Repetition – conditional probability

Definition

Let A and B be events. The **conditional probability** of A given B is defined by

$$\Pr\{A|B\} = \begin{cases} \frac{\Pr\{A\cap B\}}{\Pr\{B\}}, & \Pr\{B\} > 0, \\ \text{Not defined}, & \Pr\{B\} = 0. \end{cases}$$

Definition

Assume X and Y are jointly distributed random variables. The **conditional PMF** $p_{X|Y}(x|y)$ of X given Y is given by

$$p_{X|Y}(x|y) = \frac{\Pr\{X = x, Y = y\}}{\Pr\{Y = y\}} = \frac{p_{X,Y}(x,y)}{p_Y(y)}, \quad \text{if } p_Y(y) > 0.$$

Theorem

The conditional PMF is essential to us because we can calculate the joint PMF as

$$p_{X,Y}(x,y) = p_Y(y)p_{X|Y}(x|y).$$

Theorem

Let X and Y be random variables. Then **the law of total probability gives**

$$\begin{aligned} \Pr\{X = x\} &= \sum_y \Pr\{X = x, Y = y\} \\ &= \sum_y \Pr\{Y = y\}\Pr\{X = x|Y = y\}, \\ \Pr\{Y = y\} &= \sum_x \Pr\{Y = y, X = x\} \\ &= \sum_x \Pr\{X = x\}\Pr\{Y = y|X = x\}. \end{aligned}$$

Definition

Let X and Y be random variables, and g a real function. The **conditional expected value** of $g(X)$ given $Y = y$ is

$$\mathrm{E}[g(X)|Y = y] = \sum_x g(x)\mathrm{Pr}\{X = x|Y = y\}, \quad \text{if } \mathrm{Pr}\{Y = y\} > 0.$$

Theorem

Let X and Y be random variables, then

$$\begin{aligned}\mathrm{E}[X] &= \mathrm{E}[\mathrm{E}[X|Y]],\\ \mathrm{Var}[X] &= \mathrm{E}[\mathrm{Var}[X|Y]] + \mathrm{Var}[\mathrm{E}[X|Y]].\end{aligned}$$

The former is often called **the law of iterated expectation** and the latter **the law of total variance**.

Note: $\mathrm{E}[X|Y]$ is a random variable that takes value $\mathrm{E}[X|Y = y]$ with probability $\mathrm{Pr}\{Y = y\}$.

Note 2: The conditional variance can be written as

$$\mathrm{Var}[X|Y] = \mathrm{E}[X^2|Y] - \mathrm{E}[X|Y]^2.$$