In this appendix, we fix a symmetric monoidal category $(\mathcal{C}, \otimes, S)$ with left unitor, right unitor, associator, and symmetry isomorphisms λ, ρ, α, and τ, respectively.

0.1. **Monoid objects in a symmetric monoidal category.**

Definition 0.1. A *monoid object* (E, μ, e) is an object E in $\mathcal{C}$ along with a multiplication morphism $\mu : E \otimes E \to E$ and a unit map $e : S \to E$ such that the following diagrams commute:

$$\begin{tikzcd} E \otimes S \arrow[r, "E\otimes e"] \arrow[dr, "\rho_E"'] & E \otimes E \arrow[d, "\mu"] & S \otimes E \arrow[l, "e\otimes E"'] \arrow[dl, "\lambda_E"] \\ & E & \end{tikzcd} \qquad \begin{tikzcd} (E \otimes E) \otimes E \arrow[rr, "\mu\otimes E"] \arrow[d, "\alpha"'] & & E \otimes E \arrow[d, "\mu"] \\ E \otimes (E \otimes E) \arrow[r, "E\otimes\mu"] & E \otimes E \arrow[r, "\mu"] & E \end{tikzcd}$$

The first diagram expresses unitality, while the second expressed associativity. If in addition the following diagram commutes,

$$\begin{tikzcd} E \otimes E \arrow[rr, "\tau"] \arrow[dr, "\mu"'] & & E \otimes E \arrow[dl, "\mu"] \\ & E & \end{tikzcd}$$

then we say (E, μ, e) is a *commutative* monoid object.

Example 0.2. The object S is a monoid object, with multiplication map $\rho_S = \lambda_S : S \otimes S \to S$ and unit $\mathrm{id}_S : S \to S$.

Definition 0.3. Given two monoid objects (E_1, μ_1, e_1) and (E_2, μ_2, e_2) in a symmetric monoidal category $(\mathcal{C}, \otimes, S)$, a *monoid homomorphism* from E_1 to E_2 is a morphism $f : E_1 \to E_2$ in $\mathcal{C}$ such that the following diagrams commute:

$$\begin{tikzcd} E_1 \otimes E_1 \arrow[r, "f\otimes f"] \arrow[d, "\mu_1"'] & E_2 \otimes E_2 \arrow[d, "\mu_2"] \\ E_1 \arrow[r, "f"] & E_2 \end{tikzcd} \qquad \begin{tikzcd} & S \arrow[dl, "e_1"'] \arrow[dr, "e_2"] & \\ E_1 \arrow[rr, "f"] & & E_2 \end{tikzcd}$$

It is straightforward to show that id_{E_1} is a homomorphism of monoid objects from E_1 to itself, and that the composition of monoid homomorphisms is still a monoid homomorphism. Thus, we have categories $\mathbf{Mon}_{\mathcal{C}}$ and $\mathbf{CMon}_{\mathcal{C}}$ of monoid objects and commutative monoid objects in $\mathcal{C}$, respectively, with monoid homomorphisms between them.

Lemma 0.4. *Given two monoid objects (E_1, μ_1, e_1) and (E_2, μ_2, e_2) in a symmetric monoidal category $(\mathcal{C}, \otimes, S)$, their tensor product $E_1 \otimes E_2$ canonically becomes a monoid object in $\mathcal{C}$ with unit map*

$$e : S \xrightarrow{\cong} S \otimes S \xrightarrow{e_1\otimes e_2} E_1 \otimes E_2$$

and multiplication map

$$\mu : E_1 \otimes E_2 \otimes E_1 \otimes E_2 \xrightarrow{E_1\otimes\tau\otimes E_2} E_1 \otimes E_1 \otimes E_2 \otimes E_2 \xrightarrow{\mu_1\otimes\mu_2} E_1 \otimes E_2$$

(where here we are suppressing the associators from the notation). If in addition (E_1, μ_1, e_1) and (E_2, μ_2, e_2) are commutative *monoid objects, then $(E_1 \otimes E_2, \mu, e)$ is as well.*

Proof. Due to the size of the diagrams involved, we leave this as an exercise for the reader. It is entirely straightforward. $\square$

Lemma 0.5. *Given monoid objects (E_i, μ_i, e_i) for $i = 1, 2, 3$ in a symmetric monoidal category $\mathcal{C}$, the associator $(E_1 \otimes E_2) \otimes E_3 \xrightarrow{\cong} E_1 \otimes (E_2 \otimes E_3)$ is an isomorphism of monoid objects. In other words, up to associativity, given a collection of monoid objects $E_1, \ldots, E_n$ in $\mathcal{C}$, there is no ambiguity when talking about their tensor product $E_1 \otimes \cdots \otimes E_n$ as a monoid object.*

Proof. Clearly, up to associativity, $(E_1 \otimes E_2) \otimes E_3$ and $E_1 \otimes (E_2 \otimes E_3)$ have the same unit map $S \xrightarrow{e_1 \otimes e_2 \otimes e_3} E_1 \otimes E_2 \otimes E_3$. Thus, it remains to show that they have the same product map, up to associativity. To see this, consider the following diagram, where we've passed to a symmetric strict monoidal category:

$$\begin{tikzcd}
E_1 \otimes (E_2 \otimes E_3) \otimes E_1 \otimes (E_2 \otimes E_3) \arrow[rrrr, equal, "\alpha"] \arrow[d, "E_1 \otimes \tau_{E_2 \otimes E_3, E_1} \otimes E_2 \otimes E_3"'] & & & & (E_1 \otimes E_2) \otimes E_3 \otimes (E_1 \otimes E_2) \otimes E_3 \arrow[d, "E_1 \otimes E_2 \otimes \tau_{E_3, E_1 \otimes E_2} \otimes E_3"] \\
E_1 \otimes E_1 \otimes E_2 \otimes E_3 \otimes E_2 \otimes E_3 \arrow[d, "\mu_1 \otimes E_2 \otimes \tau \otimes E_3"'] \arrow[drr, "E_1 \otimes E_1 \otimes E_2 \otimes \tau \otimes E_3"] & & & & E_1 \otimes E_2 \otimes E_1 \otimes E_2 \otimes E_3 \otimes E_3 \arrow[d, "E_1 \otimes \tau \otimes E_2 \otimes \mu_3"] \arrow[dll, "E_1 \otimes \tau \otimes E_2 \otimes E_3 \otimes E_3"'] \\
E_1 \otimes E_2 \otimes E_2 \otimes E_3 \otimes E_3 \arrow[d, "E_1 \otimes \mu_2 \otimes \mu_3"'] & & E_1 \otimes E_1 \otimes E_2 \otimes E_2 \otimes E_3 \otimes E_3 \arrow[ll, "\mu_1 \otimes E_2 \otimes E_2 \otimes E_3 \otimes E_3"] \arrow[rr, "E_1 \otimes E_1 \otimes E_2 \otimes E_2 \otimes \mu_3"'] \arrow[dll, "\mu_1 \otimes \mu_2 \otimes \mu_3"] \arrow[drr, "\mu_1 \otimes \mu_2 \otimes \mu_3"'] & & E_1 \otimes E_1 \otimes E_2 \otimes E_2 \otimes E_3 \arrow[d, "\mu_1 \otimes \mu_2 \otimes E_3"] \\
E_1 \otimes E_2 \otimes E_3 \arrow[rrrr, equal, "\alpha"] & & & & E_1 \otimes E_2 \otimes E_3
\end{tikzcd}$$

The top pentagonal region commutes by coherence for the τ's in a symmetric monoidal category. The bottom triangle commutes by definition. The remaining four triangles commute by functoriality of $- \otimes -$. On the left is the product for $E_1 \otimes (E_2 \otimes E_3)$, while on the right is the product for $(E_1 \otimes E_2) \otimes E_3$. Thus they are equal up to associativity, as desired. $\square$

Lemma 0.6. *Let (E, μ, e) be a monoid object in $\mathcal{SH}$. Then the map $e : S \to E$ is a monoid homomorphism. Furthermore, if E is a* commutative *monoid object, then $\mu : E \otimes E \to E$ is also a monoid object homomorphism. (Here S and $E \otimes E$ are considered to be monoid objects by* **??** *and* **??***, respectively.)*

Proof. To see e is a monoid homomorphism, consider the following diagrams:

$$\begin{tikzcd}
 & S \arrow[dl, equal] \arrow[dr, "e"] & \\
S \arrow[rr, "e"] & & E
\end{tikzcd}
\qquad
\begin{tikzcd}
S \otimes S \arrow[rr, "e \otimes e"] \arrow[dr, "S \otimes e"] \arrow[dd, "\rho_S = \lambda_S"'] & & E \otimes E \arrow[dd, "\mu"] \\
 & S \otimes E \arrow[ur, "e \otimes E"] \arrow[dr, "\lambda_E"] & \\
S \arrow[rr, "e"'] & & E
\end{tikzcd}$$

The left diagram commutes by definition. The top region in the right diagram commutes by functoriality of $- \otimes -$. The right region commutes by unitality of μ. The left region commutes by naturality of λ. Thus, indeed $e : S \to E$ is a monoid object homomorphism.

Now, to see μ is a monoid object homomorphism when (E, μ, e) is a commutative monoid object, first consider the following diagram:

$$\begin{tikzcd}
& S \arrow[ddl, "e\otimes e"'] \arrow[d, "e"] \arrow[ddr, "e"] & \\
& E \arrow[dl, "E\otimes e"] \arrow[dr, equal] & \\
E \otimes E \arrow[rr, "\mu"'] & & E
\end{tikzcd}$$

The left region commutes by functoriality of $- \otimes -$, the right region commutes by definition, and the bottom region commutes by unitality of μ. Now, consider the following diagram:

$$\begin{tikzcd}
E_1 \otimes E_2 \otimes E_3 \otimes E_4 \arrow[rrr, "\mu\otimes\mu"] \arrow[dr, "E\otimes\mu\otimes E"'] \arrow[drr, "\mu\otimes E\otimes E"] \arrow[dd, "E\otimes\tau\otimes E"'] & & & E_{12} \otimes E_{34} \arrow[ddd, "\mu"] \\
& E_1 \otimes E_{23} \otimes E_4 \arrow[dr, "\mu\otimes E"] & E_{12} \otimes E_3 \otimes E_4 \arrow[ur, "E\otimes\mu"] \arrow[d, "\mu\otimes E"] & \\
E_1 \otimes E_3 \otimes E_2 \otimes E_4 \arrow[ur, "E\otimes\mu\otimes E"] \arrow[r, "\mu\otimes E\otimes E"] \arrow[d, "\mu\otimes\mu"'] & E_{13} \otimes E_2 \otimes E_4 \arrow[r, "\mu\otimes E"'] \arrow[dl, "E\otimes\mu"] & E_{123} \otimes E_4 \arrow[dr, "\mu"] & \\
E_{13} \otimes E_{24} \arrow[rrr, "\mu"] & & & E_{1234}
\end{tikzcd}$$

Here we have numbered the E's to make it clearer what's going on. The top and bottom left regions commute by functoriality of $- \otimes -$. The top left region commutes by commutativity of μ. Every other region commutes by associativity of μ. Thus, we've shown μ is a monoid object homomorphism, as desired. $\square$

Lemma 0.7. *Suppose we have some monoid object (E, μ, e) in $\mathcal{C}$ and some homomorphism of monoid objects $f : (E_1, \mu_1, e_1) \to (E_2, \mu_2, e_2)$ in $\mathbf{Mon}_{\mathcal{C}}$. Then $E \otimes f : E \otimes E_1 \to E \otimes E_2$ and $f \otimes E : E_1 \otimes E \to E_2 \otimes E$ are monoid homomorphisms, where here we are considering $E \otimes E_1$, $E \otimes E_2$, $E_1 \otimes E$, and $E_2 \otimes E$ to be monoid objects by* **??**.

Proof. We will show that $E \otimes f$ is a monoid object homomorphism, as showing $f \otimes E$ is a monoid homomorphism is entirely analagous. First consider the following diagram:

$$\begin{tikzcd}
E \otimes E_1 \otimes E \otimes E_1 \arrow[rrr, "E\otimes f\otimes E\otimes f"] \arrow[d, "E\otimes\tau\otimes E_1"'] & & & E \otimes E_2 \otimes E \otimes E_2 \arrow[d, "E\otimes\tau\otimes E_2"] \\
E \otimes E \otimes E_1 \otimes E_1 \arrow[rrr, "E\otimes E\otimes f\otimes f"] \arrow[dr, "\mu\otimes E_1\otimes E_2"] \arrow[dd, "\mu\otimes\mu_1"'] & & & E \otimes E \otimes E_2 \otimes E_2 \arrow[dl, "\mu\otimes E_2\otimes E_2"'] \arrow[dd, "\mu\otimes\mu_2"] \\
& E \otimes E_1 \otimes E_1 \arrow[r, "E\otimes f\otimes f"] \arrow[dl, "E\otimes\mu_1"] & E \otimes E_2 \otimes E_2 \arrow[dr, "E\otimes\mu_2"'] & \\
E \otimes E_1 \arrow[rrr, "E\otimes f"] & & & E \otimes E_2
\end{tikzcd}$$

The top region commutes by naturality of τ. The bottom trapezoid commutes since f is a monoid homomorphism. The remaining three regions commute by functoriality of $- \otimes -$. Now, consider the following diagram:

$$\begin{tikzcd}
& S \arrow[ddl, "e\otimes e_1"'] \arrow[d, "e"] \arrow[ddr, "e\otimes e_2"] & \\
& E \arrow[dl, "E\otimes e_1"'] \arrow[dr, "E\otimes e_2"] & \\
E\otimes E_1 \arrow[rr, "E\otimes f"] & & E\otimes E_2
\end{tikzcd}$$

The bottom region commutes since f is a monoid homomorphism. The top two regions commute by functoriality of $-\otimes-$. Thus, we've shown $E\otimes f$ is a monoid object homomorphism, as desired. $\square$

0.2. **Modules over monoid objects in a symmetric monoidal category.**

Definition 0.8. Let (E,μ,e) be a monoid object in $\mathcal{C}$. Then a *(left) module object* (N,κ) over (E,μ,e) is the data of an object N in $\mathcal{C}$ and a morphism $\kappa : E\otimes N\to N$ such that the following two diagrams commute in $\mathcal{C}$:

$$\begin{tikzcd}
S\otimes N \arrow[r, "e\otimes N"] \arrow[dr, "\lambda_N"'] & E\otimes N \arrow[d, "\kappa"] \\
& N
\end{tikzcd}
\qquad
\begin{tikzcd}
(E\otimes E)\otimes N \arrow[rr, "\mu\otimes N"] \arrow[d, "\alpha"'] & & E\otimes N \arrow[d, "\kappa"] \\
E\otimes(E\otimes N) \arrow[r, "E\otimes\kappa"] & E\otimes N \arrow[r, "\kappa"] & N
\end{tikzcd}$$

Definition 0.9. Let (E,μ,e) be a monoid object in $\mathcal{C}$, and suppose we have two (left) module objects (N,κ) and (N',κ') over (E,μ,e). Then a morphism $f : N\to N'$ is a *(left) E-module homomorphism* if the following diagram commutes in $\mathcal{C}$:

$$\begin{tikzcd}
E\otimes N \arrow[r, "E\otimes f"] \arrow[d, "\kappa"'] & E\otimes N' \arrow[d, "\kappa'"] \\
N \arrow[r, "f"] & N'
\end{tikzcd}$$

Definition 0.10. Given a monoid object (E,μ,e) in $\mathcal{C}$, we write E-**Mod** to denote the category of (left) module objects over E and E-module homomorphisms between them. We denote the homset in E-**Mod** by

$$\mathrm{Hom}_{E\text{-}\mathbf{Mod}}(M,N), \qquad \text{or simply} \qquad \mathrm{Hom}_E(M,N).$$

For our purposes, we will only consider left module objects, so we will usually drop the quanitfier "left" and just refer to them as "module objects".

Lemma 0.11. *Let (E,μ,e) be a monoid object in $\mathcal{C}$ and let (N,κ) be an E module object. Then given some object X in $\mathcal{C}$ and an isomorphism $\phi : N\xrightarrow{\cong} X$, X inherits the structure of an E-module via the action map*

$$\kappa_\phi : E\otimes X\xrightarrow{E\otimes\phi^{-1}} E\otimes N\xrightarrow{\kappa} N\xrightarrow{\phi} X.$$

Proof. We need to show the two coherence diagrams in **??** commute. To see the former commutes, consider the following diagram:

$$\begin{tikzcd}
X \arrow[rr, "e\otimes X"] \arrow[dr, "\phi^{-1}"] \arrow[dddrr, equal] & & E\otimes X \arrow[d, "E\otimes\phi^{-1}"] \\
& N \arrow[r, "e\otimes N"] \arrow[dr, equal] & E\otimes N \arrow[d, "\kappa"] \\
& & N \arrow[d, "\phi"] \\
& & X
\end{tikzcd}$$

The top trapezoid commutes by functoriality of $-\otimes-$. The middle small triangle commutes by unitality of κ. The remaining region commutes by definition. To see the second coherence diagram commutes, consider the following diagram:

$$\begin{tikzcd}
E\otimes E\otimes X \arrow[rrr, "\mu\otimes X"] \arrow[d, "E\otimes E\otimes\phi^{-1}"'] & & & E\otimes X \arrow[d, "E\otimes\phi^{-1}"] \\
E\otimes E\otimes N \arrow[rrr, "\mu\otimes N"] \arrow[d, "E\otimes\kappa"'] & & & E\otimes N \arrow[d, "\kappa"] \\
E\otimes N \arrow[rrr, "\kappa"] \arrow[d, "E\otimes\phi"'] \arrow[dr, equal] & & & N \arrow[d, "\phi"] \\
E\otimes X \arrow[r, "E\otimes\phi^{1-}"'] & E\otimes N \arrow[r, "\kappa"'] & N \arrow[r, "\phi"'] & X
\end{tikzcd}$$

The top rectangle commutes by functoriality of $-\otimes-$. The middle rectangle commutes by coherence for κ. The bottom two regions commute by definition. $\square$

Proposition 0.12. *Given a monoid object (E,μ,e) in $\mathcal{C}$, the forgetful functor E-**Mod** $\to \mathcal{C}$ has a left adjoint $\mathcal{C}\to E$-**Mod** sending an object X in $\mathcal{C}$ to $(E\otimes X,\kappa_X)$ where κ_X is the composition*

$$E\otimes(E\otimes X)\xrightarrow{\alpha^{-1}}(E\otimes E)\otimes X\xrightarrow{\mu\otimes X}E\otimes X,$$

and sending a morphism $f:X\to Y$ to $E\otimes f:E\otimes X\to E\otimes Y$.

*We call this functor $E\otimes -:\mathcal{C}\to E$-**Mod** the* free *functor, and we call E-modules in the image of the free functor* free *modules.*

Proof. In this proof, we work in a symmetric strict monoidal category. First, we wish to show that $E\otimes -:\mathcal{C}\to E$-**Mod** as constructed is well-defined. First, to see that (X,κ_X) is actually a E-module, we need to show the two diagrams in **??** commute. Indeed, consider the following diagrams:

$$\begin{tikzcd}
E\otimes X \arrow[r, "e\otimes E\otimes X"] \arrow[dr, equal] & E\otimes E\otimes X \arrow[d, "\mu\otimes X"] \\
& E\otimes X
\end{tikzcd}
\qquad
\begin{tikzcd}
E\otimes E\otimes E\otimes X \arrow[r, "\mu\otimes E\otimes X"] \arrow[d, "E\otimes\mu\otimes X"'] & E\otimes E\otimes X \arrow[d, "\mu\otimes X"] \\
E\otimes E\otimes X \arrow[r, "\mu\otimes X"'] & E\otimes X
\end{tikzcd}$$

These are precisely the diagrams obtained by applying $X\otimes -$ to the coherence diagrams for μ, so that they commute as desired. Now, suppose $f:X\to Y$ is a morphism in $\mathcal{C}$, then we would like to show that $E\otimes f : E\otimes X \to E\otimes Y$ is a morphism of E-module objects. Indeed, consider the following diagram:

$$\begin{tikzcd}
E\otimes E\otimes X \arrow[r, "E\otimes E\otimes f"] \arrow[d, "\mu\otimes X"'] & E\otimes E\otimes Y \arrow[d, "\mu\otimes Y"] \\
E\otimes X \arrow[r, "E\otimes f"] & E\otimes Y
\end{tikzcd}$$

It commutes by functoriality of $-\otimes-$, so $E\otimes f$ is indeed an E-module homomorphism as desired.

Now, in order to see that $E\otimes -$ is left adjoint to the forgetful functor, it suffices to construct a unit and counit for the adjunction and show they satisfy the zig-zag identities. Given X in $\mathcal{C}$ and (N,κ) in E-**Mod**, define $\eta_X := e\otimes X : X \to E\otimes X$ and $\varepsilon_{(N,\kappa)} := \kappa : E\otimes N \to N$. η_X is clearly natural in X by functoriality of $-\otimes-$, and $\varepsilon_{(N,\kappa)}$ is natural in (N,κ) by how morphisms in E-**Mod** are defined. Now, to see these are actually the unit and counit of an adjunction, we need to show that the following diagrams commute for all X in $\mathcal{C}$ and (N,κ) in E-**Mod**:

$$\begin{tikzcd}
E\otimes X \arrow[r, "E\otimes\eta_X = E\otimes e\otimes X"] \arrow[dr, equal] & E\otimes E\otimes X \arrow[d, "\varepsilon_{(E\otimes X,\kappa_X)} = \mu\otimes X"] \\
& E\otimes X
\end{tikzcd}
\qquad
\begin{tikzcd}
E\otimes N \arrow[d, "\varepsilon_{(N,\kappa)} = \kappa"'] & N \arrow[l, "\eta_N = e\otimes N"'] \arrow[dl, equal] \\
N &
\end{tikzcd}$$

Commutativity of the left diagram is unitality of μ, while commutativity of the right diagram is unitality of κ. Thus indeed $E\otimes - : \mathcal{C} \to E$-**Mod** is a left adjoint of the forgetful functor E-**Mod** $\to \mathcal{C}$, as desired. □

Lemma 0.13. *Let (E,μ,e) be a monoid object in $\mathcal{C}$. Further suppose we have some object X in $\mathcal{C}$ and an E-module object (N,κ), along with a commuting diagram in $\mathcal{C}$*

$$\begin{tikzcd}
X \arrow[r, "\iota"'] \arrow[rr, bend left, equal] & N \arrow[r, "r"'] & X
\end{tikzcd}$$

Then if $\ell := \iota\circ r : N \to N$ is an E-module homomorphism, then X is canonically an E-module object with structure map

$$\kappa_X : E\otimes X \xrightarrow{E\otimes\iota} E\otimes N \xrightarrow{\kappa} N \xrightarrow{r} X,$$

and furthermore, the maps $\iota : X \to N$ and $r : N \to X$ are E-module homomorphisms.

Proof. First, in order to show (X,κ_X) is an E-module, we need to show the two diagrams in **??** commute. To see the unitality diagram holds, consider the following diagram:

$$\begin{tikzcd}
S\otimes X \arrow[rrr, "e\otimes X"] \arrow[drr, "S\otimes\iota"] \arrow[ddd, "\lambda_X"'] & & & E\otimes X \arrow[d, "E\otimes\iota"] \\
& & S\otimes N \arrow[r, "e\otimes N"] \arrow[dr, "\lambda_N"'] & E\otimes N \arrow[d, "\kappa"] \\
& & & N \arrow[d, "r"] \\
X \arrow[urrr, "\iota"] \arrow[rrr, equal] & & & X
\end{tikzcd}$$

The large left triangle commutes by naturality of λ. The top trapezoid commutes by functoriality of $-\otimes-$. The small middle right triangle commutes by unitality of κ. Finally, the bottom triangle commutes by definition, since we are assuming $r \circ \iota = \mathrm{id}_X$. Now the right composition is κ_X, so we have shown $\kappa_X \circ (e \otimes X) = \lambda_X$, as desired. Now, consider the following diagram:

$$
\begin{tikzcd}
E\otimes E\otimes X \arrow[rrr, "\mu\otimes X"] \arrow[d, "E\otimes E\otimes\iota"'] \arrow[dr, "E\otimes E\otimes\iota"] & & & E\otimes X \arrow[d, "E\otimes\iota"] \\
E\otimes E\otimes N \arrow[r, "E\otimes E\otimes\ell"'] \arrow[d, "E\otimes\kappa"'] & E\otimes E\otimes N \arrow[rr, "\mu\otimes N"] \arrow[dd, "E\otimes\kappa"] & & E\otimes N \arrow[d, "\kappa"] \\
E\otimes N \arrow[d, "E\otimes r"'] \arrow[dr, "E\otimes\ell"] & & & N \arrow[d, "r"] \\
E\otimes X \arrow[r, "E\otimes\iota"'] & E\otimes N \arrow[r, "\kappa"'] & N \arrow[ur, equal] \arrow[r, "r"'] & X
\end{tikzcd}
$$

The top trapezoid commutes by funtoriality of $-\otimes-$. The top left triangle commutes by functoriality of $-\otimes-$ and the fact that $\ell\circ\iota = \iota\circ r\circ\iota = \iota\circ\mathrm{id}_X = \iota$. The middle left trapezoid commutes by since ℓ is an E-module homomorphism, by assumption. The bottom left triangle commutes by functoriality of $-\otimes-$ and the fact that $\iota\circ r = \ell$. Thus, we have shown that (X, κ_X) is an E-module object, as desired.

Now, it remains to show that $\iota : X \to N$ and $r : N \to X$ are E-module homomorphisms. To that end, consider the following two diagrams:

$$
\begin{tikzcd}
E\otimes X \arrow[r, "E\otimes\iota"] \arrow[d, "E\otimes\iota"'] & E\otimes N \arrow[ddd, "\kappa"] \\
E\otimes N \arrow[ur, "E\otimes\ell"] \arrow[d, "\kappa"'] & \\
N \arrow[d, "r"'] \arrow[dr, "\ell"] & \\
X \arrow[r, "\iota"'] & N
\end{tikzcd}
\qquad
\begin{tikzcd}
E\otimes N \arrow[r, "E\otimes r"] \arrow[dr, "E\otimes\ell"] \arrow[ddd, "\kappa"'] & E\otimes X \arrow[d, "E\otimes\iota"] \\
& E\otimes N \arrow[d, "\kappa"] \\
& N \arrow[d, "r"] \\
N \arrow[ur, "\ell"] \arrow[r, "r"] & X
\end{tikzcd}
$$

The trapezoids in each diagram commute since we are assuming ℓ is a E-module homomorphism. The four triangles commute since $\ell\circ\iota = \iota$ and $r\circ\ell = r$. Thus, we have shown that $\kappa_X\circ(E\otimes r) = r\circ\kappa$ and $\kappa\circ(E\otimes\iota) = \iota\circ\kappa_X$, so we indeed have that ι and r are E-module homomorphisms, as desired. $\square$

Proposition 0.14. *Suppose that $\mathcal{C}$ is an additive symmetric monoidal closed category. Let (E, μ, e) be a monoid object in $\mathcal{C}$, and suppose we have a family of E-module objects (N_i, κ_i) indexed by some small set I. Then $N := \bigoplus_{i\in I} N_i$ is canonically an E-module, with action map given by the composition*

$$\kappa : E \otimes \bigoplus_i N_i \xrightarrow{\cong} \bigoplus_i (E\otimes N_i) \xrightarrow{\bigoplus_i \kappa_i} \bigoplus_i N_i,$$

*where the first isomorphism is given by the fact that $E\otimes -$ preserves coproducts, since it is a left adjoint. Furthermore, N is the coproduct of all the N_i's in E-**Mod**, so that E-**Mod** has arbitrary coproducts.*

Proof. We need to show the action map κ makes the diagrams in **??** commute. To see the first (unitality) diagram commutes, consider the following diagram:

$$\begin{tikzcd}
\bigoplus_i N_i \arrow[r, "e\otimes\bigoplus_i N_i"] \arrow[dr, "\bigoplus_i(e\otimes N_i)" description] \arrow[ddr, equal] & E\otimes\bigoplus_i N_i \arrow[d, "\cong"] \\
& \bigoplus_i(E\otimes N_i) \arrow[d, "\bigoplus_i \kappa_i"] \\
& \bigoplus_i N_i
\end{tikzcd}$$

The top triangle commutes since $E\otimes -$ preserves coproducts, as it is a left adjoint. The bottom triangle commutes by unitality of each of the κ_i's. To see the second coherence diagram commutes, consider the following diagram:

$$\begin{tikzcd}
E\otimes E\otimes\bigoplus_i N_i \arrow[rr, "\mu\oplus\bigoplus_i N_i"] \arrow[d, "E\otimes\cong"'] \arrow[dr, "\cong"] & & E\otimes\bigoplus_i N_i \arrow[d, "\cong"] \\
E\otimes\bigoplus_i(E\otimes N_i) \arrow[r, "\cong"] \arrow[d, "E\otimes\bigoplus_i \kappa_i"'] & \bigoplus_i(E\otimes E\otimes N_i) \arrow[r, "\bigoplus_i(\mu\otimes N_i)"] \arrow[d, "\bigoplus_i(E\otimes\kappa_i)"'] & \bigoplus_i(E\otimes N_i) \arrow[d, "\bigoplus_i \kappa_i"] \\
E\otimes\bigoplus_i N_i \arrow[r, "\cong"'] & \bigoplus_i(E\otimes N_i) \arrow[r, "\bigoplus_i \kappa_i"'] & \bigoplus_i N_i
\end{tikzcd}$$

The bottom right square commutes by coherence for the κ_i's. Every other region commutes since $-\otimes-$ preserves colimits in each variable. Thus $N=\bigoplus_i N_i$ is indeed an E-module object, as desired.

Now, we claim that (N,κ) is the coproduct of the (N_i,κ_i)'s in E-**Mod**. First, we need to show that the canonical maps $\iota_i : N_i \hookrightarrow N$ are morphisms in E-**Mod** for all $i\in I$. To see ι_i is a homomorphism of E-module objects, consider the following diagram:

$$\begin{tikzcd}
E\otimes N_i \arrow[r, "E\otimes\iota_i"] \arrow[dr, hook, "\iota_{E\otimes N_i}" description] \arrow[dd, "\kappa_i"'] & E\otimes\bigoplus_i N_i \arrow[d, "\cong"] \\
& \bigoplus_i(E\otimes N_i) \arrow[d, "\bigoplus_i \kappa_i"] \\
N_i \arrow[r, hook, "\iota_i"'] & \bigoplus_i N_i
\end{tikzcd}$$

The top triangle commutes by additivity of $E\otimes -$. The bottom trapezoid commutes since, by univeral property of the coproduct, $\bigoplus_i \kappa_i$ is the unique arrow which makes the trapezoid commute for all $i\in I$. Now, it remains to show that given an E-module object (N',κ') and homomorphisms $f_i : N_i \to N'$ of E-module objects for all $i\in I$, that the unique arrow $f : N\to N'$ in $\mathcal{SH}$ satisfying $f\circ\iota_i = f_i$ for all $i\in I$ is a homomorphism of E-module objects, so that N is actually the coproduct of the N_i's. To see this, first let $h : \bigoplus_i(E\otimes N_i)\to E\otimes N'$ be the arrow determined by the maps

$E \otimes N_i \xrightarrow{E\otimes f_i} E \otimes N'$. Then consider the following diagram:

$$\begin{tikzcd}
E \otimes \bigoplus_i N_i \arrow[rr, "E\otimes f"] \arrow[d, "\cong"'] & & E \otimes N' \arrow[ddd, "\kappa'"] \\
\bigoplus_i (E\otimes N_i) \arrow[r, "\bigoplus_i(E\otimes f_i)"'] \arrow[urr, "h"] \arrow[dd, "\bigoplus_i \kappa_i"'] & \bigoplus_i (E \otimes N') \arrow[ur, "\nabla"'] \arrow[d, "\bigoplus_i \kappa'"] & \\
 & \bigoplus_i N' \arrow[dr, "\nabla"] & \\
\bigoplus_i N_i \arrow[ur, "\bigoplus_i f_i"] \arrow[rr, "f"'] & & N'
\end{tikzcd}$$

The top triangle commutes by additivity of $E \otimes -$. The triangle below that commutes by the universal property of the coproduct, since it is straightforward to check that $\nabla \circ \bigoplus_i (E \otimes f_i)$ and h both satisfy the universal property of the colimit. The left trapezoid commutes by functoriality of $- \oplus -$ and the fact that f_i is a homomorphism of E-module objects for all i in I. The right trapezoid commutes by naturality of ∇. Finally, the bottom triangle commutes by the universal product of the coproduct, by showing that $\nabla \circ \bigoplus_i f_i$ in place of f also satisfies the universal property of the colimit. Hence f is inded a homomorphism of E-module objects, as desired.

To recap, we have shown that given a set of E-module objects $\{(N_i, \kappa_i)\}_{i\in I}$, the inclusion maps $\iota_i : N_i \hookrightarrow \bigoplus_i N_i$ are morphisms in E-**Mod**, and that given morphisms $f_i : (N_i, \kappa_i) \to (N', \kappa')$ for all $i \in I$, the unique induced map $\bigoplus_i N_i \to N'$ is a morphism in E-**Mod**. Thus, E-**Mod** does indeed have arbitrary coproducts, and the forgetful functor E-**Mod** $\to \mathcal{SH}$ preserves them. □

Proposition 0.15. *Suppose that $\mathcal{C}$ is an additive closed symmetric monoidal category, and let (E, μ, e) be a monoid object in $\mathcal{C}$. Then E-**Mod** is itself an additive category, so that in particular the forgetful functor E-**Mod** $\to \mathcal{C}$ and the free functor $\mathcal{C} \to E$-**Mod** (**??**) are additive.*

Proof. It is a general fact that adjoint functors between additive categories are necessarily additive. In order to show E-**Mod** is an additive category, it suffices to show it has finite coproducts, that $\mathrm{Hom}_{E\text{-}\mathbf{Mod}}(N, N')$ is an abelian group for all E-modules N and N', and that composition is bilinear. We know that E-**Mod** has coproducts which are preserved by the forgetful functor E-**Mod** $\to \mathcal{C}$ by **??** (which is clearly faithful). Thus, because $\mathcal{C}$ is **Ab**-enriched and $\mathrm{Hom}_{E\text{-}\mathbf{Mod}}(N, N') \subseteq \mathcal{C}(N, N')$, it suffices to show that $\mathrm{Hom}_{E\text{-}\mathbf{Mod}}(N, N')$ is closed under addition and taking inverses. To see the former, let $f, g : N \to N'$ be E-module homomorphisms, and consider the following diagram:

$$\begin{tikzcd}
E \otimes N \arrow[r, "E\otimes\Delta_N"] \arrow[dr, "\Delta_{E\otimes N}"'] \arrow[dd, "\kappa"'] & E \otimes (N \oplus N) \arrow[r, "E\otimes(f\oplus g)"] \arrow[d, "\cong"] & E \otimes (N' \oplus N') \arrow[r, "E\otimes\nabla_{N'}"] \arrow[d, "\cong"] & E \otimes N' \arrow[dd, "\kappa'"] \\
 & (E\otimes N) \oplus (E \otimes N) \arrow[r, "(E\otimes f)\oplus(E\otimes g)"] \arrow[d, "\kappa\oplus\kappa"] & (E \otimes N') \otimes (E \otimes N') \arrow[ur, "\nabla_{E\otimes N'}"'] \arrow[d, "\kappa'\oplus\kappa'"] & \\
N \arrow[r, "\Delta_N"] & N \oplus N \arrow[r, "f\oplus g"] & N' \oplus N' \arrow[r, "\nabla_{N'}"] & N'
\end{tikzcd}$$

The outermost trapezoids commute by naturality of Δ and ∇. The triangles in the top corners and the top middle rectangle commute by additivity of $E \otimes -$. Finally, the middle bottom rectangle commutes by functoriality of $-\oplus-$ and $-\otimes-$, and the fact that f and g are E-module homomorphisms. Commutativity of the above diagram shows that $f + g$ is a homomorphism of E-modules as desired. Finally, to see $-f$ is a E-module homomorphism if f is, we would like to show that $\kappa' \circ (E \otimes (-f)) = (-f) \circ \kappa$. This follows by the fact that $\kappa' \circ (E \otimes f) = f \circ \kappa$ and additivity of $- \otimes -$ and composition. $\square$