Project 1 Notes

isakhammer

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1 Problem 1

Let normal matrices, those with diagonalization be on the form

$$A = U\Lambda U^H$$

Where Λ is a diagonal complex $n \times n$ matrix and U a unitary (complex) matrix such that $U^H U = I$ (recall that U^H is the complex conjugate of U^T).

Show that for any such matrix, one has $\|A\|_2 = \rho(A)$, where $\rho(A)$ is the spectral radius of A .

Proof. By the definition of a norm is

$$\|A\|_2^2 = \sup_{x \neq 0} \frac{\langle Ax, Ax \rangle}{\langle x, x \rangle}.$$

By taking advantage of the fact that $U^H U = I$, can we substitute $Uy = x$ such that

$$\sup_{x \neq 0} \frac{\langle Ax, Ax \rangle}{\langle x, x \rangle} = \sup_{y \neq 0} \frac{\langle AUy, AUy \rangle}{\langle Uy, Uy \rangle} = \sup_{y \neq 0} \frac{\langle U^H A^H AUy, y \rangle}{\langle y, y \rangle}$$

Since $A^H A$ is unitary can we write $U^H A^H AU = diag(\mu_1, \mu_2, \dots, \mu_n)$ which results in

Kinda sketchy argument, given in Quartentoni page 41/664. In fact, I do not believe it is true to assume A is hermetian/unitary.

$$\|A\|_2^2 = \sup_{y \neq 0} \frac{\sum_{i=1}^n \mu_i |y_i|^2}{\sum_{i=1}^n |y_i|^2} = \max_i (\mu_i) = \rho(A)^2 .$$

Here is μ_i positive eigenvalues of $A^H A$ $\square$

2 Problem 2

Consider the $n \times n$ matrix A whise nonzero elements are located on its unit subdiagonal , i.e. $A_{ij+1,i} = 1$ for $i = 1, \dots, n-1$

$$A = \begin{bmatrix} 0 & \dots & \dots & 0 \\ 1 & 0 & & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 1 & 0 \end{bmatrix}$$

a) What are the eigen values of A ? What would the Gershgorin theorem tell us abou the location of the eigenvalues of A.

b) Now construct the matrix $\hat{A}$ by adding a small number ϵ in the $(1, n)$-element of A (so that $\hat{A} = A + \epsilon e_1 e_n^T$. Show that

$$\rho\left(\hat{A}\right) = \epsilon^{\frac{1}{n}}$$

And fins an expression for the eigenvalues and eigenvectors of $\hat{A}$.

c) Derive an exact expression for the condition number $K_2\left(\hat{A}\right) = \|\hat{A}\|_2 = \|\hat{A}^{-1}\|_2$.

3 References