orbits : $\mathcal{O}(x(0)) = \{ \phi_t(x(0)) : t \in \mathbb{R} \}$

~~are~~ ~~or~~ either coincide or are disjoint.

A i) Two distinct real eigenvalues

$0 < \lambda_1 < \lambda_2$

$$\dot{x} = 2x + 2y \qquad \underline{A} = \begin{pmatrix} 2 & 2 \\ -1 & 5 \end{pmatrix}$$
$$\dot{y} = -x + 5y$$

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix}$$

find eigenvalues:

$$0 = \det(A - \lambda I) = (\lambda-2)(\lambda-5)+2 = (\lambda-3)(\lambda-4)$$

eigenvectors:

$\underline{\lambda=3}$

$$\cancel{\begin{pmatrix} 2-3 & 2 \\ -1 & 5 \end{pmatrix}} \qquad \begin{pmatrix} 2-3 & 2 \\ -1 & 5-3 \end{pmatrix}\begin{pmatrix} r^1 \\ r^2 \end{pmatrix} = 0$$

$$v_1 = \begin{pmatrix} r^1 \\ r^2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$\underline{\lambda=4}$

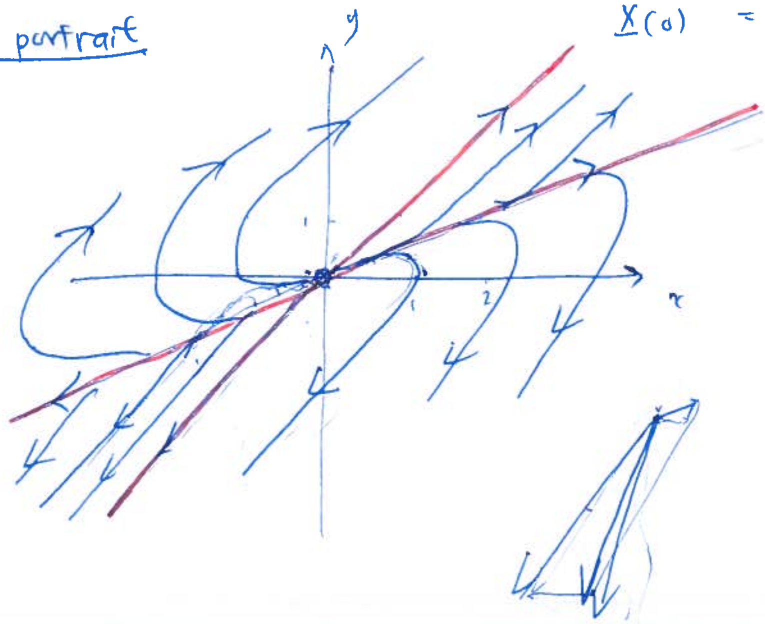
$$v_2 = \begin{pmatrix} r^1 \\ r^2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

<u>general solution</u>

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \underline{X}(t) = c_1 e^{3t}\begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2 e^{4t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

<u>phase portrait</u>

$$\underline{X}(0) = c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$



UNSTABLE NODE

~~FOCUS~~

Ⓐ ii) $\lambda_1 < \lambda_2 < 0$

$$\dot{x} = -3x + y \qquad \dot{y} = x - 3y \longrightarrow \frac{d}{dt}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} -3 & 1 \\ 1 & -3 \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$$

finding eigenvalues

$$0 = \det(A - \lambda I) = (\lambda+2)(\lambda+4)$$

finding eigenvectors

$\lambda = -2$

$$\begin{pmatrix} -3+2 & -1 \\ 1 & -3+2 \end{pmatrix} \begin{pmatrix} r^1 \\ r^2 \end{pmatrix} = 0 \longrightarrow v_1 = \begin{pmatrix} r_1 \\ r_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$ is an eigenvector

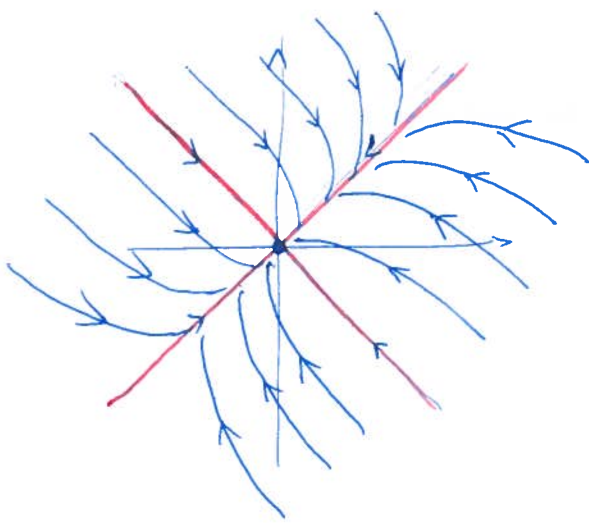
$\lambda = -4$

$v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is an eigenvector.

general solution

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = C_1 e^{-2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2 e^{-4t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

phase portrait



STABLE NODE

(iii) $\lambda_1 < 0 < \lambda_2$

$$\dot{x} = x - 2y$$
$$\dot{y} = -3x + 2y$$

$$\hookrightarrow A = \begin{pmatrix} 1 & -2 \\ -3 & 2 \end{pmatrix}$$

$$0 = \det(A - \lambda I) = (\lambda - 4)(\lambda + 1)$$

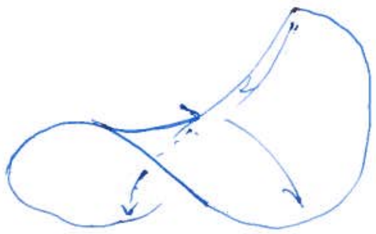
$\underline{\lambda = 4}$ $\quad v_1 = \begin{pmatrix} r^1 \\ r^2 \end{pmatrix} = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$

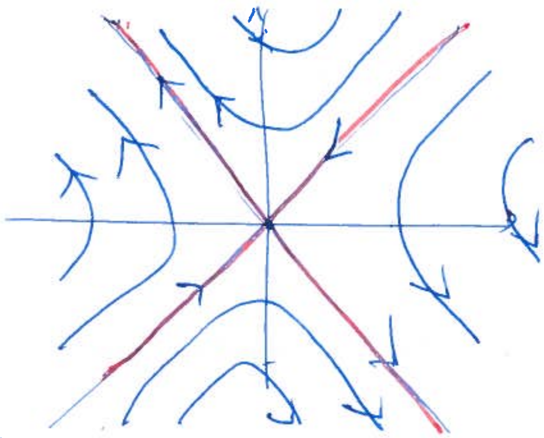
$\underline{\lambda = -1}$ $\quad v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

<u>general solution</u>.

$$\begin{pmatrix} x \\ y \end{pmatrix}(t) = c_1 e^{4t} \begin{pmatrix} -2 \\ 3 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

<u>phase portrait</u>:





SADDLE

(B) Root with degenerate multiplicity $\lambda \neq 0$

$$\dot{x} = x + y$$
$$\dot{y} = y \quad \rightarrow A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$0 = \det(A - \lambda I) = (\lambda - 1)^2.$$

~~Jordan~~ el eigenvector : $v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

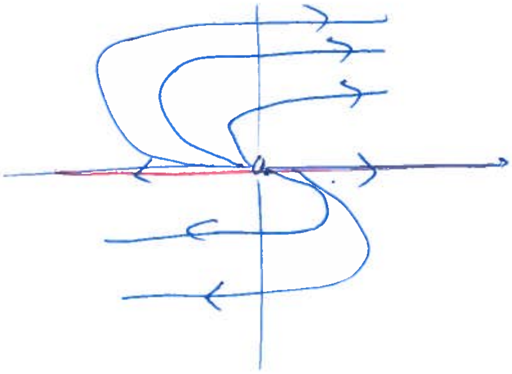
<u>Jordan chain</u>.

$$(A - \lambda 1) v_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

<u>general solution</u>

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = (c_1 + c_2 t) e^t \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 e^t \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$



when $x=0$, $y>0$

$\dot{x}>0$

when $x=0$, $y<0$

$\dot{x}<0$

DEGENERATE UNSTABLE NODE

(c) Conjugate roots

$\Re\lambda>0$

$$\dot{x} = 3x-2y \qquad \dot{y} = 2x+3y \longrightarrow A = \begin{pmatrix} 3 & -2 \\ 2 & 3 \end{pmatrix}$$

<u>finding eigenvalues</u>

$$0 = \det(A-\lambda I) = (\lambda-(3+2i))(\lambda-(3-2i))$$

<u>"eigenvectors"</u>

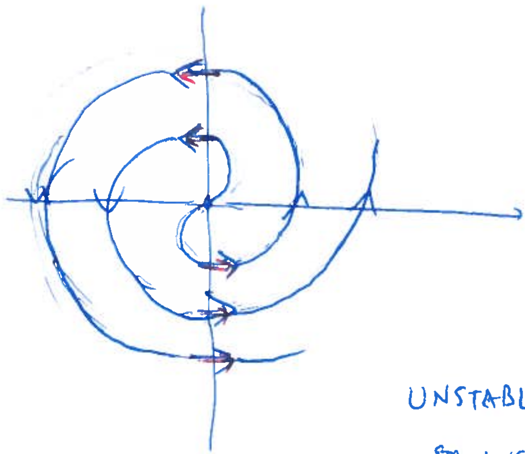
$$\begin{pmatrix} 3 & -2 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} r^1 \\ r^2 \end{pmatrix} = (3+2i)\begin{pmatrix} r^1 \\ r^2 \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} r^1 \\ r^2 \end{pmatrix} = \begin{pmatrix} 1 \\ i \end{pmatrix} \text{ is an eigenvector } (\mathbb{C}^2 \longrightarrow \mathbb{C}^2)$$

<u>general solution</u>

$$\begin{pmatrix} x \\ y \end{pmatrix}(t) = e^{3t}(K_1\cos(2t)+K_2\sin(2t))\underbrace{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}_{=\Re\begin{pmatrix} 1 \\ i \end{pmatrix}}$$
$$+ e^{3t}(K_2\cos(2t)-K_1\sin(2t))\underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{=\Im\begin{pmatrix} 1 \\ i \end{pmatrix}}$$

<u>phase portrait</u>



when $x=0$, $y>0$

$\longrightarrow \dot{x}<0$

when $x=0$, $y<0$

$\longrightarrow \dot{x}>0$

UNSTABLE FOCUS/

UNSTABLE SPIRAL

(5) Conjugate roots
$\Re\lambda > 0$

$$\dot{x} = 3x - 2y$$
$$\dot{y} = 2x + 3y \quad \rightarrow \quad \underline{\underline{A}} = \begin{pmatrix} 3 & -2 \\ 2 & 3 \end{pmatrix}$$

$$0 = \det(\underline{\underline{A}} - \lambda \underline{\underline{I}})$$
$$= (\lambda - 3)^2 + 4$$
$$= (\lambda - (3+2i))(\lambda - (3-2i))$$

"eigenvectors":

$$\begin{pmatrix} 3 & -2 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} r^1 \\ r^2 \end{pmatrix} = (3+2i)\begin{pmatrix} r^1 \\ r^2 \end{pmatrix}$$

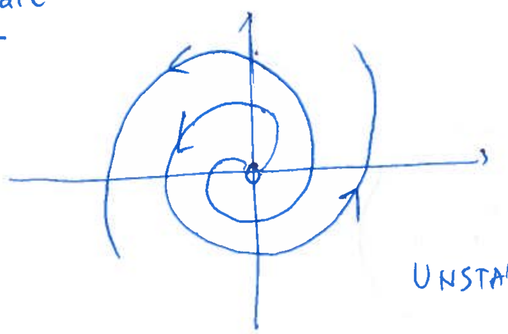
[we know the "other eigenvector" already — by conjugating the entire equation]

$$\rightarrow \begin{pmatrix} r^1 \\ r^2 \end{pmatrix} = \begin{pmatrix} 1 \\ -i \end{pmatrix}$$ is an "eigenvector".

general solution:

$$\underline{x}(t) = e^{3t}(K_1\cos(2t) + K_2\sin(2t))\begin{pmatrix} 1 \\ 0 \end{pmatrix} + e^{3t}(K_2\cos(2t) - K_1\sin(2t))\begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

phase portrait



when $x = 0$, $y > 0$, $\dot{x} < 0$
when $y = 0$, $x > 0$, $\dot{y} > 0$

UNSTABLE SPIRAL/FOCUS

(6) Conjugate Roots
$\Re\lambda < 0$

$$\dot{x} = -5x - 5y$$
$$\dot{y} = 2x + y \quad \rightarrow \quad \underline{\underline{A}} = \begin{pmatrix} -5 & -5 \\ 2 & 1 \end{pmatrix}$$

$$0 = \det(\underline{\underline{A}} - \lambda \underline{\underline{I}})$$
$$= (\lambda+5)(\lambda-1) + 10$$
$$= (\lambda - (-2+i))(\lambda + (-2-i))$$

"eigenvectors"

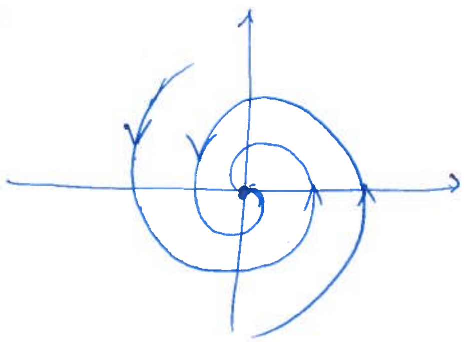
$$\begin{pmatrix} -5 & -5 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} r^1 \\ r^2 \end{pmatrix} = (-2+i)\begin{pmatrix} r^1 \\ r^2 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} r^1 \\ r^2 \end{pmatrix} = \begin{pmatrix} -2-i \\ 1+i \end{pmatrix}$$ is an "eigenvector"

general solution:

$$\underline{x}(t) = e^{-2t}(K_1\cos(t) + K_2\sin(t))\begin{pmatrix} -2 \\ 1 \end{pmatrix} + e^{-2t}(K_2\cos(t) - K_1\sin(t))\begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

phase portrait:



when $x = 0$, $y > 0$, $\dot{x} < 0$

STABLE SPIRAL/FOCUS

(c) a "degenerate" example
conjugate roots
$\Re\lambda = 0$.

7

$$\dot{x} = 2x + 4y$$
$$\dot{y} = -5x - 2y \longrightarrow A = \begin{pmatrix} 2 & 4 \\ -3 & -2 \end{pmatrix}$$

$$0 = \det(A - \lambda I)$$
$$= (\lambda^2 + 16)$$
$$= (\lambda - 4i)(\lambda + 4i)$$

"eigenvectors"

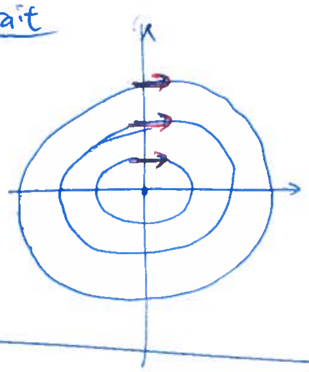
$$\begin{pmatrix} 2 & 4 \\ -5 & -2 \end{pmatrix} \begin{pmatrix} r^1 \\ r^2 \end{pmatrix} = 4i \begin{pmatrix} r^1 \\ r^2 \end{pmatrix}$$

$\rightarrow \begin{pmatrix} r^1 \\ r^2 \end{pmatrix} = \begin{pmatrix} 2-4i \\ 3+4i \end{pmatrix}$ is an "eigenvector".

general solution

$$\begin{pmatrix} x \\ y \end{pmatrix}(t) = \left(K_1 \cos(4t) + K_2 \sin(4t)\right) \underbrace{\begin{pmatrix} 2 \\ 3 \end{pmatrix}}_{= \Re \begin{pmatrix} 2-4i \\ 3+4i \end{pmatrix}}$$
$$+ \left(K_2 \cos(4t) - K_1 \sin(4t)\right) \underbrace{\begin{pmatrix} -4 \\ 4 \end{pmatrix}}_{= \Im \begin{pmatrix} 2-4i \\ 3+4i \end{pmatrix}}$$

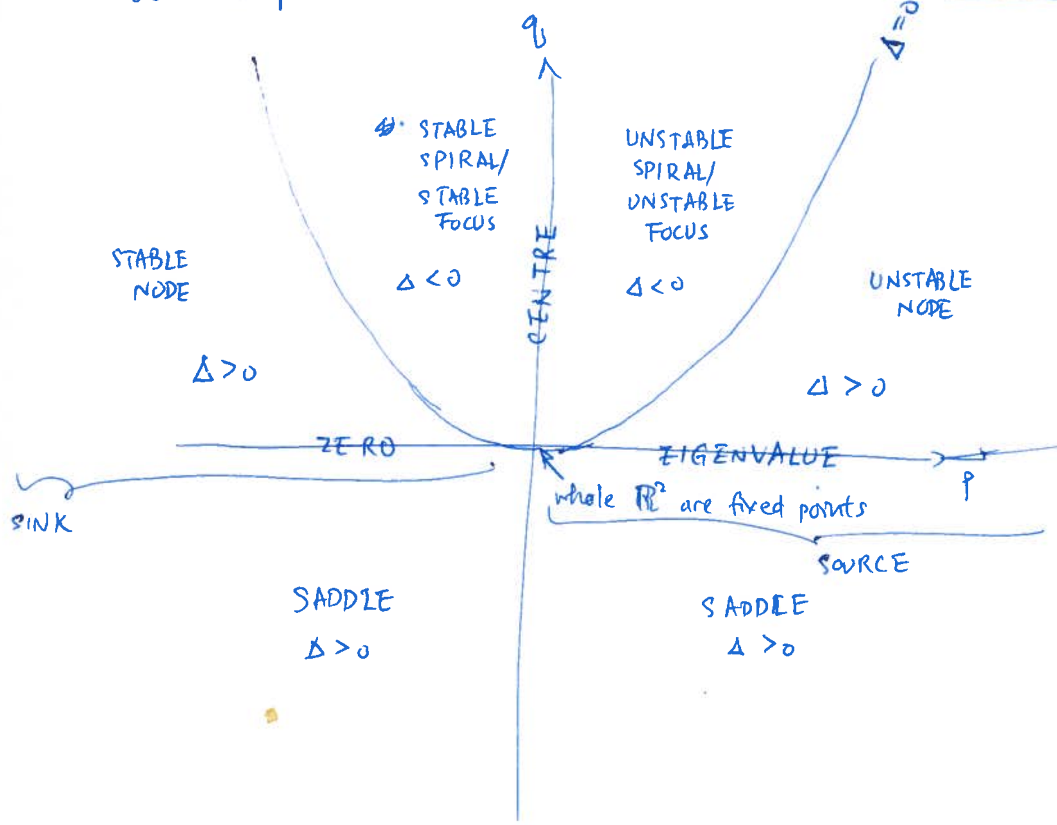
phase portrait



when $x = 0$, $y > 0$.
$\longrightarrow \dot{x} > 0$

CENTRE

SUMMARY



$$\Delta = (a+d)^2 - 4(ad - bc)$$
$$= p^2 - 4q,$$

where

$$p = a + d = \mathrm{tr}(A)$$
$$q = ad - bc = \det(A)$$

fixed points are a subspace: