Norwegian University of Science and Technology
Deptartment of Mathematical Sciences

TMA4190 Introduction to Topology Spring 2018

Solutions to exercise set 7

$\boxed{1}$ **a)** Let f and g be two smooth maps $Y \to X$. Let F be a homotopy from the identity map of X to the constant map $X \to \{x_0\}$ for some $x_0 \in X$. Then we can use F to define a homotopy from f to $Y \to \{x_0\}$ and a homotopy from $Y \to \{x_0\}$ to g. Setting these two homotopies together yields a homotopy H from f to g. It only remains to make sure that H is smooth. To achieve this we apply the technique used in the lecture. After composing with a smooth bump function, we can assume $F(x,t) = x$ for all $(x,t) \in X \times [0,1/4]$ and $F(x,t) = x_0$ for all $(x,t) \in X \times [3/4, 1]$. Then we can define H by

$$H\colon Y \times [0,1] \to X,\ (y,t) \mapsto \begin{cases} F(f(y), 2t) & t \in [0,1/2] \\ F(g(y), 2(1-t)) & t \in [1/2, 1]. \end{cases}$$

b) Let $Y = X$ and let $f\colon X \to X$ be the identity and $g\colon X \to \{x_0\} \subset X$ be the constant map for some point $x_0 \in X$. By assumption, f and g are homotopic. Hence X is contractible.

c) The map

$$F\colon \mathbb{R}^k \times [0,1] \to \mathbb{R}^k,\ (x,t) \mapsto (1-t)x$$

is a smooth homotopy from the identity map to the constant map $\mathbb{R}^k \to \{0\}$.

$\boxed{2}$ Let $f\colon S^1 \to X$ be a smooth map. Since X is contractible, there is a smooth homotopy F from the identity on X and a constant map $\{x_0\}$. The composition

$$S^1 \times [0,1] \to X,\ (x,t) \mapsto F(f(x),t)$$

defines a smooth homotopy from f to the constant map $S^1 \to \{x_0\}$.

An example which shows that the converse is false is given by $X = S^2$. The 2-sphere is simply connected, but it is not contractible. For example, the antipodal map is not homotopic to the identity. We will have to develop further techniques to be able to check all this.

$\boxed{3}$ For $k = 1$, the antipodal map is $(x,y) \mapsto (-x,-y)$. The map

$$F_1\colon S^1 \times [0,1] \to S^1,\ ((x,y),t) \mapsto \begin{pmatrix} \cos(\pi t) & -\sin(\pi t) \\ \sin(\pi t) & \cos(\pi t) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

is a smooth homotopy from the identity of S^1 to the antipodal map. To convince ourselves that $F_1(x,y,t)$ is an element in S^1, we can either just calculate its norm or observe that the matrix $\begin{pmatrix} \cos(\pi t) & -\sin(\pi t) \\ \sin(\pi t) & \cos(\pi t) \end{pmatrix}$ is an element in $O(2)$ for every t. Elements in $O(2)$ preserve the scalar product and hence the norm of vectors in $\mathbb{R}^2$.

For an arbitrary odd k, we have $S^k \subset \mathbb{R}^{k+1}$ and $k+1$ is even. Then we define a smooth homotopy from the identity in S^k to the antipodal map by

$$F_k \colon S^k \times [0,1] \to S^k,$$
$$((x_1, y_1), \ldots, (x_{(k+1)/2}, y_{(k+1)/2}), t) \mapsto (F_1(x_1, y_1, t), \ldots, F_1((x_{(k+1)/2}, y_{(k+1)/2}, t)$$

Again, for every t, $F_k(-,-,t) \colon \mathbb{R}^{k+1} \to \mathbb{R}^{k+1}$ is an element in $O(k+1)$ and preserves the norm on $\mathbb{R}^{k+1}$.

4 Given two points $x, y \in X$, we define the realtion $x \sim y$, and say x and y are path-connected, by: $x \sim y$ if and only if there is a smooth path $\gamma \colon [0,1] \to X$ with $\gamma(0) = x$ and $\gamma(1) = y$. We would like to show that $\sim$ is an equivalence relation. Therefor, we are going to link it to the homotopy relation.

Let $f \colon \{x\} \to X$, $f(x) = x$, and $g \colon \{x\} \to X$, $g(x) = y$. If $F \colon \{x\} \times [0,1] \to X$ is a smooth homotopy from f to g, then $\gamma(t) := F(x,t)$ is a smooth path from x to y. Conversely, if γ is a smooth path from x to y, then $F(x,t) := \gamma(t)$ is a smooth homotopy from f to g. Thus $x \sim y$ if and onlf if $f \sim g$.

Since homotopy is an equivalence relation, we see that path-connectedness is also an equivalence relation. Recall that the equivalence class $[x]$ of a point $x \in X$ is the set

$$[x] = \{y \in X : x \sim y\}.$$

A crucial feature of equivalence relations is that equivalence classes are either equal or disjoint, i.e. for any x and y in X we have either $[x] = [y]$ or $[x] \cap [y] = \emptyset$. We are going to use this fact in the following way: If we can show that every equivalence class $[x]$ is an open subset of X, then we know that every $[x]$ is also a closed subset. For, $X \setminus [x]$ is the union of all the other open classes and is therefore open itself (arbitrary unions of open sets are open).

So, given an arbitrary point $x_0 \in X$, we would like to show that $[x_0]$ is open. Let $x \in X$ be a point in $[x]$. Since X is a smooth manifold, there is a local parametrization $\phi \colon B_\epsilon(0) \to U$ with $\phi(0) = x$, where U is open in X and $B_\epsilon(0)$ is the open ball of radius ϵ in $\mathbb{R}^{\dim X}$. Given any $y \in U$, let $\phi^{-1}(y)$ be its preimage in $B_\epsilon(0)$. In $B_\epsilon(0)$, all points are path-connected to 0. Hence there is a smooth path

$$\gamma \colon [0,1] \to B_\epsilon(0),\ t \mapsto t \cdot \phi^{-1}(y)$$

with $\gamma(0) = 0$ and $\gamma(1) = \phi^{-1}(y)$. Since ϕ is a diffeomorphism, the composite $\phi \circ \gamma$ is a smooth path from x to y in X, i.e. $x \sim y$.

This shows that U is contained in $[x_0]$. Thus $[x_0]$ is an open subset in X, since every point $x \in [x_0]$ has an open neighborhood in X which is completely containted in $[x_0]$.

Thus $[x_0]$ is a nonempty, open and closed subset of X. Since X is connected, this implies $[x_0] = X$. Thus X is path-connected.