# 24. Lecture XXIV: One-Dimensional Global Bifurcations I

24.1. **Limit cycles of systems dependent on a parameter.** Just as we discussed linearizations about periodic orbits after analogous discussions about linearizations about critical points, we shall now discuss bifurcations at periodic orbits.

Recall that when there is a periodic orbit, one eigenvalue of $\mathrm{D}\mathbf{\Pi}(\mathbf{x}_0)$ is always equal to 1, for $\mathbf{x}_0$ a point on that periodic orbit. With reference to the stable manifold theorem for periodic orbits (Thm. 16.3) periodic orbit $\Gamma$ is nonhyperbolic when there is at least another eigenvalue on the complex unit circle.

There are three ways this can occur. Either $\mathrm{D}\mathbf{\Pi}(\mathbf{x}_0)$ has another eigenvalue of 1, or it has $-1$ for an eigenvalue, or it has a complex conjugate pair $e^{\pm i\vartheta}$ of eigenvalues.

Let us consider again the system

$$\dot{\mathbf{x}} = f(\mathbf{x}, \mu)$$

on $\mathbb{R}^d$, and suppose that at $\mu = 0$, this system has a periodic orbit given by $\gamma_0(t) = \phi_t(\mathbf{x}_0, 0)$. Let $\Sigma$ be a hyperplane through $\mathbf{x}_0$ perpendicular to $\gamma_0'(0)$. To each $\mu$ in a small neighbourhood of $\mu$, and $\mathbf{x} \in \Sigma$ in a small enough neighbourhood of $\mathbf{x}_0$, we can define a first return time $\tau(\mathbf{x}, \mu)$. This is sufficient to define a Poincaré map

$$\mathbf{\Pi} : (\mathbf{x}, \mu) \mapsto \phi_{\tau(\mathbf{x},\mu)}(\mathbf{x}, \mu).$$

In this lecture we shall be considering bifurcations about non-hyperbolic periodic orbits. This means that we require that at the bifurcation point $\mu_0$,

$$\mathbf{\Pi}(\mathbf{x}_0, \mu_0) = \mathbf{x}_0, \qquad \mathrm{D}\mathbf{\Pi}(\mathbf{x}_0, \mu_0) = 1,$$

already, analogous to the conditions presented by the centre manifold theorem in (34).

Recall from Thm.16.1 that along a periodic orbit $\gamma_\mu(t)$ of a planar $C^1$-system of period $T_\mu$, the Poincare map is given by

$$\mathrm{D}\mathbf{\Pi}(\gamma_\mu(0)) = \exp\Big(\int_0^{T_\mu} (\nabla \cdot f)(\gamma_\mu(t), \mu)\, \mathrm{d}t\Big).$$



We shall compute $\mathrm{D}\mathbf{\Pi}$ for simple examples and see how it characterizes bifurcations. Similar calculations can be made to find the sign on $a$ in (39) in our discussion on Hopf bifurcation in Lecture 23.

Another way to test for stability, in polar coordinates, is see if $\dot{r}$ has a definite sign for $r$ slightly beyond the limit cycle, and for $r$ slightly below the limit cycle. This technique is slightly more ad-hoc, and works best for limit cycles that are circles. Even if the system were written in polar coordinates, we can test the stability of limit cycles by looking at the Poincaré map using the polar coordinate transformations

$$\begin{pmatrix} \partial/\partial x \\ \partial/\partial y \end{pmatrix} = \begin{pmatrix} \cos(\vartheta) & -\sin(\vartheta) \\ -\sin(\vartheta) & \cos(\vartheta)/r \end{pmatrix} \begin{pmatrix} \partial/\partial r \\ \partial/\partial \vartheta \end{pmatrix},$$

and

$$\dot{x} = \dot{r}\cos(\vartheta) - r\sin(\vartheta)\dot{\vartheta}, \qquad \dot{y} = \dot{r}\sin(\vartheta) + r\cos(\vartheta)\dot{\vartheta},$$

so that

$$(\nabla \cdot f)((r, \vartheta), \mu) = \frac{\partial \dot{r}}{\partial r} + \frac{\partial \dot{\vartheta}}{\partial \vartheta} + \frac{\dot{r}}{r}.$$

The following examples are taken directly from section 4.5 of *Perko* and Cap. 9 of *Cain and Shaeffer*.

24.2. **Examples of simple bifurcations about nonhyperbolic periodic orbits.** *Saddle-node bifurcation*

**Example 24.1.** Consider the system

$$\begin{aligned}\dot{r} &= (r-1)^2 + \mu \\ \dot{\vartheta} &= 1.\end{aligned}$$

We find limit cycles at

$$r_{\pm}(\mu) = 1 \pm \sqrt{-\mu}$$

for $-1 < \mu < 0$. At $\mu \geq 0$, there is only one limit cycle, and at $\mu < -1$, there is again only one limit cycle because $r$ is a non-negative quantity.

We can test the stability of the limit cycles in the range $-1 < \mu < 0$. At $r_{\pm}(\mu) + \varepsilon$, we find that

$$\dot{r} = (r_{\pm}(\mu) + \varepsilon - 1)^2 + \mu = (\varepsilon \pm \sqrt{-\mu})^2 + \mu = \varepsilon^2 \pm 2\varepsilon\sqrt{-\mu}.$$

Choosing $\varepsilon$ small enough so that $0 < \varepsilon < \sqrt{-\mu}$, we find that $\dot{r}$ is positive at $r_+(\mu) + \varepsilon$ and negative at $r_-(\mu) + \varepsilon$. Choosing $\varepsilon$ small enough (in magnitude) so that $-\sqrt{-\mu} < \varepsilon < 0$, we find that $\dot{r}$ is negative at $r_+(\mu) + \varepsilon$ and positive at $r_-(\mu) + \varepsilon$.

This means that the limit cycle at $r_+(\mu)$ is unstable and the limit cycle at $r_-(\mu)$ is stable. Therefore the bifurcation at $\mu = 0$ is a saddle-node bifurcation. Moreover, at $\mu = -1$, we see a "reflected" Hopf-like bifurcation.



**Example 24.2.** Next look at the system

$$\begin{aligned}\dot{x} &= -y - x\big(\mu - (r^2 - 1)^2\big) \\ \dot{y} &= x - y\big(\mu - (r^2 - 1)^2\big)\end{aligned}$$

This is transformed into

$$\dot{r} = -r\big(\mu - (r^2 - 1)^2\big), \qquad \dot{\vartheta} = 1.$$

From this it is clear that there are limit cycles at

$$r_{\pm}(\mu) = \sqrt{1 \pm \mu^{1/2}}.$$

for $0 < \mu < 1$.

Let us test for stability with the Poincaré map this time. We can find the Poincaré map derivative using

$$(\nabla \cdot f)((r_{\pm}(\mu), \vartheta), \mu) = \frac{\partial \dot{r}}{\partial r} + \frac{\partial \dot{\vartheta}}{\partial \vartheta} + \frac{\dot{r}}{r} = 4(1 \pm \mu^{1/2})(\pm \mu^{1/2}).$$

Therefore

$$\mathrm{D}\mathbf{\Pi}((r_{\pm}(\mu), 0), \mu) = \exp\big(\pm 8\pi(1 \pm \mu^{1/2})\mu^{1/2}\big),$$

For $0 < \mu < 1$, we find that

$$\mathrm{D}\mathbf{\Pi}((r_+(\mu), 0), \mu) > 1, \qquad \mathrm{D}\mathbf{\Pi}((r_-(\mu), 0), \mu) < 1.$$

Therefore the larger limit cycle is unstable and the smaller one is stable. At $\mu = 0$, we have $\mathrm{D}\mathbf{\Pi}((1,0),0) = 1$, and so we see that a saddle-node bifurcation happens at the non-hyperbolic periodic orbit of the system at $\mu = 0$.

What we see here is that in fact, we have a nondegeneracy given by

$$\frac{\partial}{\partial \mu}\mathbf{\Pi}(\mathbf{x}_0, 0) \neq 0,$$

as a generator of this saddle-node bifurcation, for $\mathbf{x}_0$ on the periodic orbit at $\mu = 0$.

Both the saddle-node bifurcations above can be summed up graphically using bifurcation diagrams where $\mu$ is plotted against $r$.

*2. Transcritical bifurcation*

We shall see that we can likewise derive a transcritical bifurcation with a nondegeneracy condition on $\mathbf{\Pi}$ analogous to the one we had on the "centre equation" in Lecture 22 in (36).

**Example 24.3.** Let us inspect this time the system

$$\begin{aligned}\dot{x} &= -y - x(1-r^2)(1+\mu-r^2)\\ \dot{y} &= x - y(1-r^2)(1+\mu-r^2).\end{aligned}$$

In polar coordinates, we find

$$\dot{r} = -r(1-r^2)(1+\mu-r^2), \qquad \dot{\vartheta} = 1.$$

Again, it is clear that we have limit cycles at

$$r_+(\mu) = 1, \qquad r_-(\mu) = \sqrt{1+\mu}.$$

for all values of $\mu > -1$.

The divergence of the flux is

$$(\nabla \cdot f)((r_\pm(\mu),\vartheta),\mu) = \frac{\partial \dot{r}}{\partial r} + \frac{\partial \dot{\vartheta}}{\partial \vartheta} + \frac{\dot{r}}{r} = \begin{cases} 2\mu & r = r_+(\mu) \\ -2\mu(1+\mu) & r = r_-(\mu) \end{cases}.$$

So as $\mu$ increases through 0, the orbit at $r_+$ changes from being stable to being unstable, and the orbit at $r_-(\mu)$ changes from being stable to being unstable. This is characteristically a transcritical bifurcation.

Calculating the actual derivative of the Poincaré map, we find

$$\mathrm{D}\mathbf{\Pi}((r_\pm(\mu),0),\mu) = \begin{cases} e^{4\pi\mu} & r = r_+(\mu) \\ e^{-4\pi\mu(1+\mu)} & r = r_-(\mu) \end{cases}.$$

And we see that the non-degeneracy condition that would have induced this bifurcation is

$$\frac{\partial}{\partial \mu}\mathrm{D}\mathbf{\Pi}(\mathbf{x}_0, 0) \neq 0, \tag{42}$$

for $\mathbf{x}_0$ on the periodic orbit of the system at $\mu = 0$.

Again, it is possible to represent this bifurcation graphically by plotting $\mu$ against $r$.

*3. Pitchfork bifurcation*

**Example 24.4.** Finally let us look at the following system:

$$\begin{aligned}\dot{x} &= -y + x(1-r^2)\big(\mu - (r^2-1)^2\big)\\ \dot{y} &= x + y(1-r^2)\big(\mu - (r^2-1)^2\big).\end{aligned}$$

In polar coordinates, this becomes

$$\begin{aligned}\dot{r} &= r(1-r^2)\big(\mu - (r^2-1)^2\big)\\ \dot{\vartheta} &= 1.\end{aligned}$$

Again limit cycles are circles, and of radii

$$r_0(\mu) = 1, \qquad r_\pm(\mu) = \sqrt{1 \pm \mu^{1/2}}.$$

There are three limit cycles where $0 < \mu < 1$. As before, we can compute the derivative of the Poincaré map to test for stability, and we find that

$$\mathrm{D}\mathbf{\Pi}((r_0(\mu), 0), \mu) = e^{-4\pi\mu}, \qquad \mathrm{D}\mathbf{\Pi}((r_\pm(\mu), 0), \mu) = e^{4\pi\mu\left(1 \pm \mu^{1/2}\right)}.$$

This is a (reflected) subcritical pitchfork bifurcation.

24.3. **Degeneracy conditions for bifurcations.** We can sum up and fill in the various degeneracy conditions for simple bifurcations about a degenerate orbit in a theorem:

**Theorem 24.1.** *Let $\Gamma$ be a nonhyperbolic periodic orbit of a $C^2$-planar system at $\mu = \mu_0$ containing the point $\mathbf{x}_0$. Let $\mathbf{\Pi}$ be the Poincaré map defined in a neighbourhood of $(\mathbf{x}_0, \mu_0)$ for the orbit $\Gamma_0$.*

*The non-degeneracy conditions*

$$\mathrm{D}^2\mathbf{\Pi}(\mathbf{x}_0, \mu_0) \neq 0, \qquad \frac{\partial}{\partial \mu}\mathbf{\Pi}(\mathbf{x}_0, \mu_0) \neq 0$$

*induces a saddle-node bifurcation about $\Gamma_0$ at $\mu = \mu_0$.*

*The relaxed non-degeneracy conditions*

$$\frac{\partial}{\partial \mu}\mathbf{\Pi}(\mathbf{x}_0, \mu_0) = 0, \qquad \frac{\partial}{\partial \mu}\mathrm{D}\mathbf{\Pi}(\mathbf{x}_0, \mu_0) \neq 0, \qquad \mathrm{D}^2\mathbf{\Pi}(\mathbf{x}_0, \mu_0) \neq 0$$

*induces a transcritical bifurcation about $\Gamma_0$ at $\mu = \mu_0$.*

*The further relaxed non-degeneracy conditions*

$$\frac{\partial}{\partial \mu}\mathbf{\Pi}(\mathbf{x}_0, \mu_0) = 0, \quad \mathrm{D}^2\mathbf{\Pi}(\mathbf{x}_0, \mu_0) = 0, \quad \frac{\partial}{\partial \mu}\mathrm{D}\mathbf{\Pi}(\mathbf{x}_0, \mu_0) \neq 0, \quad \mathrm{D}^3\mathbf{\Pi}(\mathbf{x}_0, \mu_0) \neq 0$$

*induces a pitchfork bifurcation about $\Gamma_0$ at $\mu = \mu_0$.*