TMA 4190 Introduction to Topology
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Lecture 12[1]

12. Transversality of submanifolds

Today, we are going to study some important special cases of transversality.

First, transversality is in fact a generalization of Regularity:

Regular vs Transversal

When Z is just a single point z, its tangent space is the zero subspace of $T_z(Y)$. Thus f is transversal to $\{z\}$ if $df_x(T_x(X)) = T_z(Y)$ for all $x \in f^{-1}(z)$. This is exactly what it means to say that z is a regular value of f. So transversality includes the notion of regularity as a special case.

The second one tells us how we should actually think of and visualize transversality. Roughly speaking, we want to know how the image of f and Z meet in Y:

Intersection of submanifolds

The most important situation is the transversality of the inclusion map i of one submanifold $X \subset Y$ with another submanifold $Z \subset Y$.
To say a point $x \in X$ belongs to the preimage $i^{-1}(Z)$ simply means that x belongs to the intersection $X \cap Z$. Also, the derivative $di_x \colon T_x(X) \to T_x(Y)$ is merely the inclusion map of $T_x(X)$ into $T_x(Y)$. So $i \pitchfork Z$ **if and only if**, for every $y \in X \cap Z$,

$$\mathbf{T_y(X) + T_y(Z) = T_y(Y)}. \tag{1}$$

Notice that this equation is symmetric in X and Z. When it holds, we shall say that the **two submanifolds X and Z are transversal**, and write $X \pitchfork Z$.

Warning: For equation (1) to be true, it is **not sufficient** that $\dim T_x(X) + \dim T_x(Z) = \dim T_x(Y)$. The two subspaces must span together all of $T_x(Y)$.

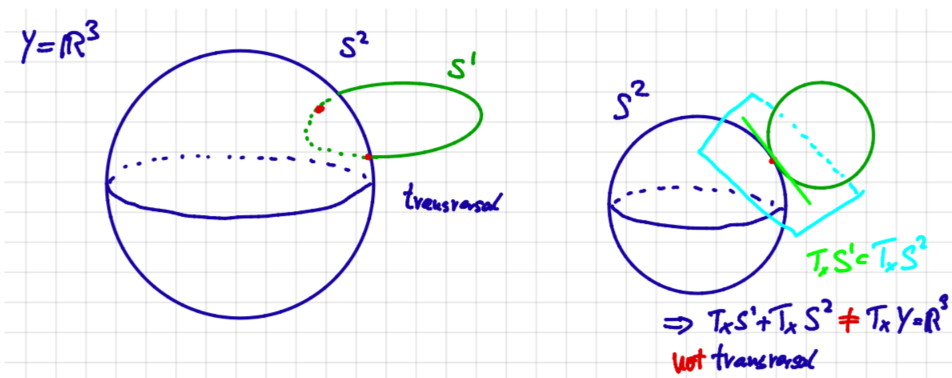
The transversality theorem for this specialize case then says:

[1]Following the books of Guillemin and Pollack: Differential Topology, and Milnor: Topology from the differentiable viewpoint.

Intersection of transversal submanifolds

The intersection of **two transversal submanifolds** X and Z of Y is again a submanifold. Moreover, the codimensions in Y satisfy

$$\operatorname{codim}(X \cap Z) = \operatorname{codim} X + \operatorname{codim} Z.$$



The additivity of codimensions follows from the codimension formula of the Transversality Theorem:

$$\begin{aligned}
\operatorname{codim} i^{-1}(Z) \text{ in } X &= \operatorname{codim} Z \text{ in } Y \\
\Rightarrow \dim X - \dim X \cap Z &= \dim Y - \dim Z \\
\Rightarrow \dim Y - \dim X \cap Z &= (\dim Y - \dim Z) + (\dim Y - \dim X) \\
\Rightarrow \operatorname{codim} X \cap Z &= \operatorname{codim} Z + \operatorname{codim} X.
\end{aligned}$$

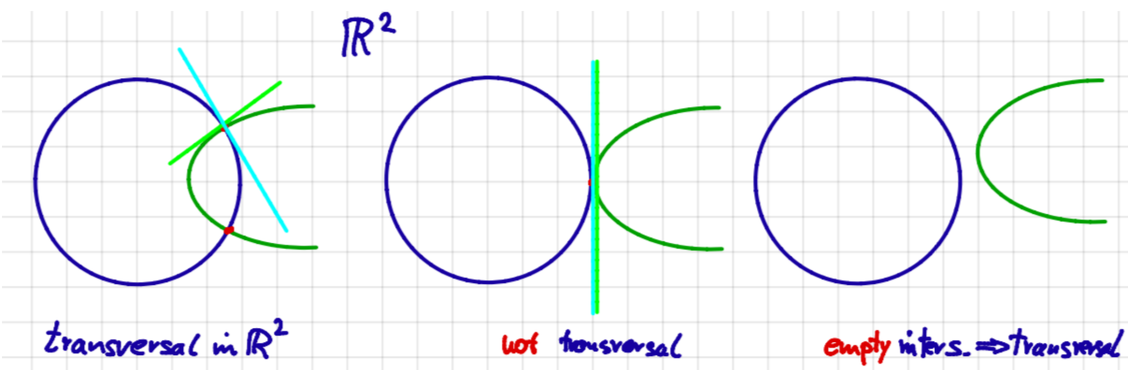
Intersect as a little as possible

We have just seen that two manifolds intersect transversally if their tangent spaces together span the whole ambient space. A different way to think of transversality is: Two manifolds intersect transversally if they intersect as little as possible at every point. And we **measure the degree of intersection in terms of tangent spaces**: If two submanifolds intersect, then they transversally if the intersection of their tangent spaces in the ambient space is minimal.

Note that the **converse of the Transversality Theorem is not true**. Weh ave seen a simple example last time: the submanifolds $X\{(x,y) \in \mathbb{R}^2 : y = x^2\}$ and $Z = \{(x,y) \in \mathbb{R}^2 : y = 0\}$ do **not intersect transversally at** 0 in $Y = \mathbb{R}^2$, but their intersection $X \cap Z = \{0\}$ is **a zero-dimensional manifold**. However, there do, of course, exist intersections which are not transversal and where the intersection is not a manifold. See the example below!

Empty intersections are transversal

It is useful to note that any smooth map $f\colon X \to Y$ whose image does not meet a submanifold Z of Y, i.e. $f^{-1}(Z) = \emptyset$, is transversal to Z for trivial reasons. For in this case **there is no condition to be satisfied**. In particular, two submanifolds which do not intersect at all, are transversal. Moreover, if $f\colon X \to Y$ is a **submersion**, then f is transversal to any submanifold Z of Y, since then $\operatorname{Im}(df_x) = T_{f(x)}(Y)$ for every x.



The ambient space matters

It is important to note that the transversality of X and Z also depends on the ambient space Y. For example, the two coordinate axes intersect transversally in $\mathbb{R}^2$, but not when considered to be submanifolds of $\mathbb{R}^3$. In general, if the dimensions of X and Z do not add up to at least the dimension of Y, then they can only intersect transversally by not intersecting at all. For example, if X and Z are curves in $\mathbb{R}^3$, then $X \pitchfork Y$ if and only if $X \cap Y = \emptyset$.

Let us have a look at an example:

Example

In $Y = \mathbb{R}^3$, we consider the two submanifolds

$$X = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 - z^2 = 1\}$$

and the sphere

$$Z_a = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = a\}.$$

We would like to understand for which a these two submanifolds intersect transversally in Y.
Therefore, we need to determine the tangent space of X and Z_a at points where they intersect. We observe that $X = f^{-1}(0)$ for the map

$$f\colon \mathbb{R}^3 \to \mathbb{R},\ (x,y,z) \mapsto x^2 + y^2 - z^2 - 1$$

and $Z_a = g^{-1}(0)$ for the map

$$g\colon \mathbb{R}^3 \to \mathbb{R},\ (x,y,z) \mapsto x^2 + y^2 + z^2 - a.$$

Since 0 is a regular value of f, the tangent space to X at a point $p = (x,y,z)$ is the kernel of the derivative of f at p (expressed as a matrix in the standard basis)

$$df_p = (2x, 2y, -2z)\colon \mathbb{R}^3 \to \mathbb{R}.$$

Hence the tangent space to X at $p = (x,y,z)$ is

$$T_p(X) = \mathrm{Ker}\,(df_p) = \mathrm{span}(\{(z,0,x),(0,z,y)\}) \subset \mathbb{R}^3.$$

Similarly, since 0 is a regular value of g, the tangent space to Z_a at a point $p = (x,y,z)$ is the kernel of the derivative of g at p (expressed as a matrix in the standard basis)

$$dg_p = (2x, 2y, 2z)\colon \mathbb{R}^3 \to \mathbb{R}.$$

Hence the tangent space to Z_a at $p = (x,y,z)$ is

$$T_p(Z_a) = \mathrm{Ker}\,(dg_p) = \mathrm{span}(\{(-z,0,x),(0,-z,y)\}) \subset \mathbb{R}^3.$$

Now X and Z intersect in the points $p = (x,y,z)$ which satisfy

$$x^2 + y^2 - z^2 - 1 = 0 = x^2 + y^2 + z^2 - a.$$

Subtracting both equations yields the condition

$$2z^2 = a - 1. \tag{2}$$

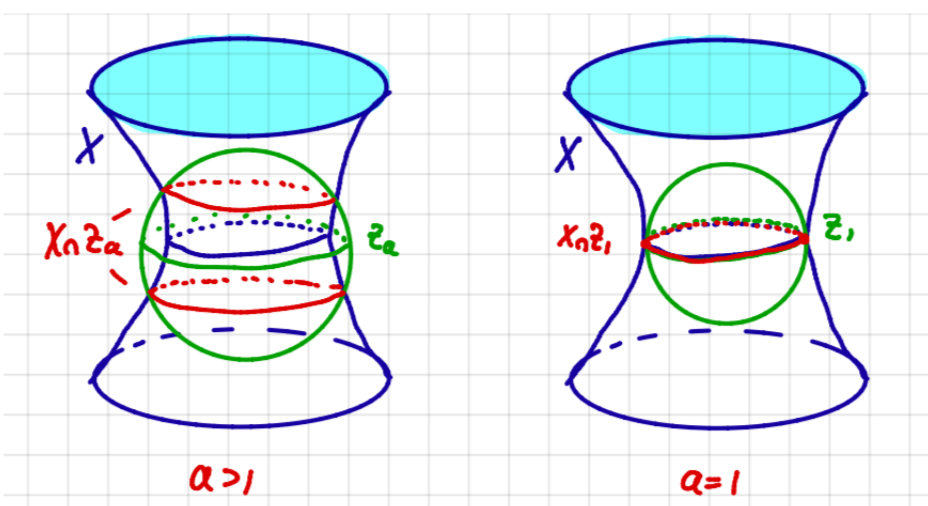
This gives us three cases for the intersection $X \cap Z_a$:

- **If** $a < 1$, then X and Z_a do not intersect, since there is no z which can satisfy condition (2): $X \cap Z_a = \emptyset$.

- **If** $a = 1$, then X and Z_1 intersect in the circle with radius 1 in the xy-plane in $\mathbb{R}^3$ with the origin as center, i.e.
$$X \cap Z_1 = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 = 1 \text{ and } z = 0\}.$$
- **If** $a > 1$, then X and Z_a intersect in two disjoint circles with lie in the planes parallel to the xy-plane in $\mathbb{R}^3$ with z-coordinate $z = \pm\sqrt{(a-1)/2}$:
$$X \cap Z_a = \{(x,y,z) \in \mathbb{R}^3 :$$
$$x^2 + y^2 = \frac{a+1}{2} \text{ and } z = \pm\sqrt{(a-1)/2}\}.$$

Now we need to check transversality (recall $T_p(\mathbb{R}^3) = \mathbb{R}^3$ at every p):

- **If** $a < 1$, then the intersection is empty and therefore **transversal**.
- **If** $a = 1$, then $T_p(X)$ and $T_p(Z_1)$ span the xy-plane in $\mathbb{R}^3$, and not all of $\mathbb{R}^3$, at every $p \in X \cap Z_1$. Thus the intersection is **not transversal**.
- **If** $a > 1$, let $p = (x,y,z) \in X \cap Z_a$. Then $T_p(X)$ and $T_p(Z_a)$ together span all of $\mathbb{R}^3$, for the vector $(-z,0,x) \in T_p(X)$ is not a linear combination of $(z,0,x)$ and $(0,z,y)$ $(z \neq 0)$. Since $T_p(Z_a)$ is 2-dimensional, this shows $Tp(X) + T_p(Z_a) = \mathbb{R}^3$ at every $p \in X \cap Z_a$. Thus the intersection is **transversal**.



Here is an example of an intersection which is not transversal and where the intersection is not a manifold:

Non-transversal intersection which is not a manifold

Let $Y = \mathbb{R}^3$ and let Z be the hyperplane defined by

$$Z = \{(x,y,z) \in \mathbb{R}^3 : x = 1\}$$

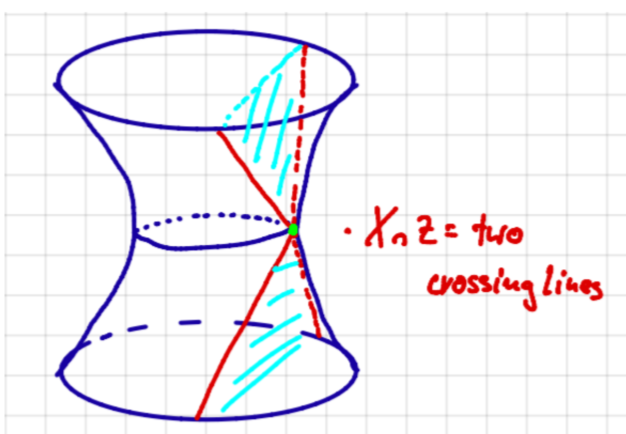
and let X be the hyperboloid defined by

$$X = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 - z^2 = 1\}.$$

The intersection of X and Z is given by the points satisfying $x = 1$ and $x^2 + y^2 - z^2 = 1$, i.e. all points such that $x = 1$ and $y^2 = z^2$. This means

$$X \cap Z = \{(x,y,z) \in \mathbb{R}^3 : x = 1,\ y = \pm z\}.$$

We have seen in one of the first lectures that a space consisting of two lines crossing each other is not a manifold. The intersection point, here the point $p = (1,0,0)$ does not have a neighborhood in $X \cap Z$ which is diffeomorphic to an open subset in Euclidean space. **Thus $X \cap Z$ is not a manifold**.
As a reality check, let us look at the tangent spaces to X and Z at p: Since Z is a parallel translate of a vector subspace of $\mathbb{R}^3$, we see that $T_p(Z)$ is the yz-plane in $\mathbb{R}^3$ (all points with $x = 0$). The tangent space to X was calculated in the previous example (and in an exercise). At $p = (1,0,0)$, $T_p(X)$ is the vector subspace in $\mathbb{R}^3$ spanned by the vectors $(0,1,0)$ and $(0,0,1)$. In other words, $T_p(X)$ is the xy-plane in $\mathbb{R}^3$. Thus $T_p(Z)$ and $T_p(X)$ do **not span** $T_p(Y) = \mathbb{R}^3$. (The problem here is that Z "is" the tangent plane to X at p.)



Codimension Formula revisited

Another way to rephrase the **codimension formula** is to say that when X is locally cut out by k independent functions and Z is locally cut out by l independent functions, then $X \cap Z$ is locally cut out by $k+l$ independent functions.

In fact, we can reprove the theorem by using independent functions:

Let y be a point in $X \cap Z \subseteq Y$. Around y, the submanifold X is cut out of Y by $k = \operatorname{codim} X$ independent functions, i.e. there is an open neighborhood $U \subseteq Y$ around y and k independent functions

$$f_1, \ldots, f_k \colon U \to \mathbb{R}$$

such that $X \cap U$ is defined by the vanishing of the f_i:

$$X \cap U = \{u \in U : f_1(u) = \cdots = f_k(u) = 0\}.$$

The independence of the f_i implies that 0 is a regular value of $f = (f_1, \ldots, f_k) \colon U \to \mathbb{R}^k$. In particular,

$$df_x \colon T_x(Y) \to \mathbb{R}^k \text{ is surjective.} \tag{3}$$

By the corollary to the Preimage Theorem we know

$$T_y(X) = \operatorname{Ker}(df_y) \subseteq T_Y(Y).$$

Then (3) implies

$$\dim \operatorname{Ker}(df_y) = \dim T_x(X) = \dim T_x(Y) - k.$$

Similarly, around y, the submanifold Z is cut out by $l = \operatorname{codim} Z$ independent functions, i.e. there is an open neighborhood $V \subseteq Y$ around y and l independent functions

$$g_1, \ldots, g_l \colon V \to \mathbb{R}$$

such that $Z \cap V$ is defined by the vanishing of the g_i:

$$Z \cap V = \{v \in V : g_1(v) = \cdots = g_l(v) = 0\}.$$

The independence of the g_i means that 0 is a regular value of $g = (g_1, \ldots, g_l) \colon V \to \mathbb{R}^l$. In particular,

$$dg_y \colon T_y(Y) \to \mathbb{R}^l \text{ is surjective.} \tag{4}$$

The tangent space to Z at y is

$$T_y(Z) = \operatorname{Ker}(dg_y) \subseteq T_Y(Y).$$

Then (4) implies

$$\dim \operatorname{Ker}(dg_y) = \dim T_y(Z) = \dim T_y(Y) - l.$$

We set $W := U \cap V$ which is an open neighborhood of y. Then, arond y, $X \cap Z$ is locally cut out by the combined collection of $k+l$ functions $f_1, \dots, f_k, g_1, \dots, g_l$, i.e.

$$\begin{aligned}&(X \cap Z) \cap W \\ &= \{w \in W : f_1(w) = \cdots = f_k(w) = g_1(w) = \cdots = g_l(w) = 0\}.\end{aligned}$$

We write h for the collection of functions f and g:

$$h = (f_1, \dots, f_k, g_1, \dots, g_l) \colon W \to \mathbb{R}^{k+l}.$$

The derivative of h at y is

$$dh_y \colon T_y(Y) \colon \mathbb{R}^{k+l},\ v \mapsto dh_y(v) = (df_y(v), dg_y(v)).$$

Now we want to relate the independence of the f_i's and g_i's to transversality:

As vector subspaces of $T_y(Y)$, $\operatorname{Ker}(df_y)$ and $\operatorname{Ker}(dg_y)$ satisfy the dimension formula

$$\begin{aligned}&\dim \operatorname{Ker}(df_y) + \dim \operatorname{Ker}(dg_y) \\ &= \dim(\operatorname{Ker}(df_y) + \operatorname{Ker}(dg_y)) + \dim(\operatorname{Ker}(df_y) \cap \operatorname{Ker}(dg_y)).\end{aligned}$$

From (3) and (4) we get that this equation is equivalent to

$$\begin{aligned}&\dim T_y(Y) - k + \dim T_y(Y) - l \\ &= \dim(\operatorname{Ker}(df_y) + \operatorname{Ker}(dg_y)) + \dim(\operatorname{Ker}(df_y) \cap \operatorname{Ker}(dg_y)).\end{aligned} \tag{5}$$

Hence the left hand side is $2 \dim T_y(Y) - (k+l)$. For the right hand side, we have

$$\dim(\operatorname{Ker}(df_y) + \operatorname{Ker}(dg_y)) \leq \dim T_y(Y) \tag{6}$$

and

$$\begin{aligned}&\dim T_y(Y) - \dim(\operatorname{Ker}(df_y) \cap \operatorname{Ker}(dg_y)) \leq k + l, \\ &\text{i.e. } \dim(\operatorname{Ker}(df_y) \cap \operatorname{Ker}(dg_y)) \geq \dim T_y(Y) - (k+l).\end{aligned} \tag{7}$$

Hence, given (5), the two inequalities (6) and (7) imply

$$\dim(\operatorname{Ker}(df_y) + \operatorname{Ker}(dg_y)) = \dim T_y(Y) \tag{8}$$

$$\iff \dim(\operatorname{Ker}(df_y) \cap \operatorname{Ker}(dg_y)) = \dim T_y(Y) - (k+l). \tag{9}$$

Now the first equation (8) means exactly that X and Z are **transversal** in Y, while the second equation (9) is true if and only if $d(h)_y$ is surjective, i.e. if and only if the $k+l$ functions $f_1, \dots, f_k, g_1, \dots, g_l$ are **independent**.

We are going to exploit what we just observed a bit further. Let us keep the above notation. **Now we assume again that X and Z meet transversally in Y.** Then 0 is a regular value of h. This implies that the tangent space to $X \cap Z$ at y equals $\mathrm{Ker}\,(dh_y)$. For $v \in T_y(Y)$, we have $dh_y(v) = 0$ if and only if both $df_y(v) = 0$ and $dg_y(v) = 0$. Thus $\mathrm{Ker}\,(dh_y)$ is the intersection of the kernel of $\mathrm{Ker}\,(df_y)$ and $\mathrm{Ker}\,(dg_y)$ in $T_y(Y)$:

$$\mathrm{Ker}\,(dh_y) = \mathrm{Ker}\,(df_y) \cap \mathrm{Ker}\,(dg_y) \text{ in } T_y(Y).$$

Thus we have proved the following useful fact:

Tangent space of intersections

If X and Z are submanifolds which meet transversally in Y, then the tangent space to the intersection $X \cap Z$ is the intersection of the tangent spaces, i.e.

$$T_y(X \cap Z)) = T_y(X) \cap T_y(Z) \text{ for all } y \in X \cap Z.$$

In the exercises for this week we prove a generalization of this fact to the preimage of a submanifold Z under a smooth map f when $f \pitchfork Z$:

Tangent space of preimages

Let $f\colon X \to Y$ be a map transversal to a submanifold Z in Y. Then $T_x(f^{-1}(Z))$ is the preimage of $T_{f(x)}(Z)$ under the linear map $df_x\colon T_x(X) \to T_{f(x)}(Y)$:

$$T_x(f^{-1}(Z)) = (df_x)^{-1}(T_{f(x)}(Z)).$$

A famous example of transversal intersections is given by Brieskorn Manifolds.

Exotic Spheres

Consider the following intersections in $\mathbb{C}^5 \setminus \{0\}$:

$$\begin{aligned} S^7_k =& \{z_1^2 + z_2^2 + z_3^2 + z_4^3 + z_5^{6k-1} = 0\} \\ & \cap \{|z_1|^2 + |z_2|^2 + |z_3|^2 + |z_4|^2 + |z_5|^2 = 1\}. \end{aligned}$$

In this week's exercises, we show that this is a transversal intersection. One can show that, for each value $k = 1, \ldots, 28$, S^7_k is a smooth manifold which is homeomorphic to S^7. But none of these manifolds are diffeomorphic. These are so called **exotic 7-spheres** were constructed by **Brieskorn** and represent each of the 28 diffeomorphism classes on S^7. That such exotic 7-spheres is a famous and groundbreaking result of **Milnor**. Milnor's work started an amazing story about the diffeomorphic structures on spheres which culminated in the solution of the **Kerviare Invariant One Problem** by **Hill, Hopkins and Ravenel** in 2009.