Repetition

Theorem

A state i is recurrent if and only if

$$\sum_{n=1}^{\infty} P_{ii}^{(n)} = \infty.$$

Equivalently, state i is transient if and only if

$$\sum_{n=1}^{\infty} P_{ii}^{(n)} < \infty.$$

Note: The theorem states that state i is recurrent if and only if the Markov chain is expected to return infinitely many times to state i. Equivalently, state i is transient if and only if the Markov chain is expected to return finitely many times to state i.

Theorem

If $i \sim j$, then state i is recurrent if and only if state j is recurrent.

Note: This means that recurrent/transient is a property shared by the entire equivalence class.

Important observations

- The definition of a **regular Markov chain** is equivalent to a Markov chain with finite state space being **irreducible** and **aperiodic** (and **positive recurrent**).
- If it is possible to leave an equivalence class, it is **transient**.
- If an equivalence class is absorbing
  - it is **recurrent** if the equivalence class is finite.
  - it may be **recurrent** or **transient** if the equivalence class is countably infinite.
- If the state space is finite, there must be at least one recurrent equivalence class.
- If the state space is infinite, there can be zero recurrent equivalence classes.

Theorem

Consider a recurrent irreducible aperiodic Markov chain with state space $\{0, 1, \ldots\}$. Then

1)
$$\lim_{n\to\infty} P_{ii}^{(n)} = \frac{1}{m_i}, \quad i = 0, 1, \ldots,$$
where $m_i = \sum_{n=0}^{\infty} n f_{ii}^{(n)}$ is the mean duration between visits to state i.

2)
$$\lim_{n\to\infty} P_{ji}^{(n)} = \lim_{n\to\infty} P_{ii}^{(n)}$$
for all states i and j.

Note: The conditions guarantee the existence of $\lim_{n\to\infty} P_{ij}^{(n)}$ for all pairs of states i and j.
Note 2: The limiting distribution fails to exist if $m_i = \infty$ for $i = 0, 1, \ldots$.

Definition

A state i is **positive recurrent** if $m_i < \infty$ and **null recurrent** if $m_i = \infty$.

Note: Positive recurrent and **null recurrent** (and **recurrent** and **transient**) are properties of the equivalence classes.

Note 2: If a recurrent equivalence class is finite, it is allways positive recurrent. Null recurrent is only relevant for countably infinite equivalence classes.

Theorem

In a positive recurrent aperiodic equivalence class with states $j = 0, 1, \ldots$,

1)

$$\lim_{n\to\infty} P_{kj}^{(n)} = \pi_j = \sum_{i=0}^{\infty} \pi_i P_{ij}, \quad k, j = 0, 1, \ldots,$$

$$\sum_{i=0}^{\infty} \pi_i = 1$$

2) $\boldsymbol{\pi} = (\pi_0, \pi_1, \ldots)$ is uniquely determined by

$$\pi_i \geq 0, \quad i = 0, 1, \ldots,$$

$$\sum_{i=0}^{\infty} \pi_i = 1,$$

$$\pi_j = \sum_{i=0}^{\infty} \pi_i P_{ij}, \quad j = 0, 1, \ldots.$$

NOTE: This is saying that each **positive recurrent aperiodic** equivalence class has its own “limiting distribution”.

Definition

Any $\boldsymbol{\pi} = (\pi_0, \pi_1, \ldots)$ such that

$$\begin{aligned} \pi_j &\geq 0, \quad j = 0, 1, \ldots, \\ \sum_{i=0}^{\infty} \pi_i &= 1, \\ \pi_j &= \sum_{i=0}^{\infty} \pi_i P_{ij}, \quad j = 0, 1, \ldots, \end{aligned}$$

is called a **stationary probability distribution**.

Note 1: The last equation is equivalent to $\mathbf{P}^{\mathrm{T}}\boldsymbol{\pi} = \boldsymbol{\pi}$.

Note 2: The existence of a **stationary distribution** is a weaker condition than the existence of a limiting distribution.

Note 3: There can be **zero**, **exactly one**, or **multiple** stationary distributions.