Linear Methods Exams

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Contents

1 Exam 18h

1.1 Problem 1

Determine whether the followeing statements are true or false. if the statement are true, no further explanation is required. If the statement is false, give a counter example.

1. The Kerner of a bounded linear operator $T : X \mapsto Y$ between normed spaces X and Y is closed.

   Answer. True

2. The range of a bounded linear operator $T : X \to Y$ between normed spaces X and Y is closed.

   Answer. False. Lets assume that X and Y is closed. Then is this true.

3. The dual space $X^{'}$ of a normed space is a Banach Space.

   Answer. True.

4. A closed subspace of a Banach Space is itself a Banach Space.

   Answer. True

1.2 Problem 2

Let $(x_k)_{k\in\mathbb{N}}$ be a sequence in a normed space $(X, \|.\|)$.

a) Prove that $(x_k)_{k\in\mathbb{N}}$ is a Cauchy sequence, then $(x_k)_{k\in\mathbb{N}}$ is bounded.

   Answer. Let v

b) Let $\|.\|_a$ and $\|.\|_b$ be equivalent norms on X and let $x \in X$. Prove that $(x_k)_{k\in\mathbb{N}}$ converges to x in $(X, \|.\|_a)$ if and only if $(x_k)\, k \in \mathbb{N}$ converges to x in $(X, \|.\|_b)$.

   Answer.

2 Appendix

2.1 Sequences in metric spaces and normed spaces

Definition 2.1 (Norm). *Criterias for norms*

(i) $\|cx\| = c\|x\|$

(ii) $\|xy\| \leq \|x\|\|y\|$

(iii) $\|x + y\| \leq \|x\| + \|y\|$

(iv) $\|x\| = 0 \quad \text{only if} \quad x = 0$

Definition 2.2 (Sequence). *Let (X, d) be a metric space. A sequence $(x_n)_{n\in\mathbb{N}}$ in X is said to* ***converge to*** *$x \in X$ for every $\epsilon > 0$ one can find $N = N(\epsilon) \in \mathbb{N}$ such that*

$$d\left(x_n, x\right) < \epsilon.$$

whenever $b \geq N$. The element x is called the ***limit*** *of the sequence $(x_n)_{n\in\mathbb{N}}$. In particular, in $(X, \|.\|)$ is a normed space. then $(x_n)_{n\in\mathbb{N}}$ converge to $x \in X$ for every $\epsilon > 0$ one can find $N = N(\epsilon) \in \mathbb{N}$ such that*

$$\|x - x_n\| < \epsilon.$$

whenever $n \geq N$..

Definition 2.3. *Given a point $x_0 \in X$ and a real number $r > 0$, we define three types of sets:*

(i) $B(x_0; r) = \{x \in X \mid d(x, x_0) < r\}$ ***(Open ball)***

(ii) $\hat{B}(x_0; r) = \{x \in X \mid d(x, x_0) \leq r\}$ ***(Closed ball)***

(iii) $S(x_0; r) = \{x \in X \mid d(x, x_0) = r\}$ ***(Sphere)***

Here is x_0 called the center and r the radius. Remark that $S(x_0, r) = \hat{B}(x_0, r) - B(x_0, r)$.

Definition 2.4 (Open and Closed Set). *A subset M of a metric space X is said to be open if it contains a ball around each of its points. A subset*

K of X is said to be closed if its complement (in X) is open, that is, $K^c = X - K$ is open.

Remark. A complement set is defined such that $A^c = U \setminus A$ or more formally $A^c = \{x \in U \mid x \notin A\}$

Lemma 2.1. *A convergent sequence in a metric space (X, d) is bounded.*

2.2 Linear Operator

Definition 2.5. *A linear operator T is an operator such that*

1. *the domain $\mathbb{D}(T)$ of T is a vector space and the range $R(T)$ lies in a vector space over the same field.*
2. *$\forall x, y \in \mathbb{D}(T)$ and scalars α*

$$T(x+y) = Tx + Ty \quad \text{and} \quad T(\alpha x) = \alpha Tx. \tag{1}$$

Definition 2.6 (Bounded Linear Operator)**.** *An linear operator $T : X \mapsto Y$ is bounded if $\forall x \in X$ and $c > 0$ such that $\|Tx\| = \|T\|\|x\| \leq c\|x\|$*

Remark. What is the smallest possible c such that $\|Tx\| \leq c\|x\|$ still hold for all non-zero $x \in \mathbb{D}(T)$? (We can leave out $x = 0$ since $Tx = 0$ for $x = 0$) By division,

$$\frac{\|Tx\|}{\|x\|} \leq c.$$

and this shows that c must be at least as big as the supremum of the expression on the left taken over the range $\mathbb{D}(T) - \{0\}$. Hense the answer to our question is that the smallest possible c is that supremum. This quantity denoted by $\|T\|$, thus

$$\|T\| = \sup_{\substack{x \in \mathbb{D}(T) \\ x \neq 0}} \frac{\|Tx\|}{\|x\|}$$

$\|T\|$ is called the **norm** of the operator T. If the range $\mathbb{D}(T) = \{0\}$, we define $\|T\| = 0$. Note that with $c = \|T\|$ is

$$\|Tx\| \leq \|T\|\|x\|$$

which is a quite frequently used formula.

Lemma 2.2. *Let T be a bounded linear operator. Then is this true,*

(i)

$$.\|T\| = \sup_{\substack{x\in\mathbb{D}(T)\\ x\neq 0}} \frac{\|Tx\|}{\|x\|} = \sup_{\substack{x\in\mathbb{D}(T)\\ x=1}} \|Tx\|$$

(ii) The norm satisfy general norm aksioms.

Proof. (i) Let $\|x\| = a$ and define $y = \frac{x}{a}$. Using this definition can we see that $\|y\| = 1$. Hense can we rewrite the definition.

$$\sup_{\substack{x\in\mathbb{D}(T)\\x\neq 0}} \frac{\|Tx\|}{\|x\|} = \sup_{\substack{x\in\mathbb{D}(T)\\x\neq 0}} \frac{\|Tx\|}{a} = \sup_{\substack{x\in\mathbb{D}(T)\\x\neq 0}} \|\frac{Tx}{a}\| = \sup_{\substack{y\in\mathbb{D}(T)\\y=1}} \|Ty\|$$

(ii) We need to prove that it satisfy the criteria $\|cT\| = c\|T\|$ and $\|T_1 + T_2\| \leq \|T_1\| + \|T_2\|$.

$$\begin{aligned}\|cT\| &= \sup_{\substack{y\in\mathbb{D}(T)\\\|y\|=1}} \|Tcy\| = \sup_{\substack{y\in\mathbb{D}(T)\\\|y\|=1}} c\|Ty\| \\ &= c\|T\|.\end{aligned}$$

$$\begin{aligned}\|T_1 + T_2\| &= \sup_{x\in\mathbb{D}(T),\|x\|=1} \|\left(T_1x + T_2x\right)\| \leq \sup_{x\in\mathbb{D}(T),\|x\|=1} \|T_1x\| + \|T_2x\| \\ &= \|T_1\| + \|T_2\|.\end{aligned}$$

□

Theorem 2.1. *Let $T : \mathbb{D} \mapsto Y$ be a linear operator where $\mathbb{D} \subset X$ and X, Y are normed spaces, then*

1. *T is continous if and only if T is bounded.*
2. *If T is continous at a single point, T is continious.*

Proof. 1. For $T = 0$ the statement is trivial. Let $T \neq 0$. Then $\|T\| \neq 0$. We Assume T To be bounded and consider any $x_0 \in \mathbb{D}(T)$. Let any $\epsilon > 0$. Then, since T is linear, for every $x \in \mathbb{D}(T)$ such that

$$\|x - x_0\| < \delta \quad where \quad \delta = \frac{\epsilon}{\|T\|}$$

we obtain

$$\|Tx - Tx_0\| = \|T\left(x - x_0\right)\| \leq \|T\|\|x - x_0\| < \|T\|\delta = \epsilon$$

. Since $x_0 \in \mathbb{D}(T)$ was arbitary, this shows that T is continous.

Conversely, assume that T is continous at an arbitary $x_0 \in \mathbb{D}(T)$ then, given any $\epsilon > 0$, there is a $\delta > 0$ such that

$$\|Tx - Tx_0\| \leq \epsilon \quad \text{for all } x \in \mathbb{D}(T) \, \text{satisfying} \quad \|x - x_0\| \leq \delta.. \tag{2}$$

We now take any $y \neq 0$ in $\mathbb{D}(T)$ and set

$$x = x_0 + \frac{\delta}{\|y\|} y. \quad \text{then} \quad x - x_0 = \frac{\delta}{\|y\|} y.$$

Hence $\|x - x_0\| = \delta$, so that we may use the result in (3) . Since T is linear we have

$$\|Tx_0 - Tx\| = \|T(x - x_0)\| = \|T\left(\frac{\delta}{\|y\|} y\right)\| = \frac{\delta}{\|y\|}\|Ty\|$$

and this implies

$$\frac{\delta}{\|y\|}\|Ty\| \leq \epsilon. \quad \text{Thus} \quad \|Ty \leq \frac{\epsilon}{\delta}\|\|y\|.$$

This can be written $\|Ty\| \leq \|y\|$, where $c = \frac{\epsilon}{\delta}$ and shows that T is bounded.

2. Continuity of T at a point implies boundedness of T by the second part of the proof of (a), which in turn implies boundedness of T by (a).

□

2.3 Banach Spaces

Definition 2.7 (Cauchy Sequence)**.** *Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in the metric space (X, d). We say that $(x_n)_{n\in\mathbb{N}}$ is* ***Cauchy Sequence*** *if for any $\epsilon > 0$ there exist an $N \in \mathbb{N}$ such that*

$$d(x_n, x_m) < \epsilon.$$

In particular if $(x_n)_{n\in\mathbb{N}}$ is a sequence in the normed space $(X, \|.\|)$, then $(x_n)_{n\in\mathbb{N}}$ is Cauchy if for any $\epsilon > 0$ there exist an $N \in \mathbb{N}$ such that

$$\|x_n - x_m\| < \epsilon, \quad s.t. \quad n, m \geq N.$$

In an inner product space $(X, \langle ., . \rangle)$, we say that a sequenxe $(x_n)_{n\in\mathbb{N}}$ is Cauchy if the sequence is Cauchy with respect to the indeuced norn $\|x\| := \langle x, x\rangle^{\frac{1}{2}}$.

Lemma 2.3. *Any Cauchy sequence in (X, d) is bounded.*

Proof. Let $(x_n)_{n\in\mathbb{N}}$ be a Cauchy sequence. Then there exist $N \in \mathbb{N}$ such that for all $m, n \geq N$ we have

$$d(x_m, x_n) < 1.$$

In particular, we have

$$d(x_N, x_m) < 1 \quad \forall \quad m \geq N.$$

Or equivalently $x_m \in B_1(x_N)$ for all $m \geq N$. Now let

$$r = max\{1, d(x_1, x_N), d(x_2, x_N), \ldots, d(x_{N-1}, x_N)\}.$$

Then for any $n \in \mathbb{N}$ we have $x_n \in B_{r+1}(x_N)$ so $(x_n)_{n\in\mathbb{N}}$ is bounded.

□

Remark. A set is **closed** if the set contains all of its boundary points (the closure of the set is equal to the set). There are some other definitions for closed also. A set is **bounded** if the distance between any two points in the set is less then some finite constant. A set in $\mathbb{R}^n$ is bounded if all of the points are contained within a disc of finite radius.

Definition 2.8 (Completeness)**.** *A sequence $(x_n)_{n\in\mathbb{N}}$ in a metric space $X = (X, d)$ is said to be Cauchy (or fundemental) if for every $\epsilon > 0$ there is an $N = N(\epsilon)$ such that $d(x_m, x_n) < \epsilon$ for every $m, n \geq N$. The space X is said to be complete if every Ceachy sequence in X converges (that is, has a limit which is an element of X).*

Remark (Procedure for Completeness proofs). To prove completeness do we choose an arbitary Cauchy sequence $(x_n)_{n\in\mathbb{N}}$ in X and show that it does converge in X. They often have the same pattern.

1. Contruct an element x (to be used as an limit).
2. Prove that x is in the space considered.
3. Prove convergence $x_n \mapsto x$

Theorem 2.2 (Convergent sequences)**.** *Every convergent sequences in a metric space is a Cauchy Sequence.*

Proof. Let $x_n \mapsto x$ for $x \in X$, then is for an $N = N(\epsilon)$

$$d(x_n, x) < \frac{\epsilon}{2} \quad \text{for any} \quad n > N.$$

To prove that this is Cauchy can we use the triangulation theorem such that

$$d(x_n, x_m) \leq d(x, x_n) + d(x, x_m) < \epsilon \quad \text{such that} \quad m, n \geq N(\epsilon)$$

This proves that $(x_n)_{n\in\mathbb{N}}$ is Cauchy.

□

Definition 2.9 (Banach Space and Hilbert Space)**.** *A metric space (X, d) is said to be complete if every Cauchy sequence $(x_n)_{x_n \in \mathbb{N}} \in X$ converges to a limit $x \in X$. A complete normed space $(X, \|.\|)$ is classed a Banach Space. Similary, a complete inner product space $(X, \langle ., . \rangle)$ is called a Hilbert space.*

Theorem 2.3. *Let (f_n) be a sequence of continious functions on $[a, b]$ which converges uniformly to a limit function f. Then f is continious on $[a, b]$.*

Proof. We want to show that for any fixed $y \in [a, b]$ and $\epsilon > 0$ we can find a $\delta > 0$ such that

$$\|x - y\| < \delta \qquad \Longrightarrow \qquad \|f(x) - f(y)\| < \epsilon$$

By the uniformly convergence (f_n) to f, there exist an N such that

$$\|f_n(x) - f(x)\| < \epsilon \quad \text{for all} \quad x \in [a, b], n \geq N.$$

Moreover, the function f_n is continious, so there exist a $\delta > 0$ such that

$$\|x - y\| < \delta \qquad \Longrightarrow \qquad \|f_N(x) - f_N(y)\| < \frac{\epsilon}{3}.$$

It follow that

$$\|f(x) - f(y)\| \leq \|f(x) - f_N(x)\| - \|f_N(x) - f_N(y)\| + \|f_N(y) - f(y)\| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.$$

whenever $\|x - y\| < \delta$ □

Theorem 2.4. *$(C[a, b], \|.\|_\infty)$ is a Banach Space*

Proof. (i) **Find a candidate for the limit**

Fix $x \in [a, b]$ and note that

$$\|f_n(x) - f(x)\| \leq \|f_n - f_m\|_\infty = \max_{a \leq x \leq b} \|f_n(x) - f_m(x)\|.$$

This if (f_n) is a Cauchy sequence in $(C[a, b], \|.\|_\infty)$, then $(f_n(x))_{n \in \mathbb{N}}$ is a Cauchy Sequence in $(\mathbb{R}, \|.\|)$. Since $(\mathbb{R}, \|.\|)$ is complete, there exist a point $f(x) \in \mathbb{R}$ such that $f_n(x) \mapsto f(x)$. A reasonable candidate for the limit is the function f given by the pointwise limits.

(ii) **Show that** $f \in C[a, b]$

We observe that the convergence of f_n to f is not onlypointwise, but in fact uniform; Since (f_n) is Cauchy, there is for every $\epsilon > 0$ an integer N such that

$$\|f_n - f\|_\infty = \max_{a \leq x \leq b} \|f_n(x) - f_m(x)\| < \frac{\epsilon}{2}, \quad n, m \geq N$$

In particular, this hold as $m \mapsto \infty$, and we get

$$\max_{a \leq x \leq b} \|f_n(x) - f(x)\| \leq \frac{\epsilon}{2} < \epsilon, \quad n \geq N \quad . \tag{3}$$

Thus, f_n converges uniformly to f on the interval $[a, b]$, and it follows by Theorem 3.13 (linear method lecture notes) that $f \in C[a, b]$.

(iii) **Show that** $f_n \mapsto f$

Follows from (3)

□

2.4 Common

Definition 2.10 (Range). *A range of a function $f : X \mapsto Y$, is denoted by range (f) or $f(X)$, is the set of all $y \in Y$ that are the image of some $x \in X$. More compact can this be written.*

$$range\,(f) = \{y \in Y \mid there\ exist\ x \in X\ such\ that\ f(x) = y\}$$

Definition 2.11. *Let $f : X \mapsto Y$ be a function.*

1. *We call f injective or one-to-one if $f(x_1) = f(x_2)$ implies $x_1 = x_2$, i.e, no two elements of the domain have tha same image. Equivalently, if $x \neq x_2$ then $f(x_1) \neq f(x_2)$.*

2. *We call f surjective or onto if range $(f) = Y$, i.e each $y \in Y$ is the image of at least one $x \in X$.*

3. *We call f bijective if f is both injective and surjective.*

Definition 2.12 (Closed Set). *Let X be a subset of a set Y. If X is closed is this true.*

(i) The compliment X^c is an open set.

(ii) X is it own set closure.

(iii) Sequences/nets/filters in X that converge do so in X.

(iv) Every point outside X has a neighbourhood disjoint from X