TMA4190 Introduction to Topology Spring 2018

Solutions to exercise set 11

$\boxed{1}$ We define

$$f\colon \mathbb{C} \to \mathbb{C},\ z \mapsto z^7 + \cos(|z|^2)(1 + 93z^4).$$

We can define a homotopy from $f_0(z) = z^7$ to $f_1(z) = f(z)$ by

$$f_t(z) = tf(z) + (1-t)z^7 = z^7 + t\cos(|z|^2)(1 + 93z^4).$$

Since $|z|^7$ dominates the absolute value of $f_t(z)$ for all $t \in [0,1]$, or in other words, since the second summand in the parantheses goes to 0 when $z \to \infty$ in

$$\frac{f_t(z)}{z^7} = 1 + t\frac{\cos(|z|^2)}{z^7}(1 + 93z^4),$$

$f_t(z)$ has no zero on the boundary of a closed ball $W \subset \mathbb{C}$ of large enough radius. Thus the homotopy

$$\frac{f_t(z)}{|f_t(z)|}\colon \partial W \to S^1$$

is defined for all t. By the homotopy invariance of $\deg_2$, we have

$$\deg_2\left(\frac{f(z)}{|f(z)|}\right) = \deg_2\left(\frac{f_0(z)}{|f_0(z)|}\right) = \deg_2(z^7) = 1 \mod 2.$$

Hence, by the result of the lecture on extending maps from the boundary, $f(z)$ must have a zero inside W.

$\boxed{2}$ **a)** Let $Z \subset Y$ be any closed submanifold with $\dim X + \dim Z = \dim Y$. For $\dim X \geq 1$, this implies $\dim Z = \dim Y - \dim X < \dim Y$.

Now let $g\colon X \to \{y\}$ be a constant map with $y \in Y$ such that $f \sim g$. In order to calculate $I_2(g, Z)$, we can assume $g \pitchfork Z$, for otherwise we can replace it by another constant map which is homotopic to g and satisfies transversality to Z. But $g \pitchfork Z$ just means $\{y\} \pitchfork Z$ in Y. Since $\dim Z < \dim Y$ and $\dim\{y\} = 0$, this can only happen if $y \notin Z$. This means $g^{-1}(Z) = \emptyset$, i.e.

$$I_2(g, Z) = \#g^{-1}(Z) = 0.$$

By the homotopy invariance of $I_2(-, Z)$ we have $I_2(f, Z) = I_2(g, Z)$, and hence $I_2(f, Z) = 0$.

b) Assume that $X = \{x\}$ is a one-point space. In particular, we have $\dim X = 0$. Thus any closed subamnifold $Z \subset Y$ with $\dim X + \dim Z = \dim Y$ is of dimension $\dim Z = \dim Y$. Hence Z intersects every submanifold of Y transversally. In particular, $\{x\} \pitchfork Z$, i.e. $f \pitchfork Z$ for every map $f\colon X = \{x\} \to Y$. Hence, by definition of intersection numbers, we can calculate $I_2(f, Z)$ by assuming $f(x) = y \in Z$ and get

$$I_2(f, Z) = \#f^{-1}(Z) = \#\{y\} = 1 \neq 0.$$

c) For every compact manifold X, the identity map satisfies $I_2(\mathrm{Id}_X, X) = 1$, since it is transversal to X and every fiber consists of one point. Applied to $\mathrm{id}\colon S^1 \to S^1$, this means $I_2(\mathrm{Id}, S^1) = 1 \neq 0$. But we just learned that if id was homotopic to a constant map, then $I_2(\mathrm{Id}, S^1)$ has to be zero. Htus there is at least one map $S^1 \to S^1$ which is not homotopic to a constant map. (In fact, up to homotopy there is a homotopy class of maps $S^1 \to^1$ for every integer $n \in \mathbb{Z}$ represented by taking nth powers in $\mathbb{C}$: $z \mapsto z^n$. You will learn more about that in Algebraic Topology.)

$\boxed{3}$ **a)** Let Y be contractible and $\dim Y > 0$. Let $f\colon X \to Y$ be a smooth map with X compact and $Z \subset Y$ closed, and $\dim X + \dim Z = \dim Y$. We proved before that Y being contractible implies that $f\colon X\ toY$ is homotopic to a constant map. Hence $I_2(f, Z) = 0$ by the previous exercise, as $\dim Y \geq 1$.

Finally, we learned before that Euclidean space $\mathbb{R}^k$ is conatractilbe.

b) If X is compact, then the intersction number $I_2(\mathrm{Id}_X, X)$ is defined and satisfies $I_2(\mathrm{Id}_X, X) = 1$. But we just learned that if X was contractible and $\dim X \geq 1$, then $I_2(\mathrm{Id}_X, X)$ must be zero. Hence if $\dim X \geq 1$, X cannot be compact and contractible.

If $\dim X$, then the only way X can be contractible is that it consists of just one point.

$\boxed{4}$ **a)** Let $f\colon X \to S^k$ be a smooth map with X compact and $0 < \dim X < k$. Let Z be a closed submanifold $Z \subset S^k$ of dimension complementary to X, i.e. $\dim X + \dim Z = \dim S^k$. The image of the derivative df_x at any point has at most dimension $\dim X = \dim T_x(X)$ in $T_{f(x)}(S^k)$. Hence, since $\dim X < k$, df_x cannot be surjectve. Hence the only points in S^k which are regular values of f are the points which are not in the image of f. By Sard's Theorem, regular values exist, Hence there must be a point $p \in S^k$ with $p \notin f(X) \cap Z$.

Now we can use stereographic projection to construct a diffeomorphism $\phi\colon S^k \setminus \{p\} \to \mathbb{R}^k$. Since p is not in the image of f, the compistion $\phi \circ f\colon X \to \mathbb{R}^k$ is defined. Since p is not in Z, $\phi_{|Z}$ is a diffeomorphism and we can $\phi(Z)$ is submanifold of dimension $\dim Z$ which is diffeomorphic to Z. This implies

$$I_2(f, Z) = I_2(\phi \circ f, \phi(Z))$$

where the latter is the intersection number mod 2 in $\mathbb{R}^k$. But since $\mathbb{R}^k$ is contractible, the previous exercise implies $I_2(\phi \circ f, \phi(Z)) = 0$. Thus $I_2(f, Z) = 0$.

b) Considering $S^1 \subset \mathbb{C}$, we can define tow submanifolds X and Z as $X = S^1 \times \{1\}$ and $Z = \{1\} \times S^1$ in $S^1 \times S^1$. They intersect in the single point $q := \{1\} \times \{1\} \in S^1 \times S^1$. The tangent space to $S^1 \times S^1$ at q is

$$T_q(S^1 \times S^1) = T_1(S^1) \times T_1(S^1) = T_1(X) \times T_1(Z).$$

Hence X and Z meet transversally in $S^1 \times S^1$. Thus their intersection number in $S^1 \times S^1$ is $I_2(X, Z) = 1$.
Now assume there was a diffeomorphism $\varphi\colon S^1 \times S^1 \to S^2$. The map $\iota\colon S^1 \hookrightarrow S^1 \times S^1$, $z \mapsto (z, 1)$ maps S^1 diffeomorphically onto X. Composition with φ gives us a map $\varphi \circ \iota\colon S^1 \to S^2$. Since φ is a diffeomorphism, $\varphi(Z)$ is the diffeomorphic image of Z, and we have

$$I_2(X, Z) = I_2(\iota, Z) = I_2(\varphi \circ \iota, \varphi(Z))$$

where the latter is the intersection number in S^2.
But by the previous point, $I_2(\varphi \circ \iota, \varphi(Z)) = 0$ since $\dim S^1 = 1 < 2$.

$\boxed{5}$ **a)** Let $I\colon X \times [0, 1] \to Y$ be a smooth homotopy with $I(x, 0) = i_0(x)$ being the embedding of X in Y and $i_1(X) = Z \subset Y$. We define a new map

$$F\colon X \times [0, 1] \to Y \times [0, 1],\ (x, t) \mapsto (I(x, t), t).$$

Since I is smooth, F is smooth.
Now we define $W := F(X \times [0, 1]) \subset Y \times [0, 1]$ to be the image of $X \times [0, 1]$ under F. We claim that W is a compact smooth manifold with boundary. First, since X is compact, $X \times [0, 1]$ is a compact, and hence its continuous image $F(X \times [0, 1])$ in $Y \times [0, 1]$ is compact.
Since $I(-, t)$ is an embedding for every $t \in [0, 1]$, and the identity map on the second component is obviously an embedding as well, F is also an embedding. Thus the image of F is a smooth manifold.
The boundary of W is then given by

$$\begin{aligned}\partial W &= W \cap \partial(Y \times [0, 1]) \\ &= F(X \times \{0\}) \cap Y \times \{0\} \cup F(X \times \{1\}) \cap Y \times \{1\} \\ &= X \times \{0\} \cup Z \times \{1\}.\end{aligned}$$

Hence W is a cobordism between X and Z.

b) Let X and Z be cobordant in Y, and let C be a compact submanifold C in Y with dimension complementary to X and Z. Let f denote the restriction to W of the projection map $Y \times [0, 1] \to Y$. Then $\partial f = f_{|\partial W}$ is a smooth map which can be extended to a map $f\colon W \to Y$. Hence, by the Boundary Theorem, $I_2(\partial f, V) = 0$ for any closed submanifold V of Y of dimension $\dim V = \dim Y - \dim \partial W$. Since $\dim \partial W = \dim X = \dim Z$, we have, in particular,

$$I_2(\partial f, C) = 0.$$

By the Transversality Homotopy Theorem, we can assume $f \pitchfork C$ and $\partial f \pitchfork C$. In particular, we can assume $X \pitchfork C$ and $Z \pitchfork C$. Hence $I_2(X, C) = \#(X \cap C)$

and $I_2(Z,C) = \#(Z \cap C)$. By definition of f, $(\partial f)^{-1}(C)$ is given by

$$\begin{aligned}(\partial f)^{-1}(C) &= \partial W \cap (C \times [0,1]) \\ &= (X \times \{0\}) \cap (C \times [0,1]) \cup (Z \times \{1\}) \cap (C \times [0,1]) \\ &= ((X \cap C) \times \{0\}) \cup ((Z \cap C) \times \{1\}).\end{aligned}$$

Thus

$$\begin{aligned}0 &= I_2(\partial f, C) \\ &= \#(\partial f)^{-1}(C) \\ &= \#((X \cap C) \times \{0\}) + \#((Z \cap C) \times \{1\}) \\ &= \#(X \cap C) + \#(Z \cap C) \\ &= I_2(X,C) + I_2(Z,C).\end{aligned}$$

Since we are working modulo 2, this implies

$$I_2(C,X) = I_2(C,Z).$$

$\boxed{6}$ Since the p_i's are all homogeneous of odd order, they satisfy

$$p_i(-x) = (-1)^{m_i} p_i(x) = -p(x_i).$$

Moreover, for any $x \in \mathbb{R}^{n+1} \setminus \{0\}$, we can consider the associated map

$$q_i \colon S^n \to \mathbb{R},\ x \mapsto p_i\left(\frac{x}{|x|}\right).$$

Since $n+1 \geq 2$, this is a smooth map. Hence we have n smooth real-valued functions $q_1, \dots, q_n$ on S^n, all satisfying the symmetry condition $q_i(-x) = p_i(-x/|x|) = -p_i(x/|x|) = -q_i(x)$.

By Coroallry 2 of the Borsuk-Ulam Theorem in the lecture, we know that these n maps $q_1, \dots, q_n$ must have a common zero on S^n, say $x_0 \in S^n$.

Since the p_i are homogeneous, $q_i(x_0) = p_i(x_0) = 0$ implies

$$p_1(x) = \cdots = p_n(x) = 0 \text{ for all } x = \lambda x_0 \text{ for some } \lambda \in \mathbb{R}.$$

Hence the line spanned by x_0 in $\mathbb{R}^{n+1}$ is the desired line on which all p_i vanish simultaneously.