2 Plane autonomous systems and linearization

• 2.1 Sketch phase diagrams for the following linear systems and classify the equilibrium point:

(i) $\dot{x} = x - 5y,\ \dot{y} = x - y$;
(ii) $\dot{x} = x + y,\ \dot{y} = x - 2y$;
(iii) $\dot{x} = -4x + 2y,\ \dot{y} = 3x - 2y$;
(iv) $\dot{x} = x + 2y,\ \dot{y} = 2x + 2y$;
(v) $\dot{x} = 4x - 2y,\ \dot{y} = 3x - y$;
(vi) $\dot{x} = 2x + y,\ \dot{y} = -x + y$.

2.1. A classification table for equilibrium points of the general linear system

$$\dot{x} = ax + by, \quad \dot{y} = cx + dy$$

is given in Section 2.5 (see also Figure 2.10 in NODE). The key parameters are $p = a + d$, $q = ad - bc$, $\Delta = p^2 - 4q$. All the systems below have an isolated equilibrium point at the origin. The scales on the axes are the same for each phase diagram but actual scales are unnecessary since the equations are homogeneous in x and y. Directions are determined by continuity from directions of $\dot{x}$ and $\dot{y}$ at convenient points in the plane.

Alternatively, classification can be decided by finding the eigenvalues of the matrix of coefficients:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

(i) $\dot{x} = x - 5y$, $\dot{y} = x - y$. The parameters are

$$p = 1 - 1 = 0, \quad q = -1 + 5 = 4 > 0, \quad \Delta = 0 - 16 = -16 < 0.$$

Therefore the origin is a centre as shown in Figure 2.1(i).

The eigenvalues of

$$A = \begin{bmatrix} 1 & -5 \\ 1 & -1 \end{bmatrix}$$

are given by

$$\begin{vmatrix} 1-\lambda & -5 \\ 1 & -1-\lambda \end{vmatrix} = \lambda^2 + 4 = 0.$$

The eigenvalues take the imaginary values $\pm 2\mathrm{i}$, which is to be expected for a centre.

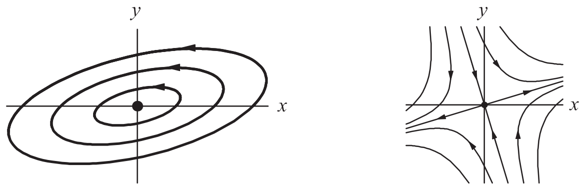

Figure 2.1 Problem 2.1(i): $\dot{x} = x - 5y$, $\dot{y} = x - y$, centre; (ii) $\dot{x} = x + y$, $\dot{y} = x - 2y$, saddle.

(ii) $\dot{x} = x + y$, $\dot{y} = x - 2y$. The parameters are

$$p = 1 - 2 = -1 < 0, \quad q = -2 - 1 = -3 < 0, \quad \Delta = 1 + 12 = 13 > 0.$$

Therefore the origin is a saddle. Its asymptotes can be found by putting $y = mx$ into the equation for the phase paths which is

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x - 2y}{x + y}.$$

The result is

$$m = \frac{1 - 2m}{1 + m}, \text{ so that } m^2 + 3m - 1 = 0.$$

Therefore the slopes of the asymptotes are

$$m_1, m_2 = \tfrac{1}{2}(-3 \pm \sqrt{13}).$$

The asymptotes and some phase paths are shown in Figure 2.1(ii).

(iii) $\dot{x} = -4x + 2y$, $\dot{y} = 3x - 2y$. The parameters are

$$p = -4 - 2 = -6 < 0, \quad q = 8 - 6 = 2 > 0, \quad \Delta = 36 - 8 = 28 > 0.$$

Therefore the origin is a stable node. The radial straight paths are given by $y = mx$ where

$$m = \frac{3 - 2m}{-4 + 2m} \text{ or } 2m^2 - 2m - 3 = 0.$$

Hence the radial paths are

$$y = m_1 x, \quad y = m_2 x, \quad \text{where } m_1, m_2 = \frac{1}{2}(1 \pm \sqrt{7}).$$

The radial paths and some phase paths are shown in Figure 2.2(iii).

(iv) $\dot{x} = x + 2y$, $\dot{y} = 2x + 2y$. The parameters are

$$p = 1 + 2 = 3 > 0, \quad q = 2 - 4 = -2 < 0, \quad \Delta = 9 + 8 = 17 > 0.$$

The origin is a saddle. The slopes of the asymptotes are

$$m_1, m_2 = \tfrac{1}{2}(1 \pm \sqrt{17}).$$

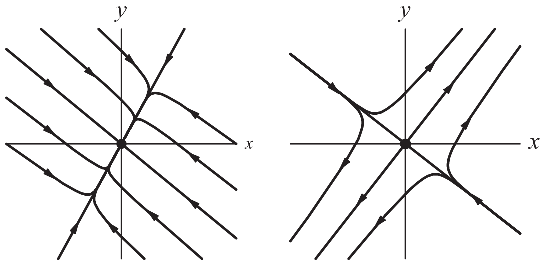

Figure 2.2 Problem 2.1(iii) :$\dot{x} = -4x + 2y$, $\dot{y} = 3x - 2y$, stable node; (iv) $\dot{x} = x + 2y$, $\dot{y} = 2x + 2y$, saddle.

See Figure 2.2(iv)

(v) $\dot{x} = 4x - 2y$, $\dot{y} = 3x - y$. The parameters are

$$p = 4 - 1 = 3 > 0, \quad q = -4 + 6 = 2 > 0, \quad \Delta = 9 - 8 = 1 > 0.$$

Therefore the origin is an unstable node. The radial paths have slopes $m_1 = \frac{1}{2}$ and $m_2 = 3$ and equations

$$y = \tfrac{1}{2}x, \quad y = 3x.$$

The phase diagram is shown in Figure 2.3(v).

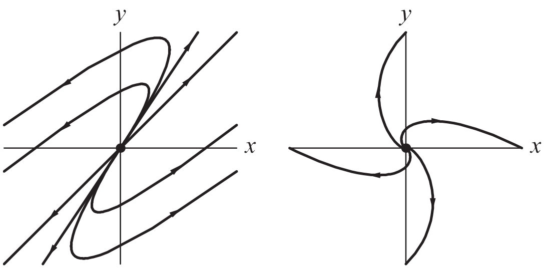

Figure 2.3 Problem 2.1(v): $\dot{x} = 4x - 2y$, $\dot{y} = 3x - y$, unstable node; (vi) $\dot{x} = 2x + y$, $\dot{y} = -x + y$, unstable spiral.

(vi) $\dot{x} = 2x + y$, $\dot{y} = -x + y$. The parameters are

$$p = 2 + 1 = 3 > 0, \quad q = 2 + 1 = 3 > 0, \quad \Delta = 9 - 12 = -3 < 0.$$

The origin is an unstable spiral. Some phase paths are shown in Figure 2.3(vi).

• 2.2 Some of the following systems either generate a single eigenvalue, or a zero eigenvalue, or in other ways vary the types illustrated in Section 2.5. Sketch their phase diagrams

(i) $\dot{x} = 3x - y$, $\dot{y} = x + y$;

(ii) $\dot{x} = x - y$, $\dot{y} = 2x - 2y$;

(iii) $\dot{x} = x$, $\dot{y} = 2x - 3y$;

(iv) $\dot{x} = x$, $\dot{y} = x + 3y$;
(v) $\dot{x} = -y$, $\dot{y} = 2x - 4y$;
(vi) $\dot{x} = x$, $\dot{y} = y$;
(vii) $\dot{x} = 0$, $\dot{y} = x$.

2.2. Note that the scales on both axes are the same.

(i) $\dot{x} = 3x - y$, $\dot{y} = x + y$. Using the classification table (See Section 2.5)

$$p = 3 + 1 = 4 > 0, \quad q = 3 + 1 = 4 > 0, \quad \Delta = 16 - 16 = 0.$$

Hence the origin is an unstable degenerate node with a repeated eigenvalue of $m = 1$. The straight line $y = x$ contains radial paths (Figure 2.4).

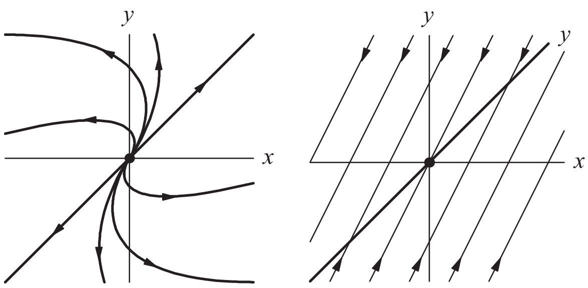

Figure 2.4 Problem 2.2(i): $\dot{x} = 3x - y$, $\dot{y} = x + y$, unstable degenerate node; (ii) $\dot{x} = x - y$, $\dot{y} = 2x - 2y$, parallel paths.

(ii) $\dot{x} = x - y$, $\dot{y} = 2x - 2y$. All points on the line $y = x$ are equilibrium points. The phase paths are given by

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2 \quad \Rightarrow \quad y = 2x + C,$$

which is a family of parallel straight lines (Figure 2.4(ii)).

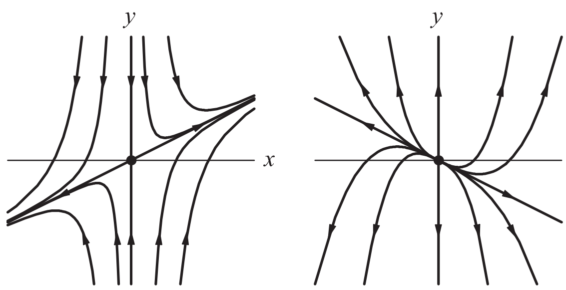

Figure 2.5 Problem 2.2(iii): $\dot{x} = x$, $\dot{y} = 2x - 3y$, saddle; (iv) $\dot{x} = x$, $\dot{y} = x + 3y$, unstable node with the y axis as radial paths.

(iii) $\dot{x} = x$, $\dot{y} = 2x - 3y$. The parameters are

$$p = 1 - 3 = -2 < 0, \quad q = -3 < 0, \quad \Delta = 4 + 12 = 16 > 0,$$

which implies that the equilibrium point is a saddle. From the first equation, the axis $x = 0$ is a solution, as also is $y = \frac{1}{2}x$. These lines are the asymptotes of the saddle point (Figure 2.5).

(iv) $\dot{x} = x$, $\dot{y} = x + 3y$. The parameters are

$$p = 1 + 3 = 4 > 0, \quad q = 3 > 0, \quad \Delta = 16 - 12 = 4 > 0,$$

which is an unstable node with radial paths along $x = 0$ and $y = -\frac{1}{2}x$ (Figure 2.5).

(v) $\dot{x} = -y$, $\dot{y} = 2x - 4y$. The parameters are

$$p = -4 < 0, \quad q = 2 > 0, \quad \Delta = 16 - 8 = 8 > 0,$$

This is a stable node but with eigenvalues $2 \pm 2\sqrt{2}$ of differing signs. This produces a similar phase diagram to that for Problem 1(iii).

(vi) $\dot{x} = x$, $\dot{y} = y$. The parameters are

$$p = 2, \quad q = 1, \quad \Delta = 4 - 4 = 0,$$

which makes it a degenerate case between an unstable node and an unstable spiral. The phase paths are given by

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y}{x} \quad \Rightarrow \quad y = Cx,$$

which is a family of radial straight lines as shown in Figure 2.6(vi): it is a it star-shaped phase diagram.

(vii) $\dot{x} = 0$, $\dot{y} = x$. All points on the y axis are equilibrium points. The parameter values are $p = q = \Delta = 0$ which makes this a degenerate case. The equations can be solved directly to give

$$x = C, \quad y = \int x\mathrm{d}t + D = \int C\mathrm{d}t + D = Ct + D.$$

Hence the phase diagram (shown in Figure 2.6(vii)) consists of all lines $x = C$ parallel to the y axis.

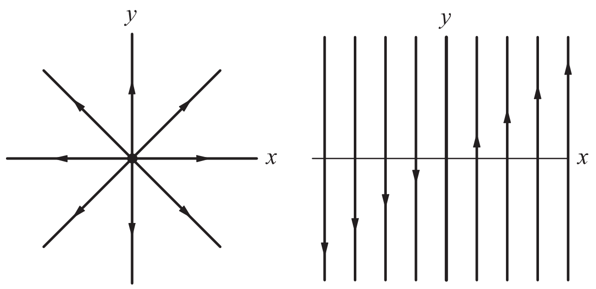

Figure 2.6 Problem 2.2(vi): $\dot{x} = x$, $\dot{y} = y$, saddle; (vii) $\dot{x} = x$, $\dot{y} = x$, unstable node with the y axis as radial paths.