We assume the reader is familiar with additive categories and (closed, symmetric) monoidal categories.

Definition 0.1. Let $\mathcal{C}$ be an additive category. Given a sequence

$$X_1 \to \cdots \to X_n$$

of morphisms in $\mathcal{C}$, we say thsi sequence is *exact* if, for any object A in $\mathcal{C}$, the induced sequence

$$\mathcal{C}(A, X_1) \to \mathcal{C}(A, X_n) \to \cdots \to \mathcal{C}(A, X_n)$$

is an exact sequence of abelian groups.

Definition 0.2. A *triangulated category* is a tuple $(\mathcal{C}, \Sigma, \Omega, \mathcal{D})$ such that

(1) $\mathcal{C}$ is an additive category.
(2) $\Sigma, \Omega : \mathcal{C} \to \mathcal{C}$ are additive functors which form an adjoint equivalence of $\mathcal{C}$ with itself. (Σ is calld the *shift functor*.)
(3) $\mathcal{D}$ is a collection of *distinguished triangles*, where a *triangle* is a diagram of the form

$$X \to Y \to Z \to \Sigma X.$$

These are also sometimes called *cofiber sequences* or *fiber sequences*.

These data must satisfy the following axioms:

TR0 Given a commutative diagram

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "\cong"'] & Y \arrow[r] \arrow[d, "\cong"'] & Z \arrow[r] \arrow[d, "\cong"'] & \Sigma X \arrow[d, "\cong"'] \\
X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & \Sigma X'
\end{tikzcd}$$

where the vertical arrows are isomorphisms, if the top row is distinguished then so is the bottom.

TR1 For any object X in $\mathcal{C}$, the diagram

$$X \xrightarrow{\mathrm{id}_X} X \to 0 \to \Sigma X$$

is a distinguished triangle.

TR2 For all $f : X \to Y$ there exists an object C_f (also sometimes denoted Y/X) called the *cofiber of f* and a distinguished triangle

$$X \xrightarrow{f} Y \to C_f \to \Sigma X.$$

TR3 Given a solid diagram

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "f"'] & Y \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d, dashed] & \Sigma X \arrow[d, "\Sigma f"] \\
X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & \Sigma X'
\end{tikzcd}$$

such that the leftmost square commmutes and both rows are distinguished, there exists a dashed arrow $Z \to Z'$ which makes the remaining two squares commute.

TR4 A triangle

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{\Sigma} X$$

is distinguished if and only if

$$Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X \xrightarrow{-\Sigma f} \Sigma Y$$

is distinguished.

TR5 (Octahedral axiom) Given three distinguished triangles

$$X \xrightarrow{f} Y \xrightarrow{h} Y/X \to \Sigma X$$

$$Y \xrightarrow{g} Z \xrightarrow{k} Z/Y \to \Sigma Y$$

$$X \xrightarrow{g\circ f} Z \xrightarrow{l} Z/X \to \Sigma X$$

there exists a distinguished triangle

$$Y/X \xrightarrow{u} Z/X \xrightarrow{v} Z/Y \xrightarrow{w} \Sigma(Y/X)$$

such that the following diagram commutes

$$\begin{tikzcd}
X \arrow[rr, "g\circ f"] \arrow[dr, "f"'] & & Z \arrow[rr, "k"] \arrow[dr, "l"] & & Z/Y \arrow[rr, "w"] \arrow[dr] & & \Sigma(Y/X) \\
& Y \arrow[ur, "g"'] \arrow[dr, "h"'] & & Z/X \arrow[ur, "v"'] \arrow[dr] & & \Sigma Y \arrow[ur, "\Sigma h"'] & \\
& & Y/X \arrow[ur, "u"'] \arrow[rr] & & \Sigma X \arrow[ur, "\Sigma f"'] & &
\end{tikzcd}$$

It turns out that the above definition is actually redundant; TR3 and TR4 follow from the remaining axioms (see Lemmas 2.2 and 2.4 in [1]). In this section, we fix a triangulated category $\mathcal{C}$, and we will always use brackets $[-,-]$ to denote the hom-set in a triangulated category $\mathcal{C}$. We now recall several important propositions for triangulated categories:

Proposition 0.3. *Let $(\mathcal{C}, \Sigma)$ be a triangulated category. Then any distinguished triangle in $\mathcal{C}$ is an exact sequence.*

Proof. Suppose we have some distinguished triangle

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X.$$

Then we would like to show that given any object A in $\mathcal{C}$, that the sequence

$$[A, X] \xrightarrow{f_*} [A, Y] \xrightarrow{g_*} [A, Z] \xrightarrow{h_*} [A, \Sigma X]$$

is exact. First we show exactness at $[A, Y]$. To see $\operatorname{im} f_* \subseteq \ker g_*$, note it suffices to show that $g \circ f = 0$. Indeed, consider the commuting diagram

$$\begin{tikzcd}
X \arrow[r, equal] \arrow[d, equal] & X \arrow[r] \arrow[d, "f"] & 0 \arrow[r] & \Sigma X \\
X \arrow[r, "f"] & Y \arrow[r, "g"] & Z \arrow[r, "h"] & \Sigma X
\end{tikzcd}$$

The top row is distinguished by axiom TR1. Thus by TR3, the following diagram commutes:

$$\begin{tikzcd}
X \arrow[r, equal] \arrow[d, equal] & X \arrow[r] \arrow[d, "f"] & 0 \arrow[r] \arrow[d] & \Sigma X \arrow[d, equal] \\
X \arrow[r, "f"] & Y \arrow[r, "g"] & Z \arrow[r, "h"] & \Sigma X
\end{tikzcd}$$

In particular, commutativity of the second square tells us that $g \circ f = 0$, as desired. Conversely, we'd liek to show that $\ker g_* \subseteq \operatorname{im} f_*$. Let $\psi : A \to Y$ be in the kernel of g_*, so that $g \circ \psi = 0$. Consider the following commutative diagram:

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "\psi"'] & 0 \arrow[r] \arrow[d] & \Sigma A \arrow[r, "-\Sigma \mathrm{id}_A"] & \Sigma A \\
Y \arrow[r, "g"] & Z \arrow[r, "h"] & \Sigma X \arrow[r, "-\Sigma f"] & \Sigma Y
\end{tikzcd}$$

The top row is distinguished by axioms TR1 and TR4. The bottom row is distinguished by axiom TR4. Thus by axiom TR3 there exists a map $\widetilde{\phi} : \Sigma A \to \Sigma X$ such that the following diagram commutes:

$$\begin{tikzcd} A \arrow[r] \arrow[d, "\psi"'] & 0 \arrow[r] \arrow[d] & \Sigma A \arrow[r, "-\Sigma\mathrm{id}_A"] \arrow[d, "\widetilde{\phi}"'] & \Sigma A \arrow[d, "\Sigma\psi"'] \\ Y \arrow[r, "g"] & Z \arrow[r, "h"] & \Sigma X \arrow[r, "-\Sigma f"] & \Sigma Y \end{tikzcd}$$

Now, since Σ is an equivalence, it is a full functor, so that in particular there exists some $\phi : A \to X$ such that $\widetilde{\phi} = \Sigma\phi$. Then by faithfullness, we may pull back the right square to get a commuting diagram

$$\begin{tikzcd} A \arrow[r, "-\mathrm{id}_A"] \arrow[d, "\phi"'] & A \arrow[d, "\psi"] \\ X \arrow[r, "-f"'] & Y \end{tikzcd}$$

Hence,

$$f_*(\phi) = f \circ \phi \overset{(*)}{=} -((-f) \circ \phi) = -(\psi \circ (-\mathrm{id}_A)) \overset{(*)}{=} \psi \circ \mathrm{id}_A = \psi,$$

where the equalities marked $(*)$ follow by bilinearity of composition in an additive category. Thus $\psi \in \operatorname{im} f_*$, as desired, meaning $\ker g_* \subseteq \operatorname{im} f_*$.

Now, we have shown that

$$[A, X] \xrightarrow{f_*} [A, Y] \xrightarrow{g_*} [A, Z] \xrightarrow{h_*} [A, \Sigma X]$$

is exact at $[A, Y]$. It remains to show exactness at $[A, Z]$. Yet this follows by the exact same argument given above applied to the sequence obtained from the shifted triangle (TR4)

$$Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X \xrightarrow{-\Sigma f} \Sigma Y.$$ □

Lemma 0.4. *Suppose we have a commutative diagram*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "j"] & Y \arrow[r, "g"] \arrow[d, "k"] & Z \arrow[r, "h"] \arrow[d, "\ell"] & \Sigma X \arrow[d, "\Sigma j"] \\ X' \arrow[r, "f'"] & Y' \arrow[r, "g'"] & Z' \arrow[r, "h'"] & \Sigma X' \end{tikzcd}$$

with both rows distinguished. Then if any two of the maps j, k, and ℓ are isomorphisms, then so is the third.

Proof. Suppose we are given any object A in $\mathcal{C}$, and consider the commutative diagram

$$\begin{tikzcd} {[A,X]} \arrow[r, "f_*"] \arrow[d, "j_*"] & {[A,Y]} \arrow[r, "g_*"] \arrow[d, "k_*"] & {[A,Z]} \arrow[r, "k_*"] \arrow[d, "\ell_*"] & {[A,\Sigma X]} \arrow[r, "-\Sigma f_*"] \arrow[d, "\Sigma j_*"] & {[A,\Sigma Y]} \arrow[r, "-\Sigma g_*"] \arrow[d, "\Sigma k_*"] & {[A,\Sigma Z]} \arrow[r, "-\Sigma h_*"] \arrow[d, "\Sigma \ell_*"] & {[A,\Sigma^2 X]} \arrow[d, "\Sigma^2 j_*"] \\ {[A,X']} \arrow[r, "f'_*"] & {[A,Y']} \arrow[r, "g'_*"] & {[A,Z']} \arrow[r, "h'_*"] & {[A,\Sigma X']} \arrow[r, "-\Sigma f'_*"] & {[A,\Sigma Y']} \arrow[r, "-\Sigma g'_*"] & {[A,\Sigma Z']} \arrow[r, "-\Sigma h'_*"] & {[A,\Sigma^2 X']} \end{tikzcd}$$

The rows are exact by Proposition 0.3 and repeated applications of axiom TR4. It follows by the five lemma that if j and k are isomorphisms, then ℓ_* is an isomorphism. Similarly, if k and ℓ are isomorphisms then Σj_* is an isomorphism. Finally, if ℓ and j are isomorphisms, then Σk_* is an isomorphism. The desired result follows by faithfullness of Σ and the Yoneda embedding. □

Proposition 0.5. *Given a map $f : X \to Y$ in a triangulated category $(\mathcal{C}, \Sigma, \Omega, \mathcal{D})$, the cofiber sequence of f is unique up to isomorphism, in the sense that given any two distinguished triangles*

$$X \xrightarrow{f} Y \to Z \to \Sigma X \qquad \textit{and} \qquad X \xrightarrow{f} Y \to Z' \to \Sigma X,$$

there exists an isomorphism $Z \to Z'$ which makes the following diagram commute:

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, equal] & Y \arrow[r] \arrow[d, equal] & Z \arrow[r] \arrow[d, dashed, "k"] & \Sigma X \arrow[d, equal] \\
X \arrow[r, "f"] & Y \arrow[r] & Z' \arrow[r] & \Sigma X
\end{tikzcd}$$

Proof. Suppose we have two distinguished triangles

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X \qquad \text{and} \qquad X \xrightarrow{f} Y \xrightarrow{g'} Z' \xrightarrow{h'} \Sigma X,$$

and consider the following commutative diagram

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, equal] & Y \arrow[r, "g"] \arrow[d, equal] & Z \arrow[r, "h"] & \Sigma X \\
X \arrow[r, "f"] & Y \arrow[r, "g'"] & Z' \arrow[r, "h'"] & \Sigma X
\end{tikzcd}$$

By axiom TR3, there exists some map $k : Z \to Z'$ which makes the following diagram commute:

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, equal] & Y \arrow[r, "g"] \arrow[d, equal] & Z \arrow[r, "h"] \arrow[d, "k"] & \Sigma X \arrow[d, equal] \\
X \arrow[r, "f"] & Y \arrow[r, "g'"] & Z' \arrow[r, "h'"] & \Sigma X
\end{tikzcd}$$

Now, by Lemma 0.4, k is an isomorphism. $\square$

Proposition 0.6. *Given an arrow $f : X \to Y$ in a triangulated category $(\mathcal{C}, \Sigma, \Omega, \mathcal{D})$, there exists an object F_f called the* fiber *of f, and a distinguished triangle*

$$F_f \to X \xrightarrow{f} Y \to \Sigma F_f (\cong C_f).$$

Proof. By axiom TR2, we have a distinguished triangle

$$X \xrightarrow{f} Y \xrightarrow{g} C_f \xrightarrow{h} \Sigma X.$$

Now, consider the commutative diagram

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, equal] & Y \arrow[r, "\widetilde{g}"] \arrow[d, equal] & \Sigma\Omega C_f \arrow[r, "\widetilde{h}"] \arrow[d, "\varepsilon_{C_f}"'] & \Sigma X \arrow[d, equal] \\
X \arrow[r, "f"] & Y \arrow[r, "g"] & C_f \arrow[r, "h"] & \Sigma X
\end{tikzcd}$$

where $\varepsilon : \Sigma\Omega \Rightarrow \mathrm{Id}_{\mathcal{C}}$ is the counit of the adjunction $\Sigma \dashv \Omega$, $\widetilde{g} = \varepsilon_{C_f}^{-1} \circ g$, and $\widetilde{h} = \varepsilon_{C_f}^{-1} \circ h$. Since each vertical map is an isomorphism and the bottom row is distinguished, the top row is also distinguished by axiom TR0. Now, since Σ is an equivalence of categories, it is faithful, so that in particular there exists some map $k : \Omega C_f \to X$ such that $\Sigma k = -\widetilde{h} \implies -\Sigma k = \widetilde{h}$. Thus, we have a distinguished triangle of the form

$$X \xrightarrow{f} Y \xrightarrow{\widetilde{g}} \Sigma\Omega C_f \xrightarrow{-\Sigma k} \Sigma X.$$

Finally, by axiom TR4, we get a distinguished triangle

$$\Omega C_f \xrightarrow{k} X \xrightarrow{f} Y \xrightarrow{\widetilde{g}} \Sigma\Omega C_f,$$

so we may define the fiber of f to be ΩC_f. $\square$

Lemma 0.7. *Given a triangle*

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X,$$

it can be shifted to the left to obtain a distinguished triangle

$$\Omega Z \xrightarrow{-\widetilde{h}} X \xrightarrow{f} Y \xrightarrow{\widetilde{g}} \Sigma\Omega Z,$$

where

$$\widetilde{h} : \Omega Z \xrightarrow{\Omega h} \Omega\Sigma X \xrightarrow{\eta_X^{-1}} X \qquad \text{and} \qquad \widetilde{g} : Y \xrightarrow{g} Z \xrightarrow{\varepsilon_Z^{-1}} \Sigma\Omega Z,$$

where ε and η are the counit and unit, respectively, of the adjunction $\Sigma \dashv \Omega$.

Proof. Consider the following diagram:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, equal] & Y \arrow[r, "\widetilde{g}"] \arrow[d, equal] & \Sigma\Omega Z \arrow[r, "\Sigma\widetilde{h}"] \arrow[d, "\varepsilon_Z"] & \Sigma X \arrow[d, equal] \\ X \arrow[r, "f"] & Y \arrow[r, "g"] & Z \arrow[r, "h"] & \Sigma X \end{tikzcd} \tag{1}$$

The left and middle squares commute by definition. To see that the right square commutes, consider the following diagram:

$$\begin{tikzcd} \Sigma\Omega Z \arrow[r, "\Sigma\Omega h"] \arrow[d, "\varepsilon_Z"'] & \Sigma\Omega\Sigma X \arrow[r, "\Sigma\eta_X^{-1}"] \arrow[dr, "\varepsilon_{\Sigma X}"] & \Sigma X \arrow[d, equal] \\ Z \arrow[rr, "h"'] & & \Sigma X \end{tikzcd}$$

By functoriality of Σ, the top composition is $\Sigma\widetilde{h}$. The left region commutes by naturality of ε. Commutativity of the right region is precisely one of the the zig-zag identities for the unit and counit of an adjunction. Hence, since diagram (1) commutes, the vertical arrows are isomorphisms, and the bottom row is distinguished, we have that the top row is distinguished as well by axiom TR0. Then by axiom TR4, since $(f, \widetilde{g}, \Sigma\widetilde{h})$ is distinguished, so is the triangle

$$\Omega Z \xrightarrow{-\widetilde{h}} X \xrightarrow{f} Y \xrightarrow{\widetilde{g}} \Sigma\Omega Z. \qquad \square$$

Lemma 0.8. *Suppose we have a distinguished triangle*

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X.$$

Then for any integer n, the sequence

$$\Sigma^n X \xrightarrow{\Sigma^n f} \Sigma^n Y \xrightarrow{\Sigma^n g} \Sigma^n Z \xrightarrow{\Sigma^n h} \Sigma\Sigma^n X$$

is exact (Definition 0.1), where $\Sigma^0 := \mathrm{Id}_{\mathcal{C}}$, and for $n > 0$, $\Sigma^{-n} := \Omega^n$.

Proof. By Proposition 0.3, this holds when $n = 0$. Now suppose we are given some $n > 0$. Using axiom TR4, by induction we have that the triangle

$$\Sigma^n X \xrightarrow{(-1)^n \Sigma^n f} \Sigma^n Y \xrightarrow{(-1)^n \Sigma^n g} \Sigma^n Z \xrightarrow{(-1)^n \Sigma^n h} \Sigma\Sigma^n X$$

is also distinguished. Thus, again by Proposition 0.3, given any object A in $\mathcal{C}$, the sequence of abelian groups

$$[A, \Sigma^n X] \xrightarrow{(-1)^n \Sigma^n f_*} [A, \Sigma^n Y] \xrightarrow{(-1)^n \Sigma^n g_*} [A, \Sigma^n Z] \xrightarrow{(-1)^n \Sigma^n h_*} [A, \Sigma\Sigma^n X]$$

is exact. A simple diagram chase yields that we can remove the signs and the sequence is still exact, so we have shown the desired statement for $n > 0$. Now, we would like to show that for $n > 0$ that the sequence

$$\Omega^n X \xrightarrow{\Omega^n f} \Omega^n Y \xrightarrow{\Omega^n g} \Omega^n Z \xrightarrow{\Omega^n h} \Sigma\Omega^n X$$

is exact. First of all, consider the commutative diagram ☐

Proposition 0.9. *Given a distinguished triangle*

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$$

and any object A in $\mathcal{C}$, there is a long exact sequence of abelian groups

$$\cdots \to [\Sigma^{n+1}A, Z] \xrightarrow{\partial} [\Sigma^n X, X] \xrightarrow{f_*} [\Sigma^n A, Y] \xrightarrow{g_*} [\Sigma^n A, Z] \xrightarrow{\partial} [\Sigma^{n-1}A, X] \to \cdots$$

extending infinitely in either direction, where for $n > 0$ we define $\Sigma^{-n} := \Omega^n$, and ∂ is the map

$$[\Sigma^{n+1}A, Z] \xrightarrow{h_*} [\Sigma^{n+1}A, \Sigma X] \cong [\Sigma^{-1}\Sigma^{n+1}A, X] \cong [\Sigma^n A, X].$$

Proof. ☐

Also important for our work is the concept of a *tensor triangulated category*, that is, a triangulated symmetric monoidal category in which the triangulated structures are compatible, in the following sense:

Definition 0.10. A *tensor triangulated category* is a triangulated symmetric monoidal category $(\mathcal{C}, \otimes, S, \Sigma, \Omega, \mathcal{D})$ such that:

TT1 For all objects X and Y in $\mathcal{C}$, there are natural isomorphisms

$$e_{X,Y} : \Sigma X \otimes Y \xrightarrow{\cong} \Sigma(X \otimes Y).$$

TT2 For each object X in $\mathcal{C}$, the functor $X \otimes (-) \cong (-) \otimes X$ is an additive functor.

TT3 For each object X in $\mathcal{C}$, the functor $X \otimes (-) \cong (-) \otimes X$ preserves distinguished triangles, in that given a distinguished triangle/(co)fiber sequence

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A,$$

then also

$$X \otimes A \xrightarrow{X \otimes f} X \otimes B \xrightarrow{X \otimes g} X \otimes C \xrightarrow{X \otimes h} \Sigma(X \otimes A)$$

and

$$A \otimes X \xrightarrow{f \otimes X} B \otimes X \xrightarrow{g \otimes X} C \otimes X \xrightarrow{h \otimes X} \Sigma(A \otimes X)$$

are distinguished triangles, where here we are being abusive and writing $X \otimes h$ and $h \otimes X$ to denote the compositions

$$X \otimes C \xrightarrow{X \otimes h} X \otimes \Sigma A \xrightarrow{\tau} \Sigma A \otimes X \xrightarrow{e_{A,X}} \Sigma(A \otimes X) \xrightarrow{\Sigma\tau} \Sigma(X \otimes A)$$

and

$$C \otimes X \xrightarrow{h \otimes X} \Sigma A \otimes X \xrightarrow{e_{A,X}} \Sigma(A \otimes X),$$

respectively.

TT4 Given objects X, Y, and Z in $\mathcal{C}$, the following diagram must commute:

$$\begin{tikzcd}
(\Sigma X \otimes Y) \otimes Z \arrow[r, "e_{X,Y} \otimes Z"] \arrow[d, "\alpha"'] & \Sigma(X \otimes Y) \otimes Z \arrow[r, "e_{X \otimes Y, Z}"] & \Sigma((X \otimes Y) \otimes Z) \arrow[d, "\alpha"] \\
\Sigma X \otimes (Y \otimes Z) \arrow[rr, "e_{X, Y \otimes Z}"'] & & \Sigma(X \otimes (Y \otimes Z))
\end{tikzcd}$$

TT5 The following diagram must commute

$$\begin{tikzcd}
\Sigma S \otimes \Sigma S \arrow[r, "e_{S,\Sigma S}"] \arrow[d, "\tau"'] & \Sigma(S \otimes \Sigma S) \arrow[r, "\Sigma\lambda_{\Sigma S}"] & \Sigma^2 S \arrow[d, "-\mathrm{id}"] \\
\Sigma S \otimes \Sigma S \arrow[r, "e_{S,\Sigma S}"] & \Sigma(S \otimes \Sigma S) \arrow[r, "\Sigma\lambda_{\Sigma S}"] & \Sigma^2 S
\end{tikzcd}$$

Usually, most tensor triangulated categories that arise in nature will satisfy additional coherence axioms (see axioms TC1–TC5 in [1]), but the above definition will suffice for our purposes. To avoid the awkwardness of saying "a tensor triangulated category which is also a closed symmetric monoidal category," we introduce the following (nonstandard) terminology:

Definition 0.11. We say a tensor triangulated category $(\mathcal{C}, \otimes, S, \Sigma, \Omega)$ is *closed* if $\mathcal{C}$ is a closed symmetric monoidal category, in the sense that for each object $X \in \mathcal{C}$, the functor $- \otimes X$ has a right adjoint $F(X, -)$.

Note that given a tensor triangulated category, we have the following characterization of the shift functor:

Proposition 0.12. *Given a tensor triangulated category* $(\mathcal{C}, \otimes, S, \Sigma, \Omega)$*, there is a canonical natural isomorphism* $\Sigma S \otimes - \cong \Sigma$.

Proof. Given an object X in $\mathcal{C}$, we have natural isomorphisms

$$\Sigma S \otimes X \xrightarrow{e_{S,X}} \Sigma(S \otimes X) \xrightarrow{\Sigma\lambda_X} \Sigma X,$$

where λ is the left unitor specified by the monoidal structure on $\mathcal{C}$. ☐

Because of the above proposition, when working with tensor triangulated categories we will often assume that $\Sigma = S^1 \otimes -$ for some object S^1. Note that in the definition of the tensor triangulated category, we chose isomorphisms

$$e_{X,Y} : \Sigma X \otimes Y \xrightarrow{\cong} \Sigma(X \otimes Y),$$

but we just as well could have chosen isomorphisms

$$e'_{X,Y} : X \otimes \Sigma Y \xrightarrow{\cong} \Sigma(X \otimes Y).$$

Remark 0.13. Given a tensor triangulated category $(\mathcal{C}, \otimes, S, \Sigma, \Omega, e)$, there are natural isomorphisms

$$e'_{X,Y} : X \otimes \Sigma Y \xrightarrow{\cong} \Sigma(X \otimes Y)$$

obtained via the composition

$$X \otimes \Sigma Y \xrightarrow{\tau} \Sigma Y \otimes X \xrightarrow{e_{Y,X}} \Sigma(Y \otimes X) \xrightarrow{\Sigma\tau} \Sigma(X \otimes Y).$$

Proposition 0.14. *The isomorphisms* $e'_{X,Y} : X \otimes \Sigma Y \to \Sigma(X \otimes Y)$ *defined in the above remark satisfy the following coherence condition for any objects* X*,* Y*, and* Z*:*

$$\begin{tikzcd}
(X \otimes Y) \otimes \Sigma Z \arrow[rr, "e'_{X\otimes Y,\Sigma Z}"] \arrow[d, "\alpha"'] & & \Sigma((X \otimes Y) \otimes Z) \arrow[d, "\Sigma\alpha"] \\
X \otimes (Y \otimes \Sigma Z) \arrow[r, "X\otimes e'_{Y,Z}"] & X \otimes \Sigma(Y \otimes Z) \arrow[r, "e'_{X,Y\otimes Z}"] & \Sigma(X \otimes (Y \otimes Z))
\end{tikzcd}$$

Proof. By the coherence theorem for monoidal categories, we may assume associativity holds up to strict equality, in which case we simply wish to show that the following diagram commutes:

$$\begin{tikzcd}
X \otimes Y \otimes \Sigma Z \arrow[r, "X\otimes e'_{Y,Z}"] \arrow[dr, "e'_{X\otimes Y,Z}"'] & X \otimes \Sigma(Y \otimes Z) \arrow[d, "e'_{X,Y\otimes Z}"] \\
& \Sigma(X \otimes Y \otimes Z)
\end{tikzcd}$$

Now consider the following diagram:

$$\begin{tikzcd}
X \otimes Y \otimes \Sigma Z \arrow[r, "X\otimes\tau_{Y,\Sigma Z}"] \arrow[d, "\tau_{X\otimes Y,\Sigma Z}"'] & X \otimes \Sigma Z \otimes Y \arrow[r, "X\otimes e_{Z,Y}"] \arrow[d, "\tau_{X,\Sigma Z\otimes Y}"] & X \otimes \Sigma(Z \otimes Y) \arrow[r, "X\otimes\Sigma\tau_{Z,Y}"] & X \otimes \Sigma(Y \otimes Z) \arrow[d, "\tau_{X,\Sigma(Y\otimes Z)}"] \\
\Sigma Z \otimes X \otimes Y \arrow[r, "\Sigma Z\otimes\tau_{X,Y}"] \arrow[dd, "e_{Z,X\otimes Y}"'] & \Sigma Z \otimes Y \otimes X \arrow[r, "e_{Z,Y}\otimes X"] \arrow[d, "e_{Z,Y\otimes X}"'] & \Sigma(Z \otimes Y) \otimes X \arrow[r, "\Sigma\tau_{Z,Y}\otimes X"] \arrow[dl, "e_{Z\otimes Y,X}"] & \Sigma(Y \otimes Z) \otimes X \arrow[d, "e_{Y\otimes Z,X}"] \\
 & \Sigma(Z \otimes Y \otimes X) \arrow[rr, "\Sigma(\tau_{Z,Y}\otimes X)"] \arrow[d, "\Sigma\tau_{Z\otimes Y,X}"] & & \Sigma(Y \otimes Z \otimes X) \arrow[d, "\Sigma\tau_{Y\otimes Z,X}"] \\
\Sigma(Z \otimes X \otimes Y) \arrow[r, "\Sigma(\tau_{Z,X}\otimes Y)"] \arrow[ur, "\Sigma(Z\otimes\tau_{X,Y})"] \arrow[rrr, bend right, "\Sigma\tau_{Z,X\otimes Y}"'] & \Sigma(X \otimes Z \otimes Y) & & \Sigma(X \otimes Y \otimes Z)
\end{tikzcd}$$

Unravelling definitions, the top composition is $e'_{X,Y\otimes Z} \circ X \otimes e'_{Y,Z}$ and the bottom composition is $e'_{X\otimes Y,Z}$, so it suffices to show this diagram commutes. The top left square commutes by coherence for symmetric monoidal categories. The trapezoid below that on the left commutes by naturality of e. The triangle below that commutes by coherence for symmetric monoidal categories. The top right rectangle commutes by functoriality of $- \otimes -$ and naturality of τ. The small triangle below that in the middle of the diagram commutes by axiom TT4 for a tensor triangulated category. Commutativity of the trapezoid on the middle right is naturality of e. Finally, the remaining region on the bottom commutes by coherence for symmetric monoidal categories. $\square$