0.1. **The category R-GrCAlg of A-graded R-commutative rings.** We will freely use the results of **??** in this section. In what follows, fix an A-graded ring R. Further suppose that for all $a, b \in A$, there exists units $\theta_{a,b} \in R_0$ such that:

- For all $a \in A$, $\theta_{a,0} = \theta_{0,a} = 1$,
- For all $a, b \in A$, $\theta_{a,b}^{-1} = \theta_{b,a}$,
- For all $a, b, c \in A$, $\theta_{a,b} \cdot \theta_{a,c} = \theta_{a,b+c}$ and $\theta_{b,a} \cdot \theta_{c,a} = \theta_{b+c,a}$.
- For all $x \in R_a$ and $y \in R_b$,
$$x \cdot y = y \cdot x \cdot \theta_{a,b}.$$

Definition 0.1. Let R-**GrCAlg** denote the following category:

- The objects are pairs (S, φ) called A-graded R-*commutative* rings, where S is an A-graded ring and $\varphi : R \to S$ is an A-graded ring homomorphism such that for all $x \in S_a$ and $y \in S_b$, we have
$$x \cdot y = y \cdot x \cdot \varphi(\theta_{a,b}),$$
- The morphisms $(S, \varphi) \to (S', \varphi')$ are A-graded ring homomorphisms $f : S \to S'$ such that $f \circ \varphi = \varphi'$.

Note that our notation for the category R-**GrCAlg** is somewhat deficient, as there may be multiple choices of families of units $\theta_{a,b} \in R_0$ satisfying the required properties which give rise to strictly different categories, as the following example illustrates. Despite this, for our purposes it will always be clear from context which collection of $\theta_{a,b}$'s we're working with.

Example 0.2. Let $A = \mathbb{Z}$, let R be the ring $\mathbb{Z}$, viewed as a $\mathbb{Z}$-graded ring concentrated in degree 0, and let $\theta_{n,m} := (-1)^{n \cdot m}$ for all $n, m \in \mathbb{Z}$. Then the category R-**GrCAlg** is simply the category of graded anticommutative rings, i.e., $\mathbb{Z}$-graded rings S such that for all homogeneous $x, y \in S$, $x \cdot y = y \cdot x \cdot (-1)^{|x||y|}$. On the other hand, if we take the same A and R, but instead we define $\theta_{n,m} = 1$ for all $n, m \in \mathbb{Z}$, then the category R-**GrCAlg** becomes the category of strictly commutative $\mathbb{Z}$-graded rings.

Proposition 0.3. *Suppose we have two morphisms $f : (B, \varphi_B) \to (C, \varphi_C)$ and $g : (B, \varphi_B) \to (D, \varphi_D)$ of A-graded R-commutative rings in R-**GrCAlg**. Then f and g make C and D both B-bimodules, respectively,*[1] *so we may form their tensor product $C \otimes_B D$, which is itself an A-graded B-bimodule (**??**). Then $C \otimes_B D$ canonically inherits the structure of an A-graded R-commutative ring with unit $1_C \otimes 1_D$ via a product*
$$(C \otimes_B D) \times (C \otimes_B D) \to C \otimes_B D$$
which sends a pair $(x \otimes y, x' \otimes y')$ of homogeneous pure tensors to the element
$$\varphi_B(\theta_{|x|,|y'|}) \cdot (xx' \otimes yy') = \varphi_C(\theta_{|x|,|y'|})xx' \otimes yy',$$
(where here $\cdot$ denotes the left module action of B on $C \otimes_B D$), and with structure map
$$\begin{aligned} \varphi : R &\to C \otimes_B D \\ r &\mapsto \varphi_B(r) \cdot (1_C \otimes 1_D) = (\varphi_C(r) \otimes 1_D) = (1_C \otimes \varphi_D(r)). \end{aligned}$$

[1]Explicitly, it is a standard fact that given a ring homomorphism $\varphi : R \to S$ that S canonically becomes an R-bimodule with left action $r \cdot s := \varphi(r)s$ and right action $s \cdot r := s\varphi(r)$, so that in particular if φ is an A-graded homomorphism of A-graded rings, then φ makes S an A-graded R-bimodule.

Proof sketch. We simply lay out everything that needs to be shown, and we leave it to the reader to fill in the details. First to show that the indicated product is actually well-defined and distributive, by **??** it suffices to show that for all homogeneous $c, c', c'' \in C$, $d, d', d'' \in D$, and $b \in B$ with $|c'| = |c''|$ and $|d'| = |d''|$, that

$$\begin{aligned}
\varphi_B(\theta_{|d|,|c'+c''|}) \cdot (c(c'+c'') \otimes dd') &= \varphi_B(\theta_{|d|,|c'|}) \cdot (cc' \otimes dd') + \varphi_B(\theta_{|d|,|c''|}) \cdot (cc'' \otimes dd') \\
\varphi_B(\theta_{|d|,|c'|}) \cdot (cc' \otimes d(d'+d'')) &= \varphi_B(\theta_{|d|,|c'|}) \cdot (cc' \otimes dd') + \varphi_B(\theta_{|d|,|c'|}) \cdot (cc' \otimes dd'') \\
\varphi_B(\theta_{|d|,|c'\cdot b|}) \cdot (c(c' \cdot b) \otimes dd') &= \varphi_B(\theta_{|d|,|c'|}) \cdot (cc' \otimes d(b \cdot d')) \\
\varphi_B(\theta_{|d'|,|c|}) \cdot ((c'+c'')c \otimes d'd) &= \varphi_B(\theta_{|d'|,|c|}) \cdot (c'c \otimes d'd) + \varphi_B(\theta_{|d'|,|c|}) \cdot (c''c \otimes d'd) \\
\varphi_B(\theta_{|d'+d''|,|c|}) \cdot (c'c \otimes (d'+d'')d) &= \varphi_B(\theta_{|d'|,|c|}) \cdot (c'c \otimes d'd) + \varphi_B(\theta_{|d''|,|c|}) \cdot (c'c \otimes d''d) \\
\varphi_B(\theta_{|d'|,|c|})((c' \cdot b)c \otimes d'd) &= \varphi_B(\theta_{|c|,|b \cdot d'|}) \cdot (c'c \otimes (b \cdot d')d).
\end{aligned}$$

These tell us that for all $x \in C \otimes_B D$ that the maps $C \otimes_B D \to C \otimes_B D$ sending $y \mapsto xy$ and $y \mapsto yx$ are well-defined A-graded homomorphisms of abelian groups, so we have a distributive product $(x, y) \mapsto xy$. Then to show that this product makes $C \otimes_B D$ an A-graded ring, by **??**, it suffices to show that for all homogeneous $x, y, z \in C \otimes_B D$ that $(xy)z = x(yz)$ and $x(1_C \otimes 1_D) = x = (1_C \otimes 1_D)x$. By distributivity, it further suffices to consider the case that x, y, and z are homogeneous *pure tensors* in $C \otimes_B D$, i.e., it suffices to show that for all homogeneous $c, c', c'' \in C$ and $d, d', d'' \in D$ that

$$((c \otimes d)(c' \otimes d'))(c'' \otimes d'') = (c \otimes d)((c' \otimes d')(c'' \otimes d''))$$

and

$$(c \otimes d)(1_C \otimes 1_D) = (c \otimes d) = (1_C \otimes 1_D)(c \otimes d).$$

Thus, we have that the given product endows $C \otimes_B D$ with the structure of an A-graded ring, as desired. Now, we wish to show that the given map $\varphi : R \to C \otimes_B D$ is a ring homomorphism. Clearly it sends 1 to $1_C \otimes 1_D$, and again by linearity, it suffices to show that given *homogeneous* $r, s \in R$ that

$$\varphi(r+s) = \varphi_B(r+s)(1_C \otimes 1_D) = \varphi_B(r)(1_C \otimes 1_D) + \varphi_B(s)(1_C \otimes 1_D) = \varphi(r) + \varphi(s)$$

and

$$\varphi(rs) = \varphi_B(rs)(1_C \otimes 1_D) = (\varphi_B(r)(1_C \otimes 1_D))(\varphi_B(s)(1_C \otimes 1_D)) = \varphi(r)\varphi(s).$$

Finally, we need to show that $C \otimes_B D$ satisfies the graded commutativity condition, for which again by linearity it suffices to show that given homogeneous $c, c' \in C$ and $d, d' \in D$ that

$$(c \otimes d)(c' \otimes d') = \varphi(\theta_{|c \otimes d|,|c' \otimes d'|})(c' \otimes d')(c \otimes d) = \varphi(\theta_{|c|+|d|,|c'|+|d'|})(c' \otimes d)(c \otimes d).$$

Showing all of these is relatively straightforward. $\square$

Proposition 0.4. *The category R-**GrCAlg** has pushouts, where given $f : (B, \varphi_B) \to (C, \varphi_C)$ and $g : (B, \varphi_B) \to (D, \varphi_D)$, their pushout is the object $(C \otimes_B D, \varphi)$ constructed in Proposition 0.3, along with the canonical maps $(C, \varphi_C) \to (C \otimes_B D, \varphi)$ sending $c \mapsto c \otimes 1_D$ and $(D, \varphi_D) \to (C \otimes_B D, \varphi)$ sending $d \mapsto 1_C \otimes d$. In particular, since (R, id_R) is initial, R-**GrCAlg** has binary coproducts.*

Proof sketch. First, we need to show that the given maps $i_C : (C, \varphi_C) \to (C \otimes_B D, \varphi)$ and $i_D : (D, \varphi_D) \to (C \otimes_B D, \varphi)$ are actually morphisms in R-**GrCAlg**, i.e., that they are ring homomorphisms and that the following diagram commutes:

$$\begin{tikzcd}
 & R \arrow[dl, "\varphi_C"'] \arrow[d, dashed, "\varphi"] \arrow[dr, "\varphi_D"] & \\
C \arrow[r, "i_C"'] & C \otimes_B D & D \arrow[l, "i_D"]
\end{tikzcd}$$

Showing this is entirely straightforward. Furthermore, i_C and i_D clearly make the following diagram commute:

$$\begin{tikzcd} B \arrow[r, "g"] \arrow[d, "f"'] & D \arrow[d, "i_D"] \\ C \arrow[r, "i_C"'] & C \otimes_B D \end{tikzcd}$$

It remains to show that i_C and i_D are the universal such arrows. Suppose we have some object (E, φ_E) in R-**GrCAlg** and a commuting diagram

$$\begin{tikzcd} B \arrow[r, "g"] \arrow[d, "f"'] & D \arrow[d, "k"] \\ C \arrow[r, "h"] & E \end{tikzcd}$$

of morphisms in R-**GrCAlg**. Then we'd like to show there exists a unique morphism $\ell : C \otimes_B D \to E$ in R-**GrCAlg** which makes the following diagram commute:

$$\begin{tikzcd} B \arrow[r, "g"] \arrow[d, "f"'] & D \arrow[d, "i_D"'] \arrow[ddr, bend left, "k"] & \\ C \arrow[r, "i_C"] \arrow[drr, bend right, "h"'] & C \otimes_B D \arrow[dr, dashed, "\ell"] & \\ & & E \end{tikzcd}$$

First we show uniqueness. Supposing such an arrow ℓ existed, given elements $c \in C$ and $d \in D$, we must have

$$\ell(c \otimes d) = \ell((c \otimes 1_D)(1_C \otimes d)) = \ell(c \otimes 1_D)\ell(1_C \otimes d) = \ell(i_C(c))\ell(i_D(d)) = h(c)k(d).$$

Since pure tensors generate $C \otimes_B D$, we have uniquely determined ℓ, and clearly it makes the above diagram commute. Now, it remains to show that as defined ℓ is a morphism in R-**GrCAlg**, i.e., that it is an A-graded ring homomorphism and that the following diagram commutes:

$$\begin{tikzcd} & R \arrow[dl, "\varphi"'] \arrow[dr, "\varphi_E"] & \\ C \otimes_B D \arrow[rr, "\ell"] & & E \end{tikzcd}$$

This is all entirely straightforward to show. $\square$

Proposition 0.5. *The assignment $(E, \mu, e) \mapsto (\pi_*(E), \pi_*(e))$ yields a functor*

$$\pi_* : \mathbf{CMon}_{\mathcal{SH}} \to \pi_*(S)\text{-}\mathbf{GrCAlg}$$

*from the category of commutative monoid objects in $\mathcal{SH}$ (**??**) to the category of A-graded $\pi_*(S)$-commutative rings (Definition 0.1).*

Proof. By **??**, we know that π_* yields a homomorphism from $\mathbf{CMon}_{\mathcal{SH}}$ to A-graded commutative rings. Furthermore, by **??**, we know that for all homogeneous $x, y \in \pi_*(E)$ that

$$x \cdot y = y \cdot x \cdot (e \circ \theta_{|x|,|y|}) = y \cdot x \cdot \pi_*(e)(\theta_{|x|,|y|}),$$

as desired. Thus, it remains to show that $\pi_*(e) : \pi_*(S) \to \pi_*(E)$ is an A-graded ring homomorphism for any (commutative) monoid object (E, μ, e) in $\mathcal{SH}$, and that given a monoid homomorphism $f : (E_1, \mu_1, e_1) \to (E_2, \mu_2, e_2)$ in $\mathbf{CMon}_{\mathcal{SH}}$, that $\pi_*(f)$ satisfies $\pi_*(f) \circ \pi_*(e_1) = \pi_*(e_2)$. The latter clearly holds, as since f is a monoid homomorphism, we have $f \circ e_1 = e_2$, so that

$$\pi_*(f) \circ \pi_*(e_1) = \pi_*(f \circ e_1) = \pi_*(e_2).$$

To see that $\pi_*(e) : \pi_*(S) \to \pi_*(E)$ is an A-graded ring homomorphism if (E, μ, e) is a monoid object, it suffices to show that $e : S \to E$ is a monoid homomorphism, since we already know π_* takes monoid homomorphisms to A-graded ring homomorphisms. Consider the following diagrams:

$$\begin{tikzcd}
S \otimes S \arrow[rr, "e\otimes e"] \arrow[dr, dashed, "S\otimes e"] \arrow[dd, "\cong"'] & & E \otimes E \arrow[dd, "\mu"] \\
& S \otimes E \arrow[ur, dashed, "e\otimes E"] \arrow[dr, dashed, "\cong"] & \\
S \arrow[rr, "e"] & & E
\end{tikzcd}
\qquad
\begin{tikzcd}
& S \arrow[dl, equal] \arrow[dr, "e"] & \\
S \arrow[rr, "e"] & & E
\end{tikzcd}$$

The right diagram commutes by definition. The top triangle in the left diagram commutes by functoriality of $- \otimes -$. The right triangle in the left diagram commutes by unitality of μ. Finally, the left triangle in the left diagram commutes by naturality of the unitors. Thus, we have shown e is a monoid object homomorphism, as desired. $\square$

0.2. A-graded commutative Hopf algebroids over R.

Definition 0.6. Let $\mathcal{C}$ be a category admitting pullbacks. A *groupoid object* in $\mathcal{C}$ consists of a pair of objects (M, O) together with five morphisms

(1) *Source and target*: $s, t : M \to O$,
(2) *Identity*: $e : O \to M$,
(3) *Composition*: $c : M \times_O M \to M$,
(4) *Inverse*: $i : M \to M$

Explicitly, $M \times_O M$ fits into the following pullback diagram:

$$\begin{tikzcd}
M \times_O M \arrow[r, "p_2"] \arrow[d, "p_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & M \arrow[d, "t"] \\
M \arrow[r, "s"'] & O
\end{tikzcd}$$

so if we're working with sets, the composition map sends a pair (g, f) such that the codomain of f is the domain of g to $g \circ f$. These data must satisfy the following diagrams:

(1) Composition works correctly:

$$\begin{tikzcd}
M \times_O M \arrow[r, "c"] \arrow[d, "p_1"'] & M \arrow[d, "t"] \\
M \arrow[r, "t"] & O
\end{tikzcd}
\qquad
\begin{tikzcd}
M & O \arrow[l, "e"'] \arrow[r, "e"] \arrow[d, equal] & M \arrow[dl, "t"] \\
& O \arrow[from=ul, "s"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
M \times_O M \arrow[r, "p_2"] \arrow[d, "c"'] & M \arrow[d, "s"] \\
M \arrow[r, "s"] & O
\end{tikzcd}$$

(2) Associativity of composition: Write $M \times_O (M \times_O M)$ and $(M \times_O M) \times_O M$ for the pullbacks of $(s, t \circ c)$ and $(s \circ c, t)$, respectively, so we have commuting diagrams

$$\begin{tikzcd}
(M \times_O M) \times_O M \arrow[rr, "p_2'"] \arrow[dr, dashed, "c\times M"] \arrow[dd, "p_1'"'] & & M \arrow[d, equal] \\
& M \times_O M \arrow[r, "p_2"] \arrow[d, "p_1"'] & M \arrow[d, "t"] \\
M \times_O M \arrow[r, "c"] & M \arrow[r, "s"] & O
\end{tikzcd}
\qquad
\begin{tikzcd}
M \times_O (M \times_O M) \arrow[rr, "p_2''"] \arrow[dr, dashed, "M\times c"] \arrow[dd, "p_1''"'] & & M \times_O M \arrow[d, "c"] \\
& M \times_O M \arrow[r, "p_2"] \arrow[d, "p_1"'] & M \arrow[d, "t"] \\
M \arrow[r, equal] & M \arrow[r, "s"] & O
\end{tikzcd}$$

where the inner and outer squares in both diagrams are pullback squares. Furthermore, assuming the diagrams in condition (1) above are satisfied, we have that $t \circ p_1 \circ p_2'' = t \circ c \circ p_2'' = s \circ p_1''$, so that by the universal property of the pullback we have a map $M \times p_1 : M \times_O (M \times_O M) \to M \times_O M$ like so:

$$
\begin{tikzcd}
M \times_O (M \times_O M) \arrow[dr, dashed, "M\times p_1"] \arrow[drr, bend left, "p_1\circ p_2''"] \arrow[ddr, bend right, "p_1''"'] & & \\
& M \times_O M \arrow[r, "p_2"] \arrow[d, "p_1"'] & M \arrow[d, "t"] \\
& M \arrow[r, "s"] & O
\end{tikzcd}
$$

Now note that again assuming composition works correctly, so $s \circ c = s \circ p_2$, we have

$$
s \circ c \circ (M \times p_1) = s \circ p_2 \circ (M \times p_1) = s \circ p_1 \circ p_2'' = t \circ p_2 \circ p_2'',
$$

so that by the unviersal property of the pullback we get a map $a : M \times_O (M \times_O M) \to (M \times_O M) \times_O M$ like so:

$$
\begin{tikzcd}
M \times_O (M \times_O M) \arrow[dr, dashed, "a"] \arrow[drrr, bend left, "p_2\circ p_2''"] \arrow[dddr, bend right, "M\times p_1"'] & & & \\
& (M \times_O M) \times_O M \arrow[rr, "p_2'"] \arrow[dd, "p_1'"'] & & M \arrow[d, equal] \\
& & & M \arrow[d, "t"] \\
& M \times_O M \arrow[r, "c"] & M \arrow[r, "s"] & O
\end{tikzcd}
$$

Then we require that the following diagram commutes:

$$
\begin{tikzcd}
M \times_O (M \times_O M) \arrow[rr, "a"] \arrow[d, "M\times c"'] & & (M \times_O M) \times_O M \arrow[d, "c\times M"] \\
M \times_O M \arrow[r, "c"] & M & M \times_O M \arrow[l, "c"']
\end{tikzcd}
$$

(3) Unitality of composition: Given the maps $(\mathrm{id}_M, e \circ t), (e \circ s, \mathrm{id}_M) : M \to M \times_O M$ defined by the universal property of $M \times_O M$:

$$
\begin{tikzcd}
M \arrow[dr, dashed, "(\mathrm{id}_M, e\circ s)"] \arrow[drr, bend left, "e\circ s"] \arrow[ddr, bend right, equal] & & \\
& M \times_O M \arrow[r, "p_2"] \arrow[d, "p_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & M \arrow[d, "t"] \\
& M \arrow[r, "s"] & O
\end{tikzcd}
\qquad
\begin{tikzcd}
M \arrow[dr, dashed, "(e\circ t, \mathrm{id}_M)"] \arrow[drr, bend left, equal] \arrow[ddr, bend right, "e\circ t"'] & & \\
& M \times_O M \arrow[r, "p_2"] \arrow[d, "p_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & M \arrow[d, "t"] \\
& M \arrow[r, "s"] & O
\end{tikzcd}
$$

the following diagram commutes:

$$\begin{tikzcd}
M \arrow[r, "{(e\circ t,\mathrm{id}_M)}"] \arrow[d, "{(\mathrm{id}_M,e\circ s)}"'] \arrow[dr, equal] & M \times_O M \arrow[d, "c"] \\
M \times_O M \arrow[r, "c"'] & M
\end{tikzcd}$$

(4) Inverse: The following diagrams must commute:

$$\begin{tikzcd}
 & M \arrow[dl, equal] \arrow[d, "i"] \\
M & M \arrow[l, "i"']
\end{tikzcd}
\qquad
\begin{tikzcd}
M \arrow[r, "{(\mathrm{id}_M,i)}"] \arrow[d, "t"'] & M \times_O M \arrow[d, "c"] & M \arrow[l, "{(i,\mathrm{id}_M)}"'] \arrow[d, "s"] \\
O \arrow[r, "e"'] & M & O \arrow[l, "e"]
\end{tikzcd}
\qquad
\begin{tikzcd}
 & M \arrow[dl, "s"'] \arrow[d, "i"] \arrow[dr, "t"] & \\
O & M \arrow[l, "t"] \arrow[r, "s"'] & O
\end{tikzcd}$$

where the arrows (id_M, i) and (i, id_M) are determined by the universal property of $M \times_O M$ like so:

$$\begin{tikzcd}
M \arrow[dr, dashed, "{(\mathrm{id}_M,i)}"] \arrow[drr, bend left, "i"] \arrow[ddr, bend right, equal] & & \\
 & M \times_O M \arrow[r, "p_2"] \arrow[d, "p_1"'] & M \arrow[d, "t"] \\
 & M \arrow[r, "s"'] & O
\end{tikzcd}
\qquad
\begin{tikzcd}
M \arrow[dr, dashed, "{(i,\mathrm{id}_M)}"] \arrow[drr, bend left, equal] \arrow[ddr, bend right, "i"'] & & \\
 & M \times_O M \arrow[r, "p_2"] \arrow[d, "p_1"'] & M \arrow[d, "t"] \\
 & M \arrow[r, "s"'] & O
\end{tikzcd}$$

Definition 0.7. An *A-graded commutative Hopf algebroid over R* is a co-groupoid object in R-**GrCAlg**, i.e., a groupoid object in $R\text{-}\mathbf{GrCAlg}^{\mathrm{op}}$. Explicitly, an A-graded commutative Hopf algebroid over E is a pair (B, Γ) of objects in R-**GrCAlg** along with morphisms

(1) *left unit*: $\eta_L : B \to \Gamma$ (corresponding to t),

(2) *right unit*: $\eta_R : B \to \Gamma$ (corresponding to s),

(3) *comultiplication*: $\Psi : \Gamma \to \Gamma \otimes_B \Gamma$ (corresponding to c),

(4) *counit*: $\varepsilon : \Gamma \to B$ (corresponding to e),

(5) *conjugation*: $c : \Gamma \to \Gamma$ (corresponding to i),

where here Γ may be viewed as a B-bimodule via η_L and a B-bimodule via η_R, so we may form their tensor product $\Gamma \otimes_B \Gamma$ with the η_R bimodule on the left and the η_L bimodule on the right, so that it fits into the following pushout diagram in R-**GrCAlg** (Proposition 0.4):

$$\begin{tikzcd}
B \arrow[r, "\eta_L"] \arrow[d, "\eta_R"'] & \Gamma \arrow[d, "g\mapsto 1\otimes g"] \\
\Gamma \arrow[r, "g\mapsto g\otimes 1"'] & \Gamma \otimes_B \Gamma
\end{tikzcd}$$

These data must satisfy the following

(1) The following diagrams must commute:

$$\begin{tikzcd}
B \arrow[r, "\eta_L"] \arrow[d, "\eta_L"'] & \Gamma \arrow[d, "\Psi"] \\
\Gamma \arrow[r, "g\mapsto g\otimes 1"'] & \Gamma \otimes_B \Gamma
\end{tikzcd}
\qquad
\begin{tikzcd}
 & B \arrow[dl, "\eta_R"'] \arrow[d, equal] \arrow[dr, "\eta_L"] & \\
\Gamma \arrow[r, "\varepsilon"'] & B & \Gamma \arrow[l, "\varepsilon"]
\end{tikzcd}
\qquad
\begin{tikzcd}
B \arrow[r, "\eta_R"] \arrow[d, "\eta_R"'] & \Gamma \arrow[d, "g\mapsto 1\otimes g"] \\
\Gamma \arrow[r, "\Psi"] & \Gamma \otimes_B \Gamma
\end{tikzcd}$$

(2) The following diagram must commute

$$\begin{tikzcd}
\Gamma\otimes_B\Gamma \arrow[d, "\Psi\otimes_B\Gamma"'] & \Gamma \arrow[l, "\Psi"'] \arrow[r, "\Psi"] & \Gamma\otimes_B\Gamma \arrow[d, "\Gamma\otimes_B\Psi"] \\
(\Gamma\otimes_B\Gamma)\otimes_B\Gamma \arrow[rr] & & \Gamma\otimes_B(\Gamma\otimes_B\Gamma)
\end{tikzcd}$$

where the bottom arrow sends $(g\otimes g')\otimes g''$ to $g\otimes(g'\otimes g'')$ and $\Psi\otimes\Gamma$ and $\Gamma\otimes\Psi$ fit into the following commutative diagrams, where both outer and inner squares in both diagrams are pushout squares in R-**GrCAlg**:

$$\begin{tikzcd}
B \arrow[r, "\eta_L"] \arrow[d, "\eta_R"'] & \Gamma \arrow[r, "\Psi"] \arrow[d, "g\mapsto 1\otimes g"] & \Gamma\otimes_B\Gamma \arrow[dd, "x\mapsto 1\otimes x"] \\
\Gamma \arrow[r, "g\mapsto g\otimes 1"'] \arrow[d, equal] & \Gamma\otimes_B\Gamma \arrow[dr, dashed, "\Gamma\otimes\Psi"] & \\
\Gamma \arrow[rr, "g\mapsto g\otimes 1"'] & & \Gamma\otimes_B(\Gamma\otimes_B\Gamma)
\end{tikzcd}
\qquad
\begin{tikzcd}
B \arrow[r, "\eta_L"] \arrow[d, "\eta_R"'] & \Gamma \arrow[r, equal] \arrow[d, "g\mapsto 1\otimes g"] & \Gamma \arrow[dd, "g\mapsto 1\otimes g"] \\
\Gamma \arrow[r, "g\mapsto g\otimes 1"'] \arrow[d, "\Psi"'] & \Gamma\otimes_B\Gamma \arrow[dr, dashed, "\Psi\otimes\Gamma"] & \\
\Gamma\otimes_B\Gamma \arrow[rr, "x\mapsto x\otimes 1"'] & & (\Gamma\otimes_B\Gamma)\otimes_B\Gamma
\end{tikzcd}$$

(3) The following diagram must commute:

$$\begin{tikzcd}
\Gamma \arrow[r, "\Psi"] \arrow[d, "\Psi"'] \arrow[dr, equal] & \Gamma\otimes_B\Gamma \arrow[d, "(\eta_L\circ\varepsilon)\cdot\mathrm{id}_\Gamma"] \\
\Gamma\otimes_B\Gamma \arrow[r, "\mathrm{id}_\Gamma\cdot(\eta_R\circ\varepsilon)"'] & \Gamma
\end{tikzcd}$$

where the right vertical arrow sends $g\otimes g'$ to $\eta_L(\varepsilon(g))g'$ and the bottom horizontal arrow sends $g\otimes g'$ to $g\eta_R(\varepsilon(g'))$.

(4) The following diagrams must commute:

$$\begin{tikzcd}
& \Gamma \arrow[d, "c"] \\
\Gamma \arrow[ur, equal] & \Gamma \arrow[l, "c"]
\end{tikzcd}
\qquad
\begin{tikzcd}
B \arrow[d, "\eta_L"'] & \Gamma \arrow[l, "\varepsilon"'] \arrow[r, "\varepsilon"] \arrow[d, "i"'] & B \arrow[d, "\eta_R"] \\
\Gamma & \Gamma\otimes_B\Gamma \arrow[l, "\mathrm{id}_\Gamma\cdot c"] \arrow[r, "c\cdot\mathrm{id}_\Gamma"'] & \Gamma
\end{tikzcd}
\qquad
\begin{tikzcd}
B \arrow[r, "\eta_L"] \arrow[dr, "\eta_R"'] & \Gamma \arrow[d, "c"'] & B \arrow[l, "\eta_R"'] \arrow[dl, "\eta_L"] \\
& \Gamma &
\end{tikzcd}$$

where the bottom left arrow in the middle diagram sends $g\otimes g'$ to $gc(g')$ and the bottom right arrow in the middle diagram sends $g\otimes g'$ to $c(g)g'$.

Proposition 0.8. *Let (E,μ,e) be a commutativite monoid object in $\mathcal{SH}$. Then the maps*

$$E\xrightarrow{\cong}E\otimes S\xrightarrow{E\otimes e}E\otimes E$$

and

$$E\xrightarrow{\cong}S\otimes E\xrightarrow{e\otimes E}E\otimes E$$

are homomorphisms of monoid objects (where here $E\otimes E$ is considered a monoid object by **??***), so that by Proposition 0.5 they induce morphisms in $\pi_*(S)$-*__GrCAlg__*:*

$$\eta_L:\pi_*(E)\to E_*(E)$$

and

$$\eta_R:\pi_*(E)\to E_*(E),$$

respectively.

Proof. We will just show $E \xrightarrow{\cong} E \otimes S \xrightarrow{E\otimes e} E \otimes E$ is a monoid object homomorphism, as showing the other is entirely analagous. First, consider the following diagram:

$$\begin{tikzcd}
E_1 \otimes E_2 \arrow[rr, "E\otimes e\otimes E\otimes e"] \arrow[dr, "E\otimes E\otimes e"'] \arrow[ddddd, "\mu"'] & & E_1 \otimes E \otimes E_2 \otimes E \arrow[ddl, "E\otimes\mu\otimes E"] \arrow[ddd, "E\otimes\tau\otimes E"] \\
& E_1 \otimes E_2 \otimes E \arrow[ur, "E\otimes e\otimes E\otimes E"] \arrow[d, equal] & \\
& E_1 \otimes E_2 \otimes E \arrow[d, equal] & \\
& E_1 \otimes E_2 \otimes E \arrow[r, "E\otimes E\otimes e\otimes E"'] \arrow[d, equal] & E_1 \otimes E_2 \otimes E \otimes E \arrow[ul, "E\otimes\mu\otimes E"'] \arrow[dl, "E\otimes E\otimes\mu"] \arrow[dd, "\mu\otimes\mu"] \\
& E_1 \otimes E_2 \otimes E \arrow[dr, "\mu\otimes E"'] & \\
E_{1,2} \arrow[rr, "E\otimes e"'] & & E_{1,2} \otimes E
\end{tikzcd}$$

The leftmost region commutes by functoriality of $- \otimes -$. The top triangle also commutes by functoriality of $- \otimes -$. The triangle below that commutes by unitality of μ. The triangle below that commutes by commutativity of μ. The next two triangles below that commutes by unitality of μ. Finally, the bottom right triangle commutes by functoriality of $- \otimes -$. Next, consider the following diagram:

$$\begin{tikzcd}
& S \arrow[dr, "\cong"] \arrow[ddl, "e"'] & & \\
& & S \otimes S \arrow[dl, "e\otimes S"'] \arrow[dr, "e\otimes e"] & \\
E \arrow[r, "\cong"] & E \otimes S \arrow[rr, "E\otimes e"'] & & E \otimes E
\end{tikzcd}$$

The leftmost region commutes by naturality of the unitors, while the rightmost region commutes by functoriality of $- \otimes -$. Hence, we have shown $E \xrightarrow{\cong} E \otimes S \xrightarrow{E\otimes e} E \otimes E$ is indeed a monoid homomorphism, as desired. $\square$

Lemma 0.9. *Let (E, μ, e) be a commutative monoid object in $\mathcal{SH}$. Then the left (resp. right) $\pi_*(E)$-module structure induced on $E_*(E)$ by the ring homomorphism η_L (resp. η_R) coincides with the canonical left (resp. right) $\pi_*(E)$-module structure on $E_*(E)$ given in* **??**.

Proof. What's going on here is a bit subtle, so we're going to be really explicit. In **??**, it was shown that $E_*(E)$ is a left $\pi_*(E)$-module via the assignment

$$\pi_*(E) \times E_*(E) \to E_*(E)$$

which sends homogeneous elements $r : S^a \to E$ and $x : S^b \to E \otimes E$ to the composition

$$S^{a+b} \xrightarrow{\cong} S^a \otimes S^b \xrightarrow{r\otimes x} E \otimes E \otimes E \xrightarrow{\mu\otimes E} E \otimes E.$$

We'd like to show that this is the same thing as the assignment $\pi_*(E) \times E_*(E) \to E_*(E)$ sending $(r, x) \mapsto \eta_L(r)x$, where $\eta_L(r)x$ denotes the product of $\eta_L(r)$ and x taken in the ring $E_*(E)$.

Explicitly, the product structure on $E_*(E) = \pi_*(E \otimes E)$ is that induced by the fact that $E \otimes E$ is a monoid object in $\mathcal{SH}$ by **??**, with product

$$E \otimes E \otimes E \otimes E \xrightarrow{E \otimes \tau \otimes E} E \otimes E \otimes E \otimes E \xrightarrow{\mu \otimes \mu} E \otimes E$$

(note the middle two factors are swapped). It is a standard fact from algebra that given a ring homomorphism $\varphi : R \to R'$, that R' is canonically a left R-module via the rule $(r, r') \mapsto \varphi(r)r'$, and a right R-module via the rule $(r', r) \mapsto r'\varphi(r)$. Thus, we can be sure that we actually have two left module actions. Furthermore, these are both clearly A-graded left module actions, so in order to show they're the same it suffices to show they agree on homogeneous elements (**??**). Now, suppose we have homogeneous elements $r : S^a \to E$ in $\pi_*(E)$ and $x : S^b \to E \otimes E$ in $E_*(E)$. Then consider the following diagram, where we've passed to a symmetric strict monoidal category:

$$\begin{tikzcd}
S^{a+b} \arrow[d, "\phi_{a,b}"] \\
S^a \otimes S^b \arrow[d, "r \otimes x"] \\
E_1 \otimes E_2 \otimes E_3 \arrow[rrrr, "\mu \otimes E"] \arrow[dd, "E \otimes e \otimes E"'] \arrow[dr, equal] & & & & E_{1,2} \otimes E_3 \arrow[dd, equal] \\
& E_1 \otimes E_2 \otimes E_3 \arrow[r, equal] & E_1 \otimes E_2 \otimes E_3 \arrow[d, "E \otimes E \otimes e \otimes E"'] \arrow[r, equal] & E_1 \otimes E_2 \otimes E_3 \arrow[dr, "\mu \otimes E"] & \\
E_1 \otimes E \otimes E_2 \otimes E_3 \arrow[ur, "E \otimes \mu \otimes E"] \arrow[rr, "E \otimes \tau \otimes E"'] & & E_1 \otimes E_2 \otimes E \otimes E_3 \arrow[ul, "E \otimes \mu \otimes E"] \arrow[ur, "E \otimes E \otimes \mu"'] \arrow[rr, "\mu \otimes \mu"'] & & E_{1,2} \otimes E_3
\end{tikzcd}$$

Here we've numbered the E's to make it clear what's going on. The bottom composition is $\eta_L(r)x$, while the top composition is the canonical left action of r on x given in **??**. The leftmost triangle commutes by unitality of μ. The triangle to the right of that commutes by commutativity of μ. The triangle to the right of that commutes by unitality of μ, as does the next triangle. The remaining triangle on the right commutes by functoriality of $- \otimes -$. Finally, the top region commutes by definition. Thus, we've shown that the left $\pi_*(E)$-module structure induced on $E_*(E)$ by η_L is in fact the canonical one. On the other hand, showing that the right $\pi_*(E)$-module structure induced on $E_*(E)$ by η_R is the canonical one is entirely analagous, and we leave it as an exercise for the reader. $\square$

Corollary 0.10. *Given a commutative monoid object (E, μ, e) in $\mathcal{SH}$, the domain of the homomorphism*

$$\Phi_E : E_*(E) \otimes_{\pi_*(E)} E_*(E) \to E_*(E \otimes E)$$

constructed in **??** *is canonically an A-graded $\pi_*(S)$-ring, and sits in the following pushout diagram in $\pi_*(S)$-***GrCAlg***:*

$$\begin{tikzcd}
\pi_*(E) \arrow[r, "\eta_L"] \arrow[d, "\eta_R"'] & E_*(E) \arrow[d, "x \mapsto 1 \otimes x"] \\
E_*(E) \arrow[r, "x \mapsto x \otimes 1"'] & E_*(E) \otimes_{\pi_*(E)} E_*(E)
\end{tikzcd}$$

Proof. By Proposition 0.4, we have a pushout diagram

$$\begin{tikzcd}
\pi_*(E) \arrow[r, "\eta_L"] \arrow[d, "\eta_R"'] & E_*(E) \arrow[d, "x \mapsto 1 \otimes x"] \\
E_*(E) \arrow[r, "x \mapsto x \otimes 1"'] & R
\end{tikzcd}$$

where the underlying A-graded abelian group of R is the tensor product over $\pi_*(E)$ of $E_*(E)$ considered as right $\pi_*(E)$-module via η_R and $E_*(E)$ considered as a left $\pi_*(E)$-module via η_L. By Lemma 0.9, as an A-graded abelian group, R is precisely the domain of Φ_E, as desired. $\square$

Lemma 0.11. *Let (E, μ, e) be a (commutative) monoid object in a symmetric monoidal category $(\mathcal{C}, \otimes, S)$. By **??**, $(E \otimes E) \otimes E$ and $E \otimes (E \otimes E)$ are canonically (commutative) monoid objects. Then the associator $\alpha : (E \otimes E) \otimes E \xrightarrow{\cong} E \otimes (E \otimes E)$ is an isomorphism of monoid objects in $\mathcal{SH}$.*

*In other words, up to associativity, given a monoid object (E, μ, e) in $\mathcal{SH}$, there is no ambiguity when considering $E \otimes E \otimes E$ as a monoid object via **??**.*

Proof. Clearly, up to associativity, $(E \otimes E) \otimes E$ and $E \otimes (E \otimes E)$ have the same unit map $S \xrightarrow{e \otimes e \otimes e} E \otimes E \otimes E$. Thus, it remains to show that they have the same product map, up to associativity. To see this, consider the following diagram, where we've passed to a symmetric strict monoidal category:

$$\begin{tikzcd}
E_1 \otimes (E_2 \otimes E_3) \otimes E_4 \otimes (E_5 \otimes E_6) \arrow[rrrr, equal, "\alpha"] \arrow[d, "E \otimes \tau_{E \otimes E, E} \otimes E \otimes E"'] & & & & (E_1 \otimes E_2) \otimes E_3 \otimes (E_4 \otimes E_5) \otimes E_6 \arrow[d, "E \otimes E \otimes \tau_{E, E \otimes E} \otimes E"] \\
E_1 \otimes E_4 \otimes E_2 \otimes E_3 \otimes E_5 \otimes E_6 \arrow[d, "\mu \otimes E \otimes \tau \otimes E"'] \arrow[drr, "E \otimes E \otimes E \otimes \tau \otimes E"] & & & & E_1 \otimes E_2 \otimes E_4 \otimes E_5 \otimes E_3 \otimes E_6 \arrow[d, "E \otimes \tau \otimes E \otimes \mu"] \arrow[dll, "E \otimes \tau \otimes E \otimes E \otimes E"'] \\
E_{1,4} \otimes E_2 \otimes E_5 \otimes E_3 \otimes E_6 \arrow[d, "E \otimes \mu \otimes \mu"'] & & E_1 \otimes E_4 \otimes E_2 \otimes E_5 \otimes E_3 \otimes E_6 \arrow[ll, "\mu \otimes E \otimes E \otimes E \otimes E"] \arrow[rr, "E \otimes E \otimes E \otimes E \otimes \mu"'] \arrow[dll, "\mu \otimes \mu \otimes \mu"] \arrow[drr, "\mu \otimes \mu \otimes \mu"'] & & E_1 \otimes E_4 \otimes E_2 \otimes E_5 \otimes E_{3,6} \arrow[d, "\mu \otimes \mu \otimes E"] \\
E_{1,4} \otimes E_{2,5} \otimes E_{3,6} \arrow[rrrr, equal, "\alpha"] & & & & E_{1,4} \otimes E_{2,5} \otimes E_{3,6}
\end{tikzcd}$$

Here we've numbered the E's to make it clearer what's going on. The top pentagonal region commutes by coherence for the τ's in a symmetric monoidal category. The bottom triangle commutes by definition. The remaining four triangles commute by functoriality of $- \otimes -$. On the left is the product for $E \otimes (E \otimes E)$, while on the right is the product for $(E \otimes E) \otimes E$. Thus they are equal up to associativity, as desired. $\square$

Lemma 0.12. *Suppose we have some monoid object (E, μ, e) in $\mathcal{C}$ and some homomorphism of monoid objects $f : (E_1, \mu_1, e_1) \to (E_2, \mu_2, e_2)$ in $\mathbf{Mon}_{\mathcal{C}}$. Then $E \otimes f : E \otimes E_1 \to E \otimes E_2$ and $f \otimes E : E_1 \otimes E \to E_2 \otimes E$ are monoid homomorphisms, where here we are considering $E \otimes E_1$, $E \otimes E_2$, $E_1 \otimes E$, and $E_2 \otimes E$ to be monoid objects by **??**.*

Proof. We will show that $E \otimes f$ is a monoid object homomorphism, as showing $f \otimes E$ is a monoid homomorphism is entirely analagous. First consider the following diagram:

$$
\begin{tikzcd}
E \otimes E_1 \otimes E \otimes E_1 \arrow[rrr, "E\otimes f\otimes E\otimes f"] \arrow[d, "E\otimes\tau\otimes E_1"'] & & & E \otimes E_2 \otimes E \otimes E_2 \arrow[d, "E\otimes\tau\otimes E_2"] \\
E \otimes E \otimes E_1 \otimes E_1 \arrow[rrr, "E\otimes E\otimes f\otimes f"] \arrow[dr, "\mu\otimes E_1\otimes E_2"] \arrow[dd, "\mu\otimes\mu_1"'] & & & E \otimes E \otimes E_2 \otimes E_2 \arrow[dl, "\mu\otimes E_2\otimes E_2"'] \arrow[dd, "\mu\otimes\mu_2"] \\
& E \otimes E_1 \otimes E_1 \arrow[r, "E\otimes f\otimes f"] \arrow[dl, "E\otimes\mu_1"] & E \otimes E_2 \otimes E_2 \arrow[dr, "E\otimes\mu_2"] & \\
E \otimes E_1 \arrow[rrr, "E\otimes f"] & & & E \otimes E_2
\end{tikzcd}
$$

The top region commutes by naturality of τ. The bottom trapezoid commutes since f is a monoid homomorphism. The remaining three regions commute by functoriality of $- \otimes -$. Now, consider the following diagram:

$$
\begin{tikzcd}
& S \arrow[dl, "e\otimes e_1"'] \arrow[d, "e"] \arrow[dr, "e\otimes e_2"] & \\
& E \arrow[dl, "E\otimes e_1"] \arrow[dr, "E\otimes e_2"'] & \\
E \otimes E_1 \arrow[rr, "E\otimes f"] & & E \otimes E_2
\end{tikzcd}
$$

The bottom region commutes since f is a monoid homomorphism. The top two regions commute by functoriality of $- \otimes -$. Thus, we've shown $E \otimes f$ is a monoid object homomorphism, as desired. ☐

Lemma 0.13. *Let (E, μ, e) be a commutative monoid object in $\mathcal{SH}$. Then the homomomorphism*

$$\Phi_E : E_*(E) \otimes_{\pi_*(E)} E_*(E) \to E_*(E \otimes E)$$

given in **??** *is a homomorphism of A-graded $\pi_*(S)$-commutative rings, where here $E_*(E) \otimes_{\pi_*(E)} E_*(E)$ is an object in $\pi_*(S)$-***GrCAlg** *by Corollary 0.10, and $E_*(E \otimes E) = \pi_*(E \otimes (E \otimes E))$ is as well by* **??** *since $E \otimes (E \otimes E)$ is a commutative monoid object by* **??**.

Proof. Consider the maps

$$f : E \otimes E \xrightarrow{e\otimes E\otimes E} E \otimes E \otimes E$$

and

$$g : E \otimes E \xrightarrow{E\otimes E\otimes e} E \otimes E \otimes E.$$

We know that the maps

$$E \xrightarrow{e\otimes E} E \otimes E \qquad \text{and} \qquad E \xrightarrow{E\otimes e} E \otimes E$$

are monoid homomorphisms by Proposition 0.8, so that f and g are monoid homomorphisms by Lemma 0.12. Furthermore, by Lemma 0.11, they are monoid homomorphisms between the same monoid objects in $\mathcal{SH}$. Finally, note that we have the following commutative diagram

$$
\begin{tikzcd}
E \arrow[r, "E\otimes e"] \arrow[d, "e\otimes E"'] \arrow[dr, "e\otimes E\otimes e"] & E \otimes E \arrow[d, "e\otimes E\otimes E"] \\
E \otimes E \arrow[r, "E\otimes E\otimes e"'] & E \otimes E \otimes E
\end{tikzcd}
$$

where the outer arrows are monoid object homomorphisms, thus, we may apply π_*, which yields the following commutative diagram in $\pi_*(S)$-**GrCAlg** (Proposition 0.5):

$$\begin{tikzcd}
\pi_*(E) \arrow[r, "\eta_L"] \arrow[d, "\eta_R"'] & E_*(E) \arrow[d, "\pi_*(f)"] \\
E_*(E) \arrow[r, "\pi_*(g)"'] & E_*(E \otimes E)
\end{tikzcd}$$

Hence by **??** and the universal property of the pushout, there exists some unique morphism $\ell : E_*(E) \otimes_{\pi_*(E)} E_*(E) \to E_*(E \otimes E)$ in $\pi_*(S)$-**GrCAlg** which makes the following diagram commute:

$$\begin{tikzcd}
\pi_*(E) \arrow[r, "\eta_L"] \arrow[d, "\eta_R"'] & E_*(E) \arrow[d, "x \mapsto 1 \otimes x"'] \arrow[ddr, bend left, "\pi_*(f)"] & \\
E_*(E) \arrow[r, "x \mapsto x \otimes 1"] \arrow[drr, bend right, "\pi_*(g)"'] & E_*(E) \otimes_{\pi_*(E)} E_*(E) \arrow[dr, dashed, "\ell"] & \\
& & E_*(E \otimes E)
\end{tikzcd}$$

Thus in order to show Φ_E is a morphism in $\pi_*(S)$-**GrCAlg**, it suffices to show that Φ_E and ℓ are the same map, since we know ℓ is a homomorphism of A-graded $\pi_*(S)$-commutative rings. Since Φ_E and ℓ are both abelian group homomorphisms, it further suffices to show they agree on homogeneous pure tensors, which generate $E_*(E) \otimes_{\pi_*(E)} E_*(E)$. Given homogeneous elements $x : S^a \to E \otimes E$ and $y : S^b \to E \otimes E$ in $E_*(E)$, unravelling how pushouts in $\pi_*(S)$-**GrCAlg** are defined (Proposition 0.4), ℓ sends the pure homogeneous tensor $x \otimes y$ to the element $\pi_*(g)(x) \cdot \pi_*(f)(y)$, where here $\cdot$ denotes the product taken in $E_*(E \otimes E) = \pi_*(E \otimes E \otimes E)$. Now, consider the following diagram:

$$\begin{tikzcd}
S^{a+b} \arrow[d, "\phi_{a,b}"] & & \\
S^a \otimes S^b \arrow[d, "x \otimes y"] & & \\
E_1 \otimes E_2 \otimes E_3 \otimes E_4 \arrow[rr, "g \otimes f = E \otimes E \otimes e \otimes e \otimes E \otimes E"] \arrow[drr, "E \otimes e \otimes E \otimes e \otimes E \otimes E"] \arrow[ddr, equal] \arrow[ddd, "E \otimes \mu \otimes E"'] & & E_1 \otimes E_2 \otimes E_a \otimes E_b \otimes E_3 \otimes E_4 \arrow[d, "E \otimes \tau_{E \otimes E, E} \otimes E \otimes E"] \\
& & E_1 \otimes E_b \otimes E_2 \otimes E_a \otimes E_3 \otimes E_4 \arrow[d, "\mu \otimes E \otimes \tau \otimes E"] \\
& E_1 \otimes E_2 \otimes E_3 \otimes E_4 \arrow[ur, "E \otimes e \otimes E \otimes e \otimes E \otimes E"] \arrow[r, "E \otimes E \otimes E \otimes e \otimes E"'] \arrow[dl, "E \otimes \mu \otimes E"] & E_1 \otimes E_2 \otimes E_3 \otimes E_a \otimes E_4 \arrow[d, "E \otimes \mu \otimes \mu"] \\
E_1 \otimes E_{2,3} \otimes E_4 \arrow[rr, equal] & & E_1 \otimes E_{2,3} \otimes E_4
\end{tikzcd}$$

Here we have labelled the E's to make things clearer. The left outside composition is $\Phi_E(x \otimes y)$, while the right composition is $\pi_*(g)(x) \cdot \pi_*(f)(y)$. The top right triangle commutes by coherence for a symmetric monoidal category. The middle tright triangle commutes by unitality of μ and coherence for a symmetric monoidal category. The bottom trapezoid commutes by unitality of μ. The rest of the diagram commutes by definition. Thus we have $\Phi_E(x \otimes y) = \pi_*(g)(x) \cdot \pi_*(f)(y)$, so that $\Phi_E = \ell$ is not just an isomorphism of left $\pi_*(E)$-modules, but an isomorphism of A-graded $\pi_*(S)$-commutative rings, as desired. $\square$

Proposition 0.14. *Let (E, μ, e) be a flat (**??**) and cellular (**??**) commutative monoid object in $\mathcal{SH}$. Then consider the map*

$$\Psi : E_*(E) \xrightarrow{\pi_*(E \otimes e \otimes E)} E_*(E \otimes E) \xrightarrow{\Phi_E^{-1}} E_*(E) \otimes_{\pi_*(E)} E_*(E),$$

*where Φ_E is the isomorphism given in **??**. Then Ψ is a homomorphism of A-graded $\pi_*(S)$-commutative rings, where here the object $E_*(E) \otimes_{\pi_*(E)} E_*(E)$ is considered an A-graded $\pi_*(S)$-commutative ring by Corollary 0.10.*

Proof. By Lemma 0.13, we know $\Phi_E : E_*(E) \otimes_{\pi_*(E)} E_*(E) \to E_*(E \otimes E)$ is a bijective homomorphism of A-graded $\pi_*(S)$-commutative rings, thus it is clearly an isomorphism in $\pi_*(S)$-**GrCAlg**, so that its inverse Φ_E^{-1} is also a homomorphism in $\pi_*(S)$-**GrCAlg**. Thus, it suffices to show that $\pi_*(E \otimes e \otimes E)$ is as well. By Proposition 0.5, it suffices to show $E \otimes e \otimes E : E \otimes E \to E \otimes E \otimes E$ is a homomorphism of monoid objects in $\mathcal{SH}$. Yet, we know this is true, as $e \otimes E : E \to E \otimes E$ is a homomorphism of monoid objects by Proposition 0.8, so that by Lemma 0.12 we have $E \otimes e \otimes E$ is also a homomorphism of monoid objects, as desired. ☐

Proposition 0.15. *Let (E, μ, e) be a commutative monoid object in $\mathcal{SH}$. Then the morphism*

$$\mu : E \otimes E \to E$$

*is a homomorphism of monoid objects (where $E \otimes E$ is considered a monoid object by **??**), so that by Proposition 0.5, under π_* it induces a homomorphism of A-graded $\pi_*(S)$-commutative rings*

$$\varepsilon : E_*(E) \to \pi_*(E).$$

Proof. Consider the following diagram:

$$\begin{tikzcd}
E \otimes E \otimes E \otimes E \arrow[rrrr, "\mu\otimes\mu"] \arrow[drrr, "\mu\otimes E\otimes E"] \arrow[dd, "E\otimes\tau\otimes E"'] \arrow[ddrr, "E\otimes\mu\otimes E"] & & & & E \otimes E \arrow[dddd, "\mu"] \\
 & & & E \otimes E \otimes E \arrow[ur, "E\otimes\mu"] \arrow[d, "\mu\otimes E"] & \\
E \otimes E \otimes E \otimes E \arrow[rr, "E\otimes\mu\otimes E"'] \arrow[dr, "E\otimes E\otimes\mu"'] \arrow[dd, "\mu\otimes\mu"'] & & E \otimes E \otimes E \arrow[r, "\mu\otimes E"] \arrow[dr, "E\otimes\mu"] & E \otimes E \arrow[ddr, "\mu"] & \\
 & E \otimes E \otimes E \arrow[rr, "E\otimes\mu"'] \arrow[dl, "\mu\otimes E"] & & E \otimes E \arrow[dr, "\mu"] & \\
E \otimes E \arrow[rrrr, "\mu"'] & & & & E
\end{tikzcd}$$

The top left triangle commutes by commutativity of μ. Every other region commutes by functoriality of $-\otimes$ and/or associativity of μ. Thus, we have shown μ satisfies the first diagram in **??** required for it to be a monoid homomorphism. To see it satisfies the second condition, consider the following diagram:

$$\begin{tikzcd}
 & S \arrow[d, "e"] \arrow[ddl, "e\otimes e"'] \arrow[ddr, "e"] & \\
 & E \arrow[dl, "E\otimes e"'] \arrow[dr, equal] & \\
E \otimes E \arrow[rr, "\mu"] & & E
\end{tikzcd}$$

The top left region commutes by functoriality of $- \otimes -$. The top right region commutes by definition. Finally, the bottom region commutes by unitality of μ. Thus we have shown μ is a monoid object homomorphism, as desired. $\square$

Proposition 0.16. *Let (E, μ, e) be a commutative monoid object in $\mathcal{SH}$. Then the morphism*

$$\tau_{E,E} : E \otimes E \to E \otimes E$$

*is a homomorphism of monoid objects (where $E \otimes E$ is considered a monoid object by **??**), so that by Proposition 0.5, under π_* it induces a homomorphism of A-graded $\pi_*(S)$-commutative rings*

$$c : E_*(E) \to E_*(E).$$

Proof. Consider the following diagram:

$$\begin{tikzcd}
E_1 \otimes E_2 \otimes E_3 \otimes E_4 \arrow[r, "\tau\otimes\tau"] \arrow[d, "E\otimes\tau\otimes E"'] & E_2 \otimes E_1 \otimes E_4 \otimes E_3 \arrow[d, "E\otimes\tau\otimes E"] \\
E_1 \otimes E_3 \otimes E_2 \otimes E_4 \arrow[r, "\tau_{E\otimes E, E\otimes E}"] \arrow[d, "\mu\otimes\mu"'] & E_2 \otimes E_4 \otimes E_1 \otimes E_3 \arrow[d, "\mu\otimes\mu"] \\
E_{1,3} \otimes E_{2,4} \arrow[r, "\tau"'] & E_{2,4} \otimes E_{1,3}
\end{tikzcd}$$

The top region commutes by coherence for the symmetries in a symmetric monoidal category, while the bottom region commutes by naturality of τ. Now, consider the following diagram:

$$\begin{tikzcd}
 & & S \arrow[dl, "\cong"'] \arrow[dr, "\cong"] & & \\
 & S \otimes S \arrow[rr, "\tau"] \arrow[dl, "e\otimes e"'] & & S \otimes S \arrow[dr, "e\otimes e"] & \\
E \otimes E \arrow[rrrr, "\tau"] & & & & E \otimes E
\end{tikzcd}$$

The top triangle commutes by coherence for a symmetric monoidal category, while the bottom region commutes by naturality of τ. Thus, we have shown $\tau_{E,E}$ is a homomorphism of monoid objects, as desired. $\square$

Proposition 0.17. *Let (E, μ, e) be a commutative monoid object in $\mathcal{SH}$ which is flat (**??**) and cellular (**??**). Then the dual E-Steenrod algebra $(E_*(E), \pi_*(E))$ with the structure maps $(\eta_L, \eta_R, \Psi, \varepsilon, c)$ constructed above is an A-graded commutative Hopf algebroid over $\pi_*(S)$ (Definition 0.7), i.e., a co-groupoid object in the category $\pi_*(S)$-**GrCAlg**.*

Proof. We need to show all the diagrams in Definition 0.7 commute. Since we are dealing with A-graded homomorphisms, when showing these diagrams commute, it always suffices to chase homogeneous elements around. To that end, we fix homogeneous elements $x : S^a \to E$ in $\pi_*(E)$ and $y : S^b \to E \otimes E$ in $E_*(E \otimes E)$ now.

First, we wish to show the outside of the following diagram commutes:

$$\begin{tikzcd}
\pi_*(E) \arrow[rr, "\eta_R"] \arrow[dd, "\eta_R"'] & & E_*(E) \arrow[dl, "\pi_*(E\otimes e\otimes E)"'] \arrow[dd, "\Psi"] \\
& E_*(E\otimes E) & \\
E_*(E) \arrow[rr, "x\mapsto 1\otimes x"'] & & E_*(E)\otimes_{\pi_*(E)} E_*(E) \arrow[ul, "\Phi_E"]
\end{tikzcd}$$

The right region commutes by how Ψ is defined (Proposition 0.14), so it suffices to show the left region commutes. To that end, consider the following diagram:

$$\begin{tikzcd}
S^a \arrow[r, "x"] \arrow[d, equals, "\phi_{0,a}=\lambda_{S^a}^{-1}"'] \arrow[dddrr, "e\otimes e\otimes x"] & E \arrow[r, "e\otimes E"] & E\otimes E \arrow[ddd, "E\otimes e\otimes E"] \\
S\otimes S^a \arrow[d, "e\otimes e\otimes x"'] & & \\
E\otimes E\otimes E \arrow[d, "E\otimes E\otimes e\otimes E"'] \arrow[drr, equals] & & \\
E\otimes E\otimes E\otimes E \arrow[rr, "E\otimes\mu\otimes E"'] & & E\otimes E\otimes E
\end{tikzcd}$$

The top composition is $\pi_*(E\otimes e\otimes E)(\eta_R(x))$, while the bottom composition is $\Phi_E(1\otimes\eta_R(x))$. The top right region commutes by functoriality of $-\otimes-$. The bottom left triangle commutes by unitality of μ. Finally, the middle triangle commutes by definition.

Now, we wish to show the following diagram commutes

$$\begin{tikzcd}
E_*(E) \arrow[dr, "\varepsilon"'] & \pi_*(E) \arrow[l, "\eta_L"'] \arrow[r, "\eta_R"] \arrow[d, equals] & E_*(E) \arrow[dl, "\varepsilon"] \\
& \pi_*(E) &
\end{tikzcd}$$

Unravelling how η_L, η_R, and ε are defined, this is the diagram obtained by applying π_* to the following diagram:

$$\begin{tikzcd}
E\otimes E \arrow[dr, "\mu"'] & E \arrow[l, "E\otimes e"'] \arrow[r, "e\otimes E"] \arrow[d, equals] & E\otimes E \arrow[dl, "\mu"] \\
& E &
\end{tikzcd}$$

This commutes by unitality of μ.

Showing that the third diagram in item (1) in Definition 0.7 is entirely analagous to how we showed the first diagram commutes.

Now, we'd like to show the following diagram commutes: $\square$