TMA4190 Introduction to Topology
Spring 2018

Norwegian University of Science and Technology
Deptartment of Mathematical Sciences

Solutions to exercise set 6

$\boxed{1}$ **a)** The tangent space to S^1 at z is $T_z(S^1) = \{v \in \mathbb{R}^2 : v \cdot z = 0\}$. The tangent space to N_z at z is $T_z(N_z) = \{(0, y) : y \in \mathbb{R}\} \subset \mathbb{R}^2$. Since $T_z(\mathbb{R}^2) = \mathbb{R}^2$, $T_z(S^1)$ and $T_z(N_z)$ span all of $T_z(\mathbb{R}^2)$ for all $z \neq (\pm 1, 0)$. Hence $S^1 \pitchfork N_z$ if and only if $z \neq (\pm 1, 0)$. (Drawing a picture explains everything.)

b) Which of the following linear spaces intersect transversally?

- The plane spanned by $\{(1, 0, 0), (2, 1, 1)\}$ and the y-axis in $\mathbb{R}^3$.
  Not transversal, since the y-axis lies in the span of $\{(1, 0, 0), (2, 1, 1)\}$.
- $\mathbb{R}^k \times \{0\}$ and $\{0\} \times \mathbb{R}^l$ in $\mathbb{R}^n$. (The answer depends on k, l, and n.)
  Transversal, if $k + l \geq n$.
- $V \times \{0\}$ and the diagonal in $V \times V$, for a real vector space V.
  Transversal, since they span all of $V \times V$: any $(v, w) \in V \times V$ is equal to the sum of $(w, w) \in \Delta_V$ and $(v - w, 0) \in V \times \{0\}$.
- The spaces of symmetric ($A^t = A$) and skew symmetric ($A^t = -A$) matrices in $M(n)$.
  Transversal, since every matrix in $M(n)$ can be written as a sum of a symmetric and an antisymmetric matrix:

$$C = \frac{1}{2}(C + C^t) + \frac{1}{2}(C - C^t) \text{ for any } C \in M(n).$$

c) $SL(n)$ and $O(n)$ do not meet transversally in $M(n)$, since $SO(n)$ is contained in $SL(n)$. Hence we also have $T_A(O(n)) = T_A(SO(n)) \subseteq T_A(SL(n))$, and these tangent spaces do not span all of $T_A(M(n)) = M(n)$.

$\boxed{2}$ **a)** Let $x \in f^{-1}(Z)$ and $y = f(x)$. We learned in the lecture that, locally around y, the submanifold Z is cut out by $k = \operatorname{codim} Z$ independent functions, i.e. there is an open neighborhood $V \subseteq Y$ around y and a smooth map $g \colon V \to \mathbb{R}^k$ such that $Z \cap V$ is defined by the vanishing of the g:

$$Z \cap V = \{v \in V : g(v) = 0\}.$$

Note that, by the Preimage Theorem, the tangent space $T_y(Z)$ is given by

$$T_y(Z) = \operatorname{Ker}(dg_y \colon T_y(Y) \to \mathbb{R}^k). \tag{1}$$

Furthermore, $f^{-1}(V)$ is an open neighborhood of x in X such that $f^{-1}(Z) \cap f^{-1}(V)$ is defined by the vanishing of $g \circ f$:

$$f^{-1}(Z) \cap f^{-1}(V) = (g \circ f)^{-1} \cap (f^{-1}(Z \cap V)) = \{u \in f^{-1}(Z \cap V) : g(f(u)) = 0\}.$$

Since f and Z are transversal, 0 is a regular value of $g \circ f$. Hence, by the Preimage Theorem, the tangent space to $f^{-1}(Z)$ at x is the subspace of $T_x(X)$ given by the kernel of $d(g \circ f)_x$.

$$\begin{aligned} T_x(f^{-1}(Z)) &= \operatorname{Ker}(d(g \circ f)_x) \\ &= \operatorname{Ker}(dg_y \circ df_x) \text{ (by the Chain Rule)} \\ &= (df_x)^{-1}(\operatorname{Ker}(dg_y)) \\ &= (df_x)^{-1}(T_y(Z)) \text{ (by (1) above).} \end{aligned}$$

b) We just need to apply the previous point to the inclusion $f = i \colon X \hookrightarrow Y$.

$\boxed{3}$ By definition, $\Gamma(A) \pitchfork \Delta$ if and only if $\Gamma(A) + \Delta(V) = V \times V$. Let (v_1, v_2) be an arbitrary element of $V \times V$. We need to check under which conditions we can find $v, w \in V$ such that

$$(v_1, v_2) = (v, v) + (w, Aw) = (v + w, v + Aw), \text{ i.e. } v_1 = v + w \text{ and } v_2 = v + Aw.$$

If we can find a suitable w, then we just set $v := v_1 - w$. Hence, by taking the difference of the two equations, we see that the question is reduced to checking whether we can find a w such that $v_2 - v_1 = Aw - w = (A - I)w$ where I is the identity map of V. But such a w exists for any choice of v_1 and v_2 if and only if $A - I$ is invertible, i.e. if and only if $\det(A - I) \neq 0$ which happens if and only if $+1$ is not an eigenvalue of A (because the eigenvalues are the λ such that $\det(A - \lambda I) = 0$).

$\boxed{4}$ Let $\Delta_X = \{(x, x) : x \in X\} \subseteq X \times X$ be the diagonal of X and $\Gamma(f) = \{(x, f(x)) : x \in X\} \subseteq X \times X$ be the graph of f. Then the set of fixed points of f in X is the intersection $\Delta_X \cap \Gamma(f)$. For

$$x = f(x) \iff (x, x) = (x, f(x)) \iff (x, x) \in \Gamma(f).$$

Recall that a 0-dimensional manifold is just a discrete set of points. Since X is compact, $X \times X$ is also compact. Hence a 0-dimensional submanifold is a discrete subset of the compact space $X \times X$ and is therefore finite. (We have used this before: discrete closed subspaces of compact spaces are finite. More generally, a comact discrete space is finite.)

Thus, in order to prove that f has only finitely many fixed points, it suffices to show that $\Delta_X \cap \Gamma(f)$ is a 0-dimensional submanifold of $X \times X$.

Hence we would like to show that Δ_X and $\Gamma(f)$ meet transversally in $X \times X$. Since both have codimension equal to $\dim X =: n$ in $X \times X$, the Transversality Theorem then implies that $\Delta_X \cap \Gamma(f)$ is a 0-dimensional submanifold of $X \times X$.

By definition, $\Delta_X \pitchfork \Gamma(f)$ means

$$T_{(x,x)}(\Gamma(f)) + T_{(x,x)}(\Delta_X) = T_{(x,x)}(X \times X)$$

for every point $(x, x) \in \Delta_X \cap \Gamma(f)$. We know $T_{(x,x)}(\Gamma(f)) = \Gamma(df_x)$ and $T_{(x,x)}(\Delta_X) = \Delta_{T_x(X)}$ by a previous exercise. Moreover, we know $T_{(x,x)}(X \times X) = T_x(X) \times T_x(X)$. Hence we need to show

$$\Gamma(df_x) + \Delta_{T_x(X)} = T_x(X) \times T_x(X) \tag{2}$$

for every point $(x, x) \in \Delta_X \cap \Gamma(f)$. This means exactly $\Gamma(df_x) \pitchfork \Delta_{T_x(X)}$ which we have shown to be true if $+1$ is not an eigenvalue of df_x in the previous exercise. Thus we can stop here, since f is Lefschetz by assumption.

But we could also just continue and give another proof as follows:

Since we know

$$\dim \Delta_{T_x(X)} = \dim \Gamma(df_x) = \dim T_x(X),$$

equality (2) will follow once we show $\Gamma(df_x) \cap \Delta_{T_x(X)} = \{0\}$. For then we have shown that $\Gamma(df_x) + \Delta_{T_x(X)}$ is a subspace of $T_x(X) \times T_x(X)$ of the same dimension as $T_x(X) \times T_x(X)$. (Recall that in a finite dimensional vector space V with subspaces U and W, the following dimension formula holds:

$$\dim U + \dim W = \dim(U + W) - \dim(U \cap W)$$

where $U + W \subseteq V$ is the subspace of V generated by U and W and $U \cap W$ is their intersection.)

Since f is a Lefschetz map, we know that for every fixed point x of f, $+1$ is not an eigenvalue of df_x. This means that df_x does not have any fixed points, for there is no $v \in T_x(X) \setminus \{0\}$ with $df_x(v) = 1 \cdot v$. As we observed for f above, this is equivalent to say that

$$\Gamma(df_x) \cap \Delta_{T_x(X)} = \{0\}.$$

$\boxed{5}$ We define the map

$$\begin{aligned} &f \colon \mathbb{C}^5 \setminus \{0\} \to \mathbb{C}, \\ &(z_1, \ldots, z_5) \mapsto z_1^2 + z_2^2 + z_3^2 + z_4^3 + z_5^{6k-1}. \end{aligned}$$

Setting $Z = f^{-1}(0)$, we need to show that Z and S^9 meet transversally. The tangent space to Z in a point $z \in Z$ is the kernel of the derivative df_z. Since f is a polynomial in the variables $z_1, \ldots, z_5$, we can use our usual rules for partial differentiation to get the following matrix for df_z (in the standard basis):

$$df_z \colon \mathbb{C}^5 \to \mathbb{C},\ df_z = \left(2z_1, 2z_2, 2z_3, 3z_4^2, (6k-1)z_5^{6k-2}\right).$$

Recall that we can represent every element $x + iy \in \mathbb{C}$ by the real 2×2-matrix $\begin{pmatrix} x & -y \\ y & x \end{pmatrix}$. Then we see that df_z is a real 2×10-matrix. Its maximal rank (as a matrix with entries in $\mathbb{R}$) is therefore 2. And, in fact, for every $z \neq 0$, df_z has rank 2, since it maps surjectively onto $\mathbb{C} \cong \mathbb{R}^2$. Thus 0 is a regular value for f and the tangent space $T_z(Z)$ is the kernel of df_z.

Writing a complex number $z = x + iy$, we can express S^9 as the fiber of the smooth map

$$\begin{aligned} &g \colon \mathbb{C}^5 \cong \mathbb{R}^{10} \to \mathbb{R}, \\ &(z_1, \ldots, z_5) \mapsto x_1^2 + y_1^2 + x_2^2 + y_2^2 + \cdots x_5^2 + y_5^2 - 1 \end{aligned}$$

at the regular value 0, i.e. $S^9 = g^{-1}(0) \subset \mathbb{C}^5 \cong \mathbb{R}^{10}$. The tangent space to S^9 at z is then given by the kernel of the derivative (in standard bases)

$$dg_z \colon \mathbb{C}^5 = \mathbb{R}^{10} \to \mathbb{R},\ dg_z = (2x_1, 2y_1, 2x_2, 2y_2, \ldots, 2x_5, 2y_5)\,.$$

Thus the tangent space $T_z(S^9)$ consists of all vectors w in $\mathbb{R}^{10}$ which are orthogonal to z, i.e. which satisfy $w \cdot z = 0$.

The tangent space of S^9 is of dimension 9 and the tangent space of $\mathbb{C}^5 \setminus \{0\}$ is of dimension 10. Hence in order to show that Z and S^9 meet transversally in $\mathbb{C}^5 \setminus \{0\}$ we need to show: For every $z \in Z \cap S^9$, there is at least vector w in $T_z(Z)$ which does not belong to $T_z S^9$. Then we have $T_z(Z) + T_z(S^9) \subseteq T_z(\mathbb{C}^5 \setminus \{0\})$ is a vector subspace of the same dimension as $T_z(\mathbb{C}^5 \setminus \{0\})$ and therefore equal $T_z(\mathbb{C}^5 \setminus \{0\})$.

So let $z = (z_1, \ldots, z_5)$ be a fixed point in $Z \cap S^9$. The tangent space $T_z(Z)$ is the kernel of df_z. Hence we need to find a vector $w \in \mathbb{C}^5 = \mathbb{R}^{10}$ with $df_z(w) = 0$ and $w \cdot z \neq 0$.

Set $m := 2 \cdot 3 \cdot (6k-1)$ and $w := (\frac{m}{2} z_1, \frac{m}{2} z_2, \frac{m}{2} z_3, \frac{m}{3} z_4, \frac{m}{6k-1} z_5)$. Then we have

$$\begin{aligned} df_z(w) &= \left(2z_1, 2z_2, 2z_3, 3z_4^2, (6k-1)z_5^{6k-2}\right) \cdot (m/2z_1, m/2z_2, m/2z_3, m/3z_4, m/(6k-1)z_5) \\ &= m\left(z_1^2 + z_2^2 + z_3^2 + z_4^3 + z_5^{6k-1}\right) \\ &= 0 \end{aligned}$$

since z is by assumption a point on Z. Hence $w \in T_z(Z)$.

Moreover, we have

$$\begin{aligned} w \cdot z &= (m/2z_1, m/2z_2, m/2z_3, m/3z_4, m/(6k-1)z_5) \cdot (z_1, z_2, z_3, z_4, z_5) \\ &= m/2z_1^2 + m/2z_2^2 + m/2z_3^2 + m/3z_4^2 + m/(6k-1)z_5^2 \\ &> 0. \end{aligned}$$

Thus w is a vector in $T_z(\mathbb{R}^{10})$ which is in $T_z(Z)$, but not in $T_z(S^9)$, and we have shown

$$T_z(Z) + T_z(S^9) = T_z(\mathbb{C}^5 \setminus \{0\}).$$

Hence Z and S^9 meet transversally in $\mathbb{C}^5 \setminus \{0\}$. The codimension of $Z \cap S^9$ in $\mathbb{C}^5 \setminus \{0\}$ is $2+1$ by the codimension formula. Thus $\dim(Z \cap S^9) = 10 - 3 = 7$.