Non-linear Stochastic Partial Differential Equations

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Article Outline

Glossary

Evolution equations Let H be a Hilbert (or Banach) space, $T > 0$ and f a mapping from $[0, T] \times H$ into H. An equation of the form

$$u'(t) = f(t, u(t)) , \quad t \in [0, T] , \tag{$*$}$$

is called an *abstract evolution equation*. If H is finite-dimensional we call $(*)$ an *ordinary evolution equation*, whereas if H is infinite-dimensional and f is a differential operator we call $(*)$ a *partial differential evolution equation*.

Continuous stochastic process Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. A *continuous stochastic process* (with values in H) is a family of (H-valued) random variables $(X(t) = X(t, \omega))_{t \geq 0}$ $(\omega \in \Omega)$ such that $X(\cdot, \omega)$ is continuous for $\mathbb{P}$-almost all $\omega \in \Omega$.

Brownian motion A real *Brownian motion* $B = (B(t))_{t \geq 0}$ on $(\Omega, \mathcal{F}, \mathbb{P})$ is a continuous real stochastic process in $[0, +\infty)$ such that (1) $B(0) = 0$ and for any $0 \leq s < t$, $B(t) - B(s)$ is a real Gaussian random variable with mean 0 and covariance $t - s$, and (2) if $0 < t_1 < \cdots < t_n$, the random variables, $B(t_1)$, $B(t_2) - B(t_1), \ldots, B(t_n) - B(t_{n-1})$ are independent.

Cylindrical Wiener process A *cylindrical Wiener process* in a Hilbert space H is a process of the form

$$W(t) = \sum_{k=1}^{\infty} e_k \beta_k(t) , \quad t \geq 0 ,$$

where (e_k) is a complete orthonormal system in H and (β_k) a sequence of mutually independent standard Brownian motions in a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

Noise Let $W(t)$ be a cylindrical Wiener process in a Hilbert space H. A noise is an expression of the form $BW(t)$, where $B \in L(H)$. If $B = I$ (the identity operator in H) the noise is called *white*.

Stochastic dynamical system This is a system governed by a partial differential evolution equation perturbed by noise.

Definition of the Subject

Several nonlinear partial differential evolution equations can be written as abstract equations in a suitable Hilbert space H (we shall denote by $|\cdot|$ the norm and by $\langle \cdot, \cdot \rangle$ the scalar product in H) of the following form [78]:

$$\begin{cases} \dfrac{dX(t)}{dt} = AX(t) + b(X(t)) , & t \geq 0 , \\ X(0) = x \in H , \end{cases} \tag{1}$$

where $A\colon D(A) \subset H \to H$ is the infinitesimal generator of a strongly continuous semigroup e^{tA} of linear bounded operators in H and $b\colon D(b) \subset H \to H$ is a nonlinear mapping.

In order to take into account unpredictable random perturbations, one is led to add to (1) a term of the form $\sigma(X(t))dW(t)$, where $\sigma\colon D(\sigma) \subset H \to L(H)$ ($L(H)$ is the space of all linear operators from H into itself) and $W(t)$ a cylindrical Wiener process in H. Then (1) is replaced by the following stochastic partial differential equation (SPDEs)

$$\begin{cases} dX(t) = (AX(t) + b(X(t)))dt + \sigma(X(t))dW(t) , \\ X(0) = x \in H , \end{cases} \tag{2}$$

whose precise meaning will be explained later.

The importance of the SPDEs is due to the fact that in the modelization of several phenomena, (2) is often more realistic than (1).

The study of SPDEs started in the 1970s and 1980s and interest is still growing. Among the early contributions we mention [7] for Navier–Stokes equations, [66] for SPDEs of monotone type, [79] for compactness methods, [57] for variational methods and [31].

Two approaches (essentially equivalent) are used for studying SPDEs: the first is based on the variational theory of partial differential equations and the second is based on the theory of strongly continuous semigroups of operators. In this article we are concerned with the latter [26,27]. For a recent introduction to the variational method see [71].

It is possible to consider perturbations of a partial differential equation by stochastic processes different from Brownian motion as jump processes, but we shall not consider this subject here.

Introduction

In this section we are concerned with (2) in the Hilbert space H, that is, with a partial differential equation of the form (1) perturbed by the noise $\sigma(X(t))dW(t)$, where $W(t)$ is a cylindrical Wiener process in H. We recall that $W(t)$ can be defined as follows:

$$W(t) = \sum_{k=1}^{\infty} e_k \beta_k(t) , \tag{3}$$

where e_k is a complete orthonormal system in H (often the system of eigenfunctions of A) and β_k is a sequence of mutually independent standard Brownian motions in a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Intuitively $W(t)$ represents a random (white) perturbation which has the same intensity in all direction of the orthonormal basis (e_k).

Remark 1 Definition (3) is formal since

$$\mathbb{E}[|W(t)|^2] = \sum_{k=1}^{\infty} t = +\infty \quad \forall\ t > 0 .$$

However, it is possible to define rigorously $W(t)$ in a suitable space larger than H [26]. We shall see in the following how to deal with $W(t)$.

The main problems we shall consider are the following:

(i) Existence and uniqueness of solutions $X(t)$ of (2).
(ii) Asymptotic behavior of $X(t)$ as $t \to \infty$.

For the sake of brevity we shall limit ourselves to stochastic perturbations of the special form $\sigma(X(t))dW(t) = BdW(t)$, where $B \in L(H)$. In this case we say that the noise involved in (2) is *additive*. The general case (*multiplicative noise*) is important and we shall give some references later. In particular a multiplicative noise is required when for physical reasons the solution $X(t)$ must belong to some convex subset of H (for instance, $X \geq 0$) [6].

In Sect. "General Problems and Results" we shall present an overview of the main problems concerning SPDEs. Section "Specific Equations" is devoted to some significant applications. We will not give complete proofs of all assertions but only a sketch in order to give an idea of the tools which are involved. The interested reader can look at the corresponding references.

We shall concentrate on some SPDEs only. We present in the following three examples the corresponding deterministic problems.

Example 2 (reaction–diffusion equations). Consider the following partial differential equation in a bounded subset $\mathcal{O}$ of $\mathbb{R}^N$ with regular boundary $\partial\mathcal{O}$:

$$\begin{cases} \partial_t X(t,\xi) = \Delta_\xi X(t,\xi) + p(X(t,\xi)) , & \xi \in \mathcal{O}, t > 0 , \\ X(t,\xi) = 0 , & t > 0, \xi \in \partial\mathcal{O} , \\ X(0,\xi) = x(\xi) , & \xi \in \mathcal{O}, x \in H , \end{cases} \tag{4}$$

where Δ_ξ is the Laplace operator and p is a polynomial of degree $d > 1$ with negative leading coefficient.

Let us write problem (4) as an abstract equation of the form (1) in the Hilbert space $H = L^2(\mathcal{O})$ (the space of all equivalence classes of square integrable real functions on $\mathcal{O}$ with respect to the Lebesgue measure).

For this purpose, we denote by A the realization of the Laplace operator with Dirichlet boundary conditions (other boundary conditions such as Neumann or Ventzell could be considered as well),

$$\begin{cases} Ax = \Delta_\xi x , \quad x \in D(A) , \\ D(A) = H^2(\mathcal{O}) \cap H_0^1(\mathcal{O}) , \end{cases}$$

and define

$$b(x)(\xi) = p(x(\xi)) , \quad x \in D(b) = L^{2d}(\mathcal{O}) .$$

For $k = 1, 2$ we denote by $H^k(\mathcal{O})$ the Sobolev space consisting of all functions which belong to $L^2(\mathcal{O})$ together with their distributional derivatives of order up to k. $H_0^1(\mathcal{O})$ is the subspace of those functions in $H^1(\mathcal{O})$ which vanish at the boundary $\partial\mathcal{O}$ of $\mathcal{O}$.

Example 3 (Burgers equation) Consider the equation in $[0, 2\pi]$ with periodic boundary conditions

$$\begin{cases} \partial_t X(t,\xi) = \partial_\xi^2 X(t,\xi) + \frac{1}{2}\partial_\xi(X^2(t,\xi)), \\ \qquad\qquad\qquad\qquad\qquad\qquad \xi \in [0,2\pi], t > 0, \\ X(t,0) = X(t,2\pi) , \quad \partial_\xi X(t,0) = \partial_\xi X(t,2\pi), \ t>0, \\ X(0,\xi) = x(\xi) , \quad \xi \in [0,2\pi] . \end{cases} \tag{5}$$

Problem (5) can be written in the abstract form (1) setting $H = L^2(0,2\pi)$,

$$Ax = D_\xi^2 x , \quad x \in D(A) = H_\#^2(0,2\pi) ,$$

and

$$b(x) = \tfrac{1}{2}\, D_\xi(x^2) , \quad x \in D(b) = H_\#^1(0,2\pi) ,$$

where

$$H_\#^1(0,2\pi) = \{x \in H^1(0,2\pi) :\ x(0) = x(2\pi)\}$$

and

$$H^2_\#(0,2\pi) = \{x \in H^2(0,2\pi)\colon x(0) = x(2\pi),\ \partial_\xi x(0) = \partial_\xi x(2\pi)\}.$$

Example 4 (2D Navier–Stokes equation). Consider the equation in the square $\mathcal{O} := [0,2\pi] \times [0,2\pi]$.

$$\begin{cases} \mathrm{d}Z = (\Delta_\xi Z - Z + D_\xi Z \cdot Z)\mathrm{d}t + \nabla p \mathrm{d}t \\ \qquad\qquad\qquad\qquad \xi \in \mathcal{O},\ t > 0, \\ \operatorname{div} Z = 0 \quad \xi \in \mathcal{O}\,,\ t > 0\,, \\ Z(t,\cdot) \text{ is periodic with period } 2\pi\,, \\ Z(0,\xi) = z(\xi) \quad \xi \in \mathcal{O}\,. \end{cases} \tag{6}$$

The unknown $Z = (Z_1, Z_2)$ represents the velocity and p the pressure of the fluid; div Z is the divergence of Z. We denote by H the space of all square integrable divergence free vectors,

$$H = \{Z = (Z_1,Z_2) \in (L^2(0,2\pi))^2\colon \operatorname{div} Z = 0\}\,,$$

and by $\mathcal{P}$ the orthogonal projector of $(L^2(0,2\pi))^2$ onto H.

Then setting $X(t,x) = \mathcal{P}Z(t,x)$, we obtain the following problem (where the pressure p has disappeared)

$$\begin{cases} \mathrm{d}X = \mathcal{P}(\Delta X - X) + \mathcal{P}(D_\xi X \cdot X) \quad \xi \in \mathcal{O}\,,\ t > 0\,, \\ X(t,\xi) \text{ is periodic with period } 2\pi\,, \quad t > 0 \\ X(0,\xi) = x \quad \xi \in \mathcal{O}\,. \end{cases}$$

Let us define the *Stokes* operator $A\colon D(A) \to H$ setting

$$Ax = \mathcal{P}(\Delta_\xi x - x)\,, \quad x \in D(A) = H^2_\#(\mathcal{O})$$

and the nonlinear operator b setting

$$b(x) = \mathcal{P}(D_\xi x \cdot x)\,, \quad x \in H^1_\#(\mathcal{O})$$

(b is more commonly written $b(x) = \mathcal{P}(x \cdot \nabla x)$). Here $H^k_\#(\mathcal{O})$, $k = 1,2$ are Sobolev spaces with periodic boundary conditions. Now, problem (5) can be written in the form (1).

We have to say that many other interesting equations have been studied recently. We mention among them the following (without claiming to present an exhaustive list).

- Cahn–Hilliard equations [21,38].
- Second-order linear SPDEs and Filtering equations; see [56,74] and references therein.
- Ginzburg–Landau equations [51,52,53].
- Kortweg–de Vries equation [36,37].
- Mathematical biology and Fleming–Viot model [48,77].
- Mathematical finance [64].
- Nonlinear Schrödinger equations [32,33,34,35].
- Porous media equations [4,30,73].
- Stochastic quantization [2,23,25,55,59,62].
- Wave equations [5,18,19,62,63,69,75].

General Problems and Results

Well-Posedness of a Stochastic Partial Differential Equation

We are here concerned with (2). The first problem is to prove the existence and uniqueness of solutions (to be defined in an appropriate way). We notice that it is not convenient to look for existence of strong solutions $X(t)$ (as for the ordinary stochastic equations) such that

$$X(t) = x + \int_0^t \Big[AX(s) + b(X(s))\Big]\mathrm{d}s + BW(t)\,, \quad t \ge 0\,.$$

This definition will require too much regularity for the solution, which does not belong to $D(A) \cap D(b)$ in general. The concept of a weaker solution has to be defined for each specific problem; see Sect. "Specific Equations".

Assume now that the existence of a unique solution $X(t,x)$ (in a suitable sense) of (2) has been proved for any $x \in H$. Then, given a real function φ in H, one is interested in the evolution in time of $\mathbb{E}[\varphi(X(t,x))]$ (the expectation of $\varphi(X(t,x))$). φ can be interpreted as an *observable*; its physical meaning could be, for instance, the mean velocity, pressure or temperature of a fluid. This leads to introduce the concept of *transition semigroup*

$$P_t\varphi(x) = \mathbb{E}[\varphi(X(t,x))]\,, \quad t \ge 0,\ x \in H,\ \varphi \in B_b(H)\,, \tag{7}$$

where $B_b(H)$ is the space of all mappings $\varphi\colon H \to \mathbb{R}$ which are bounded and Borel. It is a Banach space with the supremum norm

$$\|\varphi\|_\infty = \sup_{h \in H} |\varphi(x)|\,, \quad \varphi \in B_b(H)\,.$$

We denote by $C_b(H)$ the closed subspace of $B_b(H)$ of all uniformly continuous and bounded functions.

It is not difficult to see that P_t enjoys the semigroup property $P_{t+s} = P_tP_s$, $t,s \ge 0$, thanks to the Markovianity of the solution $X(t,x)$ to (2). However, P_t is not a strongly continuous semigroup on $C_b(H)$ in general (even when H is finite-dimensional and (2) is an ordinary differential equation).

Formally, the function $u(t,x) = P_t\varphi(x)$ is the unique solution to the following (Kolmogorov) parabolic equation:

$$\begin{cases} \dfrac{\mathrm{d}u}{\mathrm{d}t}(t,x) = K_0 u(t,x)\,, \\ u(0,x) = \varphi(x)\,, \end{cases} \tag{8}$$

where K_0 is the linear differential operator

$$K_0\varphi(x) = \tfrac{1}{2}\,\mathrm{Tr}\left[BB^* D_x^2\varphi(x)\right] + \langle Ax + b(x), D_x\varphi(x)\rangle\,, \quad x \in D(A)\cap D(b)\,, \tag{9}$$

where

$$\mathrm{Tr}\left[BB^* D_x^2\varphi(x)\right] = \sum_{h,k=1}^{\infty} \left\langle D_x^2\varphi(x)B^* e_k\,, B^* e_k\right\rangle\,,$$

and (e_k) is a complete orthonormal system on H. Notice that $K_0\varphi$ is only defined for φ of class C^2 such that the series above is convergent.

Though the semigroup P_t is not strongly continuous one can define its infinitesimal generator K, following [72],

$$K\varphi(x) = \lim_{h\to 0} \frac{1}{h}\left(P_t\varphi(x) - \varphi(x)\right)\,, \quad x \in H\,,$$

provided the limit exists and the following condition holds

$$\sup_{h\in(0,1]} \frac{1}{h}\,\|P_t\varphi - \varphi\|_\infty < +\infty\,.$$

Then an interesting problem consists in clarifying the relationship between the abstract operator K and the concrete differential operator K_0.

Let us list some important concepts related to P_t.

- P_t is called *Feller* if $P_t\varphi \in C_b(H)$ for any $\varphi \in C_b(H)$ and any $t \geq 0$.
- P_t is called *strong Feller* if $P_t\varphi \in C_b(H)$ for any $\varphi \in B_b(H)$ and any $t > 0$.
  The strong Feller property is related to a smoothing effect of the transition semigroup P_t and to the hypoellipticity of the Kolmogorov operator K_0 (when H is finite-dimensional) [75].
- P_t is called *irreducible* if $P_t 1_I(x) > 0$ for all $x \in H$ and all open sets I of H, where 1_I is the characteristic function of I ($1_I(x) = 1$ if $x \in I$, $1_I(x) = 0$ if $x \notin I$).

Irreducibility of P_t is related to the null-controllability of the deterministic system (that is, if for each $x \in H$ and $T > 0$ there is a control u such that $x(T) = 0$).

$$\begin{cases} \dfrac{\mathrm{d}x(t)}{\mathrm{d}t} = Ax(t) + b(x(t)) + Bu(t)\,, \quad t \geq 0\,, \\ x(0) = x \in H\,, \end{cases}$$

where u is a control; see, e. g Sects. 7.3 and 7.4 in [27].

Finally, we denote by $\pi_t(x, dy)$ the *law* of $X(t,x)$, so that the following integral representation for P_t,

$$P_t\varphi(x) = \int_H \varphi(y)\pi_t(x,\mathrm{d}y)\,, \quad \forall t > 0\,, x \in H\,, \quad \varphi \in B_b(H)\,, \tag{10}$$

holds.

Asymptotic Behavior of the Transition Semigroup

In order to study the asymptotic behavior of $P_t\varphi$, where $\varphi \in B_b(H)$, the concept of *invariant measure* is crucial. We say that a Borel probability measure ν on H is invariant for P_t if

$$\int_H P_t\varphi(x)\nu(\mathrm{d}x) = \int_H \varphi(x)\nu(\mathrm{d}x)\,, \quad \forall\,\varphi \in C_b(H)\,.$$

Notice that if ν is invariant and η is a random variable whose law is ν then the solution $X(t,\eta)$ to (2) with initial datum η is stationary.

In order to prove the existence of invariant measures an important tool is the following Krylov–Bogoliubov theorem; see, e. g., Theorem 3.1.1 in [27].

Theorem 5 *Assume that P_t is Feller and that for some $x \in H$ the family of probability measures $(\pi_t(x,dy))_{t\geq 0}$ is tight (that is, for any $\epsilon > 0$ there exists a compact set K_ϵ in H such that $\pi_t(x, K_\epsilon) > 1-\epsilon$ for all $t \geq 0$). Then there exists an invariant measure for P_t.*

Assume now that ν is an invariant measure for P_t and let $\varphi \in C_b(H)$. Then P_t can be uniquely extended to a contraction semigroup in $L^2(H,\nu)$. In fact, by (10) and the Hölder inequality we have

$$[P_t\varphi(x)]^2 \leq P_t(\varphi^2)(x)\,, \quad t \geq 0\,, \quad x \in H\,.$$

Integrating this inequality with respect to ν over H and taking into account that $\int_H P_t(\varphi^2)(x)\nu(\mathrm{d}x) = \int_H \varphi^2(x)\nu(\mathrm{d}x)$ by the invariance of ν yields

$$\int_H [P_t\varphi(x)]^2\nu(\mathrm{d}x) \leq \int_H \varphi^2(x)\nu(\mathrm{d}x)\,.$$

Since $C_b(H)$ is dense in $L^2(H,\nu)$, this inequality can be extended to any function of $L^2(H,\nu)$. This proves that P_t

can be uniquely extended to a contraction semigroup in $L^2(H, \nu)$.

Moreover, the following Von Neumann theorem holds [70].

Theorem 6 *Let ν be an invariant measure for P_t. Then for any $\varphi \in L^2(H, \nu)$ there exists the limit*

$$\lim_{T\to+\infty} \frac{1}{T} \int_0^T P_t\varphi(x)dt := \Pi\varphi \,,$$

where Π is a projection operator on

$$\Sigma := \{\varphi \in L^2(H, \nu) \colon P_t\varphi = \varphi, \ \forall\, t \geq 0\} \,.$$

If moreover

$$\lim_{T\to+\infty} \frac{1}{T} \int_0^T P_t\varphi(x)\mathrm{d}t = \int_H \varphi(y)\nu(dy) \quad \text{in } L^2(H, \nu),$$

then ν is called *ergodic*.

As in the deterministic case, ergodicity is a very important property of a stochastic dynamical system because it allows us to compute averages with respect to t in terms of averages with respect to $x \in H$ ("temporal" averages with "spatial" averages).

When the more stringent condition

$$\lim_{T\to+\infty} P_t\varphi(x)\mathrm{d}t = \int_H \varphi(y)\nu(\mathrm{d}y) \,, \quad \forall\, x \in H \,,$$

holds, we say that the measure ν is *strongly mixing*. In this direction we recall the following Doob's theorem; see e. g., Theorem 4.2.1 in [27].

Theorem 7 *Assume that P_t is irreducible and strongly Feller. Then P_t possesses at most one invariant measure which is ergodic and strongly mixing.*

Other Important Problems

Here we mention some additional problem which we cannot treat for space reasons. The first one concerns the so-called *small noise*. Often one is interested in studying stochastic perturbations of (1) of the form

$$\begin{cases} \mathrm{d}X(t) = (AX(t) + b(X(t)))\mathrm{d}t + \epsilon\mathrm{d}W(t) \,, \\ X(0) = x \in H \,, \end{cases} \tag{11}$$

where $\epsilon > 0$. In fact it can happens that (11) has a unique invariant measure μ_ε, whereas the corresponding deterministic system (1) has more. Then studying the limit points of μ_ε as $\varepsilon \to 0$ will help to select the "significant" invariant measures of (2) [39]. This problem also leads to study *large deviations* problems. Concerning Smoluchowski–Kramers approximations for parabolic SPDEs we mention [17].

Specific Equations

Ornstein–Uhlenbeck Equations

We are here concerned with the following stochastic differential equation:

$$\begin{cases} \mathrm{d}X(t) = AX(t)\mathrm{d}t + B\mathrm{d}W(t) \,, \quad t \geq 0 \\ X(0) = x \in H \,, \end{cases} \tag{12}$$

where $A \colon D(A) \subset H \to H$ is the infinitesimal generator of a strongly continuous semigroup e^{tA} and $B \colon H \to H$ is linear-bounded.

Formally the solution of (12) is given by the variation of constants formula

$$X(t, x) = \mathrm{e}^{tA}x + W_A(t) \,, \quad t \geq 0 \,, \tag{13}$$

where the process $W_A(t)$, called *stochastic convolution*, is given by

$$W_A(t) = \int_0^t \mathrm{e}^{(t-s)A} B\mathrm{d}W(s) \,, \quad t \geq 0 \,.$$

By a formal computation we see that

$$\mathbb{E}|W_A(t)|^2 = \sum_{k=1}^{\infty} \int_0^t |\mathrm{e}^{(t-s)A} Be_k|^2 \mathrm{d}s = \mathrm{Tr}\, Q_t \,,$$

where

$$Q_t x = \int_0^t \mathrm{e}^{sA} C \mathrm{e}^{sA^*} x\mathrm{d}s \,, \quad x \in H, \ t \geq 0 \,,$$

and $C = BB^*$. Consequently, though the definition (3) of the cylindrical Wiener process $W(t)$ is formal (see Remark 1), the stochastic convolution defined by the series

$$W_A(t) = \sum_{k=1}^{\infty} \int_0^t \mathrm{e}^{(t-s)A} Be_k \mathrm{d}\beta_k(s) \tag{14}$$

is meaningful and convergent in $L^2(\Omega, \mathcal{F}, \mathbb{P}; H)$ for all $t \geq 0$, provided $\mathrm{Tr}\, Q_t < +\infty$ for all $t > 0$. We shall assume that this hypothesis holds from now on.

It is easy to see that $W_A(t)$ is a Gaussian random variable N_{Q_t} in H. (A Borel probability measure μ in H is said to be *Gaussian* with *mean* $x \in H$ and covariance Q, where $Q \in L(H)$ is of trace class, if the Fourier transform $\hat{\mu}$ of μ is given by $\hat{\mu}(h) = \mathrm{e}^{i\langle x,h\rangle - \frac{1}{2}\langle Qh,h\rangle}$ for all $h \in H$. In this case we set $\mu = N_{x,Q}$ and $\mu = N_Q$ if $x = 0$.)

Remark 8 The limit case when $B = I$ (*white noise*) [80], is important in some application in physics, because it means that the noise acts uniformly in all directions. In this case the assumption $\operatorname{Tr} Q_t < +\infty$ is equivalent to

$$\int_0^t \operatorname{Tr}\left[e^{sA}e^{sA^*}\right] ds < +\infty\,. \tag{15}$$

Assume, for instance, that A is the Laplacian in an open bounded set $\mathcal{O} \subset \mathbb{R}^d$ with Dirichlet boundary conditions. Let $Ae_k = \alpha_k e_k$, where $(e_k)_{k\in\mathbb{N}^d}$ is the complete orthonormal system of eigenfunctions of A and $(\alpha_k)_{k\in\mathbb{N}^d}$ are the corresponding eigenvalues. Then we have

$$\operatorname{Tr} Q_t = \sum_{k\in\mathbb{N}^d} \frac{1}{\alpha_k}\left(1 - e^{-2\alpha_k t}\right), \quad t > 0\,.$$

It is well known that $\alpha_k \sim |k|^2$ and so (15) is fulfilled only if $d = 1$. To deal with the case $d > 1$ one introduces the so-called *renormalization* [68,76].

Now by (13) it follows that the law of $X(t,x)$ is given by $N_{e^{tA}x,Q_t}$, so the corresponding transition semigroup looks like

$$P_t\varphi(x) = \int_H \varphi(y) N_{e^{tA}x,Q_t}(dy)\,, \quad \varphi \in B_b(H)\,.$$

If e^{tA} is stable,

$$\|e^{tA}\| \le Me^{-\omega t}\,, \quad t \ge 0\,,$$

and for some $M, \omega > 0$ it follows that

$$\lim_{t\to+\infty} P_t\varphi(x) = \int_H \varphi(y) N_{Q_\infty}(dy)\,,$$

and so $\mu := N_{Q_\infty}$ is the unique invariant measure for P_t which is in addition ergodic and strongly mixing (for a necessary and sufficient condition for existence and uniqueness of an invariant measure for P_t see [26]).

Smooth Perturbations of System (12)

Let us consider the following stochastic differential equation

$$\begin{cases} dX(t) = (AX(t) + b(X(t))dt + BdW(t)\,, & t \ge 0\,,\\ X(0) = x \in H\,, \end{cases} \tag{16}$$

where A and B are as before and $b\colon H \to H$ is of class C^1 and Lipschitz continuous. The condition on b is very strong; however, this case is important because it can be used for approximating equations with irregular coefficients.

By a *mild* solution of problem (16) on $[0, T]$ we mean a stochastic process $X \in C_W([0, T]; H)$ such that

$$X(t) = e^{tA}x + \int_0^t e^{(t-s)A} b(X(s))ds + W_A(t)\,, \quad \mathbb{P}\text{-a.s.} \tag{17}$$

where W_A is the stochastic convolution defined by (14). By $C_W([0, T]; H)$ we mean the Banach space consisting of all continuous mappings $Y\colon [0, T] \to L^2(\Omega, \mathcal{F}, \mathbb{P}; H)$ which are adapted to W (that is, such that for all $t \in [0, T]$, $Y(t)$ is $\mathcal{F}_t$-measurable, where $\mathcal{F}_t$ is the σ-algebra generated by $\{W(s),\ s \in [0, t]\}$), endowed with the norm,

$$\|Y\|_{C_W([0,T];H))} = \left(\sup_{t\in[0,T]} \mathbb{E}\left(|Y(t)|^2\right)\right)^{1/2}.$$

It is called the space of all *mean square continuous adapted processes* on $[0, T]$ taking values on H.

Equation (17) can be solved easily by a standard fixed-point argument in the Banach space $C_W([0, T]; H)$. Thus we have the result

Proposition 9 *Equation (17) has a unique solution $X(\cdot, x)$.*

One checks easily that the corresponding transition semigroup P_t defined by (7) is Feller. When $B = I$ one can show that P_t is strongly Feller by using the *Bismut–Elworthy* formula [10,40] for the derivative of $D_x P_t\varphi(x)$ which reads as follows:

$$\langle DP_t\varphi(x), h\rangle = \frac{1}{t}\mathbb{E}\left[\varphi(X(t,x)) \int_0^t \langle D_x X(s,x)\cdot h, dW(s)\rangle\right].$$

In this case one can also show that P_t is irreducible [20]. So, in view of Doob's theorem there is at most one invariant measure.

Existence of an invariant measure can be proved under the assumption

$$\langle Ax, x\rangle + \langle b(x), x\rangle \le a - b|x|^2\,, \quad \forall\, x \in D(A)\,,$$

where $a, b > 0$. In fact in this case one finds easily by Itô's formula an estimate of the second moment of $X(t,x)$ independent of t of the form

$$\mathbb{E}|X(t,x)|^2 \le c(1 + |x|^2)\,. \tag{18}$$

By (18) the tightness of $(\pi_t(x,\cdot))_{t\ge0}$ follows for any fixed $x \in H$. We have in fact for any $R > 0$

$$\pi_t(x, B_R^c) \le \frac{1}{R^2}\int_H |y|^2 \pi_t(x, dy) = \frac{1}{R^2}\,\mathbb{E}|X(t,x)|^2 \le \frac{c}{R^2}\,(1 + |x|^2)\,,$$

where B_R^c is the complement of the ball B_R of center 0 and radius R. So, the existence of an invariant measure follows from the Krylov–Bogoliubov theorem.

Reaction–Diffusion Equations Perturbed by Noise

We consider here a stochastic perturbation of problem (4) from Example 2, but for the sake of simplicity we take $\mathcal{O} = [0,1]$ and $B = I$. So, we have $H = L^2(0,1)$ and (4) becomes

$$\begin{cases} dX(t,\xi) = \left[D_\xi^2 X(t,\xi) + p(X(t,\xi))\right] dt + dW(t,\xi)\,, \\ \hfill \xi \in [0,1]\,, \\ X(t,0) = X(t,1) = 0\,, \quad t \geq 0\,, \\ X(0,\xi) = x(\xi)\,, \quad \xi \in [0,1],\ x \in H\,, \end{cases} \tag{19}$$

where p is a polynomial of degree $d > 1$ with negative leading coefficient.

For generalizations to systems of reaction–diffusion equations in bounded subsets of $\mathbb{R}^n$ and for equations with multiplicative noise see [15,16] and references therein. For problems with reflexion see [65,82].

Now we write (19) in the mild form

$$X(t) = e^{tA}x + \int_0^t e^{(t-s)A} p(X(s))ds + W_A(t)\,, \quad t \geq 0\,, \tag{20}$$

where $W_A(t)$ is the stochastic convolution defined by (14). Moreover,

$$Ax = D_\xi^2 x\,, \quad x \in D(A) = H^2(0,1) \cap H_0^1(0,1)$$

and

$$b(x)(\xi) = p(x(\xi))\,, \quad x \in D(b) = L^{2d}(0,1)\,.$$

We notice that, since the leading coefficient of p is negative, b enjoys the basic dissipativity property

$$\langle b(x) - b(y), x - y\rangle \leq c_1|x-y|^2\,, \quad \forall\, x, y \in L^{2d}(0,1)\,,$$

where

$$c_1 = \sup_{\xi\in\mathbb{R}} p'(\xi)\,.$$

The operator A is self-adjoint and possesses a complete orthonormal system (e_k) of eigenfunctions, namely,

$$e_k(\xi) = (2/\pi)^{1/2} \sin(\pi\xi)\,, \quad \xi \in [0,1],\ k \in \mathbb{N}\,.$$

Moreover

$$Ae_k = -\pi^2 k^2 e_k\,, \quad k \in \mathbb{N}\,.$$

Therefore, we have $\langle Ax, x\rangle \leq -\pi^2|x|^2$, for all $x \in H$.

Let us give the definition of the mild solution of (19). We say that $X \in C_W([0,T];H)$ ($C_W([0,T];H)$ was defined in Sect. "Smooth Perturbations of System (12)" is a *mild* solution of problem (19) if $X(t) \in L^{2d}(0,1)$ for all $t \geq 0$ and fulfills (20).

We notice that the condition $X(t) \in L^{2d}(0,1)$ is necessary in order for the integrand in (20) be meaningful.

The basic existence and uniqueness result is the following [3,26,41].

Theorem 10 *For any $x \in L^{2d}(\mathcal{O})$, there exists a unique mild solution $X(\cdot,x)$ of problem (19).*

Proof First we consider a regular approximating problem,

$$\begin{cases} dX_\alpha(t) = (AX_\alpha(t) + b_\alpha(X_\alpha(t))dt + dW(t)\,, \\ X_\alpha(0) = x \in H\,, \end{cases} \tag{21}$$

where for any $\alpha > 0$, b_α is of class C^1, $b_\alpha(x) \to p(x)$ for all $x \in L^{2d}(0,1)$ as $\alpha \to 0$ and the basic dissipativity property,

$$\langle b_\alpha(x) - b_\alpha(y), x-y\rangle \leq c_2|x-y|^2\,, \quad \forall\, x,y \in H,\ \alpha > 0\,,$$

is fulfilled, where c_2 is a constant independent of α. By Proposition 9 we know that problem (21) has a unique mild solution X_α, that is,

$$X_\alpha(t) = e^{tA}x + \int_0^t e^{(t-s)A} b_\alpha(X(s))ds + W_A(t)\,, \quad \mathbb{P}\text{-a.s.}$$

Taking advantage of the dissipativity of b_α it is possible to find suitable estimates for X_α and to show that X_α converge, as $\alpha \to 0$, to the unique solution X of (19). □

By Theorem 10 we can construct a transition semigroup P_t on $B_b(L^{2d}(0,1))$. However, it is useful to extend this semigroup to $B_b(L^2(0,1))$. For this we need a weaker notion of solution of (19) when $x \in H = L^2(0,1)$.

Let $x \in H$. We say that $X \in C_W([0,T];H)$ is a *generalized* solution of problem (19) if there exists a sequence $(x_n) \subset L^{2d}(\mathcal{O})$, such that

$$\lim_{n\to\infty} x_n = x \quad \text{in } H\,,$$

and

$$\lim_{n\to\infty} X(\cdot,x_n) = X(\cdot,x) \quad \text{in } C_W([0,T];H)\,.$$

It is easy to see, using again dissipativity of p, that this definition does not depend on the choice of the sequence (x_n) and to prove the following result.

Corollary 11 *For any $x \in H$, there exists a unique generalized solution $X(\cdot, x)$ of problem (19)*

Now we can define the transition semigroup

$$P_t\varphi(x) = \mathbb{E}[\varphi(X(t,x))] ,$$

for all $\varphi \in B_b(H)$.

We finally discuss invariant measures of P_t assuming for simplicity that $p'(\xi) \leq 0$ so that p is dissipative. In this case we can show the following result [26].

Theorem 12 *The semigroup P_t has a unique invariant measure ν, which is ergodic and strongly mixing and,*

$$\lim_{t\to+\infty} P_t\varphi(x) = \int_H \varphi(y)\nu(dy) , \quad x \in H . \tag{22}$$

Proof Let us consider the reaction–diffusion equation starting from $-s$ where $s > 0$,

$$\begin{cases} dX(t,\xi) = \left[D_\xi^2 X(t,\xi) + p(X(t,\xi))\right] dt + dW(t,\xi) , \\ \qquad\qquad\qquad\qquad\qquad\qquad \xi \in [0,1] , \\ X(t,0) = X(t,1) = 0 , \quad t \geq 0 , \\ X(-s,\xi) = x(\xi) , \quad \xi \in [0,1] , \ x \in H . \end{cases} \tag{23}$$

Obviously we shall extend $W(t)$ to negative times setting

$$W(t) = W(-t) \quad \text{if} \ \ t \leq 0 .$$

Let us denote by $X(t,-s,x)$ the solution of (23). Then, using the dissipativity of p, one can show that there exists $\zeta \in L^2(\Omega,\mathcal{F},\mathbb{P};H)$ such that

$$\lim_{s\to+\infty} X(t,-s,x) = \zeta \quad \text{in} \ \ L^2(\Omega,\mathcal{F},\mathbb{P};H) , \quad x \in H ,$$

where ζ is independent of t. Now, using the fact that the law of $X(t,-s,x)$ coincides with that of $X(t+s,x)$, it follows that the law ν of ζ is an invariant measure for P_t.

Moreover, one can show that P_t is irreducible and strongly Feller, so the uniqueness of ν as well as (22) follow from Doob's theorem. □

Burgers Equations Perturbed by Noise

We are concerned with the Burgers equation in $[0,2\pi]$ perturbed by white noise

$$\begin{cases} dX(t,\xi) = \left(\partial_\xi^2 X(t,\xi) + \frac{1}{2}\,\partial_\xi(X^2(t,\xi))\right) dt \\ \qquad\qquad\qquad + dW(t,\xi) , \xi \in [0,2\pi], \ t > 0 , \\ X(t,0) = X(t,2\pi) , \quad t > 0 \\ X(0,\xi) = x(\xi) , \quad \xi \in [0,2\pi] . \end{cases} \tag{24}$$

We set $H = L^2(0,2\pi)$ and give the following definition. A *mild solution* in $[0,T]$ of (24) is a process $X \in C_W([0,T];H)$ such that

$$X(t) = e^{tA}x + \frac{1}{2}\int_0^t \partial_\xi e^{(t-s)A}(X^2(s))ds + W_A(t) , \quad t \geq 0 , \tag{25}$$

where the stochastic convolution $W_A(t)$ is defined by (14).

We note that (25) is meaningful thanks to the estimate

$$\left|\partial_\xi e^{(t-s)A} y\right| \leq \kappa t^{-\frac{3}{4}} |x|_1 , \quad x \in L^1(0,2\pi)$$

(which can be proved by an elementary argument of harmonic analysis).

Now we state, following [29], an existence and uniqueness result.

Theorem 13 *For any $x \in H$ and $T > 0$ there exists a unique mild solution $X \in C_W([0,T];H)$ of (24). Moreover, there exists a unique invariant measure which is ergodic and strongly mixing.*

Proof By the contraction principle it is easy to show existence and uniqueness of a local solution of (25) in a stochastic interval $[0,\tau(\omega))$ (since (25) is a semilinear equation with a locally Lipschitz nonlinearity). In order to show global existence one needs to find an a priori estimate for $\mathbb{P}$-almost all $\omega \in \Omega$. For this we first reduce (25) to a deterministic equation (more precisely to a family of deterministic equations indexed by $\omega \in \Omega$), setting

$$Y(t) = X(t) - W_A(t) .$$

$Y(t)$ fulfills in fact the equation

$$\begin{cases} \dfrac{dY(t)}{dt} = AY(t) + b(Y(t) + W_A(t)) , \quad t \in [0,T] , \\ Y(0) = x . \end{cases}$$

An a priori estimate for $Y(t)$ can be found by some manipulations using the basic property of b,

$$\langle b(x), x\rangle = 0 , \quad \forall\, x \in D(b) .$$

For details and for the existence of an invariant measure see [29], for the uniqueness see [27]. □

Another approach for studying the Burgers equation can be found in [9].

2D Navier–Stokes Equation Perturbed by Noise

We are here concerned with the equation

$$\begin{cases} dZ = (\Delta_\xi Z - Z + D_\xi Z \cdot Z)dt + \nabla p dt + B_1 dW(t) \\ \qquad\qquad\qquad\qquad \text{in } [0, +\infty) \times \mathcal{O} , \\ \operatorname{div} Z = 0 \quad \text{in } [0, +\infty) \times \mathcal{O} , \\ Z(t, \cdot) \text{ is periodic with period } 2\pi , \\ Z(0, \cdot) = z \quad \text{in } \mathcal{O} , \end{cases} \tag{26}$$

where $Z = (Z_1, Z_2)$ belongs to the space H of all square integrable divergence free vectors and $B_1 \in L(H)$.

Several papers have been devoted to stochastic 2D Navier–Stokes equations. We mention, besides the pioneering work [1,7,13,22]. Concerning 3D Navier–Stokes equations (which we do not consider here) one can look at [13,14,23,42,43,44,46,47].

We write (26) in the mild form

$$X(t) = e^{tA} x + \int_0^t e^{(t-s)A} b(X(s))ds + W_A(t) , \tag{27}$$

where the operators A and b were defined in Example 4 and the stochastic convolution $W_A(t)$ is given by (14) with $B = \mathcal{P} B_1$.

In order to study (27) it is convenient to introduce the mapping

$$\Gamma(f) = \int_0^t e^{(t-s)A} b(f(s))ds, \quad f \in L^4((0,T)\times\mathcal{O}), \quad t \geq 0,$$

and then to write (27) as

$$X(t) = e^{tA} x + \Gamma(X)(t) + W_A(t) .$$

In fact one can show [27] that Γ maps $L^4((0,T) \times \mathcal{O})$ into itself and for any $f, g \in L^4((0,T) \times \mathcal{O}) := L^4$ we have

$$|\Gamma(f) - \Gamma(g)|_{L^4} \leq 4(|f|_{L^4} + |g|_{L^4}) \, |f - g|_{L^4} .$$

Theorem 14 *For any $x \in H$ there exists a unique mild solution $X(\cdot, x)$ of (26).*

Proof One reduces (27) to a family of deterministic equations as in the case of the Burgers equation. Then one uses a fixed-point argument in the space $L^4(0,T; L^4)$ [27]. □

Existence of invariant measures is not difficult. It can be obtained, as in the previous examples, by the Krylov–Bogoliubov theorem. Uniqueness is more delicate. When the noise $BdW(t)$ is not too degenerate it was proved in [45]. Otherwise one has to use the *coupling* argument [54,58,60,81].

Future Directions

The number of papers devoted to nonlinear SPDEs is increasing. In fact, several models arising in the application are more realistic when a random perturbation is taken into account. Among new directions of research we mention the following.

- The direct study of the Kolmogorov equation (8) (considered as a parabolic equation with infinitely many variables). This can give important information about the properties of system (2) even in the case when the problem is not well posed. A result in this direction can be found in [28]. For a solution of (8) in the case of the 3D Navier–Stokes equation (with a failed attempt to prove uniqueness in law) see [24].
- Hamilton–Jacobi equations such as

$$\begin{cases} \dfrac{du(t,x)}{dt} = K_0 u(t,x) + \mathcal{H}(u_x(t,x)) , \\ \qquad\qquad\qquad\qquad x \in H, \ t \geq 0 , \\ u(0,x) = \varphi(x) , \end{cases}$$

  where $\mathcal{H}$ is a Hamiltonian. This is an important subject related to stochastic optimal control problems for SPDEs. For early results see Chap. 12 in [28]. Recent results are also connected with backward equations. Here the pioneering finite-dimensional result in [67] has been extended in infinite dimensions [50].

We believe also that an extension of the following topics to infinite dimensions would be interesting:

- Homogenization and averaging. In finite dimension see, e. g., [8,49]
- Measures solutions of Kolmogorov equations. In finite dimension this is a well-known subject, see, e. g., [11,12].

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