Repetition

Theorem

Assume $\boldsymbol{X} \sim \mathcal{N}_n(\boldsymbol{\mu}, \Sigma)$ and $\mathbf{L}$ is the Cholesky factor of Σ (i.e., $\Sigma = \mathbf{L}\mathbf{L}^{\mathrm{T}}$). Then

1) $\boldsymbol{X} \sim \mathcal{N}_n(\boldsymbol{\mu}, \Sigma)$ implies $\boldsymbol{Z} = \mathbf{L}^{-1}(\boldsymbol{X} - \boldsymbol{\mu}) \sim \mathcal{N}_n(\mathbf{0}, \mathbf{I})$.
2) $\boldsymbol{Z} \sim \mathcal{N}_n(\mathbf{0}, \mathbf{I})$ implies $\boldsymbol{X} = \mathbf{L}\boldsymbol{Z} + \boldsymbol{\mu} \sim \mathcal{N}_n(\boldsymbol{\mu}, \Sigma)$.

Note: This states that all n-dimensional multivariate Gaussian distributions are linear transformations of each other.

Theorem

If

$$\boldsymbol{X} = (\boldsymbol{X}_{\mathrm{A}}, \boldsymbol{X}_{\mathrm{B}}) \sim \mathcal{N}_{n_{\mathrm{A}}+n_{\mathrm{B}}}\left(\begin{bmatrix}\boldsymbol{\mu}_{\mathrm{A}} \\ \boldsymbol{\mu}_{\mathrm{B}}\end{bmatrix}, \begin{bmatrix}\Sigma_{\mathrm{AA}} & \Sigma_{\mathrm{AB}} \\ \Sigma_{\mathrm{BA}} & \Sigma_{\mathrm{BB}}\end{bmatrix}\right),$$

where $\boldsymbol{X}_{\mathrm{A}}$ is n_{A}-dimensional and $\boldsymbol{X}_{\mathrm{B}}$ is n_{B}-dimensional, then

$$\boldsymbol{X}_{\mathrm{A}} | \boldsymbol{X}_{\mathrm{B}} = \boldsymbol{x}_{\mathrm{B}} \sim \mathcal{N}_{n_{\mathrm{A}}}(\boldsymbol{\mu}_{\mathrm{C}}, \Sigma_{\mathrm{C}}),$$

where

$$\begin{aligned}\boldsymbol{\mu}_{\mathrm{C}} &= \boldsymbol{\mu}_{\mathrm{A}} + \Sigma_{\mathrm{AB}}\Sigma_{\mathrm{BB}}^{-1}(\boldsymbol{x}_{\mathrm{B}} - \boldsymbol{\mu}_{\mathrm{B}}) \\ \Sigma_{\mathrm{C}} &= \Sigma_{\mathrm{AA}} - \Sigma_{\mathrm{AB}}\Sigma_{\mathrm{BB}}^{-1}\Sigma_{\mathrm{BA}}.\end{aligned}$$

Note: Calculating conditional distributions is just linear algebra.
Note 2: Used **a lot** in everything from simple to complex models, but, in practice, compatitons are done by computers.

Simulation from $\mathcal{N}_n(\boldsymbol{\mu}, \Sigma)$

Input:

- n: dimension
- $\boldsymbol{\mu}$: mean vector
- Σ: covariance matrix

Algorithm:
1. calculate Cholesky factorization $\Sigma = \mathbf{L}\mathbf{L}^{\mathrm{T}}$.
2. for $i = 1 \ldots n$
3. draw $z_i \sim \mathcal{N}(0, 1)$
4. end
5. set $\boldsymbol{x} = \mathbf{L}\boldsymbol{z} + \boldsymbol{\mu}$

Output: $\boldsymbol{x}$ is a simulation from $\mathcal{N}_n(\boldsymbol{\mu}, \Sigma)$.

Definition

The stochastic process $\{B(t) : t \geq 0\}$ with state space $\mathbb{R}$ is called **Brownian motion** with **variance parameter** $\sigma^2 > 0$ if

1) $B(s+t) - B(s) \sim \mathcal{N}(0, t\sigma^2)$ for $s \geq 0$ and $t > 0$.

2) for $0 \leq t_1 < t_2 \leq t_3 < t_4$,

$$B(t_2) - B(t_1) \quad \text{and} \quad B(t_4) - B(t_3)$$

are independent.

3) $B(0) = 0$ (and the realizations are continuous).

Note: Point 2) is equivalent to independent increments when increments follow Gaussian distributions.