TMA4190 Introduction to Topology Spring 2018

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Exercise set 11

$\boxed{1}$ Show that there exists a complex number z such that

$$z^7 + \cos(|z|^2)(1 + 93z^4) = 0.$$

$\boxed{2}$ **a)** Assume $\dim X \geq 1$: Show hat if $f\colon X \to Y$ is homotopic to a constant map, then $I_2(f, Z) = 0$ for all complementary dimensional closed Z in Y.
(Hint: Show that if $\dim Z < \dim Y$, then f is homotopic to a constant $X \to \{y\}$, where $y \notin Z$.

b) For $\dim X = 0$, show that this assertion is wrong. (If X is one point, for which Z will $I_2(f, Z) \neq 0$?)

c) Show that S^1 is not simply-connected. (Recall that we call a manifold X simply-connected if it is connected and if every map of the circle S^1 into X is homotopic to a constant map.)
(Hint: Consider the identity map.)

$\boxed{3}$ **a)** Show that intersection theory is trivial in contractible boundaryless manifolds: if Y is boundaryless and contractible (i.e. its identity map is homotopic to a constant map) and $\dim Y > 0$, then $I_2(f, Z) = 0$ for every $f\colon X \to Y$, X compact and Z closed, $\dim X + \dim Z = \dim Y$. In particular, intersection theory is trivial in Euclidean space.

b) Prove that no compact boundaryless manifold - other than the one-point space - is contractible.
(Hint: Apply the previous point to the identity map.)

$\boxed{4}$ **a)** Let $f\colon X \to S^k$ be a smooth map with X compact and $0 < \dim X < k$. Show that, for all closed submanifolds $Z \subset S^k$ of dimension complementary to X, $I_2(f, Z) = 0$.
(Hint: Use Sard's Theorem to show that there exists a $p \notin f(X) \cap Z$. Now use stereographic projection and the previous exercises.)

b) Show that S^2 and the torus $T = S^1 \times S^1$ are not diffeomorphic.

$\boxed{5}$ **a)** Two compact manifolds X and Z of the same dimension in Y are called **cobordant** in Y if there exists a compact manifold with boundary $W \subset Y \times [0, 1]$

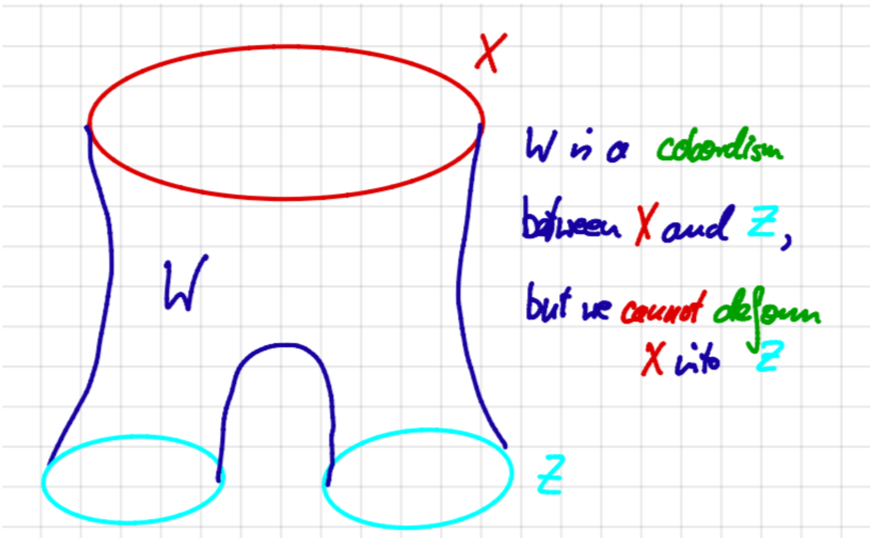
such that

$$\partial W = X \times \{0\} \cup Z \times \{1\}.$$

The manifold W is also called a **cobordism** between X and Z.

Show that if we can deform X into Z, i.e. if there is a smooth homotopy from the embedding $i_0 \colon X \hookrightarrow Y$ of X in Y to an embedding $i_1 \colon X \hookrightarrow Y$ with $i_1(X) = Z$ such that each i_t is an embedding, then X and Z are cobordant.

Note that the standard image of a cobordism, a pair of pants, illustrates that the converse is false: X and Z are cobordant, but we cannot deform X into Z, since X has one connected component whereas Z has two.



b) Show that if X and Z are cobordant in Y, then for every compact submanifold C in Y with dimension complementary to X and Z, i.e. $\dim X + \dim C = \dim Z + \dim C = \dim Y$ (where $\dim X = \dim Z$ because they are cobordant), we have

$$I_2(C, X) = I_2(C, Z).$$

(Hint: Let f be the restriction to W of the projection map $Y \times [0, 1] \to Y$, and use the Boundary Theorem.)

$\boxed{6}$ Let $p_1, \ldots, p_n$ be real polynomials in $n+1$ variables. Assume each p_i is homogeneous of odd order, i.e. there is an odd number m_i such that $p_i(\lambda x) = \lambda^{m_i} p_i(x)$ for all $\lambda \in \mathbb{R}$. We consider each p_i also as a smooth function $\mathbb{R}^{n+1} \to \mathbb{R}$ by sending x to $p_i(x)$.

Show that there is a line through the origin in $\mathbb{R}^{n+1}$ on which all the p_i's simultaneously vanish.

(Hint: Read Lecture 21 carefully.)