NTNU
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Department of Mathematical Sciences

Examination paper for **TMA4265 Stochastic Modeling**

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Examination date: August , 2019

Examination time (from–to): 09:00-13:00

Permitted examination support material: C:

- Calculator CITIZEN SR-270X, CITIZEN SR-270X College, HP30S, Casio fx-82ES PLUS with empty memory.
- Tabeller og formler i statistikk, Tapir forlag.
- K. Rottmann: Matematisk formelsamling.
- Bilingual dictionary.
- One yellow, stamped A5 sheet with own handwritten formulas and notes (on both sides).

Other information:
Note that all answers must be justified.
All ten subproblems are equally weighted.

Language: English

Number of pages: 3

Number of pages enclosed: 4

Checked by:

Informasjon om trykking av eksamensoppgave
Originalen er:
1-sidig ☐ **2-sidig** ☒
sort/hvit ☒ **farger** ☐
skal ha flervalgskjema ☐

Date Signature

Problem 1

a) Consider the Markov chain defined by the following transition matrix:

$$\mathbf{P} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{pmatrix} 0.6 & 0.4 & 0 \\ 0.3 & 0.6 & 0.1 \\ 0 & 0.9 & 0.1 \end{pmatrix} \end{array},$$

where matrix elements $P(k,l) = P(X_t = l | X_{t-1} = k)$, $X_t \in \{1,2,3\}$.

Calculate $P(X_2 = 1 | X_0 = 1)$.

Calculate $P(X_1 = 1 | X_0 = 1, X_2 = 1)$.

b)

Make rough sketches of two possible realizations from the Markov chain defined by the following transition matrix:

$$\mathbf{P} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{pmatrix} 0.3 & 0 & 0.7 \\ 0 & 1 & 0 \\ 0.5 & 0.1 & 0.4 \end{pmatrix} \end{array},$$

where matrix elements $P(k,l) = P(X_t = l | X_{t-1} = k)$, $X_t \in \{1,2,3\}$.

Starting at $X_0 = 1$, calculate the expected number of time steps until absorption in state 2.

Problem 2

We study a game with two players (A and B). If player A wins a round, he gets 1 krone from player B. Otherwise player B gets 1 krone from player A. Player A wins a round with probability $p = 0.6$, and loses a round with probability $q = 0.4$. The monetary holding of player A is denoted $X_t \in \{0, 1, \ldots, N-1, N\}$, for round $t = 0, 1 \ldots$. He loses the game if his holding equals 0 and he wins the game if his holding equals N. We specify initial state $X_0 = i$, $0 < i < N$.

a) Assume $i >> 0$ and $i << N$.

Compute $P(X_2 = i | X_0 = i)$.

Compute $P(X_4 = i|X_0 = i)$.

Compute $P(X_{10} = i|X_0 = i)$.

Define $\eta = q/p = 0.66$. Let u_i denote the probability that player A loses the game when $X_0 = i$.

b)

Derive the following result:

$$u_i = \frac{\eta^i - \eta^N}{1 - \eta^N}$$

c)

Setting $N = 10$, is there a value of $X_0 = i$ that makes this a fair game? Meaning that player A and player B have equal chances of winning the game.

Problem 3

a) Bill enjoys hunting grouses. He assumes that the **number of birds he sees** a given trip is Poisson distributed with mean μ. He further assumes that **he will hit** any of these individual birds with probability p, miss with probability $(1-p)$, and that the trials he gets are independent.

Derive the marginal probability mass function for the number of birds he hits on the trip.

b) Some say that the **number of hits** can be seen as a process $X(t)$ depending on hours t. Based on this, Bill assumes that the number of birds he hits during time interval $(0, t)$ is a Poisson process with expectation λt, and he sets $\lambda = 0.75$.

What is the probability that he hits no birds the first 4 hours?

He starts hunting at 8:00. By 12:00 he has 2 birds. What is then the probability that both were shot before 10:00?

c) Experience indicates that there are more birds in the morning, and Bill states a Poisson process model with inhomogeneous rate $\lambda(t) = 0.8 - 0.1t$, where $t \in (0, 4)$ indicates hours after 8:00.

He starts hunting at 8:00. By 12:00 he has 2 birds. What is now the probability that both were shot before 10:00?

Problem 4

In a park there are two chairs. Chair C is very comfortable, while chair H is a bit hard. We assume that people arrive to this area with rate 10 per hour. An arriving person will sit down in chair C if it is available. If a person comes and chair C is occupied, this arriving person will sit down in chair H with probability 0.5, otherwise just walk by. If a person comes and both chairs are occupied, the arriving person will just walk by. A person in chair C will leave the area at rate 2, while a person in chair H will leave at rate 5. A person who sits in chair H will not subsequently move to C if this becomes available. A person who sits down in chair C will not subsequently move to H if this becomes available.

a)

Draw the state diagram for the continuous time Markov process.

Compute the long-term probabilities of each state.

Problem 5

Paul is on vacation in Norway. He has a tight budget. When he left ($t = 0$) one EURO was equivalent of 9.0 kroner. When he comes home after $t = 50$ days one EURO is equivalent of 9.5 kroner. Paul was surprised by a rather expensive hotel after 25 days (paid with his bank card). Without any resources to check the actual currency rate at that time, he instead assumes that the currency rate $X(t)$ develops according to a Brownian motion with variance $0.05^2 t$.

a)

What is the probability that the rate was above 9.0 at the time of the expensive hotel?

Formulas: TMA4265 Stochastic Modeling:

The law of total probability

Let $B_1, B_2, \ldots$ be pairwise disjoint events with $P(\cup_{i=1}^{\infty} B_i) = 1$. Then

$$P(A|C) = \sum_{i=1}^{\infty} P(A|B_i \cap C) P(B_i|C),$$

$$E[X|C] = \sum_{i=1}^{\infty} E[X|B_i \cap C] P(B_i|C).$$

Discrete time Markov chains

Chapman-Kolmogorov equations

$$P_{ij}^{(m+n)} = \sum_{k=0}^{\infty} P_{ik}^{(m)} P_{kj}^{(n)}.$$

For an irreducible and ergodic Markov chain, $\pi_j = \lim_{n\to\infty} P_{ij}^{(n)}$ exist and is given by the equations

$$\pi_j = \sum_i \pi_i P_{ij} \quad \text{and} \quad \sum_i \pi_i = 1.$$

For transient states i, j and k, the mean passage time from i to $j \neq i$, M_{ij}, is

$$M_{ij} = 1 + \sum_k P_{ik} M_{kj}.$$

The Poisson process

The waiting time to the n-th event (the n-th arrival time), X_n, has probability density

$$f_{X_n}(t) = \frac{\lambda^n t^{n-1}}{(n-1)!} e^{-\lambda t} \quad \text{for} \quad t \geq 0.$$

Given that the number of events $N(t) = n$, the arrival times $X_1, X_2, \ldots, X_n$ have the uniform joint probability density

$$f_{X_1, X_2, \ldots, X_n | N(t)}(x_1, x_2, \ldots, x_n | n) = \frac{n!}{t^n} \quad \text{for} \quad 0 < x_1 < x_2 < \ldots < x_n \leq t.$$

Markov processes in continuous time

A (homogeneous) Markov process $X(t)$, $0 \leq t \leq \infty$, with state space $\Omega \subseteq \mathbf{Z}^+ = \{0, 1, 2, \ldots\}$, is called a birth and death process if

$$P_{i,i+1}(h) = \lambda_i h + o(h)$$

$$P_{i,i-1}(h) = \mu_i h + o(h)$$

$$P_{i,i}(h) = 1 - (\lambda_i + \mu_i)h + o(h)$$

$$P_{ij}(h) = o(h) \quad \text{for } |j - i| \geq 2$$

where $P_{ij}(s) = P(X(t+s) = j | X(t) = i)$, $i, j \in \mathbf{Z}^+$, $\lambda_i \geq 0$ are birth rates, $\mu_i \geq 0$ are death rates.

The Chapman-Kolmogorov equations

$$P_{ij}(t+s) = \sum_{k=0}^{\infty} P_{ik}(t) P_{kj}(s).$$

Limit relations

$$\lim_{h \to 0} \frac{1 - P_{ii}(h)}{h} = v_i \,, \quad \lim_{h \to 0} \frac{P_{ij}(h)}{h} = q_{ij} \,, \;\; i \neq j$$

Kolmogorov's forward equations

$$P'_{ij}(t) = \sum_{k \neq j} q_{kj} P_{ik}(t) - v_j P_{ij}(t).$$

Kolmogorov's backward equations

$$P'_{ij}(t) = \sum_{k \neq i} q_{ik} P_{kj}(t) - v_i P_{ij}(t).$$

If $P_j = \lim_{t \to \infty} P_{ij}(t)$ exist, P_j are given by

$$v_j P_j = \sum_{k \neq j} q_{kj} P_k \quad \text{and} \quad \sum_j P_j = 1.$$

In particular, for birth and death processes

$$P_0 = \frac{1}{\sum_{k=0}^{\infty} \theta_k} \quad \text{and} \quad P_k = \theta_k P_0 \quad \text{for} \quad k = 1, 2, \ldots$$

where

$$\theta_0 = 1 \quad \text{and} \quad \theta_k = \frac{\lambda_0 \lambda_1 \cdot \ldots \cdot \lambda_{k-1}}{\mu_1 \mu_2 \cdot \ldots \cdot \mu_k} \quad \text{for} \quad k = 1, 2, \ldots$$

Queueing theory

For the average number of customers in the system L, in the queue L_Q; the average amount of time a customer spends in the system W, in the queue W_Q; the service time S; the average remaining time (or work) in the system V, and the arrival rate λ_a, the following relations obtain

$$L = \lambda_a W.$$

$$L_Q = \lambda_a W_Q.$$

$$Z = \lambda_a E[S].$$

$$V = \lambda_a E[SW_Q^*] + \lambda_a E[S^2]/2.$$

Gaussian processes

The multivariate Gaussian density for $n \times 1$ random vector $\boldsymbol{x} = (x_1, \dots, x_n)$ is

$$p(\boldsymbol{x}) = \frac{1}{(2\pi)^{n/2}|\boldsymbol{\Sigma}|^{1/2}} \exp\left(-\frac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\boldsymbol{x}-\boldsymbol{\mu})\right), \quad \boldsymbol{x} \in \mathbb{R}^n,$$

where size $n \times 1$ mean vector $\boldsymbol{\mu} = (\mu_1, \dots, \mu_n)$, $E(x_i) = \mu_i$, and

$$\boldsymbol{\Sigma} = \begin{bmatrix} \Sigma_{1,1} & \dots & \Sigma_{1,n} \\ \dots & \dots & \dots \\ \Sigma_{n,1} & \dots & \Sigma_{n,n} \end{bmatrix}, \quad \Sigma_{i,j} = \mathrm{Cov}(x_i, x_j).$$

Let $\boldsymbol{x}_A = (x_{A,1}, \dots, x_{A,n_A})$ and $\boldsymbol{x}_B = (x_{B,1}, \dots, x_{B,n_B})$, be two subsets of variables, with block mean and covariance structure

$$\boldsymbol{\mu} = (\boldsymbol{\mu}_A, \boldsymbol{\mu}_B), \quad \boldsymbol{\Sigma} = \begin{bmatrix} \boldsymbol{\Sigma}_A & \boldsymbol{\Sigma}_{A,B} \\ \boldsymbol{\Sigma}_{B,A} & \boldsymbol{\Sigma}_B \end{bmatrix}.$$

The conditional density of $\boldsymbol{x}_A$, given $\boldsymbol{x}_B$, is Gaussian with

$$\begin{aligned} E(\boldsymbol{x}_A|\boldsymbol{x}_B) &= \boldsymbol{\mu}_A + \boldsymbol{\Sigma}_{A,B}\boldsymbol{\Sigma}_B^{-1}(\boldsymbol{x}_B - \boldsymbol{\mu}_B), \\ \mathrm{Var}(\boldsymbol{x}_A|\boldsymbol{x}_B) &= \boldsymbol{\Sigma}_A - \boldsymbol{\Sigma}_{A,B}\boldsymbol{\Sigma}_B^{-1}\boldsymbol{\Sigma}_{B,A}. \end{aligned}$$

The Brownian motion has increments $x(t_i) - x(t_{i-1})$ with the following properties, for any configuration of times $t_0 = 0 < t_1 < t_2 < \dots$:

- $x(t_i) - x(t_{i-1})$ and $x(t_j) - x(t_{j-1})$ are independent for all $i \neq j$.
- the distribution of $x(t_i) - x(t_{i-1})$ is identical to that of $x(t_i + s) - x(t_{i-1} + s)$, for any s.
- $x(t_i) - x(t_{i-1})$ is Gaussian distributed with 0 mean and variance $\sigma^2(t_i - t_{i-1})$.

Unless otherwise stated, $x(0) = 0$.

Some mathematical series

$$\sum_{k=0}^{n} a^k = \frac{1 - a^{n+1}}{1 - a} \quad , \qquad \sum_{k=0}^{\infty} k a^k = \frac{a}{(1-a)^2} \quad .$$