Error analysis:

Introduce $H^r(0,1) = \{ v \, ; \, v_{jx} \in L^2(0,1), \; j = 0, \cdots, r \}$

$\|v\|^2_{H^r} = \sum_{j=0}^{r} \int_0^1 v_{jx}^2 \, dx \;, \quad |v|^2_{H^r} = \int_0^1 v_{jx}^2 \, dx$

Find $u \in V$ s.t. $a(u,v) = F(v)$, $v \in V$

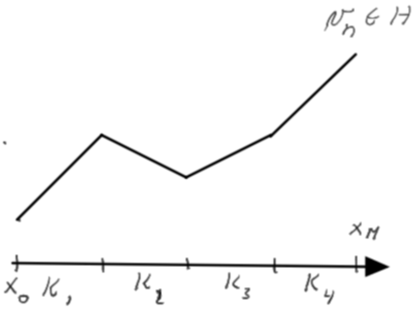
Find $u_h \in V_h$ s.t. $a(u_h, v_h) = F(v_h)$, $\forall v_h \in V_h$

What can be said about $e_h = u - u_h$?

- Assumptions: $|a(u,v)| \leq M \cdot \|u\|_V \|v\|_V$ $\qquad M, \alpha > 0.$

  $a(v,v) \geq \alpha \|v\|_V^2$

- We know $a(u - u_h, v_h) = 0 \quad \forall v_h \in V_h$



Cea's lemma: $\|u - u_h\|_V \leq \frac{M}{\alpha} \cdot \|u - v_h\|_V, \; \forall v_h \in V_h.$

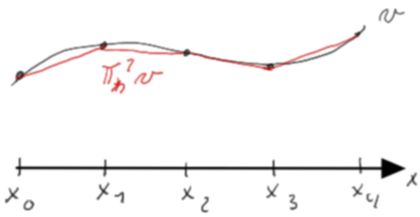
Let $V = H^1(0,1)$, $\|v\|_V^2 = \int_0^1 v^2 dx + \int_0^1 v_x^2 dx$

$V_h = X_h^1 = \{ v \in C^0[0,1], \; v|_K \in \mathbb{P}_1, \; \forall K \in \mathcal{T}_h \}$

Interpolation operator: $\Pi_h^1 : H^1(0,1) \to X_h^1$

$\Pi_h^1 v(x_i) = v(x_i), \quad i = 0, 1, \cdots, M$

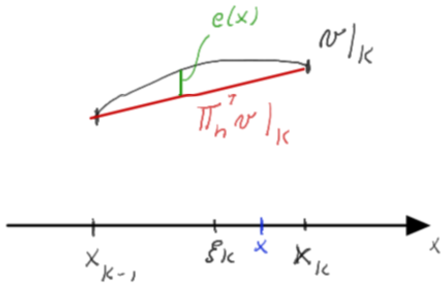
$\Pi_h^1 v = \sum_{i=0}^{M} v(x_i) \cdot \varphi_i(x)$



Interpolation error:

$e(x) = v(x) - \Pi_h^1 v(x)$

If e and e_x can be bounded then so can $\|u - \Pi_h^1 u\|$



We require a bit more smoothness

Let $v \in H^2(0,1) \subset C^1[0,1]$ (not proved)

For simplicity $v' = v_x$, $v'' = v_{xx}$

We want to bound $\|e\|_V$, and

$$\|e\|_V^2 = \int_0^1 e^2 dx + \int_0^1 e'^2 dx = \sum_K \left[\int_K e^2 dx + \int_K e'^2 dx \right]$$

so if we can bound $\int_K e^2 dx$ and $\int_K e'^2 dx$ we are done.

Since $e(x_{K-1}) = e(x_K) = 0$, $\exists \xi \in (x_{K-1}, x_K)$ s.t. $e'(\xi_K) = 0$. Then

$$e'(x) = \int_{\xi_K}^{x} e''(s) ds = \int_{\xi_K}^{x} v''(s) ds \qquad \text{since } (\Pi_h^1 v)'' = 0$$

(If $x < \xi_K$, $e'(x) = - \int_x^{\xi_K} e''(s) ds$)

$$|e'(x)| \leq \int_{K_K} |v''(s)| ds = \int_K 1 \cdot |v''(s)| ds \qquad \forall x \in (x_{K-1}, x_K)$$

$$\leq \left(\int_K 1^2 ds\right)^{1/2} \cdot \left(\int_x (v'')^2 ds\right)^{1/2}$$ Cauchy-Schwarz

$$|e'(x)|^2 \leq h_{K_K} \cdot \int_{K_K} |v''|^2 ds$$

$$h = \max_{K \in T_h} |h_K|$$

$$\int_K |e'(x)|^2 dx \leq h_K^2 \int_K |v''|^2 ds$$

$$|e'|_{H^1}^2 = \sum_K \int_K |e'(x)|^2 dx \leq \sum_K h_K^2 \cdot \int_K |v''|^2 ds$$

$$\leq h^2 \sum_K |v''|^2 ds = h^2 |v|_{H^2}^2$$

Similar, for $x \in K$

$$e(x) = \int_{g_K}^{x} e'(s) ds$$

$$|e(x)|^2 \leq h_K \cdot \int_K |e'(s)|^2 ds \leq h_K^3 \int_K |v''(s)|^2 ds$$

$$|e(x)| \leq \left|\int_g^x |e'(s)| ds\right|$$

$$\leq \int_K |e'(s)| ds$$

$$\leq h_K \cdot \left(\int_K |e'(s)|^2 ds\right)^{1/2}$$

$$\|e\|_{L^2}^2 = \sum_K \int_K e^2 dx \leq \sum_K h_K^4 \int_K |v''(s)|^2 ds$$

$$\leq h^4 |v|_{H^2}^2$$

$$|e(x)|^2 \leq h_K^2 \int_K |e'(s)|^2 ds$$

$$\leq h_K^3 \int_K |v''(s)|^2 ds$$

Interpolation error:

$$\|v - \Pi_h^1 v\|_{L^2} \leq h^2 |v|_{H^2}$$

$$|v - \Pi_h^1 v|_{H^1} \leq h \cdot |v|_{H^1}$$

and

$$\|v - \Pi_h v\|_{H^1} \leq \sqrt{h^4 + h^2} \cdot |v|_{H^2} = C \cdot h |v|_{H^2}.$$

Can be refined even more:

$$\|u - I_h^1 u\|_{L^2(K)} \leq \left(\frac{h_K}{\pi}\right)^2 \cdot |u|_{H^2}$$

$$\|u' - (I_h^1 u)'\|_{L^2(K)} \leq \left(\frac{h_K}{\pi}\right) |u|_{H^2}$$

Together with Cea's lemma we get

$$\boxed{\|u - u_h\|_{H^1} \leq \frac{M}{\alpha} \cdot C \cdot h \cdot |u|_{H^2}}$$