NTNU
Norwegian University of
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Week 46: Lecture 1

Properties of Brownian motion and some general discussion about Gaussian processes

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Information

— Digital home exam with pass/fail
— Projects must be passed to take the exam
— The numerical scores from the projects do not affect pass/fail for the final exam
— Less focus on fact-based knowledge and derivations
— More focus on calculations and explanations

Section 3.2 (Note)

Definition (Def. 1)

Let $\{X(t) : t \geq 0\}$ be a stochastic process. The **hitting time** of level $a \in \mathbb{R}$ is the stochastic variable $\tau_a = \min\{t \geq 0 : X(t) = a\}$.

Theorem (Thm. 1)

Let $\{B(t) : t \geq 0\}$ be Brownian motion with variance parameter σ^2, $s > 0$, and $a \in \mathbb{R}$. Then

$$B(t)|B(s) = a \sim \mathcal{N}(a, \sigma^2(t-s)), \quad t \geq s.$$

Theorem (Thm. 2)

Let $\{B(t) : t \geq 0\}$ *be Brownian motion with variance parameter* σ^2 *and* $a > 0$, *then*

$$\begin{aligned} \Pr\{\tau_a \leq t\} &= 2\Pr\{B(t) > a\} \\ &= 2\left(1 - \Phi\left(\frac{a}{\sqrt{t\sigma^2}}\right)\right). \end{aligned}$$

Example 1

The current stock price is 1000 kr. Let time be measured in hours. We believe the stock price will evolve as a Brownian motion with variance parameter $\sigma^2 = 10^2$. What is the probability that the stock price will reach 1050 kr within 5 hours?

Markov property for Gaussian processes

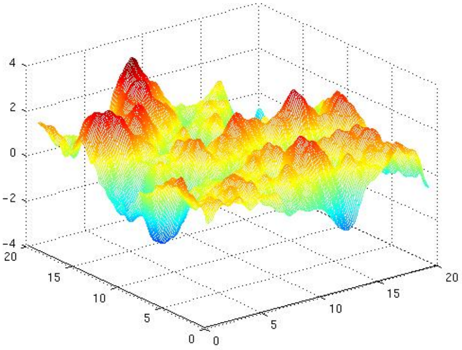
Are all Gaussian processes Markov?

Does the covariance function $C(t,s) = \sigma^2 \exp(-\phi|t-s|)$, for $t, s \geq 0$ give a Markov process?

Does the covariance function $C(t,s) = \sigma^2 \exp(-\phi(t-s)^2)$, for $t, s \geq 0$ give a Markov process?

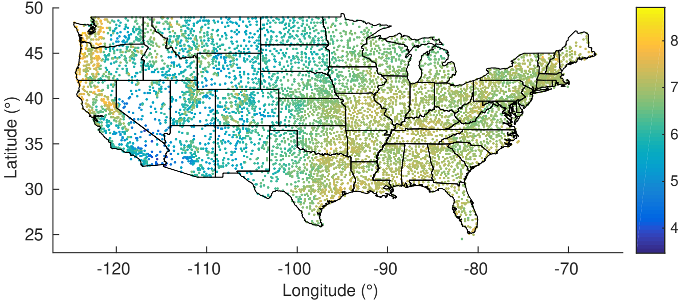
Gaussian random fields

The extension to $\{X(\boldsymbol{s}) : \boldsymbol{s} \in \mathbb{R}^2\}$ is immediate. We typically use the term **Gaussian random field.**

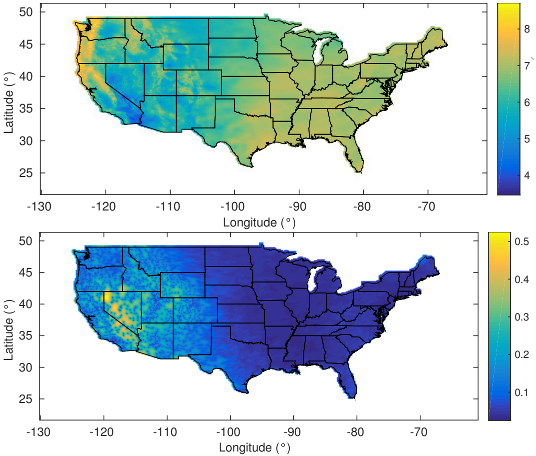


Spatial modelling

In a spatial setting we are typically interested in predicting unobserved locations



Prediction



50
45
40
35
30
25
Latitude (°)
-130
-120
-110
-100
-90
-80
-70
Longitude (°)
8
7
6
5
4
50
45
40
35
30
25
Latitude (°)
-130
-120
-110
-100
-90
-80
-70
Longitude (°)
0.5
0.4
0.3
0.2
0.1

Computational aspects

There is a difference between theory and practical applications!

— Realistic settings easily have $n = 10000$ observation locations

— Sometimes we have more than $n = 100000$ observation locations

— Dealing with 100000×100000 covariance matrices is not possible! This is 75 GBs!

— Computations are of complexity $\mathcal{O}(n^3)$

— What can we do???