25. Lecture XXV: One-Dimensional Global Bifurcations II

Last lecture, we established that there are three ways that a periodic orbit Γ_{μ_0} can be nonhyperbolic. For a point $\mathbf{x}_0 \in \Gamma_{\mu_0}$, since Γ_{μ_0} is periodic, there is always an eigenvalue of the derivative of its Poincaré map, $\mathrm{D}\mathbf{\Pi}(\mathbf{x}_0, \mu_0)$, that takes the value 1. Nonhyperbolicity then requires that either $\mathrm{D}\mathbf{\Pi}(\mathbf{x}_0, \mu_0)$ has another eigenvalue of 1, or it has a complex conjugate pair $e^{\pm i\vartheta}$ of eigenvalues, or it has the eigenvalue -1. We had only considered the first case last time. In this lecture we shall take a closer look at the remaining two cases.

25.1. **Neimark-Sacker bifurcation.** In this subsection we shall consider bifurcations about a periodic orbit where $\mathrm{D}\mathbf{\Pi}(\mathbf{x}_0, \mu_0)$ has a pair of conjugate eigenvalues $e^{\pm i\vartheta}$ in addition to the eigenvalue 1. In order to have three eigenvalues, the dynamics must take place in at least three spatial dimensions. Therefore, this sort of bifurcation does not occur on the plane. We shall see that with the correct non-degeneracy conditions, this sort of spectral (eigenvalues) behaviour leads to the periodic orbit analogue of the Hopf bifurcation (which occurs about a critical point). It is again simplest to look at an example:

Example 25.1. Consider the system:

$$\begin{aligned}
\dot{r} &= \mu(r-1) - \omega\rho - (r-1)\sqrt{(r-1)^2 + \rho^2} \\
\dot{z} &= \omega(r-1) + \mu\rho - \rho\sqrt{(r-1)^2 + \rho^2} \\
\dot{\vartheta} &= 1.
\end{aligned}$$

For any μ, we have the periodic orbit

$$r(t) = 1, \qquad \rho(t) = 0, \qquad \vartheta(t) = t,$$

where the angular variable ϑ really only needs to be defined up to multiples of 2π. It is evident that $\dot{r} = \mathrm{d}(r-1)/\mathrm{d}t$. We use r instead of $r-1$ as a variable here in order to interpret this stationary solution as a periodic orbit in $\mathbb{R}^3$, where the xy-plane is written in polar coordinates, instead of as a critical point. The final argument of the cylindrical coordinate system is z, the height from the the xy-plane.

Let $\mathbf{x}_0$ be a point on the orbit. The Poincaré map is the first return map taken at a particular angular slice, and hence ignores the angular ϑ coordinate. The linearization about the periodic orbit gives

$$\mathrm{D}\mathbf{\Pi}(\mathbf{x}_0, \mu) = \exp(2\pi \mathbf{C}_\mu), \qquad \mathbf{C}_\mu := \begin{pmatrix} \mu & -\omega \\ \omega & \mu \end{pmatrix}$$

This gives us eigenvalues of

$$e^{2\pi\mu \pm 2\pi i\omega},$$

with a bifurcation at $\mu_0 = 0$.

To see the sort of dynamics occuring, we see from the eigenvalues that for $\mu < 0$, any nearby solutions decay to $(r, z) = (1, 0)$.

For $\mu > 0$, the periodic orbit is no longer stable. However, there is an invariant manifold in the form of a torus, of major raidus 1 and minor radius $\sqrt{\mu}$ to which solutions starting near the periodic orbit converges. That is, we see that for any arbitrary phase ϕ depending on initial conditions, we have the limit cycles

$$r(t) = 1 + \sqrt{\mu}\cos(\omega t + \phi), \quad z(t) = \sqrt{\mu}\sin(\omega t + \phi), \quad \vartheta = t,$$

and these limit cycles fill up the the surface of a torus of the dimensions described.

If we collapse the angular variable ϑ, which we can do without danger of trajectories crossing paths because the remaining two equations are still an autonomous system, we find a Hopf-like bifurcation for the planar system with variables

$$\mathfrak{x} = r - 1, \qquad \mathfrak{y} = z.$$

Indeed, using the substitution on the first two equations of the system, we find

$$\dot{\mathfrak{x}} = \mu\mathfrak{x} - \omega\mathfrak{y} - \mathfrak{x}\sqrt{\mathfrak{x}^2 + \mathfrak{y}^2}$$
$$\dot{\mathfrak{y}} = \omega\mathfrak{x} + \mu\mathfrak{y} - \mathfrak{y}\sqrt{\mathfrak{x}^2 + \mathfrak{y}^2}.$$

Now using $R^2 = \mathfrak{x}^2 + \mathfrak{y}^2$ on the $\mathfrak{x}\mathfrak{y}$-plane, we find

$$2\dot{R} = \mu R - R^2,$$

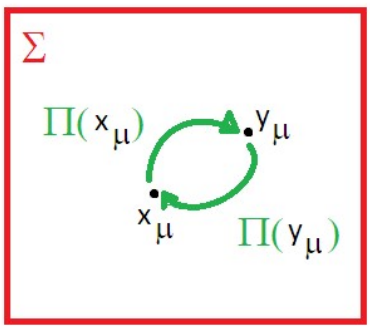
which looks like a slightly scaled transcritical bifurcation (cf. Lecture 22.1.2), but interpreted as a radial variable on a plane (i.e., $R \geq 0$), has a bifurcation diagram that coincides with the Hopf bifurcation. (When the lower half-plane is ignored, as in the radial case, a pitchfork bifurcation and a transcritical bifurcation looks the same.)

25.2. **Period-doubling bifurcation.** On the plane, we have not seen an eigenvalue of $\mathrm{D}\mathbf{\Pi}(\mathbf{x}_0, \mu_0)$ taking the value -1, either. It is, in fact, again, impossible. Let us briefly explore what an eigenvalue of -1 signifies.

Recall that $\mathrm{D}\mathbf{\Pi}(\mathbf{x}_0, \mu_0)$ always has one eigenvalue that takes the value 1 because that is the multiplier associated with the direction normal to the hyperplane Σ on which $\mathbf{\Pi}$ is defined. An eigenvalue of -1 therefore means that along some direction inside/along the hyperplane, each return of a trajectory to Σ occurs in an opposite sense from the previous one. In particular, there are trajectories Γ_μ close to our nonhyperbolic periodic orbit, which intersect Σ, say, at $\mathbf{x}_\mu$, returns at $\mathbf{y}_\mu \in \Sigma$ (along the same direction transverse to Σ) and then returns again to complete itself as a periodic orbit, intersecting Σ the second time at $\mathbf{x}_\mu$. The direction associated with the eigenvalue -1 is then $\mathbf{x}_\mu - \mathbf{y}_\mu$, which lies on Σ, because the next time it returns, i.e., on a second application of the Poincaré map, the point is mapped along this vector in the opposite direction at the same magnitude — thereby taking $\mathbf{y}_\mu$ back to $\mathbf{x}_\mu$ in the second return, and closing up a periodic orbit with *approximately* twice the period — that is,

$$\mathbf{\Pi} \circ \mathbf{\Pi}(\mathbf{x}_\mu, \mu) = \mathbf{x}_\mu$$

(see also Figure 6. on pg 371 of *Perko*):



This cannot happen for plane autonomous systems because a trajector will have to cross itself in order for a period doubling phenomenon to occur. One way to see this is, supposing such a trajectory existed on the plane, arbitrarily pick a centre $\mathcal{O}$ of the period doubling trajectory. Either $|\mathbf{x}_\mu - \mathcal{O}| < |\mathbf{y}_\mu - \mathcal{O}|$, or vice versa. As we continue along the trajectory and make our way back from $\mathbf{y}_\mu$ to $\mathbf{x}_\mu$, arbitrarily close to $\mathbf{x}_\mu$, the radius will have had to have decreased (resp. increased) back to $|\mathbf{x}_\mu - \mathcal{O}|$, which inplies a crossing.

This sort of behaviour clearly depends on the existence of periodic orbits that bend very close back on themselves at μ close to μ_0, very much like in noodle dough that is folded and stretched multiple times to make noodle. It is a behaviour closely associated with what is normally called chaotic behaviour, and can become very complicated.

Example 25.2. Let us consider a toy example of a system with $\mathbf{D\Pi}$ having an eigenvalue of -1:

$$\begin{aligned}
\dot r &= -(r-1) - \frac{1}{2}z + \mu\big(\cos^2(\vartheta/2)(r-1) + \sin(\vartheta/2)\cos(\vartheta/2)z\big) - (r-1)\sqrt{(r-1)^2+z^2}\\
\dot z &= \frac{1}{2}(r-1) - z + \mu\big(\sin^2(\vartheta/2)z + \sin(\vartheta/2)\cos(\vartheta/2)(r-1)\big) - z\sqrt{(r-1)^2+z^2}\\
\dot\vartheta &= 1.
\end{aligned}$$

Since the first two equations depend on ϑ, we can no longer collapse the final equation as we did for Example 25.1 and still retain an autonomous system. Substituting $\vartheta = t$ into the remaining two equations, we can build a Poincaré map on the slice $\vartheta = 0$.

The linearization of the non-autonomous system is

$$\begin{aligned}
\dot r &= -(r-1) - \frac{1}{2}z + \mu\big(\cos^2(t/2)(r-1) + \sin(t/2)\cos(t/2)z\big)\\
\dot z &= \frac{1}{2}(r-1) - z + \mu\big(\sin^2(t/2)z + \sin(t/2)\cos(t/2)(r-1)\big).
\end{aligned}$$

Solutions are spanned by

$$\begin{pmatrix} r(t)\\ z(t)\end{pmatrix} \in \operatorname{span}\left\{ e^{(\mu-1)t}\begin{pmatrix}\cos(t/2)\\ \sin(t/2)\end{pmatrix}, e^{-t}\begin{pmatrix}-\sin(t/2)\\ \cos(t/2)\end{pmatrix}\right\}.$$

On first return, we have $\mathbf{\Pi}(r(0), z(0); \mu) = (r(2\pi), z(2\pi))$. Therefore,

$$\mathbf{D\Pi}\begin{pmatrix} r(0)\\ 0\end{pmatrix} = -e^{2\pi(\mu-1)}\begin{pmatrix} r(0)\\ 0\end{pmatrix}, \qquad \mathbf{D\Pi}\begin{pmatrix} 0\\ z(0)\end{pmatrix} = -e^{-2\pi}\begin{pmatrix} 0\\ z(0)\end{pmatrix},$$

and the eigenvalue $-e^{2\pi(\mu-1)}$ takes the value -1 at the bifurcation point $\mu = \mu_0 = 1$.

This example is slighly artificial, however, and by inspection, it can be checked that it has similar solutions for $\mu > 1$ to Example 25.1. It often happens that period-doubling occur at a sequence of values $\{\mu_i\}_{i=0}^\infty$ for the bifurcation parameter μ, so that trajectories are, to use the noodle imagery, stretched and folded repeatedly, increasingly often at each successive bifurcation value. This phenomenon is known as the period-doubling cascade and we shall inspect it presently.

25.3. **Period doubling cascades.** Most results within our reach concerning period doubling cascades come from numerical experiments. In this subsection we shall describe two numerical examples of period doubling cascades.

Example 25.3 (Rössler's equation)**.** A simplified Lorenz system already exhibits period doubling cascades. The Rössler system is

$$\begin{aligned}
\dot x &= -y - z\\
\dot y &= x + ay\\
\dot z &= b + z(x - c).
\end{aligned}$$

We consider bifurcations of the system with fixed $a = b = 1/5$, as c varies.

It turns out that there are periodic solutions at $c > 0.4$. There is a first period-doubling bifurcation at $c_1 \approx 2.832$. The period of periodic orbits "doubles" again at $c_2 \approx 3.837$, and there is an infinite number of bifurcation values $\{c_k\}_{k=1}^\infty$ which accumulate at $c_\infty \approx 4.3$, at each of which there is a period-doubling bifurcation. (See Figure 9.15 of *Cain and Shaeffer.*)

Whilst we have been discussing dynamical systems defined by flows of ODEs, it is far simpler to look at period doubling bifurcations in systems defined by maps. That is, we consider, for example, only what happens on Σ, and not the entire trajectories themselves. Let there be a collection of maps, indexed by $\mu \in \mathbb{R}$:

$$\Psi(\cdot, \mu) : \Sigma \to \Sigma.$$

This is more general than systems defined by flows, because, for example, we can take Σ to be an interval in $\mathbb{R}$ here, even though we have shown that if Σ is a codimension one (i.e., dimension $d-1$) surface defining a Poincaré map for a system with period doubling bifurcation, we cannot have $d-1=1$.

Example 25.4 (Logistic map)**.** Consider the archetypal period-doubling discrete system:

$$x_{n+1} = \mu x_n(1-x_n).$$

Here $\mu \in (0,4]$, and $x_n \in [0,1]$. This is a discrete time version of the logistic equation used in population models.

For $\mu \in (0,1)$, as expected $x_n \to 0$, since $x_n \leq \mu^n$.

For $\mu \in (0,2)$, $x_n \to (\mu-1)/\mu$, which is easy to find simply by setting $x_\infty = \mu x_\infty(1-x_\infty)$.

For $\mu \in (2,3)$, the same behaviour holds, but there is more oscillatory behaviour in the convergence to $(\mu-1)/\mu$.

At $\mu \in (3, 1+\sqrt{6})$, we see that almost all initial values lead to permanent oscillations between two different values.

At $\mu \in (1+\sqrt{6}, \sim 3.54409)$, there is another bifurcation into oscillations between four values for almost all initial values.

As μ increases beyound $\mu_3 \approx 3.54409$, we find that there are values μ_4 at which x_n oscillates between eight values, then $\mu_5 > \mu_4$ at which x_n oscillates between sixteen values, and so on.

At $\mu \approx 3.56995$, this period-doubling cascade stops.

We think of these as period-doubling bifurcations because in higher dimensions, the return map of a trajectory to Σ is a discrete system.

It turns out that for all one-dimesional discrete systems $x_n = f(x_{n-1}, \mu)$ with period doubling bifurcations, where $f(x,\mu)$ has one quadratic maximum for each value of μ, period-doubling cascades occur at the same rate. Where the bifurcation points are at $\{\mu_i\}_{i=0}^{\infty}$, the limit

$$\delta := \lim_{n\to\infty} \frac{\mu_n - \mu_{n-1}}{\mu_{n+1} - \mu_n}$$

is universal, and is known as the FIRST FEIGENBAUM CONSTANT. The approximate value of this number is $\delta \approx 4.669201$.