NTNU
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Week 41: Lecture 2

Properties of birth and death processes

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Information

Problem 2 iv) and v) on this week's exercise sheet requires you to read Section 6.4.

Section 6.2

Example

Assume we have two individuals. Individual 1 will die after a random time ϵ_1 and individual 2 will die after a random time ϵ_2. Assume that $\epsilon_1, \epsilon_2 \overset{\text{iid}}{\sim} \mathrm{Exp}(\alpha)$, $\alpha > 0$, and define

$$X(t) = \text{“Number of individuals alive at time } t\text{”}, \quad t \geq 0.$$

a) Is $\{X(t) : t \geq 0\}$ a continuous-time Markov chain?
b) Is $\{X(t) : t \geq 0\}$ a birth and death process?
c) What are the birth and death rates?

Theorem

If $T_i \sim \mathrm{Exp}(\alpha_i)$ *with* $\alpha_i > 0$, $i = 1, 2, \ldots, n$, *and* $T_1, T_2, \ldots, T_n$ *are independent, then*

$$\min\{T_1, T_2, \ldots, T_n\} \sim \mathrm{Exp}\left(\sum_{i=1}^{n} \alpha_i\right).$$

Examples of pure death processes

1. Constant death rate: $\mu_i = \mu$ for $i \geq 1$.
2. Linear death process: $\mu_i = i\mu$ for $\mu > 0$ and $i \geq 1$.

Section 6.3

VERY IMPORTANT!

Theorem

In a birth and death process with birth rates $\lambda_0, \lambda_1, \ldots > 0$, and death rates $\mu_0 = 0$ and $\mu_1, \mu_2, \ldots > 0$, we have

1. *sojourn times are independent;*
2. *the sojourn time each time you visit state i is $\mathrm{Exp}(\lambda_i + \mu_i)$, $i = 0, 1, \ldots$.*

Note: Also valid for a finite state space $\{0, 1, \ldots, N\}$ together with $\lambda_N = 0$.

Example

Assume a birth and death process with state space $\{0, 1, 2\}$ with birth rates $\lambda_0 = 5$, $\lambda_1 = 4$ and $\lambda_2 = 0$, and death rates $\mu_0 = 0$, $\mu_1 = 3$ and $\mu_2 = 6$.

a) Which state is on average visited the longest each time?

b) Assume the initial state is 0 and rates have unit min^{-1}. What is the expected number of seconds until the second transition occurs?

VERY IMPORTANT!

Theorem

Consider a birth and death process with birth rates $\lambda_0, \lambda_1, \ldots$ *and death rates* $\mu_0, \mu_1, \ldots$. *After the sojourn time in state* i *ends, the process jumps either to state* $i-1$ *or to state* $i+1$. *The jump probabilities are*

$$\Pr\{i \to i+1\} = \frac{\lambda_i}{\lambda_i + \mu_i},$$
$$\Pr\{i \to i-1\} = \frac{\mu_i}{\lambda_i + \mu_i},$$

Definition (Alternative definition)

The birth and death process with birth rates $\lambda_0, \lambda_1, \ldots$ and death rates $\mu_0, \mu_1, \ldots$ can be constructed in the following way. Whenever, you jump to state i, two competing processes start:

1) $T_1 =$ "time until birth" $\sim \mathrm{Exp}(\lambda_i)$.
2) $T_2 =$ "time until death" $\sim \mathrm{Exp}(\mu_i)$.

If the next event is a birth, we jump to $i + 1$, and if the next event is a death we jump to $i - 1$.

Example

Consider the birth and death process illustrated above. When leaving state 1, what is the probability to jump to state 0 and what is the probability to jump to state 2?

Simulation of a birth and death process

Input:

- i_0: initial state
- B: number of jumps
- $\lambda_0, \lambda_1, \ldots$: birth rates
- $\mu_0, \mu_1, \ldots$: death rates

Simulation of a birth and death process

Algorithm:

1. set $x_0 = i_0$ and $t_0 = 0$.
2. for $b = 1 \ldots B$
3. set $i = x_{b-1}$
4. draw $s \sim \mathrm{Exp}(\mu_i + \lambda_i)$ and set $t_b = t_{b-1} + s$
5. draw $u \sim \mathcal{U}(0, 1)$
6. if $u < \lambda_i/(\lambda_i + \mu_i)$
7. set $x_b = i + 1$
8. else
9. set $x_b = i - 1$
10. end
11. end

Simulation of a birth and death process

Output:

$$x(t) = \begin{cases} x_0, & 0 \leq t < t_1 \\ x_1, & t_1 \leq t < t_2 \\ \vdots & \\ x_{B-1}, & t_{B-1} \leq t < t_B \\ x_B, & t = t_B. \end{cases}$$

Simulation example

Consider the birth and death process $\{X(t) : t \geq 0\}$ with birth rates $\lambda_0 = \lambda_1 = \ldots \lambda_{19} = 0.5$ [cars per minute] and $\lambda_{20} = 0$, and death rates $\mu_i = i/30$ [cars per minute], $i = 0, 1, \ldots, 20$.