NTNU
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Department of Mathematical Sciences

Examination paper for **TMA4190 Introduction to Topology**

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Examination time (from–to): 09:00–13:00

Permitted examination support material: D: No printed or hand-written support material is allowed. A specific basic calculator is allowed.

Other information:
All answers must be justified.

Language: English

Number of pages: 2

Number of pages enclosed: 0

Checked by:

Date Signature

Informasjon om trykking av eksamensoppgave
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Problem 1

a) Show that the map

$$f\colon \mathbb{R} \to \mathbb{R}^2,\ t \mapsto \left(\frac{e^t + e^{-t}}{2}, \frac{e^t - e^{-t}}{2}\right)$$

is an embedding.

b) We define the map g by

$$g\colon \mathbb{R}^2 \to \mathbb{R},\ (x, y) \mapsto x^2 - y^2.$$

Determine the set of regular values of g, and determine the set of critical values of g. Is g a submersion?

c) Is the set $\operatorname{Im}(f)$, the image of f in $\mathbb{R}^2$, a manifold? Is the set $(g \circ f)^{-1}(1)$ a manifold?

Problem 2 Let Z be the subset of $\mathbb{R}^4$ defined by

$$Z := \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_1 + x_2^2 + x_3^3 + x_4^4 = 0\}$$

a) Show that Z is a manifold in $\mathbb{R}^4$. What is the dimension of Z?

b) Let $S^3 \subset \mathbb{R}^4$ denote the three-dimensional sphere. Show that $Z \cap S^3$ is a manifold. What is the dimension of $Z \cap S^3$?

Problem 3 Let $X = \{(x, y) \in \mathbb{R}^2 : x \geq -1\}$, $Y = \mathbb{R}$ and

$$f\colon X \to Y,\ (x, y) \mapsto x^2 + y^2.$$

a) What is the boundary of X? Show that 1 is a regular value of f. Is 1 a regular value of $\partial f = f_{|\partial X}$?

b) Determine $\partial(f^{-1}(1))$ and $f^{-1}(1) \cap \partial X$. Why does the answer not contradict the assertion of the Preimage Theorem for manifolds with boundary?

Problem 4 Let $f\colon X \to Y$ be a smooth map between smooth manifolds with X compact, Y connnected and $\dim X = \dim Y$.

a) Show that, if $\deg_2(f) \neq 0$, then f is surjective.

b) Show that if Y is not compact, then $\deg_2(f) = 0$.

(Hint: You may use the fact that the image of a compact space under a continuous map is compact, and you may use the result of a) if needed.)

c) Let $X = Y = S^1 \subset \mathbb{R}^2$ be the unit circle and assume that $f\colon S^1 \to S^1$ is a smooth map without fixed points. Show that f is surjective.

(Hint: Show that f is homotopic to the antipodal map $\alpha\colon S^1 \to S^1$, $x \mapsto -x$. What is $\deg_2(\alpha)$? You may use the results of a) and b) if needed.)