

We assume the reader is familiar with additive categories and (closed, symmetric) monoidal categories.

Definition 0.1. A *triangulated category* is a tuple $(\mathcal{C}, \Sigma, \Omega, \mathcal{D})$ such that

- (1) \mathcal{C} is an additive category.
- (2) $\Sigma, \Omega : \mathcal{C} \rightarrow \mathcal{C}$ are additive functors which form an adjoint equivalence of \mathcal{C} with itself. (Σ is called the *shift functor*.)
- (3) \mathcal{D} is a collection of *distinguished triangles*, where a *triangle* is a diagram of the form

$$X \rightarrow Y \rightarrow Z \rightarrow \Sigma X.$$

These are also sometimes called *cofiber sequences* or *fiber sequences*.

These data must satisfy the following axioms:

TR0 Given a commutative diagram

$$\begin{array}{ccccccc} X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & \Sigma X \\ \cong \downarrow & & \cong \downarrow & & \cong \downarrow & & \cong \downarrow \\ X' & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & \Sigma X' \end{array}$$

where the vertical arrows are isomorphisms, if the top row is distinguished then so is the bottom.

TR1 For any object X in \mathcal{C} , the diagram

$$X \xrightarrow{\text{id}_X} X \rightarrow 0 \rightarrow \Sigma X$$

is a distinguished triangle.

TR2 For all $f : X \rightarrow Y$ there exists an object C_f (also sometimes denoted Y/X) called the *cofiber of f* and a distinguished triangle

$$X \xrightarrow{f} Y \rightarrow C_f \rightarrow \Sigma X.$$

TR3 Given a solid diagram with both rows commutative

$$\begin{array}{ccccccc} X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & \Sigma X \\ f \downarrow & & \downarrow & & \vdots & & \downarrow \Sigma f \\ X' & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & \Sigma X' \end{array}$$

such that the leftmost square commutes and both rows are distinguished, there exists a dashed arrow $Z \rightarrow Z'$ which makes the remaining two squares commute.

TR4 A triangle

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{\Sigma} X$$

is distinguished if and only if

$$Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X \xrightarrow{-\Sigma f} \Sigma Y$$

is distinguished.

TR5 (Octahedral axiom) Given three distinguished triangles

$$X \xrightarrow{f} Y \xrightarrow{h} Y/X \rightarrow \Sigma X$$

$$Y \xrightarrow{g} Z \xrightarrow{k} Z/Y \rightarrow \Sigma Y$$

$$X \xrightarrow{g \circ f} Z \xrightarrow{l} Z/X \rightarrow \Sigma X$$

there exists a distinguished triangle

$$Y/X \xrightarrow{u} Z/X \xrightarrow{v} Z/Y \xrightarrow{w} \Sigma(Y/X)$$

such that the following diagram commutes

$$\begin{array}{ccccccc}
X & \xrightarrow{g \circ f} & Z & \xrightarrow{k} & Z/Y & \xrightarrow{w} & \Sigma(Y/X) \\
& \searrow f & \nearrow g & \searrow l & \nearrow v & \searrow & \nearrow \Sigma h \\
& & Y & & Z/X & & \Sigma Y \\
& & \searrow h & \nearrow u & \searrow & \nearrow \Sigma f & \\
& & & Y/X & \xrightarrow{\quad} & \Sigma X &
\end{array}$$

It turns out that the above definition is actually redundant; TR3 and TR4 follow from the remaining axioms (see Lemmas 2.2 and 2.4 in [1]).

We now recall several important propositions for triangulated categories:

Proposition 0.2. *Given a map $f : X \rightarrow Y$ in a triangulated category $(\mathcal{C}, \Sigma, \Omega, \mathcal{D})$, the cofiber sequence of f is unique up to isomorphism, in the sense that given any two distinguished triangles*

$$X \xrightarrow{f} Y \rightarrow Z \rightarrow \Sigma X \quad \text{and} \quad X \xrightarrow{f} Y \rightarrow Z' \rightarrow \Sigma X,$$

there exists an isomorphism $Z \rightarrow Z'$ which makes the following diagram commute:

$$\begin{array}{ccccccc}
X & \xrightarrow{f} & Y & \longrightarrow & Z & \longrightarrow & \Sigma X \\
\parallel & & \parallel & & \downarrow k & & \parallel \\
X & \xrightarrow{f} & Y & \longrightarrow & Z' & \longrightarrow & \Sigma X
\end{array}$$

Proposition 0.3. *Given an arrow $f : X \rightarrow Y$ in a triangulated category $(\mathcal{C}, \Sigma, \Omega, \mathcal{D})$, there exists an object F_f called the fiber of f , and a distinguished triangle*

$$F_f \rightarrow X \xrightarrow{f} Y \rightarrow \Sigma F_f (\cong C_f).$$

Proposition 0.4. *Let $(\mathcal{C}, \Sigma, \Omega, \mathcal{D})$ be a triangulated category. Given a distinguished triangle*

$$X \xrightarrow{f} Y \xrightarrow{g} \Sigma X$$

and any object A in \mathcal{C} , there is a long exact sequence of abelian groups

$$\cdots \rightarrow [\Sigma^{n+1}A, Z] \xrightarrow{h_*} [\Sigma^n X, X] \xrightarrow{f_*} [\Sigma^n A, Y] \xrightarrow{g_*} [\Sigma^n A, Z] \xrightarrow{h_*} [\Sigma^{n-1}A, X] \rightarrow \cdots$$

extending infinitely in either direction, where for $n < 0$ we define $\Sigma^{-n} := \Omega^n$.

Also important for our work is the concept of a *tensor triangulated category*, that is, a triangulated symmetric monoidal category in which the triangulated structures are compatible, in the following sense:

Definition 0.5. A *tensor triangulated category* is a triangulated symmetric monoidal category $(\mathcal{C}, \otimes, S, \Sigma, \Omega, \mathcal{D})$ such that:

TT1 For all objects X and Y in \mathcal{C} , there are natural isomorphisms

$$e_{X,Y} : (\Sigma X) \otimes Y \xrightarrow{\cong} \Sigma(X \otimes Y).$$

TT2 For each object X in \mathcal{C} , the functor $X \otimes (-) \cong (-) \otimes X$ is an additive functor.

TT3 For each object X in \mathcal{C} , the functor $X \otimes (-) \cong (-) \otimes X$ preserves distinguished triangles, in that given a distinguished triangle/(co)fiber sequence

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{\Sigma} A,$$

then also

$$X \otimes A \xrightarrow{X \otimes f} X \otimes B \xrightarrow{X \otimes g} X \otimes C \xrightarrow{\Sigma(X \otimes h)} \Sigma(X \otimes A)$$

and

$$A \otimes X \xrightarrow{f \otimes X} B \otimes X \xrightarrow{g \otimes X} C \otimes X \xrightarrow{\Sigma(h \otimes X)} \Sigma(A \otimes X)$$

are distinguished triangles.

Usually, most tensor triangulated categories that arise in nature will satisfy additional coherence axioms (see axioms TC1–TC5 in [1]), but the above definition will suffice for our purposes. To avoid the awkwardness of saying “a tensor triangulated category which is also a closed symmetric monoidal category,” we introduce the following (nonstandard) terminology:

Definition 0.6. We say a tensor triangulated category $(\mathcal{C}, \otimes, S, \Sigma, \Omega)$ is *closed* if \mathcal{C} is a closed symmetric monoidal category, in the sense that for each object $X \in \mathcal{C}$, the functor $- \otimes X$ has a right adjoint $F(X, -)$.

Note that given a tensor triangulated category, we have the following characterization of the shift functor:

Proposition 0.7. *Given a tensor triangulated category $(\mathcal{C}, \otimes, S, \Sigma, \Omega)$, there is a canonical natural isomorphism $\Sigma S \otimes - \cong \Sigma$.*

Proof. Given an object X in \mathcal{C} , we have natural isomorphisms

$$\Sigma S \otimes X \xrightarrow{e_{S,X}} \Sigma(S \otimes X) \xrightarrow{\Sigma \lambda_X} \Sigma X,$$

where λ_X is the left unitor specified by the monoidal structure on \mathcal{C} . □