MA8109 Stochastic Processes and Differential Equations
Kalman Bucy Filter

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Abstract

The Kalman-Bucy filter has emerged to become a very important application of stochastic differential equations, since it is common that the problem of estimating a hidden state arise in stochastic systems. In this report, we investigate the essential mathematical theory for the linear Kalman-Bucy filter. We finally will provide a simple example.

1 Introduction

The problem of filtering often arise when we want to establish estimation of a unknown state based on noisy observations. For this reason are filtering very useful in general state estimation and calibrations problems. This leads to countless applications, such as in finance for estimation of inflation rate given noisy price fluctuations. However, solving these problems generally leads to challenging problems which requires a deep underlying mathematical framework. In this report will we focus mainly on the basic theory of linear Kalman-Bucy filter (KBF) and provide an example.

2 Filtered Probability Space

It is important to be able to organize the total information structure in the system. Firstly, we can define a given probability space $(\Omega, \mathcal{A}, P)$, which is consisting of the sample space Ω, the σ-algebra $\mathcal{A}$ generated by all events and the given probability measure P. Let is introduce the family $\underline{\mathrm{A}} = \{\mathcal{A}_t, t \leq 0\}$, where the σ-algebra $\mathcal{A}_t = \sigma\{X_s : s \in [0,t]\}$ and is generated by all the events up to time t. We can then define the filtered probability space to be $(\Omega, \mathcal{A}, \underline{\mathrm{A}}, P)$. Generally in this report is the notation is based on Platen [3].

3 The Filtering Problem

. Let the filtered probability space have the form $(\Omega, \mathcal{A}, \underline{\mathrm{A}}, P)$ and suppose that we have a unknown hidden state X_t at time t following this linear stochastic differential equation (SDE),

$$dX_t = AX_t dt + BdW_t,$$

where $t \in [0,T]$, $A \in \mathbb{R}^{d\times d}$, $X_t \in \mathbb{R}^d$ and $B \in \mathbb{R}^{d\times m}$. W_t is defined as

$$W = \{W_t = (W_t^1, \dots, W_t^m), t \in [0,T]\} \qquad (1)$$

which is a m dimensional Wiener process. In addition to this will we assume that the first value X_0 is a Gaussian random variable. The general concept is to approximate the hidden state X_t given a set of observations,

$$Y = \{Y_t = (Y_t^1, \dots, Y_t^r), t \in [0,T]\},$$

so that it is a r dimensional process. Assuming there is a linear relationship in between observations Y_t and the hidden state X_t can we define the linear observational model,

$$dY_t = HX_t dt + \Gamma dW_t^*,$$

where $H \in \mathbb{R}^{d\times e}$, $\Gamma \in \mathbb{R}^{e\times n}$ and the n-dimentional Wiener process,

$$W^* = \{W_t^* = (W_t^{*,1}, \dots, W_t^{*,n}), t \in [0,T]\}. \qquad (2)$$

Also assume that the observational model noise W^* and hiddens state model noise W is independent. For each $t \in [0,T]$, let some σ- algebra have the form

$$\mathcal{A}_t = \sigma\{X_0, Y_s, W_s, s \in [0,T]\}$$

generated by X_0, Y_s and W_s for $s \in [0,t]$ which expresses the total information up to time t. Furthermore, let the σ- algebra,

$$\mathcal{Y}_t = \{Y_s : s \in [0,t]\},$$

provide the observational information . Thus we have $\mathcal{Y}_t \subset \mathcal{A}_t$ for $t \in [0,T]$, see Platen [3, p. 420].

4 Kalman-Bucy Filter

Using the the previous results can we define the Kalman Bucy filter (KBF) as the optimal estimate,

$$\hat{X}_t = E(X_t \mid \mathcal{Y}_t).$$

We define it so the least square estimate of X_t,

$$E(\|X_t - \hat{X}_t\|^2) \leq E(\|X_t - Z_t\|^2),$$

for all $\mathcal{Y}_t$- measurable random variable Z_t. Since $\hat{X}_t$ is $\mathcal{Y}_t$ measurably must it also be $\mathcal{A}_t$ measureable. In fact, let us define the covariance matrix,

$$C_t = E((X_t - \hat{X}_t)(X_t - \hat{X}_t)^T)),$$

then can it shown in the linear case that it does satisfy the Riccati equation [3, p. 421],

$$\begin{aligned}\frac{dC_t}{dt} =& AC_t + C_t A^T + BB^T \\ &- C_t H^T(\Gamma\Gamma^T)^{-1} HC_t.\end{aligned}$$

The inital value is assumed to be $C_0 = E(X_0 X_0^T)$.

Using the results from given in Kallianpur [1], can we observe that the linear KBF estimate $\hat{X}_t$ does satisfy the SDE,

$$\begin{aligned}d\hat{X}_t =& (A - C_t H^T(\Gamma\Gamma^T)^{-1})\hat{X}_t dt \\ &+ C_t H^T(\Gamma\Gamma^T)^{-1} dY_t,\end{aligned}$$

for $t \in [0, T]$. Observing that Y_t is a driving process. However, this formulation is also equivalent with the multi-dimensional linear KBF formulated in Øksendal [2, pp. 104].

5 Orthogonal Projection in Hilbert Spaces

A more deeper intuition of KBF comes from that least square estimate of a estimate in Hilbert space with respect to a subspace is a orthogonal projection onto the subspace.

Let a square integrable process,

$$f = \{f_s, s \in [0, T]\},$$

be adapted to the total filtration $A = (\mathcal{A}_t)_{t\in[0,T]}$ and formed by the observation filtration $\mathcal{Y} = \{\mathcal{Y}\}_{t\in[0,T]}$. Additionally define the orthogonal inner product such that

$$(f, g) = E\Big(\int_0^t f_s^T g \; ds\Big) = 0.$$

Furthermore, the filter error $\varepsilon_t = X_t - \hat{X}_t$ and in a KBF shown to obtain $(\varepsilon_t, \hat{X}_t) = 0$, see Platen [3, pp. 421].

6 Example

To demonstrate the simplest case of a KBF can we assume a linear 1-dimensional filtering problem,

$$\begin{aligned}dX_t &= FX_t dt + CdW_t \\ dY_t &= GX_t dt + UdW_t^*\end{aligned}$$

where the coefficients are constant, $F, C, G, U \in \mathbb{R} \setminus \{0\}$. Then Riccati equation consequently ends up like,

$$\frac{dS}{dt} = 2FS - \frac{G^2}{D^2}S^2 + C^2, \text{ assuming } S(0) = a.$$

This can be solved analytically, see Øksendal [2, pp. 103] for more details. The solution is

$$S(t) = \frac{\alpha_1 - K\alpha_2 \exp\left(\frac{(\alpha_2-\alpha_1)G^2 t}{D^2}\right)}{1 - K\exp\left(\frac{(\alpha_2-\alpha_1)G^2 t}{D^2}\right)},$$

where the constants are defined such that

$$\begin{aligned}\alpha_1 &= G^{-2}\left(FD^2 - D\sqrt{F^2D^2 + G^2, C^2}\right), \\ \alpha_2 &= G^{-2}\left(FD^2 + D\sqrt{F^2D^2 + G^2, C^2}\right), \\ K &= \frac{a^2 - \alpha_1}{a^2 - \alpha_2}.\end{aligned}$$

The hidden state will then have the form

$$\begin{aligned}\widehat{X}_t =& \exp\left(\int_0^t H(s)ds\right)\widehat{X}_0 \\ &+ \frac{G}{D^2}\int_0^t \exp\left(\int_s^t H(u)du\right) S(s)dY_s,\end{aligned}$$

where we define the function

$$H(s) = F - \frac{G^2}{D^2}S(s).$$

Let assume a large s, then the approximation $S(s) \approx \alpha_2$ holds. Finally, this gives the optimal estimation ends up on the form

$$\begin{aligned}\widehat{X}_t \approx& \widehat{X}_0 \exp\left(\left(F - \frac{G^2\alpha_2}{D^2}\right)t\right) + \\ &\frac{G\alpha_2}{D^2}\int_0^t \exp\left(\left(F - \frac{G^2\alpha_2}{D^2}\right)(t-s)\right)dZ_s \\ =& \widehat{X}_0\exp(-\beta t) + \frac{G\alpha_2}{D^2}\exp(-\beta t)\int_0^t \exp(\beta s)dY_s.\end{aligned}$$

where $\beta = D^{-1}\sqrt{F^2D^2 + G^2C^2}$.

References

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[2] B. Øksendal. *Stochastic Differential Equations: An Introduction with Applications*. Universitext. Springer Berlin Heidelberg, 2010. URL: https://books.google.no/books?id=EQZEAAAAQBAJ.

[3] Eckhard Platen and Nicola Bruti-Liberati. "Numerical Solution of Stochastic Differential Equations with Jumps in Finance." In: Berlin Heidelberg, Germany: Springer-Verlag, 2010. URL: https://books.google.no/books?id=XjNLshgxpxUC&lpg=PP1&hl=no&pg=PP1#v=onepage&q&f=false.