計算固体力学入門 (1)

Introduction to Computational Solid Mechanics (1)

Finite Element Method
(FEM)

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Introductory remarks 1/3

- This course will be taught via Zoom (**live broadcast**).

- The recorded video will also be provided via Canvas LMS in case you miss the live session due to, for example, bad Internet connection.

  \- The video will be erased in a week.

- The presentation slides used in the course will also be provided via Canvas LMS.

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Introductory remarks 2/3

In this course, we discuss the finite element method (FEM) for basic problems in solid mechanics.

Upon completion of the course, students

- can tell "what is FEM?"
- can derive fundamental equations for FEM in solid mechanics.
- can implement your own FEM codes.

Grading

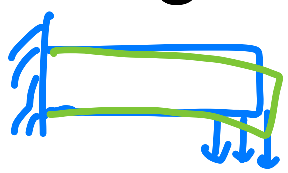
- weekly homework (40%)
- mid-term report (30%)
- final report (30%)

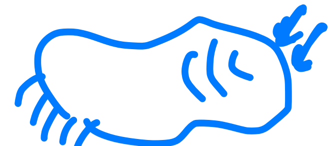
All the homeworks' and reports' assignments will be presented and should be submitted through Canvas LMS.

Introductory remarks 3/3

What is FEM used for?

For analysing mechanical (or, physical in general) phenomena such as bending of beam, propagation of vibration, etc.





$f(t)$

$Ax = b$

right-hand side vector

matrix

mechanical phenomenon $\xrightarrow{\text{modelling}}$ PDE $\xrightarrow{\text{numerical method}}$ Equations tractable by a computer $\xrightarrow{\text{analysis}}$ …

Partial Differential Equation

(Ordinary D E)

$$\frac{d^2u}{dx^2} = f$$

What computers can do

$a + b$ addition ✓

$a \times b$ multiplication ✓

Plans

The contents of this course will be as follows:

1. FEM for a one-dimensional problem.

2. Mathematical preliminaries
   - Gauss-Legendre quadrature numerical integration $\int_a^b f(x)\,dx$
   - Einstein's summation convention

3. Continuum mechanics —— 連続体の力学
   - Deformation of continuum
   - Balance of continuum
   - Basic equations

mid-term

4. Weak form

5. Discretization

6. FEM implementations

final

FEM for a 1D problem

Let us solve the following simple problem by FEM!

Problem

Solve the following ordinary differential equation: 常微分方程式

2nd order ordinary

$$\frac{\mathrm{d}^2u(x)}{\mathrm{d}x^2} = -f(x), \qquad (1)$$

(c.f 偏微分方程式 Partial D E)

B.C

$$u(0) = u(1) = 0 \qquad (2)$$

defined in $x \in [0,1]$, where f is a give function.

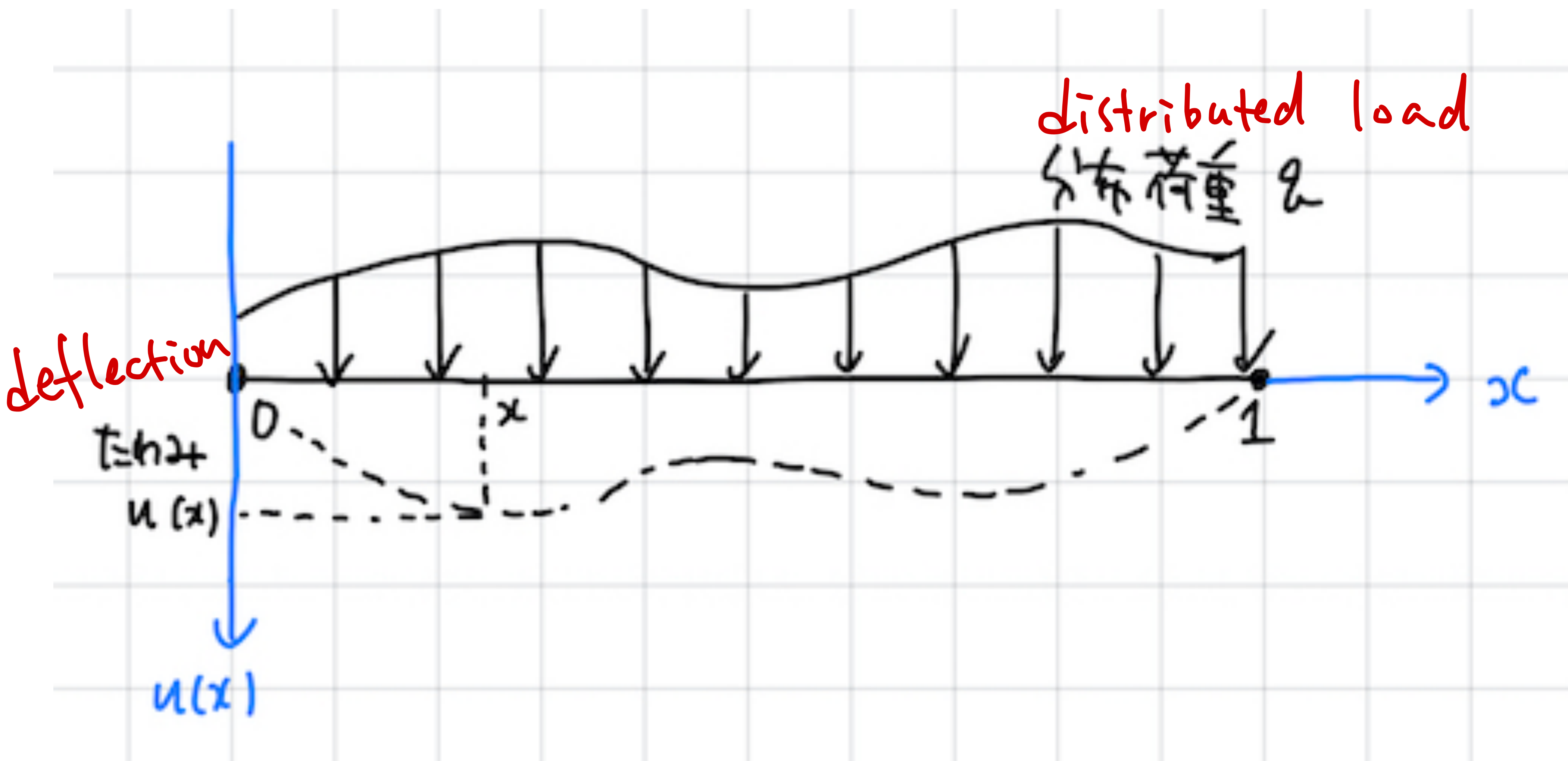
(1) (2)

Note: the set of DE and BC is called “boundary value problem”.

Modelling 1/3

First, let us confirm that the BVP (1) and (2) describes, for example, the deflection of a uniform string.

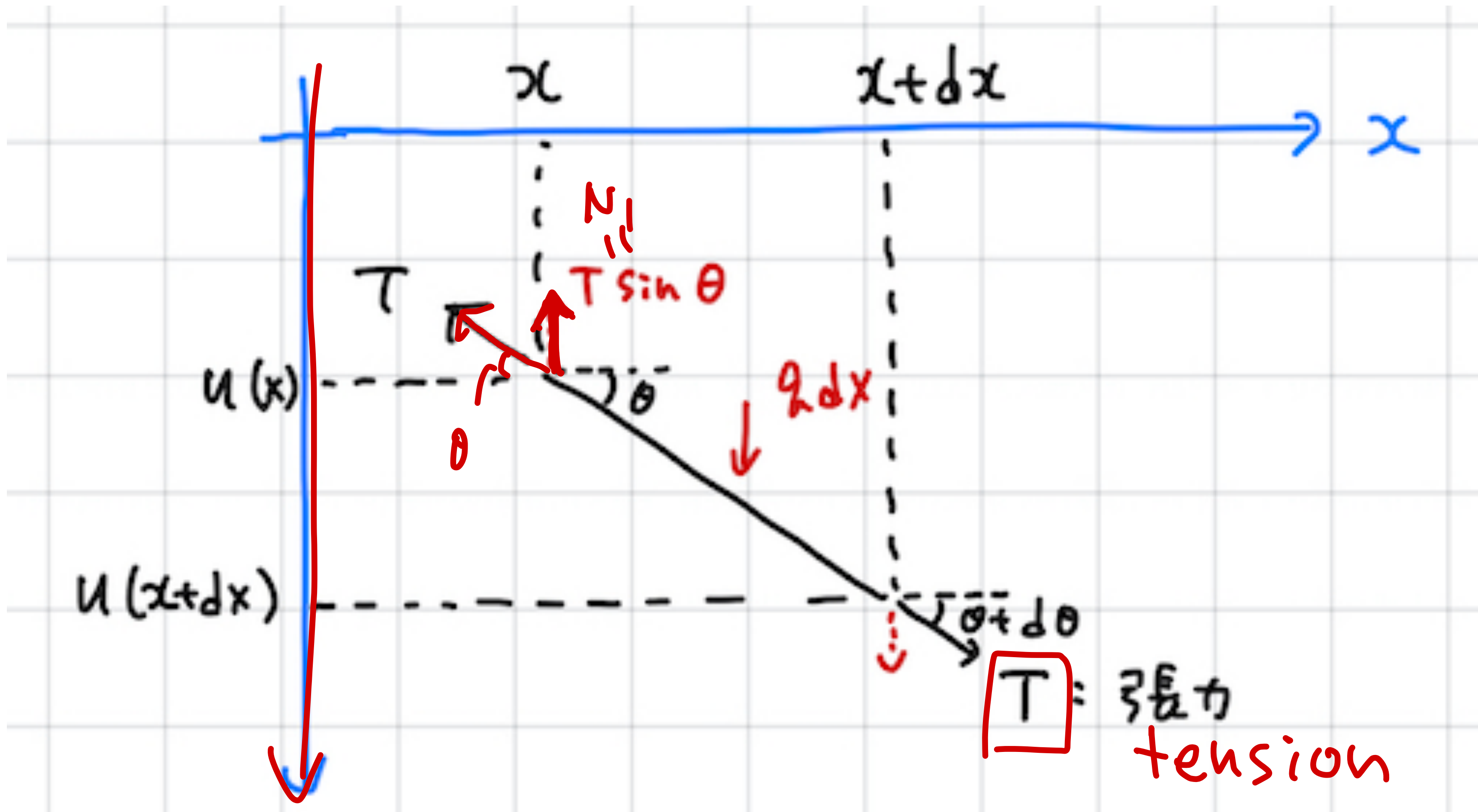
Problem: a string of unit length is subject to distributed load $q(x)$. Find the deflection $u(x)$ of the string.



assumption: the deflection is small.

Modelling 2/3

Let us take a small piece of the string to see the balance equation:



Vertical component of T at the left-hand side:

$$N_1 = -T\sin\theta \simeq -T\tan\theta = -T\frac{\mathrm{d}u(x)}{\mathrm{d}x}$$

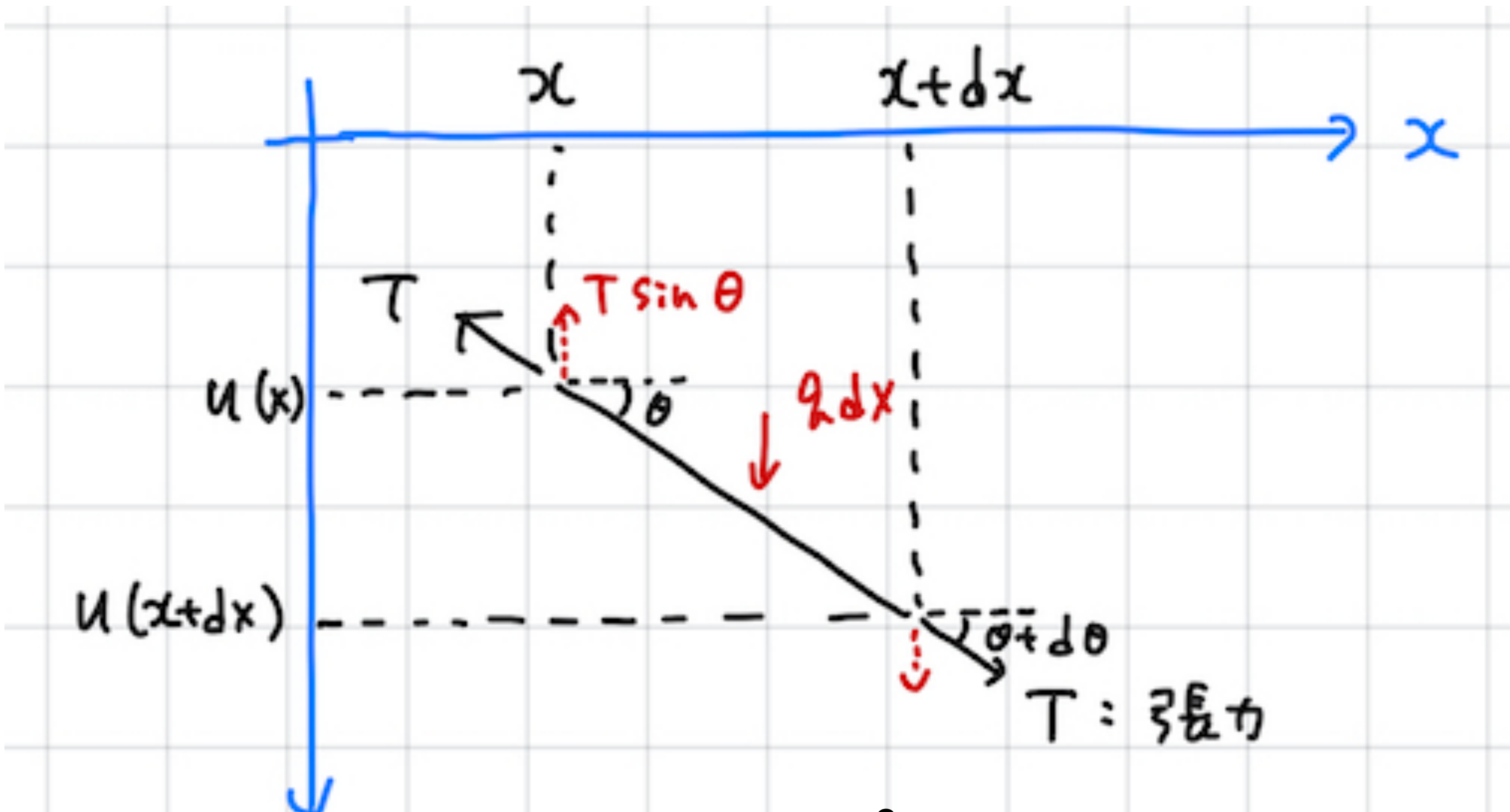
deflection angle.

That at the right-hand side:

$$N_2 = T\frac{\mathrm{d}u(x+\mathrm{d}x)}{\mathrm{d}x} \simeq T\left(\frac{\mathrm{d}u(x)}{\mathrm{d}x} + \frac{\mathrm{d}^2u(x)}{\mathrm{d}x^2}\mathrm{d}x\right)$$

Modelling 3/3

Let us take a small piece of the string to see the balance equation:



Since the internal force $N_1 + N_2 = T\dfrac{\mathrm{d}^2u(x)}{\mathrm{d}x^2}\mathrm{d}x$ should balance with the external force $q(x)\mathrm{d}x$, the equation of the mechanical system is given as

$$T\frac{\mathrm{d}^2u(x)}{\mathrm{d}x^2}\cancel{\mathrm{d}x} + q(x)\cancel{\mathrm{d}x} = 0 \xrightarrow{\text{by putting } f(x) = \frac{q(x)}{T}} \boxed{\frac{\mathrm{d}^2u}{\mathrm{d}x^2} = -f(x),} \qquad (1)$$

BVP

Thus, the problem at hand models the deflection of a supported string subject to a distributed load:

Problem

Solve the following differential equation:

$$\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} = -f(x), \quad (1)$$

$$u(0) = u(1) = 0, \quad (2)$$

defined in $x \in [0,1]$, where f is a give function.

ディリクレ

Note: (2) is often called as the BC of Dirichlet type.

ノイマン

$\boxed{\text{Neumann BC}: \ \frac{du}{dx} = 0}$

$\boxed{\text{Robin BC}: \ u + k\frac{du}{dx} = 0}$

(constant)

solve this by 11:45

Q1:
solve the BVP (1)&(2) when the RHS is given as $f(x) = 1$.

Solution of Q1

$$\begin{cases} \dfrac{d^2u(x)}{dx^2} = -1 & \text{— (1)} \\ u(0) = u(1) = 0 & \text{— (2)} \end{cases}$$

By integrating (1)

$$\frac{du}{dx} = -x + c$$

again

$$u = -\frac{1}{2}x^2 + cx + c'$$

where c & c' are unknown coefficients.

By using (2)

$$u(0) = c' = 0$$

$$\rightarrow c' = 0$$

$$u(1) = -\frac{1}{2} + c$$

$$\rightarrow c = \frac{1}{2}$$

$$\therefore u(x) = -\frac{1}{2}x^2 + \frac{1}{2}x$$

Weighted residual method 1/2

In the previous example, it was quite easy to solve the DE analytically, but this would be impossible in general.

➡ We try to find a function $\tilde{u}(x)$ approximating $u(x)$.

Requirement for $\tilde{u}(x)$

$\tilde{u}(x)$ satisfies the BC (2).

The following function can be a candidate for $\tilde{u}(x)$:

$$\tilde{u}(x) = \sum_{i=0}^{n-1} a_i g_i(x), \qquad (3)$$

where $a_i \in \mathbb{R}$ is the unknown coefficient, and $g_i : [0, 1] \to \mathbb{R}$ is a **basis function** like

$$g_i(x) = x^i \times x(1 - x)$$

Weighted residual method 2/2

Next issue:

How to determine a_i? → weighted residual method (WRM)

Recipe (WRM)

1. Prepare **test functions** (weight functions) $v_i(x)$ for $i = 0, \cdots, n-1$.
2. solve the following **weighted residual equation**:

$$\int_0^1 v_i \left(\frac{\mathrm{d}^2 \tilde{u}(x)}{\mathrm{d}x^2} + f(x) \right) \mathrm{d}x = 0. \qquad (4)$$

Q2:

Let us assume that $\tilde{u}(x) = a_0 \sin \pi x$ approximately solves

$$\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} = -1 \quad \text{in} \quad [0,1], \text{ and } u(0) = u(1) = 0\,.$$

Then, find a_0 by the WRM using the following test functions:

(a) $v_i(x) = 1$

(b) $v_i(x) = \sin \pi x$

Solution of Q2

Q2:

$\frac{d\tilde{u}}{dx} = \pi a_0 \cos\pi x \rightarrow \frac{d^2\tilde{u}}{dx^2} = -\pi^2 a_0 \sin\pi x$

Let us assume that $\tilde{u}(x) = a_0 \sin \pi x$ approximately solves

$$\frac{d^2u}{dx^2} = -1 \quad \text{in} \quad [0,1], \text{ and } u(0) = u(1) = 0 .$$

Then, find a_0 by the WRM using the following test functions:

(a) $v_i(x) = 1$
(b) $v_i(x) = \sin \pi x$

(a) From the weighted residual equation (4), we have

$$0 = \int_0^1 v_i(x)\left(\frac{d^2\tilde{u}(x)}{dx^2} + 1\right) dx$$

$$= \int_0^1 1 \cdot \left(-\pi^2 a_0 \sin\pi x + 1\right) dx$$

$$= \left[\pi a_0 \cos\pi x + x\right]_0^1$$

$$= -2a_0\pi + 1 .$$

Thus, $a_0 = \frac{1}{2\pi} \rightarrow \tilde{u}(x) = \frac{1}{2\pi}\sin\pi x$

(b) Can you Solve this problem by yourself?

$\boxed{\text{by } 12:10}$

From the weighted residual equation (4), we have

$$0 = \int_0^1 \sin\pi x \cdot \left(-\pi^2 a_0 \sin\pi x + 1\right) dx$$

$$= -\pi^2 a_0 \int_0^1 \underbrace{\sin^2\pi x}_{\frac{1-\cos 2\theta}{2}} dx + \int_0^1 \underbrace{\sin\pi x}_{-\frac{1}{\pi}\cos\pi x} dx$$

$$= -\pi^2 a_0/2 + \frac{2}{\pi} .$$

$$\Leftrightarrow a_0 = \frac{4}{\pi^3}$$

Thus, we have the following approx. solution:

$$\tilde{u}(x) = \frac{4}{\pi^3}\sin\pi x .$$

HW: plot $\tilde{u}_a(x) = \frac{1}{2\pi}\sin\pi x$, $\tilde{u}_b(x) = \frac{4}{\pi^3}\sin\pi x$ & $u(x) = -\frac{1}{2}x^2 + \frac{1}{2}x$ by using a software you like. Compare the accuracy of $\tilde{u}_a$ & $\tilde{u}_b$ against the exact solution u.

Important notes

- **The Galerkin method** uses the same test functions as the basis functions. Usually, the Galerkin method gives more accurate result than other choices of test functions. See Q2(b).

- The boundary value problem of a differential equation (1) & (2) is converted into an algebraic equation which can be solved by a computer.

Finite element method (FEM) 1/2

In the previous example, we used the weighted residual equation (4) as is.

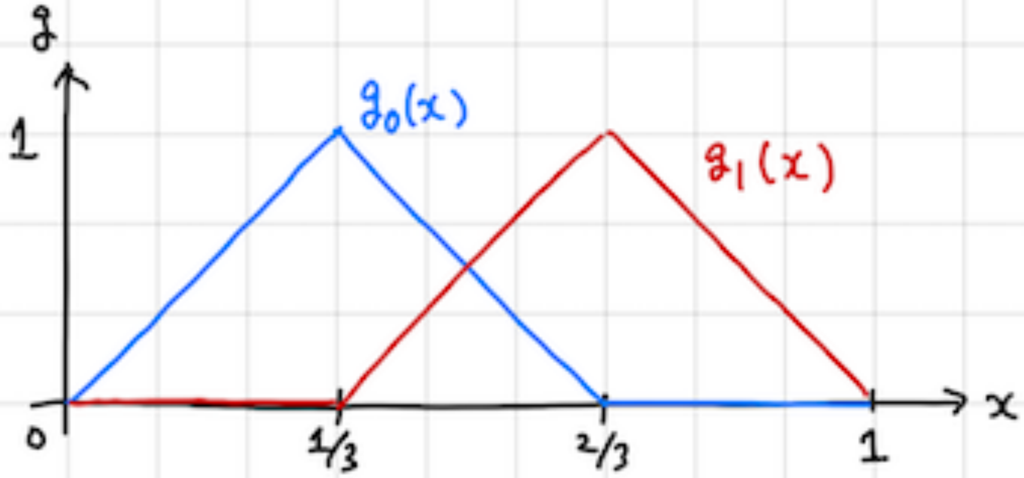
Here, we transform it a little. We here take test functions $\tilde{v}_i(x)$ satisfying the BC (2), i.e. $\tilde{v}_i(0) = \tilde{v}_i(1) = 0$. Then, (4) is rewritten by using the "integration by parts" as

$$\int_0^1 v_i'(x)u'(x)\mathrm{d}x = \int_0^1 v_i(x)f(x)\mathrm{d}x$$ **: weak form**

Note: the weak form only involves differentiation of order 1.
$\rightarrow$ linear functions can be used for $\tilde{u}$ and v !

Finite element method (FEM) 2/2

An example of the linear functions:



$$g_0(x) = \begin{cases} 3x & \text{if} \quad 0 \le x \le 1/3 \\ -3(x - 2/3) & \text{if} \quad 1/3 \le x \le 2/3 \\ 0 & \text{otherwise} \end{cases}$$

$$g_1(x) = \begin{cases} 3(x - 1/3) & \text{if} \quad 1/3 \le x \le 2/3 \\ -3(x - 1) & \text{if} \quad 2/3 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Note: they ↑ are locally supported.

The weak form discretized by the Galerkin method gives the following algebraic equation (which can be solved by a computer!)

$$\sum_{j=0}^{n-1} \left(\int_0^1 g_i'(x) g_j'(x) \mathrm{d}x \right) a_j = \int_0^1 g_i f(x) \mathrm{d}x \quad \text{for } i = 0, \cdots, n-1$$

Finite element method (FEM) 3/3

Recipe (FEM)

1. Prepare **locally supported** test functions $v_i(x)$ for $i = 0, \cdots, n-1$.
2. Prepare the **weighted residual equation (WRE)**.
3. Transform WRE into the **weak form**.
4. Discretize the weak form by **the Galerkin method**.
5. Solve the algebraic equations (by a computer).

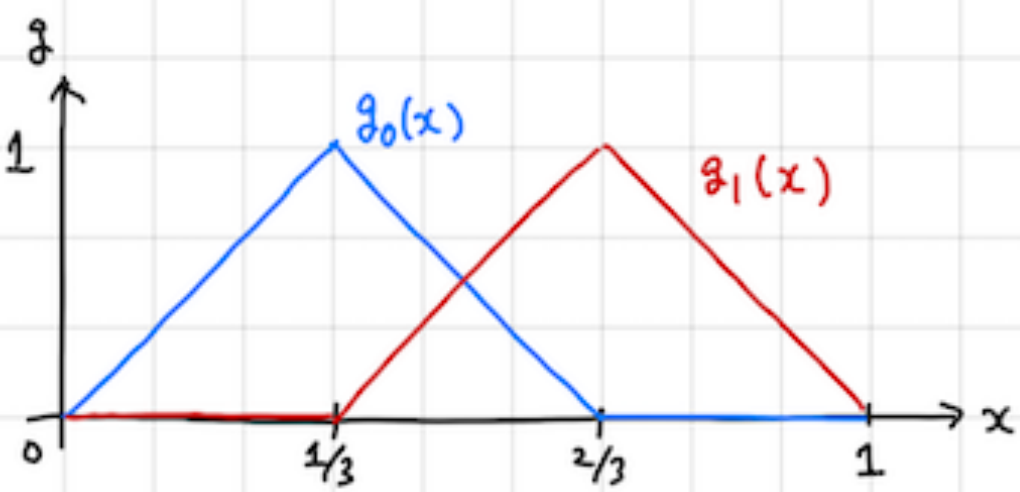
Note:
the basis functions are also locally supported by the definition of the Galerkin method.

Example 1/2

Solve the following BVP by the finite element method:

$$\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} = -1 \quad \text{in} \quad [0,1], \ \text{and} \ u(0) = u(1) = 0\,,$$

by using the following basis functions:



$$g_0(x) = \begin{cases} 3x & \text{if} \quad 0 \le x \le 1/3 \\ -3(x - 2/3) & \text{if} \quad 1/3 \le x \le 2/3 \\ 0 & \text{otherwise} \end{cases}$$

$$g_1(x) = \begin{cases} 3(x - 1/3) & \text{if} \quad 1/3 \le x \le 2/3 \\ -3(x - 1) & \text{if} \quad 2/3 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Example 2/2

Homework 1

H1. Solve the following BVP by the finite element method:

$$\frac{d^2u}{dx^2} = -1 \quad \text{in} \quad [0,1], \text{ and } u(0) = u(1) = 0\,,$$

by using the following basis functions:

$$g_0(x) = \begin{cases} 4x & \text{if } 0 \le x \le 1/4 \\ -4(x-1/2) & \text{if } 0 \le x \le 1/2 \\ 0 & \text{otherwise} \end{cases}$$

$$g_1(x) = \begin{cases} 4(x-1/4) & \text{if } 1/4 \le x \le 1/2 \\ -4(x-3/4) & \text{if } 1/2 \le x \le 3/4 \\ 0 & \text{otherwise} \end{cases}$$

$$g_2(x) = \begin{cases} 4(x-1/2) & \text{if } 1/2 \le x \le 3/4 \\ -4(x-1) & \text{if } 3/4 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Homework 1 (cont.)

H2. Solve the following BVP by the finite element method:

$$\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} = -f(x) \quad \text{in} \quad [0,1], \quad \text{and} \quad u(0) = u(1) = 0,$$

where $f(x) = \begin{cases} 1 & \text{if } \ x \leq 1/2 \\ 0 & \text{otherwise} \end{cases}$ by using appropriate basis functions.