0.1. **Triangulated categories and their basic properties.**

**Definition 0.1.** A *triangulated category* $(\mathcal{C}, \Sigma, \mathcal{D})$ is the data of:

(1) An additive category $\mathcal{C}$.

(2) An additive auto-equivalence $\Sigma : \mathcal{C} \to \mathcal{C}$ called the *shift functor*.

(3) A collection $\mathcal{D}$ of *distinguished* triangles in $\mathcal{C}$, where a *triangle* is a sequence of arrows of the form
$$X \to Y \to Z \to \Sigma X.$$
Distinguished triangles are also sometimes called *cofiber sequences* or *fiber sequences*.

These data must satisfy the following axioms:

**TR0** Given a commutative diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d, "\cong"'] & Y \arrow[r] \arrow[d, "\cong"'] & Z \arrow[r] \arrow[d, "\cong"'] & \Sigma X \arrow[d, "\cong"'] \\ X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & \Sigma X' \end{tikzcd}$$

where the vertical arrows are isomorphisms, if the top row is distinguished then so is the bottom.

**TR1** For any object $X$ in $\mathcal{C}$, the diagram
$$X \xrightarrow{\mathrm{id}_X} X \to 0 \to \Sigma X$$
is a distinguished triangle.

**TR2** For all $f : X \to Y$ there exists an object $C_f$ (also sometimes denoted $Y/X$) called the *cofiber of $f$* and a distinguished triangle
$$X \xrightarrow{f} Y \to C_f \to \Sigma X.$$

**TR3** Given a solid diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] & Y \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d, dashed] & \Sigma X \arrow[d, "\Sigma f"] \\ X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & \Sigma X' \end{tikzcd}$$

such that the leftmost square commmutes and both rows are distinguished, there exists a dashed arrow $Z \to Z'$ which makes the remaining two squares commute.

**TR4** A triangle
$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$$
is distinguished if and only if
$$Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X \xrightarrow{-\Sigma f} \Sigma Y$$
is distinguished.

**TR5** (Octahedral axiom) Given three distinguished triangles
$$X \xrightarrow{f} Y \xrightarrow{h} Y/X \to \Sigma X$$
$$Y \xrightarrow{g} Z \xrightarrow{k} Z/Y \to \Sigma Y$$
$$X \xrightarrow{g \circ f} Z \xrightarrow{l} Z/X \to \Sigma X$$

there exists a distinguished triangle

$$Y/X \xrightarrow{u} Z/X \xrightarrow{v} Z/Y \xrightarrow{w} \Sigma(Y/X)$$

such that the following diagram commutes

$$\begin{tikzcd}
X \arrow[rr, "g\circ f"] \arrow[dr, "f"'] & & Z \arrow[rr, "k"] \arrow[dr, "l"] & & Z/Y \arrow[rr, "w"] \arrow[dr] & & \Sigma(Y/X) \\
& Y \arrow[ur, "g"'] \arrow[dr, "h"'] & & Z/X \arrow[ur, "v"'] \arrow[dr] & & \Sigma Y \arrow[ur, "\Sigma h"'] & \\
& & Y/X \arrow[ur, "u"'] \arrow[rr] & & \Sigma X \arrow[ur, "\Sigma f"'] & &
\end{tikzcd}$$

It turns out that the above definition is actually redundant; TR3 and TR4 follow from the remaining axioms (see Lemmas 2.2 and 2.4 in [1]). From now on, we fix a triangulated category $(\mathcal{C}, \Sigma, \mathcal{D})$. We will denote the hom-group $\mathcal{C}(X, Y)$ by $[X, Y]$. To start, recall the following definition:

**Definition 0.2.** A sequence

$$X_1 \to X_2 \to \cdots \to X_n$$

of arrows in $\mathcal{C}$ is *exact* if, for any object $A$ in $\mathcal{C}$, the induced sequences

$$[A, X_1] \to [A, X_2] \to \cdots \to [A, X_{n-1}] \to [A, X_n]$$

and

$$[X_n, A] \to [X_{n-1}, A] \to \cdots \to [X_2, A] \to [X_1, A]$$

are exact sequences of abelian groups.

It is straightforward to verify that if we have an exact sequence in $\mathcal{C}$

$$X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \to \cdots \to X_n,$$

then the sequence remains exact if we change the signs of any of the maps involved. We will use this fact often without comment.

**Proposition 0.3.** *Every distinguished triangle is an exact sequence (in the sense of Definition 0.2).*

*Proof.* Suppose we have some distinguished triangle

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X.$$

Then first we would like to show that given any object $A$ in $\mathcal{C}$, the sequence

$$[A, X] \xrightarrow{f_*} [A, Y] \xrightarrow{g_*} [A, Z] \xrightarrow{h_*} [A, \Sigma X]$$

is exact. First we show exactness at $[A, Y]$. To see $\operatorname{im} f_* \subseteq \ker g_*$, note it suffices to show that $g \circ f = 0$. Indeed, consider the commuting diagram

$$\begin{tikzcd}
X \arrow[r, equal] \arrow[d, equal] & X \arrow[r] \arrow[d, "f"] & 0 \arrow[r] & \Sigma X \\
X \arrow[r, "f"] & Y \arrow[r, "g"] & Z \arrow[r, "h"] & \Sigma X
\end{tikzcd}$$

The top row is distinguished by axiom TR1. Thus by TR3, the following diagram commutes:

$$\begin{tikzcd}
X \arrow[r, equal] \arrow[d, equal] & X \arrow[r] \arrow[d, "f"] & 0 \arrow[r] \arrow[d] & \Sigma X \arrow[d, equal] \\
X \arrow[r, "f"] & Y \arrow[r, "g"] & Z \arrow[r, "h"] & \Sigma X
\end{tikzcd}$$

In particular, commutativity of the second square tells us that $g \circ f = 0$, as desired. Conversely, we'd like to show that $\ker g_* \subseteq \operatorname{im} f_*$. Let $\psi : A \to Y$ be in the kernel of $g_*$, so that $g \circ \psi = 0$. Consider the following commutative diagram:

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "\psi"'] & 0 \arrow[r] \arrow[d] & \Sigma A \arrow[r, "-\Sigma \mathrm{id}_A"] & \Sigma A \\
Y \arrow[r, "g"] & Z \arrow[r, "h"] & \Sigma X \arrow[r, "-\Sigma f"] & \Sigma Y
\end{tikzcd}$$

The top row is distinguished by axioms TR1 and TR4. The bottom row is distinguished by axiom TR4. Thus by axiom TR3 there exists a map $\widetilde{\phi} : \Sigma A \to \Sigma X$ such that the following diagram commutes:

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "\psi"'] & 0 \arrow[r] \arrow[d] & \Sigma A \arrow[r, "-\Sigma \mathrm{id}_A"] \arrow[d, "\widetilde{\phi}"'] & \Sigma A \arrow[d, "\Sigma \psi"'] \\
Y \arrow[r, "g"] & Z \arrow[r, "h"] & \Sigma X \arrow[r, "-\Sigma f"] & \Sigma Y
\end{tikzcd}$$

Now, since $\Sigma$ is an equivalence, it is a full functor, so that in particular there exists some $\phi : A \to X$ such that $\widetilde{\phi} = \Sigma \phi$. Then by faithfullness, we may pull back the right square to get a commuting diagram

$$\begin{tikzcd}
A \arrow[r, "-\mathrm{id}_A"] \arrow[d, "\phi"'] & A \arrow[d, "\psi"] \\
X \arrow[r, "-f"'] & Y
\end{tikzcd}$$

Hence,

$$f_*(\phi) = f \circ \phi \overset{(*)}{=} -((-f) \circ \phi) = -(\psi \circ (-\mathrm{id}_A)) \overset{(*)}{=} \psi \circ \mathrm{id}_A = \psi,$$

where the equalities marked $(*)$ follow by bilinearity of composition in an additive category. Thus $\psi \in \operatorname{im} f_*$, as desired, meaning $\ker g_* \subseteq \operatorname{im} f_*$.

Now, we have shown that

$$[A, X] \xrightarrow{f_*} [A, Y] \xrightarrow{g_*} [A, Z] \xrightarrow{h_*} [A, \Sigma X]$$

is exact at $[A, Y]$. It remains to show exactness at $[A, Z]$. Yet this follows by the exact same argument given above applied to the sequence obtained from the shifted triangle (TR4)

$$Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X \xrightarrow{-\Sigma f} \Sigma Y$$

On the other hand, we would like to show that

$$[\Sigma X, A] \xrightarrow{h^*} [Z, A] \xrightarrow{g^*} [Y, A] \xrightarrow{f^*} [X, A]$$

is exact. As above, since we can shift the triangle, it suffices to show exactness at $[Z, A]$. First, since we have shown $g \circ f = 0$, we have $f^* \circ g^* = (g \circ f)^* = 0$, so that $\operatorname{im} g^* \subseteq \ker f^*$, as desired.

Conversely, in order to see $\ker f^* \subseteq \operatorname{im} g^*$, suppose $\psi : Y \to A$ is in the kernel of $f^*$, so that $\psi \circ f = 0$. Consider the following commuting diagram:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[r, "g"] \arrow[d, "\psi"] & Z \arrow[r, "h"] & \Sigma X \\ 0 \arrow[r] & A \arrow[r, equal] & A \arrow[r] & 0 \end{tikzcd}$$

The top row is a distinguished triangle by assumption, and the bottom row is distinguished by axioms TR1 and TR4 for a triangulated category, along with the fact that $\Sigma 0 = 0$ since $\Sigma$ is additive. Thus by axiom TR3 there exists a map $\phi : Z \to A$ such that $\phi \circ g = \psi$, i.e., $g^*(\phi) = \psi$, so that $\phi \in \operatorname{im} g^*$ as desired. □

**Lemma 0.4.** *Suppose we have a commutative diagram*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "j"] & Y \arrow[r, "g"] \arrow[d, "k"] & Z \arrow[r, "h"] \arrow[d, "\ell"] & \Sigma X \arrow[d, "\Sigma j"] \\ X' \arrow[r, "f'"] & Y' \arrow[r, "g'"] & Z' \arrow[r, "h'"] & \Sigma X' \end{tikzcd}$$

*with both rows distinguished. Then if any two of the maps $j$, $k$, and $\ell$ are isomorphisms, then so is the third.*

*Proof.* Suppose we are given any object $W$ in $\mathcal{C}$, and consider the commutative diagram

$$\begin{tikzcd} {[W,X]} \arrow[r, "f_*"] \arrow[d, "j_*"] & {[W,Y]} \arrow[r, "g_*"] \arrow[d, "k_*"] & {[W,Z]} \arrow[r, "k_*"] \arrow[d, "\ell_*"] & {[W,\Sigma X]} \arrow[r, "-\Sigma f_*"] \arrow[d, "\Sigma j_*"] & {[W,\Sigma Y]} \arrow[r, "-\Sigma g_*"] \arrow[d, "\Sigma k_*"] & {[W,\Sigma Z]} \arrow[r, "-\Sigma h_*"] \arrow[d, "\Sigma \ell_*"] & {[W,\Sigma^2 X]} \arrow[d, "\Sigma^2 j_*"] \\ {[W,X']} \arrow[r, "f'_*"] & {[W,Y']} \arrow[r, "g'_*"] & {[W,Z']} \arrow[r, "h'_*"] & {[W,\Sigma X']} \arrow[r, "-\Sigma f'_*"] & {[W,\Sigma Y']} \arrow[r, "-\Sigma g'_*"] & {[W,\Sigma Z']} \arrow[r, "-\Sigma h'_*"] & {[W,\Sigma^2 X']} \end{tikzcd}$$

The rows are exact by Proposition 0.3 and repeated applications of axiom TR4. It follows by the five lemma and faithfulness of $\Sigma$ that if $j$ and $k$ are isomorphisms, then $\ell_*$ is an isomorphism. Similarly, if $k$ and $\ell$ are isomorphisms then $\Sigma j_*$ is an isomorphism. Finally, if $\ell$ and $j$ are isomorphisms, then $\Sigma k_*$ is an isomorphism. The desired result follows by faithfullness of $\Sigma$ and the Yoneda embedding. □

**Proposition 0.5.** *Given an arrow $f : X \to Y$ in $\mathcal{C}$, there exists an object $F_f$ called the* fiber *of $f$, and a distinguished triangle*

$$F_f \to X \xrightarrow{f} Y \to \Sigma F_f (\cong C_f).$$

*Proof.* Since $\Sigma$ is an equivalence, there exists some functor $\Omega : \mathcal{C} \to \mathcal{C}$ and natural isomorphisms $\varepsilon : \Omega\Sigma \Rightarrow \mathrm{Id}_{\mathcal{C}}$ and $\eta : \mathrm{Id}_{\mathcal{C}} \Rightarrow \Sigma\Omega$. By axiom TR2, we have a distinguished triangle

$$X \xrightarrow{f} Y \xrightarrow{g} C_f \xrightarrow{h} \Sigma X.$$

Now, consider the commutative diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, equal] & Y \arrow[r, "g"] \arrow[d, equal] & C_f \arrow[r, "h"] \arrow[d, "\eta_{C_f}"] & \Sigma X \arrow[d, equal] \\ X \arrow[r, "f"] & Y \arrow[r, "\widetilde{g}"] & \Sigma\Omega C_f \arrow[r, "\widetilde{h}"] & \Sigma X \end{tikzcd}$$

where $\widetilde{g} = \eta_{C_f} \circ g$, and $\widetilde{h} = h \circ \eta_{C_f}^{-1}$. Since each vertical map is an isomorphism and the top row is distinguished, the bottom row is also distinguished by axiom TR0. Now, since $\Sigma$ is an equivalence of categories, it is faithful, so that in particular there exists some map $k : \Omega C_f \to X$ such that $\Sigma k = -\widetilde{h} \implies -\Sigma k = \widetilde{h}$. Thus, we have a distinguished triangle of the form

$$X \xrightarrow{f} Y \xrightarrow{\widetilde{g}} \Sigma\Omega C_f \xrightarrow{-\Sigma k} \Sigma X.$$

Finally, by axiom TR4, we get a distinguished triangle

$$\Omega C_f \xrightarrow{k} X \xrightarrow{f} Y \xrightarrow{\widetilde{g}} \Sigma\Omega C_f,$$

so we may define the fiber of $f$ to be $\Omega C_f$. □

0.2. **Homotopy (co)limits in a triangulated category.** In this subsection, we will assume $\mathcal{C}$ has countable products and coproducts.

**Definition 0.6** ([2, Definition 1.6.4])**.** Let

$$X_0 \xrightarrow{j_1} X_1 \xrightarrow{j_2} X_2 \xrightarrow{j_3} X_3 \to \cdots$$

be a sequence of objects and morphisms in $\mathcal{C}$. The *homotopy colimit* of the sequence, denoted $\operatorname{hocolim} X_i$, is given (up to non-canonical isomorphism) as the cofiber of the map

$$\bigoplus_{i=0}^{\infty} X_i \xrightarrow{1-\text{shift}} \bigoplus_{i=0}^{\infty} X_i,$$

where the shift map $\bigoplus_{i=0}^{\infty} X_i \xrightarrow{\text{shift}} \bigoplus_{i=0}^{\infty} X_i$ is understood to be the direct sum of $j_{i+1} : X_i \to X_{i+1}$, i.e., by the universal property of the coproduct, it is induced by the maps

$$X_s \xrightarrow{j_{s+1}} X_{s+1} \hookrightarrow \bigoplus_{i=0}^{\infty} X_i.$$

Homotopy colimits in general are not colimits, but they do share some properties with the colimit. First of all, note that we have canonical $X_i \to \operatorname{hocolim} X_i$ given as the composition

$$X_i \hookrightarrow \bigoplus_{i=0}^{\infty} X_i \to \operatorname{hocolim} X_i$$

It is straightforward to see

**Proposition 0.7.** *Let*

$$X_0 \xrightarrow{j_1} X_1 \xrightarrow{j_2} X_2 \xrightarrow{j_3} X_3 \to \cdots$$

*be a sequence of objects and morphisms in $\mathcal{C}$, and suppose we have a cone under this diagram, i.e., an object $Y$ in $\mathcal{C}$ along with maps $\eta_s : X_s \to Y$ such that $\eta_s \circ j_s = \eta_{s-1}$. Then this cone factors through the homotopy colimit, i.e., there exists a map $\ell : \operatorname{hocolim} X_i \to Y$ such that for all $s \geq 0$, the following diagram commutes:*

$$\begin{tikzcd}
X_s \arrow[rr, "j_{s+1}"] \arrow[dr, hook] \arrow[dddr, bend right, "\eta_s"'] & & X_{s+1} \arrow[dl, hook'] \arrow[dddl, bend left, "\eta_{s+1}"] \\
& \bigoplus_{i=0}^{\infty} X_i \arrow[d] & \\
& \operatorname{hocolim} X_i \arrow[d, "\ell"] & \\
& Y &
\end{tikzcd}$$

*Proof sketch.* By direction inspection, the leftmost square in the following diagram commutes:

$$\begin{tikzcd}
\bigoplus_{i=0}^{\infty} X_i \arrow[r, "1-\text{shift}"] \arrow[d] & \bigoplus_{i=0}^{\infty} X_i \arrow[r] \arrow[d, "\eta"] & \operatorname{hocolim} X_i \arrow[r] \arrow[d, dashed, "\ell"] & \Sigma\left(\bigoplus_{i=0}^{\infty} X_i\right) \arrow[d] \\
0 \arrow[r] & Y \arrow[r, equal] & Y \arrow[r] & 0
\end{tikzcd}$$

The top row is distinguished by definition. The bottom row is distinguished, since it may be obtained by shifting the triangle $(\mathrm{id}_Y, 0, 0)$. Thus we get an induced dashed arrow $\ell : \operatorname{hocolim} X_i \to Y$ which makes the diagram commute, by axiom TR3. By direct inspection it clearly makes the above diagram commute. □

**Definition 0.8.** Assume that $\mathcal{C}$ has countable products, and let

$$\cdots \to X_3 \xrightarrow{j_3} X_2 \xrightarrow{j_2} X_1 \xrightarrow{j_1} X_0$$

be a sequence of objects and morphisms in $\mathcal{C}$. The *homotopy limit* of the sequence, denoted $\operatorname{holim} X_i$, is given (up to non-canonical isomorphism) as the fiber (Proposition 0.5) of the map

$$\prod_{i=0}^{\infty} X_i \xrightarrow{1-\text{shift}} \prod_{i=0}^{\infty},$$

where the shift map $\prod_{i=0}^{\infty} X_i \xrightarrow{\text{shift}} \prod_{i=0}^{\infty} X_i$ is understood to be the product of $j_i : X_i \to X_{i-1}$, i.e., by the universal property of the product, it is induced by the maps

$$\prod_{i=0}^{\infty} X_i \twoheadrightarrow X_{s+1} \xrightarrow{j_{s+1}} X_s.$$

An analagous argument to the one given above yields that $\operatorname{holim} X_i$ satisfies the same properties as the limit of the $X_i$'s, minus uniqueness.

**Proposition 0.9.** *Suppose $\mathcal{C}$ has countable coproducts, and suppose $e : X \to X$ is an idempotent in $\mathcal{C}$, so that $e \circ e = e$. Then $e$ splits in $\mathcal{C}$, i.e., $e$ factors as*

$$X \xrightarrow{r} Y \xrightarrow{\iota} X$$

*with $r \circ \iota = \mathrm{id}_Y$ and $\iota \circ r = e$. In particular, we may take $Y$ to be the homotopy colimit of*

$$X \xrightarrow{e} X \xrightarrow{e} X \xrightarrow{e} X \xrightarrow{e} \cdots .$$

*Proof.* See [2, Proposition 1.6.8]. □

0.3. **Adjointly triangulated categories.** For our purposes, we will always be dealing with triangulated categories with a bit of extra structure, in the following sense:

**Definition 0.10.** An *adjointly triangulated category* $(\mathcal{C}, \Omega, \Sigma, \eta, \varepsilon, \mathcal{D})$ is the data of a triangulated category $(\mathcal{C}, \Sigma, \mathcal{D})$ along with an inverse shift functor $\Omega : \mathcal{C} \to \mathcal{C}$ and natural isomorphisms $\eta : \mathrm{Id}_{\mathcal{C}} \Rightarrow \Sigma\Omega$ and $\varepsilon : \Omega\Sigma \Rightarrow \mathrm{Id}_{\mathcal{C}}$ such that $(\Omega, \Sigma, \eta, \varepsilon)$ forms an adjoint equivalence of $\mathcal{C}$. In other words, $\eta$ and $\varepsilon$ are natural isomorphisms which also are the unit and counit of an adjunction $\Omega \dashv \Sigma$, so they satisfy either of the following "zig-zag identities":

$$\begin{tikzcd}
\Omega \arrow[r, "\Omega\eta"] \arrow[dr, equal] & \Omega\Sigma\Omega \arrow[d, "\varepsilon\Omega"] \\
& \Omega
\end{tikzcd}
\qquad
\begin{tikzcd}
\Sigma\Omega\Sigma \arrow[d, "\Sigma\varepsilon"'] & \Sigma \arrow[l, "\eta\Sigma"'] \arrow[dl, equal] \\
\Sigma &
\end{tikzcd}$$

(Satisfying one implies the other is automatically satisfied, see [3, Lemma 3.2]).

From now on, we will assume that $\mathcal{C}$ is an *adjointly* triangulated category with inverse shift $\Omega$, unit $\eta : \mathrm{Id}_{\mathcal{C}} \Rightarrow \Sigma\Omega$, and counit $\varepsilon : \Omega\Sigma \Rightarrow \mathrm{Id}_{\mathcal{C}}$.

**Lemma 0.11.** *Given a triangle*

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X,$$

*it can be shifted to the left to obtain a distinguished triangle*

$$\Omega Z \xrightarrow{-\widetilde{h}} X \xrightarrow{f} Y \xrightarrow{\widetilde{\Omega g}} \Sigma\Omega Z,$$

*where $\widetilde{h} : \Omega Z \to X$ is the adjoint of $h : Z \to \Sigma X$ and $\widetilde{\Omega g} : Y \to \Sigma\Omega Z$ is the adjoint of $\Omega g : \Omega Y \to \Omega Z$.*

*Proof.* Note that unravelling definitions, $\widetilde{h}$ and $\widetilde{g}$ are the compositions

$$\widetilde{h} : \Omega Z \xrightarrow{\Omega h} \Omega\Sigma X \xrightarrow{\varepsilon_X} X \qquad \text{and} \qquad \widetilde{\Omega g} : Y \xrightarrow{\eta_Y} \Sigma\Omega Y \xrightarrow{\Sigma\Omega g} \Sigma\Omega Z.$$

Now consider the following diagram:

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, equal] & Y \arrow[r, "g"] \arrow[d, equal] & Z \arrow[r, "h"] \arrow[d, "\eta_Z"'] & \Sigma X \arrow[d, equal] \\
X \arrow[r, "f"] & Y \arrow[r, "\widetilde{\Omega g}"] & \Sigma\Omega Z \arrow[r, "\Sigma\widetilde{h}"] & \Sigma X
\end{tikzcd} \tag{1}$$

The left square commutes by definition. To see that the middle square commutes, expanding definitions, note it is given by the following diagram:

$$\begin{tikzcd}
Y \arrow[rr, "g"] \arrow[d, equal] & & Z \arrow[d, "\eta_Y"] \\
Y \arrow[r, "\eta_Y"] & \Sigma\Omega Y \arrow[r, "\Sigma\Omega g"] & \Sigma\Omega Z
\end{tikzcd}$$

and this commutes by naturality of $\eta$. To see that the right square commutes, consider the following diagram:

$$\begin{tikzcd}
Z \arrow[rr, "h"] \arrow[d, "\eta_Z"'] & & \Sigma X \arrow[d, equal] \arrow[dl, "\eta_{\Sigma X}"'] \\
\Sigma\Omega Z \arrow[r, "\Sigma\Omega h"'] & \Sigma\Omega\Sigma X \arrow[r, "\Sigma\varepsilon_X"'] & \Sigma X
\end{tikzcd}$$

By functoriality of $\Sigma$, the bottom composition is $\Sigma\widetilde{h}$. The left region commutes by naturality of $\eta$. Commutativity of the right region is precisely one of the the zig-zag identities. Hence, since diagram (1) commutes, the vertical arrows are isomorphisms, and the top row is distinguished, we have that the bottom row is distinguished as well by axiom TR0. Then by axiom TR4, since $(f, \widetilde{\Omega g}, \Sigma\widetilde{h})$ is distinguished, so is the triangle

$$\Omega Z \xrightarrow{-\widetilde{h}} X \xrightarrow{f} Y \xrightarrow{\widetilde{\Omega g}} \Sigma\Omega Z. \qquad \square$$

**Lemma 0.12.** *Given a distinguished triangle*

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X,$$

*for any $n > 0$, the triangle*

$$\Omega^n X \xrightarrow{(-1)^n\Omega^n f} \Omega^n Y \xrightarrow{(-1)^n\Omega^n g} \Omega^n Z \xrightarrow{(-1)^n\Omega^n h} \Omega^n\Sigma X \cong \Sigma\Omega^n X,$$

*is distinguished, where the final isomorphism is given by the composition*

$$\Omega^n\Sigma X = \Omega^{n-1}\Omega\Sigma X \xrightarrow{\Omega^{n-1}\varepsilon_X} \Omega^{n-1}X \xrightarrow{\eta_{\Omega^{n-1}X}} \Sigma\Omega\Omega^{n-1}X = \Sigma\Omega^n X.$$

*Proof.* We give a proof by induction. First we show the case $n = 1$. Note by Lemma 0.11, we have that given a distinguished triangle

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X,$$

we can shift it to the left to obtain a distinguished triangle

$$\Omega Z \xrightarrow{-\widetilde{h}} X \xrightarrow{f} Y \xrightarrow{\widetilde{\Omega g}} \Sigma\Omega Z,$$

where $\widetilde{h}$ is the adjoint of $h : Z \to \Sigma X$ and $\widetilde{\Omega g}$ is the adjoint of $\Omega g : \Omega Y \to \Omega Z$. If we apply this shifting operation again, we get the distinguished triangle

$$\Omega Y \xrightarrow{-\widetilde{\widetilde{\Omega g}}} \Omega Z \xrightarrow{-\widetilde{h}} X \xrightarrow{\widetilde{\Omega f}} \Sigma\Omega Y,$$

where unravelling definitions, $\widetilde{\Omega f}$ is the right adjoint of $\Omega f : \Omega X \to \Omega Y$ and $\widetilde{\widetilde{\Omega g}}$ is the right adjoint of $\widetilde{\Omega g}$, which itself is the left adjoint of $\Omega g$, so $\widetilde{\widetilde{\Omega g}} = \Omega g$. Hence we have a distinguished triangle

$$\Omega Y \xrightarrow{-\Omega g} \Omega Z \xrightarrow{-\widetilde{h}} X \xrightarrow{\widetilde{\Omega f}} \Sigma\Omega Y.$$

We may again shift this triangle again and the above arguments yield the distinguished triangle

$$\Omega X \xrightarrow{-\Omega f} \Omega Y \xrightarrow{-\Omega g} \Omega Z \xrightarrow{\widetilde{\Omega(-\widetilde{h})}} \Sigma\Omega X,$$

where $\widetilde{\Omega(-\widetilde{h})}$ is the right adjoint of $\Omega(-\widetilde{h}) = -\Omega\widetilde{h} : \Omega\Omega Z \to \Omega X$. Explicitly unravelling definitions, $\widetilde{\Omega(-\widetilde{h})} = \widetilde{-\Omega\widetilde{h}}$ is the composition

$$\begin{aligned}
[\Omega Z \xrightarrow{\eta_{\Omega Z}} \Sigma\Omega\Omega Z \xrightarrow{\Sigma(-\Omega\widetilde{h})} \Sigma\Omega X] &= -[\Omega Z \xrightarrow{\eta_{\Omega Z}} \Sigma\Omega\Omega Z \xrightarrow{\Sigma\Omega\widetilde{h}} \Sigma\Omega X] \\
&= -[\Omega Z \xrightarrow{\eta_{\Omega Z}} \Sigma\Omega\Omega Z \xrightarrow{\Sigma\Omega\Omega h} \Sigma\Omega\Omega\Sigma X \xrightarrow{\Sigma\Omega\varepsilon_X} \Sigma\Omega X] \\
&= -[\Omega Z \xrightarrow{\Omega h} \Omega\Sigma X \xrightarrow{\varepsilon_X} X \xrightarrow{\eta_X} \Sigma\Omega X],
\end{aligned}$$

where the first equality follows by additivity of $\Sigma$ and additivity of composition, the second follows by further unravelling how $\widetilde{h}$ is defined, and the third follows by naturality of $\eta$, which tells us the following diagram commutes:

$$\begin{tikzcd}
\Omega Z \arrow[r, "\Omega h"] \arrow[d, "\eta_{\Omega Z}"] & \Omega\Sigma X \arrow[r, "\varepsilon_X"] \arrow[d, "\eta_{\Omega\Sigma X}"] & X \arrow[d, "\eta_X"] \\
\Sigma\Omega\Omega Z \arrow[r, "\Sigma\Omega\Omega h"'] & \Sigma\Omega\Omega\Sigma X \arrow[r, "\Sigma\Omega\varepsilon_X"'] & \Sigma\Omega X
\end{tikzcd}$$

Thus indeed we have a distinguished triangle

$$\Omega X \xrightarrow{-\Omega f} \Omega Y \xrightarrow{-\Omega g} \Omega Z \xrightarrow{-\Omega h} \Omega\Sigma X \cong \Sigma\Omega X,$$

where the last isomorphism is $\eta_X \circ \varepsilon_X$, as desired.

Now, we show the inductive step. Suppose we know that given a distinguished triangle

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X,$$

that for some $n > 0$ the triangle

$$\Omega^n X \xrightarrow{(-1)^n\Omega^n f} \Omega^n Y \xrightarrow{(-1)^n\Omega^n g} \Omega^n Z \xrightarrow{(-1)^n h^n} \Sigma\Omega^n X,$$

is distinguished, where $h^n : \Omega^n Z \to \Sigma\Omega^n X$ is the composition

$$\Omega^n Z \xrightarrow{\Omega^n h} \Omega^n\Sigma X \xrightarrow{\Omega^{n-1}\varepsilon_X} \Omega^{n-1} X \xrightarrow{\eta_{\Omega^{n-1}X}} \Sigma\Omega^n X.$$

Then by applying the $n = 1$ case to this triangle, we get that the following triangle is distinguished

$$\Omega^{n+1}X \xrightarrow{-\Omega((-1)^n\Omega^n f)} \Omega^{n+1}Y \xrightarrow{-\Omega((-1)^n\Omega^n g)} \Omega^{n+1}Z \xrightarrow{-\Omega((-1)^n h^n)} \Omega\Sigma\Omega^n X \cong \Sigma\Omega^{n+1}X,$$

where the final isomorphism is the composition

$$\Omega\Sigma\Omega^n X \xrightarrow{\varepsilon_{\Omega^n X}} \Omega^n X \xrightarrow{\eta_{\Omega^n X}} \Sigma\Omega\Omega^n X = \Sigma\Omega^{n+1}X.$$

We claim that this is precisely the distinguished triangle given in the statement of the lemma for $n+1$. First of all, note that $-\Omega((-1)^n\Omega^n f) = (-1)^{n+1}\Omega^{n+1}f$, $-\Omega((-1)^n\Omega^n g) = (-1)^{n+1}\Omega^{n+1}g$, and $-\Omega((-1)^n h^n) = (-1)^{n+1}\Omega h^n$ by additivity of $\Omega$, so that the triangle becomes

$$\Omega^{n+1}X \xrightarrow{(-1)^{n+1}\Omega^{n+1}f} \Omega^{n+1}Y \xrightarrow{(-1)^{n+1}\Omega^{n+1}g} \Omega^{n+1}Z \xrightarrow{(-1)^{n+1}\Omega h^n} \Omega\Sigma\Omega^n X \cong \Sigma\Omega^{n+1}X. \tag{2}$$

Thus, in order to prove the desired characterization, it remains to show this diagram commutes:

$$\begin{tikzcd}
\Omega^{n+1}Z \arrow[r, "(-1)^{n+1}\Omega h^n"] \arrow[d, "(-1)^{n+1}\Omega^{n+1}h"'] & \Omega\Sigma\Omega^n X \arrow[r, "\varepsilon_{\Omega^n X}"] & \Omega^n X \arrow[d, "\eta_{\Omega^n X}"] \\
\Omega^{n+1}\Sigma X \arrow[r, "\Omega^n\varepsilon_X"'] & \Omega^n X \arrow[r, "\eta_{\Omega^n X}"'] & \Sigma\Omega^{n+1}X
\end{tikzcd}$$

(The top composition is the last two arrows in diagram (2), and the bottom composition is the last two arrows in the diagram in the statement of the lemma). Unravelling how $h^n$ is constructed, by additivity of $\Omega$ it further suffices to show the outside of the following diagram commutes:

$$\begin{tikzcd}
\Omega^{n+1}Z \arrow[r, "(-1)^{n+1}\Omega^{n+1}h"] \arrow[dd, "(-1)^{n+1}\Omega^{n+1}h"'] & \Omega^{n+1}\Sigma X \arrow[r, "\Omega^n\varepsilon_X"] & \Omega^n X \arrow[r, "\Omega\eta_{\Omega^{n-1}X}"] \arrow[dd, equal] & \Omega\Sigma\Omega^n X \arrow[d, "\varepsilon_{\Omega^n X}"] \\
& & & \Omega^n X \arrow[d, "\eta_{\Omega^n X}"] \\
\Omega^{n+1}\Sigma X \arrow[rr, "\Omega^n\varepsilon_X"'] & & \Omega^n X \arrow[r, "\eta_{\Omega^n X}"'] \arrow[ur, equal] & \Sigma\Omega^{n+1}X
\end{tikzcd}$$

The left rectangle and bottom right triangle commute by definition. Finally, commutativity of the top right trapezoid is precisely one of the zig-zag identities applied to $\Omega^{n-1}X$. Hence, we have shown the desired result. $\square$

**Proposition 0.13.** *Given a distinguished triangle*

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X,$$

*let $\widetilde{h} : \Omega Z \to X$ be the left adjoint of $h$. Then the following infinite sequence is exact:*

$$\begin{tikzcd}
 & & \cdots \arrow[dll] & & \\
\Omega^{n+1}Z \arrow[r, "(-1)^{n+1}\Omega^n\widetilde{h}"'] & \Omega^n X \arrow[r, "(-1)^n\Omega^n f"'] & \Omega^n Y \arrow[r, "(-1)^n\Omega^n g"] & \Omega^n Z \arrow[r, "(-1)^n\Omega^{n-1}\widetilde{h}"] & \Omega^{n-1}X \arrow[dll] \\
 & & \cdots \arrow[dll] & & \\
\Omega Z \arrow[r, "-\widetilde{h}"'] & X \arrow[r, "f"'] & Y \arrow[r, "g"] & Z \arrow[r, "h"] & \Sigma X \arrow[dll] \\
 & & \cdots \arrow[dll] & & \\
\Sigma^{n-1}Z \arrow[r, "(-1)^{n-1}\Sigma^n h"'] & \Sigma^n X \arrow[r, "(-1)^n\Sigma^n f"'] & \Sigma^n Y \arrow[r, "(-1)^n\Sigma^n g"] & \Sigma^n Z \arrow[r, "(-1)^n\Sigma^n h"] & \Sigma^{n+1}X \arrow[dll] \\
 & & \cdots & &
\end{tikzcd}$$

*In particular, it remains exact even if we remove the signs.*

*Proof.* Exactness of

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X \xrightarrow{-\Sigma f} \Sigma Y$$

is Proposition 0.3 and axiom TR4. By induction using axiom TR4, for $n > 0$ we get that each contiguous composition of three maps below is a distinguished triangle:

$$\Sigma^n X \xrightarrow{(-1)^n\Sigma^n f} \Sigma^n Y \xrightarrow{(-1)^n\Sigma^n g} \Sigma^n Z \xrightarrow{(-1)^n\Sigma^n h} \Sigma^{n+1} X \xrightarrow{(-1)^{n+1}\Sigma^{n+1} f} \Sigma^{n+1} Y,$$

thus the sequence is exact by Proposition 0.3. It remains to show exactness of the LES to the left of $Y$. It suffices to show that the row in the following diagram is exact for all $n > 0$:

$$\begin{tikzcd}
\Omega^n X \arrow[r, "(-1)^n\Omega^n f"] & \Omega^n Y \arrow[r, "(-1)^n\Omega^n g"] & \Omega^n Z \arrow[rr, "(-1)^n\Omega^{n-1}(\varepsilon_X \circ \Omega h)"] \arrow[dr, "(-1)^n\Omega^n h"'] & & \Omega^{n-1}X \arrow[r, "(-1)^{n-1}\Omega^{n-1}f"] & \Omega^{n-1}Y \\
 & & & \Omega^n\Sigma X \arrow[ur, "\Omega^{n-1}\varepsilon_X"'] & &
\end{tikzcd} \tag{3}$$

First of all, to see exactness at $\Omega^n Y$ and $\Omega^n Z$, consider the following commutative diagram:

$$\begin{tikzcd}
\Omega^n X \arrow[r, "(-1)^n\Omega^n f"] \arrow[dd, equal] & \Omega^n Y \arrow[r, "(-1)^n\Omega^n g"] \arrow[dd, equal] & \Omega^n Z \arrow[rr, "(-1)^n\Omega^{n-1}(\varepsilon_X \circ \Omega h)"] \arrow[dr, "(-1)^n\Omega^n h"'] \arrow[dd, equal] & & \Omega^{n-1}X \arrow[dd, "\eta_{\Omega^{n-1}X}"] \\
 & & & \Omega^n\Sigma X \arrow[ur, "\Omega^{n-1}\varepsilon_X"'] & \\
\Omega^n X \arrow[r, "(-1)^n\Omega^n f"] & \Omega^n Y \arrow[r, "(-1)^n\Omega^n g"] & \Omega^n Z \arrow[ur, "(-1)^n\Omega^n h"] \arrow[rr, dashed] & & \Sigma\Omega^n X
\end{tikzcd}$$

(here the dashed arrow is the morphism which makes the diagram commute). The bottom row is distinguished by Lemma 0.12. Then by axiom TR0, the top row is distinguished, and thus exact by Proposition 0.3. Thus we have shown exactness of (3) at $\Omega^n Y$ and $\Omega^n Z$. It remains to show exactness at $\Omega^{n-1}X$. In the case $n = 1$, we want to show exactness at $X$ in the following diagram:

$$\begin{tikzcd} \Omega Z \arrow[rr, "-(\varepsilon_X \circ \Omega h)"] \arrow[dr, "-\Omega h"'] & & X \arrow[r, "f"] & Y \\ & \Omega\Sigma X \arrow[ur, "\varepsilon_X"'] & & \end{tikzcd}$$

Unravelling definitions, $\varepsilon_X \circ \Omega h$ is precisely the adjoint $\widetilde{h} : \Omega Z \to X$ of $h : Z \to \Sigma X$, in which case we have that the row in the above diagram fits into a distinguished triangle by Lemma 0.11, and thus it is exact by Proposition 0.3. To see exactness at $\Omega^{n-1}X$ in diagram (3), note that if we apply Lemma 0.11 to the sequence Lemma 0.12 for $n-1$, then we get that the following composition fits into a distinguished triangle, and is thus exact:

$$\Omega^n Z \xrightarrow{-k} \Omega^{n-1}X \xrightarrow{(-1)^{n-1}\Omega^{n-1}f} \Omega^{n-1}Y,$$

where $k : \Omega(\Omega^{n-1}Z) \to \Omega^{n-1}X$ is the adjoint of the composition

$$\Omega^{n-1}Z \xrightarrow{(-1)^{n-1}\Omega^{n-1}h} \Omega^{n-1}\Sigma X \xrightarrow{\Omega^{n-2}\varepsilon_X} \Omega^{n-2}X \xrightarrow{\eta_{\Omega^{n-2}X}} \Sigma\Omega^{n-1}X.$$

Further expanding how adjoints are constructed, $k$ is the composition

$$\Omega^n Z \xrightarrow{(-1)^{n-1}\Omega^n h} \Omega^n\Sigma X \xrightarrow{\Omega^{n-1}\varepsilon_X} \Omega^{n-1}X \xrightarrow{\Omega\eta_{\Omega^{n-2}X}} \Omega\Sigma\Omega^{n-1}X \xrightarrow{\varepsilon_{\Omega^{n-1}X}} \Omega^{n-1}X.$$

Thus, in order to show exactness of (3) at $\Sigma^{n-1}X$, it suffices to show that $k = (-1)^{n-1}\Omega^{n-1}(\varepsilon_X \circ \Omega h)$. To that end, consider the following diagram:

$$\begin{tikzcd} \Omega^n Z \arrow[r, "(-1)^{n-1}\Omega^n h"] \arrow[d, "(-1)^{n-1}\Omega^n h"'] & \Omega^n\Sigma X \arrow[r, "\Omega^{n-1}\varepsilon_X"] & \Omega^{n-1}X \arrow[r, "\Omega\eta_{\Omega^{n-2}X}"] \arrow[dr, equal] & \Omega\Sigma\Omega^{n-1}X \arrow[d, "\varepsilon_{\Omega^{n-1}X}"] \\ \Omega^n\Sigma X \arrow[rrr, "\Omega^{n-1}\varepsilon_X"'] & & & \Omega^{n-1}X \end{tikzcd}$$

The top composition is $k$, while the bottom composition is $(-1)^{n-1}\Omega^{n-1}(\varepsilon_X \circ \Omega h)$. The left region commutes by definition, while commutativity of the right region is precisely one of the zig-zag identities applied to $\Omega^{n-2}X$. Thus, we have shown that $-k = (-1)^n\Omega^{n-1}(\varepsilon_X \circ \Omega h)$, so (3) is exact at $\Omega^{n-1}X$, as desired. □

0.4. **Tensor triangulated categories.** Also important for our work is the concept of a *tensor triangulated category*, that is, a triangulated symmetric monoidal category in which the triangulated structures are compatible, in the following sense:

**Definition 0.14.** A *tensor triangulated category* is a triangulated symmetric monoidal category $(\mathcal{C}, \otimes, S, \Sigma, \mathcal{D})$ such that:

- **TT1** For all objects $X$ and $Y$ in $\mathcal{C}$, there are natural isomorphisms
$$e_{X,Y} : \Sigma X \otimes Y \xrightarrow{\cong} \Sigma(X \otimes Y).$$
- **TT2** For each object $X$ in $\mathcal{C}$, the functor $X \otimes (-) \cong (-) \otimes X$ is an additive functor.
- **TT3** For each object $X$ in $\mathcal{C}$, the functor $X \otimes (-) \cong (-) \otimes X$ preserves distinguished triangles, in that given a distinguished triangle/(co)fiber sequence
$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A,$$
then also
$$X \otimes A \xrightarrow{X\otimes f} X \otimes B \xrightarrow{X\otimes g} X \otimes C \xrightarrow{X\otimes' h} \Sigma(X \otimes A)$$

and
$$A \otimes X \xrightarrow{f\otimes X} B \otimes X \xrightarrow{g\otimes X} C \otimes X \xrightarrow{h\otimes' X} \Sigma(A \otimes X)$$
are distinguished triangles, where here we writing $X \otimes' h$ and $h \otimes' X$ to denote the compositions
$$X \otimes C \xrightarrow{X\otimes h} X \otimes \Sigma A \xrightarrow{\tau} \Sigma A \otimes X \xrightarrow{e_{A,X}} \Sigma(A \otimes X) \xrightarrow{\Sigma\tau} \Sigma(X \otimes A)$$
and
$$C \otimes X \xrightarrow{h\otimes X} \Sigma A \otimes X \xrightarrow{e_{A,X}} \Sigma(A \otimes X),$$
respectively.

**TT4** Given objects $X$, $Y$, and $Z$ in $\mathcal{C}$, the following diagram must commute:
$$\begin{tikzcd}
(\Sigma X \otimes Y) \otimes Z \arrow[r, "e_{X,Y\otimes Z}"] \arrow[d, "\alpha"'] & \Sigma(X \otimes Y) \otimes Z \arrow[r, "e_{X\otimes Y,Z}"] & \Sigma((X \otimes Y) \otimes Z) \arrow[d, "\Sigma\alpha"] \\
\Sigma X \otimes (Y \otimes Z) \arrow[rr, "e_{X,Y\otimes Z}"'] & & \Sigma(X \otimes (Y \otimes Z))
\end{tikzcd}$$

Usually, most tensor triangulated categories that arise in nature will satisfy additional coherence axioms (see axioms TC1–TC5 in [1]), but the above definition will suffice for our purposes. In what follows, we fix a tensor triangulated category $(\mathcal{C}, \otimes, S, \Sigma, e, \mathcal{D})$.

**Definition 0.15.** There are natural isomorphisms
$$e'_{X,Y} : X \otimes \Sigma Y \xrightarrow{\cong} \Sigma(X \otimes Y)$$
obtained via the composition
$$X \otimes \Sigma Y \xrightarrow{\tau} \Sigma Y \otimes X \xrightarrow{e_{Y,X}} \Sigma(Y \otimes X) \xrightarrow{\Sigma\tau} \Sigma(X \otimes Y).$$

**Lemma 0.16.** *Let $A \xrightarrow{a} B \xrightarrow{b} C \xrightarrow{c} D$ be any sequence isomorphic to a distinguished triangle. Then given any $E$ in $\mathcal{C}$, the sequences*
$$E \otimes A \xrightarrow{E\otimes a} E \otimes B \xrightarrow{E\otimes b} E \otimes C \xrightarrow{E\otimes c} E \otimes D$$
*and*
$$A \otimes E \xrightarrow{a\otimes E} B \otimes E \xrightarrow{b\otimes E} C \otimes E \xrightarrow{c\otimes E} D \otimes E$$
*are exact.*

*Proof.* Since $(a, b, c)$ is isomorphic to a distinguished triangle, there exists a commuting diagram in $\mathcal{SH}$
$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\alpha"'] & Y \arrow[r, "g"] \arrow[d, "\beta"'] & Z \arrow[r, "h"] \arrow[d, "\gamma"'] & \Sigma X \arrow[d, "\delta"'] \\
A \arrow[r, "a"'] & B \arrow[r, "b"'] & C \arrow[r, "c"'] & D
\end{tikzcd}$$
where the top row is distinguished and the vertical arrows are isomorphisms. Then the following diagram commutes by functoriality of $- \otimes -$:
$$\begin{tikzcd}
E \otimes X \arrow[r, "E\otimes f"] \arrow[dd, "E\otimes\alpha"'] & E \otimes Y \arrow[r, "E\otimes g"] \arrow[dd, "E\otimes\beta"'] & E \otimes Z \arrow[rr, "E\otimes' h"] \arrow[dd, "E\otimes\gamma"'] \arrow[dr, dashed, "E\otimes h"] & & \Sigma(E \otimes X) \arrow[dd, dashed, "(E\otimes\delta)\circ(e'_{E,X})^{-1}"] \\
& & & E \otimes \Sigma X \arrow[ur, dashed, "e'_{E,X}"'] \arrow[dr, dashed, "E\otimes\delta"] & \\
E \otimes A \arrow[r, "E\otimes a"'] & E \otimes B \arrow[r, "E\otimes b"'] & E \otimes C \arrow[rr, "E\otimes c"'] & & E \otimes D
\end{tikzcd}$$

The top triangle is distinguished by axiom TT3 for a tensor triangulated category, thus exact by Proposition 0.3, so that the bottom triangle is also exact since the vertical arrows are isomorphisms and each square commutes. Similarly, the following diagram also commutes by functoriality of $- \otimes -$:

$$\begin{tikzcd}
X \otimes E \arrow[r, "f\otimes E"] \arrow[d, "\alpha\otimes E"'] & Y \otimes E \arrow[r, "g\otimes E"] \arrow[d, "\beta\otimes E"'] & Z \otimes E \arrow[rr, "h\otimes' E"] \arrow[d, "\gamma\otimes E"'] \arrow[dr, dashed, "h\otimes E"] & & \Sigma(X \otimes E) \arrow[d, "(\delta\otimes E)\circ e_{X,E}^{-1}"] \\
 & & & \Sigma X \otimes E \arrow[ur, dashed, "e_{X,E}"] \arrow[dr, dashed, "\delta\otimes E"] & \\
A \otimes E \arrow[r, "a\otimes E"'] & B \otimes E \arrow[r, "b\otimes E"'] & C \otimes E \arrow[rr, "c\otimes E"'] & & D \otimes E
\end{tikzcd}$$

The top row is distinguished by axiom TT3 for a tensor triangulated category, thus exact by Proposition 0.3, so that the bottom triangle is also exact since the vertical arrows are isomorphisms and each square commutes. $\square$

**Proposition 0.17.** *Suppose we have a distinguished triangle*

$$X \to Y \to Z \to \Sigma X$$

*in $\mathcal{C}$. Then given any object $E$ in $\mathcal{C}$, the long exact sequence given in Proposition 0.13 remains exact after applying $E \otimes -$ or $- \otimes E$.*

*Proof.* Recall that in the proof of Proposition 0.13 we showed that the sequence was exact by showing that any two consecutive maps were isomorphic to a part of a distinguished triangle. Then the desired result follows from Lemma 0.16. $\square$

**Definition 0.18.** An *adjointly tensor triangulated* category is a tensor triangulated category $(\mathcal{C}, \otimes, S, \Sigma, e, \mathcal{D})$ along with the structure of an adjointly triangulated category $(\mathcal{C}, \Omega, \Sigma, \eta, \varepsilon, \mathcal{D})$.

From now on, we fix an adjointly tensor triangulated category $(\mathcal{C}, \otimes, S, \Omega, \Sigma, \eta, \varepsilon, e, \mathcal{D})$.

**Definition 0.19.** We may define natural isomorphisms $o_{X,Y} : \Omega X \otimes Y \xrightarrow{\cong} \Omega(X \otimes Y)$ and $o'_{X,Y} : X \otimes \Omega Y \xrightarrow{\cong} \Omega(X \otimes Y)$ as the compositions

$$o_{X,Y} : \Omega X \otimes Y \xrightarrow{\varepsilon^{-1}_{\Omega X \otimes Y}} \Omega\Sigma(\Omega X \otimes Y) \xrightarrow{\Omega e^{-1}_{\Omega X, Y}} \Omega(\Sigma\Omega X \otimes Y) \xrightarrow{\Omega(\eta_X^{-1} \otimes Y)} \Omega(X \otimes Y)$$

and

$$o'_{X,Y} : X \otimes \Omega Y \xrightarrow{\tau_{X,\Omega Y}} \Omega Y \otimes X \xrightarrow{o_{Y,X}} \Omega(Y \otimes X) \xrightarrow{\Omega\tau_{Y,X}} \Omega(X \otimes Y).$$

These are both clearly natural by naturality of $\varepsilon$, $e$, $\eta$, and $\tau$.