



Please justify your answers! The most important part is *how* you arrive at an answer, not the answer itself.

- 1 Consider the exponential basis  $\{e^{2\pi i n t} : n \in \mathbb{Z}\}$  in  $(L^2[0, 1], \langle \cdot, \cdot \rangle)$ . Show that these are orthonormal vectors.
- 2 Let  $\{e_n : n \in \mathbb{N}\}$  be the standard basis in the sequence space  $\ell^p$ .
  - a) Show that the series  $\sum_{n=1}^{\infty} \alpha_n e_n$  converges in  $\ell^p$  for  $1 \leq p < \infty$  if and only if  $(\alpha_n)_{n \in \mathbb{N}} \in \ell^p$ .
  - b) Show that the series  $\sum_{n=1}^{\infty} \alpha_n e_n$  converges in  $\ell^\infty$  if and only if  $(\alpha_n)_{n \in \mathbb{N}}$  converges to zero.
- 3 Show that if a normed space  $(X, \|\cdot\|)$  has a Schauder basis, then it is separable.
- 4 Let  $L^2[-1, 1]$  be equipped with the inner product

$$\langle f, g \rangle = \int_{-1}^1 f(t) \overline{g(t)} dt.$$

Apply Gram-Schmidt's orthogonalization algorithm to the monomial basis  $\{1, x, x^2, x^3, \dots\}$  up to degree 2.

- 5 (*Exam 2014, Problem 3*)
  - a) Let  $C([0, 2] \times [0, 2], \mathbb{R})$  be an inner-product space with

$$\langle f, g \rangle = \int_0^2 \int_0^2 f(x, y) g(x, y) dx dy.$$

Find an orthogonal basis for  $\text{span}\{1, x, y\}$  in this space.

b) Find  $a, b, c \in \mathbb{R}$  such that

$$\int_0^2 \int_0^2 |xy - a - bx - cy|^2 dx dy$$

is minimal.

- 6** Let  $\|\cdot\|_a$  and  $\|\cdot\|_b$  be equivalent norms on a vector space  $X$ . Show that a sequence  $(x_n)$  in  $X$  is Cauchy with respect to the norm  $\|\cdot\|_a$  if and only if it is Cauchy with respect to the norm  $\|\cdot\|_b$ .