Norwegian University of Science and Technology
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TMA4145 Linear Methods
Fall 2018

Exercise set 10: Solutions

Please justify your answers! The most important part is *how* you arrive at an answer, not the answer itself.

$\boxed{1}$ Let X be a Hilbert space and $T : X \to X$ a bounded linear operator. Suppose x and x' are two elements in X. Show that if

$$\langle x, y\rangle = \langle x', y\rangle \quad \text{for all } y \in X,$$

then $x = x'$.

Solution. Since $\langle x, y\rangle = \langle x', y\rangle$ for all $y \in X$, we find that $0 = \langle x, y\rangle - \langle x', y\rangle = \langle x - x', y\rangle$ for all $y \in X$. In particular, this must be true for $y = x - x'$, so

$$\langle x - x', x - x'\rangle = 0.$$

This implies that $x - x' = 0$, by positive definiteness of the inner product (part (3) of definition 2.3.1), hence $x = x'$.

$\boxed{2}$ Define on $C[0, 1]$ the inner product

$$\langle f, g\rangle = \int_0^1 f(t)\overline{g(t)}\,dt$$

Show that $(C[0, 1], \langle\cdot,\cdot\rangle)$ is an inner product space, but that it is not complete with respect to the norm

$$\|f\|_2 = \left(\int_0^1 |f(t)|^2\,dt\right)^{1/2}$$

induced by the inner product.

Solution. We need to show that $\langle ., .\rangle$ defines an innerproduct on $C[0, 1]$.

Linearity

Let $f_1, f_2, g \in C[0,1]$ and let $\lambda_1, \lambda_2 \in \mathbb{C}$. We have

$$\begin{aligned}\langle \lambda_1 f_1 + \lambda_2 f_2, g\rangle &= \int_0^1 (\lambda_1 f_1(x) + \lambda_2 f_2(x))\,\overline{g(x)}dx \\ &= \int_0^1 \left(\lambda_1 f_1(x)\overline{g(x)} + \lambda_2 f_2(x)\overline{g(x)}\right) dx \\ &= \lambda_1 \int_0^1 f_1(x)\overline{g(x)}dx + \lambda_2 \int_0^1 f_2(x)\overline{g(x)}dx \\ &= \lambda_1\langle f_1, g\rangle + \lambda_2\langle f_2, g\rangle\end{aligned}$$

<u>Symmetry</u>

Let $f, g \in C[0,1]$. We have

$$\langle f, g\rangle = \int_0^1 f(x)\overline{g(x)}dx = \overline{\int_0^1 \overline{f(x)}g(x)dx} = \overline{\int_0^1 g(x)\overline{f(x)}dx} = \overline{\langle g, f\rangle}.$$

<u>Positive definiteness</u> Clearly $\langle 0, 0\rangle = \int_0^1 0\overline{0}\; dx = 0$. On the other hand, if

$$\langle f, f\rangle = \int_0^1 f(x)\overline{f(x)}\; dx = \int_0^1 |f(x)|^2\, dx = 0,$$

then $f = 0$ since f is assumed to be continuous [1].

We will now show why $(C[0,1], \langle .,.\rangle)$ is not complete with respect to the induced norm. We will do this by constructing a Cauchy sequence $\{f_n\}_{n\in\mathbb{N}}$ in $C[0,1]$ that doesn't converge in $C[0,1]$. We define (this is essentially the same example as 4a) on problem set 6)

$$f_n(x) = \begin{cases} -1, & 0 \le x < \frac{1}{2} - \frac{1}{n} \\ n(x - \frac{1}{2}), & \frac{1}{2} - \frac{1}{n} \le x < \frac{1}{2} + \frac{1}{n} \\ 1, & \frac{1}{2} + \frac{1}{n} \le x \le 1. \end{cases}$$

See also the plot below. We show that this is a Cauchy sequence. Pick an $\epsilon > 0$. Let

[1] Detailed proof: Assume that $\int_0^1 |f(x)|^2\; dx = 0$. If $f \neq 0$, there would be some $x_0 \in [0,1]$ such that $|f(x_0)|^2 > 0$. Since $|f(x)|^2$ is continuous, there would then exist some $\delta > 0$ such that $|f(x)|^2 \ge \frac{1}{2}|f(x_0)|^2$ for $|x - x_0| < \delta$ (This is a $\delta - \epsilon$ argument with $\epsilon = \frac{1}{2}|f(x_0)|^2$) But then $\int_0^1 |f(x)|^2\; dx \ge \int_{x_0-\delta}^{x_0+\delta} |f(x)|^2\; dx \ge \frac{1}{2}|f(x_0)|^2 \int_{x_0-\delta}^{x_0+\delta} \ge \frac{1}{2}|f(x_0)|^2 2\delta > 0$ – a contradiction.

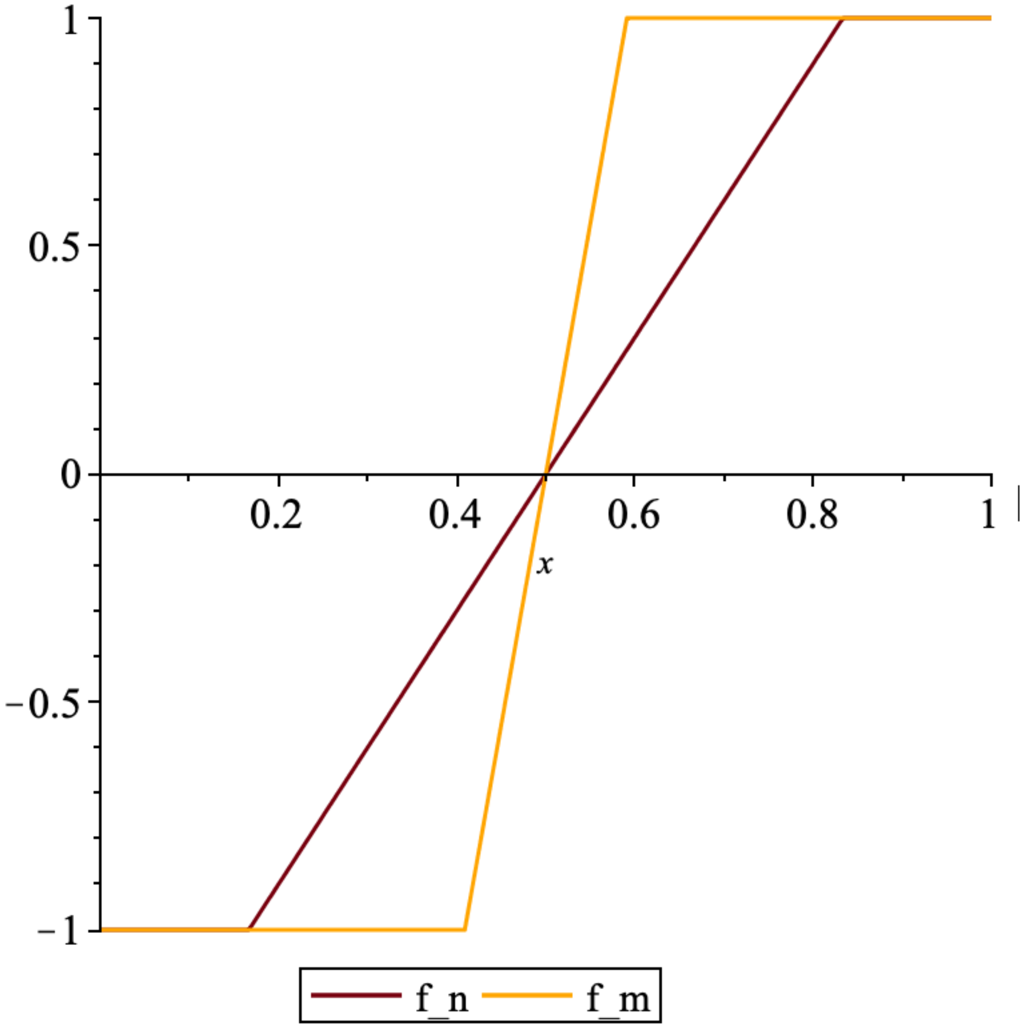

Figure 1: f_m and f_n plotted for $m = 11$ and $n = 3$. To calculate the integral $\int_0^1 |f_m(x) - f_n(x)|\, dx$, one needs to determine the area between the two graphs – essentially the area of two triangles.

$N \in \mathbb{N}$ be an integer greater than $\frac{1}{2\epsilon}$ and let $N \le n \le m \in \mathbb{N}$. We have

$$\begin{aligned}
\|f_n - f_m\| &= \sqrt{\langle f_n - f_m, f_n - f_m \rangle} \\
&= \sqrt{\int_0^1 (f_n(x) - f_m(x))^2\, dx} \\
&\le \int_0^1 |f_n(x) - f_m(x)|\, dx \\
&= \frac{1}{2}\left(\frac{1}{n} - \frac{1}{m}\right) \\
&\le \frac{1}{2n} \\
&\le \frac{1}{2N} \\
&\le \epsilon
\end{aligned}$$

and conclude that $\{f_n\}$ is a Cauchy sequence. Now assume that $f_n \to f \in C[0,1]$. We see that f_n converges point-wise to the function

$$f(x) = \begin{cases} -\frac{1}{2}, & 0 \le x < \frac{1}{2} \\ \frac{1}{2}, & \frac{1}{2} \le x \le 1 \end{cases},$$

but f is not continuous, contradicting the assumption that $f \in C[0,1]$. Hence, $C[0,1]$ is not complete.

$\boxed{3}$ Let X_1 and X_2 be two Hilbert spaces and $T \in B(X_1, X_2)$.

a) Show that there exists $T^* \in B(X_2, X_1)$ such that $\langle Tx, y\rangle_{X_2} = \langle x, T^*y\rangle_{X_1}$ for any $x \in X_1$, $y \in X_2$.
(Note: We treated the case $X_1 = X_2$ in class.)

b) Prove that $\ker T = \ker T^*T$.

Solution. a) The proof is essentially the same as the proof for $X = Y$ in the lecture notes.
Existence Fix $y \in Y$ and let $\varphi : X \to \mathbb{C}$ be defined by $\varphi(x) = \langle Tx, y\rangle$. Then φ is linear and by Cauchy-Schwarz bounded:

$$|\varphi(x)| \le |\langle Tx, y\rangle| \le \|Tx\|\|y\| \le \|T\|\|x\|\|y\|.$$

Hence φ is a bounded linear functional on X and so by the Riesz representation theorem there exists a unique $\xi \in X$ such that $\varphi(x) = \langle x, \xi\rangle$ for all $x \in X$. The vector ξ depends on the vector $y \in Y$. In order to keep track of this fact we set $T^*y := \xi$. Hence we have defined an operator T^* from Y to X based on the structure of bounded linear functionals on X. In summary, we have demonstrated the existence of an operator $T^* : Y \to X$ such that $\langle Tx, y\rangle = \langle x, T^*y\rangle$ for all $x \in X$, $y \in Y$. We need to show that T^* is bounded and linear.

Linear We have

$$\begin{aligned} \langle x, T^*(\alpha y_1 + \beta y_2)\rangle &= \langle Tx, \alpha y_1 + \beta y_2\rangle \\ &= \overline{\alpha}\langle Tx, y_1\rangle + \overline{\beta}\langle Tx, y_2\rangle \\ &= \overline{\alpha}\langle x, T^*y_1\rangle + \overline{\beta}\langle x, T^*y_2\rangle \\ &= \langle x, \alpha T^*y_1 + \beta T^*y_2\rangle \quad \text{for all } x \in X, \end{aligned}$$

and it follows by Proposition 5.8 in the notes that

$$T^*(\alpha y_1 + \beta y_2) = \alpha T^*y_1 + \beta T^*y_2.$$

<u>Bounded</u> By the Cauchy-Schwarz inequality, we get

$$\begin{aligned}\|T^*y\|^2 = \langle T^*y, T^*y\rangle &= \langle TT^*y, y\rangle \\ &\leq \|TT^*y\|\|y\| \\ &\leq \|T\|\|T^*y\|\|y\|.\end{aligned}$$

If $\|T^*y\| > 0$, we divide by $\|T^*y\|$ on both sides in the inequality and obtain

$$\|T^*y\| \leq \|T\|\|y\|.$$

This inequality is clearly also satisfied when $\|T^*y\| = 0$, so T^* is a bounded operator.

$$\|T^*\| \leq \|T\|.$$

b) It is easy to see that $\ker(T) \subset \ker(T^*T)$: if $x \in \ker(T)$, then $T^*T(x) = T^*(0) = 0$, so $x \in \ker(T^*T)$.
Then assume that $x \in \ker(T^*T)$, i.e. $T^*T(x) = 0$, which means that $T(x) \in \ker(T^*)$. Hence $T(x) \in \ker(T^*) \cap \operatorname{ran}(T)$. But by proposition 5.12 in the notes, we know that $\ker(T^*) = \operatorname{ran}(T)^\perp$, so in fact we have

$$T(x) \in \operatorname{ran}(T) \cap \operatorname{ran}(T)^\perp = \{0\},$$

which proves that $x \in \ker(T)$.
Alternative proof that $\ker(T^*T) \subset \ker(T)$*:* If $x \in \ker(T^*T)$, then

$$\begin{aligned}\|Tx\|^2 &= \langle Tx, Tx\rangle \\ &= \langle T^*Tx, x\rangle \\ &= 0\end{aligned}$$

by assumption. Therefore $Tx = 0$, so $x \in \ker(T)$.

$\boxed{4}$ Let $T : X \to X$ be a bounded linear operator on a Hilbert space X. Show that

$$\|TT^*\| = \|T^*T\| = \|T\|^2.$$

Solution. For any $x \in X$, we have

$$\|T^*Tx\| \leq \|T^*\|\ \|Tx\| = \|T^*\|\|T\|\|x\|,$$

and accordingly

$$\|T^*T\| \leq \|T^*\|\|T\| = \|T\|^2.$$

On the other hand, using the Cauchy-Schwarz inequality, we have

$$\begin{aligned}\|Tx\|^2 = \langle Tx, Tx\rangle &= \langle T^*Tx, x\rangle \\ &\leq \|T^*Tx\|\|x\| \leq \|T^*T\|\|x\|^2,\end{aligned}$$

and it follows that $\|T\| \leq \|T^*T\|^{1/2}$, or equivalently $\|T\|^2 \leq \|T^*T\|$. We conclude that

$$\|T^*T\| = \|T\|^2.$$

Finally, replacing T by T^* in the equality above, and recalling that $T^{**} = T$, we also get

$$\|TT^*\| = \|T^*\|^2 = \|T\|^2.$$

$\boxed{5}$ Consider the multiplication operator T_a on $(\ell^2, \langle \cdot, \cdot \rangle)$ given by

$$T_a x = (a_j x_j)_{j \in \mathbb{N}}$$

for a fixed sequence $a = (a_j)_{j \in \mathbb{N}} \in \ell^\infty$.

a) Determine the adjoint operator T_a^*.

b) Is T_a a normal operator? Under which condition(s) on the sequence a is T_a unitary; self-adjoint?

Solution. a) By definition, the adjoint operator T^* satisfies $\langle Tx, y \rangle = \langle x, T^* y \rangle$ for all $x, y \in \ell^2$. By writing out the definition of the inner product on ℓ^2 and T, the left hand side of this equation becomes

$$\sum_{n=1}^{\infty} a_n x_n \overline{y_n} = \sum_{n=1}^{\infty} x_n \overline{\overline{a_n} y_n} = \langle x, \overline{a} \cdot y \rangle,$$

where $\overline{a} \cdot y$ denotes the sequence $(\overline{a_n} y_n)_{n \in \mathbb{N}}$. Hence we see that if we define $T^* y = (\overline{a_n} y_n)_{n \in \mathbb{N}}$, then $\langle Tx, y \rangle = \langle x, T^* y \rangle$. Therefore $T^* y = (\overline{a_n} y_n)_{n \in \mathbb{N}}$ is the adjoint of T.

b) T **is normal** if $T^*T = TT^*$. We check whether this is true by applying both sides to an element $x \in \ell^2$:

$$T^*Tx = T^*(a_n x_n)_{n \in \mathbb{N}} = (\overline{a_n} a_n x_n)_{n \in \mathbb{N}} = (|a_n|^2 x_n)_{n \in \mathbb{N}},$$

$$TT^*x = T^*(\overline{a_n} x_n)_{n \in \mathbb{N}} = (a_n \overline{a_n} x_n)_{n \in \mathbb{N}} = (|a_n|^2 x_n)_{n \in \mathbb{N}}.$$

We see that $T^*T = TT^*$, so T is normal.

T **is self-adjoint** if $T^* = T$, so for any $x \in \ell^2$ we need $T(x) = T^*(x)$, i.e.

$$(a_n x_n)_{n \in \mathbb{N}} = (\overline{a_n} x_n)_{n \in \mathbb{N}} \quad \text{for any } x \in \ell^2.$$

This is clearly true if and only if $a_n = \overline{a_n}$ for all $n \in \mathbb{N}$.

T **is unitary** if $T^*T = TT^* = I$, i.e. $T^*T(x) = TT^*(x) = x$ for any $x \in \ell^2$. We have already seen that

$$T^*T(x) = TT^*(x) = (|a_n|^2 x_n)_{n \in \mathbb{N}} \quad \text{for any } x \in \ell^2.$$

Hence $T^*T(x) = TT^*(x) = x$ for any $x \in \ell^2$ if and only if $|a_n| = 1$ for all $n \in \mathbb{N}$.

$\boxed{6}$ Let M be a closed subspace of a Hilbert space X, which by the projection theorem is given by the direct sum

$$X = M \oplus M^{\perp}.$$

Show that the projection onto M is self-adjoint.

Solution. Let P be the projection of X onto M. We need to show that P is selfadjoint, meaning that

$$\langle Px, y \rangle = \langle x, Py \rangle \quad \text{for all} \quad x, y \in X.$$

Let $x, y \in X$. By the Projection Theorem, we can write $x = x_1 + x_2$ and $y = y_1 + y_2$, where $x_1, y_1 \in M$ and $x_2, y_2 \in M^{\perp}$. By definition, $P(x) = x_1$ and $P(y) = y_1$. We have

$$\begin{aligned} \langle Px, y \rangle &= \langle x_1, y_1 + y_2 \rangle \\ &= \langle x_1, y_1 \rangle + \langle x_1, y_2 \rangle \\ &= \langle x_1, y_1 \rangle \end{aligned}$$

since $\langle x_1, y_2 \rangle = 0$, and

$$\begin{aligned} \langle x, Py \rangle &= \langle x_1 + x_2, y_1 \rangle \\ &= \langle x_1, y_1 \rangle + \langle x_2, y_1 \rangle \\ &= \langle x_1, y_1 \rangle \end{aligned}$$

and hence $\langle Px, y \rangle = \langle x, Py \rangle$, which was what we needed to show.