2) System

$$\dot{x} = -y - xy + z - x$$
$$\dot{y} = x - y^3 + z^3 \longrightarrow \underline{0} \text{ is a fixed point}$$
$$\dot{z} = -xz - x^2z - yz^2 - z^5$$

Linearization.

$$Df(0) = \left.\begin{pmatrix} -y^2-3x^2 & -1-2xy & 2z \\ 1 & -3y^2 & 3z^2 \\ -z-2xz & -z^2 & -x-x^2-2yz-5z^4 \end{pmatrix}\right|_{(0,0,0)}$$

$$= \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

eigenvalues and eigenvectors

$$0 = \det(\lambda I - Df(0)) = \lambda^3 + \lambda$$

eigenvalues : $\lambda_1 = 0$, $\lambda_\pm = \pm i$

associated eigenvectors : $V_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, $V_\pm = \begin{pmatrix} \pm i \\ 1 \\ 0 \end{pmatrix}$

$$\mapsto \Re(V_+) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \ \Im(V_+) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

general solution to the linearized system.

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = c_1 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + (c_2 \cos(t) + c_3 \sin(t)) \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + (c_3 \cos(t) - c_2 \sin(t)) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$= c_1 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + c_2 \left(\cos(t) \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} - \sin(t) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right) + c_3 \left(\cos(t) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \sin(t) \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right)$$

$$= c_1 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \underbrace{c_2 \left(\cos(t + \tfrac{\pi}{2}) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \sin(t + \tfrac{\pi}{2}) \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right) + c_3 \left(\cos(t) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \sin(t) \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right)}_{\text{circles } 1/4 \text{ period out of phase}}$$

height above x-y plane

$\longrightarrow$ solutions do not tend to $(0,0,0)$ as $t \to \infty$,

$\longrightarrow$ stable but not asymptotically stable.

Asymptotic stability of the full system

Using the Lyapunov function $V(x,y,z) = x^2 + y^2 + z^2$, we find that

$$
\begin{aligned}
DV \cdot f &= 2x \cdot (-y - xy^2 + z^2 - x^3) \\
&\quad + 2y \cdot (x - y^3 + z^3) \\
&\quad + 2z \cdot (-xz - x^2 z - yz^2 - z^5) \\
&= -2x^2y^2 - 2y^4 - 2x^4 - 2x^2z^2 - 2z^6,
\end{aligned}
$$

which is strictly negative in $U \setminus \{(0,0,0)\}$ for any neighbourhood U of the origin.

By Lyapunov's theorem, the origin is an asymptotically stable fixed point of the system.

3) The Lorenz system is:

$$\dot{x} = \sigma(y-x)$$
$$\dot{y} = \rho x - y - xz$$
$$\dot{z} = -\beta z + xy$$

<u>fixed points</u> : nullclines: $0 = \sigma(y-x) \longrightarrow x = y$

$0 = \rho x - y - xz \longrightarrow \rho - z = \frac{y}{x} = 1$ if $x \neq 0$

$0 = -\beta z + xy \longrightarrow z = \frac{xy}{\beta} = \frac{x^2}{\beta}$

$\hookrightarrow x = \pm\sqrt{(\rho-1)\beta}$ or $x = 0$

set $\alpha := \sqrt{(\rho-1)\beta}$.

fixed points are $p_1 = (0,0,0)$

$p_2 = (\alpha, \alpha, \rho-1)$

$p_3 = (-\alpha, -\alpha, \rho-1)$.

p_2 and p_3 are fixed points only if $\rho > 1$.

so the origin is a fixed point.

<u>Linearization</u>

$$Df = \begin{pmatrix} -\sigma & \sigma & 0 \\ \rho - z & -1 & -x \\ y & x & -\beta \end{pmatrix}, \quad \text{so} \quad Df(0) = \begin{pmatrix} -\sigma & \sigma & 0 \\ \rho & -1 & 0 \\ 0 & 0 & -\beta \end{pmatrix}$$

<u>eigenvalues</u>

$$0 = \det(\lambda I - Df(0)) = (\lambda+\sigma)(\lambda+1)(\lambda+\beta) - \sigma\rho(\lambda+\beta)$$
$$= \lambda^3 + (\sigma+1+\beta)\lambda^2 + (\sigma+\beta+\sigma\beta-\sigma\rho)\lambda + \sigma\beta - \sigma\rho\beta$$

By inspection, $-\beta$ is a root, so

$$0 = (\lambda+\beta)(\lambda^2 + (\sigma+1)\lambda + (\sigma-\sigma\rho))$$

$\lambda_1 = -\beta$ or $\lambda_\pm = \dfrac{-(\sigma+1) \pm \sqrt{(\sigma+1)^2 + 4\sigma(\rho-1)}}{2}$

$\longrightarrow$ stable if $(\sigma+1)^2 < 4\sigma(1-\rho)$ as $\beta, \sigma > 0$.

$\longrightarrow$ unstable if $(\sigma+1)^2 > 4\sigma(1-\rho)$

4) Take $u_1 = x$, $u_2 = y$

$v_1 = \dot{x}$, $v_2 = \dot{y}$.

Then we postulate that

$$\dot{u}_i = \frac{\partial H}{\partial v_i}$$

$$\dot{v}_i = \frac{-\partial H}{\partial u_i}$$

for some Hamiltonian function H.

From the above we derive

$$v_i = \frac{\partial H}{\partial v_i}, \quad \frac{u_i}{(u_1^2+u_2^2)^{3/2}} = \frac{\partial H}{\partial u_i};$$

so

$$H = -\frac{1}{(u_1^2+u_2^2)^{3/2}} + \frac{v_1^2+v_2^2}{2}$$

is a suitable Hamiltonian function by which the system can be written as a Hamiltonian system.

The orthogonal system is

$$\dot{u}_i = \frac{\partial H}{\partial u_i} \implies \dot{u}_i = \frac{u_i}{(u_1^2+u_2^2)^{3/2}}$$

$$\dot{v}_i = \frac{\partial H}{\partial v_i} \qquad \dot{v}_i = v_i$$

$$\hookrightarrow \begin{cases} \ddot{x} = \dfrac{x}{(x^2+y^2)^{3/2}} \\ \ddot{y} = \dfrac{y}{(x^2+y^2)^{3/2}} \end{cases}.$$