

In this section, we will freely use the coherence theorem for a symmetric monoidal category, which says that every symmetric monoidal category is (monoidally) equivalent to a *permutative category*, that is, a symmetric monoidal category in which the associators and unitors are strict equalities.

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Definition 0.1. Let $(\mathcal{C}, \otimes, S)$ be a symmetric monoidal category with left unitor, right unitor, and associator, and symmetry isomorphism λ, ρ, α , and τ , respectively. Then a *monoid object* (E, μ, e) is an object E in \mathcal{C} along with a multiplication map $\mu : E \otimes E \rightarrow E$ and a unit map $e : S \rightarrow E$ such that the following diagram commutes:

$$\begin{array}{ccc} E \otimes S & \xrightarrow{E \otimes e} & E \otimes E \xleftarrow{e \otimes E} S \otimes E \\ & \searrow \rho & \downarrow \mu \swarrow \lambda \\ & & E \end{array} \quad \begin{array}{ccc} (E \otimes E) \otimes E & \xrightarrow{\mu \otimes E} & E \otimes E \\ \alpha \downarrow & & \downarrow \mu \\ E \otimes (E \otimes E) & \xrightarrow{E \otimes \mu} & E \otimes E \xrightarrow{\mu} E \end{array}$$

The first diagram expresses unitality, while the second expressed associativity. If in addition the following diagram commutes,

$$\begin{array}{ccc} E \otimes E & \xrightarrow{\tau} & E \otimes E \\ & \searrow \mu & \swarrow \mu \\ & & E \end{array}$$

then we say (E, μ, e) is a *commutative monoid object*.

Proposition 0.2. Let (E_1, μ_1, e_1) and (E_2, μ_2, e_2) be monoid objects in a symmetric monoidal category $(\mathcal{C}, \otimes, S)$. Then $E_1 \otimes E_2$ is canonically a ring spectrum via the maps

$$\mu : E_1 \otimes E_2 \otimes E_1 \otimes E_2 \xrightarrow{E_1 \otimes \tau \otimes E_2} E_1 \otimes E_1 \otimes E_2 \otimes E_2 \xrightarrow{\mu_1 \otimes \mu_2} E_1 \otimes E_2$$

and

$$e : S \cong S \otimes S \xrightarrow{e_1 \otimes e_2} E_1 \otimes E_2.$$

Proof.

□

todo

In what follows, fix a stable homotopy category \mathcal{SH} (??) along with the additional data there-within, and adopt the conventions outlined in ??. Further suppose we have fixed a coherent family of isomorphisms

$$\phi_{a,b} : S^{a+b} \xrightarrow{\cong} S^a \otimes S^b,$$

in the sense of ?? (the existence of such a coherent family is guaranteed by ??).

Proposition 0.3. Let (E, μ, e) be a commutative monoid object in \mathcal{SH} , and consider the multiplication map $\pi_*(E) \times \pi_*(E) \rightarrow \pi_*(E)$ which sends classes $x : S^a \rightarrow E$ and $y : S^b \rightarrow E$ to the composition

$$S^{a+b} \xrightarrow{\phi_{a,b}} S^a \otimes S^b \xrightarrow{x \otimes y} E \otimes E \xrightarrow{\mu} E.$$

Then this endows $\pi_*(E)$ with the structure of an A -graded ring with unit $e \in \pi_0(E) = [S, E]$.

Proof. In this proof, we will assume we are working in a permutative category. Suppose we have classes x, y , and z in $\pi_a(E)$, $\pi_b(E)$, and $\pi_c(E)$, respectively. To see associativity, consider the

following diagram:

$$\begin{array}{ccccc}
 S^{a+b+c} & \xrightarrow{\cong} & S^a \otimes S^b \otimes S^c & \xrightarrow{x \otimes y \otimes z} & E \otimes E \otimes E \\
 & & & & \swarrow \mu \otimes E \quad \searrow E \otimes \mu \\
 & & & & E \otimes E & \begin{array}{c} \downarrow \mu \\ E \\ \uparrow \mu \\ E \otimes E \end{array}
 \end{array}$$

(here the first arrow is the unique isomorphism obtained by composing products of $\phi_{a,b}$'s, see ??). It commutes by associativity of μ . It follows by functoriality of $- \otimes -$ that the top composition is $(x \cdot y) \cdot z$ while the bottom is $x \cdot (y \cdot z)$, so they are equal as desired. To see that $e \in \pi_0(E)$ is a left and right unit for this multiplication, consider the following diagram

$$\begin{array}{ccccc}
 & & S^a & & \\
 & \swarrow e \otimes x & \downarrow x & \searrow x \otimes e & \\
 E \otimes E & \xleftarrow{e \otimes E} & E & \xrightarrow{E \otimes e} & E \otimes E \\
 & \searrow \mu & \parallel & \swarrow \mu & \\
 & & E & &
 \end{array}$$

Commutativity of the two top triangles is functoriality of $- \otimes -$. Commutativity of the bottom two triangles is unitality of μ . Thus the diagram commutes, so $e \cdot x = x \cdot e$. Finally, to see this product is bilinear (distributive). Suppose we further have some $x' \in \pi_a(E)$ and $y' \in \pi_b(E)$, and consider the following diagrams:

$$\begin{array}{ccccccc}
 S^{a+b} & \xrightarrow{\phi_{a,b}} & S^a \otimes S^b & \xrightarrow{\Delta \otimes S^b} & (S^a \oplus S^a) \otimes S^b & \xrightarrow{(x \oplus x') \otimes y} & (E \oplus E) \otimes E \\
 \Delta \downarrow & & \downarrow \Delta & \swarrow \cong & & \swarrow \cong & \downarrow \nabla \otimes E \\
 S^{a+b} \oplus S^{a+b} & \xrightarrow{\phi_{a,b} \oplus \phi_{a,b}} & (S^a \otimes S^b) \oplus (S^a \otimes S^b) & \xrightarrow{(x \otimes y) \oplus (x' \otimes y)} & (E \otimes E) \oplus (E \otimes E) & \xrightarrow{\nabla} & E \otimes E \xrightarrow{\mu} E
 \end{array}$$

$$\begin{array}{ccccccc}
 S^{a+b} & \xrightarrow{\phi_{a,b}} & S^a \otimes S^b & \xrightarrow{S^a \otimes \Delta} & S^b \otimes (S^b \oplus S^b) & \xrightarrow{x \otimes (y \oplus y')} & E \otimes (E \oplus E) \\
 \Delta \downarrow & & \downarrow \Delta & \swarrow \cong & & \swarrow \cong & \downarrow E \otimes \nabla \\
 S^{a+b} \oplus S^{a+b} & \xrightarrow{\phi_{a,b} \oplus \phi_{a,b}} & (S^a \otimes S^b) \oplus (S^a \otimes S^b) & \xrightarrow{(x \otimes y) \oplus (x \otimes y)} & (E \otimes E) \oplus (E \otimes E) & \xrightarrow{\nabla} & E \otimes E \xrightarrow{\mu} E
 \end{array}$$

The unlabeled isomorphisms are those given by the fact that $- \otimes -$ is additive in each variable (since $S\mathcal{H}$ is tensor triangulated). Commutativity of the left squares is naturality of $\Delta : X \rightarrow X \oplus X$ in an additive category. Commutativity of the rest of the diagram follows again from the fact that $- \otimes -$ is an additive functor in each variable. Hence, by functoriality of $- \otimes -$, these diagrams tell us that $(x + x') \cdot y = x \cdot y + x' \cdot y$ and $x \cdot (y + y') = x \cdot y + x \cdot y'$, respectively. \square

Proposition 0.4. *For all $a, b \in A$ there exists an element $\theta_{a,b} \in \pi_0(S) = [S, S]$ (determined by choice of coherent family $\{\phi_{a,b}\}$) such that given any commutative monoid object (E, μ, e) in $S\mathcal{H}$, the A -graded ring structure on $\pi_*(E)$ (??) has a commutativity formula given by*

$$x \cdot y = y \cdot x \cdot (e \circ \theta_{a,b})$$

for all $x \in \pi_a(E)$ and $y \in \pi_b(E)$. In particular, $\theta_{a,b} \in \text{Aut}(S)$ is the composition

$$S \xrightarrow{\cong} S^{-a-b} \otimes S^a \otimes S^b \xrightarrow{S^{-a-b} \otimes \tau} S^{-a-b} \otimes S^b \otimes S^a \xrightarrow{\cong} S,$$

where the outermost maps are the unique maps specified by ??.

Proof. Let $\phi_{a,b}$, E , x , and y as in the statement of the proposition. Now consider the following diagram

$$\begin{array}{ccccc}
 S^{a+b} & \xrightarrow{\phi_{a,b}} & S^a \otimes S^b & \xrightarrow{x \otimes y} & E \otimes E \\
 \downarrow \phi_{b,a}^{-1} \circ \tau \circ \phi_{a,b} & & \downarrow \tau & & \downarrow \tau \\
 S^{a+b} & \xrightarrow{\phi_{b,a}} & S^b \otimes S^a & \xrightarrow{y \otimes x} & E \otimes E \\
 & & & & \searrow \mu \\
 & & & & E
 \end{array}$$

The left square commutes by definition. The middle square commutes by naturality of the symmetry isomorphism. Finally, the right square commutes by commutativity of E . Unravelling definitions, we have shown that under the product on $\pi_*(E)$ induced by the $\phi_{a,b}$'s,

$$x \cdot y = (y \cdot x) \circ (\phi_{b,a}^{-1} \circ \tau \circ \phi_{a,b}).$$

Thus, in order to show the desired result it further suffices to show that

$$(y \cdot x) \circ (\phi_{b,a}^{-1} \circ \tau \circ \phi_{a,b}) = y \cdot x \cdot (e \circ \theta_{a,b}).$$

Consider the following diagram:

$$\begin{array}{ccc}
 S^{a+b} & \xrightarrow{\phi_{a,b}} & S^a \otimes S^b \\
 \cong \downarrow & \nearrow \cong & \downarrow \tau \\
 S^b \otimes S^a \otimes S^{-a-b} \otimes S^a \otimes S^b & & S^b \otimes S^a \\
 S^b \otimes S^a \otimes S^{-a-b} \otimes \tau \downarrow & \nearrow \cong & \downarrow \phi_{b,a}^{-1} \\
 S^b \otimes S^a \otimes S^{-a-b} \otimes S^b \otimes S^a & \xrightarrow{\phi_{b,a}} & S^{a+b} \\
 \searrow y \otimes x \otimes e & \nearrow y \otimes x & \\
 E \otimes E \otimes E & \xrightarrow{E \otimes E \otimes e} & E \otimes E \\
 \mu \otimes E \downarrow & \nearrow E \otimes \mu & \parallel \\
 E \otimes E & \xrightarrow{\mu} & E
 \end{array}$$

Here any map simply labelled \cong is an appropriate composition of copies of $\phi_{a,b}$'s, associators, and their inverses, so that each of these maps are necessarily unique by ???. The two triangles in the top large rectangle commutes by coherence for the $\phi_{a,b}$'s. The parallelogram commutes by naturality of τ and coherence of the $\phi_{a,b}$'s. The middle skewed triangle commutes by functoriality of $- \otimes -$. The triangle below that commutes by unitality of μ . Finally, the bottom rectangle commutes by associativity of μ . Hence, by unravelling definitions and applying functoriality of $- \otimes -$, we get that the right composition is $(y \cdot x) \circ (\phi_{b,a}^{-1} \circ \tau \circ \phi_{a,b})$, while the left composition is $y \cdot x \cdot (e \circ \theta_{a,b})$, so they are equal as desired. \square

Proposition 0.5. *Given $a \in A$, we have $\theta_{0,a} = \theta_{a,0} = \text{id}_S$.*

Proof. Recall $\theta_{a,0}$ is the composition

$$S \xrightarrow{\phi_{-a,a}} S^{-a} \otimes S^a \xrightarrow{S^{-a} \otimes \phi_{a,0}} S^{-a} \otimes (S^a \otimes S) \xrightarrow{S^{-a} \otimes \tau} S^{-a} \otimes (S \otimes S^a) \xrightarrow{S^{-a} \otimes \phi_{0,a}^{-1}} S^{-a} \otimes S^a \xrightarrow{\phi_{-a,a}^{-1}} S$$

By the coherence theorem for symmetric monoidal categories and the fact that $\phi_{a,0}$ and $\phi_{0,a}$ coincide with the unitors, we have that the composition

$$S^a \xrightarrow{\phi_{a,0} = \rho_{S^a}^{-1}} S^a \otimes S \xrightarrow{\tau} S \otimes S^a \xrightarrow{\phi_{0,a}^{-1} = \lambda_{S^a}} S^a$$

is precisely the identity map, so by functoriality of $- \otimes -$, we have that $\theta_{a,0}$ is the composition

$$S \xrightarrow{\phi_{-a,a}} S^{-a} \otimes S^a \xrightarrow{\cong} S^{-a} \otimes S^a \xrightarrow{\phi_{-a,a}^{-1}} S,$$

so $\theta_{a,0} = \text{id}_S$, meaning

$$x \cdot y = y \cdot x \cdot (e \circ \theta_{a,0}) = y \cdot x \cdot e = y \cdot x,$$

where the last equality follows by the fact that e is the unit for the multiplication on $\pi_*(E)$. An entirely analogous argument yields that $\theta_{0,a} = \text{id}_S$. \square

Proposition 0.6. *Given some $a \in A$, the functors Σ^a and Σ^{-a} canonically form an adjoint equivalence of \mathcal{SH} .*

rewrite proof
in a permuta-
tive category

Proof. Let $X, Y \in \mathcal{SH}$. By [1, Lemma 3.2], in order to show Σ^a and Σ^{-a} are adjoint equivalences, it suffices to construct natural isomorphisms $\eta : \text{Id}_{\mathcal{SH}} \Rightarrow \Sigma^{-a} \circ \Sigma^a$ and $\varepsilon : \Sigma^a \circ \Sigma^{-a} \Rightarrow \text{Id}_{\mathcal{SH}}$ such that for all X in \mathcal{SH} , the following diagram commutes:

$$(1) \quad \begin{array}{ccc} \Sigma^a X & \xrightarrow{(\Sigma^a \eta)_X} & \Sigma^a \Sigma^{-a} \Sigma^a X \\ & \searrow & \downarrow (\varepsilon \Sigma^a)_X \\ & & \Sigma^a X \end{array}$$

Given an object X in \mathcal{SH} , define $\eta_X : X \rightarrow \Sigma^{-a} \Sigma^a X = S^{-a} \otimes S^a \otimes X$ to be the composition

$$X \xrightarrow{\lambda_X^{-1}} S \otimes X \xrightarrow{\phi_{-a,a} \otimes X} S^{-a} \otimes S^a \otimes X.$$

Clearly this is an isomorphism. To see this is natural, let $f : X \rightarrow Y$ in \mathcal{SH} . Then consider the following diagram:

$$\begin{array}{ccccc} X & \xrightarrow{\lambda_X^{-1}} & S \otimes X & \xrightarrow{\phi_{-a,a} \otimes X} & S^{-a} \otimes S^a \otimes X \\ f \downarrow & & \downarrow S \otimes f & & \downarrow S^{-a} \otimes S^a \otimes f \\ Y & \xrightarrow{\lambda_Y^{-1}} & S \otimes Y & \xrightarrow{\phi_{-a,a} \otimes Y} & S^{-a} \otimes S^a \otimes Y \end{array}$$

The left square commutes by naturality of λ . The right square commutes by functoriality of $- \otimes -$. Hence η is indeed a natural isomorphism.

On the other hand, given an object X in \mathcal{SH} , define $\varepsilon_X : \Sigma^a \Sigma^{-a} X = S^a \otimes S^{-a} \otimes X \rightarrow X$ to be the composition

$$S^a \otimes S^{-a} \otimes X \xrightarrow{\phi_{a,-a}^{-1}} S \otimes X \xrightarrow{\lambda_X} X.$$

Clearly this is an isomorphism. To see it is natural, let $f : X \rightarrow Y$ in \mathcal{SH} . Then consider the following diagram:

$$\begin{array}{ccccc} S^a \otimes S^{-a} \otimes X & \xrightarrow{\phi_{a,-a}^{-1} \otimes X} & S \otimes X & \xrightarrow{\lambda_X} & X \\ S^a \otimes S^{-a} \otimes f \downarrow & & S \otimes f \downarrow & & \downarrow f \\ S^a \otimes S^{-a} \otimes Y & \xrightarrow{\phi_{a,-a}^{-1} \otimes Y} & S \otimes Y & \xrightarrow{\lambda_Y} & Y \end{array}$$

The left square commutes by functoriality of $- \otimes -$. The right square commutes by naturality of λ . Hence, ε is natural.

Finally, let X be an object in \mathcal{SH} . Unravelling definitions, by functoriality of $- \otimes -$, in order to show that diagram (1) commutes, it suffices to show the following diagram commutes:

$$\begin{array}{ccccc}
 S^a \otimes X & \xrightarrow{S^a \otimes \lambda_X^{-1}} & S^a \otimes S \otimes X & \xrightarrow{S^a \otimes \phi_{-a,a} \otimes X} & S^a \otimes S^{-a} \otimes S^a \otimes X \\
 & \searrow & \uparrow \phi_{a,0} \otimes X & & \downarrow \phi_{a,-a}^{-1} \otimes S^a \otimes X \\
 & & & & S \otimes S^a \otimes X \\
 & & & & \downarrow \lambda_{S^a \otimes X} \\
 & & & & S^a \otimes X
 \end{array}$$

First, note that by the coherence theorem for monoidal categories, $\lambda_{S^a \otimes X} = \lambda_{S^a} \otimes X^1$. And furthermore, recall $\lambda_{S^a} = \phi_{0,a}^{-1}$. Hence, the right triangle is precisely the diagram obtained by applying $- \otimes X$ to the coherence diagram for the $\phi_{a,b}$'s, so it commutes. Commutativity of the left triangle follows by the coherence theorem for monoidal categories and the fact that $\phi_{a,0} = \lambda_{S^a}^{-1}$. Hence, the diagram commutes, so (Σ^a, Σ^{-a}) forms an adjoint equivalence of \mathcal{SH} . \square

Proposition 0.7. *Let X and Y be objects in \mathcal{SH} . Then the pairing*

$$\pi_*(X) \times \pi_*(Y) \rightarrow \pi_*(X \otimes Y)$$

sending $x : S^a \rightarrow X$ and $y : S^b \rightarrow Y$ to the composition

$$S^{a+b} \xrightarrow{\phi_{a,b}} S^a \otimes S^b \xrightarrow{x \otimes y} X \otimes Y$$

is additive in each argument.

Proof. Let $a, b \in A$, and let $x_1, x_2 : S^a \rightarrow X$ and $y : S^b \rightarrow Y$. Then consider the following diagram

$$\begin{array}{ccccc}
 S^{a+b} & \xrightarrow{\cong} & S^a \otimes S^b & \xrightarrow{\Delta \otimes S^b} & (S^a \oplus S^a) \otimes S^b \\
 & & \downarrow \Delta & \swarrow \cong & \downarrow (x_1 \oplus x_2) \otimes y \\
 & & (S^a \otimes S^b) \oplus (S^a \otimes S^b) & & (X \oplus X) \otimes Y \\
 & & \downarrow (x_1 \otimes y) \oplus (x_2 \otimes y) & \swarrow \cong & \downarrow \nabla \otimes Y \\
 & & (X \otimes Y) \oplus (X \otimes Y) & \xrightarrow{\nabla} & X \otimes Y
 \end{array}$$

The isomorphisms are given by the fact that $- \otimes -$ is additive in each variable. Both triangles and the parallelogram commute since $- \otimes -$ is additive. By functoriality of $- \otimes -$, the top composition is $(x_1 + x_2) \cdot y$ and the bottom composition is $x_1 \cdot y + x_2 \cdot y$, so they are equal, as desired. An entirely analogous argument yields that $x \cdot (y_1 + y_2) = x \cdot y_1 + x \cdot y_2$ for $x \in \pi_*(X)$ and $y_1, y_2 \in \pi_*(Y)$. \square

Proposition 0.8 ([2, Proposition 5.11]). *Let (E, μ, e) be a monoid object in \mathcal{SH} . Then $E_*(-)$ is a functor from \mathcal{SH} to left A -graded $\pi_*(E)$ -modules, where given some X in \mathcal{SH} , $E_*(X)$ may be endowed with the structure of a left A -graded $\pi_*(E)$ -module via the map*

$$\pi_*(E) \times E_*(X) \rightarrow E_*(X)$$

which given $a, b \in A$, sends $x : S^a \rightarrow E$ and $y : S^b \rightarrow E \otimes X$ to the composition

$$x \cdot y : S^{a+b} \cong S^a \otimes S^b \xrightarrow{x \otimes y} E \otimes (E \otimes X) \cong (E \otimes E) \otimes X \xrightarrow{\mu \otimes X} E \otimes X.$$

¹Technically, this equality only holds up to composition with an associator, but we are ignoring such issues.

Similarly, the assignment $X \mapsto X_*(E)$ is a functor from \mathcal{SH} to right A -graded $\pi_*(E)$ -modules, where the structure map

$$X_*(E) \times \pi_*(E) \rightarrow X_*(E)$$

sends $x : S^a \rightarrow X \otimes E$ and $y : S^b \rightarrow E$ to the composition

$$x \cdot y : S^{a+b} \cong S^a \otimes S^b \xrightarrow{x \otimes y} (X \otimes E) \otimes E \cong X \otimes (E \otimes E) \xrightarrow{X \otimes \mu} X \otimes E.$$

Finally, $E_*(E)$ is a $\pi_*(E)$ -bimodule, in the sense that the left and right actions of $\pi_*(E)$ are compatible, so that given $y, z \in \pi_*(E)$ and $x \in E_*(E)$, $y \cdot (x \cdot z) = (y \cdot x) \cdot z$.

Proof. First we show that the map $\pi_*(E) \times E_*(X) \rightarrow E_*(X)$ endows $E_*(X)$ with the structure of a left $\pi_*(E)$ -module. Let $a, b, c \in A$ and $x, x' : S^a \rightarrow E \otimes X$, $y : S^b \rightarrow E$, and $z, z' \in S^c \rightarrow E$. Then we wish to show that:

- (1) $y \cdot (x + x') = y \cdot x + y \cdot x'$,
- (2) $(z + z') \cdot x = z \cdot x + z' \cdot x$,
- (3) $(zy) \cdot x = z \cdot (y \cdot x)$,
- (4) $e \cdot x = x$.

Axioms (1) and (2) follow by the fact that $E_*(X) = \pi_*(E \otimes X)$ and **Proposition 0.7**. To see (3), consider the diagram:

$$\begin{array}{ccccc} S^{a+b+c} & \xrightarrow{\cong} & S^c \otimes S^b \otimes S^a & \xrightarrow{z \otimes y \otimes x} & E \otimes E \otimes E \otimes X \\ & & & & \nearrow E \otimes \mu \otimes X \\ & & & & E \otimes E \otimes X \\ & & & & \downarrow \mu \otimes X \\ & & & & E \otimes X \\ & & & & \uparrow \mu \otimes X \\ & & & & E \otimes E \otimes X \\ & & & & \downarrow \mu \otimes E \otimes X \end{array}$$

It commutes by associativity of μ . By functoriality of $- \otimes -$, the two outside compositions equal $z \cdot (y \cdot x)$ on the top and $(z \cdot y) \cdot x$ on the bottom. Hence, they are equal, as desired.

Next, to see (4), consider the following diagram:

$$\begin{array}{ccc} S^a & \xrightarrow{x} & E \otimes X \\ & \searrow x & \nearrow \\ & E \otimes X & \\ & \downarrow e \otimes X & \\ & E \otimes E \otimes X & \end{array}$$

$\downarrow \mu \otimes X$

The top triangle commutes by definition. The left triangle commutes by functoriality of $- \otimes -$. The right triangle commutes by unitality of μ . The top composition is x while the bottom is $e \cdot x$, thus they are necessarily equal since the diagram commutes.

Thus, we have shown that the indicated map does indeed endow $E_*(X)$ with the structure of a left $\pi_*(E)$ -module. It remains to show that $E_*(-)$ sends maps in \mathcal{SH} to A -graded homomorphisms of left A -graded $\pi_*(E)$ -modules. By definition, given $f : X \rightarrow Y$ in \mathcal{SH} , $E_*(f)$ is the map which takes a class $x : S^a \rightarrow E \otimes X$ to the composition

$$S^a \xrightarrow{x} E \otimes X \xrightarrow{E \otimes f} E \otimes Y.$$

To see this assignment is a homomorphism, suppose we are given some other $x' : S^a \rightarrow E \otimes X$ and some scalar $y : S^b \rightarrow E$. Then we would like to show $E_*(f)(x + x') = E_*(f)(x) + E_*(f)(x')$

and $E_*(f)(y \cdot x) = y \cdot E_*(f)(x)$. To see the former, consider the following diagram:

$$\begin{array}{ccc}
 S^a & \xrightarrow{\Delta} S^a \oplus S^a & \xrightarrow{x \oplus x'} (E \otimes X) \oplus (E \otimes X) \\
 & & \searrow^{(E \otimes f) \oplus (E \otimes f)} (E \otimes Y) \oplus (E \otimes Y) \\
 & & \downarrow \nabla \\
 & & E \otimes Y \\
 & & \uparrow E \otimes f \\
 & & E \otimes X \\
 & \nearrow \nabla & \\
 & & (E \otimes X) \oplus (E \otimes X)
 \end{array}$$

It commutes by naturality of ∇ in an additive category. The top composition is $E_*(f)(x) + E_*(f)(x')$, while the bottom is $E_*(f)(x+x')$, so they are equal as desired. To see that $E_*(f)(y \cdot x) = y \cdot E_*(f)(x)$, consider the following diagram:

$$\begin{array}{ccccc}
 S^{a+b} & \xrightarrow{\phi_{b,a}} S^b \otimes S^a & \xrightarrow{y \otimes x} E \otimes E \otimes X & \xrightarrow{E \otimes E \otimes f} E \otimes E \otimes Y \\
 & & \downarrow \mu \otimes X & & \downarrow \mu \otimes Y \\
 & & E \otimes X & \xrightarrow{E \otimes f} & E \otimes Y
 \end{array}$$

It commutes by functoriality of $- \otimes -$. The top composition is $E_*(f)(y \cdot x)$, while the bottom composition is $y \cdot E_*(f)(x)$, so they are equal, as desired.

Showing that $X_*(E)$ has the structure of a right $\pi_*(E)$ -module and that if $f : X \rightarrow Y$ is a morphism in \mathcal{SH} then the map

$$X_*(E) = [S^*, X \otimes E] \xrightarrow{(f \otimes E)_*} [S^*, Y \otimes E] = Y_*(E)$$

is an A -graded homomorphism of right A -graded $\pi_*(E)$ -modules is entirely analogous.

It remains to show that $E_*(E)$ is a bimodule. Let $x : S^a \rightarrow E$, $y : S^b \rightarrow E \otimes E$, and $z : S^c \rightarrow E$, and consider the following diagram:

$$\begin{array}{ccccc}
 S^{a+b+c} & \xrightarrow{\cong} S^a \otimes S^b \otimes S^c & \xrightarrow{x \otimes y \otimes z} E \otimes E \otimes E \otimes E & \xrightarrow{\mu \otimes \mu} E \otimes E \\
 & & \nearrow \mu \otimes E \otimes E & & \downarrow E \otimes \mu \\
 & & & & E \otimes E \\
 & & \searrow E \otimes E \otimes \mu & & \uparrow \mu \otimes E \\
 & & & & E \otimes E \otimes E
 \end{array}$$

Commutativity follows by functoriality of $- \otimes -$, which also tells us that the two outside compositions are $(x \cdot y) \cdot z$ (on top) and $x \cdot (y \cdot z)$ (on bottom). Hence they are equal, as desired. \square

Proposition 0.9 ([3, Proposition 2.2]). *Let (E, μ, e) be a monoid object in \mathcal{SH} and let X be any object. Then the assignment*

$$E_*(E) \times E_*(X) \rightarrow E_*(E \otimes X)$$

which sends $x : S^a \rightarrow E \otimes E$ and $y : S^b \rightarrow E \otimes X$ to the composition

$$x \cdot y : S^{a+b} \cong S^a \otimes S^b \xrightarrow{x \otimes y} E \otimes E \otimes E \otimes X \xrightarrow{E \otimes \mu \otimes X} E \otimes E \otimes X$$

induces an A -graded homomorphism of left A -graded $\pi_(E)$ -modules*

$$E_*(E) \otimes_{\pi_*(E)} E_*(X) \rightarrow E_*(E \otimes X)$$

(where here $E_(E)$ has a $\pi_*(E)$ -bimodule structure and $E_*(X)$ has a left $\pi_*(E)$ -module structure as specified by [Proposition 0.8](#), so $E_*(E) \otimes_{\pi_*(E)} E_*(X)$ is a left A -graded $\pi_*(E)$ -module by ??). Furthermore, this homomorphism is natural in X .*

Proof. First, recall by definition of the tensor product, in order to show the assignment $E_*(E) \times E_*(X) \rightarrow E_*(E \otimes X)$ induces a homomorphism $E_*(E) \otimes_{\pi_*(E)} E_*(X) \rightarrow E_*(E \otimes X)$ of A -graded abelian groups, it suffices to show that the assignment is $\pi_*(E)$ -balanced, i.e., that it is linear in each argument and satisfies $xr \cdot y = x \cdot ry$ for $x \in E_*(E)$, $y \in E_*(X)$, and $r \in \pi_*(E)$.

First, note that by the identifications $E_*(E) = \pi_*(E \otimes E)$, $E_*(X) = \pi_*(E \otimes X)$, and $E_*(E \otimes X) = \pi_*(E \otimes E \otimes X)$, and **Proposition 0.7**, it is straightforward to see that the assignment commutes with addition of maps in each argument. Now, let $a, b, c \in A$, $x : S^a \rightarrow E \otimes E$, $y : S^b \rightarrow E \otimes X$, and $z : S^c \rightarrow E$. Then we wish to show $xz \cdot y = x \cdot zy$. Consider the following diagram (where here we are passing to a permutative category):

$$\begin{array}{ccccc}
 & & & E \otimes E \otimes E \otimes X & \\
 & & E \otimes \mu \otimes E \otimes X & \nearrow & \downarrow E \otimes \mu \otimes X \\
 S^{a+b+c} & \xrightarrow{\cong} & S^a \otimes S^c \otimes S^b & \xrightarrow{x \otimes z \otimes y} & E \otimes E \otimes E \otimes E \otimes X \\
 & & E \otimes E \otimes \mu \otimes X & \searrow & \uparrow E \otimes \mu \otimes X \\
 & & & E \otimes E \otimes E \otimes X &
 \end{array}$$

It commutes by associativity of μ . By functoriality of $-\otimes-$, the top composition is given by $(xz) \cdot y$ and the bottom composition is $x \cdot (zy)$, so we have they are equal, as desired. Thus, since the map $E_*(E) \times E_*(X) \rightarrow E_*(E \otimes X)$ is $\pi_*(E)$ -balanced, we have that it induces a homomorphism of abelian groups. Furthermore, by ?? it is an A -graded homomorphism of A -graded abelian groups.

In order to see this map is furthermore a homomorphism of left $\pi_*(E)$ -modules, we must show that $z(x \cdot y) = zx \cdot y$, where x , y , and z are defined as above. Now consider the following diagram:

$$\begin{array}{ccccc}
 & & & E \otimes E \otimes E \otimes X & \\
 & & \mu \otimes E \otimes E \otimes X & \nearrow & \downarrow E \otimes \mu \otimes X \\
 S^{a+b+c} & \xrightarrow{\cong} & S^c \otimes S^a \otimes S^b & \xrightarrow{z \otimes x \otimes y} & E \otimes E \otimes E \otimes E \otimes X \\
 & & E \otimes E \otimes \mu \otimes X & \searrow & \uparrow \mu \otimes E \otimes X \\
 & & & E \otimes E \otimes E \otimes X &
 \end{array}$$

Commutativity of the triangles is functoriality of $-\otimes-$. By functoriality of $-\otimes-$, the top composition is $zx \cdot y$, and the bottom composition is $z(x \cdot y)$. Hence they are equal, as desired, so that the map we have constructed

$$E_*(E) \otimes_{\pi_*(E)} E_*(X) \rightarrow E_*(E \otimes X)$$

is indeed an A -graded homomorphism of left A -graded $\pi_*(E)$ -modules.

Next, we would like to show that this homomorphism is natural in X . Let $f : X \rightarrow Y$ in \mathcal{SH} . Then we would like to show the following diagram commutes:

$$\begin{array}{ccc}
 E_*(E) \otimes_{\pi_*(E)} E_*(X) & \xrightarrow{\Phi_X} & E_*(E \otimes X) \\
 E_*(E) \otimes_{\pi_*(E)} E_*(f) \downarrow & & \downarrow E_*(E \otimes f) \\
 E_*(E) \otimes_{\pi_*(E)} E_*(Y) & \xrightarrow{\Phi_Y} & E_*(E \otimes Y)
 \end{array} \tag{2}$$

As all the maps here are homomorphisms, it suffices to chase generators around the diagram. In particular, suppose we are given $x : S^a \rightarrow E \otimes E$ and $y : S^b \rightarrow E \otimes X$, and consider the following diagram exhibiting the two possible ways to chase $x \otimes y$ around the diagram (as usual, we are

passing to a permutative category):

$$\begin{array}{ccc}
 S^{a+b} & \xrightarrow{\phi_{a,b}} & S^a \otimes S^b \xrightarrow{x \otimes y} E \otimes E \otimes E \otimes X \xrightarrow{E \otimes \mu \otimes X} E \otimes E \otimes X \\
 & & \downarrow E \otimes E \otimes E \otimes f \quad \downarrow E \otimes E \otimes f \\
 & & E \otimes E \otimes E \otimes Y \xrightarrow{E \otimes \mu \otimes Y} E \otimes E \otimes Y
 \end{array}$$

This diagram commutes by functoriality of $- \otimes -$. Thus we have that diagram (2) does indeed commute, as desired. \square

Proposition 0.10. *Let (E, μ, e) be a flat, commutative monoid object in \mathcal{SH} (??) and let X be any cellular object in \mathcal{SH} (??). Then the natural homomorphism*

$$\Phi_X : E_*(E) \otimes E_*(X) \rightarrow E_*(E \otimes X)$$

given in Proposition 0.9 is an isomorphism of left $\pi_(E)$ -modules.*

Proof. It remains to show that if X is cellular and E is flat, then this map is an isomorphism. To start, let \mathcal{E} be the collection of objects X in \mathcal{SH} for which this map is an isomorphism. Then in order to show \mathcal{E} contains every cellular object, it suffices to show that \mathcal{E} satisfies the three conditions given for the class of cellular objects in ???. First, we need to show that Φ is an isomorphism when $X = S^a$ for some $a \in A$. Indeed, consider the map

$$\Psi : E_*(E \otimes S^a) \rightarrow E_*(E) \otimes_{\pi_*(E)} E_*(S^a)$$

which sends a class $x : S^b \rightarrow E \otimes E \otimes S^a$ in $E_b(E \otimes S^a)$ to the pure tensor $\tilde{x} \otimes \tilde{e}$, where $\tilde{x} \in E_{b-a}(E)$ is the composition

$$S^{b-a} \cong S^b \otimes S^{-a} \xrightarrow{x \otimes S^{-a}} E \otimes E \otimes S^a \otimes S^{-a} \xrightarrow{E \otimes E \otimes \phi_{a,-a}^{-1}} E \otimes E \otimes S \xrightarrow{E \otimes \rho_E} E \otimes E$$

and $\tilde{e} \in E_a(S^a)$ is the composition

$$S^a \cong S \otimes S^a \xrightarrow{e \otimes S^a} E \otimes S^a.$$

First, note Ψ is an $(A\text{-graded})$ homomorphism of abelian groups: Given $x, x' \in E_b(E \otimes S^a)$, we would like to show that $\tilde{x} \otimes \tilde{e} + \tilde{x}' \otimes \tilde{e} = \widetilde{x + x'} \otimes \tilde{e}$. It suffices to show that $\tilde{x} + \tilde{x}' = \widetilde{x + x'}$. To see this, consider the following diagram (again, we are passing to a permutative category):

$$\begin{array}{ccc}
 S^{b-a} & \xrightarrow{\Delta} & S^{b-a} \oplus S^{b-a} \\
 \downarrow \phi_{b-a} & & \downarrow \phi_{b,-a} \oplus \phi_{b,-a} \\
 S^b \otimes S^{-a} & \xrightarrow{\Delta} & (S^b \otimes S^{-a}) \oplus (S^b \otimes S^{-a}) \\
 \downarrow \Delta \otimes S^{-a} & \searrow \cong & \downarrow (x \otimes S^{-a}) \oplus (x' \otimes S^{-a}) \\
 (S^b \oplus S^b) \otimes S^{-a} & & (E \otimes E \otimes S^a \otimes S^{-a}) \oplus (E \otimes E \otimes S^a \otimes S^{-a}) \\
 \downarrow (x \oplus x') \otimes S^{-a} & \searrow \cong & \downarrow (E \otimes E \otimes \phi_{a,-a}^{-1}) \oplus (E \otimes E \otimes \phi_{a,-a}^{-1}) \\
 ((E \otimes E \otimes S^a) \oplus (E \otimes E \otimes S^a)) \otimes S^{-a} & \searrow \nabla & (E \otimes E) \oplus (E \otimes E) \\
 \downarrow \nabla \otimes S^{-a} & & \downarrow \nabla \\
 E \otimes E \otimes S^a \otimes S^{-a} & \xrightarrow{E \otimes E \otimes \phi_{a,-a}^{-1}} & E \otimes E
 \end{array}$$

The top rectangle commutes by naturality of Δ in an additive category. The bottom triangle commutes by naturality of ∇ in an additive category. Finally, the remaining regions of the diagram commute by additivity of $- \otimes -$. By functoriality of $- \otimes -$, it follows that the left

composition is $\widetilde{x + x'}$ and the right composition is $\widetilde{x} + \widetilde{x'}$, so they are equal as desired. Thus Ψ is a homomorphism of abelian groups, as desired.

Now, we claim that Ψ is an inverse to Φ , (which is enough to show Φ is an isomorphism of left $\pi_*(E)$ -modules). Since Φ and Ψ are homomorphisms it suffices to check that they are inverses on generators. First, let $x : S^b \rightarrow E \otimes E \otimes S^a$ in $E_b(E \otimes S^a)$. We would like to show that $\Phi(\Psi(x)) = x$. Consider the following diagram, where here we are passing to a permutative category:

$$\begin{array}{ccccc}
 S^b & \xrightarrow{\cong} & S^b \otimes S^{-a} \otimes S^a & & \\
 \downarrow x & & \downarrow x \otimes S^{-a} \otimes S^a & \searrow x \otimes S^{-a} \otimes e \otimes S^a & \\
 & & E \otimes E \otimes S^a \otimes S^{-a} \otimes S^a & \xrightarrow{E \otimes E \otimes S^a \otimes S^{-a} \otimes e \otimes S^a} & E \otimes E \otimes S^a \otimes S^{-a} \otimes E \otimes S^a \\
 & \nearrow E \otimes E \otimes S^a \otimes \phi_{-a,a} & \uparrow E \otimes E \otimes \phi_{a,-a} \otimes S^a & & \\
 E \otimes E \otimes S^a & \xrightarrow{E \otimes \mu \otimes S^a} & E \otimes E \otimes S^a & \xrightarrow{E \otimes E \otimes e \otimes S^a} & E \otimes E \otimes E \otimes S^a \\
 \uparrow E \otimes \mu \otimes S^a & & \downarrow E \otimes E \otimes e \otimes S^a & \nearrow E \otimes E \otimes \phi_{a,-a}^{-1} \otimes E \otimes S^a & \\
 E \otimes E \otimes E \otimes S^a & \xrightarrow{E \otimes E \otimes e \otimes S^a} & E \otimes E \otimes E \otimes S^a & &
 \end{array}$$

The top left trapezoid commutes since the isomorphism $S^b \xrightarrow{\cong} S^b \otimes S^{-a} \otimes S^a$ may be given as $S^b \otimes \phi_{-a,a}$ (see ??), in which case the trapezoid commutes by functoriality of $-\otimes-$. The triangle below that commutes by coherence for the $\phi_{a,b}$'s. The triangle below that commutes by definition. The bottom left triangle commutes by unitality for μ . The top right triangle commutes by functoriality of $-\otimes-$. Finally, the bottom right triangle commutes by functoriality of $-\otimes-$. It follows by unravelling definitions that the two outside compositions are x (on the left) and $\Phi(\Psi(x))$ (on the right), so since the diagram commutes we indeed have $\Phi(\Psi(x)) = x$, as desired.

On the other hand, suppose we are given a homogeneous pure tensor $x \otimes y$ in $E_*(E) \otimes_{\pi_*(E)} E_*(S^a)$, so $x : S^b \rightarrow E \otimes E$ and $y : S^c \rightarrow E \otimes S^a$ for some $b, c \in A$. Then we would like to show that $\Psi(\Phi(x \otimes y)) = x \otimes y$. Unravelling definitions, $\Psi(\Phi(x \otimes y))$ is the homogeneous pure tensor $\widetilde{xy} \otimes \widetilde{e}$, where $\widetilde{e} : S^a \rightarrow E \otimes S^a$ is defined above, and by functoriality of $-\otimes-$, $\widetilde{xy} : S^{b+c-a} \rightarrow E \otimes E$ is the composition

$$\begin{array}{c}
 S^{b+c-a} \\
 \downarrow \phi_{b+c,-a} \\
 S^{b+c} \otimes S^{-a} \\
 \downarrow \phi_{b,c} \otimes S^{-a} \\
 S^b \otimes S^c \otimes S^{-a} \\
 \downarrow x \otimes y \otimes S^{-a} \\
 E \otimes E \otimes E \otimes S^a \otimes S^{-a} \\
 \downarrow E \otimes \mu \otimes S^a \otimes S^{-a} \\
 E \otimes E \otimes S^a \otimes S^{-a} \\
 \downarrow E \otimes E \otimes \phi_{a,-a}^{-1} \\
 E \otimes E \otimes S \\
 \downarrow E \otimes \rho_E \\
 E \otimes E.
 \end{array}$$

In order to see $x \otimes y = \widetilde{xy} \otimes \widetilde{e}$, it suffices to show there exists some scalar $r \in \pi_{c-a}(E)$ such that $x \cdot r = \widetilde{xy}$ and $r \cdot \widetilde{e} = y$, where here \cdot denotes the right and left action of $\pi_*(E)$ on $E_*(E)$ and

$E_*(S^a)$, respectively. Now, define r to be the composition

$$S^{c-a} \cong S^c \otimes S^{-a} \xrightarrow{y \otimes S^{-a}} E \otimes S^a \otimes S^{-a} \xrightarrow{E \otimes \phi_{a,-a}^{-1}} E \otimes S \xrightarrow{\rho_E} E.$$

First, in order to see that $x \cdot r = \widetilde{xy}$, consider the following diagram, where here we are again passing to a permutative category:

$$\begin{array}{ccccccc} S^{b+c-a} & \xrightarrow{\cong} & S^b \otimes S^c \otimes S^{-a} & \xrightarrow{y \otimes S^{-a}} & E \otimes E \otimes E \otimes S^a & \xrightarrow{E \otimes \mu \otimes S^a \otimes S^{-a}} & E \otimes E \otimes S^a \otimes S^{-a} \\ & & & & \downarrow E \otimes E \otimes E \otimes \phi_{a,-a}^{-1} & \searrow E \otimes \mu \otimes \phi_{a,-a}^{-1} & \downarrow E \otimes E \otimes \phi_{a,-a}^{-1} \\ & & & & E \otimes E \otimes E & \xrightarrow{E \otimes \mu} & E \otimes E \end{array}$$

Commutativity is functoriality of $-\otimes-$, which also tells us that the two outside compositions are \widetilde{xy} (on top) and $x \cdot r$ (on the bottom), so they are equal as desired. On the other hand, in order to see that $r \cdot \tilde{e} = y$, consider the following diagram (where here we have passed to a permutative category):

$$\begin{array}{ccc} S^c & \xrightarrow{\cong} & S^c \otimes S^{-a} \otimes S^a \\ \downarrow y & & \downarrow y \otimes S^{-a} \otimes e \otimes S^a \\ E \otimes S^a & \xleftarrow{E \otimes S^a \otimes \phi_{-a,a}^{-1}} & E \otimes S^a \otimes S^{-a} \otimes S^a \\ & \searrow E \otimes S^a \otimes S^{-a} \otimes e \otimes S^a & \downarrow E \otimes \phi_{a,-a}^{-1} \otimes E \otimes S^a \\ E \otimes E \otimes S^a & \xrightarrow{\mu \otimes S^a} & E \otimes S^a \otimes S^{-a} \otimes E \otimes S^a \\ & \searrow E \otimes e \otimes S^a & \downarrow E \otimes \phi_{a,-a}^{-1} \otimes E \otimes S^a \\ E \otimes E \otimes S^a & \xrightarrow{\mu \otimes S^a} & E \otimes E \otimes S^a \end{array}$$

The top left triangle commutes since we may take the isomorphism $S^c \xrightarrow{\cong} S^c \otimes S^{-a} \otimes S^a$ to be $S^c \otimes \phi_{-a,a}$, in which case commutativity of the triangle follows by functoriality of $-\otimes-$. Commutativity of the right triangle is also functoriality of $-\otimes-$. Commutativity of the bottom triangle is unitality of μ . Finally, commutativity of the remaining middle 4-sided region is again functoriality of $-\otimes-$. It follows that y is equal to the outer composition, which is $r \cdot \tilde{e}$, as desired. Thus, we have shown that

$$\Psi(\Phi(x \otimes y)) = \widetilde{xy} \otimes \tilde{e} = (x \cdot r) \otimes \tilde{e} = x \otimes (r \cdot \tilde{e}) = x \otimes y,$$

as desired, so that for each $a \in A$, the object S^a belongs to the class \mathcal{E} .

Now, we would like to show that given a distinguished triangle

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X,$$

if two of three of the objects X , Y , and Z belong to \mathcal{E} , then so does the third. First we claim that if X belongs to \mathcal{E} , then for any $n \in \mathbb{Z}$,

a

To see this, consider the following diagram

$$\begin{array}{ccccc}
E_*(E) \otimes_{\pi_*(E)} E_*(\Sigma^a X) & \xrightarrow{\Phi_{\Sigma^a X}} & E_*(E \otimes \Sigma^a X) & \xrightarrow{E_*(\tau_{E, S^a \otimes X})} & E_*(\Sigma^a(E \otimes X)) \\
\cong \downarrow & & & & \downarrow \cong \\
E_*(E) \otimes_{\pi_*(E)} E_{*-a}(X) & \xrightarrow{\Phi_X} & E_{*-a}(E \otimes X) & \xrightarrow{\cong} & E_{*-a}(E \otimes X)
\end{array}$$

where:

- the vertical isomorphisms are induced by the isomorphism $E_*(\Sigma^a W) \cong E_{*-a}(W)$ given by the composition

$$[S^*, E \otimes S^a \otimes W] \xrightarrow{(\tau_{E \otimes S^a \otimes X})^*} [S^*, S^a \otimes E \otimes W] \cong [S^{-a} \otimes S^*, E \otimes W] \xrightarrow{(\phi_{-a, *})^*} [S^{*-a}, E \otimes W]$$

where the middle arrow is the adjunction $S^{-a} \otimes - \dashv S^a \otimes -$ given by [Proposition 0.6](#), and

- the unlabelled bottom left horizontal isomorphism ???

Thus in order to show $\Sigma^a X$ belongs to \mathcal{E} , it suffices to show diagram (??) commutes. Suppose we are given elements $x : S^b \rightarrow E \otimes E$ and $y : S^c \rightarrow E \otimes \Sigma^a X = E \otimes S^a \otimes X$. Then chasing $x \otimes y$ around diagram (??) yields the following two compositions:

$$\begin{array}{c}
S^{b+c-a} \\
\downarrow \phi_{b, c-a} \\
S^b \otimes S^{c-a} \\
\downarrow S^b \otimes \phi_{-a, c} \\
S^b \otimes S^{-a} \otimes S^c \\
\downarrow x \otimes S^{-a} \otimes y \\
E \otimes E \otimes S^{-a} \otimes E \otimes S^a \otimes X \\
\downarrow E \otimes E \otimes S^{-a} \otimes \tau_{E, S^a \otimes X} \\
E \otimes E \otimes S^{-a} \otimes S^a \otimes E \otimes X \\
\downarrow E \otimes E \otimes \phi_{-a, a}^{-1} \otimes E \otimes X \\
E \otimes E \otimes E \otimes X \\
\downarrow E \otimes \mu \otimes X \\
E \otimes E \otimes X
\end{array}
\qquad
\begin{array}{c}
S^{b+c-a} \\
\downarrow \phi_{-a, b+c} \\
S^{-a} \otimes S^{b+c} \\
\downarrow S^{-a} \otimes \phi_{b, c} \\
S^{-a} \otimes S^b \otimes S^c \\
\downarrow S^{-a} \otimes x \otimes y \\
S^{-a} \otimes E \otimes E \otimes E \otimes S^a \otimes X \\
\downarrow S^{-a} \otimes E \otimes \mu \otimes S^a \otimes X \\
S^{-a} \otimes E \otimes E \otimes S^a \otimes X \\
\downarrow S^{-a} \otimes E \otimes \tau_{E, S^a \otimes X} \\
S^{-a} \otimes E \otimes S^a \otimes E \otimes X \\
\downarrow S^{-a} \otimes \tau_{E, S^a \otimes E \otimes X} \\
S^{-a} \otimes S^a \otimes E \otimes E \otimes X \\
\downarrow \phi_{-a, a}^{-1} \otimes E \otimes E \otimes X \\
E \otimes E \otimes X
\end{array}$$

Then consider the following diagram

$$\begin{array}{ccccc}
S^{-a} \otimes S^{b+c} & \xleftarrow{\phi_{-a,b+c}} & S^{b+c} & \xleftarrow{\phi_{b,c-a}} & S^b \otimes S^{c-a} \\
\downarrow S^{-a} \otimes \phi_{b,c} & & \downarrow \phi_{b-a,c} & & \downarrow \phi_{b-a,c} \\
S^{-a} \otimes S^b \otimes S^c & \xleftarrow{\phi_{-a,b} \otimes S^c} & S^{b-a} \otimes S^c & \xleftarrow{\phi_{b,-a} \otimes S^c} & S^b \otimes S^{c-a} \\
\downarrow S^{-a} \otimes x \otimes y & \searrow S^{-a} \otimes x \otimes S^c & \searrow x \otimes S^{-a} \otimes S^c & \searrow x \otimes S^{-a} \otimes y & \downarrow S^b \otimes \phi_{-a,c} \\
S^{-a} \otimes E \otimes E \otimes E \otimes S^a \otimes X & \xrightarrow{\tau_{S^{-a}, E \otimes E \otimes E \otimes S^a \otimes X}} & E \otimes E \otimes S^{-a} \otimes S^c & \xrightarrow{\tau_{S^{-a}, E \otimes E \otimes S^c}} & E \otimes E \otimes S^{-a} \otimes E \\
\downarrow S^{-a} \otimes E \otimes \mu \otimes S^a \otimes X & \searrow S^{-a} \otimes \tau_{E \otimes E \otimes E, S^a \otimes X} & \searrow S^{-a} \otimes S^a \otimes E \otimes E \otimes E \otimes X & \xrightarrow{\tau_{S^{-a} \otimes S^a, E \otimes E \otimes E \otimes X}} & E \otimes E \otimes S^{-a} \otimes S^a \\
S^{-a} \otimes E \otimes E \otimes S^a \otimes X & \xrightarrow{S^{-a} \otimes \tau_{E \otimes E, S^a \otimes X}} & S^{-a} \otimes S^a \otimes E \otimes E \otimes E \otimes X & \xrightarrow{\phi_{-a,a}^{-1} \otimes E \otimes E \otimes E \otimes X} & E \otimes E \otimes E \otimes S^a \\
\downarrow S^{-a} \otimes E \otimes \tau_{E, S^a \otimes X} & \searrow S^{-a} \otimes \tau_{E \otimes E, S^a \otimes X} & \downarrow S^{-a} \otimes S^a \otimes E \otimes \mu \otimes X & & \downarrow E \otimes E \otimes \mu \\
S^{-a} \otimes E \otimes S^a \otimes E \otimes X & \xrightarrow{S^{-a} \otimes \tau_{E, S^a \otimes E \otimes X}} & S^{-a} \otimes S^a \otimes E \otimes E \otimes X & \xrightarrow{\phi_{-a,a}^{-1} \otimes E \otimes E \otimes X} & E \otimes E \otimes X
\end{array}$$

the map that goes here which makes the
below commute is dependent on b and c

Finally, it remains to show that \mathcal{E} is closed under taking arbitrary direct sums. Let $\{X_i\}_{i \in I}$ be a family of objects in \mathcal{E} indexed by some set I . Then note by definition, since direct sums are limits, we have that for any W in \mathcal{H} that

$$\left[W, \bigoplus_{i \in I} X_i \right] \cong \bigoplus_{i \in I} [W, X_i],$$

and furthermore this isomorphism is natural in W . Now let $X = \bigoplus_i X_i$, and consider the following diagram

$$\begin{array}{ccccc}
[S^*, E \otimes E] \otimes [S^*, E \otimes X] & \xrightarrow{\cong} & [S^*, E \otimes E], [S^*, \bigoplus_i E \otimes X_i] & \xrightarrow{\cong} & \bigoplus ([S^*, E \otimes E] \otimes [S^*, E \otimes X_i]) \\
\downarrow - \otimes - & & \downarrow - \otimes - & & \downarrow \bigoplus_i (- \otimes -) \\
[S^* \otimes S^*, E \otimes E \otimes E \otimes X] & \xrightarrow{\cong} & [S^* \otimes S^*, \bigoplus_i E \otimes E \otimes E \otimes X_i] & \xrightarrow{\cong} & \bigoplus_i [S^* \otimes S^*, E \otimes E \otimes E \otimes X_i] \\
\downarrow (\phi_{*,*})^* & & \downarrow (\phi_{*,*})^* & & \downarrow \bigoplus_i (\phi_{*,*})^* \\
[S^{*++}, E \otimes E \otimes E \otimes X] & \xrightarrow{\cong} & [S^{*++}, \bigoplus_i E \otimes E \otimes E \otimes X_i] & \xrightarrow{\cong} & \bigoplus_i [S^{*++}, E \otimes E \otimes E \otimes X_i] \\
\downarrow (E \otimes \mu \otimes X)_* & & \downarrow (E \otimes \mu \otimes X)_* & & \downarrow \bigoplus_i (E \otimes \mu \otimes X_i)_* \\
[S^{*++}, E \otimes E \otimes X] & \xrightarrow{\cong} & [S^{*++}, \bigoplus_i E \otimes E \otimes X_i] & \xrightarrow{\cong} & \bigoplus_i [S^{*++}, E \otimes E \otimes X_i]
\end{array}$$

The left squares commute by additivity of $- \otimes -$. The right squares commute by naturality of the isomorphisms given above. Since each X_i belongs to \mathcal{E} , the right vertical composition is an isomorphism, so that the left vertical composition is also an isomorphism, as desired.

To recap, we have shown that the collection of objects \mathcal{E} for which $\Phi_X : E_*(E) \otimes_{\pi_*(E)} E_*(X) \rightarrow E_*(E \otimes X)$ is an isomorphism satisfies the conditions outlined in ?? . Hence, \mathcal{E} contains every cellular object, as desired.

where did I
use cellularity
of E ?

□

In the following definition, let $\varepsilon : E_*(E) \rightarrow \pi_*(E)$ be the map which sends some $\alpha : S^a \rightarrow E \otimes E$ to the composition

$$S^a \xrightarrow{\alpha} E \otimes E \xrightarrow{\mu} E.$$

Also define $\Psi : E_*(E) \rightarrow E_*(E) \otimes_{\pi_*(E)} E_*(E)$ to be the map which factors as

$$E_*(E) \rightarrow E_*(E \otimes E) \xrightarrow{\cong} E_*(E) \otimes_{\pi_*(E)} E_*(E)$$

where the second arrow is the isomorphism prescribed by [Proposition 0.9](#), and the first arrow sends a class $\alpha : S^a \rightarrow E \otimes E$ to the composition

$$S^a \xrightarrow{\alpha} E \otimes E \cong E \otimes S \otimes E \xrightarrow{E \otimes e \otimes E} E \otimes E \otimes E.$$

Lemma 0.11 ([3, Proposition 2.30, 2.33]). *Let E be a flat commutative ring spectrum, and let X and Y be spectra such that $E_{**}(X)$ is a projective module over $\pi_{**}(E)$. Then for all $s \geq 0$ and $t, w \in \mathbb{Z}$, there is an isomorphism*

$$\Phi : [X, E \wedge Y]_{t,w} \rightarrow \text{Hom}_{E_{**}(E)}^{t,w}(E_{**}(X), E_{**}(E \wedge Y)),$$

obtained by sending a class $f : S^{t,w} \wedge X \rightarrow E \wedge Y$ in $[X, E \wedge Y]_{t,w}$ to the map

$$\Phi_f : E_{*,*}(X) \rightarrow E_{*+t,*+w}(X \wedge Y)$$

sending

$$[S^{a,b} \xrightarrow{g} E \wedge X] \mapsto [S^{a+t,b+w} \cong S^{a,b} \wedge S^{t,w} \xrightarrow{g \wedge S^{t,w}} E \wedge X \wedge S^{t,w} \cong E \wedge S^{t,w} \wedge X \xrightarrow{E \wedge f} E \wedge E \wedge Y].$$

Proof. Let $f : S^{t,w} \wedge X \rightarrow E \wedge Y$. First we want to show that Φ_f is actually an $E_{**}(E)$ -comodule homomorphism. □

finish