NTNU – Trondheim
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Department of Mathematical Sciences

Examination paper for **TMA4145 Linear Methods-Solutions**

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Permitted examination support material: D: No written or handwritten material. Calculator Casio fx-82ES PLUS, Citizen SR-270X, Hewlett Packard HP30S

Other information:
The exam consists of twelve questions, and their order is not according to the level of difficulty. All solutions should be stated in a precise and rigorous way, with any assumptions written down and arguments justified. Each solution will be graded as *rudimentary* (F), *acceptable* (D), *good* (C) or *excellent* (A). Five acceptable solutions guarantee an E; seven acceptable with at least one good a D; seven acceptable with at least five good a C; nine good with at least two excellent a B; nine good with at least seven excellent an A. These are guaranteed limits. Beyond that, the grade is based on the total achievement.

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Problem 1

a) State (without proof) whether the assertion is true or false.

1. A Lipschitz continuous function is uniformly continuous. **True**
2. The range of a linear operator T on a normed space X is always closed. **False**
3. Suppose f is a function on $\mathbb{R}$. Then $d(x, y) = |f(x) - f(y)|$ defines a metric on $\mathbb{R}$. **False**
4. $\mathbb{R}^n$ with $\|(x_1, x_2, ..., x_n)\|_\infty = \max_i |x_i|$ is complete. **True**
5. A contraction on a non-zero metric space has a unique fixed point. **False**

b) Define the following notions.

1. Define the **orthogonal complement** of a subspace M of a Hilbert space $\mathcal{H}$.
   Answer: The orthogonal complement $M^\perp$ is the following subspace of $\mathcal{H}$: $M^\perp = \{x \in \mathcal{H} : \langle x, y\rangle = 0 \text{ for all } y \in M\}$.
2. Let T be a linear operator between two normed spaces $(X, \|.\|_X)$ and $(Y, \|.\|_Y)$. Define the **operator norm** of T.
   Answer: The operator norm of T is given by $\|T\| = \sup_{x \in X} \frac{\|Tx\|_Y}{\|x\|_X}$.
3. Let T be a linear mapping on $\mathbb{C}^n$. Define the notion of a **generalized eigenvector**.
   Answer: A generalized eigenvector to an eigenvalue λ of T is a vector $x \in \mathbb{C}^n$ satisfying: $(T - \lambda I)^k x = 0$ for some positive integer k greater than 1. Equivalently, x is an element of the kernel/nullspace of $(T - \lambda I)^k$.
4. Suppose f is a function between two metric spaces (X, d_X) and (Y, d_Y). Define the notion of **uniform continuity** for f.
   Answer: The function f is called uniformly continuous if for any $\epsilon > 0$ there exists a $\delta > 0$ such that for all $x, x' \in X$ with $d_X(x, x') < \delta$ we have $d_Y(f(x), f(x')) < \epsilon$.
5. Define the notion of a **nilpotent** operator $T : V \to V$ for a finite-dimensional vector space V.
   Answer: The operator T is called nilpotent if there exists a positive integer p such that $T^p = 0$.

Problem 2 Let T be the linear operator on the space of polynomials $\mathcal{P}_2$ of degree at most 2 defined by $Tf(x) = -f(x) - f'(x)$.

a) Find the matrix representation of T with respect to the basis $1, x, x^2$ of $\mathcal{P}_2$ and its characteristic polynomial.

We identify $\mathcal{P}_2 = \{a_0 + a_1 x + a_2 x^2 : a_i \in \mathbb{C} \text{ for } i = 0,1,2\}$ with $\mathbb{C}^3$ via $f(x) = a_0 + a_1 x + a_2 x^2 \longmapsto \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix}$.

(•) Hence $Tf(x) = -f(x) - f'(x) = -(a_0 + a_1) + (-a_1 - 2a_2)x - a_2 x^2$

has the following matrix representation

$$\begin{pmatrix} -1 & -1 & 0 \\ 0 & -1 & -2 \\ 0 & 0 & -1 \end{pmatrix},$$

which describes the action of T on the coefficients of the polynomial.

$$T\begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} -1 & -1 & 0 \\ 0 & -1 & -2 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} -a_0 - a_1 \\ -a_1 - 2a_2 \\ -a_2 \end{pmatrix}.$$

(•) The characteristic polynomial of $T = \begin{pmatrix} -1 & -1 & 0 \\ 0 & -1 & -2 \\ 0 & 0 & -1 \end{pmatrix}$ is given by $p(\lambda) = \det(\lambda I - T)$ (or by $\det(T - \lambda I)$), i.e.

$$p(\lambda) = \det \begin{pmatrix} \lambda+1 & -1 & 0 \\ 0 & \lambda+1 & -2 \\ 0 & 0 & \lambda+1 \end{pmatrix} = (\lambda+1)^3.$$

b) Find the generalized eigenvectors and eigenvalues of T. Determine a basis for the space of generalized eigenvectors.

The eigenvalues of T are determined by the zeros of the characteristic polynomial, i.e. $p(\lambda) = (\lambda+1)^3 = 0$. Hence $\lambda = -1$ is the only eigenvalue of T with algebraic multiplicity 3.

The generalized eigenvectors of T are by definition the elements of the kernels of $(T-\lambda I)^2$ and $(T-\lambda I)^3$, but it is common also to consider the eigenvectors of T as generalized eigenvectors (as many of you did).

(°) eigenvectors: $\begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = 0 \rightsquigarrow \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

(') gen. eigenvector: $(T-\lambda I)^2 \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = 0$

$$\Leftrightarrow \begin{pmatrix} 0 & 0 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = 0 \rightsquigarrow \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

We have that $(T-\lambda I)^3 = 0$, i.e. $T+I$ is nilpotent of order 3. We pick $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ as an eigenvector, since it gives us a "nice" basis for the space of generalized eigenvectors.

In terms of polynomials, the generalized eigenvectors are of the form $f(x) = a_0 + a_1 x + a_2 x^2$, since $T+I$ = differentation operator D, $Df(x) = f'(x)$ which satisfies $D^3(a_0 + a_1 x + a_2 x^2) = 0 = (T+I)^3 f(x)$. $\square$

Problem 3 We consider the matrix $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$.

a) Determine the eigenvalues and eigenvectors of A^*A.

Recall A^* is defined to be the complex-transpose of a matrix A. Hence $A^* = \begin{pmatrix} 1 & 1 & -1 \\ 1 & -1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$ and so we get that $A^*A = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$.

The characteristic polynomial for A^*A is given by

$$\det\begin{pmatrix} 3-\lambda & 0 & 0 \\ 0 & 2-\lambda & 1 \\ 0 & 1 & 2-\lambda \end{pmatrix} = (3-\lambda)[(2-\lambda)(2-\lambda)-1] =$$
$$= (3-\lambda)(\lambda-3)(\lambda-1)$$

Hence the eigenvalues are $\lambda_1 = 1$ with alg. multiplicity 1 and $\lambda_2 = 3$ with alg. multiplicity 2.

(•) Eigenvectors for $\lambda_1 = 1$ satisfy $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$

$\rightsquigarrow v_1 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$.

(•) Eigenvectors for $\lambda_2 = 3$: $\begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$

$\rightsquigarrow \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = v_2$

Since the alg. multiplicity is 2 we have to deal again with generalized eigenvectors, i.e. elements in the kernel of $(A^*A - 3I)^2$.

3a) (continued)

$$(A^*A-3I)^2=\begin{pmatrix}0&0&0\\0&2&-2\\0&-2&2\end{pmatrix} \rightsquigarrow V_3=\begin{pmatrix}1\\0\\0\end{pmatrix} \left(\text{or } \begin{pmatrix}1\\1\\1\end{pmatrix}\right)$$

3b) We have to find unitary matrices U and V, s.t.

$A=U\Sigma V^*$ where Σ is ~~for~~ a diagonal matrix containing the singular values of A, which are the square roots of the eigenvalues of A^*A. In our case, $\sigma_1=\sigma_2=\sqrt{3}$ and $\sigma_3=1$ $\rightsquigarrow$ $\Sigma=\begin{pmatrix}\sqrt{3}&0&0\\0&\sqrt{3}&0\\0&0&1\end{pmatrix}$.

(•) V consists of the normalized eigenvectors of A^*A, which by Part a) are given by $v_1=\begin{pmatrix}1\\0\\0\end{pmatrix}$, $v_2=\frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\1\end{pmatrix}$, $v_3=\frac{1}{\sqrt{2}}\begin{pmatrix}0\\-1\\1\end{pmatrix}$.

$$\rightsquigarrow V=\begin{pmatrix}1&0&0\\0&1/\sqrt{2}&-1/\sqrt{2}\\0&1/\sqrt{2}&1/\sqrt{2}\end{pmatrix}$$

(•) The columns of $U=(u_1|u_2|u_3)$ are given by $u_i=\frac{1}{\sigma_i}Av_i$, $i=1,2,3$.

$$\rightsquigarrow u_2=\frac{1}{\sqrt{6}}\begin{pmatrix}2\\-1\\1\end{pmatrix},\ u_3=\frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\1\end{pmatrix},\ u_1=\frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\-1\end{pmatrix}$$

$$\rightsquigarrow U=\begin{pmatrix}1/\sqrt{3}&\sqrt{\frac{2}{3}}&0\\1/\sqrt{3}&-1/\sqrt{6}&1/\sqrt{2}\\1/\sqrt{3}&\frac{1}{\sqrt{6}}&1/\sqrt{2}\end{pmatrix}. \quad \square$$

b) Find the Singular Value Decomposition of A.

see preceding page.

Problem 4 Let T be the linear operator defined on ℓ^2 by

$$T(x_1, x_2, ...) = (0, 2x_1, x_2, 2x_3, ...).$$

a) Show that T is a bounded operator on ℓ^2 and determine the adjoint of T. (The operator may be viewed as the composition of a multiplication operator and the ~~left~~ shift operator on ℓ^2.)

Correction: right (Sorry.)

(•) We have to show that there exists a $C>0$ s.t. $\|Tx\|_2 \le C\|x\|_2$

$$\|Tx\|_2^2 = \sum_{i=1}^{\infty} 2^2|x_{2i-1}|^2 + \sum_{i=1}^{\infty}|x_{2i}|^2 \le 2^2 \sum_{i=1}^{\infty}|x_i|^2 = 4\|x\|$$

$\rightsquigarrow$ $\|Tx\|_2 \le 2\|x\|_2$

Or use the hint to split T up into $R \circ M_\lambda$, where $M_\lambda x = (2\cdot x_1, 1\cdot x_2, 2x_3, 1\cdot x_4, ...)$ is the multiplication operator for $\lambda = (2,1,2,1,...)$ and R is the right shift op $R(x_1, x_2, ...) = (0, x_1, ...)$

R & M_λ are bounded op. on ℓ^2: $\|M_\lambda x\|_2 \le 2\|x\|_2$ and $\|Rx\|_2 = \|x\|_2$ $\Rightarrow$ $\|T\|_{op} = \|RM_\lambda\|_{op} \le \|R\|_{op}\|M_\lambda\|_{op} =$

(•) By definition we have to find an operator T^* on ℓ^2 s.t. $\langle Tx, y\rangle = \langle x, T^*y\rangle$

Either by direct computation or using $R^* = L$ left shift op. ($L(x_1, x_2, ...) = (x_2, x_3, ...)$) and $M_\lambda^* = M_\lambda$ we get that $T^*x = (2x_2, x_3, 2x_4, ...)$. □

b) Determine the kernel and the range of T. Use these results to find the kernel and the range of T^*.

kernel: $\ker(T) = \{x \in \ell^2 : Tx = 0\} = \{0\} \Rightarrow T$ is injective

range: $\operatorname{ran}(T) = \{x \in \ell^2 : x_1 = 0\}$.

From $\ker(T^*) = (\operatorname{ran}(T))^\perp = \{x \in \ell^2 : x_i = 0 \text{ for } i \geq 2\}$
$= \{x \in \ell^2 : (\alpha, 0, \dots) \; \alpha \in \mathbb{C}\}$

and $(\operatorname{ran}(T^*)) = (\ker(T))^\perp = \ell^2$
(or direct computation).

Problem 5 Let $(X, \langle .,. \rangle)$ be an inner product space.

a) Define the linear functional $\varphi_y(x) := \langle x, y \rangle$ for a fixed $y \in X$. Show that $\varphi_y : X \to \mathbb{C}$ is bounded.

The problem is just a reformulation of the Cauchy-Schwarz inequality:

$|\langle x, y \rangle| \leq \|x\| \|y\|$, since this gives us

$|\varphi_y(x)| \leq \|y\| \|x\|$ for all $x \in X$, i.e.

$|\varphi_y(x)| \leq C \|x\|$ for any $C \geq \|y\|$. $\square$

b) Suppose that (x_n) and (y_n) are convergent sequences, with $\lim_n x_n = x$ and $\lim_n y_n = y$. Show that $\lim_n \langle x_n, y_n\rangle = \langle x, y\rangle$.

To show: for any $\varepsilon > 0$ there exists N s.t. for all $n \geq N$ we have $|\langle x, y\rangle - \langle x_n, y_n\rangle| < \varepsilon$, for $x_n \to x$, $y_n \to y$

$$|\langle x_n, y_n\rangle - \langle x, y\rangle| = |\langle x_n, y_n\rangle - \langle x, y_n\rangle + \langle x, y_n\rangle - \langle x, y\rangle|$$

$$= |\langle x_n - x, y_n\rangle + \langle x, y_n - y\rangle| \leq$$

$$\leq |\langle x_n - x, y_n\rangle| + |\langle x, y_n - y\rangle| \quad \text{use Cauchy-Schwarz}$$

$$\leq \|x_n - x\| \, \|y_n\| + \|x\| \, \|y_n - y\|.$$

Since $x_n \to x$ and $y_n \to y$ we have ~~that~~ for any $\varepsilon > 0$ there exists an N s.t. $\|x_n - x\| < \varepsilon/2$ and $\|y_n - y\| < \varepsilon/2$ for $n \geq N$ and (y_n) is a bounded sequence. Hence, we get that

$$|\langle x_n, y_n\rangle - \langle x, y\rangle| \leq C \|x_n - x\| + \|x\| \, \|y_n - y\| \leq K\varepsilon$$

for $n \geq N$. $\square$

Problem 6 We define the following matrix and vector:

$$A = \begin{pmatrix} 5 & 1 & 0 \\ 2 & 8 & 0 \\ 0 & 1 & 3 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}.$$

a) Use Banach's fixed point theorem to solve $Ax = b$ for the normed space $(\mathbb{R}^3, \|.\|_\infty)$. In other words, write $Ax = b$ in the form $x = Bx + c$ such that B is a contraction with a constant K on $\mathbb{R}^3$ with respect to the norm $\|y\|_\infty = \max\{|y_1|, |y_2|, |y_3|\}$. Then show that one can solve this problem by iteration starting from any $x_0 \in \mathbb{R}^3$.

In order to invoke Banach's fixed point theorem we have to check if $(\mathbb{R}^3, \|.\|_\infty)$ is complete and if we can write our problem in a way s.t. we have a contraction.

(•) $(\mathbb{R}^3, \|.\|_\infty)$ is a Banach space, i.e. a complete normed space; as shown in the course.

(•) The problem asks for a solution to

$$\begin{aligned} 5x_1 + x_2 &= 1 \\ 2x_1 + 8x_2 &= 2 \\ x_2 + 3x_3 &= 3 \end{aligned}$$

which is equivalent to the following system

$$\begin{aligned} x_1 &= \frac{1}{5} - \frac{1}{5}x_2 \\ x_2 &= \frac{1}{4} - \frac{1}{4}x_1 \\ x_3 &= 1 - \frac{1}{3}x_2 \end{aligned} \iff \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = B\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \text{ for}$$

$$B = \begin{pmatrix} 0 & -\frac{1}{5} & 0 \\ -\frac{1}{4} & 0 & 0 \\ 0 & -\frac{1}{3} & 0 \end{pmatrix}$$

The matrix B is a contraction on $(\mathbb{R}^3, \|.\|_\infty)$ since the maximal row sum is less than 1, namely $\frac{1}{3}$. $\square$

b) We denote the fixed point of the problem by $\tilde{x}$ and by (x_n) the sequence of iterations. Show that

$$d_\infty(x_n, \tilde{x}) \leq \frac{K^n}{1-K} d_\infty(x_0, x_1),$$

where K is the contraction constant K from part a).

$x_n = T^{n-1} x_0$ for some $x_0 \in \mathbb{R}^3$ and

$$d_\infty(x_n, x_{n+1}) = d_\infty(Tx_{n-1}, Tx_n) \leq K d_\infty(x_{n-1}, x_n) \leq \dots \leq K^n d_\infty(x_0, x_1)$$

and we have for $m > n$:

$$\begin{aligned} d_\infty(x_n, x_m) &\leq d_\infty(x_n, x_{n+1}) + \dots + d_\infty(x_{m-1}, x_m) \\ &\leq K^n d_\infty(x_0, x_1) + K^{n+1} d_\infty(x_0, x_1) + \dots + K^{m-1} d_\infty(x_0, x_1) \\ &= (K^n + K^{n+1} + \dots + K^{m-1}) d_\infty(x_0, x_1) \\ &\leq (K^n + K^{n-1} + \dots) d_\infty(x_0, x_1) \\ &= \frac{K^n}{1-K} d_\infty(x_0, x_1) \end{aligned}$$

Let $m \to \infty$ in the last inequality, we obtain

$$\boxed{d(x_n, \tilde{x}) \leq \frac{K^n}{1-K} d_\infty(x_0, x_1)}. \quad \square$$

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