Cheat Sheet

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1 Introduction

Theorem 1.1 (Brouwer fixed point theorem). *Let $f : D^n \to D^n$ be continious map from the (unit) disk in $\mathbb{R}^n$ to itself. Then f has a fixed point, i.e., there is some point $x \in D^n$ such that $f(x) = x$.*

Theorem 1.2 (The fundemental theorem of algebra). *A polynomial equation*

$$z^n + a_{n-1}z^{n-1} + \ldots + a_1 z + a_0 = 0$$

2 Continious maps

2.1 Metric spaces

Definition 2.1 (Metric spaces). *A metric psace (X, d) is a non-empty set X toghether with a map $d : X \times X \to \mathbb{R}$ called a metric such that the following properties hold:*

(i) ***M1*** $d(x, y) \geq 0$ *for all* $x, y \in X$, *and* $d(x, y) = 0$ *if and only if* $x = y$.

(ii) ***M2*** $d(x, y) = d(y, x)$ *for all* $x, y \in X$

(iii) ***M3*** $d(x, z) \leq d(x, y) + d(y, z)$ *for all* $x, y, z \in X$.

2.2 Continious maps between metric spaces

Definition 2.2 (Continious maps between metric spaces). *Let $d(X, d_X)$ and (Y, d_Y) be two metric spaces. A map $f : X \to Y$ is continious at $p \in X$ if for all $\varepsilon > 0$ there exists a $\delta > 0$ such that if $d_X(p, q) < \varepsilon$ then $d_Y(f(p), f(q)) < \varepsilon$. If f is continious at every point $p \in X$, we say that f is continious.*

Definition 2.3 (Open and closed balls). *Let $X(X, d)$ be a metric space, and let $a \in X$ and $r > 0$ be real number. The open ball centered at a with radius r is the subset*

$$B(a; r) = \{x \in X \mid d(x, a) < r\}$$

of X. The closed ball centered at a with radius r is the subset

$$\overline{B}(a; r) = \{x \in X \mid d(x, a) \leq r\}$$

of X.

Definition 2.4 (Open and closed sets). *Let (X, d) be a metric space. A subset $A \subseteq X$ is open in X if for every point $a \in A$, there exists an open ball $B(a; r)$ about a contained in A. We say that A is closed in X if the complement*

$A^c = X \setminus A = \{x \in X \mid x \notin A\}$ *is open*

Remark. Let $X = \{a, b, c\}$ and let $U = \{a, b\}$. Then if $\tau = \{X, \emptyset\}$, U is not open nor closed.

Lemma 2.1. *Let (X, d) be a metric space, $x \in X$ and $r > 0$ a real number. Then the open ball $B(x; r) \subseteq X$ is open in X, and the closed ball $\overline{B}(x; r) \subseteq X$ is closed in X.*

Definition 2.5 (Neightbourhoods)**.** *Let (X, d) be a metric space, A a subset of X and $x \in X$. We say that A is a neighbourhood of x if there is an open ball about x contained in A. We say that A is an open neighborhood (of x) if A itself is open.*

Theorem 2.1 (Continuity of a point)**.** *Let (X, d_X) and (Y, d_Y) be two metric spaces and let $p \in X$. A map $f : X \to Y$ is continious at p if and only if for all neighbourhoods B of $f(p)$, there is a neighbourhood A of p such that $f(A) \subseteq B$.*

Theorem 2.2 (Continious maps between metric spaces)**.** *Let (X, d_X) and (Y, d_Y) be two metric spaces. A map $f : X \to Y$ is continious if and only if for every subset $B \subseteq Y$ open in Y, the preimage of B under f,*

$$f^{-1}(B) = \{x \in X \mid f(x) \in B\} \subseteq X,$$

is open in X.

3 Topological spaces

3.1 Definitions and examples

Definition 3.1 (Topological spaces.)**.** *Recall that a topological space is a set X together with a collection Y of subsets of X that are open in X s.t.*

- ***T1**. $\emptyset, X \in \tau$*
- ***T2.** τ is closed under union if $U_\lambda \in \tau$ for all $\lambda \in \Lambda$, then*

$$\bigcup_{\lambda \in \Lambda} U_\lambda \in \tau$$

- ***T3.** τ is under finite intersections if $U_1, U_2, \ldots, U_n \in \tau$, then*

$$U_1 \cap U_2 \cap \ldots \cap U_n \in \tau$$

Theorem 3.1 (Metric spaces are topological spaces)**.** *Let (X, d) be a metrix space. Let τ_d be the collection of subsets $U \subseteq X$ with the property that $U \in \tau_d$ if and only if for each $x \in U$ there is an $r > 0$ such that $B(x; r) \subseteq U$. Then τ_d defines a topology on X.*

Theorem 3.2. *Let X be any set, and let d_1 and d_2 be two equivalent metrics on X. Then*

$$\tau_{d_1} = \tau_{d_2}.$$

Definition 3.2 (Comparable topologies)**.** *Let X be a set and suppose that τ_1 and τ_2 are two topologies on X . If $\tau_1 \subseteq \tau_2$, we say that τ_1 is coarser than τ_2 and that τ_2 is finar than τ_1. We say that τ_1 and τ_2 are comparable if either $\tau_1 \subseteq \tau_2$ or $\tau_2 \subseteq \tau_1$.*

3.2 Continious maps .

Theorem 3.3. ***Continuity between topological spaces.*** *Let X, Y be topological spaces. A map $f : X \to Y$ is said to be continious if preimages of open sets are open, i.e., if V is an open set in Y then the preimage $f^{-1}(V)$ of V is open in X.*

Theorem 3.4 (Composition of continious maps)**.** *Let X, Y and Z be topological sapces. If $f : X \to Y$ and $g : Y \to Z$ are continious maps, then the composite $g \circ f : X \to Z$ is continious.*

Definition 3.3 (Continuity at a point)**.** *Let X and Y be topological space, and let $x \in X$. A map $f : X \to Y$ is continiousat x if for all neighbourhoods V of $f(x)$ there is a neighbourhood U of x such that*

$$f(U) \subseteq V$$

Theorem 3.5. *Let X and Y be topological spaces. A map $f : X \to Y$ is continious if and only if it is continious at each $x \in X$.*

3.3 Homeomorhpism

Definition 3.4 (Homeomorphism)**.** *Let X and Y be topological spaces. A bijective map $f :\to Y$ with the property that both f and $f^{-1} : Y \to X$ are continious, is called a homeomorhpism. if there exists a homeomorhpism $f : X \to Y$, we say that X and Y are homeomorphic.*

Theorem 3.6. *Let X, Y and Z be topological spaces.*

(i) ***Reflexivity*** *: The identity map: $id : X \to X$ (where the domain and the codomain are equipped witht the same topology) , given by $id(x) = x$ for $x \in X$, is a homeomorphism.*

(ii) ***Symmentry*** *: If $f : X \to Y$ is a homeomorphis, then $f^{-1} : Y \to X$ is also a homeomorphism.*

(iii) ***Transitivity*** *: If $f : X \to Y$ and $g : Y \to Z$ are homeomorhism, then $g \circ f : X \to Z$ is also a homeomorphism.*

3.4 Closes sets

Definition 3.5 (Closed subsets)**.** *A subset K of a topological space X is closed in X if and only if the complement*

$$K^c = X \setminus K$$

is open in X .

Theorem 3.7. *Let X be a topological space.*

(i) Both $\emptyset$ and X are closed (as subsets) in X.

(ii) The intersection of any subcollection of closed sets in X is closed in X .

(iii) The union of any finite subcollection of closed sets in X is closed in X .

Definition 3.6 (Closure)**.** *Let X be a topological space, and let A be a subset of X. The closure of A, written $\overline{A}$, is the intersection of all subsets of X that constains A and which are closed in X .*

Definition 3.7 (Dense)**.** *Let X be a topological space, and let A be a subset of X . We say that A is dense in X if $\overline{A} = X$.*

Theorem 3.8. *Let $f : X \to Y$ be a map between the topological spaces. Then the following are equivalent:*

(i) f is continious.

(ii) for everry subset A of X, we have $f\left(\overline{A}\right) \subseteq \overline{f(A)}$.

(iii) for every closed subset B of Y, the preimage $f^{-1}(B)$ of B under f is closed in X .

4 Generating topologies

4.1 Generating topologies from subsets

Theorem 4.1 (The intersection of two topologies is a topology)**.** *Let X be a set, and let τ_1 and τ_2 be two topologies on X. Then $\tau_1 \cap \tau_2$ is also a topology on X.*

Definition 4.1 (Topology generated by a collection of subsets)**.** *Let X be a set, and let $\mathscr{S}$ be a collection of subsets of X. The topology generated by $\mathscr{S}$ is the topology*

$$\langle \mathscr{S} \rangle = \bigcap_{\substack{\tau \text{ topology} \\ S \subseteq \tau}} \tau$$

4.2 Basis for a topology

Definition 4.2 (Basis)**.** *Let X be a set. a **basis** for a topology on X is a collection $\mathscr{B}$ of subsets of X such that*

- ***B1**: for each $x \in X$, there is a $B \in \mathscr{B}$ such that $x \in B$*
- ***B1:** if B_1, B_2 and $x \in B_1 \cap B_2$, then there is a $B_3 \in B$ such that $x \in B_3 \subseteq B_1 \cap B_2$.*

Theorem 4.2. *Let X be a set, and let $\mathscr{B}$ be basis for a topology on X. The collection τ generated by $\mathscr{B}$ of subsets U of X with the property that for each $x \in U$ there is a basis element $B \in \mathscr{B}$ with $x \in B \subseteq U$ is a topology on X.*

Theorem 4.3. *Let X be a set, and let $\mathscr{B}$ be a basis for a topology τ on X. Then τ is equal to the collection of all unions of elements of $\mathscr{B}$.*

Theorem 4.4. *Let X be a set, and let $\mathscr{B}_1$ and $\mathscr{B}_2$ be bases for topologies τ_1 and τ_2 , respectively , on X. Then the following are equivalent.*

(i) τ_2 is finer than τ_1, i.e., $\tau \subseteq \tau_2$.

(ii) For each $B_1 \in \mathscr{B}_1$ and each $x \in B_1$, there is a $B_2 \in \mathscr{B}_2$ such that $x \in B_2 \subseteq B_1$.

4.3 Subbasis for a topology

Definition 4.3 (Subbasis)**.** *Let X be a set. A **subbasis** for a topology on X is a collection $\mathscr{S}$ whose union equals X.*

Lemma 4.1. *Let X be a set, and let $\mathscr{S}$ be a subbasis for a topology on X. The collection $\mathscr{B}$ consisting of all finite intersections of elements of $\mathscr{S}$ is a basis for a topology on X and is called the basis associated to $\mathscr{S}$.*

Definition 4.4 (Standard topology)**.** *(Not in compendium.) The standard topology on $\mathbb{R}$ is the topology generated by a basis consisting of all open intervals of $\mathbb{R}$.*

Lemma 4.2. *Let X be a set, and let $\mathscr{S}$ be a subbasis for a topology on X. The collection τ generated by $\mathscr{S}$ consisting of all unions of all basis elements of the associated basis $\mathscr{B}$ is a topology on X.*

Theorem 4.5. *Let X be a set, and let $\mathscr{S}$ be a subbasis for a topology on X. Then there exists a unique topology $\langle \mathscr{S} \rangle$ generated by $\mathscr{S}$ which is smaller than any other topology containing $\mathscr{S}$, where*

$$\langle \mathscr{S} \rangle = \left\{ \bigcup_{\lambda \in \Lambda} \bigcap_{i=1}^{n_\lambda} S_{\lambda,i} \mid S_{\lambda,i} \in \mathscr{S} \right\}$$

Theorem 4.6. *Let X and Y be topological spaces, and let $\mathscr{B}$ (resp., $\mathscr{S}$) be a basis (resp., subbasis) . Then a map $f : X \to Y$ is continious if and only if for each $B \in \mathscr{B}$ (resp. $S \in \mathscr{S}$) the preimage $f^{-1}(B)$ (resp., $f^{-1}(S)$) is open in X .*

5 Constructing topological spaces

5.1 Subspaces

Definition 5.1 (Subspace topology)**.** *Let X be a topological space, and let A be a subset of X . The collection*

$$\tau_A = \{A \cap U \mid U \text{ is open in } X\}$$

of subets of A is called the topology on A.

Lemma 5.1. *Let X be a topological space, and let A be a subsets of X. Then the collection*

$$\tau_A = \{A \cap U \mid U \text{ is open in } X\}$$

is a topology on A.

Theorem 5.1. *Let X be a topological space, and let $\mathscr{B}$ be a basis for the topology on X. If A is a subset X , the collection*

$$\mathscr{B}_A = \{A \cap B \mid B \in \mathscr{B}\}$$

is a basis for the subsapace topology on A.

Theorem 5.2. *Let X be a topological space, and let A be a subset of X . Then the subspace topology on A is the only topology on A witht the following universal property: for every topological space Y and every map :*

$$f : Y \to A$$

f is continious if and only if $i \circ f : Y \to X$ is continious where $i : A \to X$ is the inclusion map given by $i(x) = x$ for $x \in A$.

5.2 Products

Definition 5.2 (Product topology)**.** *Let X and Y be topological spaces. THe product topology on $X \times Y$ is the topology generated by the basis*

$$\mathscr{B} = \{U \times V \mid U \text{ is open in } X \text{ and } V \text{ is open in } Y\}$$

Lemma 5.2. *Let X and Y be topological spaces. Then the collection*

$$\mathscr{B} = \{U \times V \mid U \text{ is open in } X \text{ and } V \text{ is open in } Y\}$$

is a basis for a topology on $X \times Y$.

Theorem 5.3. *Let X and Y be topological paces. If $\mathscr{B}_X$ is a basis for a the topology on X and $\mathscr{B}_Y$ is a basis for the topology on Y, then the collection*

$$\mathscr{B}_{X \times Y} = \{B_X \times B_Y \mid B_X \in \mathscr{B}_X \text{ and } B_Y \in \mathscr{B}_Y\}$$

is a bsis for the product topology on $X \times Y$.

Theorem 5.4. *Let X and Y be topological spaces. Let $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ be the projections of $X \times Y$ onto its first and second factors, respectively. The product topology is the only topology on $X \times Y$ with the following universial property: for every topological space Z and every map $f : Z \to X \times Y$, f is continious if and only if $\pi_1 \circ f : Z \to X$ and $\pi_2 \circ f : Z \to Y$ are continious.*

5.3 Quotient spaces

Definition 5.3 (Equivalence classes)**.** *Let X be a set, and let $\sim$ be an equivalence relation on X . The equivalence class of $x \in X$ is the subset*

$$[x] = \{y \in X \mid x \sim y\}$$

of X . Let

$$X/\sim = \{[x] \mid x \in X\}$$

Lemma 5.3. *Let X and A be sets, and let $\pi : X \to A$ be a surjective map. Then the map*

$$\phi : X/\sim \to A$$

given by $\phi([x]) = \pi(x)$, where $x_1 \sim x_2$ if and only if $\pi(x_1) = \pi(x_2)$, is a bijection.

Definition 5.4 (Quotient space)**.** *Let X be a topological space, let A be a set, and elt $\pi : X \to A$ be a surjective map. The quotient topology on A induced by π is the collection of subsets U of A such that $\pi^{-1}(U)$ is open in X . We say* *that π is a quotient map if A is given the quotient topology, and we call A the quotient space.*

Lemma 5.4. *Let X be a topological space, let A be a set, and let $\pi : X \to A$ be a surjective map. Then the quotient topology on A induced by π is a topology and it is the finest topology on A such that π is continious.*

Definition 5.5 (Open and closed maps)**.** *Let X and Y be topological spaces, and let $f : X \to Y$ be a continious map. We say that f is an open map for each sucbset U of X that is open in X the image $f(U)$ is open in Y. Likewise, we say that f is a closed map if for each subset V of X that is closed in X the image $f(V)$ is closed in Y.*

Lemma 5.5. *Let X and Y be topological spaces, and let $\pi : X \to Y$ be a surjective continious map.*

(i) If π is in addition open then it is a quotient map.

(ii) If π is in addition closed then it is a quotient map.

Theorem 5.5. *Let X be a topological space, let A be a set, and let $\pi : X \to A$ be a surjective map. The quotient topology is the only topology on A with the following universal property: for every topological space Y and every map $f : A \to Y$, f is continious if and only if $f \circ \pi : X \to Y$ is continious.*

6 Topological properties

6.1 Connected spaces

Definition 6.1 (Connected space)**.** *Let X be a topological space. A **seperation** of X is a pair of non-empty subsets U and V that are open in X, disjoint and whose union equal X. We say that X is **connected** if there are no seperations of X. Otherwise it is **disconnected.***

Theorem 6.1 (Closed and open subsets)**.** *Let X be a topological space. Then X is connected if and only if the are no non-empty proper subsets of X that are both open and closed in X.*

Lemma 6.1 (Disconnectivity)**.** *Let X be a disconnected space with seperation U and V, and et A be a connected subspace of X. Then $A \subseteq U$ and $A \subseteq V$.*

Theorem 6.2 (Collection connectivity)**.** *Let X be a topological space, and let $\{A_\lambda\}_{\lambda \in \Lambda}$ be a collection of connected subspaces of X such that $\bigcap_{\lambda \in \Lambda} A_\lambda$ is non-empty. Then $\bigcup_{\lambda \in \Lambda} A_\lambda$ is connected.*

Definition 6.2 (Path connected space)**.** *Let X be a topological space, and let $x, y \in X$. A path from x to y is a continious map: $f : [a, b] \to X$ .t. $f(a) = x$ and $f(b) = y$ where $[a, b]$ is a subspace of $\mathbb{R}$ with the standard topology. We say that X is **path connected** if every pair of points of X can be joined by a path in X.*

Theorem 6.3 (Connectivity in product spaces)**.** *Let $X_1, X_2, \ldots, X_n$ be connected spaces. Then the product space $X_1 \times X_2 \times \ldots \times X_n$ is connected.*

Theorem 6.4 (The real numbers are connected)**.** *Let $\mathbb{R}$ be the set of real numbers equipped with the standard topology. Then $\mathbb{R}$ is connected.*

Theorem 6.5 (Generalized intermediate value theorem)**.** *Let X be a connected space and let $f : X \to \mathbb{R}$ be a continious map where $\mathbb{R}$ is given the standard topology. If $a, b \in X$ and if r is a real number that lies between $f(a)$ and $f(b)$, there is a $c \in X$ such that $f(c) = r$*

Theorem 6.6 (Connectivity)**.** *Let X be a topological space. Then X is connected if and only if the are no non-empty proper subsets of X that are both open and closed.*

Theorem 6.7 (Path connectedness implies connectedness)**.** *Let X be a path connectedness space. Then X is connected.*

6.2 Hausdorff spaces

Definition 6.3 (Hausdorff)**.** *Let X be a topological space. We say that X is **Hausdorff** if for each part of points $x, y \in X$ with $x \neq y$, there are disjoint neighborhoods U and V of x and y, respectively. In other words, for each pair of distinct point $x, y \in X$ there are open subsets U and V of X with $x \in U$ $y \in V$ where $U \cap V = \emptyset$*

Theorem 6.8. *Every metric space is Hausdorff*

Theorem 6.9. *Let X be a Hausdorff space. Then for each $x \in X$ the subset $\{x\}$ of X is closed in X.*

Theorem 6.10. *Let $X_1, X_2 \ldots, X_n$ be Hausdorff spaces. Then the product space $X_1 \times X_2 \times \ldots \times X_n$ is Hausdorff.*

Theorem 6.11. *Let X be a topological space. Then X is Hausdorff if and only if the diagonal*

$$\Delta = \{(x, x) \mid x \in X\}$$

is closed in the product space $X \times X$.

6.3 Compact spaces

Definition 6.4 (Cover of a space)**.** *Let X be a topological space, and let $\mathscr{A}$ be the collection of subsets of X. We say that $\mathscr{A}$ is a cover of X, or covering of X if $X = \bigcap_{A \in \mathscr{A}} A$. If A is also open in X for each $A \in \mathscr{A}$, we say that $\mathscr{A}$ is an **open** cover of X, or open covering of X. We say that $\mathscr{A}'$ is a subcover of $\mathscr{A}$ if $\mathscr{A}'$ is another cover of X that satisfies $\mathscr{A}' \subseteq \mathscr{A}$.*

Definition 6.5 (Compact spaces)**.** *Let X be a topological space. We say that X is **compact** if every open cover $\mathscr{A}$ of X contains a finite subcover.*

Definition 6.6 (Compact subspaces)**.** *Let X be a topological space, and let A be a subset of X. We say that A is compact in X if A is compact in the subspace topology.*

Lemma 6.2. *Let X be a topological space, and let A be a subspace of X. Then A is compact in X if and only if every cover of A by open subsets of X contains a finite subcollection that covers A.*

Theorem 6.12. *Let X be a compact space, and let A be a closed subset of X. Then A is compact in X.*

Theorem 6.13. *Let X be a Hausdorff space, and let K be a subset of X which is compact in X. Then K is closed in X.*

Theorem 6.14. *Let X be a compact space, Y a topological space and let $f : X \to Y$ be a surjective continious map. Then Y is compact.*

Lemma 6.3 (Tube lemma)**.** *Let X be a topological space, and let Y be a compact space. If $x \in X$ and U is an oppen set in the product space $X \times Y$ containing $\{x\} \times Y$, then there is a neighborhood W of x in X such that $W \times Y \subseteq U$*

Theorem 6.15. *Let $X_1, X_2, \ldots, X_n$ be compact spaces. Then the product space $X_1 \times X_2 \times \ldots \times X_n$ is compact.*

Theorem 6.16. *Let $\mathbb{R}$ be the set of real numbers equipped with the standard topology. Then every closed interval $[a, b] \in \mathbb{R}$ is compact in $\mathbb{R}$.*

Definition 6.7 (Bounded subsets)**.** *Let (X, d) be a metric space, and let A be a subset of X. We say that A is bounded if there is an $M \in \mathbb{R}$ such that $d(a_1, a_2) \leq M$ for all $a_1, a_2 \in A$.*

Theorem 6.17 (Heine- Borel)**.** *Let $\mathbb{R}^n$ be given the (Euclidian) metric topology and the Euclidian metric. A subset A of $\mathbb{R}^n$ if and only if it is closed and bounded.*

Theorem 6.18 (Generalized extreme value theorem)**.** *Let X be compact space, and let $f : X \to \mathbb{R}$ be a continious map where $\mathbb{R}$ is given the standard topology. Then there are $m, M \in X$ such that*

$$f(m) \leq f(x) \leq f(M)$$

for all $x \in X$.

7 The fundamental group

7.1 Homotopy of paths

Definition 7.1 (Homotopy)**.** *Let X and Y be topological spaces, and let $f_0, f_1 : X \to Y$ be two continious maps. Furthermotre, let $\mathbb{R}$ be the set of real numbers with the standard topology, $I = [0, 1]$ be a subsapce of $\mathbb{R}$, and let $X \times I$ be the given topology. We say that f_0 is homotopic to f_1 , written $f_0 \simeq f_1$, if there is a continious map*

$$H : X \times I \to Y$$

such that $H(x,0) = f_0(x)$ and $H(x,1) = f_1(x)$ for all $x \in X$. The map H is called a homotopy between f_0 and f_1 . If $f_0 \simeq f_1$ and f_1 is a constant map, we say that f_0 is nullhomotopic.

Lemma 7.1 (Pasting lemma)**.** *Let $X = A \cup B$ be a topological space where A and B are closed in X. Furthermore, et Y be a topological space, and assume that $f : A \to Y$ and $g : B \to Y$ are continious maps. If $f(x) = g(x)$ for all $x \in A \cap B$, then the map $h : X \to Y$ given by*

$$h(x) \begin{cases} f(x), & x \in A \\ g(x), & x \in B \end{cases}$$

is continious.

Theorem 7.1. *The relation $\simeq$ is an equivalence relation on the set of all continious maps from a topological space X to a topological space Y .*

Definition 7.2 (Homotopy classes)**.** *Let X and Y be topological spaces, and let $C(X,Y)$ be the set of continious maps from X to Y. The homotopy classes in $C(X,Y)$ are the equivalence classes under the relation $\simeq$. We write $[f]$ for the homotopy class of $f \in C(X,Y)$, i.e.,*

$$[f] = \{g \in C(X,Y) \mid f \simeq g\}$$

and we write $[X,Y]$ for the set of homotopy classes of continious maps from X to Y , i.e.,

$$[X,Y])C(X,Y) \setminus \simeq$$

Definition 7.3 (Path homotopy)**.** *Let X be a topologicala space, and let $x_0, x_1 \in X$. We say that two paths $f, g : I \to X$ in X from x_0 to x_1 are path homotopic, written $f \simeq_p g$, if there is a continious map $F : I \times I \to X$ such that*

$$H(s,0) = f(s) \quad \text{and} \quad H(s,1) = g(s)$$

for all $s \in I$, and

$$H(0,t) = x_0 \quad \text{and} \quad H(1,t) = x_1$$

for all $t \in I$. We call H a path homotopy from f to g .

Theorem 7.2. *Let X be a topological space, and let $x_0, x_1 \in X$. Then the relation $\simeq_p$ is an equivalence relation on the set of all paths from x_0 to x_1 in X .*

Definition 7.4 (Path homotopy classes)**.** *Let X be a topological space, and let $x_0, x_1 \in X$. If $f : I \to X$ is a path from x_0 to x_1, we write $[f]$ for its path homotopy class, i.e.,*

$$[f] = \{g : I \to X \mid g \text{ is a path from } x_0 \text{ to } x_1 \text{ and } f \simeq_p g\}$$

Definition 7.5 (Product of paths)**.** *Let X be a topologicala space, and let $x_0, x_1, x_2 \in X$. If $f : I \to X$ is a path from x_0 to x_1 , and $g : I \to X$ is a is a path from x_1 to x_2, we define the product of f and g as the path $f * g : I \to X$ from x_0 to x_2 given by*

$$(f * g)(s) = \begin{cases} f(2s), & 0 \le s \le \frac{1}{2} \\ g(2s-1), & \frac{1}{2} \le s \le 1. \end{cases}$$

Lemma 7.2. *Let X be a topological space, and let $x_0, x_1, x_2 \in X$. If $f : I \to X$ is a path from x_0 to x_1 and $g : I \to X$ is a path from x_1 to x_2, then the product $f * g$ induses a well-defined operation on path homotopy classes given by*

$$[f] * [g] = [f * g]$$

Theorem 7.3. *Let X be a topological space. Then the product paths, $*$, has the following properties on the set of path homotopy classes.*

(i) ***Associativity.*** *Let x_0, x_1, x_2 and x_3 be the points in X . If $f_0 : I \to X$ is a path from x_0 to x_1, $f_1 : I \to X$ is a path from x_1 to x_2, and $f_2 : I \to X$ is a path from x_2 to x_3, then*

$$([f_0] * [f_1]) * [f_2] = [f_0] * ([f_1] * [f_2])$$

(ii) ***Left and right units*** . *For $x \in X$, let $c_x : I \to X$ denote the constant path at x , given $c_x(s) = x$ for all $s \in I$. If $f : I \to X$ is a path from x_0 to x_1 then*

$$[c_{x_0}] * [f] = [f] = [f] * [c_{x_0}]$$

(iii) ***inverse*** *If $f : I \to X$ is a path from x_0 to x_1, let $\overline{f} : I \to X$ be the reverrse path from x_1 to x_0 , given by $\overline{f}(s) = f(1-s)$ for all $s \in I$. Then*

$$[f] * \left[\overline{f}\right] = [c_{x_0}] \text{ and } \quad \left[\overline{f}\right] * [f] = [c_{x_1}]$$

7.2 Definition and elementary properties of the fundamental group

Definition 7.6 (The fundemental group)**.** *Let (X, x_0) be a based space. A path $f : I \to X$ from x_0 to x_0 is called a loop in X based at x_0 . Let*

$$\pi_1(X, x_0) = \{[f] \mid f \text{ is a loop in } X \text{ based at } x_0\}$$

be the set of path hotopy classes of loops in X at x_0 . We say that $\pi_1(X, x_0)$ is the fundemental group of X based at x_0 .

Theorem 7.4. *Let (X, X_0) be a based space. THen the fundementa group $\pi_1(X, x_0)$ of X based at x_0 is, in fact, a group with product paths, $*$, as its binary operation. The identity element e is equal to the path homotopy class of the constant path at x_0, $e = [c_{x_0}]$, and the inverse of $[f]$ is $[f]^{-1} = \left[\overline{f}\right]$, where $\overline{f}$ is the reverse path of f .*

Theorem 7.5. *Let X be a path connected space, and let $x_0, x_1 \in X$. Then $\pi_1(X, x_0)$ is isomorphic to $\pi_1(X, x_1)$.*

Definition 7.7 (Simply connected spaces)**.** *Let X be a path connected sace. We say that X is simply connected if $\pi_1(X, x_0)$ is the trivial group for some $x_0 \in X$, and hence, for all $x_0 \in X$.*

Definition 7.8 (Based maps)**.** *Let (X, x_0) and (Y, y_0) be based spaces. A based map*

$$h : (X, x_0) \to (Y, y_0)$$

is a continious map $h : X \to Y$ such that $h(x_0) = y_0$.

Definition 7.9 (Homomorphism induced by based maps). *Let (X, x_0) and (Y, y_0) be based spaces, and let $h : (X, x_0) \to (Y, y_0)$ be based map. The map*

$$h_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$$

given by

$$h_*([f]) = [h \circ f]$$

is called the homomorphism indeiced by h.

Lemma 7.3. *Let (X, x_0) and (Y, y_0) be based spaces, and let $h : (X, x_0) \to (Y, y_0)$ be a based map. The map*

$$h_* : \pi_1(X, x_0) \to \pi_2(Y, y_0).$$

given by

$$h_*([f]) = [h \circ f]$$

is a homomorphism.

Theorem 7.6 (Functoriality). *Let (X, x_0) , (Y, y_0) and (Z, z_0) be based spaces, and let $h_2 : (X, x_0) \to (Y, y_0)$ and $h_2 (Y, y_0) \to (Z, z_0)$ be based maps. Then*

$$(h_2 \circ h_1) = (h_2)_* \circ (h_1)_*.$$

If $id_X : X \to X$ is the identity map, then $(id_X)_*$ is the identity automorphism of $\pi_1(X, x_0)$.

Corollary 7.1. *Let (X, x_0) and (Y, y_0) be based spaces. If $h : X \to Y$ is a homeomorphism such that $h(x_0) = y_0$, then*

$$h_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$$

is an isomorphism.

Theorem 7.7. *Let (X, x_0) and (Y, y_0) be based spaces. Then $\pi_1(X \times Y, (x_0, y_0))$ is isomorphic to the direct product $\pi_1(X, x_0) \times \pi_1(Y, y_0)$.*

7.3 Homotopy type

Lemma 7.4. *Let (X, x_0) and (Y, y_0) be based spacaes, and let $h : (X, x_0) \to (Y, y_0)$ and $k : (X, x_0) \to (Y, y_0)$ be based mmaps. If there is a homotopy $H : X \times I \to Y$ from h to k such that $H(X_0, t) = y_0$. for all $t \in I$, then the homomorphism $h_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ and $k_* : \pi_1 : (X, x_0) \to \pi(Y, y_0)$ induced by h and k , respectively , are equal.*

Definition 7.10 (Retractions). *Let X be topological space, and let A be subspace of X. We say that a continious map $r : X \to A$ is a retraction of X onto A if $r(a) = a$ for each $a \in A$. If there is a retraction of X onto A, we say that A is retract of X .*

Lemma 7.5. *Let X be topological space, and let A be a subspace of X. If $x_0 \in A$ and A is a retract of X. THen the homomorphism $i_* : \pi_1(A, x_0) \to \pi_1(X, x_0)$ induced by the inclusion map $i : A \to X$ is a monomorphism.*

Definition 7.11 (Deformation retracts). *Let X be a topological space, and let A be a subspace of X. A homotopy*

$$H : X \times I \to X$$

is called deformation retraction of X onto A if $H(x, 0) = x$ and $H(x, 1) \in A$ for all $x \in X$, and $H(a, t) = a$ for all $a \in A$ and all $t \in I$. We say that A is a deformation tract of X .

Theorem 7.8. *Let X be a topological space, and let A be a subspace of X . If $x_0 \in A$ and A is a deformation retract of X, then the homomorphism $i_* : \pi_1(A, x_0) \to \pi(X, x_0)$ indeuced by the inclusion map $i : A \to X$ is an isomorphism.*

Definition 7.12 (Homotopy equivalences). *Let X and Y be topological spaces. If $f : X \to Y$ and $g : Y \to X$ are continious maps such that $g \circ f$ is homotopic to the identity map of X, id_X, and $f \circ g$ is homotopic to the identity map of Y, id_Y , we say that f and g are homotopy equivalences. We say that each of f and g is a homotopy inverse of the other.*

Definition 7.13 (Homotopy types). *Let X and Y be topological spaces. We say that X and Y have the same homotopy type if there is a homotopy equivalence $f : X \to Y$.*

Lemma 7.6. *Let X and Y be topoplogical spaces, and let $f : X \to Y$ and $g : X \to Y$ be continious maps such that $f(x_0) = y_0$ and $g(x_0) = y_1$. if $H : X \times I \to Y$ is a homotopy from f to g, there is a path $\alpha : I \to Y$ in Y from y_0 to y_1 fiven by $\alpha(t) = H(x_0, t)$ such that $g_* = \hat{\alpha} \circ f_*$*

Theorem 7.9. *Let X and Y be topological spaces, and let $f : X \to Y$ be homotopy equivalence such that $f(x_0)\, 0 y_0$. tThen*

$$f_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$$

is an isomorphism.

8 The fundamental group of the circle

9 References

References