13. Lecture XIII: Critical Points of Planar Systems II

We shall now be looking into some dynamics around critical points that are not even qualitatively similar to any dynamics that can be observed around critical points of linear systems. From the Hartman-Grobman theorem, we know that this means we cannot be working at hyperbolic critical points. We shall be looking at some further dynamics on the centre manifold that are neither centres nor foci.

To circumscribe our discussion, we shall consider only autonomous *analytic* systems around isolated critical points. From Thm. 12.3 and Dulac's Theorem (Thm. 11.3), we know that we are not considering critical points for which the linearized system exhibits a centre. We are also not interested in critical points of systems for which the linearization exhibits nodes, foci, or saddles, as analytic systems are immediately C^2. Since we are working on real planar systems, this leaves us with critical points with zero eigenvalues in the linearization.

13.1. **Degenerate linear planar systems.** We know that critical points of degenerate linear systems are never isolated. Herein is the crux of the matter. In considering higher powers beyond the first order approximation, we can "unfold" a degenerate linearized system into exhibiting a wide variety of different behaviours.

First we take another look at degenerate linear planar systems, which we have only done in obiter.

There are three ways that a linear planar system can be degenerate. By the Jordan Normal Form Theorem (Thm. 3.1), we can assume that the linear system has been put into Jordan normal form. We can have systems governed by matrices with one zero eigenvalue, two zero eigenvalues but with geometric multiplicity one, or two zero eigenvalues with a geometric multiplicity of two, respectively:

$$\begin{pmatrix} 0 & 0 \\ 0 & \lambda \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

for $\lambda \in \mathbb{R}\backslash\{0\}$.

As we have mentioned, the final case is immediate — the entire $\mathbb{R}^2$ are fixed points. The general solution to systems governed by the first matrix is

$$\begin{pmatrix} x \\ y \end{pmatrix}(t) = C_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + C_2 e^{\lambda t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The general solution to systems governed by the second matrix is not too much more difficult to write down, recallling the Jordan chain proceedure:

$$\begin{pmatrix} x \\ y \end{pmatrix}(t) = (C_1 + C_2 t) \begin{pmatrix} 1 \\ 0 \end{pmatrix} + C_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

As we have already mentioned in Lecture 9, these rather tame behaviours are much more readily perturbed by higher order terms than when the governing matrix is not degenerate.

13.2. **Nonhyperbolic critical points of planar systems.** Next we are obliged to first consider a theorem about analytic first order systems:

Theorem 13.1. *Consider the autonomous analytic planar system given by*

$$\dot{x} = P(x, y), \qquad \dot{y} = Q(x, y).$$

Let $P_m(x, y)$ and $Q_m(x, y)$ be the m-th degree polynomials in the Taylor expansions of P and Q about an isolated fixed point at the origin. Any trajectory that approaches the origin as $t \to \infty$ either spirals towards $\mathbf{0}$ (i.e., argument tends to infinity) or approaches it along a definite direction θ_0 as $t \to \infty$.

If one trajectory spirals towards $\mathbf{0}$, then in a deleted neighbourhood $B_\delta(\mathbf{0})\backslash\{\mathbf{0}\}$, all trajectories spiral towards $\mathbf{0}$.

If there is a definite direction of approach, then either

$$\forall m \geq 1, \quad xQ_m(x,y) - yP_m(x,y) \equiv 0,$$

or along any direction θ_0 of approach,

$$\forall m \geq 1, \quad \cos(\theta_0)Q_m(\cos(\theta_0), \sin(\theta_0)) - \sin(\theta_0)P_m(\cos(\theta_0), \sin(\theta_0)) = 0.$$

We have already looked at centres and foci, and mentioned that centre-foci cannot appear for analytic planar systems.

The expressions in the theorem statements are two-dimensional versions of the cross product,

$$(x,y)^\top \wedge (P_m(x,y), Q_m(x,y))^\top.$$

It is a result of calculus that vector fields can be decomposed into divergence-free and curl-free parts. Another way of saying this is that for any fixed vector in $\mathbb{R}^d$, the subspace of vectors orthogonal to it and the subspace of vectors that has zero wedge/cross product with it are othodgonal and decompose $\mathbb{R}^d$. The result above says that unless there is a spiral, any vector field along which an approaching trajectoy approaches must be radial to arbitrary degree — that is, purely "divergence", and and any non-radial component of the approach come from the constant terms of the Taylor expansion.

It is then clear that the first non-spiral approaching possibility allowed by the theorem statement are nodes, whilst two of the the separatrices of a saddle fall under the second non-spiral approaching possibility allowed by the theorem statement as $t \to \infty$, and the remaining two are included in the same provision as $t \to -\infty$.

But the theorem also allows for more diverse behaviours under its second non-spiralling provision. It is in fact possible, as shall be demonstrated in computable examples later, that the plane gets divided into sectors separated much like saddles by separatrices that approach the critical point along definite directions as $t \to \infty$ or as $t \to -\infty$. We denominate three possible types of sectors as being HYPERBOLIC, PARABOLIC, ELLIPTIC according as there is a small enough neighbourhood about the critical point such that each trajectory in the sector not including the separatrices

(i) leaves the neighbourhood as $t \to \pm\infty$, or
(ii) leaves the neighbourhood as $t \to \infty$ and approaches the critical point as $t \to -\infty$, or vice versa, or
(iii) approaches the critical point as $t \to \pm\infty$.

[pictures]

The saddle is then seen to be a critical point with four separatrices and four hyperbolic sectors, and a node is a critical point with one single parabolic sector.

13.2.1. One zero eigenvalue.

As mentioned at the beginning of this lecture, from the Hartman-Grobman Theorem, or more directly, from Thm.12.2, we know that other sectoring behaviours are not exhibited at hyperbolic fixed points. We also have considered behaviours around critical points for which the linearized system exhibits centres, and Thm. 12.3 ensures that in the analytic case, the full dynamics about these critical points are centres or foci. This leaves critical points with one or two eigenvalues that are zero.

We first consider systems with one eigenvalue set to nought. Again, from the theorem on Jordan normal forms (Thm. 3.1), we know that the linearized system is governed by

$$\mathrm{D}f = \begin{pmatrix} 0 & 0 \\ \lambda & 0 \end{pmatrix}.$$

By scaling, we may take $\lambda = 1$, without loss of generality. If we desire that λ correspond to a stable subspace, we simply consider the system backwards in time. This compels us to consider systems of the type:

$$\dot{x} = P(x, y)$$
$$\dot{y} = y + Q(x, y),$$

where P and Q vanish to second order around the *isolated* fixed point $(x_0, y_0)^\top = \mathbf{0}$.

From the implicit function theorem (Thm.8.4), there is a function ϕ such that $y = \phi(x)$ solves $y + Q(x, y) = 0$ in a neighbourhood of $\mathbf{0}$. This is, locally, graph of the nullcline. We can write $\phi(x)$ as

$$\phi(x) = \phi(0) + \phi'(0)x + \cdots ,$$

as we are in an analytic setting. Furthermore, since P is analytic and vanishes to second order in a neighbourhood of $\mathbf{0}$, we can write

$$\psi(x) = P(x, \phi(x)) = \sum_{m \geq 2} a_m x^m.$$

It turns out that the lowest order term of ψ — P evaluated on the other nullcline — gives us further information beyond the linearization that can already classify the behaviour of the system in a neighbourhood of the critical point.

First we need to look at some classes of behaviours:

(i) a critical point is a CRITICAL POINT WITH AN ELLIPTIC DOMAIN if it has four separatrices, one elliptic sector, one hyperbolic sector, and two parabolic sectors;
(ii) a critical point is a SADDLE-NODE if it has three separatrices, two hyperbolic sectors, and one parabolic sector; and
(iii) a critical point is a CUSP if it has two separatrices and two hyperbolic sectors.

[pictures]

And we have the following theorem:

Theorem 13.2. *Let $\psi(x) = P(x, \phi(x)) = \sum_{m \geq 2} a_m x^m$ be defined as before in a neighbourhood of the origin for the planar system governed by $P(x, y)$ and $Q(x, y)$, also previously defined. Let ℓ be the smallest integer for which $a_\ell \neq 0$.*

(i) *If $\ell \equiv 1\,(\mathrm{mod.}2)$ and $a_\ell > 0$, then $\mathbf{0}$ is an unstable node,*
(ii) *if $\ell \equiv 1\,(\mathrm{mod.}2)$ and $a_\ell < 0$, then $\mathbf{0}$ is a (topological) saddle, and*
(iii) *if $\ell \equiv 0\,(\mathrm{mod.}2)$, then $\mathbf{0}$ is an saddle-node.*

13.2.2. Two zero eigenvalues with unit geometric multiplicity.

The corresponding theorem becomes considerably more complicated when both eigenvalues are zero, but the dimension of $\ker(\mathrm{D}f)$ is only one. We mention this for completeness.

In this case, it turns out that there is always a change-of co-ordinates that leaves the system in the following normal form:

$$\dot{x} = y$$
$$\dot{y} = a_k x^k + b_n x^n y + O(x^{k+1} + x^{n+1} y + y^2)$$

where the higher order terms are dominated by the first two terms in a neighbourhood of the fixed point $\mathbf{0}$.

We define two further parameters:

$$m := [k/2], \qquad \lambda := b_n^2 + (2k + 2)a_k.$$

The behaviours exhibited at the fixed point obey the following schematic: