MODEL STRUCTURES

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This document follows Mark Hovey's *Model Categories*, and its intention is to reproduce the proofs of several standard model categories in explicit detail.

1. Preliminaries

We work with von Neumann ordinals, i.e., an ordinal is a transitive set of ordinals (this definition is not circular, the empty set is an ordinal which we call "0"). In the following discussion, let α and β be ordinals. We write $\alpha + 1$ to denote the successor ordinal $\alpha \cup \{\alpha\}$. We write $\alpha < \beta$ to mean $\alpha \in \beta$, and $\alpha \leq \beta$ denotes any of the equivalent conditions: (1) $\alpha < \beta$ or $\alpha = \beta$, (2) $\alpha \in \beta + 1$, (3) $\alpha \subseteq \beta$. Given a collection of ordinals B, we write $\sup B$ or $\sup_{\beta \in B} \beta$ to denote the ordinal $\bigcup_{\beta \in B} \beta$. We define the sum of ordinals α and β recursively: $\alpha + 0 := \alpha$, $\alpha + (\beta + 1) := (\alpha + \beta) + 1$, and $\alpha + \beta := \sup_{\delta < \beta}(\alpha + \delta)$ when β is a limit ordinal. Note that addition of ordinals is not commutative, but it is associative, and continuous in its right argument: given an ordinal α and a collection of ordinals B, $\alpha + \sup B = \sup_{\beta \in B}(\alpha + \beta)$. We say an ordinal λ is a *limit ordinal* if either of the following equivalent conditions hold: (1) $\lambda = \sup_{\beta < \lambda} \beta$ or (2) $\lambda \neq \beta + 1$ for all ordinals β. Note that 0 is a limit ordinal under our definition. We may regard an ordinal α as a poset category, in which case the colimit in α is given by the supremum. We let **Ord** denote the poset category of all (small) ordinals, so there exists a unique arrow $\alpha \to \beta$ if $\alpha \leq \beta$. Given a set X, we write $|X|$ to denote its *cardinality*, i.e., $|X|$ is the least ordinal α such that there exists a bijection between α and X. A cardinal number is an an ordinal which is the cardinality of some set X.

Definition 1.1 (Hovey Definition 2.1.1)**.** Suppose $\mathcal{C}$ is a cocomplete category, and λ is an ordinal. A λ-*sequence* in $\mathcal{C}$ is a colimit-preserving functor $X : \lambda \to \mathcal{C}$, commonly written as

$$X_0 \to X_1 \to \cdots \to X_\beta \to \cdots .$$

Since X preserves colimits, for all limit ordinals $\gamma < \lambda$, the arrows $X_\alpha \to X_\gamma$ for $\alpha < \gamma$ form a colimit cone under $\{X_\alpha\}_{\alpha<\gamma}$. We refer to the map $X_0 \to \operatorname{colim}_{\beta<\lambda} X_\beta$ as the *composition* of the λ-sequence. Given a collection $\mathcal{D}$ of morphisms in $\mathcal{C}$ such that every map $X_\beta \to X_{\beta+1}$ for $\beta + 1 < \lambda$ is in $\mathcal{D}$, we refer to the composition $X_0 \to \operatorname{colim}_{\beta<\lambda} X_\beta$ as a *transfinite composition* of arrows in $\mathcal{D}$.[1]

Of particular importance to us will be collections of arrows which are *closed under transfinite composition*, i.e., collections $\mathcal{D}$ for which given any ordinal λ and λ-sequence X of arrows in $\mathcal{D}$, for any choice of colimit $\operatorname{colim} X$, the canonical map $X_0 \to \operatorname{colim} X$ is also in $\mathcal{D}$. We prove the following useful result about when a class of morphisms is closed under transfinite composition:

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[1]To be more precise, there may be different (isomorphic) choices of colimit $\operatorname{colim}_{\beta<\gamma} X_\beta$, which give rise to different choices of composition $X_0 \to \operatorname{colim}_{\beta<\gamma} X_\beta$. Thus, the composition of a λ-sequence is only unique up to composition by a unique isomorphism.

Lemma 1.2. *Let $\mathcal{C}$ be a category, and $\mathcal{D}$ a collection of arrows in $\mathcal{C}$ satisfying the following properties: $\mathcal{D}$ is closed under composition with isomorphisms, and given an ordinal λ and a λ-sequence $X : \lambda \to \mathcal{C}$ of arrows in $\mathcal{D}$ (so $X_\beta \to X_{\beta+1}$ belongs to $\mathcal{D}$ for all $\beta + 1 < \lambda$), if we then get then get for free that $X_\alpha \to X_\beta$ belongs to $\mathcal{D}$ for all $\alpha \leq \beta < \lambda$, then $\mathcal{D}$ is closed under transfinite composition.*

Proof. Let λ be an ordinal, and $X : \lambda \to \mathcal{C}$ a λ-sequence of arrows in $\mathcal{D}$. First, suppose $\lambda = \mu + 1$ is a successor ordinal. Since we know that any transfinite composition of X may be obtained from another by composing with an isomorphism and $\mathcal{D}$ is closed under composition with isomorphisms, it suffices to show there exists *some* transfinite composition of X belonging to $\mathcal{D}$. We know $\sup_{\beta<\lambda} \beta = \sup_{\beta<\mu+1} \beta = \mu$, and X is colimit preserving, so that X_μ is a colimit of the diagram X via the arrows $X_\alpha \to X_\mu$ for $\alpha < \lambda = \mu+1$. But we know in particular that $X_0 \to X_\mu$ belongs to $\mathcal{D}$, so we are done.

Conversely, suppose λ is a limit ordinal. Let $j : X \Rightarrow \underline{X_\lambda}$ be a colimit cone for X. We may use j to extend X to a $(\lambda + 1)$-sequence in the obvious way (so for $\alpha < \lambda$, the structure map $X_\alpha \to X_\lambda$ is given by j and the arrow $X_\lambda \to X_\lambda$ is the identity, as is necessary). Further note that X is still a sequence of arrows in $\mathcal{D}$, as given $\beta + 1 < \lambda + 1$, so $\beta + 1 \leq \lambda$, it is not possible that $\beta + 1 = \lambda$ as λ is a limit ordinal, in which case we know the map $X_\beta \to X_{\beta+1}$ belongs to $\mathcal{D}$ as $\beta + 1 < \lambda$. Hence, unravelling definitions and applying the asserted property of $\mathcal{D}$, we get for free that $j_0 : X_0 \to X_\lambda$ belongs to $\mathcal{D}$. $\square$

Lemma 1.3. *Given a cocomplete category $\mathcal{C}$ and a collection $\mathcal{D}$ of arrows in $\mathcal{C}$, if $\mathcal{D}$ is closed under transfinite composition, then given any limit ordinal λ and λ-sequence $X : \lambda \to \mathcal{C}$, for all $\alpha < \lambda$ the canonical map $X_\alpha \to \operatorname{colim} X$ belongs to $\mathcal{D}$.*

Proof Sketch. Let $\alpha < \lambda$, and fix a colimit cone $j : X \Rightarrow \underline{\operatorname{colim} X}$. Define $S := \{\beta : \alpha \leq \beta \leq \lambda\} \subseteq \lambda + 1$. Define a map $\phi : S \to \mathbf{Ord}$ via transfinite recursion. Let $\phi(\alpha) = 0$. Supposing $\phi(\beta)$ has been defined, let $\phi(\beta+1) = \phi(\beta)+1$. Finally, supposing $\alpha < \gamma \leq \lambda$ is a limit ordinal and $\phi(\beta)$ has been defined for $\alpha \leq \beta < \gamma$, define $\phi(\gamma) = \sup_{\alpha\leq\beta<\gamma} \phi(\beta)$. It is straightforward to verify that ϕ is order preserving, sends limit ordinals to limit ordinals, and satisfies $\alpha + \phi(\beta) = \beta$ for all $\alpha \leq \beta \leq \lambda$.

Now, construct a $\phi(\lambda)$-sequence $Y : \phi(\lambda) \to \mathcal{C}$ by $Y_\beta := X_{\alpha+\beta}$, and given $\beta \leq \beta' < \phi(\lambda)$, define the map $Y_\beta \to Y_{\beta'}$ to be the arrow $X_{\alpha+\beta} \to X_{\alpha+\beta'}$ for X. Checking that Y is functorial and colimit-preserving follows directly from the fact that X is functorial and colimit-preserving. Then it can be seen that the $j_{\alpha+\beta}$'s for $\beta < \phi(\lambda)$ restrict to a colimit cone under Y. Since Y is a $\phi(\lambda)$-sequence in $\mathcal{D}$ and $\mathcal{D}$ is closed under transfinite compositions, it follows that $j_\alpha \in \mathcal{D}$, as desired. $\square$

Definition 1.4 (Hovey Definition 2.1.2)**.** Let γ be a cardinal. An ordinal α is *γ-filtered* if it is a limit ordinal and, if $A \subseteq \alpha$ and $|A| \leq \gamma$, then $\sup A < \alpha$.

Given a cardinal γ, a γ-filtered category $\mathcal{C}$ is one such that any diagram $\mathcal{D} \to \mathcal{C}$ has a cocone when $\mathcal{D}$ has $< \gamma$ arrows. A catgory is just "filtered" if it is ω-filtered, i.e., if every finite diagram in $\mathcal{C}$ admits a cocone. Note that an ordinal α is γ-filtered precisely when it is γ-filtered as a category, and in particular every ordinal is ω-filtered.

Definition 1.5 (Hovey Definition 2.1.3)**.** Suppose $\mathcal{C}$ is a comcomplete category, $\mathcal{D} \subseteq \operatorname{Mor} \mathcal{C}$ is some collection of morphisms of $\mathcal{C}$, A is an object of $\mathcal{C}$, and κ is a cardinal. We say that A is *κ-small relative to* $\mathcal{D}$ if, for all κ-filtered ordinals λ and all λ-sequences

$$X_0 \to X_1 \to \cdots \to X_\beta \to \cdots$$

such that each map $X_\beta \to X_{\beta+1}$ is in $\mathcal{D}$ for $\beta + 1 < \lambda$, the canonical map of sets

$$\operatorname*{colim}_{\beta<\lambda} \mathcal{C}(A, X_\beta) \to \mathcal{C}(A, \operatorname*{colim}_{\beta<\lambda} X_\beta)$$

is an isomorphism. We say that A is *small relative to* $\mathcal{D}$ if it is κ-small relative to $\mathcal{D}$ for some κ. We say that A is *small* if it is small relative to $\mathcal{C}$ itself.

Definition 1.6 (Hovey Definition 2.1.4)**.** Suppose $\mathcal{C}$ is a cocomplete category, $\mathcal{D}$ is a collection of morphisms of $\mathcal{C}$, and A is an object of $\mathcal{C}$. We say that A is *finite relative to* $\mathcal{D}$ if A is κ-small relative to $\mathcal{D}$ for some finite cardinal κ. We say A is *finite* if it is finite relative to $\mathcal{C}$ itself. In this case, maps from A commute with colimits of arbitrary λ-sequences, as long as λ is a limit ordinal.

Remark 1.7. Recall that given a small category $\mathcal{D}$ and a functor $F : \mathcal{D} \to \mathbf{Set}$, we may explicitly construct the colimit of F as the set

$$\operatorname{colim} F := \left(\coprod_{d \in \mathcal{D}} F(d) \right) / \sim,$$

where the equivalence relation $\sim$ is **generated** by

$$((x \in F(d)) \sim (x' \in F(d'))) \quad \text{if} \quad (\exists (f : d \to d') \text{ with } Ff(x) = x').$$

In particular, if $\mathcal{D}$ is a filtered category then the resulting relation can be described as follows:

$$((x \in F(d)) \sim (x' \in F(d'))) \quad \text{iff} \quad (\exists\, d'', (f : d \to d''), (g : d' \to d'') \text{ with } Ff(x) = Fg(x')).$$

Then the colimit cone $\eta : F \Rightarrow \underline{\operatorname{colim} F}$ is defined by $\eta_d(x) = [x]$ for $d \in \mathcal{D}$ and $x \in F(d)$, where $[x]$ denotes the equivalence class of x in $\operatorname{colim} F$. Given a cone $\varepsilon : F \Rightarrow \underline{Y}$ under F, the unique map $\operatorname{colim} F \to Y$ maps an equivalence class $[x]$ represented by an element $x \in F(d)$ to the element $\varepsilon_d(x)$.

Now we unravel what the "canonical map" of Definition 1.5 is. Suppose we are given a cocomplete category $\mathcal{C}$, an element $A \in \mathcal{C}$, an ordinal λ, and a λ-sequence $X : \lambda \to \mathcal{C}$. For $\alpha \leq \beta < \lambda$, let $\iota_{\alpha,\beta}$ be the map $X_\alpha \to X_\beta$. Let $\eta : X \Rightarrow \underline{\operatorname{colim} X}$ be the colimit cone. By whiskering the colimit cone along the functor $\mathcal{C}(A, -)$, we get a cone $\mathcal{C}(A, \eta) : \{\mathcal{C}(A, X_\beta)\}_{\beta<\lambda} \Rightarrow \underline{\mathcal{C}(A, \operatorname{colim} X)}$. Then if we let $\varepsilon : \{\mathcal{C}(A, X_\beta)\}_{\beta<\lambda} \Rightarrow \underline{\operatorname{colim}_{\beta<\lambda} \mathcal{C}(A, X_\beta)}$ be the colimit cone, the universal property of the colimit gives us the canonical map $\ell : \operatorname{colim}_{\beta<\lambda} \mathcal{C}(A, X_\beta) \to \mathcal{C}(A, \operatorname{colim} X)$, so that the following diagram commutes:

$$\begin{tikzcd}
\mathcal{C}(A, X_0) \arrow[r, "(\iota_{0,1})_*"] \arrow[drr, "\varepsilon_0"] \arrow[ddrr, "(\eta_0)_*"'] & \mathcal{C}(A, X_1) \arrow[r, "(\iota_{1,2})_*"] \arrow[dr, "\varepsilon_1"] \arrow[ddr, "(\eta_1)_*"'] & \cdots \arrow[r] & \mathcal{C}(A, X_\beta) \arrow[r, "(\iota_{\beta,\beta+1})_*"] \arrow[dl, "\varepsilon_\beta"'] \arrow[ddl, "(\eta_\beta)_*"] & \cdots \\
& & \operatorname{colim}_{\beta<\lambda} \mathcal{C}(A, X_\beta) \arrow[d, dashed, "\ell"] & & \\
& & \mathcal{C}(A, \operatorname{colim} X) & &
\end{tikzcd}$$

In particular, by Remark 1.7, we know elements of $\operatorname{colim}_{\beta<\lambda} \mathcal{C}(A, X_\beta)$ are equivalence classes of arrows $f : A \to X_\beta$ for $\beta < \lambda$ under the relation $[f : A \to X_\beta] = [g : A \to X_{\beta'}]$ iff there exists $\beta'' \geq \beta, \beta'$ with $\iota_{\beta,\beta''} \circ f = \iota_{\beta',\beta''} \circ g$, and the map ε_β sends an arrow $f \in \mathcal{C}(A, X_\beta)$ to the element $[f]$. Then it follows that $\ell([f : A \to X_\beta]) = \eta_\beta \circ f$. Thus, this gives us the following result:

Remark 1.8. Given a cocomplete category $\mathcal{C}$, a collection $\mathcal{D}$ of arrows in $\mathcal{C}$, an object A in $\mathcal{C}$, and a cardinal κ, A is κ-small relative to $\mathcal{D}$, if, for all κ-filtered ordinals λ and all λ-sequences $X : \lambda \to \mathcal{C}$ such that the map $X_\beta \to X_{\beta+1}$ belongs to $\mathcal{D}$ for all $\beta + 1 < \lambda$, the following hold:

(i) Given arrows $f : A \to X_\alpha$ and $g : A \to X_\beta$ in $\mathcal{C}$, if f and g agree in the colimit (i.e., if the compositions $A \xrightarrow{f} X_\alpha \to \operatorname{colim} X$ and $A \xrightarrow{g} X_\beta \to \operatorname{colim} X$ are equal), then f and g are equal in some stage of the colimit (i.e., there exists $\gamma < \lambda$ with $\alpha, \beta \leq \gamma$ such that the compositions $A \xrightarrow{f} X_\alpha \to X_\gamma$ and $A \xrightarrow{g} X_\beta \to X_\gamma$ are equal).

(ii) Any arrow $f : A \to \operatorname{colim} X$ factors through some stage of the colimit (i.e., there exists $\beta < \lambda$ and an arrow $\widetilde{f} : A \to X_\beta$ such that the composition $A \xrightarrow{\widetilde{f}} X_\beta \to \operatorname{colim} X$ equals f).

In terms of the canonical map $\operatorname{colim}_{\beta<\lambda} \mathcal{C}(A, X_\beta) \to \mathcal{C}(A, \operatorname{colim} X)$, the first condition shows injectivity, while the second shows surjectivity.

We will use the characterization of smallness given by this remark whenever proving smallness arguments, as in the following example.

Example 1.9 (Hovey 2.1.5)**.** Every set is small. Indeed, if A is a set we claim that A is $|A|$-small. To see this, suppose λ is an $|A|$-filtered ordinal, and X is a λ-sequence of sets. First of all, by Remark 1.7, the elements of $\operatorname{colim} X$ are equivalence classes of elements $a \in X_\alpha$ where $a \in X_\alpha$ and $b \in X_\beta$ represent the same element of $\operatorname{colim} X$ iff there exists $\alpha, \beta \leq \gamma < \lambda$ so that a and b are sent to the same elements by the maps $X_\alpha \to X_\gamma$ and $X_\beta \to X_\gamma$, respectively. Now, we show the conditions of Remark 1.8.

First, we need to show that given $\alpha, \beta < \lambda$, if $f : A \to X_\alpha$ and $g : A \to X_\beta$ such that the compositions $\overline{f} : A \xrightarrow{f} X_\alpha \to \operatorname{colim} X$ and $\overline{g} : A \xrightarrow{g} X_\beta \to \operatorname{colim} X$ are equal, then f and g are equal in some stage of the colimit. For each $a \in A$, since $\overline{f}(a) = \overline{f}(g)$ in $\operatorname{colim} X$, by the above characterization of $\operatorname{colim} X$, there exists $\gamma_a < \lambda$ with $\alpha, \beta \leq \gamma_a$ such that $f(a)$ and $g(a)$ are sent to the same element in X_{γ_a} by the maps $X_\alpha \to X_{\gamma_a}$ and $X_\beta \to X_{\gamma_a}$, respectively. Then let $\gamma := \sup_{a \in A} \gamma_a$. Since $|\{\gamma_a\}_{a \in A}| \leq |A|$ and λ is $|A|$-filtered, necessarily $\gamma < \lambda$. Then clearly the compositions $A \xrightarrow{f} X_\alpha \to X_\gamma$ and $A \xrightarrow{g} X_\beta \to X_\gamma$ agree for all $a \in A$.

Secondly, we wish to show that given a map $f : A \to \operatorname{colim} X$, that f factors through $X_\beta \to \operatorname{colim} X$ for some $\beta < \lambda$. For each $a \in A$, by the explicit description of $\operatorname{colim} X$, there exists some $\beta_a < \lambda$ and some $x_a \in X_{\beta_a}$ such that $f(a) = [x_a]$. Then let $\beta := \sup_{a \in A} \beta_a$, so $\beta < \lambda$ as X is $|A|$-filtered. Now define $\widetilde{f} : A \to X_\beta$ like so: for $a \in A$, define $\widetilde{f}(a) \in X_\beta$ to be the image of x_a along the map $X_{\beta_a} \to X_\beta$. Then clearly the composition $f' : A \xrightarrow{\widetilde{f}} X_\beta \to \operatorname{colim} X$ is equal to f, by unravelling definitions.

Definition 1.10 (Hovey Definition 2.1.7)**.** Let I be a class of maps in a category $\mathcal{C}$.

(1) A map is *I-injective* if it has the right lifting property w.r.t. every map in I. The class of I-injective maps is denoted I-inj (or $I_\perp$).
(2) A map is *I-projective* if it has the left lifting property w.r.t. every map in I. The class of I-projective maps is denoted I-proj (or ${}_\perp I$).
(3) A map is an *I-cofibration* if it has the left lifting property w.r.t. every I-injective map. The class of I-cofibrations is the class (I-inj)-proj and is denoted I-cof (or ${}_\perp(I_\perp)$).
(4) A map is an *I-fibration* if it has the right lifting property w.r.t. every I-projective map. The class of I-fibrations is the class (I-proj)-inj and is denoted I-fib (or $({}_\perp I)_\perp$).

The following is asserted in Hovey on pg. 30 following Definition 2.1.7, but not proven. We provide a proof.

Lemma 1.11. *Given classes A and B of maps in a category $\mathcal{C}$ with $A \subseteq B$, we have $A \subseteq {}_\perp(A_\perp)$, $A \subseteq ({}_\perp A)_\perp$, $({}_\perp(A_\perp))_\perp = A_\perp$, ${}_\perp(({}_\perp A)_\perp) = {}_\perp A$, $A_\perp \supseteq B_\perp$, ${}_\perp A \supseteq {}_\perp B$, ${}_\perp(A_\perp) \subseteq {}_\perp(B_\perp)$, and $({}_\perp A)_\perp \subseteq ({}_\perp B)_\perp$.*

Proof. Each of these amount to unravelling definitions and are entirely straightforward. □

Definition 1.12 (Hovey Definition 2.1.9)**.** Let I be a set of maps in a cocomplete category $\mathcal{C}$. A *relative I-cell complex* is a transfinite composition of pushouts of elements of I. That is, if $f : A \to B$ is a relative I-cell complex, then there is an ordinal λ and a λ-sequence $X : \lambda \to \mathcal{C}$ such that f is the composition of X and such that, for each β such that $\beta + 1 < \lambda$, there is a pushout square

$$\begin{tikzcd} C_\beta \arrow[r] \arrow[d, "g_\beta"'] & X_\beta \arrow[d] \\ D_\beta \arrow[r] \arrow[ur, phantom, "\lrcorner" very near end] & X_{\beta+1} \end{tikzcd}$$

with $g_\beta \in I$. We denote the collection of relative I-cell complexes by I-cell. We say that $A \in \mathcal{C}$ is an *I-cell complex* if the map $0 \to A$ is a relative I-cell complex.

Lemma 1.13. *Let $\mathcal{C}$ be a category and I a class of morphisms in $\mathcal{C}$. Then I-*cell *is closed under composition with isomorphisms.*

Proof. Suppose that $f : B \to C$ is an element of I-cell, and $h : A \to B$ and $g : C \to D$ are isomorphisms in $\mathcal{C}$. We wish to show $f \circ h$ and $g \circ f$ are also elements of I-cell. Since $f \in I$-cell, there exists an ordinal λ, a λ-sequence X with $X_0 = B$, and a colimit cone $\eta : X \Rightarrow \underline{C}$, such that $\eta_0 = f$.

First of all, construct a new cone $\eta' : X \Rightarrow \underline{D}$ under X where $\eta'_\beta := g \circ \eta_\beta$. It is straightforward to verify that η' is a colimit cone for X since η is a colimit cone and g is an isomorphism. Thus, $g \circ f = g \circ \eta_0 = \eta'_0 \in I$-cell, as η'_0 is the composition of a sequence of pushouts of elements of I.

On the other hand, we may construct a new λ-sequence X' by defining $X'_0 = A$, $X'_\beta = X_\beta$ for all $0 < \beta < \lambda$, the map $X'_0 \to X'_\beta$ for $0 < \beta < \lambda$ to be the composition

$$A \xrightarrow{h} B = X_0 \longrightarrow X_\beta,$$

and the composition $X'_\alpha \to X'_\beta$ to simply be the same map $X_\alpha \to X_\beta$ for $0 < \alpha \leq \beta < \lambda$. It is straightforward to verify that defines a λ-sequence, and that we may define a colimit cone $\eta' : X' \Rightarrow \underline{C}$ by $\eta'_0 = \eta_0 \circ h = f \circ h$, and $\eta'_\beta = \eta_\beta$ for $0 < \beta < \lambda$. Furthermore, clearly for all $1 < \beta + 1 < \lambda$, we have the arrow $X'_\beta \to X'_{\beta+1}$ is a pushout of a map in I. Thus, in order to show $f \circ h \in I$-cell, it remains to show that the arrow $A = X'_0 \to X'_1 = X_1$ is a pushout of a map in I. Indeed, we know since $B = X_0 \to X_1$ is a pushout of a map $k : P \to Q$ in I, and it can be easily verified the diagram on the right is a pushout diagram:

$$\begin{tikzcd} P \arrow[r] \arrow[d, "k"'] & X_0 \arrow[d] \\ Q \arrow[r] & X_1 \arrow[ul, phantom, "\lrcorner", very near start] \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} P \arrow[r] \arrow[dd] & X_0 \arrow[r, "h^{-1}"] & X'_0 \arrow[d, "h"] \\ & & X_0 \arrow[d] \\ Q \arrow[rr] & & X'_1 \arrow[ul, phantom, "\lrcorner", very near start] \end{tikzcd}$$

□

Lemma 1.14 (Hovey 2.1.10). *Suppose I is a class of maps in a cocomplete category $\mathcal{C}$. Then I-cell $\subseteq {}_\perp(I_\perp)$.*

Proof. TODO. □

Theorem 1.15 (Small Object Argument, Hovey 2.1.14). *Suppose $\mathcal{C}$ is a cocomplete categroy, and I is a set of maps in $\mathcal{C}$. Suppose the domains of the maps of I are small relative to I-cell. Then there is a functorial factorization (γ, δ) on $\mathcal{C}$ such that for all morphisms $f \in \mathcal{C}$, the map $\gamma(f)$ is in I-cell and the map $\delta(f)$ is in I-inj.*

Proof. TODO. □

Corollary 1.16 (Hovey 2.1.15). *Suppose that I is a set of maps in a cocomplete category $\mathcal{C}$. Suppose as well that the domains of I are small relative to I-cell. Then given $f : A \to B$ in ${}_\perp(I_\perp)$, there is a $g : A \to C$ in I-cell such that f is a retract of g by a map which fixes A.*

Proof. TODO □

Definition 1.17 (Hovey Definition 2.1.17). Suppose $\mathcal{C}$ is a model category. We say that $\mathcal{C}$ is *cofibrantly generated* if there are sets I and J of maps such that:

1. The domains of the maps of I are small relative to I-cell;
2. The domains of the maps of J are small relative to J-cell;
3. The class of fibrations is $J_\perp$; and
4. The class of trivial fibrations is $I_\perp$.

We refer to I as the set of *generating cofibrations* and to J as the set of *generating trivial cofibrations.* A cofibrantly generated model category is *finitely generated* if we can choose the sets I and J above so that the domains and codomains of I and J are finite relative to I-cell.

Proposition 1.18 (Hovey Proposition 2.1.18). *Suppose $\mathcal{C}$ is a cofibrantly generated model category, with generating cofibrations I and generating trivial fibrations J.*

(a) The cofibrations form the class ${}_\perp(I_\perp)$.
(b) Every cofibration is a retract of a relative I-cell complex.
(c) The domains of I are small relative to the cofibrations.
(d) The trivial cofibrations form the class ${}_\perp(J_\perp)$.
(e) Every trivial cofibration is a retract of a relative J-cell complex.
(f) The domains of J are small relative to the trivial cofibrations.

If $\mathcal{C}$ is fibrantly generated, then the domains and codomains of I and J are finite relative to the cofibrations.

Proof. TODO. □

Theorem 1.19 (Hovey Theorem 2.1.19)**.** *Suppose $\mathcal{C}$ is a complete & cocomplete category. Suppose $\mathcal{W}$ is a subcategory of $\mathcal{C}$, and I and J are sets of maps of $\mathcal{C}$. Then there is a cofibrantly generated model structure on $\mathcal{C}$ with I as the set of generating cofibrations, J as the set of generating trivial fibrations, and $\mathcal{W}$ as the subcategory of weak equivalences if and only if the following conditions are satisfied.*

1. *The subcategory $\mathcal{W}$ has the 2-of-3 property and is closed under retracts.*
2. *The domains of I are small relative to I-*cell.
3. *The domains of J are small relative to J-*cell.
4. *J-cell $\subseteq \mathcal{W} \cap {}_\perp(I_\perp)$.*
5. *$I_\perp \subseteq \mathcal{W} \cap J_\perp$.*
6. *Either $\mathcal{W} \cap {}_\perp(I_\perp) \subseteq {}_\perp(J_\perp)$ or $\mathcal{W} \cap J_\perp \subseteq I_\perp$.*

Proof. TODO. □

Definition 1.20. Let $\mathcal{C}$ be a category and I a collection of morphisms in $\mathcal{C}$. Then if I is closed under transfinite composition, pushouts, and retracts then we say I is *saturated*.

2. Topological Spaces

A map $f : X \to Y$ in **Top** is an *inclusion* if it is continuous, injective, and for all $U \subseteq X$ open, there is some $V \subseteq Y$ open such that $f^{-1}(V) = U$. If f is a closed inclusion and every point in $Y \setminus f(X)$ is closed, then we call f a *closed T_1 inclusion*. We will let $\mathcal{T}$ denote the class of closed T_1 inclusions in **Top**.

The symbol D^n will denote the unit disk in $\mathbb{R}^n$, and the symbol S^{n-1} will denote the unit sphere in $\mathbb{R}^n$, so that we have the boundary inclusions $S^{n-1} \hookrightarrow D^n$. In particular, for $n = 0$ we let $D^0 = \{0\}$ and $S^{-1} = \emptyset$.

Recall: If $F : \mathcal{J} \to \mathbf{Top}$ is a functor, where $\mathcal{J}$ is a small category, the limit of F is obtained by taking the limit in the category of sets, and then topologizing it with the *initial topology*, where if $\eta : \underline{\lim F} \Rightarrow F$ is the limit cone, then the topology on $\lim F$ is that with subbasis given by sets of the form $\eta_j^{-1}(U)$ where $j \in \mathcal{J}$ and $U \subseteq F_j$ is open. Similarly, the colimit of F is obtained by taking the colimit $\operatorname{colim} F$ in the category of sets and endowing it with the *final topology*, where a set $U \subseteq \operatorname{colim} F$ is open if and only if $\varepsilon_j^{-1}(U)$ is open in F_j for all $j \in \mathcal{J}$, where $\varepsilon : F \Rightarrow \underline{\operatorname{colim} F}$ is the colimit cone.

Given a space X, we construct a functor $(-)^X : \mathbf{Top} \to \mathbf{Top}$ as follows: Given a space Y, define Y^X to be the space whose underlying set is the set $\mathbf{Top}(X, Y)$ of continuous maps $X \to Y$, and the topology on Y^X is the *compact-open topology*, i.e., the topology with subbasis given by the sets of the form

$$S(K, U) := \{f \in \mathbf{Top}(X, Y) : f(K) \subseteq U\}$$

for $K \subseteq X$ compact and $U \subseteq Z$ open. Given a continuous map $f : Y \to Z$, define the induced map $f_* : Y^X \to Z^X$ by $f_*(g) := f \circ g$. Unravelling definitions, we have that given $f : Y \to Z$ continuous, $f_*^{-1}(S(K, U)) = S(K, f^{-1}(U))$ for all $K \subseteq X$ compact and $U \subseteq Z$ open, so that f_* is continuous. Furthermore, $(-)^X$ is clearly functorial, by associativity and unitality of function composition.

Given a topological space X, we say that X is *locally compact* if for all points $x \in X$ and open neighborhoods U of x, there exists an open set $V \subseteq X$ with $x \in V$, $\overline{V} \subseteq U$, and $\overline{V}$ compact. We claim that $(-)^X$ is right adjoint to $- \times X$ when X is locally compact and Hausdorff.

Proposition 2.1. *If X is a locally compact Hausdorff space, then functor $- \times X$ is left adjoint to $(-)^X$ (so that in particular $- \times X$ preserves colimits).*

Proof. We start by constructing the counit and unit of the adjunction. Given a space Z, define the counit $\varepsilon_Z : X \times Z^X \to Z$ to be the evaluation function, taking a pair $(x, f) \mapsto f(x)$. First, we claim ε_Z is continuous. Suppose we are given an open set $V \subseteq Z$ and a point $(x, f) \in \varepsilon_Z^{-1}(U)$ (so $f(x) \in V$). Since f is continuous and X is locally compact, there exists an open set $U \subseteq X$ containing x such that $x \in U \subseteq \overline{U} \subseteq f^{-1}(V)$ with $\overline{U}$ compact. Then consider the open set $U \times S(\overline{U}, V)$ in $X \times Y^X$. First of all, $(x, f) \in U \times S(\overline{U}, V)$, as $x \in U$ and $\overline{U} \subseteq f^{-1}(V)$, so that $f(\overline{U}) \subseteq V$ meaning $f \in S(\overline{U}, V)$. Furthermore, given $(y, g) \in U \times S(\overline{U}, V)$, we have $\varepsilon_Z(y, g) = g(y) \in g(U) \subseteq g(\overline{U}) \subseteq V$, so $U \times S(\overline{U}, V)$ is an open neighborhood of x contained in $\varepsilon_Z^{-1}(V)$, as desired. Hence, ε_Z is continuous. It remains to show naturality. Given a map $f : Z \to W$, we wish to show the following diagram commutes:

$$\begin{tikzcd} X \times Z^X \arrow[r, "\varepsilon_Z"] \arrow[d, "\mathrm{id}_X \times f_*"'] & Z \arrow[d, "f"] \\ X \times W^X \arrow[r, "\varepsilon_W"] & W \end{tikzcd}$$

Indeed, chasing an element (x, g) around the diagram yields:

$$\begin{tikzcd} (x, g) \arrow[r, mapsto] \arrow[d, mapsto] & g(x) \arrow[d, mapsto] \\ (x, f \circ g) \arrow[r, mapsto] & f(g(x)) \end{tikzcd}$$

so it does indeed commute.

Now we wish to define the unit $\eta_Y : Y \to (Y \times X)^X$. Given $y \in Y$, define $\eta_Y(y) \in (Y \times X)^X$ by $\eta_Y(y)(x) := (y, x)$. First of all, for it to be true that $\eta_Y(y) \in (X \times Y)^X$, it must be true that $\eta_Y(y)$ is continuous. Indeed, this is clear as η_Y is obtained as the product map $y \times \mathrm{id}_X : X \to Y \times X$, where y represents the constant function on y (which is obviously continuous). Furthermore, η_Y itself is continuous: given $K \subseteq X$ compact and $U \subseteq Y \times X$ open, we wish to show that $\eta_Y^{-1}(S(K, U))$ is open in Y. It suffices to show that given $y \in \eta_Y^{-1}(S(K, U))$, there exists an open neighborhood W of y that is mapped by η_Y into $S(K, U)$. Since $y \in \eta_Y^{-1}(S(K, U))$, $\eta_Y(y)(K) = \{y\} \times K \subseteq U$. Then $U \cap (Y \times K)$ is an open set in the subspace $Y \times K$ containing the slice $\{y\} \times K$. By definition of the product topology, for each $k \in K$, there exist open sets $W_k \subseteq Y$ and $V_k \subseteq K$ such that $(y, k) \in W_k \times V_k \subseteq U \cap (Y \times K)$. Then the V_k's form an open cover of K, which is compact, so that there exist $k_1, \ldots, k_n \in K$ with $V_{k_1} \cup \cdots \cup V_{k_n} = K$. Hence if we define $W := W_{k_1} \cap \cdots \cap W_{k_n}$, then $\{y\} \times K \subseteq W \times K \subseteq U \cap (Y \times K)$, and W is open in Y as it is a finite intersection of open sets. Then for all $w \in W$, $\eta_Y(w)(K) = \{w\} \times K \subseteq W \times K \subseteq U$. Hence, indeed η_Y is continuous. It remains to show naturality. Given a map $f : Y \to W$, we wish to show the following diagram commutes:

$$\begin{tikzcd} Y \arrow[r, "\eta_Y"] \arrow[d, "f"'] & (Y \times X)^X \arrow[d, "(f \times \mathrm{id}_X)_*"] \\ W \arrow[r, "\eta_W"] & (W \times X)^X \end{tikzcd}$$

Indeed, chasing an element y around the top of the diagram yields the function obtained as the composition $x \mapsto (y, x) \mapsto f \times \mathrm{id}_X(y, x) = (f(y), x)$, while chasing around the bottom of the diagram more directly yields the function $x \mapsto (f(y), x)$.

Now that we have constructed the unit and counit, it remains to verify the counit-unit equations, i.e., that for each $Y \in \mathbf{Top}$ that $\varepsilon_{Y \times X} \circ (\eta_Y \times \mathrm{id}_X) = \mathrm{id}_{Y \times X}$ and $(\varepsilon_Y)_* \circ \eta_{Y^X} = \mathrm{id}_{Y^X}$. First of all, given $(y, x) \in Y \times X$, we have

$$(\varepsilon_{Y \times X} \circ (\eta_Y \times \mathrm{id}_X))(y, x) = \varepsilon_{X \times Y}(\eta_Y(y), x) = \eta_Y(y)(x) = (y, x).$$

On the other hand, given $f \in Y^X$, we have

$$(\varepsilon_Y)_*(\eta_{Y^X}(f)) = (\varepsilon_Y)_*([x \mapsto (f, x)]) = [x \mapsto (f, x) \mapsto \varepsilon_Y(f, x) = f(x)] = f.$$

Hence, indeed ε and η form the counit and unit for the adjoint pair $(- \times X, (-)^X)$. □

Now that we have gotten some topological preliminaries out of the way, we are ready to define the model structure.

Definition 2.2. A map $f : X \to Y$ in **Top** is called a *weak equivalence* if

$$\pi_n(f, x) : \pi_n(X, x) \to \pi_n(Y, f(x))$$

is an isomorphism for all $n \geq 0$ and for all $x \in X$. We will write $\mathcal{W}$ to refer to the class of all weak equivalences in **Top**.

Define the set of maps I' to consist of all the boundary inclusion $S^{n-1} \hookrightarrow D^n$ for all $n \geq 0$, and define the set J to consist of all the inclusions $D^n \hookrightarrow D^n \times I$ mapping $x \mapsto (x, 0)$ for $n \geq 0$. Then a map f will be called a *cofibration* if it is in I'-cof $= {}_\perp(I'_\perp)$, and a *fibration* if it is in J-inj $= J_\perp$.

A map in I'-cell is usually called a *relative cell complex*; a relative CW-complex is a special case of a relative cell complex, where, in particular, the cells can be attached in order of their dimension. Note that in particular maps of J are relative CW complexes, hence are relative I'-cell complexes. A fibration is often known as a *Serre fibration* in the literature.

Theorem 2.3 (Hovey Theorem 2.4.19). *There is a finitely generated model structure on* **Top** *with I' as the set of generating cofibrations, J as the set of generating trivial cofibrations, and the cofibrations, fibrations, and weak equivalences as above. Every object of* **Top** *is fibrant, and the cofibrant objects are retracts of relative cell complexes.*

Proof. We will apply Theorem 1.19 to get that there is a cofibrantly generated model structure on **Top** with I' as the set of generating cofibrations, J as the set of generating trivial fibrations, and $\mathcal{W}$ as the subcategory of weak equivalences. The six requirements outlined in the theorem will be verified like so:

1. $\mathcal{W}$ is a subcategory of $\mathcal{C}$ which has the 2-of-3 property and is closed under retracts: Lemma 2.12.
2. The domains of I' are small relative to I'-cell: Proposition 2.11.
3. The domains of J are small relative to J-cell: Proposition 2.11.
4. J-cell $\subseteq \mathcal{W} \cap {}_{\perp}(I'_{\perp})$: In Proposition 2.13, we will show ${}_{\perp}(J_{\perp}) \subseteq \mathcal{W} \cap {}_{\perp}(I'_{\perp})$, and by Lemma 1.14 J-cell $\subseteq {}_{\perp}(J_{\perp})$.
5. $I'_{\perp} \subseteq \mathcal{W} \cap J_{\perp}$: Proposition 2.14
6. $\mathcal{W} \cap J_{\perp} \subseteq I'_{\perp}$: Proposition 2.15

It will follow by the definition of a cofibrantly generated model structure (Definition 1.17) that the fibrations in this model structure are given by $J_{\perp}$, which is precisely how we defined it. By Proposition 1.18, the class of cofibrations will be given by ${}_{\perp}(I'_{\perp})$, which is likewise exactly how we defined them.

In Proposition 2.8, we will show that compact spaces are finite relative to the class $\mathcal{T}$ of closed T_1 inclusions. Hence, this model structure will be finitely generated, as the domains and codomains of I' and J are all compact, and by the reasoning given above we will have shown I'-cell $\subseteq \mathcal{T}$.

We will show that every object of **Top** is fibrant in Corollary 2.16. □

Lemma 2.4. *Let λ be an ordinal, and X a λ-sequence in* **Top**. *Then:*

(i) If X is a λ-sequence of injections, then $X_\alpha \to X_\beta$ is an injective for all $\alpha \leq \beta < \lambda$.
(ii) If X is a λ-sequence of inclusions, then the map $X_\alpha \to X_\beta$ is an inclusion for all $\alpha \leq \beta < \lambda$.
(iii) If X is a λ-sequence of closed T_1 inclusions, then the map $X_\alpha \to X_\beta$ is a closed T_1 inclusion for all $\alpha \leq \beta < \lambda$.

Proof. In what follows, given $\alpha \leq \beta < \lambda$, let $\iota_{\alpha,\beta}$ denote the map $X_\alpha \to X_\beta$.

(i) Let $\alpha < \lambda$. We perform a proof by transfinite induction on β for $\alpha \leq \beta < \lambda$ that $\iota_{\alpha,\beta} : X_\alpha \to X_\beta$ is injective. For the zero case, clearly $\iota_{\alpha,\alpha} = \mathrm{id}_\alpha$ is injective. Supposing $\iota_{\alpha,\beta}$ is injective for some $\alpha < \beta + 1 < \lambda$, we have $\iota_{\alpha,\beta+1} = \iota_{\beta,\beta+1} \circ \iota_{\alpha,\beta}$ is a composition of injections, and is therefore clearly injective itself. Finally, suppose γ is a limit ordinal with $\alpha \leq \gamma < \lambda$ such that $\iota_{\alpha,\beta}$ is injective for all $\alpha \leq \beta < \gamma$. We claim $\iota_{\alpha,\gamma}$ is injective. Since X_γ is colimit preserving and γ is a limit ordinal, X_γ is the colimit of the diagram $\{X_\beta\}_{\beta<\gamma}$ via the maps $\iota_{\beta,\gamma}$, so that in particular by Remark 1.7 and the fact that the forgetful functor **Top** $\to$ **Set** preserves colimits, given $a, b \in X_\alpha$ with $\iota_{\alpha,\gamma}(a) = \iota_{\alpha,\gamma}(b)$, there exists some $\beta < \gamma$ with $\iota_{\alpha,\beta}(a) = \iota_{\alpha,\beta}(b)$, and $\iota_{\alpha,\beta}$ is injective for all $\beta < \gamma$, so it must have been true $a = b$ in X_α.

(ii) By part(i), we know that $\iota_{\alpha,\beta}$ is injective for $\alpha \leq \beta < \lambda$. Thus it suffices to prove the following statement: For all $\alpha < \lambda$ and $U \subseteq X_\alpha$, for all $\alpha \leq \beta < \lambda$, there exists $U_\beta \subseteq X_\beta$ with $U_\alpha = U$ such that for all $\alpha \leq \beta' \leq \beta < \lambda$, $\iota^{-1}_{\beta',\beta}(U_\beta) = U_{\beta'}$. We prove this by transfinite recursion on $\alpha \leq \beta < \lambda$.

The zero case has been taken care of: $U_\alpha = U$. For the sucessor case, given $\alpha < \beta + 1 < \lambda$, supposing U_β has been defined with the desired properties, since $\iota_{\beta,\beta+1}$ is an inclusion, there exists $U_{\beta+1} \subseteq X_{\beta+1}$ with $\iota^{-1}_{\beta,\beta+1}(U_{\beta+1}) = U_\beta$. Then given $\alpha \leq \beta' \leq \beta + 1$, we have

$$\iota^{-1}_{\beta',\beta+1}(U_{\beta+1}) = (\iota_{\beta,\beta+1} \circ \iota_{\beta',\beta})^{-1}(U_{\beta+1}) = \iota^{-1}_{\beta',\beta}(\iota^{-1}_{\beta,\beta+1}(U_{\beta+1})) = \iota^{-1}_{\beta',\beta}(U_\beta) = U_{\beta'}.$$

Finally, the limit case. Suppose γ is a limit ordinal with $\alpha < \gamma \leq \lambda$, and suppose U_β has been constructed with the desired properties for $\alpha \leq \beta < \gamma$. We wish to define U_γ. Since X is colimit preserving and $\gamma = \sup_{\alpha \leq \beta < \gamma} \beta$, the maps $\iota_{\beta,\gamma}$ for $\alpha \leq \beta < \gamma$ form a colimit cone for the diagram $\{X_\beta\}_{\alpha \leq \beta < \gamma}$. Let $S = \{0, 1\}$ be the Sierpinski space whose open sets are $\{\emptyset, \{1\}, \{0, 1\}\}$. For

$\alpha \le \beta < \gamma$, define a map $s_\beta : X_\beta \to S$ mapping everything in U_β to 1 and every other point to 0. Each s_β is clearly continuous, as $s_\beta^{-1}(1) = U_\beta$. Furthermore, we claim the s_β's form a cone under the diagram $\{X_\beta\}_{\alpha\le\beta<\gamma}$, i.e., that given $\alpha \le \beta' \le \beta < \gamma$, the following diagram commutes

$$\begin{tikzcd} X_{\beta'} \arrow[rr, "\iota_{\beta',\beta}"] \arrow[dr, "s_{\beta'}"'] & & X_\beta \arrow[dl, "s_\beta"] \\ & S & \end{tikzcd}$$

To see this, let $x \in X_{\beta'}$. If $x \in U_{\beta'} = \iota_{\beta',\beta}^{-1}(U_\beta)$, then $\iota_{\beta',\beta}(x) \in U_\beta$, so $s_\beta(\iota_{\beta',\beta}(x)) = 1 = s_{\beta'}(x)$. Conversely, if $x \in X_{\beta'} \setminus U_{\beta'} = X_{\beta'} \setminus \iota_{\beta',\beta}^{-1}(U_\beta)$, then $x \notin \iota_{\beta',\beta}^{-1}(U_\beta)$, so $\iota_{\beta',\beta}(x) \notin U_\beta$, meaning $s_\beta(\iota_{\beta',\beta}(x)) = 0 = s_{\beta'}(0)$. Hence, the s_β's do indeed form a cone under $\{X_\beta\}_{\alpha\le\beta<\gamma}$, so by universal property of the colimit there exists a unique map $\ell : X_\gamma \to S$ such that $s_\beta = \ell\circ\iota_{\beta,\gamma}$ for all $\alpha \le \beta < \gamma$. Define $U_\gamma := \ell^{-1}(1)$, which is open as $\{1\}$ is open in S. It remains to show that for all $\alpha \le \beta \le \gamma$ that $\iota_{\beta,\gamma}^{-1}(U_\gamma) = U_\beta$. Indeed, we have

$$\iota_{\beta,\gamma}^{-1}(U_\gamma) = \iota_{\beta,\gamma}^{-1}(\ell^{-1}(1)) = (\ell\circ\iota_{\beta,\gamma})^{-1}(1) = s_\beta^{-1}(1) = U_\beta.$$

(iii) By part (ii), we know that $\iota_{\alpha,\beta}$ is an inclusion for $\alpha \le \beta < \lambda$.
TODO. □

This result, along with Lemma 1.2 gives the following Corollary:

Corollary 2.5. *The class of injective maps (resp. inclusions, closed T_1 inclusions) in* **Top** *is closed under transfinite composition.*

In turn, this Corollary and Lemma 1.3 gives:

Corollary 2.6. *Let λ be an ordinal, and X be a λ-sequence in* **Top**. *Then:*

(i) If X is a λ-sequence of injections, then the canonical map $X_\alpha \to \operatorname{colim} X$ is an injection for all $\alpha < \lambda$.
(ii) If X is a λ-sequence of inclusions, then the canonical map $X_\alpha \to \operatorname{colim} X$ is an inclusion for all $\alpha < \lambda$.
(iii) If X is a λ-sequence of closed T_1 inclusions, then the canonical map $X_\alpha \to \operatorname{colim} X$ is a closed T_1 inclusion for all $\alpha < \lambda$.

Lemma 2.7 (Hovey 2.4.1)**.** *Every topological space is small relative to the inclusions.*

Proof. We claim that every topological space A is $|A|$-small relative to the inclusions. We use the characterization of smallness afforded by Remark 1.8. Let λ be an $|A|$-filtered ordinal, and let $X : \lambda \to$ **Top** be a λ-sequence so that $X_\beta \to X_{\beta+1}$ is an inclusion for all $\beta+1 < \lambda$. Recall that the forgetful functor **Top** $\to$ **Set** is forgetful, so elements of $\operatorname{colim} X$ are equivalence classes of elements $a \in X_\alpha$ for $\alpha < \lambda$, where $a \in X_\alpha$ and $b \in X_\beta$ represent the same equivalence class iff there exists $\alpha, \beta \le \gamma < \lambda$ so that a and b are sent to the same element by the maps $X_\alpha \to X_\gamma$ and $X_\beta \to X_\gamma$, respectively.

First, suppose $f : A \to X_\alpha$ and $g : A \to X_\beta$ are continuous maps such that the compositions $A \xrightarrow{f} X_\alpha \to \operatorname{colim} X$ and $A \xrightarrow{g} X_\beta \to \operatorname{colim} X$ are equal. Then the same proof given in Example 1.9 works to show that f and g are equal in some stage of the colimit, as desired.

Conversely, suppose we are given a (continuous) map $f : A \to \operatorname{colim} X$. As in the proof of Example 1.9, we may find some $\beta < \lambda$ and a map of sets $\widetilde{f} : A \to X_\beta$ such that the composition $A \xrightarrow{\widetilde{f}} X_\beta \xrightarrow{j} \operatorname{colim} X$ is equal to f (note we have given the canonical map $X_\beta \to \operatorname{colim} X$ the name j). It remains to show that $\widetilde{f}$ is continuous. Let $U \subseteq X_\beta$ be open. Since j is an inclusion (Corollary 2.6), there exists $V \subseteq \operatorname{colim} X_\beta$ open such that $j^{-1}(V) = U$. Then $\widetilde{f}^{-1}(U) = \widetilde{f}^{-1}(j^{-1}(V)) = (j\circ\widetilde{f})^{-1}(V) = f^{-1}(V)$, and f is continuous, so $\widetilde{f}^{-1}(U) = f^{-1}(V)$ is open. Thus $\widetilde{f}$ is continuous, as desired. □

Proposition 2.8 (Hovey 2.4.2)**.** *Compact topological spaces are finite relative to the class $\mathcal{T}$ of closed T_1 inclusions.*

Proof. We use the characterization of smallness afforded by Remark 1.8. Let λ be a limit ordinal, and let $X : \lambda \to \mathbf{Top}$ be a λ-sequence so that $X_\beta \to X_{\beta+1}$ is a closed T_1 inclusion for all $\beta + 1 < \lambda$. Recall that the forgetful functor $\mathbf{Top} \to \mathbf{Set}$ is forgetful, so elements of $\operatorname{colim} X$ are equivalence classes of elements $a \in X_\alpha$ for $\alpha < \lambda$, where $a \in X_\alpha$ and $b \in X_\beta$ represent the same equivalence class iff there exists $\alpha, \beta \leq \gamma < \lambda$ so that a and b are sent to the same element by the maps $X_\alpha \to X_\gamma$ and $X_\beta \to X_\gamma$, respectively.

We show condition (ii) of Remark 1.8 first. Suppose for the sake of a contradiction that $f : A \to \operatorname{colim} X$ is a continuous map that does *not* factor through any X_β for $\beta < \lambda$. For each $a \in A$, the element $f(a) \in \operatorname{colim} X$ may be represented by the equivalence class of an element $x_a \in X_{\gamma_a}$ for some $\gamma_a < \lambda$. We construct a sequence $\{a_n\}_{n=0}^\infty \subseteq A$. Pick a_0 to be any point in A. Supposing a_n has been chosen, pick a_{n+1} such that $\gamma_{a_{n+1}} \geq \gamma_{a_n}$, and $x_{a_{n+1}} \in X_{\gamma_{a_{n+1}}} \setminus \iota_{\gamma_{a_n}, \gamma_{a_{n+1}}}(X_{\gamma_{a_n}})$. If no such a_{n+1} exists, then for all $a \in A$ with $\gamma_a \geq \gamma_{a_n}$, $x_a \in \iota_{\gamma_{a_n}, \gamma_a}(X_{\gamma_{a_n}})$, so f factors through $X_{\gamma_{a_n}}$, a contradiction. Let $j : X \Rightarrow \underline{\operatorname{colim} X}$ be a colimit cone. Then define $S := \{j_{\gamma_{a_n}}(x_{a_n}) : n = 1, 2, \ldots\}$ (note $j_{\gamma_{a_0}}(x_{a_0}) \notin S$). We claim that S has the discrete topology as a subset of X_λ. $\square$

Proposition 2.9 (Hovey 2.4.5 & 2.4.6). *The class $\mathcal{T}$ of closed T_1 inclusions is saturated.*

Proof. TODO. $\square$

Lemma 2.10 (Hovey 2.4.8). *$\mathcal{W} \cap \mathcal{T}$ is closed under transfinite compositions.*

Proof. TODO. $\square$

Proposition 2.11. *The domains of I' (resp. J) are small relative to I'*-cell.

Proof. By Lemma 2.7, every space is small relative to the inclusions, and in particular every space is small relative to the class $\mathcal{T}$ of closed T_1 inclusions. Hence, it suffices to show that J-cell, I'-cell $\subseteq \mathcal{T}$. We showed above in Proposition 2.9 that $\mathcal{T}$ is saturated, and clearly every map in I' and J is a closed T_1 inclusion, so the desired result follows. $\square$

Lemma 2.12 (Hovey Lemma 2.4.4). *The weak equivalences in* **Top** *are closed under retracts and satisfy 2-of-3 axiom (so that in particular the weak equivalences form a subcategory, as clearly identities are weak equivalences).*

Proof. First we show that weak equivalences satisfy 2-of-3. Let $f : X \to Y$ and $g : Y \to Z$ be continuous functions of topological spaces.

First of all, suppose f and g are both weak equivalences. Then by functoriality of π_n, since $\pi_n(f, x)$ and $\pi_n(g, f(x))$ are isomorphisms for all $x \in X$, $\pi_n(g \circ f, x) = \pi_n(g, f(x)) \circ \pi_n(f, x)$ is likewise an isomorphism for all $x \in X$, so that $g \circ f$ is a weak equivalence.

Now, suppose that $g \circ f$ and g are weak equivalences. Pick a point $x \in X$. We wish to show that $\pi_n(f, x) : \pi_n(X, x) \to \pi_n(Y, f(x))$ is an isomorphism for all $n \geq 0$. We know that $\pi_n(g \circ f, x)$ is an isomorphism, and $\pi_n(g, f(x))$ is an isomorphism, say with inverse, φ, so that

$$\varphi \circ \pi_n(g \circ f, x) = \varphi \circ \pi_n(g, f(x)) \circ \pi_n(f, x) = \pi_n(f, x)$$

is an isomorphism, as it is a composition of isomorphisms.

Now, suppose that $g \circ f$ and f are weak equivalences. Pick a point $y \in Y$. Since $\pi_0(f)$ is an isomorphism, there exists a point $x \in X$ such that $f(x)$ belongs to the path component containing y, so that there exists some $\alpha : I \to Y$ with $\alpha(0) = f(x)$ and $\alpha(1) = f(y)$. Then consider the following diagram

$$\begin{array}{ccc} \pi_n(Y, y) & \xrightarrow{\pi_n(g,y)} & \pi_n(Z, g(y)) \\ \downarrow & & \downarrow \\ \pi_n(Y, f(x)) & \xrightarrow{\pi_n(g,f(x))} & \pi_n(Z, g(f(x))) \end{array}$$

where the left arrow is the isomorphism given by conjugation by the path α, and the right arrow is the isomorphism given by conjugation by the path $g \circ \alpha$. It is tedious yet straightforward to verify that the diagram commutes. Furthermore, we know that $\pi_n(f, x)$ and $\pi_n(g \circ f, x) = \pi_n(g, f(x)) \circ \pi_n(f, x)$ are isomorphisms for all n, so that if we denote the inverse of $\pi_n(f, x)$ by φ, then

$$\pi_n(g \circ f, x) \circ \varphi = \pi_n(g, f(x)) \circ \pi_n(f, x) \circ \varphi = \pi_n(g, f(x))$$

is an isomorphism, as it is given as a composition of isomorphisms. Hence, the top arrow must likewise be an isomorphism, precisely the desired result.

The fact that weak equivalences in **Top** are closed under retracts is entirely straightforward and follows from the fact that the functors π_n preserve retract diagrams and that the class of isomorphisms in any category is closed under retracts. ☐

Proposition 2.13 (Hovey 2.4.9)**.** ${}_\perp(J_\perp) \subseteq \mathcal{W} \cap {}_\perp(I'{}_\perp)$.

Proof. First, in order to show ${}_\perp(J_\perp) \subseteq {}_\perp(I'{}_\perp)$, It suffices to show that $J \subseteq I'$-cell, as by Lemma 1.14 we would have $J \subseteq {}_\perp(I'{}_\perp)$, and

$$J \subseteq {}_\perp(I'{}_\perp) \implies {}_\perp(J_\perp) \subseteq {}_\perp(({}_\perp(I'{}_\perp))_\perp) = {}_\perp(I'{}_\perp),$$

where the implication and equality both follow from Lemma 1.11 which gives that

$$A \subseteq B \implies {}_\perp(A_\perp) \subseteq {}_\perp(B_\perp) \quad \text{ and } \quad ({}_\perp(A_\perp))_\perp = A_\perp.$$

Now, to show $J \subseteq I'$-cell, first consider the composition $j_n : D^n \hookrightarrow S^n \hookrightarrow D^{n+1}$, where the first map is the pushout

$$\begin{tikzcd} S^{n-1} \arrow[r, hook] \arrow[d, hook] & D^n \arrow[d] \\ D^n \arrow[r] \arrow[ur, phantom, "\lrcorner", very near end] & S^n \end{tikzcd}$$

obtained by gluing two copies of D^n along their boundary, and the second map map is simply the inclusion $S^n \hookrightarrow D^{n+1}$, which can be written as the pushout

$$\begin{tikzcd} S^n \arrow[r, equal] \arrow[d, hook] & S^n \arrow[d, hook] \\ D^{n+1} \arrow[r, equal] \arrow[ur, phantom, "\lrcorner", very near end] & D^{n+1} \end{tikzcd}$$

It can be seen that j_n includes D^n as a hemisphere of $S^n = \partial D^{n+1} \subseteq D^{n+1}$. Note that $D^n \times I$ is homeomorphic to D^{n+1} ("smooth out" the sharp edges of the cylinder) via some homeomorphism $h_n : D^{n+1} \to D^n \times I$, and in particular, we may define h_n so that $h_n(j_n(D^n)) = D^n \times \{0\} \subseteq D^n \times I$ by squashing the hemisphere $j_n(D^n)$ to be one of the faces of the cylinder $D^n \times I$, in which case $h_n \circ j_n : D^n \to D^n \times I$ is precisely the inclusion $D^n \hookrightarrow D^n \times I$ sending $x \mapsto (x, 0)$, and since $j_n \in I'$-cell, $h_n \circ j_n \in I'$-cell by Lemma 1.13.

Now, we claim that ${}_\perp(J_\perp) \subseteq \mathcal{W}$. First note that by Corollary 1.16 and Proposition 2.11, every map in ${}_\perp(J_\perp)$ is a retract of an element of J-cell. Furthermore, we know that $\mathcal{W}$ is closed under retracts (Lemma 2.12), so that it suffices to show that J-cell $\subseteq \mathcal{W}$. We claim it suffices to show that pushouts of maps in J are weak equivalences. Supposing we had shown this, we would have that pushouts of maps in J are weak equivalences and T_1 inclusions, as $J \subseteq \mathcal{T}$ and $\mathcal{T}$ is saturated by Proposition 2.9. Then by Lemma 2.10, we would have that J-cell $\subseteq \mathcal{W} \cap \mathcal{T}$, precisely the desired result.

Now, let $\mathcal{S}$ be the class of *inclusions of a deformation retract*, i.e., those **injective** maps $i : A \to B$ such that there exists a homotopy $H : B \times I \to B$ with $H(i(a), t) = i(a)$ for all $a \in A$, $H(b, 0) = b$ for all $b \in B$, and $H(b, 1) = i(r(b))$ for all $b \in B$ for some map $r : B \to A$[2]. We will show the following:

(1) $\mathcal{S} \subseteq \mathcal{W}$.

It suffices to show that if $i : A \to B$ belongs to $\mathcal{S}$, then i is a homotopy equivalence. Indeed, given $i : A \to B$, let $H : B \times I \to B$ and $r : B \to A$ be a homotopy and retract satisfying the conditions above. Then in particular, H is a homotopy between id_B (at time $t = 0$) and $i \circ r$ (at time $t = 1$). It remains to show that $r \circ i = \mathrm{id}_A$. First of all, note that since $H(b, 1) = i(r(b))$ for all $b \in B$, we have $H(i(a), 1) = i(r(i(a)))$. Yet, we also know that $H(i(a), t) = i(a)$ for all $t \in I$, so $i(r(i(a))) = i(a)$, and i is injective so $r(i(a)) = a$.

[2] Hovey has a typo here, namely, he does not specify that i must be injective. Without this specification, his assertion fails. For example, take $A = \mathbb{R}^2$, $B = \mathbb{R}$, $i(x, y) = x$, $H(b, t) = b$, and $r(b) = (b, 0)$. Then i is an inclusion of a deformation retract according to Hovey's "definition," but i is not injective and r is not a retract.

(2) $J \subseteq \mathcal{S}$.

For $n \geq 0$, let $j_n : D^n \hookrightarrow D^n \times I$ denote the inclusion of D^n as the subset $D^n \times \{0\}$. Define a deformation retract $H : D^n \times I \times I \to D^n \times I$ by $(x, s, t) \mapsto (x, s(1-t))$. Then indeed we have $H(j_n(x), t) = H(x, 0, t) = (x, 0) = j_n(x)$ for all $x \in D^n$, $H(x, t, 0) = (x, t(1-0)) = (x, t)$ for all $(x, t) \in D^n \times I$, and $H(x, t, 1) = (x, t(1-1)) = (x, 0) = j_n(r(x))$ for all $(x, t) \in D^n \times I$, where $r : D^n \times I \to D^n$ is the projection onto time zero sending $(x, t) \mapsto (x, 0)$. Finally, j_n is clearly injective. Thus, indeed $J \subseteq \mathcal{S}$.

(3) $\mathcal{S}$ is closed under pushouts.

Suppose we are given a pushout diagram

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "i"'] & C \arrow[d, "j"] \\
B \arrow[r, "g"'] & D \arrow[ul, phantom, "\lrcorner", very near start]
\end{tikzcd}$$

where $i \in \mathcal{S}$. Then we wish to show j in $\mathcal{S}$. First, injectivity. Suppose for the sake of a contradiction there existed nonequal $c, c' \in C$ such that $j(c) = j(c')$. Define $X := \{1, 2, 3\}$ (with the indiscrete topology, if you like), $h : C \to X$ by $c \mapsto 1$, $c' \mapsto 2$, and $C \setminus \{c, c'\} \mapsto 3$, and $k : B \to X$ by $i(f^{-1}(c)) \mapsto 1$, $i(f^{-1}(c')) \mapsto 2$, and $i(f^{-1}(C \setminus \{c, c'\})) \mapsto 3$. Then it is straightforward to see that $h \circ f = k \circ i$. Thus, there must exist a (unique) function $\ell : D \to X$ such that $\ell \circ j = h$ and $\ell \circ g = k$. But then we would have $h(c) = \ell(j(c)) = \ell(j(c')) = h(c')$ since $j(c) = j(c')$, a contradiction of the fact that $h(c) \neq h(c')$. Hence, j must be injective. Now, we look to construct H and r. Let $K : B \times I \to B$ and $r' : B \to A$ be maps satisfying the conditions for i to be an inclusion of a deformation retract.

We wish to define a homotopy $H : D \times I \to D$. Then I is a locally compact Hausdorff space (in particular, it is compact and Hausdorff), so that the functor $- \times I : \mathbf{Top} \to \mathbf{Top}$ preserves colimits (Proposition 2.1), meaning the following is a pushout diagram:

$$\begin{tikzcd}
A \times I \arrow[r, "f \times \mathrm{id}_I"] \arrow[d, "i \times \mathrm{id}_I"'] & C \times I \arrow[d, "j \times \mathrm{id}_I"] \\
B \times I \arrow[r, "g \times \mathrm{id}_I"'] & D \times I \arrow[ul, phantom, "\lrcorner", very near start]
\end{tikzcd}$$

Then by the universal property of the pushout, there is a map $H : D \times I \to D$ (the dashed line) such that the following diagram commutes

$$\begin{tikzcd}
A \times I \arrow[r, "f \times \mathrm{id}_I"] \arrow[d, "i \times \mathrm{id}_I"'] & C \times I \arrow[d, "j \times \mathrm{id}_I"] \arrow[dr, bend left, "\pi_1"] & \\
B \times I \arrow[r, "g \times \mathrm{id}_I"'] \arrow[dr, bend right, "K"'] & D \times I \arrow[ul, phantom, "\lrcorner", very near start] \arrow[dr, dashed, "H"] & C \arrow[d, "j"] \\
& B \arrow[r, "g"'] & D
\end{tikzcd}$$

Now, note $r' \circ i = \mathrm{id}_A$. Indeed, given $a \in A$, we have $i(r'(i(a))) = K(i(a), t) = i(a)$ and i is injective, so that $r'(i(a)) = a$, as desired. Hence, there exists a unique map $r : D \to C$ (the dashed line) such that the following diagram commutes:

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "i"'] & C \arrow[d, "j"] \arrow[ddr, bend left, equal] & \\
B \arrow[r, "g"'] \arrow[dr, bend right, "r'"'] & D \arrow[ul, phantom, "\lrcorner", very near start] \arrow[dr, dashed, "r"] & \\
& A \arrow[r, "f"'] & C
\end{tikzcd}$$

Now we claim that our constructions H and r endue j with the structure of an inclusion of a deformation retract, as desired. First $c \in C$, we wish to show $H(j(c), t) = j(c)$ for all t. Indeed, we

have
$$H(j(c),t) = H(j \times \mathrm{id}_I(c,t)) = j(\pi_1(c,t)) = j(c).$$
Given $d \in D$, we want to show $H(d,0) = d$. By the explicit description of the colimit in **Top**, we know that every element of D is in the image of either j or g. If $d = j(c)$ for some c, then we have just shown $H(d,0) = H(j(c),0) = j(c) = d$, as desired. On the other hand, if $d = g(b)$ for some $b \in B$ we have
$$H(d,0) = H(g \times \mathrm{id}_I(b,0)) = g(K(b,0)) = g(b) = d.$$
Finally, we claim that $H(d,1) = j(r(d))$ for all $d \in D$. If $d = j(c)$ for some $c \in C$, then we have
$$H(d,1) = H(j(c),1) = j(c) = j(r(j(c))) = j(r(d)),$$
as desired. On the other hand, if $d = g(b)$ for some $b \in B$, then
$$H(d,1) = H(g \times \mathrm{id}_I(b,1)) = g(K(b,1)) = g(i(r'(b))) = j(f(r'(b))) = j(r(g(b))) = j(r(d)).$$
□

Proposition 2.14 (Hovey 2.4.10)**.** ${I'}_\perp \subseteq \mathcal{W} \cap J_\perp$

Proof. First, by Proposition 2.13 we know ${}_\perp(J_\perp) \subseteq {}_\perp({I'}_\perp)$, and this implies ${I'}_\perp \subseteq J_\perp$, as by Lemma 1.11 we have
$${}_\perp(J_\perp) \subseteq {}_\perp({I'}_\perp) \implies J_\perp = ({}_\perp(J_\perp))_\perp \supseteq ({}_\perp({I'}_\perp))_\perp = {I'}_\perp.$$
Thus, it suffices to show that ${I'}_\perp \subseteq \mathcal{W}$. Now, suppose $p : (X, x_0) \to (Y, p(x_0))$ is in ${I'}_\perp$. We wish to show that the map $\pi_n(p, x_0) : \pi_n(X, x_0) \to \pi_n(Y, p(x_0))$ is an isomorphism for all n.

First we show that $\pi_n(p, x_0)$ is surjective. Let $g : (S^n, *) \to (Y, p(x_0))$ be a map. Then we have the following commutative diagram

$$\begin{tikzcd} * \arrow[r] \arrow[d] & X \arrow[d, "p"] \\ S^n \arrow[r, "g"] & Y \end{tikzcd}$$

where the top arrow picks out x_0. Note that the map $* \to S^n$ may be realized as a pushout of the diagram $D^n \leftarrow S^{n-1} \to *$, so that $* \to S^n$ belongs to I'-cell, and therefore ${}_\perp({I'}_\perp)$ by Lemma 1.14, and $p \in {I'}_\perp$, so $* \to S^n$ has the left lifting property against p. Thus, the above diagram has a lift $f : (S^n, *) \to (X, x_0)$ such that $p \circ f = g$, so that $\pi_n(p, x_0)([f]) = [p \circ f] = [g]$, as desired.

Finally, we show that $\pi_n(p, x_0)$ is injective. Suppose we have two maps $f, g : (S^n, *) \to (X, x_0)$ such that $p \circ f$ and $p \circ g$ represent the same element of $\pi_n(Y, p(x_0))$. Then there is a homotopy $H : S^n \times I \to Y$ such that for all $s \in S^n$ and $t \in I$, $H(s,0) = p(f(s))$, $H(s,1) = p(g(s))$, and $H(*,t) = p(x_0)$. By the universal property of the quotient, H induces a map $\overline{H} : S^n \wedge I_+ := (S^n \times I)/(* \times I)$ sending the equivalence class $[s,t] \mapsto H(s,t)$. Hence, the following diagram commutes:

$$\begin{tikzcd} S^n \vee S^n \arrow[r, "f \vee g"] \arrow[d, hook] & X \arrow[d, "p"'] \\ S^n \wedge I_+ \arrow[r, "\overline{H}"] & Y \end{tikzcd}$$

where the left arrow is an element of I'-cell, as it may be obtained by attaching an $n+1$ cell to $S^n \vee S^n$ (when $n = 0$, the attachning map is obvious; when $n > 0$, the attaching map is the quotient map $S^n \twoheadrightarrow S^n \vee S^n$ obtained by collapsing the equator). Thus, by similar reasoning to above there exists a lift $\overline{K} : S^n \wedge I_+ \to X$. Then if we define K to be the composition $S^n \times I \twoheadrightarrow S^n \wedge I_+ \xrightarrow{\overline{K}} X$, this gives us the desired homotopy between f and g: given $s \in S^n$ and $t \in I$, we have $K(s,0) = \overline{K}([s,0]) = f(s)$, $K(s,1) = \overline{K}([s,1]) = g(s)$, and $K(*,t) = \overline{K}([*,t])$ □

Proposition 2.15 (Hovey 2.4.12)**.** $\mathcal{W} \cap J_\perp \subseteq {I'}_\perp$

Proof. TODO. □

Corollary 2.16 (Hovey 2.4.14)**.** *Every topological space is fibrant, i.e., given a space X, the unique map $X \to *$ is an element of $J_\perp$.*

Proof. TODO. □

Questions/Comments:

(1)