma X]} \arrow[r, "(\nu_X)_*"] \arrow[d, "(\phi \otimes W)^*"'] & {[S^{-\mathbf{n}} \otimes S^* \otimes W, S^{\mathbf{1}} \otimes X]} \arrow[ddl, "(\phi \otimes W)^*"] & {[S^{-\mathbf{n}-\mathbf{1}} \otimes S^* \otimes W, X]} \arrow[dd, "(\phi \otimes W)^*"] \\
{[W, \Sigma X]_{*-\mathbf{n}}} \arrow[d, "(\nu_X)_*"'] & & \\
{[W, \Sigma^{\mathbf{1}} X]_{*-\mathbf{n}}} \arrow[r, "\mathrm{adj}"'] & {[S^{-\mathbf{1}} \otimes S^{*-\mathbf{n}} \otimes W, X]} \arrow[uur, "(\phi \otimes W)^*"] \arrow[r, "(\phi \otimes W)^*"'] & {[W, X]_{*-\mathbf{n}-\mathbf{1}}}
\end{tikzcd}
$$

The top right triangle commutes by how ν^{n+1} was defined. The top left oddly-shaped region commutes by functoriality of $- \otimes -$. The middle right triangle commutes by coherence for the ϕ's. The middle left rectangle commutes by naturality of the adjunction isomorphism. Commutativity of the bottom left triangle is clear (do a diagram chase). Commutativity of the bottom right triangle is coherence for the ϕ's. Finally, commutativity of the remaining region is again coherence of the ϕ's, since the adjunction isomorphisms are constructed using them (Proposition 2.4).

Now we consider the negative case: Unravelling definitions, given $n > 0$, the rightmost square in diagram (1) for $-n$ becomes

(2)
$$\begin{tikzcd}
{[W,\Omega^n Z]_*} \arrow[rr, "\Omega^{n-1}\widetilde{h}_*"] \arrow[d, "(\phi\otimes Z)_*"'] \arrow[dr, "(\phi\otimes\Omega Z)_*"] & & {[W,\Omega^{n-1}X]_*} \arrow[d, "(\phi\otimes X)_*"] \\
{[W,\Omega^{\mathbf{n}} Z]_*} \arrow[r, "(\phi\otimes Z)_*"'] \arrow[ddd, "\mathrm{adj}"'] & {[W,\Omega^{\mathbf{n-1}}\Omega Z]_*} \arrow[r, "\Omega^{\mathbf{n-1}}\widetilde{h}_*"] \arrow[d, "\mathrm{adj}"'] & {[W,\Omega^{\mathbf{n-1}}X]_*} \arrow[d, "\mathrm{adj}"] \\
 & {[S^{\mathbf{n-1}}\otimes S^*\otimes W,\Omega Z]} \arrow[r, "\widetilde{h}_*"] \arrow[d, "(\phi\otimes W)^*"'] & {[S^{\mathbf{n-1}}\otimes S^*\otimes W,X]} \arrow[ddd, "(\phi\otimes W)^*"] \\
 & {[W,\Omega Z]_{*+\mathbf{n-1}}} \arrow[d, "\mathrm{adj}"'] \arrow[ddr, "\widetilde{h}_*"] & \\
{[S^{\mathbf{n}}\otimes S^*\otimes W,Z]} \arrow[r, "(\phi\otimes W)^*"] \arrow[d, "(\phi\otimes W)^*"'] & {[S^{\mathbf{1}}\otimes S^{*+\mathbf{n-1}}\otimes W,Z]} \arrow[dl, "(\phi\otimes W)^*"'] & \\
{[W,Z]_{*+\mathbf{n}}} \arrow[rr, "\partial"'] & & {[W,X]_{*+\mathbf{n-1}}}
\end{tikzcd}$$

The top right trapezoid commutes by functoriality of $-\otimes-$. The top left triangle commutes by coherence for the ϕ's. The middle right rectangle commutes by naturality of the adjunction. The right trapezoid below that commutes obviously. The bottom left triangle commutes by coherence of the ϕ's. The large middle left rectangle commutes by coherence for the ϕ's, again since the adjunction $\Sigma^{\mathbf{n}} \dashv \Omega^{\mathbf{n}}$ is constructed using the ϕ's. Finally, to see the bottom diagram commutes, we will chase some homogeneous element $f : S^{b+\mathbf{n-1}} \otimes W \to \Omega Z$ around the region. Consider the following diagram:

$$\begin{tikzcd}
S^{\mathbf{-1}}\otimes S^{b+\mathbf{n}}\otimes W \arrow[d, "\phi\otimes W"'] & S^{b+\mathbf{n-1}}\otimes W \arrow[l, "\phi\otimes W"'] \arrow[dd, "f"] & & & \\
S^{\mathbf{-1}}\otimes S^{\mathbf{1}}\otimes S^{b+\mathbf{n+1}}\otimes W \arrow[ur, "\phi\otimes W"'] \arrow[d, "S^{\mathbf{-1}}\otimes S^{\mathbf{1}}\otimes f"'] & & & & \\
S^{\mathbf{-1}}\otimes S^{\mathbf{1}}\otimes S^{\mathbf{-1}}\otimes Z \arrow[r, "\phi\otimes Z"'] & S^{-1}\otimes Z \arrow[r, "S^{\mathbf{-1}}\otimes h"] & S^{\mathbf{-1}}\otimes\Sigma X \arrow[r, "S^{\mathbf{-1}}\otimes\nu_X"] & S^{\mathbf{-1}}\otimes S^{\mathbf{1}}\otimes X \arrow[r, "\phi\otimes X"] & X
\end{tikzcd}$$

By unravelling how the adjunction and $\widetilde{h}$ are defined, the two compositions around the outside of this diagram are the two morphisms obtained by chasing f around the bottom region in diagram (2). The top left triangle of the above diagram commutes by coherence of the ϕ's (Remark 2.3), while the bottom region commutes by functoriality of $-\otimes-$ and coherence of the ϕ's. Thus we've shown diagram (1) commutes, so the bottom row is exact, as desired. $\square$

Remark 2.6. Expressed more compactly, the above proposition says that each object W and distinguished triangle

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$$

in $\mathcal{SH}$ gives rise to the following diagram of A-graded abelian groups

$$\begin{tikzcd}
{[W,X]_*} \arrow[r, "f_*"] & {[W,Y]_*} \arrow[d, "g_*"] \\
 & {[W,Z]_*} \arrow[ul, "\partial"]
\end{tikzcd}$$

which is exact at each vertex, and where f_*, g_*, and ∂ are A-graded homomorphisms of degree 0, 0, and $-\mathbf{1}$, respectively. Explicitly, ∂ sends a class $x : S^a \otimes W \to Z$ to the composition

$$S^{a-\mathbf{1}} \otimes W \cong S^{-\mathbf{1}} \otimes S^a \otimes W \xrightarrow{S^{-\mathbf{1}} \otimes x} S^{-\mathbf{1}} \otimes Z \xrightarrow{S^{-\mathbf{1}} \otimes h} S^{-\mathbf{1}} \otimes \Sigma X \xrightarrow{S^{-\mathbf{1}} \otimes \nu_X} S^{-\mathbf{1}} \otimes S^{\mathbf{1}} \otimes X \xrightarrow{\phi^{-1}_{-\mathbf{1},\mathbf{1}} \otimes X} X.$$

2.3. **Cellular objects in $\mathcal{SH}$.** One very important class of objects in $\mathcal{SH}$ are the *cellular* objects. Intuitively, these are the objects that can be built out of the S^a's via taking coproducts and (co)fibers.

Definition 2.7. Define the class of *cellular* objects in $\mathcal{SH}$ to be the smallest class of objects such that:

(1) For all $a \in A$, the a-sphere S^a is cellular.
(2) If we have a distinguished triangle
$$X \to Y \to Z \to \Sigma X$$
such that two of the three objects X, Y, and Z are cellular, than the third object is also cellular.
(3) Given a collection of cellular objects X_i indexed by some (small) set I, the object $\bigoplus_{i \in I} X_i$ is cellular (recall we have chosen $\mathcal{SH}$ to have arbitrary coproducts).

Lemma 2.8. *Let X and Y be two isomorphic objects in $\mathcal{SH}$. Then X is cellular iff Y is cellular.*

Proof. Assume we have an isomorphism $f : X \xrightarrow{\cong} Y$ and that X is cellular. Then consider the following commutative diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, equal] & Y \arrow[r] \arrow[d, "f^{-1}"] & 0 \arrow[r] \arrow[d, equal] & \Sigma X \arrow[d, equal] \\ X \arrow[r, equal] & X \arrow[r] & 0 \arrow[r] & \Sigma X \end{tikzcd}$$

The bottom row is distinguished by axiom TR1 for a triangulated category. Hence since X is cellular, 0 is also cellular, since the class of cellular objects satisfies two-of-three for distinguished triangles. Furthermore, since the vertical arrows are all isomorphisms, the top row is distinguished as well, by axiom TR0. Thus again by two-of-three, since X and 0 are cellular, so is Y, as desired. $\square$

Lemma 2.9. *Let X and Y be cellular objects in $\mathcal{SH}$. Then $X \otimes Y$ is cellular.*

Proof. Let E be a cellular object in $\mathcal{SH}$, and let $\mathcal{E}$ be the collection of objects X in $\mathcal{SH}$ such that $E \otimes X$ is cellular. First of all, suppose we have a distinguished triangle

$$X \to Y \to Z \to \Sigma X$$

such that two of three of X, Y, and Z belong to $\mathcal{E}$. Then since $\mathcal{SH}$ is tensor triangulated, we have a distinguished triangle

$$E \otimes X \to E \otimes Y \to E \otimes Z \to \Sigma(E \otimes X).$$

Per our assumptions, two of three of $E \otimes X$, $E \otimes Y$, and $E \otimes Z$ are cellular, so that the third is by definition. Thus, all three of X, Y, and Z belong to $\mathcal{E}$ if two of them do.

Second of all, suppose we have a family X_i of objects in $\mathcal{E}$ indexed by some (small) set I, and set $X := \bigoplus_i X_i$. Then we'd like to show X belongs to $\mathcal{E}$, i.e., that $E \otimes X$ is cellular. Indeed,

$$E \otimes X = E \otimes \left(\bigoplus_i X_i \right) \cong \bigoplus_i (E \otimes X_i),$$

where the isomorphism is given by the fact that $\mathcal{SH}$ is monoidal closed, so $E \otimes -$ preserves arbitrary colimits as it is a left adjoint. Per our assumption, since each $E \otimes X_i$ is cellular, the rightmost object is cellular, since the class of cellular objects is closed under taking arbitrary coproducts, by definition. Hence $E \otimes X$ is cellular by Lemma 2.8.

Finally, we would like to show that each S^a belongs to $\mathcal{E}$, i.e., that $S^a \otimes E$ is cellular for all $a \in A$. When $E = S^b$ for some $b \in A$, this is clearly true, since $S^b \otimes S^a \cong S^{a+b}$, which is cellular by definition, so that $S^b \otimes S^a$ is cellular by Lemma 2.8. Thus by what we have shown, the class of objects X for which $S^a \otimes X$ is cellular contains every cellular object. Hence in particular $E \otimes S^a \cong S^a \otimes E$ is cellular for all $a \in A$, as desired. $\square$

Lemma 2.10. *Let W be a cellular object in $\mathcal{SH}$ such that $\pi_*(W) = 0$. Then $W \cong 0$.*

Proof. Let $\mathcal{E}$ be the collection of all X in $\mathcal{SH}$ such that $[\Sigma^n X, W] = 0$ for all $n \in \mathbb{Z}$ (where for $n > 0$, $\Sigma^{-n} := \Omega^n = (S^{-\mathbf{1}})^n \otimes -$). We claim $\mathcal{E}$ contains every cellular object in $\mathcal{SH}$. First of all, each S^a belongs to $\mathcal{E}$, as

$$[\Sigma^n S^a, W] \cong [S^{\mathbf{n}} \otimes S^a, W] \cong [S^{a+\mathbf{n}}, W] \leq \pi_*(W) = 0.$$

Furthermore, suppose we are given a distinguished triangle

$$X \to Y \to Z \to \Sigma X$$

such that two of three of X, Y, and Z belong to $\mathcal{E}$. By Proposition A.12, for all $n \in \mathbb{Z}$ we get an exact sequence

$$[\Sigma^{n+1} X, W] \to [\Sigma^n Z, W] \to [\Sigma^n Y, W] \to [\Sigma^n X, W] \to [\Sigma^{n-1} Z, W].$$

Clearly if any two of three of X, Y, and Z belong to $\mathcal{E}$, then by exactness of the above sequence all three of the middle terms will be zero, so that the third object will belong to $\mathcal{E}$ as well. Finally, suppose we have a collection of objects X_i in $\mathcal{E}$ indexed by some small set I. Then

$$\left[\Sigma^n \bigoplus_i X_i, W \right] \cong \left[\bigoplus_i \Sigma^n X_i, W \right] \cong \prod_i [\Sigma^n X_i, W] = \prod_i 0 = 0,$$

where the first isomorphism follows by the fact that Σ^n is apart of an adjoint equivalence (Proposition 2.4), so it preserves arbitrary colimits.

Thus, by definition of cellularity, $\mathcal{E}$ contains every cellular object. In particular, $\mathcal{E}$ contains W, so that $[W, W] = 0$, meaning in particular that $\mathrm{id}_W = 0$, so we have a commutative diagram

$$\begin{tikzcd} & 0 \arrow[rr, equal] \arrow[dr] & & 0 \\ W \arrow[ur] \arrow[rr, equal] & & W \arrow[ur] & \end{tikzcd}$$

Hence the diagonals exhibit isomorphisms between 0 and W, as desired. $\square$

Theorem 2.11. *Let X and Y be cellular objects in $\mathcal{SH}$, and suppose $f : X \to Y$ is a morphism such that $f_* : \pi_*(X) \to \pi_*(Y)$ is an isomorphism. Then f is an isomorphism.*

Proof. By axiom TR2 for a triangulated category (Definition A.1), we have a distinguished triangle

$$X \xrightarrow{f} Y \xrightarrow{g} C_f \xrightarrow{h} \Sigma X.$$

First of all, note that by definition since X and Y are cellular, so is C_f. Now, we claim $\pi_*(C_f) = 0$. Indeed, given $a \in A$, by Proposition A.12 we have the following exact sequence:

$$[S^a, X] \xrightarrow{f_*} [S^a, Y] \xrightarrow{g_*} [S^a, C_f] \xrightarrow{h_*} [S^a, \Sigma X] \xrightarrow{-(\Sigma f)_*} [S^a, \Sigma Y],$$

where the first arrow is an isomorphism, per our assumption that f is an isomorphism. Now consider the following diagram:

$$\begin{tikzcd}
{[S^a, \Sigma X]} \arrow[r, "(\Sigma f)_*"] \arrow[d, "(\nu_X)_*"'] & {[S^a, \Sigma Y]} \arrow[d, "(\nu_Y)_*"] \\
{[S^a, S^{\mathbf{1}} \otimes X]} \arrow[r, "(S^{\mathbf{1}} \otimes f)_*"] \arrow[d, "\cong"'] & {[S^a, S^{\mathbf{1}} \otimes Y]} \arrow[d, "\cong"] \\
{[S^{-\mathbf{1}} \otimes S^a, X]} \arrow[r, "f_*"] \arrow[d, "(\phi_{-\mathbf{1},a})^*"'] & {[S^{-\mathbf{1}} \otimes S^a, Y]} \arrow[d, "(\phi_{-\mathbf{1},a})^*"] \\
{[S^{a-\mathbf{1}}, X]} \arrow[r, "f_*"] & {[S^{a-\mathbf{1}}, Y]}
\end{tikzcd}$$

where the middle vertical arrows are the adjunction natural isomorphisms specified by Proposition 2.4. The bottom arrow is an isomorphism per our assumptions, so the top arrow is likewise an isomorphism, as desired. Hence $-(\Sigma f)_*$ is an isomorphism, so that $\operatorname{im} h_* = \ker(-(\Sigma f)_*) = 0$. Yet we also have $\ker g_* = \operatorname{im} f_* = [S^a, Y]$, so that $\ker h_* = \operatorname{im} g_* = 0$. It is only possible that $\ker h_* = \operatorname{im} h_* = 0$ if $[S^a, C_f] = 0$. Thus, we have shown $\pi_*(C_f) = 0$, and C_f is cellular, so by Lemma 2.10 there is an isomorphism $C_f \cong 0$. Now consider the following diagram:

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "f"] & Y \arrow[r] \arrow[d, equal] & C_f \arrow[r] \arrow[d, "\cong"] & \Sigma X \arrow[d, "\Sigma f"] \\
Y \arrow[r, equal] & Y \arrow[r] & 0 \arrow[r] & \Sigma Y
\end{tikzcd}$$

The middle square commutes since 0 is terminal, while the right square commutes since $C_f \cong 0$ is initial. The top row is distinguished by assumption. The bottom row is distinguished by axiom TR2. Then since the middle two vertical arrows are isomorphisms, by Lemma A.4, f is an isomorphism as well, as desired. $\square$

Lemma 2.12. *Let $e : X \to X$ be an idempotent morphism in $\mathcal{SH}$, so $e \circ e = e$. Then since $\mathcal{SH}$ is a triangulated category with arbitrary coproducts, this idempotent splits (Proposition A.8), meaning e factors as*

$$X \xrightarrow{r} Y \xrightarrow{\iota} X$$

for some object Y and morphisms r and ι with $r \circ \iota = \mathrm{id}_Y$. Then Y is cellular if X is.

Proof. It is a general categorical fact that the splitting of an idempotent, if it exists, is unique up to unique isomorphism,[2] so by Lemma 2.8, it suffices to show that e has some cellular splitting. In Proposition A.8, it is shown that we may take Y to be the homotopy colimit (Definition A.7) of the sequence

$$X \xrightarrow{e} X \xrightarrow{e} X \xrightarrow{e} X \xrightarrow{e} \cdots,$$

[2]In particular, given an idempotent $e : X \to X$ which splits as $X \xrightarrow{r} Y \xrightarrow{\iota} X$, r and ι are the coequalizer and equalizer, respectively, of e and id_X.

so there is a distinguished triangle

$$\bigoplus_{i=0}^{\infty} X \to \bigoplus_{i=0}^{\infty} X \to Y \to \Sigma\left(\bigoplus_{i=0}^{\infty} X\right).$$

Since X is cellular, by definition $\bigoplus_{i=0}^{\infty} X$ is as well. Thus by 2-of-3 for distinguished triangles for cellular objects, Y is cellular as desired. □

2.4. **Monoid objects in $\mathcal{SH}$.** Many of the proofs in this section are quite technical and not very euclidiating, so we direct the reader to the appendix for most proofs. Monoid objects in $\mathcal{SH}$ will be the focus of the rest of this paper. For a review of monoid objects in a symmetric monoidal category, see Appendix D.1. The most important example of a monoid object in $\mathcal{SH}$ is the unit S, which has multiplication map $\phi_{0,0}^{-1} = \lambda_S = \rho_S : S \otimes S \to S$ and unit map $\mathrm{id}_S : S \to S$.

Proposition 2.13 (Proposition D.17). *The assignment $(E, \mu, e) \mapsto \pi_*(E)$ is a functor π_* from the category $\mathbf{Mon}_{\mathcal{SH}}$ of monoid objects in $\mathcal{SH}$ (Definition D.2) to the category of A-graded rings. In particular, given a monoid object (E, μ, e) in $\mathcal{SH}$, $\pi_*(E)$ is canonically a ring with product $\pi_*(E) \times \pi_*(E) \to \pi_*(E)$ which sends classes $x : S^a \to E$ and $y : S^b \to E$ to the composition*

$$xy : S^{a+b} \xrightarrow{\phi_{a,b}} S^a \otimes S^b \xrightarrow{x \otimes y} E \otimes E \xrightarrow{\mu} E,$$

and the unit of this ring is $e \in \pi_0(E) = [S, E]$.

We call the ring $\pi_*(S)$ the *stable homotopy ring.* We have shown that π_* takes monoids to rings. Given a monoid object (E, μ, e) in $\mathcal{SH}$, E_* sends objects to $\pi_*(E)$-modules:

Proposition 2.14 (Proposition D.25). *Let (E, μ, e) be a monoid object in $\mathcal{SH}$. Then $E_*(-)$ is a functor from $\mathcal{SH}$ to left A-graded modules over the ring $\pi_*(E)$ (Proposition 2.13), where given some X in $\mathcal{SH}$, $E_*(X)$ may be endowed with the structure of a left A-graded $\pi_*(E)$-module via the map*

$$\pi_*(E) \times E_*(X) \to E_*(X)$$

which given $a, b \in A$, sends $x : S^a \to E$ and $y : S^b \to E \otimes X$ to the composition

$$x \cdot y : S^{a+b} \cong S^a \otimes S^b \xrightarrow{x \otimes y} E \otimes (E \otimes X) \cong (E \otimes E) \otimes X \xrightarrow{\mu \otimes X} E \otimes X.$$

Similarly, the assignment $X \mapsto X_*(E)$ is a functor from $\mathcal{SH}$ to right A-graded $\pi_*(E)$-modules, where the structure map

$$X_*(E) \times \pi_*(E) \to X_*(E)$$

sends $x : S^a \to X \otimes E$ and $y : S^b \to E$ to the composition

$$x \cdot y : S^{a+b} \cong S^a \otimes S^b \xrightarrow{x \otimes y} (X \otimes E) \otimes E \cong X \otimes (E \otimes E) \xrightarrow{X \otimes \mu} X \otimes E.$$

Finally, $E_*(E)$ is a $\pi_*(E)$-bimodule, in the sense that the left and right actions of $\pi_*(E)$ are compatible, so that given $y, z \in \pi_*(E)$ and $x \in E_*(E)$, $y \cdot (x \cdot z) = (y \cdot x) \cdot z$.

A natural question that arises is: In what sense is $\pi_*(E)$ a "graded commutative ring" if (E, μ, e) is a commutative monoid object? It turns out that $\pi_*(E)$ does satisfy a sort of graded commutativity condition, as follows:

Proposition 2.15 (Proposition D.18). *For all $a, b \in A$ there exists an element $\theta_{a,b} \in \pi_0(S) = [S, S]$ such that given any commutative monoid object (E, μ, e) in $\mathcal{SH}$, the A-graded ring structure on $\pi_*(E)$ (Proposition 2.13) has a commutativity formula given by*

$$x \cdot y = y \cdot x \cdot (e \circ \theta_{a,b})$$

for all $x \in \pi_a(E)$ and $y \in \pi_b(E)$. In particular, $\theta_{a,b} \in \mathrm{Aut}(S)$ is the composition

$$S \xrightarrow{\cong} S^{-a-b} \otimes S^a \otimes S^b \xrightarrow{S^{-a-b}\otimes\tau} S^{-a-b} \otimes S^b \otimes S^a \xrightarrow{\cong} S,$$

where the outermost maps are the unique maps specified by Remark 2.3.

Proposition 2.16. *The $\theta_{a,b}$'s described in Proposition 2.15 satisfy the following properties for all $a, b, c, d \in A$:*

(1) $\theta_{a,b} \cdot \theta_{c,d} = \theta_{a,b} \circ \theta_{c,d} = \theta_{c,d} \cdot \theta_{a,b}$ (where $\cdot$ denotes the product in $\pi_*(S)$ given in Proposition 2.13),

(2) $\theta_{a,0} = \theta_{0,a} = \mathrm{id}_S$,

(3) $\theta_{a,b} \cdot \theta_{b,a} = \mathrm{id}_S$,

(4) $\theta_{a,b} \cdot \theta_{a,c} = \theta_{a,b+c}$ and $\theta_{b,a} \cdot \theta_{c,a} = \theta_{b+c,a}$,

Proof. (1) is Lemma D.19, (2) is Lemma D.20, (3) is Lemma D.21, and (4) is Lemma D.22. □

The above proposition motivates the following definition:

Definition 2.17 (Definition E.1)**.** An *A-graded $\pi_*(S)$-commutative ring* is a pair (R, φ), where R is an A-graded ring and $\varphi : \pi_*(S) \to R$ is an A-graded ring homomorphism such that for all homogeneous $x, y \in R$,

$$x \cdot y = y \cdot x \cdot \varphi(\theta_{|x|,|y|}).$$

A homomorphism $(R, \varphi) \to (R', \varphi')$ of A-graded $\pi_*(S)$-commutative rings is an A-graded ring homomorphism $f : R \to R'$ satisfying $f \circ \varphi = \varphi'$. We write $\pi_*(S)$-**GrCAlg** to denote the category of A-graded $\pi_*(S)$-commutative rings and homomorphisms between them.

We should think of objects in $\pi_*(S)$-**GrCAlg** as "A-graded (anti)commutative rings". A useful fact is that this category has binary coproducts and pushouts (Proposition E.4). As one would expect, every commutative monoid object (E, μ, e) in $\mathcal{SH}$ gives rise to an A-graded $\pi_*(S)$-commutative ring $\pi_*(E)$:

Proposition 2.18 (Proposition E.14)**.** *The assignment $(E, \mu, e) \mapsto (\pi_*(E), \pi_*(e))$ yields a functor*

$$\pi_* : \mathbf{CMon}_{\mathcal{SH}} \to \pi_*(S)\text{-}\mathbf{GrCAlg}$$

from the category of commutative monoid objects in $\mathcal{SH}$ (Definition D.2) to the category of A-graded $\pi_*(S)$-commutative rings (Definition E.1).

Corollary 2.19. *Let (E, μ, e) be a commutative monoid object in $\mathcal{SH}$. Then by Proposition 2.18, $\pi_*(E)$ and $E_*(E) = \pi_*(E \otimes E)$ are canonically A-graded $\pi_*(S)$-commutative rings (Definition 2.17), since $E \otimes E$ is a commutative monoid object in $\mathcal{SH}$ by Lemma D.3.*

3. The Adams spectral sequence

In the sections that follow, let (E, μ, e) be a monoid object (Definition D.1) and X and Y be objects in $\mathcal{SH}$. From now on we will freely use the coherence theorem for symmetric monoidal categories without comment, in particular, we will assume unitality and associativity hold up to strict equality.

Definition 3.1. Let $\overline{E}$ be the fiber of the unit map $e : S \to E$ (Proposition A.6). Let $Y_0 := Y$ and $W_0 := E \otimes Y$. Then for $s > 0$, define

$$Y_s := \overline{E}^s \otimes Y, \qquad W_s := E \otimes Y_s = E \otimes \overline{E}^s \otimes Y,$$

where $\overline{E}^s$ denotes the s-fold tensor product $\overline{E} \otimes \cdots \otimes \overline{E}$. Then we get fiber sequences

$$Y_{s+1} \xrightarrow{i_s} Y_s \xrightarrow{j_s} W_s \xrightarrow{k_s} \Sigma Y_{s+1}$$

obtained by applying $- \otimes Y_s$ to the fiber sequence

$$\overline{E} \to S \xrightarrow{e} E \to \Sigma\overline{E}.$$

We can splice these sequences together to get the *(canonical) Adams filtration of* Y:

$$\begin{tikzcd} \cdots \arrow[r] & Y_3 \arrow[r, "i_2"] \arrow[d, "j_3"] & Y_2 \arrow[r, "i_1"] \arrow[d, "j_2"] & Y_1 \arrow[r, "i_0"] \arrow[d, "j_1"] & Y_0 = Y \arrow[d, "j_0"] \\ & W_3 & W_2 \arrow[ul, dashed, "k_2"] & W_1 \arrow[ul, dashed, "k_1"] & W_0 \arrow[ul, dashed, "k_0"] \end{tikzcd}$$

where here each k_s is of degree $-\mathbf{1}$ (in particular, the above diagram does not commute in any sense), and each i_s and j_s have degree 0. We can extend this diagram to the right by setting $Y_s = Y$, $W_s = 0$, and $i_s = \mathrm{id}_Y$ for $s < 0$. Then we may apply the functor $[X, -]_*$, and by Remark 2.6, we obtain the following A-graded unrolled exact couple (Definition B.2):

$$\begin{tikzcd} \cdots \arrow[r] & {[X, Y_{s+2}]_*} \arrow[r, "i_{s+1}"] \arrow[d, "j_{s+2}"] & {[X, Y_{s+1}]_*} \arrow[r, "i_s"] \arrow[d, "j_{s+1}"] & {[X, Y_s]_*} \arrow[r, "i_{s-1}"] \arrow[d, "j_s"] & {[X, Y_{s-1}]_*} \arrow[r] \arrow[d, "j_{s-1}"] & \cdots \\ & {[X, W_{s+2}]_*} & {[X, W_{s+1}]_*} \arrow[ul, "\partial_{s+1}"] & {[X, W_s]_*} \arrow[ul, "\partial_s"] & {[X, W_{s-1}]_*} \arrow[ul, "\partial_{s-1}"] \end{tikzcd}$$

where here we are being abusive and writing $i_s : [X, Y_{s+1}]_* \to [X, Y_s]_*$ and $j_s : [X, Y_s]_* \to [X, W_s]_*$ to denote the pushforward maps induced by $i_s : Y_{s+1} \to Y_s$ and $j_s : Y_s \to W_s$, respectively. Each i_s, j_s, and ∂_s are A-graded homomorphisms of degrees 0, 0, and $-\mathbf{1}$, respectively.

By Proposition B.3, we may associate a $\mathbb{Z} \times A$-graded spectral sequence $r \mapsto (E_r^{*,*}(X, Y), d_r)$ to the above A-graded unrolled exact couple, where d_r has $\mathbb{Z} \times A$-degree $(r, -\mathbf{1})$. We call this spectral sequence the *E-Adams spectral sequence for the computation of* $[X, Y]_*$.

For those who would rather not lose themselves in the appendix, we give a brief unravelling of how Proposition B.3 applies to the present situation. Given some $s \in \mathbb{Z}$ and some $r \geq 1$, we may define the following A-graded subgroups of $[X, W_s]$:

$$Z_r^s := \partial_s^{-1}(\mathrm{im}[i^{(r-1)} : [X, Y_{s+r}]_* \to [X, Y_{s+1}]_*])$$

and

$$B_r^s := j_s(\ker[i^{(r-1)} : [X, Y_s]_* \to [X, Y_{s-r+1}]_*]),$$

where we adopt the convention that $i^{(0)}$ is simply the identity. This yields an infinite sequence of inclusions

$$0 = B_1^s \subseteq B_2^s \subseteq B_3^s \subseteq \cdots \subseteq \mathrm{im}\, j_s = \ker \partial_s \subseteq \cdots \subseteq Z_3^s \subseteq Z_2^s \subseteq Z_1^s = [X, W_s]_*.$$

Then for $r \geq 1$, we define E_r^s to be the A-graded quotient group

$$E_r^s := Z_r^s / B_r^s.$$

Thus taking the direct sum of all the E_r^s's yields the r^{th} page of the spectral sequence

$$E_r := \bigoplus_{s \in \mathbb{Z}} E_r^s,$$

which is a $\mathbb{Z} \times A$-graded abelian group.

The differential $d_r : E_r \to E_r$ is a map of $\mathbb{Z} \times A$-degree $(r, \mathbf{1})$, and is constructed as follows: an element of $E_r^s = Z_r^s / B_r^s$ is a coset represented by some $x \in Z_r^s$, so that $\partial_s(x) = i^{(r-1)}(y)$ for some $y \in [X, Y_{s+r}]_*$. Then we define $d_r([x])$ to be the coset $[j_{s+r}(y)]$ in Z_r^{s+r}/B_r^{s+r}.

In the case $r = 1$, since $B_1^s = 0$ and $Z_1^s = [X, W_s]_*$, we have that $E_1^s = [X, W_s]_*$, and given some $x \in E_1^s = [X, W_s]_*$, the differential d_1 is given by $d_1(x) = j_{s+1}(\partial_s(x))$, so that $d_1 = j \circ \partial$.

In Appendix B.1, it is shown in explicit detail that all of these definitions make sense and are well-defined. In particular, it is shown that the differentials are well-defined A-graded homomorphisms, that $d_r \circ d_r = 0$, and that

$$\ker d_r^s / \operatorname{im} d_r^s = \frac{Z_{r+1}^s / B_r^s}{B_{r+1}^s / B_r^s} \cong Z_{r+1}^s / B_{r+1}^s = E_{r+1}^s.$$

4. The E_1 page

In this section, we aim to provide a nicer characterization of the E_1 page. Here we will often work in a symmetric strict monoidal category by the coherence theorem for symmetric monoidal categories, and we will do so without comment. Recall that by how the Adams spectral sequence for the computation of $[X, Y]_*$ is constructed, that the E_1 page is the $\mathbb{Z} \times A$-graded abelian group given by

$$E_1^{s,a}(X, Y) = [X, W_s]_a = [S^a \otimes X, \overline{E}^s \otimes Y],$$

where $\overline{E}$ is the fiber (Proposition A.6) of the unit map $e : S \to E$. In this section, we will show that under suitable conditions, these groups may alternatively be computed as hom-groups of morphisms of comodules over the dual E-Steenrod algebra (to be defined).

4.1. Flat monoid objects in $\mathcal{SH}$.

Definition 4.1. Call a monoid object (E, μ, e) in $\mathcal{SH}$ (Definition D.1) *flat* if the canonical right $\pi_*(E)$-module structure on $E_*(E)$ (Proposition 2.14) is that of a flat module.

Proposition 4.2 (Proposition D.31)**.** *Let (E, μ, e) be a monoid object (Definition D.1) and Z and W be objects in $\mathcal{SH}$. Then there is a homomorphism of abelian groups*

$$\Phi_{Z,W} : \pi_*(Z \otimes E) \otimes_{\pi_*(E)} \pi_*(E \otimes W) \to \pi_*(Z \otimes E \otimes W)$$

which given homogeneous elements $x : S^a \to Z \otimes E$ in $\pi_*(Z \otimes E)$ and $y : S^b \to E \otimes W$ in $\pi_*(E \otimes W)$, sends the homogeneous pure tensor $x \otimes y$ in $\pi_*(Z \otimes E) \otimes_{\pi_*(E)} \pi_*(E \otimes W)$ to the composition

$$S^{a+b} \xrightarrow{\phi_{a,b}} S^a \otimes S^b \xrightarrow{x \otimes y} Z \otimes E \otimes E \otimes W \xrightarrow{Z \otimes \mu \otimes W} Z \otimes E \otimes W$$

(where here we are considering the canonical A-graded right $\pi_*(E)$-module structure on $\pi_*(Z \otimes E) = Z_*(E)$ and the canonical left A-graded $\pi_*(E)$-module structure on $\pi_*(E \otimes W) = E_*(W)$ given in Proposition 2.14, so that $\pi_*(Z \otimes E) \otimes_{\pi_*(E)} \pi_*(E \otimes W)$ is a well-defined A-graded abelian group by Proposition C.18). Furthermore, this homomorphism is natural in both Z and W.

The key consequence of the assumption that E is flat in the sense of Definition 4.1 is that $\Phi_{E,W}$ is an isomorphism for cellular objects W in $\mathcal{SH}$:

Proposition 4.3 (Proposition D.34)**.** *Let (E, μ, e) be a monoid object in $\mathcal{SH}$. Then if either:*

1. *$\pi_*(Z \otimes E) = Z_*(E)$ is a flat right $\pi_*(E)$-module (via Proposition D.25) and W is cellular (Definition 2.7), or*
2. *$\pi_*(E \otimes W) = E_*(W)$ is a flat left $\pi_*(E)$-module (via Proposition D.25) and Z is cellular (Definition 2.7),*

then the natural homomorphism

$$\Phi_{Z,W} : \pi_*(Z \otimes E) \otimes_{\pi_*(E)} \pi_*(E \otimes W) \to \pi_*(Z \otimes E \otimes W)$$

given in Proposition 4.2 is an isomorphism of abelian groups.

4.2. **The dual E-Steenrod algebra.** In Section 2.4, we showed that given a monoid object (E, μ, e) in $\mathcal{SH}$, that $E_*(E)$ is canonically an A-graded bimodule over the ring $\pi_*(E)$. In this subsection, we will outline some additional structure carried by $E_*(E)$. In particular, we will show that if (E, μ, e) is a flat (Definition 4.1) commutative monoid object, then the pair $(E_*(E), \pi_*(E))$ is canonically an A-graded commutative Hopf algebroid over the stable homotopy ring $\pi_*(S)$ (Definition E.6), called the *dual E-Steenrod algebra*. To start with, we outline some structure maps relating $E_*(E)$ and $\pi_*(E)$.

Proposition 4.4. *Let (E, μ, e) be a commutative monoid object in $\mathcal{SH}$. Then the maps*

(1) $E \xrightarrow{\cong} E \otimes S \xrightarrow{E\otimes e} E \otimes E$,

(2) $E \xrightarrow{\cong} S \otimes E \xrightarrow{e\otimes E} E \otimes E$,

(3) $E \otimes E \xrightarrow{\cong} E \otimes S \otimes E \xrightarrow{E\otimes e\otimes E} E \otimes E \otimes E$,

(4) $E \otimes E \xrightarrow{\mu} E$, *and*

(5) $E \otimes E \xrightarrow{\tau_{E,E}} E \otimes E$

*are homomorphisms of monoid objects in $\mathcal{SH}$ (where here $E \otimes E$ and $E \otimes E \otimes E$ are considered as monoid objects in $\mathcal{SH}$ by Lemma D.3 and Lemma D.4, respectively), so that by Proposition 2.18, under π_* they induce morphisms in $\pi_*(S)$-**GrCAlg**:*

(1) $\eta_L : \pi_*(E) \to E_*(E)$,

(2) $\eta_R : \pi_*(E) \to E_*(E)$,

(3) $h : E_*(E) \to E_*(E \otimes E)$,

(4) $\varepsilon : E_*(E) \to \pi_*(E)$, *and*

(5) $c : E_*(E) \to E_*(E)$.

Lemma 4.5 (Lemma E.16)**.** *Let (E, μ, e) be a commutative monoid object in $\mathcal{SH}$. Then the left (resp. right) $\pi_*(E)$-module structure induced on $E_*(E)$ by the ring homomorphism η_L (resp. η_R)*[3] *coincides with the canonical left (resp. right) $\pi_*(E)$-module structure on $E_*(E)$ given in Proposition 2.14.*

Corollary 4.6 (Corollary E.17)**.** *Given a commutative monoid object (E, μ, e) in $\mathcal{SH}$, the domain of the homomorphism*

$$\Phi_{E,E} : E_*(E) \otimes_{\pi_*(E)} E_*(E) \to E_*(E \otimes E)$$

*constructed in Proposition 4.3 is canonically an A-graded $\pi_*(S)$-ring, and sits in the following pushout diagram in $\pi_*(S)$-**GrCAlg**:*

$$\begin{tikzcd}
\pi_*(E) \arrow[r, "\eta_L"] \arrow[d, "\eta_R"'] & E_*(E) \arrow[d, "x\mapsto 1\otimes x"] \\
E_*(E) \arrow[r, "x\mapsto x\otimes 1"'] & E_*(E) \otimes_{\pi_*(E)} E_*(E)
\end{tikzcd}$$

[3] Recall that given a homomorphism of rings $\varphi : R \to S$, that S canonically inherits the structure of a left (resp. right) R-module by defining $r \cdot s := \varphi(r)s$ (resp. $s \cdot r := s\varphi(r)$).

Lemma 4.7 (Lemma E.18)**.** *Let (E, μ, e) be a commutative monoid object in $\mathcal{SH}$. Then the homomorphism*

$$\Phi_{E,E} : E_*(E) \otimes_{\pi_*(E)} E_*(E) \to E_*(E \otimes E)$$

*constructed in Proposition 4.2 is a homomorphism of A-graded $\pi_*(S)$-commutative rings, i.e. a morphism in $\pi_*(S)$-**GrCAlg**, where here $E_*(E) \otimes_{\pi_*(E)} E_*(E)$ is considered as an object in $\pi_*(S)$-**GrCAlg** by Corollary 4.6, and $E_*(E \otimes E) = \pi_*(E \otimes (E \otimes E))$ is considered as an object in $\pi_*(S)$-**GrCAlg** by Proposition 2.18, since $E \otimes (E \otimes E)$ is a monoid object in $\mathcal{SH}$ by Lemma D.3.*

Definition 4.8. Let (E, μ, e) be a *commutative* monoid object (Definition D.1) which is flat (Definition 4.1) and cellular (Definition 2.7). Then the *dual E-Steenrod algebra* is the pair of A-graded abelian groups $(E_*(E), \pi_*(E))$ equipped with the following structure:

1. The A-graded $\pi_*(S)$-commutative ring structure on $\pi_*(E)$ induced from E being a commutative monoid object in $\mathcal{SH}$ (Proposition 2.18).
2. The A-graded $\pi_*(S)$-commutative ring structure on $E_*(E)$ induced from the fact that $E \otimes E$ is canonically a commutative monoid object in $\mathcal{SH}$ (Lemma D.3), so that also $E_*(E) = \pi_*(E \otimes E)$ is an A-graded $\pi_*(S)$-commutative ring (Proposition 2.18).
3. The homomorphisms of A-graded $\pi_*(S)$-commutative rings
$$\eta_L : \pi_*(E) \to E_*(E)$$
and
$$\eta_R : \pi_*(E) \to E_*(E)$$
induced under π_* by the monoid object homomorphisms
$$E \xrightarrow{\cong} E \otimes S \xrightarrow{E \otimes e} E \otimes E$$
and
$$E \xrightarrow{\cong} S \otimes E \xrightarrow{e \otimes E} E \otimes E.$$
4. The homomorphism of A-graded $\pi_*(S)$-commutative rings
$$\Psi : E_*(E) \to E_*(E) \otimes_{\pi_*(E)} E_*(E)$$
given by the composition
$$E_*(E) \xrightarrow{h} E_*(E \otimes E) \xrightarrow{\Phi_{E,E}^{-1}} E_*(E) \otimes_{\pi_*(E)} E_*(E),$$
where h is a homomorphism of A-graded $\pi_*(S)$-commutative rings induced under π_* by the monoid object homomorphism
$$E \otimes E \xrightarrow{\cong} E \otimes S \otimes E \xrightarrow{E \otimes e \otimes E} E \otimes E \otimes E,$$
and $\Phi_{E,E}$ is morphism constructed in Proposition 4.2, which is proven to be an isomorphism in Proposition 4.3 and a morphism in $\pi_*(S)$-**GrCAlg** in Lemma 4.7.
5. The homomorphism of A-graded $\pi_*(S)$-commutative rings
$$\varepsilon : E_*(E) \to \pi_*(E)$$
induced under π_* by the monoid object homomorphism
$$E \otimes E \xrightarrow{\mu} E.$$

6. The homomorphism of A-graded $\pi_*(S)$-commutative rings
$$c : E_*(E) \to E_*(E)$$
induced under π_* from the monoid object homomorphism
$$E \otimes E \xrightarrow{\tau} E \otimes E.$$

Proposition 4.9 (Proposition E.22)**.** *Let (E, μ, e) be a commutative monoid object in $\mathcal{SH}$ which is flat (Definition 4.1) and cellular (Definition 2.7). Then the dual E-Steenrod algebra $(E_*(E), \pi_*(E))$ with the structure maps $(\eta_L, \eta_R, \Psi, \varepsilon, c)$ from Definition 4.8 is an A-graded commutative Hopf algebroid over $\pi_*(S)$ (Definition E.6), i.e., a co-groupoid object in the category $\pi_*(S)$-**GrCAlg**.*

4.3. **Comodules over the dual E-Steenrod algebra.**

Lemma 4.10. *Let (E, μ, e) be a monoid object in $\mathcal{SH}$. Then for all objects X in $\mathcal{SH}$, the A-graded homomorphism*
$$E_*(E) \otimes_{\pi_*(E)} E_*(X) \xrightarrow{\Phi_{E,X}} E_*(E \otimes X)$$
is a homomorphism of left A-graded $\pi_*(E)$-module objects, where here we are considering the left E-module structure on $E_*(E) \otimes_{\pi_*(E)} E_*(X)$ induced by the canonical $\pi_*(E)$-bimodule structure on $E_*(E)$ (Proposition 2.14).

Proposition 4.11. *Let (E, μ, e) be a flat (Definition 4.1) and cellular (Definition 2.7) commutative monoid object in $\mathcal{SH}$. Then $E_*(-)$ is a functor from $\mathcal{SH}$ to the category $E_*(E)$-**CoMod** of left A-graded comodules (Definition E.11) over the dual E-Steenrod algebra, which is an A-graded commutative Hopf algebroid over $\pi_*(S)$, by Proposition 4.9.*

In particular, given an object X in $\mathcal{SH}$, we are viewing $E_*(X)$ with its canonical left $\pi_*(E)$-module structure (Proposition 2.14), and the action map
$$\Psi_X : E_*(X) \to E_*(E) \otimes_{\pi_*(E)} E_*(X)$$
is given by the composition
$$\Psi_X : E_*(X) \xrightarrow{E_*(e \otimes X)} E_*(E \otimes X) \xrightarrow{\Phi_{E,X}^{-1}} E_*(E) \otimes_{\pi_*(E)} E_*(X).$$

Proof. It is straightforward to check that the action map is a homomorphism of left $\pi_*(E)$-modules. Now, we need to show that it makes the two diagrams in Definition E.11 commute. □

5. The E_2 page

In this section, we aim to provide a nicer characterization of the E_2 page.

6. Convergence of the Adams spectral sequence

In this section, we aim to prove convergence of the spectral sequence.

7. The classical Adams spectral sequence

8. The motivic Adams spectral sequence

Completely rewrite this entire section its so bad

One of the key ideas in classical topology is that in order to study "nice spaces" like CW complexes or manifolds, we should work with a larger category $\mathcal{S}$ which has better formal properties, but not as nice of spaces. In topology, there are multiple candidates for this category, such as the category of simplicial sets or the category of compactly generated weak Hausdorff spaces. For our purposes, we will take $\mathcal{S} = \mathbf{Set}_{\boldsymbol{\Delta}}$ to be the category of simplicial sets. In this larger category, we can do homotopy theory. $\mathbb{A}^1$-homotopy theory, also called motivic homotopy theory, is motivated by applying this philosophy to the field of algebraic geometry.

8.1. **Motivic spaces.** In algebraic geometry, the key objects of study are *varieties*, i.e., smooth finite type schemes over $\operatorname{Spec} k$ for some field k. More generally, instead of considering schemes over a field k, we can consider smooth finite type schemes over some *base scheme* $\mathscr{S}$, where a "base scheme" is a Noetherian separated scheme of finite Krull dimension. We write $\mathbf{Sm}/\mathscr{S}$ to denote the category of smooth finite type schemes over $\mathscr{S}$. Often times we will write "smooth scheme over $\mathscr{S}$" or just "smooth scheme" to denote an object of $\mathbf{Sm}/\mathscr{S}$. Sadly, like the category of smooth manifolds, the category $\mathbf{Sm}/\mathscr{S}$ does not satisfy many nice formal properties, in particular, it does not have colimits, as there is no way to "glue" smooth schemes together. Taking our queue from topology, we should therefore expand the category $\mathbf{Spc}(\mathscr{S})$ to some larger category of "motivic spaces" with nice formal properties. This construction is the motivating idea behind $\mathbb{A}^1$-homotopy theory.

As it turns out, there are lots of ways to define the category of motivic spaces. We will follow the approach outlined in Section 2 of [17]. We omit many technical details, and emphasize only what we need.

Definition 8.1. A *(motivic) space* over $\mathscr{S}$ is a simplicial presheaf on $\mathbf{Sm}/\mathscr{S}$. The collection of spaces over $\mathscr{S}$ forms the category

$$\mathbf{Spc}(\mathscr{S}) := [(\mathbf{Sm}/\mathscr{S})^{\mathrm{op}}, \mathcal{S}].$$

There is already a lot we can do with this definition. Since $\mathcal{S}$ is complete and cocomplete, it follows purely formally that the category of motivic spaces is as well, so we have achieved our goal of being able to take (co)limits, which may be computed pointwise in $\mathcal{S}$. Furthermore, the requirement that objects in $\mathbf{Sm}/\mathscr{S}$ be finite type schemes over $\mathscr{S}$ ensures that $\mathbf{Sm}/\mathscr{S}$ is an essentially small category ([1]), so that $\mathbf{Spc}(\mathscr{S})$ is cartesian closed[4], and we do not have to worry about size issues (the collection of objects in $\mathbf{Spc}(\mathscr{S})$ forms a proper class).

Since $\mathcal{S} := [\mathbf{\Delta}^{\mathrm{op}}, \mathbf{Set}]$, we have an identification[5]

$$\mathbf{Spc}(\mathscr{S}) = [(\mathbf{Sm}/\mathscr{S})^{\mathrm{op}}, \mathcal{S}] = [(\mathbf{Sm}/\mathscr{S})^{\mathrm{op}}, [\mathbf{\Delta}^{\mathrm{op}}, \mathbf{Set}]] \cong [\mathbf{\Delta}^{\mathrm{op}}, [(\mathbf{Sm}/\mathscr{S})^{\mathrm{op}}, \mathbf{Set}]].$$

Hence, we may also think of motivic spaces as simplicial objects in the category of presheaves on $\mathbf{Sm}/\mathscr{S}$. In this way, by composing the Yoneda embedding with the diagonal functor, we have an embedding

$$h_{(-)} : \mathbf{Sm}/\mathscr{S} \xrightarrow{\mathcal{Y}} [(\mathbf{Sm}/\mathscr{S})^{\mathrm{op}}, \mathbf{Set}] \xrightarrow{\Delta} [\mathbf{\Delta}^{\mathrm{op}}, [(\mathbf{Sm}/\mathscr{S})^{\mathrm{op}}, \mathbf{Set}]] \cong \mathbf{Spc}(\mathscr{S})$$

taking a smooth scheme $\mathscr{X}$ to the simplicial presheaf $h_{\mathscr{X}}$ it represents. It is not hard to verify that this functor is fully faithful, since the Yoneda embedding is. Often we will not distinguish between a smooth scheme $\mathscr{X}$ and its image under this functor. We may also define based spaces:

Definition 8.2. A *based* (motivic) space is an object in the under category

$$\mathbf{Spc}_*(\mathscr{S}) := \mathbf{Spc}(S)^{\mathscr{S}/},$$

i.e., a based space is a motivic space X along with a morphism (natural transformation) $\mathscr{S} \to X$.

[4]This follows from the general categorical result that given a small category $\mathcal{C}$ and a cartesian closed complete category $\mathcal{D}$, the functor category $[\mathcal{C}, \mathcal{D}]$ is itself cartesian closed.

[5]This follows from the more general fact that given three categories $\mathcal{C}$, $\mathcal{D}$, and $\mathcal{E}$, there is a canonical isomorphism $[\mathcal{C}, [\mathcal{D}, \mathcal{E}]] \cong [\mathcal{D}, [\mathcal{C}, \mathcal{E}]]$.

This definition is motivated by the observation that $\mathscr{S}$ is the terminal motivic space. Indeed, note that by definition $\mathscr{S}$ is the terminal object in $\mathbf{Sm}/\mathscr{S}$, so that given any other smooth scheme $\mathscr{U}$ over $\mathscr{S}$, there is a unique morphism $\mathscr{U} \to \mathscr{S}$, so that $h_{\mathscr{S}}(\mathscr{U}) \cong \Delta^0$. It follows purely formally that the forgetful functor $\mathbf{Spc}_*(\mathscr{S}) \to \mathbf{Spc}(S)$ has a left adjoint $(-)_+ : \mathbf{Spc}(\mathscr{S}) \to \mathbf{Spc}_*(\mathscr{S})$ taking a motivic space X to the disjoint union $X \amalg \mathscr{S}$ obtained by freely adjoining a basepoint.

We point out a couple examples of motivic spaces which will be important. In what follows, all products are taken in the category of schemes, not in $\mathbf{Sm}/\mathscr{S}$, so that in particular given any object $\mathscr{X}$ in $\mathbf{Sm}/\mathscr{S}$, there are canonical isomorphisms $\mathscr{X} \cong \mathscr{X} \times \operatorname{Spec}\mathbb{Z}$, as $\operatorname{Spec}\mathbb{Z}$ is the terminal scheme. Let $\mathbb{A}^1$ and $\mathbb{G}_m$ denote the smooth schemes $\mathscr{S} \times \operatorname{Spec}\mathbb{Z}[x]$ and $\mathscr{S} \times \operatorname{Spec}\mathbb{Z}[x, x^{-1}]$, respectively. We may consider $\mathbb{A}^1$ as canonically based via the composition

$$\mathscr{S} \cong \mathscr{S} \times \operatorname{Spec}\mathbb{Z} \to \mathscr{S} \times \operatorname{Spec}\mathbb{Z}[x] = \mathbb{A}^1,$$

where the arrow is given by $\mathscr{S} \times f$, where $f : \operatorname{Spec}\mathbb{Z} \to \operatorname{Spec}\mathbb{Z}[x]$ corresponds to the ring morphism $\mathbb{Z}[x] \to \mathbb{Z}$ sending $x \mapsto 1$. Similarly, we may view $\mathbb{G}_m$ as canonically based via the map

$$\mathscr{S} \cong \mathscr{S} \times \operatorname{Spec}\mathbb{Z} \to \mathscr{S} \times \operatorname{Spec}\mathbb{Z}[x, x^{-1}] = \mathbb{G}_m,$$

where the arrow is similarly induced by the unique ring morphism $\mathbb{Z}[x, x^{-1}] \to \mathbb{Z}$ sending $x \mapsto 1$.

Note that in the case $\mathscr{S} = \operatorname{Spec} k$ for some field k, we have identifications $\mathbb{A}^1 \cong \mathbb{A}^1_k = \operatorname{Spec} k[x]$ and $\mathbb{G}_m \cong \operatorname{Spec} k[x, x^{-1}] \cong \mathbb{A}^1_k \setminus \{0\}$, which justifies our notation. As we will see, $\mathbb{A}^1$ will play a role similar to the interval in the homotopy theory of motivic spaces. Thought of as a motivic space, we call $\mathbb{G}_m$ the "Tate circle". It turns out that as a motivic space, $\mathbb{G}_m$ has many of the same properties that the topological circle S^1 has in the category of topological spaces. To see this, consider the case $\mathscr{S} = \operatorname{Spec}\mathbb{C}$. We have a "realization functor" $\psi : \mathbf{Sm}/\mathbb{C} \to \mathbf{Top}$ taking a scheme $\mathscr{X}$ to its set of $\mathbb{C}$-points with the analytic topology. Under this functor, $\mathbb{A}^1$ and $\mathbb{G}_m$ are taken to the spaces $\mathbb{C}$ and $\mathbb{C} \setminus \{0\}$, respectively. Note that $\mathbb{C} \setminus \{0\}$ is homotopy equivalent to the circle, which already provides one justification for thinking of $\mathbb{G}_m$ as a circle.

We can extend the realization functor $\psi : \mathbf{Sm}/\mathbb{C} \to \mathbf{Top}$ to a functor defined on all of $\mathbf{Spc}(\mathbb{C})$:

Definition 8.3. Define the *Betti realization functor* $\psi : \mathbf{Spc}(\mathbb{C}) \to \mathbf{Top}$ to be the left Kan extension of the realization functor $\psi : \mathbf{Sm}/\mathbb{C} \to \mathbf{Top}$ along the simplicial Yoneda embedding $h_{(-)} : \mathbf{Sm}/\mathbb{C} \to \mathbf{Spc}(\mathbb{C})$:

$$\begin{tikzcd} \mathbf{Sm}/\mathbb{C} \arrow[rr, "\psi"] \arrow[dr, "h_{(-)}"'] & & \mathbf{Top} \\ & \mathbf{Spc}(\mathbb{C}) \arrow[ur, dashed, "\psi"'] & \end{tikzcd}$$

Given a space X, we often denote $\psi(X)$ by $X(\mathbb{C})$.

Since $\mathbf{Sm}/\mathbb{C}$ is (essentially) small and $\mathbf{Top}$ is (small) cocomplete, it follows that this left Kan extension does in fact exist, and we may compute it via colimits ([15, Theorem 6.2.1]).

Recall that $\mathbb{S}_*$ is a symmetric monoidal category under the smash product $- \wedge -$. The unit for this monoidal structure is given by $S^0 := \Delta^0 \amalg \Delta^0$. This induces a symmetric monoidal structure on the category $\mathbf{Spc}_*(\mathscr{S})$ of based spaces over $\mathscr{S}$:

Proposition 8.4. *Given two based motivic spaces X and Y over $\mathscr{S}$, define their* smash product *$X \wedge Y$ to be the simplicial presheaf defined by*

$$(X \wedge Y)(\mathscr{U}) := X(\mathscr{U}) \wedge Y(\mathscr{U}).$$

This smash product endows $\mathbf{Spc}_*(\mathscr{S})$ *with the structure of a symmetric monoidal category, where the unit object is given by* $S^{0,0} := \mathscr{S}_+ = \mathscr{S} \amalg \mathscr{S} \cong \Delta^0 \amalg \Delta^0$.

We have shown that any smooth scheme can be viewed as a motivic space, but it also turns out that any simplicial set A can viewed as a motivic space by considering the constant functor $cA : (\mathbf{Sm}/\mathscr{S})^{\mathrm{op}} \to \mathcal{S}$ on A. As we did with objects of $\mathbf{Sm}/\mathscr{S}$, we will usually simply write A to denote the corresponding motivic space cA. Per our earlier reasoning, $\mathscr{S}$ and Δ^0 are isomorphic as motivic spaces. This observation also yields a functor $\mathcal{S}_* \to \mathbf{Spc}_*(\mathscr{S})$ taking a based simplicial set $\Delta^0 \to A$ to the based motivic space $\mathscr{S} \cong c\Delta^0 \to cA$. It follows that this functor is strongly monoidal, i.e., it preserves the monoidal unit, and given any two based simplicial sets A and B we have $cA \wedge cB \cong c(A \wedge B)$ (in fact, here this isomorhpism is an equality). Furthermore, it is relatively straightforward to check that the functor $\mathbf{Spc}_*(\mathscr{S}) \to \mathcal{S}_*$ given by evaluation at $\mathscr{S}$ is a right adjoint to c.

Interestingly, this functor yields another circle in the category of pointed spaces. We can define the *simplicial circle* to be the constant simplicial presheaf S^1 pointed at its 0-simplex. As it turns out, the simplicial circle S^1 really is an entirely distinct space from the Tate circle $\mathbb{G}_m$. So which is "the" circle? As it turns out, the approach taken in motivic homotopy theory is to view them as each equally valid, but different notions, and in fact, we obtain a bigraded family of motivic spheres $S^{p,q}$ in $\mathbf{Spc}_*(\mathscr{S})$ for $p \geq q \geq 0$ by defining

$$S^{p,q} := (S^1)^{p-q} \wedge \mathbb{G}_m^q,$$

so that $S^{1,0} \cong S^1$, $S^{1,1} \cong \mathbb{G}_m$, and $S^{0,0} \cong \mathscr{S}_+ \cong S^0$ (recall $\mathscr{S}_+ = h_{\mathscr{S}} \amalg h_{\mathscr{S}}$ is the monoidal unit in $\mathbf{Spc}_*(\mathscr{S})$). The reason for this odd grading convention has to do with the theory of *motives*, which we will not explore here.

8.2. **The unstable motivic homotopy category.** So far, we have constructed motivic spaces, and given some examples of how to work with them. Yet, we still have yet to talk about how we can "do homotopy theory" in this world. To start with, we will define the *motivic model structure* on $\mathbf{Spc}_*(\mathscr{S})$. We will define this in stages, by first defining the *projective model structure* on $\mathbf{Spc}(\mathscr{S})$ and then localizing.

Proposition 8.5. *There exists a cellular, proper, simplicial monoidal model structure on* $\mathbf{Sm}/S$ *called the* projective model structure *in which*

(1) The (global) weak equivalences are those maps $f : X \to Y$ *for which* $f(\mathscr{U}) : X(\mathscr{U}) \to Y(\mathscr{U})$ *is a weak equivalence of simplicial sets for all* $\mathscr{U}$ *in* $\mathbf{Sm}/\mathscr{S}$*,*

(2) The projective fibrations are those maps $f : X \to Y$ *for which* $f(\mathscr{U}) : X(\mathscr{U}) \to Y(\mathscr{U})$ *is a Kan fibration for all* $\mathscr{U}$ *in* $\mathbf{Sm}/\mathscr{S}$.

(3) The projective cofibrations are those maps in $\mathbf{Spc}(\mathscr{S})$ *which satisfy the left lifting property against the trivial projective fibrations.*

Of course, this also endows $\mathbf{Spc}_*(\mathscr{S})$ with a model structure, which we also call the projective model structure. There exists a Grothendieck topology on $\mathbf{Spc}_*(\mathscr{S})$ called the *Nisnevich topology.*

Definition 8.6. Given a pointed space X in $\mathbf{Spc}_*(\mathscr{S})$ and some $n \geq 0$, the n^{th} simplicial homotopy sheaf $\pi_n(X)$ of X is the Nisnevich sheaffification of the presheaf $\mathscr{U} \mapsto \pi_n(X(\mathscr{U}))$. Write W_{Nis} for the class of maps $f : X \to Y$ in $\mathbf{Spc}_*(\mathscr{S})$ for which $f_* : \pi_n(X) \to \pi_n(Y)$ is an isomorphism of Nisnevich sheaves for all $n \geq 0$.

Definition 8.7. Let $W_{\mathbb{A}^1} \subseteq \mathrm{Mor}(\mathbf{Spc}_*(\mathscr{S}))$ be the class of maps $\pi_{\mathscr{X}} : (\mathscr{X} \times \mathbb{A}^1)_+ \to \mathscr{X}_+$ for $\mathscr{X}$ in $\mathbf{Sm}/\mathscr{S}$. The *motivic model structure* on $\mathbf{Spc}_*(\mathscr{S})$ is the left Bousfield localization of the projective model structure with respect to $W_{\mathrm{Nis}} \cup W_{\mathbb{A}^1}$. This model structure is closed symmetric monoidal, pointed simplicial, left proper, and cellular. From now on, we always write $\mathbf{Spc}_*(\mathscr{S})$ to mean the model category of pointed spaces equipped with the motivic model structure. The homotopy category of $\mathbf{Spc}_*(\mathscr{S})$ is the pointed motivic homotopy category $\mathbf{H}_*(\mathscr{S})$.

8.3. **The stable motivic homotopy category.** The canonical ring morphism $\mathbb{Z}[x] \hookrightarrow \mathbb{Z}[x, x^{-1}]$ induces a map $\mathbb{G}_m \to \mathbb{A}^1$. Let T be a cofibrant replacement of the quotient simplicial sheaf $\mathbb{A}^1/\mathbb{G}_m$ in the stable model structure on $\mathbf{Spc}_*(\mathscr{S})$. We call T the *Tate sphere*. A useful fact is that the Tate sphere is equivalent to $S^1 \wedge \mathbb{G}_m$ in the motivic model structure on $\mathbf{Spc}_*(\mathscr{S})$ ([10, Lemma 3.2.15]).

It turns out that the functor $T \wedge -$ on $\mathbf{Spc}_*(\mathscr{S})$ is a left Quillen functor, and we may invert it to create the category $\mathbf{Spt}_T(\mathscr{S})$ of T-spectra. Explicitly:

Definition 8.8. A T-spectrum X is a sequence of spaces $\{X_n\}_{n\geq 0}$ in $\mathbf{Spc}_*(\mathscr{S})$ equipped with structure maps $\sigma_n : T \wedge X_n \to X_{n+1}$. A map of T-spectra $f : X \to Y$ is a collection of maps $f_n : X_n \to Y_n$ which are compatible with the structure maps in the obvious sense. We write $\mathbf{Spt}_T(\mathscr{S})$ to denote the category of T-spectra and maps between them.

Definition 8.9. Given a based space X in $\mathbf{Spc}_*(\mathscr{S})$, we can form its *suspension spectrum* $\Sigma^\infty X$ whose n^{th} term is $X \wedge T^n$ and the structure morphisms are the canonical isomorphisms. This yields a functor $\Sigma^\infty : \mathbf{Spc}_*(\mathscr{S}) \to \mathbf{Spt}_T(\mathscr{S})$, and by composing with $(-)_+ : \mathbf{Spc}(\mathscr{S}) \to \mathbf{Spc}_*(\mathscr{S})$, we get a functor $\Sigma^\infty(-)_+ : \mathbf{Spc}(\mathscr{S}) \to \mathbf{Spt}_T(\mathscr{S})$.

Now, we would like to define the stable model structure on the category of T-specra. As we did with motivic spaces, we first start with a different model structure than the one we want, and then localize to obtain the stable model structure.

Proposition 8.10. *There exists a model structure on* $\mathbf{Spt}_T(\mathscr{S})$ *called the* level model structure *in which a map* $f : X \to Y$ *is a weak equivalence (resp. a fibration) if every map* $f_n : X_n \to Y_n$ *is a weak equivalence (resp. a fibration) in the motivic model structure on* $\mathbf{Spc}_*(\mathscr{S})$*. The cofibrations are determined as those with the left lifting property against the trivial level fibrations.*

Definition 8.11. Let X be a T-spectrum. For integers p and q, the $(p, q)^{\text{th}}$ stable homotopy sheaf of X, written as $\pi_{p,q}(X)$, is the Nisnevich sheafification of the presheaf

$$\mathscr{U} \mapsto \operatorname*{colim}_n \mathbf{H}_*(\mathscr{U})(S^{p+2n,q+n}, X_n|_{\mathscr{U}})$$

(note the terms in this colimit may only be defined for large enough n). A map $f : X \to Y$ is a *stable weak equivalence* if for all integers p and q the induced maps $f_* : \pi_{p,q}(X) \to \pi_{p,q}(Y)$ are isomorphisms.

Definition 8.12. The stable model structure on $\mathbf{Spt}_T(\mathscr{S})$ is the model category where the weak equivalences are the stable weak equivalences and the cofibrations are the cofibrations in the level model structure. The fibrations are those maps with the right lifting property with respect to trivial cofibrations. We write $\mathbf{SH}_{\mathscr{S}}$ for the homotopy category of $\mathbf{Spt}_T(\mathscr{S})$ with the stable model structure.

As in the case of classical spectra, we run into the unfortunate fact that the smash product does not induce a symmetric monoidal structure on on $\mathbf{Spt}_T(\mathscr{S})$. One remedy is to use the category $\mathbf{Spt}_T^\Sigma(S)$ of symmetric T-spectra. The construction of this category is given by Hovey in [6] and Jardine in [7], and it turns out that the smash product can be used to give $\mathbf{Spt}_T^\Sigma(S)$

the structure of a symmetric monoidal category, in fact, a stable symmetric monoidal model category. It is proven in [6] that there is a zig-zag of Quillen equivalences from $\mathbf{Spt}_T^{\Sigma}(S)$ to $\mathbf{Spt}_T(\mathscr{S})$, hence $\mathbf{SH}_{\mathscr{S}}$ is equivalent to the homotopy category of $\mathbf{Spt}_T^{\Sigma}(S)$ as well. In particular, the category $\mathbf{SH}_{\mathscr{S}}$ is a tensor triangulated category where the shift functor $\Sigma := \Sigma^\infty S^{1,0} \wedge -$ is given by smashing with the suspension spectrum of $S^{1,0} = S^1 \wedge \mathscr{S}_+ \cong S^1$. The monoidal product $- \wedge -$ is induced by the smash product[6], and the monoidal unit is given by the *sphere spectrum* $S := \Sigma^\infty \mathscr{S}_+ \cong \Sigma^\infty S^0$. See the appendix for a review of (tensor) triangulated categories.

Recall earlier we defined the functor $\Sigma^\infty : \mathbf{Spc}_*(\mathscr{S}) \to \mathbf{Spt}_T(\mathscr{S})$ taking a based space to its suspension spectrum. From now on, we will instead write Σ^∞ to refer to the composition

$$\mathbf{Spc}_*(\mathscr{S}) \xrightarrow{\Sigma^\infty} \mathbf{Spt}_T(\mathscr{S}) \to \mathbf{SH}_{\mathscr{S}},$$

where the second arrow is the canonical functor from a model category to its homotopy category. A useful fact is that Σ^∞ is strict monoidal, so that there are isomorphisms

$$\Sigma^\infty X \wedge \Sigma^\infty Y \cong \Sigma^\infty (X \wedge Y)$$

in $\mathbf{SH}_{\mathscr{S}}$ for all based spaces X and Y, and furthermore, this functor factors through the unstable homotopy category $\mathbf{H}_*(\mathscr{S})$. Hence since T is weakly equivalent to $S^{2,1} = S^{1,0} \wedge S^{1,1}$ in $\mathbf{Spc}_*(\mathscr{S})$, we have the following isomorphisms in $\mathbf{SH}_{\mathscr{S}}$:

$$T \cong S^{2,1} \cong S^{1,0} \wedge S^{1,1}$$

(here we are being abusive and omitting Σ^∞'s for clarity). Almost by construction, T is invertible in $\mathbf{SH}_{\mathscr{S}}$, in the sense that there exists some object T^{-1} in $\mathbf{SH}_{\mathscr{S}}$ and an isomorphism $S \cong T^{-1} \wedge T$. Now, define the spectra

$$S^{-1,0} := T^{-1} \wedge S^{1,1} \qquad \text{and} \qquad S^{-1,-1} := T^{-1} \wedge S^{1,0} (\cong \Sigma T).$$

The notation is justified by the isomorphisms

$$\xi_1 : S \cong T^{-1} \wedge T \cong T^{-1} \wedge S^{1,1} \wedge S^{1,0} = S^{-1,0} \wedge S^{1,0}$$

and

$$\xi_2 : S \cong T^{-1} \wedge T \cong T^{-1} \wedge S^{1,1} \wedge S^{1,0} \cong T^{-1} \wedge S^{1,0} \wedge S^{1,1} = S^{-1,-1} \wedge S^{1,1}.$$

In this way, by abuse of notation, we may define $\mathbb{Z} \times \mathbb{Z}$–graded family of motivic sphere spectra in $\mathbf{SH}_{\mathscr{S}}$ by defining

$$S^{p,q} := (S^{1,0})^{p-q} \wedge (S^{1,1})^q$$

for $p, q \in \mathbb{Z}$ (recall our earlier defined conventions for powers in a monoidal category). It follows purely formally that for all $a, b \in \mathbb{Z}^2$ there exist "semi-canonical" isomorphisms[7]

$$S^{a,b} \cong S^a \wedge S^b,$$

and given $p, q \in \mathbb{Z}$, the functors $S^{p,q} \wedge -$ and $S^{-p,-q} \wedge -$ form an adjoint equivalence of $\mathbf{SH}_{\mathscr{S}}$. Given a spectrum X in $\mathbf{SH}_{\mathscr{S}}$, we write $\Sigma^{p,q}$ to denote the functor defined by $\Sigma^{p,q} X := S^{p,q} \wedge X$. In particular, the shift functor [1] in the triangulated structure on $\mathbf{SH}_{\mathscr{S}}$ is given by $\Sigma^{1,0}$, and we have canonical isomorphisms $\Sigma^{p,q} S \cong S^{p,q}$. Note that since $\Sigma^\infty : \mathbf{Spc}_*(\mathscr{S}) \to \mathbf{SH}_{\mathscr{S}}$ is strict monoidal, we have isomorphisms $\Sigma^\infty S^{p,q} \cong S^{p,q}$ for all $p \geq q \geq 0$.

Given spectra X and Y, we denote the abelian group $\mathbf{SH}_{\mathscr{S}}(X, Y)$ by $[X, Y]$[8]. We may extend $[X, Y]$ to a $\mathbb{Z}^2$-graded abelian group $[X, Y]_{**}$ by defining

$$[X, Y]_{p,q} := [\Sigma^{p,q} X, Y] = [S^{p,q} \wedge X, Y].$$

[6] Sadly, explicitly constructing the monoidal product on $\mathbf{SH}_{\mathscr{S}}$ is actually quite difficult.

[7] Explicitly, these isomorphisms are obtained by forming formal compositions of unitors, associators, and the isomorphisms $\xi_1 : S \cong S^{-1,0} \wedge S^{1,0}$ and $\xi_2 : S \cong S^{-1,-1} \wedge S^{1,1}$ and their inverses as necessary.

[8] Recall that $\mathbf{SH}_{\mathscr{S}}$ is triangulated, in particular, it is an additive category.

We denote the category of $\mathbb{Z} \times \mathbb{Z}$-graded abelian groups by $\mathbf{Ab}^{\mathbb{Z}^2}$. Given a spectrum E, it determines functors $E^{**} : \mathbf{SH}_{\mathscr{S}}^{\mathrm{op}} \to \mathbf{Ab}^{\mathbb{Z}^2}$ and $E_{**} : \mathbf{SH}_{\mathscr{S}} \to \mathbf{Ab}^{\mathbb{Z}^2}$, by defining

$$E^{p,q}(X) := [X, S^{p,q} \wedge E] = [X, E]_{-p,-q} \qquad \text{and} \qquad E_{p,q}(X) := [S^{p,q}, E \wedge X] \cong [S, E \wedge X]_{p,q}.$$

We call the functors E^{**} and E_{**} the *cohomology* and *homology* theories represented by E, respectively. One special homology theory is that represented by the sphere spectrum S, which we denote by π_{**}:

$$\pi_{p,q}(X) := [S^{p,q}, X] \cong [S^{p,q}, S \wedge X] = S_{**}(X).$$

Given a spectrum X, we refer to the collection of $\pi_{p,q}(X)$'s as the *stable homotopy groups* of X.

Note that in what happened above, we could have actually replaced $T \simeq \mathbb{A}^1/\mathbb{G}_m$ with any compact In Section A.7 of [14], the Betti realization functor (Definition 8.3) is extended to a strong symmetric monoidal functor $\psi : \mathbf{SH}_{\mathbb{C}} \to \mathbf{hoSp}$ from the motivic stable homotopy category over $\mathbb{C}$ to the classical stable homotopy category[9]. A useful fact, one which somewhat justifies the grading for the motivic spheres, is that ψ takes the T-spectrum $S^{p,q}$ to the suspension spectrum $S^p \cong \Sigma^\infty S^p$ of the p-sphere in $\mathbf{hoSp}$.

8.4. **Grading.** First, recall the standard stable homotopy category $\mathbf{hoSp}$, obtained by formally inverting the functor $\Sigma := S^1 \wedge - : \mathcal{S}_* \to \mathcal{S}_*$. It is a tensor triangulated category, where the tensor product is denoted by $-\wedge-$ and called the *smash product*. There exists a strong monoidal functor $\Sigma^\infty : \mathcal{S}_* \to \mathbf{hoSp}$. We omit Σ^∞ from the notation, and identify a space X with its suspension spectrum $\Sigma^\infty X$. The unit for the monoidal structure on $\mathbf{hoSp}$ is given by $S := S^0$. The shift functor is given by $\Sigma := S^1 \wedge -$. In particular, since the shift functor is essentially surjective, it follows that there exists some spectrum S^{-1} in $\mathbf{hoSp}$ and an isomorphism $\xi : S \cong S^{-1} \wedge S^1$. It then follows purely formally, using only the fact that $\mathbf{hoSp}$ is a symmetric monoidal category and the isomorphism $\xi : S \cong S^{-1} \wedge S^1$, that the functors $\Sigma = S^1 \wedge -$ and $\Omega = S^{-1} \wedge -$ form an adjoint equivalence of $\mathbf{hoSp}$. For each integer n, we may define

$$S^n := (S^1)^n.$$

In [3, Theorem 1.6], it is described how the chosen isomorphism $\xi : S \cong S^{-1} \wedge S^1$ determine canonical isomorphisms

$$\phi_{p,q} : S^{p+q} \xrightarrow{\cong} S^p \wedge S^q,$$

where $\phi_{p,q}$ is given simply by composing associators, unitors, and copies of ξ and ξ^{-1}. In particular, $\phi_{-1,1} = \xi$, and if p or q is zero then $\phi_{p,q}$ is precisely one of the unitor isomorphisms. As it turns out, these isomorphisms are very nice. For one, they are coherent, so that the obvious pentagonal diagrams commute for all $a, b, c \in \mathbb{Z}$:

$$\begin{tikzcd} S^{a+b} \wedge S^c \arrow[d, "\phi_{a,b} \wedge S^c"'] & S^{a+b+c} \arrow[l, "\phi_{a+b,c}"'] \arrow[r, "\phi_{a,b+c}"] & S^a \wedge S^{b+c} \arrow[d, "S^a \wedge \phi_{b,c}"] \\ (S^a \wedge S^b) \wedge S^c \arrow[rr, "\cong"] & & S^a \wedge (S^b \wedge S^c) \end{tikzcd}$$

Furthermore, these isomorphisms commute with the symmetric structure of $\mathbf{hoSp}$, like so:

[9] Explicitly, in [14, Theorem A.44], the category $\mathrm{Sp}^\Sigma(\mathbf{Top}, \mathbb{C}P^1)$ of symmetric $\mathbb{C}P^1$-spectra in $\mathbf{Top}$ is constructed, and it is shown that there is a zig-zag of Quillen equivalences between $\mathrm{Sp}^\Sigma(\mathbf{Top}, \mathbb{C}P^1)$ and the usual category of spectra $\mathrm{Sp}^\Sigma(\mathbf{Top}, S^1)$, so they have equivalent homotopy categories. Then applying the Betti realization functor levelwise yields a strict symmetric monoidal functor (Theorem A.45) from the category $\mathbf{Sp}^\Sigma_{\mathbb{P}^1}(\mathbb{C})$ of motivic symmetric $\mathbb{P}^1$-spectra to $\mathrm{Sp}^\Sigma(\mathbf{Top}, \mathbb{C}P^1)$. Finally, in the category $\mathbf{Spc}_*(\mathbb{C})$ of motivic spaces over $\mathbb{C}$, we have that T and $\mathbb{P}^1$ are equivalent, which yields a Quillen equivalence $\mathbf{Sp}^\Sigma_{\mathbb{P}^1}(\mathbb{C}) \simeq \mathbf{Sp}^\Sigma_T(\mathbb{C})$ (Theorem A.30). Putting all of this together yields the desired strong symmetric monoidal functor $\mathbf{SH}_{\mathbb{C}} \to \mathbf{hoSp}$.

Proposition 8.13 ([5, Lemma 7.1.13] or [13, Lemma 5.9]). *The following diagram is commutative for arbitrary integers p and q*

$$\begin{tikzcd}
S^{p+q} \arrow[r, "\phi_{p,q}"] \arrow[d, "(-1)^{pq}"'] & S^p \wedge S^q \arrow[d, "\tau"] \\
S^{p+q} \arrow[r, "\phi_{q,p}"] & S^q \wedge S^p
\end{tikzcd}$$

where here τ is the symmetry map specified by the symmetric monoidal structure on **hoSp**, *and*

$$(-1)^{pq} = \begin{cases} \mathrm{id} & pq \equiv 0 \bmod 2 \\ -\mathrm{id} & pq \equiv 1 \bmod 2. \end{cases}$$

(Recall that **hoSp** *is a triangulated category, and in particular an additive category, so that homsets in* **hoSp** *are abelian groups.)*

Recall that a *commutative ring spectrum* is a commutative monoid object in **hoSp**, that is, a spectrum E along with maps $\mu : E \wedge E \to E$ and $e : S \to E$ such that the following diagrams commute in **hoSp**:

$$\begin{tikzcd}
 & S \wedge E \arrow[dl, "\cong"'] \arrow[d, "e \wedge E"] \\
E & E \wedge E \arrow[l, "\mu"'] \\
 & E \wedge S \arrow[ul, "\cong"] \arrow[u, "E \wedge e"']
\end{tikzcd}
\qquad
\begin{tikzcd}
(E \wedge E) \wedge E \arrow[r, "\mu \wedge E"] \arrow[d, "\cong"'] & E \wedge E \arrow[dd, "\mu"] \\
E \wedge (E \wedge E) \arrow[d, "E \wedge \mu"'] & \\
E \wedge E \arrow[r, "\mu"] & E
\end{tikzcd}
\qquad
\begin{tikzcd}
E \wedge E \arrow[dr, "\mu"] \arrow[dd, "\tau"'] & \\
 & E \\
E \wedge E \arrow[ur, "\mu"'] &
\end{tikzcd}$$

We may define the stable homotopy groups of E to be the groups

$$\pi_n(E) := [S^n, E] \cong [\Sigma^n S, E].$$

In fact, in this setting, it turns out that the graded abelian group $\pi_*(E)$ has the structure of a graded abelian group, where we may define the product

$$\pi_p(E) \times \pi_q(E) \to \pi_{p+q}(E)$$

to send a pair $(\alpha, \beta) \in \pi_p(E) \times \pi_q(E)$ to the composition

$$S^{p+q} \xrightarrow{\phi_{p,q}} S^p \wedge S^q \xrightarrow{\alpha \wedge \beta} E \wedge E \xrightarrow{\mu} E.$$

It turns out this map is associative: Given classes α, β, and γ in $\pi_a(E)$, $\pi_b(E)$, and $\pi_c(E)$, respectively, consider the following diagram:

$$\begin{tikzcd}
S^{a+b+c} \arrow[r, "\phi_{a+b,c}"] \arrow[d, "\phi_{a,b+c}"'] & S^{a+b} \wedge S^c \arrow[r, "\phi_{a,b} \wedge S^c"] & (S^a \wedge S^b) \wedge S^c \arrow[r, "(\alpha\wedge\beta)\wedge\gamma"] \arrow[dl, "\cong"] & (E \wedge E) \wedge E \arrow[r, "\mu \wedge E"] \arrow[dl, "\cong"] & E \wedge E \arrow[d, "\mu"] \\
S^a \wedge S^{b+c} \arrow[r, "S^a \wedge \phi_{b,c}"'] & S^a \wedge (S^b \wedge S^c) \arrow[r, "\alpha \wedge (\beta\wedge\gamma)"'] & E \wedge (E \wedge E) \arrow[r, "E \wedge \mu"'] & E \wedge E \arrow[r, "\mu"'] & E
\end{tikzcd}$$

Commutativity of the left pentagon is the coherence condition for the $\phi_{p,q}$'s. Commutativity of the middle parallelogram is naturality of the associator isomorphisms. Commutativity of the right pentagon is associativity of μ. The fact that the two outside compositions equal $(\alpha \cdot \beta) \cdot \gamma$ and $\alpha \cdot (\beta \cdot \gamma)$, respectively, follows by functoriality of $- \wedge -$.

It also turns out that the map $e : S \to E$ is a unit for this multiplication. Given $\alpha \in [S^p, E]$, consider the following diagram:

$$\begin{tikzcd}
S \wedge S^p \arrow[dd, "e\wedge\alpha"'] \arrow[dr, "S\wedge\alpha"] & & S^p \arrow[ll, "\phi_{0,p}^{-1}=\rho_{S^p}^{-1}"'] \arrow[rr, "\phi_{p,0}^{-1}=\lambda_{S^p}^{-1}"] \arrow[dd, "\alpha"] & & S^p \wedge S \arrow[dl, "\alpha\wedge S"'] \arrow[dd, "\alpha\wedge e"] \\
& S \wedge E \arrow[dl, "e\wedge E"'] \arrow[dr, "\rho_E"] & & E \wedge S \arrow[dl, "\lambda_E"'] \arrow[dr, "E\wedge e"] & \\
E \wedge E \arrow[rr, "\mu"'] & & E & & E \wedge E \arrow[ll, "\mu"]
\end{tikzcd}$$

Commutativity of the top two large triangles is naturality of the unitor isomorphisms. Commutativity of the right and leftmost triangles functoriality of $- \wedge -$. Commutativity of the bottom triangles is unitality of μ. Hence, we have that $e \cdot \alpha = \alpha = \alpha \cdot e$.

This composition is also bilinear. Given $\alpha, \alpha' \in \pi_p(E)$ and $\beta, \beta' \in \pi_q(E)$, consider the following diagrams:

$$\begin{tikzcd}
S^{p+q} \arrow[r, "\phi_{p,q}"] \arrow[d, "\Delta"'] & S^p \wedge S^q \arrow[r, "\Delta\wedge S^q"] \arrow[d, "\Delta"] & (S^p \oplus S^p) \wedge S^q \arrow[r, "(\alpha\oplus\alpha')\wedge\beta"] \arrow[dl, "\cong"'] & (E \oplus E) \wedge E \arrow[dl, "\cong"'] \arrow[d, "\nabla\wedge E"] & \\
S^{p+q} \oplus S^{p+q} \arrow[r, "\phi_{p,q}\oplus\phi_{p,q}"'] & (S^p \wedge S^q) \oplus (S^p \wedge S^q) \arrow[r, "(\alpha\wedge\beta)\oplus(\alpha'\wedge\beta)"'] & (E \wedge E) \oplus (E \wedge E) \arrow[r, "\nabla"'] & E \wedge E \arrow[r, "\mu"] & E
\end{tikzcd}$$

$$\begin{tikzcd}
S^{p+q} \arrow[r, "\phi_{p,q}"] \arrow[d, "\Delta"'] & S^p \wedge S^q \arrow[r, "\Delta\wedge S^q"] \arrow[d, "\Delta"] & S^p \wedge (S^q \oplus S^q) \arrow[r, "\alpha\wedge(\beta\oplus\beta')"] \arrow[dl, "\cong"'] & E \wedge (E \oplus E) \arrow[dl, "\cong"'] \arrow[d, "\nabla\wedge E"] & \\
S^{p+q} \oplus S^{p+q} \arrow[r, "\phi_{p,q}\oplus\phi_{p,q}"'] & (S^p \wedge S^q) \oplus (S^p \wedge S^q) \arrow[r, "(\alpha\wedge\beta)\oplus(\alpha\wedge\beta')"'] & (E \wedge E) \oplus (E \wedge E) \arrow[r, "\nabla"'] & E \wedge E \arrow[r, "\mu"] & E
\end{tikzcd}$$

The unlabeled isomorphisms are those given by the fact that $- \wedge -$ is additive in each variable (since **hoSp** is tensor triangulated). Commutativity of the left squares is naturality of $\Delta_X : X \to X \oplus X$ in an additive category. Commutativity of the rest of the diagram follows again from the fact that $- \wedge -$ is an additive functor in each variable. Hence, by functoriality of $- \wedge -$, these diagrams tell us that $(\alpha + \alpha') \cdot \beta = \alpha \cdot \beta + \alpha' \cdot \beta$ and $\alpha \cdot (\beta + \beta') = \alpha \cdot \beta + \alpha \cdot \beta'$, respectively.

Finally, we have that this product is graded commutative. Given $\alpha \in \pi_p(E)$ and $\beta \in \pi_q(E)$, consider the following diagram:

$$\begin{tikzcd}
S^{p+q} \arrow[r, "\phi_{p,q}"] \arrow[d, "(-1)^{pq}"'] & S^p \wedge S^q \arrow[r, "\alpha\wedge\beta"] \arrow[d, "\tau"] & E \wedge E \arrow[d, "\tau"'] \arrow[dr, "\mu"] & \\
S^{p+q} \arrow[r, "\phi_{q,p}"] & S^q \wedge S^p \arrow[r, "\beta\wedge\alpha"] & E \wedge E \arrow[r, "\mu"'] & E
\end{tikzcd}$$

Commutativity of the left square is Proposition 8.13. Commutativity of the middle square is naturality of the symmetry isomorphisms. Finally, commutativity of the right triangle is commutativity of μ. Hence by bilinearity of $- \wedge -$, it follows that $\alpha \cdot \beta = (-1)^{pq} \beta \cdot \alpha$, as desired.

To recap, we've shown that if E is a commutative ring spectrum in the stable homotopy category, then $\pi_*(E)$ is itself canonically a *graded commutative* ring.

The natural question arises: does the same thing happen in the motivic world? In other words, if we have a monoid object (E, μ, e) in the motivic stable homotopy category $\mathbf{SH}_{\mathscr{S}}$, does the $\mathbb{Z} \times \mathbb{Z}$-graded abelian group $\pi_{**}(E)$ canonically form a bigraded ring, and furthermore if E is commutative, does the $\pi_{**}(E)$ satisfy any sort of "bigraded commutativity" condition? To answer the first question, motivated by the above work in the classical stable homotopy category, we know that to make $\pi_{**}(E)$ a $\mathbb{Z}^2$-graded ring, we need a family of isomorphisms

$$\phi_{a,b} : S^{a+b} \xrightarrow{\cong} S^a \wedge S^b$$

for each $a, b \in \mathbb{Z}^2$ such that

(1) For every $a, b, c \in \mathbb{Z}^2$, the following diagram commutes:

$$\begin{tikzcd}
S^{a+b} \wedge S^c \arrow[d, "\phi_{a,b} \wedge S^c"'] & S^{a+b+c} \arrow[l, "\phi_{a+b,c}"'] \arrow[r, "\phi_{a,b+c}"] & S^a \wedge S^{b+c} \arrow[d, "S^a \wedge \phi_{b,c}"] \\
(S^a \wedge S^b) \wedge S^c \arrow[rr, "\cong"] & & S^a \wedge (S^b \wedge S^c)
\end{tikzcd}$$

(2) For every $a \in \mathbb{Z}^2$, the isomorphisms $\phi_{(0,0),a}$ and $\phi_{a,(0,0)}$ coincide with the unital isomorphisms in $\mathbf{SH}_{\mathscr{S}}$.

We call such a family of isomorphisms $\{\phi_{a,b}\}_{a,b \in \mathbb{Z}^2}$ **coherent**. Once we have a coherent family, the exact same arguments given above for monoid objects in the classical stable homotopy category endow $\pi_{**}(E)$ with the structure of a $\mathbb{Z}^2$-graded ring. So can we find such a family? Recall that we have defined the $S^{a,b}$ as wedges of the "motivic circles" $S^{1,0}$, $S^{1,1}$, and their inverses $S^{-1,0}$ and $S^{-1,-1}$. Furthermore, by [3, Theorem 1.13], we know that the isomorphisms $\xi_1 : S \cong S^{-1,0} \wedge S^{1,0}$ and $\xi_2 : S \cong S^{-1,-1} \wedge S^{1,1}$ give rise to a canonical coherent family $\{\phi_{a,b}\}_{a,b \in \mathbb{Z}^2}$ obtained by forming formal compositions of copies of associators, unitors, ξ_1 and ξ_2, and their inverses.

cite

So, we have successfully answered our first question in the affirmative. What about the second question? As it turns out, bigraded commutativity turns out to be very subtle, but the answer is yes. First, it turns out that the functor $\mathbb{G}_m \wedge - : \mathbf{SH}_{\mathscr{S}} \to \mathbf{SH}_{\mathscr{S}}$ is an equivalence. In what follows, let $\epsilon \in [S, S] \cong [\mathbb{G}_m, \mathbb{G}_m]$ correspond to the endomorphism of

$$\mathbb{G}_m = \mathscr{S} \times \operatorname{Spec} \mathbb{Z}[x, x^{-1}]$$

induced by the ring morphism $\operatorname{Spec} \mathbb{Z}[x, x^{-1}] \to \operatorname{Spec} \mathbb{Z}[x, x^{-1}]$ sending $x \mapsto x^{-1}$. In particular, note that $\epsilon \circ \epsilon = \mathrm{id}_S$ in $\mathbf{SH}_{\mathscr{S}}$. Then the coherent family $\{\phi_{a,b}\}_{a,b \in \mathbb{Z}^2}$ induces the following bigraded commutativity condition:

Proposition 8.14. *Given a commutative ring spectrum E in $\mathbf{SH}_{\mathscr{S}}$ with unit $e \in [S, E] \cong \pi_{0,0}(E)$, the bigraded ring $\pi_{**}(E)$ is "bigraded commutative", in the sense that when $\alpha \in \pi_{a_1,a_2}(E)$ and $\beta \in \pi_{b_1,b_2}(E)$, under the product determined by the coherent family $\{\phi_{a,b}\}_{a,b \in \mathbb{Z}^2}$ described above given by [3, Theorem 1.13], we have that*

$$\alpha \cdot \beta = \beta \cdot \alpha \cdot (-e)^{(a_1 - a_2)(b_1 - b_2)} \cdot (e\epsilon)^{a_2 b_2}.$$

Proof. The proof of [3, Proposition 1.18] shows this for $E = S$. The same argument works more generally. □

Sadly, as [4] describes, this product has some issues. For one, it does not agree with the graded commutativity condition described by Voevodsky for the product on the dual motivic Steenrod algebra $\mathcal{A}_{**} := M\mathbb{Z}_{**}(M\mathbb{Z}) = \pi_{**}(M\mathbb{Z} \wedge M\mathbb{Z})$ ([16, Theorem 2.2]). Furthermore, under this grading convention, given a motivic commutative ring spectrum E over $\mathscr{S} = \operatorname{Spec} \mathbb{C}$, the map

$$\pi_{*,\star}(E) \to \pi_*(E(\mathbb{C}))$$

induced by Betti realization is not a ring homomorphism—there is an annoying sign that comes up (cf. [3, Proposition 1.19]).

Can this be fixed? According to Section 7 of [3], there are in fact more coherent families of isomorphisms $\{\phi_{a,b}\}_{a,b\in\mathbb{Z}^2}$ than just the one described above, and in fact, they give rise to non-isomorphic graded rings $\pi_{**}(E)$, in general. In [4], such a family is fixed which fixes both of the above issues:

Proposition 8.15 ([4, p. 3]). *There exists a coherent family of isomorphisms*

$$\phi_{a,b} : S^{a+b} \xrightarrow{\cong} S^a \wedge S^b$$

for $a, b \in \mathbb{Z}^2$ such that given a motivic commutative ring spectrum E with unit $e : S \to E$, the product structure induced on the bigraded abelian group $\pi_{**}(E)$ by this family is bigraded commutative in the sense that given $\alpha \in \pi_{a_1,a_2}(E)$ and $\beta \in \pi_{b_1,b_2}(E)$, the following equation holds:[10]

$$\alpha \cdot \beta = \beta \cdot \alpha \cdot (-e)^{a_1 b_1} \cdot e(-\epsilon)^{a_2 b_2}.$$

Furthermore, this product is related to the product $-\star-$ given in Proposition 8.14 by the formula:

$$\alpha \cdot \beta = \alpha \star \beta \star (-e)^{a_2(b_1-b_2)}.$$

In particular, when $e \circ \epsilon = -e$ then $e \circ (-\epsilon) = e$ and thus

$$\alpha \cdot \beta = \beta \cdot \alpha \cdot (-e)^{a_1 b_1}.$$

This is exactly Voevodsky's convention for commutativity in the dual Steenrod algebra ([16, Theorem 2.2]). Furthermore, this grading convention allows for the realization map

$$\pi_{*,\star}(E) \to \pi_*(E(\mathbb{C}))$$

to be a ring homomorphism for all commutative ring spectra E in $\mathbf{SH}_{\mathbb{C}}$.

Remark 8.16. For the rest of this paper, we will be using the coherent family $\{\phi_{a,b}\}_{a,b\in\mathbb{Z}^2}$ and the graded commutativity law specified by Proposition 8.15. Usually we will not label the maps, instead only writing $S^{a+b} \xrightarrow{\cong} S^a \wedge S^b$ or $S^{a+b} \cong S^a \wedge S^b$.

Appendix A. Triangulated categories

We assume the reader is familiar with additive categories and (closed, symmetric) monoidal categories.

A.1. Triangulated categories and their properties.

Definition A.1. A *triangulated category* $(\mathcal{C}, \Sigma, \mathcal{D})$ is the data of:

(1) An additive category $\mathcal{C}$.

(2) An additive auto-equivalence $\Sigma : \mathcal{C} \to \mathcal{C}$ called the *shift functor*.

(3) A collection $\mathcal{D}$ of *distinguished* triangles in $\mathcal{C}$, where a *triangle* is a sequence of arrows of the form

$$X \to Y \to Z \to \Sigma X.$$

Distinguished triangles are also sometimes called *cofiber sequences* or *fiber sequences*.

These data must satisfy the following axioms:

[10] We are fixing $u = -1$, in the notation of the Proposition in [4, p. 3].

TR0 Given a commutative diagram

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "\cong"'] & Y \arrow[r] \arrow[d, "\cong"'] & Z \arrow[r] \arrow[d, "\cong"'] & \Sigma X \arrow[d, "\cong"'] \\
X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & \Sigma X'
\end{tikzcd}$$

where the vertical arrows are isomorphisms, if the top row is distinguished then so is the bottom.

TR1 For any object X in $\mathcal{C}$, the diagram

$$X \xrightarrow{\mathrm{id}_X} X \to 0 \to \Sigma X$$

is a distinguished triangle.

TR2 For all $f : X \to Y$ there exists an object C_f (also sometimes denoted Y/X) called the *cofiber of f* and a distinguished triangle

$$X \xrightarrow{f} Y \to C_f \to \Sigma X.$$

TR3 Given a solid diagram

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "f"'] & Y \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d, dashed] & \Sigma X \arrow[d, "\Sigma f"] \\
X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & \Sigma X'
\end{tikzcd}$$

such that the leftmost square commmutes and both rows are distinguished, there exists a dashed arrow $Z \to Z'$ which makes the remaining two squares commute.

TR4 A triangle

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$$

is distinguished if and only if

$$Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X \xrightarrow{-\Sigma f} \Sigma Y$$

is distinguished.

TR5 (Octahedral axiom) Given three distinguished triangles

$$X \xrightarrow{f} Y \xrightarrow{h} Y/X \to \Sigma X$$

$$Y \xrightarrow{g} Z \xrightarrow{k} Z/Y \to \Sigma Y$$

$$X \xrightarrow{g \circ f} Z \xrightarrow{l} Z/X \to \Sigma X$$

there exists a distinguished triangle

$$Y/X \xrightarrow{u} Z/X \xrightarrow{v} Z/Y \xrightarrow{w} \Sigma(Y/X)$$

such that the following diagram commutes

$$\begin{tikzcd}
X \arrow[rr, "g \circ f"] \arrow[dr, "f"'] & & Z \arrow[rr, "k"] \arrow[dr, "l"'] & & Z/Y \arrow[rr, "w"] \arrow[dr] & & \Sigma(Y/X) \\
& Y \arrow[ur, "g"'] \arrow[dr, "h"'] & & Z/X \arrow[ur, "v"'] \arrow[dr] & & \Sigma Y \arrow[ur, "\Sigma h"'] & \\
& & Y/X \arrow[ur, "u"'] \arrow[rr] & & \Sigma X \arrow[ur, "\Sigma f"'] & &
\end{tikzcd}$$

It turns out that the above definition is actually redundant; TR3 and TR4 follow from the remaining axioms (see Lemmas 2.2 and 2.4 in [9]). From now on, we fix a triangulated category $(\mathcal{C}, \Sigma, \mathcal{D})$. To start, recall the following definition:

Definition A.2. A sequence

$$X_1 \to X_2 \to \cdots \to X_n$$

of arrows in $\mathcal{C}$ is *exact* if, for any object A in $\mathcal{C}$, the induced sequences

$$[A, X_1] \to [A, X_2] \to \cdots \to [A, X_{n-1}] \to [A, X_n]$$

and

$$[X_n, A] \to [X_{n-1}, A] \to \cdots \to [X_2, A] \to [X_1, A]$$

are exact sequences of abelian groups.

Proposition A.3. *Every distinguished triangle is an exact sequence (in the sense of Definition A.2).*

Proof. Suppose we have some distinguished triangle

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X.$$

Then first we would like to show that given any object A in $\mathcal{C}$, the sequence

$$[A, X] \xrightarrow{f_*} [A, Y] \xrightarrow{g_*} [A, Z] \xrightarrow{h_*} [A, \Sigma X]$$

is exact. First we show exactness at $[A, Y]$. To see $\operatorname{im} f_* \subseteq \ker g_*$, note it suffices to show that $g \circ f = 0$. Indeed, consider the commuting diagram

$$\begin{tikzcd} X \arrow[r, equal] \arrow[d, equal] & X \arrow[r] \arrow[d, "f"] & 0 \arrow[r] & \Sigma X \\ X \arrow[r, "f"] & Y \arrow[r, "g"] & Z \arrow[r, "h"] & \Sigma X \end{tikzcd}$$

The top row is distinguished by axiom TR1. Thus by TR3, the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, equal] \arrow[d, equal] & X \arrow[r] \arrow[d, "f"] & 0 \arrow[r] \arrow[d] & \Sigma X \arrow[d, equal] \\ X \arrow[r, "f"] & Y \arrow[r, "g"] & Z \arrow[r, "h"] & \Sigma X \end{tikzcd}$$

In particular, commutativity of the second square tells us that $g \circ f = 0$, as desired. Conversely, we'd like to show that $\ker g_* \subseteq \operatorname{im} f_*$. Let $\psi : A \to Y$ be in the kernel of g_*, so that $g \circ \psi = 0$. Consider the following commutative diagram:

$$\begin{tikzcd} A \arrow[r] \arrow[d, "\psi"'] & 0 \arrow[r] \arrow[d] & \Sigma A \arrow[r, "-\Sigma \mathrm{id}_A"] & \Sigma A \\ Y \arrow[r, "g"] & Z \arrow[r, "h"] & \Sigma X \arrow[r, "-\Sigma f"] & \Sigma Y \end{tikzcd}$$

The top row is distinguished by axioms TR1 and TR4. The bottom row is distinguished by axiom TR4. Thus by axiom TR3 there exists a map $\widetilde{\phi} : \Sigma A \to \Sigma X$ such that the following diagram commutes:

$$\begin{tikzcd} A \arrow[r] \arrow[d, "\psi"'] & 0 \arrow[r] \arrow[d] & \Sigma A \arrow[r, "-\Sigma \mathrm{id}_A"] \arrow[d, "\widetilde{\phi}"'] & \Sigma A \arrow[d, "\Sigma \psi"'] \\ Y \arrow[r, "g"] & Z \arrow[r, "h"] & \Sigma X \arrow[r, "-\Sigma f"] & \Sigma Y \end{tikzcd}$$

Now, since Σ is an equivalence, it is a full functor, so that in particular there exists some $\phi : A \to X$ such that $\widetilde{\phi} = \Sigma\phi$. Then by faithfullness, we may pull back the right square to get a commuting diagram

$$\begin{tikzcd} A \arrow[r, "-\mathrm{id}_A"] \arrow[d, "\phi"'] & A \arrow[d, "\psi"] \\ X \arrow[r, "-f"'] & Y \end{tikzcd}$$

Hence,

$$f_*(\phi) = f \circ \phi \overset{(*)}{=} -((-f) \circ \phi) = -(\psi \circ (-\mathrm{id}_A)) \overset{(*)}{=} \psi \circ \mathrm{id}_A = \psi,$$

where the equalities marked $(*)$ follow by bilinearity of composition in an additive category. Thus $\psi \in \operatorname{im} f_*$, as desired, meaning $\ker g_* \subseteq \operatorname{im} f_*$.

Now, we have shown that

$$[A, X] \xrightarrow{f_*} [A, Y] \xrightarrow{g_*} [A, Z] \xrightarrow{h_*} [A, \Sigma X]$$

is exact at $[A, Y]$. It remains to show exactness at $[A, Z]$. Yet this follows by the exact same argument given above applied to the sequence obtained from the shifted triangle (TR4)

$$Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X \xrightarrow{-\Sigma f} \Sigma Y$$

On the other hand, we would like to show that

$$[\Sigma X, A] \xrightarrow{h^*} [Z, A] \xrightarrow{g^*} [Y, A] \xrightarrow{f^*} [X, A]$$

is exact. As above, since we can shift the triangle, it suffices to show exactness at $[Z, A]$. First, since we have shown $g \circ f = 0$, we have $f^* \circ g^* = (g \circ f)^* = 0$, so that $\operatorname{im} g^* \subseteq \ker f^*$, as desired. Conversely, in order to see $\ker f^* \subseteq \operatorname{im} g^*$, suppose $\psi : Y \to A$ is in the kernel of f^*, so that $\psi \circ f = 0$. Consider the following commuting diagram:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[r, "g"] \arrow[d, "\psi"] & Z \arrow[r, "h"] & \Sigma X \\ 0 \arrow[r] & A \arrow[r, equal] & A \arrow[r] & 0 \end{tikzcd}$$

The top row is a distinguished triangle by assumption, and the bottom row is distinguished by axioms TR1 and TR4 for a triangulated category, along with the fact that $\Sigma 0 = 0$ since Σ is additive. Thus by axiom TR3 there exists a map $\phi : Z \to A$ such that $\phi \circ g = \psi$, i.e., $g^*(\phi) = \psi$, so that $\phi \in \operatorname{im} g^*$ as desired.

Thus, we have shown exactness of

$$[\Sigma X, A] \xrightarrow{h^*} [Z, A] \xrightarrow{g^*} [Y, A] \xrightarrow{f^*} [X, A]$$

at $[Y, A]$. To see $\ker g^* = \operatorname{im} h^*$, again the same arguments applied to the shifted triangle

$$Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X \xrightarrow{-\Sigma f} \Sigma Y$$

yield that

$$[\Sigma Y, A] \xrightarrow{(-\Sigma f)^*} [\Sigma X, A] \xrightarrow{h^*} [Z, A] \xrightarrow{g^*} [Y, A]$$

is exact at $[Z, A]$, so $\ker g^* = \operatorname{im} h^*$, as desired. □

Lemma A.4. *Suppose we have a commutative diagram*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "j"] & Y \arrow[r, "g"] \arrow[d, "k"] & Z \arrow[r, "h"] \arrow[d, "\ell"] & \Sigma X \arrow[d, "\Sigma j"] \\ X' \arrow[r, "f'"] & Y' \arrow[r, "g'"] & Z' \arrow[r, "h'"] & \Sigma X' \end{tikzcd}$$

with both rows distinguished. Then if any two of the maps j, k, and ℓ are isomorphisms, then so is the third.

Proof. Suppose we are given any object A in $\mathcal{C}$, and consider the commutative diagram

$$\begin{tikzcd} {[A,X]} \arrow[r, "f_*"] \arrow[d, "j_*"] & {[A,Y]} \arrow[r, "g_*"] \arrow[d, "k_*"] & {[A,Z]} \arrow[r, "k_*"] \arrow[d, "\ell_*"] & {[A,\Sigma X]} \arrow[r, "-\Sigma f_*"] \arrow[d, "\Sigma j_*"] & {[A,\Sigma Y]} \arrow[r, "-\Sigma g_*"] \arrow[d, "\Sigma k_*"] & {[A,\Sigma Z]} \arrow[r, "-\Sigma h_*"] \arrow[d, "\Sigma \ell_*"] & {[A,\Sigma^2 X]} \arrow[d, "\Sigma^2 j_*"] \\ {[A,X']} \arrow[r, "f'_*"] & {[A,Y']} \arrow[r, "g'_*"] & {[A,Z']} \arrow[r, "h'_*"] & {[A,\Sigma X']} \arrow[r, "-\Sigma f'_*"] & {[A,\Sigma Y']} \arrow[r, "-\Sigma g'_*"] & {[A,\Sigma Z']} \arrow[r, "-\Sigma h'_*"] & {[A,\Sigma^2 X']} \end{tikzcd}$$

The rows are exact by Proposition A.3 and repeated applications of axiom TR4. It follows by the five lemma that if j and k are isomorphisms, then ℓ_* is an isomorphism. Similarly, if k and ℓ are isomorphisms then Σj_* is an isomorphism. Finally, if ℓ and j are isomorphisms, then Σk_* is an isomorphism. The desired result follows by faithfullness of Σ and the Yoneda embedding. □

Proposition A.5. *Given a map $f : X \to Y$ in a triangulated category $(\mathcal{C}, \Sigma, \Omega, \mathcal{D})$, the cofiber sequence of f is unique up to isomorphism, in the sense that given any two distinguished triangles*

$$X \xrightarrow{f} Y \to Z \to \Sigma X \qquad \text{and} \qquad X \xrightarrow{f} Y \to Z' \to \Sigma X,$$

there exists an isomorphism $Z \to Z'$ which makes the following diagram commute:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, equal] & Y \arrow[r] \arrow[d, equal] & Z \arrow[r] \arrow[d, dashed, "k"] & \Sigma X \arrow[d, equal] \\ X \arrow[r, "f"] & Y \arrow[r] & Z' \arrow[r] & \Sigma X \end{tikzcd}$$

Proof. Suppose we have two distinguished triangles

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X \qquad \text{and} \qquad X \xrightarrow{f} Y \xrightarrow{g'} Z' \xrightarrow{h'} \Sigma X,$$

and consider the following commutative diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, equal] & Y \arrow[r, "g"] \arrow[d, equal] & Z \arrow[r, "h"] & \Sigma X \\ X \arrow[r, "f"] & Y \arrow[r, "g'"] & Z' \arrow[r, "h'"] & \Sigma X \end{tikzcd}$$

By axiom TR3, there exists some map $k : Z \to Z'$ which makes the following diagram commute:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, equal] & Y \arrow[r, "g"] \arrow[d, equal] & Z \arrow[r, "h"] \arrow[d, "k"] & \Sigma X \arrow[d, equal] \\ X \arrow[r, "f"] & Y \arrow[r, "g'"] & Z' \arrow[r, "h'"] & \Sigma X \end{tikzcd}$$

Now, by Lemma A.4, k is an isomorphism. □

Proposition A.6. *Given an arrow $f : X \to Y$ in $\mathcal{C}$, there exists an object F_f called the* fiber *of f, and a distinguished triangle*

$$F_f \to X \xrightarrow{f} Y \to \Sigma F_f (\cong C_f).$$

Proof. Since Σ is an equivalence, there exists some functor $\Omega : \mathcal{C} \to \mathcal{C}$ and natural isomorphisms $\eta : \Omega\Sigma \Rightarrow \mathrm{Id}_{\mathcal{C}}$ and $\varepsilon : \mathrm{Id}_{\mathcal{C}} \Rightarrow \Sigma\Omega$. By axiom TR2, we have a distinguished triangle

$$X \xrightarrow{f} Y \xrightarrow{g} C_f \xrightarrow{h} \Sigma X.$$

Now, consider the commutative diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, equal] & Y \arrow[r, "g"] \arrow[d, equal] & C_f \arrow[r, "h"] \arrow[d, "\eta_{C_f}"] & \Sigma X \arrow[d, equal] \\ X \arrow[r, "f"] & Y \arrow[r, "\widetilde{g}"] & \Sigma\Omega C_f \arrow[r, "\widetilde{h}"] & \Sigma X \end{tikzcd}$$

where $\widetilde{g} = \eta_{C_f} \circ g$, and $\widetilde{h} = \eta_{C_f} \circ h$. Since each vertical map is an isomorphism and the top row is distinguished, the bottom row is also distinguished by axiom TR0. Now, since Σ is an equivalence of categories, it is faithful, so that in particular there exists some map $k : \Omega C_f \to X$ such that $\Sigma k = -\widetilde{h} \implies -\Sigma k = \widetilde{h}$. Thus, we have a distinguished triangle of the form

$$X \xrightarrow{f} Y \xrightarrow{\widetilde{g}} \Sigma\Omega C_f \xrightarrow{-\Sigma k} \Sigma X.$$

Finally, by axiom TR4, we get a distinguished triangle

$$\Omega C_f \xrightarrow{k} X \xrightarrow{f} Y \xrightarrow{\widetilde{g}} \Sigma\Omega C_f,$$

so we may define the fiber of f to be ΩC_f. $\square$

A.2. **Homotopy (co)limits in a triangulated category.**

Definition A.7 ([11, Definition 1.6.4])**.** Assume that $\mathcal{C}$ has countable coproducts. Let

$$X_0 \xrightarrow{j_1} X_1 \xrightarrow{j_2} X_2 \xrightarrow{j_3} X_3 \to \cdots$$

be a sequence of objects and morphisms in $\mathcal{C}$. The *homotopy colimit* of the sequence, denoted $\operatorname{holim} X_i$, is by definition given, up to non-canonical isomorphism, as the cofiber of the map

$$\coprod_{i=0}^{\infty} X_i \xrightarrow{1-\text{shift}} \coprod_{i=0}^{\infty} X_i,$$

where the shift map $\coprod_{i=0}^{\infty} X_i \xrightarrow{\text{shift}} \coprod_{i=0}^{\infty} X_i$ is understood to be the direct sum of $j_{i+1} : X_i \to X_{i+1}$. In other words, the map $1 - \text{shift}$ is the infinite matrix

$$\begin{bmatrix} \mathrm{id}_{X_0} & 0 & 0 & 0 & \cdots \\ -j_1 & \mathrm{id}_{X_1} & 0 & 0 & \cdots \\ 0 & -j_2 & \mathrm{id}_{X_2} & 0 & \cdots \\ 0 & 0 & -j_3 & \mathrm{id}_{X_3} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Proposition A.8 ([11, Proposition 1.6.8])**.** *Suppose $\mathcal{C}$ has countable coproducts, and suppose $e : X \to X$ is an idempotent in $\mathcal{C}$, so that $e \circ e = e$. Then e splits in $\mathcal{C}$, i.e., e factors as*

$$X \xrightarrow{r} Y \xrightarrow{\iota} X$$

with $r \circ \iota = \mathrm{id}_Y$ and $\iota \circ r = e$. In particular, we may take Y to be the colimit of

$$X \xrightarrow{e} X \xrightarrow{e} X \xrightarrow{e} X \xrightarrow{e} \cdots.$$

write proof down?

Proof. See [11, Proposition 1.6.8] $\square$

A.3. **Adjointly triangulated categories.** For our purposes, we will always be dealing with triangulated categories with a bit of extra structure, in the following sense:

Definition A.9. An *adjointly triangulated category* $(\mathcal{C}, \Omega, \Sigma, \eta, \varepsilon, \mathcal{D})$ is the data of a triangulated category $(\mathcal{C}, \Sigma, \mathcal{D})$ along with an inverse shift functor $\Omega : \mathcal{C} \to \mathcal{C}$ and natural isomorphisms $\eta : \mathrm{Id}_{\mathcal{C}} \Rightarrow \Sigma\Omega$ and $\varepsilon : \Omega\Sigma \Rightarrow \mathrm{Id}_{\mathcal{C}}$ such that $(\Omega, \Sigma, \eta, \varepsilon)$ forms an adjoint equivalence of $\mathcal{C}$. In other words, η and ε are natural isomorphisms which also are the unit and counit of an adjunction $\Omega \dashv \Sigma$, so they satisfy either of the following "zig-zag identities":

$$\begin{tikzcd} \Omega \arrow[r, "\Omega\eta"] \arrow[dr, equal] & \Omega\Sigma\Omega \arrow[d, "\varepsilon\Omega"] \\ & \Omega \end{tikzcd} \qquad \begin{tikzcd} \Sigma\Omega\Sigma \arrow[d, "\Sigma\varepsilon"'] & \Sigma \arrow[l, "\eta\Sigma"'] \arrow[dl, equal] \\ \Sigma & \end{tikzcd}$$

(Satisfying one implies the other is automatically satisfied, see [12, Lemma 3.2]).

From now on, we will assume that $\mathcal{C}$ is an *adjointly* triangulated category with inverse shift Ω, unit $\eta : \mathrm{Id}_{\mathcal{C}} \Rightarrow \Sigma\Omega$, and counit $\varepsilon : \Omega\Sigma \Rightarrow \mathrm{Id}_{\mathcal{C}}$.

Lemma A.10. *Given a triangle*

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X,$$

it can be shifted to the left to obtain a distinguished triangle

$$\Omega Z \xrightarrow{-\widetilde{h}} X \xrightarrow{f} Y \xrightarrow{\widetilde{\Omega g}} \Sigma\Omega Z,$$

where $\widetilde{h} : \Omega Z \to X$ *is the adjoint of* $h : Z \to \Sigma X$ *and* $\widetilde{\Omega g} : Y \to \Sigma\Omega Z$ *is the adjoint of* $\Omega g : \Omega Y \to \Omega Z$.

Proof. Note that unravelling definitions, then $\widetilde{h}$ and $\widetilde{g}$ are the compositions

$$\widetilde{h} : \Omega Z \xrightarrow{\Omega h} \Omega\Sigma X \xrightarrow{\varepsilon_X} X \qquad \text{and} \qquad \widetilde{\Omega g} : Y \xrightarrow{\eta_Y} \Sigma\Omega Y \xrightarrow{\Sigma\Omega g} \Sigma\Omega Z.$$

Now consider the following diagram:

(3)
$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, equal] & Y \arrow[r, "g"] \arrow[d, equal] & Z \arrow[r, "h"] \arrow[d, "\eta_Z"'] & \Sigma X \arrow[d, equal] \\ X \arrow[r, "f"] & Y \arrow[r, "\widetilde{\Omega g}"] & \Sigma\Omega Z \arrow[r, "\Sigma\widetilde{h}"] & \Sigma X \end{tikzcd}$$

The left square commutes by definition. To see that the middle square commutes, expanding definitions, note it is given by the following diagram:

$$\begin{tikzcd} Y \arrow[rr, "g"] \arrow[d, equal] & & Z \arrow[d, "\eta_Y"] \\ Y \arrow[r, "\eta_Y"] & \Sigma\Omega Y \arrow[r, "\Sigma\Omega g"] & \Sigma\Omega Z \end{tikzcd}$$

and this commutes by naturality of η. To see that the right square commutes, consider the following diagram:

$$\begin{tikzcd} Z \arrow[rr, "h"] \arrow[d, "\eta_Z"'] & & \Sigma X \arrow[d, equal] \arrow[dl, "\eta_{\Sigma X}"'] \\ \Sigma\Omega Z \arrow[r, "\Sigma\Omega h"'] & \Sigma\Omega\Sigma X \arrow[r, "\Sigma\varepsilon_X"'] & \Sigma X \end{tikzcd}$$

By functoriality of Σ, the bottom composition is $\Sigma\widetilde{h}$. The left region commutes by naturality of η. Commutativity of the right region is precisely one of the the zig-zag identities. Hence, since diagram (3) commutes, the vertical arrows are isomorphisms, and the bottom row is distinguished, we have that the top row is distinguished as well by axiom TR0. Then by axiom TR4, since $(f, \widetilde{\Omega g}, \Sigma\widetilde{h})$ is distinguished, so is the triangle

$$\Omega Z \xrightarrow{-\widetilde{h}} X \xrightarrow{f} Y \xrightarrow{\widetilde{\Omega g}} \Sigma\Omega Z. \qquad \square$$

Lemma A.11. *Given a distinguished triangle*

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X,$$

for any $n > 0$, the triangle

$$\Omega^n X \xrightarrow{(-1)^n \Omega^n f} \Omega^n Y \xrightarrow{(-1)^n \Omega^n g} \Omega^n Z \xrightarrow{(-1)^n \Omega^n h} \Omega^n \Sigma X \cong \Sigma\Omega^n X,$$

is distinguished, where the final isomorphism is given by the composition

$$\Omega^n \Sigma X = \Omega^{n-1}\Omega\Sigma X \xrightarrow{\Omega^{n-1}\varepsilon_X} \Omega^{n-1} X \xrightarrow{\eta_{\Omega^{n-1}X}} \Sigma\Omega\Omega^{n-1} X = \Sigma\Omega^n X.$$

Proof. We give a proof by induction. First we show the case $n = 1$. Note by Lemma A.10, we have that given a distinguished triangle

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X,$$

we can shift it to the left to obtain a distinguished triangle

$$\Omega Z \xrightarrow{-\widetilde{h}} X \xrightarrow{f} Y \xrightarrow{\widetilde{\Omega g}} \Sigma\Omega Z,$$

where $\widetilde{h}$ is the adjoint of $h : Z \to \Sigma X$ and $\widetilde{\Omega g}$ is the adjoint of $\Omega g : \Omega Y \to \Omega Z$. If we apply this shifting operation again, we get the distinguished triangle

$$\Omega Y \xrightarrow{-\widetilde{\widetilde{\Omega g}}} \Omega Z \xrightarrow{-\widetilde{h}} X \xrightarrow{\widetilde{\Omega f}} \Sigma\Omega Y,$$

where unravelling definitions, $\widetilde{\Omega f}$ is the right adjoint of $\Omega f : \Omega X \to \Omega Y$ and $\widetilde{\widetilde{\Omega g}}$ is the right adjoint of $\widetilde{\Omega g}$, which itself is the left adjoint of Ωg, so $\widetilde{\widetilde{\Omega g}} = \Omega g$. Hence we have a distinguished triangle

$$\Omega Y \xrightarrow{-\Omega g} \Omega Z \xrightarrow{-\widetilde{h}} X \xrightarrow{\widetilde{\Omega f}} \Sigma\Omega Y.$$

We may again shift this triangle again and the above arguments yield the distinguished triangle

$$\Omega X \xrightarrow{-\Omega f} \Omega Y \xrightarrow{-\Omega g} \Omega Z \xrightarrow{\widetilde{\Omega(-\widetilde{h})}} \Sigma\Omega X,$$

where $\widetilde{\Omega(-\widetilde{h})}$ is the right adjoint of $\Omega(-\widetilde{h}) = -\Omega\widetilde{h} : \Omega\Omega Z \to \Omega X$. Explicitly unravelling definitions, $\widetilde{\Omega(-\widetilde{h})} = -\widetilde{\Omega\widetilde{h}}$ is the composition

$$\begin{aligned}
[\Omega Z \xrightarrow{\eta_{\Omega Z}} \Sigma\Omega\Omega Z \xrightarrow{\Sigma(-\Omega\widetilde{h})} \Sigma\Omega X] &= -[\Omega Z \xrightarrow{\eta_{\Omega Z}} \Sigma\Omega\Omega Z \xrightarrow{\Sigma\Omega\widetilde{h}} \Sigma\Omega X] \\
&= -[\Omega Z \xrightarrow{\eta_{\Omega Z}} \Sigma\Omega\Omega Z \xrightarrow{\Sigma\Omega\Omega h} \Sigma\Omega\Omega\Sigma X \xrightarrow{\Sigma\Omega\varepsilon_X} \Sigma\Omega X] \\
&= -[\Omega Z \xrightarrow{\Omega h} \Omega\Sigma X \xrightarrow{\varepsilon_X} X \xrightarrow{\eta_X} \Sigma\Omega X],
\end{aligned}$$

where the first equality follows by additivity of Σ and additivity of composition, the second follows by further unravelling how $\widetilde{h}$ is defined, and the third follows by naturality of η, which tells us the following diagram commutes:

$$\begin{tikzcd}
\Omega Z \arrow[r, "\Omega h"] \arrow[d, "\eta_{\Omega Z}"] & \Omega\Sigma X \arrow[r, "\varepsilon_X"] \arrow[d, "\eta_{\Omega\Sigma X}"] & X \arrow[d, "\eta_X"] \\
\Sigma\Omega\Omega Z \arrow[r, "\Sigma\Omega\Omega h"'] & \Sigma\Omega\Omega\Sigma X \arrow[r, "\Sigma\Omega\varepsilon_X"'] & \Sigma\Omega X
\end{tikzcd}$$

Thus indeed we have a distinguished triangle

$$\Omega X \xrightarrow{-\Omega f} \Omega Y \xrightarrow{-\Omega g} \Omega Z \xrightarrow{-\Omega h} \Omega\Sigma X \cong \Sigma\Omega X,$$

where the last isomorphism is $\eta_X \circ \varepsilon_X$, as desired.

Now, we show the inductive step. Suppose we know that given a distinguished triangle

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X,$$

that for some $n > 0$ the triangle

$$\Omega^n X \xrightarrow{(-1)^n\Omega^n f} \Omega^n Y \xrightarrow{(-1)^n\Omega^n g} \Omega^n Z \xrightarrow{(-1)^n h^n} \Sigma\Omega^n X,$$

is distinguished, where $h^n : \Omega^n Z \to \Sigma\Omega^n X$ is the composition

$$\Omega^n Z \xrightarrow{\Omega^n h} \Omega^n\Sigma X \xrightarrow{\Omega^{n-1}\varepsilon_X} \Omega^{n-1} X \xrightarrow{\eta_{\Omega^{n-1}X}} \Sigma\Omega^n X.$$

Then by applying the $n = 1$ case to this triangle, we get that the following triangle is distinguished

$$\Omega^{n+1} X \xrightarrow{-\Omega((-1)^n\Omega^n f)} \Omega^{n+1} Y \xrightarrow{-\Omega((-1)^n\Omega^n g)} \Omega^{n+1} Z \xrightarrow{-\Omega((-1)^n h^n)} \Omega\Sigma\Omega^n X \cong \Sigma\Omega^{n+1} X,$$

where the final isomorphism is the composition

$$\Omega\Sigma\Omega^n X \xrightarrow{\varepsilon_{\Omega^n X}} \Omega^n X \xrightarrow{\eta_{\Omega^n X}} \Sigma\Omega\Omega^n X = \Sigma\Omega^{n+1} X.$$

We claim that this is precisely the distinguished triangle given in the statement of the lemma for $n+1$. First of all, note that $-\Omega((-1)^n\Omega^n f) = (-1)^{n+1}\Omega^{n+1} f$, $-\Omega((-1)^n\Omega^n g) = (-1)^{n+1}\Omega^{n+1} g$, and $-\Omega((-1)^n h^n) = (-1)^{n+1}\Omega h^n$ by additivity of Ω, so that the triangle becomes

$$\Omega^{n+1} X \xrightarrow{(-1)^{n+1}\Omega^{n+1} f} \Omega^{n+1} Y \xrightarrow{(-1)^{n+1}\Omega^{n+1} g} \Omega^{n+1} Z \xrightarrow{(-1)^{n+1}\Omega h^n} \Omega\Sigma\Omega^n X \cong \Sigma\Omega^{n+1} X. \tag{4}$$

Thus, in order to prove the desired characterization, it remains to show this diagram commutes:

$$\begin{tikzcd}
\Omega^{n+1} Z \arrow[r, "(-1)^{n+1}\Omega h^n"] \arrow[d, "(-1)^{n+1}\Omega^{n+1} h"'] & \Omega\Sigma\Omega^n X \arrow[r, "\varepsilon_{\Omega^n X}"] & \Omega^n X \arrow[d, "\eta_{\Omega^n X}"] \\
\Omega^{n+1}\Sigma X \arrow[r, "\Omega^n\varepsilon_X"'] & \Omega^n X \arrow[r, "\eta_{\Omega^n X}"'] & \Sigma\Omega^{n+1} X
\end{tikzcd}$$

(The top composition is the last two arrows in diagram (4), and the bottom composition is the last two arrows in the diagram in the statement of the lemma). Unravelling how h^n is constructed, by additivity of Ω it further suffices to show the outside of the following diagram commutes:

$$\begin{tikzcd}
\Omega^{n+1} Z \arrow[r, "(-1)^{n+1}\Omega^{n+1} h"] \arrow[dd, "(-1)^{n+1}\Omega^{n+1} h"'] & \Omega^{n+1}\Sigma X \arrow[r, "\Omega^n\varepsilon_X"] & \Omega^n X \arrow[r, "\Omega\eta_{\Omega^{n-1}X}"] \arrow[dd, equal] & \Omega\Sigma\Omega^n X \arrow[d, "\varepsilon_{\Omega^n X}"] \\
& & & \Omega^n X \arrow[d, "\eta_{\Omega^n X}"] \\
\Omega^{n+1}\Sigma X \arrow[rr, "\Omega^n\varepsilon_X"'] & & \Omega^n X \arrow[r, "\eta_{\Omega^n X}"'] \arrow[ur, equal] & \Sigma\Omega^{n+1} X
\end{tikzcd}$$

The left rectangle and bottom right triangle commute by definition. Finally, commutativity of the top right trapezoid is precisely one of the zig-zag identities applied to $\Omega^{n-1}X$. Hence, we have shown the desired result. $\square$

Proposition A.12. *Given a distinguished triangle*

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X,$$

let $\widetilde{h} : \Omega Z \to X$ be the left adjoint of h. Then the following infinite sequence is exact:

$$\begin{tikzcd}
 & & \cdots \arrow[dll] & & \\
\Omega^{n+1}Z \arrow[r, "(-1)^{n+1}\Omega^n\widetilde{h}"'] & \Omega^n X \arrow[r, "(-1)^n\Omega^n f"'] & \Omega^n Y \arrow[r, "(-1)^n\Omega^n g"] & \Omega^n Z \arrow[r, "(-1)^n\Omega^{n-1}\widetilde{h}"] & \Omega^{n-1}X \arrow[dll] \\
 & & \cdots \arrow[dll] & & \\
\Omega Z \arrow[r, "-\widetilde{h}"'] & X \arrow[r, "f"'] & Y \arrow[r, "g"] & Z \arrow[r, "h"] & \Sigma X \arrow[dll] \\
 & & \cdots \arrow[dll] & & \\
\Sigma^{n-1}Z \arrow[r, "(-1)^{n-1}\Sigma^n h"'] & \Sigma^n X \arrow[r, "(-1)^n\Sigma^n f"'] & \Sigma^n Y \arrow[r, "(-1)^n\Sigma^n g"] & \Sigma^n Z \arrow[r, "(-1)^n\Sigma^n h"] & \Sigma^{n+1}X \arrow[dll] \\
 & & \cdots & &
\end{tikzcd}$$

In particular, it remains exact even if we remove the signs.

Proof. Exactness of

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X \xrightarrow{-\Sigma f} \Sigma Y$$

is Proposition A.3 and axiom TR4. By induction using axiom TR4, for $n > 0$ we get that each contiguous composition of three maps below is a distinguished triangle:

$$\Sigma^n X \xrightarrow{(-1)^n\Sigma^n f} \Sigma^n Y \xrightarrow{(-1)^n\Sigma^n g} \Sigma^n Z \xrightarrow{(-1)^n\Sigma^n h} \Sigma^{n+1} X \xrightarrow{(-1)^{n+1}\Sigma^{n+1} f} \Sigma^{n+1} Y,$$

thus the sequence is exact by Proposition A.3. It remains to show exactness of the LES to the left of Y. It suffices to show that the row in the following diagram is exact for all $n > 0$:

$$\begin{tikzcd}
\Omega^n X \arrow[r, "(-1)^n\Omega^n f"] & \Omega^n Y \arrow[r, "(-1)^n\Omega^n g"] & \Omega^n Z \arrow[rr, "(-1)^n\Omega^{n-1}(\varepsilon_X\circ\Omega h)"] \arrow[dr, "(-1)^n\Omega^n h"'] & & \Omega^{n-1}X \arrow[r, "(-1)^{n-1}\Omega^{n-1}f"] & \Omega^{n-1}Y \\
 & & & \Omega^n\Sigma X \arrow[ur, "\Omega^{n-1}\varepsilon_X"'] & &
\end{tikzcd} \tag{5}$$

First of all, to see exactness at $\Omega^n Y$ and $\Omega^n Z$, consider the following commutative diagram:

$$\begin{tikzcd}
\Omega^n X \arrow[r, "(-1)^n\Omega^n f"] \arrow[dd, equal] & \Omega^n Y \arrow[r, "(-1)^n\Omega^n g"] \arrow[dd, equal] & \Omega^n Z \arrow[rr, "(-1)^n\Omega^{n-1}(\varepsilon_X\circ\Omega h)"] \arrow[dd, equal] \arrow[dr, dashed, "(-1)^n\Omega^n h"'] & & \Omega^{n-1}X \arrow[dd, "\eta_{\Omega^{n-1}X}"] \\
 & & & \Omega^n\Sigma X \arrow[ur, dashed, "\Omega^{n-1}\varepsilon_X"'] & \\
\Omega^n X \arrow[r, "(-1)^n\Omega^n f"] & \Omega^n Y \arrow[r, "(-1)^n\Omega^n g"] & \Omega^n Z \arrow[rr, dashed] \arrow[ur, dashed, "(-1)^n\Omega^n h"] & & \Sigma\Omega^n X
\end{tikzcd}$$

(here the dashed arrow is the morphism which makes the diagram commute). The bottom row is distinguished by Lemma A.11. Then by axiom TR0, the top row is distinguished, and thus exact by Proposition A.3. Thus we have shown exactness of (5) at $\Omega^n Y$ and $\Omega^n Z$. It remains to show exactness at $\Omega^{n-1}X$. In the case $n=1$, we want to show exactness at X in the following diagram:

$$\begin{tikzcd}
\Omega Z \arrow[rr, "-(\varepsilon_X \circ \Omega h)"] \arrow[dr, "-\Omega h"'] & & X \arrow[r, "f"] & Y \\
& \Omega\Sigma X \arrow[ur, "\varepsilon_X"'] & &
\end{tikzcd}$$

Unravelling definitions, $\varepsilon_X \circ \Omega h$ is precisely the adjoint $\widetilde{h} : \Omega Z \to X$ of $h : Z \to \Sigma X$, in which case we have that the row in the above diagram fits into a distinguished triangle by Lemma A.10, and thus it is exact by Proposition A.3. To see exactness at $\Omega^{n-1}X$ in diagram (5), note that if we apply Lemma A.10 to the sequence Lemma A.11 for $n-1$, then we get that the following composition fits into a distinguished triangle, and is thus exact:

$$\Omega^n Z \xrightarrow{-k} \Omega^{n-1}X \xrightarrow{(-1)^{n-1}\Omega^{n-1}f} \Omega^{n-1}Y,$$

where $k : \Omega(\Omega^{n-1}Z) \to \Omega^{n-1}X$ is the adjoint of the composition

$$\Omega^{n-1}Z \xrightarrow{(-1)^{n-1}\Omega^{n-1}h} \Omega^{n-1}\Sigma X \xrightarrow{\Omega^{n-2}\varepsilon_X} \Omega^{n-2}X \xrightarrow{\eta_{\Omega^{n-2}X}} \Sigma\Omega^{n-1}X.$$

Further expanding how adjoints are constructed, k is the composition

$$\Omega^n Z \xrightarrow{(-1)^{n-1}\Omega^n h} \Omega^n\Sigma X \xrightarrow{\Omega^{n-1}\varepsilon_X} \Omega^{n-1}X \xrightarrow{\Omega\eta_{\Omega^{n-2}X}} \Omega\Sigma\Omega^{n-1}X \xrightarrow{\varepsilon_{\Omega^{n-1}X}} \Omega^{n-1}X.$$

Thus, in order to show exactness of (5) at $\Sigma^{n-1}X$, it suffices to show that $k = (-1)^{n-1}\Omega^{n-1}(\varepsilon_X \circ \Omega h)$. To that end, consider the following diagram:

$$\begin{tikzcd}
\Omega^n Z \arrow[r, "(-1)^{n-1}\Omega^n h"] \arrow[d, "(-1)^{n-1}\Omega^n h"'] & \Omega^n\Sigma X \arrow[r, "\Omega^{n-1}\varepsilon_X"] & \Omega^{n-1}X \arrow[r, "\Omega\eta_{\Omega^{n-2}X}"] \arrow[dr, equal] & \Omega\Sigma\Omega^{n-1}X \arrow[d, "\varepsilon_{\Omega^{n-1}X}"] \\
\Omega^n\Sigma X \arrow[rrr, "\Omega^{n-1}\varepsilon_X"'] & & & \Omega^{n-1}X
\end{tikzcd}$$

The top composition is k, while the bottom composition is $(-1)^{n-1}\Omega^{n-1}(\varepsilon_X \circ \Omega h)$. The left region commutes by definition, while commutativity of the right region is precisely one of the zig-zag identities applied to $\Omega^{n-2}X$. Thus, we have shown that $-k = (-1)^n\Omega^{n-1}(\varepsilon_X \circ \Omega h)$, so (5) is exact at $\Omega^{n-1}X$, as desired. □

A.4. **Tensor triangulated categories.** Also important for our work is the concept of a *tensor triangulated category*, that is, a triangulated symmetric monoidal category in which the triangulated structures are compatible, in the following sense:

Definition A.13. A *tensor triangulated category* is a triangulated symmetric monoidal category $(\mathcal{C}, \otimes, S, \Sigma, \mathcal{D})$ such that:

TT1 For all objects X and Y in $\mathcal{C}$, there are natural isomorphisms

$$e_{X,Y} : \Sigma X \otimes Y \xrightarrow{\cong} \Sigma(X \otimes Y).$$

TT2 For each object X in $\mathcal{C}$, the functor $X \otimes (-) \cong (-) \otimes X$ is an additive functor.

TT3 For each object X in $\mathcal{C}$, the functor $X \otimes (-) \cong (-) \otimes X$ preserves distinguished triangles, in that given a distinguished triangle/(co)fiber sequence

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A,$$

then also

$$X \otimes A \xrightarrow{X \otimes f} X \otimes B \xrightarrow{X \otimes g} X \otimes C \xrightarrow{X \otimes' h} \Sigma(X \otimes A)$$

and

$$A \otimes X \xrightarrow{f \otimes X} B \otimes X \xrightarrow{g \otimes X} C \otimes X \xrightarrow{h \otimes' X} \Sigma(A \otimes X)$$

are distinguished triangles, where here we writing $X \otimes' h$ and $h \otimes' X$ to denote the compositions

$$X \otimes C \xrightarrow{X \otimes h} X \otimes \Sigma A \xrightarrow{\tau} \Sigma A \otimes X \xrightarrow{e_{A,X}} \Sigma(A \otimes X) \xrightarrow{\Sigma\tau} \Sigma(X \otimes A)$$

and

$$C \otimes X \xrightarrow{h \otimes X} \Sigma A \otimes X \xrightarrow{e_{A,X}} \Sigma(A \otimes X),$$

respectively.

TT4 Given objects X, Y, and Z in $\mathcal{C}$, the following diagram must commute:

$$\begin{tikzcd}
(\Sigma X \otimes Y) \otimes Z \arrow[r, "e_{X,Y} \otimes Z"] \arrow[d, "\alpha"'] & \Sigma(X \otimes Y) \otimes Z \arrow[r, "e_{X \otimes Y, Z}"] & \Sigma((X \otimes Y) \otimes Z) \arrow[d, "\Sigma\alpha"] \\
\Sigma X \otimes (Y \otimes Z) \arrow[rr, "e_{X, Y \otimes Z}"'] & & \Sigma(X \otimes (Y \otimes Z))
\end{tikzcd}$$

Usually, most tensor triangulated categories that arise in nature will satisfy additional coherence axioms (see axioms TC1–TC5 in [9]), but the above definition will suffice for our purposes. In what follows, we fix a tensor triangulated category $(\mathcal{C}, \otimes, S, \Sigma, e, \mathcal{D})$.

Definition A.14. There are natural isomorphisms

$$e'_{X,Y} : X \otimes \Sigma Y \xrightarrow{\cong} \Sigma(X \otimes Y)$$

obtained via the composition

$$X \otimes \Sigma Y \xrightarrow{\tau} \Sigma Y \otimes X \xrightarrow{e_{Y,X}} \Sigma(Y \otimes X) \xrightarrow{\Sigma\tau} \Sigma(X \otimes Y).$$

Lemma A.15. *Let $A \xrightarrow{a} B \xrightarrow{b} C \xrightarrow{c} D$ be any sequence isomorphic to a distinguished triangle. Then given any E in $\mathcal{C}$, the sequences*

$$E \otimes A \xrightarrow{E \otimes a} E \otimes B \xrightarrow{E \otimes b} E \otimes C \xrightarrow{E \otimes c} E \otimes D$$

and

$$A \otimes E \xrightarrow{a \otimes E} B \otimes E \xrightarrow{b \otimes E} C \otimes E \xrightarrow{c \otimes E} D \otimes E$$

are exact.

Proof. Since (a, b, c) is isomorphic to a distinguished triangle, there exists a commuting diagram in $\mathcal{SH}$

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\alpha"'] & Y \arrow[r, "g"] \arrow[d, "\beta"'] & Z \arrow[r, "h"] \arrow[d, "\gamma"'] & \Sigma X \arrow[d, "\delta"'] \\
A \arrow[r, "a"'] & B \arrow[r, "b"'] & C \arrow[r, "c"'] & D
\end{tikzcd}$$

where the top row is distinguished and the vertical arrows are isomorphisms. Then the following diagram commutes by functoriality of $-\otimes-$:

$$\begin{tikzcd}
E\otimes X \arrow[r, "E\otimes f"] \arrow[d, "E\otimes\alpha"'] & E\otimes Y \arrow[r, "E\otimes g"] \arrow[d, "E\otimes\beta"'] & E\otimes Z \arrow[rr, "E\otimes' h"] \arrow[d, "E\otimes\gamma"'] \arrow[dr, "E\otimes h"] & & \Sigma(E\otimes X) \arrow[d, dashed, "(E\otimes\delta)\circ(e'_{E,X})^{-1}"] \\
 & & & E\otimes\Sigma X \arrow[ur, "e'_{E,X}"] \arrow[dr, "E\otimes\delta"] & \\
E\otimes A \arrow[r, "E\otimes a"'] & E\otimes B \arrow[r, "E\otimes b"'] & E\otimes C \arrow[rr, "E\otimes c"'] & & E\otimes D
\end{tikzcd}$$

The top triangle is distinguished by axiom TT3 for a tensor triangulated category, thus exact by Proposition A.3, so that the bottom triangle is also exact since the vertical arrows are isomorphisms and each square commutes. Similarly, the following diagram also commutes by functoriality of $-\otimes-$:

$$\begin{tikzcd}
X\otimes E \arrow[r, "f\otimes E"] \arrow[d, "\alpha\otimes E"'] & Y\otimes E \arrow[r, "g\otimes E"] \arrow[d, "\beta\otimes E"'] & Z\otimes E \arrow[rr, "h\otimes' E"] \arrow[d, "\gamma\otimes E"'] \arrow[dr, "h\otimes E"] & & \Sigma(X\otimes E) \arrow[d, "(\delta\otimes E)\circ e_{X,E}^{-1}"] \\
 & & & \Sigma X\otimes E \arrow[ur, "e_{X,E}"] \arrow[dr, "\delta\otimes E"] & \\
A\otimes E \arrow[r, "a\otimes E"'] & B\otimes E \arrow[r, "b\otimes E"'] & C\otimes E \arrow[rr, "c\otimes E"'] & & D\otimes E
\end{tikzcd}$$

The top row is distinguished by axiom TT3 for a tensor triangulated category, thus exact by Proposition A.3, so that the bottom triangle is also exact since the vertical arrows are isomorphisms and each square commutes. □

Proposition A.16. *Suppose we have a distinguished triangle*

$$X \to Y \to Z \to \Sigma X$$

in $\mathcal{C}$. Then given any object E in $\mathcal{C}$, the long exact sequence given in Proposition A.12 remains exact after applying $E\otimes -$ or $-\otimes E$.

Proof. Recall that in the proof of Proposition A.12 we showed that the sequence was exact by showing that any two consecutive maps were isomorphic to a part of a distinguished triangle. Then the desired result follows from Lemma A.15. □

Definition A.17. An *adjointly tensor triangulated* category is a tensor triangulated category $(\mathcal{C}, \otimes, S, \Sigma, e, \mathcal{D})$ along with the structure of an adjointly triangulated category $(\mathcal{C}, \Omega, \Sigma, \eta, \varepsilon, \mathcal{D})$.

From now on, we fix an adjointly tensor triangulated category $(\mathcal{C}, \otimes, S, \Omega, \Sigma, \eta, \varepsilon, e, \mathcal{D})$.

Definition A.18. We may define natural isomorphisms $o_{X,Y} : \Omega X\otimes Y \xrightarrow{\cong} \Omega(X\otimes Y)$ and $o'_{X,Y} : X\otimes\Omega Y \xrightarrow{\cong} \Omega(X\otimes Y)$ as the compositions

$$o_{X,Y} : \Omega X\otimes Y \xrightarrow{\varepsilon^{-1}_{\Omega X\otimes Y}} \Omega\Sigma(\Omega X\otimes Y) \xrightarrow{\Omega e^{-1}_{\Omega X,Y}} \Omega(\Sigma\Omega X\otimes Y) \xrightarrow{\Omega(\eta_X^{-1}\otimes Y)} \Omega(X\otimes Y)$$

and

$$o'_{X,Y} : X\otimes\Omega Y \xrightarrow{\tau_{X,\Omega Y}} \Omega Y\otimes X \xrightarrow{o_{Y,X}} \Omega(Y\otimes X) \xrightarrow{\Omega\tau_{Y,X}} \Omega(X\otimes Y).$$

These are both clearly natural by naturality of ε, e, η, and τ.

Appendix B. Spectral sequences

In what follows, we fix an abelian group A. We will freely use the theory and results of Appendix C

Definition B.1. An *A-graded spectral sequence* $(E_r, d_r)_{r \geq r_0}$ is the data of:

- A collection of A-graded abelian groups $\{E_r^*\}_{r \geq r_0}$
- A collection of A-graded homomorphisms $d_r : E_r \to E_r$ for $r \geq r_0$ (of possibly nonzero degree) such that $d_r \circ d_r = 0$
- For each $r \geq r_0$, an A-graded isomorphism $E_{r+1} \cong \ker d_r / \operatorname{im} d_r$ of degree 0 (where $\ker d_r$ and $\operatorname{im} d_r$ are canonically A-graded by Proposition C.14, and their quotient is canonically A-graded by Proposition C.16.

Typically we call a $\mathbb{Z}^2$-graded spectral sequence a *bigraded* spectral sequence, and a $\mathbb{Z}^3$-graded spectral sequence is a *trigraded* spectral sequence.

B.1. **Unrolled exact couples and their associated spectral sequences.**

For our purposes, we will only care about spectral sequences which arise from *A-graded unrolled exact couples*. In what follows, we follow [2], with minor modifications for our setting, in which everything is A-graded.

Definition B.2. An A-graded *unrolled exact couple* $(D, E; i, j, k)$ is a diagram of A-graded abelian groups and A-graded homomorphisms (of possibly non-zero degree)

$$\begin{tikzcd}
\cdots \arrow[r] & D^{s+2} \arrow[r, "i"] \arrow[d, "j"] & D^{s+1} \arrow[r, "i"] \arrow[d, "j"] & D^{s} \arrow[r, "i"] \arrow[d, "j"] & D^{s-1} \arrow[r] \arrow[d, "j"] & \cdots \\
& E^{s+2} & E^{s+1} \arrow[ul, "k"] & E^{s} \arrow[ul, "k"] & E^{s-1} \arrow[ul, "k"] &
\end{tikzcd}$$

in which each triangle $D^{s+1} \xrightarrow{i} D^s \xrightarrow{j} E_s \xrightarrow{k} D^{s+1}$ is an exact sequence. We require that each occurrence of i (resp. j, k) is of the same degree. In other words, an unrolled exact couple can be described as a tuple $(D, E; i, j, k)$ of $\mathbb{Z} \times A$-graded abelian groups and $\mathbb{Z} \times A$-graded maps $i : D \to D$, $j : D \to E$, and $k : E \to D$, such that the $\mathbb{Z}$-degrees of i, j, and k are -1, 0, and 1, respectively. Usually i and one of j or k will be of A-degree 0.

Given an A-graded unrolled exact couple $(D, E; i, j, k)$, we may define an associated $\mathbb{Z} \times A$-graded spectral sequence as follows: Given some $s \in \mathbb{Z}$ and some $r \geq 1$, we first define the following subgroups of E_s:

$$Z_r^s := k^{-1}(\operatorname{im}[i^{r-1} : D^{s+r} \to D^{s+1}]) \qquad \text{and} \qquad B_r^s := j(\ker[i^{r-1} : D^s \to D^{s-r+1}])$$

where we adopt the convention that i^0 is simply the identity. These are furthermore A-graded subgrous of E_s (by Proposition C.14 and Proposition C.15). In this way, for each $s \in \mathbb{Z}$, we get an infinite sequence of A-graded subgroups:

$$0 = B_1^s \subseteq B_2^s \subseteq B_3^s \subseteq \cdots \subseteq \operatorname{im} j = \ker k \subseteq \cdots \subseteq Z_3^s \subseteq Z_2^s \subseteq Z_1^s = E^s.$$

Now, for each $s \in \mathbb{Z}$ and $r \geq 1$, we define the A-graded abelian group

$$E_r^s := Z_r^s / B_r^s,$$

so that in particular $E_1^s = E^s$ for all $s \in \mathbb{Z}$, as $Z_1^s = k^{-1}(D^{s+1}) = E^s$ and $B_1^s = j(\ker \operatorname{id}_{D^s}) = j(0) = 0$. Now we can define differentials $d_r^s : E_r^s \to E_r^{s+r}$ to be the composition

$$E_r^s = Z_r^s / B_r^s \xrightarrow{k} \operatorname{im}[i^{r-1} : D^{s+r} \to D^{s+1}] \xrightarrow{j \circ i^{-(r-1)}} Z_r^{s+r} / B_r^{s+r} = E_r^{s+r},$$

where given some $e \in Z_r^s = k^{-1}(\operatorname{im} i^{r-1})$, the first arrow takes a class $[e] \in E_r^s$ represented by some $e \in Z_r^s$ to the element $k(e)$, which lives in $\operatorname{im} i^{r-1}$ by definition, and the second arrow takes $i^{r-1}(d)$ to the class $[j(d)]$. Note the first map is well-defined, as given $b \in B_r^s = j(\ker[i^{r-1}])$, we have $k(b) = 0$, as $b \in \operatorname{im} j = \ker k$. To see the second map is well-defined, first note that given $d \in D^{s+r}$, that

$$k(j(d)) = 0 \in \operatorname{im}[i^{r-1} : D^{s+2r} \to D^{s+r+1}],$$

so that

$$j(d) \in k^{-1}(\operatorname{im}[i^{r-1} : D^{s+2r} \to D^{s+r+1}]) = Z_r^{s+r},$$

as desired, so that given $d \in D^{s+r}$, $j(d) in Z_r^{s+r}$, so it makes sense to discuss the class $[j(d)] \in Z_r^{s+r}/B_r^{s+r} = E_r^{s+r}$. Secondly, if $i^{r-1}(d) = i^{r-1}(d')$ for some $d, d' \in D^{s+r}$, then

$$j(d) - j(d') = j(d - d') \in j(\ker[i^{r-1} : D^{s+r} \to D^{s+1}]) = B_r^{s+r},$$

so that $[j(d)] = [j(d')]$ in E_r^{s+r}, as desired. It is straightforward to check that these maps are also A-graded homomorphisms, so that by unravelling definitions d_r^s is an A-graded homomorphism of degree $\deg k - (r-1) \cdot \deg i + \deg j$ (so that in the standard case $\deg i = 0$, d_r^s simply has degree $\deg k + \deg j$).

These differentials square to zero, in the sense that for each $s \in \mathbb{Z}$ and $r \geq 1$ we have that $d_r^{s+r} \circ d_r^s : E_r^s \to E_r^{s+2r}$ is the zero map. Indeed, suppose we are given some class $[e] \in E_r^s$ represented by an element $e \in E^s$, so $k(e) = i^{r-1}(d)$ for some $d \in D^{s+r}$. Then

$$d_r^{s+r}(d_r^s([e])) = d_r^{s+r}([j(d)]) = [j(i^{-(r-1)}(k(j(d))))] = [j(i^{-(r-1)}(0))] = 0,$$

where the second-to-last equality follows by the fact that $k \circ j = 0$. Note that by unravelling definitions, $d_1^s = j \circ k$.

We claim that $\ker d_r^s = Z_{r+1}^s/B_r^s$. First of all, let $[e] \in E_r^s = Z_r^s/B_r^s$, so that $[e]$ is represented by some $e \in E^s$ with $k(e) = i^{r-1}(d)$ for some $d \in D^{s+r}$. Then if $[e] \in \ker d_r^s$, by definition this means $j(d) \in B_r^{s+r} = j(\ker[i^{r-1} : D^{s+r} \to D^{s+1}])$, so $j(d) = j(d')$ for some $d' \in D^{s+r}$ with $i^{r-1}(d') = 0$. Thus $d - d' \in \ker j = \operatorname{im} i$, so there exists some $d'' \in D^{s+r+1}$ such that $i(d'') = d - d'$. Then

$$k(e) = i^{r-1}(d) = i^{r-1}(i(d'') + d') = i^r(d'') + i^{r-1}(d'),$$

but since $i^{r-1}(d') = 0$, we have $k(e) \in \operatorname{im}[i^r : D^{s+r+1} \to D^{s+1}]$, so that $e \in Z_{r+1}^s$, meaning $[e] \in Z_{r+1}^s/B_r^s$, as desired. On the other hand, suppose we are given some class $[e] \in Z_{r+1}^s/B_r^s$, represented by $e \in Z_{r+1}^s$ with $k(e) \in \operatorname{im}[i^r : D^{s+r+1} \to D^{s+1}]$. Then if we write $k(e) = i^r(d) = i^{r-1}(i(d))$, then $d_r^s([e]) = [j(i(d))] = 0$ (since $j \circ i = 0$), as asserted.

Finally, we claim that the image of $d_r^{s-r} : E_r^{s-r} \to E_r^s$ is B_{r+1}^s/B_r^s. First, let $e \in Z_r^{s-r}$, so $k(e) = i^{r-1}(d)$ for some $d \in D^s$. Then we'd like to show that $d_r^s([e]) = [j(d)]$ belongs to B_{r+1}^s/B_r^s. It suffices to show that $d \in \ker[i^r : D^s \to D^{s-r}]$. To see this, note that

$$i^r(d) = i(i^{r-1}(d)) = i(k(e)) = 0,$$

since $i \circ k = 0$. Hence we've shown $\operatorname{im} d_r^{s-r} \subseteq B_{r+1}^s/B_r^s$. Conversely, let $j(d) \in B_{r+1}^s$, so $d \in D^s$ and $i^r(d) = 0$. Then we'd like to show that $[j(d)] \in B_{r+1}^s/B_r^s$ is in the image of d_r^{s-r}. To see this, note that

$$i^r(d) = 0 \implies i^{r-1}(d) \in \ker i = \operatorname{im} k,$$

so there exists some $e \in E^{s-r}$ such that $k(e) = i^{r-1}(d)$, so $e \in Z_r^{s-r}$. Unravelling definitions, it follows that $d_r^{s-r}([e]) = [j(d)]$, so $[j(d)]$ is indeed in the image of d_r^{s-r}, as desired.

To recap, we have constructed for each $s \in \mathbb{Z}$ and $r \geq 1$ an A-graded abelian group E_r^s along with differentials $d_r^s : E_r^s \to E_r^{s+r}$. Furthermore, if we take homology in the middle term of the following sequence

$$E_r^{s-r} \xrightarrow{d_r^{s-r}} E_r^s \xrightarrow{d_r^s} E_r^{s+r},$$

we get

$$\ker d_r^s / \operatorname{im} d_r^{s-r} = \frac{Z_{r+1}^s / B_r^s}{B_{r+1}^s / B_r^s} \cong Z_{r+1}^s / B_{r+1}^s = E_{r+1}^s.$$

Thus, we get a spectral sequence:

Proposition B.3. *We may associate a $\mathbb{Z} \times A$-graded spectral sequence $r \mapsto (E_r, d_r)$ to the A-graded unrolled exact couple $(D, E; i, j, k)$ by defining $E_r := \bigoplus_{s \in \mathbb{Z}} E_r^s$ and the differentials*

$$d_r : E_r \to E_r$$

are those constructed above, which have $\mathbb{Z} \times A$-degree $(r, \deg j - (r-1) \cdot \deg i + \deg k)$.

Appendix C. A-graded abelian groups, rings, and modules

In what follows, we fix an abelian group A. We assume the reader is familiar with the basic theory of modules over not-necessarily-commutative rings.

Definition C.1. An *A-graded abelian group* is an abelian group B along with a subgroup $B_a \leq B$ for each $a \in A$ such that the canonical map

$$\bigoplus_{a \in A} B_a \to B$$

sending $(x_a)_{a \in A}$ to $\sum_{a \in A} x_a$ is an isomorphism. Given two A-graded abelian groups B and C, a homomorphism $f : B \to C$ is a *homomorphism of A-graded abelian groups*, or just an *A-graded homomorphism*, if it preserves the grading, i.e., if it restricts to a map $B_a \to C_a$ for all $a \in A$.

It is easy to see that an A-graded abelian group B is generated by its *homogeneous* elements, that is, nonzero elements $x \in B$ such that there exists some $a \in A$ with $x \in B_a$.

Remark C.2. Clearly the condition that the canonical map $\bigoplus_{a \in A} B_a \to B$ is an isomorphism requires that $B_a \cap B_b = 0$ if $a \neq b$. In particular, given a homogeneous element $x \in B$, there exists precisely one $a \in A$ such that $x \in B_a$. We call this a the *degree* of x, and we write $|x| = a$.

Definition C.3. An *A-graded ring* is a ring R such that its underlying abelian group R is A-graded and the multiplication map $R \times R \to R$ restricts to $R_a \times R_b \to R_{a+b}$ for all $a, b \in A$. A morphism of A-graded rings is a ring homomorphism whose underlying homomorphism of abelian groups is A-graded.

Explicitly, given an A-graded ring R and homogeneous elements $x, y \in R$, we must have $|xy| = |x| + |y|$. For example, given some field k, the ring $R = k[x, y]$ is $\mathbb{Z}^2$-graded, where given $(n, m) \in \mathbb{Z}^2$, $R_{n,m}$ is the subgroup of those monomials of the form $ax^n y^m$ for some $a \in k$.

Definition C.4. Let R be an A-graded ring. A *left A-graded R-module* M is a left R-module M such that M is an A-graded abelian group and the action map $R \times M \to M$ restricts to a map $R_a \times M_b \to M_{a+b}$ for all $a, b \in A$. Right A-graded R-modules are defined similarly. Finally, an A-graded R-bimodule is an A-graded abelian group M which has the structure of both an A-graded left and right R-module such that given $r, s \in R$ and $m \in M$, $r \cdot (m \cdot s) = (r \cdot m) \cdot s$.

Morphisms between A-graded R-modules are precisely A-graded R-module homomorphisms. We write R-**GrMod** for the category of left A-graded R-modules and **GrMod**-R for the category of right A-graded R-modules.

Remark C.5. It is straightforward to see that an A-graded abelian group is equivalently an A-graded $\mathbb{Z}$-module, where here we are considering $\mathbb{Z}$ as an A-graded ring concentrated in degree 0. Thus any result below about A-graded modules applies equally to A-graded abelian groups.

Remark C.6. We often will denote an A-graded R-module M by M_*. Given some $a \in A$, we can define the shifted A-graded abelian group M_{*+a} whose b^{th} component is M_{b+a}.

Definition C.7. More generally, given two A-graded R-modules M and N and some $d \in A$, an R-module homomorphism $f : M \to N$ is *an A-graded homomorphism of degree d* if it restricts to a map $M_a \to N_{a+d}$ for all $a \in A$. Thus, an A-graded homomorphism of degree d from M to N is equivalently an A-graded homomorphism $M_* \to N_{*+d}$ or an A-graded homomorphism $M_{*-d} \to N$. Given some $a \in A$ and left (resp. right) R-modules M and N, we will write

$$\mathrm{Hom}_R^d(M, N) = \mathrm{Hom}_R(M_*, N_{*+d}) = \mathrm{Hom}_R(M_{*-d}, N_*)$$

to denote the set of A-graded homomorphisms of degree d from M to N, and simply

$$\mathrm{Hom}_R(M, N)$$

to denote the set of degree-0 A-graded homomorphisms from M to N. Clearly A-graded homomorphisms may be added and subtracted, so these are further abelian groups. Thus we have an A-graded abelian group

$$\mathrm{Hom}_R^*(M, N).$$

Unless stated otherwise, an "A-graded homomorphism" will always refer to an A-graded homomorphism of degree 0.

Oftentimes when constructing A-graded rings, we do so only by defining the product of homogeneous elements, like so:

Lemma C.8. *Suppose we have an A-graded abelian group R, a distinguished element $1 \in R_0$, and $\mathbb{Z}$-bilinear maps $m_{a,b} : R_a \times R_b \to R_{a+b}$ for all $a, b \in A$. Further suppose that for all $x \in R_a$, $y \in R_b$, and $z \in R_c$, we have*

$$m_{a+b,c}(m_{a,b}(x, y), z) = m_{a,b+c}(x, m_{b,c}(y, z)) \qquad \text{and} \qquad m_{a,0}(x, 1) = m_{0,a}(1, x) = x.$$

Then there exists a unique multiplication map $m : R \times R \to R$ which endows R with the structure of an A-graded ring and restricts to $m_{a,b}$ for all $a, b \in A$.

Proof. Given $r, s \in R$, since $R \cong \bigoplus_{a\in A} R_a$, we may uniquely decompose r and s into homogeneous elements as $r = \sum_{a\in A} r_a$ and $s = \sum_{a\in A} s_a$ with each $r_a, s_a \in R_a$ such that only finitely many of the r_a's and s_a's are nonzero. Then in order to define a distributive product $R \times R \to R$ which restricts to $m_{a,b} : R_a \times R_b \to R_{a+b}$, note we *must* define

$$r \cdot s = \left(\sum_{a\in A} r_a\right) \cdot \left(\sum_{b\in A} s_b\right) = \sum_{a,b\in A} r_a \cdot s_b = \sum_{a,b\in A} m_{a,b}(r_a, s_b).$$

Thus, we have shown uniqueness. It remains to show this product actually gives R the structure of a ring. First we claim that the sum on the right is actually finite. Note there exists only finitely many nonzero r_a's and s_b's, and if $s_b = 0$ then

$$m_{a,b}(r_a, 0) = m_{a,b}(r_a, 0 + 0) \stackrel{(*)}{=} m_{a,b}(r_a, 0) + m_{a,b}(r_a, 0) \implies m_{a,b}(r_a, 0) = 0,$$

where $(*)$ follows from bilinearity of $m_{a,b}$. A similar argument yields that $m_{a,b}(0, s_b) = 0$ for all $a, b \in A$. Hence indeed $m_{a,b}(r_a, s_b)$ is zero for all but finitely many pairs $(a, b) \in A^2$, as desired.

Observe that in particular

$$(r \cdot s)_a = \sum_{b+c=a} m_{b,c}(r_b, s_c) = \sum_{b \in A} m_{b,a-b}(r_b, s_{a-b}) = \sum_{c \in A} m_{a-c,c}(r_{a-c}, s_c).$$

Now we claim this multiplication is associative. Given $t = \sum_{a \in A} t_a \in R$, we have

$$\begin{aligned}
(r \cdot s) \cdot t &= \sum_{a,b \in A} m_{a,b}((r \cdot s)_a, t_b) \\
&= \sum_{a,b \in A} m_{a,b}\left(\sum_{c \in A} m_{a-c,c}(r_{a-c}, s_c), t_b\right) \\
&\overset{(1)}{=} \sum_{a,b,c \in A} m_{a,b}(m_{a-c,c}(r_{a-c}, s_c), t_b) \\
&\overset{(2)}{=} \sum_{a,b,c \in A} m_{c,a+b-c}(r_c, m_{a-c,b}(s_{a-c}, t_b)) \\
&\overset{(3)}{=} \sum_{a,b,c \in A} m_{a,c}(r_a, m_{b,c-b}(s_b, t_{c-b})) \\
&\overset{(1)}{=} \sum_{a,c \in A} m_{a,c}\left(r_a, \sum_{b \in A} m_{b,c-b}(s_b, t_{c-b})\right) \\
&= \sum_{a,c \in A} m_{a,c}(r_a, (s \cdot t)_c) = r \cdot (s \cdot t),
\end{aligned}$$

where each occurrence of (1) follows by bilinearity of the $m_{a,b}$'s, each occurrence of (2) is associativity of the $m_{a,b}$'s, and (3) is obtained by re-indexing by re-defining $a := c$, $b := a - c$, and $c := a + b - c$. Next, we wish to show that the distinguished element $1 \in R_0$ is a unit with respect to this multiplication. Indeed, we have

$$1 \cdot r \overset{(1)}{=} \sum_{a \in A} m_{0,a}(1, r_a) \overset{(2)}{=} \sum_{a \in A} r_a = r \qquad \text{and} \qquad r \cdot 1 \overset{(1)}{=} \sum_{a \in A} m_{a,0}(r_a, 1) \overset{(2)}{=} \sum_{a \in A} r_a = r,$$

where (1) follows by the fact that $m_{a,b}(0, -) = m_{a,b}(-, 0) = 0$, which we have shown above, and (2) follows by unitality of the $m_{0,a}$'s and $m_{0,a}$'s, respectively. Finally, we wish to show that this product is distributive. Indeed, we have

$$\begin{aligned}
r \cdot (s + t) &= \sum_{a,b \in A} m_{a,b}(r_a, (s+t)_b) \\
&= \sum_{a,b \in A} m_{a,b}(r_a, s_b + t_b) \\
&\overset{(*)}{=} \sum_{a,b \in A} m_{a,b}(r_a, s_b) + \sum_{a,b \in A} m_{a,b}(r_a, t_b) = (r \cdot s) + (r \cdot t),
\end{aligned}$$

where $(*)$ follows by bilinearity of $m_{a,b}$. An entirely analagous argument yields that $(r + s) \cdot t = (r \cdot t) + (s \cdot t)$. $\square$

Similarly, when defining A-graded modules, we will only define the action maps for homogeneous elements:

Lemma C.9. *Let R be an A-graded ring, M an A-graded abelian group, and suppose there exists $\mathbb{Z}$-bilinear maps $\kappa_{a,b} : R_a \times M_b \to M_{a+b}$ for all $a, b \in A$. Further suppose that for all $r \in R_a$,*

$r' \in R_b$, and $m \in M_c$ that

$$\kappa_{a+b,c}(r \cdot r', m) = \kappa_{a,b+c}(r, \kappa_{b,c}(r', m)) \qquad \text{and} \qquad \kappa_{0,c}(1, m) = m.$$

Then there is a unique map $\kappa : R \times M \to M$ which endows M with the structure of a left A-graded R-module and restricts to $\kappa_{a,b}$ for all $a, b \in A$.

On the other hand, suppose there exists $\mathbb{Z}$-bilinear maps $\kappa_{a,b} : M_a \times R_b \to M_{a+b}$ for all $a, b \in A$. Further suppose that for all $r \in R_a$, $r' \in R_b$, and $m \in M_c$ that

$$\kappa_{c,a+b}(m, r \cdot r') = \kappa_{c+a,b}(\kappa_{c,a}(m, r), r') \qquad \text{and} \qquad \kappa_{c,0}(m, 1) = m.$$

Then there is a unique map $\kappa : M \times R \to M$ which endows M with the structure of a right A-graded R-module and restricts to $\kappa_{a,b}$ for all $a, b \in A$.

Finally, if we have maps $\lambda_{a,b} : R_a \times M_b \to M_{a+b}$ and $\rho_{a,b} : M_a \times R_b \to M_{a+b}$ satisfying all of the above conditions, and if we further have that

$$\lambda_{a,b+c}(r, \rho_{b,c}(x, s)) = \rho_{a+b,c}(\lambda_{a,b}(r, x), s)$$

for all $r \in R_a$, $x \in M_b$, and $s \in R_c$, then the left and right A-graded R-module structures induced on M by the λ's and ρ's give M the structure of an A-graded R-bimodule.

Proof. We show the left module case, as the right module case is entirely analagous. Supposing for each $a, b \in A$ we have a map $\kappa_{a,b} : R_a \times M_b \to M_{a+b}$ satisfying the above conditions, in order to extend these to a map $R \times M \to M$, by additivity we *must* define

$$\kappa : R \times M \to M$$

to be the map sending $r = \sum_a r_a$ and $m = \sum_a m_a$ to $\sum_{a,b \in A} \kappa_{a,b}(r_a, m_b)$. Now, we need to check that for all $r, s \in R$, $x, y \in M$ that

(1) $r \cdot (x + y) = r \cdot x + r \cdot y$
(2) $(r + s) \cdot x = r \cdot x + s \cdot x$
(3) $(rs) \cdot x = r \cdot (s \cdot x)$
(4) $1 \cdot x = x$,

where above we are written $- \cdot -$ for $\kappa(-, -)$. To see the first, note

$$\begin{aligned}
\kappa(r, x + y) &= \sum_{a,b \in A} \kappa_{a,b}(r_a, (x + y)_b) \\
&= \sum_{a,b \in A} \kappa_{a,b}(r_a, x_b + y_b) \\
&= \sum_{a,b \in A} (\kappa_{a,b}(r_a, x_b) + \kappa_{a,b}(r_a, y_b)) \\
&= \sum_{a,b \in A} \kappa_{a,b}(r_a, x_b) + \sum_{a,b \in A} \kappa_{a,b}(r_a, y_b) \\
&= \kappa(r, x) + \kappa(r, y).
\end{aligned}$$

To see the second, note

$$\begin{aligned}
\kappa(r+s,x) &= \sum_{a,b\in A} \kappa_{a,b}((r+s)_a, x_b) \\
&= \sum_{a,b\in A} \kappa_{a,b}(r_a + s_a, x_b) \\
&= \sum_{a,b\in A} (\kappa_{a,b}(r_a, x_b) + \kappa_{a,b}(s_a, x_b)) \\
&= \sum_{a,b\in A} \kappa_{a,b}(r_a, x_b) + \sum_{a,b\in A} \kappa_{a,b}(s_a, x_b) \\
&= \kappa(r,x) + \kappa(s,x).
\end{aligned}$$

To see the third, note

$$\begin{aligned}
\kappa(rs,x) &= \sum_{a,b\in A} \kappa_{a,b}((rs)_a, x_b) \\
&= \sum_{a,b\in A} \kappa_{a,b}\left(\sum_{c\in A} r_c s_{a-c}, x_b\right) \\
&= \sum_{a,b,c\in A} \kappa_{a,b}(r_c s_{a-c}, x_b) \\
&= \sum_{a,b,c\in A} \kappa_{a,b}(r_c, \kappa_{a-c,b}(s_{a-c}, x_b)) \\
&=
\end{aligned}$$

FINISH

$\square$

When working with A-graded rings and modules, we will often freely use the above propositions without comment.

Recall that given a ring R, a left (resp. right) module P is *projective* if, for all diagrams of R-module homomorphisms of the form

$$\begin{tikzcd}
 & M \arrow[d, two heads, "g"] \\
P \arrow[r, "f"'] & N
\end{tikzcd}$$

with g an epimorphism, there exists a lift $h : P \to M$ satisfying $g \circ h = f$

$$\begin{tikzcd}
 & M \arrow[d, two heads, "g"] \\
P \arrow[r, "f"'] \arrow[ur, dashed, "h"] & N
\end{tikzcd}$$

(Note h is not required to be unique.)

Definition C.10. Let R be an A-graded ring, and let P be a left (resp. right) A-graded R-module. Then P is a *graded projective* module if, for all diagrams of A-graded R-module homomorphisms of the form

$$\begin{tikzcd} & M \arrow[d, two heads, "g"] \\ P \arrow[r, "f"'] & N \end{tikzcd}$$

with g an epimorphism, there exists an A-graded homomorphism $h : P \to M$ satisfying $g \circ h = f$.

$$\begin{tikzcd} & M \arrow[d, two heads, "g"] \\ P \arrow[r, "f"'] \arrow[ur, dashed, "h"] & N \end{tikzcd}$$

(Note h is not required to be unique.)

Lemma C.11. *Let R be an A-graded ring, and let M be an A-graded left (resp. right) R-module. Then for all $d \in A$, the evaluation map*

$$\begin{aligned} \mathrm{ev}_1 : \mathrm{Hom}_R^d(R, M) &\to M_d \\ \varphi &\mapsto \varphi(1) \end{aligned}$$

is an isomorphism of abelian groups.

Proof. We consider the case that M is a left A-graded R-module, as showing it when M is a right module is entirely analagous. First of all, this map is clearly a homomorphism, as given degree d A-graded homomorphisms $\varphi, \psi : R \to M$, we have

$$\mathrm{ev}_1(\varphi + \psi) = (\varphi + \psi)(1) = \varphi(1) + \psi(1) = \mathrm{ev}_1(\varphi) + \mathrm{ev}_1(\psi).$$

Now, to see it is surjective, let $m \in M_d$, and define $\varphi_m : R \to M$ to send $r \mapsto r \cdot m$. First of all, φ_m is a module homomorphism, as given $r, s \in R$,

$$\varphi_m(r + s) = (r + s) \cdot m = r \cdot m + s \cdot m = \varphi_m(r) + \varphi_m(s) \quad \text{and} \quad \varphi_m(r \cdot s) = r \cdot s \cdot m = r \cdot \varphi_m(s).$$

Furthermore, it is clearly A-graded of degree d, as given a homogeneous element $r \in R_a$ for some $a \in A$, we have $\varphi_m(r) = r \cdot m \in R_{a+d}$, since m is homogeneous of degree d. Finally, clearly

$$\mathrm{ev}_1(\varphi_m) = \varphi_m(1) = 1 \cdot m = m,$$

so indeed ev_1 is surjective. On the other hand, to see it is injective, suppose we are given $\varphi, \psi \in \mathrm{Hom}_R^d(R, M)$ such that $\varphi(1) = \psi(1)$. Then given $r \in R$, we must have

$$\varphi(r) = \varphi(r \cdot 1) = r \cdot \varphi(1) = r \cdot \psi(1) = \psi(r \cdot 1) = \psi(r),$$

so φ and ψ are exactly the same map. Thus, ev_1 is injective, as desired. □

C.1. ***A*-graded submodules and quotient modules.** In what follows, fix an A-graded ring R. We will simply say "A-graded R-module" when we are freely considering either left or right A-graded R-modules.

Definition C.12. Let M be an A-graded R-module. Then an *A-graded R-submodule* is an A-graded R-module N which is a subset of M and for which the inclusion $N \hookrightarrow M$ is an A-graded homomorphism of R-modules. Equivalently, it is a submodule N for which the canonical map

$$\bigoplus_{a \in A} N \cap M_a \to N$$

is an isomorphism.

Lemma C.13. *Let M be an A-graded R-module. Then an R-submodule $N \le M$ is an A-graded submodule if and only if it is generated as an R-module by homogeneous elements of M.*

Proof. If $N \le M$ is a A-graded submodule, it is generated by the set of all its homogeneous elements, which are also homogeneous elements in M, by definition.

Conversely, suppose $N \le M$ is a submodule which is generated by homogeneous elements of M. Then define $N_a := N \cap M_a$, and consider the canonical map

$$\Phi : \bigoplus_{a \in A} N_a \to N.$$

First of all, it is surjective, as each generator of N belongs to some N_a, by definition. To see it is injective, consider the following commutative diagram:

$$\begin{tikzcd} \bigoplus_{a\in A} N_a \arrow[r, hook] \arrow[d, "\Phi"'] & \bigoplus_{a\in A} M_a \arrow[d, "\cong"] \\ N \arrow[r, hook] & M \end{tikzcd}$$

Since Φ composes with an injection to get an injection, clearly Φ must be injective itself. We have the desired result. □

Proposition C.14. *Given two left (resp. right) A-graded R-modules M and N and an A-graded R-module homomorphism $\varphi : M \to N$ (of possibly nonzero degree), the kernel and images of φ are A-graded submodules of M and N, respectively.*

Proof. First recall that a degree d A-graded homomorphism $M \to N$ is simply an A-graded homomorphism $M_* \to N_{*+d}$, so it suffices to consider the case φ is of degree 0. Next, note that since the forgetful functor from R-modules to abelian groups preserves kernels and images, it suffices to consider the case that φ is a homomorphism of A-graded abelian groups. Finally, by Lemma C.13, it suffices to show that $\ker \varphi$ and $\operatorname{im} \varphi$ are generated by homogeneous elements of M and N, respectively.

Note that by the universal property of the coproduct in **Ab**, the data of an A-graded homomorphism of abelian groups $\varphi : M \to N$ is precisely the data of an A-indexed collection of abelian group homomorphisms $\varphi_a : M_a \to N_a$, in which case the following diagram commutes:

$$\begin{tikzcd} \bigoplus_a M_a \arrow[r, "\bigoplus_a \varphi_a"] \arrow[d, "\cong"'] & \bigoplus_a N_a \arrow[d, "\cong"] \\ M \arrow[r, "\varphi"] & N \end{tikzcd}$$

Finally, the desired result follows by the purely formal fact that taking images and kernels commutes with arbitrary direct sums. □

Proposition C.15. *Given two left (resp. right) A-graded R-modules M and N, an A-graded submodule $K \le N$, and an A-graded R-module homomorphism $\varphi : M \to N$ (of possibly nonzero degree), the submodule $\varphi^{-1}(K)$ of M is A-graded.*

Proof. Recall that a degree d A-graded homomorphism $M \to N$ is simply an A-graded homomorphism $M_* \to N_{*+d}$, so it suffices to consider the case φ is of degree 0. Now, let $x \in L := \varphi^{-1}(K)$. As an element of M, we may uniquely write $x = \sum_{a\in A} x_a$ where each $x_a \in M_a$. Similarly, if we set $y := \varphi(x)$, then we may uniquely write $y = \sum_{a\in A} y_a$ where each $y_a \in N_a$. Then since K is an A-graded submodule of N and $y \in K$, by definition, we have that $y_a \in K$ for each a. Finally, note that

$$\sum_{a \in A} y_a = y = \varphi(x) = \sum_{a \in A} \varphi(x_a),$$

so that $\varphi(x_a) = y_a \in K$ for all $a \in A$, so that $x_a \in L$ for all $a \in A$. Thus we have shown that each element in L can be written as a sum of homogeneous elements in M, as desired. □

Proposition C.16. *Given an A-graded R-module M and an A-graded subgroup $N \leq M$, the quotient M/N is canonically A-graded by defining $(M/N)_a$ to be the subgroup generated by cosets represented by homogeneous elements of degree a in M. Furthermore, the canonical maps $M_a/N_a \to (M/N)_a$ taking a coset $m + N_a$ to $m + N$ are isomorphisms.*

Proof. Consider the canonical map

$$\Phi : \bigoplus_a (M/N)_a \to M/N.$$

First of all, surjectivity of Φ follows by commutativity of the following diagram:

$$\begin{tikzcd}
\bigoplus_a M_a \arrow[r, "\cong"] \arrow[d, two heads] & M \arrow[d, two heads] \\
\bigoplus_a (M/N)_a \arrow[r, "\Phi"] & M/N
\end{tikzcd}$$

where the vertical left map sends a generator $m \in M_a$ to the coset $m + N$ in $(M/N)_a \subseteq M/N$. To see Φ is injective, suppose we are given some element $(m_a + N)_{a \in A}$ in $\bigoplus_a (M/C)_a$ such that $\sum_{a \in A}(m_a + N) = 0$ in M/N. Thus $\sum_{a \in A} m_a \in N$, and since N is A-graded this implies that each m_a belongs to $N \cap M_a = N_a$, so that in particular $m_a + N$ is zero in $(M/N)_a \subseteq M/N$, so that $(m_a + N)_{a \in A} = 0$ in $\bigoplus_a (M/N)_a$, as desired.

It remains to show that the canonical map

$$\varphi_a : M_a/N_a \to (M/N)_a$$

is an isomorphism. It is clearly surjective, as $(M/N)_a$ is generated by elements $m + N$ for $m \in M_a$, and these elements make up precisely the image of φ_a. Thus φ_a hits every generator of $(M/N)_a$, so φ_a is surjective. On the other hand, suppose we are given some $m \in M_a$ such that $\varphi(m + N_a) = m + N = 0$. Thus $m \in N$, and $m \in M_a$, so that $m \in M_a \cap N = N_a$, meaning $m + N_a = 0$ in M_a/N_a, as desired. □

C.2. **Tensor product of A-graded modules.** Recall that given a ring R, a left R-module M, a right R-module N, and an abelian group A, an *R-balanced map* $\varphi : M \times N \to B$ is one which satisifies

$$\begin{aligned}
\varphi(m, n + n') &= \varphi(m, n) + \varphi(m, n') \\
\varphi(m + m', n) &= \varphi(m, n) + \varphi(m', n) \\
\varphi(m \cdot r, n) &= \varphi(m, r \cdot n).
\end{aligned}$$

for all $m, m' \in M$, $n, n' \in N$, and $r \in R$. Then the tensor product $M \otimes_R N$ is the universal abelian group equipped with an R-balanced map $\otimes : M \times N \to M \otimes_R N$ such that for every abelian group B and every R-balanced map $\varphi : M \times N \to B$, there is a *unique* group homomorphism $\widetilde{\varphi} : M \otimes_R N \to B$ such that $\widetilde{f} \circ \otimes = f$. We call elements in the image of $\otimes : M \times N \to M \otimes_R N$ *pure tensors*. It is a standard fact that $M \otimes_R N$ is generated as an abelian group by its pure tensors.

Definition C.17. Suppose we have a right A-graded R-module M, a left A-graded R-module N, and an A-graded abelian group B. Then an *A-graded R-balanced map* $\varphi : M \times N \to B$ is an R-balanced map which restricts to $M_a \times N_b \to B_{a+b}$ for all $a, b \in A$.

Proposition C.18. *Suppose we have a right A-graded R-module M and a left A-graded R-module N. Then the tensor product*

$$M \otimes_R N$$

is naturally an A-graded abelian group by defining $(M \otimes_R N)_a$ to be the subgroup generated by homogeneous *pure tensors $m \otimes n$ with $m \in M_b$ and $n \in N_c$ such that $b + c = a$. Furthermore, if either M (resp. N) is an A-graded bimodule, then this decomposition makes $M \otimes_R N$ into a left (resp. right) A-graded R-module. In particular, if both M and N are R-bimodules, then $M \otimes_R N$ is an R-bimodule.*

Proof. By definition, since M and N are A-graded abelian groups, they are generated (as abelian groups) by their homogeneous elements. Thus it follows that $M \otimes_R N$ is generated by *homogeneous pure tensors*, that is, elements of the form $m \otimes n$ with $m \in M$ and $n \in N$ homogeneous. Now, given a homogeneous pure tensor $m \otimes n$, we define its *degree* by the formula $|m \otimes n| := |m| + |n|$. It follows this formula is well-defined by checking that given homogeneous elements $m \in M$, $n \in N$, and $r \in R$ that

$$|(m \cdot r) \otimes n| = |m \cdot r| + |n| = |m| + |r| + |n| = |m| + |r \cdot n| = |m \otimes (r \cdot n)|.$$

Thus, we may define $(M \otimes_R N)_a$ to be the subgroup of $M \otimes_R N$ generated by those pure homogeneous tensors of degree a. Now, consider the map

$$\Psi : M \times N \to \bigoplus_{a \in A} (M \otimes_R N)_a$$

which takes a pair $(m, n) = \sum_{a \in A} (m_a, n_a)$ to the element $\Psi(m, n)$ whose a^{th} component is

$$(\Psi(m, n))_a := \sum_{b+c=a} m_b \otimes n_c.$$

It is straightforward to see that this map is R-balanced, in the sense that it is additive in each argument and $\Psi(m \cdot r, n) = \Psi(m, r \cdot n)$ for all $m \in M$, $n \in N$, and $r \in R$. Thus by the universal property of $M \otimes_R N$, we get a homomorphism of abelian groups $\widetilde{\Psi} : M \otimes_R N \to \bigoplus_{a \in A} (M \otimes_R N)_a$ lifting Ψ along the canonical map $M \times N \to M \otimes_R N$. Now, also consider the canonical map

$$\Phi : \bigoplus_{a \in A} (M \otimes_R N)_a \to M \otimes_R N.$$

We would like to show $\widetilde{\Psi}$ and Φ are inverses of eah other. Since $\widetilde{\Psi}$ and Φ are both homomorphisms, it suffices to show this on generators. Let $m \otimes n$ be a homogeneous pure tensor with $m = m_a \in M_a$ and $n = n_b \in N_b$. Then we have

$$\Phi(\widetilde{\Psi}(m \otimes n)) = \Phi\left(\bigoplus_{a \in A} \sum_{b+c=a} m_b \otimes n_c \right) \overset{(*)}{=} \Phi(m \otimes n) = m \otimes n,$$

and

$$\widetilde{\Psi}(\Phi(m \otimes n)) = \widetilde{\Psi}(m \otimes n) = \bigoplus_{a \in A} \sum_{b+c=a} m_b \otimes n_c \overset{(*)}{=} m \otimes n,$$

where both occurrences of $(*)$ follow by the fact that $m_b \otimes n_c = 0$ unless $b = c = a$, in which case $m_a \otimes n_a = m \otimes n$. Thus since Φ is an isomorphism, $M \otimes_R N$ is indeed an A-graded abelian group, as desired.

Now, suppose that M is an A-graded R-bimodule, so there exists left and right A-graded actions of R on M such that given $r, s \in R$ and $m \in M$ we have $r \cdot (m \cdot s) = (r \cdot m) \cdot s$. Then we would like to show that given a left A-graded R-module N that $M \otimes_R N$ is canonically a left A-graded R-module. Indeed, define the action of R on $M \otimes_R N$ on pure tensors by the formula

$$r \cdot (m \otimes n) = (r \cdot m) \otimes n.$$

First of all, clearly this map is A-graded, as if $r \in R_a$, $m \in M_b$, and $n \in N_c$ then $(r \cdot m) \otimes n$, by definition, has degree $|r \cdot m| + |n| = |r| + |m| + |n|$ (the last equality follows since the left action of R on M is A-graded). In order to show the above map defines a left module structure, it suffices to show that given pure tensors $m \otimes n, m' \otimes n' \in M \otimes_R N$ and elements $r, r' \in R$ that

(1) $r \cdot (m \otimes n + m' \otimes n') = r \cdot (m \otimes n) + r \cdot (m' \otimes n')$,
(2) $(r + r') \cdot (m \otimes n) = r \cdot (m \otimes n) + r' \cdot (m' \otimes n')$,
(3) $(rr') \cdot (m \otimes n) = r \cdot (r' \cdot (m \otimes n))$, and
(4) $1 \cdot (m \otimes n) = m \otimes n$.

Axiom (1) holds by definition. To see (2), note that by the fact that R acts on M on the left that

$$(r + r') \cdot (m \otimes n) = ((r + r') \cdot m) \otimes n = (r \cdot m + r' \cdot m) \otimes n = r \cdot m \otimes n + r' \cdot m \otimes n.$$

That (3) and (4) hold follows similarly by the fact that $(rr') \cdot m = r \cdot (r' \cdot m)$ and $1 \cdot m = m$.

Conversely, if N is an A-graded R-bimodule, then showing $M \otimes_R N$ is canonically a right A-graded R-module via the rule

$$(m \otimes n) \cdot r = m \otimes (n \cdot r)$$

is entirely analagous.

Finally, if both M and N are R-bimodules, then by what we have shown, $M \otimes_R N$ is both a left and right R-module. To see these coincide to give $M \otimes_R N$ an R-bimodule structure, note that given $m \in M$, $n \in N$, and $r, r' \in R$ that

$$(r \cdot (m \otimes n)) \cdot r' = ((r \cdot m) \otimes n) \cdot r' = (r \cdot m) \otimes (n \cdot r') = r \cdot (m \otimes (n \cdot r')) = r \cdot ((m \otimes n) \cdot r'). \quad \square$$

Lemma C.19. *Let R be an A-graded ring, B an A-graded abelian group, M a right A-graded R-module, and N a left A-graded R-module. Further suppose we are given a map $\varphi_{a,b} : M_a \times N_b \to B_{a+b}$ for all $a, b \in A$ which commutes with addition in each argument, and such that for all $m \in M_a$, $n \in N_b$, and $r \in R_c$ that*

$$\varphi_{a+b,c}(m \cdot r, n) = \varphi_{a,b+c}(m, r \cdot n).$$

Then there is a unique A-graded R-balanced map $\varphi : M \times N \to B$ which restricts to $\varphi_{a,b}$ for all $a, b \in A$, and furthermore, the induced homorphism $\widetilde{\varphi} : M \otimes_R N \to B$ is an A-graded homomorphism of abelian groups.

Proof. $\square$ TODO

Appendix D. Monoid objects

In what follows, we fix a symmetric monoidal category $(\mathcal{C}, \otimes, S)$ with left unitor, right unitor, associator, and symmetry isomorphisms λ, ρ, α, and τ, respectively.

D.1. **Monoid objects in a symmetric monoidal category.**

Definition D.1. Let $(\mathcal{C}, \otimes, S)$ be a symmetric monoidal category with left unitor, right unitor, associator, and symmetry isomorphisms λ, ρ, α, and τ, respectively. A *monoid object* (E, μ, e) is an object E in $\mathcal{C}$ along with a multiplication morphism $\mu : E \otimes E \to E$ and a unit map $e : S \to E$ such that the following diagrams commute:

$$\begin{tikzcd} E \otimes S \arrow[r, "E\otimes e"] \arrow[dr, "\rho_E"'] & E \otimes E \arrow[d, "\mu"] & S \otimes E \arrow[l, "e\otimes E"'] \arrow[dl, "\lambda_E"] \\ & E & \end{tikzcd} \qquad \begin{tikzcd} (E \otimes E) \otimes E \arrow[rr, "\mu\otimes E"] \arrow[d, "\alpha"'] & & E \otimes E \arrow[d, "\mu"] \\ E \otimes (E \otimes E) \arrow[r, "E\otimes\mu"] & E \otimes E \arrow[r, "\mu"] & E \end{tikzcd}$$

The first diagram expresses unitality, while the second expressed associativity. If in addition the following diagram commutes,

$$\begin{tikzcd} E \otimes E \arrow[rr, "\tau"] \arrow[dr, "\mu"'] & & E \otimes E \arrow[dl, "\mu"] \\ & E & \end{tikzcd}$$

then we say (E, μ, e) is a *commutative* monoid object.

Definition D.2. Given two monoid objects (E_1, μ_1, e_1) and (E_2, μ_2, e_2) in a symmetric monoidal category $(\mathcal{C}, \otimes, S)$, a *monoid homomorphism* from E_1 to E_2 is a morphism $f : E_1 \to E_2$ in $\mathcal{C}$ such that the following diagrams commute:

$$\begin{tikzcd} E_1 \otimes E_1 \arrow[r, "f\otimes f"] \arrow[d, "\mu_1"'] & E_2 \otimes E_2 \arrow[d, "\mu_2"] \\ E_1 \arrow[r, "f"] & E_2 \end{tikzcd} \qquad \begin{tikzcd} & S \arrow[dl, "e_1"'] \arrow[dr, "e_2"] & \\ E_1 \arrow[rr, "f"] & & E_2 \end{tikzcd}$$

It is straightforward to show that id_{E_1} is a homomorphism of monoid objects from E_1 to itself, and that the composition of monoid homomorphisms is still a monoid homomorphism. Thus, we have categories $\mathbf{Mon}_{\mathcal{C}}$ and $\mathbf{CMon}_{\mathcal{C}}$ of monoid objects and commutative monoid objects in $\mathcal{C}$, respectively, with monoid homomorphisms between them.

Lemma D.3. *Given two monoid objects (E_1, μ_1, e_1) and (E_2, μ_2, e_2) in a symmetric monoidal category $(\mathcal{C}, \otimes, S)$, their tensor product $E_1 \otimes E_2$ canonically becomes a monoid object in $\mathcal{C}$ with unit map*

$$e : S \xrightarrow{\cong} S \otimes S \xrightarrow{e_1 \otimes e_2} E_1 \otimes E_2$$

and multiplication map

$$\mu : E_1 \otimes E_2 \otimes E_1 \otimes E_2 \xrightarrow{E_1 \otimes \tau \otimes E_2} E_1 \otimes E_1 \otimes E_2 \otimes E_2 \xrightarrow{\mu_1 \otimes \mu_2} E_1 \otimes E_2$$

(where here we are suppressing the associators from the notation). If in addition (E_1, μ_1, e_1) and (E_2, μ_2, e_2) are commutative *monoid objects, then $(E_1 \otimes E_2, \mu, e)$ is as well.*

Lemma D.4. *Given monoid objects (E_i, μ_i, e_i) for $i = 1, 2, 3$ in a symmetric monoidal category $\mathcal{C}$, the associator $(E_1 \otimes E_2) \otimes E_3 \xrightarrow{\cong} E_1 \otimes (E_2 \otimes E_3)$ is an isomorphism of monoid objects. In other words, up to associativity, given a collection of monoid objects $E_1, \dots, E_n$ in $\mathcal{C}$, there is no ambiguity when talking about their tensor product $E_1 \otimes \cdots \otimes E_n$ as a monoid object.*

Proof. Clearly, up to associativity, $(E_1 \otimes E_2) \otimes E_3$ and $E_1 \otimes (E_2 \otimes E_3)$ have the same unit map $S \xrightarrow{e_1 \otimes e_2 \otimes e_3} E_1 \otimes E_2 \otimes E_3$. Thus, it remains to show that they have the same product map, up to associativity. To see this, consider the following diagram, where we've passed to a symmetric strict monoidal category:

$$\begin{tikzcd}
E_1 \otimes (E_2 \otimes E_3) \otimes E_1 \otimes (E_2 \otimes E_3) \arrow[rr, equal, "\alpha"] \arrow[d, "E_1 \otimes \tau_{E_2 \otimes E_3, E_1} \otimes E_2 \otimes E_3"'] & & (E_1 \otimes E_2) \otimes E_3 \otimes (E_1 \otimes E_2) \otimes E_3 \arrow[d, "E_1 \otimes E_2 \otimes \tau_{E_3, E_1 \otimes E_2} \otimes E_3"] \\
E_1 \otimes E_1 \otimes E_2 \otimes E_3 \otimes E_2 \otimes E_3 \arrow[d, "\mu_1 \otimes E_2 \otimes \tau \otimes E_3"'] \arrow[dr, "E_1 \otimes E_1 \otimes E_2 \otimes \tau \otimes E_3"] & & E_1 \otimes E_2 \otimes E_1 \otimes E_2 \otimes E_3 \otimes E_3 \arrow[d, "E_1 \otimes \tau \otimes E_2 \otimes \mu_3"] \arrow[dl, "E_1 \otimes \tau \otimes E_2 \otimes E_3 \otimes E_3"'] \\
E_1 \otimes E_2 \otimes E_2 \otimes E_3 \otimes E_3 \arrow[d, "E_1 \otimes \mu_2 \otimes \mu_3"'] & E_1 \otimes E_1 \otimes E_2 \otimes E_2 \otimes E_3 \otimes E_3 \arrow[l, "\mu_1 \otimes E_2 \otimes E_2 \otimes E_3 \otimes E_3"'] \arrow[r, "E_1 \otimes E_2 \otimes E_2 \otimes E_2 \otimes \mu_3"] \arrow[dl, "\mu_1 \otimes \mu_2 \otimes \mu_3"] \arrow[dr, "\mu_1 \otimes \mu_2 \otimes \mu_3"'] & E_1 \otimes E_1 \otimes E_2 \otimes E_2 \otimes E_3 \arrow[d, "\mu_1 \otimes \mu_2 \otimes E_3"] \\
E_1 \otimes E_2 \otimes E_3 \arrow[rr, equal, "\alpha"] & & E_1 \otimes E_2 \otimes E_3
\end{tikzcd}$$

The top pentagonal region commutes by coherence for the τ's in a symmetric monoidal category. The bottom triangle commutes by definition. The remaining four triangles commute by functoriality of $- \otimes -$. On the left is the product for $E_1 \otimes (E_2 \otimes E_3)$, while on the right is the product for $(E_1 \otimes E_2) \otimes E_3$. Thus they are equal up to associativity, as desired. $\square$

Lemma D.5. *Suppose we have some monoid object (E, μ, e) in $\mathcal{C}$ and some homomorphism of monoid objects $f : (E_1, \mu_1, e_1) \to (E_2, \mu_2, e_2)$ in $\mathbf{Mon}_{\mathcal{C}}$. Then $E \otimes f : E \otimes E_1 \to E \otimes E_2$ and $f \otimes E : E_1 \otimes E \to E_2 \otimes E$ are monoid homomorphisms, where here we are considering $E \otimes E_1$, $E \otimes E_2$, $E_1 \otimes E$, and $E_2 \otimes E$ to be monoid objects by Lemma D.3.*

Proof. We will show that $E \otimes f$ is a monoid object homomorphism, as showing $f \otimes E$ is a monoid homomorphism is entirely analagous. First consider the following diagram:

$$\begin{tikzcd}
E \otimes E_1 \otimes E \otimes E_1 \arrow[rrr, "E \otimes f \otimes E \otimes f"] \arrow[d, "E \otimes \tau \otimes E_1"'] & & & E \otimes E_2 \otimes E \otimes E_2 \arrow[d, "E \otimes \tau \otimes E_2"] \\
E \otimes E \otimes E_1 \otimes E_1 \arrow[rrr, "E \otimes E \otimes f \otimes f"] \arrow[dr, "\mu \otimes E_1 \otimes E_2"] \arrow[dd, "\mu \otimes \mu_1"'] & & & E \otimes E \otimes E_2 \otimes E_2 \arrow[dl, "\mu \otimes E_2 \otimes E_2"'] \arrow[dd, "\mu \otimes \mu_2"] \\
 & E \otimes E_1 \otimes E_1 \arrow[r, "E \otimes f \otimes f"] \arrow[dl, "E \otimes \mu_1"'] & E \otimes E_2 \otimes E_2 \arrow[dr, "E \otimes \mu_2"] & \\
E \otimes E_1 \arrow[rrr, "E \otimes f"] & & & E \otimes E_2
\end{tikzcd}$$

The top region commutes by naturality of τ. The bottom trapezoid commutes since f is a monoid homomorphism. The remaining three regions commute by functoriality of $- \otimes -$. Now, consider the following diagram:

$$\begin{tikzcd}
 & S \arrow[dl, "e \otimes e_1"'] \arrow[d, "e"] \arrow[dr, "e \otimes e_2"] & \\
 & E \arrow[dl, "E \otimes e_1"] \arrow[dr, "E \otimes e_2"'] & \\
E \otimes E_1 \arrow[rr, "E \otimes f"] & & E \otimes E_2
\end{tikzcd}$$

The bottom region commutes since f is a monoid homomorphism. The top two regions commute by functoriality of $- \otimes -$. Thus, we've shown $E \otimes f$ is a monoid object homomorphism, as desired. $\square$

D.2. **Modules over a monoid object.**

Definition D.6. Let (E, μ, e) be a monoid object in $\mathcal{C}$. Then a *left module object* (N, κ) over (E, μ, e) is the data of an object N in $\mathcal{C}$ and a morphism $\kappa : E \otimes N \to N$ such that the following two diagrams commute in $\mathcal{C}$:

$$\begin{tikzcd}
S \otimes N \arrow[r, "e\otimes N"] \arrow[dr, "\lambda_N"'] & E \otimes N \arrow[d, "\kappa"] \\
& N
\end{tikzcd}
\qquad
\begin{tikzcd}
(E \otimes E) \otimes N \arrow[rr, "\mu\otimes N"] \arrow[d, "\alpha"'] & & E \otimes N \arrow[d, "\kappa"] \\
E \otimes (E \otimes N) \arrow[r, "E\otimes\kappa"] & E \otimes N \arrow[r, "\kappa"] & N
\end{tikzcd}$$

Definition D.7. Let (E, μ, e) be a monoid object in $\mathcal{C}$, and suppose we have two left module objects (N, κ) and (N', κ') over (E, μ, e). Then a morphism $f : N \to N'$ is a *left E-module homomorphism* if the following diagram commutes in $\mathcal{C}$:

$$\begin{tikzcd}
E \otimes N \arrow[r, "E\otimes f"] \arrow[d, "\kappa"'] & E \otimes N' \arrow[d, "\kappa'"] \\
N \arrow[r, "f"] & N'
\end{tikzcd}$$

Definition D.8. Given a monoid object (E, μ, e) in $\mathcal{C}$, we write E-**Mod** to denote the category of left module objects over E and left E-module homomorphisms between them. We denote the homset in E-**Mod** by

$$\mathrm{Hom}_{E\text{-}\mathbf{Mod}}(M, N), \qquad \text{or simply} \qquad \mathrm{Hom}_E(M, N).$$

Lemma D.9. *Let (E, μ, e) be a monoid object in $\mathcal{C}$ and let (N, κ) be a left E module object. Then given some object X in $\mathcal{C}$ and an isomorphism $\phi : N \xrightarrow{\cong} X$, X inherits the structure of a left E-module via the action map*

$$\kappa_\phi : E \otimes X \xrightarrow{E\otimes\phi^{-1}} E \otimes N \xrightarrow{\kappa} N \xrightarrow{\phi} X.$$

Proof. We need to show the two coherence diagrams in Definition D.6 commute. To see the former commutes, consider the following diagram:

$$\begin{tikzcd}
X \arrow[rr, "e\otimes X"] \arrow[dr, "\phi^{-1}"] \arrow[dddrr, equal] & & E \otimes X \arrow[d, "E\otimes\phi^{-1}"] \\
& N \arrow[r, "e\otimes N"] \arrow[dr, equal] & E \otimes N \arrow[d, "\kappa"] \\
& & N \arrow[d, "\phi"] \\
& & X
\end{tikzcd}$$

The top trapezoid commutes by functoriality of $- \otimes -$. The middle small triangle commutes by unitality of κ. The remaining region commutes by definition. To see the second coherence diagram commutes, consider the following diagram:

$$\begin{tikzcd}
E \otimes E \otimes X \arrow[rrr, "\mu\otimes X"] \arrow[d, "E\otimes E\otimes\phi^{-1}"'] & & & E \otimes X \arrow[d, "E\otimes\phi^{-1}"] \\
E \otimes E \otimes N \arrow[rrr, "\mu\otimes N"] \arrow[d, "E\otimes\kappa"'] & & & E \otimes N \arrow[d, "\kappa"] \\
E \otimes N \arrow[rrr, "\kappa"] \arrow[d, "E\otimes\phi"'] \arrow[dr, equal] & & & N \arrow[d, "\phi"] \\
E \otimes X \arrow[r, "E\otimes\phi^{1-}"'] & E \otimes N \arrow[r, "\kappa"'] & N \arrow[r, "\phi"'] & X
\end{tikzcd}$$

The top rectangle commutes by functoriality of $- \otimes -$. The middle rectangle commutes by coherence for κ. The bottom two regions commute by definition. $\square$

Proposition D.10. *Given a monoid object (E, μ, e) in $\mathcal{C}$, the forgetful functor E-**Mod** $\to \mathcal{C}$ has a left adjoint $\mathcal{C} \to E$-**Mod** sending an object $X \mapsto (E \otimes X, \kappa_X)$ where κ_X is the composition*

$$E \otimes (E \otimes X) \xrightarrow{\alpha^{-1}} (E \otimes E) \otimes X \xrightarrow{\mu\otimes X} E \otimes X,$$

and sending a morphism $f : X \to Y$ to $E \otimes f : E \otimes X \to E \otimes Y$.

Proof. In this proof, we work in a symmetric strict monoidal category. First, we wish to show that $E \otimes - : \mathcal{C} \to E$-**Mod** as constructed is well-defined. First, to see that (X, κ_X) is actually a left E-module, we need to show the two diagrams in Definition D.6 commute. Indeed, consider the following diagrams:

$$\begin{tikzcd}
E \otimes X \arrow[r, "e\otimes E\otimes X"] \arrow[dr, equal] & E \otimes E \otimes X \arrow[d, "\mu\otimes X"] \\
& E \otimes X
\end{tikzcd}
\qquad
\begin{tikzcd}
E \otimes E \otimes E \otimes X \arrow[r, "\mu\otimes E\otimes X"] \arrow[d, "E\otimes\mu\otimes X"'] & E \otimes E \otimes X \arrow[d, "\mu\otimes X"] \\
E \otimes E \otimes X \arrow[r, "\mu\otimes X"'] & E \otimes X
\end{tikzcd}$$

These are precisely the diagrams obtained by applying $X \otimes -$ to the coherence diagrams for μ, so that they commute as desired. Now, suppose $f : X \to Y$ is a morphism in $\mathcal{C}$, then we would like to show that $E \otimes f : E \otimes X \to E \otimes Y$ is a morphism of left E-module objects. Indeed, consider the following diagram:

$$\begin{tikzcd}
E \otimes E \otimes X \arrow[r, "E\otimes E\otimes f"] \arrow[d, "\mu\otimes X"'] & E \otimes E \otimes Y \arrow[d, "\mu\otimes Y"] \\
E \otimes X \arrow[r, "E\otimes f"] & E \otimes Y
\end{tikzcd}$$

It commutes by functoriality of $- \otimes -$, so $E \otimes f$ is indeed a left E-module homomorphism as desired.

Now, in order to see that $E \otimes -$ is left adjoint to the forgetful functor, it suffices to construct a unit and counit for the adjunction and show they satisfy the zig-zag identities. Given X in $\mathcal{C}$ and (N, κ) in E-**Mod**, define $\eta_X := e \otimes X : X \to E \otimes X$ and $\varepsilon_{(N,\kappa)} := \kappa : E \otimes N \to N$. η_X is clearly natural in X by functoriality of $- \otimes -$, and $\varepsilon_{(N,\kappa)}$ is natural in (N, κ) by how morphisms in E-**Mod** are defined. Now, to see these are actually the unit and counit of an adjunction, we need to show that the following diagrams commute for all X in $\mathcal{C}$ and (N,κ) in E-**Mod**:

$$\begin{tikzcd}
E\otimes X \arrow[r, "E\otimes\eta_X = E\otimes e\otimes X"] \arrow[dr, equal] & E\otimes E\otimes X \arrow[d, "\varepsilon_{(E\otimes X,\kappa_X)}=\mu\otimes X"] \\
 & E\otimes X
\end{tikzcd}
\qquad
\begin{tikzcd}
E\otimes N \arrow[d, "\varepsilon_{(N,\kappa)}=\kappa"'] & N \arrow[l, "\eta_N = e\otimes N"'] \arrow[dl, equal] \\
N &
\end{tikzcd}$$

Commutativity of the left diagram is unitality of μ, while commutativity of the right diagram is unitality of κ. Thus indeed $E\otimes - : \mathcal{C}\to E$-**Mod** is a left adjoint of the forgetful functor E-**Mod** $\to\mathcal{C}$, as desired. □

Definition D.11. We call the functor $E\otimes - : \mathcal{C}\to E$-**Mod** constructed above the *free* functor, and we call left E-modules in the image of the free functor *free modules*.

From now on we fix a monoidal closed tensor triangulated category $(\mathcal{SH},\otimes,S,\Sigma,e,\mathcal{D})$ (Definition A.13) with arbitrary (small) (co)products and sub-Picard grading $(A,\mathbf{1},h,\{S^a\},\{\phi_{a,b}\})$ (Definition 2.2), and we adopt the conventions outlined in Section 2.1. In all proofs that follow we will freely use the coherence theorem for symmetric monoidal categories. In particular, we will assume without loss of generality that the associators and unitors in $\mathcal{SH}$ are identities.

Lemma D.12. *Let (E,μ,e) be a monoid object in $\mathcal{SH}$, and suppose (N,κ) is a left module object over E. Then for all $a\in A$, $\Sigma^a N$ is canonically a left E-module object, with action map given by*

$$\kappa^a : E\otimes S^a\otimes N$$

$$\kappa^a : E\otimes\Sigma^a N = E\otimes S^a\otimes N \xrightarrow{\tau\otimes N} S^a\otimes E\otimes N \xrightarrow{S^a\otimes\kappa} S^a\otimes N = \Sigma^a N.$$

Proof. In this proof, we are assuming that unitality and associativity hold up to strict equality, by the coherence theorem for monoidal categories. In order to show $(\Sigma^a N,\kappa^a)$ is a left module object over E, we need to show κ^a makes the two coherence diagrams in Definition D.6 commute. First, to see the first diagram commutes, consider the following diagram:

$$\begin{tikzcd}
S^a\otimes N \arrow[r, "e\otimes S^a\otimes N"] \arrow[dr, dashed, "S^a\otimes e\otimes N"] \arrow[ddr, equal] & E\otimes S^a\otimes N \arrow[d, "\tau\otimes N"] \\
 & S^a\otimes E\otimes N \arrow[d, "S^a\otimes\kappa"] \\
 & S^a\otimes N
\end{tikzcd}$$

The top inner triangle commutes by coherence for a symmetric monoidal category, and the bottom inner triangle commutes by the coherence condition for κ. To see the other module condition for $\widetilde{\kappa}$, consider the following diagram:

$$\begin{tikzcd}
E\otimes E\otimes S^a\otimes N \arrow[rr, "\mu\otimes S^a\otimes N"] \arrow[d, "E\otimes\tau\otimes N"'] \arrow[dr, dashed, "\tau_{E\otimes E,S^a}\otimes N"] & & E\otimes S^a\otimes N \arrow[d, "\tau\otimes N"] \\
E\otimes S^a\otimes E\otimes N \arrow[r, "\tau\otimes E\otimes N"'] \arrow[d, "E\otimes S^a\otimes\kappa"'] & S^a\otimes E\otimes E\otimes N \arrow[r, "S^a\otimes\mu\otimes N"] \arrow[d, dashed, "S^a\otimes E\otimes\kappa"] & S^a\otimes E\otimes N \arrow[d, "S^a\otimes\kappa"] \\
E\otimes S^a\otimes N \arrow[r, "\tau\otimes N"] & S^a\otimes E\otimes N \arrow[r, "S^a\otimes\kappa"] & S^a\otimes N
\end{tikzcd}$$

The top left triangle commutes by coherence for a symmetric monoidal category. The bottom left rectangle and top right trapezoid commute by naturality of τ. Finally, the bottom right square commutes by the coherence condition for κ. □

Lemma D.13. *Given a monoid object (E, μ, e) in $\mathcal{SH}$, an object X in $\mathcal{SH}$, and some $a \in A$, the suspension of the free module $\Sigma^a(E \otimes X)$ is naturally isomorphic as a left E-module object to the free E-module $E \otimes \Sigma^a X$.*

Proof. It suffices to show the isomorphism $S^a \otimes E \otimes X \xrightarrow{\tau \otimes X} E \otimes S^a \otimes X$ is a homomorphism of left E-module objects. To see this, consider the following diagram:

$$\begin{tikzcd}
E \otimes S^a \otimes E \otimes X \arrow[rr, "E\otimes\tau\otimes X"] \arrow[d, "\tau\otimes E\otimes X"'] & & E \otimes E \otimes S^a \otimes X \arrow[dd, "\mu\otimes S^a\otimes X"] \\
S^a \otimes E \otimes E \otimes X \arrow[urr, "\tau_{S^a, E\otimes E}\otimes X"'] \arrow[d, "S^a\otimes\mu\otimes X"'] & & \\
S^a \otimes E \otimes X \arrow[rr, "\tau\otimes X"] & & E \otimes S^a \otimes X
\end{tikzcd}$$

The top triangle commutes by coherence for a symmetric monoidal category. The bottom trapezoid commutes by naturality of τ. □

Proposition D.14. *Let (E, μ, e) be a monoid object in $\mathcal{SH}$, and suppose we have a family of left E-module objects (N_i, κ_i) indexed by some small set I. Then $N := \bigoplus_{i\in I} N_i$ is canonically a left E-module, with action map given by the composition*

$$\kappa : E \otimes \bigoplus_i N_i \xrightarrow{\cong} \bigoplus_i (E \otimes N_i) \xrightarrow{\bigoplus_i \kappa_i} \bigoplus_i N_i,$$

*where the first isomorphism is given by the fact that $E \otimes -$ preserves coproducts, since it is a left adjoint as $\mathcal{SH}$ is monoidal closed. Furthermore, N is the coproduct of all the N_i's in E-**Mod**, so that E-**Mod** has arbitrary coproducts.*

Proof. We need to show the action map κ makes the diagrams in Definition D.6 commute. To see the first (unitality) diagram commutes, consider the following diagram:

$$\begin{tikzcd}
\bigoplus_i N_i \arrow[r, "e\otimes\bigoplus_i N_i"] \arrow[dr, "\bigoplus_i(e\otimes N_i)"] \arrow[ddr, equal] & E \otimes \bigoplus_i N_i \arrow[d, "\cong"] \\
& \bigoplus_i (E \otimes N_i) \arrow[d, "\bigoplus_i \kappa_i"] \\
& \bigoplus_i N_i
\end{tikzcd}$$

The top triangle commutes by additivity of $E \otimes -$ The bottom triangle commutes by unitality of each of the κ_i's. To see the second coherence diagram commutes, consider the following diagram:

$$\begin{tikzcd}
E \otimes E \otimes \bigoplus_i N_i \arrow[rr, "\mu \oplus \bigoplus_i N_i"] \arrow[d, "E\otimes\cong"'] \arrow[dr, "\cong"] & & E \otimes \bigoplus_i N_i \arrow[d, "\cong"] \\
E \otimes \bigoplus_i (E \otimes N_i) \arrow[r, "\cong"] \arrow[d, "E\otimes\bigoplus_i \kappa_i"'] & \bigoplus_i (E \otimes E \otimes N_i) \arrow[r, "\bigoplus_i(\mu\otimes N_i)"] \arrow[d, "\bigoplus_i(E\otimes\kappa_i)"'] & \bigoplus_i (E \otimes N_i) \arrow[d, "\bigoplus_i \kappa_i"] \\
E \otimes \bigoplus_i N_i \arrow[r, "\cong"'] & \bigoplus_i (E \otimes N_i) \arrow[r, "\bigoplus_i \kappa_i"'] & \bigoplus_i N_i
\end{tikzcd}$$

The bottom right square commutes by coherence for the κ_i's. Every other region commutes by additivity of $- \otimes -$ in each variable. Thus $N = \bigoplus_i N_i$ is indeed a left E-module object, as desired.

Now, we claim that (N, κ) is the coproduct of the (N_i, κ_i)'s in E-**Mod**. First, we need to show that the canonical maps $\iota_i : N_i \hookrightarrow N$ are morphisms in E-**Mod** for all $i \in I$. To see ι_i is a homomorphism of left E-module objects, consider the following diagram:

$$\begin{tikzcd}
E \otimes N_i \arrow[r, "E\otimes\iota_i"] \arrow[dr, hook, "\iota_{E\otimes N_i}"] \arrow[dd, "\kappa_i"'] & E \otimes \bigoplus_i N_i \arrow[d, "\cong"] \\
 & \bigoplus_i (E \otimes N_i) \arrow[d, "\bigoplus_i \kappa_i"] \\
N_i \arrow[r, hook, "\iota_i"'] & \bigoplus_i N_i
\end{tikzcd}$$

The top triangle commutes by additivity of $E \otimes -$. The bottom trapezoid commutes since, by univeral property of the coproduct, $\bigoplus_i \kappa_i$ is the unique arrow which makes the trapezoid commute for all $i \in I$. Now, it remains to show that given a left E-module object (N', κ') and homomorphisms $f_i : N_i \to N'$ of left E-module objects for all $i \in I$, that the unique arrow $f : N \to N'$ in $\mathcal{SH}$ satisfying $f \circ \iota_i = f_i$ for all $i \in I$ is a homomorphism of left E-module objects, so that N is actually the coproduct of the N_i's. To see this, first let $h : \bigoplus_i (E \otimes N_i) \to E \otimes N'$ be the arrow determined by the maps $E \otimes N_i \xrightarrow{E\otimes f_i} E \otimes N'$. Then consider the following diagram:

$$\begin{tikzcd}
E \otimes \bigoplus_i N_i \arrow[rr, "E\otimes f"] \arrow[d, "\cong"'] & & E \otimes N' \arrow[ddd, "\kappa'"] \\
\bigoplus_i (E \otimes N_i) \arrow[r, "\bigoplus_i(E\otimes f_i)"'] \arrow[urr, "h"] \arrow[dd, "\bigoplus_i \kappa_i"'] & \bigoplus_i (E \otimes N') \arrow[ur, "\nabla"'] \arrow[d, "\bigoplus_i \kappa'"] & \\
 & \bigoplus_i N' \arrow[dr, "\nabla"] & \\
\bigoplus_i N_i \arrow[ur, "\bigoplus_i f_i"] \arrow[rr, "f"'] & & N'
\end{tikzcd}$$

The top triangle commutes by additivity of $E \otimes -$. The triangle below that commutes by the universal property of the coproduct, since it is straightforward to check that $\nabla \circ \bigoplus_i (E \otimes f_i)$ and h both satisfy the universal property of the colimit. The left trapezoid commutes by functoriality of $- \oplus -$ and the fact that f_i is a homomorphism of left E-module objects for all i in I. The right trapezoid commutes by naturality of ∇. Finally, the bottom triangle commutes by the universal product of the coproduct, by showing that $\nabla \circ \bigoplus_i f_i$ in place of f also satisfies the universal property of the colimit. Hence f is inded a homomorphism of left E-module objects, as desired.

To recap, we have shown that given a set of left E-module objects $\{(N_i, \kappa_i)\}_{i \in I}$, that the inclusion maps $\iota_i : N_i \hookrightarrow \bigoplus_i N_i$ are morphisms in E-**Mod**, and that given morphism $f_i : (N_i, \kappa_i) \to (N', \kappa')$ for all $i \in I$, the unique induced map $\bigoplus_i N_i \to N'$ is a morphism in E-**Mod**. Thus, E-**Mod** does indeed have arbitrary coproducts, and the forgetful functor E-**Mod** $\to \mathcal{SH}$ preserves them. ☐

Proposition D.15. *Let (E, μ, e) be a monoid object in $\mathcal{SH}$. Then E-**Mod** is an additive category, so that in particular the forgetful functor E-**Mod** $\to \mathcal{SH}$ and the free functor $\mathcal{SH} \to E$-**Mod** are additive.*

Proof. It is a general fact that adjoint functors between additive categories are necessarily additive. In order to show E-**Mod** is an additive category, it suffices to show it has finite coproducts, that $\mathrm{Hom}_{E\text{-}\mathbf{Mod}}(N, N')$ is an abelian group for all left E-modules N and N', and that composition is bilinear. We know that E-**Mod** has coproducts which are preserved by the forgetful functor E-**Mod** $\to \mathcal{SH}$ by Proposition D.14 (which is clearly faithful). Thus, because $\mathcal{SH}$ is **Ab**-enriched and $\mathrm{Hom}_{E\text{-}\mathbf{Mod}}(N, N') \subseteq \mathcal{SH}(N, N')$, it suffices to show that $\mathrm{Hom}_{E\text{-}\mathbf{Mod}}(N, N')$ is closed under addition and taking inverses. To see the former, let $f, g : N \to N'$ be left E-module homomorphisms, and consider the following diagram:

$$\begin{tikzcd}
E \otimes N \arrow[r, "E\otimes\Delta_N"] \arrow[dr, "\Delta_{E\otimes N}"'] \arrow[dddd, "\kappa"'] & E \otimes (N \oplus N) \arrow[rr, "E\otimes(f\oplus g)"] \arrow[d, "\cong"'] & & E \otimes (N' \oplus N') \arrow[r, "E\otimes\nabla_{N'}"] \arrow[d, "\cong"] & E \otimes N' \arrow[dddd, "\kappa'"] \\
& (E \otimes N) \oplus (E \otimes N) \arrow[rr, "(E\otimes f)\oplus(E\otimes g)"] \arrow[dr, "(E\otimes f)\oplus(E\otimes N)"'] \arrow[ddd, "\kappa\oplus\kappa"'] & & (E \otimes N') \otimes (E \otimes N') \arrow[ur, "\nabla_{E\otimes N'}"'] \arrow[ddd, "\kappa'\oplus\kappa'"] & \\
& & (E \otimes N') \otimes (E \otimes N) \arrow[ur, "(E\otimes N')\oplus(E\otimes g)"'] \arrow[d, "\kappa'\oplus\kappa"] & & \\
& & N' \oplus N \arrow[dr, "N'\oplus g"] & & \\
N \arrow[r, "\Delta_N"] & N \oplus N \arrow[rr, "f\oplus g"] \arrow[ur, "f\oplus N"] & & N' \oplus N' \arrow[r, "\nabla_{N'}"] & N'
\end{tikzcd}$$

The outermost trapezoids commute by naturality of Δ and ∇. The triangles in the top corners and the top middle rectangle commute by additivity of $E \otimes -$. The middle triangle commutes by functoriality of $\oplus$ and $\otimes$. The middle trapezoids commute by the fact that f and g are homomorphisms of left E-modules. Finally, the middle bottom triangle commutes by functoriality of $- \oplus -$. Commutativity of the above diagram shows that $f + g$ is a homomorphism of left E-modules as desired. Finally, to see $-f$ is a left E-module homomorphism if f is, we would like to show that $\kappa' \circ (E \otimes (-f)) = (-f) \circ \kappa$. This follows by the fact that $\kappa' \circ (E \otimes f) = f \circ \kappa$ and additivity of $- \otimes -$ and composition. ☐

Definition D.16. Let (E, μ, e) be a monoid object in $\mathcal{SH}$, and suppose (N, κ) and (N', κ') are left E-module objects in E-**Mod**. Then the hom-sets in E-**Mod** can be extended to A-graded abelian groups $\mathrm{Hom}^*_{E\text{-}\mathbf{Mod}}(N, N')$, by defining

$$\mathrm{Hom}^a_{E\text{-}\mathbf{Mod}}(N, N') := \mathrm{Hom}_{E\text{-}\mathbf{Mod}}(\Sigma^a N, N')$$

for each $a \in A$ (where $\Sigma^* N$ has the left E-module structure given by Lemma D.12).

Proposition D.17. *The assignment $(E, \mu, e) \mapsto \pi_*(E)$ is a functor π_* from the category $\mathbf{Mon}_{\mathcal{SH}}$ of monoid objects in $\mathcal{SH}$ (Definition D.2) to the category of A-graded rings. In particular, given a monoid object (E, μ, e) in $\mathcal{SH}$, $\pi_*(E)$ is canonically a ring with product $\pi_*(E) \times \pi_*(E) \to \pi_*(E)$ which sends classes $x : S^a \to E$ and $y : S^b \to E$ to the composition*

$$xy : S^{a+b} \xrightarrow{\phi_{a,b}} S^a \otimes S^b \xrightarrow{x \otimes y} E \otimes E \xrightarrow{\mu} E,$$

and the unit of this ring is $e \in \pi_0(E) = [S, E]$.

Proof. First, we show that $\pi_*(E)$ is actually a ring as indicated. By Lemma C.8, in order to make the A-graded abelian group $\pi_*(E)$ into an A-graded ring, it suffices to construct an associative and unital product only with respect to homogeneous elements. Suppose we have classes x, y, and z in $\pi_a(E)$, $\pi_b(E)$, and $\pi_c(E)$, respectively. To see associativity, consider the following diagram:

$$\begin{tikzcd}
 & & & E \otimes E \arrow[d, "\mu"] \\
S^{a+b+c} \arrow[r, "\cong"] & S^a \otimes S^b \otimes S^c \arrow[r, "x \otimes y \otimes z"] & E \otimes E \otimes E \arrow[ur, "\mu \otimes E"] \arrow[dr, "E \otimes \mu"'] & E \\
 & & & E \otimes E \arrow[u, "\mu"']
\end{tikzcd}$$

(here the first arrow is the unique isomorphism obtained by composing products of $\phi_{a,b}$'s, see Remark 2.3). It commutes by associativity of μ. It follows by functoriality of $- \otimes -$ that the top composition is $(x \cdot y) \cdot z$ while the bottom is $x \cdot (y \cdot z)$, so they are equal as desired. To see that $e \in \pi_0(E)$ is a left and right unit for this multiplication, consider the following diagram

$$\begin{tikzcd}
 & S^a \arrow[dl, "e \otimes x"'] \arrow[d, "x"] \arrow[dr, "x \otimes e"] & \\
E \otimes E \arrow[dr, "\mu"'] & E \arrow[l, "e \otimes E"'] \arrow[r, "E \otimes e"] \arrow[d, equal] & E \otimes E \arrow[dl, "\mu"] \\
 & E &
\end{tikzcd}$$

Commutativity of the two top triangles is functoriality of $- \otimes -$. Commutativity of the bottom two triangles is unitality of μ. Thus the diagram commutes, so $e \cdot x = x = x \cdot e$. Finally, we wish to show this product is bilinear (distributive). Suppose we further have some $x' \in \pi_a(E)$ and $y' \in \pi_b(E)$, and consider the following diagrams:

$$\begin{tikzcd}
S^{a+b} \arrow[r, "\phi_{a,b}"] \arrow[d, "\Delta"'] & S^a \otimes S^b \arrow[r, "\Delta \otimes S^b"] \arrow[d, "\Delta"] & (S^a \oplus S^a) \otimes S^b \arrow[rr, "(x \oplus x') \otimes y"] \arrow[dl, "\cong"'] & & (E \oplus E) \otimes E \arrow[dl, "\cong"'] \arrow[d, "\nabla \otimes E"] & \\
S^{a+b} \oplus S^{a+b} \arrow[r, "\phi_{a,b} \oplus \phi_{a,b}"'] & (S^a \otimes S^b) \oplus (S^a \otimes S^b) \arrow[rr, "(x \otimes y) \oplus (x' \otimes y)"'] & & (E \otimes E) \oplus (E \otimes E) \arrow[r, "\nabla"'] & E \otimes E \arrow[r, "\mu"] & E
\end{tikzcd}$$

$$\begin{tikzcd}
S^{a+b} \arrow[r, "\phi_{a,b}"] \arrow[d, "\Delta"'] & S^a \otimes S^b \arrow[r, "S^a \otimes \Delta"] \arrow[d, "\Delta"] & S^b \otimes (S^b \oplus S^b) \arrow[rr, "x \otimes (y \oplus y')"] \arrow[dl, "\cong"'] & & E \otimes (E \oplus E) \arrow[dl, "\cong"'] \arrow[d, "E \otimes \nabla"] & \\
S^{a+b} \oplus S^{a+b} \arrow[r, "\phi_{a,b} \oplus \phi_{a,b}"'] & (S^a \otimes S^b) \oplus (S^a \otimes S^b) \arrow[rr, "(x \otimes y) \oplus (x \otimes y')"'] & & (E \otimes E) \oplus (E \otimes E) \arrow[r, "\nabla"'] & E \otimes E \arrow[r, "\mu"] & E
\end{tikzcd}$$

The unlabeled isomorphisms are those given by the fact that $- \otimes -$ is additive in each variable (since $\mathcal{SH}$ is tensor triangulated). Commutativity of the left squares is naturality of $\Delta : X \to X \oplus X$ in an additive category. Commutativity of the rest of the diagram follows again from the fact that $- \otimes -$ is an additive functor in each variable. Hence, by functoriality of $- \otimes -$, these diagrams tell us that $(x + x') \cdot y = x \cdot y + x' \cdot y$ and $x \cdot (y + y') = x \cdot y + x \cdot y'$, respectively. Thus, we have shown that if (E, μ, e) is a monoid object in $\mathcal{SH}$ then $\pi_*(E)$ is a ring, as desired.

It remains to show that given a homomorphism of monoid objects $f : (E_1, \mu_1, e_1) \to (E_2, \mu_2, e_2)$ in $\mathbf{Mon}_{\mathcal{SH}}$ that $\pi_*(f) : \pi_*(E_1) \to \pi_*(E_2)$ is an A-graded ring homomorphism. First of all, we know this is an A-graded abelian group homomorphism, since $\mathcal{SH}$ is an additive category, meaning composition with f is an abelian group homomorphism. Thus, in order to show it's a ring homomorphism, it remains to show that $\pi_*(f)(e_1) = e_2$ and that for all $x, y \in \pi_*(E)$ we have $\pi_*(f)(x \cdot y) = \pi_*(f)(x) \cdot \pi_*(f)(y)$. To see the former, note that $\pi_*(f)(e_1) = f \circ e_1$, and $f \circ e_1 = e_2$ since f is a monoid homomorphism in $\mathcal{SH}$. To see the latter, first note by distributivity of multiplication in $\pi_*(E_1)$ and $\pi_*(E_2)$ and the fact that $\pi_*(f)$ is a group homomorphism, it suffices to consider the case that x and y are homogeneous of the form $x : S^a \to E_1$ and $y : S^b \to E_2$. In this case, consider the following diagram:

$$\begin{tikzcd}
S^{a+b} \arrow[r, "\phi_{a,b}"] & S^a \otimes S^b \arrow[r, "x \otimes y"] & E_1 \otimes E_1 \arrow[r, "f \otimes f"] \arrow[d, "\mu_1"'] & E_2 \otimes E_2 \arrow[d, "\mu_2"] \\
 & & E_1 \arrow[r, "f"] & E_2
\end{tikzcd}$$

The top composition is $\pi_*(f)(x) \cdot \pi_*(f)(y)$, while the bottom composition is $\pi_*(f)(x \cdot y)$. The diagram commutes since f is a monoid object homomorphism. Thus $\pi_*(f)(x \cdot y) = \pi_*(f)(x) \cdot \pi_*(f)(y)$, as desired. □

Proposition D.18. *For all $a, b \in A$ there exists an element $\theta_{a,b} \in \pi_0(S) = [S, S]$ such that given any commutative monoid object (E, μ, e) in $\mathcal{SH}$, the A-graded ring structure on $\pi_*(E)$ (**??**) has a commutativity formula given by*

$$x \cdot y = y \cdot x \cdot (e \circ \theta_{a,b})$$

for all $x \in \pi_a(E)$ and $y \in \pi_b(E)$. In particular, $\theta_{a,b} \in \operatorname{Aut}(S)$ is the composition

$$S \xrightarrow{\cong} S^{-a-b} \otimes S^a \otimes S^b \xrightarrow{S^{-a-b} \otimes \tau} S^{-a-b} \otimes S^b \otimes S^a \xrightarrow{\cong} S,$$

where the outermost maps are the unique maps specified by Remark 2.3.

Proof. Let (E, μ, e), x, and y as in the statement of the proposition. Now consider the following diagram

$$\begin{tikzcd}
S^{a+b} \arrow[r, "\phi_{a,b}"] \arrow[d, dashed, "\phi_{b,a}^{-1} \circ \tau \circ \phi_{a,b}"'] & S^a \otimes S^b \arrow[r, "x \otimes y"] \arrow[d, "\tau"'] & E \otimes E \arrow[d, "\tau"'] \arrow[dr, "\mu"] & \\
 & & & E \\
S^{a+b} \arrow[r, "\phi_{b,a}"] & S^b \otimes S^a \arrow[r, "y \otimes x"] & E \otimes E \arrow[ur, "\mu"'] &
\end{tikzcd}$$

The left square commutes by definition. The middle square commutes by naturality of the symmetry isomorphism. Finally, the right square commutes by commutativity of E. Unravelling definitions, we have shown that under the product on $\pi_*(E)$ induced by the $\phi_{a,b}$'s,

$$x \cdot y = (y \cdot x) \circ (\phi_{b,a}^{-1} \circ \tau \circ \phi_{a,b}).$$

Thus, in order to show the desired result it further suffices to show that

$$(y \cdot x) \circ (\phi_{b,a}^{-1} \circ \tau \circ \phi_{a,b}) = y \cdot x \cdot (e \circ \theta_{a,b}).$$

Consider the following diagram:

$$\begin{tikzcd}
 & S^{a+b} \arrow[rr, "\phi_{a,b}"] \arrow[dl, "\cong"'] & & S^a \otimes S^b \arrow[d, "\tau"] \\
S^b \otimes S^a \otimes S^{-a-b} \otimes S^a \otimes S^b \arrow[urrr, "\cong"] \arrow[d, "S^b\otimes S^a\otimes S^{-a-b}\otimes\tau"'] & & & S^b \otimes S^a \arrow[d, "\phi_{b,a}^{-1}"] \arrow[dl, equal] \\
S^b \otimes S^a \otimes S^{-a-b} \otimes S^b \otimes S^a \arrow[urrr, "\cong"] \arrow[rr, "\cong"] & & S^b \otimes S^a \arrow[dr, "y\otimes x"] \arrow[ddl, "y\otimes x\otimes e"'] & S^{a+b} \arrow[l, "\phi_{b,a}"'] \\
 & & & E \otimes E \arrow[dll, "E\otimes E\otimes e"'] \arrow[d, equal] \\
 & E \otimes E \otimes E \arrow[rr, "E\otimes\mu"] \arrow[d, "\mu\otimes E"'] & & E \otimes E \arrow[d, "\mu"] \\
 & E \otimes E \arrow[rr, "\mu"'] & & E
\end{tikzcd}$$

Here any map simply labelled $\cong$ is an appropriate composition of copies of $\phi_{a,b}$'s, associators, and their inverses, so that each of these maps are necessarily unique by Remark 2.3. The triangles in the top large rectangle commutes by coherence for the $\phi_{a,b}$'s. The parallelogram commutes by naturality of τ and coherence of the of $\phi_{a,b}$'s. The middle skewed triangle commutes by functoriality of $-\otimes-$. The triangle below that commutes by unitality of μ. Finally, the bottom rectangle commmutes by associativity of μ. Hence, by unravelling definitions and applying functoriality of $-\otimes-$, we get that the right composition is $(y\cdot x)\circ(\phi_{b,a}^{-1}\circ\tau\circ\phi_{a,b})$, while the left composition is $y\cdot x\cdot(e\circ\theta_{a,b})$, so they are equal as desired. $\square$

Lemma D.19. *Suppose we have homogeneous elements $x, y \in \pi_*(S)$ with x of degree 0, then we have $x \cdot y = y \cdot x = x \circ y$ (where the $\cdot$ denotes the product given in Proposition D.17).*

Proof. As morphisms, y is an arrow $S^a \to S$ for some a in A, and x is a morphism $S \to S$. Then consider the following diagram:

$$\begin{tikzcd}
S \otimes S^a \arrow[dr, "S\otimes y"] \arrow[dd, "y\otimes x"'] & & S^a \arrow[ll, "\phi_{0,a}=\lambda_{S^a}^{-1}"'] \arrow[rr, "\phi_{a,0}=\rho_{S^a}^{-1}"] \arrow[d, "y"'] & & S^a \otimes S \arrow[dl, "y\otimes S"'] \arrow[dd, "x\otimes y"] \\
 & S \otimes S \arrow[r, "\lambda_S=\rho_S"] \arrow[dl, "x\otimes S"'] & S \arrow[d, "x"'] & S \otimes S \arrow[l, "\rho_S=\lambda_S"'] \arrow[dr, "S\otimes x"] & \\
S \otimes S \arrow[rr, "\phi_{0,0}^{-1}=\rho_S"'] & & S & & S \otimes S \arrow[ll, "\phi_{0,0}^{-1}=\lambda_S"]
\end{tikzcd}$$

The trapezoids commute by naturality of the unitors, and the triangles commute by functoriality of $-\otimes-$. The outside compositions are $y\cdot x$ on the left and $x\cdot y$ on the right, and the middle composition is $x\circ y$, so indeed we have $y\cdot x = x\cdot y = x\circ y$, as desired. $\square$

Lemma D.20. *Given $a \in A$, we have $\theta_{0,a} = \theta_{a,0} = \mathrm{id}_S$.*

Proof. Recall $\theta_{a,0}$ is the composition

$$S \xrightarrow{\phi_{-a,a}} S^{-a}\otimes S^a \xrightarrow{S^{-a}\otimes\phi_{a,0}} S^{-a}\otimes(S^a\otimes S) \xrightarrow{S^{-a}\otimes\tau} S^{-a}\otimes(S\otimes S^a) \xrightarrow{S^{-a}\otimes\phi_{0,a}^{-1}} S^{-a}\otimes S^a \xrightarrow{\phi_{-a,a}^{-1}} S$$

By the coherence theorem for symmetric monoidal categories and the fact that $\phi_{a,0}$ and $\phi_{0,a}$ coincide with the unitors, we have that the composition

$$S^a \xrightarrow{\phi_{a,0}=\rho_{S^a}^{-1}} S^a \otimes S \xrightarrow{\tau} S \otimes S^a \xrightarrow{\phi_{0,a}^{-1}=\lambda_{S^a}} S^a$$

is precisely the identity map, so by functoriality of $-\otimes-$, we have that $\theta_{a,0}$ is the composition

$$S \xrightarrow{\phi_{-a,a}} S^{-a} \otimes S^a \xrightarrow{=} S^{-a} \otimes S^a \xrightarrow{\phi_{-a,a}^{-1}} S,$$

so $\theta_{a,0} = \mathrm{id}_S$, meaning

$$x \cdot y = y \cdot x \cdot (e \circ \theta_{a,0}) = y \cdot x \cdot e = y \cdot x,$$

where the last equality follows by the fact that e is the unit for the multiplication on $\pi_*(E)$. An entirely analagous argument yields that $\theta_{0,a} = \mathrm{id}_S$. □

Lemma D.21. *Let $a, b \in A$. Then $\theta_{a,b} \cdot \theta_{b,a} = \mathrm{id}_S$.*

Proof. By Lemma D.19, it suffices to show that $\theta_{a,b} \circ \theta_{b,a} = \mathrm{id}_S$. To see this, consider the following diagram:

$$\begin{tikzcd}
S \arrow[r, "\phi"] \arrow[dddrrr, equal] & S^{-a-b} \otimes S^b \otimes S^a \arrow[r, "S^{-a-b}\otimes\tau"] \arrow[ddrr, equal] & S^{-a-b} \otimes S^a \otimes S^b \arrow[r, "\phi"] \arrow[dr, equal] & S \arrow[d, "\phi"] \\
 & & & S^{-a-b} \otimes S^a \otimes S^b \arrow[d, "S^{-a-b}\otimes\tau"] \\
 & & & S^{-a-b} \otimes S^b \otimes S^a \arrow[d, "\phi"] \\
 & & & S
\end{tikzcd}$$

Here we are suppressing associators, and any map labelled ϕ is the appropriate composition of $\phi_{a,b}$'s, unitors, associators, identities, and their inverses (see Remark 2.3). Clearly each region commutes, the middle by the fact that $\tau^2 = 0$, and the other two regions by coherence for the ϕ's. Thus we have shwon $\theta_{a,b} \cdot \theta_{b,a} = \theta_{a,b} \cdot \theta_{b,a} = \mathrm{id}_S$, as desired. □

Lemma D.22. *Let $a, b, c \in A$. Then $\theta_{a,b} \cdot \theta_{a,c} = \theta_{a,b+c}$ and $\theta_{b,a} \cdot \theta_{c,a} = \theta_{b+c,a}$.*

Proof. By Lemma D.19, it suffices to show that $\theta_{a,b} \circ \theta_{a,c} = \theta_{a,b+c}$ and $\theta_{b,a} \circ \theta_{c,a} = \theta_{b+c,a}$. First we show $\theta_{a,b} \circ \theta_{a,c} = \theta_{a,b+c}$. To see this, consider the following diagram:
(6)

$$\begin{tikzcd}
S \arrow[r, "\phi"] \arrow[d, "\phi"'] \arrow[dr, phantom, "\text{(A)}"] & S^{-a-c}S^aS^c \arrow[r, "S^{-a-c}\tau"] \arrow[d, "\phi"'] \arrow[dr, phantom, "\text{(B)}"] & S^{-a-c}S^cS^a \arrow[r, "\phi"] \arrow[d, "\phi"] \arrow[dr, phantom, "\text{(C)}"] & S \arrow[d, "\phi"] \\
S^{-a-b-c}S^aS^{b+c} \arrow[r, "\phi"] \arrow[d, "S^{-a-b-c}\tau"'] \arrow[dr, phantom, "\text{(D)}"] & S^{-a-c}S^{-b}S^aS^bS^c \arrow[r, "S^{-a-c}\tau_{S^{-b}S^aS^b,S^c}"] \arrow[d, "S^{-a-c}S^{-b}\tau_{S^a,S^bS^c}"'] \arrow[dr, phantom, "\text{(E)}"] & S^{-a-c}S^cS^{-b}S^aS^b \arrow[d, "S^{-a-c}S^cS^{-b}\tau"'] \arrow[dr, phantom, "\text{(F)}"] & S^{-a-b}S^aS^b \arrow[l, "\phi"'] \arrow[d, "S^{-a-b}\tau"] \\
S^{-a-b-c}S^{b+c}S^a \arrow[r, "\phi"] \arrow[dd, "\phi"'] & S^{-a-c}S^{-b}S^bS^cS^a \arrow[r, "S^{-a-c}\tau_{S^{-b}S^b,S^c}S^a"] \arrow[dr, phantom, "\text{(G)}"] & S^{-a-c}S^cS^{-b}S^bS^a & S^{-a-b}S^bS^a \arrow[l, "\phi"'] \arrow[dd, "\phi"] \\
& S^{-a-c}S^cS^a \arrow[r, equal] \arrow[u, "\phi"] \arrow[dr, phantom, "\text{(H)}"] & S^{-a-c}S^cS^a \arrow[u, "\phi"'] & \\
S \arrow[rrr, equal] & & & S
\end{tikzcd}$$

Here we are omitting $\otimes$ from the notation, and each occurrence of an arrow labelled ϕ indicates it is the unique arrow that can be obtained as a formal composition of tensor products of copies of $\phi_{a,b}$'s, unitors, associators, and their inverses (Remark 2.3). Clearly the composition going around the top and then the right is $\theta_{a,b} \circ \theta_{a,c}$ while the composition going left around the bottom is $\theta_{a,b+c}$. Thus, we wish to show the above diagram commutes.

Regions (A), (C), and (H) commute by coherence for the ϕ's (see previous remark). Region (E) commutes by coherence for the τ's. To see region (B) commutes, consider the following diagram, which commutes by naturality of τ:

$$\begin{tikzcd}
S^{-a-c}S^aS^c \arrow[r, "S^{-a-c}\tau"] \arrow[d, "S^{-a-c}\phi_{a-b,b}S^c"'] & S^{-a-c}S^cS^a \arrow[d, "S^{-a-c}S^c\phi_{a-b,b}"] \\
S^{-a-c}S^{a-b}S^bS^c \arrow[r, "S^{-a-c}\tau_{S^{a-b}S^b,S^c}"] \arrow[d, "S^{-a-c}\phi_{-b,a}S^bS^c"'] & S^{-a-c}S^cS^{a-b}S^b \arrow[d, "S^{-a-c}S^c\phi_{-b,a}S^b"] \\
S^{-a-c}S^{-b}S^aS^bS^c \arrow[r, "S^{-a-c}\tau_{S^{-b}S^aS^b,S^c}"] & S^{-a-c}S^cS^{-b}S^aS^b
\end{tikzcd}$$

To see region (D) commutes, note that it is simply the square

$$\begin{tikzcd}
S^{-a-b-c}S^aS^{b+c} \arrow[r, "\phi_{-a-c,-b}S^a\phi_{b,c}"] \arrow[d, "S^{-a-b-c}\tau"'] & S^{-a-c}S^{-b}S^aS^bS^c \arrow[d, "S^{-a-c}S^{-b}\tau_{S^a,S^bS^c}"] \\
S^{-a-b-c}S^{b+c}S^a \arrow[r, "\phi_{-a-c,-b}\phi_{b,c}S^a"] & S^{-a-c}S^{-b}S^bS^cS^a
\end{tikzcd}$$

This diagram commutes by naturality of τ. To see region (F) commutes, consider the following diagram, which commutes by functoriality of $- \otimes -$:

$$\begin{tikzcd}
S^{-a-c}S^cS^{-b}S^aS^b \arrow[d, "S^{-a-c}S^cS^{-b}\tau"'] & S^{-a-c}S^{c-b}S^aS^b \arrow[l, "S^{-a-c}\phi_{c,-b}S^aS^b"'] \arrow[d, "S^{-a-c}S^{c-b}\tau"] & S^{-a-b}S^aS^b \arrow[l, "\phi_{-a-c,c-b}S^aS^b"'] \arrow[d, "S^{-a-b}\tau"] \\
S^{-a-c}S^cS^{-b}S^bS^a & S^{-a-c}S^{c-b}S^bS^a \arrow[l, "S^{-a-c}\phi_{c,-b}S^bS^a"'] & S^{-a-b}S^bS^a \arrow[l, "\phi_{-a-c,c-b}S^bS^a"']
\end{tikzcd}$$

Finally, to see region (G) commutes, consider the following diagram:

$$\begin{tikzcd}
S^{-a-c}S^{-b}S^bS^cS^a \arrow[r, "S^{-a-c}\tau_{S^{-b}S^b,S^c}S^a"] & S^{-a-c}S^cS^{-b}S^bS^a \\
S^{-a-c}SS^cS^a \arrow[u, "S^{-a-c}\phi_{-b,b}S^cS^a"] \arrow[r, "S^{-a-c}\tau_{S,S^c}S^a"] & S^{-a-c}S^cSS^a \arrow[u, "S^{-a-c}S^c\phi_{-b,b}S^a"'] \\
S^{-a-c}S^cS^a \arrow[u, "S^{-a-c}\phi_{0,c}S^a=S^{-a-c}\lambda_{S^c}^{-1}S^a"] \arrow[r, equal] & S^{-a-c}S^cS^a \arrow[u, "S^{-a-c}\phi_{c,0}S^a=S^{-a-c}S\rho_{S^c}^{-1}S^a"']
\end{tikzcd}$$

The top region commutes by naturality of τ, while the bottom region commutes by coherence for a symmetric monoidal category. Thus, we have shown that diagram (6) commutes, so that $\theta_{a,b} \circ \theta_{a,c} = \theta_{a,b+c}$, as desired. Now, to see that $\theta_{b,a} \cdot \theta_{c,a} = \theta_{b+c,a}$, note that

$$\theta_{b,a} \cdot \theta_{c,a} \overset{(*)}{=} \theta_{a,b}^{-1} \cdot \theta_{a,c}^{-1} = (\theta_{a,c} \cdot \theta_{a,b})^{-1} = \theta_{a,b+c}^{-1} \overset{(*)}{=} \theta_{b+c,a},$$

where each occurrence of $(*)$ is Lemma D.21. $\square$

Lemma D.23. *Let X and Y be objects in $\mathcal{SH}$. Then the A-graded pairing*

$$\pi_*(X) \times \pi_*(Y) \to \pi_*(X \otimes Y)$$

sending $x : S^a \to X$ and $y : S^b \to Y$ to the composition

$$S^{a+b} \xrightarrow{\phi_{a,b}} S^a \otimes S^b \xrightarrow{x \otimes y} X \otimes Y$$

is additive in each argument.

Proof. Let $a, b \in A$, and let $x_1, x_2 : S^a \to X$ and $y : S^b \to Y$. Then consider the following diagram

$$\begin{tikzcd}
S^{a+b} \arrow[r, "\cong"] & S^a \otimes S^b \arrow[r, "\Delta \otimes S^b"] \arrow[d, "\Delta"'] & (S^a \oplus S^a) \otimes S^b \arrow[dl, "\cong"'] \arrow[d, "(x_1 \oplus x_2) \otimes y"] \\
& (S^a \otimes S^b) \oplus (S^a \otimes S^b) \arrow[d, "(x_1 \otimes y) \oplus (x_2 \otimes y)"'] & (X \oplus X) \otimes Y \arrow[dl, "\cong"'] \arrow[d, "\nabla \otimes Y"] \\
& (X \otimes Y) \oplus (X \otimes Y) \arrow[r, "\nabla"] & X \otimes Y
\end{tikzcd}$$

The isomorphisms are given by the fact that $- \otimes -$ is additive in each variable. Both triangles and the parallelogram commute since $- \otimes -$ is additive. By functoriality of $- \otimes -$, the top composition is $(x_1 + x_2) \cdot y$ and the bottom composition is $x_1 \cdot y + x_2 \cdot y$, so they are equal, as desired. An entirely analagous argument yields that $x \cdot (y_1 + y_2) = x \cdot y_1 + x \cdot y_2$ for $x \in \pi_*(X)$ and $y_1, y_2 \in \pi_*(Y)$. $\square$

Lemma D.24. *Let (E, μ, e) be a monoid object. Then the assignment $\pi_* : (N, \kappa) \mapsto \pi_*(N)$ yields an additive functor from E-**Mod** to A-graded left $\pi_*(E)$-modules. In particular, if (N, κ) is a left E-module in $\mathcal{SH}$ then the map*

$$\pi_*(E) \times \pi_*(N) \to \pi_*(N)$$

sending a class $r : S^a \to E$ and $x : S^b \to N$ to the composition

$$r \cdot x : S^{a+b} \xrightarrow{\phi_{a,b}} S^a \otimes S^b \xrightarrow{r \otimes x} E \otimes N \xrightarrow{\kappa} N$$

endows $\pi_*(N)$ with the structure of an A-graded left $\pi_*(E)$-module.

Proof. First let (N, κ) be an E-module object. Let $a, b, c \in A$ and $x, x' : S^a \to N$, $y : S^b \to E$, and $z, z' \in S^c \to E$. Then by Lemma C.9, it suffices to show that

(1) $y \cdot (x + x') = y \cdot x + y \cdot x'$,
(2) $(z + z') \cdot x = z \cdot x + z' \cdot x$,
(3) $(zy) \cdot x = z \cdot (y \cdot x)$,
(4) $e \cdot x = x$.

The first two axioms follow by Lemma D.23. To see (3), consider the diagram:

$$\begin{tikzcd}
 & & & E \otimes N \arrow[d, "\kappa"] \\
S^{a+b+c} \arrow[r, "\cong"] & S^c \otimes S^b \otimes S^a \arrow[r, "z \otimes y \otimes x"] & E \otimes E \otimes N \arrow[ur, "E \otimes \kappa"] \arrow[dr, "\mu \otimes N"'] & N \\
 & & & E \otimes N \arrow[u, "\kappa"']
\end{tikzcd}$$

It commutes by coherence for κ. By functoriality of $- \otimes -$, the two outside compositions equal $z \cdot (y \cdot x)$ on the top and $(z \cdot y) \cdot x$ on the bottom. Hence, they are equal, as desired.

Next, to see (4), consider the following diagram:

$$\begin{tikzcd}
S^a \arrow[rr, "x"] \arrow[dr, "x"] \arrow[ddr, bend right, "e \otimes x"'] & & N \\
 & N \arrow[ur, equal] \arrow[d, dashed, "e \otimes N"] & \\
 & E \otimes N \arrow[uur, bend right, "\kappa"'] &
\end{tikzcd}$$

The top triangle commutes by definition. The left triangle commutes by functoriality of $- \otimes -$. The right triangle commutes by unitality of κ. The top composition is x while the bottom is $e \cdot x$, thus they are necessarily equal since the diagram commutes.

Now, we'd like to show that if $f : (N, \kappa) \to (N', \kappa)$ is a homomorphism of left E-module objects, then $\pi_*(f) : \pi_*(N) \to \pi_*(N')$ is a homomorphism of left $\pi_*(E)$-modules. To see this, let $r : S^a \to E$ in $\pi_a(E)$ and $x, x : S^b \to N$ in $\pi_b(N)$. We'd like to show that $\pi_*(f)(x + x') = \pi_*(f)(x) + \pi_*(f)(x')$ and $\pi_*(f)(r \cdot x) = r \cdot \pi_*(f)(x)$. To see the former, consider the following diagram:

$$\begin{tikzcd}
 & & & N' \oplus N' \arrow[d, "\nabla"] \\
S^a \arrow[r, "\Delta"] & S^a \oplus S^a \arrow[r, "x\oplus x'"] & N \oplus N \arrow[ur, "f\oplus f"] \arrow[dr, "\nabla"'] & N' \\
 & & & N \arrow[u, "f"']
\end{tikzcd}$$

It commutes by naturality of ∇ in an additive category. The top composition is $\pi_*(f)(x) + \pi_*(f)(x')$, while the bottom is $\pi_*(f)(x+x')$, so they are equal as desired. To see that $\pi_*(f)(r\cdot x) = r \cdot \pi_*(f)(x)$, consider the following diagram:

$$\begin{tikzcd}
 & & & E \otimes N' \arrow[d, "\kappa'"] \\
S^{a+b} \arrow[r, "\phi_{b,a}"] & S^b \otimes S^a \arrow[r, "r\otimes x"] & E \otimes N \arrow[ur, "E\otimes f"] \arrow[dr, "\kappa"'] & N' \\
 & & & N \arrow[u, "f"']
\end{tikzcd}$$

It commutes by the fact that f is a homomorphism of left E-module objects. The bottom composition is $\pi_*(f)(r \cdot x)$, while the top composition is $r \cdot \pi_*(f)(x)$, so they are equal, as desired.

It remains to show this functor is additive. It suffices to show it preserves the zero object and preserves coproducts. To see the former, note that $\pi_*(0) = [S^*, 0] = 0$ by definition, since 0 is terminal. To see the latter, we need to show that given $(N,\kappa), (N', \kappa') \in E$-**Mod** that $\pi_*(N) \oplus \pi_*(N') \cong \pi_*(N \oplus N')$, and that the following diagram commutes:

$$\begin{tikzcd}
\pi_*(N) \arrow[d, "\iota_{\pi_*(N)}"'] \arrow[dr, "\pi_*(\iota_N)"] & \\
\pi_*(N) \oplus \pi_*(N') \arrow[r, "\cong"'] & \pi_*(N \oplus N')
\end{tikzcd}$$

Since each S^a is compact, for all $a, b \in A$ we have isomorphisms

$$\pi_a(N) \oplus \pi_a(N') = [S^a, N] \oplus [S^a, N'] \cong [S^a, N \oplus N'] = \pi_a(N, N'),$$

and these combine together to yield A-graded isomorphisms $\pi_*(N) \oplus \pi_*(N') \xrightarrow{\cong} \pi_*(N \oplus N')$. To see the above diagram commutes, note that since everything is an A-graded homomorphism of A-graded abelian groups, it suffices to chase homogeneous elements around to show it commutes. Indeed, it is entirely straightforward, by unravelling definitions, that both compositions around the diagram take a generator $x : S^a \to N$ in $\pi_a(N)$ to the composition

$$S^a \xrightarrow{x} N \xrightarrow{\iota_N} N \oplus N'.$$

Thus, we have shown that π_* preserves all finite coproducts, so it is additive. □

Proposition D.25. *Let (E, μ, e) be a monoid object in $\mathcal{SH}$. Then $E_*(-)$ is a functor from $\mathcal{SH}$ to left A-graded modules over the ring $\pi_*(E)$ (Proposition D.17), where given some X in $\mathcal{SH}$, $E_*(X)$ may be endowed with the structure of a left A-graded $\pi_*(E)$-module via the map*

$$\pi_*(E) \times E_*(X) \to E_*(X)$$

which given $a, b \in A$, sends $x : S^a \to E$ and $y : S^b \to E \otimes X$ to the composition

$$x \cdot y : S^{a+b} \cong S^a \otimes S^b \xrightarrow{x\otimes y} E \otimes (E \otimes X) \cong (E \otimes E) \otimes X \xrightarrow{\mu\otimes X} E \otimes X.$$

Similarly, the assignment $X \mapsto X_*(E)$ is a functor from $\mathcal{SH}$ to right A-graded $\pi_*(E)$-modules, where the structure map

$$X_*(E) \times \pi_*(E) \to X_*(E)$$

sends $x : S^a \to X \otimes E$ and $y : S^b \to E$ to the composition

$$x \cdot y : S^{a+b} \cong S^a \otimes S^b \xrightarrow{x \otimes y} (X \otimes E) \otimes E \cong X \otimes (E \otimes E) \xrightarrow{X \otimes \mu} X \otimes E.$$

Finally, $E_*(E)$ is a $\pi_*(E)$-bimodule, in the sense that the left and right actions of $\pi_*(E)$ are compatible, so that given $y, z \in \pi_*(E)$ and $x \in E_*(E)$, $y \cdot (x \cdot z) = (y \cdot x) \cdot z$.

Proof. By Lemma C.9, in order to make the A-graded abelian group $E_*(X)$ into a left A-graded module over the A-graded ring $\pi_*(E)$, it suffices to define the action map $\pi_*(E) \times E_*(X) \to E_*(X)$ only for homogeneous elements, and to show that given homogeneous elements $x, x' : S^a \to E \otimes X$ in $E_a(X)$, $y : S^b \to E$ in $\pi_b(E)$, and $z, z' : S^c \to E$ in $\pi_c(E)$, that:

(1) $y \cdot (x + x') = y \cdot x + y \cdot x'$,
(2) $(z + z') \cdot x = z \cdot x + z' \cdot x$,
(3) $(zy) \cdot x = z \cdot (y \cdot x)$,
(4) $e \cdot x = x$.

Axioms (1) and (2) follow by the fact that $E_*(X) = \pi_*(E \otimes X)$ and Lemma D.23. To see (3), consider the diagram:

$$\begin{tikzcd}
 & & & E \otimes E \otimes X \arrow[d, "\mu \otimes X"] \\
S^{a+b+c} \arrow[r, "\cong"] & S^c \otimes S^b \otimes S^a \arrow[r, "z \otimes y \otimes x"] & E \otimes E \otimes E \otimes X \arrow[ur, "E \otimes \mu \otimes X"] \arrow[dr, "\mu \otimes E \otimes X"'] & E \otimes X \\
 & & & E \otimes E \otimes X \arrow[u, "\mu \otimes X"']
\end{tikzcd}$$

It commutes by associativity of μ. By functoriality of $- \otimes -$, the two outside compositions equal $z \cdot (y \cdot x)$ on the top and $(z \cdot y) \cdot x$ on the bottom. Hence, they are equal, as desired.

Next, to see (4), consider the following diagram:

$$\begin{tikzcd}
S^a \arrow[rr, "x"] \arrow[dr, "x"'] \arrow[ddr, bend right, "e \otimes x"'] & & E \otimes X \\
 & E \otimes X \arrow[ur, equal] \arrow[d, "e \otimes E \otimes X"] & \\
 & E \otimes E \otimes X \arrow[uur, bend right, "\mu \otimes X"'] &
\end{tikzcd}$$

The top triangle commutes by definition. The left triangle commutes by functoriality of $- \otimes -$. The right triangle commutes by unitality of μ. The top composition is x while the bottom is $e \cdot x$, thus they are necessarily equal since the diagram commutes.

Thus, we have shown that the indicated map does indeed endow $E_*(X)$ with the structure of a left $\pi_*(E)$-module. It remains to show that $E_*(-)$ sends maps in $\mathcal{SH}$ to A-graded homomorphisms of left A-graded $\pi_*(E)$-modules. By definition, given $f : X \to Y$ in $\mathcal{SH}$, $E_*(f)$ is the map which takes a class $x : S^a \to E \otimes X$ to the composition

$$S^a \xrightarrow{x} E \otimes X \xrightarrow{E \otimes f} E \otimes Y.$$

To see this assignment is a homomorphism, suppose we are given some other $x' : S^a \to E \otimes X$ and some scalar $y : S^b \to E$. Then we would like to show $E_*(f)(x + x') = E_*(f)(x) + E_*(f)(x')$ and $E_*(f)(y \cdot x) = y \cdot E_*(f)(x)$. To see the former, consider the following diagram:

$$\begin{tikzcd}
 & & & (E \otimes Y) \oplus (E \otimes Y) \arrow[d, "\nabla"] \\
S^a \arrow[r, "\Delta"] & S^a \oplus S^a \arrow[r, "x \oplus x'"] & (E \otimes X) \oplus (E \otimes X) \arrow[ur, "(E\otimes f)\oplus(E\otimes f)"] \arrow[dr, "\nabla"'] & E \otimes Y \\
 & & & E \otimes X \arrow[u, "E\otimes f"']
\end{tikzcd}$$

It commutes by naturality of ∇ in an additive category. The top composition is $E_*(f)(x) + E_*(f)(x')$, while the bottom is $E_*(f)(x+x')$, so they are equal as desired. To see that $E_*(f)(y \cdot x) = y \cdot E_*(f)(x)$, consider the following diagram:

$$\begin{tikzcd}
S^{a+b} \arrow[r, "\phi_{b,a}"] & S^b \otimes S^a \arrow[r, "y \otimes x"] & E \otimes E \otimes X \arrow[r, "E\otimes E\otimes f"] \arrow[d, "\mu\otimes X"'] & E \otimes E \otimes Y \arrow[d, "\mu\otimes Y"] \\
 & & E \otimes X \arrow[r, "E\otimes f"] & E \otimes Y
\end{tikzcd}$$

It commutes by functoriality of $- \otimes -$. The top composition is $E_*(f)(y \cdot x)$, while the bottom composition is $y \cdot E_*(f)(x)$, so they are equal, as desired.

Showing that $X_*(E)$ has the structure of a right $\pi_*(E)$-module and that if $f : X \to Y$ is a morphism in $\mathcal{SH}$ then the map

$$X_*(E) = [S^*, X \otimes E] \xrightarrow{(f\otimes E)_*} [S^*, Y \otimes E] = Y_*(E)$$

is an A-graded homomorphism of right A-graded $\pi_*(E)$-modules is entirely analagous.

It remains to show that $E_*(E)$ is a $\pi_*(E)$-bimodule. Let $x : S^a \to E$, $y : S^b \to E \otimes E$, and $z : S^c \to E$, and consider the following diagram:

$$\begin{tikzcd}
 & & & E \otimes E \otimes E \arrow[d, "E\otimes\mu"] \\
S^{a+b+c} \arrow[r, "\cong"] & S^a \otimes S^b \otimes S^c \arrow[r, "x\otimes y\otimes z"] & E \otimes E \otimes E \otimes E \arrow[r, "\mu\otimes\mu"] \arrow[ur, "\mu\otimes E\otimes E"] \arrow[dr, "E\otimes E\otimes\mu"'] & E \otimes E \\
 & & & E \otimes E \otimes E \arrow[u, "\mu\otimes E"']
\end{tikzcd}$$

Commutativity follows by functoriality of $- \otimes -$, which also tells us that the two outside compositions are $(x \cdot y) \cdot z$ (on top) and $x \cdot (y \cdot z)$ (on bottom). Hence they are equal, as desired. $\square$

Lemma D.26. *Let (E, μ, e) be a monoid object in $\mathcal{SH}$, (N, κ) a left E-module, and $a \in A$. Then the assignment*

$$\mathrm{tw}^a : \pi_{*-a}(N) \to \pi_*(\Sigma^a N)$$

sending $x : S^{b-a} \to N$ to the composition

$$S^b \xrightarrow{\phi_{b-a,a}} S^{b-a} \otimes S^a \xrightarrow{x\otimes S^a} N \otimes S^a \xrightarrow{\tau} S^a \otimes N = \Sigma^a N$$

is an A-graded isomorphism of left A-graded $\pi_*(E)$-modules (where here $\pi_*(N)$ is a left $\pi_*(E)$-module by Lemma D.24, and $\pi_*(\Sigma^a N)$ has the left $\pi_*(E)$ module given by Lemma D.12 and Lemma D.24).

Proof. Unravelling definitions, the map $\mathrm{tw}^a : \pi_{*-a}(N) \to \pi_*(\Sigma^a N)$ factors as

$$\pi_{*-a}(N) = [S^{*-a}, N] \xrightarrow{-\otimes S^a} [S^{*-a}\otimes S^a, N\otimes S^a] \xrightarrow{(\phi_{*-a,a})^*} [S^*, N\otimes S^a] \xrightarrow{\tau_*} [S^*, S^a\otimes N] = \pi_*(\Sigma^a N).$$

The arrow labeled $-\otimes S^a$ is an isomorphism of abelian groups because $-\otimes S^a \cong \Sigma^a$ is an autoequivalence of $\mathcal{SH}$ (Proposition 2.4). Hence, we have shown the map is an isomorphism of abelian groups. Clearly the map preserves degree, so it is an A-graded homomorphism as desired. Finally, it remains to show that this map is a homomorphism of left $\pi_*(E)$-modules, i.e., that given $r : S^b \to E$ in $\pi_*(E)$ and $x : S^{c-a} \to N$ in $\pi_{*-a}(N)$ that $\mathrm{tw}^a(r\cdot x) = r\cdot \mathrm{tw}^a(x)$. Unravelling definitions, $\mathrm{tw}^a(r\cdot x)$ is the composition

$$S^{b+c} \xrightarrow{\cong} S^b\otimes S^{c-a}\otimes S^a \xrightarrow{r\otimes x\otimes S^a} E\otimes N\otimes S^a \xrightarrow{\kappa\otimes S^a} N\otimes S^a \xrightarrow{\tau} S^a\otimes N,$$

while on the other hand $r\cdot \mathrm{tw}^a(x)$ is the composition

$$S^{b+c} \xrightarrow{\cong} S^b\otimes S^{c-a}\otimes S^a \xrightarrow{r\otimes x\otimes S^a} E\otimes N\otimes S^a \xrightarrow{E\otimes\tau} E\otimes S^a\otimes N \xrightarrow{\tau\otimes N} S^a\otimes E\otimes N \xrightarrow{S^a\otimes\kappa} S^a\otimes N.$$

To see these are equal, consider the following diagram:

$$\begin{tikzcd}
 & & E\otimes S^a\otimes N \arrow[r, "\tau\otimes N"] & S^a\otimes E\otimes N \arrow[d, "S^a\otimes\kappa"] \\
S^{b+c} \arrow[r, "\phi"] & S^b\otimes S^{c-a}\otimes S^a \arrow[r, "r\otimes x\otimes S^a"] & E\otimes N\otimes S^a \arrow[u, "E\otimes\tau"] \arrow[ur, "\tau_{E\otimes N,S^a}"] \arrow[dr, "\kappa\otimes S^a"'] & S^a\otimes N \\
 & & & N\otimes S^a \arrow[u, "\tau"']
\end{tikzcd}$$

The top triangle commutes by coherence for a symmetric monoidal category, while the right triangle commutes by naturality of τ. □

Lemma D.27. *Let (E,μ,e) be a monoid object and (N,κ) a left E-module object in $\mathcal{SH}$. Then given a collection of $a_i \in A$ indexed by some set I, if (N,κ) is a retract of $\bigoplus_i(E\otimes S^{a_i})$ in E-**Mod**,[11] then for all left E-module objects (N',κ') in $\mathcal{SH}$, the functor π_* (Lemma D.24) induces an isomorphism*

$$\pi_* : \mathrm{Hom}_{E\text{-}\mathbf{Mod}}(N,N') \to \mathrm{Hom}_{\pi_*(E)}(\pi_*(N),\pi_*(N')).$$

Proof. First, we consider the case $N = \bigoplus_i(E\otimes S^{a_i})$. Consider the following diagram:

$$\begin{tikzcd}
\mathrm{Hom}_{E\text{-}\mathbf{Mod}}(\bigoplus_i(E\otimes S^{a_i}),N') \arrow[r, "\pi_*"] \arrow[d, "\cong"'] & \mathrm{Hom}_{\pi_*(E)}(\pi_*(\bigoplus_i(E\otimes S^{a_i})),\pi_*(N')) \arrow[d, "\cong"] \\
\prod_i \mathrm{Hom}_{E\text{-}\mathbf{Mod}}(E\otimes S^{a_i},N') \arrow[d, "\cong"'] & \mathrm{Hom}_{\pi_*(E)}(\bigoplus_i \pi_*(E\otimes S^{a_i}),\pi_*(N')) \arrow[d, "\cong"] \\
\prod_i[S^{a_i},N'] \arrow[dd, equal] & \prod_i \mathrm{Hom}_{\pi_*(E)}(\pi_*(E\otimes S^{a_i}),\pi_*(N')) \arrow[d, "\cong"] \\
 & \prod_i \mathrm{Hom}_{\pi_*(E)}(\pi_{*-a_i}(E),\pi_*(N')) \arrow[d, equal] \\
\prod_i \pi_{a_i}(N') & \prod_i \mathrm{Hom}^{a_i}_{\pi_*(E)}(\pi_*(E),\pi_*(N')) \arrow[l, "\prod_i \mathrm{ev}_1"']
\end{tikzcd}$$

[11]Here $\bigoplus_i(E\otimes S^{a_i})$ is a coproduct (Proposition D.14) of a bunch of left free E-module objects (Proposition D.10), so it is itself a left E-module object.

Here the top left vertical isomorphism exibits the universal property of the coproduct in E-**Mod**, and middle left vertical isomorphism below that is the free-forgetful adjunction for E-modules (Proposition D.10). The bottom horizontal isomorphism is the product of the evaluation-at-1 isomorphisms (Lemma C.11). On the other side, the top right vertical isomorphism is given by the fact that S^a is compact for each $a \in A$, so we have isomorphisms

$$\bigoplus_i \pi_*(E \otimes S^{a_i}) = \bigoplus_{a\in A} \bigoplus_i [S^a, E \otimes S^{a_i}] \cong \bigoplus_{a\in A} [S^a, \bigoplus_i (E \otimes S^{a_i})] = \pi_*(\bigoplus_i (E \otimes S^{a_i})),$$

where the middle isomorphism takes a generator $x : S^a \xrightarrow{E} \otimes S^{a_i}$ to the composition $S^a \xrightarrow{x} E \otimes S^{a_i} \hookrightarrow \bigoplus_i (E \otimes S^{a_i})$. The middle right vertical isomorphism exhibits the universal property of the coproduct of modules. Finally the bottom right vertical isomorphism is given by the isomorphisms

$$\pi_{*-a_i}(E \otimes S^{a_i}) = [S^{*-a_i}, E \otimes S^{a_i}] \xrightarrow{-\otimes S^{a_i}} [S^{*-a_i} \otimes S^{a_i}, E \otimes S^{a_i}] \xrightarrow{\phi^*} [S^*, E \otimes S^{a_i}] = \pi_*(E \otimes S^{a_i}),$$

where $-\otimes S^{a_i} \cong \Sigma^{a_i}$ is an isomorphism by Proposition 2.4. Now, we claim this diagram commutes. This really simply amounts to unravelling definitions, and chasing a homomorphism $f : \bigoplus_i (E \otimes S^{a_i}) \to N'$ of left E-module objects both ways around the diagram yields the composition

$$\prod_i (S^{a_i} \xrightarrow{e\otimes S^{a_i}} E \otimes S^{a_i} \hookrightarrow \bigoplus_i (E \otimes S^{a_i}) \xrightarrow{f} N').$$

Thus, since the diagram commutes, we have that

$$\pi_* : \mathrm{Hom}_{E\text{-}\mathbf{Mod}}(\bigoplus_i (E \otimes S^{a_i}), N') \to \mathrm{Hom}_{\pi_*(E)}(\pi_*(\bigoplus_i (E \otimes S^{a_i})), \pi_*(N'))$$

is an isomorphism, as desired.

Now, consider the case that N is a retract of $\bigoplus_i (E \otimes S^{a_i})$ in E-**Mod**, so there exists a commuting diagram of left E-module object homomorphisms:

$$\begin{tikzcd} N \arrow[r, "\iota"'] \arrow[rr, equal, bend left] & \bigoplus_i (E \otimes S^{a_i}) \arrow[r, "r"'] & N \end{tikzcd}$$

Now consider the following diagram:

$$\begin{tikzcd} \mathrm{Hom}^*_{E\text{-}\mathbf{Mod}}(N, N') \arrow[r, "r^*"] \arrow[d, "\pi_*"'] \arrow[rr, equal, bend left] & \mathrm{Hom}^*_{E\text{-}\mathbf{Mod}}(\bigoplus_i (E \otimes S^{a_i}), N') \arrow[r, "\iota^*"] \arrow[d, "\pi_*"] & \mathrm{Hom}^*_{E\text{-}\mathbf{Mod}}(N, N') \arrow[d, "\pi_*"'] \\ \mathrm{Hom}^*_{\pi_*(E)}(\pi_*(N), \pi_*(N')) \arrow[r, "(\pi_*(r))^*"] \arrow[rr, equal, bend right] & \mathrm{Hom}^*_{\pi_*(E)}(\pi_*(\bigoplus_i (E \otimes S^{a_i})), \pi_*(N')) \arrow[r, "(\pi_*(\iota))^*"] & \mathrm{Hom}^*_{\pi_*(E)}(\pi_*(N), \pi_*(N')) \end{tikzcd}$$

Each square commutes by functoriality of π_*. We have shown the middle vertical arrow is an isomorphism. Thus the outside arrows are isomorphisms as well, as a retract of an isomorphism is an isomorphism. $\square$

Proposition D.28. *Let (E, μ, e) be a monoid object and X an object in $\mathcal{SH}$. If there is a collection of $a_i \in A$ indexed by some set I such that $E \otimes X$ is a retract of $\bigoplus_i (E \otimes S^{a_i})$ in E-**Mod**,[12] then for all left E-module objects (N, κ) the assignment*

$$\Psi : [X, N]_* \to \mathrm{Hom}^*_{\pi_*(E)}(E_*(X), \pi_*(N))$$

[12] Here $\bigoplus_i (E \otimes S^{a_i})$ is a coproduct (Proposition D.14) of a bunch of left free E-module objects (Proposition D.10), so it is itself a left E-module object.

sending $f : S^a \otimes X \to N$ to the map $E_{*-a}(X) \to \pi_*(N)$ which sends a class $x : S^{b-a} \to E \otimes X$ to the composition

$$\Psi(f)(x) : S^b \xrightarrow{\phi} S^{b-a} \otimes S^a \xrightarrow{x \otimes S^a} E \otimes X \otimes S^a \xrightarrow{E \otimes \tau} E \otimes S^a \otimes X \xrightarrow{E \otimes f} E \otimes N \xrightarrow{\kappa} N$$

is an A-graded isomorphism of A-graded abelian groups.

Proof. Clearly as constructed, assuming $\Psi(f)$ as defined is actually a homomorphism of left $\pi_*(E)$-modules, this map is A-graded. Thus, it suffices to show that for all $a \in A$, the restriction

$$\Psi_a : [X, N]_a \to \mathrm{Hom}^a_{\pi_*(E)}(E_*(X), \pi_*(N))$$

is an isomorphism. First of all, note that Ψ_a factors as

$$\begin{tikzcd}
{[X,N]_a} \arrow[r, equal] & {[\Sigma^a X, N]} \arrow[d, "\cong"] & \\
& \mathrm{Hom}_{E\text{-}\mathbf{Mod}}(E \otimes \Sigma^a X, N) \arrow[d, "\cong"] & \\
& \mathrm{Hom}_{E\text{-}\mathbf{Mod}}(\Sigma^a(E \otimes X), N) \arrow[d, "\pi_*"] & \\
& \mathrm{Hom}_{\pi_*(E)}(\pi_*(\Sigma^a(E \otimes X)), \pi_*(N)) \arrow[d, "(\mathrm{tw}^a)^*"] & \\
& \mathrm{Hom}_{\pi_*(E)}(E_{*-a}(X), \pi_*(N)) \arrow[r, equal] & \mathrm{Hom}^a_{\pi_*(E)}(E_*(X), \pi_*(N))
\end{tikzcd}$$

where the first isomorphism is the free-forgetful adjunction for E-modules (Proposition D.10), the second isomorphism is given by Lemma D.13, the third map is that induced by the functor π_* constructed in Lemma D.24, and the final map is induced by the isomorphism $\mathrm{tw}^a : \pi_{*-a}(E \otimes X) \xrightarrow{\cong} \pi_*(\Sigma^a(E \otimes X))$ (Lemma D.26). Unravelling definitions, this composition sends a class $f : S^a \otimes X \to N$ to the map $E_{*-a}(X) \to \pi_*(N)$ which sends a class $x : S^{b-a} \to E \otimes X$ to the composition

$$S^b \xrightarrow{\cong} S^{b-a} \otimes S^a \xrightarrow{x \otimes S^a} E \otimes X \otimes S^a \xrightarrow{\tau_{E \otimes X, S^a}} S^a \otimes E \otimes X \xrightarrow{\tau \otimes X} E \otimes S^a \otimes X \xrightarrow{E \otimes f} E \otimes N \xrightarrow{\kappa} N,$$

and clearly this equals $\Psi(f)(x)$, by coherence for the symmetries. Thus, it suffices to show that

$$\pi_* : \mathrm{Hom}_{E\text{-}\mathbf{Mod}}(\Sigma^a(E \otimes X), N) \to \mathrm{Hom}_{\pi_*(E)}(\pi_*(\Sigma^a(E \otimes X)), \pi_*(N))$$

is an isomorphism when $E \otimes X$ is a retract of $\bigoplus_i (E \otimes S^{a_i})$ in E-**Mod**. This is precisely Lemma D.27. $\square$

Proposition D.29. *Let (E, μ, e) be a monoid object and (N, κ) a left E-module object in $\mathcal{SH}$. Further suppose that E and N are cellular and that $\pi_*(N)$ is a* graded projective *(Definition C.10) left $\pi_*(E)$-module (Lemma D.24). Then given some homogeneous generating set $\{x_i\}_{i \in I} \subseteq \pi_*(N)$, N is a retract of $\bigoplus_i (E \otimes S^{|x_i|})$ in E-**Mod**.*[13]

Proof. Let $M := \bigoplus_i (E \otimes S^{|x_i|})$. We have a map

$$r : M \to N$$

[13]Here $\bigoplus_i (E \otimes S^{a_i})$ is a coproduct (Proposition D.14) of a bunch of left free E-module objects (Proposition D.10), so it is itself a left E-module object.

induced by the maps

$$r_i : E \otimes S^{|x_i|} \xrightarrow{E\otimes x_i} E \otimes N \xrightarrow{\kappa} N.$$

This is a homomorphism of left E-module objects:

$$\begin{tikzcd}
E \otimes \bigoplus_i (E \otimes S^{|x_i|}) \arrow[rr, "E\otimes r"] \arrow[d, "\cong"'] \arrow[dr, "E\otimes\bigoplus_i r_i"] & & E \otimes N \arrow[dddd, "\kappa"] \\
& E \otimes \bigoplus_i N \arrow[ur, "E\otimes\nabla"] \arrow[d, "\cong"'] & \\
\bigoplus_i (E \otimes E \otimes S^{|x_i|}) \arrow[r, "\bigoplus_i (E\otimes r_i)"] \arrow[dd, "\bigoplus_i (\mu\otimes S^{|x_i|})"'] & \bigoplus_i (E \otimes N) \arrow[uur, "\nabla"'] \arrow[d, "\bigoplus_i \kappa"] & \\
& \bigoplus_i N \arrow[dr, "\nabla"] & \\
\bigoplus_i (E \otimes S^{|x_i|}) \arrow[ur, "\bigoplus_i r_i"] \arrow[rr, "r"] & & N
\end{tikzcd}$$

The right trapezoid commutes by naturality of ∇. The bottom triangle commutes by the fact that $\nabla \circ \bigoplus_i r_i$ and r satisfy the same unviersal property for the coproduct. Every other region commutes by additivity of $E \otimes -$, except the left trapezoid: Note that by expanding out how r_i is defined, it becomes

$$\begin{tikzcd}
\bigoplus_i (E \otimes E \otimes S^{|x_i|}) \arrow[r, "\bigoplus_i (E\otimes E\otimes x_i)"] \arrow[d, "\bigoplus_i (\mu\otimes S^{|x_i|})"'] & \bigoplus_i (E \otimes E \otimes N) \arrow[r, "\bigoplus_i (E\otimes\kappa)"] \arrow[d, "\bigoplus_i (\mu\otimes X)"] & \bigoplus_i (E \otimes E \otimes X) \arrow[d, "\bigoplus_i \kappa"] \\
\bigoplus_i (E \otimes S^{|x_i|}) \arrow[r, "\bigoplus_i (E\otimes x_i)"'] & \bigoplus_i (E \otimes N) \arrow[r, "\bigoplus_i \kappa"'] & \bigoplus_i (E \otimes X)
\end{tikzcd}$$

The left square commutes by functoriality of $- \otimes -$, and the right square commutes by coherence for κ. Hence, we've shown that r is a homomorphism of left E-modules, as desired. Thus, r induces a homomorphism of left $\pi_*(E)$-modules $\pi_*(r) \in \mathrm{Hom}_{\pi_*(E)}(\pi_*(M), \pi_*(N)))$ Further note that for all $i \in I$, x_i is in the image of $\pi_*(r)$, as by definition $\pi_*(r)$ sends the class

$$S^{|x_i|} \xrightarrow{e\otimes S^{|x_i|}} E \otimes S^{|x_i|} \hookrightarrow M$$

in $\pi_{|x_i|}(M)$ to the composition

$$S^{|x_i|} \xrightarrow{e\otimes S^{|x_i|}} E \otimes S^{|x_i|} \xrightarrow{E\otimes x_i} E \otimes N \xrightarrow{\kappa} N,$$

and by unitality of κ this composition is simply $x_i : S^{|x_i|} \to N$. Thus, we have constructed a surjective A-graded homomorphism $\pi_*(r) : \pi_*(M) \to \pi_*(N)$ of left $\pi_*(E)$-modules, so that since $\pi_*(N)$ is projective graded module there exists an A-graded left $\pi_*(E)$-module homomorphism $\iota : \pi_*(N) \to \pi_*(M)$ which makes the following diagram commute:

$$\begin{tikzcd}
& \pi_*(M) \arrow[d, "\pi_*(r)"] \\
\pi_*(N) \arrow[ur, "\iota"] \arrow[r, equal] & \pi_*(N)
\end{tikzcd}$$

which further induces the corresponding idempotent of left $\pi_*(E)$-modules:

$$\pi_*(M) \xrightarrow{\pi_*(r)} \pi_*(N) \xrightarrow{\iota} \pi_*(M)$$

Now, by Lemma D.27, since $M = \bigoplus_i (E \otimes S^{|x_i|})$, we have that this map is actually induced by some endomorphism $\ell : M \to M$ of left E-module objects. Now ℓ splits by Proposition A.8, meaning there exists a diagram in $\mathcal{SH}$ of the form

$$\ell : M \xrightarrow{r'} X \xrightarrow{\iota'} M$$

with $r' \circ \iota' = \mathrm{id}_X$. Note that since E and each $S^{|x_i|}$ are cellular, $E \otimes S^{|x_i|}$ is cellular for all $i \in I$ (Lemma 2.9), so that $M = \bigoplus_i (E \otimes S^{|x_i|})$ is cellular, as by definition an arbitrary coproduct of cellular objects is cellular. Thus by Lemma 2.12 X is cellular as well. Now consider the following commutative diagram

$$\begin{tikzcd}
 & & \pi_*(N) \arrow[rr, equal] \arrow[dr, "\iota"] & & \pi_*(N) \arrow[dr, "\iota"] & & \\
\pi_*(N) \arrow[r, "\iota"'] \arrow[urr, equal, bend left] & \pi_*(M) \arrow[ur, "\pi_*(r)"] \arrow[rr, "\pi_*(\ell)"] \arrow[dr, "\pi_*(r')"'] & & \pi_*(M) \arrow[ur, "\pi_*(r)"] \arrow[rr, "\pi_*(\ell)"] \arrow[dr, "\pi_*(r')"'] & & \pi_*(M) \arrow[r, "\pi_*(r')"] & \pi_*(X) \\
 & & \pi_*(X) \arrow[rr, equal] \arrow[ur, "\pi_*(\iota')"'] & & \pi_*(X) \arrow[ur, "\pi_*(\iota')"'] \arrow[urr, equal, bend right] & &
\end{tikzcd}$$

From this diagram we read off that the middle diagonal composition

$$\pi_*(X) \xrightarrow{\pi_*(\iota')} \pi_*(M) \xrightarrow{\pi_*(r)} \pi_*(N)$$

is an isomorphism with inverse $\pi_*(r') \circ \iota$. Now, since X and N are cellular, and $\pi_*(r \circ \iota')$ is an isomorphism, by Lemma 2.8 we have that $r \circ \iota'$ is an isomorphism, say with inverse p. Thus we have a commuting diagram

$$\begin{tikzcd}
N \arrow[r, "\iota' \circ p"'] \arrow[rr, equal, bend left] \arrow[dr, "p"'] & M \arrow[r, "r"'] & N \\
 & X \arrow[u, "\iota'"'] &
\end{tikzcd}$$

and the middle row exhibits N as a retract of $M = \bigoplus_i (E \otimes S^{|x_i|})$, as desired. $\square$

Corollary D.30. *Let (E, μ, e) be a monoid object and let X and Y be objects in $\mathcal{SH}$. Then if E and X are cellular and $E_*(X)$ is a graded projective (Definition C.10) left $\pi_*(E)$-module (Proposition D.25), then the map*

$$\Psi_{X,Y} : [X, E \otimes Y]_* \to \mathrm{Hom}^*_{\pi_*(E)}(E_*(X), E_*(Y))$$

sending $f : S^a \otimes X \to E \otimes Y$ to the map $E_{*-a}(X) \to E_*(Y)$ which sends a class $x : S^{b-a} \to E \otimes X$ to the composition

$$\Psi_{X,Y}(f)(x) : S^b \xrightarrow{\phi} S^{b-a} \otimes S^a \xrightarrow{x \otimes S^a} E \otimes X \otimes S^a \xrightarrow{E \otimes \tau} E \otimes S^a \otimes X \xrightarrow{E \otimes f} E \otimes E \otimes Y \xrightarrow{\mu \otimes Y} E \otimes Y$$

is an A-graded isomorphism of A-graded abelian groups.

Proof. By Proposition D.29, since $E \otimes X$ is a left E-module object (Definition D.11), $E_*(X) = \pi_*(E \otimes X)$ is a graded projective left $\pi_*(E)$-module, and $E \otimes X$ is cellular (Lemma 2.9), it follows that $E \otimes X$ is a retract of $\bigoplus_i (E \otimes S^{a_i})$ in E-**Mod** for some collection of $a_i \in A$ indexed by some set I. Thus the desired result follows by Proposition D.28 with $N = E \otimes Y$ (which is an E-module by Definition D.11). $\square$

Proposition D.31. *Let (E, μ, e) be a monoid object (Definition D.1) and Z and W be objects in $\mathcal{SH}$. Then there is a homomorphism of abelian groups*

$$\Phi_{Z,W} : \pi_*(Z \otimes E) \otimes_{\pi_*(E)} \pi_*(E \otimes W) \to \pi_*(Z \otimes E \otimes W)$$

which given homogeneous elements $x : S^a \to Z \otimes E$ in $\pi_*(Z \otimes E)$ and $y : S^b \to E \otimes W$ in $\pi_*(E \otimes W)$, sends the homogeneous pure tensor $x \otimes y$ in $\pi_*(Z \otimes E) \otimes_{\pi_*(E)} \pi_*(E \otimes W)$ to the composition

$$S^{a+b} \xrightarrow{\phi_{a,b}} S^a \otimes S^b \xrightarrow{x \otimes y} Z \otimes E \otimes E \otimes W \xrightarrow{Z \otimes \mu \otimes W} Z \otimes E \otimes W$$

(where here we are considering the canonical A-graded right $\pi_*(E)$-module structure on $\pi_*(Z \otimes E) = Z_*(E)$ and the canonical left A-graded $\pi_*(E)$-module structure on $\pi_*(E \otimes W) = E_*(W)$ given in Proposition D.25, so that $\pi_*(Z \otimes E) \otimes_{\pi_*(E)} \pi_*(E \otimes W)$ is a well-defined A-graded abelian group by Proposition C.18). Furthermore, this homomorphism is natural in both Z and W.

Proof. First, recall by definition of the tensor product, in order to show the assignment $\pi_*(Z \otimes E) \times \pi_*(E \otimes W) \to \pi_*(Z \otimes E \otimes W)$ induces a homomorphism $\pi_*(Z \otimes E) \otimes_{\pi_*(E)} \pi_*(E \otimes W) \to \pi_*(Z \otimes E \otimes W)$ of A-graded abelian groups, it suffices to show that the assignment is $\pi_*(E)$-balanced, i.e., that it is linear in each argument and satisfies $zr \cdot w = z \cdot rw$ for $z \in \pi_*(Z \otimes E)$, $w \in \pi_*(E \otimes W)$, and $r \in \pi_*(E)$.

First, note that by Lemma D.23 it is straightforward to see that the assignment commutes with addition of maps in each argument. Now, let $a, b, c \in A$, $z : S^a \to Z \otimes E$, $w : S^b \to E \otimes W$, and $r : S^c \to E$. Then we wish to show $zr \cdot w = z \cdot rw$. Consider the following diagram (where here we are passing to a symmetric strict monoidal category):

$$\begin{tikzcd}
 & & & Z \otimes E \otimes E \otimes W \arrow[d, "Z \otimes \mu \otimes W"] \\
S^{a+b+c} \arrow[r, "\cong"] & S^a \otimes S^c \otimes S^b \arrow[r, "z \otimes r \otimes w"] & Z \otimes E \otimes E \otimes E \otimes W \arrow[ur, "Z \otimes \mu \otimes E \otimes W"] \arrow[dr, "Z \otimes E \otimes \mu \otimes W"'] & Z \otimes E \otimes W \\
 & & & Z \otimes E \otimes E \otimes W \arrow[u, "Z \otimes \mu \otimes W"']
\end{tikzcd}$$

It commutes by associativity of μ. By functoriality of $- \otimes -$, the top composition is given by $(zr) \cdot w$ and the bottom composition is $z \cdot (rw)$, so we have they are equal, as desired. Thus, by Lemma C.19 we get the desired A-graded homomorphism $\pi_*(Z \otimes E) \otimes_{\pi_*(E)} \pi_*(E \otimes W) \to \pi_*(Z \otimes E \otimes W)$.

Next, we would like to show that this homomorphism is natural in Z. Let $f : Z \to Z'$ in $\mathcal{SH}$. Then we would like to show the following diagram commutes:

$$\begin{tikzcd}
\pi_*(Z \otimes E) \otimes_{\pi_*(E)} \pi_*(E \otimes W) \arrow[r, "\Phi_{Z,W}"] \arrow[d, "\pi_*(f \otimes E) \otimes \pi_*(E \otimes W)"'] & \pi_*(Z \otimes E \otimes W) \arrow[d, "\pi_*(f \otimes E \otimes W)"] \\
\pi_*(Z' \otimes E) \otimes_{\pi_*(E)} \pi_*(E \otimes W) \arrow[r, "\Phi_{Z',W}"] & \pi_*(Z' \otimes E \otimes W)
\end{tikzcd} \tag{7}$$

As all the maps here are homomorphisms, in order to show it commutes, it suffices to chase generators around the diagram. In particular, suppose we are given $z : S^a \to Z \otimes E$ and $w : S^b \to E \otimes W$, and consider the following diagram exhibiting the two possible ways to chase $z \otimes w$ around the diagram (as usual, we are passing to a symmetric strict monoidal category):

$$\begin{tikzcd}
S^{a+b} \arrow[r, "\phi_{a,b}"] & S^a \otimes S^b \arrow[r, "z \otimes w"] & Z \otimes E \otimes E \otimes W \arrow[r, "Z \otimes \mu \otimes W"] \arrow[d, "f \otimes E \otimes E \otimes W"'] & Z \otimes E \otimes W \arrow[d, "f \otimes E \otimes W"] \\
 & & Z \otimes E \otimes E \otimes W \arrow[r, "Z \otimes \mu \otimes W"] & Z' \otimes E \otimes W
\end{tikzcd}$$

This diagram commutes by functoriality of $-\otimes-$. Thus we have that diagram (7) does indeed commute, so that $\Phi_{Z,W}$ is natural in Z as desired. Showing that $\Phi_{Z,W}$ is natural in W is entirely analagous. $\square$

Lemma D.32. *Let E and X be objects in $\mathcal{SH}$. Then for all $a \in A$, there is an A-graded isomorphism of A-graded abelian groups*

$$t_X^a : E_*(\Sigma^a X) \cong E_{*-a}(X)$$

which sends a class $S^b \to E \otimes \Sigma^a X = E \otimes S^a \otimes X$ to the composition

$$S^{b-a} \xrightarrow{\phi_{b,-a}} S^b \otimes S^{-a} \xrightarrow{x\otimes S^{-a}} E\otimes S^a \otimes X \otimes S^{-a} \xrightarrow{E\otimes S^a\otimes \tau_{X,S^{-a}}} E\otimes S^a\otimes S^{-a}\otimes X \xrightarrow{E\otimes \phi_{a,-a}^{-1}\otimes X} E\otimes X$$

(where here we are ignoring associators and unitors). Furthermore this isomorphism is natural in X, and if E is a monoid object in $\mathcal{SH}$ then it is a natural isomorphism of $\pi_*(E)$-modules.

Proof. Expressed in terms of hom-sets, t_X^a is precisely the composition

$$\begin{tikzcd}
E_*(\Sigma^a X) \arrow[r, equal] & {[S^*, E\otimes S^a\otimes X]} \arrow[d, "-\otimes S^{-a}"] & \\
 & {[S^*\otimes S^{-a}, E\otimes S^a\otimes X\otimes S^{-a}]} \arrow[d, "(\phi_{*,-a})^*"] & \\
 & {[S^{*-a}, E\otimes S^a\otimes X\otimes S^{-a}]} \arrow[d, "(E\otimes S^a\otimes\tau)_*"] & \\
 & {[S^{*-a}, E\otimes S^a\otimes S^{-a}\otimes X]} \arrow[d, "(E\otimes\phi_{a,-a}^{-1}\otimes X)_*"] & \\
 & {[S^{*-a}, E\otimes X]} \arrow[r, equal] & E_{*-a}(E\otimes X)
\end{tikzcd}$$

We know the first vertical arrow is an isomorphism of abelian groups as $-\otimes-$ is additive in each variable (since $\mathcal{SH}$ is tensor triangulated) and $\Omega^a \cong -\otimes S^{-a}$ is an autoequivalence of $\mathcal{SH}$ by Proposition 2.4. The three other vertical arrows are given by composing with an isomorphism in an additive category, so they are also isomorphisms.

To see t_X^a is a homomorphism of left $\pi_*(E)$-modules, suppose we are given classes $r : S^b \to E$ $\pi_b(E)$ and $x : S^c \to E \otimes S^a \otimes X$ in $E_c(\Sigma^a X)$. Then we wish to show that $t_X^a(r\cdot x) = r \cdot t_X^a(x)$. Consider the following diagram:

$$\begin{tikzcd}
S^{b+c-a} \arrow[d, "\cong"] & E\otimes S^a\otimes X\otimes S^{-a} \arrow[r, "E\otimes S^a\otimes\tau_{X,S^{-a}}"] & E\otimes S^a\otimes S^{-a}\otimes X \arrow[d, "E\otimes\phi_{a,-a}^{-1}\otimes X"] \\
S^b\otimes S^c\otimes S^{-a} \arrow[r, "r\otimes x\otimes S^{-a}"] & E\otimes E\otimes S^a\otimes X\otimes S^{-a} \arrow[u, "\mu\otimes S^a\otimes X\otimes S^{-a}"'] \arrow[d, "E\otimes E\otimes S^a\otimes\tau_{X,S^{-a}}"'] & E\otimes X \\
 & E\otimes E\otimes S^a\otimes S^{-a}\otimes X \arrow[r, "E\otimes E\otimes\phi_{a,-a}^{-1}\otimes X"'] \arrow[uur, "\mu\otimes S^a\otimes S^{-a}\otimes X"'] & E\otimes E\otimes X \arrow[u, "\mu\otimes X"']
\end{tikzcd}$$

Both triangles commute by functoriality of $-\otimes-$. The top composition is $t_X^a(r\cdot x)$ while the bottom is $r\cdot t_X^a(x)$, so they are equal as desired.

It remains to show t^a_X is natural in X. let $f : X \to Y$ in $\mathcal{SH}$, then we would like to show the following diagram commutes:

$$\begin{tikzcd} E_*(\Sigma^a X) \arrow[r, "t^a_X"] \arrow[d, "E_*(\Sigma^a f)"'] & E_{*-a}(X) \arrow[d, "E_{*-a}(f)"] \\ E_*(\Sigma^a Y) \arrow[r, "t^a_Y"] & E_{*-a}(Y) \end{tikzcd} \tag{8}$$

We may chase a generator around the diagram since all the arrows here are homomorphisms. Let $x : S^b \to E \otimes S^a \otimes X$ in $E_*(\Sigma^a X)$. Then consider the following diagram:

$$\begin{tikzcd} S^{b-a} \arrow[r, "\cong"] & S^b \otimes S^{-a} \arrow[r, "x\otimes S^{-a}"] & E \otimes S^a \otimes X \otimes S^{-a} \arrow[r, "E\otimes S^a\otimes\tau"] \arrow[d, "E\otimes S^a\otimes f\otimes S^{-a}"'] & E \otimes S^a \otimes S^{-a} \otimes X \arrow[r, "E\otimes\phi^{-1}_{a,-a}\otimes X"] \arrow[d, "E\otimes S^a\otimes S^{-a}\otimes f"'] & E \otimes X \arrow[d, "E\otimes f"] \\ & & E \otimes S^a \otimes Y \otimes S^{-a} \arrow[r, "E\otimes S^a\otimes\tau"'] & E \otimes S^a \otimes S^{-a} \otimes Y \arrow[r, "E\otimes\phi^{-1}_{a,-a}\otimes Y"'] & E \otimes Y \end{tikzcd}$$

The left rectangle commutes by naturality of τ, while the right rectangle commutes by functoriality of $- \otimes -$. The two outside compositions are the two ways to chase x around diagram (8), so the diagram commutes as desired. □

Lemma D.33. *Let (E, μ, e) be a monoid object and Z and W objects in $\mathcal{SH}$, and suppose the map $\Phi_{Z,W}$ constructed in Proposition D.31 is an isomorphism. Then $\Phi_{\Sigma^a Z,W}$ and $\Phi_{Z,\Sigma^a W}$ are isomorphisms for all $a \in A$. In particular, $\Phi_{\Sigma Z,W}$ an $\Phi_{Z,\Sigma W}$ are isomorphisms.*

Proof. First to see $\Phi_{Z,\Sigma^a W}$ is an isomorphism, consider the following diagram

$$\begin{tikzcd} Z_*(E) \otimes_{\pi_*(E)} E_*(\Sigma^a W) \arrow[r, "Z_*(E)\otimes_{\pi_*(E)} t^W_a"] \arrow[d, "\Phi_{Z,\Sigma^a W}"'] & Z_*(E) \otimes_{\pi_*(E)} E_{*-a}(W) \arrow[d, "\Phi_{Z,W}"] \\ \pi_*(Z \otimes E \otimes \Sigma^a W) = (Z \otimes E)_*(\Sigma^a W) \arrow[r, "t^W_a"] & (Z \otimes E)_{*-a}(W) = Z\pi_{*-a}(Z \otimes E \otimes W) \end{tikzcd}$$

where the maps t_a are defined in Lemma D.32. The top arrow is well-defined since $t^W_a : E_*(\Sigma^a W) \to E_{*-a}(W)$ is a degree $-a$ isomorphism of left $\pi_*(E)$-modules by the aforementioned lemma. In order to show the left vertical arrow is an isomorphism, it suffices to show the diagram commutes, as all the other arrows are isomorphisms. To see this, note it suffices to chase a homogeneous pure tensor around the diagram, as all the maps here are homomorphisms. So let $x : S^b \to Z \otimes E$ in $Z_*(E)$ and $y : S^c \to E \otimes S^a \otimes W$ in $E_*(\Sigma^a W)$, and consider the following diagram exhibiting the two ways to chase $x \otimes y$ around:

$$\begin{tikzcd} S^{b+c-a} \arrow[d, "\phi"] & Z \otimes E \otimes E \otimes S^a \otimes S^{-a} \otimes W \arrow[r, "Z\otimes E\otimes E\otimes\phi^{-1}_{a,-a}\otimes W"] \arrow[rd, "Z\otimes\mu\otimes S^a\otimes S^{-a}\otimes W"] & Z \otimes E \otimes E \otimes W \arrow[d, "Z\otimes\mu\otimes W"] \\ S^b \otimes S^c \otimes S^{-a} \arrow[r, "x\otimes y\otimes S^{-a}"] & Z \otimes E \otimes E \otimes S^a \otimes W \otimes S^{-a} \arrow[u, "Z\otimes E\otimes E\otimes S^a\otimes\tau"] \arrow[d, "Z\otimes\mu\otimes S^a\otimes W\otimes S^{-a}"'] & Z \otimes E \otimes W \\ & Z \otimes E \otimes S^a \otimes W \otimes S^{-a} \arrow[r, "Z\otimes E\otimes S^a\otimes\tau"'] & Z \otimes E \otimes S^a \otimes S^{-a} \otimes W \arrow[u, "Z\otimes E\otimes\phi^{-1}_{a,-a}\otimes W"'] \end{tikzcd}$$

Each triangle commutes by functoriality of $- \otimes -$, so the diagram commutes as desired. Thus, we've shown $\Phi_{Z,\Sigma^a W}$ is an isomorphism for all $a \in A$.

On the other hand, in order to see $\Phi_{\Sigma^a Z,W}$ is an isomorphism, consider the following diagram:

$$\begin{tikzcd}
\pi_*(\Sigma^a Z \otimes E) \otimes_{\pi_*(E)} \pi_*(E \otimes W) \arrow[r, "\Phi_{\Sigma^a Z,W}"] \arrow[d, "\mathrm{adj} \otimes_{\pi_*(E)} \pi_*(E \otimes W)"'] & \pi_*(\Sigma^a Z \otimes E \otimes W) \arrow[d, "\mathrm{adj}"] \\
\pi_{*-a}(Z \otimes E) \otimes_{\pi_*(E)} \pi_*(E \otimes W) \arrow[r, "\Phi_{Z,W}"] & \pi_{*-a}(Z \otimes E \otimes W)
\end{tikzcd} \tag{9}$$

Here the left vertical arrow is constructed via the map $\mathrm{adj} : \pi_*(\Sigma^a Z \otimes E) \to \pi_{*-a}(Z \otimes E)$ which is given as as the composition

$$\mathrm{adj} : [S^*, S^a \otimes Z \otimes E] \xrightarrow{\cong} [S^{-a} \otimes S^*, Z \otimes E] \xrightarrow{(\phi^{-1}_{-a,*})^*} [S^{*-a}, Z \otimes E]$$

where the first arrow is the adjunction isomorphism from Proposition 2.4. Explicitly, this map sends a class $x : S^b \to S^a \otimes Z \otimes E$ to the composition

$$S^{b-a} \xrightarrow{\phi_{-a,b}} S^{-a} \otimes S^b \xrightarrow{S^{-a} \otimes x} S^{-a} \otimes S^a \otimes Z \otimes E \xrightarrow{\phi^{-1}_{-a,a} \otimes Z \otimes E} Z \otimes E.$$

In order to show $\mathrm{adj} \otimes_{\pi_*(E)} \pi_*(E \otimes W)$ is well defined, it suffices to show adj is a degree $-a$ A-graded homomorphism of right A-graded $\pi_*(E)$-modules. It is clearly additive, as any adjunction between additive categories is automatically additive, as is composing with an morphism in an additive category. Thus, it remains to show adj commutes with scalar multiplication. By additivity, it suffices to consider only homogeneous elements. Let $x : S^a \to S^{\mathbf{1}} \otimes Z \otimes E$ in $\pi_*(S^{\mathbf{1}} \otimes Z \otimes E)$ and $r : S^b \to E$ in $\pi_*(E)$. Then we'd like to show that $\mathrm{adj}(x \cdot r) = \mathrm{adj}(x) \cdot r$. To see this, consider the following diagram:

$$\begin{tikzcd}
S^{a+b-\mathbf{1}} \arrow[r, "\phi"] & S^{-\mathbf{1}} \otimes S^a \otimes S^b \arrow[r, "S^{-\mathbf{1}} \otimes x \otimes y"] & S^{-\mathbf{1}} \otimes S^{\mathbf{1}} \otimes Z \otimes E \otimes E \arrow[r, "\phi^{-1}_{-\mathbf{1},\mathbf{1}} \otimes Z \otimes E \otimes E"] \arrow[d, "S^{-\mathbf{1}} \otimes S^{\mathbf{1}} \otimes Z \otimes \mu"'] & Z \otimes E \otimes E \arrow[d, "Z \otimes \mu"] \\
& & S^{-\mathbf{1}} \otimes S^{\mathbf{1}} \otimes Z \otimes E \arrow[r, "\phi^{-1}_{-\mathbf{1},\mathbf{1}} \otimes Z \otimes E"'] & Z \otimes E
\end{tikzcd}$$

The top composition is $\mathrm{adj}(x) \cdot r$, while the bottom composition is $\mathrm{adj}(x \cdot r)$. The diagram commutes by functoriality of $- \otimes -$. Thus, it follows that $\mathrm{adj}(x) \cdot r = \mathrm{adj}(x \cdot r)$, so that adj is indeed an homomorphism of right $\pi_*(E)$-modules, in fact, an isomorphism as desired. Thus, since every arrow in diagram (9) is an isomorphism of abelian groups except the top arrow, in order to show $\Phi_{\Sigma^a Z,W}$ is an isomorphism, it suffices to show the diagram commutes. To that end, since all the arrows are homomorphisms, it suffices to chase a pure homogeneous tensor. So let $x : S^b \to \Sigma^a Z \otimes E$ and $y : S^c \to E \otimes W$, and consider the following diagram whose outside compositions exhibit the two ways to chase the pure tensor $x \otimes y$ around diagrama (9):

$$\begin{tikzcd}
S^{b+c-a} \arrow[r, "\phi"] & S^{-a} \otimes S^b \otimes S^c \arrow[r, "S^{-a} \otimes x \otimes y"] & S^{-a} \otimes S^a \otimes Z \otimes E \otimes E \otimes W \arrow[r, "S^{-a} \otimes S^a \otimes Z \otimes \mu \otimes W"] \arrow[d, "\phi^{-1}_{-a,a} \otimes Z \otimes E \otimes E \otimes W"'] & S^{-a} \otimes S^a \otimes Z \otimes E \otimes W \arrow[d, "\phi^{-1}_{-a,a} \otimes Z \otimes E \otimes W"] \\
& & Z \otimes E \otimes E \otimes W \arrow[r, "Z \otimes \mu \otimes W"] & Z \otimes E \otimes W
\end{tikzcd}$$

The diagram clearly commutes by functoriality of $- \otimes -$, so that indeed diagram (9) commutes, so that $\Phi_{\Sigma^a Z,W}$ is indeed an isomorphism as desired.

Now, it remains to show that $\Phi_{Z,\Sigma W}$ and $\Phi_{\Sigma Z,W}$ are isomorphisms. To that end, consider the following diagram:

$$\begin{tikzcd}
\pi_*(Z\otimes E)\otimes_{\pi_*(E)}\pi_*(E\otimes\Sigma W) \arrow[r, "\Phi_{Z,\Sigma W}"] \arrow[d, "\pi_*(Z\otimes E)\otimes_{\pi_*(E)}\pi_*(E\otimes\nu_W)"'] & \pi_*(Z\otimes E\otimes\Sigma W) \arrow[d, "\pi_*(Z\otimes E\otimes\nu_W)"] \\
\pi_*(Z\otimes E)\otimes_{\pi_*(E)}\pi_*(E\otimes\Sigma^{\mathbf{1}}W) \arrow[r, "\Phi_{Z,\Sigma^{\mathbf{1}}W}"'] & \pi_*(Z\otimes E\otimes\Sigma^{\mathbf{1}}W)
\end{tikzcd}$$

It commutes by naturality of Φ. Furthermore, assuming $\Phi_{Z,W}$ is an isomorphism, by what we have shown above we know that $\Phi_{Z,\Sigma^{\mathbf{1}}W}$ is an isomorphism, and since ν_W is an isomorphism, it follows that the above diagram commutes and all arrows except $\Phi_{Z,\Sigma W}$ are isomorphisms, so that $\Phi_{Z,\Sigma W}$ must be an isomorphism itself. Finally, an entirely analagous argument using naturality of Φ with respect to ν_Z yields that $\Phi_{\Sigma Z,W}$ is an isomorphism as well. $\square$

Proposition D.34. *Let (E,μ,e) be a monoid object in $\mathcal{SH}$. Then if either:*

(1) $\pi_*(Z\otimes E)=Z_*(E)$ is a flat right $\pi_*(E)$-module (via Proposition D.25) and W is cellular (Definition 2.7), or

(2) $\pi_*(E\otimes W)=E_*(W)$ is a flat left $\pi_*(E)$-module (via Proposition D.25) and Z is cellular (Definition 2.7),

then the natural homomorphism

$$\Phi_{Z,W}:\pi_*(Z\otimes E)\otimes_{\pi_*(E)}\pi_*(E\otimes W)\to\pi_*(Z\otimes E\otimes W)$$

given in Proposition D.31 is an isomorphism of abelian groups.

Proof. In this proof, we will freely employ the coherence theorem for symmetric monoidal categories, and we will assume that associativity and unitality of $-\otimes-$ holds up to strict equality. First we will consider the case that $\pi_*(Z\otimes E)=Z_*(E)$ is a flat right $\pi_*(E)$-module and W is cellular. To start, let $\mathcal{E}$ be the collection of objects W in $\mathcal{SH}$ for which this map is an isomorphism. Then in order to show $\mathcal{E}$ contains every cellular object, it suffices to show that $\mathcal{E}$ satisfies the three conditions given for the class of cellular objects in Definition 2.7. First, we need to show that $\Phi_{Z,W}$ is an isomorphism when $W=S^a$ for some $a\in A$. Indeed, consider the A-graded homomorphism

$$\Psi:\pi_*(Z\otimes E\otimes S^a)\to\pi_*(Z\otimes E)\otimes_{\pi_*(E)}\pi_*(E\otimes S^a)$$

which sends a class $x:S^b\to Z\otimes E\otimes S^a$ in $\pi_b(Z\otimes E\otimes S^a)$ to the pure tensor $\widetilde{x}\otimes\widetilde{e}$, where $\widetilde{x}\in\pi_{b-a}(Z\otimes E)$ is the composition

$$S^{b-a}\xrightarrow{\phi_{b,-a}}S^b\otimes S^{-a}\xrightarrow{x\otimes S^{-a}}Z\otimes E\otimes S^a\otimes S^{-a}\xrightarrow{Z\otimes E\otimes\phi^{-1}_{a,-a}}Z\otimes E$$

and $\widetilde{e}\in\pi_a(E\otimes S^a)$ is the composition

$$S^a\xrightarrow{e\otimes S^a}E\otimes S^a.$$

In order to see Ψ is an (A-graded) homomorphism of abelian groups: Given $x,x'\in\pi_b(Z\otimes E\otimes S^a)$, we would like to show that $\widetilde{x}\otimes\widetilde{e}+\widetilde{x}'\otimes\widetilde{e}=\widetilde{x+x'}\otimes\widetilde{e}$. It suffices to show that $\widetilde{x}+\widetilde{x}'=\widetilde{x+x'}$. To see this, consider the following diagram (again, we are passing to a symmetric strict monoidal category):

$$\begin{tikzcd}
S^{b-a} \arrow[r, "\Delta"] \arrow[d, "\phi_{b-a}"'] & S^{b-a} \oplus S^{b-a} \arrow[d, "\phi_{b,-a} \oplus \phi_{b,-a}"] \\
S^b \otimes S^{-a} \arrow[r, "\Delta"] \arrow[d, "\Delta \otimes S^{-a}"'] & (S^b \otimes S^{-a}) \oplus (S^b \otimes S^{-a}) \arrow[d, "(x \otimes S^{-a}) \oplus (x' \otimes S^{-a})"] \\
(S^b \oplus S^b) \otimes S^{-a} \arrow[ur, "\cong"] \arrow[d, "(x \oplus x') \otimes S^{-a}"'] & (Z \otimes E \otimes S^a \otimes S^{-a}) \oplus (Z \otimes E \otimes S^a \otimes S^{-a}) \arrow[d, "(Z \otimes E \otimes \phi^{-1}_{a,-a}) \oplus (Z \otimes E \otimes \phi^{-1}_{a,-a})"] \arrow[ddl, "\nabla"] \\
((Z \otimes E \otimes S^a) \oplus (Z \otimes E \otimes S^a)) \otimes S^{-a} \arrow[ur, "\cong"] \arrow[d, "\nabla \otimes S^{-a}"'] & (Z \otimes E) \oplus (Z \otimes E) \arrow[d, "\nabla"] \\
Z \otimes E \otimes S^a \otimes S^{-a} \arrow[r, "Z \otimes E \otimes \phi^{-1}_{a,-a}"'] & Z \otimes E
\end{tikzcd}$$

The top rectangle commutes by naturality of Δ in an additive category. The bottom triangle commutes by naturality of ∇ in an additive category. Finally, the remaining regions of the diagram commute by additivity of $- \otimes -$. By functoriality of $- \otimes -$, it follows that the left composition is $\widetilde{x + x'}$ and the right composition is $\widetilde{x} + \widetilde{x}'$, so they are equal as desired. Thus Ψ is a homomorphism of abelian groups, as desired.

Now, we claim that Ψ is an inverse to Φ_{Z,S^a}. Since Φ_{Z,S^a} and Ψ are homomorphisms it suffices to check that they are inverses on generators. First, let $x : S^b \to Z \otimes E \otimes S^a$ in $\pi_b(Z \otimes E \otimes S^a)$. We would like to show that $\Phi_{Z,S^a}(\Psi(x)) = x$. Consider the following diagram, where here we are passing to a symmetric strict monoidal category:

$$\begin{tikzcd}
S^b \arrow[r, "\cong"] \arrow[dd, "x"'] & S^b \otimes S^{-a} \otimes S^a \arrow[d, dashed, "x \otimes S^{-a} \otimes S^a"] \arrow[dr, "x \otimes S^{-a} \otimes e \otimes S^a"] & \\
 & Z \otimes E \otimes S^a \otimes S^{-a} \otimes S^a \arrow[r, "Z \otimes E \otimes S^a \otimes S^{-a} \otimes e \otimes S^a"'] & Z \otimes E \otimes S^a \otimes S^{-a} \otimes E \otimes S^a \arrow[ddl, "Z \otimes E \otimes \phi^{-1}_{a,-a} \otimes E \otimes S^a"] \\
Z \otimes E \otimes S^a \arrow[ur, "Z \otimes E \otimes S^a \otimes \phi_{-a,a}"] \arrow[r, equal] & Z \otimes E \otimes S^a \arrow[u, dashed, "Z \otimes E \otimes \phi_{a,-a} \otimes S^a"'] \arrow[d, dashed, "Z \otimes E \otimes e \otimes S^a"] & \\
 & Z \otimes E \otimes E \otimes S^a \arrow[ul, "Z \otimes \mu \otimes S^a"] &
\end{tikzcd}$$

The top left trapezoid commutes since the isomorphism $S^b \xrightarrow{\cong} S^b \otimes S^{-a} \otimes S^a$ may be given as $S^b \otimes \phi_{-a,a}$ (see Remark 2.3), in which case the trapezoid commmutes by functoriality of $- \otimes -$. The triangle below that commutes by coherence for the $\phi_{a,b}$'s. The bottom left triangle commutes by unitality for μ. The top right triangle commutes by functoriality of $- \otimes -$. Finally, the bottom right triangle commutes by functoriality of $- \otimes -$. It follows by unravelling definitions that the two outside compositions are x and $\Phi_{Z,S^a}(\Psi(x))$, so indeed we have $\Phi_{Z,S^a}(\Psi(x)) = x$ since the diagram commutes.

On the other hand, suppose we are given a homogeneous pure tensor $x \otimes y$ in $\pi_*(Z \otimes E) \otimes_{\pi_*(E)} \pi_*(E \otimes S^a)$, so $x : S^b \to Z \otimes E$ and $y : S^c \to E \otimes S^a$ for some $b, c \in A$. Then we would like to show that $\Psi(\Phi_{Z,S^a}(x \otimes y)) = x \otimes y$. Unravelling definitions, $\Psi(\Phi_{Z,S^a}(x \otimes y))$ is the homogeneous pure tensor $\widetilde{xy} \otimes \widetilde{e}$, where $\widetilde{e}$ is the map $e \otimes S^a : S^a \to E \otimes S^a$ is defined above, and by functoriality of $-\otimes -$, $\widetilde{xy} : S^{b+c-a} \to Z \otimes E$ is the composition

$$
\begin{tikzcd}
S^{b+c-a} \arrow[d, "\cong"] \\
S^b \otimes S^c \otimes S^{-a} \arrow[d, "x \otimes y \otimes S^{-a}"] \\
Z \otimes E \otimes E \otimes S^a \otimes S^{-a} \arrow[d, "Z \otimes \mu \otimes S^a \otimes S^{-a}"] \\
Z \otimes E \otimes S^a \otimes S^{-a} \arrow[d, "Z \otimes E \otimes \phi_{a,-a}^{-1}"] \\
Z \otimes E
\end{tikzcd}
$$

Now, define $r \in \pi_{c-a}(E)$ to be the composition

$$S^{c-a} \cong S^c \otimes S^{-a} \xrightarrow{y \otimes S^{-a}} E \otimes S^a \otimes S^{-a} \xrightarrow{E \otimes \phi_{a,-a}^{-1}} E.$$

First, we claim that $x \cdot r = \widetilde{xy}$. To that end, consider the following diagram, where here we are again passing to a symmetric strict monoidal category:

$$
\begin{tikzcd}
S^{b+c-a} \arrow[r, "\cong"] & S^b \otimes S^c \otimes S^{-a} \arrow[r, "x \otimes y \otimes S^{-a}"] & Z \otimes E \otimes E \otimes S^a \otimes S^{-a} \arrow[r, "Z \otimes \mu \otimes S^a \otimes S^{-a}"] \arrow[d, "Z \otimes E \otimes E \otimes \phi_{a,-a}^{-1}"'] & Z \otimes E \otimes S^a \otimes S^{-a} \arrow[d, "Z \otimes E \otimes \phi_{a,-a}^{-1}"] \\
& & Z \otimes E \otimes E \arrow[r, "Z \otimes \mu"'] & Z \otimes E
\end{tikzcd}
$$

Commutativity is functoriality of $-\otimes -$, which also tells us that the two outside compositions are $\widetilde{xy}$ (on top) and $x \cdot r$ (on the bottom), so they are equal as desired. On the other hand, we claim that $r \cdot \widetilde{e} = y$. To see this, consider the following diagram:

$$
\begin{tikzcd}
S^c \arrow[rr, "\cong"] \arrow[dd, "y"'] & & S^c \otimes S^{-a} \otimes S^a \arrow[dl, "y \otimes S^{-a} \otimes S^a"'] \arrow[d, "y \otimes S^{-a} \otimes e \otimes S^a"] \\
& E \otimes S^a \otimes S^{-a} \otimes S^a \arrow[r, "E \otimes S^a \otimes S^{-a} \otimes e \otimes S^a"'] \arrow[dl, "E \otimes S^a \otimes \phi_{-a,a}^{-1}"'] \arrow[d, "E \otimes \phi_{a,-a}^{-1} \otimes S^a"] & E \otimes S^a \otimes S^{-a} \otimes E \otimes S^a \arrow[dd, "E \otimes \phi_{a,-a}^{-1} \otimes E \otimes S^a"] \\
E \otimes S^a \arrow[r, equal] & E \otimes S^a \arrow[dr, "E \otimes e \otimes S^a"] & \\
E \otimes E \otimes S^a \arrow[u, "\mu \otimes S^a"] \arrow[rr, equal] & & E \otimes E \otimes S^a
\end{tikzcd}
$$

By Remark 2.3, we may take the top arrow to be $S^c \otimes \phi_{-a,a}$, in which case the top left triangle commutes by functoriality of $-\otimes -$. The bottom trapezoid commutes by unitality of μ. Every other region commutes either by definition or by functoriality of $-\otimes -$. The top composition is $r \cdot \widetilde{e}$, so we have shown $r \cdot \widetilde{e} = y$ as desired. Thus, we have that

$$\Psi(\Phi_{Z,S^a}(x \otimes y)) = \widetilde{xy} \otimes \widetilde{e} = x \cdot r \otimes \widetilde{e} = x \otimes r \cdot \widetilde{e} = x \otimes y,$$

as desired. Hence we have shown Ψ is both a left and right inverse for Φ_{Z,S^a}, so that indeed S^a belongs to $\mathcal{E}$ as desired.

Now, we would like to show that given a distinguished triangle in $\mathcal{SH}$

$$X \xrightarrow{f} Y \xrightarrow{g} W \xrightarrow{h} \Sigma X,$$

if two of three of the objects X, Y, and W belong to $\mathcal{E}$, then so does the third. From now on, write L_*^E to denote the functor from $\mathcal{SH}$ to A-graded abelian groups sending $X \mapsto \pi_*(Z \otimes E) \otimes_{\pi_*(E)} \pi_*(E \otimes X)$. Then $\Phi_{Z,-}$ is a natural transformation $L_*^E \Rightarrow \pi_*(Z \otimes E \otimes -) = Z_*(E \otimes -)$. First, recall that it follows generally that in an adjointly tensor triangulated category (Definition A.9), which $\mathcal{SH}$ is by Proposition 2.4, given a distinguished triangle (f, g, h) we have a long exact sequence (see Definition A.2 for the definition of an exact sequence in an additive category, and see Proposition A.12 for the explicit contruction of the LES associated to a distinguished triangle in an adjointly triangulated category):

$$\Omega Y \xrightarrow{\Omega g} \Omega W \xrightarrow{\widetilde{h}} X \xrightarrow{f} Y \xrightarrow{g} W \xrightarrow{h} \Sigma X \xrightarrow{\Sigma f} \Sigma Y,$$

where $\widetilde{h} : \Omega W \to X$ is the adjoint of $h : W \to \Sigma X$. Then since $\mathcal{SH}$ is further a tensor triangulated category (Definition A.13), we have that the above sequence remains exact even after tensoring by E on the left (see Proposition A.16 for details), so we have the following exact sequence in $\mathcal{SH}$:

$$E \otimes \Omega Y \xrightarrow{E \otimes \Omega g} E \otimes \Omega W \xrightarrow{E \otimes \widetilde{h}} E \otimes X \xrightarrow{E \otimes f} E \otimes Y \xrightarrow{E \otimes g} E \otimes W \xrightarrow{E \otimes h} E \otimes \Sigma X \xrightarrow{E \otimes \Sigma f} E \otimes \Sigma Y.$$

We can then apply $[S^*, -] = \pi_*(-)$ to it, which yields the following exact sequence of A-graded abelian groups:

$$E_*(\Omega Y) \xrightarrow{E_*(\Omega g)} E_*(\Omega W) \xrightarrow{E_*(\widetilde{h})} E_*(X) \xrightarrow{E_*(f)} E_*(Y) \xrightarrow{E_*(g)} E_*(W) \xrightarrow{E_*(h)} E_*(\Sigma X) \xrightarrow{E_*(f)} E_*(\Sigma Y).$$

Now, we can tensor this sequence with $\pi_*(Z \otimes E)$ on the left over $\pi_*(E)$, and since $\pi_*(Z \otimes E)$ is a flat right $\pi_*(E)$ module, we get that the top row in the following diagram is exact:

$$\begin{tikzcd}
L_*^E(\Omega Y) \arrow[r, "L_*^E(\Omega g)"] \arrow[d, "\Phi_{Z,\Omega Y}"'] & L_*^E(\Omega W) \arrow[r, "L_*^E(\widetilde{h})"] \arrow[d, "\Phi_{Z,\Omega W}"'] & L_*^E(X) \arrow[r, "L_*^E(f)"] \arrow[d, "\Phi_{Z,X}"'] & L_*^E(Y) \arrow[r, "L_*^E(g)"] \arrow[d, "\Phi_{Z,Y}"'] & L_*^E(W) \arrow[r, "L_*^E(h)"] \arrow[d, "\Phi_{Z,W}"'] & L_*^E(\Sigma X) \arrow[r, "L_*^E(\Sigma f)"] \arrow[d, "\Phi_{Z,\Sigma X}"'] & L_*^E(\Sigma Y) \arrow[d, "\Phi_{Z,\Sigma Y}"'] \\
Z_*(E \otimes \Omega Y) \arrow[r, "Z_*(E \otimes \Omega g)"'] & Z_*(E \otimes \Omega W) \arrow[r, "Z_*(E \otimes \widetilde{h})"'] & Z_*(E \otimes X) \arrow[r, "Z_*(E \otimes f)"'] & Z_*(E \otimes Y) \arrow[r, "Z_*(E \otimes g)"'] & Z_*(E \otimes W) \arrow[r, "Z_*(E \otimes h)"'] & Z_*(E \otimes \Sigma X) \arrow[r, "Z_*(E \otimes \Sigma f)"'] & Z_*(E \otimes \Sigma Y)
\end{tikzcd}$$

This diagram further commutes by naturality of $\Phi_{Z,-}$. Now, supposing that two of three of X, Y, and W belong to $\mathcal{E}$, by Lemma D.33, if $\Phi_{Z,V}$ is an isomorphism for some object V in $\mathcal{SH}$ then $\Phi_{Z,\Omega V}$ and $\Phi_{Z,\Sigma V}$ are. Thus by the five lemma, it follows that the middle three vertical arrows in the above diagram are necessarily all isomorphisms if any two of them are, so we have shown that $\mathcal{E}$ is closed under two-of-three for exact triangles, as desired.

Finally, it remains to show that $\mathcal{E}$ is closed under arbitrary coproducts. Let $\{W_i\}_{i \in I}$ be a collection of objects in $\mathcal{E}$ indexed by some (small) set I. Then we'd like to show that $W := \bigoplus_i W_i$ belongs to $\mathcal{E}$. First of all, note that $- \otimes -$ preserves arbitrary coproducts in each argument, as it has a right adjoint $F(-,-)$. Thus without loss of generality, given any object X in $\mathcal{SH}$, we may take $\bigoplus_i X \otimes W_i = X \otimes \bigoplus_i W_i$ (as $X \otimes \bigoplus_i W_i$ *is* a coproduct of all the $X \otimes W_i$'s). Now, recall that we have chosen each S^a to be a compact object (Definition 2.1), so that given any object X and collection of objects $\{Y_i\}_{i \in I}$ in $\mathcal{SH}$, if $Y := \bigoplus_{i \in I} Y_i$, then the canonical map

$$s_{X,Y_i} : \bigoplus_i X_*(Y_i) = \bigoplus_i [S^*, X \otimes Y_i] \to [S^*, \bigoplus_i X \otimes Y_i] = [S^*, X \otimes Y] = X_*(Y)$$

is an isomorphism, natural in Y_i for each i. Note in particular that s_{E,W_i} is an isomorphism of left $\pi_*(E)$-modules. To see this, first note by additivity of s_{E,W_i}, it suffices to check that $s_{E,W_i}(r \cdot x) = r \cdot s_{E,W_i}(x)$ for each homogeneous $r \in \pi_*(E)$ and homogeneous $x \in E_*(W_i)$ for some i, as such x generate $\bigoplus_i E_*(W_i)$ by definition. Then given $r : S^a \to E$ and $x : S^b \to E \otimes W_i$, consider the following diagram

$$\begin{tikzcd}
S^{a+b} \arrow[r, "\phi_{a,b}"] & S^a \otimes S^b \arrow[r, "x\otimes y"] & E \otimes E \otimes W_i \arrow[r, "E\otimes \iota_{E\otimes W_i}"] \arrow[dr, "E\otimes E\otimes \iota_{W_i}"] \arrow[ddd, "\mu\otimes W_i"'] & E \otimes \bigoplus_i (E \otimes W_i) \arrow[d, equal] \\
& & & E \otimes E \otimes W \arrow[d, "\mu\otimes W"] \\
& & & E \otimes W \arrow[d, equal] \\
& & E \otimes W_i \arrow[ur, "E\otimes \iota_{W_i}"] \arrow[r, "\iota_{E\otimes W_i}"'] & \bigoplus_i (E \otimes W_i)
\end{tikzcd}$$

where $\iota_{E\otimes W_i} : E \otimes W_i \hookrightarrow \bigoplus_i (E \otimes W_i)$ and $\iota_{W_i} : W_i \hookrightarrow \bigoplus_i W_i$ are the maps determined by the definition of the coproduct. Commutativity of the two triangles is by the fact that $E \otimes -$ is colimit preserving. Commutativity of the trapezoid is functoriality of $- \otimes -$. Thus, since s_{E,W_i} is a homomorphism of left A-graded $\pi_*(E)$-modules, the top right arrow in the following diagram is well-defined:

$$\begin{tikzcd}
\bigoplus_i Z_*(E) \otimes_{\pi_*(E)} E_*(W_i) \arrow[r, equal] \arrow[d, "\bigoplus_i \Phi_{Z,W_i}"'] & Z_*(E) \otimes_{\pi_*(E)} \bigoplus_i E_*(W_i) \arrow[r, "Z_*(E)\otimes_{\pi_*(E)} s_{E,W_i}"] & Z_*(E) \otimes_{\pi_*(E)} E_*(W) \arrow[d, "\Phi_{Z,W}"] \\
\bigoplus_i Z_*(E \otimes W_i) \arrow[r, "s_{Z,E\otimes W_i}"] & Z_*(\bigoplus_i E \otimes W_i) \arrow[r, equal] & Z_*(E \otimes W)
\end{tikzcd} \tag{10}$$

We wish to show this diagram commutes. Again, since each map here is a homomorphism, it suffices to chase generators. By definition, a generator of the top left element is a homogeneous pure tensor in $E_*(E) \otimes_{\pi_*(E)} E_*(W_i)$ for some i in I. Given classes $x : S^a \to E \otimes E$ and $y : S^b \to E \otimes W_i$, consider the following diagram:

$$\begin{tikzcd}
S^{a+b} \arrow[r, "\phi_{a,b}"] & S^a \otimes S^b \arrow[r, "x\otimes y"] & Z \otimes E \otimes E \otimes W_i \arrow[r, "Z\otimes E\otimes \iota_{E\otimes W_i}"] \arrow[d, "Z\otimes\mu\otimes W_i"'] \arrow[dr, "Z\otimes E\otimes E\otimes \iota_{W_i}"] & Z \otimes E \otimes \bigoplus_i E \otimes W_i \arrow[d, equal] \\
& & Z \otimes E \otimes W_i \arrow[d, "\iota_{Z\otimes E\otimes W_i}"'] \arrow[dr, "Z\otimes E\otimes \iota_{W_i}"] & Z \otimes E \otimes E \otimes W \arrow[d, "Z\otimes\mu\otimes W"] \\
& & \bigoplus_i Z \otimes E \otimes W_i \arrow[r, equal] & Z \otimes E \otimes W
\end{tikzcd}$$

Unravelling definitions, the two outside compositions are the two ways to chase $x \otimes y$ around diagram (10). The two triangles commute again by the fact that $- \otimes -$ preserves colimits in each argument. Commutativity of the inner parallelogram is functoriality of $- \otimes -$. Thus diagram (10) tells us $\Phi_{Z,W}$ is an isomorphism, since s_{E,W_i} and $s_{Z,E\otimes W_i}$ are isomorphisms, and Φ_{Z,W_i} is an isomorphism for each i in I, meaning $\bigoplus_i \Phi_{W_i}$ is as well.

Thus, we've shown the class $\mathcal{E}$ of objects W for which $\Phi_{Z,W}$ is an isomorphism contains the S^a's, is closed under two-of-three for distinguished triangles, and is closed under arbitrary coproducts. Thus, it follows that $\mathcal{E}$ contains the class of all cellular objects in $\mathcal{SH}$, as desired.

Now, suppose that $\pi_*(E \otimes W)$ is a flat left $\pi_*(E)$-module, then we'd like to show $\Phi_{Z,W}$ is an isomorphism for all cellular Z in $\mathcal{SH}$. Showing this is entirely analagous to above, so we only outline the argument. Let $\mathcal{E}$ be the class of Z in $\mathcal{SH}$ such that $\Phi_{Z,W}$ is an isomorphism. Then in order to show $\mathcal{E}$ contains every cellular object, it suffices to show it contains the S^a's, is closed under two-of-three for distinguished triangles, and is closed under arbitrary coproducts.

To see $\mathcal{E}$ contains the S^a's, consider the map

$$\Psi : \pi_*(S^a \otimes E \otimes W) \to \pi_*(S^a \otimes E) \otimes_{\pi_*(E)} \pi_*(E \otimes W)$$

sending $x : S^b \to S^a \otimes E \otimes W$ to $\widetilde{e} \otimes \widetilde{x}$, where $\widetilde{e} \in \pi_a(S^a \otimes E)$ is the map $S^a \otimes e : S^a \to S^a \otimes E$, and $\widetilde{x} \in \pi_{b-a}(E \otimes W)$ is the map

$$\widetilde{x} : S^{b-a} \xrightarrow{\phi_{-a,b}} S^{-a} \otimes S^b \xrightarrow{S^{-a} \otimes x} S^{-a} \otimes S^a \otimes E \otimes W \xrightarrow{\phi_{-a,a}^{-1} \otimes E \otimes W} E \otimes W.$$

Then checking that Ψ is a left and right inverse to $\Phi_{S^a,W}$ is entirely analagous, so that S^a belongs to $\mathcal{E}$ as desired.

To see $\mathcal{E}$ is closed under two-of-three for distinguished triangles, let

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$$

be a distinguished triangle in $\mathcal{SH}$. Then an analagous argument as above (using Proposition A.12 and Proposition A.16) yields a long exact sequence of A-graded abelian groups

$$\begin{tikzcd}
& \pi_*(\Omega Y \otimes E) \arrow[r, "\pi_*(\Omega g \otimes E)"] & \pi_*(\Omega Z \otimes E) \arrow[dll, "\pi_*(\widetilde{h} \otimes E)"'] \\
\pi_*(X \otimes E) \arrow[r, "\pi_*(f \otimes E)"'] & \pi_*(Y \otimes E) \arrow[r, "\pi_*(g \otimes E)"] & \pi_*(Z \otimes E) \arrow[dll, "\pi_*(h \otimes E)"'] \\
\pi_*(\Sigma X \otimes E) \arrow[r, "\pi_*(\Sigma f \otimes E)"'] & \pi_*(\Sigma Y \otimes E) &
\end{tikzcd}$$

Then since $\pi_*(E \otimes W)$ is a flat left $\pi_*(E)$-module, we can tensor the above long exact sequence with $\pi_*(E \otimes W)$ on the right to obtain a long exact sequence which fits in the left column of the following commuting diagram:

$$\begin{tikzcd}
R_*^E(\Omega Y) \arrow[r, "\Phi_{\Omega Y,W}"] \arrow[d, "R_*^E(\Omega g)"'] & \pi_*(\Omega Y \otimes E \otimes W) \arrow[d, "\pi_*(\Omega g \otimes E \otimes W)"] \\
R_*^E(\Omega Z) \arrow[r, "\Phi_{\Omega Z,W}"] \arrow[d, "R_*^E(\widetilde{h})"'] & \pi_*(\Omega Z \otimes E \otimes W) \arrow[d, "\pi_*(\widetilde{h} \otimes E \otimes W)"] \\
R_*^E(X) \arrow[r, "\Phi_{X,W}"] \arrow[d, "R_*^E(f)"'] & \pi_*(X \otimes E \otimes W) \arrow[d, "\pi_*(f \otimes E \otimes W)"] \\
R_*^E(Y) \arrow[r, "\Phi_{Y,W}"] \arrow[d, "R_*^E(g)"'] & \pi_*(Y \otimes E \otimes W) \arrow[d, "\pi_*(g \otimes E \otimes W)"] \\
R_*^E(Z) \arrow[r, "\Phi_{Z,W}"] \arrow[d, "R_*^E(h)"'] & \pi_*(Z \otimes E \otimes W) \arrow[d, "\pi_*(h \otimes E \otimes W)"] \\
R_*^E(\Sigma X) \arrow[r, "\Phi_{\Sigma X,W}"] \arrow[d, "R_*^E(\Sigma f)"'] & \pi_*(\Sigma X \otimes E \otimes W) \arrow[d, "\pi_*(\Sigma f \otimes E \otimes W)"] \\
R_*^E(\Sigma Y) \arrow[r, "\Phi_{\Sigma Y,W}"] & \pi_*(\Sigma Y \otimes E \otimes W)
\end{tikzcd}$$

where R_*^E denotes the functor from $\mathcal{SH}$ to A-graded abelian groups sending $X \mapsto \pi_*(X \otimes E) \otimes_{\pi_*(E)} \pi_*(E \otimes W)$, so that $\Phi_{-,W}$ is a natural homomorphism $R_*^E(-) \Rightarrow \pi_*(- \otimes E \otimes W)$. Then finally by Lemma D.33 and the five lemma, if any two of three of the middle three horizontal arrows are isomorphisms, then all three of the horizontal arrows are isomorphisms, as desired.

Finally, in order to show $\mathcal{E}$ is closed under arbitrary coproducts, suppose we have a collection of objects $\{Z_i\}_{i \in I}$ in $\mathcal{E}$ indexed by some (small) set $\mathcal{E}$. Then we'd like to show $Z := \bigoplus_{i \in I} Z_i$ also belongs to $\mathcal{E}$. First note that since the S^a's are compact, for any object Y we have isomorphisms

$$s_{Z_i,Y} : \bigoplus_i Z_{i*}(Y) = \bigoplus_i [S^*, Z_i \otimes Y] \to [S^*, \bigoplus_i (Z_i \otimes Y)] = [S^*, Z \otimes Y] = Z_*(Y).$$

It is straightforward to verify that $s_{Z_i,E} : \bigoplus_i Z_{i*}(E) \to Z_*(E)$ is not only an isomorphism of abelian groups, but an isomorphism of right A-graded $\pi_*(E)$-modules, so that the top arrow in the following diagram is well-defined:

$$\begin{tikzcd}
\bigoplus_i \big(Z_{i*}(E) \otimes_{\pi_*(E)} E_*(W)\big) \arrow[r, equal] \arrow[d, "\bigoplus_i \Phi_{Z_i,W}"'] & \bigoplus_i (Z_{i*}(E)) \otimes_{\pi_*(E)} E_*(W_i) \arrow[r, "s_{Z_i,E}"] & Z_*(E) \otimes_{\pi_*(E)} E_*(W) \arrow[d, "\Phi_{Z,W}"] \\
\bigoplus_i Z_{i*}(E \otimes W) \arrow[rr, "s_{Z_i,E\otimes W}"] & & Z_*(E \otimes W)
\end{tikzcd}$$

Then a simple diagram chase yields the diagram commutes, so that $\Phi_{Z,W}$ is an isomorphism, assuming all the $\Phi_{Z_i,W}$'s are. $\square$

Proposition D.35. *Let (E, μ, e) be a ring spectrum in $\mathcal{SH}$, and let X and Y be two objects in $\mathcal{SH}$ such that E and X are both cellular (Definition 2.7) and $E_*(X)$ is a projective left $\pi_*(E)$-module (Proposition D.25). Then the map*

$$[X, E \otimes Y]_* \to \mathrm{Hom}^*_{\pi_*(E)}(E_*(X), E_*(Y))$$

which sends a generator $f : S^a \otimes X \to E \otimes Y$ in $[X, E \otimes Y]_*$ to the assignment which sends a generator $x : S^b \to E \otimes X$ in $E_*(X)$ to the composition

$$S^{a+b} \to S^a \otimes S^b \xrightarrow{S^a \otimes x} S^a \otimes E \otimes X \xrightarrow{\tau \otimes X} E \otimes S^a \otimes X \xrightarrow{E \otimes f} E \otimes E \otimes Y \xrightarrow{\mu} E \otimes Y$$

is an A-graded isomorphism of A-graded abelian groups.

Appendix E. Hopf algebroids

E.1. **The category R-GrCAlg of A-graded R-commutative rings.** We will freely use the results of Appendix C in this section. In what follows, fix an A-graded ring R. Further suppose that for all $a, b \in A$, there exists units $\theta_{a,b} \in R_0$ such that:

- For all $a \in A$, $\theta_{a,0} = \theta_{0,a} = 1$,
- For all $a, b \in A$, $\theta_{a,b}^{-1} = \theta_{b,a}$,
- For all $a, b, c \in A$, $\theta_{a,b} \cdot \theta_{a,c} = \theta_{a,b+c}$ and $\theta_{b,a} \cdot \theta_{c,a} = \theta_{b+c,a}$.
- For all $x \in R_a$ and $y \in R_b$,
$$x \cdot y = y \cdot x \cdot \theta_{a,b}.$$

Definition E.1. Let R-**GrCAlg** denote the following category:

- The objects are pairs (S, φ) called A-graded R-*commutative* rings, where S is an A-graded ring and $\varphi : R \to S$ is an A-graded ring homomorphism such that for all $x \in S_a$ and $y \in S_b$, we have
$$x \cdot y = y \cdot x \cdot \varphi(\theta_{a,b}),$$
- The morphisms $(S, \varphi) \to (S', \varphi')$ are A-graded ring homomorphisms $f : S \to S'$ such that $f \circ \varphi = \varphi'$.

Note that our notation for the category R-**GrCAlg** is somewhat deficient, as there may be multiple choices of families of units $\theta_{a,b} \in R_0$ satisfying the required properties which give rise to strictly different categories, as the following example illustrates. Despite this, for our purposes it will always be clear from context which collection of $\theta_{a,b}$'s we're working with.

Example E.2. Let $A = \mathbb{Z}$, let R be the ring $\mathbb{Z}$, viewed as a $\mathbb{Z}$-graded ring concentrated in degree 0, and let $\theta_{n,m} := (-1)^{n \cdot m}$ for all $n, m \in \mathbb{Z}$. Then the category R-**GrCAlg** is simply the category of graded anticommutative rings, i.e., $\mathbb{Z}$-graded rings S such that for all homogeneous $x, y \in S$, $x \cdot y = y \cdot x \cdot (-1)^{|x||y|}$. On the other hand, if we take the same A and R, but instead we define $\theta_{n,m} = 1$ for all $n, m \in \mathbb{Z}$, then the category R-**GrCAlg** becomes the category of strictly commutative $\mathbb{Z}$-graded rings.

Proposition E.3. *Suppose we have two morphisms $f : (B, \varphi_B) \to (C, \varphi_C)$ and $g : (B, \varphi_B) \to (D, \varphi_D)$ of A-graded R-commutative rings in R-**GrCAlg**. Then f and g make C and D both B-bimodules, respectively,*[14] *so we may form their tensor product $C \otimes_B D$, which is itself an A-graded B-bimodule (Proposition C.18). Then $C \otimes_B D$ canonically inherits the structure of an A-graded R-commutative ring with unit $1_C \otimes 1_D$ via a product*

$$(C \otimes_B D) \times (C \otimes_B D) \to C \otimes_B D$$

which sends a pair $(x \otimes y, x' \otimes y')$ of homogeneous pure tensors to the element

$$\varphi_B(\theta_{|x|,|y'|}) \cdot (xx' \otimes yy') = \varphi_C(\theta_{|x|,|y'|})xx' \otimes yy',$$

(where here $\cdot$ denotes the left module action of B on $C \otimes_B D$), and with structure map

$$\begin{aligned} \varphi : R &\to C \otimes_B D \\ r &\mapsto \varphi_B(r) \cdot (1_C \otimes 1_D) = (\varphi_C(r) \otimes 1_D) = (1_C \otimes \varphi_D(r)). \end{aligned}$$

Proof sketch. We simply lay out everything that needs to be shown, and we leave it to the reader to fill in the details. First to show that the indicated product is actually well-defined and distributive, by Lemma C.19 it suffices to show that for all homogeneous $c, c', c'' \in C$, $d, d', d'' \in D$, and $b \in B$ with $|c'| = |c''|$ and $|d'| = |d''|$, that

$$\begin{aligned}
\varphi_B(\theta_{|d|,|c'+c''|}) \cdot (c(c' + c'') \otimes dd') &= \varphi_B(\theta_{|d|,|c'|}) \cdot (cc' \otimes dd') + \varphi_B(\theta_{|d|,|c''|}) \cdot (cc'' \otimes dd') \\
\varphi_B(\theta_{|d|,|c'|}) \cdot (cc' \otimes d(d' + d'')) &= \varphi_B(\theta_{|d|,|c'|}) \cdot (cc' \otimes dd') + \varphi_B(\theta_{|d|,|c'|}) \cdot (cc' \otimes dd'') \\
\varphi_B(\theta_{|d|,|c' \cdot b|}) \cdot (c(c' \cdot b) \otimes dd') &= \varphi_B(\theta_{|d|,|c'|}) \cdot (cc' \otimes d(b \cdot d')) \\
\varphi_B(\theta_{|d'|,|c|}) \cdot ((c' + c'')c \otimes d'd) &= \varphi_B(\theta_{|d'|,|c|}) \cdot (c'c \otimes d'd) + \varphi_B(\theta_{|d'|,|c|}) \cdot (c''c \otimes d'd) \\
\varphi_B(\theta_{|d'+d''|,|c|}) \cdot (c'c \otimes (d' + d'')d) &= \varphi_B(\theta_{|d'|,|c|}) \cdot (c'c \otimes d'd) + \varphi_B(\theta_{|d''|,|c|}) \cdot (c'c \otimes d''d) \\
\varphi_B(\theta_{|d'|,|c|})((c' \cdot b)c \otimes d'd) &= \varphi_B(\theta_{|c|,|b \cdot d'|}) \cdot (c'c \otimes (b \cdot d')d).
\end{aligned}$$

These tell us that for all $x \in C \otimes_B D$ that the maps $C \otimes_B D \to C \otimes_B D$ sending $y \mapsto xy$ and $y \mapsto yx$ are well-defined A-graded homomorphisms of abelian groups, so we have a distributive product $(x, y) \mapsto xy$. Then to show that this product makes $C \otimes_B D$ an A-graded ring, by Lemma C.8, it suffices to show that for all homogeneous $x, y, z \in C \otimes_B D$ that $(xy)z = x(yz)$ and $x(1_C \otimes 1_D) = x = (1_C \otimes 1_D)x$. By distributivity, it further suffices to consider the case that x, y, and z are homogeneous *pure tensors* in $C \otimes_B D$, i.e., it suffices to show that for all homogeneous $c, c', c'' \in C$ and $d, d', d'' \in D$ that

$$((c \otimes d)(c' \otimes d'))(c'' \otimes d'') = (c \otimes d)((c' \otimes d')(c'' \otimes d''))$$

[14]Explicitly, it is a standard fact that given a ring homomorphism $\varphi : R \to S$ that S canonically becomes an R-bimodule with left action $r \cdot s := \varphi(r)s$ and right action $s \cdot r := s\varphi(r)$, so that in particular if φ is an A-graded homomorphism of A-graded rings, then φ makes S an A-graded R-bimodule.

and

$$(c \otimes d)(1_C \otimes 1_D) = (c \otimes d) = (1_C \otimes 1_D)(c \otimes d).$$

Thus, we have that the given product endows $C \otimes_B D$ with the structure of an A-graded ring, as desired. Now, we wish to show that the given map $\varphi : R \to C \otimes_B D$ is a ring homomorphism. Clearly it sends 1 to $1_C \otimes 1_D$, and again by linearity, it suffices to show that given *homogeneous* $r, s \in R$ that

$$\varphi(r+s) = \varphi_B(r+s)(1_C \otimes 1_D) = \varphi_B(r)(1_C \otimes 1_D) + \varphi_B(s)(1_C \otimes 1_D) = \varphi(r) + \varphi(s)$$

and

$$\varphi(rs) = \varphi_B(rs)(1_C \otimes 1_D) = (\varphi_B(r)(1_C \otimes 1_D))(\varphi_B(s)(1_C \otimes 1_D)) = \varphi(r)\varphi(s).$$

Finally, we need to show that $C \otimes_B D$ satisfies the graded commutativity condition, for which again by linearity it suffices to show that given homogeneous $c, c' \in C$ and $d, d' \in D$ that

$$(c \otimes d)(c' \otimes d') = \varphi(\theta_{|c \otimes d|, |c' \otimes d'|})(c' \otimes d')(c \otimes d) = \varphi(\theta_{|c|+|d|, |c'|+|d'|})(c' \otimes d)(c \otimes d).$$

Showing all of these is relatively straightforward. □

Proposition E.4. *The category R-**GrCAlg** has pushouts, where given $f : (B, \varphi_B) \to (C, \varphi_C)$ and $g : (B, \varphi_B) \to (D, \varphi_D)$, their pushout is the object $(C \otimes_B D, \varphi)$ constructed in Proposition E.3, along with the canonical maps $(C, \varphi_C) \to (C \otimes_B D, \varphi)$ sending $c \mapsto c \otimes 1_D$ and $(D, \varphi_D) \to (C \otimes_B D, \varphi)$ sending $d \mapsto 1_C \otimes d$. In particular, since (R, id_R) is initial, R-**GrCAlg** has binary coproducts.*

Proof sketch. First, we need to show that the given maps $i_C : (C, \varphi_C) \to (C \otimes_B D, \varphi)$ and $i_D : (D, \varphi_D) \to (C \otimes_B D, \varphi)$ are actually morphisms in R-**GrCAlg**, i.e., that they are ring homomorphisms and that the following diagram commutes:

$$\begin{tikzcd} & R \arrow[dl, "\varphi_C"'] \arrow[d, dashed, "\varphi"] \arrow[dr, "\varphi_D"] & \\ C \arrow[r, "i_C"'] & C \otimes_B D & D \arrow[l, "i_D"] \end{tikzcd}$$

Showing this is entirely straightforward. Furthermore, i_C and i_D clearly make the following diagram commute:

$$\begin{tikzcd} B \arrow[r, "g"] \arrow[d, "f"'] & D \arrow[d, "i_D"] \\ C \arrow[r, "i_C"'] & C \otimes_B D \end{tikzcd}$$

It remains to show that i_C and i_D are the universal such arrows. Suppose we have some object (E, φ_E) in R-**GrCAlg** and a commuting diagram

$$\begin{tikzcd} B \arrow[r, "g"] \arrow[d, "f"'] & D \arrow[d, "k"] \\ C \arrow[r, "h"] & E \end{tikzcd}$$

of morphisms in R-**GrCAlg**. Then we'd like to show there exists a unique morphism $\ell : C \otimes_B D \to E$ in R-**GrCAlg** which makes the following diagram commute:

$$\begin{tikzcd} B \arrow[r, "g"] \arrow[d, "f"'] & D \arrow[d, "i_D"'] \arrow[ddr, bend left, "k"] & \\ C \arrow[r, "i_C"] \arrow[drr, bend right, "h"'] & C \otimes_B D \arrow[dr, dashed, "\ell"] & \\ & & E \end{tikzcd}$$

First we show uniqueness. Supposing such an arrow ℓ existed, given elements $c \in C$ and $d \in D$, we must have

$$\ell(c \otimes d) = \ell((c \otimes 1_D)(1_C \otimes d)) = \ell(c \otimes 1_D)\ell(1_C \otimes d) = \ell(i_C(c))\ell(i_D(d)) = h(c)k(d).$$

Since pure tensors generate $C \otimes_B D$, we have uniquely determined ℓ, and clearly it makes the above diagram commute. Now, it remains to show that as defined ℓ is a morphism in R-**GrCAlg**, i.e., that it is an A-graded ring homomorphism and that the following diagram commutes:

$$\begin{tikzcd} & R \arrow[dl, "\varphi"'] \arrow[dr, "\varphi_E"] & \\ C \otimes_B D \arrow[rr, "\ell"] & & E \end{tikzcd}$$

This is all entirely straightforward to show. $\square$

E.2. ***A*-graded commutative Hopf algebroids over *R*.**

Definition E.5. Let $\mathcal{C}$ be a category admitting pullbacks. A *groupoid object* in $\mathcal{C}$ consists of a pair of objects (M, O) together with five morphisms

(1) *Source and target*: $s, t : M \to O$,
(2) *Identity*: $e : O \to M$,
(3) *Composition*: $c : M \times_O M \to M$,
(4) *Inverse*: $i : M \to M$

Explicitly, $M \times_O M$ fits into the following pullback diagram:

$$\begin{tikzcd} M \times_O M \arrow[r, "p_2"] \arrow[d, "p_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & M \arrow[d, "t"] \\ M \arrow[r, "s"'] & O \end{tikzcd}$$

so if we're working with sets, the composition map sends a pair (g, f) such that the codomain of f is the domain of g to $g \circ f$. These data must satisfy the following diagrams:

(1) Composition works correctly:

$$\begin{tikzcd} M \times_O M \arrow[r, "c"] \arrow[d, "p_1"'] & M \arrow[d, "t"] \\ M \arrow[r, "t"] & O \end{tikzcd} \qquad \begin{tikzcd} M \arrow[dr, "s"'] & O \arrow[l, "e"'] \arrow[r, "e"] \arrow[d, equal] & M \arrow[dl, "t"] \\ & O & \end{tikzcd} \qquad \begin{tikzcd} M \times_O M \arrow[r, "p_2"] \arrow[d, "c"'] & M \arrow[d, "s"] \\ M \arrow[r, "s"] & O \end{tikzcd}$$

The first diagram says that the codomain of $g \circ f$ is the codomain of g. The second diagram says that the domain and codomain of the identity on some object x is x. The third diagram says that the domain of $g \circ f$ is the domain of f.

(2) Associativity of composition: Write $M \times_O (M \times_O M)$ and $(M \times_O M) \times_O M$ for the pullbacks of $(s, t \circ c)$ and $(s \circ c, t)$, respectively, so we have commuting diagrams

$$\begin{tikzcd}
(M \times_O M) \times_O M \arrow[rr, "p_2'"] \arrow[dr, dashed, "c\times M"'] \arrow[dd, "p_1'"'] & & M \arrow[d, equal] \\
& M \times_O M \arrow[r, "p_2"] \arrow[d, "p_1"'] & M \arrow[d, "t"] \\
M \times_O M \arrow[r, "c"] & M \arrow[r, "s"] & O
\end{tikzcd}
\qquad
\begin{tikzcd}
M \times_O (M \times_O M) \arrow[rr, "p_2''"] \arrow[dr, dashed, "M\times c"'] \arrow[dd, "p_1''"'] & & M \times_O M \arrow[d, "c"] \\
& M \times_O M \arrow[r, "p_2"] \arrow[d, "p_1"'] & M \arrow[d, "t"] \\
M \arrow[r, equal] & M \arrow[r, "s"] & O
\end{tikzcd}$$

where the inner and outer squares in both diagrams are pullback squares. Furthermore, assuming the diagrams in condition (1) above are satisfied, we have that $t \circ p_1 \circ p_2'' = t \circ c \circ p_2'' = s \circ p_1''$, so that by the universal property of the pullback we have a map $M \times p_1 : M \times_O (M \times_O M) \to M \times_O M$ like so:

$$\begin{tikzcd}
M \times_O (M \times_O M) \arrow[dr, dashed, "M\times p_1"] \arrow[drr, bend left, "p_1\circ p_2''"] \arrow[ddr, bend right, "p_1''"'] & & \\
& M \times_O M \arrow[r, "p_2"] \arrow[d, "p_1"'] & M \arrow[d, "t"] \\
& M \arrow[r, "s"] & O
\end{tikzcd}$$

Now note that again assuming composition works correctly, so $s \circ c = s \circ p_2$, we have

$$s \circ c \circ (M \times p_1) = s \circ p_2 \circ (M \times p_1) = s \circ p_1 \circ p_2'' = t \circ p_2 \circ p_2'',$$

so that by the unviersal property of the pullback we get a map $a : M \times_O (M \times_O M) \to (M \times_O M) \times_O M$ like so:

$$\begin{tikzcd}
M \times_O (M \times_O M) \arrow[dr, dashed, "a"] \arrow[drrr, bend left, "p_2\circ p_2''"] \arrow[dddr, bend right, "M\times p_1"'] & & & \\
& (M \times_O M) \times_O M \arrow[rr, "p_2'"] \arrow[dd, "p_1'"'] & & M \arrow[d, equal] \\
& & & M \arrow[d, "t"] \\
& M \times_O M \arrow[r, "c"] & M \arrow[r, "s"] & O
\end{tikzcd}$$

Then we require that the following diagram commutes:

$$\begin{tikzcd}
M \times_O (M \times_O M) \arrow[rr, "a"] \arrow[d, "M\times c"'] & & (M \times_O M) \times_O M \arrow[d, "c\times M"] \\
M \times_O M \arrow[r, "c"] & M & M \times_O M \arrow[l, "c"']
\end{tikzcd}$$

This diagram says $h \circ (g \circ f) = (h \circ g) \circ f$.

(3) Unitality of composition: Given the maps $(\mathrm{id}_M, e\circ t), (e\circ s, \mathrm{id}_M) : M \to M\times_O M$ defined by the universal property of $M\times_O M$:

$$\begin{tikzcd}
M \arrow[dr, dashed, "(\mathrm{id}_M, e\circ s)"] \arrow[drr, bend left, "e\circ s"] \arrow[ddr, bend right, equal] & & \\
& M\times_O M \arrow[r, "p_2"] \arrow[d, "p_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & M \arrow[d, "t"] \\
& M \arrow[r, "s"] & O
\end{tikzcd}
\qquad
\begin{tikzcd}
M \arrow[dr, dashed, "(e\circ t, \mathrm{id}_M)"] \arrow[drr, bend left, equal] \arrow[ddr, bend right, "e\circ t"'] & & \\
& M\times_O M \arrow[r, "p_2"] \arrow[d, "p_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & M \arrow[d, "t"] \\
& M \arrow[r, "s"] & O
\end{tikzcd}$$

the following diagram commutes:

$$\begin{tikzcd}
M \arrow[r, "(e\circ t, \mathrm{id}_M)"] \arrow[d, "(\mathrm{id}_M, e\circ s)"'] \arrow[dr, equal] & M\times_O M \arrow[d, "c"] \\
M\times_O M \arrow[r, "c"'] & M
\end{tikzcd}$$

This diagram says that given $f : x \to y$ in M, that $f\circ \mathrm{id}_x = f$ and $\mathrm{id}_y \circ f = f$.

(4) Inverse: The following diagrams must commute:

$$\begin{tikzcd}
& M \arrow[d, "i"] \arrow[dl, equal] \\
M & M \arrow[l, "i"]
\end{tikzcd}
\qquad
\begin{tikzcd}
M \arrow[r, "(\mathrm{id}_M, i)"] \arrow[d, "t"'] & M\times_O M \arrow[d, "c"] & M \arrow[l, "(i, \mathrm{id}_M)"'] \arrow[d, "s"] \\
O \arrow[r, "e"] & M & O \arrow[l, "e"']
\end{tikzcd}
\qquad
\begin{tikzcd}
& M \arrow[dl, "s"'] \arrow[d, "i"] \arrow[dr, "t"] & \\
O & M \arrow[l, "t"'] \arrow[r, "s"] & O
\end{tikzcd}$$

where the arrows (id_M, i) and (i, id_M) are determined by the universal property of $M\times_O M$ like so:

$$\begin{tikzcd}
M \arrow[dr, dashed, "(\mathrm{id}_M, i)"] \arrow[drr, bend left, "i"] \arrow[ddr, bend right, equal] & & \\
& M\times_O M \arrow[r, "p_2"] \arrow[d, "p_1"'] & M \arrow[d, "t"] \\
& M \arrow[r, "s"] & O
\end{tikzcd}
\qquad
\begin{tikzcd}
M \arrow[dr, dashed, "(i, \mathrm{id}_M)"] \arrow[drr, bend left, equal] \arrow[ddr, bend right, "i"'] & & \\
& M\times_O M \arrow[r, "p_2"] \arrow[d, "p_1"'] & M \arrow[d, "t"] \\
& M \arrow[r, "s"] & O
\end{tikzcd}$$

Given $f : x \to y$ in M, the first diagram says that $(f^{-1})^{-1} = f$. The second says that $f\circ f^{-1} = \mathrm{id}_y$ and $f^{-1}\circ f = \mathrm{id}_x$. The last diagram says that the domain and codomain of f^{-1} are y and x, respectively.

Definition E.6. An *A-graded commutative Hopf algebroid over R* is a co-groupoid object in R-**GrCAlg**, i.e., a groupoid object in R-$\mathbf{GrCAlg}^{\mathrm{op}}$. Explicitly, an A-graded commutative Hopf algebroid over E is a pair (B, Γ) of objects in R-**GrCAlg** along with morphisms

(1) *left unit*: $\eta_L : B \to \Gamma$ (corresponding to t),

(2) *right unit*: $\eta_R : B \to \Gamma$ (corresponding to s),

(3) *comultiplication*: $\Psi : \Gamma \to \Gamma\otimes_B \Gamma$ (corresponding to c),

(4) *counit*: $\varepsilon : \Gamma \to B$ (corresponding to e),

(5) *conjugation*: $c : \Gamma \to \Gamma$ (corresponding to i),

where here Γ may be viewed as a B-bimodule with left B-module structure induced by η_L and right B-module structure induced by η_R, so we may form the tensor product of bimodules $\Gamma\otimes_B\Gamma$, which further may be given the structure of an A-graded R-commutative ring (by Proposition E.3), and fits into the following pushout diagram in R-**GrCAlg** (Proposition E.4):

$$\begin{tikzcd} B \arrow[r, "\eta_L"] \arrow[d, "\eta_R"'] & \Gamma \arrow[d, "g\mapsto 1\otimes g"] \\ \Gamma \arrow[r, "g\mapsto g\otimes 1"'] & \Gamma\otimes_B\Gamma \end{tikzcd}$$

These data must satisfy the following

(1) The following diagrams must commute:

$$\begin{tikzcd} B \arrow[r, "\eta_L"] \arrow[d, "\eta_L"'] & \Gamma \arrow[d, "\Psi"] \\ \Gamma \arrow[r, "g\mapsto g\otimes 1"'] & \Gamma\otimes_B\Gamma \end{tikzcd} \qquad \begin{tikzcd} & B \arrow[dl, "\eta_R"'] \arrow[d, equal] \arrow[dr, "\eta_L"] & \\ \Gamma \arrow[r, "\varepsilon"'] & B & \Gamma \arrow[l, "\varepsilon"] \end{tikzcd} \qquad \begin{tikzcd} B \arrow[r, "\eta_R"] \arrow[d, "\eta_R"'] & \Gamma \arrow[d, "g\mapsto 1\otimes g"] \\ \Gamma \arrow[r, "\Psi"] & \Gamma\otimes_B\Gamma \end{tikzcd}$$

(2) (Coassociativity) The following diagram must commute

$$\begin{tikzcd} \Gamma\otimes_B\Gamma \arrow[d, "\Psi\otimes_B\Gamma"'] & \Gamma \arrow[l, "\Psi"'] \arrow[r, "\Psi"] & \Gamma\otimes_B\Gamma \arrow[d, "\Gamma\otimes_B\Psi"] \\ (\Gamma\otimes_B\Gamma)\otimes_B\Gamma \arrow[rr] & & \Gamma\otimes_B(\Gamma\otimes_B\Gamma) \end{tikzcd}$$

where the bottom arrow sends $(g\otimes g')\otimes g''$ to $g\otimes(g'\otimes g'')$ and $\Psi\otimes\Gamma$ and $\Gamma\otimes\Psi$ fit into the following commutative diagrams, where both outer and inner squares in both diagrams are pushout squares in R-**GrCAlg**:

$$\begin{tikzcd} B \arrow[r, "\eta_L"] \arrow[d, "\eta_R"'] & \Gamma \arrow[r, "\Psi"] \arrow[d, "g\mapsto 1\otimes g"] & \Gamma\otimes_B\Gamma \arrow[dd, "x\mapsto 1\otimes x"] \\ \Gamma \arrow[r, "g\mapsto g\otimes 1"'] \arrow[d, equal] & \Gamma\otimes_B\Gamma \arrow[dr, dashed, "\Gamma\otimes\Psi"] & \\ \Gamma \arrow[rr, "g\mapsto g\otimes 1"'] & & \Gamma\otimes_B(\Gamma\otimes_B\Gamma) \end{tikzcd} \qquad \begin{tikzcd} B \arrow[r, "\eta_L"] \arrow[d, "\eta_R"'] & \Gamma \arrow[r, equal] \arrow[d, "g\mapsto 1\otimes g"] & \Gamma \arrow[dd, "g\mapsto 1\otimes g"] \\ \Gamma \arrow[r, "g\mapsto g\otimes 1"'] \arrow[d, "\Psi"'] & \Gamma\otimes_B\Gamma \arrow[dr, dashed, "\Psi\otimes\Gamma"] & \\ \Gamma\otimes_B\Gamma \arrow[rr, "x\mapsto x\otimes 1"'] & & (\Gamma\otimes_B\Gamma)\otimes_B\Gamma \end{tikzcd}$$

(3) The following diagram must commute:

$$\begin{tikzcd} \Gamma \arrow[r, "\Psi"] \arrow[d, "\Psi"'] \arrow[dr, equal] & \Gamma\otimes_B\Gamma \arrow[d, "(\eta_L\circ\varepsilon)\cdot\mathrm{id}_\Gamma"] \\ \Gamma\otimes_B\Gamma \arrow[r, "\mathrm{id}_\Gamma\cdot(\eta_R\circ\varepsilon)"'] & \Gamma \end{tikzcd}$$

where the right vertical arrow sends $g\otimes g'$ to $\eta_L(\varepsilon(g))g'$ and the bottom horizontal arrow sends $g\otimes g'$ to $g\eta_R(\varepsilon(g'))$.

(4) The following diagrams must commute:

$$\begin{tikzcd} & \Gamma \arrow[dl, equal] \arrow[d, "c"] \\ \Gamma & \Gamma \arrow[l, "c"] \end{tikzcd} \qquad \begin{tikzcd} B \arrow[d, "\eta_L"'] & \Gamma \arrow[l, "\varepsilon"'] \arrow[r, "\varepsilon"] \arrow[d, "i"'] & B \arrow[d, "\eta_R"] \\ \Gamma & \Gamma\otimes_B\Gamma \arrow[l, "\mathrm{id}_\Gamma\cdot c"] \arrow[r, "c\cdot\mathrm{id}_\Gamma"'] & \Gamma \end{tikzcd} \qquad \begin{tikzcd} B \arrow[r, "\eta_L"] \arrow[dr, "\eta_R"'] & \Gamma \arrow[d, "c"'] & B \arrow[l, "\eta_R"'] \arrow[dl, "\eta_L"] \\ & \Gamma & \end{tikzcd}$$

where the bottom left arrow in the middle diagram sends $g\otimes g'$ to $gc(g')$ and the bottom right arrow in the middle diagram sends $g\otimes g'$ to $c(g)g'$.

Proposition E.7. *Suppose we have an A-graded commutative Hopf algebroid (B, Γ) over R with structure maps η_L, η_R, Ψ, ε, and c. Recall in Definition E.6 we considered $\Gamma \otimes_B \Gamma$ to be the A-graded R-commutative ring whose underlying abelian group was given by the tensor product of B-bimodules, where Γ has left B-module structure induced by η_L and right B-module structure induced by η_R. Then this left (resp. right) B-module structure on $\Gamma \otimes_B \Gamma$ coincides with that induced by the ring homomorphism $\Psi \circ \eta_L$ (resp. $\Psi \circ \eta_R$).*

Proof. First we show the left module structures coincide. By additivity, in order to show the module structures coincide, it suffices to show that given a homogeneous pure tensor $g \otimes g'$ in $\Gamma \otimes_B \Gamma$ and some $b \in B$ that $\Psi(\eta_L(b)) \cdot (g \otimes g') = (\eta_L(b) \cdot g) \otimes g'$, where $\cdot$ on the left denotes the product in $\Gamma \otimes_B \Gamma$ and the $\cdot$ on the right denotes the product in Γ. By the axioms for a Hopf algebroid, we have that $\Psi(\eta_L(b)) = \eta_L(b) \otimes 1$. Thus by how the product in $\Gamma \otimes_B \Gamma$ is defined (Proposition E.3), we have that

$$\Psi(\eta_L(b)) \cdot (g \otimes g') = (\eta_L(b) \otimes 1) \cdot (g \otimes g') = (\varphi_\Gamma(\theta_{0,|g|}) \cdot \eta_L(b) \cdot g) \otimes (g' \cdot 1) = (\eta_L(b) \cdot g) \otimes g',$$

where $\varphi_\Gamma : R \to \Gamma$ is the structure map, and the last equality follows by the fact that $\theta_{0,|g|} = 1$. An entirely analagous argument yields that the canonical right module structure on $\Gamma \otimes_B \Gamma$ coincides with that induced by $\Psi \circ \eta_R$. ◻

Remark E.8. By Proposition E.7, given an A-graded commutative Hopf algebroid (B, Γ) over R, there is no ambiguity when discussing the objects $\Gamma \otimes_B (\Gamma \otimes_B \Gamma)$ and $(\Gamma \otimes_B \Gamma) \otimes_B \Gamma$ — they may both be considered as the threefold tensor product of the B-bimodule Γ with itself. In particular, we have a canonical isomorphism of B-bimodules

$$(\Gamma \otimes_B \Gamma) \otimes_B \Gamma \to \Gamma \otimes_B (\Gamma \otimes_B \Gamma)$$

sending $(g \otimes g') \otimes g''$ to $g \otimes (g' \otimes g'')$, and this is precisely the isomorphism in the coassociativity diagram in the definition of a Hopf algebroid (Definition E.6).

Proposition E.9. *Suppose we have an A-graded commutative Hopf algebroid (B, Γ) over R with structure maps η_L, η_R, Ψ, ε, and c. Then η_L is a homomorphism of left B-modules, η_R is a homomorphism of right B-modules, and Ψ and ε are homomorphisms of B-bimodules.*

Proof. Since the left (resp. right) B-module structure on Γ is induced by η_L (resp. η_R), the map η_L (resp. η_R) is a homomorphism of left (resp. right) B-modules by definition.

Next, we want to show Ψ is a homomorphism of B-bimodules. Note that given $b, b' \in B$ and $g \in \Gamma$, since Ψ is a ring homomorphism, we have that

$$\Psi(\eta_L(b) g \eta_R(b')) = \Psi(\eta_L(b))\Psi(g)\Psi(\eta_R(b')).$$

By Proposition E.7, we know that $\Psi(\eta_L(b))\Psi(g)\Psi(\eta_R(b')) = b \cdot \Psi(g) \cdot b'$, where the first and second $\cdot$ denotes the left and right action of B on $\Gamma \otimes_B \Gamma$, respectively.

Lastly, we claim that $\varepsilon : \Gamma \to B$ is a homomorphism of B-bimodules. We need to show that given $g \in \Gamma$ and $b, b' \in B$ that $\varepsilon(\eta_L(b) g \eta_R(g')) = b\varepsilon(g)b'$. This follows from the fact that $\varepsilon \circ \eta_L = \varepsilon \circ \eta_R = \mathrm{id}_B$. ◻

E.3. **Comodules over a Hopf algebroid.** In what follows, fix an A-graded commutative Hopf algebroid (B, Γ) over R with structure maps η_L, η_R, Ψ, ε, and c. We will always view Γ as a B-bimodule, with left B-module structure induced by η_L, and right B-module structure induced by η_R.

Lemma E.10. *Let N be an A-graded left B-module. Then we have an A-graded isomorphism of left B-modules*

$$(\Gamma \otimes_B \Gamma) \otimes_B N \xrightarrow{\cong} \Gamma \otimes_B (\Gamma \otimes_B N)$$

sending a pure tensor $(g \otimes g') \otimes n$ to $g \otimes (g' \otimes n)$.

Proof. □

aghh i hate this

Definition E.11. A *left comodule over* Γ is a pair (N, Ψ_N), where N is a left A-graded B-module and $\Psi_N : N \to \Gamma \otimes_B N$ is an A-graded homomorphism of left A-graded B-modules (where here we view Γ as a B-bimodule with its left module structure induced by η_L, and its right module structure induced by η_R). These data are required to make the following diagrams commute

$$\begin{tikzcd} N \arrow[r, "\Psi_N"] \arrow[dr, "\cong"'] & \Gamma \otimes_B N \arrow[d, "\varepsilon \otimes N"] \\ & B \otimes_B N \end{tikzcd} \qquad \begin{tikzcd} \Gamma \otimes_B N \arrow[d, "\Psi \otimes N"'] & N \arrow[l, "\Psi_N"'] \arrow[r, "\Psi_N"] & \Gamma \otimes_B N \arrow[d, "\Gamma \otimes \Psi_N"] \\ (\Gamma \otimes_B \Gamma) \otimes_B N \arrow[rr, "\cong"] & & \Gamma \otimes_B (\Gamma \otimes_B N) \end{tikzcd}$$

The maps $\varepsilon \otimes N$ and $\Psi \otimes N$ are well-defined by **??**, and the bottom isomorphism in the right diagram is that given in Lemma E.10.

Given two left A-graded Γ-comodules (N_1, Ψ_{N_1}) and (N_2, Ψ_{N_2}), a homomorphism of left A-graded comodules $f : N_1 \to N_2$ is an A-graded homomorphism of the underlying left B-modules such that the following diagram commutes:

$$\begin{tikzcd} N_1 \arrow[r, "f"] \arrow[d, "\Psi_{N_1}"'] & N_2 \arrow[d, "\Psi_{N_2}"] \\ \Gamma \otimes_B N_1 \arrow[r, "\Gamma \otimes f"] & \Gamma \otimes_B N_2 \end{tikzcd}$$

We write Γ-**CoMod** for the resulting category of left A-graded comodules over Γ. In the above definition, we required A-graded left Γ-comodule homomorphisms to strictly preserve the grading, but we could have instead considered left Γ-comodule homomorphisms which are of degree d for some $d \in A$, or equivalently, the set of degree zero A-graded Γ-comodule homomorphisms from N_1 to the shifted comodule $(N_2)_{*+d}$. We denote the hom-set of degree-d A-graded left Γ-comodule homomorphisms from (N_1, Ψ_{N_1}) to (N_2, Ψ_{N_2}) by

$$\mathrm{Hom}^d_{\Gamma\text{-}\mathbf{CoMod}}(N_1, N_2) \qquad \text{or usually just} \qquad \mathrm{Hom}^d_\Gamma(N_1, N_2).$$

In particular, we simply write $\mathrm{Hom}_{\Gamma\text{-}\mathbf{CoMod}}(N_1, N_2)$ or $\mathrm{Hom}_\Gamma(N_1, N_2)$ for the set of degree 0 A-graded left Γ-comodule homomorphisms from (N_1, Ψ_{N_1}) to (N_2, Ψ_{N_2}).

Proposition E.12. *The category* Γ-**CoMod** *is an additive category.*

Proof. First, we show the category is **Ab**-enriched. As a subcategory of B-**Mod**, it suffices to show that hom-sets in Γ-**CoMod** are closed under addition and taking inverses. To that end, suppose we have two A-graded left Γ-comodule homomorphisms $f, g : (N_1, \Psi_{N_1}) \to (N_2, \Psi_{N_2})$, then we have

$$\begin{aligned} \Psi_{N_2} \circ (f + g) &= (\Psi_{N_2} \circ f) + (\Psi_{N_2} \circ g) \\ &= ((\Gamma \otimes_B f) \circ \Psi_{N_1}) + ((\Gamma \otimes_B g) \circ \Psi_{N_1}) \\ &= ((\Gamma \otimes_B f) + (\Gamma \otimes_B g)) \circ \Psi_{N_1} \\ &= (\Gamma \otimes_B (f + g)) \circ \Psi_{N_1}, \end{aligned}$$

where the first equality follows since Ψ_{N_2} is a homomorphism of modules, the second follows since f and g are left Γ-comodule homomorphisms, the third follows since Ψ_{N_1} is a homomorphism of modules, and the last equality follows by definition of the tensor product of modules. Hence $f+g$ is indeed an A-graded left Γ-comodule homomorphism, as desired. Now, we also claim $-f$ is an A-graded left Γ-comodule homomorphism. To that end, note that

$$\Psi_{N_2} \circ (-f) = -\Psi_{N_2} \circ f = -(\Gamma \otimes_B f) \circ \Psi_{N_1} = (\Gamma \otimes_B (-f)) \circ \Psi_{N_1},$$

where the first equality follows since Ψ_{N_2} is a module homomorphism, the second follows since f is an A-graded left Γ-comodule homomorphism, and the third equality follows by definition of the tensor product.

Thus, we've shown that the hom-sets in Γ-**CoMod** are abelian groups, and composition is clearly bilinear, so that Γ-**CoMod** is indeed **Ab**-enriched.

Now, in order to show Γ-**CoMod** is additive, it suffices to show that it contains a zero object and has binary coproducts. First, note that the zero left B-module is clearly an A-graded left Γ-comodule with structure map the unique map $0 \to \Gamma \otimes_B 0 \cong 0$, and that given any other A-graded left Γ-comodule (N, Ψ_N), the unique homomorphisms of left B-modules $0 \to N$ and $N \to 0$ are left comodule homomorphisms.

Now, suppose we have two A-graded left Γ-comodules (N_1, Ψ_{N_1}) and (N_2, Ψ_{N_2}). First, we claim their direct sum as left B-modules $N_1 \oplus N_2$ is canonically an A-graded left Γ-comodule. We know that $N_1 \oplus N_2$ is an A-graded left B-mod □

Proposition E.13. *Suppose that Γ is flat as a right B-module (with its canonical right B-module structure induced by η_R). Then the category Γ-**CoMod** is an abelian category.*

E.4. **The dual E-Steenrod algebra is a Hopf algebroid.** In this subsection, we fix a monoidal closed tensor triangulated category $\mathcal{SH}$ with arbitrary (small) (co)products and sub-Picard grading $(A, \mathbf{1}, \{S^a\}, \{\phi_{a,b}\})$ (Definition 2.2), and we adopt the conventions outlined in Section 2.

Proposition E.14. *The assignment $(E, \mu, e) \mapsto (\pi_*(E), \pi_*(e))$ yields a functor*

$$\pi_* : \mathbf{CMon}_{\mathcal{SH}} \to \pi_*(S)\text{-}\mathbf{GrCAlg}$$

from the category of commutative monoid objects in $\mathcal{SH}$ (Definition D.2) to the category of A-graded $\pi_*(S)$-commutative rings (Definition E.1).

Proof. By **??**, we know that π_* yields a homomorphism from $\mathbf{CMon}_{\mathcal{SH}}$ to A-graded commutative rings. Furthermore, by Proposition D.18, we know that for all homogeneous $x, y \in \pi_*(E)$ that

$$x \cdot y = y \cdot x \cdot (e \circ \theta_{|x|,|y|}) = y \cdot x \cdot \pi_*(e)(\theta_{|x|,|y|}),$$

as desired. Thus, it remains to show that $\pi_*(e) : \pi_*(S) \to \pi_*(E)$ is an A-graded ring homomorphism for any (commutative) monoid object (E, μ, e) in $\mathcal{SH}$, and that given a monoid homomorphism $f : (E_1, \mu_1, e_1) \to (E_2, \mu_2, e_2)$ in $\mathbf{CMon}_{\mathcal{SH}}$, that $\pi_*(f)$ satisfies $\pi_*(f) \circ \pi_*(e_1) = \pi_*(e_2)$. The latter clearly holds, as since f is a monoid homomorphism, we have $f \circ e_1 = e_2$, so that

$$\pi_*(f) \circ \pi_*(e_1) = \pi_*(f \circ e_1) = \pi_*(e_2).$$

To see that $\pi_*(e) : \pi_*(S) \to \pi_*(E)$ is an A-graded ring homomorphism if (E, μ, e) is a monoid object, it suffices to show that $e : S \to E$ is a monoid homomorphism, since we already know π_* takes monoid homomorphisms to A-graded ring homomorphisms. Consider the following diagrams:

$$\begin{tikzcd}
S \otimes S \arrow[rr, "e\otimes e"] \arrow[dr, "S\otimes e"] \arrow[dd, "\cong"'] & & E \otimes E \arrow[dd, "\mu"] \\
& S \otimes E \arrow[ur, "e\otimes E"] \arrow[dr, "\cong"] & \\
S \arrow[rr, "e"] & & E
\end{tikzcd}
\qquad
\begin{tikzcd}
& S \arrow[dl, equal] \arrow[dr, "e"] & \\
S \arrow[rr, "e"] & & E
\end{tikzcd}$$

The right diagram commutes by definition. The top triangle in the left diagram commutes by functoriality of $-\otimes-$. The right triangle in the left diagram commutes by unitality of μ. Finally, the left triangle in the left diagram commutes by naturality of the unitors. Thus, we have shown e is a monoid object homomorphism, as desired. □

Proposition E.15. *Let (E, μ, e) be a commutative monoid object in $\mathcal{SH}$. Then the maps*

(1) $E \xrightarrow{\cong} E \otimes S \xrightarrow{E\otimes e} E \otimes E$,

(2) $E \xrightarrow{\cong} S \otimes E \xrightarrow{e\otimes E} E \otimes E$,

(3) $E \otimes E \xrightarrow{\cong} E \otimes S \otimes E \xrightarrow{E\otimes e\otimes E} E \otimes E \otimes E$,

(4) $E \otimes E \xrightarrow{\mu} E$, *and*

(5) $E \otimes E \xrightarrow{\tau_{E,E}} E \otimes E$

*are homomorphisms of monoid objects in $\mathcal{SH}$ (where here $E \otimes E$ and $E \otimes E \otimes E$ are considered as monoid objects in $\mathcal{SH}$ by Lemma D.3 and Lemma D.4, respectively), so that by Proposition 2.18, under π_* they induce morphisms in $\pi_*(S)$-**GrCAlg**:*

(1) $\eta_L : \pi_*(E) \to E_*(E)$,

(2) $\eta_R : \pi_*(E) \to E_*(E)$,

(3) $h : E_*(E) \to E_*(E \otimes E)$,

(4) $\varepsilon : E_*(E) \to \pi_*(E)$, *and*

(5) $c : E_*(E) \to E_*(E)$.

Proof. To start with, we will show $E \xrightarrow{\cong} E \otimes S \xrightarrow{E \otimes e} E \otimes E$ is a monoid object homomorphism. First, consider the following diagram:

$$\begin{tikzcd}
E_1 \otimes E_2 \arrow[rr, "E\otimes e\otimes E\otimes e"] \arrow[dr, "E\otimes E\otimes e"] \arrow[ddddd, "\mu"'] & & E_1 \otimes E \otimes E_2 \otimes E \arrow[ddl, "E\otimes\mu\otimes E"] \arrow[ddd, "E\otimes\tau\otimes E"] \\
& E_1 \otimes E_2 \otimes E \arrow[ur, "E\otimes e\otimes E\otimes E"'] \arrow[d, equal] & \\
& E_1 \otimes E_2 \otimes E \arrow[d, equal] & \\
& E_1 \otimes E_2 \otimes E \arrow[r, "E\otimes E\otimes e\otimes E"'] \arrow[d, equal] & E_1 \otimes E_2 \otimes E \otimes E \arrow[ul, "E\otimes\mu\otimes E"'] \arrow[dl, "E\otimes E\otimes\mu"] \arrow[dd, "\mu\otimes\mu"] \\
& E_1 \otimes E_2 \otimes E \arrow[dr, "\mu\otimes E"'] & \\
E_{1,2} \arrow[rr, "E\otimes e"'] & & E_{1,2} \otimes E
\end{tikzcd}$$

The leftmost region commutes by functoriality of $- \otimes -$. The top triangle also commutes by functoriality of $- \otimes -$. The triangle below that commutes by unitality of μ. The triangle below that commutes by commutativity of μ. The next two triangles below that commutes by unitality of μ. Finally, the bottom right triangle commutes by functoriality of $- \otimes -$. Next, consider the following diagram:

$$\begin{tikzcd}
& S \arrow[dr, "\cong"] \arrow[ddl, "e"'] & & \\
& & S \otimes S \arrow[dl, "e\otimes S"'] \arrow[dr, "e\otimes e"] & \\
E \arrow[r, "\cong"] & E \otimes S \arrow[rr, "E\otimes e"'] & & E \otimes E
\end{tikzcd}$$

The leftmost region commutes by naturality of the unitors, while the rightmost region commutes by functoriality of $-\otimes-$. Hence, we have shown $E \xrightarrow{\cong} E\otimes S \xrightarrow{E\otimes e} E\otimes E$ is indeed a monoid homomorphism, as desired. Showing that $E \xrightarrow{\cong} S \otimes E \xrightarrow{e\otimes E} E \otimes E$ is a monoid object homomorphism is entirely analagous.

Next, we will show that $E \otimes E \xrightarrow{\cong} E \otimes S \otimes E \xrightarrow{E\otimes e\otimes E}$ is a monoid object homomorphism. $\square$

Lemma E.16. *Let (E, μ, e) be a commutative monoid object in $\mathcal{SH}$. Then the left (resp. right) $\pi_*(E)$-module structure induced on $E_*(E)$ by the ring homomorphism η_L (resp. η_R) coincides with the canonical left (resp. right) $\pi_*(E)$-module structure on $E_*(E)$ given in Proposition 2.14.*

Proof. What's going on here is a bit subtle, so we're going to be really explicit. In Proposition 2.14, it was shown that $E_*(E)$ is a left $\pi_*(E)$-module via the assignment

$$\pi_*(E) \times E_*(E) \to E_*(E)$$

which sends homogeneous elements $r : S^a \to E$ and $x : S^b \to E \otimes E$ to the composition

$$S^{a+b} \xrightarrow{\cong} S^a \otimes S^b \xrightarrow{r\otimes x} E \otimes E \otimes E \xrightarrow{\mu\otimes E} E \otimes E.$$

We'd like to show that this is the same thing as the assignment $\pi_*(E) \times E_*(E) \to E_*(E)$ sending $(r, x) \mapsto \eta_L(r)x$, where $\eta_L(r)x$ denotes the product of $\eta_L(r)$ and x taken in the ring $E_*(E)$.

Explicitly, the product structure on $E_*(E) = \pi_*(E \otimes E)$ is that induced by the fact that $E \otimes E$ is a monoid object in $\mathcal{SH}$ by Lemma D.3, with product

$$E \otimes E \otimes E \otimes E \xrightarrow{E \otimes \tau \otimes E} E \otimes E \otimes E \otimes E \xrightarrow{\mu \otimes \mu} E \otimes E$$

(note the middle two factors are swapped). It is a standard fact from algebra that given a ring homomorphism $\varphi : R \to R'$, that R' is canonically a left R-module via the rule $(r, r') \mapsto \varphi(r)r'$, and a right R-module via the rule $(r', r) \mapsto r'\varphi(r)$. Thus, we can be sure that we actually have two left module actions. Furthermore, these are both clearly A-graded left module actions, so in order to show they're the same it suffices to show they agree on homogeneous elements (Lemma C.9). Now, suppose we have homogeneous elements $r : S^a \to E$ in $\pi_*(E)$ and $x : S^b \to E \otimes E$ in $E_*(E)$. Then consider the following diagram, where we've passed to a symmetric strict monoidal category:

$$\begin{tikzcd}
S^{a+b} \arrow[d, "\phi_{a,b}"] & & & & \\
S^a \otimes S^b \arrow[d, "r \otimes x"] & & & & \\
E_1 \otimes E_2 \otimes E_3 \arrow[rrrr, "\mu \otimes E"] \arrow[dr, equal] \arrow[dd, "E \otimes e \otimes E"'] & & & & E_{1,2} \otimes E_3 \arrow[dd, equal] \\
& E_1 \otimes E_2 \otimes E_3 \arrow[r, equal] & E_1 \otimes E_2 \otimes E_3 \arrow[r, equal] \arrow[d, "E \otimes E \otimes e \otimes E"] & E_1 \otimes E_2 \otimes E_3 \arrow[dr, "\mu \otimes E"] & \\
E_1 \otimes E \otimes E_2 \otimes E_3 \arrow[ur, "E \otimes \mu \otimes E"] \arrow[rr, "E \otimes \tau \otimes E"'] & & E_1 \otimes E_2 \otimes E \otimes E_3 \arrow[ul, "E \otimes \mu \otimes E"] \arrow[ur, "E \otimes E \otimes \mu"'] \arrow[rr, "\mu \otimes \mu"'] & & E_{1,2} \otimes E_3
\end{tikzcd}$$

Here we've numbered the E's to make it clear what's going on. The bottom composition is $\eta_L(r)x$, while the top composition is the canonical left action of r on x given in Proposition 2.14. The leftmost triangle commutes by unitality of μ. The triangle to the right of that commutes by commutativity of μ. The triangle to the right of that commutes by unitality of μ, as does the next triangle. The remaining triangle on the right commutes by functoriality of $- \otimes -$. Finally, the top region commutes by definition. Thus, we've shown that the left $\pi_*(E)$-module structure induced on $E_*(E)$ by η_L is in fact the canonical one. On the other hand, showing that the right $\pi_*(E)$-module structure induced on $E_*(E)$ by η_R is the canonical one is entirely analagous, and we leave it as an exercise for the reader. $\square$

Corollary E.17. *Given a commutative monoid object (E, μ, e) in $\mathcal{SH}$, the domain of the homomorphism*

$$\Phi_E : E_*(E) \otimes_{\pi_*(E)} E_*(E) \to E_*(E \otimes E)$$

*constructed in Proposition 4.3 is canonically an A-graded $\pi_*(S)$-ring, and sits in the following pushout diagram in $\pi_*(S)$-**GrCAlg**:*

$$\begin{tikzcd}
\pi_*(E) \arrow[r, "\eta_L"] \arrow[d, "\eta_R"'] & E_*(E) \arrow[d, "x \mapsto 1 \otimes x"] \\
E_*(E) \arrow[r, "x \mapsto x \otimes 1"'] & E_*(E) \otimes_{\pi_*(E)} E_*(E)
\end{tikzcd}$$

Proof. By Proposition E.4, we have a pushout diagram

$$\begin{tikzcd} \pi_*(E) \arrow[r, "\eta_L"] \arrow[d, "\eta_R"'] & E_*(E) \arrow[d, "x\mapsto 1\otimes x"] \\ E_*(E) \arrow[r, "x\mapsto x\otimes 1"'] & R \end{tikzcd}$$

where the underlying A-graded abelian group of R is the tensor product over $\pi_*(E)$ of $E_*(E)$ considered as right $\pi_*(E)$-module via η_R and $E_*(E)$ considered as a left $\pi_*(E)$-module via η_L. By Lemma E.16, as an A-graded abelian group, R is precisely the domain of Φ_E, as desired. $\square$

Lemma E.18. *Let (E, μ, e) be a commutative monoid object in $\mathcal{SH}$. Then the homomorphism*

$$\Phi_{E,E} : E_*(E) \otimes_{\pi_*(E)} E_*(E) \to E_*(E \otimes E)$$

*constructed in Proposition 4.2 is a homomorphism of A-graded $\pi_*(S)$-commutative rings, i.e. a morphism in $\pi_*(S)$-**GrCAlg**, where here $E_*(E) \otimes_{\pi_*(E)} E_*(E)$ is considered as an object in $\pi_*(S)$-**GrCAlg** by Corollary 4.6, and $E_*(E\otimes E) = \pi_*(E\otimes(E\otimes E))$ is considered as an object in $\pi_*(S)$-**GrCAlg** by Proposition 2.18, since $E\otimes(E\otimes E)$ is a monoid object in $\mathcal{SH}$ by Lemma D.3.*

Proof. Consider the maps

$$f : E\otimes E \xrightarrow{e\otimes E\otimes E} E\otimes E\otimes E$$

and

$$g : E\otimes E \xrightarrow{E\otimes E\otimes e} E\otimes E\otimes E.$$

We know that the maps

$$E \xrightarrow{e\otimes E} E\otimes E \qquad \text{and} \qquad E \xrightarrow{E\otimes e} E\otimes E$$

are monoid homomorphisms by **??**, so that f and g are monoid homomorphisms by Lemma D.5. Furthermore, by **??**, they are monoid homomorphisms between the same monoid objects in $\mathcal{SH}$. Finally, note that we have the following commutative diagram

$$\begin{tikzcd} E \arrow[r, "E\otimes e"] \arrow[d, "e\otimes E"'] \arrow[dr, "e\otimes E\otimes e"] & E\otimes E \arrow[d, "e\otimes E\otimes E"] \\ E\otimes E \arrow[r, "E\otimes E\otimes e"'] & E\otimes E\otimes E \end{tikzcd}$$

where the outer arrows are monoid object homomorphisms, thus, we may apply π_*, which yields the following commutative diagram in $\pi_*(S)$-**GrCAlg** (Proposition E.14):

$$\begin{tikzcd} \pi_*(E) \arrow[r, "\eta_L"] \arrow[d, "\eta_R"'] & E_*(E) \arrow[d, "\pi_*(f)"] \\ E_*(E) \arrow[r, "\pi_*(g)"'] & E_*(E\otimes E) \end{tikzcd}$$

Hence by **??** and the universal property of the pushout, there exists some unique morphism $\ell : E_*(E) \otimes_{\pi_*(E)} E_*(E) \to E_*(E\otimes E)$ in $\pi_*(S)$-**GrCAlg** which makes the following diagram commute:

$$\begin{tikzcd}
\pi_*(E) \arrow[r, "\eta_L"] \arrow[d, "\eta_R"'] & E_*(E) \arrow[d, "x\mapsto 1\otimes x"'] \arrow[dr, bend left, "\pi_*(f)"] & \\
E_*(E) \arrow[r, "x\mapsto x\otimes 1"] \arrow[drr, bend right, "\pi_*(g)"'] & E_*(E)\otimes_{\pi_*(E)} E_*(E) \arrow[dr, dashed, "\ell"] & \\
 & & E_*(E\otimes E)
\end{tikzcd}$$

Thus in order to show Φ_E is a morphism in $\pi_*(S)$-**GrCAlg**, it suffices to show that Φ_E and ℓ are the same map, since we know ℓ is a homomorphism of A-graded $\pi_*(S)$-commutative rings. Since Φ_E and ℓ are both abelian group homomorphisms, it further suffices to show they agree on homogeneous pure tensors, which generate $E_*(E)\otimes_{\pi_*(E)} E_*(E)$. Given homogeneous elements $x : S^a \to E\otimes E$ and $y : S^b \to E\otimes E$ in $E_*(E)$, unravelling how pushouts in $\pi_*(S)$-**GrCAlg** are defined (Proposition E.4), ℓ sends the pure homogeneous tensor $x\otimes y$ to the element $\pi_*(g)(x)\cdot\pi_*(f)(y)$, where here $\cdot$ denotes the product taken in $E_*(E\otimes E) = \pi_*(E\otimes E\otimes E)$. Now, consider the following diagram:

$$\begin{tikzcd}
S^{a+b} \arrow[d, "\phi_{a,b}"] & & \\
S^a\otimes S^b \arrow[d, "x\otimes y"] & & \\
E_1\otimes E_2\otimes E_3\otimes E_4 \arrow[rr, "g\otimes f = E\otimes E\otimes e\otimes e\otimes E\otimes E"] \arrow[drr, "E\otimes e\otimes E\otimes e\otimes E\otimes E"] \arrow[ddr, equal] \arrow[ddd, "E\otimes\mu\otimes E"'] & & E_1\otimes E_2\otimes E_a\otimes E_b\otimes E_3\otimes E_4 \arrow[d, "E\otimes\tau_{E\otimes E,E}\otimes E\otimes E"] \\
 & & E_1\otimes E_b\otimes E_2\otimes E_a\otimes E_3\otimes E_4 \arrow[d, "\mu\otimes E\otimes\tau\otimes E"] \\
 & E_1\otimes E_2\otimes E_3\otimes E_4 \arrow[ur, "E\otimes e\otimes E\otimes e\otimes E\otimes E"] \arrow[r, "E\otimes E\otimes E\otimes e\otimes E"'] \arrow[dl, "E\otimes\mu\otimes E"'] & E_1\otimes E_2\otimes E_3\otimes E_a\otimes E_4 \arrow[d, "E\otimes\mu\otimes\mu"] \\
E_1\otimes E_{2,3}\otimes E_4 \arrow[rr, equal] & & E_1\otimes E_{2,3}\otimes E_4
\end{tikzcd}$$

Here we have labelled the E's to make things clearer. The left outside composition is $\Phi_E(x\otimes y)$, while the right composition is $\pi_*(g)(x)\cdot\pi_*(f)(y)$. The top right triangle commutes by coherence for a symmetric monoidal category. The middle tright triangle commutes by unitality of μ and coherence for a symmetric monoidal category. The bottom trapezoid commutes by unitality of μ. The rest of the diagram commutes by definition. Thus we have $\Phi_E(x\otimes y) = \pi_*(g)(x)\cdot\pi_*(f)(y)$, so that $\Phi_E = \ell$ is not just an isomorphism of left $\pi_*(E)$-modules, but an isomorphism of A-graded $\pi_*(S)$-commutative rings, as desired. $\square$

Proposition E.19. *Let (E,μ,e) be a flat (Definition 4.1) and cellular (Definition 2.7) commutative monoid object in $\mathcal{SH}$. Then consider the map*

$$\Psi : E_*(E) \xrightarrow{\pi_*(E\otimes e\otimes E)} E_*(E\otimes E) \xrightarrow{\Phi_E^{-1}} E_*(E)\otimes_{\pi_*(E)} E_*(E),$$

where Φ_E is the isomorphism given in Proposition 4.3. Then Ψ is a homomorphism of A-graded $\pi_*(S)$-commutative rings, where here the object $E_*(E)\otimes_{\pi_*(E)} E_*(E)$ is considered an A-graded $\pi_*(S)$-commutative ring by Corollary E.17.

Proof. By **??**, we know $\Phi_E : E_*(E) \otimes_{\pi_*(E)} E_*(E) \to E_*(E \otimes E)$ is a bijective homomorphism of A-graded $\pi_*(S)$-commutative rings, thus it is clearly an isomorphism in $\pi_*(S)$-**GrCAlg**, so that its inverse Φ_E^{-1} is also a homomorphism in $\pi_*(S)$-**GrCAlg**. Thus, it suffices to show that $\pi_*(E \otimes e \otimes E)$ is as well. By Proposition E.14, it suffices to show $E \otimes e \otimes E : E \otimes E \to E \otimes E \otimes E$ is a homomorphism of monoid objects in $\mathcal{SH}$. Yet, we know this is true, as $e \otimes E : E \to E \otimes E$ is a homomorphism of monoid objects by **??**, so that by Lemma D.5 we have $E \otimes e \otimes E$ is also a homomorphism of monoid objects, as desired. ☐

Proposition E.20. *Let (E, μ, e) be a commutative monoid object in $\mathcal{SH}$. Then the morphism*

$$\mu : E \otimes E \to E$$

is a homomorphism of monoid objects (where $E \otimes E$ is considered a monoid object by Lemma D.3), so that by Proposition E.14, under π_* it induces a homomorphism of A-graded $\pi_*(S)$-commutative rings

$$\varepsilon : E_*(E) \to \pi_*(E).$$

Proof. Consider the following diagram:

$$\begin{tikzcd}
E \otimes E \otimes E \otimes E \arrow[rrrr, "\mu\otimes\mu"] \arrow[drrr, "\mu\otimes E\otimes E"] \arrow[dd, "E\otimes\tau\otimes E"'] \arrow[ddrr, "E\otimes\mu\otimes E"] & & & & E \otimes E \arrow[dddd, "\mu"] \\
& & & E \otimes E \otimes E \arrow[ur, "E\otimes\mu"] \arrow[d, "\mu\otimes E"] & \\
E \otimes E \otimes E \otimes E \arrow[rr, "E\otimes\mu\otimes E"'] \arrow[drr, "E\otimes E\otimes\mu"'] \arrow[dd, "\mu\otimes\mu"'] & & E \otimes E \otimes E \arrow[r, "\mu\otimes E"] \arrow[dr, "E\otimes\mu"] & E \otimes E \arrow[ddr, "\mu"] & \\
& & E \otimes E \otimes E \arrow[r, "E\otimes\mu"'] \arrow[dll, "\mu\otimes E"] & E \otimes E \arrow[dr, "\mu"] & \\
E \otimes E \arrow[rrrr, "\mu"'] & & & & E
\end{tikzcd}$$

The top left triangle commutes by commutativity of μ. Every other region commutes by functoriality of $- \otimes -$ and/or associativity of μ. Thus, we have shown μ satisfies the first diagram in Definition D.2 required for it to be a monoid homomorphism. To see it satisfies the second condition, consider the following diagram:

$$\begin{tikzcd}
& S \arrow[ddl, "e\otimes e"'] \arrow[d, "e"] \arrow[ddr, "e"] & \\
& E \arrow[dl, "E\otimes e"'] \arrow[dr, equal] & \\
E \otimes E \arrow[rr, "\mu"] & & E
\end{tikzcd}$$

The top left region commutes by functoriality of $- \otimes -$. The top right region commutes by definition. Finally, the bottom region commutes by unitality of μ. Thus we have shown μ is a monoid object homomorphism, as desired. ☐

Proposition E.21. *Let (E, μ, e) be a commutative monoid object in $\mathcal{SH}$. Then the morphism*

$$\tau_{E,E} : E \otimes E \to E \otimes E$$

is a homomorphism of monoid objects (where $E \otimes E$ is considered a monoid object by Lemma D.3), so that by Proposition E.14, under π_* it induces a homomorphism of A-graded $\pi_*(S)$-commutative rings

$$c : E_*(E) \to E_*(E).$$

Proof. Consider the following diagram:

$$\begin{tikzcd}
E_1 \otimes E_2 \otimes E_3 \otimes E_4 \arrow[r, "\tau\otimes\tau"] \arrow[d, "E\otimes\tau\otimes E"'] & E_2 \otimes E_1 \otimes E_4 \otimes E_3 \arrow[d, "E\otimes\tau\otimes E"] \\
E_1 \otimes E_3 \otimes E_2 \otimes E_4 \arrow[r, "\tau_{E\otimes E, E\otimes E}"] \arrow[d, "\mu\otimes\mu"'] & E_2 \otimes E_4 \otimes E_1 \otimes E_3 \arrow[d, "\mu\otimes\mu"] \\
E_{1,3} \otimes E_{2,4} \arrow[r, "\tau"'] & E_{2,4} \otimes E_{1,3}
\end{tikzcd}$$

The top region commutes by coherence for the symmetries in a symmetric monoidal category, while the bottom region commutes by naturality of τ. Now, consider the following diagram:

$$\begin{tikzcd}
& & S \arrow[dl, "\cong"'] \arrow[dr, "\cong"] & & \\
& S \otimes S \arrow[rr, "\tau"] \arrow[dl, "e\otimes e"'] & & S \otimes S \arrow[dr, "e\otimes e"] & \\
E \otimes E \arrow[rrrr, "\tau"] & & & & E \otimes E
\end{tikzcd}$$

The top triangle commutes by coherence for a symmetric monoidal category, while the bottom region commutes by naturality of τ. Thus, we have shown $\tau_{E,E}$ is a homomorphism of monoid objects, as desired. ☐

Proposition E.22. *Let (E, μ, e) be a commutative monoid object in $\mathcal{SH}$ which is flat (Definition 4.1) and cellular (Definition 2.7). Then the dual E-Steenrod algebra $(E_*(E), \pi_*(E))$ with the structure maps $(\eta_L, \eta_R, \Psi, \varepsilon, c)$ constructed above is an A-graded commutative Hopf algebroid over $\pi_*(S)$ (Definition E.6), i.e., a co-groupoid object in the category $\pi_*(S)$-*$\mathbf{GrCAlg}$.

Proof. We need to show all the diagrams in Definition E.6 commute. Since we are dealing with A-graded homomorphisms, when showing these diagrams commute, it always suffices to chase homogeneous elements around. To that end, we fix homogeneous elements $x : S^a \to E$ in $\pi_*(E)$ and $y : S^b \to E \otimes E$ in $E_*(E \otimes E)$ now.

First, we wish to show the outside of the following diagram commutes:

$$\begin{tikzcd}
\pi_*(E) \arrow[rr, "\eta_R"] \arrow[dd, "\eta_R"'] & & E_*(E) \arrow[dl, "\pi_*(E\otimes e\otimes E)"'] \arrow[dd, "\Psi"] \\
& E_*(E \otimes E) & \\
E_*(E) \arrow[rr, "x\mapsto 1\otimes x"'] & & E_*(E) \otimes_{\pi_*(E)} E_*(E) \arrow[ul, "\Phi_{E,E}"]
\end{tikzcd}$$

The right region commutes by how Ψ is defined (Proposition E.19), so it suffices to show the left region commutes. To that end, consider the following diagram:

$$\begin{tikzcd}
S^a \arrow[r, "x"] \arrow[d, equal, "\phi_{0,a}=\lambda_{S^a}^{-1}"'] \arrow[rrddd, "e\otimes e\otimes x"] & E \arrow[r, "e\otimes E"] & E\otimes E \arrow[ddd, "E\otimes e\otimes E"] \\
S\otimes S^a \arrow[d, "e\otimes e\otimes x"'] & & \\
E\otimes E\otimes E \arrow[d, "E\otimes E\otimes e\otimes E"'] \arrow[rrd, equal] & & \\
E\otimes E\otimes E\otimes E \arrow[rr, "E\otimes\mu\otimes E"'] & & E\otimes E\otimes E
\end{tikzcd}$$

The top composition is $\pi_*(E\otimes e\otimes E)(\eta_R(x))$, while the bottom composition is $\Phi_{E,E}(1\otimes\eta_R(x))$. The top right region commutes by functoriality of $-\otimes-$. The bottom left triangle commutes by unitality of μ. Finally, the middle triangle commutes by definition.

Now, we wish to show the following diagram commutes

$$\begin{tikzcd}
E_*(E) \arrow[dr, "\varepsilon"'] & \pi_*(E) \arrow[l, "\eta_L"'] \arrow[r, "\eta_R"] \arrow[d, equal] & E_*(E) \arrow[dl, "\varepsilon"] \\
& \pi_*(E) &
\end{tikzcd}$$

Unravelling how η_L, η_R, and ε are defined, this is the diagram obtained by applying π_* to the following diagram:

$$\begin{tikzcd}
E\otimes E \arrow[dr, "\mu"'] & E \arrow[l, "E\otimes e"'] \arrow[r, "e\otimes E"] \arrow[d, equal] & E\otimes E \arrow[dl, "\mu"] \\
& E &
\end{tikzcd}$$

This commutes by unitality of μ.

Showing that the third diagram in item (1) in Definition E.6 is entirely analagous to how we showed the first diagram commutes.

Now, we'd like to show the following diagram commutes: □

References

[1] HeinrichD (https://mathoverflow.net/users/98306/heinrichd). *Smallness of the category of schemes of finite type*. MathOverflow. URL:https://mathoverflow.net/q/251044 (version: 2016-09-30). eprint: https://mathoverflow.net/q/251044. URL: https://mathoverflow.net/q/251044.

[2] J. Michael Boardman. "Conditionally convergent spectral sequences". In: *Homotopy Invariant Algebraic Structures* 239 (1999), pp. 49–84. DOI: 10.1090/conm/239/03597.

[3] Daniel Dugger. "Coherence for invertible objects and multigraded homotopy rings". In: *Algebraic & Geometric Topology* 14.2 (Mar. 2014), pp. 1055–1106. DOI: 10.2140/agt.2014.14.1055. URL: https://doi.org/10.2140%2Fagt.2014.14.1055.

[4] Daniel Dugger et al. *The Multiplicative Structures on Motivic Homotopy Groups*. 2022. arXiv: 2212.03938 [math.AG].

[5] Mark Hovey. *Model Categories*. American Mathematical Society, 1999.

[6] Mark Hovey. “Spectra and symmetric spectra in general model categories”. In: *Journal of Pure and Applied Algebra* 165.1 (2001), pp. 63–127. ISSN: 0022-4049. DOI: https://doi.org/10.1016/S0022-4049(00)00172-9. URL: https://www.sciencedirect.com/science/article/pii/S0022404900001729.

[7] J.F. Jardine. “Motivic symmetric spectra.” eng. In: *Documenta Mathematica* 5 (2000), pp. 445–553. URL: http://eudml.org/doc/121235.

[8] Saunders MacLane. “Natural Associativity and Commutativity”. In: *Rice Institute Pamphlet - Rice University Studies* 4 (1963). URL: https://hdl.handle.net/1911/62865.

[9] J.P. May. “The Additivity of Traces in Triangulated Categories”. In: *Advances in Mathematics* 163.1 (2001), pp. 34–73. ISSN: 0001-8708. DOI: https://doi.org/10.1006/aima.2001.1995. URL: https://www.sciencedirect.com/science/article/pii/S0001870801919954.

[10] Fabien Morel and Vladimir Voevodsky. “A1-homotopy theory of schemes”. In: *Publications Mathématiques de l’Institut des Hautes Études Scientifiques* 90.1 (Dec. 1999), pp. 45–143. ISSN: 1618-1913. DOI: 10.1007/BF02698831. URL: https://doi.org/10.1007/BF02698831.

[11] Amnon Neeman. *Triangulated Categories*. Vol. 148. Princeton Univ. Press, 2001.

[12] nLab authors. *adjoint equivalence*. https://ncatlab.org/nlab/show/adjoint+equivalence. Revision 17. July 2023.

[13] nLab authors. *Introduction to Stable homotopy theory – 1-2*. https://ncatlab.org/nlab/show/Introduction+to+Stable+homotopy+theory+--+1-2. Revision 77. July 2023.

[14] I. Panin, K. Pimenov, and O. Röndigs. *On Voevodsky’s algebraic K-theory spectrum BGL*. 2008. arXiv: 0709.3905 [math.AG].

[15] Emily Riehl. *Category Theory in Context*. Dover Publications, Inc, 2016.

[16] Vladimir Voevodsky. *Reduced power operations in motivic cohomology*. 2001. arXiv: math/0107109 [math.AG].

[17] Glen Matthew Wilson and Paul Østvær. “Two-complete stable motivic stems over finite fields”. In: *Algebraic & Geometric Topology* 17.2 (2017), pp. 1059–1104. DOI: 10.2140/agt.2017.17.1059. URL: https://doi.org/10.2140/agt.2017.17.1059.