Norwegian University of Science and Technology
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TMA4212 Numerical solution of differential equations by difference methods Spring 2021

Exercise set 1

$\boxed{1}$ Consider the following tridiagonal matrix

$$A = \begin{pmatrix} a & b & 0 & \dots & 0 \\ c & a & b & \ddots & 0 \\ \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & \ddots & c & a & b \\ 0 & \dots & 0 & c & a \end{pmatrix} = \text{tridiag}(c, a, b) \in \mathbb{R}^{M\times M},\ M \geq 4,$$

where we assume $bc > 0$. It is known that the right eigenvectors $\boldsymbol{x}^{(k)}(k = 1, ..., M)$ and the associated eigenvalues $\lambda_k(k = 1, ..., M)$ are given by

$$x_j^{(k)} = \left(\frac{b}{c}\right)^{j/2} \sin\left(\frac{jk\pi}{M+1}\right),\quad \lambda_k = a + 2\sqrt{bc}\cos\left(\frac{k\pi}{M+1}\right),$$

where $x_j^{(k)}$ is the jth element of the vector $\boldsymbol{x}^{(k)}$; $A\boldsymbol{x}^{(k)} = \lambda_k \boldsymbol{x}^{(k)}$. You can verify this by simply inserting.

a) What are the left eigenvectors $\boldsymbol{y}^{(k)}(k = 1, ..., M)$ and associated eigenvalues $\beta_k(k = 1, ..., M)$ of A which satisfy $\boldsymbol{y}^{(k)}A = \beta_k \boldsymbol{y}^{(k)}$? Note that $\boldsymbol{y}^{(k)}$'s are row vectors.

b) Assume $a > 2\sqrt{bc} > 0$ and $b = c$, calculate the following quantity (called the ℓ_2 condition number):

$$\left\|A^{-1}\right\|_2 \|A\|_2 .$$

(Hint: look at the text "finite difference methods" by Brynjulf Owren, Section 3.1.)

c) Let A_h be

$$A_h = \frac{1}{h^2}\text{tridiag}(-1, 2, -1) \in \mathbb{R}^{M\times M},$$

where $h = 1/(M + 1)$. Calculate the following quantity

$$\lim_{M\to\infty} \left\|A_h^{-1}\right\|_2 .$$

(Hint: look at the same section of the text as above.)

$\boxed{2}$ Consider a function $u(x)$ defined on $[0, 1]$. We want to approximate the derivative $u_x(x)$ by using function values of $u(x)$ on equidistant points

$$x_0 = 0,\ x_1 = \frac{1}{M+1}, ...,\ x_M = \frac{M}{M+1},\ x_{M+1} = 1.$$

Let $h = 1/(M+1)$.

a) Consider two different approximation methods:

$$u_x(x) \approx \frac{u(x+h) - u(x)}{h} \quad \text{(Forward difference)},$$

for $x = x_0, ..., x_M$, and

$$u_x(x) \approx \frac{u(x+h/2) - u(x-h/2)}{h} \quad \text{(Central difference)},$$

for $x = x_0 + h/2, ..., x_M + h/2$ (so that we only use function values on x_i's). Calculate the convergence order of these methods in terms of h. Then write down the approximation as a matrix-vector multiplication:

$$\boldsymbol{u}_x = A_h \boldsymbol{u},$$

where $\boldsymbol{u}_x$ is a vector comprised of approximated values of $u_x(x)$, A_h is an $(M+1) \times (M+1)$ matrix, and $\boldsymbol{u}$ is a vector comprised of function values of $u(x)$.

b) We define matrix exponentials by

$$\exp(A) := \sum_{k=0}^{\infty} \frac{A^k}{k!}.$$

Calculate the eigenvalues of $\exp(B_h)$ for

$$B_h = \frac{1}{h^2} \text{tridiag}(-1, 2, -1).$$