TMA 4190 Introduction to Topology
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Lecture 06[1]

6. Immersions and Embeddings

Last time we studied immersions. Recall:

Local nature

To be an immersion is a **local condition**. For example, if $\dim X = \dim Y$, then being an immersion means being a local diffeomorohism. Hence in order to say more about f we need to **add some (more global) topological properties to the local differential data**.
For example, for a **local** diffeomorphism to be a **global** one, it has to be one-to-one and onto.

Let us look at the image of an immersion. The nicest possible case is the image of the canonical immersion $\mathbb{R}^n \hookrightarrow \mathbb{R}^m$. The Local Immersion Theorem tells us that **locally** any immersion looks like **the canonical one**. But we are now going to see:

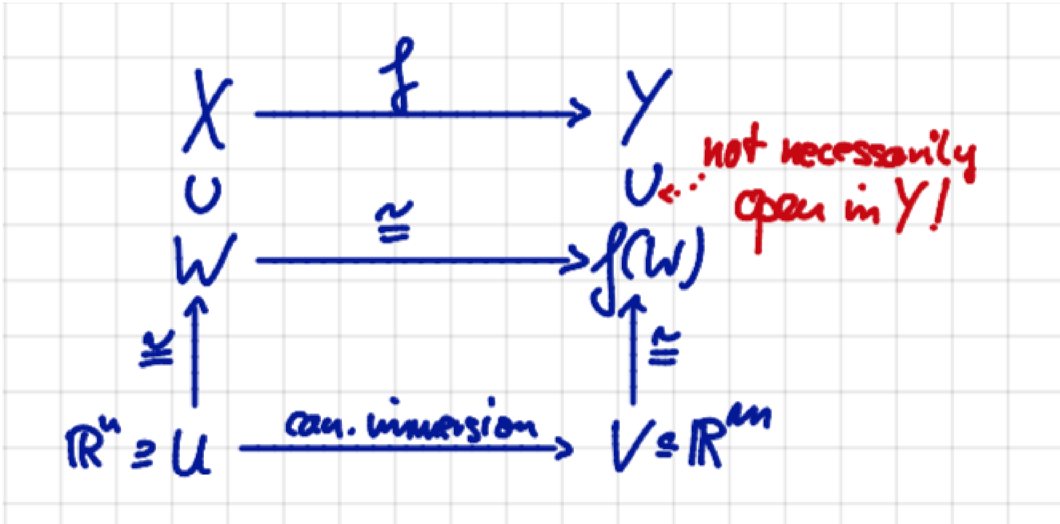
Be aware!

The image of an immersion is **not always a submanifold**.

Let us try to understand **what can go wrong**:

Let $f \colon X \to Y$ be an immersion. Then we know from the Local Immersion Theorem that f maps any sufficiently **small neighborhood** W of any point x in X **diffeomorphically onto its image** $f(W) \subset Y$. (By the LIT, W is diffeomorohic to a $U \subset \mathbb{R}^n$ which sits canonically in $V \subset \mathbb{R}^m$ which is diffeomorphic to $f(W)$, see the picture.)

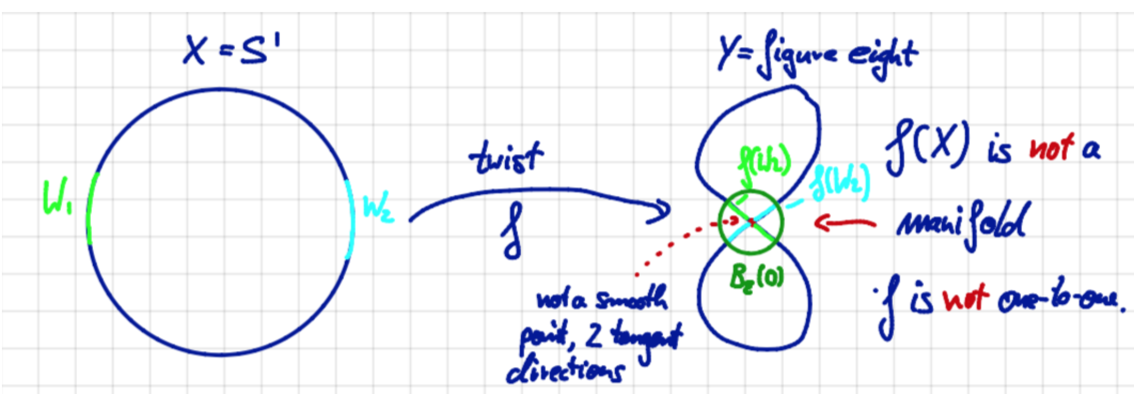
[1]Following the books of Guillemin and Pollack: Differential Topology, and Milnor: Topology from the differentiable viewpoint.



Not open in Y?

Hence every point in $f(X)$ lies in a subset which is diffeomorphic to an open subset in $\mathbb{R}^n$. Isn't that the definition of $f(X)$ being a submanifold?
No. The problem is that $f(W)$ does **not need to be open in** Y. Hence we cannot garantuee that points in $f(X)$ are in **parametrizable open neighborhoods**. UGH!

Before we try to find a global condition to fix this issue, let us look at **some examples of immersions whose image is not a submanifold**.



In the example above, f is not one-to-one and $f(X)$ has a point that is not smooth.

But even when f is one-to-one, this can happen, as the next example demonstrates. The image $f(X)$ is the same as above and not a manifold.

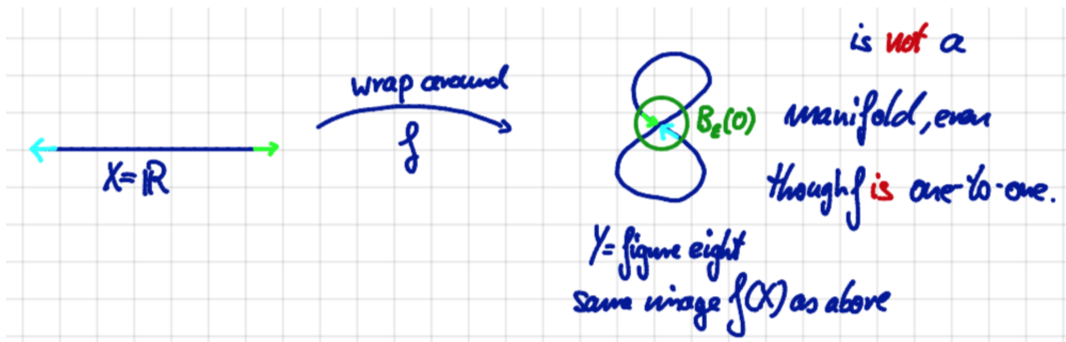


Figure eight immersion

In this example, the map f can be defined as

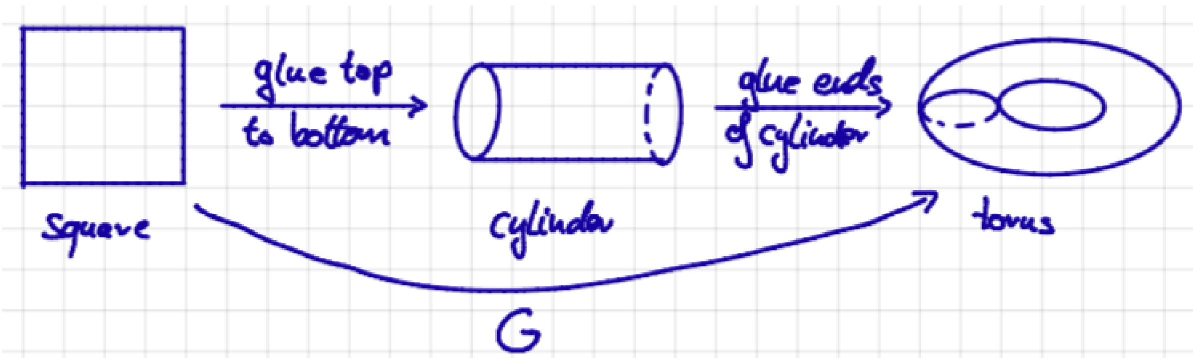
$$f\colon \mathbb{R} \to \mathbb{R}^2,\ t \mapsto (\sin(4\arctan t)), \sin(2\arctan t)).$$

(The image of f is called a lemniscate, the locus of points (x,y) satisfying $x^2 = 4y^2(1-y^2)$.)
We can check that f is **smooth**, **one-to-one** and **an immersion** (df_t is never zero and hence as a linear map between one-dimensional vector spaces an isomorphism).
But $f(X)$ is not a submanifold and f is not a diffeomorphism onto its image, because $f(X)$ is compact while X is not (an open interval in $\mathbb{R}$).

Torus by gluing:
Let $g\colon \mathbb{R} \to S^1$ be the local diffeomorphism $t \mapsto (\cos(2\pi t), \sin(2\pi t))$. We define

$$G\colon \mathbb{R}^2 \to S^1 \times S^1 =: T^2,\ G(x,y) = (g(x), g(y))$$

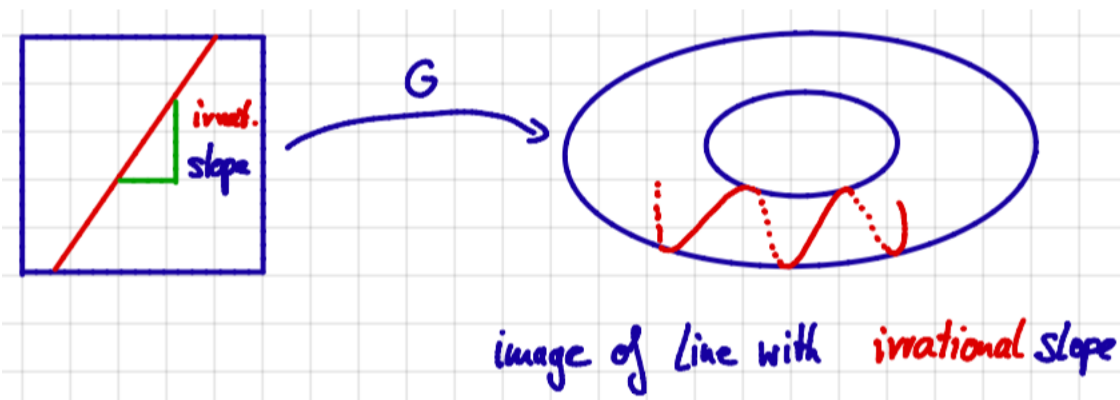
The map G is a local diffeomorphism from the plane onto the torus T^2. (G "glues" opposite sides of the square together, see the picture.)



We define the map γ by

$$\gamma\colon \mathbb{R} \to T^2,\ \gamma(t) = (g(t), g(\alpha \cdot t))$$

where α is an **irrational** number.



Note that another way to describe $\gamma = \gamma_\alpha$ would be to define it by

$$\gamma_\alpha\colon \mathbb{R} \to S^1 \times S^1,\ t \mapsto (e^{2\pi i t}, e^{2\pi i \alpha t})$$

where we consider S^1 as a subset of $\mathbb{C} \cong \mathbb{R}^2$. Then we require that the quotient α is irrational.

Image of a line with irrational slope

The map γ is an **immersion** because $d\gamma_t$ **is nonzero for every** t (and as before a nonzero linear map from a one-dimensional vector space to another is automatically injective; its image is a line in that other vector space).

And γ **is injective**, since $\gamma(t_1) = \gamma(t_2)$ implies

$$
\begin{aligned}
&g(t_1) = g(t_2) \text{ and } g(\alpha t_1) = g(\alpha t_2)\\
\Rightarrow &\cos(2\pi t_1) = \cos(2\pi t_2) \text{ and } \cos(2\pi\alpha t_1) = \cos(2\pi\alpha t_2)\\
\Rightarrow &t_1 - t_2 \in \mathbb{Z} \text{ and } \alpha(t_1 - t_2) \in \mathbb{Z}
\end{aligned}
$$

which is impossible, since α **is irrational**, unless $t_1 = t_2$.
Actually, one can show that the image of γ is a dense subset in T^2. But γ is **not a diffeomorphism onto its image**, since it is not even a homeomorphism:
For, look at the set $\gamma(\mathbb{Z}) = \{\gamma(n) : n \in \mathbb{Z}\}$. By Dirichlet's approximation theorem, for every $\epsilon > 0$, there are integers n and m such that

$$|\alpha n - m| < \epsilon.$$

Since the line segment between two points $(\cos t_1, \sin t_1)$ and $(\cos t_2, \sin t_2)$ on the unit circle is shorter than the circular arc of length $|t_1 - t_2|$ we have

$$
\begin{aligned}
&|(\cos(2\pi\alpha n), \sin(2\pi\alpha n)) - (1{,}0)|\\
=&|(\cos(2\pi\alpha n), \sin(2\pi\alpha n)) - (\cos(2\pi m), \sin(2\pi m))|\\
\leq&2\pi|\alpha n - m|\\
\leq&2\pi\epsilon.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&|\gamma(n) - \gamma(0)|\\
=&|((g(n),g(\alpha n)) - (g(0),g(0))|\\
=&|\,((1{,}0),(\cos(2\pi\alpha n), \sin(2\pi\alpha n))) - ((1{,}0),(1{,}0))\,|\\
=&|(\cos(2\pi\alpha n), \sin(2\pi\alpha n)) - (\cos(2\pi m), \sin(2\pi m))|\\
\leq&2\pi|\alpha n - m|\\
\leq&2\pi\epsilon.
\end{aligned}
$$

Thus, there is a sequence of integers such that $\gamma(n)$ converges to $\gamma(0)$, i.e. $\gamma(0)$ is a limit point in $\gamma(\mathbb{Z})$. But $\mathbb{Z}$ does not have any limit points in $\mathbb{R}$.
But note that the image of a convergent sequence under a continuous map is again a convergent sequence. Hence if γ^{-1} was continuous, then $0 = \gamma^{-1}(\gamma(0))$ had to be a limit point as well. Hence γ is **not a homeomorphism onto its image**.

Aside: LIT for the above example

Let $t_0 = 0$ for simplicity. We apply the LIT to the map
$$\gamma\colon \mathbb{R} \to S^1 \times S^1$$
above. First, we parametrize $\mathbb{R}$ by the identity and pick some $U = (-1,1)$. Then we parametrize $S^1 \times S^1$ around $\gamma(0) = (1,0,1,0)$ by
$$\begin{aligned}\psi\colon V = (-1,1) \times (-1,1) &\to S^1 \times S^1,\\ (x,y) &\mapsto (\sqrt{1-x^2},x,\sqrt{1-y^2},y).\end{aligned}$$
The corresponding map $\theta\colon U \to V$ is then
$$t \mapsto (\sin(2\pi t), \sin(2\pi\alpha t)).$$
Now we would like to modify the local parametrization ψ around $\gamma(0)$ such that θ becomes
$$U \to U \times \mathbb{R},\; t \mapsto (t,0).$$
For that we define a new map
$$\Theta\colon U \times \mathbb{R} \to \mathbb{R}^2,\; (t,s) \mapsto \theta(t) + (0,s).$$
Then we compose ψ with Θ to get a new local parametrization around $\gamma(0)$:
$$\begin{aligned}\psi \circ \Theta\colon (t,s) \mapsto &\Big(\sqrt{1-\sin^2(2\pi t)}, \sin(2\pi t),\\ &\sqrt{1-(\sin(2\pi\alpha t)+s)^2}, \sin(2\pi\alpha t)+s\Big)\\ =&(\cos(2\pi t), \sin(2\pi t),\\ &\sqrt{1-(\sin(2\pi\alpha t)+s)^2}, \sin(2\pi\alpha t)+s).\end{aligned}$$
Finally, in order to make everything work, we have to make U and V small enough such that $\sin(2\pi t)$ and $\sin(2\pi\alpha t) + s$ stay in $(-1,1)$ for all $t \in U$ and $\theta(t) + (0,s) \in V$.

The pathologies of the last two examples arise because the map sends **points near infinity** in $\mathbb{R}$ into **small regions of the image**. So if we want to tame our immersions we have to try to avoid such a behavior. It will turn out that this is the only problem.

The topological analog of **points near infinity** in a topological space X is the exterior or complement of a compact set.

Proper maps

A map $f\colon X \to Y$ between topological spaces is said to be **proper** if the **preimage** of any compact subset is a compact subset.
(Recall: For a general continuous map, the image of any compact set is compact. Check that you understand why!)

Let $f\colon X \to Y$ be a proper map and let $Z \subset Y$ be a compact subset of Y. Then $f^{-1}(Z) \subset X$ is a compact subset of X, since f is proper. The complement $X \setminus f^{-1}(Z)$ of $f^{-1}(Z)$ in X is the largest subset of X which is not mapped to Z under f. Since f is proper, every point $x \in X \setminus f^{-1}(Z)$ is contained in the complement of a compact set and $f(x) \notin Z$. Thus f sends x to the complement of a compact subset in Y. Therefore, morally speaking, a proper map sends the complement of a compact set to the complement of a compact set. In other words:

Proper maps respect infinity

Proper maps send **points near infinity** to **points near infinity**.

Let us give proper immersions a name:

Embeddings

An immersion that is **one-to-one and proper** is called an **embedding**.

Properness turns out to be a sufficient global topological constraint for a local immersion. For proper maps we have the following extension of the Local Immersion Theorem.

Embedding theorem

An embedding $f\colon X \to Y$ maps X diffeomorphically onto a **submanifold** of Y.

Proof of the theorem:
By the assumption of f being a one-to-one immersion, we know that f is a **local diffeomorphism** from X to $f(X)$. Moreover, $f\colon X \to f(X)$ is **bijective** (injective by assumption and obviously surjective onto its image), and the inverse

f^{-1} exists as a map of sets. But locally f^{-1} **is smooth**, since f is a local diffeomorphism.

Hence in order to prove that $f(X)$ is a manifold, it remains to show that **the image of any open subset W of X is an open subset of $f(X)$**. For then f maps local parametrizations diffeomorphically to local parametrizations. Hence we need to show the general statement: **A bijective proper map is a homeomorphism**.

If $f(W)$ was not an open subset, then there would be a point $y \in f(W)$ and an open neighborhood of y which is not contained in $f(W)$. In different words, there would be a point $y \in f(W)$ such that in any small neighborhood of y there would be points y_i which are not in $f(W)$. We can rephrase this by saying:

If $f(W)$ is not an open subset, then there **exists a sequence of points $y_i \in f(X)$ that do not belong to $f(W)$, but converge to a point y in $f(W)$**.

The set $S := \{y, y_i\}_i$ is compact (a countable union of compact sets). Since **f is proper**, the preimage $f^{-1}(S)$ of S in X must be compact, too.

Since **f is injective**, there is exactly one preimage x of y in X and exaclty one preimage x_i for each y_i. Since $y \in f(W)$, x must belong to W.

Since **$f^{-1}(S) = \{x, x_i\}_i$ is compact**, after possibly restricting to a subsequence, we may assume that **the sequence of the x_i converges to a point $z \in X$**, we write $x_i \to z$. That implies $f(x_i) \to f(z)$ (since f is continuous). But since $f(x_i) \to f(x)$, the **injectivity of f implies** $x = z$.

Now **W is open**, which implies that, for large i, $x_i \in W$. But this implies $y_i = f(x_i) \in W$ and **contradicts** $y_i \notin f(W)$. Hence $f(W)$ is open in Y, and $f(X)$ is indeed a manifold. QED

A corollary for compact domains

If X is compact, then any continuous map $f\colon X \to Y$ is proper (closed subsets of compact sets are compact).
Hence, for compact X, every one-to-one immersion $f\colon X \to Y$ is an embedding and f maps X diffeomorphically onto a submanifold of Y.