**TMA 4190 Introduction to Topology**
Lecturer: Gereon Quick
**Lecture 02**[1]

## 2. Lecture: Topology in $\mathbb{R}^n$ and smooth maps

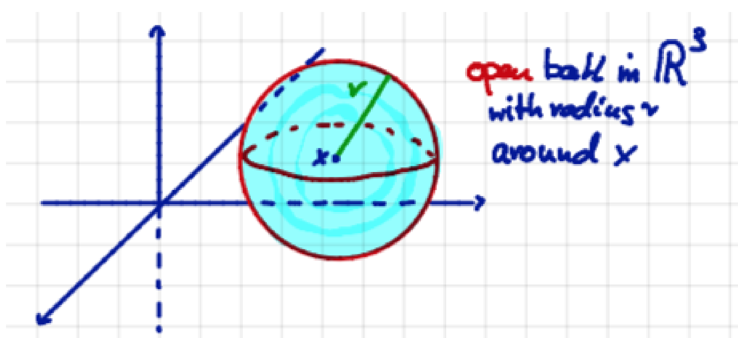
Recall from Calculus 2 that the norm of a vector $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ is defined by

$$|x| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2} \in \mathbb{R}.$$

For any $n$, the space $\mathbb{R}^n$ with this norm is called **$n$-dimensional Euclidean space**. It is a **topological space** in the following way:

### Open sets in $\mathbb{R}^n$

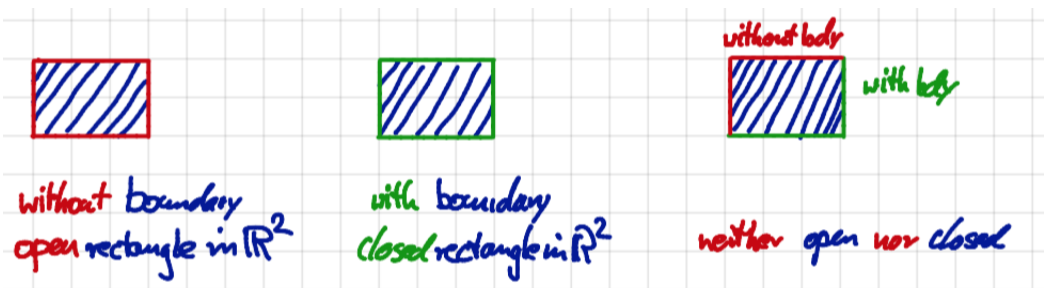
- Let $x$ be a point in $\mathbb{R}^n$ and $r > 0$ a real number. The ball
$$B_r(x) = \{y \in \mathbb{R}^n : |x - y| < r\}$$
with radius $\epsilon$ around $x$ is an **open** set in $\mathbb{R}^n$.
- The open balls $B_r(x)$ are the prototypes of open sets in $\mathbb{R}^n$.
- A subset $U \subseteq \mathbb{R}^n$ is called **open** if for every point $x \in U$ there exists a real number $\epsilon > 0$ such that $B_\epsilon(x)$ is contained in $U$.
- A subset $Z \subseteq \mathbb{R}^n$ is called **closed** if its complement $\mathbb{R}^n \setminus Z$ is open in $\mathbb{R}^n$.




- Familiar examples of open sets in $\mathbb{R}$ are open intervals, e.g. (0,1) etc.
- The cartesian product of $n$ open intervals (an open rectangle) is open in $\mathbb{R}^n$.
- Similarly, closed intervals are examples of closed sets in $\mathbb{R}$.

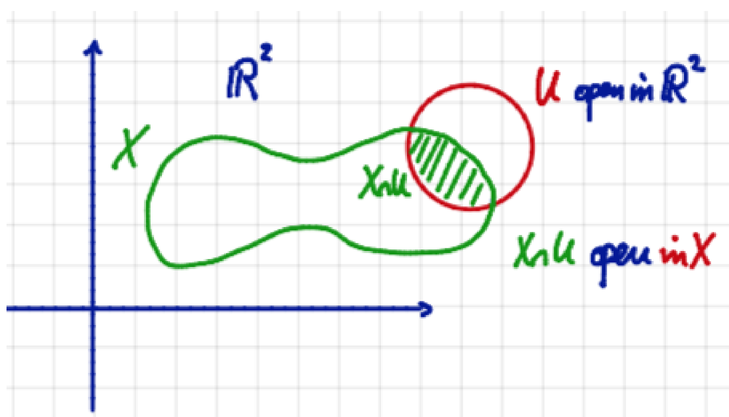
[1]Following the books of Guillemin and Pollack: Differential Topology, and Milnor: Topology from the differentiable viewpoint.

- The cartesian product of $n$ closed intervals (a closed rectangle) is closed in $\mathbb{R}^n$.
- The empty set $\emptyset$ and $\mathbb{R}^n$ itself are by both open and closed sets.
- Not every subset of $\mathbb{R}^n$ is open or closed. There are a lot of subsets which are neither open nor closed. For example, the interval (0,1] in $\mathbb{R}$; the product of an open and a closed interval in $\mathbb{R}^2$.



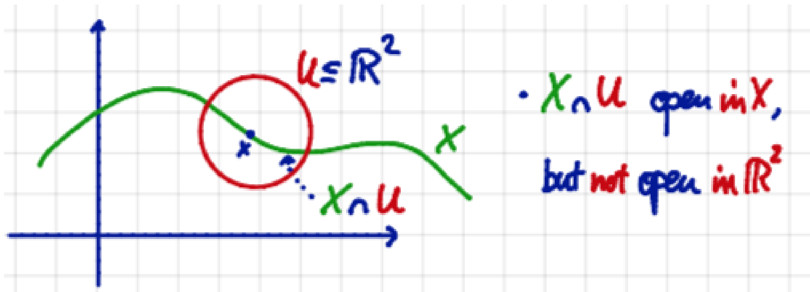

## Relative open sets

Let $X$ be a subset in $\mathbb{R}^n$. Then we say that $V \subseteq X$ is **open in $X$** (or **relatively open**) if there a an open subset $U \in \mathbb{R}^n$ with $V = U \cap X$.




## Warning

It is important to note that that the property of being **an open subset** really depends on the bigger space we are looking at. Hence **open** always refers to being **open in** some given space.
For example, a set can be open in a space $X \subset \mathbb{R}^2$, but not be open in $\mathbb{R}^2$, see the picture.




Open sets are nice for a lot of reasons. First of all, they provide us with a way to talk about things that happen **close to** a point.

## Open neighborhoods

We say that a subset $V \subseteq X$ containing a point $x \in X$ is a **neighborhood of $x$** if there is an open subset $U \subseteq V$ with $x \in U$. If $V$ itself is open, we call $V$ an **open neighborhood**.

Second, the collection of all open subsets in a set $X$, define a **topology** on $X$. A set together with a topology, is called a **topological space**.

We observe here that the word **"topology"** is used in different ways. On the one hand, it is the name of a **whole area** in mathematics. On the other hand, it is the name for a certain **structure on a set**.
We see that phenomenon happen quite often. For example,

- the term **"algebra"** denotes both a field in mathematics and a certain type of structure on a set;
- the term **"medicine"** denotes the field, but a doctor can also prescribe a specific medicine to cure a desease.

The type of maps that preserve open sets are the continuous maps:

## Continuous maps

Let $A$ be a subset in $\mathbb{R}^n$. A map $f\colon A \to \mathbb{R}^m$ is called **continuous at $a$** if for every $\epsilon > 0$, there is a $\delta > 0$ such that

$$0 < |x - a| < \delta \Rightarrow |f(x) - f(a)| < \epsilon.$$

In our new fancy notation, we can reformulate the last condition as: given any $\epsilon > 0$,

$$\text{there is a } \delta > 0 \text{ such that } \mathbf{x} \in \mathbf{B}_\delta(\mathbf{a}) \cap \mathbf{A} \Rightarrow \mathbf{f}(\mathbf{x}) \in \mathbf{B}_\epsilon(\mathbf{f}(\mathbf{a})).$$

Finally, in terms of limits, we could say: $f$ is continuous at $a$ if

$$\lim_{x \to a} f(x) = f(a).$$

The map $f$ is called **continuous** if it is continuous at every $a \in A$.

A more intrinsic characterization that serves as a definition for arbitrary topological spaces is the following.

## Continuous maps: a more general characterization

A map $f\colon A \to \mathbb{R}^m$ is **continuous** if and only if, for every open subset $U \subseteq \mathbb{R}^m$, there is some open subset $V \subseteq \mathbb{R}^n$ with $f^{-1}(U) = V \cap A$ (in other words $f^{-1}(U)$ is open in $A$).

**Proof**:

First, assume $f$ is continuous. Let $U \subseteq \mathbb{R}^m$ be an open set in $\mathbb{R}^m$. If $f^{-1}(U)$ is empty, it is open by definition. So let $a \in f^{-1}(U)$ be a point in $f^{-1}(U)$. The fact that $U$ is open means that there is an $\epsilon > 0$ such that $B_\epsilon(f(a)) \subset U$. Given this $\epsilon$, the fact that $f$ is continuous means that

$$\text{there is a } \delta > 0 \text{ such that } x \in B_\delta(a) \cap A \Rightarrow f(x) \in B_\epsilon(f(a)).$$

But

$$f(x) \in B_\epsilon(f(a)) \text{ implies } f(x) \in U \text{ which implies } x \in f^{-1}(U) \cap A.$$

Since $x$ was arbitrary in $B_\delta(a) \cap A$ this means $B_\delta(a) \cap A \subseteq f^{-1}(U)$.

Second, assume that $f^{-1}(U)$ is open in $A$ for every open subset $U \subseteq \mathbb{R}^m$. Given $a \in A$ and $\epsilon > 0$, let $B_\epsilon(f(a)) \subset \mathbb{R}^m$ be the open ball around $f(a)$ with radius $\epsilon$. Since $B_\epsilon(f(a))$ is open in $\mathbb{R}^m$, our assumption tells us that $f^{-1}(B_\epsilon(f(a)))$ is open in $A$. Since $a \in f^{-1}(B_\epsilon(f(a)))$ this means that

$$\text{there is a } \delta > 0 \text{ such that } B_\delta(a) \subseteq f^{-1}(B_\epsilon(f(a))).$$

But that means

$$x \in B_\delta(a) \Rightarrow f(x) \in B_\epsilon(f(a)).$$

Hence $f$ is continuous at $a$. Since $a$ was arbitrary, $f$ is continuous. **QED**

## Homeomorphisms

A continuous map $f\colon X \to Y$ is a **homeomorphism** if one-to-one and onto, and its inverse $f^{-1}$ is continuous as well. Homeomorphisms preserve the topology in the sense that $U \subset X$ is open in $X$ if and only if $f(U) \subset Y$ is open in $Y$.

**Examples**:

- $\tan\colon (-\pi/2,\pi/2) \to \mathbb{R}$ is a homeomorphism.
- $f\colon \mathbb{R} \to \mathbb{R}$, $x \mapsto x^3$ is a homeomorphism.

## Example: Bijection which is not a homeomorphism

Let

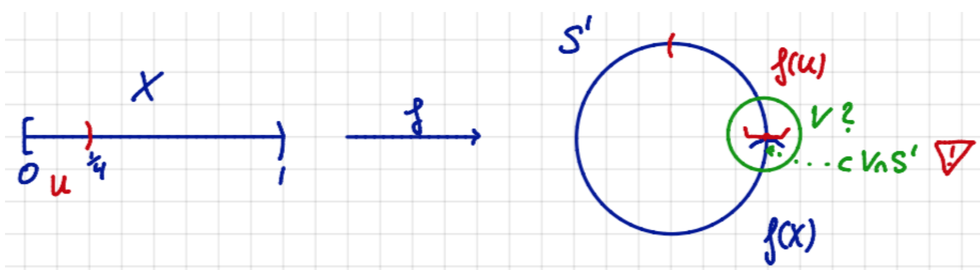
$$S^1 = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\} \subset \mathbb{R}^2$$

be the unit circle considered as a subspace of $\mathbb{R}^2$. Define a map

$$f\colon [0,1) \to S^1,\ t \mapsto (\cos(2\pi t), \sin(2\pi t)).$$

We know that $f$ is bijective and continuous from Calculs and Trigonometry class. But the function $f^{-1}$ **is not continuous**. For example, the image under $f$ of the open subset $U = [0,\frac{1}{4})$ (open **in** $[0,1)$!) is not open in $S^1$. For the point $y = f(0)$ does not lie in any open subset $V$ of $\mathbb{R}^2$ such that

$$V \cap S^1 = f(U).$$




## Spaces

From now on, when we talk about a **space** we mean a set together with a specified topology or collection of open subsets.

**Remark**:
For topological spaces $X$ and $Y$, a map $f\colon X \to Y$ is defined to be **continuous** if for every open set $U \subseteq Y$ the subset $f^{-1}(U)$ is open in $X$. Just in case you have heard of categories before: **Topological spaces** form a **category** with morphisms given by continuous maps.

Here is another extremely important property a subset in a topological space can have. We are going to use it quite often in fact.

## Compact sets in $\mathbb{R}^n$

- A subset $Z$ in a topological space is called **compact** if every open cover $\{U_i\}_i$ of $Z$ has a *finite* subcover. That is, among the $\{U_i\}_i$ it is always possible to pick $U_{i_1}, \ldots, U_{i_n}$ with
$$Z = U_{i_1} \cup \ldots \cup U_{i_n}.$$
- By the Theorem of Heine-Borel, a subset $Z \subset \mathbb{R}^n$ is **compact** if and only if it is **closed and bounded**. Being bounded means, that there is some (possibly huge) $r >> 0$ such that $Z \subset B_r(0)$.

Compactness is an important example of a topological property:

## Homeomorphisms preserve

Slogan: **Topology** is the study of properties which are preserved under homeomorphisms. From this point of view, a **topological property** is by definition a property that is preserved under homeomorphisms. For example, if $f\colon X \to Y$ is a homeomorphism, then $Z \subseteq X$ is compact if and only if $f(Z) \subseteq Y$ is compact.

Finally, open sets are nice because we can say what it means to be differentiable on an open set.

## Recall: Smooth maps on open subsets

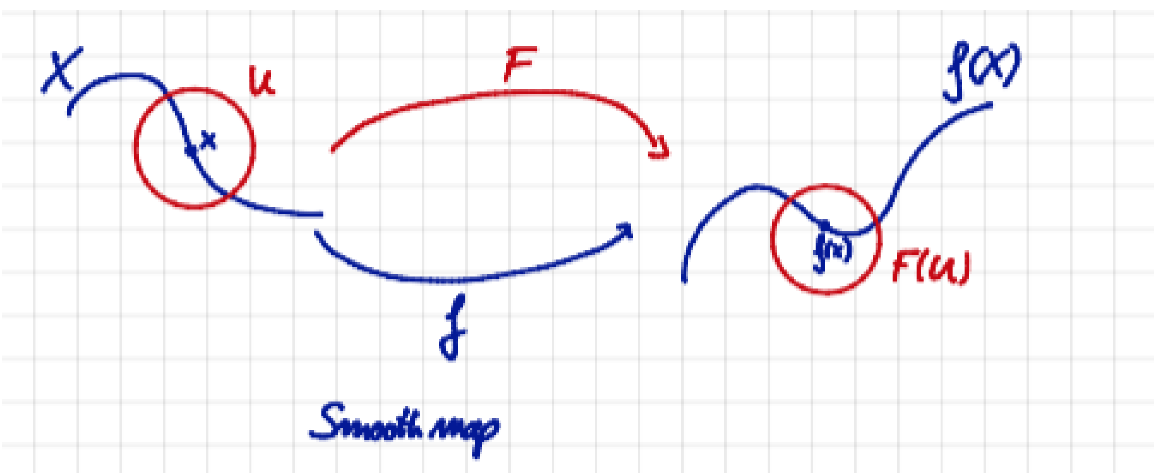
Let $U \subseteq \mathbb{R}^n$ and $V \subseteq \mathbb{R}^m$ be open sets. A map $f\colon U \to V$ is called **smooth** if it has continuous partial derivatives of **all orders** (i.e. all the partial derivatives $\partial^k f_j/\partial x_{i_1} \ldots \partial x_{i_k}$ exist and are continuous for **all** $k \geq 1$).
Recall also: another way to say that $f$ is **differentiable at** $a \in U$ if there

is a linear map $\lambda \colon R^n \to \mathbb{R}^m$ such that
$$\lim_{h \to 0} \frac{|f(a+h) - f(a) - \lambda(h)|}{|h|} = 0.$$
Note that if such a $\lambda$ exists, it is unique and is often denoted $df_a$.

Note that a smooth map is in particular also continuous. More generally, we can define smoothness for maps between arbitrary sets subsets of $\mathbb{R}^n$:

## Smooth maps

Let $X \subseteq \mathbb{R}^n$ and $Y \subseteq \mathbb{R}^m$ be arbitrary subsets. A map $f \colon X \to Y$ is called smooth if for each $x \in X$ there exist an open subset $U \subseteq \mathbb{R}^n$ and a smooth map $F \colon U \to \mathbb{R}^m$ that coincides with $f$ on all of $X \cap U$, i.e. $F_{X \cap U} = f$.



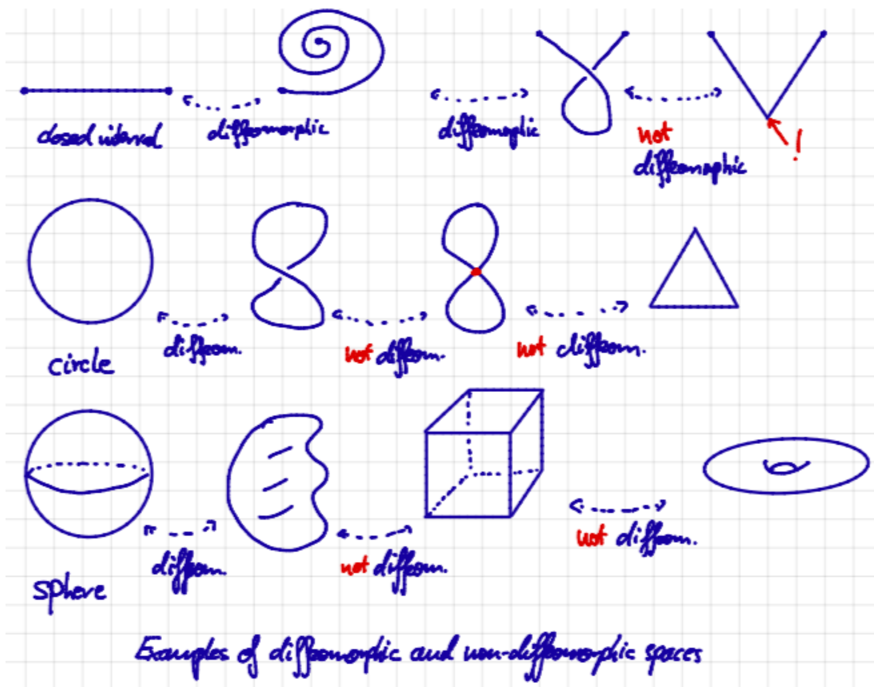

- The identity map of any set $X$ is of course smooth.
- If $f \colon X \to Y$ and $g \colon Y \to Z$ are smooth, then the composition $g \circ f$ is also smooth.
- Note that **smoothness is a local property**, that means we need to check it only in a small neighborhood of any point.

## Diffeomorphisms

A smooth map $f \colon X \to Y$ is called a **diffeomorphism** if $f$ one-to-one and onto, and its inverse $f^{-1}$ is smooth as well.

We say that $X$ and $Y$ are **diffeomorphic** if there exists a diffeomorphism $f\colon X \to Y$.

Note that every diffeomorphism is a homeomorphism, but not the other way around. For example, $f\colon \mathbb{R} \to \mathbb{R}$, $x \to x^3$ is a homeomorphism, but not a diffeomorphism. Exercise!




## Diffeomorphic spaces are "equivalent"

Differential topology is the study of those properties of spaces which do not change under diffeomorphisms. In other words, from the point of view of differential topology, diffeomorphic spaces are equivalent, and we may (and will) consider them as copies of the same abstract space, which may happen to be differently situated in their surrounding Euclidean spaces.