Norwegian University of Science and Technology
Deptartment of Mathematical Sciences

TMA4190 Introduction to Topology
Spring 2018

Exercise set 10

1 Prove the Theorem of Perron-Frobenius: An $n \times n$-matrix A with only nonnegative entries, must have a real nonnegative eigenvalue.

(Hint: It suffices to assume A nonsingular, otherwise O is an eigenvalue. Let A also denote the associated linear map of $\mathbb{R}^n$, and consider the map $v \to Av/|Av|$ restricted to $S^{n-1} \to S^{n-1}$. Show that this maps the first quadrant

$$Q = \{(x_l, \dots, x_n) \in S^{n-1} : \text{ all } x_i \geq 0\}$$

into itself. Now use the fact that there is a homeomorphism $B^{n-1} \to Q$, to get a continuous map $B^{n-1} \to B^{n-1}$.)

2 Let X and Y be submanifolds of $\mathbb{R}^N$. Show that for almost every $a \in \mathbb{R}^N$ the translate $X + a$ intersects Y transversally.

3

a) Let Y be a compact submanifold of $\mathbb{R}^M$, and $w \in \mathbb{R}^M$. Show that there exists a (not necessarily unique) point $y \in Y$ closest to w, and prove that $w - y \in N_y(Y)$. (Hint: If $c(t)$ is a curve on Y with $c(0) = y$, then the smooth function $|w - c(t)|^2$ has a minimum at 0. Now use that we have shown on Exercise Set 2 that there is a unique correspondence between tangent vectors at y and velocity vectors at 0 of curves $c\colon (-a, a) \to Y$ with $c(0) = y$.)

b) Use the previous point to show: Let Y be a compact submanifold of $\mathbb{R}^M$, and $w \in \mathbb{R}^M$. Let $h\colon N(Y) \to \mathbb{R}^M$, $h(y, v) = y + v$, be the map used in the proof of the ϵ-Neighborhood Theorem in the lecture. We know that h maps a neighborhood of Y in $N(Y)$ diffeomorphically onto $Y^\epsilon \subset \mathbb{R}^M$, where $\epsilon > 0$ is constant. Prove that if $w \in Y^\epsilon$, then $\pi(w)$ is the unique point of Y closest to w, where $\pi = \sigma \circ h^{-1}$.

4 Let X be a submanifold of $\mathbb{R}^N$. Show that "almost every" vector space V of any fixed dimension k in R^N intersects X transversally, i.e.

$$V + T_x(X) = \mathbb{R}^N \text{ for every } x \in X.$$

(Hint: Use the fact that the set $S \subset \left(\mathbb{R}^N\right)^k$ consisting of all linearly independent k-tuples of vectors in $\mathbb{R}^N$ is open in R^{Nk}. Show that the map $\mathbb{R}^k \times S \to \mathbb{R}^N$ defined by

$$((t_1, \dots, t_k), v_1, \dots, v_k) \mapsto t_1 v_1 + \dots + t_k v_k$$

is a submersion, and apply the results of the lecture.)

$\boxed{5}$ This is a harder problem, but it is an interesting application of the Transversality Theorem and ϵ-neighborhoods. So try it!

a) Suppose that $f\colon \mathbb{R}^n \to \mathbb{R}^n$ is a smooth map with $n > 1$, and let $K \subset \mathbb{R}^n$ be compact and $\epsilon > 0$. Show that there exists a map $g\colon \mathbb{R}^n \to \mathbb{R}^n$ such that dg_x is never 0, and $|f(x) - g(x)| < \epsilon$ for all $x \in K$.

(Hint: Let $M(n)$ be the space of $n \times n$-matrices. Show that the map $F\colon R^n \times M(n) \to M(n)$, defined by $F(x, A) = df_x + A$, is a submersion. Pick A so that $F_A \pitchfork \{0\}$ for $F_A\colon x \mapsto (x, A)$ as in the lecture. Now use this knowledge to construct g. At some point along this way you will have used $n > 1$. Make sure you see where and how it has been used.)

b) Show that this result is false for $n = 1$ (i.e. find f, ϵ, $K \subset \mathbb{R}$ such that we cannot find such a g).

(Hint: You could contemplate on the Mean Value Theory.)