In this appendix, we will define the notion of *A-graded anticommutative Hopf algebroids* (Definition 0.2) over an A-graded anticommutative ring R (**??**), and left comodules over them (Definition 0.6).

0.1. **A-graded anticommutative Hopf algebroids over R.** Given an A-graded anticommutative ring R, we will define an A-graded anticommutative Hopf algebroid over R to be a cogroupoid object in R-$\mathbf{GCA}^A$, i.e., a groupoid object in $(R\text{-}\mathbf{GCA}^A)^{\mathrm{op}}$. First, recall the definition of a *groupoid object* in a category with pullbacks:

Definition 0.1. Let $\mathcal{C}$ be a category with pullbacks. A *groupoid object* in $\mathcal{C}$ consists of a pair of objects (M, O) together with five morphisms

(1) *Source and target*: $s, t : M \to O$,
(2) *Identity*: $e : O \to M$,
(3) *Composition*: $c : M \times_O M \to M$,
(4) *Inverse*: $i : M \to M$

Where $M \times_O M$ will always refer to the object which into the following pullback diagram in $\mathcal{C}$:

$$\begin{tikzcd}
M \times_O M \arrow[r, "p_2"] \arrow[d, "p_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & M \arrow[d, "t"] \\
M \arrow[r, "s"'] & O
\end{tikzcd}$$

For example, if we're working in $\mathcal{C} = \mathbf{Set}$, we should think of M as a set of morphisms, and O as a set of objects. The functions s and t take a morphism to their domain and codomain, respectively, and $M \times_O M$ is the collection of pairs of morphisms $(g, f) \in M \times M$ such that $t(f) = s(g)$, and the composition map $c : M \times_O M \to M$ takes such a pair to the element $g \circ f \in M$. We think of the identity $e : O \to M$ as taking some object $x \in O$ to the identity morphism $e(x) = \mathrm{id}_x \in M$ on x, and the inverse map $i : M \to M$ takes a morphism f to its inverse f^{-1}. These data are required to make the following diagrams commute:

(1) Composition works correctly:

$$\begin{tikzcd}
M \times_O M \arrow[r, "c"] \arrow[d, "p_1"'] & M \arrow[d, "t"] \\
M \arrow[r, "t"] & O
\end{tikzcd}
\qquad
\begin{tikzcd}
M \arrow[dr, "s"'] & O \arrow[l, "e"'] \arrow[r, "e"] \arrow[d, equal] & M \arrow[dl, "t"] \\
& O &
\end{tikzcd}
\qquad
\begin{tikzcd}
M \times_O M \arrow[r, "p_2"] \arrow[d, "c"'] & M \arrow[d, "s"] \\
M \arrow[r, "s"] & O
\end{tikzcd}$$

Expressed in terms of sets, the first diagram says that the target of $g \circ f$ is the target of g. The second diagram says that the domain and codomain of the identity on some object x is x. The third diagram says that the domain of $g \circ f$ is the domain of f.

(2) Associativity of composition: Write $M \times_O (M \times_O M)$ and $(M \times_O M) \times_O M$ for the pullbacks of $(s, t \circ c)$ and $(s \circ c, t)$, respectively, so we have commuting diagrams

$$\begin{tikzcd}
(M \times_O M) \times_O M \arrow[rr, "p_2'"] \arrow[dr, dashed, "c \times M"'] \arrow[dd, "p_1'"'] & & M \arrow[d, equal] \\
& M \times_O M \arrow[r, "p_2"] \arrow[d, "p_1"'] & M \arrow[d, "t"] \\
M \times_O M \arrow[r, "c"] & M \arrow[r, "s"] & O
\end{tikzcd}
\qquad
\begin{tikzcd}
M \times_O (M \times_O M) \arrow[rr, "p_2''"] \arrow[dr, dashed, "M \times c"'] \arrow[dd, "p_1''"'] & & M \times_O M \arrow[d, "c"] \\
& M \times_O M \arrow[r, "p_2"] \arrow[d, "p_1"'] & M \arrow[d, "t"] \\
M \arrow[r, equal] & M \arrow[r, "s"] & O
\end{tikzcd}$$

where the inner and outer squares in both diagrams are pullback squares. Furthermore, assuming the diagrams in condition (1) above are satisfied, we have that $t \circ p_1 \circ p_2'' = t \circ c \circ p_2'' = s \circ p_1''$, so that by the universal property of the pullback we have a map $M \times p_1 : M \times_O (M \times_O M) \to M \times_O M$ like so:

$$\begin{tikzcd}
M \times_O (M \times_O M) \arrow[dr, dashed, "M \times p_1"] \arrow[drr, bend left, "p_1 \circ p_2''"] \arrow[ddr, bend right, "p_1''"'] & & \\
& M \times_O M \arrow[r, "p_2"] \arrow[d, "p_1"'] & M \arrow[d, "t"] \\
& M \arrow[r, "s"] & O
\end{tikzcd}$$

Now note that again assuming the diagrams above in (1) commute, we have $s \circ c = s \circ p_2$, so that

$$s \circ c \circ (M \times p_1) = s \circ p_2 \circ (M \times p_1) = s \circ p_1 \circ p_2'' = t \circ p_2 \circ p_2''.$$

Then by the unviersal property of the pullback we get a map $a : M \times_O (M \times_O M) \to (M \times_O M) \times_O M$ like so:

$$\begin{tikzcd}
M \times_O (M \times_O M) \arrow[dr, dashed, "a"] \arrow[drrr, bend left, "p_2 \circ p_2''"] \arrow[dddr, bend right, "M \times p_1"'] & & & \\
& (M \times_O M) \times_O M \arrow[rr, "p_2'"] \arrow[dd, "p_1'"'] & & M \arrow[d, equal] \\
& & & M \arrow[d, "t"] \\
& M \times_O M \arrow[r, "c"] & M \arrow[r, "s"] & O
\end{tikzcd}$$

Exercise: Show that this map a is an isomorphism. Then we require that the following diagram commutes:

$$\begin{tikzcd}
M \times_O (M \times_O M) \arrow[rr, "a"] \arrow[d, "M \times c"'] & & (M \times_O M) \times_O M \arrow[d, "c \times M"] \\
M \times_O M \arrow[r, "c"] & M & M \times_O M \arrow[l, "c"']
\end{tikzcd}$$

Expressed in terms of sets, this diagram says $h \circ (g \circ f) = (h \circ g) \circ f$.

(3) Unitality of composition: Given the maps $(\mathrm{id}_M, e \circ t), (e \circ s, \mathrm{id}_M) : M \to M \times_O M$ defined by the universal property of $M \times_O M$:

$$\begin{tikzcd}
M \arrow[dr, dashed, "(\mathrm{id}_M, e \circ s)"] \arrow[drr, bend left, "e \circ s"] \arrow[ddr, bend right, equal] & & \\
& M \times_O M \arrow[r, "p_2"] \arrow[d, "p_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & M \arrow[d, "t"] \\
& M \arrow[r, "s"] & O
\end{tikzcd}
\qquad
\begin{tikzcd}
M \arrow[dr, dashed, "(e \circ t, \mathrm{id}_M)"] \arrow[drr, bend left, equal] \arrow[ddr, bend right, "e \circ t"'] & & \\
& M \times_O M \arrow[r, "p_2"] \arrow[d, "p_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & M \arrow[d, "t"] \\
& M \arrow[r, "s"] & O
\end{tikzcd}$$

the following diagram commutes:

$$\begin{tikzcd}
M \arrow[r, "(e\circ t, \mathrm{id}_M)"] \arrow[d, "(\mathrm{id}_M, e\circ s)"'] \arrow[dr, equal] & M \times_O M \arrow[d, "c"] \\
M \times_O M \arrow[r, "c"'] & M
\end{tikzcd}$$

Expressed in terms of sets, this diagram says that given $f \in M$ with $s(f) = x$ and $t(f) = y$, that $f \circ \mathrm{id}_x = f$ and $\mathrm{id}_y \circ f = f$.

(4) Inverse: The following diagrams must commute:

$$\begin{tikzcd}
& M \arrow[d, "i"] \arrow[dl, equal] \\
M & M \arrow[l, "i"]
\end{tikzcd}
\qquad
\begin{tikzcd}
M \arrow[r, "(\mathrm{id}_M, i)"] \arrow[d, "t"'] & M \times_O M \arrow[d, "c"] & M \arrow[l, "(i, \mathrm{id}_M)"'] \arrow[d, "s"] \\
O \arrow[r, "e"] & M & O \arrow[l, "e"']
\end{tikzcd}
\qquad
\begin{tikzcd}
& M \arrow[dl, "s"'] \arrow[d, "i"] \arrow[dr, "t"] & \\
O & M \arrow[l, "t"'] \arrow[r, "s"] & O
\end{tikzcd}$$

where the arrows (id_M, i) and (i, id_M) are determined by the universal property of $M \times_O M$ like so:

$$\begin{tikzcd}
M \arrow[drr, bend left, "i"] \arrow[dr, dashed, "(\mathrm{id}_M, i)"] \arrow[ddr, bend right, equal] & & \\
& M \times_O M \arrow[r, "p_2"] \arrow[d, "p_1"'] & M \arrow[d, "t"] \\
& M \arrow[r, "s"] & O
\end{tikzcd}
\qquad
\begin{tikzcd}
M \arrow[drr, bend left, equal] \arrow[dr, dashed, "(i, \mathrm{id}_M)"] \arrow[ddr, bend right, "i"'] & & \\
& M \times_O M \arrow[r, "p_2"] \arrow[d, "p_1"'] & M \arrow[d, "t"] \\
& M \arrow[r, "s"] & O
\end{tikzcd}$$

Expressed in terms of sets, given $f \in M$ with $s(f) = x$ and $t(f) = y$, the first diagram says that $(f^{-1})^{-1} = f$, the second says that $f \circ f^{-1} = \mathrm{id}_y$ and $f^{-1} \circ f = \mathrm{id}_x$, and the last diagram says that the domain and codomain of f^{-1} are y and x, respectively.

It can be seen that groupoid objects in $\mathcal{C} = \mathbf{Set}$ are precisely (small) groupoids. Now, we can state and unravel the definition of a Hopf algebroid:

Definition 0.2. Given an A-graded anticommutative ring R (**??**), an *A-graded anticommutative Hopf algebroid over R* is a co-groupoid object in R-$\mathbf{GCA}^A$, i.e., a groupoid object in $(R\text{-}A\mathbf{GrCAlg})^{\mathrm{op}}$. Explicitly, an A-graded anticommutative Hopf algebroid over E is a pair (Γ, B) of objects in R-$A\mathbf{GrCAlg}$ along with morphisms

(1) *left unit*: $\eta_L : B \to \Gamma$ (corresponding to t),
(2) *right unit*: $\eta_R : B \to \Gamma$ (corresponding to s),
(3) *comultiplication*: $\Psi : \Gamma \to \Gamma \otimes_B \Gamma$ (corresponding to c),
(4) *counit*: $\epsilon : \Gamma \to B$ (corresponding to e),
(5) *conjugation*: $c : \Gamma \to \Gamma$ (corresponding to i),

where here Γ may be viewed as a B-bimodule with left B-module structure induced by η_L and right B-module structure induced by η_R, so we may form the tensor product of bimodules $\Gamma \otimes_B \Gamma$, which further may be given the structure of an A-graded anticommutative R-algebra (by **??**), and fits into the following pushout diagram in R-$\mathbf{GCA}^A g$ (**??**):

$$\begin{tikzcd}
B \arrow[r, "\eta_L"] \arrow[d, "\eta_R"'] & \Gamma \arrow[d, "g\mapsto 1\otimes g"] \\
\Gamma \arrow[r, "g\mapsto g\otimes 1"'] & \Gamma\otimes_B\Gamma
\end{tikzcd}$$

These data must make the following diagrams commute:

(1) (Composition works correctly)

$$\begin{tikzcd}
B \arrow[r, "\eta_L"] \arrow[d, "\eta_L"'] & \Gamma \arrow[d, "\Psi"] \\
\Gamma \arrow[r, "g\mapsto g\otimes 1"'] & \Gamma\otimes_B\Gamma
\end{tikzcd}
\qquad
\begin{tikzcd}
 & B \arrow[dl, "\eta_R"'] \arrow[dr, "\eta_L"] \arrow[d, equal] & \\
\Gamma \arrow[r, "\epsilon"'] & B & \Gamma \arrow[l, "\epsilon"]
\end{tikzcd}
\qquad
\begin{tikzcd}
B \arrow[r, "\eta_R"] \arrow[d, "\eta_R"'] & \Gamma \arrow[d, "g\mapsto 1\otimes g"] \\
\Gamma \arrow[r, "\Psi"] & \Gamma\otimes_B\Gamma
\end{tikzcd}$$

(2) (Coassociativity) The following diagram must commute

$$\begin{tikzcd}
\Gamma\otimes_B\Gamma \arrow[d, "\Psi\otimes_B\Gamma"'] & \Gamma \arrow[l, "\Psi"'] \arrow[r, "\Psi"] & \Gamma\otimes_B\Gamma \arrow[d, "\Gamma\otimes_B\Psi"] \\
(\Gamma\otimes_B\Gamma)\otimes_B\Gamma \arrow[rr, "\cong"] & & \Gamma\otimes_B(\Gamma\otimes_B\Gamma)
\end{tikzcd}$$

where $(\Gamma\otimes_B\Gamma)\otimes_B\Gamma$ and $\Gamma\otimes_B(\Gamma\otimes_B\Gamma)$ denote the rings which fit into the following pushout diagrams in R-$\mathbf{GCA}^A$:

$$\begin{tikzcd}
B \arrow[r, "\eta_L"] \arrow[d, "\eta_R"'] & \Gamma \arrow[dd, "g\mapsto(1\otimes1)\otimes g"] \\
\Gamma \arrow[d, "\Psi"'] & \\
\Gamma\otimes_B\Gamma \arrow[r, "g\otimes g'\mapsto(g\otimes g')\otimes 1"'] & (\Gamma\otimes_B\Gamma)\otimes_B\Gamma
\end{tikzcd}
\qquad
\begin{tikzcd}
B \arrow[r, "\eta_L"] \arrow[d, "\eta_R"'] & \Gamma \arrow[r, "\Psi"] & \Gamma\otimes_B\Gamma \arrow[d, "(g\otimes g')\mapsto 1\otimes(g\otimes g')"] \\
\Gamma \arrow[rr, "g\mapsto g\otimes(1\otimes1)"'] & & \Gamma\otimes_B(\Gamma\otimes_B\Gamma)
\end{tikzcd}$$

and the isomorphism $(\Gamma\otimes_B\Gamma)\otimes_B\Gamma\to\Gamma\otimes_B(\Gamma\otimes_B\Gamma)$ sends $(g\otimes g')\otimes g''$ to $g\otimes(g'\otimes g'')$, the left vertical arrow $\Psi\otimes\Gamma$ sends $g\otimes g'$ to $\Psi(g)\otimes g$, and the right vertical arrow $\Gamma\otimes\Psi$ sends $g\otimes g'$ to $g\otimes\Psi(g')$.

(3) (Co-unitality):

$$\begin{tikzcd}
\Gamma \arrow[r, "\Psi"] \arrow[d, "\Psi"'] \arrow[dr, equal] & \Gamma\otimes_B\Gamma \arrow[d, "(\eta_L\circ\epsilon)\cdot\mathrm{id}_\Gamma"] \\
\Gamma\otimes_B\Gamma \arrow[r, "\mathrm{id}_\Gamma\cdot(\eta_R\circ\epsilon)"'] & \Gamma
\end{tikzcd}$$

where the right vertical arrow sends $g\otimes g'$ to $\eta_L(\epsilon(g))g'$ and the bottom horizontal arrow sends $g\otimes g'$ to $g\eta_R(\epsilon(g'))$.

(4) (Convolution):

$$\begin{tikzcd}
 & \Gamma \arrow[dl, equal] \arrow[d, "c"] \\
\Gamma & \Gamma \arrow[l, "c"]
\end{tikzcd}
\qquad
\begin{tikzcd}
B \arrow[d, "\eta_L"'] & \Gamma \arrow[l, "\epsilon"'] \arrow[r, "\epsilon"] \arrow[d, "i"'] & B \arrow[d, "\eta_R"] \\
\Gamma & \Gamma\otimes_B\Gamma \arrow[l, "\mathrm{id}_\Gamma\cdot c"] \arrow[r, "c\cdot\mathrm{id}_\Gamma"'] & \Gamma
\end{tikzcd}
\qquad
\begin{tikzcd}
B \arrow[r, "\eta_L"] \arrow[dr, "\eta_R"'] & \Gamma \arrow[d, "c"'] & B \arrow[l, "\eta_R"'] \arrow[dl, "\eta_L"] \\
 & \Gamma &
\end{tikzcd}$$

where the bottom left arrow in the middle diagram sends $g \otimes g'$ to $gc(g')$ and the bottom right arrow in the middle diagram sends $g \otimes g'$ to $c(g)g'$.

The remainder of this subsection is devoted to proving some technical lemmas about A-graded anticommutative Hopf algebroids.

Proposition 0.3. *Suppose we have an A-graded anticommutative Hopf algebroid (Γ, B) over (R, θ) with structure maps η_L, η_R, Ψ, ϵ, and c (Definition 0.2). Recall in the definition, we considered $\Gamma \otimes_B \Gamma$ to be the A-graded R-commutative ring whose underlying abelian group was given by the tensor product of B-bimodules, where Γ has left B-module structure induced by η_L and right B-module structure induced by η_R. Thus $\Gamma \otimes_B \Gamma$ is canonically a B-bimodule, as it is a tensor product of B-bimodules. Then the canonical left (resp. right) B-module structure on $\Gamma \otimes_B \Gamma$ coincides with that induced by the ring homomorphism $\Psi \circ \eta_L$ (resp. $\Psi \circ \eta_R$).*

Proof. First we show the left module structures coincide. By additivity, in order to show the module structures coincide, it suffices to show that given a homogeneous pure tensor $g \otimes g'$ in $\Gamma \otimes_B \Gamma$ and some $b \in B$ that $\Psi(\eta_L(b)) \cdot (g \otimes g') = (\eta_L(b) \cdot g) \otimes g'$, where $\cdot$ on the left denotes the product in $\Gamma \otimes_B \Gamma$ and the $\cdot$ on the right denotes the product in Γ. By the axioms for a Hopf algebroid, we have that $\Psi(\eta_L(b)) = \eta_L(b) \otimes 1$. Thus by how the product in $\Gamma \otimes_B \Gamma$ is defined (**??**), we have that

$$\Psi(\eta_L(b)) \cdot (g \otimes g') = (\eta_L(b) \otimes 1) \cdot (g \otimes g') = (\varphi_\Gamma(\theta_{0,|g|}) \cdot \eta_L(b) \cdot g) \otimes (g' \cdot 1) = (\eta_L(b) \cdot g) \otimes g',$$

where $\varphi_\Gamma : R \to \Gamma$ is the structure map, and the last equality follows by the fact that $\theta_{0,|g|} = 1$. An entirely analagous argument yields that the canonical right module structure on $\Gamma \otimes_B \Gamma$ coincides with that induced by $\Psi \circ \eta_R$, since $\Psi \circ \eta_R = 1 \otimes \eta_R$. ☐

Remark 0.4. By the above proposition, given an A-graded commutative Hopf algebroid (Γ, B) over R, there is no ambiguity when discussing the objects $\Gamma \otimes_B (\Gamma \otimes_B \Gamma)$ and $(\Gamma \otimes_B \Gamma) \otimes_B \Gamma$ — they may both be considered as the threefold tensor product of the B-bimodule Γ with itself. In particular, we have a canonical isomorphism of B-bimodules

$$(\Gamma \otimes_B \Gamma) \otimes_B \Gamma \to \Gamma \otimes_B (\Gamma \otimes_B \Gamma)$$

sending $(g \otimes g') \otimes g''$ to $g \otimes (g' \otimes g'')$, and this is precisely the isomorphism in the coassociativity diagram in the definition of a Hopf algebroid (Definition 0.2).

Proposition 0.5. *Suppose we have an A-graded commutative Hopf algebroid (Γ, B) over R with structure maps η_L, η_R, Ψ, ϵ, and c. Then $\eta_L : B \to \Gamma$ is a homomorphism of left B-modules, $\eta_R : B \to \Gamma$ is a homomorphism of right B-modules, and $\Psi : \Gamma \to \Gamma \otimes_B \Gamma$ and $\epsilon : \Gamma \to B$ are homomorphisms of B-bimodules.*

Proof. Since the left (resp. right) B-module structure on Γ is induced by η_L (resp. η_R), the map η_L (resp. η_R) is a homomorphism of left (resp. right) B-modules by definition.

Next, we want to show Ψ is a homomorphism of B-bimodules. The left (resp. right) B-module structure on Γ is that induced by η_L (resp. η_R), and in Proposition 0.3, we showed that the left (resp. right) B-module structure on $\Gamma \otimes_B \Gamma$ is that induced by $\Psi \circ \eta_L$ (resp. $\Psi \circ \eta_R$), so that by definition $\Psi : \Gamma \to \Gamma \otimes_B \Gamma$ is a homomorphism of left (resp. right) B-modules.

Lastly, we claim that $\epsilon : \Gamma \to B$ is a homomorphism of B-bimodules. We need to show that given $g \in \Gamma$ and $b, b' \in B$ that $\epsilon(\eta_L(b) g \eta_R(g')) = b\epsilon(g)b'$. This follows from the fact that ϵ is a ring homomorphism satisfying $\epsilon \circ \eta_L = \epsilon \circ \eta_R = \mathrm{id}_B$. ☐

0.2. **Comodules over a Hopf algebroid.** In what follows, fix an A-graded anticommutative ring (R, θ) and an A-graded anticommutative Hopf algebroid (Γ, B) over R with structure maps η_L, η_R, Ψ, ϵ, and c. We will always view Γ with its *canonical* B-bimodule structure, with left B-module structure induced by η_L, and right B-module structure induced by η_R. In particular, any tensor product over B involving Γ will always refer to Γ with this bimodule structure.

Definition 0.6. A *left comodule over* Γ is a pair (N, Ψ_N), where N is a left A-graded B-module and $\Psi_N : N \to \Gamma \otimes_B N$ is an A-graded homomorphism of left A-graded B-modules. These data are required to make the following diagrams commute

$$\begin{tikzcd} N \arrow[r, "\Psi_N"] \arrow[dr, "\cong"'] & \Gamma \otimes_B N \arrow[d, "\epsilon\otimes N"] \\ & B \otimes_B N \end{tikzcd} \qquad \begin{tikzcd} \Gamma \otimes_B N \arrow[d, "\Psi\otimes N"'] & N \arrow[l, "\Psi_N"'] \arrow[r, "\Psi_N"] & \Gamma \otimes_B N \arrow[d, "\Gamma\otimes\Psi_N"] \\ (\Gamma \otimes_B \Gamma) \otimes_B N \arrow[rr, "\cong"] & & \Gamma \otimes_B (\Gamma \otimes_B N) \end{tikzcd}$$

The maps $\epsilon \otimes N$ and $\Psi \otimes N$ are well-defined by Proposition 0.5, and the bottom isomorphism in the right diagram is the canonical one sending $(g \otimes g') \otimes n \mapsto g \otimes (g' \otimes n)$.

Given two left A-graded Γ-comodules (N_1, Ψ_{N_1}) and (N_2, Ψ_{N_2}), a homomorphism of left A-graded comodules $f : N_1 \to N_2$ is an A-graded homomorphism of the underlying left B-modules such that the following diagram commutes:

$$\begin{tikzcd} N_1 \arrow[r, "f"] \arrow[d, "\Psi_{N_1}"'] & N_2 \arrow[d, "\Psi_{N_2}"] \\ \Gamma \otimes_B N_1 \arrow[r, "\Gamma\otimes f"] & \Gamma \otimes_B N_2 \end{tikzcd}$$

We write Γ-$\mathbf{CoMod}^A$ for the resulting category of left A-graded comodules over Γ. In the above definition, we required A-graded left Γ-comodule homomorphisms to strictly preserve the grading, but we could have instead considered left Γ-comodule homomorphisms which are of degree d for some $d \in A$, or equivalently, the set of degree zero A-graded Γ-comodule homomorphisms from N_1 to the shifted comodule $(N_2)_{*+d}$. We denote the hom-set of degree-d A-graded left Γ-comodule homomorphisms from (N_1, Ψ_{N_1}) to (N_2, Ψ_{N_2}) by

$$\mathrm{Hom}^d_{\Gamma\text{-}\mathbf{CoMod}^A}(N_1, N_2) \qquad \text{or usually just} \qquad \mathrm{Hom}^d_\Gamma(N_1, N_2).$$

In particular, write $\mathrm{Hom}_{\Gamma\text{-}\mathbf{CoMod}^A}(N_1, N_2)$ or just $\mathrm{Hom}_\Gamma(N_1, N_2)$ to mean the set of strictly degree preserving (degree 0) A-graded left Γ-comodule homomorphisms from (N_1, Ψ_{N_1}) to (N_2, Ψ_{N_2}).

Proposition 0.7. *The category* Γ-$\mathbf{CoMod}^A$ *is an additive category.*

Proof. First, we show the category is $\mathbf{Ab}$-enriched. Since the forgetful functor Γ-$\mathbf{CoMod}^A \to B$-$\mathbf{Mod}^A$ is clearly faithful, we may view hom-sets in Γ-$\mathbf{CoMod}^A$ as subsets of hom-groups in B-$\mathbf{Mod}^A$, so that in order to show Γ-$\mathbf{CoMod}^A$ is $\mathbf{Ab}$-enriched, it suffices to show that hom-sets in Γ-$\mathbf{CoMod}^A$ are closed under addition of module homomorphisms and taking inverses. To that end, suppose we have two A-graded left Γ-comodule homomorphisms $f, g : (N_1, \Psi_{N_1}) \to (N_2, \Psi_{N_2})$, then we have

$$\begin{aligned} \Psi_{N_2} \circ (f + g) &= (\Psi_{N_2} \circ f) + (\Psi_{N_2} \circ g) \\ &= ((\Gamma \otimes_B f) \circ \Psi_{N_1}) + ((\Gamma \otimes_B g) \circ \Psi_{N_1}) \\ &= ((\Gamma \otimes_B f) + (\Gamma \otimes_B g)) \circ \Psi_{N_1} \\ &= (\Gamma \otimes_B (f + g)) \circ \Psi_{N_1}, \end{aligned}$$

where the first equality follows since Ψ_{N_2} is a homomorphism, the second follows since f and g are left Γ-comodule homomorphisms, the third follows since Ψ_{N_1} is a homomorphism, and the last equality follows by definition of the tensor product of modules. Hence $f+g$ is indeed an A-graded left Γ-comodule homomorphism, as desired. Now, we also claim $-f$ is an A-graded left Γ-comodule homomorphism. To that end, note that

$$\Psi_{N_2} \circ (-f) = -\Psi_{N_2} \circ f = -(\Gamma \otimes_B f) \circ \Psi_{N_1} = (\Gamma \otimes_B (-f)) \circ \Psi_{N_1},$$

where the first equality follows since Ψ_{N_2} is a homomorphism, the second follows since f is an A-graded left Γ-comodule homomorphism, and the third equality follows by definition of the tensor product.

Thus, we've shown that the hom-sets in Γ-$\mathbf{CoMod}^A$ are abelian groups, and composition is clearly bilinear, so that Γ-$\mathbf{CoMod}^A$ is indeed $\mathbf{Ab}$-enriched.

Now, in order to show Γ-$\mathbf{CoMod}^A$ is additive, it suffices to show that it contains a zero object and has binary coproducts. First of all, it is straightforward to check that the zero left B-module is clearly an A-graded left Γ-comodule with structure map the unique map $0 \to \Gamma \otimes_B 0 \cong 0$, and that given any other A-graded left Γ-comodule (N, Ψ_N), the unique homomorphisms of left B-modules $0 \to N$ and $N \to 0$ are left comodule homomorphisms.

Now, suppose we have two A-graded left Γ-comodules (N_1, Ψ_{N_1}) and (N_2, Ψ_{N_2}). First, we claim their direct sum as left B-modules $N_1 \oplus N_2$ is canonically an A-graded left Γ-comodule. We know that $N_1 \oplus N_2$ is an A-graded left B-module by **??**, and we can define the structure map

$$\Psi_{N_1 \oplus N_2} : N_1 \oplus N_2 \xrightarrow{\Psi_{N_1} \oplus \Psi_{N_2}} (\Gamma \otimes_B N_1) \oplus (\Gamma \otimes_B N_2) \cong \Gamma \otimes_B (N_1 \oplus N_2),$$

where the final isomorphism is the canonical one sending $(g_1 \otimes n_1) \oplus (g_2 \otimes n_2)$ to $(g_1 \otimes n_1) + (g_2 \otimes n_2)$. Then to see this is in fact a left Γ-comodule, first consider the following diagram:

$$\begin{tikzcd}
N_1 \oplus N_2 \arrow[r, "\Psi_{N_1} \oplus \Psi_{N_2}"] \arrow[dr, "\cong"] \arrow[ddr, bend right] & (\Gamma \otimes_B N_1) \oplus (\Gamma \otimes_B N_2) \arrow[r, "\cong"] \arrow[d, dashed, "(\epsilon \otimes N_1) \oplus (\epsilon \otimes N_2)"] & \Gamma \otimes_B (N_1 \oplus N_2) \arrow[dd, "\epsilon \otimes (N_1 \oplus N_2)"] \\
 & (B \otimes_B N_1) \oplus (B \otimes_B N_2) \arrow[dr, "\cong"] & \\
 & & B \otimes_B (N_1 \oplus N_2)
\end{tikzcd}$$

A simple diagram chase yields the left and rightmost regions commute. The top left region commutes since (N_1, Ψ_{N_1}) and (N_2, Ψ_{N_2}) are left Γ-comodules. Now, consider the following diagram:

$$
\begin{tikzcd}
\Gamma \otimes_B (N_1 \oplus N_2) \arrow[r, "\Gamma\otimes_B(\Psi_{N_1}\oplus\Psi_{N_2})"] & \Gamma \otimes_B ((\Gamma \otimes_B N_1) \oplus (\Gamma \otimes_B N_2)) \arrow[r, "\Gamma\otimes_B\cong"] & \Gamma \otimes_B (\Gamma \otimes_B (N_1 \oplus N_2)) \\
(\Gamma \otimes_B N_1) \oplus (\Gamma \otimes_B N_2) \arrow[u, "\cong"] \arrow[r, "(\Gamma\otimes\Psi_{N_1})\oplus(\Gamma\otimes\Psi_{N_2})"] & (\Gamma \otimes_B (\Gamma \otimes_B N_1)) \oplus (\Gamma \otimes_B (\Gamma \otimes_B N_2)) \arrow[u, "\cong"'] \arrow[dd, "\cong\oplus\cong"] & \\
N_1 \oplus N_2 \arrow[u, "\Psi_{N_1}\oplus\Psi_{N_2}"] \arrow[d, "\Psi_{N_1}\oplus\Psi_{N_2}"'] & & \\
(\Gamma \otimes_B N_1) \oplus (\Gamma \otimes_B N_2) \arrow[r, "(\Psi\otimes N_1)\oplus(\Psi\otimes N_2)"] \arrow[d, "\cong"'] & ((\Gamma \otimes_B \Gamma) \otimes_B N_1) \oplus ((\Gamma \otimes_B \Gamma) \otimes_B N_2) \arrow[dr, "\cong"'] & \\
\Gamma \otimes_B (N_1 \oplus N_2) \arrow[rr, "\Psi\otimes(N_1\oplus N_2)"'] & & (\Gamma \otimes_B \Gamma) \otimes_B (N_1 \oplus N_2) \arrow[uuuu, "\cong"']
\end{tikzcd}
$$

The middle left region commutes since (N_1, Ψ_{N_1}) and (N_2, Ψ_{N_2}) are left Γ-comodules. Each other region in the diagram can be seen to commute by a straightforward diagram chase.

Thus, we have shown that $N_1 \oplus N_2$ is indeed canonically an A-graded left Γ-comodule. Then it remains to show that the canonical inclusions $\iota_i : N_i \hookrightarrow N_1 \oplus N_2$ are Γ-comodule homomorphisms for $i = 1, 2$, and that given Γ-comodule homomorphisms $(N_1, \Psi_{N_1}) \to (N, \Psi_N)$ and $(N_2, \Psi_{N_2}) \to (N, \Psi_N)$, that the map $N_1 \oplus N_2 \to N$ induced by the universal property of the coproduct in B-$\mathbf{Mod}^A$ is a Γ-comodule homomorphism. This is all entirely straightforward to check by doing a few simple diagram chases. $\square$

Proposition 0.8. *The forgetful functor* Γ-$\mathbf{CoMod}^A \to B$-$\mathbf{Mod}^A$ *(where here* B-$\mathbf{Mod}^A$ *is the category of* A*-graded left* B*-modules and degree-preserving module homomorphisms between them) has a right adjoint* $\Gamma \otimes_B - : B$-$\mathbf{Mod}^A \to \Gamma$-$\mathbf{CoMod}^A$ *called the* co-free construction*, where the co-free left* A*-graded* Γ*-comodule on a left* A*-graded* B*-module* M *is the* B*-module* $\Gamma \otimes_B M$ *equipped with the coaction*

$$\Psi_{\Gamma\otimes_B M} : \Gamma \otimes_B M \xrightarrow{\Psi\otimes_B M} (\Gamma \otimes_B \Gamma) \otimes_B M \xrightarrow{\cong} \Gamma \otimes_B (\Gamma \otimes_B M).$$

Explicitly, given some (N, Ψ_N) *in* Γ-$\mathbf{CoMod}$ *and some* M *in* B-$\mathbf{Mod}^A$*, the counit and unit of this adjunction are given by*

$$\eta_{(N,\Psi_N)} : N \xrightarrow{\Psi_N} \Gamma \otimes_B N$$

and

$$\varepsilon_M : \Gamma \otimes_B M \xrightarrow{\epsilon\otimes_B M} B \otimes_B M \xrightarrow{\cong} M.$$

Proof. First, we need to show that given a left A-graded B-module that the given map $\Psi_{\Gamma\otimes_B M} : \Gamma \otimes_B M \to \Gamma \otimes_B (\Gamma \otimes_B M)$ endows B with the structure of a left Γ-comodule. To that end, first consider the following diagram:

$$
\begin{tikzcd}
\Gamma \otimes_B M \arrow[r, "\Psi\otimes M"] \arrow[dr, equal] & (\Gamma \otimes_B \Gamma) \otimes_B M \arrow[rr, "\cong"] \arrow[d, "((\eta_L\circ\epsilon)\cdot\mathrm{id}_\Gamma)\otimes M"'] \arrow[dr, "(\epsilon\otimes\Gamma)\otimes M"] & & \Gamma \otimes_B (\Gamma \otimes_B M) \arrow[dd, "\epsilon\otimes(\Gamma\otimes M)"] \\
& \Gamma \otimes_B M \arrow[drr, "\cong"'] & (B \otimes_B \Gamma) \otimes_B M \arrow[l, "\eta_L\cdot\mathrm{id}_\Gamma"'] \arrow[dr, "\cong"] & \\
& & & B \otimes_B (\Gamma \otimes_B M)
\end{tikzcd}
$$

The top left region commutes by the co-unitality axiom for a Hopf algebroid. A simple diagram chase yields commutativity of every other diagram (in particular, the bottom region commutes since the left B-module structure on Γ is that induced by η_L). Now, consider the following diagram:

$$
\begin{tikzcd}
\Gamma \otimes_B (\Gamma \otimes_B M) \arrow[rr, "\Gamma\otimes(\Psi\otimes M)"] & & \Gamma \otimes_B (\Gamma \otimes_B (\Gamma \otimes_B M)) \arrow[dddd, "\Gamma\otimes\cong"] \\
(\Gamma \otimes_B \Gamma) \otimes_B M \arrow[u, "\cong"] \arrow[r, "(\Gamma\otimes\Psi)\otimes M"] & (\Gamma \otimes_B (\Gamma \otimes_B \Gamma)) \otimes_B M \arrow[ur, "\cong"] & \\
\Gamma \otimes_B M \arrow[u, "\Psi\otimes M"] \arrow[d, "\Psi\otimes M"'] & & \\
(\Gamma \otimes_B \Gamma) \otimes_B M \arrow[r, "(\Psi\otimes\Gamma)\otimes M"] \arrow[d, "\cong"'] & ((\Gamma \otimes_B \Gamma) \otimes_B \Gamma) \otimes_B M \arrow[uu, "\cong"] & \\
\Gamma \otimes_B (\Gamma \otimes_B M) \arrow[r, "\Psi\otimes(\Gamma\otimes M)"] & (\Gamma \otimes_B \Gamma) \otimes_B (\Gamma \otimes_B M) \arrow[u, "\cong"] \arrow[r, "\cong"] & \Gamma \otimes_B ((\Gamma \otimes_B \Gamma) \otimes_B M)
\end{tikzcd}
$$

The left region commutes since Ψ is co-associative. A simple diagram chase yields the commutativity of every other diagram. Thus, we have indeed shown that $(\Gamma \otimes_B M, \Psi_{\Gamma\otimes_B M})$ is an A-graded left Γ-comodule, as desired.

Now, we need to show that η and ε are natural transformations which satisy the zig-zag identities. The maps η is clearly natural by how morphisms in Γ-$\mathbf{CoMod}^A$ are defined. It is also clear that ε is natural by functoriality of $-\otimes_B -$. Thus, it remains to show the following two diagrams commute for all M in B-$\mathbf{Mod}^A$ and (N, Ψ_N) in Γ-$\mathbf{CoMod}^A$:

$$
\begin{tikzcd}
N \arrow[r, "\eta_{(N,\Psi_N)}"] \arrow[dr, equal] & \Gamma \otimes_B N \arrow[d, "\varepsilon_N"] \\
& N
\end{tikzcd}
\qquad
\begin{tikzcd}
\Gamma \otimes_B (\Gamma \otimes_B M) \arrow[d, "\Gamma\otimes_B\varepsilon_M"'] & \Gamma \otimes_B M \arrow[l, "\eta_{(\Gamma\otimes_B M, \Psi_{\Gamma\otimes_B M})}"'] \arrow[dl, equal] \\
\Gamma \otimes_B M &
\end{tikzcd}
$$

Unravelling definitions, the left diagram becomes:

$$\begin{tikzcd}
N \arrow[r, "\Psi_N"] \arrow[dr, "\cong"] \arrow[ddr, equal] & \Gamma \otimes_B N \arrow[d, "\epsilon\otimes_B N"] \\
& B \otimes_B N \arrow[d, "\cong"] \\
& N
\end{tikzcd}$$

This commutes since (N, Ψ_N) is a left Γ-comodule. On the other hand, the right diagram becomes:

$$\begin{tikzcd}
\Gamma \otimes_B (\Gamma \otimes_B M) \arrow[d, "\Gamma\otimes(\epsilon\otimes M)"'] & (\Gamma \otimes_B \Gamma) \otimes_B M \arrow[l, "\cong"'] \arrow[d, "(\Gamma\otimes_B\epsilon)\otimes M"'] \arrow[ddr, "(\mathrm{id}_\Gamma\cdot(\eta_R\circ\epsilon))\otimes M"] & \Gamma \otimes_B M \arrow[l, "\Psi\otimes M"'] \arrow[dd, equal] \\
\Gamma \otimes_B (B \otimes_B M) \arrow[d, "\Gamma\otimes\cong"'] & (\Gamma \otimes_B B) \otimes_B M \arrow[l, "\cong"'] \arrow[dl, "(\mathrm{id}_\Gamma\cdot\eta_R)\otimes M"'] & \\
\Gamma \otimes_B M \arrow[rr, equal] & & \Gamma \otimes_B M
\end{tikzcd}$$

The rightmost region commutes by co-unitality of Ψ, while a simple diagram chase yields commutativity of the remaining regions (in particular, the bottom let region commutes because the right B-module structure on Γ is induced by η_R). $\square$

Proposition 0.9. *Suppose that Γ is flat as a right B-module, i.e., suppose $\eta_R : B \to \Gamma$ is a flat ring homomorphism. Then the category Γ-$\mathbf{CoMod}^A$ is an abelian category and has enough injectives.*

Proof. In Proposition 0.7, we showed that Γ-$\mathbf{CoMod}^A$ is an additive category, so it remains to show that it has all kernels and cokernels, and that for all morphisms f in Γ-$\mathbf{CoMod}^A$ that the comparison morphism

$$\operatorname{coker}(\ker f) \to \ker(\operatorname{coker} f)$$

is an isomorphism. First, let $f : (N_1, \Psi_{N_1}) \to (N_2, \Psi_{N_2})$ be a morphism in Γ-$\mathbf{CoMod}^A$, and consider the following diagram:

$$\begin{tikzcd}
\ker f \arrow[r, hook] \arrow[d, dashed] & N_1 \arrow[r, "f"] \arrow[d, "\Psi_{N_1}"'] & N_2 \arrow[d, "\Psi_{N_2}"] \\
\Gamma \otimes_B \ker f \arrow[r, hook] & \Gamma \otimes_B N_1 \arrow[r, "\Gamma\otimes f"'] & \Gamma \otimes_B N_2
\end{tikzcd}$$

By the assumption that Γ is flat as a right B-module, we have that $\Gamma \otimes_B -$ is exact, so that in particular it preserves kernels, meaning $\Gamma \otimes_B \ker f = \ker(\Gamma \otimes_B f)$. This gives the bottom left horizontal arrow. Then by the universal property of the kernel in B-$\mathbf{Mod}^A$ and the fact that the right square commutes, we get the vertical dashed arrow which makes the left square commute, as desired, and that $\ker f$ with this structure map is indeed the kernel of f in Γ-$\mathbf{CoMod}$. Showing that this structure map makes the two diagrams in Definition 0.6 commute is an exercise in diagram chasing and applying universal properties. Now, showing that the cokernel of f belongs to Γ-$\mathbf{CoMod}^A$ is formally dual. Finally, it follows from construction that the comparison morphism

$$\operatorname{coker}(\ker f) \to \ker(\operatorname{coker} f)$$

formed in Γ-$\mathbf{CoMod}^A$ is precisely the comparison morphism in B-$\mathbf{Mod}$, which is an isomorphism, and thus clearly an isomorphism in Γ-$\mathbf{CoMod}^A$ as well. Thus Γ-$\mathbf{CoMod}^A$ is indeed abelian, as desired. $\square$

Proposition 0.10 ([1, Lemma 3.5])**.** *Suppose that Γ is flat as a right B-module, i.e., suppose $\eta_R : B \to \Gamma$ is a flat ring homomorphism. Let P be an A-graded left Γ-comodule in Γ-$\mathbf{CoMod}^A$ such that the underlying A-graded B-module is a graded projective module. Then every co-free module (Proposition 0.8) is an F-acyclic object (**??**) for the covariant hom functor* $\mathrm{Hom}_\Gamma(P, -)$.

Proof. We need to show that $\mathrm{Ext}^n_\Gamma(N, \Gamma \otimes_B M)$ vanishes for all A-graded B-modules M. First of all, let $i : M \to I^*$ be an injective resolution of M in B-$\mathbf{Mod}^A$, so we have an exact sequence of A-graded B-modules

$$0 \longrightarrow M \xrightarrow{i} I^0 \xrightarrow{d^0} I^1 \xrightarrow{d^1} I^2 \xrightarrow{d^2} I^3 \longrightarrow \cdots .$$

Then Γ is flat as a right B-module, the sequence remains exact after we tensor it with Γ on the left. Furthermore, it is a general categorical fact that right adjoints between abelian catgories preserve injective objects. Thus $\Gamma \otimes i : \Gamma \otimes_B M \to \Gamma \otimes_B I^*$ is an injective resolution in Γ-$\mathbf{CoMod}^A$. Then for $n > 0$, we have

$$\mathrm{Ext}^n_\Gamma(N, \Gamma \otimes_B M) \cong H^n(\mathrm{Hom}_\Gamma(N, \Gamma \otimes_B I^*)) \cong H^n(\mathrm{Hom}_B(N, I^*)) \cong 0,$$

where the first isomorphism follows by the forgetful-cofree adjunction for comodules over a Hopf algebroid (Proposition 0.8), and the final isomorphism follows by the fact that N is a graded projective module, i.e., a projective object in the abelian category B-$\mathbf{Mod}^A$, so that $\mathrm{Hom}_B(N, -)$ is an exact functor. $\square$