

In what follows, we fix an abelian group  $A$ . We will freely use the theory and results of ??

**Definition 0.1.** An  $A$ -graded spectral sequence  $(E_r, d_r)_{r \geq r_0}$  is the data of:

- A collection of  $A$ -graded abelian groups  $\{E_r^*\}_{r \geq r_0}$
- A collection of  $A$ -graded homomorphisms  $d_r : E_r \rightarrow E_r$  for  $r \geq r_0$  (of possibly nonzero degree) such that  $d_r \circ d_r = 0$
- For each  $r \geq r_0$ , an  $A$ -graded isomorphism  $E_{r+1} \cong \ker d_r / \operatorname{im} d_r$  of degree 0 (where  $\ker d_r$  and  $\operatorname{im} d_r$  are canonically  $A$ -graded by ??, and their quotient is canonically  $A$ -graded by ??).

Typically we call a  $\mathbb{Z}^2$ -graded spectral sequence a *bigraded* spectral sequence, and a  $\mathbb{Z}^3$ -graded spectral sequence is a *trigraded* spectral sequence.

For our purposes, we will only care about spectral sequences which arise from  $A$ -graded *unrolled exact couples*. In what follows, we follow [1], with minor modifications for our setting, in which everything is  $A$ -graded.

**Definition 0.2.** An  $A$ -graded *unrolled exact couple*  $(D, E; i, j, k)$  is a diagram of  $A$ -graded abelian groups and  $A$ -graded homomorphisms (of possibly non-zero degree)

$$\begin{array}{ccccccc} \cdots & \longrightarrow & D^{s+2} & \xrightarrow{i} & D^{s+1} & \xrightarrow{i} & D^s & \xrightarrow{i} & D^{s-1} & \longrightarrow & \cdots \\ & & \downarrow j & \swarrow k & \downarrow j & \swarrow k & \downarrow j & \swarrow k & \downarrow j & & \\ & & E^{s+2} & & E^{s+1} & & E^s & & E^{s-1} & & \end{array}$$

in which each triangle  $D^{s+1} \xrightarrow{i} D^s \xrightarrow{j} E^s \xrightarrow{k} D^{s+1}$  is an exact sequence. We require that each occurrence of  $i$  (resp.  $j$ ,  $k$ ) is of the same degree. In other words, an unrolled exact couple can be described as a tuple  $(D, E; i, j, k)$  of  $\mathbb{Z} \times A$ -graded abelian groups and  $\mathbb{Z} \times A$ -graded maps  $i : D \rightarrow D$ ,  $j : D \rightarrow E$ , and  $k : E \rightarrow D$ , such that the  $\mathbb{Z}$ -degrees of  $i$ ,  $j$ , and  $k$  are  $-1$ ,  $0$ , and  $1$ , respectively. Usually  $i$  and one of  $j$  or  $k$  will be of  $A$ -degree 0.

Given an  $A$ -graded unrolled exact couple  $(D, E; i, j, k)$ , we may define an associated  $\mathbb{Z} \times A$ -graded spectral sequence as follows: Given some  $s \in \mathbb{Z}$  and some  $r \geq 1$ , we first define the following subgroups of  $E_s$ :

$$Z_r^s := k^{-1}(\operatorname{im}[i^{r-1} : D^{s+r} \rightarrow D^{s+1}]) \quad \text{and} \quad B_r^s := j(\ker[i^{r-1} : D^s \rightarrow D^{s-r+1}])$$

where we adopt the convention that  $i^0$  is simply the identity. These are furthermore  $A$ -graded subgroups of  $E_s$  (by ?? and ??). In this way, for each  $s \in \mathbb{Z}$ , we get an infinite sequence of  $A$ -graded subgroups:

$$0 = B_1^s \subseteq B_2^s \subseteq B_3^s \subseteq \cdots \subseteq \operatorname{im} j = \ker k \subseteq \cdots \subseteq Z_3^s \subseteq Z_2^s \subseteq Z_1^s = E^s.$$

Now, for each  $s \in \mathbb{Z}$  and  $r \geq 1$ , we define the  $A$ -graded abelian group

$$E_r^s := Z_r^s / B_r^s,$$

so that in particular  $E_1^s = E^s$  for all  $s \in \mathbb{Z}$ , as  $Z_1^s = k^{-1}(D^{s+1}) = E^s$  and  $B_1^s = j(\ker \operatorname{id}_{D^s}) = j(0) = 0$ . Now we can define differentials  $d_r^s : E_r^s \rightarrow E_r^{s+r}$  to be the composition

$$E_r^s = Z_r^s / B_r^s \xrightarrow{k} \operatorname{im}[i^{r-1} : D^{s+r} \rightarrow D^{s+1}] \xrightarrow{j \circ i^{-(r-1)}} Z_r^{s+r} / B_r^{s+r} = E_r^{s+r},$$

where given some  $e \in Z_r^s = k^{-1}(\operatorname{im} i^{r-1})$ , the first arrow takes a class  $[e] \in E_r^s$  represented by some  $e \in Z_r^s$  to the element  $k(e)$ , which lives in  $\operatorname{im} i^{r-1}$  by definition, and the second arrow takes  $i^{r-1}(d)$  to the class  $[j(d)]$ . Note the first map is well-defined, as given  $b \in B_r^s = j(\ker[i^{r-1}])$ , we

have  $k(b) = 0$ , as  $b \in \text{im } j = \ker k$ . To see the second map is well-defined, first note that given  $d \in D^{s+r}$ , that

$$k(j(d)) = 0 \in \text{im}[i^{r-1} : D^{s+2r} \rightarrow D^{s+r+1}],$$

so that

$$j(d) \in k^{-1}(\text{im}[i^{r-1} : D^{s+2r} \rightarrow D^{s+r+1}]) = Z_r^{s+r},$$

as desired, so that given  $d \in D^{s+r}$ ,  $j(d) \in Z_r^{s+r}$ , so it makes sense to discuss the class  $[j(d)] \in Z_r^{s+r}/B_r^{s+r} = E_r^{s+r}$ . Secondly, if  $i^{r-1}(d) = i^{r-1}(d')$  for some  $d, d' \in D^{s+r}$ , then

$$j(d) - j(d') = j(d - d') \in j(\ker[i^{r-1} : D^{s+r} \rightarrow D^{s+1}]) = B_r^{s+r},$$

so that  $[j(d)] = [j(d')]$  in  $E_r^{s+r}$ , as desired. It is straightforward to check that these maps are also  $A$ -graded homomorphisms, so that by unravelling definitions  $d_r^s$  is an  $A$ -graded homomorphism of degree  $\deg k - (r-1) \cdot \deg i + \deg j$  (so that in the standard case  $\deg i = 0$ ,  $d_r^s$  simply has degree  $\deg k + \deg j$ ).

These differentials square to zero, in the sense that for each  $s \in \mathbb{Z}$  and  $r \geq 1$  we have that  $d_r^{s+r} \circ d_r^s : E_r^s \rightarrow E_r^{s+2r}$  is the zero map. Indeed, suppose we are given some class  $[e] \in E_r^s$  represented by an element  $e \in E^s$ , so  $k(e) = i^{r-1}(d)$  for some  $d \in D^{s+r}$ . Then

$$d_r^{s+r}(d_r^s([e])) = d_r^{s+r}([j(d)]) = [j(i^{-(r-1)}(k(j(d))))] = [j(i^{-(r-1)}(0))] = 0,$$

where the second-to-last equality follows by the fact that  $k \circ j = 0$ . Note that by unravelling definitions,  $d_1^s = j \circ k$ .

We claim that  $\ker d_r^s = Z_{r+1}^s/B_r^s$ . First of all, let  $[e] \in E_r^s = Z_r^s/B_r^s$ , so that  $[e]$  is represented by some  $e \in E^s$  with  $k(e) = i^{r-1}(d)$  for some  $d \in D^{s+r}$ . Then if  $[e] \in \ker d_r^s$ , by definition this means  $j(d) \in B_r^{s+r} = j(\ker[i^{r-1} : D^{s+r} \rightarrow D^{s+1}])$ , so  $j(d) = j(d')$  for some  $d' \in D^{s+r}$  with  $i^{r-1}(d') = 0$ . Thus  $d - d' \in \ker j = \text{im } i$ , so there exists some  $d'' \in D^{s+r+1}$  such that  $i(d'') = d - d'$ . Then

$$k(e) = i^{r-1}(d) = i^{r-1}(i(d'') + d') = i^r(d'') + i^{r-1}(d'),$$

but since  $i^{r-1}(d') = 0$ , we have  $k(e) \in \text{im}[i^r : D^{s+r+1} \rightarrow D^{s+1}]$ , so that  $e \in Z_{r+1}^s$ , meaning  $[e] \in Z_{r+1}^s/B_r^s$ , as desired. On the other hand, suppose we are given some class  $[e] \in Z_{r+1}^s/B_r^s$ , represented by  $e \in Z_{r+1}^s$  with  $k(e) \in \text{im}[i^r : D^{s+r+1} \rightarrow D^{s+1}]$ . Then if we write  $k(e) = i^r(d) = i^{r-1}(i(d))$ , then  $d_r^s([e]) = [j(i(d))] = 0$  (since  $j \circ i = 0$ ), as asserted.

Finally, we claim that the image of  $d_r^{s-r} : E_r^{s-r} \rightarrow E_r^s$  is  $B_{r+1}^s/B_r^s$ . First, let  $e \in Z_r^{s-r}$ , so  $k(e) = i^{r-1}(d)$  for some  $d \in D^s$ . Then we'd like to show that  $d_r^s([e]) = [j(d)]$  belongs to  $B_{r+1}^s/B_r^s$ . It suffices to show that  $d \in \ker[i^r : D^s \rightarrow D^{s-r}]$ . To see this, note that

$$i^r(d) = i(i^{r-1}(d)) = i(k(e)) = 0,$$

since  $i \circ k = 0$ . Hence we've shown  $\text{im } d_r^{s-r} \subseteq B_{r+1}^s/B_r^s$ . Conversely, let  $j(d) \in B_{r+1}^s$ , so  $d \in D^s$  and  $i^r(d) = 0$ . Then we'd like to show that  $[j(d)] \in B_{r+1}^s/B_r^s$  is in the image of  $d_r^{s-r}$ . To see this, note that

$$i^r(d) = 0 \implies i^{r-1}(d) \in \ker i = \text{im } k,$$

so there exists some  $e \in E_r^{s-r}$  such that  $k(e) = i^{r-1}(d)$ , so  $e \in Z_r^{s-r}$ . Unravelling definitions, it follows that  $d_r^{s-r}([e]) = [j(d)]$ , so  $[j(d)]$  is indeed in the image of  $d_r^{s-r}$ , as desired.

To recap, we have constructed for each  $s \in \mathbb{Z}$  and  $r \geq 1$  an  $A$ -graded abelian group  $E_r^s$  along with differentials  $d_r^s : E_r^s \rightarrow E_r^{s+r}$ . Furthermore, if we take homology in the middle term of the following sequence

$$E_r^{s-r} \xrightarrow{d_r^{s-r}} E_r^s \xrightarrow{d_r^s} E_r^{s+r},$$

we get

$$\ker d_r^s / \operatorname{im} d_r^{s-r} = \frac{Z_{r+1}^s / B_r^s}{B_{r+1}^s / B_r^s} \cong Z_{r+1}^s / B_{r+1}^s = E_{r+1}^s.$$

Thus, we get a spectral sequence:

**Proposition 0.3.** *We may associate a  $\mathbb{Z} \times A$ -graded spectral sequence  $r \mapsto (E_r, d_r)$  to the  $A$ -graded unrolled exact couple  $(D, E; i, j, k)$  by defining  $E_r := \bigoplus_{s \in \mathbb{Z}} E_r^s$  and the differentials*

$$d_r : E_r \rightarrow E_r$$

*are those constructed above, which have  $\mathbb{Z} \times A$ -degree  $(r, \deg j - (r - 1) \cdot \deg i + \deg k)$ .*