Norwegian University of Science and Technology
Department of Mathematical Sciences

TMA4183 Optimisation II Spring 2020

Exercise set 5

1 Assume that $\vec{b} \in C^1(\bar{\Omega}; \mathbb{R}^d)$ is a continuously differentiable, divergence free vector field on Ω (that is, $\operatorname{div} \vec{b} = 0$ in Ω) and that $f \in L^2(\Omega)$. Consider the PDE

$$\begin{aligned} \operatorname{div}(\vec{b}u) - \Delta u &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \Gamma. \end{aligned} \tag{1}$$

a) Provide a weak formulation of this PDE and show that it has a unique solution in $H_0^1(\Omega)$.

Hint: Recall the Poincaré inequality $\|u\|_{H^1} \le C_\Omega \|\nabla u\|_{L^2}$ *for* $u \in H_0^1(\Omega)$.

b) Assume that $f_k \rightharpoonup f$ in $L^2(\Omega)$ and denote by u_k the solution of (1) with right hand side f_k, and by u the solution of (1) with right hand side f. Show that $u_k \rightharpoonup u$ in $H^1(\Omega)$.

2 We now consider the same basic PDE (1) but add a non-linear sink term: We assume that $g \in L^2(\Omega)$ with $g(x) \ge 0$ for a.e. x and consider the PDE

$$\begin{aligned} g \arctan u + \operatorname{div}(\vec{b}u) - \Delta u &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \Gamma. \end{aligned} \tag{2}$$

a) Provide a weak formulation of this PDE and show that it has a unique solution in $H_0^1(\Omega)$.

b) Assume that $g_k(x) \ge 0$ for a.e. $x \in \Omega$ and that $g_k \rightharpoonup g$ in $L^2(\Omega)$. Denote by u_k the solution of (2) with sink term $g_k \arctan u$, and by u the solution of (2) with sink term $g \arctan u$. Show that $u_k \rightharpoonup u$ in $H^1(\Omega)$.

Hint: At some point it might help to verify that the convergence $u_k \rightharpoonup u$ *weakly in* $H^1(\Omega)$ *implies that* $\arctan u_k \to \arctan u$ *strongly in* $L^q(\Omega)$ *for every* $1 \le q < \infty$.