A Fisher–Geometric Action for the Isotropic Empirical Alignment Field

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Empirical data generically deform the intrinsic Fisher–Rao geometry of statistical models, producing anisotropic score covariances and heterogeneous sensitivity profiles across parameter space. While such deformations can be quantified pointwise through invariant diagnostics constructed from the Fisher-normalized empirical covariance operator, static diagnostics alone provide neither a variational principle nor a geometrically intrinsic mechanism for regularization.

In this work we construct a minimal variational theory on statistical manifolds by promoting the isotropic component of empirical alignment to an auxiliary scalar field defined over parameter space and coupled to an observable, data-dependent source. The resulting Fisher–geometric action is local, reparametrization invariant, and introduces no structure beyond that induced by the Fisher–Rao metric itself. Its Euler–Lagrange equation is a Poisson equation on the statistical manifold, describing intrinsic geometric relaxation and smoothing of empirical deformation measured in Fisher–Rao distance.

The theory is intentionally restricted to the scalar (spin-0) sector of empirical geometric deformation. This choice isolates the leading-order isotropic component of mismatch between model and data geometry, providing a principled and interpretable geometric regularization framework while leaving anisotropic tensorial extensions as a natural direction for future work. The LaTeX source and all figures associated with this work are publicly available at https://github.com/isaidcornejo/coherence-field-action.

Keywords: Information geometry; Fisher–Rao metric; Empirical alignment; Geometric regularization; Statistical manifolds

I. INTRODUCTION

The Fisher–Rao metric endows parametric statistical models with a canonical Riemannian structure, encoding infinitesimal statistical distinguishability between nearby parameter values [1–4]. This intrinsic geometry underlies asymptotic inference, natural-gradient methods [5], and modern geometric formulations of learning and optimization [6].

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Throughout this work, the statistical manifold is denoted by Θ, with local coordinates $\theta \in \Theta$ and dimension $D = \dim(\Theta)$. While some treatments distinguish between a parameter space and an abstract manifold, we adopt the notation Θ to emphasize the dual role of model parameters as coordinates on an intrinsic information-geometric manifold endowed with a unique, reparametrization-invariant metric structure.

In idealized settings where the data-generating distribution coincides with the assumed model family, the Fisher–Rao metric fully characterizes local statistical sensitivity. In empirical practice, however, the observed data distribution $q(x)$ generically deviates from the assumed model $p(x \mid \theta)$. This mismatch induces anisotropic empirical score covariances, dominant sensitivity directions, and effective dimensional reduction in parameter space, phenomena that are well documented in modern statistical models and learning systems [5, 6]. From an information-geometric perspective, these effects correspond to a deformation of the intrinsic Fisher geometry induced by empirical data.

Specifically, the empirical score covariance

$$C_{ij}(\theta; q) = \mathbb{E}_q[\partial_i \log p(x \mid \theta)\, \partial_j \log p(x \mid \theta)] \tag{1}$$

does not, in general, coincide with the Fisher–Rao metric

$$G_{ij}(\theta) = \mathbb{E}_{p(\cdot|\theta)}[\partial_i \log p(x \mid \theta)\, \partial_j \log p(x \mid \theta)]\,. \tag{2}$$

Here $q(x)$ may denote either the true data-generating distribution or the empirical measure induced by finite samples. Since only products of first-order score functions appear, the empirical covariance remains well defined even when q is atomic [3, 4].

The discrepancy between C_{ij} and G_{ij} encodes how empirical data selectively reinforce or suppress sensitivity directions relative to the model's intrinsic information geometry. Quantifying this deformation invariantly, and constructing a principled mechanism to relax and geometrically smooth its empirical manifestations across parameter space, constitute the central motivation of the present work.

The present work should be read as a geometric completion of the scalar diagnostic introduced in Ref. [7]. While the diagnostic $A(\theta; q)$ provides a pointwise, reparametrization-invariant measure of empirical deformation, it does not define a global or variational structure. Here we promote this diagnostic to a source for an intrinsic geometric variational theory, yielding a globally consistent and Fisher–Rao–aware regularization mechanism. The LaTeX source of the manuscript and all figures are publicly available at `https://github.com/isaidcornejo/coherence-field-action`.

II. INVARIANT DIAGNOSTIC OF EMPIRICAL DEFORMATION

A coordinate-invariant measure of empirical geometric deformation is obtained by forming the mixed-index alignment operator

$$H^i{}_j(\theta; q) = (G^{-1})^{ik}(\theta)\, C_{kj}(\theta; q), \tag{3}$$

which compares empirical and model sensitivities in Fisher-normalized units. By construction, $H^i{}_j$ transforms as a $(1,1)$ tensor under smooth reparametrizations of Θ, and its spectrum provides a direction-wise comparison between empirical and intrinsic Fisher geometry [3, 4].

From this operator one constructs the scalar diagnostic

$$A(\theta; q) = \mathrm{Tr}(H(\theta; q)) - D, \tag{4}$$

where $D = \dim(\Theta)$. This quantity vanishes at Fisher equilibrium and aggregates reinforced and suppressed sensitivity directions into a single reparametrization-invariant scalar observable. Importantly, $A(\theta; q)$ depends only on the trace of the alignment operator and is therefore insensitive to the detailed directional structure of empirical anisotropy [7].

The diagnostic $A(\theta; q)$ isolates the isotropic component of empirical geometric deformation, analogous to a geometric pressure acting uniformly on the statistical manifold. In contrast, the traceless part of $H^i{}_j$ encodes anisotropic distortions that may be interpreted as shear- or stress-like deformations of Fisher geometry, in direct analogy with the scalar–tensor decomposition of geometric response fields [4, 8]. While these tensorial components carry additional directional information, they are not captured by a single scalar invariant and would require a higher-rank geometric description.

The present work therefore focuses exclusively on the scalar (spin-0, pressure-like) sector of empirical deformation. This choice yields the minimal and most robust invariant diagnostic compatible with reparametrization invariance and locality, and provides a natural entry point for a variational formulation without introducing additional geometric structure.

III. FROM DIAGNOSTIC TO ALIGNMENT FIELD

The central conceptual step of this work is to treat empirical geometric deformation not merely as a pointwise diagnostic, but as a geometric field defined intrinsically over the statistical manifold Θ.

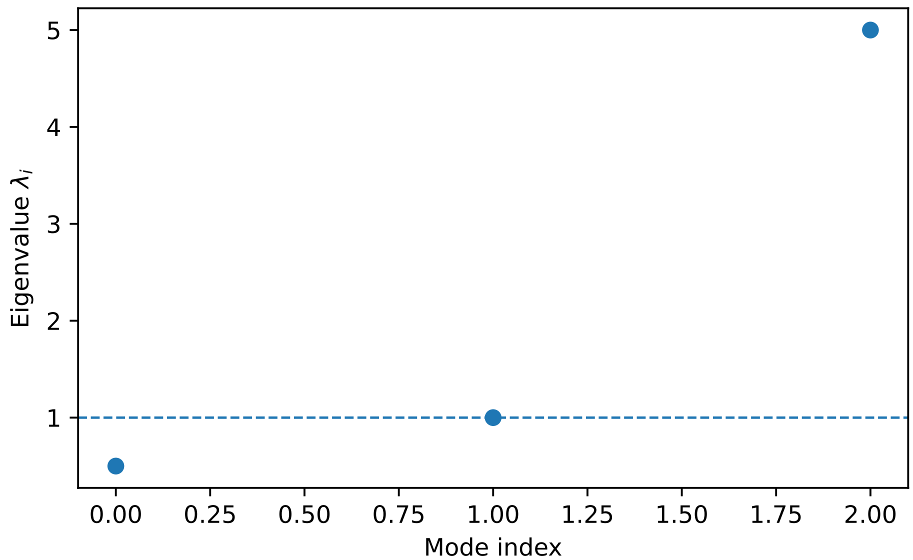

FIG. 1. Schematic spectrum of the alignment operator $H^i{}_j$. The trace captures the isotropic (pressure-like) component of empirical deformation, while traceless anisotropic modes correspond to shear-like distortions not addressed in the present scalar theory.

While one could in principle apply ad hoc smoothing procedures directly to the scalar diagnostic $A(\theta; q)$, such approaches lack a variational characterization and depend on extrinsic regularization choices. In contrast, introducing an auxiliary field allows the smoothing mechanism to be derived uniquely from the intrinsic Fisher–Rao geometry of the statistical manifold.

We therefore introduce a scalar *alignment field* $\phi(\theta)$, defined on Θ, which represents a smooth geometric response to empirical deformation. The empirical diagnostic $A(\theta; q)$ enters the theory as an externally observed, data-dependent source.

Throughout this work, the term “field” is used in the minimal geometric sense of an auxiliary scalar function defined on a statistical manifold. No implication of physical spacetime dynamics, propagating degrees of freedom, or temporal evolution is intended.

The field ϕ is not an observable quantity. Rather, it is an auxiliary geometric field whose role is to encode a regularized, manifold-aware response to empirical mismatch. No *a priori* identification between ϕ and A is imposed; their relationship emerges from a variational principle intrinsic to Fisher geometry and fixed uniquely by reparametrization invariance and locality.

This construction is intentionally restricted to the scalar (spin-0) sector, which captures the leading isotropic component of empirical deformation compatible with reparametrization invariance and minimal geometric structure. Anisotropic, higher-rank tensorial extensions are left as a natural direction for future work.

IV. INTERPRETATION OF GEOMETRIC DYNAMICS

The term "dynamics" is used here in a purely geometric sense. The coordinates θ label statistically distinguishable models on the statistical manifold Θ, not points in physical spacetime. Consequently, derivatives with respect to θ describe variation across model space rather than temporal evolution.

Within this interpretation, the quadratic form

$$G^{ij}\partial_i\phi\,\partial_j\phi \tag{5}$$

penalizes sharp spatial variations of the alignment field between statistically nearby models, as measured by Fisher–Rao distance. The Fisher geometry therefore defines the intrinsic notion of locality and smoothness appropriate for empirical deformation on parameter space.

a. Relation to regularization theory. Formally, the variational structure introduced here is related to manifold regularization and Laplacian smoothing [9, 10]. The crucial distinction is that the smoothing operator is not introduced algorithmically or heuristically. Instead, it emerges uniquely as the Laplace–Beltrami operator associated with the Fisher–Rao metric, derived from a variational principle constrained solely by reparametrization invariance, locality, and the absence of additional geometric structure [4, 8].

V. VARIATIONAL PRINCIPLE AND FISHER–GEOMETRIC ACTION

A. Minimal variational requirements

We seek a variational formulation for the alignment field $\phi(\theta)$ satisfying the following conditions:

- invariance under smooth reparametrizations of Θ,
- locality on the statistical manifold,
- minimal derivative order,
- absence of any geometric structure beyond the Fisher–Rao metric.

The Fisher–Rao metric uniquely determines the invariant volume element $\sqrt{\det G}\,\mathrm{d}^D\theta$ and the Laplace–Beltrami operator

$$\Delta_G = \frac{1}{\sqrt{\det G}}\,\partial_i\Big(\sqrt{\det G}\,G^{ij}\partial_j\Big)\,. \tag{6}$$

No additional geometric tensors, background structures, or intrinsic scales are introduced.

We emphasize that no explicit curvature couplings (such as $\phi\, R(G)$) are included. While such terms are compatible with reparametrization invariance, they introduce additional geometric structure beyond that strictly required to encode empirical alignment. The present construction is intentionally restricted to the minimal scalar sector generated solely by the Fisher–Rao metric and its associated Laplace–Beltrami operator.

B. Empirically coupled action

The minimal action coupling the alignment field to empirical data is

$$\boxed{S[\phi; q] = \int_\Theta \mathrm{d}^D\theta\, \sqrt{\det G}\, \left[\frac{1}{2}\, G^{ij} \partial_i \phi\, \partial_j \phi + \gamma\, A(\theta; q)\, \phi\right]}\,. \tag{7}$$

This action is uniquely fixed by the requirements of locality on Θ, reparametrization invariance, minimal derivative order, and the absence of additional geometric structure beyond the Fisher–Rao metric. In particular, the linear coupling $A(\theta; q)\, \phi$ is the only admissible lowest-order interaction between the empirical source and the scalar field. Higher-order or nonlinear couplings would introduce non-minimal empirical scales without improving interpretability or geometric consistency.

The coupling constant γ fixes only the overall normalization of the auxiliary field ϕ and introduces no intrinsic scale. Since ϕ is not an observable quantity, γ carries no independent geometric or statistical content and may be absorbed into a rescaling of ϕ. Its explicit presence is retained solely for notational clarity when comparing different empirical sources.

This action defines a purely geometric relaxation theory on the statistical manifold. The alignment field ϕ is an auxiliary variable whose equilibrium configuration encodes a smooth, Fisher–geometrically consistent response to empirical deformation.

VI. EULER–LAGRANGE EQUATION AND CONSISTENCY CONDITIONS

A. Field equation

Variation of the action (7) with respect to ϕ yields the Euler–Lagrange equation

$$\boxed{-\Delta_G \phi(\theta) = -\gamma\, A(\theta; q).} \tag{8}$$

This equation is a Poisson equation on the Fisher–Rao manifold (Θ, G), with the empirical diagnostic acting as a geometric source. Its structure is completely fixed by the Fisher geometry and the variational principle.

B. Global consistency and zero-mode structure

On compact statistical manifolds, the Laplace–Beltrami operator admits a constant zero mode. As a direct consequence, Eq. (8) admits solutions if and only if the empirical source satisfies the global consistency condition

$$\int_\Theta \mathrm{d}^D\theta \, \sqrt{\det G}\, A(\theta; q) = 0. \tag{9}$$

This condition is not an additional assumption but a structural consequence of the Laplace–Beltrami operator on compact manifolds. Geometrically, it reflects the fact that only the spatially varying component of empirical deformation can be relaxed by Fisher–geometric smoothing. A uniform isotropic offset corresponds to a global shift that cannot be resolved locally and must therefore be fixed by convention.

On non-compact statistical manifolds, the constraint (9) is replaced by appropriate decay conditions at large Fisher–Rao distance, which ensure both existence and uniqueness of solutions as well as finiteness of the energy functional. In practice, statistical manifolds of interest—such as exponential-family models with open parameter domains—admit natural decay conditions inherited from the Fisher volume measure.

In the univariate Gaussian example discussed in Sec. IX, the Poincaré half-plane geometry guarantees sufficient decay of admissible solutions at infinity, thereby ensuring a well-posed Poisson problem without the need for additional infrared regularization.

C. Gauge fixing and normalization

The existence of a zero mode implies a gauge freedom under constant shifts $\phi \mapsto \phi + \mathrm{const}$. This freedom may be fixed by imposing a normalization condition, such as

$$\int_\Theta \mathrm{d}^D\theta \, \sqrt{\det G}\, \phi(\theta) = 0. \tag{10}$$

This choice fixes the reference level of the alignment field without affecting any observable geometric gradients or relative structure.

VII. GREEN-FUNCTION REPRESENTATION

Formally, solutions of Eq. (8) may be expressed in terms of the Green function $\mathcal{G}(\theta, \theta')$ of the Laplace–Beltrami operator [8],

$$-\Delta_G \, \mathcal{G}(\theta, \theta') = \frac{1}{\sqrt{\det G}} \delta(\theta - \theta'), \tag{11}$$

subject to the chosen boundary or normalization conditions.

Here $\mathcal{G}(\theta, \theta')$ denotes the Green function of the Laplace–Beltrami operator and should not be confused with the Fisher–Rao metric [8].

The alignment field then admits the representation

$$\phi(\theta) = \gamma \int_{\Theta} \mathrm{d}^D \theta' \, \sqrt{\det G(\theta')} \, \mathcal{G}(\theta, \theta') \, A(\theta'; q). \tag{12}$$

This expression makes explicit that the alignment field is a Fisher–geometrically weighted average of the empirical source. The smoothing kernel is entirely determined by the intrinsic geometry of the statistical manifold [4, 8].

VIII. RECOVERY OF THE DIAGNOSTIC AND INTERPRETATION

The Poisson equation (8) ensures that ϕ coincides with the empirical diagnostic only in a weak, distributional sense. In regions where $A(\theta; q)$ varies slowly on the Fisher–Rao scale, the alignment field tracks the local structure of the empirical deformation while suppressing high-frequency geometric noise.

In this sense, the variational theory recovers the original diagnostic as a local limit while extending it to a smooth, globally consistent geometric field.

IX. ILLUSTRATIVE EXAMPLE: UNIVARIATE GAUSSIAN FAMILY

We now make the construction explicit for the univariate normal family

$$p(x \mid \mu, \sigma) = \mathcal{N}(\mu, \sigma^2),$$

whose Fisher–Rao geometry is isometric to the Poincaré half-plane and therefore possesses constant negative scalar curvature [4, 11, 12].

A. Fisher–Rao metric and volume element

In coordinates $\theta = (\mu, \sigma)$, the Fisher–Rao metric takes the form

$$G_{ij}(\mu, \sigma) = \begin{pmatrix} \sigma^{-2} & 0 \\ 0 & 2\,\sigma^{-2} \end{pmatrix}, \tag{13}$$

with inverse

$$G^{ij}(\mu, \sigma) = \begin{pmatrix} \sigma^{2} & 0 \\ 0 & \frac{1}{2}\sigma^{2} \end{pmatrix}. \tag{14}$$

The corresponding invariant volume element is

$$\sqrt{\det G}\, \mathrm{d}\mu\, \mathrm{d}\sigma = \sqrt{2}\, \sigma^{-2}\, \mathrm{d}\mu\, \mathrm{d}\sigma. \tag{15}$$

B. Laplace–Beltrami operator

The Laplace–Beltrami operator associated with this metric reads

$$\Delta_G = \sigma^2 \partial_\mu^2 + \frac{\sigma^2}{2} \partial_\sigma^2 - \sigma\, \partial_\sigma, \tag{16}$$

as obtained from the canonical Laplace–Beltrami construction on a Riemannian manifold [8].

C. Poisson equation for empirical alignment

In this setting, the alignment field satisfies the Poisson equation

$$\left(-\sigma^2 \partial_\mu^2 - \frac{\sigma^2}{2} \partial_\sigma^2 + \sigma\, \partial_\sigma \right) \phi(\mu, \sigma) = -\gamma\, A(\mu, \sigma; q). \tag{17}$$

For concreteness, consider a contaminated data distribution of the form

$$q(x) = (1 - \epsilon)\, \mathcal{N}(\mu_0, \sigma_0^2) + \epsilon\, r(x), \tag{18}$$

where $r(x)$ represents outliers or structured noise. The resulting empirical score covariance induces a smooth but nontrivial source term $A(\mu, \sigma; q)$ defined over the Poincaré half-plane.

The solution $\phi(\mu, \sigma)$ corresponds to the Fisher–geometrically relaxed response to this empirical deformation. Unlike screened constructions, the propagation of alignment is governed entirely by the intrinsic geometry of the statistical manifold and by the imposed boundary or normalization conditions [4, 8]. Regions that are nearby in Fisher–Rao distance contribute most strongly to the geometric averaging of the empirical source.

This example illustrates how the abstract variational framework reduces, in a concrete and analytically tractable case, to a well-posed Poisson problem on a canonical statistical manifold.

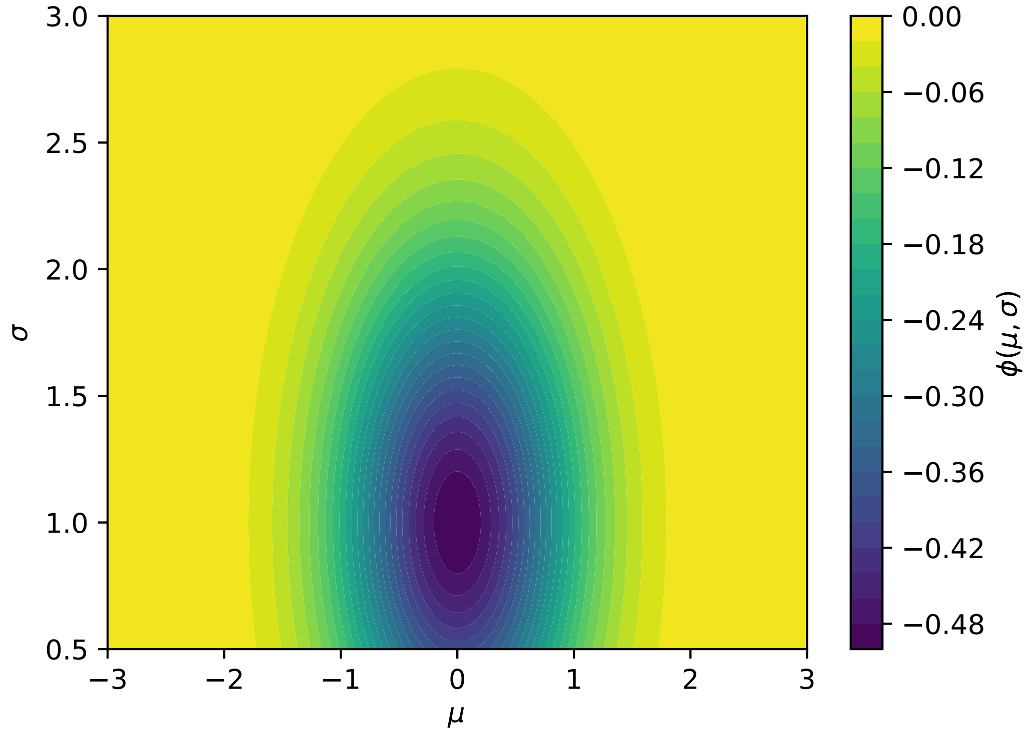

FIG. 2. Alignment field $\phi(\mu, \sigma)$ on the univariate Gaussian Fisher manifold. The field illustrates the Fisher–geometrically relaxed response to empirical deformation induced by a contaminated data distribution.

X. CONCLUSION

We have constructed a minimal, reparametrization-invariant variational theory describing isotropic empirical deformation of Fisher geometry on statistical manifolds. By elevating a static alignment diagnostic to an auxiliary geometric field, the framework embeds empirical mismatch within a principled mechanism of geometric relaxation and smoothing.

The resulting theory is entirely fixed by Fisher–Rao geometry and introduces no additional scales or structures. The alignment field satisfies a Poisson equation on the statistical manifold, with empirical deformation acting as a geometric source. Global consistency conditions and zero-mode structure replace the need for ad hoc infrared regulators and admit a natural interpretation in terms of gauge fixing and geometric normalization [8].

The theory is intentionally restricted to the scalar (spin-0) sector, which may be interpreted as the isotropic geometric pressure induced by empirical data. Anisotropic, stress-like deformations correspond to higher-rank tensorial sectors that lie beyond the scope of the present work but represent a natural direction for future research.

Within its domain of validity, the present construction provides a clean bridge between information geometry, variational regularization, and classical geometric field theory, offering a foundation that can be naturally connected to diffusion processes, potential theory, and emergent physical descriptions on statistical manifolds [4, 10]. The LaTeX source of this manuscript and all figures are available at `https://github.com/isaidcornejo/coherence-field-action`.

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See `https://www.informationphysicsinstitute.org`.

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