TMA 4190 Introduction to Topology
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Lecture 23[1]

23. Intersection Theory

The assumptions for our intersection theory to work will be always:

Assumptions for intersection theory

- We consider a smooth map $f\colon X \to Y$, where X, Y are boundaryless smooth manifolds, $Z \subset Y$ is a boundaryless submanifold.
- The dimensions are complementary, i.e. $\dim X + \dim Z = \dim Y$.
- X will always be assumed to be compact.
- All manifolds are oriented, i.e. they are orientable and we have chosen an orientation.

The idea for the new intersection number is now very simple:

If $f\colon X \to Y$ is transversal to Z, then $f^{-1}(Z)$ consists of a **finite number of points** (since $f^{-1}(Z)$ is zero-dimensional and compact becasue of the assumptions on X, Z and the dimensions; the assumptions are all important). Each point in $f^{-1}(Z)$ has an orientation number ± 1 provided by the **preimage orientation**.

If $x \in f^{-1}(Z)$ is a point in the preimage, the orientation number at x is determined as follows. If $f(x) = z \in Z$, then transversality implies $df_x(T_x(X)) + T_z(Z) = T_z(Y)$. But since the dimensions are complementary, this sum must be **direct**, i.e.,

$$df_x(T_x(X)) \cap T_z(Z) = \{0\}, \text{ and } df_x(T_x(X)) \oplus T_z(Z) = T_z(Y). \tag{1}$$

This direct sum decompostion implies that

$$\dim T_x(X) = \dim df_x(T_x(X)),$$

since $\dim T_x(X) = \dim T_z(Y) - \dim T_z(Z)$. Thus df_x must be an **isomorphism onto its image**. In particular, the orientation of $T_x(X)$ provides an **orientation of $df_x(T_x(X))$**.

[1] Following the books of Guillemin and Pollack: Differential Topology, and Milnor: Topology from the differentiable viewpoint.

Then the **orientation number at x** is $+1$ if the orientation of $T_z(Y)$ as the direct sum in (1) induced by the orientations on $df_x(T_x(X))$ and $T_z(Z)$ agrees with the given orientation of $T_z(Y)$. And it is -1 if the induced orientation disagrees.

Intersection numbers as sums of orientation numbers

If $f \pitchfork Z$, we define the **intersection number** $I(f,Z)$ to be the **sum of the orientation numbers** at the finitely many points $x \in f^{-1}(Z)$.

We claimed that introducing orientations would yield **homotopy invariant** intersection numbers in $\mathbb{Z}$. Now we we have to demonstrate that this claim holds. This will then allow us to define intersection numbers for nontransversal intersections.

Suppose that $X = \partial W$ is the **boundary** of a compact W and that f extends to a smooth map $F \colon W \to Y$, i.e. $f = \partial F = F_{|\partial W}$.

By the **Extension Theorem**, we may assume $F \pitchfork Z$. Thus, by the Preimage Theorem for manifolds with boundary, $F^{-1}(Z)$ is a compact oriented manifold with boundary $\partial F^{-1}(Z) = f^{-1}(Z)$. Since $\operatorname{codim} \partial W = 1$ in W, we have $\operatorname{codim} F^{-1}(Z) = 1$ in Y, and hence

$$\begin{aligned} \dim W - \dim F^{-1}(Z) &= \operatorname{codim} F^{-1}(Z) \text{ in } W \\ &= \operatorname{codim} Z \text{ in } Y = \dim Y - \dim Z = \dim X. \end{aligned}$$

But $\dim W = \dim X + 1$, and **thus** $\dim F^{-1}(Z) = 1$. Hence $F^{-1}(Z)$ is a **compact oriented one-manifold with boundary**. As we learned in the previous lecture, the sum of the orientation numbers at points in the boundary $f^{-1}(Z)$ must be **zero**.

As a consequence we get:

Intersection numbers for maps on boundaries

If $f \pitchfork Z$ and $X = \partial W$ is the **boundary** of a compact W and that f extends to a smooth map $F \colon W \to Y$, then the sum of orientation numbers of points in $f^{-1}(Z)$ is zero, i.e. $I(f,Z) = 0$.

This enables us to prove the key fact:

Homotopy invariance for transversal maps

Let f_0 and f_1 be two homotopic maps $X \to Y$ which are both transversal to Z. Then $I(f_0,Z) = I(f_1,Z)$.

Proof: Let $F\colon X \times [0,1] \to Y$ be a homotopy between them. Then we just learned that $I(\partial F,Z) = 0$. The boundary map ∂F is just f_0 on the copy X_0 at 0 and f_1 on the copy X_1 at 1. Now recall that the orientations of X_0 and X_1 as the boundary of $X \times [0,1]$ are given by

$$\partial(X \times [0,1]) = X_1 - X_0.$$

Hence as oriented manifolds we get

$$\partial F^{-1}(Z) = f_1^{-1}(Z) - f_0^{-1}(Z).$$

By our definition of **intersction numbers as sums of orientation numbers**, this implies

$$0 = I(\partial F,Z) = I(f_1,Z) - I(f_0,Z).$$

QED

As in the mod 2-theory, the previous theorem allows us to define intersection numbers for arbitrary maps.

Intersection numbers for arbitrary maps

Let $g\colon X \to Y$ be any smooth map. By the Transversality Homotopy Theorem, we can **choose** a smooth map $f\colon X \to Y$ which is homotopic to g and transversal to Z. Then we **define** $I(g,Z)$ to be $I(f,z)$, i.e.

$$I(g,Z) := I(f,Z).$$

We just shows that the definition does not depend on the choice of f. Moreover, all homotopic maps have equal intersection numbers:

All homotopic maps have equal Intersection Numbers

If $g_0\colon X \to Y$ and $g_1\colon X \to Y$ are arbitrary **homotopic** maps, then $I(g_0,Z) = I(g_1,Z)$.

Proof: The proof is the same is in the mod 2-case. We can choose maps $f_0 \pitchfork Z$ and $f_1 \pitchfork Z$ such that $g_0 \sim f_0$, $I(g_0,Z) = I(f_0,Z)$, and $g_1 \sim f_1$, $I(g_1,Z) = I(f_1,Z)$.

Since homotopy is a **transitive** relation, we have

$$f_0 \sim g_0 \sim g_1 \sim f_1, \text{ and hence } f_0 \sim f_1.$$

By the previous theorem, this implies

$$I(g_0,Z) = I(f_0,Z) = I(f_1,Z) = I(g_1,Z).$$

QED

The Brouwer degree

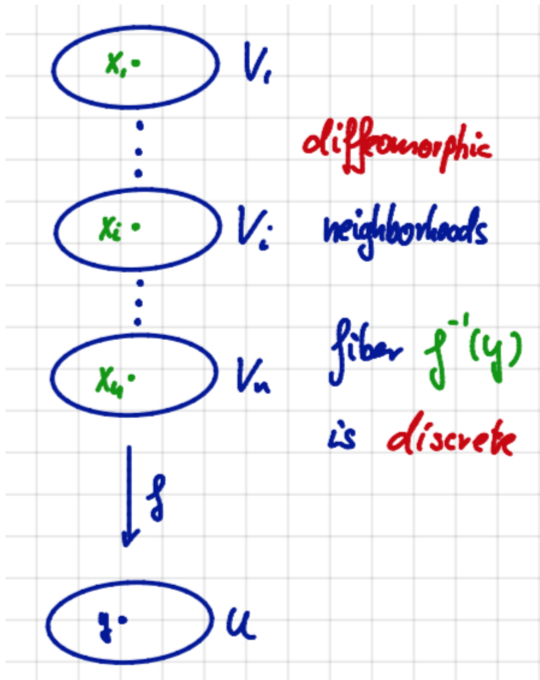
Let us look again at the special case when $\dim X = \dim Y$:

The Brouwer degree

Let $f\colon X \to Y$ be a smooth map with $\dim X = \dim Y$, X compact, and Y connected. We define the **degree of f**, denoted by $\deg(f)$, to be the intersection number $I(f,\{y\})$ at any regular value $y \in Y$ of f. In particular, we claim that the integer $I(f,\{y\})$ does not depend on the choice of the regular value y.
The degree is homotopy invariant, i.e. $f_0 \sim f_1$ implies $\deg(f_0) = \deg(f_1)$.

Proof of the claim of independence: Actually, the proof in the mod 2-case gave us this result already. But only observed the weaker consequence for mod 2-intersction numbers. To be sure, let us go through it again.

Given any $y \in Y$, we can assume that f is **transversal to** $\{y\}$. For otherwise we can replace it with a homotopic map which is transversal by the **Transversality Homotopy Theorem**. Now by the **Stack of Records Theorem**, we can find a neighborhood U of y such that the preimage $f^{-1}(U)$ is a disjoint union $V_1 \cup \cdots \cup V_n$, where each V_i is an open set in X mapped by f diffeomorphically onto U:



Hence, for all points $z \in U$, we have $\#f^{-1}(\{z\}) = n$. But this is not enough for knowing that the intersection numbers agree. For we we have to take orientations into account.

Since $f_{|V_i} \colon V_i \to U$ is a diffeomorphism, we know that

$$df_{x_i} \colon T_{x_i}(X) \to T_y(Y)$$

is an isommorphism. Now both $T_{x_i}(X)$ and $T_y(Y)$ are oriented, and hence df_{x_i} is either orientation preserving or reversing. But by our definition of orientations on manifolds, we have **either**

- $\det(df_{x_i}) > 0$ and hence, for all $z \in U$, $\det(df_{w_i}) > 0$, where w_i is the unique point in V_i with $f(w_i) = z$; in other words, df_{w_i} preserves orientations for all points $w_i \in V_i$;
- **or** $\det(df_{x_i}) < 0$ and hence, for all $z \in U$, $\det(df_{w_i}) < 0$, where w_i is the unique point in V_i with $f(w_i) = z$; in other words, df_{w_i} reverses orientations for all points $w_i \in V_i$.

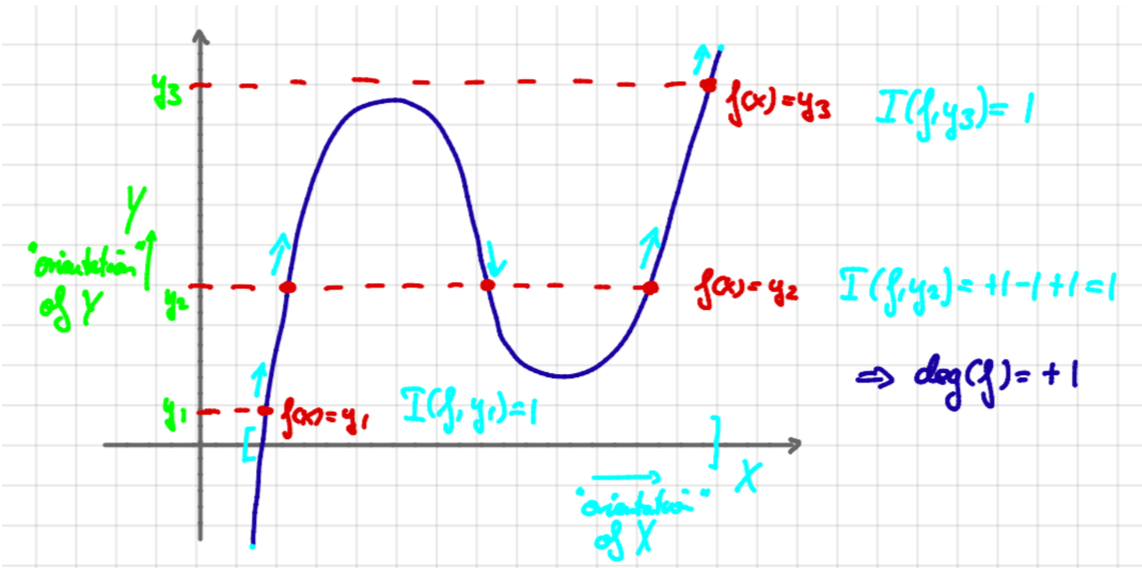
Thus the orientation number is the same for all points in V_i. Hence the sum of orientation numbers of the points in $f^{-1}(z)$ is the same for all points $z \in U$.

Consequently, the function

$$Y \to \mathbb{Z},\ y \mapsto I(f,\{y\})$$

is **locally constant**. Since Y is **connected**, it must be **globally constant**. **QED**

Here is a simple example of how to calculate a degree:



Degree of a diffeomorphism

A special case of the situation $\dim X = \dim Y$ is that of a **diffeomorphism** $f\colon X \to Y$. It follows immedtiately from the definition that f has **degree** $+1$ **or** -1 according to if f preserves or reverses orientation. In particular, we get:
An **orientation reversing diffeomorphism** of a compact boundaryless manifold is **not** smoothly homotopic to the identity.
An example of such an orientation reversing diffeomorphism is provided by the **reflection** $r_i\colon S^n \to S^n$ which we have seen in the Exercises before:

$$r_i(x_1, \ldots, x_{n+1}) = (x_1, \ldots, -x_i, \ldots, x_{n+1}).$$

As in the mod 2-case, the boundary result for intersection numbers imply the following fact on extensions of maps.

Extendable maps on boundaries have degree zero

Suppose that $f\colon X \to Y$ is a smooth map of compact oriented manifolds having the same dimension and that $X = \partial W$ is the boundary of a compact manifold W. If f can be extended to all of W, then $\deg(f) = 0$.

Example: Degree of self-maps of S^1

Recall that the restriction of complex multiplication $z \to z^m$ defines a smooth map $f_m\colon S^1 \to S^1$ for every $m \in \mathbb{Z}$. For $m \neq 0$, let us calculate the derivative $d(f_m)_z\colon T_z(S^1) \to T_{f_m(z)}(S^1)$.

We use the parametrization $\phi t \mapsto (\cos t, \sin t)$. We have the commutative diagram

$$\begin{tikzcd} S^1 \arrow[r, "f_m"] & S^1 \\ \mathbb{R} \arrow[u, "\phi"] \arrow[r, "t\mapsto mt"'] & \mathbb{R}. \arrow[u, "\phi"'] \end{tikzcd}$$

Taking derivatives yields, where we note that $t \mapsto mt$ is a linear map and therefore equal to its derivative:

$$\begin{tikzcd} T_z(S^1) \arrow[r, dotted, "d(f_m)_z"] & T_{z^m}(S^1) \\ \mathbb{R} \arrow[u, "d\phi_t"] \arrow[r, "t\mapsto mt"'] & \mathbb{R}. \arrow[u, "d\phi_{mt}"'] \end{tikzcd}$$

In order to determine $d(f_m)_z$, recall

$$d\phi_t\colon \mathbb{R} \to \mathbb{R}^2,\ s \mapsto (-\sin t, \cos t) \cdot s$$

and, hence at $z = \phi(t)$ (we have done this a long time ago):

$$T_z(S^1) = (-\sin t, \cos t) \cdot \mathbb{R}.$$

Putting these information together we obtain we get

$$\begin{aligned} d(f_m)_z &\colon T_z(S^1) \to T_{z^m}(S^1), \\ (-\sin t, \cos t) \cdot s &\mapsto m(-\sin(mt), \cos(mt)) \cdot s. \end{aligned}$$

Hence, when $m > 0$, f_m wraps the circle uniformly around itself m times preserving orientation. The map is everywhere regular and orientation preserving, so its degree is the number of preimages of any point, that is m.

Similarly, when $m < 0$ the map is everywhere regular but orientation reversing. As each point has $|m|$ preimages, the degree is $-|m| = m$.

Finally,when $m = 0$ the map is constant, so its degree is zero.

One homotopy class $S^1 \to S^1$ for every integer

One immediate consequence of this calculation (which could not have been proven with mod 2 theory) is the interesting fact that the circle admits an **infinite number** of homotopically distinct mappings. For since $\deg(z^m) = m$, none of these maps can be homotopic to another one.

Application: The Fundamental Theorm of Algebra - again

Now we can finish the proof of the Fundamental Theorem of Algebra using degrees. Remember that mod 2-degrees were only good enough for polynomials of odd order. Now we can deal with all of them.

So let

$$p(z) = z^m + a_1 z^{m-1} + \cdots + a_m$$

be a monic complex polynomial. For the argument in the case m odd, we used the homotopy from $p_0(z) = z^m$ to $p_1(z) = p(z)$ defined by

$$p_t(z) = tp(z) + (1-t)z^m = z^m + t(a_1 z^{m-1} + \cdots + a_m).$$

We observed that, if W is a closed ball around the origin in $\mathbb{C}$ with sufficiently large radius, none of the p_t has a zero on ∂W.

Thus the homotopy

$$\frac{p_t}{|p_t|} : \partial W \to S^1$$

is defined for all $t \in [0,1]$. Thus

$$\deg\left(\frac{p}{|p|}\right) = \deg\left(\frac{p_0}{|p_0|}\right).$$

Since $p_0(z) = z^m$, the degree of $p_0/|p_0|$ is the same as $\deg(z^m) = m$, and hence

$$\deg\left(\frac{p}{|p|}\right) = m.$$

Thus, if $m > 0$, $p/|p|$ does not extend to all of W, since otherwise its degree had to be zero. Hence p must have a zero inside W.

Hopf Degree Theorem in dimension one

We return our attention to self-maps of S^1. We learned that there is a homotopy class of maps $S^1 \to S^1$ for every integer m. Actually, the following theorem, the one-dimensional case of a famous theorem of Hopf, shows that the degree is a **bijective map**

$$\deg\colon [S^1,S^1] \to \mathbb{Z},\ f \mapsto \deg(f),$$

where $[S^1,S^1] = \mathrm{Hom}(S^1,S^1)/\sim$ denotes the set of equivalence classes of maps from S^1 to S^1 modulo the homotopy relation.

The same is true for every $n \geq 1$: For every $m \in \mathbb{Z}$, there is exactly one homotopy class of maps $S^n \to S^n$. We will get back to this important result later. Today we show:

Hopf Degree Theorem in dimension one

Two maps $f_0, f_1\colon S^1 \to S^1$ are homotopic if and only if they have the same degree.

Proof: We already know that if f_0 and f_1 are homotopic, then $\deg(f_0) = \deg(f_1)$.

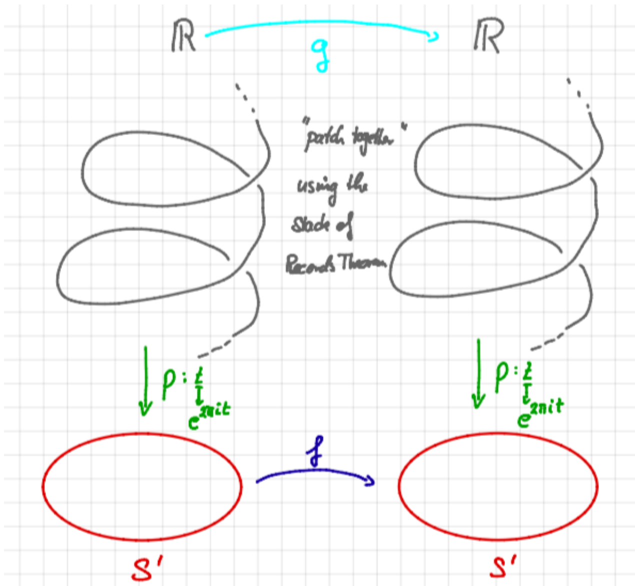
So assume $\deg(f_0) = \deg(f_1)$, and we want to show $f_0 \sim f_1$.

Remember that earlier we used the map p defined by

$$p\colon \mathbb{R} \to S^1,\ t \mapsto e^{2\pi i t},$$

and remarked that every smooth map $f\colon S^1 \to S^1$ can be lifted (lift piecewise and then patch together) to a map $g\colon \mathbb{R} \to \mathbb{R}$ with

$$g(t+1) = g(t) + q \text{ for some } q \in \mathbb{Z} \text{ such that } f(p(t)) = p(g(t)).$$



If we can show $q = \deg(f)$, then we **get a homotopy** $f_0 \sim f_1$ **as follows**:

Let g_0 and g_1 be smooth maps $\mathbb{R} \to \mathbb{R}$ with $g_0(t+1) = g_0(t) + q$, $g_1(t+1) = g_1(t) + q$ and $f_0(p(t)) = p(g_0(t))$, $f_1(p(t)) = p(g_1(t))$. Then the map $g_s(t) := sg_1 + (1-s)g_0$ also satifsies $g_s(t+1) = g_s(t) + q$. Note $g_s(t)$ defines a homotopy G from g_0 to g_1 by $G(t,s) = g_s(t)$.

But any homotopy

$$G \colon \mathbb{R} \times [0,1] \to \mathbb{R} \text{ with } G(t+1,s) = G(t,s) + q \text{ for all } t,s$$

induces a well-defined homotopy

$$F \colon S^1 \times [0,1] \to S^1, (z,s) \mapsto p(G(t,s)) \text{ for any } t \in p^{-1}(z).$$

Hence the above $g_s(t)$ induces a homotopy from

$$f_0 = p \circ g_0 \text{ to } p \circ g_1 = f_1.$$

It remains to show:

Claim: $q = \deg(f)$**.**

First, note that if f is **not surjective**, then we can pick a point $y \notin f(S^1)$. This y is automatically a regular value. Since $\#f^{-1}(p) = 0$, we must have $\deg(f) = 0$. In this case, we need to have $q = 0$, i.e. $g(t+1) = g(t)$. For otherwise $p \circ g$ was surjective and hence f would be surjective.

Note that, since the stereographic projection map $S^1 \setminus \{y\} \to \mathbb{R}$ is a diffeomorphism and $\mathbb{R}$ is contractible, this shows directly that $S^1 \setminus \{y\}$ is contractible. Hence f is a map to a contractible space and therefore **homotopic to a constant map**.

Now we assume that f is **surjective**. Let $y \in S^1$ be a regular value of f, and let $z \in f^{-1}(y)$. Since p is surjective, there is a $t \in \mathbb{R}$ with $p(t) = z$. Since y is a regular value, f is a local diffeomorphism around z. Its derivative is related to the one of g by the chain rule

$$df_z \circ dp_t = dp_{g(t)} \circ dg_t.$$

The derivative of $p\colon \mathbb{R} \to S^1$ at any t is

$$dp_t\colon \mathbb{R} \to T_{p(t)}(S^1),\ w \mapsto 2\pi(-\sin(2\pi t), \cos(2\pi t)) \cdot w.$$

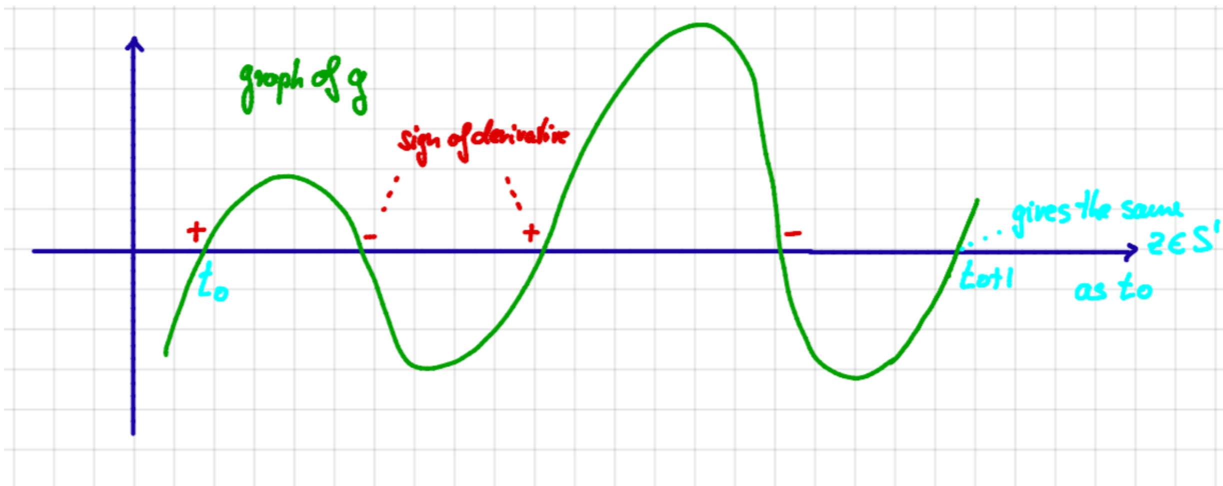
Hence the determinant of dp_t at any t is positive (in fact equal $+2\pi$). Thus the sign of the determinant of df_z equals the sign of $dg_t \in \mathbb{R}$.

As above, let $y \in S^1$ be a regular value of f and $z \in f^{-1}(y)$. Let us fix a $t_0 \in \mathbb{R}$ with $p(t_0) = z$. When we walk from t_0 to $t_0 + 1$ we need to **count** how many preimages of y we collect along the way, **with their orientation (!)**.

We start with the case $q = 0$, i.e. $g(t+1) = g(t)$. It will actually teach us all we need to remember from this proof.

We need to count how often $g(s) = g(t_0)$ with $dg_s = g'(s) > 0$ and how often $g(s) = g(t_0)$ with $dg_s = g'(s) > 0$. Note that since y is **regular**, dg_s is always $\neq 0$ at such those s.

Since g is a smooth function $\mathbb{R} \to \mathbb{R}$, this is now just an exercise from Calculus. Using the periodicity of g, i.e., that $g'(t_0)$ must have the same sign as $g'(t_0+1)$, we see that there are **exactly as many** points s with $g(s) = g(t_0)$ and $dg_s = g'(s) > 0$ as there are points with $g(s) = g(t_0)$ and $dg_s = g'(s) > 0$. Thus $\deg(f) = 0$.



Now assume $q > 0$, and $g(t+1) = g(t) + q$.

Again, we walk from t_0 to t_0+1 and sum up the orientation numbers of all the preimages of y that we collect along the way. This corresponds to counting how often we have $g(s) = g(t_0) + i$ for some $i = 0,1,\dots,q-1$ and $s \in [t_0,t_0+1]$.

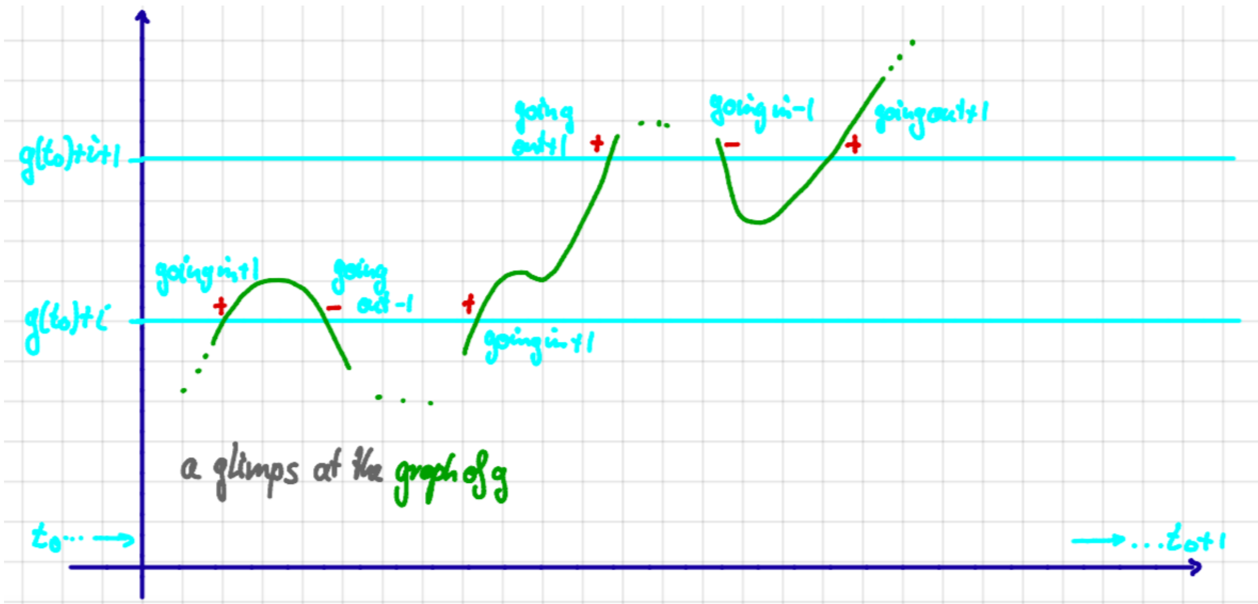
Let us look at one interval $[g(t_0)+i, g(t_0)+i+1]$ at a time. We would like to know how many $s \in [t_0,t_0+1]$ are sent to either $g(t_0)+i$ or $g(t_0)+i+1$ together with the sign of the derivative.

Therefore we look at the preimage

$$g^{-1}([g(t_0)+i, g(t_0)+i+1]).$$

This set is a disjoint union of closed intervals. For each of these intervals the start and endpoints are sent to either $g(t_0)+i$ or $g(t_0)+i+1$.

Let us think of the graph of g passing $g(t_0)+i$with a positive sign of the derivative as **going in with** $+1$ and passing $g(t_0)+i+1$ with a positive sign of the derivative as **going out** $+1$, and the other two alternatives as the ones with -1. Then we see that the graph has to go in with $+1$ for a first time, and has to go out with $+1$ for a last time (since the graph starts at $g(t_0) \leq g(t_0)+i$ and ends at $g(t_0)+q \geq g(t_0)+i+1$). In between those two points, the graph is going out with -1 as often as it goes in $+1$ and goes in with -1 as often as it goes out with $+1$.



Thus in total the orientation numbers for $g^{-1}([g(t_0)+i, g(t_0)+i+1])$ add up to $+2$. Repeating this for all $i = 0,1,\ldots,q-1$ gives a sum of orientation numbers equal to q, since we have to account for that we counted the inner points twice.

Since the sum of orientation numbers of f equals the one of g, this shows $\deg(f) = q$.

If $q < 0$, the same argument works with signs and directions reversed. **QED**