NTNU
Norwegian University of
Science and Technology

Department of Mathematical Sciences

Examination paper for **TMA4145 Linear Methods**

Academic contact during examination: Franz Luef

Phone: 40614405

Examination date: 5.12.2016

Examination time (from–to): 09:00–13:00

Permitted examination support material: D:No written or handwritten material. Calculator Casio fx-82ES PLUS, Citizen SR-270X, Hewlett Packard HP30S

Other information:
The exam consists of 12 problems, but the grade is based on your 10 best solutions. Although the problems are only formulated in English you can answer either in English or your favorite Norwegian language.

Language: English

Number of pages: 2

Number of pages enclosed: 0

Checked by:

Informasjon om trykking av eksamensoppgave
Originalen er:
1-sidig ☐ **2-sidig** ☐
sort/hvit ☐ **farger** ☐
skal ha flervalgskjema ☐

Date Signature

Problem 1 Let A be a non-empty subset of the real line $\mathbb{R}$.

a) Define the following notions: (a) the **infimum** of A; (b) the **supremum** of A; (3) the **closure** of A; (4) the **interior** of A; (5) the **boundary** of A.

b) Assume that A is *bounded from above*. Show that the supremum of A lies in the closure of A.

Problem 2 Consider the initial value problem:

$$\frac{dx}{dt} = f(t,x), \quad \text{and} \quad x(t_0) = x_0,$$

where f is a function $f : U \times V \to \mathbb{R}$ defined on $U \times V$ of $\mathbb{R}^2$ such that t_0 lies in the interior of the interval U and x_0 in the interior of the interval V, respectively.

a) Formulate the theorem of Picard-Lindelöf.

b) Solve the initial value problem

$$\frac{dx}{dt} = 2t(1+x), \quad \text{and} \quad x(0) = 0,$$

by applying the theorem of Picard-Lindelöf. Compute the first three Picard iterations $x_1(t), x_2(t)$ and $x_3(t)$ starting from $x_0(t) = 0$.

Problem 3 Given the matrix

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 2 \\ 2 & 1 \end{pmatrix}.$$

a) Compute the singular value decomposition of A.

b) Use the result of a) to find:
(1) Bases for the following vector spaces: $\ker(A), \ker(A^*), \operatorname{ran}(A), \operatorname{ran}(A^*)$.
(2) the polar decomposition of A, i.e. find a unitary matrix Q and a positive definite matrix H such that $A = QH$.
(3) The pseudo-inverse of A.

Problem 4 Let $\|.\|_a$ and $\|.\|_b$ be two norms on a vector space X.

a) Show that $\|x\| := (\|x\|_a^2 + \|x\|_b^2)^{1/2}$ is a norm on X. Furthermore, if a sequence (x_n) converges in $(X, \|.\|)$, then it converges in $(X, \|.\|_a)$ and in $(X, \|.\|_b)$.

b) Suppose there exist constants $C_1, C_2 > 0$ such that

$$C_1\|x\|_b \leq \|x\|_a \leq C_2\|x\|_b$$

holds for all $x \in X$, i.e. $\|.\|_a$ and $\|.\|_b$ are equivalent norms on X.

Show that there exist constants $C_1', C_2' > 0$ such that

$$C_1'\|x\|_a \leq \|x\|_b \leq C_2'\|x\|_a$$

holds for all $x \in X$.

Determine the constants C_1 and C_2 for the sup-norm $\|.\|_\infty$ and $\|.\|_p$-norm, $1 \leq p < \infty$. on $\mathbb{R}^n$:

$$C_1\|x\|_\infty \leq \|x\|_p \leq C_2\|x\|_\infty.$$

Problem 5 Let M be the subspace of ℓ^2 defined by

$$M = \{x = (x_n)_{n\in\mathbb{N}} \in \ell^2 : x_{2n} = 0 \quad \text{for} \quad n = 1, 2, ...\}.$$

a) Show that M is a closed subspace of ℓ^2 and determine its orthogonal complement $M^\perp$.

b) Determine the orthogonal projection P from ℓ^2 onto M without using the pojection theorem and show that $P = P^*$ and its operator norm $\|P\| = 1$.

Problem 6 Let X be a separable Hilbert space and $\{e_k : k = 0, 1, 2, ...\}$ an orthonormal basis for X. We define the operator S by $S(e_k) = e_{k+1}$ for $k = 0, 1, 2, ...$.

a) Suppose $a = (a_0, a_1, ...) \in \ell^2$ is the coefficient sequence of $x \in X$:

$$x = \sum_{k=0}^{\infty} a_k e_k.$$

Describe the operator S in terms of the coefficient sequence $(a_0, a_1, ...)$, i.e. as an operator on ℓ^2. Determine S^* on ℓ^2 and find $S^*(e_k)$ for $k = 0, 1, ...$ Compute the operator norm of S.

b) Determine if S and S^* are injective and/or surjective, respectively. Determine S^*S and SS^*, their kernels and ranges, respectively.