THE MOTIVIC ADAMS SPECTRAL SEQUENCE

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test

1. Introduction

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2.1. **Setup.** In order to construct an abstract version of the Adams spectral sequence, we need to work in some axiomatic version of a stable homotopy category $\mathcal{SH}$ which acts like the familiar classical stable homotopy category **hoSp** (Section 3) or the motivic stable homotopy category $\mathbf{SH}_{\mathscr{S}}$ over some base scheme $\mathscr{S}$ (Section 4):

Definition 2.1. A *stable homotopy category* $\mathcal{SH}$ is the following data:

- An additive closed symmetric monoidal category $(\mathcal{SH}, \otimes, S)$ with arbitrary (small) (co)products.
- A pointed abelian group $(A, \mathbf{1})$ along with a homomorphism $h : A \to \operatorname{Pic} \mathcal{SH}$.
- For each $a \in A$, a chosen representative S^a in the isomorphism class $h(a)$ such that $S^0 = S$.
- For each $a, b \in A$, an isomorphism $\phi_{a,b} : S^{a+b} \to S^a \otimes S^b$.
- A collection $\mathcal{D}$ of triangles in $\mathcal{C}$ of the following form

$$X \to Y \to Z \to \Sigma X.$$

Date: August 10, 2023.

These data are subject to the following additional conditions. First, we require that the $\phi_{a,b}$'s are *coherent*, in the following sense:

- For all $a \in A$, we must have $\phi_{a,0} = \rho_{S^a}^{-1} : S^a \to S^a \otimes S$ and $\phi_{0,a} = \lambda_{S^a}^{-1} : S^a \to S \otimes S^a$.
- For all $a, b, c \in A$, the following diagram must commute:

$$\begin{tikzcd} S^{a+b} \otimes S^c \arrow[d, "\phi_{a,b} \otimes S^c"'] & S^{a+b+c} \arrow[l, "\phi_{a+b,c}"'] \arrow[r, "\phi_{a,b+c}"] & S^a \otimes S^{b+c} \arrow[d, "S^a \otimes \phi_{b,c}"] \\ (S^a \otimes S^b) \otimes S^c \arrow[rr, "\cong"] & & S^a \otimes (S^b \otimes S^c) \end{tikzcd}$$

Now, given $a \in A$, define $\Sigma^a := S^a \otimes -$ and $\Omega^a := \Sigma^{-a} = S^{-a} \otimes -$. In particular, let $\Sigma := \Sigma^{\mathbf{1}}$ and $\Omega := \Omega^{\mathbf{1}}$. Then the above two conditions give that Σ^a and Ω^a form an adjoint equivalence of $\mathcal{SH}$, with unit η^a and counit ε^a defined by

$$\eta_X^a : X \xrightarrow{\cong} S \otimes X \xrightarrow{\phi_{-a,a} \otimes X} (S^{-a} \otimes S^a) \otimes X \xrightarrow{\cong} S^{-a} \otimes (S^a \otimes X) = \Omega^a \Sigma^a X$$

and

$$\varepsilon_X^a : \Sigma^a \Omega^a X = S^a \otimes (S^{-a} \otimes X) \xrightarrow{\cong} (S^a \otimes S^{-a}) \otimes X \xrightarrow{\phi_{a,-a}^{-1} \otimes X} S \otimes X \xrightarrow{\cong} X.$$

Thus, further require the following:

- $(\mathcal{SH}, \Sigma, \Omega, \eta^{\mathbf{1}}, \varepsilon^{\mathbf{1}}, \mathcal{D})$ is a tensor triangulated category, where the isomorphisms $\Sigma X \otimes Y \cong \Sigma(X \otimes Y)$ are given by the associator:

$$\Sigma X \otimes Y = (S^{\mathbf{1}} \otimes X) \otimes Y \xrightarrow{\cong} S^{\mathbf{1}} \otimes (X \otimes Y) = \Sigma(X \otimes Y).$$

(See Definition A.2 for the definition of a triangulated category, and Definition A.11 for the definition of a tensor triangulated category.)

From now on we fix the data given in the above definition, and we establish the following conventions:

- Given objects $X_1, \dots, X_n$ in $\mathcal{SH}$, we write $X_1 \otimes \cdots \otimes X_n$ to denote the object

$$X_1 \otimes (X_2 \otimes \cdots (X_{n-1} \otimes X_n)).$$

In particular, given an object X and a natural number $n > 0$, we write

$$X^n := \overbrace{X \otimes \cdots \otimes X}^{n \text{ times}} \qquad \text{and} \qquad X^0 := S.$$

- We denote the associator, symmetry, left unitor, and right unitor isomorphisms in $\mathcal{SH}$ by

$$\alpha_{X,Y,Z} : (X \otimes Y) \otimes Z \xrightarrow{\cong} X \otimes (Y \otimes Z) \qquad \tau_{X,Y} : X \otimes Y \xrightarrow{\cong} Y \otimes X$$
$$\lambda_X : S \otimes X \xrightarrow{\cong} X \qquad \rho_X : X \otimes S \xrightarrow{\cong} X.$$

- Given two objects X and Y, we denote the hom-abelian group of morphisms from X to Y in $\mathcal{SH}$ by $[X, Y]$, and we denote the internal hom object by $F(X, Y)$. We will often refer to morphisms in $\mathcal{SH}$ as *classes*, as we will think of them as representing homotopy classes of maps.
- Given two objects X and Y in $\mathcal{SH}$, we may extend the abelian group $[X, Y]$ to an A-graded abelian group $[X, Y]_*$ defined by

$$[X, Y]_a := [\Sigma^a X, Y] = [S^a \otimes X, Y].$$

(See Appendix C for a review of the theory of A-graded abelian groups, rings, modules, etc.)

- Given an object X in $\mathcal{SH}$ and some $a \in A$, define the abelian group
$$\pi_a(X) := [S^a, X],$$
and write $\pi_*(X)$ for the associated A-graded abelian group $\bigoplus_{a \in A} \pi_a(X)$. We call $\pi_a(X)$ the a^{th} *stable homotopy group of* X.
- Given two objects E and X in $\mathcal{SH}$, we define the A-graded abelian groups $E_*(X)$ and $E^*(X)$ by
$$E_a(X) := \pi_a(E \otimes X) = [S^a, E \otimes X] \qquad \text{and} \qquad E^a(X) := [X, S^a \otimes E].$$
We refer to the functor $E_*(-)$ as the *homology theory represented by* E, or just E-homology, and we refer to $E^*(-)$ as the *cohomology theory represented by* E, or just E-cohomology.

Proposition 2.2. *For each $a \in A$, the isomorphisms $\eta^a_X : X \to \Omega^a\Sigma^a X$ and $\varepsilon^a_X : \Sigma^a\Omega^a X \to X$ defined in Definition 2.1 exhibit an adjoint autoequivalence $(\Sigma^a, \Omega^a, \eta^a, \varepsilon^a)$ of $\mathcal{SH}$.*

Proof. In this proof, we will freely employ the coherence theorem for monoidal categories (see [2]), which essentially tells us that we may assume we are working in a strict monoidal category (i.e., that the associators and unitors and are identities). Then η^a_X and ε^a_X become simply the maps
$$\eta^a_X : X \xrightarrow{\phi_{-a,a}\otimes X} S^{-a} \otimes S^a \otimes X \qquad \text{and} \qquad \varepsilon^a_X : S^a \otimes S^{-a} \otimes X \xrightarrow{\phi^{-1}_{a,-a}\otimes X} X.$$
That these maps are natural in X follows by functoriality of $- \otimes -$. Now, recall that in order to show that these natural isomorphisms form an *adjoint* equivalence, it suffices to show that the natural isomorphisms $\eta^a : \mathrm{Id}_{\mathcal{SH}} \Rightarrow \Omega^a\Sigma^a$ and $\varepsilon^a : \Sigma^a\Omega^a \Rightarrow \mathrm{Id}_{\mathcal{SH}}$ satisfy one of the two zig-zag identities:
$$\begin{tikzcd} \Sigma^a \arrow[r, "\Sigma^a\eta^a"] \arrow[dr, equal] & \Sigma^a\Omega^a\Sigma^a \arrow[d, "\varepsilon^a\Sigma^a"] \\ & \Sigma^a \end{tikzcd} \qquad \begin{tikzcd} \Omega^a\Sigma^a\Omega^a \arrow[d, "\Omega^a\varepsilon^a"'] & \Omega^a \arrow[l, "\eta^a\Omega^a"'] \arrow[dl, equal] \\ \Omega^a & \end{tikzcd}$$
(see [4, Lemma 3.2]). We will show that the left is satisfied. Unravelling definitions, we simply wish to show that the following diagram commutes for all X in $\mathcal{SH}$:
$$\begin{tikzcd} S^a \otimes X \arrow[r, "S^a\otimes\phi_{-a,a}\otimes X"] \arrow[dr, equal] & S^a \otimes S^{-a} \otimes S^a \otimes X \arrow[d, "\phi^{-1}_{a,-a}\otimes S^a\otimes X"] \\ & S^a \otimes X \end{tikzcd}$$
Yet this is simply the diagram obtained by applying $- \otimes X$ to the associativity coherence diagram for the $\phi_{a,b}$'s, so it does commute, as desired. □

From now on, we fix the data of a stable homotopy category $\mathcal{SH}$ given above once and for all. Observe that for all $a, b \in A$, the objects S^{a+b} and $S^a \otimes S^b$ are isomorphic, since $h : A \to \mathrm{Pic}(\mathcal{SH})$ is a group homomorphism. Hence given a monoid object (E, μ, e) in $\mathcal{SH}$ (Definition D.1), supposing we had fixed isomorphisms $S^{a+b} \cong S^a \otimes S^b$ for all $a, b \in A$, we get a multiplication map $\pi_*(E) \times \pi_*(E) \to \pi_*(E)$ which sends classes $x : S^a \to E$ and $y : S^b \to E$ to the product
$$S^{a+b} \cong S^a \otimes S^b \xrightarrow{x\otimes y} E \otimes E \xrightarrow{\mu} E.$$
Naturally, we would like this product to make $\pi_*(E)$ into an A-graded ring (with unit $e \in \pi_0(E) = [S, E]$), rather than just an A-graded abelian group. Whether or not this happens is essentially the entire discussion of Dugger's paper [1], and as it turns out, $\pi_*(E)$ is in fact a graded ring provided we can choose these morphisms to be *coherent*, in the following sense:

Definition 2.3. Suppose we have a family of isomorphisms

$$\phi_{a,b} : S^{a+b} \xrightarrow{\cong} S^a \otimes S^b$$

for all $a, b \in A$. We say this family is *coherent* if:

(1) For all $a \in A$, we have equalities $\phi_{a,0} = \rho_{S^a}^{-1} : S^a \to S^a \otimes S$ and $\phi_{0,a} = \lambda_{S^a}^{-1} : S^a \to S \otimes S^a$.
(2) For all $a, b, c \in A$, the following diagram commutes:

$$\begin{tikzcd} S^{a+b} \otimes S^c \arrow[d, "\phi_{a,b}\otimes S^c"'] & S^{a+b+c} \arrow[l, "\phi_{a+b,c}"'] \arrow[r, "\phi_{a,b+c}"] & S^a \otimes S^{b+c} \arrow[d, "S^a \otimes \phi_{b,c}"] \\ (S^a \otimes S^b) \otimes S^c \arrow[rr, "\cong"] & & S^a \otimes (S^b \otimes S^c) \end{tikzcd}$$

Furthermore, Dugger gaurantees that we can always find such a coherent family:

Theorem 2.4 ([1, Proposition 7.1])**.** *There exists a coherent family of isomorphisms*

$$\phi_{a,b} : S^{a+b} \xrightarrow{\cong} S^a \otimes S^b$$

in the sense of Definition 2.3, and in particular, the set of such coherent families is in bijective correspondence with the set of normalized 2*-cocycles* $Z^2(A; \mathrm{Aut}(S))_{norm}$*, i.e., the set of functions* $\alpha : A \times A \to \mathrm{Aut}(S)$ *such that* $\alpha(0,0) = \mathrm{id}_S$ *and for all* $a, b, c \in A$*,* $\alpha(a+b,c) \cdot \alpha(a,b) = \alpha(b,c) \cdot \alpha(a, b+c)$*.*

Thus, from now on we will suppose once and for all we have fixed a coherent family $\{\phi_{a,b}\}_{a,b\in A}$. Such a coherent family has very nice properties, in particular:

Remark 2.5. Note that by induction the coherence conditions say that given any $a_1, \dots, a_n \in A$ and $b_1, \dots, b_m \in A$ such that $a_1 + \cdots + a_n = b_1 + \cdots + b_m$ and any fixed parenthesizations of $X = S^{a_1} \otimes \cdots \otimes S^{a_b}$ and $Y = S^{b_1} \otimes \cdots \otimes S^{b_m}$, there is a *unique* isomorphism $X \to Y$ that can be obtained by forming formal compositions of tensor products of $\phi_{a,b}$, identities, associators, and their inverses.

Of course, we get our desired result: $\pi_*(E)$ is indeed an A-graded ring if E is a monoid object.

Proposition 2.6. *Let* (E, μ, e) *be a commutative monoid object in* $\mathcal{SH}$*, and consider the multiplication map* $\pi_*(E) \times \pi_*(E) \to \pi_*(E)$ *which sends classes* $x : S^a \to E$ *and* $y : S^b \to E$ *to the composition*

$$S^{a+b} \xrightarrow{\phi_{a,b}} S^a \otimes S^b \xrightarrow{x\otimes y} E \otimes E \xrightarrow{\mu} E.$$

Then this endows $\pi_*(E)$ *with the structure of an* A*-graded ring with unit* $e \in \pi_0(E) = [S, E]$*.*

Proof. See Proposition D.3. □

Furthermore, it turns out that if E is a *commutative* monoid object in $\mathcal{SH}$, then $\pi_*(E)$ is "A-graded commutative," in the following sense:

Proposition 2.7. *For all* $a, b \in A$ *there exists an element* $\theta_{a,b} \in \pi_0(S) = [S, S]$ *(determined by choice of coherent family* $\{\phi_{a,b}\}$*) such that given any commutative monoid object* (E, μ, e) *in* $\mathcal{SH}$*, the* A*-graded ring structure on* $\pi_*(E)$ *(Proposition 2.6) has a commutativity formula given by*

$$x \cdot y = y \cdot x \cdot (e \circ \theta_{a,b})$$

for all $x \in \pi_a(E)$ *and* $y \in \pi_b(E)$*.*

Furthermore, $\theta_{0,a} = \theta_{a,0} = \mathrm{id}_S$ *for all* $a \in A$*, so that if either* x *or* y *has degree* 0*,* $x \cdot y = y \cdot x$*.*

Proof. See Proposition D.4 and Proposition D.5. □

The last ingredient in order to develop the Adams spectral sequence abstractly is a notion of *cellularity* in $\mathcal{SH}$:

Definition 2.8. Define the class of *cellular* objects in $\mathcal{SH}$ to be the smallest class of objects such that:

(1) For all $a \in A$, S^a is cellular.

(2) If we have a distinguished triangle
$$X \to Y \to Z \to \Sigma X (= S^{\mathbf{1}} \otimes X)$$
such that two of the three objects X, Y, and Z are cellular, than the third object is also cellular.

(3) Given a collection of cellular objects X_i indexed by some small set I, $\bigoplus_{i \in I} X_i$ is cellular.

2.2. **Construction of the Adams spectral sequence.** In what follows, let E be a commutative monoid object in $\mathcal{SH}$.

Definition 2.9. Let $\overline{E}$ be the fiber of the unit map $e : S \to E$ (Proposition A.6), and for $s \geq 0$ define
$$Y_s := \overline{E}^s \otimes Y, \qquad W_s = E \otimes Y_s = E \otimes (\overline{E}^s \otimes Y),$$
where recall for $s > 0$, $\overline{E}^s$ denotes the s-fold product parenthesized as $\overline{E} \otimes (\overline{E} \otimes \cdots (\overline{E} \otimes \overline{E}))$, and $\overline{E}^0 := S$. Then we get fiber sequences
$$Y_{s+1} \xrightarrow{i_s} Y_s \xrightarrow{j_s} W_s \xrightarrow{k_s} \Sigma Y_{s+1} (= S^{\mathbf{1}} \otimes Y_{s+1})$$
obtained by applying $- \otimes Y_s$ to the sequence
$$\overline{E} \to S \xrightarrow{e} E \to \Sigma \overline{E}$$
(and applying the necessary associator and unitor isomorphisms). These sequences can be spliced together to form the *(canonical) Adams filtration of* Y:
$$\begin{tikzcd}
\cdots \arrow[r] & Y_3 \arrow[r, "i_2"] \arrow[d, "j_3"] & Y_2 \arrow[r, "i_1"] \arrow[d, "j_2"] & Y_1 \arrow[r, "i_0"] \arrow[d, "j_1"] & Y_0 = Y \arrow[d, "j_0"] \\
& W_3 & W_2 \arrow[ul, dashed, "k_2"'] & W_1 \arrow[ul, dashed, "k_1"'] & W_0 \arrow[ul, dashed, "k_0"']
\end{tikzcd}$$
where the diagonal dashed arrows are of degree $-\mathbf{1}$ (note these triangles do NOT commute in any sense). Now we may apply the functor $[X, -]_*$, and by Proposition A.10 we obtain an exact couple of $\mathbb{N} \times A$-graded abelian groups:
$$\begin{tikzcd}
{[X, Y_*]_*} \arrow[rr, "i_{**}"] & & {[X, Y_*]_*} \arrow[d, "j_{**}"] \\
& & {[X, W_*]_*} \arrow[ull, "k_{**}"]
\end{tikzcd}$$
where i_{**}, j_{**}, and k_{**} have $\mathbb{Z} \times A$-degree $(-1, 0)$, $(0, 0)$, and $(1, -\mathbf{1})$, respectively[1]. The standard argument yields an $\mathbb{N} \times A$-graded spectral sequence called from this exact couple (cf. Section 5.9 of [7]) with E_1 page given by
$$E_1^{s,a} = [X, W_s]_a$$
and r^{th} differential of $\mathbb{Z} \times A$-degree $(r, -\mathbf{1})$:
$$d_r : E_r^{s,a} \to E_r^{s+r, a-\mathbf{1}}.$$
A priori, this is all $\mathbb{N} \times A$-graded, but we regard it as being $\mathbb{Z} \times A$-graded by setting $E_r^{s,a} := 0$ for $s < 0$ and trivially extending the definition of the differentials to these zero groups. This spectral

[1]Explicitly, the map $k_{s,a} : [X, W_s]_a \to [X, Y_{s+1}]_{a-\mathbf{1}}$ sends a map $f : S^a \otimes X \to W_s$ to the map $S^{a-\mathbf{1}} \otimes X \to Y_{s+1}$ corresponding under the isomorphism $[X, \Sigma Y_{s+1}]_* \cong [X, Y_{s+1}]_{*-\mathbf{1}}$ to the composition $k_s \circ f : S^a \otimes X \to \Sigma Y_{s+1}$.

sequence is called the *E-Adams spectral sequence* for the computation of $[X,Y]_*$. The index s is called the *Adams filtration* and a is the *stem*.

2.3. **Monoid objects in $\mathcal{SH}$.** We have constructed an Adams spectral sequence, but as it currently stands we do not yet know why it is useful. To start with, we'd like to provide a characterization of its E_1 and E_2 pages in terms of something more algebraic. To start, we first need to develop some theory of the algebra of monoid objects in $\mathcal{SH}$. Much of this work is entirely straightforward although tedious to verify, so we relegate most of the proofs in this section to Appendix D.

Proposition 2.10. *Let (E,μ,e) be a monoid object in $\mathcal{SH}$. Then $E_*(-)$ is a functor from $\mathcal{SH}$ to left A-graded $\pi_*(E)$-modules, where given some X in $\mathcal{SH}$, $E_*(X)$ may be endowed with the structure of a left A-graded $\pi_*(E)$-module via the map*

$$\pi_*(E)\times E_*(X)\to E_*(X)$$

which given $a,b\in A$, sends $x:S^a\to E$ and $y:S^b\to E\otimes X$ to the composition

$$x\cdot y : S^{a+b}\cong S^a\otimes S^b \xrightarrow{x\otimes y} E\otimes(E\otimes X)\cong(E\otimes E)\otimes X\xrightarrow{\mu\otimes X} E\otimes X.$$

Similarly, the assignment $X\mapsto X_*(E)$ is a functor from $\mathcal{SH}$ to right A-graded $\pi_*(E)$-modules, where the structure map

$$X_*(E)\times\pi_*(E)\to X_*(E)$$

sends $x:S^a\to X\otimes E$ and $y:S^b\to E$ to the composition

$$x\cdot y : S^{a+b}\cong S^a\otimes S^b \xrightarrow{x\otimes y}(X\otimes E)\otimes E\cong X\otimes(E\otimes E)\xrightarrow{X\otimes\mu} X\otimes E.$$

Finally, $E_*(E)$ is a $\pi_*(E)$-bimodule, in the sense that the left and right actions of $\pi_*(E)$ are compatible, so that given $y,z\in\pi_*(E)$ and $x\in E_*(E)$, $y\cdot(x\cdot z)=(y\cdot x)\cdot z$.

Proof. See Proposition D.7. □

Definition 2.11. Given a monoid object E in $\mathcal{SH}$, we say E is *flat* if the canonical right $\pi_*(E)$-module structure on $E_*(E)$ (see the above proposition) is that of a flat module.

2.4. **The E_1 page.** The goal of this subsection is to provide the following characterization for the E_1 page of the Adams spectral sequence:

Theorem 2.12. *Let E be a flat commutative monoid object in $\mathcal{SH}$, and let X and Y be two objects in $\mathcal{SH}$ such that $E_*(X)$ is a projective module over $\pi_*(E)$. Then for all $s\geq 0$ and $a\in A$, we have isomorphisms in the associated E-Adams spectral sequence*

$$E_1^{s,a}\cong \mathrm{Hom}^a_{E_*(E)}(E_*(X),E_*(W_s))$$

Furthermore, under these isomorphisms, the differential $d_1:E_1^{s,a}\to E_1^{s+1,a-\mathbf{1}}$ corresponds to the map

$$\mathrm{Hom}^a_{E_*(E)}(E_*(X),E_*(W_s))\to \mathrm{Hom}^{a-\mathbf{1}}_{E_*(E)}(E_*(X),E_*(X\otimes W_{s+1}))$$

which sends a map $f:E_*(X)\to E_{*+a}(W_s)$ to the composition

$$E_*(X)\xrightarrow{f}E_{*+a}(W_s)\xrightarrow{(X\otimes h_s)_*}E_{*+a-\mathbf{1}}(X\otimes Y_{s+1})\xrightarrow{(X\otimes j_{s+1})_*}E_{*+a-\mathbf{1}}(X\otimes W_{s+1}).$$

Proof. By Lemma D.10, for all $s\geq 0$ and $t,w\in\mathbb{Z}$, we have isomorphisms

$$[X,E\otimes Y_s]_{t,w}\cong\mathrm{Hom}^{t,w}_{E_*(E)}(E_*(X),E_*(E\otimes Y_s)).$$

since $W_s=E\otimes Y_s$, we have that

$$E_1^{s,(t,w)}=[X,W_s]_{t,w}\cong\mathrm{Hom}^{t,w}_{E_*(E)}(E_*(X),E_*(W_s)),$$

as desired. □

Definition 2.13. Let (E, μ, e) be a monoid object in $\mathcal{SH}$. We say E is *flat* if the canonical right $\pi_*(E)$-module structure on $E_*(E)$ is that of a flat module.

2.5. The E_2 page.

2.6. Convergence.
convergence of spectral sequences

3. The classical Adams spectral sequence

4. The motivic Adams spectral sequence

Appendix A. Triangulated categories

We assume the reader is familiar with additive categories and (closed, symmetric) monoidal categories.

Definition A.1. Let $\mathcal{C}$ be an additive category. Given a sequence

$$X_1 \to \cdots \to X_n$$

of morphisms in $\mathcal{C}$, we say this sequence is *exact* if, for any object A in $\mathcal{C}$, the induced sequence

$$\mathcal{C}(A, X_1) \to \mathcal{C}(A, X_n) \to \cdots \to \mathcal{C}(A, X_n)$$

is an exact sequence of abelian groups.

Definition A.2. A *triangulated category* is a tuple $(\mathcal{C}, \Sigma, \Omega, \eta, \varepsilon, \mathcal{D})$ such that

1. $\mathcal{C}$ is an additive category.
2. An adjoint pair of additive functors $\Sigma, \Omega : \mathcal{C} \to \mathcal{C}$ such that the unit $\eta : \mathrm{Id}_{\mathcal{C}} \Rightarrow \Omega\Sigma$ and counit $\varepsilon : \Sigma\Omega \Rightarrow \mathrm{Id}_{\mathcal{C}}$ are natural isomorphisms, i.e., the tuple $(\Sigma, \Omega, \eta, \varepsilon)$ forms an adjoint autoequivalence of $\mathcal{C}$. Usually Σ is called the *shift functor*.
3. $\mathcal{D}$ is a collection of *distinguished triangles*, where a *triangle* is a diagram of the form
$$X \to Y \to Z \to \Sigma X.$$
These are also sometimes called *cofiber sequences* or *fiber sequences*.

These data must satisfy the following axioms:

TR0 Given a commutative diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d, "\cong"'] & Y \arrow[r] \arrow[d, "\cong"'] & Z \arrow[r] \arrow[d, "\cong"'] & \Sigma X \arrow[d, "\cong"'] \\ X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & \Sigma X' \end{tikzcd}$$

where the vertical arrows are isomorphisms, if the top row is distinguished then so is the bottom.

TR1 For any object X in $\mathcal{C}$, the diagram

$$X \xrightarrow{\mathrm{id}_X} X \to 0 \to \Sigma X$$

is a distinguished triangle.

TR2 For all $f : X \to Y$ there exists an object C_f (also sometimes denoted Y/X) called the *cofiber of f* and a distinguished triangle

$$X \xrightarrow{f} Y \to C_f \to \Sigma X.$$

TR3 Given a solid diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] & Y \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d, dashed] & \Sigma X \arrow[d, "\Sigma f"] \\ X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & \Sigma X' \end{tikzcd}$$

such that the leftmost square commmutes and both rows are distinguished, there exists a dashed arrow $Z \to Z'$ which makes the remaining two squares commute.

TR4 A triangle

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{\Sigma} X$$

is distinguished if and only if

$$Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X \xrightarrow{-\Sigma f} \Sigma Y$$

is distinguished.

TR5 (Octahedral axiom) Given three distinguished triangles

$$X \xrightarrow{f} Y \xrightarrow{h} Y/X \to \Sigma X$$

$$Y \xrightarrow{g} Z \xrightarrow{k} Z/Y \to \Sigma Y$$

$$X \xrightarrow{g\circ f} Z \xrightarrow{l} Z/X \to \Sigma X$$

there exists a distinguished triangle

$$Y/X \xrightarrow{u} Z/X \xrightarrow{v} Z/Y \xrightarrow{w} \Sigma(Y/X)$$

such that the following diagram commutes

$$\begin{tikzcd}
X \arrow[rr, "g\circ f"] \arrow[dr, "f"'] & & Z \arrow[rr, "k"] \arrow[dr, "l"'] & & Z/Y \arrow[rr, "w"] \arrow[dr] & & \Sigma(Y/X) \\
& Y \arrow[ur, "g"'] \arrow[dr, "h"'] & & Z/X \arrow[ur, "v"'] \arrow[dr] & & \Sigma Y \arrow[ur, "\Sigma h"'] & \\
& & Y/X \arrow[ur, "u"'] \arrow[rr] & & \Sigma X \arrow[ur, "\Sigma f"'] & &
\end{tikzcd}$$

It turns out that the above definition is actually redundant; TR3 and TR4 follow from the remaining axioms (see Lemmas 2.2 and 2.4 in [3]).

In this section, we fix a triangulated category $\mathcal{C}$, and we will always use brackets $[-,-]$ to denote the homset in $\mathcal{C}$. Note our definition of a triangulated category is slightly nonstandard, namely, we require that the shift functor gives an *adjoint* autoequivalence of $\mathcal{C}$, not just an equivalence. In other words, we require that the natural isomorphisms $\eta : \mathrm{Id}_{\mathcal{C}} \Rightarrow \Sigma\Omega$ and $\varepsilon : \Sigma\Omega \Rightarrow \mathrm{Id}_{\mathcal{C}}$ satisfy either of the following zig-zag identities:

$$\begin{tikzcd}
\Sigma \arrow[r, "\Sigma\eta"] \arrow[dr, equal] & \Sigma\Omega\Sigma \arrow[d, "\varepsilon\Sigma"] \\
& \Sigma
\end{tikzcd}
\qquad
\begin{tikzcd}
\Omega\Sigma\Omega \arrow[d, "\Omega\varepsilon"'] & \Omega \arrow[l, "\eta\Omega"'] \arrow[dl, equal] \\
\Omega &
\end{tikzcd}$$

(Satisfying one implies the other is automatically satisfied, see [4, Lemma 3.2]). We have also fixed the data of natural isomorphisms η and ε exhibiting the adjoint equivalence $(\Sigma, \Omega, \eta, \varepsilon)$ in our definition. We very well could have only required that Σ is *an* adjoint equivalence, and everything we will do will go through all the same.

Proposition A.3. *Let $(\mathcal{C}, \Sigma)$ be a triangulated category. Then any distinguished triangle in $\mathcal{C}$ is an exact sequence (in the sense of Definition A.1).*

Proof. Suppose we have some distinguished triangle

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X.$$

Then we would like to show that given any object A in $\mathcal{C}$, the sequence

$$[A, X] \xrightarrow{f_*} [A, Y] \xrightarrow{g_*} [A, Z] \xrightarrow{h_*} [A, \Sigma X]$$

is exact. First we show exactness at $[A, Y]$. To see $\operatorname{im} f_* \subseteq \ker g_*$, note it suffices to show that $g \circ f = 0$. Indeed, consider the commuting diagram

$$\begin{tikzcd} X \arrow[r, equal] \arrow[d, equal] & X \arrow[r] \arrow[d, "f"] & 0 \arrow[r] & \Sigma X \\ X \arrow[r, "f"] & Y \arrow[r, "g"] & Z \arrow[r, "h"] & \Sigma X \end{tikzcd}$$

The top row is distinguished by axiom TR1. Thus by TR3, the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, equal] \arrow[d, equal] & X \arrow[r] \arrow[d, "f"] & 0 \arrow[r] \arrow[d] & \Sigma X \arrow[d, equal] \\ X \arrow[r, "f"] & Y \arrow[r, "g"] & Z \arrow[r, "h"] & \Sigma X \end{tikzcd}$$

In particular, commutativity of the second square tells us that $g \circ f = 0$, as desired. Conversely, we'd liek to show that $\ker g_* \subseteq \operatorname{im} f_*$. Let $\psi : A \to Y$ be in the kernel of g_*, so that $g \circ \psi = 0$. Consider the following commutative diagram:

$$\begin{tikzcd} A \arrow[r] \arrow[d, "\psi"'] & 0 \arrow[r] \arrow[d] & \Sigma A \arrow[r, "-\Sigma \mathrm{id}_A"] & \Sigma A \\ Y \arrow[r, "g"] & Z \arrow[r, "h"] & \Sigma X \arrow[r, "-\Sigma f"] & \Sigma Y \end{tikzcd}$$

The top row is distinguished by axioms TR1 and TR4. The bottom row is distinguished by axiom TR4. Thus by axiom TR3 there exists a map $\widetilde{\phi} : \Sigma A \to \Sigma X$ such that the following diagram commutes:

$$\begin{tikzcd} A \arrow[r] \arrow[d, "\psi"'] & 0 \arrow[r] \arrow[d] & \Sigma A \arrow[r, "-\Sigma \mathrm{id}_A"] \arrow[d, "\widetilde{\phi}"'] & \Sigma A \arrow[d, "\Sigma\psi"'] \\ Y \arrow[r, "g"] & Z \arrow[r, "h"] & \Sigma X \arrow[r, "-\Sigma f"] & \Sigma Y \end{tikzcd}$$

Now, since Σ is an equivalence, it is a full functor, so that in particular there exists some $\phi : A \to X$ such that $\widetilde{\phi} = \Sigma\phi$. Then by faithfullness, we may pull back the right square to get a commuting diagram

$$\begin{tikzcd} A \arrow[r, "-\mathrm{id}_A"] \arrow[d, "\phi"'] & A \arrow[d, "\psi"] \\ X \arrow[r, "-f"'] & Y \end{tikzcd}$$

Hence,

$$f_*(\phi) = f \circ \phi \overset{(*)}{=} -((-f) \circ \phi) = -(\psi \circ (-\mathrm{id}_A)) \overset{(*)}{=} \psi \circ \mathrm{id}_A = \psi,$$

where the equalities marked $(*)$ follow by bilinearity of composition in an additive category. Thus $\psi \in \operatorname{im} f_*$, as desired, meaning $\ker g_* \subseteq \operatorname{im} f_*$.

Now, we have shown that

$$[A, X] \xrightarrow{f_*} [A, Y] \xrightarrow{g_*} [A, Z] \xrightarrow{h_*} [A, \Sigma X]$$

is exact at $[A, Y]$. It remains to show exactness at $[A, Z]$. Yet this follows by the exact same argument given above applied to the sequence obtained from the shifted triangle (TR4)

$$Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X \xrightarrow{-\Sigma f} \Sigma Y. \qquad \square$$

Lemma A.4. *Suppose we have a commutative diagram*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "j"] & Y \arrow[r, "g"] \arrow[d, "k"] & Z \arrow[r, "h"] \arrow[d, "\ell"] & \Sigma X \arrow[d, "\Sigma j"] \\ X' \arrow[r, "f'"] & Y' \arrow[r, "g'"] & Z' \arrow[r, "h'"] & \Sigma X' \end{tikzcd}$$

with both rows distinguished. Then if any two of the maps j, k, and ℓ are isomorphisms, then so is the third.

Proof. Suppose we are given any object A in $\mathcal{C}$, and consider the commutative diagram

$$\begin{tikzcd} {[A,X]} \arrow[r, "f_*"] \arrow[d, "j_*"] & {[A,Y]} \arrow[r, "g_*"] \arrow[d, "k_*"] & {[A,Z]} \arrow[r, "k_*"] \arrow[d, "\ell_*"] & {[A,\Sigma X]} \arrow[r, "-\Sigma f_*"] \arrow[d, "\Sigma j_*"] & {[A,\Sigma Y]} \arrow[r, "-\Sigma g_*"] \arrow[d, "\Sigma k_*"] & {[A,\Sigma Z]} \arrow[r, "-\Sigma h_*"] \arrow[d, "\Sigma \ell_*"] & {[A,\Sigma^2 X]} \arrow[d, "\Sigma^2 j_*"] \\ {[A,X']} \arrow[r, "f'_*"] & {[A,Y']} \arrow[r, "g'_*"] & {[A,Z']} \arrow[r, "h'_*"] & {[A,\Sigma X']} \arrow[r, "-\Sigma f'_*"] & {[A,\Sigma Y']} \arrow[r, "-\Sigma g'_*"] & {[A,\Sigma Z']} \arrow[r, "-\Sigma h'_*"] & {[A,\Sigma^2 X']} \end{tikzcd}$$

The rows are exact by Proposition A.3 and repeated applications of axiom TR4. It follows by the five lemma that if j and k are isomorphisms, then ℓ_* is an isomorphism. Similarly, if k and ℓ are isomorphisms then Σj_* is an isomorphism. Finally, if ℓ and j are isomorphisms, then Σk_* is an isomorphism. The desired result follows by faithfullness of Σ and the Yoneda embedding. □

Proposition A.5. *Given a map $f : X \to Y$ in a triangulated category $(\mathcal{C}, \Sigma, \Omega, \mathcal{D})$, the cofiber sequence of f is unique up to isomorphism, in the sense that given any two distinguished triangles*

$$X \xrightarrow{f} Y \to Z \to \Sigma X \qquad \textit{and} \qquad X \xrightarrow{f} Y \to Z' \to \Sigma X,$$

there exists an isomorphism $Z \to Z'$ which makes the following diagram commute:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, equal] & Y \arrow[r] \arrow[d, equal] & Z \arrow[r] \arrow[d, dashed, "k"] & \Sigma X \arrow[d, equal] \\ X \arrow[r, "f"] & Y \arrow[r] & Z' \arrow[r] & \Sigma X \end{tikzcd}$$

Proof. Suppose we have two distinguished triangles

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X \qquad \text{and} \qquad X \xrightarrow{f} Y \xrightarrow{g'} Z' \xrightarrow{h'} \Sigma X,$$

and consider the following commutative diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, equal] & Y \arrow[r, "g"] \arrow[d, equal] & Z \arrow[r, "h"] & \Sigma X \\ X \arrow[r, "f"] & Y \arrow[r, "g'"] & Z' \arrow[r, "h'"] & \Sigma X \end{tikzcd}$$

By axiom TR3, there exists some map $k : Z \to Z'$ which makes the following diagram commute:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, equal] & Y \arrow[r, "g"] \arrow[d, equal] & Z \arrow[r, "h"] \arrow[d, "k"] & \Sigma X \arrow[d, equal] \\ X \arrow[r, "f"] & Y \arrow[r, "g'"] & Z' \arrow[r, "h'"] & \Sigma X \end{tikzcd}$$

Now, by Lemma A.4, k is an isomorphism. □

Proposition A.6. *Given an arrow $f : X \to Y$ in a triangulated category $(\mathcal{C}, \Sigma, \Omega, \mathcal{D})$, there exists an object F_f called the* fiber *of f, and a distinguished triangle*

$$F_f \to X \xrightarrow{f} Y \to \Sigma F_f (\cong C_f).$$

Proof. By axiom TR2, we have a distinguished triangle

$$X \xrightarrow{f} Y \xrightarrow{g} C_f \xrightarrow{h} \Sigma X.$$

Now, consider the commutative diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, equal] & Y \arrow[r, "\widetilde{g}"] \arrow[d, equal] & \Sigma\Omega C_f \arrow[r, "\widetilde{h}"] \arrow[d, "\varepsilon_{C_f}"'] & \Sigma X \arrow[d, equal] \\ X \arrow[r, "f"] & Y \arrow[r, "g"] & C_f \arrow[r, "h"] & \Sigma X \end{tikzcd}$$

where $\varepsilon : \Sigma\Omega \Rightarrow \mathrm{Id}_{\mathcal{C}}$ is the counit of the adjunction $\Sigma \dashv \Omega$, $\widetilde{g} = \varepsilon_{C_f}^{-1} \circ g$, and $\widetilde{h} = \varepsilon_{C_f}^{-1} \circ h$. Since each vertical map is an isomorphism and the bottom row is distinguished, the top row is also distinguished by axiom TR0. Now, since Σ is an equivalence of categories, it is faithful, so that in particular there exists some map $k : \Omega C_f \to X$ such that $\Sigma k = -\widetilde{h} \implies -\Sigma k = \widetilde{h}$. Thus, we have a distinguished triangle of the form

$$X \xrightarrow{f} Y \xrightarrow{\widetilde{g}} \Sigma\Omega C_f \xrightarrow{-\Sigma k} \Sigma X.$$

Finally, by axiom TR4, we get a distinguished triangle

$$\Omega C_f \xrightarrow{k} X \xrightarrow{f} Y \xrightarrow{\widetilde{g}} \Sigma\Omega C_f,$$

so we may define the fiber of f to be ΩC_f. □

Lemma A.7. *Given a triangle*

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X,$$

it can be shifted to the left to obtain a distinguished triangle

$$\Omega Z \xrightarrow{-\widetilde{h}} X \xrightarrow{f} Y \xrightarrow{\widetilde{\Omega g}} \Sigma\Omega Z,$$

where $\widetilde{h} : \Omega Z \to X$ is the adjoint of $h : Z \to \Sigma X$ and $\widetilde{\Omega g} : Y \to \Sigma\Omega Z$ is the adjoint of $\Omega g : \Omega Y \to \Omega Z$.

Proof. Note that unravelling definitions, if ε and η are the counit and unit of the adjoint equivalence $\Sigma \dashv \Omega$, respectively (so η^{-1} and ε^{-1} are the counit and unit of the adjunction $\Omega \dashv \Sigma$), then $\widetilde{h}$ and $\widetilde{g}$ are the compositions

$$\widetilde{h} : \Omega Z \xrightarrow{\Omega h} \Omega\Sigma X \xrightarrow{\eta_X^{-1}} X \qquad \text{and} \qquad \widetilde{\Omega g} : Y \xrightarrow{\varepsilon_Y^{-1}} \Sigma\Omega Y \xrightarrow{\Sigma\Omega g} \Sigma\Omega Z.$$

Now consider the following diagram:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, equal] & Y \arrow[r, "\widetilde{\Omega g}"] \arrow[d, equal] & \Sigma\Omega Z \arrow[r, "\Sigma\widetilde{h}"] \arrow[d, "\varepsilon_Z"] & \Sigma X \arrow[d, equal] \\ X \arrow[r, "f"] & Y \arrow[r, "g"] & Z \arrow[r, "h"] & \Sigma X \end{tikzcd}$$

The left square commutes by definition. To see that the middle square commutes, expanding definitions, note it is given by the following diagram:

$$\begin{tikzcd} Y \arrow[r, "\varepsilon_Y^{-1}"] \arrow[d, equal] & \Sigma\Omega Y \arrow[r, "\Sigma\Omega g"] & \Sigma\Omega Z \arrow[d, "\varepsilon_Y"] \\ Y \arrow[rr, "g"] & & Z \end{tikzcd}$$

and this commutes by naturality of ε. To see that the right square commutes, consider the following diagram:

$$\begin{tikzcd} \Sigma\Omega Z \arrow[r, "\Sigma\Omega h"] \arrow[d, "\varepsilon_Z"'] & \Sigma\Omega\Sigma X \arrow[r, "\Sigma\eta_X^{-1}"] \arrow[dr, "\varepsilon_{\Sigma X}"] & \Sigma X \arrow[d, equal] \\ Z \arrow[rr, "h"'] & & \Sigma X \end{tikzcd}$$

By functoriality of Σ, the top composition is $\Sigma\widetilde{h}$. The left region commutes by naturality of ε. Commutativity of the right region is precisely one of the the zig-zag identities for the unit and counit of an adjunction. Hence, since diagram (**??**) commutes, the vertical arrows are isomorphisms, and the bottom row is distinguished, we have that the top row is distinguished as well by axiom TR0. Then by axiom TR4, since $(f, \widetilde{\Omega g}, \Sigma\widetilde{h})$ is distinguished, so is the triangle

$$\Omega Z \xrightarrow{-\widetilde{h}} X \xrightarrow{f} Y \xrightarrow{\widetilde{\Omega g}} \Sigma\Omega Z. \qquad \square$$

Lemma A.8. *Given a distinguished triangle*

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X,$$

for any $n > 0$, the triangle

$$\Omega^n X \xrightarrow{(-1)^n\Omega^n f} \Omega^n Y \xrightarrow{(-1)^n\Omega^n g} \Omega^n Z \xrightarrow{(-1)^n\Omega^n h} \Omega^n\Sigma X \cong \Sigma\Omega^n X,$$

is distinguished, where the final isomorphism is given by the composition

$$\Omega^n\Sigma X = \Omega^{n-1}\Omega\Sigma X \xrightarrow{\Omega^{n-1}\eta_X^{-1}} \Omega^{n-1} X \xrightarrow{\varepsilon^{-1}_{\Omega^{n-1}X}} \Sigma\Omega\Omega^{n-1}X = \Sigma\Omega^n X,$$

where ε and η are the counit and unit of the adjunction $\Sigma \dashv \Omega$, respectively.

Proof. We give a proof by induction. First we show the case $n = 1$. Note by Lemma A.7, we have that given a distinguished triangle

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X,$$

we can shift it to the left to obtain a distinguished triangle

$$\Omega Z \xrightarrow{-\widetilde{h}} X \xrightarrow{f} Y \xrightarrow{\widetilde{\Omega g}} \Sigma\Omega Z,$$

where $\widetilde{h} : \Omega Z \to X$ is the adjoint of $h : Z \to \Sigma X$ and $\widetilde{\Omega g}$ is the adjoint of $\Omega g : \Omega Y \to \Omega Z$. If we apply this shifting operation again, we get the distinguished triangle

$$\Omega Y \xrightarrow{-\widetilde{\widetilde{\Omega g}}} \Omega Z \xrightarrow{-\widetilde{h}} X \xrightarrow{\widetilde{\Omega f}} \Sigma\Omega Y,$$

where unravelling definitions, $\widetilde{\Omega f}$ is the right adjoint of $\Omega f : \Omega X \to \Omega Y$ and $\widetilde{\widetilde{\Omega g}}$ is the right adjoint of $\widetilde{\Omega g}$, which itself is the left adjoint of Ωg, so $\widetilde{\widetilde{\Omega g}} = \Omega g$. Hence we have a distinguished triangle

$$\Omega Y \xrightarrow{-\Omega g} \Omega Z \xrightarrow{-\widetilde{h}} X \xrightarrow{\widetilde{\Omega f}} \Sigma\Omega Y.$$

We may again shift this triangle again and the above arguments yield the distinguished triangle

$$\Omega X \xrightarrow{-\Omega f} \Omega Y \xrightarrow{-\Omega g} \Omega Z \xrightarrow{\widetilde{\Omega(-\widetilde{h})}} \Sigma\Omega X,$$

where $\widetilde{\Omega(-\widetilde{h})}$ is the right adjoint of $\Omega(-\widetilde{h}) = -\Omega\widetilde{h} : \Omega\Omega Z \to \Omega X$. Explicitly unravelling definitions, $\widetilde{\Omega(-\widetilde{h})} = \widetilde{-\Omega\widetilde{h}}$ is the composition

$$
\begin{aligned}
[\Omega Z \xrightarrow{\varepsilon_{\Omega Z}^{-1}} \Sigma\Omega\Omega Z \xrightarrow{\Sigma(-\Omega\widetilde{h})} \Sigma\Omega X] &= -[\Omega Z \xrightarrow{\varepsilon_{\Omega Z}^{-1}} \Sigma\Omega\Omega Z \xrightarrow{\Sigma\Omega\widetilde{h}} \Sigma\Omega X] \\
&= -[\Omega Z \xrightarrow{\varepsilon_{\Omega Z}^{-1}} \Sigma\Omega\Omega Z \xrightarrow{\Sigma\Omega\Omega h} \Sigma\Omega\Omega\Sigma X \xrightarrow{\Sigma\Omega\eta_X^{-1}} \Sigma\Omega X] \\
&= -[\Omega Z \xrightarrow{\Omega h} \Omega\Sigma X \xrightarrow{\eta_X^{-1}} X \xrightarrow{\varepsilon_X^{-1}} \Sigma\Omega X],
\end{aligned}
$$

where the first equality follows by additivity of Σ and additivity of composition, the second follows by further unravelling how $\widetilde{h}$ is defined, and the third follows by naturality of ε, which tells us the following diagram commutes:

$$
\begin{tikzcd}
\Omega Z \arrow[r, "\Omega h"] \arrow[d, "\varepsilon_{\Omega Z}^{-1}"] & \Omega\Sigma X \arrow[r, "\eta_X^{-1}"] \arrow[d, "\varepsilon_{\Omega\Sigma X}^{-1}"] & X \arrow[d, "\varepsilon_X^{-1}"] \\
\Sigma\Omega\Omega Z \arrow[r, "\Sigma\Omega\Omega h"'] & \Sigma\Omega\Omega\Sigma X \arrow[r, "\Sigma\Omega\eta_X^{-1}"'] & \Sigma\Omega X
\end{tikzcd}
$$

Thus indeed we have a distinguished triangle

$$\Omega X \xrightarrow{-\Omega f} \Omega Y \xrightarrow{-\Omega g} \Omega Z \xrightarrow{-\Omega h} \Omega\Sigma X \cong \Sigma\Omega X,$$

where the last isomorphism is $\varepsilon_X^{-1} \circ \eta_X^{-1}$, as desired.

Now, we show the inductive step. Suppose we know that given a distinguished triangle

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X,$$

that for some $n > 0$ the triangle

$$\Omega^n X \xrightarrow{(-1)^n\Omega^n f} \Omega^n Y \xrightarrow{(-1)^n\Omega^n g} \Omega^n Z \xrightarrow{(-1)^n h^n} \Sigma\Omega^n X,$$

is distinguished, where $h^n : \Omega^n Z \to \Sigma\Omega^n X$ is the composition

$$\Omega^n Z \xrightarrow{\Omega^n h} \Omega^n\Sigma X \xrightarrow{\Omega^{n-1}\eta_X^{-1}} \Omega^{n-1} X \xrightarrow{\varepsilon_{\Omega^{n-1}X}^{-1}} \Sigma\Omega^n X.$$

Then by applying the $n = 1$ case to this triangle, we get that the following triangle is distinguished

$$\Omega^{n+1} X \xrightarrow{-\Omega((-1)^n\Omega^n f)} \Omega^{n+1} Y \xrightarrow{-\Omega((-1)^n\Omega^n g)} \Omega^{n+1} Z \xrightarrow{-\Omega((-1)^n h^n)} \Omega\Sigma\Omega^n X \cong \Sigma\Omega^{n+1} X,$$

where the final isomorphism is the composition

$$\Omega\Sigma\Omega^n X \xrightarrow{\eta_{\Omega^n X}^{-1}} \Omega^n X \xrightarrow{\varepsilon_{\Omega^n X}^{-1}} \Sigma\Omega\Omega^n X = \Sigma\Omega^{n+1} X.$$

We claim that this is precisely the distinguished triangle given in the statement of the lemma for $n+1$. First of all, note that $-\Omega((-1)^n\Omega^n f) = (-1)^{n+1}\Omega^{n+1} f$, $-\Omega((-1)^n\Omega^n g) = (-1)^{n+1}\Omega^{n+1} g$, and $-\Omega((-1)^n h^n) = (-1)^{n+1}\Omega h^n$ by additivity of Ω, so that the triangle becomes

$$\Omega^{n+1} X \xrightarrow{(-1)^{n+1}\Omega^{n+1} f} \Omega^{n+1} Y \xrightarrow{(-1)^{n+1}\Omega^{n+1} g} \Omega^{n+1} Z \xrightarrow{(-1)^{n+1}\Omega h^n} \Omega\Sigma\Omega^n X \cong \Sigma\Omega^{n+1} X. \tag{1}$$

Thus, in order to prove the desired characterization, it remains to show this diagram commutes:

$$
\begin{tikzcd}
\Omega^{n+1} Z \arrow[r, "(-1)^{n+1}\Omega h^n"] \arrow[d, "(-1)^{n+1}\Omega^{n+1} h"'] & \Omega\Sigma\Omega^n X \arrow[r, "\eta_{\Omega^n X}^{-1}"] & \Omega^n X \arrow[d, "\varepsilon_{\Omega^n X}^{-1}"] \\
\Omega^{n+1}\Sigma X \arrow[r, "\Omega^n \eta_X^{-1}"'] & \Omega^n X \arrow[r, "\varepsilon_{\Omega^n X}^{-1}"'] & \Sigma\Omega^{n+1} X
\end{tikzcd}
$$

(The top composition is the last two arrows in diagram (1), and the bottom composition is the last two arrows in the diagram). Unravelling how h^n is constructed, by additivity of Ω it further suffices to show the outside of the following diagram commutes:

$$\begin{tikzcd}
\Omega^{n+1}Z \arrow[r, "(-1)^{n+1}\Omega^{n+1}h"] \arrow[dd, "(-1)^{n+1}\Omega^{n+1}h"'] & \Omega^{n+1}\Sigma X \arrow[r, "\Omega^n\eta_X^{-1}"] & \Omega^n X \arrow[r, "\Omega\varepsilon^{-1}_{\Omega^{n-1}X}"] \arrow[dd, equal] & \Omega\Sigma\Omega^n X \arrow[d, "\eta^{-1}_{\Omega^n X}"] \\
 & & & \Omega^n X \arrow[d, "\varepsilon^{-1}_{\Omega^n X}"] \\
\Omega^{n+1}\Sigma X \arrow[rr, "\Omega^n\eta_X^{-1}"'] & & \Omega^n X \arrow[r, "\varepsilon^{-1}_{\Omega^n X}"'] \arrow[ur, equal] & \Sigma\Omega^{n+1}X
\end{tikzcd}$$

The left rectangle and bottom right triangle commute by definition. Finally, commutativity of the top right trapezoid is precisely one of the zig-zag identities for the adjunction $\Omega \dashv \Sigma$ applied to $\Omega^{n-1}X$. Hence, we have shown the desired result. $\square$

Lemma A.9. *Suppose we have a distinguished triangle*

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X.$$

Then for any integer $n \geq 0$, the sequence

$$\Sigma^n X \xrightarrow{\Sigma^n f} \Sigma^n Y \xrightarrow{\Sigma^n g} \Sigma^n Z \xrightarrow{\Sigma^n h} \Sigma^{n+1} X$$

is exact. Similarly, for any integer $n > 0$, the sequence

$$\Omega^n X \xrightarrow{\Omega^n f} \Omega^n Y \xrightarrow{\Omega^n g} \Omega^n Z \xrightarrow{\Omega^n h} \Omega^n \Sigma X \cong \Omega^{n-1} X$$

are exact (Definition A.1), where the isomorphism is given by $\Omega^{n-1}\eta_X^{-1}$, where $\eta : \mathrm{Id}_{\mathcal{C}} \to \Omega\Sigma$ is the unit of the adjunction $\Sigma \dashv \Omega$.

Proof. By Proposition A.3, the first statement holds when $n = 0$. Now suppose we are given some $n > 0$. Using axiom TR4, by induction we have that the triangle

$$\Sigma^n X \xrightarrow{(-1)^n \Sigma^n f} \Sigma^n Y \xrightarrow{(-1)^n \Sigma^n g} \Sigma^n Z \xrightarrow{(-1)^n \Sigma^n h} \Sigma^{n+1} X$$

is also distinguished. Thus, again by Proposition A.3, given any object A in $\mathcal{C}$, the sequence of abelian groups

$$[A, \Sigma^n X] \xrightarrow{(-1)^n \Sigma^n f_*} [A, \Sigma^n Y] \xrightarrow{(-1)^n \Sigma^n g_*} [A, \Sigma^n Z] \xrightarrow{(-1)^n \Sigma^n h_*} [A, \Sigma^{n+1} X]$$

is exact. A simple diagram chase yields that we can remove the signs and the sequence is still exact, so we have shown the desired statement for Σ. Now, we would like to show for $n > 0$ that the sequence

$$\Omega^n X \xrightarrow{\Omega^n f} \Omega^n Y \xrightarrow{\Omega^n g} \Omega^n Z \xrightarrow{\Omega^n h} \Omega^n \Sigma X \cong \Omega^{n-1} X$$

is exact, where the final isomorphism is $\Omega^{n-1}\eta_X^{-1}$. Now, consider the following diagram:

$$\begin{tikzcd}
\Omega^n X \arrow[r, "(-1)^n\Omega^n f"] \arrow[dd, equal] & \Omega^n Y \arrow[r, "(-1)^n\Omega^n g"] \arrow[dd, equal] & \Omega^n Z \arrow[rr, dashed] \arrow[dr, "(-1)^n\Omega^n h"] \arrow[dd, equal] & & \Sigma\Omega^n X \\
 & & & \Omega^n\Sigma X \arrow[dr, "\Omega^{n-1}\eta_X^{-1}"] & \\
\Omega^n X \arrow[r, "(-1)^n\Omega^n f"] & \Omega^n Y \arrow[r, "(-1)^n\Omega^n g"] & \Omega^n Z \arrow[rr, dashed] \arrow[ur, "(-1)^n\Omega^n h"] & & \Omega^{n-1} X \arrow[uu, "\varepsilon^{-1}_{\Omega^{n-1}X}"']
\end{tikzcd}$$

where the dashed arrows are the necessary compositions which makes this diagram commute. By Lemma A.8, the top row is distinguished, thus exact (by Proposition A.3). Thus, since the diagram commutes and the vertical arrows are isomorphisms, it follows that the bottom row is exact. Again, a simple diagram chase yields that we can forget the signs and the sequence is still exact, so we get precisely the desired result. $\square$

Proposition A.10. *Given a distinguished triangle*

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X,$$

the sequence

$$\cdots \to \Sigma^{n-1} Z \xrightarrow{\Sigma^{n-1} h} \Sigma^n X \xrightarrow{\Sigma^n f} \Sigma^n Y \xrightarrow{\Sigma^n g} \Sigma^n Z \xrightarrow{\Sigma^n h} \Sigma^{n+1} X \to \cdots$$

extending infinitely in either direction is exact (in the sense of Definition A.1), where for $n > 0$ we are defining $\Sigma^{-n} := \Omega^n$, and by abusive of notation we are writing $\Sigma^{-n} h$ to mean the composition

$$\Sigma^{-n} Z = \Omega^n Z \xrightarrow{\Omega^n h} \Omega^n \Sigma X \xrightarrow{\Omega^{n-1} \eta_X^{-1}} \Omega^{n-1} X = \Sigma^{-n+1} X,$$

where $\eta : \mathrm{Id}_{\mathcal{C}} \Rightarrow \Omega\Sigma$ is the unit of the adjunction $\Sigma \dashv \Omega$.

Proof. Exactness of

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X \xrightarrow{-\Sigma f} \Sigma Y$$

is Proposition A.3 and axiom TR4. A simple diagram chase yields that the sequence is still exact after removing the sign (change $-\Sigma f$ to Σf). Similarly, for $n > 0$, exactness of

$$\Sigma^n X \xrightarrow{\Sigma^n f} \Sigma^n Y \xrightarrow{\Sigma^n g} \Sigma^n Z \xrightarrow{\Sigma^n h} \Sigma^n X \xrightarrow{-\Sigma^n f} \Sigma^n Y$$

is Lemma A.9 combined with axiom TR4. Again, we can remove the sign and it is still exact. It remains to show exactness to the left. It suffices to show that the row in the following diagram is exact for all $n > 0$:

$$\begin{tikzcd} \Omega^n X \arrow[r, "\Omega^n f"] & \Omega^n Y \arrow[r, "\Omega^n g"] & \Omega^n Z \arrow[rr, "\Omega^{n-1}(\eta_X^{-1} \circ \Omega h)"] \arrow[dr, "\Omega^n h"'] & & \Omega^{n-1} X \arrow[r, "\Omega^{n-1} f"] & \Omega^{n-1} Y \\ & & & \Omega^n \Sigma X \arrow[ur, "\Omega^{n-1} \eta_X^{-1}"'] & & \end{tikzcd}$$

Exactness of the first three arrows is Lemma A.9. Thus, we would like to show that for all $n > 0$ that the row in the following diagram is exact:

(2)
$$\begin{tikzcd} \Omega^n Z \arrow[rr, "\Omega^{n-1}(\eta_X^{-1} \circ \Omega h)"] \arrow[dr, "\Omega^n h"'] & & \Omega^{n-1} X \arrow[r, "\Omega^{n-1} f"] & \Omega^{n-1} Y \\ & \Omega^n \Sigma X \arrow[ur, "\Omega^{n-1} \eta_X^{-1}"'] & & \end{tikzcd}$$

By Lemma A.7, the row in the following commutative diagram is a distinguished triangle:

$$\begin{tikzcd} \Omega Z \arrow[rr, "-\widetilde{h}"] \arrow[dr, "-\Omega h"'] & & X \arrow[r, "f"] & Y \arrow[r, "\widetilde{\Omega g}"] & \Sigma \Omega Z \\ & \Omega \Sigma X \arrow[ur, "\eta_X^{-1}"'] & & & \end{tikzcd}$$

Thus the row is exact by Proposition A.3, and again we can remove the sign to get that the row in (2) is exact when $n = 1$. Finally, we can apply Lemma A.9 to the distinguished triangle in the above diagram to get exactness of (2) when $n > 1$. $\square$

Also important for our work is the concept of a *tensor triangulated category*, that is, a triangulated symmetric monoidal category in which the triangulated structures are compatible, in the following sense:

Definition A.11. A *tensor triangulated category* is a triangulated symmetric monoidal category $(\mathcal{C}, \otimes, S, \Sigma, \Omega, \mathcal{D})$ such that:

TT1 For all objects X and Y in $\mathcal{C}$, there are natural isomorphisms
$$e_{X,Y} : \Sigma X \otimes Y \xrightarrow{\cong} \Sigma(X \otimes Y).$$

TT2 For each object X in $\mathcal{C}$, the functor $X \otimes (-) \cong (-) \otimes X$ is an additive functor.

TT3 For each object X in $\mathcal{C}$, the functor $X \otimes (-) \cong (-) \otimes X$ preserves distinguished triangles, in that given a distinguished triangle/(co)fiber sequence
$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A,$$
then also
$$X \otimes A \xrightarrow{X \otimes f} X \otimes B \xrightarrow{X \otimes g} X \otimes C \xrightarrow{X \otimes h} \Sigma(X \otimes A)$$
and
$$A \otimes X \xrightarrow{f \otimes X} B \otimes X \xrightarrow{g \otimes X} C \otimes X \xrightarrow{h \otimes X} \Sigma(A \otimes X)$$
are distinguished triangles, where here we are being abusive and writing $X \otimes h$ and $h \otimes X$ to denote the compositions
$$X \otimes C \xrightarrow{X \otimes h} X \otimes \Sigma A \xrightarrow{\tau} \Sigma A \otimes X \xrightarrow{e_{A,X}} \Sigma(A \otimes X) \xrightarrow{\Sigma\tau} \Sigma(X \otimes A)$$
and
$$C \otimes X \xrightarrow{h \otimes X} \Sigma A \otimes X \xrightarrow{e_{A,X}} \Sigma(A \otimes X),$$
respectively.

TT4 Given objects X, Y, and Z in $\mathcal{C}$, the following diagram must commute:
$$\begin{tikzcd} (\Sigma X \otimes Y) \otimes Z \arrow[r, "e_{X,Y} \otimes Z"] \arrow[d, "\alpha"'] & \Sigma(X \otimes Y) \otimes Z \arrow[r, "e_{X \otimes Y, Z}"] & \Sigma((X \otimes Y) \otimes Z) \arrow[d, "\alpha"] \\ \Sigma X \otimes (Y \otimes Z) \arrow[rr, "e_{X, Y \otimes Z}"'] & & \Sigma(X \otimes (Y \otimes Z)) \end{tikzcd}$$

TT5 The following diagram must commute for all $n, m \in \mathbb{Z}$:
$$\begin{tikzcd} \Sigma S \otimes \Sigma S \arrow[r, "e_{S,S}"] \arrow[d, "\tau"'] & \Sigma(S \otimes \Sigma S) \arrow[r, "\Sigma\lambda_{\Sigma S}"] & \Sigma^2 S \arrow[d, "-1"] \\ \Sigma S \otimes \Sigma S \arrow[r, "e_{S,S}"] & \Sigma(S \otimes \Sigma S) \arrow[r, "\Sigma\lambda_{\Sigma S}"] & \Sigma^2 S \end{tikzcd}$$

Usually, most tensor triangulated categories that arise in nature will satisfy additional coherence axioms (see axioms TC1–TC5 in [3]), but the above definition will suffice for our purposes. Note that in the definition of the tensor triangulated category, we chose isomorphisms
$$e_{X,Y} : \Sigma X \otimes Y \xrightarrow{\cong} \Sigma(X \otimes Y),$$
but we just as well could have chosen isomorphisms
$$e'_{X,Y} : X \otimes \Sigma Y \xrightarrow{\cong} \Sigma(X \otimes Y).$$

Definition A.12. Given a tensor triangulated category $(\mathcal{C}, \otimes, S, \Sigma, \Omega, e)$, there are natural isomorphisms
$$e'_{X,Y} : X \otimes \Sigma Y \xrightarrow{\cong} \Sigma(X \otimes Y)$$
obtained via the composition
$$X \otimes \Sigma Y \xrightarrow{\tau} \Sigma Y \otimes X \xrightarrow{e_{Y,X}} \Sigma(Y \otimes X) \xrightarrow{\Sigma\tau} \Sigma(X \otimes Y).$$

Proposition A.13. *The isomorphisms $e'_{X,Y} : X \otimes \Sigma Y \to \Sigma(X \otimes Y)$ defined above satisfy the following coherence condition for any objects X, Y, and Z:*

$$\begin{tikzcd}
(X \otimes Y) \otimes \Sigma Z \arrow[rr, "e'_{X\otimes Y,\Sigma Z}"] \arrow[d, "\alpha"'] & & \Sigma((X \otimes Y) \otimes Z) \arrow[d, "\Sigma\alpha"] \\
X \otimes (Y \otimes \Sigma Z) \arrow[r, "X \otimes e'_{Y,Z}"] & X \otimes \Sigma(Y \otimes Z) \arrow[r, "e'_{X,Y\otimes Z}"] & \Sigma(X \otimes (Y \otimes Z))
\end{tikzcd}$$

Proof. By the coherence theorem for monoidal categories, we may assume associativity holds up to strict equality, in which case we simply wish to show that the following diagram commutes:

$$\begin{tikzcd}
X \otimes Y \otimes \Sigma Z \arrow[r, "X \otimes e'_{Y,Z}"] \arrow[dr, "e'_{X\otimes Y,Z}"'] & X \otimes \Sigma(Y \otimes Z) \arrow[d, "e'_{X,Y\otimes Z}"] \\
& \Sigma(X \otimes Y \otimes Z)
\end{tikzcd}$$

Now consider the following diagram:

$$\begin{tikzcd}
X \otimes Y \otimes \Sigma Z \arrow[r, "X\otimes\tau_{Y,\Sigma Z}"] \arrow[d, "\tau_{X\otimes Y,\Sigma Z}"'] & X \otimes \Sigma Z \otimes Y \arrow[r, "X\otimes e_{Z,Y}"] \arrow[d, "\tau_{X,\Sigma Z\otimes Y}"] & X \otimes \Sigma(Z \otimes Y) \arrow[r, "X\otimes\Sigma\tau_{Z,Y}"] & X \otimes \Sigma(Y \otimes Z) \arrow[d, "\tau_{X,\Sigma(Y\otimes Z)}"] \\
\Sigma Z \otimes X \otimes Y \arrow[r, "\Sigma Z\otimes\tau_{X,Y}"] \arrow[dd, "e_{Z,X\otimes Y}"'] & \Sigma Z \otimes Y \otimes X \arrow[r, "e_{Z,Y}\otimes X"] \arrow[d, "e_{Z,Y\otimes X}"'] & \Sigma(Z \otimes Y) \otimes X \arrow[r, "\Sigma\tau_{Z,Y}\otimes X"] \arrow[dl, "e_{Z\otimes Y,X}"] & \Sigma(Y \otimes Z) \otimes X \arrow[d, "e_{Y\otimes Z,X}"] \\
& \Sigma(Z \otimes Y \otimes X) \arrow[rr, "\Sigma(\tau_{Z,Y}\otimes X)"] \arrow[d, "\Sigma\tau_{Z\otimes Y,X}"] & & \Sigma(Y \otimes Z \otimes X) \arrow[d, "\Sigma\tau_{Y\otimes Z,X}"] \\
\Sigma(Z \otimes X \otimes Y) \arrow[r, "\Sigma(\tau_{Z,X}\otimes Y)"] \arrow[ur, "\Sigma(Z\otimes\tau_{X,Y})"] \arrow[rrr, bend right, "\Sigma\tau_{Z,X\otimes Y}"'] & \Sigma(X \otimes Z \otimes Y) \arrow[rr, "\Sigma(X\otimes\tau_{Z,Y})"] & & \Sigma(X \otimes Y \otimes Z)
\end{tikzcd}$$

Unravelling definitions, the top composition is $e'_{X,Y\otimes Z} \circ X \otimes e'_{Y,Z}$ and the bottom composition is $e'_{X\otimes Y,Z}$, so it suffices to show this diagram commutes. The top left square commutes by coherence for symmetric monoidal categories. The trapezoid below that on the left commutes by naturality of e. The triangle below that commutes by coherence for symmetric monoidal categories. The top right rectangle commutes by functoriality of $- \otimes -$ and naturality of τ. The small triangle below that in the middle of the diagram commutes by axiom TT4 for a tensor triangulated category. Commutativity of the trapezoid on the middle right is naturality of e. Finally, the remaining two regions on the bottom commutes by coherence for symmetric monoidal categories. □

Appendix B. Spectral sequences

In what follows, we fix an abelian group A. We will freely use the theory and results of Appendix C

Definition B.1. An *A-graded spectral sequence* is the data of a collection of A-graded abelian groups $\{E_r^*\}_{r \geq r_0}$ along with homomorphisms of A-graded abelian groups $d_r : E_r \to E_r$ (usually of nonzero degree) such that $d_r \circ d_r = 0$ and $E_{r+1} = \ker d_r / \operatorname{im} d_r$.

Appendix C. (Co)algebra

C.1. **Grading.** First, we develop the theory of things graded by an abelian group. In what follows, we fix an abelian group A. We assume the reader is familiar with the basic theory of modules over non-commutative rings.

Definition C.1. An *A-graded abelian group* is an abelian group B along with a subgroup $B_a \leq B$ for each $a \in A$ such that the canonical map

$$\bigoplus_{a\in A} B_a \to B$$

sending $(x_a)_{a\in A}$ to $\sum_{a\in A} x_a$ is an isomorphism. Given two A-graded abelian groups B and C, a homomorphism $f : B \to C$ is a *homomorphism of A-graded abelian groups* if it preserves the grading, i.e., if it restricts to a map $B_a \to C_a$ for all $a \in A$.

Remark C.2. We often will denote an A-graded abelian group B by B_*. Given some $a \in A$, we can define the shifted A-graded abelian group B_{*+a} whose b^{th} component is B_{b+a}.

Definition C.3. More generally, given two A-graded abelian groups B and C and some $d \in A$, a group homomorphism $f : B \to C$ is *an A-graded homomorphism of degree d* if it restricts to a map $B_a \to C_{a+d}$ for all $a \in A$. Thus, an A-graded homomorphism of degree d from B to C is equivalently an A-graded homomorphism $B_* \to C_{*+d}$.

Unless stated otherwise, an "A-graded homomorphism" will always refer to an A-graded homomorphism of degree 0. It is easy to see that an A-graded abelian group B is generated by its *homogeneous* elements, that is, nonzero elements $x \in B$ such that there exists some $a \in A$ with $x \in B_a$.

Remark C.4. Clearly the condition that the canonical map $\bigoplus_{a\in A} B_a \to B$ is an isomorphism requires that $B_a \cap B_b = 0$ if $a \neq b$. In particular, given a homogeneous element $x \in B$, there exists precisely one $a \in A$ such that $x \in B_a$. We call this a the *degree* of x, and we write $|x| = a$.

Lemma C.5. *Given two A-graded abelian groups B and C, their product $B \oplus C$ is naturally an A-graded abelian group by defining*

$$(B \oplus C)_a := \bigoplus_{b+c=a} B_b \oplus C_c.$$

Proof. This is entirely straightforward, as

$$B \oplus C \cong \left(\bigoplus_{b\in A} B_b\right) \oplus \left(\bigoplus_{c\in A} C_c\right) \cong \bigoplus_{b,c\in A} B_b \oplus C_c \cong \bigoplus_{a\in A}\bigoplus_{b\in A} B_b \oplus C_{a-b} \cong \bigoplus_{a\in A}\left(\bigoplus_{b+c=a} B_b \oplus C_c\right).$$

$\square$

Definition C.6. An A-graded ring R is a ring such that is underlying abelian group is A-graded, and the multiplication map $R \times R \to R$ is a (degree 0) homomorphism of A-graded abelian groups (here R has the structure of an A-graded abelian group by Lemma C.5).

Definition C.7. Let R be an A-graded ring. A left A-graded R-module M is a left R-module M such that M is an A-graded abelian group, and the multiplication map $R \times M \to M$ is a homomorphism of A-graded abelian groups (i.e., for all $a, b \in A$ this map must restrict to $R_a \times M_b \to M_{a+b}$). Right A-graded R-modules are defined similarly. Finally, an A-graded R-bimodule is an A-graded abelian group M along with action maps

$$R \times M \to M \qquad \text{and} \qquad M \times R \to M$$

which endow M with the structure of a left and right A-graded R-module, respectively, such that given $r, s \in R$ and $m \in M$, $r \cdot (m \cdot s) = (r \cdot m) \cdot s$.

Definition C.8. An A-graded map of A-graded rings (resp. left/right A-graded R-modules) is a homomorphism of rings (resp. left/right R-modules) such that the underlying homomorphism of abelian groups is A-graded.

Explicitly, given an A-graded ring R and homogeneous elements $x, y \in R$, we must have $|xy| = |x| + |y|$. For example, given some field k, the ring $R = k[x,y]$ is $\mathbb{Z}^2$-graded, where given $(n,m) \in \mathbb{Z}^2$, $R_{n,m}$ is the subgroup of those monomials of the form ax^ny^m for some $a \in k$. Oftentimes when constructing A-graded rings, we do so only by defining the product of homogeneous elements, like so:

Proposition C.9. *Given an A-graded abelian group R, a distinguished element $1 \in R_0$, and $\mathbb{Z}$-bilinear maps $m_{a,b} : R_a \times R_b \to R_{a+b}$ for all $a, b \in A$ such that given $x \in R_a$, $y \in R_b$, and $z \in R_c$,*

$$m_{a+b,c}(m_{a,b}(x,y),z) = m_{a,b+c}(x, m_{b,c}(y,z)) \qquad \text{and} \qquad m_{a,0}(x,1) = m_{0,a}(1,x) = x,$$

there exists a unique multiplication map $m : R \times R \to R$ which endows R with the structure of an A-graded ring and restricts to $m_{a,b}$ for all $a, b \in A$.

Proof. Given $r, s \in R$, since $R \cong \bigoplus_{a \in A} R_a$, we may uniquely decompose r and s into homogeneous elements as $r = \sum_{a \in A} r_a$ and $s = \sum_{a \in A} s_a$ with each $r_a, s_a \in R_a$ such that only finitely many of the r_a's and s_a's are nonzero. Then in order to define a distributive product $R \times R \to R$ which restricts to $m_{a,b} : R_a \times R_b \to R_{a+b}$, note we *must* define

$$r \cdot s = \left(\sum_{a \in A} r_a\right) \cdot \left(\sum_{b \in A} s_b\right) = \sum_{a,b \in A} r_a \cdot s_b = \sum_{a,b \in A} m_{a,b}(r_a, s_b).$$

Thus, we have shown uniqueness. It remains to show this product actually gives R the structure of a ring. First we claim that the sum on the right is actually finite. Note there exists only finitely many nonzero r_a's and s_b's, and if $s_b = 0$ then

$$m_{a,b}(r_a, 0) = m_{a,b}(r_a, 0+0) \stackrel{(*)}{=} m_{a,b}(r_a, 0) + m_{a,b}(r_a, 0) \implies m_{a,b}(r_a, 0) = 0,$$

where $(*)$ follows from bilinearity of $m_{a,b}$. A similar argument yields that $m_{a,b}(0, r_b) = 0$ for all $a, b \in A$. Hence indeed $m_{a,b}(r_a, s_b)$ is zero for all but finitely many pairs $(a,b) \in A^2$, as desired. Observe that in particular

$$(r \cdot s)_a = \sum_{b+c=a} m_{b,c}(r_b, s_c) = \sum_{b \in A} m_{b,a-b}(r_b, s_{a-b}) = \sum_{c \in A} m_{a-c,c}(r_{a-c}, s_c).$$

Now we claim this multiplication is associative. Given $t = \sum_{a \in A} t_a \in R$, we have

$$\begin{aligned}
(r \cdot s) \cdot t &= \sum_{a,b \in A} m_{a,b}((r \cdot s)_a, t_b) \\
&= \sum_{a,b \in A} m_{a,b}\left(\sum_{c \in A} m_{a-c,c}(r_{a-c}, s_c), t_b\right) \\
&\stackrel{(1)}{=} \sum_{a,b,c \in A} m_{a,b}(m_{a-c,c}(r_{a-c}, s_c), t_b) \\
&\stackrel{(2)}{=} \sum_{a,b,c \in A} m_{c,a+b-c}(r_c, m_{a-c,b}(s_{a-c}, t_b)) \\
&\stackrel{(3)}{=} \sum_{a,b,c \in A} m_{a,c}(r_a, m_{b,c-b}(s_b, t_{c-b})) \\
&\stackrel{(1)}{=} \sum_{a,c \in A} m_{a,c}\left(r_a, \sum_{b \in A} m_{b,c-b}(s_b, t_{c-b})\right) \\
&= \sum_{a,c \in A} m_{a,c}(r_a, (s \cdot t)_c) = r \cdot (s \cdot t),
\end{aligned}$$

where each occurrence of (1) follows by bilinearity of the $m_{a,b}$'s, each occurrence of (2) is associativity of the $m_{a,b}$'s, and (3) is obtained by re-indexing by re-defining $a := c$, $b := a - c$, and $c := a + b - c$. Next, we wish to show that the distinguished element $1 \in R_0$ is a unit with respect to this multiplication. Indeed, we have

$$1 \cdot r \overset{(1)}{=} \sum_{a \in A} m_{0,a}(1, r_a) \overset{(2)}{=} \sum_{a \in A} r_a = r$$

and

$$r \cdot 1 \overset{(1)}{=} \sum_{a \in A} m_{a,0}(r_a, 1) \overset{(2)}{=} \sum_{a \in A} r_a = r,$$

where (1) follows by the fact that $m_{a,b}(0, -) = m_{a,b}(-, 0) = 0$, which we have shown above, and (2) follows by unitality of the $m_{0,a}$'s and $m_{0,a}$'s, respectively. Finally, we wish to show that this product is distributive. Indeed, we have

$$\begin{aligned} r \cdot (s + t) &= \sum_{a,b \in A} m_{a,b}(r_a, (s + t)_b) \\ &= \sum_{a,b \in A} m_{a,b}(r_a, s_b + t_b) \\ &\overset{(*)}{=} \sum_{a,b \in A} m_{a,b}(r_a, s_b) + \sum_{a,b \in A} m_{a,b}(r_a, t_b) = (r \cdot s) + (r \cdot t), \end{aligned}$$

where $(*)$ follows by bilinearity of $m_{a,b}$. An entirely analagous argument yields that $(r + s) \cdot t = (r \cdot t) + (s \cdot t)$. $\square$

When working with A-graded abelian groups, we will freely use the above proposition without comment.

Proposition C.10. *Let R be an A-graded ring, and suppose we have a right A-graded R-module M and a left A-graded R-module N. Then the tensor product*

$$M \otimes_R N$$

is naturally an A-graded abelian group by defining $(M \otimes_R N)_a$ to be the subgroup generated by homogeneous pure tensors $m \otimes n$ with $m \in M_b$ and $n \in N_c$ such that $b + c = a$. Furthermore, if either M (resp. N) is an A-graded bimodule, then $M \otimes_R N$ is naturally a left (resp. right) A-graded R-module

Proof. By definition, since M and N are A-graded abelian groups, they are generated (as abelian groups) by their homogeneous elements. Thus it follows that $M \otimes_R N$ is generated by *homogeneous pure tensors*, that is, elements of the form $m \otimes n$ with $m \in M$ and $n \in N$ homogeneous. Now, given a homogeneous pure tensor $m \otimes n$, we define its *degree* by the formula $|m \otimes n| := |m| + |n|$. It follows this formula is well-defined by checking that given homogeneous elements $m \in M$, $n \in N$, and $r \in R$ that

$$|(m \cdot r) \otimes n| = |m \cdot r| + |n| = |m| + |r| + |n| = |m| + |r \cdot n| = |m \otimes (r \cdot n)|.$$

Thus, we may define $(M \otimes_R N)_a$ to be the subgroup of $M \otimes_R N$ generated by those pure homogeneous tensors of degree a. Now, we construct a map

$$\Phi : M \times N \to \bigoplus_{a \in A} (M \otimes_R N)_a$$

which takes a pair $(m, n) = \sum_{a \in A} (m_a, n_a)$ to the element $\Phi(m, n)$ whose a^{th} component is

$$(\Phi(m, n))_a := \sum_{b+c=a} m_a \otimes n_a.$$

It is straightforward to see that this map is R-balanced, in the sense that it is additive in each argument and $\Phi(m \cdot r, n) = \Phi(m, r \cdot n)$ for all $m \in M$, $n \in N$, and $r \in R$. Thus by the universal property of $M \otimes_R N$, we get a lift $\widetilde{\Phi} : M \otimes_R N \to \bigoplus_{a \in A} (M \otimes_R N)_a$. Now, also consider the canonical map

$$\Psi : \bigoplus_{a \in A} (M \otimes_R N)_a \to M \otimes_R N.$$

We would like to show $\widetilde{\Phi}$ and Ψ are inverses of eah other. It suffices to show this on generators. Let $m \otimes n$ be a pure homogeneous tensor with $m = m_a \in M_a$ and $n = n_a \in N_b$. Then we have

$$\Psi(\widetilde{\Phi}(m \otimes n)) = \Psi\left(\bigoplus_{a \in A} \sum_{b+c=a} m_b \otimes n_c\right) \overset{(*)}{=} \Psi(m \otimes n) = m \otimes n,$$

and

$$\widetilde{\Phi}(\Psi(m \otimes n)) = \widetilde{\Phi}(m \otimes n) = \bigoplus_{a \in A} \sum_{b+c=a} m_b \otimes n_c \overset{(*)}{=} m \otimes n,$$

where both occurrences of $(*)$ follow by the fact that $m_b \otimes n_c = 0$ unless $b = c = a$, in which case $m_a \otimes n_a = m \otimes n$. Thus since Ψ is an isomorphism, $M \otimes_R N$ is indeed an A-graded abelian group, as desired.

Now, suppose that M is an A-graded R-bimodule, so there exists a left and right action of R on M such that given $r, s \in R$ and $m \in M$ we have $r \cdot (m \cdot s) = (r \cdot m) \cdot s$. Then we would like to show that given a left A-graded R-module N that $M \otimes_R N$ is canonically a left A-graded R-module. Indeed, define the action of R on $M \otimes_R N$ on pure tensors by the formula

$$r \cdot (m \otimes n) = (r \cdot m) \otimes n.$$

First of all, clearly this map is A-graded, as if $r \in R_a$, $m \in M_b$, and $n \in N_c$ then $(r \cdot m) \otimes n$, by definition, has degree $|r \cdot m| + |n| = |r| + |m| + |n|$ (the last equality follows since the left action of R on M is A-graded). In order to show the above map defines a left module structure, it suffices to show that given pure tensors $m \otimes n, m' \otimes n' \in M \otimes_R N$ and elements $r, r' \in R$ that

(1) $r \cdot (m \otimes n + m' \otimes n') = r \cdot (m \otimes n) + r \cdot (m' \otimes n')$,
(2) $(r + r') \cdot (m \otimes n) = r \cdot (m \otimes n) + r' \cdot (m' \otimes n')$,
(3) $(rr') \cdot (m \otimes n) = r \cdot (r' \cdot (m \otimes n))$, and
(4) $1 \cdot (m \otimes n) = m \otimes n$.

Axiom (1) holds by definition. To see (2), note that by the fact that R acts on M on the left that

$$(r + r') \cdot (m \otimes n) = ((r + r') \cdot m) \otimes n = (r \cdot m + r' \cdot m) \otimes n = r \cdot m \otimes n + r' \cdot m \otimes n.$$

That (3) and (4) hold follows similarly by the fact that $(rr') \cdot m = r \cdot (r' \cdot m)$ and $1 \cdot m = m$.

Conversely, if N is an A-graded R-bimodule, then showing $M \otimes_R N$ is canonically a right A-graded R-module via the rule

$$(m \otimes n) \cdot r = m \otimes (n \cdot r)$$

is entirely analagous. $\square$

Lemma C.11. *Let R be an A-graded ring, and suppose we have a right A-graded R-module M and a left A-graded R-module N. Then given an A-graded abelian group B and an A-graded R-balanced map*

$$\varphi : M \times N \to B$$

(here $M \times N$ is regarded as an A-graded abelian group by Lemma C.5), the lift

$$\widetilde{\varphi} : M \otimes_R N \to B$$

determined by the universal property of $M \otimes_R N$ is an A-graded map.

Proof. This simply amounts to unravelling definitions. Recall that the subgroup of homogeneous elements of degree a in $M \otimes_R N$ is that generated by pure tensors $m \otimes n$ with m and n homogeneous satisfying $|m| + |n| = a$. Thus, in order to show $\widetilde{\varphi}$ is an A-graded homomorphism, it suffices to show that given homogeneous $m \in M$ and $n \in N$, we have

$$|\widetilde{\varphi}(m \otimes n)| = |m \otimes n| = |m| + |n|.$$

Indeed, given two such elements, consider the following diagram

$$\begin{tikzcd} M \otimes_R N \arrow[dr, "\widetilde{\varphi}"] & \\ M \times N \arrow[u] \arrow[r, "\varphi"'] & B \end{tikzcd}$$

This diagram commutes by universal property of $- \otimes_R -$. Note that the element $m \otimes n$ is mapped to by the pair (m, n) along the left vertical map. Hence by commutativity, we necessarily have

$$|\widetilde{\varphi}(m \otimes n)| = |\varphi(m, n)| \overset{(*)}{=} |(m, n)| = |m| + |n|,$$

where $(*)$ follows by the fact that φ is an A-graded map. □

Lemma C.12. *Let R be an A-graded ring, and suppose we have an A-graded R-bimodule M. Then for all $a \in A$, we have an A-graded isomorphism of left A-graded R-modules*

$$M \otimes_R R_{*+a} \cong M_{*+a}$$

induced by the assignment

$$M \times R_{*+a} \to M_{*+a}$$

sending $m \in M_b$ and $r \in R_{c+a}$ to $m \cdot r \in M_{b+c+a}$ (where here $M \otimes_R R$ has the structure of a left A-graded R-module by Proposition C.10, and $m \cdot r$ denotes the right action of r on m).

Proof. First of all, note that if you ignore the grading then the map $M \times R_{*+a} \to M_{*+a}$ is simply the structure map for the right action of R on M. In particular, by the module axioms this map is R-balanced, so it does indeed induce an A-graded homomorphism of A-graded abelian groups $\varphi : M \otimes_R R_{*+a} \to M_{*+a}$. Furthermore, note this map is actually a homomorphism of left A-graded R-modules, as given $m \in M$ and $r, r' \in R$, we have $r \cdot (m \cdot r') = (r \cdot m) \cdot r'$, since M is a bimodule. Now, to see this map is an isomorphism, it suffices to construct an inverse. Indeed, define the map

$$\psi : M_{*+a} \to M \otimes_R R_{*+a}$$

to send $m \mapsto m \otimes 1$. First of all note this map is A-graded, as given $m \in M_{b+a}$, we have $\psi(m) = m \otimes 1$ has degree $|m| + |1| = |m| = b + a$, by definition of the graded structure on $M \otimes_R R_{*+a}$. Note that it is a homomorphism of left R-modules, as given $m, m' \in M$ and $r, r' \in R$ we have

$$\psi(rm + r'm') = (rm + r'm') \otimes 1 = r(m \otimes 1) + r'(m' \otimes 1) = r\psi(m) + r'\psi(m').$$

Now, to see ψ and φ are inverses, note first that given $m \in M_{*+a}$ that

$$\varphi(\psi(m)) = \varphi(m \otimes 1) = m \cdot 1 = m,$$

and given $m \otimes r \in M \otimes_R R_{*+a}$,

$$\psi(\varphi(m \otimes r)) = \psi(m \cdot r) = (m \cdot r) \otimes 1 = m \otimes (r \cdot 1) = m \otimes r.$$

□

Appendix D. Monoid objects in a stable homotopy category

In this section, we will freely use the coherence theorem for a symmetric monoidal category, which says that every symmetric monoidal category is (monoidally) equivalent to a *permutative category*, that is, a symmetric monoidal category in which the associators and unitors are strict equalities. add ref

Definition D.1. Let $(\mathcal{C}, \otimes, S)$ be a symmetric monoidal category with left unitor, right unitor, and associator, and symmetry isomorphism λ, ρ, α, and τ, respectively. Then a *monoid object* (E, μ, e) is an object E in $\mathcal{C}$ along with a multiplication map $\mu : E \otimes E \to E$ and a unit map $e : S \to E$ such that the following diagram commutes:

$$\begin{tikzcd} E \otimes S \arrow[r, "E\otimes e"] \arrow[dr, "\rho"'] & E \otimes E \arrow[d, "\mu"] & S \otimes E \arrow[l, "e\otimes E"'] \arrow[dl, "\lambda"] \\ & E & \end{tikzcd} \qquad \begin{tikzcd} (E\otimes E)\otimes E \arrow[rr, "\mu\otimes E"] \arrow[d, "\alpha"] & & E\otimes E \arrow[d, "\mu"] \\ E\otimes(E\otimes E) \arrow[r, "E\otimes\mu"] & E\otimes E \arrow[r, "\mu"] & E \end{tikzcd}$$

The first diagram expresses unitality, while the second expressed associativity. If in addition the following diagram commutes,

$$\begin{tikzcd} E\otimes E \arrow[rr, "\tau"] \arrow[dr, "\mu"'] & & E\otimes E \arrow[dl, "\mu"] \\ & E & \end{tikzcd}$$

then we say (E, μ, e) is a *commutative* monoid object.

Proposition D.2. *Let (E_1, μ_1, e_1) and (E_2, μ_2, e_2) be monoid objects in a symmetric monoidal category $(\mathcal{C}, \otimes, S)$. Then $E_1 \otimes E_2$ is canonically a ring spectrum via the maps*

$$\mu : E_1 \otimes E_2 \otimes E_1 \otimes E_2 \xrightarrow{E_1\otimes\tau\otimes E_2} E_1 \otimes E_1 \otimes E_2 \otimes E_2 \xrightarrow{\mu_1\otimes\mu_2} E_1 \otimes E_2$$

and

$$e : S \cong S \otimes S \xrightarrow{e_1\otimes e_2} E_1 \otimes E_2.$$

Proof. □ todo

In what follows, fix a stable homotopy category $\mathcal{SH}$ (**??**) along with the additional data therewithin, and adopt the conventions outlined in Section 2.1. Further suppose we have fixed a coherent family of isomorphisms

$$\phi_{a,b} : S^{a+b} \xrightarrow{\cong} S^a \otimes S^b,$$

in the sense of Definition 2.3 (the existence of such a coherent family is guaranteed by Theorem 2.4).

Proposition D.3. *Let (E, μ, e) be a commutative monoid object in $\mathcal{SH}$, and consider the multiplication map $\pi_*(E) \times \pi_*(E) \to \pi_*(E)$ which sends classes $x : S^a \to E$ and $y : S^b \to E$ to the composition*

$$S^{a+b} \xrightarrow{\phi_{a,b}} S^a \otimes S^b \xrightarrow{x\otimes y} E \otimes E \xrightarrow{\mu} E.$$

Then this endows $\pi_*(E)$ with the structure of an A-graded ring with unit $e \in \pi_0(E) = [S, E]$.

Proof. In this proof, we will assume we are working in a permutative category. Suppose we have classes x, y, and z in $\pi_a(E)$, $\pi_b(E)$, and $\pi_c(E)$, respectively. To see associativity, consider the following diagram:

$$\begin{tikzcd}
 & & & E\otimes E \arrow[dr, phantom, ""] \arrow[d, "\mu"] \\
S^{a+b+c} \arrow[r, "\cong"] & S^a\otimes S^b\otimes S^c \arrow[r, "x\otimes y\otimes z"] & E\otimes E\otimes E \arrow[ur, "\mu\otimes E"] \arrow[dr, "E\otimes\mu"'] & E \\
 & & & E\otimes E \arrow[u, "\mu"']
\end{tikzcd}$$

(here the first arrow is the unique isomorphism obtained by composing products of $\phi_{a,b}$'s, see Remark 2.5). It commutes by associativity of μ. It follows by functoriality of $-\otimes-$ that the top composition is $(x\cdot y)\cdot z$ while the bottom is $x\cdot(y\cdot z)$, so they are equal as desired. To see that $e\in\pi_0(E)$ is a left and right unit for this multiplication, consider the following diagram

$$\begin{tikzcd}
 & S^a \arrow[dl, "e\otimes x"'] \arrow[d, "x"] \arrow[dr, "x\otimes e"] & \\
E\otimes E \arrow[dr, "\mu"'] & E \arrow[l, "e\otimes E"'] \arrow[r, "E\otimes e"] \arrow[d, equal] & E\otimes E \arrow[dl, "\mu"] \\
 & E &
\end{tikzcd}$$

Commutativity of the two top triangles is functoriality of $-\otimes-$. Commutativity of the bottom two triangles is unitality of μ. Thus the diagram commutes, so $e\cdot x = x\cdot e$. Finally, to see this product is bilinear (distributive). Suppose we further have some $x'\in\pi_a(E)$ and $y'\in\pi_b(E)$, and consider the following diagrams:

$$\begin{tikzcd}
S^{a+b} \arrow[r, "\phi_{a,b}"] \arrow[d, "\Delta"'] & S^a\otimes S^b \arrow[r, "\Delta\otimes S^b"] \arrow[d, "\Delta"] & (S^a\oplus S^a)\otimes S^b \arrow[r, "(x\oplus x')\otimes y"] \arrow[dl, "\cong"'] & (E\oplus E)\otimes E \arrow[dl, "\cong"'] \arrow[d, "\nabla\otimes E"] & \\
S^{a+b}\oplus S^{a+b} \arrow[r, "\phi_{a,b}\oplus\phi_{a,b}"'] & (S^a\otimes S^b)\oplus(S^a\otimes S^b) \arrow[r, "(x\otimes y)\oplus(x'\otimes y)"'] & (E\otimes E)\oplus(E\otimes E) \arrow[r, "\nabla"'] & E\otimes E \arrow[r, "\mu"] & E
\end{tikzcd}$$

$$\begin{tikzcd}
S^{a+b} \arrow[r, "\phi_{a,b}"] \arrow[d, "\Delta"'] & S^a\otimes S^b \arrow[r, "S^a\otimes\Delta"] \arrow[d, "\Delta"] & S^b\otimes(S^b\oplus S^b) \arrow[r, "x\otimes(y\oplus y')"] \arrow[dl, "\cong"'] & E\otimes(E\oplus E) \arrow[dl, "\cong"'] \arrow[d, "E\otimes\nabla"] & \\
S^{a+b}\oplus S^{a+b} \arrow[r, "\phi_{a,b}\oplus\phi_{a,b}"'] & (S^a\otimes S^b)\oplus(S^a\otimes S^b) \arrow[r, "(x\otimes y)\oplus(x\otimes y')"'] & (E\otimes E)\oplus(E\otimes E) \arrow[r, "\nabla"'] & E\otimes E \arrow[r, "\mu"] & E
\end{tikzcd}$$

The unlabeled isomorphisms are those given by the fact that $-\otimes-$ is additive in each variable (since $\mathcal{SH}$ is tensor triangulated). Commutativity of the left squares is naturality of $\Delta : X\to X\oplus X$ in an additive category. Commutativity of the rest of the diagram follows again from the fact that $-\otimes-$ is an additive functor in each variable. Hence, by functoriality of $-\otimes-$, these diagrams tell us that $(x+x')\cdot y = x\cdot y + x'\cdot y$ and $x\cdot(y+y') = x\cdot y + x\cdot y'$, respectively. $\square$

Proposition D.4. *For all $a,b\in A$ there exists an element $\theta_{a,b}\in\pi_0(S)=[S,S]$ (determined by choice of coherent family $\{\phi_{a,b}\}$) such that given any commutative monoid object (E,μ,e) in $\mathcal{SH}$, the A-graded ring structure on $\pi_*(E)$ (Proposition 2.6) has a commutativity formula given by*

$$x\cdot y = y\cdot x\cdot(e\circ\theta_{a,b})$$

for all $x\in\pi_a(E)$ and $y\in\pi_b(E)$. In particular, $\theta_{a,b}\in\mathrm{Aut}(S)$ is the composition

$$S\xrightarrow{\cong} S^{-a-b}\otimes S^a\otimes S^b \xrightarrow{S^{-a-b}\otimes\tau} S^{-a-b}\otimes S^b\otimes S^a \xrightarrow{\cong} S,$$

where the outermost maps are the unique maps specified by Remark 2.5.

Proof. Let $\phi_{a,b}$, E, x, and y as in the statement of the proposition. Now consider the following diagram

$$
\begin{tikzcd}
S^{a+b} \arrow[r, "\phi_{a,b}"] \arrow[dd, dashed, "\phi_{b,a}^{-1}\circ\tau\circ\phi_{a,b}"'] & S^a \otimes S^b \arrow[r, "x\otimes y"] \arrow[dd, "\tau"'] & E \otimes E \arrow[dd, "\tau"'] \arrow[dr, "\mu"] & \\
& & & E \\
S^{a+b} \arrow[r, "\phi_{b,a}"] & S^b \otimes S^a \arrow[r, "y\otimes x"] & E \otimes E \arrow[ur, "\mu"'] &
\end{tikzcd}
$$

The left square commutes by definition. The middle square commutes by naturality of the symmetry isomorphism. Finally, the right square commutes by commutativity of E. Unravelling definitions, we have shown that under the product on $\pi_*(E)$ induced by the $\phi_{a,b}$'s,

$$x \cdot y = (y \cdot x) \circ (\phi_{b,a}^{-1} \circ \tau \circ \phi_{a,b}).$$

Thus, in order to show the desired result it further suffices to show that

$$(y \cdot x) \circ (\phi_{b,a}^{-1} \circ \tau \circ \phi_{a,b}) = y \cdot x \cdot (e \circ \theta_{a,b}).$$

Consider the following diagram:

$$
\begin{tikzcd}
S^{a+b} \arrow[rr, "\phi_{a,b}"] \arrow[d, "\cong"'] & & S^a \otimes S^b \arrow[d, "\tau"] \\
S^b \otimes S^a \otimes S^{-a-b} \otimes S^a \otimes S^b \arrow[urr, "\cong"'] \arrow[d, "S^b\otimes S^a\otimes S^{-a-b}\otimes\tau"'] & & S^b \otimes S^a \arrow[d, "\phi_{b,a}^{-1}"] \arrow[dl, equal] \\
S^b \otimes S^a \otimes S^{-a-b} \otimes S^b \otimes S^a \arrow[r, "\cong"] \arrow[urr, "\cong"] & S^b \otimes S^a \arrow[dr, "y\otimes x"] \arrow[ddl, "y\otimes x\otimes e"'] & S^{a+b} \arrow[l, "\phi_{b,a}"'] \\
& & E \otimes E \arrow[d, equal] \arrow[dll, "E\otimes E\otimes e"'] \\
E \otimes E \otimes E \arrow[rr, "E\otimes\mu"] \arrow[d, "\mu\otimes E"'] & & E \otimes E \arrow[d, "\mu"] \\
E \otimes E \arrow[rr, "\mu"'] & & E
\end{tikzcd}
$$

Here any map simply labelled $\cong$ is an appropriate composition of copies of $\phi_{a,b}$'s, associators, and their inverses, so that each of these maps are necessarily unique by Remark 2.5. The two triangles in the top large rectangle commutes by coherence for the $\phi_{a,b}$'s. The parallelogram commutes by naturality of τ and coherence of the of $\phi_{a,b}$'s. The middle skewed triangle commutes by functoriality of $- \otimes -$. The triangle below that commutes by unitality of μ. Finally, the bottom rectangle commmutes by associativity of μ. Hence, by unravelling definitions and applying functoriality of $- \otimes -$, we get that the right composition is $(y \cdot x) \circ (\phi_{b,a}^{-1} \circ \tau \circ \phi_{a,b})$, while the left composition is $y \cdot x \cdot (e \circ \theta_{a,b})$, so they are equal as desired. ∎

Proposition D.5. *Given $a \in A$, we have $\theta_{0,a} = \theta_{a,0} = \mathrm{id}_S$.*

Proof. Recall $\theta_{a,0}$ is the composition

$$S \xrightarrow{\phi_{-a,a}} S^{-a} \otimes S^a \xrightarrow{S^{-a}\otimes\phi_{a,0}} S^{-a} \otimes (S^a \otimes S) \xrightarrow{S^{-a}\otimes\tau} S^{-a} \otimes (S \otimes S^a) \xrightarrow{S^{-a}\otimes\phi_{0,a}^{-1}} S^{-a} \otimes S^a \xrightarrow{\phi_{-a,a}^{-1}} S$$

By the coherence theorem for symmetric monoidal categories and the fact that $\phi_{a,0}$ and $\phi_{0,a}$ coincide with the unitors, we have that the composition

$$S^a \xrightarrow{\phi_{a,0}=\rho_{S^a}^{-1}} S^a \otimes S \xrightarrow{\tau} S \otimes S^a \xrightarrow{\phi_{0,a}^{-1}=\lambda_{S^a}} S^a$$

is precisely the identity map, so by functoriality of $-\otimes -$, we have that $\theta_{a,0}$ is the composition

$$S \xrightarrow{\phi_{-a,a}} S^{-a}\otimes S^a \xrightarrow{=} S^{-a}\otimes S^a \xrightarrow{\phi^{-1}_{-a,a}} S,$$

so $\theta_{a,0} = \mathrm{id}_S$, meaning

$$x\cdot y = y\cdot x\cdot (e\circ \theta_{a,0}) = y\cdot x\cdot e = y\cdot x,$$

where the last equality follows by the fact that e is the unit for the multiplication on $\pi_*(E)$. An entirely analagous argument yields that $\theta_{0,a} = \mathrm{id}_S$. □

Proposition D.6. *Let X and Y be objects in $\mathcal{SH}$. Then the pairing*

$$\pi_*(X)\times \pi_*(Y) \to \pi_*(X\otimes Y)$$

sending $x : S^a \to X$ and $y : S^b \to Y$ to the composition

$$S^{a+b} \xrightarrow{\phi_{a,b}} S^a\otimes S^b \xrightarrow{x\otimes y} X\otimes Y$$

is additive in each argument.

Proof. Let $a, b\in A$, and let $x_1, x_2 : S^a \to X$ and $y : S^b \to Y$. Then consider the following diagram

$$\begin{tikzcd}
S^{a+b} \arrow[r, "\cong"] & S^a\otimes S^b \arrow[r, "\Delta\otimes S^b"] \arrow[d, "\Delta"'] & (S^a\oplus S^a)\otimes S^b \arrow[d, "(x_1\oplus x_2)\otimes y"] \arrow[dl, "\cong"'] \\
 & (S^a\otimes S^b)\oplus(S^a\otimes S^b) \arrow[d, "(x_1\otimes y)\oplus(x_2\otimes y)"'] & (X\oplus X)\otimes Y \arrow[d, "\nabla\otimes Y"] \arrow[dl, "\cong"'] \\
 & (X\otimes Y)\oplus(X\otimes Y) \arrow[r, "\nabla"] & X\otimes Y
\end{tikzcd}$$

The isomorphisms are given by the fact that $-\otimes -$ is additive in each variable. Both triangles and the parallelogram commute since $-\otimes -$ is additive. By functoriality of $-\otimes -$, the top composition is $(x_1 + x_2)\cdot y$ and the bottom composition is $x_1\cdot y + x_2\cdot y$, so they are equal, as desired. An entirely analagous argument yields that $x\cdot(y_1 + y_2) = x\cdot y_1 + x\cdot y_2$ for $x\in \pi_*(X)$ and $y_1, y_2\in \pi_*(Y)$. □

Proposition D.7 ([5, Proposition 5.11])**.** *Let (E, μ, e) be a monoid object in $\mathcal{SH}$. Then $E_*(-)$ is a functor from $\mathcal{SH}$ to left A-graded $\pi_*(E)$-modules, where given some X in $\mathcal{SH}$, $E_*(X)$ may be endowed with the structure of a left A-graded $\pi_*(E)$-module via the map*

$$\pi_*(E)\times E_*(X) \to E_*(X)$$

which given $a, b\in A$, sends $x : S^a \to E$ and $y : S^b \to E\otimes X$ to the composition

$$x\cdot y : S^{a+b} \cong S^a\otimes S^b \xrightarrow{x\otimes y} E\otimes(E\otimes X) \cong (E\otimes E)\otimes X \xrightarrow{\mu\otimes X} E\otimes X.$$

Similarly, the assignment $X \mapsto X_*(E)$ is a functor from $\mathcal{SH}$ to right A-graded $\pi_*(E)$-modules, where the structure map

$$X_*(E)\times \pi_*(E) \to X_*(E)$$

sends $x : S^a \to X\otimes E$ and $y : S^b \to E$ to the composition

$$x\cdot y : S^{a+b} \cong S^a\otimes S^b \xrightarrow{x\otimes y} (X\otimes E)\otimes E \cong X\otimes(E\otimes E) \xrightarrow{X\otimes\mu} X\otimes E.$$

Finally, $E_*(E)$ is a $\pi_*(E)$-bimodule, in the sense that the left and right actions of $\pi_*(E)$ are compatible, so that given $y, z\in \pi_*(E)$ and $x\in E_*(E)$, $y\cdot(x\cdot z) = (y\cdot x)\cdot z$.

Proof. First we show that the map $\pi_*(E)\times E_*(X) \to E_*(X)$ endows $E_*(X)$ with the structure of a left $\pi_*(E)$-module. Let $a, b, c\in A$ and $x, x' : S^a \to E\otimes X$, $y : S^b \to E$, and $z, z'\in S^c \to E$. Then we wish to show that:

(1) $y \cdot (x + x') = y \cdot x + y \cdot x'$,
(2) $(z + z') \cdot x = z \cdot x + z' \cdot x$,
(3) $(zy) \cdot x = z \cdot (y \cdot x)$,
(4) $e \cdot x = x$.

Axioms (1) and (2) follow by the fact that $E_*(X) = \pi_*(E \otimes X)$ and Proposition D.6. To see (3), consider the diagram:

$$\begin{tikzcd}
& & & E \otimes E \otimes X \arrow[d, "\mu \otimes X"] \\
S^{a+b+c} \arrow[r, "\cong"] & S^c \otimes S^b \otimes S^a \arrow[r, "z \otimes y \otimes x"] & E \otimes E \otimes E \otimes X \arrow[ur, "E \otimes \mu \otimes X"] \arrow[dr, "\mu \otimes E \otimes X"'] & E \otimes X \\
& & & E \otimes E \otimes X \arrow[u, "\mu \otimes X"']
\end{tikzcd}$$

It commutes by associativity of μ. By functoriality of $- \otimes -$, the two outside compositions equal $z \cdot (y \cdot x)$ on the top and $(z \cdot y) \cdot x$ on the bottom. Hence, they are equal, as desired.

Next, to see (4), consider the following diagram:

$$\begin{tikzcd}
S^a \arrow[rr, "x"] \arrow[dr, "x"'] \arrow[ddr, bend right, "e \otimes x"'] & & E \otimes X \\
& E \otimes X \arrow[ur, equal] \arrow[d, "e \otimes E \otimes X"] & \\
& E \otimes E \otimes X \arrow[uur, bend right, "\mu \otimes X"'] &
\end{tikzcd}$$

The top triangle commutes by definition. The left triangle commutes by functoriality of $- \otimes -$. The right triangle commutes by unitality of μ. The top composition is x while the bottom is $e \cdot x$, thus they are necessarily equal since the diagram commutes.

Thus, we have shown that the indicated map does indeed endow $E_*(X)$ with the structure of a left $\pi_*(E)$-module. It remains to show that $E_*(-)$ sends maps in $\mathcal{SH}$ to A-graded homomorphisms of left A-graded $\pi_*(E)$-modules. By definition, given $f : X \to Y$ in $\mathcal{SH}$, $E_*(f)$ is the map which takes a class $x : S^a \to E \otimes X$ to the composition

$$S^a \xrightarrow{x} E \otimes X \xrightarrow{E \otimes f} E \otimes Y.$$

To see this assignment is a homomorphism, suppose we are given some other $x' : S^a \to E \otimes X$ and some scalar $y : S^b \to E$. Then we would like to show $E_*(f)(x + x') = E_*(f)(x) + E_*(f)(x')$ and $E_*(f)(y \cdot x) = y \cdot E_*(f)(x)$. To see the former, consider the following diagram:

$$\begin{tikzcd}
& & & (E \otimes Y) \oplus (E \otimes Y) \arrow[d, "\nabla"] \\
S^a \arrow[r, "\Delta"] & S^a \oplus S^a \arrow[r, "x \oplus x'"] & (E \otimes X) \oplus (E \otimes X) \arrow[ur, "(E \otimes f) \oplus (E \otimes f)"] \arrow[dr, "\nabla"'] & E \otimes Y \\
& & & E \otimes X \arrow[u, "E \otimes f"']
\end{tikzcd}$$

It commutes by naturality of ∇ in an additive category. The top composition is $E_*(f)(x) + E_*(f)(x')$, while the bottom is $E_*(f)(x+x')$, so they are equal as desired. To see that $E_*(f)(y \cdot x) =$

$y \cdot E_*(f)(x)$, consider the following diagram:

$$\begin{tikzcd} S^{a+b} \arrow[r, "\phi_{b,a}"] & S^b \otimes S^a \arrow[r, "y\otimes x"] & E \otimes E \otimes X \arrow[r, "E\otimes E\otimes f"] \arrow[d, "\mu\otimes X"'] & E \otimes E \otimes Y \arrow[d, "\mu\otimes Y"] \\ & & E \otimes X \arrow[r, "E\otimes f"] & E \otimes Y \end{tikzcd}$$

It commutes by functoriality of $- \otimes -$. The top composition is $E_*(f)(y \cdot x)$, while the bottom composition is $y \cdot E_*(f)(x)$, so they are equal, as desired.

Showing that $X_*(E)$ has the structure of a right $\pi_*(E)$-module and that if $f : X \to Y$ is a morphism in $\mathcal{SH}$ then the map

$$X_*(E) = [S^*, X \otimes E] \xrightarrow{(f\otimes E)_*} [S^*, Y \otimes E] = Y_*(E)$$

is an A-graded homomorphism of right A-graded $\pi_*(E)$-modules is entirely analagous.

It remains to show that $E_*(E)$ is a bimodule. Let $x : S^a \to E$, $y : S^b \to E\otimes E$, and $z : S^c \to E$, and consider the following diagram:

$$\begin{tikzcd} & & & E \otimes E \otimes E \arrow[d, "E\otimes\mu"] \\ S^{a+b+c} \arrow[r, "\cong"] & S^a \otimes S^b \otimes S^c \arrow[r, "x\otimes y\otimes z"] & E \otimes E \otimes E \otimes E \arrow[r, "\mu\otimes\mu"] \arrow[ur, "\mu\otimes E\otimes E"] \arrow[dr, "E\otimes E\otimes\mu"'] & E \otimes E \\ & & & E \otimes E \otimes E \arrow[u, "\mu\otimes E"'] \end{tikzcd}$$

Commutativity follows by functoriality of $- \otimes -$, which also tells us that the two outside compositions are $(x \cdot y) \cdot z$ (on top) and $x \cdot (y \cdot z)$ (on bottom). Hence they are equal, as desired. $\square$

Proposition D.8 ([6, Proposition 2.2]). *Let (E, μ, e) be a monoid object in $\mathcal{SH}$ and let X be any object. Then the assignment*

$$E_*(E) \times E_*(X) \to E_*(E \otimes X)$$

which sends $x : S^a \to E \otimes E$ and $y : S^b \to E \otimes X$ to the composition

$$x \cdot y : S^{a+b} \cong S^a \otimes S^b \xrightarrow{x\otimes y} E \otimes E \otimes E \otimes X \xrightarrow{E\otimes\mu\otimes X} E \otimes E \otimes X$$

induces an A-graded homomorphism of left A-graded $\pi_*(E)$-modules

$$E_*(E) \otimes_{\pi_*(E)} E_*(X) \to E_*(E \otimes X)$$

(where here $E_*(E)$ has a $\pi_*(E)$-bimodule structure and $E_*(X)$ has a left $\pi_*(E)$-module structure as specified by Proposition D.7, so $E_*(E) \otimes_{\pi_*(E)} E_*(X)$ is a left A-graded $\pi_*(E)$-module by Proposition C.10). Furthermore, this homomorphism is natural in X.

Proof. First, recall by definition of the tensor product, in order to show the assignment $E_*(E) \times E_*(X) \to E_*(E \otimes X)$ induces a homomorphism $E_*(E) \otimes_{\pi_*(E)} E_*(X) \to E_*(E \otimes X)$ of A-graded abelian groups, it suffices to show that the assignment is $\pi_*(E)$-balanced, i.e., that it is linear in each argument and satisfies $xr \cdot y = x \cdot ry$ for $x \in E_*(E)$, $y \in E_*(X)$, and $r \in \pi_*(E)$.

First, note that by the identifications $E_*(E) = \pi_*(E \otimes E)$, $E_*(X) = \pi_*(E \otimes X)$, and $E_*(E \otimes X) = \pi_*(E \otimes E \otimes X)$, and Proposition D.6, it is straightforward to see that the assignment commutes with addition of maps in each argument. Now, let $a, b, c \in A$, $x : S^a \to E \otimes E$, $y : S^b \to E \otimes X$, and $z : S^c \to E$. Then we wish to show $xz \cdot y = x \cdot zy$. Consider the following diagram (where here we are passing to a permutative category):

$$\begin{tikzcd}
 & & & E\otimes E\otimes E\otimes X \arrow[d, "E\otimes\mu\otimes X"] \\
S^{a+b+c} \arrow[r, "\cong"] & S^a\otimes S^c\otimes S^b \arrow[r, "x\otimes z\otimes y"] & E\otimes E\otimes E\otimes E\otimes X \arrow[ur, "E\otimes\mu\otimes E\otimes X"] \arrow[dr, "E\otimes E\otimes\mu\otimes X"'] & E\otimes E\otimes X \\
 & & & E\otimes E\otimes E\otimes X \arrow[u, "E\otimes\mu\otimes X"']
\end{tikzcd}$$

It commutes by associativity of μ. By functoriality of $-\otimes-$, the top composition is given by $(xz)\cdot y$ and the bottom composition is $x\cdot(zy)$, so we have they are equal, as desired. Thus, since the map $E_*(E)\times E_*(X)\to E_*(E\otimes X)$ is $\pi_*(E)$-balanced, we have that it induces a homomorphism of abelian groups. Furthermore, by Lemma C.11 it is an A-graded homomorphism of A-graded abelian groups.

In order to see this map is furthermore a homomorphism of left $\pi_*(E)$-modules, we must show that $z(x\cdot y)=zx\cdot y$, where x, y, and z are defined as above. Now consider the following diagram:

$$\begin{tikzcd}
 & & & E\otimes E\otimes E\otimes X \arrow[d, "E\otimes\mu\otimes X"] \\
S^{a+b+c} \arrow[r, "\cong"] & S^c\otimes S^a\otimes S^b \arrow[r, "z\otimes x\otimes y"] & E\otimes E\otimes E\otimes E\otimes X \arrow[ur, "\mu\otimes E\otimes E\otimes X"] \arrow[dr, "E\otimes E\otimes\mu\otimes X"'] \arrow[r, "\mu\otimes\mu\otimes X"] & E\otimes E\otimes X \\
 & & & E\otimes E\otimes E\otimes X \arrow[u, "\mu\otimes E\otimes X"']
\end{tikzcd}$$

Commutativity of the triangles is functoriality of $-\otimes-$. By functoriality of $-\otimes-$, the top composition is $zx\cdot y$, and the bottom composition is $z(x\cdot y)$. Hence they are equal, as desired, so that the map we have constructed

$$E_*(E)\otimes_{\pi_*(E)}E_*(X)\to E_*(E\otimes X)$$

is indeed an A-graded homomorphism of left A-graded $\pi_*(E)$-modules.

Next, we would like to show that this homomorphism is natural in X. Let $f:X\to Y$ in $\mathcal{SH}$. Then we would like to show the following diagram commutes:

$$\begin{tikzcd}
E_*(E)\otimes_{\pi_*(E)}E_*(X) \arrow[r, "\Phi_X"] \arrow[d, "E_*(E)\otimes_{\pi_*(E)}E_*(f)"'] & E_*(E\otimes X) \arrow[d, "E_*(E\otimes f)"] \\
E_*(E)\otimes_{\pi_*(E)}E_*(Y) \arrow[r, "\Phi_Y"] & E_*(E\otimes Y)
\end{tikzcd} \tag{3}$$

As all the maps here are homomorphisms, it suffices to chase generators around the diagram. In particular, suppose we are given $x:S^a\to E\otimes E$ and $y:S^b\to E\otimes X$, and consider the following diagram exhibiting the two possible ways to chase $x\otimes y$ around the diagram (as usual, we are passing to a permutative category):

$$\begin{tikzcd}
S^{a+b} \arrow[r, "\phi_{a,b}"] & S^a\otimes S^b \arrow[r, "x\otimes y"] & E\otimes E\otimes E\otimes X \arrow[r, "E\otimes\mu\otimes X"] \arrow[d, "E\otimes E\otimes E\otimes f"'] & E\otimes E\otimes X \arrow[d, "E\otimes E\otimes f"] \\
 & & E\otimes E\otimes E\otimes Y \arrow[r, "E\otimes\mu\otimes Y"] & E\otimes E\otimes Y
\end{tikzcd}$$

This diagram commutes by functoriality of $-\otimes-$. Thus we have that diagram (3) does indeed commute, as desired. □

Proposition D.9. *Let (E,μ,e) be a* flat, commutative *monoid object in $\mathcal{SH}$ (Definition 2.11) and let X be any* cellular *object in $\mathcal{SH}$ (Definition 2.8). Then the natural homomorphism*

$$\Phi_X : E_*(E)\otimes E_*(X)\to E_*(E\otimes X)$$

given in Proposition D.8 is an isomorphism of left $\pi_*(E)$-modules.

Proof. It remains to show that if X is cellular and E is flat, then this map is an isomorphism. To start, let $\mathcal{E}$ be the collection of objects X in $\mathcal{SH}$ for which this map is an isomorphism. Then in order to show $\mathcal{E}$ contains every cellular object, it suffices to show that $\mathcal{E}$ satisfies the three conditions given for the class of cellular objects in Definition 2.8. First, we need to show that Φ is an isomorphism when $X = S^a$ for some $a\in A$. Indeed, consider the map

$$\Psi : E_*(E\otimes S^a)\to E_*(E)\otimes_{\pi_*(E)} E_*(S^a)$$

which sends a class $x : S^b\to E\otimes E\otimes S^a$ in $E_b(E\otimes S^a)$ to the pure tensor $\widetilde{x}\otimes\widetilde{e}$, where $\widetilde{x}\in E_{b-a}(E)$ is the composition

$$S^{b-a}\cong S^b\otimes S^{-a}\xrightarrow{x\otimes S^{-a}} E\otimes E\otimes S^a\otimes S^{-a}\xrightarrow{E\otimes E\otimes\phi^{-1}_{a,-a}} E\otimes E\otimes S\xrightarrow{E\otimes\rho_E} E\otimes E$$

and $\widetilde{e}\in E_a(S^a)$ is the composition

$$S^a\cong S\otimes S^a\xrightarrow{e\otimes S^a} E\otimes S^a.$$

First, note Ψ is an (A-graded) homomorphism of abelian groups: Given $x,x'\in E_b(E\otimes S^a)$, we would like to show that $\widetilde{x}\otimes\widetilde{e}+\widetilde{x}'\otimes\widetilde{e}=\widetilde{x+x'}\otimes\widetilde{e}$. It suffices to show that $\widetilde{x}+\widetilde{x}'=\widetilde{x+x'}$. To see this, consider the following diagram (again, we are passing to a permutative category):

$$\begin{tikzcd}
S^{b-a} \arrow[r, "\Delta"] \arrow[d, "\phi_{b-a}"'] & S^{b-a}\oplus S^{b-a} \arrow[d, "\phi_{b,-a}\oplus\phi_{b,-a}"] \\
S^b\otimes S^{-a} \arrow[r, "\Delta"] \arrow[d, "\Delta\otimes S^{-a}"'] & (S^b\otimes S^{-a})\oplus(S^b\otimes S^{-a}) \arrow[d, "(x\otimes S^{-a})\oplus(x'\otimes S^{-a})"] \\
(S^b\oplus S^b)\otimes S^{-a} \arrow[ur, "\cong"] \arrow[d, "(x\oplus x')\otimes S^{-a}"'] & (E\otimes E\otimes S^a\otimes S^{-a})\oplus(E\otimes E\otimes S^a\otimes S^{-a}) \arrow[d, "(E\otimes E\otimes\phi^{-1}_{a,-a})\oplus(E\otimes E\otimes\phi^{-1}_{a,-a})"] \arrow[ddl, "\nabla"] \\
((E\otimes E\otimes S^a)\oplus(E\otimes E\otimes S^a))\otimes S^{-a} \arrow[ur, "\cong"] \arrow[d, "\nabla\otimes S^{-a}"'] & (E\otimes E)\oplus(E\otimes E) \arrow[d, "\nabla"] \\
E\otimes E\otimes S^a\otimes S^{-a} \arrow[r, "E\otimes E\otimes\phi^{-1}_{a,-a}"'] & E\otimes E
\end{tikzcd}$$

The top rectangle commutes by naturality of Δ in an additive category. The bottom triangle commutes by naturality of ∇ in an additive category. Finally, the remaining regions of the diagram commute by additivity of $-\otimes-$. By functoriality of $-\otimes-$, it follows that the left composition is $\widetilde{x+x'}$ and the right composition is $\widetilde{x}+\widetilde{x}'$, so they are equal as desired. Thus Ψ is a homomorphism of abelian groups, as desired.

Now, we claim that Ψ is an inverse to Φ, (which is enough to show Φ is an isomorphism of left $\pi_*(E)$-modules). Since Φ and Ψ are homomorphisms it suffices to check that they are inverses on generators. First, let $x : S^b\to E\otimes E\otimes S^a$ in $E_b(E\otimes S^a)$. We would like to show that $\Phi(\Psi(x)) = x$.

Consider the following diagram, where here we are passing to a permutative category:

$$\begin{tikzcd}
S^b \arrow[rr, "\cong"] \arrow[dd, "x"'] & & S^b \otimes S^{-a} \otimes S^a \arrow[d, dashed, "x\otimes S^{-a}\otimes S^a"] \arrow[rr, "x\otimes S^{-a}\otimes e\otimes S^a"] & & {} \\
& & E \otimes E \otimes S^a \otimes S^{-a} \otimes S^a \arrow[rr, "E\otimes E\otimes S^a\otimes S^{-a}\otimes e\otimes S^a"'] & & E \otimes E \otimes S^a \otimes S^{-a} \otimes E \otimes S^a \arrow[ddll, "E\otimes E\otimes \phi^{-1}_{a,-a}\otimes E\otimes S^a"] \\
E \otimes E \otimes S^a \arrow[rr, equal] \arrow[urr, "E\otimes E\otimes S^a\otimes \phi_{-a,a}"] \arrow[drr, "E\otimes E\otimes e\otimes S^a"'] & & E \otimes E \otimes S^a \arrow[u, dashed, "E\otimes E\otimes \phi_{a,-a}\otimes S^a"'] \arrow[d, dashed, "E\otimes E\otimes e\otimes S^a"] & & \\
E \otimes E \otimes E \otimes S^a \arrow[u, "E\otimes \mu\otimes S^a"] \arrow[rr, equal] & & E \otimes E \otimes E \otimes S^a & &
\end{tikzcd}$$

The top left trapezoid commutes since the isomorphism $S^b \xrightarrow{\cong} S^b \otimes S^{-a} \otimes S^a$ may be given as $S^b \otimes \phi_{-a,a}$ (see Remark 2.5), in which case the trapezoid commmutes by functoriality of $- \otimes -$. The triangle below that commutes by coherence for the $\phi_{a,b}$'s. The triangle below that commutes by definition. The bottom left triangle commutes by unitality for μ. The top right triangle commutes by functoriality of $- \otimes -$. Finally, the bottom right triangle commutes by functoriality of $- \otimes -$. It follows by unravelling definitions that the two outside compositions are x (on the left) and $\Phi(\Psi(x))$ (on the right), so since the diagram commutes we indeed have $\Phi(\Psi(x)) = x$, as desired.

On the other hand, suppose we are given a homogeneous pure tensor $x \otimes y$ in $E_*(E) \otimes_{\pi_*(E)} E_*(S^a)$, so $x : S^b \to E \otimes E$ and $y : S^c \to E \otimes S^a$ for some $b, c \in A$. Then we would like to show that $\Psi(\Phi(x \otimes y)) = x \otimes y$. Unravelling definitions, $\Psi(\Phi(x \otimes y))$ is the homogeneous pure tensor $\widetilde{xy} \otimes \widetilde{e}$, where $\widetilde{e} : S^a \to E \otimes S^a$ is defined above, and by functoriality of $- \otimes -$, $\widetilde{xy} : S^{b+c-a} \to E \otimes E$ is the composition

$$\begin{tikzcd}
S^{b+c-a} \arrow[d, "\phi_{b+c,-a}"] \\
S^{b+c} \otimes S^{-a} \arrow[d, "\phi_{b,c}\otimes S^{-a}"] \\
S^b \otimes S^c \otimes S^{-a} \arrow[d, "x\otimes y\otimes S^{-a}"] \\
E \otimes E \otimes E \otimes S^a \otimes S^{-a} \arrow[d, "E\otimes \mu\otimes S^a\otimes S^{-a}"] \\
E \otimes E \otimes S^a \otimes S^{-a} \arrow[d, "E\otimes E\otimes \phi^{-1}_{a,-a}"] \\
E \otimes E \otimes S \arrow[d, "E\otimes \rho_E"] \\
E \otimes E.
\end{tikzcd}$$

In order to see $x \otimes y = \widetilde{xy} \otimes \widetilde{e}$, it suffices to show there exists some scalar $r \in \pi_{c-a}(E)$ such that $x \cdot r = \widetilde{xy}$ and $r \cdot \widetilde{e} = y$, where here $\cdot$ denotes the right and left action of $\pi_*(E)$ on $E_*(E)$ and $E_*(S^a)$, respectively. Now, define r to be the composition

$$S^{c-a} \cong S^c \otimes S^{-a} \xrightarrow{y\otimes S^{-a}} E \otimes S^a \otimes S^{-a} \xrightarrow{E\otimes \phi^{-1}_{a,-a}} E \otimes S \xrightarrow{\rho_E} E.$$

First, in order to see that $x \cdot r = \widetilde{xy}$, consider the following diagram, where here we are again passing to a permutative category:

$$\begin{tikzcd}
S^{b+c-a} \arrow[r, "\cong"] & S^b \otimes S^c \otimes S^{-a} \arrow[r, "x\otimes y\otimes S^{-a}"] & E \otimes E \otimes E \otimes S^a \otimes S^{-a} \arrow[r, "E\otimes\mu\otimes S^a\otimes S^{-a}"] \arrow[d, "E\otimes E\otimes E\otimes\phi_{a,-a}^{-1}"'] \arrow[dr, "E\otimes\mu\otimes\phi_{a,-a}^{-1}"] & E \otimes E \otimes S^a \otimes S^{-a} \arrow[d, "E\otimes E\otimes\phi_{a,-a}^{-1}"] \\
 & & E \otimes E \otimes E \arrow[r, "E\otimes\mu"'] & E \otimes E
\end{tikzcd}$$

Commutativity is functoriality of $- \otimes -$, which also tells us that the two outside compositions are $\widetilde{xy}$ (on top) and $x \cdot r$ (on the bottom), so they are equal as desired. On the other hand, in order to see that $r \cdot \widetilde{e} = y$, consider the following diagram (where here we have passed to a permutative category):

$$\begin{tikzcd}
S^c \arrow[rr, "\cong"] \arrow[dd, "y"'] & & S^c \otimes S^{-a} \otimes S^a \arrow[dl, "y\otimes S^{-a}\otimes S^a"'] \arrow[dd, "y\otimes S^{-a}\otimes e\otimes S^a"] \\
 & E \otimes S^a \otimes S^{-a} \otimes S^a \arrow[dl, "E\otimes S^a\otimes\phi_{-a,a}^{-1}"'] \arrow[dr, "E\otimes S^a\otimes S^{-a}\otimes e\otimes S^a"] & \\
E \otimes S^a \arrow[drr, "E\otimes e\otimes S^a"] & & E \otimes S^a \otimes S^{-a} \otimes E \otimes S^a \arrow[d, "E\otimes\phi_{a,-a}^{-1}\otimes E\otimes S^a"] \\
E \otimes E \otimes S^a \arrow[u, "\mu\otimes S^a"] \arrow[rr, equal] & & E \otimes E \otimes S^a
\end{tikzcd}$$

The top left triangle commutes since we may take the isomorphism $S^c \xrightarrow{\cong} S^c \otimes S^{-a} \otimes S^a$ to be $S^c \otimes \phi_{-a,a}$, in which case commutativity of the triangle follows by functoriality of $- \otimes -$. Commutativity of the right triangle is also functoriality of $- \otimes -$. Commutativity of the bottom triangle is unitality of μ. Finally, commutativity of the remaining middle 4-sided region is again functoriality of $- \otimes -$. It follows that y is equal to the outer composition, which is $r \cdot \widetilde{e}$, as desired. Thus, we have shown that

$$\Psi(\Phi(x \otimes y)) = \widetilde{xy} \otimes \widetilde{e} = (x \cdot r) \otimes \widetilde{e} = x \otimes (r \cdot \widetilde{e}) = x \otimes y,$$

as desired, so that for each $a \in A$, the object S^a belongs to the class $\mathcal{E}$.

Now, we would like to show that given a distinguished triangle

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X,$$

if two of three of the objects X, Y, and Z belong to $\mathcal{E}$, then so does the third. First we claim that if X belongs to $\mathcal{E}$, then for any $n \in \mathbb{Z}$,

a

a

a

To see this, consider the following diagram

$$\begin{tikzcd}
E_*(E) \otimes_{\pi_*(E)} E_*(\Sigma^a X) \arrow[r, "\Phi_{\Sigma^a X}"] \arrow[d, "\cong"'] & E_*(E \otimes \Sigma^a X) \arrow[r, "E_*(\tau_{E,S^a} \otimes X)"] & E_*(\Sigma^a(E \otimes X)) \arrow[d, "\cong"] \\
E_*(E) \otimes_{\pi_*(E)} E_{*-a}(X) \arrow[r, "\Phi_X"] & E_{*-a}(E \otimes X) \arrow[r, "\cong"] & E_{*-a}(E \otimes X)
\end{tikzcd}$$

where:

- the vertical isomorphisms are induced by the isomorphism $E_*(\Sigma^a W) \cong E_{*-a}(W)$ given by the composition

$$[S^*, E \otimes S^a \otimes W] \xrightarrow{(\tau_{E \otimes S^a} \otimes X)_*} [S^*, S^a \otimes E \otimes W] \cong [S^{-a} \otimes S^*, E \otimes W] \xrightarrow{(\phi_{-a,*})^*} [S^{*-a}, E \otimes W]$$

  where the middle arrow is the adjunction $S^{-a} \otimes - \dashv S^a \otimes -$ given by Proposition 2.2, and
- the unlabelled bottom left horizontal isomorphism ???

Thus in order to show $\Sigma^a X$ belongs to $\mathcal{E}$, it suffices to show diagram (**??**) commutes. Suppose we are given elements $x : S^b \to E \otimes E$ and $y : S^c \to E \otimes \Sigma^a X = E \otimes S^a \otimes X$. Then chasing $x \otimes y$ around diagram (**??**) yields the following two compositions:

$$\begin{tikzcd}
S^{b+c-a} \arrow[d, "\phi_{b,c-a}"'] \\
S^b \otimes S^{c-a} \arrow[d, "S^b \otimes \phi_{-a,c}"'] \\
S^b \otimes S^{-a} \otimes S^c \arrow[d, "x \otimes S^{-a} \otimes y"'] \\
E \otimes E \otimes S^{-a} \otimes E \otimes S^a \otimes X \arrow[d, "E \otimes E \otimes S^{-a} \otimes \tau_{E,S^a} \otimes X"'] \\
E \otimes E \otimes S^{-a} \otimes S^a \otimes E \otimes X \arrow[d, "E \otimes E \otimes \phi^{-1}_{-a,a} \otimes E \otimes X"'] \\
E \otimes E \otimes E \otimes X \arrow[d, "E \otimes \mu \otimes X"'] \\
E \otimes E \otimes X
\end{tikzcd}
\qquad
\begin{tikzcd}
S^{b+c-a} \arrow[d, "\phi_{-a,b+c}"] \\
S^{-a} \otimes S^{b+c} \arrow[d, "S^{-a} \otimes \phi_{b,c}"] \\
S^{-a} \otimes S^b \otimes S^c \arrow[d, "S^{-a} \otimes x \otimes y"] \\
S^{-a} \otimes E \otimes E \otimes E \otimes S^a \otimes X \arrow[d, "S^{-a} \otimes E \otimes \mu \otimes S^a \otimes X"] \\
S^{-a} \otimes E \otimes E \otimes S^a \otimes X \arrow[d, "S^{-a} \otimes E \otimes \tau_{E,S^a} \otimes X"] \\
S^{-a} \otimes E \otimes S^a \otimes E \otimes X \arrow[d, "S^{-a} \otimes \tau_{E,S^a} \otimes E \otimes X"] \\
S^{-a} \otimes S^a \otimes E \otimes E \otimes X \arrow[d, "\phi^{-1}_{-a,a} \otimes E \otimes E \otimes X"] \\
E \otimes E \otimes X
\end{tikzcd}$$

Then consider the following diagram

$$
\begin{tikzcd}
S^{-a} \otimes S^{b+c} \arrow[d, "S^{-a}\otimes\phi_{b,c}"] & S^{b+c-a} \arrow[l, "\phi_{-a,b+c}"'] \arrow[d, "\phi_{b-a,c}"'] & S^{b+c-a} \arrow[l, "\text{the map that goes here which makes the below commute is dependent on b and c!}"'] \arrow[d, "\phi_{b-a,c}"] \arrow[r, "\phi_{b,c-a}"] & S^b \otimes S^{c-a} \arrow[d, "S^b\otimes\phi_{-a,c}"] \\
 & S^{b-a} \otimes S^c \arrow[dl, "\phi_{-a,b}\otimes S^c"'] & S^{b-a} \otimes S^c \arrow[dr, "\phi_{b,-a}\otimes S^c"] & \\
S^{-a} \otimes S^b \otimes S^c \arrow[rrr, "\tau_{S^{-a},S^b}\otimes S^c"'] \arrow[dr, "S^{-a}\otimes x\otimes S^c"] \arrow[dd, "S^{-a}\otimes x\otimes y"] & & & S^b \otimes S^{-a} \otimes \arrow[dl, "x\otimes S^{-a}\otimes S^c"'] \arrow[dd, "x\otimes S^{-a}\otimes y"] \\
 & S^{-a} \otimes E \otimes E \otimes S^c \arrow[r, "\tau_{S^{-a},E\otimes E}\otimes S^c"] \arrow[dl, "S^{-a}\otimes E\otimes E\otimes y"'] & E \otimes E \otimes S^{-a} \otimes S^c \arrow[dr, "E\otimes E\otimes S^{-a}\otimes y"] & \\
S^{-a} \otimes E \otimes E \otimes E \otimes S^a \otimes X \arrow[rrr, "\tau_{S^{-a},E\otimes E}\otimes E\otimes S^a\otimes X"] \arrow[d, "S^{-a}\otimes E\otimes\mu\otimes S^a\otimes X"'] \arrow[dr, "S^{-a}\otimes\tau_{E\otimes E\otimes E,S^a}\otimes X"] & & & E \otimes E \otimes S^{-a} \otimes E \arrow[d, "E\otimes E"] \\
S^{-a} \otimes E \otimes E \otimes S^a \otimes X \arrow[dd, "S^{-a}\otimes E\otimes\tau_{E,S^a}\otimes X"'] \arrow[ddr, "S^{-a}\otimes\tau_{E\otimes E,S^a}\otimes X"] & S^{-a} \otimes S^a \otimes E \otimes E \otimes E \otimes X \arrow[rr, "\tau_{S^{-a}\otimes S^a,E\otimes E}\otimes E\otimes X"] \arrow[drr, "\phi^{-1}_{-a,a}\otimes E\otimes E\otimes E\otimes X"] \arrow[dd, "S^{-a}\otimes S^a\otimes E\otimes\mu\otimes X"] & & E \otimes E \otimes S^{-a} \otimes S^a \arrow[d, "E\otimes E"] \\
 & & & E \otimes E \otimes E \arrow[d, "E\otimes\mu"] \\
S^{-a} \otimes E \otimes S^a \otimes E \otimes X \arrow[r, "S^{-a}\otimes\tau_{E,S^a}\otimes E\otimes X"'] & S^{-a} \otimes S^a \otimes E \otimes E \otimes X \arrow[rr, "\phi^{-1}_{-a,a}\otimes E\otimes E\otimes X"] & & E \otimes E \otimes X
\end{tikzcd}
$$

Finally, it remains to show that $\mathcal{E}$ is closed under taking arbitrary direct sums. Let $\{X_i\}_{i\in I}$ be a family of objects in $\mathcal{E}$ indexed by some set I. Then note by definition, since direct sums are limits, we have that for any W in $\mathcal{SH}$ that

$$\left[W, \bigoplus_{i\in I} X_i\right] \cong \bigoplus_{i\in I}[W, X_i],$$

and furthermore this isomorphism is natural in W. Now let $X = \bigoplus_i X_i$, and consider the following diagram

$$
\begin{tikzcd}
{[S^*, E\otimes E]\otimes[S^\star, E\otimes X]} \arrow[r, "\cong"'] \arrow[d, "-\otimes-"] & {[S^*, E\otimes E], [S^\star, \bigoplus_i E\otimes X_i]} \arrow[r, "\cong"'] \arrow[d, "-\otimes-"] & \bigoplus([S^*, E\otimes E]\otimes[S^\star, E\otimes X_i]) \arrow[d, "\bigoplus_i(-\otimes-)"] \\
{[S^*\otimes S^\star, E\otimes E\otimes E\otimes X]} \arrow[r, "\cong"'] \arrow[d, "(\phi_{*,\star})^*"] & {[S^*\otimes S^\star, \bigoplus_i E\otimes E\otimes E\otimes X_i]} \arrow[r, "\cong"'] \arrow[d, "(\phi_{*,\star})^*"] & \bigoplus_i[S^*\otimes S^\star, E\otimes E\otimes E\otimes X_i] \arrow[d, "\bigoplus_i(\phi_{*,\star})^*"] \\
{[S^{*+\star}, E\otimes E\otimes E\otimes X]} \arrow[r, "\cong"'] \arrow[d, "(E\otimes\mu\otimes X)_*"] & {[S^{*+\star}, \bigoplus_i E\otimes E\otimes E\otimes X_i]} \arrow[r, "\cong"'] \arrow[d, "(E\otimes\mu\otimes X)_*"] & \bigoplus_i[S^{*+\star}, E\otimes E\otimes E\otimes X_i] \arrow[d, "\bigoplus_i(E\otimes\mu\otimes X_i)"] \\
{[S^{*+\star}, E\otimes E\otimes X]} \arrow[r, "\cong"'] & {[S^{*+\star}, \bigoplus_i E\otimes E\otimes X_i]} \arrow[r, "\cong"'] & \bigoplus_i[S^{*+\star}, E\otimes E\otimes X_i]
\end{tikzcd}
$$

The left squares commute by additivity of $-\otimes-$. The right squares commute by naturality of the isomorphisms given above. Since each X_i belongs to $\mathcal{E}$, the right vertical composition is an isomorphism, so that the left vertical composition is also an isomorphism, as desired.

To recap, we have shown that the collection of objects $\mathcal{E}$ for which $\Phi_X : E_*(E)\otimes_{\pi_*(E)} E_*(X) \to E_*(E\otimes X)$ is an isomorphism satisfies the conditions outlined in Definition 2.8. Hence, $\mathcal{E}$ contains every cellular object, as desired.

where did I use cellularity of E?

□

In the following definition, let $\varepsilon : E_*(E) \to \pi_*(E)$ be the map which sends some $\alpha : S^a \to E\otimes E$ to the composition
$$S^a \xrightarrow{\alpha} E \otimes E \xrightarrow{\mu} E.$$
Also define $\Psi : E_*(E) \to E_*(E) \otimes_{\pi_*(E)} E_*(E)$ to be the map which factors as
$$E_*(E) \to E_*(E \otimes E) \xrightarrow{\cong} E_*(E) \otimes_{\pi_*(E)} E_*(E)$$
where the second arrow is the isomorphism prescribed by Proposition D.8, and the first arrow sends a class $\alpha : S^a \to E \otimes E$ to the composition
$$S^a \xrightarrow{\alpha} E \otimes E \cong E \otimes S \otimes E \xrightarrow{E\otimes e\otimes E} E \otimes E \otimes E.$$

Lemma D.10 ([6, Proposition 2.30, 2.33])**.** *Let E be a flat commutative ring spectrum, and let X and Y be spectra such that $E_{**}(X)$ is a projective module over $\pi_{**}(E)$. Then for all $s \geq 0$ and $t, w \in \mathbb{Z}$, there is an isomorphism*
$$\Phi : [X, E \wedge Y]_{t,w} \to \mathrm{Hom}^{t,w}_{E_{**}(E)}(E_{**}(X), E_{**}(E \wedge Y)),$$
obtained by sending a class $f : S^{t,w} \wedge X \to E \wedge Y$ in $[X, E \wedge Y]_{t,w}$ to the map
$$\Phi_f : E_{*,*}(X) \to E_{*+t,*+w}(X \wedge Y)$$
sending
$$[S^{a,b} \xrightarrow{g} E \wedge X] \mapsto [S^{a+t,b+w} \cong S^{a,b} \wedge S^{t,w} \xrightarrow{g\wedge S^{t,w}} E \wedge X \wedge S^{t,w} \cong E \wedge S^{t,w} \wedge X \xrightarrow{E\wedge f} E \wedge E \wedge Y].$$

Proof. Let $f : S^{t,w} \wedge X \to E \wedge Y$. First we want to show that Φ_f is actually an $E_{**}(E)$-comodule homomorphism. □

finish

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