Repetition

Definition

A **discrete-time stochastic process** is a family of random variables $\{X_t : t \in T\}$ where T is discrete.

In this course $T = \{0, 1, 2, \ldots\}$, and X_n is called the **state** at time $n = 0, 1, 2, \ldots$. The set of all possible states is called the **state space**.

Definition

A **discrete-time Markov chain** is a discrete-time stochastic process $\{X_n : n = 0, 1, \ldots\}$ that satisfies the **Markov property**:

$$\begin{aligned}
&\Pr\{X_{n+1} = j | X_n = i, X_{n-1} = i_{n-1}, \ldots, X_0 = i_0\} \\
&\qquad = \Pr\{X_{n+1} = j | X_n = i\}, \qquad\qquad n = 0, 1, \ldots, \forall i, j.
\end{aligned}$$

Unless otherwise stated, the state space is $\{0, 1, \ldots, N\}$ or $\{0, 1, 2, \ldots\}$.

Definition

For discrete-time Markov chain $\{X_n : n = 0, 1, \ldots\}$, we call $P_{ij}^{n,n+1} = \Pr\{X_{n+1} = j, X_n = i\}$ the **one-step transition probabilities**.

We will allways assume **stationary transition probabilities**, i.e., that $P_{ij}^{n,n+1} = P_{ij}$ for $n = 0, 1, 2, \ldots$ and for all states i and j.

Definition

For a discrete-time Markov chain with state space $\{0, 1, \ldots, N\}$, we call

$$\mathbf{P} = \begin{bmatrix} P_{0,0} & P_{0,1} & \cdots & P_{0,N} \\ P_{1,0} & P_{1,1} & & \vdots \\ \vdots & & \ddots & \\ P_{N,0} & \cdots & & P_{N,N} \end{bmatrix}$$

the **transition probability matrix**.

Definition

Let $\{X_n : n = 0, 1, \ldots\}$ be a discrete-time Markov chain. A **(state) transition diagram** visualizes the transition probabilities as a weighted directed graph where the nodes are the states and the edges are the possible transitions marked with the transition probabilities.

Theorem

For a Markov chain $\{X_n : n = 0, 1, \ldots\}$, we have

$$\Pr\{X_{m+n} = j | X_m = i\} = P_{ij}^{(n)} = \sum_{k=0}^{\infty} P_{ik} P_{kj}^{(n-1)}, \quad n > 0,$$

where we define

$$P_{ij}^{(0)} = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases}$$

We call $P_{ij}^{(n)}$ the n**-step transition probabilities**.

Theorem

The n-step transition probabilities can be computed by matrix multiplication. If $\mathbf{P}^{(n)} = [P_{ij}^{(n)}]$, then

$$\mathbf{P}^{(n)} = \underbrace{\mathbf{P} \cdot \mathbf{P} \cdots \mathbf{P}}_{n} = \mathbf{P}^n.$$