Norwegian University of Science and Technology
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TMA4212
Numerical solution of differential equations by difference methods
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Solutions to exercise set 1

Exercise set 1, Problem 5

It is assumed that you have worked through the steps described in detail in Exercise 1, problem 4 and is ready to finalize the task by writing the results as a theorem, with proof, and are ready to do the numerical verification of the results.

Consider the boundary value problem

$$-\mu u_{xx} + b u_x = f, \qquad 0 < x < 1, \tag{1}$$

with boundary conditions

$$u(0) = g_0, \qquad u(1) = g_1. \tag{2}$$

with $\mu > 0$. An approximation to this problem is found using a finite difference scheme on an equidistributed grid, with $h = 1/M$ and gridpoints $x_m = mh$, $m = 0, \dots, M$:

$$L_h U_m = -\frac{\mu}{h^2}(U_{m-1} - 2U_m + U_{m+1}) + \frac{b}{2h}(-U_{m-1} + U_{m+1}) = f_m, \qquad m = 1, \dots, M-1, \tag{3}$$

together with the boundary values $U_0 = g_0$ and $U_M = g_1$.

Theorem 1. *Given the equation* (1),(2)*, with* $f \in C^2(0,1)$*, solved by the finite difference scheme. Assuming* $h \le 2\mu/|b|$*, the global error* $e_m = u_m - U_m$*,* $m = 1, 2, \dots, M-1$ *satisfies the error bound*

$$|e_m| \le Ch^2, \qquad \textit{with} \qquad C = \frac{1}{24}\max_{x\in(0,1)}|\partial^4 u(x)| + \frac{|b|}{6\mu}\max_{x\in(0,1)}|\partial^3 u(x)|.$$

Proof. Insert the exact solution into (3) to find the truncation error $=\tau_m$:

$$\begin{aligned}\tau_m = L_h u_m - f_m &= -\frac{\mu}{h^2}(u_{m-1} - 2u_m + u_{m+1}) + \frac{b}{2h}(-u_{m-1} + u_{m+1}) - f_m \qquad (4)\\ &= \overbrace{-\mu\partial_x^2 u_m + b\partial_x u_m - f_m}^{0} - \mu\Big(\frac{1}{12}\partial_x^4 u(\xi_m) - \frac{b}{12\mu}\partial_x^3 u(\eta_m)\Big)h^2.\end{aligned}$$

where $\eta_m, \xi_m \in (x_m - h, x_m + h)$. The bound for the truncation error is

$$|\tau_m| \le \mu\hat{C}h^2, \qquad \hat{C} = \frac{1}{12}\max_{x\in(0,1)}|\partial_x^4 u(x)| + \frac{|b|}{6\mu}\max_{x\in(0,1)}|\partial_x^3 u(x)|.$$

Subtract (3) from (4) gives $L_h e_m = \tau_m$, written out as

$$\frac{\mu}{h^2}\Big(-(1+\frac{bh}{2\mu})e_{m-1} + 2e_m - (1-\frac{bh}{2\mu})e_{m+1}\Big) = \tau_m \tag{5}$$

Let

$$\phi(x) = Ch^2 \cdot \begin{cases} \frac{1}{2}x^2 - x & \text{if } b \geq 0 \\ \frac{1}{2}(x^2-1) & \text{if } b < 0 \end{cases}$$

ensuring that $b\phi_x \leq 0$ and $\phi_{xx} = \hat{C}h^2$. Using the fact that the difference approximations used here are exact for second order polynomials, we get

$$L_h(\phi_m \pm e_m) = -\hat{C}h^2 + b\phi_x \pm \tau_m \leq 0$$

and by the assumption $h \leq 2\mu/|b|$ the discrete maximum principle applies. So, using $e_0 = e_M = 0$ we get

$$\phi_m \pm e_m \leq \max\{0, \phi(0), \phi(1)\}$$

and

$$\pm e_m \leq -\phi_m \leq \frac{1}{2}\hat{C}h^2, \qquad m = 1, \ldots, M-1,$$

which proves the result. □

The numerical verification of the order is left for you, it is just quite straightforward modifications of the code in Problem 4.

Notice one thing: The error depends on the factor $|b|/\mu$, and there is also a restriction on the stepsize based on this factor in the theorem. What happens if this restriction is no longer fullfilled? Choose some $\mu << 1$ and $b = 1$ and try it out.