TMA4212
Numerical solution of differential equations by difference methods
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Norwegian University of Science and Technology
Department of Mathematical Sciences

Solutions to exercise set 1

1 **a)** First, note that by defining $x_0^{(s)} = \sin(0\phi_s) = 0$ and $x_{n+1}^{(s)} = \sin((N+1)\phi_s) = 0$, we can use the same argument for all rows.
We have, for $i \in 1, \dots, n$, that

$$(Ax^{(s)})_i = cx_{i-1}^{(s)} + ax_i^{(s)} + bx_{i+1}^{(s)}.$$

Using the definition of $x^{(s)}$ this can be written as

$$\begin{aligned}(Ax^{(s)})_i =& c\big(\frac{c}{b}\big)^{\frac{i-1}{2}} \sin((i-1)\frac{s\pi}{n+1}) && (1)\\ &+a\big(\frac{c}{b}\big)^{\frac{i}{2}} \sin(i\frac{s\pi}{n+1}) && (2)\\ &+b\big(\frac{c}{b}\big)^{\frac{i+1}{2}} \sin((i+1)\frac{s\pi}{n+1}). && (3)\end{aligned}$$

For (1) we have, by using that $c\big(\frac{c}{b}\big)^{\frac{-1}{2}} = \sqrt{bc}$ and the trigonometric identity for angle addition in sinuses,

$$\begin{aligned}c\big(\frac{c}{b}\big)^{\frac{i-1}{2}} \sin((i-1)\frac{s\pi}{n+1}) = \sqrt{bc}\big(\frac{c}{b}\big)^{\frac{i}{2}}\big(&\sin(i\frac{s\pi}{n+1})\cos(\frac{s\pi}{n+1})\\ &-\sin(\frac{s\pi}{n+1})\cos(i\frac{s\pi}{n+1})\big).\end{aligned}$$

Similarly for (3) we get

$$\begin{aligned}b\big(\frac{c}{b}\big)^{\frac{i+1}{2}} \sin((i+1)\frac{s\pi}{n+1}) = \sqrt{bc}\big(\frac{c}{b}\big)^{\frac{i}{2}}\big(&\sin(i\frac{s\pi}{n+1})\cos(\frac{s\pi}{n+1})\\ &+\sin(\frac{s\pi}{n+1})\cos(i\frac{s\pi}{n+1})\big).\end{aligned}$$

Now, adding the three terms and noting that the second part of (1) and (2) cancel, we end up with

$$\begin{aligned}(1)+(2)+(3) &= \big(\frac{c}{b}\big)^{\frac{i}{2}} \sin(i\frac{s\pi}{n+1})\big(\sqrt{bc}\cos(\frac{s\pi}{n+1}) + a + \sqrt{bc}\cos(\frac{s\pi}{n+1})\big)\\ &= \big(\frac{c}{b}\big)^{\frac{i}{2}} \sin(i\frac{s\pi}{n+1})\big(2\sqrt{bc}\cos(\frac{s\pi}{n+1}) + a\big) = x_i^{(s)}\lambda_s.\end{aligned}$$

b) By insertion, we get the eigenvalues

$$\lambda_s = 2 + 2\cos(\frac{s\pi}{n+1}).$$

$\boxed{2}$ **a)** Forward difference:

Do a Taylor expansion of u around x_0 evaluated in $x_0 + h$:

$$u(x_0 + h) = u(x_0) + u_x(x_0)(x_0 + h - x_0) + \frac{u_{xx}(\xi)}{2}(x_0 + h - x_0)^2,$$

where $\xi \in (x_0, x_0 + h)$. Rearrange to get

$$\frac{u(x_0 + h) - u(x_0)}{h} = u_x(x_0) + \frac{u_{xx}(\xi)}{2}h.$$

Then $\tau = \frac{u_{xx}(\xi)}{2}h$, so the error term is of first order.

Backward difference:

Do a Taylor expansion of u around x_0 evaluated in $x_0 - h$:

$$u(x_0 - h) = u(x_0) + u_x(x_0)(x_0 - h - x_0) + \frac{u_{xx}(\xi)}{2}(x_0 - h - x_0)^2,$$

where $\xi \in (x_0, x_0 + h)$. Rearrange to get

$$\frac{u(x_0) - u(x_0 - h)}{h} = u_x(x_0) - \frac{u_{xx}(\xi)}{2}h.$$

Then $\tau = -\frac{u_{xx}(\xi)}{2}h$, so the error term is also of first order.

Central difference:

Use the Taylor expansions from the two above differences, but include one more term to get

$$\begin{aligned}u(x_0 + h) = u(x_0) + u_x(x_0)(x_0 + h - x_0) &+ \frac{u_{xx}(x_0)}{2}(x_0 + h - x_0)^2 \\ &+ \frac{u_{xxx}(\xi_1)}{6}(x_0 + h - x_0)^3\end{aligned}$$

and

$$\begin{aligned}u(x_0 - h) = u(x_0) + u_x(x_0)(x_0 - h - x_0) &+ \frac{u_{xx}(x_0)}{2}(x_0 - h - x_0)^2 \\ &+ \frac{u_{xxx}(\xi_2)}{6}(x_0 - h - x_0)^3.\end{aligned}$$

Adding these and rearranging and using the mean value theorem, gives (note that the first order terms cancel)

$$\begin{aligned}\frac{u(x_0 + h) - u(x_0 - h)}{2h} &= u_x(x_0) + \frac{u_{xxx}(\xi_1) + u_{xxx}(\xi_2)}{12}h^2 \\ &= u_x(x_0) + \frac{u_{xxx}(\xi_3)}{6}h^2.\end{aligned}$$

Central difference for second derivatives:

Use the Taylor approximations from central difference scheme, but instead of subtracting them, add them. Also, use one more term in the Taylor expansion. This gives (now the first and the third order terms cancel)

$$\begin{aligned}\frac{u(x_0+h)-2u(x_0)+u(x_0-h)}{h^2} &= u_{xx}(x_0)+\frac{u_{xxxx}(\xi_1)+u_{xxxx}(\xi_2)}{48}h^2\\ &= u_{xx}(x_0)+\frac{u_{xxxx}(\xi_3)}{24}h^2.\end{aligned}$$

b) Taylor expansions of $u(x_0+h)$ and $u(x_0+2h)$ yield

$$\begin{aligned}u(x_0+h) = u(x_0)+u_x(x_0)(x_0+h-x_0)+\frac{u_{xx}(x_0)}{2}(x_0+h-x_0)^2\\ +\frac{u_{xxx}(\xi_1)}{6}(x_0+h-x_0)^3,\end{aligned}$$

and

$$\begin{aligned}u(x_0+2h) = u(x_0)+u_x(x_0)(x_0+2h-x_0)+\frac{u_{xx}(x_0)}{2}(x_0+2h-x_0)^2\\ +\frac{u_{xxx}(\xi_2)}{6}(x_0+2h-x_0)^3.\end{aligned}$$

We want to find the coefficients (these may depend on h) a, b, c such that

$$au(x_0)+bu(x_0+h)+cu(x_0+2h) = u_x(x_0)+\mathcal{O}(h^2).$$

Insert the Taylor expansions for $u(x_0+h)$ and $u(x_0+2h)$

$$\begin{aligned}&au(x_0)+b(u(x_0)+u_x(x_0)(x_0+h-x_0)+\frac{u_{xx}(x_0)}{2}(x_0+h-x_0)^2\\ &+\frac{u_{xxx}(\xi_1)}{6}(x_0+h-x_0)^3)\\ &+c(u(x_0)+u_x(x_0)(x_0+2h-x_0)+\frac{u_{xx}(x_0)}{2}(x_0+2h-x_0)^2\\ &+\frac{u_{xxx}(\xi_2)}{6}(x_0+2h-x_0)^3)\\ &=(a+b+c)u(x_0)+(b+2c)u_x(x_0)h \qquad (4)\\ &+(\frac{b}{2}+2c)u_{xx}(x_0)h^2+(\frac{b}{6}u_{xxx}(\xi_1)+\frac{4}{3}cu_{xxx}(\xi_2))h^3. \qquad (5)\end{aligned}$$

We want the right hand side to consist only of $u_x(x_0)$ and second order terms. Thus we get the conditions

$$\begin{aligned}a+b+c&=0\\ (b+2c)h&=1.\end{aligned}$$

The second equations says that a, b and c all scale like $1/h$, so the second order term in (5) will reduce to a first order term. Thus we want

$$\frac{b}{2}+2c=0.$$

Solving these three equations yields the coefficients

$$a = -\frac{3}{2}h$$
$$b = \frac{2}{h}$$
$$c = -\frac{1}{2h}.$$

$\boxed{3}$ This proof is basically the same as in the lecture.

If all $v_m \leq 0$, then the statement is clearly true (since the max will always be 0). Assume there is a number p, $1 \leq p \leq M-1$, such that $v_p > 0$ is the maximum (but is not on the boundary of the interval), i.e.

$$v_p = \max_{m=0,\ldots,M} v_m.$$

Then, by the inequality given in the exercise,

$$v_p \leq \frac{1}{\beta}(\alpha v_{p-1} + \gamma v_{p+1}).$$

Since all other numbers are smaller than v_p, and $\beta \geq \alpha + \gamma$ by assumption, we have the inequalities

$$\frac{1}{\beta}(\alpha v_{p-1} + \gamma v_{p+1}) \leq \frac{\alpha + \gamma}{\beta} v_p \leq v_p.$$

If we have a strict inequality, $\beta > \alpha + \gamma$, the statement above must be false, so v_p can not be a maximum point. If $\beta = \alpha + \gamma$, the inequality can only be true if $v_{p-1} = v_p = v_{p+1}$. This means that all points are equal, and thus also the boundary points have the same value as v_p.