8. Lecture VIII: Embedded Submanifolds of $\mathbb{R}^d$

We shall now delve deeper into the qualitative local theory of autonomous ODEs and understand in exactly what way the linear system

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x}(t) = \mathrm{D}f(\mathbf{x}_0)\mathbf{x}$$

approximates the system

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x}(t) = f(\mathbf{x}(t))$$

near a hyperbolic critical point. This discussion will be quite geometrical and we shall acquire some language before the undertaking.

8.1. **Manifolds.** In this subsection we shall be discussing some tools that shall allow us to talk about the dynamics of one system approximating the dynamics of another system, as well as allow us to write down a structure theorem for nonlinear autonomous dynamical systems around critical points, similar to Theorem 3.3 for linear systems. This subsection is not per se examinable.

Without digressing inordinately to define and familarize ourselves with the notion of topology and a topological space in general, let us simply say that one can think of a "topology" as the structures on a set that characterize, or are characterized by, how convergence takes place. We are generally only concerned with $\mathbb{R}^d$ in this module, and when we need to consider convergence on other spaces (such as $C(J)$ or the space of matrices (which is essentially $\mathbb{R}^{d\times d}$)), we shall make a digression for that particular space.

The important thing to notice here is that if $f : \mathbb{R}^d \to \mathbb{R}^n$ is a continuous function, then that continuity is connected with how sequences converge in one space as compared to how they converge in the other space. Suppose $\mathbf{x}_n \to \mathbf{x}$ is a converging sequence in $\mathbb{R}^d$. Then the continuity of f implies the convergence of $\mathbf{y}_n = f(\mathbf{x}_n)$ to $\mathbf{y} = f(\mathbf{x})$. Conversely, the continuity of f is implied by the property that it transfers convergence from one space to the other. After all, continuity means that if the pre-image changes a little, then the image also changes just a little. Topology lets us talk about "nearness" on a space.

Proof. If $\mathbf{x}_n \to \mathbf{x}$, then for every $\delta > 0$, there is an N such that $n > N$ implies $|\mathbf{x}_n - \mathbf{x}| < \delta$.

If f is continuous, then for every $\varepsilon > 0$, there is a $\delta > 0$ such that $|\mathbf{x}-\mathbf{y}| < \delta$ implies $|f(\mathbf{x})-f(\mathbf{y})| < \varepsilon$.

Therefore, fixing $\varepsilon > 0$, we can find a large enough N such that $|\mathbf{x}_n-\mathbf{x}| < \delta$, and $|f(\mathbf{x}_n)-f(\mathbf{x})| < \varepsilon$. This implies that $f(\mathbf{x}_n) \to f(\mathbf{x})$. $\square$

Therefore, the structures that allow for convergence — the topology — can be characterized thus: we say that two topological spaces X and Y are topologically the same, or HOMEOMORPHIC, if there is a continuous map $H : X \to Y$ that is invertible, and its inverse $H^{-1} : Y \to X$ is also continuous. Such a map is called a HOMEOMORPHISM. On $\mathbb{R}^d$ we can put more structures on H. A C^k-DIFFEOMORPHISM is a k-times continuously differentiable map with an inverse that is also k-times continuously differentiable. If k is unbounded, we say that the diffeomorphism is SMOOTH.

On $\mathbb{R}^d$, $\mathbf{x}_n \to \mathbf{x}$ if on any open ball $B_\delta(\mathbf{x})$, one finds that $\mathbf{x}_n \in B_\delta(\mathbf{x})$ if $n > N$ for some $N = N(\delta)$. Therefore, the collection of open balls on $\mathbb{R}^d$ is the structure that controls convergence, or nearness. All $\mathbf{x}_n$ with $n > N$ are δ-near to $\mathbf{x}$. An arbitrary union of open balls is an open set, and not unreasonably an open set containing $\mathbf{x}$ is known as a NEIGHBOURHOOD of $\mathbf{x}$.

Heuristically speaking, an m-dimensional manifold is a topological space M that *looks locally like* $\mathbb{R}^m$. That is, around each point $p \in M$, there is a neighbourhood $U \subseteq M$ and a homeomorphism $\varphi_U : U \to V \subseteq \mathbb{R}^m$ (φ is continuous and invertible and its inverse is continuous), where V is a neighbourhood in $\mathbb{R}^m$. Homeomorphisms defined on overlapping neighbourhoods are related in ways that encode how M is patched together from little pieces of $\mathbb{R}^d$, much like a globe is patched together with flat strips and two caps.

In this module we shall not be needing an intrinsic definition of a manifold. Insofar as we are concerned, we shall use the word "manifold" to mean a level set of a continuous function $h : \mathbb{R}^d \to \mathbb{R}^n$ because all our manifolds are naturally embedded in phase space, which is $\mathbb{R}^d$. That is, by MANIFOLD, we shall mean a subset $X \subseteq \mathbb{R}^d$ defined by a C^1 function $h : \mathbb{R}^d \to \mathbb{R}^n$ of constant rank k, and a constant $\mathbf{c} \in \mathbb{R}^n$ thus:

$$X = \{\mathbf{x} \in \mathbb{R}^d : h(\mathbf{x}) = \mathbf{c}\},$$

and inheriting the metric properties of $\mathbb{R}^d$ as a subspace. The RANK of a function $h : \mathbb{R}^d \to \mathbb{R}^n$ at the point $p \in \mathbb{R}^d$ is the the dimension of the column space of its gradient, $\mathrm{D}h$, which in geometry is also variously denoted ∇h, grad$h$, dh, or h_*.

By the zero map $\mathbf{Z} : \mathbb{R}^d \to \mathbb{R}$, given by $\mathbf{Z}(\mathbf{x}) = 0$, it is immediate that $\mathbb{R}^d$ itself is a manifold — as it must be even by the more general heuristic definition of a manifold earlier given — $\mathbb{R}^d$ looks (globally and) locally like $\mathbb{R}^d$.

To get a quick idea of what X must look like and why the constant rank condition is important, we need look no further than the rank theorem:

Theorem 8.1. *Let M and N be C^1-manifolds of dimensions m and n, respectively, and let $F : M \to N$ be a C^1-map of constant rank $k < m$. For each $p \in M$, there exist C^1-diffeomorphisms $\varphi : M \to M$ defined on a neighbourhood of p and $\psi : N \to N$ defined on a neighbourhood of $F(p)$ such that*

$$\psi \circ F(\varphi^{-1}(p)) = ((\varphi^{-1})^1(p), (\varphi^{-1})^2(p), \ldots, (\varphi^{-1})^k(p), 0, \ldots, 0).$$

From this theorem it is immediate that

Corollary 8.2. *If $h : \mathbb{R}^d \to \mathbb{R}^n$ is a C^1-map of constant rank k, then each level set of h is a manifold of codimension k.*

The rank theorem is a ready consequence of the the Inverse Function Theorem below. These theorems will then also be useful for helping us understand how to associate linearized dynamics with the full dynamics of a nonlinear system.

Theorem 8.3 (Inverse Function Theorem)**.** *Let U and V be open subsets of $\mathbb{R}^n$ and $F : U \to V$ be a C^1-map. If $\mathrm{D}F(p)$ is non-singular at some point $p \in U$ then there are connected neighbourhoods $U_0 \subseteq U$ of p and $V_0 \subseteq V$ of $F(p)$ such that $F|_{U_0} : U_0 \to V_0$ is a C^1-diffeomorphism.*

Whlist we shall not prove this theorem here, let us mention that it can be deduced with the aid of the contraction mapping principle. Related to the Inverse Function Theorem is the Implicit Function Theorem:

Theorem 8.4 (Implicit Function Theorem)**.** *Let $U \subseteq \mathbb{R}^n \times \mathbb{R}^k$ be an open set, let $(\mathbf{x}, \mathbf{y}) = (x^1, \ldots, x^n, y^1, \ldots y^k)$ be the standard coordinates on U. Suppose $\Phi : U \to \mathbb{R}^k$ is a C^1-map, $(\mathbf{a}, \mathbf{b}) \in U$, and $\Phi(\mathbf{a}, \mathbf{b}) = \mathbf{c}$. If the matrix $\mathrm{D}\Phi(\mathbf{a}, \mathbf{b})$ is non-singular, then there exist neighbourhoods $V_0 \subseteq \mathbb{R}^n$ of $\mathbf{a}$ and $W_0 \subseteq \mathbb{R}^k$ of $\mathbf{b}$ and a smooth map $F : V_0 \to W_0$ such that*

$$\Phi(\mathbf{x}, \mathbf{y}) = \mathbf{c} \quad \leftrightarrow \quad \mathbf{y} = F(\mathbf{x}).$$

Finally we need to introduce the tangent space. Let M be a manifold of dimension n defined by $\{\mathbf{x} \in \mathbb{R}^d : h(\mathbf{x}) = \mathbf{c}\}$, where $h : \mathbb{R}^d \to \mathbb{R}^{d-n}$ is a C^1 map of constant rank. Let $p \in M$. The TANGENT SPACE of M at p is denoted by T_pM. Naturally all the vectors that can be imagined as having one end at p and "lying in M" would be directions along which h does not change, because M is by definition a level set. Therefore, we can identify the tangent space T_pM with

$$T_pM = \{\mathbf{y} \in \mathbb{R}^d : \mathbf{y} - p \in \ker \mathrm{D}h\}.$$

This is reduces to the usual definition of a tangent space if we take M to be of codimension one, or in fact, a tangent plane if $d = 3$, and $n = 2$. Since $\mathrm{D}h$ is zero in the n directions of M and nonzero in the remaining $d - n$ dimensions, we see that T_pM is in fact isomorphic to the vector space $\mathbb{R}^n$.