Stochastic Modelling

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Contents

1 Lecture 1

1.1 Practical Information

Two projects

- The projects count 20% and exam 80%.
- Must be done with two people.
- If you want to do statistics is it worth learning R.

Course Overview

- Markov chains for discret time and discrete outcome.
  - Set of states and discrete time points.
  - Transition between states
  - Future depends on the present, but not the past.
- Continious time Markoc chains. (continious time and discrete toutcome.
- Brownian motion and Gaussian processes (continionus time and continious outcome.)

1.2 Mathematical description

Definition 1.1. *A* ***stochastic process*** $\{x(t), t \in T\}$ *is a family of random variables, where* T *is a set of indicies, and* $X(t)$ *is a random variable for each value of* t*.*

1.3 Recall from Statistics Course

A random experiment is perfomed the outcome of the experiment is random.

- THe set of possible outcomes is the **sample space** ω
  - An **event** $A \subset \omega$ if the outcome is contained in A
  - The **complement** of an event A is $A^c = \omega \setminus A$
  - The **null event** $\emptyset$ is the empty set $\emptyset = \omega \setminus \omega$

1.3.1 Combining Event

Let A and B be events

- The **union** $A \cup B$ is the event that at least one of A and B occur.
- the **intersection** $A \cap B$ is the event that both A and B occur.

The events $A_1, A_2, \ldots$ are called disjoint (or **mutually exclusive**) if $A_i \cap A_j = \emptyset$ for $i \neq j$

1.3.2 Probability

Pr is called a probability on ω if

- $\Pr\{\omega\} = 1$
- $0 \leq P\{A\} \leq 1$ for all events A
- For $A_1, A_2, \ldots$ that are mutually exclusive

$$P\left\{\bigcup_{i=1}^{\infty} A_i\right\} = \sum_{i=1}^{\infty} P\{A_i\}$$

We call $P\{A\}$ the probability of A.

1.3.3 Law of total probability

Let $A_1, A_2, \ldots$ be a partition of ω ie

- $\omega = \bigcup_{i=1}^{\infty} A_i$
- $A_1, A_2, A_3, \ldots$ are mutually exclusive.

Then for any event B

$$P\{B\} = \sum_{i=1}^{\infty} P\{B \cap A_i\}$$

This concept is very important.

1.3.4 Independence

Event A and B are independent of

$$P\{A \cap B\} = P\{A\} P\{B\}$$

Events $A_1, \ldots, A_n$ are independent if for any subset

$$P\left\{\bigcap_{j=1}^{k} A_{i_j}\right\} = \prod_{j=1}^{k} P\{A_{i_j}\}$$

In this case $P\{\bigcap_{i=1}^{n} A_1\} = \prod_{i=1}^{n} P\{A_i\}$

1.3.5 Random Variables

Definition 1.2. *A* ***random variable*** *is a real-vaued function on the sample space. Informally: A random variable is a real valued variable that takes on its value by chance.*

Example.

- Throw two dice. $X =$ sum of the two dice
- Throw a coin. X is 1 for heads and X is 0 for tails.

1.3.6 Notation for random variables

We use

- upper case letters such at X, Y and Z to represent random variables.
- lower case letters as x, y, z to denote the real-valued realized value of a the random variable.

Expression such as $\{X \leq x\}$ denators the event that X assumes a valye less than or earl to the real number x.

1.3.7 Discrete random variables

The random variable X is **discrete** if it has a finite or countablle number of possible outcomes $x_1, x_2, \ldots$

- The **probability mass function** $p_x(x)$ is given by

  $$p_x(x) = P\{X = x\}$$

  and satisfies

  $$\sum_{i=1}^{\infty} p_x(x_i) = 1 \quad \text{and} \quad 0 \leq p_x(x_i) \leq 1$$

- The **cumulative distribution function** (CDF) a of X can be written

  $$F_x(x) = P\{X \leq x\} = \sum_{i:x_i \leq x} p_x(x_i)$$

1.3.8 CFD

The CDF of X may also be called the **distrobution function** of X

Let $F_x(x)$ be the CDF of X, then

- $F_x(x)$ is monetonaly increasing.
- F_x is a stepfunction, which is a pieace-wise constant with jumps at x_i.
- $\lim_{x\to\infty} F_x(x) = 1$
- $\lim_{x\to-\infty} F_x(x) = 0$

1.3.9 Continious random vairbales

A **continious** random variables takes value o a continious scale.

- The CDF, $F_x(x) = P(X \leq x)$ is continious.
- The **probability density function** (PDF) $f_x(x) = F_x'(x)$ can be used to calculate probablities

$$\begin{aligned} Pr\{a < X < b\} &= Pr\{a \leq X < b\} = Pr\{a < X \leq b\} \\ &= Pr\{a \leq X \leq b\} = \int_a^b f_x(x)\,dx \end{aligned}$$

1.3.10 Important properties

- CDF:
  - Monotonely increaing
  - continious
  - $\lim_{x\to\infty} F_x = 1$ and $\lim_{x\to-\infty} F_x(x) = 0$
- PDF
  - $f_x(x) \geq 0$ for $x \in \mathbb{R}$
  - $\int_{-\infty}^{\infty} f_x(x)\,dx = 1$

1.3.11 Expectation

Let $g : \mathbb{R} \to \mathbb{R}$ be a function and X be a random variable.

- If X is discrete, the expected value of $g(X)$ is

$$E[g(X)] = \sum_{x:p_x(x)>0} g(x)\,p_x(x)$$

- If X is continous, the expected value of $g(X)$ is

$$E[g(X)] = \int_{-\infty}^{\infty} g(x)\,f_x(x)\,dx$$

1.3.12 Variance

The variance of the random variable X is

$$Var\left[X\right] = E\left[\left(X - E\left[X\right]\right)^2\right] = E\left[X^2\right] - E\left[X\right]^2$$

Important properties of expectation and variance.

- Expectations is linear

$$E\left[aX + bY + c\right] = aE\left[X\right] + bE\left[Y\right] + c.$$

- Variance scales quadratically and is invaraient to the addition of constants

$$Var\left[aX + b\right] = a^2 Var\left[X\right]$$

- fir independent stochastic variables.

$$Var\left[X + Y\right] = Var\left[X\right] + Var\left[Y\right]$$

1.3.13 Joint CDF

If (X, Y) is a pair for random variables, their **joint comulative distribution function** is given by

$$F_{X,Y} = F\left(x, y\right) = Pr\left\{X \leq x \cap Y \leq y\right\}$$

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1.3.14 Joint distrubution for discrete random variables

If X and Y are discrete, the **joint probability mass function** $p_{x,y} = Pr\left\{X = x, Y = y\right\}$. can be used to compute probabilities

$$Pr\left\{a < X < b, c < Y \leq d\right\} = \sum_{a<x\leq b} \sum_{c<y\leq d} p_{X,Y}\left(x, y\right)$$

1.3.15 Joint distrubution for continous random variables

If X and Y are continious the **joint probability density function**

$$.f_{X,Y}\left(x, y\right) = f\left(x, y\right) = \frac{\partial^2}{\partial x \partial y} F\left(x, y\right)$$

can be used to compute probabilities

$$Pr\left\{a < X \leq b, \quad c < Y \leq d\right\} = \int_a^b \int_c^d f\left(x, y\right) dx dy$$

1.3.16 Independence

The random variables X and Y are independent if

$$Pr\left\{X \leq a, Y \leq b\right\} = Pr\left\{X \leq a\right\} \cdot Pr\left\{Y \leq b\right\}, \quad \forall a, b \in \mathbb{R}$$

In terms of CDFs: $F_{X,Y}(a,b) = F_X\left(a\right) \cdot F_Y\left(b\right) \quad \forall a, b \in \mathbb{R}$
Thus we have

- $p_{X,Y}\left(x,y\right) = p_X\left(x\right) \cdot p_Y\left(Y\right)$ for discrete random variables
- $f_{X,Y}\left(x,y\right) = f_X\left(x\right) \cdot f_Y\left(Y\right)$ for continuous random variables.

2 Lecture 2

Definition 2.1. *Let A and B be events. The conditionally pprobability of A fiven B is defined by*

$$Pr\{A \mid B\} = \begin{cases} \frac{Pr\{A \cap B\}}{Pr\{B\}}, & Pr\{B\} > 0, \\ \textit{Not defined} \quad Pr\{B\} = 0 & \end{cases}$$

Check if Lecture two is in google cal

Example. Throw one die and let X denote the number of eyes. Find $Pr\{X \geq 5 \mid X \geq 3\}$.

$$\begin{aligned} Pr\{X \geq 5 \mid \geq 3\} &= \frac{Pr\{X \geq 5\}}{Pr\{X \geq 3\}} \\ &= \frac{\frac{2}{6}}{\frac{4}{6}} = \frac{1}{2} \end{aligned}$$

Definition 2.2. ***Conditional EMF*** *(Conditionally probability mass function PMF). Assume X and Y are jointly distributed random variables. The Conditional PMF. $p_{x|y}$ of X given Y given by*

$$p_{X|Y}(x \mid y) = \frac{Pr\{X = x, Y = y\}}{Pr\{Y = y\}} = \frac{p_{X,Y}(x, y)}{p_Y}, \quad p_y(y) > 0$$

Remark. $\{X = x, Y = x\}$ is shorthand for $\{(X = x) \cap (Y = y)\}$

Remark.

- $p_{X|Y}(x \mid y)$ is a pmf for

$$x \implies \sum_x p_{X|Y}(x \mid y) = 1 \quad \forall y$$

- $P_{X|Y}(x|y)$ is not a pmf for

$$y \implies \sum_{X|Y} \neq 1 \quad \text{In General}$$

Example. Throw die and let

$$X = \text{Number of eyes}$$
$$Y = \begin{cases} 0, & \text{if} \quad X \geq 2 \\ 1, & \text{if} \quad X \geq 3 \end{cases}$$

Find the conditionally **PMF** $p_{X|Y}$
Solution. For $y = 0$

$$p_{x|y}(x \mid y) = \begin{cases} \frac{1}{2}, & x = 1, 2 \\ 0, & x = 3, 4, 5, 6 \end{cases}$$

For $y = 1$

$$p_{x|y}(x \mid y) = \begin{cases} 0, & x = 1, 2 \\ \frac{1}{4}, & x = 3, 4, 5, 6 \end{cases}$$

Didnt quite understand this example

- $\sum_x p_{X|Y}(X \mid 0) = 1$
- **Noob mistake**

$$p_{X|Y} = (1 \mid 0) + p_{X|Y}(1 \mid 1) = \frac{1}{2} + 0 \neq 1$$

2.1 Joint Distrobution

THe conditional **PMF** is essential to us because we can siplify the joint **PMF** as

$$\begin{aligned} p_{X|Y}(x, y) &= Pr\{X = x, Y = y\} \\ &= Pr\{U = y\} Pr\{X = x \mid Y = y\} \\ &= p_Y(y) p_{X|Y}(x \mid y) \end{aligned}$$

Remark. if X and Y are independent then is

$$\begin{aligned} p_{X|Y}(x \mid y) = p_X(x) \quad &\text{if} \quad p_Y(y) > 0 \\ &\implies p_{X|Y}(x \mid y) = p_X(x) \cdot p_Y(Y) \end{aligned}$$

2.2 Simplified Notation

Unless it will cause confusion, we typically write

- $p(x)$ instead of $p_X(x)$

- $p(y)$ instead of $p_Y(y)$
- $p(x, y)$ instead of $p_{X,Y}(x, y)$
- $p(x \mid Y = y)$ instead of $p_{X|Y}(X \mid y)$

2.3 Marginalization

THe law of total probability gives

$$\begin{aligned} Pr\{X = x\} &= \sum_y Pr\{X = x, Y = y\} \\ &= \sum_y Pr\{Y = y\} Pr\{X = x \mid Y = y\} \end{aligned}$$

Example. A hunter ecounter N birds. For each burd, he gets one shot and either or misses. Assume the probability of hitting is p for each bird and that the shots are independent. Additionally, assume that the number of birds ancountered in Poission distrubuted with mean λ I.e $N \sim \text{Passion}(\lambda)$. Find the **EMF** of the number of birds hit.
Solution.

i Notation.

$$\text{Let} \quad I_i = \begin{cases} 0, & \text{Miss bird} \quad i \\ 1, & \text{Hit bird} \quad i \end{cases} \quad \text{for} \quad i = 1, 2, 3, 4, \ldots$$

Let $X =$ Number of birds hit Target is $p(x), x = 0, 1, 2, \ldots$

i Condition on N

$$(X|N = n) = \begin{cases} 0, & n = 0 \\ \sum_{i=1}^{n} I_i, & n > 0 \end{cases}$$

. We know

$$\begin{aligned} (X \mid N = n) &\sim Binomial\,(n, p) \\ &\implies Pr\{X = x \mid N = n\} \\ &= \binom{n}{x} p^x (1-p)^{n-x} \quad x = 0, 1, \ldots, n \end{aligned}$$

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iii)

$$\begin{aligned} Pr\{X = x\} &= \sum_{n=0}^{\infty} Pr\{X = x, N = n\} \\ &= \sum_{n=x}^{\infty} Pr\{N = n\} Pr\{X = x \mid N = n\} \\ &= \sum_{n=x}^{\infty} \frac{\lambda^n e^{-\lambda}}{n!} \frac{p^x (1-p)^{n-x} n!}{x! (n-x)!} \\ &= \lambda^x \frac{e^{-\lambda} p^x}{x!} \sum_{n=x}^{\infty} \frac{(1-p)^{n-x}}{(n-x)!} \lambda^{n-x}, \\ &\quad \text{hint} \quad \sum_{k=0}^{\infty} \frac{a^k}{k!} = e^a \\ &= (\lambda p)^x e^{-\lambda} \frac{1}{x!} \sum_{n=x}^{\infty} \frac{[\lambda (1-p)]^{n-x}}{(n-x)!} \\ &= (\lambda p)^x e^{-\lambda p} \frac{1}{x!}, \quad x = 0, 1, 2, 3 \ldots \\ &\implies \sim Possion\,(\lambda p) \end{aligned}$$

2.4 Conditional Expectation

Let X and Y be random variables and g a real function. The **Conditional expected value** of $g(X)$ given $Y = y$ is

$$E[g(X) \mid Y = y] = \sum_x g(x) \, Pr\{X = x \mid Y = y\}, \quad \text{if} \quad Pr\{Y = y\} > 0$$

Remark.
- Note that $E[g(X) \mid Y]$ is a stochastic variable variable!
- $E[g(X) \mid Y = y]$ has probability $Pr\{Y = y\}$

Theorem 2.1 (Law of iterated expectations). *Let X and Y be random variables such that $E[|g(X)|] < \infty$, and let g be a real function. then*

$$E[g(X)] = E[E[g(x) \mid Y]].$$

Proof.

$$\begin{aligned}
E[E[g(X) \mid Y]] &= \sum_y E[g(X) \mid Y = y] \\
&= \sum_y \left\{ \sum_x g(x) \cdot Pr\{X = x \mid Y = y\} \right\} x Pr\{Y = y\} \\
&= \sum_y \sum_x g(x) \cdot Pr\{X = x, Y = y\} \\
&= \sum_x g(x) \sum_y Pr\{X = x, Y = y\} \\
&= \sum_x g(x) \, Pr\{X = x\} \\
&= E[g(X)]
\end{aligned}$$

□

Theorem 2.2 (Law of total variance). *Let X and Y be random variables such that $E\left[X^2\right] < \infty$, then*

$$Var[X] = E[Var[X \mid Y]] + Var[E[X \mid Y]]$$

Revisited Example. A hunter ecounter N birds. For each burd, he gets one shot and either or misses. Assume the probability of hitting is p for each bird and that the shots are independent. Additionally, assume that the number of birds ancountered in Poission distrubuted with mean λ I.e $N \sim \text{Passion}(\lambda)$. Find the expected value and the variance of the number of birds hit.

$$\begin{aligned} E[X] &= E[E[X \mid Y]] \\ Var[X] &= E[var[X \mid Y]] + Var[E[E \mid Y]] \end{aligned}$$

Solution .

$$\begin{aligned} E[X \mid N=n] &= np \\ Var[X \mid N=n] &= np(1-p) \\ &\implies Var[X \mid N] = Np(1-p) \end{aligned}$$

We get

- $$\begin{aligned} E[X] &= E[E[X \mid N]] \\ &= E[Np] = pE[N] \\ &= p\lambda \end{aligned}$$

- $$\begin{aligned} Var[X] &= E[Var[X \mid N]] + Var[E[X \mid N]] \\ &= E[Np(1-p)] + Var[Np] \\ &= p(1-p)\lambda + p^2 Var[N] \\ &= p(1-p)\lambda + p^2\lambda \\ &= \lambda p \end{aligned}$$

3 Lecture 3

3.1 Randoms sum

Building on the hunter example from last week. we can more generally consider random sums

$$X = \begin{cases} 0, & N = 0 \\ \zeta_1 + \zeta_2 + \ldots + \zeta_N, \quad N > 0 \end{cases}$$

where

- N is a discrete random variable with values $0, 1, \ldots$
- $\zeta_1, \zeta_2, \ldots$ are independent random variables
- N is independent of $\zeta_1, \zeta_2 + \ldots + \zeta_N$
- **Notation** $X = \sum_{i=1}^{N} \zeta_i = \zeta_1 + \zeta_2 + \ldots + \zeta_N$

Example.

1. Insurance company

$$N: \text{ Number of claims.}$$

$$\zeta_1, \zeta_2, \ldots \quad : \quad \text{Sizes of the claims}$$

Total liabilility:

$$X = \zeta_1 + \zeta_2 + \ldots + \zeta_N$$

2. Be careful!

$$\begin{aligned} \overbrace{E\left[\sum_{i=1}^{N} \zeta_i\right]}^{\neq \sum_{i=1}^{N} E[\zeta_i]} &= E\left[E\left[\sum_{i=1}^{N} \zeta_i \mid N\right]\right] \\ &= E\left[\sum_{i=1}^{N} E\left[\zeta_i \mid N\right]\right] \end{aligned}$$

3.2 Self Study

Section 2.2, 2.3, 2.4

3.3 Stochastic process in descrete time

Definition 3.1. *A* ***discrete-time stochastic process*** *is a family of random variables* $[X_t : t \in T]$ *where* T *is discrete.*

- *We use* $T = \{0, 1, 2, ..\}$ *and write* X_n *instead of* X_t
- *we call* X_n *the* ***state*** *at time* $n = 0, 1, 2, 3, \ldots$
- *We call the set of all possible states the* ***state space***

Table 1: Table for example

Day	$n = 0$	$n = 1$	$n = 2$	$\ldots$
Random Variable	X_0	X_1	X_2	$\ldots$
Realization 1	$x_0 = 0$	$x_1 = 1$	$x_2 = 1$	$\ldots$
Realization 2	$x_0 = 1$	$x_1 = 1$	$x_2 = 1$	$\ldots$

Example.

$$X_n = \begin{cases} 1, & \text{if it rains on day } n \\ 0, & \text{no rain on day } n \end{cases}$$

State space $= \{0, 1\}$
We have a problem. Need

$$Pr\left\{X_n = x_n \mid X_{n-1} = x_n, X_{n-2} = x_{n-2}, \ldots, X_0 = x_0\right\}.$$

for all $n = 0, 1, 2, \ldots$

3.4 Markov chain

Definition 3.2 (Discrete time Markov Chain)**.** *A* ***Discrete time markoc chain*** *is a discrete time stochastic process* $\{X_n : n = 0, 1, \ldots\}$ *that statisfied the* ***markov property*** *such that*

$$\begin{aligned} &Pr\left\{X_{n-1} = j \mid X_n = i, X_{n-1} = i_{n-1}, \ldots, X_0 = i_0\right\} \\ &= Pr\left\{X_{n+1} = j \mid X_n = i\right\} \end{aligned}$$

for $n = 0, 1, 2, 3, \ldots$ *and for all states* i *and* j

Definition 3.3 (One-step transition probabilities)**.** *We can define it as*

- *For a discrete Markov chain* $\{X_n : n = 0, 1, 2, \ldots\}$ *we call* $P_{ij}^{n,n+1} = Pr\left\{X_{n+1} = j, X_n = i\right\}$ *the* ***one step trainsition probabilities*** *.*

- *We will assume* ***stationary transition probabilities*** *, i.e that*

$$P_{ij}^{n,n+1} = P_{ij}$$

for $n = 0, 1, 2, \dots$ *and all states* i *and* j *.*

Some of the properties

1. "You will always go somewhere"

$$\sum_j P_{ij} = 1 \quad \forall i$$

2. The markov chain can be described as follows.

$$\begin{aligned} &Pr\{X_0 = i_0, X_1 = i_1, \dots, X_n = i_n\} \\ &= Pr\{X_0 = i_0\} Pr\{X_1 = i_1 \mid X_0 = i_0\} \dots \\ &\quad Pr\{X_n = x_n \mid X_{n-1} = i_{n-1} \dots X_0 = i_0\} \\ &\vdots \quad \text{Markov step} \\ &= Pr\{X_0 = i_0\} \cdot Pr\{X_1 = i_1 \mid X_0 = i_0\} \dots \\ &\quad Pr\{X_n = x_n \mid X_{n-1} = i_{n-1}\} \\ &= Pr\{X_0 = i_0\} P_{i_0,i_1} \cdot P_{i_1,i_2} \dots P_{i_{n-1},i_n} \end{aligned}$$

Which is a major simplification.

Definition 3.4 (Transition Probability Matrix)**.** *For a discrete time markov-chain with state space* $\{0, 1, \dots, N\}$ *we call*

$$\mathbf{P} = \begin{bmatrix} P_{00} & \dots & P_{0N} \\ P_{10} & \dots & \\ \vdots & & \ddots \\ P_{N0} & \dots & P_{NN} \end{bmatrix}$$

Is the transition matrix. For statespace $\{0, 1, 2, \dots\}$ *we envision an infinitely sized matrix.*

Example.

- Markoc chain : $\{X_n : n = 0, 1, 2, \ldots\}$
- State space $= \{0, 1\}$
- Transition Matrix

$$\mathbf{P} = \begin{bmatrix} 0.9 & 0.1 \\ 0.6 & 0.4 \end{bmatrix}$$

We can compute

$$\begin{aligned} Pr\{X_3 = 1 \mid X_2 = 0\} &= p_{01} \\ &= 0.1 \\ Pr\{X_{10} = 0 \mid X_9 = 1\} &= P_{10} \\ &= 0.6 \end{aligned}$$

Definition 3.5 (Transition Diagram)**.** *Let $\{X_n : n = 0, 1, \ldots\}$ be a discrete time Markov chain. A **state trasnistion diagram** visualizes the transition probabilities as a weighted directed graph where the nodes are the states and the edges are the possible transitions marked with the transistion probabilities.*

Example. State space $= \{0, 1, 2\}$ and

$$P = \begin{bmatrix} 0.95 & 0.05 & 9 \\ 0 & 0.9 & 0.1 \\ 0.01 & 0 & 0.99 \end{bmatrix}$$

Transisition diagram

Nice figure of the diagram

3.5 Doing n transitions.

Theorem 3.1. *For a Markoc chain $\{X_n : n = 0, 1, \ldots\}$ and any $m \geq 0$ we have*

$$Pr\{X_{m-n} = j \mid X_m = i\} = P_{ij}^{(n)} = \sum_{k=0}^{\infty} P_{ik} P_{kj}^{(n-1)}, \quad n > 0$$

where we define

$$P_{ij}^{(0)} = \begin{cases} 1, & i = j \\ 0, i \neq j \end{cases}$$

Proof. Set $m = 0$ then is

$$\begin{aligned} P_{ij}^{(n+1)} &= Pr\{X_{n+1} = j \mid X_0 = i\} \\ &= \sum_k Pr\{X_{n+1} = j, X_1 = k \mid X_0 = i\} \\ &= \sum_k Pr\{X_{n+1} = j \mid X_1 = k, X_0 = i\} \cdot Pr\{X_1 = k \mid X_0 = i\} \\ &= \sum_k P_{kj}^{(h)} \cdot P_{ik} = \sum_k P_{ik} P_{kj}^{(h)} \end{aligned}$$

□

Example. $\{X_n : n = 0, 1, 2, \ldots\}$ is a markoc chain and

$$P = \begin{bmatrix} 0.1 & 0.9 \\ 0.6 & 0.4 \end{bmatrix}$$

Find $P_{01}^{(4)}$. **Solution**.

$$P^2 = \begin{bmatrix} 0.55 & 0.45 \\ 0.30 & 0.70 \end{bmatrix}$$

So by doing matrix multiplication and we end up with

$$P^4 = P^2 \cdot P^2 = \begin{bmatrix} 0.4375 & 0.5625 \\ 0.3750 & 0.6250 \end{bmatrix}$$

Which therefore ends up with the answer

$$P_{01}^{(4)} = 0.5625$$

4 References