Norwegian University of Science and Technology
Deptartment of Mathematical Sciences

TMA4190 Introduction to Topology Spring 2018

Exercise set 12

1 Let $\beta = (v_1, \dots, v_k)$ be an ordered basis of a vector space V.

a) Show that replacing one v_i by a multiple cv_i yields an equivalently oriented ordered basis if $c > 0$, and an oppositely oriented one if $c < 0$.

b) Show that transposing two elements, i.e., interchanging the places of v_i and v_j for $i \neq j$, yields an oppositely oriented ordered basis.

c) Show that subtracting from one v_i a linear combination of the others yields an equivalently oriented ordered basis.

d) Suppose that V is the direct sum of V_1 and V_2. Show that the direct sum orientation of V from $V_1 \oplus V_2$ equals $(-1)^{(\dim V_1)(\dim V_2)}$ times the orientation from $V_2 \oplus V_1$.

2 The upper half space $\mathbb{H}^k$ is oriented by the standard orientation of $\mathbb{R}^k$. Thus $\partial\mathbb{H}^k$ acquires a boundary orientation. But $\partial\mathbb{H}^k$ may be identified with $\mathbb{R}^{k-1}$. Show that the boundary orientation agrees with the standard orientation of $\mathbb{R}^{k-1}$ if and only if k is even.

3 **a)** Write down the orientation of S^2 as the boundary of the closed unit ball B^3 in $\mathbb{R}^3$, by specifying a positively oriented ordered basis for the tangent space at each $(a, b, c) \in S^2$.

b) Show that the boundary orientation of S^k equals the orientation of $S^k = g^{-1}(1)$ as the preimage under the map

$$g\colon \mathbb{R}^{k+1} \to \mathbb{R}, \; x \mapsto |x|^2.$$

4 Suppose that $f\colon X \to Y$ is a diffeomorphism of connected oriented manifolds with boundary. Show that if $df_x\colon T_x(X) \to T_{f(x)}(Y)$ preserves orientation at one point x, then f preserves orientation globally.

5 Let X and Z be transversal submanifolds in Y and assume X, Z and Y are oriented. Let $i\colon X \hookrightarrow Y$ be the inclusion of X into Y, $j\colon Z \hookrightarrow Y$ be the inclusion of Z into Y. We orient the intersection $X \cap Z$ as the preimage $i^{-1}(Z)$, and the intersection $Z \cap X$

as the preimage $j^{-1}(X)$. Show that the orientations of $X \cap Z$ and $Z \cap X$ are related by

$$X \cap Z = (-1)^{(\operatorname{codim} X)(\operatorname{codim} Z)} Z \cap X.$$

(Hint: Show that the orientation of $S = X \cap Z$ at any y is induced by the direct sum

$$(N_y(S, X) \oplus N_y(S, Z)) \oplus T_y(S) = T_y(Y).$$

What happens when you consider $Z \cap X$ instead?)

6 **a)** Let V be a vector space. Show that both orientations on V define the same product orientation on $V \times V$.

b) Let X be an orientable manifold. Show that the product orientation on $X \times X$ is the same for all choices of orientation on X.

c) Suppose that X is not orientable. Show that $X \times Y$ is never orientable, no matter what manifold Y may be. In particular, $X \times X$ is not orientable.
(Hint: First show that $X \times \mathbb{R}^m$ is not orientable, and then use that every Y has an open subset diffeomorphic to $\mathbb{R}^m$.)

d) Prove that there exists a natural orientation on some neighborhood of the diagonal Δ in $X \times X$, whether or not X can be oriented.
But note that Δ itself is orientable if and only if $X \times X$ is orientable. Why?
(Hint: Cover a neighborhood of Δ by local parametrizations $\phi \times \phi \colon U \times U \to X \times X$, where $\phi \colon U \to X$ is a local parametrization of X, then apply the previous observations.)