Example 10.2. Consider the second order equation

$$\ddot{x} + q(x) = 0.$$

We can reduce it to a first-order system:

$$\dot{x} = y, \qquad \dot{y} = -q(x).$$

This is known as a conservative system — essentially Newton's second law in which the force is a conservative force — it can be written as a gradient of a potential, or an "energy". For this reason it is also known as a gradient system, of which we shall see more in the next lecture. Such systems have Lyapunov functions that are easy to find if the energy as usually defined has a sign — the energy itself. We require $xq(x) > 0$, so that the force is restorative.

Suppose $q(x_0) = 0$, so that $(x_0, 0)$ is a fixed point. Set

$$V(x,y) = \frac{1}{2}y^2 + \int_{x_0}^{x} q(r)\,\mathrm{d}r.$$

Then $V(x,y) > 0$ in a neighbourhood of $(x_0, 0)$ and

$$\frac{\mathrm{d}}{\mathrm{d}t}V(x(t), y(t)) = q(x)y + y(-q(x)) = 0.$$

Therefore the equilibrium $(x_0, 0)$ is stable.

11. Lecture XI: Gradient and Hamiltonian Systems

11.1. **Elementary Hamiltonian Dynamics.** A full discussion of Hamiltonian dynamics requires an introduction to manifolds with a symplectic structure, a sort of anti-symmetric inner product over the tangent bundle of spaces T_pM as p ranges over M. But it is nevertheless natural to give a superficial account of Hamiltonian systems after a discussion of the method of Lyapunov. We shall focus on select topics of the theory immediately pertainent to our ongoing discussion on fixed points.

A Hamiltonian system is a system over $\mathbb{R}^{2d}$, for which there is a function $H \in C^1(\mathbb{R}^d \times \mathbb{R}^d; \mathbb{R})$ such that the system can be written as

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x} = \frac{\partial H}{\partial \mathbf{y}}, \qquad \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{y} = -\frac{\partial H}{\partial \mathbf{x}},$$

where $\mathbf{x}$ and $\mathbf{y}$ take values in $\mathbb{R}^d$. This function H is known as the Hamiltonian (function).

The first thing to notice about Hamiltonian systems is that the Hamiltonian is a conserved quantity of the dynamics:

$$\frac{\mathrm{d}H(\mathbf{x},\mathbf{y})}{\mathrm{d}t} = \frac{\partial H}{\mathrm{d}\mathbf{x}}\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} + \frac{\partial H}{\mathrm{d}\mathbf{y}}\frac{\mathrm{d}\mathbf{y}}{\mathrm{d}t} = 0.$$

Therefore H is a Lyapunov function for the corresponding Hamiltonian system if $H(\mathbf{y}) > H(\mathbf{x}_0)$ for every $\mathbf{y}$ in a neighbourhood of a fixed point $\mathbf{x}_0$, allowing us to conclude that that fixed point is stable. We have not excluded the possibility that we can find a better Lyapunov function that is strictly decreasing along trajectories. We shall be spending a little effort a little later on on telling when it is that a critical point is stable but not asymptotically stable.

Hamiltonian systems arise naturally in modelling physical systems in which no dissipative effects are present. The Hamiltonian of system is often interpretable as its total energy, or potential of some sort (gravitational potential, hydrolic potential, electric potential, etc.). The "classical mechanics way" to think about Hamiltonian systems is that $\mathbf{y} = \dot{\mathbf{x}}$, so that where $\mathbf{x}$ is the position of a particle, $\mathbf{y}$ is its (normalized mass) momentum. This is not always the case, of course, but where it holds true,

$$\ddot{\mathbf{x}} = -\frac{\partial H}{\mathrm{d}\mathbf{x}}.$$

We call systems of the form

$$\ddot{\mathbf{x}} = -f(\mathbf{x}), \qquad \mathbf{x} \in \mathbb{R}^d$$

Newton systems if f can be written as a gradient because they are Newton's second law with normalized mass and a conservative force given by $f(\mathbf{x})$. This is a very specific type of Hamiltonian system because if $f = \nabla_{\mathbf{x}} V$, then the Hamiltonian H is given by

$$\frac{\partial H}{\mathrm{d}\mathbf{x}} = f(\mathbf{x}) = \nabla_{\mathbf{x}} V \implies H(\mathbf{x},\mathbf{y}) = V(\mathbf{x}) + G(\mathbf{y}),$$

and depends on the momentum variables $\mathbf{y}$ in a special way. The part dependent on $\mathbf{x}$ is the potential energy and the part dependent on $\mathbf{y}$ is the kinetic energy. In fact, taking

$$\nabla_{\mathbf{y}} G(\mathbf{y}) = \partial H/\partial \mathbf{y} = \dot{\mathbf{x}} = \mathbf{y},$$

we see that $G(\mathbf{y}) = |\mathbf{y}|^2/2$, as expected. With $d = 1$, we know we can always write f as a gradient/derivative, simply by integrating f, but this is not always so for higher-dimensional f because the integral of f might not be path-independent.

11.2. **Planar Hamiltonian Systems.** The analysis of critical points for Hamiltonian systems for $d = 1$ is simpler also. We shall refine our analysis by giving geometric descriptions/definitions four types of behaviour around critical points in planar systems already familiar to us from the linear theory:

(i) A critical point $\mathbf{x}_0 \in \mathbb{R}^2$ for an autonomous system is a FOCUS (or SPIRAL) if there exists a $\delta > 0$ such that for $\mathbf{y}_0$ with $0 < |\mathbf{x}_0 - \mathbf{y}_0| < \delta$, $|\phi_t(\mathbf{y}_0) - \mathbf{x}_0| \to 0$ and $|\arg(\phi_t(\mathbf{y}_0) - \mathbf{x}_0)| \to \infty$ as $t \to \infty$ (STABLE) or as $t \to -\infty$ (UNSTABLE).

(ii) A critical point $\mathbf{x}_0 \in \mathbb{R}^2$ for an autonomous system is a CENTRE if there exists a $\delta > 0$ such that every trajectory in $B_\delta(\mathbf{x}_0)\backslash\{\mathbf{x}_0\}$ is a closed curve.

(iii) A critical point $\mathbf{x}_0 \in \mathbb{R}^2$ for an autnomous system is a CENTRE-FOCUS if there exists an sequence of closed curves $\{\Gamma_n\}$ and a sequence of number $r_n \to 0$ such that Γ_{n+1} is in the open set enclosed by Γ_n and $\Gamma_n \subseteq B_{r_n}(\mathbf{x}_0)$, and every trajectory between Γ_n and Γ_{n+1} tends to one closed curve or the other as $t \to \pm\infty$. These closed curves Γ_n are known as LIMIT CYCLES.

(iv) A critical point $\mathbf{x}_0 \in \mathbb{R}^2$ for an autonomous system is a TOPOLOGICAL SADDLE if there exist two trajectories which approach $\mathbf{x}_0$ as $t \to \infty$, and two trajectories that approach $\mathbf{x}_0$ as $t \to -\infty$, and if there exists a $\delta > 0$ such that all other trajectories in $B_\delta(\mathbf{x}_0)\backslash\{\mathbf{x}_0\}$ leave $B_\delta(\mathbf{x}_0)$ as $t \to \pm\infty$. We call the four special trajectories SEPARATRICES.

We are ready to state a lemma:

Lemma 11.1. *If (x_0, y_0) is a focus of the planar Hamiltonian system*

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\partial H}{\partial y}, \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = -\frac{\partial H}{\partial x},$$

then (x_0, y_0) is not a strict maximum or strict minimum of the Hamiltonian function H.

As can be deduced from this theorem statement, Hamiltonian functions do not have to be Lyapunov functions in the strict sense we have required. However, the other case excluded by the statement is not at all exotic. If a function V attains a maximum at a critical point $\mathbf{x}_0$, and $\mathrm{D}V \cdot f \geq 0$ in a neighbourhood of $\mathbf{x}_0$, then clearly $\mathbf{x}_0$ is stable because $-V$ would supply us with a suitable Lapunov function.

Since H is conserved by the flow (invariant along trajectories), it is quite evident that if H attained a strict maximum or strict minimum at the critical point (x_0, y_0), and is C^1, then a trajectory cannot connect any points in a neighbourhood of (x_0, y_0) with the critical point itself, for then H would have to increase or decrease to the value it attains at the critical point. One instance in which H might have a focus at a critical point, then, can be that H vanishes to second order at the critical point.

In fact, this is the crux of the argument demonstrating the statement of the Lemma: If there is a focus, then there is a trajectory $\phi_t(u_0, v_0)$ that tends to the critical point (x_0, y_0) such that $H(\phi_t(u_0, v_0))$ is constant. This means that H cannot attain a strict maximm or minimum in any neighbourhood of (x_0, y_0).

Before we state a more general theorem characterizing the behaviour of planar Hamiltonian systems around critical points, we shall impose two further requirements on the Hamiltonian functions of planar Hamiltonian systems. The first requirement that they be real analytic — that is, at any point, there is a neighbourhood such that H can be expressed as a convergent series of its Taylor expansion about that point, and in particular, H is smooth. There are smooth functions that are not real analytic — the most commonly cited example of which is

$$h(x) = \exp(1/x)\mathbb{1}_{[0,\infty)},$$

which is smooth not expressible as the convergent sum of its Taylor series in any neighbourhood around 0. We have seen that Lipschitz regularity gave us existence and uniqueness, and it should be no surprise that higher regularity/smoothness requirements yield similar dividends and rule out some behaviours that are difficult to analyse.

The second requirement is that ∇H is of full rank around $\mathbf{x}_0$. We say that the critical point $\mathbf{x}_0$ of a system

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x} = f(\mathbf{x}),$$

in $\mathbb{R}^d$ is NON-DEGENERATE if $\mathrm{D}f(\mathbf{x}_0)$ is non-degenerate/non-singular/does not have 0 as an eigenvalue. In the context of planar Hamiltonian systems, we see that this means the matrix

$$\mathrm{D}f = \begin{pmatrix} \partial^2 H/\partial y \partial x & \partial^2 H/\partial y^2 \\ -\partial^2 H/\partial x^2 & -\partial^2 H/\partial y \partial x \end{pmatrix}$$

is non-singular at a critical point (x_0, y_0). Notice that this matrix can be obtained from the Hessian $\nabla^2 H$ of H by left multiplication with

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

which implies that $\det(\mathrm{D}f) = \det(\nabla^2 H)$. That is, we require that H does not vanish to second order.

From elementary calculus, if $H \in C^2(\mathbb{R}^2)$, then at a point (x_0, y_0) for which $\nabla H = (0,0)^\top$, one can deduce if (x_0, y_0) is a maximum, a minimum, or a saddle point according as $\mathrm{D}f(x_0, y_0)$ is negative definite, positive definite, or has both positive and negative eigenvalues. In particular, if H has a saddle point at (x_0, y_0), then $\det(\nabla^2 H)(x_0, y_0) < 0$.

We state now our first result on concrete behaviour of trajectories near nonhyperbolic critical points for general planar autonomous systems:

Theorem 11.2. *Let $U \subseteq \mathbb{R}^2$ be a neighbourhood of a critical point $\mathbf{x}_0$ of an autonomous system $\dot{\mathbf{x}} = f(\mathbf{x})$, $f \in C^1(U)$. Suppose $\mathbf{x}_0$ is a centre for the linearized dynamics. Then $\mathbf{x}_0$ is either a centre, a centre-focus, or a focus for the original autonomous system.*

We shall defer this proof to the next lecture. Meanwhile we state another result due in part to Dulac which we shall discuss a few more lectures thence:

Theorem 11.3 (Dulac)**.** *In any bounded region of the plane, an analytic planar system has at most a finite number of limit cycles.*

This readily implies that for analytic f, the autonomous system $\dot{\mathbf{x}} = f(\mathbf{x})$ cannot have centre-foci. Lemma 11.1 also rules out foci under certain situations. This brings us to the main theorem of this lecture:

Theorem 11.4. *Let $\mathbf{x}_0$ be a nondegenerate critical point of a planar analytic Hamiltonian system. Then $\mathbf{x}_0$ is a topological saddle of the system if it is a saddle for the Hamiltonian function, and $\mathbf{x}_0$ is a centre if it is a strict local maximum or minimum for the Hamiltonian function.*

The non-degeneracy and strict optimum conditions rule out focus behaviour by Lemma 11.1, and the analyticity assumption implies that a centre for the linearized system is a centre for the full system via Thm. 11.2 and Dulac's Theorem. Saddles are hyperbolic, and the the statement in the theorem pertaining to them follows from the Hartman-Grobman Theorem.