Stochastic Modelling

isakhammer

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Contents

1 Lecture 1

1.1 Practical Information

Two projects

- The projects count 20% and exam 80%.
- Must be done with two people.
- If you want to do statistics is it worth learning R.

Course Overview

- Markov chains for discret time and discrete outcome.
  - Set of states and discrete time points.
  - Transition between states
  - Future depends on the present, but not the past.
- Continious time Markoc chains. (continious time and discrete toutcome.
- Brownian motion and Gaussian processes (continionus time and continious outcome.)

1.2 Mathematical description

Definition 1.1. *A* ***stochastic process*** $\{x(t), t \in T\}$ *is a family of random variables, where* T *is a set of indicies, and* $X(t)$ *is a random variable for each value of* t*.*

1.3 Recall from Statistics Course

A random experiment is perfomed the outcome of the experiment is random.

- THe set of possible outcomes is the **sample space** ω
  - An **event** $A \subset \omega$ if the outcome is contained in A
  - The **complement** of an event A is $A^c = \omega \setminus A$
  - The **null event** $\emptyset$ is the empty set $\emptyset = \omega \setminus \omega$

1.3.1 Combining Event

Let A and B be events

- The **union** $A \cup B$ is the event that at least one of A and B occur.
- the **intersection** $A \cap B$ is the event that both A and B occur.

The events $A_1, A_2, \ldots$ are called disjoint (or **mutually exclusive**) if $A_i \cap A_j = \emptyset$ for $i \neq j$

1.3.2 Probability

Pr is called a probability on ω if

- $\Pr\{\omega\} = 1$
- $0 \leq P\{A\} \leq 1$ for all events A
- For $A_1, A_2, \ldots$ that are mutually exclusive

$$P\left\{\bigcup_{i=1}^{\infty} A_i\right\} = \sum_{i=1}^{\infty} P\{A_i\}$$

We call $P\{A\}$ the probability of A.

1.3.3 Law of total probability

Let $A_1, A_2, \ldots$ be a partition of ω ie

- $\omega = \bigcup_{i=1}^{\infty} A_i$
- $A_1, A_2, A_3, \ldots$ are mutually exclusive.

Then for any event B

$$P\{B\} = \sum_{i=1}^{\infty} P\{B \cap A_i\}$$

This concept is very important.

1.3.4 Independence

Event A and B are independent of

$$P\{A \cap B\} = P\{A\} P\{B\}$$

Events $A_1, \ldots, A_n$ are independent if for any subset

$$P\left\{\bigcap_{j=1}^{k} A_{i_j}\right\} = \prod_{j=1}^{k} P\{A_{i_j}\}$$

In this case $P\{\bigcap_{i=1}^{n} A_1\} = \prod_{i=1}^{n} P\{A_i\}$

1.3.5 Random Variables

Definition 1.2. *A* ***random variable*** *is a real-vaued function on the sample space. Informally: A random variable is a real valued variable that takes on its value by chance.*

Example.

- Throw two dice. $X =$ sum of the two dice
- Throw a coin. X is 1 for heads and X is 0 for tails.

1.3.6 Notation for random variables

We use

- upper case letters such at X, Y and Z to represent random variables.
- lower case letters as x, y, z to denote the real-valued realized value of a the random variable.

Expression such as $\{X \leq x\}$ denators the event that X assumes a valye less than or earl to the real number x.

1.3.7 Discrete random variables

The random variable X is **discrete** if it has a finite or countablle number of possible outcomes $x_1, x_2, \ldots$

- The **probability mass function** $p_x(x)$ is given by

  $$p_x(x) = P\{X = x\}$$

  and satisfies

  $$\sum_{i=1}^{\infty} p_x(x_i) = 1 \quad \text{and} \quad 0 \leq p_x(x_i) \leq 1$$

- The **cumulative distribution function** (CDF) a of X can be written

  $$F_x(x) = P\{X \leq x\} = \sum_{i: x_i \leq x} p_x(x_i)$$

1.3.8 CFD

The CDF of X may also be called the **distrobution function** of X

Let $F_x(x)$ be the CDF of X, then

- $F_x(x)$ is monetonaly increasing.
- F_x is a stepfunction, which is a pieace-wise constant with jumps at x_i.
- $\lim_{x\to\infty} F_x(x) = 1$
- $\lim_{x\to-\infty} F_x(x) = 0$

1.3.9 Continious random vairbales

A **continious** random variables takes value o a continious scale.

- The CDF, $F_x(x) = P(X \leq x)$ is continious.
- The **probability density function** (PDF) $f_x(x) = F_x'(x)$ can be used to calculate probablities

$$\begin{aligned} Pr\{a < X < b\} &= Pr\{a \leq X < b\} = Pr\{a < X \leq b\} \\ &= Pr\{a \leq X \leq b\} = \int_a^b f_x(x)\,dx \end{aligned}$$

1.3.10 Important properties

- CDF:
  - Monotonely increaing
  - continious
  - $\lim_{x\to\infty} F_x = 1$ and $\lim_{x\to-\infty} F_x(x) = 0$
- PDF
  - $f_x(x) \geq 0$ for $x \in \mathbb{R}$
  - $\int_{-\infty}^{\infty} f_x(x)\,dx = 1$

1.3.11 Expectation

Let $g : \mathbb{R} \to \mathbb{R}$ be a function and X be a random variable.

- If X is discrete, the expected value of $g(X)$ is

$$E[g(X)] = \sum_{x:p_x(x)>0} g(x)\,p_x(x)$$

- If X is continous, the expected value of $g(X)$ is

$$E[g(X)] = \int_{-\infty}^{\infty} g(x)\,f_x(x)\,dx$$

1.3.12 Variance

The variance of the random variable X is

$$Var\left[X\right] = E\left[\left(X - E\left[X\right]\right)^2\right] = E\left[X^2\right] - E\left[X\right]^2$$

Important properties of expectation and variance.

- Expectations is linear

$$E\left[aX + bY + c\right] = aE\left[X\right] + bE\left[Y\right] + c.$$

- Variance scales quadratically and is invaraient to the addition of constants

$$Var\left[aX + b\right] = a^2 Var\left[X\right]$$

- fir independent stochastic variables.

$$Var\left[X + Y\right] = Var\left[X\right] + Var\left[Y\right]$$

1.3.13 Joint CDF

If (X, Y) is a pair for random variables, their **joint comulative distribution function** is given by

$$F_{X,Y} = F\left(x, y\right) = Pr\left\{X \leq x \cap Y \leq y\right\}$$

.

1.3.14 Joint distrubution for discrete random variables

If X and Y are discrete, the **joint probability mass function** $p_{x,y} = Pr\left\{X = x, Y = y\right\}$. can be used to compute probabilities

$$Pr\left\{a < X < b, c < Y \leq d\right\} = \sum_{a < x \leq b} \sum_{c < y \leq d} p_{X,Y}\left(x, y\right)$$

1.3.15 Joint distrubution for continous random variables

If X and Y are continious the **joint probability density function**

$$.f_{X,Y}\left(x, y\right) = f\left(x, y\right) = \frac{\partial^2}{\partial x \partial y} F\left(x, y\right)$$

can be used to compute probabilities

$$Pr\left\{a < X \leq b, \quad c < Y \leq d\right\} = \int_a^b \int_c^d f\left(x, y\right) dx dy$$

1.3.16 Independence

The random variables X and Y are independent if

$$Pr\left\{X \leq a, Y \leq b\right\} = Pr\left\{X \leq a\right\} \cdot Pr\left\{Y \leq b\right\}, \quad \forall a, b \in \mathbb{R}$$

In terms of CDFs: $F_{X,Y}(a, b) = F_X(a) \cdot F_Y(b) \quad \forall a, b \in \mathbb{R}$
Thus we have

- $p_{X,Y}(x, y) = p_X(x) \cdot p_Y(Y)$ for discrete random variables
- $f_{X,Y}(x, y) = f_X(x) \cdot f_Y(Y)$ for continuous random variables.

2 Lecture 3

2.1 Randoms sum

Building on the hunter example from last week. we can more generally consider random sums

$$X = \begin{cases} 0, & N = 0 \\ \zeta_1 + \zeta_2 + \ldots + \zeta_N, & N > 0 \end{cases}$$

where

- N is a discrete random variable with values $0, 1, \ldots$
- $\zeta_1, \zeta_2, \ldots$ are independent random variables
- N is independent of $\zeta_1, \zeta_2 + \ldots + \zeta_N$
- **Notation** $X = \sum_{i=1}^{N} \zeta_i = \zeta_1 + \zeta_2 + \ldots + \zeta_N$

Example.

1. Insurance company

$$N: \text{ Number of claims.}$$

$$\zeta_1, \zeta_2, \ldots \quad : \quad \text{Sizes of the claims}$$

Total liabilility:

$$X = \zeta_1 + \zeta_2 + \ldots + \zeta_N$$

2. Be careful!

$$\begin{aligned} \overbrace{E\left[\sum_{i=1}^{N} \zeta_i\right]}^{\neq \sum_{i=1}^{N} E[\zeta_i]} &= E\left[E\left[\sum_{i=1}^{N} \zeta_i \mid N\right]\right] \\ &= E\left[\sum_{i=1}^{N} E\left[\zeta_i \mid N\right]\right] \end{aligned}$$

2.2 Self Study

Section 2.2, 2.3, 2.4

2.3 Stochastic process in descrete time

Definition 2.1. *A* ***discrete-time stochastic process*** *is a family of random variables* $[X_t : t \in T]$ *where* T *is discrete.*

- *We use* $T = \{0, 1, 2, ..\}$ *and write* X_n *instead of* X_t
- *we call* X_n *the* ***state*** *at time* $n = 0, 1, 2, 3, \ldots$
- *We call the set of all possible states the* ***state space***

Table 1: Table for example

Day	$n = 0$	$n = 1$	$n = 2$	$\ldots$
Random Variable	X_0	X_1	X_2	$\ldots$
Realization 1	$x_0 = 0$	$x_1 = 1$	$x_2 = 1$	$\ldots$
Realization 2	$x_0 = 1$	$x_1 = 1$	$x_2 = 1$	$\ldots$

Example.

$$X_n = \begin{cases} 1, & \text{if it rains on day } n \\ 0, & \text{no rain on day } n \end{cases}$$

State space $= \{0, 1\}$
We have a problem. Need

$$Pr\{X_n = x_n \mid X_{n-1} = x_n, X_{n-2} = x_{n-2}, \ldots, X_0 = x_0\}.$$

for all $n = 0, 1, 2, \ldots$

2.4 Markov chain

Definition 2.2 (Discrete time Markov Chain)**.** *A* ***Discrete time markoc chain*** *is a discrete time stochastic process* $\{X_n : n = 0, 1, \ldots\}$ *that statisfied the* ***markov property*** *such that*

$$\begin{aligned} &Pr\{X_{n-1} = j \mid X_n = i, X_{n-1} = i_{n-1}, \ldots, X_0 = i_0\} \\ &= Pr\{X_{n+1} = j \mid X_n = i\} \end{aligned}$$

for $n = 0, 1, 2, 3, \ldots$ *and for all states* i *and* j

Definition 2.3 (One-step transition probabilities)**.** *We can define it as*

- *For a discrete Markov chain* $\{X_n : n = 0, 1, 2, \ldots\}$ *we call* $P_{ij}^{n,n+1} = Pr\{X_{n+1} = j, X_n = i\}$ *the* ***one step trainsition probabilities*** *.*

- *We will assume* ***stationary transition probabilities*** *, i.e that*

$$P_{ij}^{n,n+1} = P_{ij}$$

for $n = 0, 1, 2, \dots$ *and all states* i *and* j *.*

Some of the properties

1. "You will always go somewhere"

$$\sum_j P_{ij} = 1 \quad \forall i$$

2. The markov chain can be described as follows.

$$\begin{aligned}
&Pr\left\{X_0 = i_0, X_1 = i_1, \dots, X_n = i_n\right\} \\
&= Pr\left\{X_0 = i_0\right\} Pr\left\{X_1 = i_1 \mid X_0 = i_0\right\} \dots \\
&\qquad Pr\left\{X_n = x_n \mid X_{n-1} = i_{n-1} \dots X_0 = i_0\right\} \\
&\ \vdots \ \text{ Markov step} \\
&= Pr\left\{X_0 = i_0\right\} \cdot Pr\left\{X_1 = i_1 \mid X_0 = i_0\right\} \dots \\
&\qquad Pr\left\{X_n = x_n \mid X_{n-1} = i_{n-1}\right\} \\
&= Pr\left\{X_0 = i_0\right\} P_{i_0,i_1} \cdot P_{i_1,i_2} \dots P_{i_{n-1},i_n}
\end{aligned}$$

Which is a major simplification.

Definition 2.4 (Transition Probability Matrix)**.** *For a discrete time markov-chain with state space* $\{0, 1, \dots, N\}$ *we call*

$$\mathbf{P} = \begin{bmatrix} P_{00} & \dots & P_{0N} \\ P_{10} & \dots & \\ \vdots & & \ddots \\ P_{N0} & \dots & P_{NN} \end{bmatrix}$$

Is the transition matrix. For statespace $\{0, 1, 2, \dots\}$ *we envision an infinitely sized matrix.*

Example.

- Markoc chain : $\{X_n : n = 0, 1, 2, \dots\}$
- State space = $\{0, 1\}$
- Transition Matrix

$$\mathbf{P} = \begin{bmatrix} 0.9 & 0.1 \\ 0.6 & 0.4 \end{bmatrix}$$

We can compute

$$\begin{aligned} Pr\{X_3 = 1 \mid X_2 = 0\} &= p_{01} \\ &= 0.1 \\ Pr\{X_{10} = 0 \mid X_9 = 1\} &= P_{10} \\ &= 0.6 \end{aligned}$$

Definition 2.5 (Transition Diagram)**.** *Let $\{X_n : n = 0, 1, \dots\}$ be a discrete time Markov chain. A **state trasnistion diagram** visualizes the transition probabilities as a weighted directed graph where the nodes are the states and the edges are the possible transitions marked with the transistion probabilities.*

Example. State space = $\{0, 1, 2\}$ and

$$P = \begin{bmatrix} 0.95 & 0.05 & 9 \\ 0 & 0.9 & 0.1 \\ 0.01 & 0 & 0.99 \end{bmatrix}$$

Transisition diagram

Nice figure of the diagram

2.5 Doing n transitions.

Theorem 2.1. *For a Markoc chain $\{X_n : n = 0, 1, \dots\}$ and any $m \geq 0$ we have*

$$Pr\{X_{m-n} = j \mid X_m = i\} = P_{ij}^{(n)} = \sum_{k=0}^{\infty} P_{ik} P_{kj}^{(n-1)}, \quad n > 0$$

where we define

$$P_{ij}^{(0)} = \begin{cases} 1, & i = j \\ 0, i \neq j \end{cases}$$

Proof. Set $m = 0$ then is

$$\begin{aligned} P_{ij}^{(n+1)} &= Pr\left\{X_{n+1} = j \mid X_0 = i\right\} \\ &= \sum_k Pr\left\{X_{n+1} = j, X_1 = k \mid X_0 = i\right\} \\ &= \sum_k Pr\left\{X_{n+1} = j \mid X_1 = k, X_0 = i\right\} \cdot Pr\left\{X_1 = k \mid X_0 = i\right\} \\ &= \sum_k P_{kj}^{(h)} \cdot P_{ik} = \sum_k P_{ik} P_{kj}^{(h)} \end{aligned}$$

□

Example. $\{X_n : n = 0, 1, 2, \ldots\}$ is a markoc chain and

$$P = \begin{bmatrix} 0.1 & 0.9 \\ 0.6 & 0.4 \end{bmatrix}$$

Find $P_{01}^{(4)}$. **Solution**.

$$P^2 = \begin{bmatrix} 0.55 & 0.45 \\ 0.30 & 0.70 \end{bmatrix}$$

So by doing matrix multiplication and we end up with

$$P^4 = P^2 \cdot P^2 = \begin{bmatrix} 0.4375 & 0.5625 \\ 0.3750 & 0.6250 \end{bmatrix}$$

Which therefore ends up with the answer

$$P_{01}^{(4)} = 0.5625$$

3 Lecture 4

3.1 Introduction to first step analysis

Input

- i_0 : starting state
- P : transition probability matrix
- T: number of time steps

Algorithm

1. Set $x_0 = i_0$
2. for $n = 1 \dots T$
3. Simulate x_n from $X_n \mid X_{n-1} = x_{n-1}$
4. end

output : One realization $x_0, x_1, \dots, x_T$

Example.

$$P = \begin{pmatrix} 0.95 & 0.05 & 0 \\ 0 & 0.90 & 0.10 \\ 0.01 & 0 & 0.99 \end{pmatrix}$$

Let $x_0 = 0$

1. $x_0 = 0$
2. $$\begin{aligned} Pr\{X_1 = 0 | X_0 = 0\} =& P_{00} = 0.95 \\ Pr\{X_1 \mid X_0\} =& P_{01} = 0.05 \\ Pr\{X_1 \mid X_0 = 0\} =& P_{02} = 0 \\ & . \end{aligned}$$

   Assume we get $x_1 = 1$
3. States
   - $$\begin{aligned} 0 &: P_{10} = 0 \\ 1 &: P_{11} = 0.90 \\ 2 &: P_{12} = 0.10 \\ &\vdots \end{aligned}$$

General notes on simulation

- $Pr\left\{A\right\} \approx \frac{\text{times A occure}}{\text{Simulations}}$
- $E\left[X\right] \approx \frac{1}{N}\sum_{i=1}^{B} x_i$

Example. We have $N = 100$ divided into two containers labelled A and b. At each time n, one ball is selected at random and moved to the container. Let Y_n denote the number of balls in container A at time n, and define $X_n = Y_n - 50$. Find the transition probabilities and simulate and plot one realization of

$$\{X_n : n = 0, 1, \ldots, 500\}$$

Answer

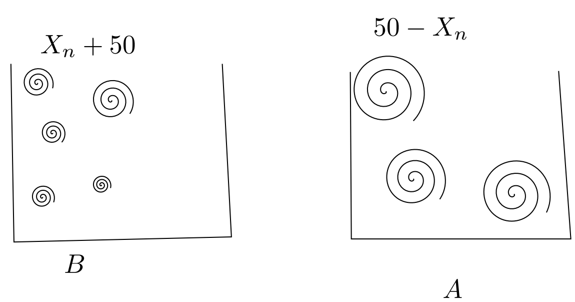

Figure 1: balls

- Only move One ball
- Can move only from i to $j = i - 1$ or $ji + 1$

$$P_{ij} = \begin{cases} \frac{50-i}{100} & , \quad j = i+1 \\ \frac{50+i}{100} & , j = i-1 \\ 0 & , \text{otherwise.} \end{cases}$$

Motivation

Definition 3.1. *For a markov chain, a state i sich that* $P_{ij} = 0 \forall j \neq i$ *is* *called **absorbing.***

Example. Let $\{X_n\}$ be a Markov chain woth transition probability matrix

$$\mathbf{P} = \begin{pmatrix} 1 & 0 & 0 \\ \alpha & \beta & \gamma \\ 0 & 0 & 1 \end{pmatrix}$$

where $\alpha, \beta, \gamma > 0$ and $\beta = 1 - \alpha - \gamma$. Assume $x_0 = 1$

1. What is the expected time until absortion ?
2. What is the probability to be absorbed in state 0 ?

Realization .

$$\overbrace{1, 1, 1, 1, 1, 2}^{\text{4 steps to absorption}} \quad , 2, 2 \ldots$$

Mathematically

Let $T = \min \{n \geq 0 : X_n = 0 \quad \text{or} \quad X_n = 2\}$. Then is

$$\begin{aligned} Q1 :& \quad E\left[T \mid X_0 = 1\right] \\ Q2 :& \quad Pr\left\{X_T = 0 \mid X_0 = 1\right\} \end{aligned}$$

The idea of first step analysis is to define

- $T^{(n)} = \min \{n \geq :: X_{m \times n} = 0 \quad \text{or} \quad X_{m+b} = 2\}$
- $T = T^{(0)}$
- $v_i^{(m)} = E\left[T^{(m)} \mid X_m = i\right]$
- $v_i = v_i^{(0)}$

Table 2: Let m be timesteps

m	0	2	3	4	5
$v_0^{(m)}$	0	0	0	0	0
$v_1^{(m)}$	v_1	v_1	v_1	v_1	v_1
$v_2^{(m)}$	0	0	0	0	0

First step analysis for Q1

$$
\begin{aligned}
v_i &= \sum_{k=0}^{2} Pr\left\{X_1 = k \mid X_0 = i\right\}(1 + v_k) \\
&= \sum_{k=0}^{2} P_{ik}(1 + v_k) = \sum_{k=0}^{2} P_{ik} v_k + 1 \quad \text{which is true for} \quad i = 0, 1, 2
\end{aligned}
$$

Which is reduced to linear algebra. Solving it by

$$
\begin{aligned}
& v_0 = v_2 = 0 \\
\implies & v_1 = \alpha v_0 + \beta v_1 + \gamma v_2 + 1 \\
\implies & v_1 = \frac{1}{1-\beta} \quad \text{[Q1]}
\end{aligned}
$$

$P_{ij} \implies i = \text{row}, \quad j = \text{column}$

First step analyis and let

$$
\begin{aligned}
u_i &= Pr\left\{X_T = 0 \mid X_0 = i\right\} \\
&\downarrow \\
u_i &= \sum_{k=0}^{2} P_{ik} u_k, \quad i = 0, 1, 2
\end{aligned}
$$

- Easy: $u_0 = 1, u_2 = 0$
- Harder: $u_1 = \alpha u_0 + \beta u_1 + \gamma u_2$ such that

$$
u_1 = \alpha \frac{1}{1-\beta} = \frac{\alpha}{\alpha - \beta} \quad \text{[Q2]}
$$

Example. let $[X_n]$ be a markov chain with transition matrix

$$\mathbf{P} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0.4 & 0.3 & 0.2 & 0.1 \\ 0.1 & 0.3 & 0.3 & 0.3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The starting state is $x_0 = 1$. Calculate the probability to be absorbed in the state D.

1. Define $u_i = Pr\left\{\text{absorbed in state } 0 \mid X_0 = i\right\}$ for $i = 0, 1, 2, 3$
2. Get the easy ones out of the way. In this case $u_0 = 1$ and $u_3 = 0$
3. $$\begin{aligned} u_1 &= P_{10}u_0 + P_{11}u_1 + P_{12}u_2 + P_{13}u_3 \\ &= 0.4 + 0.3u_1 + 0.2u_2 \\ u_2 &= P_{20}u_0 + P_{21}u_1 + P_{22}u_2 + P_{23}u_3 \\ &= 0.1 + 0.3u_1 + 0.3u_2 \end{aligned}$$
4. Solve for u_1 and u_2

4 Lecture 5

Example. Let P be the matrix

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0.4 & 0.3 & 0.2 & 0.1 \\ 0.1 & 0.3 & 0.3 & 0.3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

With starting state $x_0 = 1$

1. Define $T = \min_{n \geq 0 : X_n = 0 \quad X_n = 3}$ and $v_i = E\left[T \mid X_0 = i\right]$ for $i = 0, 1, 2, 3$
2. Set $v_0 = v_3 = 0$
3. $$v_1 = P_{10}v_0 + P_{11}v_1 + P_{12}v_2 + P_{13}v_3 = 0.3v_1 + 0.2_{v2} + 1$$

   and

   $$v_2 = P_{20}v_0 + P_{21}v_1 + P_{22}v_2 + P_{23}v_3 + 1 = 0.3v_1 + 0.3v_2 + 1$$
4. Solve the equations and end up with

   $$v_1 = \frac{90}{43} \quad \text{and} \quad v_2 \frac{100}{43}$$

Theorem 4.1. *Let $\{X_n\}$ be a discrete time Markov chain with state space $S = \{0, 1, \dots, N\}$ and transition probability matrx $\mathbf{P}$. Let $A \subset S$ be the set of absorbing state. Then*

1. *If v_i is the expected time to absorption conditional on $X_0 = i$ then*

$$v_i = 0, \quad i \in A$$
$$v_i = 1 + \sum_{i \in \mathbb{R}} P_{ik}v_k \quad i \in A^c$$

Example. A gambler has 10\$ and bets 1\$ If he wins the round, his fortune increases 1\$. The probability of winning each round is $0 < p < 1$ and the probability of losing each round is $q = 1-p$. The gambler will continue gambling until his fortine is \$ N or 0\$ where $N > 10$. What is the probability the gambler will be ruined.

1. Extract the essential stuff.

$$X_n = \text{Fortune at time} \quad n, \quad n = 0, 1, 2, \dots$$
$$\text{State space } = \{0, 1, \dots, N\}$$
$$\text{Target: } u_k = Pr\left\{\text{Absorption in state } 0 \mid X_0 = k\right\}, \quad k = 0, 1, \dots, N$$

2. Visualize the transitions. Insert figure of tranistions.

3. Make the eprobability matrix. The rows are "to" and the columns are "1"

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 & \dots & 0 \\ q & 0 & p & 0 & \dots & 0 \\ 0 & q & 0 & p & \dots & \\ \vdots & & \ddots & & & \\ & & & q & 0 & p \\ 0 & 0 & \dots & & & 1 \end{bmatrix}$$

4. Set up the iteration

$$\begin{aligned} u_0 &= 1, \quad u_N = 0, \quad \text{Easy} \\ u_i &= P_{i,i.1} u_{i-1} + P_{i,i+1} u_{i+1} \\ &= q u_{i-1} + p u_{i+1}, \quad i = 1, 2, \dots, N-1 \end{aligned}$$

5. (a)

$$\begin{aligned} \overbrace{(p+q)}^{=1} u_i &= q u_{i-1} + p u_{i+1} \\ q\left[u_i - u_{i-1}\right] &= p\left[u_{i+1} - u_i\right] \\ &\downarrow \quad \text{Trick} \quad \chi_i = u_i - u_{i-1} \\ q\chi_1 &= p\chi_{i+1}, \quad \implies \chi_{i+1} = \frac{q}{p}\chi_i \quad i = 1, 2, \dots, N \end{aligned}$$

(b)

$$\chi_1 + \chi_2 + \dots + \chi_k = [u - u_0] + [u_2 - u_1] + \dots + [u_k - u_{k-1}]$$
$$\downarrow \quad \text{Telescoping sum}$$
$$\chi_1 \left[1 + \frac{q}{p} + \left(\frac{q}{p}\right)^2 + \dots + \left(\frac{q}{p}\right)^{k-1}\right] = u_k - 1, \quad k = 1, 2, 3, \dots, N$$

For $k = N$:

$$\begin{aligned} \chi_1 &= \frac{u_N - 1}{\sum_{k=0}^{N-1} \left(\frac{q}{p}\right)^k} = \frac{-1}{\sum_{k=0}^{N-1} \left(\frac{q}{p}\right)^k} \\ &= \begin{cases} -\frac{1}{N} & , q = p = \frac{1}{2} \\ \frac{-\left(1-\frac{q}{p}\right)}{\left(1-\left(\frac{q}{p}\right)\right)} & q \neq p \end{cases} \end{aligned}$$

(c) From the telescoping sum

$$u_k = 1 + \chi_1 \sum_{i=0}^{k-1} \left(\frac{q}{p}\right)^i$$
$$= \begin{cases} 1 - \frac{1}{N} \cdot k = \frac{N-k}{N}, & p = q = \frac{1}{2} \\ 1 - \frac{1-\left(\frac{q}{p}\right)^k}{1-\left(\frac{q}{p}\right)^N} = \frac{\left(\frac{q}{p}\right)^k - \left(\frac{q}{p}\right)^N}{1-\left(\frac{q}{p}\right)^N}, & p \neq q \end{cases}, \quad \text{where} \quad k = 1, 2, \ldots.$$

.

6. The final step
$$u_{10} = \begin{cases} \frac{N-10}{N}, & p = q = \frac{1}{2} \\ \frac{\left(\frac{q}{p}\right)^{10} - \left(\frac{q}{p}\right)^N}{1-\left(\frac{q}{p}\right)^N}, & q \neq p \end{cases}$$

Remark.

- When $N \to \infty$

$$q \geq p \implies \text{Almost certain you will loose.}$$
$$q < p \implies P\,(\text{ruined}) = \left(\frac{q}{p}\right)^{10}$$

4.1 Markov Chain in infinitive time

Definition 4.1. ***Regular Markov Chain*** . *Consider a Markov chain $\{X_n : \quad n = 0, 1, \ldots\}$ with finite state space $\{0, 1, 2, \ldots\}$ and transition matrix $\mathbf{P}$. IF there exists an integer $k > 0$ so that all regular elements $\mathbf{P}^k$ are strictly positive, we call $\mathbf{P}$ and $\{X_n\}$ regular.*

Remark.

1. P is regular means that it exists an $k > 0$ so that $P_{ij}^{(k)} > 0 \quad \forall i, j$

2. If $P_{ij}^{(k)} \quad \forall i, j$, then is $P_{ij}^{(k)} > 0 \quad \forall i, j$ and $K \geq k$

5 Lecture 2020-09-14

Find Stationary distributions

(i) $\mathbf{P} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$

- Positive recurrent, aperiodic and irreducible.
- $\implies$ Limiting distribution:

$$\pi = \left(\frac{1}{2}, \frac{1}{2}\right)$$

(ii) $\mathbf{P} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

- Positive recurrent and irreducible.
- unique stationary distrobution.
- $\pi = \left(\frac{1}{2}, \frac{1}{2}\right)$

(iii) $\mathbf{P} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

- Reducible!
- Part 1:

$$\begin{aligned} \pi_0 &= 1\pi_0 = 0\pi_1 = \pi_0 \\ \pi_1 &= 0\pi_0 + 1\pi_1 = \pi_1 \\ \implies \pi &= 1 - \pi_1 \\ \implies \pi &= (\pi_1, 1 - \pi_1) \end{aligned}$$

- Part 2:

  Must have

$$\begin{aligned} \pi_0 &\geq 0 \\ \pi_1 &\geq 0 \\ \implies \pi &= (\pi_0, 1 - \pi_0), \quad 0 \leq \pi_0 \leq 1 \end{aligned}$$

5.0.1 Section 4.5

Read it yourself .

5.1 Why do we care so much about markov chains?

(i) Importance goes far beyond statisical modelling of physical phenomena.

(ii) In the end of the 80s and start of 90s the computationally power was growing stronger.

(iii) We realized that we could sample from difficult distribution by constructing Makov chains whose stationionary matched desired target distrobution.

(iv) The theory we have descussed of the theory developed to show that these methods worked.

5.2 Continuous Time Markov Chain

Definition 5.1. *The stochastic variable X has a* ***Poission distribution*** *with (mean) parameter $\mu > 0$ if*

$$p(x) = \frac{\mu^x}{x!} e^{-\mu}$$

We write $X \sim Possion(\mu)$

Remark. $X \sim Poission(10)$

(i) $E[X] = \mu$

(ii) $Var[X] = \mu$

(iii) $SD[X] \sqrt{\mu}$

Theorem 5.1. *If $X \sim Possion(\mu)$, $\quad Y \sim (\chi)$ and Y are independent.*

Theorem 5.2. *If $N \sim Possion(\mu)$ and $M \mid N \sim Binomial(N, p)$ then*

$$M \sim Poission(\mu P)$$

Remark. (i) $M = \sum_{k=1}^{N} I_k$, where $I_1, I_2, \ldots \sim Bernoulli(p)$ and $I_1, I_2, \ldots$ and N are independent.

(ii) This is called **thinning**.

5.2.1 Section 5.1.2

Definition 5.2. *A* ***Possion process*** *with rate* ***inensity*** $\lambda > 0$ *is an integet-valued stochastic process* $\{X(t) : t \geq 0\}$ *0 for which.*

- *For any* $n > 0$ *and any time point* $0 < t_0 < t_1 < \ldots < t_n$ *the increments*

$$X(t_1) - X(t_0), X(t_2) - X(t_1), \ldots, \ldots X(t_n) - X(t_{n-1})$$

  are independent

- *For* $s \geq 0$ *and* $t > 0$

$$X(s+t) - X(s) \sim Poission(\lambda t)$$

- $X(0) = 0$

Remark.

- 1. is called independent increments
- In 2, we have
$$X(s + \Delta t) - X(s) \sim Possion(\lambda \Delta t)$$
- Illustration

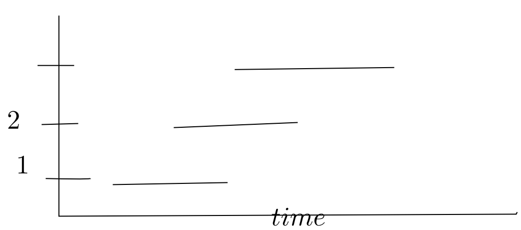

Figure 2: iillustration

- $X(t) = X(t) - X(0) \sim Possion(\lambda t)$

Example. We assume the arrival of customers to a store follows a Poission process with rate $\lambda = 4$ customers per hours. The stor opens a 09:00. What is the probability that exactly one customer has arrived by 09:30 and exactly five customers have arrived bu 11:30.

Answer. Let $X(t)$ = arrivals by time t For $t \geq 0$ (measured in hours). Then is the question

$$Pr\left\{X\left(\frac{1}{2}\right) = 1, X\left(\frac{5}{2}\right) = 5\right.$$

$$\begin{aligned}
&\downarrow = \text{ Rephrase as incements} \\
&= Pr\left\{X\left(\frac{1}{2}\right) - X(0) = 1, X\left(\frac{5}{2}\right) - X\left(\frac{1}{2}\right) = 4\right\} \\
&\downarrow\text{Independent increments} \\
&= Pr\underbrace{\left\{X\left(\frac{1}{2}\right) - X(0) = 1\right\}}_{Poission(\frac{1}{2}\lambda)} \cdot Pr\overbrace{\left\{X\left(\frac{5}{2}\right) - X\left(\frac{1}{2}\right)\right\}}^{Possion(2\lambda)} = 4 \\
&= \frac{2^1}{1!}e^{-2} \cdot \frac{8^4}{4!}e^{-8} \\
&= 0.0155
\end{aligned}$$

Example. Assume the arrical of customers to follows an inhomogenous Poission process with rate $\lambda(t) = t$, $t \geq 0$. Assume the store opens at $09:00$. What is the probability that no-one has arrived at $10:00$.

Answer.

$$X(1) - X(0) \sim Poission\overbrace{\left(\int_0^1 tdt\right)}^{=\frac{1}{2}}$$

6 Lecture 08/09/20

Equivalent classes and classifications of states in Markov chains.

Things to check

- Understand why regularity fails.
- Extend regularity to infinite spaces.

Example Let $\{X_n : 0, 1, \ldots, N\}$ be a markov chain.

(a) It can go from $0 \to 0$ and $1 \to$ with probabilities $p_{00} = p_{11} = 1$, two seperate markov chains. Realizations :

$$0, 0, 0, 0, 0, 0, \ldots$$
$$1, 1, 1, 1, 1, 1, \ldots$$
$$P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \implies P^n = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Definition 6.1. *Let $\{X_n : 0, 1, \ldots\}$ be a Markov chain with state space $\{0, 1, \ldots\}$ then is*

(i) State j is ***accessible*** *from state i if $\exists n \geq 0$ so that $P^{(n)} > 0$*

(ii) If states i and j are accessable from each other they are said to ***communcate*** *we write $i \sim j$. If states i and j do not communcate we write $i \not\sim j$*

Remark. If $i \not\sim j$, then either (or both)

(a) (i) $P_{ij}^{(n)} = 0, \quad \forall n \geq 0$
   (ii) $P_{ji} = 0, \quad \forall n \geq 0$

(b) Only the graph matters, not the values of the edges.

(c) $P_{ij}^{(0)} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$

Theorem 6.1. *Communication is an* ***equicalence relation***

(i) ***reflexive*** *, i j*

(ii) ***symmentric*** $i \sim j \implies j \sim i$

(iii) ***Transitive*** $i \sim j$ *and* $j \sim k$ *implies* $i \sim k$

A equivalence relation induces ***equivalence classes*** *of sets of states that communicate.*

Proof. (i) $P_{ii}^{(0)} = 1 \quad \implies i \sim i$

(ii) By definition is this true.

(iii) (a) $i \sim j \implies \quad \exists n \geq 0 : P_{ij}^{(n)} > 0$

$$j \sim k \implies \exists m \geq 0 : P_{jk}^{(m)} > 0$$

(b) Chapman-kilogram

$$P_{ik}^{(n+m)} = \sum_{r=0}^{\infty} P_{ir}^{(n)} P_{rj}^{(m)} \geq P_{ij}^{(n)} P_{jk}^{(m)}$$

$\implies$ k is accessible from i.

(c) Show yourself

i **is accessible from** k

□

Definition 6.2. *A Markov chain is* ***irreducible*** *if $\sim$ (communication) induces exactly one equivalent class. If not, it is called reducible.*

Definition 6.3. *The* ***period*** *of state i, written as $d(i)$ is*

$$d(i) = gcd\left\{n \geq 1 : P_{ii}^{(n)} > 0\right\}$$

If $P_{ii}^{(n)} = 0$ for all $n \geq 1$, we define $d(i) = 0$. If $d(i) = 1$, we call the state i is ***aperiodic.***

Theorem 6.2. *if $i \sim j$, then $d(i) = d(j)$*

Remark. The period is a property of the equivalence class.

Notation THe state space may be infinite: $\{0, 1, \ldots\}$. We introduce

(i) The probability the first return happend after exactly n steps

$$f_{ii}^{(n)} = Pr\{X_n = i, X_\mu \neq i, i = 1, 2, \ldots, n-1 \mid X_0 = i\} \quad n > 0$$

We will define $f_{ii}^{(0)} = 0$

(ii) The probability of returning at some time

$$f_{ii} = \sum_{k=0}^{\infty} f_{ii}^{(k)} = \lim_{n \to \infty} \sum_{k=0}^{n} f_{ii}^{(k)}.$$

Remark. $f_{ii} < i \leftrightarrow$ Positive probability of never returning to i

Definition 6.4. *State i is* ***recurrent*** *if the probabilitu of retunging to sate i in a finite number of timesteps is one $f_{ii} = 1$. A state that is not recurrent $f_{ii} < 1$ is called* ***transient*** *.*

Theorem 6.3. *A state i is recurrent if and only if*

$$\sum_{n=1}^{\infty} P_{ii}^{(n)} = \infty$$

Equivalently, state i is transient if and only if

$$\sum_{n=1}^{\infty} P_{ii}^{(n)} < \infty$$

Proof. (i)

$$\begin{aligned}
\sum_{n=1}^{\infty} P_{ii}^{(n)} &= \sum_{n=1}^{\infty} E\left[\mathbb{I}\left\{X_n = j\right\} \mid X_0 = j\right] \\
&= E\left[\sum_{n=1}^{\infty} \mathbb{I}\left\{X_n = i \mid X_0 = i\right\}\right] \\
&= E\left[M \mid X_0 = i\right] \\
M &\rightarrow \text{Returns to state.}
\end{aligned}$$

(ii) $E\left[M \mid X_0 = i\right] = \begin{cases} f_{ii} \frac{1}{1 - f_{ii}}, & f_{ii} < 1 \\ \infty, & f_{ii} = 1 \end{cases}$

$\square$

7 Lecture 2020-09-18

Read Section 5.1.4 by yourself.

Section 5.2 Motivation

(a) $\{X(t) : t \geq 0\}$ with rate $\lambda_1 = 5, \quad 0 \leq t \leq 10$

$$E[X(t)] = \lambda t = 5t,$$

(b) $\{Y(t) : t \geq 0\}$ with rate $\lambda_2 = t, \quad 0 \geq t \leq 10$

$$E[Y(t)] = \frac{t^2}{2}$$

Do scatterplot on the project when working on poission distribution.

Theorem 7.1. *Let* $p_1, p_2, \ldots \in [0,1]$ *be a sequence such that* $\lim_{n\to\infty} np_n = \lambda < \infty$*, then*

$$\lim_{n\to\infty} \binom{n}{k} p_n^k (1 - p_n)^{n-k} = \lambda^n \frac{1}{k!} e^{-\lambda}, \quad k = 0, 1, \ldots$$

Remark. In TMA4295 Statistical Inference we will say that $Binomial(n, p_n)$ converges in $Possion(\lambda)$ is $n \to \infty$.

Remark. .

(i) $p_n \to 0$, but $n \to \infty$. $np_n \to \lambda$ when $n \to \lambda$

(ii) Many trials $(n \gg 1)$ and success is rare $(p \ll 1) \implies$ Nr of Successes Poission distribution.

Typical examples

- Customers arrivals.
- Car accident.
- Telephone calls.

7.0.1 Little Oh notation

(i) You may be familar with the expessions such as

$$n = o\left(n^2\right), \quad \text{as } n \to \infty$$

May be thought as "n is much smallet than n^2 as $n \to \infty$"

(ii) We are going to mostly work with expressions of the form

$$h^2 = o(h), \quad h \to 0^+$$

May be thought as "h^2 is much smaller than h as $h \to 0^+$"

Definition 7.1. *Let f and g be real functions. We use* ***little-oh-notation*** *in the two following ways*

(i) $f(n) = o(g(n)), \quad n \to \infty \implies \lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$

(ii) $f(h) = o(g(h)) \quad h \to 0^+ \implies \lim_{n\to\infty} \frac{f(h)}{g(h)} = 0$

Example. Are the following statements false or true?

(i) $h^2 = o(h) \quad h \to 0+$

$$\lim_{n\to 0^+} \frac{h^2}{h} = 0$$

True

(ii) $h^2 = o(h) \quad h \to \infty$

$$\lim_{h\to\infty} \frac{h^2}{h} = \lim_{n\to\infty} h = \infty$$

False

(iii) $\sqrt{h} = o(h) \quad h \to 0^+$

$$\lim_{h\to 0^+} \frac{\sqrt{h}}{h} = \infty$$

False

(iv) $h \to o(1) \quad h \to 0^+$

$$\lim_{h\to 0^+} \frac{h}{1} = 0$$

True

Remark.

$$h^p = o(h) \quad h \to 0^+ \quad \implies p > 1$$

Definition 7.2. *A* ***Cprocess*** *is a stochastic process $\{N(t) : t \geq 0\}$ so that*

(i) $N(t)$ *is a integer for* $t \geq 0$

(ii) $N(t) \geq 0$, *for* $t \geq 0$

(iii) *If* $s \geq t$, *then* $N(s) \leq N(t)$

We sometimes write

$$N(a,b) = N(b) - N(a) = \textit{Number or events in } (a,b], \quad 0 \leq a \leq$$

However, the notation will not be used in the lecture.

Definition 7.3. *Let* $\{N(t) : t \geq 0\}$ *be a counting process. Then* $\{N(t) : t \geq 0\}$ *is a* ***Poission proccess*** *with* ***rate (intensity)*** $\lambda > 0$ *if*

(i) *For every integer* $m > 1$ *for any timepoints*

$$0 = t_0 < t_1 < \ldots < t_m$$
$$N(t_1) - N(t_0), N(t_2) - N(t_1), \ldots N(t_m) - N(t_{m-1})$$

"independent increments"

(ii) *For* $t \geq 0$ *and* $h > 0$, *the distrobution of* $N(t+h) - N(t)$ *only depends on* h *and* t. *"Stationary Increnements"*

(iii) $Pr\{N(t+h) - N(t) = 1\} = \lambda h + o(h), \quad h \to 0^+ \quad \forall t \geq 0$

(iv) $Pr\{N(t+h) - N(t) = 0\} = 1 - \lambda h + o(h), \quad h \to 0^+ \quad \forall t \geq 0$

(v) $N(0) = 0$

For def iii and iv can be described as

$$\begin{aligned} \Longrightarrow \ Pr\{N(t+h) - N(t) \geq 2\} &= 1 - \overbrace{[\lambda h + o(h)]}^{1\ event} - \overbrace{[1 - \lambda h + o(h)]}^{0\ events} \\ &= o(h) \\ &\Longrightarrow \textit{Events cannout occur at the same time} \\ &\Longrightarrow \ \textit{Jumps are of size 1} \end{aligned}$$

Recall

Definition 7.4. *(Simplified version.) A* ***Poission process*** *with rate* ***rate*** $\lambda > 0$ *is an integer valued stochastic process* $\{N(t) : t \geq 0\}$ *for which*

(i) *Increments are independent,*

(ii) *For* $s \geq 0$ *and* $t > 0$

$$N(s+t) - N(s) \sim Possion(\lambda t)$$

(iii) $N(0) = 0$

Theorem 7.2. *Definition of simplified and genreal of a Poission process are equivalent.*

Proof. Lets call the simplified version P1 and the general version P2, then we need to prove

- Prove that $P1 \implies P2$: i),ii) and v) is proved by definition.

$$\begin{aligned} Pr\left\{N(t+h) + N(t) = 1\right\} &= \frac{(\lambda h)^1}{1} e^{\lambda h} \\ &= \lambda h \left(1 - \lambda h o(h)\right), \quad \text{as } h \to 0^+ \\ &= \lambda h - \lambda^2 h^2 + \lambda h o(h) \\ &= \lambda h + o(h) \end{aligned}$$

  This type of manipulations are importan on the exam.

  For iv):

$$\begin{aligned} Pr\left\{N(t+h) - N(t) = 0\right\} &= \frac{(\lambda h)^0}{0!} e^{-\lambda h} \\ &= 1 \cdot (1\lambda h + o(h)) \\ &= 1 - \lambda h + o(h), \quad \forall t \geq 0 \end{aligned}$$

- Prove that $P2 \implies P1$: i) and iii) are proved by definition.

  For ii): Set $s = 0$ Ned to show that

$$N(h) - N(0) \sim Poission(\lambda h)$$

  (i) Divide $(0, h]$ into equal size sub-intervals.

$$\implies t_i = \frac{i}{m}, \quad i = 0, 1, \ldots, m.$$

  (ii) Let

$$\varepsilon = \begin{cases} 1, & \text{at least one event in } (t_{i-1}, t_i] \\ 0, & \text{Otherwise} \end{cases} \quad , \quad i = 1, 2, \ldots, m$$

  Then we can let $S_m = \sum_{i=1}^{m} \varepsilon_i$.

(iii) $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_m \sim Bernoulli\left(p_m\right)$ where $p_m = \frac{\lambda h}{m} + o\left(\frac{h}{m}\right)$ as $m \to \infty$.

Let $S = \lim_{m\to\infty} S_m a$ we get

$$\lim_{m\to\infty} mo_m = \lim_{m\to\infty}\left(\lambda h + o\left(1\right)\right) = \lambda h$$

This is calles the "Law of rare events" $S \sim Possion\left(\lambda h\right)$.

(iv) $$\begin{aligned} Pr\left\{N\left(h\right) - N\left(0\right) \neq S_m\right\} &\leq \sum_{i=1}^{m} Pr\left\{N\left(t_i\right) - N\left(t_{i-1}\right) \geq 2\right\} \\ &\leq \sum_{i=1}^{m} o\left(\frac{h}{m}\right) \\ &= m \cdot o\left(\frac{h}{m}\right) \\ &= ho\left(1\right) \\ &\to_{m\to\infty} 0 \\ &\downarrow \\ N\left(h\right) - N\left(0\right) &= S \sim Poission\left(\lambda h\right) \end{aligned}$$

$\square$

8 Lecture 2020-09-21

Example. Is it reasonable to model the following phenomena as Poission processes ?

(a) Cases of a non-infectious rare disease.

- Independent incremens: Yes, people are independent.
- Stationary increments: Yes. Few people get sick.
- Many trials, "success " is rare: Yes. many people get sick.

(b) Calls going through a phone central.

- Yes. For specific time intervals.

(c) Goals in football.

- No. Number of goals are not independent.

8.1 Properties of the Poission process

Definition 8.1. *Let $\{N(t) : t \geq 0\}$ be a Poission process. Tee* ***waiting time*** *W_n is the time of occurance of the n-th event. We define $W_0 = 0$*

Definition 8.2. *The difference $S_n = W_{n+1} - W_n$ are called the* ***sojurn times*** *(interarrival times.)*

Remark. .

(i) S_n = Time spent in stationary.

(ii) Two viewpoints.

  (a) Possion process $\{N(t) : t \geq 0\}$

  (b) Poission point process. $(W_1, W_2, W_3, \ldots)$

Definition 8.3. *The stocastic variable Y has an* ***exponantial distrobution*** *with the rate parameter $\lambda > 0$*

$$f(y) = \lambda e^{-\lambda y}, \quad y > 0$$

We write $Y \sim Exp(\lambda)$.

Remark. • We will always use this parameterization.

- Other: Scale paremter $\beta > 0$:

$$f(y) = \frac{1}{\beta}, \quad y > 0$$

Theorem 8.1. *Let $\{N(t) : t \geq 0\}$ be a Poission process with rate λ. Then $S_0, S_1, \ldots, S_{n-1} \sim Exp(\lambda)$*

Proof. For $n = 1$

(i) $Pr\{S_0 > s_0\} = Pr\{N(s_0) - N(0) = 0)$

(ii) n= 2

(a) $S_0 \sim Exp(\lambda)$

(b) $Pr\{S_1 > s_1 \mid S_0 = s_0\} = Pr\{N(s_0 + s_1) - N(s_0) = 0 \mid S_0 = s_0\}$

$$\begin{aligned} &\downarrow \text{Independent increments} \implies \text{Markov} \\ &= Pr\{N(s_0 + s_1) - N(s_0) = 0\} \\ &\downarrow \text{Stationary increments} \\ &= Pr\{N(s_1) - N(0) = 0\} \\ &= e^{-\lambda s_1}, \quad s_1 > 0 \end{aligned}$$

(c) $S \sim Exp(\lambda)$ and S_0 and S_1 are independent.

(iii) For $n = 3, 4, \ldots$

$$\text{Markoc property} \implies \text{independence..}$$

$Exp(\lambda)$ as for S_0 and S_1.

□

Remark. Alternatice definition of the possion process:

(i) Start in 0

(ii) Spend a time $Exp(\lambda)$ in each state.

Definition 8.4. *The stochastic variable Y has a* ***gamma distrobution*** *with* ***shape parameter*** *$\alpha > 0$ and* ***rate parameter*** *$\lambda > 0$ if*

$$f(y) = \frac{\lambda^{\lambda}}{\Gamma(\alpha)} y^{n-1} e^{-\lambda y}, \quad y > 0$$

We write $Y \sim Gamma(\alpha, \lambda)$

Remark. (i) Check which parametrization which is used.

(ii) Scale parameter: $\beta = \frac{1}{\lambda}$ is very common.

(iii) We will use shape and rate.

(iv) $Gamma\left(1, \lambda\right) = Exp\left(\lambda\right)$

Theorem 8.2. *For a Possion process with rate $\lambda > 0$ $W_n \sim Gamma\left(n, \lambda\right)$ for all integers $n > 0$.*

Proof. (i) $S_0, S_1, \ldots, S_{n-1} \sim Exp\left(\lambda\right)$

(ii) $W_n = S_0 + S_1 + \ldots + S_{n-1}$

$$\begin{aligned} &\downarrow \\ &\sim Gamma\left(\sum_{i=1}^{n} 1, \lambda\right) \\ &= Gamma\left(n, \lambda\right) \end{aligned}$$

□

Example. Assume the occurance of a rare disease follows a Poission process with rate $\lambda = 2$

(a) What is the probability that the first case accurs after 1 month?

(i) Let $S_0 \sim Exp\left(2\right)$

$$Pr\left\{S_0 > 1\right\} = \int_1^\infty 2e^{-2t}dt = e^{-2} \approx 0.135$$

Where $Pr\left\{N\left(1\right) - N\left(0\right)\right\}$

(b) What is the expected time until the 10th case occurs?

(i) Let $W_{10} \sim Gamma\left(10, 2\right)$

$$E\left[W_{10}\right] = \frac{10}{2} = 5, \quad \text{months.}$$

Example. Let $\left\{X\left(t\right) : t \geq 0\right\}$ is a Poission process with rate $\lambda > 0$. Determine the distrobution of $W_1 \mid X\left(t\right) = 1$

9 References