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Exercise set 3

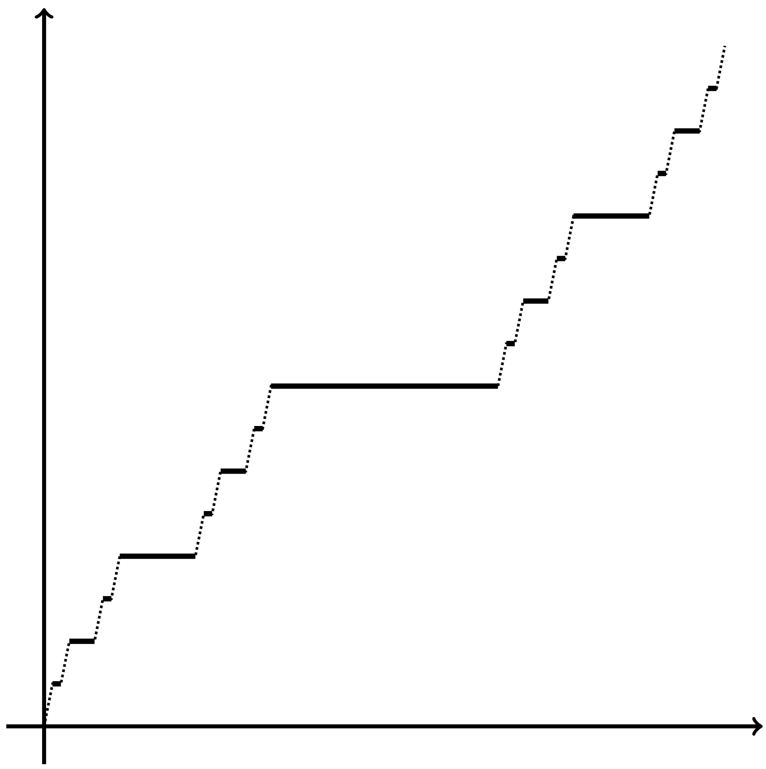
$\boxed{1}$ Recall the Cantor set from exercise sheet 1, which is constructed from the unit interval by removing first the central interval $(1/3, 2/3)$, then the two intervals $(1/9, 2/9)$ and $(7/9, 8/9)$, then four (central) intervals of length $1/27$, and so on.

We now define *Lebesgue's singular function* (sometimes called *Devil's staircase*) by setting

$$f(x) = \begin{cases} 1/2 & \text{for } 1/3 \leq x \leq 2/3, \\ 1/4 & \text{for } 1/9 \leq x \leq 2/9, \\ 3/4 & \text{for } 7/9 \leq x \leq 8/9, \\ \vdots & \end{cases}$$

Or, using again ternary expansions,[1] we have

$$f(x) = \sum_{k=1}^{\infty} \lfloor a_k/2 \rfloor 2^{-k} \text{ for } x = \sum_{k=1}^{\infty} a_k 3^{-k}.$$



[1] Here we have to use finite expansions whenever possible.

a) Show that f is continuous.

Hint: either show that f is the uniform limit of a sequence of continuous functions, or use the fact that the function f is a monotone continuation of the function defined in exercise 1c) on exercise sheet 1.

b) Show that f is differentiable with $f'(x) = 0$ in every point $x \notin C$.

c) Conclude that $f \notin H^1(0,1)$.

- *Possible solution:*

a) The function f is by construction non-decreasing, and from problem 1c) on exercise sheet 1 it follows that f is surjective. As a consequence, it is continuous.

(Assume that $0 < x < 1$. Then the monotonicity of f implies that the left and right limits

$$f(x^-) := \lim_{\substack{\varepsilon \to 0 \\ \varepsilon > 0}} f(x - \varepsilon) \qquad \text{and} \qquad f(x^+) := \lim_{\substack{\varepsilon \to 0 \\ \varepsilon > 0}} f(x + \varepsilon)$$

exist and satisfy

$$f(x^-) \leq f(x) \leq f(x^+).$$

For proving continuity, we have to show that all these three terms are equal. However, if there exists $c \in [0,1]$ such that $f(x^-) < c < f(x)$, then c cannot be contained in the image of f, since $f(y) > c$ for all $y \geq x$ and $f(y) < c$ for all $y < x$. Thus $f(x^-) = f(x)$, and similarly $f(x) = f(x^+)$.)

b) The set $[0,1] \setminus C$ is the union of open sets of the form $(\ell/3^k, (\ell+1)/3^k)$ (cf. the solution of problem 1b) on exercise sheet 1), and on each of these sets the function f is constant, and thus differentiable with $f' = 0$.

c) Assume that f is weakly differentiable with weak derivative $g \in L^2([0,1])$. Since f is differentiable on the open set $[0,1] \setminus C$ with $f'(x) = 0$ for every $x \in [0,1] \setminus C$, it follows that $g(x) = 0$ for almost every $x \in [0,1] \setminus C$. Since C is negligible, this implies that, actually, $g(x) = 0$ for almost every $x \in [0,1]$. Now consider any function $\varphi \in C_0^1([0,1])$ such that

$$\varphi(x) = \begin{cases} 0 & \text{for } x \leq 1/9, \\ 1 & \text{for } 2/9 \leq x \leq 7/9, \\ 0 & \text{for } x \geq 8/9. \end{cases}$$

Then the weak differentiability of f with weak derivative $g = 0$ implies that

$$0 = \int_0^1 \varphi(x) g(x)\, dx = -\int_0^1 \varphi'(x) f(x)\, dx.$$

On the other hand, we have that $\varphi'(x) = 0$ for $x \notin (1/9, 2/9) \cup (7/9, 8/9)$. Because f is classically differentiable in these intervals with $f' = 0$ and $f(x) = 1/4$ for $x \in (1/9, 2/9)$ and $f(x) = 3/4$ for $x \in (7/9, 8/9)$, it follows that

$$\begin{aligned} \int_0^1 \varphi'(x) f(x)\, dx &= \int_{1/9}^{2/9} \varphi'(x) f(x)\, dx + \int_{7/9}^{8/9} \varphi'(x) f(x)\, dx \\ &= \frac{1}{4} \int_{1/9}^{2/9} \varphi'(x)\, dx + \frac{3}{4} \int_{7/9}^{8/9} \varphi'(x)\, dx \\ &= \frac{1}{4} \varphi(x)\Big|_{1/9}^{2/9} + \frac{3}{4} \varphi(x)\Big|_{7/9}^{8/9} \\ &= \frac{1}{4} - \frac{3}{4} = -\frac{1}{2} \neq 0. \end{aligned}$$

Thus f cannot be weakly differentiable.

$\boxed{2}$ The *brachistochrone problem*[2] is one of the classical problems in the calculus of variations, and was posed (and solved) by Bernoulli in 1696. Given two points A and B, it asks for the path from A to B along which an object would slide in the shortest possible time, if it is originally at rest and is only accelerated by gravity (we disregard friction).

We can assume without loss of generality that $A = (0,0)$. In order to simplify the notation, we will assume moreover that the y-axis points downwards and that the point B is situated at $B = (a,b)$ with $a > 0$ and $b > 0$. We can then write the path as the graph of a function $y\colon [0,a] \to \mathbb{R}$ with $y(0) = 0$ and $y(a) = b$.[3]

In the following, we will derive an explicit formula for the travel time along the path y. To that end, we will denote by v the speed of the object.

a) Using the principle of conservation of energy, show that v and y satisfy the relation

$$v(x) = \sqrt{2gy(x)},$$

where g is the gravitational acceleration at the earth's surface. Show moreover that $y(x) \geq 0$ for all x.

b) The length of the path between 0 and x is given by $s(x) = \int_0^x \sqrt{1 + y'(\hat{x})^2}\, d\hat{x}$. Use this and the formula for the speed in order to show that the travel time along the path y is given by

$$T(y) = \int_0^a \frac{\sqrt{1 + y'(x)^2}}{\sqrt{2gy(x)}}\, dx.$$

As a consequence, we have obtained a reformulation of the brachistochrone problem as a typical variational problem of the form: "Minimise $T(y)$ subject to the constraints $y(0) = 0$ and $y(a) = b$."

c) Formulate (and simplify) the Euler–Lagrange equation for the brachistochrone problem and show that the solution satisfies

$$y + y(y')^2 = C$$

for some constant $C > 0$.

d) Show that the solution can be parameterised as

$$t \mapsto \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = C \begin{pmatrix} t - \frac{1}{2}\sin 2t \\ \frac{1}{2} - \frac{1}{2}\cos 2t \end{pmatrix}.$$

- *Possible solution:*

a) Conservation of energy in this (frictionless, mechanical) system requires that the sum of potential energy E_{pot} and kinetic energy E_{kin} remains constant. Here we have (up to shifts in the potential energy)

$$E_{\text{pot}}(x) = -mgy(x) \qquad \text{and} \qquad E_{\text{kin}}(x) = \frac{1}{2}mv(x)^2.$$

[2] The term is composed of the greek words *brachistos* meaning shortest and *chronos* meaning time. Thus it literally translates to *shortest time problem*.

[3] Try to argue why this must be the case!

Since at time $t = 0$ (and position $x = 0$) we have $y(0) = v(0) = 0$, it follows that

$$mgy(x) = \frac{1}{2}mv(x)^2$$

for all x. In particular, $y(x) \geq 0$ for all x and we have

$$v(x) = \sqrt{2gy(x)}.$$

b) Differentiating the path length with respect to x, we obtain that

$$\frac{ds}{dx} = \sqrt{1 + y'(x)^2}.$$

Moreover, the length L of the whole path is given by

$$L = \int_0^a \sqrt{1 + y'(x)^2}\, dx.$$

Now we switch from the space variable x to the time variable t. By definition of velocity, we have

$$v(t) = \frac{ds}{dt}$$

or

$$\frac{dt}{ds} = \frac{1}{v(s)}.$$

Therefore, if we denote by T the total travel time, we obtain (after some changes of variables)

$$T = \int_0^T dt = \int_0^L \frac{1}{v(s)}\, ds = \int_0^a \frac{1}{\sqrt{2g\,y(x)}} \sqrt{1 + y'(x)^2}\, dx.$$

Thus we can formulate the brachistochrone problem as the minimization of the functional

$$T(y) := \int_0^a \frac{\sqrt{1 + y'(x)^2}}{\sqrt{2g\,y(x)}}\, dx$$

subject to the constraints $y(0) = 0$ and $y(a) = b$.

c) Denote the integrand in the functional T by f, that is,

$$f(x, y, y') = \sqrt{\frac{1 + y'^2}{2g\,y}}.$$

Then the Euler–Lagrange equation for this problem is the equation

$$\frac{\partial}{\partial y} f(x, y, y') = \frac{d}{dx} \frac{\partial}{\partial y'} f(x, y, y')$$

with boundary conditions $y(0) = 0$ and $y(a) = b$. We have

$$\frac{\partial}{\partial y} f(x, y, y') = -\frac{1}{2} \sqrt{\frac{1 + y'^2}{2g}} \frac{1}{y^{3/2}}$$

and

$$\frac{\partial}{\partial y'} f(x, y, y') = \frac{1}{\sqrt{2g\,y}} \frac{y'}{\sqrt{1 + y'^2}}.$$

Multiplying everything with the constant factor $\sqrt{2g}$, we thus obtain the equation[4]

$$-\frac{1}{2}\sqrt{\frac{1+y'^2}{y^3}} = \frac{d}{dx}\frac{y'}{\sqrt{y(1+y'^2)}}.$$

The total derivative on the right hand side can be expanded as

$$\frac{d}{dx}\frac{y'}{\sqrt{y(1+y'^2)}} = \frac{y''}{\sqrt{y(1+y'^2)}} - \frac{1}{2}\frac{y'^2}{\sqrt{y^3(1+y'^2)}} - \frac{y'^2y''}{\sqrt{y(1+y'^2)^3}}.$$

Thus we obtain the equation

$$-\frac{1}{2}\sqrt{\frac{1+y'^2}{y^3}} = \frac{y''}{\sqrt{y(1+y'^2)}} - \frac{1}{2}\frac{y'^2}{\sqrt{y^3(1+y'^2)}} - \frac{y'^2y''}{\sqrt{y(1+y'^2)^3}}.$$

Multiplying this equation with $\sqrt{y(1+y'^2)}$, this becomes

$$-\frac{1}{2}\frac{1+y'^2}{y} = y'' - \frac{1}{2}\frac{y'^2}{y} - \frac{y'^2y''}{1+y'^2},$$

which can be simplified to

$$-\frac{1}{2y} = \frac{y''}{1+y'^2}$$

or

$$1 + 2yy'' + y'^2 = 0.$$

Multiplication with y' gives

$$y' + 2yy'y'' + y'^3 = 0.$$

Now note that the left hand side of this equation is actually the derivative of the function

$$y + yy'^2.$$

Thus it follows that

$$y + yy'^2 = C$$

for some constant $C > 0$.

d) Solving for y' gives

$$y' = \sqrt{\frac{C-y}{y}}.$$

With separation of variables, we now obtain

$$\sqrt{\frac{y}{C-y}}dy = dx,$$

which can be integrated to

$$x = \int\sqrt{\frac{y}{C-y}}dy + D$$

for some constant $D \in \mathbb{R}$. Now we substitute

$$y = C\sin^2 t$$

[4]There exist more elegant ways for solving this problem...

with $0 < t < \pi/2$ and obtain (since $dy = 2C \sin t \cos t dt$)

$$\int \sqrt{\frac{y}{C-y}} dy = \int \sqrt{\frac{C \sin^2 t}{C - C \sin^2 t}} 2C \sin t \cos t \, dt = 2C \int \sin^2 t \, dt.$$

Using the fact that

$$\sin^2 t = \frac{1}{2} - \frac{1}{2} \cos 2t,$$

we obtain

$$\int \sin^2 t \, dt = \int \frac{1}{2} - \frac{1}{2} \cos 2t \, dt = \frac{t}{2} - \frac{1}{4} \sin 2t.$$

Thus we get (in the variable t)

$$x(t) = Ct - \frac{C}{2} \sin 2t + D$$

with

$$y(t) = C \sin^2 t = \frac{C}{2} - \frac{C}{2} \cos 2t.$$

Since the path has to pass through the point $(0,0)$, it follows that $D = 0$. Thus the path can be parametrized as

$$t \mapsto \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = C \begin{pmatrix} t - \frac{1}{2} \sin 2t \\ \frac{1}{2} - \frac{1}{2} \cos 2t \end{pmatrix}$$

with $C > 0$ being such that it passes through the point (a, b). Note that this path is actually a Cycloid.