0.1. **Grading.** First, we develop the theory of things graded by an abelian group. In what follows, we fix an abelian group A. We assume the reader is familiar with the basic theory of modules over non-commutative rings.

Definition 0.1. An *A-graded abelian group* is an abelian group B along with a subgroup $B_a \leq B$ for each $a \in A$ such that the canonical map

$$\bigoplus_{a \in A} B_a \to B$$

sending $(x_a)_{a \in A}$ to $\sum_{a \in A} x_a$ is an isomorphism. Given two A-graded abelian groups B and C, a homomorphism $f : B \to C$ is a *homomorphism of A-graded abelian groups* if it preserves the grading, i.e., if it restricts to a map $B_a \to C_a$ for all $a \in A$.

Remark 0.2. We often will denote an A-graded abelian group B by B_*. Given some $a \in A$, we can define the shifted A-graded abelian group B_{*+a} whose b^{th} component is B_{b+a}.

Definition 0.3. More generally, given two A-graded abelian groups B and C and some $d \in A$, a group homomorphism $f : B \to C$ is *an A-graded homomorphism of degree d* if it restricts to a map $B_a \to C_{a+d}$ for all $a \in A$. Thus, an A-graded homomorphism of degree d from B to C is equivalently an A-graded homomorphism $B_* \to C_{*+d}$.

Unless stated otherwise, an "A-graded homomorphism" will always refer to an A-graded homomorphism of degree 0. It is easy to see that an A-graded abelian group B is generated by its *homogeneous* elements, that is, nonzero elements $x \in B$ such that there exists some $a \in A$ with $x \in B_a$.

Remark 0.4. Clearly the condition that the canonical map $\bigoplus_{a \in A} B_a \to B$ is an isomorphism requires that $B_a \cap B_b = 0$ if $a \neq b$. In particular, given a homogeneous element $x \in B$, there exists precisely one $a \in A$ such that $x \in B_a$. We call this a the *degree* of x, and we write $|x| = a$.

Lemma 0.5. *Given two A-graded abelian groups B and C, their product $B \oplus C$ is naturally an A-graded abelian group by defining*

$$(B \oplus C)_a := \bigoplus_{b+c=a} B_b \oplus C_c.$$

Proof. This is entirely straightforward, as

$$B \oplus C \cong \left(\bigoplus_{b \in A} B_b\right) \oplus \left(\bigoplus_{c \in A} C_c\right) \cong \bigoplus_{b,c \in A} B_b \oplus C_c \cong \bigoplus_{a \in A} \bigoplus_{b \in A} B_b \oplus C_{a-b} \cong \bigoplus_{a \in A} \left(\bigoplus_{b+c=a} B_b \oplus C_c\right).$$

$\square$

Definition 0.6. An A-graded ring R is a ring such that is underlying abelian group is A-graded, and the multiplication map $R \times R \to R$ is a (degree 0) homomorphism of A-graded abelian groups (here R has the structure of an A-graded abelian group by Lemma 0.5).

Definition 0.7. Let R be an A-graded ring. A left A-graded R-module M is a left R-module M such that M is an A-graded abelian group, and the multiplication map $R \times M \to M$ is a homomorphism of A-graded abelian groups (i.e., for all $a, b \in A$ this map must restrict to $R_a \times M_b \to M_{a+b}$). Right A-graded R-modules are defined similarly. Finally, an A-graded R-bimodule is an A-graded abelian group M along with action maps

$$R \times M \to M \qquad \text{and} \qquad M \times R \to M$$

which endow M with the structure of a left and right A-graded R-module, respectively, such that given $r, s \in R$ and $m \in M$, $r \cdot (m \cdot s) = (r \cdot m) \cdot s$.

Definition 0.8. An A-graded map of A-graded rings (resp. left/right A-graded R-modules) is a homomorphism of rings (resp. left/right R-modules) such that the underlying homomorphism of abelian groups is A-graded.

Explicitly, given an A-graded ring R and homogeneous elements $x, y \in R$, we must have $|xy| = |x| + |y|$. For example, given some field k, the ring $R = k[x, y]$ is $\mathbb{Z}^2$-graded, where given $(n, m) \in \mathbb{Z}^2$, $R_{n,m}$ is the subgroup of those monomials of the form $ax^n y^m$ for some $a \in k$. Oftentimes when constructing A-graded rings, we do so only by defining the product of homogeneous elements, like so:

Proposition 0.9. *Given an A-graded abelian group R, a distinguished element $1 \in R_0$, and $\mathbb{Z}$-bilinear maps $m_{a,b} : R_a \times R_b \to R_{a+b}$ for all $a, b \in A$ such that given $x \in R_a$, $y \in R_b$, and $z \in R_c$,*

$$m_{a+b,c}(m_{a,b}(x, y), z) = m_{a,b+c}(x, m_{b,c}(y, z)) \qquad \textit{and} \qquad m_{a,0}(x, 1) = m_{0,a}(1, x) = x,$$

there exists a unique multiplication map $m : R \times R \to R$ which endows R with the structure of an A-graded ring and restricts to $m_{a,b}$ for all $a, b \in A$.

Proof. Given $r, s \in R$, since $R \cong \bigoplus_{a \in A} R_a$, we may uniquely decompose r and s into homogeneous elements as $r = \sum_{a \in A} r_a$ and $s = \sum_{a \in A} s_a$ with each $r_a, s_a \in R_a$ such that only finitely many of the r_a's and s_a's are nonzero. Then in order to define a distributive product $R \times R \to R$ which restricts to $m_{a,b} : R_a \times R_b \to R_{a+b}$, note we *must* define

$$r \cdot s = \left(\sum_{a \in A} r_a \right) \cdot \left(\sum_{b \in A} s_b \right) = \sum_{a,b \in A} r_a \cdot s_b = \sum_{a,b \in A} m_{a,b}(r_a, s_b).$$

Thus, we have shown uniqueness. It remains to show this product actually gives R the structure of a ring. First we claim that the sum on the right is actually finite. Note there exists only finitely many nonzero r_a's and s_b's, and if $s_b = 0$ then

$$m_{a,b}(r_a, 0) = m_{a,b}(r_a, 0 + 0) \overset{(*)}{=} m_{a,b}(r_a, 0) + m_{a,b}(r_a, 0) \implies m_{a,b}(r_a, 0) = 0,$$

where $(*)$ follows from bilinearity of $m_{a,b}$. A similar argument yields that $m_{a,b}(0, r_b) = 0$ for all $a, b \in A$. Hence indeed $m_{a,b}(r_a, s_b)$ is zero for all but finitely many pairs $(a, b) \in A^2$, as desired. Observe that in particular

$$(r \cdot s)_a = \sum_{b+c=a} m_{b,c}(r_b, s_c) = \sum_{b \in A} m_{b,a-b}(r_b, s_{a-b}) = \sum_{c \in A} m_{a-c,c}(r_{a-c}, s_c).$$

Now we claim this multiplication is associative. Given $t = \sum_{a\in A} t_a \in R$, we have

$$\begin{aligned}
(r\cdot s)\cdot t &= \sum_{a,b\in A} m_{a,b}((r\cdot s)_a, t_b)\\
&= \sum_{a,b\in A} m_{a,b}\left(\sum_{c\in A} m_{a-c,c}(r_{a-c}, s_c), t_b\right)\\
&\overset{(1)}{=} \sum_{a,b,c\in A} m_{a,b}(m_{a-c,c}(r_{a-c}, s_c), t_b)\\
&\overset{(2)}{=} \sum_{a,b,c\in A} m_{c,a+b-c}(r_c, m_{a-c,b}(s_{a-c}, t_b))\\
&\overset{(3)}{=} \sum_{a,b,c\in A} m_{a,c}(r_a, m_{b,c-b}(s_b, t_{c-b}))\\
&\overset{(1)}{=} \sum_{a,c\in A} m_{a,c}\left(r_a, \sum_{b\in A} m_{b,c-b}(s_b, t_{c-b})\right)\\
&= \sum_{a,c\in A} m_{a,c}(r_a, (s\cdot t)_c) = r\cdot(s\cdot t),
\end{aligned}$$

where each occurrence of (1) follows by bilinearity of the $m_{a,b}$'s, each occurrence of (2) is associativity of the $m_{a,b}$'s, and (3) is obtained by re-indexing by re-defining $a := c$, $b := a-c$, and $c := a+b-c$. Next, we wish to show that the distinguished element $1 \in R_0$ is a unit with respect to this multiplication. Indeed, we have

$$1\cdot r \overset{(1)}{=} \sum_{a\in A} m_{0,a}(1, r_a) \overset{(2)}{=} \sum_{a\in A} r_a = r$$

and

$$r\cdot 1 \overset{(1)}{=} \sum_{a\in A} m_{a,0}(r_a, 1) \overset{(2)}{=} \sum_{a\in A} r_a = r,$$

where (1) follows by the fact that $m_{a,b}(0,-) = m_{a,b}(-,0) = 0$, which we have shown above, and (2) follows by unitality of the $m_{0,a}$'s and $m_{0,a}$'s, respectively. Finally, we wish to show that this product is distributive. Indeed, we have

$$\begin{aligned}
r\cdot(s+t) &= \sum_{a,b\in A} m_{a,b}(r_a, (s+t)_b)\\
&= \sum_{a,b\in A} m_{a,b}(r_a, s_b + t_b)\\
&\overset{(*)}{=} \sum_{a,b\in A} m_{a,b}(r_a, s_b) + \sum_{a,b\in A} m_{a,b}(r_a, t_b) = (r\cdot s) + (r\cdot t),
\end{aligned}$$

where $(*)$ follows by bilinearity of $m_{a,b}$. An entirely analagous argument yields that $(r+s)\cdot t = (r\cdot t) + (s\cdot t)$. $\square$

When working with A-graded abelian groups, we will freely use the above proposition without comment.

Proposition 0.10. *Let R be an A-graded ring, and suppose we have a right A-graded R-module M and a left A-graded R-module N. Then the tensor product*

$$M \otimes_R N$$

is naturally an A-graded abelian group by defining $(M \otimes_R N)_a$ to be the subgroup generated by homogeneous pure tensors $m \otimes n$ with $m \in M_b$ and $n \in N_c$ such that $b + c = a$. Furthermore, if either M (resp. N) is an A-graded bimodule, then $M \otimes_R N$ is naturally a left (resp. right) A-graded R-module

Proof. By definition, since M and N are A-graded abelian groups, they are generated (as abelian groups) by their homogeneous elements. Thus it follows that $M \otimes_R N$ is generated by *homogeneous pure tensors*, that is, elements of the form $m \otimes n$ with $m \in M$ and $n \in N$ homogeneous. Now, given a homogeneous pure tensor $m \otimes n$, we define its *degree* by the formula $|m \otimes n| := |m| + |n|$. It follows this formula is well-defined by checking that given homogeneous elements $m \in M$, $n \in N$, and $r \in R$ that

$$|(m \cdot r) \otimes n| = |m \cdot r| + |n| = |m| + |r| + |n| = |m| + |r \cdot n| = |m \otimes (r \cdot n)|.$$

Thus, we may define $(M \otimes_R N)_a$ to be the subgroup of $M \otimes_R N$ generated by those pure homogeneous tensors of degree a. Now, we construct a map

$$\Phi : M \times N \to \bigoplus_{a \in A} (M \otimes_R N)_a$$

which takes a pair $(m, n) = \sum_{a \in A} (m_a, n_a)$ to the element $\Phi(m, n)$ whose a^{th} component is

$$(\Phi(m, n))_a := \sum_{b+c=a} m_a \otimes n_a.$$

It is straightforward to see that this map is R-balanced, in the sense that it is additive in each argument and $\Phi(m \cdot r, n) = \Phi(m, r \cdot n)$ for all $m \in M$, $n \in N$, and $r \in R$. Thus by the universal property of $M \otimes_R N$, we get a lift $\widetilde{\Phi} : M \otimes_R N \to \bigoplus_{a \in A} (M \otimes_R N)_a$. Now, also consider the canonical map

$$\Psi : \bigoplus_{a \in A} (M \otimes_R N)_a \to M \otimes_R N.$$

We would like to show $\widetilde{\Phi}$ and Ψ are inverses of eah other. It suffices to show this on generators. Let $m \otimes n$ be a pure homogeneous tensor with $m = m_a \in M_a$ and $n = n_a \in N_b$. Then we have

$$\Psi(\widetilde{\Phi}(m \otimes n)) = \Psi\left(\bigoplus_{a \in A} \sum_{b+c=a} m_b \otimes n_c\right) \overset{(*)}{=} \Psi(m \otimes n) = m \otimes n,$$

and

$$\widetilde{\Phi}(\Psi(m \otimes n)) = \widetilde{\Phi}(m \otimes n) = \bigoplus_{a \in A} \sum_{b+c=a} m_b \otimes n_c \overset{(*)}{=} m \otimes n,$$

where both occurrences of $(*)$ follow by the fact that $m_b \otimes n_c = 0$ unless $b = c = a$, in which case $m_a \otimes n_a = m \otimes n$. Thus since Ψ is an isomorphism, $M \otimes_R N$ is indeed an A-graded abelian group, as desired.

Now, suppose that M is an A-graded R-bimodule, so there exists a left and right action of R on M such that given $r, s \in R$ and $m \in M$ we have $r \cdot (m \cdot s) = (r \cdot m) \cdot s$. Then we would like to show that given a left A-graded R-module N that $M \otimes_R N$ is canonically a left A-graded R-module. Indeed, define the action of R on $M \otimes_R N$ on pure tensors by the formula

$$r \cdot (m \otimes n) = (r \cdot m) \otimes n.$$

First of all, clearly this map is A-graded, as if $r \in R_a$, $m \in M_b$, and $n \in N_c$ then $(r \cdot m) \otimes n$, by definition, has degree $|r \cdot m| + |n| = |r| + |m| + |n|$ (the last equality follows since the left action of R on M is A-graded). In order to show the above map defines a left module structure, it suffices to show that given pure tensors $m \otimes n, m' \otimes n' \in M \otimes_R N$ and elements $r, r' \in R$ that

(1) $r \cdot (m \otimes n + m' \otimes n') = r \cdot (m \otimes n) + r \cdot (m' \otimes n')$,

(2) $(r + r') \cdot (m \otimes n) = r \cdot (m \otimes n) + r' \cdot (m' \otimes n')$,
(3) $(rr') \cdot (m \otimes n) = r \cdot (r' \cdot (m \otimes n))$, and
(4) $1 \cdot (m \otimes n) = m \otimes n$.

Axiom (1) holds by definition. To see (2), note that by the fact that R acts on M on the left that

$$(r + r') \cdot (m \otimes n) = ((r + r') \cdot m) \otimes n = (r \cdot m + r' \cdot m) \otimes n = r \cdot m \otimes n + r' \cdot m \otimes n.$$

That (3) and (4) hold follows similarly by the fact that $(rr') \cdot m = r \cdot (r' \cdot m)$ and $1 \cdot m = m$.

Conversely, if N is an A-graded R-bimodule, then showing $M \otimes_R N$ is canonically a right A-graded R-module via the rule

$$(m \otimes n) \cdot r = m \otimes (n \cdot r)$$

is entirely analagous. □

Lemma 0.11. *Let R be an A-graded ring, and suppose we have a right A-graded R-module M and a left A-graded R-module N. Then given an A-graded abelian group B and an A-graded R-balanced map*

$$\varphi : M \times N \to B$$

(here $M \times N$ is regarded as an A-graded abelian group by Lemma 0.5), the lift

$$\widetilde{\varphi} : M \otimes_R N \to B$$

determined by the universal property of $M \otimes_R N$ is an A-graded map.

Proof. This simply amounts to unravelling definitions. Recall that the subgroup of homogeneous elements of degree a in $M \otimes_R N$ is that generated by pure tensors $m \otimes n$ with m and n homogeneous satisfying $|m| + |n| = a$. Thus, in order to show $\widetilde{\varphi}$ is an A-graded homomorphism, it suffices to show that given homogeneous $m \in M$ and $n \in N$, we have

$$|\widetilde{\varphi}(m \otimes n)| = |m \otimes n| = |m| + |n|.$$

Indeed, given two such elements, consider the following diagram

$$\begin{tikzcd}
M \otimes_R N \arrow[dr, "\widetilde{\varphi}"] & \\
M \times N \arrow[u] \arrow[r, "\varphi"'] & B
\end{tikzcd}$$

This diagram commutes by universal property of $- \otimes_R -$. Note that the element $m \otimes n$ is mapped to by the pair (m, n) along the left vertical map. Hence by commutativity, we necessarily have

$$|\widetilde{\varphi}(m \otimes n)| = |\varphi(m, n)| \overset{(*)}{=} |(m, n)| = |m| + |n|,$$

where $(*)$ follows by the fact that φ is an A-graded map. □

Lemma 0.12. *Let R be an A-graded ring, and suppose we have an A-graded R-bimodule M. Then for all $a \in A$, we have an A-graded isomorphism of left A-graded R-modules*

$$M \otimes_R R_{*+a} \cong M_{*+a}$$

induced by the assignment

$$M \times R_{*+a} \to M_{*+a}$$

sending $m \in M_b$ and $r \in R_{c+a}$ to $m \cdot r \in M_{b+c+a}$ (where here $M \otimes_R R$ has the structure of a left A-graded R-module by Proposition 0.10, and $m \cdot r$ denotes the right action of r on m).

Proof. First of all, note that if you ignore the grading then the map $M \times R_{*+a} \to M_{*+a}$ is simply the structure map for the right action of R on M. In particular, by the module axioms this map is R-balanced, so it does indeed induce an A-graded homomorphism of A-graded abelian groups $\varphi : M \otimes_R R_{*+a} \to M_{*+a}$. Furthermore, note this map is actually a homomorphism of left A-graded R-modules, as given $m \in M$ and $r, r' \in R$, we have $r \cdot (m \cdot r') = (r \cdot m) \cdot r'$, since M is a bimodule. Now, to see this map is an isomorphism, it suffices to construct an inverse. Indeed, define the map

$$\psi : M_{*+a} \to M \otimes_R R_{*+a}$$

to send $m \mapsto m \otimes 1$. First of all note this map is A-graded, as given $m \in M_{b+a}$, we have $\psi(m) = m \otimes 1$ has degree $|m| + |1| = |m| = b + a$, by definition of the graded structure on $M \otimes_R R_{*+a}$. Note that it is a homomorphism of left R-modules, as given $m, m' \in M$ and $r, r' \in R$ we have

$$\psi(rm + r'm') = (rm + r'm') \otimes 1 = r(m \otimes 1) + r'(m' \otimes 1) = r\psi(m) + r'\psi(m').$$

Now, to see ψ and φ are inverses, note first that given $m \in M_{*+a}$ that

$$\varphi(\psi(m)) = \varphi(m \otimes 1) = m \cdot 1 = m,$$

and given $m \otimes r \in M \otimes_R R_{*+a}$,

$$\psi(\varphi(m \otimes r)) = \psi(m \cdot r) = (m \cdot r) \otimes 1 = m \otimes (r \cdot 1) = m \otimes r.$$

□