NTNU
Norwegian University of Science and Technology

Department of Mathematical Sciences

Examination paper for **TMA4145 Linear Methods–Solutions**

Academic contact during examination: Franz Luef

Phone: 406 14 405

Examination date: 21.12.2017

Examination time (from–to): 09:00-13:00

Permitted examination support material: D:No written or handwritten material. Calculator Casio fx-82ES PLUS, Citizen SR-270X, Hewlett Packard HP30S

Other information:
There are 5 problems on the exam and each problem counts for 20 points. All solutions should be stated in a precise and rigorous way, with any assumptions written down and arguments justified, except Problem 3.

Language: English

Number of pages: 10

Number of pages enclosed: 0

Checked by:

Informasjon om trykking av eksamensoppgave	
Originalen er:	
1-sidig ☐	**2-sidig** ☐
sort/hvit ☐	**farger** ☐
skal ha flervalgskjema ☐	

Date Signature

Disclaimer: I present one way to solve these problems although there are other possible solutions.

Problem 1

a) (1) Find the singular value decomposition for the matrix

$$A = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \end{pmatrix}.$$

Solution: Let us compute the singular values of A. Recall these are the non-zero eigenvalues of the selfadjoint matrix $AA^* = \begin{pmatrix} 3 & 3 \\ 3 & 3 \end{pmatrix}$ or $A^*A = \begin{pmatrix} 2 & 2 & -2 \\ 2 & 2 & -2 \\ -2 & -2 & 2 \end{pmatrix}$. In the first case we get a 2×2-matrix and in the second case we get a 3×3-matrix, so we use AA^* for the computation of the singular values. The eigenvalues of $\begin{pmatrix} 3 & 3 \\ 3 & 3 \end{pmatrix}$ are 6 and 0. For those, who have decided to use A^*A: the eigenvalues are $6, 0, 0$. Hence $\sigma_1 = \sqrt{6}$ is the only singular value of A, which fits very well with the fact that A has rank one.

Consequently Σ is given by

$$\Sigma = \begin{pmatrix} \sqrt{6} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Let us look at the eigenvectors of A^*A. A little bit of computation yields

$$v_1 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \; v_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \; v_3 = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}.$$

The set $\{v_1, v_2, v_3\}$ is an orthonormal basis of $\mathbb{R}^3$ and yield the columns of V:

$$V = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{-1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \end{pmatrix}.$$

Now we get the columns of U, which is an orthonormal basis for $\mathbb{R}^2$, by

$$u_1 = \frac{1}{\sigma_1} A v_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

and choosing another vector orthogonal to v_1, such as $u_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and thus

$$U = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}.$$

Hence $A = U\Sigma V^* = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \sqrt{6} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} \end{pmatrix}$

(2) The linear system:

$$\begin{aligned} x_1 + x_2 - x_3 &= 1 \\ x_1 + x_2 - x_3 &= 1 \end{aligned}$$

has infinitely many solutions. Determine the one with the minimal Euclidean norm $\|.\|_2$.

The linear system

$$\begin{aligned} x_1 + x_2 - x_3 &= 1 \\ x_1 + x_2 - x_3 &= 2 \end{aligned}$$

has no solution. Determine the least squares solution of the linear system.

Hint: The pseudoinverse of the matrix related to the linear system might be useful.

Solution: We first compute the pseudoinverse of A: The pseudoinverse in terms of the SVD is given by $A^+ = V\Sigma^+ U^*$, which gives

$$A^+ = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{6}} & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} = \frac{1}{6} \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ -1 & -1 \end{pmatrix}.$$

The system

$$x_1 + x_2 - x_3 = 1$$
$$x_1 + x_2 - x_3 = 1$$

has infinitely many solutions and we have learned that $A^+ \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$ is the solution of minimal $\|.\|_2$-norm. The system

$$x_1 + x_2 - x_3 = 1$$
$$x_1 + x_2 - x_3 = 2$$

has no solution, but $A^+ \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$ is the best approximation to a solution having minimal norm.

b) Given a $n \times n$-matrix A of rank n. Prove that A has a polar decomposition using the singular value decomposition of A. Hence, show that there exist an $n \times n$ unitary matrix W and a positive definite $n \times n$ matrix P such that $A = WP$.

Solution: The SVD decomposition gives us unitary $n \times n$ matrices U and V such that

$$A = U\Sigma V^* = UV^* V\Sigma V^*.$$

Note that UV^* is unitary as a product of two unitary matrices and $V\Sigma V^*$ is positive definite, since Σ is positive definite. Hence $V\Sigma V^*$ is the replacement of the length of a complex number and UV^* the one for the phase factor.

Problem 2

a) Let T be the linear transformation $T(x) = Ax$ on $\mathbb{R}^3$ for the matrix

$$A = \begin{pmatrix} 0 & 1/2 & 1/3 \\ 1/4 & 0 & 1/5 \\ 1/5 & \alpha & 0 \end{pmatrix},$$

where α is a real number.

(1) Determine the operator norm of $T : (\mathbb{R}^3, \|.\|_1) \to (\mathbb{R}^3, \|.\|_1)$. Note that the result depends on the parameter α.

Solution: The operator norm of $T(x) = Ax$ as a mapping on $(\mathbb{R}^3, \|.\|_1)$ is given by the maximal column sum of a matrix A. Let $A = (a_1|a_2|a_3)$ be partioned into its columns. Then we have for the operator norm

$$\|T\| = \max_{1\leq j\leq 3} |a_j|_1 = \max_{1\leq j\leq 3} \sum_{i=1}^{3} |a_{ij}|.$$

By definition of the operator norm we have

$$\|T\| = \max_{\|x\|_1=1} \|Ax\|_1 = \max_{\|x\|_1=1} \|x_1a_1 + \cdots + x_3a_3\|_1.$$

By the triangle inequality and the homogenity for the $\|.\|_1$-norm we get

$$\max_{\|x\|_1=1} \|Ax\|_1 \leq \max_{\|x\|_1=1} (|x_1|\|a_1\|_1 + \cdots + |x_3|\|a_3\|_1).$$

Let j be chosen such that $\max_{1\leq i\leq 3} \|a_i\|_1 = \|a_j\|_1$. Then we get

$$\max_{\|x\|_1=1} \|Ax\|_1 \leq \max_{\|x\|_1=1} (|x_1| + \cdots + |x_3|)\|a_j\|_1 = \|a_j\|_1.$$

We denote by $\{e_1, e_2, e_3\}$ the standard basis for $\mathbb{R}^3$. Then $\|a_j\|_1 = \|Ae_j\|_1 \leq \max_{\|x\|_1=1} \|Ax\|_1$. Let us combine our two inequalities:

$$\|a_j\|_1 \leq \max_{\|x\|_1=1} \|Ax\|_1 \leq \|a_j\|_1.$$

Consequently, we have

$$\|T\| = \max_{1\leq j\leq 3} \sum_{i=1}^{3} |a_{ij}|.$$

Now, we apply this statement to the given linear transformation:

$$\| \begin{pmatrix} 0 & 1/2 & 1/3 \\ 1/4 & 0 & 1/5 \\ 1/5 & \alpha & 0 \end{pmatrix} \| = \max\{9/20, 1/2 + |\alpha|, 8/15\}$$

. Hence for $|\alpha| \leq 1/30$ we have $\|T\| = 8/15$ and for $|\alpha| > 1/30$ we have $\|T\| = 1/2 + |\alpha|$.

(2) Determine those α's such that T is a contraction on $(\mathbb{R}^3, \|.\|_1)$.

Solution: By our computation in (1) we have that this is the case for $|\alpha| < 1/2$.

b) Rewrite the linear system

$$\begin{aligned} 3x_1 - \tfrac{3}{2}x_2 - x_3 &= 1 \\ -x_1 + 4x_2 - \tfrac{4}{5}x_3 &= 2 \\ -\tfrac{2}{5}x_1 - \tfrac{1}{2}x_2 + 2x_3 &= 4 \end{aligned}$$

as a fixed point problem and show that one can use Banach's fixed point theorem to prove the existence of a solution. Compute the first three iterations $x^{(1)}, x^{(2)}, x^{(3)}$ for the starting point $x_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$.

Solution: The system of equations is equivalent to

$$\begin{aligned} x_1 &= 0 \cdot x_1 + \frac{1}{2}x_2 + \frac{1}{3}x_3 + \frac{1}{3} \\ x_2 &= \frac{1}{4}x_1 + 0 \cdot x_2 + \frac{1}{5}x_3 + \frac{1}{2} \\ x_3 &= \frac{1}{5}x_1 + \frac{1}{4}x_2 + 0 \cdot x_3 + 2 \end{aligned}$$

which we may write in matrix form as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 & 1/2 & 1/3 \\ 1/4 & 0 & 1/5 \\ 1/5 & \alpha & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1/3 \\ 1/2 \\ 2 \end{bmatrix}$$

If we define

$$A = \begin{bmatrix} 0 & 1/2 & 1/3 \\ 1/4 & 0 & 1/5 \\ 1/5 & 1/4 & 0 \end{bmatrix} \text{ and } b = \begin{bmatrix} 1/3 \\ 1/2 \\ 2 \end{bmatrix}$$

our problem becomes solving $x = Ax + b$ – a fixed point problem. In order to apply Banach's fixed point theorem, we need to have a contraction. In this case we need that

$$\|Ax + b - (Ay + b)\| = \|A(x - y)\| \leq K\|x - y\|$$

for any $x, y \in \mathbb{R}^3$ in some norm $\|.\|$ on $\mathbb{R}^3$. Let us use the $\|.\|_1$ on $\mathbb{R}^3$. From the first part of the problem, we know that the operator norm of the operator $T : \mathbb{R}^3 \to \mathbb{R}^3$ given by $Tx = Ax$ is the maximal column sum of the matrix A. In this case the maximal row sum appears in row 2 and equals $1/2+1/4 = 3/4$. Hence $\|T\| = 3/4$. But this means that

$$\|A(x-y)\|_1 = \|T(x-y)\|_1 \leq \|T\|\|x-y\|_1 = \frac{3}{4}\|x-y\|_1.$$

Hence we have a contraction with $K = \frac{3}{4}$. By Banach's fixed point theorem we may choose any $x_0 \in \mathbb{R}^3$, and the iteration procedure $x_n = Ax_{n-1} + b$ will always converge to a solution x of $x = Ax + b$. Let us for instance pick

$$x_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

Then the first few iterations give

$$x_1 = \begin{bmatrix} 1/3 \\ 3/4 \\ 11/5 \end{bmatrix}, \; x_2 = \begin{bmatrix} 1.3694 \\ 1.0233 \\ 2.6166 \end{bmatrix}, \; x_3 = \begin{bmatrix} 1.717183 \\ 1.36568 \\ 2.59705 \end{bmatrix}.$$

Problem 3

a) (1) Suppose $(X, \|.\|_X)$ and $(Y, \|.\|_Y)$ are normed spaces. Define the notions of a *continuous* and of a *Lipschitz continuous* function $f : X \to Y$.

Solution: (i) We discussed several definitions and we just state the one in terms of $\epsilon - \delta$: We say that $f : X \to Y$ is **continuous** if or each $x_0 \in X$ and each $\varepsilon > 0$ there is a $\delta > 0$ such that

$$\|x - x_0\|_X < \delta \Rightarrow \|f(x) - f(x_0)\|_Y < \varepsilon.$$

(ii) A function $f : X \to Y$ is called **Lipschitz continuous** if there exists a finite constant L such that

$$\|f(x) - f(x')\|_Y \leq L\,\|x - x'\|_X \quad \text{for all } x, x' \in X.$$

(2) Let X be a vector space and T a linear map between the vector spaces $T : X \to X$. Define the notion of a *T-invariant subspace* of X.

Solution: A subspace M of X is called **T-invariant** if for any $x \in M$ we have also that $Tx \in M$.

(3) Let $(X, \|.\|)$ be a normed space. Define the notion of a *dense subset* of X and define when X is *separable*.

Solution: (i) A subset A of $(X, \|.\|)$ is said to be *dense* in X if for each $x \in X$ and each $\varepsilon > 0$ there exists a vector $y \in A$ such that $\|x - y\| < \varepsilon$.

(ii) A normed space X is called *separable*, if it contains a coutable dense subset.

(4) Let X be a vector space and $T : X \to X$ a linear transformation. Define the notion of a *generalized eigenspace* for an eigenvalue λ of T and the *minimal polynomial* of a $n \times n$-matrix A.

Solution: (i) A **generalized eigenspace** of λ is $\ker(T - \lambda I)^k$ for some $k > 1$.

The **minimal polynomial** of A is the among all annihiliating polynomials of A the one with the smallest degree.

(5) Define the notions of a *Cauchy sequence* and of *completeness* for normed space.

Solution: (i) Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $(X, \|.\|)$. Then we call $(x_n)_{n\in\mathbb{N}}$ a **Cauchy sequence** if for any $\varepsilon > 0$ there exists an $N \in \mathbb{N}$ such that for all $m, n \geq N$ we have

$$\|x_n - x_m\| < \varepsilon.$$

(ii) A normed space $(X, \|.\|)$ is **complete** if every Cauchy sequence in X converges to an element in X.

b) Determine if the following statements are true or false and if the statement is not true, give a counterexample.

(1) Any linear map on a normed space is bounded.

Solution: No. For example, the multiplication operator $Tx = (x_1, 2x_2, 3x_3, ...)$ is unbounded on ℓ^p for $p \in [1, \infty]$. Another well-known example is the differentiation operator $Tf = f'$ on $(C[0, 1], \|.\|_\infty)$.

(2) Any linear transformation on a finite-dimensional complex vector space has a non-trivial invariant subspace.

Solution: Yes.

(3) The set of sequences with finitely many non-zero elements is dense in the space of bounded sequences ℓ^∞.

Solution: No. For example, take the constant sequence $(1, 1, 1, ...)$ cannot approximated arbitrarily closed by elements from c_f.

(4) The orthogonal complement of any subset of an innerproduct space is closed.

Solution: Yes.

(5) The range of any bounded linear map on an infinite-dimensional vector space is closed.

Solution: No. Example: The operator $T : \ell^2 \to \ell^2$ defined by $T(x_1, x_2, ...) = (x_1, \frac{x_2}{2}, \frac{x_3}{3}, ...)$ does not have closed range.

Problem 4 For $a = (a_n)_{n\in\mathbb{N}} \in \ell^\infty$ we define the linear operator $T_a : \ell^2 \to \ell^2$ by $T_a(x_1, x_2, ...) = (a_1x_1, 0, a_3x_3, 0, ...)$ for $(x_n) \in \ell^2$.

(1) Show that T_a is bounded on ℓ^2.

Solution: $\|T_a x\|_2^2 = |a_1x_1|^2 + |a_3x_3|^2 + \cdots \leq \|(a_{2n-1})_{n\in\mathbb{N}}\|_\infty^2 \|x\|_2^2$ and hence

$$\|T_a x\|_2 \leq \|(a_{2n-1})_{n\in\mathbb{N}}\|_\infty \|x\|_2.$$

Here $(a_{2n-1})_{n\in\mathbb{N}}$ is the odd part of the sequence a, i.e. the sequence $(a_1, a_3, a_5, ...)$.

(2) Determine the operator norm of T_a.

Solution: $\|T_a\| \leq \|(a_{2n-1})_{n\in\mathbb{N}}\|_\infty$, because

$$\|T_a\| = \sup_{\|x\|_2=1} \|T_a x\|_2 \leq \sup_{\|x\|_2=1} \left(\|(a_{2n-1})_{n\in\mathbb{N}}\|_\infty \|x\|_2\right) = \|(a_{2n-1})_{n\in\mathbb{N}}\|_\infty.$$

Hence $\|(a_{2n-1})_{n\in\mathbb{N}}\|_\infty$ is an upper bound for $\{\|T_a x\|_2 : \|x\|_2 = 1\}$. Now we show that it is the least upper bound for $\{\|T_a x\|_2 : \|x\|_2 = 1\}$. Namely, for every $\varepsilon > 0$ there exists some $x^\varepsilon \in \ell^2$ with $\|x^\varepsilon\|_2 = 1$ such that

$$\|T_a x^\varepsilon\|_2 > \|x^\varepsilon\|_2 - \varepsilon.$$

For every $\varepsilon > 0$ there exists a index k_ε such that $|a_{2k_\varepsilon-1}| > \|(a_{2n-1})\|_\infty - \varepsilon$ (which follows from the definition of the supremum of the sequence (a_{2n-1})) and take $x^\varepsilon = (0, ..., 0, 1, 0, ...)$ where the 1 is in the $(2k_\varepsilon - 1)$th component. Then $T_a x_\varepsilon = |a_{2k_\varepsilon-1}| > |(a_{2n-1})\|_\infty - \varepsilon$. Hence we have $\|T_a\| = \|(a_{2n-1})\|_\infty$.

(3) Show that the range of T_a is closed.

Solution: The range of T_a is $\{x \in \ell^2 : (x_1, 0, x_3, 0, ...)\}$. There are (at least) two strategies: (i) show directly that $\{x \in \ell^2 : (x_1, 0, x_3, 0, ...)\}$ is closed; or (ii) note that $\{x \in \ell^2 : (x_1, 0, x_3, 0, ...)\}$ is the kernel of a the operator P given by $Px = (0, x_2, 0, x_4, 0, ...)$: P is linear and bounded: $\|Px\|_2 \leq \|x\|_2$ and we have $\ker(P) = \mathrm{range}(T_a)$.

(4) Determine the orthogonal complement of $\ker(T_a)$.

Solution: $\ker(T_a)$ is the subspace $\{x \in \ell^2 : (0, x_2, 0, x_4, 0, ...)\}$. By definition

$$\ker(T_a)^\perp = \{y \in \ell^2 : \langle y, x\rangle = 0 \ \text{ for all } \ x \in \ker(T_a)\},$$

i.e. we have

$$\ker(T_a)^\perp = \{y \in \ell^2 : \sum_{i=1}^{\infty} x_{2i}\overline{y_{2i}} = 0 \ \text{ for all } \ x \in \ell^2\}.$$

The expression $\sum_{i=1}^\infty x_{2i}\overline{y_{2i}} = 0$ for all $x \in \ell^2$ if and only if $y = (y_1, 0, y_3, 0, y_5, ...)$. Consequently, $\ker(T_a)^\perp = \{x \in \ell^2 : x = (x_1, 0, x_3, 0, x_5, ...)\}$.

(5) Determine for which sequences $a \in \ell^\infty$ the operator T_a satisfies $T_a^2 = T_a$.

Solution: $T_a^2 x = (a_1^2 x_1, 0, a_3^2 x_3, 0, ...)$ and thus $T_a^2 = T_a$ is equivalent to $a_i^2 = a_i$ for all $i = 1, 2, 3, ...$, which holds only for $a_{2i-1} \in \{0, 1\}$ for all $i = 1, 2, 3, ...$

Problem 5 Let $\{e_n\}_{n\in\mathbb{N}}$ be an orthonormal system in a Hilbert space X and $(\alpha_n)_{n\in\mathbb{N}}$ a sequence of complex numbers.

Show that the series $\sum_{n\in\mathbb{N}} \alpha_n e_n$ converges in X if and only if $(\alpha_n)_{n\in\mathbb{N}} \in \ell^2$.

Solution: For any finite orthonormal system $\{e_1, ..., e_n\}$ we have

$$\begin{aligned}\|\sum_{j=1}^{n} \alpha_j e_j\|^2 &= \langle \sum_{j=1}^{n} \alpha_j e_j, \sum_{j=1}^{n} \alpha_j e_j \rangle \\ &= \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i \overline{\alpha_j} \langle e_i, e_j \rangle \\ &= \sum_{j=1}^{n} |\alpha_j|^2.\end{aligned}$$

for any scalars $\alpha_1, ..., \alpha_n$. Hence the partial sums $s_n = \sum_{k=1}^{n} \alpha_k e_k$ satisfy $(s_n)_n$ for $n > m$

$$\|s_n - s_m\|^2 = \sum_{k=m+1}^{n} |\alpha_k|^2.$$

Hence (s_n) is a Cauchy sequence in X if and only if $(\|\alpha_n\|^2)_n$ is a Cauchy sequence in $\mathbb{R}$. Since X and $\mathbb{R}$ are both complete, these two sequences converge or divergence simultaneously. In the case of convergence, we take the limit $n \to \infty$ and obtain the desired claim.