Norwegian University of Science and Technology
Department of Mathematical Sciences

TMA4145 Linear Methods

Fall 2018

Exercise set 3

1 **a)** Determine if the following expressions are norms for $\mathbb{R}^3$.

1. $f(x_1, x_2, x_3) = |x_1| + |x_2|$;
2. $f(x_1, x_2, x_3) = |x_1| + (|x_2|^2 + |x_3|^2)^{1/2}$;
3. $f(x_1, x_2, x_3) = (w_1|x_1|^3 + w_2|x_2|^2 + w_3|x_3|)^{1/2}$ for some positive real numbers w_1, w_2, w_3.

b) Determine $\|z\|_1$, $\|z\|_2$ and $\|z\|_\infty$ for $z = (1+i, 1-i)$ and $z = (e^{i\pi/2}, e^{3i\pi/2})$ in $\mathbb{C}^2$.

2

1. What is an *open* ball $B_1(x_0)$ in $\mathbb{C}$ with the metric induced by the norm $\|x\| = \sqrt{\mathrm{Re}(x)^2 + \mathrm{Im}(x)^2}$?
2. What is a *closed* ball $\overline{B_1}(f_0)$ in the space of bounded and continuous real-valued functions on $[a, b]$, denoted $BC([a, b], \mathbb{R})$, with the metric induced by the supremum norm?

3 Recall that any set X can be endowed with the discrete metric

$$d(x, y) := \begin{cases} 1, & x \neq y, \\ 0, & x = y. \end{cases}$$

Show that in a discrete metric space, every subset is both open and closed.

4 Let X be a vector space and $\|.\|_a$ and $\|.\|_b$ norms on x. Show that $\|x\| := (\|x\|_a^2 + \|x\|_b^2)^{1/2}$ defines a norm on X.

Challenge (voluntary): Try to define a variant of this norm for $p \neq 2$ and contemplate about a possible proof of this statement.

5 Let $M_n(\mathbb{R})$ be the vector space of $n \times n$ matrices. Define for $A \in M_n(\mathbb{R})$ the function $\|A\|_2 = (\sum_{i,j=1}^n |a_{ij}|^2)^{1/2}$. Show that $\|.\|_2$ is a norm on $M_n(\mathbb{R})$. The trace of a matrix $A \in M_n(\mathbb{R})$ is defined as the sum of its diagonal elements, $\mathrm{tr}(A) = a_{11} + \cdots + a_{nn}$. Prove that $\|A\|_2^2 = \mathrm{tr}(A^T A)$. If the general case is too difficult, try to do it for $n = 3$.

$\boxed{6}$ Let $(X, \|.\|)$ be a normed vector space. Show that for any $x, y \in X$ we have

$$|\|x\| - \|y\|| \leq \|x - y\|.$$

This is known as the *reversed triangle inequality.*