Repetition

Definition

Let $\{X(t) : t \geq 0\}$ be a continuous-time Markov chain with states $\{0, 1, \ldots\}$ and stationary transition probabilities.

Then $\{X(t) : t \geq 0\}$ is a **birth and death process** with **birth rates** $\lambda_0, \lambda_1, \ldots$ and **death rates** $\mu_0, \mu_1, \ldots$ if

1. $P_{i,i+1}(h) = \lambda_i h + o(h) \quad (\text{as } h \to 0^+)$ for $i \geq 0$.
2. $P_{i,i-1}(h) = \mu_i h + o(h) \quad (\text{as } h \to 0^+)$ for $i \geq 1$.
3. $P_{i,i}(h) = 1 - (\lambda_i + \mu_i)h + o(h) \quad (\text{as } h \to 0^+)$ for $i \geq 0$.
4. $$P_{ij}(0) = \delta_{ij} = \begin{cases} 0, & i \neq j \\ 1, & i = j, \end{cases} \quad \text{for } i, j \geq 0.$$
5. $\mu_0 = 0, \lambda_0 > 0$ and $\mu_i, \lambda_i > 0$ for $i \geq 1$.

Note: $P_{ij}(t) = \Pr\{X(t) = j | X(0) = i\}$, $t \geq 0$, for states i and j.

Definition

A **pure birth process** is a birth and death process where $\mu_i = 0, i \geq 0$. A **pure death process** is a birth and death process where $\lambda_i = 0, i \geq 0$.

Note: A pure birth process models reproduction in the abscence of death and migration, and a pure death process models deaths in the absecence of births and migration.

Theorem

The transition probability functions of a continuous-time Markov chain with state space $\{0, 1, \ldots\}$ and stationary transition probabilities, satisfy

$$P_{ij}(t+s) = \sum_{k=0}^{\infty} P_{ik}(t)P_{kj}(s), \quad t, s \geq 0,$$

for all states i and j. This result is called the **Chapman-Kolmogorov** equation.

Note: This is a direct consequence of the Markov property.