

# MODEL STRUCTURES

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This document follows Mark Hovey's *Model Categories*, and its intention is to reproduce the proofs of several standard model categories in explicit detail.

## 1. PRELIMINARIES

We work with von Neumann ordinals, i.e., an ordinal is a transitive set of ordinals (this definition is not circular, the empty set is an ordinal which we call “0”). In the following discussion, let  $\alpha$  and  $\beta$  be ordinals. We write  $\alpha + 1$  to denote the successor ordinal  $\alpha \cup \{\alpha\}$ . We write  $\alpha < \beta$  to mean  $\alpha \in \beta$ , and  $\alpha \leq \beta$  denotes any of the equivalent conditions: (1)  $\alpha < \beta$  or  $\alpha = \beta$ , (2)  $\alpha \in \beta + 1$ , (3)  $\alpha \subseteq \beta$ . Given a collection of ordinals  $B$ , we write  $\sup B$  or  $\sup_{\beta \in B} \beta$  to denote the ordinal  $\bigcup_{\beta \in B} \beta$ . We define the sum of ordinals  $\alpha$  and  $\beta$  recursively:  $\alpha + 0 := \alpha$ ,  $\alpha + (\beta + 1) := (\alpha + \beta) + 1$ , and  $\alpha + \beta := \sup_{\delta < \beta} (\alpha + \delta)$  when  $\beta$  is a limit ordinal. Note that addition of ordinals is not commutative, but it is associative, and continuous in its right argument: given an ordinal  $\alpha$  and a collection of ordinals  $B$ ,  $\alpha + \sup B = \sup_{\beta \in B} (\alpha + \beta)$ . We say an ordinal  $\lambda$  is a *limit ordinal* if either of the following equivalent conditions hold: (1)  $\lambda = \sup_{\beta < \lambda} \beta$  or (2)  $\lambda \neq \beta + 1$  for all ordinals  $\beta$ . Note that 0 is a limit ordinal under our definition. We may regard an ordinal  $\alpha$  as a poset category, in which case the colimit in  $\alpha$  is given by the supremum. We let **Ord** denote the poset category of all (small) ordinals, so there exists a unique arrow  $\alpha \rightarrow \beta$  if  $\alpha \leq \beta$ . Given a set  $X$ , we write  $|X|$  to denote its *cardinality*, i.e.,  $|X|$  is the least ordinal  $\alpha$  such that there exists a bijection between  $\alpha$  and  $X$ . A cardinal number is an ordinal which is the cardinality of some set  $X$ .

**Definition 1.1** (Hovey Definition 2.1.1). Suppose  $\mathcal{C}$  is a cocomplete category, and  $\lambda$  is an ordinal. A  $\lambda$ -sequence in  $\mathcal{C}$  is a colimit-preserving functor  $X : \lambda \rightarrow \mathcal{C}$ , commonly written as

$$X_0 \rightarrow X_1 \rightarrow \cdots \rightarrow X_\beta \rightarrow \cdots.$$

Since  $X$  preserves colimits, for all limit ordinals  $\gamma < \lambda$ , the arrows  $X_\alpha \rightarrow X_\gamma$  for  $\alpha < \gamma$  form a colimit cone under  $\{X_\alpha\}_{\alpha < \gamma}$ . We refer to the map  $X_0 \rightarrow \operatorname{colim}_{\beta < \lambda} X_\beta$  as the *composition* of the  $\lambda$ -sequence. Given a collection  $\mathcal{D}$  of morphisms in  $\mathcal{C}$  such that every map  $X_\beta \rightarrow X_{\beta+1}$  for  $\beta + 1 < \lambda$  is in  $\mathcal{D}$ , we refer to the composition  $X_0 \rightarrow \operatorname{colim}_{\beta < \lambda} X_\beta$  as a *transfinite composition* of arrows in  $\mathcal{D}$ .<sup>1</sup>

Of particular importance to us will be collections of arrows which are *closed under transfinite composition*, i.e., collections  $\mathcal{D}$  for which given any ordinal  $\lambda$  and  $\lambda$ -sequence  $X$  of arrows in  $\mathcal{D}$ , for any choice of colimit  $\operatorname{colim} X$ , the canonical map  $X_0 \rightarrow \operatorname{colim} X$  is also in  $\mathcal{D}$ . We prove the following useful result about when a class of morphisms is closed under transfinite composition:

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*Date:* May 1, 2023.

<sup>1</sup>To be more precise, there may be different (isomorphic) choices of colimit  $\operatorname{colim}_{\beta < \gamma} X_\beta$ , which give rise to different choices of composition  $X_0 \rightarrow \operatorname{colim}_{\beta < \gamma} X_\beta$ . Thus, the composition of a  $\lambda$ -sequence is only unique up to composition by a unique isomorphism.

**Lemma 1.2.** *Let  $\mathcal{C}$  be a category, and  $\mathcal{D}$  a collection of arrows in  $\mathcal{C}$  satisfying the following properties:  $\mathcal{D}$  is closed under composition with isomorphisms, and given an ordinal  $\lambda$  and a  $\lambda$ -sequence  $X : \lambda \rightarrow \mathcal{C}$  of arrows in  $\mathcal{D}$  (so  $X_\beta \rightarrow X_{\beta+1}$  belongs to  $\mathcal{D}$  for all  $\beta + 1 < \lambda$ ), if we then get for free that  $X_\alpha \rightarrow X_\beta$  belongs to  $\mathcal{D}$  for all  $\alpha \leq \beta < \lambda$ , then  $\mathcal{D}$  is closed under transfinite composition.*

*Proof.* Let  $\lambda$  be an ordinal, and  $X : \lambda \rightarrow \mathcal{C}$  a  $\lambda$ -sequence of arrows in  $\mathcal{D}$ . First, suppose  $\lambda = \mu + 1$  is a successor ordinal. Since we know that any transfinite composition of  $X$  may be obtained from another by composing with an isomorphism and  $\mathcal{D}$  is closed under composition with isomorphisms, it suffices to show there exists *some* transfinite composition of  $X$  belonging to  $\mathcal{D}$ . We know  $\sup_{\beta < \lambda} \beta = \sup_{\beta < \mu+1} \beta = \mu$ , and  $X$  is colimit preserving, so that  $X_\mu$  is a colimit of the diagram  $X$  via the arrows  $X_\alpha \rightarrow X_\mu$  for  $\alpha < \lambda = \mu + 1$ . But we know in particular that  $X_0 \rightarrow X_\mu$  belongs to  $\mathcal{D}$ , so we are done.

Conversely, suppose  $\lambda$  is a limit ordinal. Let  $j : X \Rightarrow \underline{X}_\lambda$  be a colimit cone for  $X$ . We may use  $j$  to extend  $X$  to a  $(\lambda + 1)$ -sequence in the obvious way (so for  $\alpha < \lambda$ , the structure map  $X_\alpha \rightarrow X_\lambda$  is given by  $j$  and the arrow  $X_\lambda \rightarrow X_\lambda$  is the identity, as is necessary). Further note that  $X$  is still a sequence of arrows in  $\mathcal{D}$ , as given  $\beta + 1 < \lambda + 1$ , so  $\beta + 1 \leq \lambda$ , it is not possible that  $\beta + 1 = \lambda$  as  $\lambda$  is a limit ordinal, in which case we know the map  $X_\beta \rightarrow X_{\beta+1}$  belongs to  $\mathcal{D}$  as  $\beta + 1 < \lambda$ . Hence, unravelling definitions and applying the asserted property of  $\mathcal{D}$ , we get for free that  $j_0 : X_0 \rightarrow X_\lambda$  belongs to  $\mathcal{D}$ .  $\square$

**Lemma 1.3.** *Given a cocomplete category  $\mathcal{C}$  and a collection  $\mathcal{D}$  of arrows in  $\mathcal{C}$ , if  $\mathcal{D}$  is closed under transfinite composition, then given any limit ordinal  $\lambda$  and  $\lambda$ -sequence  $X : \lambda \rightarrow \mathcal{C}$ , for all  $\alpha < \lambda$  the canonical map  $X_\alpha \rightarrow \text{colim } X$  belongs to  $\mathcal{D}$ .*

*Proof Sketch.* Let  $\alpha < \lambda$ , and fix a colimit cone  $j : X \Rightarrow \text{colim } X$ . Define  $S := \{\beta : \alpha \leq \beta \leq \lambda\} \subseteq \lambda + 1$ . Define a map  $\phi : S \rightarrow \mathbf{Ord}$  via transfinite recursion. Let  $\phi(\alpha) = 0$ . Supposing  $\phi(\beta)$  has been defined, let  $\phi(\beta + 1) = \phi(\beta) + 1$ . Finally, supposing  $\alpha < \gamma \leq \lambda$  is a limit ordinal and  $\phi(\beta)$  has been defined for  $\alpha \leq \beta < \gamma$ , define  $\phi(\gamma) = \sup_{\alpha \leq \beta < \gamma} \phi(\beta)$ . It is straightforward to verify that  $\phi$  is order preserving, sends limit ordinals to limit ordinals, and satisfies  $\alpha + \phi(\beta) = \beta$  for all  $\alpha \leq \beta \leq \lambda$ .

Now, construct a  $\phi(\lambda)$ -sequence  $Y : \phi(\lambda) \rightarrow \mathcal{C}$  by  $Y_\beta := X_{\alpha+\beta}$ , and given  $\beta \leq \beta' < \phi(\lambda)$ , define the map  $Y_\beta \rightarrow Y_{\beta'}$  to be the arrow  $X_{\alpha+\beta} \rightarrow X_{\alpha+\beta'}$  for  $X$ . Checking that  $Y$  is functorial and colimit-preserving follows directly from the fact that  $X$  is functorial and colimit-preserving. Then it can be seen that the  $j_{\alpha+\beta}$ 's for  $0 \leq \beta < \phi(\lambda)$  restrict to a colimit cone under  $Y$ . Since  $Y$  is a  $\phi(\lambda)$ -sequence in  $\mathcal{D}$  and  $\mathcal{D}$  is closed under transfinite compositions, it follows that  $j_\alpha \in \mathcal{D}$ , as desired.  $\square$

**Definition 1.4** (Hovey Definition 2.1.2). Let  $\gamma$  be a cardinal. An ordinal  $\alpha$  is  $\gamma$ -filtered if it is a limit ordinal and, if  $A \subseteq \alpha$  and  $|A| \leq \gamma$ , then  $\sup A < \alpha$ .

Given a cardinal  $\gamma$ , a  $\gamma$ -filtered category  $\mathcal{C}$  is one such that any diagram  $\mathcal{D} \rightarrow \mathcal{C}$  has a cocone when  $\mathcal{D}$  has  $< \gamma$  arrows. A category is just “filtered” if it is  $\omega$ -filtered, i.e., if every finite diagram in  $\mathcal{C}$  admits a cocone. Note that an ordinal  $\alpha$  is  $\gamma$ -filtered precisely when it is  $\gamma$ -filtered as a category, and in particular every ordinal is  $\omega$ -filtered.

**Definition 1.5** (Hovey Definition 2.1.3). Suppose  $\mathcal{C}$  is a comcomplete category,  $\mathcal{D} \subseteq \text{Mor } \mathcal{C}$  is some collection of morphisms of  $\mathcal{C}$ ,  $A$  is an object of  $\mathcal{C}$ , and  $\kappa$  is a cardinal. We say that  $A$  is  $\kappa$ -small relative to  $\mathcal{D}$  if, for all  $\kappa$ -filtered ordinals  $\lambda$  and all  $\lambda$ -sequences

$$X_0 \rightarrow X_1 \rightarrow \cdots \rightarrow X_\beta \rightarrow \cdots$$

such that each map  $X_\beta \rightarrow X_{\beta+1}$  is in  $\mathcal{D}$  for  $\beta + 1 < \lambda$ , the canonical map of sets

$$\text{colim}_{\beta < \lambda} \mathcal{C}(A, X_\beta) \rightarrow \mathcal{C}(A, \text{colim}_{\beta < \lambda} X_\beta)$$

is an isomorphism. We say that  $A$  is *small relative to  $\mathcal{D}$*  if it is  $\kappa$ -small relative to  $\mathcal{D}$  for some  $\kappa$ . We say that  $A$  is *small* if it is small relative to  $\mathcal{C}$  itself.

**Definition 1.6** (Hovey Definition 2.1.4). Suppose  $\mathcal{C}$  is a cocomplete category,  $\mathcal{D}$  is a collection of morphisms of  $\mathcal{C}$ , and  $A$  is an object of  $\mathcal{C}$ . We say that  $A$  is *finite relative to  $\mathcal{D}$*  if  $A$  is  $\kappa$ -small relative to  $\mathcal{D}$  for some finite cardinal  $\kappa$ . We say  $A$  is *finite* if it is finite relative to  $\mathcal{C}$  itself. In particular, since *every* limit ordinal is  $\kappa$ -filtered for any finite cardinal  $\kappa$ , for an object  $A$  to be finite relative to  $\mathcal{D}$ , maps from  $A$  must commute with colimits of *arbitrary*  $\lambda$ -sequences for every limit ordinal  $\lambda$ .

**Remark 1.7.** Recall that given a small category  $\mathcal{D}$  and a functor  $F : \mathcal{D} \rightarrow \mathbf{Set}$ , we may explicitly construct the colimit of  $F$  as the set

$$\operatorname{colim} F := \left( \coprod_{d \in \mathcal{D}} F(d) \right) / \sim,$$

where the equivalence relation  $\sim$  is **generated** by

$$((x \in F(d)) \sim (x' \in F(d'))) \quad \text{if} \quad (\exists (f : d \rightarrow d') \text{ with } Ff(x) = x').$$

In particular, if  $\mathcal{D}$  is a filtered category then the resulting relation can be described as follows:

$$((x \in F(d)) \sim (x' \in F(d'))) \quad \text{iff} \quad (\exists d'', (f : d \rightarrow d'), (g : d' \rightarrow d'') \text{ with } Ff(x) = Fg(x')).$$

Then the colimit cone  $\eta : F \Rightarrow \operatorname{colim} F$  is defined by  $\eta_d(x) = [x]$  for  $d \in \mathcal{D}$  and  $x \in F(d)$ , where  $[x]$  denotes the equivalence class of  $x$  in  $\operatorname{colim} F$ . Given a cone  $\varepsilon : F \Rightarrow \underline{Y}$  under  $F$ , the unique map  $\operatorname{colim} F \rightarrow Y$  maps an equivalence class  $[x]$  represented by an element  $x \in F(d)$  to the element  $\varepsilon_d(x)$ .

Similarly, we may explicitly construct the limit of a functor  $F : \mathcal{D} \rightarrow \mathbf{Set}$  as the subset

$$\lim F = \left\{ (x_d)_{d \in \mathcal{D}} \in \prod_{d \in \mathcal{D}} F(d) : \forall (d_i \xrightarrow{\alpha} d_j) \in \mathcal{D}, F(\alpha)(x_{d_i}) = x_{d_j} \right\},$$

in which case the limit cone is simply the restriction of the projection maps for  $\prod_{d \in \mathcal{D}} F(d)$  to  $\lim F$ .

Now we unravel what the “canonical map” of [Definition 1.5](#) is. Suppose we are given a cocomplete category  $\mathcal{C}$ , an element  $A \in \mathcal{C}$ , an ordinal  $\lambda$ , and a  $\lambda$ -sequence  $X : \lambda \rightarrow \mathcal{C}$ . For  $\alpha \leq \beta < \lambda$ , let  $\iota_{\alpha, \beta}$  be the map  $X_\alpha \rightarrow X_\beta$ . Let  $\eta : X \Rightarrow \operatorname{colim} X$  be the colimit cone. By whiskering the colimit cone along the functor  $\mathcal{C}(A, -)$ , we get a cone  $\mathcal{C}(A, \eta) : \{\mathcal{C}(A, X_\beta)\}_{\beta < \lambda} \Rightarrow \mathcal{C}(A, \operatorname{colim} X)$ . Then if we let  $\varepsilon : \{\mathcal{C}(A, X_\beta)\}_{\beta < \lambda} \Rightarrow \operatorname{colim}_{\beta < \lambda} \mathcal{C}(A, X_\beta)$  be the colimit cone, the universal property of the colimit gives us the canonical map  $\ell : \operatorname{colim}_{\beta < \lambda} \mathcal{C}(A, X_\beta) \rightarrow \mathcal{C}(A, \operatorname{colim} X)$ , so that the following diagram commutes:

$$\begin{array}{ccccccc} \mathcal{C}(A, X_0) & \xrightarrow{(\iota_{0,1})_*} & \mathcal{C}(A, X_1) & \xrightarrow{(\iota_{1,2})_*} & \dots & \xrightarrow{(\iota_{\beta, \beta+1})_*} & \mathcal{C}(A, X_\beta) \\ & \searrow \varepsilon_0 & \searrow \varepsilon_1 & & & & \searrow \varepsilon_\beta \\ & & & & & & \operatorname{colim}_{\beta < \lambda} \mathcal{C}(A, X_\beta) \\ & \searrow (\eta_0)_* & \searrow (\eta_1)_* & & & & \searrow (\eta_\beta)_* \\ & & & & & & \downarrow \ell \\ & & & & & & \mathcal{C}(A, \operatorname{colim} X) \end{array}$$

In particular, by [Remark 1.7](#), we know elements of  $\operatorname{colim}_{\beta < \lambda} \mathcal{C}(A, X_\beta)$  are equivalence classes of arrows  $f : A \rightarrow X_\beta$  for  $\beta < \lambda$  under the relation  $[f : A \rightarrow X_\beta] = [g : A \rightarrow X_{\beta'}]$  iff there exists  $\beta'' \geq \beta, \beta'$  with  $\iota_{\beta, \beta''} \circ f = \iota_{\beta', \beta''} \circ g$ , and the map  $\varepsilon_\beta$  sends an arrow  $f \in \mathcal{C}(A, X_\beta)$  to the element  $[f]$ . Then it follows that  $\ell([f : A \rightarrow X_\beta]) = \eta_\beta \circ f$ . Thus, this gives us the following result:

**Proposition 1.8.** *Given a cocomplete category  $\mathcal{C}$ , a collection  $\mathcal{D}$  of arrows in  $\mathcal{C}$ , an object  $A$  in  $\mathcal{C}$ , and a cardinal  $\kappa$ ,  $A$  is  $\kappa$ -small relative to  $\mathcal{D}$ , if, for all  $\kappa$ -filtered ordinals  $\lambda$  and all  $\lambda$ -sequences  $X : \lambda \rightarrow \mathcal{C}$  such that the map  $X_\beta \rightarrow X_{\beta+1}$  belongs to  $\mathcal{D}$  for all  $\beta + 1 < \lambda$ , given any colimit  $\operatorname{colim} X$  for  $X$ , the following holds:*

- (i) *Given arrows  $f : A \rightarrow X_\alpha$  and  $g : A \rightarrow X_\beta$  in  $\mathcal{C}$ , if  $f$  and  $g$  agree in the colimit (i.e., if the compositions  $A \xrightarrow{f} X_\alpha \rightarrow \operatorname{colim} X$  and  $A \xrightarrow{g} X_\beta \rightarrow \operatorname{colim} X$  are equal), then  $f$  and  $g$  are equal in some stage of the colimit (i.e., there exists  $\gamma < \lambda$  with  $\alpha, \beta \leq \gamma$  such that the compositions  $A \xrightarrow{f} X_\alpha \rightarrow X_\gamma$  and  $A \xrightarrow{g} X_\beta \rightarrow X_\gamma$  are equal).*
- (ii) *Any arrow  $f : A \rightarrow \operatorname{colim} X$  factors through some stage of the colimit (i.e., there exists  $\beta < \lambda$  and an arrow  $\tilde{f} : A \rightarrow X_\beta$  such that the composition  $A \xrightarrow{\tilde{f}} X_\beta \rightarrow \operatorname{colim} X$  equals  $f$ ).*

In terms of the canonical map  $\operatorname{colim}_{\beta < \lambda} \mathcal{C}(A, X_\beta) \rightarrow \mathcal{C}(A, \operatorname{colim} X)$ , the first condition shows injectivity, while the second shows surjectivity.

We will use the characterization of smallness given by this remark whenever proving smallness arguments, as in the following example.

**Example 1.9** (Hovey 2.1.5). Every set is small. Indeed, if  $A$  is a set we claim that  $A$  is  $|A|$ -small. To see this, suppose  $\lambda$  is an  $|A|$ -filtered ordinal, and  $X$  is a  $\lambda$ -sequence of sets. First of all, by [Remark 1.7](#), the elements of  $\text{colim } X$  are equivalence classes of elements  $a \in X_\alpha$  where  $a \in X_\alpha$  and  $b \in X_\beta$  represent the same element of  $\text{colim } X$  iff there exists  $\alpha, \beta \leq \gamma < \lambda$  so that  $a$  and  $b$  are sent to the same elements by the maps  $X_\alpha \rightarrow X_\gamma$  and  $X_\beta \rightarrow X_\gamma$ , respectively. Now, we show the conditions of [Proposition 1.8](#).

First, we need to show that given  $\alpha, \beta < \lambda$ , if  $f : A \rightarrow X_\alpha$  and  $g : A \rightarrow X_\beta$  such that the compositions  $\bar{f} : A \xrightarrow{f} X_\alpha \rightarrow \text{colim } X$  and  $\bar{g} : A \xrightarrow{g} X_\beta \rightarrow \text{colim } X$  are equal, then  $f$  and  $g$  are equal in some stage of the colimit. For each  $a \in A$ , since  $\bar{f}(a) = \bar{g}(a)$  in  $\text{colim } X$ , by the above characterization of  $\text{colim } X$ , there exists  $\gamma_a < \lambda$  with  $\alpha, \beta \leq \gamma_a$  such that  $f(a)$  and  $g(a)$  are sent to the same element in  $X_{\gamma_a}$  by the maps  $X_\alpha \rightarrow X_{\gamma_a}$  and  $X_\beta \rightarrow X_{\gamma_a}$ , respectively. Then let  $\gamma := \sup_{a \in A} \gamma_a$ . Since  $|\{\gamma_a\}_{a \in A}| \leq |A|$  and  $\lambda$  is  $|A|$ -filtered, necessarily  $\gamma < \lambda$ . Then clearly the compositions  $A \xrightarrow{f} X_\alpha \rightarrow X_\gamma$  and  $A \xrightarrow{g} X_\beta \rightarrow X_\gamma$  agree for all  $a \in A$ .

Secondly, we wish to show that given a map  $f : A \rightarrow \text{colim } X$ , that  $f$  factors through  $X_\beta \rightarrow \text{colim } X$  for some  $\beta < \lambda$ . For each  $a \in A$ , by the explicit description of  $\text{colim } X$ , there exists some  $\beta_a < \lambda$  and some  $x_a \in X_{\beta_a}$  such that  $f(a) = [x_a]$ . Then let  $\beta := \sup_{a \in A} \beta_a$ , so  $\beta < \lambda$  as  $X$  is  $|A|$ -filtered. Now define  $\tilde{f} : A \rightarrow X_\beta$  like so: for  $a \in A$ , define  $\tilde{f}(a) \in X_\beta$  to be the image of  $x_a$  along the map  $X_{\beta_a} \rightarrow X_\beta$ . Then clearly the composition  $f' : A \xrightarrow{\tilde{f}} X_\beta \rightarrow \text{colim } X$  is equal to  $f$ , by unravelling definitions.

**Definition 1.10** (Hovey Definition 2.1.7). Let  $I$  be a class of maps in a category  $\mathcal{C}$ .

- (1) A map is *I-injective* if it has the right lifting property w.r.t. every map in  $I$ . The class of *I-injective* maps is denoted  $I\text{-inj}$  (or  $I_\perp$ ).
- (2) A map is *I-projective* if it has the left lifting property w.r.t. every map in  $I$ . The class of *I-projective* maps is denoted  $I\text{-proj}$  (or  ${}_\perp I$ ).
- (3) A map is an *I-cofibration* if it has the left lifting property w.r.t. every *I-injective* map. The class of *I-cofibrations* is the class  $(I\text{-inj})\text{-proj}$  and is denoted  $I\text{-cof}$  (or  ${}_\perp(I_\perp)$ ).
- (4) A map is an *I-fibration* if it has the right lifting property w.r.t. every *I-projective* map. The class of *I-fibrations* is the class  $(I\text{-proj})\text{-inj}$  and is denoted  $I\text{-fib}$  (or  $({}_\perp I)_\perp$ ).

The following is asserted in Hovey on pg. 30 following Definition 2.1.7, but not proven. We provide a proof.

**Lemma 1.11.** *Given classes  $A$  and  $B$  of maps in a category  $\mathcal{C}$  with  $A \subseteq B$ , we have  $A \subseteq {}_\perp(A_\perp)$ ,  $A \subseteq ({}_\perp A)_\perp$ ,  $({}_\perp(A_\perp))_\perp = A_\perp$ ,  ${}_\perp({}_\perp A)_\perp = {}_\perp A$ ,  $A_\perp \supseteq B_\perp$ ,  ${}_\perp A \supseteq {}_\perp B$ ,  ${}_\perp(A_\perp) \subseteq {}_\perp(B_\perp)$ , and  $({}_\perp A)_\perp \subseteq ({}_\perp B)_\perp$ .*

*Proof.* Each of these amount to unravelling definitions and are entirely straightforward.  $\square$

**Definition 1.12** (Hovey Definition 2.1.9). Let  $I$  be a set of maps in a cocomplete category  $\mathcal{C}$ . A *relative I-cell complex* is a transfinite composition of pushouts of elements of  $I$ . That is, if  $f : A \rightarrow B$  is a relative *I-cell complex*, then there is an ordinal  $\lambda$  and a  $\lambda$ -sequence  $X : \lambda \rightarrow \mathcal{C}$  such that  $f$  is the composition of  $X$  and such that, for each  $\beta$  such that  $\beta + 1 < \lambda$ , there is a pushout square

$$\begin{array}{ccc} C_\beta & \longrightarrow & X_\beta \\ g_\beta \downarrow & & \downarrow \\ D_\beta & \longrightarrow & X_{\beta+1} \end{array}$$

with  $g_\beta \in I$ . We denote the collection of relative *I-cell complexes* by  $I\text{-cell}$ . We say that  $A \in \mathcal{C}$  is an *I-cell complex* if the map  $0 \rightarrow A$  is a relative *I-cell complex*.

**Lemma 1.13.** *Let  $\mathcal{C}$  be a category and  $I$  a class of morphisms in  $\mathcal{C}$ . Then  $I\text{-cell}$  is closed under composition with isomorphisms.*

*Proof Sketch.* Suppose that  $f : B \rightarrow C$  is an element of  $I\text{-cell}$ , and  $h : A \rightarrow B$  and  $g : C \rightarrow D$  are isomorphisms in  $\mathcal{C}$ . We wish to show  $f \circ h$  and  $g \circ f$  are also elements of  $I\text{-cell}$ . Since  $f \in I\text{-cell}$ , there exists an ordinal  $\lambda$ , a  $\lambda$ -sequence  $X$  with  $X_0 = B$ , and a colimit cone  $\eta : X \Rightarrow \underline{C}$ , such that  $\eta_0 = f$ .

First of all, construct a new cone  $\eta' : X \Rightarrow \underline{D}$  under  $X$  where  $\eta'_\beta := g \circ \eta_\beta$ . It is straightforward to verify that  $\eta'$  is a colimit cone for  $X$  since  $\eta$  is a colimit cone and  $g$  is an isomorphism. Thus,  $g \circ f = g \circ \eta_0 = \eta'_0 \in I\text{-cell}$ , as  $\eta'_0$  is the composition of a sequence of pushouts of elements of  $I$ .

On the other hand, we may construct a new  $\lambda$ -sequence  $X'$  by defining  $X'_0 = A$ ,  $X'_\beta = X_\beta$  for all  $0 < \beta < \lambda$ , the map  $X'_0 \rightarrow X'_\beta$  for  $0 < \beta < \lambda$  to be the composition

$$A \xrightarrow{h} B = X_0 \longrightarrow X_\beta,$$

and the composition  $X'_\alpha \rightarrow X'_\beta$  to simply be the same map  $X_\alpha \rightarrow X_\beta$  for  $0 < \alpha \leq \beta < \lambda$ . It is straightforward to verify that defines a  $\lambda$ -sequence, and that we may define a colimit cone  $\eta' : X' \Rightarrow \underline{C}$  by  $\eta'_0 = \eta_0 \circ h = f \circ h$ , and  $\eta'_\beta = \eta_\beta$  for  $0 < \beta < \lambda$ . Furthermore, clearly for all  $1 < \beta + 1 < \lambda$ , we have the arrow  $X'_\beta \rightarrow X'_{\beta+1}$  is a pushout of a map in  $I$ . Thus, in order to show  $f \circ h \in I\text{-cell}$ , it remains to show that the arrow  $X'_0 = A \rightarrow X_1 = X'_1$  is a pushout of a map in  $I$ . Indeed, we know  $B = X_0 \rightarrow X_1$  is a pushout of a map  $k : P \rightarrow Q$  in  $I$ , and it can be easily verified the diagram on the right is a pushout diagram as the left diagram is a pushout diagram and  $h$  is an isomorphism

$$\begin{array}{ccc} P & \longrightarrow & X_0 \\ \downarrow k & & \downarrow \\ Q & \longrightarrow & X_1 \end{array} \quad \rightsquigarrow \quad \begin{array}{ccc} P & \longrightarrow & X_0 \xrightarrow{h^{-1}} X'_0 \\ \downarrow & & \downarrow h \\ & & X_0 \\ & & \downarrow \\ Q & \longrightarrow & X'_1 \end{array}$$

□

**Definition 1.14.** Let  $\mathcal{C}$  be a category and  $I$  a collection of morphisms in  $\mathcal{C}$ . Then if  $I$  is closed under transfinite composition, pushouts, and retracts then we say  $I$  is *saturated*.

**Lemma 1.15.** Suppose  $I$  is a class of maps in a cocomplete category  $\mathcal{C}$ . Then  $\perp I$  is saturated.

*Proof.* **TODO.**

□

This yields the following Corollary:

**Corollary 1.16** (Hovey 2.1.10). Given a cocomplete category  $\mathcal{C}$  and a class of maps  $I$  in  $\mathcal{C}$ ,  $I\text{-cell} \subseteq \perp(I_\perp)$ .

**Theorem 1.17** (Small Object Argument, Hovey 2.1.14). Suppose  $\mathcal{C}$  is a cocomplete category, and  $I$  is a set of maps in  $\mathcal{C}$ . Suppose the domains of the maps of  $I$  are small relative to  $I\text{-cell}$ . Then there is a functorial factorization  $(\gamma, \delta)$  on  $\mathcal{C}$  such that for all morphisms  $f \in \mathcal{C}$ , the map  $\gamma(f)$  is in  $I\text{-cell}$  and the map  $\delta(f)$  is in  $I\text{-inj}$ .

*Proof.* **TODO.**

□

**Corollary 1.18** (Hovey 2.1.15). Suppose that  $I$  is a set of maps in a cocomplete category  $\mathcal{C}$ . Suppose as well that the domains of  $I$  are small relative to  $I\text{-cell}$ . Then given  $f : A \rightarrow B$  in  $\perp(I_\perp)$ , there is a  $g : A \rightarrow C$  in  $I\text{-cell}$  such that  $f$  is a retract of  $g$  by a map which fixes  $A$ .

*Proof.* **TODO**

□

**Definition 1.19** (Hovey Definition 2.1.17). Suppose  $\mathcal{C}$  is a model category. We say that  $\mathcal{C}$  is *cofibrantly generated* if there are sets  $I$  and  $J$  of maps such that:

1. The domains of the maps of  $I$  are small relative to  $I\text{-cell}$ ;
2. The domains of the maps of  $J$  are small relative to  $J\text{-cell}$ ;
3. The class of fibrations is  $J_\perp$ ; and
4. The class of trivial fibrations is  $I_\perp$ .

We refer to  $I$  as the set of *generating cofibrations* and to  $J$  as the set of *generating trivial cofibrations*. A cofibrantly generated model category is *finitely generated* if we can choose the sets  $I$  and  $J$  above so that the domains and codomains of  $I$  and  $J$  are finite relative to  $I\text{-cell}$ .

**Proposition 1.20** (Hovey Proposition 2.1.18). Suppose  $\mathcal{C}$  is a cofibrantly generated model category, with generating cofibrations  $I$  and generating trivial fibrations  $J$ .

- (a) The cofibrations form the class  $\perp(I_\perp)$ .

- (b) Every cofibration is a retract of a relative  $I$ -cell complex.
- (c) The domains of  $I$  are small relative to the cofibrations.
- (d) The trivial cofibrations form the class  ${}_{\perp}(J_{\perp})$ .
- (e) Every trivial cofibration is a retract of a relative  $J$ -cell complex.
- (f) The domains of  $J$  are small relative to the trivial cofibrations.

If  $\mathcal{C}$  is fibrantly generated, then the domains and codomains of  $I$  and  $J$  are finite relative to the cofibrations.

*Proof.* **TODO.** □

**Theorem 1.21** (Hovey Theorem 2.1.19). *Suppose  $\mathcal{C}$  is a complete  $\mathcal{E}$  cocomplete category. Suppose  $\mathcal{W}$  is a subcategory of  $\mathcal{C}$ , and  $I$  and  $J$  are sets of maps of  $\mathcal{C}$ . Then there is a cofibrantly generated model structure on  $\mathcal{C}$  with  $I$  as the set of generating cofibrations,  $J$  as the set of generating trivial fibrations, and  $\mathcal{W}$  as the subcategory of weak equivalences if and only if the following conditions are satisfied.*

1. The subcategory  $\mathcal{W}$  has the 2-of-3 property and is closed under retracts.
2. The domains of  $I$  are small relative to  $I$ -cell.
3. The domains of  $J$  are small relative to  $J$ -cell.
4.  $J\text{-cell} \subseteq \mathcal{W} \cap {}_{\perp}(I_{\perp})$ .
5.  $I_{\perp} \subseteq \mathcal{W} \cap J_{\perp}$ .
6. Either  $\mathcal{W} \cap {}_{\perp}(I_{\perp}) \subseteq {}_{\perp}(J_{\perp})$  or  $\mathcal{W} \cap J_{\perp} \subseteq I_{\perp}$ .

*Proof.* **TODO.** □

We establish some notation for the following results. Given an adjunction  $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ , we will use “ $(-)^{\sharp}$ ” and “ $(-)^{\flat}$ ” to decorate a pair of adjoint arrows  $f^{\sharp} : F(C) \rightarrow D$  and  $f^{\flat} : C \rightarrow G(D)$ . Occasionally we will write  $g^{\sharp}$  or  $g^{\flat}$  to denote the transpose of a morphism  $g : C \rightarrow G(D)$  or  $g : F(C) \rightarrow D$  not already written in this form.

Given a complete and cocomplete category  $\mathcal{C}$  and arrows  $i : A \rightarrow B$ ,  $j : C \rightarrow D$ ,  $p : X \rightarrow Y$ , where  $C$  and  $D$  are exponentiable,<sup>2</sup> define  $Q(j, p)$  to be the fiber product  $(j^*, p_*)$  which fits into the following fiber diagram:

$$\begin{array}{ccc}
 X^D & \xrightarrow{p_*} & Y^D \\
 \downarrow j^* & \searrow Q(j, p) & \downarrow j^* \\
 X^C \times_{Y^C} Y^D & \xrightarrow{\quad} & Y^D \\
 \downarrow & \lrcorner & \downarrow j^* \\
 X^C & \xrightarrow{p_*} & Y^C
 \end{array}$$

where given an object  $Z$ , the pullback map  $j^* : Z^D \rightarrow Z^C$  is obtained as the adjoint of the composition

$$Z^D \times C \xrightarrow{\text{id} \times j} Z^D \times D \xrightarrow{\varepsilon_Z} Z,$$

where  $\varepsilon$  is the counit of the adjunction  $- \times D \dashv (-)^D$ .

Similarly, write  $i \wedge j := (i \times \text{id}_D, \text{id}_B \times j)$  to be the arrow which fits into the following pushout diagram:

$$\begin{array}{ccc}
 A \times C & \xrightarrow{i \times \text{id}_C} & B \times C \\
 \downarrow \text{id}_A \times j & \lrcorner & \downarrow \\
 A \times D & \xrightarrow{\quad} & A \times D \amalg_{A \times C} B \times C \\
 & \searrow i \times \text{id}_D & \downarrow \text{id}_B \times j \\
 & & B \times D
 \end{array}$$

**Proposition 1.22.** *Let  $\mathcal{C}$  be a complete and cocomplete category, and suppose we are given arrows  $i : A \rightarrow B$ ,  $j : C \rightarrow D$ , and  $p : X \rightarrow Y$  with  $C$  and  $D$  exponentiable objects in  $\mathcal{C}$ . Then there is a bijective correspondence*

<sup>2</sup>Explicitly, the functors  $- \times C$  and  $- \times D$  admit right adjoints  $(-)^C$  and  $(-)^D$ , respectively.

between lifting problems of the form

$$(1) \quad \begin{array}{ccc} A \times D \amalg_{A \times C} B \times C & \longrightarrow & X \\ i \wedge j \downarrow & & \downarrow p \\ B \times D & \longrightarrow & Y \end{array}$$

and lifting problems of the form

$$(2) \quad \begin{array}{ccc} A & \longrightarrow & X^D \\ i \downarrow & & \downarrow Q(j,p) \\ B & \longrightarrow & X^C \times_{Y^C} Y^D. \end{array}$$

Moreover, this bijection extends to a bijection between the solutions of (1) and the solutions of (2).

Before we prove this proposition, we first recall two results

**Lemma 1.23.** *Given a pair of adjoint functors  $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$  ( $F$  is the left adjoint), for any morphisms with domains and codomains as displayed below*

$$\begin{array}{ccc} F(C) & \xrightarrow{f^\sharp} & D \\ Fh \downarrow & & \downarrow k \\ F(C') & \xrightarrow{g^\sharp} & D' \end{array} \quad \rightsquigarrow \quad \begin{array}{ccc} C & \xrightarrow{f^\flat} & G(D) \\ h \downarrow & & \downarrow Gk \\ C' & \xrightarrow{g^\flat} & G(D') \end{array}$$

the left-hand square commutes in  $\mathcal{D}$  iff the right-hand transposed square commutes in  $\mathcal{C}$ .

*Proof.* This is Lemma 4.1.3 from Riehl. **TODO: ADD REFERENCE** □

**Lemma 1.24.** *Let  $\mathcal{C}$  be a complete and cocomplete category, and suppose we are given morphisms  $j : C \rightarrow D$  and  $k^\sharp : Z \times D \rightarrow W$  in  $\mathcal{C}$  with  $C$  and  $D$  exponentiable. Then the following two diagrams commute*

$$\begin{array}{ccc} Z \times C & \xrightarrow{\text{id} \times j} & Z \times D \\ k^\flat \times \text{id} \downarrow & & \downarrow k^\sharp \\ W^D \times C & \xrightarrow{(j^*)^\sharp} & W \end{array} \quad \begin{array}{ccc} Z & \xrightarrow{k^\flat} & W^D \\ (\text{id}_Z \times j)^\flat \downarrow & & \downarrow j^* \\ (Z \times D)^C & \xrightarrow{(k^\sharp)_*} & W^C \end{array}$$

*Proof.* By Lemma 1.23 it suffices to show that either diagram commutes in order to show they both commute. We will show the left diagram commutes. Recall that by how the pullback  $j^*$  is defined, that  $(j^*)^\sharp = \varepsilon_W \circ (\text{id}_{W^D} \times j)$ . Thus, it suffices to show the following diagram commutes:

$$(3) \quad \begin{array}{ccc} Z \times C & \xrightarrow{\text{id} \times j} & Z \times D \\ k^\flat \times \text{id} \downarrow & & \downarrow k^\sharp \\ W^D \times C & \xrightarrow{\text{id} \times j} & W^D \times D \xrightarrow{\varepsilon_D} & W \end{array}$$

It is straightforward to see by the universal property of the product that  $(\text{id}_{W^D} \times j) \circ (k^\flat \times \text{id}_C) = (k^\flat \times \text{id}_D) \circ (\text{id}_Z \times j)$ . Then by Lemma 1.23 applied to the following diagram

$$\begin{array}{ccc} Z & \xrightarrow{k^\flat} & W^D \\ k^\flat \downarrow & & \parallel \\ W^D & \xlongequal{\quad} & W^D \end{array}$$

we get that  $\varepsilon_D \circ (k^\flat \times \text{id}_D) = (\text{id}_{W^D})^\flat \circ (k^\flat \times \text{id}_D) = \text{id}_W \circ k^\sharp = k^\sharp$ . To summarize, we get that

$$\varepsilon_D \circ (\text{id}_{W^D} \times j) \circ (k^\flat \times \text{id}_C) = \varepsilon_D \circ (k^\flat \times \text{id}_D) \circ (\text{id}_Z \times j) = k^\sharp \circ (\text{id}_Z \times j),$$

so diagram (3) commutes, as desired. □

Now we prove the proposition.

*Proof.* Unravelling definitions, a lifting problem of the form (1) amounts to the data of maps  $f^\sharp : A \times D \rightarrow X$ ,  $g^\sharp : B \times C \rightarrow X$ , and  $h^\sharp : B \times D \rightarrow Y$  such that the following three diagrams commute:

$$(4) \quad \begin{array}{ccc} A \times C & \xrightarrow{i \times \text{id}_C} & B \times C \\ \text{id}_A \times j \downarrow & & \downarrow g^\sharp \\ A \times D & \xrightarrow{f^\sharp} & X \end{array} \quad \begin{array}{ccc} A \times D & \xrightarrow{f^\sharp} & X \\ i \times \text{id}_D \downarrow & & \downarrow p \\ B \times D & \xrightarrow{h^\sharp} & Y \end{array} \quad \begin{array}{ccc} B \times C & \xrightarrow{g^\sharp} & X \\ \text{id}_B \times j \downarrow & & \downarrow p \\ B \times D & \xrightarrow{h^\sharp} & Y \end{array}$$

(The left diagram is the data of a morphism  $A \times D \coprod_{A \times C} B \times C \rightarrow X$  and the other two diagrams assert commutativity of the lifting problem). We label these diagrams (4A), (4B), and (4C), respectively. In terms of these data, a solution to the lifting problem is a single arrow  $\ell^\sharp : B \times D \rightarrow X$  which serves as a lift for both the diagrams (4B) and (4C).

Conversely, a lifting problem of the form (2) is the data of maps  $f^b : A \rightarrow X^D$ ,  $g^b : B \rightarrow X^C$ , and  $h^b : B \rightarrow Y^D$  such that the following three diagrams commute:

$$(5) \quad \begin{array}{ccc} B & \xrightarrow{g^b} & X^C \\ h^b \downarrow & & \downarrow p_* \\ Y^D & \xrightarrow{j^*} & Y^C \end{array} \quad \begin{array}{ccc} A & \xrightarrow{f^b} & X^D \\ i \downarrow & & \downarrow p_* \\ B & \xrightarrow{h^b} & Y^D \end{array} \quad \begin{array}{ccc} A & \xrightarrow{f^b} & X^D \\ i \downarrow & & \downarrow j^* \\ B & \xrightarrow{g^b} & X^C \end{array}$$

(The left diagram is the data of a morphism  $B \rightarrow X^C \times_{Y^C} Y^D$  and the other two diagrams assert commutativity of the lifting problem). We label these diagrams (5A), (5B), and (5C), respectively. In terms of these data, a solution to the lifting problem is a single arrow  $\ell^b : B \rightarrow X^D$  which serves as a lift for both the diagrams (5B) and (5C).

Thus, in order to show the desired statement it suffices to show given arrows  $f^\sharp : A \times D \rightarrow X$ ,  $g^\sharp : B \times C \rightarrow X$ ,  $h^\sharp : B \times D \rightarrow Y$ , and  $\ell^\sharp : B \times D \rightarrow X$ , that:

- (4B) commutes iff (5B) commutes,
- (4A) commutes iff (5C) commutes,
- (4C) commutes iff (5A) commutes,
- $\ell^\sharp$  is a lift of (4B) iff  $\ell^b$  is a lift of (5B), and
- Assuming  $\ell^\sharp$  is a lift of (4B) and  $\ell^b$  is a lift of (5B),  $\ell^\sharp$  is a lift of (4C) iff  $\ell^b$  is a lift of (5C).

To start with, note that (4B) commutes iff (5B) commutes, by [Lemma 1.23](#).

Next, we claim that (4A) commutes iff (5C) commutes. By [Lemma 1.23](#), (4A) commutes iff the following diagram commutes

$$\begin{array}{ccc} A & \xrightarrow{i} & B \\ (\text{id}_A \times j)^b \downarrow & & \downarrow g^b \\ (A \times D)^C & \xrightarrow{(f^\sharp)_*} & X^C \end{array}$$

By [Lemma 1.24](#) (the second diagram),  $(f^\sharp)_* \circ (\text{id}_A \times j)^b = j^* \circ f^b$ , so this diagram commutes iff the following diagram commutes:

$$\begin{array}{ccc} A & \xrightarrow{i} & B \\ f^b \downarrow & & \downarrow g^b \\ X^D & \xrightarrow{j^*} & X^C \end{array}$$

This is precisely (5C) (but flipped), as desired.

Next, we claim that (4C) commutes iff (5A) commutes. By [Lemma 1.23](#), (5A) commutes iff the following diagram commutes

$$\begin{array}{ccc} B \times C & \xrightarrow{g^\sharp} & X \\ h^b \times \text{id} \downarrow & & \downarrow p \\ Y^D \times C & \xrightarrow{(j^*)^\sharp} & Y \end{array}$$



By [Lemma 1.24](#) (the first diagram),  $(j^*)^\sharp \circ (h^\flat \times \text{id}_C) = h^\sharp \circ (\text{id}_B \times j)$ , so this diagram commutes iff the following diagram commutes:

$$\begin{array}{ccc} B \times C & \xrightarrow{g^\sharp} & X \\ \text{id}_B \times j \downarrow & & \downarrow p \\ B \times D & \xrightarrow{h^\sharp} & Y \end{array}$$

This is precisely (4C), as desired.

Now, we claim that  $\ell$  is a lift for (4B) iff it is a lift for (5B). Indeed, by [Lemma 1.23](#),  $f^\sharp = \ell^\sharp \circ (i \times \text{id}_D)$  iff  $f^\flat = \ell^\flat \circ i$  and  $p \circ \ell^\sharp = h^\sharp$  iff  $p_* \circ \ell^\flat = h^\flat$ :

$$\begin{array}{ccccc} A \times D & \xrightarrow{f^\sharp} & X & & A & \xrightarrow{f^\flat} & X^D \\ \text{id}_A \times \text{id}_D \downarrow & & \parallel & \rightsquigarrow & \downarrow i & & \parallel \\ B \times D & \xrightarrow{\ell^\sharp} & X & & B & \xrightarrow{\ell^\flat} & X^D \end{array} \quad \begin{array}{ccccc} B \times D & \xrightarrow{\ell^\sharp} & X & & B & \xrightarrow{\ell^\flat} & X^D \\ \parallel & & \downarrow p & \rightsquigarrow & \parallel & & \downarrow p_* \\ B \times D & \xrightarrow{h^\sharp} & Y & & B & \xrightarrow{h^\flat} & Y^D \end{array}$$

In other words,  $\ell^\sharp$  makes the top (resp. bottom) triangle of (4B) commute iff  $\ell^\flat$  makes the top (resp. bottom) triangle of (5B) commute, as desired.

Finally, it remains to show that if  $\ell^\sharp$  and  $\ell^\flat$  determine lifts of (4B) and (5B), respectively, then  $\ell^\sharp$  is a lift for (4C) iff it is a lift for (5C). First of all, since  $\ell^\sharp$  and  $\ell^\flat$  define lifts of (4B) and (5B), we already have  $p \circ \ell^\sharp = h^\sharp$  and  $\ell^\flat \circ i = f^\flat$ , so it is sufficient (and necessary) to show that  $\ell^\sharp \circ (\text{id}_B \times j) = g^\sharp$  iff  $j^* \circ \ell^\flat = g^\flat$ . Note by [Lemma 1.24](#) (second diagram),  $j^* \circ \ell^\flat = (\ell^\sharp)_* \circ (\text{id}_Z \times j)^\flat$ , so it suffices to show that  $\ell^\sharp \circ (\text{id}_B \times j) = g^\sharp$  iff  $(\ell^\sharp)_* \circ (\text{id}_Z \times j)^\flat = g^\flat$ . Indeed, this follows by [Lemma 1.23](#):

$$\begin{array}{ccc} B \times C & \xlongequal{\quad} & B \times C \\ \text{id}_B \times j \downarrow & & \downarrow g^\sharp \\ B \times D & \xrightarrow{\ell^\sharp} & X \end{array} \rightsquigarrow \begin{array}{ccc} B & \xlongequal{\quad} & B \\ (\text{id}_B \times j)^\flat \downarrow & & \downarrow g^\flat \\ (B \times D)^C & \xrightarrow{(\ell^\sharp)_*} & X^C \end{array} \quad \square$$

**Lemma 1.25.** *Let  $\mathcal{C}$  be a complete and cocomplete category, and suppose we are given exponentiable objects  $A, B, C, D \in \mathcal{C}$  which fit into a pushout diagram like so:*

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & \lrcorner & \downarrow \\ C & \longrightarrow & D \end{array}$$

*Then the following is a pullback diagram*

$$\begin{array}{ccc} Z^D & \longrightarrow & Z^B \\ \downarrow & \lrcorner & \downarrow \\ Z^C & \longrightarrow & Z^A \end{array}$$

*Proof.* **TODO.** □

## 2. TOPOLOGICAL SPACES

A map  $f : X \rightarrow Y$  in **Top** is an *inclusion* if it is continuous, injective, and for all  $U \subseteq X$  open, there is some  $V \subseteq Y$  open such that  $f^{-1}(V) = U$ . If  $f$  is a closed inclusion and every point in  $Y \setminus f(X)$  is closed, then we call  $f$  a *closed  $T_1$  inclusion*. We will let  $\mathcal{T}$  denote the class of closed  $T_1$  inclusions in **Top**.

The symbol  $D^n$  will denote the unit disk in  $\mathbb{R}^n$ , and the symbol  $S^{n-1}$  will denote the unit sphere in  $\mathbb{R}^n$ , so that we have the boundary inclusions  $S^{n-1} \hookrightarrow D^n$ . In particular, for  $n = 0$  we let  $D^0 = \{0\}$  and  $S^{-1} = \emptyset$ .

Recall: If  $F : \mathcal{J} \rightarrow \mathbf{Top}$  is a functor, where  $\mathcal{J}$  is a small category, the limit of  $F$  is obtained by taking the limit in the category of sets, and then topologizing it with the *initial topology*, where if  $\eta : \varinjlim F \Rightarrow F$  is the limit cone, then the topology on  $\varinjlim F$  is that with subbasis given by sets of the form  $\eta_j^{-1}(U)$  where  $j \in \mathcal{J}$  and  $U \subseteq F_j$  is open. Similarly, the colimit of  $F$  is obtained by taking the colimit  $\varinjlim F$  in the category of

sets and endowing it with the *final topology*, where a set  $U \subseteq \operatorname{colim} F$  is open if and only if  $\varepsilon_j^{-1}(U)$  is open in  $F_j$  for all  $j \in \mathcal{J}$ , where  $\varepsilon : F \Rightarrow \operatorname{colim} F$  is the colimit cone (equivalently, a set  $C \subseteq \operatorname{colim} F$  is closed if and only if  $\varepsilon_j^{-1}(C)$  is closed in  $F_j$  for all  $j \in \mathcal{J}$ ).

Given a space  $X$ , we construct a functor  $(-)^X : \mathbf{Top} \rightarrow \mathbf{Top}$  as follows: Given a space  $Y$ , define  $Y^X$  to be the space whose underlying set is the set  $\mathbf{Top}(X, Y)$  of continuous maps  $X \rightarrow Y$ , and the topology on  $Y^X$  is the *compact-open topology*, i.e., the topology with subbasis given by the sets of the form

$$S(K, U) := \{f \in \mathbf{Top}(X, Y) : f(K) \subseteq U\}$$

for  $K \subseteq X$  compact and  $U \subseteq Z$  open. Given a continuous map  $f : Y \rightarrow Z$ , define the induced map  $f_* : Y^X \rightarrow Z^X$  by  $f_*(g) := f \circ g$ . Unravelling definitions, we have that given  $f : Y \rightarrow Z$  continuous,  $f_*^{-1}(S(K, U)) = S(K, f^{-1}(U))$  for all  $K \subseteq X$  compact and  $U \subseteq Z$  open, so that  $f_*$  is continuous. Furthermore,  $(-)^X$  is clearly functorial, by associativity and unitality of function composition.

Given a topological space  $X$ , we say that  $X$  is *locally compact* if for all points  $x \in X$  and open neighborhoods  $U$  of  $x$ , there exists an open set  $V \subseteq X$  with  $x \in V$ ,  $\bar{V} \subseteq U$ , and  $\bar{V}$  compact. We claim that  $(-)^X$  is right adjoint to  $- \times X$  when  $X$  is locally compact and Hausdorff.

**Proposition 2.1.** *If  $X$  is a locally compact Hausdorff space, then functor  $- \times X$  is left adjoint to  $(-)^X$  (so that in particular  $- \times X$  preserves colimits).*

*Proof.* We start by constructing the counit and unit of the adjunction. Given a space  $Z$ , define the counit  $\varepsilon_Z : X \times Z^X \rightarrow Z$  to be the evaluation function, taking a pair  $(x, f) \mapsto f(x)$ . First, we claim  $\varepsilon_Z$  is continuous. Suppose we are given an open set  $V \subseteq Z$  and a point  $(x, f) \in \varepsilon_Z^{-1}(V)$  (so  $f(x) \in V$ ). Since  $f$  is continuous and  $X$  is locally compact, there exists an open set  $U \subseteq X$  containing  $x$  such that  $x \in U \subseteq \bar{U} \subseteq f^{-1}(V)$  with  $\bar{U}$  compact. Then consider the open set  $U \times S(\bar{U}, V)$  in  $X \times Y^X$ . First of all,  $(x, f) \in U \times S(\bar{U}, V)$ , as  $x \in U$  and  $\bar{U} \subseteq f^{-1}(V)$ , so that  $f(\bar{U}) \subseteq V$  meaning  $f \in S(\bar{U}, V)$ . Furthermore, given  $(y, g) \in U \times S(\bar{U}, V)$ , we have  $\varepsilon_Z(y, g) = g(y) \in g(U) \subseteq g(\bar{U}) \subseteq V$ , so  $U \times S(\bar{U}, V)$  is an open neighborhood of  $x$  contained in  $\varepsilon_Z^{-1}(V)$ , as desired. Hence,  $\varepsilon_Z$  is continuous. It remains to show naturality. Given a map  $f : Z \rightarrow W$ , we wish to show the following diagram commutes:

$$\begin{array}{ccc} X \times Z^X & \xrightarrow{\varepsilon_Z} & Z \\ \text{id}_X \times f_* \downarrow & & \downarrow f \\ X \times W^X & \xrightarrow{\varepsilon_W} & W \end{array}$$

Indeed, chasing an element  $(x, g)$  around the diagram yields:

$$\begin{array}{ccc} (x, g) & \xrightarrow{\quad} & g(x) \\ \downarrow & & \downarrow \\ (x, f \circ g) & \xrightarrow{\quad} & f(g(x)) \end{array}$$

so it does indeed commute.

Now we wish to define the unit  $\eta_Y : Y \rightarrow (Y \times X)^X$ . Given  $y \in Y$ , define  $\eta_Y(y) \in (Y \times X)^X$  by  $\eta_Y(y)(x) := (y, x)$ . First of all, for it to be true that  $\eta_Y(y) \in (X \times Y)^X$ , it must be true that  $\eta_Y(y)$  is continuous. Indeed, this is clear as  $\eta_Y$  is obtained as the product map  $y \times \text{id}_X : X \rightarrow Y \times X$ , where  $y$  represents the constant function on  $y$  (which is obviously continuous). Furthermore,  $\eta_Y$  itself is continuous: given  $K \subseteq X$  compact and  $U \subseteq Y \times X$  open, we wish to show that  $\eta_Y^{-1}(S(K, U))$  is open in  $Y$ . It suffices to show that given  $y \in \eta_Y^{-1}(S(K, U))$ , there exists an open neighborhood  $W$  of  $y$  that is mapped by  $\eta_Y$  into  $S(K, U)$ . Since  $y \in \eta_Y^{-1}(S(K, U))$ ,  $\eta_Y(y)(K) = \{y\} \times K \subseteq U$ . Then  $U \cap (Y \times K)$  is an open set in the subspace  $Y \times K$  containing the slice  $\{y\} \times K$ . By definition of the product topology, for each  $k \in K$ , there exist open sets  $W_k \subseteq Y$  and  $V_k \subseteq K$  such that  $(y, k) \in W_k \times V_k \subseteq U \cap (Y \times K)$ . Then the  $V_k$ 's form an open cover of  $K$ , which is compact, so that there exist  $k_1, \dots, k_n \in K$  with  $V_{k_1} \cup \dots \cup V_{k_n} = K$ . Hence if we define  $W := W_{k_1} \cap \dots \cap W_{k_n}$ , then  $\{y\} \times K \subseteq W \times K \subseteq U \cap (Y \times K)$ , and  $W$  is open in  $Y$  as it is a finite intersection of open sets. Then for all  $w \in W$ ,  $\eta_Y(w)(K) = \{w\} \times K \subseteq W \times K \subseteq U$ . Hence, indeed  $\eta_Y$  is continuous. It remains to show naturality. Given a map  $f : Y \rightarrow W$ , we wish to show the following diagram

commutes:

$$\begin{array}{ccc} Y & \xrightarrow{\eta_Y} & (Y \times X)^X \\ f \downarrow & & \downarrow (f \times \text{id}_X)_* \\ W & \xrightarrow{\eta_W} & (W \times X)^X \end{array}$$

Indeed, chasing an element  $y$  around the top of the diagram yields the function obtained as the composition  $x \mapsto (y, x) \mapsto f \times \text{id}_X(y, x) = (f(y), x)$ , while chasing around the bottom of the diagram more directly yields the function  $x \mapsto (f(y), x)$ .

Now that we have constructed the unit and counit, it remains to verify the counit-unit equations, i.e., that for each  $Y \in \mathbf{Top}$  that  $\varepsilon_{Y \times X} \circ (\eta_Y \times \text{id}_X) = \text{id}_{Y \times X}$  and  $(\varepsilon_Y)_* \circ \eta_{Y^X} = \text{id}_{Y^X}$ . First of all, given  $(y, x) \in Y \times X$ , we have

$$(\varepsilon_{Y \times X} \circ (\eta_Y \times \text{id}_X))(y, x) = \varepsilon_{Y \times X}(\eta_Y(y), x) = \eta_Y(y)(x) = (y, x).$$

On the other hand, given  $f \in Y^X$ , we have

$$(\varepsilon_Y)_*(\eta_{Y^X}(f)) = (\varepsilon_Y)_*([x \mapsto (f, x)]) = [x \mapsto (f, x) \mapsto \varepsilon_Y(f, x) = f(x)] = f.$$

Hence, indeed  $\varepsilon$  and  $\eta$  form the counit and unit for the adjoint pair  $(- \times X, (-)^X)$ .  $\square$

Now that we have gotten some topological preliminaries out of the way, we are ready to define the model structure.

**Definition 2.2.** A map  $f : X \rightarrow Y$  in  $\mathbf{Top}$  is called a *weak equivalence* if

$$\pi_n(f, x) : \pi_n(X, x) \rightarrow \pi_n(Y, f(x))$$

is an isomorphism for all  $n \geq 0$  and for all  $x \in X$ . We will write  $\mathcal{W}$  to refer to the class of all weak equivalences in  $\mathbf{Top}$ .

Define the set of maps  $I'$  to consist of all the boundary inclusion  $S^{n-1} \hookrightarrow D^n$  for all  $n \geq 0$ , and define the set  $J$  to consist of all the inclusions  $D^n \hookrightarrow D^n \times I$  mapping  $x \mapsto (x, 0)$  for  $n \geq 0$ . Then a map  $f$  will be called a *cofibration* if it is in  $I'$ -cof  $= {}_\perp(I'_\perp)$ , and a *fibration* if it is in  $J$ -inj  $= J_\perp$ .

A map in  $I'$ -cell is usually called a *relative cell complex*; a relative CW-complex is a special case of a relative cell complex, where, in particular, the cells can be attached in order of their dimension. Note that in particular maps of  $J$  are relative CW complexes, hence are relative  $I'$ -cell complexes. A fibration is often known as a *Serre fibration* in the literature.

**Theorem 2.3** (Hovey Theorem 2.4.19). *There is a finitely generated model structure on  $\mathbf{Top}$  with  $I'$  as the set of generating cofibrations,  $J$  as the set of generating trivial cofibrations, and the cofibrations, fibrations, and weak equivalences as above. Every object of  $\mathbf{Top}$  is fibrant, and the cofibrant objects are retracts of relative cell complexes.*

*Proof.* We will apply [Theorem 1.21](#) to get that there is a cofibrantly generated model structure on  $\mathbf{Top}$  with  $I'$  as the set of generating cofibrations,  $J$  as the set of generating trivial fibrations, and  $\mathcal{W}$  as the subcategory of weak equivalences. The six requirements outlined in the theorem will be verified like so:

1.  $\mathcal{W}$  is a subcategory of  $\mathcal{C}$  which has the 2-of-3 property and is closed under retracts: [Lemma 2.12](#).
2. The domains of  $I'$  are small relative to  $I'$ -cell: [Proposition 2.11](#).
3. The domains of  $J$  are small relative to  $J$ -cell: [Proposition 2.11](#).
4.  $J$ -cell  $\subseteq \mathcal{W} \cap {}_\perp(I'_\perp)$ : In [Proposition 2.13](#), we will show  ${}_\perp(J_\perp) \subseteq \mathcal{W} \cap {}_\perp(I'_\perp)$ , and by [Corollary 1.16](#)  $J$ -cell  $\subseteq {}_\perp(J_\perp)$ .
5.  $I'_\perp \subseteq \mathcal{W} \cap J_\perp$ : [Proposition 2.14](#)
6.  $\mathcal{W} \cap J_\perp \subseteq I'_\perp$ : [Proposition 2.22](#)

It will follow by the definition of a cofibrantly generated model structure ([Definition 1.19](#)) that the fibrations in this model structure are given by  $J_\perp$ , which is precisely how we defined it. By [Proposition 1.20](#), the class of cofibrations will be given by  ${}_\perp(I'_\perp)$ , which is likewise exactly how we defined them.

In [Proposition 2.8](#), we will show that compact spaces are finite relative to the class  $\mathcal{T}$  of closed  $T_1$  inclusions. Hence, this model structure will be finitely generated, as the domains and codomains of  $I'$  and  $J$  are all compact, and by the reasoning given above we will have shown  $I'$ -cell  $\subseteq \mathcal{T}$ .

We will show that every object of  $\mathbf{Top}$  is fibrant in [Corollary 2.17](#).  $\square$

**Lemma 2.4.** *Let  $\lambda$  be an ordinal, and  $X$  a  $\lambda$ -sequence in **Top**. Then:*

- (i) *If  $X$  is a  $\lambda$ -sequence of injections, then  $X_\alpha \rightarrow X_\beta$  is an injective for all  $\alpha \leq \beta < \lambda$ .*
- (ii) *If  $X$  is a  $\lambda$ -sequence of inclusions, then the map  $X_\alpha \rightarrow X_\beta$  is an inclusion for all  $\alpha \leq \beta < \lambda$ .*
- (iii) *If  $X$  is a  $\lambda$ -sequence of closed  $T_1$  inclusions, then the map  $X_\alpha \rightarrow X_\beta$  is a closed  $T_1$  inclusion for all  $\alpha \leq \beta < \lambda$ .*

*Proof.* In what follows, given  $\alpha \leq \beta < \lambda$ , let  $\iota_{\alpha,\beta}$  denote the map  $X_\alpha \rightarrow X_\beta$ .

- (i) Let  $\alpha < \lambda$ . We perform a proof by transfinite induction on  $\beta$  for  $\alpha \leq \beta < \lambda$  that  $\iota_{\alpha,\beta} : X_\alpha \rightarrow X_\beta$  is injective. For the zero case, clearly  $\iota_{\alpha,\alpha} = \text{id}_{X_\alpha}$  is injective. Supposing  $\iota_{\alpha,\beta}$  is injective for some  $\alpha < \beta + 1 < \lambda$ , we have  $\iota_{\alpha,\beta+1} = \iota_{\beta,\beta+1} \circ \iota_{\alpha,\beta}$  is a composition of injections, and is therefore clearly injective itself. Finally, suppose  $\gamma$  is a limit ordinal with  $\alpha \leq \gamma < \lambda$  such that  $\iota_{\alpha,\beta}$  is injective for all  $\alpha \leq \beta < \gamma$ . We claim  $\iota_{\alpha,\gamma}$  is injective. Since  $X_\gamma$  is colimit preserving and  $\gamma$  is a limit ordinal,  $X_\gamma$  is the colimit of the diagram  $\{X_\beta\}_{\beta < \gamma}$  via the maps  $\iota_{\beta,\gamma}$ , so that in particular by [Remark 1.7](#) and the fact that the forgetful functor **Top**  $\rightarrow$  **Set** preserves colimits, given  $a, b \in X_\alpha$  with  $\iota_{\alpha,\gamma}(a) = \iota_{\alpha,\gamma}(b)$ , there exists some  $\beta < \gamma$  with  $\iota_{\alpha,\beta}(a) = \iota_{\alpha,\beta}(b)$ , and  $\iota_{\alpha,\beta}$  is injective for all  $\beta < \gamma$ , so it must have been true  $a = b$  in  $X_\alpha$ .
- (ii) By part(i), we know that  $\iota_{\alpha,\beta}$  is injective for  $\alpha \leq \beta < \lambda$ . Thus it suffices to prove the following statement: For all  $\alpha < \lambda$  and  $U \subseteq X_\alpha$ , for all  $\alpha \leq \beta < \lambda$ , there exists  $U_\beta \subseteq X_\beta$  with  $U_\alpha = U$  such that for all  $\alpha \leq \beta' \leq \beta < \lambda$ ,  $\iota_{\beta',\beta}^{-1}(U_\beta) = U_{\beta'}$ . We prove this by transfinite recursion on  $\alpha \leq \beta < \lambda$ .

The zero case has been taken care of:  $U_\alpha = U$ . For the successor case, given  $\alpha < \beta + 1 < \lambda$ , supposing  $U_\beta$  has been defined with the desired properties, since  $\iota_{\beta,\beta+1}$  is an inclusion, there exists  $U_{\beta+1} \subseteq X_{\beta+1}$  with  $\iota_{\beta,\beta+1}^{-1}(U_{\beta+1}) = U_\beta$ . Then given  $\alpha \leq \beta' \leq \beta + 1$ , we have

$$\iota_{\beta',\beta+1}^{-1}(U_{\beta+1}) = (\iota_{\beta,\beta+1} \circ \iota_{\beta',\beta})^{-1}(U_{\beta+1}) = \iota_{\beta',\beta}^{-1}(\iota_{\beta,\beta+1}^{-1}(U_{\beta+1})) = \iota_{\beta',\beta}^{-1}(U_\beta) = U_{\beta'}.$$

Finally, the limit case. Suppose  $\gamma$  is a limit ordinal with  $\alpha < \gamma \leq \lambda$ , and suppose  $U_\beta$  has been constructed with the desired properties for  $\alpha \leq \beta < \gamma$ . We wish to define  $U_\gamma$ . Since  $X$  is colimit preserving and  $\gamma = \sup_{\alpha \leq \beta < \gamma} \beta$ , the maps  $\iota_{\beta,\gamma}$  for  $\alpha \leq \beta < \gamma$  form a colimit cone for the diagram  $\{X_\beta\}_{\alpha \leq \beta < \gamma}$ . Let  $S = \{0, 1\}$  be the Sierpinski space whose open sets are  $\{\emptyset, \{1\}, \{0, 1\}\}$ . For  $\alpha \leq \beta < \gamma$ , define a map  $s_\beta : X_\beta \rightarrow S$  mapping everything in  $U_\beta$  to 1 and every other point to 0. Each  $s_\beta$  is clearly continuous, as  $s_\beta^{-1}(1) = U_\beta$ . Furthermore, we claim the  $s_\beta$ 's form a cone under the diagram  $\{X_\beta\}_{\alpha \leq \beta < \gamma}$ , i.e., that given  $\alpha \leq \beta' \leq \beta < \gamma$ , the following diagram commutes

$$\begin{array}{ccc} X_{\beta'} & \xrightarrow{\iota_{\beta',\beta}} & X_\beta \\ & \searrow s_{\beta'} & \swarrow s_\beta \\ & S & \end{array}$$

To see this, let  $x \in X_{\beta'}$ . If  $x \in U_{\beta'} = \iota_{\beta',\beta}^{-1}(U_\beta)$ , then  $\iota_{\beta',\beta}(x) \in U_\beta$ , so  $s_\beta(\iota_{\beta',\beta}(x)) = 1 = s_{\beta'}(x)$ . Conversely, if  $x \in X_{\beta'} \setminus U_{\beta'} = X_{\beta'} \setminus \iota_{\beta',\beta}^{-1}(U_\beta)$ , then  $x \notin \iota_{\beta',\beta}^{-1}(U_\beta)$ , so  $\iota_{\beta',\beta}(x) \notin U_\beta$ , meaning  $s_\beta(\iota_{\beta',\beta}(x)) = 0 = s_{\beta'}(x)$ . Hence, the  $s_\beta$ 's do indeed form a cone under  $\{X_\beta\}_{\alpha \leq \beta < \gamma}$ , so by universal property of the colimit there exists a unique map  $\ell : X_\gamma \rightarrow S$  such that  $s_\beta = \ell \circ \iota_{\beta,\gamma}$  for all  $\alpha \leq \beta < \gamma$ . Define  $U_\gamma := \ell^{-1}(1)$ , which is open as  $\{1\}$  is open in  $S$ . It remains to show that for all  $\alpha \leq \beta \leq \gamma$  that  $\iota_{\beta,\gamma}^{-1}(U_\gamma) = U_\beta$ . Indeed, we have

$$\iota_{\beta,\gamma}^{-1}(U_\gamma) = \iota_{\beta,\gamma}^{-1}(\ell^{-1}(1)) = (\ell \circ \iota_{\beta,\gamma})^{-1}(1) = s_\beta^{-1}(1) = U_\beta.$$

- (iii) By part (ii), we know that  $\iota_{\alpha,\beta}$  is an inclusion for  $\alpha \leq \beta < \lambda$ . Fix  $\alpha < \lambda$ . We perform transfinite induction on  $\alpha \leq \beta < \lambda$  to show that  $\iota_{\alpha,\beta}$  is a closed  $T_1$  inclusion, assuming it is already an inclusion. For the zero case, clearly  $\iota_{\alpha,\alpha} = \text{id}_{X_\alpha}$  is closed, and vacuously every point in  $X_\alpha \setminus \iota_{\alpha,\alpha}(X_\alpha) = \emptyset$  is a closed point. For the successor case, supposing  $\iota_{\alpha,\beta} : X_\alpha \rightarrow X_\beta$  is a closed  $T_1$  inclusion, we wish to show that  $\iota_{\alpha,\beta+1} : X_\alpha \rightarrow X_{\beta+1}$  is a closed  $T_1$  inclusion. Since  $\iota_{\alpha,\beta+1} = \iota_{\beta,\beta+1} \circ \iota_{\alpha,\beta}$  is a composition of closed  $T_1$  inclusions, it is clearly closed. It remains to show that every point in  $X_{\beta+1} \setminus \iota_{\alpha,\beta+1}(X_\alpha)$  is closed in  $X_{\beta+1}$ . Indeed, let  $x \in X_{\beta+1} \setminus \iota_{\alpha,\beta+1}(X_\alpha)$ . First, if  $x \in X_{\beta+1} \setminus \iota_{\beta,\beta+1}(X_\beta)$ , we are done, as  $\iota_{\beta,\beta+1}$  is a closed  $T_1$  inclusion. Hence, we may assume that  $x \in \iota_{\beta,\beta+1}(X_\beta)$ , so there exists some  $y \in X_\beta$  such that  $\iota_{\beta,\beta+1}(y) = x$ . Since  $\iota_{\beta,\beta+1}$  is closed, in order to show  $x$  is a closed point in  $X_{\beta+1}$ , it suffices to show that  $y$  is a closed point in  $X_\beta$ . Since  $\iota_{\alpha,\beta}$  is a closed  $T_1$  inclusion, it further suffices

to show that  $y$  is not in the image of  $\iota_{\alpha,\beta}$ . Suppose for the sake of a contradiction that there existed  $z \in X_\alpha$  with  $\iota_{\alpha,\beta}(z) = y$ . Then we would have

$$\iota_{\alpha,\beta+1}(z) = \iota_{\beta,\beta+1}(\iota_{\alpha,\beta}(z)) = \iota_{\beta,\beta+1}(y) = x,$$

a contradiction of the fact that  $x \in X_{\beta+1} \setminus \iota_{\alpha,\beta+1}(X_\alpha)$ . Hence, it must have been true that  $y$  is not in the image of  $\iota_{\alpha,\beta}$  in the first place, the desired result. Finally, the limit case. Suppose  $\gamma$  is a limit ordinal with  $\alpha < \gamma \leq \lambda$  such that  $\iota_{\alpha,\beta}$  is a closed  $T_1$  inclusion for all  $\alpha \leq \beta < \gamma$ . Then we wish to show  $\iota_{\alpha,\gamma}$  is a closed  $T_1$  inclusion.

First, we show  $\iota_{\alpha,\gamma}$  is closed. Let  $C \subseteq X_\alpha$  be closed. Since  $\gamma = \sup_{\alpha \leq \beta < \gamma} \beta$  and  $X$  is colimit-preserving,  $X_\gamma$  is the colimit of the  $X_\beta$ 's for  $\alpha \leq \beta < \gamma$  via the maps  $\iota_{\beta,\gamma}$ , and the topology on  $X_\gamma$  is the final topology induced by these maps. Hence, in order to show  $\iota_{\alpha,\gamma}(C)$  is closed in  $X_\gamma$ , it suffices to show that  $\iota_{\beta,\gamma}^{-1}(\iota_{\alpha,\gamma}(C))$  is closed in  $X_\beta$  for all  $\alpha \leq \beta < \gamma$ . It further suffices to show that  $\iota_{\beta,\gamma}^{-1}(\iota_{\alpha,\gamma}(C)) = \iota_{\alpha,\beta}(C)$ , as  $\iota_{\alpha,\beta}$  is closed. First, suppose  $x \in \iota_{\beta,\gamma}^{-1}(\iota_{\alpha,\gamma}(C))$ , so  $\iota_{\beta,\gamma}(x) = \iota_{\alpha,\gamma}(c)$  for some  $c \in C$ . Then since the forgetful functor  $\mathbf{Top} \rightarrow \mathbf{Set}$  preserves colimits, by the explicit description of the colimit in **Set** (Remark 1.7), there exists  $\mu$  with  $\alpha, \beta \leq \mu < \gamma$  such that  $\iota_{\beta,\mu}(x) = \iota_{\alpha,\mu}(c)$ . But  $\iota_{\alpha,\mu} = \iota_{\beta,\mu} \circ \iota_{\alpha,\beta}$ , and  $\iota_{\beta,\mu}$  is injective (by (i)) so  $x = \iota_{\alpha,\beta}(c)$ , meaning  $x \in \iota_{\alpha,\beta}(C)$ , as desired. Conversely, suppose we are given  $c \in C$ , then we wish to show  $\iota_{\alpha,\beta}(c) \in \iota_{\beta,\gamma}^{-1}(\iota_{\alpha,\gamma}(C))$ , i.e., that  $\iota_{\beta,\gamma}(\iota_{\alpha,\beta}(c)) \in \iota_{\alpha,\gamma}(C)$ . This follows immediately as  $\iota_{\beta,\gamma} \circ \iota_{\alpha,\beta} = \iota_{\alpha,\gamma}$ .

Lastly, we show that for all  $x \in X_\gamma \setminus \iota_{\alpha,\gamma}(X_\alpha)$  that  $x$  is a closed point in  $X_\gamma$ . Again by the description of the colimit in **Set** (Remark 1.7), the fact that the forgetful functor  $\mathbf{Top} \rightarrow \mathbf{Set}$  preserves colimits, and that  $X$  preserves colimits, we know that every point in  $X_\gamma$  is in the image of some  $\iota_{\beta,\gamma}$  for some  $\alpha \leq \beta < \gamma$ . Hence, there exists some  $\alpha < \beta < \gamma$  and a point  $y \in X_\beta$  with  $\iota_{\beta,\gamma}(y) = x$ . By the preceding paragraph,  $\iota_{\beta,\gamma}$  is closed, so in order to show  $x$  is a closed point in  $X_\gamma$  it suffices to show that  $y$  is a closed point in  $X_\beta$ . It further suffices to show that  $y \in X_\beta \setminus \iota_{\alpha,\beta}(X_\alpha)$ , as  $\iota_{\alpha,\beta}$  is a closed  $T_1$  inclusion. Suppose for the sake of a contradiction that there existed some  $z \in X_\alpha$  such that  $\iota_{\alpha,\beta}(z) = y$ . Then we would have

$$\iota_{\alpha,\gamma}(z) = \iota_{\beta,\gamma}(\iota_{\alpha,\beta}(z)) = \iota_{\beta,\gamma}(y) = x,$$

a contradiction of the fact that  $x \in X_\gamma \setminus \iota_{\alpha,\gamma}(X_\alpha)$ . Hence,  $y$  must not have been in the image of  $\iota_{\alpha,\beta}$  in the first place, as desired.  $\square$

This result, by Lemma 1.2 and Lemma 1.3, gives the following corollaries:

**Corollary 2.5.** *The class of injective maps (resp. inclusions, closed  $T_1$  inclusions) in **Top** is closed under transfinite composition.*

**Corollary 2.6.** *Let  $\lambda$  be an ordinal, and  $X$  be a  $\lambda$ -sequence in **Top**. Then:*

- (i) *If  $X$  is a  $\lambda$ -sequence of injections, then the canonical map  $X_\alpha \rightarrow \text{colim } X$  is an injection for all  $\alpha < \lambda$ .*
- (ii) *If  $X$  is a  $\lambda$ -sequence of inclusions, then the canonical map  $X_\alpha \rightarrow \text{colim } X$  is an inclusion for all  $\alpha < \lambda$ .*
- (iii) *If  $X$  is a  $\lambda$ -sequence of closed  $T_1$  inclusions, then the canonical map  $X_\alpha \rightarrow \text{colim } X$  is a closed  $T_1$  inclusion for all  $\alpha < \lambda$ .*

**Lemma 2.7** (Hovey 2.4.1). *Every topological space is small relative to the inclusions.*

*Proof.* We claim that every topological space  $A$  is  $|A|$ -small relative to the inclusions. We use the characterization of smallness afforded by Proposition 1.8. Let  $\lambda$  be an  $|A|$ -filtered ordinal, and let  $X : \lambda \rightarrow \mathbf{Top}$  be a  $\lambda$ -sequence so that  $X_\beta \rightarrow X_{\beta+1}$  is an inclusion for all  $\beta + 1 < \lambda$ . Recall that the forgetful functor  $\mathbf{Top} \rightarrow \mathbf{Set}$  is forgetful, so elements of  $\text{colim } X$  are equivalence classes of elements  $a \in X_\alpha$  for  $\alpha < \lambda$ , where  $a \in X_\alpha$  and  $b \in X_\beta$  represent the same equivalence class iff there exists  $\alpha, \beta \leq \gamma < \lambda$  so that  $a$  and  $b$  are sent to the same element by the maps  $X_\alpha \rightarrow X_\gamma$  and  $X_\beta \rightarrow X_\gamma$ , respectively.

First, suppose  $f : A \rightarrow X_\alpha$  and  $g : A \rightarrow X_\beta$  are continuous maps such that the compositions  $A \xrightarrow{f} X_\alpha \rightarrow \text{colim } X$  and  $A \xrightarrow{g} X_\beta \rightarrow \text{colim } X$  are equal. Then the same proof given in Example 1.9 works to show that  $f$  and  $g$  are equal in some stage of the colimit, as desired.

Conversely, suppose we are given a (continuous) map  $f : A \rightarrow \operatorname{colim} X$ . As in the proof of [Example 1.9](#), we may find some  $\beta < \lambda$  and a map of sets  $\tilde{f} : A \rightarrow X_\beta$  such that the composition  $A \xrightarrow{\tilde{f}} X_\beta \xrightarrow{j} \operatorname{colim} X$  is equal to  $f$  (note we have given the canonical map  $X_\beta \rightarrow \operatorname{colim} X$  the name  $j$ ). It remains to show that  $\tilde{f}$  is continuous. Let  $U \subseteq X_\beta$  be open. Since  $j$  is an inclusion ([Corollary 2.6](#)), there exists  $V \subseteq \operatorname{colim} X_\beta$  open such that  $j^{-1}(V) = U$ . Then  $\tilde{f}^{-1}(U) = \tilde{f}^{-1}(j^{-1}(V)) = (j \circ \tilde{f})^{-1}(V) = f^{-1}(V)$ , and  $f$  is continuous, so  $\tilde{f}^{-1}(U) = f^{-1}(V)$  is open. Thus  $\tilde{f}$  is continuous, as desired.  $\square$

**Proposition 2.8** (Hovey 2.4.2). *Compact topological spaces are finite relative to the class  $\mathcal{T}$  of closed  $T_1$  inclusions.*

*Proof.* We use the characterization of smallness afforded by [Proposition 1.8](#). Let  $\lambda$  be a limit ordinal, and let  $X : \lambda \rightarrow \mathbf{Top}$  be a  $\lambda$ -sequence so that  $X_\beta \rightarrow X_{\beta+1}$  is a closed  $T_1$  inclusion for all  $\beta + 1 < \lambda$ . Let  $j : X \Rightarrow \operatorname{colim} X$  is a colimit cone for  $X$ . Recall that the forgetful functor  $\mathbf{Top} \rightarrow \mathbf{Set}$  is forgetful, so by [Remark 1.7](#) elements of  $\operatorname{colim} X$  are equivalence classes of elements  $a \in X_\alpha$  for  $\alpha < \lambda$ , where  $a \in X_\alpha$  and  $b \in X_\beta$  represent the same equivalence class iff there exists  $\alpha, \beta \leq \gamma < \lambda$  so that  $a$  and  $b$  are sent to the same element by the maps  $X_\alpha \rightarrow X_\gamma$  and  $X_\beta \rightarrow X_\gamma$ , respectively.

First, we show condition (i) of [Proposition 1.8](#). Let  $j : X \Rightarrow \operatorname{colim} X$  be a colimit cone for  $X$ , and suppose we are given maps  $f : A \rightarrow X_\alpha$  and  $g : A \rightarrow X_\beta$  such that  $j_\alpha \circ f = j_\beta \circ g$ . WLOG, suppose  $\alpha \leq \beta$ . Then

$$j_\beta \circ \iota_{\alpha, \beta} \circ f = j_\alpha \circ f = j_\beta \circ g,$$

and  $j_\beta$  is injective ([Corollary 2.6](#)) and therefore a monomorphism in  $\mathbf{Top}$ , so  $\iota_{\alpha, \beta} \circ f = g$ , meaning  $f$  and  $g$  do indeed agree in some stage of the colimit, as desired.

Now we show condition (ii) of [Proposition 1.8](#). Let  $f : A \rightarrow \operatorname{colim} X$  be a continuous map. In order to show  $f$  factors through some  $X_\alpha$ , we first claim it is sufficient for there to be some  $\alpha < \lambda$  with  $f(A) \subseteq j_\alpha(X_\alpha)$ . Given an ordinal  $\alpha < \lambda$ , for each  $a \in A$ , there exists  $\tilde{f}(a) \in X_\alpha$  such that  $j_\alpha(\tilde{f}(a)) = f(a)$ . Thus we have defined a function  $\tilde{f} : A \rightarrow X_\alpha$  such that  $j_\alpha \circ \tilde{f} = f$ . It remains to show that  $\tilde{f}$  is continuous. Indeed, we know  $j_\alpha$  is an inclusion ([Corollary 2.6](#)), so given  $U \subseteq X_\alpha$  open, there exists  $V \subseteq \operatorname{colim} X$  open with  $j_\alpha^{-1}(V) = U$ , in which case

$$\tilde{f}^{-1}(U) = \tilde{f}^{-1}(j_\alpha^{-1}(V)) = (j_\alpha \circ \tilde{f})^{-1}(V) = f^{-1}(V),$$

which is open as  $f$  is continuous. Hence,  $\tilde{f}$  is continuous, as desired.

Now, suppose for the sake of a contradiction that for all  $\alpha < \lambda$ ,  $f(A) \not\subseteq j_\alpha(X_\alpha)$ . Thus we may construct a **strictly increasing** sequence  $\{\alpha_n\}_{n=0}^\infty \subseteq \lambda$  such that for  $n > 0$ , there exists  $x_n \in j_{\alpha_n}(X_{\alpha_n}) \setminus j_{\alpha_{n-1}}(X_{\alpha_{n-1}})$  with  $x_n \in f(A)$ . Thus for each  $n > 0$ , there exists  $y_n \in X_{\alpha_n}$  such that  $j_{\alpha_n}(y_n) = x_n$ . Note in particular that given  $0 \leq m < n$ ,  $y_n$  is not in the image of  $\iota_{\alpha_m, \alpha_n}$ . Suppose for the sake of a contradiction that  $y_n = \iota_{\alpha_m, \alpha_n}(z)$  for some  $z \in X_{\alpha_m}$  and  $0 \leq m < n$ . Then we know  $j_{\alpha_m}(z) = j_{\alpha_n}(\iota_{\alpha_m, \alpha_n}(z)) = j_{\alpha_n}(y_n) = x_n$ , and

$$x_n \in j_{\alpha_n}(X_{\alpha_n}) \setminus j_{\alpha_{n-1}}(X_{\alpha_{n-1}}) \supseteq j_{\alpha_n}(X_{\alpha_n}) \setminus j_{\alpha_{n-1}}(\iota_{\alpha_m, \alpha_{n-1}}(X_{\alpha_m})) = j_{\alpha_n}(X_{\alpha_n}) \setminus j_{\alpha_m}(X_{\alpha_m}).$$

Hence we reach a contradiction, as  $j_{\alpha_m}(z) = x_m$  but  $x_n$  is not in the image of  $j_{\alpha_m}$ . Let  $\mu := \sup_{n=1}^\infty \alpha_n$ . Clearly  $\mu \leq \lambda$ ; if  $\mu = \lambda$ , define  $X_\mu := \operatorname{colim} X$ ,  $j_\mu := \operatorname{id}_{X_\mu}$ , and for  $\alpha < \lambda$  define  $\iota_{\alpha, \mu} := j_\alpha$ . Let  $K := \{\iota_{\alpha_n, \mu}(y_n)\}_{n=1}^\infty \subseteq X_\mu$ . We claim every subset of  $K$  is closed in  $X_\mu$ . Since  $X$  is colimit preserving and  $\mu = \sup_{n=1}^\infty \alpha_n$ , the topology on  $X_\mu$  is the final topology induced by the maps  $\iota_{\alpha_n, \mu} : X_{\alpha_n} \rightarrow X_\mu$  for  $n = 1, 2, \dots$ . Thus, given a subset  $C \subseteq K$ , in order to show that  $C$  is closed in  $X_\mu$ , it is sufficient (and necessary) for  $\iota_{\alpha_n, \mu}^{-1}(C)$  to be closed in  $X_{\alpha_n}$  for  $n = 1, 2, \dots$ . Let  $n > 0$ . Given  $y \in \iota_{\alpha_n, \mu}^{-1}(C)$ , then  $\iota_{\alpha_n, \mu}(y) \in C \subseteq K$ , so that in particular  $\iota_{\alpha_n, \mu}(y) = \iota_{\alpha_m, \mu}(y_m)$  for some  $m = 1, 2, \dots$ . We claim  $m \leq n$ . Suppose for the sake of a contradiction that  $m > n$ , then we would have

$$\iota_{\alpha_m, \mu}(y_m) = \iota_{\alpha_n, \mu}(y) = \iota_{\alpha_m, \mu}(\iota_{\alpha_n, \alpha_m}(y)),$$

and  $\iota_{\alpha_m, \mu}$  is injective (by either [Lemma 2.4](#) if  $\mu < \lambda$  or by [Corollary 2.6](#) if  $\mu = \lambda$ , in which case recall we defined  $\iota_{\alpha_m, \mu} = j_{\alpha_m}$ ), thus  $y_m = \iota_{\alpha_n, \alpha_m}(y)$ , meaning  $y_m$  is in the image of  $\iota_{\alpha_n, \alpha_m}$  for  $m > n$ , a contradiction, as we showed earlier this is impossible. Thus it must have been true that  $m \leq n$  in the first place, so

$$\iota_{\alpha_n, \mu}(y) \in \{\iota_{\alpha_m, \mu}(y_m)\}_{m=1}^n \implies y \in \iota_{\alpha_n, \mu}^{-1}(\{\iota_{\alpha_m, \mu}(y_m)\}_{m=1}^n).$$



We further claim  $\iota_{\alpha_n, \mu}^{-1}(\{\iota_{\alpha_m, \mu}(y_m)\}_{m=1}^n) = \{\iota_{\alpha_m, \alpha_n}(y_m)\}_{m=1}^n$ . To see the inclusion  $\subseteq$ , suppose  $z \in X_{\alpha_n}$  with  $\iota_{\alpha_n, \mu}(z) = \iota_{\alpha_m, \mu}(y_m)$  for some  $m \leq n$ . Then  $\iota_{\alpha_n, \mu}(z) = \iota_{\alpha_n, \mu}(\iota_{\alpha_m, \alpha_n}(y_m))$  and  $\iota_{\alpha_n, \mu}$  is injective (Lemma 2.4 if  $\mu < \lambda$  and Corollary 2.6 if  $\mu = \lambda$ ), so  $z = \iota_{\alpha_m, \alpha_n}(y_m)$ , as desired. To see the opposite inclusion, given  $m \leq n$ , we have  $\iota_{\alpha_n, \mu}(\iota_{\alpha_m, \alpha_n}(y_m)) = \iota_{\alpha_m, \mu}(y_m)$ , so  $\iota_{\alpha_m, \alpha_n}(y_m) \in \iota_{\alpha_n, \mu}^{-1}(\{\iota_{\alpha_m, \mu}(y_m)\}_{m=1}^n)$ , as desired. Thus, we have shown  $y \in \{\iota_{\alpha_m, \alpha_n}(y_m)\}_{m=1}^n$ . Recall our choice of  $y \in \iota_{\alpha_n, \mu}^{-1}(C)$  was arbitrary, so  $\iota_{\alpha_n, \mu}^{-1}(C)$  is contained in  $\{\iota_{\alpha_m, \alpha_n}(y_m)\}_{m=1}^n$ . Thus, because  $\{\iota_{\alpha_m, \alpha_n}(y_m)\}_{m=1}^n$  is finite, in order to show  $\iota_{\alpha_n, \mu}^{-1}(C)$  is closed in  $X_{\alpha_n}$ , it suffices to show that  $\iota_{\alpha_m, \alpha_n}(y_m)$  is a closed point in  $X_{\alpha_n}$  for  $m = 1, \dots, n$ . As we have shown above,  $y_m$  is not in the image of  $\iota_{\alpha_0, \alpha_m}$  for any  $m \geq 1$ , and  $\iota_{\alpha_0, \alpha_m}$  is a closed  $T_1$  inclusion (Lemma 2.4), so  $y_m$  is a closed point of  $X_{\alpha_m}$  for  $m = 1, \dots, n$ . Then since  $\iota_{\alpha_m, \alpha_n}$  is closed (again by Lemma 2.4),  $\iota_{\alpha_m, \alpha_n}(y_m)$  is closed in  $X_{\alpha_n}$  for  $m = 1, \dots, n$ , precisely the desired result.

Now, we have shown that every subset of  $K$  is closed in  $X_\mu$ . Then  $j_\mu : X_\mu \rightarrow \text{colim } X$  is a closed and injective (this follows by Corollary 2.6 if  $\mu < \lambda$ , and if  $\mu = \lambda$ ,  $X_\mu = \text{colim } X$ , in which case  $j_\mu$  is the identity), so every subset of  $S := j_\mu(K)$  is closed in  $\text{colim } X$ . Note that

$$S = \{j_\mu(\iota_{\alpha_n, \mu}(y_n))\}_{n=1}^\infty = \{j_{\alpha_n}(y_n)\}_{n=1}^\infty = \{x_n\}_{n=1}^\infty \subseteq f(A),$$

Then for  $n = 1, 2, \dots$ , define  $U_n := f(A) \setminus (S \setminus \{x_n\})$ . Each  $U_n$  is open in  $f(A)$  (as  $S \setminus \{x_n\}$  is a subset of  $S$  and is therefore closed in  $\text{colim } X$ , thus in  $f(A)$ ), and the collection  $\{U_n\}_{n=1}^\infty$  forms an infinite open cover of  $f(A)$ . Finally, this open cover has no finite subcover, as  $U_n$  is the only element of the cover containing  $x_n$  for  $n = 1, 2, \dots$ . Hence we reach a contradiction, as  $f$  is continuous and  $A$  is compact, so  $f(A)$  is compact, but we have found an infinite open cover of  $f(A)$  which has no finite subcover. Thus, there must have existed soem  $\alpha < \lambda$  with  $f(A) \subseteq j_\alpha(X_\alpha)$  in the first place, in which case, as we have shown, this implies  $f$  factors through  $X_\alpha$  via a continuous map, as desired.  $\square$

**Proposition 2.9** (Hovey 2.4.5 & 2.4.6). *The class of injective maps (resp. inclusions, closed  $T_1$  inclusions) in **Top** is saturated.*

*Proof.* We know these three classes are closed under transfinite compositions (Corollary 2.5), so it suffices to show these classes are closed under pushouts and retracts. In what follows, fix a pushout diagram and a retract diagram of the following form:

$$\begin{array}{ccc} A & \xrightarrow{f} & C \\ i \downarrow & & \downarrow j \\ B & \xrightarrow{g} & D \end{array} \qquad \begin{array}{ccccc} & \curvearrowright & & \curvearrowright & \\ A & \xrightarrow{f} & B & \xrightarrow{g} & A \\ i \downarrow & & \downarrow i & & \downarrow j \\ C & \xrightarrow{h} & D & \xrightarrow{k} & C \\ & \curvearrowright & & \curvearrowright & \end{array}$$

- (i) First, consider the pushout diagram, and suppose  $i$  is injective. We wish to show  $j$  is injective. Suppose for the sake of a contradiction there existed distinct points  $c_1, c_2 \in C$  such that  $j(c_1) = j(c_2)$ . Define  $h : C \rightarrow \{1, 2, 3\}$  mapping  $c_1 \mapsto 1$ ,  $c_2 \mapsto 2$ , and every other point in  $C$  maps to 3. Define  $k : B \rightarrow \{1, 2, 3\}$  to map every point in  $i(f^{-1}(c_1))$  to 1, every point in  $i(f^{-1}(c_2))$  to 2, and every other point to 3. We may give  $\{1, 2, 3\}$  the indiscrete topology so that  $h$  and  $k$  are continuous. Then clearly  $h \circ f = k \circ i$ , so there exists a map  $\ell : D \rightarrow \{1, 2, 3\}$  such that  $\ell \circ j = h$  and  $\ell \circ g = k$ . But we reach a contradiction, as  $\ell(j(c_1)) = \ell(j(c_2))$ , but  $h(c_1) \neq h(c_2)$ .

Now, consider the retract diagram, and suppose  $i$  is injective. We wish to show  $j$  is injective. Suppose  $a_1, a_2 \in A$  such that  $j(a_1) = j(a_2)$ . Then  $h(j(a_1)) = h(j(a_2))$ , and  $h \circ j = i \circ f$ , so  $i(f(a_1)) = i(f(a_2))$ . Note that since  $g \circ f = \text{id}_A$ , necessarily  $f$  is injective, and we are assuming  $i$  is injective, so  $i(f(a_1)) = i(f(a_2)) \implies a_1 = a_2$ .

- (ii) First, consider the pushout diagram, and suppose  $i$  is an inclusion. We wish to show  $j$  is an inclusion. We know  $j$  is injective by (i). It remains to show that given  $U \subseteq C$  open, there exists  $V \subseteq D$  open with  $j^{-1}(V) = U$ . Given  $U \subseteq C$  open,  $f^{-1}(U)$  is open in  $A$  as  $f$  is continuous, and  $i$  is an inclusion, so there exists  $W \subseteq B$  open with  $i^{-1}(W) = f^{-1}(U)$ . Now let  $S = \{0, 1\}$  be the Sierpinski space with open sets  $\{\emptyset, \{1\}, \{0, 1\}\}$ . Define  $h : C \rightarrow S$  to map every point of  $U$  to 1, and every other point to 0. Define  $k : B \rightarrow S$  to map every point of  $W$  to 1, and every other point to 0. Clearly  $h$  and  $k$  are continuous. We claim  $h \circ f = k \circ i$ . Indeed, let  $a \in A$ . If  $a \in f^{-1}(U)$ , then  $h(f(a)) = 1$ , as  $f(a) \in U$ , while  $k(i(a)) = 1$ , as  $a \in U = i^{-1}(W)$ , so  $i(a) \in W$ . Conversely, if  $a \notin f^{-1}(U)$ , then  $h(f(a)) = 0$  as  $f(a) \notin U$ , while  $k(i(a)) = 0$ , as  $a \notin i^{-1}(W)$  meaning  $i(a) \notin W$ . Hence, there exists a (unique)

continuous map  $\ell : D \rightarrow S$  with  $\ell \circ j = h$  and  $\ell \circ g = k$ . Define  $V := \ell^{-1}(1)$ , which is open in  $D$  as  $\{1\}$  is open in  $S$ . Then finally, we claim  $j^{-1}(V) = U$ . Indeed,

$$j^{-1}(V) = j^{-1}(\ell^{-1}(1)) = (\ell \circ j)^{-1}(1) = h^{-1}(1) = U.$$

Thus  $j$  is an inclusion, as desired.

Now, consider the retract diagram, and suppose  $i$  is an inclusion. We wish to show  $j$  is an inclusion. We know  $j$  is injective by (i). It remains to show that given  $U \subseteq A$  open, there exists  $V \subseteq C$  open with  $j^{-1}(V) = U$ . Then since  $g$  is continuous,  $g^{-1}(U)$  is open in  $B$ , and  $i$  is an inclusion, so there exists  $W \subseteq D$  open with  $i^{-1}(W) = g^{-1}(U)$ . Then

$$j^{-1}(h^{-1}(W)) = (h \circ j)^{-1}(W) = (i \circ f)^{-1}(W) = f^{-1}(i^{-1}(W)) = f^{-1}(g^{-1}(U)) = (g \circ f)^{-1}(U) = U,$$

and  $h^{-1}(W)$  is open as  $h$  is continuous and  $W$  is open, so we are done, as if we set  $V := h^{-1}(W)$ , then we have shown  $V$  is open in  $C$  and  $j^{-1}(V) = U$ , as desired.

- (iii) First, consider the pushout diagram, and suppose  $i$  is a closed  $T_1$  inclusion. We wish to show  $j$  is a closed  $T_1$  inclusion. We know  $j$  is an inclusion by (ii). It remains to show  $j$  is closed, and every point of  $D \setminus j(C)$  is closed in  $D$ . First, we show closedness. Let  $V \subseteq C$  be a closed set, we want to show  $j(C)$  is closed in  $D$ . By definition of the colimit topology on  $D$  (which is the final topology on  $D$  induced by  $j$  and  $g$  by the discussion at the beginning of this chapter), in order to show  $j(V)$  is closed in  $D$  it suffices to show that  $j^{-1}(j(V))$  is closed in  $C$  and  $g^{-1}(j(V))$  is closed in  $B$ . Since  $j$  is injective by (i),  $j^{-1}(j(V)) = V$ , which we have defined to be closed in  $A$ . Now, to show  $g^{-1}(j(V))$  is closed, since  $i$  is a closed map and  $f$  is continuous, it suffices to show  $i(f^{-1}(V)) = g^{-1}(j(V))$ . First of all, let  $b \in i(f^{-1}(V))$ , then  $b = i(a)$  for some  $a \in A$  with  $f(a) \in V$ . Then  $g(b) = g(i(a)) = j(f(a)) \in j(V)$ , so  $b \in g^{-1}(j(V))$ . Conversely, let  $b \in g^{-1}(j(V))$ , so  $g(b) = j(c)$  for some  $c \in V$ . Then by the explicit description of the colimit in **Set** and the fact that the forgetful functor **Top**  $\rightarrow$  **Set** preserves colimits, there exists  $a \in A$  such that  $f(a) = c$  and  $i(a) = b$ . Then in particular,  $f(a) = c \in V$ , so  $a \in f^{-1}(V)$ , so  $b \in i(f^{-1}(V))$  as desired. Hence,  $j$  is indeed a closed map. It remains to show that for all  $d \in D \setminus j(C)$ ,  $d$  is a closed point in  $D$ . Given  $d \in D \setminus j(C)$ , by the explicit characterization of the colimit topology, it suffices to show that  $j^{-1}(d)$  and  $g^{-1}(d)$  are closed in  $C$  and  $B$ , respectively. First of all,  $j^{-1}(d) = \emptyset$ , which is closed. Now, since  $d$  is not in the image of  $j$ ,  $d$  is not in the image of  $g \circ i = j \circ f$ , so  $g^{-1}(d) \subseteq B \setminus i(A)$ . Thus since  $i$  is a closed  $T_1$  inclusion, in order to show  $g^{-1}(d)$  is closed, it suffices to show that  $g^{-1}(d)$  is a singleton. Suppose for the sake of a contradiction there existed distinct points  $x, y \in B$  with  $g(x) = g(y) = d$ . Define  $h : C \rightarrow \{1, 2, 3\}$  to map every point to 3. Define  $k : B \rightarrow \{1, 2, 3\}$  to map  $x \mapsto 1$ ,  $y \mapsto 2$ , and every other point in  $B$  maps to 3. Endow  $\{1, 2, 3\}$  with the indiscrete topology so that  $h$  and  $k$  are continuous. Note  $h \circ f = k \circ i$ : given  $a \in A$ , since  $x, y \in g^{-1}(d) \subseteq B \setminus i(A)$ ,  $i(a)$  does not equal  $x$  or  $y$ , thus  $k(i(a)) = 3 = h(f(a))$ . Hence by the definition of the colimit, there exists a map  $\ell : D \rightarrow X$  such that  $\ell \circ g = k$  and  $\ell \circ j = h$ . Then we reach a contradiction, as  $k(x) \neq k(y)$  but  $\ell(g(x)) = \ell(g(y)) = \ell(d)$ . Hence,  $g^{-1}(d) \notin i(A)$  must have been a singleton in the first place, meaning  $g^{-1}(d)$  is closed in  $B$  as desired.

Now, consider the retract diagram, and suppose  $i$  is a closed  $T_1$  inclusion. We wish to show  $j$  is a closed  $T_1$  inclusion. We know  $j$  is an inclusion by (ii). It remains to show  $j$  is closed, and every point of  $C \setminus j(A)$  is closed in  $C$ . First, we show closedness. Let  $V \subseteq A$  be closed. First, we claim  $j(V) = h^{-1}(i(g^{-1}(V)))$ . Given  $c \in j(V)$ ,  $c = j(a)$  for some  $a \in V$ , in which we have  $h(c) = h(j(a)) = i(f(a))$ , and  $g(f(a)) = a \in V$ , so  $f(a) \in g^{-1}(V)$ , meaning  $h(c) \in i(g^{-1}(V))$ , so  $c \in h^{-1}(i(g^{-1}(V)))$ . Conversely, given  $c \in h^{-1}(i(g^{-1}(V)))$ , so  $h(c) = i(b)$  for some  $b \in B$  with  $g(b) \in V$ . Then

$$c = k(h(c)) = k(i(b)) = j(g(b)) \in j(V),$$

as desired. Thus, we have shown  $j(V) = h^{-1}(i(g^{-1}(V)))$ . Note that since  $V \subseteq A$  is closed and  $g$  is continuous,  $g^{-1}(V)$  is closed in  $B$ . Since  $i$  is a closed map,  $i(g^{-1}(V))$  is closed in  $D$ . Finally, since  $h$  is continuous,  $h^{-1}(i(g^{-1}(V))) = j(V)$  is closed in  $C$ , as desired. It remains to show that for all  $c \in C \setminus j(A)$  that  $c$  is a closed point in  $C$ . Given  $c \in C \setminus j(A)$ , note that  $h(c) \notin i(B)$ , as if  $h(c) = i(b)$  for some  $b \in B$ , then we would have  $c = k(h(c)) = k(i(b)) = j(g(b))$ , yet we chose  $c$  not in the image of  $j$ . Hence, since  $i$  is a closed  $T_1$  inclusion, and  $h(c)$  is not in the image of  $i$ ,  $h(c)$  is a closed point



in  $D$ . Then note that since  $k \circ h = \text{id}_C$ ,  $h$  is injective, so  $c = h^{-1}(h(c))$  is a closed point, as it is the preimage of the closed set  $\{h(c)\}$  along the continuous map  $h$ .  $\square$

**Lemma 2.10** (Hovey 2.4.8).  $\mathcal{W} \cap \mathcal{T}$  is closed under transfinite compositions, where  $\mathcal{T}$  denotes the class of closed  $T_1$  inclusions.

*Proof.* Let  $\lambda$  be an ordinal, and let  $X : \lambda \rightarrow \mathbf{Top}$  be a  $\lambda$ -sequence such that for all  $\beta + 1 < \lambda$ , the map  $X_\beta \rightarrow X_{\beta+1}$  belongs to  $\mathcal{W} \cap \mathcal{T}$ . Let  $j : X \rightarrow \underline{X}_\lambda$  be a colimit cone for  $X$ . By [Corollary 2.5](#), we know that  $j_0 : X \rightarrow X_\lambda$  is a closed  $T_1$  inclusion, so it remains to show that  $\pi_n(j_0, x_0) : \pi_n(X_0, x_0) \rightarrow \pi_n(X_\lambda, j_0(x_0))$  is an isomorphism for all  $n \geq 0$  and  $x_0 \in X_0$ .

First we show surjectivity. Suppose we are given  $x_0 \in X_0$  and a continuous map  $f : (S^n, *) \rightarrow (X_\lambda, j_0(x_0))$ . Since  $S^n$  is compact, by ??  $\square$

**Proposition 2.11.** *The domains of  $I'$  (resp.  $J$ ) are small relative to  $I'$ -cell.*

*Proof.* By [Lemma 2.7](#), every space is small relative to the inclusions, and in particular every space is small relative to the class  $\mathcal{T}$  of closed  $T_1$  inclusions. Hence, it suffices to show that  $J\text{-cell}, I'\text{-cell} \subseteq \mathcal{T}$ . We showed above in [Proposition 2.9](#) that  $\mathcal{T}$  is closed under transfinite composition and pushouts, and clearly every map in  $I'$  and  $J$  is a closed  $T_1$  inclusion, so the desired result follows.  $\square$

**Lemma 2.12** (Hovey Lemma 2.4.4). *The weak equivalences in  $\mathbf{Top}$  are closed under retracts and satisfy 2-of-3 axiom (so that in particular the weak equivalences form a subcategory, as clearly identities are weak equivalences).*

*Proof.* First we show that weak equivalences satisfy 2-of-3. Let  $f : X \rightarrow Y$  and  $g : Y \rightarrow Z$  be continuous functions of topological spaces.

First of all, suppose  $f$  and  $g$  are both weak equivalences. Then by functoriality of  $\pi_n$ , since  $\pi_n(f, x)$  and  $\pi_n(g, f(x))$  are isomorphisms for all  $x \in X$ ,  $\pi_n(g \circ f, x) = \pi_n(g, f(x)) \circ \pi_n(f, x)$  is likewise an isomorphism for all  $x \in X$ , so that  $g \circ f$  is a weak equivalence.

Now, suppose that  $g \circ f$  and  $g$  are weak equivalences. Pick a point  $x \in X$ . We wish to show that  $\pi_n(f, x) : \pi_n(X, x) \rightarrow \pi_n(Y, f(x))$  is an isomorphism for all  $n \geq 0$ . We know that  $\pi_n(g \circ f, x)$  is an isomorphism, and  $\pi_n(g, f(x))$  is an isomorphism, say with inverse,  $\varphi$ , so that

$$\varphi \circ \pi_n(g \circ f, x) = \varphi \circ \pi_n(g, f(x)) \circ \pi_n(f, x) = \pi_n(f, x)$$

is an isomorphism, as it is a composition of isomorphisms.

Now, suppose that  $g \circ f$  and  $f$  are weak equivalences. Pick a point  $y \in Y$ . Since  $\pi_0(f)$  is an isomorphism, there exists a point  $x \in X$  such that  $f(x)$  belongs to the path component containing  $y$ , so that there exists some  $\alpha : I \rightarrow Y$  with  $\alpha(0) = f(x)$  and  $\alpha(1) = y$ . Then consider the following diagram

$$\begin{array}{ccc} \pi_n(Y, y) & \xrightarrow{\pi_n(g, y)} & \pi_n(Z, g(y)) \\ \downarrow & & \downarrow \\ \pi_n(Y, f(x)) & \xrightarrow{\pi_n(g, f(x))} & \pi_n(Z, g(f(x))) \end{array}$$

where the left arrow is the isomorphism given by conjugation by the path  $\alpha$ , and the right arrow is the isomorphism given by conjugation by the path  $g \circ \alpha$ . It is tedious yet straightforward to verify that the diagram commutes. Furthermore, we know that  $\pi_n(f, x)$  and  $\pi_n(g \circ f, x) = \pi_n(g, f(x)) \circ \pi_n(f, x)$  are isomorphisms for all  $n$ , so that if we denote the inverse of  $\pi_n(f, x)$  by  $\varphi$ , then

$$\pi_n(g \circ f, x) \circ \varphi = \pi_n(g, f(x)) \circ \pi_n(f, x) \circ \varphi = \pi_n(g, f(x))$$

is an isomorphism, as it is given as a composition of isomorphisms. Hence, the top arrow must likewise be an isomorphism, precisely the desired result.

The fact that weak equivalences in  $\mathbf{Top}$  are closed under retracts is entirely straightforward and follows from the fact that the functors  $\pi_n$  preserve retract diagrams and that the class of isomorphisms in any category is closed under retracts.  $\square$

**Proposition 2.13** (Hovey 2.4.9).  $\perp(J_\perp) \subseteq \mathcal{W} \cap \perp(I'_\perp)$ .

*Proof.* First, in order to show  $\perp(J_\perp) \subseteq \perp(I'_\perp)$ , It suffices to show that  $J \subseteq I'$ -cell, as by [Corollary 1.16](#) we would have  $J \subseteq \perp(I'_\perp)$ , and

$$J \subseteq \perp(I'_\perp) \implies \perp(J_\perp) \subseteq \perp((\perp(I'_\perp))_\perp) = \perp(I'_\perp),$$

where the implication and equality both follow from [Lemma 1.11](#) which gives that

$$A \subseteq B \implies \perp(A_\perp) \subseteq \perp(B_\perp) \quad \text{and} \quad (\perp(A_\perp))_\perp = A_\perp.$$

Now, to show  $J \subseteq I'$ -cell, first consider the composition  $j_n : D^n \hookrightarrow S^n \hookrightarrow D^{n+1}$ , where the first map is the pushout

$$\begin{array}{ccc} S^{n-1} & \hookrightarrow & D^n \\ \downarrow & & \downarrow \\ D^n & \longrightarrow & S^n \end{array}$$

obtained by gluing two copies of  $D^n$  along their boundary, and the second map is simply the inclusion  $S^n \hookrightarrow D^{n+1}$ , which can be written as the pushout

$$\begin{array}{ccc} S^n & \xlongequal{\quad} & S^n \\ \downarrow & & \downarrow \\ D^{n+1} & \xlongequal{\quad} & D^{n+1} \end{array}$$

It can be seen that  $j_n$  includes  $D^n$  as a hemisphere of  $S^n = \partial D^{n+1} \subseteq D^{n+1}$ . Note that  $D^n \times I$  is homeomorphic to  $D^{n+1}$  (“smooth out” the sharp edges of the cylinder) via some homeomorphism  $h_n : D^{n+1} \rightarrow D^n \times I$ , and in particular, we may define  $h_n$  so that  $h_n(j_n(D^n)) = D^n \times \{0\} \subseteq D^n \times I$  by squashing the hemisphere  $j_n(D^n)$  to be one of the faces of the cylinder  $D^n \times I$ , in which case  $h_n \circ j_n : D^n \rightarrow D^n \times I$  is precisely the inclusion  $D^n \hookrightarrow D^n \times I$  sending  $x \mapsto (x, 0)$ , and since  $j_n \in I'$ -cell,  $h_n \circ j_n \in I'$ -cell by [Lemma 1.13](#).

Now, we claim that  $\perp(J_\perp) \subseteq \mathcal{W}$ . First note that by [Corollary 1.18](#) and [Proposition 2.11](#), every map in  $\perp(J_\perp)$  is a retract of an element of  $J$ -cell. Furthermore, we know that  $\mathcal{W}$  is closed under retracts ([Lemma 2.12](#)), so that it suffices to show that  $J$ -cell  $\subseteq \mathcal{W}$ . We claim it suffices to show that pushouts of maps in  $J$  are weak equivalences. Supposing we had shown this, we would have that pushouts of maps in  $J$  are weak equivalences and  $T_1$  inclusions, as  $J \subseteq \mathcal{T}$  and  $\mathcal{T}$  is saturated by [Proposition 2.9](#). Then by [Lemma 2.10](#), we would have that  $J$ -cell  $\subseteq \mathcal{W} \cap \mathcal{T}$ , precisely the desired result.

Now, let  $\mathcal{S}$  be the class of *inclusions of a deformation retract*, i.e., those **injective** maps  $i : A \rightarrow B$  such that there exists a homotopy  $H : B \times I \rightarrow B$  with  $H(i(a), t) = i(a)$  for all  $a \in A$ ,  $H(b, 0) = b$  for all  $b \in B$ , and  $H(b, 1) = i(r(b))$  for all  $b \in B$  for some map  $r : B \rightarrow A$ .<sup>3</sup> We will show the following:

- (1)  $\mathcal{S} \subseteq \mathcal{W}$ .

It suffices to show that if  $i : A \rightarrow B$  belongs to  $\mathcal{S}$ , then  $i$  is a homotopy equivalence. Indeed, given  $i : A \rightarrow B$ , let  $H : B \times I \rightarrow B$  and  $r : B \rightarrow A$  be a homotopy and retract satisfying the conditions above. Then in particular,  $H$  is a homotopy between  $\text{id}_B$  (at time  $t = 0$ ) and  $i \circ r$  (at time  $t = 1$ ). It remains to show that  $r \circ i = \text{id}_A$ . First of all, note that since  $H(b, 1) = i(r(b))$  for all  $b \in B$ , we have  $H(i(a), 1) = i(r(i(a)))$ . Yet, we also know that  $H(i(a), t) = i(a)$  for all  $t \in I$ , so  $i(r(i(a))) = i(a)$ , and  $i$  is injective so  $r(i(a)) = a$ .

- (2)  $J \subseteq \mathcal{S}$ .

For  $n \geq 0$ , let  $j_n : D^n \hookrightarrow D^n \times I$  denote the inclusion of  $D^n$  as the subset  $D^n \times \{0\}$ . Define a deformation retract  $H : D^n \times I \times I \rightarrow D^n \times I$  by  $(x, s, t) \mapsto (x, s(1 - t))$ . Then indeed we have  $H(j_n(x), t) = H(x, 0, t) = (x, 0) = j_n(x)$  for all  $x \in D^n$ ,  $H(x, t, 0) = (x, t(1 - 0)) = (x, t)$  for all  $(x, t) \in D^n \times I$ , and  $H(x, t, 1) = (x, t(1 - 1)) = (x, 0) = j_n(r(x))$  for all  $(x, t) \in D^n \times I$ , where  $r : D^n \times I \rightarrow D^n$  is the projection onto time zero sending  $(x, t) \mapsto (x, 0)$ . Finally,  $j_n$  is clearly injective. Thus, indeed  $J \subseteq \mathcal{S}$ .

<sup>3</sup>Hovey has a typo here, namely, he does not specify that  $i$  must be injective. Without this specification, his assertion fails. For example, take  $A = \mathbb{R}^2$ ,  $B = \mathbb{R}$ ,  $i(x, y) = x$ ,  $H(b, t) = b$ , and  $r(b) = (b, 0)$ . Then  $i$  is an inclusion of a deformation retract according to Hovey’s “definition,” but  $i$  is not injective and  $r$  is not a retract.

(3)  $\mathcal{S}$  is closed under pushouts.

Suppose we are given a pushout diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & C \\ i \downarrow & \lrcorner & \downarrow j \\ B & \xrightarrow{g} & D \end{array}$$

where  $i \in \mathcal{S}$ . Then we wish to show  $j \in \mathcal{S}$ . First, we know  $j$  is injective by [Proposition 2.9](#). Now, we look to construct  $H$  and  $r$ . Let  $K : B \times I \rightarrow B$  and  $r' : B \rightarrow A$  be maps satisfying the conditions for  $i$  to be an inclusion of a deformation retract.

We wish to define a homotopy  $H : D \times I \rightarrow D$ . Then  $I$  is a locally compact Hausdorff space (in particular, it is compact and Hausdorff), so that the functor  $- \times I : \mathbf{Top} \rightarrow \mathbf{Top}$  preserves colimits ([Proposition 2.1](#)), meaning the following is a pushout diagram:

$$\begin{array}{ccc} A \times I & \xrightarrow{f \times \text{id}_I} & C \times I \\ i \times \text{id}_I \downarrow & \lrcorner & \downarrow j \times \text{id}_I \\ B \times I & \xrightarrow{g \times \text{id}_I} & D \times I \end{array}$$

Then by the universal property of the pushout, there is a map  $H : D \times I \rightarrow D$  (the dashed line) such that the following diagram commutes

$$\begin{array}{ccccc} A \times I & \xrightarrow{f \times \text{id}_I} & C \times I & & \\ i \times \text{id}_I \downarrow & \lrcorner & \downarrow j \times \text{id}_I & \searrow \pi_1 & \\ B \times I & \xrightarrow{g \times \text{id}_I} & D \times I & \xrightarrow{H} & C \\ & \searrow K & \downarrow g & \downarrow j & \\ & & B & \xrightarrow{g} & D \end{array}$$

Now, note  $r' \circ i = \text{id}_A$ . Indeed, given  $a \in A$ , we have  $i(r'(i(a))) = K(i(a), t) = i(a)$  and  $i$  is injective, so that  $r'(i(a)) = a$ , as desired. Hence, there exists a unique map  $r : D \rightarrow C$  (the dashed line) such that the following diagram commutes:

$$\begin{array}{ccc} A & \xrightarrow{f} & C \\ i \downarrow & \lrcorner & \downarrow j \\ B & \xrightarrow{g} & D \\ & \searrow r' & \downarrow r \\ & & A \xrightarrow{f} C \end{array}$$

Now we claim that our constructions  $H$  and  $r$  endow  $j$  with the structure of an inclusion of a deformation retract, as desired. First  $c \in C$ , we wish to show  $H(j(c), t) = j(c)$  for all  $t$ . Indeed, we have

$$H(j(c), t) = H(j \times \text{id}_I(c, t)) = j(\pi_1(c, t)) = j(c).$$

Given  $d \in D$ , we want to show  $H(d, 0) = d$ . By the explicit description of the colimit in  $\mathbf{Top}$ , we know that every element of  $D$  is in the image of either  $j$  or  $g$ . If  $d = j(c)$  for some  $c$ , then we have just shown  $H(d, 0) = H(j(c), 0) = j(c) = d$ , as desired. On the other hand, if  $d = g(b)$  for some  $b \in B$  we have

$$H(d, 0) = H(g \times \text{id}_I(b, 0)) = g(K(b, 0)) = g(b) = d.$$

Finally, we claim that  $H(d, 1) = j(r(d))$  for all  $d \in D$ . If  $d = j(c)$  for some  $c \in C$ , then we have

$$H(d, 1) = H(j(c), 1) = j(c) = j(r(j(c))) = j(r(d)),$$

as desired. On the other hand, if  $d = g(b)$  for some  $b \in B$ , then

$$H(d, 1) = H(g \times \text{id}_I(b, 1)) = g(K(b, 1)) = g(i(r'(b))) = j(f(r'(b))) = j(r(g(b))) = j(r(d)). \quad \square$$

**Proposition 2.14** (Hovey 2.4.10).  $I'_\perp \subseteq \mathcal{W} \cap J_\perp$

*Proof.* First, by [Proposition 2.13](#) we know  ${}_\perp(J_\perp) \subseteq {}_\perp(I'_\perp)$ , and this implies  $I'_\perp \subseteq J_\perp$ , as by [Lemma 1.11](#) we have

$${}_\perp(J_\perp) \subseteq {}_\perp(I'_\perp) \implies J_\perp = ({}_\perp(J_\perp))_\perp \supseteq ({}_\perp(I'_\perp))_\perp = I'_\perp.$$

Thus, it suffices to show that  $I'_\perp \subseteq \mathcal{W}$ . Now, suppose  $p : (X, x_0) \rightarrow (Y, p(x_0))$  is in  $I'_\perp$ . We wish to show that the map  $\pi_n(p, x_0) : \pi_n(X, x_0) \rightarrow \pi_n(Y, p(x_0))$  is an isomorphism for all  $n$ .

First we show that  $\pi_n(p, x_0)$  is surjective. Let  $g : (S^n, *) \rightarrow (Y, p(x_0))$  be a map. Then we have the following commutative diagram

$$\begin{array}{ccc} * & \longrightarrow & X \\ \downarrow & & \downarrow p \\ S^n & \xrightarrow{g} & Y \end{array}$$

where the top arrow picks out  $x_0$ . Note that the map  $* \rightarrow S^n$  may be realized as a pushout of the diagram  $D^n \leftarrow S^{n-1} \rightarrow *$ , so that  $* \rightarrow S^n$  belongs to  $I'$ -cell, and therefore  ${}_\perp(I'_\perp)$  by [Corollary 1.16](#), and  $p \in I'_\perp$ , so  $* \rightarrow S^n$  has the left lifting property against  $p$ . Thus, the above diagram has a lift  $f : (S^n, *) \rightarrow (X, x_0)$  such that  $p \circ f = g$ , so that  $\pi_n(p, x_0)([f]) = [p \circ f] = [g]$ , as desired.

Finally, we show that  $\pi_n(p, x_0)$  is injective. Suppose we have two maps  $f, g : (S^n, *) \rightarrow (X, x_0)$  such that  $p \circ f$  and  $p \circ g$  represent the same element of  $\pi_n(Y, p(x_0))$ . Then there is a homotopy  $H : S^n \times I \rightarrow Y$  such that for all  $s \in S^n$  and  $t \in I$ ,  $H(s, 0) = p(f(s))$ ,  $H(s, 1) = p(g(s))$ , and  $H(*, t) = p(x_0)$ . By the universal property of the quotient,  $H$  induces a map  $\bar{H} : S^n \wedge I_+ := (S^n \times I)/(* \times I)$  sending the equivalence class  $[s, t] \mapsto H(s, t)$ . Hence, the following diagram commutes:

$$\begin{array}{ccc} S^n \vee S^n & \xrightarrow{f \vee g} & X \\ \downarrow & & \downarrow p \\ S^n \wedge I_+ & \xrightarrow{\bar{H}} & Y \end{array}$$

where the left arrow is an element of  $I'$ -cell, as it may be obtained by attaching an  $n+1$  cell to  $S^n \vee S^n$  (when  $n = 0$ , the attaching map is obvious; when  $n > 0$ , the attaching map is the quotient map  $S^n \rightarrow S^n \vee S^n$  obtained by collapsing the equator). Thus, by similar reasoning to above there exists a lift  $\bar{K} : S^n \wedge I_+ \rightarrow X$ .

Then if we define  $K$  to be the composition  $S^n \times I \twoheadrightarrow S^n \wedge I_+ \xrightarrow{\bar{K}} X$ , this gives us the desired homotopy between  $f$  and  $g$ : given  $s \in S^n$  and  $t \in I$ , we have  $K(s, 0) = \bar{K}([s, 0]) = f(s)$ ,  $K(s, 1) = \bar{K}([s, 1]) = g(s)$ , and  $K(*, t) = \bar{K}([*, t]) = \bar{K}([*, 0]) = (*) = x_0$ .  $\square$

In what follows, given continuous maps  $p : X \rightarrow Y$  and  $i : A \rightarrow B$ , let  $Q(i, p) : X^B \rightarrow P(i, p) := X^A \times_{Y^A} Y^B$  denote the map obtained by the universal property of the fiber product (pullback) via the maps  $i^* \circ p_* : X^B \rightarrow Y^A$  and  $p_* \circ i^* : X^B \rightarrow Y^A$ .

**Lemma 2.15** (Hovey 2.4.11). *Suppose  $p : X \rightarrow Y$  is a map. Then  $p \in I'_\perp$  if and only if the map  $Q(i, p) : X^B \rightarrow P(i, p) := X^A \times_{Y^A} Y^B$  is surjective for all maps  $i : A \rightarrow B$  in  $I'$ . In particular, if  $Q(i, p) \in \mathcal{W} \cap J_\perp$  for all  $i \in I'$ , then  $p \in I'_\perp$ .*

*Proof.* First of all, suppose  $p \in I'_\perp$ , and let  $i : A \rightarrow B$  in  $I'$ . We wish to show  $Q(i, p)$  is surjective. We know the forgetful functor  $\mathbf{Top} \rightarrow \mathbf{Set}$  preserves limits, so by the explicit description of limits in  $\mathbf{Set}$  and

[Remark 1.7](#), the set  $P(i, p)$  is given by pairs of maps  $(A \xrightarrow{\alpha} X, B \xrightarrow{\beta} Y)$  such that  $p \circ \alpha = \beta \circ i$ , and the map  $Q(i, p) : X^B \rightarrow P(i, p)$  sends an arrow  $\ell : B \rightarrow X$  to the pair  $(\ell \circ i, p \circ \ell)$ . Now to see  $Q(i, p)$  is surjective, suppose there exists  $(\alpha, \beta) \in P(i, p)$  so that the following diagram commutes:

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & X \\ i \downarrow & & \downarrow p \\ B & \xrightarrow{\beta} & Y \end{array}$$

Then because  $p \in I'_\perp$  and  $i \in I'$ , there exists a lift  $\ell : B \rightarrow X$ , in which case  $Q(i, p)(\ell) = (\ell \circ i, p \circ \ell) = (\alpha, \beta)$ , as desired.

Conversely, suppose that we are given a map  $p : X \rightarrow Y$  such that  $Q(i, p)$  is surjective for all maps  $i : A \rightarrow B$  in  $I'$ . Then we wish to show that  $p \in I'_\perp$ . Let  $i : A \rightarrow B$  belong to  $I'$ , and suppose we are given a lifting problem of the same form as the above diagram. Then since  $p \circ \alpha = \beta \circ i$ , the pair  $(\alpha, \beta)$  is an element of  $P(i, p)$ . Since  $Q(i, p)$  is surjective, there exists an arrow  $\ell : B \rightarrow X$  such that  $Q(i, p)(\ell) = (\alpha, \beta)$ . But  $Q(i, p)(\ell) = (\ell \circ i, p \circ \ell)$ , so  $\ell \circ i = \alpha$  and  $p \circ \ell = \beta$ , meaning  $\ell$  is a lift, as desired. Thus indeed  $p \in I'_\perp$ .

Thus, in order to show the second part, it suffices to show that any map belonging to  $\mathcal{W} \cap J_\perp$  is surjective. To see this, suppose  $q : W \rightarrow Z$  belongs to  $\mathcal{W} \cap J_\perp$ , and let  $z \in Z$ . Pick any point  $w \in W$ , and consider the map  $\beta : S^0 = \{0, 1\} \rightarrow Z$  sending  $0 \mapsto q(w)$  and  $1 \mapsto z$ . Then since  $q \in \mathcal{W}$ , the homomorphism  $\pi_0(q, w) : (W, w) \rightarrow (Z, q(w))$  is surjective, so there exists a map  $\alpha : (S^0, 0) \rightarrow (W, w)$  and a homotopy  $H : S^0 \times I \rightarrow Z$  such that  $H(s, 0) = q(\alpha(s))$ ,  $H(s, 1) = \beta(s)$ , and  $H(0, t) = q(w)$  for all  $s \in S^0$  and  $t \in I$ . Then we have a lifting problem of the following form

$$\begin{array}{ccc} * & \xrightarrow{\alpha(1)} & W \\ 0 \downarrow & & \downarrow q \\ I & \xrightarrow{H(1, -)} & Z \end{array}$$

$H(1, -)$  is continuous as it is  $H^b(1)$ , where  $H^b : S^0 \rightarrow W^I$  is the adjoint of  $H : S^0 \times I \rightarrow W$  (see the proof of [Proposition 2.1](#) and recall how to construct the adjoint of a morphism given the unit and counit of the adjunction). Then we may identify the left arrow with the map  $D^0 \rightarrow D^0 \times I$ , in which case since  $q \in J_\perp$  there exists a lift  $\ell : I \rightarrow W$  such that  $q \circ \ell = H(1, -)$ . In particular,  $z = \beta(1) = H(1, 1) = q(\ell(1))$ , so that  $z$  does belong to the image of  $q$ , as desired.  $\square$

**Lemma 2.16** (Hovey 2.4.13). *Suppose  $p : X \rightarrow Y$  belongs to  $J_\perp$  and  $i : S^{n-1} \hookrightarrow D^n$  belongs to  $I'$ . Then the map  $Q(i, p)$  belongs to  $J_\perp$ .*

*Proof.* Suppose we are given a map  $j : D^m \times \{0\} \hookrightarrow D^m \times I$  belonging to  $J$  and a lifting problem of the following form

$$\begin{array}{ccc} D^m & \xrightarrow{\quad} & X^{D^n} \\ j \downarrow & & \downarrow Q(i, p) \\ D^m \times I & \xrightarrow{\quad} & P(i, p) = X^{S^{n-1}} \times_{Y^{S^{n-1}}} Y^{D^n} \end{array}$$

In order to show that any such diagram has a lift, by [Proposition 2.1](#) and [Proposition 1.22](#), since  $S^{n-1}$  and  $D^n$  are locally compact Hausdorff, it suffices to show that any lifting problem of the following form has a solution:

$$\begin{array}{ccc} D^m \times \{0\} \times D^n \coprod_{D^m \times \{0\} \times S^{n-1}} D^m \times I \times S^{n-1} & \xrightarrow{\quad} & X \\ (i \times \text{id}_{D^n}, \text{id}_{D^m \times I} \times i) \downarrow & & \downarrow p \\ D^m \times I \times D^n & \xrightarrow{\quad} & Y \end{array}$$

For ease of notation, write  $f$  for the map on the left. Since  $p \in J_\perp$ , in order to show that the diagram has a lift, it suffices to show that  $f \in {}_\perp(J_\perp)$ . Note that since  ${}_\perp(J_\perp)$  is characterized by a lifting property, it is closed under composition with isomorphisms (homeomorphisms). Furthermore, since  $J \subseteq {}_\perp(J_\perp)$ , it suffices to show that  $f$  is homeomorphic to the second map in the factorization  $D^{m+n} \cong D^{m+n} \times \{0\} \hookrightarrow D^{m+n} \times I \in J$ . Note that  $f = \text{id}_{D^m} \times g$ , where

$$g : \{0\} \times D^n \coprod_{\{0\} \times S^{n-1}} I \times S^{n-1} = (\{0\} \times D^n) \cup (I \times S^{n-1}) \hookrightarrow I \times D^n.$$

The space  $(\{0\} \times D^n) \cup (I \times S^{n-1})$  may be obtained by removing one of the ends of the hollow cylinder  $\partial(I \times D^n)$ . By flattening the edges of this cylinder-with-a-missing-end, we get a disk homeomorphic to  $D^n$ . Thus  $g$  is homeomorphic to the map  $\{0\} \times D^n \hookrightarrow I \times D^n$ , so that  $f = \text{id}_{D^m} \times g$  is homeomorphic to the inclusion  $D^{m+n} \times \{0\} \hookrightarrow D^{m+n} \times I$  as desired.  $\square$

**Corollary 2.17** (Hovey 2.4.14). *Every topological space is fibrant, i.e., given a space  $X$ , the unique map  $X \rightarrow *$  is an element of  $J_\perp$ . In particular, the map  $X^{D^n} \rightarrow X^{S^{n-1}}$  belongs to  $J_\perp$  for all  $n \geq 0$ .*

*Proof.* Suppose we are given a space  $X$ , a map  $j : D^n \hookrightarrow D^n \times I$  belonging to  $J$ , and a lifting problem of the following form

$$\begin{array}{ccc} D^n & \xrightarrow{f} & X \\ j \downarrow & & \downarrow \\ D^n \times I & \longrightarrow & * \end{array}$$

Then it is straightforward to see that the composition  $f \circ \pi_1$ , where  $\pi_1 : D^n \times I \rightarrow D^n$  is the canonical projection, is a lift for the diagram. Hence,  $X$  is indeed fibrant as desired. Now, suppose we are given a map  $i : S^{n-1} \hookrightarrow D^n$  belonging to  $I'$ . It is straightforward to check that the following is a pullback diagram:

$$\begin{array}{ccc} X^{S^{n-1}} & \longrightarrow & *^{D^n} \\ \parallel & \lrcorner & \downarrow \\ X^{S^{n-1}} & \longrightarrow & *^{S^{n-1}} \end{array}$$

where the top arrow sends a map  $f : S^{n-1} \rightarrow X$  to unique arrow  $D^n \rightarrow *$  (it is continuous as  $*^{D^n}$  is a singleton). Hence, by [Lemma 2.16](#) the map  $X^{D^n} \rightarrow P(i, p) = X^{S^{n-1}}$  sending  $f : D^n \rightarrow X$  to the composition  $S^{n-1} \rightarrow D^n \xrightarrow{f} X$  is fibrant (belongs to  $J_\perp$ ), as desired.  $\square$

**Lemma 2.18** (Hovey 2.4.15). *If  $p : X \rightarrow Y$  is a weak equivalence (belongs to  $\mathcal{W}$ ), then  $p^{D^n} : X^{D^n} \rightarrow Y^{D^n}$  is also a weak equivalence.*

*Proof.* We claim it suffices to show that the map  $j_Z : Z \rightarrow Z^{D^n}$  that takes  $z$  to the constant map at  $z$  is a homotopy equivalence for all spaces  $Z$ . Indeed, supposing this had been shown, then consider the following diagram

$$\begin{array}{ccc} X & \xrightarrow{p} & Y \\ j_X \downarrow & & \downarrow j_Y \\ X^{D^n} & \xrightarrow{p_*} & Y^{D^n} \end{array}$$

First of all, it is straightforward to see that this diagram commutes. By 2-of-3 ([Lemma 2.12](#)) we know that  $p_* \circ j_X = j_Y \circ p$  is a weak equivalence, if  $j_X$  is, as  $p$  is a weak equivalence. Furthermore, if  $j_X$  is a weak equivalence, then by 2-of-3 again, it would have to hold that  $p_*$  is a weak equivalence, as desired.

Now, we claim that  $j_Z$  is homotopy equivalence. First of all, it is well-defined, as given  $z \in Z$ ,  $j_Z(z) : D^n \rightarrow Z$  is a constant map, which is always continuous, so that  $j_Z(z)$  is indeed an element of  $Z^{D^n}$ . Now, to see that  $j_Z$  is continuous, recall the topology on  $Z^{D^n}$  is that with subbasis given by sets of the form

$$S(K, U) := \{f \in \mathbf{Top}(D^n, Z) : f(K) \subseteq U\}$$

for  $K \subseteq D^n$  compact and  $U \subseteq Z$  open. Hence in order to show  $j_Z$  is continuous, it suffices to show  $j_Z^{-1}(S(K, U))$  is open in  $Z$  for all  $K \subseteq D^n$  compact and  $U \subseteq Z$  open. Indeed, we have

$$j_Z^{-1}(S(K, U)) = \{z \in Z : j_Z(z)(K) \subseteq U\} = \{z \in Z : z \in U\} = U,$$

which is open as desired. Hence,  $j_Z$  is well-defined and continuous. Now, it remains to show that it is a homotopy equivalence. Define the map  $q_Z : Z^{D^n} \rightarrow Z$  to be evaluation at 0 (where by 0 we mean the origin in  $D^n$ ), so that  $q_Z(f : D^n \rightarrow Z) := f(0)$ . To see this is continuous, note that given  $U \subseteq Z$  open, that

$$q_Z^{-1}(U) := \{f \in \mathbf{Top}(D^n, Z) : f(0) \in U\} = S(\{0\}, U),$$

which is open in  $Z^{D^n}$ , as desired. Now, the composite  $q_Z \circ j_Z$  is the identity, and the composite  $j_Z \circ q_Z$  is homotopic to the identity by the homotopy  $H : Z^{D^n} \times I \rightarrow Z^{D^n}$  defined by  $H(f, t)(x) = f(tx)$ . It remains to show  $H$  is continuous. To do so, it suffices to show that  $H^\sharp : Z^{D^n} \times I \times D^n \rightarrow Z$  mapping  $(f, t, x) \mapsto f(tx)$  is continuous, as  $D^n$  is locally compact Hausdorff and  $H$  is the adjoint of  $H^\sharp$  ([Proposition 2.1](#)). Note that  $H^\sharp$  factors as

$$Z^{D^n} \times I \times D^n \longrightarrow Z^{D^n} \times D^n \longrightarrow Z$$

$$(f, t, x) \longrightarrow (f, tx) \longrightarrow f(tx),$$

and this composition is continuous as the first arrow is the product of  $\text{id}_{Z^{D^n}}$  with the multiplication map  $I \times D^n \rightarrow D^n$  sending  $(t, x) \mapsto tx$ , and the second arrow is simply the evaluation map, which is also continuous (see the proof of [Proposition 2.1](#)).  $\square$

**Lemma 2.19** (Hovey 2.4.16). *Suppose  $p : X \rightarrow Y$  belongs to  $J_\perp$ , and  $x \in X$ . Let  $F := p^{-1}(p(x))$ , and  $i : F \hookrightarrow X$  denote the inclusion. Then there is a long exact sequences*

$$\begin{array}{ccccccc} & & \cdots & \longrightarrow & \pi_{n+1}(Y, p(x)) & & \\ & & & \nearrow d_n & & & \\ \pi_n(F, x) & \xleftarrow{\pi_n(i, x)} & \pi_n(X, x) & \xrightarrow{\pi_n(p, x)} & \pi_n(Y, p(x)) & & \\ & & & \nwarrow d_{n+1} & & & \\ \pi_{n-1}(F, x) & \xleftarrow{\pi_{n-1}(i, x)} & \cdots & \xrightarrow{\pi_0(p, x)} & \pi_0(Y, p(x)) & & \end{array}$$

which is natural with respect to commutative squares

$$\begin{array}{ccc} X & \longrightarrow & X' \\ p \downarrow & & \downarrow p' \\ Y & \longrightarrow & Y' \end{array}$$

where  $p, p' \in J_\perp$ . Here  $d_n$  is a group homomorphism  $\pi_n(Y, p(x)) \rightarrow \pi_{n-1}(F, x)$  when  $n > 1$ .

*Proof.* First, fix  $n \geq 0$ . We define  $d_{n+1} : \pi_{n+1}(Y, p(x)) \rightarrow \pi_n(F, x)$ . To start with, fix a homeomorphism  $k_n : D^{n+1} \cong D^n \times I$ , and write  $\partial k_n : \partial(D^{n+1}) = S^n \cong \partial(D^n \times I)$  for the restriction of  $k_n$  to  $\partial(D^{n+1}) = S^n$ . We may represent an element of  $\pi_{n+1}(Y, p(x))$  by a map  $f : (D^{n+1}, S^n) \rightarrow (Y, p(x))$ . Then we may construct the following lifting problem

$$\begin{array}{ccc} D^n & \xrightarrow{d \mapsto x} & X \\ d \mapsto (d, 0) \downarrow & & \downarrow p \\ D^n \times I & \xrightarrow{f \circ k_n^{-1}} & Y \end{array}$$

Since  $p \in J_\perp$ , this diagram has a lift  $D^n \times I \rightarrow X$ . Let  $\ell$  denote the restriction of this lift to  $\partial(D^n \times I)$ . Note that the image of  $\ell$  is contained in  $F$ , as given  $z \in \partial(D^n \times I)$ , we have  $p(\ell(z)) = f(k_n^{-1}(z))$ . Then define  $d_{n+1}f := \ell|_{\partial(D^n \times I)} \circ \partial k_n$ . First, note that  $d_{n+1}f : S^n \rightarrow F$

**FINISH**  $\square$

**Lemma 2.20** (Hovey 2.4.17). *Suppose  $p : X \rightarrow Y$  is a weak equivalence. Then  $p^{S^n} : X^{S^n} \rightarrow Y^{S^n}$  is a weak equivalence for all  $n \geq -1$ , where  $S^{-1} = \emptyset$ .*

*Proof.* We give a proof by induction on  $n$ . In the case  $n = -1$ , since  $S^{-1} = \emptyset$ ,  $X^{S^{-1}} = Y^{S^{-1}} = *$ , as  $S^{-1}$  is initial in **Top**, and  $*$  is terminal, so that the map  $p^{S^{-1}} : * \rightarrow *$  is a homeomorphism and in particular a weak-equivalence.

Now, supposing we have shown  $p^{S^{n-1}}$  is a weak equivalence, we wish to show  $p^{S^n}$  is likewise. Consider the following pushout diagram

$$\begin{array}{ccc} S^{n-1} & \longrightarrow & D^n \\ \downarrow & & \downarrow \\ * & \longrightarrow & S^n \end{array}$$

Since  $S^{n-1}$ ,  $D^n$ ,  $S^n$ , and  $*$  are all clearly LCH, by [Lemma 1.25](#) we get that the following diagram is a pullback square for all spaces  $Z$

$$\begin{array}{ccc} Z^{S^n} & \longrightarrow & Z^{D^n} \\ \downarrow & \lrcorner & \downarrow \\ Z & \longrightarrow & Z^{S^{n-1}} \end{array}$$

(It is straightforward to see by how the compact-open topology is defined that  $Z^* \cong Z$ ). We know by [Corollary 2.17](#) that the map on the right belongs to  $J_\perp$ , so that the arrow on the left is as well (it is

straightforward to see that since  $J_\perp$  is characterized by a right lifting property, it is closed under taking pullbacks). The map  $p$  induces a map between this pullback square with  $Z = X$  and this pullback square with  $Z = Y$ .  $\square$

**Proposition 2.21** (Hovey 2.4.18). *Suppose we have a pullback square*

$$\begin{array}{ccc} W & \xrightarrow{f} & X \\ q \downarrow & \lrcorner & \downarrow p \\ Z & \xrightarrow{g} & Y \end{array}$$

in **Top**, where  $p \in J_\perp$  and  $g$  is a weak equivalence. Then  $f$  is a weak equivalence.

*Proof.* **Lemma 1.25**  $\square$

**Proposition 2.22** (Hovey 2.4.12).  $\mathcal{W} \cap J_\perp \subseteq I'_\perp$

*Proof.* Let  $p : X \rightarrow Y$  belong to  $\mathcal{W} \cap J_\perp$ . By **Lemma 2.15**, it suffices to show the map  $Q(i, p)$  belongs to  $\mathcal{W} \cap J_\perp$  for all boundary inclusions  $i : S^n \hookrightarrow D^n$ . By **Lemma 2.16**,  $Q(i, p) \in J_\perp$ . Consider the pullback square

$$\begin{array}{ccc} P(i, p) & \longrightarrow & Y^{D^n} \\ \downarrow & \lrcorner & \downarrow \\ X^{S^{n-1}} & \longrightarrow & Y^{S^{n-1}} \end{array}$$

By **Corollary 2.17**, the right-hand vertical map belongs to  $J_\perp$ .  $\square$

#### Questions/Comments:

- (1) Hovey's proof of **Lemma 2.20** starts by asserting that the pushout square

$$\begin{array}{ccc} S^{n-1} & \longrightarrow & D^n \\ \downarrow & & \downarrow \\ * & \longrightarrow & S^n \end{array}$$

gives rise to a pullback square

$$\begin{array}{ccc} Z^{S^n} & \longrightarrow & Z^{D^n} \\ \downarrow & \lrcorner & \downarrow \\ Z^* & \longrightarrow & Z^{S^{n-1}} \end{array}$$

for **all** spaces  $Z$ . How do I show this? More generally, I would like to think there is a result which says something like: given a complete, cocomplete category  $\mathcal{C}$  and exponentiable objects  $A, B, C, D \in \mathcal{C}$  which fit into a pullback diagram like so:

$$(6) \quad \begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ C & \longrightarrow & D \end{array}$$

then for all objects  $Z \in \mathcal{C}$ , we get a pullback diagram like so:

$$(7) \quad \begin{array}{ccc} Z^D & \longrightarrow & Z^B \\ \downarrow & \lrcorner & \downarrow \\ Z^C & \longrightarrow & Z^A \end{array}$$

Now, clearly I can see that a cones over the diagram (7) with nadir  $X$  are in bijection with cones under the following diagram with nadir  $Z$ :

$$(8) \quad \begin{array}{ccc} X \times A & \longrightarrow & X \times B \\ \downarrow & & \downarrow \\ X \times C & \longrightarrow & X \times D \end{array}$$



So it follows that diagram (7) is a pullback diagram iff diagram (8) is pushout diagram for all  $X \in \mathcal{C}$ . Now, in the case  $\mathcal{C} = \mathbf{Top}$ , in general the functor  $X \times -$  does not preserve coproducts (although it does if  $X$  is exponentiable, in particular, LCH). So presumably I'll have to use that  $A$ ,  $B$ ,  $C$ , and  $D$  are themselves exponentiable in order to show that (8) is a pushout diagram. How do I do this?