Notes

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Contents

1 Exercise Week 35

1.1 Problem B3.6

Burger equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0. \tag{1}$$

(a) Use the method of characteristics as described in Sect 3.4 to find a formula for the solution $u(t,x)$ given the inital condition

$$u(0,x) = \begin{cases} 0, & x \leq 0 \\ \frac{x}{a}, & 0 < x < a \\ 1, & x \geq a \end{cases}$$

(b) Suppose that $a > b$ and

$$u(0,x) = \begin{cases} a, & x \leq 0, \\ a(1-x) + bx, & 0 < x < 1, \\ b, & x \geq 1 \end{cases}$$

Show that all of th characteristics originating from $x_0 \in [0,1]$ meet at the same point.

1.2 Problem B3.7

Theorem 1.1. *Suppose that $u \in C^1([0,T] \times \Omega)$ is a solution of*

$$\frac{\partial u}{\partial t} + \mathbf{a}(u) \cdot \nabla u = 0$$

For some region $\omega \subset \mathbb{R}^n$ with $\mathbf{a} \in C^1(\mathbb{R};\mathbb{R}^n)$. Then for each $\mathbf{x}_0 \in \Omega$, u is a constant along the characteristic line defined by

$$\mathbf{x}(t) = \mathbf{x}_0 + \mathbf{a}(u(0,\mathbf{x}_0))\,t$$

Let the Hamilton equation be

$$\frac{\partial u}{\partial t} + \frac{1}{2}\left(\frac{\partial u}{\partial x}\right)^2 = 0. \tag{2}$$

Assume that $u \in C^1([0,\infty] \times \mathbb{R}^n)$ is a solution. By analogy with Theorem 1.1, a characteristic of the equation is defined as a solution of

$$\frac{dx}{dt}(t) = \frac{\partial u}{\partial x}(t, x(t)), \quad x(0) = x_0. \tag{3}$$

(a) Assuming that $x(t)$ solves (3), use the chain rule to compute $\frac{d^2x}{dt^2}$.

> **Answer. Haralds Solution**. If we let
>
> $$u_t + \frac{1}{2}u_x^2 = 0$$
>
> and
>
> $$\dot{x} = u_x$$
>
> Then can we write
>
> $$\ddot{x} = u_{xt} + \dot{x}u_{xx} = u_{xt} + u_x u_{xx} = u_{xt} + \frac{1}{2}\left(u_x^2\right)_x$$
>
> I did not get this derivation.

(b) Differentiate (2) with respect to x and then restrict the results to $(t, x(t))$ where $x(t)$ solves (3). Conclude from (a) that to

$$\frac{d^2x}{dt^2} = 0$$

Hense, for some constant v_0 (which depends on the characteristic) ,

$$x(t) = x_0 + v_0 t$$

> **Answer. Haralds Solution.** Derivation of (2) with x gives
>
> $$u_{xt} + u_{xx}u_x = 0.$$
>
> Since $u \in C^2$ is $u_{tx} \approx u_{xt} \approx 0$. So (a) gives us $\ddot{x} = 0$, and that is why $x(t) = x_0 + v_0 t$ der $(x_0,\, v_0$

(c) Show that the Lagrangian derivative of u along $x(t)$ satisfies

$$\frac{Du}{Dt} = \frac{1}{2}v_0^2$$

Implying that

$$(t, x_0 + v_0 t) = u(0, x_0) + \frac{1}{2}v_0^2 t$$

Answer. Harald solution.

$$\begin{aligned}\frac{Du}{Dt} &= \frac{d}{dt}u\left(t,x\left(t\right)\right) = \frac{d}{dt}u\left(t,x_0+v_0t\right) \\ &= u_t+v_0u_x = -\frac{1}{2}u_x^2+u_x^2 \\ &= \frac{1}{2}u_x^2 = \frac{1}{2}v_0^2 \\ &\implies u\left(t,x\left(t\right)\right) = u\left(0,x_0\right)+\frac{1}{2}v_0^2t\end{aligned}$$

Nb! $v_0 = u_x$ evaluated in $t=0$ given $v_0 = u_x\left(0,x_0\right)$

(d) Use this approach to find the solution $u\left(t,x\right)$ under the inital condition

$$u\left(0,x\right) = x^2$$

(For the characteristic starting at $(0,x_0)$, note that you can compute v_0 by evaluation (3)

Answer. Derivation

$$x \approx \lambda \text{ let alt,so}$$

1.3 Problem B4.1

Theorem 1.2. ***Wave Equation*** *is on the form*

$$\frac{\partial^2 u}{\partial t^2} - c^2\frac{\partial^2 u}{\partial x^2} = 0.. \tag{4}$$

Suppose $u\left(t,x\right)$ satisfies (4) for $x\in\mathbb{R}$. Let $\mathcal{P}$ be a parallellogram in the (t,x) plane whose sides are characteristic lines. Show that the valie of u at each vertex $\mathcal{P}$ is determined y the values at the other three vertices.

1.4 Problem 4.2

$$u\left(0,x\right) = g\left(x\right), \quad \frac{\partial u}{\partial t}\left(0,x\right) = h\left(x\right). \tag{5}$$

$$u\left(t,x\right) = \frac{1}{2}\left[g\left(x+ct\right)0g\left(x-ct\right)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} h\left(\tau\right)d\tau. \tag{6}$$

The weave equation (4) is an appropriate model for the longitudal vibrations of a spring. In this application $u\left(t,x\right)$ represents displacement parallel to the spring. Suppose that spring has length l and is free at the ends. This corresponds to the Neumann boundary conditions

$$\frac{\partial u}{\partial x}(t,0)=\frac{\partial u}{\partial x}(t,l)=0, \quad \forall t\geq 0$$

Assume the inital conditions are g and h as in (5), which also satisfu Neumann boundary condition on $[0,l]$. Determine th appropriate extension of g and h from $[0,l]$ to $\mathbb{R}$ so that the solution $u(t,x)$ given by (6) will satisfy Neumann boundary problem for all t.

2 References