We assume the reader is familiar with additive categories and (closed, symmetric) monoidal categories.

Definition 0.1. A *triangulated category* is a tuple $(\mathcal{C}, \Sigma, \Omega, \mathcal{D})$ such that

(1) $\mathcal{C}$ is an additive category.
(2) $\Sigma, \Omega : \mathcal{C} \to \mathcal{C}$ are additive functors which form an adjoint equivalence of $\mathcal{C}$ with itself. (Σ is calld the *shift functor*.)
(3) $\mathcal{D}$ is a collection of *distinguished triangles*, where a *triangle* is a diagram of the form

$$X \to Y \to Z \to \Sigma X.$$

These are also sometimes called *cofiber sequences* or *fiber sequences*.

These data must satisfy the following axioms:

TR0 Given a commutative diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d, "\cong"'] & Y \arrow[r] \arrow[d, "\cong"'] & Z \arrow[r] \arrow[d, "\cong"'] & \Sigma X \arrow[d, "\cong"'] \\ X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & \Sigma X' \end{tikzcd}$$

where the vertical arrows are isomorphisms, if the top row is distinguished then so is the bottom.

TR1 For any object X in $\mathcal{C}$, the diagram

$$X \xrightarrow{\mathrm{id}_X} X \to 0 \to \Sigma X$$

is a distinguished triangle.

TR2 For all $f : X \to Y$ there exists an object C_f (also sometimes denoted Y/X) called the *cofiber of* f and a distinguished triangle

$$X \xrightarrow{f} Y \to C_f \to \Sigma X.$$

TR3 Given a solid diagram with both rows commutative

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] & Y \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d, dashed] & \Sigma X \arrow[d, "\Sigma f"] \\ X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & \Sigma X' \end{tikzcd}$$

such that the leftmost square commmutes and both rows are distinguished, there exists a dashed arrow $Z \to Z'$ which makes the remaining two squares commute.

TR4 A triangle

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{\Sigma} X$$

is distinguished if and only if

$$Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X \xrightarrow{-\Sigma f} \Sigma Y$$

is distinguished.

TR5 (Octahedral axiom) Given three distinguished triangles

$$X \xrightarrow{f} Y \xrightarrow{h} Y/X \to \Sigma X$$

$$Y \xrightarrow{g} Z \xrightarrow{k} Z/Y \to \Sigma Y$$

$$X \xrightarrow{g \circ f} Z \xrightarrow{l} Z/X \to \Sigma X$$

there exists a distinguished triangle

$$Y/X \xrightarrow{u} Z/X \xrightarrow{v} Z/Y \xrightarrow{w} \Sigma(Y/X)$$

such that the following diagram commutes

$$\begin{tikzcd}
X \arrow[rr, "g\circ f"] \arrow[dr, "f"'] & & Z \arrow[rr, "k"] \arrow[dr, "l"'] & & Z/Y \arrow[rr, "w"] \arrow[dr] & & \Sigma(Y/X) \\
& Y \arrow[ur, "g"'] \arrow[dr, "h"'] & & Z/X \arrow[ur, "v"'] \arrow[dr] & & \Sigma Y \arrow[ur, "\Sigma h"'] & \\
& & Y/X \arrow[ur, "u"'] \arrow[rr] & & \Sigma X \arrow[ur, "\Sigma f"'] & &
\end{tikzcd}$$

It turns out that the above definition is actually redundant; TR3 and TR4 follow from the remaining axioms (see Lemmas 2.2 and 2.4 in [1]).

We now recall several important propositions for triangulated categories:

Proposition 0.2. *Given a map $f : X \to Y$ in a triangulated category $(\mathcal{C}, \Sigma, \Omega, \mathcal{D})$, the cofiber sequence of f is unique up to isomorphism, in the sense that given any two distinguished triangles*

$$X \xrightarrow{f} Y \to Z \to \Sigma X \qquad \textit{and} \qquad X \xrightarrow{f} Y \to Z' \to \Sigma X,$$

there exists an isomorphism $Z \to Z'$ which makes the following diagram commute:

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, equal] & Y \arrow[r] \arrow[d, equal] & Z \arrow[r] \arrow[d, dashed, "k"] & \Sigma X \arrow[d, equal] \\
X \arrow[r, "f"] & Y \arrow[r] & Z' \arrow[r] & \Sigma X
\end{tikzcd}$$

Proposition 0.3. *Given an arrow $f : X \to Y$ in a triangulated category $(\mathcal{C}, \Sigma, \Omega, \mathcal{D})$, there exists an object F_f called the* fiber *of f, and a distinguished triangle*

$$F_f \to X \xrightarrow{f} Y \to \Sigma F_f (\cong C_f).$$

Proposition 0.4. *Let $(\mathcal{C}, \Sigma, \Omega, \mathcal{D})$ be a triangulated category. Given a distinguished triangle*

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$$

and any object A in $\mathcal{C}$, there is a long exact sequence of abelian groups

$$\cdots \to [\Sigma^{n+1}A, Z] \xrightarrow{\partial} [\Sigma^n X, X] \xrightarrow{f_*} [\Sigma^n A, Y] \xrightarrow{g_*} [\Sigma^n A, Z] \xrightarrow{\partial} [\Sigma^{n-1}A, X] \to \cdots$$

extending infinitely in either direction, where for $n < 0$ we define $\Sigma^{-n} := \Omega^n$, and ∂ is the map

$$[\Sigma^{n+1}A, Z] \xrightarrow{h_*} [\Sigma^{n+1}A, \Sigma X] \cong [\Sigma^{-1}\Sigma^{n+1}A, X] \cong [\Sigma^n A, X].$$

Also important for our work is the concept of a *tensor triangulated category*, that is, a triangulated symmetric monoidal category in which the triangulated structures are compatible, in the following sense:

Definition 0.5. A *tensor triangulated category* is a triangulated symmetric monoidal category $(\mathcal{C}, \otimes, S, \Sigma, \Omega, \mathcal{D})$ such that:

TT1 For all objects X and Y in $\mathcal{C}$, there are natural isomorphisms

$$e_{X,Y} : \Sigma X \otimes Y \xrightarrow{\cong} \Sigma(X \otimes Y).$$

TT2 For each object X in $\mathcal{C}$, the functor $X \otimes (-) \cong (-) \otimes X$ is an additive functor.

TT3 For each object X in $\mathcal{C}$, the functor $X \otimes (-) \cong (-) \otimes X$ preserves distinguished triangles, in that given a distinguished triangle/(co)fiber sequence

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A,$$

then also

$$X \otimes A \xrightarrow{X\otimes f} X \otimes B \xrightarrow{X\otimes g} X \otimes C \xrightarrow{X\otimes h} \Sigma(X \otimes A)$$

and

$$A \otimes X \xrightarrow{f\otimes X} B \otimes X \xrightarrow{g\otimes X} C \otimes X \xrightarrow{h\otimes X} \Sigma(A \otimes X)$$

are distinguished triangles, where here we are being abusive and writing $X \otimes h$ and $h \otimes X$ to denote the compositions

$$X \otimes C \xrightarrow{X\otimes h} X \otimes \Sigma A \xrightarrow{\tau} \Sigma A \otimes X \xrightarrow{e_{A,X}} \Sigma(A \otimes X) \xrightarrow{\Sigma\tau} \Sigma(X \otimes A)$$

and

$$C \otimes X \xrightarrow{h\otimes X} \Sigma A \otimes X \xrightarrow{e_{A,X}} \Sigma(A \otimes X),$$

respectively.

TT4 Given objects X, Y, and Z in $\mathcal{C}$, the following diagram must commute:

$$\begin{tikzcd}
(\Sigma X \otimes Y) \otimes Z \arrow[r, "e_{X,Y} \otimes Z"] \arrow[d, "\alpha"'] & \Sigma(X \otimes Y) \otimes Z \arrow[r, "e_{X\otimes Y,Z}"] & \Sigma((X \otimes Y) \otimes Z) \arrow[d, "\alpha"] \\
\Sigma X \otimes (Y \otimes Z) \arrow[rr, "e_{X,Y\otimes Z}"'] & & \Sigma(X \otimes (Y \otimes Z))
\end{tikzcd}$$

TT5 The following diagram must commute

$$\begin{tikzcd}
\Sigma S \otimes \Sigma S \arrow[r, "e_{S,\Sigma S}"] \arrow[d, "\tau"'] & \Sigma(S \otimes \Sigma S) \arrow[r, "\Sigma\lambda_{\Sigma S}"] & \Sigma^2 S \arrow[d, "-\mathrm{id}"] \\
\Sigma S \otimes \Sigma S \arrow[r, "e_{S,\Sigma S}"] & \Sigma(S \otimes \Sigma S) \arrow[r, "\Sigma\lambda_{\Sigma S}"] & \Sigma^2 S
\end{tikzcd}$$

Usually, most tensor triangulated categories that arise in nature will satisfy additional coherence axioms (see axioms TC1–TC5 in [1]), but the above definition will suffice for our purposes. To avoid the awkwardness of saying "a tensor triangulated category which is also a closed symmetric monoidal category," we introduce the following (nonstandard) terminology:

Definition 0.6. We say a tensor triangulated category $(\mathcal{C}, \otimes, S, \Sigma, \Omega)$ is *closed* if $\mathcal{C}$ is a closed symmetric monoidal category, in the sense that for each object $X \in \mathcal{C}$, the functor $- \otimes X$ has a right adjoint $F(X, -)$.

Note that given a tensor triangulated category, we have the following characterization of the shift functor:

Proposition 0.7. *Given a tensor triangulated category* $(\mathcal{C}, \otimes, S, \Sigma, \Omega)$*, there is a canonical natural isomorphism* $\Sigma S \otimes - \cong \Sigma$.

Proof. Given an object X in $\mathcal{C}$, we have natural isomorphisms

$$\Sigma S \otimes X \xrightarrow{e_{S,X}} \Sigma(S \otimes X) \xrightarrow{\Sigma\lambda_X} \Sigma X,$$

where λ is the left unitor specified by the monoidal structure on $\mathcal{C}$. □

Because of the above proposition, when working with tensor triangulated categories we will often assume that $\Sigma = S^1 \otimes -$ for some object S^1. Note that in the definition of the tensor triangulated category, we chose isomorphisms

$$e_{X,Y} : \Sigma X \otimes Y \xrightarrow{\cong} \Sigma(X \otimes Y),$$

but we just as well could have chosen isomorphisms

$$e'_{X,Y} : X \otimes \Sigma Y \xrightarrow{\cong} \Sigma(X \otimes Y).$$

Remark 0.8. Given a tensor triangulated category $(\mathcal{C}, \otimes, S, \Sigma, \Omega, e)$, there are natural isomorphisms

$$e'_{X,Y} : X \otimes \Sigma Y \xrightarrow{\cong} \Sigma(X \otimes Y)$$

obtained via the composition

$$X \otimes \Sigma Y \xrightarrow{\tau} \Sigma Y \otimes X \xrightarrow{e_{Y,X}} \Sigma(Y \otimes X) \xrightarrow{\Sigma\tau} \Sigma(X \otimes Y).$$

Proposition 0.9. *The isomorphisms $e'_{X,Y} : X \otimes \Sigma Y \to \Sigma(X \otimes Y)$ defined in the above remark satisfy the following coherence condition for any objects X, Y, and Z:*

$$\begin{tikzcd}
(X \otimes Y) \otimes \Sigma Z \arrow[rr, "e'_{X \otimes Y, \Sigma Z}"] \arrow[d, "\alpha"'] & & \Sigma((X \otimes Y) \otimes Z) \arrow[d, "\Sigma\alpha"] \\
X \otimes (Y \otimes \Sigma Z) \arrow[r, "X \otimes e'_{Y,Z}"] & X \otimes \Sigma(Y \otimes Z) \arrow[r, "e'_{X, Y \otimes Z}"] & \Sigma(X \otimes (Y \otimes Z))
\end{tikzcd}$$

Proof. By the coherence theorem for monoidal categories, we may assume associativity holds up to strict equality, in which case we simply wish to show that the following diagram commutes:

$$\begin{tikzcd}
X \otimes Y \otimes \Sigma Z \arrow[r, "X \otimes e'_{Y,Z}"] \arrow[dr, "e'_{X \otimes Y, Z}"'] & X \otimes \Sigma(Y \otimes Z) \arrow[d, "e'_{X, Y \otimes Z}"] \\
 & \Sigma(X \otimes Y \otimes Z)
\end{tikzcd}$$

Now consider the following diagram:

$$\begin{tikzcd}
X \otimes Y \otimes \Sigma Z \arrow[r, "X \otimes \tau_{Y, \Sigma Z}"] \arrow[d, "\tau_{X \otimes Y, \Sigma Z}"'] & X \otimes \Sigma Z \otimes Y \arrow[r, "X \otimes e_{Z,Y}"] \arrow[d, "\tau_{X, \Sigma Z \otimes Y}"] & X \otimes \Sigma(Z \otimes Y) \arrow[r, "X \otimes \Sigma\tau_{Z,Y}"] & X \otimes \Sigma(Y \otimes Z) \arrow[d, "\tau_{X, \Sigma(Y \otimes Z)}"] \\
\Sigma Z \otimes X \otimes Y \arrow[r, "\Sigma Z \otimes \tau_{X,Y}"] \arrow[dd, "e_{Z, X \otimes Y}"'] & \Sigma Z \otimes Y \otimes X \arrow[r, "e_{Z, Y \otimes X}"] \arrow[d, "e_{Z, Y \otimes X}"'] & \Sigma(Z \otimes Y) \otimes X \arrow[r, "\Sigma\tau_{Z,Y} \otimes X"] \arrow[dl, "e_{Z \otimes Y, X}"] & \Sigma(Y \otimes Z) \otimes X \arrow[d, "e_{Y \otimes Z, X}"] \\
 & \Sigma(Z \otimes Y \otimes X) \arrow[rr, "\Sigma(\tau_{Z,Y} \otimes X)"] \arrow[d, "\Sigma\tau_{Z \otimes Y, X}"] & & \Sigma(Y \otimes Z \otimes X) \arrow[d, "\Sigma\tau_{Y \otimes Z, X}"] \\
\Sigma(Z \otimes X \otimes Y) \arrow[r, "\Sigma(\tau_{Z,X} \otimes Y)"] \arrow[ur, "\Sigma(Z \otimes \tau_{X,Y})"] \arrow[rrr, bend right, "\Sigma\tau_{Z, X \otimes Y}"'] & \Sigma(X \otimes Z \otimes Y) & & \Sigma(X \otimes Y \otimes Z)
\end{tikzcd}$$

Unravelling definitions, the top composition is $e'_{X,Y \otimes Z} \circ X \otimes e'_{Y,Z}$ and the bottom composition is $e'_{X \otimes Y, Z}$, so it suffices to show this diagram commutes. The top left square commutes by coherence for symmetric monoidal categories. The trapezoid below that on the left commutes by naturality of e. The triangle below that commutes by coherence for symmetric monoidal categories. The top right rectangle commutes by functoriality of $- \otimes -$ and naturality of τ. The small triangle below that in the middle of the diagram commutes by axiom TT4 for a tensor triangulated category. Commutativity of the trapezoid on the middle right is naturality of e. Finally, the remaining region on the bottom commutes by coherence for symmetric monoidal categories. $\square$