TMA4183
Optimisation II
Spring 2020

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Exercise set 4

$\boxed{1}$ For the following variational problems, determine whether they admit a solution and, if yes, whether the solution is unique. Moreover, derive the associated Euler–Lagrange equation and discuss its solvability. (Note that this may depend on the dimension of the set Ω.)

Here
$$X := \{u \in H^1(\Omega) : u|_\Gamma = g\},$$
where we always assume that X is non-empty, that is, that there exists a function $u \in H^1(\Omega)$ with $u|_\Gamma = g$.

a) The problem
$$\int_\Omega \frac{1}{4}u^4 + \frac{1}{2}|\nabla u(x)|^2\,dx \to \min_{u\in X}.$$

b) The problem
$$\int_\Omega c(x)|\nabla u(x)|^2\,dx \to \min_{u\in X}.$$
Here $c \in C^1(\bar{\Omega})$ satisfies $c(x) > 0$ for all $x \in \bar{\Omega}$.

c) The problem
$$\frac{1}{2}\int_\Omega (1 + u(x)^2)|\nabla u(x)|^2\,dx \to \min_{u\in X}.$$

- *Possible solution:*

a) The functional we want to minimise has the form
$$J(u) = \int_\Omega \varphi(x, u(x), \nabla u(x))\,dx$$
with integrand
$$\varphi(x, t, \xi) = \frac{1}{4}t^4 + \frac{1}{2}|\xi|^2.$$
The integrand is obviously strictly convex (simultaneously in t and ξ) and satisfies the growth condition
$$\varphi(x, t, \xi) \ge \frac{1}{4}t^2 + \frac{1}{2}|\xi|^2 - \frac{1}{4}.$$
Thus the functional J is coercive and weakly lower semi-continuous on $H^1(\Omega)$. Because X is non-empty, closed and convex, it follows that the variational problem has a unique solution.

The formal Lagrange equation for this problem is the PDE
$$\begin{aligned} u^3 - \Delta u &= 0 \quad \text{in } \Omega,\\ u &= g \quad \text{on } \Gamma. \end{aligned}$$

In dimensions $d \leq 4$, the t-derivative of the integrand satisfies the required growth condition

$$|\partial_t \varphi(x,t,\xi)| = |t|^3 \leq |t|^{p-1}$$

with $p \leq 2d/(d-2)$, and the ξ-derivative is linear. Thus the functional J is Gâteaux differentiable (again in dimension $d \leq 4$) and the Euler–Lagrange equation is the first order necessary optimality condition. Since the functional J is strictly convex and attains a minimiser, it follows that the Euler–Lagrange equation has a unique (weak) solution.

b) The continuity of c on $\bar{\Omega}$ together with its positivity imply that there exists $\underline{c} > 0$ such that

$$c(x) \geq \underline{c} > 0$$

for every $x \in \bar{\Omega}$. As a consequence,

$$J(u) = \int_\Omega c(x)|\nabla u(x)|^2\,dx \geq \underline{c}\|\nabla u\|_{L^2}^2$$

for every $u \in X$.

In the lecture, we have only discussed the Poincaré inequality with homogeneous boundary conditions, that is, for $u \in H_0^1(\Omega)$. However, it is easy to derive an inequality for the nonhomogeneous case: Assume to that end that $\hat{u} \in H^1(\Omega)$ is any function satisfying $\hat{u}|_\Gamma = g$. Then $u - \hat{u} \in H_0^1(\Omega)$ for every $u \in X$ and thus we can apply the Poincaré inequality to the difference $u - \hat{u}$ and obtain

$$\begin{aligned}\|\nabla u\|_{L^2} &\geq \|\nabla(u-\hat{u})\|_{L^2} - \|\nabla\hat{u}\|_{L^2}\\ &\geq C_\Omega\|u-\hat{u}\|_{H^1} - \|\nabla\hat{u}\|_{L^2} \geq C_\Omega\|u\|_{H^1} - C_\Omega\|\hat{u}\|_{H^1} - \|\nabla\hat{u}\|_{L^2}.\end{aligned}$$

Using this modified Poincaré inequality, we obtain that

$$J(u) \geq \hat{C}(\|u\|_{H^1}^2 - 1)$$

for some constant $\hat{C}$ and all $u \in X$. Thus J is coercive. Since the integrand is obviously strictly convex, it follows that the functional has a unique minimiser in X.

For the Euler–Lagrange equation we simply obtain the PDE

$$\begin{aligned} -\operatorname{div}\big(c(x)\nabla u(x)\big) &= 0 && \text{in } \Omega,\\ u &= g && \text{on } \Gamma. \end{aligned}$$

Since J is quadratic, it is actually Fréchet differentiable (cf. the discussion of Fréchet differentiability in the lecture) and the Euler–Lagrange equation is the first order necessary optimality condition for the optimisation problem. Because of the strict convexity of J, this has a unique solution.

c) Here we have

$$J(u) = \int_\Omega \varphi(x, u(x), \nabla u(x))\,dx$$

with integrand

$$\varphi(x,t,\xi) = (1+t^2)|\xi|^2.$$

Since $\varphi(x,t,\xi) \geq |\xi|^2$, we obtain coercivity similarly as in problem **b)** from the (modified) Poincaré inequality. Moreover, the integrand φ is convex in its last variable, implying that the functional J is weakly lower semi-continuous. Thus it attains its minimum in X. However, the integrand is *not* convex in (t,ξ),[1] and thus the minimiser need not be unique.

[1] It is convex separately in the t-variable and in the ξ-variable, but not simultaneously in both — this can be easily verified by computing the Hessian of φ.

For the Euler–Lagrange equation we compute the partial derivatives of the integrand as

$$\partial_t \varphi(x,t,\xi) = 2t|\xi|^2,$$
$$\partial_\xi \varphi(x,t,\xi) = 2(1+t^2)\xi,$$

and thus obtain the PDE

$$\begin{aligned} 2u(1+|\nabla u|^2) - 2\operatorname{div}\big((1+u^2)\nabla u\big) &= 0 \quad \text{in } \Omega,\\ u &= g \quad \text{on } \Gamma, \end{aligned}$$

which can be rewritten as

$$\begin{aligned} 2u(1-|\nabla u|^2) - 2(1+u^2)\Delta u &= 0 \quad \text{in } \Omega,\\ u &= g \quad \text{on } \Gamma. \end{aligned} \tag{1}$$

Because of the coupling of t and ξ in the ξ-derivative of φ, the growth condition for this derivative are not met, and therefore we cannot conclude easily that this PDE is actually the first order optimality condition for our variational problems. Thus existence of a solution is not obvious. Moreover, we cannot expect any uniqueness, as the variational problem might have multiple local minima because of its non-convexity.

- *Note:* For these particular problems, one can refine the analysis and show that the Euler–Lagrange equations for the first and third problem are actually solvable in all dimensions (for the third functional, we still have uniqueness issues). To that end, we make use of the fact that for every $K \geq 0$ and $u \in H^1(\Omega)$ the function

$$u_K := \max\{-K, \min\{K, u(x)\}\}$$

(where we cut off all values above K and below $-K$) is again in $H^1(\Omega)$ and satisfies for a.e. x that

$$\nabla u_k(x) = \begin{cases} \nabla u(x) & \text{if } |u(x)| < K,\\ 0 & \text{if } |u(x)| \geq K. \end{cases}$$

Denote now

$$K := \|g\|_\infty.$$

Then we have for every $u \in X$ that the corresponding function u_K still is contained in X, as the boundary values are unchanged. However, it is easy to see that the functional J in all three cases satisfies

$$J(u_K) \leq J(u)$$

with equality if and only if $u_K = u$. In particular, the solutions u of the three variational problems satisfy a maximum principle of the form

$$\|u\|_{L^\infty(\Omega)} \leq \|u|_\Gamma\|_{L^\infty(\Gamma)} = \|g\|_{L^\infty(\Gamma)}.$$

However, this also implies that, for the actual optimisation of J, the integrand $\varphi(x,t,\xi)$ is only relevant for $|t| \leq K$, and we can modify it outside of this region in such a way that the minimum is unchanged but the growth required growth conditions for the Euler–Lagrange equation are satisfied. For instance, we can replace the integrand in the third case by

$$\phi_K(x,t,\xi) = \begin{cases} (1+t^2)|\xi|^2 & \text{if } |t| \leq K,\\ (1+2K|t|-K^2)|\xi|^2 & \text{if } |t| > K. \end{cases}$$

The resulting functional J_K has the same minimisers as J but is Gâteaux differentiable, which implies that the corresponding Euler–Lagrange equation is actually solvable. However, at the solution u the Euler–Lagrange equations for J and J_K coincide, which implies that (1) has a solution.

$\boxed{2}$ Assume that $f \in C^1(\bar{\Omega})$ is a fixed, given function such that $f(x) < 0$ for all $x \in \partial\Omega$. Define moreover

$$X := \{u \in H_0^1(\Omega) : u(x) \geq f(x) \text{ for a.e. } x \in \Omega\}.$$

Show that the *obstacle problem*

$$\min_{u \in X} \frac{1}{2} \int_\Omega |\nabla u(x)|^2 \, dx$$

admits a unique solution.

Try to find optimality conditions for this problem!

- *Possible solution:* The coercivity of the functional follows by means of the Poincaré inequality, which also implies that the obviously convex functional is actually strictly convex. Thus it remains to show that the set X is convex, closed, and non-empty. The convexity and closedness are obvious from the definition of X. For the non-emptyness we use that $f < 0$ on $\partial\Omega$, which implies that the function $\hat{u} := \max\{0, f\}$ satisfies homogeneous Dirichlet boundary conditions. Since this function is also in $H^1(\Omega)$ because f is C^1, it follows that X is non-empty. As a consequence, we have a unique minimiser.

  Since the functional
$$J(u) := \frac{1}{2} \int_\Omega |\nabla u(x)|^2 \, dx$$
is Fréchet differentiable and strictly convex, and the set X is convex, it follows that the solution is uniquely characterised by the first order optimality condition
$$DJ(u)(v - u) \geq 0 \qquad \text{for all } v \in X.$$
(See e.g. Optimisation I, the note on convex optimisation, Proposition 4. Note that the proof remains valid in Hilbert spaces.) For this specific functional, we obtain the condition (*variational inequality*)
$$\int_\Omega \nabla u(x) \cdot \nabla(v(x) - u(x)) \, dx \geq 0 \qquad \text{for all } v \in H_0^1(\Omega) \quad \text{with } v(x) \geq f(x) \text{ for a.e. } x.$$
In the following, we will reformulate this as a more tangible condition. Assuming sufficient regularity of the solution u, we can integrate by parts and obtain
$$-\int_\Omega \Delta u(x)(v(x) - u(x)) \, dx \geq 0 \qquad \text{for all } v \in H_0^1(\Omega) \quad \text{with } v(x) \geq f(x) \text{ for a.e. } x.$$
Assume first that $x_0 \in \Omega$ is such that $u(x_0) > f(x_0)$. Then (if u is continuous) there exists some ball $B_\varepsilon(x_0)$ of radius $\varepsilon > 0$ around x_0 and some $c > 0$ such that $u(x) > f(x) + c$ for all $x \in B_\varepsilon(x_0)$. In particular, every test function v of the form $v = u \pm w$ with w supported in $B_\varepsilon(x_0)$ and $\|w\|_\infty < c$ is admissible, and we obtain that
$$\pm \int_{B_\varepsilon(x_0)} \Delta u(x) w(x) \, dx \geq 0$$
for all such w. This, however, implies that
$$\Delta u(x_0) = 0.$$
Now assume that $x_0 \in \Omega$ is such that $u(x_0) = f(x_0)$. Then we can choose still test functions of the form $v = u + w$ with w supported in $B_\varepsilon(x_0)$ and $w \geq 0$ and obtain that
$$-\int_{B_\varepsilon(x_0)} \Delta u(x) w(x) \, dx \geq 0 \qquad \text{for all } w \geq 0.$$

Letting ε tend to zero, this implies that

$$-\Delta u(x_0) \geq 0,$$

but the opposite inequality need no longer hold.
To summarise, we obtain the conditions

$$\begin{aligned}
\Delta u(x) &= 0 && \text{if } x \in \Omega \text{ and } u(x) > f(x), \\
\Delta u(x) &\leq 0 && \text{if } x \in \Omega \text{ and } u(x) = f(x), \\
u(x) &\geq f(x) && \text{in } \Omega, \\
u(x) &= 0 && \text{on } \Gamma.
\end{aligned}$$