Compulsory Assignment 1

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Problem 1

Let $\mu = E(X) = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$ and $\Sigma = cov(X) = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$ s.t.

$$Y = \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} X$$

1a

(i) We want to find the mean vector and the covariance vector of Y.

$$E(Y) = E(AX) = AE(X) = \begin{pmatrix} -\frac{2}{\sqrt{2}} \\ \frac{2}{\sqrt{2}} \end{pmatrix}$$

$$\begin{aligned} cov(Y) &= cov(AX) = Acov(X)A^T \\ &= \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix} \end{aligned}$$

(ii) The distribution of Y is a bivariate normal distribution, where

$$Y \sim N(E(Y), cov(Y))$$

(iii) We can observe that Y_1 and Y_2 is independent since

$$cov(Y_1, Y_2) = 0$$

1b

Let the pdf be given as the equation of a ellipse s.t.

$$\begin{aligned} f(x) &= a, \quad a > 0 \\ (x-\mu)^T \Sigma^{-1} (x-\mu) &= b. \end{aligned}$$

The relation of b and a can be derived as follows,

$$\begin{aligned} f(x) = k \cdot \exp\left(-(x-\mu)^T \Sigma (x-\mu) \right) &= a \\ \ln k - \ln a = (x-\mu)^T \Sigma (x-\mu) \end{aligned}$$

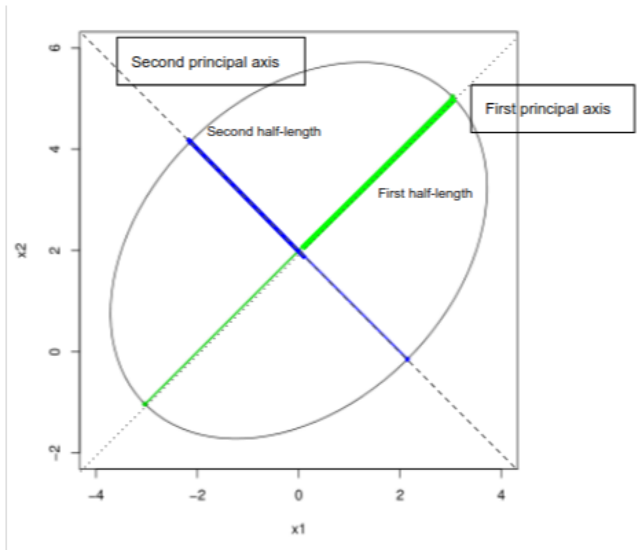
Thus $b = \ln k - \ln a$, where $k = \dfrac{1}{2\pi |\Sigma|}$. Clearly, can we observe that the aligment of the ellipse is oriented along the eigenvectors of Σ . Furthermore, the half lengths is described by the scalar b and eigenvalues

$$l_1 = \sqrt{b}\sqrt{\lambda_1} \quad \text{and} \quad l_2 = \sqrt{b}\sqrt{\lambda_2}.$$

Since $(x-\mu)\,\Sigma^{-1}\,(x-\mu)$ is a sum of normal distributed variables can we compute the probability a random variable being inside the ellipse α by using the fact that

$$(x-\mu)^{-1}\,\Sigma\,(x-\mu) \sim \chi^2_2.$$

Hence, the probability can be computed using $\chi^2_2\,(\alpha) \leq b \iff \alpha \approx 0.9$.



Problem 2

2a

Let $X = [X_1, X_2, X_3, \dots, X_n]^T$ be a stochastic vector and a vector of ones $\mathbf{1} = \mathbf{1}_{n\times 1}$.

(i)

$$\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i = \frac{1}{n}\mathbf{1}^T \left[X_1, \dots, X_n\right]^T = \frac{1}{n}\mathbf{1}^T X$$

(ii)

$$\begin{aligned}
S^2 &= \frac{1}{(n-1)} X^T C X = \frac{1}{(n-1)} X^T C C X \\
&= \frac{1}{(n-1)} \left(CX\right)^T \left(CX\right) \\
&= \frac{1}{(n-1)} \left(X - \mathbf{1}\overline{X}\right)^T \left(X - \mathbf{1}\overline{X}\right) \\
&= \frac{1}{(n-1)} \sum_{i=1}^{n} \left(X_i - \overline{X}\right)\left(X_i - \overline{X}\right)
\end{aligned}$$

2b

We want to show the independence of $\overline{X}$ and S^2. Firstly, let us emphasize the result that

$$\frac{1}{n}\mathbf{1}^T \left(C\right) = \frac{1}{n}\mathbf{1}^T \left(I - \frac{\mathbf{1}\mathbf{1}^T}{n}\right) = \frac{1}{n}\mathbf{1}^T - \frac{1}{n}\mathbf{1}^T = 0.$$

And utilize the fact that

$$cov\left(\overline{X}, S^2\right) = cov\left(\frac{1}{n}\mathbf{1}^{\mathbf{T}} X, CX\right) = \frac{1}{n}\mathbf{1}^{\mathbf{T}}\sigma I C = \sigma \cdot 0.$$

Hence, $\overline{X}$ and S^2 are independent.

2c

Let $Y = \Sigma^{-1}(X-\mu)$ and

$$\begin{aligned} Y^T C Y &= \left(\Sigma^{-1}(X-\mu)\right)^T C \left(\Sigma^{-1}(X-\mu)\right) = \frac{1}{\sigma^2}(X-\mu)^T C (X-\mu) \\ &= \frac{1}{\sigma^2} X^T C X \end{aligned}$$

The last step comes from that $(C\mu)_i = 0$. Recall from the exercise description that $Y = N(0,\Sigma)$ implies $YRY^T \sim \chi_r^2$, where R is idempotent with the rank r. Thus can we compute

$$X^T C X \sim \chi_r^2 \sigma^2$$

Hence, can we use the fact that

$$S^2 = \frac{1}{n-1} X^T C X \implies (n-1)\frac{S^2}{\sigma} \sim \chi_r^2$$

Utilizing the fact that the rank of C is the sum of the trace $tr(C) = n-1$ can we conclude that

$$(n-1)\frac{S^2}{\sigma^2} \sim \chi_{n-1}^2$$

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