Stochastic Modelling

isakhammer

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Contents

1 Lecture 1

1.1 Practical Information

Two projects

- The projects count 20% and exam 80%.
- Must be done with two people.
- If you want to do statistics is it worth learning R.

Course Overview

- Markov chains for discret time and discrete outcome.
  - Set of states and discrete time points.
  - Transition between states
  - Future depends on the present, but not the past.
- Continious time Markoc chains. (continious time and discrete toutcome.
- Brownian motion and Gaussian processes (continionus time and continious outcome.)

1.2 Mathematical description

Definition 1.1. *A **stochastic process** $\{x(t), t \in T\}$ is a family of random variables, where T is a set of indicies, and $X(t)$ is a random variable for each value of t.*

1.3 Recall from Statistics Course

A random experiment is perfomed the outcome of the experiment is random.

- THe set of possible outcomes is the **sample space** ω
  - An **event** $A \subset \omega$ if the outcome is contained in A
  - The **complement** of an event A is $A^c = \omega \setminus A$
  - The **null event** $\emptyset$ is the empty set $\emptyset = \omega \setminus \omega$

1.3.1 Combining Event

Let A and B be events

- The **union** $A \cup B$ is the event that at least one of A and B occur.
- the **intersection** $A \cap B$ is the event that both A and B occur.

The events $A_1, A_2, \ldots$ are called disjoint (or **mutually exclusive**) if $A_i \cap A_j = \emptyset$ for $i \neq j$

1.3.2 Probability

Pr is called a probability on ω if

- $\Pr\{\omega\} = 1$
- $0 \leq P\{A\} \leq 1$ for all events A
- For $A_1, A_2, \ldots$ that are mutually exclusive

$$P\left\{\bigcup_{i=1}^{\infty} A_i\right\} = \sum_{i=1}^{\infty} P\{A_i\}$$

We call $P\{A\}$ the probability of A.

1.3.3 Law of total probability

Let $A_1, A_2, \ldots$ be a partition of ω ie

- $\omega = \bigcup_{i=1}^{\infty} A_i$
- $A_1, A_2, A_3, \ldots$ are mutually exclusive.

Then for any event B

$$P\{B\} = \sum_{i=1}^{\infty} P\{B \cap A_i\}$$

This concept is very important.

1.3.4 Independence

Event A and B are independent of

$$P\{A \cap B\} = P\{A\} P\{B\}$$

Events $A_1, \ldots, A_n$ are independent if for any subset

$$P\left\{\bigcap_{j=1}^{k} A_{i_j}\right\} = \prod_{j=1}^{k} P\{A_{i_j}\}$$

In this case $P\{\bigcap_{i=1}^{n} A_1\} = \prod_{i=1}^{n} P\{A_i\}$

1.3.5 Random Variables

Definition 1.2. *A* ***random variable*** *is a real-vaued function on the sample space. Informally: A random variable is a real valued variable that takes on its value by chance.*

> **Example.**
>
> - Throw two dice. X = sum of the two dice
> - Throw a coin. X is 1 for heads and X is 0 for tails.

1.3.6 Notation for random variables

We use

- upper case letters such at X, Y and Z to represent random variables.
- lower case letters as x, y, z to denote the real-valued realized value of a the random variable.

Expression such as $\{X \leq x\}$ denators the event that X assumes a valye less than or earl to the real number x.

1.3.7 Discrete random variables

The random variable X is **discrete** if it has a finite or countablle number of possible outcomes $x_1, x_2, \ldots$

- The **probability mass function** $p_x(x)$ is given by

  $$p_x(x) = P\{X = x\}$$

  and satisfies

  $$\sum_{i=1}^{\infty} p_x(x_i) = 1 \quad \text{and} \quad 0 \leq p_x(x_i) \leq 1$$

- The **cumulative distribution function** (CDF) a of X can be written

  $$F_x(x) = P\{X \leq x\} = \sum_{i:x_i \leq x} p_x(x_i)$$

1.3.8 CFD

The CDF of X may also be called the **distrobution function** of X

Let $F_x(x)$ be the CDF of X, then

- $F_x(x)$ is monetonaly increasing.
- F_x is a stepfunction, which is a pieace-wise constant with jumps at x_i.
- $\lim_{x\to\infty} F_x(x) = 1$
- $\lim_{x\to-\infty} F_x(x) = 0$

1.3.9 Continious random vairbales

A **continious** random variables takes value o a continious scale.

- The CDF, $F_x(x) = P(X \leq x)$ is continious.
- The **probability density function** (PDF) $f_x(x) = F_x'(x)$ can be used to calculate probablities

$$\begin{aligned} Pr\{a < X < b\} &= Pr\{a \leq X < b\} = Pr\{a < X \leq b\} \\ &= Pr\{a \leq X \leq b\} = \int_a^b f_x(x)\,dx \end{aligned}$$

1.3.10 Important properties

- CDF:
  - Monotonely increaing
  - continious
  - $\lim_{x\to\infty} F_x = 1$ and $\lim_{x\to-\infty} F_x(x) = 0$
- PDF
  - $f_x(x) \geq 0$ for $x \in \mathbb{R}$
  - $\int_{-\infty}^{\infty} f_x(x)\,dx = 1$

1.3.11 Expectation

Let $g : \mathbb{R} \to \mathbb{R}$ be a function and X be a random variable.

- If X is discrete, the expected value of $g(X)$ is

$$E[g(X)] = \sum_{x:p_x(x)>0} g(x)\,p_x(x)$$

- If X is continous, the expected value of $g(X)$ is

$$E[g(X)] = \int_{-\infty}^{\infty} g(x)\,f_x(x)\,dx$$

1.3.12 Variance

The variance of the random variable X is

$$Var[X] = E\left[(X - E[X])^2\right] = E\left[X^2\right] - E[X]^2$$

Important properties of expectation and variance.

- Expectations is linear

$$E\left[aX+bY+c\right]=aE\left[X\right]+bE\left[Y\right]+c.$$

- Variance scales quadratically and is invaraient to the addition of constants

$$Var\left[aX+b\right]=a^{2}Var\left[X\right]$$

- fir independent stochastic variables.

$$Var\left[X+Y\right]=Var\left[X\right]+Var\left[Y\right]$$

1.3.13 Joint CDF

If (X,Y) is a pair for random variables, their **joint comulative distribution function** is given by

$$F_{X,Y}=F\left(x,y\right)=Pr\left\{X\leq x\cap Y\leq y\right\}$$

.

1.3.14 Joint distrubution for discrete random variables

If X and Y are discrete, the **joint probability mass function** $p_{x,y}=Pr\left\{X=x,Y=y\right\}$. can be used to compute probabilities

$$Pr\left\{a<X<b,c<Y\leq d\right\}=\sum_{a<x\leq b}\sum_{c<y\leq d}p_{X,Y}\left(x,y\right)$$

1.3.15 Joint distrubution for continous random variables

If X and Y are continious the **joint probability density function** $f_{X,Y}\left(x,y\right)=f\left(x,y\right)=\frac{\partial^{2}}{\partial x\partial y}F\left(x,y\right)$ can be used to compute probabilities

$$Pr\left\{a<X\leq b,\quad c<Y\leq d\right\}=\int_{a}^{b}\int_{c}^{d}f\left(x,y\right)dxdy$$

1.3.16 Independence

The random variables X and Y are independent if

$$Pr\left\{X\leq a,Y\leq b\right\}=Pr\left\{X\leq a\right\}\cdot Pr\left\{Y\leq b\right\},\quad\forall a,b\in\mathbb{R}$$

In terms of CDFs: $F_{X,Y}(a,b)=F_{X}\left(a\right)\cdot F_{Y}\left(b\right)\quad\forall a,b\in\mathbb{R}$

Thus we have

- $p_{X,Y}\left(x,y\right)=p_{X}\left(x\right)\cdot p_{Y}\left(Y\right)$ for discrete random variables
- $f_{X,Y}\left(x,y\right)=f_{X}\left(x\right)\cdot f_{Y}\left(Y\right)$ for continuous random variables.