Norwegian University of Science and Technology
Department of Mathematical Sciences

TMA4145 Linear Methods
Fall 2018

Exercise set 2

Please justify your answers! Note that *how* you arrive at an answer is more important than the answer itself.

1 **a)** Determine the following numbers and decide in each case whether "supremum" can be replaced by "maximum":

1. $\sup_{x\in(0,\infty)} \frac{1}{x^2}$;
2. $\sup_{x\in\mathbb{R}} e^{-2|x|}$;
3. $\sup_{n\in\mathbb{N}} \frac{n^2+3}{n^2+1}$;
4. $\sup_{n\in\mathbb{N}}(-1)^n \frac{n+3}{n^2+1}$.

b) Determine the following numbers and decide in each case whether "infimum" can be replaced by "minimum":

1. $\inf_{x\in(0,\infty)} \frac{1}{x^2}$;
2. $\inf_{x\in\mathbb{R}} e^{-2|x|}$;
3. $\inf_{n\in\mathbb{N}} \frac{n^2+3}{n^2+1}$;
4. $\inf_{n\in\mathbb{N}}(-1)^n \frac{n+3}{n^2+1}$.

2 Let A be bounded above. Show that the supremum of A is unique.

3 Let A be a bounded subset of $\mathbb{R}$, and let cA be its dilate by a positive constant $c > 0$. Show that

$$\inf cA = c \inf A.$$

4 Let X be a vector space.

1. Prove that the additive inverse is unique (meaning for any $x \in X$ there exists a unique vector $y \in X$ such that $x + y = 0$; we denote the additive inverse of x by $-x$.)
2. Show that for every $x \in X$ we have $(-1)x = -x$. In words multiplication by the scalar -1 gives the additive inverse of a vector.

5 A function $f : \mathbb{R} \to \mathbb{R}$ is called *odd* if $f(t) = -f(-t)$ for all $t \in \mathbb{R}$.

Prove or disprove that the set of odd functions $\mathbb{R} \to \mathbb{R}$ with component-wise addition

$$(f + g)(t) = f(t) + g(t),$$

and scalar multiplication

$$(\lambda f)(t) = \lambda f(t), \quad \forall\, \lambda \in \mathbb{R},$$

form a real vector space.

Hint: you need to find a zero vector, an additive inverse of each element f, and check the axioms of a real vector space. Most importantly, check that the operations of addition and scalar multiplication does not "lead out of space", i.e., that they are indeed operations $V \times V \to V$ and $\mathbb{R} \times V \to V$, respectively.

6 Let X be a vector space and T a linear mapping $T : X \to X$.

1. Show that the range of T is a subspace of X.
2. Let D be the differentiation operator $Df(x) = f'(x)$. Determine the kernel and the range of $Tf = f' - 3f$ for $f \in C^{(1)}(\mathbb{R})$, the space of continuously differentiable functions on $\mathbb{R}$.