## 9. Lecture IX: Stable Manifold and Hartman-Grobman Theorems

9.1. **Stable Manifold Theorem.** Now we are ready to discuss the ideas of the stable manifold theorem.

Recall that for a linear system, by Thm.3.3, we can decompose phase space about its critical points to $\mathbb{R} = E^s \oplus E^c \oplus E^u$, where

$$\begin{aligned} E^s &= \operatorname{span} \bigcup_{\{n:\Re\lambda_n<0\}} V_n \\ E^c &= \operatorname{span} \bigcup_{\{n:\Re\lambda_n=0\}} V_n \\ E^u &= \operatorname{span} \bigcup_{\{n:\Re\lambda_n>0\}} V_n, \end{aligned}$$

and $V_n$ were the vectors spanning the general eigenspace associated with $\lambda_n$. Now if $\mathbf{0}$ is a hyperbolic critical point, by definition, $\mathrm{D}f(\mathbf{0})$ does not have eigenvalues with $\Re\lambda_n = 0$. Therefore, for the linear system

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x}(t) = \mathrm{D}f(\mathbf{0})\mathbf{x}(t),$$

we have a decomposition of phase space into $\mathbb{R}^d = E^s \oplus E^u$.

The stable manifold theorem is as follows:



**Theorem 9.1** (Stable Manifold Theorem)**.** *Let $U$ be an open subset of $\mathbb{R}^d$ containing the origin. Let $f \in C^1(U;\mathbb{R}^d)$, and let $\phi_t$ be the flow of the nonlinear system*

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x} = f(\mathbf{x}).$$

*Suppose that $f(\mathbf{0}) = \mathbf{0}$ and $\mathrm{D}f(\mathbf{0})$ has $k$ eigenvalues (counting multiplicity) with negative real parts and $d-k$ eigenvalues with positive real parts. Then*

(i) *there exists a dimension $k$ $C^1$-manifold $M_s$ tangent to $E^s$ of the linearized system*

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x} = \mathrm{D}f(\mathbf{0})\mathbf{x}$$

*at $\mathbf{0}$ such that for all $t \geq 0$, $\phi_t(M_s) \subseteq M_s$ and for all $\mathbf{y} \in M_s$,*

$$\lim_{t\to\infty} \phi_t(\mathbf{y}) = \mathbf{0};$$

*and*

(ii) *there exists a dimension $d-k$ $C^1$-manifold $M_u$ tangent to $E^u$ of the linearized system at $\mathbf{0}$ such that for all $t \leq 0$, $\phi_t(M_u) \subseteq M_u$ and for all $\mathbf{y} \in M_u$,*

$$\lim_{t\to-\infty} \phi_t(\mathbf{y}) = \mathbf{0}.$$

**Example 9.1.** We look at a nonlinear system that we can solve explicitly:

$$\begin{aligned} \dot{x}_1 &= -x_1 \\ \dot{x}_2 &= -x_2 + x_1^2 \\ \dot{x}_3 &= x_3 + x_1^2. \end{aligned}$$

There is one fixed point, which is the origin. The linearization is given by

$$\mathrm{D}f(\mathbf{0}) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

and so $\mathbf{0}$ is a hyperbolic critical point. The eigenvalues and eigenvectors are readily deducible, and we see that

$$E^s = \{\mathbf{x} : x_2 = 0\} = \operatorname{span}\{(1,0,0)^\top, (0,1,0)^\top\}, \qquad E^u = \{\mathbf{x} : x_1 = x_2 = 0\} = \operatorname{span}\{(0,0,1)^\top\}.$$

The equations can also be integrated by hand, giving

$$\begin{aligned} x_1(t) &= y_1 e^{-t} \\ x_2(t) &= y_2 e^{-t} + y_1^2(e^{-t} - e^{-2t}) \\ x_3(t) &= y_3 e^{t} + y_1^2(e^{t} - e^{-2t})/3, \end{aligned}$$

with $\mathbf{x}(0) = \mathbf{y} = (y_1, y_2, y_3)^\top$.

We see that $\lim_{t\to\infty} \phi_t(\mathbf{y}) = \mathbf{0}$ if, and only if, $y_1^2/3 + y_3 = 0$, and so

$$M_s = \{\mathbf{y} \in \mathbb{R}^3 : y_1^2 + 3y_3 = 0\}.$$

Likewise, $\lim_{t\to-\infty} \phi_t(\mathbf{y}) = \mathbf{0}$ if, and only if, $y_1 = y_2 = 0$, and so

$$M_u = \{\mathbf{y} \in \mathbb{R}^3 : y_1 = y_2 = 0\}.$$

It is clear that $M_u$ is tangent to $E^u$ at $\mathbf{0}$ because they coincide entirely. Taking the derivative of $h(y_1, y_2, y_3) = y_1^2 + 3y_3$, for which $M_s$ is the level set at 0, we find

$$\nabla h \Big|_{\mathbf{0}} = (2y_1, 0, 3)^\top|_{(0,0,0)} = (0, 0, 3),$$

which is indeed perpendicular to $E^s$, and so $S$ and $E^s$ are tangent at $\mathbf{0}$, as expected.

The way that the Stable Manifold Theorem is usually proven gives us insight into the structure of nonlinear systems. And whilst we shall not be proving the Stable Manifold Theorem, it is of benefit to discuss some elements of its proof. First notice that for a general first order, autonomous nonlinear system, we have the following Taylor's expansion around a hyperbolic critical point $\mathbf{x}_0$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x}(t) = \mathrm{D}f(\mathbf{x}_0)\mathbf{x}(t) + \mathbf{G}(\mathbf{x}),$$

where $\mathbf{G}$ has zero first derivative at $\mathbf{x}_0$. This means that whilst $\mathbf{G}$ might not be "second-order" in $\mathbf{x}$, for every $\varepsilon$, there is a $\delta$ such that if $|\mathbf{x} - \mathbf{x}_0| < \delta$, and $|\mathbf{y} - \mathbf{x}_0| < \delta$,

$$|\mathbf{G}(\mathbf{x}) - \mathbf{G}(\mathbf{y})| \le \varepsilon |\mathbf{x} - \mathbf{y}|.$$

By applying the Jordan Normal Form Theorem (Thm. 3.1), we can assume that $\mathrm{D}f(\mathbf{x}_0)$ is of the form

$$\mathrm{D}f(\mathbf{x}_0) = \begin{pmatrix} P & 0 \\ 0 & Q \end{pmatrix},$$

where $P$ is a matrix in Jordan normal form with only eigenvalues of negative real parts, and $Q$ is a matrix in Jordan normal form with only eigenvalues of positive real parts. The linear system can be solved by exponentiating $\mathrm{D}f(\mathbf{x}_0)$ so that where

$$W(t) = \begin{pmatrix} e^{Pt} & 0 \\ 0 & 0 \end{pmatrix}, \qquad Z(t) = \begin{pmatrix} 0 & 0 \\ 0 & e^{Qt} \end{pmatrix},$$

the flow of the linearized system is

$$e^{\mathrm{D}f(\mathbf{x}_0)t} = W(t) + Z(t).$$

Using the Duhamel representation to treat the term $\mathbf{G}$ as an inhomogeneity, we find

$$\mathbf{x}(t) = e^{\mathrm{D}f(\mathbf{x}_0)t}\mathbf{x}(0) + \int_0^t e^{\mathrm{D}f(\mathbf{x}_0)(t-s)} G(\mathbf{x}(s))\, \mathrm{d}s$$

If we look at solutions that start on what might potentially be the stable manifold, we have solutions of the form

$$\mathbf{x}(t) = W(t)\mathbf{x}(0) + \int_0^t W(t-s)G(\mathbf{x}(s))\,\mathrm{d}s - \int_t^\infty Z(t-s)G(\mathbf{x}(s))\,\mathrm{d}s.$$

Now we can take Picard iterations and exploit the signs of the real parts of the eigenvalues to bound $W$ and $Z$ to reach our conclusions. A similar proceedure can be done for the unstable manifold, except we then run time in the backwards direction using the reversal $t \mapsto -t$.

The Stable Manifold Theorem only defines $M_s$ and $M_u$ on a small neighbourhood of the hyperbolic critical point. To supplement their definition in the theorem we also introduce the GLOBAL STABLE AND UNSTABLE MANIFOLDS at $\mathbf{0}$ if it is a hyperbolic fixed point:

$$W^s(\mathbf{0}) = \bigcup_{t\leq 0} \phi_t(M_s)$$
$$W^u(\mathbf{0}) = \bigcup_{t\geq 0} \phi_t(M_u).$$

These may not be manifolds in the sense we have defined, or in the more general sense conventionally used, except restricted to a neighbourhod of the hyperbolic critical point, but they are flow invariant, and satisfy the properties respectively ascribed to $M_s$ and $M_u$ in the Stable Manifold Theoremm. This is primarily because the function of which they are level sets can fail to be constant rank, and the "manifold" can intersect itself, so when we say "$C^k$-manifold" below, we mean essentially that it is $C^k$ on neighbourhoods where the function defining it has the same rank.

We are also in a position to speak briefly of non-hyperbolic critical points:

**Theorem 9.2** (Centre Manifold Theorem). *Let $f \in C^1(U;\mathbb{R}^d)$ and $f(\mathbf{0}) = \mathbf{0}$. Suppose $\mathrm{D}f(\mathbf{0})$ has $k$ eigenvalues with negative real parts, $m$ eigenvalues with zero real parts, and $(d-k-m)$ eigenvalues with positive real parts. There exists*

(i) *an $m$-dimensional $C^1$-CENTRE MANIFOLD $W^c(\mathbf{0})$ tangent to the centre subspace $E^c$ of the linearized system at $\mathbf{0}$,*
(ii) *a $k$-dimensional $C^1$ stable manifold $W^s(\mathbf{0})$ tangent to the stable subspace $E^s$ of the linearized system at $\mathbf{0}$, and*
(iii) *a $(d-k-m)$-dimensional $C^1$ unstable manifold $W^u(\mathbf{0})$ tangent to the unstable subspace $E^u$ of the linearized system at $\mathbf{0}$.*

*These three subsets of $\mathbb{R}^d$ are invariant under the flow $\phi_t$.*

What happens on the centre manifold shall remain a mystery to us as long as we are only willing to look at approximations to first order because of another topological fact, this time of the real numbers. If $\lambda = \sigma + i\tau$, and $\sigma \neq 0$, then there is always a small enough perturbation of $\lambda$ by $h \in \mathbb{C}$ such that the sign of $\Re(\lambda + h)$ is the same as the sign of $\sigma$. Not being zero is an open condition. But if $\sigma = 0$, any (general) perturbation of $\lambda$ will give $\sigma$ a sign. Therefore, we see that what determines the behaviour on the centre manifold is determined by how the nonlinear terms perturb the system *spectrally* in a neighbourhood of a critical point. We shall find that at nonhyperbolic critical points, completely novel behaviours can arise because of nonlinearity.

9.2. **Hartman-Grobman Theorem.** First we seek to exhaust the facilities of first order approximative methods as well as we can with the tools available to us.

The Hartman-Grobman Theorem is a partial generalization of the Fundamental Theorem of Linear Systems, and gives us more precise information about trajectories near hyperbolic critical points. Recall that having defined the exponential of a matrix, the Fundamental Theorem allowed us to write a solution to the autonomous linear system

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x} = \mathbf{A}\mathbf{x}(t), \qquad \mathbf{x}(0) = \mathbf{b}$$

as

$$\mathbf{x}(t) = \exp(\mathbf{A}t)\mathbf{b}.$$

The Hartman-Grobman theorem allows us to write the solution to the nonlinear system near a hyperbolic critical point $\mathbf{x}_0$ as an approximation of

$$\exp(\mathrm{D}f(\mathbf{x}_0)t)\mathbf{b},$$

if the initial condition $\mathbf{x}(0) = \mathbf{b}$ is sufficiently close to $\mathbf{x}_0$.

To understand the contents of the precise statement, we define two terms. Suppose we have two first order autnomous systems in $\mathbb{R}^d$, shifted so that one of their critical points each are at $\mathbf{0}$. Let one system have a flow $\phi_t$ and the other have a flow $\psi_t$. These two autonomous first order systems are said to be TOPOLOGICALLY EQUIVALENT in a neighbourhood of $\mathbf{0}$ if there are two open sets $U$ and $V$, both containing $\mathbf{0}$, a homeomorphism $H : U \to V$ and a monotonically increasing function $\eta : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that

$$\psi_{\eta(t)} \circ H(\mathbf{x}) = H \circ \phi_t(\mathbf{x}).$$

If we can choose $\eta$ (by changing $H$, say) so that $\eta = \mathrm{Id}_{\mathbb{R}}$, then the systems are said to be TOPOLOGICALLY CONJUGATE in a neighbourhood of $\mathbf{0}$.

**Theorem 9.3** (Hartman-Grobman Theorem)**.** *Let $\mathbf{x}_0 \in \mathbb{R}^d$ be a hyperbolic critical point of the system*

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x}(t) = f(\mathbf{x}(t)),$$

*with flow $\phi_t$. There exists neighbourhoods $U$ and $V$ of $\mathbf{x}_0$, a homeomorphism $H : U \to V$ and an interval $I \in \mathbb{R}$ containing $0$ such that for every $\mathbf{y} \in U$ and $t \in I$,*

$$H \circ \phi_t(\mathbf{y}) = \exp(\mathrm{D}f(\mathbf{x}_0)t)H(\mathbf{y}).$$

Hartman also showed that if $f \in C^2$, then we can find $H$ as above which is additionally a $C^1$-diffeomorphism.