In this appendix, we will define the notion of A-graded anticommutative Hopf algebroids (Definition 0.2) over an A-graded anticommutative ring R (??), and left comodules over them (Definition 0.6).

0.1. A-graded anticommutative Hopf algebroids over R. Given an A-graded anticommutative ring R, we will define an A-graded anticommutative Hopf algebroid over R to be a cogroupoid object in R- \mathbf{GCA}^A , i.e., a groupoid object in (R- $\mathbf{GCA}^A)^{\mathrm{op}}$. First, recall the definition of a groupoid object in a category with pullbacks:

Definition 0.1. Let \mathcal{C} be a category with pullbacks. A groupoid object in \mathcal{C} consists of a pair of objects (M, O) together with five morphisms

- (1) Source and target: $s, t: M \to O$,
- (2) Identity: $e: O \to M$,
- (3) Composition: $c: M \times_O M \to M$,
- (4) Inverse: $i: M \to M$

Where $M \times_O M$ will always refer to the object which into the following pullback diagram in \mathcal{C} :

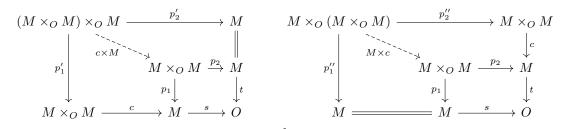
$$\begin{array}{ccc} M \times_O M & \xrightarrow{p_2} & M \\ & \downarrow t & & \downarrow t \\ M & \xrightarrow{s} & O \end{array}$$

For example, if we're working in $\mathcal{C} = \mathbf{Set}$, we should think of M as a set of morphisms, and O as a set of objects. The functions s and t take a morphism to their domain and codomain, respectively, and $M \times_O M$ is the collection of pairs of morphisms $(g, f) \in M \times M$ such that t(f) = s(g), and the composition map $c: M \times_O M \to M$ takes such a pair to the element $g \circ f \in M$. We think of the identity $e: O \to M$ as taking some object $x \in O$ to the identity morphism $e(x) = \mathrm{id}_x \in M$ on x, and the inverse map $i: M \to M$ takes a morphism f to its inverse f^{-1} . These data are required to make the following diagrams commute:

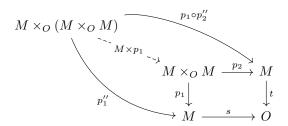
(1) Composition works correctly:

Expressed in terms of sets, the first diagram says that the target of $g \circ f$ is the target of g. The second diagram says that the domain and codomain of the identity on some object x is x. The third diagram says that the domain of $g \circ f$ is the domain of f.

(2) Associativity of composition: Write $M \times_O (M \times_O M)$ and $(M \times_O M) \times_O M$ for the pullbacks of $(s, t \circ c)$ and $(s \circ c, t)$, respectively, so we have commuting diagrams



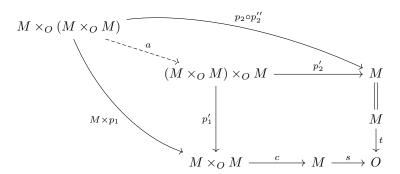
where the inner and outer squares in both diagrams are pullback squares. Furthermore, assuming the diagrams in condition (1) above are satisfied, we have that $t \circ p_1 \circ p_2'' =$ $t \circ c \circ p_2'' = s \circ p_1''$, so that by the universal property of the pullback we have a map $M \times p_1 : M \times_O (M \times_O M) \to M \times_O M$ like so:



Now note that again assuming the diagrams above in (1) commute, we have $s \circ c = s \circ p_2$, so that

$$s \circ c \circ (M \times p_1) = s \circ p_2 \circ (M \times p_1) = s \circ p_1 \circ p_2'' = t \circ p_2 \circ p_2''.$$

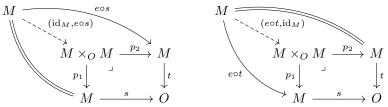
Then by the unviersal property of the pullback we get a map $a: M \times_O (M \times_O M) \to$ $(M \times_O M) \times_O M$ like so:

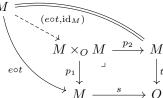


Exercise: Show that this map a is an isomorphism. Then we require that the following diagram commutes:

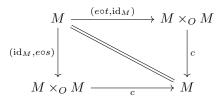
Expressed in terms of sets, this diagram says $h \circ (g \circ f) = (h \circ g) \circ f$.

(3) Unitality of composition: Given the maps $(\mathrm{id}_M, e \circ t), (e \circ s, \mathrm{id}_M) : M \to M \times_O M$ defined by the universal property of $M \times_O M$:



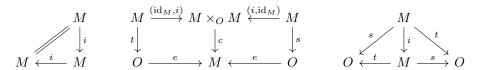


the following diagram commutes:

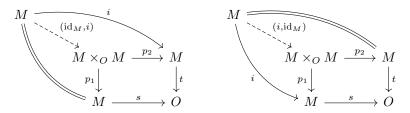


Expressed in terms of sets, this diagram says that given $f \in M$ with s(f) = x and t(f) = y, that $f \circ id_x = f$ and $id_y \circ f = f$.

(4) Inverse: The following diagrams must commute:



where the arrows (id_M, i) and (i, id_M) are determined by the universal property of $M \times_O M$ like so:



Expressed in terms of sets, given $f \in M$ with s(f) = x and t(f) = y, the first diagram says that $(f^{-1})^{-1} = f$, the second says that $f \circ f^{-1} = \mathrm{id}_y$ and $f^{-1} \circ f = \mathrm{id}_x$, and the last diagram says that the domain and codomain of f^{-1} are g and g, respectively.

It can be seen that groupoid objects in $\mathcal{C} = \mathbf{Set}$ are precisely (small) groupoids. Now, we can state and unravel the definition of a Hopf algebroid:

Definition 0.2. Given an A-graded anticommutative ring R (??), an A-graded anticommutative Hopf algebroid over R is a co-groupoid object in R- \mathbf{GCA}^A , i.e., a groupoid object in (R- $\mathbf{AGrCAlg})^{\mathrm{op}}$. Explicitly, an A-graded anticommutative Hopf algebroid over E is a pair (Γ, B) of objects in R- $\mathbf{AGrCAlg}$ along with morphisms

- (1) left unit: $\eta_L: B \to \Gamma$ (corresponding to t),
- (2) right unit: $\eta_R: B \to \Gamma$ (corresponding to s),
- (3) comultiplication: $\Psi: \Gamma \to \Gamma \otimes_B \Gamma$ (corresponding to c),
- (4) counit: $\epsilon: \Gamma \to B$ (corresponding to e),
- (5) conjugation: $c: \Gamma \to \Gamma$ (corresponding to i),

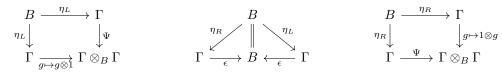
where here Γ may be viewed as a *B*-bimodule with left *B*-module structure induced by η_L and right *B*-module structure induced by η_R , so we may form the tensor product of bimodules $\Gamma \otimes_B \Gamma$, which further may be given the structure of an *A*-graded anticommutative *R*-algebra (by ??), and

fits into the following pushout diagram in R- $\mathbf{GCA}^A g$ (??):

$$\begin{array}{ccc} B & \stackrel{\eta_L}{\longrightarrow} & \Gamma \\ \downarrow^{\eta_R} \downarrow & & \downarrow^{g \mapsto 1 \otimes g} \\ \Gamma & \xrightarrow[g \mapsto g \otimes 1]{} & \Gamma \otimes_B \Gamma \end{array}$$

These data must make the following diagrams commute:

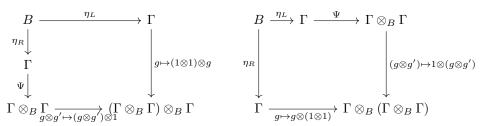
(1) (Composition works correctly)



(2) (Coassociativity) The following diagram must commute

$$\begin{array}{cccc}
\Gamma \otimes_B \Gamma & \stackrel{\Psi}{\longleftarrow} & \Gamma & \stackrel{\Psi}{\longrightarrow} & \Gamma \otimes_B \Gamma \\
\downarrow^{\Psi \otimes_B \Gamma} \downarrow & & & \downarrow^{\Gamma \otimes_B \Psi} \\
(\Gamma \otimes_B \Gamma) \otimes_B \Gamma & \stackrel{\cong}{\longrightarrow} & \Gamma \otimes_B (\Gamma \otimes_B \Gamma)
\end{array}$$

where $(\Gamma \otimes_B \Gamma) \otimes_B \Gamma$ and $\Gamma \otimes_B (\Gamma \otimes_B \Gamma)$ denote the rings which fit into the following pushout diagrams in R-**GCA**^A:

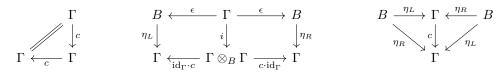


and the isomorphism $(\Gamma \otimes_B \Gamma) \otimes_B \Gamma \to \Gamma \otimes_B (\Gamma \otimes_B \Gamma)$ sends $(g \otimes g') \otimes g''$ to $g \otimes (g' \otimes g'')$, the left vertical arrow $\Psi \otimes \Gamma$ sends $g \otimes g'$ to $\Psi(g) \otimes g$, and the right vertical arrow $\Gamma \otimes \Psi$ sends $g \otimes g'$ to $g \otimes \Psi(g')$.

(3) (Co-unitality):

where the right vertical arrow sends $g \otimes g'$ to $\eta_L(\epsilon(g))g'$ and the bottom horizontal arrow sends $g \otimes g'$ to $g\eta_R(\epsilon(g'))$.

(4) (Convolution):



where the bottom left arrow in the middle diagram sends $g \otimes g'$ to gc(g') and the bottom right arrow in the middle diagram sends $g \otimes g'$ to c(g)g'.

The remainder of this subsection is devoted to proving some technical lemmas about A-graded anticommutative Hopf algebroids.

Proposition 0.3. Suppose we have an A-graded anticommutative Hopf algebroid (Γ, B) over (R, θ) with structure maps η_L , η_R , Ψ , ϵ , and c (Definition 0.2). Recall in the definition, we considered $\Gamma \otimes_B \Gamma$ to be the A-graded R-commutative ring whose underlying abelian group was given by the tensor product of B-bimodules, where Γ has left B-module structure induced by η_L and right B-module structure induced by η_R . Thus $\Gamma \otimes_B \Gamma$ is canonically a B-bimodule, as it is a tensor product of B-bimodules. Then the canonical left (resp. right) B-module structure on $\Gamma \otimes_B \Gamma$ coincides with that induced by the ring homomorphism $\Psi \circ \eta_L$ (resp. $\Psi \circ \eta_R$).

Proof. First we show the left module structures coincide. By additivity, in order to show the module structures coincide, it suffices to show that given a homogeneous pure tensor $g \otimes g'$ in $\Gamma \otimes_B \Gamma$ and some $b \in B$ that $\Psi(\eta_L(b)) \cdot (g \otimes g') = (\eta_L(b) \cdot g) \otimes g'$, where \cdot on the left denotes the product in $\Gamma \otimes_B \Gamma$ and the \cdot on the right denotes the product in Γ . By the axioms for a Hopf algebroid, we have that $\Psi(\eta_L(b)) = \eta_L(b) \otimes 1$. Thus by how the product in $\Gamma \otimes_B \Gamma$ is defined (??), we have that

$$\Psi(\eta_L(b)) \cdot (g \otimes g') = (\eta_L(b) \otimes 1) \cdot (g \otimes g') = (\varphi_{\Gamma}(\theta_{0,|g|}) \cdot \eta_L(b) \cdot g) \otimes (g' \cdot 1) = (\eta_L(b) \cdot g) \otimes g',$$

where $\varphi_{\Gamma}: R \to \Gamma$ is the structure map, and the last equality follows by the fact that $\theta_{0,|g|} = 1$. An entirely analogous argument yields that the canonical right module structure on $\Gamma \otimes_B \Gamma$ coincides with that induced by $\Psi \circ \eta_R$, since $\Psi \circ \eta_R = 1 \otimes \eta_R$.

Remark 0.4. By the above proposition, given an A-graded commutative Hopf algebroid (Γ, B) over R, there is no ambiguity when discussing the objects $\Gamma \otimes_B (\Gamma \otimes_B \Gamma)$ and $(\Gamma \otimes_B \Gamma) \otimes_B \Gamma$ —they may both be considered as the threefold tensor product of the B-bimodule Γ with itself. In particular, we have a canonical isomorphism of B-bimodules

$$(\Gamma \otimes_B \Gamma) \otimes_B \Gamma \to \Gamma \otimes_B (\Gamma \otimes_B \Gamma)$$

sending $(g \otimes g') \otimes g''$ to $g \otimes (g' \otimes g'')$, and this is precisely the isomorphism in the coassociativity diagram in the definition of a Hopf algebroid (Definition 0.2).

Proposition 0.5. Suppose we have an A-graded commutative Hopf algebroid (Γ, B) over R with structure maps η_L , η_R , Ψ , ϵ , and c. Then $\eta_L : B \to \Gamma$ is a homomorphism of left B-modules, $\eta_R : B \to \Gamma$ is a homomorphism of right B-modules, and $\Psi : \Gamma \to \Gamma \otimes_B \Gamma$ and $\epsilon : \Gamma \to B$ are homomorphisms of B-bimodules.

Proof. Since the left (resp. right) B-module structure on Γ is induced by η_L (resp. η_R), the map η_L (resp. η_R) is a homomorphism of left (resp. right) B-modules by definition.

Next, we want to show Ψ is a homomorphism of B-bimodules. The left (resp. right) B-module structure on Γ is that induced by η_L (resp. η_R), and in Proposition 0.3, we showed that the left (resp. right) B-module structure on $\Gamma \otimes_B \Gamma$ is that induced by $\Psi \circ \eta_L$ (resp. $\Psi \circ \eta_R$), so that by definition $\Psi : \Gamma \to \Gamma \otimes_B \Gamma$ is a homomorphism of left (resp. right) B-modules.

Lastly, we claim that $\epsilon : \Gamma \to B$ is a homomorphism of B-bimodules. We need to show that given $g \in \Gamma$ and $b, b' \in B$ that $\epsilon(\eta_L(b)g\eta_R(g')) = b\epsilon(g)b'$. This follows from the fact that ϵ is a ring homomorphism satisfying $\epsilon \circ \eta_L = \epsilon \circ \eta_R = \mathrm{id}_B$.

0.2. Comodules over a Hopf algebroid. In what follows, fix an A-graded anticommutative ring (R, θ) and an A-graded anticommutative Hopf algebroid (Γ, B) over R with structure maps η_L , η_R , Ψ , ϵ , and c. We will always view Γ with its canonical B-bimodule structure, with left B-module structure induced by η_L , and right B-module structure induced by η_R . In particular, any tensor product over B involving Γ will always refer to Γ with this bimodule structure.

Definition 0.6. A left comodule over Γ is a pair (N, Ψ_N) , where N is a left A-graded B-module and $\Psi_N : N \to \Gamma \otimes_B N$ is an A-graded homomorphism of left A-graded B-modules. These data are required to make the following diagrams commute

$$N \xrightarrow{\Psi_N} \Gamma \otimes_B N \qquad \qquad \Gamma \otimes_B N \xleftarrow{\Psi_N} \qquad N \xrightarrow{\Psi_N} \Gamma \otimes_B N \qquad \qquad \downarrow^{\Gamma \otimes \Psi_N} \qquad \downarrow^{\Gamma \otimes \Psi_N}$$

The maps $\epsilon \otimes N$ and $\Psi \otimes N$ are well-defined by Proposition 0.5, and the bottom isomorphism in the right diagram is the canonical one sending $(g \otimes g') \otimes n \mapsto g \otimes (g' \otimes n)$.

Given two left A-graded Γ -comodules (N_1, Ψ_{N_1}) and (N_2, Ψ_{N_2}) , a homomorphism of left A-graded comodules $f: N_1 \to N_2$ is an A-graded homomorphism of the underlying left B-modules such that the following diagram commutes:

$$\begin{array}{ccc} N_1 & \xrightarrow{f} & N_2 \\ \Psi_{N_1} \downarrow & & \downarrow \Psi_{N_2} \\ \Gamma \otimes_B N_1 & \xrightarrow{\Gamma \otimes f} & \Gamma \otimes_B N_2 \end{array}$$

We write Γ -**CoMod**^A for the resulting category of left A-graded comodules over Γ . In the above definition, we required A-graded left Γ -comodule homomorphisms to strictly preserve the grading, but we could have instead considered left Γ -comodule homomorphisms which are of degree d for some $d \in A$, or equivalently, the set of degree zero A-graded Γ -comodule homomorphisms from N_1 to the shifted comodule $(N_2)_{*+d}$. We denote the hom-set of degree-d A-graded left Γ -comodule homomorphisms from (N_1, Ψ_{N_1}) to (N_2, Ψ_{N_2}) by

$$\operatorname{Hom}_{\Gamma\text{-}\mathbf{CoMod}^A}^d(N_1, N_2)$$
 or usually just $\operatorname{Hom}_{\Gamma}^d(N_1, N_2)$.

In particular, we simply write $\operatorname{Hom}_{\Gamma\text{-}\mathbf{CoMod}^A}(N_1, N_2)$ or $\operatorname{Hom}_{\Gamma}(N_1, N_2)$ for the set of strictly degree preserving (degree 0) A-graded left Γ -comodule homomorphisms from (N_1, Ψ_{N_1}) to (N_2, Ψ_{N_2}) .

Proposition 0.7. The category Γ -CoMod^A is an additive category.

Proof. First, we show the category is **Ab**-enriched. Since the forgetful functor Γ -**CoMod**^A \to B-**Mod**^A is clearly faithful, we may view hom-sets in Γ -**CoMod**^A as subsets of hom-groups in B-**Mod**^A, so that in order to show Γ -**CoMod**^A is **Ab**-enriched, it suffices to show that hom-sets in Γ -**CoMod**^A are closed under addition of module homomorphisms and taking inverses. To that end, suppose we have two A-graded left Γ -comodule homomorphisms $f, g: (N_1, \Psi_{N_1}) \to (N_2, \Psi_{N_2})$, then we have

$$\begin{split} \Psi_{N_2} \circ (f+g) &= (\Psi_{N_2} \circ f) + (\Psi_{N_2} \circ g) \\ &= ((\Gamma \otimes_B f) \circ \Psi_{N_1}) + ((\Gamma \otimes_B g) \circ \Psi_{N_1}) \\ &= ((\Gamma \otimes_B f) + (\Gamma \otimes_B g)) \circ \Psi_{N_1} \\ &= (\Gamma \otimes_B (f+g)) \circ \Psi_{N_1}, \end{split}$$

finish

where the first equality follows since Ψ_{N_2} is a homomorphism, the second follows since f and g are left Γ -comodule homomorphisms, the third follows since Ψ_{N_1} is a homomorphism, and the last equality follows by definition of the tensor product of modules. Hence f+g is indeed an A-graded left Γ -comodule homomorphism, as desired. Now, we also claim -f is an A-graded left Γ -comodule homomorphism. To that end, note that

$$\Psi_{N_2} \circ (-f) = -\Psi_{N_2} \circ f = -(\Gamma \otimes_B f) \circ \Psi_{N_1} = (\Gamma \otimes_B (-f)) \circ \Psi_{N_1},$$

where the first equality follows since Ψ_{N_2} is a homomorphism, the second follows since f is an A-graded left Γ -comodule homomorphism, and the third equality follows by definition of the tensor product.

Thus, we've shown that the hom-sets in Γ -CoMod^A are abelian groups, and composition is clearly bilinear, so that Γ -CoMod^A is indeed Ab-enriched.

Now, in order to show Γ -**CoMod**^A is additive, it suffices to show that it contains a zero object and has binary coproducts. First of all, it is straightforward to check that the zero left B-module is clearly an A-graded left Γ -comodule with structure map the unique map $0 \to \Gamma \otimes_B 0 \cong 0$, and that given any other A-graded left Γ -comodule (N, Ψ_N) , the unique homomorphisms of left B-modules $0 \to N$ and $N \to 0$ are left comodule homomorphisms.

Now, suppose we have two A-graded left Γ -comodules (N_1, Ψ_{N_1}) and (N_2, Ψ_{N_2}) . First, we claim their direct sum as left B-modules $N_1 \oplus N_2$ is canonically an A-graded left Γ -comodule. We know that $N_1 \oplus N_2$ is an A-graded left B-module by ??, and we can define the structure map

$$\Psi_{N_1 \oplus N_2} : N_1 \oplus N_2 \xrightarrow{\Psi_{N_1} \oplus \Psi_{N_2}} (\Gamma \otimes_B N_1) \oplus (\Gamma \otimes_B N_2) \cong \Gamma \otimes_B (N_1 \oplus N_2),$$

where the final isomorphism is the canonical one sending $(g_1 \otimes n_1) \oplus (g_2 \otimes n_2)$ to $(g_1 \otimes n_1) + (g_2 \otimes n_2)$. Then

Proposition 0.8. The forgetful functor Γ -CoMod^A \rightarrow B-Mod^A (where here B-Mod^A is the category of A-graded left B-modules and degree-preserving module homomorphisms between them) has a right adjoint $\Gamma \otimes_B - : B\text{-Mod}^A \rightarrow \Gamma\text{-CoMod}^A$ called the co-free construction, where the co-free left A-graded Γ -comodule on a left A-graded B-module M is the B-module $\Gamma \otimes_B M$ equipped with the coaction

$$\Psi_{\Gamma \otimes_B M} : \Gamma \otimes_B M \xrightarrow{\Psi \otimes_B M} (\Gamma \otimes_B \Gamma) \otimes_B M \xrightarrow{\cong} \Gamma \otimes_B (\Gamma \otimes_B M).$$

Explicitly, given some (N, Ψ_N) in Γ -CoMod and some M in B-Mod^A, the counit and unit of this adjunction are given by

$$\eta_{(N,\Psi_N)}: N \xrightarrow{\Psi_N} \Gamma \otimes_B N$$

and

$$\varepsilon_M : \Gamma \otimes_B M \xrightarrow{\epsilon \otimes_B M} B \otimes_B M \xrightarrow{\cong} M.$$

Proof.

Proposition 0.9. Suppose that Γ is flat as a right B-module, i.e., suppose $\eta_R : B \to \Gamma$ is a flat ring homomorphism. Then the category Γ -CoMod^A is an abelian category and has enough injectives.

Proof. \Box finish

Proposition 0.10 ([1, Lemma 3.5]). Suppose that Γ is flat as a right B-module, i.e., suppose $\eta_R: B \to \Gamma$ is a flat ring homomorphism. Let P be an A-graded left Γ -comodule in Γ -CoMod^A such that the underlying A-graded B-module is a graded projective module. Then every co-free module (Proposition 0.8) is an F-acyclic object (??) for the covariant hom functor $\operatorname{Hom}_{\Gamma}(P, -)$.

Proof. We need to show that $\operatorname{Ext}^n_{\Gamma}(N, \Gamma \otimes_B M)$ vanishes for all A-graded B-modules M. First of all, let $i: M \to I^*$ be an injective resolution of M in B- Mod^A , so we have an exact sequence of A-graded B-modules

$$0 \longrightarrow M \xrightarrow{i} I^0 \xrightarrow{d^0} I^1 \xrightarrow{d^1} I^2 \xrightarrow{d^2} I^3 \longrightarrow \cdots.$$

Then Γ is flat as a right B-module, the sequence remains exact after we tensor it with Γ on the left. Furthermore, it is a general categorical fact that right adjoints between abelian categories preserve injective objects. Thus $\Gamma \otimes i : \Gamma \otimes_B M \to \Gamma \otimes_B I^*$ is an injective resolution in Γ -CoMod^A. Then for n > 0, we have

$$\operatorname{Ext}^n_{\Gamma}(N,\Gamma\otimes_B M)\cong H^n(\operatorname{Hom}_{\Gamma}(N,\Gamma\otimes_B I^*))\cong H^n(\operatorname{Hom}_B(N,I^*))\cong 0,$$

where the first isomorphism follows by the forgetful-cofree adjunction for comodules over a Hopf algebroid (Proposition 0.8), and the final isomorphism follows by the fact that N is a graded projective module, i.e., a projective object in the abelian category $B\text{-}\mathbf{Mod}^A$, so that $\mathrm{Hom}_B(N,-)$ is an exact functor.