MODEL STRUCTURES

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Contents

1. Preliminaries

Definition 1.1 (Hovey Definition 2.1.1)**.** Suppose $\mathcal{C}$ is a cocomplete category, and λ is an ordinal. A *λ-sequence* in $\mathcal{C}$ is a colimit-preserving functor $X : \lambda \to \mathcal{C}$, commonly written as

$$X_0 \to X_1 \to \cdots \to X_\beta \to \cdots .$$

Since X preserves colimits, for all limit ordinals $\gamma < \lambda$, the induced map

$$\operatorname*{colim}_{\beta<\gamma} X_\beta \to X_\gamma$$

is an isomorphism. We refer to the map $X_0 \to \operatorname{colim}_{\beta<\lambda} X_\beta$ as the *composition* of the λ-sequence. Given a collection $\mathcal{D}$ of morphisms in $\mathcal{C}$ such that every map $X_\beta \to X_{\beta+1}$ for $\beta + 1 < \lambda$ is in $\mathcal{D}$, we refer to the composition $X_0 \to \operatorname{colim}_{\beta<\lambda} X_\beta$ as a *transfinite composition* of maps in $\mathcal{D}$.[1]

Definition 1.2 (Hovey Definition 2.1.2)**.** Let γ be a cardinal. An ordinal α is *γ-filtered* if it is a limit ordinal and, if $A \subseteq \alpha$ and $|A| \leq \gamma$, then $\sup A < \alpha$.

Given a cardinal γ, a γ-filtered category is one such that any diagram $\mathcal{D} \to \mathcal{C}$ has a cocone where $\mathcal{D}$ has $< \gamma$ arrows. A catgory is just "filtered" if it is ω-filtered, i.e., if every finite diagram in $\mathcal{C}$ admits a cocone. Note that an ordinal α is γ-filtered precisely when it is γ-filtered as a category, and in particular every ordinal is ω-filtered.

Definition 1.3 (Hovey Definition 2.1.3)**.** Suppose $\mathcal{C}$ is a comcomplete category, $\mathcal{D} \subseteq \operatorname{Mor}\mathcal{C}$ is some collection of morphisms of $\mathcal{C}$, A is an object of $\mathcal{C}$, and κ is a cardinal. We say that A is *κ-small relative to* $\mathcal{D}$ if, for all κ-filtered ordinals λ and all λ-sequences

$$X_0 \to X_1 \to \cdots \to X_\beta \to \cdots$$

such that each map $X_\beta \to X_{\beta+1}$ is in $\mathcal{D}$ for $\beta + 1 < \lambda$, the map of sets

$$\operatorname*{colim}_{\beta<\lambda} \mathcal{C}(A, X_\beta) \to \mathcal{C}(A, \operatorname*{colim}_{\beta<\lambda} X_\beta)$$

is an isomorphism. We say that A is *small relative to* $\mathcal{D}$ if it is κ-small relative to $\mathcal{D}$ for some κ. We say that A is *small* if it is small relative to $\mathcal{C}$ itself.

Recall that given a small category $\mathcal{D}$ and a functor $F : \mathcal{D} \to \mathrm{Set}$, we may explicitly construct the colimit of F as the set

$$\operatorname{colim} F := \left(\coprod_{d\in\mathcal{D}} F(d)\right) / \sim,$$

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[1] To be more precise, there may be different (isomorphic) choices of colimit $\operatorname{colim}_{\beta<\gamma} X_\beta$, which give rise to different choices of composition $X_0 \to \operatorname{colim}_{\beta<\gamma} X_\beta$. Thus, the composition of a λ-sequence is only unique up to composition by a unique isomorphism.

where the equivalence relation $\sim$ is **generated** by

$$((x \in F(d)) \sim (x' \in F(d'))) \quad \text{if} \quad (\exists (f : d \to d') \text{ with } Ff(x) = x').$$

In particular, if $\mathcal{D}$ is a filtered category then the resulting relation can be described as follows:

$$((x \in F(d)) \sim (x' \in F(d'))) \quad \text{iff} \quad (\exists\ d'',\ (f : d \to d''),\ (g : d' \to d'') \text{ with } Ff(x) = Fg(x')). \tag{1}$$

Given a cone $\eta : F \Rightarrow \underline{Y}$ under F, the unique map $\operatorname{colim} F \to Y$ maps the equivalence class of $x \in F(d)$ to the element $\eta_d(x) \in X$. We will use this characterization of the colimit in the following example.

Example 1.4 (Hovey 2.1.5)**.** Every set is small. Indeed, if A is a set we claim that A is $|A|$-small. To see this, suppose λ is an $|A|$-filtered ordinal, and X is a λ-sequence of sets. Given $\alpha < \beta < \lambda$, let $\iota_{\alpha,\beta} : X_\alpha \to X_\beta$ denote the induced morphism. We will write $X_\lambda := \operatorname{colim}_{\beta<\lambda} X_\beta$, and let $\iota : X \Rightarrow X_\lambda$ be the colimit cone, so that given $\beta < \lambda$, $\iota_\beta : X_\beta \to X_\lambda$ is the leg of the colimit cone at X_β. By composing with the functor $\mathcal{C}(A, -) : \text{Set} \to \text{Set}$, we get another λ-sequence $\{\mathcal{C}(X_\beta, A)\}_{\beta<\lambda}$. The cone ι under X induces a cone ι_* under $\mathcal{C}(X_\beta, A)$ with nadir $\mathcal{C}(A, X_\lambda)$. Let $\eta : \mathcal{C}(X_\beta, A) \Rightarrow \underline{\operatorname{colim}_{\beta<\lambda} \mathcal{C}(X_\beta, A)}$ be the colimit cone, and let $\ell : \operatorname{colim}_{\beta<\lambda} \mathcal{C}(A, X_\lambda) \to \mathcal{C}(A, X_\lambda)$ be the unique morphism of cones so that the following diagram commutes

$$\begin{tikzcd}
\mathcal{C}(A,X_0) \arrow[r, "(\iota_{0,1})_*"] \arrow[drr, "\eta_0"] \arrow[ddrr, "(\iota_0)_*"'] & \mathcal{C}(A,X_1) \arrow[r, "(\iota_{1,2})_*"] \arrow[dr, "\eta_1"] \arrow[ddr, "(\iota_1)_*"'] & \cdots \arrow[r] & \mathcal{C}(A,X_\beta) \arrow[r, "(\iota_{\beta,\beta+1})_*"] \arrow[dl, "\eta_\beta"'] \arrow[ddl, "(\iota_\beta)_*"] & \cdots \\
& & \operatorname{colim}_{\beta<\lambda} \mathcal{C}(A,X_\beta) \arrow[d, dashed, "\ell"] & & \\
& & \mathcal{C}(A,X_\lambda) & &
\end{tikzcd}$$

First, we wish to show that ℓ is surjective. Indeed, let $f : A \to X_\lambda$. For each $a \in A$, there exists some $\beta_a \in \lambda$ and some $a' \in X_{\beta_a}$ such that $f(a) = \eta_{\beta_a}(a')$ (see the preceeding discussion). Then let $\gamma := \sup_{a\in A} \beta_a$. Since $|\{\beta_a\}_{a\in A}| \leq |A|$ and λ is $|A|$-filtered, necessarily $\gamma < \lambda$. Now, define $g : A \to X_\gamma$ like so: for $a \in A$, define $g(a) := \iota_{\beta_a,\gamma}(a')$, where $a' \in X_{\beta_a}$ was chosen earlier so that $\iota_{\beta_a}(a') = f(a)$. Then we claim that $\ell(\eta_\gamma(g)) = f$. Indeed, as ℓ is a morphism of cocones, $\ell \circ \eta = \iota_*$, so that we have

$$\ell(\eta_\gamma(g)) = (\iota_\gamma)_*(g) = \iota_\gamma \circ g,$$

and given $a \in A$ we have

$$\iota_\gamma(g(a)) = \iota_\gamma(\iota_{\beta_a,\gamma}(a')).$$

By definition of a cone, $\iota_\gamma \circ \iota_{\beta_a,\gamma} = \iota_{\beta_a}$, so that

$$\ell(\eta_\gamma(g))(a) = \iota_\gamma(\iota_{\beta_a,\gamma}(a')) = \iota_{\beta_a}(a') = f(a),$$

so that indeed $\ell(\eta_\gamma(g)) = f$.

It remains to show ℓ is injective. Suppose we are given $[f], [g] \in \operatorname{colim}_{\beta<\lambda} \mathcal{C}(A, X_\beta)$ such that $\ell([f]) = \ell([g])$. Then by the preceeding discussion, there exists $\alpha, \beta < \lambda$, $f \in \mathcal{C}(A, X_\alpha)$, and $g \in \mathcal{C}(A, X_\beta)$ such that $\eta_\alpha(f) = [f]$ and $\eta_\beta(g) = [g]$. Then since $\ell \circ \eta = \iota_*$, we have

$$\ell([f]) = \ell([g]) \implies \iota_\alpha \circ f = (\iota_\alpha)_*(f) = \ell(\eta_\alpha(f)) = \ell(\eta_\beta(g)) = (\iota_\beta)_*(g) = \iota_\beta \circ g.$$

For each $a \in A$, since $\iota_\alpha(f(a)) = \iota_\beta(g(a))$, by Equation 1 there exists γ_a with $\alpha, \beta \leq \gamma_a$ such that $\iota_{\alpha,\gamma_a}(f(a)) = \iota_{\beta,\gamma_a}(g(a))$. Then let $\gamma := \sup_{a\in A} \gamma_a$. Since $|\{\gamma_a\}_{a\in A}| \leq |A|$ and λ is $|A|$-filtered, necessarily $\gamma < \lambda$. Now, in order to show $[f] = [g]$, by Equation 1 it suffices to show that $(\iota_{\alpha,\gamma})_*(f) = (\iota_{\beta,\gamma})_*(g)$. Indeed, given $a \in A$, we have

$$(\iota_{\alpha,\gamma})_*(f)(a) = \iota_{\alpha,\gamma}(f(a)) = \iota_{\gamma_a,\gamma} \circ \iota_{\alpha,\gamma_a}(f(a)) = \iota_{\gamma_a,\gamma} \circ \iota_{\beta,\gamma_a}(g(a)) = \iota_{\beta,\gamma}(g(a)) = (\iota_{\beta,\gamma})_*(g)(a),$$

precisely the desired result..

Definition 1.5 (Hovey Definition 2.1.7)**.** Let I be a class of maps in a category $\mathcal{C}$.

(1) A map is I-*injective* if it has the right lifting property w.r.t. every map in I. The class of I-injective maps is denoted I-inj (or $I_\perp$).

(2) A map is *I-projective* if it has the left lifting property w.r.t. every map in I. The class of I-projective maps is denoted I-proj (or ${}_\perp I$).
(3) A map is an *I-cofibration* if it has the left lifting property w.r.t. every I-injective map. The class of I-cofibrations is the class (I-inj)-proj and is denoted I-cof (or ${}_\perp(I_\perp)$).
(4) A map is an *I-fibration* if it has the right lifting property w.r.t. every I-projective map. The class of I-fibrations is the class (I-proj)-inj and is denoted I-fib (or $({}_\perp I)_\perp$).

The following is asserted in Hovey on pg. 30 following Definition 2.1.7, but not proven. We provide a proof.

Lemma 1.6. *Given classes A and B of maps in a category $\mathcal{C}$ with $A \subseteq B$, we have $A \subseteq {}_\perp(A_\perp)$, $A \subseteq ({}_\perp A)_\perp$, $({}_\perp(A_\perp))_\perp = A_\perp$, ${}_\perp(({}_\perp A)_\perp) = {}_\perp A$, $A_\perp \supseteq B_\perp$, ${}_\perp A \supseteq {}_\perp B$, ${}_\perp(A_\perp) \subseteq {}_\perp(B_\perp)$, and $({}_\perp A)_\perp \subseteq ({}_\perp B)_\perp$.*

Proof. TODO. □

Definition 1.7 (Hovey Definition 2.1.9)**.** Let I be a set of maps in a cocomplete category $\mathcal{C}$. A *relative I-cell complex* is a transfinite composition of pushouts of elements of I. That is, if $f : A \to B$ is a relative I-cell complex, then there is an ordinal λ and a λ-sequence $X : \lambda \to \mathcal{C}$ such that f is the composition of X and such that, for each β such that $\beta + 1 < \lambda$, there is a pushout square

$$\begin{tikzcd} C_\beta \arrow[r] \arrow[d, "g_\beta"'] & X_\beta \arrow[d] \\ D_\beta \arrow[r] & X_{\beta+1} \end{tikzcd}$$

with $g_\beta \in I$. We denote the collection of relative I-cell complexes by I-cell. We say that $A \in \mathcal{C}$ is an *I-cell complex* if the map $0 \to A$ is a relative I-cell complex.

Lemma 1.8. *Let $\mathcal{C}$ be a category and I a class of morphisms in $\mathcal{C}$. Then* I-cell *is closed under composition with isomorphisms.*

Proof. Suppose that $f : B \to C$ is an element of I-cell, and $h : A \to B$ and $g : C \to D$ are isomorphisms in $\mathcal{C}$. We wish to show $f \circ h$ and $g \circ f$ are also elements of I-cell. Since $f \in I$-cell, there exists an ordinal λ, a λ-sequence X with $X_0 = B$, and a colimit cone $\eta : X \Rightarrow \underline{C}$, such that $\eta_0 = f$.

First of all, construct a new cone $\eta' : X \Rightarrow \underline{D}$ under X where $\eta'_\beta := g \circ \eta_\beta$. It is straightforward to verify that η' is a colimit cone for X since η is a colimit cone and g is an isomorphism. Thus, $g \circ f = g \circ \eta_0 = \eta'_0 \in I$-cell, as η'_0 is the composition of a sequence of pushouts of elements of I.

On the other hand, we may construct a new λ-sequence X' by defining $X'_0 = A$, $X'_\beta = X_\beta$ for all $0 < \beta < \lambda$, the map $X'_0 \to X'_\beta$ for $0 < \beta < \lambda$ to be the composition

$$A \xrightarrow{h} B = X_0 \longrightarrow X_\beta,$$

and the composition $X'_\alpha \to X'_\beta$ to simply be the same map $X_\alpha \to X_\beta$ for $0 < \alpha \le \beta < \lambda$. It is straightforward to verify that defines a λ-sequence, and that we may define a colimit cone $\eta' : X' \Rightarrow \underline{C}$ by $\eta'_0 = \eta_0 \circ h = f \circ h$, and $\eta'_\beta = \eta_\beta$ for $0 < \beta < \lambda$. Furthermore, clearly for all $1 < \beta + 1 < \lambda$, we have the arrow $X'_\beta \to X'_{\beta+1}$ is a pushout of a map in I. Thus, in order to show $f \circ h \in I$-cell, it remains to show that the arrow $A = X'_0 \to X'_1 = X_1$ is a pushout of a map in I. Indeed, we know since $B = X_0 \to X_1$ is a pushout of a map $k : P \to Q$ in I, and it can be easily verified the diagram on the right is a pushout diagram:

$$\begin{tikzcd} P \arrow[r] \arrow[d, "k"'] & X_0 \\ Q \arrow[r] & X_1 \arrow[from=u] \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} P \arrow[r] \arrow[dd] & X_0 \arrow[r, "h^{-1}"] & X'_0 \arrow[d, "h"] \\ & & X_0 \arrow[d] \\ Q \arrow[rr] & & X'_1 \end{tikzcd}$$

□

Lemma 1.9 (Hovey 2.1.10)**.** *Suppose I is a class of maps in a cocomplete category $\mathcal{C}$. Then* I-cell $\subseteq {}_\perp(I_\perp)$.

Proof. TODO. □

Theorem 1.10 (Small Object Argument, Hovey 2.1.14)**.** *Suppose $\mathcal{C}$ is a cocomplete categroy, and I is a set of maps in $\mathcal{C}$. Suppose the domains of the maps of I are small relative to I-*cell*. Then there is a functorial factorization (γ, δ) on $\mathcal{C}$ such that for all morphisms $f \in \mathcal{C}$, the map $\gamma(f)$ is in I-*cell *and the map $\delta(f)$ is in I-*inj.

Proof. TODO. □

Corollary 1.11 (Hovey 2.1.15)**.** *Suppose that I is a set of maps in a cocomplete category $\mathcal{C}$. Suppose as well that the domains of I are small relative to I-*cell*. Then given $f : A \to B$ in ${}_{\perp}(I_{\perp})$, there is a $g : A \to C$ in I-*cell *such that f is a retract of g by a map which fixes A.*

Proof. TODO □

Definition 1.12 (Hovey Definition 2.1.17)**.** Suppose $\mathcal{C}$ is a model category. We say that $\mathcal{C}$ is *cofibrantly generated* if there are sets I and J of maps such that:

1. The domains of the maps of I are small relative to I-cell;
2. The domains of the maps of J are small relative to J-cell;
3. The class of fibrations is $J_{\perp}$; and
4. The class of trivial fibrations is $I_{\perp}$.

We refer to I as the set of *generating cofibrations* and to J as the set of *generating trivial cofibrations*. A cofibrantly generated model category is *finitely generated* if we can choose the sets I and J above so that the domains and codomains of I and J are finite relative to I-cell.

Proposition 1.13 (Hovey Proposition 2.1.18)**.** *Suppose $\mathcal{C}$ is a cofibrantly generated model category, with generating cofibrations I and generating trivial fibrations J.*

(a) The cofibrations form the class ${}_{\perp}(I_{\perp})$.
(b) Every cofibration is a retract of a relative I-cell complex.
(c) The domains of I are small relative to the cofibrations.
(d) The trivial cofibrations form the class ${}_{\perp}(J_{\perp})$.
(e) Every trivial cofibration is a retract of a relative J-cell complex.
(f) The domains of J are small relative to the trivial cofibrations.

If $\mathcal{C}$ is fibrantly generated, then the domains and codomains of I and J are finite relative to the cofibrations.

Proof. TODO. □

Theorem 1.14 (Hovey Theorem 2.1.19)**.** *Suppose $\mathcal{C}$ is a complete & cocomplete category. Suppose $\mathcal{W}$ is a subcategory of $\mathcal{C}$, and I and J are sets of maps of $\mathcal{C}$. Then there is a cofibrantly generated model structure on $\mathcal{C}$ with I as the set of generating cofibrations, J as the set of generating trivial fibrations, and $\mathcal{W}$ as the subcategory of weak equivalences if and only if the following conditions are satisfied.*

1. *The subcategory $\mathcal{W}$ has the 2-of-3 property and is closed under retracts.*
2. *The domains of I are small relative to I-*cell.
3. *The domains of J are small relative to J-*cell.
4. *J-cell $\subseteq \mathcal{W} \cap {}_{\perp}(I_{\perp})$.*
5. *$I_{\perp} \subseteq \mathcal{W} \cap J_{\perp}$.*
6. *Either $\mathcal{W} \cap {}_{\perp}(I_{\perp}) \subseteq {}_{\perp}(J_{\perp})$ or $\mathcal{W} \cap J_{\perp} \subseteq I_{\perp}$.*

Proof. TODO. □

Definition 1.15. Let $\mathcal{C}$ be a category and I a collection of morphisms in $\mathcal{C}$. Then if I is closed under transfinite composition, pushouts, and retracts then we say I is *saturated*.

2. Topological Spaces

An injective map $f : X \to Y$ in **Top** is an *inclusion* if U is open in X if and only if there is a V open in Y such that $f^{-1}(V) = U$. If f is a closed inclusion and every point in $Y \setminus f(X)$ is closed, then we call f a *closed T_1 inclusion*. We will let $\mathcal{T}$ denote the class of closed T_1 inclusions in **Top**.

The symbol D^n will denote the unit disk in $\mathbb{R}^n$, and the symbol S^{n-1} will denote the unit sphere in $\mathbb{R}^n$, so that we have the boundary inclusions $S^{n-1} \hookrightarrow D^n$. In particular, for $n = 0$ we let $D^0 = \{0\}$ and $S^{-1} = \emptyset$.

Recall: If $F : \mathcal{J} \to \mathbf{Top}$ is a functor, where $\mathcal{J}$ is a small category, the limit of F is obtained by taking the limit in the category of sets, and then topologizing it with the *initial topology*, where if $\eta : \underline{\lim F} \Rightarrow F$ is the limit cone, then the open sets in $\lim F$ are precisely the sets of the form $\eta_j^{-1}(U)$ where $j \in \mathcal{J}$ and $U \subseteq F_j$ is open. Similarly, the colimit of F is obtained by taking the colimit $\operatorname{colim} F$ in the category of sets, and declaring a set $U \subseteq \operatorname{colim} F$ to be open if and only if $\varepsilon_j^{-1}(U)$ is open in F_j for all $j \in \mathcal{J}$, where $\varepsilon : F \Rightarrow \underline{\operatorname{colim} F}$ is the colimit cone.

Given a space $X \in \mathbf{Top}$, we say that X is compactly generated or a k-space if for every subset $A \subseteq X$, A is closed in X if and only if $A \cap K$ is closed in K for all compact subspaces $K \subseteq X$.

Proposition 2.1. *If X is a compactly generated Hausdorff space, then the functor $- \times X : \mathbf{Top} \to \mathbf{Top}$ has a right adjoint (so that in particular, $- \times X$ preserves colimits).*

Proof. TODO. □

Definition 2.2. A map $f : X \to Y$ in $\mathbf{Top}$ is called a *weak equivalence* if

$$\pi_n(f, x) : \pi_n(X, x) \to \pi_n(Y, f(x))$$

is an isomorphism for all $n \geq 0$ and for all $x \in X$. We will write $\mathcal{W}$ to refer to the class of all weak equivalences in $\mathbf{Top}$.

Define the set of maps I' to consist of all the boundary inclusion $S^{n-1} \hookrightarrow D^n$ for all $n \geq 0$, and define the set J to consist of all the inclusions $D^n \hookrightarrow D^n \times I$ mapping $x \mapsto (x, 0)$ for $n \geq 0$. Then a map f will be called a *cofibration* if it is in I-cof $= {}_\perp(I'_\perp)$, and a *fibration* if it is in J-inj $= J_\perp$.

A map in I'-cell is usually called a *relative cell complex*; a relative CW-complex is a special case of a relative cell complex, where, in particular, the cells can be attached in order of their dimension. Note that in particular maps of J are relative CW complexes, hence are relative I'-cell complexes. A fibration is often known as a *Serre fibration* in the literature.

Theorem 2.3 (Hovey Theorem 2.4.19)**.** *There is a finitely generated model structure on* $\mathbf{Top}$ *with I' as the set of generating cofibrations, J as the set of generating trivial cofibrations, and the cofibrations, fibrations, and weak equivalences as above. Every object of* $\mathbf{Top}$ *is fibrant, and the cofibrant objects are retracts of relative cell complexes.*

Proof. We will apply Theorem 1.14 to get that there is a cofibrantly generated model structure on $\mathbf{Top}$ with I' as the set of generating cofibrations, J as the set of generating trivial fibrations, and $\mathcal{W}$ as the subcategory of weak equivalences. The six requirements outlined in the theorem will be verified like so:

1. $\mathcal{W}$ is a subcategory of $\mathcal{C}$ which has the 2-of-3 property and is closed under retracts: Lemma 2.9.
2. The domains of I' are small relative to I'-cell: Proposition 2.8.
3. The domains of J are small relative to J-cell: Proposition 2.8.
4. J-cell $\subseteq \mathcal{W} \cap {}_\perp(I'_\perp)$: In Proposition 2.10, we will show ${}_\perp(J_\perp) \subseteq \mathcal{W} \cap {}_\perp(I'_\perp)$, and by Lemma 1.9 J-cell $\subseteq {}_\perp(J_\perp)$.
5. $I'_\perp \subseteq \mathcal{W} \cap J_\perp$: Proposition 2.11
6. $\mathcal{W} \cap J_\perp \subseteq I'_\perp$: Proposition 2.12

It will follow by the definition of a cofibrantly generated model structure (Definition 1.12) that the fibrations in this model structure are given by $J_\perp$, which is precisely how we defined it. By Proposition 1.13, the class of cofibrations will be given by ${}_\perp(I'_\perp)$, which is likewise exactly how we defined them.

In Proposition 2.5, we will show that compact spaces are finite relative to the class $\mathcal{T}$ of closed T_1 inclusions. Hence, this model structure will be finitely generated, as the domains and codomains of I' and J are all compact, and by the reasoning given above we will have shown I'-cell $\subseteq \mathcal{T}$.

We will show that every object of $\mathbf{Top}$ is fibrant in Corollary 2.13. □

Lemma 2.4 (Hovey 2.4.1)**.** *Every topological space is small relative to the inclusions.*

Proof. As with the case of sets, we claim that every topological space X is $|X|$-small relative to the inclusions. Indeed, suppose X is a λ-sequence of inclusions in $\mathbf{Top}$. First, we claim that each map $\iota_{\alpha,\beta} : X_\alpha \to X_\beta$ is an inclusion for $\alpha \leq \beta < \lambda$. We do so by presuming $\alpha < \lambda$ fixed and performing transfinite induction on β. First of all, in the case $\beta = \alpha$, $\iota_{\alpha,\alpha}$ is the identity and therefore clearly an inclusion. Now, suppose that $\iota_{\alpha,\beta}$ is an inclusion, then we wish to show that $\iota_{\alpha,\beta+1}$ is an inclusion. Since $\iota_{\alpha,\beta+1} = \iota_{\beta,\beta+1} \circ \iota_{\alpha,\beta}$ the composition of inclusions, it too is clearly an inclusion. Finally, suppose that γ is a limit ordinal, and that the map $\iota_{\alpha,\beta}$ is an inclusion for all $\alpha \leq \beta < \gamma$. We wish to show that the map $\iota_{\alpha,\gamma}$ is an inclusion. First, we claim this map is an injection. Since γ is a limit ordinal and X is colimit-preserving, X_γ is the colimit of the diagram X restricted to those X_β such that $\beta < \gamma$, so that in particular by Equation 1 and the discussion at the beginning of this section, given $a, b \in X_\alpha$, $\iota_{\alpha,\gamma}(a) = \iota_{\alpha,\gamma}(b)$ iff $\iota_{\alpha,\beta}(a) = \iota_{\alpha,\beta}(b)$ for some $\alpha \leq \beta < \gamma$. But we know the map $\iota_{\alpha,\beta}$ is an inclusion, so that if $\iota_{\alpha,\beta}(a) = \iota_{\alpha,\beta}(b)$, then it must have been true $a = b$ in X_α. Hence, $\iota_{\alpha,\gamma}$ is injective. Finally, we wish to show that $U \subseteq X_\alpha$ is open if and only if there is some $V \subseteq X_\gamma$ open such that $\iota_{\alpha,\gamma}^{-1}(V) = U$. The backwards direction is clear as $\iota_{\alpha,\gamma}$ is continuous. Now suppose, $U \subseteq X_\alpha$ is open. Then since $\iota_{\alpha,\beta}$ is an inclusion for all $\alpha \leq \beta < \gamma$, for $\alpha \leq \beta$ there exists $V_\beta \subseteq X_\beta$ open such that $\iota_{\alpha,\beta}^{-1}(V_\beta) = U$. Now, define

$$V := \bigcup_{\alpha \leq \beta < \gamma} \iota_{\beta,\gamma}(V_\beta).$$

First of all, we claim that $\iota_{\beta,\gamma}^{-1}(V) = V_\beta$ for all $\beta < \gamma$. □

Proposition 2.5 (Hovey 2.4.2)**.** *Compact topological spaces are finite relative to the class $\mathcal{T}$ of closed T_1 inclusions.*

Proof. TODO. □

Proposition 2.6 (Hovey 2.4.5 & 2.4.6)**.** *The class $\mathcal{T}$ of closed T_1 inclusions is saturated.*

Proof. TODO. □

Lemma 2.7 (Hovey 2.4.8)**.** *$\mathcal{W} \cap \mathcal{T}$ is closed under transfinite compositions.*

Proof. TODO. □

Proposition 2.8. *The domains of I' (resp. J) are small relative to I'-*cell.

Proof. By Lemma 2.4, every space is small relative to the inclusions, and in particular every space is small relative to the class $\mathcal{T}$ of closed T_1 inclusions. Hence, it suffices to show that J-cell, I'-cell $\subseteq \mathcal{T}$. We showed above in Proposition 2.6 that $\mathcal{T}$ is saturated, and clearly every map in I' and J is a closed T_1 inclusion, so the desired result follows. □

Lemma 2.9 (Hovey Lemma 2.4.4)**.** *The weak equivalences in* **Top** *are closed under retracts and satisfy 2-of-3 axiom (so that in particular the weak equivalences form a subcategory, as clearly identities are weak equivalences).*

Proof. First we show that weak equivalences satisfy 2-of-3. Let $f : X \to Y$ and $g : Y \to Z$ be continuous functions of topological spaces.

First of all, suppose f and g are both weak equivalences. Then by functoriality of π_n, since $\pi_n(f, x)$ and $\pi_n(g, f(x))$ are isomorphisms for all $x \in X$, $\pi_n(g \circ f, x) = \pi_n(g, f(x)) \circ \pi_n(f, x)$ is likewise an isomorphism for all $x \in X$, so that $g \circ f$ is a weak equivalence.

Now, suppose that $g \circ f$ and g are weak equivalences. Pick a point $x \in X$. We wish to show that $\pi_n(f, x) : \pi_n(X, x) \to \pi_n(Y, f(x))$ is an isomorphism for all $n \geq 0$. We know that $\pi_n(g \circ f, x)$ is an isomorphism, and $\pi_n(g, f(x))$ is an isomorphism, say with inverse, φ, so that

$$\varphi \circ \pi_n(g \circ f, x) = \varphi \circ \pi_n(g, f(x)) \circ \pi_n(f, x) = \pi_n(f, x)$$

is an isomorphism, as it is a composition of isomorphisms.

Now, suppose that $g \circ f$ and f are weak equivalences. Pick a point $y \in Y$. Since $\pi_0(f)$ is an isomorphism, there exists a point $x \in X$ such that $f(x)$ belongs to the path component containing y, so that there exists some $\alpha : I \to Y$ with $\alpha(0) = f(x)$ and $\alpha(1) = f(y)$. Then consider the following diagram

$$\begin{tikzcd}
\pi_n(Y, y) \arrow[r, "\pi_n(g,y)"] \arrow[d] & \pi_n(Z, g(y)) \arrow[d] \\
\pi_n(Y, f(x)) \arrow[r, "\pi_n(g, f(x))"] & \pi_n(Z, g(f(x)))
\end{tikzcd}$$

where the left arrow is the isomorphism given by conjugation by the path α, and the right arrow is the isomorphism given by conjugation by the path $g \circ \alpha$. It is tedious yet straightforward to verify that the diagram commutes. Furthermore, we know that $\pi_n(f, x)$ and $\pi_n(g \circ f, x) = \pi_n(g, f(x)) \circ \pi_n(f, x)$ are isomorphisms for all n, so that if we denote the inverse of $\pi_n(f, x)$ by φ, then

$$\pi_n(g \circ f, x) \circ \varphi = \pi_n(g, f(x)) \circ \pi_n(f, x) \circ \varphi = \pi_n(g, f(x))$$

is an isomorphism, as it is given as a composition of isomorphisms. Hence, the top arrow must likewise be an isomorphism, precisely the desired result.

The fact that weak equivalences in **Top** are closed under retracts is entirely straightforward and follows from the fact that the functors π_n preserve retract diagrams and that the class of isomorphisms in any category is closed under retracts. $\square$

Proposition 2.10 (Hovey 2.4.9)**.** $_\perp(J_\perp) \subseteq \mathcal{W} \cap {}_\perp(I'_\perp)$.

Proof. First, in order to show $_\perp(J_\perp) \subseteq {}_\perp(I'_\perp)$, It suffices to show that $J \subseteq I'$-cell, as by Lemma 1.9 we would have $J \subseteq {}_\perp(I'_\perp)$, and

$$J \subseteq {}_\perp(I'_\perp) \implies {}_\perp(J_\perp) \subseteq {}_\perp(({}_\perp(I'_\perp))_\perp) = {}_\perp(I'_\perp),$$

where the implication and equality both follow from Lemma 1.6 which asserts that

$$A \subseteq B \implies {}_\perp(A_\perp) \subseteq {}_\perp(B_\perp) \quad \text{and} \quad ({}_\perp(A_\perp))_\perp = A_\perp.$$

Now, to show $J \subseteq I'$-cell, first consider the composition $j_n : D^n \hookrightarrow S^n \hookrightarrow D^{n+1}$, where the first map is the pushout

$$\begin{tikzcd} S^{n-1} \arrow[r, hook] \arrow[d, hook] & D^n \arrow[d] \\ D^n \arrow[r] \arrow[ur, phantom, "\lrcorner"] & S^n \end{tikzcd}$$

obtained by gluing two copies of D^n along their boundary, and the second map map is simply the inclusion $S^n \hookrightarrow D^{n+1}$, which can be written as the pushout

$$\begin{tikzcd} S^n \arrow[r, equal] \arrow[d, hook] & S^n \arrow[d, hook] \\ D^{n+1} \arrow[r, equal] \arrow[ur, phantom, "\lrcorner"] & D^{n+1} \end{tikzcd}$$

It can be seen that j_n includes D^n as a hemisphere of $S^n = \partial D^{n+1} \subseteq D^{n+1}$. Note that $D^n \times I$ is homeomorphic to D^{n+1} ("smooth out" the sharp edges of the cylinder) via some homeomorphism $h_n : D^{n+1} \to D^n \times I$, and in particular, we may define h_n so that $h_n(j_n(D^n)) = D^n \times \{0\} \subseteq D^n \times I$ by squashing the hemisphere $j_n(D^n)$ to be one of the faces of the cylinder $D^n \times I$, in which case $h_n \circ j_n : D^n \to D^n \times I$ is precisely the inclusion $D^n \hookrightarrow D^n \times I$ sending $x \mapsto (x, 0)$, and since $j_n \in I'$-cell, $h_n \circ j_n \in I'$-cell by Lemma 1.8.

Now, we claim that $_\perp(J_\perp) \subseteq \mathcal{W}$. First note that by Corollary 1.11 and Proposition 2.8, every map in $_\perp(J_\perp)$ is a retract of an element of J-cell. Furthermore, we know that $\mathcal{W}$ is closed under retracts (Lemma 2.9), so that it suffices to show that J-cell $\subseteq \mathcal{W}$. We claim it suffices to show that pushouts of maps in J are weak equivalences. Supposing we had shown this, we would have that pushouts of maps in J are weak equivalences and T_1 inclusions, as $J \subseteq \mathcal{T}$ and $\mathcal{T}$ is saturated by Proposition 2.6. Then by Lemma 2.7, we would have that J-cell $\subseteq \mathcal{W} \cap \mathcal{T}$, precisely the desired result.

Now, let $\mathcal{S}$ be the class of *inclusions of a deformation retract*, i.e., those **injective** maps $i : A \to B$ such that there exists a homotopy $H : B \times I \to B$ with $H(i(a), t) = i(a)$ for all $a \in A$, $H(b, 0) = b$ for all $b \in B$, and $H(b, 1) = i(r(b))$ for all $b \in B$ for some map $r : B \to A$[2]. We will show the following:

(1) $\mathcal{S} \subseteq \mathcal{W}$.

It suffices to show that if $i : A \to B$ belongs to $\mathcal{S}$, then i is a homotopy equivalence. Indeed, given $i : A \to B$, let $H : B \times I \to B$ and $r : B \to A$ be a homotopy and retract satisfying the conditions above. Then in particular, H is a homotopy between id_B (at time $t = 0$) and $i \circ r$ (at time $t = 1$). It

[2]Hovey has a typo here, namely, he does not specify that i must be injective. Without this specification, his assertion fails. For example, take $A = \mathbb{R}^2$, $B = \mathbb{R}$, $i(x, y) = x$, $H(b, t) = b$, and $r(b) = (b, 0)$. Then i is an inclusion of a deformation retract according to Hovey's "definition," but i is not injective and r is not a retract.

remains to show that $r \circ i = \mathrm{id}_A$. First of all, note that since $H(b, 1) = i(r(b))$ for all $b \in B$, we have $H(i(a), 1) = i(r(i(a)))$. Yet, we also know that $H(i(a), t) = i(a)$ for all $t \in I$, so $i(r(i(a))) = i(a)$, and i is injective so $r(i(a)) = a$.

(2) $J \subseteq \mathcal{S}$.

For $n \geq 0$, let $j_n : D^n \hookrightarrow D^n \times I$ denote the inclusion of D^n as the subset $D^n \times \{0\}$. Define a deformation retract $H : D^n \times I \times I \to D^n \times I$ by $(x, s, t) \mapsto (x, s(1 - t))$. Then indeed we have $H(j_n(x), t) = H(x, 0, t) = (x, 0) = j_n(x)$ for all $x \in D^n$, $H(x, t, 0) = (x, t(1 - 0)) = (x, t)$ for all $(x, t) \in D^n \times I$, and $H(x, t, 1) = (x, t(1 - 1)) = (x, 0) = j_n(r(x))$ for all $(x, t) \in D^n \times I$, where $r : D^n \times I \to D^n$ is the projection onto time zero sending $(x, t) \mapsto (x, 0)$. Finally, j_n is clearly injective. Thus, indeed $J \subseteq \mathcal{S}$.

(3) $\mathcal{S}$ is closed under pushouts.

Suppose we are given a pushout diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"'] & C \arrow[d, "j"] \\ B \arrow[r, "g"'] & D \arrow[ul, phantom, "\lrcorner", very near start] \end{tikzcd}$$

where $i \in \mathcal{S}$. Then we wish to show j in $\mathcal{S}$. First, injectivity. Suppose for the sake of a contradiction there existed nonequal $c, c' \in C$ such that $j(c) = j(c')$. Define $X := \{1, 2, 3\}$ (with the indiscrete topology, if you like), $h : C \to X$ by $c \mapsto 1$, $c' \mapsto 2$, and $C \setminus \{c, c'\} \mapsto 3$, and $k : B \to X$ by $i(f^{-1}(c)) \mapsto 1$, $i(f^{-1}(c')) \mapsto 2$, and $i(f^{-1}(C \setminus \{c, c'\})) \mapsto 3$. Then it is straightforward to see that $h \circ f = k \circ i$. Thus, there must exist a (unique) function $\ell : D \to X$ such that $\ell \circ j = h$ and $\ell \circ g = k$. But then we would have $h(c) = \ell(j(c)) = \ell(j(c')) = h(c')$ since $j(c) = j(c')$, a contradiction of the fact that $h(c) \neq h(c')$. Hence, j must be injective. Now, we look to construct H and r. Let $K : B \times I \to B$ and $r' : B \to A$ be maps satisfying the conditions for i to be an inclusion of a deformation retract.

We wish to define a homotopy $H : D \times I \to D$. Then I is a compactly generated Hausdorff space (in particular, it is compact and Hausdorff), so that the functor $- \times I : \mathbf{Top} \to \mathbf{Top}$ preserves colimits (Proposition 2.1), meaning the following is a pushout diagram:

$$\begin{tikzcd} A \times I \arrow[r, "f \times \mathrm{id}_I"] \arrow[d, "i \times \mathrm{id}_I"'] & C \times I \arrow[d, "j \times \mathrm{id}_I"] \\ B \times I \arrow[r, "g \times \mathrm{id}_I"'] & D \times I \arrow[ul, phantom, "\lrcorner", very near start] \end{tikzcd}$$

Then by the universal property of the pushout, there is a map $H : D \times I \to D$ (the dashed line) such that the following diagram commutes

$$\begin{tikzcd} A \times I \arrow[r, "f \times \mathrm{id}_I"] \arrow[d, "i \times \mathrm{id}_I"'] & C \times I \arrow[d, "j \times \mathrm{id}_I"] \arrow[dr, bend left, "\pi_1"] & \\ B \times I \arrow[r, "g \times \mathrm{id}_I"'] \arrow[dr, bend right, "K"'] & D \times I \arrow[ul, phantom, "\lrcorner", very near start] \arrow[dr, dashed, "H"] & C \arrow[d, "j"] \\ & B \arrow[r, "g"'] & D \end{tikzcd}$$

Now, note $r' \circ i = \mathrm{id}_A$. Indeed, given $a \in A$, we have $i(r'(i(a))) = K(i(a), t) = i(a)$ and i is injective, so that $r'(i(a)) = a$, as desired. Hence, there exists a unique map $r : D \to C$ (the dashed line) such that the following diagram commutes:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"'] & C \arrow[d, "j"] \arrow[ddr, bend left, equal] & \\ B \arrow[r, "g"'] \arrow[dr, bend right, "r'"'] & D \arrow[ul, phantom, "\lrcorner", very near start] \arrow[dr, dashed, "r"] & \\ & A \arrow[r, "f"'] & C \end{tikzcd}$$

Now we claim that our constructions H and r endue j with the structure of an inclusion of a deformation retract, as desired. First $c \in C$, we wish to show $H(j(c), t) = j(c)$ for all t. Indeed, we have

$$H(j(c), t) = H(j \times \mathrm{id}_I(c, t)) = j(\pi_1(c, t)) = j(c).$$

Given $d \in D$, we want to show $H(d, 0) = d$. By the explicit description of the colimit in **Top**, we know that every element of D is in the image of either j or g. If $d = j(c)$ for some c, then we have just shown $H(d, 0) = H(j(c), 0) = j(c) = d$, as desired. On the other hand, if $d = g(b)$ for some $b \in B$ we have

$$H(d, 0) = H(g \times \mathrm{id}_I(b, 0)) = g(K(b, 0)) = g(b) = d.$$

Finally, we claim that $H(d, 1) = j(r(d))$ for all $d \in D$. If $d = j(c)$ for some $c \in C$, then we have

$$H(d, 1) = H(j(c), 1) = j(c) = j(r(j(c))) = j(r(d)),$$

as desired. On the other hand, if $d = g(b)$ for some $b \in B$, then

$$H(d, 1) = H(g \times \mathrm{id}_I(b, 1)) = g(K(b, 1)) = g(i(r'(b))) = j(f(r'(b))) = j(r(g(b))) = j(r(d)). \qquad \square$$

Proposition 2.11 (Hovey 2.4.10)**.** ${I'}_\perp \subseteq \mathcal{W} \cap J_\perp$

Proof. First, by Proposition 2.10 we know ${}_\perp(J_\perp) \subseteq {}_\perp({I'}_\perp)$, and this implies ${I'}_\perp \subseteq J_\perp$, as by Lemma 1.6 we have

$${}_\perp(J_\perp) \subseteq {}_\perp({I'}_\perp) \implies J_\perp = ({}_\perp(J_\perp))_\perp \supseteq ({}_\perp({I'}_\perp))_\perp = {I'}_\perp.$$

Thus, it suffices to show that ${I'}_\perp \subseteq \mathcal{W}$. Now, suppose $p : X \to Y$ is in ${I'}_\perp$, and $x \in X$. We wish to show that the map $\pi_n(p, x) : \pi_n(X, x) \to \pi_n(Y, p(x))$ is an isomorphism for all n.

First we show that $\pi_n(p, x)$ is surjective. Let $g : (S^n, *) \to (Y, p(x))$ be a map. Then we have the following commutative diagram

$$\begin{tikzcd} * \arrow[r] \arrow[d] & X \arrow[d, "p"] \\ S^n \arrow[r, "g"] & Y \end{tikzcd}$$

where the top arrow picks out x. Note that the map $* \to S^n$ may be realized as a pushout of the diagram $D^n \leftarrow S^{n-1} \to *$, so that $* \to S^n$ belongs to I'-cell, and therefore ${}_\perp({I'}_\perp)$ by Lemma 1.9. Then by Lemma 1.6, $({}_\perp({I'}_\perp))_\perp = {I'}_\perp$, and $p \in {I'}_\perp$, so that p has the right lifting property with respect to every element of ${}_\perp({I'}_\perp)$, and in particular, the map $* \to S^n$. Thus, the above diagram has a lift $f : (S^n, *) \to (X, x)$ such that $p \circ f = g$, so that $\pi_n(p, x)([f]) = [p \circ f] = [g]$, as desired.

Finally, we show that $\pi_n(p, x)$ is injective. Suppose we have two maps $f, g : (S^n, *) \to (X, x)$ such that $p \circ f$ and $p \circ g$ represent the same element of $\pi_n(Y, p(x))$. Then there is a homotopy $H : S^n \times I \to Y$ such that $H(x, 0) = p(f(x))$, $H(x, 1) = p(g(x))$, and $H(*, t) = p(x)$ for all t. We construct the following pushouts:

FINISH. $\square$

Proposition 2.12 (Hovey 2.4.12)**.** $\mathcal{W} \cap J_\perp \subseteq {I'}_\perp$

Proof. TODO. $\square$

Corollary 2.13 (Hovey 2.4.14)**.** *Every topological space is fibrant, i.e., given a space X, the unique map $X \to *$ is an element of $J_\perp$.*

Proof. TODO. $\square$

Questions:

(1) How to construct the map $S^n \vee S^n \to S^n \wedge I_+$ as abstractly as possible?