TMA 4190 Introduction to Topology
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Lecture 09[1]

9. A brief excursion into Lie groups - Part 1

Lie groups

A **Lie group** is a group G which is also a smooth manifold such that the two maps

$$\mu\colon G \times G \to G,\ (g,h) \mapsto g \cdot h$$

and

$$\iota\colon G \to G,\ g \mapsto g^{-1}$$

corresponding to the two group operations of multiplication and taking inverses, respectively, are both smooth. (We usually omit the dot and just write gh instead of $g \cdot h$.)
In fact, we can summarize the condition that μ and ι are smooth by requiring that

$$G \times G \to G,\ (g,h) \mapsto gh^{-1}$$

is smooth.

If G is a Lie group, then any element $g \in G$ defines maps

$$L_g \text{ and } R_g\colon G \to G,$$

called **left translation** and **right translation**, respectively, by

$$L_g(h) = gh \text{ and } R_g(h) = hg.$$

Since L_g can be expressed as the composition of smooth maps

$$G \xrightarrow{i_g} G \times G \xrightarrow{\mu} G,$$

with $i_g(h) = (g,h)$, it follows that L_g is smooth. It is actually a **diffeomorphism of G**, because $L_{g^{-1}}$ is a smooth inverse for it. Similarly, $R_g\colon G \to G$ is a diffeomorphism. In fact, many of the important properties of Lie groups follow from the fact that we can systematically map any point to any other by such a global diffeomorphism. This translation makes the study of Lie groups much more accessible compared to arbitrary smooth manifolds. In particular, we can move

[1] Following the books of Guillemin and Pollack: Differential Topology; by Lee: Introduction to Smooth Manifolds; and by Tu: An Introduction to Manifolds.

an open neighborhood around any point in G to make it an open neighborhood of the identity element. Hence, in a Lie group, we basically only need to study neighborhoods of the identity element.

Here are some simple examples of Lie groups:

- The real numbers $\mathbb{R}$ and Euclidean space $\mathbb{R}^n$ are Lie groups under addition, because the coordinates of $x-y$ are linear and therefore smooth functions of (x,y).
- Similarly, $\mathbb{C}$ and $\mathbb{C}^n$ are Lie groups under addition.
- Any finite group with the discrete topology is a (compact) Lie group.
- Suppose G is a Lie group and $H \subseteq G$ is an open subgroup (i.e. a subgroup which is also an open subspace). Then H is a Lie group as well.
- The set $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$ of nonzero real numbers is a 1-dimensional Lie group under multiplication. The subset $\mathbb{R}^+$ of positive real numbers is an open subgroup, and is thus itself a 1-dimensional Lie group (still under multiplication).
- The set $\mathbb{C}^*$ of nonzero complex numbers is a 2-dimensional Lie group under complex multiplication.
- The unit circle $S^1 \subset \mathbb{C}^*$ is a Lie group under the operations induced by multiplication of complex numbers.
- A finite product of k copies of S^1 is a Lie group. We denote it by $\mathbb{T}^k$. In particular, the 2-dimensional torus $\mathbb{T}^2 = S^1 \times S^1$ is a Lie group.
- More generally, the product of Lie groups is again a Lie group.

We will see more examples below. But before, we introduce the notion of maps between Lie groups which respect the Lie group structure.

Lie group homomorphisms

If G and H are Lie groups, a **Lie group homomorphism** from G to H is a smooth map $F: G \to H$ that is also a group homomorphism. It is called a **Lie group isomorphism** if it is also a diffeomorphism, which implies that it has an inverse that is also a Lie group homomorphism. In this case, we say that G and H are isomorphic Lie groups.

Here are some examples of Lie group homomorphisms:

- The inclusion map $S^1 \hookrightarrow \mathbb{C}$ is a Lie group homomorphism.
- Considering $\mathbb{R}$ as a Lie group under addition, and R^* as a Lie group under multiplication, the map

$$\exp\colon \mathbb{R} \to \mathbb{R}^*,\ t \mapsto e^t$$

is smooth, and is a Lie group homomorphism, since $e^{s+t} = e^s e^t$. The image of exp is the open subgroup R^+ consisting of positive real numbers. In fact, $\exp\colon \mathbb{R} \to \mathbb{R}^+$ is a Lie group isomorphism with inverse $\log\colon \mathbb{R}^+ \to \mathbb{R}$.
- Similarly, $\exp\colon \mathbb{C} \to \mathbb{C}^*$ given by $\exp(z) = e^z$ is a Lie group homomorphism. It is surjective but not injective, because its kernel consists of the complex numbers of the form $2\pi i k$, where k is an integer.
- The map
$$\epsilon\colon \mathbb{R} \to S^1,\ t \mapsto e^{2\pi i t}$$
is a Lie group homomorphism whose kernel is the set $\mathbb{Z}$ of integers.
- Similarly, the map
$$\epsilon^n\colon \mathbb{R}^n \to \mathbb{T}^n,\ (t_1, \dots, t_n) \mapsto (e^{2\pi i t_1}, \dots, e^{2\pi i t_n})$$
is a Lie group homomorphism whose kernel is $\mathbb{Z}^n$.
- If G is a Lie group and $g \in G$, **conjugation by g** is the map $C_g\colon G \to G$ given by $C_g(h) = ghg^{-1}$. Because group multiplication and inversion are smooth, C_g is smooth and it is a group homomorphism:
$$C_g(hh') = gh_1hh'g^{-1} = (ghg^{-1})(gh'g^{-1}) = C_g(h)C_g(h').$$
In fact, it is a **Lie group isomorphism**, because it has $C_{g^{-1}}$ as an inverse. A subgroup $H \subseteq G$ is said to be **normal** if $C_g(H) = H$ for every $g \in G$.

Here is an important theorem about Lie group homomorphisms:

Constant Rank Theorem

Let $f\colon G \to H$ be a Lie group homomorphism. Then the derivative df_g has the same rank (as a linear map) for all $g \in G$.

Proof: Let e_G and e_H denote the identity elements in G and H, respectively. Suppose g_0 is an arbitrary element of G. We will show that df_{g_0} has the same rank as df_e. The fact that f is a homomorphism means that for all $g \in G$,
$$f(L_{g_0}(g)) = f(g_0 g) = f(g_0)f(g) = L_{f(g_0)}(f(g));$$
or in other words, $f \circ L_{g_0} = L_{f(g_0)} \circ f$. Taking differentials of both sides at the identity and using the chain rule yields
$$df_{g_0} \circ d(L_{g_0})_{e_G} = d(L_{f(g_0)})_{e_H} \circ df_{e_G}.$$

Recall that left multiplication by any element of a Lie group is a diffeomorphism, so both $d(L_{g_0})_{e_G}$ and $d(L_{f(g_0)})_{e_H}$ are isomorphisms. Because composing with an isomorphism does not change the rank of a linear map, it follows that df_{g_0} and df_{e_G} have the same rank. **QED**

Lie group isomorphisms revisited

Every bijective Lie group homomorphism $f\colon G \to H$ is automatically a Lie group isomorphism.
For, there must be a point $g \in G$ where df_g is an isomorphism. Otherwise the Local Immersion and Submersion Theorems would imply that f looked like the canonical immersion or submersion, respectively, and f would not be bijective. By the previous theorem, this implies that df_g is an isomorphism for all $g \in G$. Hence it is a bijective local diffeomorphism everywhere. Bijective local diffeomorphisms are global diffeomorphisms. Since the map is a Lie group homomorphism, it is a Lie group isomorphism.

Now let us study some more interesting examples:

The General Linear Group

The general linear group

$$GL(n) = \{A \in M(n) : \det A \neq 0\}$$

of all invertible $n \times n$-matrices with entries in $\mathbb{R}$, is a smooth manifold of dimension n^2, since it is an **open** subset of $M(n) \cong \mathbb{R}^{n^2}$. To check that it is open, look at its complement

$$M(n) \setminus GL(n) = \{A \in M(n) : \det A = 0\} = \det^{-1}(0).$$

Since $\det\colon M(n) \to \mathbb{R}$ is continuous (it is a polynomial in the entries of the matrix) and since $\{0\}$ is a closed subset of $\mathbb{R}$, $\det^{-1}(0)$ is closed in $M(n)$.

We claim that $GL(n)$ is a Lie group. To show this we need to check that multiplication and taking inverses are smooth operations. Given two matrices A and B in $GL(n)$, the entry in position (i,j) in AB is given by

$$(AB)_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}.$$

Hence $(AB)_{ij}$ is a polynomial in the coordinates of A and B. Thus matrix multiplication

$$\mu\colon GL(n) \times GL(n) \to GL(n)$$

is a smooth map.

Recall that the (i,j)-minor of a matrix A is the determinant of the submatrix of A obtained by deleting the ith row and the jth column of A. By **Cramer's**

rule from linear algebra, the (i,j)-entry of A^{-1} is

$$(A^{-1})_{ij} = \frac{1}{\det A} \cdot (1)^{i+j}((j,i)\text{-minor of } A),$$

which is a smooth function of the a_{ij}'s provided $\det A \neq 0$, i.e. the map

$$M(n) \to \mathbb{R},\ A \mapsto (A^{-1})_{ij}$$

is smooth because it depends smoothly on the entries of A. Therefore, the map of taking inverses

$$\iota \colon GL(n) \to GL(n)$$

is also smooth.

$GL(n)$ exists over many bases

In fact, we can matrices with entries in any ring K. We denote the corresponding matrix groups by $M(n,K), GL(n,K), \ldots$. Since $K = \mathbb{R}$ is the most important case for us, we omit mentioning the base when it is clear that we work over $\mathbb{R}$.
Another very important case is $K = \mathbb{C}$. The complex general linear group $GL(n,\mathbb{C})$ is also a Lie group. It is a group under matrix multiplication, and it is an open submanifold of $M(n,\mathbb{C})$ and thus a $2n^2$-dimensional smooth manifold. It is a Lie group, since matrix products and inverses are smooth functions of the real and imaginary parts of the matrix entries.
Note that the determinant is a Lie group homomorphism for both $\mathbb{R}$ and $\mathbb{C}$:

$$\det \colon GL(n,\mathbb{R}) \to \mathbb{R}^* \text{ and } \det \colon GL(n,\mathbb{C}) \to \mathbb{C}^*.$$

For $n = 1$, we just have $GL(1,\mathbb{R}) = \mathbb{R}^*$ and $GL(1,\mathbb{C}) = \mathbb{C}^*$.

The Special Linear Group

Another example of a Lie group is the **special linear group**

$$SL(n) = \{A \in M(n) : \det A = 1\}.$$

Note that $SL(n)$ consists of all transformations of $\mathbb{R}^n$ into itself which preserve volumes and orientations. (We will discuss orientations later.)

In order to show that $SL(n)$ is a manifold, we would like to use the preimage theorem for regular values of the map

$$\det \colon M(n) \to \mathbb{R}.$$

For $SL(n) = \det^{-1}(1)$. To do this, we need to show that 1 is a regular value of det. In fact, we are going to show that 0 is the only critical value of det.

As a preparation, we are going to look at the following general situation.

Euler's identity for homogeneous polynomials

Let $P(x_1, \dots, x_k)$ be a homogeneous polynomial of degree m in k variables. First, we are going to show Euler's identity

$$\sum_i x_i \partial P/\partial x_i = mP. \tag{1}$$

Define a new function Q by

$$Q(x_1, \dots, x_k, t) := P(tx_1, \dots, tx_k) - t^m P(x_1, \dots, x_k).$$

Since P is homogeneous, we know Q is always 0. Hence its derivative with respect to t is zero as well. Hence we get

$$0 = \partial Q/\partial t = \sum_i x_i \partial P/\partial x_i(tx_1, \dots, tx_k) - mt^{m-1} P(tx_1, \dots, tx_k) \tag{2}$$

where we apply the chain rule to the first summand of Q which is the composite $t \mapsto tx \mapsto P(tx)$. Setting $t = 1$ in (2) yields (1).

Fibers of homogeneous polynomials form manifolds

Now we consider our homogeneous polynomial P as a map

$$\mathbb{R}^k \to \mathbb{R}, \ (x_1, \dots, x_k) \mapsto P(x_1, \dots, x_k).$$

We claim that **0 is the only critical value of P**.

The derivative of P at a point $(x_1, \dots, x_k)$ is

$$dP_x \colon \mathbb{R}^k \to \mathbb{R}, \ (z_1, \dots, z_k) \mapsto (\partial P/\partial x_1(x) \dots \partial P/\partial x_k(x)) \cdot \begin{pmatrix} z_1 \\ \vdots \\ z_k \end{pmatrix}$$
$$= \sum_i z_i \partial P/\partial x_i(x).$$

To show that dP_x is nonsingluar, i.e. surjective, it suffices to show that dP_x is nontrivial. But applying dP_x to x and using Euler's identity yields

$$dP_x(x) = \sum_i x_i \partial P/\partial x_i(x_1, \dots, x_k) = mP(x_1, \dots, x_k).$$

Hence **if $x = (x_1, \dots, x_k)$ is not a zero of P, then $dP_x(x)$ is nonzero**. Hence only points in the fiber over 0 might be critical points, and all nonzero real numebrs are regular values of P. This shows that $P^{-1}(a)$ is a $k-1$-dimensional submanifold of $\mathbb{R}^k$ for all $a \neq 0$.

Given two real numbers $a,b > 0$, then $(b/a)^{1/m}$ exists and we if $P(x) = a$, we have

$$P((b/a)^{1/m}x_1, \dots, (b/a)^{1/m}x_k) = b/aP(x_1, \dots, x_k) = b.$$

Multiplying each coordinate with $(b/a)^{1/m}$ corresponds to multiplicatin with the diagonal matrix with $(b/a)^{1/m}$ on the diagonal. This map is a linear isomorphism of $\mathbb{R}^k$ to itself. Hence we have the diffeomorphism

$$P^{-1}(a) \to P^{-1}(b),\ (x_1, \dots, x_k) \mapsto ((b/a)^{1/m}x_1, \dots, (b/a)^{1/m}x_k).$$

Similarly, if both $a,b < 0$ are negative, then $(b/a)^{1/m}$ exists and the same argument shows that $P^{-1}(a)$ and $P^{-1}(b)$ are diffeomorphic.

Algebraic Geometry in a nutshell

The study of the zeroes of polynomials is the central theme in Algebraic Geometry. This is a classical and fascinating part of pure mathematics. In the past 2-3 decades, strong and fascinating conncetions between Algebraic Geometry and Homotopy Theory have been developed, summarized in the field of Motivic Homotopy Theory. Just ask to learn more about it.

Back to matrices: If we think of the entries in an $n \times n$-matrix A as variables, then $\det A$ is a **homogeneous polynomial of degree** n. It is given by Leibniz' formula

$$\det(A) = \sum_{\sigma} (\operatorname{sgn}(\sigma) \prod_{i=1}^{n} a_{i\sigma(i)}) \tag{3}$$

where the sum runs over all permutations of the set $\{1, \dots, n\}$ and $\operatorname{sgn}(\sigma)$ denotes the sign of the permutation σ. Hence we can apply the previous argument to

$$P = \det \colon M(n) = \mathbb{R}^{n^2} \to \mathbb{R}$$

and get that 0 is the only critical value of det. **Thus the special linear group** $SL(n) = \det^{-1}(1)$ **is a smooth submanifold of dimension** $n^2 - 1$ **in** $M(n)$**.**