Problem 1

(i) True

(ii) False; Let T be the multiplication operator $Tx = (x_n/n)$ on ℓ^∞. Then T is a bounded operator, but the range of T is not closed: The sequence

$$x^n = (1, \sqrt{2}, \sqrt{3}, \ldots, \sqrt{n}, 0, 0, \ldots) \in \ell^\infty$$

is mapped to the sequence

$$y^n = \left(1, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{3}}, \ldots, \frac{1}{\sqrt{n}}, 0, 0, \ldots\right).$$

Hence $y^n \in T(\ell^\infty)$ for every n. It is easily verified that $y^n \to y$, where

$$y = \left(1, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{3}}, \ldots, \frac{1}{\sqrt{n}}, \ldots\right) \in \ell^\infty.$$

However, $y \notin \operatorname{ran}(T)$. The only sequence x which can possibly satisfy $Tx = y$ is

$$x = (1, \sqrt{2}, \sqrt{3}, \ldots, \sqrt{n}, \ldots),$$

but this is not an element of ℓ^∞. Hence $\operatorname{ran}(T)$ is not closed.

(iii) True

(iv) True

Problem 2 Let $(x_k)_{k\in\mathbb{N}}$ be a sequence in a normed space $(X, \|\cdot\|)$.

a) We want to prove that if $(x_k)_{k\in\mathbb{N}}$ is a Cauchy sequence, then $(x_k)_{k\in\mathbb{N}}$ is bounded (note that this was an exercise in Problem set 6). Since $(x_k)_{k\in\mathbb{N}}$ is Cauchy, there exists an $N \in \mathbb{N}$ such that

$$d(x_n, x_m) = \|x_n - x_m\| < 1 \quad \text{for all } n, m \geq N.$$

In particular, putting $m = N$, we have

$$d(x_n, x_N) = \|x_n - x_N\| < 1 \quad \text{for all } n \geq N. \tag{1}$$

We want to show that $(x_k)_{k\in\mathbb{N}}$ is bounded, meaning we can find a radius $r > 0$ and a point $x \in X$ such that $(x_k)_{k\in\mathbb{N}} \subset B_r(x)$. So let

$$\begin{aligned} r &:= \max\{1, d(x_1, x_N), \ldots, d(x_{N-1}, x_N)\} \\ &= \max\{1, \|x_1 - x_N\|, \ldots \|x_{N-1} - x_N\|\}. \end{aligned}$$

We claim that $B_{r+1}(x_N)$ contains every element of the sequence $(x_k)_{k\in\mathbb{N}}$, or equivalently $d(x_k, x_N) < r + 1$ for every $k \in \mathbb{N}$:

- If $k < N$, then our definition of r ensures that $r \geq d(x_k, x_N)$, and thus $d(x_k, x_N) \leq r < r + 1$.
- If $k \geq N$, then $d(x_k, x_N) < 1$ by (1), and since $r + 1 \geq 2$ we again have $d(x_k, x_N) \leq r + 1$.

This shows that $(x_k)_{k\in\mathbb{N}} \subset B_{r+1}(x_N)$, and thus the sequence is bounded.

b) Let $\|\cdot\|_a$ and $\|\cdot\|_b$ be equivalent norms on X, and let $x \in X$. We want to show that $(x_k)_{k\in\mathbb{N}}$ converges to x in $(X, \|\cdot\|_a)$ if and only if $(x_k)_{k\in\mathbb{N}}$ converges to x in $(X, \|\cdot\|_b)$. Note that this problem is very similar to exercise 6 in Problem set 11.

Since $\|\cdot\|_a$ and $\|\cdot\|_b$ are equivalent, there exist constants $C_1, C_2 > 0$ such that

$$C_1\|x\|_b \leq \|x\|_a \leq C_2\|x\|_b \quad \text{for all } x \in X. \tag{2}$$

Suppose first that $x_k \to x$ in $(X, \|\cdot\|_a)$. Then for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$\|x_k - x\|_a < C_1\varepsilon \quad \text{for all } k > N.$$

Using the left hand side inequality in (2), it follows that

$$\|x_k - x\|_b \leq \frac{1}{C_1}\|x_k - x\|_a < \varepsilon \quad \text{for all } k > N.$$

This shows that $x_k \to x$ also in $(X, \|\cdot\|_b)$.

Now suppose that $x_k \to x$ in $(X, \|\cdot\|_b)$. Then for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$\|x_k - x\|_b < \frac{\varepsilon}{C_2} \quad \text{for all } k > N.$$

Using the right hand side inequality in (2), it follows that

$$\|x_k - x\|_a \leq C_2\|x_k - x\|_b < \varepsilon \quad \text{for all } k > N,$$

so $x_k \to x$ also in $(X, \|\cdot\|_a)$.

Problem 3 Let $(\ell^2, \langle \cdot, \cdot \rangle)$ be the inner product space of complex-valued sequences $x = (x_k)_{k \in \mathbb{N}}$ equipped with the standard inner product

$$\langle x, y \rangle = \sum_{k=1}^{\infty} x_k \overline{y_k}, \quad x, y \in \ell^2,$$

and let $T : \ell^2 \to \ell^2$ be the multiplication operator given by

$$Tx = \left(\frac{i^k x_k}{k} \right)_{k \in \mathbb{N}},$$

where $i = \sqrt{-1}$.

a) *Showing T is linear:*

Let $\alpha, \beta \in \mathbb{C}$ and $x, y \in \ell^2$. We then have

$$\begin{aligned} T(\alpha x + \beta y) &= \left(\frac{i^k(\alpha x_k + \beta y_k)}{k} \right)_{k \in \mathbb{N}} \\ &= \left(\frac{i^k \alpha x_k}{k} + \frac{i^k \beta y_k}{k} \right)_{k \in \mathbb{N}} \\ &= \alpha \left(\frac{i^k x_k}{k} \right)_{k \in \mathbb{N}} + \beta \left(\frac{i^k y_k}{k} \right)_{k \in \mathbb{N}} = \alpha Tx + \beta Ty. \end{aligned}$$

Showing T is bounded:

We have that

$$\|Tx\|_2^2 = \sum_{k=1}^{\infty} \left| \frac{i^k x_k}{k} \right|^2 = \sum_{k=1}^{\infty} \frac{|x_k|^2}{k^2} \leq \sum_{k=1}^{\infty} |x_k|^2 = \|x\|_2^2.$$

Thus, we have $\|Tx\|_2 \leq \|x\|_2$, and this shows T is bounded.

Determining the operator norm:

The norm of T is defined as

$$\|T\| = \sup_{x \neq 0} \frac{\|Tx\|_2}{\|x\|_2}.$$

We have already seen that $\|Tx\|_2 \leq \|x\|_2$, so we immediately get

$$\|T\| \leq 1.$$

On the other hand, if we let $y = (1, 0, 0, \ldots) \in \ell^2$, then $Ty = (i, 0, 0, \ldots)$ and

$$\|y\|_2 = \|Ty\|_2 = 1.$$

It follows that

$$\|T\| = \sup_{x \neq 0} \frac{\|Tx\|_2}{\|x\|_2} \geq \frac{\|Ty\|_2}{\|y\|_2} = 1.$$

We thus have $\|T\| = 1$.

b) *Determining the adjoint operator T^*:*

The adjoint T^* is the bounded linear operator satisfying

$$\langle Tx, y \rangle = \langle x, T^* y \rangle.$$

We have that

$$\langle Tx, y \rangle = \sum_{k=1}^{\infty} \frac{i^k x_k}{k} \overline{y_k} = \sum_{k=1}^{\infty} x_k \overline{\frac{(-i)^k y_k}{k}} = \langle x, T^* y \rangle,$$

where

$$T^* y = \left(\frac{(-i)^k y_k}{k} \right)_{k \in \mathbb{N}}.$$

Determining if T is normal:

An operator is normal if $T^*T = TT^*$. In our case, we have that

$$T^*Tx = \left(\frac{x_k}{k^2} \right)_{k \in \mathbb{N}} = TT^*x,$$

so T is normal.

c) *Showing that the range of T is dense in ℓ^2:*

We have several possible approaches. One is to recall from the curriculum that the range of an operator $T : X \to X$, where X is a Hilbert space, is dense in X if and only if $\ker(T^*) = \{0\}$. Here we have $X = \ell^2$, which is a Hilbert space. Moreover, it is clear from the definition of T^* in **b)** that $T^*x = 0$ if and only if $x = 0$. Thus $\ker(T^*) = \{0\}$, and it follows that $\operatorname{ran}(T)$ is dense in ℓ^2.

Another approach is to show that for any $x \in \ell^2$ there exists a sequence $(x^n)_{n \in \mathbb{N}} \subset \operatorname{ran}(T)$ converging to x. For fixed $x = (x_1, x_2, \ldots) \in \ell^2$ we let x^n be the truncated sequence

$$x^n = (x_1, x_2, \ldots, x_n, 0, 0, \ldots).$$

Then $x^n \in \operatorname{ran}(T)$, because the sequence

$$y^n = (-ix_1, 2x_2, -3ix_3, \dots, n(-i)^n x_n, 0, 0, \dots)$$

belongs to ℓ^2 and satisfies $Ty^n = x^n$. On the other hand, it is clear that $x^n \to x$, as

$$\|x - x^n\|_2^2 = \sum_{k=n+1}^{\infty} |x_k|^2 \to 0 \quad \text{as } n \to \infty \quad (\text{since } x \in \ell^2).$$

This proves that $\operatorname{ran}(T)$ is dense in ℓ^2.

Problem 4 Let

$$A = \begin{bmatrix} 2 & 2 \\ 2 & 2 \\ -1 & -1 \end{bmatrix}.$$

a) *Note that there are many possible singular value decompositions $A = U\Sigma V^*$. Only the matrix Σ is uniquely defined. All correct SVDs are of course equally good answers.*

Seeking a singular value decomposition of A, we first find

$$A^*A = \begin{bmatrix} 9 & 9 \\ 9 & 9 \end{bmatrix}.$$

This matrix has one positive eigenvalue $\sigma_1^2 = 18$, and one eigenvalue $\sigma_2^2 = 0$. The eigenvectors corresponding to these eigenvalues are v_1 and v_2 satisfying

$$A^*Av_1 = 18v_1 \quad \Rightarrow \quad v_1 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

and

$$A^*Av_2 = 0 \quad \Rightarrow \quad v_2 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix},$$

where both eigenvectors are normalized to length one. We thus have

$$\Sigma = \begin{bmatrix} 3\sqrt{2} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad V = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}. \tag{3}$$

Finally, we find the matrix U. Its first column is u_1 given by

$$u_1 = \frac{1}{\sigma_1} Av_1 = \frac{1}{3\sqrt{2}} \begin{pmatrix} 2\sqrt{2} \\ 2\sqrt{2} \\ -\sqrt{2} \end{pmatrix} = \begin{pmatrix} \frac{2}{3} \\ \frac{2}{3} \\ -\frac{1}{3} \end{pmatrix}.$$

We can now complete the matrix U by choosing any two orthonormal vectors orthogonal to u_1. For $w \in \mathbb{C}^3$, we have that

$$\langle u_1, w \rangle = 0 \quad \Rightarrow \quad w = \begin{pmatrix} -s+t \\ s \\ 2t \end{pmatrix}, \quad s, t \in \mathbb{C}.$$

We see that by choosing (for example) $(s = 1, t = 0)$ and $(s = 1, t = 2)$, we find the orthogonal vectors

$$\tilde{u}_2 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \quad \text{and } \tilde{u}_3 = \begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix}.$$

Letting $u_2 = \tilde{u}_2/\|\tilde{u}_2\|$ and $u_3 = \tilde{u}_3/\|\tilde{u}_3\|$, we finally get

$$U = \begin{bmatrix} \frac{2}{3} & -\frac{1}{\sqrt{2}} & \frac{1}{3\sqrt{2}} \\ \frac{2}{3} & \frac{1}{\sqrt{2}} & \frac{1}{3\sqrt{2}} \\ -\frac{1}{3} & 0 & \frac{4}{3\sqrt{2}} \end{bmatrix}. \tag{4}$$

We thus have the singular value decomposition $A = U\Sigma V^*$, with U given in (4) and Σ and V given in (3) (note that only Σ is unique).

b) The pseudoinverse A^+ of A is given by $A^+ = V\Sigma^+U^*$, where

$$\Sigma^+ = \begin{bmatrix} \frac{1}{3\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

We get

$$A^+ = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{3\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{3\sqrt{2}} & \frac{1}{3\sqrt{2}} & \frac{4}{3\sqrt{2}} \end{bmatrix} = \frac{1}{18} \begin{bmatrix} 2 & 2 & -1 \\ 2 & 2 & -1 \end{bmatrix}$$

The best approximation to a solution of the inconsistent system

$$\begin{aligned} 2x_1 + 2x_2 &= 3 \\ 2x_1 + 2x_2 &= 4 \\ -x_1 - x_2 &= -4 \end{aligned}$$

is thus given by

$$z = A^+ \begin{pmatrix} 3 \\ 4 \\ -4 \end{pmatrix} = \frac{1}{18} \begin{bmatrix} 2 & 2 & -1 \\ 2 & 2 & -1 \end{bmatrix} \begin{pmatrix} 3 \\ 4 \\ -4 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Problem 5 Let $f_1(t) = \sin t$ and $f_2(t) = \sin 2t$. We have that

$$\langle f_1, f_2 \rangle = \int_0^{2\pi} \sin t \sin 2t \, dt = 2 \int_0^{2\pi} \sin^2 t \cos t \, dt = \left[\frac{2}{3} \sin^3 t \right]_0^{2\pi} = 0,$$

so f_1 and f_2 are orthogonal in $L^2[0, 2\pi]$. Moreover, we have that

$$\|f_1\|_2^2 = \int_0^{2\pi} \sin^2 t \, dt = \frac{1}{2} \int_0^{2\pi} (1 - \cos 2t) \, dt = \pi$$

and

$$\|f_2\|_2^2 = \int_0^{2\pi} \sin^2 2t \, dt = \frac{1}{2} \int_0^{2\pi} (1 - \cos 4t) \, dt = \pi,$$

so

$$e_1 := \frac{1}{\sqrt{\pi}} f_1 \quad \text{and} \quad e_2 := \frac{1}{\sqrt{\pi}} f_2$$

are orthonormal elements of $L^2[0, 2\pi]$. Now observe that

$$\int_0^{2\pi} |t - a \sin t - b \sin 2t|^2 \, dt = \|t - a f_1 - b f_2\|_2^2 = \|t - a\sqrt{\pi} e_1 - b\sqrt{\pi} e_2\|_2^2. \tag{5}$$

For a finite orthonormal system $\{e_1, e_2\}$ in $L^2[0, 2\pi]$ we have a unique closest point property, meaning that the right hand side of (5) is minimal if

$$a\sqrt{\pi} = \langle t, e_1 \rangle = \frac{1}{\sqrt{\pi}} \int_0^{2\pi} t \sin t \, dt = -2\sqrt{\pi}$$

and

$$b\sqrt{\pi} = \langle t, e_2 \rangle = \frac{1}{\sqrt{\pi}} \int_0^{2\pi} t \sin 2t \, dt = -\sqrt{\pi}.$$

Thus (5) is minimal if $a = -2$ and $b = -1$.

Problem 6

a) Let $X \neq \emptyset$ be a complete metric space, and suppose $T : X \to X$ is a mapping such that $T^k : X \to X$ is a contraction for some natural number $k > 1$. Then by Banach's fixed point theorem there exists a unique point $x^* \in X$ such that $T^k x^* = x^*$.

The point x^* is also a fixed point of T: Applying T to both sides of the equation $T^k x^* = x^*$, we get

$$T(T^k x^*) = T^k(T x^*) = T x^*.$$

This shows that Tx^* is a fixed point of T^k. Since x^* is the *unique* fixed point of T^k, this implies

$$Tx^* = x^*,$$

so x^* is a fixed point of T.

The fixed point x^* is unique also for T: Let x be any fixed point of T, i.e. any point satisfying $Tx = x$. We have that

$$T^k x = T^{k-1}(Tx) = T^{k-1}x = T^{k-2}(Tx) = T^{k-2}x = \cdots = Tx = x,$$

so any fixed point of T is also a fixed point of T^k. But the fixed point of T^k is known to be unique, so this implies that $x = x^*$.

b) We have that

$$\begin{aligned}
|Tf(t) - Tg(t)| &= \left| \int_0^t f(s) - g(s)\, ds \right| \\
&\leq \int_0^t |f(s) - g(s)|\, ds \\
&\leq \|f - g\|_\infty \int_0^t ds = t\|f - g\|_\infty,
\end{aligned}$$

and since $0 \leq t \leq 1$, we cannot conclude from the above that T is a contraction. However, we observe that

$$\begin{aligned}
|T^2 f(t) - T^2 g(t)| &= \left| \int_0^t Tf(s) - Tg(s)\, ds \right| \\
&\leq \int_0^t |Tf(s) - Tg(s)|\, ds \leq \int_0^t s\|f - g\|_\infty\, ds \\
&= \|f - g\|_\infty \int_0^t s\, ds = \frac{1}{2}t^2\|f - g\|_\infty \leq \frac{1}{2}\|f - g\|_\infty.
\end{aligned}$$

It follows that

$$\|T^2 f - T^2 g\|_\infty \leq \frac{1}{2}\|f - g\|_\infty,$$

so T^2 is a contraction on the complete normed space $(C[0,1], \|\cdot\|_\infty)$, and from the result in **a)** we conclude that T has a unique fixed point.

Let us now find this fixed point by iteration, starting with $f_0(t) = 1$. We get

$$\begin{aligned}
f_1(t) &= 1 - \int_0^t ds = 1 - t \\
f_2(t) &= 1 - \int_0^t (1 - s)\, ds = 1 - t + \frac{1}{2}t^2 \\
&\vdots \\
f_n(t) &= \sum_{k=0}^n \frac{(-t)^k}{k!}
\end{aligned}$$

and when $n \to \infty$ we see that

$$f_n(t) \to \sum_{k=0}^{\infty} \frac{(-t)^k}{k!} = e^{-t}.$$

Thus, the unique fixed point of T is $f(t) = e^{-t}$ (and it is easily checked that $Tf = f$).