NTNU
Norwegian University of
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Department of Mathematical Sciences

Examination paper for **TMA4265 Stochastic Modeling**

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Examination date: August 8, 2018
Examination time (from–to): 09:00-13:00
Permitted examination support material: C:

- Calculator CITIZEN SR-270X, CITIZEN SR-270X College, HP30S, Casio fx-82ES PLUS with empty memory.
- Tabeller og formler i statistikk, Tapir forlag.
- K. Rottmann: Matematisk formelsamling.
- Bilingual dictionary.
- One yellow, stamped A5 sheet with own handwritten formulas and notes (on both sides).

Other information:
Note that all answers must be justified.
All ten subproblems are equally weighted.

Language: English
Number of pages: 3
Number of pages enclosed: 4

Checked by:

Informasjon om trykking av eksamensoppgave	
Originalen er:	
1-sidig ☐	**2-sidig** ☒
sort/hvit ☒	**farger** ☐
skal ha flervalgskjema ☐	

Date Signature

Problem 1

We look at the development of a non-deadly disease after diagnosis. Denote the disease state of a patient at time t by X_t. Here, $X_t \in \{1, 2, 3, 4\}$, where 1 represents healthy, 2 is sick, 3 is very sick, and 4 is extremely sick. The disease is always detected in the extremely sick state, so at time $t = 0$ we set $X_0 = 4$. When a patient gets medicine, there is a tendency of getting better. Patients are monitored every 6 months, and this indicates the time step, so $t = 1$ is 6 months after detection, $t = 2$ is one year after detection, and so on. The transition dynamics are modeled by a Markov chain with transition matrix given by:

$$\mathbf{P} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0.3 & 0.6 & 0.1 & 0 \\ 0 & 0.5 & 0.4 & 0.1 \\ 0 & 0 & 0.9 & 0.1 \end{pmatrix} \end{array},$$

where element $P_{k,l} = P(X_t = l | X_{t-1} = k)$, $k, l = 1, 2, 3, 4$.

a) Note that $P(1, 1) = 1$. What do we call state 1 in this case?

Compute $P(X_2 = 3 | X_0 = 4)$.

b) What is the expected number of time steps until a patient first leaves state 4?

Find the expected time number of time steps until a patient reaches the healthy state (an absorbing state).

c) According to the model, what is the fastest time a patient can get healthy? What is the probability of this happening for a patient?

In what ways can the patient get healthy at time step $t = 4$? Compute the probability of this happening.

Problem 2

In a buliding at NTNU they are planning to have unisex restrooms for the students. Each restroom has several toilets. In a), b) and c) we will only look at arrivals to a restroom. They are assumed to be distributed according to a homogeneous Poisson process, with arrival rate $\lambda = 0.2$ per minute.

a) Assume there are no persons in the restroom at 12:00. What is the probability that no person arrives before 12:15?

What is the probability of having 2 arrival between 12:00 and 12:15?

b) What is the expected number of arrivals between 12:00 and 12:30?

Assume there have been seven arrivals from 12:00 to 12:30. What is then the probability that five arrived before 12:15?

c) Discuss the assumptions about a homogeneous Poisson process for the arrivals in this case.

In the following we consider arrivals and departures from the restroom. The time a student stays in the toilet is exponential distributed with mean $1/\mu = 5$ minutes. Toilet times are independent. There are N toilets in the restroom. If all toilets are occupied, an arriving student would just walk on by, to find another free toilet in another restroom at NTNU.

d) Assume there are $N = 3$ toilets in the restroom. We are interested in the process describing the number of people in the restroom over time.

Sketch the transition diagram for different states of this process, and indicate the infinitesimal rates between states.

Use the infinitesimal rates between states to compute the long term probability for each state.

e) One is considering having $N = 4$ toilets in the restroom, instead of three. This means additional costs for maintenance, but improved worktime and less irritation from students walking on to find another restroom. We set the cost of having four toilets to 1 kr per minute more than that of having three toilets. The irritation cost is set to 100 kr per arrival to a fully occupied restroom.

Is it optimal to have three or four toilets in terms of expected long-term costs?

Problem 3

Assume that we must optimize a function $x(t)$, $t \in (0, 1)$ which is costly to evaluate. To find useful evaluation points, we model the function as a Gaussian process. We assume that the process initially has constant mean 10 and variance 1, for all t, and a Matern correlation function such that $\text{Corr}\,(x(t), x(s)) = (1+5|t-s|)\exp(-5|t-s|)$.

The function is evaluated at $t = 0.4$, and the function is $x(0.4) = 11.5$.

a) Use the formula for Gaussian means and covariances to find the conditional mean $m(r) = E\,[x(r)|x(0.4) = 11.5]$ and standard deviation $\sigma(r) = \sqrt{\text{Var}\,[x(r)|x(0.4) = 11.5]}$, $r \in (0, 1)$.

Compute the probabilities $P(x(0.5) > 11.5|x(0.4) = 11.5)$ and $P(x(0.6) > 11.5|x(0.4) = 11.5)$.

b) Expected improvement is a criteria which is often used to determine the next evaluation point in sequential function optimization. It represents the expected linear excess over the current maximum evaluation value. Given the information $x(0.4) = 11.5$, the expected improvement at a point r is defined by $EI(r) = E[\max\{x(r) - 11.5, 0\}\,|x(0.4) = 11.5)]$. Here, $\max\{x - 11.5, 0\} = x - 11.5$ for $x \geq 11.5$, while $\max\{x - 11.5, 0\} = 0$ for $x < 11.5$.

Show that the expected improvement for this Gaussian process model, given $x(0.4) = 11.5$, is $EI(r) = [m(r) - 11.5]\Phi(\nu) + \sigma(r)\phi(\nu)$, for $\nu = [m(r) - 11.5]/\sigma(r)$, and with mean $m(r)$ and standard deviation $\sigma(r)$ from a). Further, $\phi(z) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{z^2}{2}\right)$ and $\Phi(z) = \int_{-\infty}^{z} \phi(z)dz$.

Interpret the result. How does the expected improvement relate to $m(r)$ and $\sigma(r)$? Where is the expected improvement larger, at $t = 0.5$ or $t = 0.6$?

Hint: Use that $\int z\phi(z)dz = -\phi(z)+\text{const}$ for a standard normal variable $z = [x(r) - m(r)]/\sigma(r)$, symmetry properties of the standard Gaussian density function $\phi(-z) = \phi(z)$, and the cumulative distribution function $\Phi(-z) = 1 - \Phi(z)$.

Formulas: TMA4265 Stochastic Modeling:

The law of total probability

Let $B_1, B_2, \ldots$ be pairwise disjoint events with $P(\cup_{i=1}^{\infty} B_i) = 1$. Then

$$P(A|C) = \sum_{i=1}^{\infty} P(A|B_i \cap C)P(B_i|C),$$

$$E[X|C] = \sum_{i=1}^{\infty} E[X|B_i \cap C]P(B_i|C).$$

Discrete time Markov chains

Chapman-Kolmogorov equations

$$P_{ij}^{(m+n)} = \sum_{k=0}^{\infty} P_{ik}^{(m)} P_{kj}^{(n)}.$$

For an irreducible and ergodic Markov chain, $\pi_j = \lim_{n\to\infty} P_{ij}^{(n)}$ exist and is given by the equations

$$\pi_j = \sum_i \pi_i P_{ij} \quad \text{and} \quad \sum_i \pi_i = 1.$$

For transient states i, j and k, the mean passage time from i to $j \neq i$, M_{ij}, is

$$M_{ij} = 1 + \sum_k P_{ik} M_{kj}.$$

The Poisson process

The waiting time to the n-th event (the n-th arrival time), X_n, has probability density

$$f_{X_n}(t) = \frac{\lambda^n t^{n-1}}{(n-1)!} e^{-\lambda t} \quad \text{for} \quad t \geq 0.$$

Given that the number of events $N(t) = n$, the arrival times $X_1, X_2, \ldots, X_n$ have the uniform joint probability density

$$f_{X_1,X_2,\ldots,X_n|N(t)}(x_1, x_2, \ldots, x_n|n) = \frac{n!}{t^n} \quad \text{for} \quad 0 < x_1 < x_2 < \ldots < x_n \leq t.$$

Markov processes in continuous time

A (homogeneous) Markov process $X(t)$, $0 \leq t \leq \infty$, with state space $\Omega \subseteq \mathbf{Z}^+ = \{0, 1, 2, \ldots\}$, is called a birth and death process if

$$P_{i,i+1}(h) = \lambda_i h + o(h)$$

$$P_{i,i-1}(h) = \mu_i h + o(h)$$

$$P_{i,i}(h) = 1 - (\lambda_i + \mu_i)h + o(h)$$

$$P_{ij}(h) = o(h) \quad \text{for } |j - i| \geq 2$$

where $P_{ij}(s) = P(X(t+s) = j | X(t) = i)$, $i, j \in \mathbf{Z}^+$, $\lambda_i \geq 0$ are birth rates, $\mu_i \geq 0$ are death rates.

The Chapman-Kolmogorov equations

$$P_{ij}(t+s) = \sum_{k=0}^{\infty} P_{ik}(t) P_{kj}(s).$$

Limit relations

$$\lim_{h \to 0} \frac{1 - P_{ii}(h)}{h} = v_i \, , \quad \lim_{h \to 0} \frac{P_{ij}(h)}{h} = q_{ij} \, , \;\; i \neq j$$

Kolmogorov's forward equations

$$P'_{ij}(t) = \sum_{k \neq j} q_{kj} P_{ik}(t) - v_j P_{ij}(t).$$

Kolmogorov's backward equations

$$P'_{ij}(t) = \sum_{k \neq i} q_{ik} P_{kj}(t) - v_i P_{ij}(t).$$

If $P_j = \lim_{t \to \infty} P_{ij}(t)$ exist, P_j are given by

$$v_j P_j = \sum_{k \neq j} q_{kj} P_k \quad \text{and} \quad \sum_j P_j = 1.$$

In particular, for birth and death processes

$$P_0 = \frac{1}{\sum_{k=0}^{\infty} \theta_k} \quad \text{and} \quad P_k = \theta_k P_0 \quad \text{for} \quad k = 1, 2, \ldots$$

where

$$\theta_0 = 1 \quad \text{and} \quad \theta_k = \frac{\lambda_0 \lambda_1 \cdot \ldots \cdot \lambda_{k-1}}{\mu_1 \mu_2 \cdot \ldots \cdot \mu_k} \quad \text{for} \quad k = 1, 2, \ldots$$

Queueing theory

For the average number of customers in the system L, in the queue L_Q; the average amount of time a customer spends in the system W, in the queue W_Q; the service time S; the average remaining time (or work) in the system V, and the arrival rate λ_a, the following relations obtain

$$L = \lambda_a W.$$

$$L_Q = \lambda_a W_Q.$$

$$Z = \lambda_a E[S].$$

$$V = \lambda_a E[S W_Q^*] + \lambda_a E[S^2]/2.$$

Gaussian processes

The multivariate Gaussian density for $n \times 1$ random vector $\boldsymbol{x} = (x_1, \dots, x_n)$ is

$$p(\boldsymbol{x}) = \frac{1}{(2\pi)^{n/2}|\boldsymbol{\Sigma}|^{1/2}} \exp\left(-\frac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\boldsymbol{x}-\boldsymbol{\mu})\right), \quad \boldsymbol{x} \in \mathbb{R}^n,$$

where size $n \times 1$ mean vector $\boldsymbol{\mu} = (\mu_1, \dots, \mu_n)$, $E(x_i) = \mu_i$, and

$$\boldsymbol{\Sigma} = \begin{bmatrix} \Sigma_{1,1} & \dots & \Sigma_{1,n} \\ \dots & \dots & \dots \\ \Sigma_{n,1} & \dots & \Sigma_{n,n} \end{bmatrix}, \quad \Sigma_{i,j} = \mathrm{Cov}(x_i, x_j).$$

Let $\boldsymbol{x}_A = (x_{A,1}, \dots, x_{A,n_A})$ and $\boldsymbol{x}_B = (x_{B,1}, \dots, x_{B,n_B})$, be two subsets of variables, with block mean and covariance structure

$$\boldsymbol{\mu} = (\boldsymbol{\mu}_A, \boldsymbol{\mu}_B), \quad \boldsymbol{\Sigma} = \begin{bmatrix} \boldsymbol{\Sigma}_A & \boldsymbol{\Sigma}_{A,B} \\ \boldsymbol{\Sigma}_{B,A} & \boldsymbol{\Sigma}_B \end{bmatrix}.$$

The conditional density of $\boldsymbol{x}_A$, given $\boldsymbol{x}_B$, is Gaussian with

$$\begin{aligned} E(\boldsymbol{x}_A|\boldsymbol{x}_B) &= \boldsymbol{\mu}_A + \boldsymbol{\Sigma}_{A,B}\boldsymbol{\Sigma}_B^{-1}(\boldsymbol{x}_B - \boldsymbol{\mu}_B), \\ \mathrm{Var}(\boldsymbol{x}_A|\boldsymbol{x}_B) &= \boldsymbol{\Sigma}_A - \boldsymbol{\Sigma}_{A,B}\boldsymbol{\Sigma}_B^{-1}\boldsymbol{\Sigma}_{B,A}. \end{aligned}$$

The Brownian motion has increments $x(t_i) - x(t_{i-1})$ with the following properties, for any configuration of times $t_0 = 0 < t_1 < t_2 < \dots$:

- $x(t_i) - x(t_{i-1})$ and $x(t_j) - x(t_{j-1})$ are independent for all $i \neq j$.
- the distribution of $x(t_i) - x(t_{i-1})$ is identical to that of $x(t_i + s) - x(t_{i-1} + s)$, for any s.
- $x(t_i) - x(t_{i-1})$ is Gaussian distributed with 0 mean and variance $\sigma^2(t_i - t_{i-1})$.

Unless otherwise stated, $x(0) = 0$.

Some mathematical series

$$\sum_{k=0}^{n} a^k = \frac{1-a^{n+1}}{1-a} \quad , \qquad \sum_{k=0}^{\infty} k a^k = \frac{a}{(1-a)^2} \quad .$$