Problem 1 Determine whether the following statements are true or false. If the statement is true, no further explanation is required. If the statement is false, give a counterexample.

(i) The kernel of a bounded linear operator $T : X \to Y$ between normed spaces X and Y is closed.

(ii) The range of a bounded linear operator $T : X \to Y$ between normed spaces X and Y is closed.

(iii) The dual space X' of a normed space X is a Banach space.

(iv) A closed subspace of a Banach space is itself a Banach space.

Problem 2 Let $(x_k)_{k\in\mathbb{N}}$ be a sequence in a normed space $(X, \|\cdot\|)$.

a) Prove that if $(x_k)_{k\in\mathbb{N}}$ is a Cauchy sequence, then $(x_k)_{k\in\mathbb{N}}$ is bounded.

b) Let $\|\cdot\|_a$ and $\|\cdot\|_b$ be equivalent norms on X, and let $x \in X$. Prove that $(x_k)_{k\in\mathbb{N}}$ converges to x in $(X, \|\cdot\|_a)$ if and only if $(x_k)_{k\in\mathbb{N}}$ converges to x in $(X, \|\cdot\|_b)$.

Problem 3 Let $(\ell^2, \langle\cdot,\cdot\rangle)$ be the inner product space of complex-valued sequences $x = (x_k)_{k\in\mathbb{N}}$ equipped with the standard inner product

$$\langle x, y\rangle = \sum_{k=1}^{\infty} x_k \overline{y_k}, \quad x, y \in \ell^2,$$

and let $T : \ell^2 \to \ell^2$ be the multiplication operator given by

$$Tx = \left(\frac{i^k x_k}{k}\right)_{k\in\mathbb{N}},$$

where $i = \sqrt{-1}$.

a) Show that T is a bounded linear operator on ℓ^2, and determine the operator norm $\|T\|$.

b) Determine the adjoint operator T^*. State what it means for an operator to be normal, and determine whether or not T is normal.

c) Show that the range of T is dense in ℓ^2.

Problem 4 Let

$$A = \begin{bmatrix} 2 & 2 \\ 2 & 2 \\ -1 & -1 \end{bmatrix}.$$

a) Find a singular value decomposition of A.

b) Find the pseudoinverse A^+ of A, and use it to find the best approximation to a solution of the inconsistent system:

$$\begin{aligned} 2x_1 + 2x_2 &= 3 \\ 2x_1 + 2x_2 &= 4 \\ -x_1 - x_2 &= -4 \end{aligned}$$

Problem 5 Find $a, b \in \mathbb{C}$ such that

$$\int_0^{2\pi} |t - a\sin t - b\sin 2t|^2 \, dt$$

is minimal.

Tip: You might find the formula $\sin^2 t = (1 - \cos 2t)/2$ *useful.*

Problem 6

a) Show that if $X \neq \emptyset$ is a complete metric space, and $T : X \to X$ is a mapping such that

$$T^k = \underbrace{T \circ T \circ \cdots \circ T}_{k \text{ times}}$$

is a contraction for some natural number $k > 1$, then T has a unique fixed point.

b) Consider the space of continuous functions $C[0,1]$ equipped with the metric induced by the supremum norm

$$d(f,g) = \|f - g\|_\infty = \sup_{0 \le t \le 1} |f(t) - g(t)|,$$

and let $T : C[0,1] \to C[0,1]$ be given by

$$(Tf)(t) = 1 - \int_0^t f(s)\, ds, \quad 0 \le t \le 1.$$

Show that T has a unique fixed point, and use iteration to find it starting with $f_0(t) = 1$.

Tip: You can use the result from a) even if you did not solve this problem.