TMA 4190 Introduction to Topology
Lecturer: Gereon Quick
Lecture 08[1]

8. Milnor's proof of the Fundamental Theorem of Algebra

Last time, we forgot to mention a useful fact about tangent spaces of submanifold given as the preimage of a regular value. We remedy this sin of omission today before we move on.

Tangent space of regular fibers

Let Z be the preimage of a regular value $y \in Y$ under the smooth map $f\colon X \to Y$. Then the kernel of the derivative

$$df_x\colon T_x(X) \to T_y(Y)$$

at any point $x \in Z$ is the tangent space to $T_x(Z)$.

Proof: Since $f(Z) = y$, f is constant on Z. Therefore, df_x vanishes on the subspace $T_x(Z) \subset T_x(X)$. Hence df_x sends all of $T_x(Z)$ to zero. In other words, $T_x(Z) \subseteq \operatorname{Ker} df_x$.

But df_x is surjective, since f is a submersion at any regular point. Hence the dimension of the kernel of df_x is

$$\dim T_x(X) - \dim T_y(Y) = \dim X - \dim Y = \dim Z.$$

Hence $T_x(Z)$ is a subspace of the kernel of df_x of the same dimension as $\operatorname{Ker} df_x$. Thus $T_x(Z) = \operatorname{Ker} df_x$. **QED**

The stack of records theorem

In order to make the final preparations for Milnor's proof, we have a closer look at a specific situation for regular values.

Suppose $f\colon X \to Y$ is a smooth map with $\dim X = \dim Y$ and X **compact**. Let $y \in Y$ **be a regular value** for f.

Let x be a point in $f^{-1}(y)$. Since y is a regular value, x is a regular point, i.e. df_x is surjective. But, since $\dim X = \dim Y$, this implies df_x is an isomorphism. Hence f **is a local diffeomorphism at** x.

[1]Following the books of Guillemin and Pollack: Differential Topology, and Milnor: Topology from the differentiable viewpoint.

Let $V \subset X$ and $U \subset Y$ be open neighborhoods around x and y, respectively, such that **$f_{|V}\colon V \to U$ is a diffeomorphism**.

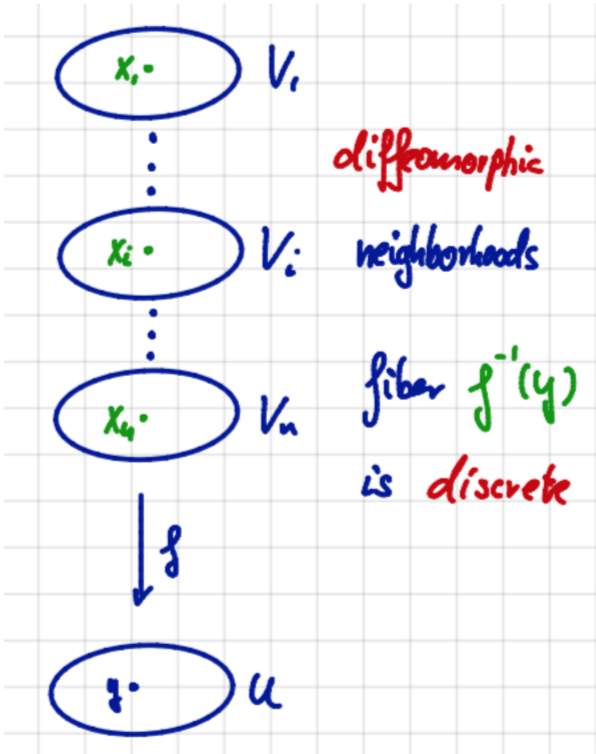
Now suppose x' is another point in $f^{-1}(y)$ with $x \neq x'$. Then $df_{x'}$ is an isomorphism as well, and we can choose an open neighborhood $V' \subset X$ around x' such that $f_{|V'}$ is a diffeomorphism onto an open subset $U' \subset Y$ containing y.

Then **V and V' are disjoint**. For, if $V \cap V' \neq \emptyset$, then f restricts to a diffeomorphism from $V \cap V'$ onto $U \cap U'$. Since $y \in U \cap U'$ and $f(x) = y = f(x')$, this would imply $x = x' \in V \cap V'$. So if $x \neq x'$, we must have $V \cap V' \emptyset$.

Hence all **the points in $f^{-1}(y)$ lie in pairwise disjoint open subsets of X**. We conclude that $f^{-1}(y)$ is discrete. Since the subset $\{y\}$ is closed in Y, the fiber $f^{-1}(y)$ is a closed subset of X. Since X is compact, this implies that **$f^{-1}(y)$ is compact** as well (closed subsets in compact spaces are compact). Hence as a compact and discrete space, **$f^{-1}(y)$ is a finite set**.

(For, given a compact discrete subset S in $\mathbb{R}^n$. Assume S was not finite. Since S is bounded, there is an $\epsilon > 0$ such that S is contained in the n-dimensional box with edges of length ϵ and center 0. Divide this box into 2^n n-dimensional boxes of equal size. The lenght of their edge is $\epsilon/2$. If S was infinite there must be at least one small box which still contains infinitely many points of S. We take this box and divide it into 2^n n-dimensional boxes of equal size. The lenght of their edges is now $\epsilon/4$. Again, if S was infinite there must be at least one of the smaller boxes which still contains infinitely many points of S.
By repeating the argument, we see that we can find an infinite sequence of points in S which converges. Since S is closed, any convergent infinite sequence of points in S must have a limit in S. Call this limit s. But then the subset $\{s\}$ would not be open in S, since every open subset of $\mathbb{R}^n$ containing s would also contain other points of S. Hence S would not be discrete. QED)

Let $f^{-1}(y) = \{x_1, \ldots, x_n\}$. We can pick finitely many open subets $W_1, \ldots, W_n$ in X with $x_i \in W_i$ which map diffeomorphically onto open subsets $U_1, \ldots, U_n$ in Y each containing y. The finite intersection $U := U_1 \cap \cdots \cap U_n$ is open in Y and with $y \in U$. The inverse image $f^{-1}(U)$ is a disjoint union of open subsets $V_1, \ldots, V_n$ and each V_i is mapped by f diffeomorphically onto U and $x_i \in V_i$.



Hence we have shown the following very useful result:

Stack of Records Theorem

Suppose $\dim X = \dim Y$, $f\colon X \to Y$ is a smooth map and X is compact. Let $y \in Y$ be a regular value for f. Then the set $f^{-1}(y)$ is a discrete finite subset $\{x_1, \ldots, x_n\}$ of X, and we can choose an open neighborhood $U \subset Y$ around y such that $f^{-1}(U) \subset X$ is the disjoint union $V_1 \cup \cdots \cup V_n$ of open subsets of X with $x_i \in V_i$ and f maps each V_i diffeomorphically onto U.

Aside

If in addition to the assumptions of the theorem all values in Y are regular, then $X \to Y$ is an example of a covering. In Topology, a continuous map $f\colon X \to Y$ is an (unramified) covering if every point in Y has an open neighborhood U such that $f^{-1}(U)$ is the disjoint union of open sets V_i such that f maps each V_i homeomorphically onto U. Coverings play an important role in Topology and Homotopy Theory.

Since $f^{-1}(y)$ is finite, it makes sense to talk about the number of elements in $f^{-1}(y)$ which we denote by $\#f^{-1}(y)$.

Locally constant fiber

The function $y \mapsto \#f^{-1}(y)$ on the set of regular points for f is **locally constant**, i.e. for every regular value y there is an open neighborhood $U \subset Y$ of y such that $\#f^{-1}(y) = \#f^{-1}(y')$ for all $y' \in U$.

Proof: Given a regular value y, let $x_1, \ldots, x_n$ be the points in $f^{-1}(y)$. We just learned that there is an open neighborhood U of y such that $f^{-1}(U) = V_1 \cup \cdots \cup V_n$ is the pairwise disjoint union of open neighborhoods V_i of x_i which all map diffeomorphically onto open subset U. This means that for every point $y' \in U$, there is exactly one point in V_i which maps to y'. And these are the only points which map onto y'. Hence $\#f^{-1}(y') = \#f^{-1}(y)$. **QED**

A short detour to general topology

To know that a function is locally constant can be very convenient in many situations. For example, locally constant functions on connected spaces are constant.

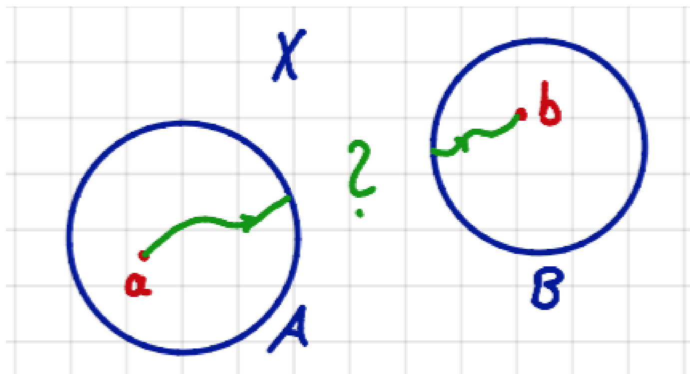
Recall that a topological space X is called **connected** if X cannot be written as the union of two nonempty disjoint open subsets; or equivalently, if X and $\emptyset$ are the only subsets which are both open and closed in X.

Connectedness is a “global” property of a topological space, i.e. it is invariant under homeomorphisms. In particular, two spaces cannot be homeomorphic if one is connected and the other is not. Familiar examples of connected spaces are intervals in $\mathbb{R}$. For example, the closed interval $[0,1]$ is connected.

The criterion for connectedness is rather elegant to state, but it does not tell us if we can actually “walk” from one point to another, as one would expect for a connected space. This is the point of a related and more concrete property. A topological space X is called **path-connected** if for any two points $x,y \in X$ there is a continuous map $\gamma\colon [0,1] \to X$ with $\gamma(0) = x$ and $\gamma(1) = y$. Again, path-connectedness is a topological property, i.e. it is preserved under homeomorphisms.

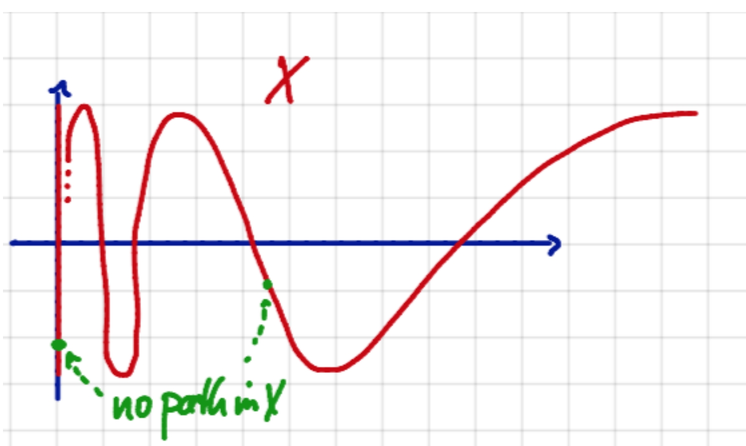
Path-connectedness is the stronger property, i.e. **if a space is path-connected, then it is also connected**. For, suppose X is path-connected. If X was not connected, then there would be two disjoint nonempty open subsets A and B with $X = A \cup B$. Since A and B are nonempty, we can choose two points $a \in A$

and $b \in B$. Since X is path-connected, there is a continuous map $\gamma\colon [0,1] \to X$ with $\gamma(0) = a$ and $\gamma(1) = b$. Hence $0 \in \gamma^{-1}(A) \subset [0,1]$ and $1 \in \gamma^{-1}(B) \subset [0,1]$. Since A and B are disjoint and open, both $\gamma^{-1}(A)$ and $\gamma^{-1}(B)$ are disjoint and open in $[0,1]$. Since $X = A \cup B$, we would have $[0,1] = \gamma^{-1}(A) \cup \gamma^{-1}(B)$ which contradicts the fact that $[0,1]$ is connected. Hence X must be connected.



But be aware that **there are connected spaces which are not path-connected**. A standard example is the subspace

$$X = \{(x, \sin(\log x)) \in \mathbb{R}^2 : x > 0\} \cup (0 \times [-1,1]).$$



Though the usual examples of connected spaces we will meet are path-connected. For example, every sphere is path-connected, and every sphere with finitely many points removed is still path-connected.

We conclude our detour with a lemma we will use in the next section. Given a map $f\colon X \to S$ from a topological space X to any set$S$. Recall that f is called

locally constant if for every $x \in X$ there is an open neighborhood $U_x \subset X$ such that $f_{|U_x}$ is constant.

A useful lemma

Let X be a connected space and $f\colon X \to S$ be locally constant. Then f is constant.

Proof: Let $s \in S$ be a value of f, i.e. $s = f(x)$ for some $x \in X$. We can write X as the **disjoint union** of the sets

$$A = \{x \in X : f(x) = s\} \text{ and } B = \{x \in X : f(x) \neq s\}.$$

Since f is locally constant, both A and B are open. For if $a \in A$, then there is an open neighborhood $U_a \subset A$ with $f(U_a) = \{s\}$, i.e. $U_a \subset A$. Similarly, if $b \in B$, then there is an open neighborhood $U_b \subset X$ with $f(U_b) = \{f(b)\}$, i.e. $U_b \subset B$. But since X is connected and $A \neq \emptyset$, we must have $A = X$, and f is constant. **QED**

Milnor's proof of the Fundamental Theorem of Algebra

Now we are ready to see how Milnor used the previous ideas for a simple proof of the following important result:

Fundamental Theorem of Algebra

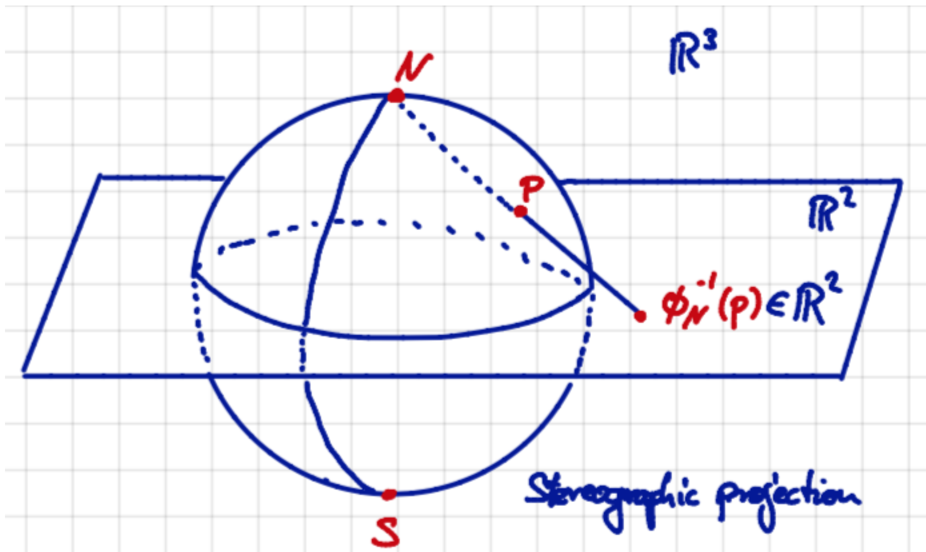
Every nonconstant complex polynomial

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$$

with $a_n \neq 0$ must have a zero.

As a consequence, **$P(z)$ must have exactly n zeroes** when we count them with multiplicities.

We are going to identify the complex numbers $\mathbb{C}$ with the points in real plane $\mathbb{R}^2$, but we keep in mind how that we can mutliply and form inverses for points in $\mathbb{C}$. To prove the theorem we need to extend the map $P\colon \mathbb{C} \to \mathbb{C}$ to a map on a **compact** space. Recall that S^2 is a compact subspace of $\mathbb{R}^3$ and that we can relate S^2 and the real plane $\mathbb{R}^2$ via stereographic projection:



The formulae for the projection from the northpole $N = (0,0,1) \in S^2$ are

$$\phi_N^{-1} \colon S^2 \setminus \{N\} \to \mathbb{R}^2,\ (x_1,x_2,x_3) \mapsto \frac{1}{1-x_3}(x_1,x_2) \text{ and}$$

$$\phi_N \colon \mathbb{R}^2 \to S^2 \setminus \{N\},\ (x_1,x_2) \mapsto \frac{1}{1+|x|^2}\left(2x_1,2x_2,|x|^2-1\right).$$

The formulae for the projection from the south pole $S = (0,0,-1) \in S^2$:

$$\phi_S^{-1} \colon S^2 \setminus \{S\} \to \mathbb{R}^2,\ (x_1,x_2,x_3) \mapsto \frac{1}{1+x_3}(x_1,x_2) \text{ and}$$

$$\phi_S \colon \mathbb{R}^2 \to S^2 \setminus \{S\},\ (x_1,x_2) \mapsto \frac{1}{1+|x|^2}\left(2x_1,2x_2,1-|x|^2\right).$$

Considering our polynomial P as a map from $\mathbb{R}^2$ to $\mathbb{R}^2$ we define a new map

$$\mathbf{f} \colon \mathbf{S^2} \to \mathbf{S^2},\ \begin{cases} f(x) := \phi_N \circ P \circ \phi_N^{-1}(x) & \text{for all } x \in S^2 \setminus \{N\} \\ f(N) := N & \text{for } x = N. \end{cases}$$

Claim: The map f is smooth.

Since ϕ_N and ϕ_N^{-1} are smooth and polynomials are smooth as well, it is clear that f is smooth at all points which are not the northpole. It remains to show that it is also smooth in a neighborhood of N.

In order to do this we use the projection from the south pole and define a map

$$Q \colon \mathbb{C} \to \mathbb{C} \text{ by } Q := \phi_S^{-1} \circ f \circ \phi_S.$$

Comparing the definitions of f and Q, we need to calculate

$$\begin{aligned}\phi_N^{-1}\circ\phi_S(x_1,x_2) &= \phi_N^{-1}\left(\frac{1}{1+|x|^2}(2x_1,2x_2,|x|^2-1)\right)\\ &= \frac{1}{1-\frac{1}{1+|x|^2}}\left(\frac{2x_1}{1+|x|^2},\frac{2x_2}{1+|x|^2}\right)\\ &= \frac{1+|x|^2}{2|x|^2}\left(\frac{2x_1}{1+|x|^2},\frac{2x_2}{1+|x|^2}\right)\\ &= \frac{1}{|x|^2}(x_1,x_2).\end{aligned}$$

Remembering complex conjugation $z\mapsto\bar z$ on $\mathbb{C}$, we can rewrite this as:

$$\phi_N^{-1}\circ\phi_S(z) = \frac{z}{|z|^2} = 1/\bar z \text{ for all } z\in\mathbb{C}\setminus\{0\}.$$

Similarly, we also get

$$\phi_S^{-1}\circ\phi_N(z) = \frac{z}{|z|^2} = 1/\bar z \text{ for all } z\in\mathbb{C}\setminus\{0\}.$$

Thus we get

$$\begin{aligned}Q(z) &= \phi_S^{-1}\circ\phi_N\circ P\circ\phi_N^{-1}\circ\phi_S(z)\\ &= \phi_S^{-1}\circ\phi_N(P(1/\bar z))\\ &= \phi_S^{-1}\circ\phi_N(a_n\bar z^{-n}+a_{n-1}\bar z^{n-1}+\cdots+a_1\bar z^{-1}+a_0)\\ &= 1/\left(\bar a_n z^{-n}+\bar a_{n-1}z^{n-1}+\cdots+\bar a_1 z^{-1}+\bar a_0\right)\\ &= z^n/\left(\bar a_n+\bar a_{n-1}z+\cdots+\bar a_1 z^{n-1}+\bar a_0 z^n\right).\end{aligned}$$

This shows that Q is smooth at $z=0$ for

$$Q(0) = \phi_S^{-1}(f(\phi_S(0)) = \phi_S^{-1}(f(N)) = \phi_S^{-1}(N) = 0$$

and hence

$$\begin{aligned}&\lim_{h\to 0}\frac{Q(h)-Q(0)}{h}\\ =&\lim_{h\to 0}\frac{h^n/\left(\bar a_n+\bar a_{n-1}h+\cdots+\bar a_0h^n\right)-0}{h}\\ =&\lim_{h\to 0} h^{n-1}/\left(\bar a_n+\bar a_{n-1}h+\cdots+\bar a_0h^n\right)\\ =&\,0.\end{aligned}$$

Since smoothness is a local property, Q is smooth in a small open neighborhood of 0. Since ϕ_S and ϕ_S^{-1} are diffeomorphisms and since ϕ_S sends an open neighborhood of N in S^2 to an open neighborhood of 0 in $\mathbb{C}$, this implies that

$$f = \phi_S^{-1}\circ Q\circ\phi_S$$

f is smooth in an open neighborhhod of N.

Next, we observe that the smooth map $f\colon S^2 \to S^2$ has **only finitely many critical points**, i.e. points x where df_x fails to be surjective. For, since ϕ_N and ϕ_N^{-1} are diffeomorphisms, the only points that might be critical for f are the points where P fails to be a local diffeomorphism, and possibly N. But the derivative of P is given by the polynomial

$$dP_z = P'(z) = \sum_{j=1}^{n} j a_j z^{j-1}$$

which has at most $n-1$ zeroes. Hence there are only finitely many z where dP_z is not an isomorphism.

Thus the set R of regular values for f is S^2 with finitely many points removed and is therefore **connected**. This implies that the function

$$R \to \mathbb{Z},\ y \mapsto \# f^{-1}(y),$$

which we have seen is locally constant, must be **constant**.

This enables us to show:

Claim: f is onto.

For, assume there is a $y_0 \in S^2$ with $f^{-1}(y_0) = \emptyset$, i.e. $\# f^{-1}(y_0) = 0$. Then y_0 is a regular value for f by definition. Since the function $y \mapsto \# f^{-1}(y)$ is constant on the set of regular values, it would have to be zero for every regular value. Hence $\# f^{-1}(y)$ would be nonzero only for cricital values y. But that would mean that **f had only finitely many values**. Since f is continuous and S^2 connected, this would imply that f is constant. (If f had different values $y_1, \ldots, y_k \in S^2$, then $S^2 = f^{-1}(y_1) \cup \cdots \cup f^{-1}(y_k)$ with $f^{-1}(y_i) \cap f^{-1}(y_j) \neq \emptyset$ and each $f^{-1}(y_1)$ would be nonempty and open (and closed), since f is continuous. That is not possible, since S^2 is connected.) But P is not constant, and ϕ_N and ϕ_N^{-1} are diffeomorphisms. Thus **f is not constant**. We conclude that **f must be onto**.

Conclusion: In particular, $f^{-1}(S) \neq \emptyset$ and there must be at least one point $p \in S^2$ with $f(p) = S$. Since ϕ_N is a diffeomorphism and $\phi_N(0) = S$, p must satisfy $P(\phi_N^{-1}(p)) = 0$. Hence **$z := \phi_N^{-1}(p) \in \mathbb{C}$ is a zero of P. QED**