0.1. **Setup.** In order to construct an abstract version of the Adams spectral sequence, we need to work in some axiomatic version of a stable homotopy category $\mathcal{SH}$ which acts like the familiar classical stable homotopy category **hoSp** (**??**) or the motivic stable homotopy category $\mathbf{SH}_{\mathscr{S}}$ over some base scheme $\mathscr{S}$ (**??**). As it turns out, practically all the data we need is the following:

Definition 0.1. A *stable homotopy category* is the following data:

- A closed tensor triangulated category $(\mathcal{SH}, \otimes, S, \Sigma, \Omega)$ with arbitrary small (co)products.
- A pointed abelian group $(A, \mathbf{1})$ and a homomorphism $h : (A, \mathbf{1}) \to (\mathrm{Pic}(\mathcal{SH}), \Sigma S)$ of pointed groups (i.e., $\mathbf{1}$ is sent to the isomorphism class of ΣS), where $\mathrm{Pic}(\mathcal{SH})$ is the group of isomorphism classes of invertible objects in $\mathcal{SH}$[1].
- For each $a \in A$, a chosen object S^a in the isomorphism class $h(a)$.

Given an abstract stable homotopy category as above, we will always assume without loss of generality that $S^0 = S$ and $\Sigma = S^{\mathbf{1}} \otimes -$ (by **??**). we establish the following conventions:

- Given objects $X_1, \dots, X_n$ in $\mathcal{SH}$, we write $X_1 \otimes \cdots \otimes X_n$ to denote the object
$$X_1 \otimes (X_2 \otimes \cdots (X_{n-1} \otimes X_n)).$$
In particular, given an object X and a natural number $n > 0$, we write
$$X^n := \overbrace{X \otimes \cdots \otimes X}^{n \text{ times}} \qquad \text{and} \qquad X^0 := S.$$
- We denote the associator, symmetry, left unitor, and right unitor isomorphisms in $\mathcal{SH}$ by
$$\alpha_{X,Y,Z} : (X \otimes Y) \otimes Z \xrightarrow{\cong} X \otimes (Y \otimes Z) \qquad \tau_{X,Y} : X \otimes Y \xrightarrow{\cong} Y \otimes X$$
$$\lambda_X : S \otimes X \xrightarrow{\cong} X \qquad \rho_X : X \otimes S \xrightarrow{\cong} X.$$
Often we will suppress these isomorphisms from the notation (particularly the associators), choosing instead to denote them without their subscripts or simply with the symbol $\cong$.
- Given some $a \in A$, we define the functor $\Sigma^a := S^a \otimes -$, so that in particular $\Sigma^{\mathbf{1}} = \Sigma$.
- Given two objects X and Y, we denote the hom-abelian group of morphisms from X to Y in $\mathcal{SH}$ by $[X, Y]$, and we denote the internal hom object by $F(X, Y)$. We will often refer to morphisms in $\mathcal{SH}$ as *classes*, as we will think of them as representing homotopy classes of maps.
- Given two objects X and Y in $\mathcal{SH}$, we may extend the abelian group $[X, Y]$ to an A-graded abelian group $[X, Y]_*$ defined by
$$[X, Y]_a := [\Sigma^a X, Y] = [S^a \otimes X, Y].$$
(See **??** for a review of the theory of A-graded abelian groups, rings, modules, etc.)
- Given an object X in $\mathcal{SH}$ and some $a \in A$, define the abelian group
$$\pi_a(X) := [S^a, X],$$
and write $\pi_*(X)$ for the associated A-graded abelian group $\bigoplus_{a \in A} \pi_a(X)$. We call $\pi_a(X)$ the a^{th} *stable homotopy group of* X.

[1]Recall an object X in a symmetric monoidal category is *invertible* if there exists some object Y in $\mathcal{SH}$ and an isomorphism $S \cong Y \otimes X$. To see ΣS is invertible, note that we have isomorphisms
$$\Sigma S \otimes \Omega S \cong \Sigma(S \otimes \Omega S) \cong \Sigma(\Omega S \otimes S) \cong \Sigma\Omega S \otimes S \cong S \otimes S \cong S,$$
where the first isomorphism is axiom TT1 for a tensor triangulated category (**??**), the second isomorphism is given by the symmetry in $\mathcal{SH}$, the third isomorphism is again axiom TT1, the fourth isomorphism is the fact that Σ and Ω for an adjoint equivalence, and finally the last isomorphism follows by the fact that S is the monoidal unit in $\mathcal{SH}$.

- Given two objects E and X in $\mathcal{SH}$, we define the A-graded abelian groups $E_*(X)$ and $E^*(X)$ by

$$E_a(X) := \pi_a(E \otimes X) = [S^a, E \otimes X] \qquad \text{and} \qquad E^a(X) := [X, S^a \otimes E].$$

  We refer to the functor $E_*(-)$ as the *homology theory represented by* E, or just E-homology, and we refer to $E^*(-)$ as the *cohomology theory represented by* E, or just E-cohomology.

From now on, we fix the data of a stable homotopy category $\mathcal{SH}$ given above once and for all. Observe that for all $a, b \in A$, the objects S^{a+b} and $S^a \otimes S^b$ are isomorphic, since $h : A \to \mathrm{Pic}(\mathcal{SH})$ is a group homomorphism. Hence given a monoid object (E, μ, e) in $\mathcal{SH}$ (**??**), supposing we had fixed isomorphisms $S^{a+b} \cong S^a \otimes S^b$ for all $a, b \in A$, we get a multiplication map $\pi_*(E) \times \pi_*(E) \to \pi_*(E)$ which sends classes $x : S^a \to E$ and $y : S^b \to E$ to the product

$$S^{a+b} \cong S^a \otimes S^b \xrightarrow{x \otimes y} E \otimes E \xrightarrow{\mu} E.$$

Naturally, we would like this product to make $\pi_*(E)$ into an A-graded ring (with unit $e \in \pi_0(E) = [S, E]$), rather than just an A-graded abelian group. This is essentially the entire discussion of Dugger's paper [1], and as it turns out, $\pi_*(E)$ is in fact a graded ring provided we can choose these morphisms to be *coherent*, in the following sense:

Definition 0.2. Suppose we have a family of isomorphisms

$$\phi_{a,b} : S^{a+b} \xrightarrow{\cong} S^a \otimes S^b$$

for all $a, b \in A$. We say this family is *coherent* if:

(1) For all $a \in A$, we have equalities $\phi_{a,0} = \rho_{S^a}^{-1} : S^a \to S^a \otimes S$ and $\phi_{0,a} = \lambda_{S^a}^{-1} : S^a \to S \otimes S^a$.
(2) For all $a, b, c \in A$, the following diagram commutes:

$$\begin{tikzcd}
S^{a+b} \otimes S^c \arrow[d, "\phi_{a,b} \otimes S^c"'] & S^{a+b+c} \arrow[l, "\phi_{a+b,c}"'] \arrow[r, "\phi_{a,b+c}"] & S^a \otimes S^{b+c} \arrow[d, "S^a \otimes \phi_{b,c}"] \\
(S^a \otimes S^b) \otimes S^c \arrow[rr, "\cong"] & & S^a \otimes (S^b \otimes S^c)
\end{tikzcd}$$

Furthermore, Dugger gaurantees that we can always find such a coherent family:

Theorem 0.3 ([1, Proposition 7.1])**.** *There exists a coherent family of isomorphisms*

$$\phi_{a,b} : S^{a+b} \xrightarrow{\cong} S^a \otimes S^b$$

in the sense of Definition 0.2, and in particular, the set of such coherent families is in bijective correspondence with the set of normalized 2*-cocycles* $Z^2(A; \mathrm{Aut}(S))_{norm}$*, i.e., the set of functions* $\alpha : A \times A \to \mathrm{Aut}(S)$ *such that* $\alpha(0,0) = \mathrm{id}_S$ *and for all* $a, b, c \in A$*,* $\alpha(a+b, c) \cdot \alpha(a, b) = \alpha(b, c) \cdot \alpha(a, b+c)$*.*

Thus, from now on we will suppose once and for all we have fixed a coherent family $\{\phi_{a,b}\}_{a,b \in A}$. Such a coherent family has very nice properties, in particular:

Remark 0.4. Note that by induction the coherence conditions say that given any $a_1, \ldots, a_n \in A$ and $b_1, \ldots, b_m \in A$ such that $a_1 + \cdots + a_n = b_1 + \cdots + b_m$ and any fixed parenthesizations of $X = S^{a_1} \otimes \cdots \otimes S^{a_b}$ and $Y = S^{b_1} \otimes \cdots \otimes S^{b_m}$, there is a *unique* isomorphism $X \to Y$ that can be obtained by forming formal compositions of tensor products of $\phi_{a,b}$, associators, and their inverses.

Of course, we get our desired result: $\pi_*(E)$ is indeed an A-graded ring if E is a monoid object.

Proposition 0.5. *Let (E, μ, e) be a commutative monoid object in $\mathcal{SH}$, and consider the multiplication map $\pi_*(E) \times \pi_*(E) \to \pi_*(E)$ which sends classes $x : S^a \to E$ and $y : S^b \to E$ to the composition*

$$S^{a+b} \xrightarrow{\phi_{a,b}} S^a \otimes S^b \xrightarrow{x \otimes y} E \otimes E \xrightarrow{\mu} E.$$

Then this endows $\pi_*(E)$ with the structure of an A-graded ring with unit $e \in \pi_0(E) = [S, E]$.

Proof. See **??**. □

Furthermore, it turns out that if E is a *commutative* monoid object in $\mathcal{SH}$, then $\pi_*(E)$ is "A-graded commutative," in the following sense:

Proposition 0.6. *For all $a, b \in A$ there exists an element $\theta_{a,b} \in \pi_0(S) = [S, S]$ (determined by choice of coherent family $\{\phi_{a,b}\}$) such that given any commutative monoid object (E, μ, e) in $\mathcal{SH}$, the A-graded ring structure on $\pi_*(E)$ (Proposition 0.5) has a commutativity formula given by*

$$x \cdot y = y \cdot x \cdot (e \circ \theta_{a,b})$$

for all $x \in \pi_a(E)$ and $y \in \pi_b(E)$.

Furthermore, $\theta_{0,a} = \theta_{a,0} = \mathrm{id}_S$ for all $a \in A$, so that if either x or y has degree 0, $x \cdot y = y \cdot x$.

Proof. See **??** and **??**. □

We also have the following result:

Proposition 0.7. *Given some $a \in A$, the functors Σ^a and Σ^{-a} canonically form an adjoint equivalence of $\mathcal{SH}$.*

Proof. See **??**. □

In particular, note that this tells us that given objects E and X in $\mathcal{SH}$, we have isomorphisms

$$E^*(X) = [X, S^* \otimes X] \cong [S^{-*} \otimes X, E] \cong [S^{-*}, F(X, E)] = \pi_{-*}(F(X, E)).$$

Similarly, given any objects X and Y in $\mathcal{SH}$, we have isomorphisms of A-graded abelian groups

$$[X, \Sigma Y]_* = [S^* \otimes X, S^{\mathbf{1}} \otimes Y] \cong [S^{-\mathbf{1}} \otimes S^* \otimes X, Y] \cong [S^{*-\mathbf{1}} \otimes X, Y] = [X, Y]_{*-\mathbf{1}},$$

where the first isomorphism is the adjunction specified by the above proposition, and the second isomorphism is induced by the isomorphism

$$S^{*-\mathbf{1}} \otimes X \xrightarrow{\phi_{-\mathbf{1},*} \otimes X} S^{-\mathbf{1}} \otimes S^* \otimes X.$$

The last ingredient in order to develop the Adams spectral sequence abstractly is a notion of *cellularity* in $\mathcal{SH}$:

Definition 0.8. Define the class of *cellular* objects in $\mathcal{SH}$ to be the smallest class of objects such that:

(1) For all $a \in A$, S^a is cellular.
(2) If we have a distinguished triangle

$$X \to Y \to Z \to \Sigma X (= S^{\mathbf{1}} \otimes X)$$

such that two of the three objects X, Y, and Z are cellular, than the third object is also cellular.
(3) Given a collection of cellular objects X_i indexed by some small set I, $\bigoplus_{i \in I} X_i$ is cellular.

0.2. **Monoid objects in $\mathcal{SH}$.** We have constructed an Adams spectral sequence, but as it currently stands we do not yet know why it is useful. To start with, we'd like to provide a characterization of its E_1 and E_2 pages in terms of something more algebraic. To start, we first need to develop some theory of the algebra of monoid objects in $\mathcal{SH}$. Much of this work is entirely straightforward although tedious to verify, so we relegate most of the proofs in this section to **??**.

Proposition 0.9. *Let (E, μ, e) be a monoid object in $\mathcal{SH}$. Then for any object X in $\mathcal{SH}$, $E_*(X)$ canonically inherits the structure of a left A-graded module over $\pi_*(E)$ (which recall is an A-graded ring by Proposition 0.5) via the map*

$$\pi_*(E) \times E_*(X) \to E_*(X)$$

which given $a, b \in A$, sends $x : S^a \to E$ and $y : S^b \to E$ to the composition

$$x \cdot y : S^{a+b} \cong S^a \otimes S^b \xrightarrow{x \otimes y} E \otimes (E \otimes X) \cong (E \otimes E) \otimes X \xrightarrow{\mu \otimes X} E \otimes X.$$

Similarly, $X_*(E)$ canonically inherits the structure of a right graded $\pi_*(E)$-module via the map

$$X_*(E) \times \pi_*(E) \to X_*(E)$$

which given $a, b \in A$, sends $x : S^a \to X \otimes E$ and $y : S^b \to E$ to the composition

$$x \cdot y : S^{a+b} \cong S^a \otimes S^b \xrightarrow{x \otimes y} (X \otimes E) \otimes E \cong X \otimes (E \otimes E) \xrightarrow{X \otimes \mu} X \otimes E.$$

Proof. See **??**. $\square$

Definition 0.10. Given a monoid object E in $\mathcal{SH}$, we say E is *flat* if the canonical right $\pi_*(E)$-module structure on $E_*(E)$ (see the above proposition) is that of a flat module.

0.3. **Construction of the Adams spectral sequence.** In what follows, let E be a commutative monoid object in $\mathcal{SH}$.

Definition 0.11. Let $\overline{E}$ be the fiber of the unit map $e : S \to E$ (**??**), and for $s \geq 0$ define

$$Y_s := \overline{E}^s \otimes Y, \qquad W_s = E \otimes Y_s = E \otimes (\overline{E}^s \otimes Y),$$

where recall for $s > 0$, $\overline{E}^s$ denotes the s-fold product parenthesized as $\overline{E} \otimes (\overline{E} \otimes \cdots (\overline{E} \otimes \overline{E}))$, and $\overline{E}^0 := S$. Then we get fiber sequences

$$Y_{s+1} \xrightarrow{i_s} Y_s \xrightarrow{j_s} W_s \xrightarrow{k_s} \Sigma Y_{s+1} (= S^{\mathbf{1}} \otimes Y_{s+1})$$

obtained by applying $- \otimes Y_s$ to the sequence

$$\overline{E} \to S \xrightarrow{e} E \to \Sigma \overline{E}$$

(and applying the necessary associator isomorphisms). These sequences can be spliced together to form the *(canonical) Adams filtration of Y*:

$$\begin{tikzcd}
\cdots \arrow[r] & Y_3 \arrow[r, "i_2"] \arrow[d, "j_3"] & Y_2 \arrow[r, "i_1"] \arrow[d, "j_2"] & Y_1 \arrow[r, "i_0"] \arrow[d, "j_1"] & Y_0 = Y \arrow[d, "j_0"] \\
& W_3 & W_2 \arrow[ul, dashed, "k_2"'] & W_1 \arrow[ul, dashed, "k_1"'] & W_0 \arrow[ul, dashed, "k_0"']
\end{tikzcd}$$

where the diagonal dashed arrows are of degree $-\mathbf{1}$ (note these triangles do NOT commute in any sense). Now we may apply the functor $[X, -]_*$, and by **??** we obtain an exact couple of

$\mathbb{N} \times A$-graded abelian groups:

$$\begin{tikzcd} {[X, Y_*]_*} \arrow[rr, "i_{**}"] & & {[X, Y_*]_*} \arrow[d, "j_{**}"] \\ & & {[X, W_*]_*} \arrow[ull, "k_{**}"] \end{tikzcd}$$

where i_{**}, j_{**}, and k_{**} have $\mathbb{Z} \times A$-degree $(-1, 0)$, $(0, 0)$, and $(1, -\mathbf{1})$, respectively[2]. The standard argument yields a $\mathbb{N} \times A$-graded spectral sequence called from this exact couple (cf. Section 5.9 of [2]) with E_1 page given by

$$E_1^{s,a} = [X, W_s]_a$$

and r^{th} differential of $\mathbb{Z} \times A$-degree $(r, -\mathbf{1})$:

$$d_r : E_r^{s,a} \to E_r^{s+r, a-\mathbf{1}}.$$

A priori, this is all $\mathbb{N} \times A$-graded, but we regard it as being $\mathbb{Z} \times A$-graded by setting $E_r^{s,a} := 0$ for $s < 0$ and trivially extending the definition of the differentials to these zero groups. This spectral sequence is called the *E-Adams spectral sequence* for the computation of $[X, Y]_*$. The index s is called the *Adams filtration* and a is the *stem*.

0.4. **The E_1 page.** The goal of this subsection is to provide the following characterization for the E_1 page of the Adams spectral sequence:

Theorem 0.12. *Let E be a flat commutative monoid object in $\mathcal{SH}$, and let X and Y be two objects in $\mathcal{SH}$ such that $E_*(X)$ is a projective module over $\pi_*(E)$. Then for all $s \geq 0$ and $a \in A$, we have isomorphisms in the associated E-Adams spectral sequence*

$$E_1^{s,a} \cong \mathrm{Hom}^a_{E_*(E)}(E_*(X), E_*(W_s))$$

Furthermore, under these isomorphisms, the differential $d_1 : E_1^{s,a} \to E_1^{s+1, a-\mathbf{1}}$ corresponds to the map

$$\mathrm{Hom}^a_{E_*(E)}(E_*(X), E_*(W_s)) \to \mathrm{Hom}^{a-\mathbf{1}}_{E_*(E)}(E_*(X), E_*(X \otimes W_{s+1}))$$

which sends a map $f : E_*(X) \to E_{*+a}(W_s)$ to the composition

$$E_*(X) \xrightarrow{f} E_{*+a}(W_s) \xrightarrow{(X \otimes h_s)_*} E_{*+a-\mathbf{1}}(X \otimes Y_{s+1}) \xrightarrow{(X \otimes j_{s+1})_*} E_{*+a-\mathbf{1}}(X \otimes W_{s+1}).$$

Proof. By **??**, for all $s \geq 0$ and $t, w \in \mathbb{Z}$, we have isomorphisms

$$[X, E \otimes Y_s]_{t,w} \cong \mathrm{Hom}^{t,w}_{E_*(E)}(E_*(X), E_*(E \otimes Y_s)).$$

since $W_s = E \otimes Y_s$, we have that

$$E_1^{s,(t,w)} = [X, W_s]_{t,w} \cong \mathrm{Hom}^{t,w}_{E_*(E)}(E_*(X), E_*(W_s)),$$

as desired. □

Definition 0.13. Let (E, μ, e) be a monoid object in $\mathcal{SH}$. We say E is *flat* if the canonical right $\pi_*(E)$-module structure on $E_*(E)$ is that of a flat module.

0.5. **The E_2 page.**

0.6. **Convergence.** convergence of spectral sequences

[2]Explicitly, the map $k_{s,a} : [X, W_s]_a \to [X, Y_{s+1}]_{a-\mathbf{1}}$ sends a map $f : S^a \otimes X \to W_s$ to the map $S^{a-\mathbf{1}} \otimes X \to Y_{s+1}$ corresponding under the isomorphism $[X, \Sigma Y_{s+1}]_* \cong [X, Y_{s+1}]_{*-\mathbf{1}}$ to the composition $k_s \circ f : S^a \otimes X \to \Sigma Y_{s+1}$.