TMA4190 Introduction to Topology
Spring 2018

Solutions to exercise set 5

Norwegian University of Science and Technology
Deptartment of Mathematical Sciences

$\boxed{1}$

a) Since f is bijective, it has an inverse $f^{-1}\colon Y \to X$. We need to show that f^{-1} is smooth. Let y be a point in Y. Since f is a local diffeomorphism, there is an open neighborhood $U \subset X$ around the ponit $f^{-1}(y)$ and an open neighborhood $V \subset Y$ around y such that $f_{|U}\colon U \to V$ is a diffeomorphism. Hence there is a smooth inverse $(f_{|U})^{-1}\colon V \to U$. Since inverses are unique (as maps of sets), $(f^{-1})_{|V}$ must agree with $(f_{|U})^{-1}$. Hence f^{-1} is a smooth map on an open neighborhhod of y. Since y was arbitrary, we see that f^{-1} is smooth at every point and therefore smooth.

b) Since f is one-to-one, it is a bijection from X onto its image $\mathrm{Im}\,(f) \subseteq Y$. Since it is a local diffeomorphism, $f\colon X \to \mathrm{Im}\,(f)$ is a bijective local diffeomorphism. By the previous point, it is a diffeomorphism.

c) We would like to show $\dim X = \mathrm{rank}(f) = \dim Y$. Because then the Inverse Function Theorem implies that f is a local diffeomorphism, and, since f is also bijective, f would be a diffeomorphism by the first point and we were done.

Assume $X \subseteq \mathbb{R}^M$ and $Y \subseteq R^N$, $\dim X = m$, $\dim Y = n$, and set $r := \mathrm{rank}(f)$. By definition of the rank, we have $m \geq r$ and $n \geq r$. We want to show $m = r = n$.

For any point $x \in X$, the linear map df_x has rank r. Recall that for a linear map $L\colon \mathbb{R}^m \to \mathbb{R}^n$ of rank r, we can choose a basis of $\mathbb{R}^n$ such that the first r basis vectors $b_1, \ldots, b_r$ span the image of L and the remaining $n - r$ basis vectors $b_{r+1}, \ldots, b_n$ span the orthogonal complement of L in $\mathbb{R}^n$. Then we choose a basis of $\mathbb{R}^m$ such that the ith basis vector is sent to b_i. The matrix representing L in these bases has the $r \times r$-identity matrix sitting in the upper left corner and zeros elsewhere. Then, as in the proof of the Local Immersion (or Submersion) Theorem, we can choose local parametrizations $\phi\colon U \to X$ around x and $\psi\colon V \to Y$ around y such that the map $\theta\colon U \to V$ in the commutative diagram

$$\begin{tikzcd}
X \arrow[r, "f"] & Y \\
U \arrow[u, "\phi"] \arrow[r, "\theta=\psi^{-1}\circ f\circ\phi"'] & V \arrow[u, "\psi"']
\end{tikzcd}$$

has the form $\theta(x_,1 \ldots, x_m) = (x_1, \ldots, x_r, 0) \in \mathbb{R}^n$. (Note that the 0 at the end of $\theta(x)$ only occur if $r < n$.)

If $m > r$, then for a sufficiently small $\epsilon > 0$, $\theta(x_,1 \ldots, x_r, \epsilon, 0) = (x_1, \ldots, x_r, 0)$ and θ is not injective. Since ϕ and ψ are diffeomorphisms, this would imply that f is not injective which contradicts that f is bijective. Hence we can assume $m = r$ and f is an immersion.

Assume we had $r < n$. Then, after possibly shrinking U, we can assume that U is a small open $B_\epsilon(0)$ around 0 in $\mathbb{R}^m$ and that $\theta(\bar{B}_\epsilon(0)) \subseteq V$ (where $\bar{B}_\epsilon(0)$ denotes the closed ball of radius ϵ: $\bar{B}_\epsilon(0) = \{x \in \mathbb{R}^m : |x| \leq \epsilon\}$). Since $\bar{B}_\epsilon(0)$ is compact, so is $\theta(\bar{B}_\epsilon(0))$. Hence $\theta(\bar{B}_\epsilon(0))$ is closed in V and is contained in $V \cap (\mathbb{R}^r \times \{0\})$. Hence $\theta(\bar{B}_\epsilon(0))$ does not contain any open subsets of V. Since ϕ and ψ are diffeomorphisms, this implies that $f(\phi(\bar{B}_\epsilon(0)))$ is closed and does not contain any nonempty open subset of Y. Since we can cover X by such local parametrizations, we see that $f(X)$ is the union of subsets which do not contain any nonempty open subset of Y.

Now if X could be assumed to be compact, then $f(X)$ is compact, and $f(X)$ can be covered by finitely many closed subsets which do not contain any nonempty open subset of Y. That would imply that $f(X)$ is itself a closed subset which does not contain any nonempty open subset of Y. Hence $f(X)$ cannot be all of Y, and f would not be surjective.

In general, for any open cover of a subspace in $\mathbb{R}^M$, we can always choose a countable subcover. This implies that $f(X)$ is the countable union of subsets which do not contain any nonempty open subset of Y. By Baire's category theorem, this implies that $f(X)$ does not contain any nonempty open subset of Y. Hence $f(X)$ cannot be equal Y and f would not be surjective.

d) We learned in the lecture that a Lie group homomorphism has constant rank. Hence we just need to apply the previous point.

$\boxed{2}$ For any $g \in G$, left multiplication $L_g \colon G \to G$ by g maps the subgroup H to the left coset $gH = \{gh : h \in H\}$. Since H is open and L_g is a diffeomorphism, the coset gH is open. Thus, G can be written as the union of the open subsets gH where g ranges over all elements in G. But since cosets are pairwise disjoint, this would give us a way to write G as the union of nonempty disjoint open subsets. Since G is connected, there can be only one coset. Therefore, $H = G$.

$\boxed{3}$ **a)** Let $g, h \in G$ be any fixed elements. Let $j \colon G \to G \times G$ be the map $j(g) = (g, h)$. Note that the composite $\mu \circ j = R_h$ is right translation by h.

For $x \in G$, let $\phi_x \colon U_x \to G$ be a local parametrization around x with $\phi(0) = x$. Then we get the diagram

$$\begin{tikzcd} G \arrow[r, "j"] & G \times G \arrow[r, "\mu"] & G \\ U_g \arrow[u, "\phi_g"] \arrow[r, "\gamma"'] & U_g \times U_h \arrow[u, "\phi_g \times \phi_h"'] \arrow[r, "\theta"'] & U_{gh}. \arrow[u, "\phi_{gh}"'] \end{tikzcd}$$

where we define the maps γ and θ such that the diagram commutes. Since $\phi_g(0) = g$ and $\phi_h(0) = h$, we must have $\gamma(u) = (u, 0) \in U_g \times U_h$ to make the left hand diagram commute. Moreover, we must have $\theta(0, 0) = 0 \in U_{gh}$.

Taking derivatives at g and using $T_{(g,h)}(G \times G) = T_g(G) \times T_h(G)$ gives

$$\begin{tikzcd}
T_g(G) \arrow[r, "dj_g"] & T_g(G) \times T_h(G) \arrow[r, "d\mu_{(g,h)}"] & T_{gh}(G) \\
\mathbb{R}^n \arrow[u, "d(\phi_g)_0"] \arrow[r, "d\gamma_0"'] & \mathbb{R}^n \times \mathbb{R}^n \arrow[u, "d(\phi_g)_0 \times d(\phi_h)_0"] \arrow[r, "d\theta_0"'] & \mathbb{R}^n. \arrow[u, "d(\phi_{gh})_0"']
\end{tikzcd}$$

Since $\gamma(u) = (u, 0)$, we have $d\gamma_0(v) = (v, 0)$ and hence $dj_g(X) = (X, 0)$. Since $\mu \circ j = R_h$, we have $d\mu_{(g,h)} \circ dj_g = d(R_h)_g$. Thus

$$d\mu_{(g,h)}(X, 0) = d\mu_{(g,h)}(dj_g(X)) = d(R_h)_g(X).$$

Repeating this argument with j replaced with $j \colon h \mapsto (g, h)$ yields

$$d\mu_{(g,h)}(0, Y) = d(L_g)_h(Y).$$

Since $d\mu_{(g,h)}$ is linear, it satisfies

$$d\mu_{(g,h)}(X, Y) = d\mu_{(g,h)}(X, 0) + d\mu_{(g,h)}(0, Y) = d(R_h)_g(X) + d(L_g)_h(Y).$$

b) Consider the map

$$G \xrightarrow{(\mathrm{Id}, \iota)} G \times G \xrightarrow{\mu} G, \; g \mapsto (g, g^{-1}) \mapsto gg^{-1} = e.$$

Since this map is constant, its derivative at e vanishes. Hence we get

$$T_e(G) \xrightarrow{(d\mathrm{Id}_e, d\iota_e)} T_e(G) \times T_e(G) \xrightarrow{d\mu_{(e,e)}} T_e(G), \; X \mapsto (X, d\iota_e(X)) \mapsto 0.$$

As we have just learned $d\mu_{(e,e)}(X, d\iota_e(X)) = X + d\iota_e(X) = 0$, and hence $d\iota_e(X) = -X$.

c) Given $g \in G$, we consider the diagram

$$\begin{tikzcd}
G \arrow[r, "\iota"] \arrow[d, "L_{g^{-1}}"'] & G \\
G \arrow[r, "\iota"'] & G. \arrow[u, "R_{g^{-1}}"']
\end{tikzcd}$$

One easily checks that it commutes. Taking the derivative at g of the top map yields a commutative diagram of derivatives

$$\begin{tikzcd}
T_g(G) \arrow[r, "d\iota_g"] \arrow[d, "d(L_{g^{-1}})_g"'] & T_{g^{-1}}(G) \\
T_e(G) \arrow[r, "d\iota_e"'] & T_e(G). \arrow[u, "d(R_{g^{-1}})_e"']
\end{tikzcd}$$

We just calculated the effect of the map $d\iota_e \colon T_e(G) \to T_e(G)$ as $X \mapsto -X$. Hence, since all maps in the above diagram are linear, we get

$$d\iota_g \colon T_g(G) \to T_{g^{-1}}, \; Y \mapsto -d(R_{g^{-1}})_e(d(L_{g^{-1}})_g(Y)).$$

$\boxed{4}$ Given elements $g, h \in G$. Let $R_{h^{-1}}$ denote the right translation with h^{-1}. We define the smooth map j_h by

$$j_h \colon G \to G \times G, \ x \mapsto (R_{h^{-1}}(x), h).$$

Note that $j_h(gh) = (ghh^{-1}, h) = (g, h) \in G \times G$. For the tangent spaces we get

$$T_{j_h(gh)}(G \times G) = T_{(g,h)}(G \times G) \cong T_g(G) \times T_h(G).$$

The composite of the map

$$G \xrightarrow{j_h} G \times G \xrightarrow{\mu} G, \ x \mapsto (R_{h^{-1}}(x), h) \mapsto \mu(xh^{-1}, h) = x$$

is the identity of G. Taking derivatives at gh yields

$$T_{gh}(G) \xrightarrow{d(j_h)_{gh}} T_g(G) \times T_h(G) \xrightarrow{d\mu_{(g,h)}} T_{gh}(G).$$

Since $\mu \circ j_h = \mathrm{Id}_G$, we also have $d\mu_{(g,h)} \circ d(j_h)_{gh} = \mathrm{Id}_{T_{gh}(G)}$. In particular,

$$d\mu_{(g,h)} \colon T_{(g,h)}(G \times G) \to T_{gh}(G)$$

is surjective. Since we started with arbirtrary elements g and h, this shows that μ is a submersion.

$\boxed{5}$ For any matrices $A \in GL(n)$ and $B \in M(n)$, we have

$$\det(B) = (\det A) \cdot \det(A^{-1}B) = (\det A) \cdot \det(L_{A^{-1}}B).$$

Taking the derivative at A and remembering the chain rule yields

$$\begin{aligned}
d(\det)_A(B) &= d((\det A) \cdot \det \circ L_{A^{-1}})(B) \\
&= (\det A) \cdot d(\det \circ L_{A^{-1}})B \\
&= (\det A) \cdot d(\det)_{(L_{A^{-1}}A)} \circ d(L_{A^{-1}})_A(B) \\
&= (\det A) \cdot d(\det)_I(A^{-1}B) \\
&= (\det A) \cdot \operatorname{tr}(A^{-1}B),
\end{aligned}$$

where we have used $d(L_{A^{-1}})_A(B) = A^{-1}B$ which can be easily checked, since matrix multiplicaion is linear. Hence, after replacing B with AB, we get

$$d(\det)_A(AB) = (\det A) \cdot (\operatorname{tr} B) \text{ for all } B \in M(n).$$