Linear Methods Lecture

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1 Lecture 1

1.1 Set Theory

Definition 1.1. *A **set** is a collection of distinct objects, its elements.*

$$x \in X \quad x \text{ is a element of the set } X$$

and similary

$$x \notin X \quad x \text{ is not an element of } X$$

Two sets are identical $X = Y$, if

$$x \in X \leftrightarrow x \in Y$$

for any element x .

Definition 1.2. *Y is a subset of X, $Y \mathbb{C} X$ if for all $y \in X$. If $Y \subset X$ and $Y \neq X$, we write $y \subset X$ (or $Y \not\subset X$). Y is then a proper subset of X . Showing to sets are equal,*

- $x \in X \leftrightarrow x \in Y$
- $x \subset Y$ *and* $y \subset X$

The empty set are denoted by null.

Example 1.

- $\mathbb{N} = \{1, 2, 3, 4, 5, \ldots\}$
- $\mathbb{Z} = \{\ldots, -1, 0, 1, \ldots\}$
- $\mathbb{Q} = \left\{\frac{p}{q} : p, q \in \mathbb{Z}, q \neq 0\right\}$
- $\mathbb{R} = \text{reals}$
- $\mathbb{C}$: Complex numbers $\quad a + ib$
- Finite set $\{3, 4, 5, 6\}$
- Intervals in $\mathbb{R}$ For real numbers $a < b < \infty$

$$\begin{gathered} (a, b) \\ [a, b] \\ (a, b], \quad [a, b). \end{gathered}$$

Definition 1.3. *Let X and Y be two sets then*

- *Union.* $X \cup Y = \{z \mid z \in X \quad or \quad z \in Y\}$

$$\bigcup_{i \in I} X_i = \{z \mid z \in X_i \quad \text{for some} \quad i \in I\}$$

- *Intersection if* $\bigcap_{i \in I} = \{z \mid z \in X_i \quad \text{For every} \quad i \in I\}$
- *Complement if S is a subset of X , then the complement of S is*

$$X \setminus S = S^c = \{x \in X : x \notin S\}.$$

- *Cartesian product*

$$X \times Y = \{(x, y) : x \in X, \quad y \in Y\}$$

Lemma 1.1.
- $x \cap (Y \cup Z) = (X \cap Y) \cup (X \cap Z)$ *and*

$$X \cup (Y \cap Z) = (X \cup Y) \cap (X \cup Z)$$

- $(X \cup Y)^c = X^c \cap Y^c$
- $(X \cap Y)^c = X^c \cup Y^c$
- *Demo organs law*

$$X \setminus (Y \cup Z) = (X \setminus Y) \cap (X \setminus Z)$$

.

- $(X^c)^c = X$

Proof. Proof of $(X \cup Y)^c) = X^c \cap Y^c$

$$\begin{aligned} x \in (X \cup Y)^c \to x \in X \cup U \\ x \notin X \quad \text{and} \quad x \notin Y \\ x \in X^c \quad \text{and} \quad x \in Y^{\cdot} \\ x \in X^c \cap Y^c \end{aligned}$$

□

1.2 Functions

Let X, Y be sets. A function f from X to Y, denoted $f : X \to Y$, is defined by a set G of ordered pairs (x, y), where $x \in X, \quad y \in Y$ and with the property that;

For each set is there a unique $y \in Y \quad$ s.t. $\quad (x, y) \in G$. We write $f(x) = y$.

- We say that X is the domain and Y is the codomain.
- The (direct) image of a set $A \subset X$ under f is

$$f(A) = \{f(t) : t \in A\} \subset Y$$

- The **inverse image** of a set $B \subset Y$ under f is

$$f^{-1}(B) = \{x \in X \mid f(x) \in B\} \subset X$$

- The **range** if f is the image of its domain X is

$$ran(f) = f(X) = \{f(t) : t \in X\}$$

Example 2. Let $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = max\{x, 0\} = x^+$$

Then is the $ran(f) = [0, \infty)$. The inverse is $f^{-1}(\{y\}) = \{y\}$ and $f^{-1}(\{0\}) = (-\infty, 0]$ and

$$f^{-1}(\{y\}) = \text{NULL} \quad \text{if} \quad y < 0$$

Definition 1.4. *Let $f : X \to Y$ be a function*

- *f is **injective** or **one-to-one** if $f(x_1) \to x_1 = x_1$*
- *f is **surjective** or **onto** if $ran(f) = y$*
- *f is **bijective** if it is both surjective and injective.*

Example 3. Lets continue the example.

- Let $f : \mathbb{R} \to \mathbb{R}$, $f(x) = max\{x, 0\}$. Injective? No; $f(x_1) = \underbrace{f(x_2)}_{=0}$ for any two $x_1, x_1 < 0$.
- A **bijection** $f : X \in Y$ has a **inverse** function $f^{-1} : Y \to X$, defined by $f^{-1}(y) = x$ if $f(x) = y$.

  THe inverse function f^{-1} is also a bijection.

Remark. Not to be confused with the inverse image of a set $f^{-1}(B)$ introduced earlier.

2 Lecture 2

2.1 Recall

Let $f : X \to Y$ then is

i) Inective: $f(x_1) = f(x_2) \to \quad x_1 = x_2$

ii) Surjective: For all y in Y there is a x in X s.t. $f(x) = y$.

iii) Bijective if i) and ii) holds.

- If $F : X \to Y$ is a bijective then there is an inverse

$$f^{-1} : Y \to X$$

  Given by

$$f^{-1}(y) = x \quad \text{if} \quad f(x) = y$$

- Identify function/map
  - id: $X \to X$
  - $id_x(x) = x$ for all $x \in X$
- The composition of a function

$$g : Y \to Z \quad \text{with} \quad f : X \to X$$

  is the function $g \cdot f : X \to Y$ defined by

$$(g \cdot f)(x) = g(f(x)) \quad \text{for} \quad x \in X$$

Definition 2.1. *Anternative version. Given a bijection $f : X \to Y$ the inverse function $f^{-1} : Y \to X$ is the unique function satisftying $f^{-1} \cdot f = id_x$ and $f \cdot f^{-1} = id_y$*

Example 4. $\frac{d}{dx} : C^1(\mathbb{R}, \mathbb{R}) \to C(\mathbb{R}, \mathbb{R})$. Inverse? no.
Let $g \in C^1(\mathbb{R}, \mathbb{R})$. Then is

$$\frac{d(g + c)}{dx} = \frac{dg}{dx} \quad \text{where c is the constant.}$$

It is surjective because given any $f \in C(\mathbb{R}, \mathbb{R})$ we can define $F \in C^1(\mathbb{R}, \mathbb{R})$ by

$$F : X \to \int_0^x f(t)\,dt$$

and

$$\frac{dF}{dx} = f \quad \text{fundamental theorem of calculus.}$$

2.2 Cardinality

Cardinality is a tool for comparing the sizes of sets.

Definition 2.2. *We say that two sets A and B has the same cardinality if there exist a bijection between A and B.*

Example.

i) The two inervals $[0, 2]$ and $[0, 1]$ have the same cardinality.

$$f : [0, 2] \to [0, 1]$$
$$f(t) = \frac{t}{2}$$

ii) Let $\mathbb{N} = \{1, 2, 3, 4, \ldots\}$ and $\mathbb{N} \setminus \{1\} = \{2, 3, 4, 5, \ldots\}$ have the same cardinality

$$f(n) = n + 1$$

iii) n is finite integer. Then there is no bijection

$$f : \{1, 2, 3, \ldots, n\} \to \mathbb{N}$$

These two sets **do not** have the same cardinality.

Definition 2.3. *Let X be a set. We say X is* ***finite*** *if either $X = NULL$ or there exist $n \in \mathbb{N}$ s. T. X has the same cardinality as $\{1, 2, 3, 4, \ldots, n\}$ if*

$$\textit{There exist} \quad f : \{1, 2, 3, \ldots mb\} \to X \quad \textit{for some} \quad n$$

X is ***infinite*** *if it is not finite.*

Definition 2.4. *A set X is*

- *Countable infinite if it has the same cardinality as $\mathbb{N}$.*

$$\exists \textit{bijection} \quad f : X \to \mathbb{N}$$

- *Countable if it is either countably infinite or finite. or equivalently*
  - *if $\exists$ injection $f : X \to \mathbb{N}$*
  - *$\exists$ surjection $f : \mathbb{N} \to X$*
- *Uncountable if it is not countable.*

Example.

- Any finitie set is, e.g. $\{2, 5, 9\}$
- $X = \{1, 4, 9, 16, \ldots, n^2, \ldots\}$ such that

$$f : \mathbb{N} \to X, \quad f(n) = n^2$$

- $\mathbb{N} \times \mathbb{N}$ is countable ;
  We arrange $N \times N$ in a table.

$$\begin{gathered} f : \mathbb{N} \to \mathbb{N} \times \mathbb{N} \\ f(1) = (1, 1) \\ f(2) = (2, 1) \\ f(3) = (1, 2) \\ f(4) = (3, 1) \\ \vdots \end{gathered}$$

- $\mathbb{Z}$ and $\mathbb{Q}$ are countable (Prob set 1).
- If X and Y are countable, then so is $X \cup Y$.

2.3 Schroeder Bernstein Theorem

Let X and Y by two be two sets. Suppose there are injective maps $f : X \to Y$ and $g : Y \to X$. Then there exists a bijection between X and Y.

Example. The interval $(0,1) \subseteq \mathbb{R}$. Claim it is uncountable.

Proof. The Cantor diagonalization argument. Suppose that $(0,1)$ is countable.

$$
\begin{aligned}
(0,1) = &\{x_1, x_2, x_3, x_4, \ldots\} \\
& f(1), f(2), f(3), \ldots
\end{aligned}
$$

$$
\begin{aligned}
f &: \mathbb{N} \to (0,1) \\
x_i &= 0{,}x_{i1}, x_{i2}, x_{i3}, \ldots
\end{aligned}
$$

Now let

$$a = 0, a_1, a_2, a_3, a_4, a_5, \ldots$$

where

$$a_i = \begin{cases} 3 & \text{if} \quad x_{ii} \neq 3 \\ 1 & \text{if} \quad x_{ii} = 3 \end{cases}$$

Then $a_i \neq x_{ii}$, so by construction $a \neq x_i$ for all i. Moreover, we must have $a \in (0,1)$. This is a contradiction, so $(0,1)$ cannot be countable. □

Example. The set of all binary sequences $X = \{(x_1, x_2, x_3, \ldots)\}$: $x_i \in \{0,1\}$ is uncountable .

Proof. Problem set 2. □

Lemma 2.1. *Let X and Y be sets. Then*

- *If X is countable and $Y \subseteq X$, then Y is also countable.*

$$\{1, 2, 3, 4, 5, \dots\} \to \{x_1, x_2, x_3, x_4, \dots\}$$

- *If X is uncountable and $X \subseteq Y$, then Y is uncountable.*
- *If X is countable and there is an injection*

$$f : Y \to X$$

then Y is countable.

- *If X is uncountable and*

$$\exists \quad \textit{injective} \quad f : X \to Y,$$

then Y is uncountable.

Example. Have proved formally that $(0, 1) \subseteq \mathbb{R}$ is countable $\overbrace{\to}^{\text{ii)}} \mathbb{R}$ must be uncountable

$$R \subset \mathbb{C} \overbrace{\longrightarrow}^{\text{ii)}} \mathbb{C} \quad \text{is uncountable}$$

Example. $R = \mathbb{Q} \cup \mathbb{I}$. Know: $\mathbb{Q}$ countable.
Assume $\mathbb{I}$ countable. Then $R \cup \mathbb{I}$ which is a contradiction. So $\mathbb{I}$ is uncountable