Norwegian University of Science and Technology
Deptartment of Mathematical Sciences

TMA4190 Introduction to Topology Spring 2018

Solutions to exercise set 12

1 Let $\beta = (v_1, \dots, v_k)$ be an ordered basis of a vector space V.

a) Replacing v_i by a multiple cv_i corresponds to multiplying β with the matrix which equals the identity matrix except at the ith position on the diagonal where 1 is replaced with c. The determinant of this matrix is equal to c. Hence $(v_1, \dots, v_k)$ and $(v_1, \dots, cv_i, \dots, v_k)$ are in the same equivalence class if and only if $c > 0$. If $c < 0$, they have opposite orientations.

b) Interchanging the places of v_i and v_j for $i \neq j$ corresponds to multiplying β with the matrix which equals the identity matrix with the ith and jth rows switched. We know from Linear Algebra that the determinant of this matrix is -1.

c) Subtracting from one v_i a linear combination of the others corresponds to multiplying β with a matrix that we obtain from the identity matrix by subtracting the corresponding linear combination of rows from the ith row. We know from Linear Algebra that this operation does not change the determinant of the matrix. Hence the determinant of the change-of-basis-matrix is still $+1$.

d) Suppose that V is the direct sum of V_1 and V_2. Let $(v_1, \dots, v_k)$ be an ordered positively oriented basis of V_1 and $(w_1, \dots, w_m)$ an ordered positively oriented basis of V_2. Then $(v_1, \dots, v_k, w_1, \dots, w_m)$ is an ordered positively oriented basis of $V_1 \oplus V_2$, and $(w_1, \dots, w_m, v_1, \dots, v_k)$ is an ordered positively oriented basis of $V_2 \oplus V_1$. Switching from the given positive basis of $V_1 \oplus V_2$ to the positive basis of $V_2 \oplus V_1$ corresponds to transposing exactly $(\dim V_1)(\dim V_2)$ many elements in the basis. Hence the determinant of the change-of-basis-matrix is $(-1)^{(\dim V_1)(\dim V_2)}$.

2 Let $(e_1, \dots, e_k)$ be the ordered basis of $\mathbb{R}^k$ which defines the standard orientation of $\mathbb{R}^k$. The orientation of $\mathbb{H}^k$ is given by the standard orientation of $\mathbb{R}^k$ restricted to the subspace $\mathbb{H}^k \subset \mathbb{R}^k$. The boundary orientation of $\partial\mathbb{H}^k$ is given by requiring that, at any point $x \in \partial\mathbb{H}^k$, the outward pointing unit normal vector $n_x = -e_k$ fits into a positively oriented basis for $\mathbb{R}^k$

$$(n_x, e_1, \dots, e_{k-1}) = (-e_k, e_1, \dots, e_{k-1}).$$

But the matrix which sends $(e_1, \dots, e_k)$ to $(-e_k, e_1, \dots, e_{k-1})$ is given by

$$A = \begin{pmatrix} 0 & 1 & & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & & 1 \\ -1 & 0 & \dots & & 0 \end{pmatrix}.$$

The matrix A can be transormed into the diagonal matrix

$$D = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & -1 \end{pmatrix}$$

by interchanging two columns exactly $k-1$ times. Hence $\det(D) = (-1)^{k-1}\det(A)$. But $\det(D) = -1$. Thus $\det(A) = 1 > 0$ if and only if $(-1)^k = 1$, i.e., if k is even.

3 **a)** At $x = (a, b, c) \in S^2$, the tangent space $T_x(S^2)$ is the two-dimensional vector subspace of $\mathbb{R}^3$ which is orthogonal to x. Since $(a, b, c) \neq (0, 0, 0)$, let us assume that, say, $b \neq 0$. A basis for $T_x(S^2)$ is given by, for example, $v = (-b, a, 0)$ and $v = (0, c, -b)$. The outward pointing normal vector is given by $n_x = (a, b, c)$. The boundary orientation of S^2, is the orientation of $T_x(S^2)$ determined by the basis (n_x, v, w). This basis is positively oriented in $T_x(\mathbb{R}^3) = \mathbb{R}^3$ if and only if the matrix $A = \begin{pmatrix} a & -b & 0 \\ b & a & c \\ c & 0 & -b \end{pmatrix}$ has positive determinant, since this is the matrix that transforms the standard basis of $\mathbb{R}^3$ into the basis (n_x, v, w). The determinant of A is

$$\det(A) = -a^2b - b^3 - bc^2 = -b(a^2 + b^2 + c^2) = -b.$$

Thus, if $b < 0$, (n_x, v, w) is a positively oriented basis of $T_x(S^2)$. If $b > 0$, we take the basis (n_x, w, v). And if $b = 0$, we start over with either a or c replacing b.

b) The boundary orientation of S^k is, at any point $x \in S^k$, given on $T_x(S^k)$ by chosing the ordered basis $(n_x, v_1, \dots, v_k)$ to be positively oriented where n_x is the outward pointing unit normal vector in $T_x(\mathbb{R}^{k+1}) = \mathbb{R}^{k+1}$ and $(v_1, \dots, v_k)$ is an ordered basis of $T_x(S^k)$. But since $S^k \subset \mathbb{R}^{k+1}$ is of codimension one, n_x spans the orthogonal complement $N_x(S^k, \mathbb{R}^{k+1})$ of $T_x(S^k)$ in $\mathbb{R}^{k+1}$. Hence the orientation of $T_x(S^k)$ induced by the direct sum

$$N_x(S^k, \mathbb{R}^{k+1}) \oplus T_x(S^k) = T_x(\mathbb{R}^{k+1}) = \mathbb{R}^{k+1}$$

equals the orientation of S^k as the preimage under g.

4 Assume that $df_{x_0} \colon T_{x_0}(X) \to T_{f(x_0)}(Y)$ preserves orientation at some point $x_0 \in X$. Since f is a diffeomorphism, df_x is an isomorphism for all $x \in X$. Hence $\det(df_x) \neq 0$ for all $x \in X$. In particular, the two disjoint open subsets $U := \{x \in X : \det(df_x) > 0\}$ and $V := \{x \in X : \det(df_x) < 0\}$ cover X. By assumption $x_0 \in U$, and hence U is nonempty. Since X is connected, this implies $U = X$.

5 Let X and Z be transversal submanifolds in Y and assume X, Z and Y are oriented. Let $i \colon X \hookrightarrow Y$ be the inclusion of X into Y. The intersection $X \cap Z$ equals the

preimage $i^{-1}(Z)$. By the lecture, the preimage orientation on $S := i^{-1}(Z)$ is induced, at any $y \in X \cap Z$, by the direct sum

$$N_y(S, X) \oplus T_y(S) = T_y(X),$$

where $N_y(S, X)$ is the orthogonal complement of $T_y(S)$ in $T_y(X)$. The orientation on $N_y(S, X)$ is induced by the direct sum

$$di_y(N_y(S, X)) \oplus T_y(Z) = T_y(Y),$$

and the fact that $d(i_y)_{|N_y(S,X)}$ is an isomorphism onto its image. Since all these vector spaces are subspaces in $T_y(Y)$, and are oriented as subspaces of $T_y(Y)$, we can identify $N_y(S, X)$ with its image under di_y in $T_y(Y)$ and can rewrite this equation as

$$N_y(S, X) \oplus T_y(Z) = T_y(Y).$$

Now let $N_y(S, Z)$ be the orthogonal complement of $T_y(S)$ in $T_y(Z)$. Then the orientation of $T_y(S)$ is determined by the direct sum

$$N_y(S, X) \oplus N_y(S, Z) \oplus T_y(S) = T_y(Y).$$

Now if we start the inclusion $j \colon Z \hookrightarrow Y$ of Z in Y instead, we get that the orientation of S considered as the preimage $j^{-1}(X)$ in Z, is determined by the direct sum

$$N_y(S, Z) \oplus N_y(S, X) \oplus T_y(S) = T_y(Y).$$

We learned in the first exercise that the signs of the orientations of $N_y(S, X) \oplus N_y(S, Z)$ and $N_y(S, Z) \oplus N_y(S, X)$ differ by $(-1)^{(\dim N_y(S,X))(\dim N_y(S,Z))}$. Now it remains to remark that, by defintion of the normal spaces as orthogonal complements, we have

$$\begin{aligned} \dim N_y(S, X) &= \operatorname{codim} X \cap Z \text{ in } X = \operatorname{codim} Z \text{ in } Y, \text{ and} \\ \dim N_y(S, Z) &= \operatorname{codim} X \cap Z \text{ in } Z = \operatorname{codim} X \text{ in } Y. \end{aligned}$$

6

a) Any basis of $V \times V$ consists of the product $(\alpha \times 0, 0 \times \beta)$ where α and β are ordered bases of V. The sign of this basis satisfies

$$\operatorname{sign}(\alpha \times 0, 0 \times \beta) = \operatorname{sign}(\alpha) \cdot \operatorname{sign}(\beta).$$

Switching the orientation of V changes both signs, $\operatorname{sign}(\alpha)$ and $\operatorname{sign}(\beta)$. Changing both signs simultaneously results in multiplying with $(-1)^2 = 1$. Hence the sign of the basis of $V \times V$ is independent of the choice of orientation for V.

b) Let X be an orientable manifold. The orientation of $X \times X$ is given by a smooth choice of orientation of each tangent space

$$T_{(x,y)}(X \times X) = T_x(X) \times T_y(X).$$

Changing the orientation of X means changing the orientation of both $T_x(X)$ and $T_y(X)$. As in the previous point, this means multiplying the sign of any ordered basis of $T_{(x,y)}(X \times X)$ by $+1$. Hence the product orientation on $X \times X$ is the same for all choices of orientation on X.

c) Let X be a smooth manifold which is not orientable. Any Euclidean space $\mathbb{R}^m$ is oriented as a manifold by the choice of the standard orientation of the tangent space $T_z(\mathbb{R}^m) = \mathbb{R}^m$ for any $z \in \mathbb{R}^m$. For any points $x \in X$ and $vzin\mathbb{R}^m$, the tangent space $T_{(x,z)}(X \times \mathbb{R}^m)$ is just $T_x(X) \times \mathbb{R}^m$. If there was a smooth choice for an orientation of $X \times \mathbb{R}^m$, then each tangent space $T_x(X)$ of X would inherit a smooth choice of orientation from the product $T_x(X) \times \mathbb{R}^m$. This contradicts the non-orientability of X.

Now let Y by any smooth manifold. If $X \times Y$ was orientable, then also $X \times U$ for an open subspace $U \subset Y$ which is diffeomorphic to some $\mathbb{R}^m$. But then $X \times \mathbb{R}^m$ would also inherit an orientation which is not possible. Applied to $Y = X$, we see that $X \times X$ is not orientable.

d) We can cover X by local parametrizations $\phi\colon U \to X$. The union of the images of the maps $\phi \times \phi\colon U \times U \to X \times X$ is then an open subspace V of $X \times X$ which includes Δ. We orient each individual $\phi(U)$ by requiring the diffeomorphism $\phi\colon U \to \phi(U)$ to be orientation preserving. This induces an orientation on $(\phi \times \phi)(U \times U) = \phi(U) \times \phi(U)$. As we argued before, changing the orientation on $\phi(U)$ does not change the orientation on the product $\phi(U) \times \phi(U)$, since we multiply the signs of all tangent spaces by $+1$. Hence there is a well-defined orientation on $\phi(U) \times \phi(U)$ which is independent on the local parametrizations chosen. Thus V which is an open neighborhood of Δ in $X \times X$ is orientable.

However, this does not mean that Δ is always orientable. For, the tangent space to Δ at any point (x, x) is the diagonal of $T_x(X) \times T_x(X)$. This diagonal is isomorphic to $T_x(X)$. Hence changing the orientation of $T_x(X)$ does change the orientation of the diagonal in $T_x(X) \times T_x(X)$. Thus if we had a smooth choice of orientations for all diagonals in $T_x(X) \times T_x(X)$, then we had a smooth choice of orientations for all $T_x(X)$. In other words, Δ is orientable if and only if X is orientable.