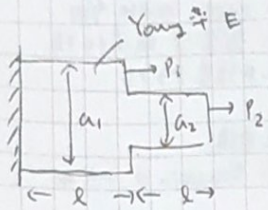
力学的システム準備 4/28(水)



Find $\mathbf{a}=(a_1,a_2)^t$ such that $\min_{\mathbf{a}} f(\mathbf{a})=\mathbf{u}\cdot\mathbf{P}$. — ①

subject to $K(\mathbf{a})\,\mathbf{u}(\mathbf{a})=\mathbf{P}$ — ②　$\mathbf{u}(\mathbf{a})=\begin{pmatrix}u_1(\mathbf{a})\\u_2(\mathbf{a})\end{pmatrix}$

$(a_1,a_2)\begin{pmatrix}l\\l\end{pmatrix}\le c$. — ③

しばらく、不等式制約なしの問題を考えよう。

Find $\mathbf{a}=(a_1,a_2)^t$ such that $\min f(\mathbf{a})=\mathbf{u}\cdot\mathbf{P}$ — ④

subject to $K(\mathbf{a})\,\mathbf{u}(\mathbf{a})=\mathbf{P}$. — ⑤

$$\begin{pmatrix}u_1\\u_2\end{pmatrix}=\frac{l}{E}\begin{pmatrix}\frac{P_1+P_2}{a_1}\\\frac{P_1+P_2}{a_1}+\frac{P_2}{a_2}\end{pmatrix}$$

(復習) 仮想仕事の原理を使うと ⑤ を具体的に書くことができる。

$$⑤\Leftrightarrow\underbrace{\frac{E}{l}\begin{pmatrix}a_1+a_2 & -a_2\\-a_2 & a_2\end{pmatrix}}_{=K}\begin{pmatrix}u_1\\u_2\end{pmatrix}=\begin{pmatrix}P_1\\P_2\end{pmatrix}\quad\Rightarrow\quad K^{-1}=\frac{l}{E}\begin{pmatrix}\frac{1}{a_1} & \frac{1}{a_1}\\\frac{1}{a_1} & \frac{1}{a_1}+\frac{1}{a_2}\end{pmatrix}\text{より}$$

最適化問題をとくためには ⑤をみたす $\mathbf{u}$ に対して ∇f を計算する必要がある。

(i) 代入法

$$\begin{aligned}f(\mathbf{a})&=\mathbf{u}\cdot\mathbf{P}\\&=\frac{l}{E}\left(\frac{P_1+P_2}{a_1},\ \frac{P_1+P_2}{a_1}+\frac{P_2}{a_2}\right)\begin{pmatrix}P_1\\P_2\end{pmatrix}\\&=\frac{l}{E}\left(\frac{P_1(P_1+P_2)}{a_1}+\frac{(P_1+P_2)P_1}{a_1}+\frac{P_2^2}{a_2}\right)\\&=\frac{l}{E}\left(\frac{(P_1+P_2)^2}{a_1}+\frac{P_2^2}{a_2}\right)\end{aligned}$$

$$\therefore\ \frac{\partial f}{\partial a_1}=-\frac{l}{E}\frac{(P_1+P_2)^2}{a_1^2}$$

$$\frac{\partial f}{\partial a_2}=-\frac{l}{E}\frac{P_2^2}{a_2^2}$$

$$\boxed{\Leftrightarrow\ \nabla f=-\frac{l}{E}\begin{pmatrix}(P_1+P_2)^2/a_1^2\\P_2^2/a_2^2\end{pmatrix}.}\ \text{— ⑥}$$

代入法は設計変数の数が増えると辛い…

(ii) 直接微分法

注: $\mathbf{P}$ は $\mathbf{a}$ に依存しない

$$f(\mathbf{a})=\mathbf{u}\cdot\mathbf{P}.\ \rightarrow\ \frac{\partial f}{\partial a_i}=\frac{\partial\mathbf{u}}{\partial a_i}\cdot\mathbf{P}\quad\left(=\left(\frac{\partial u_1}{\partial a_i}\ \ \frac{\partial u_2}{\partial a_i}\right)\begin{pmatrix}P_1\\P_2\end{pmatrix}\right)$$

なので、$\frac{\partial\mathbf{u}}{\partial a_i}$ が計算できれば良さそう。→ $K(\mathbf{a})\,\mathbf{u}(\mathbf{a})=\mathbf{P}$ の両辺を a_i で微分する

$$\frac{\partial K(\mathbf{a})}{\partial a_i}\mathbf{u}(\mathbf{a})+K(\mathbf{a})\frac{\partial\mathbf{u}(\mathbf{a})}{\partial a_i}=0\ \Leftrightarrow\ K(\mathbf{a})\frac{\partial\mathbf{u}}{\partial a_i}=-\frac{\partial K}{\partial a_i}\mathbf{u}.$$

$$\Leftrightarrow\ \frac{\partial\mathbf{u}(\mathbf{a})}{\partial a_i}=-\underset{\substack{\uparrow\\\text{知ってる。}}}{K^{-1}(\mathbf{a})}\ \underset{\substack{\uparrow\\\text{知らない}}}{\frac{\partial K(\mathbf{a})}{\partial a_i}}\ \mathbf{u}(\mathbf{a})\ \text{— ⑦.}$$

$$\frac{\partial K}{\partial a_1} = \frac{E}{l}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \frac{\partial K}{\partial a_2} = \frac{E}{l}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$ と簡単に計算できる.

⑦より

$$\frac{\partial \mathbf{u}(\mathbf{a})}{\partial a_1} = -K^{-1}\frac{\partial K}{\partial a_1}\mathbf{u}$$

$$= -\begin{pmatrix} \frac{1}{a_1} & \frac{1}{a_1} \\ \frac{1}{a_1} & \frac{1}{a_1}+\frac{1}{a_2} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

$$= -\begin{pmatrix} & \\ & \end{pmatrix}\begin{pmatrix} u_1 \\ 0 \end{pmatrix}$$

$$= -\begin{pmatrix} u_1/a_1 \\ u_1/a_1 \end{pmatrix}$$

$$\rightarrow \frac{\partial f}{\partial a_1} = \frac{\partial \mathbf{u}}{\partial a_1}\cdot \mathbf{P}$$

$$= -\begin{pmatrix} \frac{u_1}{a_1} & \frac{u_1}{a_1} \end{pmatrix}\begin{pmatrix} P_1 \\ P_2 \end{pmatrix}$$

$$= -\frac{u_1}{a_1}(P_1+P_2).$$

$u_1 = \frac{l}{E}\frac{P_1+P_2}{a_1}$ より

$$\boxed{\frac{\partial f}{\partial a_1} = -\frac{l}{E}\frac{(P_1+P_2)^2}{a_1^2}} \quad —⑧$$

$$\frac{\partial \mathbf{u}(\mathbf{a})}{\partial a_2} = -\begin{pmatrix} & \\ & \end{pmatrix}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

$$= -\begin{pmatrix} 1/a_1 & 1/a_1 \\ 1/a_1 & 1/a_1+1/a_2 \end{pmatrix}\begin{pmatrix} u_1-u_2 \\ -u_1+u_2 \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{u_1-u_2}{a_1} - \frac{-u_1+u_2}{a_1} \\ -\frac{u_1-u_2}{a_1} - \frac{-u_1+u_2}{a_1} - \frac{-u_1+u_2}{a_2} \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ \frac{u_1-u_2}{a_2} \end{pmatrix}.$$

$$\frac{\partial f}{\partial a_2} = \frac{\partial \mathbf{u}}{\partial a_2}\cdot \mathbf{P}$$

$$= \begin{pmatrix} 0 & \frac{u_1-u_2}{a_2} \end{pmatrix}\begin{pmatrix} P_1 \\ P_2 \end{pmatrix}$$

$$= \frac{u_1-u_2}{a_2}P_2$$

$u_1 - u_2 = -\frac{l}{E}\frac{P_2}{a_2}$ より $\boxed{\frac{\partial f}{\partial a_2} = -\frac{l}{E}\frac{P_2^2}{a_2^2}} \quad —⑨$

⑥と⑧⑨は一致する

② 直接微分法まとめ

0. $K\mathbf{u} = \mathbf{P}$ を解く. (m, 1回)

制約条件の数　　　設計変数の数

1. ⑦ $\Leftrightarrow$ $K(\mathbf{a})\frac{\partial \mathbf{u}}{\partial a_i} = -\frac{\partial K}{\partial a_i}\mathbf{u}$ を解く (m元連立方程式を n回 解く).

2. $\frac{\partial f}{\partial a_i} = \frac{\partial \mathbf{u}}{\partial a_i}\cdot \mathbf{P}$ を計算する.

$$\boxed{\text{連立方程式を } n+1 \text{ 回解く必要がある}}$$

③ 随伴変数法

最適化問題をLagrange乗数 $\tilde{\mathbf{u}} = (\tilde{u}_1, \tilde{u}_2)^t$ を使って次のように書く

Find $\mathbf{u}$ & $\tilde{\mathbf{u}}$ such that $\min_{\mathbf{a}, \tilde{\mathbf{u}}} \mathcal{L}(\mathbf{a}, \tilde{\mathbf{u}}) = f(\mathbf{a}) - \tilde{\mathbf{u}}^t(K\mathbf{u} - \mathbf{P})$

$K\mathbf{u} = \mathbf{P}$ より $\mathcal{L}(\mathbf{a}, \tilde{\mathbf{u}}) = f(\mathbf{a})$

$\nabla_{\mathbf{a}} \mathcal{L} = \nabla f \longrightarrow \frac{\partial f}{\partial a_i}$ を求めればOK

$$\frac{\partial L}{\partial a_i} = \frac{\partial f}{\partial a_i} - \tilde{u}^T\left(\frac{\partial K}{\partial a_i}u + K\frac{\partial u}{\partial a_i}\right).$$

$f = u^T p = p^T u$ より $\frac{\partial f}{\partial a_i} = p^T \frac{\partial u}{\partial a_i}$

$$= p^T \frac{\partial u}{\partial a_i} - \tilde{u}^T \frac{\partial K}{\partial a_i} u - \tilde{u}^T K \frac{\partial u}{\partial a_i}$$

$$= \underbrace{\left(p^T - \tilde{u}^T K\right)}_{=0} \frac{\partial u}{\partial a_i} - \tilde{u}^T \frac{\partial K}{\partial a_i} u.$$

ここで $\tilde{u}$ を、〰〰 $= 0^t$ となるように選ぶ。

$$\left(p^T - \tilde{u}^T K\right)^T = 0. \Leftrightarrow K^T \tilde{u} - p = 0.$$

と、$\frac{\partial L}{\partial a_i} = -\tilde{u}^T \frac{\partial K}{\partial a_i} u$

一方、今、$K^T = K$ より $\tilde{u} = u$. なので

$$= -u^T \frac{\partial K}{\partial a_i} u.$$

$Ku = p$ に注意すると $u = K^{-1} p. \to u^T = p^T K^{-T} = p^T K^{-1}.$

$$\boxed{\therefore \frac{\partial L}{\partial a_i} = p \cdot \left(-K^{-1}\frac{\partial K}{\partial a_i} u.\right) = p \cdot \frac{\partial u}{\partial a_i}}$$ ← 直接微分と一致

ずいはん変数法まとめ

⓪ $Ku = p$ をとく。(m元；1回)

① $K^T \tilde{u} = p$ をとく。(m元，1回)．

② $\frac{\partial f}{\partial a_i} = \frac{\partial L}{\partial a_i} = -\tilde{u}^T \frac{\partial K}{\partial a_i} u$ を計算

せっけいへんすうの数に[illegible]

連立方程式を2回解けばOK

$K = K^T$ のとき，$u = \tilde{u}$ なので 1回でOK

$\tilde{u}$：随伴変数，自己随伴 (self-adjoint)