NTNU

Norwegian University of
Science and Technology

Department of Mathematical Sciences

Examination paper for **TMA4145 Linear Methods**

Academic contact during examination: Franz Luef

Phone: 406 14 405

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Examination time (from–to): 09:00-13:00

Permitted examination support material: D:No written or handwritten material. Calculator Casio fx-82ES PLUS, Citizen SR-270X, Hewlett Packard HP30S

Other information:
There are 5 problems on the exam and each problem counts for 20 points. All solutions should be stated in a precise and rigorous way, with any assumptions written down and arguments justified, except Problem 3.

Language: English

Number of pages: 3

Number of pages enclosed: 0

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Informasjon om trykking av eksamensoppgave	
Originalen er:	
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Problem 1

a) (1) Find the singular value decomposition for the matrix

$$A = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \end{pmatrix}.$$

(2) The linear system:

$$\begin{aligned} x_1 + x_2 - x_3 &= 1 \\ x_1 + x_2 - x_3 &= 1 \end{aligned}$$

has infinitely many solutions. Determine the one with the minimal Euclidean norm $\|.\|_2$.

The linear system

$$\begin{aligned} x_1 + x_2 - x_3 &= 1 \\ x_1 + x_2 - x_3 &= 2 \end{aligned}$$

has no solution. Determine the least squares solution of the linear system.

Hint: The pseudoinverse of the matrix related to the linear system might be useful.

b) Given a $n \times n$-matrix A of rank n. Prove that A has a polar decomposition using the singular value decomposition of A. Hence, show that there exist an $n \times n$ unitary matrix W and a positive definite $n \times n$ matrix P such that $A = WP$.

Problem 2

a) Let T be the linear transformation $T(x) = Ax$ on $\mathbb{R}^3$ for the matrix

$$A = \begin{pmatrix} 0 & 1/2 & 1/3 \\ 1/4 & 0 & 1/5 \\ 1/5 & \alpha & 0 \end{pmatrix},$$

where α is a real number.

(1) Determine the operator norm of $T : (\mathbb{R}^3, \|.\|_1) \to (\mathbb{R}^3, \|.\|_1)$. Note that the result depends on the parameter α.

(2) Determine those α's such that T is a contraction on $(\mathbb{R}^3, \|.\|_1)$.

b) Rewrite the linear system

$$\begin{aligned} 3x_1 - \tfrac{3}{2}x_2 - x_3 &= 1 \\ -x_1 + 4x_2 - \tfrac{4}{5}x_3 &= 2 \\ -\tfrac{2}{5}x_1 - \tfrac{1}{2}x_2 + 2x_3 &= 4 \end{aligned}$$

as a fixed point problem and show that one can use Banach's fixed point theorem to prove the existence of a solution. Compute the first three iterations $x^{(1)}, x^{(2)}, x^{(3)}$ for the starting point $x_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$.

Problem 3

a) (1) Suppose $(X, \|.\|_X)$ and $(Y, \|.\|_Y)$ are normed spaces. Define the notions of a *continuous* and of a *Lipschitz continuous* function $f : X \to Y$.

(2) Let X be a vector space and T a linear map between the vector spaces $T : X \to X$. Define the notion of a *T-invariant subspace* of X.

(3) Let $(X, \|.\|)$ be a normed space$X$. Define the notion of a *dense subset* of X and define when X is *separable*.

(4) Let X be a vector space and $T : X \to X$ a linear transformation. Define the notion of a *generalized eigenspace* for an eigenvalue λ of T and the *minimal polynomial* of a $n \times n$-matrix A.

(5) Define the notions of a *Cauchy sequence* and of *completeness* for normed space.

b) Determine if the following statements are true or false and if the statement is not true, give a counterexample.

(1) Any linear map on a normed space is bounded.

(2) Any linear transformation on a finite-dimensional complex vector space has a non-trivial invariant subspace.

(3) The set of sequences with finitely many non-zero elements is dense in the space of bounded sequences ℓ^∞.

(4) The orthogonal complement of any subset of an innerproduct space is closed.

(5) The range of any bounded linear map on an infinite-dimensional vector space is closed.

Problem 4 For $a = (a_n)_{n\in\mathbb{N}} \in \ell^\infty$ we define the linear operator $T_a : \ell^2 \to \ell^2$ by $T_a(x_1, x_2, ...) = (a_1x_1, 0, a_3x_3, 0, ...)$ for $(x_n) \in \ell^2$.

(1) Show that T_a is bounded on ℓ^2.

(2) Determine the operator norm of T_a.

(3) Show that the range of T_a is closed.

(4) Determine the orthogonal complement of $\ker(T_a)$.

(5) Determine for which sequences $a \in \ell^\infty$ the operator T_a satisfies $T_a^2 = T_a$.

Problem 5 Let $\{e_n\}_{n\in\mathbb{N}}$ be an orthonormal system in a Hilbert space X and $(\alpha_n)_{n\in\mathbb{N}}$ a sequence of complex numbers.

Show that the series $\sum_{n\in\mathbb{N}} \alpha_n e_n$ converges in X if and only if $(\alpha_n)_{n\in\mathbb{N}} \in \ell^2$.