In this section, we will freely use the coherence theorem for a symmetric monoidal category, which says that every symmetric monoidal category is (monoidally) equivalent to a *permutative category*, that is, a symmetric monoidal category in which the associators and unitors are strict equalities.

add ref

Definition 0.1. Let $(\mathcal{C}, \otimes, S)$ be a symmetric monoidal category with left unitor, right unitor, and associator, and symmetry isomorphism λ, ρ, α, and τ, respectively. Then a *monoid object* (E, μ, e) is an object E in $\mathcal{C}$ along with a multiplication map $\mu : E \otimes E \to E$ and a unit map $e : S \to E$ such that the following diagram commutes:

$$\begin{tikzcd} E \otimes S \arrow[r, "E\otimes e"] \arrow[dr, "\rho"'] & E \otimes E \arrow[d, "\mu"] & S \otimes E \arrow[l, "e\otimes E"'] \arrow[dl, "\lambda"] \\ & E & \end{tikzcd} \qquad \begin{tikzcd} (E \otimes E) \otimes E \arrow[rr, "\mu\otimes E"] \arrow[d, "\alpha"'] & & E \otimes E \arrow[d, "\mu"] \\ E \otimes (E \otimes E) \arrow[r, "E\otimes\mu"] & E \otimes E \arrow[r, "\mu"] & E \end{tikzcd}$$

The first diagram expresses unitality, while the second expressed associativity. If in addition the following diagram commutes,

$$\begin{tikzcd} E \otimes E \arrow[rr, "\tau"] \arrow[dr, "\mu"'] & & E \otimes E \arrow[dl, "\mu"] \\ & E & \end{tikzcd}$$

then we say (E, μ, e) is a *commutative* monoid object.

Proposition 0.2. *Let (E_1, μ_1, e_1) and (E_2, μ_2, e_2) be monoid objects in a symmetric monoidal category $(\mathcal{C}, \otimes, S)$. Then $E_1 \otimes E_2$ is canonically a ring spectrum via the maps*

$$\mu : E_1 \otimes E_2 \otimes E_1 \otimes E_2 \xrightarrow{E_1\otimes\tau\otimes E_2} E_1 \otimes E_1 \otimes E_2 \otimes E_2 \xrightarrow{\mu_1\otimes\mu_2} E_1 \otimes E_2$$

and

$$e : S \cong S \otimes S \xrightarrow{e_1\otimes e_2} E_1 \otimes E_2.$$

Proof. □

todo

In what follows, fix a stable homotopy category $\mathcal{SH}$ (**??**) along with the additional data therewithin, and adopt the conventions outlined in **??**. Further suppose we have fixed a coherent family of isomorphisms

$$\phi_{a,b} : S^{a+b} \xrightarrow{\cong} S^a \otimes S^b,$$

in the sense of **??** (the existence of such a coherent family is guaranteed by **??**).

Proposition 0.3. *Let (E, μ, e) be a commutative monoid object in $\mathcal{SH}$, and consider the multiplication map $\pi_*(E) \times \pi_*(E) \to \pi_*(E)$ which sends classes $x : S^a \to E$ and $y : S^b \to E$ to the composition*

$$S^{a+b} \xrightarrow{\phi_{a,b}} S^a \otimes S^b \xrightarrow{x\otimes y} E \otimes E \xrightarrow{\mu} E.$$

Then this endows $\pi_*(E)$ with the structure of an A-graded ring with unit $e \in \pi_0(E) = [S, E]$.

Proof. In this proof, we will assume we are working in a permutative category. Suppose we have classes x, y, and z in $\pi_a(E)$, $\pi_b(E)$, and $\pi_c(E)$, respectively. To see associativity, consider the following diagram:

$$\begin{tikzcd}
 & & & E\otimes E \arrow[d, "\mu"] \\
S^{a+b+c} \arrow[r, "\cong"] & S^a\otimes S^b\otimes S^c \arrow[r, "x\otimes y\otimes z"] & E\otimes E\otimes E \arrow[ur, "\mu\otimes E"] \arrow[dr, "E\otimes\mu"'] & E \\
 & & & E\otimes E \arrow[u, "\mu"']
\end{tikzcd}$$

(here the first arrow is the unique isomorphism obtained by composing products of $\phi_{a,b}$'s, see **??**). It commutes by associativity of μ. It follows by functoriality of $-\otimes-$ that the top composition is $(x\cdot y)\cdot z$ while the bottom is $x\cdot(y\cdot z)$, so they are equal as desired. To see that $e\in\pi_0(E)$ is a left and right unit for this multiplication, consider the following diagram

$$\begin{tikzcd}
 & S^a \arrow[dl, "e\otimes x"'] \arrow[d, "x"] \arrow[dr, "x\otimes e"] & \\
E\otimes E \arrow[dr, "\mu"'] & E \arrow[l, "e\otimes E"'] \arrow[r, "E\otimes e"] \arrow[d, equal] & E\otimes E \arrow[dl, "\mu"] \\
 & E &
\end{tikzcd}$$

Commutativity of the two top triangles is functoriality of $-\otimes-$. Commutativity of the bottom two triangles is unitality of μ. Thus the diagram commutes, so $e\cdot x=x\cdot e$. Finally, to see this product is bilinear (distributive). Suppose we further have some $x'\in\pi_a(E)$ and $y'\in\pi_b(E)$, and consider the following diagrams:

$$\begin{tikzcd}
S^{a+b} \arrow[r, "\phi_{a,b}"] \arrow[d, "\Delta"'] & S^a\otimes S^b \arrow[r, "\Delta\otimes S^b"] \arrow[d, "\Delta"] & (S^a\oplus S^a)\otimes S^b \arrow[r, "(x\oplus x')\otimes y"] \arrow[dl, "\cong"'] & (E\oplus E)\otimes E \arrow[dl, "\cong"'] \arrow[d, "\nabla\otimes E"] & \\
S^{a+b}\oplus S^{a+b} \arrow[r, "\phi_{a,b}\oplus\phi_{a,b}"'] & (S^a\otimes S^b)\oplus(S^a\otimes S^b) \arrow[r, "(x\otimes y)\oplus(x'\otimes y)"'] & (E\otimes E)\oplus(E\otimes E) \arrow[r, "\nabla"'] & E\otimes E \arrow[r, "\mu"] & E
\end{tikzcd}$$

$$\begin{tikzcd}
S^{a+b} \arrow[r, "\phi_{a,b}"] \arrow[d, "\Delta"'] & S^a\otimes S^b \arrow[r, "S^a\otimes\Delta"] \arrow[d, "\Delta"] & S^b\otimes(S^b\oplus S^b) \arrow[r, "x\otimes(y\oplus y')"] \arrow[dl, "\cong"'] & E\otimes(E\oplus E) \arrow[dl, "\cong"'] \arrow[d, "E\otimes\nabla"] & \\
S^{a+b}\oplus S^{a+b} \arrow[r, "\phi_{a,b}\oplus\phi_{a,b}"'] & (S^a\otimes S^b)\oplus(S^a\otimes S^b) \arrow[r, "(x\otimes y)\oplus(x\otimes y')"'] & (E\otimes E)\oplus(E\otimes E) \arrow[r, "\nabla"'] & E\otimes E \arrow[r, "\mu"] & E
\end{tikzcd}$$

The unlabeled isomorphisms are those given by the fact that $-\otimes-$ is additive in each variable (since $\mathcal{SH}$ is tensor triangulated). Commutativity of the left squares is naturality of $\Delta : X\to X\oplus X$ in an additive category. Commutativity of the rest of the diagram follows again from the fact that $-\otimes-$ is an additive functor in each variable. Hence, by functoriality of $-\otimes-$, these diagrams tell us that $(x+x')\cdot y=x\cdot y+x'\cdot y$ and $x\cdot(y+y')=x\cdot y+x\cdot y'$, respectively. $\square$

Proposition 0.4. *For all $a,b\in A$ there exists an element $\theta_{a,b}\in\pi_0(S)=[S,S]$ (determined by choice of coherent family $\{\phi_{a,b}\}$) such that given any commutative monoid object (E,μ,e) in $\mathcal{SH}$, the A-graded ring structure on $\pi_*(E)$ (**??**) has a commutativity formula given by*

$$x\cdot y=y\cdot x\cdot(e\circ\theta_{a,b})$$

for all $x\in\pi_a(E)$ and $y\in\pi_b(E)$. In particular, $\theta_{a,b}\in\operatorname{Aut}(S)$ is the composition

$$S\xrightarrow{\cong}S^{-a-b}\otimes S^a\otimes S^b\xrightarrow{S^{-a-b}\otimes\tau}S^{-a-b}\otimes S^b\otimes S^a\xrightarrow{\cong}S,$$

*where the outermost maps are the unique maps specified by **??**.*

Proof. Let $\phi_{a,b}$, E, x, and y as in the statement of the proposition. Now consider the following diagram

$$\begin{tikzcd}
S^{a+b} \arrow[r, "\phi_{a,b}"] \arrow[dd, dashed, "\phi_{b,a}^{-1}\circ\tau\circ\phi_{a,b}"'] & S^a \otimes S^b \arrow[r, "x\otimes y"] \arrow[dd, "\tau"'] & E \otimes E \arrow[dr, "\mu"] \arrow[dd, "\tau"'] & \\
& & & E \\
S^{a+b} \arrow[r, "\phi_{b,a}"] & S^b \otimes S^a \arrow[r, "y\otimes x"] & E \otimes E \arrow[ur, "\mu"'] &
\end{tikzcd}$$

The left square commutes by definition. The middle square commutes by naturality of the symmetry isomorphism. Finally, the right square commutes by commutativity of E. Unravelling definitions, we have shown that under the product on $\pi_*(E)$ induced by the $\phi_{a,b}$'s,

$$x \cdot y = (y \cdot x) \circ (\phi_{b,a}^{-1} \circ \tau \circ \phi_{a,b}).$$

Thus, in order to show the desired result it further suffices to show that

$$(y \cdot x) \circ (\phi_{b,a}^{-1} \circ \tau \circ \phi_{a,b}) = y \cdot x \cdot (e \circ \theta_{a,b}).$$

Consider the following diagram:

$$\begin{tikzcd}
S^{a+b} \arrow[rr, "\phi_{a,b}"] \arrow[d, "\cong"'] & & S^a \otimes S^b \arrow[d, "\tau"] \\
S^b \otimes S^a \otimes S^{-a-b} \otimes S^a \otimes S^b \arrow[urr, "\cong"'] \arrow[d, "S^b\otimes S^a\otimes S^{-a-b}\otimes\tau"'] & & S^b \otimes S^a \arrow[d, "\phi_{b,a}^{-1}"] \arrow[dl, equal] \\
S^b \otimes S^a \otimes S^{-a-b} \otimes S^b \otimes S^a \arrow[r, "\cong"] \arrow[urr, "\cong"] & S^b \otimes S^a \arrow[dr, "y\otimes x"] \arrow[ddl, "y\otimes x\otimes e"'] & S^{a+b} \arrow[l, "\phi_{b,a}"'] \\
& & E \otimes E \arrow[dll, "E\otimes E\otimes e"'] \arrow[d, equal] \\
E \otimes E \otimes E \arrow[rr, "E\otimes\mu"] \arrow[d, "\mu\otimes E"'] & & E \otimes E \arrow[d, "\mu"] \\
E \otimes E \arrow[rr, "\mu"'] & & E
\end{tikzcd}$$

Here any map simply labelled $\cong$ is an appropriate composition of copies of $\phi_{a,b}$'s, associators, and their inverses, so that each of these maps are necessarily unique by **??**. The two triangles in the top large rectangle commutes by coherence for the $\phi_{a,b}$'s. The parallelogram commutes by naturality of τ and coherence of the of $\phi_{a,b}$'s. The middle skewed triangle commutes by functoriality of $- \otimes -$. The triangle below that commutes by unitality of μ. Finally, the bottom rectangle commmutes by associativity of μ. Hence, by unravelling definitions and applying functoriality of $- \otimes -$, we get that the right composition is $(y \cdot x) \circ (\phi_{b,a}^{-1} \circ \tau \circ \phi_{a,b})$, while the left composition is $y \cdot x \cdot (e \circ \theta_{a,b})$, so they are equal as desired. $\square$

Proposition 0.5. *Given $a \in A$, we have $\theta_{0,a} = \theta_{a,0} = \mathrm{id}_S$.*

Proof. Recall $\theta_{a,0}$ is the composition

$$S \xrightarrow{\phi_{-a,a}} S^{-a} \otimes S^a \xrightarrow{S^{-a}\otimes\phi_{a,0}} S^{-a} \otimes (S^a \otimes S) \xrightarrow{S^{-a}\otimes\tau} S^{-a} \otimes (S \otimes S^a) \xrightarrow{S^{-a}\otimes\phi_{0,a}^{-1}} S^{-a} \otimes S^a \xrightarrow{\phi_{-a,a}^{-1}} S$$

By the coherence theorem for symmetric monoidal categories and the fact that $\phi_{a,0}$ and $\phi_{0,a}$ coincide with the unitors, we have that the composition

$$S^a \xrightarrow{\phi_{a,0}=\rho_{S^a}^{-1}} S^a \otimes S \xrightarrow{\tau} S \otimes S^a \xrightarrow{\phi_{0,a}^{-1}=\lambda_{S^a}} S^a$$

is precisely the identity map, so by functoriality of $- \otimes -$, we have that $\theta_{a,0}$ is the composition

$$S \xrightarrow{\phi_{-a,a}} S^{-a} \otimes S^a \xrightarrow{=} S^{-a} \otimes S^a \xrightarrow{\phi_{-a,a}^{-1}} S,$$

so $\theta_{a,0} = \mathrm{id}_S$, meaning

$$x \cdot y = y \cdot x \cdot (e \circ \theta_{a,0}) = y \cdot x \cdot e = y \cdot x,$$

where the last equality follows by the fact that e is the unit for the multiplication on $\pi_*(E)$. An entirely analagous argument yields that $\theta_{0,a} = \mathrm{id}_S$. $\square$

Proposition 0.6. *Given some $a \in A$, the functors Σ^a and Σ^{-a} canonically form an adjoint equivalence of $\mathcal{SH}$.*

rewrite proof in a permutative category

Proof. Let $X, Y \in \mathcal{SH}$. By [1, Lemma 3.2], in order to show Σ^a and Σ^{-a} are adjoint equivalences, it suffices to construct natural isomorphisms $\eta : \mathrm{Id}_{\mathcal{SH}} \Rightarrow \Sigma^{-a} \circ \Sigma^a$ and $\varepsilon : \Sigma^a \circ \Sigma^{-a} \Rightarrow \mathrm{Id}_{\mathcal{SH}}$ such that for all X in $\mathcal{SH}$, the following diagram commutes:

$$\begin{tikzcd} \Sigma^a X \arrow[r, "(\Sigma^a \eta)_X"] \arrow[dr] & \Sigma^a \Sigma^{-a} \Sigma^a X \arrow[d, "(\varepsilon \Sigma^a)_X"] \\ & \Sigma^a X \end{tikzcd} \tag{1}$$

Given an object X in $\mathcal{SH}$, define $\eta_X : X \to \Sigma^{-a}\Sigma^a X = S^{-a} \otimes S^a \otimes X$ to be the composition

$$X \xrightarrow{\lambda_X^{-1}} S \otimes X \xrightarrow{\phi_{-a,a} \otimes X} S^{-a} \otimes S^a \otimes X.$$

Clearly this is an isomorphism. To see this is natural, let $f : X \to Y$ in $\mathcal{SH}$. Then consider the following diagram:

$$\begin{tikzcd} X \arrow[r, "\lambda_X^{-1}"] \arrow[d, "f"'] & S \otimes X \arrow[r, "\phi_{-a,a} \otimes X"] \arrow[d, "S \otimes f"] & S^{-a} \otimes S^a \otimes X \arrow[d, "S^{-a} \otimes S^a \otimes f"] \\ Y \arrow[r, "\lambda_Y^{-1}"'] & S \otimes Y \arrow[r, "\phi_{-a,a} \otimes Y"'] & S^{-a} \otimes S^a \otimes Y \end{tikzcd}$$

The left square commutes by naturality of λ. The right square commutes by functoriality of $- \otimes -$. Hence η is indeed a natural isomorphism.

On the other hand, given an object X in $\mathcal{SH}$, define $\varepsilon_X : \Sigma^a \Sigma^{-a} X = S^a \otimes S^{-a} \otimes X \to X$ to be the composition

$$S^a \otimes S^{-a} \otimes X \xrightarrow{\phi_{a,-a}^{-1}} S \otimes X \xrightarrow{\lambda_X} X.$$

Clearly this is an isomorphism. To see it is natural, let $f : X \to Y$ in $\mathcal{SH}$. Then consider the following diagram:

$$\begin{tikzcd} S^a \otimes S^{-a} \otimes X \arrow[r, "\phi_{a,-a}^{-1} \otimes X"] \arrow[d, "S^a \otimes S^{-a} \otimes f"'] & S \otimes X \arrow[r, "\lambda_X"] \arrow[d, "S \otimes f"'] & X \arrow[d, "f"] \\ S^a \otimes S^{-a} \otimes Y \arrow[r, "\phi_{a,-a}^{-1} \otimes Y"'] & S \otimes Y \arrow[r, "\lambda_Y"'] & Y \end{tikzcd}$$

The left square commutes by functoriality of $- \otimes -$. The right square commutes by naturality of λ. Hence, ε is natural.

Finally, let X be an object in $\mathcal{SH}$. Unravelling definitions, by functoriality of $- \otimes -$, in order to show that diagram (1) commutes, it suffices to show the following diagram commutes:

$$\begin{tikzcd}
S^a \otimes X \arrow[r, "S^a \otimes \lambda_X^{-1}"] \arrow[ddrr, equal] & S^a \otimes S \otimes X \arrow[r, "S^a \otimes \phi_{-a,a} \otimes X"] & S^a \otimes S^{-a} \otimes S^a \otimes X \arrow[d, "\phi_{a,-a}^{-1} \otimes S^a \otimes X"] \\
& & S \otimes S^a \otimes X \arrow[d, "\lambda_{S^a \otimes X}"] \\
& & S^a \otimes X \arrow[uul, "\phi_{a,0} \otimes X"]
\end{tikzcd}$$

First, note that by the coherence theorem for monoidal categories, $\lambda_{S^a \otimes X} = \lambda_{S^a} \otimes X$[1]. And furthermore, recall $\lambda_{S^a} = \phi_{0,a}^{-1}$. Hence, the right triangle is precisely the diagram obtained by applying $- \otimes X$ to the coherence diagram for the $\phi_{a,b}$'s, so it commutes. Commutativity of the left triangle follows by the coherence theorem for monoidal categories and the fact that $\phi_{a,0} = \lambda_{S^a}^{-1}$. Hence, the diagram commutes, so (Σ^a, Σ^{-a}) forms an adjoint equivalence of $\mathcal{SH}$. □

Proposition 0.7. *Let X and Y be objects in $\mathcal{SH}$. Then the pairing*

$$\pi_*(X) \times \pi_*(Y) \to \pi_*(X \otimes Y)$$

sending $x : S^a \to X$ and $y : S^b \to Y$ to the composition

$$S^{a+b} \xrightarrow{\phi_{a,b}} S^a \otimes S^b \xrightarrow{x \otimes y} X \otimes Y$$

is additive in each argument.

Proof. Let $a, b \in A$, and let $x_1, x_2 : S^a \to X$ and $y : S^b \to Y$. Then consider the following diagram

$$\begin{tikzcd}
S^{a+b} \arrow[r, "\cong"] & S^a \otimes S^b \arrow[r, "\Delta \otimes S^b"] \arrow[d, "\Delta"'] & (S^a \oplus S^a) \otimes S^b \arrow[dl, "\cong"'] \arrow[d, "(x_1 \oplus x_2) \otimes y"] \\
& (S^a \otimes S^b) \oplus (S^a \otimes S^b) \arrow[d, "(x_1 \otimes y) \oplus (x_2 \otimes y)"'] & (X \oplus X) \otimes Y \arrow[dl, "\cong"'] \arrow[d, "\nabla \otimes Y"] \\
& (X \otimes Y) \oplus (X \otimes Y) \arrow[r, "\nabla"] & X \otimes Y
\end{tikzcd}$$

The isomorphisms are given by the fact that $- \otimes -$ is additive in each variable. Both triangles and the parallelogram commute since $- \otimes -$ is additive. By functoriality of $- \otimes -$, the top composition is $(x_1 + x_2) \cdot y$ and the bottom composition is $x_1 \cdot y + x_2 \cdot y$, so they are equal, as desired. An entirely analagous argument yields that $x \cdot (y_1 + y_2) = x \cdot y_1 + x \cdot y_2$ for $x \in \pi_*(X)$ and $y_1, y_2 \in \pi_*(Y)$. □

Proposition 0.8 ([2, Proposition 5.11])**.** *Let (E, μ, e) be a monoid object in $\mathcal{SH}$. Then $E_*(-)$ is a functor from $\mathcal{SH}$ to left A-graded $\pi_*(E)$-modules, where given some X in $\mathcal{SH}$, $E_*(X)$ may be endowed with the structure of a left A-graded $\pi_*(E)$-module via the map*

$$\pi_*(E) \times E_*(X) \to E_*(X)$$

which given $a, b \in A$, sends $x : S^a \to E$ and $y : S^b \to E \otimes X$ to the composition

$$x \cdot y : S^{a+b} \cong S^a \otimes S^b \xrightarrow{x \otimes y} E \otimes (E \otimes X) \cong (E \otimes E) \otimes X \xrightarrow{\mu \otimes X} E \otimes X.$$

[1]Technically, this equality only holds up to composition with an associator, but we are ignoring such issues.

Similarly, the assignment $X \mapsto X_*(E)$ is a functor from $\mathcal{SH}$ to right A-graded $\pi_*(E)$-modules, where the structure map

$$X_*(E) \times \pi_*(E) \to X_*(E)$$

sends $x : S^a \to X \otimes E$ and $y : S^b \to E$ to the composition

$$x \cdot y : S^{a+b} \cong S^a \otimes S^b \xrightarrow{x \otimes y} (X \otimes E) \otimes E \cong X \otimes (E \otimes E) \xrightarrow{X \otimes \mu} X \otimes E.$$

Finally, $E_*(E)$ is a $\pi_*(E)$-bimodule, in the sense that the left and right actions of $\pi_*(E)$ are compatible, so that given $y, z \in \pi_*(E)$ and $x \in E_*(E)$, $y \cdot (x \cdot z) = (y \cdot x) \cdot z$.

Proof. First we show that the map $\pi_*(E) \times E_*(X) \to E_*(X)$ endows $E_*(X)$ with the structure of a left $\pi_*(E)$-module. Let $a, b, c \in A$ and $x, x' : S^a \to E \otimes X$, $y : S^b \to E$, and $z, z' \in S^c \to E$. Then we wish to show that:

(1) $y \cdot (x + x') = y \cdot x + y \cdot x'$,
(2) $(z + z') \cdot x = z \cdot x + z' \cdot x$,
(3) $(zy) \cdot x = z \cdot (y \cdot x)$,
(4) $e \cdot x = x$.

Axioms (1) and (2) follow by the fact that $E_*(X) = \pi_*(E \otimes X)$ and Proposition 0.7. To see (3), consider the diagram:

$$\begin{tikzcd}
& & & E \otimes E \otimes X \arrow[d, "\mu \otimes X"] \\
S^{a+b+c} \arrow[r, "\cong"] & S^c \otimes S^b \otimes S^a \arrow[r, "z \otimes y \otimes x"] & E \otimes E \otimes E \otimes X \arrow[ur, "E \otimes \mu \otimes X"] \arrow[dr, "\mu \otimes E \otimes X"'] & E \otimes X \\
& & & E \otimes E \otimes X \arrow[u, "\mu \otimes X"']
\end{tikzcd}$$

It commutes by associativity of μ. By functoriality of $- \otimes -$, the two outside compositions equal $z \cdot (y \cdot x)$ on the top and $(z \cdot y) \cdot x$ on the bottom. Hence, they are equal, as desired.

Next, to see (4), consider the following diagram:

$$\begin{tikzcd}
S^a \arrow[rr, "x"] \arrow[dr, "x"'] \arrow[ddr, bend right, "e \otimes x"'] & & E \otimes X \\
& E \otimes X \arrow[ur, equal] \arrow[d, "e \otimes E \otimes X"] & \\
& E \otimes E \otimes X \arrow[uur, bend right, "\mu \otimes X"'] &
\end{tikzcd}$$

The top triangle commutes by definition. The left triangle commutes by functoriality of $- \otimes -$. The right triangle commutes by unitality of μ. The top composition is x while the bottom is $e \cdot x$, thus they are necessarily equal since the diagram commutes.

Thus, we have shown that the indicated map does indeed endow $E_*(X)$ with the structure of a left $\pi_*(E)$-module. It remains to show that $E_*(-)$ sends maps in $\mathcal{SH}$ to A-graded homomorphisms of left A-graded $\pi_*(E)$-modules. By definition, given $f : X \to Y$ in $\mathcal{SH}$, $E_*(f)$ is the map which takes a class $x : S^a \to E \otimes X$ to the composition

$$S^a \xrightarrow{x} E \otimes X \xrightarrow{E \otimes f} E \otimes Y.$$

To see this assignment is a homomorphism, suppose we are given some other $x' : S^a \to E \otimes X$ and some scalar $y : S^b \to E$. Then we would like to show $E_*(f)(x + x') = E_*(f)(x) + E_*(f)(x')$

and $E_*(f)(y \cdot x) = y \cdot E_*(f)(x)$. To see the former, consider the following diagram:

$$\begin{tikzcd}
 & & & (E\otimes Y)\oplus(E\otimes Y) \arrow[d, "\nabla"] \\
S^a \arrow[r, "\Delta"] & S^a \oplus S^a \arrow[r, "x\oplus x'"] & (E\otimes X)\oplus(E\otimes X) \arrow[ur, "(E\otimes f)\oplus(E\otimes f)"] \arrow[dr, "\nabla"'] & E\otimes Y \\
 & & & E\otimes X \arrow[u, "E\otimes f"']
\end{tikzcd}$$

It commutes by naturality of ∇ in an additive category. The top composition is $E_*(f)(x) + E_*(f)(x')$, while the bottom is $E_*(f)(x+x')$, so they are equal as desired. To see that $E_*(f)(y \cdot x) = y \cdot E_*(f)(x)$, consider the following diagram:

$$\begin{tikzcd}
S^{a+b} \arrow[r, "\phi_{b,a}"] & S^b \otimes S^a \arrow[r, "y\otimes x"] & E\otimes E\otimes X \arrow[r, "E\otimes E\otimes f"] \arrow[d, "\mu\otimes X"'] & E\otimes E\otimes Y \arrow[d, "\mu\otimes Y"] \\
 & & E\otimes X \arrow[r, "E\otimes f"] & E\otimes Y
\end{tikzcd}$$

It commutes by functoriality of $-\otimes-$. The top composition is $E_*(f)(y \cdot x)$, while the bottom composition is $y \cdot E_*(f)(x)$, so they are equal, as desired.

Showing that $X_*(E)$ has the structure of a right $\pi_*(E)$-module and that if $f : X \to Y$ is a morphism in $\mathcal{SH}$ then the map

$$X_*(E) = [S^*, X \otimes E] \xrightarrow{(f\otimes E)_*} [S^*, Y \otimes E] = Y_*(E)$$

is an A-graded homomorphism of right A-graded $\pi_*(E)$-modules is entirely analagous.

It remains to show that $E_*(E)$ is a bimodule. Let $x : S^a \to E$, $y : S^b \to E \otimes E$, and $z : S^c \to E$, and consider the following diagram:

$$\begin{tikzcd}
 & & & E\otimes E\otimes E \arrow[d, "E\otimes\mu"] \\
S^{a+b+c} \arrow[r, "\cong"] & S^a\otimes S^b\otimes S^c \arrow[r, "x\otimes y\otimes z"] & E\otimes E\otimes E\otimes E \arrow[r, "\mu\otimes\mu"] \arrow[ur, "\mu\otimes E\otimes E"] \arrow[dr, "E\otimes E\otimes\mu"'] & E\otimes E \\
 & & & E\otimes E\otimes E \arrow[u, "\mu\otimes E"']
\end{tikzcd}$$

Commutativity follows by functoriality of $-\otimes-$, which also tells us that the two outside compositions are $(x \cdot y) \cdot z$ (on top) and $x \cdot (y \cdot z)$ (on bottom). Hence they are equal, as desired. □

Proposition 0.9 ([3, Proposition 2.2]). *Let (E, μ, e) be a monoid object in $\mathcal{SH}$ and let X be any object. Then the assignment*

$$E_*(E) \times E_*(X) \to E_*(E \otimes X)$$

which sends $x : S^a \to E \otimes E$ and $y : S^b \to E \otimes X$ to the composition

$$x \cdot y : S^{a+b} \cong S^a \otimes S^b \xrightarrow{x\otimes y} E \otimes E \otimes E \otimes X \xrightarrow{E\otimes\mu\otimes X} E \otimes E \otimes X$$

induces an A-graded homomorphism of left A-graded $\pi_*(E)$-modules

$$E_*(E) \otimes_{\pi_*(E)} E_*(X) \to E_*(E \otimes X)$$

(where here $E_*(E)$ has a $\pi_*(E)$-bimodule structure and $E_*(X)$ has a left $\pi_*(E)$-module structure as specified by Proposition 0.8, so $E_*(E) \otimes_{\pi_*(E)} E_*(X)$ is a left A-graded $\pi_*(E)$-module by **??***). Furthermore, this homomorphism is natural in X.*

Proof. First, recall by definition of the tensor product, in order to show the assignment $E_*(E) \times E_*(X) \to E_*(E \otimes X)$ induces a homomorphism $E_*(E) \otimes_{\pi_*(E)} E_*(X) \to E_*(E \otimes X)$ of A-graded abelian groups, it suffices to show that the assignment is $\pi_*(E)$-balanced, i.e., that it is linear in each argument and satisfies $xr \cdot y = x \cdot ry$ for $x \in E_*(E)$, $y \in E_*(X)$, and $r \in \pi_*(E)$.

First, note that by the identifications $E_*(E) = \pi_*(E \otimes E)$, $E_*(X) = \pi_*(E \otimes X)$, and $E_*(E \otimes X) = \pi_*(E \otimes E \otimes X)$, and Proposition 0.7, it is straightforward to see that the assignment commutes with addition of maps in each argument. Now, let $a, b, c \in A$, $x : S^a \to E \otimes E$, $y : S^b \to E \otimes X$, and $z : S^c \to E$. Then we wish to show $xz \cdot y = x \cdot zy$. Consider the following diagram (where here we are passing to a permutative category):

$$\begin{tikzcd}
& & & E \otimes E \otimes E \otimes X \arrow[d, "E \otimes \mu \otimes X"] \\
S^{a+b+c} \arrow[r, "\cong"] & S^a \otimes S^c \otimes S^b \arrow[r, "x \otimes z \otimes y"] & E \otimes E \otimes E \otimes E \otimes X \arrow[ur, "E \otimes \mu \otimes E \otimes X"] \arrow[dr, "E \otimes E \otimes \mu \otimes X"'] & E \otimes E \otimes X \\
& & & E \otimes E \otimes E \otimes X \arrow[u, "E \otimes \mu \otimes X"']
\end{tikzcd}$$

It commutes by associativity of μ. By functoriality of $-\otimes-$, the top composition is given by $(xz) \cdot y$ and the bottom composition is $x \cdot (zy)$, so we have they are equal, as desired. Thus, since the map $E_*(E) \times E_*(X) \to E_*(E \otimes X)$ is $\pi_*(E)$-balanced, we have that it induces a homomorphism of abelian groups. Furthermore, by **??** it is an A-graded homomorphism of A-graded abelian groups.

In order to see this map is furthermore a homomorphism of left $\pi_*(E)$-modules, we must show that $z(x \cdot y) = zx \cdot y$, where x, y, and z are defined as above. Now consider the following diagram:

$$\begin{tikzcd}
& & & E \otimes E \otimes E \otimes X \arrow[d, "E \otimes \mu \otimes X"] \\
S^{a+b+c} \arrow[r, "\cong"] & S^c \otimes S^a \otimes S^b \arrow[r, "z \otimes x \otimes y"] & E \otimes E \otimes E \otimes E \otimes X \arrow[ur, "\mu \otimes E \otimes E \otimes X"] \arrow[r, "\mu \otimes \mu \otimes X"] \arrow[dr, "E \otimes E \otimes \mu \otimes X"'] & E \otimes E \otimes X \\
& & & E \otimes E \otimes E \otimes X \arrow[u, "\mu \otimes E \otimes X"']
\end{tikzcd}$$

Commutativity of the triangles is functoriality of $- \otimes -$. By functoriality of $- \otimes -$, the top composition is $zx \cdot y$, and the bottom composition is $z(x \cdot y)$. Hence they are equal, as desired, so that the map we have constructed

$$E_*(E) \otimes_{\pi_*(E)} E_*(X) \to E_*(E \otimes X)$$

is indeed an A-graded homomorphism of left A-graded $\pi_*(E)$-modules.

Next, we would like to show that this homomorphism is natural in X. Let $f : X \to Y$ in $\mathcal{SH}$. Then we would like to show the following diagram commutes:

$$\begin{tikzcd}
E_*(E) \otimes_{\pi_*(E)} E_*(X) \arrow[r, "\Phi_X"] \arrow[d, "E_*(E) \otimes_{\pi_*(E)} E_*(f)"'] & E_*(E \otimes X) \arrow[d, "E_*(E \otimes f)"] \\
E_*(E) \otimes_{\pi_*(E)} E_*(Y) \arrow[r, "\Phi_Y"] & E_*(E \otimes Y)
\end{tikzcd} \tag{2}$$

As all the maps here are homomorphisms, it suffices to chase generators around the diagram. In particular, suppose we are given $x : S^a \to E \otimes E$ and $y : S^b \to E \otimes X$, and consider the following diagram exhibiting the two possible ways to chase $x \otimes y$ around the diagram (as usual, we are passing to a permutative category):

$$\begin{tikzcd}
S^{a+b} \arrow[r, "\phi_{a,b}"] & S^a \otimes S^b \arrow[r, "x\otimes y"] & E\otimes E\otimes E\otimes X \arrow[r, "E\otimes\mu\otimes X"] \arrow[d, "E\otimes E\otimes E\otimes f"'] & E\otimes E\otimes X \arrow[d, "E\otimes E\otimes f"] \\
 & & E\otimes E\otimes E\otimes Y \arrow[r, "E\otimes\mu\otimes Y"] & E\otimes E\otimes Y
\end{tikzcd}$$

This diagram commutes by functoriality of $-\otimes-$. Thus we have that diagram (2) does indeed commute, as desired. ☐

Proposition 0.10. *Let (E,μ,e) be a* flat *monoid object in $\mathcal{SH}$ (??) and let X be any* cellular *object in $\mathcal{SH}$ (??). Then the natural homomorphism*

$$\Phi_X : E_*(E)\otimes E_*(X) \to E_*(E\otimes X)$$

given in Proposition 0.9 is an isomorphism of left $\pi_*(E)$-modules.

Proof. It remains to show that if X is cellular and E is flat, then this map is an isomorphism. To start, let $\mathcal{E}$ be the collection of objects X in $\mathcal{SH}$ for which this map is an isomorphism. Then in order to show $\mathcal{E}$ contains every cellular object, it suffices to show that $\mathcal{E}$ satisfies the three conditions given for the class of cellular objects in **??**. First, we need to show that Φ is an isomorphism when $X = S^a$ for some $a \in A$. Indeed, consider the map

$$\Psi : E_*(E\otimes S^a) \to E_*(E)\otimes_{\pi_*(E)} E_*(S^a)$$

which sends a class $x : S^b \to E\otimes E\otimes S^a$ in $E_b(E\otimes S^a)$ to the pure tensor $\widetilde{x}\otimes\widetilde{e}$, where $\widetilde{x}\in E_{b-a}(E)$ is the composition

$$S^{b-a} \cong S^b\otimes S^{-a} \xrightarrow{x\otimes S^{-a}} E\otimes E\otimes S^a\otimes S^{-a} \xrightarrow{E\otimes E\otimes\phi^{-1}_{a,-a}} E\otimes E\otimes S \xrightarrow{E\otimes\rho_E} E\otimes E$$

and $\widetilde{e}\in E_a(S^a)$ is the composition

$$S^a \cong S\otimes S^a \xrightarrow{e\otimes S^a} E\otimes S^a.$$

First, note Ψ is an (A-graded) homomorphism of abelian groups: Given $x, x' \in E_b(E\otimes S^a)$, we would like to show that $\widetilde{x}\otimes\widetilde{e} + \widetilde{x}'\otimes\widetilde{e} = \widetilde{x+x'}\otimes\widetilde{e}$. It suffices to show that $\widetilde{x}+\widetilde{x}' = \widetilde{x+x'}$. To see this, consider the following diagram (again, we are passing to a permutative category):

$$\begin{tikzcd}
S^{b-a} \arrow[r, "\Delta"] \arrow[d, "\phi_{b-a}"'] & S^{b-a}\oplus S^{b-a} \arrow[d, "\phi_{b,-a}\oplus\phi_{b,-a}"] \\
S^b\otimes S^{-a} \arrow[r, "\Delta"] \arrow[d, "\Delta\otimes S^{-a}"'] & (S^b\otimes S^{-a})\oplus(S^b\otimes S^{-a}) \arrow[d, "(x\otimes S^{-a})\oplus(x'\otimes S^{-a})"] \\
(S^b\oplus S^b)\otimes S^{-a} \arrow[ur, "\cong"] \arrow[d, "(x\oplus x')\otimes S^{-a}"'] & (E\otimes E\otimes S^a\otimes S^{-a})\oplus(E\otimes E\otimes S^a\otimes S^{-a}) \arrow[d, "(E\otimes E\otimes\phi^{-1}_{a,-a})\oplus(E\otimes E\otimes\phi^{-1}_{a,-a})"] \arrow[ddl, "\nabla"] \\
((E\otimes E\otimes S^a)\oplus(E\otimes E\otimes S^a))\otimes S^{-a} \arrow[ur, "\cong"] \arrow[d, "\nabla\otimes S^{-a}"'] & (E\otimes E)\oplus(E\otimes E) \arrow[d, "\nabla"] \\
E\otimes E\otimes S^a\otimes S^{-a} \arrow[r, "E\otimes E\otimes\phi^{-1}_{a,-a}"'] & E\otimes E
\end{tikzcd}$$

The top rectangle commutes by naturality of Δ in an additive category. The bottom triangle commutes by naturality of ∇ in an additive category. Finally, the remaining regions of the diagram commute by additivity of $-\otimes-$. By functoriality of $-\otimes-$, it follows that the left composition is $\widetilde{x + x'}$ and the right composition is $\widetilde{x} + \widetilde{x}'$, so they are equal as desired. Thus Ψ is a homomorphism of abelian groups, as desired.

Now, we claim that Ψ is an inverse to Φ, (which is enough to show Φ is an isomorphism of left $\pi_*(E)$-modules). Since Φ and Ψ are homomorphisms it suffices to check that they are inverses on generators. First, let $x : S^b \to E \otimes E \otimes S^a$ in $E_b(E \otimes S^a)$. We would like to show that $\Phi(\Psi(x)) = x$. Consider the following diagram, where here we are passing to a permutative category:

$$\begin{tikzcd}
S^b \arrow[r, "\cong"] \arrow[dd, "x"'] & S^b \otimes S^{-a} \otimes S^a \arrow[d, dashed, "x \otimes S^{-a} \otimes S^a"] \arrow[dr, "x \otimes S^{-a} \otimes e \otimes S^a"] & \\
& E \otimes E \otimes S^a \otimes S^{-a} \otimes S^a \arrow[r, "E \otimes E \otimes S^a \otimes S^{-a} \otimes e \otimes S^a"'] & E \otimes E \otimes S^a \otimes S^{-a} \otimes E \otimes S^a \arrow[ddl, "E \otimes E \otimes \phi_{a,-a}^{-1} \otimes E \otimes S^a"] \\
E \otimes E \otimes S^a \arrow[ur, "E \otimes E \otimes S^a \otimes \phi_{-a,a}"] \arrow[r, equal] \arrow[dr, "E \otimes E \otimes e \otimes S^a"] & E \otimes E \otimes S^a \arrow[u, dashed, "E \otimes E \otimes \phi_{a,-a} \otimes S^a"] \arrow[d, dashed, "E \otimes E \otimes e \otimes S^a"] & \\
E \otimes E \otimes E \otimes S^a \arrow[u, "E \otimes \mu \otimes S^a"] \arrow[r, equal] & E \otimes E \otimes E \otimes S^a &
\end{tikzcd}$$

The top left trapezoid commutes since the isomorphism $S^b \xrightarrow{\cong} S^b \otimes S^{-a} \otimes S^a$ may be given as $S^b \otimes \phi_{-a,a}$ (see **??**), in which case the trapezoid commmutes by functoriality of $- \otimes -$. The triangle below that commutes by coherence for the $\phi_{a,b}$'s. The triangle below that commutes by definition. The bottom left triangle commutes by unitality for μ. The top right triangle commutes by functoriality of $- \otimes -$. Finally, the bottom right triangle commutes by functoriality of $- \otimes -$. It follows by unravelling definitions that the two outside compositions are x (on the left) and $\Phi(\Psi(x))$ (on the right), so since the diagram commutes we indeed have $\Phi(\Psi(x)) = x$, as desired.

On the other hand, suppose we are given a homogeneous pure tensor $x \otimes y$ in $E_*(E) \otimes_{\pi_*(E)} E_*(S^a)$, so $x : S^b \to E \otimes E$ and $y : S^c \to E \otimes S^a$ for some $b, c \in A$. Then we would like to show that $\Psi(\Phi(x \otimes y)) = x \otimes y$. Unravelling definitions, $\Psi(\Phi(x \otimes y))$ is the homogeneous pure tensor $\widetilde{xy} \otimes \widetilde{e}$, where $\widetilde{e} : S^a \to E \otimes S^a$ is defined above, and by functoriality of $- \otimes -$, $\widetilde{xy} : S^{b+c-a} \to E \otimes E$ is the composition

$$\begin{tikzcd}
S^{b+c-a} \arrow[d, "\phi_{b+c,-a}"] \\
S^{b+c} \otimes S^{-a} \arrow[d, "\phi_{b,c} \otimes S^{-a}"] \\
S^b \otimes S^c \otimes S^{-a} \arrow[d, "x \otimes y \otimes S^{-a}"] \\
E \otimes E \otimes E \otimes S^a \otimes S^{-a} \arrow[d, "E \otimes \mu \otimes S^a \otimes S^{-a}"] \\
E \otimes E \otimes S^a \otimes S^{-a} \arrow[d, "E \otimes E \otimes \phi_{a,-a}^{-1}"] \\
E \otimes E \otimes S \arrow[d, "E \otimes \rho_E"] \\
E \otimes E.
\end{tikzcd}$$

In order to see $x \otimes y = \widetilde{xy} \otimes \widetilde{e}$, it suffices to show there exists some scalar $r \in \pi_{c-a}(E)$ such that $x \cdot r = \widetilde{xy}$ and $r \cdot \widetilde{e} = y$, where here $\cdot$ denotes the right and left action of $\pi_*(E)$ on $E_*(E)$ and

$E_*(S^a)$, respectively. Now, define r to be the composition

$$S^{c-a} \cong S^c \otimes S^{-a} \xrightarrow{y \otimes S^{-a}} E \otimes S^a \otimes S^{-a} \xrightarrow{E \otimes \phi^{-1}_{a,-a}} E \otimes S \xrightarrow{\rho_E} E.$$

First, in order to see that $x \cdot r = \widetilde{xy}$, consider the following diagram, where here we are again passing to a permutative category:

$$\begin{tikzcd}
S^{b+c-a} \arrow[r, "\cong"] & S^b \otimes S^c \otimes S^{-a} \arrow[r, "x \otimes y \otimes S^{-a}"] & E \otimes E \otimes E \otimes S^a \otimes S^{-a} \arrow[r, "E \otimes \mu \otimes S^a \otimes S^{-a}"] \arrow[d, "E \otimes E \otimes E \otimes \phi^{-1}_{a,-a}"'] \arrow[dr, "E \otimes \mu \otimes \phi^{-1}_{a,-a}"] & E \otimes E \otimes S^a \otimes S^{-a} \arrow[d, "E \otimes E \otimes \phi^{-1}_{a,-a}"] \\
 & & E \otimes E \otimes E \arrow[r, "E \otimes \mu"'] & E \otimes E
\end{tikzcd}$$

Commutativity is functoriality of $- \otimes -$, which also tells us that the two outside compositions are $\widetilde{xy}$ (on top) and $x \cdot r$ (on the bottom), so they are equal as desired. On the other hand, in order to see that $r \cdot \widetilde{e} = y$, consider the following diagram (where here we have passed to a permutative category):

$$\begin{tikzcd}
S^c \arrow[rr, "\cong"] \arrow[dd, "y"'] & & S^c \otimes S^{-a} \otimes S^a \arrow[dl, "y \otimes S^{-a} \otimes S^a"'] \arrow[dd, "y \otimes S^{-a} \otimes e \otimes S^a"] \\
 & E \otimes S^a \otimes S^{-a} \otimes S^a \arrow[dl, "E \otimes S^a \otimes \phi^{-1}_{-a,a}"'] \arrow[dr, "E \otimes S^a \otimes S^{-a} \otimes e \otimes S^a"] & \\
E \otimes S^a \arrow[drr, "E \otimes e \otimes S^a"] & & E \otimes S^a \otimes S^{-a} \otimes E \otimes S^a \arrow[d, "E \otimes \phi^{-1}_{a,-a} \otimes E \otimes S^a"] \\
E \otimes E \otimes S^a \arrow[u, "\mu \otimes S^a"] \arrow[rr, equal] & & E \otimes E \otimes S^a
\end{tikzcd}$$

The top left triangle commutes since we may take the isomorphism $S^c \xrightarrow{\cong} S^c \otimes S^{-a} \otimes S^a$ to be $S^c \otimes \phi_{-a,a}$, in which case commutativity of the triangle follows by functoriality of $- \otimes -$. Commutativity of the right triangle is also functoriality of $- \otimes -$. Commutativity of the bottom triangle is unitality of μ. Finally, commutativity of the remaining middle 4-sided region is again functoriality of $- \otimes -$. It follows that y is equal to the outer composition, which is $r \cdot \widetilde{e}$, as desired. Thus, we have shown that

$$\Psi(\Phi(x \otimes y)) = \widetilde{xy} \otimes \widetilde{e} = (x \cdot r) \otimes \widetilde{e} = x \otimes (r \cdot \widetilde{e}) = x \otimes y,$$

as desired, so that for each $a \in A$, the object S^a belongs to the class $\mathcal{E}$.

Now, we would like to show that given a distinguished triangle

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X,$$

if two of three of the objects X, Y, and Z belong to $\mathcal{E}$, then so does the third. Indeed, supposing this is true, note first of all that since $\mathcal{SH}$ is tensor triangulated, by axiom TT3 (**??**), the following triangle is also distinguished:

$$E \otimes X \xrightarrow{E \otimes f} E \otimes Y \xrightarrow{E \otimes g} E \otimes Z \xrightarrow{E \otimes h} \Sigma(E \otimes X),$$

where here we are being abusive and writing $E \otimes h$ for the composition

$$E \otimes X \xrightarrow{E \otimes h} E \otimes \Sigma X \xrightarrow{e_{E,X}} \Sigma(E \otimes X).$$

Thus, by **??** we get a long exact sequence of A-graded abelian groups

$$
\begin{tikzcd}
\cdots \arrow[r] & {[\Sigma^{n+1}S^*, E \otimes Y]} \arrow[r, "(E\otimes g)_*"] & {[\Sigma^{n+1}S^*, E \otimes Z]} \arrow[dll, "\partial"'] \\
{[\Sigma^{n}S^*, E \otimes X]} \arrow[r, "(E\otimes f)_*"'] & {[\Sigma^{n}S^*, E \otimes Y]} \arrow[r, "(E\otimes g)_*"] & {[\Sigma^{n}S^*, E \otimes Z]} \arrow[dll, "\partial"'] \\
{[\Sigma^{n-1}S^*, E \otimes X]} \arrow[r, "(E\otimes f)_*"'] & {[\Sigma^{n-1}S^*, E \otimes Y]} \arrow[r] & \cdots
\end{tikzcd}
\tag{3}
$$

where for $n > 0$ we define $\Sigma^n := (\Sigma)^n = (\Sigma^{\mathbf{1}})^n$ and $\Sigma^{-n} = (\Sigma^{-\mathbf{1}})^n$, $\Sigma^0 = \mathrm{Id}_{\mathcal{SH}}$, and the maps ∂ are given by the compositions

$$[\Sigma^{n+1}S^*, E \otimes Z] \xrightarrow{(E\otimes h)_*} [\Sigma^{n+1}S^*, \Sigma(E \otimes X)] \cong [\Sigma^{-\mathbf{1}}\Sigma^{n+1}S^*, E \otimes X] \cong [\Sigma^{n}S^*, E \otimes X]$$

(here we are using the fact that the pair $(\Sigma^{-\mathbf{1}}, \Sigma^{\mathbf{1}})$ forms an adjoint equivalence (Proposition 0.6) and the fact that $\Sigma = \Sigma^{\mathbf{1}} = S^{\mathbf{1}} \otimes -$). Now, given some nonnegative integer n, we will write $\mathbf{n}$ for the corresponding element in A, i.e., $\mathbf{0} = 0 \in A$, and if $n > 0$ then

$$\mathbf{n} = \overbrace{\mathbf{1} + \cdots + \mathbf{1}}^{n \text{ times}}.$$

Then note that $\Sigma^0 S^* = S^*$ and for $n > 0$ we have isomorphisms

$$\Sigma^n S^* = \overbrace{S^{\mathbf{1}} \otimes \cdots \otimes S^{\mathbf{1}}}^{n \text{ times}} \otimes S^* \cong S^{*+\mathbf{n}}$$

and

$$\Sigma^{-n} S^* = \overbrace{S^{-\mathbf{1}} \otimes \cdots \otimes S^{-\mathbf{1}}}^{n \text{ times}} \otimes S^* \cong S^{*-\mathbf{n}},$$

where the isomorphisms are the unique ones obtained by composing products of copies of $\phi_{a,b}$'s, identities, associators, and their inverses (**??**). Thus for $n \in \mathbb{Z}$ we have identifications $[\Sigma^n S^*, E \otimes W] \cong [S^{*+\mathbf{n}}, E \otimes W] = E_{*+\mathbf{n}}(W)$, and it is straightforward to check that these yield an isomorphism of long exact sequences between (3) and the following:

$$
\begin{tikzcd}
\cdots \arrow[r] & E_{*+\mathbf{n+1}}(Y) \arrow[r, "g_*"] & E_{*+\mathbf{n+1}}(Z) \arrow[dll, "\partial"'] \\
E_{*+\mathbf{1}}(X) \arrow[r, "f_*"'] & E_{*+\mathbf{1}}(Y) \arrow[r, "g_*"] & E_{*+\mathbf{1}}(Z) \arrow[dll, "\partial"] \\
E_{*+\mathbf{n-1}}(X) \arrow[r, "f_*"'] & E_{*+\mathbf{n-1}}(Y) \arrow[r] & \cdots
\end{tikzcd}
\tag{4}
$$

where here the maps ∂ are the compositions

$$[S^{*+\mathbf{n+1}}, E \otimes Z] \xrightarrow{(E\otimes h)_*} [S^{*+\mathbf{n+1}}, \Sigma(E \otimes X)] \cong [\Sigma^{-\mathbf{1}}S^{*+\mathbf{n+1}}, E \otimes X] \cong [S^{*+\mathbf{n}}, E \otimes X].$$

An entirely analagous argument applied to the distinguished triangle

$$E \otimes E \otimes X \xrightarrow{E\otimes E\otimes f} E \otimes E \otimes Y \xrightarrow{E\otimes E\otimes g} E \otimes E \otimes Z \xrightarrow{E\otimes E\otimes h} \Sigma(E \otimes E \otimes X)$$

yields a long exact sequence

$$
\begin{tikzcd}
 & \cdots \arrow[r] & E_{*+\mathbf{n+1}}(E \otimes Y) \arrow[r, "(E\otimes g)_*"] & E_{*+\mathbf{n+1}}(E \otimes Z) \arrow[dll, "\partial"] \\
 & E_{*+\mathbf{n+1}}(E \otimes X) \arrow[r, "(E\otimes f)_*"'] & E_{*+\mathbf{n+1}}(E \otimes Y) \arrow[r, "(E\otimes g)_*"] & E_{*+\mathbf{n+1}}(E \otimes Z) \arrow[dll, "\partial"] \\
 & E_{*+\mathbf{n-1}}(E \otimes X) \arrow[r, "(E\otimes f)_*"'] & E_{*+\mathbf{n-1}}(E \otimes Y) \arrow[r] & \cdots
\end{tikzcd}
\tag{5}
$$

Now, we may apply the functor $E_*(E) \otimes_{\pi_*(E)} -$ (which is exact since we are assuming $E_*(E)$ is a flat right $\pi_*(E)$-module) to the long exact sequence (4), and we further get the following long exact sequence of A-graded left $\pi_*(E)$-modules:

$$
\begin{tikzcd}
 & \cdots \arrow[r] & L^E_{*+\mathbf{n+1}}(Y) \arrow[r, "L^E_{*+\mathbf{n+1}}(g)"] & L^E_{*+\mathbf{n+1}}(Z) \arrow[dll, "E_*(E)\otimes\partial"] \\
 & L^E_{*+\mathbf{n}}(X) \arrow[r, "L^E_{*+\mathbf{n}}(f)"'] & L^E_{*+\mathbf{n}}(Y) \arrow[r, "L^E_{*+\mathbf{n}}(g)"] & L^E_{*+\mathbf{n}}(Z) \arrow[dll, "E_*(E)\otimes\partial"] \\
 & L^E_{*+\mathbf{n-1}}(X) \arrow[r, "L^E_{*+\mathbf{n-1}}(f)"'] & L^E_{*+\mathbf{n-1}}(Y) \arrow[r] & \cdots
\end{tikzcd}
\tag{6}
$$

where here $L^E_*(-)$ is shorthand for the functor $X \mapsto E_*(E) \otimes_{\pi_*(E)} E_*(X)$. Now, we claim that the natural map $\Phi : L^E_*(X) \to E_*(E \otimes X)$ yields a chain map between (6) and (5). Since Φ is natural, we know that the following commutes for all $n \in \mathbb{Z}$:

$$
\begin{tikzcd}
L^E_{*+\mathbf{n}}(X) \arrow[r, "L^E_{*+\mathbf{n}}(f)"] \arrow[d, "\Phi_X"'] & L^E_{*+\mathbf{n}}(Y) \arrow[r, "L^E_{*+\mathbf{n}}(g)"] \arrow[d, "\Phi_Y"'] & L^E_{*+\mathbf{n}}(Z) \arrow[d, "\Phi_Z"'] \\
E_{*+\mathbf{n}}(E \otimes X) \arrow[r, "(E\otimes f)_*"'] & E_{*+\mathbf{n}}(E \otimes Y) \arrow[r, "(E\otimes g)_*"'] & E_{*+\mathbf{n}}(E \otimes Z)
\end{tikzcd}
$$

Thus it remains to show that the following diagram commutes for all $n \in \mathbb{Z}$:

$$
\begin{tikzcd}
L^E_{*+\mathbf{n+1}}(Z) \arrow[r, "E_*(E)\otimes\partial"] \arrow[d, "\Phi_Z"'] & L^E_{*+\mathbf{n}}(X) \arrow[d, "\Phi_X"] \\
E_{*+\mathbf{n+1}}(E \otimes Z) \arrow[r, "\partial"] & E_{*+\mathbf{n}}(E \otimes X)
\end{tikzcd}
$$

Let's chase a generator around, so suppose we are given a homogeneous pure tensor $x \otimes y$ in $L^E_{*+\mathbf{n}}(Z) = E_*(E) \otimes_{\pi_*(E)} E_{*+\mathbf{n+1}}(Z)$, so x and y are maps $S^a \to E \otimes E$ and $S^{b+\mathbf{n+1}} \to E \otimes Z$, respectively, for some $a, b \in A$. Then unravelling definitions, chasing $x \otimes y$ around the diagram yields the following two compositions:

$$
\begin{tikzcd}
S^{a+b+\mathbf{n}} \arrow[d, "\phi_{-\mathbf{1},a+b+\mathbf{n}+\mathbf{1}}"] \\
\Sigma^{-\mathbf{1}}(S^{a+b+\mathbf{n}+\mathbf{1}}) \arrow[d, "\Sigma^{-\mathbf{1}}\phi_{a,b+\mathbf{n}+\mathbf{1}}"] \\
\Sigma^{-\mathbf{1}}(S^a \otimes S^{b+\mathbf{n}+\mathbf{1}}) \arrow[d, "\Sigma^{-\mathbf{1}}(x\otimes y)"] \\
\Sigma^{-\mathbf{1}}(E \otimes E \otimes E \otimes Z) \arrow[d, "\Sigma^{-\mathbf{1}}(E\otimes\mu\otimes Z)"] \\
\Sigma^{-\mathbf{1}}(E \otimes E \otimes Z) \arrow[d, "\Sigma^{-\mathbf{1}}(E\otimes E\otimes h)"] \\
\Sigma^{-\mathbf{1}}(E \otimes E \otimes \Sigma X) \arrow[d, "\Sigma^{-\mathbf{1}}(e_{E\otimes E,X})"] \\
\Sigma^{-\mathbf{1}}(\Sigma(E \otimes E \otimes X)) \arrow[d, "\phi_{-\mathbf{1},\mathbf{1}}\otimes E\otimes E\otimes X"] \\
S \otimes E \otimes E \otimes X \arrow[d, "\lambda_{E\otimes E\otimes X}"] \\
E \otimes E \otimes X
\end{tikzcd}
\qquad
\begin{tikzcd}
S^{a+b+\mathbf{n}} \arrow[d, "\phi_{a,b+\mathbf{n}}"] \\
S^a \otimes S^{b+\mathbf{n}} \arrow[d, "S^a\otimes\phi_{-\mathbf{1},b+\mathbf{n}+\mathbf{1}}"] \\
S^a \otimes \Sigma^{-\mathbf{1}} S^{b+\mathbf{n}+\mathbf{1}} \arrow[d, "x\otimes y"] \\
E \otimes E \otimes \Sigma^{-\mathbf{1}}(E \otimes Z) \arrow[d, "E\otimes E\otimes\Sigma^{-\mathbf{1}}(E\otimes h)"] \\
E \otimes E \otimes \Sigma^{-\mathbf{1}}(E \otimes \Sigma X) \arrow[d, "E\otimes E\otimes\Sigma^{-\mathbf{1}}(e_{E,X})"] \\
E \otimes E \otimes \Sigma^{-\mathbf{1}}(\Sigma(E \otimes X)) \arrow[d, "E\otimes E\otimes\phi^{-1}_{-\mathbf{1},\mathbf{1}}"] \\
E \otimes E \otimes S \otimes E \otimes X \arrow[d, "E\otimes E\otimes\lambda_E\otimes X"] \\
E \otimes E \otimes E \otimes X \arrow[d, "E\otimes\mu\otimes X"] \\
E \otimes E \otimes X
\end{tikzcd}
$$

(left is bottom, right is top). Now we pass to a permutative category, and consider the following diagram:I'm stuck here, I don't know how to show these two compositions are equal.

Assuming two out of three of the objects X, Y, and Z belong to $\mathcal{E}$, by the five lemma applied to the above diagram, it follows that the third object belongs to $\mathcal{E}$ as well.

Finally, it remains to show that $\mathcal{E}$ is closed under taking arbitrary direct sums. Let $\{X_i\}_{i\in I}$ be a family of objects in $\mathcal{E}$ indexed by some set I. Then note by definition, since direct sums are limits, we have that for any W in $\mathcal{SH}$ that

$$\left[W, \bigoplus_{i\in I} X_i\right] \cong \bigoplus_{i\in I}[W, X_i],$$

and furthermore this isomorphism is natural in W. Now let $X = \bigoplus_i X_i$, and consider the following diagram

$$
\begin{tikzcd}
{[S^*, E\otimes E]\otimes[S^\star, E\otimes X]} \arrow[r, "\cong"'] \arrow[d, "-\otimes-"] & {[S^*, E\otimes E], [S^\star, \bigoplus_i E\otimes X_i]} \arrow[r, "\cong"'] \arrow[d, "-\otimes-"] & \bigoplus([S^*, E\otimes E]\otimes[S^\star, E\otimes X_i]) \arrow[d, "\bigoplus_i(-\otimes-)"] \\
{[S^*\otimes S^\star, E\otimes E\otimes E\otimes X]} \arrow[r, "\cong"'] \arrow[d, "(\phi_{*,\star})^*"] & {[S^*\otimes S^\star, \bigoplus_i E\otimes E\otimes E\otimes X_i]} \arrow[r, "\cong"'] \arrow[d, "(\phi_{*,\star})^*"] & \bigoplus_i[S^*\otimes S^\star, E\otimes E\otimes E\otimes X_i] \arrow[d, "\bigoplus_i(\phi_{*,\star})^*"] \\
{[S^{*+\star}, E\otimes E\otimes E\otimes X]} \arrow[r, "\cong"'] \arrow[d, "(E\otimes\mu\otimes X)_*"] & {[S^{*+\star}, \bigoplus_i E\otimes E\otimes E\otimes X_i]} \arrow[r, "\cong"'] \arrow[d, "(E\otimes\mu\otimes X)_*"] & \bigoplus_i[S^{*+\star}, E\otimes E\otimes E\otimes X_i] \arrow[d, "\bigoplus_i(E\otimes\mu\otimes X_i)"] \\
{[S^{*+\star}, E\otimes E\otimes X]} \arrow[r, "\cong"'] & {[S^{*+\star}, \bigoplus_i E\otimes E\otimes X_i]} \arrow[r, "\cong"'] & \bigoplus_i[S^{*+\star}, E\otimes E\otimes X_i]
\end{tikzcd}
$$

The left squares commute by additivity of $- \otimes -$. The right squares commute by naturality of the isomorphisms given above. Since each X_i belongs to $\mathcal{E}$, the right vertical composition is an isomorphism, so that the left vertical composition is also an isomorphism, as desired.

To recap, we have shown that the collection of objects $\mathcal{E}$ for which $\Phi_X : E_*(E) \otimes_{\pi_*(E)} E_*(X) \to E_*(E \otimes X)$ is an isomorphism satisfies the conditions outlined in **??**. Hence, $\mathcal{E}$ contains every cellular object, as desired. $\square$

where did I use cellularity of E?

In the following definition, let $\varepsilon : E_*(E) \to \pi_*(E)$ be the map which sends some $\alpha : S^a \to E \otimes E$ to the composition

$$S^a \xrightarrow{\alpha} E \otimes E \xrightarrow{\mu} E.$$

Also define $\Psi : E_*(E) \to E_*(E) \otimes_{\pi_*(E)} E_*(E)$ to be the map which factors as

$$E_*(E) \to E_*(E \otimes E) \xrightarrow{\cong} E_*(E) \otimes_{\pi_*(E)} E_*(E)$$

where the second arrow is the isomorphism prescribed by Proposition 0.9, and the first arrow sends a class $\alpha : S^a \to E \otimes E$ to the composition

$$S^a \xrightarrow{\alpha} E \otimes E \cong E \otimes S \otimes E \xrightarrow{E \otimes e \otimes E} E \otimes E \otimes E.$$

Lemma 0.11 ([3, Proposition 2.30, 2.33])**.** *Let E be a flat commutative ring spectrum, and let X and Y be spectra such that $E_{**}(X)$ is a projective module over $\pi_{**}(E)$. Then for all $s \geq 0$ and $t, w \in \mathbb{Z}$, there is an isomorphism*

$$\Phi : [X, E \wedge Y]_{t,w} \to \mathrm{Hom}^{t,w}_{E_{**}(E)}(E_{**}(X), E_{**}(E \wedge Y)),$$

obtained by sending a class $f : S^{t,w} \wedge X \to E \wedge Y$ in $[X, E \wedge Y]_{t,w}$ to the map

$$\Phi_f : E_{*,*}(X) \to E_{*+t,*+w}(X \wedge Y)$$

sending

$$[S^{a,b} \xrightarrow{g} E \wedge X] \mapsto [S^{a+t,b+w} \cong S^{a,b} \wedge S^{t,w} \xrightarrow{g \wedge S^{t,w}} E \wedge X \wedge S^{t,w} \cong E \wedge S^{t,w} \wedge X \xrightarrow{E \wedge f} E \wedge E \wedge Y].$$

Proof. Let $f : S^{t,w} \wedge X \to E \wedge Y$. First we want to show that Φ_f is actually an $E_{**}(E)$-comodule homomorphism. $\square$

finish