0.1. **Setup.** In order to construct an abstract version of the Adams spectral sequence, we need to work in some axiomatic version of a stable homotopy category $\mathcal{SH}$ which acts like the familiar classical stable homotopy category **hoSp** (**??**) or the motivic stable homotopy category $\mathbf{SH}_{\mathscr{S}}$ over some base scheme $\mathscr{S}$ (**??**):

Definition 0.1. A *stable homotopy category* $\mathcal{SH}$ is the following data:

- A closed tensor triangulated category $(\mathcal{SH}, \otimes, S, \Sigma, \mathcal{D})$ with arbitrary (small) (co)products.
- A pointed abelian group $(A, \mathbf{1})$ along with a homomorphism of pointed groups $h : (A, \mathbf{1}) \to (\operatorname{Pic}\mathcal{SH}, \Sigma S)$.
- For each $a \in A$, a chosen representative S^a in the isomorphism class $h(a)$ such that $S^0 = S$.
- For each $a, b \in A$, an isomorphism $\phi_{a,b} : S^{a+b} \to S^a \otimes S^b$. This family of isomorphisms is required to be *coherent*, in the following sense:
  - For all $a \in A$, we must have that $\phi_{a,0}$ coincides with the right unitor $S^a \xrightarrow{\cong} S^a \otimes S$ and $\phi_{0,a}$ coincides the left unitor $S^a \xrightarrow{\cong} S \otimes S^a$.
  - For all $a, b, c \in A$, the following diagram must commute:

$$\begin{tikzcd}
S^{a+b} \otimes S^c \arrow[d, "\phi_{a,b}\otimes S^c"'] & S^{a+b+c} \arrow[l, "\phi_{a+b,c}"'] \arrow[r, "\phi_{a,b+c}"] & S^a \otimes S^{b+c} \arrow[d, "S^a\otimes\phi_{b,c}"] \\
(S^a \otimes S^b) \otimes S^c \arrow[rr, "\cong"] & & S^a \otimes (S^b \otimes S^c)
\end{tikzcd}$$

From now on we fix the data given in the above definition, and we establish some conventions. First of all, given objects $X_1, \dots, X_n$ in $\mathcal{SH}$, we write $X_1 \otimes \cdots \otimes X_n$ to denote the object

$$X_1 \otimes (X_2 \otimes \cdots (X_{n-1} \otimes X_n)).$$

In particular, given an object X and a natural number $n > 0$, we write

$$X^n := \overbrace{X \otimes \cdots \otimes X}^{n \text{ times}} \qquad \text{and} \qquad X^0 := S.$$

We denote the associator, symmetry, left unitor, and right unitor isomorphisms in $\mathcal{SH}$ by

$$\alpha_{X,Y,Z} : (X \otimes Y) \otimes Z \xrightarrow{\cong} X \otimes (Y \otimes Z) \qquad \tau_{X,Y} : X \otimes Y \xrightarrow{\cong} Y \otimes X$$
$$\lambda_X : S \otimes X \xrightarrow{\cong} X \qquad \rho_X : X \otimes S \xrightarrow{\cong} X.$$

Note that since $S^{\mathbf{1}}$ belongs to the isomorphism class of ΣS, there exists some isomorphism $t : \Sigma S \xrightarrow{\cong} S^{\mathbf{1}}$, which we can use to construct a natural isomorphism $S^{\mathbf{1}} \otimes - \cong \Sigma$:

$$S^{\mathbf{1}} \otimes X \xrightarrow{t\otimes X} \Sigma S \otimes X \xrightarrow{e_{S,X}} \Sigma(S \otimes X) \xrightarrow{\Sigma\lambda_X} \Sigma X.$$

The last two arrows are natural in X by definition. The first arrow is natural in X by functoriality of $- \otimes -$. Furthermore, under this isomorphism $e_{X,Y} : \Sigma X \otimes Y \xrightarrow{\cong} \Sigma(X \otimes Y)$ corresponds to the associator, by commutativity of the following diagram:

$$\begin{tikzcd}
(S^{\mathbf{1}} \otimes X) \otimes Y \arrow[r, "(t\otimes X)\otimes Y"] \arrow[dd, "\alpha"] & (\Sigma S \otimes X) \otimes Y \arrow[r, "e_{S,X}\otimes Y"] \arrow[dd, "\alpha"] & \Sigma(S \otimes X) \otimes Y \arrow[r, "\Sigma\lambda_X\otimes Y"] \arrow[d, "e_{S\otimes X,Y}"'] & \Sigma X \otimes Y \arrow[dd, "e_{X,Y}"] \\
& & \Sigma((S \otimes X) \otimes Y) \arrow[d, "\Sigma\alpha"'] \arrow[dr, "\Sigma(\lambda_X\otimes Y)"] & \\
S^{\mathbf{1}} \otimes (X \otimes Y) \arrow[r, "t\otimes(X\otimes Y)"'] & \Sigma S \otimes (X \otimes Y) \arrow[r, "e_{S,X\otimes Y}"'] & \Sigma(S \otimes (X \otimes Y)) \arrow[r, "\Sigma\lambda_{X\otimes Y}"'] & \Sigma(X \otimes Y)
\end{tikzcd}$$

The left square commutes by naturality of α. Commutativity of the middle square is axiom TT4 for a tensor triangulated category. Commutativity of the right trapezoid is naturality of e. Finally the bottom triangle commutes by coherence for monoidal categories and functoriality of Σ.

Remark 0.2. In light of the above discussion, from now on we will always assume that $\Sigma = S^{\mathbf{1}} \otimes -$ and $e_{X,Y} : \Sigma X \otimes Y \xrightarrow{\cong} \Sigma(X \otimes Y)$ is the associator $\alpha : (S^{\mathbf{1}} \otimes X) \otimes Y \xrightarrow{\alpha} S^{\mathbf{1}} \otimes (X \otimes Y)$.

Given some $a \in A$, we define $\Sigma^a := S^a \otimes -$ and $\Omega^a := \Sigma^{-a} = S^{-a} \otimes -$. We specifically define $\Omega := \Omega^{\mathbf{1}}$, and we are assuming $\Sigma = \Sigma^{\mathbf{1}}$. Then it turns out that Ω^a and Σ^a form an adjoint equivalence of $\mathcal{SH}$:

Proposition 0.3. *For each $a \in A$, the isomorphisms $\eta^a_X : X \to \Sigma^a \Omega^a X$ and $\varepsilon^a_X : \Omega^a \Sigma^a X \to X$ defined in Definition 0.1 exhibit an adjoint autoequivalence $(\Omega^a, \Sigma^a, \eta^a, \varepsilon^a)$ of $\mathcal{SH}$.*

Proof. In this proof, we will freely employ the coherence theorem for monoidal categories (see [1]), which essentially tells us that we may assume we are working in a strict monoidal category (i.e., that the associators and unitors and are identities). Then η^a_X and ε^a_X become simply the maps

$$\eta^a_X : X \xrightarrow{\phi_{a,-a} \otimes X} S^a \otimes S^{-a} \otimes X \qquad \text{and} \qquad \varepsilon^a_X : S^{-a} \otimes S^a \otimes X \xrightarrow{\phi^{-1}_{-a,a} \otimes X} X.$$

That these maps are natural in X follows by functoriality of $- \otimes -$. Now, recall that in order to show that these natural isomorphisms form an *adjoint* equivalence, it suffices to show that the natural isomorphisms $\eta^a : \mathrm{Id}_{\mathcal{SH}} \Rightarrow \Omega^a \Sigma^a$ and $\varepsilon^a : \Sigma^a \Omega^a \Rightarrow \mathrm{Id}_{\mathcal{SH}}$ satisfy one of the two zig-zag identities:

$$\begin{tikzcd}
\Omega^a \arrow[r, "\Omega^a \eta^a"] \arrow[dr, equal] & \Omega^a \Sigma^a \Omega^a \arrow[d, "\varepsilon^a \Omega^a"] \\
& \Omega^a
\end{tikzcd}
\qquad
\begin{tikzcd}
\Sigma^a \Omega^a \Sigma^a \arrow[d, "\Sigma^a \varepsilon^a"'] & \Sigma^a \arrow[l, "\eta^a \Sigma^a"'] \arrow[dl, equal] \\
\Sigma^a &
\end{tikzcd}$$

(see [2, Lemma 3.2]). We will show that the left is satisfied. Unravelling definitions, we simply wish to show that the following diagram commutes for all X in $\mathcal{SH}$:

$$\begin{tikzcd}
S^{-a} \otimes X \arrow[r, "S^{-a} \otimes \phi_{a,-a} \otimes X"] \arrow[dr, equal] & S^{-a} \otimes S^a \otimes S^{-a} \otimes X \arrow[d, "\phi^{-1}_{-a,a} \otimes S^{-a} \otimes X"] \\
& S^{-a} \otimes X
\end{tikzcd}$$

Yet this is simply the diagram obtained by applying $- \otimes X$ to the associativity coherence diagram for the $\phi_{a,b}$'s, so it does commute, as desired. ☐

Given two objects X and Y in $\mathcal{SH}$, we extend the abelian group $[X, Y]$ into an A-graded abelian group $[X, Y]_*$ by defining $[X, Y]_a := [S^a \otimes X, Y]$.

- Given some $a \in A$, we will define $\Sigma^a := S^a \otimes -$ and $\Omega^a := \Sigma^{-a} = S^{-a} \otimes -$, so that in particular $\Sigma = \Sigma^{\mathbf{1}}$.
- Given two objects X and Y, we denote the hom-abelian group of morphisms from X to Y in $\mathcal{SH}$ by $[X, Y]$, and we denote the internal hom object by $F(X, Y)$. We will often refer to morphisms in $\mathcal{SH}$ as *classes*, as we will think of them as representing homotopy classes of maps.
- Given two objects X and Y in $\mathcal{SH}$, we may extend the abelian group $[X, Y]$ to an A-graded abelian group $[X, Y]_*$ defined by

$$[X, Y]_a := [\Sigma^a X, Y] = [S^a \otimes X, Y].$$

(See **??** for a review of the theory of A-graded abelian groups, rings, modules, etc.)

- Given an object X in $\mathcal{SH}$ and some $a \in A$, define the abelian group
$$\pi_a(X) := [S^a, X],$$
and write $\pi_*(X)$ for the associated A-graded abelian group $\bigoplus_{a \in A} \pi_a(X)$. We call $\pi_a(X)$ the a^{th} *stable homotopy group of* X.
- Given two objects E and X in $\mathcal{SH}$, we define the A-graded abelian groups $E_*(X)$ and $E^*(X)$ by
$$E_a(X) := \pi_a(E \otimes X) = [S^a, E \otimes X] \qquad \text{and} \qquad E^a(X) := [X, S^a \otimes E].$$
We refer to the functor $E_*(-)$ as the *homology theory represented by* E, or just E-homology, and we refer to $E^*(-)$ as the *cohomology theory represented by* E, or just E-cohomology.

From now on, we fix the data of a stable homotopy category $\mathcal{SH}$ given above once and for all. We first would like to make some remarks on the above definition. To start with, note that

Remark 0.4. Note that by induction the coherence conditions say that given any $a_1, \ldots, a_n \in A$ and $b_1, \ldots, b_m \in A$ such that $a_1 + \cdots + a_n = b_1 + \cdots + b_m$ and any fixed parenthesizations of $X = S^{a_1} \otimes \cdots \otimes S^{a_b}$ and $Y = S^{b_1} \otimes \cdots \otimes S^{b_m}$, there is a *unique* isomorphism $X \to Y$ that can be obtained by forming formal compositions of products of $\phi_{a,b}$, identities, associators, and their inverses.

Of course, we get our desired result: $\pi_*(E)$ is indeed an A-graded ring if E is a monoid object.

Proposition 0.5. *Let (E, μ, e) be a commutative monoid object in $\mathcal{SH}$, and consider the multiplication map $\pi_*(E) \times \pi_*(E) \to \pi_*(E)$ which sends classes $x : S^a \to E$ and $y : S^b \to E$ to the composition*
$$S^{a+b} \xrightarrow{\phi_{a,b}} S^a \otimes S^b \xrightarrow{x \otimes y} E \otimes E \xrightarrow{\mu} E.$$
Then this endows $\pi_*(E)$ with the structure of an A-graded ring with unit $e \in \pi_0(E) = [S, E]$.

Proof. See **??**. $\square$

Furthermore, it turns out that if E is a *commutative* monoid object in $\mathcal{SH}$, then $\pi_*(E)$ is "A-graded commutative," in the following sense:

Proposition 0.6. *For all $a, b \in A$ there exists an element $\theta_{a,b} \in \pi_0(S) = [S, S]$ (determined by choice of coherent family $\{\phi_{a,b}\}$) such that given any commutative monoid object (E, μ, e) in $\mathcal{SH}$, the A-graded ring structure on $\pi_*(E)$ (Proposition 0.5) has a commutativity formula given by*
$$x \cdot y = y \cdot x \cdot (e \circ \theta_{a,b})$$
for all $x \in \pi_a(E)$ and $y \in \pi_b(E)$.

Furthermore, $\theta_{0,a} = \theta_{a,0} = \mathrm{id}_S$ for all $a \in A$, so that if either x or y has degree 0, $x \cdot y = y \cdot x$.

Proof. See **??** and **??**. $\square$

The last ingredient in order to develop the Adams spectral sequence abstractly is a notion of *cellularity* in $\mathcal{SH}$:

Definition 0.7. Define the class of *cellular* objects in $\mathcal{SH}$ to be the smallest class of objects such that:

(1) For all $a \in A$, S^a is cellular.
(2) If we have a distinguished triangle
$$X \to Y \to Z \to \Sigma X (= S^{\mathbf{1}} \otimes X)$$
such that two of the three objects X, Y, and Z are cellular, than the third object is also cellular.
(3) Given a collection of cellular objects X_i indexed by some small set I, $\bigoplus_{i \in I} X_i$ is cellular.

0.2. **Construction of the Adams spectral sequence.** In what follows, let E be a commutative monoid object in $\mathcal{SH}$.

Definition 0.8. Let $\overline{E}$ be the fiber of the unit map $e : S \to E$ (**??**), and for $s \geq 0$ define

$$Y_s := \overline{E}^s \otimes Y, \qquad W_s = E \otimes Y_s = E \otimes (\overline{E}^s \otimes Y),$$

where recall for $s > 0$, $\overline{E}^s$ denotes the s-fold product parenthesized as $\overline{E} \otimes (\overline{E} \otimes \cdots (\overline{E} \otimes \overline{E}))$, and $\overline{E}^0 := S$. Then we get fiber sequences

$$Y_{s+1} \xrightarrow{i_s} Y_s \xrightarrow{j_s} W_s \xrightarrow{k_s} \Sigma Y_{s+1} (= S^{\mathbf{1}} \otimes Y_{s+1})$$

obtained by applying $- \otimes Y_s$ to the sequence

$$\overline{E} \to S \xrightarrow{e} E \to \Sigma \overline{E}$$

(and applying the necessary associator and unitor isomorphisms). These sequences can be spliced together to form the *(canonical) Adams filtration of* Y:

$$\begin{tikzcd}
\cdots \arrow[r] & Y_3 \arrow[r, "i_2"] \arrow[d, "j_3"] & Y_2 \arrow[r, "i_1"] \arrow[d, "j_2"] & Y_1 \arrow[r, "i_0"] \arrow[d, "j_1"] & Y_0 = Y \arrow[d, "j_0"] \\
& W_3 & W_2 \arrow[ul, dashed, "k_2"'] & W_1 \arrow[ul, dashed, "k_1"'] & W_0 \arrow[ul, dashed, "k_0"']
\end{tikzcd}$$

where the diagonal dashed arrows are of degree $-\mathbf{1}$ (note these triangles do NOT commute in any sense). Now we may apply the functor $[X, -]_*$, and by **??** we obtain an exact couple of $\mathbb{N} \times A$-graded abelian groups:

$$\begin{tikzcd}
{[X, Y_*]_*} \arrow[rr, "i_{**}"] & & {[X, Y_*]_*} \arrow[d, "j_{**}"] \\
& & {[X, W_*]_*} \arrow[ull, "k_{**}"]
\end{tikzcd}$$

where i_{**}, j_{**}, and k_{**} have $\mathbb{Z} \times A$-degree $(-1, 0)$, $(0, 0)$, and $(1, -\mathbf{1})$, respectively[1]. The standard argument yields an $\mathbb{N} \times A$-graded spectral sequence called from this exact couple (cf. Section 5.9 of [3]) with E_1 page given by

$$E_1^{s,a} = [X, W_s]_a$$

and r^{th} differential of $\mathbb{Z} \times A$-degree $(r, -\mathbf{1})$:

$$d_r : E_r^{s,a} \to E_r^{s+r, a-\mathbf{1}}.$$

A priori, this is all $\mathbb{N} \times A$-graded, but we regard it as being $\mathbb{Z} \times A$-graded by setting $E_r^{s,a} := 0$ for $s < 0$ and trivially extending the definition of the differentials to these zero groups. This spectral sequence is called the *E-Adams spectral sequence* for the computation of $[X, Y]_*$. The index s is called the *Adams filtration* and a is the *stem*.

[1] Explicitly, the map $k_{s,a} : [X, W_s]_a \to [X, Y_{s+1}]_{a-\mathbf{1}}$ sends a map $f : S^a \otimes X \to W_s$ to the map $S^{a-\mathbf{1}} \otimes X \to Y_{s+1}$ corresponding under the isomorphism $[X, \Sigma Y_{s+1}]_* \cong [X, Y_{s+1}]_{*-\mathbf{1}}$ to the composition $k_s \circ f : S^a \otimes X \to \Sigma Y_{s+1}$.

0.3. **Monoid objects in $\mathcal{SH}$.** We have constructed an Adams spectral sequence, but as it currently stands we do not yet know why it is useful. To start with, we'd like to provide a characterization of its E_1 and E_2 pages in terms of something more algebraic. To start, we first need to develop some theory of the algebra of monoid objects in $\mathcal{SH}$. Much of this work is entirely straightforward although tedious to verify, so we relegate most of the proofs in this section to **??**.

Proposition 0.9. *Let (E,μ,e) be a monoid object in $\mathcal{SH}$. Then $E_*(-)$ is a functor from $\mathcal{SH}$ to left A-graded $\pi_*(E)$-modules, where given some X in $\mathcal{SH}$, $E_*(X)$ may be endowed with the structure of a left A-graded $\pi_*(E)$-module via the map*

$$\pi_*(E)\times E_*(X)\to E_*(X)$$

which given $a,b\in A$, sends $x:S^a\to E$ and $y:S^b\to E\otimes X$ to the composition

$$x\cdot y: S^{a+b}\cong S^a\otimes S^b\xrightarrow{x\otimes y}E\otimes(E\otimes X)\cong(E\otimes E)\otimes X\xrightarrow{\mu\otimes X}E\otimes X.$$

Similarly, the assignment $X\mapsto X_*(E)$ is a functor from $\mathcal{SH}$ to right A-graded $\pi_*(E)$-modules, where the structure map

$$X_*(E)\times\pi_*(E)\to X_*(E)$$

sends $x:S^a\to X\otimes E$ and $y:S^b\to E$ to the composition

$$x\cdot y: S^{a+b}\cong S^a\otimes S^b\xrightarrow{x\otimes y}(X\otimes E)\otimes E\cong X\otimes(E\otimes E)\xrightarrow{X\otimes\mu}X\otimes E.$$

Finally, $E_*(E)$ is a $\pi_*(E)$-bimodule, in the sense that the left and right actions of $\pi_*(E)$ are compatible, so that given $y,z\in\pi_*(E)$ and $x\in E_*(E)$, $y\cdot(x\cdot z)=(y\cdot x)\cdot z$.

Proof. See **??**. □

Definition 0.10. Given a monoid object E in $\mathcal{SH}$, we say E is *flat* if the canonical right $\pi_*(E)$-module structure on $E_*(E)$ (see the above proposition) is that of a flat module.

0.4. **The E_1 page.** The goal of this subsection is to provide the following characterization for the E_1 page of the Adams spectral sequence:

Theorem 0.11. *Let E be a flat commutative monoid object in $\mathcal{SH}$, and let X and Y be two objects in $\mathcal{SH}$ such that $E_*(X)$ is a projective module over $\pi_*(E)$. Then for all $s\geq 0$ and $a\in A$, we have isomorphisms in the associated E-Adams spectral sequence*

$$E_1^{s,a}\cong\mathrm{Hom}^a_{E_*(E)}(E_*(X),E_*(W_s))$$

Furthermore, under these isomorphisms, the differential $d_1:E_1^{s,a}\to E_1^{s+1,a-\mathbf{1}}$ corresponds to the map

$$\mathrm{Hom}^a_{E_*(E)}(E_*(X),E_*(W_s))\to\mathrm{Hom}^{a-\mathbf{1}}_{E_*(E)}(E_*(X),E_*(X\otimes W_{s+1}))$$

which sends a map $f:E_*(X)\to E_{*+a}(W_s)$ to the composition

$$E_*(X)\xrightarrow{f}E_{*+a}(W_s)\xrightarrow{(X\otimes h_s)_*}E_{*+a-\mathbf{1}}(X\otimes Y_{s+1})\xrightarrow{(X\otimes j_{s+1})_*}E_{*+a-\mathbf{1}}(X\otimes W_{s+1}).$$

Proof. By **??**, for all $s\geq 0$ and $t,w\in\mathbb{Z}$, we have isomorphisms

$$[X,E\otimes Y_s]_{t,w}\cong\mathrm{Hom}^{t,w}_{E_*(E)}(E_*(X),E_*(E\otimes Y_s)).$$

since $W_s=E\otimes Y_s$, we have that

$$E_1^{s,(t,w)}=[X,W_s]_{t,w}\cong\mathrm{Hom}^{t,w}_{E_*(E)}(E_*(X),E_*(W_s)),$$

as desired. □

Definition 0.12. Let (E,μ,e) be a monoid object in $\mathcal{SH}$. We say E is *flat* if the canonical right $\pi_*(E)$-module structure on $E_*(E)$ is that of a flat module.

0.5. **The E_2 page.**

0.6. Convergence. convergence of spectral sequences

$$\begin{tikzcd}
S^{-(1,0)} \otimes S^{a+b+(1,0)} \arrow[dd, "S^{-(1,0)}\otimes\sigma_{a,b+(1,0)}"'] & S^{a+b} \arrow[l, "\sigma_{-(1,0),a+b+(1,0)}"'] \arrow[r, equal] \arrow[d, "\sigma_{-(1,0)+a,b+(1,0)}"'] & S^{a+b} \arrow[r, "\sigma_{a,b}"] \arrow[d, "\sigma_{-(1,0)+a,b+(1,0)}"] & S^a \otimes S^b \arrow[dd, "S^a\otimes\sigma_{-(1,0),b+(1,0)}"] \\
 & S^{-(1,0)+a} \otimes S^{b+(1,0)} \arrow[dl, "\sigma_{-(1,0),a}\otimes S^{b+(1,0)}"'] & S^{-(1,0)+a} \otimes S^{b+(1,0)} \arrow[dr, "\sigma_{a,-(1,0)}\otimes S^{b+(1,0)}"] & \\
S^{-(1,0)} \otimes S^a \otimes S^{b+(1,0)} \arrow[rrr, "\tau_{S^{-(1,0)},S^a}\otimes S^{b+(1,0)}"] \arrow[dr, "S^{-(1,0)}\otimes S^a\otimes y"] \arrow[dd, "S^{-(1,0)}\otimes x\otimes y"'] & & & S^a \otimes S^{-(1,0)} \otimes S^{b+(1,0)} \arrow[dl, "S^a\otimes S^{-(1,0)}\otimes y"'] \arrow[dd, "x\otimes S^{-(1,0)}\otimes y"] \\
 & S^{-(1,0)} \otimes S^a \otimes E \otimes Z \arrow[r, "\tau_{S^{-(1,0)},S^a}\otimes E\otimes Z"] \arrow[dl, "S^{-(1,0)}\otimes x\otimes E\otimes Z"'] & S^a \otimes S^{-(1,0)} \otimes E \otimes Z \arrow[dr, "x\otimes S^{-(1,0)}\otimes E\otimes Z"] & \\
S^{-(1,0)} \otimes E \otimes E \otimes E \otimes Z \arrow[rrr, "\tau_{S^{-(1,0)},E\otimes E}\otimes E\otimes Z"] \arrow[d, "S^{-(1,0)}\otimes E\otimes\mu\otimes Z"'] \arrow[dr, "S^{-(1,0)}\otimes E\otimes E\otimes E\otimes h"] & & & E \otimes E \otimes S^{-(1,0)} \otimes E \otimes Z \arrow[d, "E\otimes E\otimes S^{-(1,0)}\otimes E\otimes h"] \\
S^{-(1,0)} \otimes E \otimes E \otimes Z \arrow[d, "S^{-(1,0)}\otimes E\otimes E\otimes h"'] & S^{-(1,0)} \otimes E \otimes E \otimes E \otimes S^{(1,0)} \otimes X \arrow[rr, "\tau_{S^{-(1,0)},E\otimes E}\otimes E\otimes S^{(1,0)}\otimes X"] \arrow[dl, "S^{-(1,0)}\otimes E\otimes\mu\otimes S^{(1,0)}\otimes X"'] \arrow[d, "S^{-(1,0)}\otimes\tau_{E\otimes E\otimes E\otimes,S^{(1,0)}}\otimes X"] & & E \otimes E \otimes S^{-(1,0)} \otimes E \otimes S^{(1,0)} \otimes X \arrow[d, "E\otimes E\otimes S^{-(1,0)}\otimes\tau_{E,S^{\mathbf{1}}}\otimes X"] \\
S^{-(1,0)} \otimes E \otimes E \otimes S^{(1,0)} \otimes X \arrow[dd, "S^{-(1,0)}\otimes E\otimes\tau_{E,S^{(1,0)}}\otimes X"'] \arrow[ddr, "S^{-(1,0)}\otimes\tau_{E\otimes E,S^{(1,0)}}\otimes X"] & S^{-(1,0)} \otimes S^{(1,0)} \otimes E \otimes E \otimes E \otimes X \arrow[rr, "\tau_{S^{-(1,0)}\otimes S^{(1,0)},E\otimes E}\otimes E\otimes X"] \arrow[drr, "\sigma^{-1}_{-(1,0),(1,0)}\otimes E\otimes E\otimes E\otimes X"] \arrow[dd, "S^{-(1,0)}\otimes S^{(1,0)}\otimes E\otimes\mu\otimes X"] & & E \otimes E \otimes S^{-(1,0)} \otimes S^{(1,0)} \otimes E \otimes X \arrow[d, "E\otimes E\otimes\sigma^{-1}_{-(1,0),(1,0)}\otimes E\otimes X"] \\
 & & & E \otimes E \otimes E \otimes X \arrow[d, "E\otimes\mu\otimes X"] \\
S^{-(1,0)} \otimes E \otimes S^{(1,0)} \otimes E \otimes X \arrow[r, "S^{-(1,0)}\otimes\tau_{E,S^{(1,0)}}\otimes E\otimes X"'] & S^{-(1,0)} \otimes S^{(1,0)} \otimes E \otimes E \otimes X \arrow[rr, "\sigma^{-1}_{-(1,0),(1,0)}\otimes E\otimes E\otimes X"'] & & E \otimes E \otimes X
\end{tikzcd}$$