In **??**, we showed that given a monoid object $(E,\mu,e)$ in $\mathcal{SH}$, that $E_*(E)$ is canonically an $A$-graded bimodule over the ring $\pi_*(E)$. In this subsection, we will outline some additional structure carried by the pair $(E_*(E),\pi_*(E))$. In particular, we will show that if $(E,\mu,e)$ is a flat (Definition 0.5) commutative monoid object, then this pair, called the *dual $E$-Steenrod algebra*, is canonically an *$A$-graded anticommutative Hopf algebroid* over the stable homotopy ring $\pi_*(S)$ (**??**). To start with, we outline some structure maps relating $E_*(E)$ and $\pi_*(E)$.

First, recall that given a monoid object $(E,\mu,e)$ in $\mathcal{SH}$, $\pi_*(E)$ is canonically an $A$-graded ring by **??**, and so is $E_*(E)=\pi_*(E\otimes E)$ and $E_*(E\otimes E)=\pi_*(E\otimes E\otimes E)$, since the tensor product of monoid objects in a symmetric monoidal category is again a monoid object (**??**).

**Proposition 0.1.** *Let $(E,\mu,e)$ be a commutative monoid object in $\mathcal{SH}$. Then the maps*

*(1) $E\xrightarrow{\cong}E\otimes S\xrightarrow{E\otimes e}E\otimes E$,*

*(2) $E\xrightarrow{\cong}S\otimes E\xrightarrow{e\otimes E}E\otimes E$,*

*(3) $E\otimes E\xrightarrow{\cong}E\otimes S\otimes E\xrightarrow{E\otimes e\otimes E}E\otimes E\otimes E$,*

*(4) $E\otimes E\xrightarrow{\mu}E$, and*

*(5) $E\otimes E\xrightarrow{\tau_{E,E}}E\otimes E$*

*are homomorphisms of monoid objects in $\mathcal{SH}$ (where here $E\otimes E$ and $E\otimes E\otimes E$ are considered as monoid objects in $\mathcal{SH}$ by **??** and **??**, respectively), so that by **??**, under $\pi_*$ they induce morphisms in $\pi_*(S)$-$\mathbf{GCA}^A$:*

*(1) $\eta_L:\pi_*(E)\to E_*(E)$,*

*(2) $\eta_R:\pi_*(E)\to E_*(E)$,*

*(3) $h:E_*(E)\to E_*(E\otimes E)$,*

*(4) $\epsilon:E_*(E)\to\pi_*(E)$, and*

*(5) $c:E_*(E)\to E_*(E)$.*

*Proof.* It is a general fact that the unit and multiplication maps $e:S\to E$ and $\mu:E\otimes E\to E$ for a monoid are monoid homomorphisms when $(E,\mu,e)$ is a commutative monoid object (**??**), so that the maps $E\otimes e$, and $e\otimes E$ from $E$ to $E\otimes E$ are monoid homomorphisms, by **??**. Similarly, $E\otimes e\otimes E:E\otimes E\to E\otimes E\otimes E$ is a monoid homomorphism. Thus, it remains to show that $\tau_{E,E}:E\otimes E\to E\otimes E$ is a monoid homomorphism. First, consider the following diagram:

$$\begin{tikzcd}
E_1\otimes E_2\otimes E_3\otimes E_4 \arrow[r, "\tau\otimes\tau"] \arrow[d, "E\otimes\tau\otimes E"'] & E_2\otimes E_1\otimes E_4\otimes E_3 \arrow[d, "E\otimes\tau\otimes E"] \\
E_1\otimes E_3\otimes E_2\otimes E_4 \arrow[r, "\tau_{E\otimes E,E\otimes E}"] \arrow[d, "\mu\otimes\mu"'] & E_2\otimes E_4\otimes E_1\otimes E_3 \arrow[d, "\mu\otimes\mu"] \\
E_{1,3}\otimes E_{2,4} \arrow[r, "\tau"'] & E_{2,4}\otimes E_{1,3}
\end{tikzcd}$$

(Here we've labelled the $E$'s to make the action of the braidings clearer). The top region commutes by coherence for the symmetries in a symmetric monoidal category, while the bottom region commutes by naturality of $\tau$. Now, consider the following diagram:

$$\begin{tikzcd}
&& S \arrow[dl, "\cong"'] \arrow[dr, "\cong"] && \\
& S \otimes S \arrow[rr, "\tau"] \arrow[dl, "e\otimes e"'] && S \otimes S \arrow[dr, "e\otimes e"] & \\
E \otimes E \arrow[rrrr, "\tau"] &&&& E \otimes E
\end{tikzcd}$$

The top triangle commutes by coherence for a symmetric monoidal category, while the bottom region commutes by naturality of $\tau$. Thus, we have shown $\tau_{E,E}$ is a homomorphism of monoid objects, as desired. $\square$

Recall a that given a homomorphism of rings $f : R \to R'$, $R'$ canonically becomes an $R$-bimodule with left action $r \cdot x := f(r)x$ and right action $x \cdot r := xf(r)$. In particular, the ring homomorphisms $\eta_L : \pi_*(E) \to E_*(E)$ and $\eta_R : \pi_*(E) \to E_*(E)$ endow $E_*(E)$ with the structure of a bimodule over $\pi_*(E)$. Naturally, one may ask in what sense these bimodule structures coincide with the canonical one (from **??**). The following lemma tells us that the canonical $\pi_*(E)$-bimodule structure on $E_*(E)$ is that with left action induced by $\eta_L$ and right action induced by $\eta_R$:

**Lemma 0.2.** *Let $(E, \mu, e)$ be a commutative monoid object in $\mathcal{SH}$. Then the left (resp. right) $\pi_*(E)$-module structure induced on $E_*(E)$ by the ring homomorphism $\eta_L$ (resp. $\eta_R$) coincides with the canonical left (resp. right) $\pi_*(E)$-module structure on $E_*(E)$ given in* **??**.

*Proof.* What's going on here is a bit subtle, so we're going to be really explicit. In **??**, it was shown that $E_*(E)$ is a left $\pi_*(E)$-module via the assignment

$$\pi_*(E) \times E_*(E) \to E_*(E)$$

which sends homogeneous elements $r : S^a \to E$ and $x : S^b \to E \otimes E$ to the composition

$$S^{a+b} \xrightarrow{\cong} S^a \otimes S^b \xrightarrow{r\otimes x} E \otimes E \otimes E \xrightarrow{\mu\otimes E} E \otimes E.$$

We'd like to show that this is the same thing as the assignment $\pi_*(E) \times E_*(E) \to E_*(E)$ sending $(r, x) \mapsto \eta_L(r)x$, where $\eta_L(r)x$ denotes the product of $\eta_L(r)$ and $x$ taken in the ring $E_*(E)$. Explicitly, the product structure on $E_*(E) = \pi_*(E \otimes E)$ is that induced by the fact that $E \otimes E$ is a monoid object in $\mathcal{SH}$ (by **??**), with product

$$E \otimes E \otimes E \otimes E \xrightarrow{E\otimes\tau\otimes E} E \otimes E \otimes E \otimes E \xrightarrow{\mu\otimes\mu} E \otimes E$$

(note the middle two factors are swapped). By linearity of module actions, in order to show the canonical left $\pi_*(E)$-module structure on $E_*(E)$ agrees with that induced by $\eta_L$, it suffices to show the module actions agree on homogeneous elements. Now, suppose we have homogeneous elements $r : S^a \to E$ in $\pi_*(E)$ and $x : S^b \to E \otimes E$ in $E_*(E)$, and consider the following diagram, where we've passed to a symmetric strict monoidal category:

$$\begin{tikzcd}
S^{a+b} \arrow[d, "\phi_{a,b}"] \\
S^a \otimes S^b \arrow[d, "r\otimes x"] \\
E_1\otimes E_2\otimes E_3 \arrow[rrrr, "\mu\otimes E"] \arrow[dd, "E\otimes e\otimes E"'] \arrow[dr, equal] & & & & E_{1,2}\otimes E_3 \arrow[dd, equal] \\
& E_1\otimes E_2\otimes E_3 \arrow[r, equal] & E_1\otimes E_2\otimes E_3 \arrow[r, equal] \arrow[d, "E\otimes E\otimes e\otimes E"] & E_1\otimes E_2\otimes E_3 \arrow[dr, "\mu\otimes E"] & \\
E_1\otimes E\otimes E_2\otimes E_3 \arrow[ur, "E\otimes\mu\otimes E"] \arrow[rr, "E\otimes\tau\otimes E"'] & & E_1\otimes E_2\otimes E\otimes E_3 \arrow[ul, "E\otimes\mu\otimes E"] \arrow[ur, "E\otimes E\otimes\mu"'] \arrow[rr, "\mu\otimes\mu"'] & & E_{1,2}\otimes E_3
\end{tikzcd}$$

Here we've numbered the $E$'s to make it clear what's going on. The bottom composition is $\eta_L(r)x$, while the top composition is the canonical left action of $r$ on $x$ given in **??**. The leftmost triangle commutes by unitality of $\mu$. The triangle to the right of that commutes by commutativity of $\mu$. The triangle to the right of that commutes by unitality of $\mu$, as does the next triangle. The remaining triangle on the right commutes by functoriality of $-\otimes-$. Finally, the top region commutes by definition. Thus, we've shown that the left $\pi_*(E)$-module structure induced on $E_*(E)$ by $\eta_L$ is in fact the canonical one. On the other hand, showing that the right $\pi_*(E)$-module structure induced on $E_*(E)$ by $\eta_R$ is the canonical one is entirely analagous, and we leave it as an exercise for the reader. $\square$

Recall (**??**) that the pushout of two morphisms $f : B \to C$ and $g : B \to D$ in $R$-$\mathbf{GCA}^A$ is obtained by taking the tensor product of $B$-modules $C \otimes_B D$, where $C$ has right $B$-module action induced by $f$, and $D$ has left $B$-module action induced by $g$, and giving it an anticommutative product which makes $C \otimes_B D$ a ring. Thus, by the above lemma, we may view the tensor product of bimodules $E_*(E) \otimes_{\pi_*(E)} E_*(E)$ (where $E_*(E)$ is considered with its canonical $\pi_*(E)$-bimodule structure from **??**) as not just an $A$-graded abelian group or a $\pi_*(E)$-bimodule, but as an $A$-graded anticommutative $\pi_*(S)$-algebra:

**Corollary 0.3.** *Given a commutative monoid object $(E, \mu, e)$ in $\mathcal{SH}$, the domain of the homomorphism*

$$\Phi_{E,E} : E_*(E) \otimes_{\pi_*(E)} E_*(E) \to E_*(E \otimes E)$$

*constructed in* **??** *is canonically an $A$-graded $\pi_*(S)$-ring, and sits in the following pushout diagram in $\pi_*(S)$-$\mathbf{GCA}^A$:*

$$\begin{tikzcd}
\pi_*(E) \arrow[r, "\eta_L"] \arrow[d, "\eta_R"'] & E_*(E) \arrow[d, "x\mapsto 1\otimes x"] \\
E_*(E) \arrow[r, "x\mapsto x\otimes 1"'] & E_*(E) \otimes_{\pi_*(E)} E_*(E)
\end{tikzcd}$$

Furthermore, with respect to this ring structure, $\Phi_{E,E}$ is a homomorphism of rings:

**Lemma 0.4.** *Let $(E, \mu, e)$ be a commutative monoid object in $\mathcal{SH}$. Then the homomorphism*

$$\Phi_{E,E} : E_*(E) \otimes_{\pi_*(E)} E_*(E) \to E_*(E \otimes E)$$

*constructed in* **??** *is a homomorphism of $A$-graded anticommutative $\pi_*(S)$-algebras.*

*Proof.* Consider the maps

$$f : E \otimes E \xrightarrow{e\otimes E\otimes E} E \otimes E \otimes E$$

and

$$g : E \otimes E \xrightarrow{E\otimes E\otimes e} E \otimes E \otimes E.$$

We know that the maps

$$E \xrightarrow{e\otimes E} E \otimes E \qquad \text{and} \qquad E \xrightarrow{E\otimes e} E \otimes E$$

are monoid homomorphisms by Proposition 0.1, so that $f$ and $g$ are monoid homomorphisms by **??**. Furthermore, by **??**, they are monoid homomorphisms between the same monoid objects in $\mathcal{SH}$ (up to associativity). Finally, note that we have the following commutative diagram

$$\begin{tikzcd}
E \arrow[r, "E\otimes e"] \arrow[d, "e\otimes E"'] \arrow[dr, "e\otimes E\otimes e"] & E \otimes E \arrow[d, "e\otimes E\otimes E"] \\
E \otimes E \arrow[r, "E\otimes E\otimes e"'] & E \otimes E \otimes E
\end{tikzcd}$$

where the outer arrows are monoid object homomorphisms, thus, we may apply $\pi_*$, which yields the following commutative diagram in $\pi_*(S)$-$\mathbf{GCA}^A$ (**??**):

$$\begin{tikzcd}
\pi_*(E) \arrow[r, "\eta_L"] \arrow[d, "\eta_R"'] & E_*(E) \arrow[d, "\pi_*(f)"] \\
E_*(E) \arrow[r, "\pi_*(g)"'] & E_*(E \otimes E)
\end{tikzcd}$$

Hence by Lemma 0.4 and the universal property of the pushout, there exists some unique morphism $\ell : E_*(E) \otimes_{\pi_*(E)} E_*(E) \to E_*(E \otimes E)$ in $\pi_*(S)$-$\mathbf{GCA}^A$ which makes the following diagram commute:

$$\begin{tikzcd}
\pi_*(E) \arrow[r, "\eta_L"] \arrow[d, "\eta_R"'] & E_*(E) \arrow[d, "x\mapsto 1\otimes x"] \arrow[ddr, bend left, "\pi_*(f)"] & \\
E_*(E) \arrow[r, "x\mapsto x\otimes 1"] \arrow[drr, bend right, "\pi_*(g)"'] & E_*(E) \otimes_{\pi_*(E)} E_*(E) \arrow[dr, dashed, "\ell"] & \\
& & E_*(E \otimes E)
\end{tikzcd}$$

Thus in order to show $\Phi_E$ is a morphism in $\pi_*(S)$-$\mathbf{GCA}^A$, it suffices to show that $\Phi_E$ and $\ell$ are the same map, since we know $\ell$ is a homomorphism of $A$-graded $\pi_*(S)$-commutative rings. Since $\Phi_E$ and $\ell$ are both abelian group homomorphisms, it further suffices to show they agree on homogeneous pure tensors, which generate $E_*(E) \otimes_{\pi_*(E)} E_*(E)$ as an abelian group. Given homogeneous elements $x : S^a \to E \otimes E$ and $y : S^b \to E \otimes E$ in $E_*(E)$, unravelling how pushouts in $\pi_*(S)$-$\mathbf{GCA}^A$ are defined (**??**), $\ell$ sends the pure homogeneous tensor $x \otimes y$ to the element $\pi_*(g)(x) \cdot \pi_*(f)(y)$, where here $\cdot$ denotes the product taken in $E_*(E \otimes E) = \pi_*(E \otimes E \otimes E)$. Now, consider the following diagram:

$$
\begin{tikzcd}
S^{a+b} \arrow[d, "\phi_{a,b}"] & & \\
S^a \otimes S^b \arrow[d, "x \otimes y"] & & \\
E_1 \otimes E_2 \otimes E_3 \otimes E_4 \arrow[rr, "g \otimes f = E \otimes E \otimes e \otimes e \otimes E \otimes E"] \arrow[rrd, "E \otimes e \otimes E \otimes e \otimes E \otimes E"'] \arrow[ddd, "E \otimes \mu \otimes E"'] \arrow[rdd, equal] & & E_1 \otimes E_2 \otimes E_a \otimes E_b \otimes E_3 \otimes E_4 \arrow[d, "E \otimes \tau_{E \otimes E, E} \otimes E \otimes E"] \\
& & E_1 \otimes E_b \otimes E_2 \otimes E_a \otimes E_3 \otimes E_4 \arrow[d, "\mu \otimes E \otimes \tau \otimes E"] \\
& E_1 \otimes E_2 \otimes E_3 \otimes E_4 \arrow[ru, "E \otimes e \otimes E \otimes e \otimes E \otimes E"] \arrow[r, "E \otimes E \otimes E \otimes e \otimes E"'] \arrow[ld, "E \otimes \mu \otimes E"] & E_1 \otimes E_2 \otimes E_3 \otimes E_a \otimes E_4 \arrow[d, "E \otimes \mu \otimes \mu"] \\
E_1 \otimes E_{2,3} \otimes E_4 \arrow[rr, equal] & & E_1 \otimes E_{2,3} \otimes E_4
\end{tikzcd}
$$

Here we have labelled the $E$'s to make things clearer. The left outside composition is $\Phi_E(x \otimes y)$, while the right composition is $\pi_*(g)(x) \cdot \pi_*(f)(y)$. The top right triangle commutes by coherence for a symmetric monoidal category. The middle tright triangle commutes by unitality of $\mu$ and coherence for a symmetric monoidal category. The bottom trapezoid commutes by unitality of $\mu$. The rest of the diagram commutes by definition. Thus we have $\Phi_E(x \otimes y) = \pi_*(g)(x) \cdot \pi_*(f)(y)$, so that $\Phi_E = \ell$ is not just an isomorphism of left $\pi_*(E)$-modules, but an isomorphism of $A$-graded anticommutative $\pi_*(S)$-algebras, as desired. □

For the sake of conciseness, we make the following definition:

**Definition 0.5.** We say that a monoid object $(E, \mu, e)$ in $\mathcal{SH}$ is *flat* if the canonical right $\pi_*(E)$-module structure on $E_*(E)$ from **??** is that of a flat module, or equivalently by Lemma 0.2, if the the map $\eta_R : \pi_*(E) \to E_*(E)$ constructed in Proposition 0.1 is a flat ring homomorphism.

Finally, we can package all of this information into an object called the *dual $E$-Steenrod algebra*:

**Definition 0.6.** Let $(E, \mu, e)$ be a *commutative* monoid object in $\mathcal{SH}$ which is flat (Definition 0.5) and cellular (**??**). Then the *dual $E$-Steenrod algebra* is the pair of $A$-graded abelian groups $(E_*(E), \pi_*(E))$ equipped with the following structure:

1. The $A$-graded $\pi_*(S)$-commutative ring structure on $\pi_*(E)$ induced from $E$ being a commutative monoid object in $\mathcal{SH}$ (**??**).
2. The $A$-graded $\pi_*(S)$-commutative ring structure on $E_*(E)$ induced from the fact that $E \otimes E$ is canonically a commutative monoid object in $\mathcal{SH}$ (**??**), so that also $E_*(E) = \pi_*(E \otimes E)$ is an $A$-graded $\pi_*(S)$-commutative ring (**??**).
3. The homomorphisms of $A$-graded $\pi_*(S)$-commutative rings
$$\eta_L : \pi_*(E) \to E_*(E)$$
and
$$\eta_R : \pi_*(E) \to E_*(E)$$
induced under $\pi_*$ by the monoid object homomorphisms
$$E \xrightarrow{\cong} E \otimes S \xrightarrow{E \otimes e} E \otimes E$$

and
$$E \xrightarrow{\cong} S \otimes E \xrightarrow{e \otimes E} E \otimes E.$$

4. The homomorphism of $A$-graded $\pi_*(S)$-commutative rings
$$\Psi_E : E_*(E) \to E_*(E) \otimes_{\pi_*(E)} E_*(E)$$
given by the composition
$$E_*(E) \xrightarrow{h} E_*(E \otimes E) \xrightarrow{\Phi_{E,E}^{-1}} E_*(E) \otimes_{\pi_*(E)} E_*(E),$$
where $h$ is a homomorphism of $A$-graded $\pi_*(S)$-commutative rings induced under $\pi_*$ by the monoid object homomorphism
$$E \otimes E \xrightarrow{\cong} E \otimes S \otimes E \xrightarrow{E \otimes e \otimes E} E \otimes E \otimes E,$$
and $\Phi_{E,E}$ is morphism constructed in **??**, which is proven to be an isomorphism in **??**, and furthermore an isomorphism in $\pi_*(S)$-$\mathbf{GCA}^A$ by Lemma 0.4.
5. The homomorphism of $A$-graded $\pi_*(S)$-commutative rings
$$\epsilon : E_*(E) \to \pi_*(E)$$
induced under $\pi_*$ by the monoid object homomorphism
$$E \otimes E \xrightarrow{\mu} E.$$
6. The homomorphism of $A$-graded $\pi_*(S)$-commutative rings
$$c : E_*(E) \to E_*(E)$$
induced under $\pi_*$ from the monoid object homomorphism
$$E \otimes E \xrightarrow{\tau} E \otimes E.$$

The curious reader may wonder why we call $(E_*(E), \pi_*(E))$ the *dual* $E$-Steenrod algebra. The "dual" is there because the $E$-Steenrod algebra refers instead to the $E$-self cohomology $E^*(E) \cong [E, E]_{-*}$. Clasically, the Adams spectral sequence was originally constructed in such a way that the $E_1$ and $E_2$ pages could be characterized in terms of cohomology and the $E$-Steenrod algebra, but it turns out that our approach using homology and the dual $E$-Steenrod algebra is somewhat better behaved, at least when $E$ is flat in the sense of Definition 0.5.

0.1. **The dual $E$-Steenrod algebra is a Hopf algebroid.** Above, given a flat and cellular commutative monoid object $(E, \mu, e)$ in $\mathcal{SH}$, we constructed an algebraic gadget $(E_*(E), \pi_*(E))$ in the category $\pi_*(S)$-$\mathbf{GCA}^A$ of $A$-graded anticommutative $\pi_*(S)$-algebras called the *dual $E$-Steenrod algebra*. In this subsection, we will show this object is an example of the general notion of an *$A$-graded anticommutative Hopf algebroid*:

**Proposition 0.7.** *Let $(E, \mu, e)$ be a commutative monoid object in $\mathcal{SH}$ which is flat (Definition 0.5) and cellular (**??**). Then the dual $E$-Steenrod algebra $(E_*(E), \pi_*(E))$ with the structure maps $(\eta_L, \eta_R, \Psi, \epsilon, c)$ from Definition 0.6 is an $A$-graded anticommutative Hopf algebroid over $\pi_*(S)$ (**??**), i.e., a co-groupoid object in the category $\pi_*(S)$-$\mathbf{GCA}^A$.*

*Proof.* All that needs to be done is to show all the diagrams in **??** commute. This is nearly all entirely straightforward, the only difficulty that arises is showing the co-associativity diagram holds. The argument is sketched in the case $\mathcal{SH}$ is the classical stable homotopy category in sufficent detail in Lecture 3 of the article [1] by Adams. The argument given there works essentially the exact same way here in our more general setting. □

### 0.2. **Comodules over the dual $E$-Steenrod algebra.**

**Proposition 0.8.** *Let $(E, \mu, e)$ be a flat (Definition 0.5) and cellular (**??**) commutative monoid object in $\mathcal{SH}$. Then $E_*(-)$ is an additive functor from the full subcategory $\mathcal{SH}$-**Cell** of cellular objects in $\mathcal{SH}$ to the category $E_*(E)$-$\mathbf{CoMod}^A$ of left $A$-graded comodules (**??**) over the dual $E$-Steenrod algebra, which is an $A$-graded commutative Hopf algebroid over $\pi_*(S)$, by Proposition 0.7.*

*In particular, given an object $X$ in $\mathcal{SH}$-**Cell**, we are viewing $E_*(X)$ with its canonical left $\pi_*(E)$-module structure (**??**), and the action map*

$$\Psi_X : E_*(X) \to E_*(E) \otimes_{\pi_*(E)} E_*(X)$$

*is given by the composition*

$$\Psi_X : E_*(X) \xrightarrow{E_*(e\otimes X)} E_*(E \otimes X) \xrightarrow{\Phi_{E,X}^{-1}} E_*(E) \otimes_{\pi_*(E)} E_*(X).$$

*Proof.* Again, we refer the reader to Lecture 3 in [1]. □

**Proposition 0.9.** *Let $(E, \mu, e)$ be a flat (Definition 0.5) and cellular (**??**) commutative monoid object in $\mathcal{SH}$. Then given an object $X$ in $\mathcal{SH}$, the map*

$$\Phi_{E,X} : E_*(E) \otimes_{\pi_*(E)} E_*(X) \to E_*(E \otimes X)$$

*constructed in **??** is a homomorphism of $A$-graded left $\Gamma$-comodules, where here by **??** we are viewing $E_*(E) \otimes_{\pi_*(E)} E_*(X)$ as the co-free $E_*(E)$-comodule on $E_*(X)$ with its canonical $A$-graded left $\pi_*(E)$-module structure (from **??**), and $E_*(E \otimes X)$ with its canonical left $E_*(E)$-comodule structure from Proposition 0.8.*

*Proof.* Consider the following diagram:

$$\begin{tikzcd}
E_*(E) \otimes_{\pi_*(E)} E_*(X) \arrow[rr, "\Psi_{E_*(E)\otimes E_*(X)}"] \arrow[ddr, bend left, "E_*(e\otimes E)"'] \arrow[dddddd, "\Phi_{E,X}"'] & & E_*(E) \otimes_{\pi_*(E)} (E_*(E) \otimes_{\pi_*(E)} E_*(X)) \arrow[dddddd, "E_*(E)\otimes\Phi_{E,X}"] \\
& (E_*(E) \otimes_{\pi_*(E)} E_*(E)) \otimes_{\pi_*(E)} E_*(X) \arrow[ur, "\cong"'] \arrow[d, "\Phi_{E,E\otimes E_*(X)}"] & \\
& E_*(E \otimes E) \otimes_{\pi_*(E)} E_*(X) \arrow[d, equal] & \\
& (E \otimes E)_*(E) \otimes_{\pi_*(E)} E_*(X) \arrow[d, "\Phi_{E,X}"] & \\
& \pi_*(E \otimes E \otimes E \otimes X) \arrow[d, equal] & \\
& E_*(E \otimes E \otimes X) & \\
E_*(E \otimes X) \arrow[rr, "\Psi_{E\otimes X}"] \arrow[ur, "E_*(e\otimes E\otimes X)"] & & E_*(E) \otimes_{\pi_*(E)} E_*(E \otimes X) \arrow[ul, "\Phi_{E,E\otimes X}"']
\end{tikzcd}$$

The top and bottom regions commute by definition. The left region commutes by naturality of $\Phi_{E,X}$. Thus, it remains to show the rightmost region commutes. To that end, since all the arrows involved are homomorphisms, it suffices to chase a homogeneous pure tensor around. Let $x : S^a \to E \otimes E$, $y : S^b \to E \otimes E$, and $z : S^c \to E \otimes X$, and consider the following diagram:

$$
\begin{tikzcd}
S^{a+b+c} \arrow[d, "\phi"'] & \\
S^a \otimes S^b \otimes S^c \arrow[d, "x \otimes y \otimes z"'] & \\
E \otimes E \otimes E \otimes E \otimes E \otimes X \arrow[r, "E\otimes\mu\otimes E\otimes E\otimes X"] \arrow[d, "E\otimes E\otimes E\otimes\mu\otimes X"'] & E \otimes E \otimes E \otimes E \otimes X \arrow[d, "E\otimes E\otimes\mu\otimes X"] \\
E \otimes E \otimes E \otimes E \otimes X \arrow[r, "E\otimes\mu\otimes E\otimes X"] & E \otimes E \otimes E \otimes X
\end{tikzcd}
$$

The two compositions are the two results of chasing $(x \otimes y) \otimes z$ around the rightmost region in the above diagram. It clearly commutes by functoriality of $- \otimes -$. Hence, indeed we have that $\Phi_{E,X}$ is a homomorphism of left $E_*(E)$-comodules, as desired. □

**Lemma 0.10.** *Let $(E, \mu, e)$ be a flat (Definition 0.5) and cellular (**??**) commutative monoid object in $\mathcal{SH}$. Then the isomorphism*

$$t_X^a : E_*(\Sigma^a X) \to E_{*-a}(X)$$

*from **??** is an $A$-graded isomorphism of left $E_*(E)$-comodules.*

*Proof.* We know that $t_X^a : E_*(\Sigma^a X) \to E_{*-a}(X)$ is already an $A$-graded isomorphism of left $\pi_*(E)$-modules, so clearly it simply suffices to show that $t_X^a$ is a homomorphism of left $E_*(E)$-comodules. To that end, consider the following diagram:

(1)

$$
\begin{tikzcd}
E_*(\Sigma^a X) \arrow[rr, "\Psi_{\Sigma^a X}"] \arrow[dr, "E_*(e\otimes\Sigma^a X)"'] \arrow[dddd, "t_X^a"'] & & E_*(E) \otimes_{\pi_*(E)} E_*(\Sigma^a X) \arrow[dl, "\Phi_{E,\Sigma^a X}"] \arrow[dddd, "E_*(E)\otimes t_X^a"] \\
& E_*(E \otimes \Sigma^a X) \arrow[d, dashed, "E_*(\tau_{E,S^a}\otimes X)"] & \\
& E_*(\Sigma^a(E \otimes X)) \arrow[d, dashed, "t_{E\otimes X}^a"] & \\
& E_{*-a}(E \otimes X) & \\
E_{*-a}(X) \arrow[rr, "\Psi_X"'] \arrow[ur, "E_{*-a}(e\otimes X)"] & & E_*(E) \otimes_{\pi_*(E)} E_{*-a}(X) \arrow[ul, "\Phi_{E,X}"']
\end{tikzcd}
$$

The top and bottom regions commute by definition. To see the left and right regions commute, we'll do a diagram chase of homogeneous elements. First of all, let $x : S^b \to E \otimes S^a \otimes X$ in $E_*(\Sigma^a X)$, and consider the following diagram exhibiting the two ways to chase $x$ around the leftmost region:

$$\begin{tikzcd}
S^{b-a} \arrow[d, "\phi_{b,-a}"'] & & \\
S^b \otimes S^{-a} \arrow[d, "x\otimes S^{-a}"'] & & \\
E \otimes S^a \otimes X \otimes S^{-a} \arrow[r, "E\otimes e\otimes S^a\otimes X\otimes S^{-a}"] \arrow[dd, "E\otimes\tau\otimes S^{-a}"'] & E \otimes E \otimes S^a \otimes X \otimes S^{-a} \arrow[r, "E\otimes\tau\otimes X\otimes S^{-a}"] \arrow[dr, "E\otimes E\otimes\tau\otimes S^{-a}"'] & E \otimes S^a \otimes E \otimes X \otimes S^{-a} \arrow[d, "E\otimes\tau_{S^a,E\otimes X}\otimes S^{-a}"] \\
 & & E \otimes E \otimes X \otimes S^a \otimes S^{-a} \arrow[d, "E\otimes E\otimes X\otimes\phi^{-1}_{a,-a}"] \\
E \otimes X \otimes S^a \otimes S^{-a} \arrow[r, "E\otimes X\otimes\phi^{-1}_{a,-a}"'] \arrow[urr, "E\otimes e\otimes X\otimes S^a\otimes S^{-a}"] & E \otimes X \arrow[r, "E\otimes e\otimes X"'] & E \otimes E \otimes X
\end{tikzcd}$$

The top right region commutes by coherence for the symmetries, while the other two regions commute by functoriality of $-\otimes-$. Thus, it remains to show the rightmost region in diagram (1) commutes. To that end, let $x : S^b \to E\otimes E$ in $E_*(E)$ and $y : S^c \to E\otimes S^a\otimes X$ in $E_*(\Sigma^a X)$, and consider the following diagram, which exhibits the two ways to chase $x\otimes y$ around the rightmost region of diagram (1):

$$\begin{tikzcd}
S^{b+c-a} \arrow[d, "\phi"] & & \\
S^b \otimes S^c \otimes S^{-a} \arrow[d, "x\otimes y\otimes S^{-a}"] & & \\
E \otimes E \otimes E \otimes S^a \otimes X \otimes S^{-a} \arrow[r, "E\otimes\mu\otimes S^a\otimes X\otimes S^{-a}"] \arrow[d, "E\otimes E\otimes E\otimes\tau\otimes S^{-a}"'] & E \otimes E \otimes S^a \otimes X \otimes S^{-a} \arrow[r, "E\otimes\tau\otimes X\otimes S^{-a}"] \arrow[dr, "E\otimes E\otimes\tau\otimes S^{-a}"'] & E \otimes S^a \otimes E \otimes X \otimes S^{-a} \arrow[d, "E\otimes\tau_{S^a,E\otimes X} S^{-a}"] \\
E \otimes E \otimes E \otimes X \otimes S^a \otimes S^{-a} \arrow[rr, "E\otimes\mu\otimes X\otimes S^a\otimes S^{-a}"] \arrow[d, "E\otimes E\otimes E\otimes X\otimes\phi^{-1}_{a,-a}"'] & & E \otimes E \otimes X \otimes S^a \otimes S^{-a} \arrow[d, "E\otimes E\otimes X\otimes\phi^{-1}_{a,-a}"] \\
E \otimes E \otimes E \otimes X \arrow[rr, "E\otimes\mu\otimes X"] & & E \otimes E \otimes X
\end{tikzcd}$$

The top right region commutes by coherence for the symmetries. The remaining two regions commute by functoriality of $-\otimes-$. Thus, indeed we have that diagram (1) commutes, so $t^a_X$ is a homomorphism of left $E_*(E)$-comodules, as desired. $\square$