計算固体力学入門 (6)

Introduction to Computational Solid Mechanics (6)

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Today's topic

The contents of this course will be as follows:

1. FEM for a one-dimensional problem.

2. Mathematical preliminaries
   - Gauss-Legendre quadrature
   - Einstein's summation convention

3. Continuum mechanics
   - Deformation of continuum
   - Balance of continuum
   - Basic equations

4. **Weak form**

5. Discretisation

6. FEM implementations

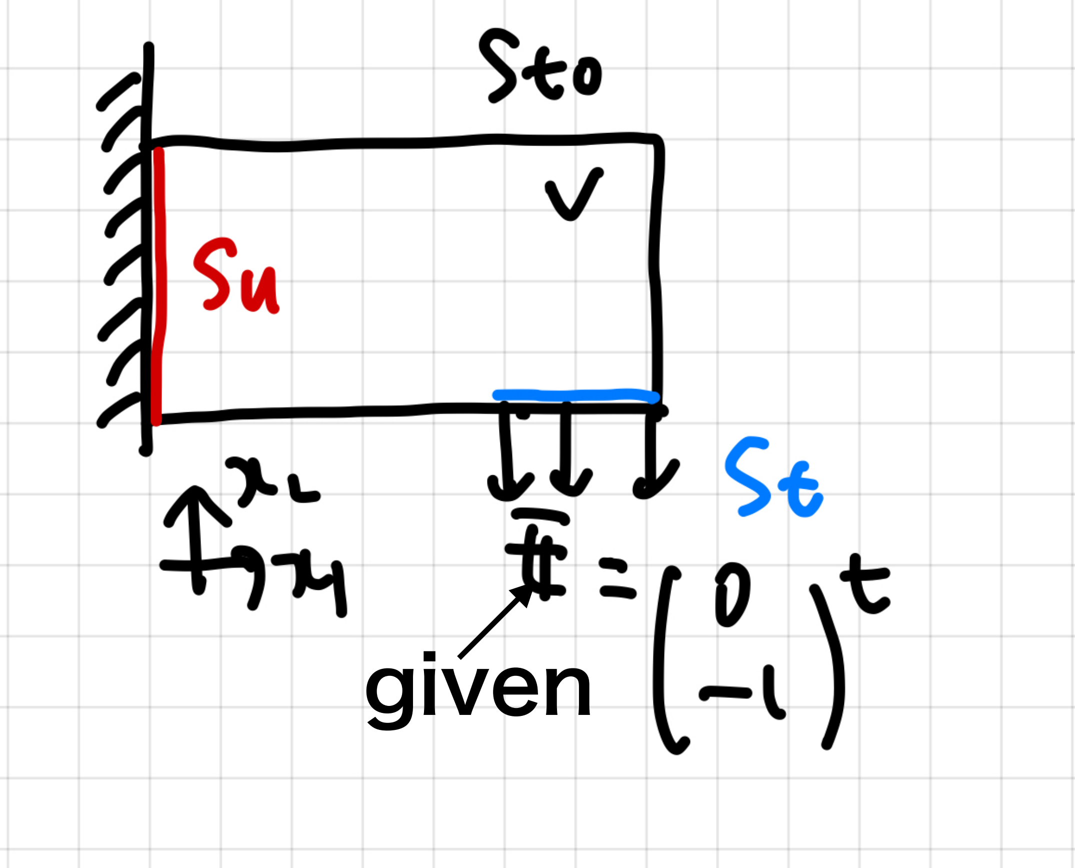
Boundary value problem

In the previous lecture, we studied the basic equations for elasticity in two dimensions.

We henceforth discuss a two-dimensional object modelled as a linear isotropic elastic material in plain-strain state, i.e. the constitutive equation is given as

$$\sigma_{ij} = \lambda \delta_{ij} \varepsilon_{kk} + 2\mu \varepsilon_{ij}\,.$$

Let us consider the following boundary value problem:



$$\sigma_{ji,j}(\boldsymbol{x}) = 0 \quad \boldsymbol{x} \in V, \qquad \cdots (1)$$

$$u_i(\boldsymbol{x}) = 0 \quad \boldsymbol{x} \in S_u, \qquad \cdots (2)$$

$$t_i(\boldsymbol{x}) = \begin{cases} 0 & \boldsymbol{x} \in S_{t0} \qquad \cdots (3) \\ \bar{t}_i & \boldsymbol{x} \in S_t \qquad \cdots (4) \end{cases}$$

where $t_i(\boldsymbol{x}) = \sigma_{ji}(\boldsymbol{x}) n_j(\boldsymbol{x})$ is the traction.

Note: indices range from 1 to 2.

Navier's equation

Let us find the displacement $\boldsymbol{u}(\boldsymbol{x})$ solving the BVP (1)-(4).

Quiz: rewrite the governing eq. (1) in terms of $u_i(\boldsymbol{x})$.

The governing equation (1) is the 2nd order DE w.r.t $u_i(\boldsymbol{x})$.

→ Let us use the FEM to solve it!

An FEM for 2D elasticity

To solve the BVP (1)-(4) by the finite element method, we start from the weighted residual equation with the (vector-valued) test function $\tilde{\boldsymbol{u}}(\boldsymbol{x})$:

$$
\begin{aligned}
0 &= \int_V \tilde{u}_i \sigma_{ji,j} \mathrm{d}V \\
&= \int_V (\tilde{u}_i \sigma_{ji})_{,j} \mathrm{d}V - \int_V \tilde{u}_{i,j} \sigma_{ji} \mathrm{d}V \\
&= \int_{S_u \cup S_t \cup S_{t0}} \tilde{u}_i \underbrace{\boxed{\sigma_{ji} n_j}}_{= t_i\text{: traction}} \mathrm{d}S - \int_V \tilde{u}_{i,j} \sigma_{ji} \mathrm{d}V \qquad \cdots (5)
\end{aligned}
$$

Let us suppose that $\tilde{\boldsymbol{u}} = \boldsymbol{0}$ on S_u, then the above (5) becomes

Weak form

$$
\int_V \tilde{u}_{i,j} \sigma_{ji} \mathrm{d}V = \int_{S_t} \tilde{u}_i \bar{t}_i \mathrm{d}S \qquad \cdots (6)
$$

The Voigt representation

The weak form (6) is slightly rewritten as

$$\int_V \tilde{\varepsilon}_{ij}\sigma_{ji}\mathrm{d}V = \int_{S_t} \tilde{u}_i t_i \mathrm{d}S \qquad \cdots (7)$$

where $\tilde{\varepsilon}_{ij} = \dfrac{1}{2}(\tilde{u}_{i,j} + \tilde{u}_{j,i})$ is the “strain” corresponding to the test function.

Let us here introduce the Voigt representation

$$\begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{pmatrix} = \begin{pmatrix} \lambda + 2\mu & \lambda & 0 \\ \lambda & \lambda + 2\mu & 0 \\ 0 & 0 & \mu \end{pmatrix} \begin{pmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{pmatrix} \qquad \cdots (8)$$

for the constitutive equation, where $\gamma_{12} = 2\varepsilon_{12}$ is the engineering strain (工学ひずみ) to see $\tilde{\varepsilon}_{ij}\sigma_{ji}$ is expanded as

The Galerkin method 1/5

Thus, the weak form becomes as

$$\int_V (\tilde{\varepsilon}_{11},\ \tilde{\varepsilon}_{22},\ \tilde{\gamma}_{12}) \begin{pmatrix} \lambda + 2\mu & \lambda & 0 \\ \lambda & \lambda + 2\mu & 0 \\ 0 & 0 & \mu \end{pmatrix} \begin{pmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{pmatrix} \mathrm{d}V = \int_{S_t} (\tilde{u}_1 \bar{t}_1 + \tilde{u}_1 \bar{t}_2) \mathrm{d}S \quad \cdots \text{(8)}$$

As in the case of 1D problem, we discretise the weak form by the Galerkin method. First, we expand the displacement

$$\boldsymbol{u}(\boldsymbol{x}) \simeq \sum_{\ell=1}^{N} \boldsymbol{a}^{\ell} N^{\ell}(\boldsymbol{x}), \quad \cdots \text{(9)}$$

where $N^{\ell}(\boldsymbol{x})$ (for $i = 1, \cdots, N$) are the basis functions, and $\boldsymbol{a}^{\ell} \in \mathbb{R}^2$ are the (vector-valued) weights.

The Galerkin method 2/5

(9) $\boldsymbol{u}(\boldsymbol{x}) \simeq \sum_{\ell=1}^{N} \boldsymbol{a}^{\ell} N^{\ell}(\boldsymbol{x}) \Rightarrow$

$$\varepsilon_{11}(\boldsymbol{x}) \simeq \sum_{\ell}^{N} a_1^{\ell} N_{,1}^{\ell}(\boldsymbol{x}) \quad \cdots(10)$$

$$\varepsilon_{22}(\boldsymbol{x}) \simeq \sum_{\ell}^{N} a_2^{\ell} N_{,2}^{\ell}(\boldsymbol{x}) \quad \cdots(11)$$

$$\gamma_{12}(\boldsymbol{x}) \simeq \sum_{\ell}^{N} \left(a_1^{\ell} N_{,2}^{\ell}(\boldsymbol{x}) + a_2^{\ell} N_{,1}^{\ell}(\boldsymbol{x}) \right) \quad \cdots(12)$$

We then choose $\tilde{u}_i(\boldsymbol{x}) = \delta_{i1} N^m(\boldsymbol{x})$ as the test function. The corresponding strains are written as

$$\tilde{\varepsilon}_{11}(\boldsymbol{x}) = N_{,1}^{m}(\boldsymbol{x}), \quad \cdots(13)$$

$$\tilde{\varepsilon}_{22}(\boldsymbol{x}) \simeq 0, \quad \cdots(14)$$

$$\tilde{\gamma}_{12}(\boldsymbol{x}) \simeq N_{,2}^{m}(\boldsymbol{x}) \quad \cdots(15)$$

The Galerkin method 3/5

We then substitute (10)-(15) into the weak form (8) to discretise it into

$$(8) \Rightarrow \int_V (N_{,1}\,,\ 0,\ N_{,2}) \begin{pmatrix} \lambda + 2\mu & \lambda & 0 \\ \lambda & \lambda + 2\mu & 0 \\ 0 & 0 & \mu \end{pmatrix} \begin{pmatrix} \sum_{\ell=1}^{N} a_1^\ell N_{,1}^\ell \\ \sum_{\ell=1}^{N} a_2^\ell N_{,2}^\ell \\ \sum_{\ell=1}^{N} (a_1^\ell N_{,2}^\ell + a_2^\ell N_{,1}^\ell) \end{pmatrix} \mathrm{d}V = \int_{S_t} N^m \bar{t}_1 \mathrm{d}S$$

$$\Leftrightarrow \sum_{\ell=1}^{N} \int_V (N_{,1}\,,\ 0,\ N_{,2}) \begin{pmatrix} (\lambda + 2\mu) N_{,1}^\ell & \lambda N_{,2}^\ell \\ \lambda N_{,1}^\ell & (\lambda + 2\mu) N_{,2}^\ell \\ \mu N_{,2}^\ell & \mu N_{,1}^\ell \end{pmatrix} \begin{pmatrix} a_1^\ell \\ a_2^\ell \end{pmatrix} \mathrm{d}V = \int_{S_t} N^m \bar{t}_1 \mathrm{d}S$$

$$\Leftrightarrow \sum_{\ell=1}^{N} \int_V ((\lambda + 2\mu) N_{,1}^m N_{,1}^\ell + \mu N_{,2}^m N_{,2}^\ell\,,\ \lambda N_{,1}^m N_{,2}^\ell + \mu N_{,2}^m N_{,1}^\ell) \begin{pmatrix} a_1^\ell \\ a_2^\ell \end{pmatrix} \mathrm{d}V = \int_{S_t} N^m \bar{t}_1 \mathrm{d}S$$

$\cdots$(16)

The Galerkin method 4/5

Quiz: repeat the same procedure with the test function $v_i(\boldsymbol{x}) = \delta_{i2} N^m(\boldsymbol{x})$ to see that (8) is discretised as

$$\sum_{\ell=1}^{N} \int_V (\lambda N_{,2}^m N_{,1}^\ell + \mu N_{,1}^m N_{,2}^\ell,\ \mu N_{,1}^m N_{,1}^\ell + (\lambda + 2\mu) N_{,2}^m N_{,2}^\ell) \begin{pmatrix} a_1^\ell \\ a_2^\ell \end{pmatrix} \mathrm{d}V = \int_{S_t} N^m \bar{t}_2 \mathrm{d}S \quad \cdots (17)$$

The Galerkin method 5/5

By combining the derived equations

$$\sum_{\ell=1}^{N}\int_V \left((\lambda+2\mu)N_{,1}^{m}N_{,1}^{\ell}+\mu N_{,2}^{m}N_{,2}^{\ell},\ \lambda N_{,1}^{m}N_{,2}^{\ell}+\mu N_{,2}^{m}N_{,1}^{\ell}\right)\begin{pmatrix}a_1^{\ell}\\ a_2^{\ell}\end{pmatrix}\mathrm{d}V=\int_{S_t}N^m\bar{t}_1\mathrm{d}S \quad \cdots(16)$$

$$\sum_{\ell=1}^{N}\int_V \left(\lambda N_{,2}^{m}N_{,1}^{\ell}+\mu N_{,1}^{m}N_{,2}^{\ell},\ \mu N_{,1}^{m}N_{,1}^{\ell}+(\lambda+2\mu)N_{,2}^{m}N_{,2}^{\ell}\right)\begin{pmatrix}a_1^{\ell}\\ a_2^{\ell}\end{pmatrix}\mathrm{d}V=\int_{S_t}N^m\bar{t}_2\mathrm{d}S \quad \cdots(17)$$

We obtain the following algebraic equations:

$$\sum_{\ell=1}^{N}\begin{pmatrix}k_{11}^{m\ell} & k_{12}^{m\ell}\\ k_{21}^{m\ell} & k_{22}^{m\ell}\end{pmatrix}\begin{pmatrix}a_1^{\ell}\\ a_2^{\ell}\end{pmatrix}=\begin{pmatrix}q_1^{\ell}\\ q_2^{\ell}\end{pmatrix} \text{ for } m=1,\cdots,N$$

$$\Leftrightarrow \mathsf{K}\boldsymbol{a}=\boldsymbol{q} \quad \cdots(18)$$

where $\mathsf{K}\in\mathbb{R}^{2N\times 2N}$, $\boldsymbol{a}\in\mathbb{R}^{2N}$, and $\boldsymbol{q}\in\mathbb{R}^{2N}$ are defined as, for example, $a_{2i-1}=a_1^i$, $a_{2i}=a_2^i$. (18) are solved by a computer, and the sol. is substituted into (9) to obtain the approx. sol. for the BVP (1)-(4).