Norwegian University of Science and Technology
Department of Mathematical Sciences

TMA4212
Numerical solution of differential equations by difference methods
Spring 2020

Solutions to exercise set 2

1 Use the idea of a false boundary: Extend the solution to include $u(-h)$. At the boundary $x = 0$ we now have two equations, the PDE and the boundary condition. Using central differences for both of them:

$$-\frac{U_{-1} - 2U_0 + U_1}{h^2} + \frac{U_1 - U_{-1}}{2h} = f_0$$
$$\frac{U_1 - U_{-1}}{2h} + U_0 = g_0$$

which, by elminating U_{-1} gives the following equation for the boundary point $m = 0$:

$$-\frac{2U_1 - 2(1 - h + h^2)U_0}{h^2} = f_0 - (1 + \frac{2}{h} g_0).$$

Implementation and testing is again left to you.

2 **a)** Let $V = H^1(0, 1)$. Multiply the equation by a test function $v \in V$ and integrate the second order term by parts to get the bilinear and linear forms

$$a(u, v) = \int_0^1 (u_x v_x + u_x v + uv)\, dx$$
$$F(v) = \int_0^1 f v\, dx.$$

The weak formulation then reads: find $u \in V$ such that $a(u, v) = F(v)$ holds for all $v \in V$.

b) We must prove that the bilinear form $a(\cdot, \cdot)$ is continuous and coercive, and that F is a continuous linear functional.

Continuity:
We want, for $u, v \in V$, the inequality

$$|a(u, v)| \le M \|u\|_{H^1} \|v\|_{H^1} \tag{1}$$

for some positive constant M.

We have

$$
\begin{aligned}
|a(u,v)| &= |\int_0^1 (u_x v_x + u_x v + uv)\,dx| \\
&\le |\int_0^1 (u_x v_x + uv)\,dx| + |\int_0^1 u_x v\,dx| \\
&= |\langle u, v\rangle_{H^1}| + |\int_0^1 u_x v\,dx| \\
&\le \|u\|_{H^1}\|v\|_{H^1} + \|u_x\|_{L^2}\|v\|_{L^2} \\
&\le 2\|u\|_{H^1}\|v\|_{H^1}.
\end{aligned}
$$

The second last inequality follows from using Schwarz' inequality on the H^1 and $L2$ inner products.

Coersivity:
We want, for $v \in V$,

$$a(v,v) \ge \alpha \|v\|_{H^1}^2$$

for some positive constant α.
Again, use our definition of the bilinear form to get

$$
\begin{aligned}
a(v,v) &= \int_0^1 (v_x^2 + v_x v + v^2)\,dx \\
&= \frac{1}{2}\int_0^1 (v_x + v)^2\,dx + \frac{1}{2}\int_0^1 (v_x^2 + v^2)\,dx \\
&\ge \frac{1}{2}\|v\|_{H^1}^2.
\end{aligned}
$$

The last inequality is due to that $\frac{1}{2}\int_0^1 (v_x + v)^2\,dx \ge 0$.

c) We follow section 2.3 in CC, but use the weak formulation for our problem. Write the discrete solution u and the discrete test function v as a linear combinations of the linear basis functions, $u(x) = \sum_i u_i \phi_i(x)$ and $v(x) = \sum_j v_j \phi_j(x)$. The discrete variational problem is: Find the vector of coefficients u such that for all basis functions ϕ_j the following holds

$$
\begin{aligned}
\int_0^1 \big(&(\sum_i u_i \phi_i'(x))\phi_j'(x) \\
&+ (\sum_i u_i \phi_i'(x))\phi_j(x) \\
&+ (\sum_i u_i \phi_i(x))\phi_j(x)\big)\,dx \\
&= \int_0^1 f(x)\phi_j(x)\,dx.
\end{aligned}
$$

Moving the sums outside the integral gives

$$\sum_i u_i \int_0^1 \big(\phi_i'(x)\phi_j'(x) + \phi_i'(x)\phi_j(x) + \phi_i(x)\phi_j(x)\big)\,dx = \int_0^1 f(x)\phi_j(x)\,dx.$$

In matrix form this is: find u such that

$$Au = F,$$

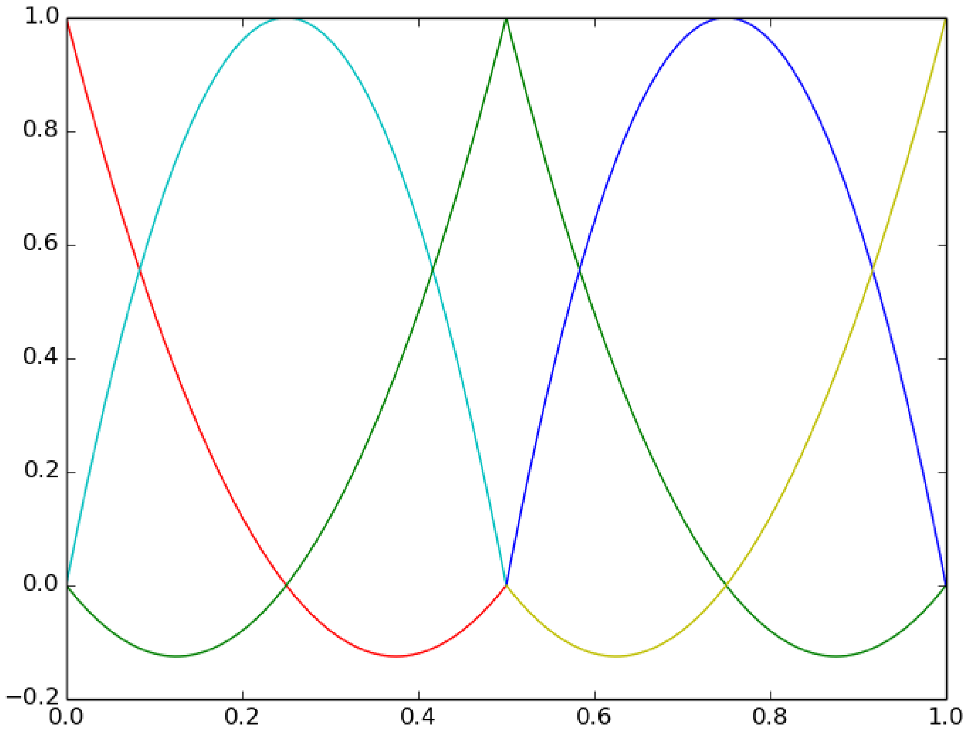
where the stiffness matrix A is given by $A_{ij} = \int_0^1 \left(\phi_i'(x)\phi_j'(x) + \phi_i'(x)\phi_j(x) + \phi_i(x)\phi_j(x)\right) dx$ and the load vector $F_j = \int_0^1 f(x)\phi_j(x)\, dx$.

To compute each entry in the stiffness matrix we sum over all the elements, and note that for linear elements on $(0, 1)$ element number k only contributes to entries k and $k + 1$. The element stiffness matrix is then given below. This is found by computing the above integral on one element (i.e. from x_k to x_{k+1}) and use the linear elements as described in CC.

$$\frac{1}{h}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix} + \frac{h}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

The element load vector is found in the same way as described in the notes.

d)



3 **a)**

b) The variational formulation of the problem is (using that the boundaries are zero)

$$\int_0^1 u_x v_x \, dx = \int_0^1 f v \, dx.$$

Then the discrete variational formulation is: Find the vector u such that for all (basis functions) $\phi_j \in X_h^2$

$$\sum_i u_i \int_0^1 \phi_i'(x)\phi_j'(x) \, dx = \int_0^1 f(x)\phi_j(x) \, dx.$$

To compute the element stiffness matrix we compute the above integral for one element only, and since only three of the basis functions are defined on the element, we get a 3x3 matrix.

On the reference element $\hat{K}$ defined on $[0,1]$ the $\mathcal{P}_2$ shape functions are given by

$$\begin{aligned}
\psi_0(\xi) &= 2(\xi - 1/2)(\xi - 1) \\
\psi_1(\xi) &= -4\xi(\xi - 1) \\
\psi_2(\xi) &= 2\xi(\xi - 1/2).
\end{aligned}$$

These are the cardinal functions on the grid points $(0, 0.5, 1)$.

Similarly, for element K, the shape functions ϕ_0^k(x), $\phi_1^k(x)$ and $\phi_2^k(x)$ are found in the same way, but with grid points $(x_k, x_k + h_k/2, x_{k+1})$.

For element K the element matrix is given by

$$\tilde{A}^k_{h_k,\alpha,\beta} = \int_{x_k}^{x_{k+1}} \frac{d\phi^k_\alpha}{dx}\frac{d\phi^k_\beta}{dx}\,dx.$$

We map from K to the reference element $\hat{K}$ by

$$\xi(x) = \Phi_k^{-1}(x) = \frac{x - x_k}{h_k}$$
$$x(\xi) = \Phi_k(x) = x_k + h_k\xi.$$

Then the shape functions on K can be written as

$$\phi^k_\alpha(x) = \psi(\xi(x)) = \psi_\alpha(\Phi_k^{-1}(x)), \qquad \alpha = 0, 1, 2.$$

By a change of variable in the integral above, we find

$$\begin{aligned}\tilde{A}^k_{h_k,\alpha,\beta} &= \int_{x_k}^{x_{k+1}} \frac{d\phi^k_\alpha}{dx}\frac{d\phi^k_\beta}{dx}\,dx \\ &= \int_0^1 \Big(\frac{d\psi_\alpha}{d\xi}\frac{d\Phi_k^{-1}}{dx}\Big)\Big(\frac{d\psi_\beta}{d\xi}\frac{d\Phi_k^{-1}}{dx}\Big)\frac{d\Phi_k}{dx}d\xi \\ &= \frac{1}{h_k}\int_0^1 \frac{d\psi_\alpha}{d\xi}\frac{d\psi_\beta}{d\xi}d\xi.\end{aligned}$$

Computing these integrals for $\alpha = 0, 1, 2$ and $\beta = 0, 1, 2$ yields the matrix

$$\tilde{A}^k_h = \frac{1}{h_k}\begin{bmatrix} \frac{7}{3} & -\frac{8}{3} & \frac{1}{3} \\ -\frac{8}{3} & \frac{16}{3} & -\frac{8}{3} \\ \frac{1}{3} & -\frac{8}{3} & \frac{7}{3} \end{bmatrix}$$

To assembly the complete matrix, a local-to-global mapping is needed. For α, β in element k the value $\tilde{A}^k_{h_k,\alpha,\beta}$ should be added to the index (i, j) (when working in one dimension), where $i = 2(k-1) + \alpha$, $j = 2(k-1) + \beta$.
Using the computed element matrix from above, and assuming that we use a equidistributed mesh ($h_k = h$) then gives

$$A_h = \frac{1}{h}\begin{bmatrix} \frac{7}{3} & -\frac{8}{3} & \frac{1}{3} & 0 & \cdots & 0 \\ -\frac{8}{3} & \frac{16}{3} & -\frac{8}{3} & 0 & \cdots & 0 \\ \frac{1}{3} & -\frac{8}{3} & \frac{14}{3} & -\frac{8}{3} & \frac{1}{3} & 0 \\ 0 & 0 & -\frac{8}{3} & \frac{16}{3} & \frac{-1}{8} & 0 \\ \cdots & & & & & \end{bmatrix}$$

For $f = 1$ the local load vector is given by

$$\tilde{F}^k_h = \begin{bmatrix} \int_{x_k}^{x_{k+1}} \phi^k_0\,dx \\ \int_{x_k}^{x_{k+1}} \phi^k_1\,dx \\ \int_{x_k}^{x_{k+1}} \phi^k_2\,dx \\ \end{bmatrix}$$

Do the same change of variable as above and compute the integrals of the three different shape functions on the reference element to get

$$\tilde{F}_h^k = \frac{1}{h} \begin{bmatrix} 1/6 \\ 2/3 \\ 1/6 \end{bmatrix}$$

Assembling the global vector in the same way as for the matrix gives

$$F_h = \frac{1}{h} \begin{bmatrix} 1/6 \\ 2/3 \\ 2/6 \\ 2/3 \\ 2/6 \\ \vdots \\ 2/3 \\ 1/6 \end{bmatrix}$$