Project 1 Notes

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2020

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1 Problem 1

Let normal matrices, those with diagonalization be on the form

$$A = U\Lambda U^H$$

Where Λ is a diagonal complex $n \times n$ matrix and U a unitary (complex) matrix such that $U^H U = I$ (recall that U^H is the complex conjugate of U^T).

Show that for any such matrix, one has $\|A\|_2 = \rho(A)$, where $\rho(A)$ is the spectral radius of A .

Answer.

Ideas

- Complex conjugate
$$A^H = \overline{A^T}$$
- Definition of spectral radius is denoted by
$$\rho(A) = \max_{\lambda_i \in \sigma(A)} \{\lambda_i\}$$
of the eigenvalues λ_i in the eigenvalue spectrum $\sigma(A)$.
- $UAU^H = \Lambda = diag(\sigma_1, \ldots, \sigma_n)$
- Let
$$\|A\|_2^2 = \sup_{x\neq 0} \sqrt{\frac{\|A\|_2^2}{\|x\|_2^2}} = \sup_{x\neq 0} \sqrt{\frac{\langle Ax, Ax\rangle}{\|x\|_2^2}} = \sup_{x\neq 0} \sqrt{\frac{\langle A^H Ax, x\rangle}{\|x\|_2^2}}$$
If we use the fact that $A = U^H \Lambda U$, can we substitue $y = U^H x$, such that
$$\|A\|_2^2 = \sup_{x\neq 0} \sqrt{\frac{\|Ax\|_2^2}{\|x\|_2^2}} = \sup_{y\neq 0} \sqrt{\frac{\langle AUy, AUy\rangle}{\|y\|_2^2}}$$
$$\implies \|y\|_2^2 = \left\langle U^H x, U^H x\right\rangle = \left\langle UU^H x, x\right\rangle = \|x\|_2^2$$
$$\implies \langle AUy, AUy\rangle = \left\langle (AU)^H AUy, y\right\rangle = \left\langle U^H A^H AUy, y\right\rangle$$
Since $A^H A$ is unitary can we write $U^H A^H AU = diag(\mu_1, \mu_2, \ldots, \mu_n)$. Which then ends up with the relationship
$$\|A\|_2^2 = \sup_{y\neq 0} \frac{\sum_{i=1}^n \mu_i \cdot |y_i|^2}{\sum_{i=1} |y_i|^2} = \max_i (\mu_i)$$

2 References