Exploring the Solar System: Planetary Orbits and ODE Solvers

Aaby I, Steinnes L, Rashid S.*

Department of physics, University of Oslo, P.O. Box 1048 Blindern, N-0316 Oslo, Norway

Abstract

Exploration of celestial orbits for planets in our Solar system is in agreement with Newton's laws. Numerical results shows Earth escaping the Sun for velocity at 8.8 AU/year witch agrees with analytical results. Applying non-square inverse law with $\beta = 3$ sees Earth spiralling towards the Sun. Implementation of Forward-Euler, Euler-Chromer and Velocity-Verlet gives conservation of angular momentum, while only Velocity-Verlet conserves the total energy as expected. Calculated perihelion precession of Mercury gives 25000″ after 100 years. Calculated CPU time for Forward-Euler, Euler-Chromer and Velocity-Verlet for all planets with $k = 10^6$, yields 12191 (ms), 16439 (ms) and 11696 (ms). For different values of number of iterations k, Velocity-Verlet is most stable when comparing with circular orbits.

I Introduction

Our place in the universe has always been a question of wonder and controversy. Up until the Renaissance, a majority believed in a geocentric universe, with Ptolemy's model being the most influential [1]. However at the end of his life, in 1543, the mathematician and astronomer Nicolaus Copernicus published the groundbreaking title "On the Revolutions of the Celestial Spheres" [2].

For the first time in the history of man, a heliocentric system was not only postulated, but also described with a rigorous geometric foundation. Drawing inspiration from Cicero, Plutarch and celestial observations by Arabic astronomers, ideas was set in a wider theoretical framework [3] . Yet inaccurate in it's prediction of planetary positions, it explained phenomenons such as the parallax effect, the distance between planets and the Sun, and the seasons on Earth [2]. Hence, it marked a crossroad in scientific history, and began a revolution which spanned a century [4].

*isakaab@student.matnat.uio.no, lasse.steinnes@fys.uio.no, sidrar@student.matnat.uio.no [October 26, 2020]

The Copernican revolution was first dismissed by the most renowned astronomer at the time, Tycho Brahe [5]. Nevertheless, with Johannes Kepler's introduction of elliptical orbits, improved telescope technology and Sir Isaac Newton's gravitational law, the heliocentric model persisted .

Today, the mechanical forces which governs the celestial orbits of the Solar System is well known to Physics, and they can be described as ordinary differential equations (ODEs). However, there lies a challenge in solving the Newtonian equations analytically for more than two bodies [6]. This paper will therefore offer a solution to the many-body problem, focusing on numerical methods to solve ODEs.

The aim of this paper is to explore celestial orbits of the Solar System by applying both analytical and numerical tools. It is laid out by first giving an overview of the mechanical laws of the universe, and how these equations can be solved numerically. The methods applied are Forward-Euler, Euler-Chromer and Velocity-Verlet. Thereafter, results are presented with graphs and benchmarks.

Different circular and elliptical orbits are investigated, together with effects on energy and angular momentum. Consequences and deviations of the inverse square law are then described. Lastly, a model with the 10 largest celestial bodies on our solar system is presented, and perihelion precession as a relativistic effect is explored for the Mercury-Sun system.

A presentation of the results is followed by a discussion and critical evaluation of the numerical methods and their application to these problems. The paper ends with some brief concluding remarks.

All programmes of relevance to solving the Solar System's celestial orbits are available at `https://github.com/isakaaby/FYS4150-Project3`.

II Theory and Methods

Note that celestial bodies are here simplified to point particles, thus reducing the gravitational force to act on a mass with no extent in space. This is reasonable - assuming approximately spherical symmetrical planetary bodies, gravitational pull is concentrated at its centre.

The theory on ODE solvers (sec. II.II) is mainly based upon Morten Hjorth-Jensen's lecture notes in the Computational Physics course (FYS4150) at the University of Oslo [7].

II.I Mechanical Laws of the Universe

II.I.1 Kepler's Laws

Kepler's laws of orbital motion [8] are the following

(i) The orbit of a planet is an ellipse with the Sun in one of its foci.

(ii) A line between the Sun and its planet sweeps out equal areas in equal time intervals.

(iii) An orbital around the Sun and the semimajor axis of the ellipse share the relation $T^2 = a^3$, where T is the period and a is the extent of the semimajor axis.

Thus, mathematically an orbit can be described by the formula

$$\frac{x^2}{b_1} + \frac{y^2}{b_2} = 1 \tag{1}$$

where a transition can be done between cartesian and polar coordinates, but it will not be elaborated upon here.

II.I.2 Newton's Laws

According to modern Physics, the motion in the universe is governed by Einstein's theory of relativity, with Newtonian mechanics as its limit for non-relativistic velocities. Isaac Newton's discovery of a universal gravitational force enabled him to accurately deduce Kepler's laws from fundamental principles [9]. As a consequence, one can easily predict past and future positions of celestial bodies.

Newton's 2.nd law presents itself as a second order ODE,

$$\vec{F} = m\vec{a} = m\sum_{i=0}^{n} \frac{d^2 x_i}{dt^2}\hat{i} \tag{2}$$

here given for a n dimensional system (x :position, a: acceleration, t: time). In a system of p objects working on each other with gravitational force, $\vec{F} = \vec{F}_g$, then a particular object of mass m and position r is subject to

$$\vec{F}_g = -G \sum_{\substack{j=1 \\ i \neq j}}^{p} \frac{mM_j}{|\vec{r} - \vec{r}_j|^2}. \tag{3}$$

where $\vec{r} \in \mathbb{R}^n$, M_j is the mass of object j and G is the gravitational constant. Eq. 3 is the inverse square law applying to gravitation. However, one could replace the power 2 with an arbitrary factor β to describe any central force, which will be explored further in this article. Applying N2L for a chosen direction on eq. 3, and dividing by m yields

$$a_i = -G \sum_{\substack{j=1 \\ i \neq j}}^{p} \frac{x_i - x_j}{|\vec{r} - \vec{r}_j|^3} M_j, \tag{4}$$

since $\widehat{i} = x_i/|\vec{r} - \vec{r}_j|$. From eq. 4, solving the equations of motion can be performed analytically, or numerically as described in sec. II.II. In the simple case of circular orbit, the centripetal force can be described as

$$F_{c,i} = -ma_{c,i} = -m\frac{v^2}{r}, \tag{5}$$

in one chosen dimension i, in which v is the tangential velocity of the orbit and r is the distance between two objects. Thus, for a system with the Sun as origin and Earth as its only orbiting planet, equating eq. 4 and eq. 5, yields

$$\frac{v_c^2}{r} = F/m_{\text{earth}} = \frac{GM_\odot}{r^2}. \tag{6}$$

Using the Solar mass as a reference $M_\odot = 1$ [solar masses], and the distance between the Earth and Sun as $r = 1$ [AU], rewriting eq. 6 results in

$$v_c = \sqrt{\frac{GM_\odot}{r}} = 2\pi. \qquad \text{[AU/year]} \tag{7}$$

As a consequence, a planet or an object orbiting close enough to a large mass (in this case $M_\odot$) with initial velocity as described in eq. 7 will have circular motion. That is, under the assumption that other forces on the orbiting object are negligible.

II.I.3 Conservative Forces; Energy and Escape Velocity

If the work of moving an object between two points is independent of path taken, it is subject to a conservational force. Mathematically

$$W \equiv \oint_c \vec{F} \cdot d\vec{r} = 0. \tag{8}$$

Since energy is directly connected to work by the work-energy theorem, equivalently

$$(K+V)_{initial} = (K+V)_{final}. \tag{9}$$

A central, spherically symmetric force is conservative by eq. 8, including gravitational forces. Assume a planet P has escaped its orbit around a massive object O, with mass M, and reached velocity $v = 0$ [AU/year] at infinite distance r [AU] away from O. As a consequence

$$(K+V)_{final} = \frac{1}{2}mv^2 - \frac{GmM}{r} = 0, \tag{10}$$

and from rewriting eq. 9 the escape velocity becomes

$$v_e = \sqrt{\frac{2GM}{r}} \overset{M_\odot, r}{=} 2\sqrt{2}\pi. \qquad \text{[AU/year]} \tag{11}$$

II.I.4 Conservation of Angular Momentum

Angular momentum is defined as $\vec{L} = \vec{r} \times \vec{p}$ where $\vec{r}, \vec{p} \in \mathbb{R}^n$ and $\vec{p} = m\vec{v}$. In celestial mechanics, $\vec{h} = \vec{r} \times \vec{v}$ is often used instead of $\vec{L}$. It describes the momentum associated with rotation around a given axis for an object. $\vec{L}$ is closely related to the torque, defined by $\vec{\tau} = \vec{r} \times \vec{F}$. This is the force initializing rotational movement. Analogous to N2L for linear forces,

$$\vec{\tau} = \frac{d\vec{L}}{dt} = \frac{d}{dt}(\vec{r} \times m\vec{v}). \tag{12}$$

The total angular momentum of a closed system is constant, thus this is a conserved quantity. In addition, for a central potential force like gravity, then $\vec{r}$ and $\vec{F}$ points in the same direction for a two-body system. As a consequence, $\vec{\tau} = 0$ and by eq. 12, angular momentum remains constant, i.e. $\vec{L}$ is in this instance conserved.

For celestial orbits, one can show that $\vec{L}$ is conserved as a result of Kepler's 2. law (K2L) (sec. II.I.1). From polar coordinates, the area swept by an arc (dA) is described by

$$dA = \frac{1}{2}r^2 d\theta + \frac{1}{2} r d\theta dr. \tag{13}$$

The first contribution comes from the area spanned by a circular arc, while the second contribution describes the addition area from a changing r. Dividing both sides by dt, then when $dr \to 0$, $dt \to 0$ and $dr/dt = 0$,

$$\frac{dA}{dt} = \frac{1}{2} r^2 \frac{d\theta}{dt} \overset{K2L}{=} C. \tag{14}$$

Since $|\vec{h}| = |r \times \dot{r}| = r^2 \dot{\theta}$, eq. 14 $\propto h$. Thus angular momentum is conserved by K2L.

II.I.5 Relativistic effects

In the classical Newtonian two-body problem of Mercury and the Sun, with Mercury orbiting around the Sun, the path in which the planet is taking is an ellipse with the Sun in one of its focal points. The closest distance to the Sun during the elliptical path is called the perihelion.

In classical physics this particular distance would be fixed. However the theory of general relativity tells us that the elliptical orientation of Mercury is slowly rotating when applying a relativstic correction to the newtonian gravitational force

$$F_G = \frac{GM_{\text{sun}} M_{\text{mercury}}}{r^2} \left[1 + \frac{3l^2}{r^2 c^2} \right] \tag{15}$$

where M_{sun} is the mass of the sun, M_{mercury} is the mass of Mercury, r is the distance to Mercury relative to the sun, $l = |\vec{r} - \vec{v}|$ is Mercury's angular momentum per unit mass relative to the sun and c is the speed of light in vacuum. This correction will result in a shift of the perihelion of the planet, so that the perihelion angle θ_p from

$$\tan \theta_p = \frac{y_p}{x_p} \tag{16}$$

will change for each orbit around the Sun. The rotation of the elliptical orientation happens due to the curvature of spacetime between Mercury and the Sun.

II.II ODE Solvers

A n-body problem ($n > 2$) has no known closed form solution, since the gravitational forces of three or more bodies lead to a non-repeating dynamic system. Thus it has no analytical solution on closed form. Numerical methods are necessary to approximate such systems. Here, Forward-Euler, Euler-Chromer and Velocity-Verlet are considered.

For each method we investigate the stability with different values of number of iterations k. For a given k and time T we calculate the numerical steplength h given by

$$h = \frac{T}{k-1} \tag{17}$$

which are used in the various numerical methods.

The Velocity Verlet method (algo. 1 is presented below, whereas the two former methods are given in sec. VI.

Algorithm 1 Velocity-Verlet: The basic outline of Velocity-Verlet algorithm for solving an ODE problem with n particles and k timesteps.

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for int j=0; j < k-1; j++ **do** ▷ For each timestep
  for int i=0; j < n; i++ **do** ▷ For each particle
    $x_{i,j+1} = x_{i,j} + hv_{x,i,j} + (1/2)h^2 a_{x,i,j}$
    $y_{i,j+1} = y_{i,j} + hv_{y,i,j} + (1/2)h^2 a_{y,i,j}$ ▷ Find positions at timestep $j+1$
    $z_{i,j+1} = z_{i,j} + hv_{z,i,j} + (1/2)h^2 a_{z,i,j}$
  end for
  for int i=0; j < n; i++ **do** ▷ For each particle
    $a_{x,i,j+1} = F_x(x_{i,j+1}, y_{i,j+1}, z_{i,j+1})$ ▷ Find acceleration at timestep j+1
    $a_{y,i,j+1} = F_y(x_{i,j+1}, y_{i,j+1}, z_{i,j+1})$ ▷ Using newtonian gravity force per mass F
    $a_{z,i,j+1} = F_z(x_{i,j+1}, y_{i,j+1}, z_{i,j+1})$

    $v_{x,i,j+1} = v_{x,i,j} + (1/2)h(a_{x,i,j+1} + a_{x,i,j})$
    $v_{y,i,j+1} = v_{y,i,j} + (1/2)h(a_{y,i,j+1} + a_{y,i,j})$ ▷ Find velocity at timestep $j+1$
    $v_{z,i,j+1} = v_{z,i,j} + (1/2)h(a_{z,i,j+1} + a_{z,i,j})$
  end for
end for

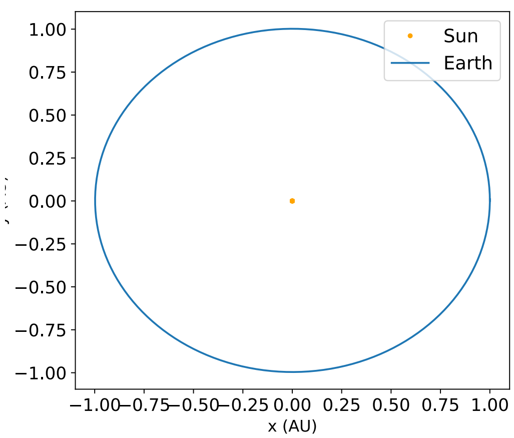
II.III Benchmarks

The methods are bench-marked with time usage and a convergence test, as well as methods which checks if energy and angular momentum are conserved properties. In addition, a test for circular orbit is also applied on the Earth-Sun system. A more detailed description of benchmarks is given in the Appendix (sec. VI) and in the README at github [10].

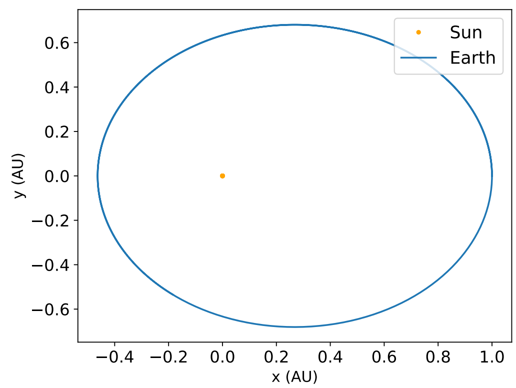
III Results

The results from applying Forward-Euler, Euler-Chromer and Velocity-Verlet to solve the ODE equations arising from Newtonian mechanics (eq. 3), are presented below. If not stated otherwise, Velocity-Verlet is used. All relevant benchmark results are presented in the appendix (sec. VI). Elliptical and circular motions are explored for the Earth-Sun system in fig.1. Then an investigation is made on the inverse square law (fig. 3).

A three-body system composed of the Sun, Earth and Jupiter is presented in fig. 4. An overview of a numerical solution of celestial orbits for all planets, including Pluto, is given in fig 5. Thereafter, relativistic effects on perihelion precession is investigated (fig. 6). A table of benchmark outcomes for energy and angular momentum is given at the end of this section.



(a) $v_y = 2\pi$ [AU/year]



(b) $v_y = 5$ [AU/year]

Figure 1: **Celestial Orbits for Sun-Earth system**: **a)** Initializing the earth at $x = 1$ [AU] and the given velocity results in a circular orbit. **b)** Another velocity, results in an elliptical orbit with the Sun in one of its foci. Parameters: $T = 1$ year , $\beta = 2$, number of iteration points $k = 10^4$.

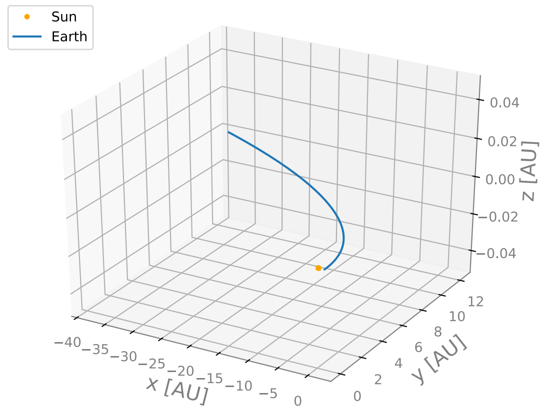


Figure 2: **Escape velocity for Sun-Earth system**: The Earth reaches its escape velocity at 8.8 AU/year. Parameters: $T = 1$ years, $\beta = 2$, number of iteration points $k = 10^4$.

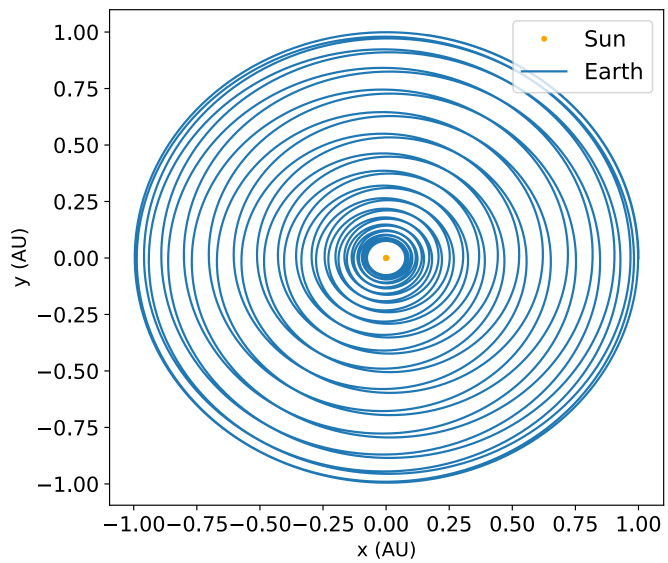
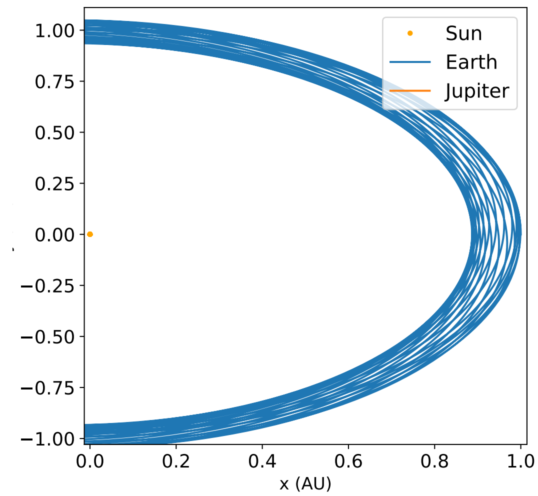
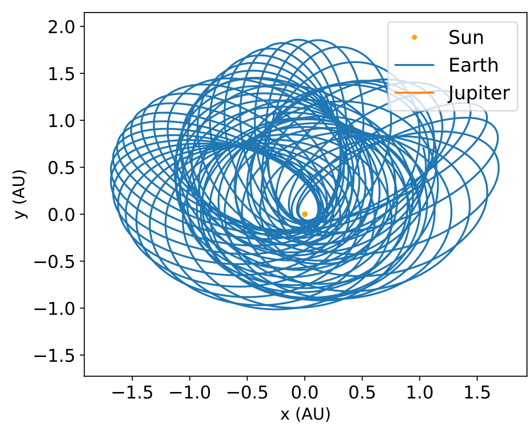


Figure 3: **Celestial Orbits for Sun-Earth system with a non-square law**: When β approaches 3.0, initializing the Earth at $x = 1$ [AU] with circular orbit ($v_y = 2\pi$), leads to the Earth ($m = 3.0 \cdot 10^{-6}$ [$M_\odot$]) spiralling toward the more massive sun. The same effects can be observed for $\beta = (2, 3]$, however with a less dramatic curvature. Parameters: $T = 10$ years , $\beta = 3.0$, number of iteration points $k = 10^4$.

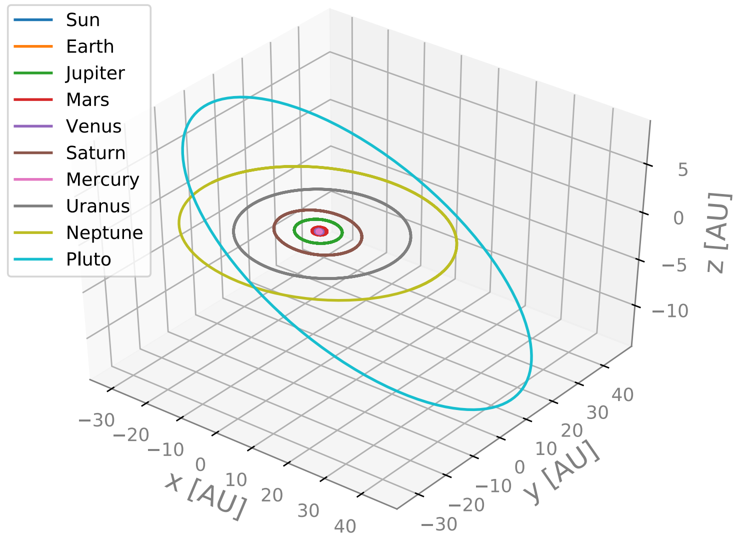


(a) $m = m_j \cdot 100$ [$M_\odot$]

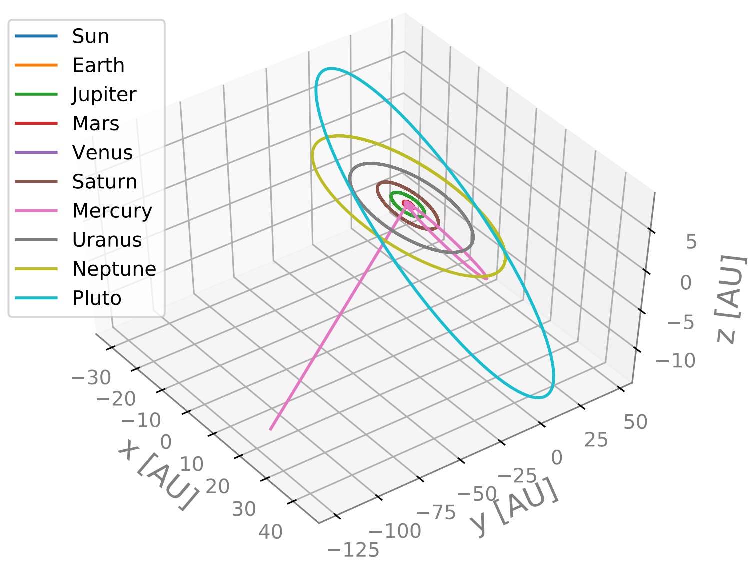


(b) $m = m_j \cdot 1000$ [$M_\odot$]

Figure 4: **Celestial Orbits for Earth-Jupyter system**: Initializing the Earth with orbital $x = 1$ [AU] and $v_y = 2\pi$ [AU/years], Jupiter at $x = 6.2$ [AU] with $v_y = 2.76$ [AU/year] and the sun held fixed at center gives a circular orbit for the Earth. However, with a larger mass, Jupiter's motions takes Earth away from its otherwise circular path. **a)** The gravitational pull from Jupiter changes the curvature of Earths orbit from smooth to irregular. **b)** With an even larger mass, Earth's orbit follows that of Jupiter. Parameters: $T = 50$ years , $\beta = 2$, number of iteration points $k = 10^5$.

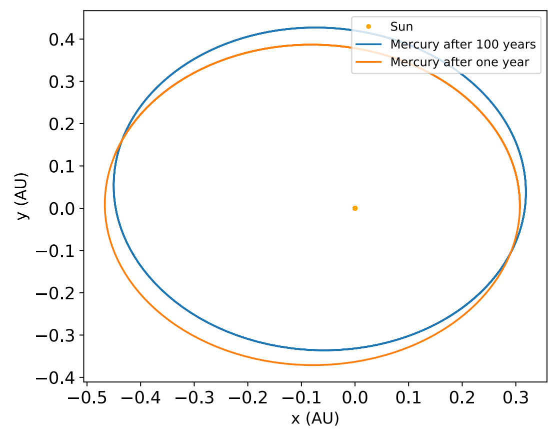


(a) method: Velocity-Verlet

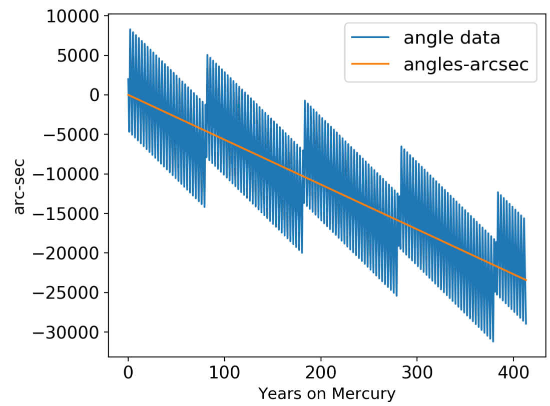


(b) method: Euler-Chromer

Figure 5: **Celestial Orbits for all planets including Pluto**: Initializing the planets with data from NASA. **a)** With the Velocity-Verlet method, all planets are kept in their respective orbits **b)** Mercury, which has the highest orbit velocity (10.09 [AU/year]) [11] is observed escaping the solar system using Euler-Chromer as a solver. Parameters: $T = 250$ years , $\beta = 2$, number of iteration points $k = 10^4$.



(a) Orbits of Mercury



(b) Perihelion angle

Figure 6: **Perihelion precession for Mercury**: With a relativistic force correction, Mercury has a precession angle. **a)** After 100 years, the orbit has rotated with an angle compared to its orbital position at year 1. This subplot exaggerates the effects, to show its precession (number of iteration points $= 10^5$). **b)** With linear interpolation, Mercury gets a perihelion precession of about 25000´´ (arcseconds) after 100 years (Earth years, corresponding 414 years on Mercury. Parameters: $T = 100$ years , $\beta = 2$, number of iteration points $= 10^6$.

Table 1: **Overview of Conservation Benchmark Outcomes**: The Forward-Euler, Euler-Chromer and Velocity-Verlet method is used as ODE solver applied on celestial mechanics. Below are the outcomes of unit tests for energy (tol: 1×10^{-7}) and angular momentum (L) (tol:1×10^{-12}) for different systems of planets orbiting the Sun (10^5 iteration points).

System	Method	Total Energy constant	L constant
Earth	Forward-Euler	No (5×10^-5)	Yes
	Euler-Chromer	No (7×10^{-7})	Yes
	Velocity-Verlet	Yes	Yes
Earth ($\beta : 3$)	Velocity- Verlet	Yes	Yes
Earth-Jupiter	Velocity-Verlet	No (7×10^{-6})	Yes

IV Discussion

As mentioned in the introduction of this paper, there exists no closed form solutions to many-body problems [6]. Thus, numerical models using ODE solvers are an excellent tool to investigate such dynamical systems. The results described above (sec. III) are therefore discussed in the context of reproducing physical phenomena of celestial orbits. It is accompanied with commentary on method performance.

The mechanical laws affecting the two-body Earth-Sun system is fulfilled with Velocity-Verlet. As expected from analytical results (eq. 6), an initial tangential velocity of $2\pi \approx 6.28$ [AU/year] gives circular orbits (fig. 1a), whereas other initial velocities results in elliptical orbits (1b). In accordance with theory, both angular momentum and energy are conserved in both cases (tab. 1). The kinetic and potential energy were almost constant for a circular orbit, while this is not the case for elliptical orbits (figs. 7-8). The escape velocity is also as expected (fig. 2). Thus, the Verlet method retains these quantities. However, the other methods did not conserve energy (tab. 1). This is in agreement with Hairer et al. [12], which explain the energy-conserving properties of geomtric numerical integrators, like that of Velocity-Verlet (also known as Störmer-Verlet).

A non-square inverse force, with $\beta = (2, 3]$ results in the Earth spiralling toward the more massive Sun (fig. 3). In this instance, the angular momentum is conserved (tab. 1), however, the orbit of the planet, was not a closed ellipsis. Thus, the model reproduces what Newton discovered over 300 years ago - that Kepler's 1.st and 3.law are dependent on an inverse square law, but the 2nd. law still holds [9]. However, with a stronger attractive force between planets, the solar system would not have the celestial arrangement observed today.

Likewise, if the nearby planets of the Earth was more massive, the stability of Earths orbit would have been altered, as is seen in fig. 4, further emphasizing the importance of the relation between distance (r) and mass (m) in Newton's gravitational law. For all planets, with an inverse square law and correct masses, stable elliptical orbits are reproduced with Velocity-Verlet as a method (fig. 5). In contrast, Euler-Cromer fails to update the planetary positions of Mercury correctly for a resolution of 10^4 iteration points, again demonstrating the benefits of applying a geometric numerical integrator. The most likely reason for this occurring with Mercury specifically, is that is has the highest orbital velocity. Thus, updating its position is more of a challenge compared to other planets. Being the better option in regards to energy-conservation, Velocity-Verlet nevertheless seem to experience more fluctuation in energies with many planets (tab. 1).

It proves a challenge to predict the perihelion precession of Mercury as a relativistic effect with the methods applied here (fig. 6). The effect is indeed visible, however an estimate of 25 000´´ (arc seconds) corresponds poorly with the 43´´ predicted by Einstein's theory of general relativity [13]. With trial and error, an increased number of iteration points lowered the estimate. However, this high resolution also gives numerical artefacts, as seen in by the jump in angles roughly every hundred Mercury years. This suggests that higher precision methods must be used in order to describe the precession correctly. For further reading on numerical models of perihelion precession of Mercury see Vankov [14].

As seen throughout this paper, a challenge lies in energy conservation and making the planets stay on their orbital paths for a given iterative method. Regarding energy conservation, the oscillations for Velocity-Verlet is smaller for a range of chosen iteration points, compared to the two other ODE solvers (figs. 10-9). This makes Velocity-Verlet slightly more stable. In agreement with [12], it has some oscillations, but around a constant value for each planet. Forward-Euler, Euler Cromer and Velocity-Verlet all has a convergence rate of 1.0 (tab. 3), suggesting a global error of O(h) for this application, despite having local error O(h^2) and O(h^3) respectively [7].

A trade-off in relative error with number of iteration points is prevalent when applying the ODE solvers on a circular motion (tab 3), in which Forward-Euler has highest precision. However, this analysis was performed with $T = 1$ year. Since Forward-Euler does not conserve energy, Velocity-Verlet is expected to perfom best over longer time periods, as seen in fig. 5b. The trade-off seen in tab. 3 can be explained 17 with lack of numerical precision when the step size gets smaller than 10^6. for The methods has all approximately a logarithmic increase in CPU-time usage with a growing number of iteration points (base 10).

V Conclusion

We have now explored celestial orbits of the Solar System by applying the various numerical methods Forward-Euler, Euler-Chromer and Velocity-Verlet. When applying the classical newtonian force 3, results show that the orbits of Earth and the other planets agrees with Newton's laws discussed in sec II.I. Results also shows that for different circular and elliptical orbits, the total angular momentum is conserved as expected. In addition to this the total energy is only conserved when applying Velocity-Verlet, which also agrees with Hairer et al. [12]. An non-square inverse force instead of the classical newtonian force results in an increasing force, where the Sun pulls the Earth closer to itself.

Applying the relativistic correction 15 to Mercury failed to give a perihelion precession in agreement with [13], suggesting in the need of higher precision. Comparing the stability of the three ODE solver, shows that Velocity-Verlet gives more stable results when predicting closed orbits and energy conservation. However results also indicates that for number of iterations $k = 10^6$, the relative error is increasing, suggesting in loss of numerical precision when applying the different numerical methods for higher k.

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VI Appendix

VI..1 Forward-Euler and Euler-Cromer

Algorithm 2 Forward-Euler: The basic outline of Forward-Euler algorithm for solving an ODE problem with n particles and k timesteps.

for int j=0; j < k-1; j++ **do** ▷ For each timestep
    for int i=0; j < n; i++ **do** ▷ For each particle
        $a_{x,i,j} = F_x(x_{i,j}, y_{i,j}, z_{i,j})$
        $a_{y,i,j} = F_y(x_{i,j}, y_{i,j}, z_{i,j})$
        $a_{z,i,j} = F_z(x_{i,j}, y_{i,j}, z_{i,j})$

        $v_{x,i,j+1} = v_{x,i,j} + ha_{x,i,j}$
        $v_{y,i,j+1} = v_{y,i,j} + ha_{y,i,j}$ ▷ Find velocity at timestep $j+1$
        $v_{z,i,j+1} = v_{z,i,j} + ha_{z,i,j}$

        $x_{i,j+1} = x_{i,j} + hv_{x,i,j}$
        $x_{i,j+1} = x_{i,j} + hv_{x,i,j}$ ▷ Find positions at timestep $j+1$
        $x_{i,j+1} = x_{i,j} + hv_{x,i,j}$
    end for
end for

Algorithm 3 Euler-Cromer: The basic outline of Euler-Cromer algorithm for solving an ODE problem with n particles and k timesteps.

for int j=0; j < k-1; j++ **do** ▷ For each timestep
    for int i=0; j < n; i++ **do** ▷ For each particle
        $a_{x,i,j} = F_x(x_{i,j}, y_{i,j}, z_{i,j})$
        $a_{y,i,j} = F_y(x_{i,j}, y_{i,j}, z_{i,j})$
        $a_{z,i,j} = F_z(x_{i,j}, y_{i,j}, z_{i,j})$

        $v_{x,i,j+1} = v_{x,i,j} + ha_{x,i,j}$
        $v_{y,i,j+1} = v_{y,i,j} + ha_{y,i,j}$ ▷ Find velocity at timestep $j+1$
        $v_{z,i,j+1} = v_{z,i,j} + ha_{z,i,j}$

        $x_{i,j+1} = x_{i,j} + hv_{x,i,j+1}$
        $x_{i,j+1} = x_{i,j} + hv_{x,i,j+1}$ ▷ Find positions at timestep $j+1$
        $x_{i,j+1} = x_{i,j} + hv_{x,i,j+1}$
    end for
end for

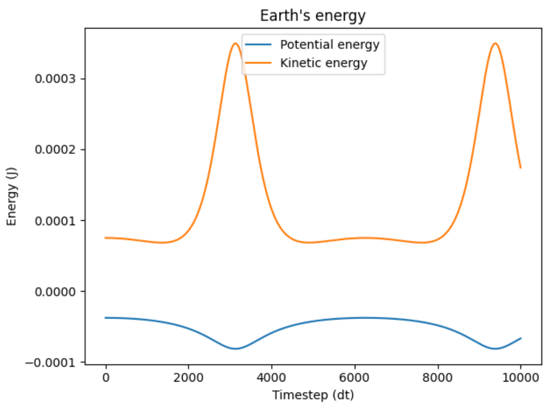
VI.I Benchmarks

VI.I.1 CPU-time

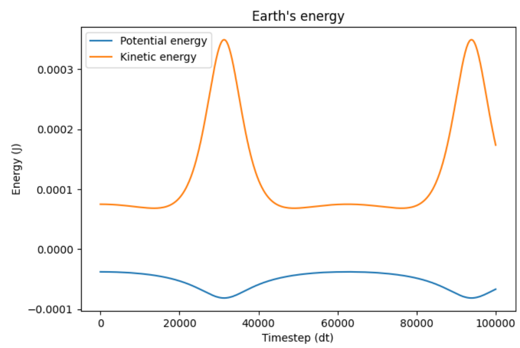
Table 2: **CPU runtimes for ODE methods**: CPU times for Forward-Euler, Euler-Chromer and Velocity-Verlet methods for number of iterations $k = 10^4$, 10^5 and 10^6.

k	Forward-Euler (ms)	Euler-Chromer (ms)	Velocity-Verlet (ms)
10^4	101	108	100
10^5	1351	1037	1340
10^6	12191	16439	11696

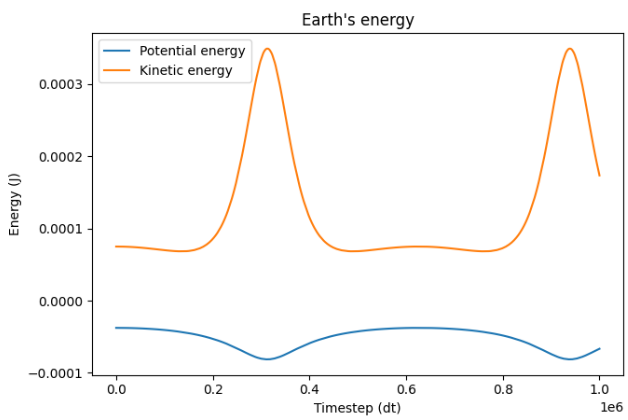
VI.I.2 Energy Conservation



(a) For $k = [0, 10^4]$

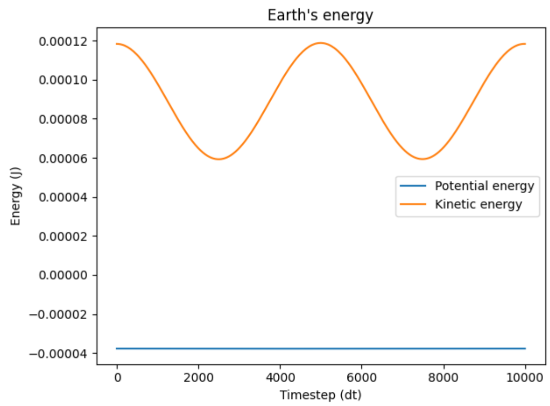


(b) For $k = [0, 10^5]$

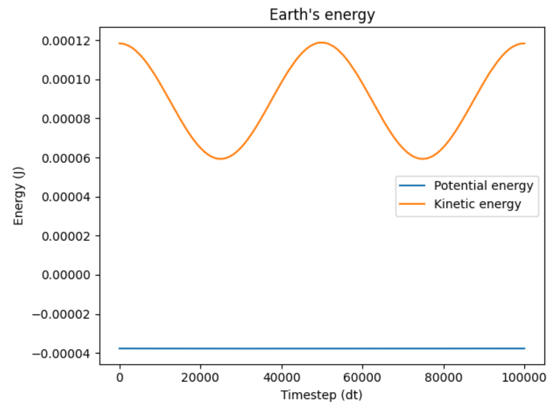


(c) For $k = [0, 10^6]$

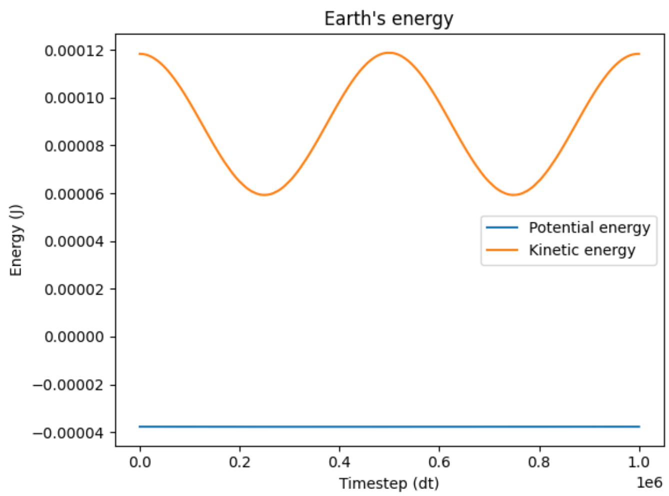
Figure 7: **Kinetic and potential energy for Earth's ellipical orbit**: Potential and kinetic energy are plotted as a function of iterative steps using Velocity-Verlet. **a)** Potential and kinetic energy for $k = [0, 10^4]$. **b)** Potential and kinetic energy for $k = [0, 10^5]$**c)** Potential and kinetic energy for $k = [0, 10^6]$. Parameters: $T = 1$ years , $\beta = 2$.



(a) For $k = [0, 10^4]$

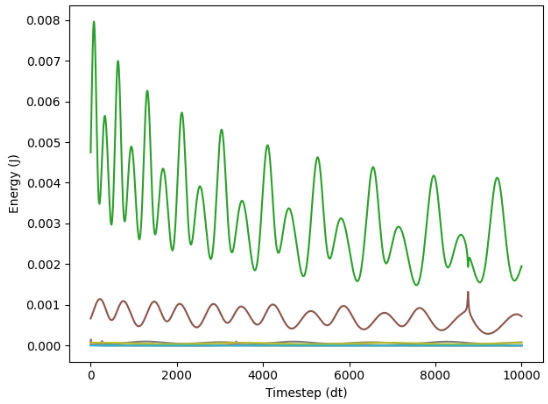


(b) For $k = [0, 10^5]$

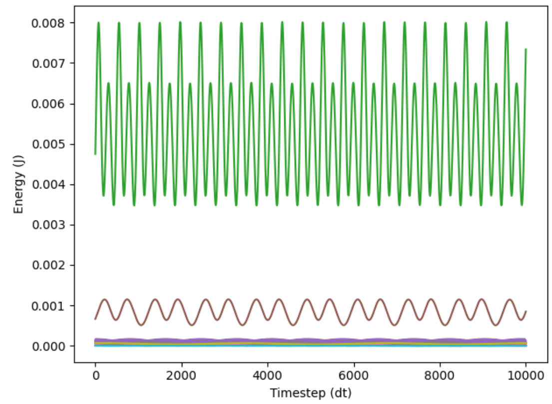


(c) For $k = [0, 10^6]$

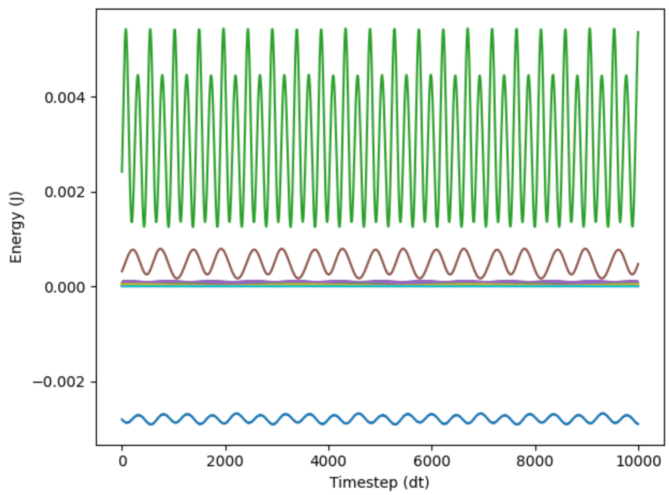
Figure 8: **Kinetic and potential energy for Earth's circular orbit**: Potential and kinetic energy are plotted as a function of iterative steps using Velocity-Verlet for a circular orbit. **a)** Potential and kinetic energy for $k = [0, 10^4]$. **b)** Potential and kinetic energy for $k = [0, 10^5]$**c)** Potential and kinetic energy for $k = [0, 10^6]$. Parameters: $T = 1$ years , $\beta = 2$.



(a) Method: Forward Euler

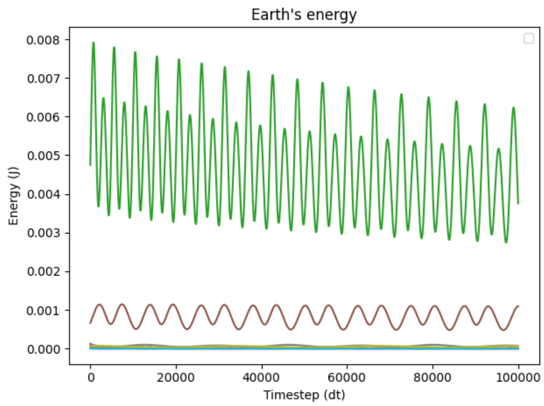


(b) Method: Euler-Cromer

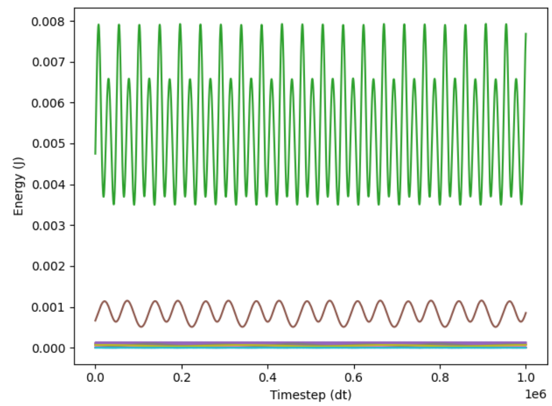


(c) Method: Velocity-Verlet

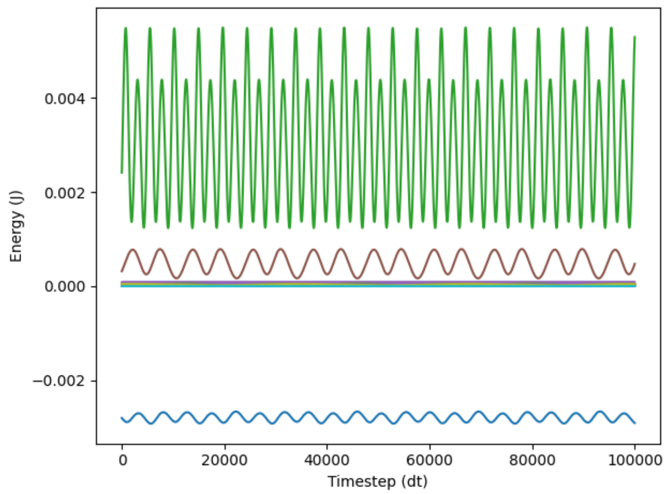
Figure 9: **Total energy for all seven planets with lower resolution**: Total energy is plotted as a function of iterative steps for three ODE solvers for a larger step size (less iteration numbers) than in fig. 10. **a)** Forward Euler gives large fluctuations in total energy, including a large drift in average energy for $k = 10^4$. Thus it is not energy conserving. **b)** Euler-Cromer has large fluctuations in energy, but has no large drift from the average.**c)** Velocity-Verlet has small fluctuations around average energies. Thus it better describes systems where energy should be conserved. Compared to 10 it also remain stable for lower k-values. Parameters: $T = 250$ years , $\beta = 2$, number of iteration points $= 10^4$.



(a) Method: Forward Euler



(b) Method: Euler-Chromer



(c) Method: Velocity-Verlet

Figure 10: **Total energy for all seven planets**: Total energy is plotted as a function of iterative steps for three ODE solvers. **a)** Forward Euler gives large fluctuations in total energy, including a drift in average energy. Thus it is not energy conserving. **b)** Euler-Chromer has large fluctuations in energy, but has no large drift from the average.**c)** Velocity-Verlet has small fluctuations around average energies. Thus it better describes systems where energy should be conserved. Parameters: $T = 250$ years , $\beta = 2$, number of iteration points $= 10^5$.

VI.I.3 Deviation from circular orbit of Earth around the Sun

Table 3: **Test outcomes for circular orbit**: Different number of iteration points (k) are used to check relative error of ODE methods for the Earth-Sun system. Lowest error is highlighted in red. In addition, convergence test outcomes are presented below. Parameters: $T = 1$ year, $\beta = 2$.

k	Forward-Euler	Euler-Chromer	Velocity-Verlet
10^3	0.0076	0.0018	0.002
10^4	0.0016	0.0014	0.001
10^5	0.00003	0.0018	0.0002
10^6	0.0006	0.0017	0.0008
Convergence rate	1.0	1.0	1.0