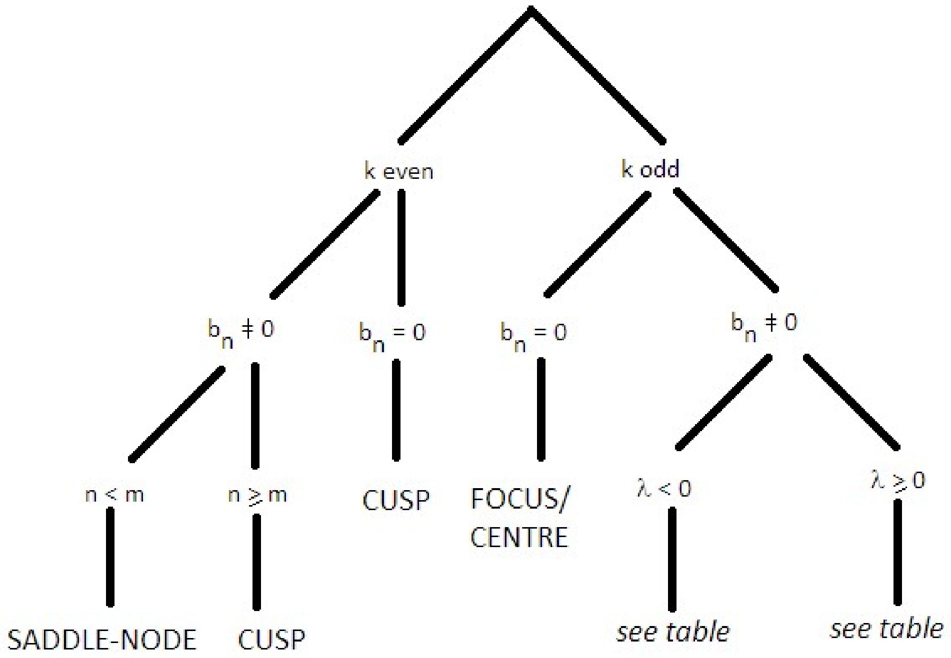

FIGURE 2. behaviours at critical points

The schematic above must also be supplemented by the following tables:
For $\lambda < 0$,

$b_n \neq 0$, $\lambda < 0$	$n < m$	$n = m$	$n > m$
n even	node	focus/centre	focus/centre
n odd	elliptic domain	focus/centre	focus/centre

and for $\lambda \geq 0$,

$b_n \neq 0$, $\lambda \geq 0$	$n < m$	$n = m$	$n > m$
n even	node	node	focus/centre
n odd	elliptic domain	elliptic domain	focus/centre

.

[There is no expectation that this schematic and its associated tables be committed to memory.]

13.2.3. Two zero eigenvalues with geometric multiplicity of two.

In this case, there are very few general theorems, and the behaviour on the centre manifold can be very complicated. In particular, if P and Q both vanish to order m around $\mathbf{0}$, then the plane is locally split into $(2m+1)$ sectors. We shall look at this case in greater detail in our discussion on index theory.

13.3. **Examples.**

Example 13.1. The system

$$\dot{x} = x^2, \qquad \dot{y} = y,$$

has a linearization governed by

$$\mathrm{D}f(0,0) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

and so has only one zero eigenvalue. It falls into case (iii) of Thm. 13.2. Therefore we expect a saddle-node.

Example 13.2. The system
$$\dot{x} = y, \qquad \dot{y} = -x^3 + 4xy,$$
has a linearization governed by
$$\mathrm{D}f(0,0) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$
and so has two zero eigenvalue, and $\dim\ker(\mathrm{D}f) = 1$. We find that $k = 3$, $m = 1$, $n = 1$, $a_3 = -1$, $b_1 = 4 \neq 0$, and $\lambda = b_1^2 - (2k+2)a_k = 24 > 0$. Consulting the schematic, we expect a critical point with an elliptic domain.

14. Lecture XIV: Centre Manifold Theory

14.1. **The Local Centre Manifold Theorem.** In this halfway lecture, we shall discuss a landmark structure theorem in our understanding of the local theory of first order autonomous systems, a culmination of our work so far. We shall turn our attention back to the problem in $\mathbb{R}^d$. We begin as before by noting that our first-order approach was based on the observation that around an isolated fixed point at the origin, we can decompose a general C^1-first-order autonomous system into

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{C}\mathbf{x} + F(\mathbf{x},\mathbf{y},\mathbf{z}) \\ \dot{\mathbf{y}} &= \mathbf{P}\mathbf{y} + G(\mathbf{x},\mathbf{y},\mathbf{z}), \\ \dot{\mathbf{z}} &= \mathbf{Q}\mathbf{z} + H(\mathbf{x},\mathbf{y},\mathbf{z}) \end{aligned} \tag{21}$$

with $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$ taking values in $\mathbb{R}^c$, $\mathbb{R}^s$, and $\mathbb{R}^u$, respectively, where $c+s+u=d$, and $\mathbf{C} \in \mathbb{C}^{c\times c}$ has c eigenvalues with zero real parts, $\mathbf{P} \in \mathbb{C}^{s\times s}$ has s eigenvalues with negative real parts, and $\mathbf{Q} \in \mathbb{C}^{u\times u}$ has u eigenvalues with positive real parts. All eigenvalues are counted with multiplicities, and complex eigenvalues come in conjugate pairs. Finally, $F(\mathbf{0}) = G(\mathbf{0}) = H(\mathbf{0}) = \mathbf{0}$.

Recall (Lecture 9) that two systems on $\mathbb{R}^d$ with flows ϕ_t and ψ_t and fixed points $\mathbf{x}_0$ and $\mathbf{y}_0$ are TOPOLOGICALLY CONJUGATE if there exist neighbourhoods U of $\mathbf{0}$ and V of $\mathbf{0}$, and a homeomorphism $h : U \to V$ for which $h(\mathbf{0}) = \mathbf{0}$ and

$$\psi_t \circ h = h \circ \phi_t.$$

The stable manifold theorem (Thm.9.1) and the Hartman-Grobman theorem (Thm.9.3) ensure us that in a neighbourhood of a hyperbolic fixed point, a C^1-first order autonomous system is topologically conjugate to its linearization. The centre manifold theorem (Thm.9.2) established the existence of a local centre manifold in addition to the stable and unstable manifolds described by the stable manifold theorem. All three local manifolds are invariant manifolds under the flow of the dynamical system. For planar systems, we have also established some very specific results regarding nonhyperbolic fixed points. We shall now put these together in the the following:

Theorem 14.1 (Local Centre Manifold Theorem)**.** *On a neighbourhood $U \subseteq \mathbb{R}^d$ about the origin, there exist functions $h_1 : B_\delta(\mathbf{0}) \subseteq \mathbb{R}^c \to B_{\delta'}(\mathbf{0}) \subseteq \mathbb{R}^s$ and $h_2 : B_\delta(\mathbf{0}) \subseteq \mathbb{R}^c \to B_{\delta'}(\mathbf{0}) \subseteq \mathbb{R}^u$, satisfying*

$$\begin{aligned} \mathrm{D}h_1(\mathbf{w})(\mathbf{C}\mathbf{w} + F(\mathbf{w}, h_1(\mathbf{w}), h_2(\mathbf{w}))) - \mathbf{P}h_1(\mathbf{w}) - G(\mathbf{w}, h_1(\mathbf{w}), h_2(\mathbf{w})) = 0, \\ \mathrm{D}h_2(\mathbf{w})(\mathbf{C}\mathbf{w} + F(\mathbf{w}, h_1(\mathbf{w}), h_2(\mathbf{w}))) - \mathbf{Q}h_2(\mathbf{w}) - H(\mathbf{w}, h_1(\mathbf{w}), h_2(\mathbf{w})) = 0, \end{aligned} \tag{22}$$

for all $\mathbf{w} \in B_\delta(\mathbf{0}) \subseteq \mathbb{R}^c$, such that the system (21) is topologically conjugate to

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{C}\mathbf{x} + F(\mathbf{x}, h_1(\mathbf{x}), h_2(\mathbf{x})) \\ \dot{\mathbf{y}} &= \mathbf{P}\mathbf{y} \\ \dot{\mathbf{z}} &= \mathbf{Q}\mathbf{z} \end{aligned},$$

in a neighbourhood of $\mathbf{0}$. If $F, G, H \in C^r(U)$, then h_1 and h_2 can be chosen to be r-times continuously differentiable.

This result, sometimes instead of Thm.9.2, and sometimes in conjunction with it, is often known simply as the "Centre Manifold Theorem".

The functions h_1 and h_2 are locally the equations that determine the centre manifold via:

$$W^c(\mathbf{0}) = \{(\mathbf{x},\mathbf{y},\mathbf{z}) \in \mathbb{R}^d : \quad \mathbf{y} = h_1(\mathbf{x}),\ \mathbf{z} = h_2(\mathbf{x})\}.$$

The equations (22) then come from the dynamics on the centre manifold given by

$$\begin{aligned} \dot{\mathbf{y}} &= \mathrm{D}h_1(\mathbf{x})\dot{\mathbf{x}} \\ \dot{\mathbf{z}} &= \mathrm{D}h_2(\mathbf{x})\dot{\mathbf{x}}, \end{aligned}$$

[which gives us the evolution of $\mathbf{y}$ and $\mathbf{z}$ restricted to the centre manifold.] The theorem stated above allows us to describe dynamics in $\mathbb{R}^d$ very precisely, given what we already know about planar dynamics, if the centre manifold is of dimension two.

The equations (22) can be impossible to solve exactly for h_1 and h_2, being a nonlinear coupled system of partial differential equations. Nevertheless if r is sufficiently large, we can approximate h_1 and h_2 by power series to a high degree of accuracy, allowing us to make higher-order approximations to the dynamical system.

Example 14.1. Let us consider another dynamical system in a neighbourhood of an isolated fixed point for which the centre manifold is locally planar:

$$\begin{aligned}
\dot{x}_1 &= x_1 y - x_1 x_2^2 = F_1(x_1, x_2, y)\\
\dot{x}_2 &= x_2 y - x_1^2 x_2 = F_2(x_1, x_2, y)\\
\dot{y} &= -y + x_1^2 + x_2^2 = -y + G(x_1, x_2, y).
\end{aligned}$$

Here, $c = 2$ and $s = 1$, $\mathbf{C}$ is the 2×2 zero matrix, and $\mathbf{P}$ is the 1×1 matrix -1. The centre subspace at the fixed point $\mathbf{0}$ is $E^c = \{(x_1, x_2, y) : y = 0\}$, and the stable subspace at $\mathbf{0}$ is $E^s = \{(x_1, x_2, y) : x_1 = x_2 = 0\}$.

Since $u = 0$, there is no h_2 to compute. Since the first order approximation is the zero matrix, h_1 vanishes to second order, and considering the highest powers in the nonlinear terms, we see that we shall not be needing terms higher than second order, either, so we have an ansatz of the form

$$h_1(x_1, x_2) = a x_1^2 + b x_1 x_2 + c x_2^2 + O(|\mathbf{x}|^3).$$

From this we have

$$\mathrm{D}h_1(x_1, x_2) = \begin{pmatrix} 2a x_1 + b x_2 \\ b x_1 + 2c x_2 \end{pmatrix} + O(|\mathbf{x}|^2).$$

Putting this into the first equation of (22), we have that

$$\begin{aligned}
0 &= \mathrm{D}h_1(x_1, x_2) \cdot F(x_1, x_2, h_1(x_2, x_2)) - (-1)h_1(x_1, x_2) - G(x_1, x_2)\\
&= \begin{pmatrix} 2a x_1 + b x_2 \\ b x_1 + 2c x_2 \end{pmatrix} \cdot \begin{pmatrix} x_1 y - x_1 x_2^2 \\ x_2 y - x_1^2 x_2 \end{pmatrix}\Bigg|_{y = h_1(x_1, x_2)}\\
&\quad + a x_1^2 + b x_1 x_2 + c x_2^2 - (x_1^2 + x_2^2) + O(|\mathbf{x}|^3).
\end{aligned}$$

For $\mathbf{x}$ sufficiently small, collecting like terms, we find

$$a = 1, \qquad b = 0, \qquad c = 1.$$

This means

$$h_1(x_1, x_2) = x_1^2 + x_2^2 + O(|\mathbf{x}|^3),$$

and by the local centre manifold theorem, the flow on the centre manifold is determined by

$$\begin{aligned}
\dot{x}_1 &= F_1(x_1, x_2, h_1(x_1, x_2)) = x_1(x_1^2 + x_2^2) - x_1 x_2^2 + O(|\mathbf{x}|^4) = x_1^3 + O(|\mathbf{x}|^4)\\
\dot{x}_2 &= F_2(x_1, x_2, h_1(x_1, x_2)) = x_2(x_1^2 + x_2^2) - x_1^2 x_2 + O(|\mathbf{x}|^4) = x_2^3 + O(|\mathbf{x}|^4).
\end{aligned}$$

This is a planar system with two zero eigenvalues and geometric multiplicity of two. Turning to polar coordinates,

$$r\dot{r} = x_1^4 + x_2^4 + O(r^5) > 0$$

for sufficiently small r. Therefore the origin is unstable.

This contrasts with an analysis we would have done if we only took the stable subspace approximation and set $y = 0$, resulting in

$$\dot{x}_1 = -x_1 x_2^2 \dot{x}_2 = -x_1^2 x_2,$$

from which we would have incorrectly concluded that the origin is stable, again by considering polar coordinates.