0.1. A-graded anticommutative rings and their algebras.

Proposition 0.1. *Suppose we have an A-graded anticommutative ring R (**??**) and two morphisms $f : (B, \varphi_B) \to (C, \varphi_C)$ and $g : (B, \varphi_B) \to (D, \varphi_D)$ in R-A**GrCAlg** (**??**). Then f and g make C and D both B-bimodules, respectively,[1] so we may form their tensor product $C \otimes_B D$, which is itself an A-graded B-bimodule (**??**). Then $C \otimes_B D$ canonically inherits the structure of an A-graded R-commutative ring with unit $1_C \otimes 1_D$ via a product*

$$(C \otimes_B D) \times (C \otimes_B D) \to C \otimes_B D$$

which sends a pair $(x \otimes y, x' \otimes y')$ of homogeneous pure tensors to the element

$$\varphi_B(\theta_{|x|,|y'|}) \cdot (xx' \otimes yy') = \varphi_C(\theta_{|x|,|y'|})xx' \otimes yy',$$

(where here $\cdot$ denotes the left module action of B on $C \otimes_B D$), and with structure map

$$\begin{aligned} \varphi : R &\to C \otimes_B D \\ r &\mapsto \varphi_B(r) \cdot (1_C \otimes 1_D) = (\varphi_C(r) \otimes 1_D) = (1_C \otimes \varphi_D(r)). \end{aligned}$$

Proof sketch. We simply lay out everything that needs to be shown, and we leave it to the reader to fill in the details. First to show that the indicated product is actually well-defined and distributive, by **??** it suffices to show that for all homogeneous $c, c', c'' \in C$, $d, d', d'' \in D$, and $b \in B$ with $|c'| = |c''|$ and $|d'| = |d''|$, that

$$\begin{aligned} \varphi_B(\theta_{|d|,|c'+c''|}) \cdot (c(c'+c'') \otimes dd') &= \varphi_B(\theta_{|d|,|c'|}) \cdot (cc' \otimes dd') + \varphi_B(\theta_{|d|,|c''|}) \cdot (cc'' \otimes dd') \\ \varphi_B(\theta_{|d|,|c'|}) \cdot (cc' \otimes d(d'+d'')) &= \varphi_B(\theta_{|d|,|c'|}) \cdot (cc' \otimes dd') + \varphi_B(\theta_{|d|,|c'|}) \cdot (cc' \otimes dd'') \\ \varphi_B(\theta_{|d|,|c' \cdot b|}) \cdot (c(c' \cdot b) \otimes dd') &= \varphi_B(\theta_{|d|,|c'|}) \cdot (cc' \otimes d(b \cdot d')) \\ \varphi_B(\theta_{|d'|,|c|}) \cdot ((c'+c'')c \otimes d'd) &= \varphi_B(\theta_{|d'|,|c|}) \cdot (c'c \otimes d'd) + \varphi_B(\theta_{|d'|,|c|}) \cdot (c''c \otimes d'd) \\ \varphi_B(\theta_{|d'+d''|,|c|}) \cdot (c'c \otimes (d'+d'')d) &= \varphi_B(\theta_{|d'|,|c|}) \cdot (c'c \otimes d'd) + \varphi_B(\theta_{|d''|,|c|}) \cdot (c'c \otimes d''d) \\ \varphi_B(\theta_{|d'|,|c|})((c' \cdot b)c \otimes d'd) &= \varphi_B(\theta_{|c|,|b \cdot d'|}) \cdot (c'c \otimes (b \cdot d')d), \end{aligned}$$

where each occurrence of $\cdot$ denotes the left or right module action of B. These tell us that for all $x \in C \otimes_B D$ that the maps $C \otimes_B D \to C \otimes_B D$ sending $y \mapsto xy$ and $y \mapsto yx$ are well-defined A-graded homomorphisms of abelian groups, so we have a distributive product $(x, y) \mapsto xy$. Then to show that this product makes $C \otimes_B D$ an A-graded ring, we need to show it is associative and unital. By **??**, it suffices to show that for all *homogeneous* $x, y, z \in C \otimes_B D$ that $(xy)z = x(yz)$ and $x(1_C \otimes 1_D) = x = (1_C \otimes 1_D)x$. By distributivity, it further suffices to consider the case that x, y, and z are homogeneous *pure tensors* in $C \otimes_B D$, i.e., it suffices to show that for all homogeneous $c, c', c'' \in C$ and $d, d', d'' \in D$ that

$$((c \otimes d)(c' \otimes d'))(c'' \otimes d'') = (c \otimes d)((c' \otimes d')(c'' \otimes d''))$$

and

$$(c \otimes d)(1_C \otimes 1_D) = (c \otimes d) = (1_C \otimes 1_D)(c \otimes d).$$

Thus, proving these hold will show $C \otimes_B D$ has the structure of an A-graded ring, as desired. Now, we wish to show that the given map $\varphi : R \to C \otimes_B D$ is a ring homomorphism. Clearly it sends 1 to $1_C \otimes 1_D$, and again by linearity, it suffices to show that given *homogeneous* $r, s \in R$ that

$$\varphi(r+s) = \varphi_B(r+s)(1_C \otimes 1_D) = \varphi_B(r)(1_C \otimes 1_D) + \varphi_B(s)(1_C \otimes 1_D) = \varphi(r) + \varphi(s)$$

[1] Explicitly, it is a standard fact that given a ring homomorphism $\varphi : R \to S$ that S canonically becomes an R-bimodule with left action $r \cdot s := \varphi(r)s$ and right action $s \cdot r := s\varphi(r)$, so that in particular if φ is an A-graded homomorphism of A-graded rings, then φ makes S an A-graded R-bimodule.

and
$$\varphi(rs) = \varphi_B(rs)(1_C \otimes 1_D) = (\varphi_B(r)(1_C \otimes 1_D))(\varphi_B(s)(1_C \otimes 1_D)) = \varphi(r)\varphi(s).$$
Finally, we need to show that $C \otimes_B D$ satisfies the graded commutativity condition, for which again by linearity it suffices to show that given homogeneous $c, c' \in C$ and $d, d' \in D$ that
$$(c \otimes d)(c' \otimes d') = \varphi(\theta_{|c\otimes d|,|c'\otimes d'|})(c' \otimes d')(c \otimes d) = \varphi(\theta_{|c|+|d|,|c'|+|d'|})(c' \otimes d')(c \otimes d).$$
Showing all of these is relatively straightforward. □

Proposition 0.2. *Given an A-graded anticommutative ring (R, θ), the category R-A**GrCAlg** has pushouts, where given $f : (B, \varphi_B) \to (C, \varphi_C)$ and $g : (B, \varphi_B) \to (D, \varphi_D)$, their pushout is the object $(C \otimes_B D, \varphi)$ constructed in Proposition 0.1, along with the canonical maps $(C, \varphi_C) \to (C \otimes_B D, \varphi)$ sending $c \mapsto c \otimes 1_D$ and $(D, \varphi_D) \to (C \otimes_B D, \varphi)$ sending $d \mapsto 1_C \otimes d$. In particular, since (R, id_R) is initial, R-A**GrCAlg** has binary coproducts.*

Proof sketch. First, we need to show that the given maps $i_C : (C, \varphi_C) \to (C \otimes_B D, \varphi)$ and $i_D : (D, \varphi_D) \to (C \otimes_B D, \varphi)$ are actually morphisms in **GrCAlg**R, i.e., that they are ring homomorphisms and that the following diagram commutes:
$$\begin{tikzcd} & R \arrow[dl, "\varphi_C"'] \arrow[d, dashed, "\varphi"] \arrow[dr, "\varphi_D"] & \\ C \arrow[r, "i_C"'] & C \otimes_B D & D \arrow[l, "i_D"] \end{tikzcd}$$
Showing this is entirely straightforward. Furthermore, i_C and i_D clearly make the following diagram commute:
$$\begin{tikzcd} B \arrow[r, "g"] \arrow[d, "f"'] & D \arrow[d, "i_D"] \\ C \arrow[r, "i_C"'] & C \otimes_B D \end{tikzcd}$$
It remains to show that i_C and i_D are the universal such arrows. Suppose we have some object (E, φ_E) in R-A**GrCAlg** and a commuting diagram
$$\begin{tikzcd} B \arrow[r, "g"] \arrow[d, "f"'] & D \arrow[d, "k"] \\ C \arrow[r, "h"] & E \end{tikzcd}$$
of morphisms in R-A**GrCAlg**. Then we'd like to show there exists a unique morphism $\ell : C \otimes_B D \to E$ in R-A**GrCAlg** which makes the following diagram commute:
$$\begin{tikzcd} B \arrow[r, "g"] \arrow[d, "f"'] & D \arrow[d, "i_D"'] \arrow[ddr, bend left, "k"] & \\ C \arrow[r, "i_C"] \arrow[drr, bend right, "h"'] & C \otimes_B D \arrow[dr, dashed, "\ell"] & \\ & & E \end{tikzcd}$$
First we show uniqueness. Supposing such an arrow ℓ existed, given elements $c \in C$ and $d \in D$, we must have
$$\ell(c \otimes d) = \ell((c \otimes 1_D)(1_C \otimes d)) = \ell(c \otimes 1_D)\ell(1_C \otimes d) = \ell(i_C(c))\ell(i_D(d)) = h(c)k(d).$$

Since pure tensors generate $C \otimes_B D$, we have uniquely determined ℓ, and clearly it makes the above diagram commute. Now, it remains to show that as defined ℓ is a morphism in R-A**GrCAlg**, i.e., that it is an A-graded ring homomorphism and that the following diagram commutes:

$$\begin{tikzcd} & R \arrow[dl, "\varphi"'] \arrow[dr, "\varphi_E"] & \\ C \otimes_B D \arrow[rr, "\ell"] & & E \end{tikzcd}$$

This is all entirely straightforward to show. $\square$

0.2. A-graded commutative Hopf algebroids over R.

Definition 0.3. Let $\mathcal{C}$ be a category admitting pullbacks. A *groupoid object* in $\mathcal{C}$ consists of a pair of objects (M, O) together with five morphisms

(1) *Source and target*: $s, t : M \to O$,

(2) *Identity*: $e : O \to M$,

(3) *Composition*: $c : M \times_O M \to M$,

(4) *Inverse*: $i : M \to M$

Explicitly, $M \times_O M$ fits into the following pullback diagram:

$$\begin{tikzcd} M \times_O M \arrow[r, "p_2"] \arrow[d, "p_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & M \arrow[d, "t"] \\ M \arrow[r, "s"'] & O \end{tikzcd}$$

so if we're working with sets, the composition map sends a pair (g, f) such that the codomain of f is the domain of g to $g \circ f$. These data must satisfy the following diagrams:

(1) Composition works correctly:

$$\begin{tikzcd} M \times_O M \arrow[r, "c"] \arrow[d, "p_1"'] & M \arrow[d, "t"] \\ M \arrow[r, "t"] & O \end{tikzcd} \qquad \begin{tikzcd} M \arrow[dr, "s"'] & O \arrow[l, "e"'] \arrow[r, "e"] \arrow[d, equal] & M \arrow[dl, "t"] \\ & O & \end{tikzcd} \qquad \begin{tikzcd} M \times_O M \arrow[r, "p_2"] \arrow[d, "c"'] & M \arrow[d, "s"] \\ M \arrow[r, "s"] & O \end{tikzcd}$$

The first diagram says that the codomain of $g \circ f$ is the codomain of g. The second diagram says that the domain and codomain of the identity on some object x is x. The third diagram says that the domain of $g \circ f$ is the domain of f.

(2) Associativity of composition: Write $M \times_O (M \times_O M)$ and $(M \times_O M) \times_O M$ for the pullbacks of $(s, t \circ c)$ and $(s \circ c, t)$, respectively, so we have commuting diagrams

$$\begin{tikzcd} (M \times_O M) \times_O M \arrow[rr, "p_2'"] \arrow[dd, "p_1'"'] \arrow[dr, dashed, "c \times M"'] & & M \arrow[d, equal] \\ & M \times_O M \arrow[r, "p_2"] \arrow[d, "p_1"'] & M \arrow[d, "t"] \\ M \times_O M \arrow[r, "c"] & M \arrow[r, "s"] & O \end{tikzcd} \qquad \begin{tikzcd} M \times_O (M \times_O M) \arrow[rr, "p_2''"] \arrow[dd, "p_1''"'] \arrow[dr, dashed, "M \times c"'] & & M \times_O M \arrow[d, "c"] \\ & M \times_O M \arrow[r, "p_2"] \arrow[d, "p_1"'] & M \arrow[d, "t"] \\ M \arrow[r, equal] & M \arrow[r, "s"] & O \end{tikzcd}$$

where the inner and outer squares in both diagrams are pullback squares. Furthermore, assuming the diagrams in condition (1) above are satisfied, we have that $t \circ p_1 \circ p_2'' =$

$t \circ c \circ p_2'' = s \circ p_1''$, so that by the universal property of the pullback we have a map $M \times p_1 : M \times_O (M \times_O M) \to M \times_O M$ like so:

$$\begin{tikzcd}
M \times_O (M \times_O M) \arrow[dr, dashed, "M\times p_1"] \arrow[drr, bend left, "p_1\circ p_2''"] \arrow[ddr, bend right, "p_1''"'] & & \\
& M \times_O M \arrow[r, "p_2"] \arrow[d, "p_1"'] & M \arrow[d, "t"] \\
& M \arrow[r, "s"] & O
\end{tikzcd}$$

Now note that again assuming composition works correctly, so $s \circ c = s \circ p_2$, we have

$$s \circ c \circ (M \times p_1) = s \circ p_2 \circ (M \times p_1) = s \circ p_1 \circ p_2'' = t \circ p_2 \circ p_2'',$$

so that by the unviersal property of the pullback we get a map $a : M \times_O (M \times_O M) \to (M \times_O M) \times_O M$ like so:

$$\begin{tikzcd}
M \times_O (M \times_O M) \arrow[dr, dashed, "a"] \arrow[drrr, bend left, "p_2\circ p_2''"] \arrow[dddr, bend right, "M\times p_1"'] & & & \\
& (M \times_O M) \times_O M \arrow[rr, "p_2'"] \arrow[dd, "p_1'"'] & & M \arrow[d, equal] \\
& & & M \arrow[d, "t"] \\
& M \times_O M \arrow[r, "c"] & M \arrow[r, "s"] & O
\end{tikzcd}$$

Then we require that the following diagram commutes:

$$\begin{tikzcd}
M \times_O (M \times_O M) \arrow[rr, "a"] \arrow[d, "M\times c"'] & & (M \times_O M) \times_O M \arrow[d, "c\times M"] \\
M \times_O M \arrow[r, "c"] & M & M \times_O M \arrow[l, "c"']
\end{tikzcd}$$

This diagram says $h \circ (g \circ f) = (h \circ g) \circ f$.

(3) Unitality of composition: Given the maps $(\mathrm{id}_M, e \circ t), (e \circ s, \mathrm{id}_M) : M \to M \times_O M$ defined by the universal property of $M \times_O M$:

$$\begin{tikzcd}
M \arrow[dr, dashed, "(\mathrm{id}_M, e\circ s)"] \arrow[drr, bend left, "e\circ s"] \arrow[ddr, bend right, equal] & & \\
& M \times_O M \arrow[r, "p_2"] \arrow[d, "p_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & M \arrow[d, "t"] \\
& M \arrow[r, "s"] & O
\end{tikzcd}
\qquad
\begin{tikzcd}
M \arrow[dr, dashed, "(e\circ t, \mathrm{id}_M)"] \arrow[drr, bend left, equal] \arrow[ddr, bend right, "e\circ t"'] & & \\
& M \times_O M \arrow[r, "p_2"] \arrow[d, "p_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & M \arrow[d, "t"] \\
& M \arrow[r, "s"] & O
\end{tikzcd}$$

the following diagram commutes:

$$\begin{tikzcd}
M \arrow[r, "(e\circ t, \mathrm{id}_M)"] \arrow[d, "(\mathrm{id}_M, e\circ s)"'] \arrow[dr, equal] & M \times_O M \arrow[d, "c"] \\
M \times_O M \arrow[r, "c"'] & M
\end{tikzcd}$$

This diagram says that given $f : x \to y$ in M, that $f \circ \mathrm{id}_x = f$ and $\mathrm{id}_y \circ f = f$.

(4) Inverse: The following diagrams must commute:

$$\begin{tikzcd} & M \arrow[dl, equal] \arrow[d, "i"] \\ M & M \arrow[l, "i"'] \end{tikzcd} \qquad \begin{tikzcd} M \arrow[r, "(\mathrm{id}_M, i)"] \arrow[d, "t"'] & M \times_O M \arrow[d, "c"] & M \arrow[l, "(i, \mathrm{id}_M)"'] \arrow[d, "s"] \\ O \arrow[r, "e"] & M & O \arrow[l, "e"'] \end{tikzcd} \qquad \begin{tikzcd} & M \arrow[dl, "s"'] \arrow[d, "i"] \arrow[dr, "t"] & \\ O & M \arrow[l, "t"'] \arrow[r, "s"] & O \end{tikzcd}$$

where the arrows (id_M, i) and (i, id_M) are determined by the universal property of $M \times_O M$ like so:

$$\begin{tikzcd} M \arrow[dr, dashed, "(\mathrm{id}_M, i)"] \arrow[drr, bend left, "i"] \arrow[ddr, bend right, equal] & & \\ & M \times_O M \arrow[r, "p_2"] \arrow[d, "p_1"'] & M \arrow[d, "t"] \\ & M \arrow[r, "s"] & O \end{tikzcd} \qquad \begin{tikzcd} M \arrow[dr, dashed, "(i, \mathrm{id}_M)"] \arrow[drr, bend left, equal] \arrow[ddr, bend right, "i"'] & & \\ & M \times_O M \arrow[r, "p_2"] \arrow[d, "p_1"'] & M \arrow[d, "t"] \\ & M \arrow[r, "s"] & O \end{tikzcd}$$

Given $f : x \to y$ in M, the first diagram says that $\left(f^{-1}\right)^{-1} = f$. The second says that $f \circ f^{-1} = \mathrm{id}_y$ and $f^{-1} \circ f = \mathrm{id}_x$. The last diagram says that the domain and codomain of f^{-1} are y and x, respectively.

Definition 0.4. Given an A-graded anticommutative ring (R, θ), an *A-graded anticommutative Hopf algebroid over R* is a co-groupoid object in R-A**GrCAlg**R, i.e., a groupoid object in $(R\text{-}A\mathbf{GrCAlg})^{\mathrm{op}}$. Explicitly, an A-graded anticommutative Hopf algebroid over E is a pair (B, Γ) of objects in R-A**GrCAlg** along with morphisms

(1) *left unit*: $\eta_L : B \to \Gamma$ (corresponding to t),
(2) *right unit*: $\eta_R : B \to \Gamma$ (corresponding to s),
(3) *comultiplication*: $\Psi : \Gamma \to \Gamma \otimes_B \Gamma$ (corresponding to c),
(4) *counit*: $\epsilon : \Gamma \to B$ (corresponding to e),
(5) *conjugation*: $c : \Gamma \to \Gamma$ (corresponding to i),

where here Γ may be viewed as a B-bimodule with left B-module structure induced by η_L and right B-module structure induced by η_R, so we may form the tensor product of bimodules $\Gamma \otimes_B \Gamma$, which further may be given the structure of an A-graded anticommutative R-algebra (by Proposition 0.1), and fits into the following pushout diagram in R-A**GrCAlg** (Proposition 0.2):

$$\begin{tikzcd} B \arrow[r, "\eta_L"] \arrow[d, "\eta_R"'] & \Gamma \arrow[d, "g \mapsto 1 \otimes g"] \\ \Gamma \arrow[r, "g \mapsto g \otimes 1"'] & \Gamma \otimes_B \Gamma \end{tikzcd}$$

These data must satisfy the following

(1) The following diagrams must commute:

$$\begin{tikzcd} B \arrow[r, "\eta_L"] \arrow[d, "\eta_L"'] & \Gamma \arrow[d, "\Psi"] \\ \Gamma \arrow[r, "g \mapsto g \otimes 1"'] & \Gamma \otimes_B \Gamma \end{tikzcd} \qquad \begin{tikzcd} & B \arrow[dl, "\eta_R"'] \arrow[d, equal] \arrow[dr, "\eta_L"] & \\ \Gamma \arrow[r, "\epsilon"'] & B & \Gamma \arrow[l, "\epsilon"] \end{tikzcd} \qquad \begin{tikzcd} B \arrow[r, "\eta_R"] \arrow[d, "\eta_R"'] & \Gamma \arrow[d, "g \mapsto 1 \otimes g"] \\ \Gamma \arrow[r, "\Psi"] & \Gamma \otimes_B \Gamma \end{tikzcd}$$

(2) (Coassociativity) The following diagram must commute

$$\begin{tikzcd} \Gamma \otimes_B \Gamma \arrow[d, "\Psi\otimes_B\Gamma"'] & \Gamma \arrow[l, "\Psi"'] \arrow[r, "\Psi"] & \Gamma \otimes_B \Gamma \arrow[d, "\Gamma\otimes_B\Psi"] \\ (\Gamma \otimes_B \Gamma) \otimes_B \Gamma \arrow[rr] & & \Gamma \otimes_B (\Gamma \otimes_B \Gamma) \end{tikzcd}$$

where the bottom arrow sends $(g \otimes g') \otimes g''$ to $g \otimes (g' \otimes g'')$, the left vertical arrow sends $g \otimes g'$ to $\Psi(g) \otimes g$, and the right vertical arrow sends $g \otimes g'$ to $g \otimes \Psi(g')$

(3) The following diagram must commute:

$$\begin{tikzcd} \Gamma \arrow[r, "\Psi"] \arrow[d, "\Psi"'] \arrow[dr, equal] & \Gamma \otimes_B \Gamma \arrow[d, "(\eta_L\circ\epsilon)\cdot\mathrm{id}_\Gamma"] \\ \Gamma \otimes_B \Gamma \arrow[r, "\mathrm{id}_\Gamma\cdot(\eta_R\circ\epsilon)"'] & \Gamma \end{tikzcd}$$

where the right vertical arrow sends $g \otimes g'$ to $\eta_L(\epsilon(g))g'$ and the bottom horizontal arrow sends $g \otimes g'$ to $g\eta_R(\epsilon(g'))$.

(4) The following diagrams must commute:

$$\begin{tikzcd} & \Gamma \arrow[d, "c"] \\ \Gamma \arrow[ur, equal] & \Gamma \arrow[l, "c"] \end{tikzcd} \qquad \begin{tikzcd} B \arrow[d, "\eta_L"'] & \Gamma \arrow[l, "\epsilon"'] \arrow[r, "\epsilon"] \arrow[d, "i"'] & B \arrow[d, "\eta_R"] \\ \Gamma & \Gamma \otimes_B \Gamma \arrow[l, "\mathrm{id}_\Gamma\cdot c"] \arrow[r, "c\cdot\mathrm{id}_\Gamma"'] & \Gamma \end{tikzcd} \qquad \begin{tikzcd} B \arrow[r, "\eta_L"] \arrow[dr, "\eta_R"'] & \Gamma \arrow[d, "c"'] & B \arrow[l, "\eta_R"'] \arrow[dl, "\eta_L"] \\ & \Gamma & \end{tikzcd}$$

where the bottom left arrow in the middle diagram sends $g \otimes g'$ to $gc(g')$ and the bottom right arrow in the middle diagram sends $g \otimes g'$ to $c(g)g'$.

Proposition 0.5. *Suppose we have an A-graded anticommutative Hopf algebroid (B, Γ) over R with structure maps η_L, η_R, Ψ, ϵ, and c. Recall in Definition 0.4 we considered $\Gamma \otimes_B \Gamma$ to be the A-graded R-commutative ring whose underlying abelian group was given by the tensor product of B-bimodules, where Γ has left B-module structure induced by η_L and right B-module structure induced by η_R. Thus $\Gamma \otimes_B \Gamma$ is canonically a B-bimodule, as it is a tensor product of B-bimodules. Then the canonical left (resp. right) B-module structure on $\Gamma \otimes_B \Gamma$ coincides with that induced by the ring homomorphism $\Psi \circ \eta_L$ (resp. $\Psi \circ \eta_R$).*

Proof. First we show the left module structures coincide. By additivity, in order to show the module structures coincide, it suffices to show that given a homogeneous pure tensor $g \otimes g'$ in $\Gamma \otimes_B \Gamma$ and some $b \in B$ that $\Psi(\eta_L(b)) \cdot (g \otimes g') = (\eta_L(b) \cdot g) \otimes g'$, where $\cdot$ on the left denotes the product in $\Gamma \otimes_B \Gamma$ and the $\cdot$ on the right denotes the product in Γ. By the axioms for a Hopf algebroid, we have that $\Psi(\eta_L(b)) = \eta_L(b) \otimes 1$. Thus by how the product in $\Gamma \otimes_B \Gamma$ is defined (Proposition 0.1), we have that

$$\Psi(\eta_L(b)) \cdot (g \otimes g') = (\eta_L(b) \otimes 1) \cdot (g \otimes g') = (\varphi_\Gamma(\theta_{0,|g|}) \cdot \eta_L(b) \cdot g) \otimes (g' \cdot 1) = (\eta_L(b) \cdot g) \otimes g',$$

where $\varphi_\Gamma : R \to \Gamma$ is the structure map, and the last equality follows by the fact that $\theta_{0,|g|} = 1$. An entirely analagous argument yields that the canonical right module structure on $\Gamma \otimes_B \Gamma$ coincides with that induced by $\Psi \circ \eta_R$, since $\Psi \circ \eta_R = 1 \otimes \eta_R$. $\square$

Remark 0.6. By the above proposition, given an A-graded commutative Hopf algebroid (B, Γ) over R, there is no ambiguity when discussing the objects $\Gamma \otimes_B (\Gamma \otimes_B \Gamma)$ and $(\Gamma \otimes_B \Gamma) \otimes_B \Gamma$ —

they may both be considered as the threefold tensor product of the B-bimodule Γ with itself. In particular, we have a canonical isomorphism of B-bimodules

$$(\Gamma \otimes_B \Gamma) \otimes_B \Gamma \to \Gamma \otimes_B (\Gamma \otimes_B \Gamma)$$

sending $(g \otimes g') \otimes g''$ to $g \otimes (g' \otimes g'')$, and this is precisely the isomorphism in the coassociativity diagram in the definition of a Hopf algebroid (Definition 0.4).

Proposition 0.7. *Suppose we have an A-graded commutative Hopf algebroid (B, Γ) over R with structure maps η_L, η_R, Ψ, ϵ, and c. Then $\eta_L : B \to \Gamma$ is a homomorphism of left B-modules, $\eta_R : B \to \Gamma$ is a homomorphism of right B-modules, and $\Psi : \Gamma \to \Gamma \otimes_B \Gamma$ and $\epsilon : \Gamma \to B$ are homomorphisms of B-bimodules.*

Proof. Since the left (resp. right) B-module structure on Γ is induced by η_L (resp. η_R), the map η_L (resp. η_R) is a homomorphism of left (resp. right) B-modules by definition.

Next, we want to show Ψ is a homomorphism of B-bimodules. In **??**, we showed that the left (resp. right) B-module structure on Γ is that induced by η_L (resp. η_R), and in Proposition 0.5, we showed that the left (resp. right) B-module structure on $\Gamma \otimes_B \Gamma$ is that induced by $\Psi \circ \eta_L$ (resp. $\Psi \circ \eta_R$), so that by definition $\Psi : \Gamma \to \Gamma \otimes_B \Gamma$ is a homomorphism of left (resp. right) B-modules.

Lastly, we claim that $\epsilon : \Gamma \to B$ is a homomorphism of B-bimodules. We need to show that given $g \in \Gamma$ and $b, b' \in B$ that $\epsilon(\eta_L(b) g \eta_R(g')) = b\epsilon(g)b'$. This follows from the fact that ϵ is a ring homomorphism satisfying $\epsilon \circ \eta_L = \epsilon \circ \eta_R = \mathrm{id}_B$. ☐

0.3. **Comodules over a Hopf algebroid.** In what follows, fix an A-graded commutative Hopf algebroid (B, Γ) over R with structure maps η_L, η_R, Ψ, ϵ, and c. We will always view Γ with its *canonical* B-bimodule structure, with left B-module structure induced by η_L, and right B-module structure induced by η_R. In particular, any tensor product over B involving Γ will always refer to Γ with this bimodule structure.

Definition 0.8. A *left comodule over* Γ is a pair (N, Ψ_N), where N is a left A-graded B-module and $\Psi_N : N \to \Gamma \otimes_B N$ is an A-graded homomorphism of left A-graded B-modules (where here we view Γ as a B-bimodule with its left module structure induced by η_L, and its right module structure induced by η_R). These data are required to make the following diagrams commute

$$\begin{tikzcd} N \arrow[r, "\Psi_N"] \arrow[dr, "\cong"'] & \Gamma \otimes_B N \arrow[d, "\epsilon \otimes N"] \\ & B \otimes_B N \end{tikzcd} \qquad \begin{tikzcd} \Gamma \otimes_B N \arrow[d, "\Psi \otimes N"'] & N \arrow[l, "\Psi_N"'] \arrow[r, "\Psi_N"] & \Gamma \otimes_B N \arrow[d, "\Gamma \otimes \Psi_N"] \\ (\Gamma \otimes_B \Gamma) \otimes_B N \arrow[rr, "\cong"] & & \Gamma \otimes_B (\Gamma \otimes_B N) \end{tikzcd}$$

The maps $\epsilon \otimes N$ and $\Psi \otimes N$ are well-defined by **??**, and the bottom isomorphism in the right diagram is the canonical one sending $(g \otimes g') \otimes n \mapsto g \otimes (g' \otimes n)$.

Given two left A-graded Γ-comodules (N_1, Ψ_{N_1}) and (N_2, Ψ_{N_2}), a homomorphism of left A-graded comodules $f : N_1 \to N_2$ is an A-graded homomorphism of the underlying left B-modules such that the following diagram commutes:

$$\begin{tikzcd} N_1 \arrow[r, "f"] \arrow[d, "\Psi_{N_1}"'] & N_2 \arrow[d, "\Psi_{N_2}"] \\ \Gamma \otimes_B N_1 \arrow[r, "\Gamma \otimes f"] & \Gamma \otimes_B N_2 \end{tikzcd}$$

We write Γ-$\mathbf{CoMod}^A$ for the resulting category of left A-graded comodules over Γ. In the above definition, we required A-graded left Γ-comodule homomorphisms to strictly preserve the grading, but we could have instead considered left Γ-comodule homomorphisms which are of degree d for some $d \in A$, or equivalently, the set of degree zero A-graded Γ-comodule homomorphisms from N_1 to the shifted comodule $(N_2)_{*+d}$. We denote the hom-set of degree-d A-graded left Γ-comodule homomorphisms from (N_1, Ψ_{N_1}) to (N_2, Ψ_{N_2}) by

$$\mathrm{Hom}^d_{\Gamma\text{-}\mathbf{CoMod}^A}(N_1, N_2) \qquad \text{or usually just} \qquad \mathrm{Hom}^d_\Gamma(N_1, N_2).$$

In particular, we simply write $\mathrm{Hom}_{\Gamma\text{-}\mathbf{CoMod}^A}(N_1, N_2)$ or $\mathrm{Hom}_\Gamma(N_1, N_2)$ for the set of strictly degree preserving (degree 0) A-graded left Γ-comodule homomorphisms from (N_1, Ψ_{N_1}) to (N_2, Ψ_{N_2}).

Proposition 0.9. *The forgetful functor $\Gamma\text{-}\mathbf{CoMod}^A \to B\text{-}\mathbf{Mod}^A$ (where here $B\text{-}\mathbf{Mod}^A$ is the category of A-graded left B-modules and A-graded module homomorphisms between them) has a right adjoint $\Gamma \otimes_B - : B\text{-}\mathbf{Mod}^A \to \Gamma\text{-}\mathbf{CoMod}^A$ called the* co-free construction*, where the co-free left A-graded Γ-comodule on a left A-graded B-module M is the B-module $\Gamma \otimes_B M$ equipped with the coaction*

$$\Psi_{\Gamma\otimes_B M} : \Gamma \otimes_B M \xrightarrow{\Psi\otimes_B M} (\Gamma \otimes_B \Gamma) \otimes_B M \xrightarrow{\cong} \Gamma \otimes_B (\Gamma \otimes_B M).$$

In particular, for all objects (N, Ψ_N) in $\Gamma\text{-}\mathbf{CoMod}^A$ and M in $B\text{-}\mathbf{Mod}^A$, we have isomorphisms of A-graded abelian groups

$$\mathrm{Hom}^*_{B\text{-}\mathbf{Mod}^A}(N, M) \cong \mathrm{Hom}^*_{\Gamma\text{-}\mathbf{CoMod}^A}(N, \Gamma \otimes_B M)$$

via the identifications

$$\mathrm{Hom}^a_{\Gamma\text{-}\mathbf{CoMod}^A}(N_*, N'_*) = \mathrm{Hom}_{\Gamma\text{-}\mathbf{CoMod}^A}(N_{*-a}, N'_*)$$

and

$$\mathrm{Hom}^a_{B\text{-}\mathbf{Mod}^A}(M_*, M'_*) = \mathrm{Hom}_{B\text{-}\mathbf{Mod}^A}(M_{*-a}, M'_*).$$

Explicitly, given some (N, Ψ_N) in $\Gamma\text{-}\mathbf{CoMod}$ and some M in $B\text{-}\mathbf{Mod}^A$, the counit and unit of this adjunction are given by

$$\eta_{(N,\Psi_N)} : N \xrightarrow{\Psi_N} \Gamma \otimes_B N$$

and

$$\varepsilon_M : \Gamma \otimes_B M \xrightarrow{\epsilon\otimes_B M} B \otimes_B M \xrightarrow{\cong} M.$$

todo

Proof. $\square$

Proposition 0.10. *The category $\Gamma\text{-}\mathbf{CoMod}^A$ is an additive category.*

Proof. First, we show the category is $\mathbf{Ab}$-enriched. Since the forgetful functor $\Gamma\text{-}\mathbf{CoMod}^A \to B\text{-}\mathbf{Mod}^A$ is clearly faithful, we may view hom-sets in $\Gamma\text{-}\mathbf{CoMod}^A$ as subsets of hom-groups in $B\text{-}\mathbf{Mod}^A$, so that in order to show $\Gamma\text{-}\mathbf{CoMod}^A$ is $\mathbf{Ab}$-enriched, it suffices to show that hom-sets in $\Gamma\text{-}\mathbf{CoMod}^A$ are closed under addition of module homomorphisms and taking inverses. To that end, suppose we have two A-graded left Γ-comodule homomorphisms $f, g : (N_1, \Psi_{N_1}) \to (N_2, \Psi_{N_2})$, then we have

$$\begin{aligned}
\Psi_{N_2} \circ (f + g) &= (\Psi_{N_2} \circ f) + (\Psi_{N_2} \circ g) \\
&= ((\Gamma \otimes_B f) \circ \Psi_{N_1}) + ((\Gamma \otimes_B g) \circ \Psi_{N_1}) \\
&= ((\Gamma \otimes_B f) + (\Gamma \otimes_B g)) \circ \Psi_{N_1} \\
&= (\Gamma \otimes_B (f + g)) \circ \Psi_{N_1},
\end{aligned}$$

where the first equality follows since Ψ_{N_2} is a homomorphism of modules, the second follows since f and g are left Γ-comodule homomorphisms, the third follows since Ψ_{N_1} is a homomorphism of modules, and the last equality follows by definition of the tensor product of modules. Hence $f + g$

is indeed an A-graded left Γ-comodule homomorphism, as desired. Now, we also claim $-f$ is an A-graded left Γ-comodule homomorphism. To that end, note that

$$\Psi_{N_2} \circ (-f) = -\Psi_{N_2} \circ f = -(\Gamma \otimes_B f) \circ \Psi_{N_1} = (\Gamma \otimes_B (-f)) \circ \Psi_{N_1},$$

where the first equality follows since Ψ_{N_2} is a module homomorphism, the second follows since f is an A-graded left Γ-comodule homomorphism, and the third equality follows by definition of the tensor product.

Thus, we've shown that the hom-sets in Γ-$\mathbf{CoMod}^A$ are abelian groups, and composition is clearly bilinear, so that Γ-$\mathbf{CoMod}^A$ is indeed $\mathbf{Ab}$-enriched.

Now, in order to show Γ-$\mathbf{CoMod}^A$ is additive, it suffices to show that it contains a zero object and has binary coproducts. First, note that the zero left B-module is clearly an A-graded left Γ-comodule with structure map the unique map $0 \to \Gamma \otimes_B 0 \cong 0$, and that given any other A-graded left Γ-comodule (N, Ψ_N), the unique homomorphisms of left B-modules $0 \to N$ and $N \to 0$ are left comodule homomorphisms.

Now, suppose we have two A-graded left Γ-comodules (N_1, Ψ_{N_1}) and (N_2, Ψ_{N_2}). First, we claim their direct sum as left B-modules $N_1 \oplus N_2$ is canonically an A-graded left Γ-comodule. We know that $N_1 \oplus N_2$ is an A-graded left B-module by **??**, and we can define the structure map

$$\Psi_{N_1 \oplus N_2} : N_1 \oplus N_2 \xrightarrow{\Psi_{N_1} \oplus \Psi_{N_2}} (\Gamma \otimes_B N_1) \oplus (\Gamma \otimes_B N_2) \cong \Gamma \otimes_B (N_1 \oplus N_2),$$

where the final isomorphism is the canonical one sending $(g_1 \otimes n_1) \oplus (g_2 \otimes n_2)$ to $(g_1 \otimes n_1) + (g_2 \otimes n_2)$. Then □ finish

Proposition 0.11. *Suppose that Γ is flat as a right B-module (with its canonical right B-module structure induced by η_R). Then the category Γ-$\mathbf{CoMod}^A$ is an abelian category.*

Proof. □ finish