

# THE MOTIVIC ADAMS SPECTRAL SEQUENCE

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## 1. INTRODUCTION

### 2. THE ADAMS SPECTRAL SEQUENCE

**2.1. Setup.** In order to construct an abstract version of the Adams spectral sequence, we need to work in some axiomatic version of a stable homotopy category  $\mathcal{SH}$  which acts like the familiar classical stable homotopy category  $\mathbf{hoSp}$  (Section 3) or the motivic stable homotopy category  $\mathbf{SH}_{\mathcal{S}}$  over some base scheme  $\mathcal{S}$  (Section 4). As it turns out, practically all the data we need is the following:

**Definition 2.1.** A *stable homotopy category* is the following data:

- A closed tensor triangulated category  $(\mathcal{SH}, \otimes, S, \Sigma, \Omega)$  with arbitrary small (co)products.
- A pointed abelian group  $(A, \mathbf{1})$  and a homomorphism  $h : (A, \mathbf{1}) \rightarrow (\mathrm{Pic}(\mathcal{SH}), \Sigma S)$  of pointed groups (i.e.,  $\mathbf{1}$  is sent to the isomorphism class of  $\Sigma S$ ), where  $\mathrm{Pic}(\mathcal{SH})$  is the group of isomorphism classes of invertible objects in  $\mathcal{SH}$ <sup>1</sup>.
- For each  $a \in A$ , a chosen object  $S^a$  in the isomorphism class  $h(a)$ .

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<sup>1</sup>Recall an object  $X$  in a symmetric monoidal category is *invertible* if there exists some object  $Y$  in  $\mathcal{SH}$  and an isomorphism  $S \cong Y \otimes X$ . To see  $\Sigma S$  is invertible, note that we have isomorphisms

$$\Sigma S \otimes \Omega S \cong \Sigma(S \otimes \Omega S) \cong \Sigma(\Omega S \otimes S) \cong \Sigma \Omega S \otimes S \cong S \otimes S \cong S,$$

Given an abstract stable homotopy category as above, we will always assume without loss of generality that  $S^0 = S$  and  $\Sigma = S^1 \otimes -$  (by [Proposition A.7](#)). we establish the following conventions:

- Given objects  $X_1, \dots, X_n$  in  $\mathcal{SH}$ , we write  $X_1 \otimes \dots \otimes X_n$  to denote the object

$$X_1 \otimes (X_2 \otimes \dots \otimes (X_{n-1} \otimes X_n)).$$

In particular, given an object  $X$  and a natural number  $n > 0$ , we write

$$X^n := \overbrace{X \otimes \dots \otimes X}^{n \text{ times}} \quad \text{and} \quad X^0 := S.$$

- We denote the associator, symmetry, left unitor, and right unitor isomorphisms in  $\mathcal{SH}$  by

$$\begin{aligned} \alpha_{X,Y,Z} : (X \otimes Y) \otimes Z &\xrightarrow{\cong} X \otimes (Y \otimes Z) & \tau_{X,Y} : X \otimes Y &\xrightarrow{\cong} Y \otimes X \\ \lambda_X : S \otimes X &\xrightarrow{\cong} X & \rho_X : X \otimes S &\xrightarrow{\cong} X. \end{aligned}$$

Often we will suppress these isomorphisms from the notation (particularly the associators), choosing instead to denote them without their subscripts or simply with the symbol  $\cong$ .

- Given some  $a \in A$ , we define the functor  $\Sigma^a := S^a \otimes -$ , so that in particular  $\Sigma^1 = \Sigma$ .
- Given two objects  $X$  and  $Y$ , we denote the hom-abelian group of morphisms from  $X$  to  $Y$  in  $\mathcal{SH}$  by  $[X, Y]$ , and we denote the internal hom object by  $F(X, Y)$ . We will often refer to morphisms in  $\mathcal{SH}$  as *classes*, as we will think of them as representing homotopy classes of maps.
- Given two objects  $X$  and  $Y$  in  $\mathcal{SH}$ , we may extend the abelian group  $[X, Y]$  to an  $A$ -graded abelian group  $[X, Y]_*$  defined by

$$[X, Y]_a := [\Sigma^a X, Y] = [S^a \otimes X, Y].$$

(See [Appendix C](#) for a review of the theory of  $A$ -graded abelian groups, rings, modules, etc.)

- Given an object  $X$  in  $\mathcal{SH}$  and some  $a \in A$ , define the abelian group

$$\pi_a(X) := [S^a, X],$$

and write  $\pi_*(X)$  for the associated  $A$ -graded abelian group  $\bigoplus_{a \in A} \pi_a(X)$ . We call  $\pi_a(X)$  the  $a^{\text{th}}$  *stable homotopy group of  $X$* .

- Given two objects  $E$  and  $X$  in  $\mathcal{SH}$ , we define the  $A$ -graded abelian groups  $E_*(X)$  and  $E^*(X)$  by

$$E_a(X) := \pi_a(E \otimes X) = [S^a, E \otimes X] \quad \text{and} \quad E^a(X) := [X, S^a \otimes E].$$

We refer to the functor  $E_*(-)$  as the *homology theory represented by  $E$* , or just  $E$ -homology, and we refer to  $E^*(-)$  as the *cohomology theory represented by  $E$* , or just  $E$ -cohomology.

From now on, we fix the data of a stable homotopy category  $\mathcal{SH}$  given above once and for all. Observe that for all  $a, b \in A$ , the objects  $S^{a+b}$  and  $S^a \otimes S^b$  are isomorphic, since  $h : A \rightarrow \text{Pic}(\mathcal{SH})$  is a group homomorphism. Hence given a monoid object  $(E, \mu, e)$  in  $\mathcal{SH}$  ([Definition D.1](#)), supposing we had fixed isomorphisms  $S^{a+b} \cong S^a \otimes S^b$  for all  $a, b \in A$ , we get a multiplication map  $\pi_*(E) \times \pi_*(E) \rightarrow \pi_*(E)$  which sends classes  $x : S^a \rightarrow E$  and  $y : S^b \rightarrow E$  to the product

$$S^{a+b} \cong S^a \otimes S^b \xrightarrow{x \otimes y} E \otimes E \xrightarrow{\mu} E.$$

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where the first isomorphism is axiom TT1 for a tensor triangulated category ([Definition A.5](#)), the second isomorphism is given by the symmetry in  $\mathcal{SH}$ , the third isomorphism is again axiom TT1, the fourth isomorphism is the fact that  $\Sigma$  and  $\Omega$  for an adjoint equivalence, and finally the last isomorphism follows by the fact that  $S$  is the monoidal unit in  $\mathcal{SH}$ .

Naturally, we would like this product to make  $\pi_*(E)$  into an  $A$ -graded ring (with unit  $e \in \pi_0(E) = [S, E]$ ), rather than just an  $A$ -graded abelian group. This is essentially the entire discussion of Dugger's paper [1], and as it turns out,  $\pi_*(E)$  is in fact a graded ring provided we can choose these morphisms to be *coherent*, in the following sense:

**Definition 2.2.** Suppose we have a family of isomorphisms

$$\phi_{a,b} : S^{a+b} \xrightarrow{\cong} S^a \otimes S^b$$

for all  $a, b \in A$ . We say this family is *coherent* if:

- (1) For all  $a \in A$ , we have equalities  $\phi_{a,0} = \rho_{S^a}^{-1} : S^a \rightarrow S^a \otimes S$  and  $\phi_{0,a} = \lambda_{S^a}^{-1} : S^a \rightarrow S \otimes S^a$ .
- (2) For all  $a, b, c \in A$ , the following diagram commutes:

$$\begin{array}{ccccc} S^{a+b} \otimes S^c & \xleftarrow{\phi_{a+b,c}} & S^{a+b+c} & \xrightarrow{\phi_{a,b+c}} & S^a \otimes S^{b+c} \\ \phi_{a,b} \otimes S^c \downarrow & & & & \downarrow S^a \otimes \phi_{b,c} \\ (S^a \otimes S^b) \otimes S^c & \xrightarrow{\cong} & & & S^a \otimes (S^b \otimes S^c) \end{array}$$

Furthermore, Dugger guarantees that we can always find such a coherent family:

**Theorem 2.3** ([1, Proposition 7.1]). *There exists a coherent family of isomorphisms*

$$\phi_{a,b} : S^{a+b} \xrightarrow{\cong} S^a \otimes S^b$$

in the sense of [Definition 2.2](#), and in particular, the set of such coherent families is in bijective correspondence with the set of normalized 2-cocycles  $Z^2(A; \text{Aut}(S))_{\text{norm}}$ , i.e., the set of functions  $\alpha : A \times A \rightarrow \text{Aut}(S)$  such that  $\alpha(0,0) = \text{id}_S$  and for all  $a, b, c \in A$ ,  $\alpha(a+b, c) \cdot \alpha(a, b) = \alpha(b, c) \cdot \alpha(a, b+c)$ .

Thus, from now on we will suppose once and for all we have fixed a coherent family  $\{\phi_{a,b}\}_{a,b \in A}$ . Such a coherent family has very nice properties, in particular:

**Remark 2.4.** Note that by induction the coherence conditions say that given any  $a_1, \dots, a_n \in A$  and  $b_1, \dots, b_m \in A$  such that  $a_1 + \dots + a_n = b_1 + \dots + b_m$  and any fixed parenthesizations of  $X = S^{a_1} \otimes \dots \otimes S^{a_n}$  and  $Y = S^{b_1} \otimes \dots \otimes S^{b_m}$ , there is a *unique* isomorphism  $X \rightarrow Y$  that can be obtained by forming formal compositions of tensor products of  $\phi_{a,b}$ , associators, and their inverses.

Of course, we get our desired result:  $\pi_*(E)$  is indeed an  $A$ -graded ring if  $E$  is a monoid object.

**Proposition 2.5.** *Let  $(E, \mu, e)$  be a commutative monoid object in  $\mathcal{SH}$ , and consider the multiplication map  $\pi_*(E) \times \pi_*(E) \rightarrow \pi_*(E)$  which sends classes  $x : S^a \rightarrow E$  and  $y : S^b \rightarrow E$  to the composition*

$$S^{a+b} \xrightarrow{\phi_{a,b}} S^a \otimes S^b \xrightarrow{x \otimes y} E \otimes E \xrightarrow{\mu} E.$$

*Then this endows  $\pi_*(E)$  with the structure of an  $A$ -graded ring with unit  $e \in \pi_0(E) = [S, E]$ .*

*Proof.* See [Proposition D.3](#). □

Furthermore, it turns out that if  $E$  is a *commutative* monoid object in  $\mathcal{SH}$ , then  $\pi_*(E)$  is “ $A$ -graded commutative,” in the following sense:

**Proposition 2.6.** *For all  $a, b \in A$  there exists an element  $\theta_{a,b} \in \pi_0(S) = [S, S]$  (determined by choice of coherent family  $\{\phi_{a,b}\}$ ) such that given any commutative monoid object  $(E, \mu, e)$  in  $\mathcal{SH}$ , the  $A$ -graded ring structure on  $\pi_*(E)$  ([Proposition 2.5](#)) has a commutativity formula given by*

$$x \cdot y = y \cdot x \cdot (e \circ \theta_{a,b})$$

for all  $x \in \pi_a(E)$  and  $y \in \pi_b(E)$ .

Furthermore,  $\theta_{0,a} = \theta_{a,0} = \text{id}_S$  for all  $a \in A$ , so that if either  $x$  or  $y$  has degree 0,  $x \cdot y = y \cdot x$ .

*Proof.* See [Proposition D.4](#) and [Proposition D.5](#).  $\square$

We also have the following result:

**Proposition 2.7.** *Given some  $a \in A$ , the functors  $\Sigma^a$  and  $\Sigma^{-a}$  canonically form an adjoint equivalence of  $\mathcal{SH}$ .*

*Proof.* See [Proposition D.6](#).  $\square$

In particular, note that this tells us that given objects  $E$  and  $X$  in  $\mathcal{SH}$ , we have isomorphisms

$$E^*(X) = [X, S^* \otimes X] \cong [S^{-*} \otimes X, E] \cong [S^{-*}, F(X, E)] = \pi_{-*}(F(X, E)).$$

Similarly, given any objects  $X$  and  $Y$  in  $\mathcal{SH}$ , we have isomorphisms of  $A$ -graded abelian groups

$$[X, \Sigma Y]_* = [S^* \otimes X, S^1 \otimes Y] \cong [S^{-1} \otimes S^* \otimes X, Y] \cong [S^{*-1} \otimes X, Y] = [X, Y]_{*-1},$$

where the first isomorphism is the adjunction specified by the above proposition, and the second isomorphism is induced by the isomorphism

$$S^{*-1} \otimes X \xrightarrow{\phi_{-1,*} \otimes X} S^{-1} \otimes S^* \otimes X.$$

The last ingredient in order to develop the Adams spectral sequence abstractly is a notion of *cellularity* in  $\mathcal{SH}$ :

**Definition 2.8.** Define the class of *cellular* objects in  $\mathcal{SH}$  to be the smallest class of objects such that:

- (1) For all  $a \in A$ ,  $S^a$  is cellular.
- (2) If we have a distinguished triangle

$$X \rightarrow Y \rightarrow Z \rightarrow \Sigma X (= S^1 \otimes X)$$

such that two of the three objects  $X$ ,  $Y$ , and  $Z$  are cellular, then the third object is also cellular.

- (3) Given a collection of cellular objects  $X_i$  indexed by some small set  $I$ ,  $\bigoplus_{i \in I} X_i$  is cellular.

**2.2. Monoid objects in  $\mathcal{SH}$ .** We have constructed an Adams spectral sequence, but as it currently stands we do not yet know why it is useful. To start with, we'd like to provide a characterization of its  $E_1$  and  $E_2$  pages in terms of something more algebraic. To start, we first need to develop some theory of the algebra of monoid objects in  $\mathcal{SH}$ . Much of this work is entirely straightforward although tedious to verify, so we relegate most of the proofs in this section to [Appendix D](#).

**Proposition 2.9.** *Let  $(E, \mu, e)$  be a monoid object in  $\mathcal{SH}$ . Then for any object  $X$  in  $\mathcal{SH}$ ,  $E_*(X)$  canonically inherits the structure of a left  $A$ -graded module over  $\pi_*(E)$  (which recall is an  $A$ -graded ring by [Proposition 2.5](#)) via the map*

$$\pi_*(E) \times E_*(X) \rightarrow E_*(X)$$

which given  $a, b \in A$ , sends  $x : S^a \rightarrow E$  and  $y : S^b \rightarrow E$  to the composition

$$x \cdot y : S^{a+b} \cong S^a \otimes S^b \xrightarrow{x \otimes y} E \otimes (E \otimes X) \cong (E \otimes E) \otimes X \xrightarrow{\mu \otimes X} E \otimes X.$$

Similarly,  $X_*(E)$  canonically inherits the structure of a right graded  $\pi_*(E)$ -module via the map

$$X_*(E) \times \pi_*(E) \rightarrow X_*(E)$$

which given  $a, b \in A$ , sends  $x : S^a \rightarrow X \otimes E$  and  $y : S^b \rightarrow E$  to the composition

$$x \cdot y : S^{a+b} \cong S^a \otimes S^b \xrightarrow{x \otimes y} (X \otimes E) \otimes E \cong X \otimes (E \otimes E) \xrightarrow{X \otimes \mu} X \otimes E.$$

*Proof.* See [Proposition D.8](#).  $\square$

**Definition 2.10.** Given a monoid object  $E$  in  $\mathcal{SH}$ , we say  $E$  is *flat* if the canonical right  $\pi_*(E)$ -module structure on  $E_*(E)$  (see the above proposition) is that of a flat module.

**2.3. Construction of the Adams spectral sequence.** In what follows, let  $E$  be a commutative monoid object in  $\mathcal{SH}$ .

**Definition 2.11.** Let  $\overline{E}$  be the fiber of the unit map  $e : S \rightarrow E$  (Proposition A.3), and for  $s \geq 0$  define

$$Y_s := \overline{E}^s \otimes Y, \quad W_s = E \otimes Y_s = E \otimes (\overline{E}^s \otimes Y),$$

where recall for  $s > 0$ ,  $\overline{E}^s$  denotes the  $s$ -fold product parenthesized as  $\overline{E} \otimes (\overline{E} \otimes \cdots (\overline{E} \otimes \overline{E}))$ , and  $\overline{E}^0 := S$ . Then we get fiber sequences

$$Y_{s+1} \xrightarrow{i_s} Y_s \xrightarrow{j_s} W_s \xrightarrow{k_s} \Sigma Y_{s+1} (= S^1 \otimes Y_{s+1})$$

obtained by applying  $- \otimes Y_s$  to the sequence

$$\overline{E} \rightarrow S \xrightarrow{e} E \rightarrow \Sigma \overline{E}$$

(and applying the necessary associator isomorphisms). These sequences can be spliced together to form the (*canonical*) *Adams filtration of  $Y$* :

$$\begin{array}{ccccccc} \cdots & \longrightarrow & Y_3 & \xrightarrow{i_2} & Y_2 & \xrightarrow{i_1} & Y_1 & \xrightarrow{i_0} & Y_0 = Y \\ & & \downarrow j_3 & \swarrow k_2 & \downarrow j_2 & \swarrow k_1 & \downarrow j_1 & \swarrow k_0 & \downarrow j_0 \\ & & W_3 & & W_2 & & W_1 & & W_0 \end{array}$$

where the diagonal dashed arrows are of degree  $-1$  (note these triangles do NOT commute in any sense). Now we may apply the functor  $[X, -]_*$ , and by Proposition A.4 we obtain an exact couple of  $\mathbb{N} \times A$ -graded abelian groups:

$$\begin{array}{ccc} [X, Y_*]_* & \xrightarrow{i_{**}} & [X, Y_*]_* \\ & \swarrow k_{**} & \downarrow j_{**} \\ & & [X, W_*]_* \end{array}$$

where  $i_{**}$ ,  $j_{**}$ , and  $k_{**}$  have  $\mathbb{Z} \times A$ -degree  $(-1, 0)$ ,  $(0, 0)$ , and  $(1, -1)$ , respectively<sup>2</sup>. The standard argument yields a  $\mathbb{N} \times A$ -graded spectral sequence called from this exact couple (cf. Section 5.9 of [6]) with  $E_1$  page given by

$$E_1^{s,a} = [X, W_s]_a$$

and  $r^{\text{th}}$  differential of  $\mathbb{Z} \times A$ -degree  $(r, -1)$ :

$$d_r : E_r^{s,a} \rightarrow E_r^{s+r,a-1}.$$

A priori, this is all  $\mathbb{N} \times A$ -graded, but we regard it as being  $\mathbb{Z} \times A$ -graded by setting  $E_r^{s,a} := 0$  for  $s < 0$  and trivially extending the definition of the differentials to these zero groups. This spectral sequence is called the *E-Adams spectral sequence* for the computation of  $[X, Y]_*$ . The index  $s$  is called the *Adams filtration* and  $a$  is the *stem*.

<sup>2</sup>Explicitly, the map  $k_{s,a} : [X, W_s]_a \rightarrow [X, Y_{s+1}]_{a-1}$  sends a map  $f : S^a \otimes X \rightarrow W_s$  to the map  $S^{a-1} \otimes X \rightarrow Y_{s+1}$  corresponding under the isomorphism  $[X, \Sigma Y_{s+1}]_* \cong [X, Y_{s+1}]_{*-1}$  to the composition  $k_s \circ f : S^a \otimes X \rightarrow \Sigma Y_{s+1}$ .

**2.4. The  $E_1$  page.** The goal of this subsection is to provide the following characterization for the  $E_1$  page of the Adams spectral sequence:

**Theorem 2.12.** *Let  $E$  be a flat commutative monoid object in  $\mathcal{SH}$ , and let  $X$  and  $Y$  be two objects in  $\mathcal{SH}$  such that  $E_*(X)$  is a projective module over  $\pi_*(E)$ . Then for all  $s \geq 0$  and  $a \in \mathbb{Z}$ , we have isomorphisms in the associated  $E$ -Adams spectral sequence*

$$E_1^{s,a} \cong \mathrm{Hom}_{E_*(E)}^a(E_*(X), E_*(W_s))$$

Furthermore, under these isomorphisms, the differential  $d_1 : E_1^{s,a} \rightarrow E_1^{s+1,a-1}$  corresponds to the map

$$\mathrm{Hom}_{E_*(E)}^a(E_*(X), E_*(W_s)) \rightarrow \mathrm{Hom}_{E_*(E)}^{a-1}(E_*(X), E_*(X \otimes W_{s+1}))$$

which sends a map  $f : E_*(X) \rightarrow E_{*+a}(W_s)$  to the composition

$$E_*(X) \xrightarrow{f} E_{*+a}(W_s) \xrightarrow{(X \otimes h_s)_*} E_{*+a-1}(X \otimes Y_{s+1}) \xrightarrow{(X \otimes j_{s+1})_*} E_{*+a-1}(X \otimes W_{s+1}).$$

*Proof.* By Lemma D.10, for all  $s \geq 0$  and  $t, w \in \mathbb{Z}$ , we have isomorphisms

$$[X, E \otimes Y_s]_{t,w} \cong \mathrm{Hom}_{E_*(E)}^{t,w}(E_*(X), E_*(E \otimes Y_s)).$$

since  $W_s = E \otimes Y_s$ , we have that

$$E_1^{s,(t,w)} = [X, W_s]_{t,w} \cong \mathrm{Hom}_{E_*(E)}^{t,w}(E_*(X), E_*(W_s)),$$

as desired.  $\square$

**Definition 2.13.** Let  $(E, \mu, e)$  be a monoid object in  $\mathcal{SH}$ . We say  $E$  is *flat* if the canonical right  $\pi_*(E)$ -module structure on  $E_*(E)$  is that of a flat module.

**2.5. The  $E_2$  page.**

**2.6. Convergence.** convergence of spectral sequences

### 3. THE CLASSICAL ADAMS SPECTRAL SEQUENCE

### 4. THE MOTIVIC ADAMS SPECTRAL SEQUENCE

### APPENDIX A. TRIANGULATED CATEGORIES

We assume the reader is familiar with additive categories and (closed, symmetric) monoidal categories.

**Definition A.1.** A *triangulated category* is a tuple  $(\mathcal{C}, \Sigma, \Omega, \mathcal{D})$  such that

- (1)  $\mathcal{C}$  is an additive category.
- (2)  $\Sigma, \Omega : \mathcal{C} \rightarrow \mathcal{C}$  form an adjoint equivalence of  $\mathcal{C}$  with itself. ( $\Sigma$  is called the *shift functor*.)
- (3)  $\mathcal{D}$  is a collection of *distinguished triangles*, where a *triangle* is a diagram of the form

$$X \rightarrow Y \rightarrow Z \rightarrow \Sigma X.$$

These are also sometimes called *cofiber sequences* or *fiber sequences*.

These data must satisfy the following axioms:

**TR0** Given a commutative diagram

$$\begin{array}{ccccccc} X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & \Sigma X \\ \cong \downarrow & & \cong \downarrow & & \cong \downarrow & & \cong \downarrow \\ X' & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & \Sigma X' \end{array}$$

where the vertical arrows are isomorphisms, if the top row is distinguished then so is the bottom.

**TR1** For any object  $X$  in  $\mathcal{C}$ , the diagram

$$X \xrightarrow{\text{id}_X} X \rightarrow 0 \rightarrow \Sigma X$$

is a distinguished triangle.

**TR2** For all  $f : X \rightarrow Y$  there exists an object  $C_f$  (also sometimes denoted  $Y/X$ ) called the *cofiber of  $f$*  and a distinguished triangle

$$X \xrightarrow{f} Y \rightarrow C_f \rightarrow \Sigma X.$$

**TR3** Given a solid diagram with both rows commutative

$$\begin{array}{ccccccc} X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & \Sigma X \\ f \downarrow & & \downarrow & & \vdots & & \downarrow \Sigma f \\ X' & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & \Sigma X' \end{array}$$

such that the leftmost square commutes and both rows are distinguished, there exists a dashed arrow  $Z \rightarrow Z'$  which makes the remaining two squares commute.

**TR4** A triangle

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{\Sigma} X$$

is distinguished if and only if

$$Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X \xrightarrow{-\Sigma f} \Sigma Y$$

is distinguished.

**TR5** (Octahedral axiom) Given three distinguished triangles

$$X \xrightarrow{f} Y \xrightarrow{h} Y/X \rightarrow \Sigma X$$

$$Y \xrightarrow{g} Z \xrightarrow{k} Z/Y \rightarrow \Sigma Y$$

$$X \xrightarrow{g \circ f} Z \xrightarrow{l} Z/X \rightarrow \Sigma X$$

there exists a distinguished triangle

$$Y/X \xrightarrow{u} Z/X \xrightarrow{v} Z/Y \xrightarrow{w} \Sigma(Y/X)$$

such that the following diagram commutes

$$\begin{array}{ccccccc} X & \xrightarrow{g \circ f} & Z & \xrightarrow{k} & Z/Y & \xrightarrow{w} & \Sigma(Y/X) \\ & \searrow f & \nearrow g & \searrow l & \nearrow v & \searrow & \nearrow \Sigma h \\ & Y & & Z/X & & \Sigma Y & \\ & \searrow h & \nearrow u & \searrow & \nearrow \Sigma f & & \\ & Y/X & \xrightarrow{\quad} & \Sigma X & & & \end{array}$$

It turns out that the above definition is actually redundant; TR3 and TR4 follow from the remaining axioms (see Lemmas 2.2 and 2.4 in [2]).

We now recall several important propositions for triangulated categories:

**Proposition A.2.** *Given a map  $f : X \rightarrow Y$  in a triangulated category  $(\mathcal{C}, \Sigma, \Omega, \mathcal{D})$ , the cofiber sequence of  $f$  is unique up to isomorphism, in the sense that given any two distinguished triangles*

$$X \xrightarrow{f} Y \rightarrow Z \rightarrow \Sigma X \quad \text{and} \quad X \xrightarrow{f} Y \rightarrow Z' \rightarrow \Sigma X,$$

there exists an isomorphism  $Z \rightarrow Z'$  which makes the following diagram commute:

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \longrightarrow & Z & \longrightarrow & \Sigma X \\ \parallel & & \parallel & & \downarrow k & & \parallel \\ X & \xrightarrow{f} & Y & \longrightarrow & Z' & \longrightarrow & \Sigma X \end{array}$$

**Proposition A.3.** *Given an arrow  $f : X \rightarrow Y$  in a triangulated category  $(\mathcal{C}, \Sigma, \Omega, \mathcal{D})$ , there exists an object  $F_f$  called the fiber of  $f$ , and a distinguished triangle*

$$F_f \rightarrow X \xrightarrow{f} Y \rightarrow \Sigma F_f (\cong C_f).$$

**Proposition A.4.** *Let  $(\mathcal{C}, \Sigma, \Omega, \mathcal{D})$  be a triangulated category. Given a distinguished triangle*

$$X \xrightarrow{f} Y \xrightarrow{g} \Sigma X$$

*and any object  $A$  in  $\mathcal{C}$ , there is a long exact sequence of abelian groups*

$$\cdots \rightarrow [\Sigma^{n+1} A, Z] \xrightarrow{h_*} [\Sigma^n X, X] \xrightarrow{f_*} [\Sigma^n A, Y] \xrightarrow{g_*} [\Sigma^n A, Z] \xrightarrow{h_*} [\Sigma^{n-1} A, X] \rightarrow \cdots$$

*extending infinitely in either direction, where for  $n < 0$  we define  $\Sigma^{-n} := \Omega^n$ .*

Also important for our work is the concept of a *tensor triangulated category*, that is, a triangulated symmetric monoidal category in which the triangulated structures are compatible, in the following sense:

**Definition A.5.** A *tensor triangulated category* is a triangulated symmetric monoidal category  $(\mathcal{C}, \otimes, S, \Sigma, \Omega, \mathcal{D})$  such that:

**TT1** For all objects  $X$  and  $Y$  in  $\mathcal{C}$ , there are natural isomorphisms

$$e_{X,Y} : (\Sigma X) \otimes Y \xrightarrow{\cong} \Sigma(X \otimes Y).$$

**TT2** For each object  $X$  in  $\mathcal{C}$ , the functor  $X \otimes (-) \cong (-) \otimes X$  is an additive functor.

**TT3** For each object  $X$  in  $\mathcal{C}$ , the functor  $X \otimes (-) \cong (-) \otimes X$  preserves distinguished triangles, in that given a distinguished triangle/(co)fiber sequence

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{\Sigma} A,$$

then also

$$X \otimes A \xrightarrow{X \otimes f} X \otimes B \xrightarrow{X \otimes g} X \otimes C \xrightarrow{\Sigma(X \otimes h)} \Sigma(X \otimes A)$$

and

$$A \otimes X \xrightarrow{f \otimes X} B \otimes X \xrightarrow{g \otimes X} C \otimes X \xrightarrow{\Sigma(h \otimes X)} \Sigma(A \otimes X)$$

are distinguished triangles.

Usually, most tensor triangulated categories that arise in nature will satisfy additional coherence axioms (see axioms TC1–TC5 in [2]), but the above definition will suffice for our purposes. To avoid the awkwardness of saying “a tensor triangulated category which is also a closed symmetric monoidal category,” we introduce the following (nonstandard) terminology:

**Definition A.6.** We say a tensor triangulated category  $(\mathcal{C}, \otimes, S, \Sigma, \Omega)$  is *closed* if  $\mathcal{C}$  is a closed symmetric monoidal category, in the sense that for each object  $X \in \mathcal{C}$ , the functor  $- \otimes X$  has a right adjoint  $F(X, -)$ .

Note that given a tensor triangulated category, we have the following characterization of the shift functor:

**Proposition A.7.** *Given a tensor triangulated category  $(\mathcal{C}, \otimes, S, \Sigma, \Omega)$ , there is a canonical natural isomorphism  $\Sigma S \otimes - \cong \Sigma$ .*



*Proof.* Given an object  $X$  in  $\mathcal{C}$ , we have natural isomorphisms

$$\Sigma S \otimes X \xrightarrow{e_{S,X}} \Sigma(S \otimes X) \xrightarrow{\Sigma \lambda_X} \Sigma X,$$

where  $\lambda_X$  is the left unitor specified by the monoidal structure on  $\mathcal{C}$ .  $\square$

## APPENDIX B. SPECTRAL SEQUENCES

In what follows, we fix an abelian group  $A$ . We will freely use the theory and results of [Appendix C](#)

**Definition B.1.** An *A-graded spectral sequence* is the data of a collection of  $A$ -graded abelian groups  $\{E_r^*\}_{r \geq r_0}$  along with homomorphisms of  $A$ -graded abelian groups (possibly of nonzero degree)  $d_r : E_r \rightarrow E_r$  such that  $d_r \circ d_r = 0$  and  $E_{r+1} = \ker d_r / \text{im } d_r$ .

## APPENDIX C. (CO)ALGEBRA

**C.1. Grading.** First, we develop the theory of things graded by an abelian group. In what follows, we fix an abelian group  $A$ .

**Definition C.1.** An *A-graded abelian group* is an abelian group  $B$  along with a subgroup  $B_a \leq B$  for each  $a \in A$  such that the canonical map

$$\bigoplus_{a \in A} B_a \rightarrow B$$

sending  $(x_a)_{a \in A}$  to  $\sum_{a \in A} x_a$  is an isomorphism. Given two  $A$ -graded abelian groups  $B$  and  $C$ , a homomorphism  $f : B \rightarrow C$  is a *homomorphism of A-graded abelian groups* if it preserves the grading, i.e., if it restricts to a map  $B_a \rightarrow C_a$  for all  $a \in A$ .

**Definition C.2.** More generally, given two  $A$ -graded abelian groups  $B$  and  $C$  and some  $d \in A$ , a group homomorphism  $f : B \rightarrow C$  is an *A-graded homomorphism of degree d* if it restricts to a map  $B_a \rightarrow C_{a+d}$  for all  $a \in A$ .

Unless stated otherwise, an “ $A$ -graded homomorphism” will always refer to an  $A$ -graded homomorphism of degree 0. It is easy to see that an  $A$ -graded abelian group  $B$  is generated by its *homogeneous* elements, that is, nonzero elements  $x \in B$  such that there exists some  $a \in A$  with  $x \in B_a$ .

**Remark C.3.** Clearly the condition that the canonical map  $\bigoplus_{a \in A} B_a \rightarrow B$  is an isomorphism requires that  $B_a \cap B_b = 0$  if  $a \neq b$ . In particular, given a homogeneous element  $x \in B$ , there exists precisely one  $a \in A$  such that  $x \in B_a$ . We call this  $a$  the *degree* of  $x$ , and we write  $|x| = a$ .

**Lemma C.4.** Given two  $A$ -graded abelian groups  $B$  and  $C$ , their product  $B \oplus C$  is naturally an  $A$ -graded abelian group by defining

$$(B \oplus C)_a := \bigoplus_{b+c=a} B_b \oplus C_c.$$

*Proof.* This is entirely straightforward, as

$$B \otimes C \cong \left( \bigoplus_{b \in A} B_b \right) \oplus \left( \bigoplus_{c \in A} C_c \right) \cong \bigoplus_{b,c \in A} B_b \oplus C_c$$

$\square$

**Definition C.5.** An *A-graded ring*  $R$  is the data of a ring  $R$  such that:

- (1) The underlying abelian group of  $R$  is  $A$ -graded;

(2) For all  $a, b \in A$ , the multiplication map  $R \times R \rightarrow R$  restricts to a map

$$R_a \times R_b \rightarrow R_{a+b},$$

i.e.,  $|x \cdot y| = |x| + |y|$  for all nonzero  $x, y \in R$ .

For example, given some field  $k$ , the ring  $R = k[x, y]$  is  $\mathbb{Z}^2$ -graded, where given  $(n, m) \in \mathbb{Z}^2$ ,  $R_{n,m}$  is the subgroup of those monomials of the form  $ax^ny^m$  for some  $a \in k$ . Oftentimes we constructing  $A$ -graded rings, we do so only by defining the product of homogeneous elements, like so:

**Proposition C.6.** *Given an  $A$ -graded abelian group  $R$ , a distinguished element  $1 \in R_0$ , and  $\mathbb{Z}$ -bilinear maps  $m_{a,b} : R_a \times R_b \rightarrow R_{a+b}$  for all  $a, b \in A$  such that given  $x \in R_a$ ,  $y \in R_b$ , and  $z \in R_c$ ,*

$$m_{a+b,c}(m_{a,b}(x, y), z) = m_{a,b+c}(x, m_{b,c}(y, z)) \quad \text{and} \quad m_{a,0}(x, 1) = m_{0,a}(1, x) = x,$$

*there exists a unique multiplication map  $m : R \times R \rightarrow R$  which endows  $R$  with the structure of an  $A$ -graded ring and restricts to  $m_{a,b}$  for all  $a, b \in A$ .*

*Proof.* Given  $r, s \in R$ , since  $R \cong \bigoplus_{a \in A} R_a$ , we may uniquely decompose  $r$  and  $s$  into homogeneous elements as  $r = \sum_{a \in A} r_a$  and  $s = \sum_{a \in A} s_a$  with each  $r_a, s_a \in R_a$  such that only finitely many of the  $r_a$ 's and  $s_a$ 's are nonzero. Then in order to define a distributive product  $R \times R \rightarrow R$  which restricts to  $m_{a,b} : R_a \times R_b \rightarrow R_{a+b}$ , note we *must* define

$$r \cdot s = \left( \sum_{a \in A} r_a \right) \cdot \left( \sum_{b \in A} s_b \right) = \sum_{a,b \in A} r_a \cdot s_b = \sum_{a,b \in A} m_{a,b}(r_a, s_b).$$

Thus, we have shown uniqueness. It remains to show this product actually gives  $R$  the structure of a ring. First we claim that the sum on the right is actually finite. Note there exists only finitely many nonzero  $r_a$ 's and  $s_b$ 's, and if  $s_b = 0$  then

$$m_{a,b}(r_a, 0) = m_{a,b}(r_a, 0 + 0) \stackrel{(*)}{=} m_{a,b}(r_a, 0) + m_{a,b}(r_a, 0) \implies m_{a,b}(r_a, 0) = 0,$$

where  $(*)$  follows from bilinearity of  $m_{a,b}$ . A similar argument yields that  $m_{a,b}(0, r_b) = 0$  for all  $a, b \in A$ . Hence indeed  $m_{a,b}(r_a, s_b)$  is zero for all but finitely many pairs  $(a, b) \in A^2$ , as desired. Observe that in particular

$$(r \cdot s)_a = \sum_{b+c=a} m_{b,c}(r_b, s_c) = \sum_{b \in A} m_{b,a-b}(r_b, s_{a-b}) = \sum_{c \in A} m_{a-c,c}(r_{a-c}, s_c).$$

Now we claim this multiplication is associative. Given  $t = \sum_{a \in A} t_a \in R$ , we have

$$\begin{aligned}
(r \cdot s) \cdot t &= \sum_{a, b \in A} m_{a, b}((r \cdot s)_a, t_b) \\
&= \sum_{a, b \in A} m_{a, b} \left( \sum_{c \in A} m_{a-c, c}(r_{a-c}, s_c), t_b \right) \\
&\stackrel{(1)}{=} \sum_{a, b, c \in A} m_{a, b}(m_{a-c, c}(r_{a-c}, s_c), t_b) \\
&\stackrel{(2)}{=} \sum_{a, b, c \in A} m_{c, a+b-c}(r_c, m_{a-c, b}(s_{a-c}, t_b)) \\
&\stackrel{(3)}{=} \sum_{a, b, c \in A} m_{a, c}(r_a, m_{b, c-b}(s_b, t_{c-b})) \\
&\stackrel{(1)}{=} \sum_{a, c \in A} m_{a, c} \left( r_a, \sum_{b \in A} m_{b, c-b}(s_b, t_{c-b}) \right) \\
&= \sum_{a, c \in A} m_{a, c}(r_a, (s \cdot t)_c) = r \cdot (s \cdot t),
\end{aligned}$$

where each occurrence of (1) follows by bilinearity of the  $m_{a, b}$ 's, each occurrence of (2) is associativity of the  $m_{a, b}$ 's, and (3) is obtained by re-indexing by re-defining  $a := c$ ,  $b := a - c$ , and  $c := a + b - c$ . Next, we wish to show that the distinguished element  $1 \in R_0$  is a unit with respect to this multiplication. Indeed, we have

$$1 \cdot r \stackrel{(1)}{=} \sum_{a \in A} m_{0, a}(1, r_a) \stackrel{(2)}{=} \sum_{a \in A} r_a = r$$

and

$$r \cdot 1 \stackrel{(1)}{=} \sum_{a \in A} m_{a, 0}(r_a, 1) \stackrel{(2)}{=} \sum_{a \in A} r_a = r,$$

where (1) follows by the fact that  $m_{a, b}(0, -) = m_{a, b}(-, 0) = 0$ , which we have shown above, and (2) follows by unitality of the  $m_{0, a}$ 's and  $m_{a, 0}$ 's, respectively. Finally, we wish to show that this product is distributive. Indeed, we have

$$\begin{aligned}
r \cdot (s + t) &= \sum_{a, b \in A} m_{a, b}(r_a, (s + t)_b) \\
&= \sum_{a, b \in A} m_{a, b}(r_a, s_b + t_b) \\
&\stackrel{(*)}{=} \sum_{a, b \in A} m_{a, b}(r_a, s_b) + \sum_{a, b \in A} m_{a, b}(r_a, t_b) = (r \cdot s) + (r \cdot t),
\end{aligned}$$

where (\*) follows by bilinearity of  $m_{a, b}$ . An entirely analogous argument yields that  $(r + s) \cdot t = (r \cdot t) + (s \cdot t)$ .  $\square$

When working with  $A$ -graded abelian groups, we will freely use the above proposition without comment. Given an  $A$ -graded ring  $R$ , we may talk about  $A$ -graded  $R$ -modules:

**Definition C.7.** Let  $R$  be an  $A$ -graded ring. A left  $A$ -graded  $R$ -module  $M$  is a left  $R$ -module  $M$  such that  $M$  is an  $A$ -graded abelian group, and for all  $a, b \in A$ , the action map  $R \times M \rightarrow M$  restricts to a map  $R_a \times M_b \rightarrow M_{a+b}$ .

Right  $A$ -graded  $R$ -modules are defined similarly. Finally, an  $A$ -graded  $R$ -bimodule is an  $A$ -graded abelian group  $M$  along with action maps

$$R \times M \rightarrow M \quad \text{and} \quad M \times R \rightarrow M$$

which endow  $M$  with the structure of a left and right  $A$ -graded  $R$ -module, respectively, such that given  $r, s \in R$  and  $m \in M$ ,  $r \cdot (m \cdot s) = (r \cdot m) \cdot s$ .

**Proposition C.8.** *Let  $R$  be an  $A$ -graded ring, and suppose we have a right  $A$ -graded  $R$ -module  $M$  and a left  $A$ -graded  $R$ -module  $N$ . Then the tensor product*

$$M \otimes_R N$$

*is naturally an  $A$ -graded abelian group by defining  $(M \otimes_R N)_a$  to be the subgroup generated by homogeneous pure tensors  $m \otimes n$  with  $m \in M_b$  and  $n \in N_c$  such that  $b + c = a$ . Furthermore, if either  $M$  (resp.  $N$ ) is an  $A$ -graded bimodule, then  $M \otimes_R N$  is naturally a left (resp. right)  $A$ -graded  $R$ -module*

*Proof.* By definition, since  $M$  and  $N$  are  $A$ -graded abelian groups, they are generated (as abelian groups) by their homogeneous elements. Thus it follows that  $M \otimes_R N$  is generated by *homogeneous pure tensors*, that is, elements of the form  $m \otimes n$  with  $m \in M$  and  $n \in N$  homogeneous. Now, given a homogeneous pure tensor  $m \otimes n$ , we define its *degree* by the formula  $|m \otimes n| := |m| + |n|$ . It follows this formula is well-defined by checking that given homogeneous elements  $m \in M$ ,  $n \in N$ , and  $r \in R$  that

$$|(m \cdot r) \otimes n| = |m \cdot r| + |n| = |m| + |r| + |n| = |m| + |r \cdot n| = |m \otimes (r \cdot n)|.$$

Thus, we may define  $(M \otimes_R N)_a$  to be the subgroup of  $M \otimes_R N$  generated by those pure homogeneous tensors of degree  $a$ . Now, we construct a map

$$\Phi : M \times N \rightarrow \bigoplus_{a \in A} (M \otimes_R N)_a$$

which takes a pair  $(m, n) = \sum_{a \in A} (m_a, n_a)$  to the element  $\Phi(m, n)$  whose  $a^{\text{th}}$  component is

$$(\Phi(m, n))_a := \sum_{b+c=a} m_b \otimes n_c.$$

It is straightforward to see that this map is  $R$ -linear in both arguments. Thus by the universal property of  $M \otimes_R N$ , we get a lift  $\tilde{\Phi} : M \otimes_R N \rightarrow \bigoplus_{a \in A} (M \otimes_R N)_a$ . Now, also consider the canonical map

$$\Psi : \bigoplus_{a \in A} (M \otimes_R N)_a \rightarrow M \otimes_R N.$$

We would like to show  $\tilde{\Phi}$  and  $\Psi$  are inverses of each other. It suffices to show this on generators. Let  $m \otimes n$  be a pure homogeneous tensor with  $m = m_a \in M_a$  and  $n = n_b \in N_b$ . Then we have

$$\Psi(\tilde{\Phi}(m \otimes n)) = \Psi \left( \bigoplus_{a \in A} \sum_{b+c=a} m_b \otimes n_c \right) \stackrel{(*)}{=} \Psi(m \otimes n) = m \otimes n,$$

and

$$\tilde{\Phi}(\Psi(m \otimes n)) = \tilde{\Phi}(m \otimes n) = \bigoplus_{a \in A} \sum_{b+c=a} m_b \otimes n_c \stackrel{(*)}{=} m \otimes n,$$

where both occurrences of  $(*)$  follow by the fact that  $m_b \otimes n_c = 0$  unless  $b = c = a$ , in which case  $m_a \otimes n_a = m \otimes n$ . Thus since  $\Psi$  is an isomorphism,  $M \otimes_R N$  is indeed an  $A$ -graded abelian group, as desired.

Now, suppose that  $M$  is an  $A$ -graded  $R$ -bimodule, so there exists a left and right action of  $R$  on  $M$  such that given  $r, s \in R$  and  $m \in M$  we have  $r \cdot (m \cdot s) = (r \cdot m) \cdot s$ . Then we would

like to show that given a left  $A$ -graded  $R$ -module  $N$  that  $M \otimes_R N$  is canonically a left  $A$ -graded  $R$ -module. Indeed, define the action of  $R$  on  $M \otimes_R N$  on pure tensors by the formula

$$r \cdot (m \otimes n) = (r \cdot m) \otimes n.$$

First of all, clearly this map is  $A$ -graded, as if  $r \in R_a$ ,  $m \in M_b$ , and  $n \in N_c$  then  $(r \cdot m) \otimes n$ , by definition, has degree  $|r \cdot m| + |n| = |r| + |m| + |n|$  (the last equality follows since the left action of  $R$  on  $M$  is  $A$ -graded). In order to show the above map defines a left module structure, it suffices to show that given pure tensors  $m \otimes n, m' \otimes n' \in M \otimes_R N$  and elements  $r, r' \in R$  that

- (1)  $r \cdot (m \otimes n + m' \otimes n') = r \cdot (m \otimes n) + r \cdot (m' \otimes n')$ ,
- (2)  $(r + r') \cdot (m \otimes n) = r \cdot (m \otimes n) + r' \cdot (m \otimes n)$ ,
- (3)  $(rr') \cdot (m \otimes n) = r \cdot (r' \cdot (m \otimes n))$ , and
- (4)  $1 \cdot (m \otimes n) = m \otimes n$ .

Axiom (1) holds by definition. To see (2), note that by the fact that  $R$  acts on  $M$  on the left that

$$(r + r') \cdot (m \otimes n) = ((r + r') \cdot m) \otimes n = (r \cdot m + r' \cdot m) \otimes n = r \cdot m \otimes n + r' \cdot m \otimes n.$$

That (3) and (4) hold follows similarly by the fact that  $(rr') \cdot m = r \cdot (r' \cdot m)$  and  $1 \cdot m = m$ .

Conversely, if  $N$  is an  $A$ -graded  $R$ -bimodule, then showing  $M \otimes_R N$  is canonically a right  $A$ -graded  $R$ -module via the rule

$$(m \otimes n) \cdot r = m \otimes (n \cdot r)$$

is entirely analogous. □

**Lemma C.9.** *Let  $R$  be an  $A$ -graded ring, and suppose we have a right  $A$ -graded  $R$ -module  $M$  and a left  $A$ -graded  $R$ -module  $N$ . Then given an  $A$ -graded abelian group  $B$  and an  $A$ -graded  $R$ -bilinear map*

$$\varphi : M \times N \rightarrow B$$

(here  $M \times N$  is regarded as an  $A$ -graded abelian group by ??), the lift

$$\tilde{\varphi} : M \otimes_R N \rightarrow B$$

determined by the universal property of  $M \otimes_R N$  is an  $A$ -graded map.

#### APPENDIX D. MONOID OBJECTS IN A STABLE HOMOTOPY CATEGORY

**Definition D.1.** Let  $(\mathcal{C}, \otimes, S)$  be a symmetric monoidal category with left unitor, right unitor, and associator, and symmetry isomorphism  $\lambda$ ,  $\rho$ ,  $\alpha$ , and  $\tau$ , respectively. Then a *monoid object*  $(E, \mu, e)$  is an object  $E$  in  $\mathcal{C}$  along with a multiplication map  $\mu : E \otimes E \rightarrow E$  and a unit map  $e : S \rightarrow E$  such that the following diagram commutes:

$$\begin{array}{ccc} E \otimes S & \xrightarrow{E \otimes e} & E \otimes E \xleftarrow{e \otimes E} S \otimes E \\ & \searrow \rho & \downarrow \mu \swarrow \lambda \\ & & E \end{array} \quad \begin{array}{ccc} (E \otimes E) \otimes E & \xrightarrow{\mu \otimes E} & E \otimes E \\ \alpha \downarrow & & \downarrow \mu \\ E \otimes (E \otimes E) & \xrightarrow{E \otimes \mu} & E \otimes E \xrightarrow{\mu} E \end{array}$$

The first diagram expresses unitality, while the second expressed associativity. If in addition the following diagram commutes,

$$\begin{array}{ccc} E \otimes E & \xrightarrow{\tau} & E \otimes E \\ & \searrow \mu \swarrow \mu & \\ & E & \end{array}$$

then we say  $(E, \mu, e)$  is a *commutative monoid object*.

**Proposition D.2.** *Let  $(E_1, \mu_1, e_1)$  and  $(E_2, \mu_2, e_2)$  be monoid objects in a symmetric monoidal category  $(\mathcal{C}, \otimes, S)$ . Then  $E_1 \otimes E_2$  is canonically a ring spectrum via the maps*

$$\mu : E_1 \otimes E_2 \otimes E_1 \otimes E_2 \xrightarrow{E_1 \otimes \tau \otimes E_2} E_1 \otimes E_1 \otimes E_2 \otimes E_2 \xrightarrow{\mu_1 \otimes \mu_2} E_1 \otimes E_2$$

and

$$e : S \cong S \otimes S \xrightarrow{e_1 \otimes e_2} E_1 \otimes E_2.$$

todo

*Proof.*

□

In what follows, fix a stable homotopy category  $\mathcal{SH}$  (Definition 2.1) along with the additional data therewithin, and adopt the conventions outlined in Section 2.1. Further suppose we have fixed a coherent family of isomorphisms

$$\phi_{a,b} : S^{a+b} \xrightarrow{\cong} S^a \otimes S^b,$$

in the sense of Definition 2.2 (the existence of such a coherent family is guaranteed by Theorem 2.3).

**Proposition D.3.** *Let  $(E, \mu, e)$  be a commutative monoid object in  $\mathcal{SH}$ , and consider the multiplication map  $\pi_*(E) \times \pi_*(E) \rightarrow \pi_*(E)$  which sends classes  $x : S^a \rightarrow E$  and  $y : S^b \rightarrow E$  to the composition*

$$S^{a+b} \xrightarrow{\phi_{a,b}} S^a \otimes S^b \xrightarrow{x \otimes y} E \otimes E \xrightarrow{\mu} E.$$

Then this endows  $\pi_*(E)$  with the structure of an  $A$ -graded ring with unit  $e \in \pi_0(E) = [S, E]$ .

*Proof.* First we show this map is associative: Given classes  $x, y$ , and  $z$  in  $\pi_a(E)$ ,  $\pi_b(E)$ , and  $\pi_c(E)$ , respectively, consider the following diagram:

$$\begin{array}{ccccccc} S^{a+b+c} & \xrightarrow{\phi_{a+b,c}} & S^{a+b} \otimes S^c & \xrightarrow{\phi_{a,b} \otimes S^c} & (S^a \otimes S^b) \otimes S^c & \xrightarrow{(x \otimes y) \otimes z} & (E \otimes E) \otimes E \xrightarrow{\mu \otimes E} E \otimes E \\ \phi_{a,b+c} \downarrow & & \swarrow \cong & & \swarrow \cong & & \downarrow \mu \\ S^a \otimes S^{b+c} & \xrightarrow{S^a \otimes \phi_{b,c}} & S^a \otimes (S^b \otimes S^c) & \xrightarrow{x \otimes (y \otimes z)} & E \otimes (E \otimes E) & \xrightarrow{E \otimes \mu} & E \otimes E \xrightarrow{\mu} E \end{array}$$

Commutativity of the left pentagon is the coherence condition for the  $\phi_{a,b}$ 's. Commutativity of the middle parallelogram is naturality of the associator isomorphisms. Commutativity of the right pentagon is associativity of  $\mu$ . The fact that the two outside compositions equal  $(x \cdot y) \cdot z$  and  $x \cdot (y \cdot z)$ , respectively, follows by functoriality of  $- \otimes -$ .

Next we claim the map  $e : S \rightarrow E$  is a unit for this multiplication. Given  $x \in \pi_a(E)$ , consider the following diagram:

$$\begin{array}{ccccc} S \otimes S^a & \xleftarrow{\phi_{0,a} = \lambda_{S^a}^{-1}} & S^a & \xrightarrow{\phi_{a,0} = \rho_{S^a}^{-1}} & S^a \otimes S \\ \downarrow e \otimes x & \searrow S \otimes x & \downarrow x & \swarrow x \otimes S & \downarrow x \otimes e \\ & S \otimes E & E \otimes S & & \\ e \otimes E \swarrow & \lambda_E \searrow & \rho_E \swarrow & E \otimes e \searrow & \\ E \otimes E & \xrightarrow{\mu} & E & \xleftarrow{\mu} & E \otimes E \end{array}$$

Commutativity of the top two large triangles is naturality of the unitor isomorphisms. Commutativity of the right and leftmost triangles is functoriality of  $- \otimes -$ . Commutativity of the bottom triangles is unitality of  $\mu$ . Hence, we have that  $e \cdot x = x = x \cdot e$ .

This product is also bilinear (distributive). Given  $x, x' \in \pi_a(E)$  and  $y, y' \in \pi_b(E)$ , consider the following diagrams:

$$\begin{array}{ccccccc}
S^{a+b} & \xrightarrow{\phi_{a,b}} & S^a \otimes S^b & \xrightarrow{\Delta \otimes S^b} & (S^a \oplus S^a) \otimes S^b & \xrightarrow{(x \oplus x') \otimes y} & (E \oplus E) \otimes E \\
\Delta \downarrow & & \downarrow \Delta & \swarrow \cong & & \swarrow \cong & \downarrow \nabla \otimes E \\
S^{a+b} \oplus S^{a+b} & \xrightarrow[\phi_{a,b} \oplus \phi_{a,b}]{} & (S^a \otimes S^b) \oplus (S^a \otimes S^b) & \xrightarrow[(x \otimes y) \oplus (x \otimes y)]{} & (E \otimes E) \oplus (E \otimes E) & \xrightarrow[\nabla]{} & E \otimes E \xrightarrow{\mu} E
\end{array}$$
  

$$\begin{array}{ccccccc}
S^{a+b} & \xrightarrow{\phi_{a,b}} & S^a \otimes S^b & \xrightarrow{S^a \otimes \Delta} & S^b \otimes (S^b \oplus S^b) & \xrightarrow{x \otimes (y \oplus y')} & E \otimes (E \oplus E) \\
\Delta \downarrow & & \downarrow \Delta & \swarrow \cong & & \swarrow \cong & \downarrow E \otimes \nabla \\
S^{a+b} \oplus S^{a+b} & \xrightarrow[\phi_{a,b} \oplus \phi_{a,b}]{} & (S^a \otimes S^b) \oplus (S^a \otimes S^b) & \xrightarrow[(x \otimes y) \oplus (x \otimes y)]{} & (E \otimes E) \oplus (E \otimes E) & \xrightarrow[\nabla]{} & E \otimes E \xrightarrow{\mu} E
\end{array}$$

The unlabeled isomorphisms are those given by the fact that  $- \otimes -$  is additive in each variable (since  $S\mathcal{H}$  is tensor triangulated). Commutativity of the left squares is naturality of  $\Delta : X \rightarrow X \oplus X$  in an additive category. Commutativity of the rest of the diagram follows again from the fact that  $- \otimes -$  is an additive functor in each variable. Hence, by functoriality of  $- \otimes -$ , these diagrams tell us that  $(x + x') \cdot y = x \cdot y + x' \cdot y$  and  $x \cdot (y + y') = x \cdot y + x \cdot y'$ , respectively.  $\square$

**Proposition D.4.** *For all  $a, b \in A$  there exists an element  $\theta_{a,b} \in \pi_0(S) = [S, S]$  (determined by choice of coherent family  $\{\phi_{a,b}\}$ ) such that given any commutative monoid object  $(E, \mu, e)$  in  $S\mathcal{H}$ , the  $A$ -graded ring structure on  $\pi_*(E)$  ([Proposition 2.5](#)) has a commutativity formula given by*

$$x \cdot y = y \cdot x \cdot (e \circ \theta_{a,b})$$

for all  $x \in \pi_a(E)$  and  $y \in \pi_b(E)$ . In particular,  $\theta_{a,b} \in \text{Aut}(S)$  is the composition

$$S \xrightarrow{\cong} S^{-a-b} \otimes S^a \otimes S^b \xrightarrow{S^{-a-b} \otimes \tau} S^{-a-b} \otimes S^b \otimes S^a \xrightarrow{\cong} S,$$

where the outermost maps are the unique maps specified by [Remark 2.4](#).

*Proof.* Let  $\phi_{a,b}$ ,  $E$ ,  $x$ , and  $y$  as in the statement of the proposition. Now consider the following diagram

$$\begin{array}{ccccc}
S^{a+b} & \xrightarrow{\phi_{a,b}} & S^a \otimes S^b & \xrightarrow{x \otimes y} & E \otimes E \\
\downarrow \phi_{b,a}^{-1} \circ \tau \circ \phi_{a,b} & & \downarrow \tau & & \downarrow \tau \\
S^{a+b} & \xrightarrow{\phi_{b,a}} & S^b \otimes S^a & \xrightarrow{y \otimes x} & E \otimes E
\end{array}
\begin{array}{c}
\searrow \mu \\
\nearrow \mu
\end{array}
\rightarrow E$$

The left square commutes by definition. The middle square commutes by naturality of the symmetry isomorphism. Finally, the right square commutes by commutativity of  $E$ . Unravelling definitions, we have shown that under the product on  $\pi_*(E)$  induced by the  $\phi_{a,b}$ 's,

$$x \cdot y = (y \cdot x) \circ (\phi_{b,a}^{-1} \circ \tau \circ \phi_{a,b}).$$

Thus, in order to show the desired result it further suffices to show that

$$(y \cdot x) \circ (\phi_{b,a}^{-1} \circ \tau \circ \phi_{a,b}) = y \cdot x \cdot (e \circ \theta_{a,b}).$$

Consider the following diagram:

$$\begin{array}{ccc}
S^{a+b} & \xrightarrow{\phi_{a,b}} & S^a \otimes S^b \\
\cong \downarrow & \nearrow \cong & \downarrow \tau \\
S^b \otimes S^a \otimes S^{-a-b} \otimes S^a \otimes S^b & & S^b \otimes S^a \\
S^b \otimes S^a \otimes S^{-a-b} \otimes \tau \downarrow & \nearrow \cong & \downarrow \phi_{b,a}^{-1} \\
S^b \otimes S^a \otimes S^{-a-b} \otimes S^b \otimes S^a & & S^{a+b} \\
\cong \downarrow & & \downarrow \phi_{b,a} \\
S^b \otimes S^a \otimes S & \xrightarrow{\cong} & S^b \otimes S^a \\
\downarrow x \otimes y \otimes e & \searrow y \otimes x \otimes S & \downarrow y \otimes x \\
E \otimes E \otimes E & \xrightarrow{E \otimes E \otimes e} E \otimes E \otimes S & \downarrow \rho \\
& \xrightarrow{E \otimes \mu} & \downarrow \mu \\
& & E \otimes E \\
& \xrightarrow{\mu} & \downarrow \mu \\
& & E
\end{array}$$

Here we are suppressing associators from the notation, and any map simply labelled  $\cong$  is an appropriate composition of copies of  $\phi_{a,b}$ 's, associators, and their inverses, so that each of these maps are necessarily unique by [Remark 2.4](#). The top triangle commutes by coherence for the  $\phi_{a,b}$ 's. The parallelogram commutes by naturality of  $\tau$  and coherence of the  $\phi_{a,b}$ 's. The trapezoid commutes again by coherence for the  $\phi_{a,b}$ 's. The middle right large triangle commutes by naturality of the unitors (and the fact that  $S^b \otimes \phi_{a,0}$  coincides with the unitor  $S^b \otimes S^a \otimes S \rightarrow S^b \otimes S^a$ ). The middle left triangle commutes by functoriality of  $- \otimes -$ . The middle triangle commutes by unitality of  $\mu$ . Finally, the bottom rectangle commutes by associativity of  $\mu$ . Hence, by unravelling definitions and applying functoriality of  $- \otimes -$ , we get that the top composition is  $(y \cdot x) \circ (\phi_{b,a}^{-1} \circ \tau \circ \phi_{a,b})$ , while the bottom composition is  $y \cdot x \cdot (e \circ \theta_{a,b})$ , so they are equal as desired.  $\square$

**Proposition D.5.** *Given  $a \in A$ , we have  $\theta_{0,a} = \theta_{a,0} = \text{id}_S$ .*

*Proof.* Recall  $\theta_{a,0}$  is the composition

$$S \xrightarrow{\phi_{-a,a}} S^{-a} \otimes S^a \xrightarrow{S^{-a} \otimes \phi_{a,0}} S^{-a} \otimes (S^a \otimes S) \xrightarrow{S^{-a} \otimes \tau} S^{-a} \otimes (S \otimes S^a) \xrightarrow{S^{-a} \otimes \phi_{0,a}^{-1}} S^{-a} \otimes S^a \xrightarrow{\phi_{-a,a}^{-1}} S$$

By the coherence theorem for symmetric monoidal categories and the fact that  $\phi_{a,0}$  and  $\phi_{0,a}$  coincide with the unitors, we have that the composition

$$S^a \xrightarrow{\phi_{a,0} = \rho_{S^a}^{-1}} S^a \otimes S \xrightarrow{\tau} S \otimes S^a \xrightarrow{\phi_{0,a}^{-1} = \lambda_{S^a}} S^a$$

is precisely the identity map, so by functoriality of  $- \otimes -$ , we have that  $\theta_{a,0}$  is the composition

$$S \xrightarrow{\phi_{-a,a}} S^{-a} \otimes S^a \xrightarrow{\cong} S^{-a} \otimes S^a \xrightarrow{\phi_{-a,a}^{-1}} S,$$

so  $\theta_{a,0} = \text{id}_S$ , meaning

$$x \cdot y = y \cdot x \cdot (e \circ \theta_{a,0}) = y \cdot x \cdot e = y \cdot x,$$

where the last equality follows by the fact that  $e$  is the unit for the multiplication on  $\pi_*(E)$ . An entirely analogous argument yields that  $\theta_{0,a} = \text{id}_S$ .  $\square$



**Proposition D.6.** *Given some  $a \in A$ , the functors  $\Sigma^a$  and  $\Sigma^{-a}$  canonically form an adjoint equivalence of  $\mathcal{SH}$ .*

*Proof.* Let  $X, Y \in \mathcal{SH}$ . By [3, Lemma 3.2], in order to show  $\Sigma^a$  and  $\Sigma^{-a}$  are adjoint equivalences, it suffices to construct natural isomorphisms  $\eta : \text{Id}_{\mathcal{SH}} \Rightarrow \Sigma^{-a} \circ \Sigma^a$  and  $\varepsilon : \Sigma^a \circ \Sigma^{-a} \Rightarrow \text{Id}_{\mathcal{SH}}$  such that for all  $X$  in  $\mathcal{SH}$ , the following diagram commutes:

$$(1) \quad \begin{array}{ccc} \Sigma^a X & \xrightarrow{(\Sigma^a \eta)_X} & \Sigma^a \Sigma^{-a} \Sigma^a X \\ & \searrow & \downarrow (\varepsilon \Sigma^a)_X \\ & & \Sigma^a X \end{array}$$

Given an object  $X$  in  $\mathcal{SH}$ , define  $\eta_X : X \rightarrow \Sigma^{-a} \Sigma^a X = S^{-a} \otimes S^a \otimes X$  to be the composition

$$X \xrightarrow{\lambda_X^{-1}} S \otimes X \xrightarrow{\phi_{-a,a} \otimes X} S^{-a} \otimes S^a \otimes X.$$

Clearly this is an isomorphism. To see this is natural, let  $f : X \rightarrow Y$  in  $\mathcal{SH}$ . Then consider the following diagram:

$$\begin{array}{ccccc} X & \xrightarrow{\lambda_X^{-1}} & S \otimes X & \xrightarrow{\phi_{-a,a} \otimes X} & S^{-a} \otimes S^a \otimes X \\ f \downarrow & & \downarrow S \otimes f & & \downarrow S^{-a} \otimes S^a \otimes f \\ Y & \xrightarrow{\lambda_Y^{-1}} & S \otimes Y & \xrightarrow{\phi_{-a,a} \otimes Y} & S^{-a} \otimes S^a \otimes Y \end{array}$$

The left square commutes by naturality of  $\lambda$ . The right square commutes by functoriality of  $- \otimes -$ . Hence  $\eta$  is indeed a natural isomorphism.

On the other hand, given an object  $X$  in  $\mathcal{SH}$ , define  $\varepsilon_X : \Sigma^a \Sigma^{-a} X = S^a \otimes S^{-a} \otimes X \rightarrow X$  to be the composition

$$S^a \otimes S^{-a} \otimes X \xrightarrow{\phi_{a,-a}^{-1}} S \otimes X \xrightarrow{\lambda_X} X.$$

Clearly this is an isomorphism. To see it is natural, let  $f : X \rightarrow Y$  in  $\mathcal{SH}$ . Then consider the following diagram:

$$\begin{array}{ccccc} S^a \otimes S^{-a} \otimes X & \xrightarrow{\phi_{a,-a}^{-1} \otimes X} & S \otimes X & \xrightarrow{\lambda_X} & X \\ S^a \otimes S^{-a} \otimes f \downarrow & & S \otimes f \downarrow & & \downarrow f \\ S^a \otimes S^{-a} \otimes Y & \xrightarrow{\phi_{a,-a}^{-1} \otimes Y} & S \otimes Y & \xrightarrow{\lambda_Y} & Y \end{array}$$

The left square commutes by functoriality of  $- \otimes -$ . The right square commutes by naturality of  $\lambda$ . Hence,  $\varepsilon$  is natural.

Finally, let  $X$  be an object in  $\mathcal{SH}$ . Unravelling definitions, by functoriality of  $- \otimes -$ , in order to show that diagram (1) commutes, it suffices to show the following diagram commutes:

$$\begin{array}{ccccc} S^a \otimes X & \xrightarrow{S^a \otimes \lambda_X^{-1}} & S^a \otimes S \otimes X & \xrightarrow{S^a \otimes \phi_{-a,a} \otimes X} & S^a \otimes S^{-a} \otimes S^a \otimes X \\ & \searrow & \swarrow \phi_{a,0} \otimes X & & \downarrow \phi_{a,-a}^{-1} \otimes S^a \otimes X \\ & & & & S \otimes S^a \otimes X \\ & & & & \downarrow \lambda_{S^a \otimes X} \\ & & & & S^a \otimes X \end{array}$$

First, note that by the coherence theorem for monoidal categories,  $\lambda_{S^a \otimes X} = \lambda_{S^a} \otimes X^3$ . And furthermore, recall  $\lambda_{S^a} = \phi_{0,a}^{-1}$ . Hence, the right triangle is precisely the diagram obtained by applying  $-\otimes X$  to the coherence diagram for the  $\phi_{a,b}$ 's, so it commutes. Commutativity of the left triangle follows by the coherence theorem for monoidal categories and the fact that  $\phi_{a,0} = \lambda_{S^a}^{-1}$ . Hence, the diagram commutes, so  $(\Sigma^a, \Sigma^{-a})$  forms an adjoint equivalence of  $\mathcal{SH}$ .  $\square$

**Proposition D.7.** *Let  $X$  and  $Y$  be objects in  $\mathcal{SH}$ . Then the pairing*

$$\pi_*(X) \times \pi_*(Y) \rightarrow \pi_*(X \otimes Y)$$

*sending  $x : S^a \rightarrow X$  and  $y : S^b \rightarrow Y$  to the composition*

$$S^{a+b} \xrightarrow{\phi_{a,b}} S^a \otimes S^b \xrightarrow{x \otimes y} X \otimes Y$$

*is bilinear.*

*Proof.* Let  $a, b \in A$ , and let  $x_1, x_2 : S^a \rightarrow X$  and  $y : S^b \rightarrow Y$ . Then consider the following diagram

$$\begin{array}{ccccc} S^{a+b} & \xrightarrow{\cong} & S^a \otimes S^b & \xrightarrow{\Delta \otimes S^b} & (S^a \oplus S^a) \otimes S^b \\ & & \Delta \downarrow & \swarrow \cong & \downarrow (x_1 \oplus x_2) \otimes y \\ & & (S^a \otimes S^b) \oplus (S^a \otimes S^b) & & (X \oplus X) \otimes Y \\ & & \downarrow (x_1 \otimes y) \oplus (x_2 \otimes y) & \swarrow \cong & \downarrow \nabla \otimes Y \\ & & (X \otimes Y) \oplus (X \otimes Y) & \xrightarrow{\nabla} & X \otimes Y \end{array}$$

The isomorphisms are given by the fact that  $-\otimes -$  is additive in each variable. Both triangles and the parallelogram commute since  $-\otimes -$  is additive. By functoriality of  $-\otimes -$ , the top composition is  $(x_1 + x_2) \cdot y$  and the bottom composition is  $x_1 \cdot y + x_2 \cdot y$ , so they are equal, as desired. An entirely analogous argument yields that  $x \cdot (y_1 + y_2) = x \cdot y_1 + x \cdot y_2$  for  $x \in \pi_*(X)$  and  $y_1, y_2 \in \pi_*(Y)$ .  $\square$

**Proposition D.8** ([4, Proposition 5.11]). *Let  $(E, \mu, e)$  be a monoid object in  $\mathcal{SH}$ . Then for any object  $X$  in  $\mathcal{SH}$ ,  $E_*(X)$  canonically inherits the structure of a left  $A$ -graded  $\pi_*(E)$ -module via the map*

$$\pi_*(E) \times E_*(X) \rightarrow E_*(X)$$

*which given  $a, b \in A$ , sends  $x : S^a \rightarrow E$  and  $y : S^b \rightarrow E \otimes X$  to the composition*

$$x \cdot y : S^{a+b} \cong S^a \otimes S^b \xrightarrow{x \otimes y} E \otimes (E \otimes X) \cong (E \otimes E) \otimes X \xrightarrow{\mu \otimes X} E \otimes X.$$

*Similarly  $X_*(E)$  canonically inherits the structure of a right  $A$ -graded  $\pi_*(E)$ -module via the map*

$$X_*(E) \times \pi_*(E) \rightarrow X_*(E)$$

*which given  $a, b \in A$ , sends  $x : S^a \rightarrow X \otimes E$  and  $y : S^b \rightarrow E$  to the composition*

$$x \cdot y : S^{a+b} \cong S^a \otimes S^b \xrightarrow{x \otimes y} (X \otimes E) \otimes E \cong X \otimes (E \otimes E) \xrightarrow{X \otimes \mu} X \otimes E.$$

*In particular,  $E_*(E)$  is a  $\pi_*(E)$ -bimodule, in the sense that the left and right actions of  $\pi_*(E)$  are compatible, so that given  $y, z \in \pi_*(E)$  and  $x \in E_*(E)$ ,  $y \cdot (x \cdot z) = (y \cdot x) \cdot z$ .*

*Proof.* First we show that the map  $\pi_*(E) \times E_*(X) \rightarrow E_*(X)$  endows  $E_*(X)$  with the structure of a left  $\pi_*(E)$ -module. Let  $a, b, c \in A$  and  $x, x' : S^a \rightarrow E \otimes X$ ,  $y : S^b \rightarrow E$ , and  $z, z' : S^c \rightarrow E$ . Then we wish to show that:

- (1)  $y \cdot (x + x') = y \cdot x + y \cdot x'$ ,
- (2)  $(z + z') \cdot x = z \cdot x + z' \cdot x$ ,

<sup>3</sup>Technically, this equality only holds up to composition with an associator, but we are ignoring such issues.

- (3)  $(zy) \cdot x = z \cdot (y \cdot x)$ ,  
 (4)  $e \cdot x = x$ .

Axioms (1) and (2) follow by the fact that  $E_*(X) = \pi_*(E \otimes X)$  and [Proposition D.7](#). To see (3), consider the diagram:

$$\begin{array}{ccccc}
 S^{a+b+c} & \xrightarrow{\cong} & S^{c+b} \otimes S^a & & \\
 \downarrow \cong & & \downarrow \cong & & \\
 S^c \otimes S^{b+a} & & & & \\
 \downarrow \cong & & & & \\
 S^c \otimes (S^b \otimes S^a) & \xleftarrow{\cong} & (S^c \otimes S^b) \otimes S^a & & \\
 \downarrow z \otimes (y \otimes x) & & \downarrow (z \otimes y) \otimes x & & \\
 E \otimes (E \otimes (E \otimes X)) & \xleftarrow{\cong} & (E \otimes E) \otimes (E \otimes X) & \xrightarrow{\mu \otimes (E \otimes X)} & E \otimes (E \otimes X) \\
 \downarrow \cong & & \uparrow \cong & & \downarrow \cong \\
 & & ((E \otimes E) \otimes E) \otimes X & \xrightarrow{(\mu \otimes E) \otimes X} & (E \otimes E) \otimes X \\
 \downarrow \cong & & \downarrow \cong & & \downarrow \mu \otimes X \\
 E \otimes ((E \otimes E) \otimes X) & \xrightarrow{\cong} & (E \otimes (E \otimes E)) \otimes X & & \\
 \downarrow E \otimes (\mu \otimes X) & & \downarrow (E \otimes \mu) \otimes X & & \\
 E \otimes (E \otimes X) & \xrightarrow{\cong} & (E \otimes E) \otimes X & \xrightarrow{\mu \otimes X} & E \otimes X
 \end{array}$$

The top square commutes by coherence of the isomorphisms  $S^{a+b} \cong S^a \otimes S^b$  ([Definition 2.2](#)). The second square from the top on the left commutes by naturality of the associators. The square below that commutes by the coherence axiom for the associators in a monoidal category. The bottom left square commutes again by naturality of the associator isomorphisms. The bottom right square commutes by associativity for  $\mu$  and functoriality of  $- \otimes X$ . Finally, the square above that commutes again by naturality of the associator isomorphism. By functoriality of  $- \otimes -$ , the two outside compositions equal  $(z \cdot y) \cdot x$  on the top and  $z \cdot (y \cdot x)$  on the bottom. Hence, they are equal, as desired.

Next, to see (4), consider the following diagram:

$$\begin{array}{ccccc}
 S^a & \xrightarrow{x} & E \otimes X & & \\
 \downarrow \phi_{0,a} = \lambda_{S^a}^{-1} & & \downarrow \lambda_{E \otimes X} & & \\
 S \otimes S^a & \xrightarrow{S \otimes x} & S \otimes (E \otimes X) & \xrightarrow{\lambda_{E \otimes X}} & E \otimes X \\
 \downarrow e \otimes x & & \downarrow e \otimes (E \otimes X) & & \downarrow \mu \otimes X \\
 & & (S \otimes E) \otimes X & \xrightarrow{(e \otimes E) \otimes X} & (E \otimes E) \otimes X \\
 & & \uparrow \cong & & \uparrow \mu \otimes X \\
 E \otimes (E \otimes X) & \xrightarrow{\cong} & (E \otimes E) \otimes X & & 
 \end{array}$$

Commutativity of the top trapezoid is naturality of the unitor. Commutativity of the left triangle is functoriality of  $- \otimes -$ . Commutativity of the bottom triangle is naturality of the associator isomorphisms. Commutativity of the right triangle is unitality of  $\mu$  and functoriality of  $- \otimes X$ . Finally, commutativity of the remaining crooked triangle follows by coherence for monoidal categories. The two outer compositions  $S^a \rightarrow E \otimes X$  are  $x$  and  $e \cdot x$ , and by commutativity they are necessarily equal.

Thus, we have shown that the indicated map does indeed endow  $E_*(X)$  with the structure of a left  $\pi_*(E)$ -module. Showing that  $X_*(E)$  has the structure of a right  $\pi_*(E)$ -module is entirely analogous.

It remains to show that  $E_*(E)$  is a bimodule. Let  $x : S^a \rightarrow E$ ,  $y : S^b \rightarrow E \otimes E$ , and  $z : S^c \rightarrow E$ , and consider the following diagram:

$$\begin{array}{ccccc}
 & & & E \otimes E \otimes E & \\
 & & \mu \otimes E \otimes E \nearrow & & \downarrow E \otimes \mu \\
 S^{a+b+c} & \xrightarrow{\cong} & S^a \otimes S^b \otimes S^c & \xrightarrow{x \otimes y \otimes z} & E \otimes E \otimes E \otimes E & \xrightarrow{\mu \otimes \mu} & E \otimes E \\
 & & & & \downarrow E \otimes E \otimes \mu & & \uparrow \mu \otimes E \\
 & & & & E \otimes E \otimes E & & 
 \end{array}$$

We are suppressing the associators here. Commutativity follows by functoriality of  $- \otimes -$ , which also tells us that the two outside compositions are  $(x \cdot y) \cdot z$  (on top) and  $x \cdot (y \cdot z)$  (on bottom). Hence they are equal, as desired.  $\square$

**Proposition D.9** ([5, Proposition 2.2]). *Let  $(E, \mu, e)$  be a commutative monoid object in  $\mathcal{SH}$  and let  $X$  be any spectrum. Then the assignment*

$$E_*(E) \times E_*(X) \rightarrow E_*(E \otimes X)$$

*which sends  $x : S^a \rightarrow E \otimes E$  and  $y : S^{c,d} \rightarrow E \otimes X$  to the composition*

$$x \cdot y : S^{a+c,b+d} \cong S^{a,b} \otimes S^{c,d} \xrightarrow{x \otimes y} E \otimes E \otimes E \otimes X \xrightarrow{E \otimes \mu \otimes X} E \otimes E \otimes X$$

*induces a homomorphism of bigraded abelian groups*

$$E_*(E) \otimes_{\pi_*(E)} E_*(X) \rightarrow E_*(E \otimes X)$$

*(where here  $E_*(E)$  has a right  $\pi_*(E)$ -module structure and  $E_*(X)$  has a left  $\pi_*(E)$ -module structure as specified by Proposition D.8). Furthermore, if  $X$  is cellular and  $E$  is a cellular flat commutative ring spectrum (Definition 2.8, Definition 2.10), then this map is an isomorphism.*

*Proof.* First we show that this map is  $\pi_*(E)$ -bilinear. By the identifications  $E_*(E) = \pi_*(E \otimes E)$ ,  $E_*(X) = \pi_*(E \otimes X)$ , and  $E_*(E \otimes X) = \pi_*(E \otimes E \otimes X)$ , we know this map commutes with addition of maps in each argument by Proposition D.7. Now, let  $a, b, c \in \mathbb{Z}^2$ ,  $x : S^a \rightarrow E \otimes E$ ,  $y : S^b \rightarrow E \otimes X$ , and  $z : S^c \rightarrow E$ . Then we wish to show  $xz \cdot y = x \cdot zy$ . Consider the following diagram

$$\begin{array}{ccccc}
 & & & E \otimes E \otimes E \otimes X & \\
 & & E \otimes E \otimes \mu \otimes X \uparrow & & \searrow E \otimes \mu \otimes X \\
 S^{a+b+c} & \xrightarrow{\cong} & S^a \otimes S^b \otimes S^c & \xrightarrow{x \otimes y \otimes z} & E \otimes E \otimes E \otimes E \otimes X & \xrightarrow{\mu \otimes \mu} & E \otimes E \otimes X \\
 & & & & \downarrow E \otimes \mu \otimes E \otimes X & & \nearrow E \otimes \mu \otimes X \\
 & & & & E \otimes E \otimes E \otimes X & & 
 \end{array}$$

(we have suppressed the associators from the notation). The top left triangle commutes by coherence for the isomorphisms  $S^{a+b} \cong S^a \otimes S^b$ . The middle parallelogram commutes by naturality of the associators. Finally, the bottom right triangle is obtained by applying  $E \otimes - \otimes X$  to the associativity diagram for  $\mu$ , so by functoriality it commutes. Again by functoriality of  $- \otimes -$ , the bottom composition is given by  $(x \cdot z) \cdot y$  and the top composition is  $x \cdot (zy)$ , so we have the desired equality.

It remains to show that if  $X$  is cellular and  $E$  is cellular flat commutative, then this map is an isomorphism.  $\square$

finish or cite

In the following definition, let  $\varepsilon : E_*(E) \rightarrow \pi_*(E)$  be the map which sends some  $\alpha : S^a \rightarrow E \otimes E$  to the composition

$$S^a \xrightarrow{\alpha} E \otimes E \xrightarrow{\mu} E.$$

Also define  $\Psi : E_*(E) \rightarrow E_*(E) \otimes_{\pi_*(E)} E_*(E)$  to be the map which factors as

$$E_*(E) \rightarrow E_*(E \otimes E) \xrightarrow{\cong} E_*(E) \otimes_{\pi_*(E)} E_*(E)$$

where the second arrow is the isomorphism prescribed by [Proposition D.9](#), and the first arrow sends a class  $\alpha : S^a \rightarrow E \otimes E$  to the composition

$$S^a \xrightarrow{\alpha} E \otimes E \cong E \otimes S \otimes E \xrightarrow{E \otimes e \otimes E} E \otimes E \otimes E.$$

**Lemma D.10** ([5, Proposition 2.30, 2.33]). *Let  $E$  be a flat commutative ring spectrum, and let  $X$  and  $Y$  be spectra such that  $E_{**}(X)$  is a projective module over  $\pi_{**}(E)$ . Then for all  $s \geq 0$  and  $t, w \in \mathbb{Z}$ , there is an isomorphism*

$$\Phi : [X, E \wedge Y]_{t,w} \rightarrow \mathrm{Hom}_{E_{**}(E)}^{t,w}(E_{**}(X), E_{**}(E \wedge Y)),$$

obtained by sending a class  $f : S^{t,w} \wedge X \rightarrow E \wedge Y$  in  $[X, E \wedge Y]_{t,w}$  to the map

$$\Phi_f : E_{*,*}(X) \rightarrow E_{*+t,*+w}(X \wedge Y)$$

sending

$$[S^{a,b} \xrightarrow{g} E \wedge X] \mapsto [S^{a+t,b+w} \cong S^{a,b} \wedge S^{t,w} \xrightarrow{g \wedge S^{t,w}} E \wedge X \wedge S^{t,w} \cong E \wedge S^{t,w} \wedge X \xrightarrow{E \wedge f} E \wedge E \wedge Y].$$

*Proof.* Let  $f : S^{t,w} \wedge X \rightarrow E \wedge Y$ . First we want to show that  $\Phi_f$  is actually an  $E_{**}(E)$ -comodule homomorphism. □

finish

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