NTNU – Trondheim
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Department of Mathematical Sciences

Examination paper for **TMA4145 Linear methods**

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Examination date: Saturday, 20 December 2014

Examination time (from–to): 9:00-13:00

Permitted examination support material: D: No written or handwritten material are allowed. Calculators Casio fx-82ES PLUS, Citizen SR-270X or Citizen SR-270X College, Hewlett Packard HP30S are allowed

Other information:
The exam consists of twelve questions, the order is according to the topics in the course not to the level of difficulty. All solutions should be stated in a precise and rigorous way, with any assumptions written down and arguments justified. Each solution will be graded as *rudimentary* (F), *acceptable* (E), *good* (C) or *excellent* (A). Five acceptable solutions guarantee an E; seven acceptable with at least one good a D; seven acceptable with at least five good a C; nine good with at least two excellent a B; nine good with at least seven excellent an A. These are guaranteed limits. Beyond that, the grade is based on the total achievement.

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Date Signature

Problem 1 Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be norms on a vector space V.

a) Show that $\|x\| = \|x\|_1 + \|x\|_2$ is also a norm. Assume that $\{x_n\}$ is a Cauchy sequence in $(V, \|\cdot\|)$ and prove that $\{x_n\}$ is a Cauchy sequence in $(V, \|\cdot\|_1)$.

b) Give an example of a vector space V, two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on V, and a sequence $\{x_n\}$ such that $\{x_n\}$ is a Cauchy sequence in $(V, \|\cdot\|_1)$ but not in $(V, \|\cdot\|)$, where $\|\cdot\|$ was defined in **a)**. Prove that the dimension of V has to be infinite for such an example.

Problem 2 Let

$$A = \begin{bmatrix} 8 & 0 & -1 \\ -2 & 5 & 0 \\ 0 & -4 & 7 \end{bmatrix}, \quad b = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$$

a) Find an LU-decomposition of A and solve the linear system $Ax = b$.

b) Rewrite the system $Ax = b$ in the form $x = Bx + c$ such that $B : \mathbb{R}^3 \to \mathbb{R}^3$ is a contraction in the norm $\|x\|_\infty = \max\{|x_1|, |x_2|, |x_3|\}$, $x = (x_1, x_2, x_3) \in \mathbb{R}^3$. Show how the new system may be solved by iteration starting from any $x_0 \in \mathbb{R}^3$.

Problem 3

a) Let $C([0,2] \times [0,2], \mathbb{R})$ be an inner-product space with

$$\langle f, g \rangle = \int_0^2 \int_0^2 f(x,y) g(x,y) dx dy.$$

Find an orthogonal basis for $\operatorname{span}\{1, x, y\}$ in this space.

b) Find $a, b, c \in \mathbb{R}$ such that $\displaystyle\int_0^2 \int_0^2 |xy - a - bx - cy|^2 dx dy$ is minimal.

Problem 4

a) Let M be a closed subspace of a Hilbert space H. For each $x \in H$ denote by $P_M(x)$ the orthogonal projection of x onto M. Prove that $P_M^2 = P_M$, $P_M^* = P_M$ and $\|P_M\| = 1$.

b) Let H be a Hilbert space and $P : H \to H$ be a bounded linear transformation that satisfy $P = P^*$ and $P^2 = P$. Prove that P is the orthogonal projection on some closed subspace M of H.

Problem 5 Let X, Y be Banach spaces and $T : X \to Y$ be a bounded linear transformation.

a) Prove that the kernel of T is a closed subspace of X.

b) Give an example of two Banach spaces X and Y and a bounded linear transformation T for which the range of T is not closed.

Problem 6 Let

$$A = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 1 & -1 & 3 \end{bmatrix}, \quad x_0 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$$

a) Show that A has two eigenvalues $\lambda_1 = 2$ and $\lambda_2 = 3$ and find the Jordan normal form of A, determine both the matrix J and the change-of-basis matrix T in $A = TJT^{-1}$.

b) Solve the initial-value problem $\dot{x} = Ax, \; x(0) = x_0$.