Lecture 5d

Algebraic Multiplicity and Geometric Multiplicity

(pages 296-7)

Let us consider our example matrix $B = \begin{bmatrix} 3 & 0 & 0 & 0 \\ -6 & 4 & 1 & 5 \\ 2 & 1 & 4 & -1 \\ 4 & 0 & 0 & -3 \end{bmatrix}$ again. We found that B had three eigenvalues, even though it is a 4×4 matrix. This is because $\lambda = 3$ was a double root of the characteristic polynomial for B. Now, if the eigenspace corresponding to $\lambda = 3$ also had two basis vectors, this wouldn't have been so strange, but instead the eigenspace corresponding to $\lambda = 3$ was the span of only one vector. This will turn out to be a less than ideal situation, but in order to study this further we will need some more definitions.

Definition: Let A be an $n \times n$ matrix with eigenvalue λ. The **algebraic multiplicity** of λ is the number of times λ is repeated as a root of the characteristic polynomial.

Definition: Let A be an $n \times n$ matrix with eigenvalue λ. The **geometric multiplicity** of λ is the dimension of the eigenspace of λ.

Example: Let $B = \begin{bmatrix} 3 & 0 & 0 & 0 \\ -6 & 4 & 1 & 5 \\ 2 & 1 & 4 & -1 \\ 4 & 0 & 0 & -3 \end{bmatrix}$, as in our previous examples. Then the algebraic multiplicity of $\lambda = 3$ is 2, but the geometric multiplicity of $\lambda = 3$ is 1. Both $\lambda = -3$ and $\lambda = 5$ have algebraic multiplicity 1 and geometric multiplicity 1.

Example: Let $A = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}$, as in our previous examples. Then both $\lambda = 2$ and $\lambda = 5$ have algebraic multiplicity 1 and geometric multiplicity 1.

Example: Let $C = \begin{bmatrix} 2 & 0 & 0 \\ 4 & 2 & 0 \\ 6 & 0 & 2 \end{bmatrix}$. Then $C - \lambda I = \begin{bmatrix} 2-\lambda & 0 & 0 \\ 4 & 2-\lambda & 0 \\ 6 & 0 & 2-\lambda \end{bmatrix}$, and so $\det(C - \lambda I) = (2-\lambda)^3$ (since $C - \lambda I$ is a triangular matrix). So, $\lambda = 2$ is the only eigenvalue of C, and we see that $\lambda = 2$ has algebraic multiplicity 3. To find the geometric multiplicity of $\lambda = 2$, we first need to find its eigenspace. To do that, we will need to row reduce $C - 2I = \begin{bmatrix} 2-2 & 0 & 0 \\ 4 & 2-2 & 0 \\ 6 & 0 & 2-2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 4 & 0 & 0 \\ 6 & 0 & 0 \end{bmatrix}$. We row reduce as follows:

$$\begin{bmatrix} 0 & 0 & 0 \\ 4 & 0 & 0 \\ 6 & 0 & 0 \end{bmatrix} \; (1/4)R_2 \; \sim \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 6 & 0 & 0 \end{bmatrix} \; R_3 - 6R_2 \; \sim \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \; R_1 \updownarrow R_2 \; \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

So we see that the homogeneous system $(C-2I)\vec{v} = \vec{0}$ is equivalent to the equation $v_1 = 0$. Replacing v_2 with the parameter s, and replacing v_3 with the parameter t, we see that the general solution to the system is

$$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ s \\ t \end{bmatrix} = s \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \text{Span} \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

And so we see that the eigenspace for $\lambda = 2$ is $\text{Span} \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$. As such, the geometric multiplicity of $\lambda = 2$ is 2.