We will freely use the results of **??** in this section. In what follows, fix an A-graded ring R. Further suppose that for all $a, b \in A$, there exists units $\theta_{a,b} \in R_0$ such that:

- For all $a \in A$, $\theta_{a,0} = \theta_{0,a} = 1$,
- For all $a, b \in A$, $\theta_{a,b}^{-1} = \theta_{b,a}$,
- For all $a, b, c \in A$, $\theta_{a,b} \cdot \theta_{a,c} = \theta_{a,b+c}$ and $\theta_{b,a} \cdot \theta_{c,a} = \theta_{b+c,a}$.
- For all $x \in R_a$ and $y \in R_b$,
$$x \cdot y = y \cdot x \cdot \theta_{a,b}.$$

Definition 0.1. Let R-**GrCAlg** denote the following category:

- The objects are pairs (S, φ) where S is an A-graded ring and $\varphi : R \to S$ is an A-graded ring homomorphism such that for all $x \in S_a$ and $y \in S_b$, we have
$$x \cdot y = y \cdot x \cdot \varphi(\theta_{a,b}),$$
- The morphisms $(S, \varphi) \to (S', \varphi')$ are A-graded ring homomorphisms $f : S \to S'$ such that $f \circ \varphi = \varphi'$.

Lemma 0.2. *Let (S, φ) be an object in R-**GrCAlg**. Then S is canonically an A-graded R-bimodule via the action maps $(r, s) \mapsto \varphi(r) \cdot s$ and $(s, r) \mapsto s \cdot \varphi(r)$ for $r \in R$ and $s \in S$.*

Proof. □ TODO

Proposition 0.3. *Let (B, φ_B) and (C, φ_C) in R-**GrCAlg**. Then since B and C are A-graded R-bimodules (Lemma 0.2), their R-tensor product $B \otimes_R C$ is also an A-graded R-bimodule (**??**). Then $B \otimes_R C$ is canonically an element of R-**GrCAlg**, with product*
$$(B \otimes_R C) \times (B \otimes_R C) \to (B \otimes_R C)$$
sending pure homogeneous tensors $(b \otimes c, b' \otimes c')$ to the element
$$\theta_{|c|,|b'|} \cdot (bb' \otimes cc'),$$
(where here $\cdot$ denotes the left module action of R on $B \otimes_R C$), and with structure map
$$\begin{aligned} \varphi : R &\to B \otimes_R C \\ r &\mapsto \varphi_B(r) \otimes 1_C. \end{aligned}$$

Proof. First we claim the indicated map is well-defined and actually gives $B \otimes_R C$ the structure of an A-graded ring. Here we are employing **??**, so it suffices to check the map is well-defined, unital, associative, and distributive with respect to homogeneous elements. By distributivity, it further suffices to check this for pure homogeneous tensors. First, to see that this product is well-defined and distributive, it suffices to check that for all homogeneous $b \in B$ and $c \in C$ that the assignments $B \times C \to B \otimes_R C$
$$(b', c') \mapsto (b \otimes c)(b' \otimes c') = \theta_{|c|,|b'|} \cdot (bb' \otimes cc') \quad \text{and} \quad (b', c') \mapsto (b' \otimes c')(b \otimes c) = \theta_{|c'|,|b|} \cdot (b'b \otimes c'c)$$
are R-balanced, as then we get that the product $(B \otimes_R C) \times (B \otimes_R C) \to B \otimes_R C$ is an A-graded homomorphism of abelian groups in each argument, as desired. We show that the first assignment is R-balanced, showing the second is entirely analagous. To see this, note that given $b, b', b'' \in B$,

$c, c', c'' \in C$, and $r \in R$ homogeneous with $|b'| = |b''|$ and $|c'| = |c''|$, that

$$\begin{aligned}(b \otimes c)((b' + b'') \otimes c') &= \theta_{|c|,|b'+b''|} \cdot ((b(b' + b'')) \otimes cc') \\ &= \theta_{|c|,|b'|} \cdot (bb' \otimes cc') + \theta_{|c|,|b''|} \cdot (bb'' \otimes cc') \\ &= (b \otimes c)(b' \otimes c') + (b \otimes c)(b'' \otimes c'),\end{aligned}$$

where in the second equality we are using that $|b' + b''| = |b'| = |b''|$,

$$\begin{aligned}(b \otimes c)(b' \otimes (c' + c'')) &= \theta_{|c|,|b'|} \cdot (bb' \otimes c(c' + c'')) \\ &= \theta_{|c|,|b'|} \cdot (bb' \otimes cc') + \theta_{|c|,|b'|} \cdot (bb' \otimes cc'') \\ &= (b \otimes c)(b' \otimes c') + (b \otimes c)(b' \otimes c''),\end{aligned}$$

and

$$\begin{aligned}(b \otimes c)(b'r \otimes c') &= \theta_{|c|,|b'r|}(bb'r \otimes cc') \\ &\overset{(1)}{=} \theta_{|c|,|b'|} \cdot \theta_{|c|,|r|} \cdot (bb' \otimes rcc') \\ &\overset{(2)}{=} \theta_{|c|,|b'|} \cdot \theta_{|c|,|r|} \cdot (bb' \otimes cr\varphi_C(\theta_{|r|,|c|})c') \\ &\overset{(3)}{=} \theta_{|c|,|b'|} \cdot \theta_{|c|,|r|} \cdot (bb' \otimes \varphi_C(\theta_{|r|,|c|})crc') \\ &\overset{(4)}{=} \theta_{|c|,|b'|} \cdot \theta_{|c|,|r|} \cdot (bb'\varphi_B(\theta_{|r|,|c|}) \otimes crc') \\ &\overset{(3)}{=} \theta_{|c|,|b'|} \cdot \theta_{|c|,|r|} \cdot (\varphi_B(\theta_{|r|,|c|})bb' \otimes crc') \\ &\overset{(5)}{=} \theta_{|c|,|b'|} \cdot \theta_{|c|,|r|} \cdot \theta_{|r|,|c|} \cdot (bb' \otimes crc') \\ &\overset{(6)}{=} \theta_{|c|,|b'|} \cdot (bb' \otimes crc') \\ &= (b \otimes c)(b' \otimes rc'),\end{aligned}$$

where:

- (1) follows by the fact that $|b'r| = |b'| + |r|$, by definition, so that $\theta_{|c|,|b'r|} = \theta_{|c|,|b'|+|r|} = \theta_{|c|,|b'|} \cdot \theta_{|c|,|r|}$,
- (2) follows by the fact that since (C, φ_C) is an object in R-**GrCAlg** that $rcc' = cr\varphi_C(\theta_{|r|,|c|})c'$,
- Each occurrence of (3) follows by the fact that since $\theta_{|r|,|c|}$ is of degree 0, it commutes with everything, since $\theta_{|r|,|c|} \cdot x = x \cdot \theta_{|r|,|c|} \cdot \theta_{0,x} = x \cdot \theta_{|r|,|c|} \cdot 1 = x \cdot \theta_{|r|,|c|}$,
- (4) follows by definition of the tensor product,
- (5) follows by definition of the R-bimodule structure on $B \otimes_R C$, and
- (6) follows by the fact that $\theta^{-1}_{|c|,|r|} = \theta_{|r|,|c|}$.

Now, to see this product is associative, let $b, b', b'' \in B$ and $c, c', c'' \in C$ homogeneous. Then

$$\begin{aligned}((b \otimes c)(b' \otimes c'))(b'' \otimes c'') &= (\theta_{|c|,|b'|} \cdot (bb' \otimes cc'))(b'' \otimes c'') \\ &= \theta_{|cc'|,|b''|} \cdot \theta_{|c|,|b'|} \cdot (bb'b'' \otimes cc'c'') \\ &\overset{(1)}{=} \theta_{|c|,|b''|} \cdot \theta_{|c'|,|b''|} \cdot \theta_{|c|,|b'|} \cdot (bb'b'' \otimes cc'c'') \\ &\overset{(2)}{=} \theta_{|c|,|b''|+|b'|} \cdot \theta_{|c'|,|b''|} \cdot (bb'b'' \otimes cc'c'') \\ &= (b \otimes c)(\theta_{|c'|,|b''|}(b'b'' \otimes c'c'')) \\ &= (b \otimes c)((b' \otimes c')(b'' \otimes c'')),\end{aligned}$$

as desired. $\square$

Definition 0.4. An *R-ring* is a monoid object (**??**) in the category of A-graded R-bimodules. An *R-coring* is a monoid object in the opposite category of the category of A-graded R-bimodules.

Definition 0.5. A *right R-bialgebroid B* consists of an R-bimodule B with the structure of an $R \otimes R^{\mathrm{op}}$-ring (B, s, t) and an R-coring (B, Δ, ε) such that:

(i) The bimodule structure in the R-coring (B, Δ, ε) is related to the $R \otimes R^{\mathrm{op}}$ ring (B, s, t) via

$$r \cdot b \cdot r' = b \cdot s(r') \cdot t(r), \qquad \text{for } r, r' \in R,\, b \in B.$$

Definition 0.6. A *Hopf algebroid*