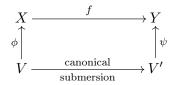


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TMA4190 Introduction to Topology Spring 2018

Solutions to exercise set 4

1 We can assume that U=X, since we just need to consider the map $U\subseteq X\xrightarrow{f}Y$. Let y be any point in f(X). We need to show that there is an open nieghborhood W around y which is contained in f(X). Let x be a point in X with f(x)=y which exists since $y\in f(X)$. By the Local Submersion Theorem, we can choose local parametrizations $\phi\colon V\to X$ around x with $Y\subset \mathbb{R}^n$ open and $\psi\colon Y'\to Y$ around y with $Y'\subset \mathbb{R}^m$ open such that the induced map $Y\to Y'$ is the canonical submersion:



By possibly shrinking V and V', we can assume that $V = B_{\epsilon}(0) \subset \mathbb{R}^n$ and $V' = B_{\epsilon}(0) \subset \mathbb{R}^m$. Then the canonical submersion maps V onto V'. Since the above diagram commutes, we see that $W := \psi(V')$ is contained in f(X). Since ψ is a diffeomorphism onto its image in Y, W is open in Y.

- a) Let $f: X \to Y$ be a submersion. We obviously have $Y = f(X) \cup (Y \setminus f(X))$. Since X is compact and f is continuous, we know f(X) is compact and therefore closed in Y. Since X is open in X and f is a submersion, the previous exercise shows that f(X) is open in Y. Hence f(X) is both open and closed in Y. Hence f(X) must be either Y or \emptyset . Assuming f is nontrivial, f(X) must be all of Y.
 - b) Given a compact smooth manifold X. Assume we had a submersion $f: X \to \mathbb{R}^n$ with for some n. By the previous point, we would have $f(X) = \mathbb{R}^n$. But since X is compact, f(X) is compact too. But \mathbb{R}^n is not compact. Hence such a submersion cannot exist.
- Given $A = (a_{ij}) \in O(n)$. Unfolding the matrix-multiplication, we see that $\sum_j a_{ij}^2$ is the *i*th diagonal entry in AA^t . Moreover, $\sum_j a_{ij}^2$ is also the square of the norm of the *i*th row vector of A. Since $A \in O(n)$, we have $AA^t = I$ and the *i*th diagonal entry in AA^t is equal 1. This shows that O(n) is contained in the product of n spheres $\prod S^{n-1}$ in $\prod \mathbb{R}^n = \mathbb{R}^{n^2} = M(n)$. Hence O(n) is bounded. But O(n) is also closed in \mathbb{R}^{n^2} , since we can define it as the inverse image of the closed point $I \in S(n)$ under the map $M(n) \to S(n)$ sending A to AA^t . Thus O(n) is closed and bounded in \mathbb{R}^{n^2} and therefore compact.

- We consider O(n) as a subspace in $M(n) = \mathbb{R}^{n^2}$. We defined O(n) as $f^{-1}(I)$ under the map $f \colon M(n) \to S(n)$, $f(A) = AA^t$. We have checked in the lecture that I is a regular value for f. By the first propostion of Lecture 8, this implies that $T_I(O(n))$ equals the kernel of $df_I \colon M(n) = T_I(M(n)) \to T_I(S(n)) = S(n)$. We calculated the derivative df_A for any $A \in O(n)$ in the lecture. It is given by $df_A(B) = BA^t + AB^t$. For A = I, this gives $df_I(B) = B + B^t$. Hence the kernel of df_I is the space of matrices satisfying $B B^t = 0$, i.e. $B^t = -B$.
- As an open subspace of M(n), the space of nonzero 2×2 -matrices $M(2) \setminus \{0\}$ is a manifold of dimension 4. A 2×2 -matrix A has rank 0 if and only if it is the zero matrix. Thus $A \in M(2) \setminus \{0\}$ has rank 1 if and only if it does not have rank 2, i.e. if and only if it is not invertible. Hence $A \in M(2) \setminus \{0\}$ has rank 1 if and only if det A = 0. Thus it suffices to show that the determinant function is a submersion $M(2) \setminus \{0\} \to \mathbb{R}$. For then $R_1 = \det^{-1}(0)$ is a submanifold of dimension 4 1 = 3.

We need to show that the derivative of the determinant at a matrix $A \in R_1$ is surjective. Since $d(\det)_A$ is a linear map $\mathbb{R}^4 \to \mathbb{R}$, it suffices to show $d(\det)_A$ is nonzero. Therefore, it suffices to show that $d(\det)_A(B) \neq 0$ for some matrix B.

Since $A \neq 0$, there is at least one entry in A which is nonzero. Assume that $a_{11} \neq 0$ (for the other cases the argument is similar). Then we take the matrix $E_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$:

$$d(\det)_A(E_{22}) = \lim_{s \to 0} \frac{\det(A + sE_{22}) - \det A}{s}$$

$$= \lim_{s \to 0} \frac{\det(A + sE_{22}) - 0}{s}$$

$$= \lim_{s \to 0} \frac{(a_{11})(a_{22} + s) - a_{12}a_{21}}{s}$$

$$= \lim_{s \to 0} \frac{a_{11}a_{22} + a_{11}s - a_{12}a_{21}}{s}$$

$$= \lim_{s \to 0} \frac{a_{11}s + \det A}{s}$$

$$= \lim_{s \to 0} \frac{a_{11}s}{s} = a_{11} \neq 0.$$