Norwegian University of Science and Technology
Department of Mathematical Sciences

TMA4183 Optimisation II Spring 2020

Exercise set 6

1 Assume that $\vec{b} \in C^1(\bar{\Omega}; \mathbb{R}^d)$ is a continuously differentiable, divergence free vector field on Ω and that $y_d \in L^2(\Omega)$ is some given (target) function. We consider the optimal control problem

$$J(y,u) = \frac{1}{2}\int_\Omega (y - y_d)^2\,dx + \frac{\alpha}{2}\int_\Omega u^2\,dx$$

subject to the state equation

$$\begin{aligned} y + \operatorname{div}(\vec{b}y) - \Delta y &= u && \text{in } \Omega,\\ y &= 0 && \text{on } \Gamma, \end{aligned}$$

where $\alpha \geq 0$ is some parameter and $u \in L^2(\Omega)$ the (distributed) control we optimise for.

a) Assume that $\alpha > 0$. Show that this control problem has a unique solution $(\bar{y}, \bar{u}) \in H_0^1(\Omega) \times L^2(\Omega)$.

b) Derive the optimality conditions for this optimal control problem.

c) We now add the additional state constraints

$$u_a \leq u(x) \leq u_b \qquad \text{for a.e. } x \in \Omega,$$

where $-\infty < u_a < u_b < +\infty$ are some fixed parameters, but allow for α to be equal to zero. Show that the resulting problem still has a unique solution and derive the corresponding optimality system.

2 We consider the parabolic problem with boundary controls

$$J(y,u) = \frac{1}{2}\int_\Omega (y(T,x) - y_d(x))^2\,dx + \frac{\alpha}{2}\int_0^T\int_\Gamma u(t,x)^2\,dx\,dt \to \min$$

subject to the constraints

$$\begin{aligned} y_t &= \Delta y && \text{in } (0,T)\times\Omega,\\ y(0,x) &= y_0(x) && \text{in } \Omega,\\ \partial_\nu y(t,x) &= u(x,t) - y(t,x) && \text{in } (0,T)\times\Gamma. \end{aligned}$$

Here $\alpha > 0$, and y_d, $y_0 \in L^2(\Omega)$ are given functions. Derive the formal optimality conditions for this optimal control problem. In particular, formulate the adjoint PDE as a (classical) final value problem.