Norwegian University of Science and Technology
Department of Mathematical Sciences

TMA4183
Optimisation II
Spring 2020

Exercise set 2

$\boxed{1}$ Consider the sequence of functions $u_k \in L^1([0,1])$

$$u_k(x) = \begin{cases} k & \text{if } 0 < x < 1/k, \\ 0 & \text{else.} \end{cases}$$

Show that the sequence u_k is bounded in $L^1([0,1])$, but that it does not admit any weakly convergent subsequence.

$\boxed{2}$ Consider the sequence of functions $u_k \in L^1(\mathbb{R})$,

$$u_k(x) = \begin{cases} 1 & \text{if } k < x < k+1, \\ 0 & \text{else.} \end{cases}$$

Show that $\int_E u_k(x)\,dx \to 0$ whenever $E \subset \mathbb{R}$ is measurable and satisfies $\mathcal{L}^1(E) < \infty$, but that u_k does not converge weakly to 0 in $L^1(\mathbb{R})$.

$\boxed{3}$ Let $1 < p < +\infty$ and assume that $C \subset L^p(E)$ is closed and convex. Given $v \in L^p(E)$, we define the (L^p-)projection $\pi_C(v)$ of v onto C as the solution of the optimisation problem

$$\min_{u \in C} \|u - v\|_{L^p}. \tag{1}$$

Show that the projection is well-defined, that is, that the optimisation problem (1) admits for each $v \in L^p(E)$ a unique solution.

$\boxed{4}$ Show that the set

$$C := \Big\{ u \in L^1([0,1]) : u \geq 0 \text{ and } \int_0^1 x u(x)\,dx \geq 1 \Big\}.$$

is convex and closed in $L^1([0,1])$, but that the optimisation problem

$$\min_{u \in C} \|u\|_{L^1}$$

does not admit a solution. (That is, the L^1-projection of $v = 0$ onto C does not exist!)