Repetition

Definition

Consider a Markov chain $\{X_n : n = 0, 1, \ldots\}$ with state space $\{0, 1, \ldots\}$ and transitition probability matrix $\mathbf{P}$. We call $\boldsymbol{\pi} = (\pi_0, \pi_1, \ldots)$ the **limiting distribution** of $\{X_n\}$ if the following holds:

1. $$\pi_j = \lim_{n\to\infty} P_{ij}^{(n)}, \quad j = 0, 1, \ldots,$$

   exist and do not depend on the initial state i.

2. $$\sum_{j=0}^{\infty} \pi_j = 1.$$

Note: For finite state spaces 1. is sufficient, but for infinite state spaces both 1. and 2. are needed.
Note 2: π_j is the probability to be in state j after a large number of steps.

Theorem

Let $\{X_n : n = 0, 1, \ldots\}$ be a regular Markov chain with state space $\{0, 1, \ldots, N\}$ and transition probability matrix $\mathbf{P}$. Then the limiting distribution $\boldsymbol{\pi} = (\pi_0, \pi_1, \ldots, \pi_N)$

1. exists and satisfies (for any initial state i)

$$\pi_j = \lim_{n\to\infty} P_{ij}^{(n)} > 0, \quad j = 0, 1, \ldots, N,$$

   (and also $\sum_{k=0}^{N} \pi_k = 1$).

2. is the unique non-negative solution of the equations

$$\begin{aligned} \pi_j &= \sum_{k=0}^{N} \pi_k P_{kj}, \quad j = 0, 1, \ldots, N, \\ \sum_{k=0}^{N} \pi_k &= 1. \end{aligned}$$

Note: If the Markov chain is not regular, the system of equations may have multiple solutions. Even in the case that the solution is unique, the solution is not guaranteed to be a limiting distribution.

Definition

The transition probability matrix $\mathbf{P}$ is called **doubly stochastic** if $\sum_k P_{ik} = \sum_k P_{kj} = 1$ for all states i and j.

Theorem

Let the Markov chain $\{X_n : n = 0, 1, \ldots\}$ be regular with finite state space $\{0, 1, \ldots, N\}$. If the transition probability matrix $\mathbf{P}$ is doubly stochastic, the limiting distribution is

$$\boldsymbol{\pi} = \left(\frac{1}{N+1}, \frac{1}{N+1}, \ldots, \frac{1}{N+1}\right).$$

Theorem

In a regular Markov chain $\{X_n : n = 0, 1, \ldots\}$, the limiting distribution $\boldsymbol{\pi} = (\pi_0, \pi_1, \ldots, \pi_N)$ gives the **long-run mean fraction of time** spent in each state. I.e.,

$$\pi_j = \lim_{n \to \infty} \mathrm{E}\left[\frac{1}{n} \sum_{k=0}^{n-1} \mathbb{1}\{X_k = j\} \middle| X_0 = i \right]$$

for any initial state i.

Note: This means that π_j describes the proportion of time spent in state j in the long run.