NTNU
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Week 38: Lecture 2

The infinitesimal definition of the Poisson process

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September 16, 2020

Information

- Minutes from first reference group meeting is available under "Course information".
- The project will be available after this lecture.
- You need to register in one of the groups called "Project group" to be able to see the project and to submit.
- You can receive help during exercise classes in weeks 39 and 40 in R2.
- **No lectures in week 40 (September 28 and September 30).**
- There will be physical guidance on September 28 and digital guidance on September 30. Information will come next week.

Section 5.1.4

Read this yourselves.

Section 5.2 (Motivation)

Consider two Poisson processes:

a) $\{X(t) : t \geq 0\}$ with rate $\lambda_1(t) = 5$, $0 \leq t \leq 10$.

b) $\{Y(t) : t \geq 0\}$ with rate $\lambda_2(t) = t$, $0 \leq t \leq 10$.

Theorem (The law of rare events)

Let $p_1, p_2, \ldots \in [0, 1]$ *be a sequence such that* $\lim_{n\to\infty} np_n = \lambda < \infty$, *then*

$$\lim_{n\to\infty} \binom{n}{k} p_n^k (1-p_n)^{n-k} = \frac{\lambda^k}{k!}\mathrm{e}^{-\lambda}, \quad k = 0, 1, \ldots.$$

Comment: In TMA4295 Statistical Inference: we will say that $\mathrm{Binomial}(n, p_n)$ **converges in probability** to $\mathrm{Poisson}(\lambda)$ is $n \to \infty$.

Little-oh notation

You may be familiar with expressions such as

$$n = o(n^2) \quad (\text{as } n \to \infty).$$

We are going to mostly work with expressions of the form

$$h^2 = o(h) \quad (\text{as } h \to 0^+).$$

Definition

Let f and g be real functions. We use **little-oh notation** in the two following ways

$$f(n) = o(g(n)) \quad (\text{as } n \to \infty) \quad \iff \quad \lim_{n\to\infty} \frac{f(n)}{g(n)} = 0,$$

$$f(h) = o(g(h) \quad (\text{as } h \to 0^+) \quad \iff \quad \lim_{h\to 0^+} \frac{f(h)}{g(h)} = 0.$$

Example

Are the following statements true or false?

a) $h^2 = o(h) \quad (\text{as } h \to 0^+)$.

b) $h^2 = o(h) \quad (\text{as } h \to \infty)$.

c) $\sqrt{h} = o(h) \quad (\text{as } h \to 0^+)$.

d) $h = o(1) \quad (\text{as } h \to 0^+)$.

Definition

A **counting process** is a stochastic process $\{N(t) : t \geq 0\}$ so that

1. $N(t)$ is integer for $t \geq 0$.
2. $N(t) \geq 0$ for $t \geq 0$.
3. If $s \leq t$, then $N(s) \leq N(t)$.

Comment: We sometimes write
$N((a, b]) = N(b) - N(a) =$ "Number of events in $(a, b]$", $0 \leq a < b$.
I will not use this notation in the lectures.

Definition (P2)

Let $\{N(t) : t \geq 0\}$ be a counting process. Then $\{N(t) : t \geq 0\}$ is a **Poisson process** with **rate (intensity)** $\lambda > 0$ if

1. For every integer $m > 1$, for any time points $0 = t_0 < t_1 < \cdots < t_m$,

$$N(t_1) - N(t_0), N(t_2) - N(t_1), \ldots, N(t_m) - N(t_{m-1})$$

are independent.

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5. $N(0) = 0.$

Definition (P2, simplified)

Let $\{N(t) : t \geq 0\}$ be a counting process. Then $\{N(t) : t \geq 0\}$ is a **Poisson process** with **rate (intensity)** $\lambda > 0$ if

1. it has independent increments.
2. it has stationary increments.
3. $\Pr\{N(t+h) - N(t) = 1\} = \lambda h + o(h) \quad (\text{as } h \to 0^+), \quad \forall t \geq 0.$
4. $\Pr\{N(t+h) - N(t) = 0\} = 1 - \lambda h + o(h) \quad (\text{as } h \to 0^+), \quad \forall t \geq 0.$
5. $N(0) = 0.$

Repetition

Definition (P1, simplified)

A **Poisson process** with **rate (intensity)** $\lambda > 0$ is an integer-valued stochastic process $\{N(t) : t \geq 0\}$ for which

1. increments are independent.
2. for $s \geq 0$ and $t > 0$,

$$N(s+t) - N(s) \sim \mathrm{Poisson}(\lambda t).$$

3. $N(0) = 0$.

Theorem

Definition P1 and Definition P2 of a Poisson process are equivalent.