NTNU

Norwegian University of
Science and Technology

Department of Mathematical Sciences

Examination paper for **TMA4265 Stochastic Modeling**

Academic contact during examination: Geir-Arne Fuglstad

Phone: 452 70 806

Examination date: December 5, 2019

Examination time (from–to): 09:00–13:00

Permitted examination support material: C:

- Tabeller og formler i statistikk, Tapir forlag
- K. Rottmann: Matematisk formelsamling
- Bilingual dictionary
- One yellow, stamped A5 sheet with personal handwritten formulas and notes (on both sides)
- One specific basic calculator

Other information:
All answers must be justified, and necessary derivations and calculations must be included.
All ten subproblems are equally weighted.
Read all ten subproblems before you start.

Language: English

Number of pages: 4

Number of pages enclosed: 4

Checked by:

Informasjon om trykking av eksamensoppgave
Originalen er:
1-sidig ☐ **2-sidig** ☒
sort/hvit ☒ **farger** ☐
skal ha flervalgskjema ☐

Date Signature

Problem 1 An individual is at each day in one of three states: "susceptible", "infected", and "immune". Each day, a susceptible individual has a probability of 0.05 to become infected, an infected individual has a probability of 0.1 to become immune, and an immune individual has a probability of 0.01 to lose his immunity and become susceptible. The only other possible transitions are that the individual remains in the same state tomorrow as today.

We model this using a discrete-time stochastic process $\{X_t : t = 0, 1, 2, \ldots\}$, where X_t denotes the state (1 = "susceptible", 2 = "infected" or 3 = "immune") of the individual at day t. This stochastic process is a discrete-time Markov chain defined by the transition probability matrix

$$\mathbf{P} = \begin{bmatrix} 0.95 & 0.05 & 0 \\ 0 & 0.9 & 0.1 \\ 0.01 & 0 & 0.99 \end{bmatrix}.$$

a)
- Calculate $\Pr\{X_{100} = 3 | X_{99} = 2, X_{98} = 2\}$.
- Calculate $\Pr\{X_{100} = 3 | X_{99} \neq 2, X_{98} = 2\}$.
- Calculate $\Pr\{X_{100} = 3 | X_{98} = 2\}$.

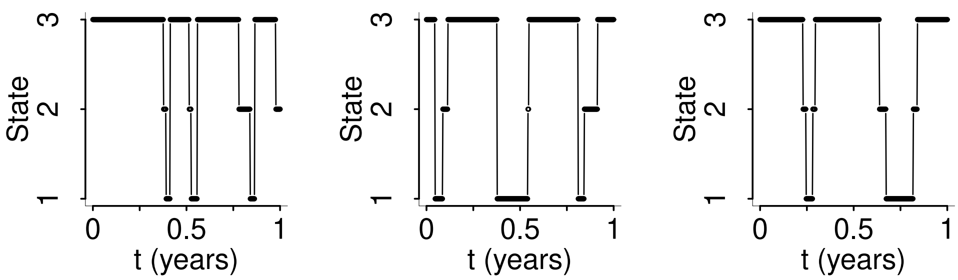
b) Assume that one year consists of 365 days.

- Calculate the long-run mean fraction of time spent in each state.
- On average, how many days per year is the individual "infected"?

c) A long time has passed and the Markov chain has reached its stationary distribution.

- Calculate the probability that the individual was "susceptible" yesterday given that he is "immune" today.
- Calculate the probability that the individual was "infected" yesterday given that he is "immune" today.
- Figure 1 shows three different realizations from discrete-time Markov chains. One of the subfigures shows a simulation over one year from the stationary distribution of $\{X_t : t = 0, 1, 2, \ldots\}$. Which subfigure is this and why can you be certain of your choice?

d) Consider now two individuals that, independently of each other, each follow a discrete-time Markov chain defined by the transition probability matrix $\mathbf{P}$ given above. Assume both individuals are "susceptible" at day $t = 0$.



(a) (b) (c)

Figure 1: Realizations of discrete-time Markov chains where states 1, 2 and 3 correspond to "susceptible", "infected" and "immune", respectively.

- What is the probability that both individuals are "infected" tomorrow given that both individuals are "susceptible" today?
- What is the probability that both individuals are "infected" tomorrow given that both individuals are "immune" today?
- Consider the discrete-time stochastic process $\{I_t : t = 0, 1, \ldots\}$, where I_t denotes the number of infected individuals at day t. The state space of the stochastic process is $\{0, 1, 2\}$. Is this a discrete-time Markov chain? Justify your answer.

Problem 2 Consider two roads, road A and road B, that both lead to a parking garage. The arrival of cars from road A to the parking garage is modelled by a Poisson process with rate $\lambda_{\text{A}} = 10$ cars per hour.

a) Consider the cars arriving from road A to the parking garage.

- What is the expected number of cars arriving between 12:00 and 12:30?
- What is the probability that 2 or fewer cars arrive from 12:00 to 12:30?

b) Each car entering the parking garage from road A must pay a parking fee. This fee varies depending on the length of the stay, but the cars pay on average 20 kr and the standard deviation is 10 kr. Assume the lengths of stay are independent between cars and that the parking fee is pre-paid immediately when arriving to the parking garage.

- Calculate the expected income from 10:00 to 16:00.
- Calculate the variance of the income from 10:00 to 16:00.

c) The arrival of cars from road B to the parking garage is modelled by a Poisson process with rate $\lambda_{\mathrm{B}} = 30$ cars per hour. Assume that the arrival of cars from road B to the parking garage is independent of the arrival of cars from road A to the parking garage.

- Give a definition of Poisson processes.
- Prove that the arrival of cars (from both road A and road B) to the parking garage satisfies the definition that you have given and is a Poisson process.
- What is the rate of this Poisson process?
- Assume that 10 cars arrive to the parking garage between 12:00 and 12:30. What is the probability that 7 of these cars arrive from road B?

Problem 3 A machine has three possible states: "flawless", "damaged", and "broken beyond repair". A flawless machine will become damaged after a time that follows an exponential distribution with expected value $1/\lambda = 10$ days. When the machine is damaged, the machine will become broken beyond repair after a time that follows an exponential distribution with expected value $1/\gamma = 10$ days. This can only be avoided by repairing the damaged machine before it is broken beyond repair. When the machine is damaged, a repairman will immediately start repairs. The time of these repairs follows an exponential distribution with expected value $1/\mu = 2$ days. If the repairs to the damaged machine finish before the machine is broken beyond repair, the machine is again flawless, but if the machine is broken beyond repair before the repairs are finished, the machine can never be repaired.

a)
- Draw the transition diagram for the three states.
- Calculate the expected time until a flawless machine becomes broken beyond repair.

Problem 4 We model the air temperature at NTNU by a Gaussian process $\{X(t) : t \geq 0\}$, where $X(t)$ denotes the air temperature at time t. Assume that the time t is measured in hours, and that the mean is given by $\mathrm{E}[X(t)] = 15$, $t \geq 0$, and that the covariance function is given by

$$\mathrm{Cov}[X(t), X(s)] = 6.25(1 + |t - s|)\exp(-|t - s|), \quad t, s \geq 0.$$

a) We observe the air temperature at time $t = 0$ to be $x(0) = 20$.

- Find an expression for the conditional mean as a function of t,

$$m(t) = \mathrm{E}[X(t)|X(0) = 20], \quad t \geq 0.$$

- Find an expression for the conditional standard deviation as a function of t,

$$\sigma(t) = \sqrt{\mathrm{Var}[X(t)|X(0) = 20]}, \quad t \geq 0.$$

- Calculate $\Pr\{X(2) > 20|X(0) = 20\}$.

b) For this subproblem we again consider the unconditional Gaussian process $\{X(t) : t \geq 0\}$. We are interested in describing how quickly the air temperature is changing with time. We do this by considering the stochastic process $\{Y(t) : t \geq 0\}$, defined by $Y(t) = (X(t+h) - X(t))/h$ for $h = 0.01$.

- Calculate $\mathrm{Var}[Y(t)]$ for $t \geq 0$.
- Prove that $\{Y(t) : t \geq 0\}$ is a Gaussian process.
- The construction $X'(t) = \lim_{h\to 0^+}(X(t+h) - X(t))/h$, for $t \geq 0$, can be taken as the definition of the derivative of the original Gaussian process. Do you expect $\{X'(t) : t \geq 0\}$ to be a Gaussian process? Briefly justify your answer.

Formulas: TMA4265 Stochastic Modeling:

The law of total probability

Let $B_1, B_2, \ldots$ be pairwise disjoint events with $P(\cup_{i=1}^{\infty} B_i) = 1$. Then

$$P(A|C) = \sum_{i=1}^{\infty} P(A|B_i \cap C)P(B_i|C),$$

$$E[X|C] = \sum_{i=1}^{\infty} E[X|B_i \cap C]P(B_i|C).$$

Discrete time Markov chains

Chapman-Kolmogorov equations

$$P_{ij}^{(m+n)} = \sum_{k=0}^{\infty} P_{ik}^{(m)} P_{kj}^{(n)}.$$

For an irreducible and ergodic Markov chain, $\pi_j = \lim_{n\to\infty} P_{ij}^{(n)}$ exist and is given by the equations

$$\pi_j = \sum_i \pi_i P_{ij} \quad \text{and} \quad \sum_i \pi_i = 1.$$

For transient states i, j and k, the mean passage time from i to $j \neq i$, M_{ij}, is

$$M_{ij} = 1 + \sum_k P_{ik} M_{kj}.$$

The Poisson process

The waiting time to the n-th event (the n-th arrival time), X_n, has probability density

$$f_{X_n}(t) = \frac{\lambda^n t^{n-1}}{(n-1)!} e^{-\lambda t} \quad \text{for} \quad t \geq 0.$$

Given that the number of events $N(t) = n$, the arrival times $X_1, X_2, \ldots, X_n$ have the uniform joint probability density

$$f_{X_1,X_2,\ldots,X_n|N(t)}(x_1, x_2, \ldots, x_n|n) = \frac{n!}{t^n} \quad \text{for} \quad 0 < x_1 < x_2 < \ldots < x_n \leq t.$$

Markov processes in continuous time

A (homogeneous) Markov process $X(t)$, $0 \le t \le \infty$, with state space $\Omega \subseteq \mathbf{Z}^+ = \{0, 1, 2, \ldots\}$, is called a birth and death process if

$$
\begin{aligned}
P_{i,i+1}(h) &= \lambda_i h + o(h) \\
P_{i,i-1}(h) &= \mu_i h + o(h) \\
P_{i,i}(h) &= 1 - (\lambda_i + \mu_i)h + o(h) \\
P_{ij}(h) &= o(h) \quad \text{for } |j-i| \ge 2
\end{aligned}
$$

where $P_{ij}(s) = P(X(t+s) = j | X(t) = i)$, $i, j \in \mathbf{Z}^+$, $\lambda_i \ge 0$ are birth rates, $\mu_i \ge 0$ are death rates.

The Chapman-Kolmogorov equations

$$P_{ij}(t+s) = \sum_{k=0}^{\infty} P_{ik}(t) P_{kj}(s).$$

Limit relations

$$\lim_{h \to 0} \frac{1 - P_{ii}(h)}{h} = v_i \,, \quad \lim_{h \to 0} \frac{P_{ij}(h)}{h} = q_{ij} \,, \;\; i \ne j$$

Kolmogorov's forward equations

$$P'_{ij}(t) = \sum_{k \ne j} q_{kj} P_{ik}(t) - v_j P_{ij}(t).$$

Kolmogorov's backward equations

$$P'_{ij}(t) = \sum_{k \ne i} q_{ik} P_{kj}(t) - v_i P_{ij}(t).$$

If $P_j = \lim_{t \to \infty} P_{ij}(t)$ exist, P_j are given by

$$v_j P_j = \sum_{k \ne j} q_{kj} P_k \quad \text{and} \quad \sum_j P_j = 1.$$

In particular, for birth and death processes

$$P_0 = \frac{1}{\sum_{k=0}^{\infty} \theta_k} \quad \text{and} \quad P_k = \theta_k P_0 \quad \text{for} \quad k = 1, 2, \ldots$$

where

$$\theta_0 = 1 \quad \text{and} \quad \theta_k = \frac{\lambda_0 \lambda_1 \cdot \ldots \cdot \lambda_{k-1}}{\mu_1 \mu_2 \cdot \ldots \cdot \mu_k} \quad \text{for} \quad k = 1, 2, \ldots$$

Queueing theory

For the average number of customers in the system L, in the queue L_Q; the average amount of time a customer spends in the system W, in the queue W_Q; the service time S; the average remaining time (or work) in the system V, and the arrival rate λ_a, the following relations obtain

$$L = \lambda_a W.$$

$$L_Q = \lambda_a W_Q.$$

$$Z = \lambda_a E[S].$$

$$V = \lambda_a E[SW_Q^*] + \lambda_a E[S^2]/2.$$

Gaussian processes

The multivariate Gaussian density for $n \times 1$ random vector $\boldsymbol{x} = (x_1, \ldots, x_n)$ is

$$p(\boldsymbol{x}) = \frac{1}{(2\pi)^{n/2}|\boldsymbol{\Sigma}|^{1/2}} \exp\left(-\frac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\boldsymbol{x}-\boldsymbol{\mu})\right), \quad \boldsymbol{x} \in \mathbb{R}^n,$$

where size $n \times 1$ mean vector $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_n)$, $E(x_i) = \mu_i$, and

$$\boldsymbol{\Sigma} = \begin{bmatrix} \Sigma_{1,1} & \ldots & \Sigma_{1,n} \\ \ldots & \ldots & \ldots \\ \Sigma_{n,1} & \ldots & \Sigma_{n,n} \end{bmatrix}, \quad \Sigma_{i,j} = \mathrm{Cov}(x_i, x_j).$$

Let $\boldsymbol{x}_A = (x_{A,1}, \ldots, x_{A,n_A})$ and $\boldsymbol{x}_B = (x_{B,1}, \ldots, x_{B,n_B})$, be two subsets of variables, with block mean and covariance structure

$$\boldsymbol{\mu} = (\boldsymbol{\mu}_A, \boldsymbol{\mu}_B), \quad \boldsymbol{\Sigma} = \begin{bmatrix} \boldsymbol{\Sigma}_A & \boldsymbol{\Sigma}_{A,B} \\ \boldsymbol{\Sigma}_{B,A} & \boldsymbol{\Sigma}_B \end{bmatrix}.$$

The conditional density of $\boldsymbol{x}_A$, given $\boldsymbol{x}_B$, is Gaussian with

$$\begin{aligned} E(\boldsymbol{x}_A|\boldsymbol{x}_B) &= \boldsymbol{\mu}_A + \boldsymbol{\Sigma}_{A,B}\boldsymbol{\Sigma}_B^{-1}(\boldsymbol{x}_B - \boldsymbol{\mu}_B), \\ \mathrm{Var}(\boldsymbol{x}_A|\boldsymbol{x}_B) &= \boldsymbol{\Sigma}_A - \boldsymbol{\Sigma}_{A,B}\boldsymbol{\Sigma}_B^{-1}\boldsymbol{\Sigma}_{B,A}. \end{aligned}$$

The Brownian motion has increments $x(t_i) - x(t_{i-1})$ with the following properties, for any configuration of times $t_0 = 0 < t_1 < t_2 < \ldots$:

- $x(t_i) - x(t_{i-1})$ and $x(t_j) - x(t_{j-1})$ are independent for all $i \neq j$.

- the distribution of $x(t_i) - x(t_{i-1})$ is identical to that of $x(t_i + s) - x(t_{i-1} + s)$, for any s.
- $x(t_i) - x(t_{i-1})$ is Gaussian distributed with 0 mean and variance $\sigma^2(t_i - t_{i-1})$.

Unless otherwise stated, $x(0) = 0$.

Some mathematical series

$$\sum_{k=0}^{n} a^k = \frac{1 - a^{n+1}}{1 - a} \quad , \qquad \sum_{k=0}^{\infty} k a^k = \frac{a}{(1-a)^2} \quad .$$