Mathemathical Modelling

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Contents

1 Lecture 1

1.1 Practical Information

You need to know

- Separable 1. order equations.
- Linear 1. order equations.
- 2. order linear equations with constant coefficients.

1.2 Dimensional Analysis

Basic facts

- Any physical relation has to make sense dimensionally.
- Any physical relation must be valid for any choice of fundamental units.

Remark.

Make sure remark looks better

- **Forbidden** $3m + 2kg = ?$
- $m = f(x, t)$ is legal
- e^{-t} and $s = 5t^2$, is nonsense
- **Dimension** is length, mass , energy, etc.
- **Unit** is meter, feet, year, etc

Given a variable R, we write $R = \overbrace{v(R)}^{\text{numerical value}} \underbrace{[R]}_{\text{unit}}$.

If we have a physical relation that is dimensionall correct that

$$f(R_1, R_2, \ldots, R_n) = 0 \quad \rightarrow \quad f(v(R_1), v(R_2), \ldots, v(R_n)) = 0$$

1.3 Fundamental Units

Given units $F_1, F_2, \ldots, F_m$ for fundamental if

$$F_1^{\alpha_1}, F_2^{\alpha_2}, \ldots, F_m^{\alpha m} = 0 \quad \rightarrow \quad \alpha_1 = \alpha_2 = \ldots = 0$$

This units are then independent.

Example 1. The units kg, m, s are independent.

Example 2. In a right angle triangle with angle α and hypothenus c. We know the area A is uniquely determined by α and c

$$A = f(c, \alpha)$$

α is dimensialless since $\alpha = \frac{s}{r}$. Since A scales as the square of the length, then is

$$f(ac, \alpha) = a^2 f(c, \alpha)$$
$$c = 1 \rightarrow f(a, \alpha) = a^2 f(1, \alpha) = a^2 h(\alpha)$$

Which then ends up with the relation

$$A = a^2 h(\alpha)$$

Make corollary environmet

Lets derive $A = a^2 h(\alpha)$ somwhat differently. We know there is a relation $f(A, c, \alpha) = 0$. We want to introduce new variables.

$$\Pi_1 = \frac{A}{c^2}, \quad c = c_1, \quad \alpha = \alpha_1$$

which means $f\left(c^2\Pi_1, c, \alpha\right) = 0$ and $h(\Pi_1, \alpha, c) = 0$. h must be dimensially consistent $\rightarrow$ h must be independent of c.

$$\begin{aligned} h(\Pi_1, \alpha) = 0 \leftrightarrow \Pi_1 = k(\alpha) \\ \rightarrow \frac{A}{c^2} = k(\alpha) \quad \leftrightarrow \quad A = c^2 k(\alpha) \end{aligned}.$$

1.4 Trinity of the first atomic blast

We assume there is a relation

$$f(E, \rho, r, t) = 0$$

- Energy: E, $[E] = kgm^2s^{-2}$
- Mass density of air: ρ, $[\rho] = kg^{-3}$
- Radius: r , $[r] = m$
- Time: t, $[t] = s$

We choose 3 independent variables, say r, t, ρ. Also we call r, t, ρ **core variables**. Let is define a dimensionalless number Π_1 such that

$$[\Pi_1] = 0$$

The relation is now given by $h(\Pi, t, r, \rho) = 0$, where h is independent of t , r and ρ. Which in fact is $h(\Pi) = 0$, where $\Pi_1 = c$ s.t. $[c] = 1$.

Given by the definitnion is

$$\frac{Et^2}{\rho r^5} = c \quad \rightarrow \quad E = \frac{c\rho r^5}{t^2}$$

Using $\rho = 12kgm^{-3}$, $r = 110m$, $t = 6 \cdot 10^{-3}$ do we end up with the relation

$$E = c \cdot 7.5 \cdot 10^{13} J$$

1.5 Steady-state single phase flow in a uniform straight pipeline

Figure of a pipe

Pipe with flow u, length L and pressure drop Δp Then there is a relation between

- L : length, $[L] = m$
- D: diameter $[D] = m$
- u: flow rate $[u] = ms^{-1}$
- Δp: Pressure drop, $\left[\Delta kgm^{-1}s^{-2}\right]$
- μ: (Shear) viscousity $[\mu] = kgm^{-1}s^{-1}$
- ρ: mass density: $[\rho] = kgm^{-3}$
- E: Wall roughness: $[E] = m$

We have to choose 3 core variables and they are not unique. Since we have 3 independent units ρ, u, D are independent such that it can be a core variable:

$$\Pi_1 = \frac{L}{D} \quad , \quad \Pi_2 = \frac{\Delta p}{\rho u^2} \quad , \quad \Pi_3 = \frac{\rho}{\mu} \quad , \quad \Pi_4 = \frac{E}{D}$$

Then the relation is

$$f\left(\Pi_1, \Pi_2, \Pi_3, \Pi^4, \rho, D, u\right) = 0 \quad \Pi_2 = h\left(\Pi_1, \Pi_3, \Pi_4\right) \leftrightarrow \frac{\Delta p}{\rho u^2} = h\left(\Pi_1, \Pi_3, \Pi_4\right)$$

$$\rightarrow \frac{\Delta p}{u^2 \rho} = \Pi_1 k\left(\Pi_3, \Pi_4\right)$$

$$\Delta p = u^2 \rho \frac{L}{D} k\left(\frac{\rho D u}{\mu}, \frac{E}{D}\right)$$

$$\text{measure} \quad \frac{\rho D \mu}{\mu} \quad , \quad k = \frac{\Delta p D}{u^2 \rho}$$

2 Lecture 2

2.1 Practical Information

Ask for zoom meeting. ola.mahlen@ntnu.no, wednesday 13-14.

2.2 Recall

Last time did we consider steady-state single phase in a flow in a pipe.

- Assuming $f(L, \Delta p, u, \mu, D, E, \rho) = 0$ we arrive with this formula

$$\frac{\Delta p D}{u^2 \rho L} = k \left(\overbrace{\frac{\rho u D}{\mu}}^{\text{Reynhold number}}, \underbrace{\frac{E}{D}}_{\text{Relative wall roughness}} \right)$$

- Dimensionless numbers are often called **dimensionless groups**. Such numbers are independent of choice of fundamental units. They have real physical meaning. **Reynholds number** R_e essentially define what type of flow. Usually $R_e < 2000$ is it laminar flow and $R_e > 4000$ turbulent flow.

2.3 Scaling

Let a pipe have diameter D and flow rate u such that $t_v = \frac{D}{u}$. Then can we describe

$$t_\alpha = \frac{D^2}{\frac{\mu}{e}}$$

where μ is the kinematic viscosity. Then is R_e defined such that

$$R_e = \frac{t_\alpha}{t_v}$$

Assume we have the relation

$$R_1 = f(R_2, \ldots, R_m)$$

Such that it exist an

$$\Pi_1 = g(\Pi_2, \Pi_2, \ldots, \Pi_{m-k}).$$

2.4 Buckinghams Pi-Theorem

Assume we have a dimensionally valid relation $f(R_1, \ldots, R_m) = 0$ and a set of fundemental units $F_1, F_2, \ldots, F_n$ such that

$$[R_j] = F_1^{a_{j1}} F_2^{a_{j2}} \ldots F_n^{a_{jn}} \quad j = 1, 2, \ldots, m$$

This then defines the dimension matrix A given by

Table 1: caption

	F_1	F_2	$\dots$	F_n
R_1	a_{11}	a_{11}		a_{1n}
R_2	a_{21}	a_{21}		a_{2n}
$\vdots$		$\ddots$		
R_n	a_{m1}	$\dots$		a_{mn}

Fix better table environment.

Let $rank\left(A\right) = dim\left(row\left(A\right)\right) = k$. This translates to that we have k dimensionally independent variables. Choosing k linearly independent row vectors, corresponds to choosing core variables. Let this basis be $\mathbf{a}_{i1}, \mathbf{a}_{i2}, \dots, \mathbf{a}_{ik}$. Let the rest of the row vectors be

$$\mathbf{a}_{j_1}, \mathbf{a}_{j_2}, \dots, \mathbf{a}_{j_{m-k}}$$

Then is $\mathbf{a}_{j_r} = \sum_{s=1}^{k} C_{j_r,s}\mathbf{a}_{\mathbf{i_s}}$ where $r = 1, \dots, m-k$. We end up with the equation

$$\Pi_r = \frac{R_{j_r}}{R_{i_1}^{r_{j_r,1}} R_{j_2}^{a_{j_r,2}} \dots R_{j_k}^{a_{j_r,k}}}$$

Are dimensionally numbers.

Our relation becomes

$$g\left(\Pi_1, \dots, \Pi_{m-k}\right) = 0, \qquad \begin{cases} i_1, i_2, \dots, i_k \\ j_1, \dots, j_{m-k} \end{cases}$$

Example. Swinging pendulum
Assume there is a relation

$$f\left(w, \alpha_0, L, M, g, \right) = 0$$

where w is the frequenxy, g gravitational acceleration, M mass, α_0 the swinging angle. We can set L, M, g as core variables such that

$$\left[\frac{L}{g}\right] = s^2 \quad \to \quad \left[\frac{L}{g}w^2\right] = 1$$

$$f\left(w, \alpha_0, L, M, g\right) = 0 \implies \quad g\left(\alpha_0, \frac{Lw^2}{g}\right) = 0$$

2.5 Scaling

We have a problem at hand, usually differential equations. Then we tru to find representative scales for the various variables, and then write the equation on so-called fimensionless form. This has several advantages

- Our dimensionless variables are of order 1 .
- We get rid of a lot of physical constants.
- It makes us able to see what terms are "small" in the equation. The idea is to introduce dimensionless variables by introducing appropiate scales. If we have a stick of lenght L, we choose L as length scale i.e

$$x^* = Lx \quad \text{Where} \quad x \quad \text{is dimenionless}$$

Example. Heat flow in a rod with length L. Let $u^*(x^*, t^*)$ be the temperatur with the boundary conditions

$$u^*(0, t^*) = 0 \quad u^*(L, t^*) = 0$$

If we let the model be

$$\frac{\partial u^*}{\partial t^*} = D \cdot \frac{\partial^2 u^*}{\partial x^{*2}}, \quad u^*(0, t^*) = 0 \quad u^*(L, t^*) = 0$$
$$u^*(x^*, 0) = u_0 \sin\left(\pi \frac{x^*}{L}\right)$$

We fund the tune scale T by scales **balancing the equation** .
Let $x^* = Lx$, and $t^* = Tt$, where T is to be determined $u^* = u_0 u$. If we find $u(x, t)$, then the physical temperature is given by

$$u^*(x^*, t^*) = u_0 u\left(\frac{x^*}{L}, \frac{t^*}{T}\right)$$

We have $u(0, t) = u(1, t) = 0$

$$\frac{\partial u^*}{\partial t^*} = D \frac{\partial^2 u^*}{\partial x^{*2}} \quad \Longrightarrow \quad \frac{u_0}{T} \frac{\partial u}{\partial t} = \frac{u_0}{L^2} D \frac{\partial^2}{\partial x^2}$$
$$\leftrightarrow \frac{\partial u}{\partial t} = \left(\frac{TD}{L^2}\right) \frac{\partial^2 u}{\partial x^2} \quad \text{Balancing the equation}$$
$$\frac{TD}{L^2} = 1 \quad \Longrightarrow T = \frac{L^2}{D}$$
$$u^*(x^*, 0) = u_0 \sin\left(\pi \frac{x^*}{L}\right)$$
$$u(x, 0) = \sin(\pi x)$$

which fulfills the condition

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \quad , u(0, t) = u(1, t) = 0$$

3 Lecture Something

3.1 Recall

$$\begin{aligned}
\frac{\partial u^*}{\partial t^*} &= D\frac{\partial^2 u^*}{\partial x^{*2}} \\
0 \le x^* &\le L \\
x^* &= Lx \\
t^* &= Tt \\
u^* &= u_0
\end{aligned}$$

We can also recall

$$\begin{aligned}
u^*\left(x^*, t^*\right) &= u_0 u\left(\frac{x^*}{L}, \frac{t^*}{T}\right) \\
\frac{u_0}{T}\frac{\partial u}{\partial t} &= D\frac{u_0}{L^2} \implies \frac{\partial u}{\partial t} = \frac{TD}{L^2}\frac{\partial^2 u}{\partial x^2} \\
\text{Require} \quad \frac{TD}{L^2} = 1 \implies T = \frac{L^2}{D}
\end{aligned}$$

This can be generelized to

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad 0 \le x \le 1$$

3.2 Sinking Ball

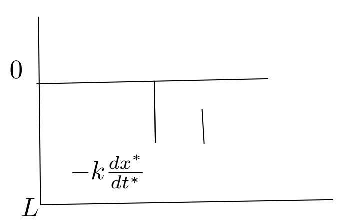

Figure 1: sinkingball

Let

- ρ_b e mass density of ball
- $\rho_f$mass density of fluid
- V Volume of ball

Then is the equation

$$\rho_b V g - \rho_f V g = V g \rho_b \left(1 - \frac{\rho_f}{\rho_b}\right)$$
$$= m\hat{g} \implies \hat{g} = g\left(1 - \frac{\rho_f}{\rho_b}\right)$$

And we then end up with the newtions law

$$m\frac{dx^{*2}}{dt^{*2}} = m\hat{g} - k\frac{dx^*}{dt}, \quad \text{Friction coefficient} \quad k$$

where

$$x^*(0) = 0, \quad \frac{dx^*}{dt^*}(0) = V$$

The cases can be described as follows

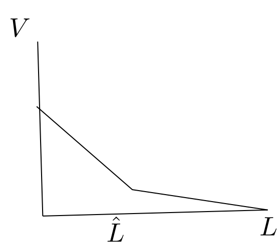

Figure 2: highV

1. High friction, not so high V. Ball will sink at constant speed most of the time.
2. Friction is low, andC not "too high". ("Free fall with V=0")
3. High V, and high friction $m\frac{d^2x^*}{dt^{*2}} = m\hat{g} - k\frac{dx^*}{dt^*}$

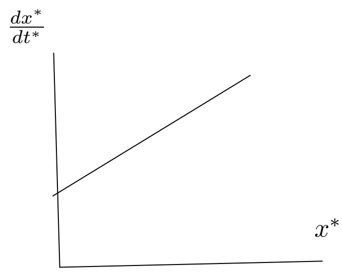

Figure 3: frefall

For this problem there is 3 characteristic speeds

1. V : initial velocity
2. v_0 : equilibrium speed in case A $v_0 = \frac{m\hat{g}}{k}$
3. v_f : free fall $v_f = \sqrt{2\hat{g}L}$

Let us put

$$\begin{aligned} \frac{d^2x^*}{dt^{*2}} = 0 &\implies k\frac{dx^*}{dt} = \hat{g}m \\ &\implies \frac{dx^*}{dt^*} = \hat{g}\frac{m}{k} = v_0 \end{aligned}$$

and put

$$\begin{aligned} x^*(0) = &\frac{dx^*}{dt^*}(0) = 0 \\ k = 0 & \end{aligned}$$

3.2.1 Scaling

1. Case A: The ball sinks at constant speed "most" of the time.
   (a) Length scale L : $x^* = Lx$. Since the ball falls with speed most of the time, a timescale would be $T = \frac{L}{v_0}$. v is not much larger than v_0

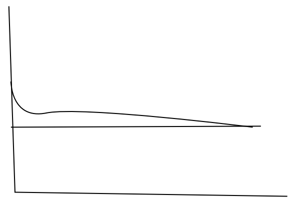

Figure 4: sinking

$\Longrightarrow$ it is not so that $v \gg v_0$

$$m\frac{L}{T^2}x^{''} = m\hat{g} - k\frac{L}{T}x^{'} \quad \text{Divide by } L$$
$$\Longrightarrow m\frac{1}{kT}x^{''} = \frac{Tm\hat{g}}{KL} - x^{'}$$
$$\frac{m}{k\frac{L}{v_0}}x^{''} = \frac{\frac{k}{v_0}m\hat{g}}{kL} - x^{'}$$
$$\Longrightarrow \frac{mv_0}{Lk}x^{''} = \frac{Lm\hat{g}}{KLv_0} - x^{'}$$

We can then derive

$$\frac{m\frac{m\hat{g}}{k}}{Lk}x^{''} = 1 - x^{'}$$
$$\Longrightarrow \frac{m^2\hat{g}}{Lk^2}x^{''} = 1 - x^{'}$$
$$\Longrightarrow \frac{m^2\hat{g}^2}{\hat{g}Lk^2}x^{''} = 1 - x^{'}$$
$$\epsilon x^{''} = 1 - x^{'} \quad \text{Where} \quad \epsilon = 2\left(\frac{v_0}{v_f}\right)^2$$

The condition are $x(0) = 0$, $\frac{L}{T}x^{'}(0) = V$ which can be rewritten to

$$x^{'}(0) = \frac{TV}{L})\frac{\frac{L}{v_0 V}}{L} = \frac{V}{v_0} = \mu$$

3.3 Let Analyze The equation

In case A is the

$$\epsilon\ddot{x} = 1 - \dot{x}$$

An approximation we can do is to put $\epsilon = 0$ such that

$$0 = 1 - \dot{x} \quad x(0) = 0, \quad \dot{x}(0) = \mu \quad \dot{x} = 0$$

unless $\mu = 1$, we cant find a solution.

3.3.1 Case B

Small friction, V is not too high. Let the lengthscale be L.

$$\frac{d^2}{dt^{*2}}x^{*2} = \hat{g}, \quad x^*(0) = \frac{dx^*}{dt^*}(0) = 0$$
$$x^*(t^*) = \frac{1}{2}\hat{g}(t^*)^2$$

Hit the bottom with speed V_f . We can choose time scale T such that

$$T = \frac{L}{v_f}$$

So gain

$$\frac{mL}{T^2}\ddot{x} = m\hat{g} - \frac{kL}{T}\dot{x}$$

What you can observe is that gravity dominates so we modify the equation to be

$$\frac{L}{\hat{g}T^2}\ddot{x} = 1 - \frac{kL}{gmT}\dot{x}$$
$$\implies 2\ddot{x} = 1 - \left(\frac{v_F}{v_0}\right), \quad \frac{K}{T}\dot{x}(0) = 0$$
$$2\ddot{x} = 1 - \epsilon\dot{x} \quad \dot{x}(0) = \frac{V}{v_f} = \mu$$

3.3.2 Case C: High V and high friction

Let us consider

$$m\frac{d^2x^*}{dt^{*2}} = -kV \quad \frac{dx^*}{dt^*} = V - \frac{kV}{m}t^* = 0$$

Where we choose the scales $t^* = \frac{m}{k} = T$, $L = \frac{Vm}{k}$, where $TV = L$.

$$\implies \ddot{x} = \epsilon - \dot{x}, \quad x(0) = 1, \quad \dot{x} = 1, \quad \epsilon = \frac{v_0}{V}$$

Example. Let

$$a\frac{d^2x^*}{dt^{*2}} + b\frac{dx^*}{dt^*} + cx^* = 0$$
$$x^*(0) = x_0, \quad \frac{dx^*}{dt^*}(0) = 0$$

Three waus to scale by balancing the equation. Last term ”small”

$$x^* = x_0 x, \quad t^* = Tt$$

Where T is to be determined.

$$a\frac{x_0}{T^2}\ddot{x} + b\frac{x_0}{T}\dot{x} + cx_0 = 0$$

$$\ddot{x} + \frac{bT}{a}\dot{x} + \frac{cT^2}{a} = 0$$

If we are smart can we choose the timescale $T = \frac{a}{b}$ then we get

$$\ddot{x} + \dot{x} + \frac{ca^2}{b^2a} = 0.$$
$$\implies \ddot{x} + \dot{x} + \left(\frac{ca}{b^2}\right)x = 0$$

3.4 Turbulence

Reynold number

$$R_e = \frac{u\rho L}{\mu} = \frac{uL}{\frac{mu}{\rho}} = \frac{uL}{\mathcal{V}}$$

Then we have

$$\frac{\partial v}{\partial t} = \mathcal{V}\frac{\partial^2 v}{\partial x^2}$$

4 References