In **??**, we showed that given a monoid object (E,μ,e) in $\mathcal{SH}$, that $E_*(E)=\pi_*(E\otimes E)$ is both a ring (since $E\otimes E$ is a monoid object if E is), and an A-graded bimodule over the ring $\pi_*(E)$. In this subsection, we will outline some additional structure carried by the pair $(E_*(E),\pi_*(E))$. Namely, we will show that if (E,μ,e) is a flat (Definition 0.5) commutative monoid object, then this pair, called the *dual E-Steenrod algebra*, is canonically an *A-graded anticommutative Hopf algebroid* over the stable homotopy ring $\pi_*(S)$ (**??**). To start with, we outline some structure maps relating $E_*(E)$ and $\pi_*(E)$.

First, recall that given a monoid object (E,μ,e) in $\mathcal{SH}$, $\pi_*(E)$ is canonically an A-graded ring by **??**, and so is $E_*(E)=\pi_*(E\otimes E)$ and $E_*(E\otimes E)=\pi_*(E\otimes E\otimes E)$, since the tensor product of monoid objects in a symmetric monoidal category is again a monoid object (**??**).

Lemma 0.1. *Let (E,μ,e) be a commutative monoid object in $\mathcal{SH}$. Then the maps*

(1) $E\xrightarrow{\cong}E\otimes S\xrightarrow{E\otimes e}E\otimes E$,

(2) $E\xrightarrow{\cong}S\otimes E\xrightarrow{e\otimes E}E\otimes E$,

(3) $E\otimes E\xrightarrow{\cong}E\otimes S\otimes E\xrightarrow{E\otimes e\otimes E}E\otimes E\otimes E$,

(4) $E\otimes E\xrightarrow{\mu}E$, *and*

(5) $E\otimes E\xrightarrow{\tau_{E,E}}E\otimes E$

*are homomorphisms of monoid objects in $\mathcal{SH}$ (where here $E\otimes E$ and $E\otimes E\otimes E$ are considered as monoid objects in $\mathcal{SH}$ by **??** and **??**, respectively), so that by **??**, under π_* they induce morphisms in $\pi_*(S)$-$\mathbf{GCA}^A$:*

(1) $\eta_L:\pi_*(E)\to E_*(E)$,

(2) $\eta_R:\pi_*(E)\to E_*(E)$,

(3) $h:E_*(E)\to E_*(E\otimes E)$,

(4) $\epsilon:E_*(E)\to\pi_*(E)$, *and*

(5) $c:E_*(E)\to E_*(E)$.

Proof. It is a general fact that the unit and multiplication maps $e:S\to E$ and $\mu:E\otimes E\to E$ for a monoid are monoid homomorphisms when (E,μ,e) is a commutative monoid object (**??**), so that the maps $E\otimes e$, and $e\otimes E$ from E to $E\otimes E$ are monoid homomorphisms, by **??**. Similarly, $E\otimes e\otimes E:E\otimes E\to E\otimes E\otimes E$ is a monoid homomorphism. Thus, it remains to show that $\tau_{E,E}:E\otimes E\to E\otimes E$ is a monoid homomorphism. First, consider the following diagram:

$$\begin{tikzcd}
E_1\otimes E_2\otimes E_3\otimes E_4 \arrow[r, "\tau\otimes\tau"] \arrow[d, "E\otimes\tau\otimes E"'] & E_2\otimes E_1\otimes E_4\otimes E_3 \arrow[d, "E\otimes\tau\otimes E"] \\
E_1\otimes E_3\otimes E_2\otimes E_4 \arrow[r, "\tau_{E\otimes E,E\otimes E}"] \arrow[d, "\mu\otimes\mu"'] & E_2\otimes E_4\otimes E_1\otimes E_3 \arrow[d, "\mu\otimes\mu"] \\
E_{1,3}\otimes E_{2,4} \arrow[r, "\tau"'] & E_{2,4}\otimes E_{1,3}
\end{tikzcd}$$

(Here we've labelled the E's to make the action of the braidings clearer). The top region commutes by coherence for the symmetries in a symmetric monoidal category, while the bottom region commutes by naturality of τ. Now, consider the following diagram:

$$\begin{tikzcd}
& & S \arrow[dl, "\cong"'] \arrow[dr, "\cong"] & & \\
& S \otimes S \arrow[rr, "\tau"] \arrow[dl, "e\otimes e"'] & & S \otimes S \arrow[dr, "e\otimes e"] & \\
E \otimes E \arrow[rrrr, "\tau"] & & & & E \otimes E
\end{tikzcd}$$

The top triangle commutes by coherence for a symmetric monoidal category, while the bottom region commutes by naturality of τ. Thus, we have shown $\tau_{E,E}$ is a homomorphism of monoid objects, as desired. $\square$

Recall a that given a homomorphism of rings $f : R \to R'$, the ring R' canonically becomes an R-bimodule with left action $r \cdot x := f(r)x$ and right action $x \cdot r := xf(r)$. In particular, the ring homomorphisms $\eta_L : \pi_*(E) \to E_*(E)$ and $\eta_R : \pi_*(E) \to E_*(E)$ endow $E_*(E)$ with the structure of a bimodule over $\pi_*(E)$. Naturally, one may ask in what sense these bimodule structures coincide with the canonical one (from **??**). The following lemma tells us that the canonical $\pi_*(E)$-bimodule structure on $E_*(E)$ is that with left action induced by η_L and right action induced by η_R:

Lemma 0.2. *Let (E, μ, e) be a commutative monoid object in $\mathcal{SH}$. Then the left (resp. right) $\pi_*(E)$-module structure induced on $E_*(E)$ by the ring homomorphism η_L (resp. η_R) coincides with the canonical left (resp. right) $\pi_*(E)$-module structure on $E_*(E)$ given in* **??**.

Proof. What's going on here is a bit subtle, so we're going to be really explicit. In **??**, it was shown that $E_*(E)$ is a left $\pi_*(E)$-module via the assignment

$$\pi_*(E) \times E_*(E) \to E_*(E)$$

which sends homogeneous elements $r : S^a \to E$ and $x : S^b \to E \otimes E$ to the composition

$$S^{a+b} \xrightarrow{\cong} S^a \otimes S^b \xrightarrow{r\otimes x} E \otimes E \otimes E \xrightarrow{\mu\otimes E} E \otimes E.$$

We'd like to show that this is the same thing as the assignment $\pi_*(E) \times E_*(E) \to E_*(E)$ sending $(r, x) \mapsto \eta_L(r)x$, where $\eta_L(r)x$ denotes the product of $\eta_L(r)$ and x taken in the ring $E_*(E)$. Explicitly, the product structure on $E_*(E) = \pi_*(E \otimes E)$ is that induced by the fact that $E \otimes E$ is a monoid object in $\mathcal{SH}$ (by **??**), with product

$$E \otimes E \otimes E \otimes E \xrightarrow{E\otimes\tau\otimes E} E \otimes E \otimes E \otimes E \xrightarrow{\mu\otimes\mu} E \otimes E$$

(note the middle two factors are swapped). By linearity of module actions, in order to show the canonical left $\pi_*(E)$-module structure on $E_*(E)$ agrees with that induced by η_L, it suffices to show the module actions agree on homogeneous elements. Now, suppose we have homogeneous elements $r : S^a \to E$ in $\pi_*(E)$ and $x : S^b \to E \otimes E$ in $E_*(E)$, and consider the following diagram, where we've passed to a symmetric strict monoidal category:

$$\begin{tikzcd}
S^{a+b} \arrow[d, "\phi_{a,b}"] \\
S^a \otimes S^b \arrow[d, "r\otimes x"] \\
E_1 \otimes E_2 \otimes E_3 \arrow[rrrr, "\mu\otimes E"] \arrow[dd, "E\otimes e\otimes E"'] \arrow[dr, equal] & & & & E_{1,2} \otimes E_3 \arrow[dd, equal] \\
& E_1 \otimes E_2 \otimes E_3 \arrow[r, equal] & E_1 \otimes E_2 \otimes E_3 \arrow[r, equal] \arrow[d, "E\otimes E\otimes e\otimes E"] & E_1 \otimes E_2 \otimes E_3 \arrow[dr, "\mu\otimes E"] & \\
E_1 \otimes E \otimes E_2 \otimes E_3 \arrow[ur, "E\otimes\mu\otimes E"] \arrow[rr, "E\otimes\tau\otimes E"'] & & E_1 \otimes E_2 \otimes E \otimes E_3 \arrow[ul, "E\otimes\mu\otimes E"] \arrow[ur, "E\otimes E\otimes\mu"'] \arrow[rr, "\mu\otimes\mu"'] & & E_{1,2} \otimes E_3
\end{tikzcd}$$

Here we've numbered the E's to make it clear what's going on. The bottom composition is $\eta_L(r)x$, while the top composition is the canonical left action of r on x given in **??**. The leftmost triangle commutes by unitality of μ. The triangle to the right of that commutes by commutativity of μ. The triangle to the right of that commutes by unitality of μ, as does the next triangle. The remaining triangle on the right commutes by functoriality of $- \otimes -$. Finally, the top region commutes by definition. Thus, we've shown that the left $\pi_*(E)$-module structure induced on $E_*(E)$ by η_L is in fact the canonical one. On the other hand, showing that the right $\pi_*(E)$-module structure induced on $E_*(E)$ by η_R is the canonical one is entirely analagous, and we leave it as an exercise for the reader. □

Recall (**??**) that the pushout of two morphisms $f : B \to C$ and $g : B \to D$ in R-$\mathbf{GCA}^A$ is obtained by taking the tensor product of B-modules $C \otimes_B D$, where C has right B-module action induced by f, and D has left B-module action induced by g, and giving it an anticommutative product which makes $C \otimes_B D$ a ring. Thus, by the above lemma, we may view the tensor product of bimodules $E_*(E) \otimes_{\pi_*(E)} E_*(E)$ (where $E_*(E)$ is considered with its canonical $\pi_*(E)$-bimodule structure from **??**) as not just an A-graded abelian group or a $\pi_*(E)$-bimodule, but as an A-graded anticommutative $\pi_*(S)$-algebra:

Corollary 0.3. *Given a commutative monoid object (E, μ, e) in $\mathcal{SH}$, the domain of the homomorphism*

$$\Phi_{E,E} : E_*(E) \otimes_{\pi_*(E)} E_*(E) \to E_*(E \otimes E)$$

constructed in **??** *is canonically an A-graded anticommutative $\pi_*(S)$-algebra, and sits in the following pushout diagram in $\pi_*(S)$-$\mathbf{GCA}^A$:*

$$\begin{tikzcd}
\pi_*(E) \arrow[r, "\eta_L"] \arrow[d, "\eta_R"'] & E_*(E) \arrow[d, "x\mapsto 1\otimes x"] \\
E_*(E) \arrow[r, "x\mapsto x\otimes 1"'] & E_*(E) \otimes_{\pi_*(E)} E_*(E)
\end{tikzcd}$$

Furthermore, with respect to this ring structure, $\Phi_{E,E}$ is a homomorphism of rings:

Lemma 0.4. *Let (E, μ, e) be a commutative monoid object in $\mathcal{SH}$. Then the homomorphism*

$$\Phi_{E,E} : E_*(E) \otimes_{\pi_*(E)} E_*(E) \to E_*(E \otimes E)$$

constructed in **??** *is a homomorphism of A-graded anticommutative $\pi_*(S)$-algebras.*

Proof. Consider the maps

$$f : E \otimes E \xrightarrow{e \otimes E \otimes E} E \otimes E \otimes E$$

and

$$g : E \otimes E \xrightarrow{E \otimes E \otimes e} E \otimes E \otimes E.$$

We know that the maps

$$E \xrightarrow{e \otimes E} E \otimes E \qquad \text{and} \qquad E \xrightarrow{E \otimes e} E \otimes E$$

are monoid homomorphisms by Lemma 0.1, so that f and g are monoid homomorphisms by **??**. Furthermore, by **??**, they are monoid homomorphisms between the same monoid objects in $\mathcal{SH}$ (when we assume that strict associativity holds). Finally, note that we have the following commutative diagram

$$\begin{tikzcd}
E \arrow[r, "E \otimes e"] \arrow[d, "e \otimes E"'] \arrow[dr, "e \otimes E \otimes e"] & E \otimes E \arrow[d, "e \otimes E \otimes E"] \\
E \otimes E \arrow[r, "E \otimes E \otimes e"'] & E \otimes E \otimes E
\end{tikzcd}$$

where the outer arrows are monoid object homomorphisms, thus, we may apply π_*, which yields the following commutative diagram in $\pi_*(S)$-$\mathbf{GCA}^A$ (**??**):

$$\begin{tikzcd}
\pi_*(E) \arrow[r, "\eta_L"] \arrow[d, "\eta_R"'] & E_*(E) \arrow[d, "\pi_*(f)"] \\
E_*(E) \arrow[r, "\pi_*(g)"'] & E_*(E \otimes E)
\end{tikzcd}$$

Hence by Lemma 0.4 and the universal property of the pushout, there exists some unique morphism $\ell : E_*(E) \otimes_{\pi_*(E)} E_*(E) \to E_*(E \otimes E)$ in $\pi_*(S)$-$\mathbf{GCA}^A$ which makes the following diagram commute:

$$\begin{tikzcd}
\pi_*(E) \arrow[r, "\eta_L"] \arrow[d, "\eta_R"'] & E_*(E) \arrow[d, "x \mapsto 1 \otimes x"'] \arrow[ddr, bend left, "\pi_*(f)"] & \\
E_*(E) \arrow[r, "x \mapsto x \otimes 1"] \arrow[drr, bend right, "\pi_*(g)"'] & E_*(E) \otimes_{\pi_*(E)} E_*(E) \arrow[dr, dashed, "\ell"] & \\
& & E_*(E \otimes E)
\end{tikzcd}$$

Thus in order to show $\Phi_{E,E}$ is a morphism in $\pi_*(S)$-$\mathbf{GCA}^A$, it suffices to show that $\Phi_{E,E}$ and ℓ are the same map, since we know ℓ is a homomorphism of A-graded anticommutative $\pi_*(S)$-algebras. Since $\Phi_{E,E}$ and ℓ are both abelian group homomorphisms, it further suffices to show they agree on homogeneous pure tensors, which generate $E_*(E) \otimes_{\pi_*(E)} E_*(E)$ as an abelian group. Given homogeneous elements $x : S^a \to E \otimes E$ and $y : S^b \to E \otimes E$ in $E_*(E)$, unravelling how pushouts in $\pi_*(S)$-$\mathbf{GCA}^A$ are defined (**??**), ℓ sends the pure homogeneous tensor $x \otimes y$ to the element $\pi_*(g)(x) \cdot \pi_*(f)(y)$, where here $\cdot$ denotes the product taken in $E_*(E \otimes E) = \pi_*(E \otimes E \otimes E)$. Now, consider the following diagram:

$$
\begin{tikzcd}
S^{a+b} \arrow[d, "\phi_{a,b}"] & & \\
S^a \otimes S^b \arrow[d, "x\otimes y"] & & \\
E_1 \otimes E_2 \otimes E_3 \otimes E_4 \arrow[rr, "g\otimes f = E\otimes E\otimes e\otimes e\otimes E\otimes E"] \arrow[drr, "E\otimes e\otimes E\otimes e\otimes E\otimes E"] \arrow[ddd, "E\otimes\mu\otimes E"'] \arrow[ddr, equal] & & E_1 \otimes E_2 \otimes E_a \otimes E_b \otimes E_3 \otimes E_4 \arrow[d, "E\otimes\tau_{E\otimes E,E}\otimes E\otimes E"] \\
 & & E_1 \otimes E_b \otimes E_2 \otimes E_a \otimes E_3 \otimes E_4 \arrow[d, "\mu\otimes E\otimes\tau\otimes E"] \\
 & E_1 \otimes E_2 \otimes E_3 \otimes E_4 \arrow[r, "E\otimes E\otimes E\otimes e\otimes E"'] \arrow[ur, "E\otimes e\otimes E\otimes e\otimes E\otimes E"] \arrow[dl, "E\otimes\mu\otimes E"] & E_1 \otimes E_2 \otimes E_3 \otimes E_a \otimes E_4 \arrow[d, "E\otimes\mu\otimes\mu"] \\
E_1 \otimes E_{2,3} \otimes E_4 \arrow[rr, equal] & & E_1 \otimes E_{2,3} \otimes E_4
\end{tikzcd}
$$

Here we have labelled the E's to make things clearer. The left outside composition is $\Phi_{E,E}(x\otimes y)$, while the right composition is $\pi_*(g)(x)\cdot\pi_*(f)(y)$. The top right triangle commutes by coherence for a symmetric monoidal category. The middle tright triangle commutes by unitality of μ and coherence for a symmetric monoidal category. The bottom trapezoid commutes by unitality of μ. The rest of the diagram commutes by definition. Thus we have $\Phi_{E,E}(x\otimes y) = \pi_*(g)(x)\cdot\pi_*(f)(y)$, so that $\Phi_{E,E} = \ell$ is not just an isomorphism of left $\pi_*(E)$-modules, but an isomorphism of A-graded anticommutative $\pi_*(S)$-algebras, as desired. $\square$

For the sake of conciseness, we make the following definition:

Definition 0.5. We say that a monoid object (E,μ,e) in $\mathcal{SH}$ is *flat* if the canonical right $\pi_*(E)$-module structure on $E_*(E)$ from **??** is that of a flat module, or equivalently by Lemma 0.2, if the the map $\eta_R : \pi_*(E) \to E_*(E)$ constructed in Lemma 0.1 is a flat ring homomorphism.

Finally, we can package all of this information into an object called the *dual E-Steenrod algebra*:

Definition 0.6. Let (E,μ,e) be a *commutative* monoid object in $\mathcal{SH}$ which is flat (Definition 0.5) and cellular (**??**). Then the *dual E-Steenrod algebra* is the pair of A-graded abelian groups $(E_*(E),\pi_*(E))$ equipped with the following structure:

1. The A-graded anticommutative $\pi_*(S)$-algebra structure on $\pi_*(E)$ induced from E being a commutative monoid object in $\mathcal{SH}$ (**??**).
2. The A-graded anticommutative $\pi_*(S)$-algebra structure on $E_*(E)$ induced from the fact that $E\otimes E$ is canonically a commutative monoid object in $\mathcal{SH}$ (**??**), so that also $E_*(E) = \pi_*(E\otimes E)$ is an A-graded anticommutative $\pi_*(S)$-algebra (**??**).
3. The homomorphisms of A-graded anticommutative $\pi_*(S)$-algebras
$$\eta_L : \pi_*(E) \to E_*(E)$$
and
$$\eta_R : \pi_*(E) \to E_*(E)$$
induced under π_* by the monoid object homomorphisms
$$E \xrightarrow{\cong} E\otimes S \xrightarrow{E\otimes e} E\otimes E$$

and

$$E \xrightarrow{\cong} S \otimes E \xrightarrow{e \otimes E} E \otimes E.$$

4. The homomorphism of A-graded anticommutative $\pi_*(S)$-algebras

$$\Psi_E : E_*(E) \to E_*(E) \otimes_{\pi_*(E)} E_*(E)$$

given by the composition

$$E_*(E) \xrightarrow{h} E_*(E \otimes E) \xrightarrow{\Phi_{E,E}^{-1}} E_*(E) \otimes_{\pi_*(E)} E_*(E),$$

where h is a homomorphism of A-graded anticommutative $\pi_*(S)$-algebras induced under π_* by the monoid object homomorphism

$$E \otimes E \xrightarrow{\cong} E \otimes S \otimes E \xrightarrow{E \otimes e \otimes E} E \otimes E \otimes E,$$

and $\Phi_{E,E}$ is morphism constructed in **??**, which is proven to be an isomorphism in **??** (since E is flat and cellular), and furthermore an isomorphism in $\pi_*(S)$-$\mathbf{GCA}^A$ by Lemma 0.4.

5. The homomorphism of A-graded anticommutative $\pi_*(S)$-algebras

$$\epsilon : E_*(E) \to \pi_*(E)$$

induced under π_* by the monoid object homomorphism

$$E \otimes E \xrightarrow{\mu} E.$$

6. The homomorphism of A-graded anticommutative $\pi_*(S)$-algebras

$$c : E_*(E) \to E_*(E)$$

induced under π_* from the monoid object homomorphism

$$E \otimes E \xrightarrow{\tau} E \otimes E.$$

The curious reader may wonder why we call $(E_*(E), \pi_*(E))$ the *dual* E-Steenrod algebra. The "dual" is there because the E-Steenrod algebra refers instead to the E-self cohomology $E^*(E) \cong [E, E]_{-*}$. Clasically, the Adams spectral sequence was originally constructed in such a way that the E_1 and E_2 pages could be characterized in terms of cohomology groups as modules over the E-Steenrod algebra, but it turns out that our approach using homology groups as comodules over the dual E-Steenrod algebra is somewhat better behaved in practice.

0.1. **The dual E-Steenrod algebra is a Hopf algebroid.** Above, given a flat and cellular commutative monoid object (E, μ, e) in $\mathcal{SH}$, we constructed an algebraic gadget $(E_*(E), \pi_*(E))$ in the category $\pi_*(S)$-$\mathbf{GCA}^A$ of A-graded anticommutative $\pi_*(S)$-algebras called the *dual E-Steenrod algebra*. In this subsection, we will show this object is an example of the general notion of an *A-graded anticommutative Hopf algebroid*:

Proposition 0.7. *Let (E, μ, e) be a commutative monoid object in $\mathcal{SH}$ which is flat (Definition 0.5) and cellular (**??**). Then the dual E-Steenrod algebra $(E_*(E), \pi_*(E))$ with the structure maps $(\eta_L, \eta_R, \Psi, \epsilon, c)$ from Definition 0.6 is an A-graded anticommutative Hopf algebroid over $\pi_*(S)$ (**??**), i.e., a co-groupoid object in the category $\pi_*(S)$-$\mathbf{GCA}^A$.*

Proof. All that needs to be done is to show all the diagrams in **??** commute. This is nearly all entirely straightforward, the only real difficulty that arises is showing the co-associativity diagram holds. The argument is sketched in the case $\mathcal{SH}$ is the classical stable homotopy category in sufficent detail in Lecture 3 of the article [1] by Adams. The argument given there works essentially the exact same way here in our more general setting. □

0.2. **Comodules over the dual E-Steenrod algebra.** Finally, we can identify some additional structure on E-homology groups of (cellular) objects in $\mathcal{SH}$ in terms of the Hopf algebroid structure on the dual E-Steenrod algebra.

Proposition 0.8. *Let (E, μ, e) be a flat (Definition 0.5) and cellular (**??**) commutative monoid object in $\mathcal{SH}$. Then $E_*(-)$ is an additive functor from the full subcategory $\mathcal{SH}$-**Cell** of cellular objects in $\mathcal{SH}$ to the category $E_*(E)$-$\mathbf{CoMod}^A$ of left A-graded comodules (**??**) over the dual E-Steenrod algebra, which is an A-graded commutative Hopf algebroid over $\pi_*(S)$, by Proposition 0.7.*

*In particular, given an object X in $\mathcal{SH}$-**Cell**, we are viewing $E_*(X)$ with its canonical left $\pi_*(E)$-module structure (**??**), and the action map is given by the composition*

$$\Psi_X : E_*(X) \xrightarrow{E_*(e \otimes X)} E_*(E \otimes X) \xrightarrow{\Phi_{E,X}^{-1}} E_*(E) \otimes_{\pi_*(E)} E_*(X).$$

Proof. Again, we refer the reader to Lecture 3 in [1], where this is shown in the classical stable homotopy category (although the proof carries over basically verbatim to our setting). □

Now, we can use this structure in order to identify the group of maps $X \to E \otimes Y$ with graded $E_*(E)$-comodule homomorphisms from $E_*(X)$ to $E_*(Y)$. First, we need the following two technical lemmas:

Lemma 0.9. *Let (E, μ, e) be a flat (Definition 0.5) and cellular (**??**) commutative monoid object in $\mathcal{SH}$. Then given an object X in $\mathcal{SH}$, the map*

$$\Phi_{E,X} : E_*(E) \otimes_{\pi_*(E)} E_*(X) \to E_*(E \otimes X)$$

*constructed in **??** is a homomorphism of A-graded left Γ-comodules, where here by **??** we are viewing $E_*(E) \otimes_{\pi_*(E)} E_*(X)$ as the co-free $E_*(E)$-comodule on $E_*(X)$ with its canonical A-graded left $\pi_*(E)$-module structure (from **??**), and $E_*(E \otimes X)$ with its canonical left $E_*(E)$-comodule structure from Proposition 0.8.*

Proof. Consider the following diagram:

$$\begin{tikzcd}
E_*(E) \otimes_{\pi_*(E)} E_*(X) \arrow[rr, "\Psi_{E_*(E)\otimes E_*(X)}"] \arrow[dddddd, "\Phi_{E,X}"'] \arrow[ddr, bend right, "E_*(e\otimes E)"'] & & E_*(E) \otimes_{\pi_*(E)} (E_*(E) \otimes_{\pi_*(E)} E_*(X)) \arrow[dddddd, "E_*(E)\otimes\Phi_{E,X}"] \\
& (E_*(E) \otimes_{\pi_*(E)} E_*(E)) \otimes_{\pi_*(E)} E_*(X) \arrow[ur, "\cong"'] \arrow[d, "\Phi_{E,E}\otimes E_*(X)"] & \\
& E_*(E \otimes E) \otimes_{\pi_*(E)} E_*(X) \arrow[d, equal] & \\
& (E \otimes E)_*(E) \otimes_{\pi_*(E)} E_*(X) \arrow[d, "\Phi_{E,X}"] & \\
& \pi_*(E \otimes E \otimes E \otimes X) \arrow[d, equal] & \\
& E_*(E \otimes E \otimes X) & \\
E_*(E \otimes X) \arrow[ur, "E_*(e\otimes E\otimes X)"] \arrow[rr, "\Psi_{E\otimes X}"] & & E_*(E) \otimes_{\pi_*(E)} E_*(E \otimes X) \arrow[ul, "\Phi_{E,E\otimes X}"']
\end{tikzcd}$$

The top and bottom regions commute by definition. The left region commutes by naturality of $\Phi_{E,X}$. Thus, it remains to show the rightmost region commutes. To that end, since all the arrows involved are homomorphisms, it suffices to chase a homogeneous pure tensor around. Let $x : S^a \to E \otimes E$, $y : S^b \to E \otimes E$, and $z : S^c \to E \otimes X$, and consider the following diagram:

$$\begin{tikzcd}
S^{a+b+c} \arrow[d, "\phi"'] & \\
S^a \otimes S^b \otimes S^c \arrow[d, "x\otimes y\otimes z"'] & \\
E \otimes E \otimes E \otimes E \otimes E \otimes X \arrow[r, "E\otimes\mu\otimes E\otimes E\otimes X"] \arrow[d, "E\otimes E\otimes E\otimes\mu\otimes X"'] & E \otimes E \otimes E \otimes E \otimes X \arrow[d, "E\otimes E\otimes\mu\otimes X"] \\
E \otimes E \otimes E \otimes E \otimes X \arrow[r, "E\otimes\mu\otimes E\otimes X"] & E \otimes E \otimes E \otimes X
\end{tikzcd}$$

The two compositions are the two results of chasing $(x \otimes y) \otimes z$ around the rightmost region in the above diagram. It clearly commutes by functoriality of $- \otimes -$. Hence, indeed we have that $\Phi_{E,X}$ is a homomorphism of left $E_*(E)$-comodules, as desired. $\square$

Lemma 0.10. *Let (E, μ, e) be a flat (Definition 0.5) and cellular (**??**) commutative monoid object in $\mathcal{SH}$. Then the isomorphism*

$$t^a_X : E_*(\Sigma^a X) \to E_{*-a}(X)$$

*from **??** is an A-graded isomorphism of left $E_*(E)$-comodules.*

Proof. We know that $t^a_X : E_*(\Sigma^a X) \to E_{*-a}(X)$ is already an A-graded isomorphism of left $\pi_*(E)$-modules, so clearly it simply suffices to show that t^a_X is a homomorphism of left $E_*(E)$-comodules. To that end, consider the following diagram:

$$\begin{tikzcd}
E_*(\Sigma^a X) \arrow[rr, "\Psi_{\Sigma^a X}"] \arrow[dr, "E_*(e\otimes\Sigma^a X)"] \arrow[dddd, "t^a_X"'] & & E_*(E) \otimes_{\pi_*(E)} E_*(\Sigma^a X) \arrow[dl, "\Phi_{E,\Sigma^a X}"'] \arrow[dddd, "E_*(E)\otimes t^a_X"] \\
& E_*(E \otimes \Sigma^a X) \arrow[d, dashed, "E_*(\tau_{E,S^a}\otimes X)"] & \\
& E_*(\Sigma^a(E \otimes X)) \arrow[d, dashed, "t^a_{E\otimes X}"] & \\
& E_{*-a}(E \otimes X) & \\
E_{*-a}(X) \arrow[rr, "\Psi_X"'] \arrow[ur, "E_{*-a}(e\otimes X)"] & & E_*(E) \otimes_{\pi_*(E)} E_{*-a}(X) \arrow[ul, "\Phi_{E,X}"']
\end{tikzcd} \tag{1}$$

The top and bottom regions commute by definition. To see the left and right regions commute, we'll do a diagram chase of homogeneous elements. First of all, let $x : S^b \to E \otimes S^a \otimes X$ in $E_*(\Sigma^a X)$, and consider the following diagram exhibiting the two ways to chase x around the leftmost region:

$$\begin{tikzcd}
S^{b-a} \arrow[d, "\phi_{b,-a}"'] & & \\
S^b \otimes S^{-a} \arrow[d, "x\otimes S^{-a}"'] & & \\
E \otimes S^a \otimes X \otimes S^{-a} \arrow[r, "E\otimes e\otimes S^a\otimes X\otimes S^{-a}"] \arrow[dd, "E\otimes\tau\otimes S^{-a}"'] & E \otimes E \otimes S^a \otimes X \otimes S^{-a} \arrow[r, "E\otimes\tau\otimes X\otimes S^{-a}"] \arrow[dr, "E\otimes E\otimes\tau\otimes S^{-a}"] & E \otimes S^a \otimes E \otimes X \otimes S^{-a} \arrow[d, "E\otimes\tau_{S^a,E\otimes X}\otimes S^{-a}"] \\
 & & E \otimes E \otimes X \otimes S^a \otimes S^{-a} \arrow[d, "E\otimes E\otimes X\otimes\phi^{-1}_{a,-a}"] \\
E \otimes X \otimes S^a \otimes S^{-a} \arrow[r, "E\otimes X\otimes\phi^{-1}_{a,-a}"'] \arrow[urr, "E\otimes e\otimes X\otimes S^a\otimes S^{-a}"] & E \otimes X \arrow[r, "E\otimes e\otimes X"'] & E \otimes E \otimes X
\end{tikzcd}$$

The top right region commutes by coherence for the symmetries, while the other two regions commute by functoriality of $-\otimes-$. Thus, it remains to show the rightmost region in diagram (1) commutes. To that end, let $x : S^b \to E \otimes E$ in $E_*(E)$ and $y : S^c \to E \otimes S^a \otimes X$ in $E_*(\Sigma^a X)$, and consider the following diagram, which exhibits the two ways to chase $x \otimes y$ around the rightmost region of diagram (1):

$$\begin{tikzcd}
S^{b+c-a} \arrow[d, "\phi"] & & \\
S^b \otimes S^c \otimes S^{-a} \arrow[d, "x\otimes y\otimes S^{-a}"] & & \\
E \otimes E \otimes E \otimes S^a \otimes X \otimes S^{-a} \arrow[r, "E\otimes\mu\otimes S^a\otimes X\otimes S^{-a}"] \arrow[d, "E\otimes E\otimes E\otimes\tau\otimes S^{-a}"'] & E \otimes E \otimes S^a \otimes X \otimes S^{-a} \arrow[r, "E\otimes\tau\otimes X\otimes S^{-a}"] \arrow[dr, "E\otimes E\otimes\tau\otimes S^{-a}"] & E \otimes S^a \otimes E \otimes X \otimes S^{-a} \arrow[d, "E\otimes\tau_{S^a,E\otimes X}S^{-a}"] \\
E \otimes E \otimes E \otimes X \otimes S^a \otimes S^{-a} \arrow[rr, "E\otimes\mu\otimes X\otimes S^a\otimes S^{-a}"] \arrow[d, "E\otimes E\otimes E\otimes X\otimes\phi^{-1}_{a,-a}"'] & & E \otimes E \otimes X \otimes S^a \otimes S^{-a} \arrow[d, "E\otimes E\otimes X\otimes\phi^{-1}_{a,-a}"] \\
E \otimes E \otimes E \otimes X \arrow[rr, "E\otimes\mu\otimes X"] & & E \otimes E \otimes X
\end{tikzcd}$$

The top right region commutes by coherence for the symmetries. The remaining two regions commute by functoriality of $-\otimes-$. Thus, indeed we have that diagram (1) commutes, so t^a_X is a homomorphism of left $E_*(E)$-comodules, as desired. $\square$

Now we may prove the theorem.

Theorem 0.11. *Let (E, μ, e) be a commutative monoid object, and X and Y objects in $\mathcal{SH}$. Suppose further that:*

- *E is flat (Definition 0.5) and cellular (**??**),*
- *X is cellular and $E_*(X)$ is a graded projective left $\pi_*(E)$-module (via **??**), and*
- *Y is cellular.*

Then the assignment

$$E_*(-) : [X, E \otimes Y] \to \mathrm{Hom}_{E_*(E)}(E_*(X), E_*(E \otimes Y)), \qquad f \mapsto E_*(f)$$

induced by the functor $E_*(-) : \mathcal{SH}\text{-}\mathbf{Cell} \to E_*(E)\text{-}\mathbf{CoMod}^A$ (Proposition 0.8) is an isomorphism of abelian groups.

Proof. Since X is cellular, by Proposition 0.8 we have that $E_*(X)$ is canonically an A-graded left $E_*(E)$-comodule. Similarly, since E and Y are cellular, we know that $E \otimes Y$ is cellular, so that $E_*(E \otimes Y)$ is also canonically an $E_*(E)$-comodule. Thus, we have a well-defined assignment

$$[X, E \otimes Y] \xrightarrow{E_*(-)} \mathrm{Hom}_{E_*(E)}(E_*(X), E_*(E \otimes Y)).$$

To see this arrow is an isomorphism, consider the following diagram:

$$\begin{tikzcd}
{[X, E \otimes Y]} \arrow[rr, "E_*(-)"] \arrow[d, "\pi_*(\mu\otimes Y)\circ E_*(-)"'] & & \mathrm{Hom}_{E_*(E)}(E_*(X), E_*(E \otimes Y)) \arrow[dll, "\pi_*(\mu\otimes Y)\circ(-)"'] \\
\mathrm{Hom}_{\pi_*(E)}(E_*(X), E_*(Y)) & & \mathrm{Hom}_{E_*(E)}(E_*(X), E_*(E) \otimes_{\pi_*(E)} E_*(Y)) \arrow[ll, "\mathrm{adj}"'] \arrow[u, "(\Phi_{E,Y})_*"']
\end{tikzcd}$$

We know the left vertical map is an isomorphism by **??**, and the bottom horizontal isomorphism is the forgetful-cofree adjunction (**??**) for A-graded left comodules over the dual E-Steenrod algebra. The right vertical arrow is a well-defined isomorphism, as $\Phi_{E,Y}$ is a homomorphism of A-graded left $E_*(E)$-comodules (Lemma 0.9), and in fact it is an isomorphism by **??**, since $E_*(E)$ is flat and Y is cellular. Thus in order to see the top arrow is an isomorphism, it suffices to show that the diagram commutes. The left triangle clearly commutes; to see the right triangle commutes, recall that by how the how forgetful-cofree adjunction for left comodules over a Hopf algebroid is defined, that the bottom vertical arrow sends an A-graded homomorphism of left $E_*(E)$-comodules $\psi : E_*(X) \to E_*(E) \otimes_{\pi_*(E)} E_*(Y)$ to the composition

$$E_*(X) \xrightarrow{\psi} E_*(E) \otimes_{\pi_*(E)} E_*(Y) \xrightarrow{\pi_*(\mu)\otimes E_*(Y)} \pi_*(E) \otimes_{\pi_*(E)} E_*(Y) \xrightarrow{\cong} E_*(Y).$$

Thus, in order to show that this composition equals $\pi_*(\mu \otimes Y) \circ \Phi_{E,Y} \circ \psi$, it suffices to show the following diagram commutes:

$$\begin{tikzcd}
E_*(E) \otimes_{\pi_*(E)} E_*(Y) \arrow[r, "\pi_*(\mu)\otimes E_*(Y)"] \arrow[d, "\Phi_{E,Y}"'] & \pi_*(E) \otimes_{\pi_*(E)} E_*(Y) \arrow[d, "\cong"] \\
E_*(E \otimes Y) \arrow[r, "\pi_*(\mu\otimes Y)"] & E_*(Y)
\end{tikzcd}$$

Since all the arrows here are homomorphisms of abelian groups, in order to show the diagram commutes, it suffices to chase pure homogeneous tensors around. To that end, let $x : S^a \to E \otimes E$ and $y : S^b \to E \otimes Y$, and consider the following diagram exhibiting the two ways to chase $x \otimes y$ around:

$$\begin{tikzcd}
S^{a+b} \arrow[r, "\phi_{a,b}"] & S^a \otimes S^b \arrow[r, "x\otimes y"] & E \otimes E \otimes E \otimes Y \arrow[r, "\mu\otimes E\otimes Y"] \arrow[d, "E\otimes\mu\otimes Y"'] & E \otimes E \otimes Y \arrow[d, "\mu\otimes Y"] \\
 & & E \otimes E \otimes Y \arrow[r, "\mu\otimes Y"] & E \otimes Y
\end{tikzcd}$$

The diagram commutes by associtiavity of μ. Thus, we have indeed shown that

$$E_*(-) : [X, E \otimes Y] \to \mathrm{Hom}_{E_*(E)}(E_*(X), E_*(E \otimes Y))$$

is an isomorphism of abelian groups, as desired. $\square$