6.

7. Lecture VII: Hyperbolic Critical Points

Starting this lecture and for the subsequent few lectures, we shall be looking at what may be known as the local theory of nonlinear systems. The word "local" here is used in a sense different to "local" well-posedness. By "local" here we mean "local in the phase space" and not "local in time".

7.1. **Linearization.** Having deduced the existence and uniqueness of a solution to the system

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x} = \tilde{f}(t, \mathbf{x}), \qquad \mathbf{x}(0) = \mathbf{b}$$

where $\tilde{f}$ is Lipschitz in $\mathbf{x}$ and continuous in t, and explored questions of continuous dependence, we shall now look again at asymptotic behaviour of solutions. To do so we shall have to focus on $\tilde{f}$ for which $\tilde{f}$ has a bounded Lipschitz constant over all of $\mathbb{R}^d$ and not just on an open set $U \subseteq \mathbb{R}^d$ around the initial value. We shall also restrict our attention to autonomous systems again.

Looking at asymptotic behaviours compels us to consider again the behaviour of the set of fixed, or equilibrium points, of the system. That is, recalling the three possible behaviour as $t \to \infty$ — that of escape to infinity (along, perhaps, a particular direction), limit at a point, or limit in a more general set (such as around a periodic orbit), we shall first discuss the second case. A fixed, or critical, or equilibrium point of the dynamics is a point $\mathbf{x}_0 \in \mathbb{R}^d$ for which

$$\tilde{f}(\mathbf{x}_0) = 0.$$

Starting from such a point, we see that $\mathrm{d}\mathbf{x}/\mathrm{d}t = 0$, so that $\mathbf{x}(t) = \mathbf{x}_0$ for all time. This point is then a fixed point of the flow because $\phi_t(\mathbf{x}_0) = \mathbf{x}_0$ for every t. At each coordinate, $\mathbf{x}_0$ solves $\tilde{f}^i(\mathbf{x}) = 0$. The codimension one surface $\tilde{f}^i(\mathbf{x}) = 0$ is known as a nullcline, and the fixed point is at the intersection of all the nullclines.

By Taylor's theorem, where f is continuously differentiable around $\mathbf{x}_0$ up to all orders, we can expand f about $\mathbf{x}_0$ as

$$\tilde{f}(\mathbf{x}) = \mathrm{D}\tilde{f}(\mathbf{x}_0)(\mathbf{x} - \mathbf{x}_0) + \frac{1}{2}\mathrm{D}^2\tilde{f}(\mathbf{x}_0) : (\mathbf{x} - \mathbf{x}_0) \otimes (\mathbf{x} - \mathbf{x}_0) + \dots.$$

The derivative $\mathrm{D}\tilde{f}(\mathbf{x}_0)$ is the matrix with entries

$$(\mathrm{D}\tilde{f}\big|_p)^j_i = \frac{\partial \tilde{f}^j}{\partial x^i}\bigg|_{\mathbf{x}_0}.$$

Therefore one can make the approximation

$$\tilde{f}(\mathbf{x}) \approx \mathrm{D}\tilde{f}(\mathbf{x}_0)(\mathbf{x} - \mathbf{x}_0)$$

in a neighbourhood of $\mathbf{x}_0$ the sense of the usual vector norm in $\mathbb{R}^d$, the remaining terms being of order $O(|\mathbf{x} - \mathbf{x}_0|^2)$.

It turns out that the *dynamics* of the *linear* system

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x} = A(\mathbf{x} - \mathbf{x}_0), \qquad A = D\tilde{f}(\mathbf{x}_0)$$

also approximates the dynamics of the full nonlinear system in a neighbourhood of $\mathbf{x}_0$ in a sense to be made precise later. We shall also see that this is not as absolute an approximation as pointwise as vectors, because this approximation-in-dynamics can fail spectacularly when A does not generate a hyperbolic flow. (Recall from §3.2 that a flow is hyperbolic when A does not have eigenvalues with zero real parts.)

To this end we shall begin by restricting our attention even more narrowly and consider autonomous first order systems around critical points $\mathbf{x}_0$ at which the eigenvalues of $\mathrm{D}\tilde{f}(\mathbf{x}_0)$ all have non-zero real parts. We shall call these critical points hyperbolic critical points. Our first task will be to make a small linear adjustment to our problem, by shifting everything by $\mathbf{x}_0$ so that the critical point of interest is the origin:

Let $f(\mathbf{x}) = \tilde{f}(\mathbf{x} + \mathbf{x}_0)$, then

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x}(t) = f(\mathbf{x}(t))$$

will have a critical point at $\mathbf{0} \in \mathbb{R}^d$ if, and only if, $\tilde{f}(\mathbf{x}_0) = 0$.

7.2. **Examples.** In the following we shall consider a few examples of hyperbolic critical points. These examples are chosen because we shall revisit them. They are all systems with dependence on at least one parameter other than time.

Example 7.1 (Van der Pol's Equation)**.** The first example we inspect is known as van der Pol's Equation/Oscillator, which models oscillatory currents in electrical circuits with vacuum tubes — essentially and oscillator with non-linear damping. For $\beta \in \mathbb{R}$, the Van der Pol Equation is

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -\beta(x^2 - 1)y - x. \end{aligned} \tag{15}$$

This equation arose in is often stated in another form. We can find its fixed points relatively easily. In fact it only has one:

$$0 = y, \qquad 0 = -\beta(x^2 - 1)y - x \qquad \Longrightarrow x = y = 0.$$

We can find $\mathrm{D}f$:

$$\mathrm{D}f = \begin{pmatrix} \partial f^1/\partial x & \partial f^1/\partial y \\ \partial f^2/\partial x & \partial f^2/\partial y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -2\beta xy - 1 & \beta \end{pmatrix}.$$

Therefore

$$\mathrm{D}f\Big|_{(0,0)} = \begin{pmatrix} 0 & 1 \\ -1 & \beta \end{pmatrix},$$

and $\mathrm{D}f|_{(0,0)}$ is non-singular for $\beta \neq 0$.

Its eigenvalues are

$$\lambda_{\pm} = \frac{\beta \pm \sqrt{\beta^2 - 4}}{2}.$$

This means that the system

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x}(t) = \mathrm{D}f|_{(0,0)}\mathbf{x}(t)$$

has an unstable focus when $0 < \beta < 2$, and an unstable node when $2 < \beta$. Either way it is a SOURCE when $\beta > 0$.

Because of the physical system which it models one does not often consider the case $\beta < 0$. But we are free to do so here: When $-2 < \beta < 0$, the linearized system has a stable focus and when $\beta < -2$, the linearized system has a stable node. That is, the system is a SINK when $\beta < 0$.

Therefore we should expect that the dynamics of the nonlinear system exhibit the respective behaviours in these ranges of β in a small region around $(x, y) = (0, 0)$.

Example 7.2 (Duffing's Equation)**.** The Duffing Equation/Oscillator models damped oscillators:

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= x - x^3 - \beta y. \end{aligned} \tag{16}$$

Here $\beta > 0$ is the damping parameter. This equation is usually written with an inhomogeneity that also models a driving force. But without such a forcing term, we have an autonomous system whose fixed points are at

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} y \\ x - x^3 - \beta y \end{pmatrix} \qquad \Longrightarrow \qquad (x, y) \in \{(0,0), (\pm 1, 0)\}.$$

The first order approximation is

$$\mathrm{D}f = \begin{pmatrix} \partial f^1/\partial x & \partial f^1/\partial y \\ \partial f^2/\partial x & \partial f^2/\partial y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1-3x^2 & -\beta \end{pmatrix}$$

At the fixed point $(x, y) = (0, 0)$,

$$\mathrm{D}f|_{(0,0)} = \begin{pmatrix} 0 & 1 \\ 1 & -\beta \end{pmatrix}.$$

and its eigenvalues are

$$\lambda_\pm = \frac{-\beta \pm \sqrt{\beta^2+4}}{2}.$$

This implies that the linearized system has a saddle for all $\beta > 0$.

At the fixed points $(x, y) = (\pm 1, 0)$,

$$\mathrm{D}f|_{(\pm 1,0)} = \begin{pmatrix} 0 & 1 \\ -2 & -\beta \end{pmatrix}.$$

and its eigenvalues are

$$\lambda_\pm = \frac{-\beta \pm \sqrt{\beta^2-8}}{2}.$$

This is an SINK for all $\beta > 0$, and so the linearized system exhibits a stable node when $\beta > \sqrt{8}$, and a stable focus when $\beta < \sqrt{8}$.

Stability is what we expect of damping, and we expect that the dynamics of the nonlinear system exhibit the respective behaviours in these ranges of β in a small region around each fixed point.

Example 7.3 (Lotka-Volterra model)**.** The second example we consider ithe Lotka-Volterra model. This equation models predator-prey populations dynamics:

$$\begin{aligned} \dot{x} &= x - xy \\ \dot{y} &= \rho(xy - y). \end{aligned} \tag{17}$$

The prey concentration is modelled by x, and the predator concentration is modelled by y. Interaction/predation is modelled by xy. The parameter ρ is positive.

First we seek the critical points:

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} x - xy \\ \rho(xy-y) \end{pmatrix} \implies x = y = \pm 1 \quad \text{or} \quad x = y = 0.$$

Again it is simple to find the linearization here:

$$\mathrm{D}f = \begin{pmatrix} 1-y & -x \\ \rho y & \rho(x-1) \end{pmatrix}.$$

At the fixed points, we find:

$$\mathrm{D}f|_{(1,1)} = \begin{pmatrix} 0 & -1 \\ \rho & 0 \end{pmatrix}, \qquad \mathrm{D}f|_{(0,0)} = \begin{pmatrix} 1 & 0 \\ 0 & -\rho \end{pmatrix}, \qquad \mathrm{D}f|_{(-1,-1)} = \begin{pmatrix} 2 & 1 \\ -\rho & 2\rho \end{pmatrix}.$$

The eigenvalues of $\mathrm{D}f$ at $(1, 1)$ are $\lambda_\pm = \pm i\sqrt{\rho}$. Therefore $(1, 1)$ is not a hyperbolic critical point.

At $(0, 0)$, the eigenvalues of $\mathrm{D}f$ are $\lambda_1 = 1$ and $\lambda_2 = -\rho$. Eigenvectors $(0, 1)^\top$ and $(1, 0)^\top$, respectively, are also more readily found than in the previous two examples. Therefore the behaviour of the linearized is that of a saddle for all $\rho > 0$, and we expect the behaviour of the nonlinear system to be the same around the point $(0, 0)$.

It makes little sense to study the equation at $(x, y) = (-1, -1)$ if we consider the model, but we are not thereby hindered. The eigenvalues of $\mathrm{D}f|_{(-1,-1)}$ are $\lambda_\pm = 1 + \rho \pm \sqrt{\rho^2 - 3\rho + 1}$. Therefore according as $\rho \in (3 - \sqrt{5}, 3 + \sqrt{5})$ or $\rho \in \mathbb{R}_{>0} \backslash [3 - \sqrt{5}, 3 + \sqrt{5}]$, the linearized system is an unstable focus or node. And we expect the nonlinear system to behave likewise in a small region around $(-1, -1)$ when ρ are in these intervals.

Example 7.4 (Augmented Lotka-Volterra model)**.** The augmented Lotka-Volterra model amends some deficiencies in the Lotka-Volterra model that we can glean from our analysis in the previous example.

Since the model is a saddle around $(0,0)$, (with eigenvectors $(0,1)^\top$ and $(1,0)^\top$, respectively) we see that no matter how small x or y gets, as long as they are positive, extinction does not happen.

Secondly, without predators ($y(0)=0$), the prey population/concentration can grow indefinitely.

We introduce two parameters — a self-sustaining population parameter $0<\varepsilon<1$ and a carrying capacity parameter $K>\varepsilon$ — to rectify these two problems in the modified equation:

$$\begin{aligned} \dot{x} &= x\left(\frac{x-\varepsilon}{x+\varepsilon}\right)\left(1-\frac{x}{K}\right) - xy \\ \dot{y} &= \rho(xy-y). \end{aligned} \tag{18}$$

Let us write ϕ for

$$\frac{x-\varepsilon}{x+\varepsilon}\left(1-\frac{x}{K}\right).$$

The critical points in $\mathbb{R}_{>0}\times\mathbb{R}_{>0}$ are at

$$x\phi(x)=xy, \qquad xy=y \qquad \Longrightarrow \qquad (x,y)\in\{(0,0),(\varepsilon,0),(K,0),(1,\phi(1))\}.$$

The gradient is

$$\mathrm{D}f = \begin{pmatrix} x\phi'(x)+\phi(x)-y & -y \\ \rho y & \rho(x-1) \end{pmatrix}.$$

and

$$\phi'(x) = \frac{-1}{K}\frac{x-\varepsilon}{x+\varepsilon} + \frac{2\varepsilon}{(x+\varepsilon)^2}\left(1-\frac{x}{K}\right).$$

At $(x,y)=(0,0)$, both species are extinct — this is the extinction equilibrium. The eigenvalues here are $\lambda_1=-1$ and $\lambda_2=-\rho$. This implies that the linearised dynamics has a stable node, and the nonlinear dynamics has one in a vicinity of $(x,y)=(0,0)$.

At $(x,y)=(\varepsilon,0)$ the predators are absent and the prey are at extinction threshold. The eigenvalues here are $\lambda_1=(1-\varepsilon/K)/2$ and $\lambda_2=-\rho(1-\varepsilon)$. This implies that the linearised dynamics has a saddle, and the nonlinear dynamics has one in a vicinity of $(x,y)=(\varepsilon,0)$.

At $(x,y)=(K,0)$ the predators are absent and the prey are at a healthy concentration. The eigenvalues here are $\lambda_1=-(K-\varepsilon)/(K+\varepsilon)$ and $\lambda_2=\rho(K-1)$. This implies that the linearised dynamics has a stable node or a saddle, according as $K<1$ or $K>1$, and the nonlinear dynamics has one in a vicinity of $(x,y)=(K,0)$.

At $(x,y)=(1,\phi(1))$ the predators are conextant — this is the coexistence equilibrium. The eigenvalues here are

$$\lambda_\pm = \frac{\phi'(1)\pm\sqrt{(\phi'(1))^2-4\rho\phi^2(1)}}{2}, \qquad \phi'(1)=\frac{2\varepsilon-(1+2\varepsilon-\varepsilon^2)/K}{(1+\varepsilon)^2},$$

which can still be hyperbolic, and is a sink or source according as $2\varepsilon-(1+2\varepsilon-\varepsilon^2)/K$ is negative or positive. It is a furthermore node or focus depending on whether $\phi'(1))^2-4\rho\phi^2(1)$ is positive or negative. The nonlinear dynamics has corresponding behaviour in a vicinity of $(x,y)=(1,\phi(1))$.

Example 7.5 (Activator-Inhibitor systems)**.** The activator-inhibitor models the concentrations of two species of chemicals, the one modelled by x is produced more rapidly as its concentration increases, and is thereby self-promoting, and the one modelled by y inhibits the production of x. This model is given by the equation:

$$\begin{aligned} \dot{x} &= \sigma \frac{x^2}{1+y} - x \\ \dot{y} &= \rho(x^2 - y). \end{aligned} \tag{19}$$

The parameters σ and ρ are positive.

Again we can find the fixed points, and they are:

$$(x, y) \in \{(0,0), (r_+, r_+^2), (r_-, r_-^2)\},$$

where

$$y = \sigma x - 1, \quad y = x^2 \qquad \Longrightarrow \qquad r_\pm = \frac{\sigma \pm \sqrt{\sigma^2 - 4}}{2}.$$

Therefore the fixed points not at zero only exist in the phase space when $\sigma > 2$.

The linearized system is determined by

$$\mathrm{D}f = \begin{pmatrix} 2\sigma x/(1+y) - 1 & -\sigma x^2/(1+y)^2 \\ 2\rho x & -\rho \end{pmatrix}.$$

At $(x, y) = (0, 0)$, we see that $\mathrm{D}f|_{(0,0)}$ has eigenvalues $\lambda_1 = -1$ and $\lambda_2 = \rho$. This implies a saddle for the linearized dynamics and a saddle for the nonlinear dynamics near $(0, 0)$.

Suppose $\sigma > 2$. Notice that on the nullcline $y = \sigma x - 1$,

$$\mathrm{D}f\Big|_{y=\sigma x-1} = \begin{pmatrix} 1 & -1/\sigma \\ 2\rho x & -\rho \end{pmatrix}.$$

That means

$$\mathrm{D}f\Big|_{(r_\pm, r_\pm^2)} = \begin{pmatrix} 1 & -1/\sigma \\ 2\rho r_\pm & -\rho \end{pmatrix}.$$

This has eigenvalues

$$\lambda_1^\pm = \frac{(1-\rho) + \sqrt{(1-\rho)^2 \mp 4\rho\sqrt{1 - 4/\sigma^2}}}{2}, \qquad \lambda_2^\pm = \frac{(1-\rho) - \sqrt{(1-\rho)^2 \mp 4\rho\sqrt{1 - 4/\sigma^2}}}{2}.$$

Therefore the behaviour at $(x, y) = (r_-, r_-^2)$ is approximated by a saddle. The behaviour at $(x, y) = (r_+, r_+^2)$ is a source or a sink according as $\rho < 1$ or $\rho > 1$.