0.1. **Triangulated categories with sub-Picard grading.** In order to construct an abstract version of the Adams spectral sequence, we need to work in some axiomatic version of a stable homotopy category $\mathcal{SH}$ which acts like the familiar classical stable homotopy category **hoSp** (**??**) or the motivic stable homotopy category $\mathbf{SH}_{\mathscr{S}}$ over some base scheme $\mathscr{S}$ (**??**). This setting will be a category which is:

(1) Symmetric monoidal closed

(2) Triangulated

(3) Graded by (some collection of) its invertible objects

These data are furthermore required to be compatible.

Definition 0.1. Let $\mathcal{C}$ be an additive category with arbitrary (small) coproducts. Then an object X in $\mathcal{C}$ is *compact* if, for any collection of objects Y_i in $\mathcal{C}$ indexed by some (small) set I, the canonical map

$$\bigoplus_i \mathcal{C}(X, Y_i) \to \mathcal{C}(X, \bigoplus_i Y_i)$$

is an isomorphism of abelian groups. (Explicitly, the above map takes a generator $x \in \mathcal{C}(X, Y_i)$ to the composition $X \xrightarrow{x} Y_i \hookrightarrow \bigoplus_i Y_i$.)

Definition 0.2. Given a tensor triangulated category $(\mathcal{C}, \otimes, S, \Sigma, e, \mathcal{D})$ (**??**), a *sub-Picard grading* on $\mathcal{C}$ is the following data:

- A pointed abelian group $(A, \mathbf{1})$ along with a homomorphism of pointed groups $h : (A, \mathbf{1}) \to (\operatorname{Pic}\mathcal{C}, \Sigma S)$, where $\operatorname{Pic}\mathcal{C}$ is the *Picard group* of isomorphism classes of invertible objects in $\mathcal{C}$.[1]
- For each $a \in A$, a chosen representative S^a called the *a-sphere* in the isomorphism class $h(a)$ such that each S^a is a compact object (Definition 0.1) and $S^0 = S$.
- For each $a, b \in A$, an isomorphism $\phi_{a,b} : S^{a+b} \to S^a \otimes S^b$. This family of isomorphisms is required to be *coherent*, in the following sense:
  - For all $a \in A$, we must have that $\phi_{a,0}$ coincides with the right unitor $\rho_{S^a}^{-1} : S^a \xrightarrow{\cong} S^a \otimes S$ and $\phi_{0,a}$ coincides the left unitor $\lambda_{S^a}^{-1} : S^a \xrightarrow{\cong} S \otimes S^a$.
  - For all $a, b, c \in A$, the following "associativity diagram" must commute:

$$\begin{tikzcd} S^{a+b} \otimes S^c \arrow[d, "\phi_{a,b} \otimes S^c"'] & S^{a+b+c} \arrow[l, "\phi_{a+b,c}"'] \arrow[r, "\phi_{a,b+c}"] & S^a \otimes S^{b+c} \arrow[d, "S^a \otimes \phi_{b,c}"] \\ (S^a \otimes S^b) \otimes S^c \arrow[rr, "\cong"] & & S^a \otimes (S^b \otimes S^c) \end{tikzcd}$$

From now on we fix a monoidal closed tensor triangulated category $(\mathcal{SH}, \otimes, S, \Sigma, e, \mathcal{D})$ with arbitrary (small) (co)products and sub-Picard grading $(A, \mathbf{1}, h, \{S^a\}, \{\phi_{a,b}\})$. We also fix an isomorphism $\nu : \Sigma S \xrightarrow{\cong} S^{\mathbf{1}}$ once and for all. We establish conventions. First, observe the following remark:

Remark 0.3. Note that by induction the coherence conditions for the $\phi_{a,b}$'s in the above definition say that given any $a_1, \ldots, a_n \in A$ and $b_1, \ldots, b_m \in A$ such that $a_1 + \cdots + a_n = b_1 + \cdots + b_m$ and any fixed parenthesizations of $X = S^{a_1} \otimes \cdots \otimes S^{a_b}$ and $Y = S^{b_1} \otimes \cdots \otimes S^{b_m}$, there is a *unique*

[1] Recall an object X is a symmetric monoidal category is *invertible* if there exists some object Y and an isomorphism $S \cong X \otimes Y$.

isomorphism $X \to Y$ that can be obtained by forming formal compositions of products of $\phi_{a,b}$, identities, associators, unitors, and their inverses (but not symmetries).

In light of this remark, we will usually simply write ϕ or even just $\cong$ for any isomorphism that is built by taking compositions of products of $\phi_{a,b}$'s, unitors, associators, identities, and their inverses. Given an object X and a natural number $n > 0$, we write

$$X^n := \overbrace{X \otimes \cdots \otimes X}^{n \text{ times}} \qquad \text{and} \qquad X^0 := S.$$

We denote the associator, symmetry, left unitor, and right unitor isomorphisms in $\mathcal{SH}$ by

$$\alpha_{X,Y,Z} : (X \otimes Y) \otimes Z \xrightarrow{\cong} X \otimes (Y \otimes Z) \qquad \tau_{X,Y} : X \otimes Y \xrightarrow{\cong} Y \otimes X$$
$$\lambda_X : S \otimes X \xrightarrow{\cong} X \qquad \rho_X : X \otimes S \xrightarrow{\cong} X.$$

Often we will drop the subscripts. Furthermore, by the coherence theorem for symmetric monoidal categories ([1]), we will often assume α, ρ, and λ are actual equalities.

Given some integer $n \in \mathbb{Z}$, we will write a bold $\mathbf{n}$ to denote the element $n \cdot \mathbf{1}$ in A. Note that we can use the isomorphism $\nu : \Sigma S \xrightarrow{\cong} S^{\mathbf{1}}$ to construct a natural isomorphism $\Sigma \cong S^{\mathbf{1}} \otimes -$:

$$\Sigma X \xrightarrow{\Sigma \lambda_X^{-1}} \Sigma(S \otimes X) \xrightarrow{e_{S,X}^{-1}} \Sigma S \otimes X \xrightarrow{\nu \otimes X} S^{\mathbf{1}} \otimes X,$$

where $e_{X,Y} : \Sigma X \otimes Y \to \Sigma(X \otimes Y)$ is the isomorphism specified by the fact that $\mathcal{SH}$ is tensor-triangulated. The first two arrows are natural in X by definition. The last arrow is natural in X by functoriality of $- \otimes -$. By abuse of notation, we will always use ν to denote this natural isomorphism, rather than the isomorphism $\Sigma S \xrightarrow{\cong} S^{\mathbf{1}}$.

Given some $a \in A$, we define $\Sigma^a := S^a \otimes -$ and $\Omega^a := \Sigma^{-a} = S^{-a} \otimes -$. We specifically define $\Omega := \Omega^{\mathbf{1}}$. We say "the a^{th} suspension of X" to denote $\Sigma^a X$. It turns out that Σ^a is an autoequivalence of $\mathcal{SH}$ for each $a \in A$, and furthermore, Ω^a and Σ^a form an adjoint equivalence of $\mathcal{SH}$ for all a in A:

Proposition 0.4. *For each $a \in A$, the isomorphisms*

$$\eta_X^a : X \xrightarrow{\lambda_X^{-1}} S \otimes X \xrightarrow{\phi_{a,-a} \otimes X} (S^a \otimes S^{-a}) \otimes X \xrightarrow{\alpha} S^a \otimes (S^{-a} \otimes X) = \Sigma^a \Omega^a X$$

and

$$\varepsilon_X^a : \Omega^a \Sigma^a X = S^{-a} \otimes (S^a \otimes X) \xrightarrow{\alpha^{-1}} (S^{-a} \otimes S^a) \otimes X \xrightarrow{\phi_{-a,a}^{-1} \otimes X} S \otimes X \xrightarrow{\lambda_X} X$$

are natural in X, and furthermore, they are the unit and counit respectively of the adjoint autoequivalence $(\Omega^a, \Sigma^a, \eta^a, \varepsilon^a)$ of $\mathcal{SH}$. In particular, since $\Sigma \cong \Sigma^{\mathbf{1}}$, $\Omega := \Omega^{\mathbf{1}}$ is a left adjoint for Σ, so that $(\mathcal{SH}, \Omega, \Sigma, \eta, \varepsilon, \mathcal{D})$ is an adjointly *triangulated category (**??**), where η and ε are the compositions*

$$\eta : \mathrm{Id}_{\mathcal{SH}} \xRightarrow{\eta^{\mathbf{1}}} \Sigma^{\mathbf{1}} \Omega \xRightarrow{\nu^{-1}\Omega} \Sigma\Omega \qquad \textit{and} \qquad \varepsilon : \Omega\Sigma \xRightarrow{\Omega\nu} \Omega\Sigma^{\mathbf{1}} \xRightarrow{\varepsilon^{\mathbf{1}}} \mathrm{Id}_{\mathcal{SH}}.$$

Proof. In this proof, we will freely employ the coherence theorem for monoidal categories (see [1]), which essentially tells us that we may assume we are working in a strict monoidal category (i.e., that the associators and unitors and are identities). Then η_X^a and ε_X^a become simply the maps

$$\eta_X^a : X \xrightarrow{\phi_{a,-a} \otimes X} S^a \otimes S^{-a} \otimes X \qquad \text{and} \qquad \varepsilon_X^a : S^{-a} \otimes S^a \otimes X \xrightarrow{\phi_{-a,a}^{-1} \otimes X} X.$$

That these maps are natural in X follows by functoriality of $- \otimes -$. Now, recall that in order to show that these natural isomorphisms form an *adjoint* equivalence, it suffices to show that the natural isomorphisms $\eta^a : \mathrm{Id}_{\mathcal{SH}} \Rightarrow \Omega^a\Sigma^a$ and $\varepsilon^a : \Sigma^a\Omega^a \Rightarrow \mathrm{Id}_{\mathcal{SH}}$ satisfy one of the two zig-zag identities:

$$\begin{tikzcd} \Omega^a \arrow[r, "\Omega^a\eta^a"] \arrow[dr, equal] & \Omega^a\Sigma^a\Omega^a \arrow[d, "\varepsilon^a\Omega^a"] \\ & \Omega^a \end{tikzcd} \qquad \begin{tikzcd} \Sigma^a\Omega^a\Sigma^a \arrow[d, "\Sigma^a\varepsilon^a"'] & \Sigma^a \arrow[l, "\eta^a\Sigma^a"'] \arrow[dl, equal] \\ \Sigma^a & \end{tikzcd}$$

(that it suffices to show only one is [2, Lemma 3.2]). We will show that the left is satisfied. Unravelling definitions, we simply wish to show that the following diagram commutes for all X in $\mathcal{SH}$:

$$\begin{tikzcd} S^{-a} \otimes X \arrow[r, "S^{-a}\otimes\phi_{a,-a}\otimes X"] \arrow[dr, equal] & S^{-a} \otimes S^a \otimes S^{-a} \otimes X \arrow[d, "\phi^{-1}_{-a,a}\otimes S^{-a}\otimes X"] \\ & S^{-a} \otimes X \end{tikzcd}$$

Yet this is simply the diagram obtained by applying $-\otimes X$ to the associativity coherence diagram for the $\phi_{a,b}$'s (since $\phi_{a,0}$ and $\phi_{0,a}$ coincide with the unitors, and here we are taking the unitors and associators to be equalities), so it does commute, as desired. $\square$

Given two objects X and Y in $\mathcal{SH}$, we will denote the hom-abelian group of morphisms from X to Y in $\mathcal{SH}$ by $[X,Y]$, and the internal hom object by $F(X,Y)$. We can extend the abelian group $[X,Y]$ into an A-graded abelian group $[X,Y]_*$ by defining $[X,Y]_a := [S^a \otimes X, Y]$.

Given an object X in $\mathcal{SH}$ and some $a \in A$, we can define the abelian group

$$\pi_a(X) := [S^a, X],$$

which we call the a^{th} *(stable) homotopy group of* X. We write $\pi_*(X)$ for the A-graded abelian group $\bigoplus_{a\in A} \pi_a(X)$, so that in particular we have a canonical isomorphism

$$\pi_*(X) = [S^*, X] \cong [S, X]_*.$$

Given some other object E, we can define the A-graded abelian groups $E_*(X)$ and $E^*(X)$ by the formulas

$$E_a(X) := \pi_a(E \otimes X) = [S^a, E \otimes X] \qquad \text{and} \qquad E^a(X) := [X, S^a \otimes E].$$

We refer to the functor $E_*(-)$ as the *homology theory represented by* E, or just E-homology, and we refer to $E^*(-)$ as the *cohomology theory represented by* E, or just E-cohomology.

A nice result is that in $\mathcal{SH}$, (co)fiber sequences (distinguished triangles) give rise to homotopy long exact sequences. To see this, we first need the following lemma:

Definition 0.5. For all X, Y in $\mathcal{SH}$ and $a \in A$, there are A-graded isomorphisms

$$s^a_{X,Y} : [X, \Sigma^a Y]_* \to [X, Y]_{*-a}$$

sending $x : S^b \otimes X \to S^a \otimes Y$ in $[X, \Sigma^a Y]_*$ to the composition

$$S^{b-a} \otimes X \xrightarrow{\phi_{-a,b}\otimes X} S^{-a} \otimes S^b \otimes X \xrightarrow{S^{-a}\otimes x} S^{-a} \otimes S^a \otimes Y \xrightarrow{\phi^{-1}_{-a,a}\otimes Y} Y.$$

Furthermore, these isomorphisms are natural in both X and Y.

In particular, for each $a \in A$ and object X in $\mathcal{SH}$, we have natural isomorphisms

$$s^a_X : \pi_*(\Sigma^a X) = [S^*, \Sigma^a X] \xrightarrow{\cong} [S, \Sigma^a X]_* \xrightarrow{s^a_{S,X}} [S, X]_{*-a} \xrightarrow{\cong} \pi_{*-a}(X)$$

sending $x : S^b \to S^a \otimes X$ in $\pi_*(\Sigma^a X)$ to the composition

$$S^{b-a} \xrightarrow{\phi_{-a,b}} S^{-a} \otimes S^b \xrightarrow{S^{-a}\otimes x} S^{-a} \otimes S^a \otimes X \xrightarrow{\phi^{-1}_{-a,a}\otimes X} X.$$

Proof. First, by unravelling definitions, note that $s^a_{X,Y}$ is precisely the composition

$$[X,\Sigma^a Y]_* = [S^* \otimes X, S^a \otimes Y] \xrightarrow{\text{adj}} [S^{-a} \otimes S^* \otimes X, Y] \xrightarrow{(\phi_{-a,*}\otimes X)^*} [S^{*-a} \otimes X, Y] = [X,Y]_{*-a},$$

where the adjunction is that from Proposition 0.4. The adjunction is natural in $S^* \otimes X$ and Y by definition, so that in particular it is natural in X and Y. It is furthermore straightforward to see by functoriality of $- \otimes -$ that the second arrow is natural in both X and Y. Thus $s^a_{X,Y}$ is natural in X and Y, as desired. $\square$

Proposition 0.6. *Suppose we are given a distinguished triangle*

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$$

and an object W in $\mathcal{SH}$. Then there exists a "connecting homomorphism" of degree $-\mathbf{1}$

$$\partial : [W,Z]_* \to [W,X]_{*-\mathbf{1}}$$

such that the following triangle is exact at each vertex:

$$\begin{tikzcd} {[W,X]_*} \arrow[r, "f_*"] & {[W,Y]_*} \arrow[d, "g_*"] \\ & {[W,Z]_*} \arrow[ul, "\partial"] \end{tikzcd}$$

Proof. By axiom TR4 for a triangulated category and the fact that distinguished triangles are exact (**??**), we have the following exact sequence in $\mathcal{SH}$

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X \xrightarrow{\Sigma f} \Sigma Y.$$

Thus, we may apply $[W,-]_*$ to get an exact sequence of A-graded abelian groups which fits into the top row in the following diagram:

$$\begin{tikzcd} {[W,X]_*} \arrow[r, "f_*"] \arrow[dd, equal] & {[W,Y]_*} \arrow[r, "g_*"] \arrow[dd, equal] & {[W,Z]_*} \arrow[r, "h_*"] \arrow[dd, equal] & {[W,\Sigma X]_*} \arrow[r, "\Sigma f_*"] \arrow[d, "(\nu_X)_*"] & {[W,\Sigma Y]_*} \arrow[d, "(\nu_Y)_*"] \\ & & & {[W,\Sigma^{\mathbf{1}} X]_*} \arrow[r, "\Sigma^{\mathbf{1}} f_*"] \arrow[d, "s^{\mathbf{1}}_{W,X}"] & {[W,\Sigma^{\mathbf{1}} Y]_*} \arrow[d, "s^{\mathbf{1}}_{W,Y}"] \\ {[W,X]_*} \arrow[r, "f_*"] & {[W,Y]_*} \arrow[r, "g_*"] & {[W,Z]_*} \arrow[r, dashed, "\partial"] & {[W,X]_{*-\mathbf{1}}} \arrow[r, "f_*"] & {[W,Y]_{*-\mathbf{1}}} \end{tikzcd}$$

where here we define $\partial : [W,Z]_* \to [W,X]_{*-\mathbf{1}}$ to be the composition which makes the diagram commute. The diagram commutes by naturality of ν and $s^{\mathbf{1}}$, so that the bottom row is exact since the top row is exact and the vertical arrows are isomorphisms. Hence may roll the bottom row up to get the desired exact triangle:

$$\begin{tikzcd} {[W,X]_*} \arrow[r, "f_*"] & {[W,Y]_*} \arrow[d, "g_*"] \\ & {[W,Z]_*} \arrow[ul, "\partial"] \end{tikzcd}$$

$\square$

Similarly, we have homology long exact sequences:

Proposition 0.7. *Suppose we are given a distinguished triangle*

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$$

and an object E in $\mathcal{SH}$. Then there exists a "connecting homomorphism" of degree $-\mathbf{1}$

$$\partial : E_*(Z) \to E_{*-1}(X)$$

such that the following triangle is exact at each vertex:

$$\begin{tikzcd} E_*(X) \arrow[r, "E_*(f)"] & E_*(Y) \arrow[d, "E_*(g)"] \\ & E_*(Z) \arrow[ul, "\partial"] \end{tikzcd}$$

Furthermore, if (E, μ, e) is a monoid object in $\mathcal{SH}$, then ∂ is a homomorphism of left $\pi_*(E)$-modules.

Proof. By axiom TR4 for a triangulated category, axiom TT3 for a tensor triangulated category, and the fact that distinguished triangles are exact (**??**), we have that the following sequence in $\mathcal{SH}$ is exact:

$$E \otimes X \xrightarrow{E \otimes f} E \otimes Y \xrightarrow{E \otimes g} E \otimes Z \xrightarrow{E \otimes h} E \otimes \Sigma X \xrightarrow{E \otimes \Sigma f} E \otimes \Sigma Y.$$

Thus, we may apply the functor $\pi_*(-) = [S^*, -]$ to get a long exact sequence which fits into the top row in the following diagram:

$$\begin{tikzcd} E_*(X) \arrow[r, "E_*(f)"] \arrow[dd, equal] & E_*(Y) \arrow[r, "E_*(g)"] \arrow[dd, equal] & E_*(Z) \arrow[r, "E_*(h)"] \arrow[dd, equal] & E_*(\Sigma X) \arrow[r, "E_*(\Sigma f)"] \arrow[d, "E_*(\nu_X)"'] & E_*(\Sigma Y) \arrow[d, "E_*(\nu_Y)"] \\ & & & E_*(\Sigma^{\mathbf{1}} X) \arrow[r, "E_*(\Sigma^{\mathbf{1}} f)"] \arrow[d, "t^{\mathbf{1}}_X"'] & E_*(\Sigma^{\mathbf{1}} Y) \arrow[d, "t^{\mathbf{1}}_Y"] \\ E_*(X) \arrow[r, "E_*(f)"] & E_*(Y) \arrow[r, "E_*(g)"] & E_*(Z) \arrow[r, "E_*(h)"] & E_{*-\mathbf{1}}(X) \arrow[r, "E_{*-\mathbf{1}}(f)"] & E_{*-\mathbf{1}}(Y) \end{tikzcd}$$

□