Exercise set 13: Solutions

Please justify your answers! The most important part is *how* you arrive at an answer, not the answer itself.

$\boxed{1}$ Let $A \in \mathcal{M}_{m\times n}(\mathbb{C})$ and $B \in \mathcal{M}_{n\times m}(\mathbb{C})$, and let $\lambda \in \mathbb{C}$ be any nonzero scalar. Show that λ is an eigenvalue of AB if and only if λ is an eigenvalue of BA.

Solution. Assume that $\lambda \neq 0$ is an eigenvalue for AB with eigenvector $v \in \mathbb{C}^m$: $ABv = \lambda v$. By multiplying both sides of this inequality with the matrix B from the left, we get

$$(BA)Bv = \lambda Bv.$$

If $Bv \neq 0$, this shows that λ is an eigenvalue of BA, with eigenvector Bv (recall that eigenvectors are non-zero by definition). If we assume that $Bv = 0$, then our assumption $ABv = \lambda v$ gives

$$\begin{aligned}\lambda v &= ABv\\ &= A(Bv) = 0,\end{aligned}$$

which is a contradiction since we assumed $\lambda \neq 0$ and $v \neq 0$ since v is an eigenvector. Therefore $Bv \neq 0$. The converse (any non-zero eigenvalue of BA is an eigenvalue of AB) is proved in the same way.

$\boxed{2}$ Suppose that A and B are *unitarily equivalent*, meaning that there exists a unitary matrix U such that

$$B = U^*AU.$$

Prove that A is positive definite (semi-definite) if and only if B is positive definite (semi-definite).

Solution. A is positive definite if

$$\langle Av, v\rangle > 0 \quad \text{for any } v \in \mathbb{C}^n, \tag{1}$$

and positive semi-definite if

$$\langle Av, v\rangle \geq 0 \quad \text{for any } v \in \mathbb{C}^n. \tag{2}$$

Then assume that $B = U^*AU$, where U is unitary. We can use the definition of the adjoint to write for $u \in \mathbb{C}^m$

$$\begin{aligned}\langle Bu, u\rangle &= \langle U^*AUu, u\rangle \\ &= \langle AUu, Uu\rangle.\end{aligned}$$

If A is positive definite, then $\langle AUu, Uu\rangle > 0$ for any u (pick $v = Uu$ in (1)), and therefore $\langle Bu, u\rangle > 0$ for any u – hence B is positive definite. Similarly B is positive semi-definite if A is positive semi-definite.
Note that $B = U^*AU$ implies that $A = UBU^*$ since U is unitary (multiply the first equation with U from the left and $U^* = U^{-1}$ from the right). The same proof as above therefore shows that A is positive (semi-)definite if B is positive (semi-)definite.

$\boxed{3}$ Let $A \in \mathcal{M}_{n\times n}(\mathbb{C})$ be a normal matrix. Prove that

$$\det(A) = \prod_{j=1}^{n} \lambda_j,$$

where the λ_j's are the (not necessarily distinct) eigenvalues of A.

Solution. By the spectral theorem, there exists a unitary matrix $U \in \mathcal{M}_{n\times n}$ such that

$$U^*AU = \Lambda,$$

where Λ is the diagonal matrix with the eigenvalues of A along the diagonal. Since the determinant function is multiplicative, we have that

$$\det(\Lambda) = \det(U^*AU) = \det(U^*)\det(A)\det(U) = \det(U^*)\det(U)\det(A) = \det(A).$$

We have used that $\det(U^*)\det(U) = \det(U^*U) = \det(I) = 1$, where I is the identity matrix. Clearly

$$\det(\Lambda) = \prod_{j=1}^{n} \lambda_j,$$

which completes the proof.

$\boxed{4}$ *(Exam 2017, Problem 1a)*

a) Find the singular value decomposition for the matrix

$$A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \end{bmatrix}.$$

b) The linear system

$$\begin{aligned} x_1 + x_2 - x_3 &= 1 \\ x_1 + x_2 - x_3 &= 1 \end{aligned}$$

has infinitely many solutions. Find the solution with minimal Euclidean norm $\|\cdot\|_2$.

c) The linear system

$$x_1 + x_2 - x_3 = 1$$
$$x_1 + x_2 - x_3 = 2$$

is inconsistent, and has no solution. Find the unique best approximation to a solution having minimum norm.

d) Prove that an $(n \times n)$ matrix A of full rank has a polar decomposition using the singular value decomposition of A. Hence, show that there exists an $(n \times n)$ unitary matrix W and a positive definite (not just semi-definite) $(n \times n)$ matrix P such that $A = WP$.

Solution a) Let us compute the singular values of A. Recall these are the square roots of the non-zero eigenvalues of the selfadjoint matrix $AA^* = \begin{pmatrix} 3 & 3 \\ 3 & 3 \end{pmatrix}$ or $A^*A = \begin{pmatrix} 2 & 2 & -2 \\ 2 & 2 & -2 \\ -2 & -2 & 2 \end{pmatrix}$. In the first case we get a 2×2-matrix and in the second case we get a 3×3-matrix, so for simplicity we use AA^* for the computation of the singular values. The eigenvalues of $\begin{pmatrix} 3 & 3 \\ 3 & 3 \end{pmatrix}$ are 6 and 0. For those, who have decided to use A^*A: the eigenvalues are $6, 0, 0$. Hence $\sigma_1 = \sqrt{6}$ is the only singular value of A, which fits very well with the fact that A has rank one.

Consequently Σ is given by

$$\Sigma = \begin{pmatrix} \sqrt{6} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Let us look at the eigenvectors of A^*A. A little bit of computation yields

$$v_1 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \ v_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \ v_3 = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}.$$

The set $\{v_1, v_2, v_3\}$ is an orthonormal basis of $\mathbb{R}^3$ and yield the columns of V:

$$V = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{-1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \end{pmatrix}.$$

Now we get the columns of U. The first column is given by

$$u_1 = \frac{1}{\sigma_1} A v_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Since we only had one singular value σ_1, the second column of U is obtained by completing $\{u_1\}$ to an orthonormal basis for $\mathbb{C}^2$. This is achieved by choosing u_2 orthogonal to u_1, and by inspection we see that we can choose $u_2 = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$. Thus

$$U = \begin{pmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\end{pmatrix}.$$

Hence $A = U\Sigma V^* = \begin{pmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\end{pmatrix}\begin{pmatrix}\sqrt{6} & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{3}}\\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0\\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}}\end{pmatrix}$

b) We first compute the pseudoinverse of A: The pseudoinverse in terms of the SVD is given by $A^+ = V\Sigma^+U^*$, which gives

$$A^+ = \begin{pmatrix}\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}}\\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0\\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}}\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt{6}} & 0\\ 0 & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\end{pmatrix} = \frac{1}{6}\begin{pmatrix}1 & 1\\ 1 & 1\\ -1 & -1\end{pmatrix}.$$

The system

$$\begin{aligned}x_1 + x_2 - x_3 &= 1\\ x_1 + x_2 - x_3 &= 1\end{aligned}$$

has infinitely many solutions and we have learned that $A^+\begin{pmatrix}1\\1\end{pmatrix} = \frac{1}{3}\begin{pmatrix}1\\1\\-1\end{pmatrix}$ is the solution of minimal $\|.\|_2$-norm.

c) The system

$$\begin{aligned}x_1 + x_2 - x_3 &= 1\\ x_1 + x_2 - x_3 &= 2\end{aligned}$$

has no solution, but $A^+\begin{pmatrix}1\\2\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1\\1\\-1\end{pmatrix}$ is the best approximation to a solution having minimal norm.

d) The SVD decomposition gives us unitary $n \times n$ matrices U and V such that

$$A = U\Sigma V^* = UV^*V\Sigma V^*.$$

Note that UV^* is unitary as a product of two unitary matrices and $V\Sigma V^*$ is positive definite by problem 2, since Σ is positive definite.