Norwegian University of Science and Technology
Deptartment of Mathematical Sciences

TMA4190 Introduction to Topology Spring 2018

Exercise set 5

1 **a)** Show that a local diffeomorphism $f: X \to Y$ which is bijective is a diffeomorphism.

b) Show that a local diffeomorphism $f: X \to Y$ which is one-to-one is a diffeomorphism of X onto an open subset of Y.

c) Show that a bijective smooth map $f: X \to Y$ of constant rank is a diffeomorphism.
(Comment: You can assume that f is a submersion to simplify things. If you want to challenge yourself, you could only assume that X is compact. Showing that f also is a submersion in general requires the use of Baire's category theorem.)

d) Show that a bijective Lie group homomorphism is a Lie group isomorphism.

2 Show that an open subgroup H, i.e. a subgroup which is also an open subset, of a connected Lie group G is equal to G.

3 Let G be a Lie group and let $e \in G$ be the identity element.

a) Let $\mu: G \times G \to G$ denote the multiplication map, and let $g, h \in G$. Recall that we denote by L_g the left translation in G by g, and by R_h the right translation by h. Using the identification $T_{(g,h)}(G \times G) = T_g(G) \times T_h(G)$, show that the differential of μ at (g, h)

$$d\mu_{(g,h)}: T_g(G) \times T_h(G) \to T_{gh}(G)$$

is given by

$$d\mu_{(g,h)}(X, Y) = d\mu_{(g,h)}(X, 0) + d\mu_{(g,h)}(0, Y) = d(R_h)_g(X) + d(L_g)_h(Y).$$

(Hint: Calculate $d\mu_{(g,h)}(X, 0)$ and $d\mu_{(g,h)}(0, Y)$ separately.)

b) Let $\iota: G \to G$ denote the inversion map. Show that $d\iota_e: T_e(G) \to T_e(G)$ is given by $d\iota_e(X) = -X$.

c) Use the previous point to show that, for any $g \in G$, the derivative of ι at g is given by

$$d\iota_g: T_g(G) \to T_{g^{-1}},\ Y \mapsto -d(R_{g^{-1}})_e(d(L_{g^{-1}})_g(Y)) \text{ for all } Y \in T_g(G).$$

4 Show that for any Lie group G, the multiplication map $\mu\colon G \times \to G$ is a submersion.

5 Show that the differential of the determinant map $\det\colon GL(n,\mathbb{R}) \to \mathbb{R}$ at $A \in GL(n,\mathbb{R})$ is given by

$$d(\det)_A(B) = (\det A) \cdot (\operatorname{tr} A^{-1}B) \text{ for all } B \in M(n).$$

In particular, $d(\det)_A(AB) = (\det A) \cdot (\operatorname{tr} AB)$ for all $B \in M(n)$.