0.1. **Setup of $\mathcal{SH}$.** In order to construct an abstract version of the Adams spectral sequence, we need to work in some axiomatic version of a stable homotopy category $\mathcal{SH}$ which acts like the familiar classical stable homotopy category **hoSp** (**??**) or the motivic stable homotopy category $\mathbf{SH}_{\mathscr{S}}$ over some base scheme $\mathscr{S}$ (**??**).

Definition 0.1. Let $\mathcal{C}$ be an additive category with arbitrary (small) coproducts. Then an object X in $\mathcal{C}$ is *compact* if, for any collection of objects Y_i in $\mathcal{C}$ indexed by some (small) set I, the canonical map

$$\bigoplus_i \mathcal{C}(X, Y_i) \to \mathcal{C}(X, \bigoplus_i Y_i)$$

is an isomorphism of abelian groups. (Explicitly, the above map takes a generator $x \in \mathcal{C}(X, Y_i)$ to the composition $X \xrightarrow{x} Y_i \hookrightarrow \bigoplus_i Y_i$.)

Definition 0.2. Given a tensor triangulated category $(\mathcal{C}, \otimes, S, \Sigma, e, \mathcal{D})$ (**??**), a *sub-Picard grading* on $\mathcal{C}$ is the following data:

- A pointed abelian group $(A, \mathbf{1})$ along with a homomorphism of pointed groups $h : (A, \mathbf{1}) \to (\operatorname{Pic}\mathcal{C}, \Sigma S)$, where $\operatorname{Pic}\mathcal{C}$ is the *Picard group* of isomorphism classes of invertible objects in $\mathcal{C}$.[1]
- For each $a \in A$, a chosen representative S^a called the *a-sphere* in the isomorphism class $h(a)$ such that each S^a is a com pact object (Definition 0.1) and $S^0 = S$.
- For each $a, b \in A$, an isomorphism $\phi_{a,b} : S^{a+b} \to S^a \otimes S^b$. This family of isomorphisms is required to be *coherent*, in the following sense:
  - For all $a \in A$, we must have that $\phi_{a,0}$ coincides with the right unitor $\rho_{S^a}^{-1} : S^a \xrightarrow{\cong} S^a \otimes S$ and $\phi_{0,a}$ coincides the left unitor $\lambda_{S^a}^{-1} : S^a \xrightarrow{\cong} S \otimes S^a$.
  - For all $a, b, c \in A$, the following "associativity diagram" must commute:

$$\begin{tikzcd} S^{a+b} \otimes S^c \arrow[d, "\phi_{a,b} \otimes S^c"'] & S^{a+b+c} \arrow[l, "\phi_{a+b,c}"'] \arrow[r, "\phi_{a,b+c}"] & S^a \otimes S^{b+c} \arrow[d, "S^a \otimes \phi_{b,c}"] \\ (S^a \otimes S^b) \otimes S^c \arrow[rr, "\cong"] & & S^a \otimes (S^b \otimes S^c) \end{tikzcd}$$

From now on we fix a monoidal closed tensor triangulated category $(\mathcal{SH}, \otimes, S, \Sigma, e, \mathcal{D})$ with arbitrary (small) (co)products and sub-Picard grading $(A, \mathbf{1}, h, \{S^a\}, \{\phi_{a,b}\})$. We also fix an isomorphism $\nu : \Sigma S \xrightarrow{\cong} S^{\mathbf{1}}$ once and for all. We establish conventions. First, observe the following remark:

Remark 0.3. Note that by induction the coherence conditions for the $\phi_{a,b}$'s in the above definition say that given any $a_1, \ldots, a_n \in A$ and $b_1, \ldots, b_m \in A$ such that $a_1 + \cdots + a_n = b_1 + \cdots + b_m$ and any fixed parenthesizations of $X = S^{a_1} \otimes \cdots \otimes S^{a_b}$ and $Y = S^{b_1} \otimes \cdots \otimes S^{b_m}$, there is a *unique* isomorphism $X \to Y$ that can be obtained by forming formal compositions of products of $\phi_{a,b}$, identities, associators, unitors, and their inverses (but not symmetries).

In light of this remark, we will usually simply write ϕ or even just $\cong$ for any isomorphism that is built by taking compositions of products of $\phi_{a,b}$'s, unitors, associators, identities, and their

[1] Recall an object X is a symmetric monoidal category is *invertible* if there exists some object Y and an isomorphism $S \cong X \otimes Y$.

inverses. Given an object X and a natural number $n > 0$, we write

$$X^n := \overbrace{X \otimes \cdots \otimes X}^{n \text{ times}} \qquad \text{and} \qquad X^0 := S.$$

We denote the associator, symmetry, left unitor, and right unitor isomorphisms in $\mathcal{SH}$ by

$$\alpha_{X,Y,Z} : (X \otimes Y) \otimes Z \xrightarrow{\cong} X \otimes (Y \otimes Z) \qquad \tau_{X,Y} : X \otimes Y \xrightarrow{\cong} Y \otimes X$$
$$\lambda_X : S \otimes X \xrightarrow{\cong} X \qquad \rho_X : X \otimes S \xrightarrow{\cong} X.$$

Often we will drop the subscripts. Furthermore, by the coherence theorem for symmetric monoidal categories ([1]), we will often assume α, ρ, and λ are actual equalities.

Given some integer $n \in \mathbb{Z}$, we will write a bold $\mathbf{n}$ to denote the element $n \cdot \mathbf{1}$ in A. Note that we can use the isomorphism $\nu : \Sigma S \xrightarrow{\cong} S^{\mathbf{1}}$ to construct a natural isomorphism $\Sigma \cong S^{\mathbf{1}} \otimes -$:

$$\Sigma X \xrightarrow{\Sigma\lambda_X^{-1}} \Sigma(S \otimes X) \xrightarrow{e_{S,X}^{-1}} \Sigma S \otimes X \xrightarrow{\nu \otimes X} S^{\mathbf{1}} \otimes X.$$

The first two arrows are natural in X by definition. The last arrow is natural in X by functoriality of $- \otimes -$. By abuse of notation, we will also use ν to denote this natural isomorphism.

Given some $a \in A$, we define $\Sigma^a := S^a \otimes -$ and $\Omega^a := \Sigma^{-a} = S^{-a} \otimes -$. We specifically define $\Omega := \Omega^{\mathbf{1}}$. We say "the a^{th} suspension of X" to denote $\Sigma^a X$. It turns out that Σ^a is an autoequivalence of $\mathcal{SH}$ for each $a \in A$, and furthermore, Ω^a and Σ^a form an adjoint equivalence of $\mathcal{SH}$ for all a in A:

Proposition 0.4. *For each $a \in A$, the isomorphisms*

$$\eta_X^a : X \xrightarrow{\lambda_X^{-1}} S \otimes X \xrightarrow{\phi_{a,-a} \otimes X} (S^a \otimes S^{-a}) \otimes X \xrightarrow{\alpha} S^a \otimes (S^{-a} \otimes X) = \Sigma^a \Omega^a X$$

and

$$\varepsilon_X^a : \Omega^a \Sigma^a X = S^{-a} \otimes (S^a \otimes X) \xrightarrow{\alpha^{-1}} (S^{-a} \otimes S^a) \otimes X \xrightarrow{\phi_{-a,a}^{-1} \otimes X} S \otimes X \xrightarrow{\lambda_X} X$$

are natural in X, and furthermore, they are the unit and counit respectively of the adjoint autoequivalence $(\Omega^a, \Sigma^a, \eta^a, \varepsilon^a)$ of $\mathcal{SH}$. In particular, since $\Sigma \cong \Sigma^{\mathbf{1}}$, $\Omega := \Omega^{\mathbf{1}}$ is a left adjoint for Σ, so that $(\mathcal{SH}, \Omega, \Sigma, \eta, \varepsilon, \mathcal{D})$ is an adjointly *triangulated category (**??**), where η and ε are the compositions*

$$\eta : \mathrm{Id}_{\mathcal{SH}} \xRightarrow{\eta^{\mathbf{1}}} \Sigma^{\mathbf{1}}\Omega \xRightarrow{\nu^{-1}\Omega} \Sigma\Omega \qquad \textit{and} \qquad \varepsilon : \Omega\Sigma \xRightarrow{\Omega\nu} \Omega\Sigma^{\mathbf{1}} \xRightarrow{\varepsilon^{\mathbf{1}}} \mathrm{Id}_{\mathcal{SH}}.$$

Proof. In this proof, we will freely employ the coherence theorem for monoidal categories (see [1]), which essentially tells us that we may assume we are working in a strict monoidal category (i.e., that the associators and unitors and are identities). Then η_X^a and ε_X^a become simply the maps

$$\eta_X^a : X \xrightarrow{\phi_{a,-a} \otimes X} S^a \otimes S^{-a} \otimes X \qquad \text{and} \qquad \varepsilon_X^a : S^{-a} \otimes S^a \otimes X \xrightarrow{\phi_{-a,a}^{-1} \otimes X} X.$$

That these maps are natural in X follows by functoriality of $- \otimes -$. Now, recall that in order to show that these natural isomorphisms form an *adjoint* equivalence, it suffices to show that the natural isomorphisms $\eta^a : \mathrm{Id}_{\mathcal{SH}} \Rightarrow \Omega^a \Sigma^a$ and $\varepsilon^a : \Sigma^a \Omega^a \Rightarrow \mathrm{Id}_{\mathcal{SH}}$ satisfy one of the two zig-zag identities:

$$\begin{tikzcd} \Omega^a \arrow[r, "\Omega^a \eta^a"] \arrow[dr, equal] & \Omega^a \Sigma^a \Omega^a \arrow[d, "\varepsilon^a \Omega^a"] \\ & \Omega^a \end{tikzcd} \qquad \begin{tikzcd} \Sigma^a \Omega^a \Sigma^a \arrow[d, "\Sigma^a \varepsilon^a"'] & \Sigma^a \arrow[l, "\eta^a \Sigma^a"'] \arrow[dl, equal] \\ \Sigma^a & \end{tikzcd}$$

(that it suffices to show only one is [2, Lemma 3.2]). We will show that the left is satisfied. Unravelling definitions, we simply wish to show that the following diagram commutes for all X in $\mathcal{SH}$:

$$\begin{tikzcd} S^{-a}\otimes X \arrow[r, "S^{-a}\otimes\phi_{a,-a}\otimes X"] \arrow[dr, equal] & S^{-a}\otimes S^{a}\otimes S^{-a}\otimes X \arrow[d, "\phi^{-1}_{-a,a}\otimes S^{-a}\otimes X"] \\ & S^{-a}\otimes X \end{tikzcd}$$

Yet this is simply the diagram obtained by applying $-\otimes X$ to the associativity coherence diagram for the $\phi_{a,b}$'s (since $\phi_{a,0}$ and $\phi_{0,a}$ coincide with the unitors, and here we are taking the unitors and associators to be equalities), so it does commute, as desired. $\square$

Given two objects X and Y in $\mathcal{SH}$, we will denote the hom-abelian group of morphisms from X to Y in $\mathcal{SH}$ by $[X,Y]$, and the internal hom object by $F(X,Y)$. We can extend the abelian group $[X,Y]$ into an A-graded abelian group $[X,Y]_*$ by defining $[X,Y]_a := [S^a\otimes X, Y]$. Given an object X in $\mathcal{SH}$ and some $a\in A$, we can define the abelian group

$$\pi_a(X) := [S^a, X],$$

which we call the a^{th} *(stable) homotopy group of* X. We write $\pi_*(X)$ for the A-graded abelian group $\bigoplus_{a\in A}\pi_a(X)$, so that in particular we have a canonical isomorphism

$$\pi_*(X) = [S^*, X] \cong [S,X]_*.$$

Given some other object E, we can define the A-graded abelian groups $E_*(X)$ and $E^*(X)$ by the formulas

$$E_a(X) := \pi_a(E\otimes X) = [S^a, E\otimes X] \qquad \text{and} \qquad E^a(X) := [X, S^a\otimes E].$$

We refer to the functor $E_*(-)$ as the *homology theory represented by* E, or just E-homology, and we refer to $E^*(-)$ as the *cohomology theory represented by* E, or just E-cohomology.

0.2. **The long exact sequence associated to a distinguished triangle.**

Proposition 0.5. *Suppose we are given a distinguished triangle*

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$$

and some object W in $\mathcal{SH}$. Then there is an infinite long exact sequence of A-graded abelian groups:

$$\cdots \to [W,Z]_{*+\mathbf{n+1}} \xrightarrow{\partial} [W,X]_{*+\mathbf{n}} \xrightarrow{f_*} [W,Y]_{*+\mathbf{n}} \xrightarrow{g_*} [W,Z]_{*+\mathbf{n}} \xrightarrow{\partial} [W,Z]_{*+\mathbf{n-1}} \to \cdots,$$

where $\partial : [W,Z]_{*+\mathbf{n+1}} \to [W,X]_{*+\mathbf{n}}$ sends a class $x : S^{a+\mathbf{n+1}}\otimes W \to Z$ to the composition

$$S^{a+\mathbf{n}}\otimes W \xrightarrow{\phi_{-\mathbf{1},a+\mathbf{n+1}}} S^{-\mathbf{1}}\otimes S^{a+\mathbf{n+1}}\otimes W \xrightarrow{S^{-\mathbf{1}}\otimes x} S^{-\mathbf{1}}\otimes Z \xrightarrow{\widetilde{h}} X,$$

where here we are suppressing the associator from the notation, and $\widetilde{h} : \Omega Z = S^{-\mathbf{1}}\otimes Z \to X$ is the adjoint (Proposition 0.4) of $h : Z\to \Sigma X$.

Proof. In this proof, we will freely employ the coherence theorem for a symmetric monoidal category, which tells us we may assume associativity and unitality of $-\otimes-$ holds up to strict equality. Furthermore, we will simply write ϕ to refer to any isomorphism that can be constructed by composing copies of products of $\phi_{a,b}$'s, unitors, identities, associators, and their inverses (see Remark 0.3). Finally, given $n>0$, we will write Σ^{-n} to denote the functor $\Omega^n = (S^{-\mathbf{1}})^n\otimes -$.

For all $n>0$, the $\phi_{a,b}$'s yield natural isomorphisms

$$s_X^{-n} : \Sigma^{-n}X = (S^{-\mathbf{1}})^n\otimes X \xrightarrow{\phi\otimes X} S^{-\mathbf{n}}\otimes X = \Omega^{\mathbf{n}}X.$$

and

$$s^n_X : \Sigma^n X \xrightarrow{\nu^n_X} (S^{\mathbf{1}})^n \otimes X \xrightarrow{\phi\otimes X} S^{\mathbf{n}} \otimes X = \Sigma^{\mathbf{n}} X,$$

where we recursively define $\nu^1 := \nu$ and ν^{n+1} is given by the composition

$$\nu^{n+1}_X : \Sigma^{n+1} X = \Sigma^n \Sigma X \xrightarrow{\nu^n_{\Sigma X}} (S^{\mathbf{1}})^n \otimes \Sigma X \xrightarrow{(S^{\mathbf{1}})^n \otimes \nu_X} (S^{\mathbf{1}})^n \otimes S^{\mathbf{1}} \otimes X = (S^{\mathbf{1}})^{n+1} \otimes X.$$

Finally, we define s^0 to be the identity natural transformation on $\mathcal{SH}$. Then we get the following natural isomorphisms of A-graded abelian groups for all $n \in \mathbb{Z}$

$$\ell^n_V : [W, \Sigma^n V]_* \xrightarrow{(s^n_V)_*} [W, \Sigma^{\mathbf{n}} V]_* \xrightarrow{r^{\mathbf{n}}_{W,V}} [W, V]_{*-\mathbf{n}},$$

where $r^{\mathbf{n}}_{W,V}$ is the natural isomorphism given as the composition

$$[W, \Sigma^{\mathbf{n}} V]_* \xrightarrow{\cong} [S^{-\mathbf{n}} \otimes S^* \otimes W, V] \xrightarrow{(\phi\otimes W)^*} [S^{*-\mathbf{n}} \otimes W, V] = [W, V]_{*-\mathbf{n}},$$

where the first isomorphism is the adjunction $\Omega^{\mathbf{n}} \dashv \Sigma^{\mathbf{n}}$ (Proposition 0.4). Now, given $n \in \mathbb{Z}$, consider the following diagram

$$\begin{tikzcd}
{[W, \Sigma^{n-1} Z]_*} \arrow[r, "h_{n-1}"] \arrow[d, "\ell^{n-1}_Z"'] & {[W, \Sigma^n X]_*} \arrow[r, "\Sigma^n f_*"] \arrow[d, "\ell^n_X"'] & {[W, \Sigma^n Y]_*} \arrow[r, "\Sigma^n g_*"] \arrow[d, "\ell^n_Y"'] & {[W, \Sigma^n Z]_*} \arrow[r, "h_n"] \arrow[d, "\ell^n_Z"'] & {[W, \Sigma^{n+1} X]_*} \arrow[d, "\ell^{n+1}_X"] \\
{[W, Z]_{*-\mathbf{n}+\mathbf{1}}} \arrow[r, "\partial"'] & {[W, X]_{*-\mathbf{n}}} \arrow[r, "f_*"'] & {[W, Y]_{*-\mathbf{n}}} \arrow[r, "g_*"'] & {[W, Z]_{*-\mathbf{n}}} \arrow[r, "\partial"'] & {[W, X]_{*-\mathbf{n}-\mathbf{1}}}
\end{tikzcd} \tag{1}$$

where for $n \geq 0$, $h_n = \Sigma^n h$, and for $n > 0$, $h_{-n} = \Omega^{n-1}\widetilde{h}$ (where $\widetilde{h} : \Omega Z \to X$ is the adjoint of $h : Z \to \Sigma X$). We would like to show the bottom row is exact. The top row is exact since it is obtained by applying $[W, -]_*$ to a fiber sequence (see **??** for full details), and we have constructed the vertical arrows to be isomorphisms. Thus it suffices to show each square commutes. The inner two squares commute by naturality of ℓ^n. Thus, it further suffices to show the outermost squares commute. Since our choice of $n \in \mathbb{Z}$ is arbitrary, we can just show the right square commutes. First consider the case that $n \geq 0$, and consider the following diagram:

$$\begin{tikzcd}
{[W, \Sigma^n Z]_*} \arrow[rrr, "\Sigma^n h_*"] \arrow[dddd, "\ell^n_Z"'] & & & {[W, \Sigma^{n+1} X]_*} \arrow[dddd, "\ell^{n+1}_X"] \arrow[dll, "\ell^n_{\Sigma X}"] \\
& {[W, \Sigma X]_{*-\mathbf{n}}} \arrow[dr, "(\nu_X)_*"] & & \\
& & {[W, \Sigma^{\mathbf{1}} X]_{*-\mathbf{n}}} \arrow[ddr, "r^{\mathbf{1}}_{W,X}"] & \\
& & & \\
{[W, Z]_{*-\mathbf{n}}} \arrow[uuur, "h_*"] \arrow[rrr, "\partial"'] & & & {[W, X]_{*-\mathbf{n}-\mathbf{1}}}
\end{tikzcd}$$

The leftmost region commutes by naturality of ℓ. By unravelling how $r^{\mathbf{1}}_{W,X}$ and the adjoint $\widetilde{h}$ used in the definition of ∂ are defined, a simple diagram chase yields that the bottom triangle commutes. Thus, it remains to show the rightmost triangle in the above diagram commutes. To see this, note that by unravelling how ℓ and r are defined, this triangle becomes

$$\begin{tikzcd}
{[W,(S^{\mathbf{1}})^n\otimes\Sigma X]} \arrow[d, "(\phi\otimes\Sigma X)_*"'] \arrow[drr, "((S^{\mathbf{1}})^n\otimes\nu_X)_*"'] & & {[W,\Sigma^{n+1}X]_*} \arrow[ll, "(\nu^n_{\Sigma X})_*"'] \arrow[d, "(\nu_X^{n+1})_*"] \\
& & {[W,(S^{\mathbf{1}})^{n+1}\otimes X]_*} \arrow[dl, "(\phi\otimes X)_*"'] \arrow[d, "(\phi\otimes X)_*"] \\
{[W,\Sigma^{\mathbf{n}}\Sigma X]_*} \arrow[r, "(\Sigma^{\mathbf{n}}\nu_X)_*"] \arrow[d, "\mathrm{adj}"'] & {[W,\Sigma^{\mathbf{n}}\Sigma^{\mathbf{1}}X]_*} \arrow[r, "(\phi\otimes X)_*"'] \arrow[d, "\mathrm{adj}"] & {[W,\Sigma^{\mathbf{n+1}}X]_*} \arrow[d, "\mathrm{adj}"] \\
{[S^{-\mathbf{n}}\otimes S^*\otimes W,\Sigma X]} \arrow[r, "(\nu_X)_*"] \arrow[d, "(\phi\otimes W)^*"'] & {[S^{-\mathbf{n}}\otimes S^*\otimes W,S^{\mathbf{1}}\otimes X]} \arrow[ddl, "(\phi\otimes W)^*"'] & {[S^{-\mathbf{n-1}}\otimes S^*\otimes W,X]} \arrow[dd, "(\phi\otimes W)^*"] \\
{[W,\Sigma X]_{*-\mathbf{n}}} \arrow[d, "(\nu_X)_*"'] & & \\
{[W,\Sigma^{\mathbf{1}}X]_{*-\mathbf{n}}} \arrow[r, "\mathrm{adj}"'] & {[S^{-\mathbf{1}}\otimes S^{*-\mathbf{n}}\otimes W,X]} \arrow[r, "(\phi\otimes W)^*"'] \arrow[uur, "(\phi\otimes W)^*"] & {[W,X]_{*-\mathbf{n-1}}}
\end{tikzcd}$$

The top right triangle commutes by how ν^{n+1} was defined. The top left oddly-shaped region commutes by functoriality of $-\otimes-$. The middle right triangle commutes by coherence for the ϕ's. The middle left rectangle commutes by naturality of the adjunction isomorphism. Commutativity of the bottom left triangle is clear (do a diagram chase). Commutativity of the bottom right triangle is coherence for the ϕ's. Finally, commutativity of the remaining region is again coherence of the ϕ's, since the adjunction isomorphisms are constructed using them (Proposition 0.4).

Now we consider the negative case: Unravelling definitions, given $n>0$, the rightmost square in diagram (1) for $-n$ becomes

$$\begin{tikzcd}
{[W,\Omega^n Z]_*} \arrow[rr, "\Omega^{n-1}\widetilde{h}_*"] \arrow[d, "(\phi\otimes Z)_*"'] \arrow[dr, "(\phi\otimes\Omega Z)_*"] & & {[W,\Omega^{n-1}X]_*} \arrow[d, "(\phi\otimes X)_*"] \\
{[W,\Omega^{\mathbf{n}}Z]_*} \arrow[r, "(\phi\otimes Z)_*"'] \arrow[dddd, "\mathrm{adj}"'] & {[W,\Omega^{\mathbf{n-1}}\Omega Z]_*} \arrow[r, "\Omega^{\mathbf{n-1}}\widetilde{h}_*"] \arrow[d, "\mathrm{adj}"'] & {[W,\Omega^{\mathbf{n-1}}X]_*} \arrow[d, "\mathrm{adj}"] \\
& {[S^{\mathbf{n-1}}\otimes S^*\otimes W,\Omega Z]} \arrow[r, "\widetilde{h}_*"] \arrow[d, "(\phi\otimes W)^*"'] & {[S^{\mathbf{n-1}}\otimes S^*\otimes W,X]} \arrow[ddd, "(\phi\otimes W)^*"] \\
& {[W,\Omega Z]_{*+\mathbf{n-1}}} \arrow[d, "\mathrm{adj}"'] \arrow[ddr, "\widetilde{h}_*"] & \\
{[S^{\mathbf{n}}\otimes S^*\otimes W,Z]} \arrow[r, "(\phi\otimes W)^*"] \arrow[d, "(\phi\otimes W)^*"'] & {[S^{\mathbf{1}}\otimes S^{*+\mathbf{n-1}}\otimes W,Z]} \arrow[dl, "(\phi\otimes W)^*"] & \\
{[W,Z]_{*+\mathbf{n}}} \arrow[rr, "\partial"'] & & {[W,X]_{*+\mathbf{n-1}}}
\end{tikzcd} \tag{2}$$

The top right trapezoid commutes by functoriality of $-\otimes-$. The top left triangle commutes by coherence for the ϕ's. The middle right rectangle commutes by naturality of the adjunction. The right trapezoid below that commutes obviously. The bottom left triangle commutes by coherence of the ϕ's. The large middle left rectangle commutes by coherence for the ϕ's, again since the adjunction $\Sigma^{\mathbf{n}}\dashv\Omega^{\mathbf{n}}$ is constructed using the ϕ's. Finally, to see the bottom diagram commutes, we will chase some homogeneous element $f : S^{b+\mathbf{n}-\mathbf{1}} \otimes W \to \Omega Z$ around the region. Consider the following diagram:

$$\begin{tikzcd}
S^{-\mathbf{1}} \otimes S^{b+\mathbf{n}} \otimes W \arrow[d, "\phi\otimes W"'] & S^{b+\mathbf{n}-\mathbf{1}} \otimes W \arrow[l, "\phi\otimes W"'] \arrow[dd, "f"] & & & \\
S^{-\mathbf{1}} \otimes S^{\mathbf{1}} \otimes S^{b+\mathbf{n}+\mathbf{1}} \otimes W \arrow[ur, "\phi\otimes W"] \arrow[d, "S^{-\mathbf{1}}\otimes S^{\mathbf{1}}\otimes f"'] & & & & \\
S^{-\mathbf{1}} \otimes S^{\mathbf{1}} \otimes S^{-\mathbf{1}} \otimes Z \arrow[r, "\phi\otimes Z"'] & S^{-1} \otimes Z \arrow[r, "S^{-\mathbf{1}}\otimes h"] & S^{-\mathbf{1}} \otimes \Sigma X \arrow[r, "S^{-\mathbf{1}}\otimes \nu_X"] & S^{-\mathbf{1}} \otimes S^{\mathbf{1}} \otimes X \arrow[r, "\phi\otimes X"] & X
\end{tikzcd}$$

By unravelling how the adjunction and $\widetilde{h}$ are defined, the two compositions around the outside of this diagram are the two morphisms obtained by chasing f around the bottom region in diagram (2). The top left triangle of the above diagram commutes by coherence of the ϕ's (Remark 0.3), while the bottom region commutes by functoriality of $-\otimes-$ and coherence of the ϕ's. Thus we've shown diagram (1) commutes, so the bottom row is exact, as desired. $\square$

Remark 0.6. Expressed more compactly, the above proposition says that for each object W and distinguished triangle

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$$

in $\mathcal{SH}$ gives rise to the following diagram of A-graded abelian groups

$$\begin{tikzcd}
{[W,X]_*} \arrow[r, "f_*"] & {[W,Y]_*} \arrow[d, "g_*"] \\
& {[W,Z]_*} \arrow[ul, "\partial"]
\end{tikzcd}$$

which is exact at each vertex, and where f_*, g_*, and ∂ are A-graded homomorphisms of degree 0, 0, and $-\mathbf{1}$, respectively. Explicitly, ∂ sends a class $x : S^a \otimes W \to Z$ to the composition

$$S^{a-\mathbf{1}} \otimes W \cong S^{-\mathbf{1}} \otimes S^a \otimes W \xrightarrow{S^{-\mathbf{1}}\otimes x} S^{-\mathbf{1}} \otimes Z \xrightarrow{S^{-\mathbf{1}}\otimes h} S^{-\mathbf{1}} \otimes \Sigma X \xrightarrow{S^{-\mathbf{1}}\otimes \nu_X} S^{-\mathbf{1}} \otimes S^{\mathbf{1}} \otimes X \xrightarrow{\phi^{-1}_{-\mathbf{1},\mathbf{1}}\otimes X} X.$$

0.3. **Cellular objects in $\mathcal{SH}$.** One very important class of objects in $\mathcal{SH}$ are the *cellular* objects. Intuitively, these are the objects that can be built out of spheres via taking coproducts and (co)fibers.

Definition 0.7. Define the class of *cellular* objects in $\mathcal{SH}$ to be the smallest class of objects such that:

(1) For all $a \in A$, the a-sphere S^a is cellular.

(2) If we have a distinguished triangle

$$X \to Y \to Z \to \Sigma X$$

such that two of the three objects X, Y, and Z are cellular, than the third object is also cellular.

(3) Given a collection of cellular objects X_i indexed by some (small) set I, the object $\bigoplus_{i\in I} X_i$ is cellular (recall we have chosen $\mathcal{SH}$ to have arbitrary coproducts).

Lemma 0.8. *Let X and Y be two isomorphic objects in $\mathcal{SH}$. Then X is cellular iff Y is cellular.*

Proof. Assume we have an isomorphism $f : X \xrightarrow{\cong} Y$ and that X is cellular. Then consider the following commutative diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, equal] & Y \arrow[r] \arrow[d, "f^{-1}"] & 0 \arrow[r] \arrow[d, equal] & \Sigma X \arrow[d, equal] \\ X \arrow[r, equal] & X \arrow[r] & 0 \arrow[r] & \Sigma X \end{tikzcd}$$

The bottom row is distinguished by axiom TR1 for a triangulated category. Hence since X is cellular, 0 is also cellular, since the class of cellular objects satisfies two-of-three for distinguished triangles. Furthermore, since the vertical arrows are all isomorphisms, the top row is distinguished as well, by axiom TR0. Thus again by two-of-three, since X and 0 are cellular, so is Y, as desired. $\square$

Lemma 0.9. *Let X and Y be cellular objects in $\mathcal{SH}$. Then $X \otimes Y$ is cellular.*

Proof. Let E be a cellular object in $\mathcal{SH}$, and let $\mathcal{E}$ be the collection of objects X in $\mathcal{SH}$ such that $E \otimes X$ is cellular. First of all, suppose we have a distinguished triangle

$$X \to Y \to Z \to \Sigma X$$

such that two of three of X, Y, and Z belong to $\mathcal{E}$. Then since $\mathcal{SH}$ is tensor triangulated, we have a distinguished triangle

$$E \otimes X \to E \otimes Y \to E \otimes Z \to \Sigma(E \otimes X).$$

Per our assumptions, two of three of $E \otimes X$, $E \otimes Y$, and $E \otimes Z$ are cellular, so that the third is by definition. Thus, all three of X, Y, and Z belong to $\mathcal{E}$ if two of them do.

Second of all, suppose we have a family X_i of objects in $\mathcal{E}$ indexed by some (small) set I, and set $X := \bigoplus_i X_i$. Then we'd like to show X belongs to $\mathcal{E}$, i.e., that $E \otimes X$ is cellular. Indeed,

$$E \otimes X = E \otimes \left(\bigoplus_i X_i \right) \cong \bigoplus_i (E \otimes X_i),$$

where the isomorphism is given by the fact that $\mathcal{SH}$ is monoidal closed, so $E \otimes -$ preserves arbitrary colimits as it is a left adjoint. Per our assumption, since each $E \otimes X_i$ is cellular, the rightmost object is cellular, since the class of cellular objects is closed under taking arbitrary coproducts, by definition. Hence $E \otimes X$ is cellular by Lemma 0.8.

Finally, we would like to show that each S^a belongs to $\mathcal{E}$, i.e., that $S^a \otimes E$ is cellular for all $a \in A$. When $E = S^b$ for some $b \in A$, this is clearly true, since $S^b \otimes S^a \cong S^{a+b}$, which is cellular by definition, so that $S^b \otimes S^a$ is cellular by Lemma 0.8. Thus by what we have shown, the class of objects X for which $S^a \otimes X$ is cellular contains every cellular object. Hence in particular $E \otimes S^a \cong S^a \otimes E$ is cellular for all $a \in A$, as desired. $\square$

Lemma 0.10. *Let W be a cellular object in $\mathcal{SH}$ such that $\pi_*(W) = 0$. Then $W \cong 0$.*

Proof. Let $\mathcal{E}$ be the collection of all X in $\mathcal{SH}$ such that $[\Sigma^n X, W] = 0$ for all $n \in \mathbb{Z}$ (where for $n > 0$, $\Sigma^{-n} := \Omega^n = (S^{-\mathbf{1}})^n \otimes -$). We claim $\mathcal{E}$ contains every cellular object in $\mathcal{SH}$. First of all, each S^a belongs to $\mathcal{E}$, as

$$[\Sigma^n S^a, W] \cong [S^{\mathbf{n}} \otimes S^a, W] \cong [S^{a+\mathbf{n}}, W] \leq \pi_*(W) = 0.$$

Furthermore, suppose we are given a distinguished triangle

$$X \to Y \to Z \to \Sigma X$$

such that two of three of X, Y, and Z belong to $\mathcal{E}$. By **??**, for all $n \in \mathbb{Z}$ we get an exact sequence

$$[\Sigma^{n+1}X, W] \to [\Sigma^n Z, W] \to [\Sigma^n Y, W] \to [\Sigma^n X, W] \to [\Sigma^{n-1}Z, W].$$

Clearly if any two of three of X, Y, and Z belong to $\mathcal{E}$, then by exactness of the above sequence all three of the middle terms will be zero, so that the third object will belong to $\mathcal{E}$ as well. Finally, suppose we have a collection of objects X_i in $\mathcal{E}$ indexed by some small set I. Then

$$\left[\Sigma^n \bigoplus_i X_i, W\right] \cong \left[\bigoplus_i \Sigma^n X_i, W\right] \cong \prod_i [\Sigma^n X_i, W] = \prod_i 0 = 0,$$

where the first isomorphism follows by the fact that Σ^n is apart of an adjoint equivalence (Proposition 0.4), so it preserves arbitrary colimits.

Thus, by definition of cellularity, $\mathcal{E}$ contains every cellular object. In particular, $\mathcal{E}$ contains W, so that $[W, W] = 0$, meaning in particular that $\mathrm{id}_W = 0$, so we have a commutative diagram

$$\begin{tikzcd} & 0 \arrow[rr, equal] \arrow[dr] & & 0 \\ W \arrow[ur] \arrow[rr, equal] & & W \arrow[ur] & \end{tikzcd}$$

Hence the diagonals exhibit isomorphisms between 0 and W, as desired. ☐

Theorem 0.11. *Let X and Y be cellular objects in $\mathcal{SH}$, and suppose $f : X \to Y$ is a morphism such that $f_* : \pi_*(X) \to \pi_*(Y)$ is an isomorphism. Then f is an isomorphism.*

Proof. By axiom TR2 for a triangulated category (**??**), we have a distinguished triangle

$$X \xrightarrow{f} Y \xrightarrow{g} C_f \xrightarrow{h} \Sigma X.$$

First of all, note that by definition since X and Y are cellular, so is C_f. Now, we claim $\pi_*(C_f) = 0$. Indeed, given $a \in A$, by **??** we have the following exact sequence:

$$[S^a, X] \xrightarrow{f_*} [S^a, Y] \xrightarrow{g_*} [S^a, C_f] \xrightarrow{h_*} [S^a, \Sigma X] \xrightarrow{-(\Sigma f)_*} [S^a, \Sigma Y],$$

where the first arrow is an isomorphism, per our assumption that f_* is an isomorphism. To see the last arrow is an isomorphism, consider the following diagram:

$$\begin{tikzcd} {[S^a, \Sigma X]} \arrow[r, "(\Sigma f)_*"] \arrow[d, "(\nu_X)_*"'] & {[S^a, \Sigma Y]} \arrow[d, "(\nu_Y)_*"] \\ {[S^a, S^{\mathbf{1}} \otimes X]} \arrow[r, "(S^{\mathbf{1}} \otimes f)_*"] \arrow[d, "\cong"'] & {[S^a, S^{\mathbf{1}} \otimes Y]} \arrow[d, "\cong"] \\ {[S^{-\mathbf{1}} \otimes S^a, X]} \arrow[r, "f_*"] \arrow[d, "(\phi_{-\mathbf{1},a})^*"'] & {[S^{-\mathbf{1}} \otimes S^a, Y]} \arrow[d, "(\phi_{-\mathbf{1},a})^*"] \\ {[S^{a-\mathbf{1}}, X]} \arrow[r, "f_*"] & {[S^{a-\mathbf{1}}, Y]} \end{tikzcd}$$

where the middle vertical arrows are the adjunction natural isomorphisms specified by Proposition 0.4. The bottom arrow is an isomorphism per our assumptions, so the top arrow is likewise an isomorphism, as desired. Thus $\operatorname{im} h_* = \ker -(\Sigma f)_* = 0$, and $\ker g_* = \operatorname{im} f_* = [S^a, Y]$, so that $\ker h_* = \operatorname{im} g_* = 0$. It is only possible that $\ker h_* = \operatorname{im} h_* = 0$ if $[S^a, C_f] = 0$. Thus, we have shown $\pi_*(C_f) = 0$, and C_f is cellular, so by Lemma 0.10 there is an isomorphism $C_f \cong 0$. Now consider the following commuting diagram:

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "f"] & Y \arrow[r] \arrow[d, equal] & C_f \arrow[r] \arrow[d, "\cong"] & \Sigma X \arrow[d, "\Sigma f"] \\
Y \arrow[r, equal] & Y \arrow[r] & 0 \arrow[r] & \Sigma Y
\end{tikzcd}$$

The top row is distinguished by assumption. The bottom row is distinguished by axiom TR2. Then since the middle two vertical arrows are isomorphisms, by **??**, f is an isomorphism as well, as desired. □

Lemma 0.12. *Let $e : X \to X$ be an idempotent morphism in $\mathcal{SH}$, so $e \circ e = e$. Then since $\mathcal{SH}$ is a triangulated category with arbitrary coproducts, this idempotent splits (**??**), meaning e factors as*

$$X \xrightarrow{r} Y \xrightarrow{\iota} X$$

for some object Y and morphisms r and ι with $r \circ \iota = \mathrm{id}_Y$. Then Y is cellular if X is.

Proof. It is a general categorical fact that the splitting of an idempotent, if it exists, is unique up to unique isomorphism,[2] so by Lemma 0.8, it suffices to show that e has some cellular splitting. In **??**, it is shown that we may take Y to be the homotopy colimit (**??**) of the sequence

$$X \xrightarrow{e} X \xrightarrow{e} X \xrightarrow{e} X \xrightarrow{e} \cdots,$$

so there is a distinguished triangle

$$\bigoplus_{i=0}^{\infty} X \to \bigoplus_{i=0}^{\infty} X \to Y \to \Sigma\left(\bigoplus_{i=0}^{\infty} X\right).$$

Since X is cellular, by definition $\bigoplus_{i=0}^{\infty} X$ is as well. Thus by 2-of-3 for distinguished triangles for cellular objects, Y is cellular as desired. □

0.4. **Monoid objects in $\mathcal{SH}$.** Many of the proofs in this section are quite technical and not very euclidiating, so we direct the reader to the appendix for most proofs. To start with, we recall the following definitions:

Definition 0.13. Let $(\mathcal{C}, \otimes, S)$ be a symmetric monoidal category with left unitor, right unitor, associator, and symmetry isomorphisms λ, ρ, α, and τ, respectively. A *monoid object* (E, μ, e) is an object E in $\mathcal{C}$ along with a multiplication morphism $\mu : E \otimes E \to E$ and a unit map $e : S \to E$ such that the following diagrams commute:

$$\begin{tikzcd}
E \otimes S \arrow[r, "E\otimes e"] \arrow[dr, "\rho_E"'] & E \otimes E \arrow[d, "\mu"] & S \otimes E \arrow[l, "e\otimes E"'] \arrow[dl, "\lambda_E"] \\
& E &
\end{tikzcd}
\qquad
\begin{tikzcd}
(E \otimes E) \otimes E \arrow[rr, "\mu\otimes E"] \arrow[d, "\alpha"'] & & E \otimes E \arrow[d, "\mu"] \\
E \otimes (E \otimes E) \arrow[r, "E\otimes\mu"] & E \otimes E \arrow[r, "\mu"] & E
\end{tikzcd}$$

The first diagram expresses unitality, while the second expressed associativity. If in addition the following diagram commutes,

$$\begin{tikzcd}
E \otimes E \arrow[rr, "\tau"] \arrow[dr, "\mu"'] & & E \otimes E \arrow[dl, "\mu"] \\
& E &
\end{tikzcd}$$

[2] In particular, given an idempotent $e : X \to X$ which splits as $X \xrightarrow{r} Y \xrightarrow{\iota} X$, r and ι are the coequalizer and equalizer, respectively, of e and id_X.

then we say (E, μ, e) is a *commutative* monoid object.

Definition 0.14. Given two monoid objects (E_1, μ_1, e_1) and (E_2, μ_2, e_2) in a symmetric monoidal category $(\mathcal{C}, \otimes, S)$, a *monoid homomorphism* from E_1 to E_2 is a morphism $f : E_1 \to E_2$ in $\mathcal{C}$ such that the following diagrams commute:

$$\begin{tikzcd} E_1 \otimes E_1 \arrow[r, "f\otimes f"] \arrow[d, "\mu_1"'] & E_2 \otimes E_2 \arrow[d, "\mu_2"] \\ E_1 \arrow[r, "f"] & E_2 \end{tikzcd} \qquad \begin{tikzcd} & S \arrow[dl, "e_1"'] \arrow[dr, "e_2"] & \\ E_1 \arrow[rr, "f"] & & E_2 \end{tikzcd}$$

It is straightforward to show that id_{E_1} is a homomorphism of monoid objects from E_1 to itself, and that the composition of monoid homomorphisms is still a monoid homomorphism. Thus, we have categories $\mathbf{Mon}_{\mathcal{C}}$ and $\mathbf{CMon}_{\mathcal{C}}$ of monoid objects and commutative monoid objects, respectively, with monoid homomorphisms between them.

Proposition 0.15. *Given two monoid objects (E_1, μ_1, e_2) and (E_2, μ_2, e_2) in a symmetric monoidal category $(\mathcal{C}, \otimes, S)$, their tensor product $E_1 \otimes E_2$ canonically becomes a monoid object in $\mathcal{C}$ with unit map*

$$e : S \xrightarrow{\cong} S \otimes S \xrightarrow{e_1 \otimes e_2} E_1 \otimes E_2$$

and multiplication map

$$\mu : E_1 \otimes E_2 \otimes E_1 \otimes E_2 \xrightarrow{E_1 \otimes \tau \otimes E_2} E_1 \otimes E_1 \otimes E_2 \otimes E_2 \xrightarrow{\mu_1 \otimes \mu_2} E_1 \otimes E_2$$

(where here we are suppressing the associators from the notation). If in addition (E_1, μ_1, e_1) and (E_2, μ_2, e_2) are commutative *monoid objects, then $(E_1 \otimes E_2, \mu, e)$ is as well.*

Proof. Due to the size of the diagrams involved, we leave this as an exercise to the reader. It is entirely straightforward, especially provided one works in a symmetric strict monoidal category. $\square$

Monoid objects in $\mathcal{SH}$ will be the focus of the rest of this paper. The most important example of a monoid object in $\mathcal{SH}$ is the unit S, which has multiplication map $\phi_{0,0}^{-1} = \lambda_S = \rho_S : S \otimes S \to S$ and unit map $\mathrm{id}_S : S \to S$.

Proposition 0.16 (??). *The assignment $(E, \mu, e) \mapsto \pi_*(E)$ is a functor π_* from the category $\mathbf{Mon}_{\mathcal{SH}}$ of monoid objects in $\mathcal{SH}$ (Definition 0.14) to the category of A-graded rings. In particular, given a monoid object (E, μ, e) in $\mathcal{SH}$, $\pi_*(E)$ is canonically a ring with product $\pi_*(E) \times \pi_*(E) \to \pi_*(E)$ which sends classes $x : S^a \to E$ and $y : S^b \to E$ to the composition*

$$xy : S^{a+b} \xrightarrow{\phi_{a,b}} S^a \otimes S^b \xrightarrow{x \otimes y} E \otimes E \xrightarrow{\mu} E,$$

and the unit of this ring is $e \in \pi_0(E) = [S, E]$.

We call the ring $\pi_*(S)$ the *stable homotopy ring*. We have shown that π_* takes monoids to rings. Given a monoid object (E, μ, e) in $\mathcal{SH}$, E_* sends objects to $\pi_*(E)$-modules:

Proposition 0.17 (??). *Let (E, μ, e) be a monoid object in $\mathcal{SH}$. Then $E_*(-)$ is a functor from $\mathcal{SH}$ to left A-graded modules over the ring $\pi_*(E)$ (Proposition 0.16), where given some X in $\mathcal{SH}$, $E_*(X)$ may be endowed with the structure of a left A-graded $\pi_*(E)$-module via the map*

$$\pi_*(E) \times E_*(X) \to E_*(X)$$

which given $a, b \in A$, sends $x : S^a \to E$ and $y : S^b \to E \otimes X$ to the composition

$$x \cdot y : S^{a+b} \cong S^a \otimes S^b \xrightarrow{x \otimes y} E \otimes (E \otimes X) \cong (E \otimes E) \otimes X \xrightarrow{\mu \otimes X} E \otimes X.$$

Similarly, the assignment $X \mapsto X_*(E)$ is a functor from $\mathcal{SH}$ to right A-graded $\pi_*(E)$-modules, where the structure map

$$X_*(E) \times \pi_*(E) \to X_*(E)$$

sends $x : S^a \to X \otimes E$ and $y : S^b \to E$ to the composition

$$x \cdot y : S^{a+b} \cong S^a \otimes S^b \xrightarrow{x \otimes y} (X \otimes E) \otimes E \cong X \otimes (E \otimes E) \xrightarrow{X \otimes \mu} X \otimes E.$$

Finally, $E_*(E)$ is a $\pi_*(E)$-bimodule, in the sense that the left and right actions of $\pi_*(E)$ are compatible, so that given $y, z \in \pi_*(E)$ and $x \in E_*(E)$, $y \cdot (x \cdot z) = (y \cdot x) \cdot z$.

A natural question that arises is: In what sense is $\pi_*(E)$ a "graded commutative ring" if (E, μ, e) is a commutative monoid object? It turns out that $\pi_*(E)$ does satisfy a sort of graded commutativity condition, as follows:

Proposition 0.18 (**??**)**.** *For all $a, b \in A$ there exists an element $\theta_{a,b} \in \pi_0(S) = [S, S]$ such that given any commutative monoid object (E, μ, e) in $\mathcal{SH}$, the A-graded ring structure on $\pi_*(E)$ (Proposition 0.16) has a commutativity formula given by*

$$x \cdot y = y \cdot x \cdot (e \circ \theta_{a,b})$$

for all $x \in \pi_a(E)$ and $y \in \pi_b(E)$. In particular, $\theta_{a,b} \in \mathrm{Aut}(S)$ is the composition

$$S \xrightarrow{\cong} S^{-a-b} \otimes S^a \otimes S^b \xrightarrow{S^{-a-b} \otimes \tau} S^{-a-b} \otimes S^b \otimes S^a \xrightarrow{\cong} S,$$

where the outermost maps are the unique maps specified by Remark 0.3.

Proposition 0.19. *The $\theta_{a,b}$'s described in Proposition 0.18 satisfy the following properties for all $a, b, c, d \in A$:*

(1) $\theta_{a,b} \cdot \theta_{c,d} = \theta_{a,b} \circ \theta_{c,d} = \theta_{c,d} \cdot \theta_{a,b}$ (where $\cdot$ denotes the product in $\pi_*(S)$ given in Proposition 0.16),

(2) $\theta_{a,0} = \theta_{0,a} = \mathrm{id}_S$,

(3) $\theta_{a,b} \cdot \theta_{b,a} = \mathrm{id}_S$,

(4) $\theta_{a,b} \cdot \theta_{a,c} = \theta_{a,b+c}$ and $\theta_{b,a} \cdot \theta_{c,a} = \theta_{b+c,a}$,

Proof. (1) is **??**, (2) is **??**, (3) is **??**, and (4) is **??**. □

The above proposition motivates the following definition:

Definition 0.20 (**??**)**.** A *A-graded $\pi_*(S)$-commutative ring* is a pair (R, φ), where R is an A-graded ring and $\varphi : \pi_*(S) \to R$ is an A-graded ring homomorphism such that for all homogeneous $x, y \in R$,

$$x \cdot y = y \cdot x \cdot \varphi(\theta_{|x|,|y|}).$$

A homomorphism $(R, \varphi) \to (R', \varphi')$ of A-graded $\pi_*(S)$-commutative rings is an A-graded ring homomorphism $f : R \to R'$ satisfying $f \circ \varphi = \varphi'$. We write $\pi_*(S)$-**GrCAlg** to denote the category of A-graded $\pi_*(S)$-commutative rings and homomorphisms between them.

We should think of objects in $\pi_*(S)$-**GrCAlg** as "A-graded (anti)commutative rings". As one would expect, every commutative monoid object (E, μ, e) in $\mathcal{SH}$ gives rise to an A-graded $\pi_*(S)$-commutative ring $\pi_*(E)$:

Proposition 0.21 (**??**). *The assignment* $(E, \mu, e) \mapsto (\pi_*(E), \pi_*(e))$ *yields a functor*

$$\pi_* : \mathbf{CMon}_{\mathcal{SH}} \to \pi_*(S)\text{-}\mathbf{GrCAlg}$$

from the category of commutative monoid objects in $\mathcal{SH}$ *(Definition 0.14) to the category of* A*-graded* $\pi_*(S)$*-commutative rings (**??**).*

Corollary 0.22. *Let* (E, μ, e) *be a commutative monoid object in* $\mathcal{SH}$*. Then by **??**,* $\pi_*(E)$ *and* $E_*(E) = \pi_*(E \otimes E)$ *are canonically* A*-graded* $\pi_*(S)$*-commutative rings (**??**), since* $E \otimes E$ *is a commutative monoid object in* $\mathcal{SH}$ *by Proposition 0.15.*