3. Lecture III: Linear Systems on $\mathbb{R}^d$

In this lecture, we look at linear systems more generally.

3.1. **Autonomous linear systems on $\mathbb{R}^d$.** We consider again the system

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x}(t) = \mathbf{A}\mathbf{x}(t),$$

where $\mathbf{A}$ is now a $d \times d$ matrix. Now as we saw, the matrix $\mathbf{A}$ is not in general diagonalizable. However, it can always be put into a standard, or *normal* form, from which we can determine which eigenvalues have strictly greater algebraic multiplicities than they have geometric multiplicities. We have the following useful theorem:

Theorem 3.1 (Jordan's Normal Form Theorem)**.** *Let $\mathbf{B}_m$ for $1 \le m \le s$ be a $d_m \times d_m$ matrix given by*

$$\big(\mathbf{B}_m\big)_i^j = \lambda_m \delta_i^j + \delta_{i-1}^j.$$

For every $d \times d$ matrix $\mathbf{A}$, there exist $d_m \times d_m$ matrices $\mathbf{B}_m$ with $\sum_{1\le m\le s} d_m = d$, and a $d \times d$ matrix $\mathbf{J}$ given by

$$\mathbf{J} = \begin{pmatrix} \mathbf{B}_1 & & \\ & \ddots & \\ & & \mathbf{B}_s \end{pmatrix}$$

for which

$$\mathbf{A} \sim \mathbf{J}.$$

The matrix $\mathbf{J}$ is unique up to the ordering of the blocks.

Recall that similarity is an equivalence relation.

The matrix $\mathbf{J}$ is the Jordan normal form of $\mathbf{A}$. We sometimes write this as $\mathbf{J} = \mathrm{diag}(\mathbf{B}_m)$. The matrices $\mathbf{B}_m$ with λ_m on the diagonal and 1 on the superdiagonal are known as the Jordan blocks. The number of Jordan blocks with λ on the diagonal is the geometric multiplicity of that eigenvalue and the number of times λ appears in $\mathbf{J}$ itself is its algebraic multiplicity.

As $\mathbf{A}$ is real, its characteristic equation also only has real coefficients, and therefore complex roots only appear in conjugate pairs.

This normal form representation allows us to decouple the first-order system into blocks with distinct eigenvalues of different multiplicities. We shall now repeat in general dimensions our deductions in the case $d = 2$. In particular, from the Fundamental Theorem for Linear Systems (Thm.1.2) it can be seen (cf. (6)) that where $\mathbf{A} = \mathbf{P}^{-1}\mathbf{J}\mathbf{P}$,

$$\mathbf{x}(t) = \mathbf{P}\exp\big(\mathbf{J}t\big)\mathbf{P}^{-1}\mathbf{x}(0).$$

One advantage of decoupling the equations is that, just as diagonal matrices are easy to exponentiate, it is also relatively benign a calculation that determines $\exp\big(\mathbf{J}t\big)$. First, a direct calculation shows that for any power $n \in \mathbf{N}$,

$$\mathbf{J}^n = \begin{pmatrix} \mathbf{B}_1^n & & \\ & \ddots & \\ & & \mathbf{B}_s^n \end{pmatrix},$$

so that by the series representation of the exponential,

$$\exp\big(\mathbf{J}t\big) = \exp\big(\mathrm{diag}(\mathbf{B}_m t)\big) = \mathrm{diag}\big(\exp(\mathbf{B}_m t)\big).$$

Next it remains to find $\exp\big(\mathbf{B}_m t\big)$. Writing $\mathbf{N}_m$ for the $d_m \times d_m$ matrix with 1s along the superdiagonal (with entries δ_{i-1}^j), we find

$$\exp(\mathbf{B}_m) = \exp(\lambda_m \mathbf{I}_{d_m} t)\exp(\mathbf{N}_m t) = e^{\lambda_m t}\exp(\mathbf{N}_m t),$$

as the identity matrix commutes with any other matrix.

Therefore we need only determine $\exp(\mathbf{N}_m t)$. We find that $\mathbf{N}_m$ is nilpotent with power d_m, as, by direct calculation (or by induction), for any power $n \in \mathbb{N}$,

$$\left(\mathbf{N}_m^n\right)_i^j = \delta_{i-n}^j.$$

Putting this into the series expansion for $\exp(\mathbf{N}_m t)$ we find

$$\Big(\exp(\mathbf{N}_m t)\Big)_i^j = \sum_{n=0}^{d_m-1} \frac{t^n}{n!}\delta_{i-n}^j,$$

i.e., $\exp(\mathbf{N}_m t)$ is an upper-diagonal matrix that looks like

$$\exp(\mathbf{N}_m t) = \begin{pmatrix} 1 & t & t^2/2! & \cdots & t^{d_m-1}/(d_m-1)! \\ 0 & 1 & t & \cdots & t^{d_m-2}/(d_m-2)! \\ 0 & 0 & 1 & \cdots & t^{d_m-3}/(d_m-3)! \\ \cdots & & & & \\ 0 & \cdots & & 1 & t \\ 0 & \cdots & & 0 & 1 \end{pmatrix}.$$

Recall that the algebraic multiplicity of an eigenvalue λ is the number of times that λ is a root of $p(x) = \det(\mathbf{A} - x\mathbf{I}) = 0$, and the geometric multiplicity is the dimension of $\ker(\mathbf{A} - \lambda\mathbf{I})$.

Associated with each *distinct* eigenvalue λ_n of $\mathbf{A}$, of algebraic multiplicity μ_n is a generalized eigenspace spanned by the μ_n linearly independent generalized eigenvectors $V_n = \{\mathbf{v}_n^1, \ldots, \mathbf{v}_n^{\mu_n}\}$. Of these, a subset will be actual eigenvectors. The rest are found by the Jordan chain proceedure. The Jordan chain proceedure by which the bases V_n are found is given in *Perko* on p.43. Note that the algorithm/theorem in Perko is written in such a way as to avoid vectors with complex entries.

The idea is that if the dimension of $\ker(\mathbf{A} - \lambda\mathbf{I})$ is not great enough, we look among $\ker(\mathbf{A} - \lambda\mathbf{I})^2$, and increasingly higher powers, until we find d linearly independent vectors.

The generalized eigenvectors give the full generalized basis by which $\mathbf{P}$ is constructed so that $\mathbf{A} = \mathbf{P}\mathbf{J}\mathbf{P}^{-1}$.

Example 3.1. Examples of matrices in Jordan normal form:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Example 3.2. Consider the Cauchy problem:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x}(t) = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & -1 & 3 \end{pmatrix}\mathbf{x}(t), \qquad \mathbf{x}(0) = \mathbf{b}.$$

Let the matrix be called $\mathbf{A}$.

One can find that the eigenvalues of the matrix are $\lambda_1 = 2$, with algebraic multiplicity 2, and $\lambda_2 = 3$.

However, λ_1 only has geometric multiplicity of 1, and its associated eigenvector is $\mathbf{v}_1 = (1, 0, 0)^\top$. The eigenvector associated with λ_2 is $\mathbf{v}_2 = (0, 0, 1)^\top$.

This tells us that the Jordan normal form is

$$\mathbf{J} = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$

We can also find the non-singular matrices $\mathbf{P}$ that puts $\mathbf{A}$ into its Jordan normal form. The Jordan chain proceedure involves finding an linearly independent vector $\mathbf{v}_3$ in the kernel of $(\mathbf{A} - \lambda_1\mathbf{I}_3)^2$ as

$\ker(\mathbf{A} - \lambda \mathbf{I}_3)$ has been exhausted. This is done by setting

$$(\mathbf{A} - \lambda_1 \mathbf{I}_3)\mathbf{v}_3 = \mathbf{v}_1.$$

Doing so we find that $\mathbf{v}_3 = (0, 1, 1)^\top$ is a solution.

The subspace decomposition then *suggests* that

$$\mathbf{x}(t) = (C_1 + C_2 t)e^{2t}\mathbf{v}_1 + C_2 e^{2t}\mathbf{v}_3 + C_3 e^{3t}\mathbf{v}_2.$$

The reason for finding extra linearly independent vectors to be generalized eigenvectors in this way is that, setting

$$\mathbf{P} = (\mathbf{v}_1, \mathbf{v}_3, \mathbf{v}_2) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix},$$

we have

$$\mathbf{J} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}.$$

Therefore with the change-of-variables $\mathbf{y} = \mathbf{P}^{-1}\mathbf{x}$, we can transform the Cauchy problem to

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{y} = \mathbf{J}\mathbf{y}, \qquad \mathbf{y}(0) = \mathbf{P}^{-1}\mathbf{b}.$$

This can be exponentiated easily:

$$\mathbf{y}(t) = (C_1 + C_2 t)e^{2t}\mathbf{P}^{-1}\mathbf{v}_1 + C_2 e^{2t}\mathbf{P}^{-1}\mathbf{v}_3 + C_3 e^{3t}\mathbf{P}^{-1}\mathbf{v}_2,$$

(because if $\mathbf{w}$ is an eigenvector of $\mathbf{A}$, then $\mathbf{A}\mathbf{P}\mathbf{P}^{-1}\mathbf{w} = \lambda\mathbf{w}$, and $\mathbf{P}^{-1}\mathbf{w}$ is the eigenvector of $\mathbf{P}^{-1}\mathbf{A}\mathbf{P}$).

We can recover $\mathbf{x}$ via $\mathbf{x} = \mathbf{P}\mathbf{y}$, verifying our intuitions.

3.2. **Linear stability theory.** The main advantage of decoupling the system by casting it into the Jordan normal form, however, is that we can state a sort of "structure theorem" for all linear systems.

Let V_n be the collection of generalized eigenvectors associated with the eigenvalue $\lambda_n \in \mathbb{C}$. We can group the collections V_n into three classes: Set

$$\begin{aligned} E^s &= \operatorname{span} \bigcup_{\{n:\Re\lambda_n<0\}} V_n \\ E^c &= \operatorname{span} \bigcup_{\{n:\Re\lambda_n=0\}} V_n \\ E^u &= \operatorname{span} \bigcup_{\{n:\Re\lambda_n>0\}} V_n. \end{aligned} \tag{10}$$

We call E^s the STABLE SUBSPACE, E^c the CENTRE SUBSPACE, and E^u the UNSTABLE SUBSPACE. They are so named because E^s consist of the directions along which the dynamics of the system enforces decay to $\mathbf{0}$ and E^u consist of the directions along which $\mathbf{x}(t)$ tends to infinity as $t \to \infty$.

We say that a subspace $E \subseteq \mathbb{R}^n$ is INVARIANT with respect to the flow ϕ_t if $\phi_t E \subseteq E$ for all time $t \in \mathbb{R}$. That is, the (forward and backward) orbit(s) of each point within E remains in E.

For us, the flow, as identified following (5) is $\exp(\mathbf{A}t)$. If all eigenvalues of $\mathbf{A}$ have non-zero real parts (i.e., the centre manifold is trivial), then we say that the flow, and the linear system, is HYPERBOLIC.

From the definition of the generalized eigenvectors that span E^s, E^c, and E^u, we can deduce that:

Lemma 3.2. *Let E be the generalized eigenspace of $\mathbf{A}$ corresponding to an eigenvalue λ. Then $\mathbf{A}E \subseteq E$.*

Let $\hat{E}$ be the subspace of the generalised eigenvectors E that is spanned by the eigenvectors of $\mathbf{A}$. This lemma boils down to the fact that if $\mathbf{u} \in E$, then it can be written as a linear combination of eigenvectors $\mathbf{e}_i$ or generalized eigenvectors $\mathbf{v}_i$, for which $\mathbf{A}^k \mathbf{v}_i \in \hat{E}$ for high enough k. Now for every k , including $k = 1$, $\mathbf{v}_i$ being defined using a Jordan chain, must still satisfy that $\mathbf{A}\mathbf{v}_i \in \lambda \mathbf{v}_i + E$. And for eigenvectors $\mathbf{e}_i$, it is even better — $\mathbf{A}\hat{E} = \hat{E}$, unless 0 is an eigenvalue, in which case the equality again reverts to an inclusion.

Since $\mathbf{A}$ acts linearly over the linear combination, $\mathbf{A}E \subseteq E$.

Iterating the theorem we see that

$$\mathbf{A}^n E \subseteq \mathbf{A}^{n-1} E \subseteq \ldots \subseteq \mathbf{A}E \subseteq E.$$

So using the partial sums of series representation of the flow $\exp(\mathbf{A}t)$, and taking the limit, it further holds that the subspaces are invariant with respect to the flow also. This invariance means that in general, the Jordan normal form decouples the system into invariant subspaces, and these invariant subspaces span the entire phase space:

Theorem 3.3. *Let $\mathbf{A}$ be a real $d \times d$ matrix. Then the phase space $\mathbb{R}^d$ can be decomposed thus:*

$$\mathbb{R}^d = E^s \oplus E^c \oplus E^u,$$

which are defined in (10)*. Furthermore, these subspaces are each invariant with respect to the flow* $\exp(\mathbf{A}t)$.