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Solutions to exercise set 8

1 Let $k > 1$ and $f: S^1 \to S^k$ be a smooth map. At any point $x \in S^1$, the derivative $df_x: T_x(S^1) \to T_{f(x)}(S^k)$ is a linear map from a one-dimensional space to a k-dimensional space. Hence df_x is not surjective for any x. Thus the only way $p \in S^k$ can be a regular value is when it is not in the image of f. But, by Sard's Theorem, almost every $p \in S^k$ is a regular value. Hence there must be a point $p \in S^k \setminus f(S^1)$. Fix such a point p. Then, after possibly rotating our coordinate system, we can use p as a center for stereographic projection. This gives us a diffeomorphism $S^k \setminus \{p\} \to \mathbb{R}^k$. Since $\mathbb{R}^k$ is contractible, every smooth map into $\mathbb{R}^k$ is homotopic to a constant map. Since the image of f is contained in $S^k \setminus \{p\} \cong \mathbb{R}^k$, f is homootopic to a constant map. Since S^k is connected, this shows that S^k is simply connected.

2 First, we know that 0 is the only critical value of det. This tells us that the crticial points must be among those with determinat 0. Laplace's formula for the determinant of a matrix A gives us for any fixed $1 \leq j \leq n$:

$$\det(A) = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \cdot \det(A_{ij})$$

where A_{ij} is the $(n-1) \times (n-1)$-submatrix of A with ith row and jth column removed.

Since the (i,j)-minor of A does not depend on a_{ij}, this implies that the ijth partial derivative of det is

$$\frac{\partial \det}{\partial a_{ij}}(A) = (-1)^{i+j} \det(A_{ij}).$$

This shows that $d(\det)_A$, which has the partial derivatives as entries, is zero if and only if all $\det A_{ij}$ vanish. But this is equivalent to saying that the rank of A is $< n-1$. Hence A is a critical point of det if and only if the rank of A is $< n-1$.

For $n = 2$, this is only the case if A is the zero matrix, since all other matrices have rank at least 1. Thus, for $n = 2$, the zero matrix is the only critical point of det. Moreover, the Hessian matrix of det at 0 is

$$H(\det)_0 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

which is invertible. Hence the zero matrix is a nondegenerate critical point, and det is a Morse function on $M(2)$.

For $n > 2$, there are nonzero matrices of rank $n-2$. Given any such matrix A, every matrix in the subspace in $M(n)$ spanned by A has rank $n-2$. Hence in any small neighborhood of a critical point in $M(n)$, there are other critical points. Hence the critical points of det are not isolated in $M(n)$. Hence they cannot be all nondegenerated, and det is not a Morse function for $n > 2$.

$\boxed{3}$ The height function h is just the restriction of the projection onto the last coordinate $\mathbb{R}^{k+1} \to \mathbb{R}$. Hence, at any $x \in S^k$, the derivative of $dh_x \colon T_x(S^k) \to \mathbb{R}$ is also just the restriction of the projection onto the last coordinate. Hence $dh_x = 0$ if x is a point with $T_x(S^k) \subseteq \mathbb{R}^k \times \{0\}$. Defining S^k to be $f^{-1}(0)$ for $f \colon R^{k+1} \to \mathbb{R}, x \mapsto |x|^2+1$, we have $T_x(S^k) = \operatorname{Ker}(df_x)$. At $x = (x_1, \ldots, x_{k+1})$, we have $df_x = 2(x_1 \ldots x_{k+1})$. Hence the $k+1$st coordinate of points in $T_x(S^k)$ is forced to be 0 if $df_x = 2(0, \ldots, 0, \pm 1)$. Hence $dh_x = 0$ if and only if $x = (0, \ldots, 0, \pm 1)$, i.e. if and only if x is either the north pole N or the south pole S on S^k.

To show that N and S are nondegenerate critical points, we need to choose local parametrizations of S^k around these points. Stereographic projection makes the formulae rather complicated. Therefore, we use

$$\phi_+ \colon B_1(0) \to S^k, \ (x_1, \ldots, x_k) \mapsto (x_1, \ldots, x_k, \sqrt{1-|x|^2}).$$

The composite $h \circ \phi_+$ is

$$h \circ \phi_+ \colon R^k \to \mathbb{R}, \ x \mapsto \sqrt{1-|x|^2}.$$

The second partial derivatives are

$$\frac{\partial^2 (h \circ \phi_+)}{x_i x_j} = -\frac{x_i x_j}{(1-|x|^2)^{3/2}} \text{ and } \frac{\partial^2 (h \circ \phi_+)}{x_i x_i} = -\frac{x_i^2 + (1-|x|^2)}{(1-|x|^2)^{3/2}}$$

Thus the Hessian of $h \circ \phi_+$ at $0 \in \mathbb{R}^k$ is $-I$, where I denotes the identity matrix in $M(k)$. Thus N is a nondegenerate critical point, and h has a maximum at N (using what we learend in Calculus 2).

Similarly, using

$$\phi_- \colon B_1(0) \to S^k, \ (x_1, \ldots, x_k) \mapsto (x_1, \ldots, x_k, -\sqrt{1-|x|^2}).$$

as a local parametrization around S, one can show that the Hessian of $h \circ \phi_-$ at 0 is I. Thus S is a nondegenerate critical point, and h has a minimum at S.

$\boxed{4}$ A vector field on X is a smooth section of π, i.e. a smooth map $\sigma \colon X \to T(X)$ such that $\pi \circ \sigma = \mathrm{Id}_X$. An equivalent way to describe such a section is to give a map $s \colon X \to \mathbb{R}^N$ such that $s(x) \in T_x(X)$ for all x. A point $x \in X$ is a zero of the vector field if $s(x) = 0$.

a) If k is odd, then $k+1$ is even and we can define the map

$$s \colon S^k \to \mathbb{R}^{k+1}, (x_1, \ldots, x_{k+1}) \mapsto (-x_2, x_1, -x_3, x_4, \ldots, -x_{k+1}, x_k).$$

This map can be extended to a linear map $\mathbb{R}^{k+1} \to \mathbb{R}^{k+1}$ and therefore s is smooth. For each $x \in S^k$, $s(x)$ is nonzero and satisfies $x \perp s(x)$. Thus $s(x)$ is a tangent vector at x, i.e. $s(x) \in T_x(S^k) \setminus \{0\}$. Hence

$$\sigma \colon S^k \to T(S^k),\ \sigma(x) := (x, s(x))$$

is the desired nonvanishing vector field on S^k.

b) Given a vector field $\sigma \colon S^k \to T(S^k)$ which has no zeros. Let $\sigma(x) = (x, s(x))$. Since $s(x) \neq 0$ for every $x \in S^k$, we can define a new vector field by

$$x \mapsto \frac{s(x)}{|s(x)|}.$$

By replacing s with this new nonvanishing vector field, we can assume $|s(x)| = 1$. Hence we can assume $s(x) \in S^k$ and $s(X) \cdot x$ for every $x \in S^k$.
Now we define the map

$$F \colon S^k \times [0,1] \to S^k,\ (x,t) \mapsto \cos(\pi t)x + \sin(\pi t)s(x).$$

We need to check that $F(x,t)$ is in fact an element in S^k for every $x \in S^k$:

$$\begin{aligned} F(x,t) \cdot F(x,t) &= (\cos(\pi t)x + \sin(\pi t)s(x)) \cdot (\cos(\pi t)x + \sin(\pi t)s(x)) \\ &= \cos^2(\pi t)(x \cdot x) + 2\cos(\pi t)\sin(\pi t)(x \cdot s(x)) + \sin^2(\pi t)(s(x) \cdot s(x)) \\ &= \cos^2(\pi t) + \sin^2(\pi t) \\ &= 1 \end{aligned}$$

where we use $x \cdot x = 1 = s(x) \cdot s(x)$ and $x \cdot s(x) = 0$. Thus $F(x,t)$ is a vector of norm 1 for every x and every t. Moreover, F is a smooth map with $F(x,0) = x$ and $F(,1) = -x$, i.e. F is a smooth homotopy from the identity to the antipodal map on S^k.

c) For $1 \leq i \leq k+1$, let r_i be the reflection map on the ith coordinate:

$$r_i \colon S^k \to S^k, (x_1, \ldots, x_{k+1}) \mapsto (x_1, \ldots, -x_i, \ldots, x_{k+1}).$$

Then the map $S^k \times [0,1] \to S^k$ defined by sending $(x_1, \ldots, x_{k+1}, t)$ to

$$(x_1, \ldots, x_{i-1}, \cos(\pi t)x_i - \sin(\pi t)x_{i+1}, \sin(\pi t)x_i + \cos(\pi t)x_{i+1}, x_{i+2}, \ldots, x_{k+1})$$

is a homotopy from the identity on S^k to the map $r_i \circ r_{i+1} \colon S^k \to S^k$.
The antipodal map is equal to the composition of reflections $r_1 \circ r_2 \circ \cdots \circ r_{k+1}$. Since k is even $r_2 \circ \cdots \circ r_{k+1}$ is homotopic to the identity. Thus the antipodal map is homotopic to the reflection r_1.

$\boxed{5}$ Let X be the set of all straight lines in $\mathbb{R}^2$ (not just lines through the origin).

a) A line in $\mathbb{R}^2$ is determined by an equation of the form $ax + by + c = 0$ with fixed $(a,b,c) \in \mathbb{R}^3$. Since an equation of the form $ax + by + c = 0$ with $a = b = 0$ does not define a line, we have to exclude triples of the form $(0,0,c)$. Moreover, the equations $ax + by + c = 0$ and $(\lambda a)x_(\lambda b)y + (\lambda c) = 0$ with $\lambda \neq 0$ determine the same line. Hence X can be identified with the set of equivalence classes

$$X = (\mathbb{R}^3 \setminus \{(0,0,0),(0,0,1)\})/\sim$$

where $\sim$ is the relation defined by

$$(a,b,c) \sim (\lambda a, \lambda b, \lambda c) \text{ if there is a } \lambda \neq 0.$$

But this is the subspace of $\mathbb{R}\mathrm{P}^2$ given by removing the point $[0:0:1]$. Since any subspace consisting of just one point is closed in $\mathbb{R}\mathrm{P}^2$, we have shown that X can be identified with an open subset of $\mathbb{R}\mathrm{P}^2$:

$$X = \mathbb{R}\mathrm{P}^2 \setminus \{[0:0:1]\}.$$

b) Every line in $\mathbb{R}^2$ is determined by the point where it crosses the x-axis and a direction which can be expressed by an angle $\in [0, 2\pi]$. Since we have not specified a direction for the line, two angles which differ by adding π determine the same line. Any angle between 0 and 2π can be described by a point on the unit circle, where the points s and $-s$ on S^1 correspond to angles which differ by adding π. Hence any line in $\mathbb{R}^2$ is determined by a $(s,x) \in S^1 \times \mathbb{R}$ where s is uniquely determined up to multupying with ± 1.