Project 1 Notes

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1 Problem 1

Let normal matrices, those with diagonalization be on the form

$$A = U\Lambda U^H$$

Where Λ is a diagonal complex $n \times n$ matrix and U a unitary (complex) matrix such that $U^H U = I$ (recall that U^H is the complex conjugate of U^T).

Show that for any such matrix, one has $\|A\|_2 = \rho(A)$, where $\rho(A)$ is the spectral radius of A .

1.1 Proof

Answer.

Proof. Starting with the definition of a subordinate matrix norm [2] can we let

$$\|A\|_2^2 = \sup_{x \neq 0} \frac{\langle Ax, Ax \rangle}{\langle x, x \rangle}.$$

Note sure if this is the correct notation for 2-norm. Indeed, by using the assumption that $U^H U = I$ and substituting $Uy = x$ can we show that

$$\|A\|^2 = \sup_{x \neq 0} \frac{\langle Ax, Ax \rangle}{\langle x, x \rangle} = \sup_{y \neq 0} \frac{\langle AUy, AUy \rangle}{\langle Uy, Uy \rangle} = \sup_{y \neq 0} \frac{\left\langle U^H A^H AUy, y \right\rangle}{\langle y, y \rangle}$$

Recall the property $A = U\Lambda U^H$ and thus

$$\begin{aligned} A^H A &= U\Lambda^H U^H U \Lambda U^H \\ &= U\Lambda^H \Lambda U^H. \end{aligned}$$

As a consequence do we end up with

$$\begin{aligned} \sup_{y \neq 0} \frac{\left\langle U^H A^H AUy, y \right\rangle}{\langle y, y \rangle} &= \sup_{y \neq 0} \frac{\left\langle U^H U \Lambda^H \Lambda U^H Uy, y \right\rangle}{\langle y, y \rangle} \\ &= \sup_{y \neq 0} \frac{\left\langle \Lambda^H \Lambda y, y \right\rangle}{\langle y, y \rangle} \\ &= \sup_{y \neq 0} \frac{\sum_{i=1}^n |\lambda_i|^2 |y_i|^2}{\sum_{i=1}^n |y_i|^2} = \max_i \left(|\lambda_i|^2 \right) \end{aligned}.$$

Given the definition of a spectral radius [1] characterized by

$$\rho(A) = \max_i |\lambda_i|.$$

Which completes the proof of $\|A\|_2 = \rho(A)$.

$\square$

2 Problem 2

Consider the $n \times n$ matrix A whise nonzero elements are located on its unit subdiagonal , i.e. $A_{i+1,i} = 1$ for $i = 1, \ldots, n-1$

$$A = \begin{bmatrix} 0 & \ldots & \ldots & 0 \\ 1 & 0 & & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \ldots & 1 & 0 \end{bmatrix}$$

(a) What are the eigenvalues of A ? What would the Gershgorin theorem tell us about the location of the eigenvalues of A.

(b) Now construct the matrix $\hat{A}$ by adding a small number ϵ in the $(1, n)$-element of A (so that $\hat{A} = A + \epsilon e_1 e_n^T$. Show that

$$\rho\left(\hat{A}\right) = \epsilon^{\frac{1}{n}}$$

And fins an expression for the eigenvalues and eigenvectors of $\hat{A}$.

(c) Derive an exact expression for the condition number $K_2\left(\hat{A}\right) = \|\hat{A}\|_2 = \|\hat{A}^{-1}\|_2$.

2.1 Answer a

Answer.

The eigenvalues can be computed such that

$$\begin{aligned} det\left(A - \lambda\right) &= \begin{vmatrix} -\lambda & & & \ldots & 0 & 0 \\ 1 & -\lambda & \ldots & & & 0 \\ 0 & 1 & -\lambda & \ldots & & 0 \\ \vdots & & & \ddots & & \\ 0 & \ldots & & & 1 & -\lambda \end{vmatrix} \\ &= -\lambda \begin{vmatrix} -\lambda & \ldots & & 0 \\ 1 & -\lambda & & \\ \vdots & & \ddots & \\ 0 & \ldots & 1 & -\lambda \end{vmatrix} \\ &= \left(-1\right)^n \lambda^n = 0 \quad \Longrightarrow \quad \lambda = 0 \end{aligned}$$

Which concludes that we did only find a trivial solutions. By applying Gershgoring theorem [2] characterized by these definitions.

Definition 2.1 (Gerschgorin discs)**.** $D_i, \quad i = 1, 2, 3 \ldots, n$ *of the matrix* A *are defined as the closed circular regions*

$$D_i = \{z \in \mathbb{C} \mid |z - a_{ii}| \leq R_i\}$$

In the complex plane where

$$R = \sum_{\substack{i=1 \\ j \neq 1}} |a_{ij}|$$

is the radius of D_i*.*

Theorem 2.1 (Gerschgorin Theorem)**.** *Let* $n \geq 2$ *and* $A \in \mathbb{C}^{n \times n}$ *. All eigenvaluesof the matrix* A *lie in the region* $D = \bigcup_{i=1}^{n} D_i$ *, where* D_i*,* $i = 1, 2, \ldots, n$*, are Gerschgorin disc defined by in the definition 2.1.*

Using the theorem 2.1 and the matrix A can we conlude it had to be some eigenvalues in within a disc of zero radius.

2.2 Answer b

Answer. Lets assume $\varepsilon \in \mathbb{C}$ and and we can then observe that

$$\hat{A} = A + \varepsilon e_1 e_n^T = \begin{pmatrix} 0 & & \ldots & & \varepsilon \\ 1 & 0 & \ldots & & \\ 0 & 1 & \ddots & & \\ \vdots & & \ddots & \ddots & \\ 0 & \ldots & & 1 & 0 \end{pmatrix}$$

We can then find the eigenvalues by solving the condition,

$$det\left(\hat{A} - \lambda\right) = \begin{vmatrix} -\lambda & & \ldots & & \varepsilon \\ 1 & -\lambda & \ldots & & \\ 0 & 1 & \ddots & & \\ \vdots & & \ddots & \ddots & \\ 0 & \ldots & & 1 & -\lambda \end{vmatrix}$$

$$= (-1)^n \lambda^n + (-1)^{n+1} \varepsilon \begin{vmatrix} 1 & -\lambda & \ldots & & \\ 0 & 1 & \ddots & & \\ \vdots & & \ddots & \ddots & \\ 0 & \ldots & & 1 & -\lambda \end{vmatrix}$$

$$= (-1)^n \lambda^n + (-1)^{n+1} \varepsilon = 0.$$

We can see that the eigenvalues λ have several complex solutions depending on the value n. However, we can conclude that any solution will satisfy $|\lambda| = \varepsilon^{\frac{1}{n}}$, thus

$$\rho\left(\hat{A}\right) = \varepsilon^{\frac{1}{n}}.$$

In general can the n eigenvalues be expressed as

$$\lambda_k = \varepsilon^{\frac{1}{n}} e^{i2\pi k/n}, \quad \text{where } k = 0, 1, \ldots, n-1.$$

The procedure to compute the eigenvectors is then to find all solutions which satisfies

$$\hat{A} v_k = \lambda_k v_k.$$

Using a recursive strategy,

$$\implies \quad \begin{bmatrix} \varepsilon v_{k,n} \\ v_{k,0} \\ \vdots \\ v_{k,n-2} \\ v_{k,n-1} \end{bmatrix} = \begin{bmatrix} v_{k,0}\lambda_k \\ v_{k,1}\lambda_k \\ \vdots \\ v_{k,n-1}\lambda_k \\ v_{k,n}\lambda_k \end{bmatrix} = \begin{bmatrix} v_{k,n}\lambda_k^n \\ v_{k,n}\lambda_k^{n-1} \\ \vdots \\ v_{k,n}\lambda_k^2 \\ v_{k,n}\lambda_k^1 \end{bmatrix},$$

does we end up with the relationship,

$$v_{k,n}\varepsilon = v_{k,n}\lambda_k^n = v_{k,n}\left(\varepsilon^{\frac{1}{n}} e^{\frac{i2\pi k}{n}}\right)^n = v_{k,n}\varepsilon \cdot e^{i2\pi k}.$$

If we choose the scale of the eigenvector to be such that the element $v_{k,n} = 1$ for all k, can we determine v_i for all $i = \{0, 1, \ldots, n-1\}$ such that the expression of the eigenvector v_k ends up to be

$$v_k = \begin{bmatrix} v_{k,0} \\ v_{k,1} \\ \vdots \\ v_{k,n-1} \\ v_{k,n} \end{bmatrix} = \begin{bmatrix} \lambda_k^n \\ \lambda_k^{n-1} \\ \vdots \\ \lambda_k \\ 1 \end{bmatrix} = \begin{bmatrix} \left(\varepsilon^{\frac{1}{n}} e^{\frac{2\pi k}{n}}\right)^n \\ \left(\varepsilon^{\frac{1}{n}} e^{\frac{2\pi k}{n}}\right)^{n-1} \\ \vdots \\ \varepsilon^{\frac{1}{n}} e^{\frac{2\pi k}{n}} \\ 1 \end{bmatrix}$$

2.3 Answer c

3 References

References

[1] Alfio Quarteroni, Riccardo Sacco, and Fausto Saleri. *Numerical mathematics*, volume 37. Springer Science & Business Media, 2010.

[2] Endre Süli and David F Mayers. *An introduction to numerical analysis.* Cambridge university press, 2003.