TMA 4190 Introduction to Topology
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Lecture 03[1]

3. Smooth manifolds

Recall that we defined what it means for subset $X \subseteq \mathbb{R}^n$ to be open. One reason why open sets are useful is that give us a way to talk about things that happen **close to** a point. In order to stress this way of thinking we are going to use the following way of speaking:

Open neighborhoods

We say that a subset $V \subseteq X$ containing a point $x \in X$ is a **neighborhood of x** if there is an open subset $U \subseteq V$ with $x \in U$. If V itself is open, we call V an **open neighborhood**.

Local properties

If we refer to something that happens in the neighborhood of a point $x \in X$, then we are often going to say that it happens **locally** (at x). Moreover, a property of a space or a function that we only need to **test for a neighborhood of each point** is a **local property**. For example, smoothness of a map is a local porperty (for we test it in a neighborhood of each point). In contrast, there are **global** properties which are properties that describe the **whole space**.

Manifolds are now spaces that **locally look like Euclidean spaces** in the following sense.

Smooth manifolds

Let $\mathbb{R}^N$ be some big Euclidean space.

- A subset $X \subseteq \mathbb{R}^N$ is a **k-dimensional smooth manifold** if it is locally diffeomorphic to $\mathbb{R}^k$. The latter means that for every point $x \in X$ there is an open subset $V \subset X$ containing x and an open subset $U \subseteq \mathbb{R}^k$ such that U and V are diffeomorphic.

[1]Following the books of Guillemin and Pollack: Differential Topology, and Milnor: Topology from the differentiable viewpoint.

- Any such diffeomorphism $\phi\colon U \to V$ is called a **(local) parametrization**.
- The inverse diffeomorphism $\phi^{-1}\colon V \to U$ is called a **(local) coordinate system on V**.

The natural number N in the previous definition is not specified. We just assume that there is some $\mathbb{R}^N$ big enough to fit X into it. We are going to discuss what we can say about the minmal N later. It is actually a very interesting question.

Remember that U is a subset of $\mathbb{R}^k$. Hence it makes sense to express a point $u \in U$ by its coordinates $u = (u_1, u_2, \ldots, u_k)$. Hence, given a coordinate system $\phi^{-1}\colon V \to U$ on V, we can talk about the coordinates $\phi_1^{-1}(x), \phi_2^{-1}(x), \ldots, \phi_k^{-1}(x))$ of a point $x \in V$. Writing $u_i(x) = \phi_i^{-1}(x)$ for $i = 1,\ldots,k$, we usually drop mentioning ϕ^{-1} and just talk about the coordinates $(u_1(x), u_2(x), \ldots, u_k(x))$ of x. Hence the $u_1, \ldots, u_k$ are really **coordinate functions**.)

First examples

- An obvious example of a k-dimensional manifold is an open subset $U \subseteq \mathbb{R}^k$. The identity map $U \to U$ is a parametrization of all of U. For example, any k-dimensional open ball $B_r(x)$ around some point is a manifold of dimension k.
- A 0-dimensional manifold M just consists of a collection of discrete points. Given $x \in M$, the set $\{x\} \subset M$ consisting of x alone is open in M and is diffeomorphic to the one-point set $\mathbb{R}^0$.

A fundamental example that will play an important role during the whole semester is the n-dimensional sphere.

The unit circle

We start with $n = 1$: Let

$$S^1 = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\} \subset \mathbb{R}^2$$

be the unit circle. We are going to show that **S^1 is a 1-dimensional manifold**.

First, suppose that (x,y) lies in the upper semicircle where $y > 0$. Then

$$\boldsymbol{\phi_1(x) = (x, \sqrt{1 - x^2})}$$

maps the open interval $W = (-1,1)$ bijectively onto the upper semicircle. Its inverse is the projection map

$$\boldsymbol{\phi_1^{-1}(x,y) = x}$$

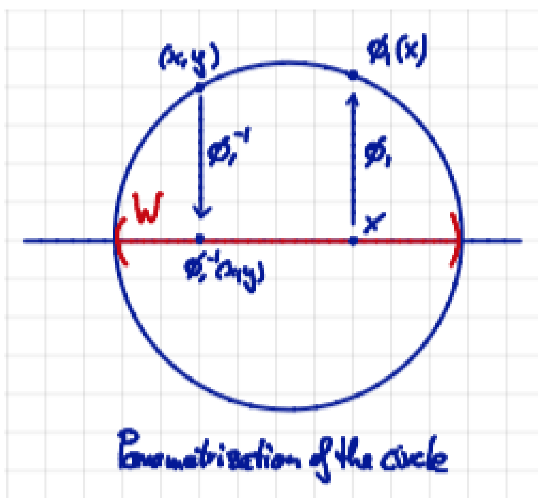
which is defined on the upper semicircle. This ϕ_1^{-1} is smooth, since it extends to a smooth map of all of $\mathbb{R}^2$ to $\mathbb{R}^1$. Therefore, ϕ_1 is a parametrization.
A parametrization of the lower semicircle where $y < 0$ is similarly defined by

$$\phi_2(x) = (x, -\sqrt{1-x^2}) \text{ with inverse } \phi_2^{-1}(x,y) = x.$$

These two maps give local parametrizations of S^1 around any point except the two points $(1,0)$ and $(-1,0)$. To cover these points, we can use the maps

$$\phi_3(y) = (\sqrt{1-y^2}, y) \text{ and } \phi_4(y) = (-\sqrt{1-y^2}, y)$$

which map W to the right and left semicircles, respectively.
This shows that S^1 is a 1-dimensional manifold.



Need at least 2 parametrizations

Note that we have used 4 parametrization maps in the above example. It is an exercise to show that it is possible to cover S^1 with only two parametrizations. (But just one parametrization cannot be enough, because S^1 is compact. For, if such a difeomorphism $\phi\colon S^1 \to U \subset \mathbb{R}^1$ to an open subset existed, it would mean that U is compact contradicting the Theorem of Heine-Borel.)

More generally:

n-sphere

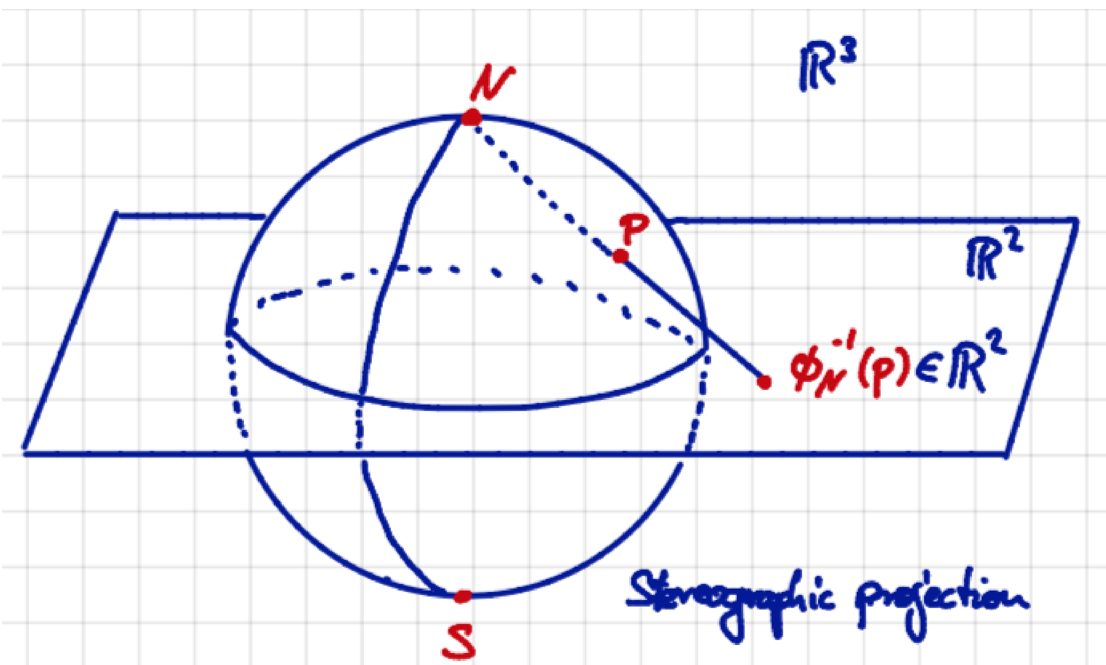
The n-sphere

$$S^n = \{x \in \mathbb{R}^{n+1} : |x| = 1\} \subset \mathbb{R}^{n+1}$$

is an n-dimensional smooth manifold.

Stereographic projection

The method of stereographic projection yields a cover of the k-sphere with only two parametrizations. It is an exercise to find the formulae for the corresponding diffeomorphisms.



Submanifolds

If Z and X are both manifolds in $\mathbb{R}^N$ and $Z \subset X$, then Z is a **submanifold** of X. In particular, X itself is a submanifold of $\mathbb{R}^N$. Any open subset of X is a submanifold of X.

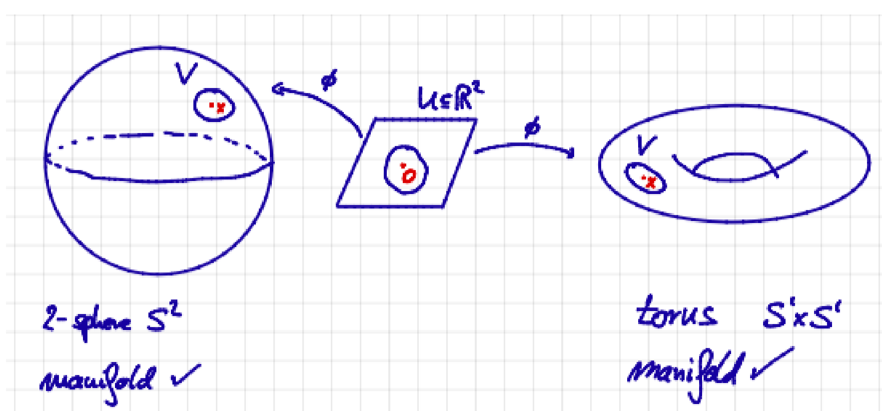
Creating new manifolds out of old ones

Let $X \subseteq \mathbb{R}^N$ and $Y \subseteq \mathbb{R}^M$ be manifolds of dimensions k and l, respectively. Then **$X \times Y \subseteq \mathbb{R}^{N+M}$ is a manifold of dimension $k+l$**. For let $W \subset \mathbb{R}^k$ an open set with $\phi\colon W \to X$ a local parametrization around $x \in X$, and $U \subset \mathbb{R}^k$ an open set with $\psi\colon U \to Y$ a local parametrization around $y \in Y$. Then we can define the map

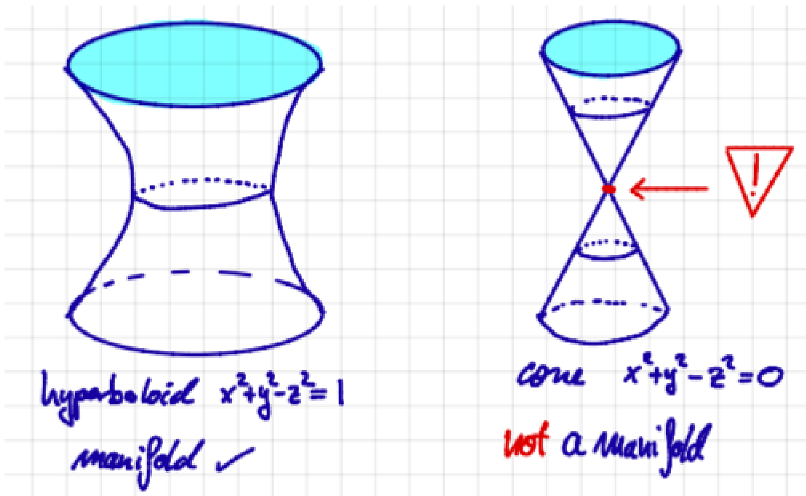
$$\phi \times \psi\colon W \times U \to X \times Y,\ \phi \times \psi(w,u) = (\phi(w), \psi(u)).$$

from the open set $W \times U \subseteq \mathbb{R}^k \times \mathbb{R}^l = \mathbb{R}^{k+l}$ to $X \times Y$. This map defines a local parametrization around (x,y). (Check this!)

Here is a picture of two smooth manifolds:



And a picture of a **hyperboloid (a manifold)** and **a cone (not a manifold)**, see the exercises.



Coordinate axes in $\mathbb{R}^2$

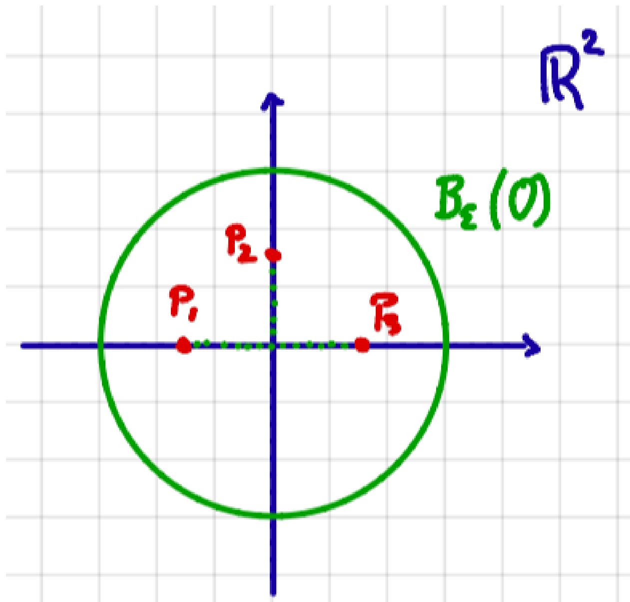
Let us show that the union of the two coordinate axes in $\mathbb{R}^2$ is **not** a manifold.
Let us call the union X. The critical point is of course the origing (0,0), since every other point on X has an open neighborhood which is diffeomorphic to an open intervall in $\mathbb{R}$. But no point in $\mathbb{R}^d$ with $d \geq 2$ has an open neighborhood homeomorphic to an open intervall. Hence X could only be 1-dimensional.
Now let us check the point $O = (0{,}0)$. **If X was a manifold**, there would be an open subset $V \subseteq X$ around O **diffeomorphic to an open intervall** in $\mathbb{R}$. By definition of open sets in a subset of $\mathbb{R}^2$, there must be an open ball $B_\epsilon(O)$ such that $B_\epsilon(O) \cap X$ contained in V. Let I be the open intervall in $\mathbb{R}$ homeomorphic to $B_\epsilon(O) \cap X$.
The subset $B_\epsilon(O) \cap X$ contains, in particular, the points

$$P_1 = (-\epsilon/2{,}0),\ P_2 = (0{,}\epsilon/2),\ \text{and}\ P_3 = (\epsilon/2{,}0).$$

In $B_\epsilon(O) \cap X$, there are paths

- from P_1 to P_2 **not passing through P_3**
- from P_1 to P_3 **not passing through P_2**
- from P_2 to P_3 **not passing through P_1**.

But there is **no triple of distinct points with this property in the open intervall $I \subset \mathbb{R}$**. Hence I **cannot** be homoeomorphic to $B_\epsilon(O) \cap X$.
Hence O does not have a neighborhood homeomorphic to an open intervall in $\mathbb{R}$, and X is not a manifold.



$\mathbb{R}^2$
$B_\varepsilon(0)$
p_1
p_2
p_3