TMA 4190 Introduction to Topology
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Lecture 10[1]

10. A brief excursion into Lie groups - Part 2

The Special Linear Group

We continue our study of the **special linear group**

$$SL(n) = \{A \in M(n) : \det A = 1\}.$$

Last time, we learned that $SL(n)$ is a smooth manifold of dimension $n^2 - 1$. The same argument as for $GL(n)$ shows that it even is a Lie group. We will see another argument for that today.

But first we would like to calculate the **tangent space of $SL(n)$ at the identity matrix**.

This space plays a special role for any Lie group. In fact, the translation property of Lie groups implies that the tangent to a Lie group G at any matrix in G is isomorphic to tangent space to G at the identity element. It carries an additional structure and is an example of a **Lie algebra**.

To determine the tangent space at the identity, we use a result we proved last week which said: if $Z = f^{-1}(y) \subseteq X$ is a submanifold defined by a regular value y of a smooth map $f\colon X \to Y$, then $T_x(Z) = \mathrm{Ker}\,(df_x) \subseteq T_x(X)$.

Hence we need to calculate the **derivative of $\det$ at the identity**.

Recall that the determinant of a matrix A is given by Leibniz' formula

$$\det(B) = \sum_{\sigma} (\mathrm{sgn}(\sigma) \prod_{i=1}^{n} b_{i\sigma(i)}) \tag{1}$$

where the sum runs over all permutations of the set $\{1, \dots, n\}$ and $\mathrm{sgn}(\sigma)$ denotes the sign of the permutation σ.

Given a matrix A, in the determinant of $B := I + sA$, every summand contains at least a factor s^2 unless it is the product of at least $n-1$ diagonal entries $b_{ii} = 1 + sa_{ii}$ (because we need $n-1$ factors **not containing s** which is only possible when we multiply $n-1$ times 1). But if a permutation $\{1, \dots, n\}$ leaves

[1] Following the books of Guillemin and Pollack: Differential Topology; by Lee: Introduction to Smooth Manifolds; and by Tu: An Introduction to Manifolds.

$n-1$ numbers fixed, it also has to leave the remaining one fixed. Hence the only summand in (1) which does not contain a factor s^2 is the summand

$$\prod_{i=1}^{n}(1+sa_{ii}) = (1+sa_{11})\cdots(1+sa_{nn}) = 1 + s\cdot \operatorname{tr}(A) + O(s^2).$$

The derivative of the determnninant at the identity

$$d(\det)_I \colon T_I(M(n)) = M(n) \to T_1(\mathbb{R}) = \mathbb{R}$$

is then given by

$$\begin{aligned} d(\det)_I(A) &= \lim_{s\to 0} \frac{\det(I+sA) - \det I}{s} \\ &= \lim_{s\to 0} \frac{1 + s\cdot \operatorname{tr}(A) + O(s^2) - 1}{s} \\ &= \lim_{s\to 0} \frac{s\cdot \operatorname{tr}(A) + O(s^2)}{s} \\ &= \lim_{s\to 0} \operatorname{tr}(A) + O(s) \\ &= \operatorname{tr}(A). \end{aligned}$$

By the result from the previous lecture, we get

$$T_I(SL(n)) = \operatorname{Ker}(d(\det)_I) = \{A \in M(n) : \operatorname{tr}(A) = 0\}.$$

In other words, the tangent space to $SL(n)$ at the identity is the space of matrices whose trace vanishes.

The Special Orthogonal Group

Recall that the orthogonal group $O(n)$ is defined as the subset of matrices A in $M(n)$ such $AA^t = I$. This equation implies, in particular, that every $A \in O(n)$ is invertible with $A^{-1} = A^t$. Hence the determinant of an $A \in O(n)$ must satisfy $(\det A)^2 = 1$, i.e. $\det A = \pm 1$. Thus, $O(n)$ splits into two disjoint parts, the subset of matrices with determinant $+1$ and the subset of matrices with determinant -1.

If A and B have determinant -1, then their product AB has determinant $+1$. Hence the subset of matrices with determinant -1 is not closed under multiplication and therefore not a subgroup of $O(n)$. But the other part is a Lie subgroup of $O(n)$ and is called the **Special Orthogonal Group** $SO(n)$:

$$SO(n) = \{A \in O(n) : \det A = 1\} \subset O(n).$$

Unitary and Special Unitary Groups

The **unitary group** $U(n)$ is defined to be

$$U(n) := \{A \in GL(n,\mathbb{C}) : \bar{A}^t A = I\},$$

where $\bar{A}$ denotes the complex conjugate of A, the matrix obtained from A by conjugating every entry of A. A similar argument as for $O(n)$ shows that $U(n)$ is a submanifold of $GL(n,\mathbb{C})$ and that $\dim U(n) = n^2$.

The **special unitary group** $SU(n)$ is defined to be the subgroup of $U(n)$ of matrices of determinant 1.

Some identities

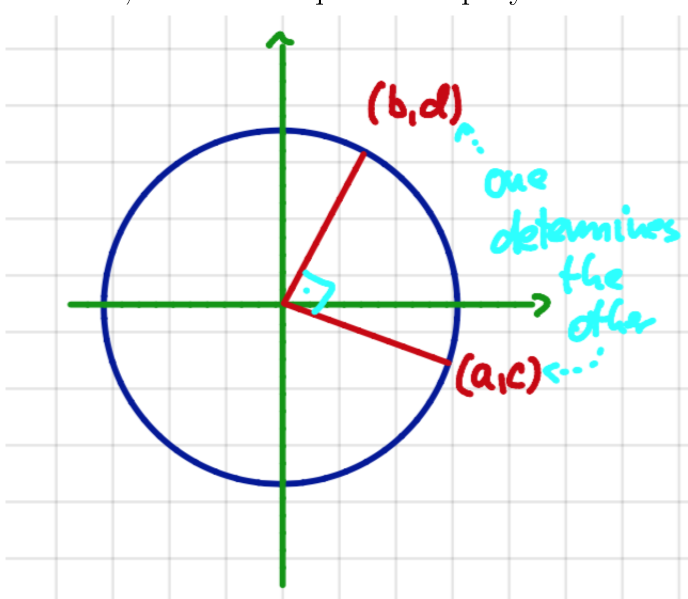
There are a couple of identities, most of which are incidental and do not reflect any deeper pattern. They are interesting nevertheless. For example:

(a) For $n = 1$, $O(1)$ consists of just two points: $O(1) = \{-1, +1\}$.
(b) For $n = 2$, $SO(2)$ is diffeomorphic to S^1:

For, any $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SO(2)$ satisfies

$$A^t A = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a^2 + c^2 & ab + cd \\ ab + cd & b^2 + d^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Hence A corresponds to two points (a,c) and (b,d) on $S^1 \subset \mathbb{R}^2$ whose corresponding vectors are orthongonal to each other. Since we also know $\det A = ad - bc = 1$, one of these points uniquely determines the other



and we can write A as $\begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$ for some real number t. Now one can check that the map

$$S^1 \to SO(2),\ (\cos t, \sin t) \mapsto \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$$

is a diffeomorphism and Lie group isomorphism.

(c) For $n = 2$, $SU(2)$ is diffeomorphic to S^3: Any $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SU(2)$ satisfies

$$\bar{A}^t A = \begin{pmatrix} \bar{a} & \bar{c} \\ \bar{b} & \bar{d} \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \bar{a}a + \bar{c}c & \bar{a}b + \bar{c}d \\ \bar{b}a + \bar{d}c & \bar{b}b + \bar{d}d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Together with $\det A = ad - bc = 1$ we get four linear equations for the complex numbers a, b, c, d, and their complex conjugates. Unraveling these equations shows that we can write A as

$$A = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \text{ with } a\bar{a} + b\bar{b} = 1.$$

Hence A corresponds uniquely to a pair of complex numbers (a,b) which satisfies $a\bar{a} + b\bar{b} = 1$. Since this is exactly the defining condition for elements of $S^3 \subset \mathbb{C}^2$, we see that

$$S^3 \to SU(2), (a,b) \mapsto \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}$$

is a diffeomorphism.

Spin groups

There are other important examples of Lie groups which, in general, do not arise as closed subgroups of $GL(n,\mathbb{R})$ or $GL(n,\mathbb{C})$. For example, the nth **Spin group** $\mathrm{Spin}(n)$ is the n-dimensional Lie group which is a double cover of $SO(n)$. The latter means that $\mathrm{Spin}(n)$ is equipped with a smooth surjective map $\pi \colon \mathrm{Spin}(n) \to SO(n)$ such that each point in $SO(n)$ has an open neighborhood U such that $\pi^{-1}(U)$ is a disjoint union of open subsets in $Spin(n)$ each of which is mapped diffeomorphically onto U by π. (We have seen covering spaces when we discussed the Stack of Records Theorem.) The map π is part of a short exact sequence of groups

$$1 \to \mathbb{Z}/2 \to \mathrm{Spin}(n) \to SO(n) \to 1.$$

Spin groups can be constructed for example via Clifford algebras. However, there are some exceptional isomorphisms in low dimensions which we can write down:

$$\begin{aligned}
\mathrm{Spin}(1) &\cong O(1),\\
\mathrm{Spin}(2) &\cong SO(2),\\
\mathrm{Spin}(3) &\cong SU(2),\\
\mathrm{Spin}(4) &\cong SU(2) \times SU(2),\\
\mathrm{Spin}(6) &\cong SU(4).
\end{aligned}$$

Topology of Lie groups

Just as $O(n)$ (this was an exercise), $SO(n)$ is compact (whereas $GL(n)$ is not compact as an open subset of $M(n)$). Similarly, $U(n)$ and $SU(n)$ are compact.

Moreover, note that both $SO(n)$ and its complement are both open and closed in $O(n)$. They are the **two connected components of $O(n)$**. In particular, there is no continuous path in $O(n)$ from a matrix with determinant $+1$ to one with determinant -1. In fact, there is no such path in $GL(n)$:

The real general linear group is not connected

Let γ be a path in $GL(n)$, i.e. a continuous map

$$\gamma\colon [0,1] \to GL(n).$$

Since γ and det are continuous, so is their composite

$$\det \circ \gamma\colon [0,1] \xrightarrow{\gamma} GL(n) \xrightarrow{\det} \mathbb{R}.$$

Hence if $\det(\gamma(0)) > 0$ and $\det(\gamma(1)) < 0$, then the Intermediate Value Theorem from Calculus implies that there must be a real number $t_0 \in (0,1)$ such that $\det(\gamma(t_0)) = 0 \notin GL(n)$. Hence γ would have to leave $GL(n)$.

Thus also $GL(n)$ has two connected components, one of which is an open subgroup consisting to all matrices A with $\det A > 0$. The other one is just an open subset consisting to all matrices A with $\det A < 0$.

The complex general linear group is connected

However, $GL(n,\mathbb{C})$ is path-connected. We see the difference between $GL(n,\mathbb{R})$ and $GL(n,\mathbb{C})$ most clearly for the case $n = 1$: $GL(1,\mathbb{R}) = \mathbb{R}^*$

is not path-connected, since we cannot cross 0; whereas $GL(1,\mathbb{C}) = \mathbb{C}^*$ is path-connected, since we can just walk around 0 in the plane.
More generally, to show that $GL(n,\mathbb{C})$ is path-connected, it suffices to show that there is path from any matrix $A \in GL(n,\mathbb{C})$ to the identity matrix $I \in GL(n,\mathbb{C})$. Therefore, we define first the function

$$P\colon \mathbb{C} \to \mathbb{C},\ z \mapsto \det(A + z(I - A)).$$

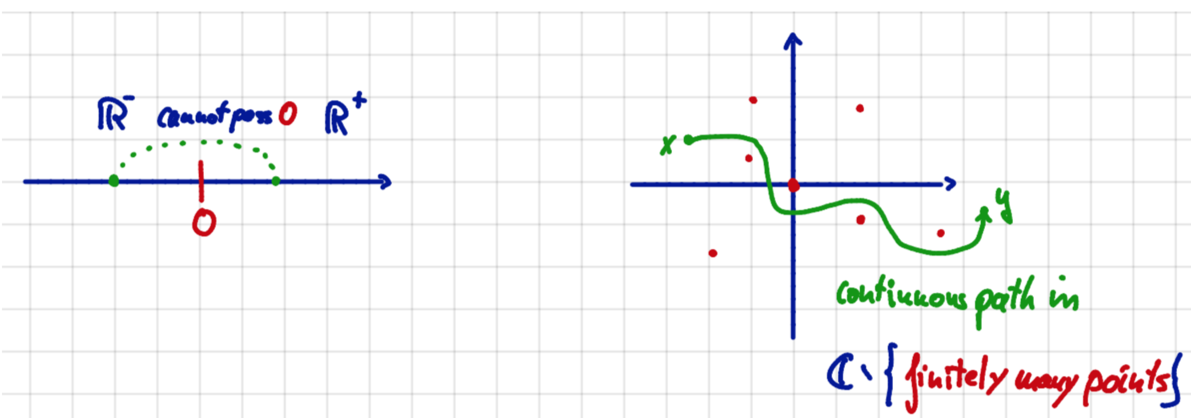
Then we have $P(0) = \det A \neq 0$ and $P(1) = \det I = 1 \neq 0$. Since P is a polynomial of degree n, it has only finitely many zeroes. Since $\mathbb{C} \setminus \{\text{set of finitely many points}\}$ is path-connected, we can find a path $\gamma\colon [0,1] \to \mathbb{C}$ with $\gamma(0) = 1$, $\gamma(1)$ and which avoids the zeroes of P, i.e.

$$P(\gamma(t)) \neq 0 \text{ for all } t.$$

Then the continuous map

$$\Gamma = P \circ \gamma\colon [0,1] \to GL(n,\mathbb{C}),\ t \mapsto A + \gamma(t)(I - A)$$

is the desired path from A to I.



The fact that $\mathrm{GL}(n,\mathbb{C})$ is connected while $GL(n,\mathbb{R})$ is not plays a crucial role for orientations of vector spaces, vector bundles, manifolds etc. For, every complex vector space, complex vector bundle, complex manifold, etc has a **natural orientation**. We will get back to this later.

Open neighborhoods of the identity.

Recall that if G is a group and $S \subset G$ is a subset, the **subgroup generated by S** is the smallest subgroup containing S, i.e., the intersection of all subgroups containing S. One can check that the subgroup generated by S is equal to the set of all elements of G that can be expressed as finite products of elements of S and their inverses.

Neighborhoods of the identity

Suppose G is a Lie group, and $W \subset G$ is any neighborhood of the identity. Then
(a) W generates an open subgroup of G.
(b) If G is connected, then W generates G. In particular, an open subgroup in a connected Lie group must be equal to the whole group.

Proof: Let $W \subset G$ be any neighborhood of the identity, and let H be the subgroup generated by W. To simplify notation, if A and B are subsets of G, we write

$$AB := \{ab : a \in A, b \in B\}, \text{ and } A^{-1} := \{a^{-1} : a \in A\}.$$

For each positive integer k, let W_k denote the set of all elements of G that can be expressed as products of k or fewer elements of $W \cup W^{-1}$. As mentioned above, H is the union of all the sets W_k as k ranges over the positive integers.

Now, $W^{-1}1$ is open because it is the image of W under the inversion map, which is a diffeomorphism. Thus, $W_1 = W \cup W^{-1}$ is open, and, for each $k > 1$, we have

$$W_k = W_1 W_{k-1} = \cup_{g \in W_1} L_g(W_{k-1}).$$

Because each L_g is a diffeomorphism, it follows by induction that each W_k is open, and thus H is open as a union of open subsets.

(b) Assume G is connected. We just showed that H is an open subgroup of G. It is an exercise to show that an open subgroup in a connected Lie group is equal to the whole group. **QED**

Lie subgroups

In the previous paragraph we talked about subgroups of a Lie group. But we did not disucss how the subgroup structure relates to the structure as a smooth manifold. Actually, this is a subtle and interesting point that illustrates the importance of the distinction between immersions and embeddings once again. So here is the definition of a Lie subgroup:

Definition of Lie subgroups

A **Lie subgroup** of a Lie group G is an abstract subgroup H such that if there exists a smooth manifold X and an **immersion** $f\colon X \to G$ from X to G such that $H = \mathrm{Im}\,(f) \subseteq G$ is the image of f, and the group operations on H are smooth, in the sense that $X \times X \xrightarrow{f\times f} G \times G \xrightarrow{\mu} G$ and $X \xrightarrow{f} G \xrightarrow{\iota} G$ are smooth.

Let us have a closer look at this rather complicated definition:

An "abstract subgroup simply means a subgroup in the algebraic sense. The group operations on the subgroup H are the restrictions of the multiplication map μ and the inverse map ι from G to H.

If H were defined to be a submanifold of G, then the multiplication map $H \times H \to H$ and similarly the inverse map $H \to H$ would automatically be smooth, and the definition would be much shorter. But since a Lie subgroup is defined to be an "immersed submanifold", it is necessary to impose the last condition.

If H is in fact also a submanifold, then life is easier:

Embedded Lie subgroups

If H is an abstract subgroup and a submanifold of a Lie group G, then it is a Lie subgroup of G. In this case, the inclusion map $H \hookrightarrow G$ is an embedding, and we call H an **embedded subgroup**.

Proof: Since H is a subgroup, multiplication and taking inverses in H are just the restrictions of multiplication and taking inverses in G and both have image in H. Since H is a submanifold we can take $X = H$ in the above definiton, the restrcitions of smooth maps to H are again smooth. **QED**

For example, the subgroups $SL(n)$ and $O(n)$ of $GL(n)$ are both submanifolds, and therefore embedded Lie subgroups. Another example is given as follows:

Complex vs Real

One easily verifies that

$$\mathbb{C} \to M(2,\mathbb{R}),\ z = x + iy \mapsto \begin{pmatrix} x & y \\ -y & x \end{pmatrix}$$

is an embedding. More generally, this map induces an embedding

$$GL(n,\mathbb{C}) \hookrightarrow GL(2n,\mathbb{R})$$

by replacing each entry $z = x + iy$ in $A \in GL(n,\mathbb{C})$ by the block $\begin{pmatrix} x & y \\ -y & x \end{pmatrix}$:

$$\begin{pmatrix} x_{11} + iy_{11} & \dots & x_{1n} + iy_{1n} \\ \vdots & \ddots & \vdots \\ x_{n1} + iy_{n1} & \dots & x_{nn} + iy_{nn} \end{pmatrix} \mapsto \begin{pmatrix} x_{11} & -y_{11} & \multirow{2}{*}{\dots} & x_{1n} & -y_{1n} \\ y_{11} & x_{11} & & y_{1n} & x_{1n} \\ \vdots & & \ddots & & \vdots \\ x_{n1} & -y_{n1} & \multirow{2}{*}{\dots} & x_{nn} & -y_{nn} \\ y_{n1} & x_{n1} & & y_{nn} & x_{nn} \end{pmatrix}$$

This way, $GL(n,\mathbb{C})$ is an embedded Lie subgroup of $GL(2n,\mathbb{R})$.

Now let us get back to understanding the definition of a Lie subgroup. The subleties of immersed and embedded subgroups can be illustrated by a familiar example:

Example of an immersed but not embedded Lie subgroup

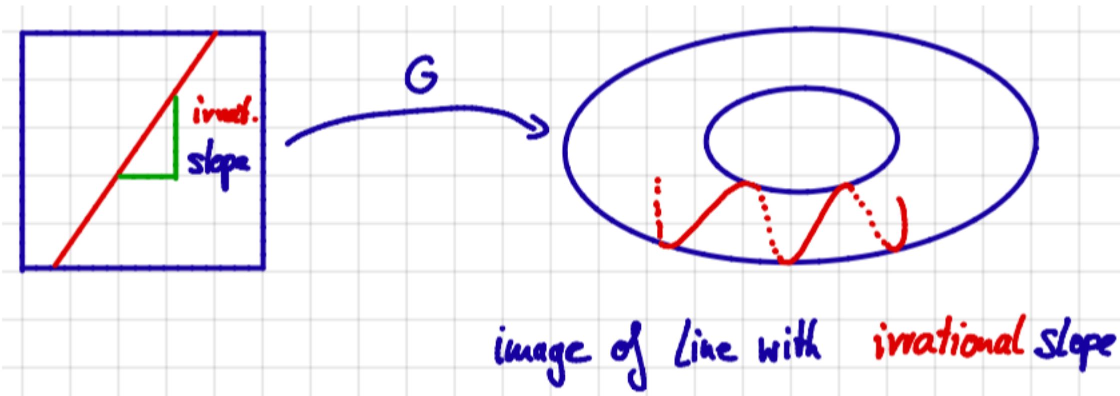
Recall the maps $g\colon \mathbb{R} \to S^1$, $t \mapsto (\cos(2\pi t), \sin(2\pi t))$, and

$$G\colon \mathbb{R}^2 \to S^1 \times S^1 = \mathbb{T}^2,\ G(x,y) = (g(x),g(y))$$

The map G is a local diffeomorphism from the plane onto the torus T^2.
Given a real number α, we defined the map γ_α by

$$\gamma_\alpha\colon \mathbb{R} \to \mathbb{T}^2,\ \gamma(t) = (g(t),g(\alpha \cdot t)).$$

We learned that γ_α is always an immersion, but its image is **not a submanifold** of $\mathbb{T}^2$ ifα is an **irrational** number. However, when α is rational, then $\gamma_\alpha(\mathbb{R})$ is a submanifold of T^2.
After checking that $\gamma_\alpha(\mathbb{R})$ is an abstract subgroup, we see that $\gamma_\alpha(\mathbb{R})$ is in fact a **Lie subgroup of $\mathbb{T}^2$** for every real number α. (Note that, in this example, the smooth manifold X and the smooth map $f\colon X \to G$ in the definition of Lie subgroups is $X = \mathbb{R}$, $f = \gamma_\alpha$, and $H = \gamma_\alpha(\mathbb{R})$.)



For an explanation of why a Lie subgroup is defined in such a complicated way, we refer to a fact we will only be able to appreciate later when we learn more about Lie theory:

Why so complicated?

A fundamental theorem in Lie group theory asserts the existence of a **one-to-one correspondence** between the connected Lie subgroups of a Lie group G and the Lie subalgebras of its Lie algebra $\mathfrak{g}$ (tangent space at the identity with its Lie bracket):

$$\{\text{connected Lie subgroups in } G\} \overset{1-1}{\longleftrightarrow} \{\text{Lie subalgebras in } \mathfrak{g}\}.$$

In the previous example, the Lie algebra of $\mathbb{T}^2$ has $\mathbb{R}^2$ as the underlying vector space, and the one-dimensional Lie subalgebras are all the lines through the origin (with addtion as group operation). Such a line is determined by its slope α. Hence **every** α should correspond to a **Lie subgroup** $\gamma_\alpha(\mathbb{R})$ in $\mathbb{T}^2$.

However, if a Lie subgroup had been defined as a subgroup that is also a submanifold, then one would have to exclude all the lines with irrational slopes as Lie subgroups of the torus. In this case it would not be possible to have a one-to-one correspondence between the connected subgroups of a Lie group and the Lie subalgebras of its Lie algebra. But this correspondence is extremely useful in Lie theory.

The following theorem is a very useful fact which we state here without proof (you can find it in Lee's book, Chapter 7, Theorem 7.21):

Closed Subgroup Theorem

Suppose G is a Lie group and $H \subseteq G$ is a Lie subgroup. Then H is closed in G if and only if it is an embedded Lie subgroup.