TMA 4190 Introduction to Topology
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Lecture 11[1]

11. Transversality

Cut out submanifolds as zeros of functions

In order to prepare the following discussion of transversality, let us have another look at the conditions of when preimages are submanifolds.

Question

Suppose that $g_1, \dots, g_k$ are smooth, real-valued functions on a manifold X of dimension $n > k$ (each g_i is a smooth function $X \to \mathbb{R}$). Under what conditions is the set Z of common zeros a reasonable geometric object? In particular, when is Z a manifold?

We have seen an answer to this question. Collect the n functions to define the map

$$g = (g_1, \dots, g_k) \colon X \to \mathbb{R}^k.$$

Then we know that $Z = g^{-1}(0)$ is a submanifold of X **if 0 is a regular value of g**.

Remark

Historically, the study of zero sets of collections of functions has been of considerable mathematical interest. For, think of the zeroes as solutions to equations. Solving equations is a fundamental goal in mathematics (though not the only one!). In classical algebraic geometry, for example, one studies sets cut out in (complex) Euclidean space as the zero sets of polynomials (in several complex variables).

In order to make it easier to find an answer to our question, we would like to reformulate the regularity condition for 0 directly in terms of the functions g_i. Since each g_i is a smooth map of X into $\mathbb{R}$, its derivative at a point x is a linear map

$$d(g_i)x : T_x(X) \to \mathbb{R}.$$

[1] Following the books of Guillemin and Pollack: Differential Topology, and Milnor: Topology from the differentiable viewpoint.

We call such a map a **linear functional** on the vector space $T_x(X)$. The set

$$T_x(X)^* := \mathrm{Hom}_{\mathbb{R}}(T_x(X),\mathbb{R})$$

of all linear functionals on $T_x(X)$ is a vector space with pointwise addition and scalar multiplication.

The derivative of g

$$dg_x \colon T_x(X) \to \mathbb{R}^k$$

equals the k-tuple of the linear functionals $(d(g_1)_x, \dots, d(g_k)_x)$. For, each $(d(g_i)_x$ is a $(1 \times n)$-matrix which is the ith row of the matrix representing dg_x.

Hence, as a linear map to a k-dimensional vector space, we see that

$$\begin{aligned} & dg_x \text{ is surjective} \\ \iff & dg_x \text{ has full rank} \\ \iff & \text{the row vectors } d(g_1)_x, \dots, d(g_k)_x \text{ are linearly independent.} \end{aligned}$$

This is the same as to say that $d(g_1)_x, \dots, d(g_k)_x$ **are linearly independent** in the vector space $T_x(X)^*$ of linear functionals on $T_x(X)$. We are going to rephrase this condition by saying that the k functions $g_1, \dots, g_k$ **are independent at** x.

This yields another way of stating the Preimage Theorem:

Preimage Theorem revisited

If the smooth, real-valued functions $g_1, \dots, g_k$ on X are **independent at each point** x where they all vanish (i.e. $g_1(x) = \cdots = g_k(x) = 0$), then the set Z of **common zeros is a submanifold** of X with dimension equal to $\dim X - k$.

It is convenient here to define the **codimension** of an arbitrary submanifold Z of X by the formula

$$\operatorname{codim} Z = \dim X - \dim Z.$$

We can think of the codimension as a **measure of how much bigger** X **is compared to** Z. In particular, note that the codimension depends not only on Z, but also on the surrounding manifold X. Hence we should always speak of the **codimension of** Z **in** X. However, the number of functions we use to cut out a submanifold determines the codimension, independently of the size of X:

Cut out manfiolds

Thus k **independent functions** on X cut out a submanifold of **codimension k**.

Once again, a natural question arises:

Question

Can every submanifold Z of X be "cut out" by independent functions?

Answer

The answer is **no, in general**.

However, there are two useful partial converses:

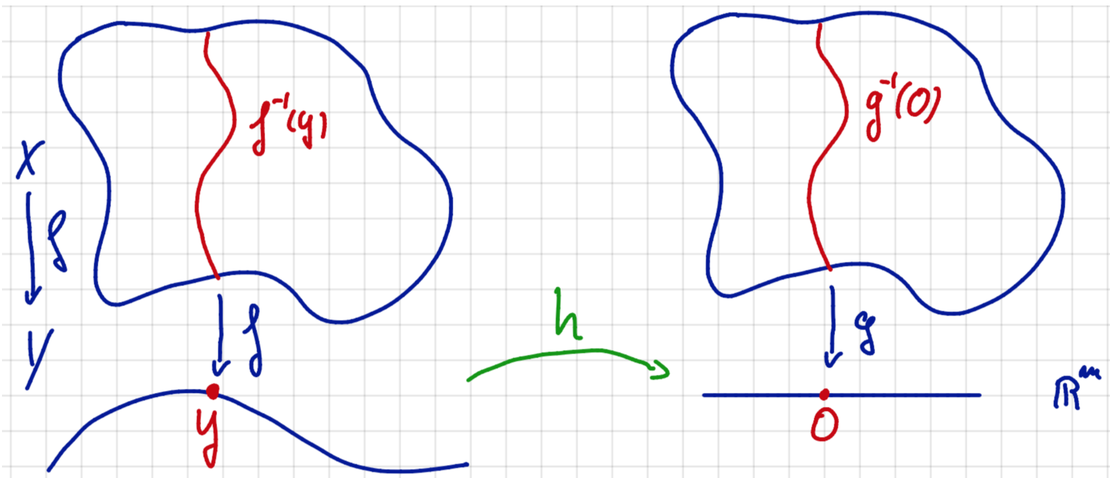
Cut out manfiolds: Partial Converse 1

If y is a regular value of a smooth map $f\colon X \to Y$, then the preimage submanifold $f^{-1}(y)$ can be cut out by independent functions.

Note that **the point** here is that we express $f^{-1}(y)$ as the set of **common zeros** for some function, not just as the preimage of some value in Y.

Proof: Assuming $\dim Y = m$, we just need to choose local coordinates around y, i.e. a diffeomorphism $h\colon W \to V$ with $W \subset Y$ and $V \subset \mathbb{R}^m$ open and $h(y) = 0$. Then we define the new map

$$g = h \circ f\colon f^{-1}(W) \to \mathbb{R}^m \text{ with } g^{-1}(0) = f^{-1}(h^{-1}(0)) = f^{-1}(y) \subseteq X.$$



The origin $0 \in \mathbb{R}^m$ is a regular value for g, for if $x \in g^{-1}(0)$ then

$$dg_x = dh_{f(x)} \circ df_x$$

is surjective, since $dh_{f(x)}$ is an isomorphism and df_x is surjective (x being a regular point for f). Hence every point in $g^{-1}(0)$ is regular, and 0 is a regular value for g. Thus the components $g_1, \ldots, g_m$ of g with $g_i \colon X \to \mathbb{R}$ are independent functions which cut out $f^{-1}(y)$. **QED**

Simple Example

In many cases, the result does not tell us too much new. It is just convenient to know that we can choose 0 as the regular value.
A simple example is given by defining S^n as $f^{-1}(0)$ of the map

$$g \colon \mathbb{R}^{n+1} \to \mathbb{R},\ x_1^2 + \cdots + x_{n+1}^2 \mathbf{-1}.$$

As we pointed out, it is not possible to write every submanifold as the zero set of some map. But locally we can!

Cut out manfiolds: Partial Converse 2

Every submanifold Z of X is **locally cut out by independent functions**. More specifically, let m be the codimension of Z in X, and let z be any point of Z. Then there exist m independent functions $g_1. \ldots, g_m$ defined on some open neighborhood W of z in X such that $Z \cap W$ is the common vanishing set of the g_i. In other words, $Z \cap W$ is cut out be independent functions in W.

Proof: This is just Exercise 5 of Exercise Set 3 applied to the immersion $Z \to W$. The idea is to use the Local Immersion Theorem and pick local coordinate functions $g_1, \dots, g_n$ ($n = \dim X$) defined on W such that $Z \cap W$ is the set of common zeros of the m functions $g_{n-m+1}, \dots, g_n$, i.e.

$$Z \cap W = \{x \in W : g_{n-m+1}(x) = 0, \dots, g_n(x) = 0\}.$$

QED

As a consequence we see that every manifold can be cut out locally by independent functions on Euclidean space (but not globally in general!)

Cut out manifolds by smooth conditions

Now we would like to understand what happens when we do not take the preimage of just a single point, but of a whole submanifold (not an arbitrary subset, since we need some control).

Given a smooth map $f\colon X \to Y$ between smooth manifolds. Assume that $Z \subseteq Y$ is a submanifold of Y. We would like to understand:

Question

Under which conditions is the subset $f^{-1}(Z) \subseteq X$ an interesting geometric object? In particular, when is $f^{-1}(Z)$ a manifold, and therefore a submanifold of X?

Note that $f^{-1}(Z)$ is the set of all $x \in X$ such that $f(x) \in Z$. In other words, it is the colllection of all the fibers $f^{-1}(z)$ for all $z \in Z$. This gives us a hint to how we can answer the question. We look at the points $z \in Z$ each at a time. This fits nicely into our general strategy: whether a space is a manifold or not is determined by the neighborhoods of points.

Strategy

More precisely, in order to check that $f^{-1}(Z)$ is a manifold, it suffices to check that for each point $x \in f^{-1}(Z)$ there is an open neighborhood $U \subset X$ of x in X such that $f^{-1}(Z) \cap U$ is a manifold. For then $f^{-1}(Z) \cap U$ **inherits the local coordinate functions** from U (by restricting them to the subset $f^{-1}(Z) \cap U$).

So let us pick a point $z \in Z$ and let $x \in X$ satisfy $f(x) = z$. We have just learned that we can write Z in a neighborhood $W \subseteq Y$ around z as the **zero set of independent functions** $g_1, \dots, g_k$, where k denotes the codimension of Z in Y. This means:

$$W \cap Z = \{w \in W : g_1(w) = \cdots = g_k(w) = 0\} \tag{1}$$

and $d(g_1)_w, \dots, d(g_k)_w$ are linearly independent in $T_w(Y)^*$ for all $w \in W \cap Z$.

We set $U := f^{-1}(W)$ which is an open neighborhood of x in X. Since taking preimages of sets commmutes with intersecting sets, we have

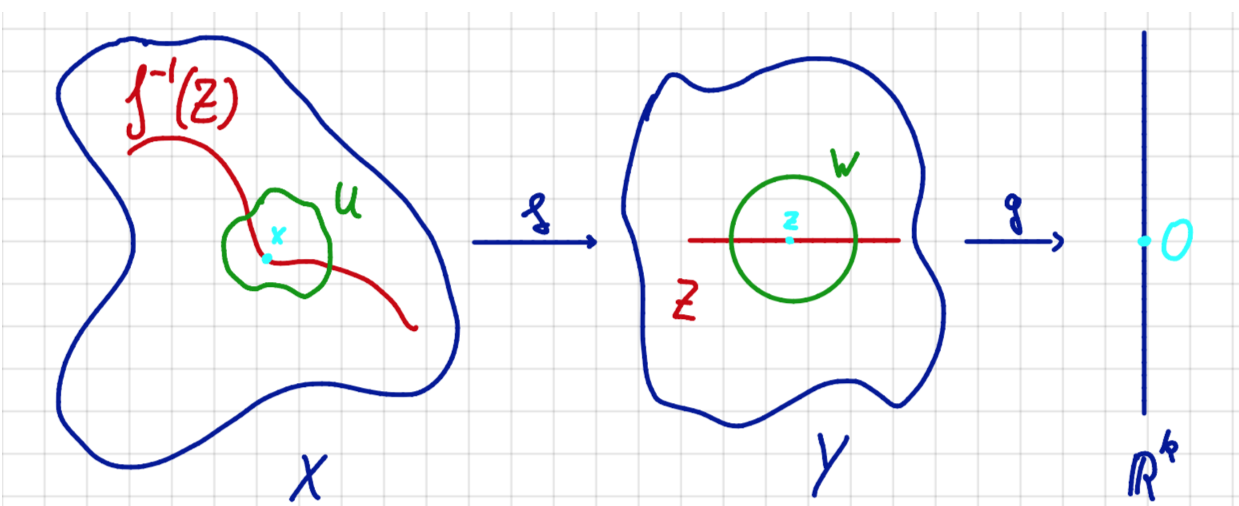
$$f^{-1}(W \cap Z) = f^{-1}(W) \cap f^{-1}(Z) = U \cap f^{-1}(Z).$$

Hence equation (1) implies that, near x, the preimage $f^{-1}(Z)$ **is the zero set of the functions** $g_1 \circ f, \dots, g_k \circ f$ in U:

$$U \cap f^{-1}(Z) = \{u \in U : (g_1 \circ f)(u) = \cdots = (g_k \circ f)(u) = 0\}.$$

Let $g \colon W \to \mathbb{R}^k$ denote the submersion $(g_1, \dots, g_k)$ defined around z. Then the Preimage Theorem applied to the composite smooth map $g \circ f \colon U \to \mathbb{R}^k$ gives us:

$$U \cap f^{-1}(Z) = (g \circ f)^{-1}(0) \text{ is a manifold } \textbf{if } \mathbf{0} \text{ is a regular value of } \mathbf{g \circ f}.$$



Hence in order to show that $f^{-1}(Z)$ is a manifold we need to understand when 0 is a regular value of $g \circ f$.

So what does it mean that 0 is a regular value of the composite $g \circ f$? The chain rule tells us

$$d(g \circ f)_x = dg_z \circ df_x.$$

Thus, the linear map

$$d(g\circ f)_x\colon T_x(X)\to\mathbb{R}^k \text{ is surjective}$$
$$\iff dg_z \text{ maps the image of } df_x \text{ onto } \mathbb{R}^k.$$

We know that $dg_z\colon T_z(Y)\to\mathbb{R}^k$, on the **whole** tangent space to Y at z, is a surjective linear map whose kernel is the subspace $T_z(Z)$. Thus dg_z induces an isomorphism

$$d\bar{g}_z\colon T_z(Y)/T_z(Z)\xrightarrow{\cong}\mathbb{R}^k.$$

In particular, $(dg_z)_{|\mathrm{Im}\,(df_x)}$ can only be **surjective if** $\mathrm{Im}\,(df_x)$ **and** $T_z(Z)$ **together span all of** $T_z(Y)$.

We conclude that $g\circ f$ **is a submersion** at $x\in f^{-1}(Z)$ **if and only if**

$$\mathrm{Im}\,(df_x)+T_z(Z)=T_z(Y).$$

We give this condition a name:

Transversality

Let $f\colon X\to Y$ be a smooth map and $Z\subseteq Y$ a submanifold. Then f is said to be **transversal to the submanifold** Z, denoted $f\pitchfork Z$, if

$$\mathrm{Im}\,(\mathbf{df_x})+\mathbf{T_{f(x)}(Z)}=\mathbf{T_{f(x)}(Y)}$$

at each point $x\in f^{-1}(Z)$ in the preimage of Z.

The above discussion then shows

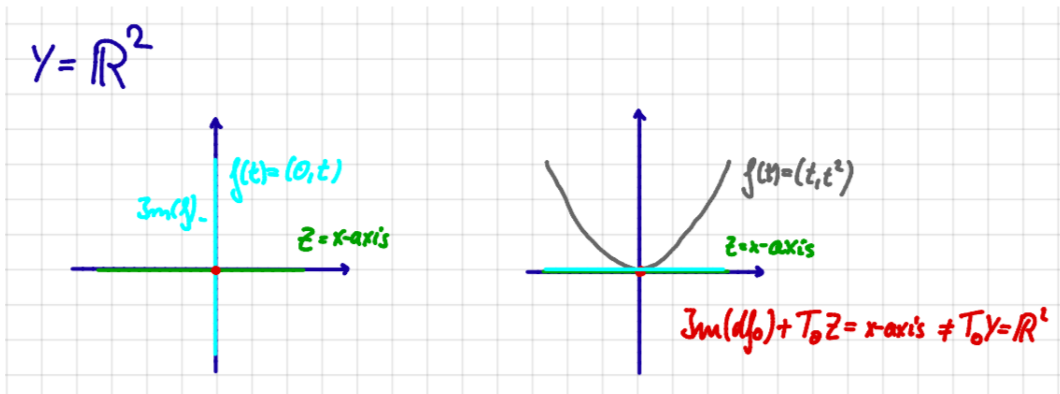
Transversality Theorem

If the smooth map $f\colon X\to Y$ is **transversal** to a submanifold $Z\subseteq Y$, then $f^{-1}(Z)$ **is a submanifold of** X. Moreover, the codimension of $f^{-1}(Z)$ in X equals the codimension of Z in Y.

Note that the number of independent functions $g_1,\dots,g_k$ we needed to locally write Z as a zero set in Y, is the same as the number of independent functions $g_1\circ f,\dots,g_k\circ f$ we needed to locally write $f^{-1}(Z)$ as a zero set in X. Therefore the **codimension of** $f^{-1}(Z)$ **in** X is **equal** the **codimension of** Z **in** Y.

For some very simple examples of transversality and non-transversality, consider $Y=\mathbb{R}^2$ with the submanifold Z being the x-axis. Then

- The map $f\colon \mathbb{R}^1 \to \mathbb{R}^2$ defined by $f(t) = (0,t)$ is transversal to Z.
- The map $f\colon \mathbb{R}^1 \to \mathbb{R}^2$ defined by $f(t) = (t,t^2)$, however, is **not** transversal to Z.



- The map $f\colon \mathbb{R}^1 \to \mathbb{R}^2$ defined by $f(t) = (t,t^2 - 1)$ is transversal to Z.
- The map $f\colon \mathbb{R}^1 \to \mathbb{R}^2$ defined by $f(t) = (t, \cos t - 1)$ is **not** transversal to Z.