Norwegian University of Science and Technology
Deptartment of Mathematical Sciences

TMA4190 Introduction to Topology
Spring 2018

Exercise set 7

1 A manifold X is *contractible* if its identity map is homotopic to some constant map $X \to \{x\}$ where x is any point of X.

a) Show that if X is contractible, then all maps of an arbitrary manifold Y into X are homotopic.

b) Conversely, show that if all maps of an arbitrary manifold Y into X are homotopic, then X is contractible.

c) Show that $\mathbb{R}^k$ is contractible.

2 A manifold X is *simply connected* if it is connected and if every smooth map from the circle S^1 into X is homotopic to a constant map. Show that all contractible spaces are simply connected. (Note that the converse is false.)

3 Show that the antipodal map $S^k \to S^k$, $x \mapsto -x$, is homotopic to the identity if k is odd. (We will see later that this is not true if n is even.)

(Hint: Start off with $k = 1$ by using the linear maps defined by

$$[0,1] \to M(2),\ t \mapsto \begin{pmatrix} \cos(\pi t) & -\sin(\pi t) \\ \sin(\pi t) & \cos(\pi t) \end{pmatrix}.)$$

4 Show that every connected manifold X is path-connected, i.e. given any two points $x_0, x_1 \in X$, there exists a smooth curve $f\colon [0,1] \to X$ with $f(0) = x_0$ and $f(1) = x_1$.

(Hint: Use the fact that homotopy is an equivalence relation to show that the relation "x_0 and x_1 can be joined by a smooth curve" is an equivalence relation on X. Then show that the equivalence classes are both open and closed subsets of X.)