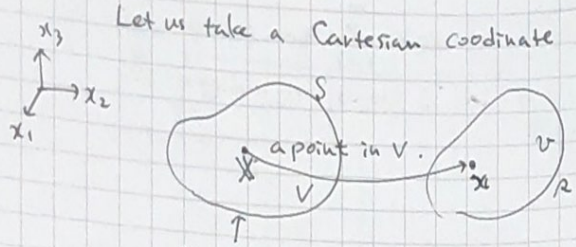
5/4(水) CM 講義 : Continuum Mechanics. 1

1. What is continuum?

2. displacement

Let us take a Cartesian coodinate x_1, x_2 & x_3 in $\mathbb{R}^3$.



an object of finite size.

The object deforms as $V \to v$. $u := x - X$ is called displacement. (1)

cannot tell how V deforms.

In the follwing, the "map" $X \leftrightarrow x$ is "one-to-one"
写像
i.e. we can defined the "inverse" $x \to X$
and the map is sufficiently smooth
(i.e. we can define $\frac{d^n}{dX_{i_1} dX_{i_2} \cdots dX_{i_n}} x(X)$ for arbitrary n).

3. Deformation gradient tensor.

Let us consider a "short" vector $dX = Y - X$ embedded in V, and assume that dX becomes as dx after the deformation.
$\parallel$
$y - x$

We can use the map $x(X)$ to evaluate the 1st order approx. of dx as

$$
\begin{aligned}
dx_i &= x_i(Y) - x_i(X) \\
&= x_i(X + dX) - x_i(X) \\
&\simeq x_i(X) + \frac{\partial x_i(X)}{\partial X_j} \cdot dX_j - x_i(X) \qquad (2) \\
&= \frac{\partial x_i}{\partial X_j} dX_j \quad \leftarrow \text{Einstein's summation conv. is used.}
\end{aligned}
$$

Let us henthforce denote $\frac{\partial x_i}{\partial X_j}$ as F_{ij}, and call F_{ij} as "deformation gradient tensor".

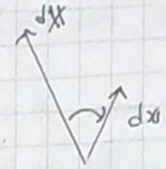
Fが正定値を示す?
$\det F (dX \cdot (dY \times dZ))$
$\parallel$
$dx \cdot (dy \times dz)$

$$\boxed{\text{Q1} \quad F_{ij} = \frac{\partial}{\partial X_j}(x_i) = \frac{\partial}{\partial X_j}(X_i + u_i) = \delta_{ij} + \frac{\partial u_i}{\partial X_j}}$$

Q2. If F_{ij} is ij comp. of orthogonal matrix,

$$
\begin{aligned}
dx_i dx_i &= F_{ij} dX_j F_{ik} dX_k \\
&= dX_j \underbrace{F^T_{ji} F_{ik}}_{=\delta_{jk}} dX_k = dX_j dX_j
\end{aligned}
$$

$dx = F\,dX$



F describes the rotation of dX

as well as the deformation.

→ We want to decompose F into two parts.

deformation & rotation.

<u>Theorem</u>. For any F of positive definite matrix, there exist

$\begin{cases} R : \text{orthogonal matrix} \\ U \,\&\, V : \text{positive definite \& symmetric matrix} \end{cases}$

such that $F = RU = VR \quad \cdots (3)$

R : Rotation

Also, such R, U & V are unique.

First, we prove the following lemma:

<u>lemma</u> For any positive definite & symmetric matrix A,

there exist positive definite & symmetric matrix B

such that $A = B^2$, and such a B is unique.

($\because$) A can be diagonalized with an orthogonal matrix P as

$$P^t A P = \Lambda = \begin{pmatrix} \lambda_1 & \cdot & \cdot \\ \cdot & \lambda_2 & \cdot \\ \cdot & \cdot & \lambda_3 \end{pmatrix} \quad \cdots (4)$$

where $\lambda_i > 0$ because A is positive definite

Let us define B as $B = P \begin{pmatrix} \sqrt{\lambda_1} & 0 & 0 \\ 0 & \sqrt{\lambda_2} & 0 \\ 0 & 0 & \sqrt{\lambda_3} \end{pmatrix} P^T$, then

it is obvious that $B^2 = P(\quad)P^t P(\quad)P^t = P \Lambda P^t = A$. — (5)

we have proved the existence of B.

We then need to show the uniqueness. To this end let us define

another P.D & S matrix $\tilde{B}$ such that $A = \tilde{B}^2$, & $\tilde{B} \neq B$.

Let us take an eigenpair $(\sqrt{\tilde{\lambda}}, \tilde{p})$ of $\tilde{B}$, to see

$$\tilde{B}\tilde{p} = \sqrt{\tilde{\lambda}}\,\tilde{p} \Leftrightarrow \tilde{B}^2 \tilde{p} = \tilde{B} \tilde{\lambda} \tilde{p} = \tilde{\lambda}\tilde{p}$$

$$\Leftrightarrow A\tilde{p} = \tilde{\lambda}\tilde{p}.$$

Thus, A & $\tilde{B}$ share the eigenvector, which means that P in (4)

diagonalize $\tilde{B}$ as well as $P^t \tilde{B} P = \begin{pmatrix} \sqrt{\tilde{\lambda}_1} & & \\ & \ddots & \\ & & \sqrt{\tilde{\lambda}_3} \end{pmatrix}$. $\quad \tilde{\lambda}_i^2 = \lambda_i^2 \Rightarrow \tilde{\lambda}_i = \lambda_i$.

$$\Leftrightarrow P^t \tilde{B}^2 P = \begin{pmatrix} \tilde{\lambda}_1 & & \\ & \tilde{\lambda}_2 & \\ & & \tilde{\lambda}_3 \end{pmatrix} \quad -(6)$$

$\therefore \tilde{B} = B$.

(Proof of the main theorem).

First note that, when F is positive definite, F^TF is also positive definite. This can be seen as

$$\det F^tF = \det F^t \det F^t = (\det F)^2 > 0.$$

$$(\because F \text{ is positive definite})$$

F^TF is also symmetric because

$$(F^TF)_{ij} = F^T_{ik} F_{kj} = F_{ki} F_{kj} = F_{ki} F^T_{jk} = F^T_{jk} F_{ki} = (F^TF)_{ji}$$

holds. According to the lemma, for F^TF, there is an unique positive definite & symmetric matrix U such that

$$F^tF = U^2$$

Exist because U is positive def.

Then, we can define $R := FU^{-1} \Leftrightarrow F = RU. \text{———} \star$

$R^TR = U^{-1}F^tFU^{-1} = I$ ($\because$ U는 대칭)

The factorisation $\star$ is unique because of the uniqueness of U.

($F = VR$ can be proved in the similar manner. starting from FF^T, which is also p.d.)

Note: $C = F^TF$ is called right Cauchy-Green tensor

$B = FF^T$ is left.

Let us now compute the length of $d\mathbb{X}$ & $d\mathbb{x}$.

$$|d\mathbb{x}|^2 = dx_i \, dx_i$$
$$= F_{ij} \, dX_j \, F_{ik} dX_k$$
$$= dX_j \, F^T_{ji} F_{ik} \, dX_k$$
$$= dX_j \, C_{jk} \, dX_k$$

$$|d\mathbb{x}|^2 - |d\mathbb{X}|^2 = dX_j \, C_{jk} \, dX_k - dX_j \, dX_j$$
$$= dX_j \, C_{jk} \, dX_k - dX_j \, \delta_{jk} \, dX_k$$
$$= dX_j \, (C_{jk} - \delta_{jk}) \, dX_k.$$
$$= d\mathbb{X} \cdot (C - I) \, d\mathbb{X}.$$

Note: $C - I = F^T F - I = U^2 - I$ $\leftarrow$ doesnot involve R (rotation)

$E = \frac{1}{2}(C - I)$ Green's strain tensor.

On the other hand,

$$|d\mathbb{x}|^2 - |d\mathbb{X}|^2 = dx_j \, dx_j - F^{-1}_{ij} \, dx_j \, F^{-1}_{ik} \, dx_k$$
$$= dx_j \left(\delta_{jk} - F^{-t}_{ji} F^{-1}_{ik} \right) dx_k$$
$$= dx_j \left(\delta_{jk} - (F F^t)^{-1}_{jk} \right) dx_k$$
$$= d\mathbb{x} \cdot (I - B^{-1}) \, d\mathbb{x}$$

$I - B^{-1} = I - V^2$ doesnot involve R either

$e = \frac{1}{2}(I - B^{-1})$: Almansi's strain tensor infinitesimal deformation theory. $F_{ij} = \delta_{ij} + u_{i,j}$

~~$x(X,t) \simeq X(t=0) + \Delta t \frac{\partial x}{\partial t}$~~

$$E_{ij} = \frac{1}{2}(C_{ij} - \delta_{ij})$$
$$= \frac{1}{2}(F^T_{ik}F_{kj} - \delta_{ij})$$
$$= \frac{1}{2}\left[\left(\delta_{ki} + \frac{\partial u_k}{\partial X_i}\right)\left(\delta_{kj} + \frac{\partial u_k}{\partial X_j}\right) - \delta_{ij}\right]$$
$$= \frac{1}{2}\left(\frac{\partial u_j}{\partial X_i} + \frac{\partial u_i}{\partial X_j} + \frac{\partial u_k}{\partial X_i}\frac{\partial u_k}{\partial X_j}\right).$$

$F_{ij} = \delta_{ij} + \varepsilon u_{i,j}$

$=$

$B = FF^T$

$B^{-1} = F^{-T} F^{-1}$

$F_{ij} F_{jk} = (\delta_{ij} + \varepsilon \quad)(\quad - \quad)$

$$e_{ij} = \frac{1}{2}(\delta_{ij} - B^{-1}_{ij})$$
$$= \frac{1}{2}(\delta_{ij} - F^{-1}_{ki}F^{-1}_{kj})$$
$$= \frac{1}{2}\left(\delta_{ij} - \left(\delta_{ki} - \frac{\partial u_k}{\partial x_i}\right)\left(\delta_{kj} - \frac{\partial u_k}{\partial x_j}\right)\right)$$
$$= \frac{1}{2}\left(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j} - \frac{\partial u_k}{\partial x_i}\frac{\partial u_k}{\partial x_j}\right).$$

$$\frac{\partial u_i}{\partial X_j} = \eta \frac{\partial \tilde{u}_i}{\partial X_j}$$
$$= \eta \frac{\partial \tilde{u}_i}{\partial x_k}\frac{\partial x_k}{\partial X_j}$$
$$= \eta \frac{\partial \tilde{u}_i}{\partial x_k}\left(\delta_{kj} + \eta \frac{\partial \tilde{u}_k}{\partial X_j}\right)$$
$$= \eta \frac{\partial \tilde{u}_i}{\partial x_j} + \cancel{\eta^2 \frac{\partial \tilde{u}_i}{\partial x_k}\frac{\partial \tilde{u}_k}{\partial X_j}} \to 0 = \frac{\partial u_i}{\partial x_j}$$