Norwegian University of Science and Technology
Department of Mathematical Sciences

TMA4145 Linear Methods
Fall 2018

Exercise set 5:Solutions

Please justify your answers! The most important part is *how* you arrive at an answer, not the answer itself.

$\boxed{1}$ Let $(x_n)_{n\in\mathbb{N}}$ be a convergent sequence in a metric space (X, d).

a) Show that $(x_n)_{n\in\mathbb{N}}$ is a bounded subset of X.

b) Show that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence.

Solution. **a)** Denote the limit of $(x_n)_{n\in\mathbb{N}}$ by x. We will find an $r > 0$ such that $(x_n)_{n\in\mathbb{N}} \subset B_r(x)$. Since the sequence converges, we can find an $N \in \mathbb{N}$ such that $d(x, x_n) < 1$ for any $n \geq N$. This is simply the definition of convergence in a metric space, with $\epsilon = 1$. Hence all but the first N elements of the sequence $(x_n)_{n\in\mathbb{N}}$ lie in the ball $B_1(x)$. Now define

$$r = \max\{1, d(x_1, x) + 1, d(x_2, x) + 1, \ldots, d(x_N, x) + 1\}.$$

Then we claim that $(x_n)_{n\in\mathbb{N}} \subset B_r(x)$, which would imply that the sequence is bounded. To see that this claim is true, note that

- If $n \geq N$, then $d(x_n, x) < 1 \leq r$ by the way we picked N. Hence $x_n \in B_r(x)$ for $n \geq N$.
- If $n < N$, then $d(x_n, x) < r$ since we defined r to be greater than the maximum of such distances. Hence $x_n \in B_r(x)$ for $n < N$, so $(x_n)_{n\in\mathbb{N}} \subset B_r(x)$.

b) To show that the sequence is Cauchy we need, for every $\epsilon > 0$, to find $N \in \mathbb{N}$ such that $d(x_m, x_n) < \epsilon$ whenever $m, n \geq N$. Let us therefore fix some arbitrary $\epsilon > 0$. Since $x_n \to x$, we can find $N \in \mathbb{N}$ such that $d(x, x_n) < \frac{\epsilon}{2}$ for every $n \geq N$. Then, if $m, n \geq N$, we have

$$d(x_m, x_n) \leq d(x_m, x) + d(x, x_n) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

by the triangle inequality. Hence this N works.

$\boxed{2}$ Let $(X, \langle \cdot, \cdot \rangle)$ be an inner product space, and suppose that (x_n) and (y_n) are convergent sequences with $\lim_n x_n = x$ and $\lim_n y_n = y$. Show that

$$\lim_{n\to\infty} \langle x_n, y_n \rangle = \langle x, y \rangle.$$

(This was exam problem 5b in Spring 2015.)

Solution. By the definition of limits in normed spaces, we need to show that for any $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $|\langle x, y\rangle - \langle x_n, y_n\rangle| < \epsilon$ whenever $n \geq N$. By adding and subtracting $\langle x_n, y\rangle$ (i.e. adding 0) to this expression, we use the triangle inequality to find that that

$$\begin{aligned} |\langle x, y\rangle - \langle x_n, y_n\rangle| &= |\langle x, y\rangle - \langle x_n, y\rangle + \langle x_n, y\rangle - \langle x_n, y_n\rangle| \\ &\leq |\langle x, y\rangle - \langle x, y_n\rangle| + |\langle x, y_n\rangle - \langle x_n, y_n\rangle| \\ &= |\langle x, y - y_n\rangle| + |\langle x - x_n, y_n\rangle| \end{aligned}$$

Let us now apply the Cauchy-Schwarz inequality to each of these summands. The first summand gives

$$|\langle x, y - y_n\rangle| \leq \|x\|\|y - y_n\|,$$

and the second summand

$$|\langle x - x_n, y_n\rangle| \leq \|x - x_n\|\|y_n\| \leq \|x - x_n\|C.$$

In the last inequality we use part a) of problem 1 to get that $(y_n)_{n\in\mathbb{N}}$ is bounded, i.e. $\|y_n\| \leq C$ for some positive constant C. If we put these two estimates back into our first calculation, we have that

$$|\langle x, y\rangle - \langle x_n, y_n\rangle| \leq \|x\|\|y - y_n\| + \|x - x_n\|C. \tag{1}$$

This equation suggest how we can pick the desired N. Since $x_n \to x$, there is (by the definition of limits) an $N_1 \in \mathbb{N}$ such that

$$\|x - x_n\| < \frac{\epsilon}{2C}$$

whenever $n \geq N_1$. Since $y_n \to y$, we similarly have $N_2 \in \mathbb{N}$ such that[1]

$$\|y - y_n\| \leq \frac{\epsilon}{2\|x\| + 1}$$

whenever $n \geq N_2$.
If we then let $N = \max\{N_1, N_2\}$, it is clear by equation (1) that for $n \geq N$ we have

$$|\langle x, y\rangle - \langle x_n, y_n\rangle| \leq \|x\|\frac{\epsilon}{2\|x\|} + C\frac{\epsilon}{2C} = \epsilon.$$

Note. Once you get more comfortable with these arguments, you could conclude straight from equation (1) that the result holds. After all, the right hand side of

[1]Why do we add a 1 to the denominator? It is simply to avoid dividing by 0 if $\|x\| = 0$.

that equation converges to 0 by assumption, and hence the left hand side must also converge to 0 – and that is what we want to show. However, to take this shortcut, you should be able to fill in all the details using the definition of the limit, as we did above!

3 We denote by c_f the vector space of all sequences with only finitely many non-zero terms. Show that c_f is not a Banach space with the norm $\|\cdot\|_\infty$. As usual, $\|\cdot\|_\infty$ is defined by

$$\|x\|_\infty = \sup_{n\in\mathbb{N}} |x_n|$$

for a sequence $x = (x_n)_{n\in\mathbb{N}} \in c_f$.

Solution. We will find a sequence $y_n \in c_f$ that is Cauchy yet not convergent (note that y_n is a sequence for each value of n – we have a sequence of sequences). Define

$$y_n = (1, \frac{1}{2}, \frac{1}{3}, ..., \frac{1}{n}, 0, 0, ...).$$

If we now consider y_n to be elements of the larger space ℓ^∞, we have that $y_n \to y$ where

$$y = (1, \frac{1}{2}, \frac{1}{3}, ..., \frac{1}{1000}, \frac{1}{1001}, ...).$$

To prove that $y_n \to y$, simply note that $\|y - y_n\|_\infty = \sup_{k\geq n} \frac{1}{k} = \frac{1}{n}$, and clearly $\frac{1}{n} \to 0$. The "problem" in this case is that $y_n \in c_f$, yet clearly $y \notin c_f$. Since the sequence y_n converges, it is Cauchy by problem 1b. However, since the limit is unique and $y_n \to y$ in ℓ^∞, the sequence y_n cannot converge in c_f (if it did converge to some element y' in c_f, the sequence would have two different limits $y \neq y'$ in ℓ^∞).

4 For each $n \in \mathbb{N}$, let

$$x^{(n)} := (1, \frac{1}{2}, \ldots, \frac{1}{n}, 0, 0, \ldots),$$

which we regard as an element of the space $\ell^p(\mathbb{R})$ (for any given $p \in [1, \infty]$).

a) Find the limit of the sequence $\left(x^{(n)}\right)_{n\geq 1}$ in $(\ell^\infty(\mathbb{R}), \|\cdot\|_\infty)$. Prove your claim.

b) Does $\left(x^{(n)}\right)_{n\geq 1}$ have a limit in $(\ell^1(\mathbb{R}), \|\cdot\|_1)$? If the limit exists, find it and prove that it is the limit.

c) Does $\left(x^{(n)}\right)_{n\geq 1}$ have a limit in $(\ell^2(\mathbb{R}), \|\cdot\|_2)$? If the limit exists, find it and prove that it is the limit.

Solution. a) Let $x := \left(1, \frac{1}{2}, \ldots, \frac{1}{n}, \frac{1}{n+1}, \ldots\right)$. Then clearly $x \in \ell^\infty(\mathbb{R})$.

We show that $x^{(n)} \to x$ with respect to the $\|\cdot\|_\infty$ norm. It is enough to show that

$$\|x^{(n)} - x\|_\infty \to 0\,.$$

But

$$x^{(n)} - x = \left(0, \ldots, 0, -\frac{1}{n+1}, -\frac{1}{n+2}, \ldots\right)\,,$$

so

$$\|x^{(n)} - x\|_\infty = \frac{1}{n+1} \to 0 \quad \text{as } n \to \infty\,.$$

b) Let us assume that $\left(x^{(n)}\right)$ has a limit in $(\ell^1(\mathbb{R}), \|\cdot\|_1)$, and let us denote that limit by y.

It follows from proposition 3.1 in the notes that

$$\|x^{(n)}\|_1 \to \|y\|_1.$$

But

$$\|x^{(n)}\|_1 = 1 + \frac{1}{2} + \ldots + \frac{1}{n} \to \infty \quad \text{as } n \to \infty\,,$$

so $\|y\|_1 = \infty$. But then $y \notin \ell^1(\mathbb{R})$, so the sequence $\left(x^{(n)}\right)$ cannot have a limit in $(\ell^1(\mathbb{R}), \|\cdot\|_1)$.

c) We show that the sequence $\left(x^{(n)}\right)$ converges to the vector x defined in part a) also in $(\ell^2(\mathbb{R}), \|\cdot\|_2)$. First note that $x \in \ell^2(\mathbb{R})$, since

$$\|x\|_2^2 = \sum_{j=1}^{\infty} \left(\frac{1}{j}\right)^2 = \sum_{j=1}^{\infty} \frac{1}{j^2} < \infty\,.$$

It is enough to show that $\|x^{(n)} - x\|_2 \to 0$. We have:

$$\|x^{(n)} - x\|_2^2 = \frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + \ldots \to 0 \quad \text{as } n \to \infty\,,$$

because this is the *tail* of the convergent series $\sum_{j=1}^{\infty} \frac{1}{j^2}$.

$\boxed{5}$ Let $C[a,b]$ be the vector space of all continuous functions $f\colon [a,b] \to \mathbb{R}$.

We will consider two norms on this space, $\|\cdot\|_1$ and $\|\cdot\|_\infty$.

a) Prove that for all $f \in C[a,b]$ we have

$$\|f\|_1 \le (b-a)\,\|f\|_\infty\,.$$

b) Let (f_n) be a sequence in $C[a,b]$.
Prove that if $f_n \to f$ with respect to $\|\cdot\|_\infty$ then $f_n \to f$ with respect to $\|\cdot\|_1$.

c) Show that the reverse of the statement in b) is not always true.

Solution. **a)** Since for all $x \in [a,b]$, we have $|f(x)| \leq \|f\|_\infty$, we derive the following:

$$\|f\|_1 = \int_a^b |f(x)|\, dx \leq \int_a^b \|f\|_\infty \, dx = (b-a)\, \|f\|_\infty \, .$$

b) If $f_n \to f$ with respect to $\|\cdot\|_\infty$ then $\|f_n - f\|_\infty \to 0$ – this is more or less the definition of convergence in a normed space.

But from part a) of this problem, $\|f_n - f\|_1 \leq (b-a)\, \|f_n - f\|_\infty$.

Then by the squeeze test, we must also have that $\|f_n - f\|_1 \to 0$, showing, again by problem 1 part a), that $f_n \to f$ with respect to $\|\cdot\|_1$.

c) For simplicity, let us work with $C[0,1]$.

We define $f\colon [0,1] \to \mathbb{R}$ as $f(x) = 1$ for all x.

Moreover, we define the sequence of functions $(f_n)_{n\geq 1}$ as follows:

$$f_n(x) = \begin{cases} nx, & \text{if } x \in [0, \frac{1}{n}] \\ 1, & \text{if } x \in [\frac{1}{n}, 1] \, . \end{cases}$$

Make sure you draw a picture of these functions, it is more important than their actual formulas. An easy calculation then shows that

$$\|f_n - f\|_1 = \int_0^1 |f_n(x) - f(x)|\, dx = \frac{1}{2}\,(1 \times \frac{1}{n}) = \frac{1}{2n} \to 0\,,$$

so $f_n \to f$ with respect to the $\|\cdot\|_1$ norm.

On the other hand, $f(0) = 1$ and for all $n \geq 1$ we have $f_n(0) = 0$, so $|f_n(0) - f(0)| = 1$. This shows that

$$\|f_n - f\|_\infty \geq 1 \quad \text{for all } \; n \geq 1\,,$$

hence f_n cannot converge to f with respect to the $\|\cdot\|_\infty$ norm.