We assume the reader is familiar with additive categories and (closed, symmetric) monoidal categories.

Definition 0.1. A *triangulated category* is a tuple $(\mathcal{C}, \Sigma, \Omega, \mathcal{D})$ such that

(1) $\mathcal{C}$ is an additive category.
(2) $\Sigma, \Omega : \mathcal{C} \to \mathcal{C}$ are additive functors which form an adjoint equivalence of $\mathcal{C}$ with itself. (Σ is calld the *shift functor*.)
(3) $\mathcal{D}$ is a collection of *distinguished triangles*, where a *triangle* is a diagram of the form
$$X \to Y \to Z \to \Sigma X.$$
These are also sometimes called *cofiber sequences* or *fiber sequences*.

These data must satisfy the following axioms:

TR0 Given a commutative diagram
$$\begin{tikzcd} X \arrow[r] \arrow[d, "\cong"'] & Y \arrow[r] \arrow[d, "\cong"'] & Z \arrow[r] \arrow[d, "\cong"'] & \Sigma X \arrow[d, "\cong"'] \\ X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & \Sigma X' \end{tikzcd}$$
where the vertical arrows are isomorphisms, if the top row is distinguished then so is the bottom.

TR1 For any object X in $\mathcal{C}$, the diagram
$$X \xrightarrow{\mathrm{id}_X} X \to 0 \to \Sigma X$$
is a distinguished triangle.

TR2 For all $f : X \to Y$ there exists an object C_f (also sometimes denoted Y/X) called the *cofiber of* f and a distinguished triangle
$$X \xrightarrow{f} Y \to C_f \to \Sigma X.$$

TR3 Given a solid diagram with both rows commutative
$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] & Y \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d, dashed] & \Sigma X \arrow[d, "\Sigma f"] \\ X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & \Sigma X' \end{tikzcd}$$
such that the leftmost square commmutes and both rows are distinguished, there exists a dashed arrow $Z \to Z'$ which makes the remaining two squares commute.

TR4 A triangle
$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{\Sigma} X$$
is distinguished if and only if
$$Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X \xrightarrow{-\Sigma f} \Sigma Y$$
is distinguished.

TR5 (Octahedral axiom) Given three distinguished triangles
$$X \xrightarrow{f} Y \xrightarrow{h} Y/X \to \Sigma X$$
$$Y \xrightarrow{g} Z \xrightarrow{k} Z/Y \to \Sigma Y$$
$$X \xrightarrow{g \circ f} Z \xrightarrow{l} Z/X \to \Sigma X$$
there exists a distinguished triangle
$$Y/X \xrightarrow{u} Z/X \xrightarrow{v} Z/Y \xrightarrow{w} \Sigma(Y/X)$$

such that the following diagram commutes

$$\begin{tikzcd}
X \arrow[rr, "g\circ f"] \arrow[dr, "f"'] & & Z \arrow[rr, "k"] \arrow[dr, "l"'] & & Z/Y \arrow[rr, "w"] \arrow[dr] & & \Sigma(Y/X) \\
& Y \arrow[ur, "g"'] \arrow[dr, "h"'] & & Z/X \arrow[ur, "v"'] \arrow[dr] & & \Sigma Y \arrow[ur, "\Sigma h"'] & \\
& & Y/X \arrow[ur, "u"'] \arrow[rr] & & \Sigma X \arrow[ur, "\Sigma f"'] & &
\end{tikzcd}$$

It turns out that the above definition is actually redundant; TR3 and TR4 follow from the remaining axioms (see Lemmas 2.2 and 2.4 in [1]).

We now recall several important propositions for triangulated categories:

Proposition 0.2. *Given a map $f : X \to Y$ in a triangulated category $(\mathcal{C}, \Sigma, \Omega, \mathcal{D})$, the cofiber sequence of f is unique up to isomorphism, in the sense that given any two distinguished triangles*

$$X \xrightarrow{f} Y \to Z \to \Sigma X \qquad \textit{and} \qquad X \xrightarrow{f} Y \to Z' \to \Sigma X,$$

there exists an isomorphism $Z \to Z'$ which makes the following diagram commute:

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, equal] & Y \arrow[r] \arrow[d, equal] & Z \arrow[r] \arrow[d, dashed, "k"] & \Sigma X \arrow[d, equal] \\
X \arrow[r, "f"] & Y \arrow[r] & Z' \arrow[r] & \Sigma X
\end{tikzcd}$$

Proposition 0.3. *Given an arrow $f : X \to Y$ in a triangulated category $(\mathcal{C}, \Sigma, \Omega, \mathcal{D})$, there exists an object F_f called the* fiber *of f, and a distinguished triangle*

$$F_f \to X \xrightarrow{f} Y \to \Sigma F_f (\cong C_f).$$

Proposition 0.4. *Let $(\mathcal{C}, \Sigma, \Omega, \mathcal{D})$ be a triangulated category. Given a distinguished triangle*

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$$

and any object A in $\mathcal{C}$, there is a long exact sequence of abelian groups

$$\cdots \to [\Sigma^{n+1} A, Z] \xrightarrow{\partial} [\Sigma^n X, X] \xrightarrow{f_*} [\Sigma^n A, Y] \xrightarrow{g_*} [\Sigma^n A, Z] \xrightarrow{\partial} [\Sigma^{n-1} A, X] \to \cdots$$

extending infinitely in either direction, where for $n < 0$ we define $\Sigma^{-n} := \Omega^n$, and ∂ is the map

$$[\Sigma^{n+1} A, Z] \xrightarrow{h_*} [\Sigma^{n+1} A, \Sigma X] \cong [\Sigma^{-1}\Sigma^{n+1} A, X] \cong [\Sigma^n A, X].$$

Also important for our work is the concept of a *tensor triangulated category*, that is, a triangulated symmetric monoidal category in which the triangulated structures are compatible, in the following sense:

Definition 0.5. A *tensor triangulated category* is a triangulated symmetric monoidal category $(\mathcal{C}, \otimes, S, \Sigma, \Omega, \mathcal{D})$ such that:

TT1 For all objects X and Y in $\mathcal{C}$, there are natural isomorphisms

$$e_{X,Y} : X \otimes (\Sigma Y) \xrightarrow{\cong} \Sigma(X \otimes Y).$$

TT2 For each object X in $\mathcal{C}$, the functor $X \otimes (-) \cong (-) \otimes X$ is an additive functor.

TT3 For each object X in $\mathcal{C}$, the functor $X \otimes (-) \cong (-) \otimes X$ preserves distinguished triangles, in that given a distinguished triangle/(co)fiber sequence

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A,$$

then also

$$X \otimes A \xrightarrow{X\otimes f} X \otimes B \xrightarrow{X\otimes g} X \otimes C \xrightarrow{X\otimes h} \Sigma(X \otimes A)$$

and

$$A \otimes X \xrightarrow{f\otimes X} B \otimes X \xrightarrow{g\otimes X} C \otimes X \xrightarrow{h\otimes X} \Sigma(A \otimes X)$$

are distinguished triangles, where here we are being abusive and writing $X \otimes h$ and $h \otimes X$ to denote the compositions

$$X \otimes C \xrightarrow{X\otimes h} X \otimes (\Sigma A) \xrightarrow{e_{X,A}} \Sigma(X \otimes A)$$

and

$$C \otimes X \xrightarrow{h\otimes X} (\Sigma A) \otimes X \xrightarrow{\tau} X \otimes (\Sigma A) \xrightarrow{e_{X,A}} \Sigma(X \otimes A) \xrightarrow{\Sigma\tau} \Sigma(A \otimes X),$$

respectively.

Usually, most tensor triangulated categories that arise in nature will satisfy additional coherence axioms (see axioms TC1–TC5 in [1]), but the above definition will suffice for our purposes. To avoid the awkwardness of saying "a tensor triangulated category which is also a closed symmetric monoidal category," we introduce the following (nonstandard) terminology:

Definition 0.6. We say a tensor triangulated category $(\mathcal{C}, \otimes, S, \Sigma, \Omega)$ is *closed* if $\mathcal{C}$ is a closed symmetric monoidal category, in the sense that for each object $X \in \mathcal{C}$, the functor $- \otimes X$ has a right adjoint $F(X, -)$.

Note that given a tensor triangulated category, we have the following characterization of the shift functor:

Proposition 0.7. *Given a tensor triangulated category* $(\mathcal{C}, \otimes, S, \Sigma, \Omega)$*, there is a canonical natural isomorphism* $\Sigma S \otimes - \cong \Sigma$.

Proof. Given an object X in $\mathcal{C}$, we have natural isomorphisms

$$\Sigma S \otimes X \xrightarrow{\tau} X \otimes \Sigma S \xrightarrow{e_{X,S}} \Sigma(X \otimes S) \xrightarrow{\Sigma\rho_X} \Sigma X,$$

where ρ is the right unitor specified by the monoidal structure on $\mathcal{C}$. $\square$

Because of the above proposition, when working with tensor triangulated categories we will often assume that $\Sigma = S^1 \otimes -$ for some object S^1.