NTNU
Norwegian University of
Science and Technology

Week 35: Lecture 1

Introduction to Markov chains

Geir-Arne Fuglstad

August 24, 2020

Information

1. We need one more person for the reference group.
2. You can ask questions either publicly or privately in the chat. Or by voice if you prefer.
3. The first exercise class is tomorrow at 08:15 in R2.

Random sums [Section 2.3]

Building on the hunter example from last week, we can more generally consider random sums

$$X = \begin{cases} 0, & N = 0, \\ \xi_1 + \xi_2 + \ldots + \xi_N, & N > 0, \end{cases}$$

where

— N is a discrete random variable taking values $0, 1, \ldots$,

— $\xi_1, \xi_2, \ldots$ are independent random variables

— N is independent of $\xi_1, \xi_2, \ldots$

Notation: $X = \sum_{i=1}^{N} \xi_i = \xi_1 + \xi_2 + \ldots + \xi_N$

Read yourselves

Sections 2.2, 2.3, 2.4.

Section 3.1: Definitions

Stochastic process in discrete time

Definition (Discrete-time stochastic process)

A **discrete-time stochastic process** is a family of random variables $\{X_t : t \in T\}$ where T is discrete.

Comments:

- We use $T = \{0, 1, 2, \ldots\}$, and write X_n instead of X_t.
- We call X_n the **state** at time $n = 0, 1, 2, \ldots$.
- We call the set of all possible states the **state space**.

Markov property

Definition (Discrete-time Markov chain)

A **discrete-time Markov chain** is a discrete-time stochastic process $\{X_n : n = 0, 1, \ldots\}$ that satisfies the **Markov property**:

$$\begin{aligned}\Pr\{X_{n+1} = j | X_n = i, X_{n-1} = i_{n-1}, \ldots, X_0 = i_0\} \\ = \Pr\{X_{n+1} = j | X_n = i\},\end{aligned}$$

for $n = 0, 1, 2, \ldots$, and for all states i and j.

Comments:

— Unless otherwise specified, the state space is $\{0, 1, \ldots, N\}$ or $\{0, 1, 2, \ldots\}$.

Transition probabilities

Definition (One-step transition probabilities)

1. For a discrete-time Markov chain $\{X_n : n = 0, 1, \ldots\}$, we call $P_{ij}^{n,n+1} = \Pr\{X_{n+1} = j | X_n = i\}$ the **one-step transition probabilities.**

2. We will allways assume **stationary transition probabilities**, i.e., that $P_{ij}^{n,n+1} = P_{ij}$ for $n = 0, 1, 2, \ldots$, and all states i and j.

Transition probability matrices

Definition (Transition probability matrix)

For a discrete-time Markov chain with state space $\{0, 1, \ldots, N\}$, we call

$$\mathbf{P} = \begin{bmatrix} P_{0,0} & P_{0,1} & \cdots & P_{0,N} \\ P_{1,0} & P_{1,1} & & \vdots \\ \vdots & & \ddots & \\ P_{N,0} & \cdots & & P_{N,N} \end{bmatrix}$$

the **transition probability matrix**.

Comment: For state space $\{0, 1, 2, \ldots\}$ we envision an infinitely-sized matrix.

Transition diagrams

Definition (Transition diagram)

Let $\{X_n : n = 0, 1, \ldots\}$ be a discrete-time Markov chain. A **(state) transition diagram** visualizes the transition probabilities as a weighted directed graph where the nodes are the states and the edges are the possible transitions marked with the transition probabilities.

Section 3.2: Using transition probability matrices

n-step transitions

Theorem (Theorem 3.1)

For a Markov chain $\{X_n : n = 0, 1, \dots\}$ and any $m \geq 0$, we have

$$\Pr\{X_{m+n} = j | X_m = i\} = P_{ij}^{(n)} = \sum_{k=0}^{\infty} P_{ik} P_{kj}^{(n-1)}, \quad n > 0,$$

where we define

$$P_{ij}^{(0)} = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases}$$

*$P_{ij}^{(n)}$ are called the **n-step transition probabilities**.*

Matrix calculations

Theorem

The n-step transition probabilities can be computed by matrix multiplication. If $\mathbf{P}^{(n)} = [P_{ij}^{(n)}]$*, then*

$$\mathbf{P}^{(n)} = \underbrace{\mathbf{P} \cdot \mathbf{P} \cdots \mathbf{P}}_{n} = \mathbf{P}^{n}.$$