Norwegian University of Science and Technology
Deptartment of Mathematical Sciences

TMA4190 Introduction to Topology Spring 2018

Exercise set 9

$\boxed{1}$ If $U \subset \mathbb{R}^k$ and $V \subset \mathbb{H}^k$ are open neighborhoods of 0, prove that there exists no diffeomorphism of V with U. (Hint: Inverse Function Theorem.)

$\boxed{2}$ Prove that if $f\colon X \to Y$ is a diffeomorphism of manifolds with boundary, then ∂f maps ∂X diffeomorphically onto ∂Y. (Hint: Inverse Function Theorem.)

$\boxed{3}$ We define the smooth maps

$$F\colon \mathbb{R} \times [-1/2, 1/2] \to \mathbb{R}^3,\ (t,s) \mapsto (\cos t, \sin t, s),\ \text{and}$$
$$G\colon \mathbb{R} \times [-1/2, 1/2] \to \mathbb{R}^3,\ (t,s) \mapsto ((1 + s\cos(t/2))\cos t, (1 + s\cos(t/2))\sin t, s\sin(t/2)).$$

We define X to be the image of F in $\mathbb{R}^3$, and Y to be the image of G in $\mathbb{R}^3$.

a) Show that X is a 2-dimensional manifold with boundary whose boundary is diffeomorphic to the disjoint union of two copies of the unit circle. (Convince yourself that X is a cylinder obtained by starting with a rectangular surface and then glueing two opposite edges together.)

b) Show that Y is a 2-dimensional manifold with boundary whose boundary is diffeomorphic to just one copy of the unit circle. (Convince yourself that Y is a Möbius band obtained by starting with a rectangular surface and then glueing two opposite edges after twisting one edge once. If you do not get through all the formulae, make sure you understand the answer visually at least.)

$\boxed{4}$ Suppose that X is a manifold with boundary and $x \in \partial X$. Let $\phi\colon U \to X$ be a local parametrization with $\phi(0) = x$, where U is an open subset of $\mathbb{H}^k$. Then $d\phi_0\colon \mathbb{R}^k \to T_x(X)$ is an isomorphism. Define the upper halfspace $H_x(X)$ in $T_x(X)$ to be the image of $\mathbb{H}^k$ under $d\phi_0$, $H_x(X) := d\phi_0(\mathbb{H}^k)$.

a) Prove that $H_x(X)$ does not depend on the choice of local parametrization.

b) Show that there are precisely two unit vectors in $T_x(X)$ that are perpendicular to $T_x(\partial X)$ and that one lies inside $H_x(X)$, the other outside. The one in $H_x(X)$ is called the inward unit normal vector to the boundary, and the other is the outward unit normal vector to the boundary. Denote the outward unit normal vector by $n(x)$.

c) If $X \subset \mathbb{R}^N$, we consider $n(x)$ as an element in $\mathbb{R}^N$ and get a map $n : \partial X \to \mathbb{R}^N$. Show that n is smooth.