NTNU
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Week 44: Lecture 2

Gaussian processes and covariance functions

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Information

- **No lectures** on November 2 and 4 (week 45)
- We finish the curriculum on November 9.
- November 11: we look at winter 2019 exam together.
- November 16: Questions and answers
- November 18: **No lecture**

Section 4 (Note)

Definition (Def. 1)

The stochastic process $\{X(t) : t \geq 0\}$ with state space $\mathbb{R}$ is called a **Gaussian process** on $[0, \infty)$ if for all $m \geq 1$, for all $0 \leq t_1 < t_2 < \cdots < t_m$,

$$(X(t_1), X(t_2), \ldots, X(t_m))$$

has an m-dimensional multivariate Gaussian distribution.

Example 1

Prove that Brownian motion is a Gaussian process on $[0, \infty)$.

Theorem (Thm. 1)

A Gaussian process $\{X(t) : t \in T\}$ is fully determined by two functions:

1) *a **mean function** $m : T \to \mathbb{R}$ so that*

$$\mathrm{E}[X(t)] = m(t), \quad t \in T.$$

2) *a **covariance function** $C : T \times T \to \mathbb{R}$ so that*

$$\mathrm{Cov}[X(t_1), X(t_2)] = C(t_1, t_2), \quad t_1, t_2 \in T.$$

Example 2

Find the mean function and the covariance function of Brownian motion with variance parameter $\sigma^2 > 0$. Are these enough to fully specify Brownian motion with variance parameter $\sigma^2 > 0$?

Definition (Def. 2)

Let $\{X(t) : t \in T\}$ be a stochastic process. The **correlation function** $r : T \times T \to [-1, 1]$ is defined by

$$
\begin{aligned}
r(t_1, t_2) &= \mathrm{Corr}[X(t_1), X(t_2)] \\
&= \frac{\mathrm{Cov}[X(t_1), X(t_2)]}{\sqrt{\mathrm{Var}[X(t_1)]\mathrm{Var}[X(t_2)]}} \\
&= \frac{C(t_1, t_2)}{\sqrt{C(t_1, t_1)C(t_2, t_2)}},
\end{aligned}
$$

where $C : T \times T \to \mathbb{R}$ is the covariance function.

Definition (Def. 3)

A stochastic process on $[0, \infty)$ is **stationary** if

1) $m(t) = \mu_0$ for $t \in [0, \infty)$
2) $C(t_1, t_2) = \sigma^2 r(|t_1 - t_2|)$ for $t_1, t_2 \in [0, \infty)$

Here $\sigma^2 > 0$ is called the **marginal variance**, and $r : [0, \infty) \to [-1, 1]$ is called a **stationary correlation function** and satisfies $r(0) = 1$.

Common stationary covariance functions

— Exponential:

$$C(t_1, t_2) = \sigma^2 \exp(-\phi_{\mathrm{E}}|t_1 - t_2|), \quad t_1, t_2 \in \mathbb{R}.$$

— Gaussian:

$$C(t_1, t_2) = \sigma^2 \exp(-\phi_{\mathrm{G}}(t_1 - t_2)^2), \quad t_1, t_2 \in \mathbb{R}.$$

— Matérn-type:

$$C(t_1, t_2) = \sigma^2(1 + \phi_{\mathrm{M}}|t_1 - t_2|) \exp(-\phi_{\mathrm{M}}|t_1 - t_2|), \quad t_1, t_2 \in \mathbb{R}.$$

Example 3

Demonstration of simulation and covariance functions using `R`.

Section 4.2 (Note)

Simulation of Gaussian process

Input:

- $[a, b]$: interval of interest
- m: mean function
- C: covariance function

Algorithm:

1. make grid $a = t_1 < t_2 < \cdots < t_n = b$
2. set $\boldsymbol{\mu} = (m(t_1), m(t_2), \ldots, m(t_n))$
3. set $\Sigma_{ij} = C(t_i, t_j)$ for $i, j = 1, 2, \ldots, n$
4. draw $\boldsymbol{x} \sim \mathcal{N}_n(\boldsymbol{\mu}, \Sigma)$

Output:

We have simulated values $\boldsymbol{x} = (x(t_1), x(t_2), \ldots, x(t_n))$.

Section 4.3 (Note)

Example 4

Let $\{B(t) : t \geq 0\}$ be standard Brownian motion, and let $X(t) = (B(t)|B(1) = 1)$ for $0 < t < 1$. Find $\mu(t) = \mathrm{E}[X(t)]$ and $\sigma(t)^2 = \mathrm{Var}[X(t)]$ for $0 < t < 1$.

Conditional Gaussian processes

Let $\{X(t) : t \geq 0\}$ be a Gaussian process. Assume that the process has been observed at locations $B = \{s_1 < s_2 < \cdots < s_m\}$ and let $\boldsymbol{X}_B = (X(s_1), X(s_2), \ldots, X(s_m))$.

Then for any set of locations $A = \{t_1 < t_2 < \cdots < t_n\}$, let $\boldsymbol{X}_A = (X(t_1), X(t_2), \ldots, X(t_n))$. We have

$$\boldsymbol{X}_A | \boldsymbol{X}_B = \boldsymbol{x}_B \sim \mathcal{N}_n(\boldsymbol{\mu}_\mathrm{C}, \Sigma_\mathrm{C}),$$

where

$$\begin{aligned}\boldsymbol{\mu}_\mathrm{C} &= \boldsymbol{\mu}_A + \Sigma_\mathrm{AB}\Sigma_\mathrm{BB}^{-1}(\boldsymbol{x}_B - \boldsymbol{\mu}_B)\\ \Sigma_\mathrm{C} &= \Sigma_\mathrm{AA} - \Sigma_\mathrm{AB}\Sigma_\mathrm{BB}^{-1}\Sigma_\mathrm{BA}.\end{aligned}$$

Example 5

Let $\{B(t) : t \geq 0\}$ be standard Brownian motion, and let $X(t) = (B(t)|B(1) = 1)$. Calculate $\Pr\{X(1/2) > 1\}$.