MODEL STRUCTURES

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Contents

1. Preliminaries

Definition 1.1 (Hovey Definition 2.1.1)**.** Suppose $\mathcal{C}$ is a cocomplete category, and λ is an ordinal. A λ-*sequence* in $\mathcal{C}$ is a colimit-preserving functor $X : \lambda \to \mathcal{C}$, commonly written as

$$X_0 \to X_1 \to \cdots \to X_\beta \to \cdots .$$

Since X preserves colimits, for all limit ordinals $\gamma < \lambda$, the induced map

$$\operatorname*{colim}_{\beta<\gamma} X_\beta \to X_\gamma$$

is an isomorphism. We refer to the map $X_0 \to \operatorname{colim}_{\beta<\lambda} X_\beta$ as the *composition* of the λ-sequence. Given a collection $\mathcal{D}$ of morphisms in $\mathcal{C}$ such that every map $X_\beta \to X_{\beta+1}$ for $\beta + 1 < \lambda$ is in $\mathcal{D}$, we refer to the composition $X_0 \to \operatorname{colim}_{\beta<\lambda} X_\beta$ as a *transfinite composition* of maps in $\mathcal{D}$.[1]

Definition 1.2 (Hovey Definition 2.1.2)**.** Let γ be a cardinal. An ordinal α is γ-*filtered* if it is a limit ordinal and, if $A \subseteq \alpha$ and $|A| \leq \gamma$, then $\sup A < \alpha$.

Given a cardinal γ, a γ-filtered category $\mathcal{C}$ is one such that any diagram $\mathcal{D} \to \mathcal{C}$ has a cocone when $\mathcal{D}$ has $< \gamma$ arrows. A catgory is just "filtered" if it is ω-filtered, i.e., if every finite diagram in $\mathcal{C}$ admits a cocone. Note that an ordinal α is γ-filtered precisely when it is γ-filtered as a category, and in particular every ordinal is ω-filtered.

Definition 1.3 (Hovey Definition 2.1.3)**.** Suppose $\mathcal{C}$ is a comcomplete category, $\mathcal{D} \subseteq \operatorname{Mor} \mathcal{C}$ is some collection of morphisms of $\mathcal{C}$, A is an object of $\mathcal{C}$, and κ is a cardinal. We say that A is κ-*small relative to* $\mathcal{D}$ if, for all κ-filtered ordinals λ and all λ-sequences

$$X_0 \to X_1 \to \cdots \to X_\beta \to \cdots$$

such that each map $X_\beta \to X_{\beta+1}$ is in $\mathcal{D}$ for $\beta + 1 < \lambda$, the map of sets

$$\operatorname*{colim}_{\beta<\lambda} \mathcal{C}(A, X_\beta) \to \mathcal{C}(A, \operatorname*{colim}_{\beta<\lambda} X_\beta)$$

is an isomorphism. We say that A is *small relative to* $\mathcal{D}$ if it is κ-small relative to $\mathcal{D}$ for some κ. We say that A is *small* if it is small relative to $\mathcal{C}$ itself.

Remark 1.4. Recall that given a small category $\mathcal{D}$ and a functor $F : \mathcal{D} \to \mathrm{Set}$, we may explicitly construct the colimit of F as the set

$$\operatorname{colim} F := \left(\coprod_{d\in\mathcal{D}} F(d) \right) / \sim,$$

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[1] To be more precise, there may be different (isomorphic) choices of colimit $\operatorname{colim}_{\beta<\gamma} X_\beta$, which give rise to different choices of composition $X_0 \to \operatorname{colim}_{\beta<\gamma} X_\beta$. Thus, the composition of a λ-sequence is only unique up to composition by a unique isomorphism.

where the equivalence relation $\sim$ is **generated** by

$$((x \in F(d)) \sim (x' \in F(d'))) \quad \text{if} \quad (\exists (f : d \to d') \text{ with } Ff(x) = x').$$

In particular, if $\mathcal{D}$ is a filtered category then the resulting relation can be described as follows:

$$((x \in F(d)) \sim (x' \in F(d'))) \quad \text{iff} \quad (\exists\, d'', (f : d \to d''), (g : d' \to d'') \text{ with } Ff(x) = Fg(x')).$$

Given a cone $\eta : F \Rightarrow \underline{Y}$ under F, the unique map $\operatorname{colim} F \to Y$ maps the equivalence class of $x \in F(d)$ to the element $\eta_d(x) \in X$.

We will use the characterization of the colimit afforded by Remark 1.4 in the following example.

Example 1.5 (Hovey 2.1.5)**.** Every set is small. Indeed, if A is a set we claim that A is $|A|$-small. To see this, suppose λ is an $|A|$-filtered ordinal, and X is a λ-sequence of sets. Given $\alpha < \beta < \lambda$, let $\iota_{\alpha,\beta} : X_\alpha \to X_\beta$ denote the induced morphism. We will write $X_\lambda := \operatorname{colim}_{\beta<\lambda} X_\beta$, and let $\iota : X \Rightarrow X_\lambda$ be the colimit cone, so that given $\beta < \lambda$, $\iota_\beta : X_\beta \to X_\lambda$ is the leg of the colimit cone at X_β. By composing with the functor $\mathcal{C}(A, -) : \mathrm{Set} \to \mathrm{Set}$, we get another λ-sequence $\{\mathcal{C}(X_\beta, A)\}_{\beta<\lambda}$. The cone ι under X induces a cone ι_* under $\mathcal{C}(X_\beta, A)$ with nadir $\mathcal{C}(A, X_\lambda)$. Let $\eta : \mathcal{C}(X_\beta, A) \Rightarrow \underline{\operatorname{colim}_{\beta<\lambda} \mathcal{C}(X_\beta, A)}$ be the colimit cone, and let $\ell : \operatorname{colim}_{\beta<\lambda} \mathcal{C}(A, X_\lambda) \to \mathcal{C}(A, X_\lambda)$ be the unique morphism of cones so that the following diagram commutes

$$\begin{tikzcd}
\mathcal{C}(A,X_0) \arrow[r, "(\iota_{0,1})_*"] \arrow[drr, "\eta_0"] \arrow[ddrr, "(\iota_0)_*"'] & \mathcal{C}(A,X_1) \arrow[r, "(\iota_{1,2})_*"] \arrow[dr, "\eta_1"] \arrow[ddr, "(\iota_1)_*"'] & \cdots \arrow[r] & \mathcal{C}(A,X_\beta) \arrow[r, "(\iota_{\beta,\beta+1})_*"] \arrow[dl, "\eta_\beta"'] \arrow[ddl, "(\iota_\beta)_*"] & \cdots \\
 & & \operatorname{colim}_{\beta<\lambda} \mathcal{C}(A,X_\beta) \arrow[d, dashed, "\ell"] & & \\
 & & \mathcal{C}(A,X_\lambda) & &
\end{tikzcd}$$

First, we wish to show that ℓ is surjective. Indeed, let $f : A \to X_\lambda$. For each $a \in A$, there exists some $\beta_a \in \lambda$ and some $a' \in X_{\beta_a}$ such that $f(a) = \eta_{\beta_a}(a')$ (see the preceeding discussion). Then let $\gamma := \sup_{a\in A} \beta_a$. Since $|\{\beta_a\}_{a\in A}| \le |A|$ and λ is $|A|$-filtered, necessarily $\gamma < \lambda$. Now, define $g : A \to X_\gamma$ like so: for $a \in A$, define $g(a) := \iota_{\beta_a,\gamma}(a')$, where $a' \in X_{\beta_a}$ was chosen earlier so that $\iota_{\beta_a}(a') = f(a)$. Then we claim that $\ell(\eta_\gamma(g)) = f$. Indeed, as ℓ is a morphism of cocones, $\ell \circ \eta = \iota_*$, so that we have

$$\ell(\eta_\gamma(g)) = (\iota_\gamma)_*(g) = \iota_\gamma \circ g,$$

and given $a \in A$ we have

$$\iota_\gamma(g(a)) = \iota_\gamma(\iota_{\beta_a,\gamma}(a')).$$

By definition of a cone, $\iota_\gamma \circ \iota_{\beta_a,\gamma} = \iota_{\beta_a}$, so that

$$\ell(\eta_\gamma(g))(a) = \iota_\gamma(\iota_{\beta_a,\gamma}(a')) = \iota_{\beta_a}(a') = f(a),$$

so that indeed $\ell(\eta_\gamma(g)) = f$.

It remains to show ℓ is injective. Suppose we are given $[f], [g] \in \operatorname{colim}_{\beta<\lambda} \mathcal{C}(A, X_\beta)$ such that $\ell([f]) = \ell([g])$. Then by the preceeding discussion, there exists $\alpha, \beta < \lambda$, $f \in \mathcal{C}(A, X_\alpha)$, and $g \in \mathcal{C}(A, X_\beta)$ such that $\eta_\alpha(f) = [f]$ and $\eta_\beta(g) = [g]$. Then since $\ell \circ \eta = \iota_*$, we have

$$\ell([f]) = \ell([g]) \implies \iota_\alpha \circ f = (\iota_\alpha)_*(f) = \ell(\eta_\alpha(f)) = \ell(\eta_\beta(g)) = (\iota_\beta)_*(g) = \iota_\beta \circ g.$$

For each $a \in A$, since $\iota_\alpha(f(a)) = \iota_\beta(g(a))$, by Remark 1.4 there exists γ_a with $\alpha, \beta \le \gamma_a$ such that $\iota_{\alpha,\gamma_a}(f(a)) = \iota_{\beta,\gamma_a}(g(a))$. Then let $\gamma := \sup_{a\in A} \gamma_a$. Since $|\{\gamma_a\}_{a\in A}| \le |A|$ and λ is $|A|$-filtered, necessarily $\gamma < \lambda$. Now, in order to show $[f] = [g]$, by Remark 1.4 it suffices to show that $(\iota_{\alpha,\gamma})_*(f) = (\iota_{\beta,\gamma})_*(g)$. Indeed, given $a \in A$, we have

$$(\iota_{\alpha,\gamma})_*(f)(a) = \iota_{\alpha,\gamma}(f(a)) = \iota_{\gamma_a,\gamma} \circ \iota_{\alpha,\gamma_a}(f(a)) = \iota_{\gamma_a,\gamma} \circ \iota_{\beta,\gamma_a}(g(a)) = \iota_{\beta,\gamma}(g(a)) = (\iota_{\beta,\gamma})_*(g)(a),$$

precisely the desired result.

Lemma 1.6. *Let $\mathcal{J}$ be a directed set (so $\mathcal{J}$ is a nonempty set with a relation $\leq$ such that $\leq$ is reflexive, transitive, and given $x, y \in \mathcal{J}$, there exists $z \in \mathcal{J}$ with $x \leq z$ and $y \leq z$). Then given any object $x \in \mathcal{J}$, define $\mathcal{J}_x$ to be the subcategory of $\mathcal{J}$ containing precisely those elements $y \in \mathcal{J}$ with $y \geq x$. Then given a functor $F : \mathcal{J} \to \mathcal{C}$ and an element $x \in \mathcal{J}$, if η is a colimit cone for F, then η restricts to a colimit cone for $F|_{\mathcal{J}_x}$ for all $x \in \mathcal{J}$.*

Proof Sketch. Let $\eta : F \Rightarrow \underline{X}$ be a colimit cone for F, then it clearly restricts to a cone under $F|_{\mathcal{J}_x}$. In order to show that η restricts to a colimit cone, it suffices to show that any cone under $F|_{\mathcal{J}_x}$ extends uniquely to a cone under F. This can be easily checked. □

Definition 1.7 (Hovey Definition 2.1.7)**.** Let I be a class of maps in a category $\mathcal{C}$.

(1) A map is *I-injective* if it has the right lifting property w.r.t. every map in I. The class of I-injective maps is denoted I-inj (or $I_\perp$).
(2) A map is *I-projective* if it has the left lifting property w.r.t. every map in I. The class of I-projective maps is denoted I-proj (or ${}_\perp I$).
(3) A map is an *I-cofibration* if it has the left lifting property w.r.t. every I-injective map. The class of I-cofibrations is the class (I-inj)-proj and is denoted I-cof (or ${}_\perp(I_\perp)$).
(4) A map is an *I-fibration* if it has the right lifting property w.r.t. every I-projective map. The class of I-fibrations is the class (I-proj)-inj and is denoted I-fib (or $({}_\perp I)_\perp$).

The following is asserted in Hovey on pg. 30 following Definition 2.1.7, but not proven. We provide a proof.

Lemma 1.8. *Given classes A and B of maps in a category $\mathcal{C}$ with $A \subseteq B$, we have $A \subseteq {}_\perp(A_\perp)$, $A \subseteq ({}_\perp A)_\perp$, $({}_\perp(A_\perp))_\perp = A_\perp$, ${}_\perp(({}_\perp A)_\perp) = {}_\perp A$, $A_\perp \supseteq B_\perp$, ${}_\perp A \supseteq {}_\perp B$, ${}_\perp(A_\perp) \subseteq {}_\perp(B_\perp)$, and $({}_\perp A)_\perp \subseteq ({}_\perp B)_\perp$.*

Proof. Each of these amount to unravelling definitions and are entirely straightforward. □

Definition 1.9 (Hovey Definition 2.1.9)**.** Let I be a set of maps in a cocomplete category $\mathcal{C}$. A *relative I-cell complex* is a transfinite composition of pushouts of elements of I. That is, if $f : A \to B$ is a relative I-cell complex, then there is an ordinal λ and a λ-sequence $X : \lambda \to \mathcal{C}$ such that f is the composition of X and such that, for each β such that $\beta + 1 < \lambda$, there is a pushout square

$$\begin{tikzcd} C_\beta \arrow[r] \arrow[d, "g_\beta"'] & X_\beta \arrow[d] \\ D_\beta \arrow[r] \arrow[ur, phantom, "\lrcorner" very near end] & X_{\beta+1} \end{tikzcd}$$

with $g_\beta \in I$. We denote the collection of relative I-cell complexes by I-cell. We say that $A \in \mathcal{C}$ is an *I-cell complex* if the map $0 \to A$ is a relative I-cell complex.

Lemma 1.10. *Let $\mathcal{C}$ be a category and I a class of morphisms in $\mathcal{C}$. Then I-*cell *is closed under composition with isomorphisms.*

Proof. Suppose that $f : B \to C$ is an element of I-cell, and $h : A \to B$ and $g : C \to D$ are isomorphisms in $\mathcal{C}$. We wish to show $f \circ h$ and $g \circ f$ are also elements of I-cell. Since $f \in I$-cell, there exists an ordinal λ, a λ-sequence X with $X_0 = B$, and a colimit cone $\eta : X \Rightarrow \underline{C}$, such that $\eta_0 = f$.

First of all, construct a new cone $\eta' : X \Rightarrow \underline{D}$ under X where $\eta'_\beta := g \circ \eta_\beta$. It is straightforward to verify that η' is a colimit cone for X since η is a colimit cone and g is an isomorphism. Thus, $g \circ f = g \circ \eta_0 = \eta'_0 \in I$-cell, as η'_0 is the composition of a sequence of pushouts of elements of I.

On the other hand, we may construct a new λ-sequence X' by defining $X'_0 = A$, $X'_\beta = X_\beta$ for all $0 < \beta < \lambda$, the map $X'_0 \to X'_\beta$ for $0 < \beta < \lambda$ to be the composition

$$A \xrightarrow{h} B = X_0 \longrightarrow X_\beta,$$

and the composition $X'_\alpha \to X'_\beta$ to simply be the same map $X_\alpha \to X_\beta$ for $0 < \alpha \leq \beta < \lambda$. It is straightforward to verify that defines a λ-sequence, and that we may define a colimit cone $\eta' : X' \Rightarrow \underline{C}$ by $\eta'_0 = \eta_0 \circ h = f \circ h$, and $\eta'_\beta = \eta_\beta$ for $0 < \beta < \lambda$. Furthermore, clearly for all $1 < \beta + 1 < \lambda$, we have the arrow $X'_\beta \to X'_{\beta+1}$ is a pushout of a map in I. Thus, in order to show $f \circ h \in I$-cell, it remains to show that the arrow

$A = X_0' \to X_1' = X_1$ is a pushout of a map in I. Indeed, we know since $B = X_0 \to X_1$ is a pushout of a map $k : P \to Q$ in I, and it can be easily verified the diagram on the right is a pushout diagram:

$$\begin{tikzcd} P \arrow[r] \arrow[d, "k"'] & X_0 \arrow[d] \\ Q \arrow[r] & X_1 \arrow[ul, phantom, "\lrcorner", very near start] \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} P \arrow[r] \arrow[dd] & X_0 \arrow[r, "h^{-1}"] & X_0' \arrow[d, "h"] \\ & & X_0 \arrow[d] \\ Q \arrow[rr] & & X_1' \arrow[ul, phantom, "\lrcorner", very near start] \end{tikzcd}$$

□

Lemma 1.11 (Hovey 2.1.10). *Suppose I is a class of maps in a cocomplete category $\mathcal{C}$. Then $I\text{-cell} \subseteq {}_\perp(I_\perp)$.*

Proof. TODO. □

Theorem 1.12 (Small Object Argument, Hovey 2.1.14). *Suppose $\mathcal{C}$ is a cocomplete categroy, and I is a set of maps in $\mathcal{C}$. Suppose the domains of the maps of I are small relative to I-cell. Then there is a functorial factorization (γ, δ) on $\mathcal{C}$ such that for all morphisms $f \in \mathcal{C}$, the map $\gamma(f)$ is in I-cell and the map $\delta(f)$ is in I-inj.*

Proof. TODO. □

Corollary 1.13 (Hovey 2.1.15). *Suppose that I is a set of maps in a cocomplete category $\mathcal{C}$. Suppose as well that the domains of I are small relative to I-cell. Then given $f : A \to B$ in ${}_\perp(I_\perp)$, there is a $g : A \to C$ in I-cell such that f is a retract of g by a map which fixes A.*

Proof. TODO □

Definition 1.14 (Hovey Definition 2.1.17). Suppose $\mathcal{C}$ is a model category. We say that $\mathcal{C}$ is *cofibrantly generated* if there are sets I and J of maps such that:

1. The domains of the maps of I are small relative to I-cell;
2. The domains of the maps of J are small relative to J-cell;
3. The class of fibrations is $J_\perp$; and
4. The class of trivial fibrations is $I_\perp$.

We refer to I as the set of *generating cofibrations* and to J as the set of *generating trivial cofibrations*. A cofibrantly generated model category is *finitely generated* if we can choose the sets I and J above so that the domains and codomains of I and J are finite relative to I-cell.

Proposition 1.15 (Hovey Proposition 2.1.18). *Suppose $\mathcal{C}$ is a cofibrantly generated model category, with generating cofibrations I and generating trivial fibrations J.*

(a) The cofibrations form the class ${}_\perp(I_\perp)$.
(b) Every cofibration is a retract of a relative I-cell complex.
(c) The domains of I are small relative to the cofibrations.
(d) The trivial cofibrations form the class ${}_\perp(J_\perp)$.
(e) Every trivial cofibration is a retract of a relative J-cell complex.
(f) The domains of J are small relative to the trivial cofibrations.

If $\mathcal{C}$ is fibrantly generated, then the domains and codomains of I and J are finite relative to the cofibrations.

Proof. TODO. □

Theorem 1.16 (Hovey Theorem 2.1.19). *Suppose $\mathcal{C}$ is a complete & cocomplete category. Suppose $\mathcal{W}$ is a subcategory of $\mathcal{C}$, and I and J are sets of maps of $\mathcal{C}$. Then there is a cofibrantly generated model structure on $\mathcal{C}$ with I as the set of generating cofibrations, J as the set of generating trivial fibrations, and $\mathcal{W}$ as the subcategory of weak equivalences if and only if the following conditions are satisfied.*

1. *The subcategory $\mathcal{W}$ has the 2-of-3 property and is closed under retracts.*
2. *The domains of I are small relative to I-cell.*
3. *The domains of J are small relative to J-cell.*
4. *$J\text{-cell} \subseteq \mathcal{W} \cap {}_\perp(I_\perp)$.*
5. *$I_\perp \subseteq \mathcal{W} \cap J_\perp$.*

6. *Either* $\mathcal{W} \cap {}_\perp(I_\perp) \subseteq {}_\perp(J_\perp)$ *or* $\mathcal{W} \cap J_\perp \subseteq I_\perp$.

Proof. TODO. □

Definition 1.17. Let $\mathcal{C}$ be a category and I a collection of morphisms in $\mathcal{C}$. Then if I is closed under transfinite composition, pushouts, and retracts then we say I is *saturated*.

2. Topological Spaces

A map $f : X \to Y$ in **Top** is an *inclusion* if it is continuous, injective, and for all $U \subseteq X$ open, there is some $V \subseteq Y$ open such that $f^{-1}(V) = U$. If f is a closed inclusion and every point in $Y \setminus f(X)$ is closed, then we call f a *closed* T_1 *inclusion*. We will let $\mathcal{T}$ denote the class of closed T_1 inclusions in **Top**.

The symbol D^n will denote the unit disk in $\mathbb{R}^n$, and the symbol S^{n-1} will denote the unit sphere in $\mathbb{R}^n$, so that we have the boundary inclusions $S^{n-1} \hookrightarrow D^n$. In particular, for $n = 0$ we let $D^0 = \{0\}$ and $S^{-1} = \emptyset$.

Recall: If $F : \mathcal{J} \to \mathbf{Top}$ is a functor, where $\mathcal{J}$ is a small category, the limit of F is obtained by taking the limit in the category of sets, and then topologizing it with the *initial topology*, where if $\eta : \underline{\lim F} \Rightarrow F$ is the limit cone, then the topology on $\lim F$ is that with subbasis given by sets of the form $\eta_j^{-1}(U)$ where $j \in \mathcal{J}$ and $U \subseteq F_j$ is open. Similarly, the colimit of F is obtained by taking the colimit $\operatorname{colim} F$ in the category of sets and endowing it with the *final topology*, where a set $U \subseteq \operatorname{colim} F$ is open if and only if $\varepsilon_j^{-1}(U)$ is open in F_j for all $j \in \mathcal{J}$, where $\varepsilon : F \Rightarrow \underline{\operatorname{colim} F}$ is the colimit cone.

Given a space X, we construct a functor $(-)^X : \mathbf{Top} \to \mathbf{Top}$ as follows: Given a space Y, define Y^X to be the space whose underlying set is the set $\mathbf{Top}(X, Y)$ of continuous maps $X \to Y$, and the topology on Y^X is the *compact-open topology*, i.e., the topology with subbasis given by the sets of the form

$$S(K, U) := \{f \in \mathbf{Top}(X, Y) : f(K) \subseteq U\}$$

for $K \subseteq X$ compact and $U \subseteq Z$ open. Given a continuous map $f : Y \to Z$, define the induced map $f_* : Y^X \to Z^X$ by $f_*(g) := f \circ g$. Unravelling definitions, we have that given $f : Y \to Z$ continuous, $f_*^{-1}(S(K, U)) = S(K, f^{-1}(U))$ for all $K \subseteq X$ compact and $U \subseteq Z$ open, so that f_* is continuous. Furthermore, $(-)^X$ is clearly functorial, by associativity and unitality of function composition.

Given a topological space X, we say that X is *locally compact* if for all points $x \in X$ and open neighborhoods U of x, there exists an open set $V \subseteq X$ with $x \in V$, $\overline{V} \subseteq U$, and $\overline{V}$ compact. We claim that $(-)^X$ is right adjoint to $- \times X$ when X is locally compact and Hausdorff.

Proposition 2.1. *If X is a locally compact Hausdorff space, then functor $- \times X$ is left adjoint to $(-)^X$ (so that in particular $- \times X$ preserves colimits).*

Proof. We start by constructing the counit and unit of the adjunction. Given a space Z, define the counit $\varepsilon_Z : X \times Z^X \to Z$ to be the evaluation function, taking a pair $(x, f) \mapsto f(x)$. First, we claim ε_Z is continuous. Suppose we are given an open set $V \subseteq Z$ and a point $(x, f) \in \varepsilon_Z^{-1}(U)$ (so $f(x) \in V$). Since f is continuous and X is locally compact, there exists an open set $U \subseteq X$ containing x such that $x \in U \subseteq \overline{U} \subseteq f^{-1}(V)$ with $\overline{U}$ compact. Then consider the open set $U \times S(\overline{U}, V)$ in $X \times Y^X$. First of all, $(x, f) \in U \times S(\overline{U}, V)$, as $x \in U$ and $\overline{U} \subseteq f^{-1}(V)$, so that $f(\overline{U}) \subseteq V$ meaning $f \in S(\overline{U}, V)$. Furthermore, given $(y, g) \in U \times S(\overline{U}, V)$, we have $\varepsilon_Z(y, g) = g(y) \in g(U) \subseteq g(\overline{U}) \subseteq V$, so $U \times S(\overline{U}, V)$ is an open neighborhood of x contained in $\varepsilon_Z^{-1}(V)$, as desired. Hence, ε_Z is continuous. It remains to show naturality. Given a map $f : Z \to W$, we wish to show the following diagram commutes:

$$\begin{tikzcd} X \times Z^X \arrow[r, "\varepsilon_Z"] \arrow[d, "\mathrm{id}_X \times f_*"'] & Z \arrow[d, "f"] \\ X \times W^X \arrow[r, "\varepsilon_W"] & W \end{tikzcd}$$

Indeed, chasing an element (x, g) around the diagram yields:

$$\begin{tikzcd} (x, g) \arrow[r, mapsto] \arrow[d, mapsto] & g(x) \arrow[d, mapsto] \\ (x, f \circ g) \arrow[r, mapsto] & f(g(x)) \end{tikzcd}$$

so it does indeed commute.

Now we wish to define the unit $\eta_Y : Y \to (Y \times X)^X$. Given $y \in Y$, define $\eta_Y(y) \in (Y \times X)^X$ by $\eta_Y(y)(x) := (y, x)$. First of all, for it to be true that $\eta_Y(y) \in (X \times Y)^X$, it must be true that $\eta_Y(y)$ is continuous. Indeed, this is clear as η_Y is obtained as the product map $y \times \mathrm{id}_X : X \to Y \times X$, where y represents the constant function on y (which is obviously continuous). Furthermore, η_Y itself is continuous: given $K \subseteq X$ compact and $U \subseteq Y \times X$ open, we wish to show that $\eta_Y^{-1}(S(K,U))$ is open in Y. It suffices to show that given $y \in \eta_Y^{-1}(S(K,U))$, there exists an open neighborhood W of y that is mapped by η_Y into $S(K,U)$. Since $y \in \eta_Y^{-1}(S(K,U))$, $\eta_Y(y)(K) = \{y\} \times K \subseteq U$. Then $U \cap (Y \times K)$ is an open set in the subspace $Y \times K$ containing the slice $\{y\} \times K$. By definition of the product topology, for each $k \in K$, there exist open sets $W_k \subseteq Y$ and $V_k \subseteq K$ such that $(y,k) \in W_k \times V_k \subseteq U \cap (Y \times K)$. Then the V_k's form an open cover of K, which is compact, so that there exist $k_1, \ldots, k_n \in K$ with $V_{k_1} \cup \cdots \cup V_{k_n} = K$. Hence if we define $W := W_{k_1} \cap \cdots \cap W_{k_n}$, then $\{y\} \times K \subseteq W \times K \subseteq U \cap (Y \times K)$, and W is open in Y as it is a finite intersection of open sets. Then for all $w \in W$, $\eta_Y(w)(K) = \{w\} \times K \subseteq W \times K \subseteq U$. Hence, indeed η_Y is continuous. It remains to show naturality. Given a map $f : Y \to W$, we wish to show the following diagram commutes:

$$\begin{array}{ccc} Y & \xrightarrow{\eta_Y} & (Y \times X)^X \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle (f \times \mathrm{id}_X)_*} \\ W & \xrightarrow{\eta_W} & (W \times X)^X \end{array}$$

Indeed, chasing an element y around the top of the diagram yields the function obtained as the composition $x \mapsto (y,x) \mapsto f \times \mathrm{id}_X(y,x) = (f(y), x)$, while chasing around the bottom of the diagram more directly yields the function $x \mapsto (f(y), x)$.

Now that we have constructed the unit and counit, it remains to verify the counit-unit equations, i.e., that for each $Y \in \mathbf{Top}$ that $\varepsilon_{Y \times X} \circ (\eta_Y \times \mathrm{id}_X) = \mathrm{id}_{Y \times X}$ and $(\varepsilon_Y)_* \circ \eta_{Y^X} = \mathrm{id}_{Y^X}$. First of all, given $(y,x) \in Y \times X$, we have

$$(\varepsilon_{Y \times X} \circ (\eta_Y \times \mathrm{id}_X))(y,x) = \varepsilon_{X \times Y}(\eta_Y(y), x) = \eta_Y(y)(x) = (y,x).$$

On the other hand, given $f \in Y^X$, we have

$$(\varepsilon_Y)_*(\eta_{Y^X}(f)) = (\varepsilon_Y)_*([x \mapsto (f,x)]) = [x \mapsto (f,x) \mapsto \varepsilon_Y(f,x) = f(x)] = f.$$

Hence, indeed ε and η form the counit and unit for the adjoint pair $(- \times X, (-)^X)$. $\square$

Definition 2.2. A map $f : X \to Y$ in $\mathbf{Top}$ is called a *weak equivalence* if

$$\pi_n(f,x) : \pi_n(X,x) \to \pi_n(Y, f(x))$$

is an isomorphism for all $n \geq 0$ and for all $x \in X$. We will write $\mathcal{W}$ to refer to the class of all weak equivalences in $\mathbf{Top}$.

Define the set of maps I' to consist of all the boundary inclusion $S^{n-1} \hookrightarrow D^n$ for all $n \geq 0$, and define the set J to consist of all the inclusions $D^n \hookrightarrow D^n \times I$ mapping $x \mapsto (x, 0)$ for $n \geq 0$. Then a map f will be called a *cofibration* if it is in I-cof $= {}_\perp(I'_\perp)$, and a *fibration* if it is in J-inj $= J_\perp$.

A map in I'-cell is usually called a *relative cell complex*; a relative CW-complex is a special case of a relative cell complex, where, in particular, the cells can be attached in order of their dimension. Note that in particular maps of J are relative CW complexes, hence are relative I'-cell complexes. A fibration is often known as a *Serre fibration* in the literature.

Theorem 2.3 (Hovey Theorem 2.4.19)**.** *There is a finitely generated model structure on* $\mathbf{Top}$ *with* I' *as the set of generating cofibrations,* J *as the set of generating trivial cofibrations, and the cofibrations, fibrations, and weak equivalences as above. Every object of* $\mathbf{Top}$ *is fibrant, and the cofibrant objects are retracts of relative cell complexes.*

Proof. We will apply Theorem 1.16 to get that there is a cofibrantly generated model structure on $\mathbf{Top}$ with I' as the set of generating cofibrations, J as the set of generating trivial fibrations, and $\mathcal{W}$ as the subcategory of weak equivalences. The six requirements outlined in the theorem will be verified like so:

1. $\mathcal{W}$ is a subcategory of $\mathcal{C}$ which has the 2-of-3 property and is closed under retracts: Lemma 2.11.
2. The domains of I' are small relative to I'-cell: Proposition 2.10.
3. The domains of J are small relative to J-cell: Proposition 2.10.

4. J-cell $\subseteq \mathcal{W} \cap {}_{\perp}(I'_{\perp})$: In Proposition 2.12, we will show ${}_{\perp}(J_{\perp}) \subseteq \mathcal{W} \cap {}_{\perp}(I'_{\perp})$, and by Lemma 1.11 J-cell $\subseteq {}_{\perp}(J_{\perp})$.
5. $I'_{\perp} \subseteq \mathcal{W} \cap J_{\perp}$: Proposition 2.13
6. $\mathcal{W} \cap J_{\perp} \subseteq I'_{\perp}$: Proposition 2.14

It will follow by the definition of a cofibrantly generated model structure (Definition 1.14) that the fibrations in this model structure are given by $J_{\perp}$, which is precisely how we defined it. By Proposition 1.15, the class of cofibrations will be given by ${}_{\perp}(I'_{\perp})$, which is likewise exactly how we defined them.

In Proposition 2.7, we will show that compact spaces are finite relative to the class $\mathcal{T}$ of closed T_1 inclusions. Hence, this model structure will be finitely generated, as the domains and codomains of I' and J are all compact, and by the reasoning given above we will have shown I'-cell $\subseteq \mathcal{T}$.

We will show that every object of **Top** is fibrant in Corollary 2.15. □

Lemma 2.4. *Let λ be an ordinal, and X a λ-sequence of inclusions in* **Top**. *Then the map $X_\alpha \to X_\beta$ is an inclusion for all $\alpha \leq \beta < \gamma$.*

Given $\alpha \leq \beta < \gamma$, let $\iota_{\alpha,\beta}$ denote the map $X_\alpha \to X_\beta$. In what follows, given a continuous map $f : A \to B$ of topological spaces, let $(*)$ be the property "for all $U \subseteq A$ open, there exists $V \subseteq B$ open with $f^{-1}(V) = U$" (so an inclusion is an injective continuous map satisfying $(*)$). Hence to prove the above Lemma, suffices to prove the following two statements separately:

(1) If $\iota_{\beta,\beta+1}$ is injective for all $\beta + 1 < \lambda$, then $\iota_{\alpha,\beta}$ is injective for all $\alpha \leq \beta < \lambda$.

Proof. Let $\alpha < \lambda$. We perform a proof by transfinite induction on β for $\alpha \leq \beta < \lambda$ that $\iota_{\alpha,\beta}$ is injective. For the zero case, clearly $\iota_{\alpha,\alpha} = \mathrm{id}_\alpha$ is injective. Supposing $\iota_{\alpha,\beta}$ is injective for some $\alpha < \beta + 1 < \lambda$, we have $\iota_{\alpha,\beta+1} = \iota_{\beta,\beta+1} \circ \iota_{\alpha,\beta}$ is a composition of injections, and is therefore clearly injective itself. Finally, suppose γ is a limit ordinal with $\alpha \leq \gamma < \lambda$ such that $\iota_{\alpha,\beta}$ is injective for all $\alpha \leq \beta < \gamma$. We claim $\iota_{\alpha,\gamma}$ is injective. Since X_γ is colimit preserving and γ is a limit ordinal, X_γ is the colimit of the diagram $\{X_\beta\}_{\beta<\gamma}$ via the maps $\iota_{\beta,\gamma}$, so that in particular by Remark 1.4 and the discussion at the beginning of this section, given $a, b \in X_\alpha$ with $\iota_{\alpha,\gamma}(a) = \iota_{\alpha,\gamma}(b)$, there exists some $\beta < \gamma$ with $\iota_{\alpha,\beta}(a) = \iota_{\alpha,\beta}(b)$, and $\iota_{\alpha,\beta}$ is injective for all $\beta < \gamma$, so it must have been true $a = b$ in X_α. □

(2) If $\iota_{\beta,\beta+1}$ satisfies $(*)$ for all $\beta + 1 < \lambda$, then $\iota_{\alpha,\beta}$ satisfies $(*)$ for $\alpha \leq \beta < \lambda$.

Proof. We will prove the following slightly stronger statement: for each $\alpha < \lambda$ and $U \subseteq X_\alpha$ open, there exist open sets $V_\beta \subseteq X_\beta$ for $\alpha \leq \beta < \lambda$ with $V_\alpha = U$ such that for all $\alpha \leq \beta' \leq \beta$, $\iota^{-1}_{\beta',\beta}(V_\beta) = V_{\beta'}$ (so that in particular for all $\alpha \leq \beta < \lambda$, $\iota^{-1}_{\alpha,\beta}(V_\beta) = U$). We perform transfinite induction on β, viewing α and U as fixed.

The zero case has been taken care of: $V_\alpha = U$. For the sucessor case, given $\alpha < \beta + 1 < \lambda$, supposing V_β has been defined with the desired properties, since $\iota_{\beta,\beta+1}$ is an inclusion, there exists $V_{\beta+1} \subseteq X_{\beta+1}$ with $\iota^{-1}_{\beta,\beta+1}(V_{\beta+1}) = V_\beta$. Then given $\alpha \leq \beta' \leq \beta$, we have

$$\iota^{-1}_{\beta',\beta}(V_\beta) = (\iota_{\beta,\beta+1} \circ \iota_{\alpha,\beta})^{-1}(V_{\beta+1}) = \iota^{-1}_{\alpha,\beta}(\iota^{-1}_{\beta,\beta+1}(V_{\beta+1})) = \iota^{-1}_{\alpha,\beta}(V_\beta) = U,$$

and given $\alpha \leq \beta' \leq \beta + 1$, we have

$$\iota^{-1}_{\beta',\beta+1}(V_{\beta+1}) = (\iota_{\beta,\beta+1} \circ \iota_{\beta',\beta})^{-1}(V_{\beta+1}) = \iota^{-1}_{\beta',\beta}(\iota_{\beta,\beta'}(V_{\beta+1})) = \iota^{-1}_{\beta',\beta}(V_\beta) = V_{\beta'}.$$

Finally, the limit case. Suppose γ is a limit ordinal with $\alpha < \gamma \leq \lambda$, and suppose V_β has been constructed with the desired properties for $\alpha \leq \beta < \gamma$. We wish to define V_γ. Since X is colimit preserving and γ is a limit ordinal, the maps $\iota_{\beta,\gamma}$ for $\beta < \gamma$ form a colimit cone for the diagram $\{X_\beta\}_{\beta<\gamma}$. Then by Lemma 1.6, the $\iota_{\beta,\gamma}$'s for $\alpha \leq \beta < \gamma$ form a colimit cone for the diagram $\{X_\beta\}_{\alpha\leq\beta<\gamma}$. Let $S = \{0, 1\}$ be the Sierpinski space whose open sets are $\{\emptyset, \{1\}, \{0, 1\}\}$. For $\alpha \leq \beta < \gamma$, define a map $j_\beta : X_\beta \to S$ mapping everything in V_β to 1 and every other point to 0. Each j_β is clearly continuous, as $j_\beta^{-1}(1) = V_\beta$. Furthermore, we claim the j_β's form a cone under the diagram $\{X_\beta\}_{\alpha\leq\beta<\gamma}$, i.e., that given $\alpha \leq \beta' \leq \beta < \gamma$, the following diagram commutes

$$\begin{tikzcd} X_{\beta'} \arrow[rr, "\iota_{\beta',\beta}"] \arrow[dr, "j_{\beta'}"'] & & X_\beta \arrow[dl, "j_\beta"] \\ & S & \end{tikzcd}$$

To see this, let $x \in X_{\beta'}$. If $x \in V_{\beta'} = \iota^{-1}_{\beta',\beta}(V_\beta)$, then $\iota_{\beta',\beta}(x) \in V_\beta$, so $j_\beta(\iota_{\beta',\beta}(x)) = 1 = j_{\beta'}(x)$. Conversely, if $x \in X_{\beta'} \setminus V_{\beta'}$, then $x \notin \iota^{-1}_{\beta',\beta}(V_\beta)$, so $\iota_{\beta',\beta}(x) \notin V_\beta$, meaning $j_\beta(\iota_{\beta',\beta}(x)) = 0 = j_{\beta'}(0)$. Hence, the j_β's do indeed form a cone under $\{X_\beta\}_{\alpha\leq\beta<\gamma}$, so by universal property of the colimit there exists a unique map $\ell : X_\gamma \to S$ such that $j_\beta = \ell \circ \iota_{\beta,\gamma}$ for all $\alpha \leq \beta < \gamma$. Define $V_\gamma := \ell^{-1}(1)$. It remains to show that for all $\alpha \leq \beta \leq \gamma$ that $\iota^{-1}_{\beta,\gamma}(V_\gamma) = V_\beta$. Indeed, we have

$$\iota^{-1}_{\beta,\gamma}(V_\gamma) = \iota^{-1}_{\beta,\gamma}(\ell^{-1}(1)) = (\ell \circ \iota^{-1}_{\beta,\gamma})^{-1}(1) = j_\beta^{-1}(1) = V_\beta.$$

□

Corollary 2.5. *Let λ be an ordinal X be a λ-sequence of inclusions in* **Top**. *Then for all $\alpha < \lambda$, the map $X_\alpha \to \operatorname{colim} X$ is an inclusion.*

Proof. Given a λ-sequence X of inclusions where λ is an $|X|$-filtered ordinal, we wish to show the natural map of sets $\operatorname{colim}_{\beta<\lambda} \mathbf{Top}(X, X_\beta) \to \mathbf{Top}(A, \operatorname{colim} X)$ is a bijection. We know by Remark 1.4 that an element of $\operatorname{colim}_{\beta<\lambda} \mathbf{Top}(X, X_\beta)$ may be written as an equivalence class $[f]$ represented by some map $f : X \to X_\beta$ for some

FINISH. □

Lemma 2.6 (Hovey 2.4.1)**.** *Every topological space is small relative to the inclusions.*

Proof. We claim that every topological space X is $|X|$-small. Suppose X is a λ-sequence of inclusions in **Top** where λ is an $|X|$-filtered ordinal. We know that X is $|X|$-small as a set (Example 1.5), so there exists a bijection □

Proposition 2.7 (Hovey 2.4.2)**.** *Compact topological spaces are finite relative to the class $\mathcal{T}$ of closed T_1 inclusions.*

Proof. TODO. □

Proposition 2.8 (Hovey 2.4.5 & 2.4.6)**.** *The class $\mathcal{T}$ of closed T_1 inclusions is saturated.*

Proof. TODO. □

Lemma 2.9 (Hovey 2.4.8)**.** *$\mathcal{W} \cap \mathcal{T}$ is closed under transfinite compositions.*

Proof. TODO. □

Proposition 2.10. *The domains of I' (resp. J) are small relative to I'-*cell.

Proof. By Lemma 2.6, every space is small relative to the inclusions, and in particular every space is small relative to the class $\mathcal{T}$ of closed T_1 inclusions. Hence, it suffices to show that $J\text{-cell}, I'\text{-cell} \subseteq \mathcal{T}$. We showed above in Proposition 2.8 that $\mathcal{T}$ is saturated, and clearly every map in I' and J is a closed T_1 inclusion, so the desired result follows. □

Lemma 2.11 (Hovey Lemma 2.4.4)**.** *The weak equivalences in* **Top** *are closed under retracts and satisfy 2-of-3 axiom (so that in particular the weak equivalences form a subcategory, as clearly identities are weak equivalences).*

Proof. First we show that weak equivalences satisfy 2-of-3. Let $f : X \to Y$ and $g : Y \to Z$ be continuous functions of topological spaces.

First of all, suppose f and g are both weak equivalences. Then by functoriality of π_n, since $\pi_n(f, x)$ and $\pi_n(g, f(x))$ are isomorphisms for all $x \in X$, $\pi_n(g \circ f, x) = \pi_n(g, f(x)) \circ \pi_n(f, x)$ is likewise an isomorphism for all $x \in X$, so that $g \circ f$ is a weak equivalence.

Now, suppose that $g \circ f$ and g are weak equivalences. Pick a point $x \in X$. We wish to show that $\pi_n(f, x) : \pi_n(X, x) \to \pi_n(Y, f(x))$ is an isomorphism for all $n \geq 0$. We know that $\pi_n(g \circ f, x)$ is an isomorphism, and $\pi_n(g, f(x))$ is an isomorphism, say with inverse, φ, so that

$$\varphi \circ \pi_n(g \circ f, x) = \varphi \circ \pi_n(g, f(x)) \circ \pi_n(f, x) = \pi_n(f, x)$$

is an isomorphism, as it is a composition of isomorphisms.

Now, suppose that $g \circ f$ and f are weak equivalences. Pick a point $y \in Y$. Since $\pi_0(f)$ is an isomorphism, there exists a point $x \in X$ such that $f(x)$ belongs to the path component containing y, so that there exists some $\alpha : I \to Y$ with $\alpha(0) = f(x)$ and $\alpha(1) = f(y)$. Then consider the following diagram

$$\begin{tikzcd}
\pi_n(Y, y) \arrow[r, "\pi_n(g,y)"] \arrow[d] & \pi_n(Z, g(y)) \arrow[d] \\
\pi_n(Y, f(x)) \arrow[r, "\pi_n(g,f(x))"] & \pi_n(Z, g(f(x)))
\end{tikzcd}$$

where the left arrow is the isomorphism given by conjugation by the path α, and the right arrow is the isomorphism given by conjugation by the path $g \circ \alpha$. It is tedious yet straightforward to verify that the diagram commutes. Furthermore, we know that $\pi_n(f, x)$ and $\pi_n(g \circ f, x) = \pi_n(g, f(x)) \circ \pi_n(f, x)$ are isomorphisms for all n, so that if we denote the inverse of $\pi_n(f, x)$ by φ, then

$$\pi_n(g \circ f, x) \circ \varphi = \pi_n(g, f(x)) \circ \pi_n(f, x) \circ \varphi = \pi_n(g, f(x))$$

is an isomorphism, as it is given as a composition of isomorphisms. Hence, the top arrow must likewise be an isomorphism, precisely the desired result.

The fact that weak equivalences in **Top** are closed under retracts is entirely straightforward and follows from the fact that the functors π_n preserve retract diagrams and that the class of isomorphisms in any category is closed under retracts. □

Proposition 2.12 (Hovey 2.4.9)**.** ${}_\perp(J_\perp) \subseteq \mathcal{W} \cap {}_\perp(I'_\perp)$.

Proof. First, in order to show ${}_\perp(J_\perp) \subseteq {}_\perp(I'_\perp)$, It suffices to show that $J \subseteq I'$-cell, as by Lemma 1.11 we would have $J \subseteq {}_\perp(I'_\perp)$, and

$$J \subseteq {}_\perp(I'_\perp) \implies {}_\perp(J_\perp) \subseteq {}_\perp(({}_\perp(I'_\perp))_\perp) = {}_\perp(I'_\perp),$$

where the implication and equality both follow from Lemma 1.8 which gives that

$$A \subseteq B \implies {}_\perp(A_\perp) \subseteq {}_\perp(B_\perp) \quad \text{and} \quad ({}_\perp(A_\perp))_\perp = A_\perp.$$

Now, to show $J \subseteq I'$-cell, first consider the composition $j_n : D^n \hookrightarrow S^n \hookrightarrow D^{n+1}$, where the first map is the pushout

$$\begin{tikzcd} S^{n-1} \arrow[r, hook] \arrow[d, hook] & D^n \arrow[d] \\ D^n \arrow[r] \arrow[ur, phantom, "\lrcorner"] & S^n \end{tikzcd}$$

obtained by gluing two copies of D^n along their boundary, and the second map map is simply the inclusion $S^n \hookrightarrow D^{n+1}$, which can be written as the pushout

$$\begin{tikzcd} S^n \arrow[r, equal] \arrow[d, hook] & S^n \arrow[d, hook] \\ D^{n+1} \arrow[r, equal] \arrow[ur, phantom, "\lrcorner"] & D^{n+1} \end{tikzcd}$$

It can be seen that j_n includes D^n as a hemisphere of $S^n = \partial D^{n+1} \subseteq D^{n+1}$. Note that $D^n \times I$ is homeomorphic to D^{n+1} ("smooth out" the sharp edges of the cylinder) via some homeomorphism $h_n : D^{n+1} \to D^n \times I$, and in particular, we may define h_n so that $h_n(j_n(D^n)) = D^n \times \{0\} \subseteq D^n \times I$ by squashing the hemisphere $j_n(D^n)$ to be one of the faces of the cylinder $D^n \times I$, in which case $h_n \circ j_n : D^n \to D^n \times I$ is precisely the inclusion $D^n \hookrightarrow D^n \times I$ sending $x \mapsto (x, 0)$, and since $j_n \in I'$-cell, $h_n \circ j_n \in I'$-cell by Lemma 1.10.

Now, we claim that ${}_\perp(J_\perp) \subseteq \mathcal{W}$. First note that by Corollary 1.13 and Proposition 2.10, every map in ${}_\perp(J_\perp)$ is a retract of an element of J-cell. Furthermore, we know that $\mathcal{W}$ is closed under retracts (Lemma 2.11), so that it suffices to show that J-cell $\subseteq \mathcal{W}$. We claim it suffices to show that pushouts of maps in J are weak equivalences. Supposing we had shown this, we would have that pushouts of maps in J are weak equivalences and T_1 inclusions, as $J \subseteq \mathcal{T}$ and $\mathcal{T}$ is saturated by Proposition 2.8. Then by Lemma 2.9, we would have that J-cell $\subseteq \mathcal{W} \cap \mathcal{T}$, precisely the desired result.

Now, let $\mathcal{S}$ be the class of *inclusions of a deformation retract*, i.e., those **injective** maps $i : A \to B$ such that there exists a homotopy $H : B \times I \to B$ with $H(i(a), t) = i(a)$ for all $a \in A$, $H(b, 0) = b$ for all $b \in B$, and $H(b, 1) = i(r(b))$ for all $b \in B$ for some map $r : B \to A$[2]. We will show the following:

(1) $\mathcal{S} \subseteq \mathcal{W}$.

It suffices to show that if $i : A \to B$ belongs to $\mathcal{S}$, then i is a homotopy equivalence. Indeed, given $i : A \to B$, let $H : B \times I \to B$ and $r : B \to A$ be a homotopy and retract satisfying the conditions above. Then in particular, H is a homotopy between id_B (at time $t = 0$) and $i \circ r$ (at time $t = 1$). It

[2]Hovey has a typo here, namely, he does not specify that i must be injective. Without this specification, his assertion fails. For example, take $A = \mathbb{R}^2$, $B = \mathbb{R}$, $i(x, y) = x$, $H(b, t) = b$, and $r(b) = (b, 0)$. Then i is an inclusion of a deformation retract according to Hovey's "definition," but i is not injective and r is not a retract.

remains to show that $r \circ i = \mathrm{id}_A$. First of all, note that since $H(b,1) = i(r(b))$ for all $b \in B$, we have $H(i(a),1) = i(r(i(a)))$. Yet, we also know that $H(i(a),t) = i(a)$ for all $t \in I$, so $i(r(i(a))) = i(a)$, and i is injective so $r(i(a)) = a$.

(2) $J \subseteq \mathcal{S}$.

For $n \geq 0$, let $j_n : D^n \hookrightarrow D^n \times I$ denote the inclusion of D^n as the subset $D^n \times \{0\}$. Define a deformation retract $H : D^n \times I \times I \to D^n \times I$ by $(x,s,t) \mapsto (x, s(1-t))$. Then indeed we have $H(j_n(x),t) = H(x,0,t) = (x,0) = j_n(x)$ for all $x \in D^n$, $H(x,t,0) = (x,t(1-0)) = (x,t)$ for all $(x,t) \in D^n \times I$, and $H(x,t,1) = (x,t(1-1)) = (x,0) = j_n(r(x))$ for all $(x,t) \in D^n \times I$, where $r : D^n \times I \to D^n$ is the projection onto time zero sending $(x,t) \mapsto (x,0)$. Finally, j_n is clearly injective. Thus, indeed $J \subseteq \mathcal{S}$.

(3) $\mathcal{S}$ is closed under pushouts.

Suppose we are given a pushout diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"'] & C \arrow[d, "j"] \\ B \arrow[r, "g"'] & D \arrow[ul, phantom, "\lrcorner", very near start] \end{tikzcd}$$

where $i \in \mathcal{S}$. Then we wish to show j in $\mathcal{S}$. First, injectivity. Suppose for the sake of a contradiction there existed nonequal $c, c' \in C$ such that $j(c) = j(c')$. Define $X := \{1,2,3\}$ (with the indiscrete topology, if you like), $h : C \to X$ by $c \mapsto 1$, $c' \mapsto 2$, and $C \setminus \{c,c'\} \mapsto 3$, and $k : B \to X$ by $i(f^{-1}(c)) \mapsto 1$, $i(f^{-1}(c')) \mapsto 2$, and $i(f^{-1}(C \setminus \{c,c'\})) \mapsto 3$. Then it is straightforward to see that $h \circ f = k \circ i$. Thus, there must exist a (unique) function $\ell : D \to X$ such that $\ell \circ j = h$ and $\ell \circ g = k$. But then we would have $h(c) = \ell(j(c)) = \ell(j(c')) = h(c')$ since $j(c) = j(c')$, a contradiction of the fact that $h(c) \neq h(c')$. Hence, j must be injective. Now, we look to construct H and r. Let $K : B \times I \to B$ and $r' : B \to A$ be maps satisfying the conditions for i to be an inclusion of a deformation retract.

We wish to define a homotopy $H : D \times I \to D$. Then I is a locally compact Hausdorff space (in particular, it is compact and Hausdorff), so that the functor $- \times I : \mathbf{Top} \to \mathbf{Top}$ preserves colimits (Proposition 2.1), meaning the following is a pushout diagram:

$$\begin{tikzcd} A \times I \arrow[r, "f \times \mathrm{id}_I"] \arrow[d, "i \times \mathrm{id}_I"'] & C \times I \arrow[d, "j \times \mathrm{id}_I"] \\ B \times I \arrow[r, "g \times \mathrm{id}_I"'] & D \times I \arrow[ul, phantom, "\lrcorner", very near start] \end{tikzcd}$$

Then by the universal property of the pushout, there is a map $H : D \times I \to D$ (the dashed line) such that the following diagram commutes

$$\begin{tikzcd} A \times I \arrow[r, "f \times \mathrm{id}_I"] \arrow[d, "i \times \mathrm{id}_I"'] & C \times I \arrow[d, "j \times \mathrm{id}_I"] \arrow[dr, bend left, "\pi_1"] & \\ B \times I \arrow[r, "g \times \mathrm{id}_I"'] \arrow[dr, bend right, "K"'] & D \times I \arrow[ul, phantom, "\lrcorner", very near start] \arrow[dr, dashed, "H"] & C \arrow[d, "j"] \\ & B \arrow[r, "g"'] & D \end{tikzcd}$$

Now, note $r' \circ i = \mathrm{id}_A$. Indeed, given $a \in A$, we have $i(r'(i(a))) = K(i(a),t) = i(a)$ and i is injective, so that $r'(i(a)) = a$, as desired. Hence, there exists a unique map $r : D \to C$ (the dashed line) such that the following diagram commutes:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"'] & C \arrow[d, "j"] \arrow[ddr, bend left, equal] & \\ B \arrow[r, "g"'] \arrow[dr, bend right, "r'"'] & D \arrow[ul, phantom, "\lrcorner", very near start] \arrow[dr, dashed, "r"] & \\ & A \arrow[r, "f"'] & C \end{tikzcd}$$

Now we claim that our constructions H and r endue j with the structure of an inclusion of a deformation retract, as desired. First $c \in C$, we wish to show $H(j(c), t) = j(c)$ for all t. Indeed, we have

$$H(j(c), t) = H(j \times \mathrm{id}_I(c, t)) = j(\pi_1(c, t)) = j(c).$$

Given $d \in D$, we want to show $H(d, 0) = d$. By the explicit description of the colimit in **Top**, we know that every element of D is in the image of either j or g. If $d = j(c)$ for some c, then we have just shown $H(d, 0) = H(j(c), 0) = j(c) = d$, as desired. On the other hand, if $d = g(b)$ for some $b \in B$ we have

$$H(d, 0) = H(g \times \mathrm{id}_I(b, 0)) = g(K(b, 0)) = g(b) = d.$$

Finally, we claim that $H(d, 1) = j(r(d))$ for all $d \in D$. If $d = j(c)$ for some $c \in C$, then we have

$$H(d, 1) = H(j(c), 1) = j(c) = j(r(j(c))) = j(r(d)),$$

as desired. On the other hand, if $d = g(b)$ for some $b \in B$, then

$$H(d, 1) = H(g \times \mathrm{id}_I(b, 1)) = g(K(b, 1)) = g(i(r'(b))) = j(f(r'(b))) = j(r(g(b))) = j(r(d)).$$ ☐

Proposition 2.13 (Hovey 2.4.10)**.** $I'_\perp \subseteq \mathcal{W} \cap J_\perp$

Proof. First, by Proposition 2.12 we know ${}_\perp(J_\perp) \subseteq {}_\perp(I'_\perp)$, and this implies $I'_\perp \subseteq J_\perp$, as by Lemma 1.8 we have

$${}_\perp(J_\perp) \subseteq {}_\perp(I'_\perp) \implies J_\perp = ({}_\perp(J_\perp))_\perp \supseteq ({}_\perp(I'_\perp))_\perp = I'_\perp.$$

Thus, it suffices to show that $I'_\perp \subseteq \mathcal{W}$. Now, suppose $p : (X, x_0) \to (Y, p(x_0))$ is in $I'_\perp$. We wish to show that the map $\pi_n(p, x_0) : \pi_n(X, x_0) \to \pi_n(Y, p(x_0))$ is an isomorphism for all n.

First we show that $\pi_n(p, x_0)$ is surjective. Let $g : (S^n, *) \to (Y, p(x_0))$ be a map. Then we have the following commutative diagram

$$\begin{tikzcd} * \arrow[r] \arrow[d] & X \arrow[d, "p"] \\ S^n \arrow[r, "g"] & Y \end{tikzcd}$$

where the top arrow picks out x_0. Note that the map $* \to S^n$ may be realized as a pushout of the diagram $D^n \leftarrow S^{n-1} \to *$, so that $* \to S^n$ belongs to I'-cell, and therefore ${}_\perp(I'_\perp)$ by Lemma 1.11, and $p \in I'_\perp$, so $* \to S^n$ has the left lifting property against p. Thus, the above diagram has a lift $f : (S^n, *) \to (X, x_0)$ such that $p \circ f = g$, so that $\pi_n(p, x_0)([f]) = [p \circ f] = [g]$, as desired.

Finally, we show that $\pi_n(p, x_0)$ is injective. Suppose we have two maps $f, g : (S^n, *) \to (X, x_0)$ such that $p \circ f$ and $p \circ g$ represent the same element of $\pi_n(Y, p(x_0))$. Then there is a homotopy $H : S^n \times I \to Y$ such that for all $s \in S^n$ and $t \in I$, $H(s, 0) = p(f(s))$, $H(s, 1) = p(g(s))$, and $H(*, t) = p(x_0)$. By the universal property of the quotient, H induces a map $\overline{H} : S^n \wedge I_+ := (S^n \times I)/(* \times I)$ sending the equivalence class $[s, t] \mapsto H(s, t)$. Hence, the following diagram commutes:

$$\begin{tikzcd} S^n \vee S^n \arrow[r, "f \vee g"] \arrow[d, hook] & X \arrow[d, "p"'] \\ S^n \wedge I_+ \arrow[r, "\overline{H}"] & Y \end{tikzcd}$$

where the left arrow is an element of I'-cell, as it may be obtained by attaching an $n+1$ cell to $S^n \vee S^n$ (when $n = 0$, the attachning map is obvious; when $n > 0$, the attaching map is the quotient map $S^n \twoheadrightarrow S^n \vee S^n$ obtained by collapsing the equator). Thus, by similar reasoning to above there exists a lift $\overline{K} : S^n \wedge I_+ \to X$. Then if we define K to be the composition $S^n \times I \twoheadrightarrow S^n \wedge I_+ \xrightarrow{\overline{K}} X$, this gives us the desired homotopy between f and g: given $s \in S^n$ and $t \in I$, we have $K(s, 0) = \overline{K}([s, 0]) = f(s)$, $K(s, 1) = \overline{K}([s, 1]) = g(s)$, and $K(*, t) = \overline{K}([*, t])$ ☐

Proposition 2.14 (Hovey 2.4.12)**.** $\mathcal{W} \cap J_\perp \subseteq I'_\perp$

Proof. TODO. ☐

Corollary 2.15 (Hovey 2.4.14)**.** *Every topological space is fibrant, i.e., given a space X, the unique map $X \to *$ is an element of $J_\perp$.*

Proof. TODO. □

Questions/Comments:

(1) It bother me that the only explanation Hovey gives for what I proved in Lemma 2.4 is that it "follows by transfinite induction" (pg. 49 in "proof" of Lemma 2.4.1). Also, my original construction of V did NOT work.

(2)