Cheat Sheet

isakhammer

2020

1 Introduction

2 Continious maps

3 Topological spaces

Definition 3.1 (Topological spaces.). *Recall that a topological space is a set X together with a collection Y of subsets of X that are open in X s.t.*

- ***T1***. $\emptyset, X \in \tau$
- ***T2.*** *τ is closed under union if $U_\lambda \in \tau$ for all $\lambda \in \Lambda$, then*

$$\bigcup_{\lambda \in \Lambda} U_\lambda \in \tau$$

- ***T3.*** *τ is under finite intersections if $U_1, U_2, \dots, U_n \in \tau$, then*

$$U_1 \cap U_2 \cap \ldots \cap U_n \in \tau$$

Definition 3.2 (Open and closed sets). . *Let (X, τ), $U \subseteq X$*

- ***Open set.*** *If $U \in \tau$, then is U open.*
- ***Closed set.*** *If $U^c = X - U \in \tau$, then is U closed*

Remark. Let $X = \{a, b, c\}$ and let $U = \{a, b\}$. Then if $\tau = \{X, \emptyset\}$, U is not open nor closed.

Definition 3.3 (Neighbourhoods). *Let X be a topological space, U a subset* *of X and $x \in X$. We say U is a neighborhood of x if $x \in U$ and U is open in X.*

Theorem 3.1. ***Continuity between topological spaces.*** *Let X, Y be topological spaces. A map $f : X \to Y$ is said to be continious if preimages of open sats are open, i.e., if V is an open set in Y then the preimage $f^{-1}(V)$ of V is open in X.*

4 Generating topologies

4.1 Generating topologies from subsets

Theorem 4.1 (The intersection of two topologies is a topology)**.** *Let X be a set, and let τ_1 and τ_2 be two topologies on X. Then $\tau_1 \cap \tau_2$ is also a topology on X.*

Definition 4.1 (Topology generated by a collection of subsets)**.** *Let X be a set, and let $\mathscr{S}$ be a collection of subsets of X. The topology generated by $\mathscr{S}$ is the topology*

$$\langle \mathscr{S} \rangle = \bigcap_{\substack{\tau \text{ topology} \\ S \subseteq \tau}} \tau$$

4.2 Basis for a topology

Definition 4.2 (Basis)**.** *Let X be a set. a **basis** for a topology on X is a collection $\mathscr{B}$ of subsets of X such that*

- ***B1**: for each $x \in X$, there is a $B \in \mathscr{B}$ such that $x \in B$*
- ***B1:** if B_1, B_2 and $x \in B_1 \cap B_2$, then there is a $B_3 \in B$ such that $x \in B_3 \subseteq B_1 \cap B_2$.*

Theorem 4.2. *Let X be a set, and let $\mathscr{B}$ be basis for a topology on X. The collection τ generated by $\mathscr{B}$ of subsets U of X with the property that for each $x \in U$ there is a basis element $B \in \mathscr{B}$ with $x \in B \subseteq U$ is a topology on X.*

Theorem 4.3. *Let X be a set, and let $\mathscr{B}$ be a basis for a topology τ on X. Then τ is equal to the collection of all unions of elements of $\mathscr{B}$.*

Theorem 4.4. *Let X be a set, and let $\mathscr{B}_1$ and $\mathscr{B}_2$ be bases for topologies τ_1 and τ_2 , respectively , on X. Then the following are equivalent.*

(i) τ_2 *is finer than* τ_1, *i.e.,* $\tau \subseteq \tau_2$.

(ii) *For each* $B_1 \in \mathscr{B}_1$ *and each* $x \in B_1$, *there is a* $B_2 \in \mathscr{B}_2$ *such that* $x \in B_2 \subseteq B_1$.

4.3 Subbasis for a topology

Definition 4.3 (Subbasis)**.** *Let X be a set. A **subbasis** for a topology on X is a collection $\mathscr{S}$ whose union equals X.*

Lemma 4.1. *Let X be a set, and let $\mathscr{S}$ be a subbasis for a topology on X. The collection $\mathscr{B}$ consisting of all finite intersections of elements of $\mathscr{S}$ is a basis for a topology on X and is called the basis associated to $\mathscr{S}$.*

Definition 4.4 (Standard topology)**.** *(Not in compendium.) The standard topology on $\mathbb{R}$ is the topology generated by a basis consisting of all open intervals of $\mathbb{R}$.*

Lemma 4.2. *Let X be a set, and let $\mathscr{S}$ be a subbasis for a topology on X. The collection τ generated by $\mathscr{S}$ consisting of all unions of all basis elements of the associated basis $\mathscr{B}$ is a topology on X.*

Theorem 4.5. *Let X be a set, and let $\mathscr{S}$ be a subbasis for a topology on X. Then there exists a unique topology $\langle \mathscr{S} \rangle$ generated by $\mathscr{S}$ which is smaller than any other topology containing $\mathscr{S}$, where*

$$\langle \mathscr{S} \rangle = \left\{ \bigcup_{\lambda \in \Lambda} \bigcap_{i=1}^{n_\lambda} S_{\lambda,i} \mid S_{\lambda,i} \in \mathscr{S} \right\}$$

Theorem 4.6. *Let X and Y be topological spaces, and let $\mathscr{B}$ (resp., $\mathscr{S}$) be a basis (resp., subbasis) . Then a map $f : X \to Y$ is continious if and only if for each $B \in \mathscr{B}$ (resp. $S \in \mathscr{S}$) the preimage $f^{-1}(B)$ (resp., $f^{-1}(S)$) is open in X .*

5 Constructing topological spaces

5.1 Subspaces

Definition 5.1 (Substapce topology)**.** *Let X be a topological space, and let A be a subset of X . The collection*

$$\tau_A = \{A \cap U \mid U \text{ is open in } X\}$$

of subets of A is called the topology on A.

Lemma 5.1. *Let X be a topological space, and let A be a subsets of X. Then the collection*

$$\tau_A)\{A \cap U \mid U \text{ is open in } X\}$$

is a topology on A.

Theorem 5.1. *Let X be a topological space, and let $\mathscr{B}$ be a basis for the topology on X. If A is a subset X , the collection*

$$\mathscr{B}_A = \{A \cap B \mid B \in \mathscr{B}\}$$

is a basis for the subsapace topology on A.

Theorem 5.2. *Let X be a topological space, and let A be a subset of X . THen the subspace topology on A is the only topology on A witht the following universal property: for every topological space Y and every map :*

$$f : Y \to A$$

f is continious if and only if $i \circ f : Y \to X$ is continious where $i : A \to X$ is the inclusion map given by $i(x) = x$ for $x \in A$.

5.2 Products

Definition 5.2 (Product topology)**.** *Let X and Y be topological spaces. THe product topology on $X \times Y$ is the topology generated by the basis*

$$\mathscr{B} = \{U \times V \mid U \text{ is open in } X \text{ and } V \text{ is open in } Y\}$$

Lemma 5.2. *Let X and Y be topological spaces. Then the collection*

$$\mathscr{B})\,\{U \times V \mid U \text{ is open in } X \text{ and } V \text{ is open in } Y\}$$

is a bsis for a topology on $X \times Y$.

Theorem 5.3. *Let X and Y be topological paces. If $\mathscr{B}_X$ is a basis for a the topology on X and $\mathscr{B}_Y$ is a basis for the topology on Y, then the collection*

$$\mathscr{B}_{X\times Y} = \{B_X \times B_Y \mid B_X \in \mathscr{B}_X \text{ and } B_Y \in \mathscr{B}_Y\}$$

is a bsis for the product topology on $X \times Y$.

Theorem 5.4. *Let X and Y be topological spaces. Let $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ be the projections of $X \times Y$ onto its first and second factors, respectively. The product topology is the only topology on $X \times Y$ with the following universal property: for every topological space Z and every map $f : Z \to X \times Y$, f is continious if and only if $\pi_1 \circ f : Z \to X$ and $\pi_2 \circ f : Z \to Y$ are continious.*

5.3 Quotient spaces

Definition 5.3 (Equivalence classes)**.** *Let X be a set, and let $\sim$ be an equivalence relation on X . The equivalence class of $x \in X$ is the subset*

$$[x] = \{y \in X \mid x \sim y\}$$

of X . Let

$$X/\sim = \{[x] \mid x \in X\}$$

Lemma 5.3. *Let X and A be sets, and let $\pi : X \to A$ be a surjective map. Then the map*

$$\phi : X/\sim \to A$$

given by $\phi([x]) = \pi(x)$, where $x_1 \sim x_2$ if and only if $\pi(x_1) = \pi(x_2)$, is a bijection.

Definition 5.4 (Quotient space)**.** *Let X be a topological space, let A be a set, and elt $\pi : X \to A$ be a surjective map. The quotient topology on A induced by π is the collection of subsets U of A such that $\pi^{-1}(U)$ is open in X . We say that π is a quotient map if A is given the quotient topology, and we call A the quotient space.*

Lemma 5.4. *Let X be a topological space, let A be a set, and let $\pi : X \to A$ be a surjective map. Then the quotient topology on A induced by π is a topology and it is the finest topology on A such that π is continious.*

Definition 5.5 (Open and closed maps)**.** *Let X and Y be topological spaces, and let $f : X \to Y$ be a continious map. We say that f is an open map for each sucbset U of X that is open in X the image $f(U)$ is open in Y . Likewise, we say that f is a closed map if for each subset V of X that is closed in X the image $f(V)$ is closed in Y .*

Lemma 5.5. *Let X and Y be topological spaces, and let $\pi : X \to Y$ be a surjective continious map.*

(i) If π is in addition open then it is a quotient map.

(ii) If π is in addition closed then it is a quotient map.

Theorem 5.5. *Let X be a topological space, let A be a set, and let $\pi : X \to A$ be a surjective map. The quotient topology is the only topology on A with the following universal property: for every topological space Y and every map $f : A \to Y$, f is continious if and only if $f \circ \pi : X \to Y$ is continious.*

6 Topological properties

6.1 Connected spaces

Definition 6.1 (Connected space)**.** *Let X be a topological space. A* ***seperation*** *of X is a pair of non-empty subsets U and V that are open in X, disjoint and whose union equal X. We say that X is* ***connected*** *if there are no seperations of X. Otherwise it is* ***disconnected.***

Theorem 6.1 (Closed and open subsets)**.** *Let X be a topological space. Then X is connected if and only if the are no non-empty proper subsets of X that are both open and closed in X.*

Lemma 6.1 (Disconnectivity)**.** *Let X be a disconnected space with seperation U and V, and et A be a connected subspace of X. Then $A \subseteq U$ and $A \subseteq V$.*

Theorem 6.2 (Collection connectivity)**.** *Let X be a topological space, and let $\{A_\lambda\}_{\lambda\in\Lambda}$ be a collection of connected subspaces of X such that $\bigcap_{\lambda\in\Lambda} A_\lambda$ is non-empty. Then $\bigcup_{\lambda\in\Lambda} A_\lambda$ is connected.*

Definition 6.2 (Path connected space)**.** *Let X be a topological space, and let $x, y \in X$. A path from x to y is a continious map: $f : [a, b] \to X$ .t. $f(a) = x$ and $f(b) = y$ where $[a, b]$ is a subspace of $\mathbb{R}$ with the standard topology. We say that X is* ***path connected*** *if every pair of points of X can be joined by a path in X.*

Theorem 6.3 (Connectivity in product spaces)**.** *Let $X_1, X_2, \ldots, X_n$ be connected spaces. Then the product space $X_1 \times X_2 \times \ldots \times X_n$ is connected.*

Theorem 6.4 (The real numbers are connected)**.** *Let $\mathbb{R}$ be the set of real numbers equipped with the standard topology. Then $\mathbb{R}$ is connected.*

Theorem 6.5 (Generalized intermediate value theorem)**.** *Let X be a connected space and let $f : X \to \mathbb{R}$ be a continious map where $\mathbb{R}$ is given the standard topology. If $a, b \in X$ and if r is a real number that lies between $f(a)$ and $f(b)$, there is a $c \in X$ such that $f(c) = r$*

Theorem 6.6 (Connectivity). *Let X be a topological space. Then X is connected if and only if the are no non-empty proper subsets of X that are both open and closed.*

Theorem 6.7 (Path connectedness implies connectedness). *Let X be a path connectedness space. Then X is connected.*

6.2 Hausdorff spaces

Definition 6.3 (Hausdorff). *Let X be a topological space. We say that X is* ***Hausdorff*** *if for each part of points $x, y \in X$ with $x \neq y$, there are disjoint neighborhoods U and V of x and y, respectively. In other words, for each pair of distinct point $x, y \in X$ there are open subsets U and V of X with $x \in U$ $y \in V$ where $U \cap V = \emptyset$*

Theorem 6.8. *Every metric space is Hausdorff*

Theorem 6.9. *Let X be a Hausdorff space. Then for each $x \in X$ the subset $\{x\}$ of X is closed in X.*

Theorem 6.10. *Let $X_1, X_2 \ldots, X_n$ be Hausdorff spaces. Then the product space $X_1 \times X_2 \times \ldots \times X_n$ is Hausdorff.*

Theorem 6.11. *Let X be a topological space. Then X is Hausdorff if and only if the diagonal*

$$\Delta = \{(x, x) \mid x \in X\}$$

is closed in the product space $X \times X$.

6.3 Compact spaces

Definition 6.4 (Cover of a space). *Let X be a topological space, and let $\mathscr{A}$ be the collection of subsets of X. We say that $\mathscr{A}$ is a cover of X, or covering of X if $X = \bigcap_{A \in \mathscr{A}} A$. If A is also open in X for each $A \in \mathscr{A}$, we* *say that $\mathscr{A}$ is an **open** cover of X, or open covering of X. We say that $\mathscr{A}'$ is a subcover of $\mathscr{A}$ if $\mathscr{A}'$ is another cover of X that satisfies $\mathscr{A}' \subseteq \mathscr{A}$.*

Definition 6.5 (Compact spaces)**.** *Let X be a topological space. We say that X is **compact** if every open cover $\mathscr{A}$ of X contains a finite subcover.*

Definition 6.6 (Compact subspaces)**.** *Let X be a topological space, and let A be a subset of X. We say that A is compact in X if A is compact in the subspace topology.*

Lemma 6.2. *Let X be a topological space, and let A be a subspace of X. Then A is compact in X if and only if every cover of A by open subsets of X contains a finite subcollection that covers A.*

Theorem 6.12. *Let X be a compact space, and let A be a closed subset of X. Then A is compact in X.*

Theorem 6.13. *Let X be a Hausdorff space, and let K be a subset of X which is compact in X. Then K is closed in X.*

Theorem 6.14. *Let X be a compact space, Y a topological space and let $f : X \to Y$ be a surjective continious map. Then Y is compact.*

Lemma 6.3 (Tube lemma)**.** *Let X be a topological space, and let Y be a compact space. If $x \in X$ and U is an oppen set in the product space $X \times Y$ containing $\{x\} \times Y$, then there is a neighborhood W of x in X such that $W \times Y \subseteq U$*

Theorem 6.15. *Let $X_1, X_2, \ldots, X_n$ be compact spaces. Then the product space $X_1 \times X_2 \times \ldots \times X_n$ is compact.*

Theorem 6.16. *Let $\mathbb{R}$ be the set of real numbers equipped with the standard topology. Then every closed interval $[a, b] \in \mathbb{R}$ is compact in $\mathbb{R}$.*

Definition 6.7 (Bounded subsets)**.** *Let (X, d) be a metric space, and let A be a subset of X. We say that A is bounded if there is an $M \in \mathbb{R}$ such that $d(a_1, a_2) \leq M$ for all $a_1, a_2 \in A$.*

Theorem 6.17 (Heine- Borel)**.** *Let $\mathbb{R}^n$ be given the (Euclidian) metric topology and the Euclidian metric. A subset A of $\mathbb{R}^n$ if and only if it is closed and bounded.*

Theorem 6.18 (Generalized extreme value theorem)**.** *Let X be compact space, and let $f : X \to \mathbb{R}$ be a continious map where $\mathbb{R}$ is given the standard topology. Then there are $m, M \in X$ such that*

$$f(m) \leq f(x) \leq f(M)$$

for all $x \in X$.

7 The fundamental group

8 The fundamental group of the circle

9 References

References