MODEL PREDICTIVE CONTROL

HYBRID MPC

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COURSE STRUCTURE

- ✔ **Basic concepts** of model predictive control (MPC) and **linear MPC**
- ✔ **Linear time-varying** and **nonlinear** MPC
- ✔ MPC computations: **quadratic programming** (QP), **explicit** MPC
- Hybrid MPC
- Stochastic MPC
- Data-driven MPC

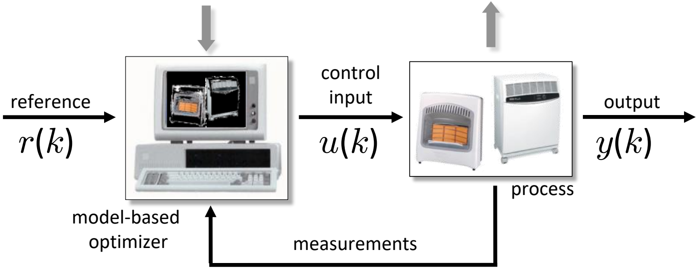
Course page:

`http://cse.lab.imtlucca.it/~bemporad/mpc_course.html`

HYBRID MPC

HYBRID MODEL PREDICTIVE CONTROL

$$\begin{cases} x(k+1) & = & Ax(k) + B_1u(k) + B_2\delta(k) + B_3z(k) + B_5 \\ y(k) & = & Cx(k) + D_1u(k) + D_2\delta(k) + D_3z(k) + D_5 \\ E_2\delta(k) & + & E_3z(k) \leq E_4x(k) + E_1u(k) + E_5 \end{cases}$$



Use a hybrid dynamical model of the process to predict its future evolution and choose the "best" control action

MIQP FORMULATION OF HYBRID MPC

(Bemporad, Morari, 1999)

- Finite-horizon optimal control problem (regulation)

$$\begin{array}{ll} \min & \displaystyle\sum_{k=0}^{N-1} y_k' Q y_k + u_k' R u_k \\ \text{s.t.} & \left\{\begin{array}{rcl} x_{k+1} & = & Ax_k + B_1 u_k + B_2 \delta_k + B_3 z_k + B_5 \\ y_k & = & Cx_k + D_1 u_k + D_2 \delta_k + D_3 z_k + D_5 \\ E_2 \delta_k & + & E_3 z_k \leq E_4 x_k + E_1 u_k + E_5 \\ x_0 & = & x(t) \end{array}\right. \end{array}$$

$Q = Q' \succ 0, R = R' \succ 0$

- Treat u_k, δ_k, z_k as free decision variables, $k = 0, \ldots, N-1$

- Predictions can be constructed **exactly as in the linear case**

$$x_k = A^k x_0 + \sum_{j=0}^{k-1} A^j (B_1 u_{k-1-j} + B_2 \delta_{k-1-j} + B_3 z_{k-1-j} + B_5)$$

MIQP FORMULATION OF HYBRID MPC

(Bemporad, Morari, 1999)

- After substituting x_k, y_k the resulting optimization problem becomes the following **Mixed-Integer Quadratic Programming (MIQP)** problem

$$\begin{array}{ll} \min_{\xi} & \frac{1}{2}\xi' H\xi + x'(t)F'\xi + \frac{1}{2}x'(t)Yx(t) \\ \text{s.t.} & G\xi \leq W + Sx(t) \end{array}$$

- The optimization vector $\xi = [u_0, \ldots, u_{N-1}, \delta_0, \ldots, \delta_{N-1}, z_0, \ldots, z_{N-1}]$ has **mixed real and binary** components

$$\begin{array}{l} u_k \in \mathbb{R}^{m_c} \times \{0,1\}^{m_b} \\ \delta_k \in \{0,1\}^{r_b} \\ z_k \in \mathbb{R}^{r_c} \end{array} \quad \Longrightarrow \quad \xi \in \mathbb{R}^{N(m_c+r_c)} \times \{0,1\}^{N(m_b+r_b)}$$

HYBRID MPC FOR REFERENCE TRACKING

- Consider the more general set-point tracking problem

$$
\begin{array}{rl}
\min_\xi & \displaystyle\sum_{k=0}^{N-1} \|y_k - r\|_Q^2 + \|u_k - u_r\|_R^2 \\
& +\sigma \left(\|x_k - x_r\|_2^2 + \|\delta_k - \delta_r\|_2^2 + \|z_k - z_r\|_2^2 \right) \\
\text{s.t.} & \text{MLD model equations} \\
& x_0 = x(t) \\
& x_N = x_r
\end{array}
$$

with $\sigma > 0$ and $\|v\|_Q^2 = v'Qv$

- The equilibrium $(x_r, u_r, \delta_r, z_r)$ corresponding to r can be obtained by solving the following mixed-integer feasibility problem

$$
\begin{array}{rcl}
x_r & = & Ax_r + B_1 u_r + B_2 \delta_r + B_3 z_r + B_5 \\
r & = & Cx_r + D_1 u_r + D_2 \delta_r + D_3 z_r + D_5 \\
E_2 \delta_r & + & E_3 z_r \leq E_4 x_r + E_1 u_r + E_5
\end{array}
$$

CLOSED-LOOP CONVERGENCE

(Bemporad, Morari, 1999)

- **Theorem**. Let $(x_r, u_r, \delta_r, z_r)$ be the equilibrium corresponding to r. Assume $x(0)$ such that the MIQP problem **is feasible at time $t = 0$**. Then $\forall Q, R \succ 0, \sigma > 0$ the hybrid MPC closed-loop **converges asymptotically**

$$\begin{array}{rcl} \lim_{t\to\infty} y(t) & = & r \\ \lim_{t\to\infty} u(t) & = & u_r \end{array} \qquad \begin{array}{rcl} \lim_{t\to\infty} x(t) & = & x_r \\ \lim_{t\to\infty} \delta(t) & = & \delta_r \\ \lim_{t\to\infty} z(t) & = & z_r \end{array}$$

  and **all constraints are fulfilled** at each time $t \geq 0$.

- The proof easily follows from standard Lyapunov arguments (see next slide)

- **Lyapunov asymptotic stability** and **exponential stability** follows if proper terminal cost and constraints are imposed (Lazar, Heemels, Weiland, Bemporad, 2006)

CONVERGENCE PROOF

- **Main idea**: Use the **value function** $V^*(x(t))$ as a **Lyapunov function**
- Let $\xi_t = [u_0^t, \ldots, u_{N-1}^t, \delta_0^t, \ldots, \delta_{N-1}^t, z_0^t, \ldots, z_{N-1}^t]$ be the optimal sequence @t
- By construction @t+1 $\bar{\xi} = [u_1^t, \ldots, u_{N-1}^t, u_r, \delta_1^t, \ldots, \delta_{N-1}^t, \delta_r, z_0^t, \ldots, z_{N-1}^t, z_r]$ is feasible, as it satisfies all MLD constraints + terminal constraint $x_N = x_r$
- The cost of $\bar{\xi}$ is $V^*(x(t)) - \|y(t)-r\|_Q^2 - \|u(t)-u_r\|_R^2$
$$-\sigma\left(\|\delta(t)-\delta_r\|_2^2 + \|z(t)-z_r\|_2^2 + \|x(t)-x_r\|_2^2\right) \geq V^*(x(t+1))$$
- $V^*(x(t))$ is monotonically decreasing and ≥ 0, so $\exists \lim_{t\to\infty} V^*(x(t)) \in \mathbb{R}$
- Hence $\|y(t)-r\|_Q^2, \|u(t)-u_r\|_R^2, \|\delta(t)-\delta_r\|_2^2, \|z(t)-z_r\|_2^2, \|x(t)-x_r\|_2^2 \to 0$
- Since $R, Q \succ 0$, $\lim_{t\to\infty} y(t) = r$ and all other variables converge. $\square$

Global optimum is not needed to prove convergence !

MILP FORMULATION OF HYBRID MPC

(Bemporad, Borrelli, Morari, 2000)

- Finite-horizon optimal control problem using infinity norms

$$\min_{\xi} \quad \sum_{k=0}^{N-1} \|Qy_k\|_\infty + \|Ru_k\|_\infty$$

$$\text{s.t.} \quad \left\{ \begin{array}{rcl} x_{k+1} & = & Ax_k + B_1u_k + B_2\delta_k + B_3z_k + B_5 \\ y_k & = & Cx_k + D_1u_k + D_2\delta_k + D_3z_k + D_5 \\ E_2\delta_k & + & E_3z_k \leq E_4x_k + E_1u_k + E_5 \\ x_0 & = & x(t) \end{array} \right.$$

$$Q \in \mathbb{R}^{m_y \times n_y}$$
$$R \in \mathbb{R}^{m_u \times n_u}$$

- Introduce additional variables $\epsilon_k^y, \epsilon_k^u, k = 0, \ldots, N-1$

$$\left\{ \begin{array}{rcl} \epsilon_k^y & \geq & \|Qy_k\|_\infty \\ \epsilon_k^u & \geq & \|Ru_k\|_\infty \end{array} \right. \Rightarrow \left\{ \begin{array}{rcl} \epsilon_k^y & \geq & \pm Q^i y_k \\ \epsilon_k^u & \geq & \pm R^i u_k \end{array} \right.$$

Q^i = ith row of Q

MILP FORMULATION OF HYBRID MPC

(Bemporad, Borrelli, Morari, 2000)

- After substituting x_k, y_k the resulting optimization problem becomes the following **Mixed-Integer Linear Programming (MILP)** problem

$$
\begin{array}{ll}
\min_\xi & \displaystyle\sum_{k=0}^{N-1} \epsilon_k^y + \epsilon_k^u \\
\text{s.t.} & G\xi \leq W + Sx(t)
\end{array}
$$

- $\xi = [u_0, \ldots, u_{N-1}, \delta_0, \ldots, \delta_{N-1}, z_0, \ldots, z_{N-1}, \epsilon_0^y, \epsilon_0^u, \ldots, \epsilon_{N-1}^y, \epsilon_{N-1}^u]$ is the optimization vector, with **mixed real and binary** components

$$
\begin{array}{l}
u_k \in \mathbb{R}^{m_c} \times \{0,1\}^{m_b} \\
\delta_k \in \{0,1\}^{r_b} \\
z_k \in \mathbb{R}^{r_c} \\
\epsilon_k^y, \epsilon_k^u \in \mathbb{R}
\end{array}
\quad \Longrightarrow \quad
\xi \in \mathbb{R}^{N(m_c+r_c+2)} \times \{0,1\}^{N(m_b+r_b)}
$$

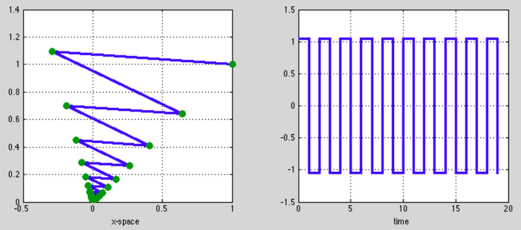
- Same approach applies to any **convex piecewise affine** stage cost

HYBRID MPC EXAMPLE

- PWA system:

$$\begin{cases} x(t+1) &= 0.8 \begin{bmatrix} \cos\alpha(t) & -\sin\alpha(t) \\ \sin\alpha(t) & \cos\alpha(t) \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \\ y(t) &= \begin{bmatrix} 0 & 1 \end{bmatrix} x(t) \\ \alpha(t) &= \begin{cases} \frac{\pi}{3} & \text{if } \begin{bmatrix} 1 & 0 \end{bmatrix} x(t) \geq 0 \\ -\frac{\pi}{3} & \text{if } \begin{bmatrix} 1 & 0 \end{bmatrix} x(t) < 0 \end{cases} \end{cases}$$

- Open-loop simulation:



go to demo `demos/hybrid/bm99sim.m`

HYBRID MPC EXAMPLE

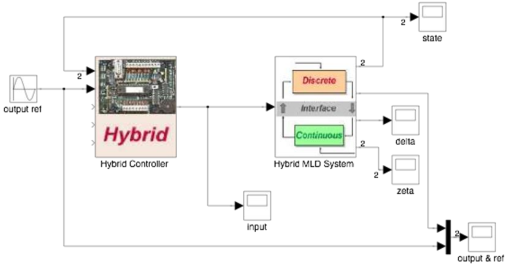
```
/* 2x2 PWA system  - Example from the paper
   A. Bemporad and M. Morari, ``Control of systems integrating logic, dynamics,
   and constraints,'' Automatica, vol. 35, no. 3, pp. 407-427, 1999.
   (C) 2003 by A. Bemporad, 2003 */

SYSTEM pwa {

INTERFACE {
            STATE { REAL x1 [-10,10];
                    REAL x2 [-10,10];}

            INPUT { REAL u [-1.1,1.1];}

            OUTPUT{ REAL y;}

            PARAMETER {
              REAL alpha = 1.0472; /* 60 deg in radiants */
              REAL C = cos(alpha);
              REAL S = sin(alpha);}
           }

IMPLEMENTATION {
            AUX { REAL z1,z2;
                  BOOL sign; }
            AD  { sign = x1<=0; }

            DA  { z1 = {IF sign THEN 0.8*(C*x1+S*x2)
                        ELSE 0.8*(C*x1-S*x2) };
                  z2 = {IF sign THEN 0.8*(-S*x1+C*x2)
                        ELSE 0.8*(S*x1+C*x2) };  }

            CONTINUOUS {x1 = z1;
                        x2 = z2+u; }

            OUTPUT { y = x2;  }
              }
}
```

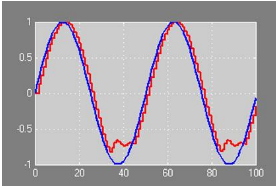
go to `demos/hybrid/bm99.hys`

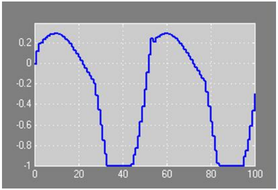
HYBRID MPC EXAMPLE

- Closed-loop MPC results:

$$\begin{array}{ll} \min & \displaystyle\sum_{k=1}^{2} |y_k - r(t)| \\ \text{s.t.} & -1 \leq u_k \leq 1,\ i = 0, 1 \end{array}$$







- Average CPU time to solve MILP: $\approx$ 1 ms/step

(Macbook Pro 3GHz Intel Core i7 using GLPK)

HYBRID MPC – TEMPERATURE CONTROL

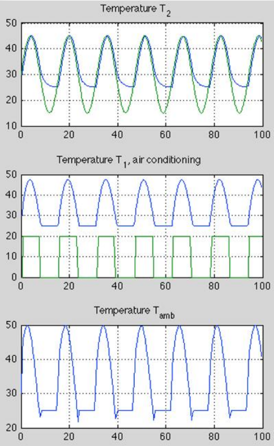
```
>> refs.x=2;            % just weight state #2
>> Q.x=1;             % unit weight on state #2
>> Q.rho=Inf;                % hard constraints
>> Q.norm=Inf;                 % infinity norms
>> N=2;                     % prediction horizon
>> limits.xmin=[25;-Inf];
```

```
>> C=hybcon(S,Q,N,limits,refs);
```

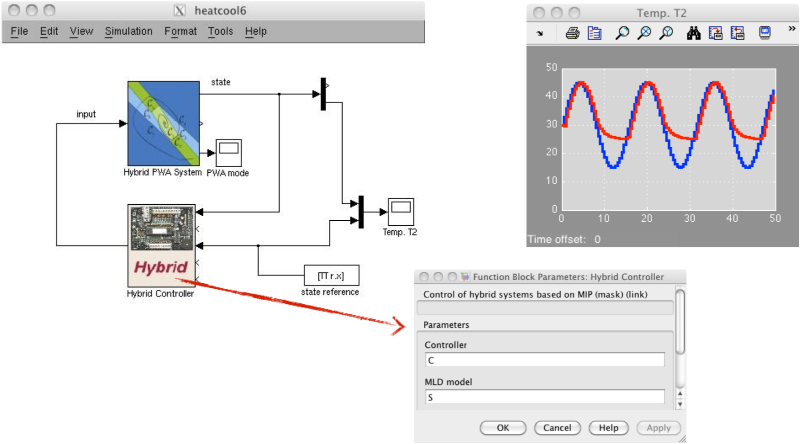
```
>> C

Hybrid controller based on MLD model S <heatcoolmodel.hys> [Inf-norm]

2 state measurement(s)
0 output reference(s)
0 input reference(s)
1 state reference(s)
0 reference(s) on auxiliary continuous z-variables

20 optimization variable(s) (8 continuous, 12 binary)
46 mixed-integer linear inequalities
sampling time = 0.5, MILP solver = 'glpk'

Type "struct(C)" for more details.
>>
```

```
>> [XX,UU,DD,ZZ,TT]=sim(C,S,r,x0,Tstop);
```

$$\begin{array}{ll} \min & \displaystyle\sum_{k=1}^{2} \|x_{2k} - r(t)\|_\infty \\ \text{s.t.} & \begin{cases} x_{1k} \geq 25, \ k = 1, 2 \\ \text{MLD model} \end{cases} \end{array}$$



HYBRID MPC — TEMPERATURE CONTROL



- Average CPU time to solve MILP: $\approx$ 1 ms/step

  (Macbook Pro 3GHz Intel Core i7 using GLPK)

MIXED-INTEGER PROGRAMMING SOLVERS

- Binary constraints make Mixed-Integer Programming (MIP) a hard problem ($\mathcal{NP}$-complete)

- However, excellent general purpose **branch & bound** / **branch & cut** solvers available for MILP and MIQP (Gurobi, CPLEX, FICO Xpress, GLPK, CBC, ...)

  (more solvers/benchmarks: see `http://plato.la.asu.edu/bench.html`)

- MIQP approaches tailored to embedded hybrid MPC applications:
  - B&B + (dual) active set methods for QP

    (Leyffer, Fletcher, 1998) (Axehill, Hansson, 2006) (Bemporad, 2015) (Bemporad, Naik, 2018)
  - B&B + interior point methods: (Frick, Domahidi, Morari, 2015)
  - B&B + fast gradient projection: (Naik, Bemporad, 2017)
  - B&B + ADMM: (Stellato, Naik, Bemporad, Goulart, Boyd, 2018)

- No need to reach global optimum (see convergence proof), although performance may deteriorate

BRANCH & BOUND METHOD FOR MIQP

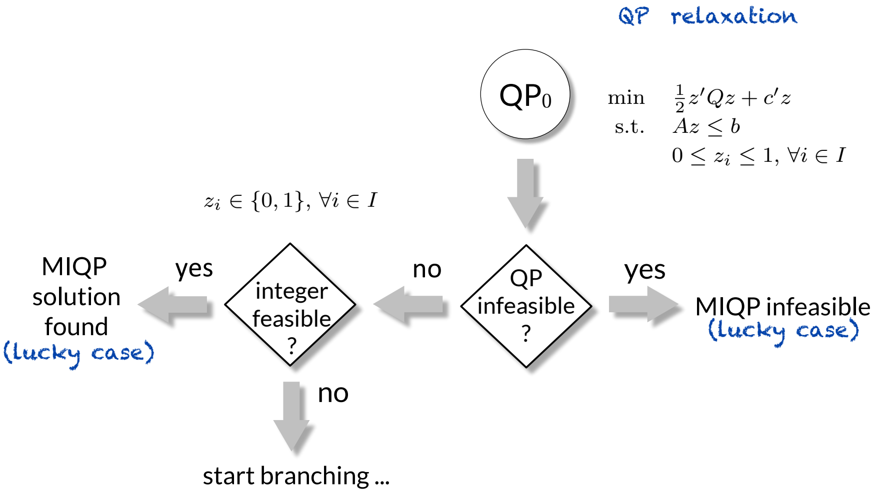
(Dakin, 1965)

- We want to solve the following MIQP

$$\begin{array}{rl} \min & V(z) \triangleq \frac{1}{2}z'Qz + c'z \\ \text{s.t.} & Az \leq b \\ & z_i \in \{0,1\},\ \forall i \in I \end{array} \qquad \begin{array}{l} z \in \mathbb{R}^n \\ Q = Q' \succeq 0 \\ I \subseteq \{1,\ldots,n\} \end{array}$$

- **Branch & Bound (B&B)** is the simplest (and most popular) approach to solve the problem to optimality
- **Key idea**:
  - for each binary variable $z_i, i \in I$, either set $z_i = 0$, or $z_i = 1$, or $z_i \in [0,1]$
  - solve the corresponding **QP relaxation** of the MIQP problem
  - use QP result to decide the next combination of fixed/relaxed variables

BRANCH & BOUND METHOD FOR MIQP



- **Branching rule**: pick the index i such that z_i is closest to $\frac{1}{2}$ (max fractional part)

  (Breu, Burdet, 1974)

- Solve two new QP relaxations

QP_0

$$\begin{array}{ll} \min & \frac{1}{2}z'Qz + c'z \\ \text{s.t.} & Az \le b \\ & 0 \le z_i \le 1,\ \forall i \in I \end{array}$$

QP_1

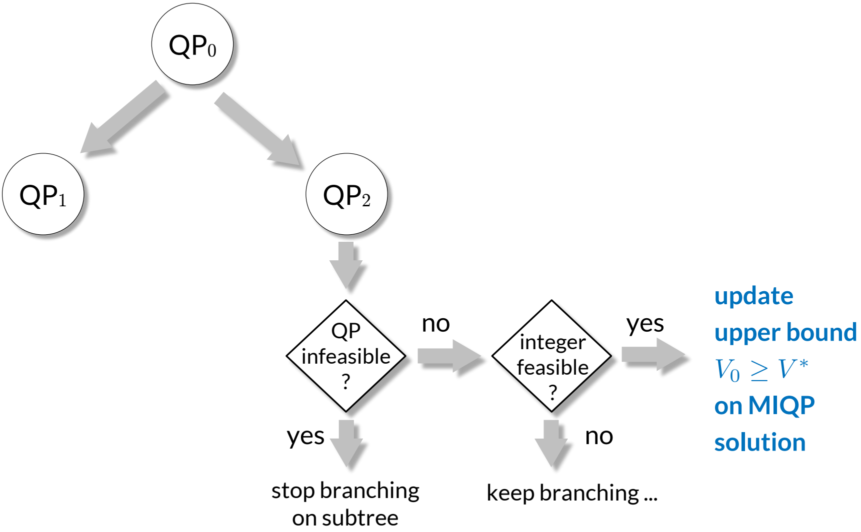
$$\begin{array}{ll} \min & \frac{1}{2}z'Qz + c'z \\ \text{s.t.} & Az \le b \\ & z_i = 0 \\ & 0 \le z_j \le 1,\ \forall j \in I,\ j \ne i \end{array}$$

QP_2

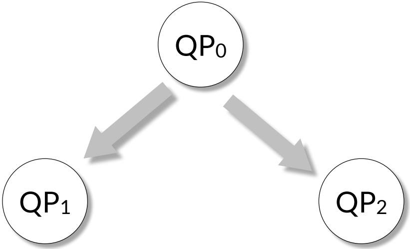
$$\begin{array}{ll} \min & \frac{1}{2}z'Qz + c'z \\ \text{s.t.} & Az \le b \\ & z_i = 1 \\ & 0 \le z_j \le 1,\ \forall j \in I,\ j \ne i \end{array}$$

- Possibly exploit **warm starting** from QP_0 when solving new relaxations QP_1 and QP_2

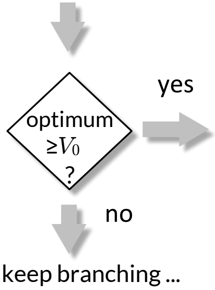
BRANCH & BOUND METHOD FOR MIQP



BRANCH & BOUND METHOD FOR MIQP



The cost V_0 of the best integer-feasible solution found so fare gives an upper bound $V_0 \geq V^*$ on MIQP solution



stop branching

(adding further equality constraints can only increase the optimal cost)

BRANCH & BOUND METHOD FOR MIQP

- While solving the QP relaxation, if the **dual cost** is available it gives a **lower bound** to the solution of the relaxed problem

- The QP solver can be stopped whenever the dual cost $\geq V_0$!

  This may save a lot of computations

- When no further branching is possible, either the MIQP problem is recognized infeasible or the optimal solution z^* has been found

SOLVING MIQP VIA NNLS

(Bemporad, 2015)

- B&B method + QP solver based on **nonnegative least squares** applied to solving the MIQP

$$
\begin{array}{ll}
\min\limits_{z} & V(z) \triangleq \frac{1}{2} z'Qz + c'z \\
\text{s.t.} & \ell \leq Az \leq u \\
 & Gz = g \\
 & \bar{A}_i z \in \{\bar{\ell}_i, \bar{u}_i\},\ i = 1, \ldots, q
\end{array}
\qquad Q = Q' \succ 0
$$

- Binary constraints on z are a special case: $\bar{\ell}_i = 0, \bar{u}_i = 1$, $\bar{A}_i = [0 \ldots 0\, 1\, 0 \ldots 0]$
- Warm starting from parent node exploited when solving new QP relaxation
- QP solver interrupted when dual cost larger than best known upper-bound

- **Worst-case** CPU time (ms) on **random MIQP** problems:

n	m	q	NNLS$_{LDL}$	NNLS$_{QR}$	GUROBI	CPLEX
10	5	2	2.3	1.2	1.4	8.0
10	100	2	5.7	3.3	6.1	31.4
50	25	5	4.2	6.1	14.1	30.1
50	200	10	68.8	104.4	114.6	294.1
100	50	2	4.6	10.2	37.2	69.2
100	200	15	137.5	365.7	259.8	547.8
150	100	5	15.6	49.2	157.2	260.1
150	300	20	1174.4	3970.4	1296.1	2123.9

n = # variables
m = # inequalities
q = # binary vars
(no equalities)

Compiled Embedded MATLAB code (QP solver) + MATLAB code (B&B)
CPU results measured on Macbook Pro 3GHz Intel Core i7

NNLS-LDL = recursive LDL' factorization used to solve least-square problems in QP solver

NNLS-QR = recursive QR factorization used instead (numerically more robust)

SOLVING MIQP VIA NNLS

- **Worst-case** CPU time (ms) on **random purely binary QP** problems:

n	m	q	NNLS_{LDL}	NNLS_{QR}	GUROBI	CPLEX
2	10	2	5.1	4.0	0.7	8.4
4	20	4	8.9	4.3	4.5	16.7
8	40	8	19.2	18.0	37.1	14.7
12	60	12	59.7	57.8	82.3	47.9
20	100	20	483.5	457.7	566.8	99.6
25	250	25	110.4	93.3	1054.4	169.4
30	150	30	1645.4	1415.8	2156.2	184.5

- Worst-case CPU time (ms) on a **hybrid MPC** problem

N = prediction horizon

MIQP regularized to make Q strictly $\succ 0$ (solution difference is negligible)

N	NNLS_{LDL}	NNLS_{QR}	GUROBI	CPLEX
2	2.2	2.3	1.2	3.0
3	3.4	3.9	2.0	6.5
4	5.0	6.5	2.6	8.1
5	7.6	9.8	3.7	9.0
6	12.3	17.7	4.3	11.0
7	20.5	30.5	5.8	13.1
8	28.9	47.1	7.3	17.3
9	38.8	62.5	9.5	18.9
10	55.4	98.2	10.9	22.4

SOLVING MIQP VIA NNLS AND PROXIMAL-POINT ITERATIONS

(Bemporad, Naik, 2018)

- **Robustified approach**: use **NNLS + proximal-point iterations** to solve QP relaxations (Bemporad, 2018)

$$\begin{array}{rl} z_{k+1} = \arg\min_z & \frac{1}{2}z'Qz + c'z + \frac{\epsilon}{2}\|z-z_k\|_2^2 \\ \text{s.t.} & \ell \leq Az \leq u \\ & Gz = g \end{array}$$

- CPU time (ms) on **MIQP** coming from hybrid MPC (`bm99` demo):

For $N = 10$:
30 real vars
10 binary vars
160 inequalities

prox-NNLS* = warm start of binary vars exploited

N	prox-NNLS		prox-NNLS*		GUROBI		CPLEX	
	avg	max	avg	max	avg	max	avg	max
2	2.0	2.6	2.0	2.6	1.6	2.0	3.1	6.0
4	5.3	8.8	3.1	6.9	3.1	3.9	8.9	15.7
8	29.7	71.0	8.1	43.4	7.2	13.2	15.5	80.2
10	76.2	146.1	14.4	103.2	11.1	17.6	35.1	95.3
12	155.8	410.8	26.9	263.4	14.9	31.2	61.7	103.7
15	484.2	1242.3	61.7	766.9	25.9	109.8	89.9	181.1

CPU time measured on Intel Core i7-4700MQ CPU 2.40 GHz

FAST GRADIENT PROJECTION FOR MIQP

(Naik, Bemporad, 2017)

- Consider again the MIQP problem with Hessian $Q = Q' \succ 0$

$$\begin{aligned} \min_z \quad & V(z) \triangleq \frac{1}{2} z'Qz + c'z \\ \text{s.t.} \quad & \ell \leq Az \leq u \\ & Gz = g \\ & \bar{A}_i z \in \{\bar{\ell}_i, \bar{u}_i\},\ i = 1, \ldots, p \end{aligned}$$

$$\begin{aligned} w^k &= y^k + \beta_k (y^k - y^{k-1}) \\ z^k &= -Kw^k - Jx \\ s^k &= \ldots \\ y_i^{k+1} &= \max\left\{w_i^k + s_i^k, 0\right\},\ i \in I_{\rm ineq} \end{aligned}$$

- Use B&B and **fast gradient projection** to solve dual of QP relaxation

constraint is relaxed	$\bar{A}_i z \leq \bar{u}_i$	$\rightarrow$	$y_i^{k+1} = \max\left\{w_i^k + s_i^k, 0\right\}$	$(y_i \geq 0)$
constraint is fixed	$\bar{A}_i z = \bar{u}_i$	$\rightarrow$	$y_i^{k+1} = w_i^k + s_i^k$	$(y_i \lessgtr 0)$
constraint is ignored	$\bar{A}_i z = \bar{\ell}_i$	$\rightarrow$	$y_i^{k+1} = 0$	$(y_i = 0)$

FAST GRADIENT PROJECTION FOR MIQP

(Naik, Bemporad, 2017)

- **Same dual QP matrices** at each node, **preconditioning** computed only once
- **Warm-start** exploited, **dual cost** used to stop QP relaxations earlier
- Criterion based on Farkas lemma to detect **QP infeasibility**
- Numerical results (time in ms):

n	m	p	q	miqpGPAD	GUROBI
10	100	2	2	15.6	6.56
50	25	5	3	3.44	8.74
50	150	10	5	63.22	46.25
100	50	2	5	6.22	26.24
100	200	15	5	164.06	188.42
150	100	5	5	31.26	88.13
150	200	20	5	258.80	274.06
200	50	15	6	35.08	144.38

n = # variables
m = # inequality constraints
p = # binary constraints
q = # equality constraints

CPU time measured on Intel Core i7-4700MQ CPU 2.40 GHz

MIQP AND ADMM

- B&B + ADMM: solve QP relaxations via ADMM

  (Stellato, Naik, Bemporad, Goulart, Boyd, 2018)

$$
\begin{array}{ll}
\min & \frac{1}{2}x'Qx + c'x \\
\text{s.t.} & \ell \leq Ax \leq u \\
 & A_i x \in \{\ell_i, u_i\},\ i \in I
\end{array}
$$

- Simpler **heuristic** approach: only perform one set of ADMM iterations

  (Takapoui, Moehle, Boyd, Bemporad, 2017)

quantization step

$$
\begin{array}{rcl}
x^{k+1} & = & -(Q + \rho A^T A)^{-1}(\rho A^T(y^k - z^k) + c) \\
z^{k+1} & = & \min\{\max\{Ax^{k+1} + y^k, \ell\}, u\} \\
z_i^{k+1} & = & \begin{cases} \ell_i & \text{if } z_i^{k+1} < \frac{\ell_i + u_i}{2} \\ u_i & \text{if } z_i^{k+1} \geq \frac{\ell_i + u_i}{2} \end{cases},\ i \in I \\
y^{k+1} & = & y^k + Ax^{k+1} - z^{k+1}
\end{array}
$$

- Iterations converge to a (local) solution

- Similar heuristic idea also applicable to fast gradient methods (Naik, Bemporad, 2017)

HEURISTIC ADMM METHOD FOR (SUBOPTIMAL) MIQP

(Takapoui, Moehle, Boyd, Bemporad, 2017)

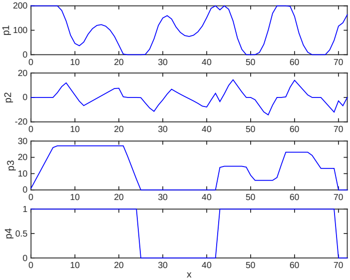
- **Example**: parallel hybrid electric vehicle control problem

engine power

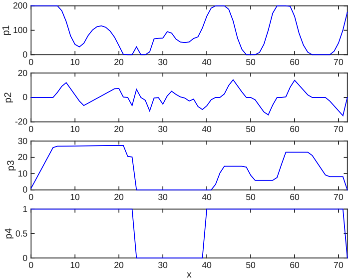
electrical power

energy stored
in battery

engine on/off



optimal solution

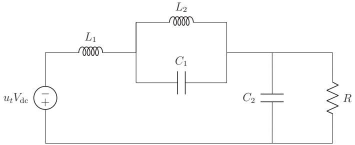


ADMM solution

HEURISTIC ADMM METHOD FOR (SUBOPTIMAL) MIQP

(Takapoui, Moehle, Boyd, Bemporad, 2017)

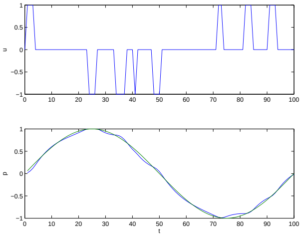
- **Example**: power converter control problem



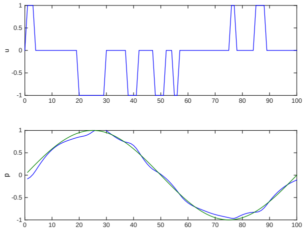
$$
\begin{array}{ll}
\text{minimize} & \sum_{t=0}^{T} (v_{2,t} - v_{\rm des})^2 + \lambda |u_t - u_{t-1}| \\
\text{subject to} & \xi_{t+1} = G\xi_t + Hu_t \\
& \xi_0 = \xi_T \\
& u_0 = u_T \\
& u_t \in \{-1, 0, 1\}
\end{array}
$$

input voltage sign u_t

output voltage v_2

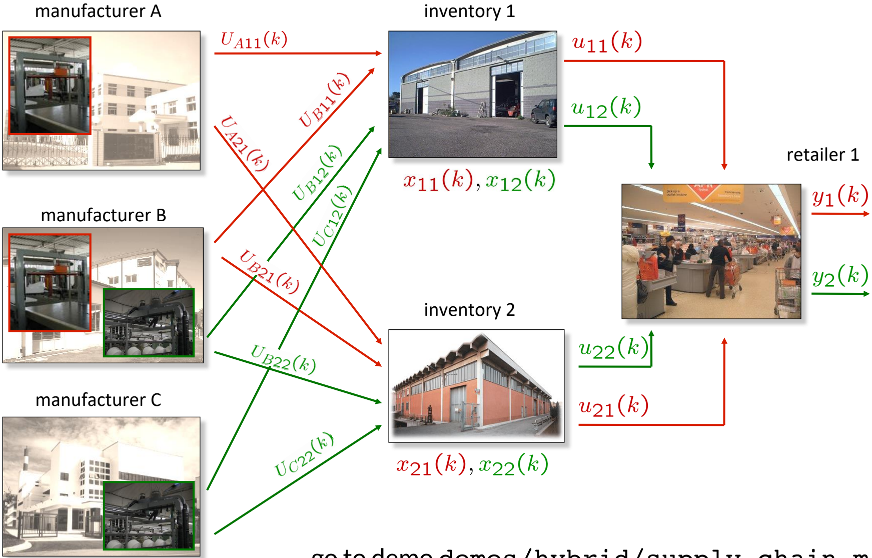


optimal solution



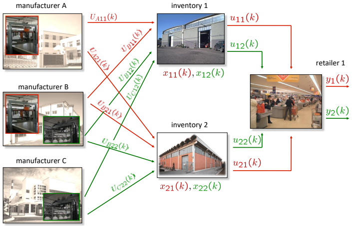
ADMM solution

A SIMPLE EXAMPLE IN SUPPLY CHAIN MANAGEMENT



go to demo `demos/hybrid/supply_chain.m`

SUPPLY CHAIN MANAGEMENT - SYSTEM VARIABLES



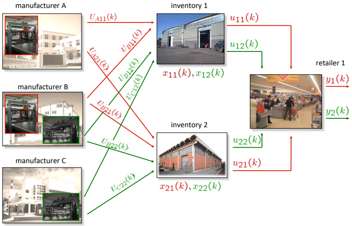
- **Continuous states**:
  $x_{ij}(k)$ = amount of j hold in inventory i at time k ($i = 1, 2, j = 1, 2$)

- **Continuous outputs**:
  $y_j(k)$ = amount of j sold at time k ($j = 1, 2$)

- **Continuous inputs**:
  $u_{ij}(k)$ = amount of j taken from inventory i at time k ($i = 1, 2, j = 1, 2$)

- **Binary inputs**:
  $U_{Xij}(k) = 1$ if manufacturer X produces and send j to inventory i at time k

SUPPLY CHAIN MANAGEMENT - CONSTRAINTS



- Max capacity of inventory i:

$$0 \leq \sum_{j=1}^{2} x_{ij} \leq x_{Mi}$$

- Max transportation from inventories:

$$0 \leq u_{ij}(k) \leq u_M$$

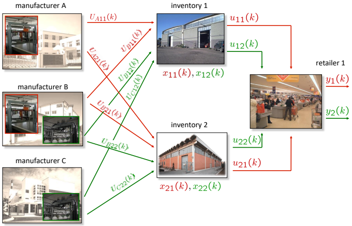
- A product can only be sent to one inventory:

$U_{A11}(k)$ and $U_{A21}(k)$ cannot be both = 1
$U_{B11}(k)$ and $U_{B21}(k)$ cannot be both = 1
$U_{B12}(k)$ and $U_{B22}(k)$ cannot be both = 1
$U_{C12}(k)$ and $U_{C22}(k)$ cannot be both = 1

- A manufacturer can only produce one type of product at one time:

$[U_{B11}(k)$ or $U_{B21}(k) = 1]$, $[U_{B12}(k)$ or $U_{B22}(k) = 1]$ cannot be both true

- Let $P_{A1}, P_{B1}, P_{B2}, P_{C2}$ = amount of product of type 1 (2) produced by A (B, C) in one time interval



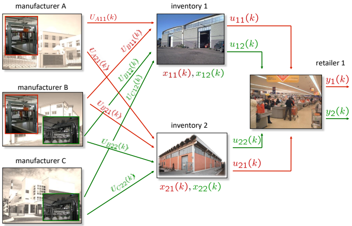
- Level of inventories

$$\left\{\begin{array}{rcl} x_{11}(k+1) & = & x_{11}(k) + P_{A1}U_{A11}(k) + P_{B1}U_{B11}(k) - u_{11}(k) \\ x_{12}(k+1) & = & x_{12}(k) + P_{B2}U_{B12}(k) + P_{C2}U_{C12}(k) - u_{12}(k) \\ x_{21}(k+1) & = & x_{21}(k) + P_{A1}U_{A21}(k) + P_{B1}U_{B21}(k) - u_{21}(k) \\ x_{22}(k+1) & = & x_{22}(k) + P_{B2}U_{B22}(k) + P_{C2}U_{C22}(k) - u_{22}(k) \end{array}\right.$$

- Retailer: all items requested from inventories are sold

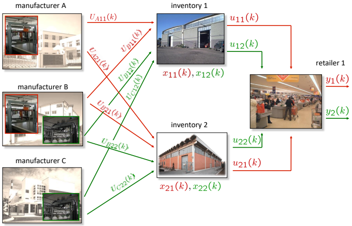
$$\left\{\begin{array}{rcl} y_1 & = & u_{11} + u_{21} \\ y_2 & = & u_{12} + u_{22} \end{array}\right.$$

SUPPLY CHAIN MANAGEMENT - HYSDEL CODE




```
SYSTEM supply_chain{
INTERFACE {
        STATE { REAL x11    [0,10];
                REAL x12    [0,10];
                REAL x21    [0,10];
                REAL x22    [0,10]; }

        INPUT { REAL u11 [0,10];
         REAL u12 [0,10];
         REAL u21 [0,10];
         REAL u22 [0,10];
         BOOL UA11,UA21,UB11,UB12,UB21,UB22,UC12,UC22; }

        OUTPUT {REAL y1,y2;}

        PARAMETER { REAL PA1,PB1,PB2,PC2,xM1,xM2;}
}
IMPLEMENTATION {

        AUX { REAL zA11, zB11, zB12, zC12, zA21, zB21, zB22, zC22;}


        DA {     zA11 = {IF UA11 THEN PA1 ELSE 0};
                 zB11 = {IF UB11 THEN PB1 ELSE 0};
                 zB12 = {IF UB12 THEN PB2 ELSE 0};
                 zC12 = {IF UC12 THEN PC2 ELSE 0};
                 zA21 = {IF UA21 THEN PA1 ELSE 0};
                 zB21 = {IF UB21 THEN PB1 ELSE 0};
                 zB22 = {IF UB22 THEN PB2 ELSE 0};
                 zC22 = {IF UC22 THEN PC2 ELSE 0}; }

        CONTINUOUS {x11 = x11 + zA11 + zB11 - u11;
                    x12 = x12 + zB12 + zC12 - u12;
                    x21 = x21 + zA21 + zB21 - u21;
                    x22 = x22 + zB22 + zC22 - u22;  }

        OUTPUT {    y1 = u11 + u21;
                    y2 = u12 + u22; }

        MUST {  ~(UA11 & UA21);
                ~(UC12 & UC22);
                ~((UB11 | UB21) & (UB12 | UB22));
                ~(UB11 & UB21);
                ~(UB12 & UB22);
                x11+x12 <= xM1;
                x11+x12 >=0;
                x21+x22 <= xM2;
                x21+x22 >=0; }
} }
```

SUPPLY CHAIN MANAGEMENT - OBJECTIVES



- Meet customer demand as much as possible:

$$y_1 \approx r_1, \quad y_2 \approx r_2$$

- Minimize transportation costs

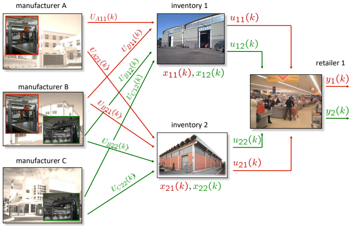
- Fulfill all constraints

SUPPLY CHAIN MANAGEMENT - PERFORMANCE INDEX

$$
\begin{aligned}
\min \sum_{k=0}^{N-1} \quad & \overbrace{10(|y_{1,k} - r_1(t)| + |y_{2,k} - r_2(t)|}^{\text{penalty on demand tracking error}} + \\
& \overbrace{4(|u_{11,k}| + |u_{12,k}|)}^{\text{shipping cost from inv. 1 to market}} \qquad + \\
& \overbrace{2(|u_{21,k}| + |u_{22,k}|)}^{\text{shipping cost from inv. 2 to market}} \qquad + \\
& \overbrace{1(|U_{A11,k}| + |U_{A21,k}|)}^{\text{cost from A to inventories}} + \\
& \overbrace{4(|U_{B11,k}| + |U_{B12,k}| + U_{B21,k}| + |U_{B22,k}|)}^{\text{cost from B to inventories}} + \\
& \overbrace{10(|U_{C12,k}| + |U_{C22,k}|)}^{\text{cost from C to inventories}}
\end{aligned}
$$

SUPPLY CHAIN MANAGEMENT - SIMULATION SETUP

```
>> refs.y=[1 2];                    % weights output2 #1, #2
>> Q.y=diag([10 10]);                        % output weights
…
>> Q.norm=Inf;                              % infinity norms
>> N=2;                                % optimization horizon
>> limits.umin=umin;                          % constraints
>> limits.umax=umax;
>> limits.xmin=xmin;                            % xij(k)>=0
>> limits.xmax=xmax;                % xij(k)<=xMi (redundant)
```

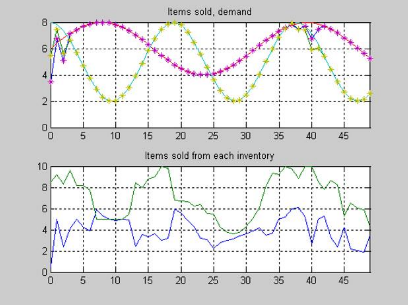


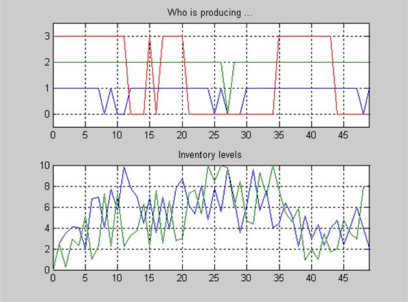

```
>> C=hybcon(S,Q,N,limits,refs);
```

```
 >> C

Hybrid controller based on MLD model S <supply_chain.hys>


[Inf-norm]

4 state measurement(s)
2 output reference(s)
12 input reference(s)
0 state reference(s)
0 reference(s) on auxiliary continuous z-variables

44 optimization variable(s) (8 continuous, 12 binary)
176 mixed-integer linear inequalities
sampling time = 1, MILP solver = 'glpk'

Type "struct(C)" for more details.
>>
```

```
>> x0=[0;0;0;0];                                % Initial condition
>> r.y=[6+2*sin((0:Tstop-1)'/5)             % Reference trajectories
     5+3*cos((0:Tstop-1)'/3)];
```

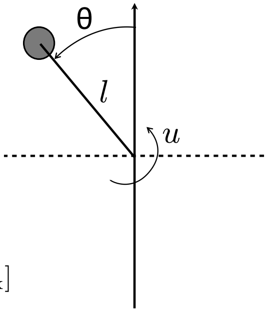
```
>> [XX,UU,DD,ZZ,TT]=sim(C,S,r,x0,Tstop);
```





CPU time: ≈ 13 ms/sample (GLPK) or 9 ms (CPLEX) on Macbook Pro 3GHz Intel Core i7

HYBRID MPC OF AN INVERTED PENDULUM



- **Goal**: swing the pendulum up

- **Non-convex** input constraint

$$u \in [-\tau_{\max}, -\tau_{\min}] \cup \{0\} \cup [\tau_{\min}, \tau_{\max}]$$

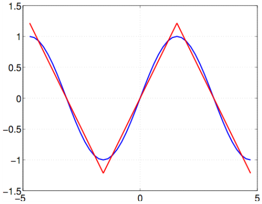
- **Nonlinear** dynamical model

$$l^2 M\ddot{\theta} = Mgl\sin\theta - \beta\dot{\theta} + u$$

INVERTED PENDULUM: NONLINEARITY

- Approximate $\sin(\theta)$ as the piecewise linear function

$$\sin\theta \approx s \triangleq \begin{cases} -\alpha\theta-\gamma & \text{if} \quad \theta \leq -\frac{\pi}{2} \\ \alpha\theta & \text{if} \quad |\theta| \leq \frac{\pi}{2} \\ -\alpha\theta+\gamma & \text{if} \quad \theta \geq \frac{\pi}{2} \end{cases}$$



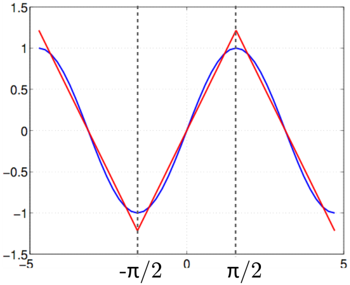
- Get optimal values for α and γ by minimizing fit error

$$\begin{aligned} \min_{\alpha} \quad & \int_0^{\frac{\pi}{2}} (\alpha\theta - \sin(\theta))^2 d\theta \\ = \quad & \frac{\theta}{2} - \frac{1}{2}\cos\theta\sin\theta - 2\alpha\sin\theta + \frac{1}{3}\alpha^2\theta^3 + 2\alpha\theta\cos\theta \Big|_0^{\frac{\pi}{2}} = \frac{1}{24}\pi^3\alpha^2 - 2\alpha + \frac{\pi}{4} \end{aligned}$$

- Zeroing the derivative with respect to α gives $\alpha = \frac{24}{\pi^3}$

- Requiring $s = 0$ for $\theta = \pi$ gives $\gamma = \frac{24}{\pi^2}$

INVERTED PENDULUM: NONLINEARITY



- Introduce the event variables

$$[\delta_3 = 1] \quad \leftrightarrow \quad [\theta \leq -\tfrac{\pi}{2}]$$
$$[\delta_4 = 1] \quad \leftrightarrow \quad [\theta \geq \tfrac{\pi}{2}]$$

  along with the logic constraint

$$[\delta_4 = 1] \rightarrow [\delta_3 = 0]$$

- Set $s = \alpha\theta + s_3 + s_4$ with

$$s_3 = \begin{cases} -2\alpha\theta - \gamma & \text{if } \delta_3 = 1 \\ 0 & \text{otherwise} \end{cases}$$
$$s_4 = \begin{cases} -2\alpha\theta + \gamma & \text{if } \delta_4 = 1 \\ 0 & \text{otherwise} \end{cases}$$

INVERTED PENDULUM: NON-CONVEX CONSTRAINT

- To model the constraint $u \in [-\tau_{\max}, -\tau_{\min}] \cup \{0\} \cup [\tau_{\min}, \tau_{\max}]$ introduce the auxiliary variable

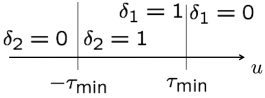
$$\tau_A = \begin{cases} u & \text{if } -\tau_{\min} \le u \le \tau_{\min} \\ 0 & \text{otherwise} \end{cases}$$

and let $u - \tau_A$ be the torque acting on the pendulum, with

$$u \in [-\tau_{\max}, \tau_{\max}]$$

- The input u has no effect on the dynamics for $u \in [-\tau_{\min}, \tau_{\min}]$. Hence, the solver will not choose values in that range if u is penalized in the MPC cost

INVERTED PENDULUM: NON-CONVEX CONSTRAINT



- Introduce new event variables

$$[\delta_1 = 1] \quad \leftrightarrow \quad [u \leq \tau_{\min}]$$

$$[\delta_2 = 1] \quad \leftrightarrow \quad [u \geq -\tau_{\min}]$$

along with the logic constraint $[\delta_1 = 0] \rightarrow [\delta_2 = 1]$ and set

$$\tau_A = \begin{cases} u & \text{if } [\delta_1 = 1] \wedge [\delta_2 = 1] \\ 0 & \text{otherwise} \end{cases}$$

so that $u - \tau_A$ is zero in for $u \in [-\tau_{\min}, \tau_{\min}]$

INVERTED PENDULUM: DYNAMICS

- Set $x \triangleq \left[\begin{smallmatrix}\theta\\ \dot\theta\end{smallmatrix}\right], y \triangleq \theta$ and transform into linear model

$$\begin{aligned}\frac{d}{dt}\begin{bmatrix}x_1\\ x_2\end{bmatrix} &= \begin{bmatrix}0 & 1\\ \frac{g}{l}\alpha & -\frac{\beta}{l^2M}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}+\begin{bmatrix}0 & 0\\ \frac{g}{l} & \frac{1}{l^2M}\end{bmatrix}\begin{bmatrix}s_3+s_4\\ u-\tau_A\end{bmatrix}\\ y &= \begin{bmatrix}1 & 0\end{bmatrix}x\end{aligned}$$

- Discretize in time with sample time $T_s = 50$ ms

$$\begin{aligned}\begin{bmatrix}x_1(k+1)\\ x_2(k+1)\end{bmatrix} &= A\begin{bmatrix}x_1(k)\\ x_2(k)\end{bmatrix}+B\begin{bmatrix}s_3(k)+s_4(k)\\ u(k)-\tau_A(k)\end{bmatrix}\\ y(k) &= \begin{bmatrix}1 & 0\end{bmatrix}x(k)\end{aligned}$$

$$A \triangleq e^{T_sA_c},\ B \triangleq \int_0^{T_s} e^{tA_c}B_c dt$$

INVERTED PENDULUM: HYSDEL MODEL

```
/* Hybrid model of a pendulum

   (C) 2012 by A. Bemporad, April 2012 */

SYSTEM hyb_pendulum {

INTERFACE {
  STATE {
    REAL th     [-2*pi,2*pi];
    REAL thdot [-20,20];
          }
  INPUT {
    REAL u [-11,11];
          }
  OUTPUT{
    REAL y;
          }
  PARAMETER {
    REAL tau_min,alpha,gamma;
    REAL a11,a12,a21,a22,b11,b12,b21,b22;
  }
}

IMPLEMENTATION {
   AUX {
     REAL tauA,s3,s4;
     BOOL d1,d2,d3,d4;
   }
   AD {
     d1 = u<=tau_min;
     d2 = u>=-tau_min;
     d3 = th <= -0.5*pi;
     d4 = th >= 0.5*pi;
   }
   DA  {
     tauA = {IF d1 & d2 THEN u ELSE 0};
     s3 = {IF d3 THEN -2*alpha*th-gamma ELSE 0};
     s4 = {IF d4 THEN -2*alpha*th+gamma ELSE 0};
   }

   CONTINUOUS {
     th    = a11*th+a12*thdot+b11*(s3+s4)+b12*(u-tauA);
     thdot = a21*th+a22*thdot+b21*(s3+s4)+b22*(u-tauA);
   }

   OUTPUT {
     y = th;
   }

   MUST {
     d4->~d3;
     ~d1->d2;
   }
 }
 }
```

```
>> S=mld('pendulum',Ts);
```

go to demo `demos/hybrid/pendulum_init.m`

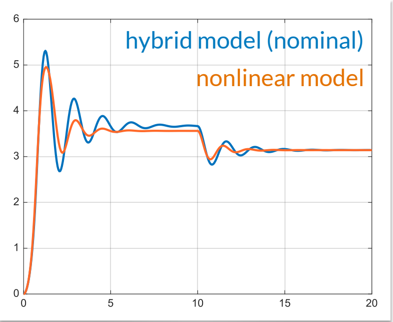
INVERTED PENDULUM: MODEL VALIDATION

- Open-loop simulation from initial condition $\theta(0) = 0, \dot{\theta}(0) = 0$
- Input torque excitation

$$u(t) = \begin{cases} 2\,\mathrm{Nm} & \text{if } 0 \leq t \leq 10\,\mathrm{s} \\ 0 & \text{otherwise} \end{cases}$$

```
>> u0=2;
>> U=[2*ones(200,1);zeros(200,1)];
>> x0=[0;0];
```

```
>> [X,T,D,Z,Y]=sim(S,x0,U);
```



INVERTED PENDULUM: MPC DESIGN

```
>> refs.y=1;
>> refs.u=1;
>> Q.y=1;
>> Q.y=0.01;
>> Q.rho=Inf;
>> Q.norm=Inf;
>> N=5;
>> limits.umin=-10;
>> limits.umax=10;
```

- MPC cost function

$$\sum_{k=0}^{4} |y_k - r(t))| + |0.01 u_k|$$

- MPC constraints $u \in [-\tau_{\max}, \tau_{\max}]$

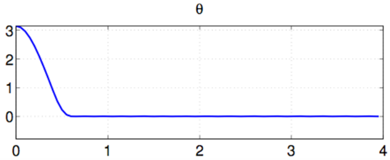
```
>> C=hybcon(S,Q,N,limits,refs);
```

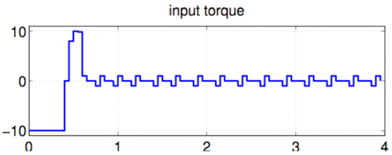
```
 >> C

Hybrid controller based on MLD model S <pendulum.hys> [Inf-norm]

2 state measurement(s)
1 output reference(s)
1 input reference(s)
0 state reference(s)
0 reference(s) on auxiliary continuous z-variables

55 optimization variable(s) (30 continuous, 25 binary)
155 mixed-integer linear inequalities
sampling time = 0.05, MILP solver = 'gurobi'

Type "struct(C)" for more details.
>>
```

INVERTED PENDULUM: CLOSED-LOOP RESULTS

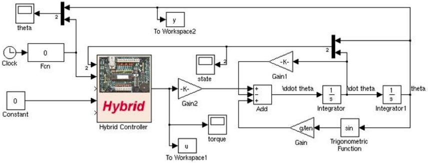
- Nominal simulation

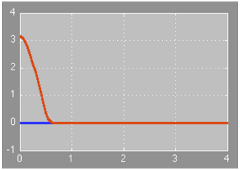
```
>> [X,U,D,Z,T,Y]=sim(C,S,r,x0,4);
```

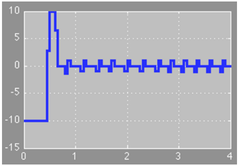




- Nonlinear simulation







CPU time:

51 ms per time step (GLPK)

22 ms per time step (CPLEX)

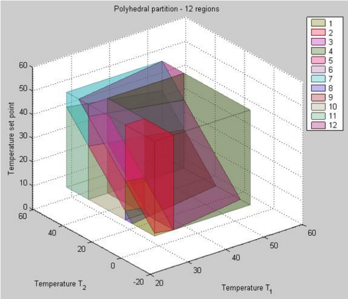
25 ms (GUROBI)

(Macbook Pro 3GHz Intel Core i7)

EXPLICIT HYBRID MPC

EXPLICIT HYBRID MPC

- It is possible to write hybrid MPC laws in explicit form too !
- The explicit MPC law is still piecewise affine on polyhedra



(Bemporad, Borrelli, Morari, 2000)

(Mayne, ECC 2001)

(Mayne, Rakovic, 2002)

(Bemporad, Hybrid Toolbox, 2003)

(Borrelli, Baotic, Bemporad, Morari, 2005)

(Alessio, Bemporad, 2006)

- The control law may be discontinuous, polyhedra may overlap
- Comparison of quadratic costs can be avoided by lifting the parameter space

(Fuchs, Axehill, Morari, 2015)

EXPLICIT HYBRID MPC (MLD FORMULATION)

$$\min_\xi J(\xi, x(t)) = \sum_{k=0}^{N-1} \|Qy_k\|_\infty + \|Ru_k\|_\infty$$

$$\text{subject to} \quad \left\{ \begin{array}{rcl} x_{k+1} & = & Ax_k + B_1 u_k + B_2 \delta_k + B_3 z_k + B_5 \\ y_k & = & Cx_k + D_1 u_k + D_2 \delta_k + D_3 z_k + D_5 \\ E_2 \delta_k + E_3 z_k & \leq & E_4 x_k + E_1 u_k + E_5 \\ x_0 & = & x(t) \end{array} \right.$$

- **On-line optimization**: solve the problem for a **given state** $x(t)$ as the **MILP**

$$\begin{array}{ll} \min_\xi & \sum_{k=0}^{N-1} \epsilon_k^y + \epsilon_k^u \\ \text{s.t.} & G\xi \leq W + S x(t) \end{array}$$

- **Off-line optimization**: solve the MILP in advance **for all states** $x(t)$

$\Rightarrow$ **multiparametric Mixed-Integer Linear Program (mp-MILP)**

MULTIPARAMETRIC MILP

- Consider the mp-MILP

$$\begin{array}{rl} \min_{\xi_c,\xi_d} & f_c'\xi_c + f_d'\xi_d \\ \text{s.t.} & G_c\xi_c + G_d\xi_d \leq W + Sx \end{array} \qquad \begin{array}{l} \xi_c \in \mathbb{R}^{n_c} \\ \xi_d \in \{0,1\}^{n_d} \\ x \in \mathbb{R}^m \end{array}$$

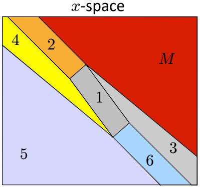
- A mp-MILP can be solved by alternating MILPs and mp-LPs

  (Dua, Pistikopoulos, 1999)

- The multiparametric solution $\xi^*(x)$ is **PWA** (but possibly discontinuous)

- The MPC controller is piecewise affine in $x = x(t)$

$$u(x) = \begin{cases} F_1x + g_1 & \text{if} \quad H_1x \leq K_1 \\ \vdots & \vdots \\ F_Mx + g_M & \text{if} \quad H_Mx \leq K_M \end{cases}$$



(More generally, the parameter vector x includes states and reference signals)

EXPLICIT HYBRID MPC (PWA FORMULATION)

- Consider the MPC formulation using a PWA prediction model

$$
\begin{array}{rl}
\min_\xi J(\xi, x(t)) & = \displaystyle\sum_{k=0}^{N-1} \|Qy_k\|_\infty + \|Ru_k\|_\infty \\
\text{subject to} & \left\{\begin{array}{rcl} x_{k+1} & = & A_{i(k)}x_k + B_{i(k)}u_k + f_{i(k)} \\ y_k & = & C_{i(k)}x_k + D_{i(k)}u_k + g_{i(k)} \\ & & i(k) \text{ such that } H_{i(k)}x_k + W_{i(k)}u_k \leq K_{i(k)} \\ x_0 & = & x(t) \end{array}\right.
\end{array}
$$

- **Method #1**: The explicit solution can be obtained by using a combination of **dynamic programming (DP)** and **mpLP** (Borrelli, Baotic, Bemporad, Morari, 2005)

- Clearly the explicit hybrid MPC law is again piecewise affine, as PWA systems$\equiv$ MLD systems

- **Method #2:** (Bemporad, Hybrid Toolbox, 2003)

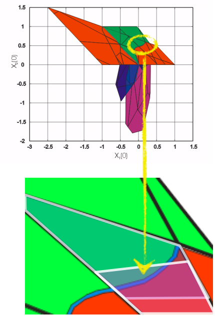
  (Alessio, Bemporad, 2006) (Mayne, ECC 2001) (Mayne, Rakovic, 2002)

  1 Use backwards (=DP) **reachability analysis** for enumerating all feasible mode sequences $I = \{i(0), i(1), \ldots, i(N)\}$

  2 For each fixed sequence I, solve the explicit finite-time optimal control problem for the corresponding linear time-varying system (**mpQP** or **mpLP**)

  3a **Case of 1 / ∞-norms** or **convex PWA costs**: Compare value functions and **split regions**

  3b **Case of quadratic costs**: the partition may not be fully polyhedral, better **keep overlapping polyhedra** and compare on-line quadratic cost functions when overlaps are detected



- Comparison of quadratic costs can be avoided by lifting the parameter space (Fuchs, Axehill, Morari, 2015)

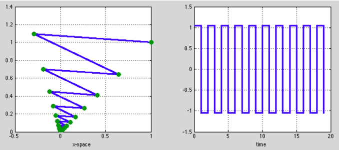
HYBRID MPC EXAMPLE - EXPLICIT VERSION

- PWA system:

$$\begin{cases} x(t+1) &= 0.8 \begin{bmatrix} \cos\alpha(t) & -\sin\alpha(t) \\ \sin\alpha(t) & \cos\alpha(t) \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \\ y(t) &= \begin{bmatrix} 0 & 1 \end{bmatrix} x(t) \\ \alpha(t) &= \begin{cases} \frac{\pi}{3} & \text{if } \begin{bmatrix} 1 & 0 \end{bmatrix} x(t) \geq 0 \\ -\frac{\pi}{3} & \text{if } \begin{bmatrix} 1 & 0 \end{bmatrix} x(t) < 0 \end{cases} \end{cases}$$

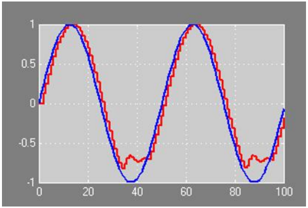
subject to $-1 \leq u(t) \leq 1$

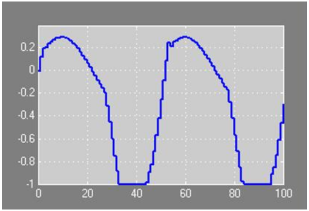
- MPC objective: $\min \sum_{k=1}^{2} |y_k - r(t)|$
- Open-loop behavior:



go to demo `demos/hybrid/bm99sim.m`

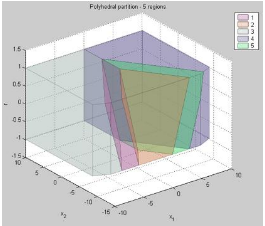
Closed-loop MPC



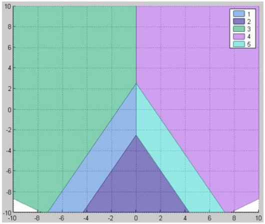


HYBRID MPC EXAMPLE - EXPLICIT VERSION

$$
u(x,r) = \begin{cases}
\begin{bmatrix} 0.6928 & -0.4 & 1 \end{bmatrix} \begin{bmatrix} x \\ r \end{bmatrix} & \text{if } \begin{bmatrix} 0.6928 & -0.4 & 1 \\ -0.4 & -0.6928 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \\ -0.6928 & 0.4 & -1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ r \end{bmatrix} \leq \begin{bmatrix} 1 \\ 10 \\ 10 \\ 1 \\ 1 \\ 1 \\ 1e{-}006 \end{bmatrix} \\ & \text{(Region \#1)} \\
1 & \text{if } \begin{bmatrix} -0.6928 & 0.4 & -1 \\ 0.6928 & 0.4 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ r \end{bmatrix} \leq \begin{bmatrix} -1 \\ -1 \\ 1 \\ 1 \\ 10 \end{bmatrix} \\ & \text{(Region \#2)} \\
-1 & \text{if } \begin{bmatrix} -0.4 & -0.6928 & 0 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0.6928 & -0.4 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ r \end{bmatrix} \leq \begin{bmatrix} 10 \\ 10 \\ 1e{-}006 \\ 10 \\ -1 \\ 1 \\ 1 \\ 10 \end{bmatrix} \\ & \text{(Region \#3)} \\
-1 & \text{if } \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0.4 & -0.6928 & 0 \\ 0 & -1 & 0 \\ -0.6928 & -0.4 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ r \end{bmatrix} \leq \begin{bmatrix} 0 \\ 10 \\ 10 \\ 10 \\ -1 \\ 1 \\ 1 \\ 10 \end{bmatrix} \\ & \text{(Region \#4)} \\
\begin{bmatrix} -0.6928 & -0.4 & 1 \end{bmatrix} \begin{bmatrix} x \\ r \end{bmatrix} & \text{if } \begin{bmatrix} -0.6928 & -0.4 & 1 \\ 0.4 & -0.6928 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \\ 0.6928 & 0.4 & -1 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ r \end{bmatrix} \leq \begin{bmatrix} 1 \\ 10 \\ 10 \\ 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} \\ & \text{(Region \#5)}
\end{cases}
$$



Section with $r = 0$



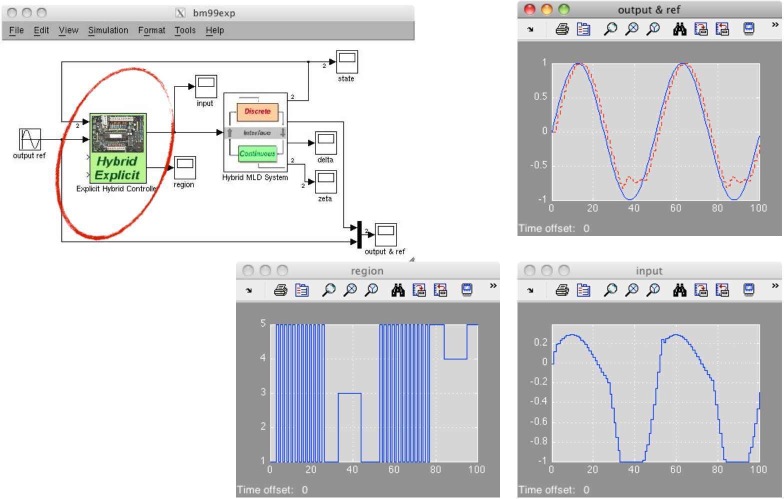
goto to `/demos/hybrid/bm99sim.m`

Offline CPU time = 1.51 s (Macbook Pro 3GHz Intel Core i7)

PWA law $\equiv$ MPC law !

HYBRID MPC EXAMPLE - EXPLICIT VERSION

- Closed-loop explicit MPC



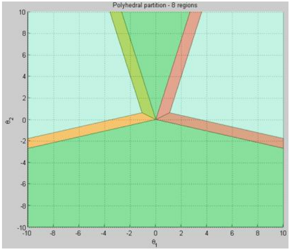
EXPLICIT PWA REGULATOR

- MPC problem:

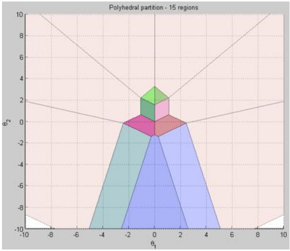
$$
\begin{array}{ll}
\min & 10\|x_N\|_\infty + \displaystyle\sum_{k=0}^{N-1} 10\|x_k\|_\infty + \|u_k\|_\infty \\
\text{s.t.} & \left\{ \begin{array}{rcccl} -1 & \leq & u_k & \leq & 1,\ k=0,\ldots,N-1 \\ -10 & \leq & x_k & \leq & 10,\ k=1,\ldots,N \end{array} \right.
\end{array}
$$

$$
\begin{array}{rcl}
Q & = & \left[\begin{smallmatrix} 10 & 0 \\ 0 & 10 \end{smallmatrix}\right] \\
R & = & 1
\end{array}
$$

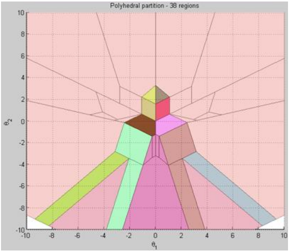
prediction horizon $N = 1$



prediction horizon $N = 2$



prediction horizon $N = 3$

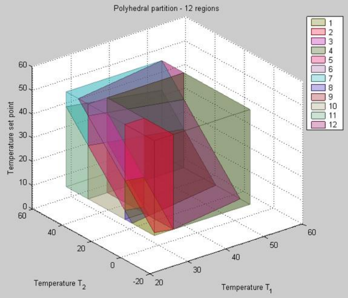


go to `demos/hybrid/bm99benchmark.m`

EXPLICIT HYBRID MPC — TEMPERATURE CONTROL

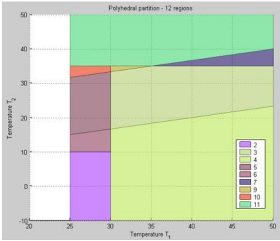
```
>> E=expcon(C,range,options);
```

```
 >> E

Explicit controller (based on hybrid controller C)
3 parameter(s)
1 input(s)
12 partition(s)
sampling time = 0.5

The controller is for hybrid systems (tracking)
This is a state-feedback controller.

Type "struct(E)" for more details.
>>
```



384 numbers to store in memory

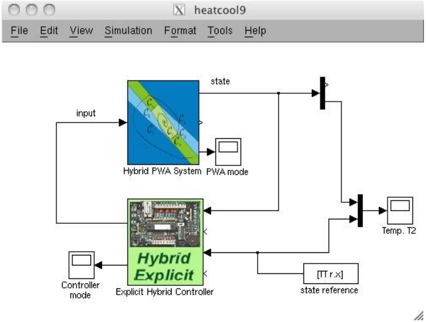
(T_1, T_2) section for $T_{\rm ref} = 30$

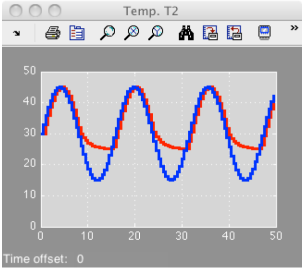


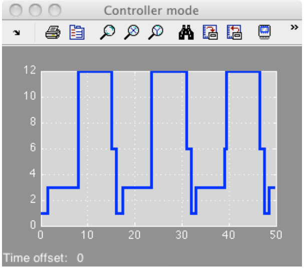
$$\min \sum_{k=0}^{2} \|x_{2k} - r(t)\|_\infty$$

$$\text{s.t.} \begin{cases} x_{1k} \geq 25,\ k = 1,2 \\ \text{hybrid model} \end{cases}$$

EXPLICIT HYBRID MPC – TEMPERATURE CONTROL







generated
C-code

`utils/expcon.h`

```c
#define EXPCON_REG 12
#define EXPCON_NTH 3
#define EXPCON_NYM 2
#define EXPCON_NH 72
#define EXPCON_NF 12
static double EXPCON_F[]={
    -1,0,0,0,-1,0,
    -1,-1,-1,-1,-1,-1,0,-3,-3,
    -3,0,-3,0,0,0,0,0,
    0,0,4,4,4,0,4,0,0,
    0,0,0,0};

static double EXPCON_G[]={
    101.6,1.6,1.6,-1.6,98.4001,0,100,51.6,
    101.6,51.6,48.4,50};

static double EXPCON_H[]={
    0,0,0,-0.00999999,0,-0.0333333,
    0.02,0.00999999,-0.02,0,0,-0.0333333,0.02,0.00999999,
    0,0,-0.02,0.02,0,-1,0.00999999,0,
```

IMPLEMENTATION ASPECTS OF HYBRID MPC

- **Alternatives**:
  1. **solve MIP** on-line
  2. **evaluate a PWA function** (explicit solution)

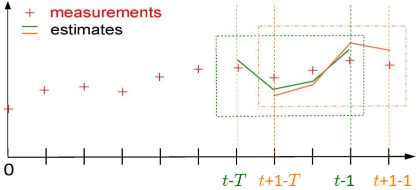
- **Small problems** (short horizon $N = 1, 2$, one or two inputs, 4-6 binary vars): explicit PWA control law is preferable
  - **CPU time** to evaluate the control law is shorter than by MIP
  - **control code** is simpler (no complex solver must be included in the control software!)
  - **more insight** in controller behavior

- **Medium/large problems** (longer horizon, many inputs and binary variables): on-line MIP is preferable

MOVING HORIZON ESTIMATION AND FAULT DETECTION

(Bemporad, Mignone, Morari, 1999) (Ferrari-Trecate, Mignone, Morari, 2002)

- **Goal**: estimate the state of a hybrid system from past I/O measurements
- **Moving horizon estimation** based on MLD models solves the problem



MLD model augmented by

- state disturbance $\xi \in \mathbb{R}^n$
- output disturbance $\zeta \in \mathbb{R}^p$

- At each time t get the estimate $\hat{x}(t)$ by solving the **MIQP**

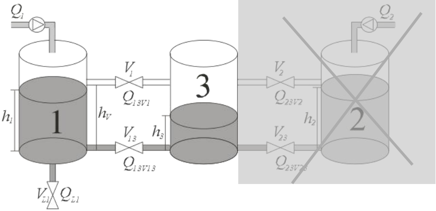
$$\min_{\hat{x}(t-T|t)} \sum_{k=0}^{T} \|\hat{y}(t-k|t) - y(t-k)\|_2^2 + \ldots$$

$$\text{s.t.} \quad \text{constraints on } \hat{x}(t-T+k|t), \hat{y}(t-T+k|t)$$

- For **fault detection** also include unknown binary disturbances $\phi \in \{0,1\}^{n_f}$

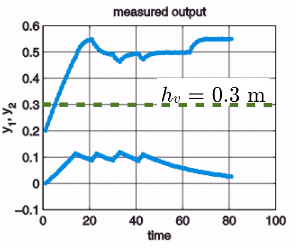
MHE EXAMPLE - THREE TANK SYSTEM

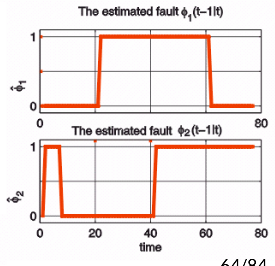
- Can only measure tank levels h_1, h_2
- The system has two faults:
  - ϕ_1: leak in tank 1 between 20 s $\leq t \leq$ 60 s
  - ϕ_2: valve V_1 blocked for $t \geq$ 40 s



(COSY benchmark problem)

- Add logic constraint
  $[h_1 \leq h_v] \to \phi_2 = 0$





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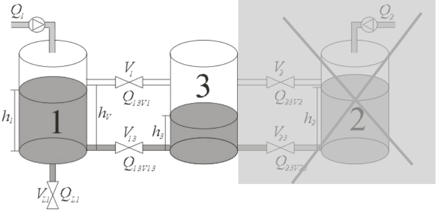

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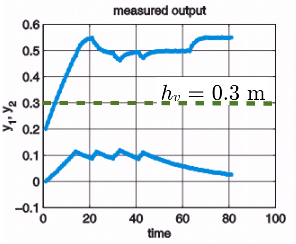
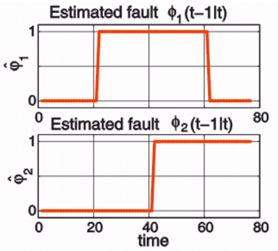




A FEW (HYBRID) MPC TRICKS

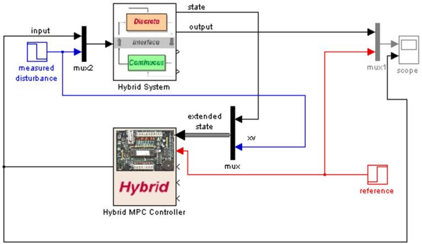
MEASURED DISTURBANCES

- A measured disturbance $v(t)$ enters the hybrid system
- Augment the hybrid prediction model with the constant state

$$\begin{array}{rcl} x^v_{k+1} & = & x^v_k \\ x^v_0 & = & v(t) \end{array}$$

- HYSDEL model

```
INTERFACE{
      STATE{
         REAL x      [-1e3, 1e3];
         REAL xv     [-1e3, 1e3];
      }
      ...
   }
IMPLEMENTATION{
      CONTINUOUS{
         x = A*x + B*u + Bv*xv
         xv= xv;
         ...
      }
}
```



- Same trick applies to linear MPC

go to demo `demos/hybrid/hyb_meas_dist.m`

REFERENCE TRACKING

- Hybrid MPC formulation for **reference tracking**

$$
\begin{array}{ll}
\min & \sum_{k=0}^{N-1} \|W^y(y_{k+1} - r(t))\|_2^2 + \|W^{\Delta u}\Delta u_k\|_2^2 \\
\text{s.t.} & \text{hybrid dynamics} \\
& \Delta u_k = u_k - u_{k-1},\ k = 0, \ldots, N-1,\ u_{-1} = u(t-1) \\
& u_{\min} \leq u_k \leq u_{\max},\ k = 0, \ldots, N-1 \\
& y_{\min} \leq y_k \leq y_{\max},\ k = 1, \ldots, N \\
& \Delta u_{\min} \leq \Delta u_k \leq \Delta u_{\max},\ k = 0, \ldots, N-1
\end{array}
$$

- The resulting optimization problem is the **MIQP**

$$
\begin{array}{ll}
\min_{\xi} & J(\xi, x(t)) = \frac{1}{2}\xi' H \xi + [x'(t)\, r'(t)\, u'(t-1)] F \xi \\
\text{s.t.} & G\xi \leq W + S \begin{bmatrix} x(t) \\ r(t) \\ u(t-1) \end{bmatrix}
\end{array}
\qquad
\xi = \begin{bmatrix} \Delta u_0 \\ \delta_0 \\ z_0 \\ \vdots \\ \Delta u_{N-1} \\ \delta_{N-1} \\ z_{N-1} \end{bmatrix}
$$

- Same trick as in linear MPC

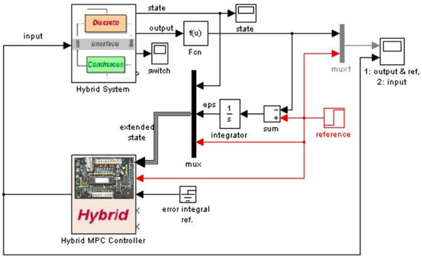
INTEGRAL ACTION

- Augment hybrid prediction model with **integrals of output tracking errors**

$$\epsilon_{k+1} = \epsilon_k + T_s(r(t) - y_k)$$

- Treat set point $r(t)$ as a measured disturbance (= constant state)
- Add weight on ϵ_k in cost function
- HYSDEL model:

```
INTERFACE{
      STATE{
         REAL x          [-100,100];
         ...
         REAL epsilon    [-1e3, 1e3];
         REAL r          [0,     100]; }
      OUTPUT {
         REAL y; }
      ... }
IMPLEMENTATION{
      CONTINUOUS{
         epsilon=epsilon+Ts*(r-(c*x));
         r=r;
         ...   }
      OUTPUT{
         y=c*x;  }    }
```



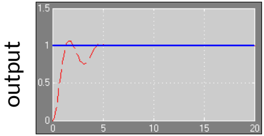
- Same trick applies to linear MPC

go to demo `demos/hybrid/hyb_integral_action.m`

TIME-VARYING CONSTRAINTS

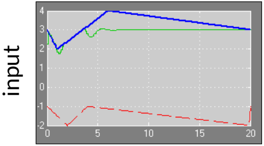
- Consider the **time-varying constraint**

$$u(t) \leq u_{\max}(t)$$



- Augment the hybrid prediction model with the constant state

$$\begin{aligned} x^u_{k+1} &= x^u_k \\ x^u_0 &= u_{\max}(t) \end{aligned}$$



  and output $y^u_k = x^u(k) - u_k$, subject to the constraint $y^u_k \geq 0, k = 0, 1, \ldots, N$

- Same trick applies to linear MPC

  go to demo `demos/linear/varbounds.m`

- **Alternative**: in HYSDEL simply impose `MUST {u <= xu;}`

REFERENCE/DISTURBANCE PREVIEW

(Bemporad, 2006)

- Measured disturbance $v(t)$ is known M steps in advance
- Augment the model with the following **buffer dynamics**

$$\left\{\begin{array}{ccc} x_{k+1}^{M-1} & = & x_k^{M-2} \\ x_{k+1}^{M-2} & = & x_k^{M-3} \\ & \vdots & \\ x_{k+1}^{1} & = & x_k^{0} \\ x_{k+1}^{0} & = & x_k^{0} \end{array}\right. \quad \text{with initial condition} \quad \left\{\begin{array}{ccc} x_0^{M-1} & = & v(t) \\ x_0^{M-2} & = & v(t+1) \\ \vdots & = & \vdots \\ x_0^{1} & = & v(t+M-2) \\ x_0^{0} & = & v(t+M-1) \end{array}\right.$$

- The predicted state x^{M-1} of the buffer is

$$x_k^{M-1} = \left\{\begin{array}{ll} v(t+k) & k = 0, \ldots, M-1 \\ v(t+M-1) & k = M, \ldots, N-1 \end{array}\right.$$

- Preview of reference signal $r(t+k)$ can be dealt with in a similar way
- Same trick applies to linear MPC

DELAYS - METHOD #1

- Hybrid model with **delays**

$$
\begin{array}{rcl}
x(t+1) & = & Ax(t) + B_1 u(t-\tau) + B_2\delta(t) + B_3 z(t) + B_5 \\
E_2\delta(t) & + & E_3 z(t) \leq E_1 u(t-\tau) + E_4 x(t) + E_5
\end{array}
$$

- Map delays to poles in $z = 0$:

$$
x_k(t) \triangleq u(t-k) \quad \Rightarrow \quad x_k(t+1) = x_{k-1}(t),\ k = 1, \ldots, \tau
$$

$$
\begin{bmatrix} x(t+1) \\ x_\tau(t+1) \\ x_{\tau-1}(t+1) \\ \vdots \\ x_1(t+1) \end{bmatrix} = \begin{bmatrix} A & B_1 & 0 & 0 & \ldots & 0 \\ 0 & 0 & I_m & 0 & \ldots & 0 \\ 0 & 0 & 0 & I_m & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ x_\tau(t) \\ x_{\tau-1}(t) \\ \vdots \\ x_1(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ I_m \end{bmatrix} u(t) + \begin{bmatrix} B_2 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \delta(t) + \begin{bmatrix} B_3 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} z(t) + \begin{bmatrix} B_5 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}
$$

- Apply MPC to the extended MLD system
- Same trick as in linear MPC

DELAYS - METHOD #2

- **Delay-free** model:

$$\bar{x}(t) \triangleq x(t+\tau) \quad \Longrightarrow \quad \begin{cases} \bar{x}(t+1) = A\bar{x}(t) + B_1 u(t) + B_2\bar{\delta}(t) + B_3\bar{z}(t) + B_5 \\ E_2\bar{\delta}(t) + E_3\bar{z}(t) \leq E_1 u(t) + E_4\bar{x}(t) + E_5 \end{cases}$$

- Design MPC for delay-free model, $u(t) = f_{\rm MPC}(\bar{x}(t))$

- Compute the predicted state

$$\bar{x}(t) = \hat{x}(t+\tau) = A^\tau x(t) + \sum_{j=1}^{\tau-1} A^j (B_1 \underbrace{u(t-1-j)}_{\text{past inputs!}} + B_2\bar{\delta}(t+j) + B_3\bar{z}(t+j) + B_5)$$

  where $\bar{\delta}(t+j)$, $\bar{z}(t+j)$ are obtained from MLD inequalities or by simulation

- Compute the MPC control move $u(t) = f_{\rm MPC}(\hat{x}(t+\tau))$

CHOICE CONSTRAINTS

- **Logic constraint**: make one or more **choices** out of a set of alternatives:
  - make **at most one** choice: $\delta_1 + \delta_2 + \delta_3 \leq 1$
  - make **at least two** choices: $\delta_1 + \delta_2 + \delta_3 \geq 2$
  - **exclusive or** constraint: $\delta_1 + \delta_2 + \delta_3 = 1$
- More generally:

$\sum_{i=1}^{N} \delta_i \leq m$	choose **at most** m items out of N
$\sum_{i=1}^{N} \delta_i = m$	choose **exactly** m items out of N
$\sum_{i=1}^{N} \delta_i \geq m$	choose **at least** m items out of N

"NO-GOOD" CONSTRAINTS

- Given a binary vector $\bar{\delta} \in \{0,1\}^n$ we want to impose the constraint

$$\boxed{\delta \neq \bar{\delta}}$$

- This may be useful for example to extract different solutions from an MIP that has multiple optima

- The **"no-good"** condition can be expressed equivalently as

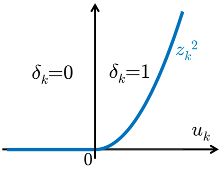
$$\sum_{i \in T} \delta_i - \sum_{i \in F} \delta_i \leq -1 + \sum_{i=1}^{n} \bar{\delta}_i \qquad \begin{array}{l} F = \{i : \bar{\delta}_i = 0\} \\ T = \{i : \bar{\delta}_i = 1\} \end{array}$$

or

$$\boxed{\sum_{i=1}^{n} (2\bar{\delta}_i - 1)\delta_i \leq \sum_{i=1}^{n} \bar{\delta}_i - 1}$$

ASYMMETRIC WEIGHTS

- **Asymmetric weight**: only weight a variable u_k if $u_k \geq 0$



- We can introduce a binary variable $[\delta_k = 1] \leftrightarrow [u_k \geq 0]$ and

$$z_k = \max\{u_k, 0\} = \begin{cases} u_k & \text{if } \delta_k = 1 \\ 0 & \text{otherwise} \end{cases}$$

  then weight z_k instead of u_k

- **Better solution**: only introduce auxiliary variable z_k and optimize

$$\begin{array}{ll} \min & (\ldots) + \sum_{k=0}^{N-1} z_k^2 \\ \text{s.t.} & z_k \geq u_k \\ & z_k \geq 0 \end{array}$$

- Similar approach when $\|\cdot\|_\infty$ or $\|\cdot\|_1$ are used as penalties

- Same trick applies to linear MPC

GENERAL REMARKS ABOUT MIP MODELING

- The complexity of solving a mixed-integer program largely depends on the number of integer (binary) variables involved in the problem

- Hence, when creating a hybrid model one has to

Be thrifty with binary variables !

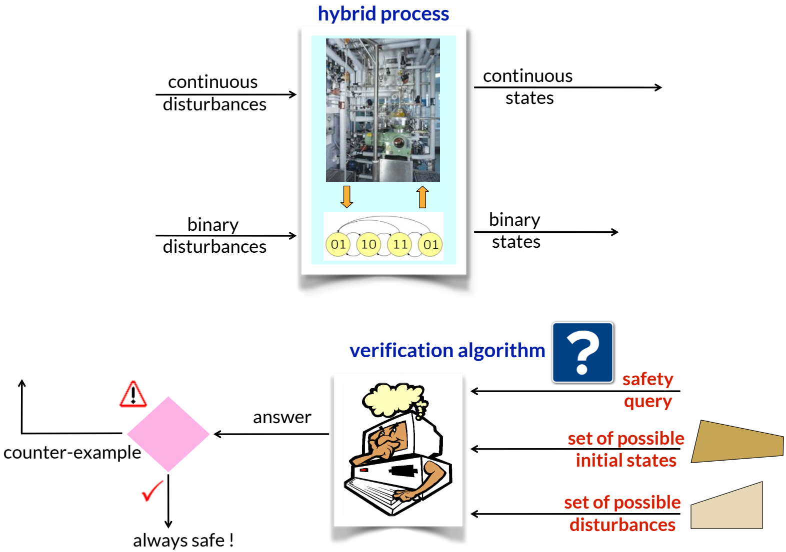
- Adding logical constraints usually helps

- Generally speaking

modeling is an art

VERIFICATION (REACHABILITY ANALYSIS)

HYBRID VERIFICATION PROBLEM



VERIFICATION ALGORITHM #1

- **Query**: Is the target set X_f reachable after N steps from some initial state $x_0 \in X_0$ for some input $u_0, \ldots, u_{N-1} \in U$?

- The query can be answered by solving the **mixed-integer feasibility test**

$$
\begin{array}{rl}
\min_{\xi} & 0 \\
\text{s.t.} & x_{k+1} = Ax_k + B_1 u_k + B_2 \delta_k + B_3 z_k + B_5 \\
& E_2 \delta_k + E_3 z_k \leq E_4 x_k + E_1 u_k + E_5 \\
& S_u u_k \leq T_u \quad (u_k \in U), \quad k = 0, 1, \ldots, N-1 \\
& S_0 x_0 \leq T_0 \quad (x_0 \in X_0) \\
& S_f x_N \leq T_f \quad (x_N \in X_f)
\end{array}
$$

with respect to $\xi = [x_0, \ldots, x_N, u_0, \ldots, u_{N-1}, \delta_0, \ldots, \delta_{N-1}, z_0, \ldots, z_{N-1}]$

- Other approaches:
  - Exploit structure and use polyhedral computation (Torrisi, 2003)
  - Use abstractions (LPs) + SAT solvers (Giorgetti, Pappas, Bemporad)

VERIFICATION EXAMPLE

- MLD model: room temperature control system
- Set of unsafe states:

$$X_f = \left\{ \left[\begin{smallmatrix} T_1 \\ T_2 \end{smallmatrix}\right] : 10 \leq T_1, T_2 \leq 15 \right\}$$

- Set of initial states:

$$X_0 = \left\{ \left[\begin{smallmatrix} T_1 \\ T_2 \end{smallmatrix}\right] : 35 \leq T_1, T_2 \leq 40 \right\}$$

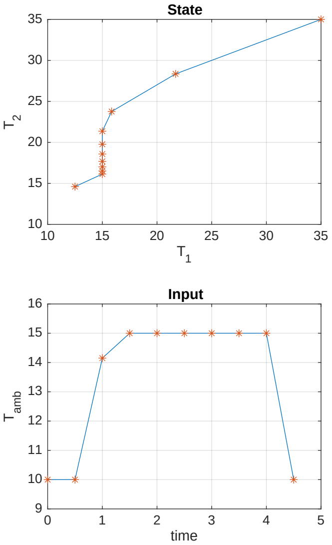
- Set of possible inputs:

$$U = \{T_{\rm amb} : 10 \leq T_{\rm amb} \leq 30\}$$

- Time horizon: $N = 10$ steps

```
>> [flag,x0,U]=reach(S,N,Xf,X0,umin,umax);
```

VERIFICATION EXAMPLE



$$U = \{T_{\rm amb}:\ 10 \le T_{\rm amb} \le 30\}$$

```
>> umin=20;
>> reach(S,N,Xf,X0,umin,umax);
Hybrid Toolbox v.1.4.2 [February 2, 2020]
Elapsed time is 0.023282 seconds.

Xf is not reachable from X0

>>
```

$$U = \{T_{\rm amb}:\ 20 \le T_{\rm amb} \le 30\}$$

VERIFICATION ALGORITHM #2

- **Query**: Is the target set X_f reachable **within** N steps from some initial state $x_0 \in X_0$ for some input $u_0, \ldots, u_{N-1} \in U$?

- Augment the MLD system to register the entrance of the target (unsafe) set $X_f = \{x : \; A_f x \leq b_f\}$:

  - Add a new variable δ_k^f, with $[\delta_k^f = 1] \to [A_f x_{k+1} \leq b_f]$

$$\underbrace{\Longrightarrow}_{\text{big-M}} \; A_f(Ax_k + B_1 u_k + B_2 \delta_k + B_3 z_k + B_5) \leq b_f + M(1 - \delta_k^f)$$

  - Add the constraint $\displaystyle\sum_{k=0}^{N-1} \delta_k^f \geq 1$ (i.e., $x_k \in X_f$ for at least one k)

  - Solve MILP feasibility test

A MORE COMPLEX VERIFICATION EXAMPLE

- States $x_1, x_2, x_3 \in \mathbb{R}$, $x_4, x_5 \in \{0,1\}$, inputs $u_1, u_2 \in \mathbb{R}$, $u_3 \in \{0,1\}$

- Events:
$$\begin{aligned}
&[\delta_1 = 1] \leftrightarrow [x_1 \le 0] \\
&[\delta_2 = 1] \leftrightarrow [x_2 \ge 1] \\
&[\delta_3 = 1] \leftrightarrow [x_3 - x_2 \le 1]
\end{aligned}$$

- Switched dynamics

$$\begin{aligned}
x_1(k+1) &= \begin{cases} 0.1x_1(k) + 0.5x_2(k) & \text{if } (\delta_1(k) \wedge \delta_2(k)) \vee x_4(k) \text{ true} \\ -0.3x_3(k) - x_1(k) + u_1(k) & \text{otherwise} \end{cases} \\
x_2(k+1) &= \begin{cases} -0.8x_1(k) + 0.7x_3(k) - u_1(k) - u_2(k) & \text{if } \delta_3(k) \vee x_5(k) \text{ true} \\ -0.7x_1(k) - 2x_2(k) & \text{otherwise} \end{cases} \\
x_3(k+1) &= \begin{cases} -0.1x_3(k) + u_2(k) & \text{if } (\delta_3(k) \wedge x_5(k)) \vee (\delta_1(k) \wedge x_4(k)) \text{ true} \\ x_3(k) - 0.5x_1(k) - 2u_1(k) & \text{otherwise} \end{cases}
\end{aligned}$$

- Automaton

$$\begin{aligned}
x_4(k+1) &= \delta_1(k) \wedge x_4(k) \\
x_5(k+1) &= ((x_4(k) \vee x_5(k)) \wedge (\delta_1(k) \vee \delta_2(k)) \vee (\delta_3(k) \wedge u_3(k))
\end{aligned}$$

A MORE COMPLEX VERIFICATION EXAMPLE

- **Query**: Verify if it possible that, starting from the set X_0

$$X_0 = \{x : -0.1 \leq x_1, x_3 \leq 0.1,\ x_2 = 0.1,\ x_4, x_5 \in \{0,1\}\}$$

the state $x(k) \in X_f$

$$X_f = \{x : -1 \leq x_1, x_3 \leq 1,\ 0.5 \leq x_2 \leq 1,\ x_4, x_5 \in \{0,1\}\}$$

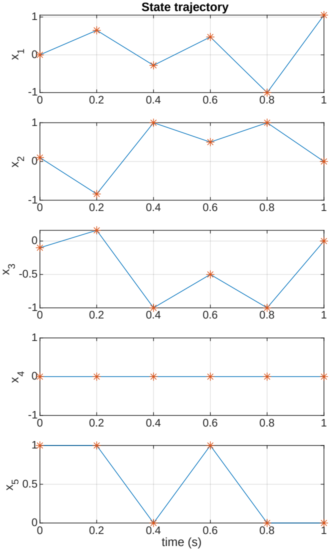
at some $k \leq N$, $N = 5$, under the restriction that $\forall k \leq N$

$$\begin{array}{l} x_3(k) + x_2(k) \leq 0 \\ \delta_1(k) \vee \delta_2(k) \vee x_5(k) = \texttt{true} \\ \neg x_4(k) \vee x_5(k) = \texttt{true} \end{array}$$

```
>> [flag,x0,U,xf,X,T,D,Z,Y,reachtime]=reach(S,[1 N],Xf,X0);
```

go to demo `demos/hybrid/reachtest.m`

A MORE COMPLEX VERIFICATION EXAMPLE




```
>> reachtest
Hybrid Toolbox v.1.4.2 [February 2, 2020]
Elapsed time is 0.038049 seconds.
>> reachtime

reachtime =


3
4


>>
```

The set X_f is reached by $x(k)$ at time steps $k = 3, 4$