In what follows, we fix an abelian group A. We will freely use the theory and results of **??**

Definition 0.1. An *A-graded spectral sequence* $(E_r, d_r)_{r\geq r_0}$ is the data of:

- A collection of A-graded abelian groups $\{E_r^*\}_{r\geq r_0}$
- A collection of A-graded homomorphisms $d_r : E_r \to E_r$ for $r \geq r_0$ (of possibly nonzero degree) such that $d_r \circ d_r = 0$
- For each $r \geq r_0$, an A-graded isomorphism $E_{r+1} \cong \ker d_r / \operatorname{im} d_r$ of degree 0 (where $\ker d_r$ and $\operatorname{im} d_r$ are canonically A-graded by **??**, and their quotient is canonically A-graded by **??**.

Typically we call a $\mathbb{Z}^2$-graded spectral sequence a *bigraded* spectral sequence, and a $\mathbb{Z}^3$-graded spectral sequence is a *trigraded* spectral sequence.

For our purposes, we will only care about spectral sequences which arise from *A-graded unrolled exact couples*. In what follows, we follow [1], with minor modifications for our setting, in which everything is A-graded.

Definition 0.2. An A-graded *unrolled exact couple* $(D, E; i, j, k)$ is a diagram of A-graded abelian groups and A-graded homomorphisms (of possibly non-zero degree)

$$\begin{tikzcd}
\cdots \arrow[r] & D^{s+2} \arrow[r, "i"] \arrow[d, "j"] & D^{s+1} \arrow[r, "i"] \arrow[d, "j"] & D^{s} \arrow[r, "i"] \arrow[d, "j"] & D^{s-1} \arrow[r] \arrow[d, "j"] & \cdots \\
& E^{s+2} & E^{s+1} \arrow[ul, "k"] & E^{s} \arrow[ul, "k"] & E^{s-1} \arrow[ul, "k"] &
\end{tikzcd}$$

in which each triangle $D^{s+1} \xrightarrow{i} D^s \xrightarrow{j} E_s \xrightarrow{k} D^{s+1}$ is an exact sequence. We require that each occurrence of i (resp. j, k) is of the same degree. In other words, an unrolled exact couple can be described as a tuple $(D, E; i, j, k)$ of $\mathbb{Z} \times A$-graded abelian groups and $\mathbb{Z} \times A$-graded maps $i : D \to D$, $j : D \to E$, and $k : E \to D$, such that the $\mathbb{Z}$-degrees of i, j, and k are -1, 0, and 1, respectively. Usually i and one of j or k will be of A-degree 0.

Given an A-graded unrolled exact couple $(D, E; i, j, k)$, we may define an associated $\mathbb{Z} \times A$-graded spectral sequence as follows: Given some $s \in \mathbb{Z}$ and some $r \geq 1$, we first define the following subgroups of E_s:

$$Z_r^s := k^{-1}(\operatorname{im}[i^{r-1} : D^{s+r} \to D^{s+1}]) \qquad \text{and} \qquad B_r^s := j(\ker[i^{r-1} : D^s \to D^{s-r+1}])$$

where we adopt the convention that i^0 is simply the identity. These are furthermore A-graded subgrous of E_s (by **??** and **??**). In this way, for each $s \in \mathbb{Z}$, we get an infinite sequence of A-graded subgroups:

$$0 = B_1^s \subseteq B_2^s \subseteq B_3^s \subseteq \cdots \subseteq \operatorname{im} j = \ker k \subseteq \cdots \subseteq Z_3^s \subseteq Z_2^s \subseteq Z_1^s = E^s.$$

Now, for each $s \in \mathbb{Z}$ and $r \geq 1$, we define the A-graded abelian group

$$E_r^s := Z_r^s / B_r^s,$$

so that in particular $E_1^s = E^s$ for all $s \in \mathbb{Z}$, as $Z_1^s = k^{-1}(D^{s+1}) = E^s$ and $B_1^s = j(\ker \operatorname{id}_{D^s}) = j(0) = 0$. Now we can define differentials $d_r^s : E_r^s \to E_r^{s+r}$ to be the composition

$$E_r^s = Z_r^s / B_r^s \xrightarrow{k} \operatorname{im}[i^{r-1} : D^{s+r} \to D^{s+1}] \xrightarrow{j \circ i^{-(r-1)}} Z_r^{s+r} / B_r^{s+r} = E_r^{s+r},$$

where given some $e \in Z_r^s = k^{-1}(\operatorname{im} i^{r-1})$, the first arrow takes a class $[e] \in E_r^s$ represented by some $e \in Z_r^s$ to the element $k(e)$, which lives in $\operatorname{im} i^{r-1}$ by definition, and the second arrow takes $i^{r-1}(d)$ to the class $[j(d)]$. Note the first map is well-defined, as given $b \in B_r^s = j(\ker[i^{r-1}])$, we have $k(b) = 0$, as $b \in \operatorname{im} j = \ker k$. To see the second map is well-defined, first note that given $d \in D^{s+r}$, that

$$k(j(d)) = 0 \in \operatorname{im}[i^{r-1} : D^{s+2r} \to D^{s+r+1}],$$

so that

$$j(d) \in k^{-1}(\operatorname{im}[i^{r-1} : D^{s+2r} \to D^{s+r+1}]) = Z_r^{s+r},$$

as desired, so that given $d \in D^{s+r}$, $j(d) in Z_r^{s+r}$, so it makes sense to discuss the class $[j(d)] \in Z_r^{s+r}/B_r^{s+r} = E_r^{s+r}$. Secondly, if $i^{r-1}(d) = i^{r-1}(d')$ for some $d, d' \in D^{s+r}$, then

$$j(d) - j(d') = j(d - d') \in j(\ker[i^{r-1} : D^{s+r} \to D^{s+1}]) = B_r^{s+r},$$

so that $[j(d)] = [j(d')]$ in E_r^{s+r}, as desired. It is straightforward to check that these maps are also A-graded homomorphisms, so that by unravelling definitions d_r^s is an A-graded homomorphism of degree $\deg k - (r-1) \cdot \deg i + \deg j$ (so that in the standard case $\deg i = 0$, d_r^s simply has degree $\deg k + \deg j$).

These differentials square to zero, in the sense that for each $s \in \mathbb{Z}$ and $r \geq 1$ we have that $d_r^{s+r} \circ d_r^s : E_r^s \to E_r^{s+2r}$ is the zero map. Indeed, suppose we are given some class $[e] \in E_r^s$ represented by an element $e \in E^s$, so $k(e) = i^{r-1}(d)$ for some $d \in D^{s+r}$. Then

$$d_r^{s+r}(d_r^s([e])) = d_r^{s+r}([j(d)]) = [j(i^{-(r-1)}(k(j(d))))] = [j(i^{-(r-1)}(0))] = 0,$$

where the second-to-last equality follows by the fact that $k \circ j = 0$. Note that by unravelling definitions, $d_1^s = j \circ k$.

We claim that $\ker d_r^s = Z_{r+1}^s/B_r^s$. First of all, let $[e] \in E_r^s = Z_r^s/B_r^s$, so that $[e]$ is represented by some $e \in E^s$ with $k(e) = i^{r-1}(d)$ for some $d \in D^{s+r}$. Then if $[e] \in \ker d_r^s$, by definition this means $j(d) \in B_r^{s+r} = j(\ker[i^{r-1} : D^{s+r} \to D^{s+1}])$, so $j(d) = j(d')$ for some $d' \in D^{s+r}$ with $i^{r-1}(d') = 0$. Thus $d - d' \in \ker j = \operatorname{im} i$, so there exists some $d'' \in D^{s+r+1}$ such that $i(d'') = d - d'$. Then

$$k(e) = i^{r-1}(d) = i^{r-1}(i(d'') + d') = i^r(d'') + i^{r-1}(d'),$$

but since $i^{r-1}(d') = 0$, we have $k(e) \in \operatorname{im}[i^r : D^{s+r+1} \to D^{s+1}]$, so that $e \in Z_{r+1}^s$, meaning $[e] \in Z_{r+1}^s/B_r^s$, as desired. On the other hand, suppose we are given some class $[e] \in Z_{r+1}^s/B_r^s$, represented by $e \in Z_{r+1}^s$ with $k(e) \in \operatorname{im}[i^r : D^{s+r+1} \to D^{s+1}]$. Then if we write $k(e) = i^r(d) = i^{r-1}(i(d))$, then $d_r^s([e]) = [j(i(d))] = 0$ (since $j \circ i = 0$), as asserted.

Finally, we claim that the image of $d_r^{s-r} : E_r^{s-r} \to E_r^s$ is B_{r+1}^s/B_r^s. First, let $e \in Z_r^{s-r}$, so $k(e) = i^{r-1}(d)$ for some $d \in D^s$. Then we'd like to show that $d_r^s([e]) = [j(d)]$ belongs to B_{r+1}^s/B_r^s. It suffices to show that $d \in \ker[i^r : D^s \to D^{s-r}]$. To see this, note that

$$i^r(d) = i(i^{r-1}(d)) = i(k(e)) = 0,$$

since $i \circ k = 0$. Hence we've shown $\operatorname{im} d_r^{s-r} \subseteq B_{r+1}^s/B_r^s$. Conversely, let $j(d) \in B_{r+1}^s$, so $d \in D^s$ and $i^r(d) = 0$. Then we'd like to show that $[j(d)] \in B_{r+1}^s/B_r^s$ is in the image of d_r^{s-r}. To see this, note that

$$i^r(d) = 0 \implies i^{r-1}(d) \in \ker i = \operatorname{im} k,$$

so there exists some $e \in E^{s-r}$ such that $k(e) = i^{r-1}(d)$, so $e \in Z_r^{s-r}$. Unravelling definitions, it follows that $d_r^{s-r}([e]) = [j(d)]$, so $[j(d)]$ is indeed in the image of d_r^{s-r}, as desired.

To recap, we have constructed for each $s \in \mathbb{Z}$ and $r \geq 1$ an A-graded abelian group E_r^s along with differentials $d_r^s : E_r^s \to E_r^{s+r}$. Furthermore, if we take homology in the middle term of the following sequence

$$E_r^{s-r} \xrightarrow{d_r^{s-r}} E_r^s \xrightarrow{d_r^s} E_r^{s+r},$$

we get

$$\ker d_r^s / \operatorname{im} d_r^{s-r} = \frac{Z_{r+1}^s / B_r^s}{B_{r+1}^s / B_r^s} \cong Z_{r+1}^s / B_{r+1}^s = E_{r+1}^s.$$

Thus, we get a spectral sequence:

Proposition 0.3. *We may associate a $\mathbb{Z} \times A$-graded spectral sequence $r \mapsto (E_r, d_r)$ to the A-graded unrolled exact couple $(D, E; i, j, k)$ by defining $E_r := \bigoplus_{s\in\mathbb{Z}} E_r^s$ and the differentials*

$$d_r : E_r \to E_r$$

are those constructed above, which have $\mathbb{Z} \times A$-degree $(r, \deg j - (r-1)\cdot \deg i + \deg k)$.

Definition 0.4 (Exact couple)**.** An A-graded *exact couple* is a tuple $\mathcal{E} = (D, E; i, j, k)$, where D and E are A-graded abelian groups and i, j, and k are A-graded homomorphisms (of possibly nonzero degree)

$$\begin{tikzcd} D_* \arrow[rr, "i"] & & D_* \arrow[dl, "j"] \\ & E_* \arrow[ul, "k"] & \end{tikzcd}$$

which form an *exact triangle*, in the sense that kernel = image at each vertex.

Definition 0.5 (Derived couple)**.** Given an exact couple $(D, E; i, j, k)$ as in the above definition, the composition $j \circ k : E \to E$ itself satisfies

$$(j \circ k) \circ (j \circ k) = j \circ (k \circ j) \circ k = j \circ 0 \circ k = 0,$$

so we may form the A-graded homology group $H(E) := \ker(j \circ k)/\operatorname{im}(j \circ k)$. Then we may form the triangle $\mathcal{E}'$

$$\begin{tikzcd} i(D) \arrow[rr, "i'"] & & i(D) \arrow[dl, "j'"] \\ & H(E) \arrow[ul, "k'"] & \end{tikzcd}$$

where i' is the restriction of i to $i(D)$, while j' and k' are given by

$$j'(i(d)) = [j(d)] \qquad \text{and} \qquad k'([e]) = k(e).$$

The map j' is well-defined since if $i(d) = i(d')$ then $i(d - d') = 0$, so that $d - d' \in \ker i = \operatorname{im} k$, meaning $d - d' = k(e)$ for some $e \in E$, so that

$$j(d) - j(d') = j(d - d') = j(k(e)) \in \operatorname{im}(j \circ k)$$

is a boundary, so that $[j(d)] = [j(d')]$. Similarly k' is well defined since if $[e] = [e']$ then $e - e' \in \operatorname{im}(j \circ k)$, which implies $e - e' = j(k(e''))$ for some $e'' \in E$, so that

$$k(e) - k(e') = k(e - e') = k(j(k(e''))) = 0,$$

where the last equality follows by the fact that $k \circ j = 0$. Further note that $i(D)$ and $H(E)$ are A-graded by **??** and **??**, in which case by unravelling definitions, each of i', j', and k' are A-graded homomorphisms with

$$\deg i' = \deg i, \qquad \deg j' = \deg j - \deg i, \qquad \text{and} \qquad \deg k' = \deg k.$$

We call $\mathcal{E}'$ the *derived couple* of $\mathcal{E}$. A diagram chase (left to the reader, or see [2, Lemma 1.10]) yields that $\mathcal{E}'$ is an exact couple.

If we iterate the process of taking the exact couple r times, the result is called the r^{th} *derived couple* $\mathcal{E}_r$ *of* $\mathcal{E}$.

$$\begin{tikzcd} D_r \arrow[rr, "i"] & & D_r \arrow[dl, "j^{(r)}"] \\ & E_r \arrow[ul, "k"] & \end{tikzcd}$$

Here $D_r = i^r(D)$ is a subgroup of D, and $E_r = H(E_{r-1})$ is a subquotient of E. The maps i and k are induced from the i and k of $\mathcal{E}$, while $j^{(r)}$ sends $[i^r(d)]$ to $[j(d)]$. In particular, by induction it can be seen that $\deg j^{(r)} = \deg j - r \cdot \deg i$, and the degrees of i and k remain unchanged as we take successive derived couples.

Definition 0.6 (The spectral sequence associated to an exact couple)**.** An A-graded exact couple $\mathcal{E} = (D, E; i, j, k)$ gives rise to a spectral sequence $(E_r, d_r)_{r\geq 0}$, where $E_0 = E$, $d_0 = j \circ k$, and for $r > 0$, E_r is defined above and d_r is the composition $j^{(r)} \circ k$.

In practice, we will always shift everything up a degree by re-defining $E_r := E_{r-1}$ and $d_r := d_{r-1}$, so we get a spectral sequence $(E_r, d_r)_{r\geq 1}$ with $E^1 = E$ and $d^1 = j \circ k$. Then it follows that the differential $d^r = j^{(r-1)} \circ k$ has degree

$$\deg j^{(r-1)} + \deg k = \deg j - (r-1) \cdot \deg i + \deg k.$$

Remark 0.7. Given an exact couple $\mathcal{E} = (D, E; i, j, k)$, we can define A-graded subgroups $Z_r = k^{-1}(i^r(D)) \subseteq E$ and $B_r = j(\ker(i^r)) \subseteq E$ for $r \geq 1$. By induction, it is straightforward to check that we have inclusions

$$B_1 \subseteq B_2 \subseteq B_3 \subseteq \cdots \subseteq \operatorname{im} j = \ker k \subseteq \cdots \subseteq Z_3 \subseteq Z_2 \subseteq Z_1$$

and that the maps

$$Z_r \to E_r$$

sending an element e to its class $[[\cdots[e]\cdots]]$ has kernel B_r, so we have identifications $E_r = Z_r/B_r$ as A-graded abelian groups. Let $e \in Z_r$, so $k(e) = i^r(d)$ for some $d \in D$. Then under this identification, it can be seen that the map $d_{r+1} : Z_r/B_r \to Z_r/B_r$ sends the coset $e + B_r \in Z_r/B_r$ to the coset $j(d) + B_r$, and that $\ker d_r = Z_r$ and $\operatorname{im} d_r = B_r$ for all $r \geq 1$.

Henceforth, we fix an A-graded exact couple $\mathcal{E} = (D, E; i, j, k)$, and we let $(E_r, d_r)_{r\geq 1}$ denote the associated A-graded spectral sequence. We make the identifications given by the above remark, so we assume E_r is the A-graded abelian group Z_r/B_r for all $r \geq 1$, so that in particular for all $a \in A$ we have identifications $E_r^a = Z_r^a/B_r^a$ (by **??**).