23. Lecture XXIII: One-Dimensional Local Bifurcations II

23.1. **Hopf Bifurcation.** We continue with our discussion on transversality/non-degeneracy as the origin of one dimensional/codimension one bifurcations.

4. Hopf bifurcation

We can think of relaxing non-degeneracy conditions as introducing symmetries. A symmetry of a system, roughly speaking,is a transformation of underlying variables that leaves the dynamics exactly unchanged. For example, $\mathbb{R}^2$, augmented with the usual Euclidean distance by which it is measured, is symmetric under rotation, reflections, and translations. If $\sigma : \mathbb{R}^2 \to \mathbb{R}^2$ is any of the three types of operations aforementioned, and $\rho(\mathbf{x}, \mathbf{y})$ the distance between two points $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$, then

$$\rho(\sigma(\mathbf{x}), \sigma(\mathbf{y})) = \rho(\mathbf{x}, \mathbf{y}).$$

If a function $V : (x, y, z) \in \mathbb{R}^3 \mapsto V(x, y, z)$ is symmetric with respect to translation in y, this can be expressed as

$$\frac{\partial V}{\partial y} = 0;$$

that is, V is left unchanged under translation in y. If this condition were restricted to a particular point — a local optimum — we can think of this condition as an infinitesimal symmetry in a vanishingly small neighbourhood of that point. Symmetry can have the effect of collapsing higher dimensional problems into lower-dimensional ones. For example, a problem with rotational symmetry in d dimensions is essentially a one-dimensional problem because the angular coordinates are all symmetrical and only dynamics along the radial coordinates matter.

This brings us to the symmetry condition that we shall use to obtain another type of bifurcation: Instead of requiring that $\mathrm{D}f(\mathbf{x}_0, \mu_0)$ from (33) in Lecture XXII have only one zero eigenvalue, let us suppose that $\mathrm{D}f$ has one and only pair of purely imaginary eigenvalues. Recall that D refers to derivative in the spatial coordinates only, and not in μ, so $\mathrm{D}f$ is a $(d \times d)$-matrix. We must in turn require that these eigenvalues be conjugates if f is real. Let $\lambda_+ = \lambda_+(\mu_0)$ and $\lambda_- = \bar{\lambda}_+$. We shall restrict this symmetry with the non-degeneracy condition:

$$\left.\frac{\mathrm{d}}{\mathrm{d}\mu}\right|_{\mu=0} \Re(\lambda) \neq 0, \tag{38}$$

where we take the critical point to be at $\mu_0 = 0$. This condition replaces (34).

Putting the "centre" part of the $(d \times d)$-system $\dot{\mathbf{x}} = f(\mathbf{x}, \mu)$ in a normal form, we know that two conjugate imaginary eigenvalues mean that the centre variables x and y satisfy

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \mu & -\omega \\ \omega & \mu \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + O(x^2, y^2, xy, \mu^2)$$

(If we append to this system the equation $\dot{\mu} = 0$ for the parameter μ as well, we have a three-dimensional centre manifold for the $(d+1)$-dimensional system.) The eigenvalues of the linearization are $\lambda_\pm(\mu) = \pm i\omega + \mu$, which reduce to $\lambda_\pm(0) = \pm i\omega$ at $\mu = 0$.

It turns out that for systems of dimension $d > 2$ with two conjugate purely imaginary eigenvalues at $\mu = 0$, with the non-degeneracy condition (38), more can be shown about higher-order terms:

(i) at the critical point $(\mathbf{0}, 0)$, the centre manifold to the $(d \times d)$-system is two dimensional, and the system on the centre manifold can be put into the normal form:

$$\begin{aligned}\dot{x} &= -y + ax(x^2 + y^2) - by(x^2 + y^2) + O(|\mathbf{x}|^4) \\ \dot{y} &= x + bx(x^2 + y^2) + ay(x^2 + y^2) + O(|\mathbf{x}|^4),\end{aligned}$$

and

(ii) any sufficiently smooth system $\dot{\mathbf{x}} = g(\mathbf{x}, \boldsymbol{\nu})$ with $\nu \in \mathbb{R}^m$ being a vector-valued parameter (or, are m independent scalar parameters) for which $\dot{x} = g(\mathbf{x}, \mathbf{0})$ is topologically equivalent to $\dot{x} = f(\mathbf{x}, 0)$ takes the following form on the local centre manifold:

$$\begin{aligned} \dot{x} &= \mu x - y + ax(x^2 + y^2) - by(x^2 + y^2) \\ \dot{y} &= x + \mu y + bx(x^2 + y^2) + ax(x^2 + y^2). \end{aligned} \tag{39}$$

Take a moment to appreciate what this means — this means that *any* system that looks like $\dot{x} = f(\mathbf{x}, 0)$ at the origin/critical point has the same centre manifold behaviour, as long as this system governed by f satisfies the non-degeneracy condition (38) for its only two purely imaginary eigenvalues.

A system $\dot{x} = g(\mathbf{x}, \boldsymbol{\nu})$ for which $\dot{x} = g(\mathbf{x}, \mathbf{0})$ is topologically equivalent to $\dot{x} = f(\mathbf{x}, 0)$ is known as an UNFOLDING of the system $\dot{x} = f(\mathbf{x}, 0)$. We think of the role of the parameters $\boldsymbol{\nu}$ as unpacking all possible degenerate behaviour that can be hidden in $\dot{x} = f(\mathbf{x}, 0)$, or otherwise that the system $\dot{x} = f(\mathbf{x}, 0)$ sits in the intersection of all its possible unfoldings. Now there are some systems with the fascinating property that it takes only finitely many parameters to characterise all possible unfoldings.

For systems with only two eigenvalues with zero real parts, and a non-degeneracy condition given by (38), for example, we see that all unfoldings look locally alike, and one parameter is enough to describe them. We call this fullest possible unfolding of $\dot{x} = f(\mathbf{x}, 0)$ its UNIVERSAL UNFOLDING.

Let us take a closer look at the centre manifold dynamics of the universal unfolding. In polar coordinates, (39) becomes:

$$\dot{r} = \mu r - ar^3 \tag{40}$$

$$\dot{\vartheta} = -1 - br^3. \tag{41}$$

We can normalize the radial equation to

$$\dot{r} = \mu r - r^3.$$

This of course exhibits a pitchfork-like bifurcation if $a \neq 0$. But the interpretation of this variable as a radial variable yields visually very different phase portrait changes as μ passes through 0, and this bifurcation is known as the HOPF BIFURCATION. In particular, we require $r > 0$. There is no trajectory with $r < 0$. This bifurcation is the sudden appearance of a periodic orbit from a focus.

We in fact have the following situation: If $a > 0$, then the bifurcation exists for $\mu > 0$, the periodic solutions are stable, and the bifurcation is said to be SUPERCRITICAL. If $a < 0$, then solutions exist for $\mu < 0$, and the periodic solutions are unstable, and the bifurcation is said to be subcritical.

The situation for $d = 2$ can be made clearer. The existence of a Hopf bifurcation requires in place of the eigenvalue condition a "genericity condition",

$$a \neq 0.$$

This arises from a Lyapunov condition on the Poincaré map, and a can be calculated from a formula. For this we refer to Theorem 1 of pg 352 in *Perko*.

The Van der Pol system exhibits a Hopf bifurcation. We leave casting the Van er Pol equations into the form (39) and calculating the Lyapunov number as an exercise.

23.2. **Examples of codimension one bifurcations.**

Example 23.1 (Augmented Lotka-Volterra model)**.** Recall the augmented Lotka-Volterra model of Lecture 7 (Example 7.4) for predator-prey dynamics:

$$\begin{aligned} \dot{x} &= x\left(\frac{x - \varepsilon}{x + \varepsilon}\right)\left(1 - \frac{x}{K}\right) - xy \\ \dot{y} &= \rho(xy - y). \end{aligned}$$

Recall that $x(t)$ modelled the prey population and $y(t)$ the predator population, that K is the carrying capacity, ρ the death rate for predators, and ε the self-sustaining population parameter, which is < 1. For simplicity, let us take $\varepsilon = 0$.

Recall also that the equilibria are at $(0, 0)$ — the extinction equilibrium, at $(K, 0)$ — the prey only equilibirum, and at $(1, 1 - 1/K)$, the coexistence equilibrium.

Taking K to be the bifurcation parameter, we find that the the coexistence equilibrium and the extinction equilibrium coincides as K varies through 1, and bifurcates from it on either side of $K = 1$. On inspection of the first equation

$$\dot{x} = x(1 - x/K) - xy,$$

we can readily see that this is a transcritical bifurcation.

Example 23.2 (Activator-Inhibitor system)**.** Recall that in Example 7.5, we considered activator-inhibitor systems modelling the concentration of two reacting species of chemicals. We also considered this system at the beginning of Lecture 22.

The activator-inhibitor system is given by the equations:

$$\begin{aligned}\dot{x} &= \sigma \frac{x^2}{1+y} - x \\ \dot{y} &= \rho(x^2 - y).\end{aligned}$$

The parameters σ and ρ are positive.

The fixed points are:

$$(x, y) \in \{(0, 0), (r_+, r_+^2), (r_-, r_-^2)\},$$

where

$$y = \sigma x - 1, \quad y = x^2 \qquad \Longrightarrow \qquad r_\pm = \frac{\sigma \pm \sqrt{\sigma^2 - 4}}{2}.$$

Therefore the fixed points not at zero only exist in the phase space when $\sigma > 2$.

Discounting the trivial equilibrium in which there are no chemicals of either species at all, as σ decreases to $\sigma = 2$, the system undergoes a saddle-node bifurcation.

When $\sigma > 2$, the equilibrium (r_-, r_-^2) is always a saddle.

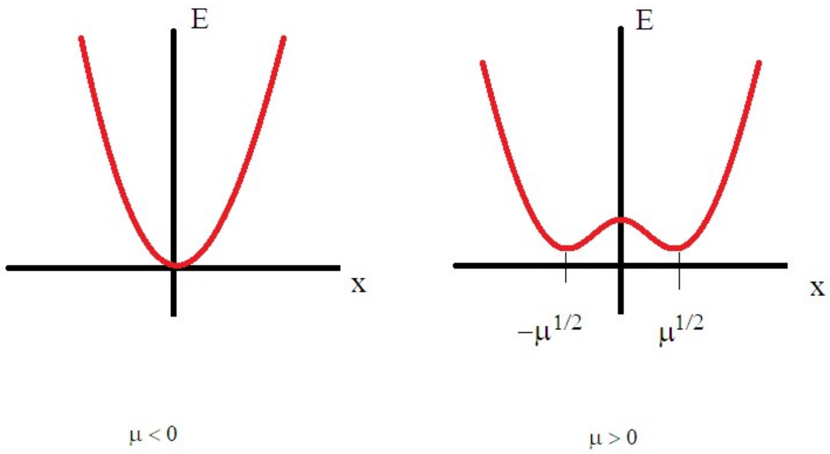
When $\sigma > 2$, the equilibrium (r_+, r_+^2) is always a node. It is a source or a sink according as $\rho < 1$ or $\rho > 1$. If $\rho < 1$, then there is in fact little change in behaviour through the bifurcation point — essentially all solutions decay to nought. But if $\rho > 1$, we see that that (r_+, r_+^2) is an attracting fixed point — a chemical equilibrium between two reacting species that suddenly fails to become a chemical equilibrium at $\sigma = 2$.

What we have seen is in fact a higher dimensional bifurcation with respect to the parameter ρ at $\rho = 1$, but we shall not pursue this discussion now.

Example 23.3. This is not so much an example as an interpretation of a pitchfork bifurcation. Recall that a pitchfork bifurcation is given in normal form by

$$\dot{x} = \mu x - x^3.$$

In fact pitchfork bifurcations, like their close cousins the Hopf bifurcation, also exhibit criticality behaviour. In the equation above, as we have seen, as μ increases through $\mu = 0$, a stable fixed point bifurcates into three — an unstable fixed point at $x = 0$, and two stable fixed points at $x = \pm\sqrt{\mu}$. This can be found in physical systems where a energy potential well changes into two potential wells:



This sort of bifurcation is known as a SUPERCRITICAL, FORWARD, or SAFE pitchfork bifurcation.

Alternatively, if the sign in front of x^3 were plus instead of minus, we see that as μ *decreases* through $\mu = 0$, un unstable fixed point bifurcates into three — a stable fixed point at $x = 0$, and two unstable fixed points at $x = \pm\sqrt{-\mu}$. This corresponds to a reflection of the potential wells above about a horizontal.

This sort of bifurcation is known as a SUBCRITICAL, BACKWARDS, INVERTED, or UNSAFE pitchfork bifurcation.