Introduction to Topology

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Version 0.1

2021

Contents

1. Introduction

These are lecture notes from the course TMA4190 Introduction to Topology given in the Spring semester 2021 at NTNU. They are intended as a supplement to the lectures and may not be entirely self-contained.

Please send me an email if you spot any errors!

What is topology?

> *Topology! The stratosphere of human thought! In the twenty-fourth century it might possibly be of use to someone. . .*
>
> — Aleksandr Solzhenitsyn

Topology is a part of mathematics concerned with the study of spaces. In topology, we consider two spaces to be *equivalent* if one can be continuously deformed into the other. Such a continuous deformation is known as a *homeomorphism*, i.e., a continuous bijection with a continuous inverse. See Figure 1.1 for an example of two homeomorphic spaces.

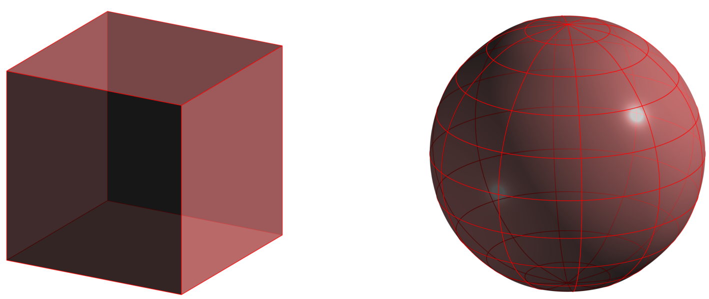

Figure 1.1: The surface of the (unit) cube and the (unit) sphere S^2 are homeomorphic.

We might ask ourselves the following question.

Question Let X and Y be two spaces. Does there exist a homeomorphism $\varphi\colon X \to Y$? In other words, are X and Y homeomorphic?

Showing that two spaces are homeomorphic involves the construction of a specific homeomorphism between them. Proving that two spaces are *not* homeomorphic is a problem of a different nature. It is a hopeless exercise to check every possible map between the two spaces for whether or not it is a homeomorphism. Instead we might check to see whether there is some "topological invariant" of spaces (where this invariant is preserved under a homeomorphism) that allows us to differentiate between the two spaces.

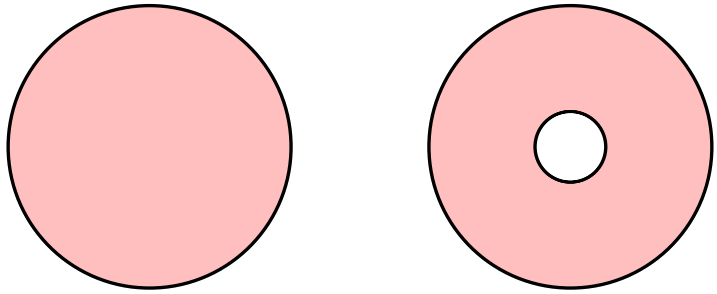

Figure 1.2: The disc D^2 and the annulus are not homeomorphic.

One instrument to help us detect topological information of a space is the *fundamental group* associated to the space. It is reasonable to expect that the disc D^2 and the annulus are not homeomorphic. The annulus has a hole through it while the disc does not, see Figure 1.2.

To detect the hole through the annulus we may use loops, i.e., continuous maps from the unit interval to the annulus with the endpoints identified. See Figure 1.3.

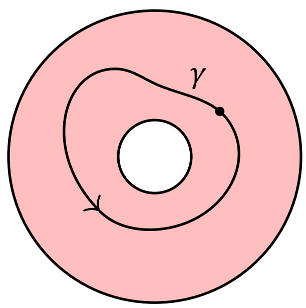

Figure 1.3: A loop.

It is then possible to construct a *group* involving such loops. This group is what is known as the fundamental group.

Some applications

To help illustrate some of the power of topology, let us consider two theorems, both of which may be proved using topology and more specifically, the fundamental group.

The first theorem is the Brouwer fixed point theorem.

Theorem 1.1 **(Brouwer fixed point theorem)** *Let $f\colon D^n \to D^n$ be a continuous map from the (unit) disk in $\mathbb{R}^n$ to itself. Then f has a fixed point, i.e., there is some point $x \in D^n$ such that $f(x) = x$.*

For $n = 1$ this is a well-known result from calculus: The graph of any continuous map $f\colon [0,1] \to [0,1]$ must cross the diagonal $y = x$ for some $x_* \in [0,1]$. Hence, $f(x_*) = x_*$. See Figure 1.4.

The second theorem is the fundamental theorem of algebra.

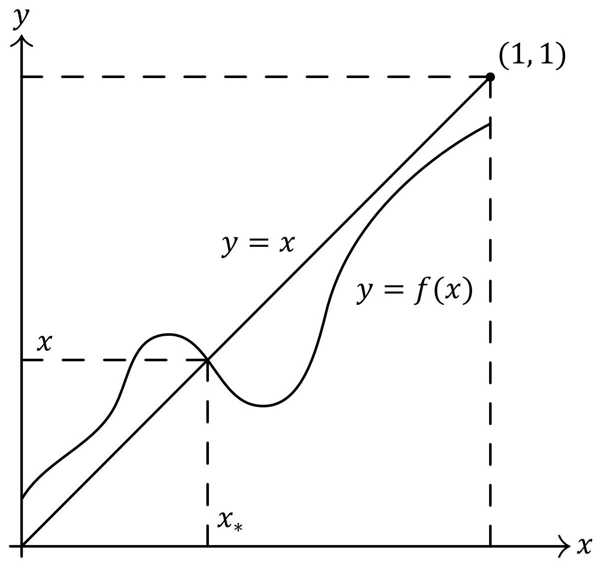

Figure 1.4: The graph of any continuous map from $[0,1]$ to $[0,1]$ must cross the diagonal.

Theorem 1.2 **(The fundamental theorem of algebra)** *A polynomial equation*

$$z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0 = 0$$

of degree $n > 0$ with complex coefficients has at least one complex root.

To prove it we will use the fact that the fundamental group of the circle is isomorphic to the group of integers. The fundamental theorem of algebra may be proved in many different ways, including using only algebraic techniques and analysis. However, the proof we will provide (based on [1]) is a fairly simple corollary of the computation of the fundamental group of the circle.

2. Continuous maps

2.1 Metric spaces

From calculus we know what to mean by a continuous map from $\mathbb{R}^n$ to $\mathbb{R}^m$: a map $f\colon \mathbb{R}^n \to \mathbb{R}^m$ is *continuous* at $p \in \mathbb{R}^n$ if for all $\epsilon > 0$ there exists a $\delta > 0$ such that if $\|p - q\|_{\mathbb{R}^n} < \delta$, then $\|f(p) - f(q)\|_{\mathbb{R}^m} < \epsilon$. Here $\|\cdot\|_{\mathbb{R}^n}$ denotes the Euclidean norm in $\mathbb{R}^n$. Similarly, $\|\cdot\|_{\mathbb{R}^m}$ denotes the Euclidean norm in $\mathbb{R}^m$.

Topological spaces provide the most general setting for which the concept of continuity makes sense. Before we get to the concept of a topological space, let us consider metric spaces. Metric spaces allow us to speak of distance between elements. Using the notion of distance between elements we can make sense of continuity of maps between metric spaces.

Definition 2.1 (Metric spaces) A *metric space* (X, d) is a non-empty set X together with a map $d\colon X \times X \to \mathbb{R}$ called a *metric* such that the following properties hold:

M1 $d(x,y) \geqslant 0$ for all $x, y \in X$, and $d(x,y) = 0$ if and only if $x = y$;

M2 $d(x,y) = d(y,x)$ for all $x, y \in X$;

M3 $d(x,z) \leqslant d(x,y) + d(y,z)$ for all $x, y, z \in X$.

The first condition says that the distance between two elements is always positive, and equal to zero if and only if the two elements are the same. The second condition says that distance is symmetric. The third condition says that the *triangle inequality* holds. The metric d is sometimes also referred to as a distance function.

Example 2.2 ($\mathbb{R}^n$ seen as a metric space) Let $X = \mathbb{R}$ and d be the map defined by $d(x,y) = |x - y| (= \sqrt{(x-y)^2})$. The first two requirements for d are clearly satisfied, and the third follows from the usual triangle inequality for real numbers,

$$d(x,z) = |x - z| = |(x-y) + (y-z)| \leqslant |x-y| + |y-z| = d(x,y) + d(y,z).$$

For $X = \mathbb{R}^n$ with $n > 0$ an integer, let $d(x,y) = \|x - y\|$ where $\|\cdot\|$ is the Euclidean norm, e.g., for $n = 2$, $d(x,y) = \|x-y\| = \sqrt{(x_1-y_1)^2 + (x_2-y_2)^2}$. Again, the first two requirements for d are clearly satisfied. The third requirement follows from the triangle inequality for vectors in $\mathbb{R}^n$.

We may equip $\mathbb{R}^n$ with other metrics than the one described in Example 2.2. For instance, for $X = \mathbb{R}^2$, let

$$d(x,y) = |x_1 - y_1| + |x_2 - y_2|.$$

This is known as the *taxi cab metric*.

We say that two metrics d_1 and d_2 on the same set X are *equivalent* if there exist constants L and M such that

$$d_1(x,y) \leqslant L d_2(x,y) \quad \text{and} \quad d_2(x,y) \leqslant M d_1(x,y)$$

for all $x, y \in X$.

Example 2.3 (Discrete metric spaces) For any set X, let $d\colon X \times X \to \mathbb{R}$ be the map given by

$$d(x,y) = \begin{cases} 1 & x \neq y, \\ 0 & x = y. \end{cases}$$

We call d the *discrete metric* on X.

Example 2.4 ($C[a,b]$) Let $X = C[a,b]$, i.e., the set of continuous maps from the interval $I = [a,b] \subseteq \mathbb{R}$ to $\mathbb{R}$, and let

$$d(x,y) = \max_{i \in I} |x(i) - y(i)|.$$

Example 2.5 If d is a metric on a set X, and $A \subseteq X$ is any subset of X, then d is also a metric on A.

2.2 Continuous maps between metric spaces

The definition of continuity of maps between metric spaces is completely analogous to the situation that we have from calculus.

Definition 2.6 (Continuous maps between metric spaces) Let (X, d_X) and (Y, d_Y) be two metric spaces. A map $f\colon X \to Y$ is *continuous* at $p \in X$ if for all $\epsilon > 0$ there exists a $\delta > 0$ such that if $d_X(p,q) < \delta$ then $d_Y(f(p), f(q)) < \epsilon$.

If f is continuous at every $p \in X$, we say that f is *continuous*.

To get us to the setting of topological spaces we will need the concept of *open* and *closed* sets.

Definition 2.7 (Open and closed balls) Let (X, d) be a metric space, and let $a \in X$ and $r > 0$ be real number. The *open ball* centered at a with radius r is the subset

$$\mathrm{B}(a;r) = \{x \in X \mid d(x,a) < r\}$$

of X. The *closed ball* centered at a with radius r is the subset

$$\overline{\mathrm{B}}(a;r) = \{x \in X \mid d(x,a) \leqslant r\}$$

of X.

In Euclidean space with the usual metric (induced from Euclidean norm), a ball (as defined above) is precisely what we think of as a ball in everyday language. Open balls are sometimes referred to as simply balls, and closed balls are sometimes referred to as discs, e.g. Theorem 1.1.

Example 2.8 (Open balls in discrete metric spaces) Let (X, d) be the metric space defined in Example 2.3. Then

$$\mathrm{B}(x; r_1) = \{x\} \quad \text{and} \quad \mathrm{B}(x; r_2) = X$$

for all $0 < r_1 \leqslant 1$ and all $r_2 > 1$.

Definition 2.9 (Open and closed sets) Let (X, d) be a metric space. A subset $A \subseteq X$ is *open* in X if for every point $a \in A$, there exists an open ball $\mathrm{B}(a; r)$ about a contained in A. We say that A is *closed* in X if the complement $A^c = X \setminus A = \{x \in X \mid x \notin A\}$ is open.

Most subsets are *neither* open nor closed. Subsets that are both open and closed are sometimes referred to as *clopen*. In particular, both $\emptyset$ and X are clopen in X.

Lemma 2.10 *Let (X, d) be a metric space, $x \in X$ and $r > 0$ a real number. Then the open ball $\mathrm{B}(x; r) \subseteq X$ is open in X, and the closed ball $\overline{\mathrm{B}}(x; r) \subseteq X$ is closed in X.*

Proof. We prove the statement about open balls. The statement about closed balls follows from a similar argument.

Assume that $y \in \mathrm{B}(x; r)$. We need to prove that there is an open ball $\mathrm{B}(y; \epsilon)$ about y that is contained in $\mathrm{B}(x; r)$. Let $\epsilon = r - d(x, y)$. By the triangle inequality of the metric d, M3, we have that for $z \in \mathrm{B}(y; \epsilon)$,

$$d(x, z) \leqslant d(x, y) + d(y, z) < d(x, y) + \epsilon = d(x, y) + r - d(x, y) = r.$$

Hence, $\mathrm{B}(y; \epsilon) \subseteq \mathrm{B}(x; r)$. □

For a metric space (X, d), a subset $A \subseteq X$ and $x \in X$, we say that: (i) x is an *interior point* of A if there is an open ball $\mathrm{B}(x; r)$ about x which is contained in A, (ii) x is an *exterior point* of A if there is an open ball $\mathrm{B}(x; r)$ which is contained in A^c and (iii) x is a *boundary point* if all open balls about x contains points in A and in A^c. Hence, A is open in X if and only if A only consists of its interior points. An interior point will *always* belong to A. An exterior point will *never* belong to A. A boundary point will some times belong to A, and some times to A^c.

Definition 2.11 (Neighborhoods) Let (X, d) be a metric space, A a subset of X and $x \in X$. We say that A is a *neighborhood of x* if there is an open ball about x that is contained in A. We say that A is an *open neighborhood* (of x) if A itself is open.

Theorem 2.12 (Continuity at a point) *Let (X, d_X) and (Y, d_Y) be two metric spaces, and let $p \in X$. A map $f : X \to Y$ is continuous at p if and only if for all neighborhoods B of $f(p)$, there is a neighborhood A of p such that $f(A) \subseteq B$.*

Proof. Assume that f is continuous at p. If B is a neighborhood of $f(p)$, then, by definition, there is an open ball $\mathrm{B}_Y(f(p); \epsilon)$ about $f(p)$ that is contained in B. Since f is continuous at p, there is a

$\delta > 0$ such that if $d_X(p,q) < \delta$, then $d_Y(f(p), f(q)) < \epsilon$. Hence, $f(\mathrm{B}_X(p;\delta)) \subseteq \mathrm{B}_Y(f(p);\epsilon) \subseteq B$. That is, if we let $A = \mathrm{B}_X(p,\delta)$, then for all neighborhoods B of $f(p)$, we have that $f(A) \subseteq B$ where A is a neighborhood of p.

Assume that for all neighborhoods B of $f(p)$, there is a neighborhood A of p such that $f(A) \subseteq B$. We need to prove that for all $\epsilon > 0$, there exists a $\delta > 0$ such that if $d_X(p,q) < \delta$, then $d_Y(f(p), f(q)) < \epsilon$. By utilizing the fact that $B = \mathrm{B}_Y(f(p);\epsilon)$ is a neighborhood of $f(p)$, then, by assumption, there must be a neighborhood A of p such that $f(A) \subseteq B$. Since A is a neighborhood of p, there is an open ball $\mathrm{B}_X(p;\delta)$ about p that is contained in A. Now assume that $d_X(p,p') < \delta$. Then $p' \in \mathrm{B}_X(p;\delta) \subseteq A$. Thus $f(p') \in B = \mathrm{B}_Y(f(p);\epsilon)$, and hence, $d_Y(f(p), f(p')) < \epsilon$. Thus f is continuous at p. □

The following theorem gives an alternative description of continuous maps between metric spaces.

Theorem 2.13 **(Continuous maps between metric spaces)** *Let (X, d_X) and (Y, d_Y) be two metric spaces. A map $f\colon X \to Y$ is continuous if and only if for every subset $B \subseteq Y$ open in Y, the* preimage *of B under f,*

$$f^{-1}(B) = \{x \in X \mid f(x) \in B\} \subseteq X,$$

is open in X.

Proof. Assume that f is continuous. For $B \subseteq Y$ open in Y, we have to prove that $f^{-1}(B) \subseteq X$ is open in X. Let $a \in f^{-1}(B)$. We want to prove that there is an open ball about a in X that is contained in $f^{-1}(B)$. By assumption, $B \subseteq Y$ is open in Y. Hence, there is a $\epsilon > 0$ such that $\mathrm{B}_Y(f(a);\epsilon) \subseteq B$. From the assumption that f is continuous there is a $\delta > 0$ such that $\mathrm{B}_X(a;\delta) \subseteq f^{-1}(\mathrm{B}_Y(f(a);\epsilon)) \subseteq f^{-1}(B)$.

We now prove the opposite implication. Assume that for every subset $B \subseteq Y$ open in Y, the preimage $f^{-1}(B)$ of B under f is open in X. Let $a \in X$ and $\epsilon > 0$ be a real number. From the first assumption it follows that $f^{-1}(\mathrm{B}_Y(f(a);\epsilon)) \subseteq X$ is open in X. As $f^{-1}(\mathrm{B}_Y(f(a);\epsilon))$ is open and contains a, there is a $\delta > 0$ such that $\mathrm{B}_X(a;\delta) \subseteq f^{-1}(\mathrm{B}_Y(f(a);\epsilon))$. Thus $x \in \mathrm{B}_X(a;\delta)$ implies that $f(x) \in \mathrm{B}_Y(f(a);\epsilon)$. Hence, $f\colon X \to Y$ is continuous. □

Note that $f^{-1}(B)$ always exists even if there is no inverse map. In the cases where f has an inverse there is no ambiguity.

2.3 Exercises

Exercise 2.1 Does $d(x,y) = (x-y)^2$ define a metric on $X = \mathbb{R}$?

Exercise 2.2 Show that $\mathbb{R}^2$ equipped with the taxi cab metric is a metric space.

Exercise 2.3 Draw a picture of the open ball $\mathrm{B}((0,0);1)$ in the metric space $(\mathbb{R}^2, d)$ with

(a) $d(x,y) = d_1(x,y) = |x_1 - y_1| + |x_2 - y_2|$;

(b) $d(x,y) = d_2(x,y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$;

(c) $d(x,y) = d_\infty(x,y) = \max\{|x_1 - y_1|, |x_2 - y_2|\}$.

Exercise 2.4 Show that d_1, d_2 and d_∞ (as defined in Exercise 2.3) are equivalent on $X = \mathbb{R}^2$.

Exercise 2.5 Show that in a discrete metric space (X, d), cf. Example 2.3, every subset is both open and closed in X.

Exercise 2.6 Show that for equivalent metrics d and d' on the set X, the open sets are the same.

Bibliography

[1] J.R. Munkres. *Topology*. Prentice Hall, Inc., Upper Saddle River, NJ, 2000. Second edition.