## 26. Lecture XXVI: Higher Dimensional Local Bifurcations

26.1. **Universal unfoldings.** There are many different topics in dynamical systems, even as they relate to flows and ordinary differential equations, that we have not had time to cover in this module. I should like bring this set of lectures to a close by considering more closely the question of universal unfoldings. We seek to understand the totality of possible ways a system can be parameterized, and the associated bifurcations as these parameters vary. Another way of looking at this is, we seek to use extra parameters to unpack all possible near-by behaviours of a system, or, to find all collections of systems that intersect at one specified system.

Recall that a UNIVERSAL UNFOLDING of a $d$-dimensional system $\dot{\mathbf{x}} = f(\mathbf{x})$ is a system $\dot{x} = g(\mathbf{x}, \boldsymbol{\nu})$, indexed by finitely many parameters $\boldsymbol{\nu} \in \mathbb{R}^m$, such that at $\boldsymbol{\nu} = \boldsymbol{\nu}_0$, $\dot{\mathbf{x}} = g(\mathbf{x}, \boldsymbol{\nu}_0)$ is topologically equivalent to $\dot{\mathbf{x}} = f(\mathbf{x})$.

In this lecture we shall look at universal unfoldings that depend on more than one parameter. We shall limit our discussion to systems in one *spatial* dimension. The dimensionality to which the title of this lecture refers are the degrees of freedom required (by the parameters) to yield universal embeddings. This is higher dimensionality is often referred to as "bifurcations of higher *co*dimensions". We are taking another look at one dimensional bifurcations from the opposite perspective, by another organization principle, where, instead of systematically relaxing non-degeneracy conditions to get individual bifurcations, we shall systematically look at more and more degenerate/nonlinear systems and ask for all possible bifurcations at their nonhyperbolic fixed points.

First let us consider the simplest nonlinear system:

$$\dot{x} = -x^2.$$



Observe that unfolding by adding higher order terms does not change the qualitative behaviour of the system about the nonhyperbolic critical point $x = 0$. The fixed points of the system

$$\dot{x} = -x^2 + \mu_3 x^3$$

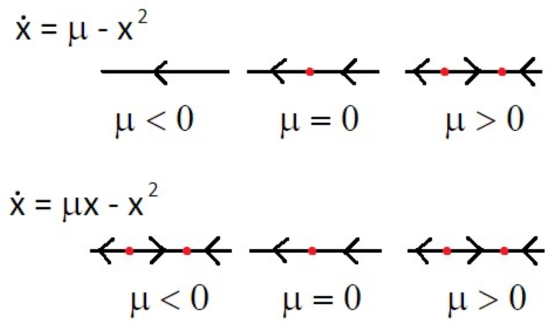
are $x = 0$ and $x = 1/\mu_3$. (We have be using $\mu_i$ to denote a sequence of bifurcation values, but here, each $\mu_i$ is not just a fixed value, but a parameter that is allowed to vary over $\mathbb{R}$.) As this unfolding approaches the original system, $\mu_3 \to 0$, and the remaining fixed point leaves any fixed neighbourhood of the nonhyperbolic fixed point. This leaves us to consider unfoldings of lower orders:

$$\dot{x} = \mu_0 + \mu_1 x - x^2$$

By an affine shift $x \mapsto (x - \mu_1/2)$, we see that this unfolding has dynamics which are topologically equivalent to

$$\dot{x} = \mu - x^2, \qquad \mu = \mu_0 - \frac{\mu_1^2}{4}.$$

This is the normal form for the saddle-node bifurcation. This verifies our earlier calculations for the saddle-node bifurcation in Lecture 22 that this normal form is the universal unfolding of the system with $f(x) = -x^2$, without non-degeneracy assumptions beyond nonhyperbolicity. In what sense is the transcritical bifurcation included in this unfolding? We need only look at the phase portraits of the normal forms expressing these two types of bifurcations to see that all behaviours/dynamics to either side of a transcritical bifurcation is included in dynamics expressed by the universal unfolding that is the normal form for a saddle-node bifurcation:




26.2. **Revisiting the pitchfork bifurcation.** Next, looking at

$$\dot{x} = -x^3,$$

we can postulate an unfolding of the form

$$\dot{x} = -x^3 + \mu_2 x^2 + \mu_1 x + \mu_0.$$

As it turns out, the normal form $\dot{x} = \mu x^2 - x^3$ for the pitchfork bifurcation is not a universal unfolding of $\dot{x} = -x^3$.

By a translation, as before, we can elimate one of the lower-order terms, so that we have the topologically equivalent system

$$\dot{x} = -x^3 + \mu_1 x + \mu_0.$$

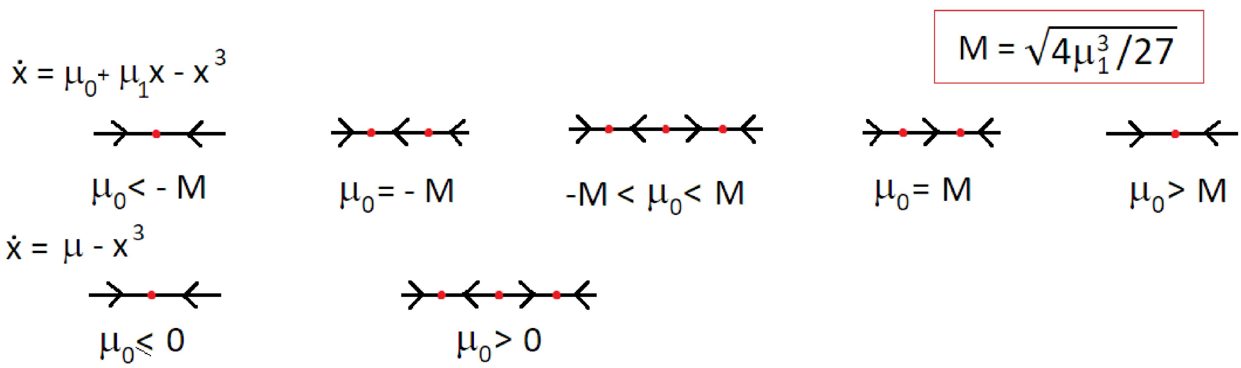
We can find the fixed points of this system as before by setting

$$0 = p(x, \mu_1, \mu_0) = -x^3 + \mu_1 x + \mu_0.$$

It is an elementary calculation to find the maxima and minima of $p$ — one minimum at $x = -\sqrt{\mu_1/3}$ and a maximum at $x = \sqrt{\mu_1/3}$. The system has three fixed points if $p$ has three roots, which is the case when $p(-\sqrt{\mu_1/3}) < 0$ and $p(\sqrt{\mu_1/3}) > 0$. This works out to be $\mu_0^2 < 4\mu_1^3/27$.

Likewise, the system has two fixed points when $\mu_1^2 = 4\mu_1^3/27$. And finally, the system has one fixed point if $\mu_0^2 > 4\mu_1^3/27$.

Its phase portrait can be compared to that of the normal form of the pitchfork bifurcation below:




So the full pitchfork bifurcation is in fact a codimension two bifurcation.

The locus of points on the $\mu_1\mu_2$-plane on which we have the phase portrait (or its reflection) not seen in the unfolding given b $\dot{x} - \mu x^3$ lies on the curve

$$\mu_0^2 = \frac{4\mu^3}{27}.$$

This sort of higher-dimensional bifurcation is known as a CUSP BIFURCATION.

This analysis for higher order systems can be continued. For $\dot{x} = -x^4$, a universal unfolding require three parameters to describe and results in a SWALLOWTAIL BIFURCATION (see Figure 2 on pg 346 of *Perko*).