NTNU
Norwegian University of
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Week 41: Lecture 1

Continuous-time Markov chains, and birth and death processes

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Information

In problem 1e) in Project 1:

It is **not correct** to say that "$\{I_n : n = 0, 1, \ldots\}$ is a not a Markov chain because we do not know S_n and R_n".

It is **correct** to say that "$\{I_n : n = 0, 1, \ldots\}$ is a not a Markov chain because our knowledge about S_n and R_n can increase by knowing previous values of I_n".

It is usually easiest to construct a counterexample against the Markov property.

Definition

Let $\{X(t) : t \geq 0\}$ be a continuous-time Markov chain with states $\{0, 1, \ldots\}$ and stationary transition probabilities.

Note: $P_{ij}(t) = \mathrm{Pr}\{X(t) = j | X(0) = i\}, t \geq 0$, for states i and j.

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Then $\{X(t) : t \geq 0\}$ is a **birth and death process** with **birth rates** $\lambda_0, \lambda_1, \ldots$ and **death rates** $\mu_0, \mu_1, \ldots$ if

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2. $P_{i,i-1}(h) = \mu_i h + o(h) \quad (\text{as } h \to 0^+)$ for $i \geq 1$.
3. $P_{i,i}(h) = 1 - (\lambda_i + \mu_i)h + o(h) \quad (\text{as } h \to 0^+)$ for $i \geq 0$.

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4. $$P_{ij}(0) = \delta_{ij} = \begin{cases} 0, & i \neq j \\ 1, & i = j, \end{cases} \quad \text{for } i, j \geq 0.$$

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4. $$P_{ij}(0) = \delta_{ij} = \begin{cases} 0, & i \neq j \\ 1, & i = j, \end{cases} \quad \text{for } i, j \geq 0.$$
5. $\mu_0 = 0, \lambda_0 > 0$ and $\mu_i, \lambda_i > 0$ for $i \geq 1$.

Note: $P_{ij}(t) = \Pr\{X(t) = j | X(0) = i\}, t \geq 0$, for states i and j.

Definition

A **pure birth process** is a birth and death process where $\mu_i = 0, i \geq 0$.

Definition

A **pure death process** is a birth and death process where $\lambda_i = 0, i \geq 0$.

Theorem (Chapman-Kolmogorov)

The transition probability functions of a continuous-time Markov chain with state space $\{0, 1, \ldots\}$ *and stationary transition probabilities, satisfy*

$$P_{ij}(t+s) = \sum_{k=0}^{\infty} P_{ik}(t)P_{kj}(s), \quad t, s \geq 0,$$

for all states i *and* j.

Example

Find the distribution of the sojourn time in state *i* for a pure birth process with birth rates $\lambda_0, \lambda_1, \ldots > 0$.

Examples of pure birth processes

1) Poisson process: $\lambda_i = \lambda > 0$ for $i \geq 0$.
2) Yule process: $\lambda_i = \beta i$ for $\beta > 0$ and $i \geq 1$ starting in state 1.
3) Explosive birth process: $\lambda_i = i^2$ for $i \geq 1$ starting in state 1.

Example

We have a pure birth process starting in state 0 where $\lambda_0 \neq \lambda_1$. Calculate the probability that the process will be in state 1 at time $t \geq 0$.