In this appendix, we will define the notion of A-graded anticommutative Hopf algebroids (Definition 0.2) over an A-graded anticommutative ring R (??), and left comodules over them (Definition 0.6).

0.1. A-graded anticommutative Hopf algebroids over R. Given an A-graded anticommutative ring R, we will define an A-graded anticommutative Hopf algebroid over R to be a cogroupoid object in R- $\mathbf{GCA}^A$ , i.e., a groupoid object in (R- $\mathbf{GCA}^A)^{\mathrm{op}}$ . First, recall the definition of a groupoid object in a category with pullbacks:

**Definition 0.1.** Let  $\mathcal{C}$  be a category with pullbacks. A groupoid object in  $\mathcal{C}$  consists of a pair of objects (M, O) together with five morphisms

- (1) Source and target:  $s, t: M \to O$ ,
- (2) Identity:  $e: O \to M$ ,
- (3) Composition:  $c: M \times_O M \to M$ ,
- (4) Inverse:  $i: M \to M$

Where  $M \times_O M$  will always refer to the object which into the following pullback diagram in  $\mathcal{C}$ :

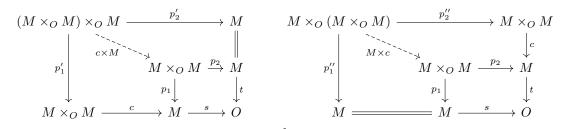
$$\begin{array}{ccc} M \times_O M & \xrightarrow{p_2} & M \\ & \downarrow t & & \downarrow t \\ M & \xrightarrow{s} & O \end{array}$$

For example, if we're working in  $\mathcal{C} = \mathbf{Set}$ , we should think of M as a set of morphisms, and O as a set of objects. The functions s and t take a morphism to their domain and codomain, respectively, and  $M \times_O M$  is the collection of pairs of morphisms  $(g, f) \in M \times M$  such that t(f) = s(g), and the composition map  $c: M \times_O M \to M$  takes such a pair to the element  $g \circ f \in M$ . We think of the identity  $e: O \to M$  as taking some object  $x \in O$  to the identity morphism  $e(x) = \mathrm{id}_x \in M$  on x, and the inverse map  $i: M \to M$  takes a morphism f to its inverse  $f^{-1}$ . These data are required to make the following diagrams commute:

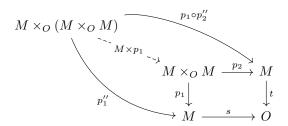
(1) Composition works correctly:

Expressed in terms of sets, the first diagram says that the target of  $g \circ f$  is the target of g. The second diagram says that the domain and codomain of the identity on some object x is x. The third diagram says that the domain of  $g \circ f$  is the domain of f.

(2) Associativity of composition: Write  $M \times_O (M \times_O M)$  and  $(M \times_O M) \times_O M$  for the pullbacks of  $(s, t \circ c)$  and  $(s \circ c, t)$ , respectively, so we have commuting diagrams



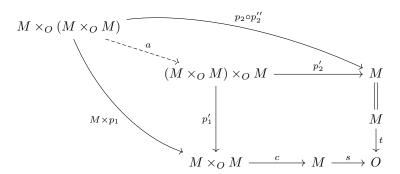
where the inner and outer squares in both diagrams are pullback squares. Furthermore, assuming the diagrams in condition (1) above are satisfied, we have that  $t \circ p_1 \circ p_2'' =$  $t \circ c \circ p_2'' = s \circ p_1''$ , so that by the universal property of the pullback we have a map  $M \times p_1 : M \times_O (M \times_O M) \to M \times_O M$  like so:



Now note that again assuming the diagrams above in (1) commute, we have  $s \circ c = s \circ p_2$ , so that

$$s \circ c \circ (M \times p_1) = s \circ p_2 \circ (M \times p_1) = s \circ p_1 \circ p_2'' = t \circ p_2 \circ p_2''.$$

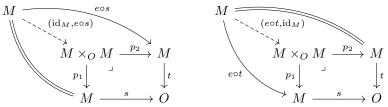
Then by the unviersal property of the pullback we get a map  $a: M \times_O (M \times_O M) \to$  $(M \times_O M) \times_O M$  like so:

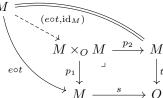


Exercise: Show that this map a is an isomorphism. Then we require that the following diagram commutes:

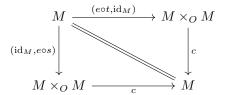
Expressed in terms of sets, this diagram says  $h \circ (g \circ f) = (h \circ g) \circ f$ .

(3) Unitality of composition: Given the maps  $(\mathrm{id}_M, e \circ t), (e \circ s, \mathrm{id}_M) : M \to M \times_O M$  defined by the universal property of  $M \times_O M$ :



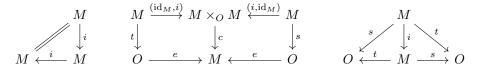


the following diagram commutes:

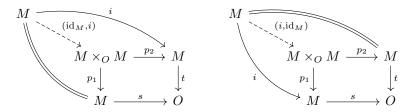


Expressed in terms of sets, this diagram says that given  $f \in M$  with s(f) = x and t(f) = y, that  $f \circ id_x = f$  and  $id_y \circ f = f$ .

(4) Inverse: The following diagrams must commute:



where the arrows  $(\mathrm{id}_M, i)$  and  $(i, \mathrm{id}_M)$  are determined by the universal property of  $M \times_O M$  like so:



Expressed in terms of sets, given  $f \in M$  with s(f) = x and t(f) = y, the first diagram says that  $(f^{-1})^{-1} = f$ , the second says that  $f \circ f^{-1} = \mathrm{id}_y$  and  $f^{-1} \circ f = \mathrm{id}_x$ , and the last diagram says that the domain and codomain of  $f^{-1}$  are g and g, respectively.

It can be seen that groupoid objects in  $\mathcal{C} = \mathbf{Set}$  are precisely (small) groupoids. Now, we can state and unravel the definition of a Hopf algebroid:

**Definition 0.2.** Given an A-graded anticommutative ring R (??), an A-graded anticommutative Hopf algebroid over R is a co-groupoid object in R- $\mathbf{GCA}^A$ , i.e., a groupoid object in (R- $A\mathbf{GrCAlg})^{\mathrm{op}}$ . Explicitly, an A-graded anticommutative Hopf algebroid over E is a pair  $(\Gamma, B)$  of objects in R- $A\mathbf{GrCAlg}$  along with morphisms

- (1) left unit:  $\eta_L: B \to \Gamma$  (corresponding to t),
- (2) right unit:  $\eta_R: B \to \Gamma$  (corresponding to s),
- (3) comultiplication:  $\Psi: \Gamma \to \Gamma \otimes_B \Gamma$  (corresponding to c),
- (4) counit:  $\epsilon: \Gamma \to B$  (corresponding to e),
- (5) conjugation:  $c: \Gamma \to \Gamma$  (corresponding to i),

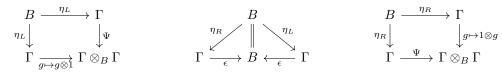
where here  $\Gamma$  may be viewed as a *B*-bimodule with left *B*-module structure induced by  $\eta_L$  and right *B*-module structure induced by  $\eta_R$ , so we may form the tensor product of bimodules  $\Gamma \otimes_B \Gamma$ , which further may be given the structure of an *A*-graded anticommutative *R*-algebra (by ??), and

fits into the following pushout diagram in R- $\mathbf{GCA}^A g$  (??):

$$\begin{array}{ccc} B & \stackrel{\eta_L}{\longrightarrow} & \Gamma \\ \downarrow^{\eta_R} \downarrow & & \downarrow^{g \mapsto 1 \otimes g} \\ \Gamma & \xrightarrow[g \mapsto g \otimes 1]{} & \Gamma \otimes_B \Gamma \end{array}$$

These data must make the following diagrams commute:

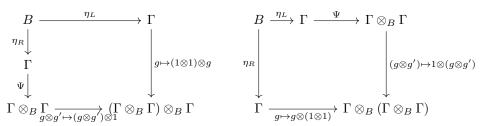
(1) (Composition works correctly)



(2) (Coassociativity) The following diagram must commute

$$\begin{array}{cccc}
\Gamma \otimes_B \Gamma & \stackrel{\Psi}{\longleftarrow} & \Gamma & \stackrel{\Psi}{\longrightarrow} & \Gamma \otimes_B \Gamma \\
\downarrow^{\Psi \otimes_B \Gamma} \downarrow & & & \downarrow^{\Gamma \otimes_B \Psi} \\
(\Gamma \otimes_B \Gamma) \otimes_B \Gamma & \stackrel{\cong}{\longrightarrow} & \Gamma \otimes_B (\Gamma \otimes_B \Gamma)
\end{array}$$

where  $(\Gamma \otimes_B \Gamma) \otimes_B \Gamma$  and  $\Gamma \otimes_B (\Gamma \otimes_B \Gamma)$  denote the rings which fit into the following pushout diagrams in R-**GCA**<sup>A</sup>:

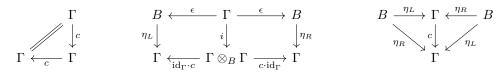


and the isomorphism  $(\Gamma \otimes_B \Gamma) \otimes_B \Gamma \to \Gamma \otimes_B (\Gamma \otimes_B \Gamma)$  sends  $(g \otimes g') \otimes g''$  to  $g \otimes (g' \otimes g'')$ , the left vertical arrow  $\Psi \otimes \Gamma$  sends  $g \otimes g'$  to  $\Psi(g) \otimes g$ , and the right vertical arrow  $\Gamma \otimes \Psi$  sends  $g \otimes g'$  to  $g \otimes \Psi(g')$ .

(3) (Co-unitality):

where the right vertical arrow sends  $g \otimes g'$  to  $\eta_L(\epsilon(g))g'$  and the bottom horizontal arrow sends  $g \otimes g'$  to  $g\eta_R(\epsilon(g'))$ .

(4) (Convolution):



where the bottom left arrow in the middle diagram sends  $g \otimes g'$  to gc(g') and the bottom right arrow in the middle diagram sends  $g \otimes g'$  to c(g)g'.

The remainder of this subsection is devoted to proving some technical lemmas about A-graded anticommutative Hopf algebroids.

**Proposition 0.3.** Suppose we have an A-graded anticommutative Hopf algebroid  $(\Gamma, B)$  over  $(R, \theta)$  with structure maps  $\eta_L$ ,  $\eta_R$ ,  $\Psi$ ,  $\epsilon$ , and c (Definition 0.2). Recall in the definition, we considered  $\Gamma \otimes_B \Gamma$  to be the A-graded R-commutative ring whose underlying abelian group was given by the tensor product of B-bimodules, where  $\Gamma$  has left B-module structure induced by  $\eta_L$  and right B-module structure induced by  $\eta_R$ . Thus  $\Gamma \otimes_B \Gamma$  is canonically a B-bimodule, as it is a tensor product of B-bimodules. Then the canonical left (resp. right) B-module structure on  $\Gamma \otimes_B \Gamma$  coincides with that induced by the ring homomorphism  $\Psi \circ \eta_L$  (resp.  $\Psi \circ \eta_R$ ).

*Proof.* First we show the left module structures coincide. By additivity, in order to show the module structures coincide, it suffices to show that given a homogeneous pure tensor  $g \otimes g'$  in  $\Gamma \otimes_B \Gamma$  and some  $b \in B$  that  $\Psi(\eta_L(b)) \cdot (g \otimes g') = (\eta_L(b) \cdot g) \otimes g'$ , where  $\cdot$  on the left denotes the product in  $\Gamma \otimes_B \Gamma$  and the  $\cdot$  on the right denotes the product in  $\Gamma$ . By the axioms for a Hopf algebroid, we have that  $\Psi(\eta_L(b)) = \eta_L(b) \otimes 1$ . Thus by how the product in  $\Gamma \otimes_B \Gamma$  is defined (??), we have that

$$\Psi(\eta_L(b)) \cdot (g \otimes g') = (\eta_L(b) \otimes 1) \cdot (g \otimes g') = (\varphi_{\Gamma}(\theta_{0,|g|}) \cdot \eta_L(b) \cdot g) \otimes (g' \cdot 1) = (\eta_L(b) \cdot g) \otimes g',$$

where  $\varphi_{\Gamma}: R \to \Gamma$  is the structure map, and the last equality follows by the fact that  $\theta_{0,|g|} = 1$ . An entirely analogous argument yields that the canonical right module structure on  $\Gamma \otimes_B \Gamma$  coincides with that induced by  $\Psi \circ \eta_R$ , since  $\Psi \circ \eta_R = 1 \otimes \eta_R$ .

**Remark 0.4.** By the above proposition, given an A-graded commutative Hopf algebroid  $(\Gamma, B)$  over R, there is no ambiguity when discussing the objects  $\Gamma \otimes_B (\Gamma \otimes_B \Gamma)$  and  $(\Gamma \otimes_B \Gamma) \otimes_B \Gamma$ —they may both be considered as the threefold tensor product of the B-bimodule  $\Gamma$  with itself. In particular, we have a canonical isomorphism of B-bimodules

$$(\Gamma \otimes_B \Gamma) \otimes_B \Gamma \to \Gamma \otimes_B (\Gamma \otimes_B \Gamma)$$

sending  $(g \otimes g') \otimes g''$  to  $g \otimes (g' \otimes g'')$ , and this is precisely the isomorphism in the coassociativity diagram in the definition of a Hopf algebroid (Definition 0.2).

**Proposition 0.5.** Suppose we have an A-graded commutative Hopf algebroid  $(\Gamma, B)$  over R with structure maps  $\eta_L$ ,  $\eta_R$ ,  $\Psi$ ,  $\epsilon$ , and c. Then  $\eta_L : B \to \Gamma$  is a homomorphism of left B-modules,  $\eta_R : B \to \Gamma$  is a homomorphism of right B-modules, and  $\Psi : \Gamma \to \Gamma \otimes_B \Gamma$  and  $\epsilon : \Gamma \to B$  are homomorphisms of B-bimodules.

*Proof.* Since the left (resp. right) B-module structure on  $\Gamma$  is induced by  $\eta_L$  (resp.  $\eta_R$ ), the map  $\eta_L$  (resp.  $\eta_R$ ) is a homomorphism of left (resp. right) B-modules by definition.

Next, we want to show  $\Psi$  is a homomorphism of B-bimodules. The left (resp. right) B-module structure on  $\Gamma$  is that induced by  $\eta_L$  (resp.  $\eta_R$ ), and in Proposition 0.3, we showed that the left (resp. right) B-module structure on  $\Gamma \otimes_B \Gamma$  is that induced by  $\Psi \circ \eta_L$  (resp.  $\Psi \circ \eta_R$ ), so that by definition  $\Psi : \Gamma \to \Gamma \otimes_B \Gamma$  is a homomorphism of left (resp. right) B-modules.

Lastly, we claim that  $\epsilon : \Gamma \to B$  is a homomorphism of B-bimodules. We need to show that given  $g \in \Gamma$  and  $b, b' \in B$  that  $\epsilon(\eta_L(b)g\eta_R(g')) = b\epsilon(g)b'$ . This follows from the fact that  $\epsilon$  is a ring homomorphism satisfying  $\epsilon \circ \eta_L = \epsilon \circ \eta_R = \mathrm{id}_B$ .

0.2. Comodules over a Hopf algebroid. In what follows, fix an A-graded anticommutative ring  $(R, \theta)$  and an A-graded anticommutative Hopf algebroid  $(\Gamma, B)$  over R with structure maps  $\eta_L$ ,  $\eta_R$ ,  $\Psi$ ,  $\epsilon$ , and c. We will always view  $\Gamma$  with its canonical B-bimodule structure, with left B-module structure induced by  $\eta_L$ , and right B-module structure induced by  $\eta_R$ . In particular, any tensor product over B involving  $\Gamma$  will always refer to  $\Gamma$  with this bimodule structure.

**Definition 0.6.** A left comodule over  $\Gamma$  is a pair  $(N, \Psi_N)$ , where N is a left A-graded B-module and  $\Psi_N : N \to \Gamma \otimes_B N$  is an A-graded homomorphism of left A-graded B-modules. These data are required to make the following diagrams commute

$$N \xrightarrow{\Psi_N} \Gamma \otimes_B N \qquad \qquad \Gamma \otimes_B N \xleftarrow{\Psi_N} \qquad N \xrightarrow{\Psi_N} \Gamma \otimes_B N \qquad \qquad \downarrow^{\Gamma \otimes \Psi_N} \qquad \downarrow^{\Gamma \otimes \Psi_N}$$

The maps  $\epsilon \otimes N$  and  $\Psi \otimes N$  are well-defined by Proposition 0.5, and the bottom isomorphism in the right diagram is the canonical one sending  $(g \otimes g') \otimes n \mapsto g \otimes (g' \otimes n)$ .

Given two left A-graded  $\Gamma$ -comodules  $(N_1, \Psi_{N_1})$  and  $(N_2, \Psi_{N_2})$ , a homomorphism of left A-graded comodules  $f: N_1 \to N_2$  is an A-graded homomorphism of the underlying left B-modules such that the following diagram commutes:

$$\begin{array}{ccc} N_1 & \xrightarrow{f} & N_2 \\ \Psi_{N_1} \downarrow & & \downarrow \Psi_{N_2} \\ \Gamma \otimes_B N_1 & \xrightarrow{\Gamma \otimes f} & \Gamma \otimes_B N_2 \end{array}$$

We write  $\Gamma$ -**CoMod**<sup>A</sup> for the resulting category of left A-graded comodules over  $\Gamma$ . In the above definition, we required A-graded left  $\Gamma$ -comodule homomorphisms to strictly preserve the grading, but we could have instead considered left  $\Gamma$ -comodule homomorphisms which are of degree d for some  $d \in A$ , or equivalently, the set of degree zero A-graded  $\Gamma$ -comodule homomorphisms from  $N_1$  to the shifted comodule  $(N_2)_{*+d}$ . We denote the hom-set of degree-d A-graded left  $\Gamma$ -comodule homomorphisms from  $(N_1, \Psi_{N_1})$  to  $(N_2, \Psi_{N_2})$  by

$$\operatorname{Hom}_{\Gamma\text{-}\mathbf{CoMod}^A}^d(N_1, N_2)$$
 or usually just  $\operatorname{Hom}_{\Gamma}^d(N_1, N_2)$ .

In particular, we simply write  $\operatorname{Hom}_{\Gamma\text{-}\mathbf{CoMod}^A}(N_1, N_2)$  or  $\operatorname{Hom}_{\Gamma}(N_1, N_2)$  for the set of strictly degree preserving (degree 0) A-graded left  $\Gamma$ -comodule homomorphisms from  $(N_1, \Psi_{N_1})$  to  $(N_2, \Psi_{N_2})$ .

**Proposition 0.7.** The category  $\Gamma$ -CoMod<sup>A</sup> is an additive category.

Proof. First, we show the category is **Ab**-enriched. Since the forgetful functor  $\Gamma$ -**CoMod**<sup>A</sup>  $\to$  B-**Mod**<sup>A</sup> is clearly faithful, we may view hom-sets in  $\Gamma$ -**CoMod**<sup>A</sup> as subsets of hom-groups in B-**Mod**<sup>A</sup>, so that in order to show  $\Gamma$ -**CoMod**<sup>A</sup> is **Ab**-enriched, it suffices to show that hom-sets in  $\Gamma$ -**CoMod**<sup>A</sup> are closed under addition of module homomorphisms and taking inverses. To that end, suppose we have two A-graded left  $\Gamma$ -comodule homomorphisms  $f, g: (N_1, \Psi_{N_1}) \to (N_2, \Psi_{N_2})$ , then we have

$$\begin{split} \Psi_{N_2} \circ (f+g) &= (\Psi_{N_2} \circ f) + (\Psi_{N_2} \circ g) \\ &= ((\Gamma \otimes_B f) \circ \Psi_{N_1}) + ((\Gamma \otimes_B g) \circ \Psi_{N_1}) \\ &= ((\Gamma \otimes_B f) + (\Gamma \otimes_B g)) \circ \Psi_{N_1} \\ &= (\Gamma \otimes_B (f+g)) \circ \Psi_{N_1}, \end{split}$$

where the first equality follows since  $\Psi_{N_2}$  is a homomorphism, the second follows since f and g are left  $\Gamma$ -comodule homomorphisms, the third follows since  $\Psi_{N_1}$  is a homomorphism, and the last equality follows by definition of the tensor product of modules. Hence f+g is indeed an A-graded left  $\Gamma$ -comodule homomorphism, as desired. Now, we also claim -f is an A-graded left  $\Gamma$ -comodule homomorphism. To that end, note that

$$\Psi_{N_2} \circ (-f) = -\Psi_{N_2} \circ f = -(\Gamma \otimes_B f) \circ \Psi_{N_1} = (\Gamma \otimes_B (-f)) \circ \Psi_{N_1},$$

where the first equality follows since  $\Psi_{N_2}$  is a homomorphism, the second follows since f is an A-graded left  $\Gamma$ -comodule homomorphism, and the third equality follows by definition of the tensor product.

Thus, we've shown that the hom-sets in  $\Gamma$ -CoMod<sup>A</sup> are abelian groups, and composition is clearly bilinear, so that  $\Gamma$ -CoMod<sup>A</sup> is indeed **Ab**-enriched.

Now, in order to show  $\Gamma$ -**CoMod**<sup>A</sup> is additive, it suffices to show that it contains a zero object and has binary coproducts. First of all, it is straightforward to check that the zero left B-module is clearly an A-graded left  $\Gamma$ -comodule with structure map the unique map  $0 \to \Gamma \otimes_B 0 \cong 0$ , and that given any other A-graded left  $\Gamma$ -comodule  $(N, \Psi_N)$ , the unique homomorphisms of left B-modules  $0 \to N$  and  $N \to 0$  are left comodule homomorphisms.

Now, suppose we have two A-graded left  $\Gamma$ -comodules  $(N_1, \Psi_{N_1})$  and  $(N_2, \Psi_{N_2})$ . First, we claim their direct sum as left B-modules  $N_1 \oplus N_2$  is canonically an A-graded left  $\Gamma$ -comodule. We know that  $N_1 \oplus N_2$  is an A-graded left B-module by ??, and we can define the structure map

$$\Psi_{N_1 \oplus N_2} : N_1 \oplus N_2 \xrightarrow{\Psi_{N_1} \oplus \Psi_{N_2}} (\Gamma \otimes_B N_1) \oplus (\Gamma \otimes_B N_2) \cong \Gamma \otimes_B (N_1 \oplus N_2),$$

where the final isomorphism is the canonical one sending  $(g_1 \otimes n_1) \oplus (g_2 \otimes n_2)$  to  $(g_1 \otimes n_1) + (g_2 \otimes n_2)$ . Then

finish

**Proposition 0.8.** The forgetful functor  $\Gamma$ -CoMod<sup>A</sup>  $\rightarrow$  B-Mod<sup>A</sup> (where here B-Mod<sup>A</sup> is the category of A-graded left B-modules and degree-preserving module homomorphisms between them) has a right adjoint  $\Gamma \otimes_B - : B\text{-Mod}^A \rightarrow \Gamma\text{-CoMod}^A$  called the co-free construction, where the co-free left A-graded  $\Gamma$ -comodule on a left A-graded B-module M is the B-module  $\Gamma \otimes_B M$  equipped with the coaction

$$\Psi_{\Gamma \otimes_B M} : \Gamma \otimes_B M \xrightarrow{\Psi \otimes_B M} (\Gamma \otimes_B \Gamma) \otimes_B M \xrightarrow{\cong} \Gamma \otimes_B (\Gamma \otimes_B M).$$

Explicitly, given some  $(N, \Psi_N)$  in  $\Gamma$ -CoMod and some M in B-Mod<sup>A</sup>, the counit and unit of this adjunction are given by

$$\eta_{(N,\Psi_N)}: N \xrightarrow{\Psi_N} \Gamma \otimes_B N$$

and

$$\varepsilon_M : \Gamma \otimes_B M \xrightarrow{\epsilon \otimes_B M} B \otimes_B M \xrightarrow{\cong} M.$$

Proof.

**Proposition 0.9.** Suppose that  $\Gamma$  is flat as a right B-module, i.e., suppose  $\eta_R : B \to \Gamma$  is a flat ring homomorphism. Then the category  $\Gamma$ -CoMod<sup>A</sup> is an abelian category.

Proof.