Norwegian University of Science and Technology
Deptartment of Mathematical Sciences

TMA4190 Introduction to Topology Spring 2018

Solutions to exercise set 2

$\boxed{1}$ We can choose any basis of V to define a linear isomorphism $\phi\colon \mathbb{R}^k \to V$ which is a diffeomorphism. Given a point $x \in V$, we modify ϕ by adding x and get a new diffeomorphism (not linear anymore!)

$$\psi\colon \mathbb{R}^k \to V,\ w \mapsto \phi(w) + x.$$

The derivative $d\psi_0$ of ψ at 0 is just $\phi\colon \mathbb{R}^k \to \mathbb{R}^N$ (indepentdent of x). The tangent space of V at x is by our definition the image of $d\psi_0$ in $\mathbb{R}^N$, which is equal to the image of ϕ in $\mathbb{R}^N$ which is by definition of ϕ equal to V.

$\boxed{2}$ We only answer the queston about the tangent space of $T(a,b)$. For all points apart from $(a+b,0,0)$ we can parametrize $T(a,b) \subset \mathbb{R}^3$ by

$$\begin{aligned}\phi\colon (0,2\pi)\times(0,2\pi) &\to T(a,b)\\ (s,t) &\mapsto ((a+b\cos s)\cos t, (a+b\cos s)\sin t, b\sin s).\end{aligned}$$

The derivative of ϕ at (s,t) is

$$d\phi_{(s,t)}\colon \mathbb{R}^2 \to \mathbb{R}^3,\ d\phi_{(s,t)} = \begin{pmatrix} -b\sin s\cos t & -(a+b\cos s)\sin t \\ -b\sin s\sin t & (a+b\cos s)\cos t \\ b\cos s & 0 \end{pmatrix}.$$

The tangent space to $T(a,b)$ at the point $\phi(s,t)$ is $d\phi_{(s,t)}(\mathbb{R}^2) \subset \mathbb{R}^3$.

Let us check that the column vectors of the matrix $d\phi_{(s,t)}$, and hence the whole tangent space, is orthogonal to the vector pointing from the center of the circle with radius b to $\phi(s,t)$. The center point is $(a\cos t, a\sin t, 0$, and hence the vector we need to look at is $(b\cos s\cos t, b\cos s\sin t, b\sin s)$. We calculate the two scalar products:

$$\begin{aligned}&(b\cos s\cos t, b\cos s\sin t, b\sin s)\cdot\begin{pmatrix} -b\sin s\cos t \\ -b\sin s\sin t \\ b\cos s\end{pmatrix}\\ =&-b^2\cos s\sin s\cos^2 t - b^2\cos s\sin s\sin^2 t + b^2\sin s\cos s\\ =&b^2\cos s\sin s(-\cos^2 t - \sin^2 t + 1)\\ =&b^2\cos s\sin s(-1+1) = 0\end{aligned}$$

and

$$\begin{aligned}&(b\cos s\cos t, b\cos s\sin t, b\sin s)\cdot\begin{pmatrix} -(a+b\cos s)\sin t \\ (a+b\cos s)\cos t \\ 0\end{pmatrix}\\ =&-b(a+b\cos s)\cos s\sin t\cos t b(a+b\cos s)\cos s\sin t\cos t + 0\\ =&0.\end{aligned}$$

In order to cover also the point $(a+b,0,0)$ it suffices to rotate our parametrization by the angle π in the $xy-$plane and use the diffeomorphism

$$\begin{aligned}\phi\colon (0,2\pi)\times(0,2\pi) &\to T(a,b)\\ (s,t) &\mapsto ((-a+b\cos s)\cos t, (-a+b\cos s)\sin t, b\sin s)\end{aligned}$$

which covers $(a+b,0,0)$ for $(s,t)=(\pi,\pi)$.

$\boxed{3}$ Around the point $(\sqrt{a},0,0)$ on H_a we can choose the local parametrization

$$\phi\colon B_{\sqrt{a}}((0,0))\to H\cap\{y^2+z^2<a\},\ (y,z)\mapsto\left(\sqrt{z^2-y^2+a},y,z\right).$$

The derivative in the standard basis at a point (y,z) is the linear map

$$d\phi_{(y,z)}\colon \mathbb{R}^2\to\mathbb{R}^3,\ d\phi_{(y,z)}=\begin{pmatrix} -\frac{y}{\sqrt{z^2-y^2+a}} & \frac{z}{\sqrt{z^2-y^2+a}}\\ 1 & 0\\ 0 & 1\end{pmatrix}.$$

Hence at a point $(u,v,w)\in H_a$ the image of the standard basis of $\mathbb{R}^2$ is

$$\begin{pmatrix}-v/u\\1\\0\end{pmatrix} \text{ and } \begin{pmatrix}w/u\\0\\1\end{pmatrix}$$

where we write $u=\sqrt{w^2-v^2+a})$. Hence the tangent space at $T_{(u,v,w)}(H_a)$ is spanned by these two vectors.
For $(u,v,w)=\sqrt{a},0,0)$ we get that $T_{(\sqrt{a},0,0)}(H_a)$ is simply spanned by

$$\begin{pmatrix}0\\1\\0\end{pmatrix} \text{ and } \begin{pmatrix}0\\0\\1\end{pmatrix}.$$

$\boxed{4}$ **a)** The inverse map to F is the projection π onto the first factor, for we obviously have $\pi\circ F=\mathrm{Id}_X$ and $F\circ\pi=\mathrm{Id}_{\Gamma(f)}$. Let $X\subset\mathbb{R}^N$, $Y\subset\mathbb{R}^M$, and let f be smooth. Then for any point $x\in X$, there is an open subset $U\subset\mathbb{R}^N$ and a smooth map $\tilde{f}\colon U\to\mathbb{R}^M$ with $\tilde{f}_{X\cap U}=f_{X\cap U}$. Then $(\tilde{f}\times\mathrm{Id})\colon U\to\mathbb{R}^N\times\mathbb{R}^M$ is a smooth extension of $F_{X\cap U}$. Hence F is smooth. The inverse map π is a smooth map, since it extends to the smooth projection on all of $\mathbb{R}^N\times\mathbb{R}^M$. Hence F is a diffeomorphism when f is smooth. Hence any local parametrization $\phi\colon V\to X$ can be extended to a local parametrization $F\circ\phi\colon V\to\Gamma(f)$. Thus the graph $\Gamma(f)$ is a manifold if X is.

b) For $\in X$, let $\phi\colon U\to X$ be a local parametrization around x with $\phi(0)=x$ and open $U\subseteq\mathbb{R}^k$. Let $\psi\colon W\to Y$ be a local parametrization around $f(x)$ with $\psi(0)=f(x)$ and open $W\subseteq\mathbb{R}^l$. Then $\phi\times\psi\colon U\times W\to X\times Y$ is a local parametrization around $(x, f(x))$ with $U \times W \subset \mathbb{R}^{k+l}$ open. Then we can construct a commuative diagram

$$\begin{tikzcd} X \arrow[r, "F"] & X \times Y \\ U \arrow[u, "\phi"] \arrow[r, "G=\psi^{-1}\circ F\circ\phi"'] & U \times W \arrow[u, "\phi\times\psi"'] \end{tikzcd}$$

where G is the map defined by $v \mapsto (v, \psi^{-1}(f(\phi(v))))$. Thus G is the map $\mathrm{Id}_V \times (\psi^{-1} \circ f\phi)$. Hence $dG_0 \colon \mathbb{R}^k \to \mathbb{R}^k \times \mathbb{R}^l$ is the linear map

$$dG_0 = \mathrm{Id}_{\mathbb{R}^k} \times (d\psi^{-1}_{f(x)} \circ df_x \circ d\phi_0).$$

Thus in the commutative diagram below, dF_x has to be defined as $\mathrm{Id}_{T_x(X)} \times df_x$:

$$\begin{tikzcd} T_x(X) \arrow[r, dotted, "dF_x=\mathrm{Id}_{T_x(X)}\times df_x"] & T_x(X) \times T_y(Y) \\ \mathbb{R}^k \arrow[u, "d\phi_0"] \arrow[r, "dG_0=\mathrm{Id}_{\mathbb{R}^k}\times(d\psi^{-1}_{f(x)}\circ df_x\circ d\phi_0)"'] & \mathbb{R}^k \times \mathbb{R}^l. \arrow[u, "d\phi_0\times d\psi_0"'] \end{tikzcd}$$

Hence $dF_x(v) = (v, df_x(v))$.

c) For $\in X$, let $\phi \colon U \to X$ be a local parametrization around x with $\phi(0) = x$ and open $U \subseteq \mathbb{R}^k$. Since F is a diffeomorphism, $\psi = F \circ \phi \colon U \to \Gamma(f)$ is then a local parametrization around $(x, f(x))$ with $\psi(0) = F(\phi(0)) = (x, f(x))$. The tangent space of $\Gamma(f)$ at $(x, f(x))$ is by definition the image of $d\psi_0 \colon \mathbb{R}^k \to \mathbb{R}^{N+M}$. By the chain rule we have

$$d\psi_0 = dF_x \circ d\phi_0 \colon \mathbb{R}^k \xrightarrow{d\phi_0} \mathbb{R}^N \xrightarrow{dF_x} \mathbb{R}^{N+M}.$$

Hence by our definition of tangent spaces:

$$T_{(x,f(x))}(\Gamma(f)) = d\psi_0(\mathbb{R}^k) = dF_x(d\phi_0(\mathbb{R}^k)) = dF_x(T_x(X)).$$

Finally, by the previous point, we know $dF_x = \mathrm{Id}_{T_x(X)} \times df_x$ and get

$$T_{(x,f(x))}(\Gamma(f)) = (\mathrm{Id}_{T_x(X)} \times df_x)(T_x(X)) = \Gamma(df_x) \subset T_x(X) \times T_{f(x)}(Y)$$

which is the graph of df_x in $T_x(X) \times T_{f(x)}(Y)$.

$\boxed{5}$ **a)** Given a smooth map $c \colon I \to \mathbb{R}^k$ with $c = (c_1, \ldots, c_k)$ and $c_i \colon I \to \mathbb{R}$ all smooth. The derivative of c at $t_0 \in I$ is a linear map $dc_{t_0} \colon T_{t_0}I = \mathbb{R} \to \mathbb{R}^k = T_{x_0}(\mathbb{R}^k)$:

$$dc_{t_0}(v) = (c_1'(t_0), \ldots, c_k'(t_0)) \cdot v.$$

Since $v \in \mathbb{R}$ is just a real number, we get

$$dc_{t_0}(1) = (c_1'(t_0), \ldots, c_k'(t_0)) \in \mathbb{R}^k.$$

b) First, assume $X = \mathbb{R}^k$ and let $w = (w_1, \dots, w_k)$ be a vecor in $T_x X = \mathbb{R}^k$. Then define the curve $c_w \colon \mathbb{R} \to \mathbb{R}^k$ by $t \mapsto t \cdot w$. The derivative of c_w at any t_0 is

$$d(c_w)_{t_0} \colon \mathbb{R} \to \mathbb{R}^k,\ t \mapsto (w_1, \dots, w_k) \cdot t.$$

Thus we have $d(c_w)_{t_0}(1) = w$.

Now let X be an arbitrary k-dimensional smooth manifold, $x \in X$, and let v be a vector in $T_x(X)$. Let $\phi \colon V \to X$ be a local parametrization around x with $\phi(0) = x$. By definition, $T_x(X) = d\phi_0(\mathbb{R}^k)$ and there is a unique vector $w \in \mathbb{R}^k$ with $d\phi_0(w) = v$. Since any open ball around the origin in $\mathbb{R}^k$ is diffeomorphic to $\mathbb{R}^k$, we can assume that w is contained in $V \subseteq \mathbb{R}^k$ (we could in fact assume $V = \mathbb{R}^k$). Let $c_w \colon \mathbb{R} \to \mathbb{R}^k$ be the linear curve in $\mathbb{R}^k$ defined in the previous point. Then we define $c \colon \mathbb{R} \to X$ by $c = \phi \circ c_w$, i.e. $c(t) = \phi(t \cdot w)$. The derivative of c at $t_0 = 0 \in \mathbb{R}$ is

$$d(c)_{t_0} = d\phi_0 \circ d(c_w)_{t_0}.$$

Thus

$$d(c)_{t_0}(1) = d\phi_0(d(c_w)_{t_0}(1)) = d\phi_0(w) = v.$$