TMA 4190 Introduction to Topology

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Lecture 04[1]

4. Tangent spaces and derivatives

Let us get back to the derivative of a smooth map $f\colon \mathbb{R}^n \to \mathbb{R}^m$. Let x be a point in the domain of f and $h \in \mathbb{R}^n$ be any vector in $\mathbb{R}^n$. Then the **derivative of f** in the direction h can be defined as the limit

$$df_x(h) = \lim_{t\to 0} \frac{f(x+th) - f(x)}{t}.$$

Hence for a fixed x, the derivative is a map

$$df_x\colon \mathbb{R}^n \to \mathbb{R}^m$$

sending a vector $h \in \mathbb{R}^n$ to the vector $df_x(h) \in \mathbb{R}^m$. In Calculus we learned that this map is **linear** (which means $df_x(h+g) = df_x(h) + df_x(g)$ and $df_x(\lambda h) = \lambda df_x(h)$ for all $h,g \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$). Note that **df_x is defined on all of $\mathbb{R}^n$ even if f is not**.

The derivative is a linear approximation

The derivative of f is a map on its own. We think of the parallel translate of df_x to x, i.e. $h \mapsto x + df_x(h)$ as the best **linear approximation** of f at x.

Note that if $f = L\colon U \to \mathbb{R}^m$ is a itself a **linear map**, then

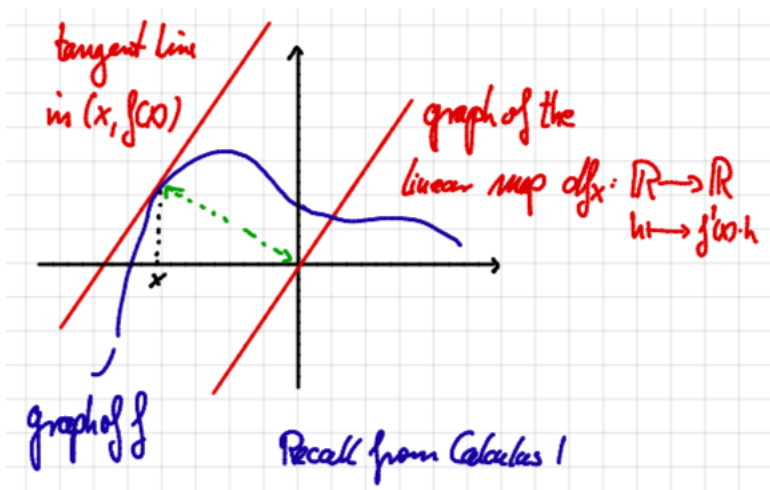
$$df_x = L \text{ for all } x \in U.$$

In particular, the derivative of the inclusion map $U \subseteq \mathbb{R}^n$ at any point $x \in U$ is the identity map on $\mathbb{R}^n$.

[1] Following the books of Guillemin and Pollack: Differential Topology, and Milnor: Topology from the differentiable viewpoint.

df_x and the tangent line

In Calculus 1, we visualized the derivative by saying that $f'(x)$ is the slope of the tangent line at the graph of f at the point $(x,f(x))$. But the derivative $f'(x)$ **really is the linear map** $df_x \colon \mathbb{R} \to \mathbb{R}$ given by multiplying with the real number $f'(x)$. The tangent line at $(x,f(x)$ corresponds to the parallel translate of the linear map df_x, whose graph is the line through the origin with slope $f'(x)$.
We observe that, in order to get a **vector space**, the tangent space to the graph of f at $(x,f(x)$ is the **image of** $\mathbb{R}$ **under** df_x **in** $\mathbb{R}^2$.



We are going to use this picture of parallel translates to define the tangent space of a manifold at a point.

Let $X \subseteq \mathbb{R}^N$ be k-dimensional manifold and $x \in X$ a point. Let $\phi \colon U \to X$ be a **local parametrization around** x (i.e. there is an open subset $V \subseteq X$ containing x and an open subset $U \subseteq \mathbb{R}^k$ together with a diffeomorphism $\phi \colon U \to V$; we then also write $\phi \colon U \to X$ for the composite $U \xrightarrow{\phi} V \hookrightarrow X$).
We assume $\phi(0) = x$.

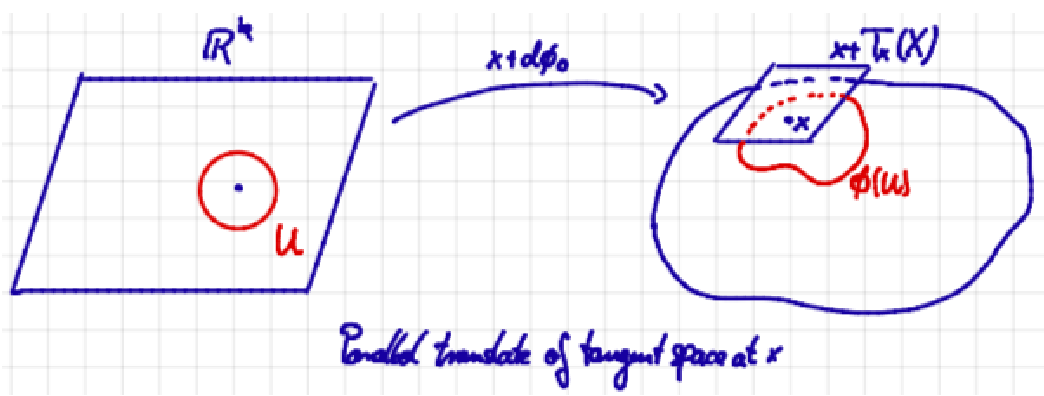
Tangent space

Then the **best linear approximation** to $\phi \colon U \to X$ at 0 is the map

$$u \mapsto \phi(0) + d\phi_0(u) = x + d\phi_0(u).$$

We define the **tangent space** $T_x(X)$ of X at x to be **the image of the linear map** $d\phi_0\colon \mathbb{R}^k \to \mathbb{R}^N$. Note that $T_x(X)$ is a **vector subspace of** $\mathbb{R}^N$.
Its parallel translate $x + T_x(X)$ **is the best linear approximation to** X **through the point** x.

By this definition, a **tangent vector** to $X \subseteq \mathbb{R}^N$ at x is a point $v \in \mathbb{R}^N$ that lies in the vector subspace $T_x(X)$ of $\mathbb{R}^N$. However, we usually picture v geometrically as the arrow running from x to $x+v$ in the translate $x+T_x(X)$.



In order to define $T_x(X)$ we made a **choice** of a parametrization ϕ. We have to check what happens if we choose a different parametrization. Are we getting the same tangent space?

$T_x(X)$ is well-defined

So let $\psi\colon V \to X$ be **another local parametrization** around x with $\psi(0) = x$. By shrinking both U and V we

$$\text{can assume } \phi(\mathbf{U}) = \psi(\mathbf{V})$$

(replace U by $\phi^{-1}(\phi(U) \cap \psi(V)) \subset U$
and V by $\psi^{-1}(\phi(U) \cap \psi(V)) \subset V$). Then the map

$$\theta := \psi^{-1} \circ \phi\colon U \to V$$

is a diffeomorphism (its the composite of two diffeomorphisms). By definition of θ, we have $\phi = \psi \circ \theta$. Differentiating yields

$$d\phi_0 = d\psi_0 \circ d\theta_0$$

(where we have used the chain rule). This implies that the image of $d\phi_0$ is contained in the image of $d\psi_0$:

$$\mathbf{d\phi_0(\mathbb{R}^k) \subseteq d\psi_0(\mathbb{R}^k) \text{ in } \mathbb{R}^N}.$$

By switching the roles of ϕ and ψ in the argument, we also get:

$$\mathbf{d\psi_0(\mathbb{R}^k) \subseteq d\phi_0(\mathbb{R}^k) \text{ in } \mathbb{R}^N}.$$

Hence $d\phi_0(\mathbb{R}^k) = d\psi_0(\mathbb{R}^k)$ in $\mathbb{R}^N$. This shows that whatever local parametrization around x we start with, the vector subspace $T_x(X) \subseteq \mathbb{R}^N$ is always the same. In mathematical terms we say that **$T_x(X)$ is well-defined**.

Dimension of $T_x(X)$

If X is a k-dimensional manifold, then **$T_x(X)$ is a k-dimensional vector space over $\mathbb{R}$**. (For we know from Calculus that the derivative of a diffeomorphism is a linear isomorphism. Hence $d\phi_0$ is an isomorphism of vector spaces $d\phi_0\colon \mathbb{R}^k \xrightarrow{\cong} T_x(X)$.)

Example: Tangent space at the unit circle

Let $p = (a,b) \in S^1$ be a point with $b > 0$. A local parametrization around p with $\phi(0) = p$ is given by

$$\phi\colon (-\epsilon,\epsilon) \to S^1,\ x \mapsto (t + a,\ \sqrt{1 - (x + a)^2}).$$

The derivative at x is the linear map

$$d\phi_x \colon \mathbb{R} \to \mathbb{R}^2,\ d\phi_x = (1, -\frac{x+a}{\sqrt{1-(x+a)^2}}).$$

Hence the image of $\mathbb{R}$ under $d\phi_0$ in $\mathbb{R}^2$ is the line spanned by $(-b,a)$ (writing $b = \sqrt{1-a^2}$).

Example: Tangent space at S^2

Let $p = (a,b,c)$ be point on S^2 which is not the north pole. Then we use the stereographic projection $\phi_N \colon \mathbb{R}^2 \to S^2$ as a local parametrization. (We do not need to translate first to get $\phi_N(0) = p$. That is up to us.)
Recall that

$$\phi_N(x,y) = \frac{1}{1+x^2+y^2}\left(2x,2y,x^2+y^2-1\right).$$

The derivative at (x,y) is the linear map $d\phi_N \colon \mathbb{R}^2 \to \mathbb{R}^3$ defined by the matrix (in the standard basis):

$$d(\phi_N)_{(x,y)} = \frac{2}{(1+x^2+y^2)^2}\begin{pmatrix} 1-x^2+y^2 & -2xy \\ -2xy & 1+x^2-y^2 \\ 2x & 2y \end{pmatrix}.$$

The image of $\mathbb{R}^2$ under the linear map $d(\phi_N)_{(x,y)}$ is the tangent space $T_{\phi_N(x,y)}S^2$. This image is spanned by the two column vectors of the matrix $d(\phi_N)_{(x,y)}$. Let us check that we get the space we would have expected, i.e. a plane which is orthogonal to the vector $\phi_N(x,y)$ (neglecting the first factors):

$$\begin{aligned}
&\left(2x,2y,x^2+y^2-1\right)\cdot\begin{pmatrix} 1-x^2+y^2 \\ -2xy \\ 2x \end{pmatrix} \\
&= 2x(1-x^2+y^2) - 2xy^2 + 2x(x^2+y^2-1) \\
&= 2x - 2x^3 + 2xy^2 - 4xy^2 + 2x^3 + 2xy^2 - 2x \\
&= 0.
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\left(2x,2y,x^2+y^2-1\right)\cdot\begin{pmatrix}-2xy\\1+x^2-y^2\\2y\end{pmatrix}\\
&=-4x^2y+2y(1+x^2-y^2)+2y(x^2+y^2-1)\\
&=-4x^2y+2y+2x^2y-2y^3+2x^2y+2y^3-2y\\
&=0.
\end{aligned}$$

Hence the plane spanned by the column vectors is orthogonal to $\phi_N(x,y)$.

The induced derivative

Now let $f\colon X\to Y$ be a smooth map from a k-dimensional smooth manifold $X\subseteq\mathbb{R}^N$ to an l-dimensional smooth manifold $Y\subseteq\mathbb{R}^M$. We would like to define a map **best linear approximation of f at a point x**. For $y=f(x)$, this should result in a **linear map** of vector spaces

$$T_x(X)\to T_y(Y).$$

Suppose that $\phi\colon U\to X$ is a local parametrization around x with $U\subseteq\mathbb{R}^k$, and $\psi\colon V\to Y$ a local parametrization around y with $V\subseteq\mathbb{R}^l$. We can assume $\phi(0)=x$ and $\psi(0)=y$. Then we define a map $\theta\colon U\to V$ by the following commutative diagram (which means that it does not matter which way we walk around from U to Y):

$$\begin{tikzcd}
X \arrow[r, "f"] & Y \\
U \arrow[u, "\phi"] \arrow[r, "\theta=\psi^{-1}\circ f\circ\phi"'] & V. \arrow[u, "\psi"']
\end{tikzcd}$$

Define df_x

Taking derivatives yields a diagram of linear maps and we define df_x to be **the linear map which makes the diagram commutative**:

$$\begin{tikzcd}
T_x(X) \arrow[r, dotted, "df_x"] & T_y(Y) \\
\mathbb{R}^k \arrow[u, "d\phi_0"] \arrow[r, "d\theta_0"'] & \mathbb{R}^l. \arrow[u, "d\psi_0"']
\end{tikzcd}$$

Since $d\phi_0$ is an isomorphism, we have to **define** df_x as

$$\mathbf{df_x := d\psi_0 \circ d\theta_0 \circ d\phi_0^{-1}}.$$

We call df_x also the **derivative of f at x**.

Again, we need to check that our definiton of df_x does not depend on the choices of parametrizations. This is left as an exercise. (See below.)

Tangent space of products

Given two smooth manifolds $X \subseteq \mathbb{R}^N$ and $Y \subseteq \mathbb{R}^M$ and points $x \in X$, $y \in Y$, then the tangent space of the product X and Y is the product of the tangent spaces, i.e.

$$T_{(x,y)}(X \times Y) = T_x(X) \times T_y(Y).$$

This follows from the fact that we can choose neighborhoods in $X \times Y$ by taking the product of neighborhoods in X and Y, respectively.
Moreover, it is easy to check that if $: X \to X'$ and $g\colon Y \to Y'$ are smooth maps, then the derivative of the prdoct map is the product of the derivatives, i.e.

$$d(f \times g)_{(x,y)} = df_x \times dg_y$$

for all $(x,y) \in X \times Y$.

Finally, we would like to have that the new derivative satisfies the chain rule. So let $g\colon Y \to Z$ be another smooth map. Let $\eta\colon W \to Z$ be a local parametrization around $z = g(y)$ with an open subset $W \subseteq \mathbb{R}^m$ and $\eta(0) = z$. Then we have a commutative diagram

$$\begin{tikzcd}
X \arrow[r, "f"] & Y \arrow[r, "g"] & Z \\
U \arrow[u, "\phi"] \arrow[r, "\theta=\psi^{-1}\circ f\circ\phi"'] & V \arrow[u, "\psi"'] \arrow[r, "\iota=\eta^{-1}\circ g\circ\psi"'] & W \arrow[u, "\eta"']
\end{tikzcd}$$

which gives us the commutative square

$$\begin{tikzcd}
X \arrow[r, "g\circ f"] & Z \\
U \arrow[u, "\phi"] \arrow[r, "\iota\circ\theta"'] & W. \arrow[u, "\eta"']
\end{tikzcd}$$

Thus, by definition,

$$d(g \circ f)_x = d\eta_0 \circ d(\iota \circ \theta)_0 \circ d\phi_0^{-1}.$$

The Chain Rule from Calculus 2 for maps of open sets of Euclidean spaces, then gives

$$d(\iota \circ \theta)_0 = (d\iota_0) \circ (d\theta_0).$$

Thus

$$d(g \circ f)_x = (d\eta_0 \circ d\iota_0 \circ d\psi_0^{-1}) \circ (d\psi_0 \circ d\theta_0 \circ d\phi_0^{-1}) = dg_y \circ df_x.$$

Hence we have in fact the desired rule.

Chain Rule

If $X \xrightarrow{f} Y \xrightarrow{g} Z$ are smooth maps of manifolds, then

$$d(g \circ f)_x = dg_{f(x)} \circ df_x.$$

Let $\phi' \colon U \to X$ and $\psi' \colon V' \to Y$ be another choice of local parametrizations around x and y, respectively. Again by shrinking both U and U', both V and V' accordingly we can assume that $\phi(U) = \phi'(U') \subseteq X$ and $\psi(V) = \psi'(V') \subseteq Y$. Then $d\phi_0$ and $d\phi'_0$ differ by a linear isomorphism of $\mathbb{R}^k$, say α: $d\phi_0 = d\phi'_0 \circ \alpha$. Similarly, there is a linear isomorphism β of $\mathbb{R}^l$ such that $d\psi_0 = d\psi'_0 \circ \beta$. Let $\theta' \colon U \to V$ be defined similarly to θ, i.e. $\theta' = \psi'^{-1} \circ f \circ \phi'$. This gives us the following diagram in which each square commutes

$$\begin{tikzcd}
T_x(X) \arrow[r, dotted, "df_x"] & T_y(Y) \\
\mathbb{R}^k \arrow[u, "d\phi'_0"'] \arrow[r, "d\theta'_0"'] & \mathbb{R}^l \arrow[u, "d\psi'_0"] \\
\mathbb{R}^k \arrow[u, "\alpha"'] \arrow[uu, bend left, "d\phi_0"] \arrow[r, "d\theta_0"'] & \mathbb{R}^l. \arrow[u, "\beta"] \arrow[uu, bend right, "d\psi_0"']
\end{tikzcd}$$

Hence we get the desired identity

$$d\psi'_0 \circ d\theta'_0 \circ {d\phi'_0}^{-1} = d\psi_0 \circ d\theta_0 \circ d\phi_0^{-1} = df_x.$$

For more examples of tangent spaces have a look at the exercises.