Norwegian University of Science and Technology
Department of Mathematical Sciences

TMA4145 Linear Methods
Fall 2018

Exercise set 5

Please justify your answers! The most important part is *how* you arrive at an answer, not the answer itself.

$\boxed{1}$ Let $(x_n)_{n\in\mathbb{N}}$ be a convergent sequence in a metric space (X, d).

a) Show that $(x_n)_{n\in\mathbb{N}}$ is a bounded subset of X.

b) Show that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence.

$\boxed{2}$ Let $(X, \langle\cdot,\cdot\rangle)$ be an inner product space, and suppose that (x_n) and (y_n) are convergent sequences with $\lim_n x_n = x$ and $\lim_n y_n = y$. Show that

$$\lim_{n\to\infty} \langle x_n, y_n\rangle = \langle x, y\rangle.$$

(This was exam problem 5b in Spring 2015.)

$\boxed{3}$ We denote by c_f the vector space of all sequences with only finitely many non-zero terms. Show that c_f is not a Banach space with the norm $\|\cdot\|_\infty$. As usual, $\|\cdot\|_\infty$ is defined by

$$\|x\|_\infty = \sup_{n\in\mathbb{N}} |x_n|$$

for a sequence $x = (x_n)_{n\in\mathbb{N}} \in c_f$.

$\boxed{4}$ For each $n \in \mathbb{N}$, let

$$x^{(n)} := \left(1, \frac{1}{2}, \ldots, \frac{1}{n}, 0, 0, \ldots\right),$$

which we regard as an element of the space $\ell^p(\mathbb{R})$ (for any given $p \in [1, \infty]$).

a) Find the limit of the sequence $\left(x^{(n)}\right)_{n\geq 1}$ in $(\ell^\infty(\mathbb{R}), \|\cdot\|_\infty)$. Prove your claim.

b) Does $\left(x^{(n)}\right)_{n\geq 1}$ have a limit in $(\ell^1(\mathbb{R}), \|\cdot\|_1)$? If the limit exists, find it and prove that it is the limit.

c) Does $\left(x^{(n)}\right)_{n\geq 1}$ have a limit in $(\ell^2(\mathbb{R}), \|\cdot\|_2)$? If the limit exists, find it and prove that it is the limit.

$\boxed{5}$ Let $C[a,b]$ be the vector space of all continuous functions $f\colon [a,b] \to \mathbb{R}$.

We will consider two norms on this space, $\|\cdot\|_1$ and $\|\cdot\|_\infty$.

a) Prove that for all $f \in C[a,b]$ we have

$$\|f\|_1 \leq (b-a)\,\|f\|_\infty\,.$$

b) Let (f_n) be a sequence in $C[a,b]$.
Prove that if $f_n \to f$ with respect to $\|\cdot\|_\infty$ then $f_n \to f$ with respect to $\|\cdot\|_1$.

c) Show that the reverse of the statement in b) is not always true.