Project 1 Notes

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2020

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1 Problem 1

Let normal matrices, those with diagonalization be on the form

$$A = U\Lambda U^H$$

Where Λ is a diagonal complex $n \times n$ matrix and U a unitary (complex) matrix such that $U^H U = I$ (recall that U^H is the complex conjugate of U^T).

Show that for any such matrix, one has $\|A\|_2 = \rho(A)$, where $\rho(A)$ is the spectral radius of A .

1.1 Proof

Answer.

Proof. Starting with the definition of a subordinate matrix norm [2] can we let

$$\|A\|_2^2 = \sup_{x \neq 0} \frac{\langle Ax, Ax \rangle}{\langle x, x \rangle}.$$

Note sure if this is the correct notation for 2-norm. Indeed, by using the assumption that $U^H U = I$ and substituting $Uy = x$ can we show that

$$\|A\|^2 = \sup_{x \neq 0} \frac{\langle Ax, Ax \rangle}{\langle x, x \rangle} = \sup_{y \neq 0} \frac{\langle AUy, AUy \rangle}{\langle Uy, Uy \rangle} = \sup_{y \neq 0} \frac{\left\langle U^H A^H AUy, y \right\rangle}{\langle y, y \rangle}$$

Recall the property $A = U\Lambda U^H$ and thus

$$\begin{aligned} A^H A &= U\Lambda^H U^H U\Lambda U^H \\ &= U\Lambda^H \Lambda U^H. \end{aligned}$$

As a consequence do we end up with

$$\begin{aligned} \sup_{y \neq 0} \frac{\left\langle U^H A^H AUy, y \right\rangle}{\langle y, y \rangle} &= \sup_{y \neq 0} \frac{\left\langle U^H U\Lambda^H \Lambda U^H Uy, y \right\rangle}{\langle y, y \rangle} \\ &= \sup_{y \neq 0} \frac{\left\langle \Lambda^H \Lambda y, y \right\rangle}{\langle y, y \rangle} \\ &= \sup_{y \neq 0} \frac{\sum_{i=1}^n |\lambda_i|^2 |y_i|^2}{\sum_{i=1}^n |y_i|^2} = \max_i \left(|\lambda_i|^2 \right) \end{aligned} .$$

Given the definition of a spectral radius [1] characterized by

$$\rho(A) = \max_i |\lambda_i|.$$

Which completes the proof of $\|A\|_2 = \rho(A)$.

□

1.2 Note

Let say $A = U\Lambda U^H$. Then is it not possible to get any useful answers.

(i) Lets compute AA

$$AA = U\Lambda U^H U\Lambda U^H = U\Lambda\Lambda U^H$$

(ii) Lets compute $A^H A$

$$A^H A = U\Lambda^H U^H U\Lambda U^H = U\Lambda^H \Lambda U^H$$

(iii) Lets compute AA^H

$$AA^H = U\Lambda U^H U\Lambda^H U^H = U\Lambda\Lambda^H U^H.$$

2 Problem 2

Consider the $n \times n$ matrix A whise nonzero elements are located on its unit subdiagonal , i.e. $A_{i+1,i} = 1$ for $i = 1, \ldots, n-1$

$$A = \begin{bmatrix} 0 & \ldots & \ldots & 0 \\ 1 & 0 & & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \ldots & 1 & 0 \end{bmatrix}$$

(a) What are the eigenvalues of A ? What would the Gershgorin theorem tell us abou the location of the eigenvalues of A.

(b) Now construct the matrix $\hat{A}$ by adding a small number ϵ in the $(1, n)$-element of A (so that $\hat{A} = A + \epsilon e_1 e_n^T$. Show that

$$\rho\left(\hat{A}\right) = \epsilon^{\frac{1}{n}}$$

And fins an expression for the eigenvalues and eigenvectors of $\hat{A}$.

(c) Derive an exact expression for the condition number $K_2\left(\hat{A}\right) = \|\hat{A}\|_2 = \|\hat{A}^{-1}\|_2$.

2.1 Answer a

Answer.

The eigenvalues can be computed such that

$$\begin{aligned} det\left(A - \lambda\right) &= \begin{vmatrix} -\lambda & & & \ldots & 0 & 0 \\ 1 & -\lambda & \ldots & & & 0 \\ 0 & 1 & -\lambda & \ldots & & 0 \\ \vdots & & & \ddots & & \\ 0 & \ldots & & & 1 & -\lambda \end{vmatrix} \\ &= -\lambda \begin{vmatrix} -\lambda & \ldots & & 0 \\ 1 & -\lambda & & \\ \vdots & & \ddots & \\ 0 & \ldots & 1 & -\lambda \end{vmatrix} \\ &= \left(-1\right)^n \lambda^n = 0 \qquad \Longrightarrow \qquad \lambda = 0 \end{aligned}$$

Which concludes that we did only find a trivial solution.

2.2 Answer b

Answer. We can observe that

$$\hat{A} = A + \varepsilon e_1 e_n^T = \begin{pmatrix} 0 & & \dots & & \varepsilon \\ 1 & 0 & \dots & & \\ 0 & 1 & \ddots & & \\ \vdots & & \ddots & \ddots & \\ 0 & \dots & & 1 & 0 \end{pmatrix}$$

We can then find the eigenvalues by computing

$$det\left(\hat{A} - \lambda\right) = \begin{vmatrix} -\lambda & & \dots & & \varepsilon \\ 1 & -\lambda & \dots & & \\ 0 & 1 & \ddots & & \\ \vdots & & \ddots & \ddots & \\ 0 & \dots & & 1 & -\lambda \end{vmatrix}$$

$$= (-1)^n \lambda^n + (-1)^{n+1} \varepsilon \begin{vmatrix} 1 & -\lambda & \dots & & \\ 0 & 1 & \ddots & & \\ \vdots & & \ddots & \ddots & \\ 0 & \dots & & 1 & -\lambda \end{vmatrix}$$

$$= (-1)^n \lambda^n + (-1)^{n+1} \varepsilon = 0$$

Solvin this equation do we obtain

$$\implies \lambda = \varepsilon^{\frac{1}{n}} = \begin{cases} , & n = 0 \\ , & n = 1 \end{cases}$$

As we can see can λ have several complex solutions depending on the value n. However, we can conclude that $|\lambda| = \varepsilon^{\frac{1}{n}}$ thus

$$\rho\left(\hat{A}\right) = \varepsilon^{\frac{1}{n}}.$$

Which completes the task.

3 References

References

[1] Alfio Quarteroni, Riccardo Sacco, and Fausto Saleri. *Numerical mathematics*, volume 37, page 14. Springer Science & Business Media, 2010.

[2] Endre Süli and David F Mayers. *An introduction to numerical analysis*, page 64. Cambridge university press, 2003.