Repetition

Theorem

If $T_i \sim \text{Exp}(\alpha_i)$ with $\alpha_i > 0$, $i = 1, 2, \ldots, n$, and $T_1, T_2, \ldots, T_n$ are independent, then

$$\min\{T_1, T_2, \ldots, T_n\} \sim \text{Exp}\left(\sum_{i=1}^{n} \alpha_i\right).$$

Theorem (IMPORTANT)

In a birth and death process with birth rates $\lambda_0, \lambda_1, \ldots > 0$, and death rates $\mu_0 = 0$ and $\mu_1, \mu_2, \ldots > 0$, we have

1. sojourn times are independent;
2. each time you visit state i, the sojourn time is $\text{Exp}(\lambda_i + \mu_i)$, $i = 0, 1, \ldots$.

Note: Also valid for a finite state space $\{0, 1, \ldots, N\}$ together with $\lambda_N = 0$.

Theorem (IMPORTANT)

Consider a birth and death process with birth rates $\lambda_0, \lambda_1, \ldots$ and death rates $\mu_0, \mu_1, \ldots$. After the sojourn time in state i ends, the process jumps either to state $i-1$ or to state $i+1$. The jump probabilities are

$$\Pr\{i \to i+1\} = \frac{\lambda_i}{\lambda_i + \mu_i},$$
$$\Pr\{i \to i-1\} = \frac{\mu_i}{\lambda_i + \mu_i}.$$

Alternative definition

The birth and death process with birth rates $\lambda_0, \lambda_1, \ldots$ and death rates $\mu_0, \mu_1, \ldots$ can be constructed in the following way. Whenever, you jump to state i, two competing and independent processes start:

1) $T_1 =$ "time until birth" $\sim \text{Exp}(\lambda_i)$.

2) $T_2 =$ "time until death" $\sim \text{Exp}(\mu_i)$.

If the next event is a birth, we jump to $i + 1$, and if the next event is a death we jump to $i - 1$.

Simulation of birth and death processes

Input:

- i_0: initial state
- B: number of jumps
- $\lambda_0, \lambda_1, \ldots$: birth rates
- $\mu_0, \mu_1, \ldots$: death rates

Algorithm:

1. set $x_0 = i_0$ and $t_0 = 0$.
2. for $b = 1 \ldots B$
3.   set $i = x_{b-1}$
4.   draw $s \sim \text{Exp}(\mu_i + \lambda_i)$ and set $t_b = t_{b-1} + s$
5.   draw $u \sim \mathcal{U}(0, 1)$
6.   if $u < \lambda_i/(\lambda_i + \mu_i)$
7.   set $x_b = i + 1$
8.   else
9.   set $x_b = i - 1$
10.   end
11. end

Output:

$$x(t) = \begin{cases} x_0, & 0 \leq t < t_1 \\ x_1, & t_1 \leq t < t_2 \\ \vdots & \\ x_{B-1}, & t_{B-1} \leq t < t_B \\ x_B, & t = t_B. \end{cases}$$

Note: You only need to store two vectors: jump times and which states you jump to. This information uniquely describes the realization $x(t)$ on $0 \leq t \leq t_B$.