Suggested solution, exam TMA4265, Stochastic Modeling, Dec 11, 2017

Task 1

a)

$$P(X_3 = 2|X_1 = 1) = \sum_{k=1}^{3} P(X_3 = 2|X_2 = k)P(X_2 = k|X_1 = 1) = 0.4\cdot 0.5 + 0.5\cdot 0.7 + 0.1\cdot 0.5 = 0.6$$

Because of the Markov property:

$$P(X_4 = 2|X_2 = 1, X_1 \neq 1) = P(X_4 = 2|X_2 = 1) = 0.6$$

$$P(X_3 = 2|X_2 \neq 1, X_1 = 1) = \frac{P(X_3 = 2, X_2 \neq 1|X_1 = 1)}{P(X_2 \neq 1|X_1 = 1)}$$

$$P(X_3 = 2, X_2 \neq 1|X_1 = 1) = \sum_{k=2}^{3} P(X_3 = 2|X_2 = k)P(X_2 = k|X_1 = 1) = 0.5\cdot 0.7 + 0.1\cdot 0.5 = 0.4$$

$$P(X_2 \neq 1|X_1 = 1) = 0.5 + 0.1 = 0.6$$

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$$P(X_3 = 2|X_2 \neq 1, X_1 = 1) = \frac{0.4}{0.6} = 0.67$$

b)

Let $\hat{\pi}_i = \lim_{t\to\infty} P(X_t = i|X_0)$. These long-run probabilities are determined by

$$\hat{\pi}_i = \sum_{j=1}^{3} \hat{\pi}_j P(j, i), \quad \sum_{i=1}^{3} \hat{\pi}_i = 1$$

We get:

$$\begin{aligned}
\hat{\pi}_1 &= 0.4\hat{\pi}_1 + 0.05\hat{\pi}_2 + 0.05\hat{\pi}_3, \;\; \hat{\pi}_3 = 12\hat{\pi}_1 - \hat{\pi}_2 \\
\hat{\pi}_2 &= 0.5\hat{\pi}_1 + 0.7\hat{\pi}_2 + 0.5(12\hat{\pi}_1 - \hat{\pi}_2), \;\; \hat{\pi}_2 = \frac{0.5 + 6}{1 - 0.7 + 0.5}\hat{\pi}_1 \\
\hat{\pi}_1 &+ \hat{\pi}_2 + \hat{\pi}_3 = \hat{\pi}_1 + 12\hat{\pi}_1 = 1,
\end{aligned}$$

$\hat{\pi}_1 = 1/13 \approx 0.08$
$\hat{\pi}_2 = \frac{0.5+6}{1-0.7+0.5}0.08 \approx 0.62$, $\hat{\pi}_3 = 12 \cdot 0.08 - 0.62 \approx 0.30$

$$P(X_{t-1} = 1 | X_t = 2) = \frac{P(X_{t-1} = 1)P(X_t = 2 | X_{t-1} = 1)}{P(X_t = 2)} = \frac{\hat{\pi}_1 P(1,2)}{\hat{\pi}_2} = 0.08 \cdot 0.5/0.62 \approx 0.065$$

$$P(X_{t-2} = 1 | X_t = 2) = \frac{P(X_{t-2} = 1)P(X_t = 2 | X_{t-2} = 1)}{P(X_t = 2)} = \frac{\hat{\pi}_1 P^2(1,2)}{\hat{\pi}_2} = 0.08 \cdot 0.6/0.62 \approx 0.077$$

where $P^2(1,2)$ is the two-generation transition probability in a).

c)

The time to leave state 1 is geometric distributed with probability $p = P(1,2) + P(1,3) = 0.6$. This means that the expected number of generations before leaving the lower class is $1/0.6 = 1.7$.

Define $u_i = E(\text{additional generations until upper class} | X_t = i)$.

By a first-step analysis:

$$\begin{aligned} u_1 &= 1 + P(1,1)u_1 + P(1,2)u_2 + P(1,3)u_3 \\ u_2 &= 1 + P(2,1)u_1 + P(2,2)u_2 + P(3,3)u_3 \\ u_3 &= 0 \end{aligned}$$

This means that

$$\begin{aligned} u_1 &= \frac{1 + 0.5u_2}{1 - 0.4} \\ u_2 &= 1 + 0.05\frac{1 + 0.5u_2}{0.6} + 0.7u_2 \end{aligned}$$

$$u_2 = \frac{1 + 0.05/0.6}{1 - 0.7 - 0.05 \cdot 0.5/0.6} \approx 4.2$$

$$u_1 = \frac{1 + 0.5u_2}{0.6} \approx 5.2$$

d)

Let $\boldsymbol{X}_t = (X_{t,1}, X_{t,2}, X_{t,3})$ denote the outcome of the three sons. From the independence assumption, we have

$$P(\boldsymbol{X}_t = (x_{t,1}, x_{t,2}, x_{t,3}) | X_{t-1} = k) = \prod_{j=1}^{3} P(k, x_{t,j})$$

We know that $\boldsymbol{X}_t = (2,2,2)$.

$$\begin{aligned} P(X_{t-1} = 1 | \boldsymbol{X}_t = (2,2,2)) &= \frac{P(X_{t-1} = 1, \boldsymbol{X}_t = (2,2,2))}{P(\boldsymbol{X}_t = (2,2,2))} = \frac{[P(1,2)]^3 \hat{\pi}_1}{\sum_{k=1}^3 [P(k,2)]^3 \hat{\pi}_k} \\ &= \frac{0.5^3 \cdot 0.08}{0.5^3 \cdot 0.08 + 0.7^3 \cdot 0.62 + 0.5^3 \cdot 0.3} \approx 0.038 \end{aligned}$$

Here the denominator denotes the long-run probability

$$P(\boldsymbol{X}_t = (2,2,2)) = 0.5^3 \cdot 0.08 + 0.7^3 \cdot 0.62 + 0.5^3 \cdot 0.3 = 0.26$$

$$\begin{aligned} P(X_{t-2} = 1 | \boldsymbol{X}_t = (2,2,2)) &= \sum_{k=1}^{3} P(X_{t-2} = 1, X_{t-1} = k | \boldsymbol{X}_t = (2,2,2)) \\ &= \sum_{k=1}^{3} \frac{P(X_{t-2} = 1) P(X_{t-1} = k | X_{t-2} = 1) [P(k,2)]^3}{P(\boldsymbol{X}_t = (2,2,2))} \\ &= \frac{0.08(0.4 \cdot 0.5^3 + 0.5 \cdot 0.7^3 + 0.1 \cdot 0.5^3)}{0.26} = 0.07 \end{aligned}$$

Task 2

a)

In the exponential distribution $E(T) = 1/\mu = 1/0.2 = 5$.

$$f(t) = \int_0^\infty f(t|\mu) p(\mu) d\mu = \int_0^\infty \mu \exp(-\mu t) \alpha \exp(-\alpha\mu) d\mu$$

We reform the integral to recognize the integral over a gamma density:

$$f(t) = \alpha \int_0^\infty \mu \exp(-(t+\alpha)\mu) d\mu = \frac{\alpha}{(t+\alpha)^2} \int_0^\infty \frac{(t+\alpha)^2}{1} \mu^{2-1} \exp(-(t+\alpha)\mu) d\mu$$

$$f(t) = \frac{\alpha}{(t+\alpha)^2}.$$

b)

As t increases the derivative goes to 0, and then $0 = -\lambda P_{0,0}(t) + \mu(1 - P_{0,0}(t))$. This gives $P_{0,0}(t) = \mu/(\mu+\lambda) = 0.2/1.2 = 0.17$. The top and bottom plots converge to this value, but there is difference in the convergence rate. We check if our parameter settings can be representative of the top plot: The theoretical derivative of $P_{0,0}(t)$ at $t = 0$ is $-\lambda = -1$, which means that the tangent line should cross the first axis near 1. This looks reasonable in the top plot. For the bottom plot the tangent line seems to cross the first axis at a much smaller value, about 0.25. The bottom plot has must have much larger parameter λ since the probabilities converge faster.

Alternatively, one can solve the differential equation to see that the exponential decline is determined by $\exp(-(\lambda + \mu)t)$, and this means that $\exp(-1.2 \cdot 1) = 0.3$, $\exp(-1.2 \cdot 3) = 0.03$, i.e. the convergence is correct in the top plot.

c)

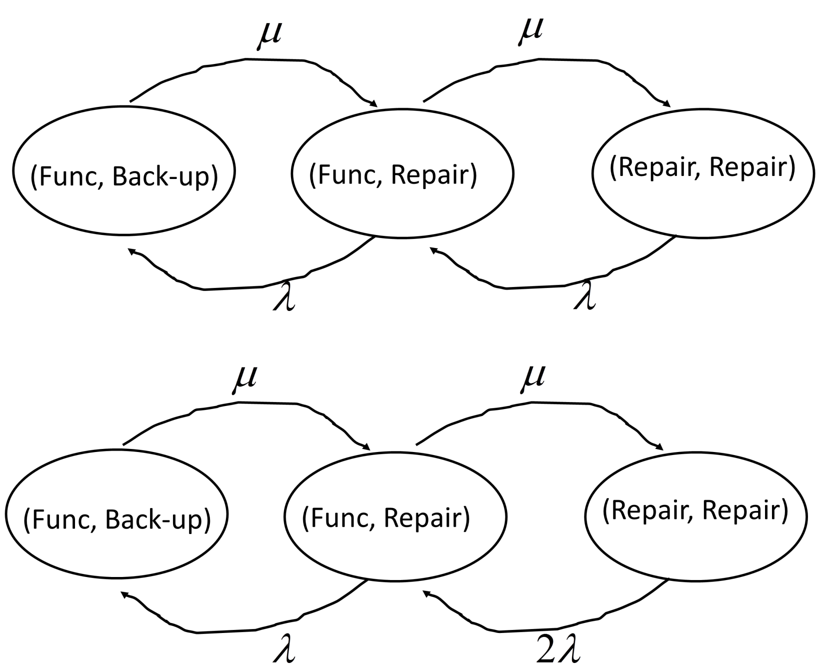

Figure 1: Transition diagrams for top: 2c) and bottom: 2d) .

Three states: (functional, back-up) (F, B), (functional, under repair) (F, R) and (under repair, under repair) (R, R). The long-term probabilities $\lim_{t\to\infty} P_{i,j}(t) = P_j$, for the three j states. The probabilities can be determined by long-term moves in and out of states

$$\begin{aligned} P_{F,B}\mu &= P_{F,R}\lambda \\ P_{F,R}\mu &= P_{R,R}\lambda \\ 1 &= P_{F,B} + P_{F,R} + P_{R,R} \end{aligned}$$

This gives

$$P_{F,R}(1+\mu/\lambda+\lambda/\mu) = 1, \quad P_{F,R} = \frac{1}{1+\mu/\lambda+\lambda/\mu} = 1/(1+0.2+5) = 0.161 \approx 0.16$$

$$P_{F,B} = 0.161 \cdot (1/0.2) = 0.807, \quad P_{R,R} = 0.161 \cdot (0.2/1) = 0.0323$$

d)

The rate from (R, R) is now doubled because there are two repair persons. The probabilities can be determined by long-term moves in and out of states

$$\begin{aligned} P_{F,B}\mu &= P_{F,R}\lambda \\ P_{F,R}\mu &= P_{R,R}2\lambda \\ 1 &= P_{F,B} + P_{F,R} + P_{R,R} \end{aligned}$$

This gives

$$P_{F,R}(1+\mu/(2\lambda)+\lambda/\mu) = 1, \quad P_{F,R} = \frac{1}{1+\mu/(2\lambda)+\lambda/\mu} = 1/(1+0.1+5) = 0.164$$

$$P_{F,B} = 0.164 \cdot (1/0.2) = 0.82, \quad P_{R,R} = 0.164 \cdot (0.2/2) = 0.0164$$

This means that the long-term time in state (R, R) is reduced from 0.0323 to 0.0164. The long-run proportion of time in the functional state is the complementary probability. We check how much the increased functional time is worth: Long-term income without extra repair person:

$$\text{Revenues} - \text{Cost} = 10000 \cdot (1 - 0.0323) - 0 = 9677$$

Long-term income with extra repair person

$$\text{Revenues} - \text{Cost} = 10000 \cdot (1 - 0.0164) - 1000 \cdot 0.0164 = 9820$$

Since $9820 > 9677$, this means that the ski team should get the second repair person when both cameras get under repair.

Task 3

a)

The conditional distribution is Gaussian with mean and variance

$$\begin{aligned} E(x_{50}|x_{45},x_{55}) &= 35 + \boldsymbol{\Sigma}_{50,(45,55)}\boldsymbol{\Sigma}^{-1}_{(45,55)}((34.4,35.1)-(35,35)), \\ \mathrm{Var}(x_{50}|x_{45},x_{55}) &= 0.5^2 - \boldsymbol{\Sigma}_{50,(45,55)}\boldsymbol{\Sigma}^{-1}_{(45,55)}\boldsymbol{\Sigma}_{(45,55),50}. \end{aligned}$$

We use the property of the inverse matrix:

$$\boldsymbol{\Sigma}_{(45,55)} = \begin{bmatrix} 0.5^2 & 0.5^2\exp(-1) \\ 0.5^2\exp(-1) & 0.5^2 \end{bmatrix}, \quad \boldsymbol{\Sigma}^{-1}_{(45,55)} = \begin{bmatrix} 4.6 & -1.7 \\ -1.7 & 4.6 \end{bmatrix}.$$

where $\exp(-1) = 0.37$ and $\exp(-2) = 0.14$.

Moreover $\exp(-0.1{\cdot}5) = 0.6$ defines the correlation from the measurement locations to depth 50, and $\mathrm{Cov}(x_{50},x_{45}) = 0.5^2 0.6 = 0.15$. Then

$$\begin{aligned} E(x_{50}|x_{45},x_{55}) &= 35 + (0.15,0.15)\begin{bmatrix} 4.6 & -1.7 \\ -1.7 & 4.6 \end{bmatrix}(-0.6,0.1)^t = 34.78, \\ \mathrm{Var}(x_{50}|x_{45},x_{55}) &= 0.5^2 - (0.15,0.15)\begin{bmatrix} 4.6 & -1.7 \\ -1.7 & 4.6 \end{bmatrix}(0.15,0.15)^t = 0.118 = 0.34^2. \end{aligned}$$

$$P(x_{50} < 35) = P(Z < \frac{35-34.78}{0.34}) = 0.74.$$

b)

It looks like the mean salinity increases as a function of depth (more fresh water at the surface). This means that $\mathrm{E}(x_t) = \mu_t$ could be a better model, where the mean varies as a function of depth, possibly linearly. It further looks like the variability is larger near the surface. This means that $\mathrm{Var}(x_t) = \sigma_t^2$ could be a better model, where the variance is a function of depth. It is difficult to tell if the correlation varies with depth.

The exponential covariance function is a Markovian process, so only the two nearest points are interesting in the conditioning. The variance would be largest 2.5 m between the sampling depths. We use the same formula as in a), but now the correlation is larger; $\exp(-0.1\cdot 5) = 0.16$ and $\exp(-0.1\cdot 2.5) = 0.78$. This means:

$$\begin{aligned} Var(x_{2.5}|x_0,x_5) &= 0.5^2 - (0.195,0.195)\begin{bmatrix} 6.3 & -3.8 \\ -3.8 & 6.3 \end{bmatrix}(0.195,0.195)^t \\ &= 0.06 = 0.24^2. \end{aligned}$$

This means that the criterion is matched.

For a model with larger variance at the top, it could be useful to focus the data gathering in shallow regions. Otherwise the criterion would not be matched at the shallow locations. From Figure 2 it is clear that the variance is already quite small for large depths.