Finally, we may construct the spectral sequence. Henceforth, we will assume the reader is familiar with the theory of spectral sequenes arising from unrolled exact couples, along with the notion of (conditional, strong) convergence of such spectral sequences to their (co)limits. The primary reference for these facts will be Boardman's paper [1] on conditionally convergent spectral sequences. When using any results from this reference, we will be sure to provide a proper citation. Note that Boardman works with $\mathbb{Z}$-graded groups, although everything he does carries through entirely the same with A-graded groups.

From now on, let (E, μ, e) be a monoid object and X and Y be objects in $\mathcal{SH}$.

0.1. **Construction of the spectral sequence.**

Definition 0.1. Let $\overline{E}$ be the fiber of the unit map $e : S \to E$ (**??**). Let $Y_0 := Y$ and $W_0 := E \otimes Y$. For $s > 0$, define

$$Y_s := \overline{E}^s \otimes Y, \qquad W_s := E \otimes Y_s = E \otimes \overline{E}^s \otimes Y,$$

where $\overline{E}^s$ denotes the s-fold tensor product $\overline{E} \otimes \cdots \otimes \overline{E}$. Then we get fiber sequences

$$Y_{s+1} \xrightarrow{i_s} Y_s \xrightarrow{j_s} W_s \xrightarrow{k_s} \Sigma Y_{s+1}$$

obtained by applying $- \otimes Y_s$ to the fiber sequence

$$\overline{E} \to S \xrightarrow{e} E \to \Sigma \overline{E}.$$

We can splice these sequences together to get the following diagram, which is called *the canonical Adams-resolution of* Y:

$$\begin{tikzcd}
\cdots \arrow[r] & Y_3 \arrow[r, "i_2"] \arrow[d, "j_3"] & Y_2 \arrow[r, "i_1"] \arrow[d, "j_2"] & Y_1 \arrow[r, "i_0"] \arrow[d, "j_1"] & Y_0 = Y \arrow[d, "j_0"] \\
& W_3 & W_2 \arrow[ul, dashed, "k_2"] & W_1 \arrow[ul, dashed, "k_1"] & W_0 \arrow[ul, dashed, "k_0"]
\end{tikzcd}$$

Here we are using dashed arrows to denote the (degree $-\mathbf{1}$) maps $k_s : W_s \to \Sigma Y_{s+1}$. In particular, the above diagram does not commute in any sense.

Now, by applying $[X, -]_*$ to the canonical E-Adams resolution of Y, we get an associated unrolled exact couple, and thus a spectral sequence:

Definition 0.2. Consider the canonical E-Adams resolution of Y from Lemma 0.3:

$$\begin{tikzcd}
\cdots \arrow[r] & Y_3 \arrow[r, "i_2"] \arrow[d, "j_3"] & Y_2 \arrow[r, "i_1"] \arrow[d, "j_2"] & Y_1 \arrow[r, "i_0"] \arrow[d, "j_1"] & Y_0 = Y \arrow[d, "j_0"] \\
& W_3 & W_2 \arrow[ul, dashed, "k_2"] & W_1 \arrow[ul, dashed, "k_1"] & W_0 \arrow[ul, dashed, "k_0"]
\end{tikzcd}$$

We can extend this diagram to the right by setting $Y_s = Y$, $W_s = 0$, and $i_s = \mathrm{id}_Y$ for $s < 0$. Then we may apply the functor $[X, -]_*$, and by **??**, we obtain the following A-graded unrolled exact couple:

$$\begin{tikzcd}
\cdots \arrow[r] & {[X, Y_{s+2}]_*} \arrow[r, "i_{s+1}"] \arrow[d, "j_{s+2}"] & {[X, Y_{s+1}]_*} \arrow[r, "i_s"] \arrow[d, "j_{s+1}"] & {[X, Y_s]_*} \arrow[r, "i_{s-1}"] \arrow[d, "j_s"] & {[X, Y_{s-1}]_*} \arrow[r] \arrow[d, "j_{s-1}"] & \cdots \\
& {[X, W_{s+2}]_*} & {[X, W_{s+1}]_*} \arrow[ul, "\partial_{s+1}"] & {[X, W_s]_*} \arrow[ul, "\partial_s"] & {[X, W_{s-1}]_*} \arrow[ul, "\partial_{s-1}"] &
\end{tikzcd}$$

where here we are being abusive and writing $i_s : [X, Y_{s+1}]_* \to [X, Y_s]_*$ and $j_s : [X, Y_s]_* \to [X, W_s]_*$ to denote the pushforward maps induced by $i_s : Y_{s+1} \to Y_s$ and $j_s : Y_s \to W_s$, respectively. Each i_s, j_s, and ∂_s are A-graded homomorphisms of degrees 0, 0, and $-\mathbf{1}$, respectively.

In [1, §0], it is described how we may associate a $\mathbb{Z} \times A$-graded spectral sequence $r \mapsto (E_r^{*,*}(X,Y), d_r)$ to the above A-graded unrolled exact couple, where d_r has $\mathbb{Z} \times A$-degree $(r, -\mathbf{1})$. We call this spectral sequence the *E-Adams spectral sequence for the computation of* $[X,Y]_*$.

For those who would rather not lose themselves in Boardman's document, we give a brief unravelling of how it applies to the present situation. Given some $s \in \mathbb{Z}$ and some $r \geq 1$, we may define the following A-graded subgroups of $[X, W_s]_*$:

$$Z_r^s := \partial_s^{-1}(\operatorname{im}[i^{(r-1)} : [X, Y_{s+r}]_* \to [X, Y_{s+1}]_*])$$

and

$$B_r^s := j_s(\ker[i^{(r-1)} : [X, Y_s]_* \to [X, Y_{s-r+1}]_*]),$$

where we adopt the convention that $i^{(0)}$ is simply the identity. This yields an infinite sequence of inclusions

$$0 = B_1^s \subseteq B_2^s \subseteq B_3^s \subseteq \cdots \subseteq \operatorname{im} j_s = \ker \partial_s \subseteq \cdots \subseteq Z_3^s \subseteq Z_2^s \subseteq Z_1^s = [X, W_s]_*.$$

Then for $r \geq 1$, we define E_r^s to be the A-graded quotient group

$$E_r^s := Z_r^s / B_r^s.$$

Thus taking the direct sum of all the E_r^s's yields the r^{th} page of the spectral sequence

$$E_r := \bigoplus_{s \in \mathbb{Z}} E_r^s,$$

which is a $\mathbb{Z} \times A$-graded abelian group.

The differential $d_r : E_r \to E_r$ is a map of $\mathbb{Z} \times A$-degree $(r, \mathbf{1})$, and is constructed as follows: an element of $E_r^s = Z_r^s / B_r^s$ is a coset represented by some $x \in Z_r^s$, so that $\partial_s(x) = i^{(r-1)}(y)$ for some $y \in [X, Y_{s+r}]_*$. Then we define $d_r([x])$ to be the coset $[j_{s+r}(y)]$ in Z_r^{s+r}/B_r^{s+r}.

In the case $r = 1$, since $B_1^s = 0$ and $Z_1^s = [X, W_s]_*$, we have that $E_1^s = [X, W_s]_*$, and given some $x \in E_1^s = [X, W_s]_*$, the differential d_1 is given by $d_1(x) = j_{s+1}(\partial_s(x))$, so that $d_1 = j \circ \partial$. Furthermore, since the unrolled exact couple which yields the spectral sequence vanishes on its negative terms, we have that $E_r^{s,a}(X,Y) = 0$ for $s < 0$. In particular, the E-Adams spectral sequence is a *half-plane spectral sequence with entering differentials*, in the sense of [1, §7].

Showing in explicit detail that all of these definitions make sense and are well-defined is relatively straightforward. Furthermore, one may check that that $d_r \circ d_r = 0$, and that

$$\ker d_r^s / \operatorname{im} d_r^s = \frac{Z_{r+1}^s / B_r^s}{B_{r+1}^s / B_r^s} \cong Z_{r+1}^s / B_{r+1}^s = E_{r+1}^s.$$

Above we constructed the spectral sequence by means of the "canonical" E-Adams resolution of Y, but one may more generally pursue the notion of *E-Adams resolutions* of the object Y, for which the canonical Adams resolution constructed above will be an example. We do not explore this generality here (although one certainly could); these are useful when one wants to construct an Adams resolution from an algebraic resolution of $E_*(Y)$, or by modifying an Adams resolution for some other object. One may find different notions of what exactly constitutes an Adams resolution in the literature (for example, see [3, Definition 2.2.1] or [4, Definition 11.3.1]), and they will always be defined so that the E-Adams spectral sequence for $[X,Y]_*$ is independent of the choice of Adams resolution for Y, at least from its E_2 page onwards. One important condition (or definitional consequence) one will always find for an E-Adams resolution is that the i's must vanish in E-homology. We can show that the canonical E-Adams resolution we have constructed satisfies this property:

Lemma 0.3. *Let i_s and j_s be as in Definition 0.1. Then the maps $j_s : Y_s \to W_s$ induce split monomorphisms $E_*(j_s)$ on E-homology, so that in particular the maps $i_s : Y_{s+1} \to Y_s$ vanish in E-homology, i.e., $E_*(i_s)$ is the zero map.*

Proof. First, note that since

$$Y_{s+1} \xrightarrow{i_s} Y_s \xrightarrow{j_s} W_s \xrightarrow{k_s} \Sigma Y_{s+1}$$

is a distinguished triangle and $\mathcal{SH}$ is tensor triangulated, there is a distinguished triangle of the form

$$E \otimes Y_{s+1} \xrightarrow{E\otimes i_s} E \otimes Y_s \xrightarrow{E\otimes j_s} E \otimes W_s \to \Sigma(E \otimes Y_{s+1}).$$

Thus, applying $\pi_*(-) \cong [S, -]_*$ to the triangle yields that the following sequence is exact (see **??** for details):

$$E_*(Y_{s+1}) \xrightarrow{E_*(i_s)} E_*(Y_s) \xrightarrow{E_*(j_s)} E_*(W_s).$$

Now, it is straightforward to verify by construction that j_s is the map $e \otimes Y_s : Y_s \to E \otimes Y_s = W_s$. Thus, by unitality of μ, we have that $E \otimes j_s : E \otimes Y_s \to E \otimes W_s$ is a split monomorphism, with right inverse $\mu \otimes Y_s : E \otimes W_s = E \otimes E \otimes Y_s \to E \otimes Y_s$. Then since any functor preserves split monomorphisms, it follows that $E_*(j_s) = \pi_*(E \otimes j_s)$ is likewise a split monomorphism, so that in particular $E_*(j_s)$ is injective. Thus $\operatorname{im} E_*(i_s) = \ker E_*(j_s) = 0$, so that i_s is indeed the zero map, as desired. $\square$

0.2. **The E_2 page.** Now, we would like to characterize the E_2 page of the spectral sequence in terms of something more concrete. Namely, we will characterize the E_2 page in terms of Ext of comodules over the dual E-Steenrod algebra. For a quick review of Ext in an abelian category and derived functors, see **??**. The goal of this subsection will be to prove the following theorem:

Theorem 0.4. *Let (E, μ, e) be a commutative monoid object, and X and Y objects in $\mathcal{SH}$. Suppose further that:*

- *E is flat (**??**) and cellular (**??**),*
- *X is cellular and $E_*(X)$ is a graded projective left $\pi_*(E)$-module (via **??**),*
- *Y is cellular.*

Then the non-vanishing entries of the second page of the E-Adams spectral sequence for the computation of $[X, Y]_*$ (Definition 0.2) are the Ext *groups of A-graded left comodules over the anticommutative Hopf algebroid structure on the dual E-Steenrod algebra (**??**), i.e., we have the following isomorphisms for all $s \geq 0$ and $a \in A$:*

$$E_2^{s,a}(X, Y) \cong \operatorname{Ext}^{s,a+\mathbf{s}}_{E_*(E)}(E_*(X), E_*(Y)) := \operatorname{Ext}^{s}_{E_*(E)}(E_*(X), E_{*+a+\mathbf{s}}(Y)).$$

Proof. By Proposition 0.8 below, for each $s \geq 0$ and $a \in A$, $E_2^{s,a}(X, Y)$ is isomorphic to the s^{th} cohomology group of the cochain complex obtained by applying $F := \operatorname{Hom}^{a+\mathbf{s}}_{E_*(E)}(E_*(X), -)$ to the complex

$$0 \longrightarrow E_*(W_0) \xrightarrow{E_*(\delta_0)} E_*(\Sigma W_1) \xrightarrow{E_*(\delta_1)} E_*(\Sigma^2 W_2) \xrightarrow{E_*(\delta_2)} E_*(\Sigma^3 W_3) \longrightarrow \cdots.$$

Furthermore, by Lemma 0.7, this complex is an F-acyclic resolution of $E_*(Y)$ (**??**). Thus, since the category of $E_*(E)$-comodules is an abelian category with enough injectives (**??**), we have by **??** that

$$E_2^{s,a}(X, Y) \cong R^s\operatorname{Hom}^{a+\mathbf{s}}_{E_*(E)}(E_*(X), -)(E_*(Y)) = \operatorname{Ext}^{s,a+\mathbf{s}}(E_*(X), E_*(Y)),$$

as desired. $\square$

As a result of this theorem, the spectral sequence is often shifted, by re-defining

$$E_r^{s,a}(X, Y)^{\text{new}} := E_r^{s,a+\mathbf{s}}(X, Y),$$

in which case the given isomorphism characterizing the E_2 page is strictly degree-preserving. This is in fact the standard convention for the classical stable homotopy category. We leave it to the reader to unravel what the differential d_2 corresponds to under this identification. The remainder of this subsection is devoted to proving Lemma 0.7 and Proposition 0.8. To start, we establish the following convention:

Definition 0.5. Given some (nonnegative integer) $n \geq 0$, define natural isomorphisms $\nu_X^n : \Sigma^{\mathbf{n}} X \to \Sigma^n X$ inductively, by setting $\nu_X^0 := \lambda_X$, $\nu_X^1 := \nu_X^{-1}$, and supposing ν_X^{n-1} has been defined for some $n > 1$, define ν_X^n to be the composition

$$\nu_X^n : \Sigma^{\mathbf{n}} X = S^{\mathbf{n}} \otimes X \xrightarrow{\phi_{\mathbf{n-1,1}} \otimes X} S^{\mathbf{n-1}} \otimes S^{\mathbf{1}} \otimes X \xrightarrow{S^{\mathbf{n-1}} \otimes \nu_X^{-1}} S^{\mathbf{n-1}} \Sigma X \xrightarrow{\nu_{\Sigma X}^{\mathbf{n-1}}} \Sigma^n X.$$

By induction, naturality of ν, and functoriality of $- \otimes -$, these isomorphisms are clearly natural in X.

Lemma 0.6. *Suppose E and Y are cellular. Then for all $s \in \mathbb{Z}$, the objects Y_s and W_s from the canonical E-Adams resolution of Y (Definition 0.1) are cellular.*

Proof. Unravelling definitions, for $s < 0$, $W_s = 0$ and $Y_s = Y$, which are both cellular.[1] For $s \geq 0$, we have $W_s = E \otimes Y_s$, so that by cellularity of E and **??**, it suffices to show that Y_s is cellular for $s \geq 0$. We know $Y_0 = Y$ is cellular by definition. For $s > 0$, Y_s is the tensor product $\overline{E}^s \otimes Y$, where $\overline{E}$ fits into the distinguished triangle

$$\overline{E} \to S \xrightarrow{e} E \to \Sigma \overline{E}.$$

By the definition of cellularity, $\overline{E}$ is cellular since S and E are. Thus $\overline{E}^s \otimes Y$ is cellular by **??**, as it is a tensor product of cellular objects in $\mathcal{SH}$. □

Lemma 0.7. *Let (E, μ, e) be a flat (**??**) and cellular (**??**) commutative monoid object and X and Y cellular objects in $\mathcal{SH}$, and for $s \geq 0$ define Y_s and W_s as in Definition 0.1. In particular, for each $s \geq 0$, $W_s = E \otimes Y_s$ and we have distinguished triangles*

$$Y_{s+1} \xrightarrow{i_s} Y_s \xrightarrow{j_s} W_s \xrightarrow{k_s} \Sigma Y_{s+1}.$$

*Then if $E_*(X)$ is a graded projective (**??**) left $\pi_*(E)$-module (via **??**) then the sequence*

$$0 \to E_*(Y) \xrightarrow{E_*(j_0)} E_*(W_0) \xrightarrow{E_*(\delta_0)} E_*(\Sigma W_1) \xrightarrow{E_*(\delta_1)} E_*(\Sigma^2 W_2) \xrightarrow{E_*(\delta_2)} E_*(\Sigma^3 W_3) \to \cdots$$

*is an F-acyclic resolution (**??**) of $E_*(Y)$ in $E_*(E)$-$\mathbf{CoMod}^A$ for*

$$F = \mathrm{Hom}^a_{E_*(E)}(E_*(X), -)$$

for all $a \in A$, where δ_s is the composition

$$\Sigma^s W_s \xrightarrow{\Sigma^s k_s} \Sigma^{s+1} Y_{s+1} \xrightarrow{\Sigma^{s+1} j_{s+1}} \Sigma^{s+1} W_{s+1}.$$

Proof. By Lemma 0.6, each W_s is cellular, so that furthermore $\Sigma^s W_s \cong S^{\mathbf{s}} \otimes W_s$ is cellular for each $s \geq 0$, by **??**. Thus, the sequence does indeed live in $E_*(E)$-$\mathbf{CoMod}^A$ by **??**, as desired. Next, we claim that $E_*(\Sigma^s W_s)$ is an F-acyclic object for each $s \geq 0$, i.e., that

$$\mathrm{Ext}^{n,a}_{E_*(E)}(E_*(X), E_*(\Sigma^s W_s)) = \mathrm{Ext}^n_{E_*(E)}(E_*(X), E_{*+a}(\Sigma^s W_s)) = 0$$

[1] 0 is cellular because it is the cofiber of the identity on S by axiom TR1 for a triangulated category (**??**), i.e., there is a distinguished triangle $S \to S \to 0 \to \Sigma S$.

for all $n > 0$, $s \geq 0$, and $a \in A$. Note that we have an A-graded isomorphism of left $E_*(E)$-comodules:

$$\begin{tikzcd}
E_*(E) \otimes_{\pi_*(E)} E_{*+a}(\Sigma^s Y_s) \arrow[r, equal] & E_*(E) \otimes_{\pi_*(E)} E_{*+a}(\Sigma^s Y_s) \arrow[d, "\Phi_{E,\Sigma^s Y_s}"] & \\
& E_*(E \otimes \Sigma^s Y_s) \arrow[d, "E_*(E \otimes (\nu^s_{Y_s})^{-1})"] & \\
& E_*(E \otimes S^{\mathbf{s}} \otimes Y_s) \arrow[d, "E_*(\tau \otimes Y_s)"] & \\
& E_*(S^{\mathbf{s}} \otimes E \otimes Y_s) \arrow[d, "E_*(\nu^s_{E \otimes Y_s})"] & \\
& E_*(\Sigma^s(E \otimes Y_s)) \arrow[r, equal] & E_*(\Sigma^s W_s)
\end{tikzcd}$$

where $\Phi_{E,\Sigma^s Y}$ is an A-graded isomorphism of abelian groups by **??**, and furthermore an isomorphism of $E_*(E)$-comodules by **??**. Every other arrow is an isomorphism of $E_*(E)$-comodules by functoriality of $E_*(-) : \mathcal{SH}$-**Cell** $\to E_*(E)$-$\mathbf{CoMod}^A$. Thus, since $E_*(\Sigma^s W_s)$ is isomorphic to $E_*(E) \otimes_{\pi_*(E)} E_{*+a}(\Sigma^s Y_s)$ in $E_*(E)$-$\mathbf{CoMod}^A$, and in particular since $\mathrm{Ext}^n_{E_*(E)}(E_*(X), -)$ is a functor, we have

$$\mathrm{Ext}^n_{E_*(E)}(E_*(X), E_{*+a}(\Sigma^s W_s)) \cong \mathrm{Ext}^n_{E_*(E)}(E_*(X), E_*(E) \otimes_{\pi_*(E)} E_{*+a}(\Sigma^s Y_s)).$$

Yet, $E_*(E) \otimes_{\pi_*(E)} E_{*+a}(\Sigma^s Y_s)$ is a co-free $E_*(E)$-comodule (**??**), in which case since $E_*(X)$ is graded projective as an object in $\pi_*(E)$-$\mathbf{Mod}^A$, we have that

$$\mathrm{Ext}^{n,a}_{E_*(E)}(E_*(X), E_*(E) \otimes_{\pi_*(E)} E_{*+a}(\Sigma^s Y_s)) = 0,$$

by **??**.

Finally, it remains to show that the sequence is exact. To that end, first note that by induction on axiom TR4 for a triangulated category and the fact that distinguished triangles are exact (**??**), the following sequence in $\mathcal{SH}$ is exact (since a sequence clearly remains exact even after changing the signs of its maps):

$$\Sigma^s Y_s \xrightarrow{\Sigma^s j_s} \Sigma^s W_s \xrightarrow{\Sigma^s k_s} \Sigma^{s+1} Y_{s+1} \xrightarrow{\Sigma^{s+1} i_s} \Sigma^{s+1} Y_s \xrightarrow{\Sigma^{s+1} j_s} \Sigma^{s+1} W_s$$

(see **??** for the definition of an exact triangle in an additive category). Furthermore, since $\mathcal{SH}$ is tensor triangulated, the sequence remains exact after applying $E \otimes -$ (see **??** for details), so that taking E-homology yields the following exact sequence of homology groups:

$$E_*(\Sigma^s Y_{s+1}) \xrightarrow{E_*(\Sigma^s i_s)} E_*(\Sigma^s Y_s) \xrightarrow{E_*(\Sigma^s j_s)} E_*(\Sigma^s W_s) \xrightarrow{E_*(\Sigma^s k_s)} E_*(\Sigma^{s+1} Y_{s+1}) \xrightarrow{E_*(\Sigma^{s+1} i_s)} E_*(\Sigma^{s+1} Y_s).$$

Then since $E_*(i_s) : E_*(Y_{s+1}) \to E_*(Y_s)$ is the zero map (by Lemma 0.3) and we have natural isomorphisms

$$E_*(\Sigma^t X) \xrightarrow{\nu^t_X} E_*(\Sigma^{\mathbf{t}} X) \xrightarrow{t^{\mathbf{t}}_X} E_{*-\mathbf{t}}(X)$$

(the first from Definition 0.5 and the latter from **??**), we have that $E_*(\Sigma^t i_s) : E_*(\Sigma^t Y_{s+1}) \to E_*(\Sigma^t Y_s)$ is the zero map for all $t \in \mathbb{Z}$, so that in particular the above exact sequence splits to yield the short exact sequence

$$0 \to E_*(\Sigma^s Y_s) \xrightarrow{E_*(\Sigma^s j_s)} E_*(\Sigma^s W_s) \xrightarrow{E_*(\Sigma^s k_s)} E_*(\Sigma^{s+1} Y_{s+1}) \to 0.$$

Then we may splice these sequences together for $s \geq 0$ to yield the following diagram:

$$\begin{tikzcd}[column sep=small]
0 \arrow[r] & E_*(Y) \arrow[r, "E_*(j_0)"] & E_*(W_0) \arrow[rr, "E_*(\delta_0)"] \arrow[dr, "E_*(k_0)"'] & & E_*(\Sigma W_1) \arrow[rr, "E_*(\delta_1)"] \arrow[dr, "E_*(\Sigma k_1)"'] & & E_*(\Sigma^2 W_2) \arrow[r] & \cdots \\
& & & E_*(\Sigma Y_1) \arrow[ur, "E_*(\Sigma j_1)"'] & & E_*(\Sigma^2 Y_2) \arrow[ur, "E_*(\Sigma^2 j_2)"'] & &
\end{tikzcd}$$

It is straightforward to check the top row is exact by exactness of the short exact sequences, as desired. $\square$

Proposition 0.8. *Let (E, μ, e) be a commutative monoid object, and X and Y objects in $\mathcal{SH}$. Suppose further that:*

- *E is flat (**??**) and cellular (**??**),*
- *X is cellular, and $E_*(X)$ is a graded projective left $\pi_*(E)$-module (via **??**), and*
- *Y is cellular.*

Then for all $s \in \mathbb{Z}$ and $a \in A$, the line in the first page of the E-Adams spectral sequence for the computation of $[X, Y]_*$ (Definition 0.2)

$$0 \to E_1^{0,a+\mathbf{s}}(X,Y) \xrightarrow{d_1} E_1^{1,a+\mathbf{s}-\mathbf{1}}(X,Y) \xrightarrow{d_1} E_1^{2,a+\mathbf{s}-\mathbf{2}}(X,Y) \to \cdots \to E_1^{s,a}(X,Y) \to \cdots$$

is isomorphic to the complex obtained by applying $\mathrm{Hom}^{a+\mathbf{s}}_{E_*(E)}(E_*(X), -)$ to the complex of A-graded left $E_*(E)$-comodules

$$0 \to E_*(W_0) \xrightarrow{E_*(\delta_0)} E_*(\Sigma W_1) \xrightarrow{E_*(\delta_1)} E_*(\Sigma^2 W_2) \to \cdots \to E_*(\Sigma^s W_s) \to \cdots$$

from Lemma 0.7.

Proof. By Lemma 0.6, since E and Y are cellular, W_t is as well for each $t \geq 0$. Furthermore, for $t > 0$, we have isomorphisms

$$S^{\mathbf{t}} \otimes W_t \xrightarrow{\nu^t_{W_t}} \Sigma^t W_t,$$

and by **??**, the object $S^{\mathbf{t}} \otimes W_t$ is cellular since $S^{\mathbf{t}}$ and W_t are. Hence, by **??**, the complex

$$0 \to E_*(W_0) \xrightarrow{E_*(\delta_0)} E_*(\Sigma W_1) \xrightarrow{E_*(\delta_1)} E_*(\Sigma^2 W_2) \to \cdots \to E_*(\Sigma^s W_s) \to \cdots$$

actually lives in $E_*(E)$-$\mathbf{CoMod}^A$, as desired. Now, let $t \geq 0$, and consider the following diagram:

$$\begin{tikzcd}
{[X, W_t]_{a+\mathbf{s}-\mathbf{t}}} \arrow[d, "(k_t)_*"'] & {[X, \Sigma^{\mathbf{t}} W_t]_{a+\mathbf{s}}} \arrow[l, "s^{\mathbf{t}}_{X,W_t}"'] \arrow[r, "(\nu^t_{W_t})_*"] \arrow[d, dashed, "(\Sigma^{\mathbf{t}} k_t)_*"] & {[X, \Sigma^t W_t]_{a+\mathbf{s}}} \arrow[dd, "(\Sigma^t k_t)_*"] \arrow[dddd, bend left, "(\delta_t)_*"] \\
{[X, \Sigma Y_{t+1}]_{a+\mathbf{s}-\mathbf{t}}} \arrow[d, "(\nu_{Y_{t+1}})_*"'] & {[X, \Sigma^{\mathbf{t}} \Sigma Y_{t+1}]_{a+\mathbf{s}}} \arrow[l, "s^{\mathbf{t}}_{X,\Sigma Y_{t+1}}"'] \arrow[d, dashed, "(\Sigma^{\mathbf{t}} \nu_{Y_{t+1}})_*"] \arrow[dr, "(\nu^t_{\Sigma Y_{t+1}})_*"] & \\
{[X, \Sigma^{\mathbf{1}} Y_{t+1}]_{a+\mathbf{s}-\mathbf{t}}} \arrow[d, "s^{\mathbf{1}}_{X,Y_{t+1}}"'] & {[X, \Sigma^{\mathbf{t}} \Sigma^{\mathbf{1}} Y_{t+1}]_{a+\mathbf{s}}} \arrow[l, "s^{\mathbf{t}}_{X,\Sigma^{\mathbf{1}} Y_{t+1}}"'] & {[X, \Sigma^{t+1} Y_{t+1}]_{a+\mathbf{s}}} \arrow[dd, "(\Sigma^{t+1} j_{t+1})_*"] \\
{[X, Y_{t+1}]_{a+\mathbf{s}-\mathbf{t}-\mathbf{1}}} \arrow[d, "(j_{t+1})_*"'] & {[X, \Sigma^{\mathbf{t}+\mathbf{1}} Y_{t+1}]_{a+\mathbf{s}}} \arrow[l, "s^{\mathbf{t}+\mathbf{1}}_{X,Y_{t+1}}"'] \arrow[u, dashed, "(\phi_{\mathbf{t},\mathbf{1}} \otimes Y_{t+1})_*"] \arrow[ur, "(\nu^{t+1}_{Y_{t+1}})_*"'] \arrow[d, dashed, "(\Sigma^{\mathbf{t}+\mathbf{1}} j_{t+1})_*"] & \\
{[X, W_{t+1}]_{a+\mathbf{s}-\mathbf{t}-\mathbf{1}}} & {[X, \Sigma^{\mathbf{t}+\mathbf{1}} W_{t+1}]_{a+\mathbf{s}}} \arrow[l, "s^{\mathbf{t}+\mathbf{1}}_{X,W_{t+1}}"'] \arrow[r, "(\nu^{t+1}_{W_{t+1}})_*"] & {[X, \Sigma^{t+1} W_{t+1}]_{a+\mathbf{s}}}
\end{tikzcd}$$

where here the $s^a_{X,Y} : [X, \Sigma^a Y]_* \cong [X, Y]_{*-a}$'s are the natural isomorphisms from **??**. By unravelling definitions, we have the top left object is $E_1^{t,a+\mathbf{s}-\mathbf{t}}(X,Y)$ and the bottom left object is $E_1^{t+1,a+\mathbf{s}-\mathbf{t}-\mathbf{1}}$, and the vertical left composition in the above diagram is the differential d_1 between them. The first, second, and fourth rectangles from the top on the left rectangle commute by naturality of the s^a's. Furthermore, a simple diagram chase and coherence of the ϕ's (**??**) yields that the third rectangle on the left commutes. The trapezoids on the right commute by naturality of ν^t and ν^{t+1}. Finally, the middle right triangle commutes by how we defined ν^{t+1} in terms of ν^t.

Now, consider the following diagram:

$$\begin{tikzcd}
E_1^{t,a+\mathbf{s}-\mathbf{t}}(X,Y) \arrow[r, "d_1"] \arrow[d, "(s^{\mathbf{t}}_{X,W_t})^{-1}"'] & E_1^{t+1,a+\mathbf{s}-\mathbf{t}-\mathbf{1}}(X,Y) \arrow[d, "(s^{\mathbf{t+1}}_{X,W_{t+1}})^{-1}"] \\
{[X,\Sigma^{\mathbf{t}}W_t]_{a+\mathbf{s}}} \arrow[d, "(\nu^t_{W_t})_*"'] & {[X,\Sigma^{\mathbf{t+1}}W_{t+1}]_{a+\mathbf{s}}} \arrow[d, "(\nu^{t+1}_{W_{t+1}})_*"] \\
{[X,\Sigma^{t}W_t]_{a+\mathbf{s}}} \arrow[r, "(\delta_t)_*"] \arrow[d, "E_*(-)"'] & {[X,\Sigma^{t+1}W_{t+1}]_{a+\mathbf{s}}} \arrow[d, "E_*(-)"] \\
\mathrm{Hom}_{E_*(E)}(E_*(\Sigma^{a+\mathbf{s}}X), E_*(\Sigma^t W_t)) \arrow[r, "E_*(\delta_t)"] \arrow[d, "((t_X^{a+\mathbf{s}})^{-1})^*"'] & \mathrm{Hom}_{E_*(E)}(E_*(\Sigma^{a+\mathbf{s}}X), E_*(\Sigma^{t+1} W_{t+1})) \arrow[d, "((t_X^{a+\mathbf{s}})^{-1})^*"] \\
\mathrm{Hom}^{a+\mathbf{s}}_{E_*(E)}(E_*(X), E_*(\Sigma^t W_t)) \arrow[r, "E_*(\delta_t)"] & \mathrm{Hom}^{a+\mathbf{s}}_{E_*(E)}(E_*(X), E_*(\Sigma^{t+1} W_{t+1}))
\end{tikzcd}$$

where here the maps $t_X^{a+\mathbf{s}} : E_*(\Sigma^a) \to E_{*-a}(X)$ are the $E_*(E)$-comodule isomorphisms from **??**. We have just shown the top region commutes. Furthermore, since X and $\Sigma^t W_t$ are cellular for all $t \geq 0$, the arrows labelled $E_*(-)$ are well-defined, and they clearly make the middle rectangle commute (a simple diagram chase suffices). The bottom rectangle also clearly commutes, Thus, it suffices to show that the maps labelled $E_*(-)$ are isomorphisms. To that end, consider the following diagram:

$$\begin{tikzcd}
{[X,\Sigma^t W_t]_{a+\mathbf{s}}} \arrow[r, "E_*(-)"] \arrow[d, "f_*"'] & \mathrm{Hom}_{E_*(E)}(E_*(\Sigma^{a+\mathbf{s}}X), E_*(\Sigma^t W_t)) \arrow[d, "E_*(f)_*"] \\
{[X, E\otimes\Sigma^{\mathbf{t}} Y_t]_{a+\mathbf{s}}} \arrow[r, "E_*(-)"] & \mathrm{Hom}_{E_*(E)}(E_*(\Sigma^{a+\mathbf{s}}X), E_*(E\otimes\Sigma^{\mathbf{t}} Y_t))
\end{tikzcd}$$

where here $f : \Sigma^t W_t \to E \otimes \Sigma^{\mathbf{t}} Y_t$ is the isomorphism

$$\Sigma^t W_t \xrightarrow{\nu^t_W} \Sigma^{\mathbf{t}} W_t = S^{\mathbf{t}} \otimes E \otimes Y_t \xrightarrow{\tau\otimes Y_t} E \otimes S^{\mathbf{t}} \otimes Y_t = E \otimes \Sigma^{\mathbf{t}} Y_t.$$

The bottom horizontal arrow is an isomorphism by **??**. Thus, the top horizontal arrow is an isomorphism, as desired. Showing

$$E_*(-) : [X, \Sigma^{t+1} W_{t+1}]_{a+\mathbf{s}} \to \mathrm{Hom}_{E_*(E)}(E_*(\Sigma^{a+\mathbf{s}}X), E_*(\Sigma^{t+1} W_{t+1}))$$

is an isomorphism is entirely analagous. Thus, for each $t \geq 0$, we have constructed isomorphisms

$$E^{t,a+\mathbf{s}-\mathbf{t}}(X,Y) \xrightarrow{\cong} \mathrm{Hom}^{a+\mathbf{s}}_{E_*(E)}(E_*(X), E_*(\Sigma^t W_t))$$

such that the following diagram commutes:

$$\begin{tikzcd}
E^{t,a+\mathbf{s}-\mathbf{t}}(X,Y) \arrow[r, "d_1"] \arrow[d, "\cong"'] & E^{t+1,a+\mathbf{s}-\mathbf{t}-\mathbf{1}}(X,Y) \arrow[d, "\cong"] \\
\mathrm{Hom}^{a+\mathbf{s}}_{E_*(E)}(E_*(X), E_*(\Sigma^t W_t)) \arrow[r, "\mathrm{Hom}^{a+\mathbf{s}}_{E_*(E)}(E_*(X),E_*(\delta_t))"] & \mathrm{Hom}^{a+\mathbf{s}}_{E_*(E)}(E_*(X), E_*(\Sigma^{t+1} W_{t+1}))
\end{tikzcd}$$

Hence, we have proven the desired result. □

0.3. **Convergence of the spectral sequence.** In this subsection, we briefly sketch some converge properties of the spectral sequence. Boardman already works quite generally in [1], so most of this is simply a review of the material contained within. From now on, we assume familiarity with derived limits of (A-graded) abelian groups (see Boardman §1), filtered (A-graded) groups (see Boardman §2), convergence of spectral sequences (Boardman Definition 5.2) and conditional convergence of a spectral sequence associated to an unrolled exact couple (Boardman Definition 5.10). We adopt his notation, writing

$$E^s_\infty(X,Y) := \left(\bigcap_{r=1}^{\infty} Z^s_r\right) / \left(\bigcup_{r=1}^{\infty} B^s_r\right) \qquad \text{and} \qquad RE_\infty(X,Y) := \operatorname*{Rlim}_r Z^s_r$$

to denote the E_∞-term and the derived E_∞-term of the spectral sequence, respectively.

Ideally, the E-Adams spectral sequence for $[X,Y]_*$ would give us information which allows us to compute the group $[X,Y]_*$. Note that $[X,Y]_*$ is the colimit of the unrolled exact couple which determines the spectral sequence, as $Y_s = Y$ for $s < 0$. Furthermore, since $(E_r(X,Y), d_r)$ is a half-plane spectral sequence with entering differentials, we may apply the results from [1, §7], where suitable conditions under which the spectral sequence converges to the colimit $[X,Y]_*$ are described (in particular, see Theorem 7.3 there). Unfortunately, in practice, the conditions outlined there are not usually satisfied for this spectral sequence, namely, in order for the spectral sequence to converge to $[X,Y]_*$, we must have that $\lim_s [X,Y_s]_* = 0$. There is no reason to believe this would be satisfied, so we must take an alternative approach. Following Section 5 of Bousfield's seminal paper [2], we can instead set up the spectral sequence by means of a tower *under* Y. First, we must define the *E-nilpotent completion of* Y:

Definition 0.9 ([2, pgs. 272–273])**.** Let (E,μ,e) be a monoid object in $\mathcal{SH}$, and Y any object. Write $\overline{E}$ for the fiber (**??**) of the unit $S \xrightarrow{e} E$, so we have a distinguished triangle

$$\overline{E} \to S \xrightarrow{e} E \to \Sigma\overline{E}.$$

Set $Y_0 := Y$ and $W_0 := Y \otimes E$, and for $s > 0$ define $Y_s := Y \otimes \overline{E}^s$ and $W_s := Y_s \otimes E$. Then since $\mathcal{SH}$ is tensor triangulated, for each $s \geq 0$ we may tensor the above sequence with Y_s on the right, which yields the following distinguished triangle

$$Y_{s+1} \xrightarrow{i} Y_s \xrightarrow{j} W_s \xrightarrow{k} \Sigma Y_{s+1}.$$

Then for $s \geq 0$, define Y/Y^s (up to non-canonical isomorphism) to be the cofiber of $i^s : Y_s \to Y_0 = Y$ (so in particular we may take $Y/Y_1 = E \otimes Y$ and $Y/Y_0 = 0$), so we have a distinguished triangle

$$Y_s \xrightarrow{i^s} Y \xrightarrow{b} Y/Y_s \xrightarrow{c} \Sigma Y_s.$$

Then for each $s \geq 0$, by the octahedral axiom (axiom TR5) for a triangulated category applied to the triangles

$$Y_{s+1} \xrightarrow{i} Y_s \xrightarrow{j} W_s \xrightarrow{k} \Sigma Y_{s+1}$$
$$Y_s \xrightarrow{i^s} Y \xrightarrow{b} Y/Y_s \xrightarrow{c} \Sigma Y_s$$
$$Y_{s+1} \xrightarrow{i^{s+1}} Y \xrightarrow{b} Y/Y_{s+1} \xrightarrow{c} \Sigma Y_{s+1},$$

there exists a distinguished triangle

$$W_s \xrightarrow{p} Y/Y_{s+1} \xrightarrow{q} Y/Y_s \xrightarrow{r} \Sigma W_s \tag{1}$$

which makes the following diagram commute:

$$\begin{tikzcd}
Y_{s+1} \arrow[rr, "i^{s+1}"] \arrow[dr, "i"'] & & Y \arrow[rr, "b"] \arrow[dr, "b"] & & Y/Y_s \arrow[rr, "r"] \arrow[dr, "c"] & & \Sigma W_s \\
& Y_s \arrow[ur, "i^s"'] \arrow[dr, "j"'] & & Y/Y_{s+1} \arrow[ur, "q"] \arrow[dr, "c"] & & \Sigma Y_s \arrow[ur, "\Sigma j"'] & \\
& & W_s \arrow[ur, "p"] \arrow[rr, "k"] & & \Sigma Y_{s+1} \arrow[ur, "\Sigma i"'] & &
\end{tikzcd} \tag{2}$$

The triangles from (1) for $s \geq 0$ may be spliced together to yield a tower $\{Y/Y_s\}_s$ under Y:

$$\begin{tikzcd}
Y \arrow[r] & \cdots \arrow[r] & Y/Y_3 \arrow[r, "q"] \arrow[d, dashed, "r"] & Y/Y_2 \arrow[r, "q"] \arrow[d, dashed, "r"] & Y/Y_1 \arrow[r, "q"] \arrow[d, dashed, "r"] & Y/Y_0 \arrow[r, equal] \arrow[d, dashed, "r"] & 0 \\
& & W_3 & W_2 \arrow[ul, "p"] & W_1 \arrow[ul, "p"] & W_0 \arrow[ul, "p"] &
\end{tikzcd}$$

where here the dashed arrows are really (degree $-\mathbf{1}$) maps $Y/Y_s \to \Sigma W_s$. The fact that this is a tower under Y follows from diagram (2), which tells us that $Y \xrightarrow{b} Y/Y_s$ factors as $Y \xrightarrow{b} Y/Y_{s+1} \xrightarrow{q} Y/Y_s$. We define the *$E$-nilpotent completion of* Y to be the object $Y_E^\wedge$ (defined up to non-canonical isomorphism) obtained as the homotopy limit of this tower (**??**):

$$Y_E^\wedge := \operatorname*{holim}_s Y_s/Y.$$

Since $Y_E^\wedge$ is the homotopy limit of a tower under Y, it comes equipped with a canonical map $Y \to Y_E^\wedge$.

Remark 0.10. In [2], the E-nilpotent completion of Y is denoted "$E^\wedge Y$", while the notation "$Y_E^\wedge$" we use here is standard in the modern literature.

It turns out that applying $[X,-]_*$ to this tower under Y yields an exact couple, the associated spectral sequence of which is precisely the E-Adams spectral sequence for $[X,Y]_*$.

Proposition 0.11. *Consider the tower under Y constructed in Definition 0.9:*

$$\begin{tikzcd}
Y \arrow[r] & \cdots \arrow[r] & Y/Y_3 \arrow[r, "q"] \arrow[d, dashed, "r"] & Y/Y_2 \arrow[r, "q"] \arrow[d, dashed, "r"] & Y/Y_1 \arrow[r, "q"] \arrow[d, dashed, "r"] & Y/Y_0 \arrow[r, equal] \arrow[d, dashed, "r"] & 0 \\
& & W_3 & W_2 \arrow[ul, "p"] & W_1 \arrow[ul, "p"] & W_0 \arrow[ul, "p"] &
\end{tikzcd}$$

*We may extend it to the right by defining $Y/Y_s = W_s = 0$ for $s < 0$. Then by **??**, we may apply the functor $[X,-]_*$ which yields the following A-graded unrolled exact couple:*

$$\begin{tikzcd}
\cdots \arrow[r] & {[X,Y/Y_{s+2}]_*} \arrow[r, "q"] \arrow[d, "\delta"] & {[X,Y/Y_{s+1}]_*} \arrow[r, "q"] \arrow[d, "\delta"] & {[X,Y/Y_s]_*} \arrow[r, "q"] \arrow[d, "\delta"] & {[X,Y/Y_{s-1}]_*} \arrow[r] \arrow[d, "\delta"] & \cdots \\
& {[X,W_{s+2}]_*} & {[X,W_{s+1}]_*} \arrow[ul, "p"] & {[X,W_s]_*} \arrow[ul, "p"] & {[X,W_{s-1}]_*} \arrow[ul, "p"] &
\end{tikzcd}$$

Thus by [1, §0], there is an induced spectral sequence. This spectral sequence is precisely the E-Adams spectral sequence for $[X,Y]_*$ (Definition 0.2).

Proof. Let $(E'_r(X,Y), d'_r)$ denote this new spectral sequence. For $s \geq 0$, define

$$f_s : [X,Y/Y_s]_* \xrightarrow{c_*} [X,\Sigma Y_s]_* \xrightarrow{(\nu_{Y_s})_*} [X,\Sigma^{\mathbf{1}} Y_s]_* \xrightarrow{s^{\mathbf{1}}_{X,Y_s}} [X,Y_s]_{*-\mathbf{1}},$$

and for $s < 0$ let it be the unique map

$$f_s : [X,Y/Y_s]_* = 0 \to [X,Y_s]_{*-\mathbf{1}} = [X,Y]_{*-1}.$$

For $s \in \mathbb{Z}$, let

$$g_s := \mathrm{id}_{W_s} : [X,W_s]_* \to [X,W_s]_*.$$

We claim these maps $(f_s, g_s)_s$ define a homomorphism of A-graded unrolled exact couples between the unrolled exact couple given above determined by the quotient tower $\{Y/Y_s\}$ under Y, and that obtained by applying $[X,-]_*$ to the canonical E-Adams resolution, i.e., that the following diagram commutes for all $s \in \mathbb{Z}$:

$$\begin{tikzcd}
{[X,Y/Y_s]_*} \arrow[r] \arrow[d, "f_s"'] & {[X,Y/Y_{s-1}]_*} \arrow[r] \arrow[d, dashed, "f_{s-1}"] & {[X,W_{s-1}]_{*-\mathbf{1}}} \arrow[r] \arrow[d, equal] & {[X,Y/Y_s]_{*-\mathbf{1}}} \arrow[d, "f_s"] \\
{[X,Y_s]_{*-\mathbf{1}}} \arrow[r] & {[X,Y_{s-1}]_{*-\mathbf{1}}} \arrow[r] & {[X,W_{s-1}]_{*-\mathbf{1}}} \arrow[r] & {[X,Y_s]_{*-\mathbf{2}}}
\end{tikzcd}$$

In the case $s \leq 0$, we know $Y/Y_s = Y/Y_{s-1} = W_{s-1} = 0$, so that the top row is entirely 0, and thus the diagram must commute. In the case $s > 0$, by unravelling definitions we have that the diagram becomes

$$\begin{tikzcd}
{[X,Y/Y_s]_*} \arrow[r, "q_*"] \arrow[d, "c_*"'] & {[X,Y/Y_{s-1}]_*} \arrow[rr, "\delta"] \arrow[d, "c_*"] \arrow[dr, "r_*"] & & {[X,W_{s-1}]_{*-\mathbf{1}}} \arrow[r, "p_*"] \arrow[ddd, equal] & {[X,Y/Y_s]_{*-\mathbf{1}}} \arrow[d, "c_*"] \\
{[X,\Sigma Y_s]_*} \arrow[r, "\Sigma i_*"] \arrow[d, "(\nu_{Y_s})_*"'] & {[X,\Sigma Y_{s-1}]_*} \arrow[r, "\Sigma j_*"] \arrow[d, "(\nu_{Y_{s-1}})_*"] & {[X,\Sigma W_{s-1}]_*} \arrow[d, "(\nu_{W_{s-1}})_*"] & & {[X,\Sigma Y_s]_{*-\mathbf{1}}} \arrow[d, "(\nu_{Y_s})_*"] \\
{[X,\Sigma^{\mathbf{1}} Y_s]_*} \arrow[r, "\Sigma^{\mathbf{1}} i_*"] \arrow[d, "s^{\mathbf{1}}_{X,Y_s}"'] & {[X,\Sigma^{\mathbf{1}} Y_{s-1}]_*} \arrow[r, "\Sigma^{\mathbf{1}} j_*"] \arrow[d, "s^{\mathbf{1}}_{X,Y_{s-1}}"] & {[X,\Sigma^{\mathbf{1}} W_{s-1}]_*} \arrow[dr, "s^{\mathbf{1}}_{X,W_{s-1}}"] & & {[X,\Sigma^{\mathbf{1}} Y_s]_{*-\mathbf{1}}} \arrow[d, "s^{\mathbf{1}}_{X,Y_s}"] \\
{[X,Y_s]_{*-\mathbf{1}}} \arrow[r, "i_*"] & {[X,Y_{s-1}]_{*-\mathbf{1}}} \arrow[rr, "j_*"] & & {[X,W_{s-1}]_{*-\mathbf{1}}} \arrow[r, "\partial_*"] \arrow[uur, "k_*"] & {[X,Y_s]_{*-\mathbf{2}}}
\end{tikzcd}$$

Clearly commutativity of this diagram yields that the given collection of maps define a homomorphism of A-graded unrolled exact couples. Each rectangular region commutes by naturality, as does the middle bottom trapezoidal region. The two regions involving δ and ∂ commute by unravelling how the differential is defined in **??**. Finally, the remaining two regions commute by commutativity of Equation 2.

Thus, we have defined a homomorphism of A-graded unrolled exact couples, and it is straightforward to check that therefore the maps g_s lift to well-defined graded homomorphisms $\widetilde{g}^s_r$:

$E_r^s(X,Y) \to E_r^{s\prime}(X,Y)$ for $s \geq 0$ sending a class $[x] \in Z_r^s/B_r^s = E_r^s(X,Y)$ to the class $\widetilde{g}_r^s([x]) := [g_s(x)]$ in $Z_r^{s\prime}/B_r^{s\prime} = E_r^{s\prime}(X,Y)$, which make the following diagrams commute for all $r \geq 1$:

$$\begin{tikzcd} E_r(X,Y) \arrow[r, "\widetilde{g}_r"] \arrow[d, "d_r"'] & E'_r(X,Y) \arrow[d, "d'_r"] \\ E_r(X,Y) \arrow[r, "\widetilde{g}_r"] & E'_r(X,Y) \end{tikzcd} \qquad \begin{tikzcd} \ker d_r \arrow[r, "\widetilde{g}_r"] \arrow[d, two heads] & \ker d'_r \arrow[d, two heads] \\ E_{r+1}(X,Y) \arrow[r, "\widetilde{g}_{r+1}"] & E'_{r+1}(X,Y) \end{tikzcd}$$

(commutativity of the first diagram implies the top arrow in the second diagram is well-defined). Yet we know that each g_s is the identity, so that we shown that $(E_r(X,Y), d_r) = (E'_r(X,Y), d'_r)$, as desired. ☐

By means of this new presentation of the spectral sequence, we may consider the sense in which the spectral sequence converges to the *limit* $\lim_s [X, Y/Y_s]_*$ of the tower $\{Y/Y_s\}_s$ under Y, by means [1, Theorem 7.4]. First of all, it is standard that since $Y_E^\wedge$ is the homotopy limit of this tower, we have a *Milnor short exact sequence*

$$0 \to \operatorname*{Rlim}_s [X, Y/Y_s]_{*+1} \to [X, Y_E^\wedge]_* \to \lim_s [X, Y/Y_s]_* \to 0$$

(the same argument given in [1, Theorem 4.9] works, although we warn the reader that Boardman has a sign error there — he writes the first term in the short exact sequence with -1, when it should be $+1$). Thus, if $\operatorname{Rlim}_s [X, Y/Y_s]_*$ vanishes, we get an identification of the limit

$$\lim_s [X, Y/Y_s]_* = [X, Y_E^\wedge]_*.$$

By [1, Theorem 7.4], this is further satisfied if the derived E_∞-term $RE_\infty(X,Y)$ is zero, in which case the spectral sequence converges strongly to the limit $[X, Y_E^\wedge]_*$, meaning in particular the natural maps

$$[X, Y_E^\wedge]_* \to \lim_s [X, Y_E^\wedge]_*/F^s[X, Y_E^\wedge]_*$$

$$F^s[X, Y_E^\wedge]_*/F^{s+1}[X, Y_E^\wedge]_* \to E_\infty^{s,*}(X,Y)$$

are isomorphisms, where here F^s is the decreasing filtration on $[X, Y_E^\wedge]_*$ given by

$$F^s[X, Y_E^\wedge]_* := \ker([X, Y_E^\wedge]_* = \lim_s [X, Y/Y_s]_* \to [X, Y/Y_s]_*).$$