Stochastic Modelling

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Contents

1 Lecture 1

1.1 Practical Information

Two projects

- The projects count 20% and exam 80%.
- Must be done with two people.
- If you want to do statistics is it worth learning R.

Course Overview

- Markov chains for discret time and discrete outcome.
  - Set of states and discrete time points.
  - Transition between states
  - Future depends on the present, but not the past.
- Continious time Markoc chains. (continious time and discrete toutcome.
- Brownian motion and Gaussian processes (continionus time and continious outcome.)

1.2 Mathematical description

> **Definition 1.1.** *A* ***stochastic process*** $\{x(t), t \in T\}$ *is a family of random variables, where T is a set of indicies, and $X(t)$ is a random variable for each value of t.*

1.3 Recall from Statistics Course

A random experiment is perfomed the outcome of the experiment is random.

- THe set of possible outcomes is the **sample space** ω
  - An **event** $A \subset \omega$ if the outcome is contained in A
  - The **complement** of an event A is $A^c = \omega \setminus A$
  - The **null event** $\emptyset$ is the empty set $\emptyset = \omega \setminus \omega$

1.3.1 Combining Event

Let A and B be events

- The **union** $A \cup B$ is the event that at least one of A and B occur.
- the **intersection** $A \cap B$ is the event that both A and B occur.

The events $A_1, A_2, \ldots$ are called disjoint (or **mutually exclusive**) if $A_i \cap A_j = \emptyset$ for $i \neq j$

1.3.2 Probability

Pr is called a probability on ω if

- $\Pr\{\omega\} = 1$
- $0 \leq P\{A\} \leq 1$ for all events A
- For $A_1, A_2, \ldots$ that are mutually exclusive

$$P\left\{\bigcup_{i=1}^{\infty} A_i\right\} = \sum_{i=1}^{\infty} P\{A_i\}$$

We call $P\{A\}$ the probability of A.

1.3.3 Law of total probability

Let $A_1, A_2, \ldots$ be a partition of ω ie

- $\omega = \bigcup_{i=1}^{\infty} A_i$
- $A_1, A_2, A_3, \ldots$ are mutually exclusive.

Then for any event B

$$P\{B\} = \sum_{i=1}^{\infty} P\{B \cap A_i\}$$

This concept is very important.

1.3.4 Independence

Event A and B are independent of

$$P\{A \cap B\} = P\{A\} P\{B\}$$

Events $A_1, \ldots, A_n$ are independent if for any subset

$$P\left\{\bigcap_{j=1}^{k} A_{i_j}\right\} = \prod_{j=1}^{k} P\{A_{i_j}\}$$

In this case $P\{\bigcap_{i=1}^{n} A_1\} = \prod_{i=1}^{n} P\{A_i\}$

1.3.5 Random Variables

Definition 1.2. *A* ***random variable*** *is a real-vaued function on the sample space. Informally: A random variable is a real valued variable that takes on its value by chance.*

Example.

- Throw two dice. $X =$ sum of the two dice
- Throw a coin. X is 1 for heads and X is 0 for tails.

1.3.6 Notation for random variables

We use

- upper case letters such at X, Y and Z to represent random variables.
- lower case letters as x, y, z to denote the real-valued realized value of a the random variable.

Expression such as $\{X \leq x\}$ denators the event that X assumes a valye less than or earl to the real number x.

1.3.7 Discrete random variables

The random variable X is **discrete** if it has a finite or countablle number of possible outcomes $x_1, x_2, \ldots$

- The **probability mass function** $p_x(x)$ is given by

  $$p_x(x) = P\{X = x\}$$

  and satisfies

  $$\sum_{i=1}^{\infty} p_x(x_i) = 1 \quad \text{and} \quad 0 \leq p_x(x_i) \leq 1$$

- The **cumulative distribution function** (CDF) a of X can be written

  $$F_x(x) = P\{X \leq x\} = \sum_{i: x_i \leq x} p_x(x_i)$$

1.3.8 CFD

The CDF of X may also be called the **distrobution function** of X

Let $F_x(x)$ be the CDF of X, then

- $F_x(x)$ is monetonaly increasing.
- F_x is a stepfunction, which is a pieace-wise constant with jumps at x_i.
- $\lim_{x\to\infty} F_x(x) = 1$
- $\lim_{x\to-\infty} F_x(x) = 0$

1.3.9 Continious random vairbales

A **continious** random variables takes value o a continious scale.

- The CDF, $F_x(x) = P(X \le x)$ is continious.
- The **probability density function** (PDF) $f_x(x) = F'_x(x)$ can be used to calculate probablities

$$\begin{aligned} Pr\{a < X < b\} &= Pr\{a \le X < b\} = Pr\{a < X \le b\} \\ &= Pr\{a \le X \le b\} = \int_a^b f_x(x)\,dx \end{aligned}$$

1.3.10 Important properties

- CDF:
  - Monotonely increaing
  - continious
  - $\lim_{x\to\infty} F_x = 1$ and $\lim_{x\to-\infty} F_x(x) = 0$
- PDF
  - $f_x(x) \ge 0$ for $x \in \mathbb{R}$
  - $\int_{-\infty}^{\infty} f_x(x)\,dx = 1$

1.3.11 Expectation

Let $g : \mathbb{R} \to \mathbb{R}$ be a function and X be a random variable.

- If X is discrete, the expected value of $g(X)$ is

$$E[g(X)] = \sum_{x:p_x(x)>0} g(x)\,p_x(x)$$

- If X is continous, the expected value of $g(X)$ is

$$E[g(X)] = \int_{-\infty}^{\infty} g(x)\,f_x(x)\,dx$$

1.3.12 Variance

The variance of the random variable X is

$$Var[X] = E\left[(X - E[X])^2\right] = E\left[X^2\right] - E[X]^2$$

Important properties of expectation and variance.

- Expectations is linear

$$E[aX + bY + c] = aE[X] + bE[Y] + c.$$

- Variance scales quadratically and is invaraient to the addition of constants

$$Var[aX + b] = a^2 Var[X]$$

- fir independent stochastic variables.

$$Var[X + Y] = Var[X] + Var[Y]$$

1.3.13 Joint CDF

If (X, Y) is a pair for random variables, their **joint comulative distribution function** is given by

$$F_{X,Y} = F(x, y) = Pr\{X \leq x \cap Y \leq y\}$$

.

1.3.14 Joint distrubution for discrete random variables

If X and Y are discrete, the **joint probability mass function** $p_{x,y} = Pr\{X = x, Y = y\}$. can be used to compute probabilities

$$Pr\{a < X < b, c < Y \leq d\} = \sum_{a<x\leq b} \sum_{c<y\leq d} p_{X,Y}(x, y)$$

1.3.15 Joint distrubution for continous random variables

If X and Y are continious the **joint probability density function**

$$.f_{X,Y}(x, y) = f(x, y) = \frac{\partial^2}{\partial x \partial y} F(x, y)$$

can be used to compute probabilities

$$Pr\{a < X \leq b, \quad c < Y \leq d\} = \int_a^b \int_c^d f(x, y)\, dx dy$$

1.3.16 Independence

The random variables X and Y are independent if

$$Pr\left\{X \leq a, Y \leq b\right\} = Pr\left\{X \leq a\right\} \cdot Pr\left\{Y \leq b\right\}, \quad \forall a, b \in \mathbb{R}$$

In terms of CDFs: $F_{X,Y}(a,b) = F_X(a) \cdot F_Y(b) \quad \forall a, b \in \mathbb{R}$

Thus we have

- $p_{X,Y}(x,y) = p_X(x) \cdot p_Y(Y)$ for discrete random variables
- $f_{X,Y}(x,y) = f_X(x) \cdot f_Y(Y)$ for continuous random variables.

2 Lecture 3

2.1 Randoms sum

Building on the hunter example from last week. we can more generally consider random sums

$$X = \begin{cases} 0, & N = 0 \\ \zeta_1 + \zeta_2 + \ldots + \zeta_N, & N > 0 \end{cases}$$

where

- N is a discrete random variable with values $0, 1, \ldots$
- $\zeta_1, \zeta_2, \ldots$ are independent random variables
- N is independent of $\zeta_1, \zeta_2 + \ldots + \zeta_N$
- **Notation** $X = \sum_{i=1}^{N} \zeta_i = \zeta_1 + \zeta_2 + \ldots + \zeta_N$

Example.

1. Insurance company

$$N: \text{ Number of claims.}$$

$$\zeta_1, \zeta_2, \ldots \quad : \quad \text{Sizes of the claims}$$

Total liabilility:

$$X = \zeta_1 + \zeta_2 + \ldots + \zeta_N$$

2. Be careful!

$$\begin{aligned} \overbrace{E\left[\sum_{i=1}^{N} \zeta_i\right]}^{\neq \sum_{i=1}^{N} E[\zeta_i]} &= E\left[E\left[\sum_{i=1}^{N} \zeta_i \mid N\right]\right] \\ &= E\left[\sum_{i=1}^{N} E\left[\zeta_i \mid N\right]\right] \end{aligned}$$

2.2 Self Study

Section 2.2, 2.3, 2.4

2.3 Stochastic process in descrete time

Definition 2.1. *A **discrete-time stochastic process** is a family of random variables* $[X_t : t \in T]$ *where* T *is discrete.*

- *We use* $T = \{0, 1, 2, ..\}$ *and write* X_n *instead of* X_t
- *we call* X_n *the **state** at time* $n = 0, 1, 2, 3, \ldots$
- *We call the set of all possible states the **state space***

Table 1: Table for example

Day	$n = 0$	$n = 1$	$n = 2$	$\ldots$
Random Variable	X_0	X_1	X_2	$\ldots$
Realization 1	$x_0 = 0$	$x_1 = 1$	$x_2 = 1$	$\ldots$
Realization 2	$x_0 = 1$	$x_1 = 1$	$x_2 = 1$	$\ldots$

Example.

$$X_n = \begin{cases} 1, & \text{if it rains on day } n \\ 0, & \text{no rain on day } n \end{cases}$$

State space $= \{0, 1\}$
We have a problem. Need

$$Pr\left\{X_n = x_n \mid X_{n-1} = x_n, X_{n-2} = x_{n-2}, \ldots, X_0 = x_0\right\}.$$

for all $n = 0, 1, 2, \ldots$

2.4 Markov chain

Definition 2.2 (Discrete time Markov Chain)**.** *A **Discrete time markoc chain** is a discrete time stochastic process* $\{X_n : n = 0, 1, \ldots\}$ *that statisfied the **markov property** such that*

$$\begin{aligned} &Pr\left\{X_{n-1} = j \mid X_n = i, X_{n-1} = i_{n-1}, \ldots, X_0 = i_0\right\} \\ &= Pr\left\{X_{n+1} = j \mid X_n = i\right\} \end{aligned}$$

for $n = 0, 1, 2, 3, \ldots$ *and for all states* i *and* j

Definition 2.3 (One-step transition probabilities)**.** *We can define it as*

- *For a discrete Markov chain* $\{X_n : n = 0, 1, 2, \ldots\}$ *we call* $P_{ij}^{n,n+1} = Pr\left\{X_{n+1} = j, X_n = i\right\}$ *the **one step trainsition probabilities** .*

- *We will assume* ***stationary transition probabilities*** *, i.e that*

$$P_{ij}^{n,n+1} = P_{ij}$$

for $n = 0, 1, 2, \dots$ *and all states* i *and* j *.*

Some of the properties

1. "You will always go somewhere"

$$\sum_j P_{ij} = 1 \quad \forall i$$

2. The markov chain can be described as follows.

$$\begin{aligned}
&Pr\{X_0 = i_0, X_1 = i_1, \dots, X_n = i_n\} \\
&= Pr\{X_0 = i_0\} Pr\{X_1 = i_1 \mid X_0 = i_0\} \dots \\
&\quad Pr\{X_n = x_n \mid X_{n-1} = i_{n-1} \dots X_0 = i_0\} \\
&\vdots \quad \text{Markov step} \\
&= Pr\{X_0 = i_0\} \cdot Pr\{X_1 = i_1 \mid X_0 = i_0\} \dots \\
&\quad Pr\{X_n = x_n \mid X_{n-1} = i_{n-1}\} \\
&= Pr\{X_0 = i_0\} P_{i_0,i_1} \cdot P_{i_1,i_2} \dots P_{i_{n-1},i_n}
\end{aligned}$$

Which is a major simplification.

Definition 2.4 (Transition Probability Matrix)**.** *For a discrete time markov-chain with state space* $\{0, 1, \dots, N\}$ *we call*

$$\mathbf{P} = \begin{bmatrix} P_{00} & \dots & P_{0N} \\ P_{10} & \dots & \\ \vdots & & \ddots \\ P_{N0} & \dots & P_{NN} \end{bmatrix}$$

Is the transition matrix. For statespace $\{0, 1, 2, \dots\}$ *we envision an infinitely sized matrix.*

Example.

- Markoc chain : $\{X_n : n = 0, 1, 2, \ldots\}$
- State space $= \{0, 1\}$
- Transition Matrix

$$\mathbf{P} = \begin{bmatrix} 0.9 & 0.1 \\ 0.6 & 0.4 \end{bmatrix}$$

We can compute

$$\begin{aligned} Pr\{X_3 = 1 \mid X_2 = 0\} &= p_{01} \\ &= 0.1 \\ Pr\{X_{10} = 0 \mid X_9 = 1\} &= P_{10} \\ &= 0.6 \end{aligned}$$

Definition 2.5 (Transition Diagram)**.** *Let $\{X_n : n = 0, 1, \ldots\}$ be a discrete time Markov chain. A **state trasnistion diagram** visualizes the transition probabilities as a weighted directed graph where the nodes are the states and the edges are the possible transitions marked with the transistion probabilities.*

Example. State space $= \{0, 1, 2\}$ and

$$P = \begin{bmatrix} 0.95 & 0.05 & 9 \\ 0 & 0.9 & 0.1 \\ 0.01 & 0 & 0.99 \end{bmatrix}$$

Transisition diagram

Nice figure of the diagram

2.5 Doing n transitions.

Theorem 2.1. *For a Markoc chain $\{X_n : n = 0, 1, \ldots\}$ and any $m \geq 0$ we have*

$$Pr\{X_{m-n} = j \mid X_m = i\} = P_{ij}^{(n)} = \sum_{k=0}^{\infty} P_{ik} P_{kj}^{(n-1)}, \quad n > 0$$

where we define

$$P_{ij}^{(0)} = \begin{cases} 1, & i = j \\ 0, i \neq j \end{cases}$$

Proof. Set $m = 0$ then is

$$\begin{aligned} P_{ij}^{(n+1)} &= Pr\left\{X_{n+1} = j \mid X_0 = i\right\} \\ &= \sum_k Pr\left\{X_{n+1} = j, X_1 = k \mid X_0 = i\right\} \\ &= \sum_k Pr\left\{X_{n+1} = j \mid X_1 = k, X_0 = i\right\} \cdot Pr\left\{X_1 = k \mid X_0 = i\right\} \\ &= \sum_k P_{kj}^{(h)} \cdot P_{ik} = \sum_k P_{ik} P_{kj}^{(h)} \end{aligned}$$

□

Example. $\{X_n : n = 0, 1, 2, \ldots\}$ is a markoc chain and

$$P = \begin{bmatrix} 0.1 & 0.9 \\ 0.6 & 0.4 \end{bmatrix}$$

Find $P_{01}^{(4)}$. **Solution**.

$$P^2 = \begin{bmatrix} 0.55 & 0.45 \\ 0.30 & 0.70 \end{bmatrix}$$

So by doing matrix multiplication and we end up with

$$P^4 = P^2 \cdot P^2 = \begin{bmatrix} 0.4375 & 0.5625 \\ 0.3750 & 0.6250 \end{bmatrix}$$

Which therefore ends up with the answer

$$P_{01}^{(4)} = 0.5625$$

3 Lecture 4

3.1 Introduction to first step analysis

Input

- i_0 : starting state
- P : transition probability matrix
- T: number of time steps

Algorithm

1. Set $x_0 = i_0$
2. for $n = 1 \ldots T$
3. Simulate x_n from $X_n \mid X_{n-1} = x_{n-1}$
4. end

output : One realization $x_0, x_1, \ldots, x_T$

Example.

$$P = \begin{pmatrix} 0.95 & 0.05 & 0 \\ 0 & 0.90 & 0.10 \\ 0.01 & 0 & 0.99 \end{pmatrix}$$

Let $x_0 = 0$

1. $x_0 = 0$
2. $$\begin{aligned} Pr\left\{X_1 = 0 | X_0 = 0\right\} =& P_{00} = 0.95 \\ Pr\left\{X_1 \mid X_0\right\} =& P_{01} = 0.05 \\ Pr\left\{X_1 \mid X_0 = 0\right\} =& P_{02} = 0 \\ & . \end{aligned}$$

   Assume we get $x_1 = 1$
3. States
   - $$\begin{aligned} 0 &: P_{10} = 0 \\ 1 &: P_{11} = 0.90 \\ 2 &: P_{12} = 0.10 \\ &\vdots \end{aligned}$$

General notes on simulation

- $Pr\{A\} \approx \frac{\text{times A occure}}{\text{Simulations}}$
- $E[X] \approx \frac{1}{N}\sum_{i=1}^{B} x_i$

Example. We have $N = 100$ divided into two containers labelled A and b. At each time n, one ball is selected at random and moved to the container. Let Y_n denote the number of balls in container A at time n, and define $X_n = Y_n - 50$. Find the transition probabilities and simulate and plot one realization of

$$\{X_n : n = 0, 1, \ldots, 500\}$$

Answer

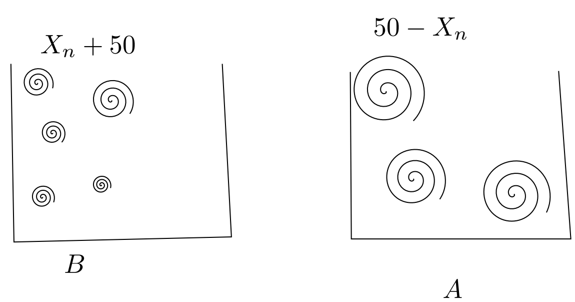

Figure 1: balls

- Only move One ball
- Can move only from i to $j = i - 1$ or $ji + 1$

$$P_{ij} = \begin{cases} \frac{50-i}{100} & , \quad j = i+1 \\ \frac{50+i}{100} & , j = i - 1 \\ 0 & , \text{otherwise.} \end{cases}$$

Motivation

Definition 3.1. *For a markov chain, a state i sich that* $P_{ij} = 0 \forall j \neq i$ *is* *called **absorbing.***

Example. Let $\{X_n\}$ be a Markov chain woth transition probability matrix

$$\mathbf{P} = \begin{pmatrix} 1 & 0 & 0 \\ \alpha & \beta & \gamma \\ 0 & 0 & 1 \end{pmatrix}$$

where $\alpha, \beta, \gamma > 0$ and $\beta = 1 - \alpha - \gamma$. Assume $x_0 = 1$

1. What is the expected time until absortion ?

2. What is the probability to be absorbed in state 0 ?

Realization .

$$\overbrace{1, 1, 1, 1, 1, 2}^{\text{4 steps to absorption}}, 2, 2 \ldots$$

Mathematically

Let $T = \min \{n \geq 0 : X_n = 0 \quad \text{or} \quad X_n = 2\}$. Then is

$$\begin{aligned} Q1 : & \quad E[T \mid X_0 = 1] \\ Q2 : & \quad Pr\{X_T = 0 \mid X_0 = 1\} \end{aligned}$$

The idea of first step analysis is to define

- $T^{(n)} = \min \{n \geq :: X_{m \times n} = 0 \quad \text{or} \quad X_{m+b} = 2\}$
- $T = T^{(0)}$
- $v_i^{(m)} = E\left[T^{(m)} \mid X_m = i\right]$
- $v_i = v_i^{(0)}$

Table 2: Let m be timesteps

m	0	2	3	4	5
$v_0^{(m)}$	0	0	0	0	0
$v_1^{(m)}$	v_1	v_1	v_1	v_1	v_1
$v_2^{(m)}$	0	0	0	0	0

First step analysis for Q1

$$v_i = \sum_{k=0}^{2} Pr\left\{X_1 = k \mid X_0 = i\right\}(1 + v_k)$$
$$= \sum_{k=0}^{2} P_{ik}(1 + v_k) = \sum_{k=0}^{2} P_{ik} v_k + 1 \quad \text{which is true for} \quad i = 0, 1, 2$$

Which is reduced to linear algebra. Solving it by

$$\begin{aligned} & v_0 = v_2 = 0 \\ \Longrightarrow\ & v_1 = \alpha v_0 + \beta v_1 + \gamma v_2 + 1 \\ \Longrightarrow\ & v_1 = \frac{1}{1-\beta} \quad \text{[Q1]} \end{aligned}$$

$P_{ij} \implies i = \text{row}, \quad j = \text{column}$

First step analyis and let

$$u_i = Pr\left\{X_T = 0 \mid X_0 = i\right\}$$
$$\downarrow$$
$$u_i = \sum_{k=0}^{2} P_{ik} u_k, \quad i = 0, 1, 2$$

- Easy: $u_0 = 1, u_2 = 0$
- Harder: $u_1 = \alpha u_0 + \beta u_1 + \gamma u_2$ such that

$$u_1 = \alpha \frac{1}{1-\beta} = \frac{\alpha}{\alpha - \beta} \quad \text{[Q2]}$$

Example. let $[X_n]$ be a markov chain with transition matrix

$$\mathbf{P} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0.4 & 0.3 & 0.2 & 0.1 \\ 0.1 & 0.3 & 0.3 & 0.3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The starting state is $x_0 = 1$. Calculate the probability to be absorbed in the state D.

1. Define $u_i = Pr\left\{\text{absorbed in state } 0 \mid X_0 = i\right\}$ for $i = 0, 1, 2, 3$

2. Get the easy ones out of the way. In this case $u_0 = 1$ and $u_3 = 0$

3. $$\begin{aligned} u_1 &= P_{10}u_0 + P_{11}u_1 + P_{12}u_2 + P_{13}u_3 \\ &= 0.4 + 0.3u_1 + 0.2u_2 \\ u_2 &= P_{20}u_0 + P_{21}u_1 + P_{22}u_2 + P_{23}u_3 \\ &= 0.1 + 0.3u_1 + 0.3u_2 \end{aligned}$$

4. Solve for u_1 and u_2

4 References