Recall that in Lecture 13, we discussed planar systems for which $\mathrm{D}f|_{\mathbf{x}_0}$ at an isolated critical point $\mathbf{x}_0$ was degenerate. It either had one zero eigenvalue, two zero eigenvalues but a geometric multiplicity of one, or two zero eigenvalues with full geometric multiplicity. Whilst we characterized behaviours in the first two cases, we left the final case open. In the final case, it is possible for there to be many (compatibly oriented) hyperbolic, elliptic, and parabolic sectors at an isolated critical point. We shall address some aspects of the final case now.

Theorem 20.7 (Bendixson's Index Formula)**.** *Let e be the number of elliptic sectors and h be the number of hyperbolic sectors at an isolated critical point $\mathbf{x}_0$ of a dynamical system governed by the analytic vector field f. Then*

$$I_f(\mathbf{x}_0) = 1 + \frac{1}{2}(e - h).$$

This theorem is a consequence of the following lemma, which is also helpful in the effective calculation of the index:

Lemma 20.8. *Let $\mathbf{x}_0$ be an isolated critical point of the planar analytic system*

$$\dot{x} = P(x,y), \qquad \dot{y} = Q(x,y).$$

Let C be a Jordan curve enclosing $\mathbf{x}_0$ and no other critical points. Let A be the number of times $Q(x,y)/P(x,y)$ jumps from $-\infty$ to ∞ and let D be the number of times $Q(x,y)/P(x,y)$ jumps from ∞ to $-\infty$ as (x,y) varies one cycle along C in the counter-clockwise sense. Then

$$I_{(P,Q)^\top}(\mathbf{x}_0) = \frac{1}{2}(D - A).$$

Proof. The index is defined by the formula

$$I_{(P,Q)^\top}(\mathbf{x}_0) := \oint_C \mathrm{d}\Big(\arctan(\frac{\mathrm{d}y}{\mathrm{d}x})\Big).$$

By L'Hospital's rule, $\mathrm{d}y/\mathrm{d}x = Q/P$, so setting

$$\Theta(x,y) = \arctan(\frac{Q(x,y)}{P(x,y)}), \tag{32}$$

we have

$$I_{(P,Q)^\top}(\mathbf{x}_0) := \frac{1}{2\pi}\oint_C \mathrm{d}\Theta(x,y).$$

Next notice that $|Q(x,y)/P(x,y)|$ is infinite on C exactly when $P(x,y) = 0$ because P and Q are never zero cotemporaneously on C.

Now every time $Q(x,y)/P(x,y)$ jumps from $-\infty$ to ∞, it crosses an angle $(2k+1)\pi/2$ clockwise (decreases across such an angle). Similarly, every time $Q(x,y)/P(x,y)$ changes from ∞ to $-\infty$, it crosses an angle $(2k+1)\pi/2$ counter clockwise (increases across such an angle).

As the integral is phase-indpendent, we can add any fixed small shift ϑ to Θ, and see that every increase in 2π of Θ involves two jumps from ∞ to $-\infty$, and every decrease in 2π of Θ involves two jumps from $-\infty$ to ∞.

□

What Bendixson's index theorem does suggest is that it is possible to calculate an index by inspecting the phase portrait, and this lemma foregoing allows us to do so algebraically from the equations.

Remark 20.1. Assuming the hypotheses of the lemma, let M and N respectively be the number of times that $Q(x,y)/P(x,y)$ changes sign along C at a zero of Q from negative to positive, and from positive to negative. Then

$$I_{(P,Q)^\top}(\mathbf{x}_0) = \frac{1}{2}(M - N).$$

■

Remark 20.2. Recall that the winding number of a curve γ on $\mathbb{C}$ about a point is defined as

$$W(\gamma, z_0) := \frac{1}{2\pi i}\int_\gamma \frac{1}{z - z_0}\,\mathrm{d}z = \frac{1}{2\pi}\int_\gamma \mathrm{d}\arg(z - z_0).$$

From the calculations in the proof above (specifically (32)), it is also apparent what is meant that if the vector field f were "straightened-out", so that corresponding contortions were introduced to a simple closed curve γ on the same plane on which lies f, and thereafter the plane were identified canonically with $\mathbf{C}^2$, then $I_f(\gamma)$ is the winding number of the contorted curve. The plane straightened out by f is simply the plane where $f(z)$ replaces z.

■

21. Lecture XXI: Index Theory II

In this lecture we shall calculate some examples where we apply Bendixson's Index Theorem, or the attendent lemma. Next we shall look at another way to technique by which to analyse planar systems globally, much like we used the polar coordinates transformation to analyse them locally.

21.1. **Examples.**

Example 21.1. Consider the system

$$\begin{aligned}\dot{x} &= P(x, y) = x,\\ \dot{y} &= Q(x, y) = -y.\end{aligned}$$

This is the archetypal saddle. There is a critical point at the origin and it is a saddle.

Using Lemma 20.8, or the remark thereafter, let us take a small simple closed curve about the origin. For concreteness, we can take the circle of radius $\delta > 0$. As we traverse this circle in the counter-clockwise sense, we encounter two places where $Q(x, y) = 0$, namely the intersections with the positive and negative parts of the x-axis. In order to avoid having any of these intersections at the end-points of our traversal on the circle, and introducing some ambiguity as to the number of intesections encountered, we can start at, say, $\vartheta = \pi/4$.

At the first encounter with $Q(x, y) = 0$, we are at $(-\delta, 0)$. The sign of $Q(x, y)/P(x, y)$ changes from positive to negative.

At the second encounter with $Q(x, y) = 0$, we are at $(\delta, 0)$. The sign of $Q(x, y)/P(x, y)$ changes from positive to negative.

We conclude that

$$I_f(\mathbf{0}) = \frac{1}{2}(0 - 2) = -1.$$

Example 21.2. Consider the nonlinear system

$$\begin{aligned}\dot{x} &= P(x, y) = x^2 + xy\\ \dot{y} &= Q(x, y) = \frac{1}{2}y^2 + xy.\end{aligned}$$

This system can be found on pp 151 - 152 of *Perko*. It has a critical point at the origin with two elliptic sectors and two parabolic sectors.

We see that $Q(x, y) = 0$ along $y = 2x$ and $y = 0$. A circle enclosing the origin traverses this line twice. Going in a counter-clockwise sense, starting at $\vartheta = \pi/4$, at the first intersection we find $Q(x, y)/P(x, y)$ changes sign from negative to positive. This happens again at each of the subsequent intersections. Therefore,

$$I_f(\mathbf{0}) = \frac{1}{2}(4 - 0) = 2,$$

also verifying Bendixson's index theorem.

21.2. **The Poincaré Index Theorem.**

The Poincaré index theorem equates the index of a system on a two-dimensional surface and the Euler-Poincaré characteristic of that surface. A two dimensional surface is a two dimensional, compact, C^2-manifold, defined in a manner described in Lecture 8. Alternatively, for us, it is the surface of a doughnut with a finite number of holes, although we amke the caveat that this alternative excludes a number of interesting two dimensional surfaces not smoothly embeddable in $\mathbb{R}^3$, in particular, to non-orientable surfaces, to which the theorem still in fact applies. It goes beyond the scope of this module to discuss what precisely orientability means. But we point out that smooth orientable 2-dimensional surfaces are all smoothly embeddable in $\mathbb{R}^3$.

On every neighbourhood of such a surface M, we can locally parameterize the surface by coordinates (x, y), as a two dimensional manifold "looks locally like $\mathbb{R}^2$". And it makes sense to discuss a two-dimensional dynamical system on this surface, just as we had done for a system on a plane.

The vector field $f(p)$ will be in a separate vector space, the tangent space T_pM. Suppose now that this vector field has finitely many critical points $\{\mathbf{p}_i\}_{i=1}^m$.

Using the local homeomorphism to $\mathbb{R}^2$, around each point $\mathbf{p}_i$, there is a small neighbourhood U_i of $\mathbf{p}_i$ and a smooth homeomorphism $\varphi_i : U_i \to V_i \subseteq \mathbb{R}^2$ between U_i and an open neighbourhood in $\mathbb{R}^2$. We can define the index $I_f(\mathbf{p}_i)$ around the critical point $\mathbf{p}_i$ on the surface M as the index $I_{f_j}(\varphi_i(\mathbf{p}_i))$ of the vector field $f_i := \varphi_{i*}f = \mathrm{D}\varphi_i f$ (the pushfoward is defined in Lecture 8, which is f, written out in local coordinates φ_i).

A heuristic way to think about this is that U_i is basically a flat patch, and the index at $\mathbf{p}_i$ can be found by drawing a small enough simple closed curve around $\mathbf{p}_i$ so that the curve fits inside U_i. We need coordinates φ_i essentially so that we can speak clearly about the line integral defining the index.

We then define the INDEX OF THE SURFACE M relative to the vector field f as

$$I_f(M) := \sum_i I_f(\mathbf{p}_i).$$

The startling fact is that $I_f(M)$ is in fact vector field independent. This number only depends on M, and is equal to the EULER-POINCARÉ CHARACTERISTIC of M, $\chi(M)$. This number $\chi(M)$ is calculated in the following way:

First we invoke the fact that any two dimensional surface can be tessellated by finitely many curvilinear triangles. These curvilinear triangles may be found by first tessellating a local patch $V_i = \varphi_i(U_i) \subseteq \mathbb{R}^2$ by actual triangles in $\mathbb{R}^2$, which defines the triangle on $U_i \subseteq M$. There are two things of which we want to be sure. The first is that these tessellations match up properly. The second is more notational — I used U_i and V_i because we had been talking about them, but in fact, we can find patches to cover all of M, and not just near critical points of some arbitrary vector field f. This tessellation is known as a TRIANGULATION. Given a triangulation Δ of a surface, we can calculate an integer

$$\chi_\Delta(M) = F - E + V,$$

where F is the number of faces, E the number of edges, and V the number of vertices.

A triangulation is not unique. However, by varying any given triangulation, say, by removing edges and vertices, it can be shown that any triangulation returns the same number $\chi_\Delta(M)$, so in fact $\chi_\Delta(M)$ is independent of the triangulation Δ, and we can call it $\chi(M)$. In fact, by removing edges, it can be shown that this number does not depend on the tessellation being one of curvilinear triangles at all. $\chi(M)$ only depends on topological properties of M. We call such a number a TOPOLOGICAL INVARIANT.

Having defined both $I_f(M)$ and $\chi(M)$, we can state the following theorem:

Theorem 21.1 (Poincaré Index Theorem). *Let M be a smooth two-dimensional surface, and f a C^1-vector field defined on it, with at most finitely many critical points. Then*

$$I_f(M) = \chi(M).$$

Refer to *Perko*, pp.307 – 310 for a sketch of a proof. The idea for orientable surfaces is to show it for a sphere by reasoning about curves on spheres and the resulting index, and then proceed to add "handles" to the sphere, where each "handle" is itself homeomorphic to a sphere.

21.3. **Compactifying the plane.** This brings us to a notion that is important in the theory of planar systems but which we shall not have time to explore in detail.

It is possible to compactify a plane by adding a point at infinity. This notion on the complex plane of adding a "point at infinity" turns the complex plane into closed surface known as the Riemann sphere. As you may also be familiar via the theory of fractional linear transformation (i.e., Möbius transformations), it is possible to project the plane onto a sphere $\{(x, y, z) : x^2 + y^2 + (z-1)^2 = 1\}$, sitting atop the plane, tangent to it at the origin, via the standard stereographic projection. This projects the point at infinity to the "north pole", and gives us a way to discuss criticality at infinity.

Where the $x-y$ plane is the phase plane of a planar system, we call this sphere the BENDIXSON SPHERE.

Another projection that is often used generates the POINCARÉ SPHERE. This projection projects $\mathbb{R}^2$ again onto the sphere $\{(x, y, z) : x^2 + y^2 + (z-1)^2 = 1\}$, but this time, we take a line from the centre of the enclosed ball, $(0, 0, 1)$. We associate the intersection of this line with the $x-y$ plane with the point of intersection of this line with the sphere that is not the north pole. For simiplicity of calculations, it helps to consider the change-of-coordinates $(x, y, z) \mapsto (X, Y, Z)$ on the sphere, where $X = x$, $Y = y$, and $Z = 1-z$. In the new coordinates, the sphere is $\{(X, Y, Z) : X^2+Y^2+Z^2 = 1\}$. Let us use small letters for the phase plane and capital letters for the coordinates on the sphere. The map $(x, y) \mapsto (X, Y, Z)$ from the intersection with the plane to the associated point on the sphere can be written out explicutly:

By looking at similar triangles, it is clear that

$$\frac{x}{1} = \frac{X}{Z}, \qquad \frac{y}{1} = \frac{Y}{Z},$$

and using $X^2 + Y^2 + Z^2 = 1$, we find that

$$(xZ)^2 + (yZ)^2 + Z^2 = 1,$$

so we can write the projection as

$$X = \frac{x}{\sqrt{1+x^2+y^2}}, \qquad Y = \frac{y}{\sqrt{1+x^2+y^2}}, \qquad Z = \frac{1}{\sqrt{1+x^2+y^2}}.$$

The idea of the Poncaré sphere is that in the Bendixson projection, the behaviour at infinity is projected to a point, but on the Poincaré sphere, it is spread out over the equator, and so may become less complicated.

Either way, we see that starting with a planar system, we can find a homeomorphic system on a compact two-dimensional surface. Therefore we have an index. Now if f on the plane had finitely many critical points, we have seen that the sum of their indices can be any integer. But the Euler-Poincaré characteristic of a sphere (as of a cube, or a dodecahedron) is $\chi(S^2) = 2$.

To reconcile the discrepancy we have to notice that it is possible to have a critical point at infinity. We should like to be able independently to calculate the index at infinity to be able to show that adding it to the indices at the remaining critical points, we in fact arrive at the Euler-Poincaré characteristic of a sphere, 2.

To this end it is actually simpler to look at not the Poincaré or Bendixson spheres, but simply an inversion of the plane by the unit circle (flipping the plane inside-out about the unit circle).

Let us define the inverted coordinates of the $x-y$ plane by

$$X = \frac{x}{x^2+y^2}, \qquad Y = \frac{-y}{x^2+y^2}.$$

The point of an inversion is that $(X^2+Y^2) \cdot (x^2+y^2) = 1$ with a reflection. So in polar coordinates (r, ϑ), an inversion is given by

$$R = \frac{1}{r}, \qquad \Theta = -\vartheta.$$

(Perhaps see Coxeter's *Introduction to Geometry* if you have not heard of inversions in Euclidean geometry classes.)

This is familar as the inversion from complex analysis if we set $z = x + iy$, $Z = X + iY$, then an inversion is

$$Z = \frac{1}{z},$$

which accounts for the prima facie superfluous reflection.

Writing $f = (P, Q)^\top$, set $g = P + iQ$, where we abuse notation so that $P(x, y) = P(z)$, and mutatis mutandis for Q, f and g. If we invert g by

$$G = \Big(\frac{P}{P^2+Q^2}, \frac{-Q}{P^2+Q^2}\Big)^\top,$$

we can readily see from the linearity of the derivative that the inverted system to

$$\dot{z} = P(z) + iQ(z)$$

is

$$\dot{Z} = G_1(1/Z) + iG_2(1/Z),$$

where, of course, $G = G_1 + iG_2$. The inverted system in polar coordinates is simply

$$\dot{R} = \frac{-\dot{r}}{r^2}, \qquad \dot{\Theta} = -\dot{\vartheta}.$$

We define the INDEX AT INFINITY, denoted by $I_f(\infty)$, as the index of the origin in the inverted system.

As expected from the Poincaré index theorem, we have the following result:

Theorem 21.2. *Let f be a C^1-vector field for which $\dot{x} = f(x)$ has m critical points, with indices at these critical points given by $\{I_i\}_{i=1}^m$. Then*

$$I_f(\infty) + \sum_i I_i = 2.$$

This turns on a simple calculation which itself turns on the fact that a simple closed curve integrated one way in the original system maps onto a simple closed curve oriented in the opposite sense in the inverted system. For details see *Jordan and Smith*, section 3.2.