TMA 4190 Introduction to Topology
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Lecture 16[1]

16. Embedding Abstract Manifolds in Euclidean Space

We start with some general facts and some terminology.

The closure of a subset

Let X be a topological space and A be an arbitrary subset. The **closure of A** in X, denoted $\overline{A}$, is the intersection of all closed subsets on X which contain A.
For example, the closure of an open ball $B_\epsilon(0)$ in $\mathbb{R}^N$ is just the closed ball

$$\overline{B_\epsilon(0)} = \{x \in \mathbb{R}^N : |x| \leq \epsilon\}.$$

We need the closure of a subset for example when we want to talk about the support of a function:

Support of a function

Let X be a smooth manifold and $f\colon X \to \mathbb{R}$ be a smooth function $f\colon X \to \mathbb{R}$. The **closed** subset

$$\mathrm{supp}(f) := \overline{\{x \in X : f(x) \neq 0\}}$$

is called the **support of f**.

We are now going to introduce a fundamental tool for studying manifolds.

Partition of unity

Let X be an abstract smooth k-manifold and let $\{U_\alpha\}$ be an open cover, i.e. a collection of open subsets in X which cover X. A sequence of smooth functions $\{\rho_i\colon X \to \mathbb{R}\}$ is called a **partition of unity subordinate to the open cover $\{U_\alpha\}$** if it has the following properties:

(a) $0 \leq \rho_i(x) \leq 1$ for all $x \in X$ and all i.

[1]Following the books of Guillemin and Pollack: Differential Topology, and Milnor: Topology from the differentiable viewpoint.

(b) Each $x \in X$ has a neighborhood on which all but finitely many functions ρ_i are identically zero.
(c) For each i, $\text{supp}(\rho_i) \subset U_\alpha$ for some α.
(d) For each $x \in X$, $\sum_i \rho_i(x) = 1$. (Note that according to (b), this sum is always finite.)

The most general existence result for partitions of unity (without assuming that each ρ_i is smooth) is that they exist on every paracompact space, i.e. spaces on which every open cover has a locally finite refinement (every point has a neighborhood that intersects only finitely many sets in the cover).

Before we prove that partictions of unity exist on manifolds, we need some preparation.

Separating closed subsets

Let A and C be disjoint closed subsets in $\mathbb{R}^N$. Then there are disjoint open subsets U and V such that $A \subset U$ and $C \subset V$.

Proof: For each $a \in A$, choose an $\epsilon_a > 0$ such that $B_{2\epsilon_a}(a) \cap C = \emptyset$. This is possible since C is closed. Similarly, for each $c \in C$, choose an $\epsilon_c > 0$ such that $B_{2\epsilon_c}(c) \cap A = \emptyset$. We define

$$U := \cup_{a \in A} B_{\epsilon_a}(a) \text{ and } V := \cup_{c \in C} B_{\epsilon_c}(c).$$

Then U and V are open subsets with $A \subset U$ and $C \subset V$. We claim that U and V are disjoint.

For, if $x \in U \cap V$, then

$$x \in B_{\epsilon_a}(a) \cap B_{\epsilon_c}(c)$$

for some $a \in A$ and $c \in C$. By the triangle inequality, this implies

$$|a - c| < \epsilon_a + \epsilon_c.$$

But, if $\epsilon_a \leq \epsilon_c$, then $|a - c| < 2\epsilon_c$ and $a \in B_{2\epsilon_c}(c)$. And, if $\epsilon_c \leq \epsilon_a$, then $|a - c| < 2\epsilon_a$ and $c \in B_{2\epsilon_a}(a)$. Both cases are impossible. **QED**

Another important tool that we wil need are smooth bump functions. We have met them in a previous lecture. Today we will need them in a slightly more interesting form:

Smooth bump functions revisited

Let $U \subset \mathbb{R}^N$ be open and $K \subset U$ be compact. Then there is a smooth function $\varphi\colon \mathbb{R}^N \to \mathbb{R}$ with $\varphi(x) = 1$ for all $x \in K$ and $\varphi(x) = 0$ for all $x \in \mathbb{R}^N \setminus C$ for some closed subset C with $K \subset C \subset U$.

Proof: Recall the smooth function

$$f\colon \mathbb{R} \to \mathbb{R},\ f(x) = \begin{cases} e^{-1/x^2} & x > 0 \\ 0 & x \le 0. \end{cases}$$

For any given $\epsilon > 0$, we define a function

$$f_\epsilon(x) := f(x)f(x-\epsilon).$$

As a product of two smooth functions, f_ϵ is smooth.

Next we define yet another function

$$g_\epsilon\colon \mathbb{R} \to \mathbb{R},\ g_\epsilon(x) := \frac{\int_0^x f_\epsilon(t)dt}{\int_0^\epsilon f_\epsilon(t)dt}.$$

By the Fundamental Theorem of Calculus, g_ϵ is smooth, nondecreasing, and

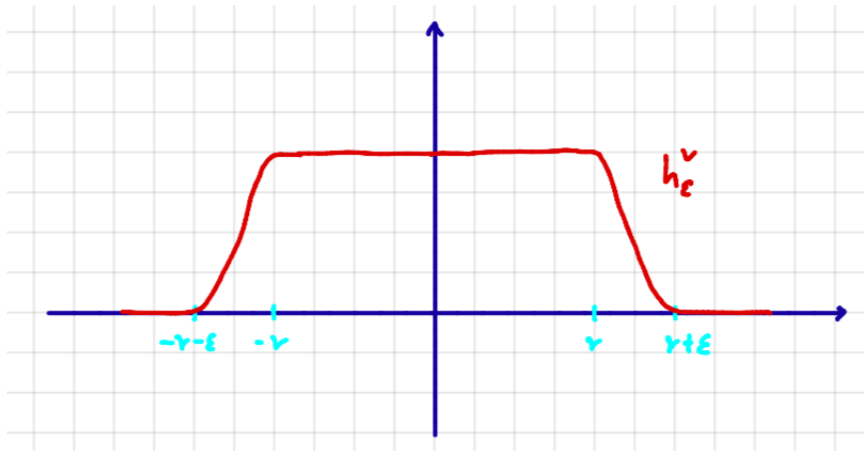
$$\begin{cases} g_\epsilon(x) = 0 & x \le 0 \\ 0 < g_\epsilon(x) < 1 & 0 < x < \epsilon \\ g_\epsilon(x) = 1 & x \ge \epsilon \end{cases}$$

Finallly, for any fixed point $a \in \mathbb{R}^N$ and for any given $r > 0$, we define

$$h_\epsilon^r\colon \mathbb{R}^N \to \mathbb{R},\ h_\epsilon^r(x) = 1 - g_\epsilon(|x-a| - r).$$

Then h_ϵ^r is smooth, nonincreasing, and

$$\begin{cases} h_\epsilon^r(x) = 1 & |x-a| \le r \\ 0 < h_\epsilon^r(x) < 1 & r < |x-a| < r+\epsilon \\ h_\epsilon^r(x) = 0 & |x-a| \ge r+\epsilon \end{cases}$$



This gives us a smooth function $\phi\colon \mathbb{R}^N \to \mathbb{R}$ which has value 1 on the compact subset $\overline{B_r(a)}$ and has value 0 outside the closed subset $\overline{B_{r+\epsilon}(a)}$.

Now let $U \subset \mathbb{R}^N$ be open and $K \subset U$ be compact. For this general situation we need to work a bit harder and rearrange the argument as follows:

Let ψ be the function

$$\psi_\epsilon\colon \mathbb{R}^N \to \mathbb{R},\ \psi_\epsilon(x) = \begin{cases} e^{-1/|x|^2} & |x| < 1 \\ 0 & |x| \geq 1. \end{cases}$$

This is a smooth map with $\int_{\mathbb{R}^N} \psi dx = 1$ (using a standard Lebesgue measure dx on $\mathbb{R}^N$).

For a given $\epsilon > 0$, define $\psi_\epsilon\colon R^N \to \mathbb{R}$ by $\psi_\epsilon(x) := \epsilon^{-N}\psi(x/\epsilon)$. This is still a smooth function with $\int_{\mathbb{R}^N} \psi_\epsilon dx = 1$.

Since $\mathbb{R}^N \setminus U$ is closed and K is compact, we can choose a small $\epsilon > 0$ such that, for each point $x \in K$, we have $B_{2\epsilon}(x) \cap U = \emptyset$. Then the $V := \cup_{x\in K} B_\epsilon(x)$ is an open set containing K with **compact closure** $\bar{V} \subset U$ contained in U.

Let χ_V be the characteristic function on V, i.e. the function

$$\chi_V\colon \mathbb{R}^N \to \mathbb{R},\ \begin{cases} \chi_V(x) = 1 & \text{for } x \in V \\ \chi_V(x) = 0 & \text{for } x \notin V. \end{cases}$$

The function χ_V is identically 1 on K and has compact support contained in U. But it is of course not smooth on $\mathbb{R}^N$, not even continuous. Hence we need to modify it, to make it smooth. The function ψ_ϵ, for the fixed ϵ, will serve as a tool to "smoothen" χ_V.

Then the desired smooth function φ is the convolution $\psi_\epsilon * \chi_V$ of χ_V and ψ_ϵ:

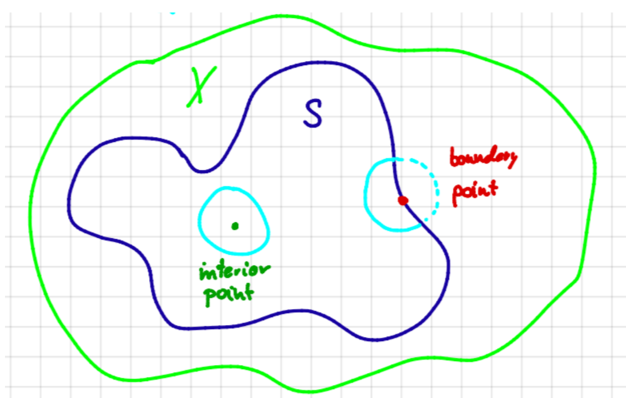
$$\varphi\colon \mathbb{R}^N \to \mathbb{R}, x \mapsto \int_{\mathbb{R}^N} \psi_\epsilon(x-y)\chi_V(y)dy.$$

Note that the integral is well-defined, since the closure of V, which is the support of χ_V, is compact. **QED**

Finally some terminology:

The interior of a set

Let X be a topological space, and S a subset of X. Then the **interior of S**, denoted $\mathrm{int}(S)$, is the union of all open subsets of X contained in S. By definition, the interior of any S is an open subset of X. In fact, it is the largest open subset of X which is contained in S.
If $S \subset \mathbb{R}^N$ then $\mathrm{int}(S)$ is the set of all points $s \in S$ such that there is a small open ball centered at x which is contained in S.
Obviously, if U is an open subset of X then $\mathrm{int}(U) = U$. In particular, if $X \subset \mathbb{R}^N$ is open then $\mathrm{int}(X) = X$. But in general $\mathrm{int}(S)$ is a proper subset of X.



Existence of partitions of unity

We are going to show that partitions of unity exist on manifolds step by step with increasing difficulty. We start with the case of compact subspaces in $\mathbb{R}^N$. Then we are going to transport this result to compact abstract smooth manifolds. Finally, we discuss arbitrary compact smooth k-manifolds. There is no need to restrict to compact manifolds. In fact, partitions of unity exist on every **para-compact** topological space (every open cover has a **locally finite refinement**), a class of spaces much larger than abstract manifolds.

First case: $X \subset \mathbb{R}^N$ compact.

Let $\{U_\alpha\}$ be an open cover of X. Since X is compact, $\{U_\alpha\}$ has a finite subcover $\{U_1, \ldots, U_n\}$. A partition of unity subordinate to the finite subcover is also a partition of unity subordinate to the original cover.

Step 1: We are going to show that we can shrink the covering to an open covering $\{V_1 \ldots, V_n\}$ such that $\bar{V}_i \subset U_i$ for each i.

Consider the closed subset

$$A := X \setminus (U_2 \cup \cdots \cup U_n)$$

of X. Since $\{U_1, \ldots, U_n\}$ cover X, we know $A \subset U_1$. Since A and $X \setminus U_1$ are closed disjoint, we can choose an open subset V_1 containing A such that V_1 is disjoint to an open subset W which contains $X \setminus U_1$. Thus V_1 is contained in the complement $X \setminus W$. Since $X \setminus W$ is a closed subset which contains V_1, we know $\bar{V}_1 \subset X \setminus W$, since the closure of V_1 is the intersection of all closed subsets which contain V_1. Since $X \setminus U_1 \subset W$ by the choice of W, we have $X \setminus W \subset X \setminus (X \setminus U_1) = U - 1$. Thus we have $\bar{V}_1 \subset U_1$. Since V_1 contains the complement of $U_2 \cup \cdots \cup U_n$ in X, the collection $\{V_1, U_2, \ldots, U_n\}$ covers X.

Now we proceed by induction as follows: Given open subsets $V_1, \ldots, V_{k-1}$ such that

$$X = \{V_1, \ldots, V_{k-1}, U_k, U_{k+1}, \ldots, U_n\},$$

let A_k be the subset

$$A_k = X \setminus (V_1 \cup \cdots \cup V_{k-1}) \cup (U_{k+1} \cup \cdots \cup U_n).$$

Then A_k is a closed subset of X which is contained in the open set U_k. Choose an open subset V_k containing A_k such that $\bar{V}_k \subset U_k$. Then $\{V_1, \ldots, V_{k-1}, V_k, U_{k+1}, \ldots, U_n\}$ covers X. At the nth step of the induction we are done.

Step 2: Given the open covering $\{U_1, \ldots, U_n\}$ of X, we use Step 1 to choose an open covering $\{V_1, \ldots, V_n\}$ of X such that $\bar{V}_i \subset U_i$ for each i. Then we repeat this process and choose an open covering $\{W_1, \ldots, W_n\}$ of X such that $\bar{W}_i \subset V_i$ for each i.

For each i, we choose a smooth bump function

$$\varphi_i \colon X \to [0,1] \text{ such that } \varphi_i(\bar{W}_i) = \{1\} \text{ and } \varphi_i(X - V_i) = \{0\}.$$

Since $\varphi_i^{-1}(\mathbb{R} \setminus \{0\}) \subset V_i$, we have

$$\operatorname{supp}(\varphi_i) \subset \bar{V}_i \subset U_i.$$

(Here is the point where see why we need to apply Step 1 twice: If we were working with the V_i's instead of W_i's, then we would have $\mathrm{supp}(\varphi) \subset \bar{U}_i$ instead of $\mathrm{supp}(\varphi) \subset U_i$ as required for a partition subordinate to the cover $\{U_i\}$.)

Since $\{W_1, \ldots, W_n\}$ covers X, we have

$$\varphi(x) := \sum_{i=1}^{n} \varphi_i(x) > 0 \text{ for all } x \in X.$$

Finally, for each i, we define

$$\rho_i(x) := \frac{\varphi_i(x)}{\varphi(x)}.$$

Second case: $X \subset \mathbb{R}^N$ and $X = X_1 \cup X_2 \cup X_3 \cup \cdots$ where each X_i is compact and $X_i \subset \mathrm{int}(X_{i+1})$.

Let $\{U_\alpha\}$ be an open cover of X. For each i, we define

$$U_\alpha^i := U_\alpha \cap (X_{i+1} \setminus \mathrm{int}(X_{i-2})).$$

Then $\{U_\alpha^i\}$ is an open cover of $Y_i := X_i \setminus \mathrm{int}(X_{i-1})$. Since $\mathrm{int}(X_{i-1})$ is an open subset, Y_i is a closed subset of X_i and therefore Y_i is also compact. Then, for each i, the first case implies that there is a partition of unity φ_α^i on Y_i subordinate to the cover $\{U_\alpha^i\}$.

For each $x \in X$, there is an i such that $x \in X_i$ and hence $\varphi_\alpha^j(x) = 0$ for all $j \geq i + 2$. Hence, for each $x \in X$, the sum

$$\varphi(x) := \sum_{\alpha,i} \varphi_\alpha^i(x)$$

is a finite sum in some open set containing x.

Now for each α we define

$$\rho_\alpha^i(x) := \frac{\varphi_\alpha^i(x)}{\varphi(x)}$$

This is partition of unity subordinate to the open cover $\{U_\alpha\}$.

Third case: $X \subset \mathbb{R}^N$ is open.

Define subsets

$$X_i := \{x \in X : |x| \leq i \text{ and the distance to } \mathbb{R}^N \setminus X \geq 1/j\}.$$

Then these subsets satisfy:

- each X_i is compact, since it is the intersection $X \cap \overline{B_i(0)} \cap (X \setminus (\cup_{p \in \mathbb{R}^N \setminus X} B_{1/i}(p))$ and therefore closed and bounded in $\mathbb{R}^N$;
- for each i: $X_i \subset \mathrm{int}(X_{i+1})$;
- $X = X_1 \cup X_2 \cup \cdots$.

Hence we can apply the second case.

Fourth case: $X \subset \mathbb{R}^N$ arbitrary.

Let $\{U_\alpha\}$ be an open cover of X. By the definition of the topology on X, for each α, there is a subset $V_\alpha \subset \mathbb{R}^N$ open in $\mathbb{R}^N$ such that $U_\alpha = X \cap V_\alpha$. Let Y be the union of all the V_α in $\mathbb{R}^N$. By the third case, there is a partition of unity on Y subordinate to the open cover $\{V_\alpha\}$. This is also a partition of unity on X subordinate to the open cover $\{U_\alpha\}$.

Last case: X is a compact abstract smooth k-manifold.

Let $\{V_\alpha\}$ be an open over of X. By intersecting with the domains of charts on X, we get a refinement of the cover. Hence we can assume that V_α are the **domains of charts** on X. Since X is **compact**, the domains of finitely many charts on X suffice to cover X. Let us lable them $(V_1,\phi_1), \ldots, (V_n,\phi_n)$. Then each $U_i = \phi_i(V_i)$ is an open subset in $\mathbb{R}^k$.

Now we can **proceed exactly as in the case of a compact subspace in $\mathbb{R}^N$** for the finite cover $\{U_1, \ldots, U_n\}$ of the space $Y := U_1 \cup \cdots \cup U_n \subset \mathbb{R}^k$. This yields a partition of unity $\{\rho_i\}$ subordinate to the cover $\{U_1, \ldots, U_n\}$. Composition of each ρ_i with ϕ_i yields a partition of unity $\{\rho_i \circ \phi_i\}$ on X subordinate to the cover $\{V_1, \ldots, V_n\}$. **QED**

Now we are ready to prove the following embedding result.

Embedding abstract manifolds into Euclidean space

Let X be a compact abstract smooth k-manifold. Then there is an embedding, i.e. an injective proper map, $X \hookrightarrow \mathbb{R}^N$ for some large N.

Proof:

The collection of all V_α for all charts (V_α,ϕ_α) is an open cover of X. Since X is compact, we can cover X by the images of a **finite number of charts** $V_1, \ldots, V_n$.

Let $\{\rho_i\}$ be a partition of unity subordinate to the open cover defined by the V_i's.

For a chart $\phi_i \colon V_i \to U_i \subset \mathbb{R}^k$, we define a new map

$$g_i \colon X \to \mathbb{R}^k,\ g_i(x) = \begin{cases} \rho_i(x) \cdot \phi_i(x) & \text{for } x \in V_i \\ 0 & \text{for } x \in X \setminus \operatorname{supp}(\rho_i). \end{cases}$$

The map g_i is well-defined, since if $x \in V_i \setminus \operatorname{supp}(\rho_i)$, then both definitions agree to be 0. Moreover, g_i is **continuous**, since its restrictions to the two **open** subsets V_i and $X \setminus \operatorname{supp}(\rho_i)$ are continuous (this is why we do not use $X \setminus V_i$ in the definition because that would be a closed subset).

Now we define a map

$$g \colon X \to \mathbb{R}^n \times \mathbb{R}^{nk},\ x \mapsto (\rho_1(x), \dots, \rho_n(x), g_1(x), \dots, g_n(x)).$$

We observe that **g is continuous**, since the g_i's and the ρ_i's are continuous.

Claim: g is an injective proper map.

Since X is compact, g is a **proper** map.

Now we show that g is injective. For assume $g(x) = g(y)$. Then $\rho_i(x) = \rho_i(y)$ for all i by the definition of g. But, by the definition of a partition of unity, for at least one i, we must have $\rho_i(x) = \rho_i(y) \neq 0$.

Thus x and y must lie in the same V_i, since ρ_i is supported on V_i, i.e. $\rho_i(x) \neq 0$ implies $x \in V_i$. Hence, since $g_i(x) = g_i(y)$ and $\rho_i(x) = \rho_i(y) \neq 0$,we must have $\phi_i(x) = \phi_i(y)$. Since ϕ_i is a bijection, this shows $x = y$. Thus **g is injective**. **QED**

Actually, g is also an immersion, but we have not defined what that means for an abstract manifold. Since this is just an exercise in translating the defintions, we omit this point and rather move on.

All manifolds can be embedded in Euclidean space

In fact, every abstract k-manifold X can be embedded in Euclidean space. One can just keep on going with the above argument in the non-compact case and use local coordinates to map pieces of X into $\mathbb{R}^k$. Though when using only finitely many copies of $\mathbb{R}^k$ to accomodate infinitely many neighborhoods of X, we loose injectivity. The key tool that restores injectivity are **partitions of unity** which even out the troubles occuring because of overlapping neighborhoods.

For this to work, it is crucial that the topology on X has a **countable** basis. This is a technical point which we did and will not discuss because it would divert us too far from the main story.
We just remark that it is possible to construct topological spaces without a countable basis which are locally homeomorphic to Euclidean space, but which cannot be embedded into Euclidean space.

Another application of the existence of partitions of unity is the following lemma which will turn out to be key tool in the proof of Whitney's Theorem.

Existence of proper functions on manifolds

On any manifold X, there is a proper map $p\colon X \to \mathbb{R}$.

Proof: Let $\{U_\alpha\}$ be the collection of open subsets of X that have **compact closure**, and let ρ_α be a subordinate partition of unity. Then

$$p(x) = \sum_{i=1}^{\infty} i\rho_i(x)$$

is a well-defined smooth function, since, in a neighborhood of every point, it is a finite sum of smooth functions.

In order to show that p is **proper**, we need to show that the preimage of any compact subset of $\mathbb{R}$ is again compact. Every compact subset $K \subset \mathbb{R}$ is contained in a closed interval of the form $[-j,j]$ for some natural number j. Hence if we can show that $p^{-1}([-j,j])$ is compact, then $p^{-1}(K)$ is a closed subset of a compact set and therefore also compact.

For given j, if for any x we had $\rho_1(x) = \cdots = \rho_j(x)$, then

$$\sum_{i=j+1}^{\infty} \rho_i(x) = 1$$

and therefore

$$p(x) \geq (j+1) \sum_{i=j+1}^{\infty} \rho_i(x).$$

This shows

$$p^{-1}([-j,j]) \subset \cup_{i=1}^{j} \{x : \rho_i(x) \neq 0\}.$$

Since $\mathrm{supp}(\rho_i) \subset U_i$ and U_i has compact closure, this shows that $p^{-1}([-j,j])$ is a closed subset in a compact set and therefore it is also compact. **QED**

Whitney's Theorem

Every smooth k-dimensional manifold $X \subset \mathbb{R}^N$ admits an embedding into $\mathbb{R}^{2k+1}$.

Recall that the strongest result is that $N = 2k$ suffices. But that is much harder. And again, this is an upper bound which works for every k-dimensional manifold. For a many manifolds, an even lower dimension suffices, e.g. $S^n \subset \mathbb{R}^{n+1}$.

Proof: The idea is to replace the injective immersion $f \colon X \hookrightarrow \mathbb{R}^N$ with the map $(f,p) \colon X \hookrightarrow \mathbb{R}^{N+1}$ with a proper $p \colon X \to \mathbb{R}$. Then (f,p) is still an injective immersion, and it is proper, since p is proper. It remains to reduce the dimension $N+1$. The details are a bit more involved:

Starting with $X \subset \mathbb{R}^N$ we have seen that we can find an injective immersion $f \colon X \to \mathbb{R}^{2k+1}$. By composing f with the injective immersion map

$$\mathbb{R}^{2k+1} \to B_1(0),\ x \mapsto \frac{x}{1+|x|^2},$$

we can assume that $|f(x)| < 1$ for all $x \in X$.

Let $p \colon X \to \mathbb{R}$ be a proper function which we know to exist by the revious lemma. We define a new injective immersion

$$F \colon X \to \mathbb{R}^{2k+2},\ x \mapsto (f(x),p(x)).$$

Since $2k+2 > 2k+1$, we can apply the argument from last time and find a nonzero vector $a \in \mathbb{R}^{2k+2}$ such that

$$\pi \circ F \colon X \to H$$

is still an injective immersion, where π is the projection onto the orthogonal complement $H = \{b \in \mathbb{R}^{2k+2} : b \perp a\}$ of a in $\mathbb{R}^{2k+2}$. By rescaling we can assume $|a| = 1$.

Since $\pi \circ F$ is an injective immersion for almost every $a \in S^{2k+1}$, we **can assume that a is neither the north nor the south pole on S^{2k+1}**. This will allow us tp show that **$\pi \circ F$ is proper**:

Claim: Given any bound c, there exists another number d such that

$$\{x \in X : |(\pi \circ F)(x)| \le c\} \subset \{x \in X : |p(x)| \le d\}.$$

As p is **proper**,

$$\{x \in X : |p(x)| \leq d\} = p^{-1}([-d,d])$$

is a **compact** subset of X.

Thus the claim implies that the preimage under $\pi \circ F$ of every closed ball in H is a compact subset of X. Sicne every compact subset K of H is a closed subset of some closed ball in X, this shows that $(\pi \circ F)^{-1}(K)$ is a closed subset of compact subset in X and therefore also compact.

If the claim is false, then there exists a c and a sequence of points $\{x_i\}$ in X for which

$$|(\pi \circ F)(x_i)| \leq c, \text{ but } |p(x_i)| \to \infty$$

(becasue there is no d bounding $|p(x_i)|$).

By definition of the projection onto an orthogonal complement, for every $z \in \mathbb{R}^{2k+2}$, $\pi(z)$ is the one point in H for which $z - \pi(z)$ is a multiple of a. In particular,

$$F(x_i) - \pi \circ F(x_i) \text{ is a multiple of } a \text{ for each } i,$$

and hence so is the vector

$$w_i := \frac{1}{p(x_i)}(F(X_i) - \pi \circ F(x_i)).$$

Let us look at what happens when $i \to \infty$:

$$\frac{F(x_i)}{p(x_i)} = \left(\frac{f(x_i)}{p(x_i)},1\right) \to (0,\ldots,0,1)$$

because $|f(x_i)| < 1$ for all i and $p(x_i) \to \infty$. We have

$$\left|\frac{\pi \circ F(x_i)}{p(x_i)}\right| \leq \frac{c}{|p(x_i)|}.$$

Thus

$$\frac{\pi \circ F(x_i)}{p(x_i)} \to 0 \Rightarrow w_i \to (0,\ldots,0,1).$$

But each w_i is a multiple of a. Hence the limit of thw w_i must be a mutiple of a. We conclude that a must be either the north or south pole of S^{k+1} which contradicts our assumption on a. This proves the claim and the theorem. **QED**