Collision of characteristics

Harald Hanche-Olsen

Time to collision

Here we consider an IVP (initial value problem) for a quasilinear equation:

$$u_t + a(u)u_x = 0, \qquad u(0,x) = g(x) \tag{1}$$

where the PDE is supposed to hold for $t > 0$, and $g\colon \mathbb{R} \to \mathbb{R}$ is given. For simplicity, we will assume that a and g are C^1 functions.

The characteristic equations are

$$\dot{x}(t) = a\big(u(t)\big), \qquad \dot{u}(t) = 0.$$

By the second equation, u is constant along any characteristic, and hence so is $\dot{x}$, by the first equation. Thus $x(t)$ has the form $x(t) = ct + \xi$ for constants c and ξ. Setting $t = 0$ and recalling that $u(t)$ should really be $u\big(t, x(t)\big)$, we obtain $c = \dot{x}(t) = \dot{x}(0) = a\big(u(0, x(0))\big) = a\big(g(\xi)\big)$. Writing

$$c(\xi) = a\big(g(\xi)\big),$$

we conclude that the characteristics have the form

$$x = c(\xi)t + \xi, \tag{2}$$

and since u is constant along this characteristic, we must have

$$u(t,x) = g(\xi). \tag{3}$$

To find $u(t,x)$ from (3), we need to solve (2) with respect to ξ for given (t,x). Taking the derivative in (2), we get

$$\frac{\partial x}{\partial \xi} = 1 + tc'(\xi).$$

If $c'(\xi) \ge 0$ for all ξ, it is clear that (2) can be solved with respect to ξ for all $x \in \mathbb{R}$ and $t \ge 0$: First, this implies $\partial x/\partial\xi \ge 1$, so x is not only an increasing function of ξ, implying that there is at most one solution – but also $x \to \pm\infty$ when $\xi \to \pm\infty$, so the intermediate value theorem from calculus implies the *existence* of a solution ξ for every x.

If $c'(\xi) < 0$ for some ξ, however, then we cannot do this when t is too large. Clearly, the critical time in this case is

$$\tau = \frac{-1}{\inf\limits_{\xi\in\mathbb{R}} c'(\xi)}.$$

When $0 < t < \tau$, we can solve (2) for ξ, while when $t > \tau$, we cannot.

Thus τ is the first time of collision of the characteristics, after which there is no longer a classical solution.