Exercise set 13

Please justify your answers! The most important part is *how* you arrive at an answer, not the answer itself.

1 Let $A \in \mathcal{M}_{m\times n}(\mathbb{C})$ and $B \in \mathcal{M}_{n\times m}(\mathbb{C})$, and let $\lambda \in \mathbb{C}$ be any nonzero scalar. Show that λ is an eigenvalue of AB if and only if λ is an eigenvalue of BA.

2 Suppose that A and B are *unitarily equivalent*, meaning that there exists a unitary matrix U such that

$$B = U^*AU.$$

Prove that A is positive definite (semi-definite) if and only if B is positive definite (semi-definite).

3 Let $A \in \mathcal{M}_{n\times n}(\mathbb{C})$ be a normal matrix. Prove that

$$\det(A) = \prod_{j=1}^{n} \lambda_j,$$

where the λ_j's are the (not necessarily distinct) eigenvalues of A.

4 *(Exam 2017, Problem 1a)*

a) Find the singular value decomposition for the matrix

$$A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \end{bmatrix}.$$

b) The linear system

$$\begin{aligned} x_1 + x_2 - x_3 &= 1 \\ x_1 + x_2 - x_3 &= 1 \end{aligned}$$

has infinitely many solutions. Find the solution with minimal Euclidean norm $\|\cdot\|_2$.

c) The linear system

$$x_1 + x_2 - x_3 = 1$$
$$x_1 + x_2 - x_3 = 2$$

is inconsistent, and has no solution. Find the unique best approximation to a solution having minimum norm.

d) Prove that an $(n \times n)$ matrix A of full rank has a polar decomposition using the singular value decomposition of A. Hence, show that there exists an $(n \times n)$ unitary matrix W and a positive definite (not just semi-definite) $(n \times n)$ matrix P such that $A = WP$.