Norwegian University of Science and Technology
Deptartment of Mathematical Sciences

TMA4190 Introduction to Topology
Spring 2018

Exercise set 2

$\boxed{1}$ Let V be a vector subspace of $\mathbb{R}^N$. Show that $T_x(V) = V$ for $x \in V$.

$\boxed{2}$ Determine the tangent space to the torus $S^1 \times S^1 \subset \mathbb{R}^4$ at an arbitrary point p. Recall the description of the torus $T(a, b) \subset \mathbb{R}^3$ from the previous exercise set. Can you describe the tangent space at a point in $T(a, b) \subset \mathbb{R}^3$?

$\boxed{3}$ Determine the tangent space to the subspace of $\mathbb{R}^3$ defined by $x^2 + y^2 - z^2 = a$ at $(\sqrt{a}, 0, 0)$ for $a > 0$.

$\boxed{4}$ The graph of a map $f: X \to Y$ is the subset of $X \times Y$ defined by

$$\Gamma(f) = \{(x, f(x)) \in X \times Y : x \in X\}.$$

Define $F: X \to \Gamma(f)$ by $F(x) = (x, f(x))$. We assume that X and Y are smooth manifolds and f is a smooth map.

a) Show F is a diffeomorphism, and conclude that $\Gamma(f)$ is a smooth manifold.

b) We also write F for the composite map $F: X \to X \times Y$, $x \mapsto (x, f(x))$. Show that $dF_x(v) = (v, df_x(v))$. (You can use $T_{(x,y)}(X \times Y) = T_x(X) \times T_y(Y)$.)

c) Show that the tangent space to $\Gamma(f)$ at the point $(x, f(x))$ is the graph of $df_x: T_x(X) \to T_{f(x)}(Y)$.

$\boxed{5}$ A curve in a manifold X is a smooth map $t \mapsto c(t)$ of an open interval of $\mathbb{R}$ into X. The velocity vector of the curve c at time t_0 in $x_0 = c(t_0)$ -denoted simply $dc/dt(t_0)$ - is defined to be the vector $dc_{t_0}(1) \in T_{x_0}(X)$, where $dc_{t_0}: \mathbb{R}^1 \to T_{x_0}(X)$.

a) For $X = \mathbb{R}^k$ and $c(t) = (c_1(t), \ldots, c_k(t))$, show that

$$\frac{dc}{dt}(t_0) = dc_{t_0}(1) = (c_1'(t_0), \ldots, c_k'(t_0)) \in T_{x_0}\mathbb{R}^k.$$

b) For an arbitrary k-dimensional smooth manifold, use the above observation and local parametrizations to prove that every vector in $T_{x_0}(X)$ is the velocity vector of some curve in X.

Aside: This shows that there is a unique correspondence between tangent vectors at $x_0 \in X$ and velocity vectors at t_0 of curves $c\colon I \to X$ with $c(t_0)$. Note that two curves $c_1\colon I \to X$ and $c_2\colon J \to X$, with I and J open in $\mathbb{R}$, have the same velocity vector in $c_1(t_1) = x_0 = c_2(t_2)$ if $d(c_1)_{t_1}(1) = d(c_2)_{t_2}(1) \in T_{x_0}(X)$. One can show that having the same velocity vector in a point of X is an equivalence relation the set of curves through x_0 in X. Using this relation, we have shown that there is a unique correspondence between tangent vectors at X in x and equivalence classes of smooth curves through x_0 in X.