計算固体力学入門 (5)

Introduction to Computational Solid Mechanics (5)

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Today's topic

The contents of this course will be as follows:

1. FEM for a one-dimensional problem.

2. Mathematical preliminaries
   - Gauss-Legendre quadrature
   - Einstein's summation convention

3. Continuum mechanics
   - Deformation of continuum
   - **Balance of continuum**
   - **Basic equations**

4. Weak form

5. Discretization

6. FEM implementations

Tensor product

Let us consider fixed two vectors $\boldsymbol{a}$ and $\boldsymbol{b} \in \mathbb{R}^3$, and another vector $\boldsymbol{u} \in \mathbb{R}^3$. By using these vectors, let us define another vector $\boldsymbol{v} \in \mathbb{R}^3$ as

$$\boldsymbol{v} = (\boldsymbol{b} \cdot \boldsymbol{u})\boldsymbol{a}. \qquad \cdots (1)$$

With this procedure, we have defined a "map": $\boldsymbol{u} \to \boldsymbol{v}$. Such a map is called a tensor product of the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, and will be denoted as

$$\boldsymbol{v} = (\boldsymbol{a} \otimes \boldsymbol{b})[\boldsymbol{u}] \ (= (\boldsymbol{b} \cdot \boldsymbol{u})\boldsymbol{a}). \qquad \cdots (2)$$

Note: We have not used the components of the relevant vectors to define (2), i.e. it can be defined without coordinate system.

Note: Tensor product is the tensor (which will be defined in the next page).

Tensor

Let us consider a map $L : \mathbb{R}^3 \ni \boldsymbol{u} \to \boldsymbol{v} \in \mathbb{R}^3$, i.e. $\boldsymbol{v}$ is defined as

$$\boldsymbol{v} = L[\boldsymbol{u}]. \qquad \cdots(3)$$

When L is linear, i.e. $L[a\boldsymbol{u} + b\boldsymbol{v}] = aL[\boldsymbol{u}] + bL[\boldsymbol{v}]$ holds, it is called the linear transform of a vector, or the (second-order) tensor. Again, L is independent of the choice of coordinate. Once we fix the coordinate system, (3) is represented as

$$v_i = L_{ij}u_j \Leftrightarrow \boldsymbol{v} = \mathsf{L}\boldsymbol{u}. \qquad \cdots(4)$$

The matrix $\mathsf{L} \in \mathbb{R}^{3\times3}$ is called the representation matrix of the tensor L.

(example)
the ij-component of representation matrix of $\boldsymbol{a} \otimes \boldsymbol{b}$ is given as $a_i b_j$.

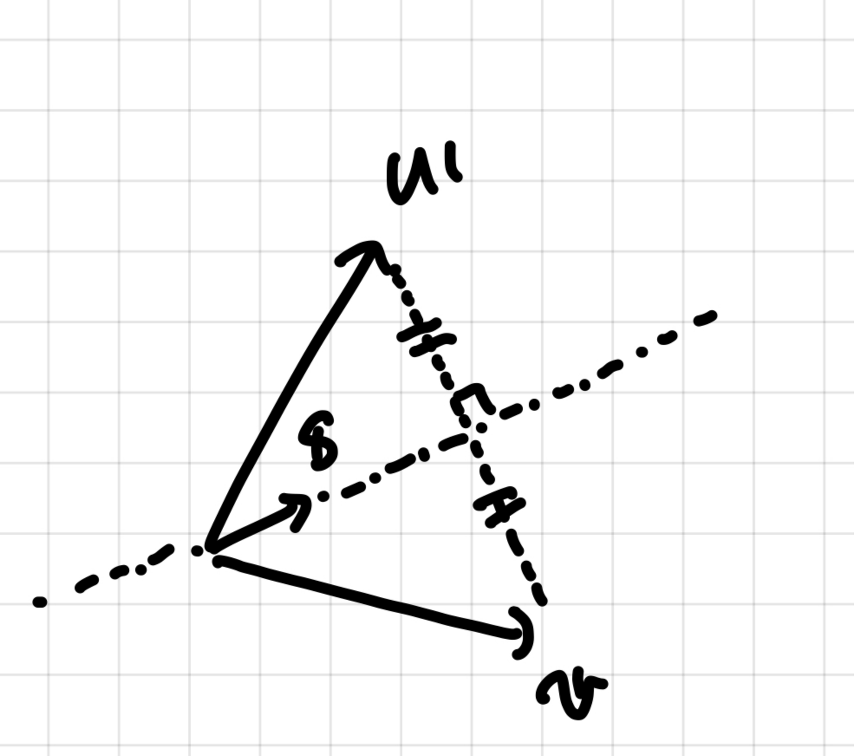
Examples

Q1:
find the representation matrix of the identity map I which maps a vector $\boldsymbol{u}$ to $\boldsymbol{u}$ itself, i.e. $\boldsymbol{u} = I[\boldsymbol{u}]$

Q2:
show that the "reflection" $\boldsymbol{v}$ of the vector $\boldsymbol{u}$ with respect to an unit vector $\boldsymbol{s}$ is given as follows:

$$\boldsymbol{v} = (2\boldsymbol{n} \otimes \boldsymbol{n} - I)\boldsymbol{u},$$

and show the corresponding representation matrix.



Adjoint linear transformation

Definition: adjoint linear transformation

The tensor $\tilde{L}$ satisfying

$$L[\boldsymbol{u}] \cdot \boldsymbol{v} = \boldsymbol{u} \cdot \tilde{L}[\boldsymbol{v}] \qquad \cdots(5)$$

is called the adjoint linear transformation.

(example):

Let us consider the tensor product $L = \boldsymbol{a} \otimes \boldsymbol{b}$. Its adjoint $\tilde{L}$ is given as $\tilde{L} = \boldsymbol{b} \otimes \boldsymbol{a}$.

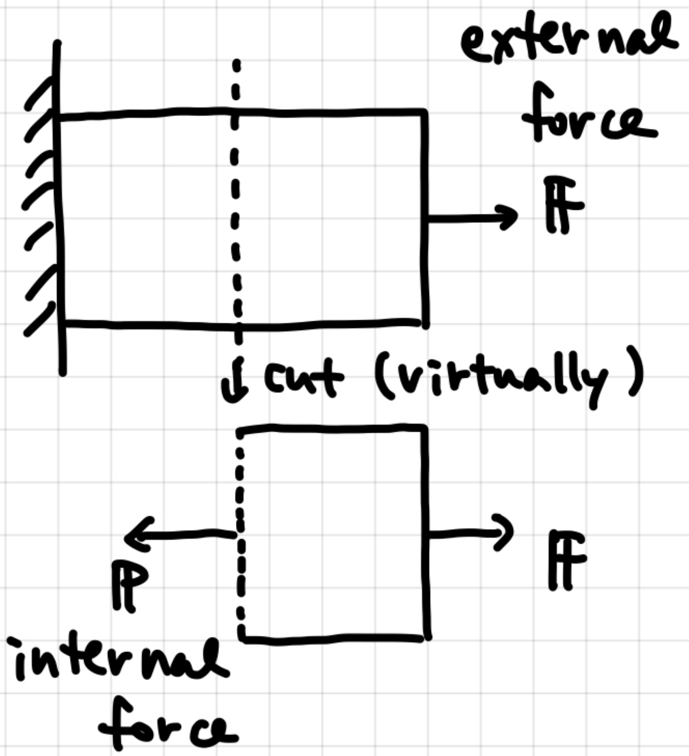
$$(L[\boldsymbol{u}] \cdot \boldsymbol{v} = (\boldsymbol{a} \otimes \boldsymbol{b})[\boldsymbol{u}] \cdot \boldsymbol{v} = ((\boldsymbol{b} \cdot \boldsymbol{u})\boldsymbol{a}) \cdot \boldsymbol{v} = \boldsymbol{u} \cdot (\boldsymbol{b}(\boldsymbol{a} \cdot \boldsymbol{v})) = \boldsymbol{u} \cdot (\boldsymbol{b} \otimes \boldsymbol{a})[\boldsymbol{v}] = \boldsymbol{u} \cdot \tilde{L}[\boldsymbol{v}])$$

(Quiz):

Show the representation matrix of $\tilde{L} = \boldsymbol{b} \otimes \boldsymbol{a}$.

Stress tensor σ 1/2

Let us consider a uniform bar subject to an external force $\boldsymbol{F}$.



Then, at an arbitrary cross section of the bar, we observe the corresponding internal force $\boldsymbol{P}$.

(Otherwise, the bar cannot be balanced!)

Stress vector (or traction)

= internal force per unit area.

Definition: Stress tensor

The linear transformation σ whose adjoint generates the traction $\boldsymbol{P}$ on a surface from the unit normal vector $\boldsymbol{n}$, i.e.

$$P = \tilde{\sigma}[\boldsymbol{n}] \qquad \cdots (6)$$

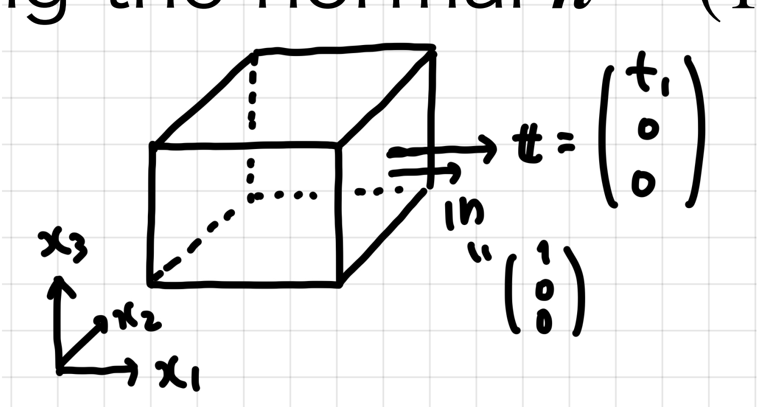
is called the stress tensor.

(6) is called the Cauchy stress formula.

Stress tensor σ 2/2

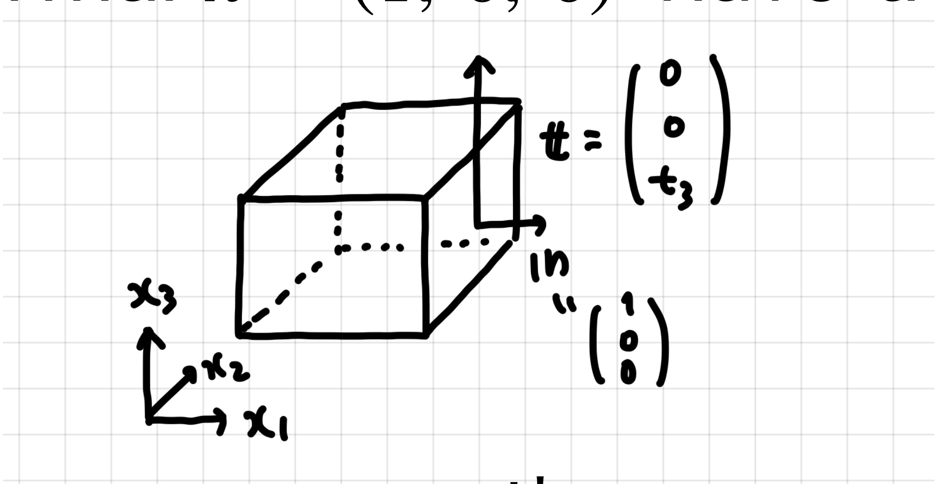
Let us consider a rectangular box (whose edges are parallel to given axes).

Let us consider the traction $\boldsymbol{t} = (t_1, 0, 0)^t$ acting on the surface having the normal $\boldsymbol{n} = (1, 0, 0)^t$. The Cauchy stress formula gives



$$t_1 = \sigma_{j1} n_j = \sigma_{11} n_1 + \sigma_{21} n_2 + \sigma_{31} n_3 = \sigma_{11}.$$

Similarly, the traction $\boldsymbol{t} = (0, 0, t_3)^t$ acting on the surface and the normal $\boldsymbol{n} = (1, 0, 0)^t$ have the following relation:

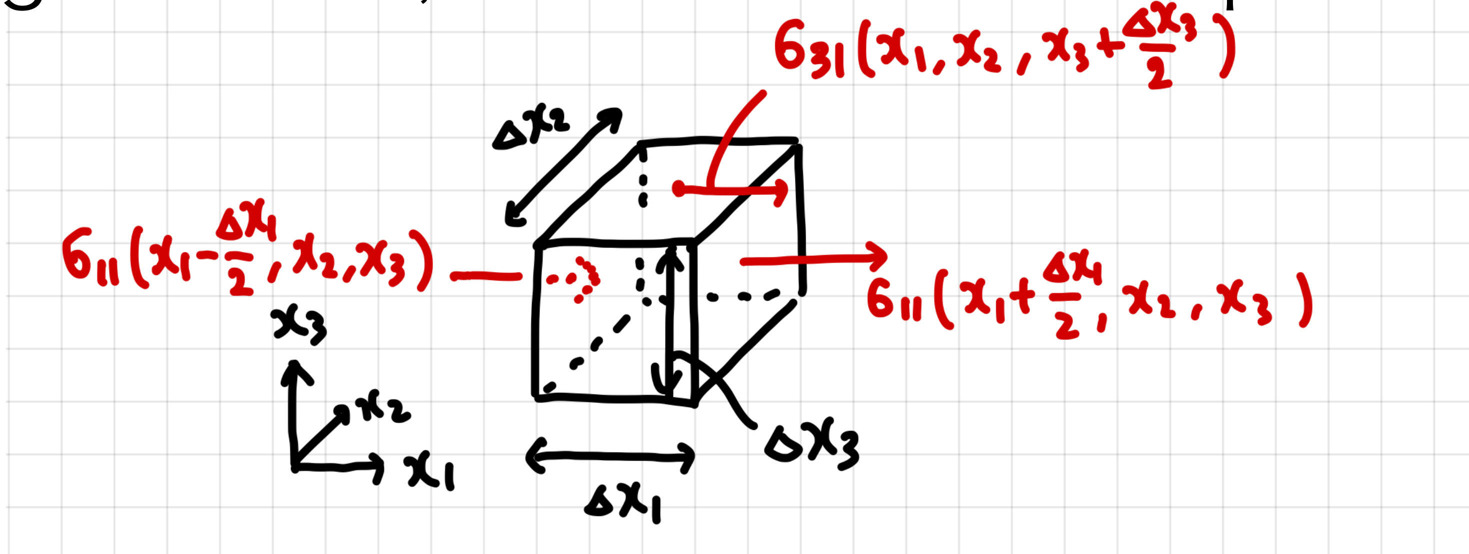


$$t_3 = \sigma_{j3} n_j = \sigma_{13} n_1 + \sigma_{23} n_2 + \sigma_{33} n_3 = \sigma_{13}.$$

$\rightarrow$ σ_{ij} provides j^{th} component of a traction acting on the surface perpendicular to x_i axis.

Balance equation 1/3

Let us consider a rectangular box (whose edges (of length Δx_i) are parallel to given axes, and the "centre" is put at the origin).



The 1st component of traction working on the "left" and "right" surface can be written as

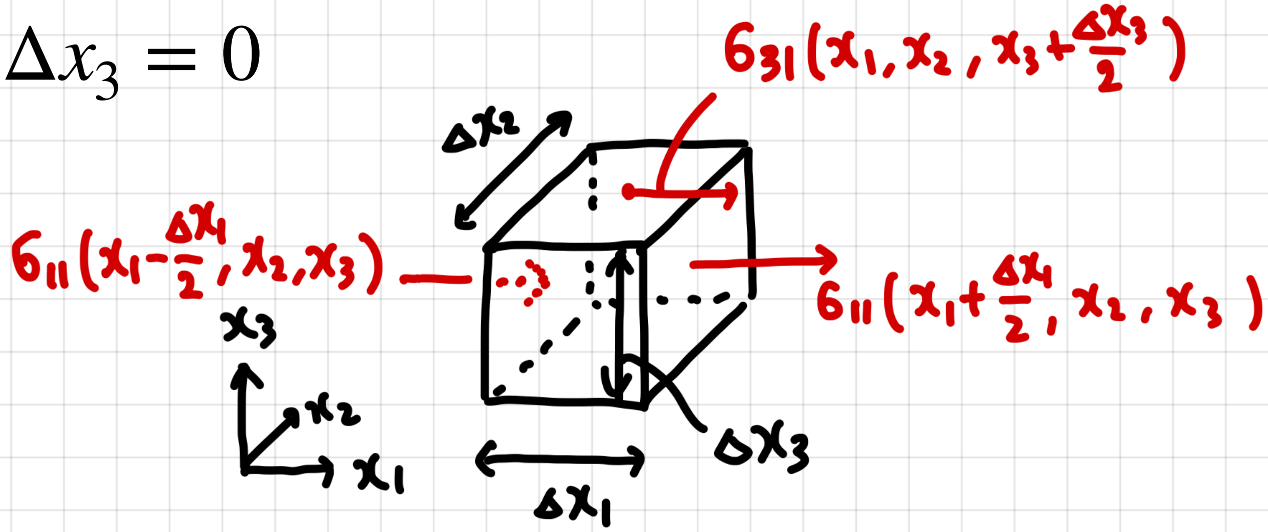
$$\sigma_{11}(x_1 \pm \Delta x_1/2,\, x_2,\, x_3) \simeq \sigma_{11}(x_1,\, x_2,\, x_3) \pm \partial\sigma_{11}(x_1,\, x_2,\, x_3)/\partial x_1 \times \Delta x_1/2$$

Quiz: present the 1st comp. of traction working on the "top", "bottom", "front" and "back" surfaces.

Balance equation 2/3

When this box is subject to a body force $\boldsymbol{F} = (F_1, F_2, F_3)^t$, the balance equation in x_1 direction can be obtained as

$$
\begin{aligned}
&\sigma_{11}(x_1 + \Delta x_1/2, x_2, x_3)\Delta x_2 \Delta x_3 - \sigma_{11}(x_1 - \Delta x_1/2, x_2, x_3)\Delta x_2 \Delta x_3 \\
&+\sigma_{21}(x_1, x_2 + \Delta x_2/2, x_3)\Delta x_3 \Delta x_1 - \sigma_{21}(x_1, x_2 - \Delta x_2/2, x_3)\Delta x_3 \Delta x_1 \\
&+\sigma_{31}(x_1, x_2, x_3 + \Delta x_3/2)\Delta x_1 \Delta x_2 - \sigma_{31}(x_1, x_2, x_3 - \Delta x_3/2)\Delta x_1 \Delta x_2 \\
&+F_i \Delta x_1 \Delta x_2 \Delta x_3 = 0
\end{aligned}
$$



By using the approximations in the previous page, and taking the limit of $\Delta x_1 \Delta x_2 \Delta x_3 \to 0$, we have the balance equation as $\sigma_{j1,j} + F_1 = 0$.

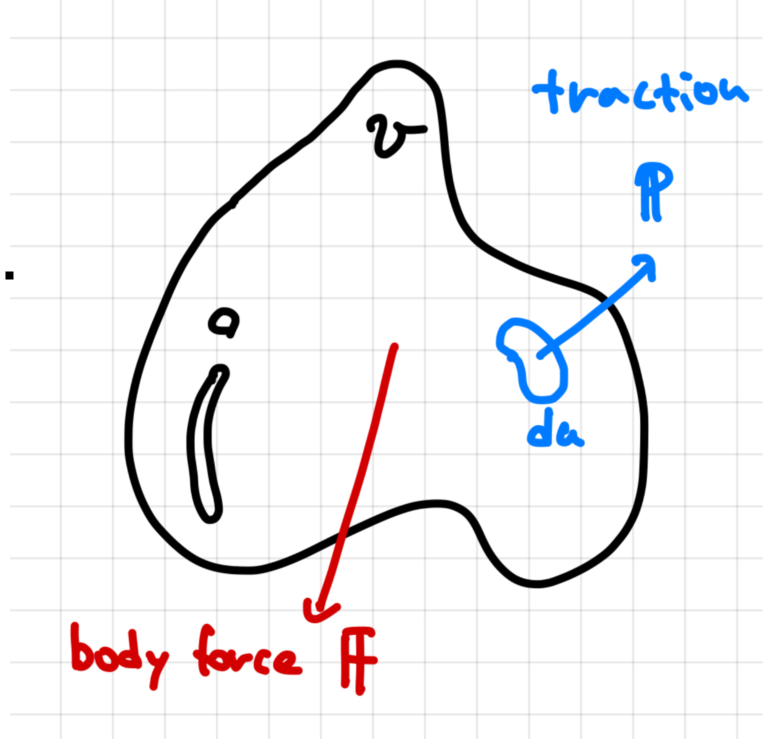
By repeating the above procedure for $i = 2$ and 3, we have the following balance equation of the continuum:

$$\sigma_{ji,j} + F_i = 0.$$

Balance equation 3/3

The balance equation can be derived by a different approach.

Let us take an arbitrary partial domain v of the continuum, and consider the forces acting on v. The forces are divided into two parts: the body force $\boldsymbol{F}$ and the traction $\boldsymbol{P}$.



The balance equation for v is written as

$$\int_{\partial v} p_i \mathrm{d}a + \int_v F_i \mathrm{d}v = 0$$

$$\Leftrightarrow \int_{\partial v} \sigma_{ji} n_j \mathrm{d}a + \int_v F_i \mathrm{d}v = 0$$

$$\Leftrightarrow \int_v \sigma_{ji,j} \mathrm{d}a + \int_v F_i \mathrm{d}v = 0.$$

$\rightarrow$ Since v is arbitrary, we have

$$\sigma_{ji,j} + F_i = 0.$$

Constitutive equation 1/2

We then discuss the constitutive equation which relates the stress tensor σ_{ij} and strain tensor ε_{ij}.

Linear elastic material

When $\sigma_{ij} = C_{ijk\ell}\varepsilon_{k\ell}$ holds in a material, such a material is called linear elastic (c.f. $f = kx$).

Note: The elastic tensor $C_{ijk\ell}$ is (a representation of) a linear map from a second order tensor ε_{ij} to another one σ_{ij}. In this sense, $C_{ijk\ell}$ is called 4th order tensor.

In this lecture, we focus on the material whose elastic tensor has the following form:

$$C_{ijk\ell} = \lambda\delta_{ij}\delta_{k\ell} + \mu\delta_{ik}\delta_{j\ell} + \mu\delta_{i\ell}\delta_{jk},$$

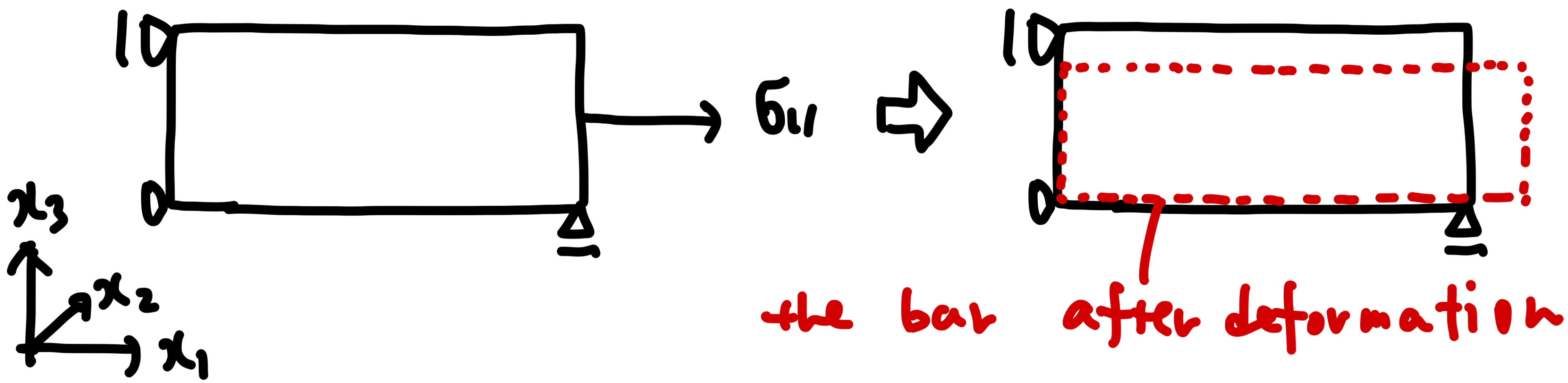
i.e. the isotropic elastic material, where λ and μ are the Lame constants.

Constitutive equation 2/2

Quiz: show that the constitutive equation for isotropic material is give as $\sigma_{ij} = \lambda \delta_{ij} \varepsilon_{kk} + 2\mu \varepsilon_{ij}$.

Poisson's ratio

Let us consider a bar subject to uniaxial tension.



The constitutive equations give

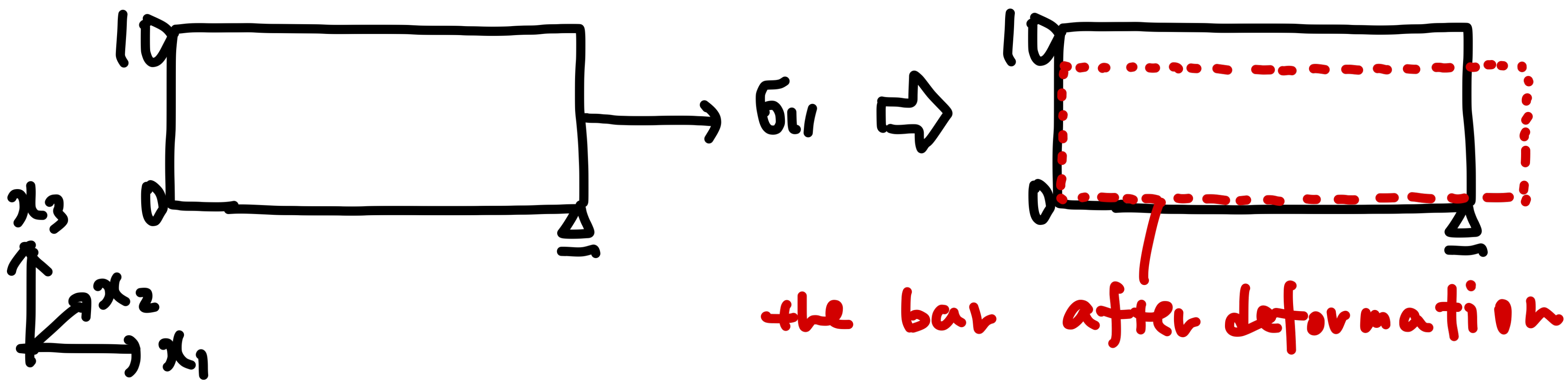
$$
\begin{aligned}
\sigma_{11} &= \lambda \varepsilon_{kk} + 2\mu\varepsilon_{11}, & \cdots(7)\\
0 = \sigma_{22} &= \lambda \varepsilon_{kk} + 2\mu\varepsilon_{22}, & \cdots(8)\\
0 = \sigma_{33} &= \lambda \varepsilon_{kk} + 2\mu\varepsilon_{33}\,. & \cdots(9)
\end{aligned}
$$

From (8) and (9), it is obvious that $\varepsilon_{22} = \varepsilon_{33}$. We then substitute this into (8) to have $\varepsilon_{22} = -\boxed{\dfrac{\lambda}{2(\lambda+\mu)}}\varepsilon_{11}$.

The poisson ratio ν

Young's modulus

Let us consider a bar subject to uniaxial tension.



The constitutive equations give

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\begin{aligned}
\sigma_{11} &= \lambda \varepsilon_{kk} + 2\mu \varepsilon_{11}, & \cdots(7)\\
0 = \sigma_{22} &= \lambda \varepsilon_{kk} + 2\mu \varepsilon_{22}, & \cdots(8)\\
0 = \sigma_{33} &= \lambda \varepsilon_{kk} + 2\mu \varepsilon_{33}\,. & \cdots(9)
\end{aligned}
$$

$\varepsilon_{22} = -\dfrac{\lambda}{2(\lambda+\mu)}\varepsilon_{11}$ is substituted into (7) to have $\sigma_{11} = \boxed{\dfrac{(3\lambda+2\mu)\mu}{\lambda+\mu}}\varepsilon_{11}$

The Young's modulus E

Note: The isotropic elastic material can be characterised by two material constants.

Two-dimensional elasticity 1/2

Suppose that $u_3(\boldsymbol{x}) = 0$, and $u_1(\boldsymbol{x})$ and $u_2(\boldsymbol{x})$ are independent of x_3.

e.g. the object is "very" thick in x_3 direction, and the external force is uniform in this direction.

→ Such a mechanical system is called in plane-strain state.

According to the definition of the plane-strain state, we have $u_3 = 0, \ \ u_{i,3} = 0$, and thus we also have $\varepsilon_{33} = \varepsilon_{13} = \varepsilon_{23} = \varepsilon_{31} = \varepsilon_{32} = 0$.

Quiz: derive the constitutive equations for such a material.

Two-dimensional elasticity 2/2

Similarly, if the stress in a mechanical system admits $\sigma_{33} = \sigma_{13} = \sigma_{23} = \sigma_{31} = \sigma_{32} = 0$, it is in plane-strain state.

Homework: derive the constitutive equations for a material in the plane-strain state.