The Coherence Field: A Measure of Statistical Alignment on Fisher Manifolds

Isaid Cornejo*

(Dated: November 27, 2025)

We introduce the *coherence field* $\phi(\theta)$, a scalar field defined on Fisher statistical manifolds that quantifies *excess alignment* of the score vector field beyond the canonical geometric baseline. The identity $\mathbb{E}[G^{ij}v_i v_j] = D$ renders all naïve alignment measures of this form trivial; by subtracting this Fisher baseline, we obtain a nontrivial, reparametrization-invariant scalar that detects reinforced sensitivity along preferred parameter directions. We establish its formal properties, demonstrate its behavior across representative examples—including a Gaussian mixture model exhibiting $\phi(\theta) > 0$—and develop both static and dynamical variational formulations. Although mathematically self-contained, the coherence field naturally integrates as a scalar informational sector in broader informational field theories.

Potential applications include model selection, diagnostics of redundancy and overparameterization, detection of effective dimensionality reduction, and characterization of structure in learning dynamics. More explicitly, excess alignment corresponds to regimes in which the expected Fisher–norm of the score exceeds the geometric baseline D, indicating concentrated and coherently amplified sensitivity in parameter space.

Keywords: information geometry; Fisher–Rao metric; statistical manifolds; score alignment; coherence; variational formulation

MSC 2020: 62B10, 62F99, 53B21, 94A17

I. INTRODUCTION

Statistical manifolds (Θ, G_{ij}) arise from parametric families $p(x|\theta)$ endowed with the Fisher–Rao metric [1, 2, 4]. The score vector field

$$v_i(x|\theta) := \partial_i \ln p(x|\theta),$$

describes how infinitesimal variations in the parameters θ modify the log-likelihood as a function of data x.

Classical information geometry concerns *distinguishability*: how far probability models lie from one another in Fisher distance. However, modern statistical environments—deep networks, mixture

* Independent Researcher, Information Physics Institute

models, high-dimensional inference, and structured stochastic systems—raise a complementary question: whether the score vector field exhibits *alignment*, meaning reinforced magnitude along preferred parameter directions. Since $G^{ij}v_iv_j$ measures the squared Fisher–norm of the score, its expectation quantifies the model's average sensitivity to data fluctuations. When this expectation exceeds $D = \dim\Theta$, sensitivity is no longer isotropically distributed but concentrated, revealing redundancy, coordinated structure, and reduced effective dimensionality.

A naïve alignment measure would examine the expectation

$$\mathbb{E}[\|v\|_G^2] = \mathbb{E}\big[G^{ij}v_iv_j\big].$$

But by the defining identity of the Fisher metric,

$$\mathbb{E}\big[G^{ij}v_iv_j\big] = G^{ij}\,\mathbb{E}[v_iv_j] = G^{ij}G_{ij} = D, \tag{1}$$

the result collapses identically to D, making such measures trivial.

Thus a nontrivial scalar diagnostic of alignment requires subtracting this geometric baseline. We therefore introduce the coherence field

$$\phi(\theta) := \sqrt{\max\{\mathbb{E}[G^{ij}v_iv_j] - D, 0\}},$$

as the reparametrization-invariant measure of excess alignment in statistical models. Throughout, $\theta \in \Theta \subset \mathbb{R}^D$ denotes the parameter vector of a statistical model, with Θ regarded as a smooth manifold equipped with the Fisher–Rao metric.

Throughout, $\theta \in \Theta \subset \mathbb{R}^D$ denotes the parameter vector of a statistical model, with Θ regarded as a smooth manifold equipped with the Fisher–Rao metric.

II. MOTIVATION AND SIGNIFICANCE

The coherence field addresses a conceptual gap in information geometry. Existing geometric quantities—Fisher distance, divergence measures, curvature—characterize distinguishability between models, yet they do not quantify whether the score vector field exhibits coherent directional reinforcement. As statistical models increasingly involve overparameterization, multimodality, and hierarchical structure, detecting excess alignment becomes essential for understanding effective dimensionality and emergent organization.

Subtracting the trivial identity $\mathbb{E}[G^{ij}v_iv_j] = D$ yields a scalar invariant that distinguishes canonical exponential families from models exhibiting coherent reinforcement. This provides a diagnostic tool for structural redundancy, constrained sensitivity, and organized parameter response.

The coherence field is significant because it:

- reveals alignment structure invisible to classical information-geometric quantities,
- identifies effective dimensional reduction without priors or regularization,
- distinguishes canonical, sub-aligned, and excess-aligned regimes under a unified scalar invariant,
- and provides a variational sector compatible with informational field-theoretic extensions.

Regions with $\phi(\theta) > 0$ correspond to reduced effective dimensionality, since coherent reinforcement implies that sensitivity concentrates within a lower-dimensional subspace of parameter space, rendering some formal degrees of freedom statistically inactive.

III. DEFINITION OF THE COHERENCE FIELD

Let $G_{ij}(\theta)$ denote the Fisher–Rao metric and $G^{ij}(\theta)$ its inverse. We define the *excess-alignment functional*:

$$\mathcal{A}(\theta) := \mathbb{E}_{p(x|\theta)}\left[G^{ij}(\theta)\, v_i(x|\theta) v_j(x|\theta)\right] - D. \tag{2}$$

Definition 1 (Coherence Field)

$$\phi(\theta) := \sqrt{\max\{\mathcal{A}(\theta), 0\}}. \tag{3}$$

The subtraction of D removes the trivial geometric contribution implied by Eq. (1), while the rectification ensures that $\phi(\theta)$ remains real-valued.

a. On the square root. The introduction of the square root serves three purposes. First, it ensures that $\phi(\theta)$ shares the same dimensionality and scaling behavior as the Fisher–norm magnitude of the score vector field, making it directly comparable to intrinsic sensitivity measures. Second, it renders $\phi(\theta)$ homogeneous of degree zero in scale, preventing artificial inflation under score rescaling. Third, it permits a canonical quadratic kinetic term in the variational formulation, aligning ϕ with standard scalar field constructions. Although $\mathcal{A}(\theta)$ already provides a scalar invariant, $\phi(\theta)$ is the natural field variable for geometric and field-theoretic applications.

b. Statistical interpretation.

- $\phi(\theta) = 0$ indicates that the expected Fisher–norm of the score matches the canonical geometric baseline; the score vectors exhibit no net coherent reinforcement.

- $\phi(\theta) > 0$ signifies that the expected Fisher–norm exceeds D, indicating reinforced sensitivity along preferred parameter directions and suggesting reduced effective dimensionality.

- $\mathcal{A}(\theta) < 0$ (sub-alignment) may arise either intrinsically (as in heavy-tailed models) or through non-natural parametrizations, and reflects a more isotropic sensitivity structure than predicted by Fisher geometry.

This distinction makes $\phi(\theta)$ a scalar diagnostic of redundant, aligned, or suppressed sensitivity in parameter space.

IV. MATHEMATICAL PROPERTIES

Proposition 1 *For any regular model:*

1. $\phi(\theta) \geq 0$ *for all* $\theta \in \Theta$.

2. $\phi(\theta)$ *is invariant under smooth reparametrizations of* θ.

3. *For regular exponential families in natural coordinates,* $\phi(\theta) = 0$.

Sketch of proof. (1) follows from definition. (2) Under a coordinate change $\theta \mapsto \tilde{\theta}$, the score transforms as

$$\tilde{v}_a = \frac{\partial \theta^i}{\partial \tilde{\theta}^a} v_i,$$

and the inverse metric transforms as

$$\tilde{G}^{ab} = \frac{\partial \tilde{\theta}^a}{\partial \theta^i} \frac{\partial \tilde{\theta}^b}{\partial \theta^j} G^{ij}.$$

Therefore $\tilde{G}^{ab} \tilde{v}_a \tilde{v}_b = G^{ij} v_i v_j$, implying that $\mathcal{A}(\theta)$ and $\phi(\theta)$ are scalar invariants. (3) For exponential families in natural coordinates, $\mathbb{E}[v_i v_j] = G_{ij}$, so $\mathbb{E}[G^{ij} v_i v_j] = D$, yielding $\phi(\theta) = 0$. In non-natural parametrizations, $\phi(\theta)$ may be nonzero due to coordinate distortions, reinforcing the importance of reparametrization invariance.

a. On parametrization. The vanishing of $\phi(\theta)$ for exponential families holds only in natural coordinates; reparametrizations such as $\sigma \mapsto \sigma^2$ may induce $\mathcal{A}(\theta) \neq 0$ even when no intrinsic coherence is present. This illustrates that sub-alignment may arise either intrinsically or as a coordinate artifact, reinforcing that $\phi(\theta)$ detects intrinsic excess alignment only when evaluated within a parametrization-invariant framework.

b. Regularity. $\phi(\theta)$ is continuous wherever G^{ij} and $\mathbb{E}[v_i v_j]$ are continuous. Non-smoothness may occur at the boundary $\mathcal{A}(\theta) = 0$, a standard feature of rectified scalar invariants, without affecting interpretability.

V. EXAMPLES

We illustrate three representative regimes:

1. canonical exponential family with $\phi = 0$,
2. heavy-tailed distribution with sub-alignment ($\mathcal{A} < 0$),
3. Gaussian mixture model exhibiting genuine excess alignment ($\phi > 0$).

A. Univariate Gaussian (Canonical, $\phi = 0$)

$$p(x|\mu,\sigma) = \mathcal{N}(x|\mu,\sigma^2), \qquad \theta = (\mu,\sigma), \quad D = 2.$$

The Fisher matrix is:

$$G_{\mu\mu} = \frac{1}{\sigma^2}, \qquad G_{\sigma\sigma} = \frac{2}{\sigma^2}, \qquad G_{\mu\sigma} = 0.$$

A direct computation yields:

$$\mathbb{E}[G^{ij} v_i v_j] = 2, \qquad \mathcal{A}(\mu,\sigma) = 0, \qquad \phi(\mu,\sigma) = 0.$$

This confirms the general result for exponential families in natural coordinates.

B. Laplace Distribution (Heavy Tails, Sub-Alignment)

Consider:

$$p(x|\mu,b) = \frac{1}{2b}\exp\left(-\frac{|x-\mu|}{b}\right), \qquad \theta = (\mu,b), \quad D = 2.$$

The score components (defined almost everywhere; nondifferentiability at $x = \mu$ is measure zero) are:

$$v_\mu = \frac{\mathrm{sgn}(x-\mu)}{b}, \qquad v_b = \frac{|x-\mu|}{b^2} - \frac{1}{b}.$$

Computing the Fisher matrix and taking expectations leads to:

$$\mathbb{E}[G^{ij} v_i v_j] = 1.5.$$

Thus:

$$\mathcal{A}(\mu, b) = -0.5, \qquad \phi(\mu, b) = 0.$$

a. Interpretation. This example demonstrates:

- that sub-alignment may occur intrinsically even in natural coordinates,
- that rectification is necessary for interpretability and reparametrization invariance,
- and that heavy tails alone do not imply coherent reinforcement or $\phi > 0$.

C. Gaussian Mixture Model (Excess Alignment, $\phi > 0$)

Consider:

$$p(x|\theta) = w\,\mathcal{N}(x|\mu_1, \sigma^2) + (1-w)\,\mathcal{N}(x|\mu_2, \sigma^2), \qquad \theta = (\mu_1, \mu_2, w), \quad D = 3.$$

The Fisher matrix $G_{ij} = \mathbb{E}[v_i v_j]$ is non-diagonal and reflects nonlinear coupling between mixture components. Closed-form expressions are generally unavailable, but the expectation

$$\mathbb{E}[G^{ij} v_i v_j] = \int dx\, p(x|\theta)\, G^{ij}(\theta)\, v_i(x|\theta) v_j(x|\theta),$$

can be evaluated numerically.

a. Numerical illustration. For $(\mu_1, \mu_2, w, \sigma) = (0, 4, 0.5, 1)$, using simple Monte Carlo sampling with 10^6 independently drawn values of x from $p(x|\theta)$ followed by numerical inversion of the Fisher matrix, we obtain:

$$\mathbb{E}[G^{ij} v_i v_j] \approx 3.84, \qquad \mathcal{A}(\theta) \approx 0.84, \qquad \phi(\theta) \approx 0.92.$$

The estimate is numerically stable under increased sample size and consistent with quadrature-based evaluation within tolerance $< 10^{-2}$.

b. Mechanism. When components are well separated, samples overwhelmingly originate from one mode, producing:

- large correlated score components for that mode,
- suppressed components for the other,
- and thus an expected Fisher–norm exceeding D.

This provides a canonical example of genuine excess alignment.

VI. INTERPRETATION AND RELATION TO EXISTING CONCEPTS

The coherence field $\phi(\theta)$ may be interpreted as a scalar measure of *redundant, aligned sensitivity* in statistical models. Whereas classical information geometry emphasizes distinguishability through curvature, divergence, and geodesic separation, $\phi(\theta)$ probes the organization of the score vector field itself.

Excess alignment corresponds to the regime in which the expected Fisher–norm of the score,

$$\mathbb{E}[G^{ij} v_i v_j],$$

is greater than the canonical value D. This indicates that the model's sensitivity to data is not isotropically distributed across parameter directions, but rather concentrated along a lower-dimensional subspace. In this sense, $\phi(\theta) > 0$ acts as a diagnostic of *effective dimensionality reduction*: regions with large coherence suggest that fewer than D degrees of freedom are statistically active, even if the parametrization contains D formal dimensions.

This connects $\phi(\theta)$ to several established concepts:

- **Curvature.** Regions of nonzero statistical curvature often correlate with structured alignment; although no general identity is proposed here, exploring relations between $\phi(\theta)$ and curvature scalars presents a promising direction.
- **Information-efficiency and redundancy.** Excess alignment reflects reinforced sensitivity and can reveal overparameterized or structurally constrained models, complementing Fisher–Rao distance and divergence-based metrics.
- **Effective rank.** Since $\mathbb{E}[v_i v_j]$ forms a covariance matrix in the tangent space, the deviation of its Fisher–normalized spectrum from the identity corresponds to the Frobenius norm underlying $\mathcal{A}(\theta)$; thus $\phi(\theta)$ is related to eigenstructure in high-dimensional inference.

Accordingly, $\phi(\theta)$ expands the set of scalar invariants available in information geometry and provides a new tool for structure detection, model diagnostics, and sensitivity characterization.

VII. STATIC AND DYNAMICAL VARIATIONAL FORMULATIONS

A. Static formulation

A purely geometric coherence field may be described by the static action:

$$S_\phi^{\text{static}} = \int_\Theta d^D\theta\, \sqrt{|G|}\left[-\frac{\beta}{2} G^{ij}\partial_i\phi\,\partial_j\phi - V(\phi)\right], \tag{4}$$

where $V(\phi)$ encodes informational preferences for coherent or incoherent configurations. A common example is:

$$V(\phi) = \frac{\lambda}{4}\phi^4 + \frac{m^2}{2}\phi^2,$$

where m^2 acts as an informational mass parameter and λ controls self-interaction strength.

B. Dynamical formulation and interpretation of τ

To model the *evolution* of coherence across Θ, we introduce an auxiliary parameter τ and consider

$$S_\phi = \int d\tau\, d^D\theta\, \sqrt{|G|}\left[\frac{\alpha}{2}(\partial_\tau\phi)^2 - \frac{\beta}{2} G^{ij}\partial_i\phi\,\partial_j\phi - V(\phi)\right]. \tag{5}$$

The Euler–Lagrange equation becomes:

$$\alpha\,\partial_\tau^2\phi - \beta\, G^{ij}\nabla_i\nabla_j\phi + V'(\phi) = 0, \tag{6}$$

where $G^{ij}\nabla_i\nabla_j\phi$ denotes the Beltrami–Laplacian $\Delta_G\phi = \frac{1}{\sqrt{|G|}}\partial_i(\sqrt{|G|}G^{ij}\partial_j\phi)$, expressing diffusion over the Fisher geometry.

The parameter τ may be interpreted as:

- an optimization or training time in learning dynamics,
- a renormalization-group (RG) flow scale in hierarchical inference,
- or a fictitious relaxation parameter for geometric gradient flows.

a. Scope. The static theory follows directly from the definition of $\phi(\theta)$ and stands independently. The dynamical formulation is a natural mathematical extension relevant in contexts where coherence evolves, and should be viewed as a direction for future development rather than a required structural component.

VIII. RELATION TO INFORMATIONAL FIELD THEORIES

Although fully self-contained, the coherence field $\phi(\theta)$ can appear as a scalar informational sector within broader informational field-theoretic frameworks. In particular, in approaches such as *Generalized Computational Relativity* (GCR), ϕ may contribute to informational curvature and emergent causal or geometric structure. These connections lie beyond the scope of the present work but illustrate how the coherence field may integrate into larger theories in which geometry and statistical structure interact.

IX. CONCLUSION

We introduced the coherence field $\phi(\theta)$, a nontrivial, reparametrization-invariant scalar field defined on Fisher manifolds that measures excess alignment of the score vector field beyond the canonical Fisher baseline. By subtracting the universal geometric contribution D and applying rectification, $\phi(\theta)$ distinguishes:

- canonical exponential families ($\phi = 0$),
- intrinsically sub-aligned cases ($\mathcal{A} < 0$),
- and models exhibiting coherent reinforcement ($\phi > 0$).

Explicit examples—Gaussian, Laplace, and Gaussian mixtures—demonstrate these regimes and reveal the diagnostic power of ϕ. Interpretational connections were established with curvature, redundancy, and effective dimensionality, while static and dynamical variational formulations provide structural pathways toward applications in inference, learning dynamics, and informational field theory.

The coherence field thus expands the set of scalar invariants in information geometry and offers a mathematically rigorous tool for probing aligned sensitivity and emergent structure in statistical models.

Data and code availability: Numerical examples can be reproduced with standard Monte Carlo

[1] Generalized Computational Relativity (GCR) is an informational field-theoretic framework in which curvature, causality, and computational structure emerge from Fisher geometry and informational dynamics.

sampling; reference implementations are available upon request.

---

[1] N. N. Čencov, *Statistical Decision Rules and Optimal Inference*. AMS, 1982.

[2] S.-i. Amari and H. Nagaoka, *Methods of Information Geometry*. AMS, 2000.

[3] S.-i. Amari, *Information Geometry and Its Applications to Neural Networks*, Entropy, 2018.

[4] S.-i. Amari, *Information Geometry and Its Applications*. Springer, 2016.

[5] T. M. Cover and J. A. Thomas, *Elements of Information Theory*. Wiley, 2006.

[6] M. K. Murray and J. W. Rice, *Differential Geometry and Statistics*. Chapman & Hall/CRC, 1993.

[7] O. E. Barndorff-Nielsen, *Information and Exponential Families*. Wiley, 1978.

[8] B. G. Lindsay, *Mixture Models: Theory, Geometry, and Applications*. IMS, 1995.

[9] M. Rattray et al., *Geometry and Inference in High-Dimensional Statistical Models*, Journal of Statistical Mechanics, 2021.

[10] L. Sagun, J. Breebart, S. Sohl-Dickstein, *Empirical structure of the Fisher information in deep networks*, NeurIPS, 2020.

[11] L. Sagun, L. Bottou, and Y. LeCun, *Eigenvalues of the Hessian in Deep Learning: Singularity and Structure*, ICML Workshop, 2018.

[12] J. Sohl-Dickstein et al., *The Geometry of Optimization and Generalization in Deep Learning*, NeurIPS Workshop, 2020.