Norwegian University of Science and Technology
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TMA4183
Optimisation II
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**Exercise set 7**

**1** Assume that $U$ and $V$ are Hilbert spaces and that $F\colon U \to V$ is bounded linear and injective. We want to solve the equation

$$Fu = y^\delta$$

given noisy data $y^\delta = \hat{y} + n^\delta$. To that end, we consider constrained regularisation

$$\min_u \frac{1}{2}\|u\|^2 \qquad \text{subject to } \|Fu - y^\delta\| \le \delta. \tag{1}$$

In this exercise, we show that this is a well-posed regularisation method.

Assume in the following that the operator $F$ has dense range. That is, for every $y \in V$ and $\varepsilon > 0$ there exists $u \in U$ with $\|Fu - y\| \le \varepsilon$.

**a)** Show that the problem (1) admits a unique solution for every $y^\delta \in V$ and every $\delta > 0$.

**b)** Assume that $\hat{u} \in U$ and $\hat{y} = F\hat{u}$. Assume moreover that $\|\hat{y} - y_k\| \le \delta_k \to 0$ and that $u_k$ solves

$$\min_u \frac{1}{2}\|u\|^2 \qquad \text{subject to } \|Fu - y_k\| \le \delta_k.$$

Show that $u_k \to \hat{u}$.

*Hint: Show that $\|u_k\| \le \|\hat{u}\|$ for all $k$ and conclude that $u_k$ is weakly convergent. Then show that the weak limit is actually $\hat{u}$ and we have strong convergence.*

**c)** Show that the solution of (1) depends for fixed $\delta > 0$ continuously on $y^\delta$. That is: Assume that $y_k \to y^\delta$ and denote by $u_k$ the solution of

$$\min_u \frac{1}{2}\|u\|^2 \qquad \text{subject to } \|Fu - y_k\| \le \delta.$$

Show that the sequence $u_k$ converges to the solution $u^\delta$ of (1).

*Hint:*

- *Show first that there exists $w \in U$ such that $\|Fw - y^\delta\| \le \delta/2$ and conclude that $\|u_k\| \le \|w\|$ for all sufficiently large $k$. Thus $u_k$ converges weakly to some $\tilde{u} \in U$.*
- *Next show that there exists a sequence $\lambda_k \to 0$ from above such that $\|Fw_k - y_k\| \le \delta$ where $w_k = \lambda_k w + (1 - \lambda_k)\tilde{u}$. Conclude that $\|u_k\| \le \|w_k\|$ and thus $\|u_k\| \to \|\tilde{u}\|$, which in turn implies $u_k \to \tilde{u}$.*
- *Finally show that there exists a sequence $\mu_k \to 0$ from above such that $\|Fv_k - y_k\| \le \delta$ where $v_k = \mu_k w + (1 - \mu_k)u^\delta$. Conclude that $\|u_k\| \le \|v_k\|$ and thus $\|\tilde{u}\| \le \|u^\delta\|$. This shows that, in fact, $\tilde{u} = u^\delta$.*