In this section, we will freely use the coherence theorem for a symmetric monoidal category, which says that every symmetric monoidal category is (monoidally) equivalent to a *permutative category*, that is, a symmetric monoidal category in which the associators and unitors are strict equalities. add ref

Definition 0.1. Let $(\mathcal{C}, \otimes, S)$ be a symmetric monoidal category with left unitor, right unitor, and associator, and symmetry isomorphism λ, ρ, α, and τ, respectively. Then a *monoid object* (E, μ, e) is an object E in $\mathcal{C}$ along with a multiplication map $\mu : E \otimes E \to E$ and a unit map $e : S \to E$ such that the following diagram commutes:

$$\begin{tikzcd} E \otimes S \arrow[r, "E\otimes e"] \arrow[dr, "\rho"'] & E \otimes E \arrow[d, "\mu"] & S \otimes E \arrow[l, "e\otimes E"'] \arrow[dl, "\lambda"] \\ & E & \end{tikzcd} \qquad \begin{tikzcd} (E \otimes E) \otimes E \arrow[rr, "\mu\otimes E"] \arrow[d, "\alpha"'] & & E \otimes E \arrow[d, "\mu"] \\ E \otimes (E \otimes E) \arrow[r, "E\otimes\mu"] & E \otimes E \arrow[r, "\mu"] & E \end{tikzcd}$$

The first diagram expresses unitality, while the second expressed associativity. If in addition the following diagram commutes,

$$\begin{tikzcd} E \otimes E \arrow[rr, "\tau"] \arrow[dr, "\mu"'] & & E \otimes E \arrow[dl, "\mu"] \\ & E & \end{tikzcd}$$

then we say (E, μ, e) is a *commutative* monoid object.

Proposition 0.2. *Let (E_1, μ_1, e_1) and (E_2, μ_2, e_2) be monoid objects in a symmetric monoidal category $(\mathcal{C}, \otimes, S)$. Then $E_1 \otimes E_2$ is canonically a ring spectrum via the maps*

$$\mu : E_1 \otimes E_2 \otimes E_1 \otimes E_2 \xrightarrow{E_1\otimes\tau\otimes E_2} E_1 \otimes E_1 \otimes E_2 \otimes E_2 \xrightarrow{\mu_1\otimes\mu_2} E_1 \otimes E_2$$

and

$$e : S \cong S \otimes S \xrightarrow{e_1\otimes e_2} E_1 \otimes E_2.$$

Proof. □ todo

In what follows, fix a stable homotopy category $\mathcal{SH}$ (**??**) along with the additional data therewithin, and adopt the conventions outlined in **??**. Further suppose we have fixed a coherent family of isomorphisms

$$\phi_{a,b} : S^{a+b} \xrightarrow{\cong} S^a \otimes S^b,$$

in the sense of **??** (the existence of such a coherent family is guaranteed by **??**).

Proposition 0.3. *Let (E, μ, e) be a commutative monoid object in $\mathcal{SH}$, and consider the multiplication map $\pi_*(E) \times \pi_*(E) \to \pi_*(E)$ which sends classes $x : S^a \to E$ and $y : S^b \to E$ to the composition*

$$S^{a+b} \xrightarrow{\phi_{a,b}} S^a \otimes S^b \xrightarrow{x\otimes y} E \otimes E \xrightarrow{\mu} E.$$

Then this endows $\pi_*(E)$ with the structure of an A-graded ring with unit $e \in \pi_0(E) = [S, E]$.

Proof. In this proof, we will assume we are working in a permutative category. Suppose we have classes x, y, and z in $\pi_a(E)$, $\pi_b(E)$, and $\pi_c(E)$, respectively. To see associativity, consider the following diagram:

$$\begin{tikzcd}
 & & & E\otimes E \arrow[d, "\mu"] \\
S^{a+b+c} \arrow[r, "\cong"] & S^a\otimes S^b\otimes S^c \arrow[r, "x\otimes y\otimes z"] & E\otimes E\otimes E \arrow[ur, "\mu\otimes E"] \arrow[dr, "E\otimes\mu"'] & E \\
 & & & E\otimes E \arrow[u, "\mu"']
\end{tikzcd}$$

(here the first arrow is the unique isomorphism obtained by composing products of $\phi_{a,b}$'s, see **??**). It commutes by associativity of μ. It follows by functoriality of $-\otimes-$ that the top composition is $(x\cdot y)\cdot z$ while the bottom is $x\cdot(y\cdot z)$, so they are equal as desired. To see that $e\in\pi_0(E)$ is a left and right unit for this multiplication, consider the following diagram

$$\begin{tikzcd}
 & S^a \arrow[dl, "e\otimes x"'] \arrow[d, "x"] \arrow[dr, "x\otimes e"] & \\
E\otimes E \arrow[dr, "\mu"'] & E \arrow[l, "e\otimes E"'] \arrow[r, "E\otimes e"] \arrow[d, equal] & E\otimes E \arrow[dl, "\mu"] \\
 & E &
\end{tikzcd}$$

Commutativity of the two top triangles is functoriality of $-\otimes-$. Commutativity of the bottom two triangles is unitality of μ. Thus the diagram commutes, so $e\cdot x = x\cdot e$. Finally, to see this product is bilinear (distributive). Suppose we further have some $x'\in\pi_a(E)$ and $y'\in\pi_b(E)$, and consider the following diagrams:

$$\begin{tikzcd}
S^{a+b} \arrow[r, "\phi_{a,b}"] \arrow[d, "\Delta"'] & S^a\otimes S^b \arrow[r, "\Delta\otimes S^b"] \arrow[d, "\Delta"] & (S^a\oplus S^a)\otimes S^b \arrow[r, "(x\oplus x')\otimes y"] \arrow[dl, "\cong"'] & (E\oplus E)\otimes E \arrow[dl, "\cong"'] \arrow[d, "\nabla\otimes E"] & \\
S^{a+b}\oplus S^{a+b} \arrow[r, "\phi_{a,b}\oplus\phi_{a,b}"'] & (S^a\otimes S^b)\oplus(S^a\otimes S^b) \arrow[r, "(x\otimes y)\oplus(x'\otimes y)"'] & (E\otimes E)\oplus(E\otimes E) \arrow[r, "\nabla"'] & E\otimes E \arrow[r, "\mu"] & E
\end{tikzcd}$$

$$\begin{tikzcd}
S^{a+b} \arrow[r, "\phi_{a,b}"] \arrow[d, "\Delta"'] & S^a\otimes S^b \arrow[r, "S^a\otimes\Delta"] \arrow[d, "\Delta"] & S^b\otimes(S^b\oplus S^b) \arrow[r, "x\otimes(y\oplus y')"] \arrow[dl, "\cong"'] & E\otimes(E\oplus E) \arrow[dl, "\cong"'] \arrow[d, "E\otimes\nabla"] & \\
S^{a+b}\oplus S^{a+b} \arrow[r, "\phi_{a,b}\oplus\phi_{a,b}"'] & (S^a\otimes S^b)\oplus(S^a\otimes S^b) \arrow[r, "(x\otimes y)\oplus(x\otimes y')"'] & (E\otimes E)\oplus(E\otimes E) \arrow[r, "\nabla"'] & E\otimes E \arrow[r, "\mu"] & E
\end{tikzcd}$$

The unlabeled isomorphisms are those given by the fact that $-\otimes-$ is additive in each variable (since $\mathcal{SH}$ is tensor triangulated). Commutativity of the left squares is naturality of $\Delta : X\to X\oplus X$ in an additive category. Commutativity of the rest of the diagram follows again from the fact that $-\otimes-$ is an additive functor in each variable. Hence, by functoriality of $-\otimes-$, these diagrams tell us that $(x+x')\cdot y = x\cdot y + x'\cdot y$ and $x\cdot(y+y') = x\cdot y + x\cdot y'$, respectively. $\square$

Proposition 0.4. *For all $a,b\in A$ there exists an element $\theta_{a,b}\in\pi_0(S)=[S,S]$ (determined by choice of coherent family $\{\phi_{a,b}\}$) such that given any commutative monoid object (E,μ,e) in $\mathcal{SH}$, the A-graded ring structure on $\pi_*(E)$ (**??**) has a commutativity formula given by*

$$x\cdot y = y\cdot x\cdot(e\circ\theta_{a,b})$$

for all $x\in\pi_a(E)$ and $y\in\pi_b(E)$. In particular, $\theta_{a,b}\in\operatorname{Aut}(S)$ is the composition

$$S\xrightarrow{\cong} S^{-a-b}\otimes S^a\otimes S^b\xrightarrow{S^{-a-b}\otimes\tau} S^{-a-b}\otimes S^b\otimes S^a\xrightarrow{\cong} S,$$

*where the outermost maps are the unique maps specified by **??**.*

Proof. Let $\phi_{a,b}$, E, x, and y as in the statement of the proposition. Now consider the following diagram

$$\begin{tikzcd}
S^{a+b} \arrow[r, "\phi_{a,b}"] \arrow[dd, dashed, "\phi_{b,a}^{-1}\circ\tau\circ\phi_{a,b}"'] & S^a \otimes S^b \arrow[r, "x\otimes y"] \arrow[dd, "\tau"'] & E \otimes E \arrow[dr, "\mu"] \arrow[dd, "\tau"'] & \\
 & & & E \\
S^{a+b} \arrow[r, "\phi_{b,a}"] & S^b \otimes S^a \arrow[r, "y\otimes x"] & E \otimes E \arrow[ur, "\mu"'] &
\end{tikzcd}$$

The left square commutes by definition. The middle square commutes by naturality of the symmetry isomorphism. Finally, the right square commutes by commutativity of E. Unravelling definitions, we have shown that under the product on $\pi_*(E)$ induced by the $\phi_{a,b}$'s,

$$x \cdot y = (y \cdot x) \circ (\phi_{b,a}^{-1} \circ \tau \circ \phi_{a,b}).$$

Thus, in order to show the desired result it further suffices to show that

$$(y \cdot x) \circ (\phi_{b,a}^{-1} \circ \tau \circ \phi_{a,b}) = y \cdot x \cdot (e \circ \theta_{a,b}).$$

Consider the following diagram:

$$\begin{tikzcd}
S^{a+b} \arrow[rr, "\phi_{a,b}"] \arrow[d, "\cong"'] & & S^a \otimes S^b \arrow[d, "\tau"] \\
S^b \otimes S^a \otimes S^{-a-b} \otimes S^a \otimes S^b \arrow[urr, "\cong"'] \arrow[d, "S^b\otimes S^a\otimes S^{-a-b}\otimes\tau"'] & & S^b \otimes S^a \arrow[d, "\phi_{b,a}^{-1}"] \\
S^b \otimes S^a \otimes S^{-a-b} \otimes S^b \otimes S^a \arrow[urr, "\cong"] \arrow[r, "\cong"] & S^b \otimes S^a \arrow[ur, equal] \arrow[dr, "y\otimes x"] \arrow[ddl, "y\otimes x\otimes e"'] & S^{a+b} \arrow[l, "\phi_{b,a}"'] \\
 & & E \otimes E \arrow[dll, "E\otimes E\otimes e"'] \arrow[d, equal] \\
E \otimes E \otimes E \arrow[rr, "E\otimes\mu"] \arrow[d, "\mu\otimes E"'] & & E \otimes E \arrow[d, "\mu"] \\
E \otimes E \arrow[rr, "\mu"'] & & E
\end{tikzcd}$$

Here any map simply labelled $\cong$ is an appropriate composition of copies of $\phi_{a,b}$'s, associators, and their inverses, so that each of these maps are necessarily unique by **??**. The two triangles in the top large rectangle commutes by coherence for the $\phi_{a,b}$'s. The parallelogram commutes by naturality of τ and coherence of the of $\phi_{a,b}$'s. The middle skewed triangle commutes by functoriality of $- \otimes -$. The triangle below that commutes by unitality of μ. Finally, the bottom rectangle commmutes by associativity of μ. Hence, by unravelling definitions and applying functoriality of $- \otimes -$, we get that the right composition is $(y \cdot x) \circ (\phi_{b,a}^{-1} \circ \tau \circ \phi_{a,b})$, while the left composition is $y \cdot x \cdot (e \circ \theta_{a,b})$, so they are equal as desired. $\square$

Proposition 0.5. *Given $a \in A$, we have $\theta_{0,a} = \theta_{a,0} = \mathrm{id}_S$.*

Proof. Recall $\theta_{a,0}$ is the composition

$$S \xrightarrow{\phi_{-a,a}} S^{-a} \otimes S^a \xrightarrow{S^{-a}\otimes\phi_{a,0}} S^{-a} \otimes (S^a \otimes S) \xrightarrow{S^{-a}\otimes\tau} S^{-a} \otimes (S \otimes S^a) \xrightarrow{S^{-a}\otimes\phi_{0,a}^{-1}} S^{-a} \otimes S^a \xrightarrow{\phi_{-a,a}^{-1}} S$$

By the coherence theorem for symmetric monoidal categories and the fact that $\phi_{a,0}$ and $\phi_{0,a}$ coincide with the unitors, we have that the composition

$$S^a \xrightarrow{\phi_{a,0}=\rho_{S^a}^{-1}} S^a \otimes S \xrightarrow{\tau} S \otimes S^a \xrightarrow{\phi_{0,a}^{-1}=\lambda_{S^a}} S^a$$

is precisely the identity map, so by functoriality of $-\otimes-$, we have that $\theta_{a,0}$ is the composition

$$S \xrightarrow{\phi_{-a,a}} S^{-a}\otimes S^{a} \xrightarrow{=} S^{-a}\otimes S^{a} \xrightarrow{\phi_{-a,a}^{-1}} S,$$

so $\theta_{a,0}=\mathrm{id}_S$, meaning

$$x\cdot y = y\cdot x\cdot(e\circ\theta_{a,0}) = y\cdot x\cdot e = y\cdot x,$$

where the last equality follows by the fact that e is the unit for the multiplication on $\pi_*(E)$. An entirely analagous argument yields that $\theta_{0,a}=\mathrm{id}_S$. □

Proposition 0.6. *Let X and Y be objects in $\mathcal{SH}$. Then the pairing*

$$\pi_*(X)\times\pi_*(Y)\to\pi_*(X\otimes Y)$$

sending $x: S^a\to X$ and $y: S^b\to Y$ to the composition

$$S^{a+b}\xrightarrow{\phi_{a,b}} S^a\otimes S^b\xrightarrow{x\otimes y} X\otimes Y$$

is additive in each argument.

Proof. Let $a,b\in A$, and let $x_1,x_2: S^a\to X$ and $y: S^b\to Y$. Then consider the following diagram

$$\begin{tikzcd}
S^{a+b} \arrow[r, "\cong"] & S^a\otimes S^b \arrow[r, "\Delta\otimes S^b"] \arrow[d, "\Delta"'] & (S^a\oplus S^a)\otimes S^b \arrow[dl, "\cong"'] \arrow[d, "(x_1\oplus x_2)\otimes y"] \\
& (S^a\otimes S^b)\oplus(S^a\otimes S^b) \arrow[d, "(x_1\otimes y)\oplus(x_2\otimes y)"'] & (X\oplus X)\otimes Y \arrow[dl, "\cong"'] \arrow[d, "\nabla\otimes Y"] \\
& (X\otimes Y)\oplus(X\otimes Y) \arrow[r, "\nabla"] & X\otimes Y
\end{tikzcd}$$

The isomorphisms are given by the fact that $-\otimes-$ is additive in each variable. Both triangles and the parallelogram commute since $-\otimes-$ is additive. By functoriality of $-\otimes-$, the top composition is $(x_1+x_2)\cdot y$ and the bottom composition is $x_1\cdot y+x_2\cdot y$, so they are equal, as desired. An entirely analagous argument yields that $x\cdot(y_1+y_2)=x\cdot y_1+x\cdot y_2$ for $x\in\pi_*(X)$ and $y_1,y_2\in\pi_*(Y)$. □

Proposition 0.7 ([1, Proposition 5.11])**.** *Let (E,μ,e) be a monoid object in $\mathcal{SH}$. Then $E_*(-)$ is a functor from $\mathcal{SH}$ to left A-graded $\pi_*(E)$-modules, where given some X in $\mathcal{SH}$, $E_*(X)$ may be endowed with the structure of a left A-graded $\pi_*(E)$-module via the map*

$$\pi_*(E)\times E_*(X)\to E_*(X)$$

which given $a,b\in A$, sends $x: S^a\to E$ and $y: S^b\to E\otimes X$ to the composition

$$x\cdot y: S^{a+b}\cong S^a\otimes S^b\xrightarrow{x\otimes y} E\otimes(E\otimes X)\cong(E\otimes E)\otimes X\xrightarrow{\mu\otimes X} E\otimes X.$$

Similarly, the assignment $X\mapsto X_*(E)$ is a functor from $\mathcal{SH}$ to right A-graded $\pi_*(E)$-modules, where the structure map

$$X_*(E)\times\pi_*(E)\to X_*(E)$$

sends $x: S^a\to X\otimes E$ and $y: S^b\to E$ to the composition

$$x\cdot y: S^{a+b}\cong S^a\otimes S^b\xrightarrow{x\otimes y}(X\otimes E)\otimes E\cong X\otimes(E\otimes E)\xrightarrow{X\otimes\mu} X\otimes E.$$

Finally, $E_*(E)$ is a $\pi_*(E)$-bimodule, in the sense that the left and right actions of $\pi_*(E)$ are compatible, so that given $y,z\in\pi_*(E)$ and $x\in E_*(E)$, $y\cdot(x\cdot z)=(y\cdot x)\cdot z$.

Proof. First we show that the map $\pi_*(E)\times E_*(X)\to E_*(X)$ endows $E_*(X)$ with the structure of a left $\pi_*(E)$-module. Let $a,b,c\in A$ and $x,x': S^a\to E\otimes X$, $y: S^b\to E$, and $z,z'\in S^c\to E$. Then we wish to show that:

(1) $y \cdot (x + x') = y \cdot x + y \cdot x'$,
(2) $(z + z') \cdot x = z \cdot x + z' \cdot x$,
(3) $(zy) \cdot x = z \cdot (y \cdot x)$,
(4) $e \cdot x = x$.

Axioms (1) and (2) follow by the fact that $E_*(X) = \pi_*(E \otimes X)$ and Proposition 0.6. To see (3), consider the diagram:

$$\begin{tikzcd}
 & & & & E \otimes E \otimes X \arrow[d, "\mu\otimes X"] \\
S^{a+b+c} \arrow[r, "\cong"] & S^c \otimes S^b \otimes S^a \arrow[r, "z\otimes y\otimes x"] & E \otimes E \otimes E \otimes X \arrow[urr, "E\otimes\mu\otimes X"] \arrow[drr, "\mu\otimes E\otimes X"'] & & E \otimes X \\
 & & & & E \otimes E \otimes X \arrow[u, "\mu\otimes X"']
\end{tikzcd}$$

It commutes by associativity of μ. By functoriality of $- \otimes -$, the two outside compositions equal $z \cdot (y \cdot x)$ on the top and $(z \cdot y) \cdot x$ on the bottom. Hence, they are equal, as desired.

Next, to see (4), consider the following diagram:

$$\begin{tikzcd}
S^a \arrow[rr, "x"] \arrow[dr, "x"'] \arrow[ddr, bend right, "e\otimes x"'] & & E \otimes X \\
 & E \otimes X \arrow[ur, equal] \arrow[d, "e\otimes E\otimes X"] & \\
 & E \otimes E \otimes X \arrow[uur, bend right, "\mu\otimes X"'] &
\end{tikzcd}$$

The top triangle commutes by definition. The left triangle commutes by functoriality of $- \otimes -$. The right triangle commutes by unitality of μ. The top composition is x while the bottom is $e \cdot x$, thus they are necessarily equal since the diagram commutes.

Thus, we have shown that the indicated map does indeed endow $E_*(X)$ with the structure of a left $\pi_*(E)$-module. It remains to show that $E_*(-)$ sends maps in $\mathcal{SH}$ to A-graded homomorphisms of left A-graded $\pi_*(E)$-modules. By definition, given $f : X \to Y$ in $\mathcal{SH}$, $E_*(f)$ is the map which takes a class $x : S^a \to E \otimes X$ to the composition

$$S^a \xrightarrow{x} E \otimes X \xrightarrow{E\otimes f} E \otimes Y.$$

To see this assignment is a homomorphism, suppose we are given some other $x' : S^a \to E \otimes X$ and some scalar $y : S^b \to E$. Then we would like to show $E_*(f)(x + x') = E_*(f)(x) + E_*(f)(x')$ and $E_*(f)(y \cdot x) = y \cdot E_*(f)(x)$. To see the former, consider the following diagram:

$$\begin{tikzcd}
 & & & & (E \otimes Y) \oplus (E \otimes Y) \arrow[d, "\nabla"] \\
S^a \arrow[r, "\Delta"] & S^a \oplus S^a \arrow[r, "x\oplus x'"] & (E \otimes X) \oplus (E \otimes X) \arrow[urr, "(E\otimes f)\oplus(E\otimes f)"] \arrow[drr, "\nabla"'] & & E \otimes Y \\
 & & & & E \otimes X \arrow[u, "E\otimes f"']
\end{tikzcd}$$

It commutes by naturality of ∇ in an additive category. The top composition is $E_*(f)(x) + E_*(f)(x')$, while the bottom is $E_*(f)(x+x')$, so they are equal as desired. To see that $E_*(f)(y{\cdot}x) =$

$y \cdot E_*(f)(x)$, consider the following diagram:

$$\begin{tikzcd} S^{a+b} \arrow[r, "\phi_{b,a}"] & S^b \otimes S^a \arrow[r, "y\otimes x"] & E \otimes E \otimes X \arrow[r, "E\otimes E\otimes f"] \arrow[d, "\mu\otimes X"'] & E \otimes E \otimes Y \arrow[d, "\mu\otimes Y"] \\ & & E \otimes X \arrow[r, "E\otimes f"] & E \otimes Y \end{tikzcd}$$

It commutes by functoriality of $- \otimes -$. The top composition is $E_*(f)(y \cdot x)$, while the bottom composition is $y \cdot E_*(f)(x)$, so they are equal, as desired.

Showing that $X_*(E)$ has the structure of a right $\pi_*(E)$-module and that if $f : X \to Y$ is a morphism in $\mathcal{SH}$ then the map

$$X_*(E) = [S^*, X \otimes E] \xrightarrow{(f\otimes E)_*} [S^*, Y \otimes E] = Y_*(E)$$

is an A-graded homomorphism of right A-graded $\pi_*(E)$-modules is entirely analagous.

It remains to show that $E_*(E)$ is a bimodule. Let $x : S^a \to E$, $y : S^b \to E\otimes E$, and $z : S^c \to E$, and consider the following diagram:

$$\begin{tikzcd} & & & E \otimes E \otimes E \arrow[d, "E\otimes\mu"] \\ S^{a+b+c} \arrow[r, "\cong"] & S^a \otimes S^b \otimes S^c \arrow[r, "x\otimes y\otimes z"] & E \otimes E \otimes E \otimes E \arrow[ur, "\mu\otimes E\otimes E"] \arrow[r, "\mu\otimes\mu"] \arrow[dr, "E\otimes E\otimes\mu"'] & E \otimes E \\ & & & E \otimes E \otimes E \arrow[u, "\mu\otimes E"'] \end{tikzcd}$$

Commutativity follows by functoriality of $- \otimes -$, which also tells us that the two outside compositions are $(x \cdot y) \cdot z$ (on top) and $x \cdot (y \cdot z)$ (on bottom). Hence they are equal, as desired. $\square$

Proposition 0.8 ([2, Proposition 2.2]). *Let (E, μ, e) be a monoid object in $\mathcal{SH}$ and let X be any object. Then the assignment*

$$E_*(E) \times E_*(X) \to E_*(E \otimes X)$$

which sends $x : S^a \to E \otimes E$ and $y : S^b \to E \otimes X$ to the composition

$$x \cdot y : S^{a+b} \cong S^a \otimes S^b \xrightarrow{x\otimes y} E \otimes E \otimes E \otimes X \xrightarrow{E\otimes\mu\otimes X} E \otimes E \otimes X$$

induces an A-graded homomorphism of left A-graded $\pi_*(E)$-modules

$$E_*(E) \otimes_{\pi_*(E)} E_*(X) \to E_*(E \otimes X)$$

*(where here $E_*(E)$ has a $\pi_*(E)$-bimodule structure and $E_*(X)$ has a left $\pi_*(E)$-module structure as specified by Proposition 0.7, so $E_*(E) \otimes_{\pi_*(E)} E_*(X)$ is a left A-graded $\pi_*(E)$-module by **??**). Furthermore, this homomorphism is natural in X.*

Proof. First, recall by definition of the tensor product, in order to show the assignment $E_*(E) \times E_*(X) \to E_*(E \otimes X)$ induces a homomorphism $E_*(E) \otimes_{\pi_*(E)} E_*(X) \to E_*(E \otimes X)$ of A-graded abelian groups, it suffices to show that the assignment is $\pi_*(E)$-balanced, i.e., that it is linear in each argument and satisfies $xr \cdot y = x \cdot ry$ for $x \in E_*(E)$, $y \in E_*(X)$, and $r \in \pi_*(E)$.

First, note that by the identifications $E_*(E) = \pi_*(E \otimes E)$, $E_*(X) = \pi_*(E \otimes X)$, and $E_*(E \otimes X) = \pi_*(E \otimes E \otimes X)$, and Proposition 0.6, it is straightforward to see that the assignment commutes with addition of maps in each argument. Now, let $a, b, c \in A$, $x : S^a \to E \otimes E$, $y : S^b \to E \otimes X$, and $z : S^c \to E$. Then we wish to show $xz \cdot y = x \cdot zy$. Consider the following diagram (where here we are passing to a permutative category):

$$\begin{tikzcd}[column sep=small] & & & E\otimes E\otimes E\otimes X \arrow[d, "E\otimes\mu\otimes X"] \\ S^{a+b+c} \arrow[r, "\cong"] & S^a\otimes S^c\otimes S^b \arrow[r, "x\otimes z\otimes y"] & E\otimes E\otimes E\otimes E\otimes X \arrow[ur, "E\otimes\mu\otimes E\otimes X"] \arrow[dr, "E\otimes E\otimes\mu\otimes X"'] & E\otimes E\otimes X \\ & & & E\otimes E\otimes E\otimes X \arrow[u, "E\otimes\mu\otimes X"'] \end{tikzcd}$$

It commutes by associativity of μ. By functoriality of $-\otimes-$, the top composition is given by $(xz)\cdot y$ and the bottom composition is $x\cdot(zy)$, so we have they are equal, as desired. Thus, since the map $E_*(E)\times E_*(X)\to E_*(E\otimes X)$ is $\pi_*(E)$-balanced, we have that it induces a homomorphism of abelian groups. Furthermore, by **??** it is an A-graded homomorphism of A-graded abelian groups.

In order to see this map is furthermore a homomorphism of left $\pi_*(E)$-modules, we must show that $z(x\cdot y)=zx\cdot y$, where x, y, and z are defined as above. Now consider the following diagram:

$$\begin{tikzcd}[column sep=small] & & & E\otimes E\otimes E\otimes X \arrow[d, "E\otimes\mu\otimes X"] \\ S^{a+b+c} \arrow[r, "\cong"] & S^c\otimes S^a\otimes S^b \arrow[r, "z\otimes x\otimes y"] & E\otimes E\otimes E\otimes E\otimes X \arrow[ur, "\mu\otimes E\otimes E\otimes X"] \arrow[dr, "E\otimes E\otimes\mu\otimes X"'] \arrow[r, "\mu\otimes\mu\otimes X"] & E\otimes E\otimes X \\ & & & E\otimes E\otimes E\otimes X \arrow[u, "\mu\otimes E\otimes X"'] \end{tikzcd}$$

Commutativity of the triangles is functoriality of $-\otimes-$. By functoriality of $-\otimes-$, the top composition is $zx\cdot y$, and the bottom composition is $z(x\cdot y)$. Hence they are equal, as desired, so that the map we have constructed

$$E_*(E)\otimes_{\pi_*(E)} E_*(X)\to E_*(E\otimes X)$$

is indeed an A-graded homomorphism of left A-graded $\pi_*(E)$-modules.

Next, we would like to show that this homomorphism is natural in X. Let $f:X\to Y$ in $\mathcal{SH}$. Then we would like to show the following diagram commutes:

$$\begin{tikzcd} E_*(E)\otimes_{\pi_*(E)} E_*(X) \arrow[r, "\Phi_X"] \arrow[d, "E_*(E)\otimes_{\pi_*(E)}E_*(f)"'] & E_*(E\otimes X) \arrow[d, "E_*(E\otimes f)"] \\ E_*(E)\otimes_{\pi_*(E)} E_*(Y) \arrow[r, "\Phi_Y"] & E_*(E\otimes Y) \end{tikzcd} \tag{1}$$

As all the maps here are homomorphisms, it suffices to chase generators around the diagram. In particular, suppose we are given $x:S^a\to E\otimes E$ and $y:S^b\to E\otimes X$, and consider the following diagram exhibiting the two possible ways to chase $x\otimes y$ around the diagram (as usual, we are passing to a permutative category):

$$\begin{tikzcd} S^{a+b} \arrow[r, "\phi_{a,b}"] & S^a\otimes S^b \arrow[r, "x\otimes y"] & E\otimes E\otimes E\otimes X \arrow[r, "E\otimes\mu\otimes X"] \arrow[d, "E\otimes E\otimes E\otimes f"'] & E\otimes E\otimes X \arrow[d, "E\otimes E\otimes f"] \\ & & E\otimes E\otimes E\otimes Y \arrow[r, "E\otimes\mu\otimes Y"] & E\otimes E\otimes Y \end{tikzcd}$$

This diagram commutes by functoriality of $-\otimes-$. Thus we have that diagram (1) does indeed commute, as desired. $\square$

Proposition 0.9. *Let (E, μ, e) be a flat monoid object in $\mathcal{SH}$ (**??**) and let X be any cellular object in $\mathcal{SH}$ (**??**). Then the natural homomorphism*

$$\Phi_X : E_*(E) \otimes E_*(X) \to E_*(E \otimes X)$$

given in Proposition 0.8 is an isomorphism of left $\pi_*(E)$-modules.

Proof. To start, let $\mathcal{E}$ be the collection of objects X in $\mathcal{SH}$ for which this map is an isomorphism. Then in order to show $\mathcal{E}$ contains every cellular object, it suffices to show that $\mathcal{E}$ satisfies the three conditions given for the class of cellular objects in **??**. First, we need to show that Φ is an isomorphism when $X = S^a$ for some $a \in A$. Indeed, consider the map

$$\Psi : E_*(E \otimes S^a) \to E_*(E) \otimes_{\pi_*(E)} E_*(S^a)$$

which sends a class $x : S^b \to E \otimes E \otimes S^a$ in $E_b(E \otimes S^a)$ to the pure tensor $\widetilde{x} \otimes \widetilde{e}$, where $\widetilde{x} \in E_{b-a}(E)$ is the composition

$$S^{b-a} \cong S^b \otimes S^{-a} \xrightarrow{x \otimes S^{-a}} E \otimes E \otimes S^a \otimes S^{-a} \xrightarrow{E \otimes E \otimes \phi^{-1}_{a,-a}} E \otimes E \otimes S \xrightarrow{E \otimes \rho_E} E \otimes E$$

and $\widetilde{e} \in E_a(S^a)$ is the composition

$$S^a \cong S \otimes S^a \xrightarrow{e \otimes S^a} E \otimes S^a.$$

First, note Ψ is an (A-graded) homomorphism of abelian groups: Given $x, x' \in E_b(E \otimes S^a)$, we would like to show that $\widetilde{x} \otimes \widetilde{e} + \widetilde{x}' \otimes \widetilde{e} = \widetilde{x + x'} \otimes \widetilde{e}$. It suffices to show that $\widetilde{x} + \widetilde{x}' = \widetilde{x + x'}$. To see this, consider the following diagram (again, we are passing to a permutative category):

$$\begin{tikzcd}
S^{b-a} \arrow[r, "\Delta"] \arrow[d, "\phi_{b-a}"'] & S^{b-a} \oplus S^{b-a} \arrow[d, "\phi_{b,-a} \oplus \phi_{b,-a}"] \\
S^b \otimes S^{-a} \arrow[r, "\Delta"] \arrow[d, "\Delta \otimes S^{-a}"'] & (S^b \otimes S^{-a}) \oplus (S^b \otimes S^{-a}) \arrow[d, "(x \otimes S^{-a}) \oplus (x' \otimes S^{-a})"] \\
(S^b \oplus S^b) \otimes S^{-a} \arrow[ur, "\cong"] \arrow[d, "(x \oplus x') \otimes S^{-a}"'] & (E \otimes E \otimes S^a \otimes S^{-a}) \oplus (E \otimes E \otimes S^a \otimes S^{-a}) \arrow[d, "(E \otimes E \otimes \phi^{-1}_{a,-a}) \oplus (E \otimes E \otimes \phi^{-1}_{a,-a})"] \arrow[ddl, "\nabla"] \\
((E \otimes E \otimes S^a) \oplus (E \otimes E \otimes S^a)) \otimes S^{-a} \arrow[ur, "\cong"] \arrow[d, "\nabla \otimes S^{-a}"'] & (E \otimes E) \oplus (E \otimes E) \arrow[d, "\nabla"] \\
E \otimes E \otimes S^a \otimes S^{-a} \arrow[r, "E \otimes E \otimes \phi^{-1}_{a,-a}"'] & E \otimes E
\end{tikzcd}$$

The top rectangle commutes by naturality of Δ in an additive category. The bottom triangle commutes by naturality of ∇ in an additive category. Finally, the remaining regions of the diagram commute by additivity of $- \otimes -$. By functoriality of $- \otimes -$, it follows that the left composition is $\widetilde{x + x'}$ and the right composition is $\widetilde{x} + \widetilde{x}'$, so they are equal as desired. Thus Ψ is a homomorphism of abelian groups, as desired.

Now, we claim that Ψ is an inverse to Φ, (which is enough to show Φ is an isomorphism of left $\pi_*(E)$-modules). Since Φ and Ψ are homomorphisms it suffices to check that they are inverses on generators. First, let $x : S^b \to E \otimes E \otimes S^a$ in $E_b(E \otimes S^a)$. We would like to show that $\Phi(\Psi(x)) = x$.

Consider the following diagram, where here we are passing to a permutative category:

$$\begin{tikzcd}
S^b \arrow[r, "\cong"] \arrow[dd, "x"'] & S^b \otimes S^{-a} \otimes S^a \arrow[d, dashed, "x\otimes S^{-a}\otimes S^a"] \arrow[dr, "x\otimes S^{-a}\otimes e\otimes S^a"] & \\
& E \otimes E \otimes S^a \otimes S^{-a} \otimes S^a \arrow[r, "E\otimes E\otimes S^a\otimes S^{-a}\otimes e\otimes S^a"'] & E \otimes E \otimes S^a \otimes S^{-a} \otimes E \otimes S^a \arrow[ddl, "E\otimes E\otimes \phi^{-1}_{a,-a}\otimes E\otimes S^a"] \\
E \otimes E \otimes S^a \arrow[ur, "E\otimes E\otimes S^a\otimes \phi_{-a,a}"] \arrow[r, equal] \arrow[dr, "E\otimes E\otimes e\otimes S^a"'] & E \otimes E \otimes S^a \arrow[u, dashed, "E\otimes E\otimes \phi_{a,-a}\otimes S^a"'] \arrow[d, dashed, "E\otimes E\otimes e\otimes S^a"] & \\
E \otimes E \otimes E \otimes S^a \arrow[u, "E\otimes \mu\otimes S^a"] \arrow[r, equal] & E \otimes E \otimes E \otimes S^a &
\end{tikzcd}$$

The top left trapezoid commutes since the isomorphism $S^b \xrightarrow{\cong} S^b \otimes S^{-a} \otimes S^a$ may be given as $S^b \otimes \phi_{-a,a}$ (see **??**), in which case the trapezoid commmutes by functoriality of $- \otimes -$. The triangle below that commutes by coherence for the $\phi_{a,b}$'s. The triangle below that commutes by definition. The bottom left triangle commutes by unitality for μ. The top right triangle commutes by functoriality of $- \otimes -$. Finally, the bottom right triangle commutes by functoriality of $- \otimes -$. It follows by unravelling definitions that the two outside compositions are x (on the left) and $\Phi(\Psi(x))$ (on the right), so since the diagram commutes we indeed have $\Phi(\Psi(x)) = x$, as desired.

On the other hand, suppose we are given a homogeneous pure tensor $x \otimes y$ in $E_*(E) \otimes_{\pi_*(E)} E_*(S^a)$, so $x : S^b \to E \otimes E$ and $y : S^c \to E \otimes S^a$ for some $b, c \in A$. Then we would like to show that $\Psi(\Phi(x \otimes y)) = x \otimes y$. Unravelling definitions, $\Psi(\Phi(x \otimes y))$ is the homogeneous pure tensor $\widetilde{xy} \otimes \widetilde{e}$, where $\widetilde{e} : S^a \to E \otimes S^a$ is defined above, and by functoriality of $- \otimes -$, $\widetilde{xy} : S^{b+c-a} \to E \otimes E$ is the composition

$$\begin{tikzcd}
S^{b+c-a} \arrow[d, "\phi_{b+c,-a}"] \\
S^{b+c} \otimes S^{-a} \arrow[d, "\phi_{b,c}\otimes S^{-a}"] \\
S^b \otimes S^c \otimes S^{-a} \arrow[d, "x\otimes y\otimes S^{-a}"] \\
E \otimes E \otimes E \otimes S^a \otimes S^{-a} \arrow[d, "E\otimes \mu\otimes S^a\otimes S^{-a}"] \\
E \otimes E \otimes S^a \otimes S^{-a} \arrow[d, "E\otimes E\otimes \phi^{-1}_{a,-a}"] \\
E \otimes E \otimes S \arrow[d, "E\otimes \rho_E"] \\
E \otimes E.
\end{tikzcd}$$

In order to see $x \otimes y = \widetilde{xy} \otimes \widetilde{e}$, it suffices to show there exists some scalar $r \in \pi_{c-a}(E)$ such that $x \cdot r = \widetilde{xy}$ and $r \cdot \widetilde{e} = y$, where here $\cdot$ denotes the right and left action of $\pi_*(E)$ on $E_*(E)$ and $E_*(S^a)$, respectively. Now, define r to be the composition

$$S^{c-a} \cong S^c \otimes S^{-a} \xrightarrow{y\otimes S^{-a}} E \otimes S^a \otimes S^{-a} \xrightarrow{E\otimes \phi^{-1}_{a,-a}} E \otimes S \xrightarrow{\rho_E} E.$$

First, in order to see that $x \cdot r = \widetilde{xy}$, consider the following diagram, where here we are again passing to a permutative category:

$$\begin{tikzcd}
S^{b+c-a} \arrow[r, "\cong"] & S^b \otimes S^c \otimes S^{-a} \arrow[r, "x \otimes y \otimes S^{-a}"] & E \otimes E \otimes E \otimes S^a \otimes S^{-a} \arrow[r, "E \otimes \mu \otimes S^a \otimes S^{-a}"] \arrow[d, "E \otimes E \otimes E \otimes \phi_{a,-a}^{-1}"'] \arrow[dr, "E \otimes \mu \otimes \phi_{a,-a}^{-1}"] & E \otimes E \otimes S^a \otimes S^{-a} \arrow[d, "E \otimes E \otimes \phi_{a,-a}^{-1}"] \\
 & & E \otimes E \otimes E \arrow[r, "E \otimes \mu"'] & E \otimes E
\end{tikzcd}$$

Commutativity is functoriality of $- \otimes -$, which also tells us that the two outside compositions are $\widetilde{xy}$ (on top) and $x \cdot r$ (on the bottom), so they are equal as desired. On the other hand, in order to see that $r \cdot \widetilde{e} = y$, consider the following diagram (where here we have passed to a permutative category):

$$\begin{tikzcd}
S^c \arrow[rr, "\cong"] \arrow[dd, "y"'] & & S^c \otimes S^{-a} \otimes S^a \arrow[dl, "y \otimes S^{-a} \otimes S^a"'] \arrow[dd, "y \otimes S^{-a} \otimes e \otimes S^a"] \\
 & E \otimes S^a \otimes S^{-a} \otimes S^a \arrow[dl, "E \otimes S^a \otimes \phi_{-a,a}^{-1}"'] \arrow[dr, "E \otimes S^a \otimes S^{-a} \otimes e \otimes S^a"] & \\
E \otimes S^a \arrow[drr, "E \otimes e \otimes S^a"] & & E \otimes S^a \otimes S^{-a} \otimes E \otimes S^a \arrow[d, "E \otimes \phi_{a,-a}^{-1} \otimes E \otimes S^a"] \\
E \otimes E \otimes S^a \arrow[u, "\mu \otimes S^a"] \arrow[rr, equal] & & E \otimes E \otimes S^a
\end{tikzcd}$$

The top left triangle commutes since we may take the isomorphism $S^c \xrightarrow{\cong} S^c \otimes S^{-a} \otimes S^a$ to be $S^c \otimes \phi_{-a,a}$, in which case commutativity of the triangle follows by functoriality of $- \otimes -$. Commutativity of the right triangle is also functoriality of $- \otimes -$. Commutativity of the bottom left triangle is unitality of μ. Finally, commutativity of the remaining middle 4-sided region is again functoriality of $- \otimes -$. It follows that y is equal to the outer composition, which is $r \cdot \widetilde{e}$, as desired. Thus, we have shown that

$$\Psi(\Phi(x \otimes y)) = \widetilde{xy} \otimes \widetilde{e} = (x \cdot r) \otimes \widetilde{e} = x \otimes (r \cdot \widetilde{e}) = x \otimes y,$$

as desired, so that for each $a \in A$, the object S^a belongs to the class $\mathcal{E}$.

Now, we would like to show that given a distinguished triangle

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X,$$

if two of three of the objects X, Y, and Z belong to $\mathcal{E}$, then so does the third. From now on, write $L_*^E(-)$ to denote the functor $X \mapsto E_*(E) \otimes_{\pi_*(E)} E_*(X)$, and let $\widetilde{h} : \Omega Z \to X$ denote the adjoint of $Z \to \Sigma X$, i.e., $\widetilde{h}$ is the composition

$$\Omega Z \xrightarrow{\Omega h} \Omega \Sigma X = S^{-\mathbf{1}} \otimes S^{\mathbf{1}} \otimes X \xrightarrow{\phi_{-\mathbf{1},\mathbf{1}}^{-1}} S \otimes X \xrightarrow{\lambda_X} X.$$

First, consider the following diagram:

$$\begin{tikzcd}
L_*^E(\Omega Y) \arrow[r, "L_*^E(\Omega g)"] \arrow[d, "\Phi_{\Omega Y}"'] & L_*^E(\Omega Z) \arrow[r, "L_*^E(\widetilde{h})"] \arrow[d, "\Phi_{\Omega Z}"'] & L_*^E(X) \arrow[r, "L_*^E(f)"] \arrow[d, "\Phi_X"'] & L_*^E(Y) \arrow[r, "L_*^E(g)"] \arrow[d, "\Phi_Y"'] & L_*^E(Z) \arrow[r, "L_*^E(h)"] \arrow[d, "\Phi_Z"'] & L_*^E(\Sigma X) \arrow[r, "L_*^E(\Sigma f)"] \arrow[d, "\Phi_{\Sigma X}"'] & L_*^E(\Sigma Y) \arrow[d, "\Phi_{\Sigma Y}"'] \\
E_*(E \otimes \Omega Y) \arrow[r, "E_*(E \otimes \Omega g)"'] & E_*(E \otimes \Omega Z) \arrow[r, "E_*(E \otimes h)"'] & E_*(E \otimes X) \arrow[r, "E_*(E \otimes f)"'] & E_*(E \otimes Y) \arrow[r, "E_*(E \otimes g)"'] & E_*(E \otimes Z) \arrow[r, "E_*(E \otimes h)"'] & E_*(E \otimes \Sigma X) \arrow[r, "E_*(E \otimes \Sigma f)"'] & E_*(E \otimes \Sigma Y)
\end{tikzcd}$$

why?

The diagram commutes since Φ is natural. The top and bottom rows are exact. Thus by the five lemma, it suffices to show that if Φ_X is an isomorphism, then $\Phi_{\Omega X}$ and $\Phi_{\Sigma X}$ are likewise isomorphisms. First, note that given an object X in $\mathcal{SH}$, we have an isomorphism $t^E_X : E_*\Omega X \to E_{*+1}X$ which sends a class $x : S^a \to E \otimes \Omega X = E \otimes (S^{-\mathbf{1}} \otimes X)$ to the composition

$$S^{a+\mathbf{1}} \cong S^a \otimes S^{\mathbf{1}} \xrightarrow{x \otimes S^{\mathbf{1}}} E \otimes S^{-\mathbf{1}} \otimes X \otimes S^{\mathbf{1}} \xrightarrow{E \otimes S^{-\mathbf{1}} \otimes \tau_{X,S^{\mathbf{1}}}} E \otimes S^{-\mathbf{1}} \otimes S^{\mathbf{1}} \otimes X \xrightarrow{E \otimes \phi^{-1}_{-\mathbf{1},\mathbf{1}} \otimes X} E \otimes X$$

(here we are suppressing the unitors and associators). Note also that t^E_X is a homomorphism of left $\pi_*(E)$-modules. Indeed, given $r : S^a \to E$ and $x : S^b \to E \otimes \Omega X = E \otimes S^{-\mathbf{1}} \otimes X$, consider the following diagram:

$$\begin{tikzcd}
 & & E \otimes S^{-\mathbf{1}} \otimes X \otimes S^{\mathbf{1}} \arrow[r, "E \otimes S^{-\mathbf{1}} \otimes \tau"] & E \otimes S^{-\mathbf{1}} \otimes S^{\mathbf{1}} \otimes X \arrow[d, "E \otimes \phi^{-1}_{-\mathbf{1},\mathbf{1}} \otimes X"] \\
S^{a+b+\mathbf{1}} \arrow[r, "\cong"] & S^a \otimes S^b \otimes S^{\mathbf{1}} \arrow[r, "r \otimes x \otimes S^{\mathbf{1}}"] & E \otimes E \otimes S^{-\mathbf{1}} \otimes X \otimes S^{\mathbf{1}} \arrow[u, "\mu \otimes S^{-\mathbf{1}} \otimes X \otimes S^{\mathbf{1}}"] \arrow[d, "E \otimes E \otimes S^{-\mathbf{1}} \otimes \tau"'] & E \otimes X \\
 & & E \otimes E \otimes S^{-\mathbf{1}} \otimes S^{\mathbf{1}} \otimes X \arrow[r, "E \otimes E \otimes \phi^{-1}_{-\mathbf{1},\mathbf{1}} \otimes X"'] \arrow[uur, "\mu \otimes S^{-\mathbf{1}} \otimes S^{\mathbf{1}} \otimes X"'] & E \otimes E \otimes X \arrow[u, "\mu \otimes X"']
\end{tikzcd}$$

Each triangle commutes by functoriality of $- \otimes -$. The top composition is $t^E_X(r \cdot x)$, while the bottom is $r \cdot t^E_X(x)$. Thus $t^E_X(r \cdot x) = r \cdot t^E_X(x)$, so t^E_X is indeed a homomorphism of left $\pi_*(E)$-modules, as desired. Thus, consider the following diagram:

$$\begin{tikzcd}
E_*(E) \otimes_{\pi_*(E)} E_*(\Omega X) \arrow[r, "E_*(E) \otimes t^E_X"] \arrow[d, "\Phi_{\Omega X}"'] & E_*(E) \otimes_{\pi_*(E)} E_{*+1}(X) \arrow[d, "\Phi_X"] \\
E_*(E \otimes \Omega X) \arrow[r, "t^{E \otimes E}_X"] & E_{*+1}(E \otimes X)
\end{tikzcd} \tag{2}$$

Since t^X_E is a homomorphism of left $\pi_*(E)$-modules, the top map is well-defined. We claim this diagram commutes. To see this, let $x : S^a \to E \otimes E$ and $y : S^b \to E \otimes \Omega X = E \otimes S^{-\mathbf{1}} \otimes X$, and consider the following diagram:

$$\begin{tikzcd}
S^{a+b+\mathbf{1}} \arrow[d, "\cong"] & E \otimes E \otimes E \otimes S^{-\mathbf{1}} \otimes S^{\mathbf{1}} \otimes X \arrow[r, "E \otimes E \otimes E \otimes \phi^{-1}_{-\mathbf{1},\mathbf{1}} \otimes X"] \arrow[ddr, "E \otimes \mu \otimes S^{-\mathbf{1}} \otimes S^{\mathbf{1}} \otimes X"] & E \otimes E \otimes E \otimes X \arrow[d, "E \otimes \mu \otimes X"] \\
S^a \otimes S^b \otimes S^{\mathbf{1}} \arrow[r, "x \otimes y \otimes S^{\mathbf{1}}"] & E \otimes E \otimes E \otimes S^{-\mathbf{1}} \otimes X \otimes S^{\mathbf{1}} \arrow[u, "E \otimes E \otimes E \otimes S^{-\mathbf{1}} \otimes \tau"'] \arrow[d, "E \otimes \mu \otimes S^{-\mathbf{1}} \otimes X \otimes S^{\mathbf{1}}"'] & E \otimes E \otimes X \\
 & E \otimes E \otimes S^{-\mathbf{1}} \otimes X \otimes S^{\mathbf{1}} \arrow[r, "E \otimes E \otimes S^{-\mathbf{1}} \otimes \tau"] & E \otimes E \otimes S^{-\mathbf{1}} \otimes S^{\mathbf{1}} \otimes X \arrow[u, "E \otimes E \otimes \phi^{-1}_{-\mathbf{1},\mathbf{1}} \otimes X"']
\end{tikzcd}$$

Each triangle commutes by functoriality of $- \otimes -$. The two outside compositions are the two ways to chase the homogeneous pure tensor $x \otimes y$ around diagram (2), so the diagram commutes, meaning $\Phi_{\Omega X}$ is an isomorphism, as desired. An entirely analagous argument (swap every occurrence of $-\mathbf{1}$ and $\mathbf{1}$ in the above) yields the map $s^E_X : E_*(\Sigma X) \to E_{*-\mathbf{1}}(X)$ sending a class $x : S^a \to E \otimes \Sigma X = E \otimes S^{\mathbf{1}} \otimes X$ to the composition

$$S^{a-\mathbf{1}} \cong S^a \otimes S^{-\mathbf{1}} \xrightarrow{x \otimes S^{-\mathbf{1}}} E \otimes S^{\mathbf{1}} \otimes X \otimes S^{-\mathbf{1}} \xrightarrow{E \otimes S^{\mathbf{1}} \otimes \tau} E \otimes S^{\mathbf{1}} \otimes S^{-\mathbf{1}} \otimes X \xrightarrow{E \otimes \phi^{-1}_{\mathbf{1},-\mathbf{1}} \otimes X} E \otimes X$$

is an isomorphism of left $\pi_*(E)$-modules, and that the following diagram commutes:

$$\begin{tikzcd}
E_*(E) \otimes_{\pi_*(E)} E_*(\Sigma X) \arrow[r, "E_*(E)\otimes s_X^E"] \arrow[d, "\Phi_{\Sigma X}"'] & E_*(E) \otimes_{\pi_*(E)} E_{*+\mathbf{1}}(X) \arrow[d, "\Phi_X"] \\
E_*(E \otimes \Sigma X) \arrow[r, "s_X^{E\otimes E}"'] & E_{*+\mathbf{1}}(E \otimes X)
\end{tikzcd}$$

Thus indeed we have shown that $\Phi_{\Omega X}$ and $\Phi_{\Sigma X}$ are isomorphisms if Φ_X is, so $\mathcal{E}$ satisfies two-of-three for distinguished triangles.

Finally, it remains to show that $\mathcal{E}$ is closed under arbitrary direct sums. Let $\{X_i\}_{i\in I}$ be a collection of objects in $\mathcal{E}$ indexed by some (small) set I. Then we'd like to show that $X := \bigoplus_i X_i$ belongs to $\mathcal{E}$. Note we have an isomorphism

$$t^E :$$

□

In the following definition, let $\varepsilon : E_*(E) \to \pi_*(E)$ be the map which sends some $\alpha : S^a \to E \otimes E$ to the composition

$$S^a \xrightarrow{\alpha} E \otimes E \xrightarrow{\mu} E.$$

Also define $\Psi : E_*(E) \to E_*(E) \otimes_{\pi_*(E)} E_*(E)$ to be the map which factors as

$$E_*(E) \to E_*(E \otimes E) \xrightarrow{\cong} E_*(E) \otimes_{\pi_*(E)} E_*(E)$$

where the second arrow is the isomorphism prescribed by Proposition 0.8, and the first arrow sends a class $\alpha : S^a \to E \otimes E$ to the composition

$$S^a \xrightarrow{\alpha} E \otimes E \cong E \otimes S \otimes E \xrightarrow{E\otimes e\otimes E} E \otimes E \otimes E.$$

Lemma 0.10 ([2, Proposition 2.30, 2.33])**.** *Let E be a flat commutative ring spectrum, and let X and Y be spectra such that $E_{**}(X)$ is a projective module over $\pi_{**}(E)$. Then for all $s \geq 0$ and $t, w \in \mathbb{Z}$, there is an isomorphism*

$$\Phi : [X, E \wedge Y]_{t,w} \to \operatorname{Hom}^{t,w}_{E_{**}(E)}(E_{**}(X), E_{**}(E \wedge Y)),$$

obtained by sending a class $f : S^{t,w} \wedge X \to E \wedge Y$ in $[X, E \wedge Y]_{t,w}$ to the map

$$\Phi_f : E_{*,*}(X) \to E_{*+t,*+w}(X \wedge Y)$$

sending

$$[S^{a,b} \xrightarrow{g} E \wedge X] \mapsto [S^{a+t,b+w} \cong S^{a,b} \wedge S^{t,w} \xrightarrow{g\wedge S^{t,w}} E \wedge X \wedge S^{t,w} \cong E \wedge S^{t,w} \wedge X \xrightarrow{E\wedge f} E \wedge E \wedge Y].$$

Proof. Let $f : S^{t,w} \wedge X \to E \wedge Y$. First we want to show that Φ_f is actually an $E_{**}(E)$-comodule homomorphism. □

finish