Introduction to Topology

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Contents

1. Introduction

These are lecture notes from the course TMA4190 Introduction to Topology given in the Spring semester 2021 at NTNU. They are intended as a supplement to the lectures and may not be entirely self-contained.

Please send me an email if you spot any errors!

What is topology?

> *Topology! The stratosphere of human thought! In the twenty-fourth century it might possibly be of use to someone. . .*
>
> — Aleksandr Solzhenitsyn

Topology is a part of mathematics concerned with the study of spaces. In topology, we consider two spaces to be *equivalent* if one can be continuously deformed into the other. Such a continuous deformation is known as a *homeomorphism*, i.e., a continuous bijection with a continuous inverse. See Figure 1.1 for an example of two homeomorphic spaces.

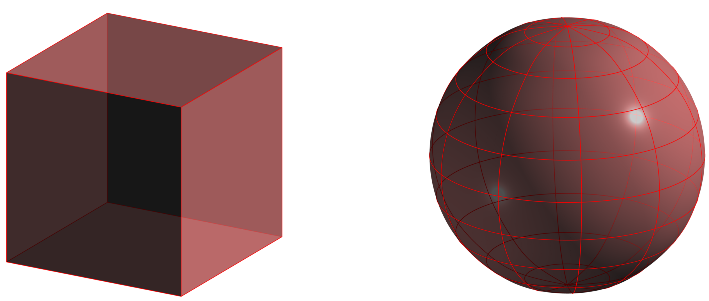

Figure 1.1: The surface of the (unit) cube and the (unit) sphere S^2 are homeomorphic.

We might ask ourselves the following question.

> **Question** Let X and Y be two spaces. Does there exist a homeomorphism $\varphi: X \to Y$? In other words, are X and Y homeomorphic?

Showing that two spaces are homeomorphic involves the construction of a specific homeomorphism between them. Proving that two spaces are *not* homeomorphic is a problem of a different nature. It is a hopeless exercise to check every possible map between the two spaces for whether or not it is a homeomorphism. Instead we might check to see whether there is some "topological invariant" of spaces (where this invariant is preserved under a homeomorphism) that allows us to differentiate between the two spaces.

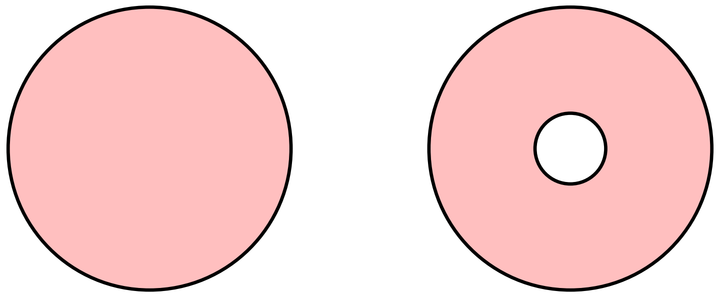

Figure 1.2: The disc D^2 and the annulus are not homeomorphic.

One instrument to help us detect topological information of a space is the *fundamental group* associated to the space. It is reasonable to expect that the disc D^2 and the annulus are not homeomorphic. The annulus has a hole through it while the disc does not, see Figure 1.2.

To detect the hole through the annulus we may use loops, i.e., continuous maps from the unit interval to the annulus with the endpoints identified. See Figure 1.3.

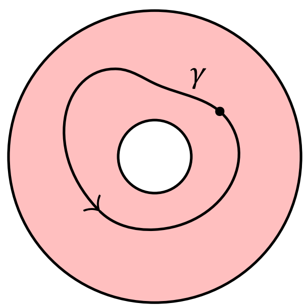

Figure 1.3: A loop.

It is then possible to construct a *group* involving such loops. This group is what is known as the fundamental group.

Some applications

To help illustrate some of the power of topology, let us consider two theorems, both of which may be proved using topology and more specifically, the fundamental group.

The first theorem is the Brouwer fixed point theorem.

Theorem 1.1 **(Brouwer fixed point theorem)** *Let $f: D^n \to D^n$ be a continuous map from the (unit) disk in $\mathbb{R}^n$ to itself. Then f has a fixed point, i.e., there is some point $x \in D^n$ such that $f(x) = x$.*

For $n = 1$ this is a well-known result from calculus: The graph of any continuous map $f: [0,1] \to [0,1]$ must cross the diagonal $y = x$ for some $x_* \in [0,1]$. Hence, $f(x_*) = x_*$. See Figure 1.4.

The second theorem is the fundamental theorem of algebra.

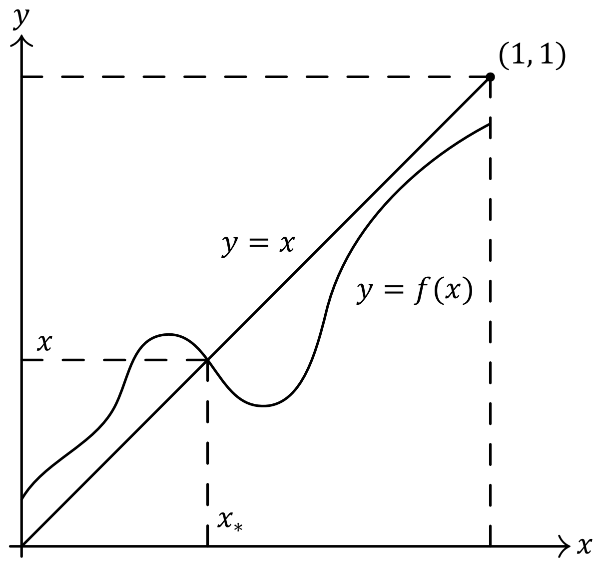

Figure 1.4: The graph of any continuous map from $[0, 1]$ to $[0, 1]$ must cross the diagonal.

Theorem 1.2 **(The fundamental theorem of algebra)** *A polynomial equation*

$$z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0 = 0$$

of degree $n > 0$ with complex coefficients has at least one complex root.

To prove it we will use the fact that the fundamental group of the circle is isomorphic to the group of integers. The fundamental theorem of algebra may be proved in many different ways, including using only algebraic techniques and analysis. However, the proof we will provide (based on [1]) is a fairly simple corollary of the computation of the fundamental group of the circle.

2. Continuous maps

2.1 Metric spaces

From calculus we know what to mean by a continuous map from $\mathbb{R}^n$ to $\mathbb{R}^m$: a map $f\colon \mathbb{R}^n \to \mathbb{R}^m$ is *continuous* at $p \in \mathbb{R}^n$ if for all $\epsilon > 0$ there exists a $\delta > 0$ such that if $\|p - q\|_{\mathbb{R}^n} < \delta$, then $\|f(p) - f(q)\|_{\mathbb{R}^m} < \epsilon$. Here $\|\cdot\|_{\mathbb{R}^n}$ denotes the Euclidean norm in $\mathbb{R}^n$. Similarly, $\|\cdot\|_{\mathbb{R}^m}$ denotes the Euclidean norm in $\mathbb{R}^m$.

Topological spaces provide the most general setting for which the concept of continuity makes sense. Before we get to the concept of a topological space, let us consider metric spaces. Metric spaces allow us to speak of distance between elements. Using the notion of distance between elements we can make sense of continuity of maps between metric spaces.

Definition 2.1 **(Metric spaces)** A *metric space* (X, d) is a non-empty set X together with a map $d\colon X \times X \to \mathbb{R}$ called a *metric* such that the following properties hold:

M1 $d(x, y) \geqslant 0$ for all $x, y \in X$, and $d(x, y) = 0$ if and only if $x = y$;

M2 $d(x, y) = d(y, x)$ for all $x, y \in X$;

M3 $d(x, z) \leqslant d(x, y) + d(y, z)$ for all $x, y, z \in X$.

The first condition says that the distance between two elements is always positive, and equal to zero if and only if the two elements are the same. The second condition says that distance is symmetric. The third condition says that the *triangle inequality* holds. The metric d is sometimes also referred to as a distance function.

Example 2.2 **($\mathbb{R}^n$ seen as a metric space)** Let $X = \mathbb{R}$ and d be the map defined by $d(x, y) = |x - y| (= \sqrt{(x - y)^2})$. The first two requirements for d are clearly satisfied, and the third follows from the usual triangle inequality for real numbers,

$$d(x, z) = |x - z| = |(x - y) + (y - z)| \leqslant |x - y| + |y - z| = d(x, y) + d(y, z).$$

For $X = \mathbb{R}^n$ with $n > 0$ an integer, let $d(x, y) = \|x - y\|$ where $\|\cdot\|$ is the Euclidean norm, e.g., for $n = 2$, $d(x, y) = \|x - y\| = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$. Again, the first two requirements for d are clearly satisfied. The third requirement follows from the triangle inequality for vectors in $\mathbb{R}^n$.

We may equip $\mathbb{R}^n$ with other metrics than the one described in Example 2.2. For instance, for $X = \mathbb{R}^2$, let

$$d(x, y) = |x_1 - y_1| + |x_2 - y_2|.$$

This is known as the *taxi cab metric*.

We say that two metrics d_1 and d_2 on the same set X are *equivalent* if there exist constants L and M such that

$$d_1(x,y) \leqslant Ld_2(x,y) \quad \text{and} \quad d_2(x,y) \leqslant Md_1(x,y)$$

for all $x, y \in X$.

Example 2.3 (Discrete metric spaces) For any set X, let $d\colon X \times X \to \mathbb{R}$ be the map given by

$$d(x,y) = \begin{cases} 1 & x \neq y, \\ 0 & x = y. \end{cases}$$

We call d the *discrete metric* on X.

Example 2.4 ($C[a,b]$) Let $X = C[a,b]$, i.e., the set of continuous maps from the interval $I = [a,b] \subseteq \mathbb{R}$ to $\mathbb{R}$, and let

$$d(x,y) = \max_{i \in I} |x(i) - y(i)|.$$

Example 2.5 If d is a metric on a set X, and $A \subseteq X$ is any subset of X, then d is also a metric on A.

2.2 Continuous maps between metric spaces

The definition of continuity of maps between metric spaces is completely analogous to the situation that we have from calculus.

Definition 2.6 (Continuous maps between metric spaces) Let (X, d_X) and (Y, d_Y) be two metric spaces. A map $f\colon X \to Y$ is *continuous* at $p \in X$ if for all $\epsilon > 0$ there is a $\delta > 0$ such that if $d_X(p,q) < \delta$ then $d_Y(f(p), f(q)) < \epsilon$.

If f is continuous at every $p \in X$, we say that f is *continuous*.

To get us to the setting of topological spaces we will need the concept of *open* and *closed* sets.

Definition 2.7 (Open and closed balls) Let (X,d) be a metric space, and let $a \in X$ and $r > 0$ be real number. The *open ball* centered at a with radius r is the subset

$$\mathrm{B}(a;r) = \{x \in X \mid d(x,a) < r\}$$

of X. The *closed ball* centered at a with radius r is the subset

$$\overline{\mathrm{B}}(a;r) = \{x \in X \mid d(x,a) \leqslant r\}$$

of X.

In Euclidean space with the usual metric (induced from Euclidean norm), a ball (as defined above) is precisely what we think of as a ball in everyday language. Open balls are sometimes referred to as simply balls, and closed balls are sometimes referred to as discs, e.g. Theorem 1.1.

Example 2.8 (Open balls in discrete metric spaces) Let (X, d) be the metric space defined in Example 2.3. Then

$$\mathrm{B}(x; r_1) = \{x\} \quad \text{and} \quad \mathrm{B}(x; r_2) = X$$

for all $0 < r_1 \leqslant 1$ and all $r_2 > 1$.

Definition 2.9 (Open and closed sets) Let (X, d) be a metric space. A subset $A \subseteq X$ is *open* in X if for every point $a \in A$, there is an open ball $\mathrm{B}(a; r)$ about a contained in A. We say that A is *closed* in X if the complement $A^c = X \setminus A = \{x \in X \mid x \notin A\}$ is open.

Most subsets are *neither* open nor closed. Subsets that are both open and closed are sometimes referred to as *clopen*. In particular, both $\emptyset$ and X are clopen in X.

Lemma 2.10 *Let (X, d) be a metric space, $x \in X$ and $r > 0$ a real number. Then the open ball $\mathrm{B}(x; r) \subseteq X$ is open in X, and the closed ball $\overline{\mathrm{B}}(x; r) \subseteq X$ is closed in X.*

Proof. We prove the statement about open balls. The statement about closed balls follows from a similar argument.

Assume that $y \in \mathrm{B}(x; r)$. We need to prove that there is an open ball $\mathrm{B}(y; \epsilon)$ about y that is contained in $\mathrm{B}(x; r)$. Let $\epsilon = r - d(x, y)$. By the triangle inequality of the metric d, M3, we have that for $z \in \mathrm{B}(y; \epsilon)$,

$$d(x, z) \leqslant d(x, y) + d(y, z) < d(x, y) + \epsilon = d(x, y) + r - d(x, y) = r.$$

Hence, $\mathrm{B}(y; \epsilon) \subseteq \mathrm{B}(x; r)$. □

For a metric space (X, d), a subset $A \subseteq X$ and $x \in X$, we say that: (i) x is an *interior point* of A if there is an open ball $\mathrm{B}(x; r)$ about x which is contained in A, (ii) x is an *exterior point* of A if there is an open ball $\mathrm{B}(x; r)$ which is contained in A^c and (iii) x is a *boundary point* if all open balls about x contains points in A and in A^c. Hence, A is open in X if and only if A only consists of its interior points. An interior point will *always* belong to A. An exterior point will *never* belong to A. A boundary point will some times belong to A, and some times to A^c.

Definition 2.11 (Neighborhoods) Let (X, d) be a metric space, A a subset of X and $x \in X$. We say that A is a *neighborhood of x* if there is an open ball about x that is contained in A. We say that A is an *open neighborhood* (of x) if A itself is open.

Theorem 2.12 (Continuity at a point) *Let (X, d_X) and (Y, d_Y) be two metric spaces, and let $p \in X$. A map $f \colon X \to Y$ is continuous at p if and only if for all neighborhoods B of $f(p)$, there is a neighborhood A of p such that $f(A) \subseteq B$.*

Proof. Assume that f is continuous at p. If B is a neighborhood of $f(p)$, then, by definition, there is an open ball $\mathrm{B}_Y(f(p); \epsilon)$ about $f(p)$ that is contained in B. Since f is continuous at p, there is a

$\delta > 0$ such that if $d_X(p,q) < \delta$, then $d_Y(f(p), f(q)) < \epsilon$. Hence, $f(\mathrm{B}_X(p;\delta)) \subseteq \mathrm{B}_Y(f(p);\epsilon) \subseteq B$. That is, if we let $A = \mathrm{B}_X(p,\delta)$, then for all neighborhoods B of $f(p)$, we have that $f(A) \subseteq B$ where A is a neighborhood of p.

Assume that for all neighborhoods B of $f(p)$, there is a neighborhood A of p such that $f(A) \subseteq B$. We need to prove that for all $\epsilon > 0$, there is a $\delta > 0$ such that if $d_X(p,q) < \delta$, then $d_Y(f(p), f(q)) < \epsilon$. By utilizing the fact that $B = \mathrm{B}_Y(f(p);\epsilon)$ is a neighborhood of $f(p)$, then, by assumption, there must be a neighborhood A of p such that $f(A) \subseteq B$. Since A is a neighborhood of p, there is an open ball $\mathrm{B}_X(p;\delta)$ about p that is contained in A. Now assume that $d_X(p,p') < \delta$. Then $p' \in \mathrm{B}_X(p;\delta) \subseteq A$. Thus $f(p') \in B = \mathrm{B}_Y(f(p);\epsilon)$, and hence, $d_Y(f(p), f(p')) < \epsilon$. Thus f is continuous at p. □

The following theorem gives an alternative description of continuous maps between metric spaces.

Theorem 2.13 **(Continuous maps between metric spaces)** *Let (X, d_X) and (Y, d_Y) be two metric spaces. A map $f\colon X \to Y$ is continuous if and only if for every subset $B \subseteq Y$ open in Y, the* preimage *of B under f,*

$$f^{-1}(B) = \{x \in X \mid f(x) \in B\} \subseteq X,$$

is open in X.

Proof. Assume that f is continuous. For $B \subseteq Y$ open in Y, we have to prove that $f^{-1}(B) \subseteq X$ is open in X. Let $a \in f^{-1}(B)$. We want to prove that there is an open ball about a in X that is contained in $f^{-1}(B)$. By assumption, $B \subseteq Y$ is open in Y. Hence, there is an $\epsilon > 0$ such that $\mathrm{B}_Y(f(a);\epsilon) \subseteq B$. From the assumption that f is continuous there is a $\delta > 0$ such that $\mathrm{B}_X(a;\delta) \subseteq f^{-1}(\mathrm{B}_Y(f(a);\epsilon)) \subseteq f^{-1}(B)$.

We now prove the opposite implication. Assume that for every subset $B \subseteq Y$ open in Y, the preimage $f^{-1}(B)$ of B under f is open in X. Let $a \in X$ and $\epsilon > 0$ be a real number. From the first assumption it follows that $f^{-1}(\mathrm{B}_Y(f(a);\epsilon)) \subseteq X$ is open in X. As $f^{-1}(\mathrm{B}_Y(f(a);\epsilon))$ is open and contains a, there is a $\delta > 0$ such that $\mathrm{B}_X(a;\delta) \subseteq f^{-1}(\mathrm{B}_Y(f(a);\epsilon))$. Thus $x \in \mathrm{B}_X(a;\delta)$ implies that $f(x) \in \mathrm{B}_Y(f(a);\epsilon)$. Hence, $f\colon X \to Y$ is continuous. □

Let A and B be sets, and let $f\colon A \to B$. Then $f^{-1}(B)$ will always exist even if there is no inverse map. In the cases where f has an inverse there is no ambiguity. If U and V are both subsets of B then

$$f^{-1}(U \cup V) = f^{-1}(U) \cup f^{-1}(V) \quad \text{and} \quad f^{-1}(U \cap V) = f^{-1}(U) \cap f^{-1}(V),$$

and furthermore, if $U \subseteq V$ then $f^{-1}(U) \subseteq f^{-1}(V)$. Let $U \subseteq A$ and $V \subseteq B$, then

$$U \subseteq f^{-1}(f(U)) \quad \text{and} \quad f(f^{-1}(V)) \subseteq V.$$

We also note that if U is a subset of B then

$$f^{-1}(B \setminus U) = f^{-1}(U^c) = (f^{-1}(U))^c = A \setminus f^{-1}(U).$$

2.3 Exercises

Exercise 2.1 Does $d(x, y) = (x - y)^2$ define a metric on $X = \mathbb{R}$?

Exercise 2.2 Show that $\mathbb{R}^2$ equipped with the taxi cab metric is a metric space.

Exercise 2.3 Let (X, d) be a metric space. Show that the map $d' : X \times X \to \mathbb{R}$ given by

$$d'(x, y) = \frac{d(x, y)}{1 + d(x, y)}$$

is also a metric on X.

Exercise 2.4 Draw a picture of the open ball $\mathrm{B}((0, 0); 1)$ in the metric space $(\mathbb{R}^2, d)$ with

(a) $d(x, y) = d_1(x, y) = |x_1 - y_1| + |x_2 - y_2|$;

(b) $d(x, y) = d_2(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$;

(c) $d(x, y) = d_\infty(x, y) = \max\{|x_1 - y_1|, |x_2 - y_2|\}$.

Exercise 2.5 Show that d_1, d_2 and d_∞ (as defined in Exercise 2.4) are equivalent on $X = \mathbb{R}^2$.

Exercise 2.6 Show that in a discrete metric space (X, d), cf. Example 2.3, every subset is both open and closed in X.

Exercise 2.7 Show that for equivalent metrics d and d' on the set X, the open sets are the same.

Exercise 2.8 Let (X, d_X) and (Y, d_Y) be metric spaces, and let $f : X \to Y$ be a map. Show that f is continuous if and only if for every subset $B \subseteq Y$ closed in Y, the preimage $f^{-1}(B)$ is closed in X.

3. Topological spaces

3.1 Definition and examples

Topological spaces are spaces constructed to support continuous maps. The definition is as follows.

Definition 3.1 **(Topological spaces)** A *topological space* is a set X together with a collection $\mathcal{T}$ of subsets of X that are called *open* in X, such that the following properties hold.

T1 The subsets $\emptyset$ and X are in $\mathcal{T}$.

T2 The union of the elements of any subcollection of $\mathcal{T}$ is in $\mathcal{T}$.

T3 The intersection of the elements of any finite subcollection of $\mathcal{T}$ is in $\mathcal{T}$.

A topological space is strictly speaking an ordered pair $(X, \mathcal{T})$. We refer to $\mathcal{T}$ as the *topology* on X. We will often omit specific mention of $\mathcal{T}$ if no confusion will arise.

The following theorem states that every metric space (X, d) is a topological space with the *metric topology* $\mathcal{T}_d$ on X.

Theorem 3.2 **(Metric spaces are topological spaces)** *Let (X, d) be a metric space. Let $\mathcal{T}_d$ be the collection of subsets $U \subseteq X$ with the property that $U \in \mathcal{T}_d$ if and only if for each $x \in U$ there is an $r > 0$ such that $\mathrm{B}(x; r) \subseteq U$. Then $\mathcal{T}_d$ defines a topology on X.*

Proof. Clearly, $\emptyset \in \mathcal{T}_d$. To show that $X \in \mathcal{T}_d$, note that for any $x \in X$, $\mathrm{B}(x; 1) \subseteq X$. Hence, $X \in \mathcal{T}_d$. Thus T1 is satisfied.

Let $\{U_\lambda\}_{\lambda \in \Lambda}$ be any subcollection of $\mathcal{T}_d$. We need to prove that $V = \bigcup_{\lambda \in \Lambda} U_\lambda \in \mathcal{T}_d$. Let $x \in V$. From $V = \bigcup_{\lambda \in \Lambda} U_\lambda$ there is $\lambda \in \Lambda$ such that $x \in U_\lambda$. By the property of U_λ satisfied by the U_λ in $\mathcal{T}_d$ there is an $r > 0$ such that $\mathrm{B}(x; r) \subseteq U_\lambda$. Hence, $\mathrm{B}(x; r) \subseteq V$. Thus $V \in \mathcal{T}_d$, and T2 is satisfied.

We prove that the intersection of two elements of $\mathcal{T}_d$ is also an element of $\mathcal{T}_d$. The general result then follows by an induction argument. Let $U_1, U_2 \in \mathcal{T}_d$. We need to prove that $U_1 \cap U_2 \in \mathcal{T}_d$. Let $x \in U_1 \cap U_2$. Since $U_1 \cap U_2 \subseteq U_i$, we have that $x \in U_i$ for $i = 1, 2$. By the defining property of $\mathcal{T}_d$ there is an $r_i > 0$ such that $\mathrm{B}(x; r_i) \subseteq U_i$ for $i = 1, 2$. Let $r = \min\{r_1, r_2\}$. Then $\mathrm{B}(x; r) \subseteq \mathrm{B}(x; r_i) \subseteq U_i$ for $i = 1, 2$. Thus $\mathrm{B}(x; r) \subseteq U_1 \cap U_2$, and so $U_1 \cap U_2 \in \mathcal{T}_d$. Hence, T3 is satisfied. □

The following theorem relates the metric topologies for two equivalent metrics.

Theorem 3.3 *Let X be any set, and let d_1 and d_2 be two equivalent metrics on X. Then $\mathcal{T}_{d_1} = \mathcal{T}_{d_2}$.*

This follows from Exercise 2.7.

Example 3.4 (Discrete topology) Let X be any set. The collection $\mathcal{T}$ of all subsets of X, i.e. the power set $\mathcal{P}(X)$ of X, is a topology on X. We refer to this topology as the *discrete topology*. A set X equipped with the discrete topology is referred to as a *discrete topological space*.

The discrete topology is the unique topology where the singletons are open. We can think of a discrete topological space as a space of separate, isolated points, with no close interaction between different points.

For any set X, the discrete topology is the largest topology we may equip X with. The smallest topology is called the indiscrete topology.

Example 3.5 (Indiscrete topology) Let X be any set. The collection $\mathcal{T}$ consisting of $\emptyset$ and X is a topology on X, referred to as the *indiscrete topology* on X. A set X equipped with the indiscrete topology is referred to as an *indiscrete topological space*.

Example 3.6 Let $X = \{a, b, c\}$. The following collections all define a topology on X.

(1) $\mathcal{T}_1 = \mathcal{T}_{\text{ind}} = \{\emptyset, X\}$

(2) $\mathcal{T}_2 = \{\emptyset, \{a\}, X\}$

(3) $\mathcal{T}_3 = \{\emptyset, \{a, b\}, X\}$

(4) $\mathcal{T}_4 = \{\emptyset, \{a\}, \{a, b\}, X\}$

(5) $\mathcal{T}_5 = \{\emptyset, \{a, b\}, \{b\}, \{b, c\}, X\}$

(6) $\mathcal{T}_6 = \mathcal{T}_{\text{disc}} = \mathcal{P}(X) = \{\emptyset, \{a\}, \{a, b\}, \{a, c\}, \{b\}, \{b, c\}, \{c\}, X\}$

There are in total 29 topologies on X. However, there are also collections of subsets of X which do not define topologies on X. None of the following collections of subsets of X define a topology on X.

(1) $\{\emptyset, \{a\}, \{b\}, X\}$

(2) $\{\emptyset, \{a\}, \{c\}, X\}$

(3) $\{\emptyset, \{a, b\}, \{b, c\}, X\}$

Definition 3.7 (Comparable topologies) Let X be any set and suppose that $\mathcal{T}_1$ and $\mathcal{T}_2$ are two topologies on X. If $\mathcal{T}_1 \subseteq \mathcal{T}_2$, we say that $\mathcal{T}_1$ is *coarser* than $\mathcal{T}_2$ and that $\mathcal{T}_2$ is *finer* than $\mathcal{T}_1$. We say that $\mathcal{T}_1$ and $\mathcal{T}_2$ are *comparable* if either $\mathcal{T}_1 \subseteq \mathcal{T}_2$ or $\mathcal{T}_2 \subseteq \mathcal{T}_1$.

Clearly, for any set X, the discrete topology $\mathcal{T}_{\text{disc}}$ contains the indiscrete topology $\mathcal{T}_{\text{ind}}$: $\mathcal{T}_{\text{disc}} \supseteq \mathcal{T}_{\text{ind}}$. Hence, the discrete topology is finer than the indiscrete topology and the indiscrete topology is coarser than the discrete topology.

Example 3.8 (Cofinite topology) Let X be any set. The collection $\mathcal{T}$ of subsets of X consisting of subsets $U \subseteq X$ such that $U^c = X \setminus U$ is either finite or all of X is a topology on X. We refer to this topology as the *cofinite topology* on X.

If X is a finite set, the cofinite topology is equal to the discrete topology. If X is an infinite set, the cofinite topology is *strictly coarser* than the discrete topology, in the sense that the cofinite topology is properly contained in the discrete topology.

We end this section with a theorem that we can use to prove that some set is open. To state the theorem we need the following definition.

Definition 3.9 (Neighborhoods) Let X be a topological space, U a subset of X and $x \in X$. We say that U is a *neighborhood of* x if $x \in U$ and U is open in X.

A neighborhood in the sense of the previous definition is sometimes referred to as an *open* neighborhood, cf. Definition 2.11.

Theorem 3.10 *Let X be a topological space. A subset U of X is open in X if and only if for every $x \in U$ there is a neighborhood U_x of x such that $U_x \subseteq U$.*

Proof. Assume that U is open in X. Then for every $x \in U$, U is a neighborhood of x that is contained in U.

We prove the other implication. Assume that for every $x \in U$ there is a $U_x \in \mathcal{T}$ such that $x \in U_x \subseteq U$, i.e., that U_x is a neighborhood of x such that $U_x \subseteq U$. To prove that $U \in \mathcal{T}$, we will prove that $U = \bigcup_{x \in U} U_x$. Assume that $x' \in U_{x'}$. Then $x' \in U_{x'} \subseteq \bigcup_{x \in U} U_x$. Furthermore, any point in $\bigcup_{x \in U} U_x$ is in U_x for some $x \in U$ so by assumption, $U_x \subseteq U$ and $x \in U_x \subseteq U$. Hence, $U = \bigcup_{x \in U} U_x$. As U is the union of open sets it must be an open set as well by T2. $\square$

3.2 Continuous maps

We know from Theorem 2.13 that a map between metric spaces is continuous if and only if the preimage of an open set is open. This motivates the following definition.

Definition 3.11 (Continuous maps between topological spaces) Let X and Y be topological spaces. A map $f: X \to Y$ is said to be *continuous* if preimages of open sets are open, i.e., if V is an open set in Y then the preimage $f^{-1}(V)$ of V under f is open in X.

Hence, all continuous maps between metric spaces (X, d_X) and (Y, d_Y) are also continuous maps between the corresponding topological spaces X and Y with the metric topologies $\mathcal{T}_{d_X}$ and $\mathcal{T}_{d_Y}$, respectively.

Example 3.12 Let X and Y be topological spaces. Then all constant maps from X to Y are continuous: the preimages are either empty or the entire space, and these are always open, cf. T1.

Example 3.13 Let X be a discrete topological space and Y a topological space. Then all maps from X to Y are continuous.

Example 3.14 Let X be any topological space and Y be an indiscrete topological space. Then all maps from X to Y are continuous.

Example 3.15 Let $\mathbb{R}_{\mathrm{disc}}$ be the discrete topological space consisting of the real numbers with the discrete topology, and let $\mathbb{R}$ be the topological space consisting of the the real numbers with the usual (Euclidean) metric topology. Then the identity map

$$\mathbb{R}_{\mathrm{disc}} \xrightarrow{\mathrm{id}} \mathbb{R}$$

is continuous by Example 3.13, while the identity map

$$\mathbb{R} \xrightarrow{\mathrm{id}} \mathbb{R}_{\mathrm{disc}}$$

is *not* continuous: singletons are open in the discrete topology but not in the (Euclidean) metric topology.

The following theorem says that the composition of two continuous maps is a continuous map.

Theorem 3.16 (Composition of continuous maps) *Let X, Y and Z be topological spaces. If $f\colon X \to Y$ and $g\colon Y \to Z$ are continuous maps, then the composite $g \circ f\colon X \to Z$ is continuous.*

Proof. Let $W \subseteq Z$ be open in Z. We need to prove that $(g \circ f)^{-1}(W)$ is open in X. Since

$$\begin{aligned}(g \circ f)^{-1}(W) &= \{x \in X \mid g(f(x)) \in W\}\\ &= \{x \in X \mid f(x) \in g^{-1}(W)\}\\ &= \{x \in X \mid x \in f^{-1}(g^{-1}(W))\} = f^{-1}(g^{-1}(W))\end{aligned}$$

and that $g^{-1}(W)$ is open in Y and $f^{-1}(g^{-1}(W))$ is open in X (by continuity of g and f), it follows that $(g \circ f)^{-1}(W)$ is open in X. Hence, $g \circ f$ is continuous. □

We can express continuity at a point for maps between topological spaces using neighborhoods. (See Theorem 2.12 for the case of metric spaces.)

Definition 3.17 (Continuity at a point) Let X and Y be topological spaces, and let $x \in X$. A map $f\colon X \to Y$ is *continuous at x* if for all neighborhoods V of $f(x)$ there is a neighborhood U of x such that $f(U) \subseteq V$.

Theorem 3.18 *Let X and Y be topological spaces. A map $f\colon X \to Y$ is continuous if and only if it is continuous at each $x \in X$.*

Proof. Assume that f is continuous, and let $x \in X$ and V be a neighborhood of $f(x)$. Then the set $U = f^{-1}(V)$ is a neighborhood of x such that $f(U) \subseteq V$.

Assume that f is continuous at each $x \in X$. Let $V \subseteq Y$ be open in Y. Choose $x \in f^{-1}(V)$. Since f is continuous at x there is neighborhood U_x of x such that $f(U_x) \subseteq V$. Hence, $U_x \subseteq f^{-1}(V)$. It follows that $f^{-1}(V)$ can be written as the union of the open sets U_x, and hence, it is open in X. Thus f is continuous. □

3.3 Homeomorphisms

We now introduce the notion of topological equivalence, also known as homeomorphism.

Definition 3.19 **(Homeomorphisms)** Let X and Y be topological spaces. A bijective map $f\colon X \to Y$ with the property that both f and $f^{-1}\colon Y \to X$ are continuous, is called a *homeomorphism*. If there exists a homeomorphism $f\colon X \to Y$, we say that X and Y are *homeomorphic* and write $X \cong Y$.

A homeomorphism $f\colon X \to Y$ gives a one-to-one correspondence between open sets in X and Y. As a result, any property of a topological space that can be expressed in terms of its elements and its open subsets is preserved by homeomorphisms. Such a property is called a *topological property*.

Example 3.20 Let $\mathbb{R}$ be the topological space of the real numbers with the (Euclidean) metric topology. The map

$$\begin{aligned} f\colon \mathbb{R} &\to \mathbb{R} \\ x &\mapsto 2x-1 \end{aligned}$$

is a homeomorphism. Let

$$\begin{aligned} g\colon \mathbb{R} &\to \mathbb{R} \\ y &\mapsto \frac{1}{2}(y+1) \end{aligned}$$

then, clearly, $g(f(x)) = x$ and $f(g(y)) = y$ for all real numbers x and y. Thus f is a bijection and $f^{-1} = g$. From calculus we know that f and g are continuous. Hence, f is a homeomorphism.

Example 3.21 Let $X = \{a,b\}$, and let $\mathcal{T}_1 = \{\emptyset, \{a\}, X\}$ and $\mathcal{T}_2 = \{\emptyset, \{b\}, X\}$ be two topologies on X. The map $f\colon X \to X$ given by $f(a) = b$ and $f(b) = a$ is clearly a continuous bijection (with the domain given $\mathcal{T}_1$ as topology, and the codomain given $\mathcal{T}_2$ as topology). Also, f is its own inverse: $f = f^{-1}$. Hence, f is a homeomorphism and $(X,\mathcal{T}_1) \cong (X,\mathcal{T}_2)$.

Homeomorphisms are continuous bijections, but the converse is not true.

Example 3.22 Let $X = \{a,b\}$. The identity map $\mathrm{id}\colon X \to X$ where the domain is given the discrete topology and the codomain is given the indiscrete topology is a continuous bijection but *not* a homeomorphism: the inverse map is not continuous.

The following theorem says that being homeomorphic is an equivalence relation on any set of topological spaces.

Theorem 3.23 *Let X, Y and Z be topological spaces.*

Reflexivity *The identity map* $\mathrm{id}\colon X \to X$ *(where the domain and the codomain are equipped with the same topology), given by* $\mathrm{id}(x) = x$ *for* $x \in X$*, is a homeomorphism.*

Symmetry *If* $f\colon X \to Y$ *is a homeomorphism, then* $f^{-1}\colon Y \to X$ *is also a homeomorphism.*

Transitivity *If* $f\colon X \to Y$ *and* $g\colon Y \to Z$ *are homeomorphisms, then* $g \circ f\colon X \to Z$ *is also a homeomorphism.*

Proof. The identity map $\mathrm{id}\colon X \to X$ (where the domain and the codomain are equipped with the same topology) is clearly continuous and bijective. As the identity map is its own inverse, then it is also a homeomorphism. Hence, $X \cong X$ and so $\cong$ satisfies the reflexivity condition for an equivalence relation.

If $f\colon X \to Y$ is a homeomorphism, then $f^{-1}\colon Y \to X$ is also a homeomorphism: f^{-1} is a continuous bijection with continuous inverse $(f^{-1})^{-1} = f\colon X \to Y$. Hence, $X \cong Y$ if and only if $Y \cong X$. Thus $\cong$ satisfies the symmetry condition for an equivalence relation.

Theorem 3.16 tells us that the composition of two homeomorphisms $f\colon X \to Y$ and $g\colon Y \to Z$ is continuous. The composition of two bijective maps is always bijective. Hence, $g \circ f$ is a continuous bijection. We need to prove that its inverse, $(g \circ f)^{-1}$, is continuous. Since $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$ is a composition of continuous maps, then by Theorem 3.16 so is $(g \circ f)^{-1}$. Thus $g \circ f$ is a homeomorphism. Hence, if $X \cong Y$ and $Y \cong Z$, then $X \cong Z$. Thus $\cong$ satisfies the transitivity condition for an equivalence relation. $\square$

3.4 Closed sets

Recall that in a topological space X, a subset A of X is an open subset if and only if A is an element of the topology of X, i.e., $A \in \mathcal{T}$.

Definition 3.24 (Closed subsets) A subset K of a topological space X is *closed* in X if and only if the complement $K^c = X \setminus K$ is open in X.

This is completely analogous to how we defined closed subsets in metric spaces, cf. Definition 2.9.

Example 3.25 Let X be a discrete topological space. Since every subset of X is open in X, it follows that every subset of X is also closed in X.

Example 3.26 Let X be an indiscrete topological space. The only subsets of X that are closed in X are $\emptyset$ and X (which are also the only subsets that are open in X).

Recall that in a discrete topological space, all the singletons are open sets. This is usually *not* the case.

Example 3.27 Let $\mathbb{R}$ be the topological space of the real numbers with the (Euclidean) metric topology. Then every subset $[a, b] = \{x \in \mathbb{R} \mid a \leqslant x \leqslant b\} \subseteq \mathbb{R}$ is closed in $\mathbb{R}$: the complement $[a, b]^c = \mathbb{R} \setminus [a, b] = (-\infty, a) \cup (b, \infty)$ is a union of open sets in $\mathbb{R}$, and hence, is open in $\mathbb{R}$. Furthermore, all the singletons are closed: the complement $\{a\}^c = \mathbb{R}\setminus\{a\} = (-\infty, a) \cup (a, \infty)$

is a union of open sets in $\mathbb{R}$, and hence, is open in $\mathbb{R}$.

By passing to complements we get the following theorem.

Theorem 3.28 *Let X be a topological space.*

(1) *Both $\emptyset$ and X are closed (as subsets) in X.*

(2) *The intersection of any subcollection of closed sets in X is closed in X.*

(3) *The union of any finite subcollection of closed sets in X is closed in X.*

It follows that we could have defined a topological space X by specifying a collection of subsets of X satisfying the three statements in Theorem 3.28 where we would say that a subset of X is open in X if its complement is closed in X.

We end this section with a theorem describing the connection between continuous maps and closed sets. We will need the following definition.

Definition 3.29 **(Closure)** Let X be a topological space, and let A be a subset of X. The *closure of* A, written $\overline{A}$, is the intersection of all subsets of X that contain A and which are closed in X.

From the definition it follows that $\overline{A}$ is the smallest subset of X that contains A and which is closed in X. Furthermore, if A is closed in X, then $\overline{A} = A$.

There is an analogous definition for open sets where we take union instead of intersection. We can define the *interior of* A, written $\mathrm{Int}(A)$, to be the union of all subsets of X that are contained in A and which are open in X. It follows that $\mathrm{Int}(A)$ is the largest subset of X that is contained in A and which is open in X. Furthermore, $\mathrm{Int}(A) \subseteq A \subseteq \overline{A}$.

Example 3.30 Let X be a topological space consisting of the set $\{a, b, c\}$ and the topology $\mathcal{T} = \{\emptyset, \{a\}, \{a, b\}, X\}$. Then the closed subsets in X are $\emptyset, \{b, c\}, \{c\}$ and X. Thus the intersection of all of the closed subsets that contain $\{b\}$ is simply $\{b, c\} \cap X = \{b, c\}$, and hence, $\overline{\{b\}} = \{b, c\}$.

Example 3.31 Let $\mathbb{R}$ be the topological space of the real numbers with the (Euclidean) metric topology. Assume that $a < b$ are real numbers. Then $\overline{(a, b]} = [a, b]$ and $\mathrm{Int}((a, b]) = (a, b)$.

Let (X, d) be a metric space. If we consider X as a topological space with the metric topology $\mathcal{T}_d$, the closure $\overline{\mathrm{B}(x; r)}$ of an open ball $\mathrm{B}(x; r)$ about $x \in X$ is, in general, *not* the same as the closed ball $\overline{\mathrm{B}}(x; r)$. If d is the discrete metric and X has at least two elements, then $\overline{\mathrm{B}(x; 1)} = \{x\}$ while $\overline{\mathrm{B}}(x; 1) = X$. It is always the case that $\overline{\mathrm{B}(x; r)} \subseteq \overline{\mathrm{B}}(x; r)$.

Definition 3.32 **(Dense)** Let X be a topological space, and let A be a subset of X. We say that A is *dense in* X if $\overline{A} = X$.

From the definition it follows that A is dense in X if and only if $A \cap U \neq \emptyset$ for every nonempty subset U of X which is open in X.

Example 3.33 Let $\mathbb{R}$ be the topological space of the real numbers with the (Euclidean) metric topology. Then the subset $\mathbb{Q}$ of rational numbers is dense in $\mathbb{R}$: $\overline{\mathbb{Q}} = \mathbb{R}$.

Example 3.34 For any topological space X, the subset X is dense in X. If X is a discrete topological space, then the subset X is the only dense subset in X.

Theorem 3.35 *Let $f\colon X \to Y$ be a map between topological spaces. Then the following are equivalent:*

(1) *f is continuous;*

(2) *for every subset A of X, we have $f(\overline{A}) \subseteq \overline{f(A)}$;*

(3) *for every closed subset B of Y, the preimage $f^{-1}(B)$ of B under f is closed in X.*

Proof. By passing to complements, it follows readily that (1) and (3) are equivalent. We will prove that (2) is equivalent to (3).

Assume (2). Let B be a subset of Y that is closed in Y, and let $A = f^{-1}(B)$. We must show that A is closed. We have $f(A) \subseteq B$. If $x \in \overline{A}$, then $f(x) \in f(\overline{A}) \subseteq \overline{f(A)} \subseteq B$. Hence, $x \in f^{-1}(B) = A$. In other words, $\overline{A} \subseteq A$. Thus $A = \overline{A}$, and hence, $f^{-1}(B)$ is closed in X.

Now assume (3). Let A be a subset of X. We must show that $f(\overline{A}) \subseteq \overline{f(A)}$. Since $\overline{f(A)}$ is closed in Y, it follows by assumption that $f^{-1}(\overline{f(A)})$ is closed in X. Furthermore, $A \subseteq f^{-1}(f(A)) \subseteq f^{-1}(\overline{f(A)})$. Since $f^{-1}(\overline{f(A)})$ is closed in X, it follows that $\overline{A} \subseteq f^{-1}(\overline{f(A)})$. Hence, $f(\overline{A}) \subseteq \overline{f(A)}$. □

3.5 Exercises

Exercise 3.1 Let $X = \{a, b, c, d\}$. Show that $\mathcal{T} = \{\emptyset, \{a\}, \{a, b\}, \{a, b, c\}, \{b, c, d\}, X\}$ is *not* a topology on X. Find a topology $\mathcal{T}'$ (different from the discrete topology) on X such that $\mathcal{T} \subseteq \mathcal{T}'$.

Exercise 3.2 Let X be a non-empty set, and let x_0 be an element of X. Show that

$$\mathcal{T} = \{U \subseteq X \mid x_0 \notin U \text{ or } X \setminus U \text{ is finite}\}$$

is a topology on X.

Exercise 3.3 Let X be a set, and let A be a subset of X. Define the coarsest topology on X such that A is open in X.

Exercise 3.4 Show that the discrete topology $\mathcal{T}_{\text{disc}}$ is finer than the cofinite topology $\mathcal{T}_{\text{cof}}$ on any set X.

Exercise 3.5 Let $X = \{a, b, c, d\}$. Find two topologies $\mathcal{T}_1$ and $\mathcal{T}_2$ with $\mathcal{T}_1 \neq \mathcal{T}_2$ such that a bijection $f\colon X \to X$ is a homeomorphism (where the domain is given $\mathcal{T}_1$ as topology and the codomain is given $\mathcal{T}_2$ as topology).

Exercise 3.6 Let X be a topological space, and let A and B be subsets of X.

(a) Assume that $A \subseteq B$. Show that $\overline{A} \subseteq \overline{B}$.

(b) Show that $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

We say that a set A *intersects* or *meets* a set B if $A \cap B \neq \emptyset$.

Exercise 3.7 Let X be a topological space, and let A be a subset of X. Show that $x \in \overline{A}$ if and only if every neighborhood of x intersects A.

Exercise 3.8 Let $X = \{a, b, c, d, e\}$, and let

$$\mathcal{T} = \{\emptyset, \{a, b\}, \{a, b, c\}, \{a, b, d, e\}, \{b\}, \{b, c\}, \{b, d, e\}, \{b, c, d, e\}, \{d, e\}, X\}$$

be a topology on X. Is the subset $\{a, b\}$ dense in X?

4. Generating topologies

4.1 Generating topologies from subsets

The following theorem tells us how we may extract a third topology from two other topologies on the same set.

Theorem 4.1 (The intersection of two topologies is a topology) *Let X be a set, and let $\mathcal{T}_1$ and $\mathcal{T}_2$ be two topologies on X. Then $\mathcal{T}_1 \cap \mathcal{T}_2$ is also a topology on X.*

Proof. Clearly, $\emptyset$ and X are in $\mathcal{T}_1 \cap \mathcal{T}_2$, so T1 is satisfied.

Let $\{U_\lambda\}_{\lambda\in\Lambda}$ be a collection of sets such that $U_\lambda \in \mathcal{T}_1 \cap \mathcal{T}_2$ for each $\lambda \in \Lambda$ where Λ is some index set. Then, for $i = 1, 2$, $U_\lambda \in \mathcal{T}_i$ for each $\lambda \in \Lambda$. Thus $\bigcup_{\lambda\in\Lambda} U_\lambda \in \mathcal{T}_i$ for $i = 1, 2$. Hence, $\bigcup_{\lambda\in\Lambda} U_\lambda \in \mathcal{T}_1 \cap \mathcal{T}_2$, and so, T2 is satisfied.

Finally, to prove that T3 is satisfied, let $U, V \in \mathcal{T}_1 \cap \mathcal{T}_2$. Thus, for $i = 1, 2$, $U, V \in \mathcal{T}_i$ implies that $U \cap V \in \mathcal{T}_i$. Hence, $U \cap V \in \mathcal{T}_1 \cap \mathcal{T}_2$. □

Theorem 4.1 may be extended to hold for a family of topologies: if $\{\mathcal{T}_\lambda\}_{\lambda\in\Lambda}$ is a family of topologies on X, then $\bigcap_{\lambda\in\Lambda} \mathcal{T}_\lambda$ is also a topology on X. If we follow the convention that for subsets S of a fixed (large) set U,

$$\bigcap_{S\in\emptyset} S = U,$$

then the extended version of Theorem 4.1 may also include an empty family $\{\mathcal{T}_\lambda\}_{\lambda\in\emptyset}$ of topologies with

$$\bigcap_{\lambda\in\emptyset} \mathcal{T}_\lambda = \mathcal{P}(X),$$

i.e., the discrete topology on X (with our fixed (large) set U being equal to $\mathcal{P}(X)$). However, not all mathematicians follow this convention. Thus we will in general not define the intersection of an empty family.

The union of two topologies is not necessarily a topology.

Example 4.2 Let $X = \{a, b, c\}$, and let $\mathcal{T}_1 = \{\emptyset, \{a\}, \{a, b\}, X\}$ and $\mathcal{T}_2 = \{\emptyset, \{c\}, X\}$ be two topologies on X. Then

$$\mathcal{T}_1 \cap \mathcal{T}_2 = \{\emptyset, X\}$$

is the indiscrete topology on X while

$$\mathcal{T}_1 \cup \mathcal{T}_2 = \{\emptyset, \{a\}, \{a, b\}, \{c\}, X\}$$

is *not* a topology on X: $\mathcal{T}_1 \cup \mathcal{T}_2$ does not satisfy T2.

Recall from Section 3.1 that for any set X the discrete topology $\mathcal{T}_{\text{disc}}$ is the largest topology we may equip X with, and the indiscrete topology $\mathcal{T}_{\text{ind}}$ is the smallest topology we may equip X with. For any topology $\mathcal{T}$ on X we have

$$\mathcal{T}_{\text{ind}} \subseteq \mathcal{T} \subseteq \mathcal{T}_{\text{disc}}.$$

That is, we have partially ordered topologies on X by inclusion.

Let X be a set. We often want to have a collection of subsets $\mathcal{S}$ of X to be the open subsets of a topology on X.

Definition 4.3 (Topology generated by a collection of subsets) Let X be a set, and let $\mathcal{S}$ be a collection of subsets of X. The *topology generated by* $\mathcal{S}$ is the topology

$$\langle \mathcal{S} \rangle = \bigcap_{\substack{\mathcal{T} \text{ topology} \\ \mathcal{S} \subseteq \mathcal{T}}} \mathcal{T}$$

on X.

In other words, $\langle \mathcal{S} \rangle$ contains $\mathcal{S}$ and for any other topology $\mathcal{T}'$ containing $\mathcal{S}$, we have $\langle \mathcal{S} \rangle \subseteq \mathcal{T}'$. Thus $\langle \mathcal{S} \rangle$ is unique.

Example 4.4 Let X be a set, and let $\mathcal{S} = \emptyset$. Then $\langle \mathcal{S} \rangle$ is the same as the indiscrete topology on X, i.e.,

$$\langle \mathcal{S} \rangle = \mathcal{T}_{\text{ind}} = \{\emptyset, X\}.$$

Example 4.5 Let X be a set, and let $\mathcal{S}$ be the collection of all the singletons of X, i.e., $\mathcal{S} = \{\{x\} \mid x \in X\}$. Then $\langle \mathcal{S} \rangle$ is the same as the discrete topology on X, i.e.,

$$\langle \mathcal{S} \rangle = \mathcal{T}_{\text{disc}} = \mathcal{P}(X).$$

4.2 Basis for a topology

It is often convenient to define a topology $\mathcal{T}$ on a set X by only specifying a subcollection $\mathcal{B}$ of $\mathcal{T}$ satisfying certain properties. The open subsets of X are then precisely the unions of subcollections of $\mathcal{B}$. In this way, we say the basis determines, or generates, the topology.

Definition 4.6 (Basis) Let X be a set. A *basis* for a topology on X is a collection $\mathcal{B}$ of subsets of X such that

B1 for each $x \in X$, there is a $B \in \mathcal{B}$ such that $x \in B$;

B2 if $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$, then there is a $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$.

The elements of $\mathcal{B}$ are sometimes referred to as *basis elements*. Basis elements are subsets of X.

Example 4.7 Let X be a set, and let $\mathcal{B}$ be the collection of all the singletons of X. Then $\mathcal{B}$ is a basis for the discrete topology on X.

Example 4.8 Let (X, d) be a metric space. Then the collection of (open) ϵ-balls

$$\mathcal{B} = \{\mathrm{B}(x; \epsilon) \mid x \in X, \epsilon > 0\}$$

is a basis for the metric topology $\mathcal{T}_d$, as defined in Theorem 3.2, on X.

The following theorem describes a topology generated by a basis.

Theorem 4.9 *Let X be a set, and let $\mathcal{B}$ be basis for a topology on X. The collection $\mathcal{T}$ generated by $\mathcal{B}$ of subsets U of X with the property that for each $x \in U$ there is a basis element $B \in \mathcal{B}$ with $x \in B \subseteq U$ is a topology on X.*

Proof. Clearly, $\emptyset$ and X are both in $\mathcal{T}$. Hence, T1 is satisfied.

Let $\{U_\lambda\}_{\lambda \in \Lambda}$ be a subcollection of $\mathcal{T}$. Let $V = \bigcup_{\lambda \in \Lambda} U_\lambda$. We need to prove that $V \in \mathcal{T}$. Let $x \in V$. Then there is a $\lambda \in \Lambda$ such that $x \in U_\lambda$. Since $U_\lambda \in \mathcal{T}$, there is a basis element $B \in \mathcal{B}$ such that $x \in B \subseteq U_\lambda$. As $U_\lambda \subseteq V$, it follows that $x \in B \subseteq V$. Hence, $V \in \mathcal{T}$, and so, T2 is satisfied.

Let $U_1, U_2 \in \mathcal{T}$. We need to prove that $U_1 \cap U_2 \in \mathcal{T}$. Let $x \in U_1 \cap U_2$. Since $U_1 \cap U_2 \subseteq U_i$ we have $x \in U_i$, and thus there is a basis element $B_i \in \mathcal{B}$ with $x \in B_i \subseteq U_i$ for $i = 1, 2$. Hence, $x \in B_1 \cap B_2 \subseteq U_1 \cap U_2$. By B2 there is a basis element $B_3 \in \mathcal{B}$ with $x \in B_3 \subseteq B_1 \cap B_2$. Thus $x \in B_3 \subseteq U_1 \cap U_2$, and hence, T3 is satisfied. □

The topology generated by a basis may also be described using the following theorem.

Theorem 4.10 *Let X be a set, and let $\mathcal{B}$ be a basis for a topology $\mathcal{T}$ on X. Then $\mathcal{T}$ is equal to the collection of all unions of elements of $\mathcal{B}$.*

Proof. Let $B \in \mathcal{B}$ be any basis element. Then for each $x \in B$ we obviously have $x \in B$ and $B \subseteq B$. Thus $B \in \mathcal{T}$. It follows that any union of basis elements is a union of elements of $\mathcal{T}$, and hence, is in $\mathcal{T}$.

Conversely, let $U \in \mathcal{T}$. For each $x \in U$ there is a $B_x \in \mathcal{B}$ with $x \in B_x$ and $B_x \subseteq U$. Then $U = \bigcup_{x \in U} B_x$, and thus, U is the union of elements of $\mathcal{B}$. □

We end this section with a theorem describing a criterion for whether one topology is finer than another when both topologies are described using bases.

Theorem 4.11 *Let X be a set, and let $\mathcal{B}_1$ and $\mathcal{B}_2$ be bases for topologies $\mathcal{T}_1$ and $\mathcal{T}_2$, respectively, on X. Then the following are equivalent:*

(1) *$\mathcal{T}_2$ is finer than $\mathcal{T}_1$, i.e., $\mathcal{T}_1 \subseteq \mathcal{T}_2$.*

(2) *For each $B_1 \in \mathcal{B}_1$ and each $x \in B_1$ there is a $B_2 \in \mathcal{B}_2$ such that $x \in B_2 \subseteq B_1$.*

Proof. Assume (1). Let $B_1 \in \mathcal{B}_1$ and $x \in B_1$. Since $B_1 \in \mathcal{T}_1$ and $\mathcal{T}_1 \subseteq \mathcal{T}_2$ we have $B_1 \in \mathcal{T}_2$. Furthermore, as $\mathcal{T}_2$ is the topology generated by $\mathcal{B}_2$ there is a $B_2 \in \mathcal{B}_2$ such that $x \in B_2$ where $B_2 \subseteq B_1$. Hence, (2) is satisfied.

Now assume (2). Let $U \in \mathcal{T}_1$. We must prove that $U \in \mathcal{T}_2$. Since $\mathcal{B}_1$ generates $\mathcal{T}_1$, then for each $x \in U$ there is a $B_1 \in \mathcal{B}_1$ such that $x \in B_1 \subseteq U$. By assumption there is a $B_2 \in \mathcal{B}_2$ such that $x \in B_2 \subseteq B_1$. Hence, $B_2 \subseteq U$, and so, $U \in \mathcal{T}_2$. Thus (1) is satisfied. □

In order to have $\mathcal{T}_1 \subseteq \mathcal{T}_2$ it is *not* necessary to have $\mathcal{B}_1 \subseteq \mathcal{B}_2$, i.e., each basis element in $\mathcal{B}_1$ need not be a basis element in $\mathcal{B}_2$. However, for each basis element $B_1 \in \mathcal{B}_1$ and each point $x \in B_1$ there should be some (possibly) smaller basis element $B_2 \in \mathcal{B}_2$ such that $x \in B_2 \subseteq B_1$.

4.3 Subbasis for a topology

Let X be a set, and let $\mathcal{S}$ be a collection of subsets of X. We can form a basis $\mathcal{B}$ for a topology by simply taking all finite intersections

$$B = \bigcap_{i=1}^{n} S_i$$

of elements of $\mathcal{S}$. Thus the open sets in the topology generated by this basis are all unions of such basis elements B, cf. Theorem 4.10. Thus the open sets are all unions of all finite intersections of elements of $\mathcal{S}$. The collection $\mathcal{S}$ is then referred to as a subbasis.

Definition 4.12 (Subbasis) Let X be a set. A *subbasis* for a topology on X is a collection $\mathcal{S}$ of subsets of X whose union equals X.

Lemma 4.13 *Let X be a set, and let $\mathcal{S}$ be a subbasis for a topology on X. The collection $\mathcal{B}$ consisting of all finite intersections of elements of $\mathcal{S}$ is a basis for a topology on X and is called the basis associated to $\mathcal{S}$.*

Proof. Each $x \in X$ must lie in some $S \in \mathcal{S}$. Hence, $x \in S$. Thus x is an element of the basis element S in $\mathcal{B}$, and so, B1 is satisfied.

Let $B_1 = \bigcap_{i=1}^{m} S_i$ and $B_2 = \bigcap_{i=1}^{n} S_i'$ be two basis elements of $\mathcal{B}$, and let $x \in B_1 \cap B_2$. We must prove that there is a basis element $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$. Let

$$B_3 = \left(\bigcap_{i=1}^{m} S_i\right) \cap \left(\bigcap_{i=1}^{n} S_i'\right).$$

Then B_3 is also a finite intersection of elements of $\mathcal{S}$, and hence, $B_3 \in \mathcal{B}$ with $x \in B_3$. Thus B2 is satisfied. $\square$

By combining the previous lemma with Theorem 4.10, we get the following lemma.

Lemma 4.14 *Let X be a set, and let $\mathcal{S}$ be a subbasis for a topology on X. The collection $\mathcal{T}$ generated by $\mathcal{S}$ consisting of all unions of all basis elements of the associated basis $\mathcal{B}$ is a topology on X.*

When referring to the topology $\mathcal{T}$ generated by the subbasis $\mathcal{S}$ we mean the topology generated by the associated basis $\mathcal{B}$. We have $\mathcal{S} \subseteq \mathcal{B} \subseteq \mathcal{T}$.

The following theorem provides an explicit description of the topology generated by a collection of subsets $\mathcal{S}$ of a set X.

Theorem 4.15 *Let X be a set, and let $\mathcal{S}$ be a subbasis for a topology on X. Then there exists a unique topology $\langle \mathcal{S} \rangle$ generated by $\mathcal{S}$ which is smaller than any other topology containing $\mathcal{S}$, where*

$$\langle \mathcal{S} \rangle = \left\{ \bigcup_{\lambda \in \Lambda} \bigcap_{i=1}^{n_\lambda} S_{\lambda,i} \,\middle|\, S_{\lambda,i} \in \mathcal{S} \right\}.$$

In other words, $\langle \mathcal{S} \rangle$ is the topology for which $\mathcal{S}$ is a subbasis.

Proof. Since the discrete topology $\mathcal{T}_{\text{disc}} = \mathcal{P}(X)$, there is at least one topology on X that contains $\mathcal{S}$. We know from Theorem 4.1 that taking the intersection of all topologies that contain $\mathcal{S}$ is again a topology which clearly still contains $\mathcal{S}$. By construction, $\langle \mathcal{S} \rangle$ is then contained in any other topology containing $\mathcal{S}$. Thus $\langle \mathcal{S} \rangle$ is the unique topology with this property.

Let

$$\mathcal{T}_{\mathcal{S}} = \left\{ \bigcup_{\lambda \in \Lambda} \bigcap_{i=1}^{n_\lambda} S_{\lambda,i} \,\middle|\, S_{\lambda,i} \in \mathcal{S} \right\}.$$

Clearly, $\mathcal{T}_{\mathcal{S}} \subseteq \langle \mathcal{S} \rangle$. We need to prove that they are equal. To do this we will prove that $\mathcal{T}_{\mathcal{S}}$ is a topology on X that contains $\mathcal{S}$. Hence, by the first part $\langle \mathcal{S} \rangle = \mathcal{T}_{\mathcal{S}}$. Since $\mathcal{S}$ is a subbasis for a topology on X, by Lemma 4.14 we know that the topology generated by $\mathcal{S}$ is equal to the collection of all unions of basis elements of the associated $\mathcal{B}$ to $\mathcal{S}$. Hence, $\mathcal{T}_{\mathcal{S}}$ is a topology on X. $\square$

We end this section with a theorem about continuity and (sub)basis.

Theorem 4.16 *Let X and Y be topological spaces, and let $\mathcal{B}$ (resp., $\mathcal{S}$) be a basis (resp., subbasis) for the topology on Y. Then a map $f: X \to Y$ is continuous if and only if for each $B \in \mathcal{B}$ (resp. $S \in \mathcal{S}$) the preimage $f^{-1}(B)$ (resp., $f^{-1}(S)$) is open in X.*

Proof. We prove the statement about basis.

Assume that f is continuous. Since each basis element $B \in \mathcal{B}$ is open in Y, then by continuity $f^{-1}(B)$ is open in X.

Assume that for each $B \in \mathcal{B}$ the preimage $f^{-1}(B)$ is open in X. Let $\mathcal{T}_Y$ be the topology on Y. Since every $V \in \mathcal{T}_Y$ is a union $V = \bigcup_{\lambda \in \Lambda} B_\lambda$ of basis elements $B_\lambda \in \mathcal{B}$, we have

$$f^{-1}(V) = \bigcup_{\lambda \in \Lambda} f^{-1}(B_\lambda).$$

Thus if each $f^{-1}(B_\lambda)$ is open in X, so is $f^{-1}(V)$. $\square$

4.4 Exercises

Exercise 4.1 Let $X = \{a, b, c, d, e\}$, and let

$$\mathcal{T} = \{\emptyset, \{a, b\}, \{a, b, d, e\}, \{b\}, \{b, d, e\}, \{b, c, d, e\}, \{c, d, e\}, \{d, e\}, X\}$$

be a topology on X. Show that $\mathcal{S} = \{\{a, b\}, \{b, d, e\}, \{c, d, e\}\}$ is a subbasis for $\mathcal{T}$. Is $\mathcal{S}' = \{\{a, b\}, \{b, c, d, e\}, \{d, e\}\}$ a subbasis for $\mathcal{T}$?

Exercise 4.2 Let $\mathcal{B}$ be the collection of all open intervals $(a,b) = \{x \in \mathbb{R} \mid a < x < b\}$ in $\mathbb{R}$.

(a) Show that $\mathcal{B}$ is a basis for a topology on $\mathbb{R}$. The topology generated by $\mathcal{B}$ is called the *standard topology* on $\mathbb{R}$ denoted by $\mathcal{T}_{\text{std}}$.

(b) Show that $\mathcal{T}_{\text{std}} = \mathcal{T}_d$ where $\mathcal{T}_d$ is the metric topology obtained from the metric $d(x,y) = |x-y|$.

Exercise 4.3 Show that

$$\mathcal{S} = \{(a,\infty) \mid a \in \mathbb{R}\} \cup \{(-\infty,b) \mid b \in \mathbb{R}\}$$

is a subbasis for the standard topology on $\mathbb{R}$.

Exercise 4.4 Let $\mathbb{Q}$ deonte the set of rational numbers, and let $\mathbb{R}$ denote the set of real numbers. Show that

$$\mathcal{B} = \{(a,b) \mid a < b, a, b \in \mathbb{Q}\}$$

is a basis for the standard topology on $\mathbb{R}$ where $(a,b) = \{x \in \mathbb{R} \mid a < x < b\}$.

Exercise 4.5 Let $\mathcal{B}$ be the collection of all half-open intervals of the form $[a,b) = \{x \in \mathbb{R} \mid a \leqslant x < b\}$ in $\mathbb{R}$.

(a) Show that $\mathcal{B}$ is a basis for a topology on $\mathbb{R}$. The topology generated by $\mathcal{B}$ is called the *lower limit topology* on $\mathbb{R}$.

(b) Find the closure of the subset $(0,1)$ of $\mathbb{R}$ given the lower limit topology.

Exercise 4.6 For each $n \in \mathbb{Z}$, let

$$B(n) = \begin{cases} \{n\} & \text{if } n \text{ is odd,} \\ \{n-1, n, n+1\} & \text{if } n \text{ is even.} \end{cases}$$

Show that the collection $\mathcal{B} = \{B(n) \mid n \in \mathbb{Z}\}$ is a basis for a topology on $\mathbb{Z}$. The topology generated by $\mathcal{B}$ is known as the *digital line topology* on $\mathbb{Z}$.

Exercise 4.7 Let $\mathcal{B}$ be the collection of all subsets of the form $A_{a,b} = \{az + b \mid z \in \mathbb{Z}\}$ of $\mathbb{Z}$, where $a, b \in \mathbb{Z}$ and $a \neq 0$. (The set $A_{a,b}$ is known as an *arithmetic progression*.)

(a) Show that $\mathcal{B}$ is a basis for a topology on $\mathbb{Z}$.

(b) Show that there are infinitely many primes by using the topology generated by $\mathcal{B}$. (This topology is known as the *arithmetic progression topology* on $\mathbb{Z}$.)

Exercise 4.8 Let X be a topological space, and let $\mathcal{B}$ be a basis for the topology on X. Show that a subset A of X is dense in X if and only if every non-empty basis element in $\mathcal{B}$ intersects A. (Recall that a set U *intersects* a set V if $U \cap V \neq \emptyset$.)

Bibliography

[1] J.R. Munkres. *Topology*. Prentice Hall, Inc., Upper Saddle River, NJ, 2000. Second edition.