Norwegian University of Science and Technology
Deptartment of Mathematical Sciences

TMA4190 Introduction to Topology
Spring 2018

Solutions to exercise set 9

$\boxed{1}$ Let $U \subset \mathbb{R}^k$ and $V \subset \mathbb{H}^k$ be open neighborhoods of 0. Suppose there was a diffeomorphism $\theta \colon U \to V$. We can assume that 0 is sent to a boundary point of V. In fact, we can assume that $\theta(0) = 0$. Otherwise we just another pick point $u \in U$ with $\theta(u) \in \partial V$. Then $d\theta_0 \colon \mathbb{R}^k \to \mathbb{R}^k$ is an isomorphism. By the Inverse Function Theorem, there are subsets W_1 and W_2 in $\mathbb{R}^k$ containing 0 which are both **open in $\mathbb{R}^k$** such that θ maps W_1 diffeomorphically onto W_2. Since W_2 is open in $\mathbb{R}^k$ and contained in the image of θ, we get that V must be **open in $\mathbb{R}^k$**. But since V contains 0, it satisfies $V \cap \partial\mathbb{H}^k \neq \emptyset$ and cannot be open **in $\mathbb{R}^k$**.

$\boxed{2}$ Let $f \colon X \to Y$ be a diffeomorphism of manifolds with boundary. Let $\phi \colon U \to X$ and $\psi \colon V \to Y$ be local parametrizations, where U and V are open subsets of $\mathbb{H}^k$ (check that you know why the dimensions of X and Y must be equal). Let $\theta \colon U \to V$ be the induced map. By shrinking U and V if necessary, we can assume that θ is a diffeomorphism with

$$f \circ \phi = \psi \circ \theta.$$

Boundary points of X are those which are in the image $\phi(\partial U) = \phi(U \cap \partial\mathbb{H}^k)$. Similarly, boundary points of X are those which are in the image $\psi(\partial V) = \psi(V \cap \partial\mathbb{H}^k)$. Hence we need to show $\theta(\partial U) \subset \partial V$, for then

$$f(\phi(\partial U)) = \psi(\theta(\partial U)) \subset \psi(\partial V) \subset \partial Y.$$

The argument is again based on the Inverse Function Theorem. Suppose there is a point $u \in \partial U$ which is mapped to an interior point $v = \theta(u)$ in V. Since θ is a diffeomorphism, the derivative $d(\theta^{-1})_v \colon \mathbb{R}^k \to \mathbb{R}^k$ of its inverse is an isomorphism. But, since $v \in \mathrm{Int}(V)$, V contains a neighborhood W of v that is open in $\mathbb{R}^k$. Thus the Inverse Function Theorem implies that $\theta^{-1}(W)$ contains a neighborhood of u that is open in $\mathbb{R}^k$. Hence u is also an interior point in U which contradicts the assumption $u \in \partial U$.

$\boxed{3}$ We define the smooth maps

$$F \colon \mathbb{R} \times [-1/2, 1/2] \to \mathbb{R}^3,\ (t,s) \mapsto (\cos t, \sin t, s),\ \text{and}$$
$$G \colon \mathbb{R} \times [-1/2, 1/2] \to \mathbb{R}^3,\ (t,s) \mapsto ((1 + s\cos(t/2))\cos(t), (1 + s\cos(t/2))\sin(t), s\sin(t/2)).$$

We define X to be the image of F in $\mathbb{R}^3$ and Y to be the image of G in $\mathbb{R}^3$.

a) The image of F is the product $S^1 \times [-1/2, 1/2]$. This is a product of a manifold without a boundary S^1 and the manifold $[-1/2, 1/2]$ with boundary. The boundary of $[-1/2, 1/2]$ constists of the disjoint union of $\{-1/2\}$ and $\{1/2\}$. By the result of the lecture, we get

$$\partial X = \partial(S^1 \times [-1/2, 1/2]) = S^1 \times \{-1/2\} \cup S^1 \times \{1/2\}.$$

b) We define the maps

$$\begin{aligned}
&\phi_+ \colon (-\pi, \pi) \times [0, 3/4) \to Y,\\
&(t, s) \mapsto ((1 + (-1/2 + s)\cos(t/2))\cos t, (1 + (-1/2 + s)\cos(t/2))\sin t, (-1/2 + s)\sin(t/2))\\
&\phi_- \colon (-\pi, \pi) \times [0, 3/4) \to Y,\\
&(t, s) \mapsto ((1 + (1/2 - s)\cos(t/2))\cos t, (1 + (1/2 - s)\cos(t/2))\sin t, (1/2 - s)\sin(t/2))\\
&\psi_+ \colon (0, 2\pi) \times [0, 3/4) \to Y,\\
&(t, s) \mapsto ((1 + (-1/2 + s)\cos(t/2))\cos t, (1 + (-1/2 + s)\cos(t/2))\sin t, (-1/2 + s)\sin(t/2))\\
&\psi_- \colon (0, 2\pi) \times [0, 3/4) \to Y,\\
&(t, s) \mapsto ((1 + (1/2 - s)\cos(t/2))\cos t, (1 + (1/2 - s)\cos(t/2))\sin t, (1/2 - s)\sin(t/2)).
\end{aligned}$$

As one can check by calculating the partial derivatives, each of these maps are diffeomorphisms, and the union of their images covers Y. Hence we can use these four maps as local parametrizations of Y.
The boundary of Y is then given by the union of the points

$$\partial Y = \phi_+((-\pi, \pi) \times \{0\}) \cup \phi_-((-\pi, \pi) \times \{0\}) \cup \psi_+((0, 2\pi) \times \{0\}) \cup \psi_-((0, 2\pi) \times \{0\}).$$

Setting $s = 0$ in the fomulae for those maps gives

$$\begin{aligned}
\partial Y =& \{((1 - \frac{1}{2}\cos(t/2))\cos t, (1 - \frac{1}{2}\cos(t/2))\sin t, -\frac{1}{2}\sin(t/2)) \in \mathbb{R}^3 : t \in \mathbb{R}\}\\
&\cup \{((1 + \frac{1}{2}\cos(t/2))\cos t, (1 + \frac{1}{2}\cos(t/2))\sin t, \frac{1}{2}\sin(t/2)) \in \mathbb{R}^3 : t \in \mathbb{R}\}.
\end{aligned}$$

But, in fact, the two sets describing ∂Y are the same which we see when we replace t with $t + 2\pi$ and use some simple trigonometric identities:

$$\begin{cases}
(1 - \frac{1}{2}\cos(\frac{t+2\pi}{2}))\cos(t + 2\pi) & = (1 + \frac{1}{2}\cos(t/2))\cos t,\\
(1 - \frac{1}{2}\cos(\frac{t+2\pi}{2}))\sin(t + 2\pi) & = ((1 + \frac{1}{2}\cos(t/2))\sin t\\
-\frac{1}{2}\sin(\frac{t+2\pi}{2})) & = \frac{1}{2}\sin(t/2).
\end{cases}$$

Hence

$$\partial Y = \{((1 + \frac{1}{2}\cos(t/2))\cos t, (1 + \frac{1}{2}\cos(t/2))\sin t, \frac{1}{2}\sin(t/2)) \in \mathbb{R}^3 : t \in \mathbb{R}\}.$$

Now we would like to show that ∂Y is diffeomorphic to S^1. Remembering the trigonometric identities

$$\sin t = 2\sin(t/2)\cos(t/2) \text{ and } \cos t = \cos^2(t/2) - \sin^2(t/2)$$

we see that the map

$$\varphi \colon \mathbb{R}^2 \to \mathbb{R}^3,\ (x, y) \mapsto ((1 + \frac{1}{2}x)(x^2 - y^2), (1 + \frac{1}{2}x)2xy, \frac{1}{2}y)$$

restricts to a bijection from S^1 onto ∂Y (for injectivity, note that the last coordinate determines y uniquely, then the circle equation determines x up to sign, and the first and/or second coordinate determine the sign of x).

It remains to check that $\varphi_{|S^1}$ is a local diffeomorphism. Since $\varphi_{|S^1}$ is a bijection onto its image, this will show that it is a diffeomorphism.

First we observe that φ is smooth, since the three functions in each coordinate are just polynomials and hence smoooth. To see that $\varphi_{|S^1}$ is a local diffeomorphism, we use the maps

$$\phi\colon U \to S^1, t \mapsto (\cos(t/2), \sin(t/2))$$

and

$$\psi\colon U \to \partial Y, t \mapsto ((1 + \frac{1}{2}\cos(t/2))\cos t, (1 + \frac{1}{2}\cos(t/2))\sin t, \frac{1}{2}\sin(t/2))$$

where $U \subset \mathbb{R}$ is some sufficiently small open subset. Then ϕ and ψ serve as local parametrizations of S^1 and ∂Y, respectively, for suitable choices of U. But the induced map $\theta\colon U \to U$ which arises as the composite $\psi^{-1} \circ \varphi_{|S^1} \circ \phi$ is just the identity $t \mapsto t$. Hence $\varphi_{|S^1}$ is a local diffeomorphism.

4 Suppose that X is a manifold with boundary and $x \in \partial X$. Let $\phi\colon U \to X$ be a local parametrization with $\phi(0) = x$, where U is an open subset of $\mathbb{H}^k$. Then $d\phi_0\colon \mathbb{R}^k \to T_x(X)$ is an isomorphism. Define the upper halfspace $H_x(X)$ in $T_x(X)$ to be the image of $\mathbb{H}^k$ under $d\phi_0$, $H_x(X) := d\phi_0(\mathbb{H}^k)$.

a) We proceed as in the lecture when we showed that tangent spaces are well-defined.

Let $\psi\colon V \to X$ be another local parametrization around x with $\psi(0) = x$, where V is an open subset of $\mathbb{H}^k$. By shrinking both U and V, we can assume $\phi(U) = \psi(V)$ (replace U by $\phi^{-1}(\phi(U) \cap \psi(V)) \subset U$ and V by $\psi^{-1}(\phi(U) \cap \psi(V)) \subset V$). Then the map

$$\theta := \psi^{-1} \circ \phi\colon U \to V$$

is a diffeomorphism (its the composite of two diffeomorphisms). By definition of θ, we have $\phi = \psi \circ \theta$. Differentiating yields

$$d\phi_0 = d\psi_0 \circ d\theta_0$$

(where we have used the chain rule). This implies that the image of $d\phi_0$ is contained in the image of $d\psi_0$:

$$d\phi_0(\mathbb{R}^k) \subseteq d\psi_0(\mathbb{R}^k) \text{ in } \mathbb{R}^N.$$

By switching the roles of ϕ and ψ in the argument, we also get:

$$d\psi_0(\mathbb{R}^k) \subseteq d\phi_0(\mathbb{R}^k) \text{ in } \mathbb{R}^N.$$

Hence $T_x(X) = d\phi_0(\mathbb{R}^k) = d\psi_0(\mathbb{R}^k)$ is well-defined in $\mathbb{R}^N$.

In particular, the image of the upper halfplane $\mathbb{H}^k \subset \mathbb{R}^k$ is well-defined:

$$H_x(X) = d\phi_0(\mathbb{H}^k) = d\psi_0(\mathbb{H}^k) \text{ in } \mathbb{R}^N.$$

b) The codimension of $T_x(\partial X)$ in $T_x(X)$ is one. Thus the orthogonal complement of $T_x(\partial X)$ is one-dimensional and is spanned by one vector. By definition of ∂X as the image of the points in ∂HH^k under local parametrizations, we know that $d\phi_0(e_k)$ spans the complement of $T_x(\partial X)$ in $T_x(X)$, since $d\phi_0$ is an isomorphism and $e_k = (0, \ldots, 0, 1)$ is nonzero and not contained in $d\phi_0(\mathbb{H}^k)$. We also know by the definition of $H_x(X)$ that $d\phi_0(e_k) \in H_x(X)$, and therefore $d\phi_0(-e_k) \notin H_x(X)$. But we do not know wheter $d\phi_0(-e_k)$ is orthogonal to $T_x(\partial X)$ in $T_x(X)$. To make $d\phi_0(-e_k)$ into a vector wich is orthogonal to $T_x(\partial X)$, we apply the Gram-Schmidt process. It produces an unit vector which is orthogonal to $T_x(\partial X)$. We denote this vector $n(x)$, this is the outward unit normal vector to ∂X. Note that $-n(x)$ is a unit vector contained in $H_x(X)$ and orthogonal to $T_x(\partial X)$, this is the inward unit normal vector to ∂X.

c) From what we have learned in the previous point, we can construct $n(x)$ by applying the Gram-Schmidt orthonormalization process to $d\phi_0(-e_k)$. This process depends smoothly on the coefficients in the matrix representing $d\phi_0$. Since the derivative $d\phi_u$ depends smoothly on u, $d\phi_u(-e_k)$ depends smoothly on u. By the independence of the choice of local parametrization, we see that $n(y) = d\phi_u(-e_k)$ for all $y \in \phi(\partial U)$ which is an open neighborhood of x in ∂X, where $\phi(u) = y$. Thus, in total we see that $n(x)$ depends smoothly on x in ∂X.