Hints for Project 1, Problem X

September 8, 2020

The proof we are asking for can be divided into two parts

1. Show that if $\|\delta A\| \cdot \|A^{-1}\| < 1$ then $A + \delta A$ is invertible. This would be the same as saying that $A + \delta A$ can only be singular if $\|\delta A\| \geq \|A^{-1}\|^{-1}$
2. Construct a matrix δA with $\|\delta A\| = \|A^{-1}\|^{-1}$ such that $A + \delta A$ is singular.

A candidate for δA is the rank one matrix $\delta A = -\|A^{-1}\|^{-1} x y^T$ where

$$\|x\| = 1, \quad \|y\|_* := \max_{z \neq 0} \frac{|y^T z|}{\|z\|} = 1 \quad \text{and } \|A^{-1}\| = y^T A^{-1} x$$

One needs to prove that such x and y can be found, but you can postpone that to begin with. It just remains to prove the two properties that this δA should have, namely that its norm is $\|A^{-1}\|^{-1}$ and that $A + \delta A$ is singular, for the latter, try to compute $(A + \delta A) \cdot A^{-1} x$.