TMA 4190 Introduction to Topology
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Lecture 18[1]

18. Brouwer Fixed Point Thereom and One-Manifolds

The following theorem gives us a complete list of smooth one-dimensional manifolds. Note that in genera, since every manifold is the disjoint union of its connected components, it suffices to classify connected manifold.

Classification of One-Manifolds

(a) Every compact, connected, one-dimensional smooth manifold without boundary is diffeomorphic to S^1.
(b) Every compact, connected, one-dimensional smooth manifold with boundary is diffeomorphic to $[0,1]$.
(c) Every noncompact, connected, one-dimensional smooth manifold with boundary is diffeomorphic to either $[0,1)$, $(0,1]$ or $(0,1)$.

The details of the proof are surprisingly complicated. We content ourselves with a rough idea.

Some heuristics on why the theorem may be true:

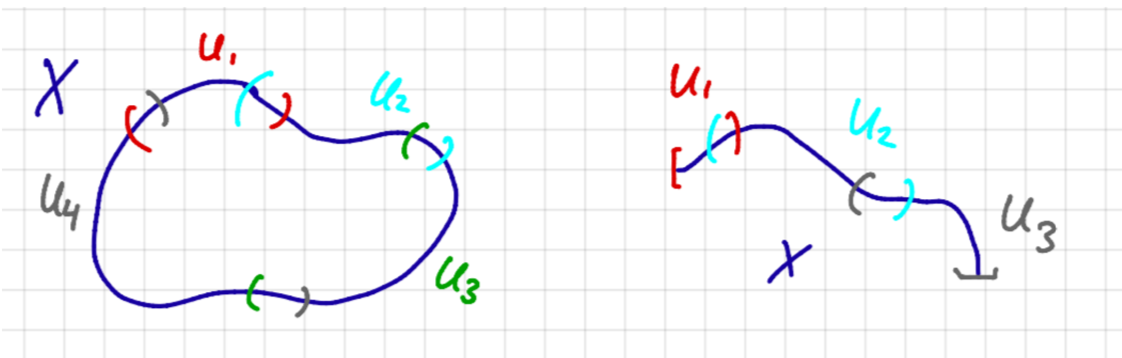
(a) Let X be a nonempty, compact, connected 1-manifold. Each point has a neighborhood diffeomorphic to $(-1,1)$. By compactness, finitely many such neighborhoods $U_1, \ldots, U_n$ cover X. If n was equal 1, then $X \cong (-1,1)$. But an open interval is not compact. Thus, there must be at least two neighborhoods. Since X is connected, these two charts must intersect. The union of these two intervals has to be either an open interval (if they intersect on one side of each) or a circle (if they intersect on both sides). But if their union is an open interval, there has to be another chart, by the compactness of X. Since there are only finitely many U_i's, we must eventually arrive at the situation where the neighborhoods intersect on both sides and form a circle. Then one has to use this to construct a diffeomorphism to S^1.

(b) Let X be a compact, connected, one-dimensional smooth manifold with boundary. Since X has at least one boundary point, there must be neighborhood in X containing that boundary point. This neighborhood must be diffeomorphic

[1]Following the books of Guillemin and Pollack: Differential Topology, and Milnor: Topology from the differentiable viewpoint.

to $[a,b)$ for some a, b. Since this interval is not compact, there must be another neighborhood in X. This neighborhood either intersects another boundary point which would yield us $X \cong [a,c]$ for some c, or it does not contain a boundary point. In the latter case, the union of the neighborhoods is diffeomorphic to a half-open interval $[a,d)$ which is not compact. Hence there has to be another neighborhood. Since X is compact, this process will end after finitely many steps when we eventually get that X is the union of neighborhoods which is diffeomorphic to a closed interval.

(c) When X is not compact, we repeat the above processes. The difference is that the process may not terminate and we end up with open or half-open intervals.



Much more interesting than the actual theorem are its consequences which are surprisingly rich.

Boundary of One-Manifolds

The boundary of any **compact** one-dimensional manifold with boundary consists of an **even** number of points.

Proof: Every compact one-manifold with boundary X is the disjoint union of **finitely many connected components**. Each component is diffeomorphic to a copy of $[0,1]$. Hence the boundary of each component consists of **two points**. The boundary of X consists of these finitely many **pairs** of points. **QED**

Retractions

Let X be a smooth manifold and $Z \subset X$ be a submanifold. Then a **retraction** is a smooth map $f\colon X \to Z$ such that $f_{|Z}$ is the identity.

There is an important restriction for the existence of such retractions for manifolds with boundary:

No retractions onto boundaries

If X is any **compact** manifold with boundary, then there is no retraction of X onto its boundary.

Proof: Suppose there is such a smooth map $g\colon X \to \partial X$ such that $\partial g\colon \partial X \to \partial X$ is the identity. By Sard's Theorem, we can choose a regular value $z \in \partial X$ of g. Since ∂g is the identity, all values in ∂X a regular for ∂g. Hence z is regular for both g and ∂g. By the Preimage Thoerem for manifolds with boundary, we know that $g^{-1}(z)$ is a submanifold of X with boundary

$$\partial(g^{-1}(z)) = g^{-1}(z) \cap \partial X.$$

Moreover, the codimension of $g^{-1}(z)$ in X equals the codimension of $\{z\}$ in ∂X, namely $\dim X - 1$ as $\{z\}$ has dimension 0. Hence $g^{-1}(z)$ is **one-dimensional**. Since it is a closed subset in the compact manifold X, it is also **compact**.

By definiton of ∂g as the restriction of g to ∂X, we have

$$(\partial g)^{-1}(z) = (g_{|\partial X})^{-1}(z) = g^{-1}(z) \cap \partial X = \partial(g^{-1}(z)).$$

But, since $\partial g = \mathrm{Id}_{\partial X}$,

$$\{z\} = (\partial g)^{-1}(z) = \partial(g^{-1}(z)).$$

This **contradicts** the previous result that the boundary $\partial(g^{-1}(z))$ of the compact one-dimensional manifold $g^{-1}(z)$ consists of an **even** number of points. **QED**

This theorem has a famous consequence:

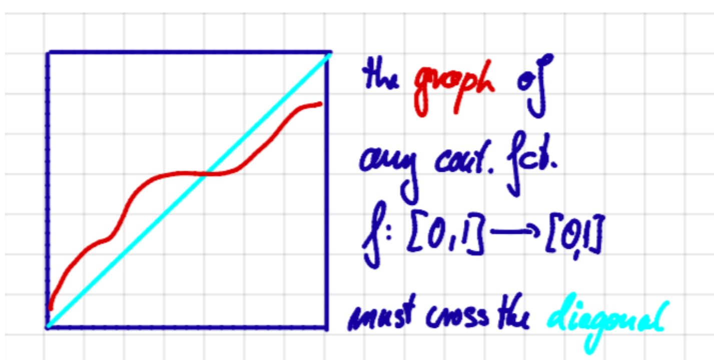
Brouwer Fixed-Point Theorem for smooth maps

Let $f\colon B^n \to B^n$ be a **smooth** map of the closed unit ball $B^n = \{x \in \mathbb{R}^n : |x| \leq 1\} \subset \mathbb{R}^n$ into itself. Then f must have a **fixed point**, i.e. there is an $x \in B^n$ with $f(x) = x$.

Before we prove the theorem, let us have a look at dimension one, where the result is very familiar:

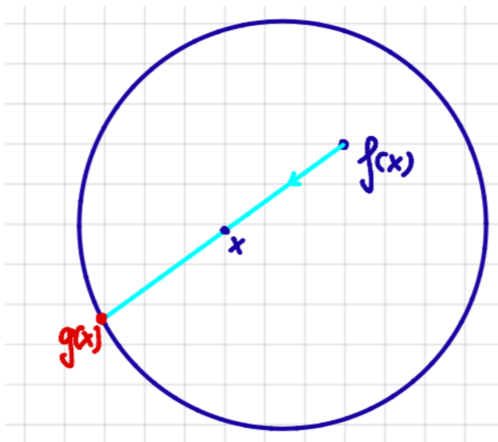
Brouwer FPT is familiar in dimension one

Note that we have seen this theorem for $n = 1$ in Calculus 1. Let $f\colon [0,1] \to [0,1]$ be a continuous map. Then it must have a fixed point. For, if not, then $g(x) = f(x) - x$ is a continuous map defined on [0,1]. We have $g(0) \geq 0$ and $g(1) \leq 0$, since $f(0) \geq 0$ and $f(1) \leq 1$. If $g(0) = 0$ or $g(1) = 1$, we are done. But if $g(0) > 0$ and $g(1) < 1$, then the Intermediate Value Theorem implies that there is an $x_0 \in (0,1)$ with $g(x_0) = 0$, i.e. $f(x_0) = x_0$.



Proof of Brouwer FPT: Suppose that there exists an f without fixed points. We will show that such an f would allow us to construct a retraction $g\colon B^n \to \partial B^n$. But, since B^n is **compact**, we have just proved that such a retraction cannot exist.

So suppose $f(x) \neq x$ for all $x \in B^n$. Then, for every $x \in B^n$, the two points x and $f(x)$ determine a line. Let $g(x)$ be the point where the line segment starting at $f(x)$ and passing through x hits the boundary ∂B^n. This defines a map $g\colon B^n \to \partial B^n$.



If $x \in \partial B^n$, then $g(x) = x$ by construction of g. Hence $g\colon B^n \to \partial B^n$ is the identity on ∂B^n. Thus, in order to show that g is a retraction, it remains to show that g is smooth.

To show this, we describe $g(x)$ explicitly. As a point on the line from $f(x)$ to x, $g(x)$ can be written in the form

$$g(x) = x + tv, \text{ where } v := \frac{x - f(x)}{|x - f(x)|}$$

for some real number t. Note that, since we assume $x \neq f(x)$, the vector v is always defined. In fact, it is the unit vector pointing from $f(x)$ to x. Moreover, since f is smooth, v depends smoothly on x.

We need to calculate t and show that **t depends smoothly on x**. Since $g(x)$ is a point on boundary of B^n, we know $|g(x)| = 1$, and t is determined by the equation

$$1 = |g(x)|^2 = (x + tv) \cdot (x + tv) = x \cdot x + 2tx \cdot v + t^2 v \cdot v$$

or, equivalently,

$$0 = (v \cdot v)t^2 + (2x \cdot v)t + x \cdot x - 1. \tag{1}$$

By definition of v, we know $v \cdot v = |v|^2 = 1$. Since v points from $f(x)$ to x, we know that t must be positive. Now we just need to find the positive solution of the quadratc equation (1) for t and get

$$\begin{aligned} t &= \frac{-2x \cdot v + \sqrt{4(x \cdot v)^2 - 4(x \cdot x - 1)}}{2} \\ &= -x \cdot v + \sqrt{(x \cdot v)^2 - x \cdot x + 1} \end{aligned}$$

where $(x \cdot v)^2 - x \cdot x + 1$ is positive, since $x \cdot x = |x|^2 \leq 1$ and $(x \cdot v)^2 > 0$. Since the scalar products and square roots involved depend smoothly on x, we see that t depends smoothly on x. Hence g is smooth. **QED**

Note that, **for** $n = 1$, in the above proof we would construct a map $g\colon [0,1] \to \{0,1\}$ which would send 0 to 0 and 1 to 1. Such a map cannot be smooth, not even continuous by the Intermediate Value Theorem.

Brouwer Fixed-Point Theorem for continuous maps

Any **continuous** map $F\colon B^n \to B^n$ must have a **fixed point**.

Proof: The idea is to reduce this theorem to the statement on smooth maps by **approximating** F by a smooth mapping. This is possible by **Weierstrass' Approximation Theorem**, an important result from Calculus, which applies as B^n is **compact** and says:

Given $\epsilon > 0$, there is a **polynomial function** $Q\colon B^n \to \mathbb{R}^n$ with

$$|Q(x) - F(x)| < \epsilon \text{ for all } x \in B^n.$$

(Recall that a *polynomial function* is a function that arises by **finitely many** additions and multiplications of the coordinate functions. Such functions are obviously **smooth**.)

However, it is possible that Q sends points in B^n to points outside of B^n. In order to remedy this defect, we replace Q with

$$P(x) := \frac{Q(x)}{1+\epsilon}.$$

Since $|F(x)| \leq 1$, this new polynomial P satisfies:

$$(1+\epsilon)|P(x)| = |Q(x)| \leq |Q(x) - F(x)| + |F(x)| < \epsilon + 1$$

where we appy the trianlge inequality. Hence $|P(x)| \leq 1$ and **P is a map $B^n \to B^n$**. Moreover,

$$\begin{aligned}(1+\epsilon)|P(x) - F(x)| &= |Q(x) - (1+\epsilon)F(x)| = |Q(x) - F(x) + \epsilon F(x)| \\ &\leq |Q(x) - F(x)| + \epsilon|F(x)| < 2\epsilon\end{aligned}$$

where we use that $|F(x)| \leq 1$. Since $1+\epsilon > 1$, this shows

$$|P(x) - F(x)| < 2\epsilon. \tag{2}$$

Now **suppose that $F(x) \neq x$** for all $x \in B^n$. Then the continuous function

$$B^n \to B^n,\, x \mapsto |F(x) - x|$$

must have a **minimum** μ, since B^n is **compact**. Since $F(x) \neq x$ for all x, we must have $\mu > 0$.

Now,for $\epsilon = \mu/2$, we choose polynomials Q and then P as above. Since $|F(x) - x| \geq \mu$ for all $x \in B^n$, the triangle inequality yields

$$\begin{aligned} \mu \leq |F(x) - x| &= |F(x) - P(x) + P(x) - x| \\ &\leq |F(x) - P(x)| + |P(x) - x|. \end{aligned}$$

But by (2), we know

$$|F(x) - P(x)| < \mu \text{ for all } x \in B^n.$$

Thus $|P(x) - x| > 0$, and therefore $P(x) \neq x$ for all $x \in B^n$.

Hence $P \colon B^n \to B^n$ is a **smooth map** from B^n to itself **without a fixed point**. This contradicts the statement on smooth maps and completes the proof. **QED**

The theorem is not true for the open ball:

Counterexamples on open balls

Let $B_1^k(0) = \{x \in \mathbb{R}^k : |x| < 1\}$ be the **open** ball in $\mathbb{R}^k$. We define the map

$$\varphi \colon B_1^k(0) \to \mathbb{R}^k,\, x \mapsto \frac{x}{\sqrt{1 - |x|^2}}.$$

This is a **smooth** map with **smooth inverse**

$$\varphi^{-1} \colon \mathbb{R}^k \to B_1^k(0),\, y \mapsto \frac{y}{\sqrt{1 + |y|^2}}$$

Thus φ is a **diffeomorphism** $B_1^k(0) \to \mathbb{R}^k$.
The translation $T \colon \mathbb{R}^k \to \mathbb{R}^k$, $x \mapsto x + 1$ does not have a fixed point. Hence the composite map

$$\varphi^{-1} \circ T \circ \varphi \colon B_1^k(0) \to B_1^k(0)$$

does **not** have a fixed point. For if it had a fixed point x, then

$$\varphi^{-1}(T(\varphi(x))) = x \Rightarrow\ T(\varphi(x)) = \varphi(x)$$

and T had a fixed point, which is not the case.

Brouwer's Fixed-Point Theorem has many important consequences. Here is one of them:

Brouwer Invariance of Domain

Let U be an **open** subset of $\mathbb{R}^n$, and let $f\colon U \to \mathbb{R}^n$ be a **continuous injective** map. Then $f(U)$ is also **open**.

Instead of studying the proof of this theorem, let us note a consequence of this result:

Topological Invariance of Dimension

If $n > m$, and U is a nonempty **open** subset of $\mathbb{R}^n$, then there is **no continuous injective** map from U to $\mathbb{R}^m$. In particular, $\mathbb{R}^n$ and $\mathbb{R}^m$ are **not homeomorphic** whenever $n \neq m$.

Even though it sounds like an obvious fact, this is a rather deep theorem. Note that there exist weird things like a continuous surjection from $\mathbb{R}^m$ to $\mathbb{R}^n$ for $n > m$ due to variants of the Peano curve construction. Hence often we have to be careful with our topological intuition.

Proof of Topological Invariance of Dimension: If there was such a continuous injective map from U to $\mathbb{R}^m$, then we could compose it with the embedding $\mathbb{R}^m \hookrightarrow (\mathbb{R}^m \times \{0\}) \subset \mathbb{R}^n$. Hence the composite would yield a **continuous injective** map from U into $\mathbb{R}^n$. By the theorem, the image would be both open in $\mathbb{R}^n$ and lie in the subspace $\mathbb{R}^m \times \{0\}$. But no **open** subset of $\mathbb{R}^n$ can be contained in $\mathbb{R}^m \times \{0\}$, since we must be able to fit at least a tiny **open ball** of $\mathbb{R}^n$ into that subset and there is no room for such a ball in the direction of the remaining $n - m$ coordinates.

Finally, a homeomorphism from $\mathbb{R}^n$ to $\mathbb{R}^m$ would be such a continuous injective map. **QED**

Note that invariance of domain and dimension for **smooth** injective maps is just a consequence of the Inverse Function Theorem. But for maps which are just continuous and injective, it is much harder to achieve.