Norwegian University of Science and Technology
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TMA4145 Linear Methods
Fall 2018

Exercise set 6

1 Let (X, d) be a metric space. Show that any Cauchy sequence $(x_n)_{n\in\mathbb{N}}$ is bounded in X.

Solution. Since (x_n) is a Cauchy sequence, by choosing say $\epsilon = 1$, there is $N \in \mathbb{N}$ such that

$$d(x_n, x_m) < 1 \quad \text{for all} \;\; n, m \geq N\,.$$

In particular, putting $m = N$ we have

$$d(x_n, x_N) < 1 \quad \text{for all} \;\; n \geq N\,. \tag{1}$$

We want to show that the sequence is bounded by showing that $(x_n)_{n\in\mathbb{N}} \subset B_r(x_N)$ for some radius r. Our strategy is to use that (1) shows that we control x_n for $n \geq N$, so we only need to control $x_1, ..., x_{N-1}$. In fact, let

$$r := \max\{1, d(x_1, x_N) + 1/2, \ldots, d(x_{N-1}) + 1/2\}.$$

Note that $r \geq 1$. Then we claim that $B_r(x_N)$ contains every element of the sequence $(x_n)_{n\in\mathbb{N}}$. To prove the claim, we need to prove that $d(x_n, x_N) < r$ for all $n \in \mathbb{N}$:

- If $n < N$, then $d(x_n, x_N) < r$ since our definition of r ensures that $r \geq d(x_n, x_N) + 1/2 > d(x_n, x_N)$.
- If $n \geq N$, then $d(x_n, x_N) < 1 \leq r$ by the way we picked N and r.

2 Let (X, d) be a metric space, and assume that $Y \subset X$ is a dense subset of X. Show that for any $x \in X$ there exists a sequence $(y_n)_{n\in\mathbb{N}} \subset Y$ such that $x = \lim_{n\to\infty} y_n$.

Solution. By the comments after the definition of a dense subset in the lecture notes, we know that for every $\epsilon > 0$ we can find some $y \in Y$ with $d(x, y) < \epsilon$. By picking $\epsilon = 1/n$ for $n \in \mathbb{N}$, we find a sequence of elements $y_n \in Y$ such that

$$d(x, y_n) < 1/n.$$

But this implies that $\lim_{n\to\infty} y_n = x$: for every $\epsilon > 0$, one can find $N \in \mathbb{N}$ such that $1/n < \epsilon$ whenever $n \geq N$. So for $n \geq N$ we have

$$d(x, y_n) < 1/n < \epsilon,$$

which shows that $y_n \to x$ by the definition of limits in metric spaces.

$\boxed{3}$ Prove the following two statements for a normed space $(X, \|.\|)$.

a) Any ball $B_r(x) = \{y \in X : \|x - y\| < r\}$ in $(X, \|.\|)$ is bounded and $\operatorname{diam}(B_r(x)) \leq 2r$.

b) If A is a bounded subset of $(X, \|.\|)$, then for any $a \in A$ we have $A \subseteq \bar{B}_{\operatorname{diam}(A)}(a)$. (Recall that the a closed ball $\bar{B}_r(x)$ is the set $\{y \in X : \|y - x\| \leq r\}$.)

Solution. Note that there are many ways of solving this, in particular because lemma 3.2 gives us a lot of equivalent statements to describe boundedness.
a) By part 2 of lemma 3.2, a set A is bounded if there is some constant $0 \leq M < \infty$ such that $\|x - y\| < M$ for every $x, y \in A$. This implies that $B_r(x)$ is bounded: if $y, z \in B_r(x)$, then

$$\|y - z\| = \|(y - x) + (x - z)\| \leq \|y - x\| + \|z - x\| \leq r + r = 2r,$$

where we have used the triangle inequality and the definition of $B_r(x)$. Hence we can pick $M = 2r$, and $B_r(x)$ is bounded. The diameter $\operatorname{diam}(B_r(x))$ is defined to be $\sup\{\|y-z\| : y, z \in B_r(x)\}$. Above we showed that $\|y-z\| \leq 2r$ for any $y, z \in B_r(x)$, hence $\operatorname{diam}(B_r(x)) = \sup\{\|y - z\| : y, z \in B_r(x)\} \leq 2r$.
b) Recall that $\operatorname{diam}(A)$ is defined to be $\sup\{\|y - z\| : y, z \in A\}$, and by part 3 of lemma 3.2 we know that $\operatorname{diam}(A) < \infty$. Now pick $a \in A$, and consider some $x \in A$. By the definition of the diameter, we have

$$\|a - x\| \leq \operatorname{diam}(A).$$

Since x was arbitrary, this shows exactly that $A \subseteq \bar{B}_{\operatorname{diam}(A)}(a)$, since $\bar{B}_{\operatorname{diam}(A)}(a) = \{x \in X : \|x - a\| \leq \operatorname{diam}(A)\}$.

$\boxed{4}$ **a)** Let $(f_n)_{n\in\mathbb{N}}$ be defined by

$$f_n(t) = \begin{cases} 0 & \text{for } a \leq t \leq \frac{a+b}{2}, \\ n(t - \frac{a+b}{2}) & \text{for } \frac{a+b}{2} < t \leq \frac{a+b}{2} + \frac{1}{n}, \\ 1 & \text{for } \frac{a+b}{2} + \frac{1}{n} \leq t \leq b. \end{cases}$$

in $C[a, b]$. Use the definition of uniform convergence to determine if $(f_n)_{n\in\mathbb{N}}$ converges uniformly on $[a, b]$.

b) Let $(f_n)_{n\in\mathbb{N}}$ be the sequence on $[0,1]$ defined by $f_n(x) = \frac{1}{1+nx}$. Use the definition of uniform convergence to determine if $(f_n)_{n\in\mathbb{N}}$ converges uniformly on $[0,1]$.

Solution. a) As on the previous problem set, we highly recommend that you sketch some of these functions! It is not hard to see that f_n converge pointwise to the function f given by

$$f(t) = \begin{cases} 0 & \text{for } a \leq t \leq \frac{a+b}{2}, \\ 1 & \text{for } \frac{a+b}{2} < t \leq b. \end{cases}$$

However, this convergence is not uniform. [1] Pick $\epsilon = \frac{1}{4}$. I claim that there is no $N \in \mathbb{N}$ such that

$$|f(x) - f_n(x)| < \frac{1}{4}$$

for any $x \in X$ whenever $n \geq N$. Since each f_n is a continuous function with $f(a) = 0$ and $f(b) = 1$, there must for each n be some $x_{1/2} \in [a,b]$ with $f_n(x_{1/2}) = \frac{1}{2}$. However, the only values of the function f are 0 and 1, so we must have $|f(x_{1/2}) - f_n(x_{1/2})| = \frac{1}{2} \not< \frac{1}{4}$. Since we could find such a point $x_{1/2}$ for any n, it is clearly not possibly to find $N \in \mathbb{N}$ such that

$$|f(x) - f_n(x)| < \frac{1}{4}$$

for any $x \in X$ whenever $n \geq N$.

b) Once again, a simple sketch is highly recommended. Clearly, for each fixed $x \in (0,1]$, we have that $\lim_{n\to\infty} f_n(x) = 0$, and $\lim_{n\to\infty} f_n(0) = 1$. hence f_n converges pointwise to

$$f(t) = \begin{cases} 1 & \text{for } x = 0, \\ 0 & \text{for } 0 < x \leq 1. \end{cases}$$

To show that the convergence is not uniform, we can use more or less the same argument as above. Each f_n is a continuous function with $f_n(0) = 1$ and $f_n(1) = \frac{1}{1+n} \leq \frac{1}{2}$, so for any n there must exist $x_{1/2} \in [a,b]$ with $f_n(x_{1/2}) = \frac{1}{2}$. As before the limit function f only takes the values 0 and 1, so by picking $\epsilon = \frac{1}{4}$ we will not be able to find $N \in \mathbb{N}$ such that

$$|f(x) - f_n(x)| < \frac{1}{4}$$

for any $x \in X$ whenever $n \geq N$.

$\boxed{5}$ Show that $(\ell^\infty(\mathbb{R}), \|\cdot\|_\infty)$ is a Banach space.

[1] Since each f_n is a continuous function, we know from theorem 3.11 that any uniform limit of the f_n would be continuous. But clearly f is not continuous - this would be a slick proof that the convergence is not uniform. However, the goal of this exercise is that you should become familiar with the definition of uniform convergence. That is why we ask that you use the definition to solve this problem.

Solution. Let $(x_n)_n$ be a Cauchy sequence in ℓ^∞. Note that $(x_n)_{n\in\mathbb{N}}$ is a *sequence of sequences*, and we will use the notation from the lecture notes:

$$x_n = (x_1^{(n)}, x_2^{(n)}, \ldots),$$

so the sequences have the following form:

$$\begin{aligned} x_1 &= (x_1^{(1)}, x_2^{(1)}, x_3^{(1)} \ldots, x_k^{(1)}, \ldots) \\ x_2 &= (x_1^{(2)}, x_2^{(2)}, x_3^{(2)} \ldots, x_k^{(2)}, \ldots) \\ x_3 &= (x_1^{(3)}, x_2^{(3)}, x_3^{(3)} \ldots, x_k^{(3)}, \ldots) \\ &\ldots \\ x_n &= (x_1^{(n)}, x_2^{(n)}, x_3^{(n)} \ldots, x_k^{(n)}, \ldots) \\ &\ldots. \end{aligned} \tag{2}$$

We need to show that $(x_n)_{n\in\mathbb{N}}$ converges to some sequence z in ℓ^∞. As we did in the completeness proofs in the lecture notes, we split the proof into three steps.

Step 1: Find a candidate for the limit x. For fixed $k \in \mathbb{N}$, consider the sequence $(x_k^{(n)})_n$. This corresponds to the elements along a vertical line in (2). We will show that $(x_k^{(n)})_n$ is a Cauchy sequence of real numbers. Since $(x_n)_n$ is a Cauchy sequence in $\ell^\infty(\mathbb{R})$, we can for every $\epsilon > 0$ find some $N \in \mathbb{N}$ such that $\|x_m - x_n\|_\infty < \epsilon$ for $m, n \geq N$. Now note that

$$|x_k^{(m)} - x_k^{(n)}| \leq \sup\{|x_k^{(m)} - x_k^{(n)}| : k \in \mathbb{N}\} = \|x_m - x_n\|_\infty.$$

Therefore, if $m, n \geq N$, we actually have that $|x_k^{(m)} - x_k^{(n)}| < \epsilon$ – hence $(x_k^{(n)})_n$ is a Cauchy sequence of real numbers. Since $\mathbb{R}$ is complete, we know that the Cauchy sequence $(x_k^{(n)})_n$ must converge to some limit $z_k \in \mathbb{R}$:

$$z_k = \lim_{n\to\infty} x_k^{(n)}.$$

Our candidate for the limit of $(x_n)_n$ is the sequence

$$z = (z_1, z_2, z_3, \ldots).$$

Note that z_k is the limit of the elements in the k'th column of (2).
Step 2: Show that z is in ℓ^∞. We need to find some positive constant C such that

$$|z_k| < C \text{ for any } k \in \mathbb{N}.$$

By our definition in step 1, $z_k = \lim_{n\to\infty} x_k^{(n)}$. Since $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\ell^\infty(\mathbb{R})$, it is also a bounded sequence in $\ell^\infty(\mathbb{R})$ (from previous problem set), meaning that there is some $K > 0$ such that $\|x_n\|_\infty < K$ for any $n \in \mathbb{N}$. Since $\|x_n\|_\infty$ is the supremum over all the elements x_k^n of $(x_n)_{n\in\mathbb{N}}$, this actually means that

$$|x_k^n| \leq \|x_n\|_\infty < K \text{ for any } k, n \in \mathbb{N}.$$

This means that any element in the infinite matrix (2) is bounded by K. Let us return to the element z_k of z, which we wanted to bound. Since $z_k = \lim_{n\to\infty} x_k^{(n)}$, we can find some $N \in \mathbb{N}$ such that $|z_k - x_k^{(n)}| < 1$ whenever $n \geq N$ [2]. But then the triangle inequality gives that

$$|z_k| \leq |x_k^N| + |z_k - x_k^{(N)}| < K + 1.$$

This upper bound, $K+1$, does not depend on k, so it is an upper bound for *all* $|z_k|$. Hence $\|z\|_\infty = \sup\{|z_k| : k \in \mathbb{N}\} \leq K + 1$, so $z \in \ell^\infty(\mathbb{R})$.

Step 3: Show the convergence. We want to prove that $\|x_n - z\|_\infty \to 0$ for $n \to \infty$.

Given $\epsilon > 0$, use the fact that $(x_n)_{n\in\mathbb{N}}$ is Cauchy to pick N so that if $m, n > N$ then

$$\|x_m - x_n\|_\infty < \epsilon.$$

Since $|x_k^{(m)} - x_k^{(n)}| \leq \|x_m - x_n\|_\infty$ for any $k \in \mathbb{N}$, this implies that

$$|x_k^{(m)} - x_k^{(n)}| < \epsilon \text{ for any } k \in \mathbb{N} \text{ and } m, n \geq N.$$

Taking limits as $m \to \infty$ we have, since $\lim_{m\to\infty} x_k^{(m)} = z_k$, that

$$|z_k - x_k^{(n)}| \leq \epsilon \text{ for any } k \in \mathbb{N} \text{ and } n \geq N.$$

Taking supremum in k, we obtain

$$\sup_k |z_k - x_k^{(n)}| \leq \epsilon \text{ for any } n \geq N,$$

i.e. $\|z - x_n\|_\infty \leq \epsilon$ for all $n > N$. But then we have actually shown that for any $\epsilon > 0$, we can find $N \in \mathbb{N}$ such that $\|z - x_n\|_\infty \leq \epsilon$ for all $n > N$ – which is exactly the definition of $x_n \to z$ in $\ell^\infty(\mathbb{R})$.

$\boxed{6}$ Let c_0 denote the space of real-valued sequences converging to zero.

a) Show that $(c_0, \|\cdot\|_\infty)$ is a subspace of $(\ell^\infty, \|\cdot\|_\infty)$.

b) Show that c_0 is closed in ℓ^∞, and conclude that $(c_0, \|\cdot\|_\infty)$ is complete.

Solution. a) To show that c_0 is a subspace, we need (by lemma 2.1) to show that $0 \in c_f$ and that c_f is closed under addition and scalar multiplication. Clearly $0 \in c_f$ – 0 is the sequence with only zeros, and this sequence clearly converges to zero. Then assume that $x, y \in c_0$, we need to show that $x + y \in c_0$. If $x = (x_n)_{n\in\mathbb{N}}$ and $y = (y_n)_{n\in\mathbb{N}}$, then $x + y = (x_n + y_n)_{n\in\mathbb{N}}$ (recall that we add sequences by adding the

[2] This is the definition of limit, with $\epsilon = 1$.

elements of each coordinate). By assumption $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = 0$, and it follows that

$$\lim_{n\to\infty}(x_n + y_n) = \lim_{n\to\infty} x_n + \lim_{n\to\infty} y_n = 0 + 0 = 0.$$

This shows that the sequence $x + y$ converges to 0, hence $x + y \in c_0$.
Similarly, if $x \in c_0$ and $\lambda \in \mathbb{R}$, we know that $\lambda x = (\lambda x_n)_{n\in\mathbb{N}}$. Since we assume that $\lim_{n\to\infty} x_n = 0$, we find that

$$\lim_{n\to\infty} \lambda x_n = \lambda \lim_{n\to\infty} x_n = \lambda 0 = 0,$$

hence $\lambda x \in c_0$.

b) To show that a subset of a metric space is closed, we need to show that it contains all its limit points (see the discussion after example 3.1.2). Hence we need to show that if $(x_n)_{n\in\mathbb{N}}$ is a sequence in c_0 such that $x_n \to z$ for some $z \in \ell^\infty(\mathbb{R})$, then $z \in c_0$. We use the notation from problem (5), so

$$x_n = (x_1^{(n)}, x_2^{(n)}, ...)$$

and

$$z = (z_1, z_2, \dots).$$

Given $\epsilon > 0$ we can find some $N \in \mathbb{N}$ such that

$$\|x_n - z\|_\infty < \frac{\epsilon}{2} \text{ for } n \geq N$$

since we assume $x_n \to z$ in $\ell^\infty(\mathbb{R})$. This means that for any $k \in \mathbb{N}$ we can use the triangle inequality to get

$$\begin{aligned} |z_k| &\leq |x_k^{(N)}| + |x_k^{(N)} - z_k| \\ &\leq |x_k^{(N)}| + \|x_N - z\|_\infty \\ &\leq |x_k^{(N)}| + \frac{\epsilon}{2}. \end{aligned}$$

Since $x_N \in c_0$, we know that $\lim_{k\to\infty} x_k^{(N)} = 0$, so we can find $M \in \mathbb{N}$ such that $|x_k^{(N)}| < \frac{\epsilon}{2}$ whenever $k \geq M$. But this means that for $k \geq M$ we have

$$\begin{aligned} |z_k| \leq |x_k^{(N)}| + \frac{\epsilon}{2} \\ \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon, \end{aligned}$$

which shows that $|z_k| \to 0$ as $k \to \infty$, hence $z \in c_0$. Thus c_0 is closed, and since it is a closed subspace of the complete space $\ell^\infty(\mathbb{R})$ it is complete by theorem 3.12.