NTNU
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Week 37: Lecture 2

The basic limit theorem of Markov chains

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Information

- Please use this Google form to provide feedback to the reference group:
  `https://s.ntnu.no/tma4265_2020_meeting1`
- Project 1 will be available on September 18 (Friday, week 38).
- The deadline will be October 4 (Sunday, week 40).
- We have no weekly exercises in weeks 39 and 40, and no lectures in week 40.
- You can use groups of **two** or **three**.
- More information will come later.

Section 4.3.3

Theorem (Theorem 4.2)

A state i is recurrent if and only if

$$\sum_{n=1}^{\infty} P_{ii}^{(n)} = \infty.$$

Equivalently, state i is transient if and only if

$$\sum_{n=1}^{\infty} P_{ii}^{(n)} < \infty.$$

Theorem (Corollary 4.1)

If $i \sim j$, then i is recurrent if and only if j is recurrent.

Example

$$\mathbf{P} = \begin{bmatrix} 1/2 & 1/2 \\ 0 & 1 \end{bmatrix}.$$

Which states are transient and which states are recurrent?

Example

$$\mathbf{P} = \begin{bmatrix} 0 & 1/2 & 1/2 & 0 \\ 1/2 & 0 & 0 & 1/2 \\ 0 & 0 & 2/3 & 1/3 \\ 0 & 0 & 1/2 & 1/2 \end{bmatrix}$$

Which states are recurrent and which states are transient?

Example

State space is $\{\ldots, -2, -1, 0, 1, 2, \ldots\}$. Are the following Markov chains transient or recurrent?

Section 4.4

Basic limit theorem of Markov chains

Theorem (Theorem 4.3)

Consider a recurrent irreducible aperiodic Markov chain. Then

1)
$$\lim_{n\to\infty} P_{ii}^{(n)} = \frac{1}{m_i}, \quad i = 0, 1, \ldots,$$

where $m_i = \sum_{n=0}^{\infty} n f_{ii}^{(n)}$ *is the mean duration between visits to state* i.

2)
$$\lim_{n\to\infty} P_{ji}^{(n)} = \lim_{n\to\infty} P_{ii}^{(n)}$$

for all states i *and* j.

Definition

A state i is **positive recurrent** if $m_i < \infty$ and **null recurrent** if $m_i = \infty$.

Theorem (Theorem 4.4)

In a positive recurrent aperiodic equivalence class with states $j = 0, 1, \ldots,$

1) $\lim_{n\to\infty} P_{jj}^{(n)} = \pi_j = \sum_{i=0}^{\infty} \pi_i P_{ij}, \quad i, j = 0, 1, \ldots,$

   and $\sum_{i=0}^{\infty} \pi_i = 1$.

2) $\boldsymbol{\pi} = (\pi_0, \pi_1, \ldots)$ *is uniquely determined by*

$$\pi_i \geq 0, \quad i = 0, 1, \ldots,$$

$$\sum_{i=0}^{\infty} \pi_i = 1,$$

$$\pi_j = \sum_{i=0}^{\infty} \pi_i P_{ij}, \quad j = 0, 1, \ldots.$$

Stationary probability distribution

Definition

Any $\boldsymbol{\pi} = (\pi_0, \pi_1, \ldots)$ such that

$$\pi_j \geq 0, \quad j = 0, 1, \ldots,$$
$$\sum_{i=0}^{\infty} \pi_i = 1,$$
$$\pi_j = \sum_{i=0}^{\infty} \pi_i P_{ij}, \quad j = 0, 1, \ldots,$$

is called a **stationary probability distribution.**