Stochastic Modelling

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2020

Contents

1 Lecture 1

1.1 Practical Information

Two projects

- The projects count 20% and exam 80%.
- Must be done with two people.
- If you want to do statistics is it worth learning R.

Course Overview

- Markov chains for discret time and discrete outcome.
  - Set of states and discrete time points.
  - Transition between states
  - Future depends on the present, but not the past.
- Continious time Markoc chains. (continious time and discrete toutcome.
- Brownian motion and Gaussian processes (continionus time and continious outcome.)

1.2 Mathematical description

Definition 1.1. *A* ***stochastic process*** $\{x(t), t \in T\}$ *is a family of random variables, where* T *is a set of indicies, and* $X(t)$ *is a random variable for each value of* t.

1.3 Recall from Statistics Course

A random experiment is perfomed the outcome of the experiment is random.

- THe set of possible outcomes is the **sample space** ω
  - An **event** $A \subset \omega$ if the outcome is contained in A
  - The **complement** of an event A is $A^c = \omega \setminus A$
  - The **null event** $\emptyset$ is the empty set $\emptyset = \omega \setminus \omega$

1.3.1 Combining Event

Let A and B be events

- The **union** $A \cup B$ is the event that at least one of A and B occur.
- the **intersection** $A \cap B$ is the event that both A and B occur.

The events $A_1, A_2, \ldots$ are called disjoint (or **mutually exclusive**) if $A_i \cap A_j = \emptyset$ for $i \neq j$

1.3.2 Probability

Pr is called a probability on ω if

- $\Pr\{\omega\} = 1$
- $0 \leq P\{A\} \leq 1$ for all events A
- For $A_1, A_2, \ldots$ that are mutually exclusive

$$P\left\{\bigcup_{i=1}^{\infty} A_i\right\} = \sum_{i=1}^{\infty} P\{A_i\}$$

We call $P\{A\}$ the probability of A.

1.3.3 Law of total probability

Let $A_1, A_2, \ldots$ be a partition of ω ie

- $\omega = \bigcup_{i=1}^{\infty} A_i$
- $A_1, A_2, A_3, \ldots$ are mutually exclusive.

Then for any event B

$$P\{B\} = \sum_{i=1}^{\infty} P\{B \cap A_i\}$$

This concept is very important.

1.3.4 Independence

Event A and B are independent of

$$P\{A \cap B\} = P\{A\} P\{B\}$$

Events $A_1, \ldots, A_n$ are independent if for any subset

$$P\left\{\bigcap_{j=1}^{k} A_{i_j}\right\} = \prod_{j=1}^{k} P\{A_{i_j}\}$$

In this case $P\{\bigcap_{i=1}^{n} A_1\} = \prod_{i=1}^{n} P\{A_i\}$

1.3.5 Random Variables

Definition 1.2. *A **random variable** is a real-vaued function on the sample space. Informally: A random variable is a real valued variable that takes on its value by chance.*

Example.

- Throw two dice. $X =$ sum of the two dice
- Throw a coin. X is 1 for heads and X is 0 for tails.

1.3.6 Notation for random variables

We use

- upper case letters such at X, Y and Z to represent random variables.
- lower case letters as x, y, z to denote the real-valued realized value of a the random variable.

Expression such as $\{X \leq x\}$ denators the event that X assumes a valye less than or earl to the real number x.

1.3.7 Discrete random variables

The random variable X is **discrete** if it has a finite or countablle number of possible outcomes $x_1, x_2, \ldots$

- The **probability mass function** $p_x(x)$ is given by

  $$p_x(x) = P\{X = x\}$$

  and satisfies

  $$\sum_{i=1}^{\infty} p_x(x_i) = 1 \quad \text{and} \quad 0 \leq p_x(x_i) \leq 1$$

- The **cumulative distribution function** (CDF) a of X can be written

  $$F_x(x) = P\{X \leq x\} = \sum_{i:x_i \leq x} p_x(x_i)$$

1.3.8 CFD

The CDF of X may also be called the **distrobution function** of X

Let $F_x(x)$ be the CDF of X, then

- $F_x(x)$ is monetonaly increasing.
- F_x is a stepfunction, which is a pieace-wise constant with jumps at x_i.
- $\lim_{x\to\infty} F_x(x) = 1$
- $\lim_{x\to-\infty} F_x(x) = 0$

1.3.9 Continious random vairbales

A **continious** random variables takes value o a continious scale.

- The CDF, $F_x(x) = P(X \leq x)$ is continious.
- The **probability density function** (PDF) $f_x(x) = F_x'(x)$ can be used to calculate probablities

$$\begin{aligned} Pr\{a < X < b\} &= Pr\{a \leq X < b\} = Pr\{a < X \leq b\} \\ &= Pr\{a \leq X \leq b\} = \int_a^b f_x(x)\,dx \end{aligned}$$

1.3.10 Important properties

- CDF:
  - Monotonely increaing
  - continious
  - $\lim_{x\to\infty} F_x = 1$ and $\lim_{x\to-\infty} F_x(x) = 0$
- PDF
  - $f_x(x) \geq 0$ for $x \in \mathbb{R}$
  - $\int_{-\infty}^{\infty} f_x(x)\,dx = 1$

1.3.11 Expectation

Let $g : \mathbb{R} \to \mathbb{R}$ be a function and X be a random variable.

- If X is discrete, the expected value of $g(X)$ is

$$E[g(X)] = \sum_{x:p_x(x)>0} g(x)\,p_x(x)$$

- If X is continous, the expected value of $g(X)$ is

$$E[g(X)] = \int_{-\infty}^{\infty} g(x)\,f_x(x)\,dx$$

1.3.12 Variance

The variance of the random variable X is

$$Var[X] = E\left[(X - E[X])^2\right] = E\left[X^2\right] - E[X]^2$$

Important properties of expectation and variance.

- Expectations is linear

$$E[aX + bY + c] = aE[X] + bE[Y] + c.$$

- Variance scales quadratically and is invaraient to the addition of constants

$$Var[aX + b] = a^2 Var[X]$$

- fir independent stochastic variables.

$$Var[X + Y] = Var[X] + Var[Y]$$

1.3.13 Joint CDF

If (X, Y) is a pair for random variables, their **joint comulative distribution function** is given by

$$F_{X,Y} = F(x, y) = Pr\{X \leq x \cap Y \leq y\}$$

.

1.3.14 Joint distrubution for discrete random variables

If X and Y are discrete, the **joint probability mass function** $p_{x,y} = Pr\{X = x, Y = y\}$. can be used to compute probabilities

$$Pr\{a < X < b, c < Y \leq d\} = \sum_{a<x\leq b} \sum_{c<y\leq d} p_{X,Y}(x, y)$$

1.3.15 Joint distrubution for continous random variables

If X and Y are continious the **joint probability density function**

$$.f_{X,Y}(x, y) = f(x, y) = \frac{\partial^2}{\partial x \partial y} F(x, y)$$

can be used to compute probabilities

$$Pr\{a < X \leq b, \quad c < Y \leq d\} = \int_a^b \int_c^d f(x, y)\, dx dy$$

1.3.16 Independence

The random variables X and Y are independent if

$$Pr\left\{X \leq a, Y \leq b\right\} = Pr\left\{X \leq a\right\} \cdot Pr\left\{Y \leq b\right\}, \quad \forall a, b \in \mathbb{R}$$

In terms of CDFs: $F_{X,Y}(a,b) = F_X\left(a\right) \cdot F_Y\left(b\right) \quad \forall a, b \in \mathbb{R}$
Thus we have

- $p_{X,Y}\left(x,y\right) = p_X\left(x\right) \cdot p_Y\left(Y\right)$ for discrete random variables
- $f_{X,Y}\left(x,y\right) = f_X\left(x\right) \cdot f_Y\left(Y\right)$ for continuous random variables.

2 Lecture 3

2.1 Randoms sum

Building on the hunter example from last week. we can more generally consider random sums

$$X = \begin{cases} 0, & N = 0 \\ \zeta_1 + \zeta_2 + \ldots + \zeta_N, & N > 0 \end{cases}$$

where

- N is a discrete random variable with values $0, 1, \ldots$
- $\zeta_1, \zeta_2, \ldots$ are independent random variables
- N is independent of $\zeta_1, \zeta_2 + \ldots + \zeta_N$
- **Notation** $X = \sum_{i=1}^{N} \zeta_i = \zeta_1 + \zeta_2 + \ldots + \zeta_N$

Example.

1. Insurance company

$$N: \text{ Number of claims.}$$

$$\zeta_1, \zeta_2, \ldots \quad : \quad \text{Sizes of the claims}$$

Total liabilility:

$$X = \zeta_1 + \zeta_2 + \ldots + \zeta_N$$

2. Be careful!

$$\begin{aligned} \overbrace{E\left[\sum_{i=1}^{N} \zeta_i\right]}^{\neq \sum_{i=1}^{N} E[\zeta_i]} &= E\left[E\left[\sum_{i=1}^{N} \zeta_i \mid N\right]\right] \\ &= E\left[\sum_{i=1}^{N} E\left[\zeta_i \mid N\right]\right] \end{aligned}$$

2.2 Self Study

Section 2.2, 2.3, 2.4

2.3 Stochastic process in descrete time

Definition 2.1. *A* ***discrete-time stochastic process*** *is a family of random variables* $[X_t : t \in T]$ *where* T *is discrete.*

- *We use* $T = \{0, 1, 2, ..\}$ *and write* X_n *instead of* X_t
- *we call* X_n *the* ***state*** *at time* $n = 0, 1, 2, 3, \ldots$
- *We call the set of all possible states the* ***state space***

Table 1: Table for example

Day	$n = 0$	$n = 1$	$n = 2$	$\ldots$
Random Variable	X_0	X_1	X_2	$\ldots$
Realization 1	$x_0 = 0$	$x_1 = 1$	$x_2 = 1$	$\ldots$
Realization 2	$x_0 = 1$	$x_1 = 1$	$x_2 = 1$	$\ldots$

Example.

$$X_n = \begin{cases} 1, & \text{if it rains on day } n \\ 0, & \text{no rain on day } n \end{cases}$$

State space $= \{0, 1\}$
We have a problem. Need

$$Pr\left\{X_n = x_n \mid X_{n-1} = x_n, X_{n-2} = x_{n-2}, \ldots, X_0 = x_0\right\}.$$

for all $n = 0, 1, 2, \ldots$

2.4 Markov chain

Definition 2.2 (Discrete time Markov Chain)**.** *A* ***Discrete time markoc chain*** *is a discrete time stochastic process* $\{X_n : n = 0, 1, \ldots\}$ *that statisfied the* ***markov property*** *such that*

$$\begin{aligned} &Pr\left\{X_{n-1} = j \mid X_n = i, X_{n-1} = i_{n-1}, \ldots, X_0 = i_0\right\} \\ &= Pr\left\{X_{n+1} = j \mid X_n = i\right\} \end{aligned}$$

for $n = 0, 1, 2, 3, \ldots$ *and for all states* i *and* j

Definition 2.3 (One-step transition probabilities)**.** *We can define it as*

- *For a discrete Markov chain* $\{X_n : n = 0, 1, 2, \ldots\}$ *we call* $P_{ij}^{n,n+1} = Pr\left\{X_{n+1} = j, X_n = i\right\}$ *the* ***one step trainsition probabilities*** *.*

- *We will assume* ***stationary transition probabilities*** *, i.e that*

$$P_{ij}^{n,n+1} = P_{ij}$$

for $n = 0, 1, 2, \ldots$ *and all states* i *and* j *.*

Some of the properties

1. "You will always go somewhere"

$$\sum_j P_{ij} = 1 \quad \forall i$$

2. The markov chain can be described as follows.

$$\begin{aligned}
&Pr\left\{X_0 = i_0, X_1 = i_1, \ldots, X_n = i_n\right\} \\
&= Pr\left\{X_0 = i_0\right\} Pr\left\{X_1 = i_1 \mid X_0 = i_0\right\} \ldots \\
&\qquad Pr\left\{X_n = x_n \mid X_{n-1} = i_{n-1} \ldots X_0 = i_0\right\} \\
&\vdots \;\; \text{Markov step} \\
&= Pr\left\{X_0 = i_0\right\} \cdot Pr\left\{X_1 = i_1 \mid X_0 = i_0\right\} \ldots \\
&\qquad Pr\left\{X_n = x_n \mid X_{n-1} = i_{n-1}\right\} \\
&= Pr\left\{X_0 = i_0\right\} P_{i_0,i_1} \cdot P_{i_1,i_2} \ldots P_{i_{n-1},i_n}
\end{aligned}$$

Which is a major simplification.

Definition 2.4 (Transition Probability Matrix)**.** *For a discrete time markov-chain with state space* $\{0, 1, \ldots, N\}$ *we call*

$$\mathbf{P} = \begin{bmatrix} P_{00} & \ldots & P_{0N} \\ P_{10} & \ldots & \\ \vdots & & \ddots \\ P_{N0} & \ldots & P_{NN} \end{bmatrix}$$

Is the transition matrix. For statespace $\{0, 1, 2, \ldots\}$ *we envision an infinitely sized matrix.*

Example.

- Markoc chain : $\{X_n : n = 0, 1, 2, \ldots\}$
- State space = $\{0, 1\}$
- Transition Matrix

$$\mathbf{P} = \begin{bmatrix} 0.9 & 0.1 \\ 0.6 & 0.4 \end{bmatrix}$$

We can compute

$$\begin{aligned} Pr\{X_3 = 1 \mid X_2 = 0\} &= p_{01} \\ &= 0.1 \\ Pr\{X_{10} = 0 \mid X_9 = 1\} &= P_{10} \\ &= 0.6 \end{aligned}$$

Definition 2.5 (Transition Diagram)**.** *Let $\{X_n : n = 0, 1, \ldots\}$ be a discrete time Markov chain. A* ***state trasnistion diagram*** *visualizes the transition probabilities as a weighted directed graph where the nodes are the states and the edges are the possible transitions marked with the transistion probabilities.*

Example. State space = $\{0, 1, 2\}$ and

$$P = \begin{bmatrix} 0.95 & 0.05 & 9 \\ 0 & 0.9 & 0.1 \\ 0.01 & 0 & 0.99 \end{bmatrix}$$

Transisition diagram

Nice figure of the diagram

2.5 Doing n transitions.

Theorem 2.1. *For a Markoc chain $\{X_n : n = 0, 1, \ldots\}$ and any $m \geq 0$ we have*

$$Pr\{X_{m-n} = j \mid X_m = i\} = P_{ij}^{(n)} = \sum_{k=0}^{\infty} P_{ik} P_{kj}^{(n-1)}, \quad n > 0$$

where we define

$$P_{ij}^{(0)} = \begin{cases} 1, & i = j \\ 0, i \neq j \end{cases}$$

Proof. Set $m = 0$ then is

$$\begin{aligned} P_{ij}^{(n+1)} &= Pr\left\{X_{n+1} = j \mid X_0 = i\right\} \\ &= \sum_k Pr\left\{X_{n+1} = j, X_1 = k \mid X_0 = i\right\} \\ &= \sum_k Pr\left\{X_{n+1} = j \mid X_1 = k, X_0 = i\right\} \cdot Pr\left\{X_1 = k \mid X_0 = i\right\} \\ &= \sum_k P_{kj}^{(h)} \cdot P_{ik} = \sum_k P_{ik} P_{kj}^{(h)} \end{aligned}$$

□

Example. $\{X_n : n = 0, 1, 2, \ldots\}$ is a markoc chain and

$$P = \begin{bmatrix} 0.1 & 0.9 \\ 0.6 & 0.4 \end{bmatrix}$$

Find $P_{01}^{(4)}$. **Solution**.

$$P^2 = \begin{bmatrix} 0.55 & 0.45 \\ 0.30 & 0.70 \end{bmatrix}$$

So by doing matrix multiplication and we end up with

$$P^4 = P^2 \cdot P^2 = \begin{bmatrix} 0.4375 & 0.5625 \\ 0.3750 & 0.6250 \end{bmatrix}$$

Which therefore ends up with the answer

$$P_{01}^{(4)} = 0.5625$$

3 Lecture 4

3.1 Introduction to first step analysis

Input

- i_0 : starting state
- P : transition probability matrix
- T: number of time steps

Algorithm

1. Set $x_0 = i_0$
2. for $n = 1 \ldots T$
3. Simulate x_n from $X_n \mid X_{n-1} = x_{n-1}$
4. end

output : One realization $x_0, x_1, \ldots, x_T$

Example.

$$P = \begin{pmatrix} 0.95 & 0.05 & 0 \\ 0 & 0.90 & 0.10 \\ 0.01 & 0 & 0.99 \end{pmatrix}$$

Let $x_0 = 0$

1. $x_0 = 0$
2. $$\begin{aligned} Pr\left\{X_1 = 0 | X_0 = 0\right\} =& P_{00} = 0.95 \\ Pr\left\{X_1 \mid X_0\right\} =& P_{01} = 0.05 \\ Pr\left\{X_1 \mid X_0 = 0\right\} =& P_{02} = 0 \\ & . \end{aligned}$$

   Assume we get $x_1 = 1$
3. States
   - $$\begin{aligned} 0 &: P_{10} = 0 \\ 1 &: P_{11} = 0.90 \\ 2 &: P_{12} = 0.10 \\ &\vdots \end{aligned}$$

General notes on simulation

- $Pr\left\{A\right\} \approx \frac{\text{times A occure}}{\text{Simulations}}$
- $E\left[X\right] \approx \frac{1}{N}\sum_{i=1}^{B} x_i$

Example. We have $N = 100$ divided into two containers labelled A and b. At each time n, one ball is selected at random and moved to the container. Let Y_n denote the number of balls in container A at time n, and define $X_n = Y_n - 50$. Find the transition probabilities and simulate and plot one realization of

$$\{X_n : n = 0, 1, \ldots, 500\}$$

Answer

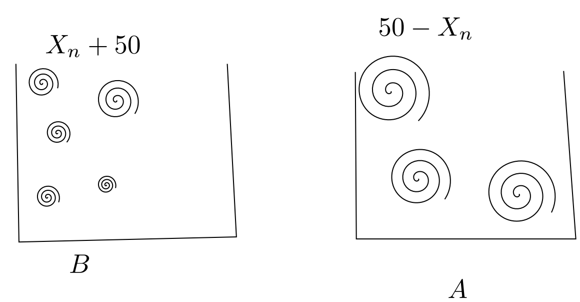

Figure 1: balls

- Only move One ball
- Can move only from i to $j = i - 1$ or $ji + 1$

$$P_{ij} = \begin{cases} \frac{50-i}{100} & , \quad j = i+1 \\ \frac{50+i}{100} & , j = i-1 \\ 0 & , \text{otherwise.} \end{cases}$$

Motivation

Definition 3.1. *For a markov chain, a state i sich that $P_{ij} = 0 \forall j \neq i$ is* *called* ***absorbing.***

Example. Let $\{X_n\}$ be a Markov chain woth transition probability matrix

$$\mathbf{P} = \begin{pmatrix} 1 & 0 & 0 \\ \alpha & \beta & \gamma \\ 0 & 0 & 1 \end{pmatrix}$$

where $\alpha, \beta, \gamma > 0$ and $\beta = 1 - \alpha - \gamma$. Assume $x_0 = 1$

1. What is the expected time until absortion ?

2. What is the probability to be absorbed in state 0 ?

Realization .

$$\overbrace{1, 1, 1, 1, 1, 2}^{\text{4 steps to absorption}} \quad , 2, 2 \ldots$$

Mathematically

Let $T = \min\{n \geq 0 : X_n = 0 \quad \text{or} \quad X_n = 2\}$. Then is

$$\begin{aligned} Q1 : &\quad E\left[T \mid X_0 = 1\right] \\ Q2 : &\quad Pr\left\{X_T = 0 \mid X_0 = 1\right\} \end{aligned}$$

The idea of first step analysis is to define

- $T^{(n)} = \min\{n \geq :: X_{m \times n} = 0 \quad \text{or} \quad X_{m+b} = 2\}$
- $T = T^{(0)}$
- $v_i^{(m)} = E\left[T^{(m)} \mid X_m = i\right]$
- $v_i = v_i^{(0)}$

Table 2: Let m be timesteps

m	0	2	3	4	5
$v_0^{(m)}$	0	0	0	0	0
$v_1^{(m)}$	v_1	v_1	v_1	v_1	v_1
$v_2^{(m)}$	0	0	0	0	0

First step analysis for Q1

$$
\begin{aligned}
v_i &= \sum_{k=0}^{2} Pr\left\{X_1 = k \mid X_0 = i\right\}(1 + v_k) \\
&= \sum_{k=0}^{2} P_{ik}(1 + v_k) = \sum_{k=0}^{2} P_{ik} v_k + 1 \quad \text{which is true for} \quad i = 0, 1, 2
\end{aligned}
$$

Which is reduced to linear algebra. Solving it by

$$
\begin{aligned}
& v_0 = v_2 = 0 \\
\implies & v_1 = \alpha v_0 + \beta v_1 + \gamma v_2 + 1 \\
\implies & v_1 = \frac{1}{1-\beta} \quad \text{[Q1]}
\end{aligned}
$$

$P_{ij} \implies i = \text{row}, \quad j = \text{column}$

First step analyis and let

$$
\begin{aligned}
u_i &= Pr\left\{X_T = 0 \mid X_0 = i\right\} \\
&\downarrow \\
u_i &= \sum_{k=0}^{2} P_{ik} u_k, \quad i = 0, 1, 2
\end{aligned}
$$

- Easy: $u_0 = 1, u_2 = 0$
- Harder: $u_1 = \alpha u_0 + \beta u_1 + \gamma u_2$ such that

$$
u_1 = \alpha \frac{1}{1-\beta} = \frac{\alpha}{\alpha - \beta} \quad \text{[Q2]}
$$

Example. let $[X_n]$ be a markov chain with transition matrix

$$\mathbf{P} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0.4 & 0.3 & 0.2 & 0.1 \\ 0.1 & 0.3 & 0.3 & 0.3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The starting state is $x_0 = 1$. Calculate the probability to be absorbed in the state D.

1. Define $u_i = Pr\left\{\text{absorbed in state } 0 \mid X_0 = i\right\}$ for $i = 0, 1, 2, 3$
2. Get the easy ones out of the way. In this case $u_0 = 1$ and $u_3 = 0$
3. $$\begin{aligned} u_1 &= P_{10}u_0 + P_{11}u_1 + P_{12}u_2 + P_{13}u_3 \\ &= 0.4 + 0.3u_1 + 0.2u_2 \\ u_2 &= P_{20}u_0 + P_{21}u_1 + P_{22}u_2 + P_{23}u_3 \\ &= 0.1 + 0.3u_1 + 0.3u_2 \end{aligned}$$
4. Solve for u_1 and u_2

4 Lecture 5

Example. Let P be the matrix

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0.4 & 0.3 & 0.2 & 0.1 \\ 0.1 & 0.3 & 0.3 & 0.3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

With starting state $x_0 = 1$

1. Define $T = \min_{n \geq 0 : X_n = 0 \quad X_n = 3}$ and $v_i = E\left[T \mid X_0 = i\right]$ for $i = 0, 1, 2, 3$
2. Set $v_0 = v_3 = 0$
3. $$v_1 = P_{10}v_0 + P_{11}v_1 + P_{12}v_2 + P_{13}v_3 = 0.3v_1 + 0.2_{v2} + 1$$

   and

   $$v_2 = P_{20}v_0 + P_{21}v_1 + P_{22}v_2 + P_{23}v_3 + 1 = 0.3v_1 + 0.3v_2 + 1$$
4. Solve the equations and end up with

   $$v_1 = \frac{90}{43} \quad \text{and} \quad v_2 \frac{100}{43}$$

Theorem 4.1. *Let $\{X_n\}$ be a discrete time Markov chain with state space $S = \{0, 1, \ldots, N\}$ and transition probability matrx $\mathbf{P}$. Let $A \subset S$ be the set of absorbing state. Then*

1. *If v_i is the expected time to absorption conditional on $X_0 = i$ then*

$$\begin{aligned} v_i &= 0, \quad i \in A \\ v_i &= 1 + \sum_{i \in \mathbb{R}} P_{ik} v_k \quad i \in A^c \end{aligned}$$

Example. A gambler has 10\$ and bets 1\$ If he wins the round, his fortune increases 1\$. The probability of winning each round is $0 < p < 1$ and the probability of losing each round is $q = 1 - p$. The gambler will continue gambling until his fortine is \$ N or 0\$ where $N > 10$. What is the probability the gambler will be ruined.

1. Extract the essential stuff.

$$\begin{aligned} X_n &= \text{Fortune at time} \quad n, \quad n = 0, 1, 2, \ldots \\ \text{State space } &= \{0, 1, \ldots, N\} \\ \text{Target: } u_k &= Pr\left\{\text{Absorption in state } 0 \mid X_0 = k\right\}, \quad k = 0, 1, \ldots, N \end{aligned}$$

2. Visualize the transitions. Insert figure of tranistions.

3. Make the eprobability matrix. The rows are "to" and the columns are "1"

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 & \dots & 0 \\ q & 0 & p & 0 & \dots & 0 \\ 0 & q & 0 & p & \dots & \\ \vdots & & \ddots & & & \\ & & & q & 0 & p \\ 0 & 0 & \dots & & & 1 \end{bmatrix}$$

4. Set up the iteration

$$\begin{aligned} u_0 &= 1, \quad u_N = 0, \quad \text{Easy} \\ u_i &= P_{i,i.1} u_{i-1} + P_{i,i+1} u_{i+1} \\ &= q u_{i-1} + p u_{i+1}, \quad i = 1, 2, \dots, N-1 \end{aligned}$$

5. (a)

$$\begin{aligned} \overbrace{(p+q)}^{=1} u_i &= q u_{i-1} + p u_{i+1} \\ q\left[u_i - u_{i-1}\right] &= p\left[u_{i+1} - u_i\right] \\ &\downarrow \quad \text{Trick} \quad \chi_i = u_i - u_{i-1} \\ q\chi_1 = p\chi_{i+1}, &\quad \implies \chi_{i+1} = \frac{q}{p}\chi_i \quad i = 1, 2, \dots, N \end{aligned}$$

(b)

$$\chi_1 + \chi_2 + \dots + \chi_k = [u - u_0] + [u_2 - u_1] + \dots + [u_k - u_{k-1}]$$

$$\downarrow \quad \text{Telescoping sum}$$

$$\chi_1 \left[1 + \frac{q}{p} + \left(\frac{q}{p}\right)^2 + \dots + \left(\frac{q}{p}\right)^{k-1}\right] = u_k - 1, \quad k = 1, 2, 3, \dots, N$$

For $k = N$:

$$\begin{aligned} \chi_1 &= \frac{u_N - 1}{\sum_{k=0}^{N-1} \left(\frac{q}{p}\right)^k} = \frac{-1}{\sum_{k=0}^{N-1} \left(\frac{q}{p}\right)^k} \\ &= \begin{cases} -\frac{1}{N} & , q = p = \frac{1}{2} \\ \frac{-\left(1 - \frac{q}{p}\right)}{\left(1 - \left(\frac{q}{p}\right)\right)} & q \neq p \end{cases} \end{aligned}$$

(c) From the telescoping sum

$$u_k = 1 + \chi_1 \sum_{i=0}^{k-1} \left(\frac{q}{p}\right)^i$$
$$= \begin{cases} 1 - \frac{1}{N} \cdot k = \frac{N-k}{N}, & p = q = \frac{1}{2} \\ 1 - \frac{1-\left(\frac{q}{p}\right)^k}{1-\left(\frac{q}{p}\right)^N} = \frac{\left(\frac{q}{p}\right)^k - \left(\frac{q}{p}\right)^N}{1-\left(\frac{q}{p}\right)^N}, & p \neq q \end{cases}, \quad \text{where} \quad k = 1, 2, \ldots.$$

.

6. The final step

$$u_{10} = \begin{cases} \frac{N-10}{N}, & p = q = \frac{1}{2} \\ \frac{\left(\frac{q}{p}\right)^{10} - \left(\frac{q}{p}\right)^N}{1-\left(\frac{q}{p}\right)^N}, & q \neq p \end{cases}$$

Remark.
- When $N \to \infty$

$$q \geq p \implies \text{Almost certain you will loose.}$$
$$q < p \implies P\,(\text{ruined}) = \left(\frac{q}{p}\right)^{10}$$

4.1 Markov Chain in infinitive time

Definition 4.1. ***Regular Markov Chain*** . *Consider a Markov chain $\{X_n : \quad n = 0, 1, \ldots\}$ with finite state space $\{0, 1, 2, \ldots\}$ and transition matrix $\mathbf{P}$. IF there exists an integer $k > 0$ so that all regular elements $\mathbf{P}^k$ are strictly positive, we call $\mathbf{P}$ and $\{X_n\}$ regular.*

Remark.
1. P is regular means that it exists an $k > 0$ so that $P_{ij}^{(k)} > 0 \quad \forall i, j$
2. If $P_{ij}^{(k)} \quad \forall i, j$, then is $P_{ij}^{(k)} > 0 \quad \forall i, j$ and $K \geq k$

5 Lecture 08/09/20

Equivalent classes and classifications of states in Markov chains.

Things to check

- Understand why regularity fails.
- Extend regularity to infinite spaces.

Example Let $\{X_n : 0, 1, \ldots, N\}$ be a markov chain.

(a) It can go from $0 \to 0$ and $1 \to$ with probabilities $p_{00} = p_{11} = 1$, two seperate markov chains. Realizations :

$$0, 0, 0, 0, 0, 0, \ldots$$
$$1, 1, 1, 1, 1, 1, \ldots$$
$$P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \implies P^n = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Definition 5.1. *Let $\{X_n : 0, 1, \ldots\}$ be a Markov chain with state space $\{0, 1, \ldots\}$ then is*

(i) State j is ***accessible*** *from state i if $\exists n \geq 0$ so that $P^{(n)} > 0$*

(ii) If states i and j are accessable from each other they are said to ***communcate*** *we write $i \sim j$. If states i and j do not communcate we write $i \nsim j$*

Remark. If $i \nsim j$, then either (or both)

(a) (i) $P_{ij}^{(n)} = 0, \quad \forall n \geq 0$
 (ii) $P_{ji} = 0, \quad \forall n \geq 0$

(b) Only the graph matters, not the values of the edges.

(c) $P_{ij}^{(0)} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$

Theorem 5.1. *Communication is an* ***equicalence relation***

(i) ***reflexive*** *, i j*

(ii) ***symmentric*** *$i \sim j \implies j \sim i$*

(iii) ***Transitive*** *$i \sim j$ and $j \sim k$ implies $i \sim k$*

A equivalence relation induces ***equivalence classes*** *of sets of states that communicate.*

Proof. (i) $P_{ii}^{(0)} = 1 \quad \Longrightarrow i \sim i$

(ii) By definition is this true.

(iii) (a) $i \sim j \implies \quad \exists n \geq 0 : P_{ij}^{(n)} > 0$

$$j \sim k \implies \exists m \geq 0 : P_{jk}^{(m)} > 0$$

(b) Chapman-kilogram

$$P_{ik}^{(n+m)} = \sum_{r=0}^{\infty} P_{ir}^{(n)} P_{rj}^{(m)} \geq P_{ij}^{(n)} P_{jk}^{(m)}$$

$\Longrightarrow$ k is accessible from i.

(c) Show yourself

i **is accessible from** k

□

Definition 5.2. *A Markov chain is* ***irreducible*** *if $\sim$ (communication) induces exactly one equivalent class. If not, it is called reducible.*

Definition 5.3. *The* ***period*** *of state i, written as $d(i)$ is*

$$d(i) = gcd\left\{n \geq 1 : P_{ii}^{(n)} > 0\right\}$$

If $P_{ii}^{(n)} = 0$ for all $n \geq 1$, we define $d(i) = 0$. If $d(i) = 1$, we call the state i is ***aperiodic.***

Theorem 5.2. *if $i \sim j$, then $d(i) = d(j)$*

Remark. The period is a property of the equivalence class.

Notation THe state space may be infinite: $\{0, 1, \ldots\}$. We introduce

(i) The probability the first return happend after exactly n steps

$$f_{ii}^{(n)} = Pr\left\{X_n = i, X_\mu \neq i, i = 1, 2, \ldots, n-1 \mid X_0 = i\right\} \quad n > 0$$

We will define $f_{ii}^{(0)} = 0$

(ii) The probability of returning at some time

$$f_{ii} = \sum_{k=0}^{\infty} f_{ii}^{(k)} = \lim_{n \to \infty} \sum_{k=0}^{n} f_{ii}^{(k)}.$$

Remark. $f_{ii} < i \leftrightarrow$ Positive probability of never returning to i

Definition 5.4. *State i is* ***recurrent*** *if the probabilitu of retunging to sate i in a finite number of timesteps is one $f_{ii} = 1$. A state that is not recurrent $f_{ii} < 1$ is called* ***transient*** *.*

Theorem 5.3. *A state i is recurrent if and only if*

$$\sum_{n=1}^{\infty} P_{ii}^{(n)} = \infty$$

Equivalently, state i is transient if and only if

$$\sum_{n=1}^{\infty} P_{ii}^{(n)} < \infty$$

Proof. (i)

$$\begin{aligned}
\sum_{n=1}^{\infty} P_{ii}^{(n)} &= \sum_{n=1}^{\infty} E\left[\mathbb{I}\left\{X_n = j\right\} \mid X_0 = j\right] \\
&= E\left[\sum_{n=1}^{\infty} \mathbb{I}\left\{X_n = i \mid X_0 = i\right\}\right] \\
&= E\left[M \mid X_0 = i\right] \\
M &\to \text{Returns to state.}
\end{aligned}$$

(ii) $E\left[M \mid X_0 = i\right] = \begin{cases} f_{ii} \frac{1}{1 - f_{ii}}, & f_{ii} < 1 \\ \infty, & f_{ii} = 1 \end{cases}$

□

6 References