Partial Differential Equations

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2020

Contents

1 Lecture 1

1.1 Praktiske Ting

- Borthwick, Introduction to Partial Differential Equations - Springer Link
- Ingen obligatoriske øvinger.

1.2 Bevaring av Konserveringslov

- **Konserveringslov**

$$
\begin{aligned}
\frac{\partial u}{\partial t} + \frac{\partial}{\partial x} f(u) &= 0 \\
u(t,x) &\quad \text{ukjent} \\
f &\quad \text{er oppgit}
\end{aligned}
$$

- **Hamilton Jacobi**

$$\frac{\partial u}{\partial t} + f\left(\frac{\partial u}{\partial x}\right) = 0$$

- **Bølgelingingen**

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0$$

- **Varmeligningen**

$$\frac{\partial u}{\partial t} - \mathbb{H} \frac{\partial^2}{\partial x^2} = f(t,x)$$

- **Possion lingingen**

$$
\begin{aligned}
-\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} &= f(x,y) \\
-\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) u &= f
\end{aligned}
$$

- **Korteweg - de vries**

$$\frac{\partial u}{\partial t} + \frac{\partial^3}{\partial x^3} - 6u \frac{\partial u}{\partial x} = 0$$

- **Navier Stokes**

$$\rho\left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \nabla \mathbf{u}\right) = -\nabla p + \mu \Delta \mathbf{u} + \rho \mathbf{g}$$

1.3 Notation

En generell pde kan beskrives som

$$F\left(t, x, y, \ldots, u_t, u_t, \ldots, u_y, u_{xy}, \ldots\right) = 0$$

og blir beskrevet som en partiell diffligning.

$$u_t + f'\left(u\right) u_x = 0$$

En **klassisk løsning** til en PDE av order m er en C^m- funksjon som i ligningen.

Example. Bølgeligningen $u_{tt} - c^2 u_{xx} = 0$ har en klassisk løsning $u\left(t, u\right) = f\left(x \pm ct\right)$ med $f \in C^2$ der $\left[f, f', f''\right]$ er kontinuerlig.

$$\begin{aligned}
u_t &= \pm c f'\left(x \pm ct\right) \\
u_{tt} &= f''\left(x \pm ct\right) \\
u_{xx} &= c^2 f''\left(x \pm ct\right) \\
u_{tt} &= f''\left(x \pm ct\right) \\
u_{tt} &= c^2 u_{xx}.
\end{aligned}$$

Der av løsningen

$$u\left(t, x\right) = f_1\left(x + ct\right) + f_2\left(x - ct\right)$$

1.4 PDE-Teori

- Fine løsninger
- Analyse
  - Velstilt.
    - Løsninger eksisteres
    - De er entydige.
    - De avhenger kontiuerlig av data.
  - General opp oppførsel

$$\begin{aligned}
u_t - u_{xx} \quad t > 0 \\
u\left(0, x\right) = u_0\left(x\right)
\end{aligned}$$

  - Tilnærmede løsninger (numerikk)

1.5 Kap 3, Transportligningen

$$u_t + v u_x = 0 \quad \text{der} \quad v\left(t, x\right) = 0 \quad , u\left(0, x\right) = u_0\left(x\right)$$

Som kan omskrives til

$$\begin{aligned}
&\frac{du\left(t,x\left(t\right)\right)}{dt}=u_t+\dot{x}u_x=0\\
&\quad\text{dersom}\quad \dot{x}=v\left(t,x\right)\quad \text{har entydig løsning gitt}\quad x\left(0\right)=x_0\\
&\text{forutsatt}\quad v\in C^1
\end{aligned}$$

derfor er $u\left(t,x\left(t\right)\right)=u\left(0,x\left(0\right)\right)=u_0\left(x_0\right)$. La oss definere $X\left(t,x_0\right)=x\left(t\right)$

$$\text{dersom}\quad x\quad \text{løser}\quad \begin{cases}\dot{x} & =v\left(t,x\right)\\ x\left(0\right) & =x_0\end{cases}$$

La $u\left(t,X\left(t,x_0\right)\right)=u_0\left(x_0\right)$. For a finne $u\left(t,x\right)$, løs $\left(X\left(t,x_0\right)\right)$ med hennold pa x_0 og sett inn

Example.

$$u_t+\left(at+b\right)u_x=0\quad a,b\quad \text{er kont}$$

Da er ligningen $\dot{x}+at=b$ slik at

$$\begin{aligned}
x&=\frac{1}{2}at^2+bt+c\\
x_0&=x-\frac{1}{2}at^2-bt\\
u\left(t,x\right)&=u_0\left(x-\frac{1}{2}at^2-bt\right)\\
u_t&=-\left(at+b\right)
\end{aligned}$$

2 Lecture 26/08

2.1 ODE teori

Theorem 2.1.

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) \quad \begin{cases} \dot{\mathbf{x}}(t) & = \mathbf{f}(t, \mathbf{x}(t)) \\ \mathbf{x}(0) = \mathbf{x}_0 \end{cases}$$

Anta ligningen er et apoent interval $o \in I$ slik at $\Omega\mathbb{R}^n$ er et område slik at $\mathbf{f} \in C^1(I \times \Omega; \mathbb{R}^n)$. Da finnes et største intervall J $o \in J \subseteq I$ moden funcksjon $\mathbf{x} : I \to \Omega$ som løser problemet. Videre er løsningen gitt.

Proof. Ideen er eksistense. Pcard iterasjon

$$\mathbf{x}_{k+1}(t) = \mathbf{x}_0 + \int_0^t f(\tau, \mathbf{x}_k(\tau))\, d\tau$$

Entydighet

Kontiuerlig /derivering avhenging av dote

□

Theorem 2.2. *Anta gitt*

$$\dot{\mathbf{x}} = f(t, \mathbf{x}, \hat{c})$$

der $\mathbf{x}(0) = \mathbf{a}$

Dersom f er C^{k+1} , så vil $\mathbf{f}$ vare en C^k funksjon av $(t, \mathbf{a}, \hat{c})$

Autonome system

$$\dot{\mathbf{x}} = f(\mathbf{x})$$

Løsningkurvene blander en en-dimensional foliering av Ω. Har en kurve gjennom hvert punkt med ingen krysninger.

Ikke autonomt system

$$\dot{\mathbf{x}} = f(t, \mathbf{x})$$

Ekvivalent

$$\begin{aligned}
\dot{\tau} &= 1 \\
\dot{\mathbf{x}} &= f(\tau,) \\
\tau(0) &= 0 \\
x^{(n)} &= f\left(t, x, \dot{x}, \ldots, x^{(n-1)}\right) \\
&.
\end{aligned}$$

Hvis vi setter

$$\begin{aligned}
\mathbf{w} &= \left(x, \dot{x}, \ldots, x^{(n-1)}\right) \\
\dot{w_1} &= w_2 \\
\dot{w_2} &= w_3 \\
&\vdots \\
\dot{w_n} &= f(t, \mathbf{w})
\end{aligned}$$

$$\dot{\mathbf{w}} = F(t, \mathbf{w}).$$

2.2 Kvasilinær Ligning

$$au_x + bu_y = c$$

a, b, c er alle funksjoner av $x, y, u(x, y)$

Grafen til u er

$$\{(x, y, z) \mid z = u(x, y)\}$$

Da antar vi en løsning u , en kurve γ i (x, y) - planet.

$$(x(\tau), y(\tau))$$

Git enn løsning $u(x, y)$ får vi en kurve i T i $\mathbb{R}^3$ i $(x(\tau), y(\tau), z(\tau))$. Da ender vi opp med

$$\begin{aligned} z(\tau) &= u(x(\tau), y(\tau)) \\ \dot{z} &= \dot{x} u_x(x, y) + \dot{y} u_y(x, y) \\ \text{anta} \quad &\begin{cases} \dot{x}(\tau) = & a(x, y, u(x, y)) = a(x, y, z) \\ \dot{y}(\tau) & = b(x, y, u(x, y)) = b(x, y, z) \end{cases} \end{aligned}$$

da blir

$$\dot{z} = a u_x + b u_y = x(x, y, u(x, y)) = c(x, y, z)$$

Vi får da

$$\begin{cases} \dot{x} & = a(x, y, z) \\ \dot{y} & = b(x, y, z) \\ \dot{z} & = c(x, y, z) \end{cases}$$

Som er kaldt de karakteristiske ligningene til

$$a u_x + b u_y = c$$

Grafen til en løsning u er an union av løsningkurven av de karakteristiske ligningene. (karakteristikk).

Ikke-karakteristiske data for ligningen har formen

$$u(x, y) = u_0(x, y) \quad \text{for} \quad (x, y) \in \gamma$$

der γ er en kurve i $\mathbb{R}^2$, slik at

$$\{(x, y, z) \mid (x, y) \in \gamma \quad \text{og} \quad z = k(x, y)\}$$

Blir en kurve Γ i $\mathbb{R}^3$ ned en tangent som stiller en parabola med

$$(a(x, y, z), b(x, y, z), c(x, y, z))$$

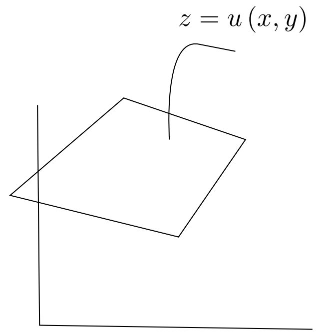

Figure 1: kurveiplan

Theorem 2.3. *Problemet har en entydig løsning $u(x,y)$ for (x,y) i et åpent område som inneholder γ .*

Konkret: Anta γ er gitt ved

$$(\chi(\sigma), \mu(\sigma), (\dot{\chi}, \dot{\mu}) \neq 0, 0)$$

Sett $\chi(\sigma) = u_0(\chi(\sigma), \mu(\sigma))$ og Γ gitt ved (χ, μ, ζ). Da løser vi k ?? med initialdata $(\chi(\sigma), \mu(\sigma), \zeta(\sigma))$. of kall resultatet

$$(x(\sigma,\tau), y(\sigma,\tau), z(\sigma,\tau))$$

Vi skal ha

$$u(x(\sigma,\tau), y(\sigma,\tau)) = z(\sigma,\tau)$$

Som er en implisitt løsning. Finn (σ,τ) skal være en funksjojn av (x,y).

Example.

$$\begin{aligned} u_t + a(t,u)\,u_x &= c(t,u) \\ u(0,x) &= u_0(x) \end{aligned}$$

Kontuerlighet slik at

$$\begin{aligned} &\dot{t} = 1, \quad t(0) = 0 \quad t(\tau) = \tau \\ &\dot{x} = a(t,x,z) \\ &\dot{z} = c(t,z) \end{aligned}$$

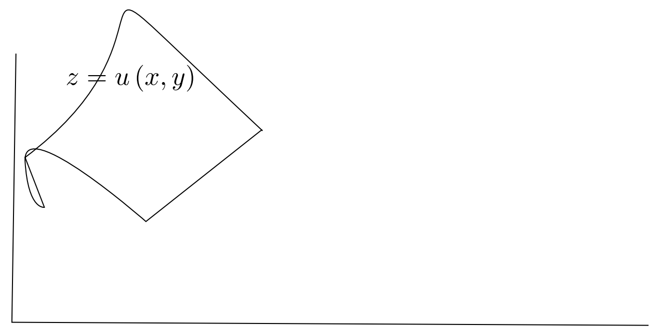

Figure 2: kurveiplan2

som kan forenkles til

$$.\begin{cases} \dot{x} & = a(t,x,z) \\ \dot{z} & = c(t,z) \end{cases}$$

Spesialtilfeller er

Tramert $a = (t,x)$

$$\text{så} \quad \begin{cases} \dot{x} & = a(t,x) \\ x(0) & = \zeta \end{cases}$$

Kan løses hver for seg of så løser vi

$$\begin{cases} \dot{z} = & c(t,x(t),z) \\ z(0) & = u_0(\zeta) \end{cases}$$

Slik at

$$\frac{Du}{Dt} = c(t,x,y) \implies u(0,\zeta) = u_0(\zeta)$$

Spesialtilfelle

$$u_t + a(u)\,u_x = 0$$
$$\dot{t} = 1, t(0) = 0, t = \tau$$
$$\dot{x} = a(z)$$
$$\dot{z} = 0$$

$x = \zeta + ta(u_0(\zeta))$ slik at $u(t,x) = u_0(\zeta)$

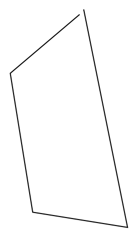

Figure 3: kdfkdkfd

Siste spesialtilfelle

$$
\begin{aligned}
&-yuu_x + xuu_y = 1, \quad u(x,0) = 0 \\
&\dot{x} = -yz \quad x(0) = \sigma \\
&\dot{y} = xz, \quad y(0) = 0 \\
&\dot{z} = 1 \quad z(0) = \sigma \quad z(\tau) = \sigma + \tau
\end{aligned}
$$

Et av resultatene er

$$\frac{d}{d\tau}\left(x^2 + y^2\right) = 2x\dot{x} + 2y\dot{y} = 0$$

Slik at $x^2 + y^2 = \text{konstant} = \sigma^2$.

La oss skrive

$$\left.\begin{aligned} x &= \sigma\cos(\phi(t)) \\ y &= \sigma\sin(\phi(t)) \end{aligned}\right\} \phi(0) = 0$$

da er

$$
\begin{aligned}
&\dot{x} = -\sigma\sin(\phi(\tau)) \cdot \phi\dot{(\tau)} = -y\phi\dot{(\tau)} \\
&\dot{y} = \sigma\cos(\phi(\tau))\dot{\phi} = x\dot{\phi}(\tau)
\end{aligned}
$$

derfor er

$$\dot{\phi}(\tau) = z = \sigma + \tau\phi = \sigma\tau + \frac{1}{2}\tau^2$$

$$\frac{1}{2}\tau^2 + \sigma\tau - \phi = 0, \quad \tau = \frac{1}{2}\left(-\sigma \pm \sqrt{\sigma^2 + 2\phi}\right)$$

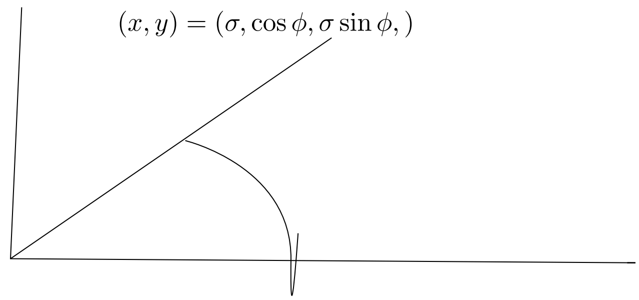

Figure 4: polarfigure

3 Lecture 02/09

3.1 Duhamds Prinsipp

$$u_t + c^2 u_{xx} = f(t, x)$$
$$u(0, x) = 0$$
$$u_t(0, x) = 0$$

La $\eta_s(t, x)$ løse

$$\eta_{s,tt} + c^2 \eta_{s,xx} = 0$$
$$\eta_s(s, x) = 0$$
$$\eta_{s,j}(s, x) = f(s, x)$$

D. Alembtert

$$\eta_s(t, x) = \frac{1}{2c} \int_{x-x(t-s)} f(s, \chi)\, d\chi$$

Duhamed

$$u(t, x) = \int_0^t \eta_s(t, x)\, ds$$

Theorem 3.1. *Hvis $f \in C_1$ så vil dette løse probleme.*

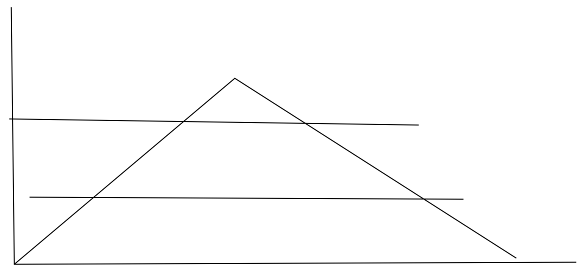

Figure 5: dkdkkd

Proof. la $u(0,x)=0$.

$$\begin{aligned} u_t &= \overbrace{\eta_t(t,x)}^{=0} + \int_0^t \eta_{s,t}(t,x)\,ds \\ u_t(0,x) &= 0 \\ u_{tt} &= \eta_{t,t}(t,x) + \int_0^t \eta_{s,tt}(t,x)\,ds \\ &= f(t,x) - c^2 \int_0^t \eta_{s,xx}(t,x)\,ds \\ &= f(t,x) - c^2 \partial_{xx} \int_0^t \eta_s(t,x)\,ds \\ &= f(t,x) - c^2 + u_{xx}(t,x) \end{aligned}$$

We trenger $\eta \in C^2$, $\eta_{s,tt}$ og $\eta_{s,xx}$ kontinuerlig mhp. s,t,x .

$$\eta_{s,x} = \frac{1}{2c}\left(f\left(x + c(t-s) - f\left(x - c(t-s)\right)\right)\right)$$

$\eta_{s,xx}, \eta_{s,tt}$ er kontinuerlig . Merk at

$$u(t,x) = \int_0^t \overbrace{\int_{x-c(t-s)}^{x+c(t-s)} f(s,\chi)\,d\chi}^{\eta_s(t,x)}\,ds$$

□

Eksempel.

La $f(t,x) = \psi(t) \cdot h(x)$ og

$$\eta_s(t,x) = \frac{1}{2c} \int_{x-x(t-s)}^{x+c(t-s)} \psi(s)\, h(\chi)\, d\chi$$

$$=$$

$frac\psi(s)2c\left(H(x+c(t-s)) - H(x-c(t-s))\right)$

Der $\dot{H}(x) = h(x)$.

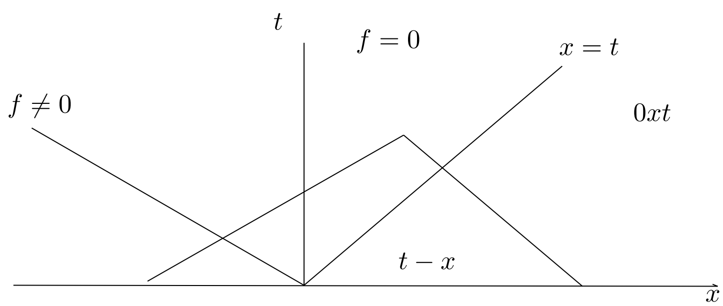

Figure 6: bølgeomraade

$$\begin{aligned}
\eta_s(t,x) &= 0 \quad \text{hvis} \quad s > t-x \\
\eta_s(t,s) &= \frac{\psi(s)}{2c}(x+t-s) \quad 0 < s < t-x \\
u(t,x) &= \int_0^{t-x} \frac{\cos s}{2}(x+t-s)\, ds \\
&= \frac{1}{2}\left[-s \sin s - \cos s + (t-x)\sin s\right]_{s=0}^{s=t-x} \\
&= \frac{1}{2}\left(1 - \cos(t-x)\right)
\end{aligned}$$

Her ble det brukt at

$$\int s \cos^2 s ds = s \sin s + \cos s$$

3.2 Randverdier

Vi kan ta bølgeligningen

$$\left.\begin{aligned} u_{tt} - c^2 u_{xx} &= 0 \\ u(0,x) &= g(x) \\ u_t(0,x) &= h(x) \\ u(t,0) &= 0 \end{aligned}\right\} x > 0$$

En forutsetning for en klassisk løsning er $g(0) = 0$ or $h(0) = 0$. Hvis ikke er ikke initalialbetingelsene konsistente.

D'Alembert

$$u(t,c) = \frac{g(x+ct) - g(x-ct)}{2} = \frac{1}{2c}\int_{x-ct}^{x+ct} h(\zeta)\, d\zeta$$

Løsning. Utvid g, h antisymmetrisk om 0

$$\begin{aligned} g(-x) &= -g(x) \\ h(-x) &= -h(x) \end{aligned}$$

Konsenvensen av antisymmentrien er at

$$u(t,-x) = -u(t,x)$$

Alternativt til randbetingelsen, også kjent som**Neumann betingelse** $u(t,0) = 0$ er

$$u_x(t,0) = 0$$

Analogien er at man har en trå festet på en ring i et stivt rør som ikke har friksjon. Da må tråen være horisontal.

Hvis vi ser på

$$\begin{aligned} &\left\{\frac{d}{dx}\left(g(x+ct) - g(x-ct)\right)\right\}_{x=0} \\ &= \dot{g}(ct) + \dot{g}(-ct) \\ &\vdots \end{aligned}$$

$$\begin{aligned} \dot{g}(ct) &= -\dot{g}(-ct) \\ g(x) &= g(-x) \\ g, h &\quad \text{symmetriske} \end{aligned}$$

$$\frac{d}{dx} h(\zeta)\, d\zeta \quad \text{for} \quad x = 0$$

Utvid g (og h) antisymmetrisk om $0, L$

$$\begin{aligned}
g(-x) &= -g(x) \\
g(L-x) &= -g(L+x) \\
g(L+x) &= g(L-x) = g(x-L) \\
g(x+L) &= g(x-L) \quad x \to x+L \\
g(x+2L) &= g(x)
\end{aligned}$$

Notater om kuler

$$\begin{aligned}
B_r(\mathbf{a}) &= \{x \in \mathbb{R}^n \mid \|\mathbf{x}-\mathbf{a}\| < r\} \\
\overline{B}_r(\mathbf{a}) &= \{x \in \mathbb{R}^n \mid \|\mathbf{x}-\mathbf{a}\| \leq r\} \\
\partial B_r(\mathbf{a}) &= \{x \in \mathbb{R}^n \mid \|\mathbf{x}-\mathbf{a}\| = r\}
\end{aligned}$$

For integratler

$$\int_{B_r(\mathbf{a})} f(\mathbf{x})\, d^n\mathbf{x} = \int_0^r \int_{\partial B_\rho(\mathbf{a})} f(\mathbf{x})\, dS dr$$

Vi kan skrive $\mathbf{x} = \mathbf{a} + \rho\mathbf{y}$ der $\|\mathbf{y}\| = 1$ og $\mathbf{y} \in S^{n-1}$ For $f = 1$

$$\begin{aligned}
\int_{B_r(\mathbf{a})} d^n\mathbf{x} &= \int_0^r \underbrace{\int_{\partial B_\rho(s)} dS dr}_{A_n \rho^{r-n}} \\
&= A_n \int_0^r \rho^{n-1} d\rho = \frac{A_n}{n} r^n
\end{aligned}$$

Der A_n er volumet av $S^{n-1} = \partial B_1(\mathbf{o}) \subseteq \mathbb{R}^n$

$$A_2 = 2\pi, \quad A_3 = 4\pi, \ldots$$

$$\int_{B_r(\mathbf{a})} f(\mathbf{x})\, d^n\mathbf{x} = \int_0^r, \int_{S^{n-1}} f(\mathbf{a} + \rho\mathbf{y})\, dS \rho^{n-1} d\rho$$

Integrasjon i generaliserte polarkoordinater.

$$\frac{d}{dr} \int_{B_r(\mathbf{a})} f(\mathbf{x})\, d^n\mathbf{x} = r^{n-1} \int_{S^{n-1}} f(\mathbf{a} + r\mathbf{y})\, dS$$

3.3 Darboux Formel

La oss ha en kule med sentrum $\mathbf{x}$ og radius ρ. Definer $\overline{f}\left(\mathbf{x};\rho\right)$ der

$$\begin{aligned}\overline{f\left(\mathbf{x};\rho\right)} &= \int_{\partial B_\rho(\mathbf{x})} f\left(\mathbf{y}\right)dS\left(\mathbf{y}\right) := \frac{1}{A_n\rho^{n-1}}\int_{\partial B_\rho(\mathbf{x})} f\left(\mathbf{y}\right)dS. \\ &= \int_{S^{n-1}} f\left(\mathbf{x}+\rho\mathbf{z}\right)dS\left(\mathbf{z}\right) := \frac{1}{A_n}\int_{S^{n-1}}\cdots \\ \Delta\overline{f}\left(\mathbf{x};\rho\right) &= \overline{\Delta f}\left(\mathbf{x};\rho\right)\end{aligned}$$

Derbours Formel

$$\left(\rho^{n-1}\overline{f}_\rho\right)_\rho = \rho^{n-1}\Delta\overline{f}$$

4 Lecture 04/09/29

Bølgeligningen

$$u_{tt} - \nabla u = 0$$

$$u\left(0, \mathbf{x}\right) = g\left(\mathbf{x}\right), \quad u_t\left(0, \mathbf{x}\right) = h\left(\mathbf{x}\right)$$

Darboux Formel. Anta $f : \mathbb{R}^n \to \mathbb{R}$

$$\begin{aligned} \overline{f}\left(\mathbf{x}, \rho\right) &= \int_{\partial B_\rho(\mathbf{x})} f\left(\mathbf{y}\right) dS\left(\mathbf{y}\right) \\ &= \int_{f(\mathbf{x}+\rho\mathbf{z})} dS\left(\mathbf{z}\right) \end{aligned}$$

$$\begin{aligned} \int_{B_\rho(\mathbf{x})} \Delta f\left(\mathbf{y}\right) d^n\mathbf{y} &= \int_{\partial B_\rho(\mathbf{x})} \mathcal{V}\nabla f\left(\mathbf{y}\right) \\ &= A_n \rho^{n-1} \int_{S^{n-1}} \mathcal{V}\nabla f\left(\mathbf{x} + \rho\mathbf{z}\right) dS\left(\xi\right) \\ &= A_n \rho^{n-1} \frac{\partial}{\partial \rho} f\left(\mathbf{x} - \rho\mathbf{z}\right) dS\left(\mathbf{z}\right) \\ &= A_n \rho^{n-1} \overline{f}_\rho \end{aligned}$$

Theorem 4.1. *Darboux*

$$r^{n-1} \Delta \overline{f} = \left(r^{n-1} \overline{f}_\rho\right)_\rho .$$

Ide! Sett

$$\overline{u}\left(t, \mathbf{x}, \rho\right) = ⨍_{\partial B_\rho(\mathbf{x})} u\left(t, \mathbf{y}\right) dS\left(\mathbf{y}\right) = ⨍_{S^{n-1}} u\left(t, \mathbf{x} + \rho\mathbf{z}\right)$$

Merk at

$$\left.\begin{aligned} \overline{u}_{tt} - \Delta\overline{u} &= 0 \\ \Delta\overline{u} &= \tfrac{1}{\rho^{n-1}} \left(\rho^{n-1} u_\rho\right)_\rho \end{aligned}\right\} \underbrace{\overline{u}_{tt} - \frac{1}{\rho^{n-1}} \left(\rho^{n-1} \overline{u}_\rho\right)_\rho}_{\overline{u}_{\rho\rho} + \frac{n-1}{\rho}\overline{u}_\rho} = 0$$

$$\begin{aligned} \left(\rho^k \overline{u}\right)_{\rho\rho} &= \left(\rho^k \overline{u}_\rho + k\rho^{k-1}\overline{u}\right)_\rho \\ &= \rho^k \overline{u}_{\rho\rho} + 2k\rho^{k-1}\overline{u}_\rho + k\left(k-1\right)\rho^{k-2}\overline{u} \end{aligned}$$

Da ender vi opp med

$$\left(\rho\overline{u}\right)_{\rho\rho} = \rho\overline{u}_{\rho\rho} + 2k\overline{u}_\rho$$

La oss definere

$$\hat{f}(\mathbf{x},\rho) = \rho\overline{f}(\mathbf{x})$$
$$\hat{u}(t,\mathbf{x},\rho) = \rho\overline{u}(t,\mathbf{x})$$

Som gjør vi ender opp i

$$\begin{aligned}\hat{u}_{\rho\rho} &= \rho\overline{u}_{\rho\rho} + 2\overline{u}_{\rho} = \rho\left(\overline{u}_{\rho\rho} + (n-1)\,u_{p}\right)\\ \rho\overline{u}_{tt} &= \hat{u}_{tt}\\ \rho\overline{u}_{tt} - \hat{u_{\rho\rho}} &= 0\end{aligned}$$

Andre interessante omskriving er

$$\hat{u}(t,\mathbf{x},\partial) = \rho\overline{u}(t,\mathbf{x},\rho) = \rho \fint_{S^{n-1}} u(t,\mathbf{x}+\rho\mathbf{z})\,dS(\mathbf{z})$$

NB! $\overline{u}(t,\mathbf{x},-\rho) = \overline{u}(t,\mathbf{x},\rho) \implies \hat{u}(t,\mathbf{x},-\rho) = -\hat{u}(t,\mathbf{x},\rho)\,\hat{u}_{tt} - \Delta\hat{u} = 0$

$$\begin{aligned}\hat{u}_{tt} - \hat{u}_{\rho\rho} &= 0\\ \hat{u}(0,\mathbf{x},\rho) &= \hat{g}(\mathbf{x},\rho)\\ \hat{u}_{t}(0,\mathbf{x},\rho) &= \hat{h}(\mathbf{x},\rho)\end{aligned}$$

dAlembert

$$\begin{aligned}\hat{u}(t,\mathbf{x},\rho) = &\frac{\hat{g}(\mathbf{x},\rho-t) + \hat{g}(\mathbf{x},\rho+t)}{2}\\ &+\frac{1}{2}\int_{\rho-t}^{\rho-t}\hat{h}(\mathbf{x},\sigma)\,d\sigma\end{aligned}$$

Ved å ta gensene

$$\begin{aligned}u(t,x) &= \lim_{\rho\to\infty}\overline{u}(t,\mathbf{x},\rho)\\ &= \lim_{\rho\to\infty}\frac{\hat{u}(t,\mathbf{x},\rho)}{\rho}\\ &= \lim_{\rho\to\infty}\left(\frac{\hat{g}(\mathbf{x},t+\rho)\,\hat{g}(\mathbf{x},t-\rho)}{2\rho} + \frac{1}{2\rho}\int_{t-\rho}^{t+\rho}\hat{h}(\mathbf{x},s)\,ds\right)\\ &= \hat{g}_{\rho}(\mathbf{x},t) + \hat{h}(\mathbf{x},t)\\ &= \partial_t\left(\hat{g}(\mathbf{x},t)\right) + \hat{h}(\mathbf{x},t)\end{aligned}$$

Theorem 4.2. *Som er kjent som Kirchoffs Integralformel for* $n = 3$

$$u(t,x) = \hat{g}(\mathbf{x},t)_t + \hat{h}(\mathbf{x},t)$$

. Method of descent

Fra $n=3$ til $n=2$. Problem

$$u_{tt}-\Delta u=0,\quad t>0,x\in\mathbb{R}^2$$

$$\left.\begin{aligned} u\left(0,\mathbf{x}\right) &= g\left(\mathbf{x}\right)\\ u_t\left(0,\mathbf{x}\right) &= h\left(\mathbf{x}\right)\end{aligned}\right\} x\in\mathbb{R}^2$$

La

$$\begin{aligned} u\left(t,\left(x_1,x_2,x_3\right)\right) &= u\left(t,x_1,x_2\right)\\ g\left(x_1,x_2,x_3\right) &= g\left(x_1,x_2\right)\\ h\left(x_1,x_2,x_3\right) &= h\left(x_1,x_2\right)\end{aligned}$$

Resultat

$$u\left(t,x\right)=\overbrace{\hat{g}\left(x,t\right),\hat{h}\left(\mathbf{x},t\right)}^{\text{Regulert i }\mathbb{R}^3}$$

$$\begin{aligned}\hat{g}\left(\mathbf{x},t\right) &= t\fint_{S^2} g\left(\mathbf{x}+\rho\mathbf{z}\right)dS\left(\mathbf{z}\right)\\ &= \frac{1}{4\pi}2t\int_{S^2} g\left(\mathbf{x}0t\mathbf{z}\right)\left[z_3>0\right]dS\left(\mathbf{z}\right)\\ &= \frac{1}{4\pi}\int_D g\left(\mathbf{x}+\mathbf{tz}\right)\frac{dz_1dz_2}{\sqrt{1-z_1^2-z_2^2}}\end{aligned}$$

Parametrisert med

$$\begin{aligned} z &= \left(z_1,z_2\right)\in D\\ dS &= \frac{dz_1dz_2}{\sqrt{1-z_1^2-z_2^2}}\\ dS &= \sqrt{1+\left(\frac{\partial z_3}{\partial z_1}\right)^2\left(\frac{\partial z_3}{\partial z_2}\right)^2}dz_1dz_2\\ \implies z_3 &= \sqrt{1-z_1^2+z_2^2}\\ \left(\frac{\partial z_3}{\partial z_1}\right)^2 &= \left(\frac{-z_1}{\sqrt{1-z_1^2+z_2^2}}\right)^2=\frac{z_1^2}{1-z_1^2+z_2^2}\\ \implies dS &= \sqrt{1+}\end{aligned}$$

Poission Integralformel for n=2

$$u\left(t,x\right)=\frac{\partial}{\partial t}\left(\int_D g\left(\mathbf{x}+t\mathbf{z}\right)\frac{1}{\sqrt{1-\|z\|^2}}dz_1dz_2\right)+\frac{t}{2\pi}\int_D\frac{h\left(\mathbf{x}+t\mathbf{z}\right)}{\sqrt{1-\|z\|^2}}dz_1dz_2$$

5 Lecture 11/09

Definition 5.1. ***Varmeligning*** .

$$u_t - \Delta u = 0, \quad \boldsymbol{x} \in \mathbb{R}^n$$
$$u_t - u_{xx} = 0$$

Anta at u løser $u_t - u_{xx} = 0$

$$u(t,x) = u\left(\lambda^2 t, \lambda x\right)$$

blir en løsning $\lambda > 0$.

$$\begin{aligned}\int_{\mathbb{R}} v(t,x)\,dx &= \int_{\mathbb{R}} \left(\lambda^2 t, \lambda x\right) \\ &= \frac{1}{\lambda}\int_{\mathbb{R}} u\left(\lambda^2 t, y\right)\end{aligned}$$

Gjør istedet

$$v(t,x) = \frac{1}{\lambda} u\left(\lambda^2 t, \lambda x\right)$$
$$\text{Så blir} \quad \int_{\mathbb{R}} v(t,x)\,dx = \int_{\mathbb{R}} u\left(\lambda^2 t, x\right) dx$$

Løsning u kalles **selv simular** dersom

$$u(t,x) = \frac{1}{\lambda} u\left(\lambda^2 t, \lambda x\right) \quad \text{for alle} \quad t > 0, x \in \mathbb{R}, \lambda > 0$$

i

Hvis u er selsimulæør er

$$u(t,x) = \frac{1}{\sqrt{t}} u\left(1, \frac{x}{\sqrt{t}}\right), \quad \lambda = \frac{1}{\sqrt{t}}$$

Prøv å finne en løsning u på formen

$$\begin{aligned}
u(t,x) &= -\frac{1}{\sqrt{t}} w\left(\frac{x^2}{t}\right) \\
u_t &= \frac{1}{\sqrt{t}} w'\left(\frac{x^2}{t}\right)\frac{-1}{t} - \frac{1}{2}t^{-2} - \frac{1}{2}t^{-\frac{3}{2}} w\left(\frac{x^2}{t}\right) \\
&= t^{\frac{-5}{2}} - \frac{1}{2}t^{-\frac{3}{2}} \\
u_x &= x^2 t^{-\frac{3}{2}} \\
u_{xx} &= 4x^2 t^{-\frac{5}{2}} w'' 2t^{-\frac{3}{2}} w' \\
u_t - u_{xx} &= -4x^2 t^{-\frac{5}{2}} w'' - \left(2t^{-\frac{3}{2}} + x^2 t^{-\frac{5}{2}}\right) w' - \frac{1}{2} t^{-\frac{3}{2}} w \mid_{\cdot - t^{\frac{3}{2}}} \\
&= 4\frac{x^2}{t} w'' - \left(2 + \frac{x^2}{t}\right) w' - \frac{1}{2} w, \quad \xi = \frac{x^2}{t} \\
&= 4\xi w''(\zeta) + 2w'(\xi) + \xi w'(\xi) + \frac{1}{2} w(\xi)
\end{aligned}$$

Videre kan vi gjøre

$$\xi\left(4w'' + w'\right) + \frac{1}{2}\left(w' + w\right) = 0$$

Dersom $w = e^{-\frac{\xi}{4}}$, så er $4w' + w = 4w'' + w = 0$.

Da blir $u\left(\frac{1}{\sqrt{t}}\right) = e^{\frac{-x^2}{4t}}$

$$\int_{-\infty}^{\infty} u(t,x)\,dx = \frac{\sqrt{4t}}{\sqrt{t}} \int_{-\infty}^{\infty} e^{-y^2} dy = \sqrt{4y}, \quad x = \sqrt{4t}, \quad dx = \sqrt{4t}dy$$

Heter **varmekjernen**(heat kernel), **varmelingningens fundamentalløsning** og kan også skrives som

$$H_t = \frac{1}{\sqrt{4\pi t}} e^{\frac{-x^2}{4t}}$$

Varmekjernen i $\mathbb{R}^n$

$$\begin{aligned}
H_t(\mathbf{x}) &= H_t(x_1) \cdot H(x_2) \ldots H_t(x_n) \\
&= \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{\|\mathbf{x}\|^2}{4t}}
\end{aligned}$$

$$\int_{\mathbb{R}^n} H_t(\mathbf{x})\, d^n x = 1, \quad (\partial_t - \Delta) H_t = 0$$

5.1 Cauchy Problemet

$$\begin{aligned}
&u_t - \Delta u = 0 \quad \text{for} \quad t > 0, \quad x \in \mathbb{R} \\
&u(0,\mathbf{x}) = g(\mathbf{x})
\end{aligned}$$

$$
\begin{aligned}
u\left(t,x\right) &= \int_{\mathbb{R}^n} H_t\left(\mathbf{x}-\xi\right) g\left(\xi\right) d^n\xi \\
&= H_t \circ g\left(\mathbf{x}\right) = \int_{\mathbb{R}_n} H_t\left(\mathbf{x}-\xi\right) \cdot g\left(\xi\right)
\end{aligned}
$$

Det kan omskrives slik at

$$
\left(\partial_t - \Delta\right) u = \int_{\mathbb{R}^n} \left(\partial_t + \Delta\right) H_t\left(\mathbf{x}-\Delta\right) H_t\left(\mathbf{x}-\xi\right) g\left(\xi\right) d^n\xi = 0
$$

Dette krever at $\partial_t H_t, \partial_{x_i} H_t, \partial_{x_i,x_i} H_t$ alle er integrerbare mhp. x.

Kan ikke brukes.

$$
u\left(0,\mathbf{x}\right) = g\left(\mathbf{x}\right)
$$

Må vise istedet at

$$
\lim_{t\to 0} u\left(t,\mathbf{x}\right) = g\left(\mathbf{x}\right)
$$

Starter med å ta differansen

$$
\begin{aligned}
u\left(t,\mathbf{x}\right) - g\left(\mathbf{x}\right) &= \int_{\mathbb{R}^n} H_t\left(\mathbf{x}-\xi\right)\left(g\left(\xi\right)-g\left(\mathbf{x}\right)\right) d^n\xi \\
\left|u\left(t,\mathbf{x}\right) - g\left(\mathbf{x}\right)\right| &= \left|\int_{\mathbb{R}^n} H_t\left(\mathbf{x}-\xi\right)\left(g\left(\xi\right)-g\left(\mathbf{x}\right)\right) d^n\xi\right| \\
&\leq \left(\int_{B_\delta(\mathbf{x})} + \int_{\mathbb{R}^n \quad B_\delta(\mathbf{x})}\right) H_t\left(\mathbf{x}-\xi\right)\left|g\left(\xi\right)-g\left(\mathbf{x}\right)\right| d^n\mathbf{x}
\end{aligned}
$$

Anta g er kontiuerlig , la $\varepsilon > 0$ velg $\delta > 0$ slik at $\|\mathbf{x}-\xi\| > \delta \implies \quad \|g\left(\mathbf{x}\right) - g\left(\xi\right)\|$,

$$
I_1 \leq \int_{B_\delta(\mathbf{x})} H_t\left(\mathbf{x}-\xi\right) d^n\xi < \int_{\mathbb{R}_n} H_t\left(\mathbf{x}-\xi\right) \cdot \varepsilon d^n\xi = \varepsilon
$$

Anta $\|g\| \leq M$ overalt

$$
\begin{aligned}
I_2 &\leq M \int_{\mathbb{R}^n - B_\delta(\mathbf{x})} H_t\left(\mathbf{x}-\xi\right) d^n\xi = M \int_{\mathbb{R}^n - B_{\delta l}(\xi)} H_t\left(\xi\right) d^n\xi \\
&= \frac{M}{\left(4\pi t\right)^{\frac{n}{2}}} \int_{\mathbb{R}^n - B_\delta(0)} e^{\frac{\|\xi\|^2}{4t}} d\xi, \quad \mu = \xi/\sqrt{4t}, \quad d^n\xi = \left(4t\right)^{\frac{n}{2}} d^n\mu
\end{aligned}
$$

Vi ender da opp med at

$$
I_2 \leq \frac{M}{\pi^{\frac{n}{2}}} \int_{B_{\frac{\delta}{\sqrt{ut}}}} e^{-\|\mu\|^2} d^n\mu \to 0 \quad \text{når} \quad t \to 0
$$

Så $I_2 \leq \varepsilon$ når t er liten nokk.

Konvolusjon.

$$f \circ g(\mathbf{x}) = \int_{\mathbb{R}^n} f(\mathbf{x} - \xi)\, g(\xi)\, d^n \xi$$

Er ok dersom $\quad f, g \in L'(\mathbb{R}^n)$ dvs

$$\int_{\mathbb{R}^n} \|f\| d^n \mathbf{x} < \infty$$

$f \circ g \in L'$

$$f \circ g = g \circ f$$
$$f \circ (g \circ h) = (f \circ g) \circ h$$

Løsning av Cauchy Problemer

$$u(t, \mathbf{x}) = H_t \circ g(\mathbf{x})$$

$$H_{\tau_2 + \tau_1} \circ g = H_{\tau_2} \circ H_{\tau_1} \circ g$$

Theorem 5.1.
$$H_{\tau_2 + \tau_1} = H_{\tau_2} \circ H_{\tau_1}$$

Semigruppe egenskaper
$$e^{a+b} = e^a + e^b$$

La A være $n \times n$ matrise

$$e^A = \sum_{n=0}^{\infty} \frac{A^n}{n!}$$

$$e^{A+B} = e^A e^B, \quad \text{er ikke alltid sant}$$
$$e^{(s+t)A} = e^{sA} e^{tA}$$

Entydighet?

$$\left.\begin{aligned} u_t - \Delta u &= 0 \\ u(0, \mathbf{x}) &= g(\mathbf{x}) \end{aligned}\right\} \text{Har ikke entydig løsning!!}$$

Men dersom g er bregrenset så finnes kun en begrenset løsning !

Hvis vi lar

$$u_t - u_{xx} = 0$$
$$u(0, \mathbf{x}) = 0$$

En typisk ikketriviell løsning:

$$u\left(t,x\right)e^{-Ax^2}$$

blir ubegrenset.

6 References