2-dim TQFTs

TQFTs are rich geometric gadgets, encoding many fundamental manifold invariants. Roughly speaking, they capture the idea of cutting a manifold into pieces (cobordisms), attaching invariants to these pieces, and then gluing these invariants together to obtain an invariant of the original manifold.

A TQFT is a symmetric monoidal functor $Z : n\mathrm{Cob} \longrightarrow \mathrm{Vect}_{\mathbb{C}}$ (linear category). When $n = 2$ these are equivalent to Frobenius algebras:

Theorem: $2\mathrm{TQFT}_{\mathbb{C}} \simeq \mathrm{cFA}_{\mathbb{C}}$

Categorical preliminaries

A category $\mathcal{C}$ consists of

- objects: $A, B, C, \dots$ $(A \in \mathcal{C})$
- morphisms ('arrows'): $A \xrightarrow{f} B$ $(f \in \mathcal{C}(A,B))$

subject to:

1) Given $A \xrightarrow{f} B$, $B \xrightarrow{g} C$ we can compose: $A \xrightarrow{g\circ f} C$
2) Composition is associative: $h\circ(g\circ f) = (h\circ g)\circ f$, $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$
3) For every $A \in \mathcal{C}$ there is a (unique) identity morphism 1_A (id_A): $f\circ 1_A = f = 1_B \circ f$.

Examples: $\mathrm{Vect}_{\mathbb{C}}$: vector spaces over $\mathbb{C}$, linear maps

Top : topological spaces, continuous maps

A functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ consists of

- map from the objects of $\mathcal{C}$ to the objects of $\mathcal{D}$
- map $F_{A,B} : \mathcal{C}(A,B) \longrightarrow \mathcal{D}(F(A),F(B))$

subject to:

1) Given $A \xrightarrow{f} B \xrightarrow{g} C$ in $\mathcal{C}$ $F_{A,C}(g\circ f) = F_{B,C}(g)\circ F_{A,B}(f)$ (F covariant)
2) $F_{A,A}(1_A) = 1_{F(A)}$ for all $A \in \mathcal{C}$.

$F, G : \mathcal{C} \longrightarrow \mathcal{D}$ functors. A natural transformation $\eta : F \Rightarrow G$ assigns to each $A \in \mathcal{C}$ a morphism $\eta(A) : F(A) \longrightarrow G(A)$ in $\mathcal{D}$ such that for each $A \xrightarrow{f} B$ in $\mathcal{C}$

$$\begin{tikzcd} F(A) \arrow[r, "F_{A,B}(f)"] \arrow[d, "\eta(A)"'] \arrow[dr, phantom, "\circ"] & F(B) \arrow[d, "\eta(B)"] \\ G(A) \arrow[r, "G_{A,B}(f)"'] & G(B). \end{tikzcd}$$

η is natural isomorphism if $\eta(A), \eta(B)$ are isomorphisms: $X \underset{\beta}{\overset{\alpha}{\rightleftarrows}} Y$ $\beta\circ\alpha = 1_X, \alpha\circ\beta = 1_Y, X \cong Y$.

We write $F \cong G$.

$\mathcal{C}$ and $\mathcal{D}$ are equivalent if there exists functors $F : \mathcal{C} \longrightarrow \mathcal{D}$, $G : \mathcal{D} \longrightarrow \mathcal{C}$ such that $1_{\mathcal{C}} \cong G\circ F$, $1_{\mathcal{D}} \cong F\circ G$.

A strict monoidal category $(\mathcal{C}, \otimes, I)$ is a category $\mathcal{C}$ with a functor $\otimes : \mathcal{C}\times\mathcal{C} \to \mathcal{C}$ which is associative and with an object $I \in \mathcal{C}$ which is a left and right unit for $\otimes$. $(\mathcal{C}, \otimes, I)$ is symmetric if for each pair of objects A, B in $\mathcal{C}$ there is a twist (braid) map $\tau_{A,B} : A\otimes B \longrightarrow B \otimes A$ subject to:

- for every pair $A \xrightarrow{f} A'$, $B \xrightarrow{g} B'$ in $\mathcal{C}$

$$\begin{tikzcd} A\otimes B \arrow[r, "\tau_{A,B}"] \arrow[d, "f\otimes g"'] \arrow[dr, phantom, "\circ"] & B\otimes A \arrow[d, "g\otimes f"] \\ A'\otimes B' \arrow[r, "\tau_{A',B'}"'] & B'\otimes A' \end{tikzcd}$$

- for every triple $A, B, C \in \mathcal{C}$

$$\begin{tikzcd} A\otimes B\otimes C \arrow[rr, "\tau_{A,B\otimes C}"] \arrow[dr, "\tau_{A,B}\otimes 1_C"'] & \circ & B\otimes C\otimes A \\ & B\otimes A\otimes C \arrow[ur, "1_B\otimes\tau_{A,C}"'] & \end{tikzcd} \qquad \begin{tikzcd} A\otimes B\otimes C \arrow[rr, "\tau_{A\otimes B,C}"] \arrow[dr, "1_A\otimes\tau_{B,C}"'] & \circ & C\otimes A\otimes B \\ & A\otimes C\otimes B \arrow[ur, "\tau_{A,C}\otimes 1_B"'] & \end{tikzcd}$$

- $\tau_{B,A}\circ\tau_{A,B} = 1_{A\otimes B}$ for every pair $A, B \in \mathcal{C}$.

Cobordisms

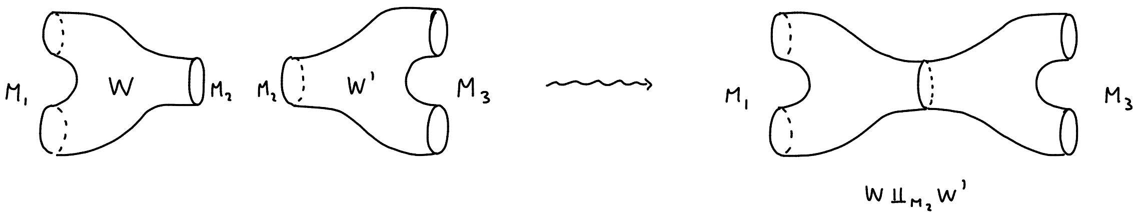
(See exercise set 11, problem 5.) We will only consider 2-dim cobordisms. (Manifolds are always assumed to be compact smooth.)

Let 2Cob be the category with

- objects : closed oriented 1-manifolds
- morphisms : $M, N \in 2\mathrm{Cob}$, a morphism from M to N is a cobordism W from M to N, i.e. W is an oriented 2-manifold equipped with an orientation-preserving diffeomorphism $\partial W \xrightarrow{\cong} M \amalg \bar{N}$.

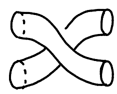
W, W' define the same morphism in 2Cob if there is an orientation-preserving diffeomorphism $W \xrightarrow{\cong} W'$ (extending $\partial W \cong M \amalg \bar{N} \cong \partial W'$). For any $M \in 2\mathrm{Cob}$, 1_M is represented by the cobordism $W = M \times I$.

$M_1, M_2, M_3 \in 2\mathrm{Cob}$, cobordisms $W : M_1 \to M_2$, $W' : M_2 \to M_3$. The composition $W' \circ W : M_1 \to M_3$ is defined to be the morphism represented by $W \amalg_{M_2} W'$.



<u>Note</u>: To give $W \amalg_{M_2} W'$ a smooth structure, we can make a choice of a smooth collar around M_2 inside of W and W'. Different choices of collars (can) lead to different smooth structures on $W \amalg_{M_2} W'$, but the resulting cobordisms are diffeomorphic (but there is no canonical diffeomorphism). See Milnor's Lectures on the h-cobordism theorem for full details.

$(2\mathrm{Cob}, \amalg, \emptyset)$ is a monoidal category.

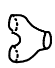
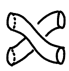
The cobordism induced by the twist diffeomorphism $M \amalg M' \longrightarrow M' \amalg M$ is the twist cobordism : 

$(2\mathrm{Cob}, \amalg, \emptyset, T)$ is a symmetric monoidal category.

2Cob can be described explicitly in terms of generators and relations, where we use the classification of surfaces.

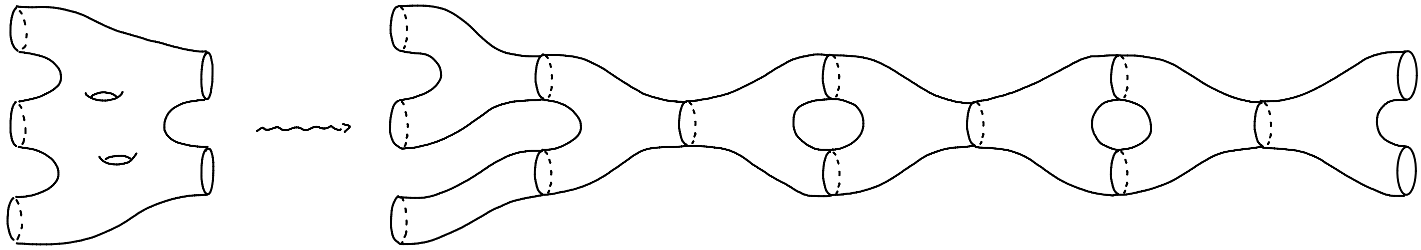
A generating set for a monoidal category is a set S of morphisms such that all morphisms in the category can be obtained from elements in S by composition and $\otimes$.

A skeleton of 2Cob (full subcategory comprising exactly one object from each isomorphism class) is the full subcategory $\{\underline{0}, \underline{1}, \underline{2}, \ldots\}$ with $\underline{n} = \amalg S^1$. Let 2Cob denote this skeleton.

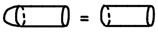
<u>Theorem</u>: 2Cob is generated by the six cobordisms:  ,  ,  ,  ,  , 

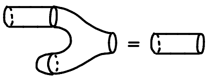
(We will use the classification of surfaces for this theorem.)

The normal form of a connected surface with m in-boundaries, n out-boundaries, genus g is a decomposition of the surface into a number of basic cobordisms.

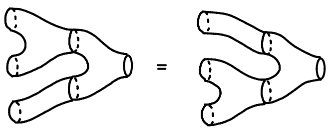


The relations we need are as follows:

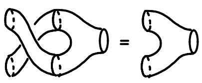
1. Identity:  etc.
2. Unit and counit:

 , etc.

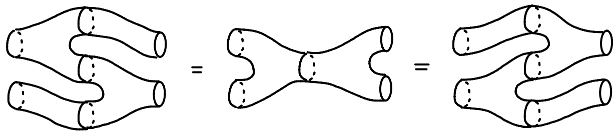
3. Associativity and coassociativity:

 , etc.

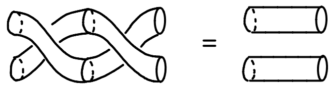
4. Commutativity and cocommutativity:

 , etc.

5. Frobenius:



6. Twisting:



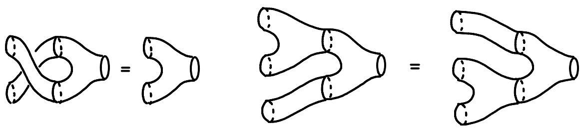
These relations are sufficient but not minimal.

2-dim TQFTs and commutative Frobenius algebras

A 2-dim TQFT is a symmetric monoidal functor $Z: 2\mathrm{Cob} \longrightarrow \mathrm{Vect}_{\mathbb{C}}$.

Let $Z(S^1) = Z(\underline{1}) = A$. Then $Z(\underline{n}) = A^{\otimes n}$. Furthermore,

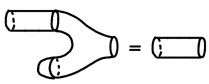
$Z($  $): A \otimes A \xrightarrow{m} A$

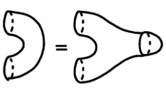
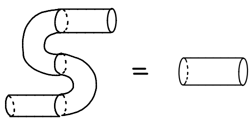
m is commutative and associative: 

Moreover,

$Z($  $) = \mathbb{C} \xrightarrow{u} A$

$Z($  $) = A \xrightarrow{tr} \mathbb{C}$,



 $: A \otimes A \xrightarrow{m} A \xrightarrow{tr} \mathbb{C}$ is nondegenerate:  (use the Frobenius relation).

A is a commutative Frobenius algebra (i.e. commutative $\mathbb{C}$-algebra together with a linear map $tr: A \longrightarrow \mathbb{C}$ such that $(a,b) \longmapsto tr(ab)$ is nondegenerate.

Example: (1) $A = M_n(\mathbb{C})$, $tr((a_{ij})) = \sum_i a_{ii}$.

(2) $A = \mathbb{C}[t]/(t^n - 1)$, $tr(1) = 1$, $tr(t^i) = 0$ for $i = 1, 2, \dots, n-1$.

Theorem: $2\mathrm{TQFT} \simeq \mathrm{cFA}_{\mathbb{C}}$.

For a proof see J. Koch's book (CUP, No. 59 of LMSST, 2003).

TQFTs produce topological invariants: every closed surface can be considered as a cobordism from $\emptyset$ to $\emptyset$, so its image under a TQFT is a linear map $\mathbb{C} \longrightarrow \mathbb{C}$ (i.e. a constant) which is a topological invariant of the surface.

<u>TQFTs and physics</u>

TQFTs posses certain features that we expect from quantum gravity.

The closed manifolds represent space. The cobordisms represents space-time. The $Z(M)$'s are the state spaces. An operator associated to a space-time is the time-evolution operator (Feynman path integral).

Topological means that these do not depend on any additional structure on space-time (e.g. Riemannian metric, curvature) but only on the topology.

See Barrett (J. Math. Phys. Vol. 36, 1995) or Freed (Bulletin AMS, 2013).

Also, Milnor's paper (Bulletin, AMS, 2015) is definitely worth reading. (No physics.)