2.3. **A word on non-homogeneity.** Only a slight modification is needed to treat systems of the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{g},$$

where $\mathbf{g} \in \mathbb{R}^d$ if zero is not an eigenvalue of $\mathbf{A}$ (that is, if $\mathbf{A}$ is invertible). If $\mathbf{A}$ is invertible, we can find a unique vector $\mathbf{h} \in \mathbb{R}^d$ such that $\mathbf{A}\mathbf{h} = \mathbf{g}$. Then using the change-of-variable $\mathbf{y} = \mathbf{x} + \mathbf{h}$ — a constant translation, we can write the system as

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{y}(t) = \mathbf{A}\mathbf{y}(t).$$

We can solve for $\mathbf{y}$ in the manner already prescribed (by exponentiation of $\mathbf{A}$), and then recover $\mathbf{x}$ via $\mathbf{x} = \mathbf{y} - \mathbf{h}$. The only difference then is that, in the case $d = 2$ for example, the fixed points are now either at $\mathbf{h}$.

If $\mathbf{A}$ is singular, assume that it has been reduced to Jordan normal form. Then either

$$\mathbf{A} = \begin{pmatrix} \lambda & 0 \\ 0 & 0 \end{pmatrix} \qquad \text{or} \qquad \mathbf{A} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Both of these cases can be solved directly.

2.4. **Phase Portraits.** As described in Lecture 1, for $d = 2$, there is a particular vivid way to represent solutions via phase portraits. By definition, for autonomous systems, the flow $\phi(t, s, \mathbf{x}(0))$ only depends on t and s via the duration $t - s$. Therefore the (forward) orbits

$$\mathrm{O}_+(\mathbf{x}(0)) = \{\phi_t(\mathbf{x}(0)) : t \in \mathbb{R}\}$$

form a set partially ordered by inclusion — for any two initial points $\mathbf{x}(0)$ and $\mathbf{y}(0)$, either

(i) $\mathrm{O}_+(\mathbf{x}(0)) \subseteq \mathrm{O}_+(\mathbf{y}(0))$, or
(ii) $\mathrm{O}_+(\mathbf{y}(0)) \subseteq \mathrm{O}_+(\mathbf{x}(0))$, or
(iii) $\mathrm{O}_+(\mathbf{x}(0)) \cap \mathrm{O}_+(\mathbf{y}(0)) = \varnothing$.

A cleaner but slightly coarser statement is that the orbits

$$\mathrm{O}(\mathbf{x}(0)) = \{\phi_t(\mathbf{x}(0)) : t \in \mathbb{R}\}$$

either coincide or are disjoint. This observation ensures that we can draw reasonably clean phase portraits for 2-dimensional linear autonomous systems. We shall now devote the remainder of this lecture to sketching phase portraits of each of the cases mentioned in the previous subsection.