In what follows, we fix an abelian group A. We will freely use the theory and results of **??**

Definition 0.1. An *A-graded spectral sequence* $(E_r, d_r)_{r \geq r_0}$ is the data of:

- A collection of A-graded abelian groups $\{E_r^*\}_{r \geq r_0}$
- A collection of A-graded homomorphisms $d_r : E_r \to E_r$ for $r \geq r_0$ (of possibly nonzero degree) such that $d_r \circ d_r = 0$
- For each $r \geq r_0$, an A-graded isomorphism $E_{r+1} \cong \ker d_r / \operatorname{im} d_r$ of degree 0 (where $\ker d_r$ and $\operatorname{im} d_r$ are canonically A-graded by **??**, and their quotient is canonically A-graded by **??**).

Typically we call a $\mathbb{Z}^2$-graded spectral sequence a *bigraded* spectral sequence, and a $\mathbb{Z}^3$-graded spectral sequence is a *trigraded* spectral sequence.

Definition 0.2. Let (E_r, d_r) and (E'_r, d'_r) be A-graded spectral sequences defined for $r \geq r_0$, then a homomorphism of spectral sequences $f : (E_r, d_r) \to (E'_r, d'_r)$ is the data of a collection A-graded homomorphisms $f_r : E_r \to E'_r$, all of the same (possibly nonzero) degree, such that for all $r \geq r_0$, the following two diagrams commute.

$$\begin{tikzcd} E_r \arrow[r, "f_r"] \arrow[d, "d_r"'] & E'_r \arrow[d, "d'_r"] \\ E_r \arrow[r, "f_r"] & E'_r \end{tikzcd} \qquad \begin{tikzcd} \ker d_r \arrow[r, "f_r"] \arrow[d, two heads] & \ker d'_r \arrow[d, two heads] \\ E_{r+1} \arrow[r, "f_{r+1}"] & E'_{r+1} \end{tikzcd}$$

(Commutativity of the first diagram guarantees the top arrow in the second diagram is well-defined.)

Proposition 0.3. *Let $f : (E_r, d_r) \to (E'_r, d'_r)$ be a homomorphism of A-graded spectral sequences. Then if $f_r : E_r \to E'_r$ is an isomorphism for some $r \geq r_0$, then $f_{r'}$ is an isomorphism for all $r' > r$.*

Proof. By induction, it suffices to show that if $f_r : E_r \to E'_r$ is an isomorphism, then so is $f_{r+1} : E_{r+1} \to E'_{r+1}$. First of all, we know the following diagram commutes:

$$\begin{tikzcd} \ker d_r \arrow[r, "f_r"] \arrow[d, two heads] & \ker d'_r \arrow[d, two heads] \\ E_{r+1} \arrow[r, "f_{r+1}"] & E'_{r+1} \end{tikzcd}$$

Since the left, top, and right arrows are surjective, it follows the bottom arrow must be surjective as well. Now, we claim that f_{r+1} is injective. Without loss of generality, we will assume that the isomorhpism $E_{r+1} \cong \ker d_r / \operatorname{im} d_r$ is an equality. Now let $x \in \ker d_r$ such that $f_{r+1}([x]) = 0$, then by commutativity of the above diagram, we have that $0 = f_{r+1}([x]) = [f_r(x)]$ in E'_{r+1}, so that $f_r(x) \in \operatorname{im} d'_r$, meaning $f_r(x) = d'_r(y)$ for some $y \in E'_r$. Then since f_r is an isomorphism, we have $x = f_r^{-1}(d'_r(y))$. Furthermore, since f is a homomorphism of spectral sequences, we know that $f_r^{-1} \circ d'_r = d_r \circ f_r^{-1}$, hence

$$x = d_r(f_r^{-1}(y)) \in \operatorname{im} d_r,$$

so that $[x]$ was 0 in E_r to begin with. Hence, f_{r+1} is an isomorphism, as desired. $\square$

0.1. **Unrolled exact couples and their associated spectral sequences.** For our purposes, we will only care about spectral sequences which arise from *A-graded unrolled exact couples.* In what follows, we follow [1], with minor modifications for our setting, in which everything is A-graded.

Definition 0.4. An A-graded *unrolled exact couple* $(D, E; i, j, k)$ is a diagram of A-graded abelian groups and A-graded homomorphisms (of possibly non-zero degree)

$$\begin{tikzcd}
\cdots \arrow[r] & D^{s+2} \arrow[r, "i"] \arrow[d, "j"] & D^{s+1} \arrow[r, "i"] \arrow[d, "j"] & D^{s} \arrow[r, "i"] \arrow[d, "j"] & D^{s-1} \arrow[r] \arrow[d, "j"] & \cdots \\
& E^{s+2} & E^{s+1} \arrow[ul, "k"] & E^{s} \arrow[ul, "k"] & E^{s-1} \arrow[ul, "k"] &
\end{tikzcd}$$

in which each triangle $D^{s+1} \xrightarrow{i} D^s \xrightarrow{j} E_s \xrightarrow{k} D^{s+1}$ is an exact sequence. We require that each occurrence of i (resp. j, k) is of the same degree. In other words, an unrolled exact couple can be described as a tuple $(D, E; i, j, k)$ of $\mathbb{Z} \times A$-graded abelian groups and $\mathbb{Z} \times A$-graded maps $i : D \to D$, $j : D \to E$, and $k : E \to D$, such that the $\mathbb{Z}$-degrees of i, j, and k are -1, 0, and 1, respectively. Usually i and one of j or k will be of A-degree 0.

Definition 0.5. Given two A-graded unrolled exact couples $(D, E; i, j, k)$ and $(D', E'; i', j', k')$, a *homomorphism of A-graded unrolled exact couples* (f, g) is the data of $\mathbb{Z} \times A$-graded maps $f : D \to D'$ and $g : E \to E'$ (of $\mathbb{Z}$-degree zero, although possibly of nonzero A-degree) such that the following diagram commutes:

$$\begin{tikzcd}
D \arrow[r, "i"] \arrow[d, "f"'] & D \arrow[r, "j"] \arrow[d, "f"] & E \arrow[r, "k"] \arrow[d, "g"] & D \arrow[d, "f"] \\
D' \arrow[r, "i'"] & D' \arrow[r, "j'"] & E' \arrow[r, "k'"] & D'
\end{tikzcd}$$

Given an A-graded unrolled exact couple $(D, E; i, j, k)$, we may define an associated $\mathbb{Z} \times A$-graded spectral sequence as follows: Given some $s \in \mathbb{Z}$ and some $r \geq 1$, we first define the following subgroups of E_s:

$$Z_r^s := k^{-1}(\mathrm{im}[i^{r-1} : D^{s+r} \to D^{s+1}]) \qquad \text{and} \qquad B_r^s := j(\ker[i^{r-1} : D^s \to D^{s-r+1}])$$

where we adopt the convention that i^0 is simply the identity. These are furthermore A-graded subgrous of E_s (by **??** and **??**). In this way, for each $s \in \mathbb{Z}$, we get an infinite sequence of A-graded subgroups:

$$0 = B_1^s \subseteq B_2^s \subseteq B_3^s \subseteq \cdots \subseteq \mathrm{im}\, j = \ker k \subseteq \cdots \subseteq Z_3^s \subseteq Z_2^s \subseteq Z_1^s = E^s.$$

Now, for each $s \in \mathbb{Z}$ and $r \geq 1$, we define the A-graded abelian group

$$E_r^s := Z_r^s / B_r^s,$$

so that in particular $E_1^s = E^s$ for all $s \in \mathbb{Z}$, as $Z_1^s = k^{-1}(D^{s+1}) = E^s$ and $B_1^s = j(\ker \mathrm{id}_{D^s}) = j(0) = 0$. Now we can define differentials $d_r^s : E_r^s \to E_r^{s+r}$ to be the composition

$$E_r^s = Z_r^s / B_r^s \xrightarrow{k} \mathrm{im}[i^{r-1} : D^{s+r} \to D^{s+1}] \xrightarrow{j \circ i^{-(r-1)}} Z_r^{s+r} / B_r^{s+r} = E_r^{s+r},$$

where given some $e \in Z_r^s = k^{-1}(\mathrm{im}\, i^{r-1})$, the first arrow takes a class $[e] \in E_r^s$ represented by some $e \in Z_r^s$ to the element $k(e)$, which lives in $\mathrm{im}\, i^{r-1}$ by definition, and the second arrow takes $i^{r-1}(d)$ to the class $[j(d)]$. Note the first map is well-defined, as given $b \in B_r^s = j(\ker[i^{r-1}])$, we have $k(b) = 0$, as $b \in \mathrm{im}\, j = \ker k$. To see the second map is well-defined, first note that given $d \in D^{s+r}$, that

$$k(j(d)) = 0 \in \mathrm{im}[i^{r-1} : D^{s+2r} \to D^{s+r+1}],$$

so that
$$j(d) \in k^{-1}(\mathrm{im}[i^{r-1} : D^{s+2r} \to D^{s+r+1}]) = Z_r^{s+r},$$
as desired, so that given $d \in D^{s+r}$, $j(d) \in Z_r^{s+r}$, so it makes sense to discuss the class $[j(d)] \in Z_r^{s+r}/B_r^{s+r} = E_r^{s+r}$. Secondly, if $i^{r-1}(d) = i^{r-1}(d')$ for some $d, d' \in D^{s+r}$, then
$$j(d) - j(d') = j(d - d') \in j(\ker[i^{r-1} : D^{s+r} \to D^{s+1}]) = B_r^{s+r},$$
so that $[j(d)] = [j(d')]$ in E_r^{s+r}, as desired. It is straightforward to check that these maps are also A-graded homomorphisms, so that by unravelling definitions d_r^s is an A-graded homomorphism of degree $\deg k - (r-1) \cdot \deg i + \deg j$ (so that in the standard case $\deg i = 0$, d_r^s simply has degree $\deg k + \deg j$).

These differentials square to zero, in the sense that for each $s \in \mathbb{Z}$ and $r \geq 1$ we have that $d_r^{s+r} \circ d_r^s : E_r^s \to E_r^{s+2r}$ is the zero map. Indeed, suppose we are given some class $[e] \in E_r^s$ represented by an element $e \in E^s$, so $k(e) = i^{r-1}(d)$ for some $d \in D^{s+r}$. Then
$$d_r^{s+r}(d_r^s([e])) = d_r^{s+r}([j(d)]) = [j(i^{-(r-1)}(k(j(d))))] = [j(i^{-(r-1)}(0))] = 0,$$
where the second-to-last equality follows by the fact that $k \circ j = 0$. Note that by unravelling definitions, $d_1^s = j \circ k$.

We claim that $\ker d_r^s = Z_{r+1}^s/B_r^s$. First of all, let $[e] \in E_r^s = Z_r^s/B_r^s$, so that $[e]$ is represented by some $e \in E^s$ with $k(e) = i^{r-1}(d)$ for some $d \in D^{s+r}$. Then if $[e] \in \ker d_r^s$, by definition this means $j(d) \in B_r^{s+r} = j(\ker[i^{r-1} : D^{s+r} \to D^{s+1}])$, so $j(d) = j(d')$ for some $d' \in D^{s+r}$ with $i^{r-1}(d') = 0$. Thus $d - d' \in \ker j = \mathrm{im}\, i$, so there exists some $d'' \in D^{s+r+1}$ such that $i(d'') = d - d'$. Then
$$k(e) = i^{r-1}(d) = i^{r-1}(i(d'') + d') = i^r(d'') + i^{r-1}(d'),$$
but since $i^{r-1}(d') = 0$, we have $k(e) \in \mathrm{im}[i^r : D^{s+r+1} \to D^{s+1}]$, so that $e \in Z_{r+1}^s$, meaning $[e] \in Z_{r+1}^s/B_r^s$, as desired. On the other hand, suppose we are given some class $[e] \in Z_{r+1}^s/B_r^s$, represented by $e \in Z_{r+1}^s$ with $k(e) \in \mathrm{im}[i^r : D^{s+r+1} \to D^{s+1}]$. Then if we write $k(e) = i^r(d) = i^{r-1}(i(d))$, then $d_r^s([e]) = [j(i(d))] = 0$ (since $j \circ i = 0$), as asserted.

Finally, we claim that the image of $d_r^{s-r} : E_r^{s-r} \to E_r^s$ is B_{r+1}^s/B_r^s. First, let $e \in Z_r^{s-r}$, so $k(e) = i^{r-1}(d)$ for some $d \in D^s$. Then we'd like to show that $d_r^s([e]) = [j(d)]$ belongs to B_{r+1}^s/B_r^s. It suffices to show that $d \in \ker[i^r : D^s \to D^{s-r}]$. To see this, note that
$$i^r(d) = i(i^{r-1}(d)) = i(k(e)) = 0,$$
since $i \circ k = 0$. Hence we've shown $\mathrm{im}\, d_r^{s-r} \subseteq B_{r+1}^s/B_r^s$. Conversely, let $j(d) \in B_{r+1}^s$, so $d \in D^s$ and $i^r(d) = 0$. Then we'd like to show that $[j(d)] \in B_{r+1}^s/B_r^s$ is in the image of d_r^{s-r}. To see this, note that
$$i^r(d) = 0 \implies i^{r-1}(d) \in \ker i = \mathrm{im}\, k,$$
so there exists some $e \in E^{s-r}$ such that $k(e) = i^{r-1}(d)$, so $e \in Z_r^{s-r}$. Unravelling definitions, it follows that $d_r^{s-r}([e]) = [j(d)]$, so $[j(d)]$ is indeed in the image of d_r^{s-r}, as desired.

To recap, we have constructed for each $s \in \mathbb{Z}$ and $r \geq 1$ an A-graded abelian group E_r^s along with differentials $d_r^s : E_r^s \to E_r^{s+r}$ which satisfy $d_r^{s+r} \circ d_r^s = 0$. Furthermore, if we take homology in the middle term of the following sequence
$$E_r^{s-r} \xrightarrow{d_r^{s-r}} E_r^s \xrightarrow{d_r^s} E_r^{s+r},$$
we get
$$\ker d_r^s / \mathrm{im}\, d_r^{s-r} = \frac{Z_{r+1}^s/B_r^s}{B_{r+1}^s/B_r^s} \cong Z_{r+1}^s/B_{r+1}^s = E_{r+1}^s.$$
Thus, we get a spectral sequence:

Proposition 0.6. *We may associate a $\mathbb{Z} \times A$-graded spectral sequence $r \mapsto (E_r, d_r)$ to the A-graded unrolled exact couple $(D, E; i, j, k)$ by defining $E_r := \bigoplus_{s \in \mathbb{Z}} E_r^s$ and the differentials*

$$d_r : E_r \to E_r$$

are those constructed above, which have $\mathbb{Z} \times A$-degree $(r, \deg j - (r-1) \cdot \deg i + \deg k)$.

Proposition 0.7. *Let $(f, g) : (D, E; i, j, k) \to (D', E'; i', j', k')$ be a homomorphism of A-graded unrolled exact couples (Definition 0.5). Then there is an induced homomorphism of $\mathbb{Z} \times A$-graded spectral sequences (Definition 0.2) $\widetilde{g} : (E_r, d_r) \to (E'_r, d'_r)$ between their associated spectral sequences. Furthermore, if $g : E \to E'$ is an isomorphism, then $\widetilde{g} : E_r \to E'_r$ is an isomorphism for all $r \geq 1$.*

Proof. To start, we define the maps $\widetilde{g}_r : E_r \to E'_r$. Recall that $E_r := Z_r/B_r$, where $B_r \subseteq Z_r \subseteq E$. Similarly $E'_r := Z'_r/B'_r$. We claim that for all $r \geq 1$, g yields a well-defined map $Z_r/B_r \to Z'_r/B'_r$. To that end, it suffices to show that $g(Z_r) \subseteq Z'_r$ and B_r is contained in the kernel of the composition

$$Z_r \xrightarrow{g} Z'_r \twoheadrightarrow Z'_r/B'_r = E'_r.$$

First, let $x \in Z_r$, so $k(x) = i^{r-1}(d)$ for some $d \in D$. Then we'd like to show that $g(x) \in Z'_r$, i.e., that there exists some $d' \in D'$ such that $k'(g(x)) = (i')^{r-1}(d')$. Indeed, since (f, g) is a homomorphism of unrolled exact couples, we have that

$$k'(g(x)) = f(k(x)) = f(i^{r-1}(d)) = (i')^{r-1}(f(d)),$$

as desired. Now, let $x \in B_r$, so that $x = j(d)$ for some $d \in D$ such that $i^{r-1}(d) = 0$. Then we'd like to show that $g(x) \in B'_r$, i.e., that $g(x) = j'(d')$ for some $d' \in D'$ such that $(i)^{r-1}(d') = 0$. To that end, note

$$g(x) = g(j(d)) = j'(f(d)), \qquad \text{and} \qquad (i')^{r-1}(f(d)) = f(i^{r-1}(d)) = f(0) = 0,$$

so that indeed $g(x) \in B'_r$ as desired. Thus, for each $r \geq 1$ we have shown g yields a well-defined assignment $\widetilde{g}_r : Z_r/B_r \to Z'_r/B'_r$ defined by $\widetilde{g}_r([x]) = [g(x)]$. They are furthermore clearly A-graded since g is. Now, it remains to show these maps $\widetilde{g}_r$ actually make a homomorphism of spectral sequences, i.e., that the following diagrams commute for all $r \geq 1$:

$$\begin{tikzcd} E_r \arrow[r, "\widetilde{g}_r"] \arrow[d, "d_r"'] & E'_r \arrow[d, "d'_r"] \\ E_r \arrow[r, "\widetilde{g}_r"] & E'_r \end{tikzcd} \qquad \begin{tikzcd} \ker d_r \arrow[r, "\widetilde{g}_r"] \arrow[d, two heads] & \ker d'_r \arrow[d, two heads] \\ E_{r+1} \arrow[r, "\widetilde{g}_{r+1}"] & E'_{r+1} \end{tikzcd}$$

To see the first diagram commutes, let $x \in Z_r$, so $k(x) = i^{r-1}(d)$ for some $d \in D$, then we'd like to show $d'_r(\widetilde{g}_r([x])) = \widetilde{g}_r(d_r([x]))$. By what we have shown above, we know that $k'(g(x)) = (i')^{r-1}(f(d))$, so that unravelling definitions we have

$$d'_r(\widetilde{g}_r([x])) = d'_r([g(x)]) = [j'((i')^{-(r-1)}(k'(g(x))))] = [j'(f(d))]$$

and

$$\widetilde{g}_r(d_r(x)) = [g(j(d))] = [j'(f(d))],$$

so the diagram does commute as desired. On the other hand, in order to see the second diagram commutes, let $x \in Z_{r+1}$, so that by our above work $[x] \in Z_{r+1}/B_r$ is precisely an element of $\ker d_r$ (and conversely every element of $\ker d_r$ is of this form). Write p_r and p'_r for the projection maps $\ker d_r \twoheadrightarrow E_{r+1}$ and $\ker d'_r \twoheadrightarrow E'_{r+1}$. Unravelling definitions, p_r takes $[x] \in Z_{r+1}/B_r$ to $[x] \in Z_{r+1}/B_{r+1} = E_{r+1}$, and p'_r is defined similarly. Then we'd like to show that $\widetilde{g}_{r+1}(p_r([x])) = p'_r(\widetilde{g}_r([x]))$. This is clear, as

$$\widetilde{g}_{r+1}(p_r([x])) = \widetilde{g}_{r+1}([x]) = [g(x)]$$

while
$$p'_r(\widetilde{g}_r([x])) = p'_r([g(x)]) = [g(x)],$$
as desired. Thus, indeed $\widetilde{g}$ is a homomorphism of spectral sequences, as desired. □

0.2. **Convergence of spectral sequences.** In what follows, we fixed an A-graded unrolled exact couple $(D, E; i, j, k)$ and its associated $\mathbb{Z} \times A$-graded spectral sequence (E_r, d_r) constructed above. In this subsection, we will outline what it means for this spectral sequence to converge to some *target* group. We will be following [1, §1–7].

For $s \in \mathbb{Z}$ and $r \geq 1$, let Z_r^s and B_r^s denote the A-graded subgroups of E_s defined above, which for each $s \in \mathbb{Z}$ satisfy
$$0 = B_1^s \subseteq B_2^s \subseteq B_3^s \subseteq \cdots \subseteq \operatorname{im} j = \ker k \subseteq \cdots \subseteq Z_3^s \subseteq Z_2^s \subseteq Z_1^s = E^s.$$
For each $s \in \mathbb{Z}$, we may further introduce the A-graded groups:
$$\begin{aligned}
Z_\infty^s &:= \bigcap_{r=1}^{\infty} Z_r^s = \lim_r Z_r^s, && \text{the group of } \textit{infinite cycles}; \\
B_\infty^s &:= \bigcup_{r=1}^{\infty} B_r^s = \operatorname*{colim}_r B_r^s, && \text{the group of } \textit{infinite boundaries}; \\
E_\infty^s &:= Z_\infty^s / B_\infty^s \cong (Z_\infty^s / B_m^s)/(B_\infty^s / B_m^s), && \text{which form the } E_\infty\textit{-term}; \\
RE_\infty^s &:= R\lim_r Z_r^s \cong R\lim_r (Z_r^s / B_m^s), && \text{which form the } \textit{derived } E_\infty\textit{-term}.
\end{aligned}$$
here the isomorphisms in the second and third lines above are given by [1, Proposition 2.4].

Definition 0.8 ([1, Definition 5.2])**.** Given an A-graded target group G with decreasing filtration $(F^s G)_{s \in \mathbb{Z}}$, we say the spectral sequence:

(i) *converges weakly to* G if $F^{-\infty} = G$ and we have isomorphisms $E_\infty^s \cong F^s G / F^{s+1}$ for all $s \in \mathbb{Z}$;

(ii) *converges to* G if (i) holds and $F^\infty = 0$;

(iii) *converges strongly to* G if (i) holds and $F^\infty = RF^\infty = 0$.