NTNU
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Week 39: Lecture 1

Properties of the Poisson process

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Information

- You need to register in one of the groups called "Project group" to be able to see the project and to submit.
- You can receive help with the project during exercise classes in R2 tomorrow.
- We have also created a Blackboard forum where you can get answers about practical questions.
- **No lectures in week 40 (September 28 and September 30).**

Definition (P2, simplified)

Let $\{N(t) : t \geq 0\}$ be a counting process. Then $\{N(t) : t \geq 0\}$ is a **Poisson process** with **rate (intensity)** $\lambda > 0$ if

1. it has independent increments.
2. it has stationary increments.
3. $\Pr\{N(t+h) - N(t) = 1\} = \lambda h + o(h) \quad (\text{as } h \to 0^+), \quad \forall t \geq 0.$
4. $\Pr\{N(t+h) - N(t) = 0\} = 1 - \lambda h + o(h) \quad (\text{as } h \to 0^+), \quad \forall t \geq 0.$
5. $N(0) = 0.$

Definition (P1, simplified)

A **Poisson process** with **rate (intensity)** $\lambda > 0$ is an integer-valued stochastic process $\{N(t) : t \geq 0\}$ for which

1. increments are independent.
2. for $s \geq 0$ and $t > 0$,
$$N(s+t) - N(s) \sim \mathrm{Poisson}(\lambda t).$$
3. $N(0) = 0$.

Theorem

Definition P1 and Definition P2 of a Poisson process are equivalent.

Example

Is it reasonable to model the following phenomena as Poisson processes?

a) Cases of a non-infectious rare disease.
b) Calls going through a phone central.
c) Goals in football.

Section 5.3

Definition

Let $\{N(t) : t \geq 0\}$ be a Poisson process. The **waiting time** W_n is the time of occurrence of the n-th event. We define $W_0 = 0$.

Definition

The differences $S_n = W_{n+1} - W_n$ are called the **sojourn times** (interarrival times).

Definition

The stochastic variable Y has an **exponential distribution** with **rate parameter** $\lambda > 0$ if

$$f(y) = \lambda e^{-\lambda y}, \quad y > 0.$$

We write $Y \sim \mathrm{Exp}(\lambda)$.

Theorem (Theorem 5.5)

Let $\{N(t) : t \geq 0\}$ be a Poisson process with rate λ. Then $S_0, S_1, \ldots, S_{n-1} \overset{\text{iid}}{\sim} \mathrm{Exp}(\lambda)$.

Definition

The stochastic variable Y has a **gamma distribution** with **shape parameter** $\alpha > 0$ and **rate parameter** $\lambda > 0$ if

$$f(y) = \frac{\lambda^\alpha}{\Gamma(\alpha)} y^{\alpha-1} \mathrm{e}^{-\lambda y}, \quad y > 0.$$

We write $Y \sim \mathrm{Gamma}(\alpha, \lambda)$.

Theorem (Theorem 5.4)

For a Poisson process with rate $\lambda > 0$, $W_n \sim \mathrm{Gamma}(n, \lambda)$ for all integers $n > 0$.

Example

Assume the occurrences of a rare disease follows a Poisson process with rate $\lambda = 2$ per month.

a) What is the probability that the first case occurs after 1 month?

b) What is the expected time until the 10th case occurs?

Section 5.4

Example

$\{X(t) : t \geq 0\}$ is a Poisson process with rate $\lambda > 0$. Determine the distribution of $W_1|X(t) = 1$.