Norwegian University of Science and Technology
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TMA4212 Numerical solution of differential equations by difference methods Spring 2021

Exercise set 1 Solution

$\boxed{1}$ Consider the following tridiagonal matrix

$$A = \begin{pmatrix} a & b & 0 & \dots & 0 \\ c & a & b & \ddots & 0 \\ \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & \ddots & c & a & b \\ 0 & \dots & 0 & c & a \end{pmatrix} = \text{tridiag}(c, a, b) \in \mathbb{R}^{M\times M},\ M \geq 4,$$

where we assume $bc > 0$. It is known that the right eigenvectors $\boldsymbol{x}^{(k)}(k = 1, ..., M)$ and the associated eigenvalues $\lambda_k(k = 1, ..., M)$ are given by

$$x_j^{(k)} = \left(\frac{b}{c}\right)^{j/2} \sin\left(\frac{jk\pi}{M+1}\right),\quad \lambda_k = a + 2\sqrt{bc}\cos\left(\frac{k\pi}{M+1}\right),$$

where $x_j^{(k)}$ is the jth element of the vector $\boldsymbol{x}^{(k)}$; $A\boldsymbol{x}^{(k)} = \lambda_k\boldsymbol{x}^{(k)}$. You can verify this by simply inserting.

a) What are the left eigenvectors $\boldsymbol{y}^{(k)}(k = 1, ..., M)$ and associated eigenvalues $\beta_k(k = 1, ..., M)$ of A which satisfy $\boldsymbol{y}^{(k)}A = \beta_k\boldsymbol{y}^{(k)}$? Note that $\boldsymbol{y}^{(k)}$'s are row vectors.
[Solution] By taking transpose, we have $(\boldsymbol{y}^{(k)}A)^T = A^T(\boldsymbol{y}^{(k)})^T = \beta_k(\boldsymbol{y}^{(k)})^T$. Therefore, we only need to know the right eigenvectors and associated eigenvalues of $A^T = \text{tridiag}(b, a, c)$. Right eigenvectors and associated eigenvalues for a tridiagonal matrix are already given above, hence

$$y_j^{(k)} = \left(\frac{c}{b}\right)^{j/2} \sin\left(\frac{jk\pi}{M+1}\right),\quad \beta_k = a + 2\sqrt{bc}\cos\left(\frac{k\pi}{M+1}\right).$$

b) Assume $a > 2\sqrt{bc} > 0$ and $b = c$, calculate the following quantity (called the ℓ_2 condition number):

$$\left\|A^{-1}\right\|_2 \|A\|_2\,.$$

(Hint: look at the text "finite difference methods" by Brynjulf Owren, Section 3.1.)
[Solution] Since $\|A_h\|_2 = \sqrt{\rho(A^TA)} = \sqrt{\rho(A^2)}$,

$$\left\|A^{-1}\right\|_2 \|A\|_2 = \frac{\max|\lambda_k|}{\min|\lambda_k|} = \frac{a + 2|b|\cos\left(\frac{\pi}{M+1}\right)}{a + 2|b|\cos\left(\frac{M\pi}{M+1}\right)}.$$

c) Let A_h be

$$A_h = \frac{1}{h^2}\text{tridiag}(-1, 2, -1) \in \mathbb{R}^{M\times M},$$

where $h = 1/(M+1)$. Calculate the following quantity

$$\lim_{M\to\infty} \left\|A_h^{-1}\right\|_2 .$$

(Hint: look at the same section of the text as above.)
[Solution] For this matrix, eigenvalues are given by

$$\frac{2 + 2\cos\left(\frac{k\pi}{M+1}\right)}{h^2}, \; k = 1, ..., M.$$

The smallest eigenvalue is when $k = M$, and this can be expanded as

$$\frac{2 + 2\cos(\pi - h\pi)}{h^2} = \pi^2 - \frac{\pi^4 h^2}{12} + \mathcal{O}(h^4).$$

Therefore,

$$\lim_{M\to\infty} \left\|A_h^{-1}\right\|_2 = \lim_{h\to 0} \frac{h^2}{2 + 2\cos(\pi - h\pi)} = \frac{1}{\pi^2}.$$

$\boxed{2}$ Consider a function $u(x)$ defined on $[0, 1]$. We want to approximate the derivative $u_x(x)$ by using function values of $u(x)$ on equidistant points

$$x_0 = 0, \; x_1 = \frac{1}{M+1}, ..., \; x_M = \frac{M}{M+1}, \; x_{M+1} = 1.$$

Let $h = 1/(M+1)$.

a) Consider two different approximation methods:

$$u_x(x) \approx \frac{u(x+h) - u(x)}{h} \quad \text{(Forward difference)},$$

for $x = x_0, ..., x_M$, and

$$u_x(x) \approx \frac{u(x+h/2) - u(x-h/2)}{h} \quad \text{(Central difference)},$$

for $x = x_0 + h/2, ..., x_M + h/2$ (so that we only use function values on x_i's). Calculate the convergence order of these methods in terms of h. Then write down the approximation as a matrix-vector multiplication:

$$\boldsymbol{u}_x = A_h \boldsymbol{u},$$

where $\boldsymbol{u}_x$ is a vector comprised of approximated values of $u_x(x)$, A_h is an $(M+1)\times(M+1)$ matrix, and $\boldsymbol{u}$ is a vector comprised of function values of $u(x)$.
[Solution] For the forward difference scheme, by using the Taylor expansion around $x_i, i = 0..., M$ we have

$$\begin{aligned}\frac{u(x_i+h) - u(x_i)}{h} &= \frac{\big(u(x_i) + u_x(x_i)(x_i + h - x_i) + u_{xx}(\xi)(x_i + h - x_i)^2/2\big) - u(x_i)}{h} \\ &= \frac{hu_x(x_i) + h^2 u_{xx}(\xi)/2}{h} = u_x(x_i) + \frac{hu_{xx}(\xi)}{2},\end{aligned}$$

for some $\xi \in [x_i, x_i + h]$. Therefore this is a first order approximation.
For the central difference scheme, by using the Taylor expansion around $x_k = x_0 + h/2, ..., x_M + h/2$ we have

$$\begin{aligned}
&u(x_k + h/2) - u(x_k - h/2) \\
= &\big(u(x_k) + u_x(x_k)(x_k + h/2 - x_k) + u_{xx}(x_k)(x_k + h/2 - x_k)^2/2 + u_{xxx}(\xi_1)(x_k + h/2 - x_k)^3/6\big) \\
&- \big(u(x_k) + u_x(x_k)(x_k - h/2 - x_k) + u_{xx}(x_k)(x_k - h/2 - x_k)^2/2 + u_{xxx}(\xi_2)(x_k - h/2 - x_k)^3/6\big) \\
&= hu_x(x_k) + \frac{h^3(u_{xxx}(\xi_1) - u_{xxx}(\xi_2))}{48},
\end{aligned}$$

for some $\xi_1 \in [x_k, x_k + h/2], \xi_1 \in [x_k - h/2, x_k]$. Dividing this by h and we conclude that this central difference gives the second order convergence.
For both schemes, we can write $\boldsymbol{u}_x = A_h \boldsymbol{u}$ with

$$A_h = \frac{1}{h}\begin{pmatrix} 1 & -1 & 0 & \dots & 0 \\ 0 & 1 & -1 & \ddots & 0 \\ \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & \ddots & 0 & 1 & -1 \\ 0 & \dots & 0 & 0 & 1 \end{pmatrix},$$

where the first and last rows may vary depending on the boundary conditions we may impose.

b) We define matrix exponentials by

$$\exp(A) := \sum_{k=0}^{\infty} \frac{A^k}{k!}.$$

Calculate the eigenvalues of $\exp(B_h)$ for

$$B_h = \frac{1}{h^2}\text{tridiag}(-1, 2, -1).$$

[Solution] B_h is a symmetric real matrix and therefore we can diagonalize it by $B_h = TDT^{-1}$ where T is some orthogonal matrix (indeed, we already know the form from the previous problem: T consists of eigenvectors of B_h and D consists of eigenvalues). Using this decomposition,

$$\begin{aligned}
\exp(B_h) := \sum_{k=0}^{\infty} \frac{B_h^k}{k!} &= T\left(\sum_{k=0}^{\infty} \frac{D^k}{k!}\right) T^{-1} \\
&= T \begin{pmatrix} \sum_{k=0}^{\infty} \frac{\lambda_1^k}{k!} & 0 & 0 & \dots & 0 \\ 0 & \sum_{k=0}^{\infty} \frac{\lambda_2^k}{k!} & 0 & \ddots & 0 \\ \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & \dots & 0 & 0 & \sum_{k=0}^{\infty} \frac{\lambda_M^k}{k!} \end{pmatrix} T^{-1},
\end{aligned}$$

and we know $\sum_{k=0}^{\infty} \frac{\lambda_i^k}{k!} = \exp(\lambda_i)$. Therefore the eigenvalues are

$$\exp(\lambda_i) = \exp\left(\frac{2 + 2\cos\left(\frac{i\pi}{M+1}\right)}{h^2}\right).$$