Norwegian University of Science and Technology
Deptartment of Mathematical Sciences

TMA4190 Introduction to Topology
Spring 2018

Solutions to exercise set 1

1 Every linear map $\mathbb{R}^n \to \mathbb{R}^m$ is smooth, and the derivative is equal to the map itself. Hence, given a k-dimensional vector subspace V of $\mathbb{R}^N$, it suffices to choose a basis in V to get a linear isomorphism $\phi\colon V \to \mathbb{R}^k$. This map serves as a parametrization, since it is a diffeomorphism.

Now given a linear map $f\colon V \to \mathbb{R}^m$, the composite $\phi \circ f\colon \mathbb{R}^k \to \mathbb{R}^m$ is linear and therefore smooth. Since ϕ is a diffeomorphism, this implies that f must be smooth too.

2 **a)** Let $a > 0$ be a real number. We want to show that the subset

$$H_a = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 - z^2 = a\} \subset \mathbb{R}^3$$

is a 2-dimensional manifold. To do that we need to find local parametrizations. First, there is a diffeomorphism

$$\phi\colon \mathbb{R}^2 \setminus \overline{B_a((0,0))} \to H \cap \{z > 0\},\ (x, y) \mapsto \left(x, y, \sqrt{x^2 + y^2 - a}\right)$$

where $\overline{B_a((0,0))}$ denotes the closed ball of radius a around the origin, i.e.

$$\overline{B_{\sqrt{a}}((0,0))} = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq a\}.$$

Note that $H \cap \{z > 0\}$ is an open subset of H, because it is equal to the intersection of H with the open subset $\{z > 0\} \subset \mathbb{R}^3$. The inverse to ϕ is the projection map

$$\phi^{-1}\colon H \cap \{z > 0\} \to \mathbb{R}^2,\ (x, y, z) \mapsto (x, y).$$

This map is smooth since it can be extended to a smooth map on the whole of $\mathbb{R}^3$ to $\mathbb{R}^2$. To check that ϕ is smooth, we look at its Jacobian matrix in the standard basis

$$\begin{aligned} d\phi_x\colon \mathbb{R}^3 \to \mathbb{R}^2,\ &\begin{pmatrix} \partial\phi_1/\partial x & \partial\phi_2/\partial x & \partial\phi_3/\partial x \\ \partial\phi_1/\partial y & \partial\phi_2/\partial y & \partial\phi_3/\partial y \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & x(x^2 + y^2 - a)^{-1/2} \\ 0 & 1 & y(x^2 + y^2 - a)^{-1/2} \end{pmatrix}. \end{aligned}$$

On the open set $\mathbb{R}^2 \setminus \overline{B_a((0,0))}$, the entries of this matrix are continuously differentiable functions.

The local parametrization for $H \cap \{z < 0\}$ is similarly given by

$$\phi\colon \mathbb{R}^2 \setminus \overline{B_a((0,0))} \to H \cap \{z < 0\},\ (x, y) \mapsto \left(x, y, -\sqrt{x^2 + y^2 - a}\right).$$

It remains to cover the points in $H \cap \{z = 0\}$. We are going to cover those points by the following four open sets together with local parametrizations:

$$B_{\sqrt{a}}((0,0)) \to H \cap \{x^2 + z^2 < a\},\ (x,z) \mapsto \left(x, \sqrt{z^2 - x^2 + a}, z\right)$$

$$B_{\sqrt{a}}((0,0)) \to H \cap \{x^2 + z^2 < a\},\ (x,z) \mapsto \left(x, -\sqrt{z^2 - x^2 + a}, z\right)$$

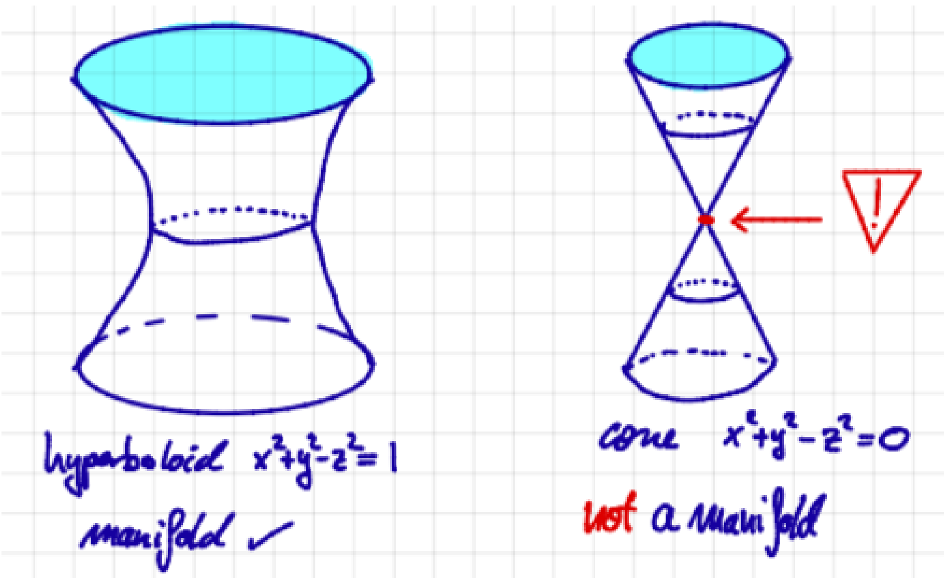
$$B_{\sqrt{a}}((0,0)) \to H \cap \{y^2 + z^2 < a\},\ (y,z) \mapsto \left(\sqrt{z^2 - y^2 + a}, y, z\right)$$

$$B_{\sqrt{a}}((0,0)) \to H \cap \{y^2 + z^2 < a\},\ (y,z) \mapsto \left(-\sqrt{z^2 - y^2 + a}, y, z\right).$$

b) If $a = 0$, then the set

$$H_0 = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 - z^2 = 0\} \subset \mathbb{R}^3$$

is **not a manifold**, since there is no local parametrization around the point $(0,0,0) \in H$. For, assume there was such a local parametrization $\phi\colon U \to V$ with both $U \subseteq \mathbb{R}^2$ and $(0,0,0) \in V \subseteq H$ open. We can assume $\phi((0,0)) = (0,0,0)$. (Remember that V being open in H means that there is an open subset $\tilde{V} \subseteq \mathbb{R}^3$ with $V = H \cap \tilde{V}$. In particular, $\tilde{V}$ must contain a small open ball $B_r((0,0,0))$ around the origin.) Then removing the point $(0,0,0)$ from H splits H and therefore V into two disjoint connected components. But $U \setminus \{(0,0)\} \subset \mathbb{R}^2$ is connected. Hence ϕ cannot be a diffeomorphism (because if it was, $\phi_{|U\setminus\{(0,0)\}}$ would also be a diffeomorphism).



$\boxed{3}$ For $0 < b < a$, let $T(a,b)$ denote the set of points in $\mathbb{R}^3$ at distance b from the circle of radius a in the xy-plane. We can parametrize these points as follows: First the points in the xy-plane which lie on the circle of radius a satisfy

$$(a\cos t, a\sin t, 0) \text{ for } t \in [0, 2\pi).$$

A point in the plane in the direction of a fixed point $(a\cos t, a\sin t, 0)$ which lies on the circle of radius b around the point $(a\cos t, a\sin t, 0)$ has coordinates $(a + b\cos s)\cos t, (a + b\cos s)\sin t, b\sin s)$ with $s \in [0, 2\pi)$. For we have

$$\begin{aligned}
&|\,((a+b\cos s)\cos t, (a+b\cos s)\sin t, b\sin s) - (a\cos t, a\sin t, 0)\,|^2 \\
=&|\,((b\cos s)\cos t, (b\cos s)\sin t, b\sin s)\,|^2 \\
=&b^2\cos^2 s(\cos^2 t + \sin^2 t) + b^2\sin^2 s = b^2.
\end{aligned}$$

Hence the points on $T(a,b) \subset \mathbb{R}^3$ are given by

$$\begin{aligned}
&T(a,b) = \\
&\qquad \{((a+b\cos s)\cos t, (a+b\cos s)\sin t, b\sin s) : s, t \in [0, 2\pi)\}.
\end{aligned}$$

The points on $S^1 \times S^1 \subset \mathbb{R}^4$ are given by

$$\begin{aligned}
S^1 \times S^1 &= \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_1^2 + x_2^2 = x_3^2 + x_4^2 = 1\} \\
&= \{(\cos t, \sin t, \cos s, \sin s) \in \mathbb{R}^4 : t \in [0, 2\pi), s \in [0, 2\pi)\}.
\end{aligned}$$

Now it is clear how we can define a continuous map

$$\begin{gathered}
\phi\colon S^1 \times S^1 \to T(a,b), \\
(\cos t, \sin t, \cos s, \sin s) \mapsto ((a+b\cos s)\cos t, (a+b\cos s)\sin t, b\sin s)\,.
\end{gathered}$$

In order to check that ϕ is smooth, we use the coordinates of $\mathbb{R}^4$ again. Then ϕ is given by

$$(x_1, x_2, x_3, x_4) \mapsto ((a+bx_3)x_1, (a+bx_3)x_2, bx_4)\,.$$

Its derivative in the standard basis at a point $p = (x_1, x_2, x_3, x_4)$ is then given by

$$d\phi_p\colon \mathbb{R}^4 \to \mathbb{R}^3,\ d\phi_p = \begin{pmatrix} a+bx_3 & 0 & bx_1 & 0 \\ 0 & a+bx_3 & bx_2 & 0 \\ 0 & 0 & 0 & b \end{pmatrix}.$$

Since all partial derivatives are smooth maps, ϕ is a smooth map.
Its inverse is the map $\psi\colon \mathbb{R}^3 \to \mathbb{R}^4$ defined by

$$(y_1, y_2, y_3,) \mapsto \left(\frac{y_1}{\sqrt{y_1^2+y_2^2}}, \frac{y_2}{\sqrt{y_1^2+y_2^2}}, \frac{\sqrt{y_1^2+y_2^2}-a}{b}, y_3/b\right).$$

The image of ψ lies in $S^1 \times S^1$ since

$$\left(\frac{y_1}{\sqrt{y_1^2+y_2^2}}\right)^2 + \left(\frac{y_2}{\sqrt{y_1^2+y_2^2}}\right)^2 = \frac{y_1^2+y_2^2}{y_1^2+y_2^2} = 1.$$

To check the other equation, we recall that y_1, y_2 and y_3 are connected by the condition of being on $T(a,b)$ which means

$$\begin{aligned}
&\left(y_1 - \frac{ay_1}{\sqrt{y_1^2+y_2^2}}\right)^2 + \left(y_2 - \frac{ay_2}{\sqrt{y_1^2+y_2^2}}\right)^2 + y_3^2 = b^2 \\
\iff &\left(\sqrt{y_1^2+y_2^2} - a\right)^2 + y_3^2 = b^2 \\
\iff &y_1^2 + y_2^2 + y_3^2 + a^2 - 2a\sqrt{y_1^2+y_2^2} = b^2.
\end{aligned}$$

Now we can calculate

$$\left(\frac{\sqrt{y_1^2+y_2^2}-a}{b}\right)^2 + (y_3/b)^2 = \frac{y_1^2+y_2^2+y_3^2+a^2-2a\sqrt{y_1^2+y_2^2}}{b^2} = 1.$$

Hence the image of (y_1, y_2, y_3) does lie on $S^1 \times S^1$.

We can easily check that $\psi \circ \phi$ is the identity. For example, using $y_1^2+y_2^2 = (a+bx_3)^2$, we get

$$\psi(\phi(x)) = \left(\frac{(a+bx_3)x_1}{\sqrt{(a+bx_3)^2}}, \frac{(a+bx_3)x_2}{\sqrt{(a+bx_3)^2}}, \frac{(a+bx_3)-a}{b}, \frac{bx_4}{b}\right) = (x_1, x_2, x_3, x_4),$$

and, setting $\sqrt{y_{12}} := \sqrt{y_1^2+y_2^2}$,

$$\begin{aligned}\phi(\psi(y)) = &\left(\left(a + b\frac{\sqrt{y_{12}}-a}{b}\right)\frac{y_1}{\sqrt{y_{12}}}, \left(a + b\frac{\sqrt{y_{12}}-a}{b}\right)\frac{y_2}{\sqrt{y_{12}}}, \frac{by_3}{b}\right)\\ =&(y_1, y_2, y_3),\end{aligned}$$

It remains to check that ψ is smooth. The derivative of ψ in the standard basis at a point $q = (y_1, y_2, y_3)$ is then given by

$$d\psi_q \colon \mathbb{R}^3 \to \mathbb{R}^4,\ d\psi_q = \begin{pmatrix} \frac{y_2^2}{(y_1^2+y_2^2)^{3/2}} & -\frac{y_1y_2}{(y_1^2+y_2^2)^{3/2}} & 0 \\ -\frac{y_1y_2}{(y_1^2+y_2^2)^{3/2}} & \frac{y_1^2}{(y_1^2+y_2^2)^{3/2}} & 0 \\ \frac{y_1}{b(y_1^2+y_2^2)^{1/2}} & \frac{y_2}{b(y_1^2+y_2^2)^{1/2}} & 0 \\ 0 & 0 & 1/b \end{pmatrix}.$$

Since $b < a$ we know $y_1^2 + y_2^2 \neq 0$ and all partial derivatives are continuous smooth functions. Hence ψ is smooth. This proves that ϕ is a global diffeomorphism $S^1 \times S^1 \to T(a,b)$ for all $0 < b < a$.

$\boxed{4}$ Let $N = (0, \dots, 0, 1) \in S^k$ be the "north pole" on the k-dimensional sphere. The stereographic projection ϕ_N^{-1} from $S^k \setminus \{N\}$ onto $\mathbb{R}^k$ is the map which sends a point p to the point at which the line through N and p intersects the subspace in $\mathbb{R}^{k+1}$ defined by $x_{k+1} = 0$. (See the picture for $k = 2$.)

a) Let $N = (0, \dots, 0, 1) \in S^k$ be the "north pole" on the k-dimensional sphere. The stereographic projection ϕ_N^{-1} from $S^k \setminus \{N\}$ onto $\mathbb{R}^k$ is the map which sends a point p to the point at which the line through N and p intersects the subspace in $\mathbb{R}^{k+1}$ defined by $x_{k+1} = 0$. In order to get from N to p, we walk in the direction of the vector

$$v := p - N = (x_1, \dots, x_{k+1} - 1).$$

To find $\phi_N^{-1}(p)$ we need to find the real number λ such that the k+1-st coordinate of $N + \lambda \cdot v$ is 0. Hence we need to solve

$$1 + \lambda(x_{k+1} - 1) = 0 \iff \lambda = \frac{1}{1 - x_{k+1}}.$$

Hence

$$\phi_N^{-1}(x_1, \dots, x_{k+1}) = \frac{1}{1 - x_{k+1}}(x_1, \dots, x_k).$$

b) The inverse ϕ_N is given as follows. Given the point $x = (x_1, \dots, x_k) \in \mathbb{R}^k$. To find its image under ϕ_N, we walk from N in the direction of the vector $w = x - N$ until we reach the sphere, i.e. we need to find $\lambda \in \mathbb{R}$ such that

$$N + \lambda \cdot w = (\lambda x_1, \dots, \lambda x_k, 1 - \lambda)$$

lies on S^k. Hence we need to solve

$$\begin{aligned}
&\lambda^2(x_1^2 + \dots + x_k^2) + (1-\lambda)^2 = 1 \\
\iff &\lambda^2|x|^2 + 1 - 2\lambda + \lambda^2 = 1 \\
\iff &\lambda(1 + |x|^2) - 2\lambda = 0 \\
\iff &\lambda = \frac{2}{1 + |x|^2}.
\end{aligned}$$

Thus

$$\phi_N(x) = \frac{1}{1 + |x|^2}\left(2x_1, \dots, 2x_k, |x|^2 - 1\right).$$

We need to check that both ϕ_S and ϕ_S^{-1} are smooth. Since all entries in ϕ_N are smooth (infinitely often differentiable functions in each variable) functions, we get that ϕ_N is smooth. But to check we calculate anyway:

$$d(\phi_N)_x = \frac{2}{(1 + |x|^2)^2}\begin{pmatrix} 1 + |x|^2 - x_1^2 & -2x_1x_2 & -2x_1x_3 & \dots & -2x_1x_k \\ -2x_2x_1 & 1 + |x|^2 - x_2^2 & -2x_2x_3 & \dots & -2x_2x_k \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -2x_kx_1 & \dots & \dots & \dots & 1 + |x|^2 - x_k^2 \\ 2x_1 & 2x_2 & \dots & \dots & 2x_k \end{pmatrix}.$$

Again, all entries are smooth functions and ϕ_N is smooth. Similary for ϕ_N^{-1}, all entries are smooth functions, since $x_{k+1} \neq 1$. Its derivative looks much nicer:

$$d(\phi_N^{-1})_x = \begin{pmatrix} \frac{1}{1-x_{k+1}} & 0 & 0 & \dots & 0 \\ 0 & \frac{1}{1-x_{k+1}} & 0 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots & \\ \frac{x_1}{(1-x_{k+1})^2} & \frac{x_2}{(1-x_{k+1})^2} & \dots & \dots & \frac{x_k}{(1-x_{k+1})^2} \end{pmatrix}.$$

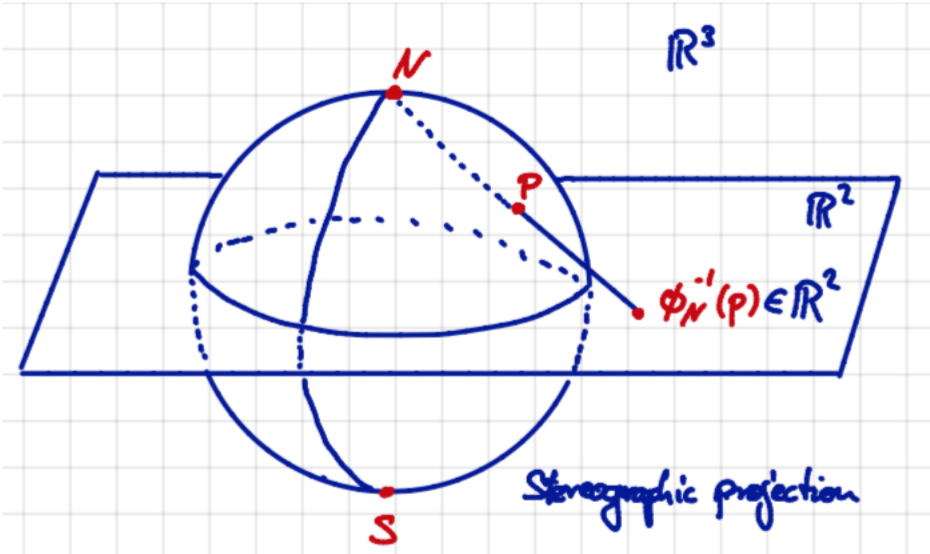
c) The formulae for the projection from the south pole $S = (0, \dots, 0, -1) \in S^k$ are similar:

$$\phi_S^{-1}(x_1, \dots, x_{k+1}) = \frac{1}{1 + x_{k+1}}(x_1, \dots, x_k).$$

and, while we get the same λ,

$$\phi_S(x) = S + \lambda(x - S) = \frac{1}{1 + |x|^2}\left(2x_1, \dots, 2x_k, 1 - |x|^2\right).$$

To check that both ϕ_S and ϕ_S^{-1} are both smooth is completely analogous. Hence, since all points are covered by these two parametrizations, S^k is a smooth manifold.


N
$\mathbb{R}^3$
p
$\mathbb{R}^2$
$\phi_N^{-1}(p) \in \mathbb{R}^2$
S
Stereographic projection