15.4. **Some results in $\mathbb{R}^2$.** In $\mathbb{R}^2$, there is a fundamental result about simple closed curves (i.e., closed curves $\gamma : [0,1] \to \mathbb{R}^2$ such that $\gamma(t) = \gamma(T)$ implies $T = 1$ and $t = 0$, or vice versa), known as the Jordan curve theorem:

Theorem 15.4 (Jordan curve theorem)**.** *Let $\gamma : [0,1] \to \mathbb{R}^2$ be a simple cloesd curve. Then $\mathbb{R}^2 \backslash \{\gamma(t) : t \in [0,1]\}$ is a disjoint union of two open sets.*

In fact, simple closed curves are also known as Jordan curves. This might seem like a trivial result, but its proof is not easy. A hint of its depth is that we can characterize the topological dimension of a space by removing objects. One of the fundamental properties distinguishing $\mathbb{R}$ from $\mathbb{R}^d$, $d > 2$, is that by removing a point from $\mathbb{R}$, it becomes disconnected — it is the disjoint union of two open sets.

We call these two disjoint parts the “interior” and the “exterior”.

It turns out that a limit cycle has to be approached maximally tangentially. We can think of a approach to a limit cycle as simliar to an approach to a point that sits at the centre of a focus, and we have the following theorem that is analogous to the second part of Thm.13.1 [(in an analytic planar system, if one trajectory spirals, then all do)]:

Theorem 15.5. *If one trajectory in the exterior of a limit cycle Γ of a planar C^1-system has Γ as its ω-limit set (resp. α-limit set), then every trajectory in some exterior neighbourhood of Γ does so also. Moreover any such trajectory spirals towards Γ as $t \to \infty$ (resp. $t \to -\infty$) in the sense that it intersects any line segment normal to Γ at a point on a sequence of times $\{t_n\}$ diverging to infinity (resp. negative infinity).*

In polar terms, from any $\mathbf{y}$ in the interior of Γ, the trajectory $\phi_t(\mathbf{x}_0)$ satisfies

$$d(\phi_t(\mathbf{x}_0), \Gamma) \searrow 0, \qquad \arg(\phi_t(\mathbf{x}_0) - \mathbf{y}) \to \infty,$$

as $t \to \infty$.

Example 15.4. Recall from Dulac’s theorem (Thm. 11.3) that analytic planar systems can have at most finitely many limit cycles. We shall look at a non-analytic planar system that exhibits a centre-focus:

$$\dot{x} = -y + x(x^2 + y^2)\sin\left(\frac{1}{\sqrt{x^2 + y^2}}\right)$$
$$\dot{y} = x + y(x^2 + y^2)\sin\left(\frac{1}{\sqrt{x^2 + y^2}}\right).$$

In polar coordinates this becomes:

$$\dot{r} = r^3 \sin(1/r), \qquad \dot{\vartheta} = 1.$$

As we can see, at $r = 1/(n\pi)$, for every $n \in \mathbb{N}$, there is a limit cycle, and between each limit cycle are alternating stable and unstable spirals that turn in the same sense.

Hamiltonian systems often have cyclic behaviour as energy passes from one mode to another and then back again, being conserved throughout the cycle. We shall consider them in greater detail in the next lecture. Recalling Example 11.1, we call the separatrices of the saddles HETEROCLINIC as they “bend towards” different limit sets. Elliptic domains would have what are known as HOMOCLINIC orbits for the same reason. It is slightly a matter of perspective whether an orbit is homo- or hetero- clinic, according as when we should like to consider different points in the same limit set.

16. Lecture XVI: Poincaré Map and Stability

We shall have a slight respite from the confinement to the plane in this lecture and consider again periodic orbits of autonomous systems in $\mathbb{R}^d$.

16.1. **The Poincaré Map.** A basic construction in the study of periodic orbits is the Poincaré map. Suppose Γ is a periodic orbit of a C^1-first order autonomous system in a neighbourhood $U \subseteq \mathbb{R}^d$. The Poincaré map is the map $\mathbf{\Pi} : U \to U$ defined thus: Let $\mathbf{x}_0$ be a point on Γ. Let Σ be the hyperplane perpendicular to Γ at $\mathbf{x}_0$. That is, if Γ is defined by $\gamma : [0,1] \to \mathbb{R}^d$ so that $\gamma(0) = \mathbf{x}_0$, then

$$\Sigma = \{\mathbf{y} \in \tilde{U} : (\mathbf{y} - \mathbf{x}_0) \cdot \gamma'(0) = 0\},$$

where $\tilde{U}$ is a small neighbourhood around $\mathbf{x}_0$ (not the entire Γ).

As Γ is a trajectory, we know that $\gamma'(0) = \mathbf{x}'(0) = f(\mathbf{x}_0)$, where f is the function defining our dynamics $\dot{\mathbf{x}} = f(\mathbf{x})$.

If $\mathbf{x}$ is a point in a small neighbourhood of $\mathbf{x}_0$, and $\mathbf{x} \in \Sigma$, then we expect $\phi_t(\mathbf{x})$ to intersect Σ again after some time $\tau(\mathbf{x})$ at $\phi_{\tau(\mathbf{x})}(\mathbf{x})$. We define $\mathbf{\Pi} : \Sigma \to \Sigma$ as the map $\mathbf{\Pi} : \mathbf{x} \to \phi_{\tau(\mathbf{x})}(\mathbf{x})$, so that in fact $\mathbf{\Pi}$ is not defined on U but on a codimension one subset Σ. We can also allow Σ to be a smooth, curved codimension one surface transverse to Γ at $\mathbf{x}_0$.

The fact that this time $\tau(\mathbf{x})$ and the point $\phi_{\tau(\mathbf{x})}(\mathbf{x})$ is well-defined in a neighbourhoood $U \cap \Sigma$ of $\mathbf{x}_0$ is a direct consequence of the implicit function theorem for the map $F : \mathbb{R}_{\geq 0} \times \Sigma \to \Sigma$ given by

$$F(t, \mathbf{x}) = (\phi_t(\mathbf{x}) - \mathbf{x}_0) \cdot f(\mathbf{x}_0).$$

The level set $F(t, \mathbf{x}) = 0$ gives us the points $\mathbf{x}$ in the neighbourhood of $\mathbf{x}_0$ on Σ and their first return times, $\tau(\mathbf{x})$.

By thinking about an iterative map, we can see that if $|\nabla \mathbf{\Pi}(\mathbf{x})| < 1$, then we have stability. In $\mathbb{R}^2$, we have the following theorem:

Theorem 16.1. *Let γ be a periodic solution of period T for the C^1- planar system $\dot{\mathbf{x}} = f(\mathbf{x})$. The derivative of the Poincaré map $\mathbf{\Pi}$ along a straight line normal to $\gamma'(0)$ is given by*

$$|\mathrm{D}\mathbf{\Pi}(\gamma(0))| = \exp\Big(\int_0^T (\nabla \cdot f)(\gamma(t)) \, \mathrm{d}t \Big).$$

Recall that $\mathbf{\Pi}$ is defined on a codimension one subset, so that $\mathbf{\Pi}$ and $\mathrm{D}\mathbf{\Pi}$ are both scalar objects for planar systems.

The reason this is result true is slightly complicated, and depends on Floquet's theorem, which we shall discuss later. But since $\nabla \cdot f = \mathrm{tr}(\mathrm{D}f)$, if we can write

$$(\mathrm{D}f)(\gamma(t)) = \frac{\mathrm{d}}{\mathrm{d}t} \log(\mathbf{M}), \tag{25}$$

for some invertible-matrix-valued $\mathbf{M}$, we can use Jacobi's formula

$$\frac{\mathrm{d}}{\mathrm{d}t}(\log(\det(\mathbf{M}))) = \mathrm{tr}(\frac{\mathrm{d}}{\mathrm{d}t} \log(\mathbf{M}))$$

to get that

$$\exp\Big(\int_0^T \mathrm{tr}(\frac{\mathrm{d}}{\mathrm{d}t} \log(\mathbf{M})) \, \mathrm{d}t \Big) = \frac{\det(\mathbf{M}(T))}{\det(\mathbf{M}(0))} = \det(\mathbf{M}(T)).$$

Notice from (25), by normalizing $\mathrm{D}f(\gamma(0))$, we have

$$\mathbf{M}(t) = \exp\Big(\int_0^t \mathrm{D}f(\gamma(s)) \, \mathrm{d}s \Big). \tag{26}$$

This can be compared to (27) below.

From the above, we see that the periodic solution is a stable or unstable limit cycle according as

$$\int_0^T (\nabla \cdot f)(\gamma(t))\,\mathrm{d}t$$

is negative or positive. If this quantity is zero, then γ belongs to a continuous band of cycles all with the same period.

We shall see that the derivative of the Poincaré map, $\mathrm{D}\mathbf{\Pi}(\mathbf{x}_0)$, for $\mathbf{x}_0$ such that $\Gamma_{\mathbf{x}_0}$ is periodic, serves very much "near" a periodic orbit, as the linearization $\mathrm{D}f$ does around a critical point. This shall be the basis for a stable manifold theorem for periodic orbits.

16.2. **Stable Manifold Theorem for Periodic Orbits.** In order to achieve higer-dimensional results analogous to the one above in which the stability of a limit cycle is seen to depend on $\mathrm{D}\mathbf{\Pi}$, with a Poincaré map $\mathbf{\Pi}$ defined along some codimension one subset Σ transverse to the limit cycle at a point $\mathbf{x}_0$, we shall attempt to recreate a first-order analysis similar to our first-order analysis about critical points.

The LINEARIZATION about a periodic orbit $\Gamma = \{\gamma(t) : t \in [0, T]\}$ of the system

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x}(t) = f(\mathbf{x}(t))$$

is defined as the *nonautonomous* system

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x}(t) = \mathrm{D}f(\gamma(t))\mathbf{x}(t).$$

The fundamental matrix solutions for the linearized nonautonomous system is a matrix valued C^1 function Φ satisfying

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi = \mathrm{D}f(\gamma(t))\Phi,$$

completely analogous to the definition in (12) of Lecture 3. And for the same reason, as there are no non-homogeneous terms, we can write

$$\mathbf{x}(t) = \Phi(t)\Phi^{-1}(0)\mathbf{x}_0.$$

Arguing especially heuristically, using integrating factors, by normalizing $\mathrm{D}f(\gamma(0))$, we have that

$$\mathbf{x}(t) = \exp\left(\int_0^t \mathrm{D}f(\gamma(s))\,\mathrm{d}s\right)\mathbf{x}_0. \tag{27}$$

We can compare this with (26) above and see that the eigenvalues of $\mathbf{M}(T)$ tells us exactly how the Poincaré map acts in a small enough neighbourhood of $\gamma(0)$ if the cycle is "hyperbolic". We shall clarify the meaning of this caveat in the main theorem of this lecture.

Recall that for autonomous linear systems, we can choose $\Phi(t) = \exp(\mathbf{A}t)$, the flow. Floquet's theorem gives us a way to write Φ in (27) in a more similar way in the nonautonomous case:

Theorem 16.2 (Floquet's Theorem)**.** *Suppose* $\mathrm{D}f(\gamma(t))$ *is a continuous, matrix-valued, periodic function with period* T*. Then for all* $t \in \mathbb{R}$*, we can write the fundamental matrix solution defined above in the form*

$$\Phi(t) = \mathbf{Q}(t)\exp(\mathbf{B}t), \tag{28}$$

where $\mathbf{Q}(t)$ *is a non-singular, matrix-valued,* T*-periodic function and* $\mathbf{B}$ *is a constant matrix.*

The proof of this theorem turns on the simple observation that if a matrix $\mathbf{C}$ is non-singular, then we can define its logarithm — that is, find a possibly complex matrix $\mathbf{B}$ such that $\mathbf{C} = \exp(\mathbf{B}T)$. The matrix $\mathbf{C}$ in this case comes directly from the representation of the solution by the fundamental matrix solution, $\mathbf{C} := \Phi^{-1}(0)\Phi(T)$. Then it holds that the remaining factor $\mathbf{Q}(t)$ must be

$$\mathbf{Q}(t) := \Phi(t)\exp(-\mathbf{B}t) = \Phi(t)\Phi^{-1}(T)\Phi(0).$$

It may be checked readily that $\mathbf{Q}$ is periodic with period T, to see that it is non-singular one has but to notice that if Φ takes the form (26), $\mathbf{Q}$ is an exponentiation.

Written out thus, Floquet's theorem seems like a mere technical result, but the consequences of this representation (28) is far-reaching. Using the change-of-variables $\mathbf{y} = \mathbf{Q}^{-1}(t)\mathbf{x}$, a direct calculation shows that the nonautonomous linearized system can be written as the autonomous system

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{y} = \mathbf{B}\mathbf{y}.$$

Now we can bring the full suite of linear methods developed thus far to bear.

We know that $\mathbf{Q}$ is non-singular and periodic (and if everything is smooth, also bounded), therefore the stability of the cycle Γ is determined by the eigenvalues of $\mathbf{B}$ — these eigenvalues are called the CHARACTERISTIC EXPONENTS of $\gamma(t)$, and are determined modulo $2\pi i$. Trajectories for which the characteristic exponents all have non-zer real parts are known as HYPERBOLIC PERIODIC ORBITS.

The eigenvalues of $\exp(\mathbf{B}T)$ are known as the CHARACTERISTIC MULTIPLIERS — these determine how far $\mathbf{x}$ moves every time it comes around to the surface Σ again, i.e., the magnitude of $\mathbf{\Pi}(\mathbf{x}) - \mathbf{x}$, or $\mathrm{D}\mathbf{\Pi}$. We shall not write this out as a theorem, but from our deductions foregoing, it may be guessed that the characteristic multipliers are the eigenvalues of $\mathrm{D}\mathbf{\Pi}$ (if one considered $\mathbf{\Pi}$ as a map $\Sigma \subseteq U \to U$ instead of $\Sigma \to \Sigma$.

Of course, $\mathbf{B}$ has to be singular and have one zero eigenvalue because $\mathbf{\Pi}(\mathbf{x})$ does not move $\mathbf{x}$ in the direction of $\gamma'(0)$, so the multiplier in that direction must be 1, whose logarithm is 0.

These observations leads us right to the main theorem of this lecture:

Theorem 16.3 (Stable Manifold Theorem for Cycles). *Let $\Gamma = \{\mathbf{x} \in \mathbb{R}^d : \mathbf{x} = \gamma(t)\}$ be a periodic orbit with period T, of a C^1 first order autonomous system with flow ϕ_t. Suppose $\gamma(t) = \phi_t(\mathbf{x}_0)$. If k characteristic exponent of $\gamma(t)$ have negative real parts, and $d-k-1$ have positive real parts, then there is a $\delta > 0$ such that the stable manifold,*

$$M_s(\Gamma) = \{\mathbf{x} \in B_\delta(\Gamma) : d(\phi_t(\mathbf{x}), \Gamma) \to 0 \ \textit{as}\ t \to \infty,\ \forall t \geq 0\,(\phi_t(\mathbf{x}) \in B_\delta(\Gamma))\},$$

is a $(k+1)$-dimensional differentiable manifold which is positively ϕ_t-invariant, and the unstable manifold,

$$M_u(\Gamma) = \{\mathbf{x} \in B_\delta(\Gamma) : d(\phi_t(\mathbf{x}), \Gamma) \to 0 \ \textit{as}\ t \to -\infty,\ \forall t \geq 0\,(\phi_t(\mathbf{x}) \in B_\delta(\Gamma))\},$$

is a $(d-k-1)$ dimensional differentiable manifold which is negatively ϕ_t-invariant. Furthermore, the stable and unstable manifolds intersect transversally.

Subanifolds $M \subseteq X$ and $N \subseteq X$ intersect TRANSVERSELY (or TRANSVERSALLY) if at every point p of their intersection , $T_pM \oplus T_pN = T_pX$. This transversality condition is analogous to the condition of tangency to invariant subspaces in the stable manifold theorem for critical points.