Project 1 - TMA4215

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1 Problem 1

1.1 Problem Description

Let normal matrices, those with diagonalization be on the form

$$A = U\Lambda U^H$$

Where Λ is a diagonal complex $n \times n$ matrix and U a unitary (complex) matrix such that $U^H U = I$ (recall that U^H is the complex conjugate of U^T).

Show that for any such matrix, one has $\|A\|_2 = \rho(A)$, where $\rho(A)$ is the spectral radius of A .

1.2 Answer 1a

Proof. Starting with the definition of a subordinate matrix norm given in Mayers [2] can we let

$$\|A\|_2^2 = \sup_{x \neq 0} \frac{\langle Ax, Ax \rangle}{\langle x, x \rangle}.$$

Indeed, by using the assumption that $U^H U = I$ and substituting $Uy = x$ can we show that

$$\|A\|^2 = \sup_{x \neq 0} \frac{\langle Ax, Ax \rangle}{\langle x, x \rangle} = \sup_{y \neq 0} \frac{\langle AUy, AUy \rangle}{\langle Uy, Uy \rangle} = \sup_{y \neq 0} \frac{\left\langle U^H A^H AUy, y \right\rangle}{\langle y, y \rangle}$$

Recall the property $A = U\Lambda U^H$ and thus

$$\begin{aligned} A^H A &= U\Lambda^H U^H U \Lambda U^H \\ &= U\Lambda^H \Lambda U^H. \end{aligned}$$

As a consequence do we end up with

$$\begin{aligned} \sup_{y \neq 0} \frac{\left\langle U^H A^H AUy, y \right\rangle}{\langle y, y \rangle} &= \sup_{y \neq 0} \frac{\left\langle U^H U \Lambda^H \Lambda U^H Uy, y \right\rangle}{\langle y, y \rangle} \\ &= \sup_{y \neq 0} \frac{\left\langle \Lambda^H \Lambda y, y \right\rangle}{\langle y, y \rangle} \\ &= \sup_{y \neq 0} \frac{\sum_{i=1}^n |\lambda_i|^2 |y_i|^2}{\sum_{i=1}^n |y_i|^2} = \max_i \left(|\lambda_i|^2 \right) \end{aligned}.$$

Given from Sacco[1], the definition of a spectral radius is characterized by

$$\rho(A) = \max_i |\lambda_i|.$$

Which completes the proof of $\|A\|_2 = \rho(A)$.

□

2 Problem 2

2.1 Problem Describtion

Consider the $n \times n$ matrix A whise nonzero elements are located on its unit subdiagonal , i.e. $A_{i+1,i} = 1$ for $i = 1, \ldots, n-1$

$$A = \begin{bmatrix} 0 & \ldots & \ldots & 0 \\ 1 & 0 & & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \ldots & 1 & 0 \end{bmatrix}$$

(a) What are the eigenvalues of A ? What would the Gershgorin theorem tell us about the location of the eigenvalues of A.

(b) Now construct the matrix $\hat{A}$ by adding a small number ϵ in the $(1, n)$-element of A (so that $\hat{A} = A + \epsilon e_1 e_n^T$). Show that

$$\rho\left(\hat{A}\right) = \epsilon^{\frac{1}{n}}$$

And find an expression for the eigenvalues and eigenvectors of $\hat{A}$.

(c) Derive an exact expression for the condition number

$$K_2\left(\hat{A}\right) = \|\hat{A}\|_2 \cdot \|\hat{A}^{-1}\|_2.$$

2.2 Answer 2a

The eigenvalues can be computed such that

$$\begin{aligned} det\left(A - \lambda\right) &= \begin{vmatrix} -\lambda & & & \ldots & 0 & 0 \\ 1 & -\lambda & \ldots & & & 0 \\ 0 & 1 & -\lambda & \ldots & & 0 \\ \vdots & & & \ddots & & \\ 0 & \ldots & & & 1 & -\lambda \end{vmatrix} \\ &= -\lambda \begin{vmatrix} -\lambda & \ldots & & 0 \\ 1 & -\lambda & & \\ \vdots & & \ddots & \\ 0 & \ldots & 1 & -\lambda \end{vmatrix} \\ &= \left(-1\right)^n \lambda^n = 0 \quad \implies \quad \lambda = 0 \end{aligned}$$

Which concludes that all eigenvalues are zero. Recall the Gershgoring Theorem, given in Mayers [2].

Definition 2.1 (Gerschgorin Discs)**.** *Suppose that* $n \geq 2$ *and* $A \in \mathbb{C}^{n\times n}$*. The* ***Gerschgoring Discs*** $D_i, i = 1, 2, 3 \ldots, n$ *of the matrix* A *are defined as the closed circular regions*

$$D_i = \{z \in \mathbb{C} : \quad |z - a_{ii}| \leq R_i\}$$

In the complex plane, where

$$R = \sum_{\substack{j=1 \\ j\neq i}}^{n} |a_{ij}|$$

is the radius of D_i*.*

Theorem 2.1 (Gerschgorin Theorem)**.** *Let* $n \geq 2$ *and* $A \in \mathbb{C}^{n\times n}$ *. All eigenvalues of the matrix* A *lie in the region* $D = \bigcup_{i=1}^{n} D_i$ *, where* D_i*,* $i = 1, 2, \ldots, n$*, are discs defined by in the definition 2.1.*

Using the Gerschgorin Theorem 2.1 on the matrix A, can we establish that every eigenvalue must be inside the union of gerschgorin discs. Note that in this example is all discs centered around the origin with a radius of 0 or 1. It is worth commenting that even though this is true, because all eigenvalues is zero, can it is worth mentioning that $r = 1$ is a fairly inaccurate estimate even though the theorem has powerful statements.

2.3 Answer 2b

All eigenvalues λ of matrix $\hat{A}$ requires that

$$det\left(\hat{A} - \lambda I\right) = 0.$$

It is true that

$$\hat{A} = A + \varepsilon e_1 e_n^T = \begin{pmatrix} 0 & & \dots & & \varepsilon \\ 1 & 0 & \dots & & \\ 0 & 1 & \ddots & & \\ \vdots & & \ddots & \ddots & \\ 0 & \dots & & 1 & 0 \end{pmatrix},$$

and therefore can obtain

$$det\left(\hat{A}-\lambda I\right)=\begin{vmatrix}-\lambda & & \dots & & \varepsilon\\ 1 & -\lambda & \dots & & \\ 0 & 1 & \ddots & & \\ \vdots & & \ddots & \ddots & \\ 0 & \dots & & 1 & -\lambda\end{vmatrix}$$

$$=(-1)^n\lambda^n+(-1)^{n+1}\varepsilon\begin{vmatrix}1 & -\lambda & \dots & & \\ 0 & 1 & \ddots & & \\ \vdots & & \ddots & \ddots & \\ 0 & \dots & & 1 & -\lambda\end{vmatrix}$$

$$=(-1)^n\lambda^n+(-1)^{n+1}\varepsilon=0.$$

We can see that the eigenvalues λ have several complex solutions depending on the value n. However, it is clear that all λ will satisfy $|\lambda|=\varepsilon^{\frac{1}{n}}$, thus

$$\rho\left(\hat{A}\right)=\varepsilon^{\frac{1}{n}}.$$

The general expression of the eigenvalues is

$$\lambda_k=\varepsilon^{\frac{1}{n}}e^{i2\pi k/n},\quad k=\{0,1,\dots,n-1\}.$$

The procedure to compute the eigenvectors is then to find all solutions which satisfies

$$\hat{A}v_k=\lambda_k v_k.$$

Using recursion do we end up with the equation,

$$\begin{bmatrix}\varepsilon v_{k,n}\\ v_{k,0}\\ \vdots\\ v_{k,n-2}\\ v_{k,n-1}\end{bmatrix}=\begin{bmatrix}v_{k,0}\lambda_k\\ v_{k,1}\lambda_k\\ \vdots\\ v_{k,n-1}\lambda_k\\ v_{k,n}\lambda_k\end{bmatrix}=\begin{bmatrix}v_{k,n}\lambda_k^n\\ v_{k,n}\lambda_k^{n-1}\\ \vdots\\ v_{k,n}\lambda_k^2\\ v_{k,n}\lambda_k^1\end{bmatrix}.$$

Observe that for the n-th element is

$$\varepsilon v_{k,n}=v_{k,n}\lambda_k^n=v_{k,n}\left(\varepsilon^{\frac{1}{n}}e^{\frac{i2\pi k}{n}}\right)^n=v_{k,n}\varepsilon\cdot e^{i2\pi k}.$$

Choosing the scale of all eigenvectors such that the element $v_{k,n}=1$, can we determine a general expression of the eigenvectors to be

$$v_k=\begin{bmatrix}v_{k,0}\\ v_{k,1}\\ \vdots\\ v_{k,n-1}\\ v_{k,n}\end{bmatrix}=\begin{bmatrix}\lambda_k^n\\ \lambda_k^{n-1}\\ \vdots\\ \lambda_k\\ 1\end{bmatrix}=\begin{bmatrix}\left(\varepsilon^{\frac{1}{n}}e^{\frac{i2\pi k}{n}}\right)^n\\ \left(\varepsilon^{\frac{1}{n}}e^{\frac{i2\pi k}{n}}\right)^{n-1}\\ \vdots\\ \varepsilon^{\frac{1}{n}}e^{\frac{i2\pi k}{n}}\\ 1\end{bmatrix},\quad k=\{0,1,2,\dots,n-1\}.$$

2.4 Answer 2c

We recall the theorem given in Mayers [2].

Theorem 2.2. *Let* $A \in \mathbb{R}^{n\times n}$ *and denote the eigenvalues of the matrix* $B = A^TA$ *by* $\lambda_k, k = 0, 1, \ldots, n-1$. *Then,*

$$\|A\|_2 = \max_k \lambda_k^{\frac{1}{2}}.$$

Note that $\hat{A}^T = \hat{A}^{-1}$, so by computing the matrix

$$\hat{A}^{-1}\hat{A} = \begin{bmatrix} 1 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & \\ \vdots & & \ddots & \\ 0 & \ldots & & \varepsilon^2 \end{bmatrix},$$

can we determine the eigenvalues.

$$\begin{aligned} det\left(\hat{A}^{-1}\hat{A} - \lambda I\right) &= \begin{vmatrix} 1-\lambda & 0 & \ldots & 0 \\ 0 & 1-\lambda & \ldots & \\ \vdots & & \ddots & \\ 0 & \ldots & & \varepsilon^2-\lambda \end{vmatrix} \\ &= \left(\varepsilon^2 - \lambda\right)(1-\lambda)^{n-1} = 0 \end{aligned}$$

Which shows that $|\lambda_{\max}|^{\frac{1}{2}} = 1$ and $|\lambda_{\min}|^{\frac{1}{2}} = \varepsilon$, since $\varepsilon < 1$. And therefore can conclude that $\|\hat{A}\|_2 = 1$. Similary, for the inverse matrix $\hat{A}^{-1}$ is $\|\hat{A}^{-1}\|_2 = \varepsilon^{-1}$. We therefore end up with

$$K_2\left(\hat{A}\right) = \frac{1}{\varepsilon}.$$

3 Problem X

3.1 Problem Description

Let A be any invertible $n \times n$- matrix. Suppose that δA is the smallest possible matrix, measured in a subordinate (natural) matrix norm $\|\cdot\|$ such that $A+\delta A$ is singular. Show that

$$\|\delta A\| = \|A^{-1}\|^{-1}$$

3.2 Answer

Assume that $A + \delta A$ is singular such that

$$det\,(A + \delta A) = 0.$$

Then can we find a vector x which satisfies

$$Ax + \delta Ax = 0.$$

It is then possible to rewrite such that

$$\begin{aligned} Ax &= -\delta Ax. \\ x &= -A^{-1}\delta Ax \\ \|x\| &= \|A^{-1}\delta Ax\| \leq \|A^{-1}\|\,\|\delta A\|\,\|x\|. \end{aligned}$$

Clearly, can it be observed that

$$\|A^{-1}\|^{-1} \leq \|\delta A\|.$$

Let us choose a candidate for δA such that $\delta A = -\|A^{-1}\|^{-1} xy^T$ where

$$\|x\| = 1, \quad \|y\|_* := \max_{z \neq 0} \frac{|y^T z|}{\|z\|} = 0 \quad \text{and} \quad \|A^{-1}\| = y^T A^{-1} x.$$

Firstly, we need to prove that $\|\delta A\| = \|A^{-1}\|$.

$$\begin{aligned} \|\delta A\| &= \|\|A^{-1}\|^{-1} xy^T\| = \|\frac{xy^T}{y^T A^{-1}}\| \\ &\leq \|A^{-1}\|^{-1}\|x\|\,\|y^T\| \\ &= \|A^{-1}\|^{-1} \end{aligned}$$

. . . Did not finish proof.

4 References

References

[1] Alfio Quarteroni, Riccardo Sacco, and Fausto Saleri. *Numerical mathematics*, volume 37. Springer Science & Business Media, 2010.

[2] Endre Süli and David F Mayers. *An introduction to numerical analysis*. Cambridge university press, 2003.