Application of stochastic modeling and simulation to vehicle system dynamics

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1 Introduction

This contribution presents numerical methods, problem statements, and ideas of stochastic modeling that can enhance the use of vehicle dynamics simulations. The illustration is meant to be clear and concise. At some points references shall be made to more thorough reviews or more complete presentations. This paper will not give a complete overall picture of the topic, but compose certain methods and methodologies that act well together. Finally, we will demonstrate some possible applications and worked examples.

Models used nowadays in vehicle dynamics simulations are mostly deterministic. In some cases, deterministic models fall short in describing the reality and are somehow limited with regard to the problems they can encounter.

Stochastic versions of those models can incorporate uncertainties. Uncertainties may originate from the

- modeling process;
- parameters used to describe model properties;
- numerical evaluation of a model; or the
- validation process.

The methods presented in this paper are focused on uncertainties that can be expressed as parametric uncertainties. They can arise from the derivation of parameters (e.g. measurement or estimation), the use as design variables, material properties, or operating conditions, and much more.

Before formally introducing terminology and methodology with regard to our understanding of stochastic modeling, we present some applications of stochastic modeling and simulation to vehicle system dynamics to clarify the underlying motivation and hypothesis.

Kewlani [KCI12] predicts the movement of an unmanned ground vehicle over rugged terrain. He employs a three degree of freedom white-box two-track model, for calculating lateral-, yaw- and roll dynamics. His model is able to incorporate and propagate uncertain vehicle and terrain parameters. Terrain parameters are drawn from sensor estimates and therefore not exactly known. Thus uncertain parameters can be unknown additional loads, height of obstacles, surface profiles and the friction coefficient. His approach shall contribute to the autonomous operation and estimation of traversability of vehicles in unstructured environments.

Bigoni [BTE14] analyzes the sensitivity of the critical speed of a fixed half-wagon with respect to suspension parameters. Above a critical speed, railway vehicles exhibit lateral oscillations, while the wheel flanges bang from one rail to the other [Coo72]. In this context, uncertainties arise from the manufacturing process, wear over time, loads, and wind gusts, e.g. because of crossing trains. The underlying motivation is to support engineers in the process of understanding and designing complex systems.

The railway industry [FPK12] proposes the use of uncertainty quantification and sensitivity analysis in the process of certification of railway vehicles. The authors focus on the representation of track irregularities (e.g. stiffness, profile, and friction coefficient), uncertain load, and varying suspension parameters to predict the vehicle and track behavior for the whole life cycle. Virtual homologation shall lead to leaner processes and better understanding of the behavior around critical situations.

All aforementioned problem statements have in common, that parametric uncertainty must be known beforehand and is propagated through dynamic systems in order to quantify and analyze effects on output metrics. Contributions of Zimmermann et al. [EMZ15]; [ZW15] take a different angle. The authors estimate subsets in the parameter space, also called *solution spaces*, which lead to an admissible model behavior. The model behavior is restricted with respect to design goals of different disciplines. Such goals may be the maximum acceleration at a frontal crash [ZH13] or the maximum side slip angle in a specific driving maneuver [ERM14]. Possible solution spaces are further reduced to an n-orthotope in order to resolve conflicts of interests. By doing so, the authors yield intervals for every parameter. Requirements for different subsystems thus can be communicated to separate departments independently during the development process. To obtain insight on the overall concept and ideas, the reader is referred to [ZW15].

In the remainder of this chapter, we will highlight the most important aspects of stochastic modeling. Readers that are interested in applications only, may skip to chapter 7 afterwards.

The solution of a deterministic model essentially is a single realization of its stochastic version. The idea behind this is the very same principle Monte Carlo methods are based on. Monte Carlo methods, also called *simulation techniques*, enables one to solve stochastic problems by repeatedly calculating model responses at random sampling points in the parameter space [SLM91]. Thus, sampling-based methods render deterministic models to propagate parametric uncertainties to its output metrics, which become time-dependent stochastic variables. By principle this workflow can be used with black-box models, as one doesn't need to modify or even know the models' equations. This applies to the models' solver as well. Figure 1 illustrates the overall idea of enhancing deterministic models.

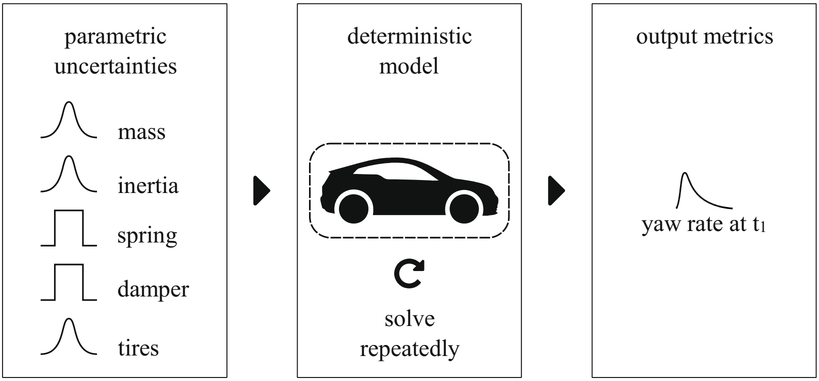

Figure 1: Model with uncertain parameters and output metrics

As there are many sampling-based methods, that generate their samples independently of one another, the whole workflow can be executed mostly parallel. Estimating output metrics in a post-processing step is relatively cheap in terms of computational effort.

Repeatedly evaluating high-fidelity models may be time-consuming. As the accuracy of bare Monte Carlo estimates is strongly dependent on the number of randomly chosen samples [SLM91], this approach might turn out to be not feasible to solving some problems in a given time frame. In the past decades, a lot of, partly complex sampling strategies evolved in order to improve convergence speed [Hal64], [Sob67], [Has70]. With faster convergence one yields better estimates with fewer sampling points, i.e. model evaluations.

Another approach to improving overall computation time is to build a meta-model, also called *high-dimensional model representation* (HDMR). One can then use this HDMR as a replacement in a Monte Carlo approach, as their evaluation is much cheaper than evaluating the original model. Computational effort is then shifted from Monte Carlo estimation to building an HDMR.

With all this laid out, we continue by illustrating the broad consensus with regard to organizing and encountering certain aspects of probabilistic modeling of mechanical systems and sensitivity analysis in general. This is followed by an outline of Monte Carlo Filtering and Polynomial Chaos expansion, with the latter being a meta-modeling technique. Following that, we will provide exemplary applications of Uncertainty Quantification, Sensitivity Analysis, and Monte Carlo Filtering based on a Polynomial Chaos expansion. This shall give an idea about the methods' use cases, as well as the potential of meta-modeling techniques.

2 Uncertainty Quantification and Sensitivity Analysis

Uncertainties in mechanical systems and their computational handling have drawn much attention in the past decade [Sud08]. With lots of contributions from various directions, such as statisticians, as well as practitioners in different domains, general-purpose terminology and methodology has evolved. To classify most of the problem statements, methods and ideas, there are two terms widely used: *Uncertainty Quantification* (UQ) and *Sensitivity Analysis* (SA). The two fields are historically quite disjunct [Sud08], but for a meaningful analysis one needs to answer questions from both.

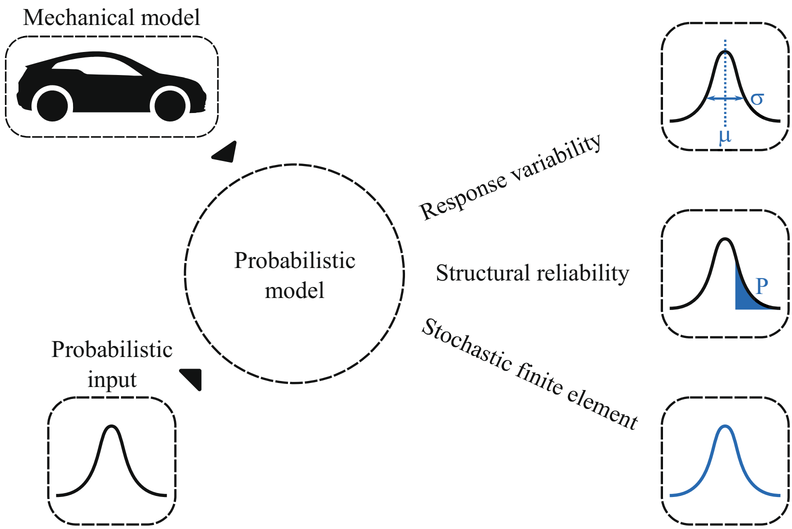

Figure 2: Classification of UQ methods in probabilistic engineering mechanics [Sud08], redrawn

Sudret [Sud08] classifies UQ methods according to the retention of information from the models response (figure 2). For this paper, we focus on *stochastic finite element methods* [GS91] that estimate the complete response probability density function (PDF). Problems of the other two classes can then be solved in a post-processing step. Ghanem and Spanos [GS91] originally extended classical finite element methods (FEM) to incorporate stochastic properties, e.g. material properties. The use of stochastic finite element methods is not limited to FEM-problems though.

Methods of UQ can therefore be seen as an answer to the question: What is the impact of parametric uncertainty to the models response? Whereas methods of SA try to indicate the relative importance of uncertain input parameters regarding that impact.

Mathematical models describing real systems usually consist of nonlinear coupled systems of equations [SvH10]. Sensitivity analysis apportions uncertainties at the output to uncertainties of the input and thus helps to analyze these equations. Furthermore, it gives an insight to the influence of each uncertain input [SRA08]. According to Saltelli [SRA08], [STC02] the applications of sensitivity analysis can be systematically divided into different *settings* whereas Hamby [Ham94] formulated *questions* that can be answered by sensitivity analysis. These definitions help to formalize SA-related studies. In the remainder of this chapter, we describe these settings and questions. In the following chapters we will attribute the presented methods to this definitions.

Saltelli's setting *Factor Prioritization* describes the problem to identify the most important parameters. Hamby's questions *'which parameters require additional research to reduce the uncertainty of the output?'* and *'which parameters contribute most to output uncertainty?'* are dealt with. The identification of parameters that almost do not influence the output variability is part of the setting *Factor Fixing* or *Factor Screening*. The detected parameters can be fixed to any value within their range and thereby used to simplify the model. The objective of sensitivity analysis within the setting *Factor Mapping* is to identify parameters that lead to an output with a predefined behavior. Additionally, the relevant spaces of the parameter domain can be determined. If the user is interested in reducing the output variance to a certain value, the setting *Variance Cutting* is adequate. Furthermore, in alignment with Saltelli's settings, Borgonovo [BP16] proposed the *Stability Setting*. It includes problems, usually in linear programming, about finding regions in the input space that do not change the optimal plan or preferred alternative.

According to Saltelli [STC02], for SA the following steps should be executed regardless of the method used:

1. Establishing the goal of the analysis
2. Selecting parameters to be considered
3. Assigning distributions to parameters
4. Choosing a sensitivity analysis method
5. Generating samples of the parameters
6. Evaluating the model on the generated samples
7. Analyzing the model output

3 Methods of Sensitivity Analysis

Driven by successful applications of sensitivity analysis in many sectors, several authors improved existing and developed new methods in the last decades [GLM14]. In general, one can classify methods for sensitivity analysis as either local or global. Using local methods, an operating point or a combination of parameters leading to a condition is selected. The influence of each parameter at this point is determined by small variations. Local methods are, for instance, suited for stability analysis. In contrast, global methods consider the whole domain of parameters. Global methods have proven best in helping understand parametric importance [SvH10].

A property of local approaches is that no distribution is assigned to the parameters. The simplest methods are based on one-at-a-time (OAT) techniques. With OAT methods, just one parameter is varied, while the others stay constant. Widely used methods are based on partial derivatives, which are intuitively understandable as sensitivity measures at a local point [BP16]. One advantage of these methods is the small number of required model evaluations. However, if the parameters are uncertain or the linearity of the model is unknown, global methods should be used instead, because of their exploration of the whole parameter space.

Figure 3 shows the standard procedure of a global sensitivity analysis according to Siebertz [SvH10].

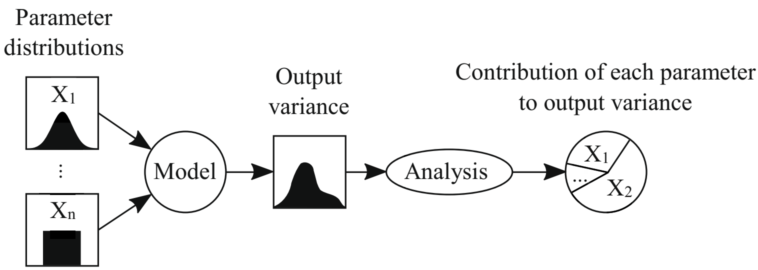

Figure 3: Global sensitivity analysis [SvH10], redrawn

In the following, we provide an intuition of variance-based methods that are related to global approaches. The underlying concept of variance based methods is to quantify the reduction of output variance when fixing parameters of interest within their range. When fixing a single parameter yields a great reduction of output variance, the parameter is said to be sensitive. This is based on the assumption, that variance is a good indicator of uncertainty. Global methods average over all possible variance reductions, when fixing a parameter to possible values in its range. By this, the question, how great the influence of each parameter on the output uncertainty is, can be answered. Thus the variance of the output is decomposed into quantified effects, depending on parameters and their combinations. Variance-based methods are allocated to the settings *Factor Prioritization, Factor Fixing* and *Variance Cutting*. Besides considering the whole parameter space, interaction between several parameters can be revealed and also grouping of parameters, where groups of parameters are treated as a single parameter to reduce computational effort, are possible options. An essential property of variance-based SA is their model-independence. That means, that non-linear and non-additive models can be analyzed properly. A drawback on the other hand, is the increased computational cost, compared to local approaches, for obtaining sensitivity measures [SRA08].

The *sensitivity indices* are measures that quantify the sensitivity of a parameter in the context of variance-based methods. The first-order index of a parameter X_i is defined as $S_i = V_i / \mathrm{Var}(Y)$, whereby $\mathrm{Var}(Y)$ is the unconditional variance of the output. The conditional variance V_i is the variance of the expectation of Y over the axis of X_i. Higher order conditional variances and sensitivity indices are constructed analogously. E.g. $S_{i,j} = V_{ij} / \mathrm{Var}(Y)$ is the second-order index of the parameters X_i and X_j which describes the part of $\mathrm{Var}(Y)$ caused by the interaction of X_i and X_j. The decomposition of variance then reads:

$$\mathrm{Var}(Y) = \sum_i V_i + \sum_i \sum_{j>i} V_{ij} + \sum_i \sum_{j>i} \sum_{m>j} V_{ijm} + \ldots + V_{123\ldots n} \tag{1}$$

Equation (1) normalized by $\mathrm{Var}(Y)$ yields normalized conditional variances on the right-hand side that are:

$$1 = \sum_i S_i + \sum_i \sum_{j>i} S_{ij} + \sum_i \sum_{j>i} \sum_{m>j} S_{ijm} + \ldots + S_{123\ldots n} \tag{2}$$

The so called total effect S_{Ti} of the parameter X_i includes all sensitivity indices of the right-hand side of equation (2) which originate, *inter alia*, from X_i. Hence S_{Ti} describes the whole contribution from X_i to $\mathrm{Var}(Y)$. For instance, $\mathrm{S}_{\mathrm{T}1} = \mathrm{S}_1 + \mathrm{S}_{12} + \mathrm{S}_{13} + \mathrm{S}_{123}$ is the total effect of X_1 out of a three-dimensional parameter space including X_1, X_2 and X_3. A small value of S_{Ti} indicates an unimportant parameter so that it can be fixed according to the setting *Factor Fixing*. Small values S_i are necessary but not sufficient for determining unimportant parameters.

In a series of contributions Cukier and Schaibly [Cuk73], [Sch73], [CSS75] developed the Fourier amplitude sensitivity testing (FAST) method. This approach was the first that enables the calculation of first-order sensitivity indices with a feasible number of model evaluations. Sobol [Sob90] provided a generalized presentation of the sensitivity indices and a MC-based implementation for their calculation. In 1999 Saltelli [STC99] generalized this method and introduced total effects.

4 Monte Carlo Methods

Mathematical methods relying on the same principle as Monte Carlo methods are significantly older than its name. Its origin dates back to the 19th century known as *statistical sampling* [NB10]. However, only since the development of mechanical calculators at the end of the 19th century and computers during World War II numerical methods became more feasible and important. The term was established 1949 in 'The Monte Carlo method' by Nicholas Metropolis and Stanislaw Ulam [MU49].

The term *Monte Carlo methods* or *Monte Carlo simulation* includes algorithms based on the law of large numbers. Today they are used in diverse fields of application, to analyze and simulate complex systems. Initially used for numerical calculation of multidimensional integrals and processes within particle physics, the Monte Carlo methods became applied e.g. in statistical mechanics, particle statistics, analysis of populations in biology and medicine, and chemical applications [Leh67].

The Monte Carlo-based approaches to sensitivity analysis require samples drawn from the parameter space. There are various strategies to generate a sample of a random variable with a specific distribution. A popular technique is *random sampling* that initially was executed with manual or physical techniques like coin flipping or a Geiger counter [RK08]. However, these approaches are not appropriate for computational simulation. They are time-consuming, the generated numbers are not repeatable, and they can exhibit bias and dependence. Nowadays *pseudo random* and *quasi random* samples are used instead [SLM91]. They are generated by deterministic algorithms and thus are reproducible. Furthermore, they have low memory requirements and work fast.

Samples, produced by pseudo random generators, are scattered over the considered domain without control. There can be clusters, which means areas with an aggregation of many points, and gaps with only a few or no points. This spread of points in a multidimensional space can be measured by *discrepancy*. The smaller the discrepancy, the better the samples become with regard to Monte Carlo methods. Quasi random sampling methods, also known as *low-discrepancy sequences*, were developed to improve discrepancy and hence convergence speed [Hal64], [Sob67]. The basic idea of these sequences is that the selection of new points is influenced by existing points to maintain an even spread of points.

5 Monte Carlo Filtering and Regionalized Sensitivity Analysis

The Monte Carlo filtering (MCF) method identifies parameters that are responsible for output realizations within a predefined region. Alternatively, one can set arbitrarily complex conditions to be satisfied for filtering output realizations. Thus the MCF fits to the setting *Factor Mapping*. Like variance based methods MCF is based on a Monte Carlo simulation. After specifying thresholds or conditions, that define an acceptable (*behavioral* B) and an unacceptable behavior (*non-behavioral* $\bar{B}$) respectively, the essential step is to filter the output realizations according to these thresholds or conditions [STC02], [SRA08]. This yields conclusions to conditional parameter sensitivities. In the following, one MCF method, more precisely the Regionalized Sensitivity Analysis (RSA) developed by Hornberger and Spear [Hor80], [Spe80], will be presented.

5.1 Filtering and Kolmogorov-Smirnov Test

After a Monte Carlo simulation with N model evaluations the output realizations are split into the subsets $(Y\,|\,B)$ and $(Y\,|\,\bar{B})$. Y denotes the whole range of values of the output. $(Y\,|\,B)$ includes all output realizations which are marked as *behavioral*, that is to say, they do not exceed the predefined thresholds or do meet the conditions. Correspondingly, all *non-behavioral* output realizations are located in $(Y\,|\,\bar{B})$. The same splitting and assignment is applied to the parameter space. All parameter combinations leading to a *behavioral* output realization are assigned to the subset $(X\,|\,B)$, whereas the combinations which lead to *non-behavioral* realizations are assigned to $(X\,|\,\bar{B})$. The relationship between Y and X is shown in figure 4. Here the function $g(X)$ represents the model which maps from the domain X to the codomain Y.

Hence, for each parameter two subsets $(X_j\,|\,B)$ and $(X_j\,|\,\bar{B})$ as well as their respective density functions $f(X_j\,|\,B)$ and $f(X_j\,|\,\bar{B})$ are obtained. A comparison of these functions allows for conclusions regarding the importance of the considered parameter. If the two distributions are dissimilar, the parameter X_j is said to have influence on the behavior of the system. The reasoning behind that is the idea, that when specific values of a parameter yield output realizations in the two filtered sets with different probabilities, that parameter and the choice of its value must make a difference. A quantitative comparison of the distributions can be performed by the Kolmogorov-Smirnov test [Spe80], [SRA08].

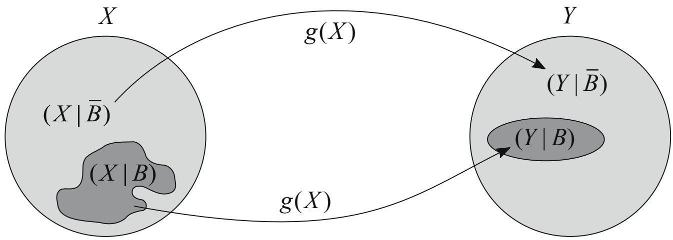

Figure 4: RSA filtering

The Kolmogorov-Smirnov test determines whether or not two samples originate from the same distribution. Instead of density functions, the cumulative distributions $F(X_j\,|\,B)$ and $F(X_j\,|\,\overline{B})$ are used for a discrimination. The null hypothesis H0 and the alternative hypothesis H1 read:

- H0: The two distributions of the two samples are equal: $F(X_j\,|\,B) = F(X_j\,|\,\overline{B})$
- H1: The two distributions of the two samples differ: $F(X_j\,|\,B) \neq F(X_j\,|\,\overline{B})$

H0 is either rejected or accepted by looking at the maximum vertical distance D between the two cumulative distributions, shown in figure 5. If D exceeds a critical value, depending on the sample size and a significance level, H0 is rejected. Otherwise it is accepted.

A rejected null hypothesis means that the output behavior is sensitive with respect to the considered parameter and the test value D can be interpreted as a sensitivity measure. The higher D is, the more important the parameter becomes.

Furthermore, the graphical representation in figure 5 shows parts in the domain of the parameter X_j, which more likely result in $(Y\,|\,B)$ or in $(Y\,|\,\overline{B})$ respectively. E.g. with a low gradient of $F(X_j\,|\,\overline{B})$ and a high gradient of $F(X_j\,|\,B)$ for X_j in the left half, low values of X_j will more likely produce a *behavioral* output.

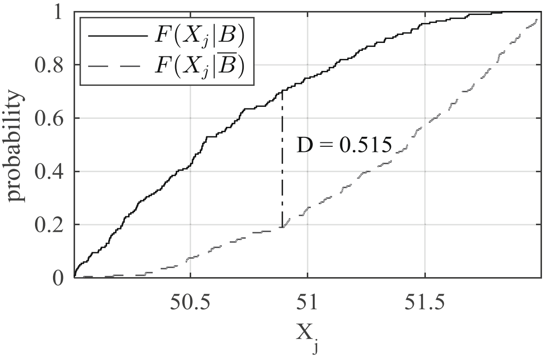

Figure 5: Kolmogorov-Smirnov test

5.2 Properties and Applications of Regionalized Sensitivity Analysis

Regionalized Sensitivity Analysis can be attributed to methods of global sensitivity analysis, since the whole parameter space is covered. Furthermore, it is *model-independent* or so called *model-free*. The acceptance of the null hypothesis does not indicate the unimportance of a parameter – this is comparable to small values of S_i (chapter 3). It is a necessary, but not a sufficient condition [Hor80], [Spe80]. Besides conditional first order effects, as described above, interaction effects until second order (chapter 3) can be estimated when a prior correlation analysis and a principal component analysis (PCA) is performed. Alternatively, the scatterplots of parameter pairs with denoted realizations out of $(X \mid B)$ and $(X \mid \overline{B})$ give an insight into output-behavior-influencing interactions. A drawback of the method is the low statistical power when the fraction of behavioral realizations is barely larger than 5% [SRA08].

Hornberger and Spear [Hor80], [Spe80] developed the RSA to identify parameters that should be examined more closely during the remaining modelling process. Several adapted applications of RSA can be found in literature: Saltelli [SRA08] links RSA to model calibration associated with the setting *Factor Mapping*. Moreover he uses the RSA to identify possible combinations of parameters of a proportional-integral controller to keep a chemical reactor stable. Also, Brockmann [BM10] determines operating conditions leading to predefined behavior in biofilm systems. The last two applications are examples of problems related to the *Stability Setting*.

Table 1 provides a comparison of different sensitivity analysis methods.

Table 1: Properties of sensitivity measures, following [STC02]

Method / measure	Model free	Global properties	1st order effect	2nd order effect	Higher order effect	Total effect	Basis
$\delta Y / \delta X_j$	N	N	Y*	N	N	N	Partial derivative
Standardized regression coefficients	N	Y	Y**	N	N	N	Regression coefficients
Variance based methods	Y	Y	Y	Y	Y	Y	Variance
Regionalized sensitivity analysis	Y	Y	Y	Y***	N	N	Thresholds or conditions and test value *D*

* local, ** if model linear, *** with PCA

6 Polynomial Chaos

The main idea behind Polynomial Chaos (PC) is to yield a deterministic representation of a stochastic process, by expanding random variables in terms of a polynomial series. The first application of a PC expansion (PCE) dates back to 1938. Wiener [Wie38] expanded normally distributed random variables in terms of Hermite polynomials of different orders. The application of *Homogeneous Chaos* to arbitrary L^2-functions is attributed to Cameron and Martin [CM47], while Ghanem and Spanos brought its use to engineering applications [GS91]. The extension of PC to common distributions, other than Gaussian, was carried out later by Xiu and Karniadakis [XK02] and is known as generalized PCE (gPCE). In the remainder of this chapter, only some basic ideas and properties of PCE will be highlighted. For more thorough reviews and presentations of the matter, the reader is advised to [O'H13] and [Xiu10], [LK10].

By creating a gPCE for an output of a computer model $Y = \mathcal{M}(X)$, one can see a gPCE also as a surrogate for a simulator $\mathcal{M}$. A gPCE not only represents the random variable of a model output, it can also be used for evaluation instead of the original model. Since polynomials are likely to be computationally much lighter to evaluate than the original model, it is sensible to use it as a surrogate for subsequent analysis, such as SA, prediction and optimization - hence the terms meta-model and HDMR mentioned before.

The gPCE is based on orthogonal polynomials. By choosing a sequence of polynomials that is orthogonal with respect to the distribution of input uncertainty, using a gPCE yields exponential convergence. The condition of orthogonality can be stated as

$$\langle \psi_n, \psi_m \rangle = \int_\Omega \psi_n(x)\psi_m(x)p(x)\mathrm{d}x = \begin{cases} \gamma_n & \text{if n} = \text{m} \\ 0 & \text{if n} \neq \text{m} \end{cases} \qquad n,m \in \mathbb{N}_0 \tag{3}$$

with a system of polynomials $\{\psi_n(x),\, n \in \mathbb{N}_0\}$ of n-th order, $p(x)$ as a density function and γ_n as a non-zero value. That is, if $\mathrm{n} \neq \mathrm{m}$ then:

$$\langle \psi_n, \psi_m \rangle = 0$$

Having found a proper sequence of polynomials, we can expand a model's output for the univariate case as

$$Y = \mathcal{M}(X) = \sum_{j=0}^{\infty} y_j \psi_j(X) \tag{4}$$

Where $\{\psi_n(X),\, n \in \mathbb{N}_0\}$ is a polynomial basis and y_j are coefficients to be computed. Having an orthogonal basis, with given $\mathcal{M}$ there is a unique expansion that represents the random variable Y.

The coefficients can be found by projecting the model response on the orthogonal basis [SK11]:

$$y_j = \langle \mathcal{M}, \psi_j \rangle / \langle \psi_j, \psi_j \rangle \tag{5}$$

In equation (5) the denominator is known analytically and the numerator can be derived by computing an integral. The choice of an orthogonal basis therefore simplifies and numerically stabilizes the calculation of coefficients, needed for a gPC representation.

An important feature of gPCE is, that basic properties of Y, such as the first two moments, can be derived analytically from the gPCE coefficients. With a random variable Y and $n \in \mathbb{N}$, the k-th moment of Y reads:

$$m_k = \mathrm{E}[Y^k] = \int_\Omega y^k \mathrm{d}F_Y(y) \tag{6}$$

With $\psi_0 \equiv 1$, for every sequence of orthogonal polynomials, the expectation and variance of the output Y can be derived as follows [Ber05]:

$$\begin{aligned} \mathrm{E}[Y] &= m_1 \\ &= \int_\Omega \left(\sum_{k=0}^{\infty} y_k \psi_k \right) \psi_0 \mathrm{d}F_Y(y) \\ &= \langle y_0 \psi_0, \psi_0 \rangle \\ &= y_0 \end{aligned} \tag{7}$$

and respectively:

$$\begin{aligned} \mathrm{Var}[Y] &= m_2 - m_1^2 \\ &= \int_\Omega \left(\sum_{k=0}^{\infty} y_k \psi_k \right) \left(\sum_{j=0}^{\infty} y_j \psi_j \right) \mathrm{d}F_Y(y) - y_0^2 \\ &= \sum_{k=1}^{\infty} y_k^2 \mathrm{E}[\psi_k^2] \end{aligned} \tag{8}$$

Sudret [Sud08] additionally derived sensitivity indices of all orders (chapter 3) analytically, based on gPCE coefficients. This renders gPCE a promising candidate for metamodeling purpose in UQ and SA.

Although with equation (5) the computation of gPCE coefficients seems quite straightforward, there are various approaches from which one can choose. There are mainly two classes of methods – *intrusive* and *non-intrusive*. The latter allows building a gPCE for black-box models, because the model does not need to be modified. As the presentation of those methods is beyond the scope of this paper, the reader is again advised to the references cited.

7 Applications to Vehicle Dynamics Simulation

In this chapter, applications and numerical results obtained using the aforementioned methods will be presented and described in detail. First, a worked example using a single-track model will be used to exploit basic properties of UQ and SA. In addition to that it will serve as a reference for a plausibility check. Second, an application to a commercial high-fidelity black-box vehicle dynamics model will be demonstrated. A comparison of a direct RSA and a PC-based RSA will be shown by use of a single-track model of an A-double heavy vehicle combination.

7.1 Single-Track Model

The transient linear single-track model (STM) that will be used in this work, is based on the work by Riekert and Schunck [RS40], who developed a simplified vehicle dynamics model consisting of a front- and a rear axle with only one wheel each. Whereby more complex dynamic effects such as rolling and pitching and thus any load transfer between front- and rear axle are neglected, by postulating zero height of the center of gravity. This consequently helps reduce the physical phenomenon's complexity to three degrees of freedom. Furthermore, in accordance with [RS40], all model equations are linear. For the remainder of this chapter we will use a widely used and accepted notation, as can be seen in [MW14].

Isermann [Ise06] devised the following state-space representation:

$$\begin{bmatrix} \dot{\boldsymbol{x}}(t) \\ \boldsymbol{y}(t) \end{bmatrix} = \begin{bmatrix} \boldsymbol{A} & \boldsymbol{b} \\ \boldsymbol{C} & \boldsymbol{d} \end{bmatrix} \cdot \begin{bmatrix} \boldsymbol{x}(t) \\ u(t) \end{bmatrix} \tag{9}$$

$$\underbrace{\begin{bmatrix} \dot{\beta} \\ \ddot{\psi} \end{bmatrix}}_{\dot{x}} = \underbrace{\begin{bmatrix} -\dfrac{c_{\alpha V} + c_{\alpha H} + m\dot{v}}{mv} & \dfrac{c_{\alpha H} l_V - c_{\alpha V} l_V}{mv^2} - 1 \\ \dfrac{c_{\alpha H} l_H - c_{\alpha V} l_V}{J_z} & -\dfrac{c_{\alpha H} l_H^2 + c_{\alpha V} l_V^2}{J_z v} \end{bmatrix}}_{A} \cdot \underbrace{\begin{bmatrix} \beta \\ \dot{\psi} \end{bmatrix}}_{x} + \underbrace{\begin{bmatrix} \dfrac{c_{\alpha V}}{mv} \\ \dfrac{c_{\alpha V} l_V}{J_z} \end{bmatrix}}_{b} \cdot \underbrace{\delta_V}_{u} \tag{10}$$

$$\underbrace{\begin{bmatrix} a_y \\ \beta \\ \dot{\psi} \end{bmatrix}}_{y} = \underbrace{\begin{bmatrix} -\dfrac{c_{\alpha V} + c_{\alpha H} + m\dot{v}}{m} & \dfrac{c_{\alpha H} l_V - c_{\alpha V} l_V}{mv} \\ 1 & 0 \\ 0 & 1 \end{bmatrix}}_{C} \cdot \underbrace{\begin{bmatrix} \beta \\ \dot{\psi} \end{bmatrix}}_{x} + \underbrace{\begin{bmatrix} \dfrac{c_{\alpha V}}{m} \\ 0 \\ 0 \end{bmatrix}}_{d} \cdot \underbrace{\delta_V}_{u} \tag{11}$$

This coupled system of differential equations (10) represents the relations between side-slip angle β, yaw rate $\dot{\psi}$ and the front wheel steering angle δ_V. It additionally allows to compute the lateral acceleration a_y by means of equation (11).

As we want to use the models' equations as a reference for a plausibility check, we additionally derive the steady-state response of the system. On steady-state driving conditions the left-hand side of equation (10) is zero.

With $\dot{v} = 0$ the response reads:

$$\boldsymbol{x} = -\boldsymbol{A}^{-1}\boldsymbol{b} \cdot u = \begin{bmatrix} \dfrac{\dfrac{ml_v}{c_{\alpha H} l} v^2 - l_h}{l + \dfrac{m}{l}\left(\dfrac{l_h}{c_{\alpha V}} - \dfrac{l_v}{c_{\alpha H}}\right) v^2} \\ \dfrac{v}{l + \dfrac{m}{l}\left(\dfrac{l_h}{c_{\alpha V}} - \dfrac{l_v}{c_{\alpha H}}\right) v^2} \end{bmatrix} \cdot u \tag{12}$$

The STM and the resulting statistics and sensitivity indices of the model outputs are tested by using two uncertain parameters, namely mass and inertia. The uncertain parameters are assumed to be independent and identically distributed (i.i.d.). All STM parameters are listed in table 2. The parameterization, including the uncertain parameters, is adopted from Kohlhuber [Koh16]. Kohlhuber analyzes the need to adapt model-based control algorithms, such as Electronic Stability Control (ESC), to varying operating conditions and hence varying STM parameters.

Table 2: STM parameters

Parameter	**Unit**	**Distribution / Value**
m	[kg]	$\mathcal{U}(500,800)$
J_z	[kg m^2]	$\mathcal{U}(500,800)$
l_v	[m]	1.155
l_h	[m]	0.945
$c_{\alpha V}$	[N/rad]	35000
$c_{\alpha H}$	[N/rad]	55000
v	[m/s]	27.78

To simulate a step steering maneuver, the input stimulus reads:

$$\delta_V(t) = \begin{cases} 0°: t < 2\,\mathrm{s} \\ 0.95°: t \geq 2\,\mathrm{s} \end{cases} \tag{13}$$

In the remainder of this chapter, quantities of interest derived by use of a gPCE will be compared to references based on Monte Carlo simulations. First we will focus on UQ-related quantities, e.g. estimations of mean and variance. Following that, we will exemplarily discuss time-dependent sensitivity indices computed with a gPCE.

Figure 6 shows the yaw rate of ten Monte Carlo realizations in a time-series graph. This gives a rough idea of the time-dependent stochastic output. At the beginning $(t < 2\text{s})$, there seems to be zero uncertainty propagated to the output. After the steering step $(t \geq 2\text{s})$, uncertainty from m and J_z is reflected in the output signal.

Figure 7 shows two estimates of the yaw rate's probability density function (PDF) at $t_{PDF} = 2.35\text{s}$. Here t_{PDF} is roughly that time, when most of the peaks of the yaw rate's Monte Carlo realizations emerge. One PDF estimate is derived from a kernel density estimation (KDE), as proposed in [Par62], based on the MC samples from the original model. The other estimation is derived from a KDE based on MC samples drawn from a fifth-order gPCE, by using it as a surrogate. By comparing these two estimates qualitatively they appear to be almost identical.

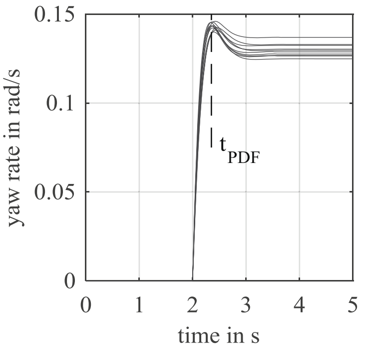

Figure 6: Exemplary MC realizations

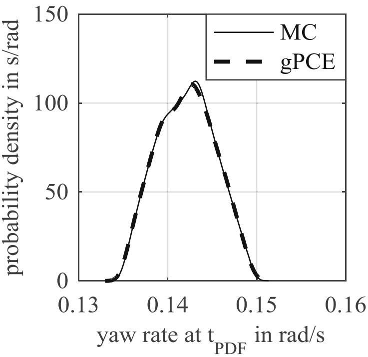

Figure 7: Estimated PDFs at $t_{PDF} = 2.35$ s

Mean and variance are closely related to statistical moments. Moments are quantitative measures of position and shape of point-sets and hence allow for a quantitative comparison of PDF estimators. Figure 8 shows convergence graphs of MC estimates of the output mean and variance at $t_{PDF} = 2.35\text{s}$. These serve as a reference and allow for comparison with the estimates, derived analytically from the coefficients of a fifth-order gPCE. The gPCE itself is based on 36 model evaluations.

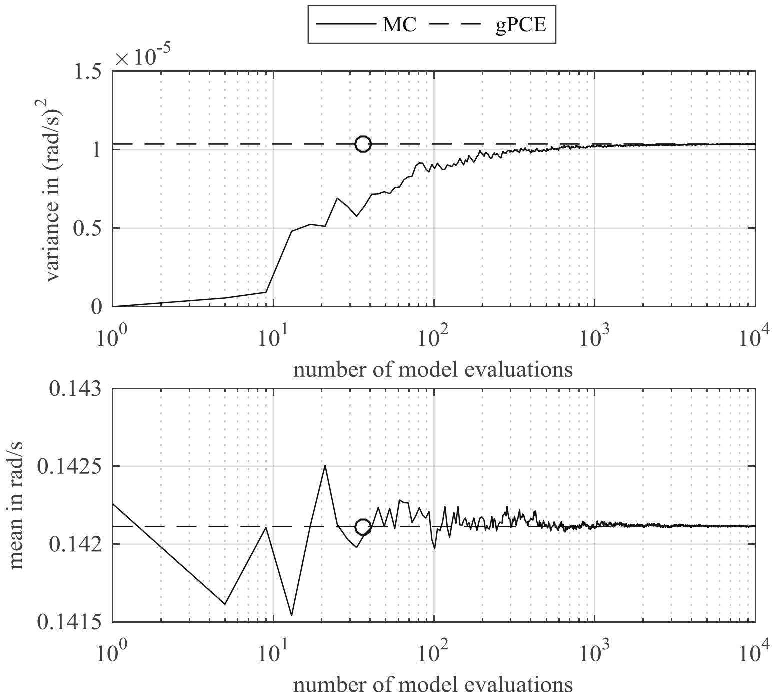

Figure 8: Convergence graphs of estimations over the number of model evaluations

Sensitivity indices, see chapter 3, are defined per model output and per time step. Since these indices are normalized and allow for a quantitative comparison, the usual graphical representations for analyzing sensitivity indices mostly use pie charts. The equivalent of a time-dependent pie chart can be an area chart, where every time step represents the same data as held by a pie chart. We therefore adopt the representation of time-dependent sensitivity indices as area charts from [STC00].

Figure 9 shows first-order sensitivity indices for the yaw rate, calculated based on the coefficients of a fifth-order gPCE. The contribution of relative parametric uncertainty to output uncertainty can be directly obtained by that chart. Following that, the output uncertainty, immediately after the steering step input ($t = 2$s), depends exclusively on the J_z distribution. Whereas at later times ($t > 3$s) yaw rate uncertainty depended solely on m.

Since the STM is a white-box model, i.e. its equations are known, we could have derived this observation also by taking a closer look at equations (10) and (12). From (10) we learn, with $\dot{\boldsymbol{x}} = \boldsymbol{x} = 0$ $(t = 2\text{s})$, that the initial change of $\ddot{\psi}$ due to an excitation does not depend on m, see vector $\boldsymbol{b}$. For steady state conditions $(t > 3\text{s})$, from (12) we learn that $\dot{\psi}$ does not depend on J_z and therefore all output uncertainty must source from m.

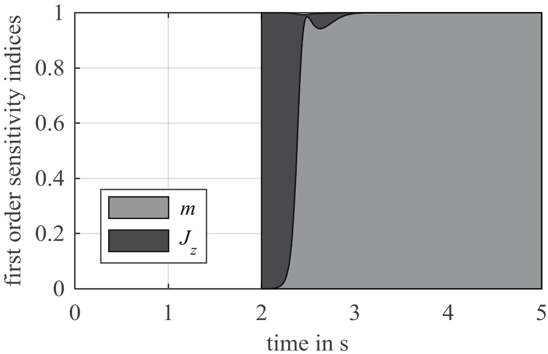

Figure 9: First-order sensitivity indices of the yaw rate

7.2 Dyna4

This chapter serves as a proof of concept by demonstrating results obtained by implementing the aforementioned methods in the simulation framework *Dyna4*. *Dyna4* is a product of *TESIS DYNAware Technische Simulation Dynamischer Systeme GmbH*.

For the sake of reproducibility the model under consideration is part of the demo project *Car Professional* delivered with Dyna4. After accelerating the vehicle to a speed of $v_{Veh} = 20$ m/s, we simulate an open-loop steering wheel step maneuver, followed by a moderate deceleration. The input stimulus reads:

$$\delta_H(t) = \begin{cases} 0°: t < 11.5\text{s} \\ 31.5°: t \geq 11.5\text{s} \end{cases} \quad \text{and} \quad p_{Br}(t) = \begin{cases} 0\,\text{bar}: t < 21\text{s} \\ 25\,\text{bar}: t \geq 21\text{s} \end{cases} \tag{14}$$

With the steering wheel angle denoted as δ_H and the wheel cylinder brake pressure as p_{Br}.

Except from uncertain parameters (table 3) we used the demo-parameterization of the Car Professional model.

Table 3: Uncertain parameters used with the Car Professional model

Parameter	**Unit**	**Distribution / Value**
Mass (*m*)	[kg]	$\mathcal{U}(1000,1500)$
Center of gravity position, z (COG_z)	[m]	$\mathcal{U}(0.15,0.3)$
Anti-roll bar (c_{ARB})	[Nm / rad]	$\mathcal{U}(10000,80000)$
Damper gain	[-]	$\mathcal{U}(0.5,2.0)$
Inertia, z (J_z)	[kg m^2]	$\mathcal{U}(2000,3000)$

The damper gain is applied to the whole characteristic curve, which is defining the damper force over the speed of deflection. As the following consideration shall serve as a proof of concept, we provide no reasoning for the choice of uncertain parameters or their distribution.

As exemplary output metrics we use the yaw rate again and additionally the roll angle (figure 10).

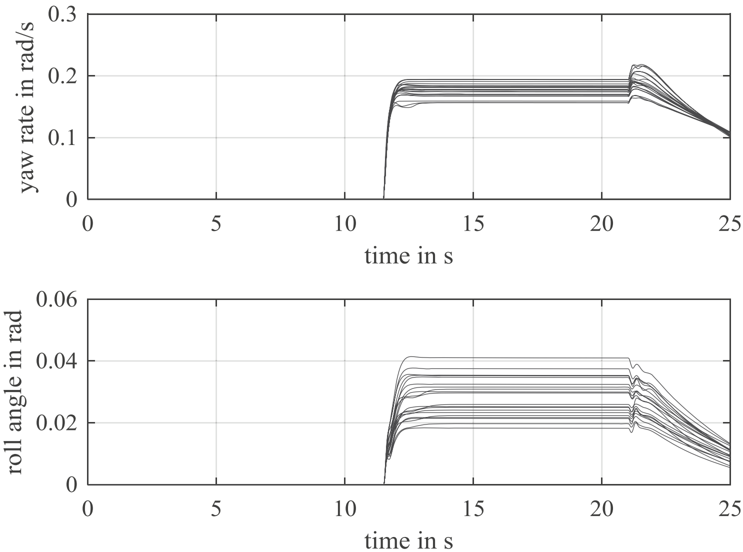

Figure 10: Exemplary MC realizations of the yaw rate (above) and the roll angle (below)

Figures 11 and 12 show area plots of first order sensitivity indices of the yaw rate and the roll angle respectively. The results were obtained by MC simulation with 10,000 model evaluations. Again the sensitivities are parallel to expert opinions. For example, the uncertain c_{ARB} causes the bulk of uncertainty of the stationary roll angle, albeit it is of minor importance to the yaw rate immediately after the steering wheel step.

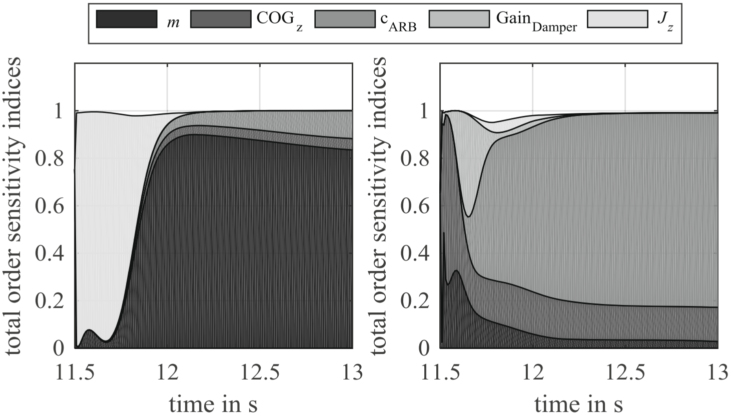

Figure 11: Yaw rate's first order sensitivities

Figure 12: Roll angle's first order sensitivities

Figure 13 and 14 compare PDFs yielded by MC estimations based on 10,000 samples, drawn from the original model (labeled as MC) and drawn from a second-order gPCE respectively. The second-order gPCE was built with only 61 model evaluations.

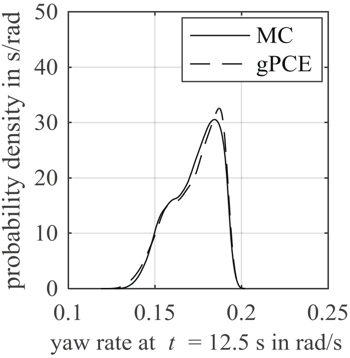

Figure 13: Estimated PDFs of the yaw rate

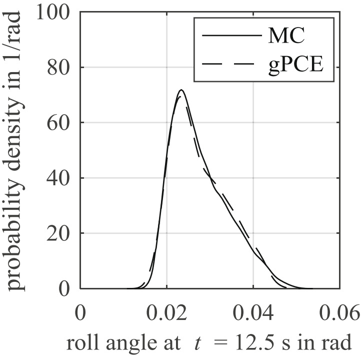

Figure 14: Estimated PDFs of the roll angle

Figure 15 gives an impression of the fast convergence of gPCE compared to pure MC. The gPCE-based estimate of the first-order sensitivity index of the mass with respect to the yaw rate was computed by using the same 61 model evaluations as above.

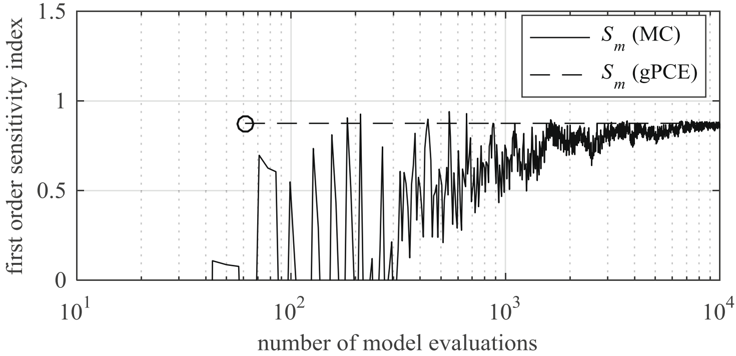

Figure 15: Convergence graph of estimations over the number of model evaluations

7.3 Long Combination Vehicle

An A-double vehicle combination consists of a tractor, a semi-trailer and a dolly followed by a second semi-trailer. It is not registered in Europe due to its length but it is the most common long combination vehicle in the USA [AW07]. Although long combinations with multiple articulations provide productivity benefits, they entail highly complex vehicle dynamics. In this example, we are interested in analyzing the driving stability by considering the Rearward Amplification (RA).

RA is defined as the ratio of the peak lateral acceleration ($a_{y,\max}$) of the towing unit to $a_{y,\max}$ of the last trailer, measured at the respective center of gravity (CG). Thus it quantifies the lateral acceleration amplification. Low values of RA indicate low probabilities of rear-trailer rollover [HB02]. Following [PRM01] lower values of RA can be achieved, e.g. by

- shorter distance from the CG of the truck to the hitch point;
- shorter coupling rear overhang;
- longer drawbar lengths on dollies;
- longer trailer wheelbase; and
- tires with higher cornering-stiffness.

We try to reproduce these statements by use of RSA. To do so, we used a single track model (STM) with trailers and choose a maximum value as a filter-condition for the RSA. What follows is a brief description of the model in use, as well as details about parameterization.

A STM of an A-double combination developed by Nilsson [NT13] is illustrated in figure 16. Beside the dimensions and the CGs denoted with the moments of inertia J_i and the masses m_i, the orientations Φ_i with respect to (w.r.t.) the global coordinate system (X, Y) and the articulation angles Θ_i, as well as the steering angle δ are labeled.

The derivation of the equations of motion and state space model, deployed in the following, is fully given in [NT13]. The longitudinal speed of the truck w.r.t. the local coordinate system (x, y) is assumed to be constant and each axle longitudinal force is moderate. Furthermore, the articulation and steering angles are assumed to be small. Finally, a state space model with the input δ and the state variables $v, \dot{\Phi}_1, \Theta_1, \dot{\Theta}_1, \Theta_2, \dot{\Theta}_2, \Theta_3, \dot{\Theta}_3$ is devised. With this, the lateral acceleration and yaw rate of each unit can be calculated.

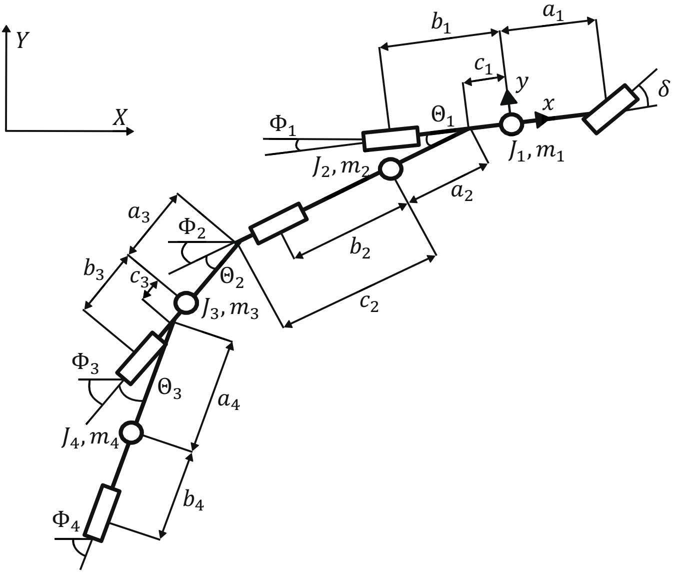

Figure 16: Parameters of a STM with three articulations

The parameterization we used for the following example is given in [NT13] and represents a fully-laden A-double combination. In order to yield statements regarding sensitive parameters, that are able to lower the values of RA, we choose uncertain parameters stated in table 4.

Table 4: Uncertain parameters used with the STM with trailers

Parameter	**Unit**	**Distribution / Value**
Overall length, dolly	[m]	$\mathcal{U}(5,7)$
c_1	[m]	$\mathcal{U}(0,2.5)$
c_2	[m]	$\mathcal{U}(4.25,7)$
c_3	[m]	$\mathcal{U}(0,0.6)$

Additionally the cornering stiffness of all axis was assumed to be uniformly distributed with 90% and 130% of the nominal value [NT13] as lower and upper bounds. In total there are 9 uncertain parameters.

For computing the RA we simulated a single-sine steer maneuver. The vehicle combination performs a single lane change at a fixed speed of 88 km/h. The sinusoidal steering wheel input was applied with an amplitude of 1.5 degree and the frequency 0.4 Hz. In the following RSA we will filter for RA values lower than 1.4.

Figure 17 gives an impression of the overall uncertainty of the truck's and the second trailer's lateral acceleration. The mean of all possible accelerations over time is plotted as lines, whereas the filled areas mark the maximum and minimum values occurred in a MC simulation with 3000 samples. Besides the range of possible outcomes for each lateral acceleration, the dynamic behavior with delayed and amplified response of the last trailer to an excitation can be studied in figure 17.

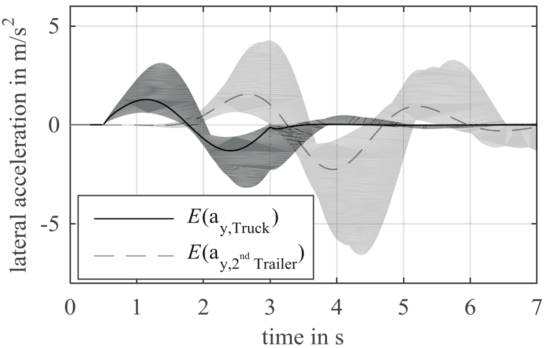

Figure 17: Maximum and mean values of lateral acceleration of the truck and the 2nd trailer in a MC simulation with 3000 samples

In figure 18, normalized histograms of MC realizations based on the original model and a second-order gPCE are shown. The second-order gPCE was computed by use of 181 model evaluations. As seen before, the gPCE represents the output PDF sufficiently. In the following RSA, values of RA will be grouped in two sets – behavioral (B) and non-behavioral $(\overline{B})$, sufficing the condition RA ≤ 1.4 and RA>1.4 respectively. Following figure 4, the filtering is mapped back to the input space of parameters subsequently. Afterwards there are two corresponding sets for every parameter.

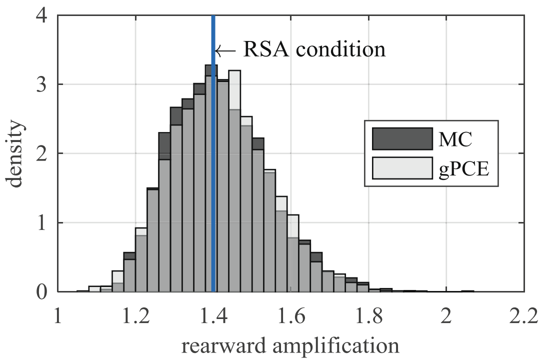

Figure 18: Normalized histogram of MC realizations based on the original model (dark gray) and a second-order gPCE (light gray)

Figure 19 shows cumulative density functions (CDF) of two, exemplarily chosen, parameters. By applying a Kolmogorov-Smirnov test to the two CDFs of the sets B and $\overline{B}$ one yields the maximum vertical distance D. In the example shown, the distributions are not equal and D can be interpreted as a measure of importance. Besides a sensitivity estimate regarding a desired outcome (small values of RA), ranges of parameters, that benefit behavioral outcome, can readily be seen in figure 19. The probability to yield a behavioral outcome for small values of c_2, that is the distance of the 1st trailers CG to the rear coupling point, is greater than to yield a non-behavioral outcome. Small values of C2r, that is the cornering stiffness of the 1st trailer, on the other hand, benefit non-behavioral outcomes. In fact, these statements do match the statements of [PRM01] for all parameters. Therefore RSA applied to a STM with trailers is able to reproduce expert statements regarding parametric impact on complex dynamic systems. The filtering condition, i.e. the description of behavioral performance may be arbitrarily complex.

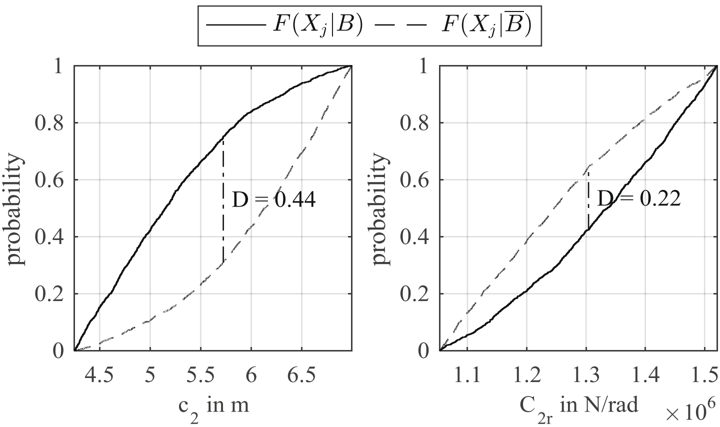

Figure 19: Cumulative distributions and Smirnov statistics of c_2 and C_{2r}

To analyze the capability of gPCE to act as surrogate for RSA, figure 20 compares CDFs of B and $\overline{B}$ exemplarily for one parameter. The CDFs were computed by 3,000 evaluations of the original model (black) and of a second-order gPCE (blue), which was built based on 181 model evaluations. Based on that qualitative assessment, gPCE seems to be sufficient to act as a model replacement in a RSA context.

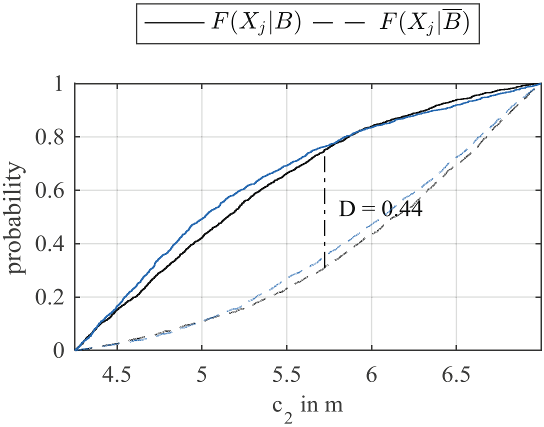

Figure 20: Cumulative distributions and Smirnov statistic of c_2 based on model evaluations (black) and gPCE evaluations (blue)

8 Summary

In this contribution, we emphasized methods on the one hand, which are relatively new to vehicle dynamics simulations and exemplary applications on the other hand.

After providing an outline of questions and problems, which cannot readily be handled by well-known and widely-used deterministic models, we presented two fields of active research that try to encounter uncertainty-related problems. In the following, we provided an overview over slightly different formularizations and definitions of abstracted problem statements. The *stability setting*, e.g. has been defined only recently [BP16]. Following that we described a comprehensive set of methods, able to treat most of the introduced abstracted problem statements.

In the second part we described three worked examples. By analyzing the equations of the single track model without trailers, we ensured plausibility of the sensitivity estimates. The following implementation to a high-fidelity black-box model proves viability of the considered methods under real-world conditions. The single track model with trailers demonstrated the ability of reproducing and hence generating expert statements regarding properties of complex systems by using the proposed methods. With all examples we tried to clarify the problems that can be treated, as well as the ability of generalized polynomial chaos to greatly reduce computational effort, by using it as a meta-modeling technique.

Contributions and Acknowledgement

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This article contains results of theses from Matthias Rowold and Alejandro Valero-Andreu. The authors would like to thank them for their contributions.

Schmeiler (corresponding author) initiated the idea of the paper and contributed to the assessment and selection process of uncertainty-related methods. He advised the Master's thesis of Alejandro Valero-Andreu, who extended a method of gPCE by estimating variance-based sensitivity measures based on sparse-grid integration. Schmeiler further developed Valero-Andreu's approach and developed an integration with the Dyna4 simulation framework.

Rowold (co-author) developed a RSA implementation and an application to a STM with trailers. He contributed to the presentation of SA, MC and RSA in this paper, as well as revised also the manuscript critically for important intellectual content.

Lienkamp contributed essentially to the conception of the research project. He revised the manuscript critically for important intellectual content. Lienkamp gave final approval of the version to be published and agrees for all aspects of the work. As a guarantor, he accepts responsibility for the overall integrity of the manuscript.

List of Abbreviations

HDMR	High-Dimensional Model Representation
UQ	Uncertainty Quantification
SA	Sensitivity Analysis
PDF	Probability Density Function
FEM	Finite Element Methods
OAT	One-At-a-Time
FAST	Fourier Amplitude Sensitivity Testing
MCF	Monte Carlo Filtering
RSA	Regionalized Sensitivity Analysis
PCA	Principal Component Analysis
STM	Single-Track Model
PC	Polynomial Chaos
PCE	Polynomial Chaos Expansion
gPCE	generalized Polynomial Chaos Expansion
MC	Monte Carlo
ESC	Electronic Stability Control

KDE Kernel Density Estimation
COG Center of Gravity
ARB Anti-Roll Bar
RA Rearward Amplification
CFD Cumulative Density Functions

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