Norwegian University of Science
and Technology
Deptartment of Mathematical
Sciences

TMA4190 Introduction to Topology Spring 2018

Solutions to exercise set 10

$\boxed{1}$ If A is not invertible, then 0 is an eigenvalue, and we are done. So assume A is nonsingular. For any vector $v \in S^{n-1} \subset \mathbb{R}^n$, the vector $Av/|Av|$ has norm one and lies on S^{n-1}. (Note that this map is not defined for $v = 0$ and we cannot continuously extend it on 0. Hence we cannot just consider this as a map $B^n \to B^n$!)

Let $g\colon S^{n-1} \to S^{n-1}$ be the map $v \mapsto Av/|Av|$. Now we use the assumption on A: if

$$v \in Q = \{(x_l, \dots, x_n) \in S^{n-1} : \text{ all } x_i \geq 0\}$$

then Av has only nonnegative entries, since the entries in v are all nonnegative and all the entries in A are by assumption nonnegative. Since $|Av| > 0$ is nonnegative as well, we know that $g(v)$ is an element in Q. Thus we can restrict g to a map $g\colon Q \to Q$.

Now we can compose with a homoemorphism $\varphi\colon Q \xrightarrow{\cong} B^{n-1}$ to get a continuous map

$$f\colon B^{n-1} \xrightarrow{\varphi} Q \xrightarrow{g} Q \xrightarrow{\varphi^{-1}} B^{n-1}.$$

By the Brouwer Fixed Point Theorem for continuous maps, f must have a fixed point $y \in B^{n-1}$ with $f(y) = y$. Hence the image of $w := \varphi^{-1}(y)$ is a vector in $S^{n-1} \subset R^n$ with

$$Aw/|Aw| = w, \text{ i.e. } Aw = |Aw| \cdot w.$$

Since w is nonzero being a point on S^{n-1}, w is an eigenvector with real nonnegative eigenvalue $|Aw|$.

$\boxed{2}$ Let X and Y be submanifolds of $\mathbb{R}^N$. As in the lecture, we define a

$$F\colon X \times \mathbb{R}^N \to \mathbb{R}^N, (x, a) \mapsto x + a.$$

The derivative of F is given by

$$dF_{(x,a)}\colon T_x(X) \times \mathbb{R}^N \to \mathbb{R}^N, (v, w) \mapsto v + w.$$

Thus $dF_{(x,a)}$ is surjective at every point (x, a). Hence F is a submersion, and therefore transversal to every submanifold of $\mathbb{R}^N$. In particular, it is transversal to both boundaryless submanifolds $\text{Int}(Y)$ and ∂Y.

By the Transversality Theorem of the lecture, the map

$$t_a\colon X \to \mathbb{R}^N, \; x \mapsto x + a$$

is transversal to each of $\text{Int}(Y)$ and ∂Y for almost every $a \in \mathbb{R}^N$. Hence it is transversal to both $\text{Int}(Y)$ and ∂Y for almost every $a \in \mathbb{R}^N$.

The derivative of the translation t_a is just

$$d(t_a)_x \colon T_x(X) \to \mathbb{R}^N, \ v \mapsto v.$$

Moreover, the tangent spaces of $X+a$ and X are equal, since any local parametrization ϕ of X defines a local parametrization $\phi + a$ of $X+a$. Since the derivatives of ϕ and $\phi + a$ are equal, we have $T_x(X) = T_{x+a}(X+a)$.

Hence the transverality $t_a \pitchfork Y$ implies

$$\mathbb{R}^N = \mathrm{Im}\,(d(t_a)_x) + T_{t_a(x)}(Y) = T_x(X) + T_{x+a}(Y) = T_{x+a}(X+a) + T_{x+a}(Y). \quad (1)$$

If $y = x+a \in Y$, then (1) means that $X+a$ and Y meet transversally in $y = x+a$. If $x+a \notin Y$, then $x+a \notin (X+a) \cap Y$, and $X+a$ and Y meet transversally in $x+a$ automatically.

$\boxed{3}$ **a)** Let Y be a compact submanifold of $\mathbb{R}^M$, and let $w \in \mathbb{R}^M$. Since Y is compact, the continuous function

$$Y \to \mathbb{R}, \ y \mapsto |w - y|^2$$

has a minimum. Let $y \in Y$ be a point, where this function has its minimum (there may be many such y's, we just pick one). Hence y is a point of Y which is closest to w.

Now let $c \colon (-a,a) \to Y$ be any smooth curve on Y with $c(0) = y$. The smooth function

$$f \colon (-a,a) \to \mathbb{R}, \ t \mapsto |w - c(t)|^2$$

then has a minimum at $t = 0$. Thus its derivative df_0 at 0 vanishes. Writing $f(t) = |w - c(t)|^2 = (w - c(t)) \cdot (w - c(t))$ using the scalar product, we see that df_0 is given by

$$df_0 = 2(w - c(0)) \cdot (-dc_0)$$

where we consider dc_0 as a vector in $\mathbb{R}^M$ (it really is a matrix with one row and M columns). In particular, we get $w - y = w - c(0)$ is orthogonal to dc_0 in $\mathbb{R}^M$. Since every tangent vector in $T_y(Y)$ is the velocity vector dc_0 for some smooth curve c on Y with $c(0) = y$, this shows $w - y \in N_y(Y)$ by definition of $N_y(Y)$ as the orthogonal complement of $T_y(Y)$ in $\mathbb{R}^M$.

b) Let $N \subset N(Y)$ be the open neighborhood of Y (or rather $= Y \times \{0\}$) in $N(Y)$ which is mapped diffeomorphically onto $Y^\epsilon \subset \mathbb{R}^M$ by h. Given $w \in Y^\epsilon$, there is unique element $n \in N$ with $h(n) = w$. Since elements in $N(Y)$ are pairs (y,v) with $v \in N_y(Y)$, there is a unique $y \in Y$ and $v \in N_y(Y)$ such that $n = (y,v)$ and $\sigma(n) = y$. Since $h(y,v) = y + v$ by definition, and $h(n) = w$ by the choice of n, we must have $v = w - y \in N_y(Y)$. Hence the pair (y,v) is uniquely determined by $w \in Y^\epsilon$.

Since we have the commutative diagram

$$\begin{tikzcd} N(Y) \arrow[r, "h"] \arrow[dr, "\sigma"'] & Y^\epsilon \arrow[d, "\pi"] \\ & Y \end{tikzcd}$$

we know $\pi(w) = \sigma(n) = y$.

By the previous point, we know that any $y_0 \in Y$ with minimal distance to w, must satisfy $w - y_0 \in N_{y_0}(Y)$. We just learned that $\sigma(n) = y \in Y$ is the unique element in Y with this property. Hence $\pi(w)$ is the unique point of Y closest to w.

4 Let X be a submanifold of $\mathbb{R}^N$. Let V be a k-dimensional vector subspace of $\mathbb{R}^N$. Every such V has a basis consisting of a k-tuple of linearly independent k-tuples of vectors in $\mathbb{R}^N$. In particular, every V is the span of such a k-tuple in $\mathbb{R}^N$.

So let $S \subset (\mathbb{R}^N)^k$ be the set consisting of all linearly independent k-tuples of vectors in $\mathbb{R}^N$. For a k-tuple of vectors $[v] := v_1, \dots, v_k$ in $\mathbb{R}^N$, let $A_{[v]}$ be the $N \times k$-matrix with the v_i's as column vectors. Then the k-tuple $v_1, \dots, v_k$ is linearly independent if and only if he $k \times k$-matrix $A^t_{[v]} A_{[v]}$ is invertible, i.e. $\det(A^t_{[v]} A_{[v]}) \neq 0$. Hence S is the inverse image of the open subset $\mathbb{R} \setminus \{0\}$ under the continuous map

$$\mathbb{R}^{Nk} \to \mathbb{R},\ [v] \mapsto \det(A^t_{[v]} A_{[v]}).$$

Thus S is an open subset in $\mathbb{R}^{Nk}$.

We define the map $\varphi\colon \mathbb{R}^k \times S \to \mathbb{R}^N$ by

$$([t],[v]) := ((t_1, \dots, t_k), v_1, \dots, v_k) \mapsto t_1 v_1 + \dots + t_k v_k.$$

Since S is open in $\mathbb{R}^{Nk}$, the tangent space to S at any $[v]$ is just $\mathbb{R}^{Nk}$. Moreover, φ is linear in each coordinate. Thus the derivative of φ at any point $([t],[v])$ is just φ. Since φ is surjective, $d\varphi_{([t],[v])}$ is surjective. Thus φ is a submersion.

Hence, the Transversality Theorem of the lecture implies shows that, for almost every $s = [v]$ in S, the map

$$\varphi_{[v]}\colon \mathbb{R}^k \to \mathbb{R}^N,\ (t_1, \dots, t_k) \mapsto t_1 v_1 + \dots + t_k v_k$$

is transversal to every submanifold in $\mathbb{R}^N$. In particular, $\varphi_{[v]} \pitchfork X$ for almost every $s = [v]$. This means

$$\mathbb{R}^N = \mathrm{Im}\,(d\varphi_{[v]}) + T_x(X) = \mathrm{Im}\,(\varphi_{[v]}) + T_x(X)$$

for every $x \in X$. But the $\mathrm{Im}\,(\varphi_{[v]})$ is by definition of φ just the span of the k-tuple $[v] = \{v_1, \dots, v_k\}$ in $\mathbb{R}^N$.

By our opening remark, every k-dimensional vector subspace in $\mathbb{R}^N$ is the span of some $[v] \in S$. Thus we have shown that for almost every $V := \mathrm{span}([v]) = \mathrm{Im}\,(\varphi_{[v]})$ in $\mathbb{R}^N$ we have

$$V + T_x(X) = \mathbb{R}^N$$

for every $x \in X$. Thus $V \pitchfork X$.

5 **a)** Suppose that $f\colon \mathbb{R}^n \to \mathbb{R}^n$ is a smooth map with $n > 1$, and let $K \subset \mathbb{R}^n$ be compact and $\epsilon > 0$. If $df_x \neq 0$ for all $x \in \mathbb{R}^n$, then we can take $g = f$.

Now assume there is an $x \in X$ such that $df_x = 0$. We would like to replace f with a suitable smooth function $g\colon R^n \to \mathbb{R}^n$ satisfying the two conditions

(a) $dg_x \neq 0$ for all $x \in X$, and
(b) $|f(x) - g(x)| < \epsilon$ for all $x \in K$.

The idea for the solution is to replace f with $f + A$ for a suitable matrix $A \in M(n)$. For any given $A \in M(n) \setminus \{0\}$, the set of norms $\{|Ax| \in \mathbb{R} : x \in K\}$ has a maximum $\mu_A > 0$.
Hence **if** $\mu_A < \epsilon$, then $|Ax| < \epsilon$ for all $x \in K$, and we can define $g \colon \mathbb{R}^n \to \mathbb{R}^n$ by $g(x) = f(x) + Ax$. This map is smooth and satisfies condition (b).
Since $\frac{\epsilon}{2\mu_A} A$ is a linear map, the derivative of g at x is $dg_x = df_x + \frac{\epsilon}{2\mu_A} A$.
In order to prove the assertion, it remains to show that we can find an $A \in M(n)$ such that $df_x + A \neq 0$ and $\mu_A < \epsilon$.
To do this, we define the map

$$F \colon R^n \times M(n) \to M(n), \; (x, A) \mapsto df_x + A.$$

The derivative $dF_{(x,A)}$ of F at a point (x, A) is the sum of the derivative of df_x and the identity map on $M(n)$. In particular, $dF_{(x,A)} \colon \mathbb{R}^n \times M(n) \to M(n)$ is always surjective. Hence F is a submersion, and thus transversal to every submanifold of $M(n)$.
By the Transversality Theorem of the lecture, this implies that, for almost all $A \in M(n)$, the map

$$F_A \colon \mathbb{R}^n \to M(n), \; x \mapsto F(x, A)$$

is transversal to the submanifold $\{0\}$ of $M(n)$.
But, **for $n > 1$**, $\dim \mathbb{R}^n = n$ is **strictly less** than $n^2 = \dim M(n)$.
Thus, since $\{0\}$ is a zero-dimensional submanifold of $M(n)$, F_A is transversal to $\{0\}$ if and only if the intersection $\mathrm{Im}\,(F_A) \cap \{0\}$ is empty, i.e. $FA(x) \neq 0$ for all $x \in X$.
The subset of matrices in $M(n)$ with $\max\{|Ax| : x \in K\} < \epsilon$ is open in $M(n)$. This implies that the intersection of its complement with any subset of measure zero in $M(n)$ has measure zero. Thus, by the Transversality Theorem, we can choose an $A \in M(n)$ with $F_A(x) = df_x + A \neq 0$ for all $x \in X$ **and** $\max\{|Ax| : x \in K\} < \epsilon$.

b) For $n = 1$, we construct a counter-example: Let $f \colon \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2$, let $K = [-2, 2] \subset \mathbb{R}$, and let $\epsilon = 1$. Let $g \colon \mathbb{R} \to \mathbb{R}$ be a smooth function with

$$|f(x) - g(x)| < 1 \text{ for all } x \in K.$$

In particular, this implies

$$3 < g(-2) < 5, \; 3 < g(2) < 5, \text{ and } -1 < g(0) < 1.$$

This shows

$$\frac{g(0) - g(-2)}{0 - (-2)} < 0 \text{ and } \frac{g(2) - g(0)}{2 - 0} > 0.$$

By the Mean Value Theorem, there are real numbers $c \in (-2, 0)$ and $e \in (0, 2)$ such that

$$g'(c) = \frac{g(0) - g(-2)}{0 - (-2)} < 0 \text{ and } g'(e) = \frac{g(2) - g(0)}{2 - 0} > 0.$$

Since g is smooth, g' is differentiable. Hence we can apply the Intermediate Value Theorem to g' and get a number $e \in (c, e)$ with $g'(d) = 0$. Hence we cannot find g with both $g'(x) \neq 0$ for all x and $|f - g| < \epsilon$ on K.