In fact, we can say a little more, though we shall not seek to justify this:

Theorem 16.4. *Under the hypotheses of the previous theorem, the magnitude of the real parts of the characteristic exponents of the T-periodic orbit Γ are all lower-bounded by $\alpha > 0$. There exists a $K > 0$ such that for each $\mathbf{x} \in M_s(\Gamma)$, there exists an asymptotic phase t_0 such that for all $t \geq 0$,*

$$|\phi_t(\mathbf{x}) - \gamma(t - t_0)| \leq Ke^{-\alpha t/T},$$

and for each $\mathbf{x} \in M_u(\Gamma)$, there exists an asymptotic phase t_0 such that for all $t \geq 0$,

$$|\phi_t(\mathbf{x}) - \gamma(t - t_0)| \leq Ke^{\alpha t/T}.$$

That is, not only do orbits approach a limit cycle Γ, they also become phase-locked with Γ. In a way this is not too surprising, because the flux f governing the dynamics is continuous.

As with the stable manifold theorem for critical points, there is an associated (weak) centre manifold theorem, asserting the existence of a centre manifold of dimension equal to one less than the number of characteristic exponents with zero real parts. But we should like to revisit the condition of Thm. 16.1,which chararcterised stability with a calculable quantity. This can be generalized to higher dimsions for reasons we have already touched upon in our deductions leading up to the stable manifold theorem for periodic orbits.

Theorem 16.5. *Let γ be a T-periodic orbit of a C^1-first order autonomous system with flux f. A necessary but not generally sufficient condition for the orbit γ to be asymptotically stable is that*

$$\int_0^T (\nabla \cdot f)(\gamma(t))\ \mathrm{d}t \leq 0.$$

The non-sufficiency comes from the fact that we need the eigenvalues of $\int_0^T \mathrm{D}f(\gamma(t))\ \mathrm{d}t$ all to be negative except the 0 eigenvalue arising from the fact that the Poincaré map maps onto a codimension one surface, whereas the condition stated in the theorem is merely the trace of this quantity. This would have been enough in dimension $d = 2$, where only one eigenvalue is unspecified apart from the 0.

17. Lecture XVII: Poincaré-Bendixson Theorem

Let us start off with a lemma on general ω-limit sets:

Lemma 17.1. *Let $\omega(\Gamma)$ be a bounded limit set. Then it is connected.*

We say that a set X is NOT CONNECTED if there are disjoint open sets U and V each of which intersects X, and $X \subseteq U \cup V$. A set is CONNECTED if it is not disconnected.

Proof. Let $\Gamma = \Gamma_{\mathbf{x}_0}$ and $\mathbf{x}(0) = \mathbf{x}_0$.

Suppose $\omega(\Gamma)$ is not connected. Then it is the union of two disjoint closed sets A and B because by Thm.15.2, $\omega(\Gamma)$ is closed. Since A and B are closed, there is a positive distance between them:

$$d(A,B) = \inf_{\mathbf{x}\in A, \mathbf{y}\in B} |\mathbf{x}-\mathbf{y}|_{\mathbb{R}^d} > \delta > 0.$$

Since A and B are in $\omega(\Gamma)$, there are aritrarily large times for which $d(\mathbf{x}(t), A) < \delta/2$, and also arbitrarily large times for which $d(\mathbf{x}(t), A) > \delta/2$. Since $t \mapsto \mathbf{x}(t)$ is continuous, so is $t \mapsto d(\mathbf{x}(t), A)$. Therefore by the intermediate value theorem, there is a sequence of times t_n for which $d(\mathbf{x}(t_n), A) \to \delta/2$. Since A and B are bounded, so is the set of points $\{\mathbf{x}(t_n)\}$. Therefore there is a subsequence such that $\mathbf{x}(t_{n_k})$ converges to a point $\mathbf{z}$, with $d(\mathbf{z}, A) = \delta/2$.

But by the triangle inequality, $\mathbf{z}$ is neither in A nor in B — a contradiction. $\square$

Cycles of finite period are always their own α- and ω- limit sets. Last time we considered the stability of cycles with finite period T. This also covered any particular cycle in an attractor with cycles of arbitrarily large periods. This time we cast our minds back to a brief discussion on separatrices, homoclines, and heteroclines, to discuss cycles that contain critical points. These are topologically, cycles, and they are often rectifiable — they are simple closed curves — but they have infinite period.

We shall call a SEPARATRIX CYCLE of a dynamical system $\dot{\mathbf{x}} = f(\mathbf{x})$ a continuous image of a circle which is a finite union of critical points and compatibly oriented separatrices connecting them. That is, a union of points $\mathbf{x}_i$ and separatrices Γ_i such that $\alpha(\Gamma_i) = \mathbf{x}_i$, and $\omega(\Gamma_i) = \mathbf{x}_{i+1}$, for $i = 1, \ldots, m$, and $\mathbf{x}_{m+1} = \mathbf{x}_1$. A GRAPHIC is a finite union of compatibly oriented separatrix cycles.

In this lecture we return to the plane and our goal here is singular: the proof of the theorem following.

Theorem 17.2 (Poincaré-Bendixson Theorem)**.** *Suppose the forward orbit Γ^+ of a trajectory Γ of a C^1-planar system is contained in a compact subset $F \subset \mathbb{R}^2$. Either $\omega(\Gamma)$ is a periodic orbit or it contains a critical point.*

It also happens that for analytic systems we can be slightly more precise, as we have seen is often the case, and the alternative provided for by the theorem for which $\omega(\Gamma)$ contains a critical point can be refined further to say that $\omega(\Gamma)$ *is* a critical point, or $\omega(\Gamma)$ is a graphic.

In order to prove the theorem we shall be needing auxiliary results. We say that a closed, (straight) line segment ℓ is a TRANSVERSAL to a system if it is transverse as a manifold to any trajectory that intersects it. In particular, it does not contain a critical point. Recall that transversality means that if the trajectory Γ and the line segment ℓ intersect at an interior point $\mathbf{x}_0$, then $T_{\mathbf{x}_0}\Gamma \otimes T_{\mathbf{x}_0}\ell = \mathbb{R}^2$. In simpler language, Γ and ℓ cannot be tangent at $\mathbf{x}_0$. Since the derivative of Γ at $\mathbf{x}_0$ is $f(\mathbf{x}_0)$, this also means that the derivative of ℓ at $\mathbf{x}_0$ is not some non-zero scalar multiple of $f(\mathbf{x}_0)$.

We say that $\mathbf{x}_0$ is a regular point of the system unless it is a critical point. Unless $\mathbf{x}_0$ is a critical point, Γ continues beyond $\mathbf{x}_0$, and therefore Γ crosses ℓ at any interior point $\mathbf{x}_0$ that is a REGULAR POINT. Our first lemma relaxes this observation slightly:

Lemma 17.3. *Let $\mathbf{y}_0$ be an interior point of a transversal ℓ and regular point of a C^1-autonomous first order system. Then for all $\varepsilon > 0$, there exists a $\delta > 0$ such that every trajectory passing through a point in $B_\delta(\mathbf{y}_0)$ crosses ℓ at some time $|t| < \varepsilon$.*

That is, if ℓ is transverse to the system at a regular point $\mathbf{y}_0$, then it is transverse to the system along a small neighbourhood $B_\delta(\mathbf{y}_0) \cap \ell$ of points.

Proof. Since limit sets are cloesd by Thm. 15.2, we know that in a small enough neighbourhood $B_\delta(\mathbf{y}_0)$ of a regular point, all points are regular. By assumption, ℓ is a transveral, so it is transverse to any trajectory that intersects it, and we only have to prove that any trajectory passing through a small enough neighbourhood intersects ℓ.

By the continuity of f and boundedness of ∇f, and the regularity of $\mathbf{y}_0$, we know the trajectory $\Gamma_{\mathbf{y}}$ through any point $\mathbf{y} \in B_\delta(\mathbf{y}_0) \cap \ell$ must be $f(\mathbf{y}) = f(\mathbf{y}_0) + \nabla f(\mathbf{z}) \cdot (\mathbf{y} - \mathbf{y}_0) \approx f(\mathbf{y}_0)$ for $\mathbf{z} \in B_\delta(\mathbf{y}_0)$ in a small neighbourhood of the non-zero vector $f(\mathbf{y}_0)$. By the continuity of these trajectories, we can find a small enough neighbourhood such that every trajectory passing through that neighbourhood intersects ℓ.

□

Lemma 17.4. *Let $\mathbf{x}_0$ be a point on a positively invariant set, and let ℓ be a transversal. Then $\Gamma^+(\mathbf{x}_0)$ intersects ℓ in a monotone sequence (that is, if $\mathbf{x}_i$ is the ith intersection, then $\mathbf{x}_i$ lies between $\mathbf{x}_{i-1}$ and $\mathbf{x}_{i+1}$ on ℓ) at only finitely many points.*

Proof. Let $\{t_i\}$ be the increasing sequence of time for which $\phi_{t_i}(\mathbf{x}_0) = \mathbf{x}_i$.

By the previous theorem, the region enclosed by the Jordan curve formed by the segment of ℓ between $\mathbf{x}_{i-1}$ and $\mathbf{x}_i$, and the arc $\{\phi_t(\mathbf{x}_0) : t_{i-1} \leq t \leq t_i\}$ is positively invariant. Therefore $\mathbf{x}_{i+1}$ must be in this region, and hence $\mathbf{x}_{i+1}$ cannot be on the segment of ℓ between $\mathbf{x}_{i-1}$ and $\mathbf{x}_i$.

Since ℓ is closed and bounded, if $\{\mathbf{x}_i\}$ are an infinite sequence, then $\mathbf{x}_i$ converges on ℓ, and this limit must be a limit point, which is excluded by construction.

□

Another way to see this is that if $\mathbf{x}_{i+1}$ were between $\mathbf{x}_{i-1}$ and $\mathbf{x}_i$, then ℓ cannot be a transversal — it must either be tangent to a trajectory at some point between $\mathbf{x}_{i-1}$ and $\mathbf{x}_i$, or contain a critical point there, by Rolle's theorem/the intermediate value theorem.

Lemma 17.5. *Let ℓ be a transversal to the system for which $\omega(\Gamma)$ is a limit set. Then $\omega(\Gamma)$ intersects ℓ at at most one point.*

Proof. Let Γ be the image of the solution curve $t \mapsto \mathbf{x}(t)$ over $\mathbb{R}$.

Suppose $\omega(\Gamma)$ intersects ℓ at two points, $\mathbf{y}_1$ and $\mathbf{y}_2$. Then there is a sequence of times $\{t_n\}$ and a sequence of time $\{s_n\}$ each tending to infinity for which $\mathbf{x}(t_n) \to \mathbf{y}_1$ and $\mathbf{x}(s_n) \to \mathbf{y}_2$. Since any neighbourhood of $\mathbf{y}_i$ intersects ℓ, and each time sequence tends to infinity, by the Lemma 17.3 we can choose $\mathbf{x}(t_n)$ and $\mathbf{x}(s_n)$ to lie on ℓ. However, this violates the monotonicity established in the lemma immediately foregoing.

□

Lemma 17.6. *If Γ and $\omega(\Gamma)$ share a point, then Γ is either a critical point or a periodic orbit.*

Proof. Suppose $\mathbf{x}_0 \in \Gamma \cap \omega(\Gamma)$. Then there is a solution $\mathbf{x}(t)$ for which $\Gamma = \{\mathbf{x}(t) : t \in \mathbb{R}\}$ and a sequence of times t_n with $n = 1, 2, \ldots$, and $t_n \to \infty$ such that $\mathbf{x}_n := \mathbf{x}(t_n) \in \Gamma$ and $\mathbf{x}_n \to \mathbf{x}_0$. As $\mathbf{x}_0 \in \Gamma$, there is a (finite) $T \in \mathbb{R}$ such that $\mathbf{x}_0 = \mathbf{x}(T)$.

Since $|\mathbf{x}(t_n) - \mathbf{x}(T)| \to 0$, by continuity, we know that there are points t'_n such that $\mathbf{x}(t'_n) = x(T)$ for sufficiently large n, and $|t_n - t'_n| \to 0$. This is the intermediate value theorem. This implies that $t \mapsto \mathbf{x}(t)$ is periodic. A critical point is a periodic orbit with period zero. □

Lemma 17.7. *If $\omega(\Gamma)$ contains no critical point but contains a periodic orbit Γ_0, then $\omega(\Gamma) = \Gamma_0$.*

Proof. Suppose $\Gamma_0 \subset \omega(\Gamma)$. Since $\omega(\Gamma)$ is connected, $A = \omega(\Gamma) \backslash \Gamma_0$ is not closed, and Γ_0 contains some limit point $\mathbf{y}_0$ of A. That means that every neighbourhood $B_\delta(\mathbf{y}_0)$ intersects A.

Let ℓ be a transversal of the system through $\mathbf{y}_0$. For a small enough neighbourhood $B_\delta(\mathbf{y}_0)$, every trajectory passing through $B_\delta(\mathbf{y}_0)$ also crosses ℓ. Suppose $\mathbf{y} \in B_\delta(\mathbf{y}_0) \cap A$ and $\Gamma_{\mathbf{y}}$ crosses ℓ at $\mathbf{y}_1$.

Since $\Gamma_{\mathbf{y}} \subseteq A \subseteq \omega(\Gamma)$, it must that $\Gamma_{\mathbf{y}}$ is a limit orbit distinct from Γ_0. Therefore $\mathbf{y}_0 \neq \mathbf{y}_1$ and $\omega(\Gamma)$ crosses a transversal at two distinct points — a contradiction.

□

Proof of Thm.17.2. Suppose $\omega(\Gamma)$ does not contain a critical point.

If Γ is a periodic orbit, then $\Gamma \subseteq \omega(\Gamma)$, then in fact $\Gamma = \omega(\Gamma)$.

If Γ is not a periodic orbit, and $\omega(\Gamma)$ only contains regular points, then choosing $\mathbf{y} \in \omega(\Gamma)$ and flowing it shows that there is a limit orbit $\Gamma_0 \subseteq \omega(\Gamma)$. We shall show that Γ_0 is periodic, from which we shall conclude that $\omega(\Gamma) = \Gamma_0$.

Since Γ^+ is contained in a compact set F, possibly by enlarging F slightly, Γ_0 is also contained with a compact set.

Since $\omega(\Gamma)$ is closed, Γ_0 has an ω-limit point in $\omega(\Gamma)$. Let ℓ be a transversal through $\mathbf{y}_0$ (i.e., $\mathbf{y}_0$ is an interior point of ℓ). By Lemma 17.3, Γ_0 must intersect ℓ at some point. Since there are finitely many intersections by lemma 17.4, and $\mathbf{y}_0$ is a limit point, it must be that Γ_0 intersects ℓ at $\mathbf{y}_0$.

Since $\mathbf{y}_0$ is an ω-limit point of Γ_0, it holds that $\mathbf{y}_0 \in \Gamma_0 \cap \omega(\Gamma_0)$, and by Lemma 17.6 and assumptions on $\omega(\Gamma)$, Γ_0 is periodic.

Finally, by Lemma 17.7, $\Gamma_0 \subseteq \omega(\Gamma)$ implies that $\omega(\Gamma) = \Gamma_0$.

□