Repetition

Definition

Let $\{X_n : n = 0, 1, \ldots\}$ be a Markov chain with state space $\{0, 1, \ldots\}$ and transition probability matrix $\mathbf{P}$.

1) State j is **accessible** from state i if $\exists n \geq 0$ so that $P_{ij}^{(n)} > 0$.

2) If states i and j are accessible from each other, they are said to **communicate** and we write $i \sim j$. If states i and j do not communicate we write $i \nsim j$.

Theorem

Communication is an **equivalence relation**, i.e.,

1) **reflexive**: $i \sim i$

2) **symmetric**: $i \sim j \Rightarrow j \sim i$

3) **transitive**: $i \sim j$ and $j \sim k$ implies $i \sim k$

Note: This means that $\sim$ induces **equivalence classes** on the state space of the Markov chain.

Definition

A Markov chain is **irreducible** if $\sim$ (communication) induces exactly one equivalence class. If not, it is called **reducible**.

Definition

The **period** of state i, written as $d(i)$, is

$$d(i) = \gcd\{n \geq 1 : P_{ii}^{(n)} > 0\}.$$

If $P_{ii}^{(n)} = 0$ for all $n \geq 1$, we define $d(i) = 0$.

If $d(i) = 1$, we say that state i is **aperiodic**.

Theorem

If $i \sim j$, then $d(i) = d(j)$.

Note: This means that the period is a property of the entire equivalence class.

Notation

The state space may be infinite: $\{0, 1, \ldots\}$.

1) The probability that the first return happens after exactly n steps:

$$f_{ii}^{(n)} = \Pr\{X_n = i, X_\nu \neq i, \nu = 1, 2, \ldots, n-1 | X_0 = i\}, \quad n > 0.$$

   We define $f_{ii}^{(0)} = 0$.

2) The probability of returning at some time:

$$f_{ii} = \sum_{k=0}^{\infty} f_{ii}^{(k)} = \lim_{n \to \infty} \sum_{k=0}^{n} f_{ii}^{(k)}.$$

Note: $f_{ii} < 1$ means that there is a positive probability of never returning to state i.

Definition

State i is **recurrent** if the probability of returning to state i in a finite number of time steps is one, i.e., $f_{ii} = 1$. A state that is not recurrent, i.e., $f_{ii} < 1$, is called **transient**.