0.1. **Setup.** In order to construct an abstract version of the Adams spectral sequence, we need to work in some axiomatic version of a stable homotopy category $\mathcal{SH}$ which acts like the familiar classical stable homotopy category **hoSp** (**??**) or the motivic stable homotopy category $\mathbf{SH}_{\mathscr{S}}$ over some base scheme $\mathscr{S}$ (**??**):

Definition 0.1. A *stable homotopy category* $\mathcal{SH}$ is the following data:

- An additive closed symmetric monoidal category $(\mathcal{SH}, \otimes, S)$ with arbitrary (small) (co)products.
- A pointed abelian group $(A, \mathbf{1})$ along with a homomorphism $h : A \to \operatorname{Pic} \mathcal{SH}$.
- For each $a \in A$, a chosen representative S^a in the isomorphism class $h(a)$ such that $S^0 = S$.
- For each $a, b \in A$, an isomorphism $\phi_{a,b} : S^{a+b} \to S^a \otimes S^b$.
- A collection $\mathcal{D}$ of triangles in $\mathcal{C}$ of the following form

$$X \to Y \to Z \to \Sigma X.$$

These data are subject to the following additional conditions. First, we require that the $\phi_{a,b}$'s are *coherent*, in the following sense:

- For all $a \in A$, we must have $\phi_{a,0} = \rho_{S^a}^{-1} : S^a \to S^a \otimes S$ and $\phi_{0,a} = \lambda_{S^a}^{-1} : S^a \to S \otimes S^a$.
- For all $a, b, c \in A$, the following diagram must commute:

$$
\begin{array}{ccc}
S^{a+b} \otimes S^c \xleftarrow{\phi_{a+b,c}} S^{a+b+c} \xrightarrow{\phi_{a,b+c}} S^a \otimes S^{b+c} \\
\phi_{a,b} \otimes S^c \downarrow \qquad\qquad\qquad\qquad \downarrow S^a \otimes \phi_{b,c} \\
(S^a \otimes S^b) \otimes S^c \xrightarrow{\quad\cong\quad} S^a \otimes (S^b \otimes S^c)
\end{array}
$$

Now, given $a \in A$, define $\Sigma^a := S^a \otimes -$ and $\Omega^a := \Sigma^{-a} = S^{-a} \otimes -$. In particular, let $\Sigma := \Sigma^{\mathbf{1}}$ and $\Omega := \Omega^{\mathbf{1}}$. Then the above two conditions give that Σ^a and Ω^a form an adjoint equivalence of $\mathcal{SH}$, with unit η^a and counit ε^a defined by

$$\eta_X^a : X \xrightarrow{\cong} S \otimes X \xrightarrow{\phi_{-a,a} \otimes X} (S^{-a} \otimes S^a) \otimes X \xrightarrow{\cong} S^{-a} \otimes (S^a \otimes X) = \Omega^a \Sigma^a X$$

and

$$\varepsilon_X^a : \Sigma^a \Omega^a X = S^a \otimes (S^{-a} \otimes X) \xrightarrow{\cong} (S^a \otimes S^{-a}) \otimes X \xrightarrow{\phi_{a,-a}^{-1} \otimes X} S \otimes X \xrightarrow{\cong} X.$$

Thus, further require the following:

- $(\mathcal{SH}, \Sigma, \Omega, \eta^{\mathbf{1}}, \varepsilon^{\mathbf{1}}, \mathcal{D})$ is a tensor triangulated category, where the isomorphisms $\Sigma X \otimes Y \cong \Sigma(X \otimes Y)$ are given by the associator:

$$\Sigma X \otimes Y = (S^{\mathbf{1}} \otimes X) \otimes Y \xrightarrow{\cong} S^{\mathbf{1}} \otimes (X \otimes Y) = \Sigma(X \otimes Y).$$

(See **??** for the definition of a triangulated category, and **??** for the definition of a tensor triangulated category.)

From now on we fix the data given in the above definition, and we establish the following conventions:

- Given objects $X_1, \ldots, X_n$ in $\mathcal{SH}$, we write $X_1 \otimes \cdots \otimes X_n$ to denote the object

$$X_1 \otimes (X_2 \otimes \cdots (X_{n-1} \otimes X_n)).$$

In particular, given an object X and a natural number $n > 0$, we write

$$X^n := \overbrace{X \otimes \cdots \otimes X}^{n \text{ times}} \qquad \text{and} \qquad X^0 := S.$$

- We denote the associator, symmetry, left unitor, and right unitor isomorphisms in $\mathcal{SH}$ by

$$\alpha_{X,Y,Z} : (X \otimes Y) \otimes Z \xrightarrow{\cong} X \otimes (Y \otimes Z) \qquad \tau_{X,Y} : X \otimes Y \xrightarrow{\cong} Y \otimes X$$

$$\lambda_X : S \otimes X \xrightarrow{\cong} X \qquad\qquad \rho_X : X \otimes S \xrightarrow{\cong} X.$$

- Given two objects X and Y, we denote the hom-abelian group of morphisms from X to Y in $\mathcal{SH}$ by $[X, Y]$, and we denote the internal hom object by $F(X, Y)$. We will often refer to morphisms in $\mathcal{SH}$ as *classes*, as we will think of them as representing homotopy classes of maps.
- Given two objects X and Y in $\mathcal{SH}$, we may extend the abelian group $[X, Y]$ to an A-graded abelian group $[X, Y]_*$ defined by

$$[X, Y]_a := [\Sigma^a X, Y] = [S^a \otimes X, Y].$$

(See **??** for a review of the theory of A-graded abelian groups, rings, modules, etc.)
- Given an object X in $\mathcal{SH}$ and some $a \in A$, define the abelian group

$$\pi_a(X) := [S^a, X],$$

and write $\pi_*(X)$ for the associated A-graded abelian group $\bigoplus_{a \in A} \pi_a(X)$. We call $\pi_a(X)$ the ath *stable homotopy group of* X.
- Given two objects E and X in $\mathcal{SH}$, we define the A-graded abelian groups $E_*(X)$ and $E^*(X)$ by

$$E_a(X) := \pi_a(E \otimes X) = [S^a, E \otimes X] \qquad \text{and} \qquad E^a(X) := [X, S^a \otimes E].$$

We refer to the functor $E_*(-)$ as the *homology theory represented by* E, or just E-homology, and we refer to $E^*(-)$ as the *cohomology theory represented by* E, or just E-cohomology.

Proposition 0.2. *For each $a \in A$, the isomorphisms $\eta_X^a : X \to \Omega^a \Sigma^a X$ and $\varepsilon_X^a : \Sigma^a \Omega^a X \to X$ defined in Definition 0.1 exhibit an adjoint autoequivalence $(\Sigma^a, \Omega^a, \eta^a, \varepsilon^a)$ of $\mathcal{SH}$.*

Proof. In this proof, we will freely employ the coherence theorem for monoidal categories (see [2]), which essentially tells us that we may assume we are working in a strict monoidal category (i.e., that the associators and unitors and are identities). Then η_X^a and ε_X^a become simply the maps

$$\eta_X^a : X \xrightarrow{\phi_{-a,a} \otimes X} S^{-a} \otimes S^a \otimes X \qquad \text{and} \qquad \varepsilon_X^a : S^a \otimes S^{-a} \otimes X \xrightarrow{\phi_{a,-a}^{-1} \otimes X} X.$$
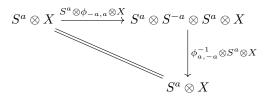
That these maps are natural in X follows by functoriality of $- \otimes -$. Now, recall that in order to show that these natural isomorphisms form an *adjoint* equivalence, it suffices to show that the natural isomorphisms $\eta^a : \mathrm{Id}_{\mathcal{SH}} \Rightarrow \Omega^a \Sigma^a$ and $\varepsilon^a : \Sigma^a \Omega^a \Rightarrow \mathrm{Id}_{\mathcal{SH}}$ satisfy one of the two zig-zag identities:

$$
\begin{array}{ccc}
\Sigma^a \xrightarrow{\Sigma^a \eta^a} \Sigma^a \Omega^a \Sigma^a & \qquad & \Omega^a \Sigma^a \Omega^a \xleftarrow{\eta^a \Omega^a} \Omega^a \\
\quad \searrow \quad \downarrow{\varepsilon^a \Sigma^a} & & \Omega^a \varepsilon^a \downarrow \quad \swarrow \quad \\
\Sigma^a & & \Omega^a
\end{array}
$$

(see [3, Lemma 3.2]). We will show that the left is satisfied. Unravelling definitions, we simply wish to show that the following diagram commutes for all X in $\mathcal{SH}$:

$$
\begin{array}{ccc}
S^a \otimes X & \xrightarrow{S^a \otimes \phi_{-a,a} \otimes X} & S^a \otimes S^{-a} \otimes S^a \otimes X \\
& \searrow & \downarrow{\phi_{a,-a}^{-1} \otimes S^a \otimes X} \\
& & S^a \otimes X
\end{array}
$$

Yet this is simply the diagram obtained by applying $-\otimes X$ to the associativity coherence diagram for the $\phi_{a,b}$'s, so it does commute, as desired. $\qquad\square$

From now on, we fix the data of a stable homotopy category $\mathcal{SH}$ given above once and for all. Observe that for all $a, b \in A$, the objects S^{a+b} and $S^a \otimes S^b$ are isomorphic, since $h : A \to \mathrm{Pic}(\mathcal{SH})$ is a group homomorphism. Hence given a monoid object (E, μ, e) in $\mathcal{SH}$ (**??**), supposing we had fixed isomorphisms $S^{a+b} \cong S^a \otimes S^b$ for all $a, b \in A$, we get a multiplication map $\pi_*(E) \times \pi_*(E) \to \pi_*(E)$ which sends classes $x : S^a \to E$ and $y : S^b \to E$ to the product

$$S^{a+b} \cong S^a \otimes S^b \xrightarrow{x \otimes y} E \otimes E \xrightarrow{\mu} E.$$

Naturally, we would like this product to make $\pi_*(E)$ into an A-graded ring (with unit $e \in \pi_0(E) = [S, E]$), rather than just an A-graded abelian group. Whether or not this happens is essentially the entire discussion of Dugger's paper [1], and as it turns out, $\pi_*(E)$ is in fact a graded ring provided we can choose these morphisms to be *coherent*, in the following sense:

Definition 0.3. Suppose we have a family of isomorphisms

$$\phi_{a,b} : S^{a+b} \xrightarrow{\cong} S^a \otimes S^b$$

for all $a, b \in A$. We say this family is *coherent* if:

(1) For all $a \in A$, we have equalities $\phi_{a,0} = \rho_{S^a}^{-1} : S^a \to S^a \otimes S$ and $\phi_{0,a} = \lambda_{S^a}^{-1} : S^a \to S \otimes S^a$.
(2) For all $a, b, c \in A$, the following diagram commutes:

$$
\begin{array}{ccccc}
S^{a+b} \otimes S^c & \xleftarrow{\phi_{a+b,c}} & S^{a+b+c} & \xrightarrow{\phi_{a,b+c}} & S^a \otimes S^{b+c} \\
{\scriptstyle \phi_{a,b} \otimes S^c} \downarrow & & & & \downarrow {\scriptstyle S^a \otimes \phi_{b,c}} \\
(S^a \otimes S^b) \otimes S^c & & \xrightarrow{\cong} & & S^a \otimes (S^b \otimes S^c)
\end{array}
$$

Furthermore, Dugger gaurantees that we can always find such a coherent family:

Theorem 0.4 ([1, Proposition 7.1]). *There exists a coherent family of isomorphisms*

$$\phi_{a,b} : S^{a+b} \xrightarrow{\cong} S^a \otimes S^b$$

in the sense of Definition 0.3, and in particular, the set of such coherent families is in bijective correspondence with the set of normalized 2-cocycles $Z^2(A; \mathrm{Aut}(S))_{norm}$, i.e., the set of functions $\alpha : A \times A \to \mathrm{Aut}(S)$ such that $\alpha(0,0) = \mathrm{id}_S$ and for all $a, b, c \in A$, $\alpha(a+b, c) \cdot \alpha(a, b) = \alpha(b, c) \cdot \alpha(a, b+c)$.

Thus, from now on we will suppose once and for all we have fixed a coherent family $\{\phi_{a,b}\}_{a,b \in A}$. Such a coherent family has very nice properties, in particular:

Remark 0.5. Note that by induction the coherence conditions say that given any $a_1, \ldots, a_n \in A$ and $b_1, \ldots, b_m \in A$ such that $a_1 + \cdots + a_n = b_1 + \cdots + b_m$ and any fixed parenthesizations of $X = S^{a_1} \otimes \cdots \otimes S^{a_b}$ and $Y = S^{b_1} \otimes \cdots \otimes S^{b_m}$, there is a *unique* isomorphism $X \to Y$ that can be obtained by forming formal compositions of tensor products of $\phi_{a,b}$, identities, associators, and their inverses.

Of course, we get our desired result: $\pi_*(E)$ is indeed an A-graded ring if E is a monoid object.

Proposition 0.6. *Let (E, μ, e) be a commutative monoid object in $\mathcal{SH}$, and consider the multiplication map $\pi_*(E) \times \pi_*(E) \to \pi_*(E)$ which sends classes $x : S^a \to E$ and $y : S^b \to E$ to the composition*

$$S^{a+b} \xrightarrow{\phi_{a,b}} S^a \otimes S^b \xrightarrow{x \otimes y} E \otimes E \xrightarrow{\mu} E.$$

Then this endows $\pi_*(E)$ with the structure of an A-graded ring with unit $e \in \pi_0(E) = [S, E]$.

Proof. See **??**. $\qquad\square$

Furthermore, it turns out that if E is a *commutative* monoid object in $\mathcal{SH}$, then $\pi_*(E)$ is "A-graded commutative," in the following sense:

Proposition 0.7. *For all $a, b \in A$ there exists an element $\theta_{a,b} \in \pi_0(S) = [S, S]$ (determined by choice of coherent family $\{\phi_{a,b}\}$) such that given any commutative monoid object (E, μ, e) in $\mathcal{SH}$, the A-graded ring structure on $\pi_*(E)$ (Proposition 0.6) has a commutativity formula given by*

$$x \cdot y = y \cdot x \cdot (e \circ \theta_{a,b})$$

for all $x \in \pi_a(E)$ and $y \in \pi_b(E)$.

Furthermore, $\theta_{0,a} = \theta_{a,0} = \mathrm{id}_S$ for all $a \in A$, so that if either x or y has degree 0, $x \cdot y = y \cdot x$.

Proof. See **??** and **??**. $\qquad\square$

The last ingredient in order to develop the Adams spectral sequence abstractly is a notion of *cellularity* in $\mathcal{SH}$:

Definition 0.8. Define the class of *cellular* objects in $\mathcal{SH}$ to be the smallest class of objects such that:

(1) For all $a \in A$, S^a is cellular.
(2) If we have a distinguished triangle

$$X \to Y \to Z \to \Sigma X (= S^{\mathbf{1}} \otimes X)$$

such that two of the three objects X, Y, and Z are cellular, than the third object is also cellular.
(3) Given a collection of cellular objects X_i indexed by some small set I, $\bigoplus_{i \in I} X_i$ is cellular.

0.2. **Construction of the Adams spectral sequence.** In what follows, let E be a commutative monoid object in $\mathcal{SH}$.

Definition 0.9. Let $\overline{E}$ be the fiber of the unit map $e : S \to E$ (**??**), and for $s \geq 0$ define

$$Y_s := \overline{E}^s \otimes Y, \qquad W_s = E \otimes Y_s = E \otimes (\overline{E}^s \otimes Y),$$

where recall for $s > 0$, $\overline{E}^s$ denotes the s-fold product parenthesized as $\overline{E} \otimes (\overline{E} \otimes \cdots (\overline{E} \otimes \overline{E}))$, and $\overline{E}^0 := S$. Then we get fiber sequences

$$Y_{s+1} \xrightarrow{i_s} Y_s \xrightarrow{j_s} W_s \xrightarrow{k_s} \Sigma Y_{s+1}(= S^{\mathbf{1}} \otimes Y_{s+1})$$

obtained by applying $- \otimes Y_s$ to the sequence

$$\overline{E} \to S \xrightarrow{e} E \to \Sigma \overline{E}$$

(and applying the necessary associator and unitor isomorphisms). These sequences can be spliced together to form the *(canonical) Adams filtration of Y*:
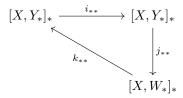
$$\cdots \longrightarrow Y_3 \xrightarrow{i_2} Y_2 \xrightarrow{i_1} Y_1 \xrightarrow{i_0} Y_0 = Y$$

with $j_3, k_2, j_2, k_1, j_1, k_0, j_0$ and W_3, W_2, W_1, W_0

where the diagonal dashed arrows are of degree $-\mathbf{1}$ (note these triangles do NOT commute in any sense). Now we may apply the functor $[X, -]_*$, and by **??** we obtain an exact couple of

$\mathbb{N} \times A$-graded abelian groups:

$$
\begin{array}{ccc}
[X, Y_*]_* & \xrightarrow{\;i_{**}\;} & [X, Y_*]_* \\
& k_{**} \nwarrow \quad \searrow & \downarrow j_{**} \\
& & [X, W_*]_*
\end{array}
$$

where i_{**}, j_{**}, and k_{**} have $\mathbb{Z} \times A$-degree $(-1, 0)$, $(0, 0)$, and $(1, -\mathbf{1})$, respectively[1]. The standard argument yields an $\mathbb{N} \times A$-graded spectral sequence called from this exact couple (cf. Section 5.9 of [4]) with E_1 page given by

$$
E_1^{s,a} = [X, W_s]_a
$$

and r^{th} differential of $\mathbb{Z} \times A$-degree $(r, -\mathbf{1})$:

$$
d_r : E_r^{s,a} \to E_r^{s+r, a-\mathbf{1}}.
$$

A priori, this is all $\mathbb{N} \times A$-graded, but we regard it as being $\mathbb{Z} \times A$-graded by setting $E_r^{s,a} := 0$ for $s < 0$ and trivially extending the definition of the differentials to these zero groups. This spectral sequence is called the *E-Adams spectral sequence* for the computation of $[X, Y]_*$. The index s is called the *Adams filtration* and a is the *stem*.

0.3. **Monoid objects in** $\mathcal{SH}$. We have constructed an Adams spectral sequence, but as it currently stands we do not yet know why it is useful. To start with, we'd like to provide a characterization of its E_1 and E_2 pages in terms of something more algebraic. To start, we first need to develop some theory of the algebra of monoid objects in $\mathcal{SH}$. Much of this work is entirely straightforward although tedious to verify, so we relegate most of the proofs in this section to **??**.

Proposition 0.10. *Let (E, μ, e) be a monoid object in $\mathcal{SH}$. Then $E_*(-)$ is a functor from $\mathcal{SH}$ to left A-graded $\pi_*(E)$-modules, where given some X in $\mathcal{SH}$, $E_*(X)$ may be endowed with the structure of a left A-graded $\pi_*(E)$-module via the map*

$$
\pi_*(E) \times E_*(X) \to E_*(X)
$$

which given $a, b \in A$, sends $x : S^a \to E$ and $y : S^b \to E \otimes X$ to the composition

$$
x \cdot y : S^{a+b} \cong S^a \otimes S^b \xrightarrow{x \otimes y} E \otimes (E \otimes X) \cong (E \otimes E) \otimes X \xrightarrow{\mu \otimes X} E \otimes X.
$$

Similarly, the assignment $X \mapsto X_*(E)$ is a functor from $\mathcal{SH}$ to right A-graded $\pi_*(E)$-modules, where the structure map

$$
X_*(E) \times \pi_*(E) \to X_*(E)
$$

sends $x : S^a \to X \otimes E$ and $y : S^b \to E$ to the composition

$$
x \cdot y : S^{a+b} \cong S^a \otimes S^b \xrightarrow{x \otimes y} (X \otimes E) \otimes E \cong X \otimes (E \otimes E) \xrightarrow{X \otimes \mu} X \otimes E.
$$

Finally, $E_*(E)$ is a $\pi_*(E)$-bimodule, in the sense that the left and right actions of $\pi_*(E)$ are compatible, so that given $y, z \in \pi_*(E)$ and $x \in E_*(E)$, $y \cdot (x \cdot z) = (y \cdot x) \cdot z$.

Proof. See **??**. $\square$

Definition 0.11. Given a monoid object E in $\mathcal{SH}$, we say E is *flat* if the canonical right $\pi_*(E)$-module structure on $E_*(E)$ (see the above proposition) is that of a flat module.

---

[1] Explicitly, the map $k_{s,a} : [X, W_s]_a \to [X, Y_{s+1}]_{a-\mathbf{1}}$ sends a map $f : S^a \otimes X \to W_s$ to the map $S^{a-\mathbf{1}} \otimes X \to Y_{s+1}$ corresponding under the isomorphism $[X, \Sigma Y_{s+1}]_* \cong [X, Y_{s+1}]_{*-\mathbf{1}}$ to the composition $k_s \circ f : S^a \otimes X \to \Sigma Y_{s+1}$.

0.4. **The E_1 page.** The goal of this subsection is to provide the following characterization for the E_1 page of the Adams spectral sequence:

Theorem 0.12. *Let E be a flat commutative monoid object in $\mathcal{SH}$, and let X and Y be two objects in $\mathcal{SH}$ such that $E_*(X)$ is a projective module over $\pi_*(E)$. Then for all $s \geq 0$ and $a \in A$, we have isomorphisms in the associated E-Adams spectral sequence*

$$E_1^{s,a} \cong \mathrm{Hom}_{E_*(E)}^a(E_*(X), E_*(W_s))$$

Furthermore, under these isomorphisms, the differential $d_1 : E_1^{s,a} \to E_1^{s+1,a-1}$ corresponds to the map

$$\mathrm{Hom}_{E_*(E)}^a(E_*(X), E_*(W_s)) \to \mathrm{Hom}_{E_*(E)}^{a-1}(E_*(X), E_*(X \otimes W_{s+1}))$$

which sends a map $f : E_*(X) \to E_{*+a}(W_s)$ to the composition

$$E_*(X) \xrightarrow{f} E_{*+a}(W_s) \xrightarrow{(X \otimes h_s)_*} E_{*+a-1}(X \otimes Y_{s+1}) \xrightarrow{(X \otimes j_{s+1})_*} E_{*+a-1}(X \otimes W_{s+1}).$$

Proof. By **??**, for all $s \geq 0$ and $t, w \in \mathbb{Z}$, we have isomorphisms

$$[X, E \otimes Y_s]_{t,w} \cong \mathrm{Hom}_{E_*(E)}^{t,w}(E_*(X), E_*(E \otimes Y_s)).$$

since $W_s = E \otimes Y_s$, we have that

$$E_1^{s,(t,w)} = [X, W_s]_{t,w} \cong \mathrm{Hom}_{E_*(E)}^{t,w}(E_*(X), E_*(W_s)),$$

as desired. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Definition 0.13. Let (E, μ, e) be a monoid object in $\mathcal{SH}$. We say E is *flat* if the canonical right $\pi_*(E)$-module structure on $E_*(E)$ is that of a flat module.

0.5. **The E_2 page.**

0.6. **Convergence.** convergence of spectral sequences