TMA4165: Problem Sheet II

published: 19/01/2020, revised: 22/01/2020

1. (June 2018 examination) Let $x^0(t)$ and $x^\eta(t)$ be solutions to the Cauchy problem

$$0 = \dot{x} - x + \frac{1}{1+r^2}x^2, \qquad x(0) = \frac{1}{2},$$

with $r = 0$ and $r = \eta > 0$, respectively.

Show that for each $r \in \mathbb{R}$, this equation has a unique solution on some interval of existence around $t = 0$, and that the dependence on the parameter r is continuous, in particular, that there is a bounded function $K : t \to \mathbb{R}$ independent of r such that

$$|x^0(t) - x^\eta(t)| \leq K(t)\eta^2.$$

2. Show that

$$T(x) = \frac{\pi}{2} + x - \arctan(x), \qquad x \in \mathbb{R}$$

has no fixed point even though

$$|T(x) - T(y)| < |x - y|$$

for every pair of distinct $x, y \in \mathbb{R}$.

Explain if this contradicts the contraction mapping principle.

3. Let $f : [0,1] \to \mathbb{R}$ be a continuous function. Show that the following *boundary value* problem for a *second-order* ODE:

$$-\frac{\mathrm{d}^2}{\mathrm{d}x^2}u(x) + \lambda \sin(u(x)) = f(x), \qquad x \in [0,1]$$
$$u(0) = 0, \qquad u(1) = 0$$

has a unique solution for every sufficiently small λ. Write out the first three Picard iterations. beginning with $u_0 \equiv 0$.

4. (*Perko* Example 2.1) Show that the Cauchy problem

$$\frac{\mathrm{d}}{\mathrm{d}t}y(t) = y^2(t) - t, \qquad y(0) = 1$$

exhibits finite-time blow-up, and that this maximal time of existence is less than $t = 1.13$.

5. The following equation for $f : [-a, a] \to \mathbb{R}$ arises in a model for one-dimensional gas dynamics:

$$f(x) = 1 + \frac{1}{\pi}\int_{-a}^{a} \frac{1}{1+(x-y)^2} f(y)\,\mathrm{d}y, \qquad x \in [-a, a].$$

Show that the integral equation has a unique, non-negative solution for every $0 < a < \infty$. What happens in the asymptotic regime as $a \to \infty$?