

In this appendix, we will define the notion of *A-graded anticommutative Hopf algebroids* (??) over an *A-graded anticommutative ring* R (??), and left comodules over them (??).

0.1. A-graded anticommutative Hopf algebroids over R . Given an *A-graded anticommutative ring* R , we will define an *A-graded anticommutative Hopf algebroid* over R to be a co-groupoid object in $R\text{-}\mathbf{GCA}^A$, i.e., a groupoid object in $(R\text{-}\mathbf{GCA}^A)^{\text{op}}$. First, recall the definition of a *groupoid object* in a category with pullbacks:

Definition 0.1. Let \mathcal{C} be a category with pullbacks. A *groupoid object* in \mathcal{C} consists of a pair of objects (M, O) together with five morphisms

- (1) *Source and target*: $s, t : M \rightarrow O$,
- (2) *Identity*: $e : O \rightarrow M$,
- (3) *Composition*: $c : M \times_O M \rightarrow M$,
- (4) *Inverse*: $i : M \rightarrow M$

Where $M \times_O M$ will always refer to the object which into the following pullback diagram in \mathcal{C} :

$$\begin{array}{ccc} M \times_O M & \xrightarrow{p_2} & M \\ p_1 \downarrow & \lrcorner & \downarrow t \\ M & \xrightarrow{s} & O \end{array}$$

For example, if we're working in $\mathcal{C} = \mathbf{Set}$, we should think of M as a set of morphisms, and O as a set of objects. The functions s and t take a morphism to their domain and codomain, respectively, and $M \times_O M$ is the collection of pairs of morphisms $(g, f) \in M \times M$ such that $t(f) = s(g)$, and the composition map $c : M \times_O M \rightarrow M$ takes such a pair to the element $g \circ f \in M$. We think of the identity $e : O \rightarrow M$ as taking some object $x \in O$ to the identity morphism $e(x) = \text{id}_x \in M$ on x , and the inverse map $i : M \rightarrow M$ takes a morphism f to its inverse f^{-1} . These data are required to make the following diagrams commute:

- (1) Composition works correctly:

$$\begin{array}{ccc} M \times_O M & \xrightarrow{c} & M \\ p_1 \downarrow & & \downarrow t \\ M & \xrightarrow{t} & O \end{array} \quad \begin{array}{ccc} M & \xleftarrow{e} & O \xrightarrow{e} M \\ s \swarrow & \parallel & \searrow t \\ & O & \end{array} \quad \begin{array}{ccc} M \times_O M & \xrightarrow{p_2} & M \\ c \downarrow & & \downarrow s \\ M & \xrightarrow{s} & O \end{array}$$

Expressed in terms of sets, the first diagram says that the target of $g \circ f$ is the target of g . The second diagram says that the domain and codomain of the identity on some object x is x . The third diagram says that the domain of $g \circ f$ is the domain of f .

- (2) Associativity of composition: Write $M \times_O (M \times_O M)$ and $(M \times_O M) \times_O M$ for the pullbacks of $(s, t \circ c)$ and $(s \circ c, t)$, respectively, so we have commuting diagrams

$$\begin{array}{ccc} (M \times_O M) \times_O M & \xrightarrow{p'_2} & M \\ p'_1 \downarrow & \searrow c \times M & \parallel \\ & M \times_O M & \xrightarrow{p_2} M \\ & p_1 \downarrow & \downarrow t \\ M \times_O M & \xrightarrow{c} & M \xrightarrow{s} O \end{array} \quad \begin{array}{ccc} M \times_O (M \times_O M) & \xrightarrow{p''_2} & M \times_O M \\ p''_1 \downarrow & \searrow M \times c & \downarrow c \\ & M \times_O M & \xrightarrow{p_2} M \\ & p_1 \downarrow & \downarrow t \\ M & \xlongequal{\quad} & M \xrightarrow{s} O \end{array}$$

where the inner and outer squares in both diagrams are pullback squares. Furthermore, assuming the diagrams in condition (1) above are satisfied, we have that $t \circ p_1 \circ p''_2 =$

$t \circ c \circ p_2'' = s \circ p_1''$, so that by the universal property of the pullback we have a map $M \times p_1 : M \times_O (M \times_O M) \rightarrow M \times_O M$ like so:

$$\begin{array}{ccccc}
 M \times_O (M \times_O M) & & & & \\
 \swarrow M \times p_1 & \searrow p_1 \circ p_2'' & & & \\
 & M \times_O M & \xrightarrow{p_2} & M & \\
 \searrow p_1'' & \downarrow p_1 & & \downarrow t & \\
 & M & \xrightarrow{s} & O &
 \end{array}$$

Now note that again assuming the diagrams above in (1) commute, we have $s \circ c = s \circ p_2$, so that

$$s \circ c \circ (M \times p_1) = s \circ p_2 \circ (M \times p_1) = s \circ p_1 \circ p_2'' = t \circ p_2 \circ p_2''.$$

Then by the universal property of the pullback we get a map $a : M \times_O (M \times_O M) \rightarrow (M \times_O M) \times_O M$ like so:

$$\begin{array}{ccccc}
 M \times_O (M \times_O M) & & & & \\
 \swarrow a & \searrow p_2 \circ p_2'' & & & \\
 & (M \times_O M) \times_O M & \xrightarrow{p_2'} & M & \\
 \searrow M \times p_1 & \downarrow p_1' & & \parallel & \\
 & M \times_O M & \xrightarrow{c} & M & \xrightarrow{s} O \\
 & & & \downarrow t &
 \end{array}$$

Exercise: Show that this map a is an isomorphism. Then we require that the following diagram commutes:

$$\begin{array}{ccc}
 M \times_O (M \times_O M) & \xrightarrow{a} & (M \times_O M) \times_O M \\
 M \times c \downarrow & & \downarrow c \times M \\
 M \times_O M & \xrightarrow{c} & M \xleftarrow{c} M \times_O M
 \end{array}$$

Expressed in terms of sets, this diagram says $h \circ (g \circ f) = (h \circ g) \circ f$.

- (3) Unitality of composition: Given the maps $(\text{id}_M, e \circ t), (e \circ s, \text{id}_M) : M \rightarrow M \times_O M$ defined by the universal property of $M \times_O M$:

$$\begin{array}{ccc}
 M & \xrightarrow{e \circ s} & M \times_O M \xrightarrow{p_2} M \\
 \swarrow (\text{id}_M, e \circ s) & & \downarrow p_1 \quad \lrcorner \quad \downarrow t \\
 & M & \xrightarrow{s} O
 \end{array}
 \qquad
 \begin{array}{ccc}
 M & \xrightarrow{e \circ t} & M \times_O M \xrightarrow{p_2} M \\
 \swarrow (e \circ t, \text{id}_M) & & \downarrow p_1 \quad \lrcorner \quad \downarrow t \\
 & M & \xrightarrow{s} O
 \end{array}$$

the following diagram commutes:

$$\begin{array}{ccc}
 M & \xrightarrow{(eot, \text{id}_M)} & M \times_O M \\
 (\text{id}_M, eos) \downarrow & \searrow & \downarrow c \\
 M \times_O M & \xrightarrow{c} & M
 \end{array}$$

Expressed in terms of sets, this diagram says that given $f \in M$ with $s(f) = x$ and $t(f) = y$, that $f \circ \text{id}_x = f$ and $\text{id}_y \circ f = f$.

(4) Inverse: The following diagrams must commute:

$$\begin{array}{ccccc}
 M & & M & \xrightarrow{(\text{id}_M, i)} & M \times_O M & \xleftarrow{(i, \text{id}_M)} & M \\
 \swarrow i & \downarrow i & t \downarrow & & \downarrow c & & \downarrow s \\
 M & \xleftarrow{i} & M & & O & \xleftarrow{e} & M & \xleftarrow{e} & O
 \end{array}
 \quad
 \begin{array}{ccccc}
 M & & & & M \\
 \swarrow s & \downarrow i & \searrow t & & \\
 O & \xleftarrow{t} & M & \xrightarrow{s} & O
 \end{array}$$

where the arrows (id_M, i) and (i, id_M) are determined by the universal property of $M \times_O M$ like so:

$$\begin{array}{ccc}
 M & \xrightarrow{i} & M \\
 (\text{id}_M, i) \searrow & & \downarrow t \\
 M \times_O M & \xrightarrow{p_2} & M \\
 p_1 \downarrow & & \downarrow t \\
 M & \xrightarrow{s} & O
 \end{array}
 \quad
 \begin{array}{ccc}
 M & \xrightarrow{(i, \text{id}_M)} & M \\
 i \searrow & & \downarrow t \\
 M \times_O M & \xrightarrow{p_2} & M \\
 p_1 \downarrow & & \downarrow t \\
 M & \xrightarrow{s} & O
 \end{array}$$

Expressed in terms of sets, given $f \in M$ with $s(f) = x$ and $t(f) = y$, the first diagram says that $(f^{-1})^{-1} = f$, the second says that $f \circ f^{-1} = \text{id}_y$ and $f^{-1} \circ f = \text{id}_x$, and the last diagram says that the domain and codomain of f^{-1} are y and x , respectively.

It can be seen that groupoid objects in $\mathcal{C} = \mathbf{Set}$ are precisely (small) groupoids. Now, we can state and unravel the definition of a Hopf algebroid:

Definition 0.2. Given an A -graded anticommutative ring R (??), an A -graded anticommutative Hopf algebroid over R is a co-groupoid object in $R\text{-GCA}^A$, i.e., a groupoid object in $(R\text{-AGrCAlg})^{\text{op}}$. Explicitly, an A -graded anticommutative Hopf algebroid over E is a pair (Γ, B) of objects in $R\text{-AGrCAlg}$ along with morphisms

- (1) *left unit*: $\eta_L : B \rightarrow \Gamma$ (corresponding to t),
- (2) *right unit*: $\eta_R : B \rightarrow \Gamma$ (corresponding to s),
- (3) *comultiplication*: $\Psi : \Gamma \rightarrow \Gamma \otimes_B \Gamma$ (corresponding to c),
- (4) *counit*: $\epsilon : \Gamma \rightarrow B$ (corresponding to e),
- (5) *conjugation*: $c : \Gamma \rightarrow \Gamma$ (corresponding to i),

where here Γ may be viewed as a B -bimodule with left B -module structure induced by η_L and right B -module structure induced by η_R , so we may form the tensor product of bimodules $\Gamma \otimes_B \Gamma$, which further may be given the structure of an A -graded anticommutative R -algebra (by ??), and

fits into the following pushout diagram in $R\text{-}\mathbf{GCA}^A g$ (??):

$$\begin{array}{ccc} B & \xrightarrow{\eta_L} & \Gamma \\ \eta_R \downarrow & & \downarrow g \mapsto 1 \otimes g \\ \Gamma & \xrightarrow{g \mapsto g \otimes 1} & \Gamma \otimes_B \Gamma \end{array}$$

These data must make the following diagrams commute:

(1) (Composition works correctly)

$$\begin{array}{ccc} B & \xrightarrow{\eta_L} & \Gamma \\ \eta_L \downarrow & & \downarrow \Psi \\ \Gamma & \xrightarrow{g \mapsto g \otimes 1} & \Gamma \otimes_B \Gamma \end{array} \quad \begin{array}{ccc} & B & \\ \eta_R \swarrow & \parallel & \searrow \eta_L \\ \Gamma & \xrightarrow{\epsilon} B \xleftarrow{\epsilon} & \Gamma \end{array} \quad \begin{array}{ccc} B & \xrightarrow{\eta_R} & \Gamma \\ \eta_R \downarrow & & \downarrow g \mapsto 1 \otimes g \\ \Gamma & \xrightarrow{\Psi} & \Gamma \otimes_B \Gamma \end{array}$$

(2) (Coassociativity) The following diagram must commute

$$\begin{array}{ccccc} \Gamma \otimes_B \Gamma & \xleftarrow{\Psi} & \Gamma & \xrightarrow{\Psi} & \Gamma \otimes_B \Gamma \\ \Psi \otimes_B \Gamma \downarrow & & & & \downarrow \Gamma \otimes_B \Psi \\ (\Gamma \otimes_B \Gamma) \otimes_B \Gamma & \xrightarrow{\cong} & & & \Gamma \otimes_B (\Gamma \otimes_B \Gamma) \end{array}$$

where $(\Gamma \otimes_B \Gamma) \otimes_B \Gamma$ and $\Gamma \otimes_B (\Gamma \otimes_B \Gamma)$ denote the rings which fit into the following pushout diagrams in $R\text{-}\mathbf{GCA}^A$:

$$\begin{array}{ccc} B & \xrightarrow{\eta_L} & \Gamma \\ \eta_R \downarrow & & \downarrow g \mapsto (1 \otimes 1) \otimes g \\ \Gamma & & \downarrow \Psi \\ \Gamma \otimes_B \Gamma & \xrightarrow{g \otimes g' \mapsto (g \otimes g') \otimes 1} & (\Gamma \otimes_B \Gamma) \otimes_B \Gamma \end{array} \quad \begin{array}{ccc} B & \xrightarrow{\eta_L} \Gamma \xrightarrow{\Psi} & \Gamma \otimes_B \Gamma \\ \eta_R \downarrow & & \downarrow (g \otimes g') \mapsto 1 \otimes (g \otimes g') \\ \Gamma & \xrightarrow{g \mapsto g \otimes (1 \otimes 1)} & \Gamma \otimes_B (\Gamma \otimes_B \Gamma) \end{array}$$

and the isomorphism $(\Gamma \otimes_B \Gamma) \otimes_B \Gamma \rightarrow \Gamma \otimes_B (\Gamma \otimes_B \Gamma)$ sends $(g \otimes g') \otimes g''$ to $g \otimes (g' \otimes g'')$, the left vertical arrow $\Psi \otimes \Gamma$ sends $g \otimes g'$ to $\Psi(g) \otimes g$, and the right vertical arrow $\Gamma \otimes \Psi$ sends $g \otimes g'$ to $g \otimes \Psi(g')$.

(3) (Co-unitality):

$$\begin{array}{ccc} \Gamma & \xrightarrow{\Psi} & \Gamma \otimes_B \Gamma \\ \Psi \downarrow & \searrow & \downarrow (\eta_L \circ \epsilon) \cdot \text{id}_\Gamma \\ \Gamma \otimes_B \Gamma & \xrightarrow{\text{id}_\Gamma \cdot (\eta_R \circ \epsilon)} & \Gamma \end{array}$$

where the right vertical arrow sends $g \otimes g'$ to $\eta_L(\epsilon(g))g'$ and the bottom horizontal arrow sends $g \otimes g'$ to $g\eta_R(\epsilon(g'))$.

(4) (Convolution):

$$\begin{array}{ccc} & \Gamma & \\ & \parallel & \downarrow c \\ \Gamma & \xleftarrow{c} & \Gamma \end{array} \quad \begin{array}{ccccc} B & \xleftarrow{\epsilon} & \Gamma & \xrightarrow{\epsilon} & B \\ \eta_L \downarrow & & \downarrow i & & \downarrow \eta_R \\ \Gamma & \xleftarrow{\text{id}_\Gamma \cdot c} & \Gamma \otimes_B \Gamma & \xrightarrow{c \cdot \text{id}_\Gamma} & \Gamma \end{array} \quad \begin{array}{ccccc} B & \xrightarrow{\eta_L} & \Gamma & \xleftarrow{\eta_R} & B \\ & \searrow \eta_R & \downarrow c & \swarrow \eta_L & \\ & & \Gamma & & \end{array}$$

where the bottom left arrow in the middle diagram sends $g \otimes g'$ to $gc(g')$ and the bottom right arrow in the middle diagram sends $g \otimes g'$ to $c(g)g'$.

The remainder of this subsection is devoted to proving some technical lemmas about A -graded anticommutative Hopf algebroids.

Proposition 0.3. *Suppose we have an A -graded anticommutative Hopf algebroid (Γ, B) over (R, θ) with structure maps η_L , η_R , Ψ , ϵ , and c (??). Recall in the definition, we considered $\Gamma \otimes_B \Gamma$ to be the A -graded R -commutative ring whose underlying abelian group was given by the tensor product of B -bimodules, where Γ has left B -module structure induced by η_L and right B -module structure induced by η_R . Thus $\Gamma \otimes_B \Gamma$ is canonically a B -bimodule, as it is a tensor product of B -bimodules. Then the canonical left (resp. right) B -module structure on $\Gamma \otimes_B \Gamma$ coincides with that induced by the ring homomorphism $\Psi \circ \eta_L$ (resp. $\Psi \circ \eta_R$).*

Proof. First we show the left module structures coincide. By additivity, in order to show the module structures coincide, it suffices to show that given a homogeneous pure tensor $g \otimes g'$ in $\Gamma \otimes_B \Gamma$ and some $b \in B$ that $\Psi(\eta_L(b)) \cdot (g \otimes g') = (\eta_L(b) \cdot g) \otimes g'$, where \cdot on the left denotes the product in $\Gamma \otimes_B \Gamma$ and the \cdot on the right denotes the product in Γ . By the axioms for a Hopf algebroid, we have that $\Psi(\eta_L(b)) = \eta_L(b) \otimes 1$. Thus by how the product in $\Gamma \otimes_B \Gamma$ is defined (??), we have that

$$\Psi(\eta_L(b)) \cdot (g \otimes g') = (\eta_L(b) \otimes 1) \cdot (g \otimes g') = (\varphi_\Gamma(\theta_{0,|g|}) \cdot \eta_L(b) \cdot g) \otimes (g' \cdot 1) = (\eta_L(b) \cdot g) \otimes g',$$

where $\varphi_\Gamma : R \rightarrow \Gamma$ is the structure map, and the last equality follows by the fact that $\theta_{0,|g|} = 1$. An entirely analagous argument yields that the canonical right module structure on $\Gamma \otimes_B \Gamma$ coincides with that induced by $\Psi \circ \eta_R$, since $\Psi \circ \eta_R = 1 \otimes \eta_R$. \square

Remark 0.4. By the above proposition, given an A -graded commutative Hopf algebroid (Γ, B) over R , there is no ambiguity when discussing the objects $\Gamma \otimes_B (\Gamma \otimes_B \Gamma)$ and $(\Gamma \otimes_B \Gamma) \otimes_B \Gamma$ — they may both be considered as the threefold tensor product of the B -bimodule Γ with itself. In particular, we have a canonical isomorphism of B -bimodules

$$(\Gamma \otimes_B \Gamma) \otimes_B \Gamma \rightarrow \Gamma \otimes_B (\Gamma \otimes_B \Gamma)$$

sending $(g \otimes g') \otimes g''$ to $g \otimes (g' \otimes g'')$, and this is precisely the isomorphism in the coassociativity diagram in the definition of a Hopf algebroid (??).

Proposition 0.5. *Suppose we have an A -graded commutative Hopf algebroid (Γ, B) over R with structure maps η_L , η_R , Ψ , ϵ , and c . Then $\eta_L : B \rightarrow \Gamma$ is a homomorphism of left B -modules, $\eta_R : B \rightarrow \Gamma$ is a homomorphism of right B -modules, and $\Psi : \Gamma \rightarrow \Gamma \otimes_B \Gamma$ and $\epsilon : \Gamma \rightarrow B$ are homomorphisms of B -bimodules.*

Proof. Since the left (resp. right) B -module structure on Γ is induced by η_L (resp. η_R), the map η_L (resp. η_R) is a homomorphism of left (resp. right) B -modules by definition.

Next, we want to show Ψ is a homomorphism of B -bimodules. The left (resp. right) B -module structure on Γ is that induced by η_L (resp. η_R), and in ??, we showed that the left (resp. right) B -module structure on $\Gamma \otimes_B \Gamma$ is that induced by $\Psi \circ \eta_L$ (resp. $\Psi \circ \eta_R$), so that by definition $\Psi : \Gamma \rightarrow \Gamma \otimes_B \Gamma$ is a homomorphism of left (resp. right) B -modules.

Lastly, we claim that $\epsilon : \Gamma \rightarrow B$ is a homomorphism of B -bimodules. We need to show that given $g \in \Gamma$ and $b, b' \in B$ that $\epsilon(\eta_L(b)g\eta_R(g')) = b\epsilon(g)b'$. This follows from the fact that ϵ is a ring homomorphism satisfying $\epsilon \circ \eta_L = \epsilon \circ \eta_R = \text{id}_B$. \square

0.2. Comodules over a Hopf algebroid. In what follows, fix an A -graded anticommutative ring (R, θ) and an A -graded anticommutative Hopf algebroid (Γ, B) over R with structure maps $\eta_L, \eta_R, \Psi, \epsilon$, and c . We will always view Γ with its *canonical* B -bimodule structure, with left B -module structure induced by η_L , and right B -module structure induced by η_R . In particular, any tensor product over B involving Γ will always refer to Γ with this bimodule structure.

Definition 0.6. A *left comodule over Γ* is a pair (N, Ψ_N) , where N is a left A -graded B -module and $\Psi_N : N \rightarrow \Gamma \otimes_B N$ is an A -graded homomorphism of left A -graded B -modules. These data are required to make the following diagrams commute

$$\begin{array}{ccc} N & \xrightarrow{\Psi_N} & \Gamma \otimes_B N \\ & \searrow \cong & \downarrow \epsilon \otimes N \\ & & B \otimes_B N \end{array} \qquad \begin{array}{ccccc} \Gamma \otimes_B N & \xleftarrow{\Psi_N} & N & \xrightarrow{\Psi_N} & \Gamma \otimes_B N \\ \Psi \otimes N \downarrow & & & & \downarrow \Gamma \otimes \Psi_N \\ (\Gamma \otimes_B \Gamma) \otimes_B N & \xrightarrow{\cong} & & & \Gamma \otimes_B (\Gamma \otimes_B N) \end{array}$$

The maps $\epsilon \otimes N$ and $\Psi \otimes N$ are well-defined by ??, and the bottom isomorphism in the right diagram is the canonical one sending $(g \otimes g') \otimes n \mapsto g \otimes (g' \otimes n)$.

Given two left A -graded Γ -comodules (N_1, Ψ_{N_1}) and (N_2, Ψ_{N_2}) , a homomorphism of left A -graded comodules $f : N_1 \rightarrow N_2$ is an A -graded homomorphism of the underlying left B -modules such that the following diagram commutes:

$$\begin{array}{ccc} N_1 & \xrightarrow{f} & N_2 \\ \Psi_{N_1} \downarrow & & \downarrow \Psi_{N_2} \\ \Gamma \otimes_B N_1 & \xrightarrow{\Gamma \otimes f} & \Gamma \otimes_B N_2 \end{array}$$

We write $\Gamma\text{-CoMod}^A$ for the resulting category of left A -graded comodules over Γ . In the above definition, we required A -graded left Γ -comodule homomorphisms to strictly preserve the grading, but we could have instead considered left Γ -comodule homomorphisms which are of degree d for some $d \in A$, or equivalently, the set of degree zero A -graded Γ -comodule homomorphisms from N_1 to the shifted comodule $(N_2)_{*+d}$. We denote the hom-set of degree- d A -graded left Γ -comodule homomorphisms from (N_1, Ψ_{N_1}) to (N_2, Ψ_{N_2}) by

$$\text{Hom}_{\Gamma\text{-CoMod}^A}^d(N_1, N_2) \quad \text{or usually just} \quad \text{Hom}_{\Gamma}^d(N_1, N_2).$$

In particular, write $\text{Hom}_{\Gamma\text{-CoMod}^A}(N_1, N_2)$ or just $\text{Hom}_{\Gamma}(N_1, N_2)$ to mean the set of strictly degree preserving (degree 0) A -graded left Γ -comodule homomorphisms from (N_1, Ψ_{N_1}) to (N_2, Ψ_{N_2}) .

Proposition 0.7. *The category $\Gamma\text{-CoMod}^A$ is an additive category.*

Proof. First, we show the category is **Ab**-enriched. Since the forgetful functor $\Gamma\text{-CoMod}^A \rightarrow B\text{-Mod}^A$ is clearly faithful, we may view hom-sets in $\Gamma\text{-CoMod}^A$ as subsets of hom-groups in $B\text{-Mod}^A$, so that in order to show $\Gamma\text{-CoMod}^A$ is **Ab**-enriched, it suffices to show that hom-sets in $\Gamma\text{-CoMod}^A$ are closed under addition of module homomorphisms and taking inverses. To that end, suppose we have two A -graded left Γ -comodule homomorphisms $f, g : (N_1, \Psi_{N_1}) \rightarrow (N_2, \Psi_{N_2})$, then we have

$$\begin{aligned} \Psi_{N_2} \circ (f + g) &= (\Psi_{N_2} \circ f) + (\Psi_{N_2} \circ g) \\ &= ((\Gamma \otimes_B f) \circ \Psi_{N_1}) + ((\Gamma \otimes_B g) \circ \Psi_{N_1}) \\ &= ((\Gamma \otimes_B f) + (\Gamma \otimes_B g)) \circ \Psi_{N_1} \\ &= (\Gamma \otimes_B (f + g)) \circ \Psi_{N_1}, \end{aligned}$$

where the first equality follows since Ψ_{N_2} is a homomorphism, the second follows since f and g are left Γ -comodule homomorphisms, the third follows since Ψ_{N_1} is a homomorphism, and the last equality follows by definition of the tensor product of modules. Hence $f + g$ is indeed an A -graded left Γ -comodule homomorphism, as desired. Now, we also claim $-f$ is an A -graded left Γ -comodule homomorphism. To that end, note that

$$\Psi_{N_2} \circ (-f) = -\Psi_{N_2} \circ f = -(\Gamma \otimes_B f) \circ \Psi_{N_1} = (\Gamma \otimes_B (-f)) \circ \Psi_{N_1},$$

where the first equality follows since Ψ_{N_2} is a homomorphism, the second follows since f is an A -graded left Γ -comodule homomorphism, and the third equality follows by definition of the tensor product.

Thus, we've shown that the hom-sets in $\Gamma\text{-CoMod}^A$ are abelian groups, and composition is clearly bilinear, so that $\Gamma\text{-CoMod}^A$ is indeed **Ab**-enriched.

Now, in order to show $\Gamma\text{-CoMod}^A$ is additive, it suffices to show that it contains a zero object and has binary coproducts. First of all, it is straightforward to check that the zero left B -module is clearly an A -graded left Γ -comodule with structure map the unique map $0 \rightarrow \Gamma \otimes_B 0 \cong 0$, and that given any other A -graded left Γ -comodule (N, Ψ_N) , the unique homomorphisms of left B -modules $0 \rightarrow N$ and $N \rightarrow 0$ are left comodule homomorphisms.

Now, suppose we have two A -graded left Γ -comodules (N_1, Ψ_{N_1}) and (N_2, Ψ_{N_2}) . First, we claim their direct sum as left B -modules $N_1 \oplus N_2$ is canonically an A -graded left Γ -comodule. We know that $N_1 \oplus N_2$ is an A -graded left B -module by ??, and we can define the structure map

$$\Psi_{N_1 \oplus N_2} : N_1 \oplus N_2 \xrightarrow{\Psi_{N_1} \oplus \Psi_{N_2}} (\Gamma \otimes_B N_1) \oplus (\Gamma \otimes_B N_2) \cong \Gamma \otimes_B (N_1 \oplus N_2),$$

where the final isomorphism is the canonical one sending $(g_1 \otimes n_1) \oplus (g_2 \otimes n_2)$ to $(g_1 \otimes n_1) + (g_2 \otimes n_2)$. Then to see this is in fact a left Γ -comodule, first consider the following diagram:

$$\begin{array}{ccccc} N_1 \oplus N_2 & \xrightarrow{\Psi_{N_1} \oplus \Psi_{N_2}} & (\Gamma \otimes_B N_1) \oplus (\Gamma \otimes_B N_2) & \xrightarrow{\cong} & \Gamma \otimes_B (N_1 \oplus N_2) \\ & \searrow \cong & \downarrow (\epsilon \otimes N_1) \oplus (\epsilon \otimes N_2) & & \downarrow \epsilon \otimes (N_1 \oplus N_2) \\ & & (B \otimes_B N_1) \oplus (B \otimes_B N_2) & \xrightarrow{\cong} & B \otimes_B (N_1 \oplus N_2) \end{array}$$

A simple diagram chase yields the left and rightmost regions commute. The top left region commutes since (N_1, Ψ_{N_1}) and (N_2, Ψ_{N_2}) are left Γ -comodules. Now, consider the following

diagram:

$$\begin{array}{ccccc}
\Gamma \otimes_B (N_1 \oplus N_2) & \xrightarrow{\Gamma \otimes_B (\Psi_{N_1} \oplus \Psi_{N_2})} & \Gamma \otimes_B ((\Gamma \otimes_B N_1) \oplus (\Gamma \otimes_B N_2)) & \xrightarrow{\Gamma \otimes_B \cong} & \Gamma \otimes_B (\Gamma \otimes_B (N_1 \oplus N_2)) \\
\cong \uparrow & & \uparrow \cong & & \uparrow \cong \\
(\Gamma \otimes_B N_1) \oplus (\Gamma \otimes_B N_2) & \xrightarrow{(\Gamma \otimes_B \Psi_{N_1}) \oplus (\Gamma \otimes_B \Psi_{N_2})} & (\Gamma \otimes_B (\Gamma \otimes_B N_1)) \oplus (\Gamma \otimes_B (\Gamma \otimes_B N_2)) & & \\
\uparrow \Psi_{N_1} \oplus \Psi_{N_2} & & \downarrow \cong \oplus \cong & & \\
N_1 \oplus N_2 & & & & \\
\downarrow \Psi_{N_1} \oplus \Psi_{N_2} & & & & \\
(\Gamma \otimes_B N_1) \oplus (\Gamma \otimes_B N_2) & \xrightarrow{(\Psi \otimes N_1) \oplus (\Psi \otimes N_2)} & ((\Gamma \otimes_B \Gamma) \otimes_B N_1) \oplus ((\Gamma \otimes_B \Gamma) \otimes_B N_2) & \xrightarrow{\cong} & (\Gamma \otimes_B \Gamma) \otimes_B (N_1 \oplus N_2) \\
\cong \downarrow & & \searrow \cong & & \uparrow \cong \\
\Gamma \otimes_B (N_1 \oplus N_2) & \xrightarrow{\Psi \otimes (N_1 \oplus N_2)} & & &
\end{array}$$

The middle left region commutes since (N_1, Ψ_{N_1}) and (N_2, Ψ_{N_2}) are left Γ -comodules. Each other region in the diagram can be seen to commute by a straightforward diagram chase.

Thus, we have shown that $N_1 \oplus N_2$ is indeed canonically an A -graded left Γ -comodule. Then it remains to show that the canonical inclusions $\iota_i : N_i \hookrightarrow N_1 \oplus N_2$ are Γ -comodule homomorphisms for $i = 1, 2$, and that given Γ -comodule homomorphisms $(N_1, \Psi_{N_1}) \rightarrow (N, \Psi_N)$ and $(N_2, \Psi_{N_2}) \rightarrow (N, \Psi_N)$, that the map $N_1 \oplus N_2 \rightarrow N$ induced by the universal property of the coproduct in $B\text{-}\mathbf{Mod}^A$ is a Γ -comodule homomorphism. This is all entirely straightforward to check by doing a few simple diagram chases. \square

Proposition 0.8. *The forgetful functor $\Gamma\text{-CoMod}^A \rightarrow B\text{-Mod}^A$ (where here $B\text{-Mod}^A$ is the category of A -graded left B -modules and degree-preserving module homomorphisms between them) has a right adjoint $\Gamma \otimes_B - : B\text{-Mod}^A \rightarrow \Gamma\text{-CoMod}^A$ called the co-free construction, where the co-free left A -graded Γ -comodule on a left A -graded B -module M is the B -module $\Gamma \otimes_B M$ equipped with the coaction*

$$\Psi_{\Gamma \otimes_B M} : \Gamma \otimes_B M \xrightarrow{\Psi \otimes_B M} (\Gamma \otimes_B \Gamma) \otimes_B M \xrightarrow{\cong} \Gamma \otimes_B (\Gamma \otimes_B M).$$

Explicitly, given some (N, Ψ_N) in $\Gamma\text{-CoMod}$ and some M in $B\text{-Mod}^A$, the counit and unit of this adjunction are given by

$$\eta_{(N, \Psi_N)} : N \xrightarrow{\Psi_N} \Gamma \otimes_B N$$

and

$$\varepsilon_M : \Gamma \otimes_B M \xrightarrow{\varepsilon \otimes_B M} B \otimes_B M \xrightarrow{\cong} M.$$

Proof. First, we need to show that given a left A -graded B -module that the given map $\Psi_{\Gamma \otimes_B M} : \Gamma \otimes_B M \rightarrow \Gamma \otimes_B (\Gamma \otimes_B M)$ endows B with the structure of a left Γ -comodule. To that end, first

consider the following diagram:

$$\begin{array}{ccccc}
 \Gamma \otimes_B M & \xrightarrow{\Psi \otimes M} & (\Gamma \otimes_B \Gamma) \otimes_B M & \xrightarrow{\cong} & \Gamma \otimes_B (\Gamma \otimes_B M) \\
 & \searrow & \downarrow ((\eta_L \circ \epsilon) \cdot \text{id}_\Gamma) \otimes M & \searrow (\epsilon \otimes \Gamma) \otimes M & \downarrow \epsilon \otimes (\Gamma \otimes M) \\
 & & \Gamma \otimes_B M & \xleftarrow{\eta_L \cdot \text{id}_\Gamma} & (B \otimes_B \Gamma) \otimes_B M \\
 & & & \searrow \cong & \downarrow \cong \\
 & & & & B \otimes_B (\Gamma \otimes_B M)
 \end{array}$$

The top left region commutes by the co-unitality axiom for a Hopf algebroid. A simple diagram chase yields commutativity of every other diagram (in particular, the bottom region commutes since the left B -module structure on Γ is that induced by η_L). Now, consider the following diagram:

$$\begin{array}{ccc}
 \Gamma \otimes_B (\Gamma \otimes_B M) & \xrightarrow{\Gamma \otimes (\Psi \otimes M)} & \Gamma \otimes_B (\Gamma \otimes_B (\Gamma \otimes_B M)) \\
 \cong \uparrow & & \searrow \cong \\
 (\Gamma \otimes_B \Gamma) \otimes_B M & \xrightarrow{(\Gamma \otimes \Psi) \otimes M} & (\Gamma \otimes_B (\Gamma \otimes_B \Gamma)) \otimes_B M \\
 \Psi \otimes M \uparrow & & \uparrow \cong \\
 \Gamma \otimes_B M & & \\
 \Psi \otimes M \downarrow & & \uparrow \cong \\
 (\Gamma \otimes_B \Gamma) \otimes_B M & \xrightarrow{(\Psi \otimes \Gamma) \otimes M} & ((\Gamma \otimes_B \Gamma) \otimes_B \Gamma) \otimes_B M \\
 \cong \downarrow & & \uparrow \cong \\
 \Gamma \otimes_B (\Gamma \otimes_B M) & \xrightarrow{\Psi \otimes (\Gamma \otimes M)} & (\Gamma \otimes_B \Gamma) \otimes_B (\Gamma \otimes_B M) \xrightarrow{\cong} \Gamma \otimes_B ((\Gamma \otimes_B \Gamma) \otimes_B M) \\
 & & \downarrow \Gamma \otimes \cong
 \end{array}$$

The left region commutes since Ψ is co-associative. A simple diagram chase yields the commutativity of every other diagram. Thus, we have indeed shown that $(\Gamma \otimes_B M, \Psi_{\Gamma \otimes_B M})$ is an A -graded left Γ -comodule, as desired.

Now, we need to show that η and ε are natural transformations which satisfy the zig-zag identities. The maps η is clearly natural by how morphisms in $\Gamma\text{-CoMod}^A$ are defined. It is also clear that ε is natural by functoriality of $- \otimes_B -$. Thus, it remains to show the following two diagrams commute for all M in $B\text{-Mod}^A$ and (N, Ψ_N) in $\Gamma\text{-CoMod}^A$:

$$\begin{array}{ccc}
 N & \xrightarrow{\eta_{(N, \Psi_N)}} & \Gamma \otimes_B N \\
 & \searrow & \downarrow \varepsilon_N \\
 & & N
 \end{array}
 \qquad
 \begin{array}{ccc}
 \Gamma \otimes_B (\Gamma \otimes_B M) & \xleftarrow{\eta_{(\Gamma \otimes_B M, \Psi_{\Gamma \otimes_B M})}} & \Gamma \otimes_B M \\
 \Gamma \otimes_B \varepsilon_M \downarrow & & \searrow \\
 \Gamma \otimes_B M & &
 \end{array}$$

Unravelling definitions, the left diagram becomes:

$$\begin{array}{ccc}
 N & \xrightarrow{\Psi_N} & \Gamma \otimes_B N \\
 \searrow \cong & & \downarrow \epsilon \otimes_B N \\
 & & B \otimes_B N \\
 \searrow \cong & & \downarrow \cong \\
 & & N
 \end{array}$$

This commutes since (N, Ψ_N) is a left Γ -comodule. On the other hand, the right diagram becomes:

$$\begin{array}{ccccc}
 \Gamma \otimes_B (\Gamma \otimes_B M) & \xleftarrow{\cong} & (\Gamma \otimes_B \Gamma) \otimes_B M & \xleftarrow{\Psi \otimes M} & \Gamma \otimes_B M \\
 \downarrow \Gamma \otimes (\epsilon \otimes M) & & \downarrow (\Gamma \otimes_B \epsilon) \otimes M & \searrow & \parallel \\
 \Gamma \otimes_B (B \otimes_B M) & \xleftarrow{\cong} & (\Gamma \otimes_B B) \otimes_B (\text{id}_{\Gamma} \cdot (\eta_R \circ \epsilon)) \otimes M & & \Gamma \otimes_B M \\
 \downarrow \Gamma \otimes \cong & \swarrow (\text{id}_{\Gamma} \cdot \eta_R) \otimes M & & \searrow & \\
 \Gamma \otimes_B M & \xleftarrow{\quad \quad \quad} & & & \Gamma \otimes_B M
 \end{array}$$

The rightmost region commutes by co-unitality of Ψ , while a simple diagram chase yields commutativity of the remaining regions (in particular, the bottom left region commutes because the right B -module structure on Γ is induced by η_R). \square

Proposition 0.9. *Suppose that Γ is flat as a right B -module, i.e., suppose $\eta_R : B \rightarrow \Gamma$ is a flat ring homomorphism. Then the category $\Gamma\text{-CoMod}^A$ is an abelian category and has enough injectives.*

Proof. In ??, we showed that $\Gamma\text{-CoMod}^A$ is an additive category, so it remains to show that it has all kernels and cokernels, and that for all morphisms f in $\Gamma\text{-CoMod}^A$ that the comparison morphism

$$\text{coker}(\ker f) \rightarrow \ker(\text{coker } f)$$

is an isomorphism. First, let $f : (N_1, \Psi_{N_1}) \rightarrow (N_2, \Psi_{N_2})$ be a morphism in $\Gamma\text{-CoMod}^A$, and consider the following diagram:

$$\begin{array}{ccccc}
 \ker f & \hookrightarrow & N_1 & \xrightarrow{f} & N_2 \\
 \downarrow \text{dashed} & & \downarrow \Psi_{N_1} & & \downarrow \Psi_{N_2} \\
 \Gamma \otimes_B \ker f & \hookrightarrow & \Gamma \otimes_B N_1 & \xrightarrow{\Gamma \otimes f} & \Gamma \otimes_B N_2
 \end{array}$$

By the assumption that Γ is flat as a right B -module, we have that $\Gamma \otimes_B -$ is exact, so that in particular it preserves kernels, meaning $\Gamma \otimes_B \ker f = \ker(\Gamma \otimes_B f)$. This gives the bottom left horizontal arrow. Then by the universal property of the kernel in $B\text{-Mod}^A$ and the fact that the right square commutes, we get the vertical dashed arrow which makes the left square commute, as desired, and that $\ker f$ with this structure map is indeed the kernel of f in $\Gamma\text{-CoMod}$. Showing that this structure map makes the two diagrams in ?? commute is an exercise in diagram chasing and applying universal properties. Now, showing that the cokernel of f belongs to $\Gamma\text{-CoMod}^A$ is formally dual. Finally, it follows from construction that the comparison morphism

$$\text{coker}(\ker f) \rightarrow \ker(\text{coker } f)$$

formed in $\Gamma\text{-CoMod}^A$ is precisely the comparison morphism in $B\text{-Mod}$, which is an isomorphism, and thus clearly an isomorphism in $\Gamma\text{-CoMod}^A$ as well. Thus $\Gamma\text{-CoMod}^A$ is indeed abelian, as desired. \square

Proposition 0.10 ([nlab:introduction to the adams spectral sequence]). *Suppose that Γ is flat as a right B -module, i.e., suppose $\eta_R : B \rightarrow \Gamma$ is a flat ring homomorphism. Let P be an A -graded left Γ -comodule in $\Gamma\text{-}\mathbf{CoMod}^A$ such that the underlying A -graded B -module is a graded projective module. Then every co-free module (??) is an F -acyclic object (??) for the covariant hom functor $\text{Hom}_\Gamma(P, -)$.*

Proof. We need to show that $\text{Ext}_\Gamma^n(N, \Gamma \otimes_B M)$ vanishes for all A -graded B -modules M . First of all, let $i : M \rightarrow I^*$ be an injective resolution of M in $B\text{-}\mathbf{Mod}^A$, so we have an exact sequence of A -graded B -modules

$$0 \longrightarrow M \xrightarrow{i} I^0 \xrightarrow{d^0} I^1 \xrightarrow{d^1} I^2 \xrightarrow{d^2} I^3 \longrightarrow \dots$$

Then Γ is flat as a right B -module, the sequence remains exact after we tensor it with Γ on the left. Furthermore, it is a general categorical fact that right adjoints between abelian categories preserve injective objects. Thus $\Gamma \otimes i : \Gamma \otimes_B M \rightarrow \Gamma \otimes_B I^*$ is an injective resolution in $\Gamma\text{-}\mathbf{CoMod}^A$. Then for $n > 0$, we have

$$\text{Ext}_\Gamma^n(N, \Gamma \otimes_B M) \cong H^n(\text{Hom}_\Gamma(N, \Gamma \otimes_B I^*)) \cong H^n(\text{Hom}_B(N, I^*)) \cong 0,$$

where the first isomorphism follows by the forgetful-cofree adjunction for comodules over a Hopf algebroid (??), and the final isomorphism follows by the fact that N is a graded projective module, i.e., a projective object in the abelian category $B\text{-}\mathbf{Mod}^A$, so that $\text{Hom}_B(N, -)$ is an exact functor. \square