Norwegian University of Science and Technology
Deptartment of Mathematical Sciences

TMA4190 Introduction to Topology Spring 2018

Solutions to exercise set 3

1 The derivative of f at $x \in \mathbb{R}^n$ is just given by the linear map $df_x : \mathbb{R}^n \to \mathbb{R}^n$, $v \mapsto A \cdot v$. Hence df_x is an isomorphism if and only if A is invertible. So f is a local diffeomorphism if and only if A is invertible. Now f is a diffeomorphism if and only if the map $x \mapsto f(x) - b = Ax$ is a diffeomorphism which is the case if and only if A is invertible. For, if this is the case then $y \mapsto A^{-1}y + A^{-1}b$ is the inverse of f. Hence f is a diffeomorphism if and only if A is invertible.

2 The map f is not an embedding, since it is not injective. But we can check it is an immersion by showing that the derivative is injective everywhere. The derivative of f at (s,t) is

$$df_{(s,t)} : \mathbb{R}^2 \to \mathbb{R}^3,\ df_{(s,t)} = \begin{pmatrix} -\sin s \cos t & -(2+\cos s)\sin t \\ -\sin s \sin t & (2+\cos s)\cos t \\ \cos s & 0 \end{pmatrix}.$$

In order to show that $df_{(s,t)}$ is injective, we need to check that it has full or maximal rank, i.e. rank 2. Hence we need to check that the two column vectors are always linearly independent. To simplify notation, we set $x = \sin s$, $y = \cos s$, $u = \sin t$, and $v = \cos t$. Now assume there are two real numbers λ and μ such that

$$\begin{aligned} \lambda(-xv) + \mu(-(2+y)u) &= 0 \\ \lambda(-xu) + \mu((2+y)u) &= 0 \\ \lambda y &= 0. \end{aligned}$$

We distinguish two cases: $y = \cos s = 0$ and $y = \cos s \neq 0$. If $y \neq 0$, then we must have $\lambda = 0$ and

$$\begin{aligned} \mu(2+y)u &= 0 \\ \mu(2+y)v &= 0. \end{aligned}$$

Since $|y| \leq 1$, we know $2 + y \neq 0$. Hence we can divide by $2 + y$. Moreover, we know that not both u and v can be 0 at the same time. This implies $\mu = 0$.

If $y = 0$, then $x = \sin s = \pm 1$ and still $2 + y \neq 0$. Hence we get the system

$$\begin{aligned} \pm\lambda v - \mu(2+y)u &= 0 \\ \pm\lambda u + \mu(2+y)v &= 0. \end{aligned}$$

But since u and v are never both 0, we know that the vectors (v,u) and (u,v) are linearly independent. Hence we must have $\lambda = 0$ and $\mu(2+y) = 0$. The latter implies $\mu = 0$, since $2 + y \neq 0$.

$\boxed{3}$ Let $\alpha = b/a$ for two relatively prime integers a, b with $a \neq 0$. Then, for $t = a$, we have

$$\gamma(a) = (e^{2\pi i a}, e^{2\pi i b}) = (e^0, e^0) = \gamma(0).$$

Hence the image of $\mathbb{R}$ under γ is equal to the image of $[0, a]$ under γ. Thus γ factors through the map

$$g\colon S^1 \to S^1 \times S^1,\ \ e^{2\pi i t} \mapsto (e^{2\pi i a t}, e^{2\pi i b t}),$$

and $f\colon \mathbb{R} \to S^1,\ t \mapsto e^{2\pi i t}$. In other words, there is a commutative diagram

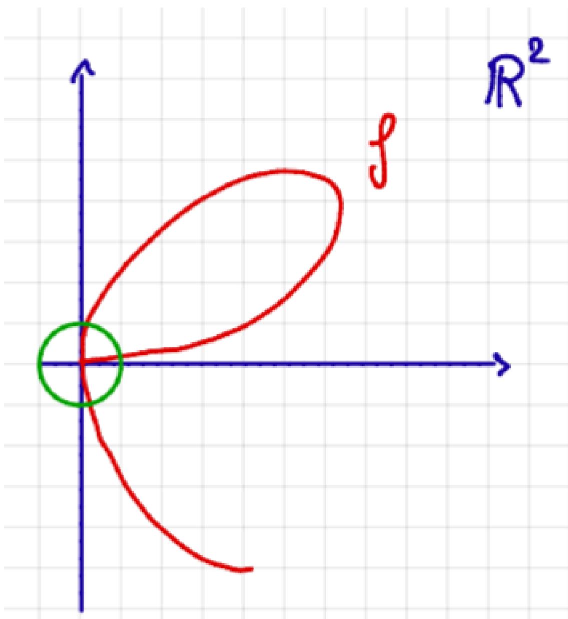
$$\begin{tikzcd} \mathbb{R} \arrow[rr, "\gamma"] \arrow[dr, "f"'] & & S^1 \times S^1 \\ & S^1. \arrow[ur, "g"'] & \end{tikzcd}$$

Since S^1 is compact and g a one-to-one immersion, g is an embedding.

$\boxed{4}$ **a)** The derivative of f at t is the map $df_t\colon \mathbb{R} \to \mathbb{R}^2$ given by the Jacobian matrix

$$\begin{aligned}
&(2\cos(2t)\cos t + \sin(2t)(-\sin t), 2\cos(2t)\sin t + \sin(2t)\cos t) \\
&= (2(\cos^2 t - \sin^2 t)\cos t - 2\sin t\cos t\sin t, 2(\cos^2 t - \sin^2 t)\sin t + 2\sin t\cos t\cos t) \\
&= 2(\cos^3 t - 2\sin^2 t\cos t, 2\cos^2 t\sin t - \sin^3 t) \\
&= 2(\cos t(\cos^2 t - 2\sin^2 t), \sin t(2\cos^2 t - \sin^2 t)) \\
&= 2(\cos t(1 - 3\sin^2 t), \sin t(3\cos^2 t - 1))
\end{aligned}$$

where we have use several trigonometric identities. Since the derivative df_t is nontrivial for all t, it is always injective as a linear map $\mathbb{R} \to \mathbb{R}^2$. Hence f is an immersion.



b) But f is not a homeomorphism onto $\mathrm{Im}\,(f)$. For, consider the open subset $(\pi/4, 3\pi/4)$ in $(0, 3\pi/4)$. If f was a homeomorphism, then $f((\pi/4, 3\pi/4))$ had to be open in $\mathrm{Im}\,(f)$ as well. That means that around any point, for example the point $f(\pi/2) = (0,0)$, there had to an open beighborhood contained in $f((\pi/4, 3\pi/4))$. By the definition of the open sets in $\mathrm{Im}\,(f)$ as a subspace of $\mathbb{R}^2$, there had to be an open ball $B_\epsilon(0,0) \in \mathbb{R}^2$ with

$$B_\epsilon(0,0) \cap \mathrm{Im}\,(f) \subset f((\pi/4, 3\pi/4)).$$

But for every $\epsilon > 0$, we have

$$B_\epsilon(0,0) \cap f((0, \pi/4)) \neq \emptyset,$$

since $|\sin(2t)(\cos t, \sin t)| < \epsilon$ for all $t < \epsilon/2$ (where we use $\sin x \leq x$ and $|(\cos t, \sin t)| = 1$). Hence f cannot be an open map and therefore not a homeomorphism.

c)
- What is the difference between $\mathrm{Im}\,(f)$ and the graph $\Gamma(f)$?
  The graph of a map $X \to Y$ is a subspace of $X \times Y$. In this case, $\Gamma(f)$ is a subspace of $(0, 3\pi/4) \times \mathbb{R}^2$, whereas $\mathrm{Im}\,(f)$ is a subspace of $\mathbb{R}^2$.
- Is the map $F\colon (0, 3\pi/4) \to (0, 3\pi/4) \times \mathbb{R}^2$ an embedding?
  Yes, because F is a diffeomorphism $(0, 3\pi/4) \to \Gamma(f)$ (since f is smooth, see previous exercise set). Hence it is in particular, a one-to-one immersion and proper.
- Would f be an embedding if it was defined on the closed interval $[0, 3\pi/4]$?
  No, because f would not be injective anymore: $f(0) = (0,0) = f(\pi/2)$.
- Is the map $g\colon (0, 3\pi/4) \to \mathbb{R}^3,\ t \mapsto \sin(2t)(\cos t, \sin t, t)$ an embedding?
  No, this map is still just an immersion and it is even one-to-one, but it is not a homeomorphism onto its image in $\mathbb{R}^3$. The same argument as in the previous point.
- Is the map $h\colon [0, 3\pi/4] \to \mathbb{R}^3,\ t \mapsto (\sin(2t)\cos t, \sin(2t)\sin t, 2t)$ an embedding?
  Yes, this time we have a map which is an immersion, it is one-to-one this time $f(0) = (0,0,0) \neq (0,0,\pi) = f(\pi/2)$, and it is defined on a compact space and is therefore a proper map.

$\boxed{5}$ By the Local Immersion Theorem, we can choose local parametrizations $\phi\colon V \to Z$ and $\psi\colon W \to X$ around z with $V \subset \mathbb{R}^k$ and $W \subset \mathbb{R}^n$ such that

$$\begin{array}{ccc} Z & \xrightarrow{\text{inclusion}} & X \\ \phi \uparrow & & \uparrow \psi \\ V & \xrightarrow[\text{immersion}]{\text{canonical}} & W \end{array}$$

commutes. The map ψ is a diffeomorphism onto its image $\psi(W) \subset X$. The inverse map $\psi^{-1}\colon \psi(W) \to W$ is a local coordinate system on the open neighborhood $\psi(W)$ around $z \in X$. We write $x_i\colon \psi(W) \to \mathbb{R}$ for the ith component of ψ^{-1}, i.e. a point $p \in \psi(W)$ has the local coordinates $(x_1(p), \ldots, x_n(p)) = (\psi_1^{-1}(p), \ldots, \psi_n^{-1}(p))$. Since the above diagram commutes and ϕ is a diffeomorphism onto its image, we have $\phi(V) =$

$Z \cap \psi(W)$. Hence, since the lower horizontal map is the canonical immersion, the points in $Z \cap \psi(W)$ are exactly those on which the coordinate functions $x_{k+1}, \dots, x_n$ vanish. Relabeling the open subset $\psi(W)$ as U we have

$$Z \cap U = \{p \in U : x_{k+1}(p) = \cdots = x_n(p) = 0\}.$$