

In what follows, we fix an abelian group A . We assume the reader is familiar with the basic theory of modules over not-necessarily-commutative rings.

Definition 0.1. An A -graded abelian group is an abelian group B along with a subgroup $B_a \leq B$ for each $a \in A$ such that the canonical map

$$\bigoplus_{a \in A} B_a \rightarrow B$$

sending $(x_a)_{a \in A}$ to $\sum_{a \in A} x_a$ is an isomorphism. Given two A -graded abelian groups B and C , a homomorphism $f : B \rightarrow C$ is a *homomorphism of A -graded abelian groups*, or just an *A -graded homomorphism*, if it preserves the grading, i.e., if it restricts to a map $B_a \rightarrow C_a$ for all $a \in A$.

It is easy to see that an A -graded abelian group B is generated by its *homogeneous* elements, that is, nonzero elements $x \in B$ such that there exists some $a \in A$ with $x \in B_a$.

Remark 0.2. Clearly the condition that the canonical map $\bigoplus_{a \in A} B_a \rightarrow B$ is an isomorphism requires that $B_a \cap B_b = 0$ if $a \neq b$. In particular, given a homogeneous element $x \in B$, there exists precisely one $a \in A$ such that $x \in B_a$. We call this a the *degree* of x , and we write $|x| = a$.

Definition 0.3. An A -graded ring is a ring R such that its underlying abelian group R is A -graded and the multiplication map $R \times R \rightarrow R$ restricts to $R_a \times R_b \rightarrow R_{a+b}$ for all $a, b \in A$. A morphism of A -graded rings is a ring homomorphism whose underlying homomorphism of abelian groups is A -graded.

Explicitly, given an A -graded ring R and homogeneous elements $x, y \in R$, we must have $|xy| = |x| + |y|$. For example, given some field k , the ring $R = k[x, y]$ is \mathbb{Z}^2 -graded, where given $(n, m) \in \mathbb{Z}^2$, $R_{n, m}$ is the subgroup of those monomials of the form $ax^n y^m$ for some $a \in k$.

Definition 0.4. Let R be an A -graded ring. A *left A -graded R -module* M is a left R -module M such that M is an A -graded abelian group and the action map $R \times M \rightarrow M$ restricts to a map $R_a \times M_b \rightarrow M_{a+b}$ for all $a, b \in A$. Right A -graded R -modules are defined similarly. Finally, an A -graded R -bimodule is an A -graded abelian group M which has the structure of both an A -graded left and right R -module such that given $r, s \in R$ and $m \in M$, $r \cdot (m \cdot s) = (r \cdot m) \cdot s$.

Morphisms between A -graded R -modules are precisely A -graded R -module homomorphisms. We write $R\text{-GrMod}$ for the category of left A -graded R -modules and $\text{GrMod-}R$ for the category of right A -graded R -modules.

Remark 0.5. It is straightforward to see that an A -graded abelian group is equivalently an A -graded \mathbb{Z} -module, where here we are considering \mathbb{Z} as an A -graded ring concentrated in degree 0. Thus any result below about A -graded modules applies equally to A -graded abelian groups.

Lemma 0.6. Given an A -graded ring R and two left (resp. right) A -graded R -modules M and N , their direct sum $M \oplus N$ is naturally a left (resp. right) A -graded R -module group by defining

$$(M \oplus N)_a := \bigoplus_{b+c=a} M_b \oplus N_c.$$

Proof. Consider the isomorphism

$$M \oplus N \cong \left(\bigoplus_{b \in A} M_b \right) \oplus \left(\bigoplus_{c \in A} N_c \right) \cong \bigoplus_{b, c \in A} M_b \oplus N_c \cong \bigoplus_{a \in A} \bigoplus_{b \in A} M_b \oplus N_{a-b} \cong \bigoplus_{a \in A} \left(\bigoplus_{b+c=a} M_b \oplus N_c \right).$$

It is entirely straightforward to see that this isomorphism is precisely the inverse of the canonical map

$$\bigoplus_{a \in A} \left(\bigoplus_{b+c=a} M_b \oplus N_c \right) \rightarrow M \oplus N,$$

so $M \oplus N$ is indeed A -graded as indicated. \square

Oftentimes when constructing A -graded rings, we do so only by defining the product of homogeneous elements, like so:

Proposition 0.7. *Suppose we have an A -graded abelian group R , a distinguished element $1 \in R_0$, and \mathbb{Z} -bilinear maps $m_{a,b} : R_a \times R_b \rightarrow R_{a+b}$ for all $a, b \in A$. Further suppose that for all $x \in R_a$, $y \in R_b$, and $z \in R_c$, we have*

$$m_{a+b,c}(m_{a,b}(x, y), z) = m_{a,b+c}(x, m_{b,c}(y, z)) \quad \text{and} \quad m_{a,0}(x, 1) = m_{0,a}(1, x) = x.$$

Then there exists a unique multiplication map $m : R \times R \rightarrow R$ which endows R with the structure of an A -graded ring and restricts to $m_{a,b}$ for all $a, b \in A$.

Proof. Given $r, s \in R$, since $R \cong \bigoplus_{a \in A} R_a$, we may uniquely decompose r and s into homogeneous elements as $r = \sum_{a \in A} r_a$ and $s = \sum_{a \in A} s_a$ with each $r_a, s_a \in R_a$ such that only finitely many of the r_a 's and s_a 's are nonzero. Then in order to define a distributive product $R \times R \rightarrow R$ which restricts to $m_{a,b} : R_a \times R_b \rightarrow R_{a+b}$, note we *must* define

$$r \cdot s = \left(\sum_{a \in A} r_a \right) \cdot \left(\sum_{b \in A} s_b \right) = \sum_{a,b \in A} r_a \cdot s_b = \sum_{a,b \in A} m_{a,b}(r_a, s_b).$$

Thus, we have shown uniqueness. It remains to show this product actually gives R the structure of a ring. First we claim that the sum on the right is actually finite. Note there exists only finitely many nonzero r_a 's and s_b 's, and if $s_b = 0$ then

$$m_{a,b}(r_a, 0) = m_{a,b}(r_a, 0 + 0) \stackrel{(*)}{=} m_{a,b}(r_a, 0) + m_{a,b}(r_a, 0) \implies m_{a,b}(r_a, 0) = 0,$$

where $(*)$ follows from bilinearity of $m_{a,b}$. A similar argument yields that $m_{a,b}(0, s_b) = 0$ for all $a, b \in A$. Hence indeed $m_{a,b}(r_a, s_b)$ is zero for all but finitely many pairs $(a, b) \in A^2$, as desired. Observe that in particular

$$(r \cdot s)_a = \sum_{b+c=a} m_{b,c}(r_b, s_c) = \sum_{b \in A} m_{b,a-b}(r_b, s_{a-b}) = \sum_{c \in A} m_{a-c,c}(r_{a-c}, s_c).$$

Now we claim this multiplication is associative. Given $t = \sum_{a \in A} t_a \in R$, we have

$$\begin{aligned}
(r \cdot s) \cdot t &= \sum_{a,b \in A} m_{a,b}((r \cdot s)_a, t_b) \\
&= \sum_{a,b \in A} m_{a,b} \left(\sum_{c \in A} m_{a-c,c}(r_{a-c}, s_c), t_b \right) \\
&\stackrel{(1)}{=} \sum_{a,b,c \in A} m_{a,b}(m_{a-c,c}(r_{a-c}, s_c), t_b) \\
&\stackrel{(2)}{=} \sum_{a,b,c \in A} m_{c,a+b-c}(r_c, m_{a-c,b}(s_{a-c}, t_b)) \\
&\stackrel{(3)}{=} \sum_{a,b,c \in A} m_{a,c}(r_a, m_{b,c-b}(s_b, t_{c-b})) \\
&\stackrel{(1)}{=} \sum_{a,c \in A} m_{a,c} \left(r_a, \sum_{b \in A} m_{b,c-b}(s_b, t_{c-b}) \right) \\
&= \sum_{a,c \in A} m_{a,c}(r_a, (s \cdot t)_c) = r \cdot (s \cdot t),
\end{aligned}$$

where each occurrence of (1) follows by bilinearity of the $m_{a,b}$'s, each occurrence of (2) is associativity of the $m_{a,b}$'s, and (3) is obtained by re-indexing by re-defining $a := c$, $b := a - c$, and $c := a + b - c$. Next, we wish to show that the distinguished element $1 \in R_0$ is a unit with respect to this multiplication. Indeed, we have

$$1 \cdot r \stackrel{(1)}{=} \sum_{a \in A} m_{0,a}(1, r_a) \stackrel{(2)}{=} \sum_{a \in A} r_a = r \quad \text{and} \quad r \cdot 1 \stackrel{(1)}{=} \sum_{a \in A} m_{a,0}(r_a, 1) \stackrel{(2)}{=} \sum_{a \in A} r_a = r,$$

where (1) follows by the fact that $m_{a,b}(0, -) = m_{a,b}(-, 0) = 0$, which we have shown above, and (2) follows by unitality of the $m_{0,a}$'s and $m_{a,0}$'s, respectively. Finally, we wish to show that this product is distributive. Indeed, we have

$$\begin{aligned}
r \cdot (s + t) &= \sum_{a,b \in A} m_{a,b}(r_a, (s + t)_b) \\
&= \sum_{a,b \in A} m_{a,b}(r_a, s_b + t_b) \\
&\stackrel{(*)}{=} \sum_{a,b \in A} m_{a,b}(r_a, s_b) + \sum_{a,b \in A} m_{a,b}(r_a, t_b) = (r \cdot s) + (r \cdot t),
\end{aligned}$$

where $(*)$ follows by bilinearity of $m_{a,b}$. An entirely analogous argument yields that $(r + s) \cdot t = (r \cdot t) + (s \cdot t)$. \square

When working with A -graded rings, we will often freely use the above proposition without comment. In what follows, fix an A -graded ring R . We will simply say “ A -graded R -module” when we are freely considering either left or right A -graded R -modules.

Remark 0.8. We often will denote an A -Graded R -module M by M_* . Given some $a \in A$, we can define the shifted A -graded abelian group M_{*+a} whose b^{th} component is M_{b+a} .

Definition 0.9. More generally, given two A -graded R -modules M and N and some $d \in A$, an R -module homomorphism $f : M \rightarrow N$ is an A -graded homomorphism of degree d if it restricts to a map $M_a \rightarrow N_{a+d}$ for all $a \in A$. Thus, an A -graded homomorphism of degree d from M

to N is equivalently an A -graded homomorphism $M_* \rightarrow N_{*+d}$ or an A -graded homomorphism $M_{*-d} \rightarrow N$. Given some $a \in A$ and left (resp. right) R -modules M and N , we will write

$$\mathrm{Hom}_R^d(M, N)$$

to denote the abelian group of A -graded homomorphisms of degree d from M to N . Thus we have an A -graded abelian group

$$\mathrm{Hom}_R^*(M, N).$$

Unless stated otherwise, an “ A -graded homomorphism” will always refer to an A -graded homomorphism of degree 0.

Definition 0.10. Let M be an A -graded R -module. Then an A -graded R -submodule is an A -graded R -module N which is a subset of M and for which the inclusion $N \hookrightarrow M$ is an A -graded homomorphism of R -modules. Equivalently, it is a submodule N one for which the canonical map

$$\bigoplus N \cap M_a \rightarrow N$$

is an isomorphism.

Lemma 0.11. Let M be an A -graded R -module. Then an R -submodule $N \leq M$ is an A -graded submodule if and only if it is generated as an R -module by homogeneous elements of M .

Proof. If $N \leq M$ is a A -graded submodule, it is generated by the set of all its homogeneous elements, which are also homogeneous elements in M , by definition.

Conversely, suppose $N \leq M$ is a submodule which is generated by homogeneous elements of M . Then define $N_a := N \cap M_a$, and consider the canonical map

$$\Phi : \bigoplus_{a \in A} N_a \rightarrow N.$$

First of all, it is surjective, as each generator of N belongs to some N_a , by definition. To see it is injective, consider the following commutative diagram:

$$\begin{array}{ccc} \bigoplus_{a \in A} N_a & \hookrightarrow & \bigoplus_{a \in A} M_a \\ \Phi \downarrow & & \downarrow \cong \\ N & \hookrightarrow & M \end{array}$$

Since Φ composes with an injection to get an injection, clearly Φ must be injective itself. We have the desired result. \square

Proposition 0.12. Given two left (resp. right) A -graded R -modules M and N and an A -graded R -module homomorphism $\varphi : M \rightarrow N$ (of possibly nonzero degree), the kernel and images of φ are A -graded submodules of M and N , respectively.

Proof. First recall that a degree d A -graded homomorphism $M \rightarrow N$ is simply an A -graded homomorphism $M_* \rightarrow N_{*+d}$, so it suffices to consider the case φ is of degree 0. Next, note that since the forgetful functor from R -modules to abelian groups preserves kernels and images, it suffices to consider the case that φ is a homomorphism of A -graded abelian groups. Finally, by [Lemma 0.11](#), it suffices to show that $\ker \varphi$ and $\mathrm{im} \varphi$ are generated by homogeneous elements of M and N , respectively.

Note that by the universal property of the coproduct in **Ab**, the data of an A -graded homomorphism of abelian groups $\varphi : M \rightarrow N$ is precisely the data of an A -indexed collection of abelian group homomorphisms $\varphi_a : M_a \rightarrow N_a$, in which case the following diagram commutes:

$$\begin{array}{ccc} \bigoplus_a M_a & \xrightarrow{\bigoplus_a \varphi_a} & \bigoplus_a N_a \\ \cong \downarrow & & \downarrow \cong \\ M & \xrightarrow{\varphi} & N \end{array}$$

Finally, the desired result follows by the purely formal fact that taking images and kernels commutes with arbitrary direct sums. \square

Proposition 0.13. *Given two left (resp. right) A -graded R -modules M and N , an A -graded submodule $K \leq N$, and an A -graded R -module homomorphism $\varphi : M \rightarrow N$ (of possibly nonzero degree), the submodule $\varphi^{-1}(K)$ of M is A -graded.*

Proof. Recall that a degree d A -graded homomorphism $M \rightarrow N$ is simply an A -graded homomorphism $M_* \rightarrow N_{*+d}$, so it suffices to consider the case φ is of degree 0. Now, let $x \in L := \varphi^{-1}(K)$. As an element of M , we may uniquely write $x = \sum_{a \in A} x_a$ where each $x_a \in M_a$. Similarly, if we set $y := \varphi(x)$, then we may uniquely write $y = \sum_{a \in A} y_a$ where each $y_a \in N_a$. Then since K is an A -graded submodule of N and $y \in K$, by definition, we have that $y_a \in K$ for each a . Finally, note that

$$\sum_{a \in A} y_a = y = \varphi(x) = \sum_{a \in A} \varphi(x_a),$$

so that $\varphi(x_a) = y_a \in K$ for all $a \in A$, so that $x_a \in L$ for all $a \in A$. Thus we have shown that each element in L can be written as a sum of homogeneous elements in M , as desired. \square

Proposition 0.14. *Given an A -graded R -module M and an A -graded subgroup $N \leq M$, the quotient M/N is canonically A -graded by defining $(M/N)_a$ to be the subgroup generated by cosets represented by homogeneous elements of degree a in M . Furthermore, the canonical maps $M_a/N_a \rightarrow (M/N)_a$ taking a coset $m + N_a$ to $m + N$ are isomorphisms.*

Proof. Consider the canonical map

$$\Phi : \bigoplus_a (M/N)_a \rightarrow M/N.$$

First of all, surjectivity of Φ follows by commutativity of the following diagram:

$$\begin{array}{ccc} \bigoplus_a M_a & \xrightarrow{\cong} & M \\ \downarrow & & \downarrow \\ \bigoplus_a (M/N)_a & \xrightarrow{\Phi} & M/N \end{array}$$

where the vertical left map sends a generator $m \in M_a$ to the coset $m + N$ in $(M/N)_a \subseteq M/N$. To see Φ is injective, suppose we are given some element $(m_a + N)_{a \in A}$ in $\bigoplus_a (M/N)_a$ such that $\sum_{a \in A} (m_a + N) = 0$ in M/N . Thus $\sum_{a \in A} m_a \in N$, and since N is A -graded this implies that each m_a belongs to $N \cap M_a = N_a$, so that in particular $m_a + N$ is zero in $(M/N)_a \subseteq M/N$, so that $(m_a + N)_{a \in A} = 0$ in $\bigoplus_a (M/N)_a$, as desired.

It remains to show that the canonical map

$$\varphi_a : M_a/N_a \rightarrow (M/N)_a$$

is an isomorphism. It is clearly surjective, as $(M/N)_a$ is generated by elements $m + N$ for $m \in M_a$, and these elements make up precisely the image of φ_a . Thus φ_a hits every generator of $(M/N)_a$, so φ_a is surjective. On the other hand, suppose we are given some $m \in M_a$ such that $\varphi(m + N_a) = m + N = 0$. Thus $m \in N$, and $m \in M_a$, so that $m \in M_a \cap N = N_a$, meaning $m + N_a = 0$ in M_a/N_a , as desired. \square

Proposition 0.15. *Suppose we have a right A -graded R -module M and a left A -graded R -module N . Then the tensor product*

$$M \otimes_R N$$

is naturally an A -graded abelian group by defining $(M \otimes_R N)_a$ to be the subgroup generated by homogeneous pure tensors $m \otimes n$ with $m \in M_b$ and $n \in N_c$ such that $b + c = a$. Furthermore, if either M (resp. N) is an A -graded bimodule, then this decomposition makes $M \otimes_R N$ into a left (resp. right) A -graded R -module

Proof. By definition, since M and N are A -graded abelian groups, they are generated (as abelian groups) by their homogeneous elements. Thus it follows that $M \otimes_R N$ is generated by *homogeneous pure tensors*, that is, elements of the form $m \otimes n$ with $m \in M$ and $n \in N$ homogeneous. Now, given a homogeneous pure tensor $m \otimes n$, we define its *degree* by the formula $|m \otimes n| := |m| + |n|$. It follows this formula is well-defined by checking that given homogeneous elements $m \in M$, $n \in N$, and $r \in R$ that

$$|(m \cdot r) \otimes n| = |m \cdot r| + |n| = |m| + |r| + |n| = |m| + |r \cdot n| = |m \otimes (r \cdot n)|.$$

Thus, we may define $(M \otimes_R N)_a$ to be the subgroup of $M \otimes_R N$ generated by those pure homogeneous tensors of degree a . Now, consider the map

$$\Psi : M \times N \rightarrow \bigoplus_{a \in A} (M \otimes_R N)_a$$

which takes a pair $(m, n) = \sum_{a \in A} (m_a, n_a)$ to the element $\Psi(m, n)$ whose a^{th} component is

$$(\Psi(m, n))_a := \sum_{b+c=a} m_b \otimes n_c.$$

It is straightforward to see that this map is R -balanced, in the sense that it is additive in each argument and $\Psi(m \cdot r, n) = \Psi(m, r \cdot n)$ for all $m \in M$, $n \in N$, and $r \in R$. Thus by the universal property of $M \otimes_R N$, we get a homomorphism of abelian groups $\tilde{\Psi} : M \otimes_R N \rightarrow \bigoplus_{a \in A} (M \otimes_R N)_a$ lifting Ψ along the canonical map $M \times N \rightarrow M \otimes_R N$. Now, also consider the canonical map

$$\Phi : \bigoplus_{a \in A} (M \otimes_R N)_a \rightarrow M \otimes_R N.$$

We would like to show $\tilde{\Psi}$ and Φ are inverses of each other. Since $\tilde{\Psi}$ and Φ are both homomorphisms, it suffices to show this on generators. Let $m \otimes n$ be a homogeneous pure tensor with $m = m_a \in M_a$ and $n = n_b \in N_b$. Then we have

$$\Phi(\tilde{\Psi}(m \otimes n)) = \Phi\left(\bigoplus_{a \in A} \sum_{b+c=a} m_b \otimes n_c\right) \stackrel{(*)}{=} \Phi(m \otimes n) = m \otimes n,$$

and

$$\tilde{\Psi}(\Phi(m \otimes n)) = \tilde{\Psi}(m \otimes n) = \bigoplus_{a \in A} \sum_{b+c=a} m_b \otimes n_c \stackrel{(*)}{=} m \otimes n,$$

where both occurrences of $(*)$ follow by the fact that $m_b \otimes n_c = 0$ unless $b = c = a$, in which case $m_a \otimes n_a = m \otimes n$. Thus since Φ is an isomorphism, $M \otimes_R N$ is indeed an A -graded abelian group, as desired.

Now, suppose that M is an A -graded R -bimodule, so there exists left and right A -graded actions of R on M such that given $r, s \in R$ and $m \in M$ we have $r \cdot (m \cdot s) = (r \cdot m) \cdot s$. Then we would like to show that given a left A -graded R -module N that $M \otimes_R N$ is canonically a left A -graded R -module. Indeed, define the action of R on $M \otimes_R N$ on pure tensors by the formula

$$r \cdot (m \otimes n) = (r \cdot m) \otimes n.$$

First of all, clearly this map is A -graded, as if $r \in R_a$, $m \in M_b$, and $n \in N_c$ then $(r \cdot m) \otimes n$, by definition, has degree $|r \cdot m| + |n| = |r| + |m| + |n|$ (the last equality follows since the left action of R on M is A -graded). In order to show the above map defines a left module structure, it suffices to show that given pure tensors $m \otimes n, m' \otimes n' \in M \otimes_R N$ and elements $r, r' \in R$ that

- (1) $r \cdot (m \otimes n + m' \otimes n') = r \cdot (m \otimes n) + r \cdot (m' \otimes n')$,
- (2) $(r + r') \cdot (m \otimes n) = r \cdot (m \otimes n) + r' \cdot (m \otimes n)$,
- (3) $(rr') \cdot (m \otimes n) = r \cdot (r' \cdot (m \otimes n))$, and
- (4) $1 \cdot (m \otimes n) = m \otimes n$.

Axiom (1) holds by definition. To see (2), note that by the fact that R acts on M on the left that

$$(r + r') \cdot (m \otimes n) = ((r + r') \cdot m) \otimes n = (r \cdot m + r' \cdot m) \otimes n = r \cdot m \otimes n + r' \cdot m \otimes n.$$

That (3) and (4) hold follows similarly by the fact that $(rr') \cdot m = r \cdot (r' \cdot m)$ and $1 \cdot m = m$.

Conversely, if N is an A -graded R -bimodule, then showing $M \otimes_R N$ is canonically a right A -graded R -module via the rule

$$(m \otimes n) \cdot r = m \otimes (n \cdot r)$$

is entirely analogous. □

Lemma 0.16. *Let R be an A -graded ring, and suppose we have a right A -graded R -module M and a left A -graded R -module N . Then given an A -graded abelian group B and an A -graded R -balanced homomorphism*

$$\varphi : M \times N \rightarrow B$$

(here $M \times N$ is regarded as an A -graded abelian group by [Lemma 0.6](#)), the lift

$$\tilde{\varphi} : M \otimes_R N \rightarrow B$$

determined by the universal property of $M \otimes_R N$ is an A -graded homomorphism.

Proof. This simply amounts to unravelling definitions. Recall that the subgroup of homogeneous elements of degree a in $M \otimes_R N$ is that generated by pure tensors $m \otimes n$ with m and n homogeneous satisfying $|m| + |n| = a$. Thus, in order to show $\tilde{\varphi}$ is an A -graded homomorphism, it suffices to show that given homogeneous $m \in M$ and $n \in N$, we have

$$|\tilde{\varphi}(m \otimes n)| = |m \otimes n| = |m| + |n|.$$

Indeed, given two such elements, consider the following diagram

$$\begin{array}{ccc} M \otimes_R N & & \\ \uparrow & \searrow \tilde{\varphi} & \\ M \times N & \xrightarrow{\varphi} & B \end{array}$$

This diagram commutes by universal property of $- \otimes_R -$. Note that the element $m \otimes n$ is mapped to by the pair (m, n) along the left vertical map. Hence by commutativity, we necessarily have

$$|\tilde{\varphi}(m \otimes n)| = |\varphi(m, n)| \stackrel{(*)}{=} |(m, n)| = |m| + |n|,$$

where $(*)$ follows by the fact that φ is an A -graded homomorphism. \square