TMA 4190 Introduction to Topology
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Lecture 20[1]

20. Intersection Numerbs and Degree modulo 2

A classical geometric approach to classifying maps is to study their fibres. This appraoch is directly related to other fundamental problems in mathematics. For example, if $f\colon X \to Y$ is a map defined by an equation and given a value $y \in Y$, the set $\{x \in X : f(x) = y\}$ is the set of solutions of the equation. In geometric terms, we could rephrase the question which x solve equation f by asking how f meets or intersects the subspace $\{y\}$ in Y.

Building on the methods we have developed so far, we are going to exploit this geometric approach to derive interesting and powerful invariants. We will start with intersection numbers modulo 2. In order to define a $\mathbb{Z}$-valued invariant we will have to introduce orientations later.

Before we get to work, here is a brief summary of the previous long lecture the results of which will play a key role today:

The previous lecture in a nutshell

We proved three key results about transversality which can be roughly summarized as follows:

(a) The **Transversality Theorem** says that when a homotopy F is transversal to Z, then, in this homotopy family, **almost every** $f_s = F(-,s)$ is **transversal to** Z.
(b) The **Transversality Homotopy Theorem** says that given a map f and a submanifold Z, then **there exists** a map g **transversal to** Z and g is **homotopic** to f.
(c) The **Extension Theorem** says that, **given** a map f which is transversal to Z on **a subset** C, then we can always **replace** f with a homotopic map g which is **transversal to** Z **everywhere** (not only on C) and $f = g$ on an open set containing C.

(a) is a generalization of Sard's Theorem. For (b) and (c), the key for the proof was the **ϵ-Neighborhood Theorem**.

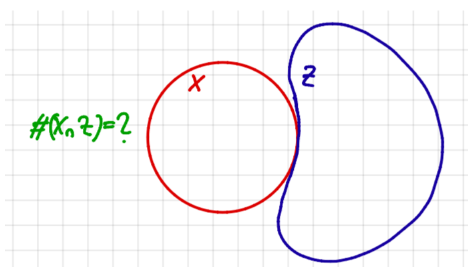
[1] Following the books of Guillemin and Pollack: Differential Topology, and Milnor: Topology from the differentiable viewpoint.

We are going to apply these results today. First let us start with a natural situation.

Intersecting manifolds

Two submanifolds X and Z inside Y have **complementary dimension** if $\dim X + \dim Z = \dim Y$. (We assume all manfifolds are boundaryless for the moment.) If $X \pitchfork Z$, the Preimage Theorem tells us that their intersection $X \cap Z$ is manifold with $\operatorname{codim}(X \cap Z)$ in X being equal to $\operatorname{codim} Z$ in Y. Since $\operatorname{codim} Z = \dim X$, **$X \cap Z$ is a zero-dimensional manifold**.
If we further assume that both X and Z are **closed** and that at least one of them, say X, is **compact**, then $X \cap Z$ must be a **finite set of points**. We are going to think of this number of points in $X \cap Z$ as the **intersection number** of X and Z, denoted by $\#(X \cap Z)$.

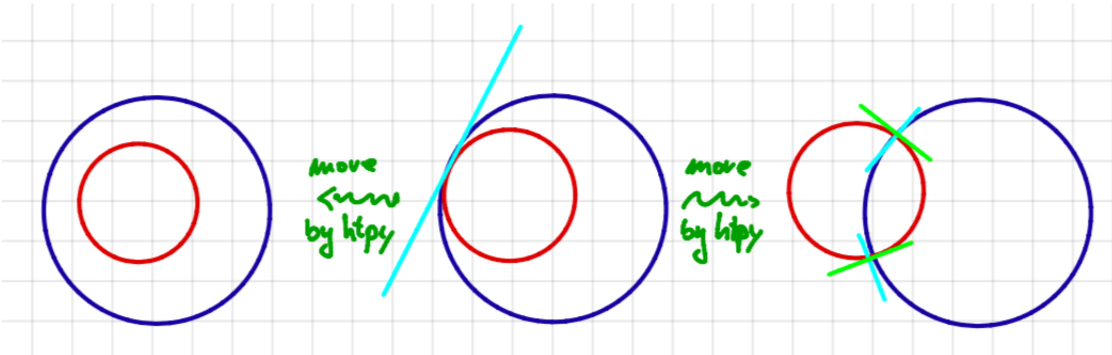
We would like to generalize the notion of intersection numbers. A first obstacle is that if X and Z do **not intersect transversally**, then it makes in general no sense to count the points in $X \cap Z$. Hence, once again, transversality is key.



Luckily, we have learned how to **move or deform** manifolds to make intersections transversal: we can alter them in **homotopic families**. And since **embeddings form a stable class** of maps, i.e. for any homotopy i_t of an embedding i_0, there is an $\epsilon > 0$ such that i_t is still an embedding for all $t < \epsilon$, any small homotopy of i gives us another embedding $X \hookrightarrow Y$ and thus produces an image manifold that is a **diffeomorphic copy** of X adjacent to the original.

But we still have to be **careful**. For the intersection number may depend on how we move or deform the manifold.

For example, take two circles in $\mathbb{R}^2$. Assume they intersect nontransversally, i.e. they touch each other in a point such that both tangent spaces agree and together just span a line. Then we can move the cricles by a simple translations $x \mapsto x + ta$ in direction a such that they either in two points or in no points. In both cases, the intersection is transversal, but the intersection numbers do not agree. But we observe that parity of the intersection numbers is preserved. i.e. up to a multiple of 2 the intersection numbers after moving into a transversal intersection agree.



This observation is the starting point for the following generalization.

Mod 2 Intersection numbers

Let X be a **compact** manifold, and let $f\colon X \to Y$ be a smooth map **transversal** to the **closed** manifold Z in Y. Assume $\dim X + \dim Z = \dim Y$. Then $f^{-1}(Z)$ is a closed submanifold of X of codimension equal to $\dim X$. Hence $f^{-1}(Z)$ is of **dimension zero**, and therefore a **finite** set. We define the **mod 2 intersection number** of the map f with Z, denoted $I_2(f,Z)$, to be **the number of points in $f^{-1}(Z)$ modulo 2**:

$$I_2(f,Z) := \# f^{-1}(Z) \mod 2.$$

For an **arbitrary** smooth map $g\colon X \to Y$, we can **choose** a map $f\colon X \to Y$ that is **homotopic to g** and **transversal** to Z by the **Transversality Homotopy Theorem**. Then we **define** $I_2(g,Z) := I_2(f,Z)$.

Of course, we need to check that the intersection number does not depend on the choice of homotopic map. The key technical result that allows us to show independence is the Extension Theorem. We did not have time to discuss the theorem and its quite technical proof in the lecture. So here is the thereom and one of its applications that will be crucial for us.

The **Extension Theorem** says the following: Let $f\colon X \to Y$ be a smooth map, Y boundaryless, and Z a closed submanifold of Y without boundary. Let C be a closed subset of X. Assume that $f \pitchfork Z$ on C and $\partial f \pitchfork Z$ on $C \cap \partial X$.

Then there exists a smooth map $g\colon X \to Y$ **homotopic to f**, such that $g \pitchfork Z$ and $\partial g \pitchfork Z$, and on a **neighborhood of C we have $g = f$**.

We apply this result in the situation we were discussing for intersection numbers, i.e. X, Y and $Z \subset Y$ are boundaryless manifolds. The product $X \times [0,1]$ is then a manifold with boundary. We let C be the boundary of $X \times [0,1]$, i.e. C is the closed subset

$$C := \partial(X \times [0,1]) = X \times \{0\} \cup X \times \{1\}.$$

Now we apply the theorem to the case of a smooth homotopy

$$F\colon X \times [0,1] \to Y.$$

Then ∂F, i.e. F restricted to the boundary of $X \times [0,1]$, is given by the two maps

$$f_0 = F(-,0)\colon X \to Y \text{ and } f_1 = F(-,1)\colon X \to Y.$$

The two conditions $F \pitchfork Z$ on C and $\partial F \pitchfork C$ on $C \cap \partial X$ are thus equivalent, and mean $f_0 \pitchfork Z$ and $f_1 \pitchfork Z$.

Hence, assuming $f_0 \pitchfork Z$ and $f_1 \pitchfork Z$, the Extension Theorem says that there is a smooth map

$$G\colon X \times [0,1] \to Y \text{ with } \mathbf{G \pitchfork Z} \text{ and } \partial G \pitchfork Z,$$

and $G = F$ on a neighborhood of C. The latter means that

$$G \text{ is still a homotopy from } f_0 = G(-,0) \text{ to } f_1 = G(-,1).$$

Mod 2 Intersection Numbers are well-defined

If $f_0\colon X \to Y$ and $f_1\colon X \to Y$ are homotopic and both transversal to Z, then $I_2(f_0,Z) = I_2(f_1,Z)$.

Proof: Let $F\colon X \times I \to Y$ be a homotopy of f_0 and f_1. By the above discussion, we may assume that $F \pitchfork Z$. By the Preimage Theorem with boundary, this implies $F^{-1}(Z)$ is a submanifold of $X \times [0,1]$ such that

$$\operatorname{codim} F^{-1}(Z) \text{ in } X \times [0,1] = \operatorname{codim} Z \text{ in } Y.$$

Hence

$$\begin{aligned}\dim F^{-1}(Z) &= \dim(X \times [0,1]) + \dim Z - \dim Y \\ &= \dim X + 1 + \dim Z - \dim Y \\ &= 1\end{aligned}$$

since we assume that $\dim X + \dim Z = \dim Y$.

Moreover, the boundary of $F^{-1}(Z)$ is

$$\partial F^{-1}(Z) = F^{-1}(Z) \cap \partial(X \times [0,1]) = f_0^{-1}(Z) \times \{0\} \cup f_1^{-1}(Z) \times \{1\}.$$

Since X is compact, $F^{-1}(Z)$ is **compact**. Hence the **classification of compact one-manifolds** implies that $\partial F^{-1}(Z)$ must have an **even** number of points. Thus

$$I_2(f_0,Z) = \# f_0^{-1}(Z) = \# f_1^{-1}(Z) = I_2(f_1,Z) \qquad \text{mod } \mathbf{2}.$$

QED

We can generalize this a bit further.

All homotopic maps have equal Intersection Numbers

If $g_0 \colon X \to Y$ and $g_1 \colon X \to Y$ are arbitrary homotopic maps, then $I_2(g_0,Z) = I_2(g_1,Z)$.

Proof: As before, we can choose maps $f_0 \pitchfork Z$ and $f_1 \pitchfork Z$ such that $g_0 \sim f_0$, $I_2(g_0,Z) = I_2(f_0,Z)$, and $g_1 \sim f_1$, $I_2(g_1,Z) = I_2(f_1,Z)$. Since homotopy is a **transitive** relation (we showed that it is, in fact, an equivalence relation), we have

$$f_0 \sim g_0 \sim g_1 \sim f_1, \text{ and hence } f_0 \sim f_1.$$

By the previous theorem, this implies

$$I_2(g_0,Z) = I_2(f_0,Z) = I_2(f_1,Z) = I_2(g_1,Z).$$

QED

Now that we have a solid notion of intersection numbers modulo 2 for maps and submanifolds, let us return to situation we started with.

mod 2 Intersction Numbers of submanifolds

Assume X is a compact submanifold of Y and Z a closed submanifold of Y. Assume the **dimensions are complementary**, i.e. $\dim X + \dim Z = \dim Y$. Then we can define the **mod 2 intersection number of X with Z**, denoted by $I_2(X,Z)$, by

$$I_2(X,Z) := I_2(i,Z)$$

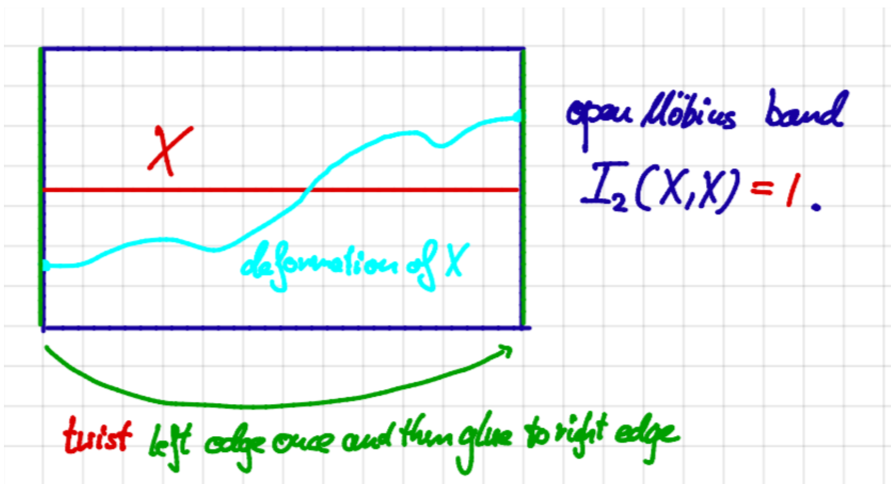
where $i \colon X \hookrightarrow Y$ is the inclusion.
Note that when $X \pitchfork Z$, then $I_2(X,Z) = \#(X \cap Z)$. In general, we have to move or deform X into a **transversal position**.

Some particular situations:

- If $I_2(X,Z) \neq 0$, then no matter how X is moved or deformed, it cannot be pulled entirely away from Z.

  For example, on the torus $Y = S^1 \times S^1$, the two circles $S^1 \times \{1\}$ and $\{1\} \times S^1$ have complimentary dimensions and nonzero mod 2 intersection number.
- If $\dim X = 2 \dim Y$, for then we may consider $I_2(X,X)$ as the **mod 2 self-intersection number** of X.

  An illustrative example is the central curve on the open Möbius band (see Exercise Set 9). Check that $I_2(X,X) = 1$.



- If X happens to be the **boundary** of some W in Y, then $I_2(X,Z)$ =0. For if $Z \pitchfork X$, then, roughly speaking, Z must "pass out" of W as often as it "enters". Hence $\#(X \cap Z)$ is **even**.

The latter case can be made rigorous as follows:

Boundary Theorem

Suppose that X is the **boundary** of some **compact** manifold W and $g\colon X \to Y$ is a smooth map. **If** g can be **extended** to all of W, **then** $I_2(g,Z) = 0$ for any closed submanifold Z in Y of **complementary dimension**, i.e. $\dim X + \dim Z = \dim Y$.

Proof: Let $G\colon W \to Y$ be an extension of g, i.e. $\partial G = g$. From the **Transversality Homotopy Theorem**, we obtain a map $F\colon W \to Y$ **homotopic** to G with $F \pitchfork Z$ and $\partial F \pitchfork Z$. We write $f := \partial F$. Then $f \sim g$ and hence

$$I_2(g,Z) = I_2(f,Z) = \#f^{-1}(Z) \mod 2.$$

Now $F^{-1}(Z)$ is a compact submanifold whose codimension in W is the same as the codimension of Z in Y. Here we use again that X is the boundary of W, for this implies $\dim W = \dim \partial W + 1 = \dim X + 1$, and hence

$$\dim F^{-1}(Z) = \dim X + 1 - \dim Y + \dim Z = 1.$$

Hence $F^{-1}(Z)$ is a **compact one-dimensional manifold with boundary**, so

$$\#\partial(F^{-1}(Z)) = \#(\partial F)^{-1}(Z) = \#f^{-1}(Z) \text{ is } \textbf{even}.$$

QED

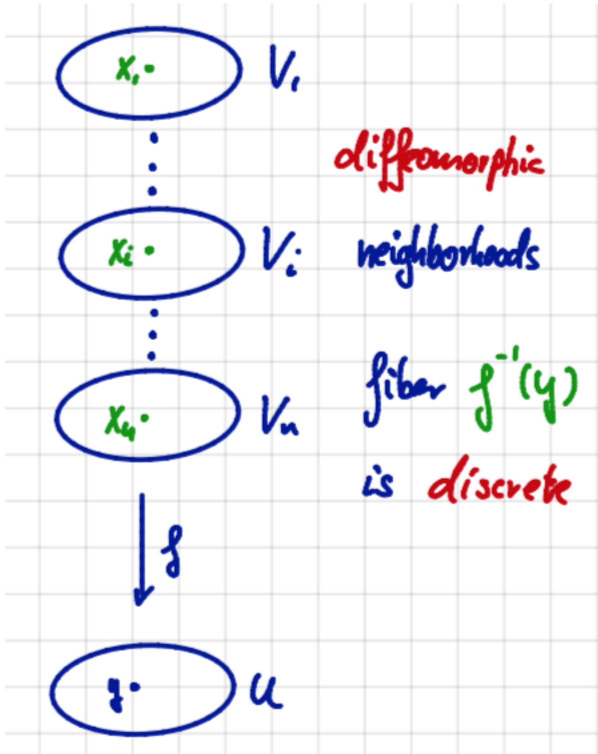
Intersection theory gives us an interesting **homotopy invariant** attached to maps between manifolds of the **same dimension**. The definition depends on the following fact.

The Degree mod 2

If $f\colon X \to Y$ is a smooth map of a **compact** manifold X into a **connected** manifold Y and $\dim X = \dim Y$, then $I_2(f,\{y\})$ is the **same for all points** $y \in Y$. This common value is called the **mod 2 degree of f**, denoted $\deg_2(f)$.

Note: The degree mod 2 is defined only when the range manifold Y is **connected**, the domain X is **compact**, and $\dim X = \dim Y$. Whenever we write $\deg_2$, we assume that these assumptions are satisfied.

Proof: Given any $y \in Y$, we can assume that f is **transversal to** $\{y\}$. For otherwise we can replace it with a homotopic map which is transversal by the **Transversality Homotopy Theorem**. Now by the **Stack of Records Theorem**, we can find a neighborhood U of y such that the preimage $f^{-1}(U)$ is a disjoint union $V_1 \cup \cdots \cup V_n$, where each V_i is an open set in X mapped by f diffeomorphically onto U:



Hence, for all points $z \in U$, we have

$$I_2(f,\{z\}) = \#f^{-1}(\{z\}) = n \mod 2.$$

Consequently, the function

$$Y \to \mathbb{Z}/2,\, y \mapsto I_2(f,\{y\})$$

is **locally constant**. Since Y is **connected**, it must be **globally constant**. **QED**

Since $\deg_2$ is defined as an intersection number, we immediately obtain the following theorems.

$\deg_2$ is a homotopy invariant

Homotopic maps have the same mod 2 degree, i.e.

$$f_0 \sim f_1 \Rightarrow \deg_2(f_0) = \deg_2(f_1).$$

Proof: If $f_0 \sim f_1$, then for every $y \in Y$:

$$\deg_2(f_0) = I_2(f_0,\{y\}) = I_2(f_1,\{y\}) = \deg_2(f_1).$$

QED

Extensions of maps on boundaries have $\deg_2$ equal zero

If $X = \partial W$ for some compact manifold W, and if $f\colon X \to Y$ can be **extended** to all of W, then $\deg_2(f) = 0$.

Note that when W is compact, then the closed subset $X = \partial W$ is also compact. Hence $\deg_2(f)$ is defined.

Proof: This is the Boundary Theorem applied to the zero-dimensional submanifold $\{y\}$ for any $y \in Y$. **QED**

This has an interesting immediate consequence:

Obstruction for extending maps

Let W be a compact manifold, and $f\colon \partial W \to Y$ a smooth map. **If** $\deg_2(f) \neq 0$, then f **cannot be extended** to a smooth map $W \to Y$ on all of W.

Now that we have the invariant $\deg_2$, there are upsides and downsides equipped to $\deg_2$:

The **good news** is that $\deg_2(f)$ is **easy to calculate**: just pick any regular value y for f and count preimage points

$$\deg_2(f) = \# f^{-1}(y) \mod 2.$$

The **bad news** is that its **power is limited**. For example, the map

$$\mathbb{C} \to \mathbb{C},\ z \mapsto z^n,$$

which wraps the circle S^1 smoothly around S^1 n times, has mod 2 degree zero if n is even, and one if n is odd. Hence $\deg_2$ **cannot distinguish** between many different maps, for example $\deg_2$ of the constant map $S^1 \to S^1$ is equal to $\deg_2$ of the map $S^1 \to S^1$ sending $z \mapsto z^2$.

We will remedy this defect soon, when we define intersection numbers and degree functions which have values in $\mathbb{Z}$. This will lead us to the notion of orientation. the idea is that, for example in the case of intersection with a boundary, we need to distinguish between points where a map “goes in” and points where it “goes out”.

Nevertheless, there are some nice and powerful applications of $\deg_2$.

Application: Existence of zeros for complex valued functions.

Suppose that $p\colon \mathbb{C} \to \mathbb{C}$ is a smooth (as a map $\mathbb{R}^2 \to \mathbb{R}^2$), complex function and $W \subset \mathbb{C}$ is a smooth compact region in the plane, i.e. a **two-dimensional compact manifold with boundary**.

Question: Is there a $z \in W$ with $p(z) = 0$?

Assume that p has **no zeros on the boundary** ∂W. Then

$$\frac{p}{|p|}\colon \partial W \to S^1$$

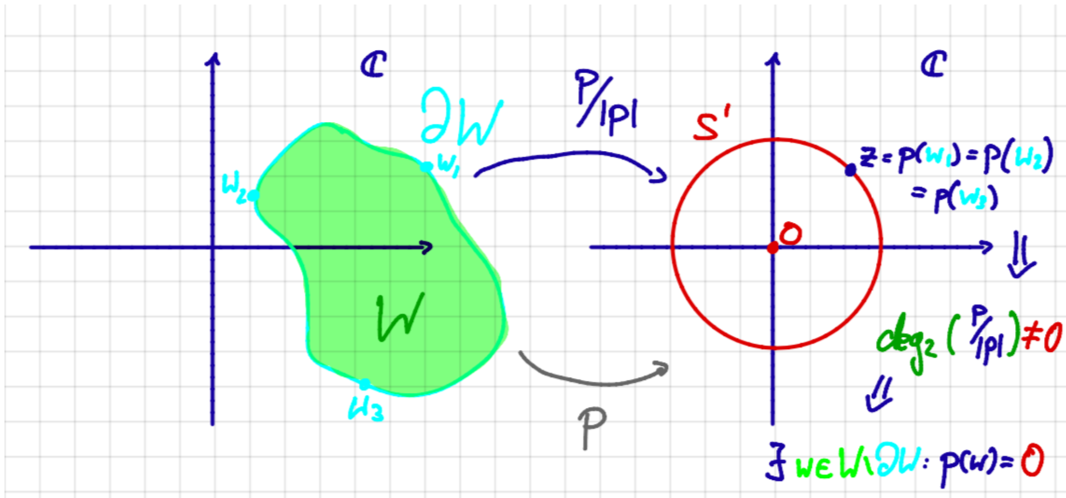
is defined and smooth as a map of **compact one-manifolds**.

Now **if p has no zeros inside W**, then $\frac{p}{|p|}$ is defined on all of W, i.e. $\frac{p}{|p|}\colon \partial W \to \mathbb{C}$ can be **extended** to a smooth map $W \to \mathbb{C}$. If this is the case, we just learned that we must have $\deg_2(\frac{p}{|p|}) = 0$. In other words:

Existence of zeros via $\deg_2$

If the mod 2 degree of $\frac{p}{|p|}\colon \partial W \to S^1$ is **nonzero**, then the function p has a **zero inside** W.

Note that calculating $\deg_2(\frac{p}{|p|})$ simply consists of picking a point $z \in S^1$, we could think of it as a direction, and just counting the number of times we find a $w \in \partial W$ with $p(w) = z$, i.e. how often $p(w)$ points in the chosen direction. The theorem tells us that **this simple procedure can tell us whether p has a zero inside W**. (If you have learned about Complex Analysis, then this should remind you of the Residue Theorem and Cauchy's formula.)



Application: Fundamental Theorem of Algebra in **odd** degrees.

The condition on the degree arises from the **defect of** $\deg_2$ that it cannot distinguish different even numbers. Since we have already seen Milnor's proof of the Fundamental Theorem of Algebra, this is in principle an old story for us. But since we have already done the hard work, so let us have a look at it anyway.

Let

$$p(z) = z^m + a_1 z^{m-1} + \cdots + a_m$$

be a monic complex polynomial. We can define a homotopy from $p_0(z) = z^m$ to $p_1(z) = p(z)$ by

$$p_t(z) = tp(z) + (1-t)z^m = z^m + t(a_1 z^{m-1} + \cdots + a_m).$$

For large z, consider

$$\frac{p(z)}{z^m} = 1 + (\frac{a_1}{z} + \cdots + \frac{a_m}{z^m}).$$

As $z \to \infty$, the term $\frac{a_1}{z} + \cdots + \frac{a_m}{z^m} \to 0$. Hence, if W is a closed ball around the origin in $\mathbb{C}$ with sufficiently large radius, none of the p_t has a zero on ∂W.

Thus the homotopy

$$\frac{p_t}{|p_t|} \colon \partial W \to S^1$$

is defined for all $t \in [0,1]$. Thus

$$\deg_2\left(\frac{p}{|p|}\right) = \deg_2\left(\frac{p_0}{|p_0|}\right).$$

Since $p_0(z) = z^m$ and $\#\{z \in \partial W : z^m = 1\}$ for closed ball $W \subset \mathbb{C}$ around 0, we have

$$\deg_2 \left(\frac{p_0}{|p_0|} \right) = m \mod 2.$$

Hence **if** m is **odd**, then $\deg_2 \left(\frac{p}{|p|} \right) \neq 0$, and there must be $w \in W$ with $p(w) = 0$ by the previous result.