Norwegian University of Science and Technology
Department of Mathematical Sciences

TMA4145 Linear Methods
Fall 2018

Exercise set 9: Solutions

Please justify your answers! The most important part is how you arrive at an answer, not the answer itself.

1 Show that point evaluation is a bounded linear functional on $C[a,b]$. That is, for some fixed $t_0 \in [a,b]$ define $f_{t_0} : C[a,b] \to \mathbb{C}$ by

$$f_{t_0}(x) = x(t_0), \quad x \in C[a,b],$$

and show that f_{t_0} is a bounded linear functional on $C[a,b]$.

Solution. By the definition of a bounded linear operator, we need to show that f_{t_0} is linear and that there is some $c > 0$ such that $|f_{t_0}(x)| \leq c\|x\|_\infty$.
Linear: If $\alpha, \beta \in \mathbb{C}$ and $x, y \in C[a,b]$, then

$$\begin{aligned} f_{t_0}(\alpha x + \beta y) &= (\alpha x + \beta y)(t_0) \\ &= \alpha x(t_0) + \beta y(t_0) \\ &= \alpha f_{t_0}(x) + \beta f_{t_0}(y), \end{aligned}$$

so f_{t_0} is linear.
Bounded: By definition, since the supremum is an upper bound, we have that

$$\|x\|_\infty = \sup_{t \in [a,b]} |x(t)| \geq |x(t_0)| = |f_{t_0}(x)|.$$

Hence we can choose $c = 1$, and f_{t_0} is bounded.

2 Let T be a bounded linear operator on a real Hilbert space X. Show that the operator norm of T can be expressed in terms of the inner product of X:

$$\|T\| = \sup\{\langle Tx, y\rangle : x, y \in X \ \text{ with } \ \|x\| = \|y\| = 1\}.$$

Solution. We will first show that

$$\sup\{\langle x, y\rangle : y \in X \ \text{ with } \ \|y\| = 1\} = \|x\| \ \text{ for all } \ x \in X. \tag{1}$$

By the Cauchy-Schwarz inequality we have

$$\langle x, y\rangle \le |\langle x, y\rangle| \le \|x\|\|y\| = \|x\|, \quad \text{for all} \;\; x, y \in X \,\text{with}\, \|y\| = 1.$$

It follows that

$$\sup\{\langle x, y\rangle : \, y \in X \;\; \text{with} \;\; \|y\| = 1\} \le \|x\| \;\; \text{for all} \;\; x \in X.$$

It remains to show the inequality

$$\sup\{\langle x, y\rangle : \, y \in X \;\; \text{with} \;\; \|y\| = 1\} \ge \|x\| \;\; \text{for all} \;\; x \in X.$$

This clearly holds when $x = 0$, since $\langle 0, y\rangle = 0$ for all $y \in X$. Now suppose $x \neq 0$. Let $y = \frac{x}{\|x\|}$ and notice that $\|y\| = 1$. We have

$$\langle x, y\rangle = \langle x, \frac{x}{\|x\|}\rangle = \frac{1}{\|x\|}\langle x, x\rangle = \frac{1}{\|x\|}\|x\|^2 = \|x\|.$$

The inequality follows.

We will now show that the norm of T can be expressed in terms of the innerproduct. We have

$$\begin{aligned} \|T\| &= \sup\{\|Tx\|, \, x \in X \;\; \text{with} \;\; \|x\| = 1\} \\ &= \sup\{\, \sup\{\langle Tx, y\rangle : \, y \in X, \, \|y\| = 1\} : \, x \in X, \, \|x\| = 1\} \quad \text{(equation (1))} \\ &= \sup\{\langle Tx, y\rangle : \, x, y \in X \;\; \text{with} \;\; \|x\| = \|y\| = 1\} \end{aligned}$$

$\boxed{3}$ Let c_f be the subspace of ℓ^2 that consists of all sequences with finitely many non-zero terms.

a) Show that best approximation fails for c_f.

b) Why does this not contradict the best approximation theorem?

Solution. a) Let $x = (1, 1/2, 1/3, ...)$. We start by showing that

$$\inf\{\|x - m\| : \, m \in c_f\} = 0.$$

Since the norm is always non-negative, we have that

$$\inf\{\|x - m\| : \, m \in M\} \ge 0.$$

We will show equality by constructing a sequence $\{m_n\}_{n \in \mathbb{N}}$ in c_f such that $\|x - m_n\| \to 0$. Let

$$m_n = (1, 1/2, 1/3, ..., 1/n, 0, 0, ...).$$

We have

$$\begin{aligned}
\lim_{n\to\infty} \|x - m_n\| &= \lim_{n\to\infty} \|(0, 0, ..., 1/(n+1), 1/(n+2), 1/(n+3), ...)\| \\
&= \lim_{n\to\infty} \sum_{k=n+1}^{\infty} \frac{1}{k^2} \\
&= \lim_{n\to\infty} \left(\sum_{k=1}^{\infty} \frac{1}{k^2} - \sum_{k=1}^{n} \frac{1}{k^2} \right) \\
&= \sum_{k=1}^{\infty} \frac{1}{k^2} - \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{k^2} \\
&= \|x\| - \|x\| \\
&= 0
\end{aligned}$$

which was what we needed to prove. Now, suppose there exist a sequence $m \in c_f$ such that $\|x - m\| = 0$. Then we have $m = x$, but x is not in c_f since it has no non-zero terms.

b) This does not contradict the best approximation theorem because the conditions for the theorem are not satisfied. Indeed, the subspace c_f is not closed. To see this, observe that that the sequence $\{m_n\}_{n\in\mathbb{N}}$ converges to x, but x is not in c_f.

$\boxed{4}$ Let M be a subspace of an inner product space $(X, \langle\cdot,\cdot\rangle)$. Show that the orthogonal complement $M^\perp$ is closed.

Solution. We want to show that $\overline{M^\perp} = M^\perp$, where $\overline{M^\perp}$ denotes the closure of $M^\perp$. We always have that a set is contained in its closure, so we only need to show that $\overline{M^\perp} \subset M$. Assume that $x \in \overline{M^\perp}$; we want to show $x \in M^\perp$. There exists a sequence (x_n) with $x_n \in M^\perp$ for each n such that $x_n \to x$ [1]. Now let $y \in M$. By assumption $\langle x_n, y\rangle = 0$ for any n, and we want to show that $\langle x, y\rangle = 0$. From problem 2 on problem set 5, the fact that $\lim_{n\to\infty} x_n = x$ implies that [2]

$$0 = \langle x_n, y\rangle \to \langle x, y\rangle \text{ as } n \to \infty,$$

and therefore $\langle x, y\rangle = 0$. Since y was *any* element from M, this shows that $x \in M^\perp$.

$\boxed{5}$ Let M be the plane of $\mathbb{R}^3$ given by $x_1 + x_2 + x_3 = 0$. Find the linear mapping that is the orthogonal projection of $\mathbb{R}^3$ onto this plane.

Solution. On page 68 in the notes, we defined the projection $P : \mathbb{R}^3 \to \mathbb{R}^3$ by decomposing each $x \in \mathbb{R}^3$ into $x = y + z$ using the projection theorem, where $y \in M$

[1] See discussion before example 3.1.3 in the notes.

[2] You may also prove this directly using the Cauchy Schwarz inequality.

and $z \in M^\perp$, and defining P by $Px = y$. So we need to find this y, and a natural starting point is to find $M^\perp$. Since $\mathbb{R}^3$ is 3-dimensional, M is 2-dimensional and $\mathbb{R}^3 = M \oplus M^\perp$ by the projection theorem, $M^\perp$ must be one-dimensional. It is not difficult to see that the vector $a = (1, 1, 1)$ belongs to $M^\perp$, since[3]

$$\langle x, a\rangle = x_1 + x_2 + x_3 = 0 \text{ if } x \in M.$$

Since $M^\perp$ is one-dimensional and contains a, it follows that $M^\perp = \{\lambda a : \lambda \in \mathbb{R}\}$. Now let $x = (x_1, x_2, x_3) \in \mathbb{R}^3$. We want to decompose $x = y + z$ with $y \in M$ and $z \in M^\perp$. Since $M^\perp = \{\lambda a : \lambda \in \mathbb{R}\}$, we must have that $z = \lambda a$ for some $\lambda \in \mathbb{C}$:

$$x = y + \lambda a,$$

so in terms of coordinates we have

$$(x_1, x_2, x_3) = (y_1, y_2, y_3) + (\lambda, \lambda, \lambda).$$

We solve this equation for (y_1, y_2, y_3):

$$(y_1, y_2, y_3) = (x_1 - \lambda, x_2 - \lambda, x_3 - \lambda).$$

Since $(y_1, y_2, y_3) \in M$ we must have that $0 = y_1 + y_2 + y_3 = x_1 - \lambda + x_2 - \lambda + x_3 - \lambda = x_1 + x_2 + x_3 - 3\lambda$. We may solve this for λ to get

$$\lambda = \frac{x_1 + x_2 + x_3}{3},$$

and inserting this back into our expression for (y_1, y_2, y_3) we find that

$$\begin{aligned}(y_1, y_2, y_3) &= (x_1 - \lambda, x_2 - \lambda, x_3 - \lambda)\\ &= \frac{1}{3}(2x_1 - x_2 - x_3, 2x_2 - x_1 - x_3, 2x_3 - x_1 - x_2).\end{aligned}$$

Since $Px = y$, this means that we have shown that

$$P(x_1, x_2, x_3) = \frac{1}{3}(2x_1 - x_2 - x_3, 2x_2 - x_1 - x_3, 2x_3 - x_1 - x_2).$$

$\boxed{6}$ (Exam 2017, problem 4) For $a = (a_n)_{n\in\mathbb{N}} \in \ell^\infty(\mathbb{R})$ we define the linear operator $T_a : \ell^2(\mathbb{R}) \to \ell^2(\mathbb{R})$ by

$$T_a(x_1, x_2, \dots) = (a_1x_1, 0, a_3x_3, 0, \dots), \quad x \in \ell^2(\mathbb{R}).$$

a) Show that T_a is bounded on $\ell^2(\mathbb{R})$.

b) Determine the operator norm of T_a.

c) ~~Show that the range of T_a is closed.~~

d) ~~Determine the orthogonal complement of $\ker(T_a)$.~~

[3] Remember that the inner product on $\mathbb{R}^3$ is just the usual dot product.

e) Determine for which sequences $a \in \ell^\infty(\mathbb{R})$ the operator T_a satisfies $T_a^2 = T_a$.

Solution.

a) $\|T_a x\|_2^2 = |a_1 x_1|^2 + |a_3 x_3|^2 + \cdots \leq \|(a_{2n-1})_{n\in\mathbb{N}}\|_\infty^2 \|x\|_2^2$ and hence

$$\|T_a x\|_2 \leq \|(a_{2n-1})_{n\in\mathbb{N}}\|_\infty \|x\|_2.$$

Here $(a_{2n-1})_{n\in\mathbb{N}}$ is the odd part of the sequence a, i.e. the sequence $(a_1, a_3, a_5, ...)$.

b) $\|T_a\| \leq \|(a_{2n-1})_{n\in\mathbb{N}}\|_\infty$, because

$$\|T_a\| = \sup_{\|x\|_2=1} \|T_a x\|_2 \leq \sup_{\|x\|_2=1} \left(\|(a_{2n-1})_{n\in\mathbb{N}}\|_\infty \|x\|_2\right) = \|(a_{2n-1})_{n\in\mathbb{N}}\|_\infty.$$

Hence $\|(a_{2n-1})_{n\in\mathbb{N}}\|_\infty$ is an upper bound for $\{\|T_a x\|_2 : \|x\|_2 = 1\}$. Now we show that it is the least upper bound for $\{\|T_a x\|_2 : \|x\|_2 = 1\}$. Namely, for every $\varepsilon > 0$ we need to show that there exists some $x^\varepsilon \in \ell^2$ with $\|x^\varepsilon\|_2 = 1$ such that

$$\|T_a x^\varepsilon\|_2 > \|(a_{2n-1})_{n\in\mathbb{N}}\|_\infty - \varepsilon.$$

For every $\varepsilon > 0$ there exists an index k_ε such that $|a_{2k_\varepsilon-1}| > \|(a_{2n-1})\|_\infty - \varepsilon$ (which follows from the definition of the supremum of the sequence (a_{2n-1})). Take $x^\varepsilon = (0, ..., 0, 1, 0, ...)$ where the 1 is in the $(2k_\varepsilon - 1)$th component. Then $\|T_a x_\varepsilon\|_2 = |a_{2k_\varepsilon-1}| > |(a_{2n-1})\|_\infty - \varepsilon$. Hence we have $\|T_a\| = \|(a_{2n-1})\|_\infty$.

c) The solution below relied on the extra assumption that the sequence a was bounded from below – this was not stated in the problem. ~~The range of T_a is clearly $\{x \in \ell^2 : (x_1, 0, x_3, 0, ...)\}$. Note that $\{x \in \ell^2 : (x_1, 0, x_3, 0, ...)\}$ is the kernel of a the operator P given by $Px = (0, x_2, 0, x_4, 0, ...)$. Furthermore P is bounded: $\|Px\|_2 \leq \|x\|_2$. Since the kernel of any bounded linear operator is closed, we get that $\ker(T) = \mathrm{ran}(T)$ is closed.~~

d) See above. ~~$\ker(T_a)$ is the subspace $\{x \in \ell^2 : (0, x_2, 0, x_4, 0, ...)\}$. By definition $\ker(\mathrm{T}_a)^\perp = \{y \in \ell^2 : \langle y, x\rangle = 0 \text{ for all } x \in \ker(T_a)\}$,i.e. we have $\ker(\mathrm{T}_a)^\perp = \{y \in \ell^2 : \sum_{i=1}^\infty x_{2i}$ expression $\sum_{i=1}^\infty x_{2i}\overline{y_{2i}} = 0$ for all $x \in \ell^2$ if and only if $y = (y_1, 0, y_3, 0, y_5, ...)$. Consequently, $\ker(T_a)^\perp = \{x \in \ell^2 : x = (x_1, 0, x_3, 0, x_5, ...)\}$.~~

e) $T_a^2 x = (a_1^2 x_1, 0, a_3^2 x_3, 0, ...)$ and thus $T_a^2 = T_a$ is equivalent to $a_i^2 = a_i$ for all $i = 1, 2, 3, ...$, which holds only for $a_{2i-1} \in \{0, 1\}$ for all $i = 1, 2, 3, ...$