

In the section that follows, let (E, μ, e) be a monoid object and X and Y be objects in \mathcal{SH} .

0.1. Construction of the spectral sequence.

Definition 0.1. Let \overline{E} be the fiber of the unit map $e : S \rightarrow E$ (??). Let $Y_0 := Y$ and $W_0 := E \otimes Y$. For $s > 0$, define

$$Y_s := \overline{E}^s \otimes Y, \quad W_s := E \otimes Y_s = E \otimes \overline{E}^s \otimes Y,$$

where \overline{E}^s denotes the s -fold tensor product $\overline{E} \otimes \cdots \otimes \overline{E}$. Then we get fiber sequences

$$Y_{s+1} \xrightarrow{i_s} Y_s \xrightarrow{j_s} W_s \xrightarrow{k_s} \Sigma Y_{s+1}$$

obtained by applying $- \otimes Y_s$ to the fiber sequence

$$\overline{E} \rightarrow S \xrightarrow{e} E \rightarrow \Sigma \overline{E}.$$

We can splice these sequences together to get the following diagram, which is called *the canonical Adams-resolution of Y* :

$$\begin{array}{ccccccc} \cdots & \longrightarrow & Y_3 & \xrightarrow{i_2} & Y_2 & \xrightarrow{i_1} & Y_1 & \xrightarrow{i_0} & Y_0 = Y \\ & & \downarrow j_3 & \swarrow k_2 & \downarrow j_2 & \swarrow k_1 & \downarrow j_1 & \swarrow k_0 & \downarrow j_0 \\ & & W_3 & & W_2 & & W_1 & & W_0 \end{array}$$

Here we are using dashed arrows to denote the (degree -1) maps $k_s : W_s \rightarrow \Sigma Y_{s+1}$, in particular, the above diagram does not commute in any sense.

Lemma 0.2. *The maps $i_s : Y_{s+1} \rightarrow Y_s$ from the canonical E -Adams resolution of Y (Lemma 0.2) vanish in E -homology, i.e., $E_*(i_s)$ is the zero map.*

Proof. First, note that since

$$Y_{s+1} \xrightarrow{i_s} Y_s \xrightarrow{j_s} W_s \xrightarrow{k_s} \Sigma Y_{s+1}$$

is a distinguished triangle and \mathcal{SH} is tensor triangulated, there is a distinguished triangle of the form

$$E \otimes Y_{s+1} \xrightarrow{E \otimes i_s} E \otimes Y_s \xrightarrow{E \otimes j_s} E \otimes W_s \rightarrow \Sigma(E \otimes Y_{s+1}).$$

Thus, applying $\pi_*(-) \cong [S, -]_*$ to the triangle yields that the following sequence is exact (see ?? for details):

$$E_*(Y_{s+1}) \xrightarrow{E_*(i_s)} E_*(Y_s) \xrightarrow{E_*(j_s)} E_*(W_s).$$

Now, it is straightforward to verify by construction that j_s is the map $e \otimes Y_s : Y_s \rightarrow E \otimes Y_s = W_s$. Thus, by unitality of μ , we have that $E \otimes j_s : E \otimes Y_s \rightarrow E \otimes W_s$ is a split monomorphism, with right inverse $\mu \otimes Y_s : E \otimes W_s = E \otimes E \otimes Y_s \rightarrow E \otimes Y_s$. Then since any functor preserves split monomorphisms, it follows that $E_*(j_s) = \pi_*(E \otimes j_s)$ is likewise a split monomorphism, so that in particular $E_*(j_s)$ is injective. Thus $\text{im } E_*(i_s) = \ker E_*(j_s) = 0$, so that i_s is indeed the zero map, as desired. \square

Now, by applying $[X, -]_*$ to the canonical E -Adams resolution of Y , we get an associated unrolled exact couple, and thus a spectral sequence:

Definition 0.3. Consider the canonical E -Adams resolution of Y from Lemma 0.2:

$$\begin{array}{ccccccc} \cdots & \longrightarrow & Y_3 & \xrightarrow{i_2} & Y_2 & \xrightarrow{i_1} & Y_1 & \xrightarrow{i_0} & Y_0 = Y \\ & & \downarrow j_3 & \swarrow k_2 & \downarrow j_2 & \swarrow k_1 & \downarrow j_1 & \swarrow k_0 & \downarrow j_0 \\ & & W_3 & & W_2 & & W_1 & & W_0 \end{array}$$

We can extend this diagram to the right by setting $Y_s = Y$, $W_s = 0$, and $i_s = \text{id}_Y$ for $s < 0$. Then we may apply the functor $[X, -]_*$, and by ??, we obtain the following A -graded unrolled exact couple (??):

$$\begin{array}{ccccccc} \cdots & \longrightarrow & [X, Y_{s+2}]_* & \xrightarrow{i_{s+1}} & [X, Y_{s+1}]_* & \xrightarrow{i_s} & [X, Y_s]_* \xrightarrow{i_{s-1}} [X, Y_{s-1}]_* \longrightarrow \cdots \\ & & \downarrow j_{s+2} & \swarrow \partial_{s+1} & \downarrow j_{s+1} & \swarrow \partial_s & \downarrow j_s \swarrow \partial_{s-1} \downarrow j_{s-1} \\ & & [X, W_{s+2}]_* & & [X, W_{s+1}]_* & & [X, W_s]_* & & [X, W_{s-1}]_* \end{array}$$

where here we are being abusive and writing $i_s : [X, Y_{s+1}]_* \rightarrow [X, Y_s]_*$ and $j_s : [X, Y_s]_* \rightarrow [X, W_s]_*$ to denote the pushforward maps induced by $i_s : Y_{s+1} \rightarrow Y_s$ and $j_s : Y_s \rightarrow W_s$, respectively. Each i_s , j_s , and ∂_s are A -graded homomorphisms of degrees 0, 0, and -1 , respectively.

By ??, we may associate a $\mathbb{Z} \times A$ -graded spectral sequence $r \mapsto (E_r^{*,*}(X, Y), d_r)$ to the above A -graded unrolled exact couple, where d_r has $\mathbb{Z} \times A$ -degree $(r, -1)$. We call this spectral sequence the *E-Adams spectral sequence for the computation of $[X, Y]_*$* .

For those who would rather not lose themselves in the appendix, we give a brief unravelling of how ?? applies to the present situation. Given some $s \in \mathbb{Z}$ and some $r \geq 1$, we may define the following A -graded subgroups of $[X, W_s]_*$:

$$Z_r^s := \partial_s^{-1}(\text{im}[i^{(r-1)} : [X, Y_{s+r}]_* \rightarrow [X, Y_{s+1}]_*])$$

and

$$B_r^s := j_s(\ker[i^{(r-1)} : [X, Y_s]_* \rightarrow [X, Y_{s-r+1}]_*]),$$

where we adopt the convention that $i^{(0)}$ is simply the identity. This yields an infinite sequence of inclusions

$$0 = B_1^s \subseteq B_2^s \subseteq B_3^s \subseteq \cdots \subseteq \text{im } j_s = \ker \partial_s \subseteq \cdots \subseteq Z_3^s \subseteq Z_2^s \subseteq Z_1^s = [X, W_s]_*.$$

Then for $r \geq 1$, we define E_r^s to be the A -graded quotient group

$$E_r^s := Z_r^s / B_r^s.$$

Thus taking the direct sum of all the E_r^s 's yields the r^{th} page of the spectral sequence

$$E_r := \bigoplus_{s \in \mathbb{Z}} E_r^s,$$

which is a $\mathbb{Z} \times A$ -graded abelian group.

The differential $d_r : E_r \rightarrow E_r$ is a map of $\mathbb{Z} \times A$ -degree $(r, 1)$, and is constructed as follows: an element of $E_r^s = Z_r^s / B_r^s$ is a coset represented by some $x \in Z_r^s$, so that $\partial_s(x) = i^{(r-1)}(y)$ for some $y \in [X, Y_{s+r}]_*$. Then we define $d_r([x])$ to be the coset $[j_{s+r}(y)]$ in Z_r^{s+r} / B_r^{s+r} .

In the case $r = 1$, since $B_1^s = 0$ and $Z_1^s = [X, W_s]_*$, we have that $E_1^s = [X, W_s]_*$, and given some $x \in E_1^s = [X, W_s]_*$, the differential d_1 is given by $d_1(x) = j_{s+1}(\partial_s(x))$, so that $d_1 = j \circ \partial$. Furthermore, since the unrolled exact couple which yields the spectral sequence vanishes on its negative terms, we have that $E_r^{s,a}(X, Y) = 0$ for $s < 0$.

In ??, it is shown in explicit detail that all of these definitions make sense and are well-defined. In particular, it is shown that the differentials are well-defined A -graded homomorphisms, that $d_r \circ d_r = 0$, and that

$$\ker d_r^s / \text{im } d_r^s = \frac{Z_{r+1}^s / B_r^s}{B_{r+1}^s / B_r^s} \cong Z_{r+1}^s / B_{r+1}^s = E_{r+1}^s.$$

0.2. The E_2 page. Now, we would like to characterize the E_2 page of the spectral sequence in terms of something more concrete. Namely, we will characterize the E_2 page in terms of Ext of comodules over the dual E -Steenrod algebra. For a quick review of Ext in an abelian category and derived functors, see ???. The goal of this subsection will be to prove the following theorem:

Theorem 0.4. *Let (E, μ, e) be a commutative monoid object, and X and Y objects in $S\mathcal{H}$. Suppose further that:*

- E is flat (??) and cellular (??),
- X is cellular and $E_*(X)$ is a graded projective left $\pi_*(E)$ -module (via ??),
- Y is cellular.

Then the non-vanishing entries of the second page of the E -Adams spectral sequence for the computation of $[X, Y]_$ (Definition 0.3) are the Ext groups of A -graded left comodules over the anticommutative Hopf algebroid structure on the dual E -Steenrod algebra (??), i.e., we have the following isomorphisms for all $s \in \mathbb{N}$ and $a \in A$:*

$$E_2^{s,a}(X, Y) \cong \text{Ext}_{E_*(E)}^{s, a+s}(E_*(X), E_*(Y)) := \text{Ext}_{E_*(E)}^s(E_*(X), E_{*+a+s}(Y)).$$

Proof. By Proposition 0.9 below, for each $s \in \mathbb{N}$ and $a \in A$, $E_2^{s,a}(X, Y)$ is isomorphic to the s^{th} cohomology group of the cochain complex obtained by applying $F := \text{Hom}_{E_*(E)}^{a+s}(E_*(X), -)$ to the complex

$$0 \longrightarrow E_*(W_0) \xrightarrow{E_*(\delta_0)} E_*(\Sigma W_1) \xrightarrow{E_*(\delta_1)} E_*(\Sigma^2 W_2) \xrightarrow{E_*(\delta_2)} E_*(\Sigma^3 W_3) \longrightarrow \dots$$

Furthermore, by Lemma 0.7, this complex is an F -acyclic resolution of $E_*(Y)$ (??). Thus, since the category of $E_*(E)$ -comodules is an abelian category with enough injectives (??), we have by ?? that

$$E_2^{s,a}(X, Y) \cong R^s \text{Hom}_{E_*(E)}^{a+s}(E_*(X), -)(E_*(Y)) = \text{Ext}_{E_*(E)}^{s, a+s}(E_*(X), E_*(Y)),$$

as desired. \square

We leave it to the reader to unravel what the differential d_2 corresponds to under this identification. The remainder of this subsection is devoted to proving Lemma 0.7 and Proposition 0.9.

Definition 0.5. Given some (nonnegative integer) $n \in \mathbb{N}$, define natural isomorphisms $\nu_X^n : \Sigma^n X \rightarrow \Sigma^n X$ inductively, by setting $\nu_X^0 := \lambda_X$, $\nu_X^1 := \nu_X^{-1}$, and supposing ν_X^{n-1} has been defined for some $n > 1$, define ν_X^n to be the composition

$$\nu_X^n : \Sigma^n X = S^n \otimes X \xrightarrow{\phi_{n-1,1} \otimes X} S^{n-1} \otimes S^1 \otimes X \xrightarrow{S^{n-1} \otimes \nu_X^{-1}} S^{n-1} \Sigma X \xrightarrow{\nu_{\Sigma X}^{n-1}} \Sigma^n X.$$

By induction, naturality of ν , and functoriality of $- \otimes -$, these isomorphisms are clearly natural in X .

Lemma 0.6. *Suppose E and Y are cellular. Then for all $s \in \mathbb{Z}$, the objects Y_s and W_s from the canonical E -Adams resolution of Y (Definition 0.1) are cellular.*

Proof. Unravelling definitions, for $s < 0$, $W_s = 0$ and $Y_s = Y$, which are both cellular.¹ For $s \geq 0$, we have $W_s = E \otimes Y_s$, so that by cellularity of E and ??, it suffices to show that Y_s is cellular

¹ 0 is cellular because it is the cofiber of the identity on S by axiom TR1 for a triangulated category (??), i.e., there is a distinguishing triangle $S \rightarrow S \rightarrow 0 \rightarrow \Sigma S$.

for $s \geq 0$. We know $Y_0 = Y$ is cellular by definition. For $s > 0$, Y_s is the tensor product $\overline{E}^s \otimes Y$, where \overline{E} fits into the distinguished triangle

$$\overline{E} \rightarrow S \xrightarrow{e} E \rightarrow \Sigma \overline{E}.$$

By the definition of cellularity, \overline{E} is cellular since S and E are. Thus $\overline{E}^s \otimes Y$ is cellular by ??, as it is a tensor product of cellular objects in \mathcal{SH} . \square

Lemma 0.7. *Let (E, μ, e) be a flat (??) and cellular (??) commutative monoid object and X and Y cellular objects in \mathcal{SH} , and for $s \geq 0$ define Y_s and W_s as in Definition 0.1. In particular, for each $s \geq 0$, $W_s = E \otimes Y_s$ and we have distinguished triangles*

$$Y_{s+1} \xrightarrow{i_s} Y_s \xrightarrow{j_s} W_s \xrightarrow{k_s} \Sigma Y_{s+1}.$$

Then if $E_*(X)$ is a graded projective (??) left $\pi_*(E)$ -module (via ??) then the sequence

$$0 \rightarrow E_*(Y) \xrightarrow{E_*(j_0)} E_*(W_0) \xrightarrow{E_*(\delta_0)} E_*(\Sigma W_1) \xrightarrow{E_*(\delta_1)} E_*(\Sigma^2 W_2) \xrightarrow{E_*(\delta_2)} E_*(\Sigma^3 W_3) \rightarrow \dots$$

is an F -acyclic resolution (??) of $E_*(Y)$ in $E_*(E)\text{-CoMod}^A$ for

$$F = \text{Hom}_{E_*(E)}^a(E_*(X), -)$$

for all $a \in A$, where δ_s is the composition

$$\Sigma^s W_s \xrightarrow{\Sigma^s k_s} \Sigma^{s+1} Y_{s+1} \xrightarrow{\Sigma^{s+1} j_{s+1}} \Sigma^{s+1} W_{s+1}.$$

Proof. By Lemma 0.6, each W_s is cellular, so that furthermore $\Sigma^s W_s \cong S^s \otimes W_s$ is cellular for each $s \geq 0$, by ??. Thus, the sequence does indeed live in $E_*(E)\text{-CoMod}^A$ by ??, as desired. Next, we claim that $E_*(\Sigma^s W_s)$ is an F -acyclic object for each $s \geq 0$, i.e., that

$$\text{Ext}_{E_*(E)}^{n,a}(E_*(X), E_*(\Sigma^s W_s)) = \text{Ext}_{E_*(E)}^n(E_*(X), E_{*+a}(\Sigma^s W_s)) = 0$$

for all $n > 0$, $s \geq 0$, and $a \in A$. Note that we have an A -graded isomorphism of left $E_*(E)$ -comodules:

$$\begin{aligned} E_*(E) \otimes_{\pi_*(E)} E_{*+a}(\Sigma^s Y_s) &= E_*(E) \otimes_{\pi_*(E)} E_{*+a}(\Sigma^s Y_s) \\ &\downarrow \Phi_{E, \Sigma^s Y_s} \\ E_*(E \otimes \Sigma^s Y_s) & \\ &\downarrow E_*(E \otimes (\nu_{Y_s}^s)^{-1}) \\ E_*(E \otimes S^s \otimes Y_s) & \\ &\downarrow E_*(\tau \otimes Y_s) \\ E_*(S^s \otimes E \otimes Y_s) & \\ &\downarrow E_*(\nu_{E \otimes Y_s}^s) \\ E_*(\Sigma^s(E \otimes Y_s)) &= E_*(\Sigma^s W_s) \end{aligned}$$

where $\Phi_{E, \Sigma^s Y}$ is an A -graded isomorphism of abelian groups by ??, and furthermore an isomorphism of $E_*(E)$ -comodules by ??. Every other arrow is an isomorphism of $E_*(E)$ -comodules by functoriality of $E_*(-) : \mathcal{SH}\text{-Cell} \rightarrow E_*(E)\text{-CoMod}^A$. Thus, since $E_*(\Sigma^s W_s)$ is isomorphic to $E_*(E) \otimes_{\pi_*(E)} E_{*+a}(\Sigma^s Y_s)$ in $E_*(E)\text{-CoMod}^A$, and in particular since $\text{Ext}_{E_*(E)}^n(E_*(X), -)$ is a functor, we have

$$\text{Ext}_{E_*(E)}^n(E_*(X), E_{*+a}(\Sigma^s W_s)) \cong \text{Ext}_{E_*(E)}^n(E_*(X), E_*(E) \otimes_{\pi_*(E)} E_{*+a}(\Sigma^s Y_s)).$$

Yet, $E_*(E) \otimes_{\pi_*(E)} E_{*+a}(\Sigma^s Y_s)$ is a co-free $E_*(E)$ -comodule (??), in which case since $E_*(X)$ is graded projective as an object in $\pi_*(E)\text{-Mod}^A$, we have that

$$\text{Ext}_{E_*(E)}^{n,a}(E_*(X), E_*(E) \otimes_{\pi_*(E)} E_{*+a}(\Sigma^s Y_s)) = 0,$$

by ??.

Finally, it remains to show that the sequence is exact. To that end, first note that by induction on axiom TR4 for a triangulated category and the fact that distinguished triangles are exact (??), the following sequence in \mathcal{SH} is exact (since a sequence clearly remains exact even after changing the signs of its maps):

$$\Sigma^s Y_s \xrightarrow{\Sigma^s j_s} \Sigma^s W_s \xrightarrow{\Sigma^s k_s} \Sigma^{s+1} Y_{s+1} \xrightarrow{\Sigma^{s+1} i_s} \Sigma^{s+1} Y_s \xrightarrow{\Sigma^{s+1} j_s} \Sigma^{s+1} W_s$$

(see ?? for the definition of an exact triangle in an additive category). Furthermore, since \mathcal{SH} is tensor triangulated, the sequence remains exact after applying $E \otimes -$ (see ?? for details), so that taking E -homology yields the following exact sequence of homology groups:

$$E_*(\Sigma^s Y_{s+1}) \xrightarrow{E_*(\Sigma^s i_s)} E_*(\Sigma^s Y_s) \xrightarrow{E_*(\Sigma^s j_s)} E_*(\Sigma^s W_s) \xrightarrow{E_*(\Sigma^s k_s)} E_*(\Sigma^{s+1} Y_{s+1}) \xrightarrow{E_*(\Sigma^{s+1} i_s)} E_*(\Sigma^{s+1} Y_s).$$

Then since $E_*(i_s) : E_*(Y_{s+1}) \rightarrow E_*(Y_s)$ is the zero map (by Lemma 0.2) and we have natural isomorphisms

$$E_*(\Sigma^t X) \xrightarrow{\nu_X^t} E_*(\Sigma^t X) \xrightarrow{t_X^t} E_{*-t}(X)$$

(the first from Definition 0.5 and the latter from ??), we have that $E_*(\Sigma^t i_s) : E_*(\Sigma^t Y_{s+1}) \rightarrow E_*(\Sigma^t Y_s)$ is the zero map for all $t \in \mathbb{Z}$, so that in particular the above exact sequence splits to yield the short exact sequence

$$0 \rightarrow E_*(\Sigma^s Y_s) \xrightarrow{E_*(\Sigma^s j_s)} E_*(\Sigma^s W_s) \xrightarrow{E_*(\Sigma^s k_s)} E_*(\Sigma^{s+1} Y_{s+1}) \rightarrow 0.$$

Then we may splice these sequences together for $s \geq 0$ to yield the following diagram:

$$\begin{array}{ccccccc} 0 & \rightarrow & E_*(Y) & \xrightarrow{E_*(j_0)} & E_*(W_0) & \xrightarrow{E_*(\delta_0)} & E_*(\Sigma W_1) & \xrightarrow{E_*(\delta_1)} & E_*(\Sigma^2 W_2) & \rightarrow & \dots \\ & & & & \searrow E_*(k_0) & & \nearrow E_*(\Sigma j_1) & & \searrow E_*(\Sigma k_1) & & \nearrow E_*(\Sigma^2 j_2) \\ & & & & & & E_*(\Sigma Y_1) & & & & E_*(\Sigma^2 Y_2) \end{array}$$

It is straightforward to check the top row is exact by exactness of the short exact sequences, as desired. \square

Lemma 0.8. *Let (E, μ, e) be a commutative monoid object, and X and Y objects in \mathcal{SH} . Suppose further that:*

- E is flat (??) and cellular (??),
- X is cellular and $E_*(X)$ is a graded projective left $\pi_*(E)$ -module (via ??), and
- Y is cellular.

Then the assignment

$$E_*(-) : [X, E \otimes Y] \rightarrow \text{Hom}_{E_*(E)}(E_*(X), E_*(E \otimes Y)), \quad f \mapsto E_*(f)$$

induced by the functor $E_*(-) : \mathcal{SH}\text{-Cell} \rightarrow E_*(E)\text{-CoMod}^A$ is an isomorphism of abelian groups.

Proof. Since X is cellular, by ?? we have that $E_*(X)$ is canonically an A -graded left $E_*(E)$ -comodule. Similarly, since E and Y are cellular, we know that $E \otimes Y$ is cellular, so that $E_*(E \otimes Y)$ is also canonically an $E_*(E)$ -comodule. Thus, we have a well-defined assignment

$$[X, E \otimes Y] \xrightarrow{E_*(-)} \text{Hom}_{E_*(E)}(E_*(X), E_*(E \otimes Y)).$$

To see this arrow is an isomorphism, consider the following diagram:

$$\begin{array}{ccc} [X, E \otimes Y] & \xrightarrow{E_*(-)} & \text{Hom}_{E_*(E)}(E_*(X), E_*(E \otimes Y)) \\ \downarrow \pi_*(\mu \otimes Y) \circ E_*(-) & \swarrow \pi_*(\mu \otimes Y) \circ (-) & \uparrow (\Phi_{E,Y})_* \\ \text{Hom}_{\pi_*(E)}(E_*(X), E_*(Y)) & \xleftarrow{\text{adj}} & \text{Hom}_{E_*(E)}(E_*(X), E_*(E) \otimes_{\pi_*(E)} E_*(Y)) \end{array}$$

We know the left vertical map is an isomorphism by ??, and the bottom horizontal isomorphism is the forgetful-cofree adjunction (??) for A -graded left comodules over the dual E -Steenrod algebra. The right vertical arrow is a well-defined isomorphism, as $\Phi_{E,Y}$ is a homomorphism of A -graded left $E_*(E)$ -comodules (??), and in fact it is an isomorphism by ??, since $E_*(E)$ is flat and Y is cellular. Thus in order to see the top arrow is an isomorphism, it suffices to show that the diagram commutes. The left triangle clearly commutes; to see the right triangle commutes, recall that by how the how forgetful-cofree adjunction for left comodules over a Hopf algebroid is defined, that the bottom vertical arrow sends an A -graded homomorphism of left $E_*(E)$ -comodules $\psi : E_*(X) \rightarrow E_*(E) \otimes_{\pi_*(E)} E_*(Y)$ to the composition

$$E_*(X) \xrightarrow{\psi} E_*(E) \otimes_{\pi_*(E)} E_*(Y) \xrightarrow{\pi_*(\mu) \otimes E_*(Y)} \pi_*(E) \otimes_{\pi_*(E)} E_*(Y) \xrightarrow{\cong} E_*(Y).$$

Thus, in order to show that this composition equals $\pi_*(\mu \otimes Y) \circ \Phi_{E,Y} \circ \psi$, it suffices to show the following diagram commutes:

$$\begin{array}{ccc} E_*(E) \otimes_{\pi_*(E)} E_*(Y) & \xrightarrow{\pi_*(\mu) \otimes E_*(Y)} & \pi_*(E) \otimes_{\pi_*(E)} E_*(Y) \\ \downarrow \Phi_{E,Y} & & \downarrow \cong \\ E_*(E \otimes Y) & \xrightarrow{\pi_*(\mu \otimes Y)} & E_*(Y) \end{array}$$

Since all the arrows here are homomorphisms of abelian groups, in order to show the diagram commutes, it suffices to chase pure homogeneous tensors around. To that end, let $x : S^a \rightarrow E \otimes E$ and $y : S^b \rightarrow E \otimes Y$, and consider the following diagram exhibiting the two ways to chase $x \otimes y$ around:

$$\begin{array}{ccccc} S^{a+b} & \xrightarrow{\phi_{a,b}} & S^a \otimes S^b & \xrightarrow{x \otimes y} & E \otimes E \otimes E \otimes Y \xrightarrow{\mu \otimes E \otimes Y} E \otimes E \otimes Y \\ & & & & \downarrow E \otimes \mu \otimes Y \quad \downarrow \mu \otimes Y \\ & & & & E \otimes E \otimes Y \xrightarrow{\mu \otimes Y} E \otimes Y \end{array}$$

The diagram commutes by associativity of μ . Thus, we have indeed show that

$$E_*(-) : [X, E \otimes Y] \rightarrow \text{Hom}_{E_*(E)}(E_*(X), E_*(Y))$$

is an isomorphism of abelian groups. □

Proposition 0.9. *Let (E, μ, e) be a commutative monoid object, and X and Y objects in $S\mathcal{H}$. Suppose further that:*

- E is flat (??) and cellular (??),

- X is cellular, and $E_*(X)$ is a graded projective left $\pi_*(E)$ -module (via ??), and
- Y is cellular.

Then for all $s \in \mathbb{Z}$ and $a \in A$, the line in the first page of the E -Adams spectral sequence for the computation of $[X, Y]_*$ (Definition 0.3)

$$0 \rightarrow E_1^{0,a+s}(X, Y) \xrightarrow{d_1} E_1^{1,a+s-1}(X, Y) \xrightarrow{d_1} E_1^{2,a+s-2}(X, Y) \rightarrow \cdots \rightarrow E_1^{s,a}(X, Y) \rightarrow \cdots$$

is isomorphic to the complex obtained by applying $\text{Hom}_{E_*(E)}^{a+s}(E_*(X), -)$ to the complex of A -graded left $E_*(E)$ -comodules

$$0 \rightarrow E_*(W_0) \xrightarrow{E_*(\delta_0)} E_*(\Sigma W_1) \xrightarrow{E_*(\delta_1)} E_*(\Sigma^2 W_2) \rightarrow \cdots \rightarrow E_*(\Sigma^s W_s) \rightarrow \cdots$$

from Lemma 0.7.

Proof. By Lemma 0.6, since E and Y are cellular, W_t is as well for each $t \in \mathbb{N}$. Furthermore, for $t > 0$, we have isomorphisms

$$S^t \otimes W_t \xrightarrow{\nu_{W_t}^t} \Sigma^t W_t,$$

and by ??, the object $S^t \otimes W_t$ is cellular since S^t and W_t are. Hence, by ??, the complex

$$0 \rightarrow E_*(W_0) \xrightarrow{E_*(\delta_0)} E_*(\Sigma W_1) \xrightarrow{E_*(\delta_1)} E_*(\Sigma^2 W_2) \rightarrow \cdots \rightarrow E_*(\Sigma^s W_s) \rightarrow \cdots$$

actually lives in $E_*(E)\text{-CoMod}^A$, as desired. Now, let $t \in \mathbb{N}$, and consider the following diagram:

$$\begin{array}{ccccc}
[X, W_t]_{a+s-t} & \xleftarrow{s_{X, W_t}^t} & [X, \Sigma^t W_t]_{a+s} & \xrightarrow{(\nu_{W_t}^t)_*} & [X, \Sigma^t W_t]_{a+s} \\
(k_t)_* \downarrow & & (\Sigma^t k_t)_* \downarrow & & \downarrow (\Sigma^t k_t)_* \\
[X, \Sigma Y_{t+1}]_{a+s-t} & \xleftarrow{s_{X, \Sigma Y_{t+1}}^t} & [X, \Sigma^t \Sigma Y_{t+1}]_{a+s} & & \downarrow (\nu_{\Sigma Y_{t+1}}^t)_* \\
(\nu_{Y_{t+1}})_* \downarrow & & (\Sigma^t \nu_{Y_{t+1}})_* \downarrow & \searrow & \downarrow (\Sigma^t k_t)_* \\
[X, \Sigma^1 Y_{t+1}]_{a+s-t} & \xleftarrow{s_{X, \Sigma^1 Y_{t+1}}^t} & [X, \Sigma^t \Sigma^1 Y_{t+1}]_{a+s} & & \downarrow (\nu_{\Sigma^1 Y_{t+1}}^t)_* \\
s_{X, Y_{t+1}}^1 \downarrow & & (\phi_{t,1} \otimes Y_{t+1})_* \downarrow & \searrow & \downarrow (\Sigma^t k_t)_* \\
[X, Y_{t+1}]_{a+s-t-1} & \xleftarrow{s_{X, Y_{t+1}}^{t+1}} & [X, \Sigma^{t+1} Y_{t+1}]_{a+s} & & \downarrow (\nu_{Y_{t+1}}^{t+1})_* \\
(j_{t+1})_* \downarrow & & (\Sigma^{t+1} j_{t+1})_* \downarrow & \searrow & \downarrow (\Sigma^{t+1} j_{t+1})_* \\
[X, W_{t+1}]_{a+s-t-1} & \xleftarrow{s_{X, W_{t+1}}^{t+1}} & [X, \Sigma^{t+1} W_{t+1}]_{a+s} & \xrightarrow{(\nu_{W_{t+1}}^{t+1})_*} & [X, \Sigma^{t+1} W_{t+1}]_{a+s}
\end{array}$$

$(\delta_t)_*$

where here the $s_{X,Y}^a : [X, \Sigma^a Y]_* \cong [X, Y]_{*-a}$'s are the natural isomorphisms from ??. By unravelling definitions, we have the top left object is $E_1^{t,a+s-t}(X, Y)$ and the bottom left object is $E_1^{t+1,a+s-t-1}(X, Y)$, and the vertical left composition in the above diagram is the differential d_1 between them. The first, second, and fourth rectangles from the top on the left rectangle commute by naturality of the s^a 's. Furthermore, a simple diagram chase and coherence of the ϕ 's (??) yields that the third rectangle on the left commutes. The trapezoids on the right commute by naturality of ν^t and ν^{t+1} . Finally, the middle right triangle commutes by how we defined ν^{t+1} in terms of ν^t .

Now, consider the following diagram:

$$\begin{array}{ccc}
E_1^{t,a+s-t}(X, Y) & \xrightarrow{d_1} & E_1^{t+1,a+s-t-1}(X, Y) \\
(s_{X, W_t}^t)^{-1} \downarrow & & \downarrow (s_{X, W_{t+1}}^{t+1})^{-1} \\
[X, \Sigma^t W_t]_{a+s} & & [X, \Sigma^{t+1} W_{t+1}]_{a+s} \\
(\nu_{W_t}^t)_* \downarrow & & \downarrow (\nu_{W_{t+1}}^{t+1})_* \\
[X, \Sigma^t W_t]_{a+s} & \xrightarrow{(\delta_t)_*} & [X, \Sigma^{t+1} W_{t+1}]_{a+s} \\
E_*(-) \downarrow & & \downarrow E_*(-) \\
\mathrm{Hom}_{E_*(E)}(E_*(\Sigma^{a+s} X), E_*(\Sigma^t W_t)) & \xrightarrow{E_*(\delta_t)} & \mathrm{Hom}_{E_*(E)}(E_*(\Sigma^{a+s} X), E_*(\Sigma^{t+1} W_{t+1})) \\
((t_X^{a+s})^{-1})^* \downarrow & & \downarrow ((t_X^{a+s})^{-1})^* \\
\mathrm{Hom}_{E_*(E)}^{a+s}(E_*(X), E_*(\Sigma^t W_t)) & \xrightarrow{E_*(\delta_t)} & \mathrm{Hom}_{E_*(E)}^{a+s}(E_*(X), E_*(\Sigma^{t+1} W_{t+1}))
\end{array}$$

where here the maps $t_X^{a+s} : E_*(\Sigma^a) \rightarrow E_{*-a}(X)$ are the $E_*(E)$ -comodule isomorphisms from ?? . We have just shown the top region commutes. Furthermore, since X and $\Sigma^t W_t$ are cellular for all $t \in \mathbb{N}$, the arrows labelled $E_*(-)$ are well-defined, and they clearly make the middle rectangle commute (a simple diagram chase suffices). The bottom rectangle also clearly commutes. Thus, it suffices to show that the maps labelled $E_*(-)$ are isomorphisms. To that end, consider the following diagram:

$$\begin{array}{ccc}
[X, \Sigma^t W_t]_{a+s} & \xrightarrow{E_*(-)} & \mathrm{Hom}_{E_*(E)}(E_*(\Sigma^{a+s} X), E_*(\Sigma^t W_t)) \\
f_* \downarrow & & \downarrow E_*(f)_* \\
[X, E \otimes \Sigma^t Y_t]_{a+s} & \xrightarrow{E_*(-)} & \mathrm{Hom}_{E_*(E)}(E_*(\Sigma^{a+s} X), E_*(E \otimes \Sigma^t Y_t))
\end{array}$$

where here $f : \Sigma^t W_t \rightarrow E \otimes \Sigma^t Y_t$ is the isomorphism

$$\Sigma^t W_t \xrightarrow{\nu_{W_t}^t} \Sigma^t W_t = S^t \otimes E \otimes Y_t \xrightarrow{\tau \otimes Y_t} E \otimes S^t \otimes Y_t = E \otimes \Sigma^t Y_t.$$

The bottom horizontal arrow is an isomorphism by [Lemma 0.8](#). Thus, the top horizontal arrow is an isomorphism, as desired. Showing

$$E_*(-) : [X, \Sigma^{t+1} W_{t+1}]_{a+s} \rightarrow \mathrm{Hom}_{E_*(E)}(E_*(\Sigma^{a+s} X), E_*(\Sigma^{t+1} W_{t+1}))$$

is an isomorphism is entirely analogous. Thus, for each $t \in \mathbb{N}$, we have constructed isomorphisms

$$E^{t,a+s-t}(X, Y) \xrightarrow{\cong} \mathrm{Hom}_{E_*(E)}^{a+s}(E_*(X), E_*(\Sigma^t W_t))$$

such that the following diagram commutes:

$$\begin{array}{ccc}
E^{t,a+s-t}(X, Y) & \xrightarrow{d_1} & E^{t+1,a+s-t-1}(X, Y) \\
\cong \downarrow & & \downarrow \cong \\
\mathrm{Hom}_{E_*(E)}^{a+s}(E_*(X), E_*(\Sigma^t W_t)) & \xrightarrow{\mathrm{Hom}_{E_*(E)}^{a+s}(E_*(X), E_*(\delta_t))} & \mathrm{Hom}_{E_*(E)}^{a+s}(E_*(X), E_*(\Sigma^{t+1} W_{t+1}))
\end{array}$$

Hence, we have proven the desired result. \square

0.3. Convergence of the spectral sequence. In this subsection, we briefly sketch some convergence properties of the spectral sequence. For the relevant definitions and theorems, we refer the reader to [1], all of the work done therein carries over entirely to our setting. Of particular relevance are his definitions of convergence of spectral sequences (see Definition 5.2) and conditional convergence of a spectral sequence associated to an unrolled exact couple (Definition 5.10). We adopt his notation, writing $E_\infty(X, Y)$ and $RE_\infty(X, Y)$ to denote the E_∞ -term and the derived E_∞ -term of the spectral sequence, respectively.

Ideally, the E -Adams spectral sequence for $[X, Y]_*$ would give us information which allows us to compute the group $[X, Y]_*$. Note that $[X, Y]_*$ is the colimit of the unrolled exact couple which determines the spectral sequence, as $Y_s = Y$ for $s < 0$. Furthermore, since $E_r^{s,a}(X, Y) = 0$ for $s < 0$, we may think of the E -Adams spectral sequence for $[X, Y]_*$ as a *half-plane spectral sequence with entering differentials*, in the sense of [1, §7], where suitable conditions under which the spectral sequence converges to the colimit $[X, Y]_*$ are described (see Theorem 7.3 there). In practice, the conditions outlined there are not usually satisfied, namely, in order for the spectral sequence to converge to $[X, Y]_*$, we must have that $\lim_s [X, Y_s]_* = 0$. In practice, there is no reason to believe this would be satisfied. Yet, there is still hope. Following [2, §5], we can instead set up the spectral sequence by means of a tower *under* Y . First, we must define the *E -nilpotent completion of Y* .

Definition 0.10 ([2, pgs. 272–273]). Let (E, μ, e) be a monoid object in \mathcal{SH} , and Y any object. Write \overline{E} for the fiber (??) of the unit $S \xrightarrow{e} E$, so we have a distinguished triangle

$$\overline{E} \rightarrow S \xrightarrow{e} E \rightarrow \Sigma \overline{E}.$$

Set $Y_0 := Y$ and $W_0 := Y \otimes E$, and for $s > 0$ define $Y_s := Y \otimes \overline{E}^s$ and $W_s := Y_s \otimes E$. Then since \mathcal{SH} is tensor triangulated, for each $s \geq 0$ we may tensor the above sequence with Y_s on the right, which yields the following distinguished triangle

$$Y_{s+1} \xrightarrow{i} Y_s \xrightarrow{j} W_s \xrightarrow{k} \Sigma Y_{s+1}.$$

Then for $s \geq 0$, define Y/Y^s (up to non-canonical isomorphism) to be the cofiber of $i^s : Y_s \rightarrow Y_0 = Y$ (so in particular we may take $Y/Y_1 = E \otimes Y$ and $Y/Y_0 = 0$), so we have a distinguished triangle

$$Y_s \xrightarrow{i^s} Y \xrightarrow{b} Y/Y_s \xrightarrow{c} \Sigma Y_s.$$

Then for each $s \geq 0$, by the octahedral axiom (axiom TR5) for a triangulated category applied to the triangles

$$\begin{aligned} Y_{s+1} &\xrightarrow{i} Y_s \xrightarrow{j} W_s \xrightarrow{k} \Sigma Y_{s+1} \\ Y_s &\xrightarrow{i^s} Y \xrightarrow{b} Y/Y_s \xrightarrow{c} \Sigma Y_s \\ Y_{s+1} &\xrightarrow{i^{s+1}} Y \xrightarrow{b} Y/Y_{s+1} \xrightarrow{c} \Sigma Y_{s+1}, \end{aligned}$$

there exists a distinguished triangle

$$(1) \quad W_s \xrightarrow{p} Y/Y_{s+1} \xrightarrow{q} Y/Y_s \xrightarrow{r} \Sigma W_s.$$

is distinguished and the following diagram commutes:

$$(2) \quad \begin{array}{ccccccc} Y_{s+1} & \xrightarrow{i^{s+1}} & Y & \xrightarrow{b} & Y/Y_s & \xrightarrow{r} & \Sigma W_s \\ & \searrow i & \nearrow i^s & \searrow b & \nearrow q & \searrow c & \nearrow \Sigma j \\ & & Y_s & & Y/Y_{s+1} & & \Sigma Y_s \\ & & \searrow j & \nearrow p & \searrow c & \nearrow \Sigma i & \\ & & & W_s & \xrightarrow{k} & \Sigma Y_{s+1} & \end{array}$$

The triangles from (1) may be spliced together to yield a tower $\{Y/Y_s\}_s$ under Y :

$$\begin{array}{ccccccc} Y & \longrightarrow & \cdots & \longrightarrow & Y/Y_3 & \xrightarrow{q} & Y/Y_2 & \xrightarrow{q} & Y/Y_1 & \xrightarrow{q} & Y/Y_0 & = 0 \\ & & & & \downarrow r & \swarrow p & \downarrow r & \swarrow p & \downarrow r & \swarrow p & \downarrow r \\ & & & & W_3 & & W_2 & & W_1 & & W_0 \end{array}$$

where here the dashed arrows are really (degree -1) maps $Y/Y_s \rightarrow \Sigma W_s$. The fact that this is a tower under Y follows from diagram (??), which tells us that $Y \xrightarrow{b} Y/Y_s$ factors as $Y \xrightarrow{b} Y/Y_{s+1} \xrightarrow{q} Y/Y_s$. We define the *E-nilpotent completion of Y* to be the object Y_E^\wedge (defined up to non-canonical isomorphism) obtained as the homotopy limit of this tower (??):

$$Y_E^\wedge := \operatorname{holim}_s Y_s/Y.$$

This comes equipped with a map $\alpha : Y \rightarrow Y_E^\wedge$.

Remark 0.11. In [2], the E -nilpotent completion of Y (Definition 0.10) is denoted “ $E^\wedge Y$ ”, while the notation “ Y_E^\wedge ” we use here is standard in the modern literature.

It turns out that applying $[X, -]_*$ to this tower under Y yields an exact couple, the associated spectral sequence of which is precisely the E -Adams spectral sequence for $[X, Y]_*$.

Proposition 0.12. Consider the tower under Y constructed in Definition 0.10:

$$\begin{array}{ccccccc} Y & \longrightarrow & \cdots & \longrightarrow & Y/Y_3 & \xrightarrow{q} & Y/Y_2 & \xrightarrow{q} & Y/Y_1 & \xrightarrow{q} & Y/Y_0 & = 0 \\ & & & & \downarrow r & \swarrow p & \downarrow r & \swarrow p & \downarrow r & \swarrow p & \downarrow r \\ & & & & W_3 & & W_2 & & W_1 & & W_0 \end{array}$$

We may extend it to the right by defining $Y/Y_s = W_s = 0$ for $s < 0$. Then by ??, we may apply the functor $[X, -]_*$ which yields the following A -graded unrolled exact couple (??):

$$\begin{array}{ccccccc} \cdots & \longrightarrow & [X, Y/Y_{s+2}]_* & \xrightarrow{q} & [X, Y/Y_{s+1}]_* & \xrightarrow{q} & [X, Y/Y_s]_* & \xrightarrow{q} & [X, Y/Y_{s-1}]_* & \longrightarrow & \cdots \\ & & \downarrow \delta & \swarrow p & \downarrow \delta & \swarrow p & \downarrow \delta & \swarrow p & \downarrow \delta & & \\ & & [X, W_{s+2}]_* & & [X, W_{s+1}]_* & & [X, W_s]_* & & [X, W_{s-1}]_* & & \end{array}$$

Thus by ??, there is an induced spectral sequence. This spectral sequence is precisely the E -Adams spectral sequence for $[X, Y]_*$ (Definition 0.3).

Proof. For $s \geq 0$, define

$$f_s : [X, Y/Y_s]_* \xrightarrow{c_*} [X, \Sigma Y_s]_* \xrightarrow{(\nu_{Y_s})_*} [X, \Sigma^1 Y_s]_* \xrightarrow{s_{X, Y_s}^1} [X, Y_s]_{*-1},$$

and for $s < 0$ let it be the unique map

$$f_s : [X, Y/Y_s]_* = 0 \rightarrow [X, Y_s]_{*-1} = [X, Y]_{*-1}.$$

For $s \in \mathbb{Z}$, let

$$g_s := \text{id}_{W_s} : [X, W_s]_* \rightarrow [X, W_s]_*.$$

We claim these maps $(f_s, g_s)_s$ define a homomorphism of A -graded unrolled exact couples (??) between the unrolled exact couple given above determined by the quotient tower $\{Y/Y_s\}$ under Y , and that obtained by applying $[X, -]_*$ to the canonical E -Adams resolution. To that end, it suffices to show that the following diagram commutes for all $s \in \mathbb{Z}$:

$$\begin{array}{ccccccc} [X, Y/Y_s]_* & \longrightarrow & [X, Y/Y_{s-1}]_* & \longrightarrow & [X, W_{s-1}]_{*-1} & \longrightarrow & [X, Y/Y_s]_{*-1} \\ f_s \downarrow & & f_{s-1} \downarrow & & \parallel & & \downarrow f_s \\ [X, Y_s]_{*-1} & \longrightarrow & [X, Y_{s-1}]_{*-1} & \longrightarrow & [X, W_{s-1}]_{*-1} & \longrightarrow & [X, Y_s]_{*-2} \end{array}$$

In the case $s \leq 0$, we know $Y/Y_s = Y/Y_{s-1} = W_{s-1} = 0$, so that the top row is entirely 0, and thus the diagram must commute. In the case $s > 0$, by unravelling definitions we have that the diagram becomes

$$\begin{array}{ccccccc} [X, Y/Y_s]_* & \xrightarrow{q_*} & [X, Y/Y_{s-1}]_* & \xrightarrow{\delta} & [X, W_{s-1}]_{*-1} & \xrightarrow{p_*} & [X, Y/Y_s]_{*-1} \\ c_* \downarrow & & \downarrow c_* & \searrow r_* & \parallel & & \downarrow c_* \\ [X, \Sigma Y_s]_* & \xrightarrow{\Sigma i_*} & [X, \Sigma Y_{s-1}]_* & \xrightarrow{\Sigma j_*} & [X, \Sigma W_{s-1}]_* & & [X, \Sigma Y_s]_{*-1} \\ (\nu_{Y_s})_* \downarrow & & \downarrow (\nu_{Y_{s-1}})_* & & \downarrow (\nu_{W_{s-1}})_* & & \downarrow (\nu_{Y_s})_* \\ [X, \Sigma^1 Y_s]_* & \xrightarrow{\Sigma^1 i_*} & [X, \Sigma^1 Y_{s-1}]_* & \xrightarrow{\Sigma^1 j_*} & [X, \Sigma^1 W_{s-1}]_* & & [X, \Sigma^1 Y_s]_{*-1} \\ s_{X, Y_s}^1 \downarrow & & \downarrow s_{X, Y_{s-1}}^1 & & \searrow s_{X, W_{s-1}}^1 & & \downarrow s_{X, Y_s}^1 \\ [X, Y_s]_{*-1} & \xrightarrow{i_*} & [X, Y_{s-1}]_{*-1} & \xrightarrow{j_*} & [X, W_{s-1}]_{*-1} & \xrightarrow{\partial_*} & [X, Y_s]_{*-2} \end{array}$$

Clearly commutativity of this diagram yields that the given collection of maps define a homomorphism of A -graded unrolled exact couples. Each rectangular region commutes by naturality, as does the middle bottom trapezoidal region. The two regions involving δ and ∂ commute by unravelling how the differential is defined in ?. Finally, the remaining two regions commute by commutativity of [Equation 2](#).

Thus, we have defined a homomorphism of A -graded unrolled exact couples, so that by ?? it induces a homomorphism of the associated spectral sequences \tilde{g} . Further unravelling how this homomorphism of spectral sequences is defined, since the homomorphism of unrolled exact couples is the identity on the $[X, W_s]_*$ terms, it follows that the two spectral sequences are strictly equal. \square

By means of this new presentation of the spectral sequence, we may consider the sense in which the spectral sequence converges to the *limit* $\lim_s [X, Y/Y_s]_*$ of the tower $\{Y/Y_s\}_s$ under Y . First of all, it is standard that since Y_E^\wedge is the homotopy limit of this tower, we have a *Milnor short exact sequence*

$$0 \rightarrow \text{Rlim}_s [X, Y/Y_s]_{*+1} \rightarrow [X, Y_E^\wedge]_* \rightarrow \lim_s [X, Y/Y_s]_* \rightarrow 0$$

(the same argument given in [1, Theorem 4.9] works, although we warn the reader that Boardmann has a sign error there — he writes the first term in the short exact sequence with -1 , when it

should be $+1$). Thus, if $\mathrm{Rlim}_s [X, Y/Y_s]_*$ vanishes, we get an identification of the limit

$$\lim_s [X, Y/Y_s]_* = [X, Y_E^\wedge]_*.$$

By [1, Theorem 7.4], this is further satisfied if the derived E_∞ -term $RE_\infty(X, Y)$ is zero, in which case the spectral sequence converges strongly to the limit $[X, Y_E^\wedge]_*$, meaning in particular the natural maps

$$\begin{aligned} [X, Y_E^\wedge]_* &\rightarrow \lim_s [X, Y_E^\wedge]_* / F^s [X, Y_E^\wedge]_* \\ F^s [X, Y_E^\wedge]_* / F^{s+1} [X, Y_E^\wedge]_* &\rightarrow E_\infty^{s,*}(X, Y) \end{aligned}$$

are isomorphisms, where here F^s is the decreasing filtration on $[X, Y_E^\wedge]_*$ given by

$$F^s [X, Y_E^\wedge]_* := \ker([X, Y_E^\wedge]_* = \lim_s [X, Y/Y_s]_* \rightarrow [X, Y/Y_s]_*).$$

Remark 0.13. In [2, §6], work is done in the classical setting to analyze the conditions under which the derived page RE_∞ vanishes, and $[X, Y_E^\wedge]_*$ tells us information about $[X, Y]_*$. Specifically, Bousfield describes some cases under which the canonical map $Y \rightarrow Y_E^\wedge$ is an isomorphism. More work should be done to see which of those results transfer over to our setting. In particular, there may be connectivity conditions one can place on Y under which one gets convergence of the spectral sequence.