In what follows, we fix an abelian group A. We assume the reader is familiar with the basic theory of modules over not-necessarily-commutative rings.

Definition 0.1. An *A-graded abelian group* is an abelian group B along with a subgroup $B_a \leq B$ for each $a \in A$ such that the canonical map

$$\bigoplus_{a \in A} B_a \to B$$

sending $(x_a)_{a \in A}$ to $\sum_{a \in A} x_a$ is an isomorphism. Given two A-graded abelian groups B and C, a homomorphism $f : B \to C$ is a *homomorphism of A-graded abelian groups*, or just an *A-graded homomorphism*, if it preserves the grading, i.e., if it restricts to a map $B_a \to C_a$ for all $a \in A$.

It is easy to see that an A-graded abelian group B is generated by its *homogeneous* elements, that is, nonzero elements $x \in B$ such that there exists some $a \in A$ with $x \in B_a$.

Remark 0.2. Clearly the condition that the canonical map $\bigoplus_{a \in A} B_a \to B$ is an isomorphism requires that $B_a \cap B_b = 0$ if $a \neq b$. In particular, given a homogeneous element $x \in B$, there exists precisely one $a \in A$ such that $x \in B_a$. We call this a the *degree* of x, and we write $|x| = a$.

Definition 0.3. An *A-graded ring* is a ring R such that its underlying abelian group R is A-graded and the multiplication map $R \times R \to R$ restricts to $R_a \times R_b \to R_{a+b}$ for all $a, b \in A$. A morphism of A-graded rings is a ring homomorphism whose underlying homomorphism of abelian groups is A-graded.

Explicitly, given an A-graded ring R and homogeneous elements $x, y \in R$, we must have $|xy| = |x| + |y|$. For example, given some field k, the ring $R = k[x, y]$ is $\mathbb{Z}^2$-graded, where given $(n, m) \in \mathbb{Z}^2$, $R_{n,m}$ is the subgroup of those monomials of the form $ax^n y^m$ for some $a \in k$.

Definition 0.4. Let R be an A-graded ring. A *left A-graded R-module* M is a left R-module M such that M is an A-graded abelian group and the action map $R \times M \to M$ restricts to a map $R_a \times M_b \to M_{a+b}$ for all $a, b \in A$. Right A-graded R-modules are defined similarly. Finally, an A-graded R-bimodule is an A-graded abelian group M which has the structure of both an A-graded left and right R-module such that given $r, s \in R$ and $m \in M$, $r \cdot (m \cdot s) = (r \cdot m) \cdot s$.

Morphisms between A-graded R-modules are precisely A-graded R-module homomorphisms. We write R-**GrMod** for the category of left A-graded R-modules and **GrMod**-R for the category of right A-graded R-modules.

Remark 0.5. It is straightforward to see that an A-graded abelian group is equivalently an A-graded $\mathbb{Z}$-module, where here we are considering $\mathbb{Z}$ as an A-graded ring concentrated in degree 0. Thus any result below about A-graded modules applies equally to A-graded abelian groups.

Lemma 0.6. *Given an A-graded ring R and two left (resp. right) A-graded R-modules M and N, their direct sum $M \oplus N$ is naturally a left (resp. right) A-graded R-module group by defining*

$$(M \oplus N)_a := M_a \oplus N_a.$$

Proof. The canonical map $\bigoplus_{a \in A}(M_a \oplus N_a) \to M \oplus N$ factors as

$$\bigoplus_{a \in A}(M_a \oplus N_a) \xrightarrow{\cong} \bigoplus_{a \in A} M_a \oplus \bigoplus_{a \in A} N_a \xrightarrow{\cong} M \oplus N. \qquad \square$$

Oftentimes when constructing A-graded rings, we do so only by defining the product of homogeneous elements, like so:

Lemma 0.7. *Suppose we have an A-graded abelian group R, a distinguished element $1 \in R_0$, and $\mathbb{Z}$-bilinear maps $m_{a,b} : R_a \times R_b \to R_{a+b}$ for all $a, b \in A$. Further suppose that for all $x \in R_a$, $y \in R_b$, and $z \in R_c$, we have*

$$m_{a+b,c}(m_{a,b}(x,y),z) = m_{a,b+c}(x, m_{b,c}(y,z)) \qquad \textit{and} \qquad m_{a,0}(x,1) = m_{0,a}(1,x) = x.$$

Then there exists a unique multiplication map $m : R \times R \to R$ which endows R with the structure of an A-graded ring and restricts to $m_{a,b}$ for all $a, b \in A$.

Proof. Given $r, s \in R$, since $R \cong \bigoplus_{a \in A} R_a$, we may uniquely decompose r and s into homogeneous elements as $r = \sum_{a \in A} r_a$ and $s = \sum_{a \in A} s_a$ with each $r_a, s_a \in R_a$ such that only finitely many of the r_a's and s_a's are nonzero. Then in order to define a distributive product $R \times R \to R$ which restricts to $m_{a,b} : R_a \times R_b \to R_{a+b}$, note we *must* define

$$r \cdot s = \left(\sum_{a \in A} r_a\right) \cdot \left(\sum_{b \in A} s_b\right) = \sum_{a,b \in A} r_a \cdot s_b = \sum_{a,b \in A} m_{a,b}(r_a, s_b).$$

Thus, we have shown uniqueness. It remains to show this product actually gives R the structure of a ring. First we claim that the sum on the right is actually finite. Note there exists only finitely many nonzero r_a's and s_b's, and if $s_b = 0$ then

$$m_{a,b}(r_a, 0) = m_{a,b}(r_a, 0+0) \stackrel{(*)}{=} m_{a,b}(r_a, 0) + m_{a,b}(r_a, 0) \implies m_{a,b}(r_a, 0) = 0,$$

where $(*)$ follows from bilinearity of $m_{a,b}$. A similar argument yields that $m_{a,b}(0, s_b) = 0$ for all $a, b \in A$. Hence indeed $m_{a,b}(r_a, s_b)$ is zero for all but finitely many pairs $(a, b) \in A^2$, as desired. Observe that in particular

$$(r \cdot s)_a = \sum_{b+c=a} m_{b,c}(r_b, s_c) = \sum_{b \in A} m_{b,a-b}(r_b, s_{a-b}) = \sum_{c \in A} m_{a-c,c}(r_{a-c}, s_c).$$

Now we claim this multiplication is associative. Given $t = \sum_{a \in A} t_a \in R$, we have

$$\begin{aligned}
(r \cdot s) \cdot t &= \sum_{a,b \in A} m_{a,b}((r \cdot s)_a, t_b) \\
&= \sum_{a,b \in A} m_{a,b}\left(\sum_{c \in A} m_{a-c,c}(r_{a-c}, s_c), t_b\right) \\
&\stackrel{(1)}{=} \sum_{a,b,c \in A} m_{a,b}(m_{a-c,c}(r_{a-c}, s_c), t_b) \\
&\stackrel{(2)}{=} \sum_{a,b,c \in A} m_{c,a+b-c}(r_c, m_{a-c,b}(s_{a-c}, t_b)) \\
&\stackrel{(3)}{=} \sum_{a,b,c \in A} m_{a,c}(r_a, m_{b,c-b}(s_b, t_{c-b})) \\
&\stackrel{(1)}{=} \sum_{a,c \in A} m_{a,c}\left(r_a, \sum_{b \in A} m_{b,c-b}(s_b, t_{c-b})\right) \\
&= \sum_{a,c \in A} m_{a,c}(r_a, (s \cdot t)_c) = r \cdot (s \cdot t),
\end{aligned}$$

where each occurrence of (1) follows by bilinearity of the $m_{a,b}$'s, each occurrence of (2) is associativity of the $m_{a,b}$'s, and (3) is obtained by re-indexing by re-defining $a := c$, $b := a - c$, and

$c := a+b-c$. Next, we wish to show that the distinguished element $1 \in R_0$ is a unit with respect to this multiplication. Indeed, we have

$$1 \cdot r \overset{(1)}{=} \sum_{a\in A} m_{0,a}(1, r_a) \overset{(2)}{=} \sum_{a\in A} r_a = r \qquad \text{and} \qquad r \cdot 1 \overset{(1)}{=} \sum_{a\in A} m_{a,0}(r_a, 1) \overset{(2)}{=} \sum_{a\in A} r_a = r,$$

where (1) follows by the fact that $m_{a,b}(0,-) = m_{a,b}(-,0) = 0$, which we have shown above, and (2) follows by unitality of the $m_{0,a}$'s and $m_{0,a}$'s, respectively. Finally, we wish to show that this product is distributive. Indeed, we have

$$\begin{aligned} r \cdot (s+t) &= \sum_{a,b\in A} m_{a,b}(r_a, (s+t)_b) \\ &= \sum_{a,b\in A} m_{a,b}(r_a, s_b + t_b) \\ &\overset{(*)}{=} \sum_{a,b\in A} m_{a,b}(r_a, s_b) + \sum_{a,b\in A} m_{a,b}(r_a, t_b) = (r\cdot s) + (r \cdot t), \end{aligned}$$

where $(*)$ follows by bilinearity of $m_{a,b}$. An entirely analagous argument yields that $(r+s)\cdot t = (r\cdot t) + (s \cdot t)$. $\square$

Lemma 0.8. *Let R be an A-graded ring, M an A-graded abelian group, and suppose there exists $\mathbb{Z}$-bilinear maps $\kappa_{a,b} : R_a \times M_b \to M_{a+b}$ for all $a, b \in A$. Further suppose that for all $r \in R_a$, $r' \in R_b$, and $m \in M_c$ that*

$$\kappa_{a+b,c}(r \cdot r', m) = \kappa_{a,b+c}(r, \kappa_{b,c}(r', m)) \qquad \textit{and} \qquad \kappa_{0,c}(1, m) = m.$$

Then there is a unique map $\kappa : R \times M \to M$ which endows M with the structure of a left A-graded R-module and restricts to $\kappa_{a,b}$ for all $a, b \in A$.

On the other hand, suppose there exists $\mathbb{Z}$-bilinear maps $\kappa_{a,b} : M_a \times R_b \to M_{a+b}$ for all $a, b \in A$. Further suppose that for all $r \in R_a$, $r' \in R_b$, and $m \in M_c$ that

$$\kappa_{c,a+b}(m, r \cdot r') = \kappa_{c+a,b}(\kappa_{c,a}(m, r), r') \qquad \textit{and} \qquad \kappa_{c,0}(m, 1) = m.$$

Then there is a unique map $\kappa : M \times R \to M$ which endows M with the structure of a right A-graded R-module and restricts to $\kappa_{a,b}$ for all $a, b \in A$.

Proof. We show the left module case, as the right module case is entirely analagous. Supposing for each $a, b \in A$ we have a map $\kappa_{a,b} : R_a \times M_b \to M_{a+b}$ satisfying the above conditions, in order to extend these to a map $R \times M \to M$, by additivity we *must* define

$$\kappa : R \times M \to M$$

to be the map sending $r = \sum_a r_a$ and $m = \sum_a m_a$ to $\sum_{a,b\in A} \kappa_{a,b}(r_a, m_b)$. Now, we need to check that for all $r, s \in R$, $x, y \in M$ that

(1) $r \cdot (x + y) = r \cdot x + r \cdot y$

(2) $(r + s) \cdot x = r \cdot x + s \cdot x$

(3) $(rs) \cdot x = r \cdot (s \cdot x)$

(4) $1 \cdot x = x$,

where above we are written $- \cdot -$ for $\kappa(-,-)$. To see the first, note

$$\begin{aligned}
\kappa(r, x+y) &= \sum_{a,b\in A} \kappa_{a,b}(r_a, (x+y)_b) \\
&= \sum_{a,b\in A} \kappa_{a,b}(r_a, x_b + y_b) \\
&= \sum_{a,b\in A} (\kappa_{a,b}(r_a, x_b) + \kappa_{a,b}(r_a, y_b)) \\
&= \sum_{a,b\in A} \kappa_{a,b}(r_a, x_b) + \sum_{a,b\in A} \kappa_{a,b}(r_a, y_b) \\
&= \kappa(r, x) + \kappa(r, y).
\end{aligned}$$

To see the second, note

$$\begin{aligned}
\kappa(r+s, x) &= \sum_{a,b\in A} \kappa_{a,b}((r+s)_a, x_b) \\
&= \sum_{a,b\in A} \kappa_{a,b}(r_a + s_a, x_b) \\
&= \sum_{a,b\in A} (\kappa_{a,b}(r_a, x_b) + \kappa_{a,b}(s_a, x_b)) \\
&= \sum_{a,b\in A} \kappa_{a,b}(r_a, x_b) + \sum_{a,b\in A} \kappa_{a,b}(s_a, x_b) \\
&= \kappa(r, x) + \kappa(s, x).
\end{aligned}$$

To see the third, note

$$\begin{aligned}
\kappa(rs, x) &= \sum_{a,b\in A} \kappa_{a,b}((rs)_a, x_b) \\
&= \sum_{a,b\in A} \kappa_{a,b}\left(\sum_{c\in A} r_c s_{a-c}, x_b\right) \\
&= \sum_{a,b,c\in A} \kappa_{a,b}(r_c s_{a-c}, x_b) \\
&= \sum_{a,b,c\in A} \kappa_{a,b}(r_c, \kappa_{a-c,b}(s_{a-c}, x_b)) \\
&=
\end{aligned}$$

FINISH

$\square$

When working with A-graded rings and modules, we will often freely use the above propositions without comment. In what follows, fix an A-graded ring R. We will simply say "A-graded R-module" when we are freely considering either left or right A-graded R-modules.

Remark 0.9. We often will denote an A-Braded R-module M by M_*. Given some $a \in A$, we can define the shifted A-graded abelian group M_{*+a} whose b^{th} component is M_{b+a}.

Definition 0.10. More generally, given two A-graded R-modules M and N and some $d \in A$, an R-module homomorphism $f : M \to N$ is *an A-graded homomorphism of degree d* if it restricts to a map $M_a \to N_{a+d}$ for all $a \in A$. Thus, an A-graded homomorphism of degree d from M

to N is equivalently an A-graded homomorphism $M_* \to N_{*+d}$ or an A-graded homomorphism $M_{*-d} \to N$. Given some $a \in A$ and left (resp. right) R-modules M and N, we will write

$$\mathrm{Hom}_R^d(M,N) = \mathrm{Hom}_R(M_*, N_{*+d}) = \mathrm{Hom}_R(M_{*-d}, N_*)$$

to denote the set of A-graded homomorphisms of degree d from M to N, and simply

$$\mathrm{Hom}_R(M,N)$$

to denote the set of degree-0 A-graded homomorphisms from M to N. Clearly A-graded homomorphisms may be added and subtracted, so these are further abelian groups. Thus we have an A-graded abelian group

$$\mathrm{Hom}_R^*(M,N).$$

Unless stated otherwise, an "A-graded homomorphism" will always refer to an A-graded homomorphism of degree 0.

Lemma 0.11. *Let R be an A-graded ring and M an A-graded left (resp. right) R-module. Then for all $d \in A$, the evaluation map*

$$\begin{aligned} \mathrm{ev}_1 : \mathrm{Hom}_R^d(R,M) &\to M_d \\ \varphi &\mapsto \varphi(1) \end{aligned}$$

is an isomorphism of abelian groups.

Proof. We consider the case that M is a left A-graded R-module, as showing it when M is a right module is entirely analagous. First of all, this map is clearly a homomorphism, as given degree d A-graded homomorphisms $\varphi, \psi : R \to M$, we have

$$\mathrm{ev}_1(\varphi + \psi) = (\varphi + \psi)(1) = \varphi(1) + \psi(1) = \mathrm{ev}_1(\varphi) + \mathrm{ev}_1(\psi).$$

Now, to see it is surjective, let $m \in M_d$, and define $\varphi_m : R \to M$ to send $r \mapsto r \cdot m$. First of all, φ_m is a module homomorphism, as given $r, s \in R$,

$$\varphi_m(r+s) = (r+s)\cdot m = r\cdot m + s \cdot m = \varphi_m(r) + \varphi_m(s) \quad \text{and} \quad \varphi_m(r \cdot s) = r \cdot s \cdot m = r \cdot \varphi_m(s).$$

Furthermore, it is clearly A-graded of degree d, as given a homogeneous element $r \in R_a$ for some $a \in A$, we have $\varphi_m(r) = r \cdot m \in R_{a+d}$, since m is homogeneous of degree d. Finally, clearly

$$\mathrm{ev}_1(\varphi_m) = \varphi_m(1) = 1 \cdot m = m,$$

so indeed ev_1 is surjective. On the other hand, to see it is injective, suppose we are given $\varphi, \psi \in \mathrm{Hom}_R^d(R,M)$ such that $\varphi(1) = \psi(1)$. Then given $r \in R$, we must have

$$\varphi(r) = \varphi(r \cdot 1) = r \cdot \varphi(1) = r \cdot \psi(1) = \psi(r \cdot 1) = \psi(r),$$

so φ and ψ are exactly the same map. Thus, ev_1 is injective, as desired. $\square$

Recall that given a ring R, a left (resp. right) module P is *projective* if, for all diagrams of R-module homomorphisms of the form

$$\begin{tikzcd} & M \arrow[d, two heads, "g"] \\ P \arrow[r, "f"'] & N \end{tikzcd}$$

with g an epimorphism, there exists a lift $h : P \to M$ satisfying $g \circ h = f$

$$\begin{tikzcd} & M \arrow[d, two heads, "g"] \\ P \arrow[r, "f"'] \arrow[ur, dashed, "h"] & N \end{tikzcd}$$

(Note h is not required to be unique.)

Definition 0.12. Let R be an A-graded ring, and let P be a left (resp. right) A-graded R-module. Then P is a *graded projective* module if, for all diagrams of A-graded R-module homomorphisms of the form

$$\begin{tikzcd} & M \arrow[d, two heads, "g"] \\ P \arrow[r, "f"'] & N \end{tikzcd}$$

with g an epimorphism, there exists an A-graded homomorphism $h : P \to M$ satisfying $g \circ h = f$.

$$\begin{tikzcd} & M \arrow[d, two heads, "g"] \\ P \arrow[r, "f"'] \arrow[ur, dashed, "h"] & N \end{tikzcd}$$

(Note h is not required to be unique.)

Definition 0.13. Let M be an A-graded R-module. Then an *A-graded R-submodule* is an A-graded R-module N which is a subset of M and for which the inclusion $N \hookrightarrow M$ is an A-graded homomorphism of R-modules. Equivalently, it is a submodule N one for which the canonical map

$$\bigoplus N \cap M_a \to N$$

is an isomorphism.

Lemma 0.14. *Let M be an A-graded R-module. Then an R-submodule $N \leq M$ is an A-graded submodule if and only if it is generated as an R-module by homogeneous elements of M.*

Proof. If $N \leq M$ is a A-graded submodule, it is generated by the set of all its homogeneous elements, which are also homogeneous elements in M, by definition.

Conversely, suppose $N \leq M$ is a submodule which is generated by homogeneous elements of M. Then define $N_a := N \cap M_a$, and consider the canonical map

$$\Phi : \bigoplus_{a \in A} N_a \to N.$$

First of all, it is surjective, as each generator of N belongs to some N_a, by definition. To see it is injective, consider the following commutative diagram:

$$\begin{tikzcd} \bigoplus_{a\in A} N_a \arrow[r, hook] \arrow[d, "\Phi"'] & \bigoplus_{a\in A} M_a \arrow[d, "\cong"] \\ N \arrow[r, hook] & M \end{tikzcd}$$

Since Φ composes with an injection to get an injection, clearly Φ must be injective itself. We have the desired result. $\square$

Proposition 0.15. *Given two left (resp. right) A-graded R-modules M and N and an A-graded R-module homomorphism $\varphi : M \to N$ (of possibly nonzero degree), the kernel and images of φ are A-graded submodules of M and N, respectively.*

Proof. First recall that a degree d A-graded homomorphism $M \to N$ is simply an A-graded homomorphism $M_* \to N_{*+d}$, so it suffices to consider the case φ is of degree 0. Next, note that since the forgetful functor from R-modules to abelian groups preserves kernels and images, it suffices to consider the case that φ is a homomorphism of A-graded abelian groups. Finally, by Lemma 0.14, it suffices to show that $\ker \varphi$ and $\operatorname{im} \varphi$ are generated by homogeneous elements of M and N, respectively.

Note that by the universal property of the coproduct in **Ab**, the data of an A-graded homomorphism of abelian groups $\varphi : M \to N$ is precisely the data of an A-indexed collection of abelian group homomorphisms $\varphi_a : M_a \to N_a$, in which case the following diagram commutes:

$$\begin{tikzcd} \bigoplus_a M_a \arrow[r, "\bigoplus_a \varphi_a"] \arrow[d, "\cong"'] & \bigoplus_a N_a \arrow[d, "\cong"] \\ M \arrow[r, "\varphi"] & N \end{tikzcd}$$

Finally, the desired result follows by the purely formal fact that taking images and kernels commutes with arbitrary direct sums. $\square$

Proposition 0.16. *Given two left (resp. right) A-graded R-modules M and N, an A-graded submodule $K \leq N$, and an A-graded R-module homomorphism $\varphi : M \to N$ (of possibly nonzero degree), the submodule $\varphi^{-1}(K)$ of M is A-graded.*

Proof. Recall that a degree d A-graded homomorphism $M \to N$ is simply an A-graded homomorphism $M_* \to N_{*+d}$, so it suffices to consider the case φ is of degree 0. Now, let $x \in L := \varphi^{-1}(K)$. As an element of M, we may uniquely write $x = \sum_{a \in A} x_a$ where each $x_a \in M_a$. Similarly, if we set $y := \varphi(x)$, then we may uniquely write $y = \sum_{a \in A} y_a$ where each $y_a \in N_a$. Then since K is an A-graded submodule of N and $y \in K$, by definition, we have that $y_a \in K$ for each a. Finally, note that

$$\sum_{a \in A} y_a = y = \varphi(x) = \sum_{a \in A} \varphi(x_a),$$

so that $\varphi(x_a) = y_a \in K$ for all $a \in A$, so that $x_a \in L$ for all $a \in A$. Thus we have shown that each element in L can be written as a sum of homogeneous elements in M, as desired. $\square$

Proposition 0.17. *Given an A-graded R-module M and an A-graded subgroup $N \leq M$, the quotient M/N is canonically A-graded by defining $(M/N)_a$ to be the subgroup generated by cosets represented by homogeneous elements of degree a in M. Furthermore, the canonical maps $M_a/N_a \to (M/N)_a$ taking a coset $m + N_a$ to $m + N$ are isomorphisms.*

Proof. Consider the canonical map

$$\Phi : \bigoplus_a (M/N)_a \to M/N.$$

First of all, surjectivity of Φ follows by commutativity of the following diagram:

$$\begin{tikzcd} \bigoplus_a M_a \arrow[r, "\cong"] \arrow[d, two heads] & M \arrow[d, two heads] \\ \bigoplus_a (M/N)_a \arrow[r, "\Phi"] & M/N \end{tikzcd}$$

where the vertical left map sends a generator $m \in M_a$ to the coset $m + N$ in $(M/N)_a \subseteq M/N$. To see Φ is injective, suppose we are given some element $(m_a + N)_{a\in A}$ in $\bigoplus_a (M/C)_a$ such that $\sum_{a\in A}(m_a + N) = 0$ in M/N. Thus $\sum_{a\in A} m_a \in N$, and since N is A-graded this implies that each m_a belongs to $N \cap M_a = N_a$, so that in particular $m_a + N$ is zero in $(M/N)_a \subseteq M/N$, so that $(m_a + N)_{a\in A} = 0$ in $\bigoplus_a (M/N)_a$, as desired.

It remains to show that the canonical map

$$\varphi_a : M_a/N_a \to (M/N)_a$$

is an isomorphism. It is clearly surjective, as $(M/N)_a$ is generated by elements $m + N$ for $m \in M_a$, and these elements make up precisely the image of φ_a. Thus φ_a hits every generator of $(M/N)_a$, so φ_a is surjective. On the other hand, suppose we are given some $m \in M_a$ such that $\varphi(m + N_a) = m + N = 0$. Thus $m \in N$, and $m \in M_a$, so that $m \in M_a \cap N = N_a$, meaning $m + N_a = 0$ in M_a/N_a, as desired. $\square$

Recall that given a ring R, a left R-module M, a right R-module N, and an abelian group A, an *R-balanced map* $\varphi : M \times N \to B$ is one which satisifies

$$\begin{aligned} \varphi(m, n + n') &= \varphi(m, n) + \varphi(m, n') \\ \varphi(m + m', n) &= \varphi(m, n) + \varphi(m', n) \\ \varphi(m \cdot r, n) &= \varphi(m, r \cdot n). \end{aligned}$$

for all $m, m' \in M$, $n, n' \in N$, and $r \in R$. Then the tensor product $M \otimes_R N$ is the universal abelian group equipped with an R-balanced map $\otimes : M \times N \to M \otimes_R N$ such that for every abelian group B and every R-balanced map $\varphi : M \times N \to B$, there is a *unique* group homomorphism $\widetilde{\varphi} : M \otimes_R N \to B$ such that $\widetilde{f} \circ \otimes = f$. We call elements in the image of $\otimes : M \times N \to M \otimes_R N$ *pure tensors*. It is a standard fact that $M \otimes_R N$ is generated as an abelian group by its pure tensors.

Definition 0.18. Suppose we have a right A-graded R-module M, a left A-graded R-module N, and an A-graded abelian group B. Then an *A-graded R-balanced map* $\varphi : M \times N \to B$ is an R-balanced map which restricts to $M_a \times N_b \to B_{a+b}$ for all $a, b \in A$.

Proposition 0.19. *Suppose we have a right A-graded R-module M and a left A-graded R-module N. Then the tensor product*

$$M \otimes_R N$$

is naturally an A-graded abelian group by defining $(M \otimes_R N)_a$ to be the subgroup generated by homogeneous *pure tensors $m \otimes n$ with $m \in M_b$ and $n \in N_c$ such that $b + c = a$. Furthermore, if either M (resp. N) is an A-graded bimodule, then this decomposition makes $M \otimes_R N$ into a left (resp. right) A-graded R-module*

Proof. By definition, since M and N are A-graded abelian groups, they are generated (as abelian groups) by their homogeneous elements. Thus it follows that $M \otimes_R N$ is generated by *homogeneous pure tensors*, that is, elements of the form $m \otimes n$ with $m \in M$ and $n \in N$ homogeneous. Now, given a homogeneous pure tensor $m \otimes n$, we define its *degree* by the formula $|m \otimes n| := |m| + |n|$. It follows this formula is well-defined by checking that given homogeneous elements $m \in M$, $n \in N$, and $r \in R$ that

$$|(m \cdot r) \otimes n| = |m \cdot r| + |n| = |m| + |r| + |n| = |m| + |r \cdot n| = |m \otimes (r \cdot n)|.$$

Thus, we may define $(M \otimes_R N)_a$ to be the subgroup of $M \otimes_R N$ generated by those pure homogeneous tensors of degree a. Now, consider the map

$$\Psi : M \times N \to \bigoplus_{a \in A} (M \otimes_R N)_a$$

which takes a pair $(m, n) = \sum_{a \in A} (m_a, n_a)$ to the element $\Psi(m, n)$ whose a^{th} component is

$$(\Psi(m, n))_a := \sum_{b+c=a} m_b \otimes n_c.$$

It is straightforward to see that this map is R-balanced, in the sense that it is additive in each argument and $\Psi(m \cdot r, n) = \Psi(m, r \cdot n)$ for all $m \in M$, $n \in N$, and $r \in R$. Thus by the universal property of $M \otimes_R N$, we get a homomorphism of abelian groups $\widetilde{\Psi} : M \otimes_R N \to \bigoplus_{a \in A} (M \otimes_R N)_a$ lifting Ψ along the canonical map $M \times N \to M \otimes_R N$. Now, also consider the canonical map

$$\Phi : \bigoplus_{a \in A} (M \otimes_R N)_a \to M \otimes_R N.$$

We would like to show $\widetilde{\Psi}$ and Φ are inverses of eah other. Since $\widetilde{\Psi}$ and Φ are both homomorphisms, it suffices to show this on generators. Let $m \otimes n$ be a homogeneous pure tensor with $m = m_a \in M_a$ and $n = n_b \in N_b$. Then we have

$$\Phi(\widetilde{\Psi}(m \otimes n)) = \Phi\left(\bigoplus_{a \in A} \sum_{b+c=a} m_b \otimes n_c \right) \stackrel{(*)}{=} \Phi(m \otimes n) = m \otimes n,$$

and

$$\widetilde{\Psi}(\Phi(m \otimes n)) = \widetilde{\Psi}(m \otimes n) = \bigoplus_{a \in A} \sum_{b+c=a} m_b \otimes n_c \stackrel{(*)}{=} m \otimes n,$$

where both occurrences of $(*)$ follow by the fact that $m_b \otimes n_c = 0$ unless $b = c = a$, in which case $m_a \otimes n_a = m \otimes n$. Thus since Φ is an isomorphism, $M \otimes_R N$ is indeed an A-graded abelian group, as desired.

Now, suppose that M is an A-graded R-bimodule, so there exists left and right A-graded actions of R on M such that given $r, s \in R$ and $m \in M$ we have $r \cdot (m \cdot s) = (r \cdot m) \cdot s$. Then we would like to show that given a left A-graded R-module N that $M \otimes_R N$ is canonically a left A-graded R-module. Indeed, define the action of R on $M \otimes_R N$ on pure tensors by the formula

$$r \cdot (m \otimes n) = (r \cdot m) \otimes n.$$

First of all, clearly this map is A-graded, as if $r \in R_a$, $m \in M_b$, and $n \in N_c$ then $(r \cdot m) \otimes n$, by definition, has degree $|r \cdot m| + |n| = |r| + |m| + |n|$ (the last equality follows since the left action of R on M is A-graded). In order to show the above map defines a left module structure, it suffices to show that given pure tensors $m \otimes n, m' \otimes n' \in M \otimes_R N$ and elements $r, r' \in R$ that

(1) $r \cdot (m \otimes n + m' \otimes n') = r \cdot (m \otimes n) + r \cdot (m' \otimes n')$,

(2) $(r + r') \cdot (m \otimes n) = r \cdot (m \otimes n) + r' \cdot (m' \otimes n')$,

(3) $(rr') \cdot (m \otimes n) = r \cdot (r' \cdot (m \otimes n))$, and

(4) $1 \cdot (m \otimes n) = m \otimes n$.

Axiom (1) holds by definition. To see (2), note that by the fact that R acts on M on the left that

$$(r + r') \cdot (m \otimes n) = ((r + r') \cdot m) \otimes n = (r \cdot m + r' \cdot m) \otimes n = r \cdot m \otimes n + r' \cdot m \otimes n.$$

That (3) and (4) hold follows similarly by the fact that $(rr') \cdot m = r \cdot (r' \cdot m)$ and $1 \cdot m = m$.

Conversely, if N is an A-graded R-bimodule, then showing $M \otimes_R N$ is canonically a right A-graded R-module via the rule

$$(m \otimes n) \cdot r = m \otimes (n \cdot r)$$

is entirely analagous. □

Lemma 0.20. *Let R be an A-graded ring, and suppose we have a right A-graded R-module M and a left A-graded R-module N. Then given an A-graded abelian group B and an A-graded R-balanced map*

$$\varphi : M \times N \to B,$$

the lift

$$\widetilde{\varphi} : M \otimes_R N \to B$$

determined by the universal property of $M \otimes_R N$ is an A-graded homomorphism.

Proof. This simply amounts to unravelling definitions. Recall that the subgroup of homogeneous elements of degree a in $M \otimes_R N$ is that generated by pure tensors $m \otimes n$ with m and n homogeneous satisfying $|m| + |n| = a$. Thus, in order to show $\widetilde{\varphi}$ is an A-graded homomorphism, it suffices to show that given homogeneous $m \in M$ and $n \in N$ that $\widetilde{\varphi}(m \otimes n)$ is homogeneous and that

$$|\widetilde{\varphi}(m \otimes n)| = |m \otimes n| = |m| + |n|.$$

Indeed, given two such elements m and n, consider the following diagram

$$\begin{tikzcd} M \otimes_R N \arrow[dr, "\widetilde{\varphi}"] & \\ M \times N \arrow[u] \arrow[r, "\varphi"'] & B \end{tikzcd}$$

This diagram commutes by universal property of $- \otimes_R -$. Note that the element $m \otimes n$ is mapped to by the pair (m, n) along the left vertical map. Hence by commutativity, we necessarily have

$$|\widetilde{\varphi}(m \otimes n)| = |\varphi(m, n)| \overset{(*)}{=} |m| + |n|,$$

where $(*)$ follows by the fact that φ is an A-graded R-balanced map. □