0.1. **Setup.** In order to construct an abstract version of the Adams spectral sequence, we need to work in some axiomatic version of a stable homotopy category $\mathcal{SH}$ which acts like the familiar classical stable homotopy category **hoSp** (**??**) or the motivic stable homotopy category $\mathbf{SH}_{\mathscr{S}}$ over some base scheme $\mathscr{S}$ (**??**).

Definition 0.1. Let $\mathcal{C}$ be an additive category with arbitrary (small) coproducts. Then an object X in $\mathcal{C}$ is *compact* if, for any collection of objects Y_i in $\mathcal{C}$ indexed by some (small) set I, the canonical map

$$\coprod_i \mathcal{C}(X, Y_i) \to \mathcal{C}(X, \coprod_i Y_i)$$

is an isomorphism of abelian groups. (Explicitly, the above map takes a generator $x \in \mathcal{C}(X, Y_i)$ to the composition $X \xrightarrow{x} Y_i \hookrightarrow \coprod_i Y_i$.)

Definition 0.2. Given a tensor triangulated category $(\mathcal{C}, \otimes, S, \Sigma, e, \mathcal{D})$ (**??**), a *sub-Picard grading* on $\mathcal{C}$ is the following data:

- A pointed abelian group $(A, \mathbf{1})$ along with a homomorphism of pointed groups $h : (A, \mathbf{1}) \to (\operatorname{Pic}\mathcal{C}, \Sigma S)$, where $\operatorname{Pic}\mathcal{C}$ is the *Picard group* of isomorphism classes of invertible objects in $\mathcal{C}$.[1]
- For each $a \in A$, a chosen representative S^a in the isomorphism class $h(a)$ such that each S^a is a compact object (Definition 0.1) and $S^0 = S$.
- A chosen isomorphism $\nu : \Sigma S \xrightarrow{\cong} S^{\mathbf{1}}$.
- For each $a, b \in A$, an isomorphism $\phi_{a,b} : S^{a+b} \to S^a \otimes S^b$. This family of isomorphisms is required to be *coherent*, in the following sense:
  - For all $a \in A$, we must have that $\phi_{a,0}$ coincides with the right unitor $S^a \xrightarrow{\cong} S^a \otimes S$ and $\phi_{0,a}$ coincides the left unitor $S^a \xrightarrow{\cong} S \otimes S^a$.
  - For all $a, b, c \in A$, the following "associativity diagram" must commute:

$$\begin{tikzcd}
S^{a+b} \otimes S^c \arrow[d, "\phi_{a,b} \otimes S^c"'] & S^{a+b+c} \arrow[l, "\phi_{a+b,c}"'] \arrow[r, "\phi_{a,b+c}"] & S^a \otimes S^{b+c} \arrow[d, "S^a \otimes \phi_{b,c}"] \\
(S^a \otimes S^b) \otimes S^c \arrow[rr, "\cong"] & & S^a \otimes (S^b \otimes S^c)
\end{tikzcd}$$

Remark 0.3. Note that by induction the coherence conditions for the $\phi_{a,b}$'s in the above definition say that given any $a_1, \ldots, a_n \in A$ and $b_1, \ldots, b_m \in A$ such that $a_1 + \cdots + a_n = b_1 + \cdots + b_m$ and any fixed parenthesizations of $X = S^{a_1} \otimes \cdots \otimes S^{a_b}$ and $Y = S^{b_1} \otimes \cdots \otimes S^{b_m}$, there is a *unique* isomorphism $X \to Y$ that can be obtained by forming formal compositions of products of $\phi_{a,b}$, identities, associators, unitors, and their inverses.

From now on we fix a monoidal closed tensor triangulated category $(\mathcal{SH}, \otimes, S, \Sigma, e, \mathcal{D})$ with arbitrary (small) (co)products and sub-Picard grading $(A, \mathbf{1}, h, \{S^a\}, \nu, \{\phi_{a,b}\})$. We establish some conventions. First of all, given an object X and a natural number $n > 0$, we write

$$X^n := \overbrace{X \otimes \cdots \otimes X}^{n \text{ times}} \qquad \text{and} \qquad X^0 := S.$$

We denote the associator, symmetry, left unitor, and right unitor isomorphisms in $\mathcal{SH}$ by

$$\alpha_{X,Y,Z} : (X \otimes Y) \otimes Z \xrightarrow{\cong} X \otimes (Y \otimes Z) \qquad \tau_{X,Y} : X \otimes Y \xrightarrow{\cong} Y \otimes X$$
$$\lambda_X : S \otimes X \xrightarrow{\cong} X \qquad \rho_X : X \otimes S \xrightarrow{\cong} X.$$

[1]Recall an object X is a symmetric monoidal category is *invertible* if there exists some object Y and an isomorphism $S \cong X \otimes Y$.

Often we will drop the subscripts. Furthermore, by the coherence theorem for symmetric monoidal categories, we will often assume α, ρ, and λ are actual equalities. Given some inte, there exists some isomorphismger $n \in \mathbb{Z}$, we will write a bold $\mathbf{n}$ to denote the element $n \cdot \mathbf{1}$ in A. Note that we can use the isomorphism $\nu : S^{\mathbf{1}} \otimes - \cong \Sigma$ to construct a natural isomorphism $S^{\mathbf{1}} \otimes - \cong \Sigma$:

$$S^{\mathbf{1}} \otimes X \xrightarrow{\nu \otimes X} \Sigma S \otimes X \xrightarrow{e_{S,X}} \Sigma(S \otimes X) \xrightarrow{\Sigma \lambda_X} \Sigma X.$$

The last two arrows are natural in X by definition. The first arrow is natural in X by functoriality of $- \otimes -$. By abuse of notation, we will also use ν to denote this natural isomorphism. Furthermore, under this isomorphism, $e_{X,Y} : \Sigma X \otimes Y \xrightarrow{\cong} \Sigma(X \otimes Y)$ corresponds to the associator, by commutativity of the following diagram:

$$\begin{tikzcd}
(S^{\mathbf{1}} \otimes X) \otimes Y \arrow[r, "(\nu \otimes X) \otimes Y"] \arrow[dd, "\alpha"'] & (\Sigma S \otimes X) \otimes Y \arrow[r, "e_{S,X} \otimes Y"] \arrow[dd, "\alpha"] & \Sigma(S \otimes X) \otimes Y \arrow[r, "\Sigma \lambda_X \otimes Y"] \arrow[d, "e_{S \otimes X, Y}"'] & \Sigma X \otimes Y \arrow[dd, "e_{X,Y}"] \\
 & & \Sigma((S \otimes X) \otimes Y) \arrow[d, "\Sigma \alpha"'] \arrow[dr, "\Sigma(\lambda_X \otimes Y)"] & \\
S^{\mathbf{1}} \otimes (X \otimes Y) \arrow[r, "\nu \otimes (X \otimes Y)"'] & \Sigma S \otimes (X \otimes Y) \arrow[r, "e_{S, X \otimes Y}"'] & \Sigma(S \otimes (X \otimes Y)) \arrow[r, "\Sigma \lambda_{X \otimes Y}"'] & \Sigma(X \otimes Y)
\end{tikzcd}$$

The left square commutes by naturality of α. Commutativity of the middle square is axiom TT4 for a tensor triangulated category. Commutativity of the right trapezoid is naturality of e. Finally the bottom triangle commutes by coherence for monoidal categories and functoriality of Σ.

Given some $a \in A$, we define $\Sigma^a := S^a \otimes -$ and $\Omega^a := \Sigma^{-a} = S^{-a} \otimes -$. We specifically define $\Omega := \Omega^{\mathbf{1}}$. We will see later that for each $a \in A$, Σ^a and Ω^a form an adjoint equivalence of $\mathcal{SH}$ (Proposition 0.5), so that in particular since Ω forms an adjoint equivalence with $\Sigma^{\mathbf{1}} \cong \Sigma$, $\mathcal{SH}$ is canonically an *adjointly* triangulated category (**??**).

Given two objects X and Y in $\mathcal{SH}$, we will denote the hom-abelian group of morphisms from X to Y in $\mathcal{SH}$ by $[X, Y]$, and the internal hom object by $F(X, Y)$. We can extend the abelian group $[X, Y]$ into an A-graded abelian group $[X, Y]_*$ by defining $[X, Y]_a := [S^a \otimes X, Y]$. Given an object X in $\mathcal{SH}$ and some $a \in A$, we can define the abelian group

$$\pi_a(X) := [S^a, X],$$

which we call the a^{th} *stable homotopy group of* X. We write $\pi_*(X)$ for the A-graded abelian group $\bigoplus_{a \in A} \pi_a(X)$, so that in particular we have a canonical isomorphism

$$\pi_*(X) = [S^*, X] \cong [S, X]_*.$$

Given some other object E, we can define the A-graded abelian groups $E_*(X)$ and $E^*(X)$ by the formulas

$$E_a(X) := \pi_a(E \otimes X) = [S^a, E \otimes X] \qquad \text{and} \qquad E^a(X) := [X, S^a \otimes E].$$

We refer to the functor $E_*(-)$ as the *homology theory represented by* E, or just E-homology, and we refer to $E^*(-)$ as the *cohomology theory represented by* E, or just E-cohomology. Finally, we state the following definition in $\mathcal{SH}$:

Definition 0.4. Define the class of *cellular* objects in $\mathcal{SH}$ to be the smallest class of objects such that:

(1) For all $a \in A$, S^a is cellular.
(2) If we have a distinguished triangle

$$X \to Y \to Z \to \Sigma X$$

such that two of the three objects X, Y, and Z are cellular, than the third object is also cellular.

(3) Given a collection of cellular objects X_i indexed by some (small) set I, the object $\coprod_{i\in I} X_i$ is cellular (recall we have chosen $\mathcal{SH}$ to have arbitrary (co)products).

0.2. **Miscellaneous facts about $\mathcal{SH}$.** what do I call this subsection?

Proposition 0.5. *For each $a \in A$, the isomorphisms*

$$\eta^a_X : X \xrightarrow{\lambda_X^{-1}} S \otimes X \xrightarrow{\Phi_{a,-a}\otimes X} (S^a \otimes S^{-a}) \otimes X \xrightarrow{\alpha} S^a \otimes (S^{-a} \otimes X) = \Sigma^a \Omega^a X$$

and

$$\varepsilon^a_X : \Omega^a \Sigma^a X = S^{-a} \otimes (S^a \otimes X) \xrightarrow{\alpha^{-1}} (S^{-a} \otimes S^a) \otimes X \xrightarrow{\phi^{-1}_{-a,a}\otimes X} S \otimes X \xrightarrow{\lambda_X} X$$

are natural in X, and furthermore, they are the unit and counit respectively of the adjoint autoequivalence $(\Omega^a, \Sigma^a, \eta^a, \varepsilon^a)$ of $\mathcal{SH}$. In particular, since $\Sigma \cong \Sigma^{\mathbf{1}}$, $\Omega := \Omega^{\mathbf{1}}$ is a left adjoint for Σ, so that $(\mathcal{SH}, \Omega, \Sigma, \eta, \varepsilon, \mathcal{D})$ is an adjointly *triangulated category (**??**), where η and ε are the compositions*

$$\eta : \mathrm{Id}_{\mathcal{SH}} \overset{\eta^{\mathbf{1}}}{\Longrightarrow} \Sigma^{\mathbf{1}}\Omega \overset{\nu\Omega}{\Longrightarrow} \Sigma\Omega \qquad \text{and} \qquad \varepsilon : \Omega\Sigma \overset{\Omega\nu^{-1}}{\Longrightarrow} \Omega\Sigma^{\mathbf{1}} \overset{\varepsilon^{\mathbf{1}}}{\Longrightarrow} \mathrm{Id}_{\mathcal{SH}}.$$

Proof. In this proof, we will freely employ the coherence theorem for monoidal categories (see [1]), which essentially tells us that we may assume we are working in a strict monoidal category (i.e., that the associators and unitors and are identities). Then η^a_X and ε^a_X become simply the maps

$$\eta^a_X : X \xrightarrow{\phi_{a,-a}\otimes X} S^a \otimes S^{-a} \otimes X \qquad \text{and} \qquad \varepsilon^a_X : S^{-a} \otimes S^a \otimes X \xrightarrow{\phi^{-1}_{-a,a}\otimes X} X.$$

That these maps are natural in X follows by functoriality of $-\otimes-$. Now, recall that in order to show that these natural isomorphisms form an *adjoint* equivalence, it suffices to show that the natural isomorphisms $\eta^a : \mathrm{Id}_{\mathcal{SH}} \Rightarrow \Omega^a\Sigma^a$ and $\varepsilon^a : \Sigma^a\Omega^a \Rightarrow \mathrm{Id}_{\mathcal{SH}}$ satisfy one of the two zig-zag identities:

$$\begin{tikzcd} \Omega^a \arrow[r, "\Omega^a\eta^a"] \arrow[dr, equal] & \Omega^a\Sigma^a\Omega^a \arrow[d, "\varepsilon^a\Omega^a"] \\ & \Omega^a \end{tikzcd} \qquad \begin{tikzcd} \Sigma^a\Omega^a\Sigma^a \arrow[d, "\Sigma^a\varepsilon^a"'] & \Sigma^a \arrow[l, "\eta^a\Sigma^a"'] \arrow[dl, equal] \\ \Sigma^a \end{tikzcd}$$

(that it suffices to show only one is [2, Lemma 3.2]). We will show that the left is satisfied. Unravelling definitions, we simply wish to show that the following diagram commutes for all X in $\mathcal{SH}$:

$$\begin{tikzcd} S^{-a} \otimes X \arrow[r, "S^{-a}\otimes\phi_{a,-a}\otimes X"] \arrow[dr, equal] & S^{-a} \otimes S^a \otimes S^{-a} \otimes X \arrow[d, "\phi^{-1}_{-a,a}\otimes S^{-a}\otimes X"] \\ & S^{-a} \otimes X \end{tikzcd}$$

Yet this is simply the diagram obtained by applying $-\otimes X$ to the associativity coherence diagram for the $\phi_{a,b}$'s (since $\phi_{a,0}$ and $\phi_{0,a}$ coincide with the unitors, and here we are taking the unitors and associators to be equalities), so it does commute, as desired. □

0.3. **Monoid objects in $\mathcal{SH}$.**

Proposition 0.6. *Let (E, μ, e) be a monoid object in $\mathcal{SH}$ (**??**). Then $\pi_*(E)$ is canonically an A-graded ring via the assignment $\pi_*(E) \times \pi_*(E) \to \pi_*(E)$ which takes classes $x : S^a \to E$ and $y : S^b \to E$ to the composition*

$$S^{a+b} \xrightarrow{\phi}_{a,b} S^a \otimes S^b \xrightarrow{x\otimes y} E \otimes E \xrightarrow{\mu} E.$$

In particular, the unit for this ring is the element $e \in [S, E] = \pi_0(E)$.

Proof. See **??**. □

Proposition 0.7. *Let (E, μ, e) be a monoid object in $\mathcal{SH}$. Then $E_*(-)$ is a functor from $\mathcal{SH}$ to left A-graded $\pi_*(E)$-modules, where given some X in $\mathcal{SH}$, $E_*(X)$ may be endowed with the structure of a left A-graded $\pi_*(E)$-module via the map*

$$\pi_*(E) \times E_*(X) \to E_*(X)$$

which given $a, b \in A$, sends $x : S^a \to E$ and $y : S^b \to E \otimes X$ to the composition

$$x \cdot y : S^{a+b} \cong S^a \otimes S^b \xrightarrow{x \otimes y} E \otimes (E \otimes X) \cong (E \otimes E) \otimes X \xrightarrow{\mu \otimes X} E \otimes X.$$

Similarly, the assignment $X \mapsto X_*(E)$ is a functor from $\mathcal{SH}$ to right A-graded $\pi_*(E)$-modules, where the structure map

$$X_*(E) \times \pi_*(E) \to X_*(E)$$

sends $x : S^a \to X \otimes E$ and $y : S^b \to E$ to the composition

$$x \cdot y : S^{a+b} \cong S^a \otimes S^b \xrightarrow{x \otimes y} (X \otimes E) \otimes E \cong X \otimes (E \otimes E) \xrightarrow{X \otimes \mu} X \otimes E.$$

Finally, $E_*(E)$ is a $\pi_*(E)$-bimodule, in the sense that the left and right actions of $\pi_*(E)$ are compatible, so that given $y, z \in \pi_*(E)$ and $x \in E_*(E)$, $y \cdot (x \cdot z) = (y \cdot x) \cdot z$.

Proof. See **??**. $\square$

Definition 0.8. Given a monoid object E in $\mathcal{SH}$, we say E is *flat* if the canonical right $\pi_*(E)$-module structure on $E_*(E)$ (see the above proposition) is that of a flat module.