計算固体力学入門 (4)

Introduction to Computational Solid Mechanics (4)

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Today's topic

The contents of this course will be as follows:

1. FEM for a one-dimensional problem.
2. Mathematical preliminaries
   - Gauss-Legendre quadrature
   - Einstein's summation convention
3. Continuum mechanics
   - **Deformation of continuum**
   - Balance of continuum
   - Basic equations
4. Weak form
5. Discretization
6. FEM implementations

What is continuum?

Continuum (連続体)

models an object in motion (運動する) of finite size with deformation (変形).

(cf.) mass:
models an object in motion of infinitesimal size.

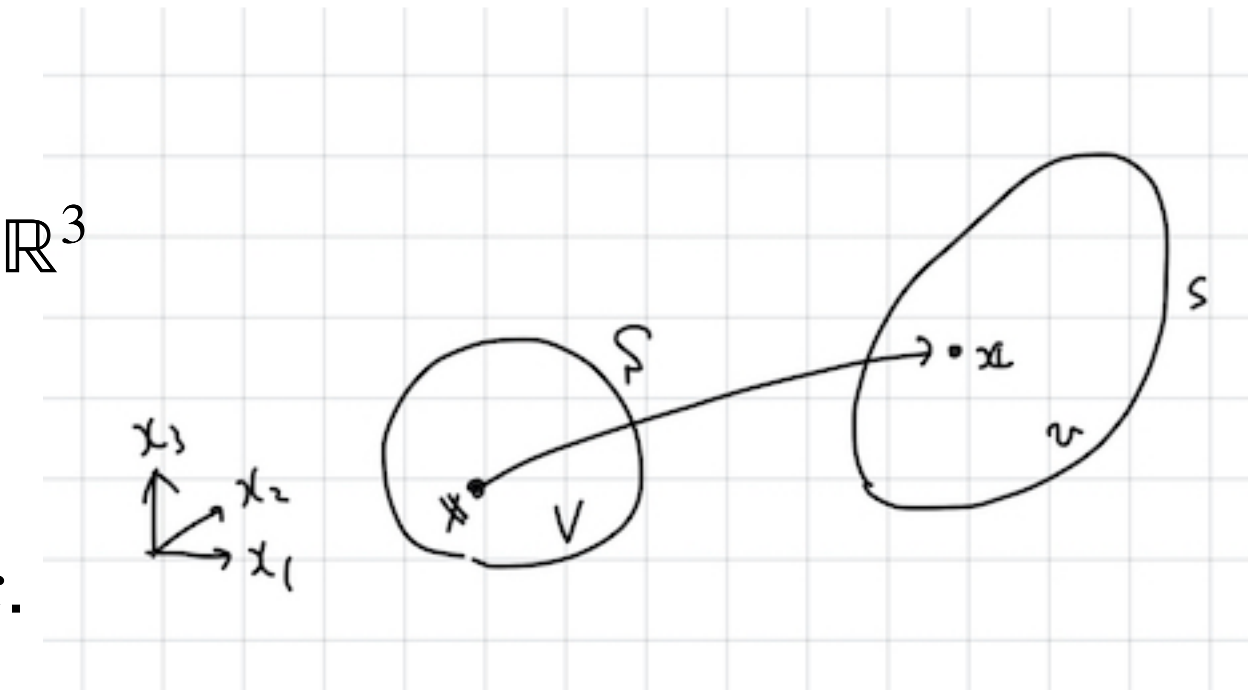
rigid body:
models an object in motion of finite size without deformation.

In this lecture, we define **strain** (ひずみ) which characterises deformation.

Note: Today, I would not distinguish second-order tensor with matrix. The rigorous definition for the tensor will be given in the next lecture.

Displacement

Let us consider an object V (with boundary S) of finite size placed in $\mathbb{R}^3$ with Cartesian coordinate (x_1, x_2, x_3), and a point $\boldsymbol{X}$ embedded in V. The object V deforms as $V,\ S,\ \boldsymbol{X} \rightarrow v,\ s,\ \boldsymbol{x}$.



displacement (変位) $\boldsymbol{u}$

$$\boldsymbol{u} := \boldsymbol{x} - \boldsymbol{X}$$

The displacement involves the information on the rigid-body translation as well as that on the deformation.

How can we extract the deformation from the displacement?

Assumption

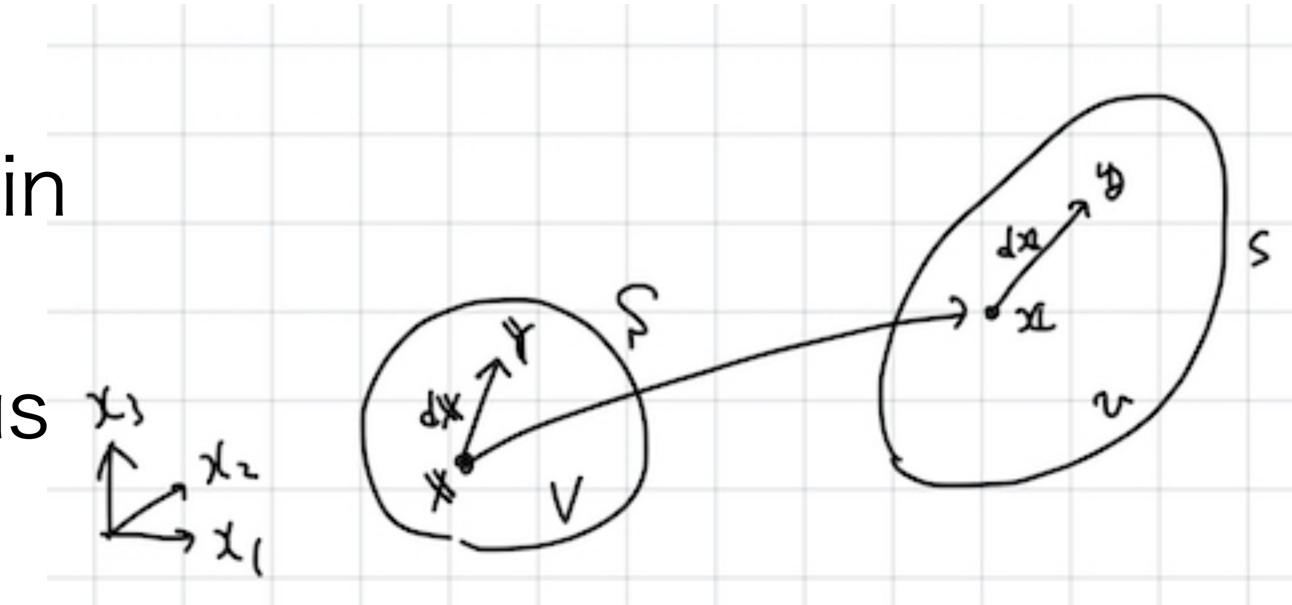
In the discussions to follow, we assume

- The map (写像) $\boldsymbol{X} \to \boldsymbol{x}$ is bijection (全単射).
  i.e. we can define the inverse $\boldsymbol{x} \to \boldsymbol{X}$.

- The map is sufficiently smooth.
  i.e. we can define $\mathrm{d}^{n_1+n_2+n_3}\boldsymbol{x}/\mathrm{d}X_1^{n_1}X_2^{n_2}X_3^{n_3}$ for arbitrary $n_1, n_2, n_3 \geq 0$.

Deformation gradient tensor 1/4

We consider a line element $\mathrm{d}\boldsymbol{X} := \boldsymbol{Y} - \boldsymbol{X}$ in V in stead of a point in V, and observe how the element is affected by the deformation. Let us denote the element after the deformation as $\mathrm{d}\boldsymbol{x} := \boldsymbol{y} - \boldsymbol{x}$.



The first order approximation of $\mathrm{d}\boldsymbol{x}$ is evaluated as

$$
\begin{aligned}
dx_i &= x_i(\boldsymbol{Y}) - x_i(\boldsymbol{X}) \\
&= x_i(\boldsymbol{X} + \mathrm{d}\boldsymbol{X}) - x_i(\boldsymbol{X}) \\
&\simeq \mathsf{F}_{ij}(\boldsymbol{X})\mathrm{d}X_j,
\end{aligned}
$$

where the deformation gradient tensor (変形勾配テンソル) $\mathsf{F}_{ij} := \dfrac{\partial x_i(\boldsymbol{X})}{\partial X_j}$ is introduced. Note: the Einstein rule is adopted!

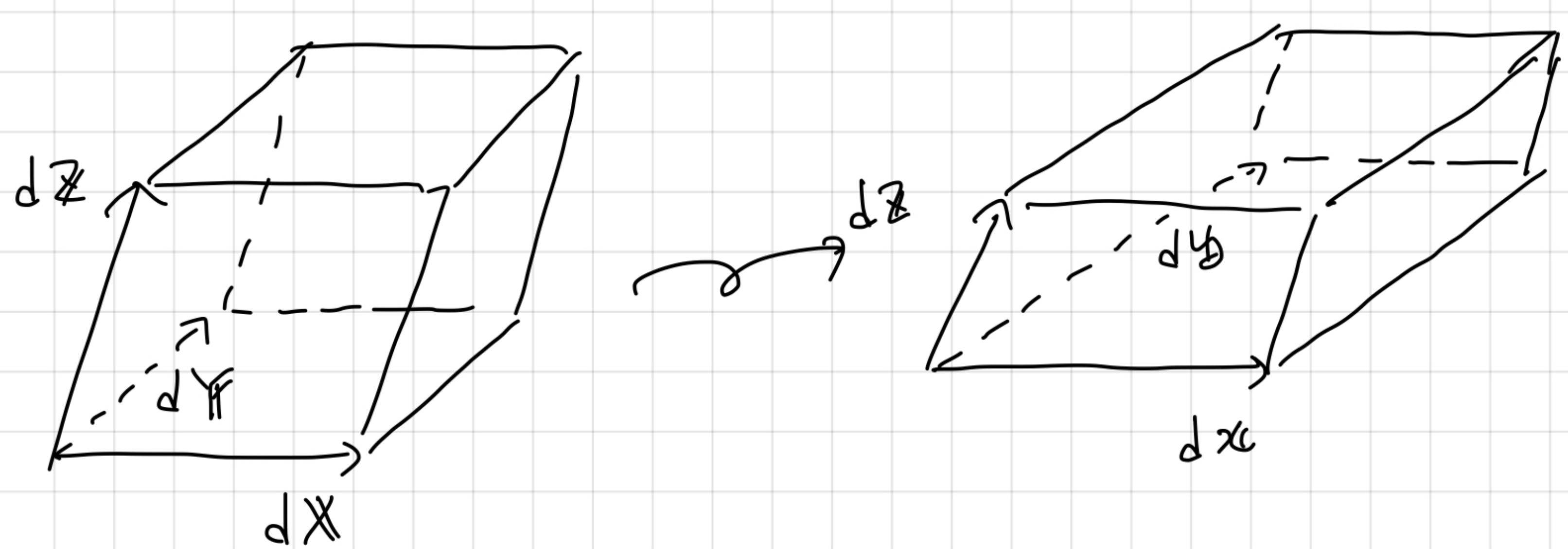
Deformation gradient tensor 2/4

The volume of parallelepiped (平行六面体) defined by three line segments $\mathrm{d}\boldsymbol{X}$, $\mathrm{d}\boldsymbol{Y}$, $\mathrm{d}\boldsymbol{Z}$

$$\mathrm{d}\boldsymbol{X} \cdot (\mathrm{d}\boldsymbol{Y} \times \mathrm{d}\boldsymbol{Z})$$

changes, after the deformation, as

$$\begin{aligned}\mathrm{d}\boldsymbol{x} \cdot (\mathrm{d}\boldsymbol{y} \times \mathrm{d}\boldsymbol{z}) &= \mathsf{F}\mathrm{d}\boldsymbol{X} \cdot (\mathsf{F}\mathrm{d}\boldsymbol{Y} \times \mathsf{F}\mathrm{d}\boldsymbol{Z}) \\ &= \det \mathsf{F} \left[\mathrm{d}\boldsymbol{X} \cdot (\mathrm{d}\boldsymbol{Y} \times \mathrm{d}\boldsymbol{Z})\right].\end{aligned}$$



Thus, the determinant of F represents the relation between the two parallelepipeds. → F should be positive definite.

Deformation gradient tensor 3/4

Quiz 1:
express F_{ij} in terms of the Kronecker's delta and the gradient of the displacement.

Quiz 2:
Let us consider that a single line segment deforms, and assume that the corresponding deformation gradient tensor is represented by an orthogonal matrix. Show that the length of the line segment does not change under such a deformation.

Deformation gradient tensor 4/4

$\mathrm{F}_{ij} = \dfrac{\partial x_i}{\partial X_j}$ describes both the deformation and rotation associated with the deformation. We want to decompose F into two parts as either

$$\mathrm{F} = \mathrm{RU}$$

or

$$\mathrm{F} = \mathrm{VR}$$

where R is an orthogonal matrix (representing the rotation), and U and V are positive definite and symmetric matrices (representing the deformation).

Decomposition of F 1/3

Theorem: polar decomposition of a positive definite matrix

For any positive definite matrix F, there exist an orthogonal matrix R, and positive definite symmetric matrices U and V such that

$$F = RU = VR,$$

and such R, U and V are unique.

Decomposition of F 2/3

To prove theorem, we first introduce the following lemma (補題).

Lemma

For any symmetric and positive definite matrix A, there exist an unique symmetric and positive definite matrix B such that

$$\mathsf{A} = \mathsf{B}^2$$

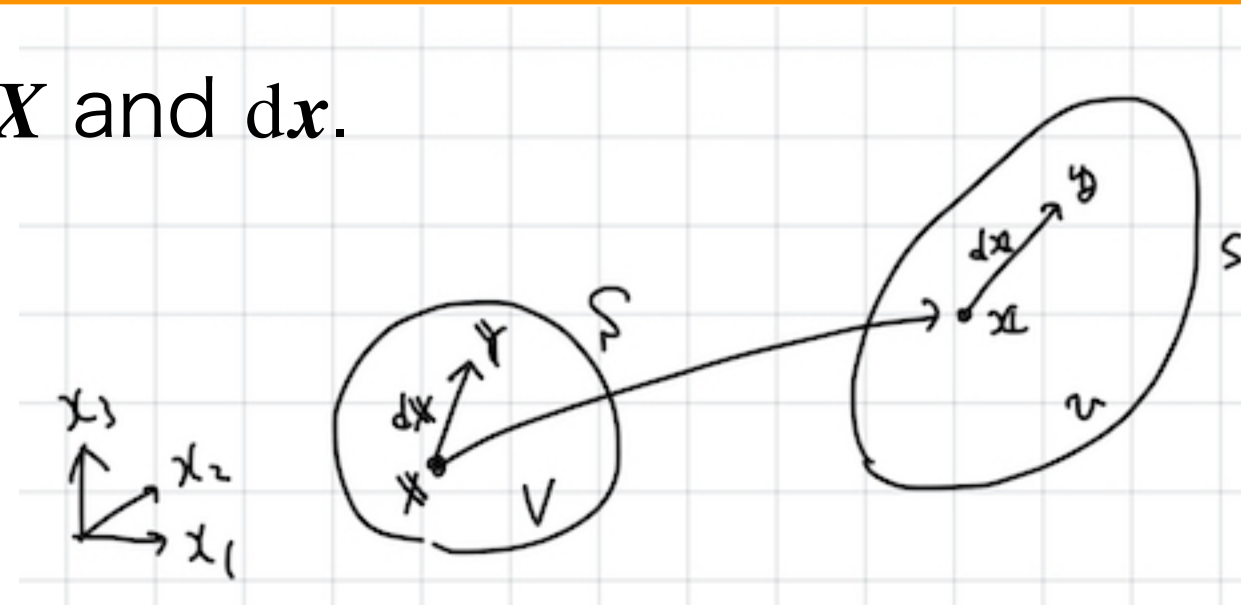
(Proof)

Decomposition of F 3/3

(Proof of the main theorem)

Note: $C = F^t F$ is called the right Cauchy-Green tensor, and $B = FF^t$ is the left Cauchy-Green tensor.

Green's strain tensor



Let us now compare the length of $\mathrm{d}\boldsymbol{X}$ and $\mathrm{d}\boldsymbol{x}$.

Since we have

$$
\begin{aligned}
|\,\mathrm{d}\boldsymbol{x}\,|^2 &= \mathrm{d}x_i \mathrm{d}x_i \\
&= \mathsf{F}_{ij}\mathrm{d}X_j \mathsf{F}_{ik}\mathrm{d}X_k \\
&= \mathrm{d}X_j \mathsf{F}^t_{ji}\mathsf{F}_{ik}\mathrm{d}X_k \\
&= \mathrm{d}X_j \mathsf{C}_{jk}\mathrm{d}X_k
\end{aligned}
$$

The change in the length of small segment can be evaluated as

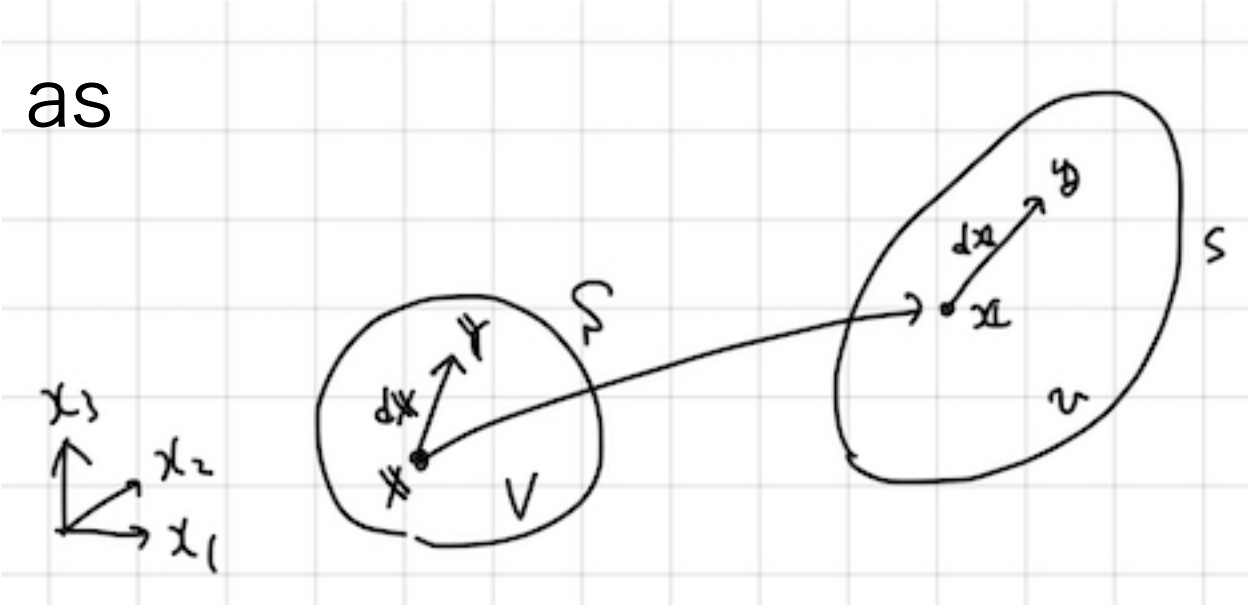
$$|\,\mathrm{d}\boldsymbol{x}\,|^2 - |\,\mathrm{d}\boldsymbol{X}\,|^2 = \mathrm{d}\boldsymbol{X}\cdot(\mathsf{C}-\mathrm{I})\mathrm{d}\boldsymbol{X}$$

Since $\mathsf{C}-\mathrm{I} = \mathsf{F}^t\mathsf{F}-\mathrm{I} = \mathsf{U}^2-\mathrm{I}$ does not involve any rotation component, it seems that this would appropriately describe deformation.

$\rightarrow$ $\mathsf{E} := \dfrac{1}{2}(\mathsf{C}-\mathrm{I})$ is called Green's strain tensor.

Almansi’s strain tensor

Another expression can be obtained as



$$|\,\mathrm{d}\boldsymbol{x}\,|^2 - |\,\mathrm{d}\boldsymbol{X}\,|^2 = \mathrm{d}\boldsymbol{x} \cdot (\mathtt{I} - \mathtt{B}^{-1})\mathrm{d}\boldsymbol{x}$$

$\mathtt{I} - \mathtt{B}^{-1} = \mathtt{I} - (\mathtt{F}\mathtt{F}^t)^{-1} = \mathtt{I} - \mathtt{V}^2$ does not involve any rotation component either. Thus, this would also be appropriately describe the deformation.

$\rightarrow$ $\mathtt{e} := \dfrac{1}{2}(\mathtt{I} - \mathtt{B}^{-1})$ is called Almansi’s strain tensor.

Infinitesimal strain 1/2

Quiz 3. Show the following:

$$\mathrm{E}_{ij} = \frac{1}{2}\left(\frac{\partial u_j}{\partial X_i} + \frac{\partial u_i}{\partial X_j} + \frac{\partial u_k}{\partial X_i}\frac{\partial u_k}{\partial X_j}\right) \text{ and } \mathrm{e}_{ij} = \frac{1}{2}\left(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j} + \frac{\partial u_k}{\partial x_i}\frac{\partial u_k}{\partial x_j}\right).$$

Infinitesimal strain 2/2

When $|\boldsymbol{u}|$ is small, i.e. we have $\boldsymbol{x} = \boldsymbol{X} + \eta\tilde{\boldsymbol{u}}$ with a tiny η, the higher order terms in the previous page can be neglected as

$$\mathsf{E}_{ij} = \frac{1}{2}\left(\frac{\partial u_j}{\partial X_i} + \frac{\partial u_i}{\partial X_j}\right) \text{ and } \mathsf{e}_{ij} = \frac{1}{2}\left(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j}\right).$$

Furthermore, we have $\mathsf{E}_{ij} = \mathsf{e}_{ij}$ because $\dfrac{\partial u_i}{\partial X_j} \simeq \dfrac{\partial u_i}{\partial x_j}$ holds.