Repetition

Theorem

Let $\{X(t) : t \geq 0\}$ be a birth and death process. Assume that state 0 is the only absorbing state, and that the probability of absorption in state 0 is 1 for all starting states. Then we can find

$$v_i = \mathrm{E}[\min\{t \geq 0 : X(t) = 0\} | X(0) = i], \quad i = 0, 1, \ldots,$$

by solving

$$\begin{aligned} v_0 &= 0, \\ v_i &= \frac{1}{\lambda_i + \mu_i} + \sum_{j \neq i} \Pr\{i \to j\} v_j, \quad i \neq 0. \end{aligned}$$

Theorem

The calculation of the probability to be absorbed in state i for a continuous-time Markov chain works exactly like for a discrete-time Markov chain with one-step transition probabilties given by $P_{ij} = \Pr\{i \to j\}$ for $i \neq j$ and $P_{ij} = 0$ for $i = j$.

Definition

A **continuous-time Markov chain** $\{X(t) : t \geq 0\}$ with state space $\{0, 1, \ldots, N\}$ and stationary transition probabilities is defined through **(transition) rates** $q_{ij} \geq 0$ for $j \neq i$.

Let $q_i = \sum_{j \neq i} q_{ij}$, $i = 0, 1, \ldots, N$, then $\{X(t) : t \geq 0\}$ is defined through

1. $P_{ij}(h) = q_{ij} h + o(h) \quad (\text{as } h \to 0^+) \;$ for $i \neq j$.
2. $P_{ii}(h) = 1 - q_i h + o(h) \quad (\text{as } h \to 0^+)$
3. $$P_{ij}(0) = \delta_{ij} = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases}$$

Constructive definition

A **continuous-time Markov chain** $\{X(t) : t \geq 0\}$ with state space $\{0, 1, \ldots, N\}$ and stationary transition probabilities is defined through **(transition) rates** $q_{ij} \geq 0$ for $j \neq i$.

Let $q_i = \sum_{j \neq i} q_{ij}$, $i = 0, 1, \ldots, N$, then each time $\{X(t) : t \geq 0\}$ jumps to a new state state i

1. the sojourn time is $\text{Exp}(q_i)$
2. after the sojourn time ends, the jump probabilities to the next state are $\Pr\{i \to j\} = \frac{q_{ij}}{q_i}$ for $j \neq i$.

Notation

We collect all the probability transition functions in a matrix

$$\mathbf{P}(t) = \begin{bmatrix} P_{0,0}(t) & P_{0,1}(t) & \cdots & P_{0,N}(t) \\ P_{1,0}(t) & P_{1,1}(t) & & \vdots \\ \vdots & & \ddots & \\ P_{N,0}(t) & \cdots & & P_{N,N}(t) \end{bmatrix}$$

and we define the **infinitesimal matrix** as

$$\mathbf{A} = \begin{bmatrix} -q_0 & q_{0,1} & \cdots & q_{0,N} \\ q_{1,0} & -q_1 & & \vdots \\ \vdots & & \ddots & \\ q_{N,0} & \cdots & & -q_N \end{bmatrix}$$

Note: The forward Kolmogorov differential equations can collectively be written as $\mathbf{P}'(t) = \mathbf{P}(t)\mathbf{A}$, $t \geq 0$, and $\mathbf{P}(0) = \mathbf{I}$.

Theorem

The stationary distributions of a continuous-time Markov chain with state space $\{0, 1, \ldots, N\}$ and stationary transition probabilities are found by solving

$$\pi_i q_i = \sum_{k \neq i} \pi_k q_{ki}, \quad i = 0, 1, \ldots, N,$$
$$\sum_{k=0}^{N} \pi_k = 1.$$

Little's law

For a (stable) queueing system

$$L = \lambda W,$$

where

L: Average number of customers in the system.

λ: Rate of arrival to the system.

W: Average time spent by a customer in the system.

Note: This result is valid very generally.