NTNU

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Department of Mathematical Sciences

Examination paper for **TMA4265 Stochastic Modeling**

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Examination date: December 11, 2017

Examination time (from–to): 09:00-13:00

Permitted examination support material: C:

- Calculator CITIZEN SR-270X, CITIZEN SR-270X College, HP30S, Casio fx-82ES PLUS with empty memory.
- Tabeller og formler i statistikk, Tapir forlag.
- K. Rottmann: Matematisk formelsamling.
- Bilingual dictionary.
- One yellow, stamped A5 sheet with own handwritten formulas and notes (on both sides).

Other information:
Note that all answers must be justified.
All ten subproblems are equally weighted.

Language: English

Number of pages: 5

Number of pages enclosed: 4

Checked by:

Informasjon om trykking av eksamensoppgave	
Originalen er:	
1-sidig ☐	**2-sidig** ☒
sort/hvit ☒	**farger** ☐
skal ha flervalgskjema ☐	

Date Signature

Problem 1

Sociologists have used Markov chains to model the social classes over successive generations in a family. Here, we assume a first-order Markov chain model, where the social class of a son depends on the father's class only. There are three classes: {1: Lower, 2: Middle, 3: Upper}, and the transition matrix is given by

$$\mathbf{P} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} \;\;1 & \;\;\;\;2 & \;\;\;\;3 \end{array} \\ \begin{pmatrix} 0.4 & 0.5 & 0.1 \\ 0.05 & 0.7 & 0.25 \\ 0.05 & 0.5 & 0.45 \end{pmatrix} \end{array},$$

where element $P_{k,l} = P(X_t = l | X_{t-1} = k)$, and $X_t \in \{1, 2, 3\}$ represents the state of the man in generation $t = 1, 2, \ldots$. We first assume only one son per father.

a)

Calculate $P(X_3 = 2 | X_1 = 1)$.

Calculate $P(X_4 = 2 | X_2 = 1, X_1 \neq 1)$.

Calculate $P(X_3 = 2 | X_2 \neq 1, X_1 = 1)$.

b)

Show that the long-run proportions for the social classes are $\hat{\pi}_1 = 0.08$, $\hat{\pi}_2 = 0.62$ and $\hat{\pi}_3 = 0.30$.

Assume, in the long-run, that a son is in the middle class. What is the probability that his father was in the lower class?

Assume, in the long-run, that a son is in the middle class. What is the probability that his grandfather was in the lower class?

c) Set $X_t = 1$.

What is the expected number of generations until the family first leaves the lower class?

Calculate the expected number of generations until the family gets a son in the upper class.

d) Assume now that a father has three sons. The class of the sons are conditionally independent, and defined by the Markov transition matrix, given the fathers class. Use the model to solve the following:

In the long-run, what is the probability that the father was in the lower class, given that all three sons are in the middle class?

In the long-run, what is the probability that the grandfather was in the lower class, given that all three sons are in the middle class?

Problem 2

A ski team has a webcamera in one of their favourite slopes. The conditions are harsh, and various elements of the camera can fail. We will assume that the time until failure is exponential distributed with rate parameter $\mu > 0$ (per week).

a)

Set $\mu = 0.2$. What is the expected time until failure?

Assume instead that μ is a random variable with an exponential density function with rate $\alpha > 0$: $f(\mu) = \alpha \exp(-\alpha\mu)$. Compute the marginal probability density function for the failure time.

Assume from here on that the failure rate is $\mu = 0.2$. When the webcamera fails, it goes immediately to repair, and the repair time is exponential distributed with rate parameter $\lambda = 1$. Denote the two states at time t by functional ($X_t = 1$) and under repair ($X_t = 0$). Kolmogorov's forward equations give transition probabilities $P(X_t = j | X_0 = i) = P_{i,j}(t)$, defined by the following differential equation:

$$P'_{0,0}(t) = -\lambda P_{0,0}(t) + \mu P_{0,1}(t), \quad P'_{1,0}(t) = -\lambda P_{1,0}(t) + \mu P_{1,1}(t), \tag{1}$$

$P_{i,0}(t) + P_{i,1}(t) = 1$, for $i = 0, 1$.

b)

Figure 1 shows three plots. Which plot represents the correct display of $P_{0,0}(t)$ and $P_{1,0}(t)$ in our case?

Use equation (1) and Figure 1 to find the long-run probabilities of the functional and under repair state.

The ski team has a back-up camera which is of the same brand, and independent of the other. When a camera fails (at rate $\mu = 0.2$), the other one is immediately installed to replace the one that fails. There is one person that repairs a camera (at rate $\lambda = 1$). She can only work on one camera at a time. We are interested in the long-run distribution of the process.

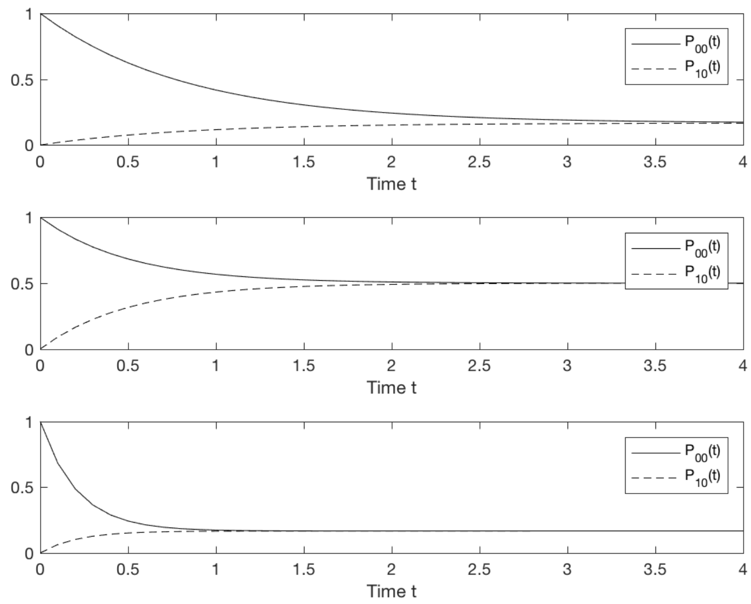

Figure 1: Transition probabilities as a function of time for three different parameter settings of μ and λ.

c)

Draw a transition diagram for the three states of the process:

- (functional, back-up): one camera is on the web, and another is ready as back-up.
- (functional, under repair): one camera is on the web, and another is under repair.
- (under repair,under repair): no camera is on the web, both are under repair.

Compute the long-run distribution for all states.

The ski team considers engaging another repair person in the situations when both cameras have failed. This extra person also repairs a camera at rate $\lambda = 1$.

d)

Compute the long-run distribution for all states in this situation with two repair persons available when both cameras have failed.

The ski team makes money on the webactivity. When the system is functional, the income is 10000 kroner per week. They have no income when both cameras are under repair. The first repair person has a fixed yearly salary, but the second repair person demands a salary of 1000 kroner per time unit (week) when she has to work. Based on maximum expected monetary value, is it affordable for the team to hire the second repair person when required? Or should they stick with only the first repair person?

Problem 3

Special sensors at a buoy can measure salinity in the water. This is done by lowering a sensor in the water, and acquiring measurement at some selected depths. Salinity X_t [g/kg] for depth t [m] is here modeled as a Gaussian process with constant mean $E(X_t) = \mu = 35$, for all depths between 0 and 100 meters, and exponential (Markovian) covariance function defined by $\mathrm{Cov}(X_t, X_s) = 0.5^2 \exp(-0.1|t-s|)$, for distance $|t-s|$ between depths t and s.

In this exercise you can use that the inverse of a 2×2 matrix is defined by:

$$\boldsymbol{A} = \begin{bmatrix} b & c \\ c & b \end{bmatrix}, \quad \boldsymbol{A}^{-1} = \begin{bmatrix} b/(b^2-c^2) & -c/(b^2-c^2) \\ -c/(b^2-c^2) & b/(b^2-c^2) \end{bmatrix}.$$

a) Salinity is measured at depths 45 and 55 m. The measurements are $x_{45} = 34.4$ and $x_{55} = 35.1$ g/kg.

Use formulas for the Gaussian process to compute the conditional mean and variance for salinity at depth 50 m.

Compute the conditional probability that salinity at depth 50 m is below 35 g/kg.

b) Figure 2 shows buoy data of salinity at 10 different days. Salinity has been measured every 5 meters between 0 and 100.

Based on this plot, are the modeling assumptions valid?

A biologist says that after these measurements are conducted, the maximum standard deviation in salinity, at any depth between 0 and 100 m, should be less than 0.3 g/kg. Using the sampling scheme of 5 meter intervals from 0

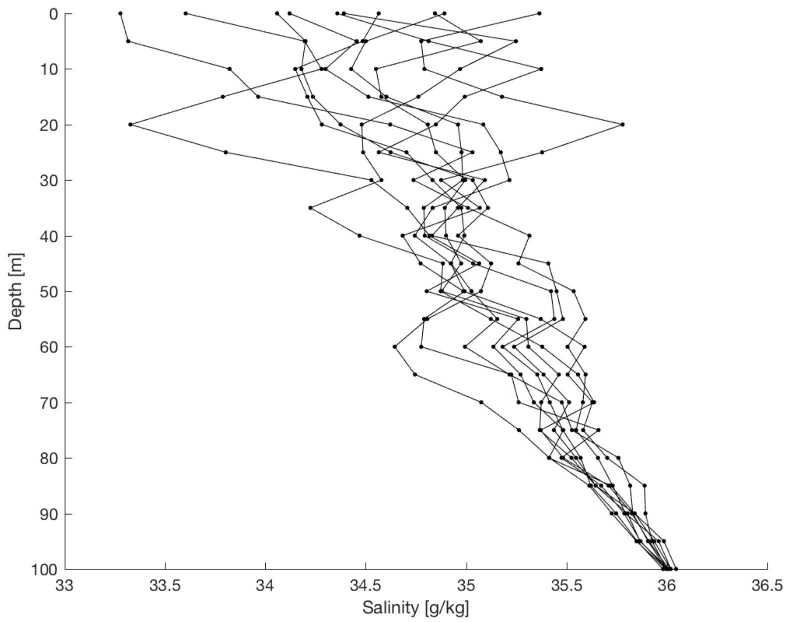

Figure 2: Salinity data for 10 different days. They are measured for different depths at a buoy.

to 100 m depth, and assuming the model specified above holds, is this level of uncertainty achieved?

What if an alternative model is used, that fits the data better: Discuss if another placement scheme for the sensors is more intelligent for getting low uncertainty at all depths.

Formulas: TMA4265 Stochastic Modeling:

The law of total probability

Let $B_1, B_2, \ldots$ be pairwise disjoint events with $P(\cup_{i=1}^{\infty} B_i) = 1$. Then

$$P(A|C) = \sum_{i=1}^{\infty} P(A|B_i \cap C)P(B_i|C),$$

$$E[X|C] = \sum_{i=1}^{\infty} E[X|B_i \cap C]P(B_i|C).$$

Discrete time Markov chains

Chapman-Kolmogorov equations

$$P_{ij}^{(m+n)} = \sum_{k=0}^{\infty} P_{ik}^{(m)} P_{kj}^{(n)}.$$

For an irreducible and ergodic Markov chain, $\pi_j = \lim_{n\to\infty} P_{ij}^{(n)}$ exist and is given by the equations

$$\pi_j = \sum_i \pi_i P_{ij} \quad \text{and} \quad \sum_i \pi_i = 1.$$

For transient states i, j and k, the mean passage time from i to $j \neq i$, M_{ij}, is

$$M_{ij} = 1 + \sum_k P_{ik} M_{kj}.$$

The Poisson process

The waiting time to the n-th event (the n-th arrival time), X_n, has probability density

$$f_{X_n}(t) = \frac{\lambda^n t^{n-1}}{(n-1)!} e^{-\lambda t} \quad \text{for} \quad t \geq 0.$$

Given that the number of events $N(t) = n$, the arrival times $X_1, X_2, \ldots, X_n$ have the uniform joint probability density

$$f_{X_1,X_2,\ldots,X_n|N(t)}(x_1, x_2, \ldots, x_n|n) = \frac{n!}{t^n} \quad \text{for} \quad 0 < x_1 < x_2 < \ldots < x_n \leq t.$$

Markov processes in continuous time

A (homogeneous) Markov process $X(t)$, $0 \leq t \leq \infty$, with state space $\Omega \subseteq \mathbf{Z}^+ = \{0, 1, 2, \ldots\}$, is called a birth and death process if

$$P_{i,i+1}(h) = \lambda_i h + o(h)$$

$$P_{i,i-1}(h) = \mu_i h + o(h)$$

$$P_{i,i}(h) = 1 - (\lambda_i + \mu_i)h + o(h)$$

$$P_{ij}(h) = o(h) \quad \text{for } |j - i| \geq 2$$

where $P_{ij}(s) = P(X(t+s) = j | X(t) = i)$, $i, j \in \mathbf{Z}^+$, $\lambda_i \geq 0$ are birth rates, $\mu_i \geq 0$ are death rates.

The Chapman-Kolmogorov equations

$$P_{ij}(t+s) = \sum_{k=0}^{\infty} P_{ik}(t) P_{kj}(s).$$

Limit relations

$$\lim_{h \to 0} \frac{1 - P_{ii}(h)}{h} = v_i\,, \quad \lim_{h \to 0} \frac{P_{ij}(h)}{h} = q_{ij}\,,\ \ i \neq j$$

Kolmogorov's forward equations

$$P'_{ij}(t) = \sum_{k \neq j} q_{kj} P_{ik}(t) - v_j P_{ij}(t).$$

Kolmogorov's backward equations

$$P'_{ij}(t) = \sum_{k \neq i} q_{ik} P_{kj}(t) - v_i P_{ij}(t).$$

If $P_j = \lim_{t \to \infty} P_{ij}(t)$ exist, P_j are given by

$$v_j P_j = \sum_{k \neq j} q_{kj} P_k \quad \text{and} \quad \sum_j P_j = 1.$$

In particular, for birth and death processes

$$P_0 = \frac{1}{\sum_{k=0}^{\infty} \theta_k} \quad \text{and} \quad P_k = \theta_k P_0 \quad \text{for} \quad k = 1, 2, \ldots$$

where

$$\theta_0 = 1 \quad \text{and} \quad \theta_k = \frac{\lambda_0 \lambda_1 \cdot \ldots \cdot \lambda_{k-1}}{\mu_1 \mu_2 \cdot \ldots \cdot \mu_k} \quad \text{for} \quad k = 1, 2, \ldots$$

Queueing theory

For the average number of customers in the system L, in the queue L_Q; the average amount of time a customer spends in the system W, in the queue W_Q; the service time S; the average remaining time (or work) in the system V, and the arrival rate λ_a, the following relations obtain

$$L = \lambda_a W.$$

$$L_Q = \lambda_a W_Q.$$

$$Z = \lambda_a E[S].$$

$$V = \lambda_a E[SW_Q^*] + \lambda_a E[S^2]/2.$$

Gaussian processes

The multivariate Gaussian density for $n \times 1$ random vector $\boldsymbol{x} = (x_1, \dots, x_n)$ is

$$p(\boldsymbol{x}) = \frac{1}{(2\pi)^{n/2}|\boldsymbol{\Sigma}|^{1/2}} \exp\left(-\frac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\boldsymbol{x}-\boldsymbol{\mu}) \right), \quad \boldsymbol{x} \in \mathbb{R}^n,$$

where size $n \times 1$ mean vector $\boldsymbol{\mu} = (\mu_1, \dots, \mu_n)$, $E(x_i) = \mu_i$, and

$$\boldsymbol{\Sigma} = \left[\begin{array}{ccc} \Sigma_{1,1} & \dots & \Sigma_{1,n} \\ \dots & \dots & \dots \\ \Sigma_{n,1} & \dots & \Sigma_{n,n} \end{array} \right], \quad \Sigma_{i,j} = \mathrm{Cov}(x_i, x_j).$$

Let $\boldsymbol{x}_A = (x_{A,1}, \dots, x_{A,n_A})$ and $\boldsymbol{x}_B = (x_{B,1}, \dots, x_{B,n_B})$, be two subsets of variables, with block mean and covariance structure

$$\boldsymbol{\mu} = (\boldsymbol{\mu}_A, \boldsymbol{\mu}_B), \quad \boldsymbol{\Sigma} = \left[\begin{array}{cc} \boldsymbol{\Sigma}_A & \boldsymbol{\Sigma}_{A,B} \\ \boldsymbol{\Sigma}_{B,A} & \boldsymbol{\Sigma}_B \end{array} \right].$$

The conditional density of $\boldsymbol{x}_A$, given $\boldsymbol{x}_B$, is Gaussian with

$$\begin{array}{rcl} E(\boldsymbol{x}_A|\boldsymbol{x}_B) & = & \boldsymbol{\mu}_A + \boldsymbol{\Sigma}_{A,B}\boldsymbol{\Sigma}_B^{-1}(\boldsymbol{x}_B - \boldsymbol{\mu}_B), \\ \mathrm{Var}(\boldsymbol{x}_A|\boldsymbol{x}_B) & = & \boldsymbol{\Sigma}_A - \boldsymbol{\Sigma}_{A,B}\boldsymbol{\Sigma}_B^{-1}\boldsymbol{\Sigma}_{B,A}. \end{array}$$

The Brownian motion has increments $x(t_i) - x(t_{i-1})$ with the following properties, for any configuration of times $t_0 = 0 < t_1 < t_2 < \dots$:

- $x(t_i) - x(t_{i-1})$ and $x(t_j) - x(t_{j-1})$ are independent for all $i \neq j$.
- the distribution of $x(t_i) - x(t_{i-1})$ is identical to that of $x(t_i + s) - x(t_{i-1} + s)$, for any s.
- $x(t_i) - x(t_{i-1})$ is Gaussian distributed with 0 mean and variance $\sigma^2(t_i - t_{i-1})$.

Unless otherwise stated, $x(0) = 0$.

Some mathematical series

$$\sum_{k=0}^{n} a^k = \frac{1-a^{n+1}}{1-a} \quad , \qquad \sum_{k=0}^{\infty} k a^k = \frac{a}{(1-a)^2} \quad .$$