Probability and Interference

April 13, 2023

1 Probability

1.1 Exercise 1: Decay of a radioactive particle

We consider the decay of a radioactive particle, without knowing the underlying microscopic mechanisms. From experiments, we know that if the particle exists at time t, then the probability that the particle has decayed at time $t + dt$ is γdt where dt is *infinitesimal*. *This process is called a Markovian Process because the probability to decay at every time-step is constant does not depend on the state or history of the system. This type of systems is much easier to study than Markovian systems.*

Q1 What must be the physical dimension of γ?

R1 As the probability has no dimensions, γ must be the inverse of a time.

Q2 What is the probability $P_s(t)$ that the particle has not yet decayed at time t? There are (at least) two ways to compute this quantity: working with discrete time-intervals or by immediately deriving the differential equation for $P_s(t)$

Q2.1 What is the probability $P_s(t)$ that the particle has not yet decayed after N time-intervals?

R2.1 The probability of not-decaying in a time-interval is $1 - \gamma dt$. After $N = t/dt$ timesteps, the product probability is $P_s(t) = (1 - \gamma dt)^N = (1 - \gamma dt)^{\frac{t}{dt}}$.

Q2.2 Assume that $dt \to 0$; what is the probability $P_s(t)$ in that case?

R2.1 By taking the exponential of the logarithm, $P_s(t) = \exp\left[\frac{t}{dt} \log(1 - \gamma dt)\right]$ and using the fact that $\log(1 - \gamma dt) \approx -\gamma dt + O(dt^2)$, we get $P_s(t) = \exp\left[-\frac{t}{dt}\gamma dt\right] = e^{-\gamma t}$

Q2.3 What is the physical dimension of P_s?

R2.3 It's adimensional (it's a probability)

Q3 What is the probability density $p(t)$ that the particle, which has not decayed yet at time $t = 0$, decays exactly at time t? What is its physical dimension?

R3 First, we need to find the probability density for a particle to decay at time t, which is given by γ.

With that, we use the composition of probabilities: the probability that the particle did not decay until time t is $P_s(t)$, and the probability density to decay is γ therefore $p(t) = \gamma P_s(t) = \gamma e^{-\gamma t}$

Q4 Show that $p(t)$ is a valid probability density.

R4 $p(t) \geq 0$ assuming that $\gamma > 0$. For the normalisation, you must compute the integral over all its domain, that is

$$\int_0^\infty p(t)dt = \int_0^\infty \gamma e^{-\gamma t} dt \tag{1}$$
$$= -\int_0^\infty \frac{d}{dt}\left(e^{-\gamma t}\right) dt \tag{2}$$
$$= \int_\infty^0 \frac{d}{dt}\left(e^{-\gamma t}\right) dt \tag{3}$$
$$= \left[(e^{-\gamma t}\right|_\infty^0 = 1 \tag{4}$$

Q5 What is the average life-time $\langle t \rangle$ of our poor particle? What is the variance of the average life-time $\mathbb{V}ar(t)$?

R5 Average life-time: we use the definition of an expectation value (first line) and then continue the calculation

$$\langle t \rangle = \int_0^\infty t p(t) dt \tag{5}$$
$$= \int_0^\infty t\gamma e^{-\gamma t} dt \tag{6}$$
$$=^{(\tau = t\gamma)} \frac{1}{\gamma}\int_0^\infty \tau e^{-\tau} d\tau \tag{7}$$
$$= \frac{1}{\gamma}\left\{\left[\tau e^{-\tau}\right|_\infty^0 - \int_0^\infty (-e^{-\tau}) d\tau\right\} \tag{8}$$
$$= \frac{1}{\gamma}\{0+1\} \tag{9}$$
$$= \frac{1}{\gamma} \tag{10}$$

The variance, instead, is given by $\mathbb{V}ar(t) = \left\langle t^2 \right\rangle - \langle t \rangle^2$, so the first term is:

$$\left\langle t^2 \right\rangle = \int_0^\infty t^2 p(t) dt \tag{11}$$
$$= \int_0^\infty t^2 \gamma e^{-\gamma t} dt \tag{12}$$
$$=^{(\tau = t\gamma)} \frac{1}{\gamma^2}\int_0^\infty \tau^2 e^{-\tau} d\tau \tag{13}$$
$$= \frac{1}{\gamma^2}\left\{\left[(-\tau e^{-\tau}\right|_\infty^0 - \int_0^\infty (-2\tau e^{-\tau}) d\tau\right\} \tag{14}$$
$$= \frac{1}{\gamma^2}\{0-2\} \tag{15}$$
$$= \frac{2}{\gamma^2} \tag{16}$$

So $\mathbb{V}ar(t) = \left\langle t^2 \right\rangle - \langle t \rangle^2 = \frac{2}{\gamma^2} - \frac{1}{\gamma^2} = \frac{1}{\gamma^2}$.

Q6 *Imagine that a new particle has recently been discovered as a by-product of a nuclear reaction, and you must design an experiment to measure its average life-time. The experiment consists in observing the particle and measuring how long it takes for it to decay.* Compute how many times (on average) $N_s(\epsilon_r)$ you should repeat the experiment to know the average life-time of the particle with a relative error ϵ_r?

- Hint: Assuming that the average time is $\langle t \rangle \pm \sigma_t$, where σ_t is the standard error, the average relative error is $\sigma_t / \langle t \rangle$.
- What is the dependency of N_s on ϵ_r? A power-law or an exponential?
- For power-law dependencies, the exponenti is commonly known as *sample complexity*, and it's the equivalent of computational complexity for sampling problems.

R6 The absolute error or standard error will be $\sigma_r = \sqrt{\mathbb{V}ar(t)/N_s} = \frac{1}{\gamma\sqrt{N_s}}$. The absolute error is $\epsilon_r = \sigma_r / \langle t \rangle = \frac{1}{\sqrt{N_s}}$.

So the number of experiments required to know the average life-time with absolute accuracy σ_r will be $N_s(\sigma_r) = (\gamma\sigma_r)^{-2}$ while the number of experiments required to known the average life-time given a target relative accuracy is $N_s(\epsilon_r) = (\epsilon_r)^{-2}$. The sample complexity is therefore 2 and we have a power-law relationship.

1.2 Exercise 2: Gaussians distribution

We consider the gaussian distribution, defined by the following probability density:

$$p(x) = Z \exp\left(-\frac{(x-x_0)^2}{2\sigma^2}\right), \tag{17}$$

where $(x, x_0, \sigma) \in \mathbb{R}$ are real numbers and σ should be assumed positive. Z is an unknown normalisation factor.

Q1 Find the normalisation factor Z for which $p(x)$ is a correctly-normalised probability density.

- Hint: this formula might be handy...

$$\int_{-\infty}^{\infty} e^{-az^2} = \sqrt{\pi/a} \tag{18}$$

R1 We compute the normalisation factor...

$$Z^{-1} = \int_{-\infty}^{\infty} \exp\left[-\frac{(x-x_0)^2}{2\sigma^2}\right] dx \tag{19}$$

We can make a substitution to simplify the integration. Let $u = \frac{x-x_0}{\sqrt{2}\sigma}$, so $x = \sqrt{2}\sigma u + x_0$. Then, $\frac{dx}{du} = \sqrt{2}\sigma$:

$$Z^{-1} = \sqrt{2}\sigma \int_{-\infty}^{\infty} \exp\left[-u^2\right] du. \tag{20}$$

We know that the integral of the standard Gaussian function is given by:

$$\int_{-\infty}^{\infty} \exp\left[-u^2\right] du = \sqrt{\pi}. \tag{21}$$

Substitute this result back into the equation for Z:

$$Z^{-1} = \sqrt{2\pi}\sigma \tag{22}$$

Q2 Assuming $Z^{-1} = \sigma\sqrt{2\pi}$, compute the average value $\langle x \rangle$

R2 Using the definition of an expectation value...

$$\langle x \rangle = Z^{-1} \int_{-\infty}^{\infty} x \exp\left[-\frac{(x-x_0)^2}{2\sigma^2}\right] dx \tag{23}$$
$$\tag{24}$$

To solve this integral, we can make a substitution. Let $u = \frac{x-x_0}{\sqrt{2}\sigma}$, so $x = \sqrt{2}\sigma u + x_0$. Then, $\frac{dx}{du} = \sqrt{2}\sigma$:

$$\langle x \rangle = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \exp\left(-u^2\right)(\sqrt{2}\sigma u + x_0) du. \tag{25}$$

Now, we can separate the integral into two parts:

$$\langle x \rangle = \frac{\sqrt{2}\sigma}{\sqrt{\pi}} \int_{-\infty}^{\infty} u \exp\left(-u^2\right) du + \frac{x_0}{\sqrt{\pi}} \int_{-\infty}^{\infty} \exp\left(-u^2\right) du. \tag{26}$$

The first integral is an odd function, so its value will be 0. The second integral is just the standard Gaussian integral, which we already know has a value of $\int_{-\infty}^{\infty} \exp\left(-u^2\right) du = \sqrt{\pi}$. Plugging these results back into the equation for $\langle x \rangle$, we get:

$$\langle x \rangle = x_0. \tag{27}$$

Therefore, the average value $\langle x \rangle$ is equal to x_0.

Q3 Compute the variance $\mathbb{V}ar(x)$.

R3 As before, we need to compute $\left\langle x^2 \right\rangle$:

$$\left\langle x^2 \right\rangle = Z^{-1} \int_{-\infty}^{\infty} x^2 \exp\left[-\frac{(x-x_0)^2}{2\sigma^2}\right] dx \tag{28}$$
$$\tag{29}$$

and using the same change of variables, we obtain

$$\left\langle x^2 \right\rangle = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-u^2} (\sqrt{2}\sigma u + x_0)^2 du \tag{30}$$

$$= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} (2\sigma^2 u^2 + \sqrt{2}\sigma u x_0 + x_0^2) e^{-u^2} du \tag{31}$$

$$= x_0^2 + \frac{2\sigma^2}{\sqrt{\pi}} \int_{-\infty}^{\infty} u^2 e^{-u^2} du \tag{32}$$

$$= x_0^2 + \frac{2\sigma^2}{\sqrt{\pi}} \int_{-\infty}^{\infty} u^2 e^{-u^2} du \tag{33}$$

$$\tag{34}$$

And this last integral can be solved by parts and equals to $\int_{-\infty}^{\infty} u^2 e^{-u^2} du = \frac{\sqrt{\pi}}{2}$, therefore we obtain:

$$\left\langle x^2 \right\rangle = x_0^2 + \sigma^2 \tag{35}$$

Therefore the variance will be

$$\mathbb{V}ar(x) = \left\langle x^2 \right\rangle - \left\langle x \right\rangle^2 = x_0^2 + \sigma^2 - x_0^2 = \sigma^2 \tag{36}$$

Q4 Assuming that x represents the position of a particle on a line, what is the physical interpretation of x_0 and σ?

R4 x_0 is the average position of the particle, while σ is the variance of the resulting distribution. This means that if I measure N times the particle, I will find the average position N with uncertainty $\sigma/\sqrt{N}$.

Q5 If I define the joint probability density for the position of two particles as $\tilde{p}(x, y) = p(x)p(y)$ are the positions of the two variables independent? Why? How can I modify $\tilde{p}(x, y)$ to correlate the two variables?

R5 The two random variables are independent by definition, as the configuration of x does not affect y. I can add any function mixing x and y to correlate the two variables, for example $\tilde{p}(x, y) = p(x)p(y)p(xy)$.

1.3 Exercise 3: Interference and Young's slit experiment

We consider the Double slit experiment of Young that was described in the main lecture. We consider a plane located at $z = 0$ with two slits separated by a distance a along the axis $\hat{x}$. A plane wave with impulse $p = \hbar/\lambda$ and energy $E = \hbar\omega$ and travelling along the direction $\hat{z}$ illuminates the plane. The slits diffract the fields, and for $z > 0$ the field is determined by the linear combination of the two wavefronts. We place a screen along the axis of propagation at position $D \gg a$ parallel to the plane with the slits.

We neglect the diffraction due to the particular shape of the slits, and assume that the field diffracted by every slit $i = \{+, -\}$has a spherical wave-front given by

$$\psi(r) = \psi_0 e^{ip_0 r/\hbar} \tag{37}$$

where r is the distance between the point considered and the source slit.

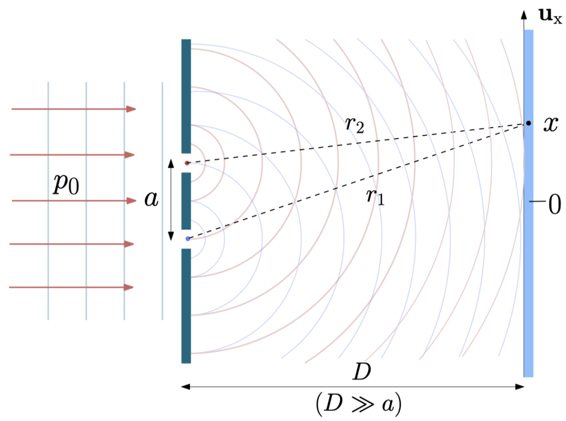

Figure 1: Schematic diagram of Young's double-slit experiment.

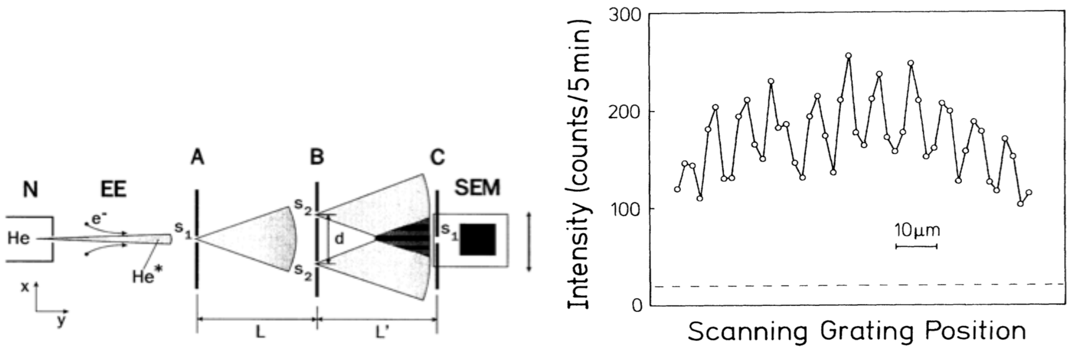

Figure 2: Schematic diagram of Young's double-slit experiment and measurement data (From PRL 66, 2689).

Q1 Working in a 2D plane, the position of the slits is therefore $\vec{a}_\pm = (0, \pm\frac{a}{2})$. Compute the field $\psi_\pm(x)$ diffracted by each slit arriving on the screen (D, x) and simplify the formula by using the fact that $D \gg a, x$

R1 Given the distance $r_\pm$ between the screen and each slit,

$$\begin{aligned} r_\pm &= \sqrt{D^2 + (x \mp \frac{a}{2})^2} && (38)\\ &= D\sqrt{1 + \frac{1}{D^2}(x \mp \frac{a}{2})^2} && (39)\\ &= D\left(1 + \frac{1}{D^2}(x \mp \frac{a}{2})^2 + O(D^{-4})\right) && (40)\\ &= D\left(1 + \frac{1}{2D^2}(x \mp \frac{a}{2})^2 + O(D^{-4})\right) && (41)\\ &= D + \frac{x^2}{2D} + \frac{a^2}{8D} \mp \frac{xa}{2D} + O(D^{-3}) && (42) \end{aligned}$$

The field $\psi_\pm(x)$ diffracted by each slit arriving on the screen (x, D) is:

$$\begin{aligned} \psi_\pm(x) &= \psi_0 e^{i\frac{p_0}{\hbar}r_\pm} && (43)\\ &\approx \psi_0 e^{ip_0 D/\hbar} exp\left[i\frac{p_0}{\hbar}\left(\frac{x^2 \mp ax + \frac{a^2}{4}}{2D}\right)\right] && (44) \end{aligned}$$

Q2 Compute the total incident field on the screen $\psi(x)$ using the same assumption as above.

First we compute the difference in the fields:

$$\Delta r = r_+ - r_- = \frac{xa}{D} \qquad (45)$$

$$\begin{aligned} \psi(x) &= \psi_+(x) + \psi_-(x) && (46)\\ &= \psi_0 e^{i\frac{p_0}{\hbar}r_+} + \psi_0 e^{i\frac{p_0}{\hbar}r_-} && (47)\\ &= \psi_0 e^{i\frac{p_0}{\hbar}r_-}\left(1 + \exp\left[i\frac{p_0}{\hbar}\Delta r\right]\right) && (48) \end{aligned}$$

Q3 Compute the probability density $P(x)$ to find a particle on the point x of the screen. What is the distance δ between the interference fringes?

R3 The total density is:

$$\begin{aligned} P(x) &= |\psi(x)|^2 && (49)\\ &= |\psi_0|^2\left(1 + \exp\left[i\frac{p_0}{\hbar}\Delta r\right]\right)\left(1 + \exp\left[-i\frac{p_0}{\hbar}\Delta r\right]\right) && (50)\\ &= |\psi_0|^2\left(2 + \exp\left[i\frac{p_0}{\hbar}\Delta r\right] + \exp\left[-i\frac{p_0}{\hbar}\Delta r\right]\right) && (51)\\ &= 2|\psi_0|^2\left(1 + \cos[\frac{p_0}{\hbar}\Delta r]\right) && (52)\\ &= 2|\psi_0|^2\left(1 + \cos[\frac{p_0}{\hbar}\frac{xa}{D}]\right) && (53)\\ &= 2|\psi_0|^2\left(1 + \cos[2\pi x\frac{p_0}{h}\frac{a}{D}]\right) && (54) \end{aligned}$$

Which means that the amplitude is modulated by $\cos[2\pi x \frac{p_0}{h} \frac{a}{D}] = \cos[2\pi \frac{x}{\delta}]$ which has a period of

$$\delta = \frac{\lambda D}{a} \tag{55}$$

where I used the De Broglie wavelength relationship $p_0 = \frac{h}{\lambda}$.

Q4 What is the momentum along the x direction transmitted by a particle with wavelength λ to the screen under a classical assumption of the particle taking a single path? Assuming that the screen is mounted on a detector, what precision δp_x is needed to discriminate if the particle comes from one or the other slit?

R4 The projection on the $\hat{x}$ axis of the momentum $\vec{p}$ of the photons hitting the screen at the point x is $p_x(x) = p_0 \sin(\theta)$ (see Fig. 3). Assuming $\theta_\pm \approx \frac{x \pm a/2}{D} + O(\frac{a}{D})$ we obtain:

$$p_x^\pm(x) = p_0 \sin\left(\frac{x \pm a/2}{D}\right) \approx p_0 \frac{x \pm a/2}{D} \tag{56}$$

which is valid to second order in $\frac{a}{D}$.

The momentum difference between p_x^+ and p_x^- is then:

$$\Delta p_x = p_x^+ - p_x^- = p_0 \frac{a}{D} \tag{57}$$

and therefore our measurement apparatus will need a precision of $\delta p_x < p_0 \frac{a}{D}$ to determine from which slit the photons came from.

Q5 Is it possible to measure at the same time the interference fringes and the slit that the particles came from?

R5 To determine the slit we need a momentum resolution of

$$\delta p_x \ll p_0 \frac{a}{D} \tag{58}$$

from Heisenberg's uncertainty principle we know that

$$\delta p_x \delta x \geq \frac{\hbar}{2} \tag{59}$$

so assuming $\delta p_x \ll p_0 \frac{a}{D}$ we have that

$$\delta x \gg \frac{\hbar}{2p_0} \frac{D}{a} \tag{60}$$

and using the fact that $p_0 = \frac{h}{\lambda}$ we obtain

$$\delta x \gg \frac{\lambda}{\pi} \frac{D}{a} = \frac{\delta}{4\pi} \tag{61}$$

But if the fringes are separated by $\delta = \frac{a}{\lambda D}$, as we computed before, this means that $\delta x > \delta$ and therefore according to the principle of Heisenberg's indetermination we will not be able to resolve the fringes.

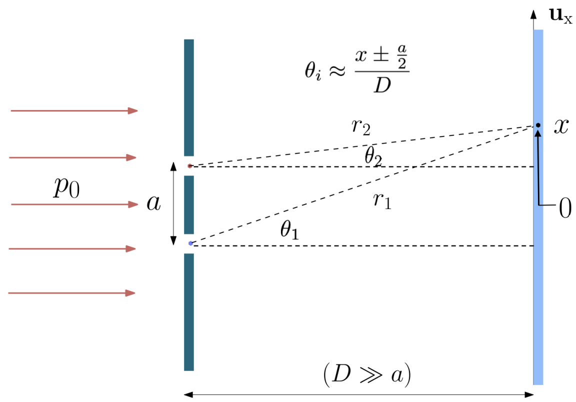

Figure 3: Incident angle of the photons on the screen.

Q6 In 1991, O. Carnal and J. Mlynek performed a Young experiment by accelerating helium atoms through two slits spaced by $a = 8\mu m$ (Phys. Rev. Lett. 66, 2689 (1991)). The helium atoms are then detected on a wall at a distance $D = 64cm$ from the slits. It is assumed that all helium atoms have the same velocity $v = 940m/s$. Knowing that the mass of a helium atom is $m = 6.64 \cdot 10^{-27}kg$, what is their de Broglie wavelength? Is this value compatible with the results of fig. 2?

R6 We have

$$a = 8\mu m \tag{62}$$
$$D = 64cm \tag{63}$$
$$\delta = \frac{\lambda D}{a} = 10\mu m \text{from the image} \tag{64}$$

So the De-Broglie wavelength is:

$$\lambda = \frac{\delta a}{D} = 0.12nm \tag{65}$$

Instead, knowing that the atoms have the properties:

$$v = 940m/s \tag{66}$$
$$m = 6.641 \cdot 10^{-27}kg \tag{67}$$

we find

$$p = mv = \frac{h}{\lambda'} \tag{68}$$

therefore

$$\lambda' = \frac{h}{mv} = 0.106nm \tag{69}$$

1.4 Excercise 4: Photo-electric effect

In 1905 Einstein predicted the corpuscular nature of light (existance of quantas of lights, the photons) and stated that a purely wave-description could not explain the photo-electric effect, which was later *experimentally observed by Millikan in 1906. The photo-electric effect predicts that when light with a sufficiently high energy hits a material, electrons are emitted with an energy proportional to the energy of the photons in the incoming light beam. The difference between the energy absorbed from a single photon and the emitted electron is equal to the binding energy of the electron.* We consider here a similar experience: a Potassium cathode is illuminated with light of different wave-lengths:

- We consider incoming light with $\lambda_1 = 253.7nm$, and we observe emitted electrons with an energy of $K_1 = 3.14eV$.
- We consider incoming light with $\lambda_1 = 589nm$, and we observe emitted electrons with an energy of $K_2 = 0.46eV$.

Now answer the following questions, considering that the speed of light is $c = 3.00 \cdot 10^8 m/s$ and that $1eV = 1.6 \cdot 10^{-19} m/s$:

Q1 Compute the Planck's constant that gives the relation between wave-length and photon energy.

R1 The energy of a single photon is

$$E_{ph} = h\nu = \frac{hc}{\lambda} \tag{70}$$

and this must match the energy of the emitted electron which is

$$E_e = E_0 + K \tag{71}$$

where E_0 is the binding energy and K is the kinetic energy. Then, we can build the system of equations

$$\begin{cases} \frac{hc}{\lambda_1} = E_0 + K_1 \\ \frac{hc}{\lambda_2} = E_0 + K_2 \end{cases} \tag{72}$$

obtaining

$$hc\left(\frac{1}{\lambda_1} - \frac{1}{\lambda_2}\right) = K_1 - K_2 \tag{73}$$

and therefore

$$h = \frac{1}{c}(K_1 - K_2)\left(\frac{1}{\lambda_1} - \frac{1}{\lambda_2}\right)^{-1} = 6.63 \cdot 10^{-34} Js \tag{74}$$

Q2 Compute the binding energy of the electrons in Potassium, E_0.

R2 Now that we know h, we can extract the binding energy which will be

$$E_0 = \frac{hc}{\lambda_1} - K_1 = 1.75eV \tag{75}$$

Q3 Compute the maximum wave-length that is necessary to observe photo-electric effect for this material.

R3 $\lambda_{max} = \frac{hc}{E_0} = 710nm$