In what follows, we fix an abelian group A. We assume the reader is familiar with the basic theory of modules over not-necessarily-commutative rings.

Definition 0.1. An *A-graded abelian group* is an abelian group B along with a subgroup $B_a \leq B$ for each $a \in A$ such that the canonical map

$$\bigoplus_{a \in A} B_a \to B$$

sending $(x_a)_{a \in A}$ to $\sum_{a \in A} x_a$ is an isomorphism. Given two A-graded abelian groups B and C, a homomorphism $f : B \to C$ is a *homomorphism of A-graded abelian groups*, or just an *A-graded homomorphism*, if it preserves the grading, i.e., if it restricts to a map $B_a \to C_a$ for all $a \in A$.

We denote the category of A-graded abelian groups and A-graded homomorphisms between them by $\mathbf{Ab}(A)$

It is easy to see that an A-graded abelian group B is generated by its *homogeneous* elements, that is, nonzero elements $x \in B$ such that there exists some $a \in A$ with $x \in B_a$.

Remark 0.2. Clearly the condition that the canonical map $\bigoplus_{a \in A} B_a \to B$ is an isomorphism requires that $B_a \cap B_b = 0$ if $a \neq b$. In particular, given a homogeneous element $x \in B$, there exists precisely one $a \in A$ such that $x \in B_a$. We call this a the *degree* of x, and we write $|x| = a$.

Definition 0.3. An *A-graded ring* is a ring R such that its underlying abelian group R is A-graded and the multiplication map $R \times R \to R$ restricts to $R_a \times R_b \to R_{a+b}$ for all $a, b \in A$. A morphism of A-graded rings is a ring homomorphism whose underlying homomorphism of abelian groups is A-graded.

Explicitly, given an A-graded ring R and homogeneous elements $x, y \in R$, we must have $|xy| = |x| + |y|$. For example, given some field k, the ring $R = k[x, y]$ is $\mathbb{Z}^2$-graded, where given $(n, m) \in \mathbb{Z}^2$, $R_{n,m}$ is the subgroup of those monomials of the form $ax^n y^m$ for some $a \in k$.

Definition 0.4. Let R be an A-graded ring. A *left A-graded R-module* M is a left R-module M such that M is an A-graded abelian group and the action map $R \times M \to M$ restricts to a map $R_a \times M_b \to M_{a+b}$ for all $a, b \in A$. Right A-graded R-modules are defined similarly. Finally, an A-graded R-bimodule is an A-graded abelian group M which has the structure of both an A-graded left and right R-module such that given $r, s \in R$ and $m \in M$, $r \cdot (m \cdot s) = (r \cdot m) \cdot s$.

Morphisms between A-graded R-modules are precisely A-graded R-module homomorphisms. We write R-$\mathbf{Mod}(A)$ for the category of left A-graded R-modules and $\mathbf{Mod}$-$R(A)$ for the category of right A-graded R-modules.

Remark 0.5. It is straightforward to see that an A-graded abelian group is equivalently an A-graded $\mathbb{Z}$-module, where here we are considering $\mathbb{Z}$ as an A-graded ring concentrated in degree 0. Thus any result below about A-graded modules applies equally to A-graded abelian groups.

Remark 0.6. We often will denote an A-graded R-module M by M_*. Given some $a \in A$, we can define the shifted A-graded abelian group M_{*+a} whose b^{th} component is M_{b+a}.

Definition 0.7. More generally, given two A-graded R-modules M and N and some $d \in A$, an R-module homomorphism $f : M \to N$ is *an A-graded homomorphism of degree d* if it restricts to a map $M_a \to N_{a+d}$ for all $a \in A$. Thus, an A-graded homomorphism of degree d from M

to N is equivalently an A-graded homomorphism $M_* \to N_{*+d}$ or an A-graded homomorphism $M_{*-d} \to N$. Given some $a \in A$ and left (resp. right) R-modules M and N, we will write

$$\mathrm{Hom}_R^d(M, N) = \mathrm{Hom}_R(M_*, N_{*+d}) = \mathrm{Hom}_R(M_{*-d}, N_*)$$

to denote the set of A-graded homomorphisms of degree d from M to N, and simply

$$\mathrm{Hom}_R(M, N)$$

to denote the set of degree-0 A-graded homomorphisms from M to N. Clearly A-graded homomorphisms may be added and subtracted, so these are further abelian groups. Thus we have an A-graded abelian group

$$\mathrm{Hom}_R^*(M, N).$$

Unless stated otherwise, an "A-graded homomorphism" will always refer to an A-graded homomorphism of degree 0.

Oftentimes when constructing A-graded rings, we do so only by defining the product of homogeneous elements, like so:

Lemma 0.8. *Suppose we have an A-graded abelian group R, a distinguished element $1 \in R_0$, and $\mathbb{Z}$-bilinear maps $m_{a,b} : R_a \times R_b \to R_{a+b}$ for all $a, b \in A$. Further suppose that for all $x \in R_a$, $y \in R_b$, and $z \in R_c$, we have*

$$m_{a+b,c}(m_{a,b}(x, y), z) = m_{a,b+c}(x, m_{b,c}(y, z)) \qquad \textit{and} \qquad m_{a,0}(x, 1) = m_{0,a}(1, x) = x.$$

Then there exists a unique multiplication map $m : R \times R \to R$ which endows R with the structure of an A-graded ring and restricts to $m_{a,b}$ for all $a, b \in A$.

Proof. Given $r, s \in R$, since $R \cong \bigoplus_{a \in A} R_a$, we may uniquely decompose r and s into homogeneous elements as $r = \sum_{a \in A} r_a$ and $s = \sum_{a \in A} s_a$ with each $r_a, s_a \in R_a$ such that only finitely many of the r_a's and s_a's are nonzero. Then in order to define a distributive product $R \times R \to R$ which restricts to $m_{a,b} : R_a \times R_b \to R_{a+b}$, note we *must* define

$$r \cdot s = \left(\sum_{a \in A} r_a\right) \cdot \left(\sum_{b \in A} s_b\right) = \sum_{a,b \in A} r_a \cdot s_b = \sum_{a,b \in A} m_{a,b}(r_a, s_b).$$

Thus, we have shown uniqueness. It remains to show this product actually gives R the structure of a ring. First we claim that the sum on the right is actually finite. Note there exists only finitely many nonzero r_a's and s_b's, and if $s_b = 0$ then

$$m_{a,b}(r_a, 0) = m_{a,b}(r_a, 0 + 0) \overset{(*)}{=} m_{a,b}(r_a, 0) + m_{a,b}(r_a, 0) \implies m_{a,b}(r_a, 0) = 0,$$

where $(*)$ follows from bilinearity of $m_{a,b}$. A similar argument yields that $m_{a,b}(0, s_b) = 0$ for all $a, b \in A$. Hence indeed $m_{a,b}(r_a, s_b)$ is zero for all but finitely many pairs $(a, b) \in A^2$, as desired. Observe that in particular

$$(r \cdot s)_a = \sum_{b+c=a} m_{b,c}(r_b, s_c) = \sum_{b \in A} m_{b,a-b}(r_b, s_{a-b}) = \sum_{c \in A} m_{a-c,c}(r_{a-c}, s_c).$$

Now we claim this multiplication is associative. Given $t = \sum_{a\in A} t_a \in R$, we have

$$\begin{aligned}
(r\cdot s)\cdot t &= \sum_{a,b\in A} m_{a,b}((r\cdot s)_a, t_b)\\
&= \sum_{a,b\in A} m_{a,b}\left(\sum_{c\in A} m_{a-c,c}(r_{a-c}, s_c), t_b\right)\\
&\overset{(1)}{=} \sum_{a,b,c\in A} m_{a,b}(m_{a-c,c}(r_{a-c}, s_c), t_b)\\
&\overset{(2)}{=} \sum_{a,b,c\in A} m_{c,a+b-c}(r_c, m_{a-c,b}(s_{a-c}, t_b))\\
&\overset{(3)}{=} \sum_{a,b,c\in A} m_{a,c}(r_a, m_{b,c-b}(s_b, t_{c-b}))\\
&\overset{(1)}{=} \sum_{a,c\in A} m_{a,c}\left(r_a, \sum_{b\in A} m_{b,c-b}(s_b, t_{c-b})\right)\\
&= \sum_{a,c\in A} m_{a,c}(r_a, (s\cdot t)_c) = r\cdot(s\cdot t),
\end{aligned}$$

where each occurrence of (1) follows by bilinearity of the $m_{a,b}$'s, each occurrence of (2) is associativity of the $m_{a,b}$'s, and (3) is obtained by re-indexing by re-defining $a := c$, $b := a - c$, and $c := a+b-c$. Next, we wish to show that the distinguished element $1 \in R_0$ is a unit with respect to this multiplication. Indeed, we have

$$1\cdot r \overset{(1)}{=} \sum_{a\in A} m_{0,a}(1, r_a) \overset{(2)}{=} \sum_{a\in A} r_a = r \qquad \text{and} \qquad r\cdot 1 \overset{(1)}{=} \sum_{a\in A} m_{a,0}(r_a, 1) \overset{(2)}{=} \sum_{a\in A} r_a = r,$$

where (1) follows by the fact that $m_{a,b}(0,-) = m_{a,b}(-,0) = 0$, which we have shown above, and (2) follows by unitality of the $m_{0,a}$'s and $m_{0,a}$'s, respectively. Finally, we wish to show that this product is distributive. Indeed, we have

$$\begin{aligned}
r\cdot(s+t) &= \sum_{a,b\in A} m_{a,b}(r_a, (s+t)_b)\\
&= \sum_{a,b\in A} m_{a,b}(r_a, s_b + t_b)\\
&\overset{(*)}{=} \sum_{a,b\in A} m_{a,b}(r_a, s_b) + \sum_{a,b\in A} m_{a,b}(r_a, t_b) = (r\cdot s) + (r\cdot t),
\end{aligned}$$

where $(*)$ follows by bilinearity of $m_{a,b}$. An entirely analagous argument yields that $(r+s)\cdot t = (r\cdot t) + (s\cdot t)$. $\square$

Similarly, when defining A-graded modules, we will only define the action maps for homogeneous elements:

Lemma 0.9. *Let R be an A-graded ring, M an A-graded abelian group, and suppose there exists $\mathbb{Z}$-bilinear maps $\kappa_{a,b} : R_a \times M_b \to M_{a+b}$ for all $a, b \in A$. Further suppose that for all $r \in R_a$, $r' \in R_b$, and $m \in M_c$ that*

$$\kappa_{a+b,c}(r\cdot r', m) = \kappa_{a,b+c}(r, \kappa_{b,c}(r', m)) \qquad \text{and} \qquad \kappa_{0,c}(1, m) = m.$$

Then there is a unique map $\kappa : R\times M \to M$ which endows M with the structure of a left A-graded R-module and restricts to $\kappa_{a,b}$ for all $a, b \in A$.

On the other hand, suppose there exists $\mathbb{Z}$-bilinear maps $\kappa_{a,b} : M_a \times R_b \to M_{a+b}$ for all $a, b \in A$. Further suppose that for all $r \in R_a$, $r' \in R_b$, and $m \in M_c$ that

$$\kappa_{c,a+b}(m, r \cdot r') = \kappa_{c+a,b}(\kappa_{c,a}(m, r), r') \qquad \text{and} \qquad \kappa_{c,0}(m, 1) = m.$$

Then there is a unique map $\kappa : M \times R \to M$ which endows M with the structure of a right A-graded R-module and restricts to $\kappa_{a,b}$ for all $a, b \in A$.

Finally, if we have maps $\lambda_{a,b} : R_a \times M_b \to M_{a+b}$ and $\rho_{a,b} : M_a \times R_b \to M_{a+b}$ satisfying all of the above conditions, and if we further have that

$$\lambda_{a,b+c}(r, \rho_{b,c}(x, s)) = \rho_{a+b,c}(\lambda_{a,b}(r, x), s)$$

for all $r \in R_a$, $x \in M_b$, and $s \in R_c$, then the left and right A-graded R-module structures induced on M by the λ's and ρ's give M the structure of an A-graded R-bimodule.

Proof. We show the left module case, as the right module case is entirely analagous. Supposing for each $a, b \in A$ we have a map $\kappa_{a,b} : R_a \times M_b \to M_{a+b}$ satisfying the above conditions, in order to extend these to a map $R \times M \to M$, by additivity we *must* define

$$\kappa : R \times M \to M$$

to be the map sending $r = \sum_a r_a$ and $m = \sum_a m_a$ to $\sum_{a,b \in A} \kappa_{a,b}(r_a, m_b)$. Now, we need to check that for all $r, s \in R$, $x, y \in M$ that

(1) $r \cdot (x + y) = r \cdot x + r \cdot y$

(2) $(r + s) \cdot x = r \cdot x + s \cdot x$

(3) $(rs) \cdot x = r \cdot (s \cdot x)$

(4) $1 \cdot x = x$,

where above we are written $- \cdot -$ for $\kappa(-, -)$. To see the first, note

$$\begin{aligned} \kappa(r, x + y) &= \sum_{a,b \in A} \kappa_{a,b}(r_a, (x + y)_b) \\ &= \sum_{a,b \in A} \kappa_{a,b}(r_a, x_b + y_b) \\ &= \sum_{a,b \in A} (\kappa_{a,b}(r_a, x_b) + \kappa_{a,b}(r_a, y_b)) \\ &= \sum_{a,b \in A} \kappa_{a,b}(r_a, x_b) + \sum_{a,b \in A} \kappa_{a,b}(r_a, y_b) \\ &= \kappa(r, x) + \kappa(r, y). \end{aligned}$$

To see the second, note

$$\begin{aligned} \kappa(r + s, x) &= \sum_{a,b \in A} \kappa_{a,b}((r + s)_a, x_b) \\ &= \sum_{a,b \in A} \kappa_{a,b}(r_a + s_a, x_b) \\ &= \sum_{a,b \in A} (\kappa_{a,b}(r_a, x_b) + \kappa_{a,b}(s_a, x_b)) \\ &= \sum_{a,b \in A} \kappa_{a,b}(r_a, x_b) + \sum_{a,b \in A} \kappa_{a,b}(s_a, x_b) \\ &= \kappa(r, x) + \kappa(s, x). \end{aligned}$$

To see the third, note

$$\begin{aligned}\kappa(rs, x) &= \sum_{a,b\in A} \kappa_{a,b}((rs)_a, x_b)\\ &= \sum_{a,b\in A} \kappa_{a,b}\left(\sum_{c\in A} r_c s_{a-c}, x_b\right)\\ &= \sum_{a,b,c\in A} \kappa_{a,b}(r_c s_{a-c}, x_b)\\ &= \sum_{a,b,c\in A} \kappa_{a,b}(r_c, \kappa_{a-c,b}(s_{a-c}, x_b))\\ &=\end{aligned}$$

□ FINISH

When working with A-graded rings and modules, we will often freely use the above propositions without comment.

Recall that given a ring R, a left (resp. right) module P is *projective* if, for all diagrams of R-module homomorphisms of the form

$$\begin{tikzcd} & M \arrow[d, two heads, "g"] \\ P \arrow[r, "f"'] & N \end{tikzcd}$$

with g an epimorphism, there exists a lift $h : P \to M$ satisfying $g \circ h = f$

$$\begin{tikzcd} & M \arrow[d, two heads, "g"] \\ P \arrow[r, "f"'] \arrow[ur, dashed, "h"] & N \end{tikzcd}$$

(Note h is not required to be unique.)

Definition 0.10. Let R be an A-graded ring, and let P be a left (resp. right) A-graded R-module. Then P is a *graded projective* module if, for all diagrams of A-graded R-module homomorphisms of the form

$$\begin{tikzcd} & M \arrow[d, two heads, "g"] \\ P \arrow[r, "f"'] & N \end{tikzcd}$$

with g an epimorphism, there exists an A-graded homomorphism $h : P \to M$ satisfying $g \circ h = f$.

$$\begin{tikzcd} & M \arrow[d, two heads, "g"] \\ P \arrow[r, "f"'] \arrow[ur, dashed, "h"] & N \end{tikzcd}$$

(Note h is not required to be unique.)

Lemma 0.11. *Given an A-graded ring R and two left (resp. right) A-graded R-modules M and N, their direct sum $M \oplus N$ is naturally a left (resp. right) A-graded R-module group by defining*

$$(M \oplus N)_a := M_a \oplus N_a.$$

Proof. The canonical map $\bigoplus_{a\in A}(M_a \oplus N_a) \to M \oplus N$ factors as

$$\bigoplus_{a\in A}(M_a \oplus N_a) \xrightarrow{\cong} \bigoplus_{a\in A} M_a \oplus \bigoplus_{a\in A} N_a \xrightarrow{\cong} M \oplus N. \qquad \square$$

Lemma 0.12. *Let R be an A-graded ring, and let M be an A-graded left (resp. right) R-module. Then for all $d \in A$, the evaluation map*

$$\begin{aligned} \mathrm{ev}_1 : \mathrm{Hom}_R^d(R, M) &\to M_d \\ \varphi &\mapsto \varphi(1) \end{aligned}$$

is an isomorphism of abelian groups.

Proof. We consider the case that M is a left A-graded R-module, as showing it when M is a right module is entirely analagous. First of all, this map is clearly a homomorphism, as given degree d A-graded homomorphisms $\varphi, \psi : R \to M$, we have

$$\mathrm{ev}_1(\varphi + \psi) = (\varphi + \psi)(1) = \varphi(1) + \psi(1) = \mathrm{ev}_1(\varphi) + \mathrm{ev}_1(\psi).$$

Now, to see it is surjective, let $m \in M_d$, and define $\varphi_m : R \to M$ to send $r \mapsto r \cdot m$. First of all, φ_m is a module homomorphism, as given $r, s \in R$,

$$\varphi_m(r + s) = (r + s) \cdot m = r \cdot m + s \cdot m = \varphi_m(r) + \varphi_m(s) \quad \text{and} \quad \varphi_m(r \cdot s) = r \cdot s \cdot m = r \cdot \varphi_m(s).$$

Furthermore, it is clearly A-graded of degree d, as given a homogeneous element $r \in R_a$ for some $a \in A$, we have $\varphi_m(r) = r \cdot m \in R_{a+d}$, since m is homogeneous of degree d. Finally, clearly

$$\mathrm{ev}_1(\varphi_m) = \varphi_m(1) = 1 \cdot m = m,$$

so indeed ev_1 is surjective. On the other hand, to see it is injective, suppose we are given $\varphi, \psi \in \mathrm{Hom}_R^d(R, M)$ such that $\varphi(1) = \psi(1)$. Then given $r \in R$, we must have

$$\varphi(r) = \varphi(r \cdot 1) = r \cdot \varphi(1) = r \cdot \psi(1) = \psi(r \cdot 1) = \psi(r),$$

so φ and ψ are exactly the same map. Thus, ev_1 is injective, as desired. $\square$

0.1. ***A*-graded submodules and quotient modules.** In what follows, fix an A-graded ring R. We will simply say "A-graded R-module" when we are freely considering either left or right A-graded R-modules.

Definition 0.13. Let M be an A-graded R-module. Then an *A-graded R-submodule* is an A-graded R-module N which is a subset of M and for which the inclusion $N \hookrightarrow M$ is an A-graded homomorphism of R-modules. Equivalently, it is a submodule N for which the canonical map

$$\bigoplus_{a\in A} N \cap M_a \to N$$

is an isomorphism.

Lemma 0.14. *Let M be an A-graded R-module. Then an R-submodule $N \leq M$ is an A-graded submodule if and only if it is generated as an R-module by homogeneous elements of M.*

Proof. If $N \leq M$ is a A-graded submodule, it is generated by the set of all its homogeneous elements, which are also homogeneous elements in M, by definition.

Conversely, suppose $N \leq M$ is a submodule which is generated by homogeneous elements of M. Then define $N_a := N \cap M_a$, and consider the canonical map

$$\Phi : \bigoplus_{a\in A} N_a \to N.$$

First of all, it is surjective, as each generator of N belongs to some N_a, by definition. To see it is injective, consider the following commutative diagram:

$$\begin{tikzcd} \bigoplus_{a\in A} N_a \arrow[r, hook] \arrow[d, "\Phi"'] & \bigoplus_{a\in A} M_a \arrow[d, "\cong"] \\ N \arrow[r, hook] & M \end{tikzcd}$$

Since Φ composes with an injection to get an injection, clearly Φ must be injective itself. We have the desired result. $\square$

Proposition 0.15. *Given two left (resp. right) A-graded R-modules M and N and an A-graded R-module homomorphism $\varphi : M \to N$ (of possibly nonzero degree), the kernel and images of φ are A-graded submodules of M and N, respectively.*

Proof. First recall that a degree d A-graded homomorphism $M \to N$ is simply an A-graded homomorphism $M_* \to N_{*+d}$, so it suffices to consider the case φ is of degree 0. Next, note that since the forgetful functor from R-modules to abelian groups preserves kernels and images, it suffices to consider the case that φ is a homomorphism of A-graded abelian groups. Finally, by Lemma 0.14, it suffices to show that $\ker\varphi$ and $\operatorname{im}\varphi$ are generated by homogeneous elements of M and N, respectively.

Note that by the universal property of the coproduct in **Ab**, the data of an A-graded homomorphism of abelian groups $\varphi : M \to N$ is precisely the data of an A-indexed collection of abelian group homomorphisms $\varphi_a : M_a \to N_a$, in which case the following diagram commutes:

$$\begin{tikzcd} \bigoplus_a M_a \arrow[r, "\bigoplus_a \varphi_a"] \arrow[d, "\cong"'] & \bigoplus_a N_a \arrow[d, "\cong"] \\ M \arrow[r, "\varphi"] & N \end{tikzcd}$$

Finally, the desired result follows by the purely formal fact that taking images and kernels commutes with arbitrary direct sums. $\square$

Proposition 0.16. *Given two left (resp. right) A-graded R-modules M and N, an A-graded submodule $K \le N$, and an A-graded R-module homomorphism $\varphi : M \to N$ (of possibly nonzero degree), the submodule $\varphi^{-1}(K)$ of M is A-graded.*

Proof. Recall that a degree d A-graded homomorphism $M \to N$ is simply an A-graded homomorphism $M_* \to N_{*+d}$, so it suffices to consider the case φ is of degree 0. Now, let $x \in L := \varphi^{-1}(K)$. As an element of M, we may uniquely write $x = \sum_{a\in A} x_a$ where each $x_a \in M_a$. Similarly, if we set $y := \varphi(x)$, then we may uniquely write $y = \sum_{a\in A} y_a$ where each $y_a \in N_a$. Then since K is an A-graded submodule of N and $y \in K$, by definition, we have that $y_a \in K$ for each a. Finally, note that

$$\sum_{a\in A} y_a = y = \varphi(x) = \sum_{a\in A} \varphi(x_a),$$

so that $\varphi(x_a) = y_a \in K$ for all $a \in A$, so that $x_a \in L$ for all $a \in A$. Thus we have shown that each element in L can be written as a sum of homogeneous elements in M, as desired. $\square$

Proposition 0.17. *Given an A-graded R-module M and an A-graded subgroup $N \le M$, the quotient M/N is canonically A-graded by defining $(M/N)_a$ to be the subgroup generated by cosets represented by homogeneous elements of degree a in M. Furthermore, the canonical maps $M_a/N_a \to (M/N)_a$ taking a coset $m + N_a$ to $m + N$ are isomorphisms.*

Proof. Consider the canonical map

$$\Phi : \bigoplus_a (M/N)_a \to M/N.$$

First of all, surjectivity of Φ follows by commutativity of the following diagram:

$$\begin{tikzcd} \bigoplus_a M_a \arrow[r, "\cong"] \arrow[d, two heads] & M \arrow[d, two heads] \\ \bigoplus_a (M/N)_a \arrow[r, "\Phi"] & M/N \end{tikzcd}$$

where the vertical left map sends a generator $m \in M_a$ to the coset $m + N$ in $(M/N)_a \subseteq M/N$. To see Φ is injective, suppose we are given some element $(m_a + N)_{a\in A}$ in $\bigoplus_a (M/C)_a$ such that $\sum_{a\in A}(m_a + N) = 0$ in M/N. Thus $\sum_{a\in A} m_a \in N$, and since N is A-graded this implies that each m_a belongs to $N \cap M_a = N_a$, so that in particular $m_a + N$ is zero in $(M/N)_a \subseteq M/N$, so that $(m_a + N)_{a\in A} = 0$ in $\bigoplus_a (M/N)_a$, as desired.

It remains to show that the canonical map

$$\varphi_a : M_a/N_a \to (M/N)_a$$

is an isomorphism. It is clearly surjective, as $(M/N)_a$ is generated by elements $m + N$ for $m \in M_a$, and these elements make up precisely the image of φ_a. Thus φ_a hits every generator of $(M/N)_a$, so φ_a is surjective. On the other hand, suppose we are given some $m \in M_a$ such that $\varphi(m + N_a) = m + N = 0$. Thus $m \in N$, and $m \in M_a$, so that $m \in M_a \cap N = N_a$, meaning $m + N_a = 0$ in M_a/N_a, as desired. $\square$

0.2. **Tensor product of A-graded modules.** Recall that given a ring R, a left R-module M, a right R-module N, and an abelian group A, an *R-balanced map* $\varphi : M \times N \to B$ is one which satisifies

$$\begin{aligned} \varphi(m, n + n') &= \varphi(m, n) + \varphi(m, n') \\ \varphi(m + m', n) &= \varphi(m, n) + \varphi(m', n) \\ \varphi(m \cdot r, n) &= \varphi(m, r \cdot n). \end{aligned}$$

for all $m, m' \in M$, $n, n' \in N$, and $r \in R$. Then the tensor product $M \otimes_R N$ is the universal abelian group equipped with an R-balanced map $\otimes : M \times N \to M \otimes_R N$ such that for every abelian group B and every R-balanced map $\varphi : M \times N \to B$, there is a *unique* group homomorphism $\widetilde{\varphi} : M \otimes_R N \to B$ such that $\widetilde{f} \circ \otimes = f$. We call elements in the image of $\otimes : M \times N \to M \otimes_R N$ *pure tensors*. It is a standard fact that $M \otimes_R N$ is generated as an abelian group by its pure tensors.

Definition 0.18. Suppose we have a right A-graded R-module M, a left A-graded R-module N, and an A-graded abelian group B. Then an *A-graded R-balanced map* $\varphi : M \times N \to B$ is an R-balanced map which restricts to $M_a \times N_b \to B_{a+b}$ for all $a, b \in A$.

Proposition 0.19. *Suppose we have a right A-graded R-module M and a left A-graded R-module N. Then the tensor product*

$$M \otimes_R N$$

is naturally an A-graded abelian group by defining $(M \otimes_R N)_a$ to be the subgroup generated by homogeneous *pure tensors $m \otimes n$ with $m \in M_b$ and $n \in N_c$ such that $b + c = a$. Furthermore, if either M (resp. N) is an A-graded bimodule, then this decomposition makes $M \otimes_R N$ into a left (resp. right) A-graded R-module. In particular, if both M and N are R-bimodules, then $M \otimes_R N$ is an R-bimodule.*

Proof. By definition, since M and N are A-graded abelian groups, they are generated (as abelian groups) by their homogeneous elements. Thus it follows that $M \otimes_R N$ is generated by *homogeneous pure tensors*, that is, elements of the form $m \otimes n$ with $m \in M$ and $n \in N$ homogeneous. Now, given a homogeneous pure tensor $m \otimes n$, we define its *degree* by the formula $|m \otimes n| := |m| + |n|$. It follows this formula is well-defined by checking that given homogeneous elements $m \in M$, $n \in N$, and $r \in R$ that

$$|(m \cdot r) \otimes n| = |m \cdot r| + |n| = |m| + |r| + |n| = |m| + |r \cdot n| = |m \otimes (r \cdot n)|.$$

Thus, we may define $(M \otimes_R N)_a$ to be the subgroup of $M \otimes_R N$ generated by those pure homogeneous tensors of degree a. Now, consider the map

$$\Psi : M \times N \to \bigoplus_{a \in A} (M \otimes_R N)_a$$

which takes a pair $(m, n) = \sum_{a \in A} (m_a, n_a)$ to the element $\Psi(m, n)$ whose a^{th} component is

$$(\Psi(m, n))_a := \sum_{b+c=a} m_b \otimes n_c.$$

It is straightforward to see that this map is R-balanced, in the sense that it is additive in each argument and $\Psi(m \cdot r, n) = \Psi(m, r \cdot n)$ for all $m \in M$, $n \in N$, and $r \in R$. Thus by the universal property of $M \otimes_R N$, we get a homomorphism of abelian groups $\widetilde{\Psi} : M \otimes_R N \to \bigoplus_{a \in A} (M \otimes_R N)_a$ lifting Ψ along the canonical map $M \times N \to M \otimes_R N$. Now, also consider the canonical map

$$\Phi : \bigoplus_{a \in A} (M \otimes_R N)_a \to M \otimes_R N.$$

We would like to show $\widetilde{\Psi}$ and Φ are inverses of eah other. Since $\widetilde{\Psi}$ and Φ are both homomorphisms, it suffices to show this on generators. Let $m \otimes n$ be a homogeneous pure tensor with $m = m_a \in M_a$ and $n = n_b \in N_b$. Then we have

$$\Phi(\widetilde{\Psi}(m \otimes n)) = \Phi\left(\bigoplus_{a \in A} \sum_{b+c=a} m_b \otimes n_c \right) \overset{(*)}{=} \Phi(m \otimes n) = m \otimes n,$$

and

$$\widetilde{\Psi}(\Phi(m \otimes n)) = \widetilde{\Psi}(m \otimes n) = \bigoplus_{a \in A} \sum_{b+c=a} m_b \otimes n_c \overset{(*)}{=} m \otimes n,$$

where both occurrences of $(*)$ follow by the fact that $m_b \otimes n_c = 0$ unless $b = c = a$, in which case $m_a \otimes n_a = m \otimes n$. Thus since Φ is an isomorphism, $M \otimes_R N$ is indeed an A-graded abelian group, as desired.

Now, suppose that M is an A-graded R-bimodule, so there exists left and right A-graded actions of R on M such that given $r, s \in R$ and $m \in M$ we have $r \cdot (m \cdot s) = (r \cdot m) \cdot s$. Then we would like to show that given a left A-graded R-module N that $M \otimes_R N$ is canonically a left A-graded R-module. Indeed, define the action of R on $M \otimes_R N$ on pure tensors by the formula

$$r \cdot (m \otimes n) = (r \cdot m) \otimes n.$$

First of all, clearly this map is A-graded, as if $r \in R_a$, $m \in M_b$, and $n \in N_c$ then $(r \cdot m) \otimes n$, by definition, has degree $|r \cdot m| + |n| = |r| + |m| + |n|$ (the last equality follows since the left action of R on M is A-graded). In order to show the above map defines a left module structure, it suffices to show that given pure tensors $m \otimes n, m' \otimes n' \in M \otimes_R N$ and elements $r, r' \in R$ that

(1) $r \cdot (m \otimes n + m' \otimes n') = r \cdot (m \otimes n) + r \cdot (m' \otimes n')$,

(2) $(r + r') \cdot (m \otimes n) = r \cdot (m \otimes n) + r' \cdot (m' \otimes n')$,

(3) $(rr') \cdot (m \otimes n) = r \cdot (r' \cdot (m \otimes n))$, and

(4) $1 \cdot (m \otimes n) = m \otimes n$.

Axiom (1) holds by definition. To see (2), note that by the fact that R acts on M on the left that

$$(r + r') \cdot (m \otimes n) = ((r + r') \cdot m) \otimes n = (r \cdot m + r' \cdot m) \otimes n = r \cdot m \otimes n + r' \cdot m \otimes n.$$

That (3) and (4) hold follows similarly by the fact that $(rr') \cdot m = r \cdot (r' \cdot m)$ and $1 \cdot m = m$.

Conversely, if N is an A-graded R-bimodule, then showing $M \otimes_R N$ is canonically a right A-graded R-module via the rule

$$(m \otimes n) \cdot r = m \otimes (n \cdot r)$$

is entirely analagous.

Finally, if both M and N are R-bimodules, then by what we have shown, $M \otimes_R N$ is both a left and right R-module. To see these coincide to give $M \otimes_R N$ an R-bimodule structure, note that given $m \in M$, $n \in N$, and $r, r' \in R$ that

$$(r \cdot (m \otimes n)) \cdot r' = ((r \cdot m) \otimes n) \cdot r' = (r \cdot m) \otimes (n \cdot r') = r \cdot (m \otimes (n \cdot r')) = r \cdot ((m \otimes n) \cdot r'). \quad \square$$

Lemma 0.20. *Let R be an A-graded ring, B an A-graded abelian group, M a right A-graded R-module, and N a left A-graded R-module. Further suppose we are given a map $\varphi_{a,b} : M_a \times N_b \to B_{a+b}$ for all $a, b \in A$ which commutes with addition in each argument, and such that for all $m \in M_a$, $n \in N_b$, and $r \in R_c$ that*

$$\varphi_{a+b,c}(m \cdot r, n) = \varphi_{a,b+c}(m, r \cdot n).$$

Then there is a unique A-graded R-balanced map $\varphi : M \times N \to B$ which restricts to $\varphi_{a,b}$ for all $a, b \in A$, and furthermore, the induced homorphism $\widetilde{\varphi} : M \otimes_R N \to B$ is an A-graded homomorphism of abelian groups.

TODO

Proof. $\square$

0.3. A-graded categories.

Definition 0.21. An *A-graded category* is an **Ab**-enriched category $\mathcal{C}$ along with, for each $a \in A$, **Ab**-enriched functors

$$\Sigma^a : \mathcal{C} \to \mathcal{C}$$

and natural isomorphisms $\lambda : \Sigma^0 \cong \mathrm{Id}_{\mathcal{C}}$ and $\phi_{a,b} : \Sigma^{a+b} \cong \Sigma^a \Sigma^b$ such that for all $a, b, c \in A$, the following diagrams in $\mathrm{End}(\mathcal{C})$ commute:

$$\begin{tikzcd}
\Sigma^a\Sigma^0 \arrow[dr, "\Sigma^a\lambda"'] & \Sigma^a \arrow[l, "\phi_{a,0}"'] \arrow[r, "\phi_{0,a}"] \arrow[d, dash] & \Sigma^0\Sigma^a \arrow[dl, "\lambda\Sigma^a"] \\
& \Sigma^a &
\end{tikzcd}
\qquad
\begin{tikzcd}
\Sigma^{a+b+c} \arrow[r, "\phi_{a+b,c}"] \arrow[d, "\phi_{a,b+c}"'] & \Sigma^{a+b}\Sigma^c \arrow[d, "\phi_{a,b}\Sigma^c"] \\
\Sigma^a\Sigma^{b+c} \arrow[r, "\Sigma^a\phi_{b,c}"'] & \Sigma^a\Sigma^b\Sigma^c
\end{tikzcd}$$

Example 0.22. Given an A-graded ring R, the category of left A-graded R-modules and degree-preserving module homomorphisms between them is canonically an A-graded category, with shift functors Σ^a taking an A-graded left R-module M to the shifted module M_{*-a}. It is clear we have strict equalities $\Sigma^0 = \mathrm{Id}_{\mathcal{C}}$ and $\Sigma^a\Sigma^b = \Sigma^{a+b}$.

Example 0.23. Given a tensor-triangulated category $(\mathcal{SH}, \otimes, S, \Sigma, \mathcal{D})$ (**??**) with sub-Picard grading $(A, \mathbf{1}, \{S^a\}, \{\phi_{a,b}\})$ (**??**), $\mathcal{SH}$ is canonically an A-graded category, with shift functors $\Sigma^a := S^a \otimes -$, and natural isomorphisms

$$\lambda : \Sigma^0 X = S^0 \otimes X = S \otimes X \xrightarrow{\cong} X$$

and
$$\phi_{a,b} : \Sigma^{a+b}X = S^{a+b} \otimes X \xrightarrow{\phi_{a,b}\otimes X} (S^a \otimes S^b) \otimes X \cong S^a \otimes (S^b \otimes X) = \Sigma^a\Sigma^b X.$$

Proposition 0.24. *Given an A-graded category $\mathcal{C}$, the shift functors Σ^{-a} and Σ^a canonically form an adjoint auto-equivalence of $\mathcal{C}$, with unit and counit*
$$\eta^a : \mathrm{Id}_{\mathcal{C}} \xrightarrow{\lambda^{-1}} \Sigma^0 \xrightarrow{\phi_{a,-a}} \Sigma^a\Sigma^{-a} \qquad \text{and} \qquad \varepsilon^a : \Sigma^{-a}\Sigma^a \xrightarrow{\phi^{-1}_{-a,a}} \Sigma^0 \xrightarrow{\lambda} \mathrm{Id}_{\mathcal{C}}.$$

Proof. Clearly η^a and ε^a are natural isomorphisms, so Σ^{-a} and Σ^a form an auto-equivalence of $\mathcal{C}$. To see that they further form an *adjoint* auto-equivalence, it suffices to show that they satisfy the following zig-zag identities:

$$\begin{tikzcd}
\Sigma^{-a} \arrow[r, "\Sigma^{-a}\eta^a"] \arrow[dr, no head] & \Sigma^{-a}\Sigma^a\Sigma^{-a} \arrow[d, "\varepsilon^a\Sigma^{-a}"] \\
& \Sigma^{-a}
\end{tikzcd}
\qquad
\begin{tikzcd}
\Sigma^a\Sigma^{-a}\Sigma^a \arrow[d, "\Sigma^a\varepsilon^a"'] & \Sigma^a \arrow[l, "\eta^a\Sigma^a"'] \arrow[dl, no head] \\
\Sigma^a &
\end{tikzcd}$$

Since ε^a and η^a are natural isomorphisms, it further suffices to show that one of these is satisfied (see [1, Lemma 3.2]). We will show the left diagram commutes. Unravelling definitions, it becomes

$$\begin{tikzcd}
\Sigma^{-a} \arrow[r, "\Sigma^{-a}\lambda^{-1}"] \arrow[dr, equal] & \Sigma^{-a}\Sigma^0 \arrow[r, "\Sigma^{-a}\phi_{a,-a}"] & \Sigma^{-a}\Sigma^a\Sigma^{-a} \arrow[d, "\phi^{-1}_{-a,a}\Sigma^{-a}"] \\
& \Sigma^{-a} \arrow[u, "\phi_{-a,0}"] \arrow[r, "\phi_{0,-a}"] \arrow[dr, equal] & \Sigma^0\Sigma^{-a} \arrow[d, "\lambda\Sigma^{-a}"] \\
& & \Sigma^{-a}
\end{tikzcd}$$

Clearly this diagram commutes, by the coherence conditions for λ and the $\phi_{a,b}$'s in Definition 0.21. $\square$

Proposition 0.25. *Given an A-graded category $\mathcal{C}$, we can form a new* $\mathbf{Ab}(A)$*-enriched category $\mathcal{C}^*$ with the same objects as $\mathcal{C}$, and whose hom-sets are the A-graded abelian groups $\mathcal{C}^*(X,Y) := \bigoplus_{a\in A}\mathcal{C}^a(X,Y)$ defined by*
$$\mathcal{C}^a(X,Y) := \mathcal{C}(\Sigma^a X, Y).$$
Composition is induced by the graded maps
$$\mathcal{C}^a(Y,Z) \times \mathcal{C}^b(X,Y) \to \mathcal{C}^{a+b}(X,Z)$$
Sending $g : \Sigma^a Y \to Z$ and $f : \Sigma^b X \to Y$ to the composition
$$g \circ f : \Sigma^{a+b}X \xrightarrow{\phi_{a,b}} \Sigma^a\Sigma^b X \xrightarrow{\Sigma^a f} \Sigma^a Y \xrightarrow{g} Z,$$
and the identity in $\mathcal{C}^*(X,X)$ is given by $\lambda_X : \Sigma^0 X \to X$ in $\mathcal{C}(\Sigma^0 X, X) = \mathcal{C}^0(X,X) \subseteq \mathcal{C}^*(X,X)$.

Proof. By Lemma 0.20, in order to show composition map be realized as an A-graded homomorphism
$$\mathcal{C}^*(Y,Z) \otimes_{\mathbb{Z}} \mathcal{C}^*(X,Y) \to \mathcal{C}(X,Z),$$
it suffices to show that for all $g, g' \in \mathcal{C}^a(Y,Z)$ and $f, f' \in \mathcal{C}^b(X,Y)$ that $(g+g')\circ f = (g\circ f)+(g\circ f')$ and $g\circ(f+f') = (g\circ f)+(g\circ f')$ (where here $-\circ-$ denotes the composition defined above). These

follow by bilinearity of composition in $\mathcal{C}$ (since $\mathcal{C}$ is **Ab**-enriched) and the fact that $\Sigma^a : \mathcal{C} \to \mathcal{C}$ is an **Ab**-enriched functor:

$$\begin{aligned}(g+g')\circ f :&= (g+g')\circ \Sigma^a f \circ (\phi_{a,b})_X \\ &= (g\circ \Sigma^a f\circ (\phi_{a,b})_X) + (g'\circ \Sigma^a f \circ (\phi_{a,b})_X) \\ &= (g\circ f) + (g'\circ f)\end{aligned}$$

and

$$\begin{aligned}g\circ (f+f') :&= g\circ \Sigma^a (f+f')\circ (\phi_{a,b})_X \\ &= g\circ (\Sigma^a f + \Sigma^a f')\circ (\phi_{a,b})_X \\ &= (g\circ \Sigma^a f\circ (\phi_{a,b})_X) + (g\circ \Sigma^a f' \circ (\phi_{a,b})_X) \\ &= (g\circ f) + (g\circ f').\end{aligned}$$

□

Thus, we have a well-defined composition map. We need to show it is associative and unital with respect to λ_X. To see associativity, let $h \in \mathcal{C}^a(Z,W)$, $g \in \mathcal{C}^b(Y,Z)$, and $f \in \mathcal{C}^c(X,Y)$, and consider the following diagram:

$$\begin{tikzcd}
\Sigma^{a+b+c}X \arrow[r, "\phi_{a,b+c}"] \arrow[d, "\phi_{a+b,c}"'] & \Sigma^a\Sigma^{b+c}X \arrow[r, "\Sigma^a\phi_{b,c}"] & \Sigma^a\Sigma^b\Sigma^c X \arrow[d, "\Sigma^a\Sigma^b f"] & & \\
\Sigma^{a+b}\Sigma^c X \arrow[r, "\Sigma^{a+b}f"'] \arrow[urr, "\phi_{a,b}\Sigma^c"] & \Sigma^{a+b}Y \arrow[r, "\phi_{a,b}"'] & \Sigma^a\Sigma^b Y \arrow[r, "\Sigma^a g"] & \Sigma^a Z \arrow[r, "h"] & W
\end{tikzcd}$$

The top composition is $h\circ(g\circ f)$, while the bottom composition is $(h\circ g)\circ f$. The left triangle commutes by coherence for the ϕ's (Definition 0.21), and the right triangle commutes by naturality of $\phi_{a,b}$. Thus composition is associative, as desired. Now, to see unitality, consider the following diagram:

$$\begin{tikzcd}
\Sigma^0\Sigma^a X \arrow[dd, "\Sigma^0 f"] \arrow[dr, "\lambda_{\Sigma^a X}"] & & \Sigma^a X \arrow[ll, "\phi_{0,a}"'] \arrow[rr, "\phi_{a,0}"] \arrow[dl, equal] \arrow[dd, "f"] \arrow[ddrr, equal] & & \Sigma^a\Sigma^0 X \arrow[dd, "\Sigma^a\lambda_X"] \\
& \Sigma^a X \arrow[dr, "f"] & & & \\
\Sigma^0 Y \arrow[rr, "\lambda_Y"] & & Y & & \Sigma^a X \arrow[ll, "f"]
\end{tikzcd}$$

The left composition is $\lambda_Y \circ f$, while the right composition is $f\circ\lambda_X$. The rightmost triangle commutes by coherence for the ϕ's (Definition 0.21), as does the top left triangle. The leftmost triangle commutes by naturality of λ. Finally, the remaining two regions commute by definition. Thus we have $\lambda_Y\circ f = f\circ\lambda_X = f$, so that $\lambda_X = \mathrm{id}_X$ in $\mathcal{C}^*$, as desired.

Proposition 0.26. *Given an A-graded category $\mathcal{C}$ and some $a\in A$, there is a canonical* **Ab**-*enriched inclusion $\iota_{\mathcal{C}} : \mathcal{C} \hookrightarrow \mathcal{C}^*$ which is the identity on objects, and sends $f\in\mathcal{C}(X,Y)$ to the composition*

$$\Sigma^0 X \xrightarrow{\lambda_X} X \xrightarrow{f} Y$$

in $\mathcal{C}(\Sigma^0 X, Y) = \mathcal{C}^0(X,Y) \subseteq \mathcal{C}^*(X,Y)$.

Proof. We need to show that $\iota_{\mathcal{C}}$ preserves identities, compositions, and addition of morphisms. The former is clear, as $\iota_{\mathcal{C}}(\mathrm{id}_X)$ is the composition

$$\Sigma^0 X \xrightarrow{\lambda_X} X \xrightarrow{\mathrm{id}_X} X,$$

so that $\iota_{\mathcal{C}}(\mathrm{id}_X) = \lambda_X$, as desired. To see it preserves composition, let $f : X \to Y$ and $g : Y \to Z$ in $\mathcal{C}$, and consider the following diagram:

$$\begin{tikzcd}
\Sigma^0 X \arrow[rrr, "\lambda_X"] \arrow[d, "\phi_{0,0}"'] \arrow[dr, equal] & & & X \arrow[d, "f"] & \\
\Sigma^0\Sigma^0 X \arrow[r, "\Sigma^0\lambda_X"'] & \Sigma^0 X \arrow[r, "\Sigma^0 f"'] & \Sigma^0 Y \arrow[r, "\lambda_Y"'] & Y \arrow[r, "g"] & Z
\end{tikzcd}$$

The top composition is $\iota_{\mathcal{C}}(g \circ f)$, while the bottom composition is $\iota_{\mathcal{C}}(g) \circ \iota_{\mathcal{C}}(f)$. The left triangle commutes by coherence for the ϕ's, while the right trapezoid commutes by naturality of λ. Thus, $\iota_{\mathcal{C}}$ preserves composition, as desired. Finally, it is clear that $\iota_{\mathcal{C}}$ preserves addition, as given $f, g \in \mathcal{C}(X, Y)$ we have

$$\iota_{\mathcal{C}}(f+g) = (f+g) \circ \lambda_X = (f \circ \lambda_X) + (g \circ \lambda_X) = \iota_{\mathcal{C}}(f) + \iota_{\mathcal{C}}(g),$$

where the middle equality follows by bilinearity of addition of morphisms in $\mathcal{C}$, and the fact that addition of morphisms in $\mathcal{C}^*$ is defined to be addition of the underlying morphisms in $\mathcal{C}$. $\square$

Definition 0.27. Given two A-graded categories $\mathcal{C}$ and $\mathcal{D}$, a *lax A-graded functor* from $\mathcal{C}$ to $\mathcal{D}$ is the data of an **Ab**-enriched functor $F : \mathcal{C} \to \mathcal{D}$ along with natural isomorphisms $t^a_X : F \circ \Sigma^a \cong \Sigma^a \circ F$ for each $a \in A$.

If, in addition, the maps t^a_X make the following diagrams commute for all X in $\mathcal{SH}$ and $a, b \in A$, we say F is a *strict* A-graded functor:

$$\begin{tikzcd}
F\Sigma^0 \arrow[r, "t^0"] \arrow[dr, "F\lambda"'] & \Sigma^0 F \arrow[d, "\lambda F"] \\
& F
\end{tikzcd}
\qquad
\begin{tikzcd}
F\Sigma^{a+b} \arrow[r, "F\phi_{a,b}"] \arrow[d, "t^{a+b}"'] & F\Sigma^a\Sigma^b \arrow[r, "t^a\Sigma^b"] & \Sigma^a F\Sigma^b \arrow[d, "\Sigma^a t^b"] \\
\Sigma^{a+b}F \arrow[rr, "\phi_{a,b}F"] & & \Sigma^a\Sigma^b F
\end{tikzcd}$$

Proposition 0.28. *Given a lax A-graded functor $F : \mathcal{C} \to \mathcal{D}$ between two A-graded categories, for each pair of objects X and Y in $\mathcal{C}$, there is a unique A-graded homomorphism*

$$F^* : \mathcal{C}^*(X, Y) \to \mathcal{D}^*(F(X), F(Y))$$

such that the following diagram commutes for all $f : \Sigma^a X \to Y$ in $\mathcal{C}^a(X, Y)$:

$$\begin{tikzcd}
F(\Sigma^a X) \arrow[d, "t^a_X"'] \arrow[dr, "F(f)"] & \\
\Sigma^a F(X) \arrow[r, "F^*(f)"'] & F(Y)
\end{tikzcd}$$

Furthermore, given X and Y in $\mathcal{C}$, the map $F^* : \mathcal{C}^*(X, Y) \to \mathcal{D}^*(F(X), F(Y))$ is injective (resp. surjective) if and only if $F : \mathcal{C}(\Sigma^a X, Y) \to \mathcal{D}^*(F(\Sigma^a X), F(Y))$ is injective (resp. surjective) for all $a \in \mathcal{C}$.

Proof. Given a homogeneous element $f \in \mathcal{C}^a(X, Y)$, we *must* define

$$F^*(f) := F(f) \circ (t^a_X)^{-1}.$$

In this way, for each $a \in A$ we have asssignments

$$F^a : \mathcal{C}^a(X, Y) \to \mathcal{D}^a(F(X), F(Y)).$$

Note that each of these maps F^a are further homomorphisms of abelian groups, as given $f, g : \Sigma^a X \to Y$, we have

$$\begin{aligned} F^a(f+g) &= F(f+g) \circ (t^a_X)^{-1} \\ &= (F(f) + F(g)) \circ (t^a_X)^{-1} \\ &= (F(f) \circ (t^a_X)^{-1}) + (F(g) \circ (t^a_X)^{-1}) \\ &= F^a(f) + F^a(g). \end{aligned}$$

Thus, by the universal property of the coproduct of abelian groups, for each X and Y in $\mathcal{C}$, the maps $F^a : \mathcal{C}^a(X,Y) \to \mathcal{D}^a(F(X), F(Y))$ extend uniquely to an A-graded homomorphism

$$F^* : \mathcal{C}^*(X,Y) \to \mathcal{D}^*(F(X), F(Y)).$$

We have made no choices so far, and we have fully determined F^* and shown it satisfies the desired properties, so that in particular we have shown uniqueness holds in the desired sense.

Note that F^* can equivalently be characterized as the unique A-graded homomorphism such that its restriction $F^a : \mathcal{C}^a(X,Y) \to \mathcal{D}^a(F(X), F(Y))$ makes the following diagram commute for all X, Y in $\mathcal{C}$ and $a \in A$:

$$\begin{tikzcd} \mathcal{C}^a(X,Y) \arrow[r, "F^a"] \arrow[dd, equal] & \mathcal{D}^a(F(X), F(Y)) \arrow[d, equal] \\ & \mathcal{D}(\Sigma^a F(X), F(Y)) \arrow[d, "(t^a_X)^*"] \\ \mathcal{C}(\Sigma^a X, Y) \arrow[r, "F"] & \mathcal{D}(F(\Sigma^a X), F(Y)) \end{tikzcd}$$

If $F : \mathcal{C}(\Sigma^a X, Y) \to \mathcal{D}(F(\Sigma^a X), F(Y))$ is injective (resp. surjective) for all $a \in A$, it clearly follows that F^a is injective (resp. surjective) for all $a \in A$, since the vertical maps in the above diagram are isomorphisms. Thus, if further follows that $F^* = \bigoplus_{a \in A} F_a$ is injective (resp. surjective), as desired. On the other hand, suppose that $F^* : \mathcal{C}^*(X,Y) \to \mathcal{D}^*(F(X), F(Y))$ is injective (resp. surjective) for some X and Y in $\mathcal{C}$. Then in particular, it follows that $F^a : \mathcal{C}^a(X,Y) \to \mathcal{D}^a(F(X), F(Y))$ is injective (resp. surjective). Then looking at the same diagram, we have that $F : \mathcal{C}(\Sigma^a X, Y) \to \mathcal{D}(F(\Sigma^a X), F(Y))$ is injective (resp. surjective) for all $a \in A$, as desired. $\square$

Lemma 0.29. *Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{E}$ be two lax A-graded functors between A-graded categories, with structure maps $t^a : F\Sigma^a \cong \Sigma^a F$ and $s^a : G\Sigma^a \cong \Sigma^a G$, respectively. Note that $G \circ F$ is a also a lax A-graded functor with structure map*

$$GF\Sigma^a \xrightarrow{Gt^a} G\Sigma^a F \xrightarrow{s^a F} \Sigma^a GF.$$

Then we have $F^* \circ G^* = (G \circ F)^*$.

Proof. Let $f : \Sigma^a X \to Y$ in $\mathcal{C}^a(X,Y)$, and consider the following diagram in $\mathcal{E}$:

$$\begin{tikzcd} GF(\Sigma^a X) \arrow[d, "Gt^a_X"'] \arrow[ddr, "GF(f)"] & \\ G\Sigma^a F(X) \arrow[d, "s^a_{F(X)}"'] \arrow[dr, "G(F^*(f))"'] & \\ \Sigma^a GF(X) \arrow[r, dashed] & GF(Y) \end{tikzcd}$$

The top inner triangle commutes by the uniqueness property of F^* and functoriality of G. Furthermore, by the uniqueness property of G^*, we know that $G^*(F^*(f))$ for the dashed line is the unique arrow which makes the bottom inner triangle commute. Finally, by the uniqueness property of $G \circ F$, we know that $(G \circ F)^*(f)$ for the dashed line is the unique arrow which makes the outside triangle commute. Hence, we know that $(G \circ F)^*(f) = G^*(F * (f))$ for all $f \in \mathcal{C}^a(X, Y)$. It follows that $(G \circ F)^*$ and $G^* \circ F^*$ agree on the entirety of $\mathcal{C}^*(X, Y)$, as they are both homomorphisms of abelian groups. $\square$

Proposition 0.30. *Let $\mathcal{C}$ and $\mathcal{D}$ be A-graded categories, and suppose we have a strict A-graded functor $F : \mathcal{C} \to \mathcal{D}$. Then there is an $\mathbf{Ab}(A)$-enriched functor*

$$F^* : \mathcal{C}^* \to \mathcal{D}^*$$

which makes the following diagram commute

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "F"] \arrow[d, hook, "\iota_{\mathcal{C}}"'] & \mathcal{D} \arrow[d, hook, "\iota_{\mathcal{D}}"] \\
\mathcal{C}^* \arrow[r, "F^*"] & \mathcal{D}^*
\end{tikzcd}$$

(see Proposition 0.26 for the definition of the vertical functors) and is given on hom-sets by the maps

$$F^* : \mathcal{C}^*(X, Y) \to \mathcal{D}^*(F(X), F(Y))$$

constructed in Proposition 0.28, so that in particular F^* is faithful (resp. full) if and only if F is, and given two composoable strict A-graded functors F and G, $(G \circ F)^* = G^* \circ F^*$ (Lemma 0.29).

Proof. We need to show that F^* is a functor, i.e., that it preserves identities and composition of morphisms. To see the former, note that given $X \in \mathcal{C}$, $F_*(\lambda_X)$ is the composition

$$F(\lambda_X) \circ (t^0_X)^{-1} = \lambda_{F(X)} = \lambda_{F^*(X)},$$

where the first equality follows since F is an A-graded functor (see the first diagram in Definition 0.27). Now, to see F^* preserves composition, since it acts via homomorphisms, it suffices to show it preserves composition of homogeneous elements. To that end, let $g \in \mathcal{C}^a(Y, Z)$ and $f \in \mathcal{C}^b(X, Y)$, and consider the following diagram in $\mathcal{D}$:

$$\begin{tikzcd}
\Sigma^{a+b}F(X) \arrow[rrr, "(t^{a+b}_X)^{-1}"] \arrow[d, "\phi_{a,b}F"'] & & & F(\Sigma^{a+b}X) \arrow[dl, "F(\phi_{a,b})"] \arrow[ddd, "F(g \circ' f)"] \\
\Sigma^a\Sigma^b F(X) \arrow[dd, "\Sigma^a F^*(f)"'] & \Sigma^a F(\Sigma^b X) \arrow[l, "\Sigma^a t^b_X"'] \arrow[ddl, "\Sigma^a F(f)"] & F(\Sigma^a\Sigma^b X) \arrow[l, "t^a_{\Sigma^b X}"'] \arrow[d, "F(\Sigma^a f)"] & \\
 & & F(\Sigma^a Y) \arrow[dll, "t^a_Y"'] \arrow[dr, "F(g)"] & \\
\Sigma^a F(Y) \arrow[rrr, "F^*(g)"'] & & & F(Z)
\end{tikzcd}$$

The bottom composition is $F^*(g) \circ' F^*(f) := F^*(g) \circ \Sigma^a F^*(f) \circ (\phi_{a,b})_{F(X)}$, while the top composition is $F^*(g \circ' f) := F(g \circ' f) \circ (t^a_X)^{-1}$. The top trapezoid commutes by coherence for the t^a's, since F is an A-graded functor (see the second diagram in Definition 0.27). The leftmost and bottom triangles commute by how we have constructed F^*. The rightmost trapezoid commutes by functoriality of F, since $g \circ' f$ is defined to be $(\phi_{a,b})_X \circ \Sigma^a f \circ g$. Finally, the middle oddly-shaped quadrilateral commutes by naturality of t^a. $\square$