

In what follows, we fix an abelian group  $A$ . We assume the reader is familiar with the basic theory of modules over not-necessarily-commutative rings.

**Definition 0.1.** An  $A$ -graded abelian group is an abelian group  $B$  along with a subgroup  $B_a \leq B$  for each  $a \in A$  such that the canonical map

$$\bigoplus_{a \in A} B_a \rightarrow B$$

sending  $(x_a)_{a \in A}$  to  $\sum_{a \in A} x_a$  is an isomorphism. Given two  $A$ -graded abelian groups  $B$  and  $C$ , a homomorphism  $f : B \rightarrow C$  is a *homomorphism of  $A$ -graded abelian groups*, or just an  *$A$ -graded homomorphism*, if it preserves the grading, i.e., if it restricts to a map  $B_a \rightarrow C_a$  for all  $a \in A$ .

It is easy to see that an  $A$ -graded abelian group  $B$  is generated by its *homogeneous* elements, that is, nonzero elements  $x \in B$  such that there exists some  $a \in A$  with  $x \in B_a$ .

**Remark 0.2.** Clearly the condition that the canonical map  $\bigoplus_{a \in A} B_a \rightarrow B$  is an isomorphism requires that  $B_a \cap B_b = 0$  if  $a \neq b$ . In particular, given a homogeneous element  $x \in B$ , there exists precisely one  $a \in A$  such that  $x \in B_a$ . We call this  $a$  the *degree* of  $x$ , and we write  $|x| = a$ .

**Definition 0.3.** An  $A$ -graded ring is a ring  $R$  such that its underlying abelian group  $R$  is  $A$ -graded and the multiplication map  $R \times R \rightarrow R$  restricts to  $R_a \times R_b \rightarrow R_{a+b}$  for all  $a, b \in A$ . A morphism of  $A$ -graded rings is a ring homomorphism whose underlying homomorphism of abelian groups is  $A$ -graded.

Explicitly, given an  $A$ -graded ring  $R$  and homogeneous elements  $x, y \in R$ , we must have  $|xy| = |x| + |y|$ . For example, given some field  $k$ , the ring  $R = k[x, y]$  is  $\mathbb{Z}^2$ -graded, where given  $(n, m) \in \mathbb{Z}^2$ ,  $R_{n, m}$  is the subgroup of those monomials of the form  $ax^n y^m$  for some  $a \in k$ .

**Definition 0.4.** Let  $R$  be an  $A$ -graded ring. A *left  $A$ -graded  $R$ -module*  $M$  is a left  $R$ -module  $M$  such that  $M$  is an  $A$ -graded abelian group and the action map  $R \times M \rightarrow M$  restricts to a map  $R_a \times M_b \rightarrow M_{a+b}$  for all  $a, b \in A$ . Right  $A$ -graded  $R$ -modules are defined similarly. Finally, an  $A$ -graded  $R$ -bimodule is an  $A$ -graded abelian group  $M$  which has the structure of both an  $A$ -graded left and right  $R$ -module such that given  $r, s \in R$  and  $m \in M$ ,  $r \cdot (m \cdot s) = (r \cdot m) \cdot s$ .

Morphisms between  $A$ -graded  $R$ -modules are precisely  $A$ -graded  $R$ -module homomorphisms. We write  $R\text{-GrMod}$  for the category of left  $A$ -graded  $R$ -modules and  $\text{GrMod-}R$  for the category of right  $A$ -graded  $R$ -modules.

**Remark 0.5.** It is straightforward to see that an  $A$ -graded abelian group is equivalently an  $A$ -graded  $\mathbb{Z}$ -module, where here we are considering  $\mathbb{Z}$  as an  $A$ -graded ring concentrated in degree 0. Thus any result below about  $A$ -graded modules applies equally to  $A$ -graded abelian groups.

**Remark 0.6.** We often will denote an  $A$ -graded  $R$ -module  $M$  by  $M_*$ . Given some  $a \in A$ , we can define the shifted  $A$ -graded abelian group  $M_{*+a}$  whose  $b^{\text{th}}$  component is  $M_{b+a}$ .

**Definition 0.7.** More generally, given two  $A$ -graded  $R$ -modules  $M$  and  $N$  and some  $d \in A$ , an  $R$ -module homomorphism  $f : M \rightarrow N$  is an  *$A$ -graded homomorphism of degree  $d$*  if it restricts to a map  $M_a \rightarrow N_{a+d}$  for all  $a \in A$ . Thus, an  $A$ -graded homomorphism of degree  $d$  from  $M$  to  $N$  is equivalently an  $A$ -graded homomorphism  $M_* \rightarrow N_{*+d}$  or an  $A$ -graded homomorphism  $M_{*-d} \rightarrow N$ . Given some  $a \in A$  and left (resp. right)  $R$ -modules  $M$  and  $N$ , we will write

$$\text{Hom}_R^d(M, N) = \text{Hom}_R(M_*, N_{*+d}) = \text{Hom}_R(M_{*-d}, N_*)$$

to denote the set of  $A$ -graded homomorphisms of degree  $d$  from  $M$  to  $N$ , and simply

$$\text{Hom}_R(M, N)$$

to denote the set of degree-0  $A$ -graded homomorphisms from  $M$  to  $N$ . Clearly  $A$ -graded homomorphisms may be added and subtracted, so these are further abelian groups. Thus we have an  $A$ -graded abelian group

$$\text{Hom}_R^*(M, N).$$

Unless stated otherwise, an “ $A$ -graded homomorphism” will always refer to an  $A$ -graded homomorphism of degree 0.

Oftentimes when constructing  $A$ -graded rings, we do so only by defining the product of homogeneous elements, like so:

**Lemma 0.8.** *Suppose we have an  $A$ -graded abelian group  $R$ , a distinguished element  $1 \in R_0$ , and  $\mathbb{Z}$ -bilinear maps  $m_{a,b} : R_a \times R_b \rightarrow R_{a+b}$  for all  $a, b \in A$ . Further suppose that for all  $x \in R_a$ ,  $y \in R_b$ , and  $z \in R_c$ , we have*

$$m_{a+b,c}(m_{a,b}(x, y), z) = m_{a,b+c}(x, m_{b,c}(y, z)) \quad \text{and} \quad m_{a,0}(x, 1) = m_{0,a}(1, x) = x.$$

*Then there exists a unique multiplication map  $m : R \times R \rightarrow R$  which endows  $R$  with the structure of an  $A$ -graded ring and restricts to  $m_{a,b}$  for all  $a, b \in A$ .*

*Proof.* Given  $r, s \in R$ , since  $R \cong \bigoplus_{a \in A} R_a$ , we may uniquely decompose  $r$  and  $s$  into homogeneous elements as  $r = \sum_{a \in A} r_a$  and  $s = \sum_{a \in A} s_a$  with each  $r_a, s_a \in R_a$  such that only finitely many of the  $r_a$ 's and  $s_a$ 's are nonzero. Then in order to define a distributive product  $R \times R \rightarrow R$  which restricts to  $m_{a,b} : R_a \times R_b \rightarrow R_{a+b}$ , note we *must* define

$$r \cdot s = \left( \sum_{a \in A} r_a \right) \cdot \left( \sum_{b \in A} s_b \right) = \sum_{a,b \in A} r_a \cdot s_b = \sum_{a,b \in A} m_{a,b}(r_a, s_b).$$

Thus, we have shown uniqueness. It remains to show this product actually gives  $R$  the structure of a ring. First we claim that the sum on the right is actually finite. Note there exists only finitely many nonzero  $r_a$ 's and  $s_b$ 's, and if  $s_b = 0$  then

$$m_{a,b}(r_a, 0) = m_{a,b}(r_a, 0 + 0) \stackrel{(*)}{=} m_{a,b}(r_a, 0) + m_{a,b}(r_a, 0) \implies m_{a,b}(r_a, 0) = 0,$$

where  $(*)$  follows from bilinearity of  $m_{a,b}$ . A similar argument yields that  $m_{a,b}(0, s_b) = 0$  for all  $a, b \in A$ . Hence indeed  $m_{a,b}(r_a, s_b)$  is zero for all but finitely many pairs  $(a, b) \in A^2$ , as desired. Observe that in particular

$$(r \cdot s)_a = \sum_{b+c=a} m_{b,c}(r_b, s_c) = \sum_{b \in A} m_{b,a-b}(r_b, s_{a-b}) = \sum_{c \in A} m_{a-c,c}(r_{a-c}, s_c).$$

Now we claim this multiplication is associative. Given  $t = \sum_{a \in A} t_a \in R$ , we have

$$\begin{aligned}
(r \cdot s) \cdot t &= \sum_{a,b \in A} m_{a,b}((r \cdot s)_a, t_b) \\
&= \sum_{a,b \in A} m_{a,b} \left( \sum_{c \in A} m_{a-c,c}(r_{a-c}, s_c), t_b \right) \\
&\stackrel{(1)}{=} \sum_{a,b,c \in A} m_{a,b}(m_{a-c,c}(r_{a-c}, s_c), t_b) \\
&\stackrel{(2)}{=} \sum_{a,b,c \in A} m_{c,a+b-c}(r_c, m_{a-c,b}(s_{a-c}, t_b)) \\
&\stackrel{(3)}{=} \sum_{a,b,c \in A} m_{a,c}(r_a, m_{b,c-b}(s_b, t_{c-b})) \\
&\stackrel{(1)}{=} \sum_{a,c \in A} m_{a,c} \left( r_a, \sum_{b \in A} m_{b,c-b}(s_b, t_{c-b}) \right) \\
&= \sum_{a,c \in A} m_{a,c}(r_a, (s \cdot t)_c) = r \cdot (s \cdot t),
\end{aligned}$$

where each occurrence of (1) follows by bilinearity of the  $m_{a,b}$ 's, each occurrence of (2) is associativity of the  $m_{a,b}$ 's, and (3) is obtained by re-indexing by re-defining  $a := c$ ,  $b := a - c$ , and  $c := a + b - c$ . Next, we wish to show that the distinguished element  $1 \in R_0$  is a unit with respect to this multiplication. Indeed, we have

$$1 \cdot r \stackrel{(1)}{=} \sum_{a \in A} m_{0,a}(1, r_a) \stackrel{(2)}{=} \sum_{a \in A} r_a = r \quad \text{and} \quad r \cdot 1 \stackrel{(1)}{=} \sum_{a \in A} m_{a,0}(r_a, 1) \stackrel{(2)}{=} \sum_{a \in A} r_a = r,$$

where (1) follows by the fact that  $m_{a,b}(0, -) = m_{a,b}(-, 0) = 0$ , which we have shown above, and (2) follows by unitality of the  $m_{0,a}$ 's and  $m_{a,0}$ 's, respectively. Finally, we wish to show that this product is distributive. Indeed, we have

$$\begin{aligned}
r \cdot (s + t) &= \sum_{a,b \in A} m_{a,b}(r_a, (s + t)_b) \\
&= \sum_{a,b \in A} m_{a,b}(r_a, s_b + t_b) \\
&\stackrel{(*)}{=} \sum_{a,b \in A} m_{a,b}(r_a, s_b) + \sum_{a,b \in A} m_{a,b}(r_a, t_b) = (r \cdot s) + (r \cdot t),
\end{aligned}$$

where  $(*)$  follows by bilinearity of  $m_{a,b}$ . An entirely analogous argument yields that  $(r + s) \cdot t = (r \cdot t) + (s \cdot t)$ .  $\square$

Similarly, when defining  $A$ -graded modules, we will only define the action maps for homogeneous elements:

**Lemma 0.9.** *Let  $R$  be an  $A$ -graded ring,  $M$  an  $A$ -graded abelian group, and suppose there exists  $\mathbb{Z}$ -bilinear maps  $\kappa_{a,b} : R_a \times M_b \rightarrow M_{a+b}$  for all  $a, b \in A$ . Further suppose that for all  $r \in R_a$ ,  $r' \in R_b$ , and  $m \in M_c$  that*

$$\kappa_{a+b,c}(r \cdot r', m) = \kappa_{a,b+c}(r, \kappa_{b,c}(r', m)) \quad \text{and} \quad \kappa_{0,c}(1, m) = m.$$

*Then there is a unique map  $\kappa : R \times M \rightarrow M$  which endows  $M$  with the structure of a left  $A$ -graded  $R$ -module and restricts to  $\kappa_{a,b}$  for all  $a, b \in A$ .*

On the other hand, suppose there exists  $\mathbb{Z}$ -bilinear maps  $\kappa_{a,b} : M_a \times R_b \rightarrow M_{a+b}$  for all  $a, b \in A$ . Further suppose that for all  $r \in R_a$ ,  $r' \in R_b$ , and  $m \in M_c$  that

$$\kappa_{c,a+b}(m, r \cdot r') = \kappa_{c+a,b}(\kappa_{c,a}(m, r), r') \quad \text{and} \quad \kappa_{c,0}(m, 1) = m.$$

Then there is a unique map  $\kappa : M \times R \rightarrow M$  which endows  $M$  with the structure of a right  $A$ -graded  $R$ -module and restricts to  $\kappa_{a,b}$  for all  $a, b \in A$ .

Finally, if we have maps  $\lambda_{a,b} : R_a \times M_b \rightarrow M_{a+b}$  and  $\rho_{a,b} : M_a \times R_b \rightarrow M_{a+b}$  satisfying all of the above conditions, and if we further have that

$$\lambda_{a,b+c}(r, \rho_{b,c}(x, s)) = \rho_{a+b,c}(\lambda_{a,b}(r, x), s)$$

for all  $r \in R_a$ ,  $x \in M_b$ , and  $s \in R_c$ , then the left and right  $A$ -graded  $R$ -module structures induced on  $M$  by the  $\lambda$ 's and  $\rho$ 's give  $M$  the structure of an  $A$ -graded  $R$ -bimodule.

*Proof.* We show the left module case, as the right module case is entirely analagous. Supposing for each  $a, b \in A$  we have a map  $\kappa_{a,b} : R_a \times M_b \rightarrow M_{a+b}$  satisfying the above conditions, in order to extend these to a map  $R \times M \rightarrow M$ , by additivity we *must* define

$$\kappa : R \times M \rightarrow M$$

to be the map sending  $r = \sum_a r_a$  and  $m = \sum_a m_a$  to  $\sum_{a,b \in A} \kappa_{a,b}(r_a, m_b)$ . Now, we need to check that for all  $r, s \in R$ ,  $x, y \in M$  that

- (1)  $r \cdot (x + y) = r \cdot x + r \cdot y$
- (2)  $(r + s) \cdot x = r \cdot x + s \cdot x$
- (3)  $(rs) \cdot x = r \cdot (s \cdot x)$
- (4)  $1 \cdot x = x$ ,

where above we are written  $- \cdot -$  for  $\kappa(-, -)$ . To see the first, note

$$\begin{aligned} \kappa(r, x + y) &= \sum_{a,b \in A} \kappa_{a,b}(r_a, (x + y)_b) \\ &= \sum_{a,b \in A} \kappa_{a,b}(r_a, x_b + y_b) \\ &= \sum_{a,b \in A} (\kappa_{a,b}(r_a, x_b) + \kappa_{a,b}(r_a, y_b)) \\ &= \sum_{a,b \in A} \kappa_{a,b}(r_a, x_b) + \sum_{a,b \in A} \kappa_{a,b}(r_a, y_b) \\ &= \kappa(r, x) + \kappa(r, y). \end{aligned}$$

To see the second, note

$$\begin{aligned} \kappa(r + s, x) &= \sum_{a,b \in A} \kappa_{a,b}((r + s)_a, x_b) \\ &= \sum_{a,b \in A} \kappa_{a,b}(r_a + s_a, x_b) \\ &= \sum_{a,b \in A} (\kappa_{a,b}(r_a, x_b) + \kappa_{a,b}(s_a, x_b)) \\ &= \sum_{a,b \in A} \kappa_{a,b}(r_a, x_b) + \sum_{a,b \in A} \kappa_{a,b}(s_a, x_b) \\ &= \kappa(r, x) + \kappa(s, x). \end{aligned}$$

To see the third, note

$$\begin{aligned}
 \kappa(rs, x) &= \sum_{a,b \in A} \kappa_{a,b}((rs)_a, x_b) \\
 &= \sum_{a,b \in A} \kappa_{a,b} \left( \sum_{c \in A} r_c s_{a-c}, x_b \right) \\
 &= \sum_{a,b,c \in A} \kappa_{a,b}(r_c s_{a-c}, x_b) \\
 &= \sum_{a,b,c \in A} \kappa_{a,b}(r_c, \kappa_{a-c,b}(s_{a-c}, x_b)) \\
 &=
 \end{aligned}$$

□

FINISH

When working with  $A$ -graded rings and modules, we will often freely use the above propositions without comment.

Recall that given a ring  $R$ , a left (resp. right) module  $P$  is *projective* if, for all diagrams of  $R$ -module homomorphisms of the form

$$\begin{array}{ccc}
 & M & \\
 & \downarrow g & \\
 P & \xrightarrow{f} & N
 \end{array}$$

with  $g$  an epimorphism, there exists a lift  $h : P \rightarrow M$  satisfying  $g \circ h = f$

$$\begin{array}{ccc}
 & M & \\
 & \downarrow g & \\
 P & \xrightarrow{f} & N \\
 \nearrow h & & 
 \end{array}$$

(Note  $h$  is not required to be unique.)

**Definition 0.10.** Let  $R$  be an  $A$ -graded ring, and let  $P$  be a left (resp. right)  $A$ -graded  $R$ -module. Then  $P$  is a *graded projective* module if, for all diagrams of  $A$ -graded  $R$ -module homomorphisms of the form

$$\begin{array}{ccc}
 & M & \\
 & \downarrow g & \\
 P & \xrightarrow{f} & N
 \end{array}$$

with  $g$  an epimorphism, there exists an  $A$ -graded homomorphism  $h : P \rightarrow M$  satisfying  $g \circ h = f$ .

$$\begin{array}{ccc}
 & M & \\
 & \downarrow g & \\
 P & \xrightarrow{f} & N \\
 \nearrow h & & 
 \end{array}$$

(Note  $h$  is not required to be unique.)

**Lemma 0.11.** *Let  $R$  be an  $A$ -graded ring, and let  $M$  be an  $A$ -graded left (resp. right)  $R$ -module. Then for all  $d \in A$ , the evaluation map*

$$\begin{aligned} \text{ev}_1 : \text{Hom}_R^d(R, M) &\rightarrow M_d \\ \varphi &\mapsto \varphi(1) \end{aligned}$$

*is an isomorphism of abelian groups.*

*Proof.* We consider the case that  $M$  is a left  $A$ -graded  $R$ -module, as showing it when  $M$  is a right module is entirely analogous. First of all, this map is clearly a homomorphism, as given degree  $d$   $A$ -graded homomorphisms  $\varphi, \psi : R \rightarrow M$ , we have

$$\text{ev}_1(\varphi + \psi) = (\varphi + \psi)(1) = \varphi(1) + \psi(1) = \text{ev}_1(\varphi) + \text{ev}_1(\psi).$$

Now, to see it is surjective, let  $m \in M_d$ , and define  $\varphi_m : R \rightarrow M$  to send  $r \mapsto r \cdot m$ . First of all,  $\varphi_m$  is a module homomorphism, as given  $r, s \in R$ ,

$$\varphi_m(r + s) = (r + s) \cdot m = r \cdot m + s \cdot m = \varphi_m(r) + \varphi_m(s) \quad \text{and} \quad \varphi_m(r \cdot s) = r \cdot s \cdot m = r \cdot \varphi_m(s).$$

Furthermore, it is clearly  $A$ -graded of degree  $d$ , as given a homogeneous element  $r \in R_a$  for some  $a \in A$ , we have  $\varphi_m(r) = r \cdot m \in R_{a+d}$ , since  $m$  is homogeneous of degree  $d$ . Finally, clearly

$$\text{ev}_1(\varphi_m) = \varphi_m(1) = 1 \cdot m = m,$$

so indeed  $\text{ev}_1$  is surjective. On the other hand, to see it is injective, suppose we are given  $\varphi, \psi \in \text{Hom}_R^d(R, M)$  such that  $\varphi(1) = \psi(1)$ . Then given  $r \in R$ , we must have

$$\varphi(r) = \varphi(r \cdot 1) = r \cdot \varphi(1) = r \cdot \psi(1) = \psi(r \cdot 1) = \psi(r),$$

so  $\varphi$  and  $\psi$  are exactly the same map. Thus,  $\text{ev}_1$  is injective, as desired.  $\square$

**0.1.  $A$ -graded submodules and quotient modules.** In what follows, fix an  $A$ -graded ring  $R$ . We will simply say “ $A$ -graded  $R$ -module” when we are freely considering either left or right  $A$ -graded  $R$ -modules.

**Definition 0.12.** Let  $M$  be an  $A$ -graded  $R$ -module. Then an  $A$ -graded  $R$ -submodule is an  $A$ -graded  $R$ -module  $N$  which is a subset of  $M$  and for which the inclusion  $N \hookrightarrow M$  is an  $A$ -graded homomorphism of  $R$ -modules. Equivalently, it is a submodule  $N$  for which the canonical map

$$\bigoplus_{a \in A} N \cap M_a \rightarrow N$$

is an isomorphism.

**Lemma 0.13.** *Let  $M$  be an  $A$ -graded  $R$ -module. Then an  $R$ -submodule  $N \leq M$  is an  $A$ -graded submodule if and only if it is generated as an  $R$ -module by homogeneous elements of  $M$ .*

*Proof.* If  $N \leq M$  is a  $A$ -graded submodule, it is generated by the set of all its homogeneous elements, which are also homogeneous elements in  $M$ , by definition.

Conversely, suppose  $N \leq M$  is a submodule which is generated by homogeneous elements of  $M$ . Then define  $N_a := N \cap M_a$ , and consider the canonical map

$$\Phi : \bigoplus_{a \in A} N_a \rightarrow N.$$

First of all, it is surjective, as each generator of  $N$  belongs to some  $N_a$ , by definition. To see it is injective, consider the following commutative diagram:

$$\begin{array}{ccc} \bigoplus_{a \in A} N_a & \hookrightarrow & \bigoplus_{a \in A} M_a \\ \Phi \downarrow & & \downarrow \cong \\ N & \hookrightarrow & M \end{array}$$

Since  $\Phi$  composes with an injection to get an injection, clearly  $\Phi$  must be injective itself. We have the desired result.  $\square$

**Proposition 0.14.** *Given two left (resp. right)  $A$ -graded  $R$ -modules  $M$  and  $N$  and an  $A$ -graded  $R$ -module homomorphism  $\varphi : M \rightarrow N$  (of possibly nonzero degree), the kernel and images of  $\varphi$  are  $A$ -graded submodules of  $M$  and  $N$ , respectively.*

*Proof.* First recall that a degree  $d$   $A$ -graded homomorphism  $M \rightarrow N$  is simply an  $A$ -graded homomorphism  $M_* \rightarrow N_{*+d}$ , so it suffices to consider the case  $\varphi$  is of degree 0. Next, note that since the forgetful functor from  $R$ -modules to abelian groups preserves kernels and images, it suffices to consider the case that  $\varphi$  is a homomorphism of  $A$ -graded abelian groups. Finally, by [Lemma 0.13](#), it suffices to show that  $\ker \varphi$  and  $\operatorname{im} \varphi$  are generated by homogeneous elements of  $M$  and  $N$ , respectively.

Note that by the universal property of the coproduct in **Ab**, the data of an  $A$ -graded homomorphism of abelian groups  $\varphi : M \rightarrow N$  is precisely the data of an  $A$ -indexed collection of abelian group homomorphisms  $\varphi_a : M_a \rightarrow N_a$ , in which case the following diagram commutes:

$$\begin{array}{ccc} \bigoplus_a M_a & \xrightarrow{\bigoplus_a \varphi_a} & \bigoplus_a N_a \\ \cong \downarrow & & \downarrow \cong \\ M & \xrightarrow{\varphi} & N \end{array}$$

Finally, the desired result follows by the purely formal fact that taking images and kernels commutes with arbitrary direct sums.  $\square$

**Proposition 0.15.** *Given two left (resp. right)  $A$ -graded  $R$ -modules  $M$  and  $N$ , an  $A$ -graded submodule  $K \leq N$ , and an  $A$ -graded  $R$ -module homomorphism  $\varphi : M \rightarrow N$  (of possibly nonzero degree), the submodule  $\varphi^{-1}(K)$  of  $M$  is  $A$ -graded.*

*Proof.* Recall that a degree  $d$   $A$ -graded homomorphism  $M \rightarrow N$  is simply an  $A$ -graded homomorphism  $M_* \rightarrow N_{*+d}$ , so it suffices to consider the case  $\varphi$  is of degree 0. Now, let  $x \in L := \varphi^{-1}(K)$ . As an element of  $M$ , we may uniquely write  $x = \sum_{a \in A} x_a$  where each  $x_a \in M_a$ . Similarly, if we set  $y := \varphi(x)$ , then we may uniquely write  $y = \sum_{a \in A} y_a$  where each  $y_a \in N_a$ . Then since  $K$  is an  $A$ -graded submodule of  $N$  and  $y \in K$ , by definition, we have that  $y_a \in K$  for each  $a$ . Finally, note that

$$\sum_{a \in A} y_a = y = \varphi(x) = \sum_{a \in A} \varphi(x_a),$$

so that  $\varphi(x_a) = y_a \in K$  for all  $a \in A$ , so that  $x_a \in L$  for all  $a \in A$ . Thus we have shown that each element in  $L$  can be written as a sum of homogeneous elements in  $M$ , as desired.  $\square$

**Proposition 0.16.** *Given an  $A$ -graded  $R$ -module  $M$  and an  $A$ -graded subgroup  $N \leq M$ , the quotient  $M/N$  is canonically  $A$ -graded by defining  $(M/N)_a$  to be the subgroup generated by cosets represented by homogeneous elements of degree  $a$  in  $M$ . Furthermore, the canonical maps  $M_a/N_a \rightarrow (M/N)_a$  taking a coset  $m + N_a$  to  $m + N$  are isomorphisms.*

*Proof.* Consider the canonical map

$$\Phi : \bigoplus_a (M/N)_a \rightarrow M/N.$$

First of all, surjectivity of  $\Phi$  follows by commutativity of the following diagram:

$$\begin{array}{ccc} \bigoplus_a M_a & \xrightarrow{\cong} & M \\ \downarrow & & \downarrow \\ \bigoplus_a (M/N)_a & \xrightarrow{\Phi} & M/N \end{array}$$

where the vertical left map sends a generator  $m \in M_a$  to the coset  $m + N$  in  $(M/N)_a \subseteq M/N$ . To see  $\Phi$  is injective, suppose we are given some element  $(m_a + N)_{a \in A}$  in  $\bigoplus_a (M/N)_a$  such that  $\sum_{a \in A} (m_a + N) = 0$  in  $M/N$ . Thus  $\sum_{a \in A} m_a \in N$ , and since  $N$  is  $A$ -graded this implies that each  $m_a$  belongs to  $N \cap M_a = N_a$ , so that in particular  $m_a + N$  is zero in  $(M/N)_a \subseteq M/N$ , so that  $(m_a + N)_{a \in A} = 0$  in  $\bigoplus_a (M/N)_a$ , as desired.

It remains to show that the canonical map

$$\varphi_a : M_a/N_a \rightarrow (M/N)_a$$

is an isomorphism. It is clearly surjective, as  $(M/N)_a$  is generated by elements  $m + N$  for  $m \in M_a$ , and these elements make up precisely the image of  $\varphi_a$ . Thus  $\varphi_a$  hits every generator of  $(M/N)_a$ , so  $\varphi_a$  is surjective. On the other hand, suppose we are given some  $m \in M_a$  such that  $\varphi(m + N_a) = m + N = 0$ . Thus  $m \in N$ , and  $m \in M_a$ , so that  $m \in M_a \cap N = N_a$ , meaning  $m + N_a = 0$  in  $M_a/N_a$ , as desired.  $\square$

**0.2. Tensor product of  $A$ -graded modules.** Recall that given a ring  $R$ , a left  $R$ -module  $M$ , a right  $R$ -module  $N$ , and an abelian group  $A$ , an  $R$ -balanced map  $\varphi : M \times N \rightarrow B$  is one which satisfies

$$\begin{aligned} \varphi(m, n + n') &= \varphi(m, n) + \varphi(m, n') \\ \varphi(m + m', n) &= \varphi(m, n) + \varphi(m', n) \\ \varphi(m \cdot r, n) &= \varphi(m, r \cdot n). \end{aligned}$$

for all  $m, m' \in M$ ,  $n, n' \in N$ , and  $r \in R$ . Then the tensor product  $M \otimes_R N$  is the universal abelian group equipped with an  $R$ -balanced map  $\otimes : M \times N \rightarrow M \otimes_R N$  such that for every abelian group  $B$  and every  $R$ -balanced map  $\varphi : M \times N \rightarrow B$ , there is a *unique* group homomorphism  $\tilde{\varphi} : M \otimes_R N \rightarrow B$  such that  $\tilde{\varphi} \circ \otimes = \varphi$ . We call elements in the image of  $\otimes : M \times N \rightarrow M \otimes_R N$  *pure tensors*. It is a standard fact that  $M \otimes_R N$  is generated as an abelian group by its pure tensors.

**Definition 0.17.** Suppose we have a right  $A$ -graded  $R$ -module  $M$ , a left  $A$ -graded  $R$ -module  $N$ , and an  $A$ -graded abelian group  $B$ . Then an  $A$ -graded  $R$ -balanced map  $\varphi : M \times N \rightarrow B$  is an  $R$ -balanced map which restricts to  $M_a \times N_b \rightarrow B_{a+b}$  for all  $a, b \in A$ .

**Proposition 0.18.** Suppose we have a right  $A$ -graded  $R$ -module  $M$  and a left  $A$ -graded  $R$ -module  $N$ . Then the tensor product

$$M \otimes_R N$$

is naturally an  $A$ -graded abelian group by defining  $(M \otimes_R N)_a$  to be the subgroup generated by homogeneous pure tensors  $m \otimes n$  with  $m \in M_b$  and  $n \in N_c$  such that  $b + c = a$ . Furthermore, if either  $M$  (resp.  $N$ ) is an  $A$ -graded bimodule, then this decomposition makes  $M \otimes_R N$  into a left (resp. right)  $A$ -graded  $R$ -module. In particular, if both  $M$  and  $N$  are  $R$ -bimodules, then  $M \otimes_R N$  is an  $R$ -bimodule.



*Proof.* By definition, since  $M$  and  $N$  are  $A$ -graded abelian groups, they are generated (as abelian groups) by their homogeneous elements. Thus it follows that  $M \otimes_R N$  is generated by *homogeneous pure tensors*, that is, elements of the form  $m \otimes n$  with  $m \in M$  and  $n \in N$  homogeneous. Now, given a homogeneous pure tensor  $m \otimes n$ , we define its *degree* by the formula  $|m \otimes n| := |m| + |n|$ . It follows this formula is well-defined by checking that given homogeneous elements  $m \in M$ ,  $n \in N$ , and  $r \in R$  that

$$|(m \cdot r) \otimes n| = |m \cdot r| + |n| = |m| + |r| + |n| = |m| + |r \cdot n| = |m \otimes (r \cdot n)|.$$

Thus, we may define  $(M \otimes_R N)_a$  to be the subgroup of  $M \otimes_R N$  generated by those pure homogeneous tensors of degree  $a$ . Now, consider the map

$$\Psi : M \times N \rightarrow \bigoplus_{a \in A} (M \otimes_R N)_a$$

which takes a pair  $(m, n) = \sum_{a \in A} (m_a, n_a)$  to the element  $\Psi(m, n)$  whose  $a^{\text{th}}$  component is

$$(\Psi(m, n))_a := \sum_{b+c=a} m_b \otimes n_c.$$

It is straightforward to see that this map is  $R$ -balanced, in the sense that it is additive in each argument and  $\Psi(m \cdot r, n) = \Psi(m, r \cdot n)$  for all  $m \in M$ ,  $n \in N$ , and  $r \in R$ . Thus by the universal property of  $M \otimes_R N$ , we get a homomorphism of abelian groups  $\tilde{\Psi} : M \otimes_R N \rightarrow \bigoplus_{a \in A} (M \otimes_R N)_a$  lifting  $\Psi$  along the canonical map  $M \times N \rightarrow M \otimes_R N$ . Now, also consider the canonical map

$$\Phi : \bigoplus_{a \in A} (M \otimes_R N)_a \rightarrow M \otimes_R N.$$

We would like to show  $\tilde{\Psi}$  and  $\Phi$  are inverses of each other. Since  $\tilde{\Psi}$  and  $\Phi$  are both homomorphisms, it suffices to show this on generators. Let  $m \otimes n$  be a homogeneous pure tensor with  $m = m_a \in M_a$  and  $n = n_b \in N_b$ . Then we have

$$\Phi(\tilde{\Psi}(m \otimes n)) = \Phi\left(\bigoplus_{a \in A} \sum_{b+c=a} m_b \otimes n_c\right) \stackrel{(*)}{=} \Phi(m \otimes n) = m \otimes n,$$

and

$$\tilde{\Psi}(\Phi(m \otimes n)) = \tilde{\Psi}(m \otimes n) = \bigoplus_{a \in A} \sum_{b+c=a} m_b \otimes n_c \stackrel{(*)}{=} m \otimes n,$$

where both occurrences of  $(*)$  follow by the fact that  $m_b \otimes n_c = 0$  unless  $b = c = a$ , in which case  $m_a \otimes n_a = m \otimes n$ . Thus since  $\Phi$  is an isomorphism,  $M \otimes_R N$  is indeed an  $A$ -graded abelian group, as desired.

Now, suppose that  $M$  is an  $A$ -graded  $R$ -bimodule, so there exists left and right  $A$ -graded actions of  $R$  on  $M$  such that given  $r, s \in R$  and  $m \in M$  we have  $r \cdot (m \cdot s) = (r \cdot m) \cdot s$ . Then we would like to show that given a left  $A$ -graded  $R$ -module  $N$  that  $M \otimes_R N$  is canonically a left  $A$ -graded  $R$ -module. Indeed, define the action of  $R$  on  $M \otimes_R N$  on pure tensors by the formula

$$r \cdot (m \otimes n) = (r \cdot m) \otimes n.$$

First of all, clearly this map is  $A$ -graded, as if  $r \in R_a$ ,  $m \in M_b$ , and  $n \in N_c$  then  $(r \cdot m) \otimes n$ , by definition, has degree  $|r \cdot m| + |n| = |r| + |m| + |n|$  (the last equality follows since the left action of  $R$  on  $M$  is  $A$ -graded). In order to show the above map defines a left module structure, it suffices to show that given pure tensors  $m \otimes n, m' \otimes n' \in M \otimes_R N$  and elements  $r, r' \in R$  that

- (1)  $r \cdot (m \otimes n + m' \otimes n') = r \cdot (m \otimes n) + r \cdot (m' \otimes n')$ ,
- (2)  $(r + r') \cdot (m \otimes n) = r \cdot (m \otimes n) + r' \cdot (m \otimes n)$ ,
- (3)  $(rr') \cdot (m \otimes n) = r \cdot (r' \cdot (m \otimes n))$ , and

$$(4) \quad 1 \cdot (m \otimes n) = m \otimes n.$$

Axiom (1) holds by definition. To see (2), note that by the fact that  $R$  acts on  $M$  on the left that

$$(r + r') \cdot (m \otimes n) = ((r + r') \cdot m) \otimes n = (r \cdot m + r' \cdot m) \otimes n = r \cdot m \otimes n + r' \cdot m \otimes n.$$

That (3) and (4) hold follows similarly by the fact that  $(rr') \cdot m = r \cdot (r' \cdot m)$  and  $1 \cdot m = m$ .

Conversely, if  $N$  is an  $A$ -graded  $R$ -bimodule, then showing  $M \otimes_R N$  is canonically a right  $A$ -graded  $R$ -module via the rule

$$(m \otimes n) \cdot r = m \otimes (n \cdot r)$$

is entirely analagous.

Finally, if both  $M$  and  $N$  are  $R$ -bimodules, then by what we have shown,  $M \otimes_R N$  is both a left and right  $R$ -module. To see these coincide to give  $M \otimes_R N$  an  $R$ -bimodule structure, note that given  $m \in M$ ,  $n \in N$ , and  $r, r' \in R$  that

$$(r \cdot (m \otimes n)) \cdot r' = ((r \cdot m) \otimes n) \cdot r' = (r \cdot m) \otimes (n \cdot r') = r \cdot (m \otimes (n \cdot r')) = r \cdot ((m \otimes n) \cdot r'). \quad \square$$

**Lemma 0.19.** *Let  $R$  be an  $A$ -graded ring,  $B$  an  $A$ -graded abelian group,  $M$  a right  $A$ -graded  $R$ -module, and  $N$  a left  $A$ -graded  $R$ -module. Further suppose we are given a map  $\varphi_{a,b} : M_a \times N_b \rightarrow B_{a+b}$  for all  $a, b \in A$  which commutes with addition in each argument, and such that for all  $m \in M_a$ ,  $n \in N_b$ , and  $r \in R_c$  that*

$$\varphi_{a+b,c}(m \cdot r, n) = \varphi_{a,b+c}(m, r \cdot n).$$

*Then there is a unique  $A$ -graded  $R$ -balanced map  $\varphi : M \times N \rightarrow B$  which restricts to  $\varphi_{a,b}$  for all  $a, b \in A$ , and furthermore, the induced homomorphism  $\tilde{\varphi} : M \otimes_R N \rightarrow B$  is an  $A$ -graded homomorphism of abelian groups.*

TODO

Proof.

□