0.1. **Construction of the spectral sequence.** In the sections that follow, let (E, μ, e) be a monoid object and X and Y be objects in $\mathcal{SH}$.

Definition 0.1 ([3, Definition 11.3.1])**.** An *E-Adams resolution of Y* $(Y_s, W_s; i, j, k)$ is a diagram of the form

$$\begin{tikzcd}
\cdots \arrow[r] & Y_{s+1} \arrow[r, "i_s"] \arrow[d, "j_{s+1}"'] & Y_s \arrow[r] \arrow[d, "j_s"] & \cdots \arrow[r] & Y_2 \arrow[r, "i_1"] & Y_1 \arrow[r, "i_0"] \arrow[d, "j_1"] & Y_0 \arrow[d, "j_0"] \\
& W_{s+1} & W_s \arrow[ul, dashed, "k_s"] & & & Y_1 \arrow[ul, dashed, "k_1"] & W_0 \arrow[ul, dashed, "k_0"]
\end{tikzcd}$$

such that the dashed arrows really stand for (degree $-\mathbf{1}$) maps $k_s : W_s \to \Sigma Y_{s+1}$, and

(1) There is an isomorphism $Y_0 \cong Y$;

(2) for each s, the sequence
$$Y_{s+1} \xrightarrow{i_s} Y_s \xrightarrow{j_s} W_s \xrightarrow{k_s} \Sigma Y_{s+1}$$
is a distinguished triangle;

(3) W_s is isomorphic to $E \otimes T_s$ for some object T_s in $\mathcal{SH}$;

(4) $E_*(i_s) : E_*(Y_{s+1}) \to E_*(Y_s)$ is zero.

It turns out that every object Y in $\mathcal{SH}$ admits a *canonical* E-Adams resolution:

Definition 0.2. Let $\overline{E}$ be the fiber of the unit map $e : S \to E$ (**??**). Let $Y_0 := Y$ and $W_0 := E \otimes Y$. For $s > 0$, define
$$Y_s := \overline{E}^s \otimes Y, \qquad W_s := E \otimes Y_s = E \otimes \overline{E}^s \otimes Y,$$
where $\overline{E}^s$ denotes the s-fold tensor product $\overline{E} \otimes \cdots \otimes \overline{E}$. Then we get fiber sequences
$$Y_{s+1} \xrightarrow{i_s} Y_s \xrightarrow{j_s} W_s \xrightarrow{k_s} \Sigma Y_{s+1}$$
obtained by applying $- \otimes Y_s$ to the fiber sequence
$$\overline{E} \to S \xrightarrow{e} E \to \Sigma \overline{E}.$$
We can splice these sequences together to get the *canonical Adams-resolution of Y*:

$$\begin{tikzcd}
\cdots \arrow[r] & Y_3 \arrow[r, "i_2"] \arrow[d, "j_3"] & Y_2 \arrow[r, "i_1"] \arrow[d, "j_2"] & Y_1 \arrow[r, "i_0"] \arrow[d, "j_1"] & Y_0 = Y \arrow[d, "j_0"] \\
& W_3 & W_2 \arrow[ul, dashed, "k_2"] & W_1 \arrow[ul, dashed, "k_1"] & W_0 \arrow[ul, dashed, "k_0"]
\end{tikzcd}$$

Proposition 0.3. *The "canonical E-Adams resolution of Y" from Definition 0.2 is in fact an E-Adams resolution of Y, in the sense of Definition 0.1*

Proof. By construction, the only thing we need to check is that $E_*(i_s) : E_*(Y_{s+1}) \to E_*(Y_s)$ is zero. First, note that since
$$Y_{s+1} \xrightarrow{i_s} Y_s \xrightarrow{j_s} W_s \xrightarrow{k_s} \Sigma Y_{s+1}$$
is a distinguished triangle and $\mathcal{SH}$ is tensor triangulated, there is a distinguished triangle of the form
$$E \otimes Y_{s+1} \xrightarrow{E \otimes i_s} E \otimes Y_s \xrightarrow{E \otimes j_s} E \otimes W_s \to \Sigma(E \otimes Y_{s+1}).$$
Thus, applying $\pi_*(-) \cong [S, -]_*$ to the triangle yields that the following sequence is exact (see **??** for details):
$$E_*(Y_{s+1}) \xrightarrow{E_*(i_s)} E_*(Y_s) \xrightarrow{E_*(j_s)} E_*(W_s).$$

Now, it is straightforward to verify by how it is constructed that j_s is the map $e \otimes Y_s : Y_s \to E \otimes Y_s = W_s$. Thus, by unitality of μ, we have that $E \otimes j_s : E \otimes Y_s \to E \otimes W_s$ is a split monomorphism, with right inverse $\mu \otimes Y_s : E \otimes W_s = E \otimes E \otimes Y_s \to E \otimes Y_s$. Then since any functor preserves split monomorphisms, it follows that $E_*(j_s) = \pi_*(E \otimes j_s)$ is likewise a split monomorphism, so that in particular $E_*(j_s)$ is injective. Thus $\operatorname{im} E_*(i_s) = \ker E_*(j_s) = 0$, so that i_s is indeed the zero map, as desired. $\square$

Now, by applying $[X, -]_*$ to an E-Adams resolution of Y, we get an associated unrolled exact couple, and thus a spectral sequence:

Definition 0.4. Suppose we have an E-Adams resolution of Y (Definition 0.1):

$$\begin{tikzcd}
\cdots \arrow[r] & Y_3 \arrow[rr, "i_2"] \arrow[d, "j_3"] & & Y_2 \arrow[rr, "i_1"] \arrow[d, "j_2"] & & Y_1 \arrow[rr, "i_0"] \arrow[d, "j_1"] & & Y_0 = Y \arrow[d, "j_0"] \\
& W_3 & & W_2 \arrow[ull, dashed, "k_2"] & & W_1 \arrow[ull, dashed, "k_1"] & & W_0 \arrow[ull, dashed, "k_0"]
\end{tikzcd}$$

We can extend this diagram to the right by setting $Y_s = Y$, $W_s = 0$, and $i_s = \mathrm{id}_Y$ for $s < 0$. Then we may apply the functor $[X, -]_*$, and by **??**, we obtain the following A-graded unrolled exact couple (**??**):

$$\begin{tikzcd}
\cdots \arrow[r] & {[X, Y_{s+2}]_*} \arrow[r, "i_{s+1}"] \arrow[d, "j_{s+2}"] & {[X, Y_{s+1}]_*} \arrow[r, "i_s"] \arrow[d, "j_{s+1}"] & {[X, Y_s]_*} \arrow[r, "i_{s-1}"] \arrow[d, "j_s"] & {[X, Y_{s-1}]_*} \arrow[r] \arrow[d, "j_{s-1}"] & \cdots \\
& {[X, W_{s+2}]_*} & {[X, W_{s+1}]_*} \arrow[ul, "\partial_{s+1}"] & {[X, W_s]_*} \arrow[ul, "\partial_s"] & {[X, W_{s-1}]_*} \arrow[ul, "\partial_{s-1}"] &
\end{tikzcd}$$

where here we are being abusive and writing $i_s : [X, Y_{s+1}]_* \to [X, Y_s]_*$ and $j_s : [X, Y_s]_* \to [X, W_s]_*$ to denote the pushforward maps induced by $i_s : Y_{s+1} \to Y_s$ and $j_s : Y_s \to W_s$, respectively. Each i_s, j_s, and ∂_s are A-graded homomorphisms of degrees 0, 0, and $-\mathbf{1}$, respectively.

By **??**, we may associate a $\mathbb{Z} \times A$-graded spectral sequence $r \mapsto (E_r^{*,*}(X, Y), d_r)$ to the above A-graded unrolled exact couple, where d_r has $\mathbb{Z} \times A$-degree $(r, -\mathbf{1})$. We call this spectral sequence the *E-Adams spectral sequence for the computation of* $[X, Y]_*$.

For those who would rather not lose themselves in the appendix, we give a brief unravelling of how **??** applies to the present situation. Given some $s \in \mathbb{Z}$ and some $r \geq 1$, we may define the following A-graded subgroups of $[X, W_s]_*$:

$$Z_r^s := \partial_s^{-1}(\operatorname{im}[i^{(r-1)} : [X, Y_{s+r}]_* \to [X, Y_{s+1}]_*])$$

and

$$B_r^s := j_s(\ker[i^{(r-1)} : [X, Y_s]_* \to [X, Y_{s-r+1}]_*]),$$

where we adopt the convention that $i^{(0)}$ is simply the identity. This yields an infinite sequence of inclusions

$$0 = B_1^s \subseteq B_2^s \subseteq B_3^s \subseteq \cdots \subseteq \operatorname{im} j_s = \ker \partial_s \subseteq \cdots \subseteq Z_3^s \subseteq Z_2^s \subseteq Z_1^s = [X, W_s]_*.$$

Then for $r \geq 1$, we define E_r^s to be the A-graded quotient group

$$E_r^s := Z_r^s / B_r^s.$$

Thus taking the direct sum of all the E_r^s's yields the r^{th} page of the spectral sequence

$$E_r := \bigoplus_{s \in \mathbb{Z}} E_r^s,$$

which is a $\mathbb{Z} \times A$-graded abelian group.

The differential $d_r : E_r \to E_r$ is a map of $\mathbb{Z} \times A$-degree $(r, \mathbf{1})$, and is constructed as follows: an element of $E_r^s = Z_r^s / B_r^s$ is a coset represented by some $x \in Z_r^s$, so that $\partial_s(x) = i^{(r-1)}(y)$ for some $y \in [X, Y_{s+r}]_*$. Then we define $d_r([x])$ to be the coset $[j_{s+r}(y)]$ in Z_r^{s+r}/B_r^{s+r}.

In the case $r = 1$, since $B_1^s = 0$ and $Z_1^s = [X, W_s]_*$, we have that $E_1^s = [X, W_s]_*$, and given some $x \in E_1^s = [X, W_s]_*$, the differential d_1 is given by $d_1(x) = j_{s+1}(\partial_s(x))$, so that $d_1 = j \circ \partial$. Furthermore, since the unrolled exact couple which yields the spectral sequence vanishes on its negative terms, we hav that $E_r^{s,a}(X, Y) = 0$ for $s < 0$.

In **??**, it is shown in explicit detail that all of these definitions make sense and are well-defined. In particular, it is shown that the differentials are well-defined A-graded homomorphisms, that $d_r \circ d_r = 0$, and that

$$\ker d_r^s / \operatorname{im} d_r^s = \frac{Z_{r+1}^s / B_r^s}{B_{r+1}^s / B_r^s} \cong Z_{r+1}^s / B_{r+1}^s = E_{r+1}^s.$$

Note, we have called the above spectral sequence *the* E-Adams spectral sequence for the computation of $[X, Y]_*$, even though it was constructed in terms of an E-Adams resolution for Y. We would like to show that, from the E_2-page onwards, that this spectral sequence is independent (up to isomorphism) of the chosen E-Adams resolution of Y. To start, we prove the following proposition:

Proposition 0.5. *[3, Proposition 11.4.1] Suppose we have E-Adams resolutions (Definition 0.1) $(Y_s, W_s; i, j, k)$ and $(Y_s', W_s'; i', j', k')$ of objects Y and Y' in $\mathcal{SH}$, respectively. Then any arrow $f : Y \to Y'$ in $\mathcal{SH}$ induces a homomorphism of unrolled exact couples $(Y_s, W_s; i, j, k) \to (Y_s', W_s'; i', j', k')$ (**??**), thus, an induced homomorphism of associated spectral sequences by **??**.*

Proof. First we need maps $f_s : Y_s \to Y_s'$ and $g_s : W_s \to W_s'$. To start with, define f_0 to be the composition

$$f_0 : Y_0 \cong Y \xrightarrow{f} Y' \cong Y_0'.$$

Now, by induction, supposing $f_0, g_0, f_1, g_1, \ldots, f_{s-2}, g_{s-2}, f_{s-1}$ have been defined for some $s > 0$, consider the following diagram:

$$\begin{tikzcd}
Y_{s+1} \arrow[dr, "i_s"] & & & & \\
& Y_s \arrow[r, "j_s"] \arrow[d, "f_s"'] & W_s \arrow[r, "k_s"] \arrow[d, dashed, "g_s"] & \Sigma Y_{s+1} \arrow[r, "-\Sigma i_s"] \arrow[d, dashed, "\Sigma f_{s+1}"] & \Sigma Y_s \arrow[d, "\Sigma f_s"] \\
& Y_s' \arrow[r, "j_s'"] & W_s' \arrow[r, "k_s'"] & \Sigma Y_{s+1}' \arrow[r, "-\Sigma i_s'"] & \Sigma Y_s'
\end{tikzcd}$$

Our goal is to construct the dashed arrows so that the diagram commutes. □

Proposition 0.6. *Suppose we have E-Adams resolutions (Definition 0.1) $(Y_s, W_s; i, j, k)$ and $(Y_s', W_s'; i', j', k')$ of objects Y and Y' in $\mathcal{SH}$, respectively. Then given an arrow $f : Y \to Y'$ in $\mathcal{SH}$ such that $E_*(f) : E_*(Y) \to E_*(Y')$ is an isomorphism of A-graded abelian groups, the induced homomorphism of spectral sequences $(E_r(X, Y), d_r) \to (E_r(X, Y'), d_r)$ from Proposition 0.5 is an isomorphism from the E_2-page onwards.*

In particular, the E-Adams spectral sequence for $[X, Y]_*$, from the E_2-page onwards, does not depend on the choice of E-Adams resolution for Y.

Proof. □ todo

As a result of this proposition, we will simply say "the E-Adams spectral sequence for $[X,Y]_*$" to generally refer to any such spectral sequence induced by any E-Adams resolution of Y.

0.2. **The E_2 page.** Now, we would like to characterize the E_2 page of the spectral sequence in terms of something more concrete. Namely, we will characterize the E_2 page in terms of Ext of comodules over the dual E-Steenrod algebra. For a quick review of Ext in an abelian category and derived functors, see **??**. The goal of this subsection will be to prove the following theorem:

Theorem 0.7. *Let (E,μ,e) be a commutative monoid object, and X and Y objects in $\mathcal{SH}$. Suppose further that:*

- *E is flat (**??**) and cellular (**??**),*
- *X is cellular and $E_*(X)$ is a graded projective left $\pi_*(E)$-module (via **??**),*
- *Y is cellular.*

Then the non-vanishing entries of the second page of the E-Adams spectral sequence for the computation of $[X,Y]_*$ (Definition 0.4) are the Ext *groups of A-graded left comodules over the anticommutative Hopf algebroid structure on the dual E-Steenrod algebra (**??**), i.e., we have the following isomorphisms for all $s\in\mathbb{N}$ and $a\in A$:*

$$E_2^{s,a}(X,Y)\cong \operatorname{Ext}^{s,a+\mathbf{s}}_{E_*(E)}(E_*(X),E_*(Y)) := \operatorname{Ext}^{s}_{E_*(E)}(E_*(X),E_{*+a+\mathbf{s}}(Y)).$$

Proof. As we have shown above in Proposition 0.6, from the E_2-page onwards, the E-Adams spectral sequence is indepdent of choice of E-Adams resolution of Y. Thus, in order to characterize the E_2 page as desired, we may assume we are working with the canonical E-Adams resolution $(Y_s,W_s;i,j,k)$ of Y from Definition 0.2.

By Proposition 0.12 below, for each $s\in\mathbb{N}$ and $a\in A$, $E_2^{s,a}(X,Y)$ is isomorphic to the s^{th} cohomology group of the cochain complex obtained by applying $F:=\operatorname{Hom}^{a+\mathbf{s}}_{E_*(E)}(E_*(X),-)$ to the complex

$$0\longrightarrow E_*(W_0)\xrightarrow{E_*(\delta_0)}E_*(\Sigma W_1)\xrightarrow{E_*(\delta_1)}E_*(\Sigma^2W_2)\xrightarrow{E_*(\delta_2)}E_*(\Sigma^3W_3)\longrightarrow\cdots.$$

Furthermore, by Lemma 0.10, this complex is an F-acyclic resolution of $E_*(Y)$ (**??**). Thus, since the category of $E_*(E)$-comodules is an abelian category with enough injectives (**??**), we have by **??** that

$$E_2^{s,a}(X,Y)\cong R^s\operatorname{Hom}^{a+\mathbf{s}}_{E_*(E)}(E_*(X),-)(E_*(Y))=\operatorname{Ext}^{s,a+\mathbf{s}}(E_*(X),E_*(Y)),$$

as desired. ☐

We leave it to the reader to unravel what the differential d_2 corresponds to under this identification.

Definition 0.8. Given some (nonnegative integer) $n\in\mathbb{N}$, define natural isomorphisms $\nu_X^n:\Sigma^{\mathbf{n}}X\to\Sigma^nX$ inductively, by setting $\nu_X^0:=\lambda_X$, $\nu_X^1:=\nu_X^{-1}$, and supposing ν_X^{n-1} has been defined for some $n>1$, define ν_X^n to be the composition

$$\nu_X^n:\Sigma^{\mathbf{n}}X=S^{\mathbf{n}}\otimes X\xrightarrow{\phi_{\mathbf{n-1},\mathbf{1}}\otimes X}S^{\mathbf{n-1}}\otimes S^{\mathbf{1}}\otimes X\xrightarrow{S^{\mathbf{n-1}}\otimes\nu_X^{-1}}S^{\mathbf{n-1}}\Sigma X\xrightarrow{\nu_{\Sigma X}^{\mathbf{n-1}}}\Sigma^nX.$$

By induction, naturality of ν, and functoriality of $-\otimes-$, these isomorphisms are clearly natural in X.

Lemma 0.9. *Let (E,μ,e) be a monoid object and X and Y objects in $\mathcal{SH}$. Further suppose E and Y are cellular. Then for all $s\in\mathbb{Z}$, the objects Y_s and W_s from the canonical E-Adams resolution of Y (Definition 0.2) are cellular.*

Proof. Unravelling definitions, for $s < 0$, $W_s = 0$ and $Y_s = Y$, which are both cellular.[1] For $s \geq 0$, we have $W_s = E \otimes Y_s$, so that by cellularity of E and **??**, it suffices to show that Y_s is cellular for $s \geq 0$. We know $Y_0 = Y$ is cellular by definition. For $s > 0$, Y_s is the tensor product $\overline{E}^s \otimes Y$, where $\overline{E}$ fits into the distinguished triangle

$$\overline{E} \to S \xrightarrow{e} E \to \Sigma\overline{E}.$$

By the definition of cellularity, $\overline{E}$ is cellular since S and E are. Thus, by the aforementioned lemma, $\overline{E}^s \otimes Y$ is cellular by **??**, as it is a tensor product of cellular objects in $\mathcal{SH}$. □

Lemma 0.10. *Let (E, μ, e) be a flat (**??**) and cellular (**??**) commutative monoid object and X and Y cellular objects in $\mathcal{SH}$, and define Y_s, W_s as in Definition 0.2. In particular, for each $s \in \mathbb{Z}$, we have distinguished triangles*

$$Y_{s+1} \xrightarrow{i_s} Y_s \xrightarrow{j_s} W_s \xrightarrow{k_s} \Sigma Y_{s+1}.$$

*Then if $E_*(X)$ is a graded projective (**??**) left $\pi_*(E)$-module (via **??**) then the sequence*

$$0 \to E_*(Y) \xrightarrow{E_*(j_0)} E_*(W_0) \xrightarrow{E_*(\delta_0)} E_*(\Sigma W_1) \xrightarrow{E_*(\delta_1)} E_*(\Sigma^2 W_2) \xrightarrow{E_*(\delta_2)} E_*(\Sigma^3 W_3) \to \cdots$$

*is an F-acyclic resolution (**??**) of $E_*(Y)$ in $E_*(E)$-$\mathbf{CoMod}^A$ for*

$$F = \mathrm{Hom}^a_{E_*(E)}(E_*(X), -)$$

for all $a \in A$, where δ_s is the composition

$$\Sigma^s W_s \xrightarrow{\Sigma^s k_s} \Sigma^{s+1} Y_{s+1} \xrightarrow{\Sigma^{s+1} j_{s+1}} \Sigma^{s+1} W_{s+1}.$$

Proof. By Lemma 0.9, each W_s is cellular, so that furthermore $\Sigma^s W_s \cong S^{\mathbf{s}} \otimes W_s$ is cellular for each $s \geq 0$, by **??**. Thus, the sequence does indeed live in $E_*(E)$-$\mathbf{CoMod}^A$ by **??**, as desired. Next, we claim that $E_*(\Sigma^s W_s)$ is an F-acyclic object for each $s \geq 0$, i.e., that

$$\mathrm{Ext}^{n,a}_{E_*(E)}(E_*(X), E_*(\Sigma^s W_s)) = \mathrm{Ext}^n_{E_*(E)}(E_*(X), E_{*+a}(\Sigma^s W_s)) = 0$$

for all $n > 0$, $s \geq 0$, and $a \in A$. Note that we have an A-graded isomorphism of left $E_*(E)$-comodules:

$$\begin{tikzcd}
E_*(E) \otimes_{\pi_*(E)} E_{*+a}(\Sigma^s Y_s) \arrow[r, equal] & E_*(E) \otimes_{\pi_*(E)} E_{*+a}(\Sigma^s Y_s) \arrow[d, "\Phi_{E,\Sigma^s Y_s}"] & \\
& E_*(E \otimes \Sigma^s Y_s) \arrow[d, "E_*(E \otimes (\nu^s_{Y_s})^{-1})"] & \\
& E_*(E \otimes S^{\mathbf{s}} \otimes Y_s) \arrow[d, "E_*(\tau \otimes Y_s)"] & \\
& E_*(S^{\mathbf{s}} \otimes E \otimes Y_s) \arrow[d, "E_*(\nu^s_{E \otimes Y_s})"] & \\
& E_*(\Sigma^s(E \otimes Y_s)) \arrow[r, equal] & E_*(\Sigma^s W_s)
\end{tikzcd}$$

where $\Phi_{E,\Sigma^s Y}$ is an A-graded isomorphism of abelian groups by **??**, and furthermore an isomorphism of $E_*(E)$-comodules by **??**. Every other arrow is an isomorphism of $E_*(E)$-comodules by functoriality of $E_*(-) : \mathcal{SH}$-$\mathbf{Cell} \to E_*(E)$-$\mathbf{CoMod}^A$. Thus, since $E_*(\Sigma^s W_s)$ is isomorphic to

[1] 0 is cellular because it is the cofiber of the identity on S by axiom TR1 for a triangulated category (**??**), i.e., there is a distinguished triangle $S \to S \to 0 \to \Sigma S$.

$E_*(E) \otimes_{\pi_*(E)} E_{*+a}(\Sigma^s Y_s)$ in $E_*(E)$-$\mathbf{CoMod}^A$, and in particular since $\mathrm{Ext}^n_{E_*(E)}(E_*(X), -)$ is a functor, we have

$$\mathrm{Ext}^n_{E_*(E)}(E_*(X), E_{*+a}(\Sigma^s W_s)) \cong \mathrm{Ext}^n_{E_*(E)}(E_*(X), E_*(E) \otimes_{\pi_*(E)} E_{*+a}(\Sigma^s Y_s)).$$

Yet, $E_*(E) \otimes_{\pi_*(E)} E_{*+a}(\Sigma^s Y_s)$ is a co-free $E_*(E)$-comodule, in which case since $E_*(X)$ is graded projective as an object in $\pi_*(E)$-$\mathbf{Mod}^A$, we have that

$$\mathrm{Ext}^{n,a}_{E_*(E)}(E_*(X), E_*(E) \otimes_{\pi_*(E)} E_{*+a}(\Sigma^s Y_s)) = 0,$$

by **??**.

Finally, it remains to show that the sequence is exact. To that end, first note that by induction on axiom TR4 for a triangulated category and the fact that distinguished triangles are exact (**??**), the following sequence in $\mathcal{SH}$ is exact (since a sequence clearly remains exact even after changing the signs of its maps):

$$\Sigma^s Y_s \xrightarrow{\Sigma^s j_s} \Sigma^s W_s \xrightarrow{\Sigma^s k_s} \Sigma^{s+1} Y_{s+1} \xrightarrow{\Sigma^{s+1} i_s} \Sigma^{s+1} Y_s \xrightarrow{\Sigma^{s+1} j_s} \Sigma^{s+1} W_s$$

(see **??** for the definition of an exact triangle in an additive category). Furthermore, since $\mathcal{SH}$ is tensor triangulated, the sequence remains exact after applying $E \otimes -$ (see **??** for details), so that taking E-homology yields the following exact sequence of homology groups:

$$E_*(\Sigma^s Y_{s+1}) \xrightarrow{E_*(\Sigma^s i_s)} E_*(\Sigma^s Y_s) \xrightarrow{E_*(\Sigma^s j_s)} E_*(\Sigma^s W_s) \xrightarrow{E_*(\Sigma^s k_s)} E_*(\Sigma^{s+1} Y_{s+1}) \xrightarrow{E_*(\Sigma^{s+1} i_s)} E_*(\Sigma^{s+1} Y_s).$$

Then since $E_*(i_s) : E_*(Y_{s+1}) \to E_*(Y_s)$ is the zero map (by Proposition 0.3) and we have natural isomorphisms

$$E_*(\Sigma^t X) \xrightarrow{\nu^t_X} E_*(\Sigma^{\mathbf{t}} X) \xrightarrow{t^{\mathbf{t}}_X} E_{*-\mathbf{t}}(X)$$

(the first from Definition 0.8 and the latter from **??**), we have that $E_*(\Sigma^t i_s) : E_*(\Sigma^t Y_{s+1}) \to E_*(\Sigma^t Y_s)$ is the zero map for all $t \in \mathbb{Z}$, so that in particular the above exact sequence splits to yield the short exact sequence

$$0 \to E_*(\Sigma^s Y_s) \xrightarrow{E_*(\Sigma^s j_s)} E_*(\Sigma^s W_s) \xrightarrow{E_*(\Sigma^s k_s)} E_*(\Sigma^{s+1} Y_{s+1}) \to 0.$$

Then we may splice these sequences together for $s \geq 0$ to yield the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & E_*(Y) \arrow[r, "E_*(j_0)"] & E_*(W_0) \arrow[rr, "E_*(\delta_0)"] \arrow[dr, "E_*(k_0)"'] & & E_*(\Sigma W_1) \arrow[rr, "E_*(\delta_1)"] \arrow[dr, "E_*(\Sigma k_1)"'] & & E_*(\Sigma^2 W_2) \arrow[r] & \cdots \\
& & & E_*(\Sigma Y_1) \arrow[ur, "E_*(\Sigma j_1)"'] & & E_*(\Sigma^2 Y_2) \arrow[ur, "E_*(\Sigma^2 j_2)"'] & &
\end{tikzcd}$$

It follows the top row is exact, as desired. □

Lemma 0.11. *Let (E, μ, e) be a commutative monoid object, and X and Y objects in $\mathcal{SH}$. Suppose further that:*

- *E is flat (**??**) and cellular (**??**),*
- *X is cellular and $E_*(X)$ is a graded projective left $\pi_*(E)$-module (via **??**), and*
- *Y is cellular.*

Then the assignment

$$E_*(-) : [X, E \otimes Y] \to \mathrm{Hom}_{E_*(E)}(E_*(X), E_*(E \otimes Y)), \qquad f \mapsto E_*(f)$$

induced by the functor $E_*(-) : \mathcal{SH}$-$\mathbf{Cell} \to E_*(E)$-$\mathbf{CoMod}^A$ is an isomorphism of abelian groups.

Proof. Since X is cellular, by **??** we have that $E_*(X)$ is canonically an A-graded left $E_*(E)$-comodule. Similarly, since E and Y are cellular, we know that $E \otimes Y$ is cellular, so that $E_*(E \otimes Y)$ is also canonically an $E_*(E)$-comodule. Thus, we have a well-defined assignment

$$[X, E \otimes Y] \xrightarrow{E_*(-)} \mathrm{Hom}_{E_*(E)}(E_*(X), E_*(E \otimes Y)).$$

To see this arrow is an isomorphism, consider the following diagram:

$$\begin{tikzcd}
{[X, E \otimes Y]} \arrow[rr, "E_*(-)"] \arrow[d, "\pi_*(\mu\otimes Y)\circ E_*(-)"'] & & \mathrm{Hom}_{E_*(E)}(E_*(X), E_*(E \otimes Y)) \arrow[dll, "\pi_*(\mu\otimes Y)\circ(-)"'] \\
\mathrm{Hom}_{\pi_*(E)}(E_*(X), E_*(Y)) & & \mathrm{Hom}_{E_*(E)}(E_*(X), E_*(E) \otimes_{\pi_*(E)} E_*(Y)) \arrow[ll, "\mathrm{adj}"'] \arrow[u, "(\Phi_{E,Y})_*"']
\end{tikzcd}$$

We know the left vertical map is an isomorphism by **??**, and the bottom horizontal isomorphism is the forgetful-cofree adjunction (**??**) for A-graded left comodules over the dual E-Steenrod algebra. The right vertical arrow is a well-defined isomorphism, as $\Phi_{E,Y}$ is a homomorphism of A-graded left $E_*(E)$-comodules (**??**), and in fact it is an isomorphism by **??**, since $E_*(E)$ is flat and Y is cellular. Thus in order to see the top arrow is an isomorphism, it suffices to show that the diagram commutes. The left triangle clearly commutes; to see the right triangle commutes, recall that by how the how forgetful-cofree adjunction for left comodules over a Hopf algebroid is defined, that the bottom vertical arrow sends an A-graded homomorphism of left $E_*(E)$-comodules $\psi : E_*(X) \to E_*(E) \otimes_{\pi_*(E)} E_*(Y)$ to the composition

$$E_*(X) \xrightarrow{\psi} E_*(E) \otimes_{\pi_*(E)} E_*(Y) \xrightarrow{\pi_*(\mu)\otimes E_*(Y)} \pi_*(E) \otimes_{\pi_*(E)} E_*(Y) \xrightarrow{\cong} E_*(Y).$$

Thus, in order to show that this composition equals $\pi_*(\mu \otimes Y) \circ \Phi_{E,Y} \circ \psi$, it suffices to show the following diagram commutes:

$$\begin{tikzcd}
E_*(E) \otimes_{\pi_*(E)} E_*(Y) \arrow[r, "\pi_*(\mu)\otimes E_*(Y)"] \arrow[d, "\Phi_{E,Y}"'] & \pi_*(E) \otimes_{\pi_*(E)} E_*(Y) \arrow[d, "\cong"] \\
E_*(E \otimes Y) \arrow[r, "\pi_*(\mu\otimes Y)"] & E_*(Y)
\end{tikzcd}$$

Since all the arrows here are homomorphisms of abelian groups, in order to show the diagram commutes, it suffices to chase pure homogeneous tensors around. To that end, let $x : S^a \to E \otimes E$ and $y : S^b \to E \otimes Y$, and consider the following diagram exhibiting the two ways to chase $x \otimes y$ around:

$$\begin{tikzcd}
S^{a+b} \arrow[r, "\phi_{a,b}"] & S^a \otimes S^b \arrow[r, "x\otimes y"] & E \otimes E \otimes E \otimes Y \arrow[r, "\mu\otimes E\otimes Y"] \arrow[d, "E\otimes\mu\otimes Y"'] & E \otimes E \otimes Y \arrow[d, "\mu\otimes Y"] \\
& & E \otimes E \otimes Y \arrow[r, "\mu\otimes Y"] & E \otimes Y
\end{tikzcd}$$

The diagram commutes by associtiavity of μ. Thus, we have indeed show that

$$E_*(-) : [X, E \otimes Y] \to \mathrm{Hom}_{E_*(E)}(E_*(X), E_*(Y))$$

is an isomorphism of abelian groups. $\square$

Proposition 0.12. *Let (E, μ, e) be a commutative monoid object, and X and Y objects in $\mathcal{SH}$. Suppose further that:*

- *E is flat (**??**) and cellular (**??**),*

- *X is cellular, and $E_*(X)$ is a graded projective left $\pi_*(E)$-module (via **??**), and*
- *Y is cellular.*

Then for all $s \in \mathbb{Z}$ and $a \in A$, the line in the first page of the E-Adams spectral sequence for the computation of $[X, Y]_*$ associated to the canonical E-Adams resolution of Y (Definition 0.2)

$$0 \to E_1^{0,a+\mathbf{s}}(X,Y) \xrightarrow{d_1} E_1^{1,a+\mathbf{s}-\mathbf{1}}(X,Y) \xrightarrow{d_1} E_1^{2,a+\mathbf{s}-\mathbf{2}}(X,Y) \to \cdots \to E_1^{s,a}(X,Y) \to \cdots$$

is isomorphic to the complex obtained by applying $\mathrm{Hom}^{a+\mathbf{s}}_{E_*(E)}(E_*(X), -)$ to the complex of A-graded left $E_*(E)$-comodules

$$0 \to E_*(W_0) \xrightarrow{E_*(\delta_0)} E_*(\Sigma W_1) \xrightarrow{E_*(\delta_1)} E_*(\Sigma^2 W_2) \to \cdots \to E_*(\Sigma^s W_s) \to \cdots$$

from Lemma 0.10.

Proof. By Lemma 0.9, since E and Y are cellular, W_t is as well for each $t \in \mathbb{N}$. Furthermore, for $t > 0$, we have isomorphisms

$$S^{\mathbf{t}} \otimes W_t \xrightarrow{\nu^t_{W_t}} \Sigma^t W_t,$$

and by **??**, the object $S^{\mathbf{t}} \otimes W_t$ is cellular since $S^{\mathbf{t}}$ and W_t are. Hence, by **??**, the complex

$$0 \to E_*(W_0) \xrightarrow{E_*(\delta_0)} E_*(\Sigma W_1) \xrightarrow{E_*(\delta_1)} E_*(\Sigma^2 W_2) \to \cdots \to E_*(\Sigma^s W_s) \to \cdots$$

actually lives in $E_*(E)$-$\mathbf{CoMod}^A$, as desired. Now, let $t \in \mathbb{N}$, and consider the following diagram:

$$\begin{tikzcd}
{[X, W_t]_{a+\mathbf{s}-\mathbf{t}}} \arrow[d, "(k_t)_*"] & {[X, \Sigma^{\mathbf{t}} W_t]_{a+\mathbf{s}}} \arrow[l, "s^{\mathbf{t}}_{X,W_t}"'] \arrow[r, "(\nu^t_{W_t})_*"] \arrow[d, dashed, "(\Sigma^{\mathbf{t}} k_t)_*"] & {[X, \Sigma^t W_t]_{a+\mathbf{s}}} \arrow[dd, "(\Sigma^t k_t)_*"] \arrow[dddd, bend left, "(\delta_t)_*"] \\
{[X, \Sigma Y_{t+1}]_{a+\mathbf{s}-\mathbf{t}}} \arrow[d, "(\nu_{Y_{t+1}})_*"] & {[X, \Sigma^{\mathbf{t}} \Sigma Y_{t+1}]_{a+\mathbf{s}}} \arrow[l, "s^{\mathbf{t}}_{X,\Sigma Y_{t+1}}"'] \arrow[d, dashed, "(\Sigma^{\mathbf{t}} \nu_{Y_{t+1}})_*"] \arrow[dr, "(\nu^t_{\Sigma Y_{t+1}})_*"] & \\
{[X, \Sigma^{\mathbf{1}} Y_{t+1}]_{a+\mathbf{s}-\mathbf{t}}} \arrow[d, "s^{\mathbf{1}}_{X,Y_{t+1}}"] & {[X, \Sigma^{\mathbf{t}} \Sigma^{\mathbf{1}} Y_{t+1}]_{a+\mathbf{s}}} \arrow[l, "s^{\mathbf{t}}_{X,\Sigma^{\mathbf{1}} Y_{t+1}}"'] & {[X, \Sigma^{t+1} Y_{t+1}]_{a+\mathbf{s}}} \arrow[dd, "(\Sigma^{t+1} j_{t+1})_*"] \\
{[X, Y_{t+1}]_{a+\mathbf{s}-\mathbf{t}-\mathbf{1}}} \arrow[d, "(j_{t+1})_*"] & {[X, \Sigma^{\mathbf{t}+\mathbf{1}} Y_{t+1}]_{a+\mathbf{s}}} \arrow[l, "s^{\mathbf{t}+\mathbf{1}}_{X,Y_{t+1}}"'] \arrow[u, dashed, "(\phi_{\mathbf{t},\mathbf{1}} \otimes Y_{t+1})_*"] \arrow[ur, "(\nu^{t+1}_{Y_{t+1}})_*"'] \arrow[d, dashed, "(\Sigma^{\mathbf{t}+\mathbf{1}} j_{t+1})_*"] & \\
{[X, W_{t+1}]_{a+\mathbf{s}-\mathbf{t}-\mathbf{1}}} & {[X, \Sigma^{\mathbf{t}+\mathbf{1}} W_{t+1}]_{a+\mathbf{s}}} \arrow[l, "s^{\mathbf{t}+\mathbf{1}}_{X,W_{t+1}}"'] \arrow[r, "(\nu^{t+1}_{W_{t+1}})_*"] & {[X, \Sigma^{t+1} W_{t+1}]_{a+\mathbf{s}}}
\end{tikzcd}$$

where here the $s^a_{X,Y} : [X, \Sigma^a Y]_* \cong [X, Y]_{*-a}$'s are the natural isomorphisms from **??**. By unravelling definitions, we have the top left object is $E_1^{t,a+\mathbf{s}-\mathbf{t}}(X,Y)$ and the bottom left object is $E_1^{t+1,a+\mathbf{s}-\mathbf{t}-\mathbf{1}}$, and the vertical left composition in the above diagram is the differential d_1 between them. The first, second, and fourth rectangles from the top on the left rectangle commute by naturality of the s^a's. Furthermore, a simple diagram chase and coherence of the ϕ's (**??**) yields that the third rectangle on the left commutes. The trapezoids on the right commute by naturality of ν^t and ν^{t+1}. Finally, the middle right triangle commutes by how we defined ν^{t+1} in terms of ν^t.

Now, consider the following diagram:

$$\begin{tikzcd}
E_1^{t,a+\mathbf{s}-\mathbf{t}}(X,Y) \arrow[rr, "d_1"] \arrow[d, "(s^{\mathbf{t}}_{X,W_t})^{-1}"'] & & E_1^{t+1,a+\mathbf{s}-\mathbf{t}-\mathbf{1}}(X,Y) \arrow[d, "(s^{\mathbf{t+1}}_{X,W_{t+1}})^{-1}"] \\
{[X,\Sigma^{\mathbf{t}}W_t]_{a+\mathbf{s}}} \arrow[d, "(\nu^t_{W_t})_*"'] & & {[X,\Sigma^{\mathbf{t+1}}W_{t+1}]_{a+\mathbf{s}}} \arrow[d, "(\nu^{t+1}_{W_{t+1}})_*"] \\
{[X,\Sigma^{t}W_t]_{a+\mathbf{s}}} \arrow[rr, "(\delta_t)_*"] \arrow[d, "E_*(-)"'] & & {[X,\Sigma^{t+1}W_{t+1}]_{a+\mathbf{s}}} \arrow[d, "E_*(-)"] \\
\mathrm{Hom}_{E_*(E)}(E_*(\Sigma^{a+\mathbf{s}}X),E_*(\Sigma^tW_t)) \arrow[rr, "E_*(\delta_t)"] \arrow[d, "((t_X^{a+\mathbf{s}})^{-1})^*"'] & & \mathrm{Hom}_{E_*(E)}(E_*(\Sigma^{a+\mathbf{s}}X),E_*(\Sigma^{t+1}W_{t+1})) \arrow[d, "((t_X^{a+\mathbf{s}})^{-1})^*"] \\
\mathrm{Hom}^{a+\mathbf{s}}_{E_*(E)}(E_*(X),E_*(\Sigma^tW_t)) \arrow[rr, "E_*(\delta_t)"] & & \mathrm{Hom}^{a+\mathbf{s}}_{E_*(E)}(E_*(X),E_*(\Sigma^{t+1}W_{t+1}))
\end{tikzcd}$$

where here the maps $t_X^{a+\mathbf{s}} : E_*(\Sigma^a) \to E_{*-a}(X)$ are the $E_*(E)$-comodule isomorphisms from **??**. We have just shown the top region commutes. Furthermore, since X and Σ^tW_t are cellular for all $t \in \mathbb{N}$, the arrows labelled $E_*(-)$ are well-defined, and they clearly make the middle rectangle commute (a simple diagram chase suffices). The bottom rectangle also clearly commutes, Thus, it suffices to show that the maps labelled $E_*(-)$ are isomorphisms. To that end, consider the following diagram:

$$\begin{tikzcd}
{[X,\Sigma^tW_t]_{a+\mathbf{s}}} \arrow[r, "E_*(-)"] \arrow[d, "f_*"'] & \mathrm{Hom}_{E_*(E)}(E_*(\Sigma^{a+\mathbf{s}}X),E_*(\Sigma^tW_t)) \arrow[d, "E_*(f)_*"] \\
{[X,E\otimes\Sigma^{\mathbf{t}}Y_t]_{a+\mathbf{s}}} \arrow[r, "E_*(-)"] & \mathrm{Hom}_{E_*(E)}(E_*(\Sigma^{a+\mathbf{s}}X),E_*(E\otimes\Sigma^{\mathbf{t}}Y_t))
\end{tikzcd}$$

where here $f : \Sigma^tW_t \to E\otimes\Sigma^{\mathbf{t}}Y_t$ is the isomorphism

$$\Sigma^tW_t \xrightarrow{\nu^t_W} \Sigma^{\mathbf{t}}W_t = S^{\mathbf{t}}\otimes E\otimes Y_t \xrightarrow{\tau\otimes Y_t} E\otimes S^{\mathbf{t}}\otimes Y_t = E\otimes\Sigma^{\mathbf{t}}Y_t.$$

The bottom horizontal arrow is an isomorphism by Lemma 0.11. Thus, the top horizontal arrow is an isomorphism, as desired. Showing

$$E_*(-) : [X,\Sigma^{t+1}W_{t+1}]_{a+\mathbf{s}} \to \mathrm{Hom}_{E_*(E)}(E_*(\Sigma^{a+\mathbf{s}}X),E_*(\Sigma^{t+1}W_{t+1}))$$

is an isomorphism is entirely analagous. Thus, for each $t \in \mathbb{N}$, we have constructed isomorphisms

$$E^{t,a+\mathbf{s}-\mathbf{t}}(X,Y) \xrightarrow{\cong} \mathrm{Hom}^{a+\mathbf{s}}_{E_*(E)}(E_*(X),E_*(\Sigma^tW_t))$$

such that the following diagram commutes:

$$\begin{tikzcd}
E^{t,a+\mathbf{s}-\mathbf{t}}(X,Y) \arrow[r, "d_1"] \arrow[d, "\cong"'] & E^{t+1,a+\mathbf{s}-\mathbf{t}-\mathbf{1}}(X,Y) \arrow[d, "\cong"] \\
\mathrm{Hom}^{a+\mathbf{s}}_{E_*(E)}(E_*(X),E_*(\Sigma^tW_t)) \arrow[r, "\mathrm{Hom}^{a+\mathbf{s}}_{E_*(E)}(E_*(X){,}E_*(\delta_t))"] & \mathrm{Hom}^{a+\mathbf{s}}_{E_*(E)}(E_*(X),E_*(\Sigma^{t+1}W_{t+1}))
\end{tikzcd}$$

Hence, we have proven the desired result. $\square$

0.3. **Convergence of the spectral sequence.** Before we can state and prove some convergence results for the spectral sequence we have constructed above, we outline a bit of the theory of *nilpotent completion* of objects in $\mathcal{SH}$. Namely, we will outline suitable conditions under which the E-Adams spectral sequence for $[X,Y]_*$ converges to the homotopy groups $[X, Y^\wedge_E]_*$, where $Y^\wedge_E$ is an *E-nilpotent completion of Y*. The main reference for this section and the next will be §5–6 in the paper [1] by Bousfield. First, we state some definitions.

Definition 0.13 ([2])**.** Given an object Y in $\mathcal{SH}$ and a monoid objedct (E,μ,e) an *E-completion* $\widehat{Y}$ of Y is an object in $\mathcal{SH}$ such that:

(a) There is a map $Y \to \widehat{Y}$ inducing an isomorphism in E_*-homology.
(b) $\widehat{Y}$ has an E-Adams resolution $(\widehat{Y}_s, \widehat{W}_s; i,j,k)$ (Definition 0.1) with $\operatorname{holim} \widehat{Y}_s = 0$ (see **??** for the definition of homotopy limits in a triangulated category with products).

Definition 0.14 ([1, pgs. 272–273])**.** Let (E,μ,e) be a monoid object in $\mathcal{SH}$, and Y any object. Write $\overline{E}$ for the homotopy fiber (**??**) of the unit $S \xrightarrow{e} E$, so we have a distinguished triangle

$$\overline{E} \to S \xrightarrow{e} E \to \Sigma\overline{E}.$$

Set $Y_0 := Y$ and $W_0 := Y \otimes E$, and for $s > 0$ define $Y_s := Y \otimes \overline{E}^s$ and $W_s := Y_s \otimes E$. Then since $\mathcal{SH}$ is tensor triangulated, for each $s \geq 0$ we may tensor the above sequence with Y_s on the right, which yields the following distinguished triangle

$$Y_{s+1} \xrightarrow{i} Y_s \xrightarrow{j} W_s \xrightarrow{k} \Sigma Y_{s+1}.$$

Then for $s \in \mathbb{N}$, define Y/Y^s to be the cofiber of $i^s : Y_s \to Y_0 = Y$ (so in particular we may take $Y/Y_1 = E \otimes Y$ and $Y/Y_0 = 0$), so we have a distinguished triangle

$$Y_s \xrightarrow{i^s} Y \xrightarrow{b} Y/Y_s \xrightarrow{c} \Sigma Y_s.$$

Then for each $s \geq 0$, by the octahedral axiom (axiom TR5) for a triangulated category applied to the triangles

$$Y_{s+1} \xrightarrow{i} Y_s \xrightarrow{j} W_s \xrightarrow{k} \Sigma Y_{s+1}$$
$$Y_s \xrightarrow{i^s} Y \xrightarrow{b} Y/Y_s \xrightarrow{c} \Sigma Y_s$$
$$Y_{s+1} \xrightarrow{i^{s+1}} Y \xrightarrow{b} Y/Y_{s+1} \xrightarrow{c} \Sigma Y_{s+1},$$

there exists a distinguished triangle

$$W_s \xrightarrow{p} Y/Y_{s+1} \xrightarrow{q} Y/Y_s \xrightarrow{r} \Sigma W_s. \tag{1}$$

is distinguished and the following diagram commutes:

$$\begin{tikzcd}
Y_{s+1} \arrow[rr, "i^{s+1}"] \arrow[dr, "i"'] & & Y \arrow[rr, "b"] \arrow[dr, "b"'] & & Y/Y_s \arrow[rr, "r"] \arrow[dr, "c"] & & \Sigma W_s \\
& Y_s \arrow[ur, "i^s"'] \arrow[dr, "j"'] & & Y/Y_{s+1} \arrow[ur, "q"'] \arrow[dr, "c"] & & \Sigma Y_s \arrow[ur, "\Sigma j"'] & \\
& & W_s \arrow[ur, "p"] \arrow[rr, "k"] & & \Sigma Y_{s+1} \arrow[ur, "\Sigma i"'] & &
\end{tikzcd} \tag{2}$$

The triangles from (1) may be spliced together to yield a tower $\{Y/Y_s\}_s$ under Y:

$$\begin{tikzcd}
Y \arrow[r] & \cdots \arrow[r] & Y/Y_3 \arrow[r, "q"] \arrow[d, dashed, "r"] & Y/Y_2 \arrow[r, "q"] \arrow[d, dashed, "r"] & Y/Y_1 \arrow[r, "q"] \arrow[d, dashed, "r"] & Y/Y_0 \arrow[r, equal] \arrow[d, dashed, "r"] & 0 \\
& & W_3 & W_2 \arrow[ul, "p"] & W_1 \arrow[ul, "p"] & W_0 \arrow[ul, "p"] &
\end{tikzcd}$$

where here the dashed arrows are really (degree $-\mathbf{1}$) maps $Y/Y_s \to \Sigma W_s$. The fact that this is a tower under Y follows from diagram (**??**), which tells us that $Y \xrightarrow{b} Y/Y_s$ factors as $Y \xrightarrow{b} Y/Y_{s+1} \xrightarrow{q} Y/Y_s$. We define the *$E$-nilpotent completion of Y* to be the object $Y^\wedge_E$ (defined up to non-canonical isomorphism) obtained as the homotopy limit of this tower (**??**):

$$Y^\wedge_E := \operatorname*{holim}_s Y_s/Y.$$

This comes equipped with a map $\alpha : Y \to Y^\wedge_E$.

Proposition 0.15. *Consider the tower under Y constructed in Definition 0.14:*

$$\begin{tikzcd}
Y \arrow[r] & \cdots \arrow[r] & Y/Y_3 \arrow[r, "q"] \arrow[d, dashed, "r"] & Y/Y_2 \arrow[r, "q"] \arrow[d, dashed, "r"] & Y/Y_1 \arrow[r, "q"] \arrow[d, dashed, "r"] & Y/Y_0 \arrow[r, equal] \arrow[d, dashed, "r"] & 0 \\
& & W_3 & W_2 \arrow[ul, "p"] & W_1 \arrow[ul, "p"] & W_0 \arrow[ul, "p"] &
\end{tikzcd}$$

We may extend it to the right by defining $Y/Y_s = W_s = 0$ for $s < 0$. Then by **??***, we may apply the functor $[X,-]_*$ which yields the following A-graded unrolled exact couple (***??***):*

$$\begin{tikzcd}
\cdots \arrow[r] & {[X, Y/Y_{s+2}]_*} \arrow[r, "q"] \arrow[d, "\delta"] & {[X, Y/Y_{s+1}]_*} \arrow[r, "q"] \arrow[d, "\delta"] & {[X, Y/Y_s]_*} \arrow[r, "q"] \arrow[d, "\delta"] & {[X, Y/Y_{s-1}]_*} \arrow[r] \arrow[d, "\delta"] & \cdots \\
& {[X, W_{s+2}]_*} & {[X, W_{s+1}]_*} \arrow[ul, "p"] & {[X, W_s]_*} \arrow[ul, "p"] & {[X, W_{s-1}]_*} \arrow[ul, "p"] &
\end{tikzcd}$$

Thus by **??***, there is an induced spectral sequence. This spectral sequence is precisely the E-Adams spectral sequence for $[X,Y]_*$ (Definition 0.4) determined by the canonical E-Adams resolution of Y (Definition 0.2).*

Proof. For $s \geq 0$, define

$$f_s : [X, Y/Y_s]_* \xrightarrow{c_*} [X, \Sigma Y_s]_* \xrightarrow{(\nu_{Y_s})_*} [X, \Sigma^{\mathbf{1}} Y_s]_* \xrightarrow{s^{\mathbf{1}}_{X,Y_s}} [X, Y_s]_{*-\mathbf{1}},$$

and for $s < 0$ let it be the unique map

$$f_s : [X, Y/Y_s]_* = 0 \to [X, Y_s]_{*-\mathbf{1}} = [X, Y]_{*-1}.$$

For $s \in \mathbb{Z}$, let

$$g_s := \mathrm{id}_{W_s} : [X, W_s]_* \to [X, W_s]_*.$$

We claim these maps $(f_s, g_s)_s$ define a homomorphism of A-graded unrolled exact couples (**??**) between the unrolled exact couple given above and that obtained by applying $[X,-]_*$ to the canonical E-Adams resolution. To that end, it suffices to show that the following diagram commutes for all $s \in \mathbb{Z}$:

$$\begin{tikzcd}
{[X, Y/Y_s]_*} \arrow[r] \arrow[d, "f_s"] & {[X, Y/Y_{s-1}]_*} \arrow[r] \arrow[d, dashed, "f_{s-1}"] & {[X, W_{s-1}]_{*-\mathbf{1}}} \arrow[r] \arrow[d, equal] & {[X, Y/Y_s]_{*-\mathbf{1}}} \arrow[d, "f_s"] \\
{[X, Y_s]_{*-\mathbf{1}}} \arrow[r] & {[X, Y_{s-1}]_{*-\mathbf{1}}} \arrow[r] & {[X, W_{s-1}]_{*-\mathbf{1}}} \arrow[r] & {[X, Y_s]_{*-\mathbf{2}}}
\end{tikzcd}$$

In the case $s \leq 0$, we know $Y/Y_s = Y/Y_{s-1} = W_{s-1} = 0$, so that the top row is entirely 0, and thus the diagram must commute. In the case $s > 0$, by unravelling definitions we have that the diagram becomes

$$\begin{tikzcd}
{[X, Y/Y_s]_*} \arrow[r, "q_*"] \arrow[d, "c_*"'] & {[X, Y/Y_{s-1}]_*} \arrow[rr, "\delta"] \arrow[d, "c_*"] \arrow[dr, "r_*"] & & {[X, W_{s-1}]_{*-\mathbf{1}}} \arrow[r, "p_*"] \arrow[ddd, equals] & {[X, Y/Y_s]_{*-\mathbf{1}}} \arrow[d, "c_*"] \\
{[X, \Sigma Y_s]_*} \arrow[r, "\Sigma i_*"] \arrow[d, "(\nu_{Y_s})_*"'] & {[X, \Sigma Y_{s-1}]_*} \arrow[r, "\Sigma j_*"] \arrow[d, "(\nu_{Y_{s-1}})_*"] & {[X, \Sigma W_{s-1}]_*} \arrow[d, "(\nu_{W_{s-1}})_*"] & & {[X, \Sigma Y_s]_{*-\mathbf{1}}} \arrow[d, "(\nu_{Y_s})_*"] \\
{[X, \Sigma^{\mathbf{1}} Y_s]_*} \arrow[r, "\Sigma^{\mathbf{1}} i_*"] \arrow[d, "s^{\mathbf{1}}_{X,Y_s}"'] & {[X, \Sigma^{\mathbf{1}} Y_{s-1}]_*} \arrow[r, "\Sigma^{\mathbf{1}} j_*"] \arrow[d, "s^{\mathbf{1}}_{X,Y_{s-1}}"] & {[X, \Sigma^{\mathbf{1}} W_{s-1}]_*} \arrow[dr, "s^{\mathbf{1}}_{X,W_{s-1}}"] & & {[X, \Sigma^{\mathbf{1}} Y_s]_{*-\mathbf{1}}} \arrow[d, "s^{\mathbf{1}}_{X,Y_s}"] \\
{[X, Y_s]_{*-\mathbf{1}}} \arrow[r, "i_*"] & {[X, Y_{s-1}]_{*-\mathbf{1}}} \arrow[rr, "j_*"] & & {[X, W_{s-1}]_{*-\mathbf{1}}} \arrow[r, "\partial_*"] \arrow[uur, "k_*"] & {[X, Y_s]_{*-\mathbf{2}}}
\end{tikzcd}$$

Clearly commutativity of this diagram yields that the given collection of maps define a homomorphism of A-graded unrolled exact couples. Each rectangular region commutes by naturality, as does the middle bottom trapezoidal region. The two regions involving δ and ∂ commute by unravelling how the differential is defined in **??**. Finally, the remaining two regions commute by commutativity of Equation 2.

Thus, we have defined a homomorphism of A-graded unrolled exact couples, so that by **??** it induces a homomorphism of the associated spectral sequences $\widetilde{g}$. Further unravelling how this homomorphism of spectral sequences is defined, since the homomorphism of unrolled exact couples is the identity on the $[X, W_s]_*$ terms, it follows that the two spectral sequences are strictly equal. ☐

Remark 0.16. In [1], the E-nilpotent completion of Y (Definition 0.14) is denoted "$E^\wedge Y$", while the notation "$Y_E^\wedge$" we use here is standard in the modern literature.

Definition 0.17. Let (E, μ, e) be a monoid object and X and Y two objects in $\mathcal{SH}$. Then we have an associated E-Adams spectral sequence $(E_r^{*,*}(X,Y), d_r)$ (Definition 0.4) and E-nilpotent completion $Y_E^\wedge$ (Definition 0.14). Then we may define a decreasing A-graded filtration of $[X, Y_E^\wedge]_*$ by defining

$$F^s[X, Y_E^\wedge]_* := \ker\left((\alpha_s)_* : [X, Y_E^\wedge]_* \to [X, \overline{E}_{s-1} \otimes Y]_*\right),$$

for $s > 0$, where α_s is the composition

$$Y_E^\wedge \to \prod_{i=0}^{\infty} (\overline{E}_i \otimes Y) \twoheadrightarrow \overline{E}_{s-1} \otimes Y$$

Note that $F^1[X, Y_E^\wedge]_* = [X, E \otimes Y]_*$. To see this, it suffices to show that α_1 is the zero map. To see this, note that by how homotopy limits are constructed in **??**, we have that the following diagram commutes:

Definition 0.18. Let (E, μ, e) be a monoid object in $\mathcal{SH}$, and X and Y any objects. Then for all