Solutions

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Contents

1 Chapter 4

1.1 Exercise 4.6

Let $\mathscr{B}$ be collection of all subsets on the form $A_{a,b} = \{az + b \mid z \in \mathbb{Z}\}$ of $\mathbb{Z}$, where $a, b \in \mathbb{Z}$ and $a \neq 0$. (The set $A_{a,b}$ is known as as an arithmetic progression)

- Show that $\mathscr{B}$ is a basis for a topology on $\mathbb{Z}$.

  Answer.

  - **B1:** For every $n \in \mathbb{Z}$, there is an
  $$a \in \mathbb{Z} \setminus \{0\}$$
  such that
  $$n \in A_{a,n} = \{az + n \mid z \in \mathbb{Z}\}$$
  Hence, **B1** holds.

  - **B2:** Let $B_1 = A_{a_1,b_1}$, $B_2 = A_{a_2,b_2}$ be two basis elements. Let $x \in B_1 \cap B_2$. Then
  $$\begin{aligned} x \in B_1 = A_{a_1,b_1} &= \{a_1 z + b_1\} \\ &= \{\ldots, -2a_1 + b_1, -a_1 + b_1, b_1, a_1 + b_1, \ldots\} \end{aligned}$$
  Let $x \in B_2 = A_{a_2,b_2} = A_{a_2,x}$. Thus if
  $$B_3 = A_{a_1,a_2}, x = \{a_1 a_2 z + x \mid z \in z\}$$
  then $x \in B_3 \subseteq B_1 \cap B_2$. Hence **B2** holds.

- Show that there are infinitely many primes by using the topology generated by $\mathscr{B}$. (This topology is known as the arithmetic progression on $\mathbb{Z}$)

Answer. We observe that $A_{a,b}$ is both open and closed: it is clearly open as it is a basis element and it is closed since

$$A_{a,b}^c = \mathbb{Z} \setminus A_{a,b}$$

is open: for $x \in A_{a,b}^c$, we have

$$A_{a,x} \subseteq A_{a,b}^c$$

.

Assume there are finitel many primes. Then

$$\bigcup_{P \text{ primes}} A_{p,a} = \mathbb{Z} \setminus \{-1, 1\}$$

is closed as it is the union of finitely many closed sets. Hence, $\{-1, 1\}$ must be open which is a contradiction: Every non-empty open set in this space is infinite.

Thus there are infinitely many prims.

Chapter 5

Ex 5.1

$\mathbb{R}$. $\mathbb{R}$ with the standard topology

$$\begin{aligned} X &= (a,b) \subseteq \quad \mathbb{R}\text{is subpace} \\ Y &= (-1,1) \subseteq \quad \mathbb{R}\text{is subspace} \\ X &\simeq Y \end{aligned}$$

Let

$$\begin{aligned} f : X &\longrightarrow Y \\ x &\longmapsto fX(x) = 2\frac{x-a}{b-a} - 1. \end{aligned}$$

Then f is a bijectictive continious map with inverse

$$\begin{aligned} f^{-1}Y : X &\longrightarrow Y \\ y &\longmapsto f^{-1}Y(y) = a + (b-a)\,\frac{y+1}{2}. \end{aligned}$$

Which is continious. Thus f is a homeomorphism.

Ex 5.2

Let

$$\begin{aligned} g : Y &\longrightarrow \mathbb{R} \\ y &\longmapsto g(y) = \tan\left(\frac{\pi}{2}y\right). \end{aligned}$$

Then g is a bijective continious map with inverse

$$\begin{aligned} g^{-1} : \mathbb{R} &\longrightarrow Y \\ t &\longmapsto g^{-1}(t) = \frac{2}{\pi}\arctan(t). \end{aligned}$$

From calculus we know that g^{-1} is continious. Hence, g is homeomorphism

$$x \simeq \mathbb{R}$$

Let

$$\begin{aligned} h : X &\longrightarrow \mathbb{R} \\ x &\longmapsto h(x) = (g \cdot f(x)) = g(f(x)) \\ &\downarrow \\ g(f(x)) &= \left(\frac{2(x-a)}{b-a} - 1\right) \\ &= \tan\left(\frac{\pi}{2}\left(\frac{2(x-a)}{b-a} - 1\right)\right) \end{aligned}$$

.

Then h is a homeomorphism as it is the composition of f and g .

Ex 5.3

- X be topological space.
- Let $Y \subseteq X$, $A \subseteq Y$ be subsets .
- τ_{X_A} subspace topology on A inherited from X .
- τ_{Y_A} subspace topology on A inherited from Y .

Let τ be the topology on X, and let τ_Y be the subspace topology on Y . Thus

$$\begin{aligned} \tau_Y &= \{Y \cap U \mid U \subseteq X \text{ is open}\} \\ \tau_{X_A} &= \{A \cap V \mid V \subseteq X \text{ is open}\} \\ \tau_{Y_A} &= \{A \cap \mid w \in \tau_Y\} \end{aligned}$$

(i) $\tau_{X_A} \subseteq \tau_{Y_A}$: Let $A \cap V \in \tau_{X_A}$. Then

$$Y \cap V \in \tau_Y$$

and so

$$A \cap V \in \tau_{Y_A}$$

Hence $\tau_{X_A} \subseteq \tau_{Y_A}$

(ii) $\tau_{Y_A} \subseteq \tau_{X_A}$: Let $A \cap W \in \tau_{Y_A}$. Then there is a $V \in \tau$ s.t.

$$W = Y \cap V$$

Hence,

$$\begin{aligned} A \cap W &= A \cap (Y \cap V) \\ &= A \cap V \in \tau_{X_A} \end{aligned}$$

$\tau_{X_A} = \tau_{Y_A}$

Ex 5.4

Let $\emptyset \subseteq \mathbb{R}$ be a subspace.

$$A = \left\{x \in \emptyset \mid -\sqrt{5} < x < \sqrt{5}\right\}$$

Fact.

(i) X topological space

(ii) $S \subseteq X$ subspace

$K \subseteq S$ is closed

$$\Longrightarrow$$

There is a $L \subseteq X$ closed in X with $K = S \cap L$

Proof. (i) $\Longrightarrow$: Assume that $K \subseteq S$ is closed in S . Then

$$V = S \setminus K$$

is open in S . Thus there is some $U \subseteq X$ open with

$$V = S \cap U.$$

Furthermore

$$L = X \setminus U$$

is closed in X. Also

$$\begin{aligned} S \cap L &= S \cap (X \setminus U) = S \setminus (S \cap U) \\ &= S \cap V = K \end{aligned}$$

(ii) $\Longleftarrow$: By asumption. $U = X \setminus L$ is open in X. Thus $V = S \cap U$ is open in S . Since

$$\begin{aligned} S \setminus K &= S \setminus (S \cap L) \\ &= S \cap (X \setminus L) \\ &= S \cap U \\ &= V \end{aligned}$$

Thus K is closed. Since $\left(-\sqrt{5}, \sqrt{5}\right) \subseteq \mathbb{R}$ is open and

$$A = \left(-\sqrt{5}, \sqrt{5}\right) \cap \mathbb{Q}$$

A is open in $\mathbb{Q}$. Similarly as

$$\left[-\sqrt{5}, \sqrt{5}\right] \subseteq \mathbb{R}$$

is closed and

$$A = \left(-\sqrt{5}, \sqrt{5}\right) \cap \mathbb{Q}$$

A is closed in $\mathbb{Q}$.

□

5.5

X, Y topological spaces. $A \subseteq X, B \subseteq Y$ subspaces. $\tau_{A\times B}$ product topology on $A \times B$. $\tau_{(X\times Y)_{A\times B}}$ the subspace topology on $A \times B$ inherited from $X \times Y$.

We will show that

$$\tau_{A\times B} = \tau_{(X\times Y)_{A\times B}}.$$

Let τ_A be the subspace topology on A , i.e.,

$$\tau_A = \{A \subseteq U \mid U \subseteq X \text{ is open}\}$$

Similarly ,

$$\tau_B = \{B \subseteq V \mid V \subseteq Y \text{ is open}\}$$

Let $\mathscr{B}_X$ be the basis for the topology on X . Let $\mathscr{B}_Y$ be the basis for the topology on Y . Then

$$\mathscr{B}_{A\times B} = \{(A \subseteq U_X) \times (B \cap U_Y) \mid U_X \in \mathscr{B}_X, U_Y \in \mathscr{B}_Y\}$$

is a basis for $\tau_{A\times B}$. From the fact that

$$\mathscr{B}_{X\times Y} = \{B_x \times B_Y \mid B_X \in \mathscr{B}_X, B_Y \in \mathscr{B}_Y\}$$

is a basis for $\tau_{X\times Y}$, it follows that

$$\mathscr{B}_{(X\times Y)_{A\times B}} = \{(A \cap B_X) \cap (B_x \times B_Y) \mid B_X \in \mathscr{B}_X, B_Y \in \mathscr{B}_Y\}$$

is a basis for $\tau_{(X\times Y)_{A\times B}}$. Since

$$(A \times B) \subseteq (B_X \times B_Y) = (A \cap B_X) \times (B \subseteq B_Y)$$

We have

$$\mathscr{B}_{A\times B} = \mathscr{B}_{(X\times Y)_{A\times B}}$$

is a basis for $\tau_{A\times B}$.

5.6

X, Y topological spaces Let

$$\pi_1 : X \times Y \to X$$
$$\pi_2 : X \times Y \to y$$

be projection maps. Let $\tau_{X\times Y}$ be the product topology $X \times Y$. By definition of the product topology, τ_1 and τ_2 are contiions.

Assume that τ is some topology on $X \times Y$ s.t. π_1 and π_2 are continious. Then,

$$\pi^{-1}(U) = U \times Y \in \tau$$
$$\pi_2^{-1}(V) = X \times V \in \tau$$

For $U \subseteq X$ is open, $V \subseteq Y$is open. Since τ is a topology

$$(U \times Y) \cap (X \times Y) = U \times V \in \tau$$

Hence ,

$$\tau_{X \times Y} \in \tau$$

Ex 5.8

Let

- $\mathbb{R}$: $\mathbb{R}$ with standard topology.
- $\pi \ \mathbb{R} \to \mathbb{Z}$
-

$$\begin{aligned} x : x, &\quad x \in X \\ n, &\quad x \in (n-1, n+1)\,, \quad n \text{ odd integer.} \end{aligned}$$

$$\tau^{\pi} = \left\{U \subseteq \mathbb{Z}, \quad \pi^{-1}(U) \text{ is open in } \mathbb{R}\right\}$$

For n an odd integet, we have

$$\pi^{-1}(\{n\}) = (n-1, n+1) \subseteq \mathbb{R}$$

is open. For n an even integer , $\pi^{-1}(\{n\}) = \{n\}$ which is not open.

The smallest open subset of $\mathbb{Z}$ that contains n is $\{n-1, n, n+1\}$ as

$$\pi^{-1}(\{n-1, n, n+1\} = (n-2, n+2))$$

is open in $\mathbb{R}$.

Hence, τ^{π} is the same as the digital line topology.

2 References