TMA4165: Problem Sheet III

published 02/02/2020

1. Find and classify the fixed points of the following systems and draw their phase portraits.

(i) $\dot{x} = x - y$, $\dot{y} = x + y - 2xy$;
(ii) $\dot{x} = 1 - xy$, $\dot{y} = (x-1)y$;
(iii) $\dot{x} = (1 + x - 2y)x$, $\dot{y} = (x-1)y$;
(iv) $\dot{x} = x - y$, $\dot{y} = x^2 - 1$;
(v) $\dot{x} = -6y + 2xy$, $\dot{y} = y^2 - x^2$;
(vi) $\dot{x} = \sin(x)\cos(y)$, $\dot{y} = \sin(y)\cos(x)$.

2. Use the Lyapunov function $V(x, y, z) = x^2 + y^2 + z^2$ to show that the origin is an asymptotically stable fixed point of the system

$$\begin{aligned}\dot{x} &= -y - xy^2 + z^2 - x^3 \\ \dot{y} &= x - y^3 + z^3 \\ \dot{z} &= -xz - x^2 z - yz^2 - z^5.\end{aligned}$$

Show that the trajectories of the linearized system near $(x, y, z) = (0, 0, 0)$ lie on circles in planes perpendicular to $(0, 0, 1)$, and so the origin is stable but not asymptotically stable for the linearized system.

3. For $\sigma, \rho, \beta > 0$, the Lorenz equations are:

$$\begin{aligned}\dot{x} &= \sigma(y - x) \\ \dot{y} &= \rho x - y - xz \\ \dot{z} &= -\beta z + xy.\end{aligned}$$

Determine conditions on the parameters for which the origin is asymptotically stable and conditions for which the origin is unstable.

4. Show that the planar two-body problem can be written as a Hamiltonian system with two degrees of freedom in $\mathbb{R}^4 \backslash \{\mathbf{0}\}$:

$$\begin{aligned}\ddot{x} &= -\frac{x}{(x^2 + y^2)^{3/2}} \\ \ddot{y} &= -\frac{y}{(x^2 + y^2)^{3/2}}.\end{aligned}$$

Find the gradient system orthogonal to this one in $\mathbb{R}^4$.