

0.1. **Setup.** In order to construct an abstract version of the Adams spectral sequence, we need to work in some axiomatic version of a stable homotopy category \mathcal{SH} which acts like the familiar classical stable homotopy category \mathbf{hoSp} (??) or the motivic stable homotopy category $\mathbf{SH}_{\mathcal{S}}$ over some base scheme \mathcal{S} (??). As it turns out, practically all the data we need is the following:

Definition 0.1. A *stable homotopy category* is the following data:

- A closed tensor triangulated category $(\mathcal{SH}, \otimes, S, \Sigma, \Omega)$ with arbitrary small (co)products.
- A pointed abelian group $(A, \mathbf{1})$ and a homomorphism $h : (A, \mathbf{1}) \rightarrow (\mathrm{Pic}(\mathcal{SH}), \Sigma S)$ of pointed groups (i.e., $\mathbf{1}$ is sent to the isomorphism class of ΣS), where $\mathrm{Pic}(\mathcal{SH})$ is the group of isomorphism classes of invertible objects in \mathcal{SH} ¹.
- For each $a \in A$, a chosen object S^a in the isomorphism class $h(a)$.

Given an abstract stable homotopy category as above, we will always assume without loss of generality that $S^0 = S$ and $\Sigma = S^1 \otimes -$ (by ??). we establish the following conventions:

- Given objects X_1, \dots, X_n in \mathcal{SH} , we write $X_1 \otimes \dots \otimes X_n$ to denote the object

$$X_1 \otimes (X_2 \otimes \dots (X_{n-1} \otimes X_n)).$$

In particular, given an object X and a natural number $n > 0$, we write

$$X^n := \overbrace{X \otimes \dots \otimes X}^{n \text{ times}} \quad \text{and} \quad X^0 := S.$$

- We denote the associator, symmetry, left unitor, and right unitor isomorphisms in \mathcal{SH} by

$$\begin{aligned} \alpha_{X,Y,Z} : (X \otimes Y) \otimes Z &\xrightarrow{\cong} X \otimes (Y \otimes Z) & \tau_{X,Y} : X \otimes Y &\xrightarrow{\cong} Y \otimes X \\ \lambda_X : S \otimes X &\xrightarrow{\cong} X & \rho_X : X \otimes S &\xrightarrow{\cong} X. \end{aligned}$$

Often we will suppress these isomorphisms from the notation (particularly the associators).

- Given some $a \in A$, we define the functor $\Sigma^a := S^a \otimes -$, so that in particular $\Sigma^1 = \Sigma$.
- Given two objects X and Y , we denote the hom-abelian group of morphisms from X to Y in \mathcal{SH} by $[X, Y]$, and we denote the internal hom object by $F(X, Y)$. We will often refer to morphisms in \mathcal{SH} as *classes*, as we will think of them as representing homotopy classes of maps.
- Given two objects X and Y in \mathcal{SH} , we may extend the abelian group $[X, Y]$ to an A -graded abelian group $[X, Y]_*$ defined by

$$[X, Y]_a := [\Sigma^a X, Y] = [S^a \otimes X, Y].$$

(See ?? for a review of the theory of A -graded abelian groups, rings, modules, etc.)

- Given an object X in \mathcal{SH} and some $a \in A$, define the abelian group

$$\pi_a(X) := [S^a, X],$$

and write $\pi_*(X)$ for the associated A -graded abelian group $\bigoplus_{a \in A} \pi_a(X)$. We call $\pi_a(X)$ the a^{th} *stable homotopy group of X* .

¹Recall an object X in a symmetric monoidal category is *invertible* if there exists some object Y in \mathcal{SH} and an isomorphism $S \cong Y \otimes X$. To see ΣS is invertible, note that we have isomorphisms

$$\Sigma S \otimes \Omega S \cong \Sigma(S \otimes \Omega S) \cong \Sigma(\Omega S \otimes S) \cong \Sigma \Omega S \otimes S \cong S \otimes S \cong S,$$

where the first isomorphism is axiom TT1 for a tensor triangulated category (??), the second isomorphism is given by the symmetry in \mathcal{SH} , the third isomorphism is again axiom TT1, the fourth isomorphism is the fact that Σ and Ω for an adjoint equivalence, and finally the last isomorphism follows by the fact that S is the monoidal unit in \mathcal{SH} .

- Given two objects E and X in \mathcal{SH} , we define the A -graded abelian groups $E_*(X)$ and $E^*(X)$ by

$$E_a(X) := \pi_a(E \otimes X) = [S^a, E \otimes X] \quad \text{and} \quad E^a(X) := [X, S^a \otimes E].$$

We refer to the functor $E_*(-)$ as the *homology theory represented by E* , or just E -homology, and we refer to $E^*(-)$ as the *cohomology theory represented by E* , or just E -cohomology.

From now on, we fix the data of a stable homotopy category \mathcal{SH} given above once and for all. Observe that for all $a, b \in A$, the objects S^{a+b} and $S^a \otimes S^b$ are isomorphic, since $h : A \rightarrow \text{Pic}(\mathcal{SH})$ is a group homomorphism. Hence given a monoid object (E, μ, e) in \mathcal{SH} (??), supposing we had fixed isomorphisms $S^{a+b} \cong S^a \otimes S^b$ for all $a, b \in A$, we get a multiplication map $\pi_*(E) \times \pi_*(E) \rightarrow \pi_*(E)$ which sends classes $x : S^a \rightarrow E$ and $y : S^b \rightarrow E$ to the product

$$S^{a+b} \cong S^a \otimes S^b \xrightarrow{x \otimes y} E \otimes E \xrightarrow{\mu} E.$$

Naturally, we would like this product to make $\pi_*(E)$ into an A -graded ring (with unit $e \in \pi_0(E) = [S, E]$), rather than just an A -graded abelian group. Whether or not this happens is essentially the entire discussion of Dugger's paper [1], and as it turns out, $\pi_*(E)$ is in fact a graded ring provided we can choose these morphisms to be *coherent*, in the following sense:

Definition 0.2. Suppose we have a family of isomorphisms

$$\phi_{a,b} : S^{a+b} \xrightarrow{\cong} S^a \otimes S^b$$

for all $a, b \in A$. We say this family is *coherent* if:

- (1) For all $a \in A$, we have equalities $\phi_{a,0} = \rho_{S^a}^{-1} : S^a \rightarrow S^a \otimes S$ and $\phi_{0,a} = \lambda_{S^a}^{-1} : S^a \rightarrow S \otimes S^a$.
- (2) For all $a, b, c \in A$, the following diagram commutes:

$$\begin{array}{ccc} S^{a+b} \otimes S^c & \xleftarrow{\phi_{a+b,c}} & S^{a+b+c} \xrightarrow{\phi_{a,b+c}} S^a \otimes S^{b+c} \\ \phi_{a,b} \otimes S^c \downarrow & & \downarrow S^a \otimes \phi_{b,c} \\ (S^a \otimes S^b) \otimes S^c & \xrightarrow{\cong} & S^a \otimes (S^b \otimes S^c) \end{array}$$

Furthermore, Dugger guarantees that we can always find such a coherent family:

Theorem 0.3 ([1, Proposition 7.1]). *There exists a coherent family of isomorphisms*

$$\phi_{a,b} : S^{a+b} \xrightarrow{\cong} S^a \otimes S^b$$

in the sense of Definition 0.2, and in particular, the set of such coherent families is in bijective correspondence with the set of normalized 2-cocycles $Z^2(A; \text{Aut}(S))_{\text{norm}}$, i.e., the set of functions $\alpha : A \times A \rightarrow \text{Aut}(S)$ such that $\alpha(0,0) = \text{id}_S$ and for all $a, b, c \in A$, $\alpha(a+b, c) \cdot \alpha(a, b) = \alpha(b, c) \cdot \alpha(a, b+c)$.

Thus, from now on we will suppose once and for all we have fixed a coherent family $\{\phi_{a,b}\}_{a,b \in A}$. Such a coherent family has very nice properties, in particular:

Remark 0.4. Note that by induction the coherence conditions say that given any $a_1, \dots, a_n \in A$ and $b_1, \dots, b_m \in A$ such that $a_1 + \dots + a_n = b_1 + \dots + b_m$ and any fixed parenthesizations of $X = S^{a_1} \otimes \dots \otimes S^{a_n}$ and $Y = S^{b_1} \otimes \dots \otimes S^{b_m}$, there is a *unique* isomorphism $X \rightarrow Y$ that can be obtained by forming formal compositions of tensor products of $\phi_{a,b}$, identities, associators, and their inverses.

Of course, we get our desired result: $\pi_*(E)$ is indeed an A -graded ring if E is a monoid object.

Proposition 0.5. *Let (E, μ, e) be a commutative monoid object in \mathcal{SH} , and consider the multiplication map $\pi_*(E) \times \pi_*(E) \rightarrow \pi_*(E)$ which sends classes $x : S^a \rightarrow E$ and $y : S^b \rightarrow E$ to the composition*

$$S^{a+b} \xrightarrow{\phi_{a,b}} S^a \otimes S^b \xrightarrow{x \otimes y} E \otimes E \xrightarrow{\mu} E.$$

Then this endows $\pi_(E)$ with the structure of an A -graded ring with unit $e \in \pi_0(E) = [S, E]$.*

Proof. See ??.

□

Furthermore, it turns out that if E is a commutative monoid object in \mathcal{SH} , then $\pi_*(E)$ is “ A -graded commutative,” in the following sense:

Proposition 0.6. *For all $a, b \in A$ there exists an element $\theta_{a,b} \in \pi_0(S) = [S, S]$ (determined by choice of coherent family $\{\phi_{a,b}\}$) such that given any commutative monoid object (E, μ, e) in \mathcal{SH} , the A -graded ring structure on $\pi_*(E)$ (Proposition 0.5) has a commutativity formula given by*

$$x \cdot y = y \cdot x \cdot (e \circ \theta_{a,b})$$

for all $x \in \pi_a(E)$ and $y \in \pi_b(E)$.

Furthermore, $\theta_{0,a} = \theta_{a,0} = \text{id}_S$ for all $a \in A$, so that if either x or y has degree 0, $x \cdot y = y \cdot x$.

Proof. See ?? and ??.

□

We also have the following result:

Proposition 0.7. *Given some $a \in A$, the functors Σ^a and Σ^{-a} canonically form an adjoint equivalence of \mathcal{SH} .*

Proof. See ??.

□

In particular, note that this tells us that given objects E and X in \mathcal{SH} , we have isomorphisms

$$E^*(X) = [X, S^* \otimes X] \cong [S^{-*} \otimes X, E] \cong [S^{-*}, F(X, E)] = \pi_{-*}(F(X, E)).$$

Similarly, given any objects X and Y in \mathcal{SH} , we have isomorphisms of A -graded abelian groups

$$[X, \Sigma Y]_* = [S^* \otimes X, S^1 \otimes Y] \cong [S^{-1} \otimes S^* \otimes X, Y] \cong [S^{*-1} \otimes X, Y] = [X, Y]_{*-1},$$

where the first isomorphism is the adjunction specified by the above proposition, and the second isomorphism is induced by the isomorphism

$$S^{*-1} \otimes X \xrightarrow{\phi_{-1,*} \otimes X} S^{-1} \otimes S^* \otimes X.$$

The last ingredient in order to develop the Adams spectral sequence abstractly is a notion of *cellularity* in \mathcal{SH} :

Definition 0.8. Define the class of *cellular* objects in \mathcal{SH} to be the smallest class of objects such that:

- (1) For all $a \in A$, S^a is cellular.
- (2) If we have a distinguished triangle

$$X \rightarrow Y \rightarrow Z \rightarrow \Sigma X (= S^1 \otimes X)$$

such that two of the three objects X , Y , and Z are cellular, then the third object is also cellular.

- (3) Given a collection of cellular objects X_i indexed by some small set I , $\bigoplus_{i \in I} X_i$ is cellular.

0.2. Construction of the Adams spectral sequence. In what follows, let E be a commutative monoid object in \mathcal{SH} .

Definition 0.9. Let \overline{E} be the fiber of the unit map $e : S \rightarrow E$ (??), and for $s \geq 0$ define

$$Y_s := \overline{E}^s \otimes Y, \quad W_s = E \otimes Y_s = E \otimes (\overline{E}^s \otimes Y),$$

where recall for $s > 0$, \overline{E}^s denotes the s -fold product parenthesized as $\overline{E} \otimes (\overline{E} \otimes \cdots (\overline{E} \otimes \overline{E}))$, and $\overline{E}^0 := S$. Then we get fiber sequences

$$Y_{s+1} \xrightarrow{i_s} Y_s \xrightarrow{j_s} W_s \xrightarrow{k_s} \Sigma Y_{s+1} (= S^1 \otimes Y_{s+1})$$

obtained by applying $-\otimes Y_s$ to the sequence

$$\overline{E} \rightarrow S \xrightarrow{e} E \rightarrow \Sigma \overline{E}$$

(and applying the necessary associator and unitor isomorphisms). These sequences can be spliced together to form the (*canonical*) *Adams filtration* of Y :

$$\begin{array}{ccccccc} \cdots & \longrightarrow & Y_3 & \xrightarrow{i_2} & Y_2 & \xrightarrow{i_1} & Y_1 & \xrightarrow{i_0} & Y_0 = Y \\ & & \downarrow j_3 & \swarrow k_2 & \downarrow j_2 & \swarrow k_1 & \downarrow j_1 & \swarrow k_0 & \downarrow j_0 \\ & & W_3 & & W_2 & & W_1 & & W_0 \end{array}$$

where the diagonal dashed arrows are of degree -1 (note these triangles do NOT commute in any sense). Now we may apply the functor $[X, -]_*$, and by ?? we obtain an exact couple of $\mathbb{N} \times A$ -graded abelian groups:

$$\begin{array}{ccc} [X, Y_*]_* & \xrightarrow{i_{**}} & [X, Y_*]_* \\ & \searrow k_{**} & \downarrow j_{**} \\ & & [X, W_*]_* \end{array}$$

where i_{**} , j_{**} , and k_{**} have $\mathbb{Z} \times A$ -degree $(-1, 0)$, $(0, 0)$, and $(1, -1)$, respectively². The standard argument yields an $\mathbb{N} \times A$ -graded spectral sequence called from this exact couple (cf. Section 5.9 of [2]) with E_1 page given by

$$E_1^{s,a} = [X, W_s]_a$$

and r^{th} differential of $\mathbb{Z} \times A$ -degree $(r, -1)$:

$$d_r : E_r^{s,a} \rightarrow E_r^{s+r,a-1}.$$

A priori, this is all $\mathbb{N} \times A$ -graded, but we regard it as being $\mathbb{Z} \times A$ -graded by setting $E_r^{s,a} := 0$ for $s < 0$ and trivially extending the definition of the differentials to these zero groups. This spectral sequence is called the *E-Adams spectral sequence* for the computation of $[X, Y]_*$. The index s is called the *Adams filtration* and a is the *stem*.

²Explicitly, the map $k_{s,a} : [X, W_s]_a \rightarrow [X, Y_{s+1}]_{a-1}$ sends a map $f : S^a \otimes X \rightarrow W_s$ to the map $S^{a-1} \otimes X \rightarrow Y_{s+1}$ corresponding under the isomorphism $[X, \Sigma Y_{s+1}]_* \cong [X, Y_{s+1}]_{*-1}$ to the composition $k_s \circ f : S^a \otimes X \rightarrow \Sigma Y_{s+1}$.

0.3. Monoid objects in \mathcal{SH} . We have constructed an Adams spectral sequence, but as it currently stands we do not yet know why it is useful. To start with, we'd like to provide a characterization of its E_1 and E_2 pages in terms of something more algebraic. To start, we first need to develop some theory of the algebra of monoid objects in \mathcal{SH} . Much of this work is entirely straightforward although tedious to verify, so we relegate most of the proofs in this section to ??.

Proposition 0.10. *Let (E, μ, e) be a monoid object in \mathcal{SH} . Then $E_*(-)$ is a functor from \mathcal{SH} to left A -graded $\pi_*(E)$ -modules, where given some X in \mathcal{SH} , $E_*(X)$ may be endowed with the structure of a left A -graded $\pi_*(E)$ -module via the map*

$$\pi_*(E) \times E_*(X) \rightarrow E_*(X)$$

which given $a, b \in A$, sends $x : S^a \rightarrow E$ and $y : S^b \rightarrow E \otimes X$ to the composition

$$x \cdot y : S^{a+b} \cong S^a \otimes S^b \xrightarrow{x \otimes y} E \otimes (E \otimes X) \cong (E \otimes E) \otimes X \xrightarrow{\mu \otimes X} E \otimes X.$$

Similarly, the assignment $X \mapsto X_*(E)$ is a functor from \mathcal{SH} to right A -graded $\pi_*(E)$ -modules, where the structure map

$$X_*(E) \times \pi_*(E) \rightarrow X_*(E)$$

sends $x : S^a \rightarrow X \otimes E$ and $y : S^b \rightarrow E$ to the composition

$$x \cdot y : S^{a+b} \cong S^a \otimes S^b \xrightarrow{x \otimes y} (X \otimes E) \otimes E \cong X \otimes (E \otimes E) \xrightarrow{X \otimes \mu} X \otimes E.$$

Finally, $E_*(E)$ is a $\pi_*(E)$ -bimodule, in the sense that the left and right actions of $\pi_*(E)$ are compatible, so that given $y, z \in \pi_*(E)$ and $x \in E_*(E)$, $y \cdot (x \cdot z) = (y \cdot x) \cdot z$.

Proof. See ??.

□

Definition 0.11. Given a monoid object E in \mathcal{SH} , we say E is *flat* if the canonical right $\pi_*(E)$ -module structure on $E_*(E)$ (see the above proposition) is that of a flat module.

0.4. The E_1 page. The goal of this subsection is to provide the following characterization for the E_1 page of the Adams spectral sequence:

Theorem 0.12. *Let E be a flat commutative monoid object in \mathcal{SH} , and let X and Y be two objects in \mathcal{SH} such that $E_*(X)$ is a projective module over $\pi_*(E)$. Then for all $s \geq 0$ and $a \in A$, we have isomorphisms in the associated E -Adams spectral sequence*

$$E_1^{s,a} \cong \text{Hom}_{E_*(E)}^a(E_*(X), E_*(W_s))$$

Furthermore, under these isomorphisms, the differential $d_1 : E_1^{s,a} \rightarrow E_1^{s+1,a-1}$ corresponds to the map

$$\text{Hom}_{E_*(E)}^a(E_*(X), E_*(W_s)) \rightarrow \text{Hom}_{E_*(E)}^{a-1}(E_*(X), E_*(X \otimes W_{s+1}))$$

which sends a map $f : E_*(X) \rightarrow E_{*+a}(W_s)$ to the composition

$$E_*(X) \xrightarrow{f} E_{*+a}(W_s) \xrightarrow{(X \otimes h_s)_*} E_{*+a-1}(X \otimes Y_{s+1}) \xrightarrow{(X \otimes j_{s+1})_*} E_{*+a-1}(X \otimes W_{s+1}).$$

Proof. By ??, for all $s \geq 0$ and $t, w \in \mathbb{Z}$, we have isomorphisms

$$[X, E \otimes Y_s]_{t,w} \cong \text{Hom}_{E_*(E)}^{t,w}(E_*(X), E_*(E \otimes Y_s)).$$

since $W_s = E \otimes Y_s$, we have that

$$E_1^{s,(t,w)} = [X, W_s]_{t,w} \cong \text{Hom}_{E_*(E)}^{t,w}(E_*(X), E_*(W_s)),$$

as desired.

□

Definition 0.13. Let (E, μ, e) be a monoid object in \mathcal{SH} . We say E is *flat* if the canonical right $\pi_*(E)$ -module structure on $E_*(E)$ is that of a flat module.

0.5. The E_2 page.

0.6. **Convergence.** convergence of spectral sequences