Mathemathical Modelling

isakhammer

2020

Contents

1 Lecture 1

1.1 Practical Information

You need to know

- Separable 1. order equations.
- Linear 1. order equations.
- 2. order linear equations with constant coefficients.

1.2 Dimensional Analysis

Basic facts

- Any physical relation has to make sense dimensionally.
- Any physical relation must be valid for any choice of fundamental units.

Remark.

Make sure remark looks better

- **Forbidden** $3m + 2kg = ?$
- $m = f(x, t)$ is legal
- e^{-t} and $s = 5t^2$, is nonsense
- **Dimension** is length, mass , energy, etc.
- **Unit** is meter, feet, year, etc

Given a variable R, we write $R = \overbrace{v(R)}^{\text{numerical value}} \underbrace{[R]}_{\text{unit}}$.

If we have a physical relation that is dimensionall correct that

$$f(R_1, R_2, \ldots, R_n) = 0 \quad \rightarrow \quad f(v(R_1), v(R_2), \ldots, v(R_n)) = 0$$

1.3 Fundamental Units

Given units $F_1, F_2, \ldots, F_m$ for fundamental if

$$F_1^{\alpha_1}, F_2^{\alpha_2}, \ldots, F_m^{\alpha m} = 0 \quad \rightarrow \quad \alpha_1 = \alpha_2 = \ldots = 0$$

This units are then independent.

Example 1. The units kg, m, s are independent.

Example 2. In a right angle triangle with angle α and hypothenus c. We know the area A is uniquely determined by α and c

$$A = f\left(c, \alpha\right)$$

α is dimensialless since $\alpha = \frac{s}{r}$. Since A scales as the square of the length, then is

$$f\left(ac, \alpha\right) = a^2 f\left(c, \alpha\right)$$
$$c = 1 \rightarrow f\left(a, \alpha\right) = a^2 f\left(1, \alpha\right) = a^2 h\left(\alpha\right)$$

Which then ends up with the relation

$$A = a^2 h\left(\alpha\right)$$

Make corollary environmet

Lets derive $A = a^2 h\left(\alpha\right)$ somwhat differently. We know there is a relation $f\left(A, c, \alpha\right) = 0$. We want to introduce new variables.

$$\Pi_1 = \frac{A}{c^2}, \quad c = c_1, \quad \alpha = \alpha_1$$

which means $f\left(c^2\Pi_1, c, \alpha\right) = 0$ and $h\left(\Pi_1, \alpha, c\right) = 0$. h must be dimensially consistent $\rightarrow$ h must be independent of c.

$$\begin{aligned} h\left(\Pi_1, \alpha\right) = 0 \leftrightarrow \Pi_1 = k\left(\alpha\right) \\ \rightarrow \frac{A}{c^2} = k\left(\alpha\right) \quad \leftrightarrow \quad A = c^2 k\left(\alpha\right) \end{aligned}.$$

1.4 Trinity of the first atomic blast

We assume there is a relation

$$f\left(E, \rho, r, t\right) = 0$$

- Energy: E, $[E] = kgm^2s^{-2}$
- Mass density of air: ρ, $[\rho] = kg^{-3}$
- Radius: r , $[r] = m$
- Time: t, $[t] = s$

We choose 3 independent variables, say r, t, ρ. Also we call r, t, ρ **core variables**. Let is define a dimensionalless number Π_1 such that

$$[\Pi_1] = 0$$

The relation is now given by $h\left(\Pi, t, r, \rho\right) = 0$, where h is independent of t , r and ρ. Which in fact is $h\left(\Pi\right) = 0$, where $\Pi_1 = c$ s.t. $[c] = 1$.

Given by the definitnion is

$$\frac{Et^2}{\rho r^5} = c \quad \to \quad E = \frac{c\rho r^5}{t^2}$$

Using $\rho = 12kgm^{-3}$, $r = 110m$, $t = 6 \cdot 10^{-3}$ do we end up with the relation

$$E = c \cdot 7.5 \cdot 10^{13} J$$

1.5 Steady-state single phase flow in a uniform straight pipeline

Figure of a pipe

Pipe with flow u, length L and pressure drop Δp Then there is a relation between

- L : length, $[L] = m$
- D: diameter $[D] = m$
- u: flow rate $[u] = ms^{-1}$
- Δp: Pressure drop, $\left[\Delta kgm^{-1}s^{-2}\right]$
- μ: (Shear) viscousity $[\mu] = kgm^{-1}s^{-1}$
- ρ: mass density: $[\rho] = kgm^{-3}$
- E: Wall roughness: $[E] = m$

We have to choose 3 core variables and they are not unique. Since we have 3 independent units ρ, u, D are independent such that it can be a core variable:

$$\Pi_1 = \frac{L}{D} \quad , \quad \Pi_2 = \frac{\Delta p}{\rho u^2} \quad , \quad \Pi_3 = \frac{\rho}{\mu} \quad , \quad \Pi_4 = \frac{E}{D}$$

Then the relation is

$$f\left(\Pi_1, \Pi_2, \Pi_3, \Pi^4, \rho, D, u\right) = 0 \quad \Pi_2 = h\left(\Pi_1, \Pi_3, \Pi_4\right) \leftrightarrow \frac{\Delta p}{\rho u^2} = h\left(\Pi_1, \Pi_3, \Pi_4\right)$$

$$\to \frac{\Delta p}{u^2 \rho} = \Pi_1 k\left(\Pi_3, \Pi_4\right)$$

$$\Delta p = u^2 \rho \frac{L}{D} k\left(\frac{\rho D u}{\mu}, \frac{E}{D}\right)$$

$$\text{measure} \quad \frac{\rho D \mu}{\mu} \quad , \quad k = \frac{\Delta p D}{u^2 \rho}$$

2 Lecture 2

2.1 Practical Information

Ask for zoom meeting. ola.mahlen@ntnu.no, wednesday 13-14.

2.2 Recall

Last time did we consider steady-state single phase in a flow in a pipe.

- Assuming $f(L, \Delta p, u, \mu, D, E, \rho) = 0$ we arrive with this formula

$$\frac{\Delta p D}{u^2 \rho L} = k\left(\overbrace{\frac{\rho u D}{\mu}}^{\text{Reynhold number}}, \underbrace{\frac{E}{D}}_{\text{Relative wall roughness}}\right)$$

- Dimensionless numbers are often called **dimensionless groups**. Such numbers are independent of choice of fundamental units. They have real physical meaning. **Reynholds number** R_e essentially define what type of flow. Usually $R_e < 2000$ is it laminar flow and $R_e > 4000$ turbulent flow.

2.3 Scaling

Let a pipe have diameter D and flow rate u such that $t_v = \frac{D}{u}$. Then can we describe

$$t_\alpha = \frac{D^2}{\frac{\mu}{e}}$$

where μ is the kinematic viscosity. Then is R_e defined such that

$$R_e = \frac{t_\alpha}{t_v}$$

Assume we have the relation

$$R_1 = f(R_2, \dots, R_m)$$

Such that it exist an

$$\Pi_1 = g(\Pi_2, \Pi_2, \dots, \Pi_{m-k}).$$

2.4 Buckinghams Pi-Theorem

Assume we have a dimensionally valid relation $f(R_1, \dots, R_m) = 0$ and a set of fundemental units $F_1, F_2, \dots, F_n$ such that

$$[R_j] = F_1^{a_{j1}} F_2^{a_{j2}} \dots F_n^{a_{jn}} \quad j = 1, 2, \dots, m$$

This then defines the dimension matrix A given by

Table 1: caption

	F_1	F_2	$\dots$	F_n
R_1	a_{11}	a_{11}		a_{1n}
R_2	a_{21}	a_{21}		a_{2n}
$\vdots$		$\ddots$		
R_n	a_{m1}	$\dots$		a_{mn}

Fix better table environment.

Let $rank\,(A) = dim\,(row\,(A)) = k$. This translates to that we have k dimensionally independent variables. Choosing k linearly independent row vectors, corresponds to choosing core variables. Let this basis be $\mathbf{a}_{i1}, \mathbf{a}_{i2}, \dots, \mathbf{a}_{ik}$. Let the rest of the row vectors be

$$\mathbf{a}_{j_1}, \mathbf{a}_{j_2}, \dots, \mathbf{a}_{j_{m-k}}$$

Then is $\mathbf{a}_{j_r} = \sum_{s=1}^{k} C_{j_r,s}\mathbf{a}_{\mathbf{i_s}}$ where $r = 1, \dots, m-k$. We end up with the equation

$$\Pi_r = \frac{R_{j_r}}{R_{i_1}^{r_{j_r,1}} R_{j_2}^{a_{j_r,2}} \dots R_{j_k}^{a_{j_r,k}}}$$

Are dimensionally numbers.

Our relation becomes

$$g\left(\Pi_1, \dots, \Pi_{m-k}\right) = 0, \qquad \begin{cases} i_1, i_2, \dots, i_k \\ j_1, \dots, j_{m-k} \end{cases}$$

Example. Swinging pendulum
Assume there is a relation

$$f\left(w, \alpha_0, L, M, g,\right) = 0$$

where w is the frequenxy, g gravitational acceleration, M mass, α_0 the swinging angle. We can set L, M, g as core variables such that

$$\left[\frac{L}{g}\right] = s^2 \quad \to \quad \left[\frac{L}{g}w^2\right] = 1$$

$$f\left(w, \alpha_0, L, M, g\right) = 0 \implies \quad g\left(\alpha_0, \frac{Lw^2}{g}\right) = 0$$

2.5 Scaling

We have a problem at hand, usually differential equations. Then we tru to find representative scales for the various variables, and then write the equation on so-called fimensionless form. This has several advantages

- Our dimensionless variables are of order 1 .
- We get rid of a lot of physical constants.
- It makes us able to see what terms are "small" in the equation. The idea is to introduce dimensionless variables by introducing appropiate scales. If we have a stick of lenght L, we choose L as length scale i.e

$$x^* = Lx \quad \text{Where} \quad x \quad \text{is dimenionless}$$

Example. Heat flow in a rod with length L. Let $u^*(x^*, t^*)$ be the temperatur with the boundary conditions

$$u^*(0, t^*) = 0 \quad u^*(L, t^*) = 0$$

If we let the model be

$$\frac{\partial u^*}{\partial t^*} = D \cdot \frac{\partial^2 u^*}{\partial {x^*}^2}, \quad u^*(0, t^*) = 0 \quad u^*(L, t^*) = 0$$
$$u^*(x^*, 0) = u_0 \sin\left(\pi \frac{x^*}{L}\right)$$

We fund the tune scale T by scales **balancing the equation** .
Let $x^* = Lx$, and $t^* = Tt$, where T is to be determined $u^* = u_0 u$. If we find $u(x, t)$, then the physical temperature is given by

$$u^*(x^*, t^*) = u_0 u\left(\frac{x^*}{L}, \frac{t^*}{T}\right)$$

We have $u(0, t) = u(1, t) = 0$

$$\frac{\partial u^*}{\partial t^*} = D \frac{\partial^2 u^*}{\partial {x^*}^2} \quad \Longrightarrow \quad \frac{u_0}{T}\frac{\partial u}{\partial t} = \frac{u_0}{L^2} D \frac{\partial^2}{\partial x^2}$$
$$\leftrightarrow \frac{\partial u}{\partial t} = \left(\frac{TD}{L^2}\right)\frac{\partial^2 u}{\partial x^2} \quad \text{Balancing the equation}$$
$$\frac{TD}{L^2} = 1 \quad \Longrightarrow T = \frac{L^2}{D}$$
$$u^*(x^*, 0) = u_0 \sin\left(\pi \frac{x^*}{L}\right)$$
$$u(x, 0) = \sin(\pi x)$$

which fulfills the condition

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \quad , u(0, t) = u(1, t) = 0$$

3 References