NTNU
Norwegian University of
Science and Technology

Week 42: Lecture 1

Limiting probabilites of continuous-time Markov chains

Geir-Arne Fuglstad

October 12, 2020

Information

In weeks 44 and 45, the statistics group offers a self-study short course in R. With in-class supervision on

- Tuesday November 3 at 10–12.
- Wednesday November 4 at 16–18.

More information will follow.

Note: this short course is not part of our course, but can be helpful for you.

Section 6.3

Theorem

For a birth and death process, under suitable regularity conditions,

$$P'_{i0}(t) = -\lambda_0 P_{i0}(t) + \mu_1 P_{i1}(t), \quad t \geq 0,$$
$$P'_{ij}(t) = \lambda_{j-1} P_{i,j-1}(t) - (\lambda_j + \mu_j) P_{i,j}(t) + \mu_{j+1} P_{i,j+1}(t), \quad t \geq 0, j > 0,$$

with initial conditions

$$P_{ij}(0) = \delta_{ij} = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases}$$

These are called the ***forward Kolmogorov differential equations****.*

Example

A machine has two states: working (1) and broken (0). When the machine is working, the time to breakdown follows an exponential distribution with rate $\mu > 0$. When the machine breaks down, the time to fix the machine follows an exponential distribution with rate $\lambda > 0$. All breakdown times and repair times are independent. The machine works at time 0. Calculate the probability that the machine works at time $t = 10$.

Section 6.4

Theorem

For a birth and death process without absorbing states, the limiting probabilities

$$\pi_j = \lim_{t\to\infty} P_{ij}(t), \quad j = 0, 1, \ldots$$

1. *exist and* $\pi_j \geq 0$, $j = 0, 1, \ldots$.
2. *are not dependent on the state* i.

Definition

If $\sum_{j=0}^{\infty} \pi_j = 1$, then $\boldsymbol{\pi} = (\pi_0, \pi_1, \ldots)$ is called the **limiting (probability) distribution.**

Theorem

Let π_j, $j = 0, 1, \ldots$, be the limiting probabilites of a birth and death process without absorbing states. If $\pi_j > 0$, $j = 0, 1, \ldots$, then $\boldsymbol{\pi} = (\pi_0, \pi_1, \ldots)$ is the unique solution of

$$\begin{aligned} \lambda_0 \pi_0 &= \mu_1 \pi_1, \\ (\lambda_j + \mu_j)\pi_j &= \lambda_{j-1}\pi_{j-1} + \mu_{j+1}\pi_{j+1}, \quad j \geq 1, \\ \sum_{j=0}^{\infty} \pi_j &= 1. \end{aligned}$$

Note: Also works for finite state spaces $\{0, 1, \ldots, N\}$ with $\lambda_N = 0$ and $\mu_N > 0$.

Example

What is the long-run mean fractions of time spent in state 0 and in state 1?

Theorem

When the solution of

$$\lambda_0 \pi_0 = \mu_1 \pi_1,$$
$$(\lambda_j + \mu_j)\pi_j = \lambda_{j-1}\pi_{j-1} + \mu_{j+1}\pi_{j+1}, \quad j \geq 1,$$
$$\sum_{j=0}^{\infty} \pi_j = 1.$$

is unique, it is given by

$$\pi_j = \frac{\theta_j}{\sum_{k=0}^{\infty} \theta_k}, \quad j = 0, 1, \ldots,$$

where

$$\theta_0 = 1, \quad \textit{and} \quad \theta_k = \frac{\lambda_0 \lambda_1 \cdots \lambda_{k-1}}{\mu_1 \mu_2 \cdots \mu_k}, \quad k = 1, 2, \ldots.$$

Example

Calculate the limiting distribution.

Section 6.5

Example

What is the expected time to reach state 0 when starting in state 2?