Norwegian University of Science and Technology
Department of Mathematical Sciences

TMA4183
Optimisation II
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Exercise set 2

$\boxed{1}$ Consider the sequence of functions $u_k \in L^1([0,1])$

$$u_k(x) = \begin{cases} k & \text{if } 0 < x < 1/k, \\ 0 & \text{else.} \end{cases}$$

Show that the sequence u_k is bounded in $L^1([0,1])$, but that it does not admit any weakly convergent subsequence.

- *Possible solution:* The boundedness of the sequence is obvious, as

$$\|u_k\|_{L^1} = \int_0^1 |u_k(x)|\,dx = \int_0^{1/k} k\,dx = 1$$

for all $k \in \mathbb{N}$. Now assume that some subsequence $\{u_{k'}\}_{k'}$ of $\{u_k\}_k$ converges weakly to some $u \in L^1([0,1])$. Then we have for every interval $(a,b) \subset [0,1]$ with $a > 0$ that

$$\int_a^b u(x)\,dx = \lim_{k'} \int_a^b u_{k'}(x)\,dx = 0,$$

as $u_{k'}(x) = 0$ for all $x > a$ as soon as $k' > 1/a$. As a consequence, it follows that $u(x) = 0$ for almost every $x > 0$. On the other hand, we have that

$$\int_0^1 u(x)\,dx = \lim_{k'} \int_0^1 u_{k'}(x)\,dx = 1,$$

which is a contradiction to u being zero. Thus the sequence cannot have any weakly convergent subsequence.

$\boxed{2}$ Consider the sequence of functions $u_k \in L^1(\mathbb{R})$,

$$u_k(x) = \begin{cases} 1 & \text{if } k < x < k+1, \\ 0 & \text{else.} \end{cases}$$

Show that $\int_E u_k(x)\,dx \to 0$ whenever $E \subset \mathbb{R}$ is measurable and satisfies $\mathcal{L}^1(E) < \infty$, but that u_k does not converge weakly to 0 in $L^1(\mathbb{R})$.

- *Possible solution:* Assume that $E \subset \mathbb{R}$ is measurable with $\mathcal{L}^1(E) < \infty$. Define $E_k := E \cap (k, k+1)$. Since E is measurable, so are the sets E_k. Moreover, they are by construction disjoint and all contained in E. Thus

$$\sum_{k=0}^{\infty} \mathcal{L}^1(E_k) = \mathcal{L}^1\Big(\bigcup_{k=0}^{\infty} E_k\Big) \le \mathcal{L}^1(E) < \infty.$$

In particular, this implies that

$$\mathcal{L}^1(E_k) \to 0 \qquad \text{as } k \to \infty.$$

However, by construction we have

$$\int_E u_k(x)\,dx = \int_{E\cap(k,k+1)} 1\,dx = \mathcal{L}^1(E_k),$$

which proves the first part of the assertion.

The second part, that is, that u_k does not converge weakly in $L^1(\mathbb{R})$ to zero follows by considering the test function $v \equiv 1 \in L^\infty(\mathbb{R})$, for which we have

$$\int_{\mathbb{R}} u_k(x)v(x)\,dx = \int_{\mathbb{R}} u_k(x)\,dx = 1 \not\to 0.$$

$\boxed{3}$ Let $1 < p < +\infty$ and assume that $C \subset L^p(E)$ is closed and convex. Given $v \in L^p(E)$, we define the (L^p-)projection $\pi_C(v)$ of v onto C as the solution of the optimisation problem

$$\min_{u\in C} \|u - v\|_{L^p}. \tag{1}$$

Show that the projection is well-defined, that is, that the optimisation problem (1) admits for each $v \in L^p(E)$ a unique solution.

- *Possible solution:* We note first that the optimisation problem (1) is equivalent to the problem

$$\min_{u\in C} \|u - v\|_{L^p}^p, \tag{2}$$

the only difference being the composition with the strictly increasing function $t \mapsto t^p$.

Let now $u_0 \in C$ be arbitrary and assume that $\{u_k\}_{k\in\mathbb{N}}$ is a minimising sequence for (2). Then

$$\lim_{k\to\infty} \|u_k - v\|_{L^p}^p = \inf_{u\in C} \|u - v\|_{L^p}^p \le \|u_0 - v\|_{L^p}^p,$$

which implies that the sequence $\{u_k\}_{k\in\mathbb{N}}$ is bounded. Since $1 < p < +\infty$, it follows that it admits a subsequence $\{u_{k'}\}_{k'}$ that converges weakly to some $\bar{u}$ in $L^p(E)$. Since C is closed and convex, it follows that $\bar{u} \in C$. Next, we have that the mapping $u \mapsto \|u - v\|_{L^p}^p$ is (strictly) convex and continuous, and therefore weakly lower semi-continuous. Thus

$$\|\bar{u} - v\|_{L^p}^p \le \liminf_{k'} \|u_k - v\|_{L^p}^p = \inf_{u\in C} \|u - v\|_{L^p}^p,$$

which shows that $\bar{u}$ solves (2). Finally, the uniqueness of the solution follows from the convexity of C together with the strict convexity of the mapping $u \mapsto \|u - v\|_{L^p}^p$.

$\boxed{4}$ Show that the set

$$C := \Big\{u \in L^1([0,1]) : u \ge 0 \text{ and } \int_0^1 xu(x)\,dx \ge 1\Big\}.$$

is convex and closed in $L^1([0,1])$, but that the optimisation problem

$$\min_{u\in C} \|u\|_{L^1}$$

does not admit a solution. (That is, the L^1-projection of $v = 0$ onto C does not exist!)

- *Possible solution:* The convexity and closedness of the set C in $L^1([0,1])$ is obvious: Any convex combination of non-negative functions is non-negative, and

$$\int_0^1 x(\lambda u(x) + (1-\lambda)v(x))\,dx = \lambda \int_0^1 xu(x)\,dx + (1-\lambda)\int_0^1 xv(x)\,dx \geq \lambda + (1-\lambda) = 1$$

whenever u, $v \in C$, which shows the convexity of C. Moreover, if the sequence $\{u_k\}_{k\in\mathbb{N}} \subset C$ converges to u in $L^1([0,1])$ then $\int_0^1 xu_k(x)\,dx \to \int_0^1 xu(x)\,dx$ —this can be seen by various approaches, the most direct being the estimate

$$\Big|\int_0^1 x(u_k(x) - u(x))\,dx\Big| \leq \int_0^1 x|u_k(x) - u(x)|\,dx \leq \int_0^1 |u_k(x) - u(x)|\,dx = \|u_k - u\|_{L^1} \to 0.$$

For showing that the optimisation problem $\min_{u\in C}\|u\|_{L^1}$ does not have a solution, we note first that all functions $u \in C$ are non-negative, and therefore

$$\|u\|_{L^1} = \int_0^1 u(x)\,dx \qquad \text{for all } u \in C.$$

Moreover, for $u \in C$ we have $\int_0^1 xu(x)\,dx \geq 1$ and therefore

$$\|u\|_{L^1} = \int_0^1 u(x)\,dx = \int_0^1 xu(x)\,dx + \int_0^1 (1-x)u(x)\,dx \geq 1 + \int_0^1 (1-x)u(x)\,dx \geq 1.$$

Moreover, equality in the last estimate holds if and only if $u = 0$, which is impossible for functions in C (as $\int_0^1 xu(x)\,dx \geq 1$). Thus we have in fact that

$$\|u\|_{L^1} > 1 \qquad \text{for all } u \in C.$$

In order to show the non-existence of a solution of the optimisation problem, it is therefore sufficient to show that $\inf_{u\in C}\|u\| = 1$, that is, to find a sequence u_k in C such that $\lim_{k\to\infty}\|u_k\|_{L^1} = 1$. To that end we define

$$u_k(x) = \begin{cases} 0 & \text{if } 0 \leq x \leq 1 - 1/k, \\ k+1 & \text{if } 1 - 1/k < x \leq 1. \end{cases}$$

Then $u_k(x) \geq 0$ for all x and

$$\int_0^1 xu_k(x)\,dx = \int_{1-1/k}^1 x(k+1)\,dx = \frac{1}{k}\Big(1 - \frac{1}{2k}\Big)(k+1) = 1 + \frac{k-1}{2k^2} \geq 1$$

for all k, showing that $u_k \in C$. Moreover,

$$\|u_k\|_{L^1} = \frac{k+1}{k} \to 1,$$

which finishes the proof.