Repetition

Definition

The stochastic process $\{X(t) : t \geq 0\}$ with state space $\mathbb{R}$ is called a **Gaussian process** on $[0, \infty)$ if for all $m \geq 1$, for all $0 \leq t_1 < t_2 < \cdots < t_m$,

$$(X(t_1), X(t_2), \ldots, X(t_m))$$

has an m-dimensional multivariate Gaussian distribution.

Theorem

A Gaussian process $\{X(t) : t \in T\}$ is fully determined by two functions:

1) a **mean function** $m : T \to \mathbb{R}$ so that

$$\mathrm{E}[X(t)] = m(t), \quad t \in T.$$

2) a **covariance function** $C : T \times T \to \mathbb{R}$ so that

$$\mathrm{Cov}[X(t_1), X(t_2)] = C(t_1, t_2), \quad t_1, t_2 \in T.$$

Definition

Let $\{X(t) : t \in T\}$ be a stochastic process. The **correlation function** $r : T \times T \to [-1, 1]$ is defined by

$$\begin{aligned} r(t_1, t_2) &= \mathrm{Corr}[X(t_1), X(t_2)] \\ &= \frac{\mathrm{Cov}[X(t_1), X(t_2)]}{\sqrt{\mathrm{Var}[X(t_1)]\mathrm{Var}[X(t_2)]}} \\ &= \frac{C(t_1, t_2)}{\sqrt{C(t_1, t_1)C(t_2, t_2)}}, \end{aligned}$$

where $C : T \times T \to \mathbb{R}$ is the covariance function.

Definition

A stochastic process on $[0, \infty)$ is **stationary** if

1) $m(t) = \mu_0$ for $t \in [0, \infty)$

2) $C(t_1, t_2) = \sigma^2 r(|t_1 - t_2|)$ for $t_1, t_2 \in [0, \infty)$

Here $\sigma^2 > 0$ is called the **marginal variance**, and $r : [0, \infty) \to [-1, 1]$ is called a **stationary correlation function** and satisfies $r(0) = 1$.

Common stationary covariance functions

- Exponential:
$$C(t_1, t_2) = \sigma^2 \exp(-\phi_{\mathrm{E}}|t_1 - t_2|), \quad t_1, t_2 \in \mathbb{R}.$$

- Gaussian:
$$C(t_1, t_2) = \sigma^2 \exp(-\phi_{\mathrm{G}}(t_1 - t_2)^2), \quad t_1, t_2 \in \mathbb{R}.$$

- Matérn-type:
$$C(t_1, t_2) = \sigma^2(1 + \phi_{\mathrm{M}}|t_1 - t_2|) \exp(-\phi_{\mathrm{M}}|t_1 - t_2|), \quad t_1, t_2 \in \mathbb{R}.$$

The properties of the realizations are controlled through:

- **Marginal variance** (σ^2): how much can the process deviate from the mean.
- **Range** (ϕ_{E}, ϕ_{M} and ϕ_{G}): how far away are things dependent.
- **Smoothness**: Realizations are 0 times differentiable for Exponential, 1 times differentiable for Matérn-type, and infinitely many times differentiable for Gaussian.

Simulation of a Gaussian process

Input:

- $[a, b]$: interval of interest
- m: mean function
- C: covariance function

Algorithm:
1. make grid $a = t_1 < t_2 < \cdots < t_n = b$
2. set $\boldsymbol{\mu} = (m(t_1), m(t_2), \ldots, m(t_n))$
3. set $\Sigma_{ij} = C(t_i, t_j)$ for $i, j = 1, 2, \ldots, n$
4. draw $\boldsymbol{x} \sim \mathcal{N}_n(\boldsymbol{\mu}, \Sigma)$

Output:
We have simulated values $\boldsymbol{x} = (x(t_1), x(t_2), \ldots, x(t_n))$.

Conditional Gaussian process

Let $\{X(t) : t \geq 0\}$ be a Gaussian process. Assume that the process has been observed at locations $B = \{s_1 < s_2 < \cdots < s_m\}$ and let $\boldsymbol{X}_B = (X(s_1), X(s_2), \ldots, X(s_m))$.

Then for any set of locations $A = \{t_1 < t_2 < \cdots < t_n\}$, let $\boldsymbol{X}_A = (X(t_1), X(t_2), \ldots, X(t_n))$. We have

$$\boldsymbol{X}_A | \boldsymbol{X}_B = \boldsymbol{x}_B \sim \mathcal{N}_n(\boldsymbol{\mu}_{\mathrm{C}}, \Sigma_{\mathrm{C}}),$$

where

$$\boldsymbol{\mu}_{\mathrm{C}} = \boldsymbol{\mu}_A + \Sigma_{\mathrm{AB}} \Sigma_{\mathrm{BB}}^{-1} (\boldsymbol{x}_B - \boldsymbol{\mu}_B)$$
$$\Sigma_{\mathrm{C}} = \Sigma_{\mathrm{AA}} - \Sigma_{\mathrm{AB}} \Sigma_{\mathrm{BB}}^{-1} \Sigma_{\mathrm{BA}}.$$