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Exercise set 4: Solutions

1 Recall that $\mathcal{P}_4$ is the space of polynomials of degree at most 4. Show that the sets $U, V \subset \mathcal{P}_4$, defined by

$$U := \{p \in \mathcal{P}_4 : \; p(-1) = p(1) = 0\},$$
$$V := \{p \in \mathcal{P}_4 : \; p(1) = p(2) = p(3) = 0\}$$

are subspaces of $\mathcal{P}_4$ and determine the subspace $U \cap V$.

Solution We show that U is a subspace of $\mathcal{P}_4$. As we have seen in the lecture notes, this amounts to showing that U is a subset of $\mathcal{P}_4$, and that for any $\alpha, \beta \in \mathbb{R}$ and $p, q \in U$ we have

$$\alpha p + \beta q \in U.$$

From the definition of U, it is clear that U is a subset of $\mathcal{P}_4$ – it is defined as a set of elements in $\mathcal{P}_4$ satisfying certain conditions. Now let $\alpha, \beta \in \mathbb{R}$ and $p, q \in U$. By definition of U, we have that $p(-1) = p(1) = 0 = q(1) = q(-1)$. Hence

$$(\alpha p + \beta q)(1) = \alpha p(1) + \beta q(1) = 0 + 0 = 0,$$

and similarly $(\alpha p + \beta q)(-1) = 0$. This shows that $\alpha p + \beta q \in U$ for any $\alpha, \beta \in \mathbb{R}$ and any $p, q \in U$, hence U is a subspace.

The same kind of argument shows that V is a subspace.

Turning to $U \cap V$, we clearly have

$$U \cap V = \{p \in \mathcal{P}_4 : \; p(-1) = p(1) = p(2) = p(3) = 0\}.$$

This is the set of all real polynomials of degree at most 4 with exactly 4 roots: $-1, 1, 2, 3$.

Let $p_0 := (x + 1)(x - 1)(x - 2)(x - 3)$.

Then $U \cap V = \{\lambda \, p_0 \colon \lambda \in \mathbb{R}\}$.

2 Let $M_n(\mathbb{C})$ be the space of $n \times n$ matrices with complex entries. For $A \in M_n(\mathbb{C})$ we define its *trace* by $\operatorname{tr}(A) = a_{11} + \cdots + a_{nn}$.

a) Show that for $A, B \in M_3(\mathbb{C})$ we have $\mathrm{tr}(AB) = \mathrm{tr}(BA)$ and try to show this property of the trace for $n \times n$ matrices.

b) Let $S \subset M_n(\mathbb{C})$ be defined as the matrices with $\mathrm{tr}(A) = 0$. Show that S is a subspace of $M_n(\mathbb{C})$.

Solution.
a) We will do the general case – the 3×3-case can also be proved by writing A and B as matrices, multiplying them and calculating the traces of AB and BA. Let $A, B \in M_n(\mathbb{C})$ be $n \times n$-matrices with entries a_{ij} and b_ij, respectively. If we let $C = AB$, then we know (or can show) that the entries of C are given by

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}. \tag{1}$$

Similarly, if $D = BA$, then the entries of D are given by

$$d_{ij} = \sum_{k=1}^{n} b_{ik} a_{kj} = \sum_{k=1}^{n} a_{kj} b_{ik}. \tag{2}$$

The trace is the sum of the diagonal elements. Hence

$$\mathrm{tr}(AB) = \mathrm{tr}(C) = \sum_{i=1}^{n} c_{ii} = \sum_{i=1}^{n} \sum_{k=1}^{n} a_{ik} b_{ki} \tag{3}$$

and

$$\mathrm{tr}(BA) = \mathrm{tr}(D) = \sum_{i=1}^{n} d_{ii} = \sum_{i=1}^{n} \sum_{k=1}^{n} a_{ki} b_{ik}. \tag{4}$$

Clearly the sums in equations (3) and (4) are equal – they are the same sum except that the *names* i and k for the variables have been switched.
b) We need to show that if $\mathrm{tr}(A) = \mathrm{tr}(B) = 0$ and $\lambda \in \mathbb{C}$, then $\mathrm{tr}(\lambda A + B) = 0$. In fact, we have that the function $\mathrm{tr} : M_n(\mathbb{C}) \to \mathbb{C}$ is a linear transformation, meaning that

$$\mathrm{tr}(\lambda A + B) = \lambda \mathrm{tr}(A) + \mathrm{tr}(B),$$

so if $\mathrm{tr}(A) = \mathrm{tr}(B) = 0$, we must have $\mathrm{tr}(\lambda A + B) = 0$. The fact that tr is linear is rather obvious, but we can show it formally. The trace is the sum of the diagonal elements, so

$$\mathrm{tr}(\lambda A + B) = \sum_{i=1}^{n} \lambda a_{ii} + b_{ii} = \lambda \sum_{i=1}^{n} a_{ii} + \sum_{i=1}^{n} b_{ii} = \lambda \mathrm{tr}(A) + \mathrm{tr}(B).$$

3 **a)** Prove that $(l^\infty(\mathbb{R}), \|\cdot\|_\infty)$ is a normed space, where for any bounded sequence $x = (x_n) \in l^\infty(\mathbb{R})$ we define

$$\|x\|_\infty := \sup_{n \in \mathbb{N}} |x_n|.$$

Is this norm associated with an inner product?

b) Show that the norm $\|\cdot\|_p$ on $\ell^p(\mathbb{R})$ does not satisfy the parallelogram law

$$\|x-y\|_p^2 + \|x+y\|_p^2 = 2\|x\|_p^2 + 2\|y\|_p^2 \quad \text{for all} \quad x, y \in X,$$

for any $p \neq 2$.

Solution.
a) We verify the three axioms of the norm. Let $x = (x_n) \in l^\infty(\mathbb{R})$.

(i) Since $|a| \geq 0$ for all real numbers a,

$$\|x\|_\infty = \sup_{n\in\mathbb{N}} |x_n| \geq 0.$$

Moreover, if $\|x\|_\infty = \sup_{n\in\mathbb{N}} |x_n| = 0$, then $|x_n| = 0$ for all $n \in \mathbb{N}$, hence $x_n = 0$ for all $n \in \mathbb{N}$. This shows that $x = (x_n) = (0)$, so x is the null vector in $l^\infty(\mathbb{R})$.

(ii) Let $\lambda \in \mathbb{R}$. Then

$$\|\lambda\, x\|_\infty = \sup_{n\in\mathbb{N}} |\lambda\, x_n| = \sup_{n\in\mathbb{N}} |\lambda|\, |x_n| = |\lambda| \sup_{n\in\mathbb{N}} |x_n| = |\lambda|\, \|x\|_\infty.$$

Note that we used the following property of the supremum: if $A \subset \mathbb{R}$ and $c \geq 0$, then

$$\sup(cA) = c\sup(A).$$

(iii) Let $x = (x_n)$ and $y = (y_n) \in l^\infty(\mathbb{R})$. Since for any two real numbers a and b, $|a+b| \leq |a| + |b|$, we have

$$\|x+y\|_\infty = \sup_{n\in\mathbb{N}} |x_n + y_n| \leq \sup_{n\in\mathbb{N}} \left(|x_n| + |y_n|\right) \leq \sup_{n\in\mathbb{N}} |x_n| + \sup_{n\in\mathbb{N}} |y_n| = \|x\|_\infty + \|y\|_\infty.$$

Note that we used the following property of the supremum: if $f, g\colon \mathbb{N} \to \mathbb{R}$, then

$$\sup_{n\in\mathbb{N}} \left(f(n) + g(n)\right) \leq \sup_{n\in\mathbb{N}} f(n) + \sup_{n\in\mathbb{N}} g(n).$$

Finally, this norm is *not* associated with an inner product because it does not satisfy the parallelogram identity. Indeed, let us consider the sequences

$$\begin{aligned} x &= (x_n) \ \text{ where } x_n = 1 + \frac{1}{n} \ \text{ for all } n \geq 1, \\ y &= (y_n) \ \text{ where } y_n = 1 - \frac{1}{n} \ \text{ for all } n \geq 1, \ \text{ so} \end{aligned}$$

$$\begin{aligned} x_n + y_n &= 2 \qquad \text{for all } n \geq 1, \\ x_n - y_n &= \frac{2}{n} \qquad \text{for all } n \geq 1. \end{aligned}$$

Then clearly $\|x\|_\infty = 2$, $\|y\|_\infty = 1$, $\|x+y\|_\infty = 2$ and $\|x-y\|_\infty = 2$, so

$$\|x+y\|_\infty^2 + \|x-y\|_\infty^2 = 8 \neq 10 = 2\|x\|_\infty^2 + 2\|y\|_\infty^2.$$

b) We have already shown that the parallelogram identity does not hold in ℓ^∞, so we will focus on $p < \infty, p \neq 2$. In this case, an even simpler counterexample than the one for $p = \infty$ works. Let

$$\begin{aligned} x &= (1, 1, 0, 0, \dots) \\ y &= (1, -1, 0, 0, \dots). \end{aligned}$$

Then we obviously have that $x, y \in \ell^p$ for any $1 \leq p \leq \infty$, since both sequences have only two non-zero elements. Note that

$$\begin{aligned} x + y &= (2, 0, 0, \dots) \\ x - y &= (0, 2, 0, \dots). \end{aligned}$$

The ℓ^p norms of the vectors $x, y, x+y, x-y$ are easily calculated for $p < \infty$:

$$\begin{aligned} \|x\| &= (1^p + 1^p)^{1/p} = 2^{1/p} \\ \|y\| &= (1^p + 1^p)^{1/p} = 2^{1/p} \\ \|x+y\| &= (2^p + 0^p)^{1/p} = 2 \\ \|x-y\| &= (0^p + 2^p)^{1/p} = 2. \end{aligned}$$

So we have that $\|x+y\|^2 + \|x-y\|^2 = 2^2 + 2^2 = 8$, whereas $2\|x\|^2 + 2\|y\|^2 = 2 \cdot 2^{2/p} + 2 \cdot 2^{2/p} = 4 \cdot 2^{2/p}$. The parallelogram identity holds if

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2,$$

which in this case means that

$$8 = 4 \cdot 2^{2/p}.$$

This holds if and only if

$$2 = 2^{2/p},$$

which happens if and only if $p = 2$. Hence the parallelogram identity does *not* hold for $p \neq 2$.

$\boxed{4}$ Find a sequence $x = (x_1, x_2, \dots)$ of real numbers which converges to 0, but which is not in any space $\ell^p(\mathbb{R})$, $1 \leq p < \infty$.

Solution. We need to find a sequence $x = (x_1, x_2, \dots)$ that converges to zero, but the convergence must be slow enough that it does not belong to any ℓ^p-space for $p < \infty$. One way to approach this problem is to look for a sequence of the form $x_n = \frac{1}{f(n)}$ for some function f – what we then need is some function $f(t)$ that grows very slowly towards infinity as $t \to \infty$. One function that grows slowly as $t \to \infty$ is the natural logarithm $f(t) = \ln(t)$, so let us consider the sequence x defined by[1]

$$x_n = \frac{1}{\ln(n+1)}.$$

Since $\ln(n+1) \to \infty$ as $n \to \infty$, the sequence x converges to zero. We also need to check that the sequence belongs to no ℓ^p-space for $1 \leq p < \infty$, so consider the sum

$$\sum_{n=1}^{\infty} |x_n|^p = \sum_{n=1}^{\infty} \frac{1}{\ln(n+1)^p};$$

we wish to show that this sum does not converge. The prototype example of a series that does not converge is the harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n+1},$$

so let us compare our series to the harmonic series for large values of n:

$$\begin{aligned}
\lim_{n\to\infty} \frac{\frac{1}{\ln(n+1)^p}}{\frac{1}{n+1}} &= \lim_{n\to\infty} \frac{n+1}{\ln(n+1)^p} \\
&= \left(\lim_{n\to\infty} \frac{(n+1)^{1/p}}{\ln(n+1)} \right)^p \\
&= \left((1/p) \lim_{n\to\infty} \frac{(n+1)^{1/p-1}}{1/(n+1)} \right)^p \\
&= \left((1/p) \lim_{n\to\infty} (n+1)^{1/p} \right)^p = \infty.
\end{aligned}$$

We used L'Hopital's rule in this calculation. By the comparison test for series and the fact that $\sum_{n=1}^{\infty} \frac{1}{n+1}$ diverges, we conclude that $\sum_{n=1}^{\infty} \frac{1}{\ln(n+1)^p}$ also diverges.

5 Suppose $(X, \langle ., . \rangle)$ is an inner product space, and let $\| \cdot \| = \langle \cdot \cdot \rangle^{1/2}$.

a) Show that $\| \cdot \|$ satisfies the parallelogram law.

b) Let ω be a n^{th} root of unity, i.e. $\omega^n = 1$ and $\omega^k \neq 1$ for $k < n$. Show that for $n \geq 3$

$$\langle x, y \rangle = \frac{1}{n} \sum_{k=1}^{n} \omega^k \|x + \omega^k y\|^2.$$

c) Show that

$$\langle x, y \rangle = \int_0^1 e^{2\pi i \varphi} \|x + e^{2\pi i \varphi} y\|^2 d\varphi.$$

[1] We use $\ln(n+1)$ instead of $\ln(n)$ to avoid dividing by $\ln(1) = 0$.

Solution. a) We need to show that $\|x+y\|^2+\|x-y\|^2 = 2\|x\|^2+2\|y\|^2$ for any $x, y \in X$. This is just a calculation where we write out the norm in terms of the inner product and use the bilinearity of the inner product:

$$\begin{aligned}\|x+y\|^2+\|x-y\|^2 &= \langle x+y, x+y\rangle + \langle x-y, x-y\rangle \\ &= \langle x,x\rangle + \langle y,y\rangle + \langle x,y\rangle + \langle y,x\rangle + \langle x,x\rangle + \langle y,y\rangle - \langle x,y\rangle - \langle y,x\rangle \\ &= 2\langle x,x\rangle + 2\langle y,y\rangle = 2\|x\|^2 + 2\|y\|^2.\end{aligned}$$

b) We will write the right hand side using inner products. We have

$$\begin{aligned}\sum_{k=1}^{n-1} \omega^k \|x+\omega^k y\|^2 &= \sum_{k=1}^{n} \omega^k \langle x+\omega^k y, x+\omega^k y\rangle \\ &= \sum_{k=1}^{n} \omega^k \left(\langle x,x\rangle + \langle \omega^k y, \omega^k y\rangle + \omega^k \langle y,x\rangle + \omega^{-k}\langle x,y\rangle\right) \\ &= \|x\|^2 \sum_{k=1}^{n} \omega^k + \|y\|^2 \sum_{k=1}^{n} \omega^k + \langle y,x\rangle \sum_{k=1}^{n} \omega^{2k} + \sum_{k=1}^{n} \langle x,y\rangle.\end{aligned}$$

The result would follow if we could show that $\sum_{k=1}^n \omega^k = \sum_{k=1}^n \omega^{2k} = 0$ – in that case the first three summands disappear and the last summand is clearly $n\langle x,y\rangle$.

Hence we need to calculate $\sum_{k=1}^n \omega^k$, where ω is an n'th root of unity. This is a geometric sum, and we know that [2]

$$\sum_{k=1}^{n} \omega^k = \frac{1-\omega^{n+1}}{1-\omega} - 1 = \frac{1-\omega}{1-\omega} - 1 = 0.$$

The -1 appears to compensate for the fact that the usual formula for a geometric sum starts summation at $k=0$. Note that we have used $\omega^{n+1} = \omega$ since ω is an n'th root of unity. The same argument also gives that [3]

$$\sum_{k=1}^{n} \omega^{2k} = \sum_{k=1}^{n} (\omega^2)^k = \frac{1-(\omega^2)^{n+1}}{1-\omega^2} - 1 = \frac{1-\omega^2}{1-\omega^2} - 1 = 0,$$

where we use that $\omega^{2(n+1)} = \omega^2(\omega^n)^2 = \omega^2$ since $\omega^n = 1$.

c) As above we write the norm using inner products, and by using exactly the same kind of simplifications as above we obtain

$$\begin{aligned}\int_0^1 e^{2\pi i\varphi}\|x+e^{2\pi i\varphi}y\|^2 d\varphi &= \int_0^1 e^{2\pi i\varphi}\left(\|x\|^2 + e^{2\pi i\varphi}\langle y,x\rangle + e^{-2\pi i\varphi}\langle x,y\rangle + \|y\|^2\right) d\varphi \\ &= \|x\|^2 \int_0^1 e^{2\pi i\varphi} d\varphi + \|y\|^2 \int_0^1 e^{2\pi i\varphi} d\varphi + \langle y,x\rangle \int_0^1 e^{4\pi i\varphi} d\varphi + \langle x,y\rangle \int_0^1 d\varphi \\ &= \langle x,y\rangle.\end{aligned}$$

The last inequality follows from calculating these integrals, which is straightforward.

[2] Here we need $n \geq 2$, since otherwise we divide by $\omega - 1 = 1 - 1 = 0$.

[3] This is where we need $n \geq 3$. If $n = 2$, we divide by $1 - \omega^2 = 0$ below.

$\boxed{6}$ Let $(\mathbb{R}^n, \|.\|_p)$ be the space of real n-tuples with p-norm $\|x\|_p = (\sum_{i=1}^n |x_i|^p)^{1/p}$ for some $1 \leq p < \infty$. Show that

$$\sum_{i=1}^n |x_i| \leq n^{(p-1)/p} (\sum_{i=1}^n |x_i|^p)^{1/p}.$$

Solution. This is an example of Hölder's inequality. Note that $\frac{1}{p} + \frac{p-1}{p} = 1$ – in the terminology of the lecture notes we have that $p/(p-1)$ is the conjugate exponent of p. Let x be the n-tuple $(x_1, x_2, ...x_n)$ and let $y = (1, 1, ..., 1)$. Hölder's inequality states that

$$\begin{aligned}\sum_{i=1}^n |x_i||y_i| &\leq \left(\sum_{i=1}^n |x_i|^p\right)^{1/p} \left(\sum_{i=1}^n 1^q\right)^{1/q} \\ &= \left(\sum_{i=1}^n |x_i|^p\right)^{1/p} n^{1/q},\end{aligned}$$

where q is the conjugate exponent of p. If we now insert that the conjugate exponent of p is $p/(p-1)$, we obtain the desired inequality.