Norwegian University of Science and Technology
Deptartment of Mathematical Sciences

TMA4190 Introduction to Topology Spring 2018

Exercise set 8

$\boxed{1}$ Recall that a manifold X is *simply connected* if it is connected and if every smooth map of the circle S^1 into X is homotopic to a constant map. Prove that the sphere S^k is simply connected if $k > 1$. (Hint: If $f\colon S^1 \to S^k$ and $k > 1$, Sard's Theorem gives us a point $p \notin f(S^1)$. Now use stereographic projection.)

$\boxed{2}$ Show that the determinant function on $M(n)$ is a Morse function if $n = 2$, but not if $n > 2$. (Hint: To find the partial derivatives of det, one can use Laplace's formula for the determinant: for any fixed j,

$$\det(A) = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \cdot \det(A_{ij})$$

where A_{ij} is the submatrix of A with ith row and jth column removed. Check if the zero matrix is nondegenerate.)

$\boxed{3}$ Show that the "height function" $h\colon S^k \to \mathbb{R}$, $(x_1, \ldots, x_{k+1}) \mapsto x_{k+1}$ on the k-sphere S^k is a Morse function with two critical points, one of which is a maximum and the other a minimum.

$\boxed{4}$ A vector field on X is a smooth section of $\pi\colon T(X) \to X$, i.e. a smooth map $\sigma\colon X \to T(X)$ such that $\pi \circ \sigma = \mathrm{Id}_X$. An equivalent way to describe such a section is to give a map $s\colon X \to \mathbb{R}^N$ such that $s(x) \in T_x(X)$ for all x (with correspinding $\sigma(x) = (x, s(x))$). A point $x \in X$ is a zero of the vector field σ if $\sigma(x) = (x, 0)$ or equivalently $s(x) = 0$.

a) Show that if k is odd, there exists a vector field on S^k having no zeros.
(Hint: For $k = 1$, use $(x_1, x_2) \mapsto (-x_2, x_1)$.)

b) Prove that if S^k has a vector field which has no zeros, then its antipodal map $x \mapsto -x$ is homotopic to the identity.
(Hint: Show that you may assume $|s(x)| = 1$ everywhere. Now contemplate about $(\cos(\pi t))x + (\sin(\pi t))s(x)$ when t varies from 0 to 1.)

c) Show that if k is even, then the antipodal map on S^k is homotopic to the reflection map

$$r\colon S^k \to S^k, (x_1, \ldots, x_{k+1}) \mapsto (-x_1, x_2, \ldots, x_{k+1}).$$

(Hint: Consider also the reflections $r_i(x_1, \ldots, x_{k+1}) = (x_1, \ldots, -x_i, \ldots, x_{k+1})$. Show that $r_i \circ r_{i+1}$ is homotopic to the identity on S^k.)

$\boxed{5}$ Let X be the set of all straight lines in $\mathbb{R}^2$ (not just lines through the origin).

a) Show that X is an abstract smooth 2-manifold by showing that we can identify X with an open subset of the real projective plane $\mathbb{RP}^2$.

(Here we use that open subsets of abstract smooth k-manifolds are again abstract smooth k-manifolds.)

b) Show that there is a bijection bewtween X and the set of equivalence classes

$$(S^1 \times \mathbb{R})/\sim$$

where $\sim$ is the equivalence relation defined by

$$(s, x) \sim (y, t) \iff t = \pm s \text{ and } y = x.$$