NTNU
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Week 43: Lecture 2

Introduction to multivariate Gaussian distributions

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October 21, 2020

Information

- Reference group meeting will be on October 22.
- You can use the Google form `https://s.ntnu.no/tma4265_Meeting2` to let the reference group know your opinion on the course.
- Scores are available for Project 1.
- Project 2 will be made available on Thursday evening.

Section 1 (Note)

There are two perspectives for Gaussian processes

1) Mathematical treatment of continuous-time Markov chains. See Sections 8.1–8.2 in the book.
2) Statistical approach to Gaussian processes. See the note on Gaussian processes.

Sections 2.1–2.2 (Note)

Definition

The stochastic variable X has a **Gaussian distribution** with **mean** $\mu \in \mathbb{R}$ and **variance** $\sigma^2 > 0$ if the probability density function is given by

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{1}{2\sigma^2}(x-\mu)^2\right), \quad x \in \mathbb{R}.$$

We will write $X \sim \mathcal{N}(\mu, \sigma^2)$.

Notation

We will use $\phi(\cdot)$ and $\Phi(\cdot)$ to denote the probability density function and cumulative distribution function, respectively, of $Z \sim \mathcal{N}(0, 1)$. This means

$$\phi(z) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{z^2}{2}\right), \quad z \in \mathbb{R},$$
$$\Phi(z) = \int_{\infty}^{z} \phi(x)\mathrm{d}x, \quad z \in \mathbb{R}.$$

Example

Let $X_t =$ "Average temperature on day t". Assume that $X_t \sim \mathcal{N}(15, 2.5^2)$ for $t = 0, 1, \ldots$.

a) Calculate the probability that the average temperature on day 10 is greater than 20.

b) Assume that $Y_1 \sim \mathcal{N}(0, \sigma^2)$ and $\epsilon_2 \sim \mathcal{N}(0, \sigma^2)$ are independent. Determine the distribution of $Y_2 = \rho Y_1 + \sqrt{1 - \rho^2}\epsilon_2$ for $-1 < \rho < 1$.

Definition

The stochastic vector $\boldsymbol{X} = (X_1, X_2, \ldots, X_n)$ has an **n-dimensional multivariate Gaussian distribution** with **mean vector** $\boldsymbol{\mu} = (\mu_1, \mu_2, \ldots, \mu_n)$ and $n \times n$ **covariance matrix** Σ if its probability density function is given by

$$f(\boldsymbol{x}) = \left(\frac{1}{\sqrt{2\pi}}\right)^n \frac{1}{|\Sigma|^{1/2}} \exp\left(-\frac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu})^{\mathrm{T}}\Sigma^{-1}(\boldsymbol{x}-\boldsymbol{\mu})\right), \quad \boldsymbol{x} \in \mathbb{R}^n.$$

We will write $\boldsymbol{X} \sim \mathcal{N}_n(\boldsymbol{\mu}, \Sigma)$.

Theorem

Let $X_1, X_2, \ldots, X_n$ be stochastic variables, $a_1, a_2, \ldots, a_n \in \mathbb{R}$, and $b_1, b_2, \ldots, b_n \in \mathbb{R}$, then

$$\mathrm{Cov}\left[\sum_{i=1}^{n} a_i X_i, \sum_{j=1}^{n} b_j X_j\right] = \sum_{i=1}^{n}\sum_{j=1}^{n} a_i b_j \mathrm{Cov}[X_i, X_j].$$

This means that $\mathrm{Cov}[\cdot,\cdot]$ is bilinear.

Example

Assume $\sigma^2 > 0$ and $-1 < \rho < 1$, and let $X_t = \rho X_{t-1} + \sqrt{1-\rho^2}\epsilon_t$, $t = 2, 3, \ldots$, where $X_1, \epsilon_2, \epsilon_3, \ldots \overset{\text{iid}}{\sim} \mathcal{N}(0, \sigma^2)$. Determine the distribution of (X_1, X_2, X_3).

Section 2.3 (Note)

Theorem

Assume $\boldsymbol{X} \sim \mathcal{N}_n(\boldsymbol{\mu}, \Sigma)$, $\mathbf{A}$ *is an* $m \times n$ *matrix, and* $\boldsymbol{b}$ *is a vector of length m. If* $\boldsymbol{Y} = \mathbf{A}\boldsymbol{X} + \boldsymbol{b}$, *then*

$$\boldsymbol{Y} \sim \mathcal{N}_m(\mathbf{A}\boldsymbol{\mu} + \boldsymbol{b}, \mathbf{A}\Sigma\mathbf{A}^{\mathrm{T}}).$$