Lecture 23

① Reminder of lecture 21:

• One-step methods (OSM): A numerical methods to compute an approximative solution to the initial value problem

$$\text{(IVP)} \quad \begin{cases} y'(t) = f(t, y(t)) \\ y(t_0) = y_0 \end{cases}$$

which is of the form:

$$y_{i+1} := y_i + \tau \Phi(t_i, y_i, y_{i+1}, \tau)$$

with an increment function $\Phi : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^+ \to \mathbb{R}$,

• Consistency error: Let $y(t)$ the exact solution to (IVP)

$$\eta(t, \Delta t) := \underbrace{y(t) + \tau \Phi(t, y(t), y(t+\tau), \tau)}_{\text{do 1-step with one-step method starting from exact value } y(t)} - \underbrace{y(t+\tau)}_{\text{exact value at } t+\tau}$$

• One-step method is consistent of order p:

$$\eta(t, \tau) = \mathcal{O}(\tau^{p+1}), \quad \tau \to 0.$$

• Global discretization error at t_i for a one-step method is defined by

$$e(t_i, \tau) := y(t_i) - y_i$$

where $t_i = t_0 + i \frac{(T - t_0)}{n} \qquad i = 0, 1, \dots, n.$

• OSM is convergent of order p:

$$\max_{i \in \{0, \dots, n\}} |e_i(t_i, \tau)| = \mathcal{O}(\tau^p) \qquad \tau \to 0.$$

Theorem 1: Let the increment function of a OSM satisfy the two Lipschitz conditions

$$|\Phi(t, u, w, \tau) - \Phi(t, v, w, \tau)| \leq L |u - v|,$$

$$|\Phi(t, w, u, \tau) - \Phi(t, w, v, \tau)| \leq L |u - v|.$$

with $L \in \mathbb{R}^+$. Then under the timestep restriction $\tau \leq L^{-1}$, we have

$$|e(t_{i+1}, \tau)| \leq \left(|e(t_0, \tau)| + \frac{(t_{i+1} - t_0)}{1 - \tau L} \cdot \frac{\eta(\tau)}{\tau} \right) \cdot \exp\left(\frac{L (t_{i+1} - t_0)}{1 - \tau L} \right)$$

where $\eta(\tau) := \max_{i \in \{0, \dots, n\}} |\eta(t_i, \tau)|.$

Corollary / Discussion of Thm I :

If a OSM

a) has a increment function Φ which is Lipschitz continuous with respect to the y_{i+1} and y_i slot

b) is consistent of order p

c) has an initial value $\hat{y}_0$ satisfying
$$\hat{y}_0 = y_0 + \mathcal{O}(\tau^p)$$

Then the OSM is convergent of order p.

Proof:

Since a) is satisfied, the OSM satisfies the error estimate given in Thm 1.

· Now c) implies that

$$|e(t_0,\tau)| = \left|\hat{y}_0 + \tau\Phi(t_0,\hat{y}_0,y_1,\tau) - y(t_1)\right| \quad \text{(by definition)}$$

$$= \Big|\underbrace{y_0 + \tau\Phi(t_0,y_0,y_1,\tau) - y(t_1)}_{=\eta(t_0,\tau)} + \underbrace{\hat{y}_0 - y_0}_{\mathcal{O}(\tau^p)} + \tau\Phi(t_0,\hat{y}_0,y_1,\tau) - \tau\Phi(t_0,y_0,y_1,\tau)\Big|$$

$$\le \eta(t_0,\tau) + \mathcal{O}(\tau^p) + L\underbrace{|\hat{y}_0 - y_0|}_{\mathcal{O}(\tau^p)} \le \mathcal{O}(\tau^{p+1}) + \mathcal{O}(\tau^p) + \mathcal{O}(\tau^p).$$

Finally, because of b) we have $\frac{\eta(\tau)}{\tau} = \mathcal{O}(\tau^p)$.

② Proof of Thm I :

Consider the error e_{i+1} for some $i \in \{0,\dots,n-1\}$. Then by definitions

$$e(t_{i+1},\tau) = y_{i+1} - y(t_{i+1})$$

$$= y_i + \tau\Phi(t_i,y_i,y_{i+1},\tau) - y(t_{i+1})$$

$$= y_i + \tau\Phi(t_i,y_i,y_{i+1},\tau)$$

$$-\left(y(t_i) + \tau\Phi(t_i,y(t_i),y(t_{i+1}),\tau)\right) \qquad = 0$$

$$+\left(y(t_i) + \tau\Phi(t_i,y(t_i),y(t_{i+1}),\tau)\right) - y(t_{i+1}) = \eta(t_i,\tau)$$

Rearranging terms
$$= \underbrace{y_i - y(t_i)}_{=e(t_i,\tau)} + \eta(t_i,\tau)$$

$$+ \tau\Phi(t_i,y_i,y_{i+1},\tau) - \tau\Phi(t_i,y(t_i),y(t_{i+1}),\tau)$$

Adding "0"
$$= e(t_i,\tau) + \eta(t_i,\tau)$$

$$+ \tau\Phi(t_i,y_i,y_{i+1},\tau) \boxed{-\tau\Phi(t_i,y(t_i),y_{i+1},\tau)} = 0$$

$$\boxed{+\tau\Phi(t_i,y(t_i),y_{i+1},\tau)} - \tau\Phi(t_i,y(t_i),y(t_{i+1}),\tau)$$

Now we can use the two Lipschitz conditions

This gives us:

$$|e(t_{i+1},\tau)| \leq |e(t_i,\tau)| + |\eta(t_i,\tau)|$$
$$+ \tau L \underbrace{|y_i - y(t_i)|}_{|e(t_i,\tau)|}$$
$$+ \tau L \underbrace{|y_{i+1} - y(t_{i+1})|}_{= |e(t_{i+1},\tau)|}.$$

If $\tau L < 1 \Leftrightarrow \tau < \frac{1}{L}$ we can rearange terms to get

$$(1-\tau L)|e(t_{i+1},\tau)| \leq (1+\tau L)|e(t_i,\tau)| + |\eta(t_i,\tau)|$$
$$\leq (1+\tau L)|e(t_i,\tau)| + \eta(\tau).$$

- So we managed to estimate the discretization error at t_{i+1} by the discretization error $e(t_i,\tau)$ at t_i plus a consistency error $\eta(\tau)$.

We will now use a little lemma known as a discrete Gronwall inequality.

Lemma (Discrete Gronwall inequality)

Let $\{e_i\}_{i=0}^n, \{\delta_i\}_{i=0}^n, \{\tilde{\eta}_i\}_{i=0}^n$ be non-negative sequences and assume that

$$e_{i+1} \leq (1+\delta_i)e_i + \tilde{\eta}_i \quad \text{holds for } i = 0,1,\dots,n-1$$

Then

$$e_{i+1} \leq \left(e_0 + \sum_{k=0}^{i} \tilde{\eta}_k\right) \cdot \exp\left(\sum_{k=0}^{i} \delta_k\right). \quad i=0,\dots n-1.$$

Finishing proof of Thm 1:

Divide by $1-\tau L > 0$ to get

$$|e(t_{i+1},\tau)| \leq \left(\frac{1+\tau L}{1-\tau L}\right)|e(t_i,\tau)| + \frac{\eta}{(1-\tau L)}$$

- Write $e_{i+1} := |e(t_{i+1},\tau)|$; $\tilde{\eta}_i := \frac{\eta}{(1-\tau L)}$

$$(1+\delta_i) = \frac{1+\tau L}{1-\tau L} \quad \text{with } \delta_i = \frac{1+\tau L}{1-\tau L} - 1 = \frac{2\tau L}{1-\tau L}$$

Then Gronwall gives

$$|e(t_{i+1},\tau)| \leq \left(e(t_0,\tau) + \sum_{k=0}^{i} \frac{\eta}{1-\tau L}\right) \exp\left(\sum_{k=0}^{i} \frac{2\tau L}{1-\tau L}\right)$$

Now $\sum_{k=0}^{i} \frac{1}{1-\tau L} = \frac{(i+1)}{1-\tau L} = \frac{1}{(1-\tau L)} \cdot \frac{t_{i+1}-t_0}{\tau}$ which leads to

$$|e(t_{i+1},\tau)| \leq \left(|e(t_0,\tau)| + \left(\frac{\eta}{1-\tau L}\right) \cdot \frac{t_{i+1}-t_0}{\tau}\right) \exp\left(\frac{2L(t_{i+1}-t_0)}{(1-\tau L)}\right)$$

(3) Proof of Gronwall's inequality:

Proof by induction on i

1) Base case $i = 0$:

$$\tilde{e}_{0+1} = \tilde{e}_1 \le (1+\delta_0)\tilde{e}_0 + \tilde{\eta}_0$$

$$\le (1+\delta_0)(\tilde{e}_0 + \tilde{\eta}_0)$$

$$\le e^{\delta_0}(\tilde{e}_0 + \tilde{\eta}_0)$$

since $(1+x) \le e^x$ for $x \ge 0$.

2) Induction step $i \mapsto i+1$:

By assumption:

$$e_{i+1} \le (1+\delta_i)e_i + \tilde{\eta}_i$$

$$\le (1+\delta_i)(e_i + \tilde{\eta}_i)$$

$(1+x)\cdot$

$\le$

By induction assumption we know that

$$e_i \le \left(e_0 + \sum_{k=0}^{i-1}\tilde{\eta}_k\right)\cdot \exp\left(\sum_{k=0}^{i-1}\delta_k\right)$$

Thus

$$e_{i+1} \le e^{\delta_i}\left(e_i + \tilde{\eta}_k\right)$$

$$\le e^{\delta_i}\left(e_0 + \sum_{k=0}^{i-1}\tilde{\eta}_k\right)\cdot \exp\left(\sum_{k=0}^{i-1}\delta_k\right) + e^{\delta_i}\tilde{\eta}_k$$

$$= \left(e_0 + \sum_{k=0}^{i-1}\eta_k\right)\exp\left(\sum_{k=0}^{i}\delta_k\right) + e^{\delta_i}\tilde{\eta}_k$$

$$\le \left(e_0 + \sum_{k=0}^{i-1}\tilde{\eta}_k + \tilde{\eta}_i\right)\exp\left(\sum_{k=0}^{i}\delta_k\right)$$

since $e^{\delta_i} \le e^{\sum_{k=0}^{i}\delta_i}$

since $\delta_i \ge 0$.

$$= \left(e_0 + \sum_{k=0}^{i}\tilde{\eta}_k\right)\exp\left(\sum_{k=0}^{i}\delta_k\right).$$

∎

④ Order conditions part I

Motivation: In lecture 2.2, we introduced Runge-Kutta methods, but we haven't said anything about their convergence order, and in particular, how the choice of the coefficients in the Butcher table $\begin{array}{c|c} c & A \\ \hline & b^T \end{array}$ impacts the convergence order.

- Thm. 1 has given us a tool to reduce the question of the global convergence order to the question of the local consistency order:

$$\boxed{\text{consistent of order } p \Rightarrow \text{convergent of order } p}$$

So this leads to the

Questions: • What kind of conditions must (A, b, c) satisfy to achieve a high consistency order?

- Given s stages in the R-K method, what is the highest order we can achieve?

We will now answer a couple of interesting questions related to consistency, order and the choice of coefficients.

Lemma 1

If an s-stage explicit Runge-Kutta methods (RKM) has consistency order p for all $f \in C^\infty(\Omega, \mathbb{R}^n)$, then

$$p \leq s.$$

In other words: The consistency order of an explicit RKM can never be higher than its number of stages.

Proof:

Consider the simple IVP (with $f(t,y) = y$):

$$y' = y \quad , \; y(0) = 1$$

with the exact solution $y(t) = e^t$.

Applying an explicit s-stage R-K method defined by

$$\begin{array}{c|cccc} c_1 & 0 & & & \\ \vdots & a_{21} & 0 & & 0 \\ & \vdots & & \ddots & \\ c_s & a_{s1} & \cdots & a_{ss-1} & 0 \\ \hline & b_1 & \cdots & & b_s \end{array}$$

gives the following expression for the stage derivatives k_i:

$k_1 = f(t_i + c_1\tau, y_i) = y_i$ which is a 0-order polynomial in τ,

$k_2 = f(t_i + c_2\tau, y_i + \tau a_{21}k_1) = y_i + \tau a_{21} y_i$ (a 1-order polynomial in τ)

$k_3 = f(t_i + c_3\tau, y_i + \tau a_{31}k_1 + \tau a_{32}k_2) = y_i + \tau a_{31} y_i + \tau a_{32}(y_i + \tau a_{21} y_i)$

which is a 2nd order polynomial in τ.

So each stage k_j will be a polynomial in τ of order at most $j-1$. In particular $y_{i+1} = y_i + \tau \sum_{j=0}^{s} b_j k_j$ will be some polynomial $p_s(\tau)$ with order at most s.

So choosing $y_0 = 1$, $t_0 = 0$, we see that for one step the consistency error satisfies

$$\eta(t_0,\tau) = |y(t_0+\tau) - y_1| = |e^{\tau} - p_s(\tau)|$$

So in order that a polynomial of order s approximates the exponential function up to order $O(\tau^{p+1})$;

$$|e^{\tau} - p_s(\tau)| = O(\tau^{p+1})$$

we must have that $p \le s$. ∎

Definition: A OSM is called consistent if the consistency error $\eta(t_i,\tau) = o(\tau)$ (little o), that is,

$$\frac{\eta(t_i,\tau)}{\tau} \longrightarrow 0 \quad \text{for } \tau \to 0.$$

Note this is the minimal requirement to ensure that a OSM $y_\tau(t)$ converges to $y(t)$ for $\tau \to 0$.

Lemma: A RKM is consistent if and only if $\boxed{\sum_{j=1}^{s} b_j = 1}$.

Let $y(t)$ be the exact solution to the $y' = f(t,y)$, $y(t_0) = y_0$.

Do a Taylor development of $y(t)$ around t_0:

$$y(t_i+\tau) = y(t_i) + \tau \underbrace{y'(t_i)}_{f(t_i,y_i)} + o(\tau).$$

Also since $k_j = f\left(t_i + c_j\tau,\ y_i + \tau \sum_{l=1}^{s} a_{jl} k_l\right)$,

we have that $k_j = k_j(\tau) \longrightarrow f(t_i,y_i)$ for $\tau \to 0$.

Thus

$$\begin{aligned}
\eta(t_i,\tau) &= y(t_i+\tau) - \left(y(t_i) + \tau \sum_{j=1}^{s} b_j k_j\right) \\
&= y(t_i) + \tau f(t_i,y_i) + o(\tau) - \left(y(t_i) + \tau \sum_{j=1}^{s} b_j k_j\right) \\
&= \tau\left(f(t_i,y_i) - \sum_{j=1}^{s} b_j k_j(\tau)\right) + o(\tau)
\end{aligned}$$

$$\begin{aligned}
\lim_{\tau\to0} \frac{\eta(t_i,\tau)}{\tau} &= \lim_{\tau\to0}\left(f(t_i,y_i) - \sum_{j=1}^{s} b_j k_j(\tau) + \frac{o(\tau)}{\tau}\right) \\
&= f(t_i,y_i) - \sum_{j=1}^{s} b_j f(t_i,y_i) = 0 \text{ if and only if } \sum_{j=1}^{s} b_j = 1.
\end{aligned}$$

Definition (Autonomization)

An IVP

$$\begin{cases} y' = f(t, y(t)) & \circledast \\ y(t_0) = y_0 \end{cases}$$

is called <u>autonomous</u> if f <u>does not explicitly</u> <u>depend on t</u>; that is, f depends only on $y(t)$:
$y' = f(y(t))$. Otherwise an IVP is called <u>non-autonomous</u>.

<u>Lemma</u>: Every non-autonomous system can be transformed into an autonomous system (autonomization).

<u>Proof</u>: Idea is that time as an extra variable.
by adding the IVP

$$s'(t) = 1 \quad s(t_0) = t_0 \quad , \text{so } s(t) = t.$$

Set $X := \begin{pmatrix} s \\ y \end{pmatrix} \in \mathbb{R}^{n+1}$, then the IVP $\circledast$ is equivalent to

$$X' = \begin{pmatrix} s' \\ y' \end{pmatrix} = \begin{pmatrix} 1 \\ f(t, y(t)) \end{pmatrix} = \begin{pmatrix} 1 \\ f(s(t), y(t)) \end{pmatrix} =: \tilde{f}(X(t))$$

So $\tilde{f}$ does not explicitly depend on t any more. ∎

Obviously, a Runge Kutta method for an autonomous system need only to specify (A, b) in the Butcher table, and not c.

- Now, one naturally property of a RKM to wish for is that it yields the same numerical result no matter whether we apply it to a non-autonomous system or its autonomized counterpart. We then say that the RKM is invariant under autonomization.

<u>Lemma</u>: A RKM is <u>invariant under</u> <u>autonomization</u> if and only if it is <u>consistent</u> and the Butcher table satisfies

$$c_i = \sum_{j=1}^{s} a_{ij} \quad \text{for } i = 1, \dots, s.$$

<u>Proof</u>: Without proof.

As a result of the previous two lemmas, one often considers only autonomous systems to ease the notation and derivations of theoretical results.

5. Multivariate Taylor Formula

In this section we assume that all IVP are written as an autonomous system

$$x' = f(x(t))$$
$$x(t_0) = x_0 . \quad \text{with } x(t) \in \mathbb{R}^n.$$

Multivariate Taylor Formula

Recall the Taylor Formula for functions $f:(a,b) \to \mathbb{R}$. of one variable:

$$f(x+h) = \sum_{n=0}^{p} \frac{f^{(n)}(x)}{n!} h^n + O(h^{p+1}).$$

- Now if $f \in C^p(\Omega, \mathbb{R}^m)$ with $\Omega \in \mathbb{R}^d$ we can define a multivariate version of Taylor's Formula for vector-valued functions of several variables:

First we define what $f^{(n)}(x)$ means:

Let $h_1, \dots, h_n \in \mathbb{R}^n$ be n vectors. Then

$$f^{(n)}(x) \cdot [h_1, \dots, h_n] := \sum_{i_1, \dots, i_n = 1}^{d} \frac{\partial^n f(x)}{\partial x_{i_1} \cdots \partial x_{i_n}} h_{1,i_1} \cdots h_{n,i_n}$$

- So for fixed $x \in \mathbb{R}^d$, $f^{(n)}(x)$ denotes a symmetric, multilinear (more exactly n-linear) mapping, and its application to its n arguments $h_1, \dots, h_n$ is denoted by $[h_1, \dots h_n]$. Note that $h_{1,m}$ denotes component m of the vector h_1.

Then we have the following

Lemma:

Let $f \in C^{p+1}(\Omega, \mathbb{R}^m)$. Then

$$f(x+h) = \sum_{n=0}^{p} \frac{1}{n!} f^{(n)}(x) \cdot [\underbrace{h, \dots, h}_{n\text{-times}}] + O(\|h\|^{p+1})$$

holds for any $x \in \Omega$ and sufficiently small $h \in \mathbb{R}^d$.

Proof: Without proof ∎

MS

- we want to derive certain conditions on the coefficients in the Butcher table/tableau which will ensure that the corresponding RK M has a certain consistency order.
- To derive these <u>order conditions</u> we will focus on autonomous systems

$$\begin{cases} x' = f(x) \\ x(t_0) = x_0 \end{cases}$$

and we will only consider consistency order ≤ 3.

- Recall that to show that a general OSM has consistency order p, we must show that

$$\eta(t,\tau) = x(t) + \tau \Phi(x(t), x(t+\tau), \tau) - x(t+\tau) = O(\tau^{p+1}).$$

. So for a RKM <u>the idea is</u> to do a Taylor development of x around t and then to do same for discrete functions $x(t) + \Phi(x(t), x(t+\tau), \tau)$ as a function of τ.

. Then we will compare the terms in front various powers of τ.

<u>Taylor expansion of $x(t)$</u>

- $x(t+\tau) = x(t) + x'(t)\tau + \dfrac{x''(t)}{2!}\tau^2 + \dfrac{x'''}{3!}\tau^3 + O(\tau^4)$.

Now we compute

- $x'(t) = f(x(t))$
- $x''(t) = \dfrac{d}{dt} x'(t) = \dfrac{d}{dt} f(x(t)) = f'(x(t))[x'(t)] = f'[f]$

where in the last step, we omit the argument $x(t)$ to simplify the notation.

- $x'''(t) = \dfrac{d}{dt} x'' = \dfrac{d}{dt} f'[f] = f''[f,f] + f'[f'[f]]$.

So this gives

$$\boxed{x(t+\tau) = x(t) + \tau f + \frac{\tau^2}{2!} f'[f] + \frac{\tau^3}{3!}\left(f''[f,f] + f'[f'[f]]\right) + O(\tau^4).}$$

- Of course we could continue this game, but we stop here.

Taylor expansion of the discrete function $X_\Delta(t+\tau)$

Recall the definition of the RKR in terms of the stage derivatives (see lecture 23):

$$k_j = f\left(x(t) + \tau \sum_{l=1}^{s} a_{jl} k_l\right) \qquad *)$$

$$X_\Delta(t+\tau) = X(t) + \tau \sum_{j=1}^{s} b_j k_j . \qquad +)$$

Now think of the k_j as a functions in τ and try to do a Taylor expansion. You might think that we will just chasing our own tail since k_j appears both on the lhs and rhs of $*)$.

But note that k_l on the rhs are multiplied, we will "bootstrap" a Taylor expansion of k_j.

- Note that $k_j(\tau) \to f(x(t))$ for $\tau \to 0$.
- Now do a first order Taylor-expansion of f to obtain

$$k_j(\tau) = f\left(x(t) + \tau \sum_{l=1}^{s} a_{jl} k_l(\tau)\right)$$

$$= f(x(t)) + \mathcal{O}(\tau).$$

Now substitute this into the rhs of $*)$, and we get

$$k_j(\tau) = f\Big(x + \underbrace{\tau \sum_{l=1}^{s} a_{jl} f + \mathcal{O}(\tau^2)}_{=:h}\Big).$$

Now do a second order Taylor expansion of $f(x+h)$:

$$k_j(\tau) = f(x+h) = f(x) + f'(x)[h] + \mathcal{O}(h^2)$$

$$= f(x) + \tau \sum_{l=1}^{s} a_{jl} f'[f] + \underbrace{f'[\mathcal{O}(\tau^2)]}_{\mathcal{O}(\tau^2)} + \mathcal{O}(\tau^2)$$

$$= f + \tau c_j f'[f] + \mathcal{O}(\tau^2)$$

where we used that $\sum_{l=1}^{s} a_{jl} = c_j$ and $\mathcal{O}(h^2) = \mathcal{O}(\tau^2)$

Inserting this again into the rhs of $*)$ gives now

$$k_j = f\Big(x + \underbrace{\tau \sum_{l=1}^{s} a_{jl}\left(f + \tau c_l f'[f]\right) + \mathcal{O}(\tau^3)}_{=:h}\Big)$$

A 3rd order Taylor expansion of f gives

$$k_j = f(x+h) = f + f'[h] + \frac{f''[h,h]}{2!} + \mathcal{O}(h^3)$$

Again $\mathcal{O}(h^3) = \mathcal{O}(\tau^3)$ by the definition of h, and moreover

$$k_j = f + \tau \overbrace{\sum_{l=1}^{s} a_{jl}}^{c_j} f'[f] + \tau^2 \sum_{l=1}^{s} a_{jl} c_l f'[f'[f]] + \mathcal{O}(\tau^3)$$

$$+ \frac{f''}{2!}\Big[\tau \underbrace{\sum_{l=1}^{s} a_{jl}}_{c_j} f, \tau \underbrace{\sum_{l=1}^{s} a_{jl}}_{c_j} f\Big] + \mathcal{O}(\tau^3)$$

$$= f + \tau c_j f'[f] + \tau^2 \sum_{l=1}^{s} a_{jl} c_l f'[f'[f]]$$

$$+ \frac{\tau^2}{2} c_j^2 f''[f,f] + \mathcal{O}(\tau^3).$$

Insert this into $*$) and rewrite it slightly to make comparisons with the Taylor expansion of $x(t+\tau)$ easier gives:

$$\boxed{\begin{aligned} x_\Delta(t+\tau) &= x(t) + \tau \sum_{j=1}^{s} b_j k_j \\ &= x + \tau \sum_{j=1}^{s} b_j f + \frac{\tau^2}{2!}\left(2 \sum_{j=1}^{s} b_j c_j f'[f]\right) \\ &+ \frac{\tau^3}{3!}\left(3 \sum_{j=1}^{s} b_j c_j^2 f''[f,f] + 6 \cdot \sum_{j=1}^{s}\sum_{\ell=1}^{s} b_j a_{j\ell} c_\ell f'[f'[f]]\right) \\ &+ \mathcal{O}(\tau^4). \end{aligned}}$$

Now comparing this with Taylor expansion of the exact solution, $x(t+\tau)$, yields:

<u>Theorem 2</u> (Order conditions)

Consider the conditions

1) $\sum_{j=1}^{s} b_j = 1$

2) $\sum_{j=1}^{s} b_j c_j = \frac{1}{2}$

3) $\sum_{j=1}^{s} b_j c_j^2 = \frac{1}{3}$

4) $\sum_{j=1}^{s}\sum_{\ell=1}^{s} b_j a_{j\ell} c_\ell = \frac{1}{6}$

Then for a smooth enough f, the RKM has consistency order

$p=1$ if condition 1) holds

$p=2$ if " 2) holds

$p=3$ if " 3) and 4) holds.

<u>Remark</u>:

Order conditions for $p>3$ can be derived in the same way by considering higher order Taylor expansion of $x(t+\tau)$ and x_Δ. but this will quickly become difficult to do the book-keeping of all the derivatives.

· There is a more advanced theory involving labeled trees, see the ODE lecture notes by Anne Kværnø.