TMA 4190 Introduction to Topology
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Lecture 19[1]

19. Transversality is generic

Today we are going to review what we have learned about transversality and show that it is actually a generic property. We start with the following extension of Sard's Theorem:

Transversality Theorem

Suppose that $F\colon X \times S \to Y$ is a smooth map of manifolds, where only X has a boundary, and let Z be any boundaryless submanifold of Y. If both F and ∂F are transversal to Z, then for almost every $s \in S$, both f_s and ∂f_s are transversal to Z (where f_s denotes the map $x \mapsto f_s(x) = F(x,s)$, and similarly $\partial f_s(x) = \partial F(x,s)$).

Note that, roughly speaking, the difference between requiring that F is transversal to Z versus f_s is transversal to Z is that for F the image of $T_{(x,s)}(X \times S)$ under $dF_{(x,s)}$ has to be big enough, whereas for f_s we look at the potentially smaller image of $T_{(x,s)}(X \times S)$ under $d(f_s)_x$. Similarly for ∂F and ∂f_s.

Proof: By the Preimage Theorem, the preimage $W := F^{-1}(Z)$ is a submanifold of $X \times S$ with boundary $\partial W = W \cap \partial(X \times S)$. Let $\pi\colon X \times S \to S$ be the natural projection map.

We will show that whenever $s \in S$ is a **regular value** for the restriction $\pi\colon W \to S$ then $f_s \pitchfork Z$, and whenever s is a regular value for $\partial\pi\colon \partial W \to S$, then $\partial f_s \pitchfork Z$. By Sard's theorem (which also holds for manifolds with boundary), almost every $s \in S$ is a regular value for both maps, so the theorem follows.

In order to show that $f_s \pitchfork Z$, suppose that $f_s(x) = z \in Z$. Because $F(x,s) = z$ and $F \pitchfork Z$, we know that

$$dF_{(x,s)}(T_{(x,s)}(X \times S)) + T_z(Z) = T_z(Y).$$

Hence, given any vector $a \in T_z(Y)$, there exists a vector $b \in T_{(x,s)}(X \times S)$ such that

$$dF_{(x,s)}(b) - a \in T_z(Z).$$

[1]Following the books of Guillemin and Pollack: Differential Topology, and Milnor: Topology from the differentiable viewpoint.

What **we need** is to find a vector $v \in T_x(X)$ such that
$$df_s(v) - a \in T_z(Z),$$
as that would show that $df_s(T_x(X)) + T_z(Z) = T_z(Y)$.

Since
$$T_{(x,s)}(X \times S) = T_x(X) \times T_s(S),$$
we can write b as a pair (w,e) for vectors $w \in T_x(X)$ and $e \in T_s(S)$.

If e was zero, we would be done, for since the restriction of F to $X \times \{s\}$ is f_s, it follows that
$$dF_{(x,s)}(w,0) = df_s(w).$$

Although e need not be zero, we may use the projection π to kill it off.

It is easy to check that
$$d\pi_{(x,s)} \colon T_x(X) \times T_s(S) \to T_s(S)$$
is just projection onto the second factor (this holds for every projection map from a product of manifolds).

Now we use the assumption that s is a regular value of π. For this implies that
$$d\pi_{(x,s)} \colon T_{(x,s)}(W) \to T_s(S)$$
is surjective. In particular, the fiber over $e \in T_s(S)$ is nonempty, and there is some vector of the form (u,e) in $T_{(x,s)}(W)$.

But $F \colon W \to Z$, so $dF_{(x,s)}(u,e)$ is an element in $T_z(Z)$. Consequently, the vector $v := w - u \in T_x(X)$ is our solution. For
$$df_s(v) - a = dF_{(x,s)}((w,e) - (u,e)) - a = (dF_{(x,s)}(w,e) - a) - dF_s(u,e),$$
and both of the latter vectors belong to $T_z(Z)$.

Precisely the same argument shows that $\partial f_s \pitchfork Z$ when s is a regular value of $\partial\pi$. **QED**

Transversality is generic - first case

Transversality for smooth maps $X \to \mathbb{R}^M$ is generic in the following sense: Let $f \colon X \to \mathbb{R}^M$ be any smooth map. Let S be an open ball in $\mathbb{R}^M$, and define
$$F \colon X \times S \to \mathbb{R}^M, \;\; F(x,s) = f(x) + s.$$

The derivative of F at (x,s) is

$$dF_{(x,s)} = (df_x, \mathrm{Id}_{\mathbb{R}^M})\colon T_x(X) \times \mathbb{R}^M \to \mathbb{R}^M.$$

Thus $dF_{(x,s)}$ is obviously surjective at any (x,s). Hence F is a **submersion**. This implies that F is **transversal to every submanifold $Z \subset \mathbb{R}^M$**. Now we can apply the **Transversality Theorem** we have just proven: Since F and ∂F are transversal to Z, for **almost every** $s \in S$, the map $f_s(x) = f(x) + s$ is transversal to Z. Thus, for any submanifold $Z \subset \mathbb{R}^M$, there is an s, with **arbitrarily small norm** in $\mathbb{R}^M$, such that f may be deformed into a map f_s transversal to Z by the **translation by s**.

This shows us that transversality is generic for maps $X \to \mathbb{R}^M$. We would like to generalize this result to an **arbitrary** boundaryless smooth manifold $Y \subset \mathbb{R}^M$ and smooth map $f\colon X \to Y$.

Given a submanifold $Z \subset Y$, we have just learned how to vary $f\colon X \to Y \subset \mathbb{R}^M$ as a family of maps $X \to \mathbb{R}^M$ such that $f_s \pitchfork Z$ for arbitrarily small s, where we consider Z as a submanifold in $\mathbb{R}^M$.

It remains to understand how we can project these maps down onto Y such that a small perturbation f_s of f remains transversal to the given submanifold $Z \subset Y$. To do so, we must understand a little of the **geometry of Y with respect to its environment**. As usual, the compact case is clearest.

ϵ-Neighborhood Theorem

For a **compact** boundaryless manifold Y in $\mathbb{R}^M$ and a positive number ϵ, let Y^ϵ be the open set of points in $\mathbb{R}^M$ with distance less than ϵ from Y. If ϵ is sufficiently small, then each point $w \in Y^\epsilon$ possesses a unique closest point in Y, denoted $\pi(w)$. Moreover, the map $\pi\colon Y^\epsilon \to Y$ is a submersion. When Y is **not compact**, there still exists a submersion $\pi\colon Y^\epsilon \to Y$ that is the identity on Y, but now ϵ must be allowed to be a positive **smooth function** $\epsilon\colon Y \to \mathbb{R}^{>0}$ on Y, and Y^ϵ is defined as

$$Y^\epsilon = \{w \in \mathbb{R}^M : |w - y| < \epsilon(y) \text{ for some } y \in Y\} \subset \mathbb{R}^M.$$

The manifold Y^ϵ is called a **tubular neighborhhood** of Y in $\mathbb{R}^M$.

Note that the important point of the theorem is not so much the existence of the Y^ϵ, but rather that they come equipped with the submersion π. As we will see in a bit, this is related to a key tool, the normal bundle.

Before we prove this theorem, we study a first consequence:

Creating families of submersions

Let $f\colon X \to Y$ be a smooth map where Y is a boundaryless manifold. Let S be the open ball in $\mathbb{R}^M$. Then there is a smooth map $F\colon X \times S \to Y$ such that $F(x,0) = f(x)$, and for any **fixed** $x \in X$, the map

$$S \to Y,\ s \mapsto F(x,s) \text{ is a submersion.}$$

In particular, both F and ∂F are submersions.

Proof: Let $Y \subset \mathbb{R}^M$ and S be the unit ball in $\mathbb{R}^M$. We define

$$F\colon X \times S \to Y,\ F(x,s) = \pi(f(x) + \epsilon(f(x))s). \tag{1}$$

Since $\pi\colon Y^\epsilon \to Y$ restricts to the identity on Y, we have

$$F(x,0) = \pi(f(x) + 0) = f(x).$$

For **fixed** x, the map

$$\varphi\colon S \to Y^\epsilon,\ s \mapsto f(x) + \epsilon(f(x))s$$

is the translation of a linear map. Thus $d\varphi_s$ is just given by multiplying a vector in $T_s(S) = \mathbb{R}^M$ by the real number $\epsilon(f(x)) > 0$ (to get a vector in $T_{\varphi(s)}(Y^\epsilon) \subset \mathbb{R}^M$). This derivative is just $\epsilon(f(x))$ times the identity of $\mathbb{R}^M$, and therefore surjective. Thus φ is a **submersion**.

As the composition of two submersions is a submersion, we get that

$$S \to Y,\ s \mapsto F(x,s) \text{ is a submersion.}$$

Hence the restriction $F_{\{x\}\times S}\colon \{x\} \times S \to Y$ of F to the submanifold $\{x\} \times S$ is submersion for every $x \in X$. Since every point of $X \times S$ lies in one of these submanifolds, **F must be a submersion** as well, since its derivative $dF_{(x,s)}$ is already surjective onto $T_{F(x,s)}$ when restricted to $T_{(x,s)}(\{x\} \times S) \subset T_{(x,s)}(X \times S)$.

The same argument applied to ∂F and ∂X, shows that ∂F is a submersion. **QED**

Now we can prove that transversality is generic:

Transversality Homotopy Theorem

For any smooth map $f\colon X \to Y$ and any boundaryless submanifold Z of the boundaryless manifold Y, there exists a smooth map $g\colon X \to Y$ homotopic to f such that $g \pitchfork Z$ and $\partial g \pitchfork Z$.

Proof: For the family of mappings F of the previous conesequence of the ϵ-Neighborhood Theorem, the Transversality Theorem implies that $f_s \pitchfork Z$ and $\partial f_s \pitchfork Z$ for almost all $s \in S$. But each f_s is homotopic to f, the homotopy being

$$X \times I \to Y,\ (x,t) \mapsto F(x,ts).$$

QED

Now we are going to prove the ϵ-Neighborhood Theorem. To do this we introduce an important geometric tool similar to the tangent bundle.

The Normal Bundle

For each $y \in Y$, define $N_y(Y)$, the normal space of Y at y, to be the orthogonal complement of $T_y(Y)$ in $\mathbb{R}^M$. The normal bundle $N(Y)$ is then defined to be the set

$$N(Y) = \{(y,v) \in Y \times \mathbb{R}^M : v \in N_y(Y)\}.$$

Note that unlike $T(Y)$, $N(Y)$ is not intrinsic to the manifold Y but depends on the specific relationship between Y and the surrounding $\mathbb{R}^M$. There is a natural projection map $\sigma\colon N(Y) \to Y$ defined by $\sigma(y,v) = y$.

The normal bundle $N(Y)$ is actually a manifold itself. In order to show this, we must recall an elementary fact from linear algebra:

Suppose that $A\colon \mathbb{R}^M \to \mathbb{R}^k$ is a linear map. Its **transpose** is a linear map $A^t\colon \mathbb{R}^k \to \mathbb{R}^M$ characterized by the dot product equation

$$Av \cdot w = v \cdot A^t w \text{ for all } v \in \mathbb{R}^M, w \in \mathbb{R}^k.$$

Claim: If A is surjective, then A^t maps $\mathbb{R}^k$ isomorphically onto the orthogonal complement of the kernel of A.

First we note that A^t is injective. For if $A^t w = 0$, then $Av \cdot w = v \cdot A^t w = 0$, so that $w \perp A(\mathbb{R}^M)$. Since A is surjective, w must be zero.

Now, if $v \in \operatorname{Ker}(A)$, i.e. $Av = 0$, then $0 = Av \cdot w = v \cdot A^t w$. Thus $A^t(\mathbb{R}^k) \perp \operatorname{Ker}(A)$. Hence A^t maps $\mathbb{R}^k$ injectively into the orthogonal complement of Ker (A). As Ker (A) has dimension $M-k$, its complement has dimension k, so A^t is surjective, too.

Normal bundles are manifolds

If $Y \subset \mathbb{R}^M$, then $N(Y)$ is a manifold of dimension M and the projection $\sigma\colon N(Y) \to Y$ is a submersion.

Proof: We need to find loal parametrizations for $N(Y)$.

Therefor, we use that we have learned that we can write every manifold locally as the zeros of a smooth function. Hence around every point in Y, there is an open neighborhood $U \subset Y$ and an open subset $\tilde{U} \subset \mathbb{R}^M$ with $U = Y \cap \tilde{U}$ such that we can write U as the zeros of a submersion

$$\varphi\colon \tilde{U} \to \mathbb{R}^k \ (k = \operatorname{codim} Y) \text{ with } U = Y \cap \tilde{U} = \varphi^{-1}(0).$$

The set $N(U)$ equals $N(Y) \cap (U \times \mathbb{R}^M)$, thus is **open in** $N(Y)$.

For each $y \in U$, $d\varphi_y\colon \mathbb{R}^M \to \mathbb{R}^k$ is **surjective** and has **kernel** $T_y(Y)$ by the Preimage Theorem.

Therefore its **transpose** maps R^k isomorphically onto the orthogonal complement of Ker $(d\varphi_y) = T_y(Y)$ which is $N_y(Y)$ by definition:

$$(d\varphi_y)^t\colon \mathbb{R}^k \xrightarrow{\cong} (T_y(Y))^\perp = N_y(Y).$$

Hence the map

$$\psi\colon U \times \mathbb{R}^k \to N(U),\ (y,v) \mapsto (y, d\varphi_y^t(v))$$

is bijective. It is also an embedding of $U \times \mathbb{R}^k$ into $U \times \mathbb{R}^M$, since it is the identity on the first factor and an injective linear map on the second factor. Hence ψ is a diffeomorphism, and $N(U)$ is a **manifold** with local parametrization ψ.

The **dimenion** of $N(U)$ is

$$\dim N(U) = \dim U + k = \dim Y + \operatorname{codim} Y = M.$$

Since every point of $N(Y)$ has such a neighborhood, $N(Y)$ is a **manifold**.

Since $\sigma \circ \psi\colon U \times \mathbb{R}^k \to U$ is just the projection onto the first factor, which is a submersion. Thus $d(\sigma \circ \psi)_{(u,w)}$, is surjective at every point (u,w). Hence $d\sigma_u$ is surjective at every u, and σ is a **submersion**. **QED**

Before the get to proof the actual theorem, we start with a lemma that will give us the existence of the ϵ-neighborhood Y^ϵ.

ϵ-Neighborhood Lemma

Let $Y \subset \mathbb{R}^M$ be a boundaryless manifold. Then any neighborhood $\tilde{U}$ of Y in $\mathbb{R}^M$, i.e. any open subset $\tilde{U}$ of $\mathbb{R}^M$ with $Y \subset \tilde{U}$, contains

$$Y^\epsilon = \{w \in \mathbb{R}^M : |w - y| < \epsilon(y) \text{ for some } y \in Y\}$$

where $\epsilon\colon Y \to \mathbb{R}^{>0}$ is a suitable smooth function. Moreover, if Y is **compact**, ϵ can be chosen **constant**.

Proof: For each point $\alpha \in Y$, we can find a small radius ϵ_α such that the open ball $B_{2\epsilon_\alpha}(\alpha) \subset \tilde{U}$ is contained in $\tilde{U}$. We set

$$U_\alpha := Y \cap B_{\epsilon_\alpha}(\alpha).$$

Claim:

$$U_\alpha^{\epsilon(\alpha)} = \{w \in \mathbb{R}^M : |w - y'| < \epsilon(\alpha) \text{ for some } y' \in U_\alpha\} \subset \tilde{U}.$$

For, $w \in U_\alpha^{\epsilon_\alpha}$ means there is an $y' \in U_\alpha$ with $|w - y'| < \epsilon_\alpha$. But $y' \in U_\alpha$ means $|y' - \alpha| < \epsilon_\alpha$. Thus the **triangle inequality** implies

$$|w - \alpha| \leq |w - y'| + |y' - \alpha| < 2\epsilon_\alpha.$$

Thus $w \in B_{2\epsilon_\alpha}(\alpha) \subset \tilde{U}$ by the choice of ϵ_α.

The collection of all U_α forms an open cover $\{U_\alpha\}$ of $Y \subset \mathbb{R}^M$. By the Theorem in the Existence of Partitions of Unity for subsets in $\mathbb{R}^M$, we can choose a **partition of unity** $\{\rho_i\}$ subordinate to the cover $\{U_\alpha\}$.

Now we define the function

$$\epsilon\colon Y \to \mathbb{R}^{>0},\ y \mapsto \sum_i \rho_i(y)\epsilon_i$$

Note that ϵ is a smooth function, since all the ρ_i's are smooth.

Claim: $Y^\epsilon \subset \tilde{U}$.

Let $w \in Y^\epsilon$. Then there is a $y \in Y$ such that $|w - y| < \epsilon(y)$. For this y, only finitely many of the numbers $\rho_i(y)$ are nonzero, say $\rho_{i_1}(y), \ldots, \rho_{i_n}(y)$. This implies $y \in U_{i_1} \cap \cdots \cap U_{i_n}$.

Let ϵ_{i_m} be the maximum of the finitely many numbers $\epsilon_{i_1}, \ldots, \epsilon_{i_n}$. Then, since $\sum_i \rho_i(y) = 1$, we have $\epsilon(y) \leq \epsilon_{i_m}$. Hence

$$|w - y| < \epsilon(y) \leq \epsilon_{i_m} \text{ implies } w \in U_{i_m}^{\epsilon_{i_m}} \subset \tilde{U}.$$

Thus $Y^\epsilon \subset \tilde{U}$.

If Y is compact, we can reduce $\{U_\alpha\}$ to a finite cover $U_{\alpha_1},\dots,U_{\alpha_n}$ and let ϵ be equal the maximum of the ϵ_{α_j}. **QED**

Now we are equipped for the proof of the ϵ-Neighborhood Theorem.

Proof of the ϵ-Neighborhood Theorem:

The idea of the proof is to use a version of the **Inverse Function Theorem** to show that the ϵ-neighborhood Y^ϵ of $Y = Y \times \{0\}$ in $\mathbb{R}^M \times \mathbb{R}^M$ is diffeomorphic to an **open subspace in the normal bundle**. Then we use the **natural submersion** $\sigma\colon N(Y) \to Y$ from the normal bundle to get the submersion $\pi\colon Y^\epsilon \to Y$:

$$\begin{tikzcd} Y^\epsilon \arrow[r, "h^{-1}"] \arrow[dr, "\pi"'] & N(Y) \arrow[d, "\sigma"] \\ & Y \end{tikzcd}$$

To make this precise, we define the map
$$h\colon N(Y) \to \mathbb{R}^M,\ (y,v) \mapsto y + v.$$

We claim that h is **regular at every point** of $Y \times \{0\}$ in $N(Y)$.

For, since h is just the restriction of the linear map
$$H\colon \mathbb{R}^M \times \mathbb{R}^M \to \mathbb{R}^M, (w,z)) \mapsto w + z,$$
the derivative of h at (y,v) is just
$$dh_{(y,v)} = H\colon \mathbb{R}^M \times \mathbb{R}^M \to \mathbb{R}^M.$$

Hence at any point (y,v) we have
$$dh_{(y,v)}(w,0) = w \text{ and } dh_{(y,v)}(0,z) = z.$$

The tangent space to $N(Y)$ at $(y,0)$ is just
$$T_{(y,0)}(N(Y)) = T_y(Y) \times \{0\} \oplus \{0\} \times N_y(Y),$$
since $T_y(Y)$ and $N_y(Y)$ are orthogonal complements in $\mathbb{R}^M$ and $\dim N(Y) = M$.

At the point $(y,0)$, $dh_{(y,0)}$ maps
$$T_{(y,0)}(Y \times \{0\}) \text{ onto } T_y(Y) \text{ in } \mathbb{R}^M,$$
and it maps
$$T_{(y,0)}(\{y\} \times N_y(Y)) = \{0\} \times N_y(Y) \text{ onto } N_y(Y) \text{ in } \mathbb{R}^M,$$

where we use that $N_y(Y)$ is a vector space and hence its own tangent space.

Hence, in total, we get

$$\begin{aligned} dh_{(y,0)}(T_{(y,0)}(N(Y))) &= T_{(y,0)}(Y \times \{0\}) + T_{(y,0)}(\{y\} \times N_y(Y)) \\ &= T_y(Y) + N_y(Y) = \mathbb{R}^M. \end{aligned}$$

Thus $dh_{(y,0)}$ is surjective and h is regular at $(y,0)$.

Since h maps $Y \times \{0\}$ diffeomorphically onto Y and is regular at each $(y,0)$, a version of the **Inverse Function Theorem** implies that h must map a **neighborhood of $Y \times \{0\}$ in $N(Y)$** diffeomorphically onto a **neighborhood of Y in $\mathbb{R}^M$**.

Now any neighborhood of Y contains some Y^ϵ by the **ϵ-Neighborhood Lemma**. Thus $h^{-1}\colon Y^\epsilon \to N(Y)$ is defined, and

$$\pi = \sigma \circ h^{-1}\colon Y^\epsilon \to Y$$

is the desired submersion.

It is an exercise to check that we can describe π for compact manifolds as given in the theorem. **QED**

In the final part of today's lecture, we look at another application of the ϵ-Nieghborhoood Theorem. In fact, there is a stronger form of the Transversality Homotopy Theorem. In order to be able to formulate it, we need some terminology.

Transversality on subsets

Let $f\colon X \to Y$ be a smooth map, $Z \subset Y$ a submanifold, and $C \subset X$ be a **subset**. We will say f is transversal to Z on C, if the transversality condition

$$\mathrm{Im}\,(df_x) + T_{f(x)}(Z) = T_{f(x)}(Y) \tag{2}$$

for all $x \in C \cap f^{-1}(Z)$.
Note that, even if C is a submanifold, this is different than requiring $f_{|C} \pitchfork Z$, since (2) involves $\mathrm{Im}\,(df_x) = df_x(T_x(X))$, not $\mathrm{Im}\,(d(f_{|C})_x) = df_x(T_x(C))$, which is smaller in general.

Now we can formulate the next important technical result.

Extension Theorem

Let $f\colon X \to Y$ be a smooth map, Y boundaryless, and Z a closed submanifold of Y without boundary. Let C be a closed subset of X. Assume that $f \pitchfork Z$ on C and $\partial f \pitchfork Z$ on $C \cap \partial X$. Then there exists a smooth map $g\colon X \to Y$ **homotopic to f**, such that $g \pitchfork Z$ and $\partial g \pitchfork Z$, and on a neighborhood of C we have $g = f$.

Since ∂X is always closed in X, we obtain the important special case:

Extension of maps on boundaries

Assume $f\colon X \to Y$ is a smooth map such that the boundary map $\partial f\colon \partial X \to Y$ is transversal to Z. Then there exists a map $g\colon X \to Y$ **homotopic to f** such that $\partial g = \partial f$ and $g \pitchfork Z$.
In particular, suppose there is a smooth map $h\colon \partial X \to Y$ transversal to Z. Then, **if** h extends to **any** map on the whole manifold $X \to Y$, it extends to a map that is **transversal to Z** on all of X.

For the proof of the Extension Theorem we need lemma first:

Lemma

If U is an open subset which contains the closed set C in X, then there exists a smooth function $\gamma\colon X \to [0,1]$ that is identically equal to one outside U but that is zero on a neighborhood of C.

Proof: Let C' be any closed set contained in U that contains C in its interior, and let $\{\rho_i\}$ be a partition of unity subordinate to the open cover $\{U, X \backslash C'\}$. Here it comes handy that we proved the existence of partition of unity for arbitrary subsets of $\mathbb{R}^N$. Then just take γ to be the sum of those ρ_i that vanish outside of $X \setminus C'$. **QED**

Proof of the Extension Theorem:

First we show that $f \pitchfork Z$ on a neighborhood of C i.e. an open subset containing C. If $x \in C$ but $x \notin f^{-1}(Z)$, then since Z is closed, $X \setminus f^{-1}(Z)$ is a neighborhood of x on which $f \pitchfork Z$ automatically.

If $x \in f^{-1}(Z)$, then there is a neighborhood W of $f(x)$ in Y and a submersion $\varphi\colon W \to \mathbb{R}^k$ such that $f \pitchfork Z$ at a point $x' \in f^{-1}(Z \cap W)$ precisely when $\varphi \circ f$ is regular at x'. But if $\varphi \circ f$ is regular at x, so it is regular in a neighborhood of x. Thus $f \pitchfork Z$ on a neighborhood of every point $x \in C$, and so

$$f \pitchfork Z \text{ on a } \textbf{neighborhood } U \text{ of } C \text{ in } X.$$

Second, let γ be the function in the above lemma for the closed subset C and the open neighborhood U of C in X. We set $\tau := \gamma^2$. Since

$$d\tau_x = 2\gamma(x)d\gamma_x, \text{ hence } \gamma(x) = 0 = \tau(x) \Rightarrow d\tau_x = 0.$$

Now we modify the map $F\colon X \times S \to Y$ which we defined in (1) in proving the Homotopy Theorem, where S is the unit ball in $\mathbb{R}^M$. and set

$$G\colon X \times S \to Y,\ G(x,s) := F(x, \tau(x)s).$$

Claim: $G \pitchfork Z$.

For suppose that $(x,s) \in G^{-1}(Z)$, and let us assume first $\tau(x) \neq 0$. Then the map

$$S \to Y,\ r \mapsto G(x,r),$$

is a submersion, since it is the composition of the

$$\text{diffeomorphism } r \mapsto \tau(x)r \text{ with the submersion } r \mapsto F(x,r).$$

Hence G is regular at (x,s), so certainly $G \pitchfork Z$ at (x,s).

To show the claim when $\tau(x) = 0$, we need to check that the image of the derivative $dG_{(x,s)}$ is big enough. To do this, we introuce the map

$$m\colon X \times S \to X \times S,\ (x,s) \mapsto (x,\tau(x)s).$$

We would like to calculate the derivative of m. Therefor, we apply the product rule to the second coordinate and remember that $\tau\colon X \to [0,1]$, i.e. $\tau(x)$ and $d\tau_x(v)$ are both in $\mathbb{R}$ for any $v \in T_x(X)$. Then we get

$$dm_{(x,s)}(v,w) = (v,\tau(x) \cdot w + d\tau_x(v) \cdot s)$$

where w and s are vectors in $\mathbb{R}^M$.

We observe that $G = F \circ m$. Hence in order to calculate the derivative of G, we can apply the chain rule. Since we are interested in the case where $\tau(x) = 0$ and $d\tau_x = 0$ we get

$$dG_{(x,s)}(v,w) = dF_{(x,s)}(v,0).$$

Moreover, since $F(x,0) = f(x)$ for all x by construction of F, we know $F_{|X\times\{0\}} = f$. This implies

$$dF_{(x,s)}(v,0) = dF_{(x,0)}(v,0) = df_x(v).$$

Hence we get

$$dG_{(x,s)}(v,w) = df_x(v)$$

and therefore

$$\mathrm{Im}\,(dG_{(x,s)}) = \mathrm{Im}\,(df_x(v)) \subset T_{f(x)}(Y). \tag{3}$$

Now $\tau(x) = 0$, implies $x \in U$ by definition of γ and τ. But by the choice of U above, this implies $f \pitchfork Z$ at x. Hence (3) implies $G \pitchfork Z$ at (x,s).

The same argument shows $\partial G \pitchfork Z$.

Now we can apply the Transversality Theorem to $G\colon X \times S \to Y$ and get that we can pick and fix an s (almost every s works) for which the map

$$g(x) := G(x,s) \text{ satisfies } g \pitchfork Z \text{ and } \partial g \pitchfork Z.$$

The map G is then a homotopy

$$f = F_{|X\times\{0\}} = G_{|X\times\{0\}} \sim G_{|X\times\{s\}} = g.$$

Finally, if x belongs to the neighborhood of C on which $\tau = 0$, then we even have $g(x) = G(x,s) = F(x,0) = f(x)$. **QED**

Let us summarize what we have done today:

This lecture in a nutshell

We have proven three key results about transversality which can be roughly summarized as follows:

(a) The **Transversality Theorem** says that when a homotopy F is transversal to Z, then, in this homotopy family, **almost every** $f_s = F(-,s)$ is transversal to Z.

(b) The **Transversality Homotopy Theorem** says that given a map f and a submanifold Z, then **there exists** a map g **transversal to** Z and g is **homotopic** to f.

(c) The **Extension Theorem** says that, **given** a map f which is transversal to Z on **a subset** C, then we can always **replace** f with a homotopic map g which is **transversal to** Z **everywhere** (not only on C) and $f = g$ on an open set containing C.

(a) is a generalization of Sard's Theorem. For (b) and (c), the key for the proof is the **ϵ-Neighborhood Theorem**.

Appendix: The Inverse Function Theorem revisited

In the course of this lecture, we used a generalization of the Inverse Function Theorem that we are now going to prove. It will also allow us to show an interesting result on normal bundles and tubular neighborhoods.

As always We start with the compact case:

Generalization of the IFT - compact case

Let $f\colon X \to Y$ be a smooth map that is one-to-one on a **compact** submanifold Z of X. Suppose that for all $x \in Z$,

$$df_x\colon T_x(X) \to T_{f(x)}(Y)$$

is an isomorphism. Then f maps an open neighborhood of Z in X diffeomorphically onto an open neighborhood of $f(Z)$ in Y. If Z is a single point, this is just the usual IFT.

Proof: First of all, we know that f maps Z diffeomorphically onto its image $f(Z)$, since $f\colon Z \to f(Z)$ is a bijective local diffeomorphism and therefore a diffeomorphism. We would like to show that we can extend this to an open neighborhood around Z.

Since df_x is an isomorphism for all $x \in Z$, for each $x \in Z$, there exists an open neighborhood U_x in X around x on which $f_{|U_x}$ is a diffeomorphism. The collection $\{U_x\}$ is an open cover of Z. Since Z is **compact**, we can choose a **finite** subcover $\{U_1, \ldots, U_n\}$. We set $U : + \cup_i U_i$. Restricted to the open subset U, $f_{|U}$ is a local diffeomorphism.

Hence, by a previous exercise, it suffices to show that there is some **open** subset V in X which contains Z such that $f_{|V}$ is injective. Then $f_{|U\cap V}$ is an injective local diffeomorphism and therefore a diffeomorphism onto its image. Since $Z \subset U$ and $Z \subset V$, we also have $Z \subset U \cap V$ and the assertion is proven.

We are going to show that V exists by assuming the contrary and then show that this leads to a contradiction to the assumptions.

So suppose such a V does not exist. That means that there exists at least one point $z \in Z$ such that in any small open neighborhood W_i of z there are points a_i and b_i with $a_i \neq b_i$, but $f(a_i) = f(b_i)$. For otherwise, every point in Z had an open neighborhood on which f was injective, and we were done.

By choosing the W_i smaller and smaller around z_0 and by choosing subsequences a_j and b_j, we can assume that both the a_i and b_i converge to z. Since

$f(a_i) = f(b_i)$ for all i and f is continuous, we have $f(a_i) \to f(z)$ and $f(b_i) \to f(z)$. But since df_z is an isomorphism, the usual Inverse Function Theorem implies that there is a small open neighborhood W_z in X around z such that $f_{|W_z}$ is a diffeomorphism. Since $a_i \to z$ and $b_i \to z$, for N large enough, we have a_i,$b_i \in W_z$ and hence $f(a_i) = f(b_i) \in f(W_z)$ for all $i \geq N$. But since $f_{|W_z}$ is injective, this implies $a_i = b_i$ for all $i \geq N$. This contradicts the choice of the a_i and b_i. **QED**

As it is often the case, it is the existence of partitions of unity that allows us to move from the compact to the general case. We use the technique to show the following lemma:

Local finiteness lemma

An open cover $\{V_\alpha\}$ of a manifold X is called **locally finite** if each point of X possesses a neighborhood that intersects only finitely many of the sets V_α. Any open cover $\{U_\alpha\}$ admits a locally finite refinement $\{V_\alpha\}$.

Proof:

QED

Generalization of the IFT - general case

Let $f\colon X \to Y$ be a smooth map that is one-to-one on a submanifold Z of X. Suppose that for all $x \in Z$,

$$df_x\colon T_x(X) \to T_{f(x)}(Y)$$

is an isomorphism. Assume that f maps Z diffeomorphically onto $f(Z)$. Then f maps an open neighborhood of Z in X diffeomorphically onto an open neighborhood of $f(Z)$ in Y.

Proof: Since df_x is an isomorphism for all $x \in Z$, for each $x \in Z$, there exists an open neighborhood V_x in X around x on which $f_{|V_x}$ is a diffeomorphism. Let $U_x = f(V_x)$ be the open image in Y. The collection of all U_x is an open cover of $f(Z)$, since each $f(x) \in f(Z)$ lies in some $U_x = f(V_x)$. By the lemma above, we can choose a locally finite subcover $\{U_i\}$ of $f(Z)$ in Y. For each U_i, there is a local inverse $g_i\colon U_i \to X$ of f.

We define

$$W := \{y \in U_i : g_i(y) = g_j(y) \text{ whenever } y \in U_i \cap U_j\}.$$

On the subset W, we can define an inverse

$$g\colon W \to X,\ g(y) = g_i(y) \text{ for any } i.$$

This is well-defined by construction of W, since $g(y) = g_i(y) = g_j(y)$ whenever $y \in U_i \cap U_j$. Since the g_i's are local inverses of f, we have $f(Z) \subset W$.

It remains to show that W contains an open subset which still contains $f(Z)$. Let $x \in Z$, and hence $f(x) \in f(Z)$. Then $f(x)$ lies at least one U_k. We fix one such U_k with $f(x) \in U_k$. After shrinking U_j if necessary, we can assume yy the local finiteness of the cover $\{U_i\}$, that there are only finitely many of the U_i's which intersect U_k, say $U_1, \ldots, U_n$. If $U \subset W$, we are done, since then every point in $f(Z)$ has an open neighborhood which is contained in W.

If U is not contained in W, then, for $i = 1, \ldots, n$, we set C_{ik} be the **closure** of the set $\{y \in U_i \cap U_k : g_i(y) \neq g_k(y)\}$. Since the union of a **finite** union of closed subsets is closed, $C_k := C_{1k} \cup \cdots \cup C_{nk}$ is closed. Hence

$$U := U_k \setminus C_k$$

is open in Y.

By definition of W and the C_k, we know $U \subset W$. It remains to make sure that we $f(x)$ is still in U, i.e. that it does not beling to one of the closures C_{ik}.

Note that $f(x)$ satisfies $g_i(f(x)) = x = g_k(f(x))$ for all $i = 1, \ldots, n$. Since df_x is an isomorphism, the usual Inverse Function Theorem implies that there is a small open neighborhood $V_\epsilon \subset U$ around x such that $f_{|V_\epsilon}$ is a diffeomorphism. Hence, for each $i = 1, \ldots, n$, we have

$$g_i(f(x')) = x' = g_k(f(x')) \text{ for all } x' \in V_\epsilon \cap g_i(U_i) \cap g_k(U_k).$$

Hence the finite intersection $f(V_\epsilon) \cap U_k \cap U_1 \cap \cdots \cap U_n$ is an open which is not contained in any of the sets $\{y \in U_i \cap U_k : g_i(y) \neq g_k(y)\}$. Thus $f(x)$ is not contained in C_k. Therefore, $U \subset W$ is an open neighorhood of $f(x)$. **QED**