NTNU

Norwegian University of Science and Technology

Department of Mathematical Sciences

Examination paper for **TMA4265 Stochastic Modeling**

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Examination date: November 28, 2018

Examination time (from–to): 09:00-13:00

Permitted examination support material: C:

- Calculator CITIZEN SR-270X, CITIZEN SR-270X College, HP30S, Casio fx-82ES PLUS with empty memory.
- Tabeller og formler i statistikk, Tapir forlag.
- K. Rottmann: Matematisk formelsamling.
- Bilingual dictionary.
- One yellow, stamped A5 sheet with own handwritten formulas and notes (on both sides).

Other information:
Note that all answers must be justified.
All ten subproblems are equally weighted.

Language: English

Number of pages: 5

Number of pages enclosed: 4

Checked by:

Informasjon om trykking av eksamensoppgave
Originalen er:
1-sidig ☐ **2-sidig** ☒
sort/hvit ☒ **farger** ☐
skal ha flervalgskjema ☐

Date Signature

Problem 1

We draw a realization from each of the following three Markov chains;

$$
\mathbf{P}_1 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \!\!\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{pmatrix} 0.6 & 0.2 & 0.2 \\ 0.1 & 0.7 & 0.2 \\ 0.1 & 0.3 & 0.6 \end{pmatrix} \end{array}, \quad
\mathbf{P}_2 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \!\!\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{pmatrix} 0.5 & 0.5 & 0 \\ 0.1 & 0.8 & 0.1 \\ 0 & 0.1 & 0.9 \end{pmatrix} \end{array}, \quad
\mathbf{P}_3 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \!\!\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{pmatrix} 0.8 & 0.2 & 0 \\ 0.4 & 0.5 & 0.1 \\ 0 & 0.3 & 0.7 \end{pmatrix} \end{array}.
$$

Here, matrix elements $P(k,l) = P(X_t = l | X_{t-1} = k)$, and $X_t \in \{1,2,3\}$, for times $t = 1, 2, \ldots, 100$. We initiate the process with $X_0 = 1$.

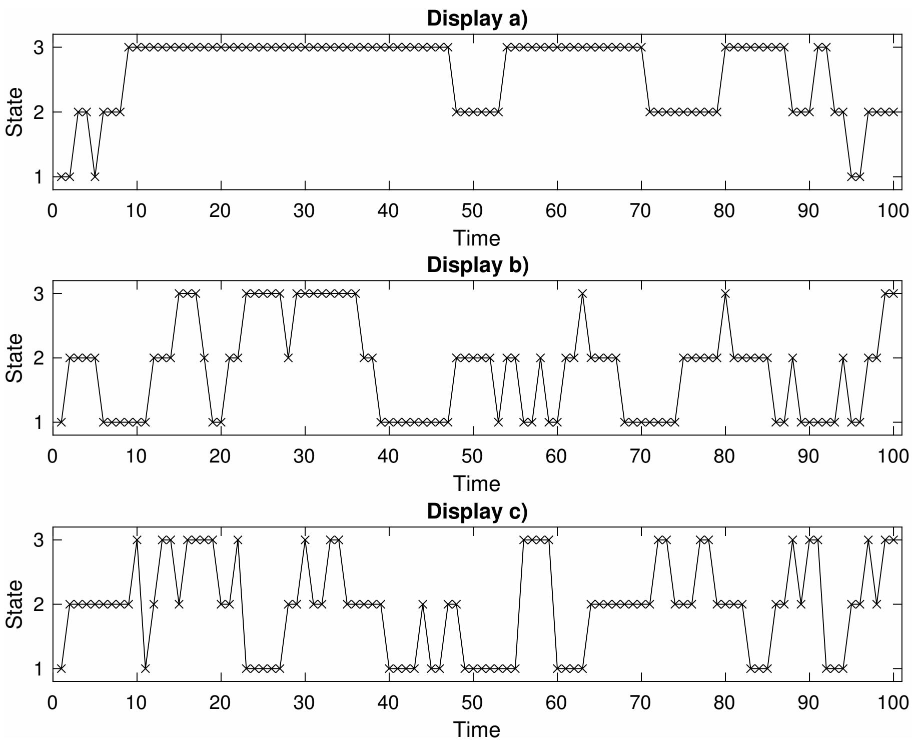

Figure 1: Markov chain realizations of different Markov transition probabilities, plotted as a function of time.

a)

Relate Markov chains defined by transition matrix $\mathbf{P}_1$, $\mathbf{P}_2$ and $\mathbf{P}_3$ to one of the displays a), b) and c) in Figure 1. Explain clearly why you can make your chosen relations.

Make rough sketches of possible realizations from the Markov chains defined by the following two transition matrices:

$$\mathbf{P}_1 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}\!\!\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{pmatrix} 0.6 & 0.4 & 0 \\ 0.3 & 0.6 & 0.1 \\ 0 & 0 & 1 \end{pmatrix} \end{array}, \quad \mathbf{P}_2 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}\!\!\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{pmatrix} 0.4 & 0.6 & 0 \\ 0 & 0.4 & 0.6 \\ 0.6 & 0 & 0.4 \end{pmatrix} \end{array},$$

starting at $X_0 = 1$, and for $t = 0, 1, 2, \ldots, 25$.

b) Consider the following Markov chain;

$$\mathbf{P} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}\!\!\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{pmatrix} 0.5 & 0.5 & 0 \\ 0.1 & 0.8 & 0.1 \\ 0 & 0.1 & 0.9 \end{pmatrix} \end{array}.$$

Calculate $P(X_3 = 2|X_1 = 2)$.

Calculate $P(X_2 = 2|X_1 = 2, X_3 = 2)$.

Calculate the long-run distribution; $\pi_j = \lim_{t\to\infty} P(X_t = j|X_1 = 1)$, where $\sum_{j=1}^{3} \pi_j = 1$.

Problem 2

We study a simple version of the game 'picture-lotto'. Ingrid and Sverre have one board each, with 9 pictures. There are 18 picture-cards (head down) on the table, matching the ones that Ingrid and Sverre have on their boards. In a stepwise procedure, they turn one and one card on the table. When one card is turned, it is placed to cover the identical picture on the right board. The winner is the first person to cover her/his entire board.

We let X_t be the number of covered pictures on Ingrid's board after step t, while $t - X_t$ is the number of covered pictures on Sverre's board after step t. We formulate this as a Markov chain process.

a)

What are the possible states for X_t, $t = 1, \ldots, 18$?

What are the transition probabilities $P_{k,l}(t) = P(X_t = l|X_{t-1} = k)$?

b)

If $X_{15} = 8$, what is the chance that Ingrid wins?

If $X_{14} = 8$, what is the chance that Ingrid wins?

If $X_{13} = 7$, what is the chance that Ingrid wins?

Problem 3

A large pipeline system has lots of components. In this problem we consider minor and major failures on such components. In practice, minor failures might consist of small leaks or partly corroded pipes, which can be fixed relatively quickly, at moderate costs. Major failures might consist of broken pipe components that require more time and money to fix.

We will treat minor and major system failures as independent. We further assume that the number of minor failures on the pipeline system follows a Poisson process with rate $\mu_1 = 0.4$ per day, while the number of major failures follows a Poisson process with rate $\mu_2 = 0.2$ per day.

a)

What is the probability that no failures of any kind have occurred after one week (7 days)?

What is the probability that the first minor failure occurs before the first major failure?

b) The number of failures (minor plus major) after the first week is 4.

What is then the probability that there are more minor failures than major failures?

c) Assume that the cost of repairing a minor failure has mean 10000 kr and standard deviation 1000 kr, and that the cost of repairing a major failure has mean 50000 kr and standard deviation 10000 kr. All repair costs are assumed to be independent of each other.

What is the expected repair costs after a week?

What is the variance of repair costs after a week?

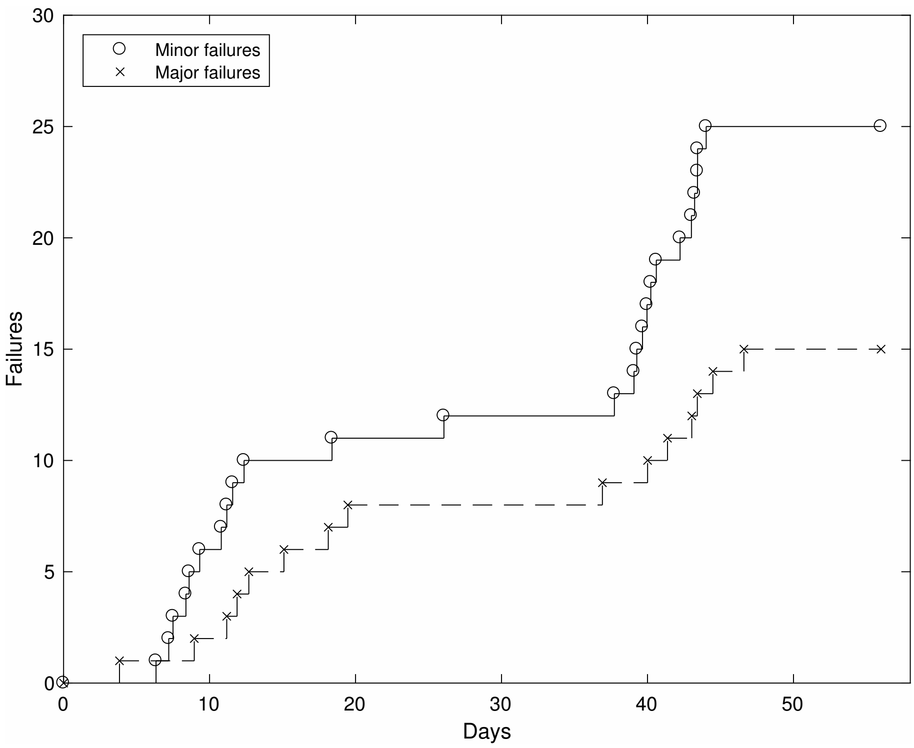

Figure 2: Number of minor (solid) and major (dashed) failures, plotted over time.

d) Figure 2 shows realized failures of minor and major types from a pipeline system.

Does it look like the modeling assumptions mentioned above hold?

Can you suggest an alternative model for the realized failures?

Problem 4

The company running the pipeline system in Problem 3 conducts extensive testing to study the repair times as well as the times to component failures. For a system in functional state (without failures), they learn that the minor failure rate is $\mu_1 = 0.4$ per day, while the major failure rate is $\mu_2 = 0.2$ per day. They further notice that a system in the state of minor repair moves to the state of major repair with rate $\mu_{1,2} = 0.3$ per day. The system cannot move directly from the state of major repair to minor repair. The repair rates, bringing the pipeline system to a functional state, are set to $\lambda_1 = 1$ per day from the state of minor failure and $\lambda_2 = 0.5$ per day from the state of major failure.

a) Set up the state diagram for the continuous time Markov process.

Compute the long-term probabilities of each state.

Problem 5

In a particular country the property prices have gone up about 5 percent per year the last decade, with about 2 percent standard deviation per year. Starting at a relative price of 1 at time $t = 0$, economists indicate that a reasonable model for the relative property price $X(t)$ is to have mean $1 + 0.05t$ over time (where t is year) and an additive error term represented by a Brownian motion with zero mean and variance $0.02^2 t$.

a)

What is the probability of having relative property price $X(5)$ larger than 1.2?

What is the probability that the price is larger after 5 years than 4 years?

Formulas: TMA4265 Stochastic Modeling:

The law of total probability

Let $B_1, B_2, \ldots$ be pairwise disjoint events with $P(\cup_{i=1}^{\infty} B_i) = 1$. Then

$$P(A|C) = \sum_{i=1}^{\infty} P(A|B_i \cap C)P(B_i|C),$$

$$E[X|C] = \sum_{i=1}^{\infty} E[X|B_i \cap C]P(B_i|C).$$

Discrete time Markov chains

Chapman-Kolmogorov equations

$$P_{ij}^{(m+n)} = \sum_{k=0}^{\infty} P_{ik}^{(m)} P_{kj}^{(n)}.$$

For an irreducible and ergodic Markov chain, $\pi_j = \lim_{n\to\infty} P_{ij}^{(n)}$ exist and is given by the equations

$$\pi_j = \sum_i \pi_i P_{ij} \quad \text{and} \quad \sum_i \pi_i = 1.$$

For transient states i, j and k, the mean passage time from i to $j \neq i$, M_{ij}, is

$$M_{ij} = 1 + \sum_k P_{ik} M_{kj}.$$

The Poisson process

The waiting time to the n-th event (the n-th arrival time), X_n, has probability density

$$f_{X_n}(t) = \frac{\lambda^n t^{n-1}}{(n-1)!} e^{-\lambda t} \quad \text{for} \quad t \geq 0.$$

Given that the number of events $N(t) = n$, the arrival times $X_1, X_2, \ldots, X_n$ have the uniform joint probability density

$$f_{X_1, X_2, \ldots, X_n | N(t)}(x_1, x_2, \ldots, x_n | n) = \frac{n!}{t^n} \quad \text{for} \quad 0 < x_1 < x_2 < \ldots < x_n \leq t.$$

Markov processes in continuous time

A (homogeneous) Markov process $X(t)$, $0 \leq t \leq \infty$, with state space $\Omega \subseteq \mathbf{Z}^+ = \{0, 1, 2, \ldots\}$, is called a birth and death process if

$$P_{i,i+1}(h) = \lambda_i h + o(h)$$

$$P_{i,i-1}(h) = \mu_i h + o(h)$$

$$P_{i,i}(h) = 1 - (\lambda_i + \mu_i)h + o(h)$$

$$P_{ij}(h) = o(h) \quad \text{for } |j - i| \geq 2$$

where $P_{ij}(s) = P(X(t+s) = j | X(t) = i)$, $i, j \in \mathbf{Z}^+$, $\lambda_i \geq 0$ are birth rates, $\mu_i \geq 0$ are death rates.

The Chapman-Kolmogorov equations

$$P_{ij}(t+s) = \sum_{k=0}^{\infty} P_{ik}(t) P_{kj}(s).$$

Limit relations

$$\lim_{h \to 0} \frac{1 - P_{ii}(h)}{h} = v_i \,, \quad \lim_{h \to 0} \frac{P_{ij}(h)}{h} = q_{ij} \,, \;\; i \neq j$$

Kolmogorov's forward equations

$$P'_{ij}(t) = \sum_{k \neq j} q_{kj} P_{ik}(t) - v_j P_{ij}(t).$$

Kolmogorov's backward equations

$$P'_{ij}(t) = \sum_{k \neq i} q_{ik} P_{kj}(t) - v_i P_{ij}(t).$$

If $P_j = \lim_{t \to \infty} P_{ij}(t)$ exist, P_j are given by

$$v_j P_j = \sum_{k \neq j} q_{kj} P_k \quad \text{and} \quad \sum_j P_j = 1.$$

In particular, for birth and death processes

$$P_0 = \frac{1}{\sum_{k=0}^{\infty} \theta_k} \quad \text{and} \quad P_k = \theta_k P_0 \quad \text{for} \quad k = 1, 2, \ldots$$

where

$$\theta_0 = 1 \quad \text{and} \quad \theta_k = \frac{\lambda_0 \lambda_1 \cdot \ldots \cdot \lambda_{k-1}}{\mu_1 \mu_2 \cdot \ldots \cdot \mu_k} \quad \text{for} \quad k = 1, 2, \ldots$$

Queueing theory

For the average number of customers in the system L, in the queue L_Q; the average amount of time a customer spends in the system W, in the queue W_Q; the service time S; the average remaining time (or work) in the system V, and the arrival rate λ_a, the following relations obtain

$$L = \lambda_a W.$$
$$L_Q = \lambda_a W_Q.$$
$$Z = \lambda_a E[S].$$
$$V = \lambda_a E[S W_Q^*] + \lambda_a E[S^2]/2.$$

Gaussian processes

The multivariate Gaussian density for $n \times 1$ random vector $\boldsymbol{x} = (x_1, \dots, x_n)$ is

$$p(\boldsymbol{x}) = \frac{1}{(2\pi)^{n/2}|\boldsymbol{\Sigma}|^{1/2}} \exp\left(-\frac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\boldsymbol{x}-\boldsymbol{\mu})\right), \quad \boldsymbol{x} \in \mathbb{R}^n,$$

where size $n \times 1$ mean vector $\boldsymbol{\mu} = (\mu_1, \dots, \mu_n)$, $E(x_i) = \mu_i$, and

$$\boldsymbol{\Sigma} = \begin{bmatrix} \Sigma_{1,1} & \dots & \Sigma_{1,n} \\ \dots & \dots & \dots \\ \Sigma_{n,1} & \dots & \Sigma_{n,n} \end{bmatrix}, \quad \Sigma_{i,j} = \mathrm{Cov}(x_i, x_j).$$

Let $\boldsymbol{x}_A = (x_{A,1}, \dots, x_{A,n_A})$ and $\boldsymbol{x}_B = (x_{B,1}, \dots, x_{B,n_B})$, be two subsets of variables, with block mean and covariance structure

$$\boldsymbol{\mu} = (\boldsymbol{\mu}_A, \boldsymbol{\mu}_B), \quad \boldsymbol{\Sigma} = \begin{bmatrix} \boldsymbol{\Sigma}_A & \boldsymbol{\Sigma}_{A,B} \\ \boldsymbol{\Sigma}_{B,A} & \boldsymbol{\Sigma}_B \end{bmatrix}.$$

The conditional density of $\boldsymbol{x}_A$, given $\boldsymbol{x}_B$, is Gaussian with

$$\begin{aligned} E(\boldsymbol{x}_A|\boldsymbol{x}_B) &= \boldsymbol{\mu}_A + \boldsymbol{\Sigma}_{A,B}\boldsymbol{\Sigma}_B^{-1}(\boldsymbol{x}_B - \boldsymbol{\mu}_B), \\ \mathrm{Var}(\boldsymbol{x}_A|\boldsymbol{x}_B) &= \boldsymbol{\Sigma}_A - \boldsymbol{\Sigma}_{A,B}\boldsymbol{\Sigma}_B^{-1}\boldsymbol{\Sigma}_{B,A}. \end{aligned}$$

The Brownian motion has increments $x(t_i) - x(t_{i-1})$ with the following properties, for any configuration of times $t_0 = 0 < t_1 < t_2 < \dots$:

- $x(t_i) - x(t_{i-1})$ and $x(t_j) - x(t_{j-1})$ are independent for all $i \neq j$.
- the distribution of $x(t_i) - x(t_{i-1})$ is identical to that of $x(t_i + s) - x(t_{i-1} + s)$, for any s.
- $x(t_i) - x(t_{i-1})$ is Gaussian distributed with 0 mean and variance $\sigma^2(t_i - t_{i-1})$.

Unless otherwise stated, $x(0) = 0$.

Some mathematical series

$$\sum_{k=0}^{n} a^k = \frac{1 - a^{n+1}}{1 - a} \quad , \qquad \sum_{k=0}^{\infty} k a^k = \frac{a}{(1-a)^2} \quad .$$