TMA 4190 Introduction to Topology
Lecturer: Gereon Quick
Lecture 07[1]

7. Submersions

Let $f\colon X \to Y$ be a smooth map between smooth manifolds. Remember that the derivative at $x \in X$, $df_x\colon T_xX \to T_{f(x)}Y$, is a linear map between vector spaces, and we are trying to answer the question:

A natural question:

How much does df_x tell us about f?

We move to the third case:

Third case: df_x is surjective

Assume $\dim X > \dim Y$. The best possible behavior of df_x is then that

$$df_x\colon T_x(X) \to T_{f(x)}(Y)$$

is a **surjective linear map**.

Again, there is a name for this case:

Submersions

If df_x is surjective, we say that f is a **submersion at x**. If f is a submersion at every point, we say that f is an **submersion**.

The **canonical submersion** for $n \geq m$ is the standard projection

$$\mathbb{R}^n \to \mathbb{R}^m,\ (a_1, \ldots, a_n) \mapsto (a_1, \ldots, a_m)$$

onto the first m coordinates (i.e. omitting the remaining $n - m$ coordinates).

Up to diffeomorphism the canonical submersion is **locally** the only submersion:

[1]Following the books of Guillemin and Pollack: Differential Topology, and Milnor: Topology from the differentiable viewpoint.

Local Submersion Theorem

Suppose that $f\colon X \to Y$ is a submersion at x, and $y = f(x)$. Then there exist local coordinates around x and y such that

$$f(x_1, \ldots, x_n) = (x_1, \ldots, x_m).$$

In other words, f is **locally equivalent to the canonical submersion**.

Proof of the Local Submersion Theorem:
As for the immersion case, we start by choosing any local parametrization $\phi\colon U \to X$ with $\phi(0) = x$ and $\psi\colon V \to Y$ with $\psi(0) = y$:

$$\begin{tikzcd} X \arrow[r, "f"] & Y \\ U \arrow[u, "\phi"] \arrow[r, "\theta=\psi^{-1}\circ f\circ\phi"'] & V \arrow[u, "\psi"'] \end{tikzcd}$$

Now we are going to manipulate ϕ and ψ such that θ becomes the canonical submersion.

By the assumption, we know $d\theta_0\colon \mathbb{R}^n \to \mathbb{R}^m$ is surjective. Hence, after choosing a suitable basis for $\mathbb{R}^m$, we can assume that $d\theta_0$ is the matrix

$$M(I_m|0)$$

which consists of the $m \times m$-identity matrix sitting in the first n columns, and the zero $n \times (n-m)$-matrix occupying the remaining columns.

Choosing a basis

This time we need to choose a suitable basis for $\mathbb{R}^n$. Let $e_1^m, \ldots, e_m^m \in \mathbb{R}^m$ be the standard basis. Since $d\theta_0$ is surjective, the induced linear map

$$d\bar{\theta}_0\colon \mathbb{R}^n/\mathrm{Ker}\,(d\theta_0) \to \mathbb{R}^m$$

from the quotient vector space $\mathbb{R}^n$ modulo the kernel of $d\theta_0$ to $\mathbb{R}^m$ is an isomorphism. Hence we can choose unique vectors $b_1, \ldots, b_m \in \mathbb{R}^n/\mathrm{Ker}\,(d\theta_0)$ with $d\bar{\theta}_0(b_i) = e_i^m$ for $i = 1, \ldots, m$, and these $b_1, \ldots, b_m$ form a basis of $\mathbb{R}^n/\mathrm{Ker}\,(d\theta_0)$. Now we choose a basis vector $b_{m+1}, \ldots, b_n$ of $\mathrm{Ker}\,(d\theta_0)$. This gives us a basis $b_1, \ldots, b_n$ of $\mathbb{R}^n$ such that $d\theta_0(b_i) = e_i^m$ for $i = 1, \ldots, m$ and $d\theta_0(b_i) = 0$ for $i = m+1, \ldots, n$. Hence in this basis for $\mathbb{R}^n$ and the standard basis for $\mathbb{R}^m$ the matrix for $d\theta_0$ is exactly $M(I_m|0)$ (remember: the columns are the images of the basis vectors).

Back to the proof: We define a new map

$$\Theta\colon U \to \mathbb{R}^n, \text{ by } \Theta(a) = (\theta(a), a_{m+1}, \ldots, a_n)$$

for a point $a = (a_1, \ldots, a_n)$. It is related to θ by the commutative diagram

$$\begin{tikzcd} U \arrow[rr, "\theta"] \arrow[dr, "\Theta"'] & & \mathbb{R}^m \\ & \mathbb{R}^n. \arrow[ur, "\text{canonical submersion}"'] & \end{tikzcd}$$

The derivative $d\Theta_0$ at 0 is given by the identity matrix I_n. Hence Θ is a **local diffeomorphism at** 0. Thus we can find a small neighborhood U' around 0 in $\mathbb{R}^n$ such that Θ^{-1} **exists** as a diffeomorphism from U' onto some small neighborhood around 0 in U.

By construction,

$$\theta = \text{canonincal submersion} \circ \Theta, \text{ i.e. } \theta \circ \boldsymbol{\Theta^{-1}} = \text{canonincal submersion}.$$

This gives us the commutative diagram

$$\begin{tikzcd} & X \arrow[r, "f"] & Y \\ & U \arrow[u, "\phi"] \arrow[r, "g"] & V. \arrow[u, "\psi"'] \\ U' \arrow[uur, bend left, "\phi\circ\Theta^{-1}"] \arrow[ur, "\Theta^{-1}"'] \arrow[urr, bend right, "\text{canonical submersion}"'] & & \end{tikzcd}$$

Hence it suffices to replace U with U' and ϕ with $\phi \circ \Theta^{-1}$ to get the desired commutative diagram

$$\begin{tikzcd} X \arrow[r, "f"] & Y \\ U' \arrow[u, "\phi\circ\Theta^{-1}"] \arrow[r, "\text{canonical}", "\text{submersion}"'] & V \arrow[u, "\psi"'] \end{tikzcd}$$

which proves the theorem. **QED**

We observe from the proof of the theorem that if $f\colon X \to Y$ is a submersion at x, then it is also a **submersion for all points in a neighborhood of** x. For, local parametrization $\phi\colon U \to X$ of the proof also parametrizes any point in the image of ϕ which is an open subset around x (open because ϕ is a diffeomorphism onto its image).

Given a map $f\colon X \to Y$ and a point $y \in Y$, we would like to study the **fiber of f over y**, i.e. the set

$$f^{-1}(y) = \{x \in X : f(x) = y\} \subseteq Y.$$

Be aware

In general, there is no reason for that set $f^{-1}(y)$ has any nice geometric structure.

But life is much nicer in the world of submersions. So suppose that $f\colon X \to Y$ is a **submersion at a point $x \in X$** with $f(x) = y$ or in other words $x \in f^{-1}(y)$. By the Local Submersion Theorem, we can choose local coordinates around x and y such that, expressed in these local coordinates, $y = (0, \dots, 0)$ and f becomes the canonical submersion. Let $V \subset X$ be the chosen local neighborhood around x on which the local coordinates are defined. We write $u_1, \dots, u_n$ for the local coordinate functions. Expressed in these local coorindates f becomes

$$f(u_1, \dots, u_n) = (u_1, \dots, u_m).$$

Moreover, still in these coordinates, the fiber over y is the set of points

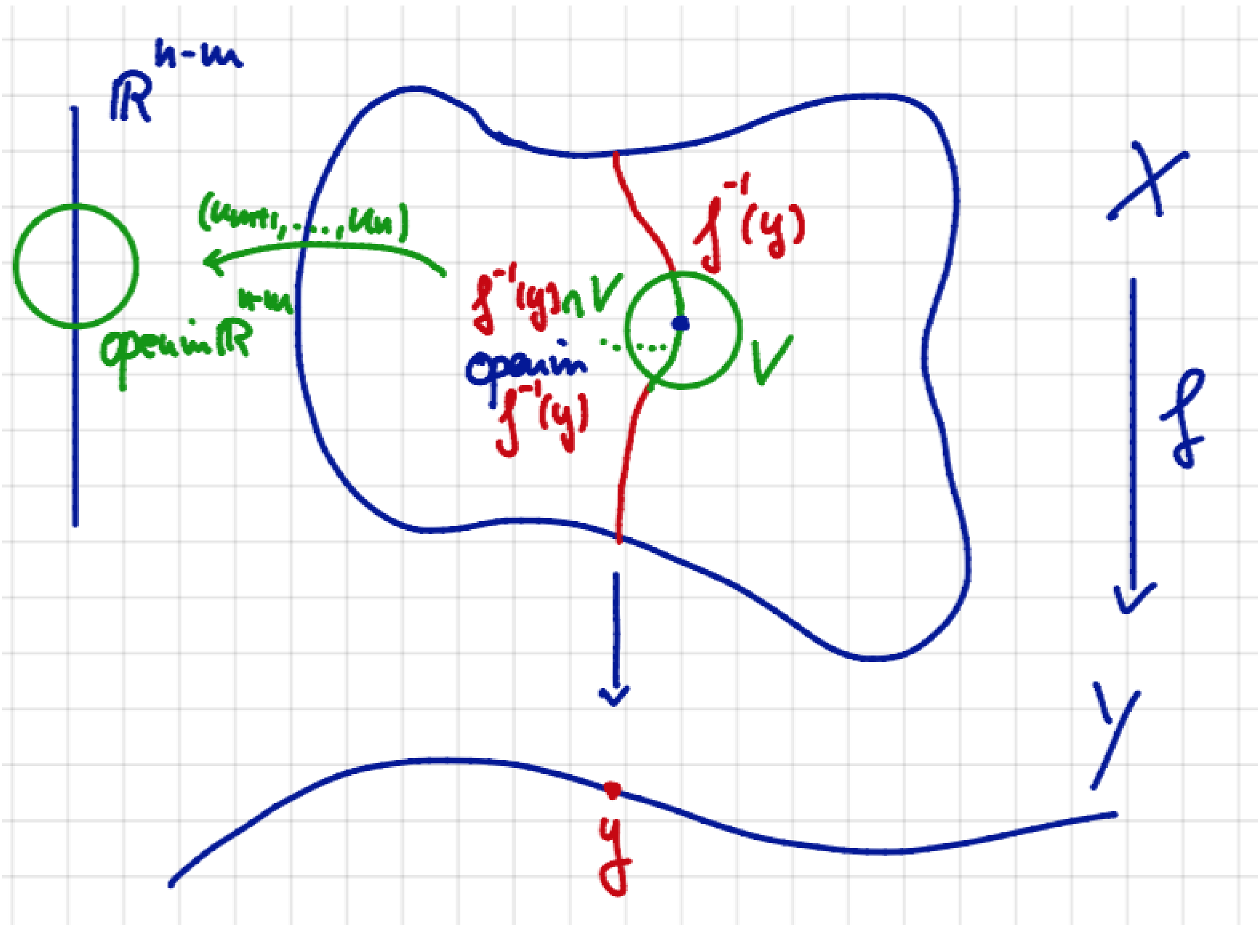
$$f^{-1}(y) \cap V = \{p \in V : u_1(p) = \cdots = u_m(p) = 0\}.$$

Hence we can use the remaining functions $u_{m+1}, \dots, u_n$ to define a local coordinate system on $f^{-1}(y) \cap V$ which is an **open subset in $f^{-1}(y)$**. With these local coordinates, $f^{-1}(y)$ looks like Euclidean space $\mathbb{R}^{m-n}$ in a neighborhood of x.

We would like this to be the case for every point in the fiber $f^{-1}(y)$. This is not always the case. So let us give the desired case a name:

Regular values

For a smooth map of manifolds $f\colon X \to Y$, a point $y \in Y$ is called a **regular value for f** if $df_x\colon T_x(X) \to T_y(Y)$ at every point $x \in X$ such that $f(x) = y$.



Then the above argument shows the following important result:

Preimage Theorem

If y is a regular value for $f\colon X \to Y$, then the fiber $f^{-1}(y)$ over y is a **submanifold of X**, with $\dim f^{-1}(y) = \dim X - \dim Y$.

As a first application, we can show once again that spheres are smooth manifolds.

Example: Spheres at preimages

Let $f\colon \mathbb{R}^{k+1} \to \mathbb{R}$ be the map

$$x = (x_1, \dots, x_{k+1}) \mapsto |x|^2 = x_1^2 + \cdots + x_{k+1}^2.$$

The derivative dfa at the point $a = (a_1, \dots, a_{k+1})$ has the matrix $(2a_1 \dots 2a_{k+1})$. Thus $df_a\colon \mathbb{R}^{k+1} \to \mathbb{R}$ is surjective unless $f(a) = 0$, so every nonzero real number is a regular value of f. In particular, we get again that the sphere $S^k = f^{-1}(1)$ is a k-dimensional manifold.

Since regular values are so nice, we also want to have a name for other values:

Critical values

For a smooth map of manifolds $f\colon X \to Y$, a point $y \in Y$ which is not a regular value, is called a **critical value for f**.
Note that critical values got their name from the fact that $f^{-1}(y)$ can be very complicated if y is critical.

Note that all values y which are not in the image of f are also regular values for f. For, if $f^{-1}(y)$ is the **empty set**, then there is no condition to be satisfied.

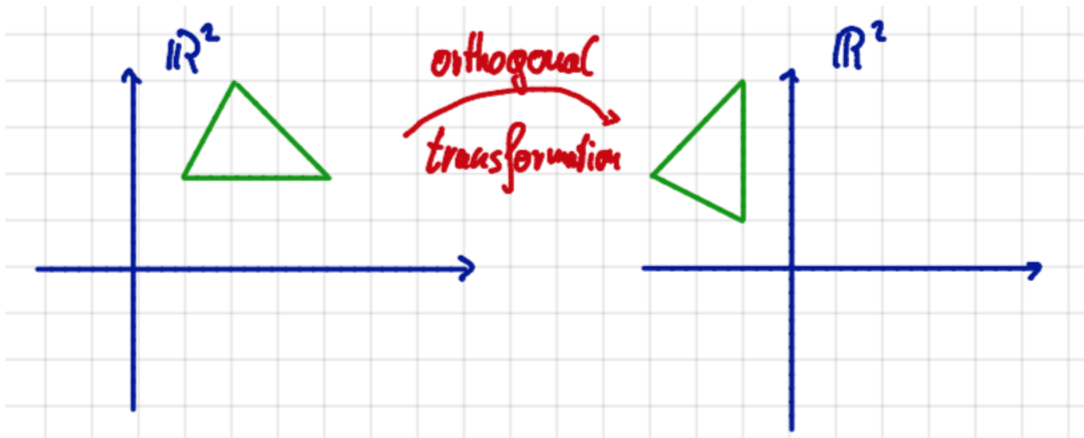
Summary for regular values

Suppose $f\colon X \to Y$ is a smooth map of manifolds. Then **y being a regular value for f** has the following meaning:

- when $\dim X > \dim Y$, then f is a submersion at each point $x \in f^{-1}(y)$;
- when $\dim X = \dim Y$, then f is a local diffeomorphism at each point $x \in f^{-1}(y)$;
- when $\dim X < \dim Y$, then y is not in the image of f; for, all values in the image are critical (df_x cannot be surjective when $\dim T_x(X) < \dim T_{f(x)}(Y)$).

Matrix subgroups are manifolds

A very important application of the Preimage Theorem, is that we can use it to show that various matrix groups sare smooth manifolds. Let $M(n)$ denote the space of real $n \times n$-matrices. It is isomorphic as a vector space to R^{n^2} (we can write every $n \times n$-matrix as a column vector of length n^2). Let $O(n)$ be the subgroup of matrices A in $M(n)$ which satisfy $AA^t = I$ where A^t denotes the transpose of A and I is the $n \times n$-identity matrix. Note that $O(n)$ is the subgroup of matrices which preserve the scalar product of vectors. In particular, matrices in $O(n)$ preserve distances in $\mathbb{R}^n$.



Our goal is to show that $O(n)$ **is a smooth manifold of dimension** $n(n-1)/2$.

First, we note that AA^t is a symmetric matrix. For

$$(AA^t)^t = (A^t)^t A^t = AA^t.$$

The subspace $S(n)$ of symmetric matrices in $M(n)$ is a smooth submanifold of $M(n)$ of dimension $\mathbb{R}^k$ with $k = n(n+1)/2$ (everything below the diagonal is determined by what happens above the diagonal such that there are $n(n+1)/2$ free entries). We define the map

$$f \colon M(n) \to S(n),\ A \mapsto AA^t.$$

This map is smooth, since multiplication of matrices is smooth and taking transposes is obviously smooth as well.

Now we observe $O(n) = f^{-1}(I)$. Hence, in order to show that $O(n)$ is a smooth manifold, we just need to show that I **is a regular value for** f. So let us compute the derivative of f at a matrix A:

$$\begin{aligned}
df_A(B) &= \lim_{s\to 0}\frac{f(A+sB)-f(A)}{s}\\
&= \lim_{s\to 0}\frac{(A+sB)(A+sB)^t - AA^t}{s}\\
&= \lim_{s\to 0}\frac{(A+sB)(A^t+sB^t) - AA^t}{s}\\
&= \lim_{s\to 0}\frac{AA^t + sBA^t + sAB^t + s^2BB^t - AA^t}{s}\\
&= \lim_{s\to 0}\frac{sBA^t + sAB^t + s^2BB^t}{s}\\
&= \lim_{s\to 0} BA^t + AB^t + sBB^t\\
&= AB^t + BA^t.
\end{aligned}$$

In order to check that I is a regular value, we need to show that

$$df_A\colon T_A(M(n)) \to T_{f(A)}(S(n))$$

is surjective for all $A \in O(n)$. Since $M(n) \cong R^{n^2}$ and $S(n) \cong \mathbb{R}^{n(n+1)/2}$ are diffeomorphic to Euclidean spaces, we have

$$T_A(M(n)) = M(n) \text{ and } T_{f(A)}(S(n)) = S(n).$$

Hence, given a matrix $C \in S(n)$, we need to show that there is a matrix $B \in M(n)$ with $df_A(B) = BA^t + AB^t = C$.

Since C is symmetric, we have $C = \frac{1}{2}(2C) = \frac{1}{2}(C + C^t)$. Since $AB^t = (BA^t)^t$, we set $B = \frac{1}{2}CA$. Then, using $AA^t = I$, we get

$$df_A(B) = (\frac{1}{2}CA)A^t + A(\frac{1}{2}CA)^t = \frac{1}{2}CAA^t + \frac{1}{2}AA^tC^t = \frac{1}{2}C + \frac{1}{2}C^t = C.$$

Thus I is a regular value, and $O(n)$ is a submanifold of $M(n)$. We can also calculate the dimension of $O(n)$:

$$\dim O(n) = \dim M(n) - \dim S(n) = n^2 - \frac{n(n+1)}{2} = \frac{n(n-1)}{2}.$$

Lie groups

The manifold $O(n)$ is an example of a very important class of smooth manifolds. For, $O(n)$ is both a smooth manifold and a group such that the group operations are smooth. For both the multiplication map

$$O(n) \times O(n) \to O(n),\ (A,B) \mapsto AB$$

and the map of forming the inverse

$$O(n) \to O(n),\ A \mapsto A^{-1}$$

are smooth (for the latter note $A^{-1} = A^t$ for $A \in O(n)$, but taking inverse is also smooth for other matrix groups).
In general, a group which is also a manifold such that the group operations are smooth is called a **Lie group**.
Lie groups are extremely interesting and important and have a rich and exciting theory. For example, the tangent space at a Lie group at the identity element is a Lie algebra, a vector space with a certain additional operation. Such Lie algebras can be classified completely. Lie groups and Lie algebras play an important role in many different areas of mathematics and physics.