**TMA 4190 Introduction to Topology**
Lecturer: Gereon Quick
**Lecture 22**[1]

## 22. Orientations

Our next goal is to improve our definition of intersection numbers and remedy the defect that they only distinguish between even and odd numbers. One of the reasons for this limitation was that a homotopy can turn a nontransversal intersection into either a empty intersection or an intersection in two points. The idea for dealing this phenomenon is to take into account in which "direction" the intersection happens. The solution to implement this idea is to introduce orientations. We will see that, unfortunately, not all manifolds are orientable. But for those manifolds that orientable, we will introduce an improved intersection theory in the next lecture.

### Orientations on vector spaces

An **orientation** for a finite dimensional real vector space $V$ is an **equivalence class of ordered bases** where the relation is defined as follows: the ordered basis $(v_1, \ldots, v_n)$ has the **same orientation** as the basis $(v_1', \ldots, v_n')$ if the matrix $A$ with

$$v_i' = Av_i \text{ for all } i \text{ has } \det(A) > 0.$$

It has the **opposite orientation** if $\det(A) < 0$.

The fact, that this an equivalence relation follows from the multiplicativity of the determinant function.

Thus each finite dimensional vector space **has precisely two orientations**, corresponding to the two equivalence classes of ordered bases.

So an orientation of $V$ is a choice of an equivalence class of ordered bases. To make it easier to talk about the choice of orientation, we attach to the chosen orientation a **positive sign** and a **negative sign** to the other orientation. We say then that an ordered basis is positively oriented (respectively negatively oriented) if its equivalence class belongs to the orientation $+1$ (respectively $-1$). We often confuse an orientations with their corresponding signs $+1$ or $-1$.

The vector space $\mathbb{R}^n$ has a **standard orientation** corresponding to the ordered basis $(e_1, \ldots, e_n)$. We always assign $+1$ to the standard orientation of $\mathbb{R}^n$.

[1] Following the books of Guillemin and Pollack: Differential Topology, and Milnor: Topology from the differentiable viewpoint.

**Warning:** The **ordering** of the basis elements **is essential**. Interchanging the positions of two basis vectors changes the sign of the orientation! Check this by calculating the determinant of the corresponding permutation matrix.

In the case of the zero dimensional vector space it is convenient to define an "orientation" as the symbol $+1$ or $-1$.

If $\varphi\colon V \to W$ is an **isomorphism** of vector spaces, then $\varphi$ **either preserves or reverses** the orientation. For, given two ordered bases $\beta$ and $\beta'$ of $V$ belonging to the the same equivalence class, the ordered bases $\varphi(\beta)$ and $\varphi(\beta')$ either still belong to the same equivalence class of ordered bases of $W$ or not. Whether $\varphi$ preserves or reverses the orientation is determined by its determinant. If $\det(\varphi)$ is positive, then $\varphi$ preserves orientations, and if $\det(\varphi)$ is negative, then $\varphi$ reserves orientations.

**Orientations on manifolds**

## Orienting manifolds

An **orientation of a smooth manifold** $X$ is a **smooth choice of orientations** for all the **tangent spaces** $T_x(X)$. That means: around each point $x \in X$ there must exist a local parametrization $\phi\colon U \to X$ such that the isomorphism $d\phi_u\colon \mathbb{R}^k \to T_{\phi(u)}(X)$ **preserves orientations at each point** $u$ of $U \subseteq \mathbb{H}^k$. The orientation on $R^k$ is alsways assumed to be the standard one.
For zero-dimensional manifolds, orientations are very simple. To each point $x \in X$ we simply assign an orientation number $+1$ or $-1$.

A manifold $X$ is called **orientable** if such a smooth choice of orientations of tangent spaces exists.

**Warning:** Not all manifolds possess orientations, the most famous example being the Möbius strip.

**Consequence: Orientability helps classifying manifolds**: there is the class of orientable manifolds, and the class of non-orientable manifolds.

A manifold is called **oriented** if it is orientable and a choice of orientation has been made. Hence an **oriented manifold** really is a **pair** consisting of a manifold together with a chosen orientation.
A smooth map $f\colon X \to Y$ between oriented manifolds is called **orientation preserving** if its derivative preserves orientations at every point.

We just learned that a manifold may or may not be orientable. To assign $+1$ or $-1$ to the orientation of $T_x(X)$ for every point is a locally constant function. If $X$ is orientable this assignment is continuous. If $X$ is in addition connected, then this assignment must be constant. Hence on every connected component of an orientable manifold, the orientation is constant $+1$ or $-1$.

Here is a rigorous proof of this fact:

**Orientable manifolds have exactly two orientations**

A connected, orientable manifold with boundary admits exactly two orientations.

**Proof:** Assume we are given two orientations on $X$. (There are at least two, since given one, we can reverse signs everywhere and get another orientation.)

We show that the **set of points at which two orientations agree** and the **set where they disagree** are **both open**. Consequently, two orientations of a **connected** manifold are either identical or opposite.

Since $X$ is **orientable**, we can choose local parametrizations $\phi\colon U \to X$ and $\phi'\colon U' \to X$ around $x \in X$ with $\phi(0) = x = \phi'(0)$ such that $d\phi_u$ preserves the first orientation and $d\phi'_{u'}$ preserves the second, for all $u \in U$ and $u' \in U'$. After possibly shrinking we can assume $\phi(U) = \phi'(U')$ (replace $U$ and $U'$ with $\phi^{-1}(\phi(U) \cap \phi'(U'))$ and $\phi'^{-1}(\phi(U) \cap \phi'(U'))$, respectively).

If the two orientations of $T_x(X)$ **agree**, then the map

$$d(\phi^{-1} \circ \phi')_0 \colon \mathbb{R}^k \to \mathbb{R}^k$$

is an orientation preserving isomorphism. Thus the determinant of $d(\phi^{-1} \circ \phi')_0$ is positive. Hence the function

$$\varphi\colon U' \to \mathbb{R},\ u' \mapsto \det(d(\phi^{-1} \circ \phi')_{u'})$$

satisfies $\varphi(0) > 0$.

Since the derivative depends continuously on $u'$ and the determinant function is continuous, $\varphi$ is **continuous**. Hence, since $\varphi(0) > 0$, there is an open neighborhood $V'$ around $0$ in $U'$ on which $\varphi > 0$. But this implies that the orientations of $T_x(X)$ induced by $\phi$ and $\phi'$, respectively, agree for all $x$ in the open subset $\phi'(V')$. Since every point on $X$ has such an open neighborhood, the set of points where the orientations agree is **open**.

If the orientations on $T_x(X)$ indced by $\phi$ and $\phi'$, respectively, disagree, the same argument shows that the set of points where the orientations disagree is open. **QED**

## Reversed orientation

Hence if $X$ is an oriented manifold $X$, then we can talk about the manifold with the **reversed orientation**. This is again an oriented manifold which we **denote by** $-X$.

It is now a long and techincal endeavour to check how orientations behave under the main constructions and relate to the concepts we have developed so far. We will go through them one by one:

**Products:**

If $X$ and $Y$ are oriented and one of them is boundaryless, then $X \times Y$ is a manifold with boundary and inherits an orientation in the following way:

At a point $(x,y) \in X \times Y$, let $\alpha = (v_1, \dots, v_k)$ be an ordered basis of $T_x(X)$, and $\beta = (w_1, \dots, w_m)$ be an ordered basis of $T_y(Y)$. We denote by $(\alpha \times 0, 0 \times \beta)$ the ordered basis $((v_1,0), \dots, (v_k,0), (0,w_1), \dots, (0,w_m))$ of $T_x(X) \times T_y(Y) = T_{(x,y)}(X \times Y)$.

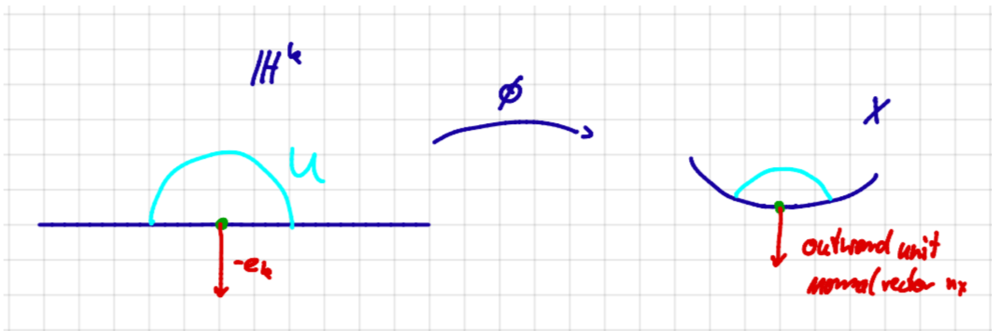
Now it comes handy that we related orientations of ordered bases to signs. For we can define the orientation of $T_x(X) \times T_y(Y)$ simply by determining a sign by setting

$$\mathrm{sign}(\alpha \times 0, 0 \times \beta) = \mathrm{sign}(\alpha) \cdot \mathrm{sign}(\beta).$$

**Induced orientation on the boundary**

Let $X$ be an oriented smooth manifold with boudary. Then $\partial X$ inherits an orientation as follows:

At every point $x \in \partial X$, $T_x(\partial X)$ is a subspace of codimension one in $T_x(X)$. Its orthogonal complement in $T_x(X)$, is a line which contains exactly two unit vectors: one is pointing **inward** into $T_x(X)$, the other one is pointing **outward** away from $T_x(X)$.




This can be made precise by choosing a local parametrization $\phi\colon U \to X$ around $x$ with $U \subset \mathbb{H}^k$ open and $\phi(0) = x$. The derivative $d\phi_0\colon R^k \to T_x(X)$ is by definition of $T_x(X)$ an isomorphism.

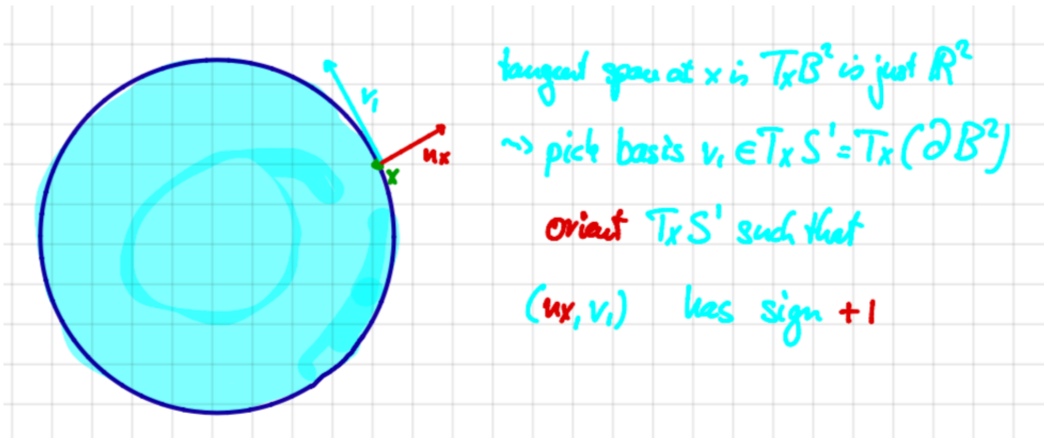
In $\mathbb{R}^k$, there are two unit vectors: $e_k = (0, \dots, 0, 1)$ one pointing into $\mathbb{H}^k$, and $-e_k = (0, \dots, 0, -1)$ pointing out of $\mathbb{H}^k$ Usng the **Gram-Schmidt process** we can orthonormalize the image of $e_k$ under $d\phi_0$ with respect to $T_x(\partial X)$ and get the inward pointing unit normal vector. The orthonormalization with respect to $T_x(\partial X)$ of $d\phi_0(-e_k)$ is the outward pointing unit normal vector. (Note that the inner product on $T_x(X)$ is induced by the standard inner product on $\mathbb{R}^N$, where $X \subset \mathbb{R}^N$ and hence $T_x(X) \subset \mathbb{R}^N$.)

We denote the **outward pointing unit normal vector by** $n_x$. We checked on Exercise Set 9 that the construction of $n_x$ does not depend on the choice of $\phi$ and that the assigment $x \mapsto n_x$ is a **smooth** map on $\partial X$.

Now we are ready to **orient** $T_x(\partial X)$ by declaring the sign of any ordered basis $(v_1, \dots, v_{k-1})$ to be the **sign of the ordered basis** $(n_x, v_1, \dots, v_{k-1})$ **for** $T_x(X)$:

$$\operatorname{sign}(v_1, \dots, v_{k-1}) := \operatorname{sign}(n_x, v_1, \dots, v_{k-1}).$$

Since both the assignment $x \mapsto n_x$ and the choice of sign for ordered bases on $T_x(X)$ are smooth, this defines an orientation of $\partial X$ which is called the **boundary orientation**.




## Orientations of One-manifolds

Let us apply what we just learned to the case of a one-manifold with boundary. The boundary $\partial X$ is zero dimensional. The orientation of the zero-dimensional vector space $T_x(\partial X)$ is equal to the sign of the basis of $T_x(X)$ consisting of the outward-pointing unit vector $n_x$.
As an example, let us look at the compact interval $X = [0,1]$ with its standard orientation inherited from being a subset in $\mathbb{R}$. Note that local parametrizations of $[0,1]$ are given by

$$\phi\colon [0,1) \to [0,1], x \mapsto x$$

around $0 \in [0,1]$ and

$$\psi\colon [0,1) \to [0,1], x \mapsto 1 - x$$

around $1 \in [0,1]$.
Hence, at $x = 1$, the outward-pointing normal vector is $1 \in \mathbb{R} = T_x(X)$. The basis consisting of this vector is positively oriented. At $x = 0$ the outward-pointing normal vector is the negatively oriented $-1 \in \mathbb{R} = T_0(X)$. Thus the orientation of $T_1(\partial X)$ is $+1$, and the orientation of $T_0(\partial X)$ is $-1$.

Reversing the orientation on [0,1] simply reverses the orientations at each boundary point. Thus the sum of both orientation numbers at the boundary points of [0,1] is always zero.

Since any compact one-manifold with boundary is diffeomorphic is the disjoint union of copies of [0,1], we conlcude:

## Boundary orientations of one-manifolds

The **sum of the orientation numbers** at the **boundary points** of any compact oriented one-dimensional manifold with boundary is **zero**.
In particular, the **boundary points of a smooth path** $\gamma$ on an oriented manifold $X$, i.e. a smooth map $\gamma\colon [0,1] \to X$, must have **opposite orientation signs**.
This will turn out to be the **crucial point** which will allow us to define **homotopy invariant intersection numbers** with values in $\mathbb{Z}$ in the next lecture.

### Oriented Homotopies

As an application of product and boundary orientations, we would like to orient the product $[0,1] \times X$ for a boundaryless smooth oriented manifold $X$ which is the domain of all homotopies on $X$. This will be crucial for the homotopy invariance of intersection numbers in the next section.

We just learned that a products and boundaries inherit orientations. For each $t \in [0,1]$, the slice $X_t := \{t\} \times X$ is diffeomorphic to $X$, and the orientation on $X_t$ should be such that the diffeomorphism

$$X \to X_t, \; x \mapsto (t,x) \text{ preserves orientations.}$$

For the future applications, we are particularly interested in the orientation of the boundary

$$\partial([0,1] \times X) = \{0\} \times X \cup \{1\} \times X.$$

So let us try to understand the induced orientation on the boundary.

We start with $X_1$: We see from the local parametrization $\psi$ above that along $X_1$ the outward-pointing normal vector is

$$n_{(1,x)} = (1,0) = (1,0,\dots,0) \in T_1([0,1]) \times T_x(X),$$

If $\beta = (v_1,\dots,v_k)$ is an ordered basis of $T_x(X)$, then $0 \times \beta = ((0,v_1),\dots,(0,v_k))$ is an ordered basis of $T_x(X_1)$. By definition of the boundary orientation, $n_{(1,0)},(0 \times \beta)$ is positively oriented if and only if $\beta$ is positively oriented, in terms of signs:

$$\operatorname{sign}(n_{(1,0)},(0 \times \beta)) = \operatorname{sign}(\beta).$$

If we calculate the orientation induced from the product structure, then we get

$$\operatorname{sign}((1,0),(0 \times \beta)) = \operatorname{sign}(1)\operatorname{sign}(\beta) = \operatorname{sign}(\beta).$$

We learn from these two equations, that the **boundary orientation of $X_1$** is just the **orientation of $X$** as a copy in the product $[0,1] \times X$.

This sounds obvious, **but pay attention**:

We see from the local parametrization $\phi$ that along $X_0$ the outward-pointing normal vector is

$$n_{(0,x)} = (-1,0) = (-1,0,\dots,0) \in T_0([0,1]) \times T_x(X).$$

Hence the orientation on $T_0([0,1])$ is opposite to the standard orientation of $\mathbb{R}$. Hence the formula for product orientations yields

$$\mathrm{sign}((-1,0),0 \times \beta)) = \mathrm{sign}(-1)\mathrm{sign}(\beta) = -\mathrm{sign}(\beta).$$

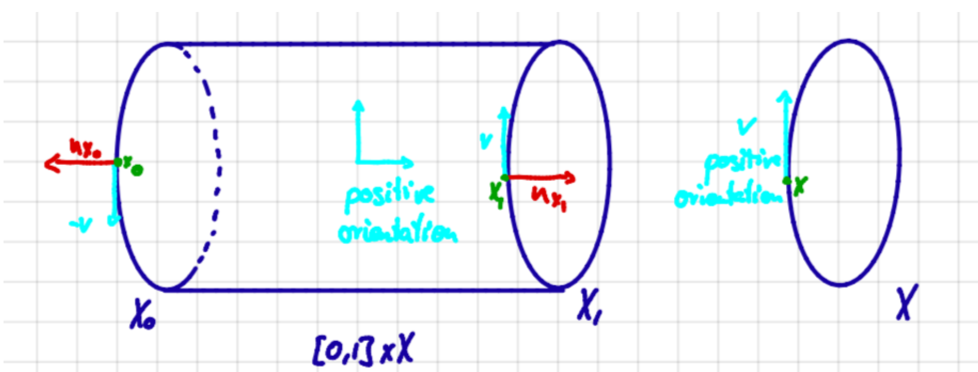
Thus the **boundary orientation on $X_0$** is the **reverse** of its **orientation as a copy of $X$** in the product $[0,1] \times X$.

Thus the orientation on the boundary is

$$\partial([0,1] \times X) = X_1 \cup (-X_0).$$

We will also express this fact by using the **notation**

$$\partial([0,1] \times X) = X_1 - X_0.$$




## Orientations on direct sums of vector spaces

Our next goal is to **orient preimages**. In order to do so, we will have to look at direct sums (not just products) of vector spaces, and we need to orient those guys.

So suppose that $V = V_1 \oplus V_2$ is a direct sum of vector spaces. Then orientations on any two of these vector spaces automatically induces a direct sum orientation on the third, as follows. Note that this not only means, orientations on $V_1$ and $V_2$

determine an orientation on $V$, but also orientations on $V$ and, say, $V_2$ determine an orientation on $V_1$.

Choose ordered bases $\beta_1$ of $V_1$ and $\beta_2$ of $V_2$. Let $\beta = (\beta_1,\beta_2)$ be the combined ordered basis of $V$ (in this order!). For orientations or signs to be compatible with the structure as a direct sum, we require the formula

$$\operatorname{sign}(\beta) = \operatorname{sign}(\beta_1) \cdot \operatorname{sign}(\beta_2).$$

It follows immediately from the way matrices on direct sums are put together that this formula determines an orientation on the third space if two orientations are given. But note again that the order of the summands $V_1$ and $V_2$ is crucial.

### Orientations of transversal preimages

Let $f\colon X \to Y$ be a smooth map with $f \pitchfork Z$ and $\partial f \pitchfork Z$, where $X$, $Y$, and $Z$ are all **oriented** and $Y$ and $Z$ are boundaryless. We would like to define a **preimage orientation** on the manifold with boundary $S = f^{-1}(Z)$.

If $f(x) = z \in Z$, then

$$T_x(S) = (df_x)^{-1}(T_z(Z)) \subset T_x(X).$$

Let $N_x(S;X)$ be the **orthogonal complement** to $T_x(S)$ in $T_x(X)$. By definition, we have a direct sum decomposition

$$N_x(S;X) \oplus T_x(S) = T_x(X).$$

Hence, by our observation on orientations on direct sums, we need only **choose an orientation on $N_x(S;X)$** to obtain a **direct sum orientation on $T_x(S)$**.

Since $f \pitchfork Z$, we have

$$\begin{aligned} T_z(Y) &= df_x(T_x(X)) + T_z(Z) \\ &= df_x(N_x(S;X) \oplus T_x(S)) + T_z(Z) \\ &= df_x(N_x(S;X)) \oplus T_z(Z) \text{ since } df_x(T_x(S)) = T_z(Z). \end{aligned}$$

Thus the orientations on $Z$ and $Y$ induce a direct image orientation on $df_x(N_x(S;X))$. It remains to show that this also induces an orientation on $N_x(S;X)$. But

$$\{0\} \subset T_z(Z) \Rightarrow \operatorname{Ker}(df_x) \subset (df_x)^{-1}(T_z(Z)) = T_x(S),$$

and hence the restriction of $df_x$ to $N_x(S;X)$ is in fact an **isomorphism onto its image**. Therefore the induced orientation on $df_x(N_x(S;X))$ defines an orientation on $N_x(S;X)$ via the isomorphism $df_x$.

Since the orientations on $X$, $Y$ and $Z$ vary smoothly and $df_x$ also depends smoothly on $x$, the induced orientation on $T_x(S)$ varies smoothly with $x$.

**Note** that we did not really use that $N_x(S;X)$ is orthogonal to $T_x(S)$. All we needed was a direct sum decomposition $H \oplus T_x(S) = T_x(X)$ with a space $H$ with an orientation induced by the orientation of $X$. We will exploit this fact in the proof below.

### Orientations on boundaries of preimages

Let $f\colon X \to Y$ be a smooth map with $f \pitchfork Z$ and $\partial f \pitchfork Z$, where $X$, $Y$, and $Z$ are all otientid and $Y$ and $Z$ are boundaryless.

Then the manifold $\partial f^{-1}(Z)$ acquires two orientations:

- one as the boundary of the manifold $f^{-1}(Z)$, and
- one as the preimage of $Z$ under the map $\partial f\colon \partial X \to Y$,

It turns out that there is a formlua that relates these two orientations:

## Orientations on boundaries of preimages

$$\partial(f^{-1}(Z)) = (-1)^{\operatorname{codim} Z}(\partial f)^{-1}(Z).$$

This means the orientations of $\partial f^{-1}(Z)$, induced by being a boundary or by being a preimage, are the same if $\operatorname{codim} Z$ is even, and opposite if $\operatorname{codim} Z$ is odd.

### Proof:

Denote $f^{-1}(Z)$ again by $S$.

Let $H$ be a subspace of $T_x(\partial X)$ complementary to $T_x(\partial S)$, i.e.

$$H \oplus T_x(\partial S) = T_x(\partial X).$$

Note that $H$ is also complementary to $T_x(S)$ in $T_x(X)$, i.e.

$$H \oplus T_x(S) = T_x(X).$$

For we have

$$H \cap T_x(S) = \{0\} \text{ and } T_x(S) \cap T_x(\partial X) = T_x(\partial S),$$

and

$$\dim H = \dim T_x(\partial X) - \dim T_x(\partial S) = \dim T_x(X) - \dim T_x(S).$$

Hence we may use $H$ to define the direct sum orientation of both $S$ and $\partial S$ at $x$.

Since $H \subset T_x(\partial X) \subset T_x(X)$, the maps $df_x$ and $d(\partial f)_x$ agree on $H$, i.e.

$$df_x(H) = d(\partial f)_x(H).$$

As in the case of $N_x(S;X)$, since

$$\{0\} \subset T_z(Z) \Rightarrow \operatorname{Ker}(df_x) \subset f^{-1}(T_z(Z)) = T_x(S),$$

the intersection $\operatorname{Ker}(df_x) \cap H$ is $\{0\}$. Hence the restrictions of $df_x$ and $d(\partial f)_x$ to $H$ are isomorphisms onto their common image.

Thus $f \pitchfork Z$ and $\partial f \pitchfork Z$ imply that we have two direct sum decompositions $df_x(H) \oplus T_z(Z) = T_z(Y) = d(\partial f)_x(H) \oplus T_z(Z)$, and the two orients of $H$ via these direct sums agree.

To conclude, we obtained that $H$ has a well-defined orientation. Hence we can use this unique orientation on $H$ to orient

$$S \text{ via } H \oplus T_x(S) = T_x(X) \text{ and } \partial S \text{ via } H \oplus T_x(\partial S) = T_x(\partial X).$$

It remains to check how this orientation of $T_x(\partial S)$ relates to the orientation of the boundary induced from the orientation of $T_x(S)$.

Let $n_x$ be the outward unit vector to $\partial S$ in $T_x(S)$, and let $\mathbb{R} \cdot n_x$ represent the one-dimensional subspace spanned by $n_x$. We orient this space by assigning the sign $+1$ to the basis $(n_x)$.

Even though $n_x$ need not be perpendicular to all of $T_x(\partial X)$, it suffices to know that $n_x$ lies in the halfspace pointing away from $T_x(X)$ to know that the orientations of $\mathbb{R} \cdot n_x$, $T_x(\partial X)$ and $T_x(X)$ are related by the direct sum

$$T_x(X) = \mathbb{R} \cdot n_x \oplus T_x(\partial X).$$

Now we use that $H$ is complementary to both $T_x(S)$ in $T_x(X)$ and $T_x(\partial S)$ in $T_x(\partial X)$ and plugg this into the above direct sum to get

$$H \oplus T_x(S) = \mathbb{R} \cdot n_x \oplus H \oplus T_x(\partial S).$$

This equation is already almost what we need, since we would like to compare the orientations $T_x(S)$ and $\mathbb{R} \cdot n_x \oplus T_x(\partial S)$. For doing so, we need to move $\mathbb{R} \cdot n_x$ passed $H$. If $\dim H = m$, $H$ has $m$ basis vectors $(w_1, \ldots, w_m)$. Remembering the rule for orienting direct sums, this means we have to apply exactly $m$ transpositions to the ordered set

$$(n_x, w_1, \ldots, w_m) \text{ to get to } (w_1, \ldots, w_m, n_x).$$

This results in $m$ shifts of signs. Hence we get

$$H \oplus T_x(S) = (-1)^{\operatorname{codim} Z} H \oplus \mathbb{R} \cdot n_x \oplus T_x(\partial S).$$

Since $H$ appears on both sides as the first summand, we get disregard it for the computation and get that if $\partial S$ is oriented as a preimage under $\partial f$, then its orientation relates to the one of $T_x(S)$ by

$$T_x(S) = (-1)^{\operatorname{codim} Z} \mathbb{R} \cdot n_x \oplus T_x(\partial S).$$

Now, if $\partial S$ is oriented as a boundary, then we have

$$T_x(S) = \mathbb{R} \cdot n_x \oplus T_x(\partial S).$$

Thus

$$\operatorname{sign}(\partial S) \text{ as a boundary} = (-1)^{\operatorname{codim} Z} \cdot \operatorname{sign}(\partial S) \text{ as a preimage.}$$

**QED**

The following theorem shows that an important class of manifolds is orientable. Recall that a manifold $X$ is called **simply-connected** if it is conneced and every smooth map $S^1 \to X$ is homotopic to a constant map.

## Simply-connected implies orientable

Every simply-connected manifold is orientable.

**Proof:**

We start by picking any point $x \in X$, and choose an orientation for the tangent space $T_x(X)$. Since $T_x(X)$ is a vector space, this is always possible.

Now let $y \in X$ be any other point in $X$. Since $X$ is simply-connected, it is in particular also connected. By a previous exercise, since $X$ is a smooth manifold, $X$ is therefore even **path-connected**. Hence there is a smooth map $\gamma\colon [0{,}1] \to X$ with $\gamma(0) = x$ and $\gamma(1) = y$. For every point in $z \in \gamma([0{,}1])$ we choose a local parametrization $\phi_z\colon V_z \to U_z$ around $z$. By shrinking $V_z$ if necessary, we can assume that each $V_z$ is an open ball in $\mathbb{R}^k$.

The sets $U_z \cap \gamma([0{,}1])$ is open in $\gamma([0{,}1])$, and the collection of $\{U_z \cap \gamma([0{,}1])\}$ for all $z \in \gamma([0{,}1])$ is an open covering of $\gamma([0{,}1])$. Since $[0{,}1]$ is compact and $\gamma$ continuous, the image $\gamma([0{,}1])$ is compact. Hence **finitely many** of the $U_z$ suffice to cover $\gamma([0{,}1])$. We label these open sets $U_1, \ldots, U_m$ and **order them** such that $U_i \cap U_{i+1} \neq \emptyset$ and $x \in U_1$, $y \in U_m$.

For $U_1$, we choose the orientation which is compatible with the chosen orientation of $T_x(X)$. That means: let $\phi_1 \colon U_1 \to X$ be the associated local parametrization with $\phi_1(0) = x$. If $d(\phi_1)_0 \colon \mathbb{R}^k \to T_x(X)$ is orientation preserving, we orient the vector space $T_a(U_1)$ such that $d(\phi_1)_{\phi_1^{-1}(z)} \colon \mathbb{R}^k \to T_a(X)$ is orientation preserving for all $a \in U_1$.

If $d(\phi_1)_0 \to \mathbb{R}^k \to T_x(X)$ reverses orientation, we first replace $\phi_1$ with $\tilde{\phi}_1 \colon V_1 \to X$, $v \mapsto \phi_1(-v)$. This new map $\tilde{\phi}_1$ is also a local parametrization of $X$ with domain $V_1$, since $V_1$ is an open ball in $\mathbb{R}^k$ and $\phi_1$ is therefore symmetric with respect to the origin.

Hence after replacing $\phi_1$ with $\tilde{\phi}_1$, we can assume that $d(\phi_1)_0$ is orientation preserving, and we orient all $T_a(U_1)$ as above. Note that switching from $\phi_1(v)$ to $\phi_1(-v)$ corresponds to switching the orientation on $\mathbb{R}^k$.

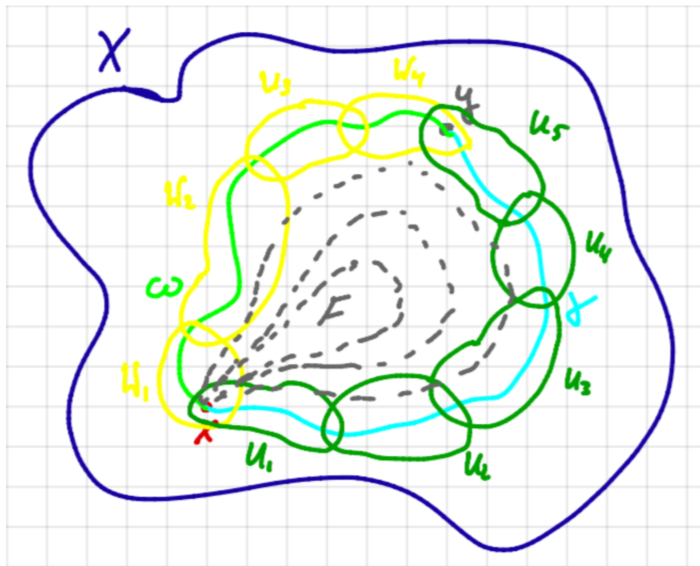
For $U_2$, we choose the orientation which is compatible with the orientation of the $T_a(X)$ for all poins $a \in U_1 \cap U_2$. That means: if $d(\phi_2)_{\phi_2^{-1}(a)}$ is orientation preserving on $T_a(X)$ for $a \in U_1 \cap U_2$, we orient $T_a(X)$ such that $d(\phi_2)_{\phi_2^{-1}(a)} \colon \mathbb{R}^k \to T_a(X)$ is orientation preserving for all $a \in U_2$. If it is not orientation preserving, then we replace $\phi_2(v)$ by $\phi_2(-v)$.

Continuing this way, we obtain an orientation for $U_m$ and therefore $T_y(X)$ after finitely many steps.

**It remains to show** that the induced orientation on $T_y(X)$ **does not depend on the choice** of $\gamma$ and the $U_i$'s.

So let $\omega \colon [0,1] \to X$ be another smooth path with $\omega(0) = x$ and $\omega(1) = y$. As for $\gamma$, we choose open sets $W_1, \ldots, W_l$ covering all points in $\omega([0,1])$ with $x \in W_1$ and $y \in W_l$ and $W_i \cap W_{i+1} \neq \emptyset$. Then we orient $T_y(X)$ following the same procedure using the $W_i$'s.

Arriving at $y$, we do not know a prioir whether the orientation of $T_y(X)$ induced by $\gamma$ and the orientation of $T_y(X)$ induced by $\omega$ agree. But now we can use that $X$ is **simply-connected**.




For, walking first along $\gamma$ and then back on $\omega$ defines, after readjusting the speed and smoothing things out, a **loop** $\alpha\colon [0,1] \to X$ with $\alpha(0) = x = \alpha(1)$, i.e. a smooth map $\alpha\colon S^1 \to X$. Walking along $\alpha$, we obtain an **isomorphism**

$$T(\alpha)\colon T_x(X) = T_{\alpha(0)}(X) \xrightarrow{\cong} T_{\alpha(1)}(X) = T_x(X)$$

by composing

$$T_x(X) \xrightarrow{d(\phi_1)_\bullet^{-1}} \mathbb{R}^k \xrightarrow{d(\phi_2)_\bullet} T_z(X) \xrightarrow{d(\phi_2)_\bullet^{-1}} \mathbb{R}^k \xrightarrow{d(\phi_2)_\bullet} \cdots \xrightarrow{d(\psi_{m-1})_\bullet^{-1}} \mathbb{R}^k \xrightarrow{d(\psi_m)_\bullet} T_x(X)$$

where the subscript $\bullet$ stands for the varying points at which we take derivatives.

Another way to picture how we get from $T_x(X)$ to $T_y(X)$ via $\gamma$ and $\omega$, respectively, is the following diagram:

$$\begin{tikzcd}
 & T_y(X) & \\
T_{b_{m-1}}(X) \arrow[ur, bend left] & & T_{a_{m-1}}(X) \arrow[ul, bend right] \\
\vdots & \mathbb{R}^k \arrow[uu, "d(\psi_m)_0"] \arrow[ul, "d(\psi_{m-1})_\bullet"'] \arrow[ur, "d(\phi_{m-1})_\bullet"] \arrow[dl, "d(\phi_{m-1})_\bullet"] \arrow[dr, "d(\phi_1)_\bullet"'] \arrow[dd, "d(\phi_1)_0"] & \vdots \\
T_{b_1}(X) \arrow[uu, bend left] & & T_{a_1}(X) \arrow[uu, bend right] \\
 & T_x(X) \arrow[ul, bend left] \arrow[ur, bend right] &
\end{tikzcd}$$

The isomorphism $T(\alpha)$ is either orientation preserving or reversing. If it preserves the orientation, then its determinant is positive, and if it reverses the orientation, then its determinant is negative. And **$T(\alpha)$ is orientation preserving** if and only if the **two orientations** on $T_y(X)$ induced by $\gamma$ and $\omega$, respectively, **agree**.

Since $X$ is **simply-connected**, $\alpha$ is homotopic to the constant map $c_x\colon S^1 \to \{x\}$.

Let $F\colon S^1\times[0{,}1] \to X$ be a homotopy from $\alpha$ to $c_x$. Since $S^1\times[0{,}1]$ is **compact**, its image in $X$ is compact and we can add **finitely many** open subsets to the collection $U_1,\dots,U_m,W_1,\dots,W_l$ to cover $F(S^1\times[0{,}1])$ with the codomains of local parametrizations.

For each $t\in[0{,}1]$, $F(-,t)$ defines a **smooth loop** from $x$ to $x$. Using the above procedure for orienting tangent spaces along a path, we obtain an isomorphism

$$T(F(-,t))\colon T_x(X) = T_{F(0,t)}(X) \xrightarrow{\cong} T_{F(1,t)}(X) = T_x(X) \text{ for each } t\in[0{,}1].$$

Taking the **determinant** of $T(F(-,t))$ defines a map

$$[0{,}1] \to \mathbb{R},\ t \mapsto \det(T(F(-,t)))$$

which is **continuous**, since each point of $X$ is contained an open neighborhood on which the orientation is determined by the derivatives of local parametrizations, and these derivatives vary smoothly with the basepoints.

Since each $T(F(-,t))$ is an isomorphism, its determinant is either strictly positive $> 0$ or strictly negative $< 0$. Since $[0,1]$ is **connected** and $t \mapsto \det(T(F(-,t)))$ is **continuous**, we have

$$\text{either } \det(T(F(-,t))) > 0 \text{ or } \det(T(F(-,t))) < 0 \text{ for all } t \in [0,1].$$

But we know that, for $t = 1$, $F(-,1) = c_x$ is the **constant loop at $x$**. Thsus

$$\det(T(F(-,1))) = \det(\mathrm{Id}_{T_x(X)}) > 0.$$

Hence we must have $\det(T(F(-,t))) > 0$ **for all** $t \in [0,1]$. In other words, $T(F(-,t))$ must be orientation preserving for all $t$, and in particular, $T(\alpha)$ **is orientation preserving**.

This shows that the orientation of $T_y(X)$ does not depend on the choice of $\gamma$. **QED**

Let us summarize the key points we shoud remember from this technical lecture.

**Key points we need to take from this lecture**

• An orientation of a vector space is a choice of a sign, $+1$ or $-1$, for an equivalence of orderings of a bases. We can think of it as choosing a positive and negative direction.
• An orientation on a manifold is a smooth choice of orientations of the tangent spaces for each point. Such a choice may or may not exist. Hence manifolds can be orientable or not.
• Orientability helps us classifying manifolds: there is a box with orientable and a box with non-orientable manifolds.
• The boundary of a cylinder has opposite orientations:

$$\partial([0,1] \times X) = X_1 - X_0.$$

• As a conseuqnce: For any compact oriented one-dimensional manifold with boundary, the sum of the orientation numbers at the boundary points is zero. This is the key point for defining homotopy invariant intersection numbers soon.
• There is a formula for the boundary of preimages:

$$\mathrm{sign}(\partial f^{-1}(Z)) \text{ as a boundary} = (-1)^{\mathrm{codim}\, Z} \cdot \mathrm{sign}(\partial f^{-1}(Z)) \text{ as a preimage}.$$

• Simply-connected manifolds are orientable.