**TMA4195 - SIMPLE ATMOSPHERIC MODELS RELATED TO GLOBAL WARMING**

In this project, we will set up simple models that could help us to understand some of the mechanisms responsible for global warming.

There is a large scientific consensus on the fact that human emissions of greenhouse gas and in particular $CO_2$ in the atmosphere play an essential role in the currently observed global warming. The goal of the first part in this project is to understand the role of $CO_2$ in the global energy balance of the earth. In this part, we will consider global averaged values and the focus is on the heat transfers due to radiations.

In the second part, we look at the role of the polar regions. Polar regions play an essential role in the energy balance by their capacity of reflecting a much larger proportion of incoming solar radiations compared to other surfaces on Earth. At the same time, the existence of polar regions covered by ice depends directly from the temperature. Therefore, a spatial model including the heat fluxes from equatorial to polar regions is necessary to estimate the extent of the polar region and therefore the Earth overall capacity to reflect solar radiations.

## 1. Earth Energy Balance

The atmosphere temperature is the result of heat transfers, as depicted in Figure 1. Solar radiation enters the atmosphere and is either absorbed by the atmosphere and the earth or reflected back by the atmosphere, the clouds and the earth surface. In this part, we will only consider radiative heat transfers even if other form of transfers, such as convection and conduction, are essential to get an accurate picture. All the energy comes from the solar radiation but the temperature gradients it creates generate fluid motions in the occean (currents) and in the atmosphere (winds) that, in turn, transport the heat in a highly non-negligible way.

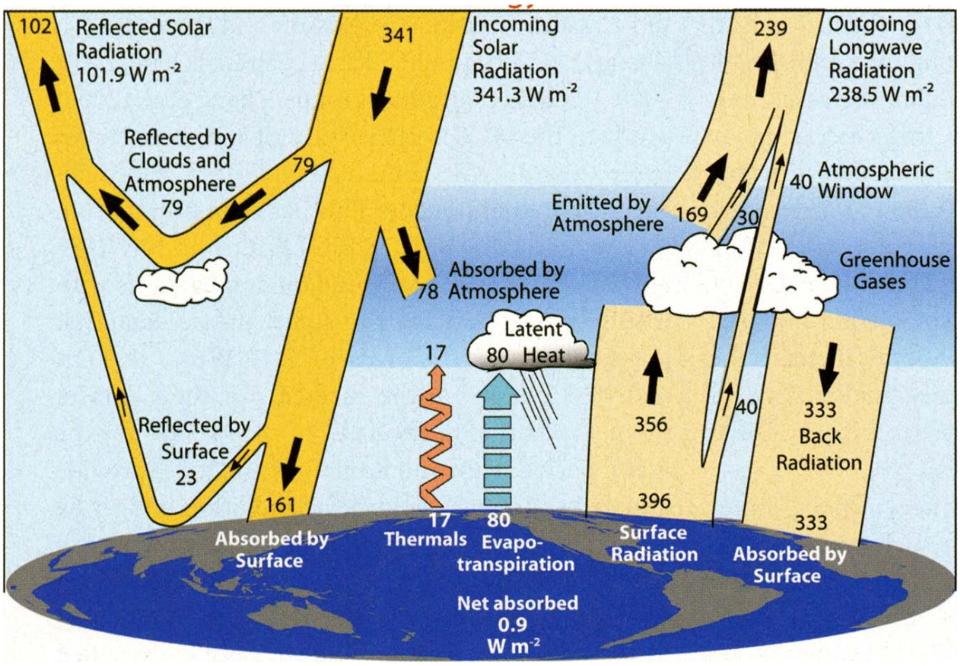


FIGURE 1. Global annual mean Earth's energy balance, from [1]

### 1.1. Background on radiation.

**1.1. Background on radiation.** In this section, we give a very short introduction to radiation, just enough to understand the basic models we will use. For more details, we highly recommend [2].

Any medium emits radiations, randomly in any direction, and the radiative power is a function of temperature. A black body corresponds to an ideal medium which does not reflect any radiation. The Planck's law

$$E_{b\nu}(T,\nu) = \frac{2\pi h\nu^3}{c_0^2(e^{\frac{h\nu}{kT}} - 1)} \tag{1}$$

gives the distribution of a black body emissive power in vaccum as a function of wavelength. It depends on the temperature and, by summing up the contribution coming from all frequencies, we obtain the Stefan-Boltzmann law for the black body emission power,

$$E_b(T) = \sigma T^4,$$

where $\sigma$ is the Stefan-Boltzmann constant, see [2, (1.20)]. This is an idealized situation and a general material is characterized by its emissivity $\varepsilon$, which is a value between zero and one, such that the emitted power is equal to $E(T) = \varepsilon\sigma T^4$, that is a given fraction of the power emitted by the black body. In our energy balance, the atmosphere and the earth will contribute to heat creation by their radiations. When radiation propagates within a participating medium, the energy is either transmitted, absorbed or scattered (reflected), see for example Figure 10-1 in [2]. If we neglect scattering effects and consider a light beam propagating in a given direction, the energy $I_\eta(s)$ at the position $s$ is given by

$$\frac{dI_\eta}{ds} = -\kappa_\eta I_\eta.$$

After integration over a given layer depth $L$, we obtain an absorption coefficient equal to $\alpha_\eta = 1 - e^{-\kappa_\eta L}$. Note that the media is also emitting its own radiations, which we have not considered. It can be proven, see [2, pages 11–47], that at local thermodynamic equilibrium, we have

$$\frac{dI_\eta}{ds} = \kappa_\eta(I_{b\eta} - I_\eta), \tag{2}$$

where $I_{b\eta}$ denotes the black-body radiation distribution given by the Planck law. For a given medium, which in our case is composed of the gases in the atmosphere, the value of $\kappa_\eta$ is strongly dependent on the frequency. Indeed, a given molecule absorbs specifically some given frequencies: This is the basis for spectroscopy. In Figure 2, we can observe directly the frequencies that have been absorbed by the molecules of the atmosphere. The Sun radiation, as observed at the top of the atmosphere, is very close to the radiation of the black body corresponding to the Sun average temperature.

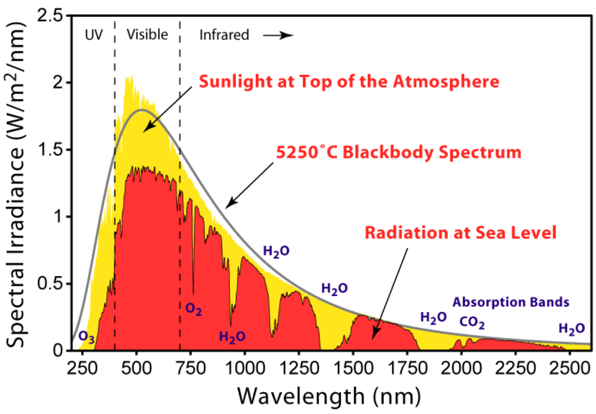


FIGURE 2. Solar irradiation onto Earth from [3].

1.2. **Modeling simplifications.** The absorption coefficient $\kappa_\eta$ depends on the frequency and on the temperature. The Sun and the Earth have very different temperature and therefore, they will emit in a very different spectrum range, as predicted by the black body distribution (1). We will not consider the whole continuous spectrum but nevertheless consider separately the short and long wave emissions. The short-waves ($sw$) come from the sun and the long-wave ($lw$) from the Earth, as depicted in Figure 1. In each of these two frequency ranges, we will consider constant effective absorption and reflection coefficients.

The computation of these effective absorption and reflection coefficients is intricate but we will try to use the data given in Table 1. An incoming energy radiation $P_{\text{in}}$ will thus be decomposed into an absorbed part, $P_{\text{abs}}$, a transmitted part $P_{\text{trans}}$ and a scattered or reflected part $P_{\text{refl}}$,

$$P_{\text{in}} = P_{\text{abs}} + P_{\text{refl}} + P_{\text{trans}} \tag{3}$$

with

$$P_{\text{abs}} = a(1-r)P_{\text{in}}, \quad P_{\text{refl}} = rP_{\text{in}} \quad \text{and} \quad P_{\text{trans}} = (1-a)(1-r)P_{\text{in}}.$$

The constants $a$ and $r$ are between 0 and 1. This approach is a simple approach based on transfer coefficients, as depicted in Figure 3

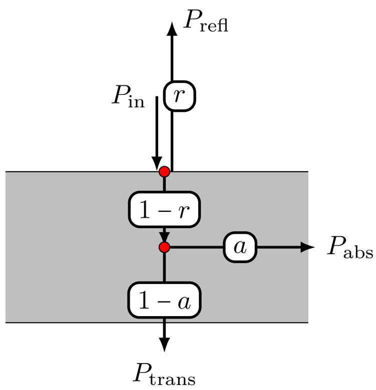


FIGURE 3. Transfer coefficient approach: We obtain any of the quantities indicated in the figure ($P_{\text{trans}}$, $P_{\text{refl}}$ and $P_{\text{abs}}$) by following the nodes in the graph and multiplying by the appropriate transfer coefficient. For example, $P_{\text{trans}}$ is equal to the power at the second node multiplied with $(1-a)$; the power at the second node is obtained by multiplying the power at the first node with $1-r$; thus we obtain $P_{\text{trans}} = (1-a)(1-r)P_{\text{ind}}$.

In the decomposition (3), the only source that is considered is the incoming radiation $P_{\text{in}}$. As we have seen in (2), the participating media, here the atmosphere, is also going to be a source of radiation, which should be considered. We propose to consider this emitting power as a separate source and propagate it using the transfer function approach. We consider two emitting regions, the atmosphere and the earth, with their respective temperature $T_A$ and $T_E$. The emissivity to compute the emitted radiation $P_A^0$ and $P_E^0$ from the atmosphere and the Earth, respectively, are given in Table 1. For the emitted power of the atmosphere, different temperature ranges (not accounted in our model) imply that there is more radiation emitted downwards than upwards. The ratio $f$ given in the same table accounts for that: From the total atmosphere emission, a fraction $f$ is emitted downwards and a fraction $1-f$ upwards.

Temperature differences induce a heat flux, which is represented by a simple linear relationship,

$$P_{E\to A} = -\alpha(T_A - T_E).$$

In fact the exchange of energy is much more complex and is responsible for the weather we observe but in the long term, we may assume that such relation may make sense.

In particular, a significant amount of heat is transported chemically through vapor, from the Earth to the atmosphere. When water condensates in the atmosphere, the heat that was stored in the water molecules at the earth surface when they were transformed in liquid form is released. We model this release process of *latent heat*, in a very rough way, also as a linear dependence,

$$P_{\text{latent}} = -\beta(T_A - T_E),$$

Question 1: ***Set up the radiation energy transfers using the simplified model using the description of Figure 2 and the data from table 1. Consider separately the short and long waves. From there, compute the energy balance and deduce a set of equation for the temperature***

You can choose the complexity of the model you want to consider so that you do not have to use all the parameters given in Table 1.

| parameter | symbol | value | unit |
|---|---|---|---|
| averaged solar flux | $P_S^0$ | 341.3 | $\mathrm{W\,m^{-2}}$ |
| Cloud cover | $C_C$ | 66.0 | % |
| *sw* molecular scattering coefficient | $r_{SM}$ | 10.65 | % |
| *sw* cloud scattering coefficient | $r_{SC}$ | 22.0 | % |
| *sw* Earth reflectivity | $r_{SE}$ | 17.0 | % |
| *sw* absorptivity: ozone | $a_{O3}$ | 8.0 | % |
| *sw* cloud absorptivity | $a_{SC}$ | 12.39 | % |
| *sw* absorptivity: H2O-CO2-CH4 | $a_{SW}$ | 14.51 | % |
| *lw* cloud scattering coefficient | $r_{LC}$ | 19.5 | % |
| *lw* Earth reflectivity | $r_{LE}$ | 0.0 | % |
| *lw* cloud absorptivity | $a_{LC}$ | 62.2 | % |
| *lw* absorptivity: H2O-CO2-CH4-O3 | $a_{LW}$ | 82.58 | % |
| Earth emissivity | $\varepsilon_E = 1 - r_{LE}$ | 100.0 | % |
| atmosph. emissivity | $\varepsilon_A$ | 87.5 | % |
| asymmetry factor | $f_A$ | 61.8 | % |
| sensible heat flux | $\alpha$ | 3 | $\mathrm{W\,m^{-2}\,K^{-1}}$ |
| latent heat flux | $\beta$ | 4 | $\mathrm{W\,m^{-2}\,K^{-1}}$ |

TABLE 1. Parameter values adapted from [1]

Question 2: ***From this set of equation, how do you obtain the sensitivity with respect to the modeling parameters? For example the sensitivity with respect to the absorption coefficients of the greenhouse gases,*** $a_{SW}$ ***and*** $a_{LW}$***.***

By sensitivity, we means the change in temperature as a response of the change of one of the parameters.

## 2. Zonal model and the role of the polar regions

In this part, we want to investigate the role of the polar regions. Polar regions are covered by ice and, for that reason, they play an essential role in the energy balance of the Earth due to their high capacity to reflect solar radiation. The ice appears in these regions because of the low temperature. Therefore, we are in presence of an amplification feedback mechanism: If the temperature increases, ice will disappear; then, less radiation will be reflected so the temperature will increase and this will induce the disappearance of more ice. Let us set up a model that attempts to determine the extend of the ice sheet by computing the temperature distribution with respect to the latitude, denoted $\phi$.

In general, the heat equation is given by

$$\frac{\partial}{\partial t}(c\rho T) = \nabla \cdot (k \nabla T) + Q \tag{4}$$

where $c$ is the specific heat capacity, $\rho$ the density, $k$ the thermal conductivity and $Q$ a volumetric heat source.

Question 3: ***Derive this equation as the expression of the conservation of energy.***

We consider the atmosphere at the surface of the earth and we will assume that its temperature $T$ depends only of the latitude. Then, we obtain the following form for the heat equation,

$$C_a \frac{\partial T}{\partial t} = D_\phi (K_a \frac{\partial T}{\partial \phi}) + q. \tag{5}$$

Question 4: ***What is the expression of the differential operator $D_\phi$? Rather obtain a physical derivation using control volume than using direct coordinate transformation formulas for the differential operators.***

We consider a source term of the form

$$q(t,x) = (1 - \alpha(x)) \frac{G_{\mathrm{SC}}}{4} S(x) - B_{\mathrm{out}} T(t,x) - A_{\mathrm{out}},$$

where $S(x)$ is the latitudinal distribution of incident solar radiation, $\alpha(x)$ is the *albido*, that is the portion of a radiation that is reflected back. The term $G_{\mathrm{SC}}$ denotes the solar constant, see [4], which we can take equal to $G_{\mathrm{SC}} = 1360\,\mathrm{W\,m^{-2}}$.

The term $A_{\mathrm{out}} + B_{\mathrm{out}} T$ is a linear approximation of the outgoing emitted long wave radiations. Let $x = \sin(\phi)$, the equilibrium equation takes the form

$$-D \frac{\partial}{\partial x}((1 - x^2) \frac{\partial T}{\partial x}) = -I(x) + QS(x) a(x, x_s). \tag{6}$$

The function $a(x, x_s)$ is the co-albedo, that is $1 - \alpha$, which gives the part of the radiation which is absorbed by the earth. The term $I = B_{\mathrm{out}} T + A_{\mathrm{out}}$ corresponds to the emission term. Here, $Q = \frac{G_{\mathrm{SC}}}{4}$ . We will take $A_{\mathrm{out}} = 201.4\,\mathrm{W\,m^{-2}}$ and $B_{\mathrm{out}} = 1.45\,\mathrm{W\,m^{-2}\,{}^\circ C^{-1}}$. The term $S(x)$ is an annual average for the given latitude of the solar exposition and we can consider the approximatio

$$S(x) = 1 + S_2 \frac{1}{2}(3x^2 - 1),$$

for $S_2 = -0.477$

Question 5: ***From pure geometrical considerations, derive an expression for $S(x)$. How does it compare with the approximation that is given here?***

The polar region is characterized by a low co-albedo. We denote by by $x_s$ the location of the ice-sheet. We can choose

$$a(x, x_s) = \begin{cases} a_u & = 0.38 \text{ if } x > x_s, \\ a_l & = 0.68 \text{ if } x_s < x. \end{cases}$$

or a smoother version, such as $a(x, x_s) = u_l(1 - H(x - x_s)) + u_u H(x - x_s)$, for $H(x) = \frac{1}{\pi}\mathrm{atan}(\frac{x}{\beta}) + \frac{1}{2}$ and $\beta > 0$ a given parameter.

Question 6: ***Explain how (6) is obtained. Let us consider the solution to (6) in $[0, x_s]$ and $[x_s, 1]$. What are the boundary conditions at 0, $x_s$ and 1? Describe a methodology to compute $x_s$***

The operator $u \mapsto -((1-x^2)u')'$ which enters equation (6) is the Legendre operator. General solutions of the Legendre operators are given by the Legendre functions, $P_n$, see [5]. In [6], you can see a plot of the first Legendre polynomials. The Legendre polynomials satisfy

$$-((1-x^2)P_n')' = n(n+1)P_n. \tag{7}$$

To solve (6) using Legendre polynomials, we proceed by using the ansatz $T(x) = \sum_{n=1}^{\infty} T_n P_n(x)$. Instead of computed $x_s$ for a given $Q$, we solve the easier problem of finding the solar energy $Q$ that results in a ice cap located at a given $x_s$. This problem is easier because $Q$ enters linearly in the equation (6), compared to $x_s$ which enters through the non-linear function $a$.

Question 7: ***Compute the solution to* (6) *for a given* $x_s$ *using Legendre polynomials. Find an expression for* $Q(x_s)$*.***

Question 8: ***Setup a numerical scheme for* (6)*. Compute the function* $Q(x_s)$ *numerically.***

We have computed an equilibrium solution and we now want to check its stability. To do so, we consider a perturbation $\delta T(t,x)$ around the equilibrium solution $T(x)$. By neglecting the higher-order term in (6), we can derive an evolution equation for $\delta T$. Note that the equation we obtain is, by construction, linear.

Question 9: ***Derive the equation for* $\delta T$*.***

This equation takes the form

$$\frac{\partial \delta T}{\partial t} = \mathcal{M}(\delta T) \tag{8}$$

for some operator $\mathcal{M}$. To check the stability we use the ansatz $\delta T(t,x) = e^{\lambda t} u(x)$ and computes the admissible values for $\lambda$ for a $u \neq 0$.

Question 10: ***Discretize the equation* (8) *you have obtained and use this result to test the stability of the solution***

## References


[1] Kevin E Trenberth, John T Fasullo, and Jeffrey Kiehl. "Earth's global energy budget". In: *Bulletin of the American Meteorological Society* 90.3 (2009), pages 311–324.

[2] Michael F Modest. *Radiative heat transfer*. Academic press, 2013.

[3] Wikipedia. [Online; accessed 15-October-2020]. URL: https://commons.wikimedia.org/wiki/File:Solar_Spectrum.png.

[4] Wikipedia. [Online; accessed 12-October-2020]. URL: https://en.wikipedia.org/wiki/Solar_constant.

[5] Kwong-Tin Tang. *Mathematical methods for engineers and scientists - Fourier analysis, partial differential equations and variational methods*. Volume 3. Springer, 2007.

[6] Wikipedia. [Online; accessed 12-October-2020]. URL: https://en.wikipedia.org/wiki/Legendre_polynomials.