TMA 4190 Introduction to Topology
Lecturer: Gereon Quick
Lecture 01[1]

1. Introduction

1.1. **Organization.** First some general info:

Lectures: Mondays 12.15-14.00 in R92,
and Thursdays 10.15-12.00 in R54.

Exercises: Thursdays 16.15-17.00 in R21, but NOT WEEKLY. We will discuss exercises futher in class.

Important: You will have to solve the exercises yourself. The exercise classes will NOT consist of me giving solutions. If nobody comes up with suggestions, there will be nothing going on. You need to work in order to learn...

General advice: Talk to each other and to me. Ask questions! Interact!!! Solve exercises!!!!!
That's how you learn. Do not sit quiet and just read.

Course webpage (on which I will try to put more information soon):

wiki.math.ntnu.no/tma4190/2018v/start

Office hours: Upon request.

Just send me an email: gereon.quick@ntnu.no

Text books: In the beginning we will follow the book

[GP] V. Guillemin and A. Pollack, Differential Topology.

Another excellent and very short book:

[M] J.W. Milnor, Topology from the Differentiable Viewpoint.

Some other useful books:

[D] B. Dundas, Differential Topology.

[T] L.W. Tu, An Introduction to Manifolds.

[1] Following the books of Guillemin and Pollack: Differential Topology, and Milnor: Topology from the differentiable viewpoint.

There are many other good books out there. Ask me if you need more.

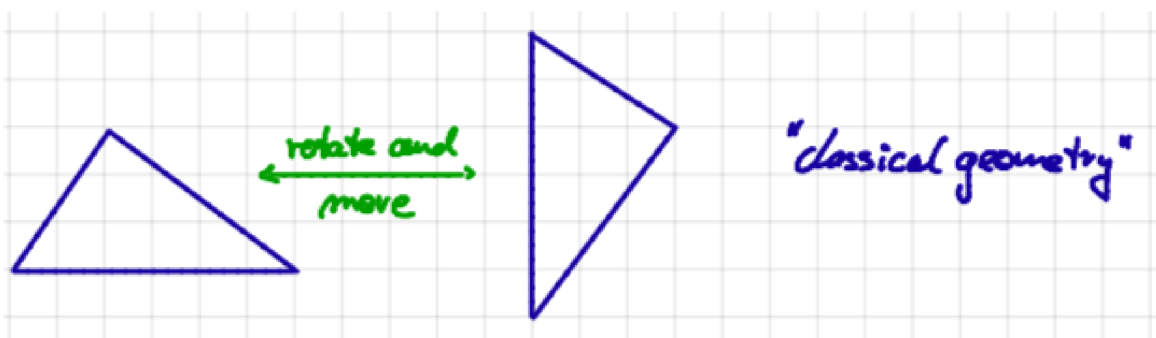
1.2. **What is required?** We will just assume some knowldge in multivariable calculus, corresponding to **Calculus 1 and 2**. For example, you should know what it means for **a map $\mathbb{R}^n \to \mathbb{R}^m$ to be smooth or differentiable**.
We will also assume knowledge on **complex numbers and linear algebra**, corresponding to what you learn in **Calculus 3**. For example, you should know what is a subspace of a vector space, what is the image of a linear map, when is a linear map invertible.
Finally, it would be desirable if you have heard the words:
open, closed, compact in connection with subsets of $\mathbb{R}^n$. Ideally, you also know, for example, what these notions have to do with convergence of sequences. But no worries, I will try to remind you of as much as I can during class. If you want to refresh your knowledge on Topology, you may want to have a look at the books

[J] K. Jänich, Topology.

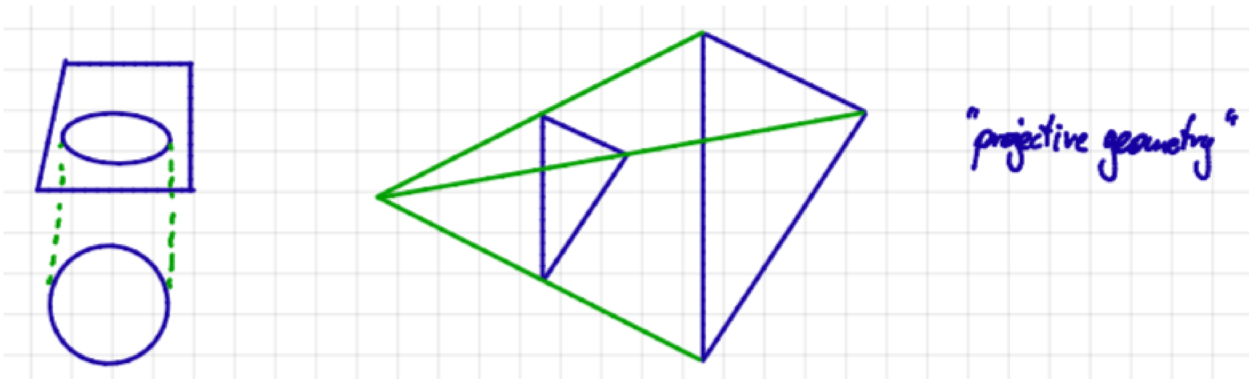
[D] B. Dundas, Appendix in Differential Topology.

As always, ASK ME if you wonder about anything!

1.3. **What this class is about?** Super roughly speaking, **Topology is some kind of Geometry**. Classical geometers were interested in measruing angles and distances. For example, two things are the "same" (congruent) in **classical geometry** if you can transform one into the other by moving or flipping them over. No stretching allowed. That means angles and lengths of edges stay the same.



A first variation to allow flexibility, is **projective geometry**: Two things are considered the same if they are both **views** of the same object. For example, an ellipse and a circle can be projectively equivalent; for one can look like the other when you look at them from the right prespective.



In **topology**, we take this idea one step further and consider two things the same if we can **continuously transform** one into the other. For example, a triangle is equivalent to a circle is equivalent to a square.

In **differential topology**, the part we will mostly be interested in, we only allow **smooth transformations**. (Then square and circle are different, because a square has edges which are not smooth.)

What Differential Topology is about:

Roughly speaking, differential topology is the study of properties that do not change under diffeomorphisms (specified transformations that are allowed).

We will make sense of all this during the course. This is just a first super rough distinction.

The goal of this class

Learn something about fundamental

- geometric objects, mostly we study smooth manifolds;
- methods and ideas in (differential) topology;
- applications of these objects and methods in different areas of mathematics.

In order to get a first idea, let's look at a fundamental example:

The Circle

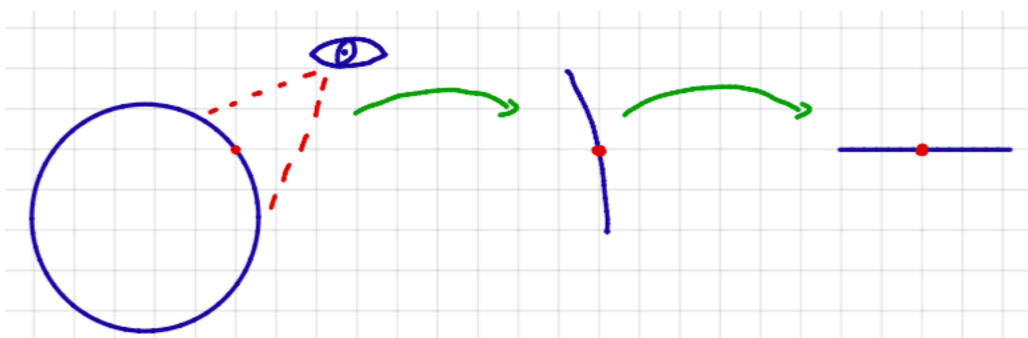
Let us start with the unit circle

$$S^1 = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\} \subset \mathbb{R}^2.$$

The circle is something on-dimensional, isn't it? But how do we describe that precisely. Well, it's clear if we **zoom in** at any point, it just looks like a bended line segment. Looking very closely it even looks almost like a striaght line segment.
So, **"locally"** (whatever that means) the circle looks like a segment of $\mathbb{R}^1$.
The unit circle S^1, more generally, the n-dimensional sphere

$$S^n = \{(x_1, \ldots, x_{n+1}) \in R^{n+1} : x_1^2 + \cdots + x_{n+1}^2 = 1\}$$

is an example of a smooth manifold.



Let us give a first working definition of what kind of objects we are going to study:

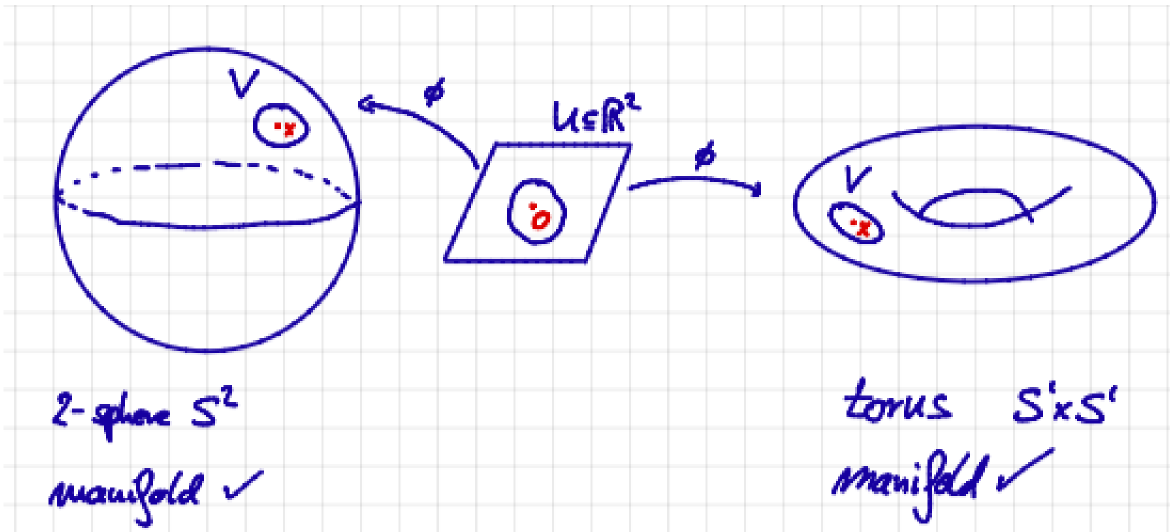
Working definition: What is a manifold

A manifold is a geometric object such that each point has a neighborhood which looks like $\mathbb{R}^n$.
We will make precise what "looks like" means. For smooth manifolds, we need a condition that takes differentiable data into account. The right notion is that of "diffeomorphism".

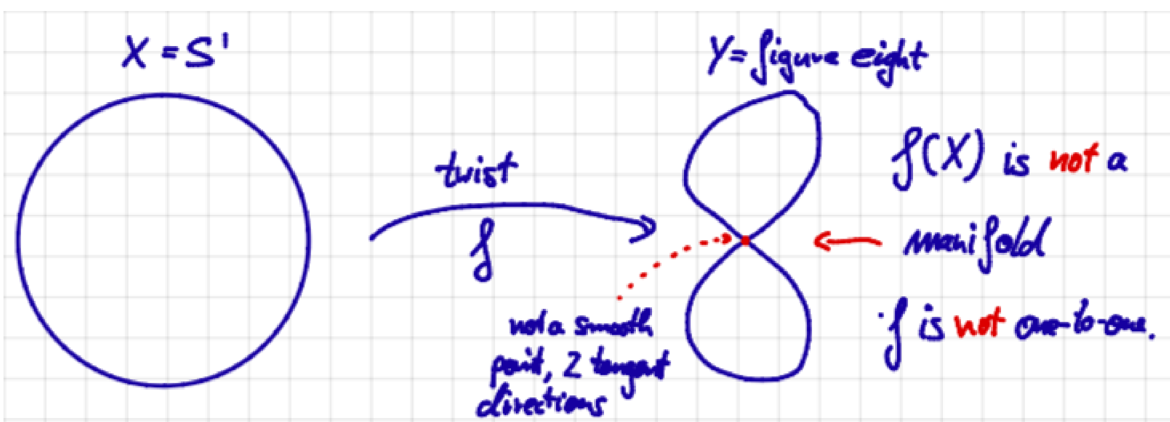
A universe of examples

The previous definition may sound quite strict. Every point looks the same in a small neighborhood. But we will see that there is a huge universe of examples of very different kind. In fact, one of the main goals in topology is to classify all types of manifolds.

Here are two more pictures of examples of smooth manifolds: one of the 2-sphere, the other of the torus:



And here is a NON-Example, the figure eight. The center point does not have any "nice" neighborhood.



1.4. **Some nice theorems.** Here are some examples of theorems we are going to prove during this class:

Fundamental Theorem of Algebra

Let $P(X) = X^n + a_{n-1}X^{n-1} + \cdots + a_1 X + a_0$ be a polynomial with complex coefficients, i.e. $a_0, \ldots, a_{n-1} \in \mathbb{C}$.
Then $P(X)$ has a zero in $\mathbb{C}$, i.e. there exists at least one complex number $z \in \mathbb{C}$ such that $P(z) = 0$.
That means of course that $P(X)$ has exaclty n zeroes in $\mathbb{C}$ (counted with multiplicities).

This has at first glance nothing to do with topology. But we can do it! (**Fundamental application**)

Brouwer Fixed Point Theorem

Every continuous map $f\colon D^n \to D^n$ has a fixed point, i.e. there is an $x \in D^n$ such that $f(x) = x$. Here D^n is the n-dimensional unit disc

$$D^n = \{(x_1, \ldots, x_n) \in R^n : x_1^2 + \cdots + x_n^2 \leq 1\}$$

This may not look so exciting, but HOW can you show that a fixed point always exists? (**Fundamental method**)

Hairy Ball Theorem

Assume you have a ball with hairs attached to it. Then it is impossible to comb the hair continuously and have all the hairs lay flat. Some hair will always be sticking right up.
A more mathematical formulation:
Every smooth vector field on a sphere has a singular point.
An even more general statement:
The n-dimensional sphere S^n admits a smooth field of nonzero tangent vectors if and only if n is odd.

This just sounds like a fun fact. But wind speeds on the surface of the earth is an example of a vector field on a sphere!
(**Fundamental object AND application**)

Something else one can prove using topological methods.

Multiplicative Structures on $\mathbb{R}^n$

Let $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ be a bilinear map with two-sided identity element $e \neq 0$ and no zero-divisors. Then $n = 1, 2, 4, \text{or } 8$.
What we are looking for is a "multiplication map". You know the cases $n = 1$ and $n = 2$ very well. It's just $\mathbb{R}$ and $\mathbb{C} \cong \mathbb{R}^2$. These are actually fields.
For $n = 4$, there are the Hamiltonians, or Quaternions, $\mathbb{H} \cong \mathbb{R}^4$ with a multiplication which as almost as good as the one in $\mathbb{C}$ and $\mathbb{R}$, but it is not commutative. (You add elements i, j, k to $\mathbb{R}$ with certain multiplication rules.)
For $n = 8$, there are the Octonions $\mathbb{O} \cong \mathbb{R}^8$. The multiplication is not associative and not commutative.
And that's it!!

This is a really deep result!
The crucial and, at first glance maybe surprising, point to prove this fundamental result is that the statement has something to do with the behavior of tangent spaces on spheres. That's a topological problem. Frank Adams was the first to solve it. The prove goes way beyond the methods of this class, unfortunately. So **stay tuned** on the Topology Chanel and lear more about it in **Advanced Aglebraic Topology...**