

MODEL STRUCTURES

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1. PRELIMINARIES

Definition 1.1 (Hovey Definition 2.1.1). Suppose \mathcal{C} is a cocomplete category, and λ is an ordinal. A λ -sequence in \mathcal{C} is a colimit-preserving functor $X : \lambda \rightarrow \mathcal{C}$, commonly written as

$$X_0 \rightarrow X_1 \rightarrow \cdots \rightarrow X_\beta \rightarrow \cdots.$$

Since X preserves colimits, for all limit ordinals $\gamma < \lambda$, the induced map

$$\operatorname{colim}_{\beta < \gamma} X_\beta \rightarrow X_\gamma$$

is an isomorphism. We refer to the map $X_0 \rightarrow \operatorname{colim}_{\beta < \lambda} X_\beta$ as the *composition* of the λ -sequence. Given a collection \mathcal{D} of morphisms in \mathcal{C} such that every map $X_\beta \rightarrow X_{\beta+1}$ for $\beta + 1 < \lambda$ is in \mathcal{D} , we refer to the composition $X_0 \rightarrow \operatorname{colim}_{\beta < \lambda} X_\beta$ as a *transfinite composition* of maps in \mathcal{D} .

Definition 1.2 (Hovey Definition 2.1.2). Let γ be a cardinal. An ordinal α is γ -filtered if it is a limit ordinal and, if $A \subseteq \alpha$ and $|A| \leq \gamma$, then $\sup A < \alpha$.

Definition 1.3 (Hovey Definition 2.1.3). Suppose \mathcal{C} is a comcomplete category, $\mathcal{D} \subseteq \operatorname{Mor} \mathcal{C}$ is some collection of morphisms of \mathcal{C} , A is an object of \mathcal{C} , and κ is a cardinal. We say that A is κ -small relative to \mathcal{D} if, for all κ -filtered ordinals λ and all λ -sequences

$$X_0 \rightarrow X_1 \rightarrow \cdots \rightarrow X_\beta \rightarrow \cdots$$

such that each map $X_\beta \rightarrow X_{\beta+1}$ is in \mathcal{D} for $\beta + 1 < \lambda$, the map of sets

$$\operatorname{colim}_{\beta < \lambda} \mathcal{C}(A, X_\beta) \rightarrow \mathcal{C}(A, \operatorname{colim}_{\beta < \lambda} X_\beta)$$

is an isomorphism. We say that A is *small relative to \mathcal{D}* if it is κ -small relative to \mathcal{D} for some κ . We say that A is *small* if it is small relative to \mathcal{C} itself.

Recall that given a small category \mathcal{D} and a functor $F : \mathcal{D} \rightarrow \operatorname{Set}$, we may explicitly construct the colimit of F as the set

$$\operatorname{colim} F := \left(\prod_{d \in \mathcal{D}} F(d) \right) / \sim,$$

where the equivalence relation \sim is **generated** by

$$((x \in F(d)) \sim (x' \in F(d'))) \quad \text{if} \quad (\exists (f : d \rightarrow d') \text{ with } Ff(x) = x').$$

In particular, if \mathcal{D} is a filtered category then the resulting relation can be described as follows:

$$(1) \quad ((x \in F(d)) \sim (x' \in F(d'))) \quad \text{iff} \quad (\exists d'', (f : d \rightarrow d''), (g : d' \rightarrow d'') \text{ with } Ff(x) = Fg(x')).$$

given a cone $\eta : F \Rightarrow \underline{Y}$ under F , the unique map $\operatorname{colim} F \rightarrow Y$ maps the equivalence class of $x \in F(d)$ to the element $\eta_d(x) \in Y$. We will use this characterization of the colimit in the following example.

Example 1.4 (Hovey 2.1.5). Every set is small. Indeed, if A is a set we claim that A is $|A|$ -small. To see this, suppose λ is an $|A|$ -filtered ordinal, and X is a λ -sequence of sets. Given $\alpha < \beta < \lambda$, let $\iota_{\alpha,\beta} : X_\alpha \rightarrow X_\beta$ denote the induced morphism. We will write $X_\lambda := \text{colim}_{\beta < \lambda} X_\beta$, and let $\iota : X \Rightarrow X_\lambda$ be the colimit cone, so that given $\beta < \lambda$, $\iota_\beta : X_\beta \rightarrow X_\lambda$ is the leg of the colimit cone at X_β . By composing with the functor $\mathcal{C}(A, -) : \text{Set} \rightarrow \text{Set}$, we get another λ -sequence $\{\mathcal{C}(X_\beta, A)\}_{\beta < \lambda}$. The cone ι under X induces a cone ι_* under $\mathcal{C}(X_\beta, A)$ with nadir $\mathcal{C}(A, X_\lambda)$. Let $\eta : \mathcal{C}(X_\beta, A) \Rightarrow \text{colim}_{\beta < \lambda} \mathcal{C}(X_\beta, A)$ be the colimit cone, and let $\ell : \text{colim}_{\beta < \lambda} \mathcal{C}(A, X_\beta) \rightarrow \mathcal{C}(A, X_\lambda)$ be the unique morphism of cones so that the following diagram commutes

$$\begin{array}{ccccccc}
 \mathcal{C}(A, X_0) & \xrightarrow{(\iota_{0,1})_*} & \mathcal{C}(A, X_1) & \xrightarrow{(\iota_{1,2})_*} & \dots & \xrightarrow{(\iota_{\beta,\beta+1})_*} & \mathcal{C}(A, X_\beta) & \xrightarrow{(\iota_{\beta,\beta+1})_*} & \dots \\
 & \searrow \eta_0 & \searrow \eta_1 & & & & \searrow \eta_\beta & & \\
 & & & & & & \text{colim}_{\beta < \lambda} \mathcal{C}(A, X_\beta) & & \\
 & \searrow (\iota_0)_* & \searrow (\iota_1)_* & & & & \searrow (\iota_\beta)_* & & \\
 & & & & & & \downarrow \ell & & \\
 & & & & & & \mathcal{C}(A, X_\lambda) & &
 \end{array}$$

First, we wish to show that ℓ is surjective. Indeed, let $f : A \rightarrow X_\lambda$. For each $a \in A$, there exists some $\beta_a \in \lambda$ and some $a' \in X_{\beta_a}$ such that $f(a) = \eta_{\beta_a}(a')$ (see the preceding discussion). Then let $\gamma := \sup_{a \in A} \beta_a$. Since $|\{\beta_a\}_{a \in A}| \leq |A|$ and λ is $|A|$ -filtered, necessarily $\gamma < \lambda$. Now, define $g : A \rightarrow X_\gamma$ like so: for $a \in A$, define $g(a) := \iota_{\beta_a,\gamma}(a')$, where $a' \in X_{\beta_a}$ was chosen earlier so that $\iota_{\beta_a}(a') = f(a)$. Then we claim that $\ell(\eta_\gamma(g)) = f$. Indeed, as ℓ is a morphism of cocones, $\ell \circ \eta = \iota_*$, so that we have

$$\ell(\eta_\gamma(g)) = (\iota_\gamma)_*(g) = \iota_\gamma \circ g,$$

and given $a \in A$ we have

$$\iota_\gamma(g(a)) = \iota_\gamma(\iota_{\beta_a,\gamma}(a')).$$

By definition of a cone, $\iota_\gamma \circ \iota_{\beta_a,\gamma} = \iota_{\beta_a}$, so that

$$\ell(\eta_\gamma(g))(a) = \iota_\gamma(\iota_{\beta_a,\gamma}(a')) = \iota_{\beta_a}(a') = f(a),$$

so that indeed $\ell(\eta_\gamma(g)) = f$.

It remains to show ℓ is injective. Suppose we are given $[f], [g] \in \text{colim}_{\beta < \lambda} \mathcal{C}(A, X_\beta)$ such that $\ell([f]) = \ell([g])$. Then by the preceding discussion, there exists $\alpha, \beta < \lambda$, $f \in \mathcal{C}(A, X_\alpha)$, and $g \in \mathcal{C}(A, X_\beta)$ such that $\eta_\alpha(f) = [f]$ and $\eta_\beta(g) = [g]$. Then since $\ell \circ \eta = \iota_*$, we have

$$\ell([f]) = \ell([g]) \implies \iota_\alpha \circ f = (\iota_\alpha)_*(f) = \ell(\eta_\alpha(f)) = \ell(\eta_\beta(g)) = (\iota_\beta)_*(g) = \iota_\beta \circ g.$$

For each $a \in A$, since $\iota_\alpha(f(a)) = \iota_\beta(g(a))$, by [Equation 1](#) there exists γ_a with $\alpha, \beta \leq \gamma_a$ such that $\iota_{\alpha,\gamma_a}(f(a)) = \iota_{\beta,\gamma_a}(g(a))$. Then let $\gamma := \sup_{a \in A} \gamma_a$. Since $|\{\gamma_a\}_{a \in A}| \leq |A|$ and λ is $|A|$ -filtered, necessarily $\gamma < \lambda$. Now, in order to show $[f] = [g]$, by [Equation 1](#) it suffices to show that $(\iota_{\alpha,\gamma})_*(f) = (\iota_{\beta,\gamma})_*(g)$. Indeed, given $a \in A$, we have

$$(\iota_{\alpha,\gamma})_*(f)(a) = \iota_{\alpha,\gamma}(f(a)) = \iota_{\gamma_a,\gamma} \circ \iota_{\alpha,\gamma_a}(f(a)) = \iota_{\gamma_a,\gamma} \circ \iota_{\beta,\gamma_a}(g(a)) = \iota_{\beta,\gamma}(g(a)) = (\iota_{\beta,\gamma})_*(g)(a),$$

precisely the desired result.

Definition 1.5 (Hovey Definition 2.1.7). Let I be a class of maps in a category \mathcal{C} .

- (1) A map is *I-injective* if it has the right lifting property w.r.t. every map in I . The class of *I-injective* maps is denoted $I\text{-inj}$ (or I_\perp).
- (2) A map is *I-projective* if it has the left lifting property w.r.t. every map in I . The class of *I-projective* maps is denoted $I\text{-proj}$ (or ${}_\perp I$).
- (3) A map is an *I-cofibration* if it has the left lifting property w.r.t. every *I-injective* map. The class of *I-cofibrations* is the class $(I\text{-inj})\text{-proj}$ and is denoted $I\text{-cof}$ (or ${}_\perp(I_\perp)$).
- (4) A map is an *I-fibration* if it has the right lifting property w.r.t. every *I-projective* map. The class of *I-fibrations* is the class $(I\text{-proj})\text{-inj}$ and is denoted $I\text{-fib}$ (or $({}_\perp I)_\perp$).

Definition 1.6 (Hovey Definition 2.1.9). Let I be a set of maps in a cocomplete category \mathcal{C} . A *relative I -cell complex* is a transfinite composition of pushouts of elements of I . That is, if $f : A \rightarrow B$ is a relative I -cell complex, then there is an ordinal λ and a λ -sequence $X : \lambda \rightarrow \mathcal{C}$ such that f is the composition of X and such that, for each β such that $\beta + 1 < \lambda$, there is a pushout square

$$\begin{array}{ccc} C_\beta & \longrightarrow & X_\beta \\ g_\beta \downarrow & \lrcorner & \downarrow \\ D_\beta & \longrightarrow & X_{\beta+1} \end{array}$$

with $g_\beta \in I$. We denote the collection of relative I -cell complexes by $I\text{-cell}$. We say that $A \in \mathcal{C}$ is an *I -cell complex* if the map $0 \rightarrow A$ is a relative I -cell complex.

Lemma 1.7 (Hovey 2.1.10). *Suppose I is a class of maps in a category \mathcal{C} with all small colimits. Then $I\text{-cell} \subseteq I\text{-cof}$.*

Definition 1.8 (Hovey Definition 2.1.17). Suppose \mathcal{C} is a model category. We say that \mathcal{C} is *cofibrantly generated* if there are sets I and J of maps such that:

1. The domains of the maps of I are small relative to $I\text{-cell}$;
2. The domains of the maps of J are small relative to $J\text{-cell}$;
3. The class of fibrations is $J\text{-inj}$; and
4. The class of trivial fibrations is $I\text{-inj}$.

We refer to I as the set of *generating cofibrations* and to J as the set of *generating trivial cofibrations*. A cofibrantly generated model category is *finitely generated* if we can choose the sets I and J above so that the domains and codomains of I and J are finite relative to $I\text{-cell}$.

Proposition 1.9 (Hovey Proposition 2.1.18). *Suppose \mathcal{C} is a cofibrantly generated model category, with generating cofibrations I and generating trivial fibrations J .*

- (a) *The cofibrations form the class $I\text{-cof}$.*
- (b) *Every cofibration is a retract of a relative I -cell complex.*
- (c) *The domains of I are small relative to the cofibrations.*
- (d) *The trivial cofibrations form the class $J\text{-cof}$.*
- (e) *Every trivial cofibration is a retract of a relative J -cell complex.*
- (f) *The domains of J are small relative to the trivial cofibrations.*

If \mathcal{C} is fibrantly generated, then the domains and codomains of I and J are finite relative to the cofibrations.

Proof. **TODO.** □

Theorem 1.10 (Hovey Theorem 2.1.19). *Suppose \mathcal{C} is a complete & cocomplete category. Suppose \mathcal{W} is a subcategory of \mathcal{C} , and I and J are sets of maps of \mathcal{C} . Then there is a cofibrantly generated model structure on \mathcal{C} with I as the set of generating cofibrations, J as the set of generating trivial fibrations, and \mathcal{W} as the subcategory of weak equivalences if and only if the following conditions are satisfied.*

1. *The subcategory \mathcal{W} has the 2-of-3 property and is closed under retracts.*
2. *The domains of I are small relative to $I\text{-cell}$.*
3. *The domains of J are small relative to $J\text{-cell}$.*
4. *$J\text{-cell} \subseteq \mathcal{W} \cap I\text{-cof}$.*
5. *$I\text{-inj} \subseteq \mathcal{W} \cap J\text{-inj}$.*
6. *Either $\mathcal{W} \cap I\text{-cof} \subseteq J\text{-cof}$ or $\mathcal{W} \cap J\text{-inj} \subseteq I\text{-inj}$.*

Proof. **TODO.** □

Definition 1.11. Let \mathcal{C} be a category and I a collection of morphisms in \mathcal{C} . Then if I is closed under transfinite composition, pushouts, and retracts then we say I is *saturated*.

2. TOPOLOGICAL SPACES

An injective map $f : X \rightarrow Y$ in **Top** is an *inclusion* if U is open in X if and only if there is a V open in Y such that $f^{-1}(V) = U$. If f is a closed inclusion and every point in $Y \setminus f(X)$ is closed, then we call f a *closed T_1 inclusion*. We will let \mathcal{T} denote the class of closed T_1 inclusions in **Top**.

The symbol D^n will denote the unit disk in \mathbb{R}^n , and the symbol S^{n-1} will denote the unit sphere in \mathbb{R}^n , so that we have the boundary inclusions $S^{n-1} \hookrightarrow D^n$. In particular, for $n = 0$ we let $D^0 = \{0\}$ and $S^{-1} = \emptyset$.

Definition 2.1. A map $f : X \rightarrow Y$ in **Top** is called a *weak equivalence* if

$$\pi_n(f, x) : \pi_n(X, x) \rightarrow \pi_n(Y, f(x))$$

is an isomorphism for all $n \geq 0$ and for all $x \in X$. We will write \mathcal{W} to refer to the class of all weak equivalences in **Top**.

Define the set of maps I' to consist of all the boundary inclusion $S^{n-1} \hookrightarrow D^n$ for all $n \geq 0$, and define the set J to consist of all the inclusions $D^n \hookrightarrow D^n \times I$ mapping $x \mapsto (x, 0)$ for $n \geq 0$. Then a map f will be called a *cofibration* if it is in $I'\text{-cof} = {}_\perp(I'_\perp)$, and a *fibration* if it is in $J\text{-inj} = J_\perp$.

A map in I' -cell is usually called a *relative cell complex*; a relative CW-complex is a special case of a relative cell complex, where, in particular, the cells can be attached in order of their dimension. Note that in particular maps of J are relative CW complexes, hence are relative I' -cell complexes. A fibration is often known as a *Serre fibration* in the literature.

Theorem 2.2 (Hovey Theorem 2.4.19). *There is a finitely generated model structure on **Top** with I' as the set of generating cofibrations, J as the set of generating trivial cofibrations, and the cofibrations, fibrations, and weak equivalences as above. Every object of **Top** is fibrant, and the cofibrant objects are retracts of relative cell complexes.*

Proof. We will apply [Theorem 1.10](#) to get that there is a cofibrantly generated model structure on **Top** with I' as the set of generating cofibrations, J as the set of generating trivial fibrations, and \mathcal{W} as the subcategory of weak equivalences. The six requirements outlined in the theorem will be verified like so:

1. \mathcal{W} is a subcategory of \mathcal{C} which has the 2-of-3 property and is closed under retracts: [Lemma 2.6](#).
2. The domains of I' are small relative to I' -cell: In [Lemma 2.3](#), we will show that every space is small relative to the inclusions, and in particular every space is small relative to the class \mathcal{T} of closed T_1 inclusions. Hence, it will suffice to show that $I'\text{-cell} \subseteq \mathcal{T}$. In [Proposition 2.5](#), we will show that \mathcal{T} is saturated, and clearly every map in I' is a closed T_1 inclusion, so the desired result follows.
3. The domains of J are small relative to J -cell: By the same argument given above, this will follow by [Lemma 2.3](#), [Proposition 2.5](#), and the fact that $J \subseteq \mathcal{T}$.
4. $J\text{-cell} \subseteq \mathcal{W} \cap I'\text{-cof}$: In [Proposition 2.7](#), we will show $J\text{-cof} \subseteq \mathcal{W} \cap I'\text{-cof}$, and by [Lemma 1.7](#) $J\text{-cell} \subseteq J\text{-cof}$.
5. $I'\text{-inj} \subseteq \mathcal{W} \cap J\text{-inj}$: [Proposition 2.8](#)
6. $\mathcal{W} \cap J\text{-inj} \subseteq I'\text{-inj}$: [Proposition 2.9](#)

It will follow by the definition of a cofibrantly generated model structure ([Definition 1.8](#)) that the fibrations in this model structure are given by $J\text{-inj}$, which is precisely how we defined it. By [Proposition 1.9](#), the class of cofibrations will be given by $I'\text{-cof}$, which is likewise exactly how we defined them.

In [Proposition 2.4](#), we will show that compact spaces are finite relative to the class \mathcal{T} of closed T_1 inclusions. Hence, this model structure will be finitely generated, as the domains and codomains of I' and J are all compact, and by the reasoning given above we will have shown $I'\text{-cell} \subseteq \mathcal{T}$.

We will show that every object of **Top** is fibrant in [Corollary 2.10](#). Finally, to see that cofibrant objects are retracts of relative cell complexes, FINISH □

Lemma 2.3 (Hovey 2.4.1). *Every topological space is small relative to the inclusions.*

Proof. TODO. □

Proposition 2.4 (Hovey 2.4.2). *Compact topological spaces are finite relative to the class \mathcal{T} of closed T_1 inclusions.*

Proof. TODO. □

Proposition 2.5 (Hovey 2.4.5 & 2.4.6). *The class of closed T_1 inclusions is saturated.*

Proof. TODO. □

Lemma 2.6 (Hovey Lemma 2.4.4). *The weak equivalences in **Top** are closed under retracts and satisfy 2-of-3 axiom (so that in particular the weak equivalences form a subcategory, as clearly identities are weak equivalences).*

Proof. First we show that weak equivalences satisfy 2-of-3. Let $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ be continuous functions of topological spaces.

First of all, suppose f and g are both weak equivalences. Then by functoriality of π_n , since $\pi_n(f, x)$ and $\pi_n(g, f(x))$ are isomorphisms for all $x \in X$, $\pi_n(g \circ f, x) = \pi_n(g, f(x)) \circ \pi_n(f, x)$ is likewise an isomorphism for all $x \in X$, so that $g \circ f$ is a weak equivalence.

Now, suppose that $g \circ f$ and g are weak equivalences. Pick a point $x \in X$. We wish to show that $\pi_n(f, x) : \pi_n(X, x) \rightarrow \pi_n(Y, f(x))$ is an isomorphism for all $n \geq 0$. We know that $\pi_n(g \circ f, x)$ is an isomorphism, and $\pi_n(g, f(x))$ is an isomorphism, say with inverse, φ , so that

$$\varphi \circ \pi_n(g \circ f, x) = \varphi \circ \pi_n(g, f(x)) \circ \pi_n(f, x) = \pi_n(f, x)$$

is an isomorphism, as it is a composition of isomorphisms.

Now, suppose that $g \circ f$ and f are weak equivalences. Pick a point $y \in Y$. Since $\pi_0(f)$ is an isomorphism, there exists a point $x \in X$ such that $f(x)$ belongs to the path component containing y , so that there exists some $\alpha : I \rightarrow Y$ with $\alpha(0) = f(x)$ and $\alpha(1) = y$. Then consider the following diagram

$$\begin{array}{ccc} \pi_n(Y, y) & \xrightarrow{\pi_n(g, y)} & \pi_n(Z, g(y)) \\ \downarrow & & \downarrow \\ \pi_n(Y, f(x)) & \xrightarrow{\pi_n(g, f(x))} & \pi_n(Z, g(f(x))) \end{array}$$

where the left arrow is the isomorphism given by conjugation by the path α , and the right arrow is the isomorphism given by conjugation by the path $g \circ \alpha$. It is tedious yet straightforward to verify that the diagram commutes. Furthermore, we know that $\pi_n(f, x)$ and $\pi_n(g \circ f, x) = \pi_n(g, f(x)) \circ \pi_n(f, x)$ are isomorphisms for all n , so that if we denote the inverse of $\pi_n(f, x)$ by φ , then

$$\pi_n(g \circ f, x) \circ \varphi = \pi_n(g, f(x)) \circ \pi_n(f, x) \circ \varphi = \pi_n(g, f(x))$$

is an isomorphism, as it is given as a composition of isomorphisms. Hence, the top arrow must likewise be an isomorphism, precisely the desired result.

The fact that weak equivalences in **Top** are closed under retracts is entirely straightforward and follows from the fact that the functors π_n preserve retract diagrams and that the class of isomorphisms in any category is closed under retracts. \square

Proposition 2.7 (Hovey 2.4.9). $J\text{-cof} \subseteq \mathcal{W} \cap I'\text{-cof}$.

Proof. **TODO.** \square

Proposition 2.8 (Hovey 2.4.10). $I'\text{-inj} \subseteq \mathcal{W} \cap J\text{-inj}$

Proof. **TODO.** \square

Proposition 2.9 (Hovey 2.4.12). $\mathcal{W} \cap J\text{-inj} \subseteq I'\text{-inj}$

Proof. **TODO.** \square

Corollary 2.10 (Hovey 2.4.14). *Every topological space is fibrant, i.e., given a space X , the unique map $X \rightarrow *$ is an element of $J\text{-inj}$.*

Proof. **TODO.** \square

Questions:

- (1) What is an example of a relative cell complex that is not a CW complex?