NTNU
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Week 36: Lecture 2

Limiting probabilities for regular Markov chains

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Information

- You can use this Google form to provide feedback to the reference group:
  `https://s.ntnu.no/tma4265_2020_meeting1`
- Time of the exam has been decided: **02.12.2020 at 09:00**.
  - You can bring one yellow, stamped A5 sheet with handwritten notes (on both sides).
  - You can bring the blue book “Tabeller og formler i statistikk”.
  - The exam will include the same formula sheet as earlier exams.

Section 4.1

Example

Is

$$\mathbf{P} = \begin{bmatrix} 1/2 & 1/2 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 1/2 \\ 0 & 0 & 1/2 & 1/2 \end{bmatrix}$$

regular?

Example

Are the following Markov chains regular?

a)

$$\mathbf{P} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

b)

$$\mathbf{P} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Definition (Limiting distribution)

Consider a Markov chain $\{X_n : n = 0, 1, \ldots\}$. We call $\boldsymbol{\pi} = (\pi_0, \pi_1, \ldots)$ the **limiting distribution** of $\{X_n\}$ if the following holds:

1. $$\pi_j = \lim_{n\to\infty} P_{ij}^{(n)}, \quad j = 0, 1, \ldots,$$
   exist and do not depend on the initial state i
2. $$\sum_{j=0}^{\infty} \pi_j = 1,$$

Theorem (Theorem 4.1)

Let $\{X_n : n = 0, 1, \ldots\}$ *be a regular Markov chain with state space* $\{0, 1, \ldots, N\}$ *and transition probability matrix* $\mathbf{P}$*. Then the limiting distribution* $\boldsymbol{\pi} = (\pi_0, \pi_1, \ldots, \pi_N)$

1. *exists and satisfies (for any initial state* i*)*

$$\pi_j = \lim_{n \to \infty} P_{ij}^{(n)} > 0, \quad j = 0, 1, \ldots, N,$$

2. *is the unique non-negative solution of the equations*

$$\pi_j = \sum_{k=0}^{N} \pi_k P_{kj}, \quad j = 0, 1, \ldots, N,$$

$$\sum_{k=0}^{N} \pi_k = 1.$$

Example

A Markov chain $\{X_n : n = 0, 1, \ldots\}$ has transition probability matrix

$$\mathbf{P} = \begin{bmatrix} 1/2 & 1/2 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 1/2 \\ 0 & 0 & 1/2 & 1/2 \end{bmatrix}.$$

Find the limiting probabilities.

Definition

The transition probability matrix $\mathbf{P}$ is called **doubly stochastic** if $\sum_k P_{ik} = \sum_k P_{kj} = 1$ for all states i and j.

Definition

The transition probability matrix **P** is called **doubly stochastic** if $\sum_k P_{ik} = \sum_k P_{kj} = 1$ for all states i and j.

Theorem

Let the Markov chain $\{X_n : n = 0, 1, \ldots\}$ be regular with finite state space $\{0, 1, \ldots, N\}$. If the transition probability matrix **P** *is doubly stochastic, the limiting distribution is*

$$\pi = \left(\frac{1}{N+1}, \frac{1}{N+1}, \ldots, \frac{1}{N+1}\right).$$

Theorem (Long-run mean fraction of time)

In a regular Markov chain $\{X_n : n = 0, 1, \ldots\}$, the limiting distribution $\boldsymbol{\pi} = (\pi_0, \pi_1, \ldots, \pi_N)$ gives the ***long-run mean fraction of time*** *spent in each state. I.e.,*

$$\pi_j = \lim_{n\to\infty} \mathrm{E}\left[\frac{1}{n}\sum_{k=0}^{n-1} \mathbb{1}\{X_k = j\} \middle| X_0 = i\right]$$

for any initial state i.

Example

A Markov chain $\{X_n : n = 0, 1, \ldots\}$ has transition probability matrix

$$\mathbf{P} = \begin{bmatrix} 1/2 & 1/2 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 1/2 \\ 0 & 0 & 1/2 & 1/2 \end{bmatrix}.$$

What fraction of time, in the long run, does the chain spend in state 0?

Section 4.2

Example

Assume the weather each day is either sunny (S) or cloudy (C), and let X_n denote the weather on day n. Assume that

$$\begin{aligned}
\mathrm{P}\{X_{n+1} = \mathrm{S} | X_n = \mathrm{S}, X_{n-1} = \mathrm{S}\} &= 0.8 \\
\mathrm{P}\{X_{n+1} = \mathrm{S} | X_n = \mathrm{S}, X_{n-1} = \mathrm{C}\} &= 0.6 \\
\mathrm{P}\{X_{n+1} = \mathrm{S} | X_n = \mathrm{C}, X_{n-1} = \mathrm{S}\} &= 0.4 \\
\mathrm{P}\{X_{n+1} = \mathrm{S} | X_n = \mathrm{C}, X_{n-1} = \mathrm{C}\} &= 0.1
\end{aligned}$$

a) Is $\{X_n : n = 0, 1, \ldots\}$ a Markov chain?

b) In the long run, what is the proportion of sunny days?