

In this appendix, we fix a symmetric monoidal category $(\mathcal{C}, \otimes, S)$ with left unitor, right unitor, associator, and symmetry isomorphisms λ , ρ , α , and τ , respectively.

0.1. Monoid objects in a symmetric monoidal category.

Definition 0.1. A *monoid object* (E, μ, e) is an object E in \mathcal{C} along with a multiplication morphism $\mu : E \otimes E \rightarrow E$ and a unit map $e : S \rightarrow E$ such that the following diagrams commute:

$$\begin{array}{ccc} E \otimes S & \xrightarrow{E \otimes e} & E \otimes E \xleftarrow{e \otimes E} S \otimes E \\ & \searrow \rho_E & \downarrow \mu \swarrow \lambda_E \\ & & E \end{array} \quad \begin{array}{ccc} (E \otimes E) \otimes E & \xrightarrow{\mu \otimes E} & E \otimes E \\ \alpha \downarrow & & \downarrow \mu \\ E \otimes (E \otimes E) & \xrightarrow{E \otimes \mu} & E \otimes E \xrightarrow{\mu} E \end{array}$$

The first diagram expresses unitality, while the second expressed associativity. If in addition the following diagram commutes,

$$\begin{array}{ccc} E \otimes E & \xrightarrow{\tau} & E \otimes E \\ & \searrow \mu & \swarrow \mu \\ & & E \end{array}$$

then we say (E, μ, e) is a *commutative monoid object*.

Example 0.2. The object S is a monoid object, with multiplication map $\rho_S = \lambda_S : S \otimes S \rightarrow S$ and unit $\text{id}_S : S \rightarrow S$.

Definition 0.3. Given two monoid objects (E_1, μ_1, e_1) and (E_2, μ_2, e_2) in a symmetric monoidal category $(\mathcal{C}, \otimes, S)$, a *monoid homomorphism* from E_1 to E_2 is a morphism $f : E_1 \rightarrow E_2$ in \mathcal{C} such that the following diagrams commute:

$$\begin{array}{ccc} E_1 \otimes E_1 & \xrightarrow{f \otimes f} & E_2 \otimes E_2 \\ \mu_1 \downarrow & & \downarrow \mu_2 \\ E_1 & \xrightarrow{f} & E_2 \end{array} \quad \begin{array}{ccc} & S & \\ e_1 \swarrow & & \searrow e_2 \\ E_1 & \xrightarrow{f} & E_2 \end{array}$$

It is straightforward to show that id_{E_1} is a homomorphism of monoid objects from E_1 to itself, and that the composition of monoid homomorphisms is still a monoid homomorphism. Thus, we have categories $\mathbf{Mon}_{\mathcal{C}}$ and $\mathbf{CMon}_{\mathcal{C}}$ of monoid objects and commutative monoid objects in \mathcal{C} , respectively, with monoid homomorphisms between them.

Lemma 0.4. Given two monoid objects (E_1, μ_1, e_1) and (E_2, μ_2, e_2) in a symmetric monoidal category $(\mathcal{C}, \otimes, S)$, their tensor product $E_1 \otimes E_2$ canonically becomes a monoid object in \mathcal{C} with unit map

$$e : S \xrightarrow{\cong} S \otimes S \xrightarrow{e_1 \otimes e_2} E_1 \otimes E_2$$

and multiplication map

$$\mu : E_1 \otimes E_2 \otimes E_1 \otimes E_2 \xrightarrow{E_1 \otimes \tau \otimes E_2} E_1 \otimes E_1 \otimes E_2 \otimes E_2 \xrightarrow{\mu_1 \otimes \mu_2} E_1 \otimes E_2$$

(where here we are suppressing the associators from the notation). If in addition (E_1, μ_1, e_1) and (E_2, μ_2, e_2) are commutative monoid objects, then $(E_1 \otimes E_2, \mu, e)$ is as well.

Proof. Due to the size of the diagrams involved, we leave this as an exercise for the reader. It is entirely straightforward. \square

Lemma 0.5. *Given monoid objects (E_i, μ_i, e_i) for $i = 1, 2, 3$ in a symmetric monoidal category \mathcal{C} , the associator $(E_1 \otimes E_2) \otimes E_3 \xrightarrow{\cong} E_1 \otimes (E_2 \otimes E_3)$ is an isomorphism of monoid objects. In other words, up to associativity, given a collection of monoid objects E_1, \dots, E_n in \mathcal{C} , there is no ambiguity when talking about their tensor product $E_1 \otimes \dots \otimes E_n$ as a monoid object.*

Proof. Clearly, up to associativity, $(E_1 \otimes E_2) \otimes E_3$ and $E_1 \otimes (E_2 \otimes E_3)$ have the same unit map $S \xrightarrow{e_1 \otimes e_2 \otimes e_3} E_1 \otimes E_2 \otimes E_3$. Thus, it remains to show that they have the same product map, up to associativity. To see this, consider the following diagram, where we've passed to a symmetric strict monoidal category:

$$\begin{array}{ccc}
E_1 \otimes (E_2 \otimes E_3) \otimes E_1 \otimes (E_2 \otimes E_3) & \xlongequal{\alpha} & (E_1 \otimes E_2) \otimes E_3 \otimes (E_1 \otimes E_2) \otimes E_3 \\
\downarrow E_1 \otimes \tau_{E_2 \otimes E_3, E_1} \otimes E_2 \otimes E_3 & & \downarrow E_1 \otimes E_2 \otimes \tau_{E_3, E_1 \otimes E_2} \otimes E_3 \\
E_1 \otimes E_1 \otimes E_2 \otimes E_3 \otimes E_2 \otimes E_3 & & E_1 \otimes E_2 \otimes E_1 \otimes E_2 \otimes E_3 \otimes E_3 \\
\downarrow \mu_1 \otimes E_2 \otimes \tau \otimes E_3 \quad \swarrow E_1 \otimes E_1 \otimes E_2 \otimes \tau \otimes E_3 & & \swarrow E_1 \otimes \tau \otimes E_2 \otimes E_3 \otimes E_3 \quad \downarrow E_1 \otimes \tau \otimes E_2 \otimes \mu_3 \\
E_1 \otimes E_2 \otimes E_2 \otimes E_3 \otimes E_3 \otimes E_2 \otimes E_3 \otimes E_3 & \xleftarrow{\mu_1 \otimes E_2 \otimes E_2 \otimes E_3 \otimes E_3} E_1 \otimes E_2 \otimes E_2 \otimes E_3 \otimes E_3 & \xrightarrow{E_1 \otimes E_2 \otimes E_2 \otimes E_3 \otimes E_3} E_1 \otimes E_1 \otimes E_2 \otimes E_2 \otimes E_3 \\
\downarrow E_1 \otimes \mu_2 \otimes \mu_3 \quad \swarrow \mu_1 \otimes \mu_2 \otimes \mu_3 & & \swarrow \mu_1 \otimes \mu_2 \otimes \mu_3 \quad \downarrow \mu_1 \otimes \mu_2 \otimes E_3 \\
E_1 \otimes E_2 \otimes E_3 & \xlongequal{\alpha} & E_1 \otimes E_2 \otimes E_3
\end{array}$$

The top pentagonal region commutes by coherence for the τ 's in a symmetric monoidal category. The bottom triangle commutes by definition. The remaining four triangles commute by functoriality of $- \otimes -$. On the left is the product for $E_1 \otimes (E_2 \otimes E_3)$, while on the right is the product for $(E_1 \otimes E_2) \otimes E_3$. Thus they are equal up to associativity, as desired. \square

Lemma 0.6. *Let (E, μ, e) be a monoid object in \mathcal{SH} . Then the map $e : S \rightarrow E$ is a monoid homomorphism. Furthermore, if E is a commutative monoid object, then $\mu : E \otimes E \rightarrow E$ is also a monoid object homomorphism. (Here S and $E \otimes E$ are considered to be monoid objects by [Example 0.2](#) and [Lemma 0.4](#), respectively.)*

Proof. To see e is a monoid homomorphism, consider the following diagrams:

$$\begin{array}{ccc}
& S & \\
& \parallel & \searrow e \\
S & \xrightarrow{e} & E
\end{array}
\quad
\begin{array}{ccccc}
S \otimes S & \xrightarrow{e \otimes e} & E \otimes E & & \\
\downarrow \rho_S = \lambda_S & \searrow S \otimes e & \nearrow e \otimes E & \downarrow \mu & \\
& S \otimes E & & & \\
& \searrow \lambda_E & \nearrow & & \\
S & \xrightarrow{e} & E
\end{array}$$

The left diagram commutes by definition. The top region in the right diagram commutes by functoriality of $- \otimes -$. The right region commutes by unitality of μ . The left region commutes by naturality of λ . Thus, indeed $e : S \rightarrow E$ is a monoid object homomorphism.

Now, to see μ is a monoid object homomorphism when (E, μ, e) is a commutative monoid object, first consider the following diagram:

$$\begin{array}{ccc}
 & S & \\
 e \otimes e \swarrow & \downarrow e & \searrow e \\
 E \otimes E & \xrightarrow{\mu} & E
 \end{array}$$

The left region commutes by functoriality of $- \otimes -$, the right region commutes by definition, and the bottom region commutes by unitality of μ . Now, consider the following diagram:

$$\begin{array}{ccccc}
 E_1 \otimes E_2 \otimes E_3 \otimes E_4 & \xrightarrow{\mu \otimes \mu} & & & E_{12} \otimes E_{34} \\
 \downarrow E \otimes \tau \otimes E & \searrow E \otimes \mu \otimes E & \searrow \mu \otimes E \otimes E & \searrow E \otimes \mu & \downarrow \mu \\
 E_1 \otimes E_3 \otimes E_2 \otimes E_4 & \xrightarrow{\mu \otimes E \otimes E} & E_{12} \otimes E_3 \otimes E_4 & & \\
 \downarrow \mu \otimes \mu & \searrow E \otimes \mu & \downarrow \mu \otimes E & \searrow \mu & \\
 E_{13} \otimes E_{24} & \xrightarrow{\mu} & E_{123} \otimes E_4 & \xrightarrow{\mu} & E_{1234}
 \end{array}$$

Here we have numbered the E 's to make it clearer what's going on. The top and bottom left regions commute by functoriality of $- \otimes -$. The top left region commutes by commutativity of μ . Every other region commutes by associativity of μ . Thus, we've shown μ is a monoid object homomorphism, as desired. \square

Lemma 0.7. Suppose we have some monoid object (E, μ, e) in \mathcal{C} and some homomorphism of monoid objects $f : (E_1, \mu_1, e_1) \rightarrow (E_2, \mu_2, e_2)$ in $\mathbf{Mon}_{\mathcal{C}}$. Then $E \otimes f : E \otimes E_1 \rightarrow E \otimes E_2$ and $f \otimes E : E_1 \otimes E \rightarrow E_2 \otimes E$ are monoid homomorphisms, where here we are considering $E \otimes E_1$, $E \otimes E_2$, $E_1 \otimes E$, and $E_2 \otimes E$ to be monoid objects by [Lemma 0.4](#).

Proof. We will show that $E \otimes f$ is a monoid object homomorphism, as showing $f \otimes E$ is a monoid homomorphism is entirely analogous. First consider the following diagram:

$$\begin{array}{ccc}
 E \otimes E_1 \otimes E \otimes E_1 & \xrightarrow{E \otimes f \otimes E \otimes f} & E \otimes E_2 \otimes E \otimes E_2 \\
 \downarrow E \otimes \tau \otimes E_1 & & \downarrow E \otimes \tau \otimes E_2 \\
 E \otimes E \otimes E_1 \otimes E_1 & \xrightarrow{E \otimes E \otimes f \otimes f} & E \otimes E \otimes E_2 \otimes E_2 \\
 \downarrow \mu \otimes \mu_1 & \searrow \mu \otimes E_1 \otimes E_2 & \searrow \mu \otimes E_2 \otimes E_2 \\
 E \otimes E_1 & \xrightarrow{E \otimes f} & E \otimes E_2
 \end{array}$$

The top region commutes by naturality of τ . The bottom trapezoid commutes since f is a monoid homomorphism. The remaining three regions commute by functoriality of $- \otimes -$. Now, consider

the following diagram:

$$\begin{array}{ccccc}
 & & S & & \\
 & \swarrow & \downarrow e & \searrow & \\
 & E \otimes E_1 & E & E \otimes E_2 & \\
 & \swarrow E \otimes e_1 & & E \otimes e_2 \searrow & \\
 E \otimes E_1 & \xrightarrow{E \otimes f} & E \otimes E_2 & &
 \end{array}$$

The bottom region commutes since f is a monoid homomorphism. The top two regions commute by functoriality of $- \otimes -$. Thus, we've shown $E \otimes f$ is a monoid object homomorphism, as desired. \square

0.2. Modules over monoid objects in a symmetric monoidal category.

Definition 0.8. Let (E, μ, e) be a monoid object in \mathcal{C} . Then a *(left) module object* (N, κ) over (E, μ, e) is the data of an object N in \mathcal{C} and a morphism $\kappa : E \otimes N \rightarrow N$ such that the following two diagrams commute in \mathcal{C} :

$$\begin{array}{ccc}
 S \otimes N & \xrightarrow{e \otimes N} & E \otimes N \\
 \searrow \lambda_N & & \downarrow \kappa \\
 & & N
 \end{array}
 \quad
 \begin{array}{ccc}
 (E \otimes E) \otimes N & \xrightarrow{\mu \otimes N} & E \otimes N \\
 \alpha \downarrow & & \downarrow \kappa \\
 E \otimes (E \otimes N) & \xrightarrow{E \otimes \kappa} & E \otimes N \xrightarrow{\kappa} N
 \end{array}$$

Definition 0.9. Let (E, μ, e) be a monoid object in \mathcal{C} , and suppose we have two (left) module objects (N, κ) and (N', κ') over (E, μ, e) . Then a morphism $f : N \rightarrow N'$ is a *(left) E -module homomorphism* if the following diagram commutes in \mathcal{C} :

$$\begin{array}{ccc}
 E \otimes N & \xrightarrow{E \otimes f} & E \otimes N' \\
 \kappa \downarrow & & \downarrow \kappa' \\
 N & \xrightarrow{f} & N'
 \end{array}$$

Definition 0.10. Given a monoid object (E, μ, e) in \mathcal{C} , we write $E\text{-}\mathbf{Mod}$ to denote the category of (left) module objects over E and E -module homomorphisms between them. We denote the homset in $E\text{-}\mathbf{Mod}$ by

$$\mathrm{Hom}_{E\text{-}\mathbf{Mod}}(M, N), \quad \text{or simply} \quad \mathrm{Hom}_E(M, N).$$

For our purposes, we will only consider left module objects, so we will usually drop the quantifier “left” and just refer to them as “module objects”.

Lemma 0.11. Let (E, μ, e) be a monoid object in \mathcal{C} and let (N, κ) be an E module object. Then given some object X in \mathcal{C} and an isomorphism $\phi : N \xrightarrow{\cong} X$, X inherits the structure of an E -module via the action map

$$\kappa_\phi : E \otimes X \xrightarrow{E \otimes \phi^{-1}} E \otimes N \xrightarrow{\kappa} N \xrightarrow{\phi} X.$$

Proof. We need to show the two coherence diagrams in [Definition 0.8](#) commute. To see the former commutes, consider the following diagram:

$$\begin{array}{ccc}
 X & \xrightarrow{e \otimes X} & E \otimes X \\
 & \searrow \phi^{-1} & \downarrow E \otimes \phi^{-1} \\
 & N & \xrightarrow{e \otimes N} E \otimes N \\
 & & \downarrow \kappa \\
 & & N \\
 & & \downarrow \phi \\
 & & X
 \end{array}$$

The top trapezoid commutes by functoriality of $- \otimes -$. The middle small triangle commutes by unitality of κ . The remaining region commutes by definition. To see the second coherence diagram commutes, consider the following diagram:

$$\begin{array}{ccccc}
 E \otimes E \otimes X & \xrightarrow{\mu \otimes X} & E \otimes X & & \\
 E \otimes E \otimes \phi^{-1} \downarrow & & \downarrow E \otimes \phi^{-1} & & \\
 E \otimes E \otimes N & \xrightarrow{\mu \otimes N} & E \otimes N & & \\
 E \otimes \kappa \downarrow & & \downarrow \kappa & & \\
 E \otimes N & \xrightarrow{\kappa} & N & & \\
 E \otimes \phi \downarrow & & \downarrow \phi & & \\
 E \otimes X & \xrightarrow{E \otimes \phi^{-1}} E \otimes N & \xrightarrow{\kappa} N & \xrightarrow{\phi} X &
 \end{array}$$

The top rectangle commutes by functoriality of $- \otimes -$. The middle rectangle commutes by coherence for κ . The bottom two regions commute by definition. \square

Proposition 0.12. *Given a monoid object (E, μ, e) in \mathcal{C} , the forgetful functor $E\text{-}\mathbf{Mod} \rightarrow \mathcal{C}$ has a left adjoint $\mathcal{C} \rightarrow E\text{-}\mathbf{Mod}$ sending an object X in \mathcal{C} to $(E \otimes X, \kappa_X)$ where κ_X is the composition*

$$E \otimes (E \otimes X) \xrightarrow{\alpha^{-1}} (E \otimes E) \otimes X \xrightarrow{\mu \otimes X} E \otimes X,$$

and sending a morphism $f : X \rightarrow Y$ to $E \otimes f : E \otimes X \rightarrow E \otimes Y$.

We call this functor $E \otimes - : \mathcal{C} \rightarrow E\text{-}\mathbf{Mod}$ the free functor, and we call E -modules in the image of the free functor free modules.

Proof. In this proof, we work in a symmetric strict monoidal category. First, we wish to show that $E \otimes - : \mathcal{C} \rightarrow E\text{-}\mathbf{Mod}$ as constructed is well-defined. First, to see that (X, κ_X) is actually a E -module, we need to show the two diagrams in [Definition 0.8](#) commute. Indeed, consider the following diagrams:

$$\begin{array}{ccc}
 E \otimes X & \xrightarrow{e \otimes E \otimes X} & E \otimes E \otimes X \\
 & \searrow & \downarrow \mu \otimes X \\
 & & E \otimes X
 \end{array}
 \qquad
 \begin{array}{ccc}
 E \otimes E \otimes E \otimes X & \xrightarrow{\mu \otimes E \otimes X} & E \otimes E \otimes X \\
 E \otimes \mu \otimes X \downarrow & & \downarrow \mu \otimes X \\
 E \otimes E \otimes X & \xrightarrow{\mu \otimes X} & E \otimes X
 \end{array}$$

These are precisely the diagrams obtained by applying $X \otimes -$ to the coherence diagrams for μ , so that they commute as desired. Now, suppose $f : X \rightarrow Y$ is a morphism in \mathcal{C} , then we would

like to show that $E \otimes f : E \otimes X \rightarrow E \otimes Y$ is a morphism of E -module objects. Indeed, consider the following diagram:

$$\begin{array}{ccc} E \otimes E \otimes X & \xrightarrow{E \otimes E \otimes f} & E \otimes E \otimes Y \\ \mu \otimes X \downarrow & & \downarrow \mu \otimes Y \\ E \otimes X & \xrightarrow{E \otimes f} & E \otimes Y \end{array}$$

It commutes by functoriality of $-\otimes-$, so $E \otimes f$ is indeed an E -module homomorphism as desired.

Now, in order to see that $E \otimes -$ is left adjoint to the forgetful functor, it suffices to construct a unit and counit for the adjunction and show they satisfy the zig-zag identities. Given X in \mathcal{C} and (N, κ) in $E\text{-}\mathbf{Mod}$, define $\eta_X := e \otimes X : X \rightarrow E \otimes X$ and $\varepsilon_{(N, \kappa)} := \kappa : E \otimes N \rightarrow N$. η_X is clearly natural in X by functoriality of $-\otimes-$, and $\varepsilon_{(N, \kappa)}$ is natural in (N, κ) by how morphisms in $E\text{-}\mathbf{Mod}$ are defined. Now, to see these are actually the unit and counit of an adjunction, we need to show that the following diagrams commute for all X in \mathcal{C} and (N, κ) in $E\text{-}\mathbf{Mod}$:

$$\begin{array}{ccc} E \otimes X & \xrightarrow{E \otimes \eta_X = E \otimes e \otimes X} & E \otimes E \otimes X \\ & \searrow & \downarrow \varepsilon_{(E \otimes X, \kappa_X)} = \mu \otimes X \\ & & E \otimes X \end{array} \quad \begin{array}{ccc} E \otimes N & \xleftarrow{\eta_N = e \otimes N} & N \\ & \searrow & \downarrow \varepsilon_{(N, \kappa)} = \kappa \\ & & N \end{array}$$

Commutativity of the left diagram is unitality of μ , while commutativity of the right diagram is unitality of κ . Thus indeed $E \otimes - : \mathcal{C} \rightarrow E\text{-}\mathbf{Mod}$ is a left adjoint of the forgetful functor $E\text{-}\mathbf{Mod} \rightarrow \mathcal{C}$, as desired. \square

Lemma 0.13. *Let (E, μ, e) be a monoid object in \mathcal{C} . Further suppose we have some object X in \mathcal{C} and an E -module object (N, κ) , along with a commuting diagram in \mathcal{C}*

$$\begin{array}{ccccc} & \curvearrowright & & & \\ X & \xrightarrow{\iota} & N & \xrightarrow{r} & X \end{array}$$

Then if $\ell := \iota \circ r : N \rightarrow N$ is an E -module homomorphism, then X is canonically an E -module object with structure map

$$\kappa_X : E \otimes X \xrightarrow{E \otimes \iota} E \otimes N \xrightarrow{\kappa} N \xrightarrow{r} X,$$

and furthermore, the maps $\iota : X \rightarrow N$ and $r : N \rightarrow X$ are E -module homomorphisms.

Proof. First, in order to show (X, κ_X) is an E -module, we need to show the two diagrams in [Definition 0.8](#) commute. To see the unitality diagram holds, consider the following diagram:

$$\begin{array}{ccccc} S \otimes X & \xrightarrow{e \otimes X} & E \otimes X & & \\ & \searrow S \otimes \iota & \downarrow E \otimes \iota & & \\ & & S \otimes N & \xrightarrow{e \otimes N} & E \otimes N \\ & & \searrow \lambda_N & \downarrow \kappa & \\ & & & N & \\ \lambda_X \downarrow & & \nearrow \iota & \downarrow r & \\ X & \xrightarrow{\quad \quad \quad} & X & & \end{array}$$

The large left triangle commutes by naturality of λ . The top trapezoid commutes by functoriality of $-\otimes-$. The small middle right triangle commutes by unitality of κ . Finally, the bottom triangle

commutes by definition, since we are assuming $r \circ \iota = \text{id}_X$. Now the right composition is κ_X , so we have shown $\kappa_X \circ (e \otimes X) = \lambda_X$, as desired. Now, consider the following diagram:

$$\begin{array}{ccccc}
 E \otimes E \otimes X & \xrightarrow{\mu \otimes X} & E \otimes X & & \\
 \downarrow E \otimes E \otimes \iota & \searrow E \otimes E \otimes \iota & \downarrow E \otimes \iota & & \\
 E \otimes E \otimes N & \xrightarrow{E \otimes E \otimes \ell} & E \otimes E \otimes N & \xrightarrow{\mu \otimes N} & E \otimes N \\
 \downarrow E \otimes \kappa & & \downarrow E \otimes \kappa & & \downarrow \kappa \\
 E \otimes N & & E \otimes N & & N \\
 \downarrow E \otimes r & \searrow E \otimes \ell & \downarrow \kappa & & \downarrow r \\
 E \otimes X & \xrightarrow{E \otimes \iota} & E \otimes N & \xrightarrow{\kappa} & N \\
 & & & & \downarrow r \\
 & & & & X
 \end{array}$$

The top trapezoid commutes by functoriality of $- \otimes -$. The top left triangle commutes by functoriality of $- \otimes -$ and the fact that $\ell \circ \iota = \iota \circ r \circ \iota = \iota \circ \text{id}_X = \iota$. The middle left trapezoid commutes by since ℓ is an E -module homomorphism, by assumption. The bottom left triangle commutes by functoriality of $- \otimes -$ and the fact that $\iota \circ r = \ell$. Thus, we have shown that (X, κ_X) is an E -module object, as desired.

Now, it remains to show that $\iota : X \rightarrow N$ and $r : N \rightarrow X$ are E -module homomorphisms. To that end, consider the following two diagrams:

$$\begin{array}{ccc}
 E \otimes X & \xrightarrow{E \otimes \iota} & E \otimes N \\
 \downarrow E \otimes \iota & \searrow E \otimes \ell & \downarrow \kappa \\
 E \otimes N & & E \otimes N \\
 \downarrow \kappa & & \downarrow \kappa \\
 N & & N \\
 \downarrow r & \searrow \ell & \downarrow r \\
 X & \xrightarrow{\iota} & N
 \end{array}
 \qquad
 \begin{array}{ccc}
 E \otimes N & \xrightarrow{E \otimes r} & E \otimes X \\
 \downarrow E \otimes \ell & \searrow E \otimes \iota & \downarrow E \otimes \iota \\
 E \otimes N & & E \otimes N \\
 \downarrow \kappa & & \downarrow \kappa \\
 N & & N \\
 \downarrow r & \searrow \ell & \downarrow r \\
 X & \xrightarrow{r} & X
 \end{array}$$

The trapezoids in each diagram commute since we are assuming ℓ is a E -module homomorphism. The four triangles commute since $\ell \circ \iota = \iota$ and $r \circ \ell = r$. Thus, we have shown that $\kappa_X \circ (E \otimes r) = r \circ \kappa$ and $\kappa \circ (E \otimes \iota) = \iota \circ \kappa_X$, so we indeed have that ι and r are E -module homomorphisms, as desired. \square

Proposition 0.14. *Suppose that \mathcal{C} is an additive symmetric monoidal closed category. Let (E, μ, e) be a monoid object in \mathcal{C} , and suppose we have a family of E -module objects (N_i, κ_i) indexed by some small set I . Then $N := \bigoplus_{i \in I} N_i$ is canonically an E -module, with action map given by the composition*

$$\kappa : E \otimes \bigoplus_i N_i \xrightarrow{\cong} \bigoplus_i (E \otimes N_i) \xrightarrow{\bigoplus_i \kappa_i} \bigoplus_i N_i,$$

where the first isomorphism is given by the fact that $E \otimes -$ preserves coproducts, since it is a left adjoint. Furthermore, N is the coproduct of all the N_i 's in $E\text{-Mod}$, so that $E\text{-Mod}$ has arbitrary coproducts.

Proof. We need to show the action map κ makes the diagrams in [Definition 0.8](#) commute. To see the first (unitality) diagram commutes, consider the following diagram:

$$\begin{array}{ccc}
 \bigoplus_i N_i & \xrightarrow{e \otimes \bigoplus_i N_i} & E \otimes \bigoplus_i N_i \\
 & \searrow \bigoplus_i (e \otimes N_i) & \downarrow \cong \\
 & & \bigoplus_i (E \otimes N_i) \\
 & & \downarrow \bigoplus_i \kappa_i \\
 & & \bigoplus_i N_i
 \end{array}$$

The top triangle commutes since $E \otimes -$ preserves coproducts, as it is a left adjoint. The bottom triangle commutes by unitality of each of the κ_i 's. To see the second coherence diagram commutes, consider the following diagram:

$$\begin{array}{ccccc}
 E \otimes E \otimes \bigoplus_i N_i & \xrightarrow{\mu \otimes \bigoplus_i N_i} & E \otimes \bigoplus_i N_i & & \\
 E \otimes \cong \downarrow & \searrow \cong & \downarrow \cong & & \\
 E \otimes \bigoplus_i (E \otimes N_i) & \xrightarrow{\cong} & \bigoplus_i (E \otimes E \otimes N_i) & \xrightarrow{\bigoplus_i (\mu \otimes N_i)} & \bigoplus_i (E \otimes N_i) \\
 E \otimes \bigoplus_i \kappa_i \downarrow & & \bigoplus_i (E \otimes \kappa_i) \downarrow & & \downarrow \bigoplus_i \kappa_i \\
 E \otimes \bigoplus_i N_i & \xrightarrow{\cong} & \bigoplus_i (E \otimes N_i) & \xrightarrow{\bigoplus_i \kappa_i} & \bigoplus_i N_i
 \end{array}$$

The bottom right square commutes by coherence for the κ_i 's. Every other region commutes since $- \otimes -$ preserves colimits in each variable. Thus $N = \bigoplus_i N_i$ is indeed an E -module object, as desired.

Now, we claim that (N, κ) is the coproduct of the (N_i, κ_i) 's in $E\text{-}\mathbf{Mod}$. First, we need to show that the canonical maps $\iota_i : N_i \hookrightarrow N$ are morphisms in $E\text{-}\mathbf{Mod}$ for all $i \in I$. To see ι_i is a homomorphism of E -module objects, consider the following diagram:

$$\begin{array}{ccc}
 E \otimes N_i & \xrightarrow{E \otimes \iota_i} & E \otimes \bigoplus_i N_i \\
 \downarrow \kappa_i & \searrow \iota_{E \otimes N_i} & \downarrow \cong \\
 & & \bigoplus_i (E \otimes N_i) \\
 & & \downarrow \bigoplus_i \kappa_i \\
 N_i & \xrightarrow{\iota_i} & \bigoplus_i N_i
 \end{array}$$

The top triangle commutes by additivity of $E \otimes -$. The bottom trapezoid commutes since, by universal property of the coproduct, $\bigoplus_i \kappa_i$ is the unique arrow which makes the trapezoid commute for all $i \in I$. Now, it remains to show that given an E -module object (N', κ') and homomorphisms $f_i : N_i \rightarrow N'$ of E -module objects for all $i \in I$, that the unique arrow $f : N \rightarrow N'$ in \mathcal{SH} satisfying $f \circ \iota_i = f_i$ for all $i \in I$ is a homomorphism of E -module objects, so that N is actually the coproduct of the N_i 's. To see this, first let $h : \bigoplus_i (E \otimes N_i) \rightarrow E \otimes N'$ be the arrow determined by the maps

$E \otimes N_i \xrightarrow{E \otimes f_i} E \otimes N'$. Then consider the following diagram:

$$\begin{array}{ccc}
 E \otimes \bigoplus_i N_i & \xrightarrow{E \otimes f} & E \otimes N' \\
 \cong \downarrow & \nearrow h & \downarrow \nabla \\
 \bigoplus_i (E \otimes N_i) & \xrightarrow{\bigoplus_i (E \otimes f_i)} & \bigoplus_i (E \otimes N') \\
 \downarrow \bigoplus_i \kappa_i & & \downarrow \bigoplus_i \kappa' \\
 \bigoplus_i N_i & \xrightarrow{\bigoplus_i f_i} & N' \\
 & \searrow f & \downarrow \kappa'
 \end{array}$$

The top triangle commutes by additivity of $E \otimes -$. The triangle below that commutes by the universal property of the coproduct, since it is straightforward to check that $\nabla \circ \bigoplus_i (E \otimes f_i)$ and h both satisfy the universal property of the colimit. The left trapezoid commutes by functoriality of $- \oplus -$ and the fact that f_i is a homomorphism of E -module objects for all i in I . The right trapezoid commutes by naturality of ∇ . Finally, the bottom triangle commutes by the universal property of the coproduct, by showing that $\nabla \circ \bigoplus_i f_i$ in place of f also satisfies the universal property of the colimit. Hence f is indeed a homomorphism of E -module objects, as desired.

To recap, we have shown that given a set of E -module objects $\{(N_i, \kappa_i)\}_{i \in I}$, the inclusion maps $\iota_i : N_i \hookrightarrow \bigoplus_i N_i$ are morphisms in $E\text{-Mod}$, and that given morphisms $f_i : (N_i, \kappa_i) \rightarrow (N', \kappa')$ for all $i \in I$, the unique induced map $\bigoplus_i N_i \rightarrow N'$ is a morphism in $E\text{-Mod}$. Thus, $E\text{-Mod}$ does indeed have arbitrary coproducts, and the forgetful functor $E\text{-Mod} \rightarrow \mathcal{H}$ preserves them. \square

Proposition 0.15. *Suppose that \mathcal{C} is an additive closed symmetric monoidal category, and let (E, μ, e) be a monoid object in \mathcal{C} . Then $E\text{-Mod}$ is itself an additive category, so that in particular the forgetful functor $E\text{-Mod} \rightarrow \mathcal{C}$ and the free functor $\mathcal{C} \rightarrow E\text{-Mod}$ (Proposition 0.12) are additive.*

Proof. It is a general fact that adjoint functors between additive categories are necessarily additive. In order to show $E\text{-Mod}$ is an additive category, it suffices to show it has finite coproducts, that $\text{Hom}_{E\text{-Mod}}(N, N')$ is an abelian group for all E -modules N and N' , and that composition is bilinear. We know that $E\text{-Mod}$ has coproducts which are preserved by the forgetful functor $E\text{-Mod} \rightarrow \mathcal{C}$ by Proposition 0.14 (which is clearly faithful). Thus, because \mathcal{C} is **Ab**-enriched and $\text{Hom}_{E\text{-Mod}}(N, N') \subseteq \mathcal{C}(N, N')$, it suffices to show that $\text{Hom}_{E\text{-Mod}}(N, N')$ is closed under addition and taking inverses. To see the former, let $f, g : N \rightarrow N'$ be E -module homomorphisms, and consider the following diagram:

$$\begin{array}{ccccccc}
 E \otimes N & \xrightarrow{E \otimes \Delta_N} & E \otimes (N \oplus N) & \xrightarrow{E \otimes (f \oplus g)} & E \otimes (N' \oplus N') & \xrightarrow{E \otimes \nabla_{N'}} & E \otimes N' \\
 \downarrow \kappa & \searrow \Delta_{E \otimes N} & \cong \downarrow & & \downarrow \cong & \nearrow \nabla_{E \otimes N'} & \downarrow \kappa' \\
 & & (E \otimes N) \oplus (E \otimes N) & \xrightarrow{(E \otimes f) \oplus (E \otimes g)} & (E \otimes N') \oplus (E \otimes N') & & \\
 & & \downarrow \kappa \oplus \kappa & & \downarrow \kappa' \oplus \kappa' & & \\
 N & \xrightarrow{\Delta_N} & N \oplus N & \xrightarrow{f \oplus g} & N' \oplus N' & \xrightarrow{\nabla_{N'}} & N'
 \end{array}$$

The outermost trapezoids commute by naturality of Δ and ∇ . The triangles in the top corners and the top middle rectangle commute by additivity of $E \otimes -$. Finally, the middle bottom rectangle commutes by functoriality of $- \oplus -$ and $- \otimes -$, and the fact that f and g are E -module homomorphisms. Commutativity of the above diagram shows that $f + g$ is a homomorphism of

E -modules as desired. Finally, to see $-f$ is a E -module homomorphism if f is, we would like to show that $\kappa' \circ (E \otimes (-f)) = (-f) \circ \kappa$. This follows by the fact that $\kappa' \circ (E \otimes f) = f \circ \kappa$ and additivity of $- \otimes -$ and composition. \square