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Exercise set 5

$\boxed{1}$ Assume that $\vec{b} \in C^1(\bar{\Omega}; \mathbb{R}^d)$ is a continuously differentiable, divergence free vector field on Ω (that is, $\operatorname{div} \vec{b} = 0$ in Ω) and that $f \in L^2(\Omega)$. Consider the PDE

$$\begin{aligned} \operatorname{div}(\vec{b}u) - \Delta u &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \Gamma. \end{aligned} \tag{1}$$

a) Provide a weak formulation of this PDE and show that it has a unique solution in $H_0^1(\Omega)$.

Hint: Recall the Poincaré inequality $\|u\|_{H^1} \leq C_\Omega \|\nabla u\|_{L^2}$ *for* $u \in H_0^1(\Omega)$.

b) Assume that $f_k \rightharpoonup f$ in $L^2(\Omega)$ and denote by u_k the solution of (1) with right hand side f_k, and by u the solution of (1) with right hand side f. Show that $u_k \rightharpoonup u$ in $H^1(\Omega)$.

- *Possible solution:*

a) For the weak formulation of the PDE, we multiply with a test function $v \in H_0^1(\Omega)$ (since we have Dirichlet boundary conditions on the whole of Γ, the test functions need to be zero on the whole boundary) and integrate, obtaining the equation

$$\int_\Omega \big(\operatorname{div}(\vec{b}u) - \Delta u\big) v \, dx = \int_\Omega f v \, dx.$$

Now we integrate the second term by parts, which gives us

$$-\int_\Omega (\Delta u) v \, dx = \int_\Omega \nabla u \cdot \nabla v \, dx + \int_\Gamma (\partial_\nu u) v \, dx.$$

However, the boundary integral is zero, because the test function v is zero on Γ. Thus we obtain the weak formulation

$$\int_\Omega \operatorname{div}(\vec{b}u) v + \nabla u \cdot \nabla v \, dx = \int_\Omega f v \, dx \qquad \text{for all } v \in H_0^1(\Omega).$$

For existence and uniqueness, we use the Lax–Milgram theorem. To that end, we write

$$\begin{aligned} a(u, v) &= \int_\Omega \operatorname{div}(\vec{b}u) v + \nabla u \cdot \nabla v \, dx, \\ \ell(v) &= \int_\Omega f v \, dx. \end{aligned}$$

Since

$$\ell(v) = \int_\Omega f v \, dx \leq \|f\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} \leq \|f\|_{L^2(\Omega)} \|v\|_{H^1(\Omega)},$$

it follows that ℓ is a bounded linear form on $H_0^1(\Omega)$. Because $\operatorname{div} \vec{b} = 0$ we have that

$$\operatorname{div}(\vec{b}u) = \operatorname{div}(\vec{b})u + \vec{b} \cdot \nabla u = \vec{b} \cdot \nabla u.$$

Thus we have

$$a(u,v) = \int_\Omega (\vec{b}\cdot\nabla u)v + \nabla u\cdot\nabla v\,dx \le \|\vec{b}\|_{L^\infty}\|\nabla u\|_{L^2}\|v\|_{L^2} + \|\nabla u\|_{L^2}\|\nabla v\|_{L^2}$$
$$\le (\|\vec{b}\|_{L^\infty}+1)\|u\|_{H^1}\|v\|_{H^1},$$

which shows that a is a bounded bilinear form on $H_0^1(\Omega)$.
It thus remains to show that a is coercive, that is, that

$$a(u,u) \ge C\|u\|_{H^1}^2$$

for some $C > 0$ and all $u \in H_0^1(\Omega)$. In order to show such an estimate, we note first that

$$\int_\Omega \operatorname{div}(\vec{b}u)u\,dx = -\int_\Omega (\vec{b}u)\cdot\nabla u\,dx + \int_\Gamma (\vec{b}u)\cdot\nu u\,dx$$
$$= -\int_\Omega (\vec{b}u)\cdot\nabla u\,dx = -\int_\Omega \operatorname{div}(\vec{b}u)u\,dx$$

for all $u \in H_0^1(\Omega)$. Here we have again used the assumption that $\operatorname{div}(\vec{b}) = 0$. This shows that, actually,

$$\int_\Omega \operatorname{div}(\vec{b}u)u\,dx = 0,$$

and therefore

$$a(u,u) = \int_\Omega \operatorname{div}(\vec{b}u)u + \nabla u\cdot\nabla u\,dx = \|\nabla u\|_{L^2}^2.$$

By the Poincaré inequality, we can estimate this further by

$$a(u,u) = \|\nabla u\|_{L^2}^2 \ge C_\Omega^2\|u\|_{H^1}^2,$$

which shows the coercivity of a.
As a consequence, a is bounded and coercive, and ℓ is bounded linear, and thus the Lax–Milgram theorem implies that the PDE has a unique solution in $H_0^1(\Omega)$.
In addition, the solution u satisfies the stability estimate

$$C_\Omega^2\|u\|_{H^1}^2 = a(u,u) = \ell(u) \le \|f\|_{L^2}\|u\|_{L^2} \le \|f\|_{L^2}\|u\|_{H^1}$$

and therefore

$$C_\Omega^2\|u\|_{H^1} \le \|f\|_{L^2}. \tag{2}$$

b) Since we have a linear PDE, the solution mapping $f \mapsto u$ is linear as well. Moreover, because of (2) it is bounded and thus weakly continuous, that is, if $f_k \rightharpoonup f$ in $L^2(\Omega)$ then the corresponding solutions u_k converge weakly to the solution u of (1).
Alternatively, one can verify the convergence $u_k \rightharpoonup u$ by hand: Because of the bound (1), the sequence u_k is bounded, and thus has a weakly convergent subsequence. Now let $\{u_{k'}\}$ be any convergent subsequence and denote its weak limit by $\tilde{u}$. We have to show that $\tilde{u}$ solves the PDE. However, if $v \in H_0^1(\Omega)$ is any test function, then the weak convergence of $u_{k'}$ to $\tilde{u}$ implies that

$$a(u_{k'},v) = \int_\Omega v\vec{b}\cdot\nabla u_{k'} + \nabla u_{k'}\cdot\nabla v\,dx \to \int_\Omega v\vec{b}\cdot\nabla\tilde{u} + \nabla\tilde{u}\cdot\nabla v\,dx = a(\tilde{u},v)$$

and

$$\int_\Omega f_{k'}v\,dx \to \int_\Omega fv\,dx.$$

Since

$$a(u_k,v) = \int_\Omega f_k v\,dx$$

for all k, this shows that

$$a(\tilde{u}, v) = \int_\Omega fv\,dx$$

and thus $\tilde{u}$ solves (1), that is, $\tilde{u} = u$ (since the solution is unique). Because this holds for any convergent subsequence, the whole sequence u_k converges weakly to u.

$\boxed{2}$ We now consider the same basic PDE (1) but add a non-linear sink term: We assume that $g \in L^2(\Omega)$ with $g(x) \geq 0$ for a.e. x and consider the PDE

$$\begin{aligned} g \arctan u + \operatorname{div}(\vec{b}u) - \Delta u &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \Gamma. \end{aligned} \tag{3}$$

a) Provide a weak formulation of this PDE and show that it has a unique solution in $H_0^1(\Omega)$.

b) Assume that $g_k(x) \geq 0$ for a.e. $x \in \Omega$ and that $g_k \rightharpoonup g$ in $L^2(\Omega)$. Denote by u_k the solution of (3) with sink term $g_k \arctan u$, and by u the solution of (3) with sink term $g \arctan u$. Show that $u_k \rightharpoonup u$ in $H^1(\Omega)$.

Hint: At some point it might help to verify that the convergence $u_k \rightharpoonup u$ weakly in $H^1(\Omega)$ implies that $\arctan u_k \to \arctan u$ *strongly in $L^q(\Omega)$ for every $1 \leq q < \infty$.*

- *Possible solution:*

a) The weak formulation for this PDE can be obtained in the same way as for the previous problem, the only difference being the additional term $g \arctan u$. We obtain

$$\int_\Omega g \arctan(u)v + \operatorname{div}(\vec{b}u)v + \nabla u \cdot \nabla v\,dx = \int_\Omega fv\,dx \qquad \text{for all } v \in H_0^1(\Omega).$$

For proving existence and uniqueness, we can no longer make use of the Lax–Milgram theorem, as we are now dealing with a non-linear PDE. Instead, we have to use the Browder–Minty theorem. The right hand side of the PDE is, as above, a bounded linear form. Thus we “only” have to show that the operator $A\colon H_0^1(\Omega) \to H_0^1(\Omega)^*$,

$$u \mapsto A(u) = \Big[v \mapsto \int_\Omega g \arctan(u)v + \operatorname{div}(\vec{b}u)v + \nabla u \cdot \nabla v\,dx\Big]$$

is coercive, strictly monotone and demi-continuous (or hemi-continuous).
We start by proving the coercivity. Since $g \geq 0$ and $\operatorname{sgn}(\arctan(t)) = \operatorname{sgn}(t)$ for all t, it follows that

$$\int_\Omega g \arctan(u)u\,dx \geq 0$$

for all u. Thus, using the results of the previous exercise,

$$A(u)(u) \geq \int_\Omega \operatorname{div}(\vec{b}u)u + \nabla u \cdot \nabla u\,dx \geq C_\Omega^2 \|u\|_{H^1}^2 \tag{4}$$

and thus

$$\lim_{\|u\|_{H^1}\to\infty} \frac{A(u)(u)}{\|u\|_{H^1}} \geq \lim_{\|u\|_{H^1}\to\infty} C_\Omega^2 \|u\|_{H^1} = +\infty,$$

which shows the coercivity of A.

Next we show the strict monotonicity of A. Here we have that

$$A(u)(u-v) - A(v)(u-v) = \int_\Omega g(\arctan(u) - \arctan(v))(u-v)\,dx + \int_\Omega \operatorname{div}(\vec{b}(u-v))(u-v) + \nabla(u-v)\cdot\nabla(u-v)\,dx.$$

As in the previous exercise, we have that

$$\int_\Omega \operatorname{div}(\vec{b}(u-v))(u-v)\,dx = 0.$$

Moreover, because $g \geq 0$ and arctan is increasing, it follows that

$$\int_\Omega g(\arctan(u) - \arctan(v))(u-v)\,dx \geq 0.$$

Thus

$$A(u)(u-v) - A(v)(u-v) \geq \int_\Omega \nabla(u-v)\cdot\nabla(u-v)\,dx = \|\nabla(u-v)\|_{L^2}^2 \geq C_\Omega^2\|u-v\|_{H^1}^2 \geq 0$$

with equality if and only if $u = v$. Thus A is strictly monotone.
Finally, we show that A is demi-continuous. To that end, we assume that $u_k \to u$ in $H_0^1(\Omega)$ and that $v \in H_0^1(\Omega)$ is fixed. Then we immediately see that

$$\int_\Omega \operatorname{div}(\vec{b}u_k)v + \nabla u_k\cdot\nabla v\,dx \to \int_\Omega \operatorname{div}(\vec{b}u)v + \nabla u\cdot\nabla v\,dx.$$

Moreover, after possibly passing to a subsequence, we have that

$$g(x)\arctan(u_k(x))v(x) \to g(x)\arctan(u(x))v(x)$$

for almost every $x \in \Omega$. In addition, $|\arctan(t)| \leq \pi/2$ for all t, and thus

$$|g\arctan(u_k)v| \leq \frac{\pi}{2}|gv|.$$

Since

$$\int_\Omega |gv|\,dx \leq \|g\|_{L^2}\|v\|_{L^2} < \infty,$$

it follows that $|gv|$ is summable. Thus we can use Lebesgue's theorem of dominated convergence and obtain that

$$\int_\Omega g\arctan(u_k)v\,dx \to \int_\Omega g\arctan(u)v\,dx.$$

This proves that A is demi-continuous, and thus the Browder–Minty theorem implies that the PDE (3) has a unique solution.

b) Because of (4), we have that

$$C_\Omega^2\|u_k\|_{H^1}^2 \leq A(u_k)(u_k) = \int_\Omega f u_k\,dx \leq \|f\|_{L^2}\|u_k\|_{L^2} \leq \|f\|_{L^2}\|u_k\|_{H^1},$$

which implies that the sequence u_k is bounded in $H^1(\Omega)$. Thus it admits a weakly convergent sub-sequence. Choose now any weakly convergent sub-sequence and denote by $\tilde{u}$ its weak limit. We need to show that $\tilde{u}$ solves the limiting PDE (3). Let therefore $v \in H_0^1(\Omega)$ be any test function. Because of the weak convergence of u_k to $\tilde{u}$ in $H_0^1(\Omega)$, we have that

$$\int_\Omega \operatorname{div}(\vec{b}u_k)v + \nabla u_k\cdot\nabla v\,dx \to \int_\Omega \operatorname{div}(\vec{b}\tilde{u})v + \nabla\tilde{u}\cdot\nabla v\,dx. \tag{5}$$

Moreover,

$$
\begin{aligned}
\Big|\int_\Omega g_k \arctan(u_k) v\,dx - \int_\Omega g \arctan(\tilde u) v\,dx\Big| \\
&\le \Big|\int_\Omega g_k(\arctan(u_k) - \arctan(\tilde u)) v\,dx\Big| + \Big|\int_\Omega (g_k - g)\arctan(\tilde u) v\,dx\Big| \\
&\le \|g_k\|_{L^2} \|(\arctan(u_k) - \arctan(\tilde u)) v\|_{L^2} + \Big|\int_\Omega (g_k - g)\arctan(\tilde u) v\,dx\Big|.
\end{aligned}
\tag{6}
$$

Because $|\arctan(t)| \le \pi/2$ and $v \in H_0^1(\Omega)$ (and thus in $L^2(\Omega)$), it follows that $\arctan(\tilde u)v \in L^2$. The weak convergence $g_k \rightharpoonup g$ in L^2 therefore implies that the last term in (6) tends to zero.

Moreover, the weak convergence $u_k \rightharpoonup \tilde u$ in $H_0^1(\Omega)$ implies strong convergence $u_k \to \tilde u$ in L^2. After passing to a subsequence, we can thus assume that $u_k(x) - \tilde u(x) \to 0$ for almost every $x \in \Omega$. In addition,

$$
(\arctan(u_k) - \arctan(\tilde u))^2 v^2 \le \frac{\pi^2}{4} v^2,
$$

which is a summable function. Lebesgue's theorem of dominated convergence implies therefore that

$$
\|(\arctan(u_k) - \arctan(\tilde u))v\|_{L^2}^2 = \int_\Omega (\arctan(u_k) - \arctan(\tilde u))^2 v^2\,dx \to 0.
$$

As a consequence, the right hand side in (6) converges to zero. Together with (5), we obtain that

$$
\int_\Omega g_k \arctan(u_k) v + \operatorname{div}(\vec b u_k) v + \nabla u_k \cdot \nabla v\,dx \to \int_\Omega g \arctan(\tilde u) v + \operatorname{div}(\vec b \tilde u) v + \nabla \tilde u \cdot \nabla v\,dx. \tag{7}
$$

Since u_k solves the PDE with sink term $g_k \arctan(u_k)$, it follows that the left hand side in (7) is equal to $\int_\Omega f v\,dx$ for all k. Thus we have that

$$
\int_\Omega g \arctan(\tilde u) v + \operatorname{div}(\vec b \tilde u) v + \nabla \tilde u \cdot \nabla v\,dx = \int_\Omega f v\,dx.
$$

Since this holds for all test functions $v \in H_0^1(\Omega)$, we obtain that $\tilde u$ solves the PDE (3). The uniqueness of the solution of (3) now implies that $\tilde u = u$ and that the whole sequence u_k converges weakly in $H^1(\Omega)$ to u.