Norwegian University of Science and Technology
Department of Mathematical Sciences

TMA4145 Linear Methods
Fall 2018

Exercise set 11: Solutions

Please justify your answers! The most important part is *how* you arrive at an answer, not the answer itself.

1 Consider the exponential basis $\{e^{2\pi int} : n \in \mathbb{Z}\}$ in $(L^2[0,1], \langle .,. \rangle)$. Verify Parseval's identity for this particular case directly.

Solution. We need to show that $\langle e^{2\pi int}, e^{2\pi imt} \rangle = 0$ if $n \neq m$, and $\langle e^{2\pi int}, e^{2\pi int} \rangle = 1$. We start with the last case.

$$\begin{aligned}
\langle e^{2\pi int}, e^{2\pi int} \rangle &= \int_0^1 e^{2\pi int}\, \overline{e^{2\pi int}}\, dt \\
&= \int_0^1 e^{2\pi int}\, e^{-2\pi int}\, dt \\
&= \int_0^1 e^{2\pi i(n-n)t}\, dt \\
&= \int_0^1 e^0\, dt \\
&= \int_0^1 1\, dt \\
&= 1.
\end{aligned}$$

Suppose $n \neq m$. We have

$$\begin{aligned}
\langle e^{2\pi int}, e^{2\pi imt} \rangle &= \int_0^1 e^{2\pi int}\, \overline{e^{2\pi imt}}\, dt \\
&= \int_0^1 e^{2\pi int}\, e^{-2\pi imt}\, dt \\
&= \int_0^1 e^{2\pi i(n-m)t}\, dt \\
&= \left.\frac{e^{2\pi i(n-m)t}}{2\pi i(n-m)}\right|_0^1 \\
&= \frac{e^{2\pi i(n-m)} - e^0}{2\pi i(n-m)} \\
&= \frac{1-1}{2\pi i(n-m)} && (e^{2\pi i(n-m)} = 1 \text{ because } n-m \in \mathbb{Z}) \\
&= 0,
\end{aligned}$$

hence the system is orthonormal.

$\boxed{2}$ Let $\{e_n : n \in \mathbb{N}\}$ be the standard basis in the sequence space ℓ^p.

a) Show that the series $\sum_{n=0}^{\infty} \alpha_n e_n$ converges in ℓ^p for $1 \le p < \infty$ if and only if $(\alpha_n)_{n\in\mathbb{N}} \in \ell^p$.

b) Show that the series $\sum_{n=0}^{\infty} \alpha_n e_n$ converges in ℓ^∞ if and only if $(\alpha_n)_{n\in\mathbb{N}}$ converges to zero.

Solution. **a)** Recall what it means for a series to converge in a normed space such as ℓ_p: we need the partial sums

$$s_N = \sum_{n=0}^{N} \alpha_n e_n$$

to form a convergent sequence in ℓ^p. Assume first that $\sum_{n=0}^{\infty} \alpha_n e_n$ converges in ℓ^p to some sequence $x = (x_1, x_2, ...) \in \ell^p$. As we have seen before, if a sequence y_n converges to some y in a normed space, then $\|y_n\|$ converges to $\|y\|$. In this case we know that $s_N = \sum_{n=0}^{N} \alpha_n e_n$ converges to $x \in \ell^p$, so

$$\|s_N\|_{\ell^p}^p = \|\sum_{n=0}^{N} \alpha_n e_n\|_{\ell^p}^p = \sum_{n=0}^{N} |\alpha_n|^p \to \|x\|_{\ell^p}^p.$$

But this means that the partial sums $\sum_{n=0}^{N} |\alpha_n|^p$ converge to the number $\|x\|_{\ell^p}^p$, which shows that the sum $\sum_{n=0}^{\infty} |\alpha_n|^p$ converges and hence $(\alpha_n)_{n\in\mathbb{N}} \in \ell^p$.

Conversely, assume that $(\alpha_n)_{n\in\mathbb{N}} \in \ell^p$. We want to show that the sum $\sum_{n=0}^{\infty} \alpha_n e_n$ converges to the sequence $\alpha = (\alpha_n)_{n\in\mathbb{N}}$ in ℓ^p. As discussed, this means that we need to show that the partial sums $s_N = \sum_{n=0}^{N} \alpha_n e_n$ converge to α in ℓ^p. Note that

$$\alpha - s_N = (0, 0, 0, ..., 0, \alpha_{N+1}, \alpha_{N+2}, ...),$$

and therefore

$$\|\alpha - s_N\|_{\ell^p}^p = \sum_{n=N+1}^{\infty} |\alpha_n|^p.$$

Since $\alpha \in \ell^p$ we know that $\sum_{n=0}^{\infty} |\alpha_n|^p$ converges, and as we have used in many previous problems sets this implies that $\sum_{n=N+1}^{\infty} |\alpha_n|^p \to 0$ as $N \to \infty$. Hence $\|\alpha - s_N\|_{\ell^p}^p \to 0$, so the partial sums s_N converge to α in ℓ^p and this means by definition that $\sum_{n=0}^{\infty} \alpha_n e_n$ converges to α in ℓ^p.

b) Start by assuming that $\alpha_n \to 0$. To show that $\sum_{n=0}^{\infty} \alpha_n e_n$ converges in ℓ^∞, we will show that the partial sums $s_N = \sum_{n=0}^{N} \alpha_n e_n$ converge to α. Clearly

$$\alpha - s_N = (\alpha_1, \alpha_2, ...) - (\alpha_1, \alpha_2, ...\alpha_N, 0, 0, ...) = (0, 0, ..., 0, \alpha_{N+1}, \alpha_{N+2}, ...)$$

so

$$\|\alpha - S_N\|_\infty = \|(0, 0, ..., 0, \alpha_{N+1}, \alpha_{N+2}, ...)\|_\infty = \sup_{n \ge N+1} |\alpha_n|.$$

Since $\alpha_n \to 0$, we know that $\sup_{n \geq N+1} |\alpha_n| \to 0$ as $N \to \infty$ (Do make sure that you understand why this is true!).

Conversely, assume that $\sum_{n=0}^{\infty} \alpha_n e_n$ converges in ℓ^∞. This means that the partial sums $s_N = \sum_{n=0}^{N} \alpha_n e_n$ converge to some element $x = (x_1, x_2, ...)$ in ℓ^∞. It is simple to see that $\alpha_i = x_i$ for each $i \in \mathbb{N}$: If $N > i$, then

$$x - s_N = (x_1 - \alpha_1, x_2 - \alpha_2, ..., x_i - \alpha_i, ..., x_N - \alpha_N, x_{N+1}, x_{N+2}, ...),$$

and so

$$|x_i - \alpha_i| \leq \sup_{j \in N} |x_j - s_N(j)| = \|x - s_N\|_\infty \to 0 \text{ as } N \to \infty$$

which implies that $x_i = \alpha_i$. Since we assumed that $s_N \to x$, we now know that $s_N \to \alpha$ in ℓ^∞. Note that

$$\alpha - s_N = (0, 0, ..., 0, \alpha_{N+1}, \alpha_{N+2} ...)$$

so

$$\|\alpha - s_N\|_\infty = \sup_{j > N} |\alpha_j|,$$

and this last expression converges to 0 since $s_N \to \alpha$. This implies that

$$|\alpha_{N+1}| \leq \sup_{j > N} |\alpha_j| = \|\alpha - s_N\|_\infty \to 0 \text{ as } N \to \infty,$$

hence $\alpha_i \to 0$.

$\boxed{3}$ Show that if a normed space $(X, \|\cdot\|)$ has a Schauder basis, then it is separable.

Solution. To prove that X is separable we need to find a countable, dense subset of X. I claim that the set $Y = \{\sum_{n=1}^{N} c_n x_n : c_n \in \mathbb{Q}, N \in \mathbb{N}\}$ satisfies these criteria, i.e. the set of finite linear combinations of basis elements with rational coefficients.

Let us first ask whether this set is countable. For this we will need three different facts on countability:

1. $\mathbb{Q}$ is countable (proposition 1.10)
2. A finite cartesian product of countable sets is countable (proposition 1.9)
3. A countable union of countable sets is countable (proposition 1.9) .

First note that we can write $Y = \cup_{N \in \mathbb{N}} Y_N$, where $Y_N = \{\sum_{n=1}^{N} c_n x_n : c_n \in \mathbb{Q}\}$. Since this is a countable union, point (3) shows that we need only show that Y_N is countable for every $N \in \mathbb{N}$ to show that Y is countable. I claim that we can find a bijection from Y_N to $\mathbb{Q}^N$, and the latter space is countable by points (1) and (2). This bijection is given by sending $\sum_{n=1}^{N} c_n x_n$ in Y_N to $(c_1, c_2, ..., c_N)$ in $\mathbb{Q}^N$. This is clearly both injective (since $(x_n)_n$ is a Schauder basis, the coefficients are unique) and surjective, so Y_N is countable. As we have seen, this implies that Y is countable.

Now we need to prove that Y is dense. Pick an $x \in X$ and an $\epsilon > 0$; we need to find $y \in Y$ such that $\|x-y\|_X < \epsilon$. Since $(x_n)_n$ is a Schauder basis, there is a finite linear combination $\sum\limits_{n=1}^{N} c_n x_n$ such that $\|x - \sum\limits_{n=1}^{N} c_n x_n\|_X < \frac{\epsilon}{2}$. It is well known that $\mathbb{Q}$ is a dense subset of $\mathbb{R}$, so for every c_n we pick a $q_n \in \mathbb{Q}$ such that $|q_n - c_n| < \frac{\epsilon}{2N\|x_n\|}$. Now let $y = \sum\limits_{n=1}^{N} q_n x_n$. We use the triangle inequality to calculate that

$$
\begin{aligned}
\|x-y\| &= \|x - \sum_{n=1}^{N} q_n x_n\| \\
&= \| \left(x - \sum_{n=1}^{N} c_n x_n \right) + \left(\sum_{n=1}^{N} c_n x_n - \sum_{n=1}^{N} q_n x_n \right) \| \\
&\leq \|x - \sum_{n=1}^{N} c_n x_n\| + \| \sum_{n=1}^{N} c_n x_n - \sum_{n=1}^{N} q_n x_n\| \\
&< \frac{\epsilon}{2} + \sum_{n=1}^{N} |c_n - q_n| \|x_n\| \\
&\leq \frac{\epsilon}{2} + \sum_{n=1}^{N} \frac{\epsilon}{2N} = \epsilon.
\end{aligned}
$$

To sum this up we have shown that an arbitrary $x \in X$ may be approximated within a distance of any $\epsilon > 0$ by points in Y, so Y is dense.

4 Let $L^2[-1,1]$ be equipped with the inner product

$$\langle f, g \rangle = \int_{-1}^{1} f(t)\overline{g(t)}dt.$$

Apply Gram-Schmidt's orthogonalization algorithm to the monomial basis $\{1, x, x^2, x^3, ...\}$ up to degree 2.

Solution. We follow the procedure in the proof of proposition 6.6.

$$\tilde{e}_1 = 1 \qquad\qquad e_1 = \frac{1}{\sqrt{\int_{-1}^{1} 1^2\,dx}} = \frac{1}{\sqrt{2}}$$

$$\tilde{e}_2 = x - \frac{1}{\sqrt{2}}\int_{-1}^{1}\frac{1}{\sqrt{2}}x\,dx = x \qquad\qquad e_2 = \frac{x}{\sqrt{\int_{-1}^{1} x^2\,dx}} = \frac{x}{\sqrt{2/3}}$$

$$\tilde{e}_3 = x - \frac{1}{\sqrt{2}}\int_{-1}^{1}\frac{1}{\sqrt{2}}x^2\,dx - \frac{x}{\sqrt{2/3}}\int_{-1}^{1}\frac{x^3}{\sqrt{2/3}}$$

$$= x^2 - \frac{1}{3} \qquad\qquad e_3 = \frac{x^2-\frac{1}{3}}{\sqrt{\int_{-1}^{1}(x^2-\frac{1}{3})^2\,dx}} = \frac{x^2-\frac{1}{3}}{\sqrt{\frac{8}{45}}}$$

$\boxed{5}$ *(Exam 2014, Problem 3)*

a) Let $C([0,2]\times[0,2],\mathbb{R})$ be an inner-product space with

$$\langle f,g\rangle = \int_0^2\int_0^2 f(x,y)g(x,y)\,dx\,dy.$$

Find an orthogonal basis for $\operatorname{span}\{1,x,y\}$ in this space.

b) Find $a,b,c\in\mathbb{R}$ such that

$$\int_0^2\int_0^2 |xy-a-bx-cy|^2\,dx\,dy$$

is minimal.

Solution. a) We use the Gram Schmidt procedure and the notation from the proof of proposition 6.6. Since $x_1 = 1$, the first step is not very complicated:

$$\tilde{e}_1 = 1 \qquad\qquad e_1 = \frac{1}{\sqrt{\int_0^2\int_0^2 1^2\,dx\,dy}} = \frac{1}{2}$$

Next $x_2 = x$. Since $\tilde{e}_2 = x_2 - \langle x_2, e_1\rangle e_1$, we calculate

$$\tilde{e}_2 = x - \frac{1}{2}\int_0^2\int_0^2 x\frac{1}{2}\;dx\;dy = x-1,$$

and since

$$\|\tilde{e}_2\| = \sqrt{\int_0^2\int_0^2 (x-1)^2\;dx\;dy} = \sqrt{\frac{4}{3}},$$

we have

$$e_2 = \frac{\sqrt{3}}{2}(x-1).$$

Finally, $x_3 = y$, and $\tilde{e}_3 = y - \langle x_3, e_1\rangle e_1 - \langle x_3, e_2\rangle e_2$, so

$$\tilde{e}_3 = y - \frac{1}{2}\int_0^2\int_0^2 y\frac{1}{2}\,dx\,dy - \frac{4}{3}\int_0^2\int_0^2 y(x-1)\,dx\,dy = y - 1,$$

since the second integral is zero, as $\int_0^2 (x-1)dx = 0$. Since $\tilde{e}_3$ has the same form as $\tilde{e}_2$, we see that

$$e_3 = \frac{\sqrt{3}}{2}(y-1).$$

The set $\{e_1, e_2, e_3\}$ is then the orthonormal basis we are looking for. Since the problem only asked for an orthogonal basis, we could also have used $\{1, x-1, y-1\}$.

b) The problem asks us to find the element in $\operatorname{span}\{1, x, y\} = \operatorname{span}\{e_1, e_2, e_3\}$ closest to xy, and corollary 6.5 says that this element is given by

$$z = \langle xy, e_1\rangle e_1 + \langle xy, e_2\rangle e_2 + \langle xy, e_3\rangle e_3.$$

We need to find some integrals.

$$\langle xy, e_1\rangle = \frac{1}{2}\int_0^2\int_0^2 xy\,dx\,dy = \frac{1}{2}4 = 2.$$

$$\langle xy, e_2\rangle = \frac{\sqrt{3}}{2}\int_0^2\int_0^2 xy(x-1)\,dx\,dy = \frac{\sqrt{3}}{2}\frac{4}{3} = \frac{2}{\sqrt{3}}.$$

By symmetry we get

$$\langle xy, e_3\rangle = \frac{2}{\sqrt{3}}.$$

Therefore

$$z = 2\frac{1}{2} + \frac{2}{\sqrt{3}}\frac{\sqrt{3}}{2}(x-1) + \frac{2}{\sqrt{3}}\frac{\sqrt{3}}{2}(y-1) = x + y - 1.$$

$\boxed{6}$ Let $\|\cdot\|_a$ and $\|\cdot\|_b$ be equivalent norms on a vector space X. Show that a sequence (x_n) in X is Cauchy with respect to the norm $\|\cdot\|_a$ if and only if it is Cauchy with respect to the norm $\|\cdot\|_b$.

Solution. By assumption, there exist (positive) constants C_1 and C_2 such that

$$C_1\|x\|_a \le \|x\|_b \le C_2\|x\|_a \quad \text{for all } x \in X.$$

Assume that (x_n) is Cauchy with respect to $\|\cdot\|_a$. We need to show that for any $\epsilon > 0$, there exists some $N \in \mathbb{N}$ such that

$$\|x_m - x_n\|_b < \epsilon \quad \text{for any } m, n \ge N.$$

Since we assume that (x_n) is Cauchy with respect to $\|\cdot\|_a$, there exists $N_a \in \mathbb{N}$ such that
$$\|x_m - x_n\|_a < \frac{\epsilon}{C_2} \quad \text{for any } m, n \geq N.$$
Hence if $m, n \geq N$, we have that
$$\|x_m - x_n\|_b \leq C_2 \|x_m - x_n\|_a < C_2 \frac{\epsilon}{C_2} = \epsilon,$$
which is what we needed to prove. The other direction (Cauchy with respect to $\|\cdot\|_b$ implies Cauchy with respect to $\|\cdot\|_a$) is exactly the same argument, this time using the inequality
$$\|x\|_a \leq \frac{\|x\|_b}{C_1}.$$