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Examination paper for
TMA4195 Mathematical Modeling, solution proposal

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Problem 1 It is reasonable to assume that the energy E of a vibrating string depends on its length L, the amplitude A, the mass density per length ρ of the string, and the tension T of the string ($[T] = \mathrm{kg\,m\,s^{-2}}$).

Set up the dimension matrix for all these variables, and explain why A, ρ, and T can be chosen as core variables (dimensionally independent) for a dimensional analysis of this problem.

Use dimensional analysis to show that E is independent of ρ, and find an expression for E in terms of L, A, and T (the expression will involve an unspecified function of a single dimensionless variable).

Solution proposal:
The dimension matrix is

$$\begin{array}{cccc} & \mathrm{m} & \mathrm{s} & \mathrm{kg} \\ E & 2 & -2 & 1 \\ L & 1 & 0 & 0 \\ A & 1 & 0 & 0 \\ \rho & -1 & 0 & 1 \\ T & 1 & -2 & 1 \end{array}$$

The 3×3 sub-matrix of the dimension matrix defined by $A, \rho,$ and T is

$$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & 1 \\ 1 & -2 & 1 \end{bmatrix},$$

which is non-singular (check the detrminant). Thus, the three varables are dimensionally independent, i.e. can be chosen as core varables.
Define one dimensionless variable for each remaining variable:

$$\pi_1 = \frac{E}{AT}, \quad \pi_2 = \frac{L}{A}.$$

Thus, a relation $k(E, L, A, \rho, T) = 0$ is equvivalent to a relation $h(\pi_1, \pi_2, A, \rho, T) = 0$ by change of variables. Since the three last variables are dimensionally independent, h is independent of them, and we really have $h(\pi_1, \pi_2) = 0 \Leftrightarrow \pi_1 = f(\pi_2) \Leftrightarrow E = ATf(\frac{L}{A})$ showing that E is independent of ρ.

Problem 2 Let m, m_1, and m_2 denote masses, E energy, ω angular frequency ($[\omega] = \mathrm{s}^{-1}$), F force, and r distance. The three physical laws; Einstein's $E = mc^2$, the Planck-Einstein relation $E = \hbar\omega$, and Newton's law of gravity $F = G\frac{m_1 m_2}{r^2}$ define the dimensions of the physical constants $c, \hbar$, and G.

Express the dimensions of $c, \hbar$, and G in terms of SI-units.

The numerical value of these three constants depends on the choice of units for length, time, and mass. If L, T, M, and $\tilde{L}, \tilde{T}, \tilde{M}$ are two systems of units for length, time, and mass, then

$$a = val(a) L^\alpha T^\beta M^\gamma = \widetilde{val}(a) \tilde{L}^\alpha \tilde{T}^\beta \tilde{M}^\gamma$$

for a physical constant a, where $val(a)$ and $\widetilde{val}(a)$ are the numerical values for a for the two systems of units.

Express, in terms of $c, \hbar$, and G, the units for length, time, and mass which makes the numerical values for $c, \hbar$, and G all equal to 1. (These are the so-called Planck units for length, time, and mass.)

Solution proposal:
We have $[E] = \mathrm{kg\,m^2\,s^{-2}}$, $[m] = \mathrm{kg}$, thus $[c] = [\sqrt{\frac{E}{m}}] = \mathrm{m\,s^{-1}}$.

Since $[f] = \mathrm{s}^{-1}$, we have $[\hbar] = [\frac{E}{f}] = \mathrm{kg\,m^2\,s^{-2}\,s} = \mathrm{kg\,m^2\,s^{-1}}$.

And finally, since $[F] = \mathrm{kg\,m\,s^{-2}}$, we have $[G] = [\frac{Fr^2}{m_1 m_2}] = \mathrm{kg\,m\,s^{-2}\,m^2\,kg^{-2}} = \mathrm{kg^{-1}\,m^3\,s^{-2}}$.

Since the numerical value of all three constants are unity, the Planck units L, T, and M must satisfy

$$c = LT^{-1}, \quad \hbar = L^2 T^{-1} M, \quad G = L^3 T^{-2} M^{-1}.$$

Multiplying the last two equations eliminates M:

$$c = LT^{-1}, \quad G\hbar = L^5 T^{-3}.$$

Eliminating L, we obtain $T = \sqrt{\frac{G\hbar}{c^5}}$. From $L = cT$ we get $L = \sqrt{\frac{G\hbar}{c^3}}$, and also find $M = \sqrt{\frac{\hbar c}{G}}$.

Note: In terms of SI-units, these Planck scales are $L = 1.616 \times 10^{-35}\,\mathrm{m}$, $T = 2.177 \times 10^{-44}\,\mathrm{s}$, and $M = 2.176 \times 10^{-8}\,\mathrm{kg}$.

Problem 3 We consider a one-dimensional model of gravitational segregation for incompressible two-phase flow in porous media, where capillary forces are neglected. Let

- the x-axis be tilted an angle α with respect to the horizontal plane.
- ρ_1 and ρ_2 be the mass densities of the phases, where $\rho_1 < \rho_2$.
- k, g, ϕ, μ_1, and μ_2 denote the absolute permeability, the gravitational acceleration, the porosity, and the two phase viscosities respectively (all constant).
- S_i be the saturation of phase $i, i = 1, 2$, (remember $S_1 + S_2 = 1$). Define $S = S_1$ from now on.
- $\lambda_i(S) = \frac{k_{ri}(S)}{\mu_i}$ be the phase mobility of phase $i, i = 1, 2$, where $k_{ri}(S)$ is the relative permeability to phase i.

The volumetric flux density (Darcy velocity) for phase i, $i = 1, 2$, is then

$$u_i = -k\lambda_i(S)\Big(\frac{\partial p}{\partial x} + \rho_i g \sin(\alpha)\Big),$$

where p is the pressure.

Note : From now on we assume countercurrent flow, that is $u_1 + u_2 = 0$.

a) Express $\frac{\partial p}{\partial x}$ as a function of S, and show that

$$u_1 = \Delta\rho\, k\, g \sin(\alpha) h(S),$$

where $h(S) = \frac{\lambda_1(S)\lambda_2(S)}{\lambda_1(S)+\lambda_2(S)}$ and $\Delta\rho = \rho_2 - \rho_1$.

Solution proposal:
We have

$$u_1 = -u_2 \Leftrightarrow -k\lambda_1(S)(\frac{\partial p}{\partial x} + \rho_1 g\sin(\alpha)) = k\lambda_2(S)(\frac{\partial p}{\partial x} + \rho_2 g\sin(\alpha)) \Leftrightarrow$$

$$\frac{\partial p}{\partial x} = -g\sin(\alpha)\frac{\rho_1\lambda_1(S) + \rho_2\lambda_2(S)}{\lambda_1(S) + \lambda_2(S)}.$$

Inserting this expression for $\frac{\partial p}{\partial x}$ into

$$u_1 = -k\lambda_1(S)(\frac{\partial p}{\partial x} + \rho_1 g\sin(\alpha)),$$

we obtain

$$u_1 = -kg\sin(\alpha)\lambda_1(S)(\rho_1 - \frac{\rho_1\lambda_1(S) + \rho_2\lambda_2(S)}{\lambda_1(S) + \lambda_2(S)} =$$

$$\Delta\rho kg\sin(\alpha)\frac{\lambda_1(S)\lambda_2(S)}{\lambda_1(S) + \lambda_2(S)} = \Delta\rho kg\sin(\alpha)h(S).$$

b) Let $g(S) = \mu_2 h(S)$, so that $g(S)$ is dimensionless. Assume L is the length scale for the problem. Find a time scale T such that the conservation law

$$\phi\frac{\partial S}{\partial t} + \frac{\partial u_1}{\partial x} = 0$$

written in dimensionless form is

$$\frac{\partial S}{\partial t} + g'(S)\frac{\partial S}{\partial x} = 0$$

(in this last equation, x and t are now dimensionless).

Solution proposal:
In physical variables the conservation equation is

$$\phi\frac{\partial S}{\partial t} + \frac{\partial u_1}{\partial x} = 0 \Leftrightarrow \phi\frac{\partial S}{\partial t} + \frac{\Delta\rho kg\sin(\alpha)}{\mu_2}g'(S)\frac{\partial S}{\partial x} = 0.$$

Setting L for length scale and T as time scale, the dimensionless equation becomes

$$\frac{\phi}{T}\frac{\partial S}{\partial t} + \frac{\Delta\rho kg\sin(\alpha)}{L\mu_2}g'(S)\frac{\partial S}{\partial x} = 0 \Leftrightarrow \frac{\partial S}{\partial t} + \frac{T\Delta\rho kg\sin(\alpha)}{\phi L\mu_2}g'(S)\frac{\partial S}{\partial x} = 0.$$

Thus, setting

$$T = \frac{\phi\mu_2 L}{\Delta\rho kg\sin(\alpha)},$$

we obtain the equation in the required dimensionless form.

c) Let $k_{r1}(S) = S$ and $k_{r2} = 1 - S$. Assume $M = \frac{\mu_1}{\mu_2} < 1$, and sketch the characteristics in the (x,t)-plane when the initial condition is

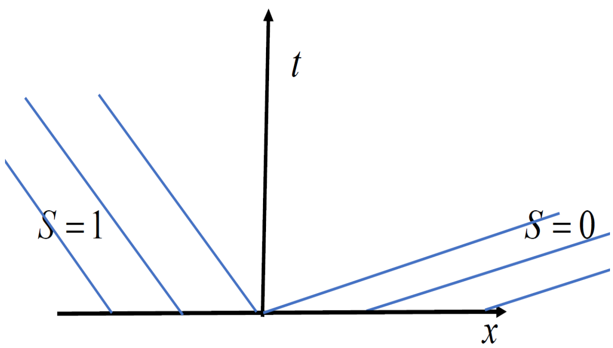
$$S(x,0) = 1 \text{ for } x < 0 \quad \text{and} \quad S(x,0) = 0 \text{ for } x \geq 0$$

Is the solution continuous as a function of x for each $t > 0$?

Solution proposal:
We have $g(S) = \mu_2 h(S) = \frac{S(1-S)}{S+M(1-S)}$, so

$$g'(S) = \frac{-(1-M)S^2 - 2Ms + M}{((1-M)S+M)^2}.$$

Thus, $g'(0) = \frac{1}{M}$ and $g'(1) = -1$. This means that the characteristics for $S = 0$ move to the right with speed $\frac{1}{M}$, and the characteristics for $S = 1$ move to the left with unit speed as illustrated



Between the lines $x = -t$ and $x = \frac{1}{M}x$ we obtain a rarefaction wave, since $g'(S)$ is monotonically decreasing in the interval $[0,1]$. This follows from calculating

$$g''(S) = -\frac{2M}{((1-M)S+M)^3} < 0.$$

Thus, since we have a rarefaction wave, the solution is continuous for each $t > 0$.

d) Now, put $\mu_1 = \mu_2$, such that the conservation equation (in dimensionless form) is

$$\frac{\partial S}{\partial t} + (1-2S)\frac{\partial S}{\partial x} = 0.$$

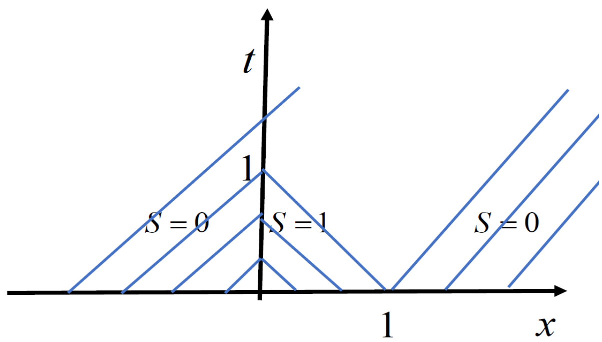
We consider new initial conditions:

$$S(x,0) = 1 \text{ for } 0 < x < 1 \quad \text{and} \quad S(x,0) = 0 \text{ for } x < 0 \text{ and } x > 1$$

(i.e. we study how a concentrated portion of the lightest phase behaves under the influence of gravity).

Sketch the characteristics, and show that we obtain a shock at $x = 0$ with velocity zero (at early times). Show that the shock starts to move at $t = 1$.

Solution proposal:
The characteristics are illustrated below. We see the characteristics collide at



$x = 0$, while at $x = 1$ we have a rarefaction wave. At the shock we have $S_L = 0$ and $S_R = 1$ at early times (none of the rarefaction characteristics have reached $x = 0$ yet). Thus, the speed of the shock is given by

$$U = \frac{g(1) - g(0)}{1 - 0} = 0,$$

that is, the shock does not move.
After the characteristic $x = -t + 1$ originating from $x = 1$ at $t = 0$ meets the shock at time $t = 1$, the values for S in the rarefaction wave will be the S_R values at the shock. Since $0 < S_R < 1$ for $t > 1$, we have $g(S_R) > 0$, and the speed is

$$U = \frac{g(S_R) - g(0)}{S_R - 0} = \frac{g(S_R)}{S_R} \neq 0,$$

and the shock begins to move.

e) Let $x = a(t)$ be the position of the shock at time $t > 1$. Show that $a(t)$ satisfies

$$a' = \frac{1}{2} + \frac{a-1}{2t}, \qquad a(1) = 0.$$

The solution is $a(t) = t + 1 - 2\sqrt{t}$ (you do not have to show this).

At a given time $t > 0$, what is S_R (the right positive value) at the shock, and how long is the interval on the x-axis where $S(x,t) > 0$?

Solution proposal:
The solution in the rarefaction wave starting at $x = 1$ is given by

$$g'(S) = \frac{x-1}{t} \quad \Leftrightarrow 1 - 2S = \frac{x-1}{t} \quad \Leftrightarrow S(x,t) = \frac{1}{2} - \frac{x-1}{2t}.$$

So, if $x = a(t)$ is the position of the shock for $t > 1$, we have $S_R = S(a(t),t)$, while $S_L = 0$. Thus, the speed of the shock is

$$a'(t) = \frac{g(S_R) - g(S_L)}{S_R - S_L} = 1 - S_R = 1 - S(a(t),t) =$$

$$1 - (\frac{1}{2} - \frac{a(t)-1}{2t}) \Leftrightarrow a'(t) = \frac{1}{2} + \frac{a(t)-1}{2t},$$

where $a(1) = 0$ since the shock starts to move from $x = 0$ at $t = 1$.

For $0 \leq t \leq 1$, $S_R = 1$ as previously proven. For $t > 1$, the position of the shock is $a(t) = t + 1 - 2\sqrt{t}$. Thus, $S_R = S(a(t),t) = \frac{1}{2} - \frac{t+1-2\sqrt{t}-1}{2t} = \frac{1}{\sqrt{t}}$ for $t > 1$.

The characteristic starting at $x = 1$ with $S = 0$ is the fastest characteristic in the rarefaction wave. Thus the position where $S(x,t) = 0$ and $S(\tilde{x},t) > 0$ for $a(t) < \tilde{x} < x$ is $x(t) = 1 + g'(0)t = 1 + t$. Thus, for $0 \leq t \leq 1$ the lenght of the interval where $S > 0$ is $x(t) - 0 = 1 + t$. For $t > 1$, this lenght is $x(t) - a(t) = 2\sqrt{t}$.

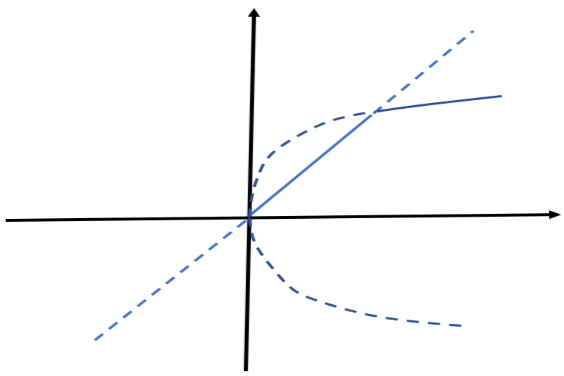
Problem 4 Draw the bifurcation diagram in the (μ, u)-plane for the equation

$$u' = (u - \mu)(u^2 - \mu).$$

Draw the parts of the diagram with stable equilibrium points as solid lines, and the unstable equilibrium points as dashed lines.

Solution proposal:
The equilibrium points are the solutions to $f(\mu, u) = 0$, where $u' = f(\mu, u)$. Thus, the equilibrium points are given by $u = \mu$ or $u^2 = \mu$.
We have $\frac{\partial f}{\partial u} = 3u^2 - 2\mu u - \mu$. Thus, $\frac{\partial f}{\partial u}(\mu, \mu) = \mu(\mu - 1) < 0$ gives stabilty on the curve $u = \mu$. Consequently, on the curve $u = \mu$, we have stability stability for $0 < \mu < 1$, and instability for $\mu < 0$ and $\mu > 1$.

On the curve $u^2 = \mu$ we have $\mu \geq 0$ with $u = \pm\sqrt{\mu}$. $\frac{\partial f}{\partial u}(\mu, \sqrt{\mu}) = 2\mu(1 - \sqrt{\mu})$. Thus, on the curve $u = \sqrt{\mu}$ we have stability when $2\mu(1 - \sqrt{\mu}) < 0 \quad \Leftrightarrow \quad \mu > 1$, and instability when $0 < \mu < 1$.
For $u = -\sqrt{\mu}$, $\frac{\partial f}{\partial u}(\mu, -\sqrt{\mu}) = 2\mu(1 + \sqrt{\mu}) > 0$ for $\mu > 0$. So we have instability on the curve $u = -\sqrt{\mu}$. The bifurcation diagram in the (μ, u) plane is



Problem 5 Let $\beta, \gamma, \kappa, \eta$, and λ be given positive constants. We consider a model of cell density $u^*(t^*)$ in tissue, where $u^*(t^*)$ satisfies

$$\frac{du^*}{dt^*} = \frac{\beta}{1 + \frac{v^*}{\gamma}} u^* - \kappa u^*.$$

Here $v^*(t^*)$ is the concentration of an inhibitor chemical which satisfies

$$\frac{dv^*}{dt^*} = \eta u^* - \lambda v^*.$$

a) Describe the meaning of the various terms on the right hand side in the two equations. Assume $\kappa \ll \lambda$. Scale the equations using $\frac{1}{\kappa}$ as scale for t^*. Find a scale for u^* and v^* such that the dimensionless equations are of the form

$$\frac{du}{dt} = \frac{\alpha}{1 + v} u - u,$$
$$\epsilon \frac{dv}{dt} = u - v.$$

What is the meaning of the dimensionless constants α and ϵ, and from the given assumptions, what can we say about the size of ϵ? Assume $\alpha > 1$. Why is this physically reasonable? Find the equilibrium points, and determine if they are unstable or not.

Solution proposal:
We see that κ is the relative death rate of cells, while β is the relative birth rate for cells when no inhibitor is present. It is reasonable that $\beta \geq \kappa$, or else the cells will die out even when no inhibitor is present. γ is the inhibitor concentration giving half the relative cell birth rate compared to when no inhibitor is present, and consequently γ should be a natural scale for v^*. In the second equation, η models the production rate of inhibitor in response to cell concentration, and λ is the relative destruction rate of the inhibitor.

As discussed, γ is a natural scale for v^*. Thus we set

$$v^* = \gamma v, \quad t^* = \frac{1}{\kappa}t, \text{ and } \quad u^* = Uu,$$

for some U to be determined by comparison with the given dimensionless equations. For the first equation in dimensionless form we obtain

$$\kappa U u_t = \frac{\beta U}{1+v} - \kappa U u \quad \Leftrightarrow u_t = \frac{\alpha}{1+v}u - u,$$

where $\alpha = \frac{\beta}{\kappa}$. For the second equation,

$$\kappa\gamma = \eta U u - \lambda\gamma v \quad \Leftrightarrow \quad \epsilon v_t = \frac{\eta U}{\lambda\gamma}u - v,$$

where $\epsilon = \frac{\kappa}{\lambda}$. Thus, choosing $U = \frac{\gamma\lambda}{\eta}$, we get the dimensionless equations in the required form.

ϵ is the ratio between the time scales $\frac{1}{\lambda}$ and $\frac{1}{\kappa}$ for inhibitor disintegration and cell death, and is consequently small (cell tissue die much slower than the time for inhibitor to disintergrate). α is also the ratio beteen two time scales, the time scales for death and birth of cells when no inhibitor is present, and should, as discussed, be greater than unity.

It is straightforward to see that $(0,0)$ and $(\alpha-1, \alpha-1)$ are the only equlibrium points. To assess the stability properties of the two equilibrium points, we consider the Jacobi matrix of the right hand side of the equations, that is, the Jacobi matrix dF of

$$\overrightarrow{F} = \begin{bmatrix} \frac{\alpha}{1+v}u - u \\ u - v \end{bmatrix},$$

which is

$$dF = \begin{bmatrix} \frac{\alpha}{1+v} - 1 & -\frac{\alpha u}{(1+v)^2} \\ 1 & -1 \end{bmatrix}.$$

Let λ_1, λ_2 be the eigenvalues for dF. Then $\lambda_1\lambda_2 = \det(dF)$ and $\lambda_1 + \lambda_2 = \text{trace}(dF)$. For $(0,0)$, the detrminant of the Jacobi matrix is $-(\alpha - 1) < 0$, hence the eigenvalues are real and have opposite signs. Consequently $(0,0)$ is unstable. For $(\alpha - 1, \alpha - 1)$ the determinant is $\frac{\alpha-1}{\alpha} > 0$, and the trace -1. Thus, both eigenvalues have negative real parts. Therefore, $(\alpha - 1, \alpha - 1)$ is asymptotically stable.

b) We consider singular perturbation for the system of equations in the parameter ϵ. Find the outer solution $(u_O(t), v_O(t))$ to order $O(1)$, where this leading order approximation to the outer solution can only be given implicitly. Show that (for this approximation) $(u_O(t), v_O(t)) \to (\alpha - 1, \alpha - 1)$ as $t \to \infty$ for all possible non-zero initial conditions $(u_O(0), v_O(0)) \neq (0,0)$.

Solution proposal
We set $\epsilon = 0$ in order to find the lowest order approximation (u_O, v_O) of the outer solution. Put $(u_O, v_O) = (u, v)$ temporarily. We find

$$u = v, \text{ giving } \frac{du}{dt} = \frac{\alpha}{1+u}u - u.$$

The last equation is separable, and using partial fractions we obtain

$$(\frac{1}{u} + \frac{\alpha}{\alpha - 1 - u})du = (\alpha - 1)dt \quad \Leftrightarrow \quad \frac{|u|}{|\alpha - 1 - u|^{\alpha}} = Ce^{(\alpha-1)t},$$

where C is a non-negative constant. If $u(0) > 0$, we see $C > 0$, and that the right hand side of the last expression approaches ∞ as $t \to \infty$ (recall $\alpha > 1$). Again, since $\alpha > 1$ the only way for the left hand side to approach infinity as $t \to \infty$ is that $u \to \alpha - 1$. Since $v = u$, we conclude that the first order approximation of the outer solution satisfies $(u_O, v_O) \to (\alpha - 1, \alpha - 1)$ as $t \to \infty$, as long as $(u_O(0), v_O(0)) \neq (0,0)$.

c) We have a boundary layer at $t = 0$. Find, again to leading order, an inner solution by defining $\delta(\epsilon)$ as new time scale, and determine $\delta(\epsilon)$ (set $\tau = \frac{t}{\delta(\epsilon)}$

as new time variable). Also determine a uniform approximate solution using the standard matching procedure, where the uniform solution is expressed in terms of $u_O(t), t, \epsilon$, and the initial conditions $(u(0), v(0))$.

Solution proposal
Expressed in the new time variable τ, the equations become

$$\frac{1}{\delta(\epsilon)}u_\tau = \frac{\alpha}{1+v}u - u,$$

$$\frac{\epsilon}{\delta(\epsilon)}v_\tau = u - v,$$

where we again temporarily write $(u_I, v_I) = (u, v)$. Thus, choosing $\delta(\epsilon) = \epsilon$, the second equation is balanced in all terms, and we obtain

$$u_\tau = \epsilon(\frac{\alpha}{1+v}u - u),$$

$$v_\tau = u - v.$$

Thus, to first order we have $u(\tau) = D$ for a constant D, and $v_\tau = D - v$. This gives $v(\tau) = D + Ee^{-\tau}$ for some constant E. Thus, the first order approximation to the inner solution is $(u_I(\tau), v_I(\tau)) = (D, D + Ee^{-\tau})$, and we see that $(u_I(\tau), v_I(\tau) \to (D, D)$ as $\tau \to \infty$.
Since $u_O(t) = v_O(t)$, the matching procedure gives $\lim_{t\to 0}(u_O(t), v_O(t)) = \lim_{\tau\to\infty}(u_I(\tau), v_I(\tau)) = (D, D)$. The first order approximation to the uniform solution is consequently

$$u_U(t) = u_O(t) + u_I(\frac{t}{\epsilon}) - D = u_O(t),$$

$$v_U(t) = v_O(t) + v_I(\frac{t}{\epsilon}) - D = u_O(t) + Ee^{-\frac{t}{\epsilon}}.$$

Using the initial conditions $(u_U(0), v_U(0)) \approx (u(0), v(0))$, gives

$$u_U(t) = u_O(t) \text{ and } v_U(t) = u_O(t) + (v(0) - u(0))e^{-\frac{t}{\epsilon}}.$$

The constant C in the implicit solution for $u_O(t)$ is determined from $u(0)$, but is not needed for writing down $(u_U(t), v_U(t))$ in the form as requested in this exercise.