計算固体力学入門 (7)

Introduction to Computational Solid Mechanics (7)

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Today's topic

The contents of this course will be as follows:

1. FEM for a one-dimensional problem.

2. Mathematical preliminaries
   - Gauss-Legendre quadrature
   - Einstein's summation convention

3. Continuum mechanics
   - Deformation of continuum
   - Balance of continuum
   - Basic equations

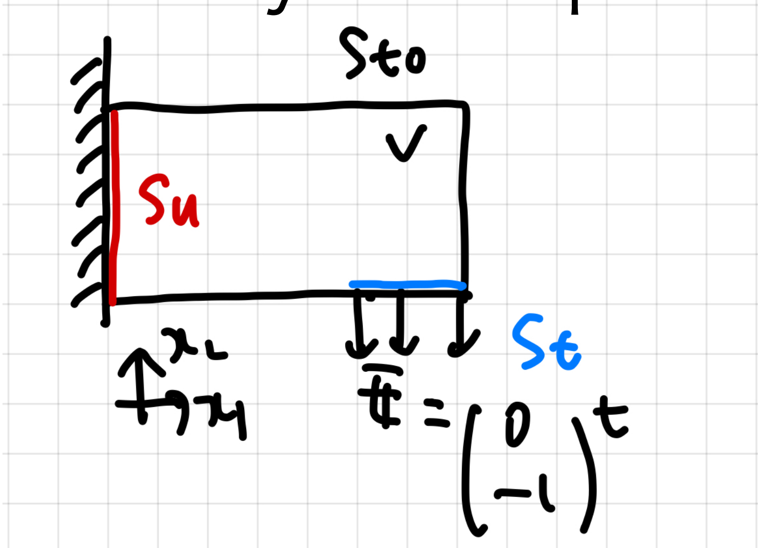
4. Weak form

5. **Discretisation**

6. FEM implementations

The weak form

The boundary value problem



$$\sigma_{ji,j}(\boldsymbol{x}) = 0 \quad \boldsymbol{x} \in V, \qquad \cdots (1)$$

$$u_i(\boldsymbol{x}) = 0 \quad \boldsymbol{x} \in S_u, \qquad \cdots (2)$$

$$t_i(\boldsymbol{x}) = \begin{cases} 0 & \boldsymbol{x} \in S_{t0} & \cdots (3) \\ \bar{t}_i & \boldsymbol{x} \in S_t & \cdots (4) \end{cases}$$

is converted to the following form:

$$\sum_{\ell=1}^{N} \int_V \left((\lambda + 2\mu)N_{,1}^{m} N_{,1}^{\ell} + \mu N_{,2}^{m} N_{,2}^{\ell},\ \lambda N_{,1}^{m} N_{,2}^{\ell} + \mu N_{,2}^{m} N_{,1}^{\ell}\right)\begin{pmatrix} a_1^{\ell} \\ a_2^{\ell} \end{pmatrix} \mathrm{d}V = \int_{S_t} N^m \bar{t}_1 \mathrm{d}S$$

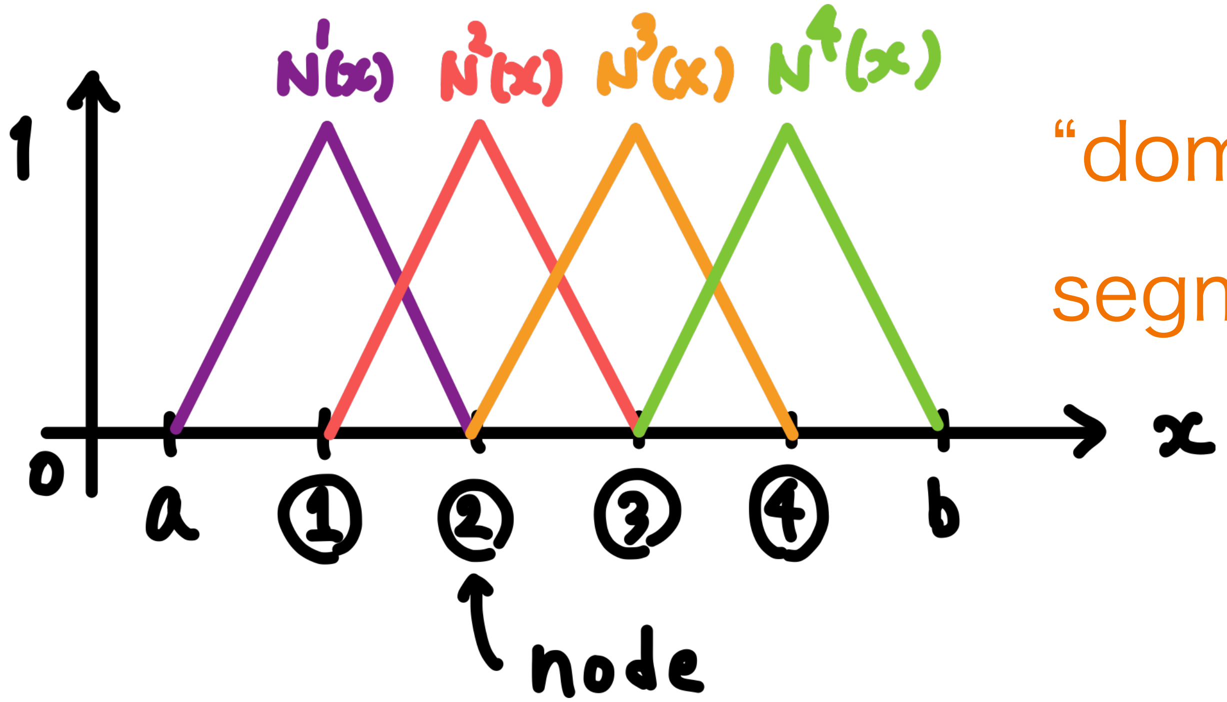
$$\sum_{\ell=1}^{N} \int_V \left(\lambda N_{,2}^{m} N_{,1}^{\ell} + \mu N_{,1}^{m} N_{,2}^{\ell},\ \mu N_{,1}^{m} N_{,1}^{\ell} + (\lambda + 2\mu) N_{,2}^{m} N_{,2}^{\ell}\right)\begin{pmatrix} a_1^{\ell} \\ a_2^{\ell} \end{pmatrix} \mathrm{d}V = \int_{S_t} N^m \bar{t}_2 \mathrm{d}S$$

$$\Leftrightarrow \sum_{\ell=1}^{N} \begin{pmatrix} k_{11}^{m\ell} & k_{12}^{m\ell} \\ k_{21}^{m\ell} & k_{22}^{m\ell} \end{pmatrix} \begin{pmatrix} a_1^{\ell} \\ a_2^{\ell} \end{pmatrix} = \begin{pmatrix} q_1^{\ell} \\ q_2^{\ell} \end{pmatrix} \text{ for } m = 1, \cdots, N$$

by assuming $u_i(\boldsymbol{x}) = \sum_{i=1}^{N} a_i^{\ell} N^{\ell}(\boldsymbol{x})$. Next topic: how to choose $N^{\ell}(\boldsymbol{x})$?

Space discretisation 1/3

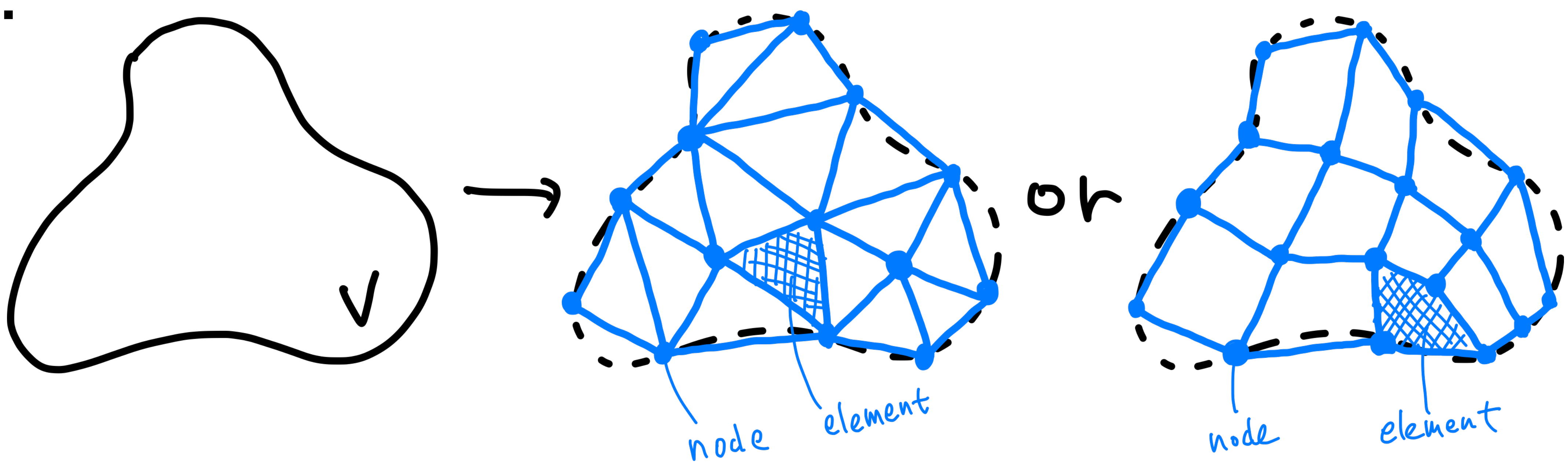
In the 1D case:



In solving $\dfrac{d^2u(x)}{dx^2} + f(x) = 0$ in $[a, b]$, the "domain" $[a, b]$ is divided into N line segments. $N^\ell(x)$ is assumed to be

- (piecewise) linear
- $N^\ell(x) = 1$ at ℓ^{th} node
- $N^\ell(x) = 0$ at the other nodes

In 2D:

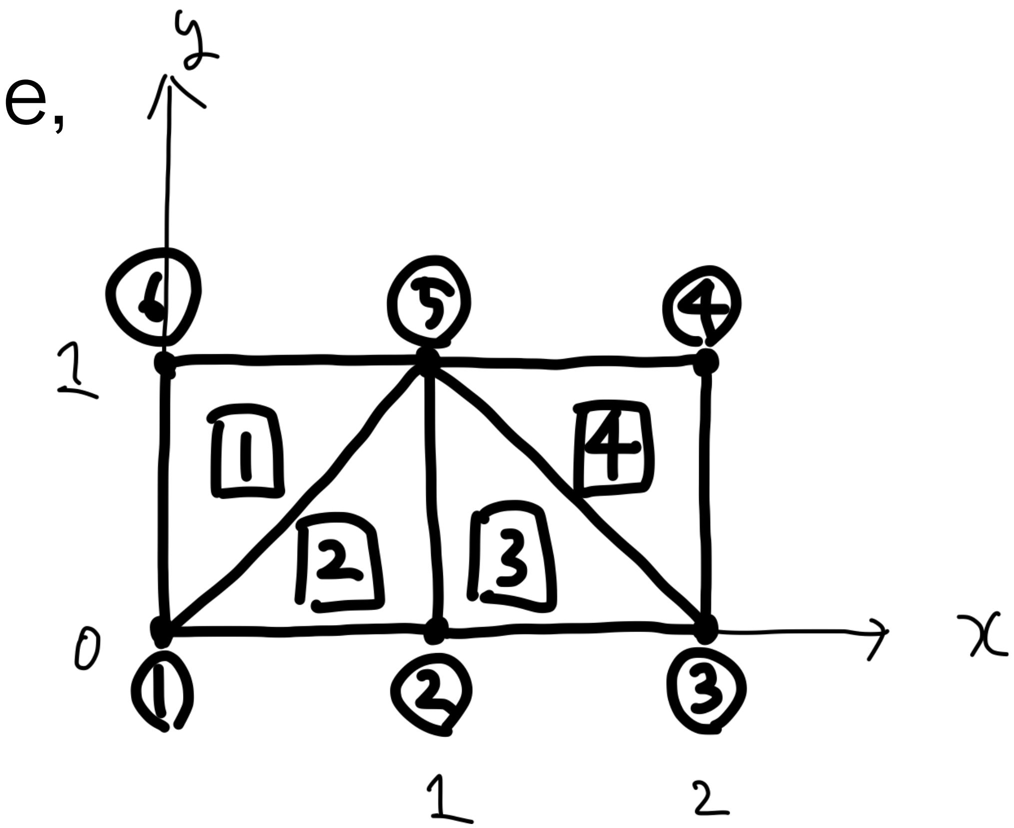


To solve $\sigma_{ji,j}(\boldsymbol{x}) = 0$ in $\boldsymbol{x} \in V$, the domain V should be divided into N patches of triangle or quadrangle. Today's focus: triangular one.

Space discretisation 2/3

How can we represent V discretised by the triangular patches?

Let us use an example that V is a rectangle, which is divided into four triangles.



It suffice to store

- coordinate of nodes ⓘ
- indices of nodes consisting of each elements $\boxed{j}$ (connectivity)

Example:

```
# nodal coodinate
# index, x, y
1, 0.0, 0.0
2, 1.0, 0.0
3, 2.0, 0.0
4, 2.0, 0.0
5, 1.0, 1.0
6, 0.0, 1.0
```

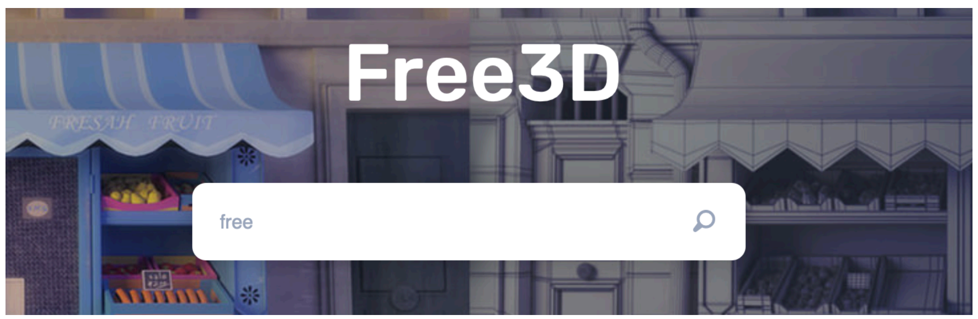
```
# connectivity
# index, 1st, 2nd, and 3rd vertex
1, 1, 5, 6
2, 1, 2, 5
3, 2, 3, 5
4, 3, 4, 5
```

Note: order of vertices is important.

Space discretisation 3/3

Note: the previous expression for a planar domain can easily be extended to the surface representation.

Examples: So called "3D model" for 3D printing:



https://free3d.com

Stanford Bunny:

The "Stanford Bunny"

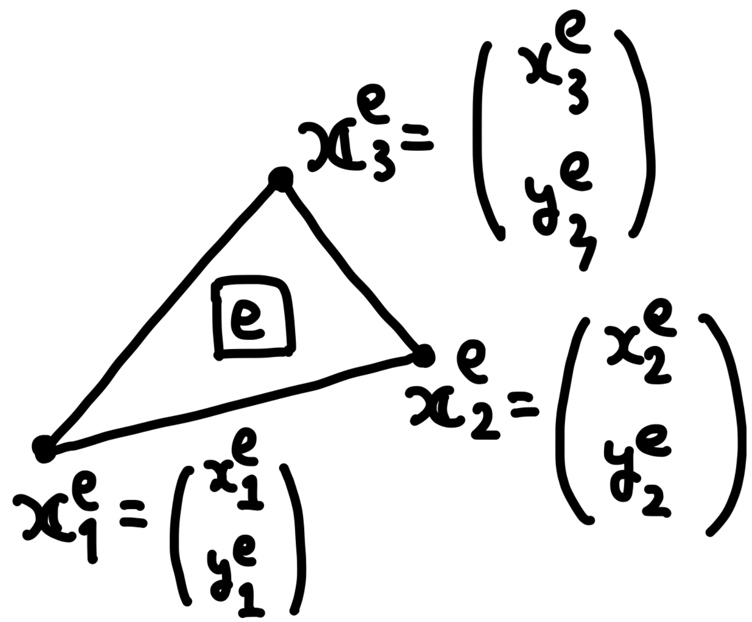
The Stanford 3D Scanning Repository

http://graphics.stanford.edu/data/3Dscanrep

There are several formats of the "3D model", such as STL, STEP, IGES, OFF, PLY, 3DS etc, but all of them just give the nodal coordinate and the connectivity. Useful free software: MeshLab

Basis function 1/4

To define the basis function, let us consider a single element:



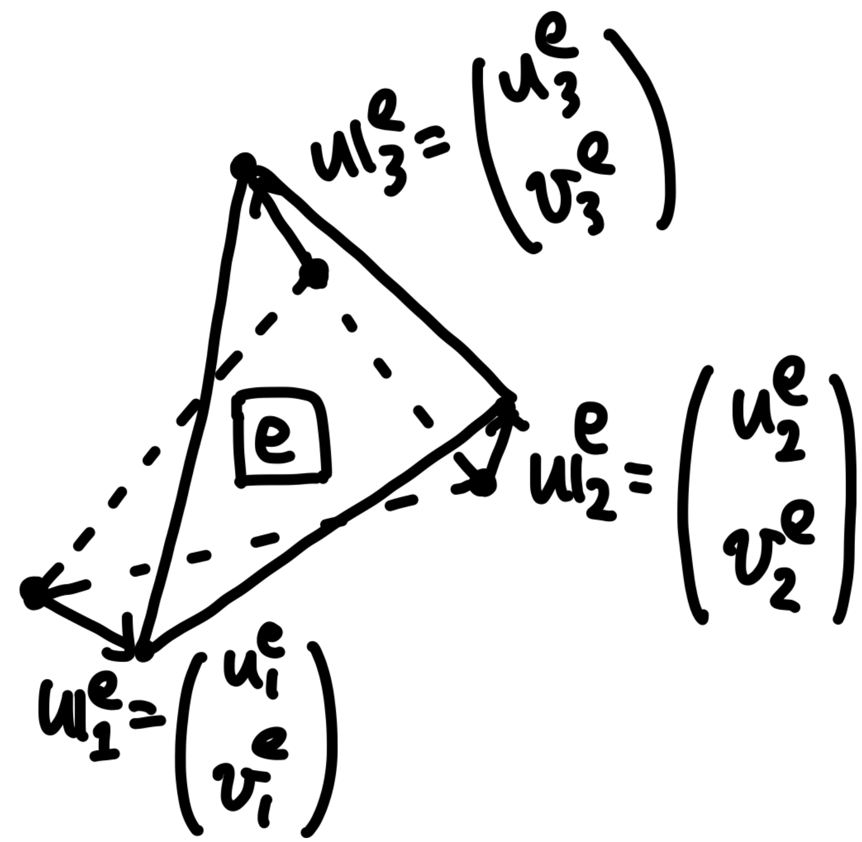
Notation:

$\boldsymbol{x}_i^e = (x_i^e,\ y_i^e)^t$: coordinate of i^{th} vertex.

$\boldsymbol{u}_i^e = (u_i^e,\ v_i^e)^t$: displacement of i^{th} vertex.

$\boldsymbol{d}^e = (u_1^e, v_1^e, u_2^e, v_2^e, u_3^e, v_3^e)^t$: element disp. vec.

A_e: area of the element $\boxed{e}$.



Note: each element has six degrees of freedom in its deformation.

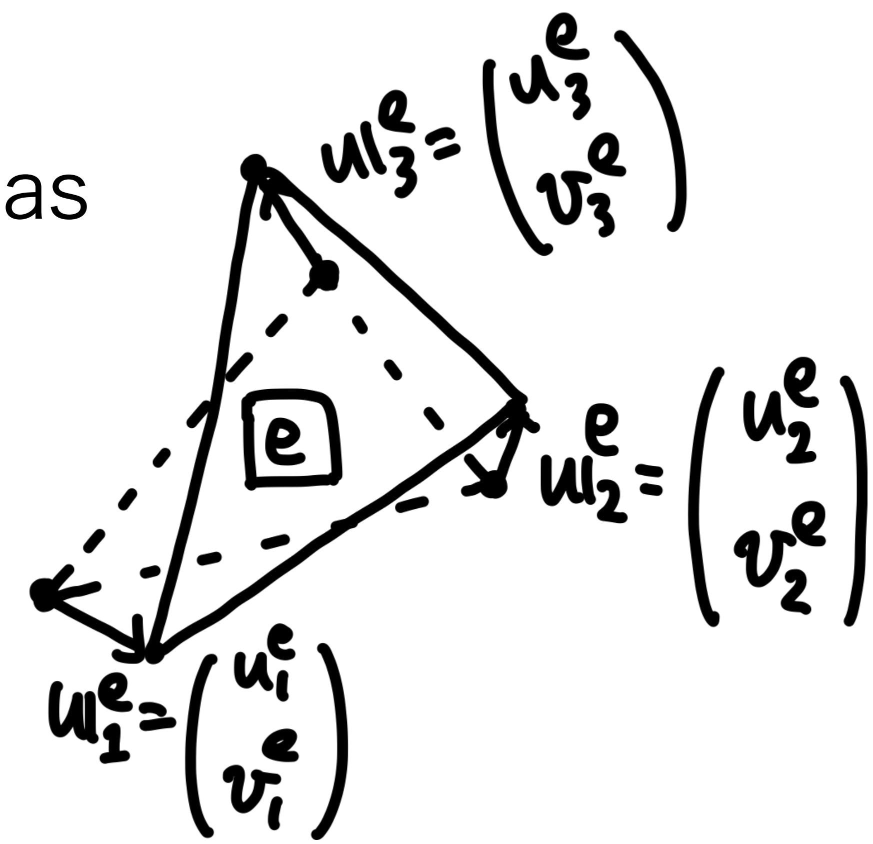
Basis function 2/4

Let us now assume that the displacement $\boldsymbol{u}(\boldsymbol{x}) = (u(\boldsymbol{x}), v(\boldsymbol{x}))^t$ is linear in the element $\boxed{e}$, i.e. $u(\boldsymbol{x})$, for example, is approximated as

$$u(\boldsymbol{x}) \simeq a_1 + a_2 x + a_3 y = (1,\ x,\ y)\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}, \qquad \cdots \text{(A)}$$

where a_i $(i = 1, 2, 3)$ are coefficients.

By (A), the nodal displacement is expressed as



$$u(\boldsymbol{x}_i^e) = u_i^e$$

$$\Leftrightarrow a_1 + a_2 x_i^e + a_3 y_i^e = u_i^e$$

$$\Leftrightarrow \begin{pmatrix} 1 & x_1^e & y_1^e \\ 1 & x_2^e & y_2^e \\ 1 & x_3^e & y_3^e \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} u_1^e \\ u_2^e \\ u_3^e \end{pmatrix} \qquad \cdots \text{(B)}$$

Basis function 3/4

(B) is substituted into (A) to have

$$u(x,y) \simeq (1,\ x,\ y)\begin{pmatrix} 1 & x_1^e & y_1^e \\ 1 & x_2^e & y_2^e \\ 1 & x_3^e & y_3^e \end{pmatrix}^{-1} \begin{pmatrix} u_1^e \\ u_2^e \\ u_3^e \end{pmatrix} = (N_1^e(\boldsymbol{x}),\ N_2^e(\boldsymbol{x}),\ N_3^e(\boldsymbol{x}))\begin{pmatrix} u_1^e \\ u_2^e \\ u_3^e \end{pmatrix}$$

where $N_i^e(\boldsymbol{x})$ is defined as

$$\begin{pmatrix} N_1^e(\boldsymbol{x}) \\ N_2^e(\boldsymbol{x}) \\ N_3^e(\boldsymbol{x}) \end{pmatrix} = \begin{pmatrix} 1 & x_1^e & y_1^e \\ 1 & x_2^e & y_2^e \\ 1 & x_3^e & y_3^e \end{pmatrix}^{-t} \begin{pmatrix} 1 \\ x \\ y \end{pmatrix}$$

$$= \frac{1}{2A_e}\begin{pmatrix} (y_2^e - y_3^e)x - (x_2^e - x_3^e)y + x_2^e y_3^e - x_3^e y_2^e \\ (y_3^e - y_1^e)x - (x_3^e - x_1^e)y + x_3^e y_1^e - x_1^e y_3^e \\ (y_1^e - y_2^e)x - (x_1^e - x_2^e)y + x_1^e y_2^e - x_2^e y_1^e \end{pmatrix}$$

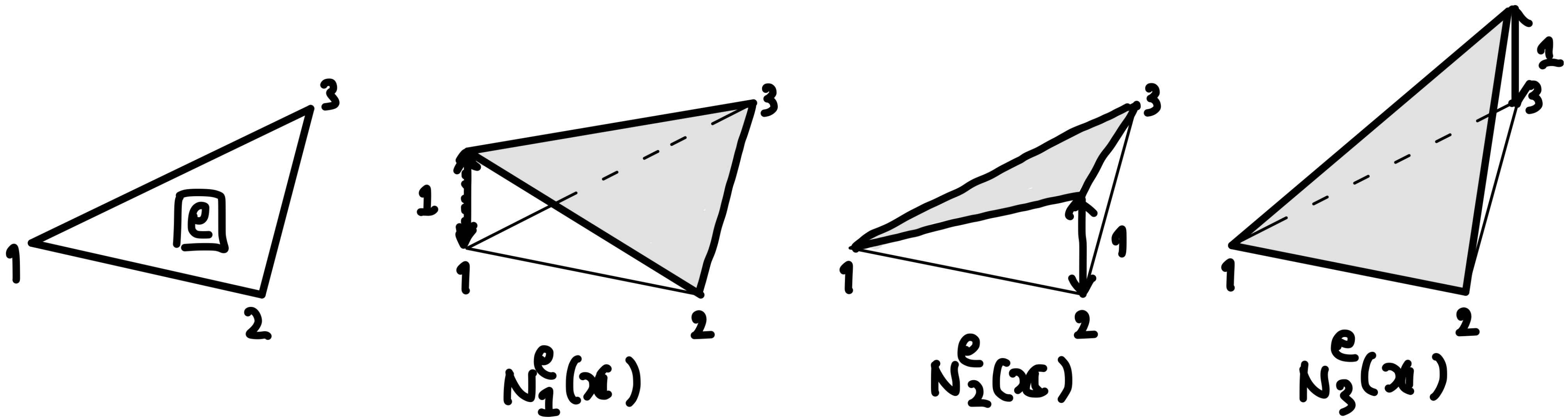
↑ Confirm this in HW.

Basis function 4/4

Also confirm the following in HW:

- Show that $N_i^e(\boldsymbol{x}_j^e) = \delta_{ij}$.
- Show that $\displaystyle\sum_{i=1}^{3} N_i^e(\boldsymbol{x}) = 1$.

Discretized displacement



With the basis functions, the displacement in the element $\boxed{e}$ is expanded as follows:

$$u(\boldsymbol{x}) = \sum_{i=1}^{3} N_i^e(\boldsymbol{x}) u_i^e \text{ and } v(\boldsymbol{x}) = \sum_{i=1}^{3} N_i^e(\boldsymbol{x}) v_i^e$$

$$\Leftrightarrow \begin{pmatrix} u(\boldsymbol{x}) \\ v(\boldsymbol{x}) \end{pmatrix} = \begin{pmatrix} N_1^e(\boldsymbol{x}) & 0 & N_2^e(\boldsymbol{x}) & 0 & N_3^e(\boldsymbol{x}) & 0 \\ 0 & N_1^e(\boldsymbol{x}) & 0 & N_2^e(\boldsymbol{x}) & 0 & N_3^e(\boldsymbol{x}) \end{pmatrix} \begin{pmatrix} u_1^e \\ v_1^e \\ u_2^e \\ v_2^e \\ u_3^e \\ v_3^e \end{pmatrix}$$

$$\Leftrightarrow \boldsymbol{u}(\boldsymbol{x}) = \mathrm{N}^e(\boldsymbol{x}) \boldsymbol{d}^e$$

Discretized strain

Accordingly, the strain in the element $\boxed{e}$ is expressed as

$$
\begin{pmatrix} \varepsilon_{11}(\boldsymbol{x}) \\ \varepsilon_{22}(\boldsymbol{x}) \\ \gamma_{12}(\boldsymbol{x}) \end{pmatrix}
=
\begin{pmatrix} \dfrac{\partial u(\boldsymbol{x})}{\partial x} \\ \dfrac{\partial v(\boldsymbol{x})}{\partial y} \\ \dfrac{\partial u(\boldsymbol{x})}{\partial y} + \dfrac{\partial v(\boldsymbol{x})}{\partial x} \end{pmatrix}
=
\begin{pmatrix} \dfrac{\partial}{\partial x} & 0 \\ 0 & \dfrac{\partial}{\partial y} \\ \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial x} \end{pmatrix}
\begin{pmatrix} u(\boldsymbol{x}) \\ v(\boldsymbol{x}) \end{pmatrix}
$$

$$
= \underbrace{\begin{pmatrix} \dfrac{\partial}{\partial x} & 0 \\ 0 & \dfrac{\partial}{\partial y} \\ \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial x} \end{pmatrix}
\begin{pmatrix} N_1^e(\boldsymbol{x}) & 0 & N_2^e(\boldsymbol{x}) & 0 & N_3^e(\boldsymbol{x}) & 0 \\ 0 & N_1^e(\boldsymbol{x}) & 0 & N_2^e(\boldsymbol{x}) & 0 & N_3^e(\boldsymbol{x}) \end{pmatrix}}_{= \mathsf{B}^e \text{ (B-matrix)}}
\begin{pmatrix} u_1^e \\ v_1^e \\ u_2^e \\ v_2^e \\ u_3^e \\ v_3^e \end{pmatrix}
$$

$$
= \mathsf{B}^e \boldsymbol{d}^e
$$

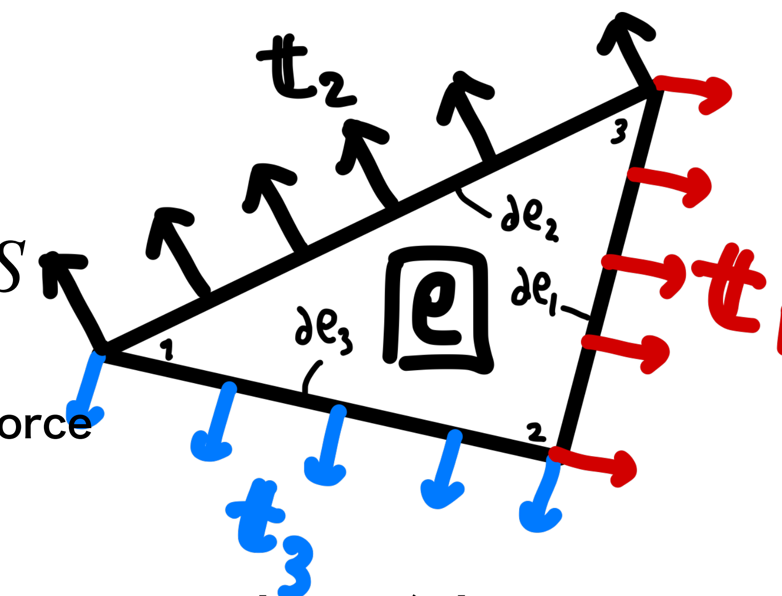
Note: B^e is a constant matrix since $N_i^e(\boldsymbol{x})$ is a liner function. That is why, the triangular element with (piecewise) linear basis function is called the CST (=constant strain triangular) element.

Element stiffness eq. 1/2

Let us consider the following weak form for the element $\boxed{e}$ (see (8) in 06slide.pdf):

$$\int_e (\tilde{\varepsilon}_{11}, \tilde{\varepsilon}_{22}, \tilde{\gamma}_{12}) \underbrace{\begin{pmatrix} \lambda + 2\mu & \lambda & 0 \\ \lambda & \lambda + 2\mu & 0 \\ 0 & 0 & \mu \end{pmatrix}}_{= \mathrm{D}} \begin{pmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{pmatrix} \mathrm{d}V = \sum_{i=1}^{3} \int_{\partial e_i} \tilde{\boldsymbol{u}}^t \boldsymbol{t}_i \mathrm{d}S$$

internal force



By using the same basis function for the strains ε_{ij} and $\tilde{\varepsilon}_{ij}$ (the Galerkin method) as

$$(\varepsilon_{11}, \varepsilon_{22}, \gamma_{12})^t \simeq (\mathrm{B}^e \boldsymbol{d}^e)^t,\ (\tilde{\varepsilon}_{11}, \tilde{\varepsilon}_{22}, \tilde{\gamma}_{12})^t \simeq (\mathrm{B}^e \tilde{\boldsymbol{d}}^e)^t \text{ and } \tilde{\boldsymbol{u}} = \mathrm{N}^e \tilde{\boldsymbol{d}}^e$$

The weak form is discretised as follows:

$$\int_e (\mathrm{B}^e \boldsymbol{d}^e)^t \mathrm{D}^e \mathrm{B}^e \boldsymbol{d}^e \mathrm{d}V = \sum_{i=1}^{3} \int_{\partial e_i} (\mathrm{N}^e \tilde{\boldsymbol{d}}^e)^t \boldsymbol{t}_i \mathrm{d}S$$

$$\Leftrightarrow \tilde{\boldsymbol{d}}^{et} \left(\int_e \mathrm{B}^{et} \mathrm{D}^e \mathrm{B}^e \mathrm{d}V \right) \boldsymbol{d}^e = \tilde{\boldsymbol{d}}^{et} \sum_{i=1}^{3} \int_{\partial e_i} \mathrm{N}^{et} \boldsymbol{t}_i \mathrm{d}S$$

Element stiffness eq. 2/2

weak form $\Leftrightarrow \tilde{\boldsymbol{d}}^{et}\left(\int_e \mathsf{B}^{et}\mathsf{D}^e\mathsf{B}^e \mathrm{d}V\right)\boldsymbol{d}^e = \tilde{\boldsymbol{d}}^{et}\sum_{i=1}^{3}\int_{\partial e_i}\mathsf{N}^{et}\boldsymbol{t}_i \mathrm{d}S$

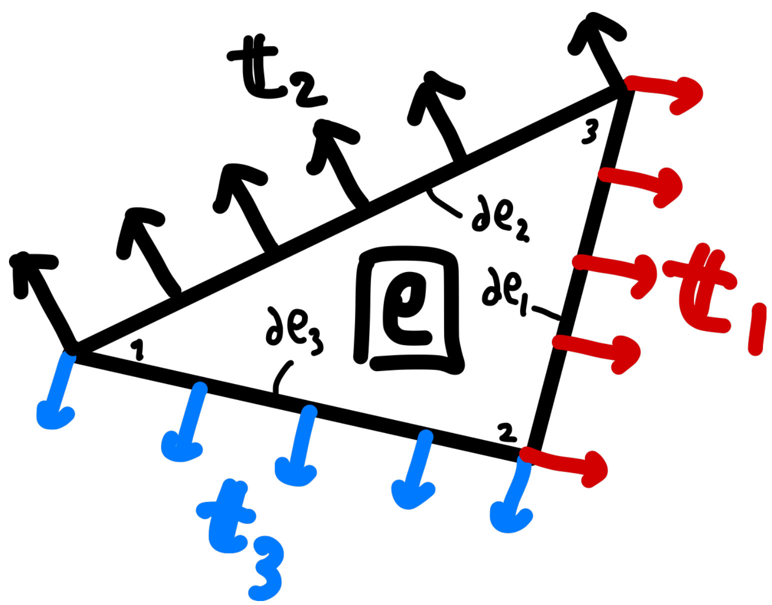
Since $\tilde{\boldsymbol{d}}^e$ can be arbitrary, we can rewrite the above as

$$\left(\int_e \mathsf{B}^{et}\mathsf{D}^e\mathsf{B}^e \mathrm{d}V\right)\boldsymbol{d}^e = \sum_{i=1}^{3}\int_{\partial e_i}\mathsf{N}^{et}\boldsymbol{t}_i \mathrm{d}S$$

$$\Leftrightarrow \mathsf{K}^e\boldsymbol{d}^e = \boldsymbol{f}^e$$

Element load vector

Element stiffness matrix



which is called the element stiffness equation.

→ Element stiffness equations for all the triangular will be combined to obtain the global stiffness equation. This procedure will be seen in an example presented afterwards.

On the computation of K^e

$$\mathsf{K}^e = \int_e \mathsf{B}^{et}\mathsf{D}^e\mathsf{B}^e \mathrm{d}V = \int_e \begin{pmatrix} \frac{\partial N_1^e}{\partial x} & 0 & \frac{\partial N_1^e}{\partial y} \\ 0 & \frac{\partial N_1^e}{\partial x} & \frac{\partial N_1^e}{\partial x} \\ \frac{\partial N_2^e}{\partial x} & 0 & \frac{\partial N_2^e}{\partial y} \\ 0 & \frac{\partial N_2^e}{\partial x} & \frac{\partial N_2^e}{\partial x} \\ \frac{\partial N_3^e}{\partial x} & 0 & \frac{\partial N_3^e}{\partial y} \\ 0 & \frac{\partial N_3^e}{\partial x} & \frac{\partial N_3^e}{\partial x} \end{pmatrix} \begin{pmatrix} \lambda + 2\mu & \lambda & 0 \\ \lambda & \lambda + 2\mu & 0 \\ 0 & 0 & \mu \end{pmatrix} \begin{pmatrix} \frac{\partial N_1^e}{\partial x} & 0 & \frac{\partial N_2^e}{\partial x} & 0 & \frac{\partial N_3^e}{\partial x} & 0 \\ 0 & \frac{\partial N_1^e}{\partial x} & 0 & \frac{\partial N_2^e}{\partial x} & 0 & \frac{\partial N_3^e}{\partial x} \\ \frac{\partial N_1^e}{\partial y} & \frac{\partial N_1^e}{\partial x} & \frac{\partial N_2^e}{\partial y} & \frac{\partial N_2^e}{\partial x} & \frac{\partial N_3^e}{\partial y} & \frac{\partial N_3^e}{\partial x} \end{pmatrix} \mathrm{d}V$$

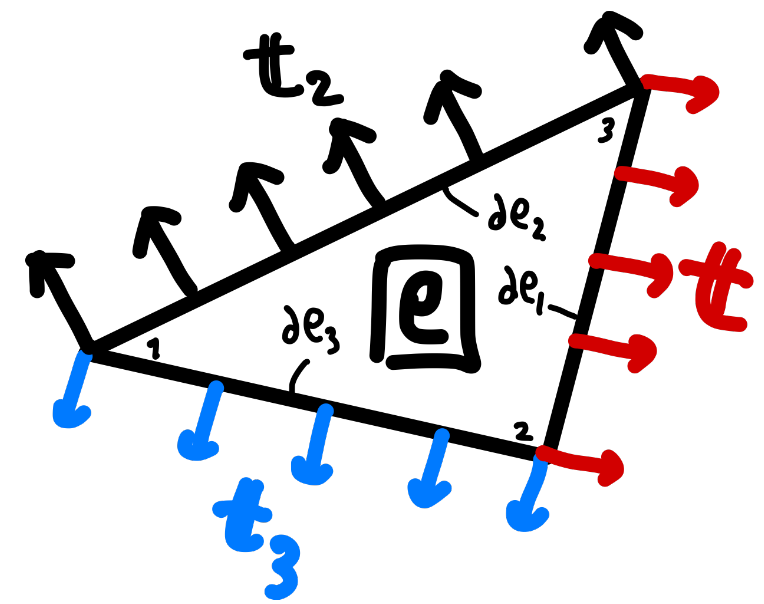
Since $\dfrac{\partial N_i^e}{\partial x}$ and $\dfrac{\partial N_i^e}{\partial y}$ are constant functions, the integral can be evaluated analytically.

On the computation of $\boldsymbol{f}^e$ 1/2

If $\boldsymbol{t}_i$ is constant on the edge ∂e_i (otherwise finer mesh should be introduced), $\boldsymbol{f}^e$ is evaluated as

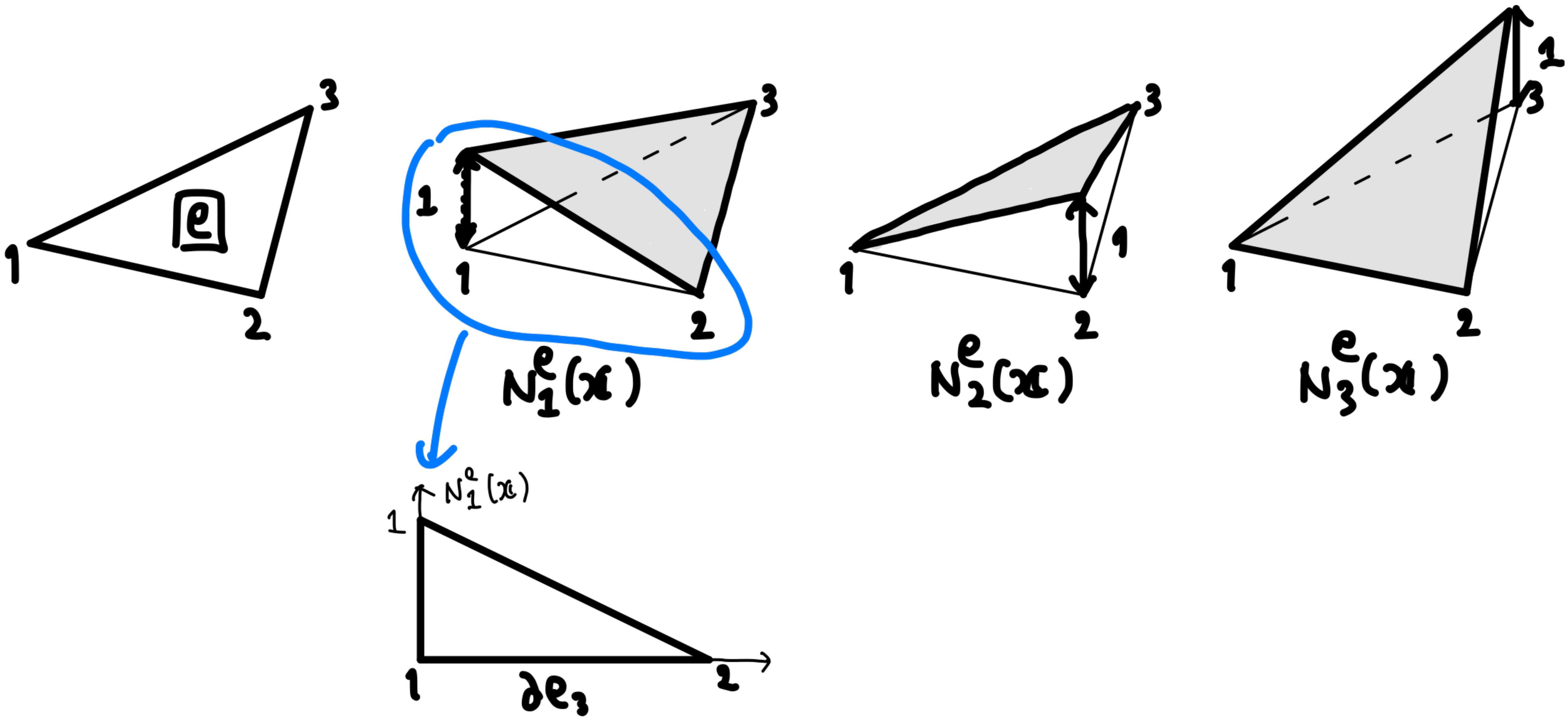
$$\boldsymbol{f}^e = \sum_{i=1}^{3} \int_{\partial e_i} \mathsf{N}^{et} \boldsymbol{t}_i \mathrm{d}S$$

$$\simeq \sum_{i=1}^{3} \int_{\partial e_i} \mathsf{N}^{et} \mathrm{d}S \; \boldsymbol{t}_i = \sum_{i=1}^{3} \int_{\partial e_i} \begin{pmatrix} N_1^e & 0 \\ 0 & N_1^e \\ N_2^e & 0 \\ 0 & N_2^e \\ N_3^e & 0 \\ 0 & N_3^e \end{pmatrix} \mathrm{d}S \; \boldsymbol{t}_i$$



How to compute $\int_{\partial e_i} N_j^e \mathrm{d}S$? → Analytically.

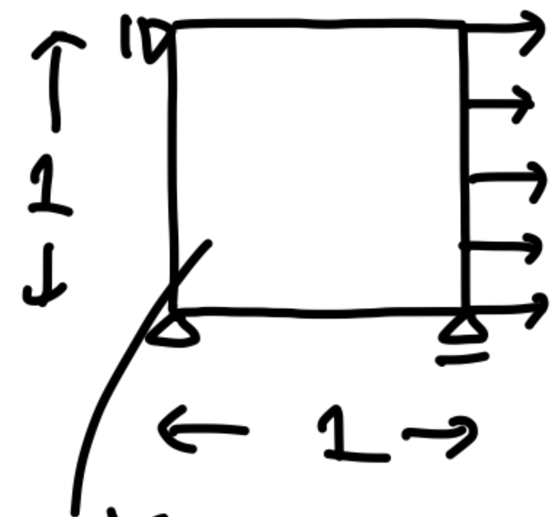
On the computation of f^e 2/2



$\int_{\partial e_3} N_1^e \mathrm{d}S$ is, for example, nothing but the area of the above triangle which is equal to a half of the length of ∂e_3.

Example 1/9

Let us consider a uniaxial tension problem. (with plane-stress assumption)

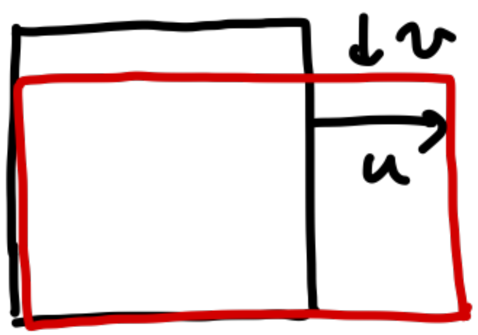


$\mathbf{t} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

Young's modulus $E = 1$

Poisson's ratio $\nu = 0.3$

The analytical sol. for this problem is obtained as



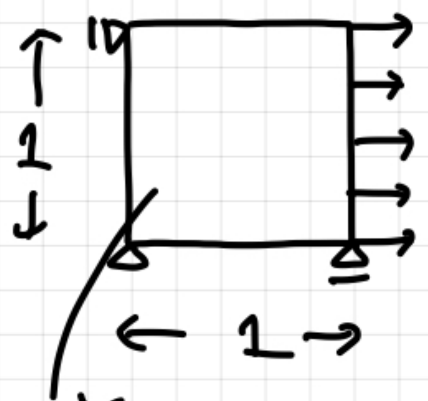
$$u = 1 \cdot \frac{t_1}{E} = 1$$

↑ "width" of the square

$$v = \nu u = 0.3$$

Let's solve this problem by FEM.

Example 2/9

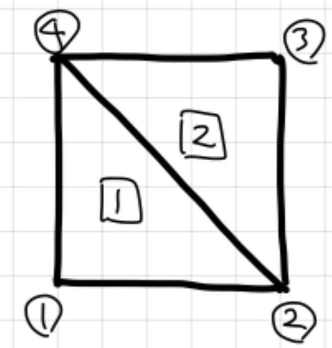


$\mathbf{t} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

Young's modulus $E = 1$

Poisson's ratio $\nu = 0.3$

divide into triangular patches $\longrightarrow$



- nodal coordinate

ind.	x	y
1	0	0
2	1	0
3	1	1
4	0	1

Connectivity

ind	1, 2, 3
$\boxed{1}$	1 2 4
$\boxed{2}$	2 3 4

Lame's consts.

$$\lambda = \frac{E\nu}{1-\nu^2} = \frac{30}{91}$$

$$\mu = \frac{5}{13}$$

$$\rightarrow \quad D = \begin{pmatrix} \lambda+2\mu & \lambda & 0 \\ \lambda & \lambda+2\mu & 0 \\ 0 & 0 & \mu \end{pmatrix}$$

$$= \begin{pmatrix} \frac{100}{91} & \frac{30}{91} & 0 \\ \frac{30}{91} & \frac{100}{91} & 0 \\ 0 & 0 & \frac{5}{13} \end{pmatrix}$$

Example 3/9

B-matrices

$$B^e = \begin{pmatrix} \frac{\partial N_1}{\partial x} & 0 & \frac{\partial N_2}{\partial x} & 0 & \frac{\partial N_3}{\partial x} & 0 \\ 0 & \frac{\partial N_1}{\partial y} & 0 & \frac{\partial N_2}{\partial y} & 0 & \frac{\partial N_3}{\partial y} \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial y} & \frac{\partial N_2}{\partial x} & \frac{\partial N_3}{\partial y} & \frac{\partial N_3}{\partial x} \end{pmatrix}$$

$$B^{\boxed{1}} = \begin{bmatrix} -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \\ -1 & -1 & 0 & 1 & 1 & 0 \end{bmatrix}$$

$$B^{\boxed{2}} = \begin{bmatrix} 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 1 & 0 & -1 \end{bmatrix}$$

Example 4/9

ind	1, 2, 3
$\boxed{1}$	1 2 4
$\boxed{2}$	2 3 4

K-matrices

$$K^{\boxed{i}} = \int_{\boxed{i}} \left(B^{\boxed{i}}\right)^t D\, B^{\boxed{i}}\, dV$$

$$= B^{\boxed{i}\,t} D\, B^{\boxed{i}} \times \frac{1}{2}$$

area of $\boxed{i}$

$$K^{\boxed{1}} = \begin{array}{c c}
\begin{array}{cccccc} \overbrace{\quad\quad\quad\quad\quad}^{1} & & \overbrace{\quad\quad\quad\quad\quad}^{2} & & \overbrace{\quad\quad\quad\quad\quad}^{4} & \end{array} & \\
\left[\begin{array}{cccccc}
\frac{135}{182} & \frac{5}{14} & -\frac{50}{91} & -\frac{5}{26} & -\frac{5}{26} & -\frac{15}{91} \\
\frac{5}{14} & \frac{135}{182} & -\frac{15}{91} & -\frac{5}{26} & -\frac{5}{26} & -\frac{50}{91} \\
-\frac{50}{91} & -\frac{15}{91} & \frac{50}{91} & 0 & 0 & \frac{15}{91} \\
-\frac{5}{26} & -\frac{5}{26} & 0 & \frac{5}{26} & \frac{5}{26} & 0 \\
-\frac{5}{26} & -\frac{5}{26} & 0 & \frac{5}{26} & \frac{5}{26} & 0 \\
-\frac{15}{91} & -\frac{50}{91} & \frac{15}{91} & 0 & 0 & \frac{50}{91}
\end{array}\right] & \begin{array}{c} \left.\begin{array}{c}\\ \\ \end{array}\right\} 1 \\ \left.\begin{array}{c}\\ \\ \end{array}\right\} 2 \\ \left.\begin{array}{c}\\ \\ \end{array}\right\} 4 \end{array}
\end{array}$$

$$K^{\boxed{2}} = \begin{array}{c c}
\begin{array}{cccccc} \overbrace{\quad\quad\quad\quad\quad}^{2} & & \overbrace{\quad\quad\quad\quad\quad}^{3} & & \overbrace{\quad\quad\quad\quad\quad}^{4} & \end{array} & \\
\left[\begin{array}{cccccc}
\frac{5}{26} & 0 & -\frac{5}{26} & -\frac{5}{26} & 0 & \frac{5}{26} \\
0 & \frac{50}{91} & -\frac{15}{91} & -\frac{50}{91} & \frac{15}{91} & 0 \\
-\frac{5}{26} & -\frac{15}{91} & \frac{135}{182} & \frac{5}{14} & -\frac{50}{91} & -\frac{5}{26} \\
-\frac{5}{26} & -\frac{50}{91} & \frac{5}{14} & \frac{135}{182} & -\frac{15}{91} & -\frac{5}{26} \\
0 & \frac{15}{91} & -\frac{50}{91} & -\frac{15}{91} & \frac{50}{91} & 0 \\
\frac{5}{26} & 0 & -\frac{5}{26} & -\frac{5}{26} & 0 & \frac{5}{26}
\end{array}\right] & \begin{array}{c} \left.\begin{array}{c}\\ \\ \end{array}\right\} 2 \\ \left.\begin{array}{c}\\ \\ \end{array}\right\} 3 \\ \left.\begin{array}{c}\\ \\ \end{array}\right\} 4 \end{array}
\end{array}$$

Example 5/9

$$K^{(1)} = \begin{array}{cc}
\begin{array}{cccccc} \multicolumn{2}{c}{1} & \multicolumn{2}{c}{2} & \multicolumn{2}{c}{4} \end{array} & \\
\begin{bmatrix}
\frac{135}{182} & \frac{5}{14} & -\frac{50}{91} & -\frac{5}{26} & -\frac{5}{26} & -\frac{15}{91} \\
\frac{5}{14} & \frac{135}{182} & -\frac{15}{91} & -\frac{5}{26} & -\frac{5}{26} & -\frac{50}{91} \\
-\frac{50}{91} & -\frac{15}{91} & \frac{50}{91} & 0 & 0 & \frac{15}{91} \\
-\frac{5}{26} & -\frac{5}{26} & 0 & \frac{5}{26} & \frac{5}{26} & 0 \\
-\frac{5}{26} & -\frac{5}{26} & 0 & \frac{5}{26} & \frac{5}{26} & 0 \\
-\frac{15}{91} & -\frac{50}{91} & \frac{15}{91} & 0 & 0 & \frac{50}{91}
\end{bmatrix} & \begin{array}{c} 1 \\ \\ 2 \\ \\ 4 \\ \\ \end{array}
\end{array}$$

$$K^{(2)} = \begin{array}{cc}
\begin{array}{cccccc} \multicolumn{2}{c}{2} & \multicolumn{2}{c}{3} & \multicolumn{2}{c}{4} \end{array} & \\
\begin{bmatrix}
\frac{5}{26} & 0 & -\frac{5}{26} & -\frac{5}{26} & 0 & \frac{5}{26} \\
0 & \frac{50}{91} & -\frac{15}{91} & -\frac{50}{91} & \frac{15}{91} & 0 \\
-\frac{5}{26} & -\frac{15}{91} & \frac{135}{182} & \frac{5}{14} & -\frac{50}{91} & -\frac{5}{26} \\
-\frac{5}{26} & -\frac{50}{91} & \frac{5}{14} & \frac{135}{182} & -\frac{15}{91} & -\frac{5}{26} \\
0 & \frac{15}{91} & -\frac{50}{91} & -\frac{15}{91} & \frac{50}{91} & 0 \\
\frac{5}{26} & 0 & -\frac{5}{26} & -\frac{5}{26} & 0 & \frac{5}{26}
\end{bmatrix} & \begin{array}{c} 2 \\ \\ 3 \\ \\ 4 \\ \\ \end{array}
\end{array}$$

global stiffness matrix

$$K = \begin{array}{cc}
\begin{array}{cccccccc} \multicolumn{2}{c}{1} & \multicolumn{2}{c}{2} & \multicolumn{2}{c}{3} & \multicolumn{2}{c}{4} \end{array} & \\
\begin{bmatrix}
\frac{135}{182} & \frac{5}{14} & -\frac{50}{91} & -\frac{5}{26} & 0 & 0 & -\frac{5}{26} & -\frac{15}{91} \\
\frac{5}{14} & \frac{135}{182} & -\frac{15}{91} & -\frac{5}{26} & 0 & 0 & -\frac{5}{26} & -\frac{50}{91} \\
-\frac{50}{91} & -\frac{15}{91} & \frac{50}{91} & 0 & 0 & 0 & 0 & \frac{15}{91} \\
-\frac{5}{26} & -\frac{5}{26} & 0 & \frac{5}{26} & 0 & 0 & \frac{5}{26} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{5}{26} & -\frac{5}{26} & 0 & \frac{5}{26} & 0 & 0 & \frac{5}{26} & 0 \\
-\frac{15}{91} & -\frac{50}{91} & \frac{15}{91} & 0 & 0 & 0 & 0 & \frac{50}{91}
\end{bmatrix} & \begin{array}{c} 1 \\ \\ 2 \\ \\ 3 \\ \\ 4 \\ \\ \end{array}
\end{array}
+
\begin{array}{cc}
\begin{array}{cccccccc} \multicolumn{2}{c}{1} & \multicolumn{2}{c}{2} & \multicolumn{2}{c}{3} & \multicolumn{2}{c}{4} \end{array} & \\
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{5}{26} & 0 & -\frac{5}{26} & -\frac{5}{26} & 0 & \frac{5}{26} \\
0 & 0 & 0 & \frac{50}{91} & -\frac{15}{91} & -\frac{50}{91} & \frac{15}{91} & 0 \\
0 & 0 & -\frac{5}{26} & -\frac{15}{91} & \frac{135}{182} & \frac{5}{14} & -\frac{50}{91} & -\frac{5}{26} \\
0 & 0 & -\frac{5}{26} & -\frac{50}{91} & \frac{5}{14} & \frac{135}{182} & -\frac{15}{91} & -\frac{5}{26} \\
0 & 0 & 0 & \frac{15}{91} & -\frac{50}{91} & -\frac{15}{91} & \frac{50}{91} & 0 \\
0 & 0 & \frac{5}{26} & 0 & -\frac{5}{26} & -\frac{5}{26} & 0 & \frac{5}{26}
\end{bmatrix} & \begin{array}{c} 1 \\ \\ 2 \\ \\ 3 \\ \\ 4 \\ \\ \end{array}
\end{array}$$

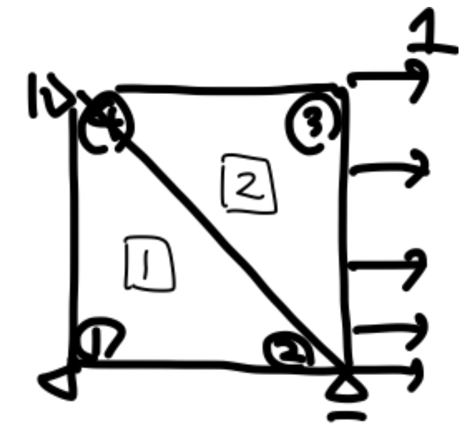
global stiffness matrix

$$
K = \begin{array}{c}
\begin{array}{cccccccc} \multicolumn{2}{c}{\textcircled{1}} & \multicolumn{2}{c}{\textcircled{2}} & \multicolumn{2}{c}{\textcircled{3}} & \multicolumn{2}{c}{\textcircled{4}} \end{array} \\
\left[\begin{array}{cccccccc}
\frac{135}{182} & \frac{5}{14} & -\frac{50}{91} & -\frac{5}{26} & 0 & 0 & -\frac{5}{26} & -\frac{15}{91} \\
\frac{5}{14} & \frac{135}{182} & -\frac{15}{91} & -\frac{5}{26} & 0 & 0 & -\frac{5}{26} & -\frac{50}{91} \\
-\frac{50}{91} & -\frac{15}{91} & \frac{50}{91} & 0 & 0 & 0 & 0 & \frac{15}{91} \\
-\frac{5}{26} & -\frac{5}{26} & 0 & \frac{5}{26} & 0 & 0 & \frac{5}{26} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{5}{26} & -\frac{5}{26} & 0 & \frac{5}{26} & 0 & 0 & \frac{5}{26} & 0 \\
-\frac{15}{91} & -\frac{50}{91} & \frac{15}{91} & 0 & 0 & 0 & 0 & \frac{50}{91}
\end{array}\right]
\end{array}
\begin{array}{l} \\ \textcircled{1} \\ \\ \textcircled{2} \\ \\ \textcircled{3} \\ \\ \textcircled{4} \\ \\ \end{array}
+
\begin{array}{c}
\begin{array}{cccccccc} \multicolumn{2}{c}{\textcircled{1}} & \multicolumn{2}{c}{\textcircled{2}} & \multicolumn{2}{c}{\textcircled{3}} & \multicolumn{2}{c}{\textcircled{4}} \end{array} \\
\left[\begin{array}{cccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{5}{26} & 0 & -\frac{5}{26} & -\frac{5}{26} & 0 & \frac{5}{26} \\
0 & 0 & 0 & \frac{50}{91} & -\frac{15}{91} & -\frac{50}{91} & \frac{15}{91} & 0 \\
0 & 0 & -\frac{5}{26} & -\frac{15}{91} & \frac{135}{182} & \frac{5}{14} & -\frac{50}{91} & -\frac{5}{26} \\
0 & 0 & -\frac{5}{26} & -\frac{50}{91} & \frac{5}{14} & \frac{135}{182} & -\frac{15}{91} & -\frac{5}{26} \\
0 & 0 & 0 & \frac{15}{91} & -\frac{50}{91} & -\frac{15}{91} & \frac{50}{91} & 0 \\
0 & 0 & \frac{5}{26} & 0 & -\frac{5}{26} & -\frac{5}{26} & 0 & \frac{5}{26}
\end{array}\right]
\end{array}
\begin{array}{l} \\ \textcircled{1} \\ \\ \textcircled{2} \\ \\ \textcircled{3} \\ \\ \textcircled{4} \\ \\ \end{array}
$$

$$
= \left[\begin{array}{cccccccc}
\frac{135}{182} & \frac{5}{14} & -\frac{50}{91} & -\frac{5}{26} & 0 & 0 & -\frac{5}{26} & -\frac{15}{91} \\
\frac{5}{14} & \frac{135}{182} & -\frac{15}{91} & -\frac{5}{26} & 0 & 0 & -\frac{5}{26} & -\frac{50}{91} \\
-\frac{50}{91} & -\frac{15}{91} & \frac{135}{182} & 0 & -\frac{5}{26} & -\frac{5}{26} & 0 & \frac{5}{14} \\
-\frac{5}{26} & -\frac{5}{26} & 0 & \frac{135}{182} & -\frac{15}{91} & -\frac{50}{91} & \frac{5}{14} & 0 \\
0 & 0 & -\frac{5}{26} & -\frac{15}{91} & \frac{135}{182} & \frac{5}{14} & -\frac{50}{91} & -\frac{5}{26} \\
0 & 0 & -\frac{5}{26} & -\frac{50}{91} & \frac{5}{14} & \frac{135}{182} & -\frac{15}{91} & -\frac{5}{26} \\
-\frac{5}{26} & -\frac{5}{26} & 0 & \frac{5}{14} & -\frac{50}{91} & -\frac{15}{91} & \frac{135}{182} & 0 \\
-\frac{15}{91} & -\frac{50}{91} & \frac{5}{14} & 0 & -\frac{5}{26} & -\frac{5}{26} & 0 & \frac{135}{182}
\end{array}\right]
$$

Example 7/9

global displacement vector :



$$\mathbf{d} = (\underset{\overset{\parallel}{0}}{u_1}, \underset{\overset{\parallel}{0}}{v_1}, u_2, \underset{\overset{\parallel}{0}}{v_2}, u_3, v_3, \underset{\overset{\parallel}{0}}{u_4}, v_4)^t$$

global load vector

traction free .

$$\mathbf{f} = (?, ?, \tfrac{1}{2}, ?, \tfrac{1}{2}, 0, ?, 0)^t$$

Reaction force
(unknown)

Example 8/9

Thus, we have $K d = f$ as

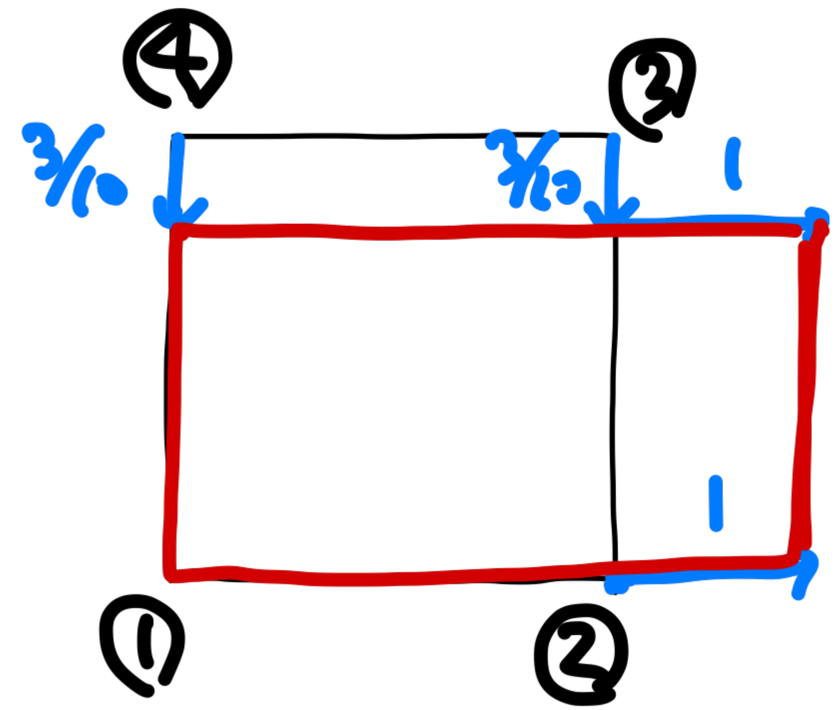
$$\begin{bmatrix} \frac{135}{182} & \frac{5}{14} & -\frac{50}{91} & -\frac{5}{26} & 0 & 0 & -\frac{5}{26} & -\frac{15}{91} \\ \frac{5}{14} & \frac{135}{182} & -\frac{15}{91} & -\frac{5}{26} & 0 & 0 & -\frac{5}{26} & -\frac{50}{91} \\ -\frac{50}{91} & -\frac{15}{91} & \frac{135}{182} & 0 & -\frac{5}{26} & -\frac{5}{26} & 0 & \frac{5}{14} \\ -\frac{5}{26} & -\frac{5}{26} & 0 & \frac{135}{182} & -\frac{15}{91} & -\frac{50}{91} & \frac{5}{14} & 0 \\ 0 & 0 & -\frac{5}{26} & -\frac{15}{91} & \frac{135}{182} & \frac{5}{14} & -\frac{50}{91} & -\frac{5}{26} \\ 0 & 0 & -\frac{5}{26} & -\frac{50}{91} & \frac{5}{14} & \frac{135}{182} & -\frac{15}{91} & -\frac{5}{26} \\ -\frac{5}{26} & -\frac{5}{26} & 0 & \frac{5}{14} & -\frac{50}{91} & -\frac{15}{91} & \frac{135}{182} & 0 \\ -\frac{15}{91} & -\frac{50}{91} & \frac{5}{14} & 0 & -\frac{5}{26} & -\frac{5}{26} & 0 & \frac{135}{182} \end{bmatrix} \begin{pmatrix} 0 \\ 0 \\ u_2 \\ 0 \\ u_3 \\ v_3 \\ 0 \\ v_4 \end{pmatrix} = \begin{pmatrix} ? \\ ? \\ \frac{1}{2} \\ ? \\ \frac{1}{2} \\ 0 \\ ? \\ 0 \end{pmatrix} \quad \cdots \mathbf{A}$$

which can be shrinked as

$$\begin{bmatrix} \frac{135}{182} & -\frac{5}{26} & -\frac{5}{26} & \frac{5}{14} \\ -\frac{5}{26} & \frac{135}{182} & \frac{5}{14} & -\frac{5}{26} \\ -\frac{5}{26} & \frac{5}{14} & \frac{135}{182} & -\frac{5}{26} \\ \frac{5}{14} & -\frac{5}{26} & -\frac{5}{26} & \frac{135}{182} \end{bmatrix} \begin{pmatrix} u_2 \\ u_3 \\ v_3 \\ v_4 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} u_2 \\ u_3 \\ v_3 \\ v_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -3/10 \\ -3/10 \end{pmatrix}$$

Example 9/9

$$\begin{pmatrix} u_2 \\ u_3 \\ v_3 \\ v_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -3/10 \\ -3/10 \end{pmatrix}$$



Consistent with the analytical sol.!

Note: The reaction force can be obtained by (A).

Strain and stress in elements can also be computed.