MODEL STRUCTURES

Contents

1. Preliminaries

Definition 1.1 (Hovey Definition 2.1.1)**.** Suppose $\mathcal{C}$ is a cocomplete category, and λ is an ordinal. A λ-*sequence* in $\mathcal{C}$ is a colimit-preserving functor $X : \lambda \to \mathcal{C}$, commonly written as

$$X_0 \to X_1 \to \cdots \to X_\beta \to \cdots .$$

Since X preserves colimits, for all limit ordinals $\gamma < \lambda$, the induced map

$$\operatorname*{colim}_{\beta<\lambda} X_\beta \to X_\gamma$$

is an isomorphism. We refer to the map $X_0 \to \operatorname{colim}_{\beta<\lambda} X_\beta$ as the *composition* of the λ-sequence. Given a collection $\mathcal{D}$ of morphisms in $\mathcal{C}$ such that every map $X_\beta \to X_{\beta+1}$ for $\beta + 1 < \lambda$ is in $\mathcal{D}$, we refer to the composition $X_0 \to \operatorname{colim}_{\beta<\lambda} X_\beta$ as a *transfinite composition* of maps in $\mathcal{D}$.

Definition 1.2 (Hovey Definition 2.1.2)**.** Let γ be a cardinal. An ordinal α is γ-*filtered* if it is a limit ordinal and, if $A \subseteq \alpha$ and $|A| \le \gamma$, then $\sup A < \alpha$.

Given a cardinal γ, a γ-filtered category is one such that any diagram $\mathcal{D} \to \mathcal{C}$ has a cocone where $\mathcal{D}$ has $< \gamma$ arrows. A catgory is just "filtered" if it is ω-filtered, i.e., if every finite diagram in $\mathcal{C}$ admits a cocone. Note that an ordinal α is γ-filtered precisely when it is γ-filtered as a category, and in particular every ordinal is ω-filtered.

Definition 1.3 (Hovey Definition 2.1.3)**.** Suppose $\mathcal{C}$ is a comcomplete category, $\mathcal{D} \subseteq \operatorname{Mor}\mathcal{C}$ is some collection of morphisms of $\mathcal{C}$, A is an object of $\mathcal{C}$, and κ is a cardinal. We say that A is κ-*small relative to* $\mathcal{D}$ if, for all κ-filtered ordinals λ and all λ-sequences

$$X_0 \to X_1 \to \cdots \to X_\beta \to \cdots$$

such that each map $X_\beta \to X_{\beta+1}$ is in $\mathcal{D}$ for $\beta + 1 < \lambda$, the map of sets

$$\operatorname*{colim}_{\beta<\lambda} \mathcal{C}(A, X_\beta) \to \mathcal{C}(A, \operatorname*{colim}_{\beta<\lambda} X_\beta)$$

is an isomorphism. We say that A is *small relative to* $\mathcal{D}$ if it is κ-small relative to $\mathcal{D}$ for some κ. We say that A is *small* if it is small relative to $\mathcal{C}$ itself.

Recall that given a small category $\mathcal{D}$ and a functor $F : \mathcal{D} \to \mathrm{Set}$, we may explicitly construct the colimit of F as the set

$$\operatorname{colim} F := \left(\coprod_{d \in \mathcal{D}} F(d) \right) / \sim,$$

where the equivalence relation $\sim$ is **generated** by

$$((x \in F(d)) \sim (x' \in F(d'))) \quad \text{if} \quad (\exists (f : d \to d') \text{ with } Ff(x) = x').$$

In particular, if $\mathcal{D}$ is a filtered category then the resulting relation can be described as follows:

$$((x \in F(d)) \sim (x' \in F(d'))) \quad \text{iff} \quad (\exists\, d'', (f : d \to d''), (g : d' \to d'') \text{ with } Ff(x) = Fg(x')). \tag{1}$$

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given a cone $\eta : F \Rightarrow \underline{Y}$ under F, the unique map $\operatorname{colim} F \to Y$ maps the equivalence class of $x \in F(d)$ to the element $\eta_d(x) \in X$. We will use this characterization of the colimit in the following example.

Example 1.4 (Hovey 2.1.5)**.** Every set is small. Indeed, if A is a set we claim that A is $|A|$-small. To see this, suppose λ is an $|A|$-filtered ordinal, and X is a λ-sequence of sets. Given $\alpha < \beta < \lambda$, let $\iota_{\alpha,\beta} : X_\alpha \to X_\beta$ denote the induced morphism. We will write $X_\lambda := \operatorname{colim}_{\beta<\lambda} X_\beta$, and let $\iota : X \Rightarrow X_\lambda$ be the colimit cone, so that given $\beta < \lambda$, $\iota_\beta : X_\beta \to X_\lambda$ is the leg of the colimit cone at X_β. By composing with the functor $\mathcal{C}(A, -) : \mathrm{Set} \to \mathrm{Set}$, we get another λ-sequence $\{\mathcal{C}(X_\beta, A)\}_{\beta<\lambda}$. The cone ι under X induces a cone ι_* under $\mathcal{C}(X_\beta, A)$ with nadir $\mathcal{C}(A, X_\lambda)$. Let $\eta : \mathcal{C}(X_\beta, A) \Rightarrow \underline{\operatorname{colim}_{\beta<\lambda} \mathcal{C}(X_\beta, A)}$ be the colimit cone, and let $\ell : \operatorname{colim}_{\beta<\lambda} \mathcal{C}(A, X_\lambda) \to \mathcal{C}(A, X_\lambda)$ be the unique morphism of cones so that the following diagram commutes

$$\begin{tikzcd}
\mathcal{C}(A,X_0) \arrow[r, "(\iota_{0,1})_*"] \arrow[drr, "\eta_0"] \arrow[ddrr, "(\iota_0)_*"'] & \mathcal{C}(A,X_1) \arrow[r, "(\iota_{1,2})_*"] \arrow[dr, "\eta_1"] \arrow[ddr, "(\iota_1)_*"'] & \cdots \arrow[r] & \mathcal{C}(A,X_\beta) \arrow[r, "(\iota_{\beta,\beta+1})_*"] \arrow[dl, "\eta_\beta"'] \arrow[ddl, "(\iota_\beta)_*"] & \cdots \\
 & & \operatorname{colim}_{\beta<\lambda} \mathcal{C}(A,X_\beta) \arrow[d, dashed, "\ell"] & & \\
 & & \mathcal{C}(A,X_\lambda) & &
\end{tikzcd}$$

First, we wish to show that ℓ is surjective. Indeed, let $f : A \to X_\lambda$. For each $a \in A$, there exists some $\beta_a \in \lambda$ and some $a' \in X_{\beta_a}$ such that $f(a) = \eta_{\beta_a}(a')$ (see the preceeding discussion). Then let $\gamma := \sup_{a\in A} \beta_a$. Since $|\{\beta_a\}_{a\in A}| \leq |A|$ and λ is $|A|$-filtered, necessarily $\gamma < \lambda$. Now, define $g : A \to X_\gamma$ like so: for $a \in A$, define $g(a) := \iota_{\beta_a,\gamma}(a')$, where $a' \in X_{\beta_a}$ was chosen earlier so that $\iota_{\beta_a}(a') = f(a)$. Then we claim that $\ell(\eta_\gamma(g)) = f$. Indeed, as ℓ is a morphism of cocones, $\ell \circ \eta = \iota_*$, so that we have

$$\ell(\eta_\gamma(g)) = (\iota_\gamma)_*(g) = \iota_\gamma \circ g,$$

and given $a \in A$ we have

$$\iota_\gamma(g(a)) = \iota_\gamma(\iota_{\beta_a,\gamma}(a')).$$

By definition of a cone, $\iota_\gamma \circ \iota_{\beta_a,\gamma} = \iota_{\beta_a}$, so that

$$\ell(\eta_\gamma(g))(a) = \iota_\gamma(\iota_{\beta_a,\gamma}(a')) = \iota_{\beta_a}(a') = f(a),$$

so that indeed $\ell(\eta_\gamma(g)) = f$.

It remains to show ℓ is injective. Suppose we are given $[f], [g] \in \operatorname{colim}_{\beta<\lambda} \mathcal{C}(A, X_\beta)$ such that $\ell([f]) = \ell([g])$. Then by the preceeding discussion, there exists $\alpha, \beta < \lambda$, $f \in \mathcal{C}(A, X_\alpha)$, and $g \in \mathcal{C}(A, X_\beta)$ such that $\eta_\alpha(f) = [f]$ and $\eta_\beta(g) = [g]$. Then since $\ell \circ \eta = \iota_*$, we have

$$\ell([f]) = \ell([g]) \implies \iota_\alpha \circ f = (\iota_\alpha)_*(f) = \ell(\eta_\alpha(f)) = \ell(\eta_\beta(g)) = (\iota_\beta)_*(g) = \iota_\beta \circ g.$$

For each $a \in A$, since $\iota_\alpha(f(a)) = \iota_\beta(g(a))$, by Equation 1 there exists γ_a with $\alpha, \beta \leq \gamma_a$ such that $\iota_{\alpha,\gamma_a}(f(a)) = \iota_{\beta,\gamma_a}(g(a))$. Then let $\gamma := \sup_{a\in A} \gamma_a$. Since $|\{\gamma_a\}_{a\in A}| \leq |A|$ and λ is $|A|$-filtered, necessarily $\gamma < \lambda$. Now, in order to show $[f] = [g]$, by Equation 1 it suffices to show that $(\iota_{\alpha,\gamma})_*(f) = (\iota_{\beta,\gamma})_*(g)$. Indeed, given $a \in A$, we have

$$(\iota_{\alpha,\gamma})_*(f)(a) = \iota_{\alpha,\gamma}(f(a)) = \iota_{\gamma_a,\gamma} \circ \iota_{\alpha,\gamma_a}(f(a)) = \iota_{\gamma_a,\gamma} \circ \iota_{\beta,\gamma_a}(g(a)) = \iota_{\beta,\gamma}(g(a)) = (\iota_{\beta,\gamma})_*(g)(a),$$

precisely the desired result..

Definition 1.5 (Hovey Definition 2.1.7)**.** Let I be a class of maps in a category $\mathcal{C}$.

(1) A map is *I-injective* if it has the right lifting property w.r.t. every map in I. The class of I-injective maps is denoted I-inj (or $I_\perp$).
(2) A map is *I-projective* if it has the left lifting property w.r.t. every map in I. The class of I-projective maps is denoted I-proj (or ${}_\perp I$).
(3) A map is an *I-cofibration* if it has the left lifting property w.r.t. every I-injective map. The class of I-cofibrations is the class (I-inj)-proj and is denoted I-cof (or ${}_\perp(I_\perp)$).

(4) A map is an I-*fibration* if it has the right lifting property w.r.t. every I-projective map. The class of I-fibrations is the class (I-proj)-inj and is denoted I-fib (or $({}_\perp I)_\perp$).

Definition 1.6 (Hovey Definition 2.1.9)**.** Let I be a set of maps in a cocomplete category $\mathcal{C}$. A *relative I-cell complex* is a transfinite composition of pushouts of elements of I. That is, if $f : A \to B$ is a relative I-cell complex, then there is an ordinal λ and a λ-sequence $X : \lambda \to \mathcal{C}$ such that f is the composition of X and such that, for each β such that $\beta + 1 < \lambda$, there is a pushout square

$$\begin{tikzcd} C_\beta \arrow[r] \arrow[d, "g_\beta"'] & X_\beta \arrow[d] \\ D_\beta \arrow[r] & X_{\beta+1} \arrow[ul, phantom, "\lrcorner", very near start] \end{tikzcd}$$

with $g_\beta \in I$. We denote the collection of relative I-cell complexes by I-cell. We say that $A \in \mathcal{C}$ is an I-*cell complex* if the map $0 \to A$ is a relative I-cell complex.

Lemma 1.7 (Hovey 2.1.10)**.** *Suppose I is a class of maps in a category $\mathcal{C}$ with all small colimits. Then* I-cell $\subseteq$ I-cof*.*

Definition 1.8 (Hovey Definition 2.1.17)**.** Suppose $\mathcal{C}$ is a model category. We say that $\mathcal{C}$ is *cofibrantly generated* if there are sets I and J of maps such that:

1. The domains of the maps of I are small relative to I-cell;
2. The domains of the maps of J are small relative to J-cell;
3. The class of fibrations is J-inj; and
4. The class of trivial fibrations is I-inj.

We refer to I as the set of *generating cofibrations* and to J as the set of *generating trivial cofibrations*. A cofibrantly generated model category is *finitely generated* if we can choose the sets I and J above so that the domains and codomains of I and J are finite relative to I-cell.

Proposition 1.9 (Hovey Proposition 2.1.18)**.** *Suppose $\mathcal{C}$ is a cofibrantly generated model category, with generating cofibrations I and generating trivial fibrations J.*

(a) The cofibrations form the class I-cof*.*
(b) Every cofibration is a retract of a relative I-cell complex.
(c) The domains of I are small relative to the cofibrations.
(d) The trivial cofibrations form the class J-cof*.*
(e) Every trivial cofibration is a retract of a relative J-cell complex.
(f) The domains of J are small relative to the trivial cofibrations.

If $\mathcal{C}$ is fibrantly generated, then the domains and codomains of I and J are finite relative to the cofibrations.

Proof. TODO. □

Theorem 1.10 (Hovey Theorem 2.1.19)**.** *Suppose $\mathcal{C}$ is a complete & cocomplete category. Suppose $\mathcal{W}$ is a subcategory of $\mathcal{C}$, and I and J are sets of maps of $\mathcal{C}$. Then there is a cofibrantly generated model structure on $\mathcal{C}$ with I as the set of generating cofibrations, J as the set of generating trivial fibrations, and $\mathcal{W}$ as the subcategory of weak equivalences if and only if the following conditions are satisfied.*

1. *The subcategory $\mathcal{W}$ has the 2-of-3 property and is closed under retracts.*
2. *The domains of I are small relative to* I-cell.
3. *The domains of J are small relative to* J-cell.
4. J-cell $\subseteq \mathcal{W} \cap I$-cof*.*
5. I-inj $\subseteq \mathcal{W} \cap J$-inj.
6. *Either* $\mathcal{W} \cap I$-cof $\subseteq J$-cof *or* $\mathcal{W} \cap J$-inj $\subseteq I$-inj.

Proof. TODO. □

Definition 1.11. Let $\mathcal{C}$ be a category and I a collection of morphisms in $\mathcal{C}$. Then if I is closed under transfinite composition, pushouts, and retracts then we say I is *saturated*.

2. Topological Spaces

An injective map $f : X \to Y$ in **Top** is an *inclusion* if U is open in X if and only if there is a V open in Y such that $f^{-1}(V) = U$. If f is a closed inclusion and every point in $Y \setminus f(X)$ is closed, then we call f a *closed T_1 inclusion*. We will let $\mathcal{T}$ denote the class of closed T_1 inclusions in **Top**.

The symbol D^n will denote the unit disk in $\mathbb{R}^n$, and the symbol S^{n-1} will denote the unit sphere in $\mathbb{R}^n$, so that we have the boundary inclusions $S^{n-1} \hookrightarrow D^n$. In particular, for $n = 0$ we let $D^0 = \{0\}$ and $S^{-1} = \emptyset$.

Definition 2.1. A map $f : X \to Y$ in **Top** is called a *weak equivalence* if

$$\pi_n(f, x) : \pi_n(X, x) \to \pi_n(Y, f(x))$$

is an isomorphism for all $n \geq 0$ and for all $x \in X$. We will write $\mathcal{W}$ to refer to the class of all weak equivalences in **Top**.

Define the set of maps I' to consist of all the boundary inclusion $S^{n-1} \hookrightarrow D^n$ for all $n \geq 0$, and define the set J to consist of all the inclusions $D^n \hookrightarrow D^n \times I$ mapping $x \mapsto (x, 0)$ for $n \geq 0$. Then a map f will be called a *cofibration* if it is in I'-cof $= {}_\perp(I'_\perp)$, and a *fibration* if it is in J-inj $= J_\perp$.

A map in I'-cell is usually called a *relative cell complex*; a relative CW-complex is a special case of a relative cell complex, where, in particular, the cells can be attached in order of their dimension. Note that in particular maps of J are relative CW complexes, hence are relative I-cell complexes. A fibration is often known as a *Serre fibration* in the literature.

Theorem 2.2 (Hovey Theorem 2.4.19)**.** *There is a finitely generated model structure on* **Top** *with I' as the set of generating cofibrations, J as the set of generating trivial cofibrations, and the cofibrations, fibrations, and weak equivalences as above. Every object of* **Top** *is fibrant, and the cofibrant objects are retracts of relative cell complexes.*

Proof. We will apply Theorem 1.10 to get that there is a cofibrantly generated model structure on **Top** with I' as the set of generating cofibrations, J as the set of generating trivial fibrations, and $\mathcal{W}$ as the subcategory of weak equivalences. The six requirements outlined in the theorem will be verified like so:

1. $\mathcal{W}$ is a subcategory of $\mathcal{C}$ which has the 2-of-3 property and is closed under retracts: Lemma 2.6.
2. The domains of I' are small relative to I'-cell: In Lemma 2.3, we will show that every space is small relative to the inclusions, and in particular every space is small relative to the class $\mathcal{T}$ of closed T_1 inclusions. Hence, it will suffice to show that I'-cell $\subseteq \mathcal{T}$. In Proposition 2.5, we will show that $\mathcal{T}$ is saturated, and clearly every map in I' is a closed T_1 inclusion, so the desired result follows.
3. The domains of J are small relative to J-cell: By the same argument given above, this will follow by Lemma 2.3, Proposition 2.5, and the fact that $J \subseteq \mathcal{T}$.
4. J-cell $\subseteq \mathcal{W} \cap I'$-cof: In Proposition 2.7, we will show J-cof $\subseteq \mathcal{W} \cap I'$-cof, and by Lemma 1.7 J-cell $\subseteq J$-cof.
5. I'-inj $\subseteq \mathcal{W} \cap J$-inj: Proposition 2.8
6. $\mathcal{W} \cap J$-inj $\subseteq I'$-inj: Proposition 2.9

It will follow by the definition of a cofibrantly generated model structure (Definition 1.8) that the fibrations in this model structure are given by J-inj, which is precisely how we defined it. By Proposition 1.9, the class of cofibrations will be given by I'-cof, which is likewise exactly how we defined them.

In Proposition 2.4, we will show that compact spaces are finite relative to the class $\mathcal{T}$ of closed T_1 inclusions. Hence, this model structure will be finitely generated, as the domains and codomains of I' and J are all compact, and by the reasoning given above we will have shown I'-cell $\subseteq \mathcal{T}$.

We will show that every object of **Top** is fibrant in Corollary 2.10. Finally, to see that cofibrant objects are retracts of relative cell complexes, FINISH □

Lemma 2.3 (Hovey 2.4.1)**.** *Every topological space is small relative to the inclusions.*

Proof. As with the case of sets, we claim that every topological space X is $|X|$-small! □

Proposition 2.4 (Hovey 2.4.2)**.** *Compact topological spaces are finite relative to the class $\mathcal{T}$ of closed T_1 inclusions.*

Proof. TODO. □

Proposition 2.5 (Hovey 2.4.5 & 2.4.6)**.** *The class of closed T_1 inclusions is saturated.*

Proof. TODO. □

Lemma 2.6 (Hovey Lemma 2.4.4)**.** *The weak equivalences in* **Top** *are closed under retracts and satisfy 2-of-3 axiom (so that in particular the weak equivalences form a subcategory, as clearly identities are weak equivalences).*

Proof. First we show that weak equivalences satisfy 2-of-3. Let $f : X \to Y$ and $g : Y \to Z$ be continuous functions of topological spaces.

First of all, suppose f and g are both weak equivalences. Then by functoriality of π_n, since $\pi_n(f, x)$ and $\pi_n(g, f(x))$ are isomorphisms for all $x \in X$, $\pi_n(g \circ f, x) = \pi_n(g, f(x)) \circ \pi_n(f, x)$ is likewise an isomorphism for all $x \in X$, so that $g \circ f$ is a weak equivalence.

Now, suppose that $g \circ f$ and g are weak equivalences. Pick a point $x \in X$. We wish to show that $\pi_n(f, x) : \pi_n(X, x) \to \pi_n(Y, f(x))$ is an isomorphism for all $n \geq 0$. We know that $\pi_n(g \circ f, x)$ is an isomorphism, and $\pi_n(g, f(x))$ is an isomorphism, say with inverse, φ, so that

$$\varphi \circ \pi_n(g \circ f, x) = \varphi \circ \pi_n(g, f(x)) \circ \pi_n(f, x) = \pi_n(f, x)$$

is an isomorphism, as it is a composition of isomorphisms.

Now, suppose that $g \circ f$ and f are weak equivalences. Pick a point $y \in Y$. Since $\pi_0(f)$ is an isomorphism, there exists a point $x \in X$ such that $f(x)$ belongs to the path component containing y, so that there exists some $\alpha : I \to Y$ with $\alpha(0) = f(x)$ and $\alpha(1) = f(y)$. Then consider the following diagram

$$\begin{tikzcd} \pi_n(Y, y) \arrow[r, "\pi_n(g,y)"] \arrow[d] & \pi_n(Z, g(y)) \arrow[d] \\ \pi_n(Y, f(x)) \arrow[r, "\pi_n(g,f(x))"] & \pi_n(Z, g(f(x))) \end{tikzcd}$$

where the left arrow is the isomorphism given by conjugation by the path α, and the right arrow is the isomorphism given by conjugation by the path $g \circ \alpha$. It is tedious yet straightforward to verify that the diagram commutes. Furthermore, we know that $\pi_n(f, x)$ and $\pi_n(g \circ f, x) = \pi_n(g, f(x)) \circ \pi_n(f, x)$ are isomorphisms for all n, so that if we denote the inverse of $\pi_n(f, x)$ by φ, then

$$\pi_n(g \circ f, x) \circ \varphi = \pi_n(g, f(x)) \circ \pi_n(f, x) \circ \varphi = \pi_n(g, f(x))$$

is an isomorphism, as it is given as a composition of isomorphisms. Hence, the top arrow must likewise be an isomorphism, precisely the desired result.

The fact that weak equivalences in **Top** are closed under retracts is entirely straightforward and follows from the fact that the functors π_n preserve retract diagrams and that the class of isomorphisms in any category is closed under retracts. □

Proposition 2.7 (Hovey 2.4.9)**.** *J-cof $\subseteq \mathcal{W} \cap I'$-cof.*

Proof. TODO. □

Proposition 2.8 (Hovey 2.4.10)**.** *I'-inj $\subseteq \mathcal{W} \cap J$-inj*

Proof. TODO. □

Proposition 2.9 (Hovey 2.4.12)**.** *$\mathcal{W} \cap J$-inj $\subseteq I'$-inj*

Proof. TODO. □

Corollary 2.10 (Hovey 2.4.14)**.** *Every topological space is fibrant, i.e., given a space X, the unique map $X \to *$ is an element of J-inj.*

Proof. TODO. □

Questions:

(1) What is an example of a relative cell complex that is not a CW complex?