TMA 4190 Introduction to Topology
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Lecture 27[1]

27. The Hopf Degree Theorem

Today we are going to generalize an important result on the homotopy classes of maps to spheres. We proved previously that there is exactly one homotopy class of maps $S^1 \to S^1$ for every integer $n \in \mathbb{Z}$. By our classification of one-manifolds, we can read this also as follows:

For every compact, connected, boundaryless one-manifold X, there is exactly one homotopy class of maps $X \to S^1$ for every integer $n \in \mathbb{Z}$.

Today we are going to prove a generalization of this result to higher dimensions. It is a famous theorem of Hopf:

The Hopf Degree Theorem

Two maps $X \to S^k$ of a compact, connected, **oriented**, boundaryless k-manifold X to S^k are **homotopic** if and only if they have the **same degree**.

Recall that the degree of a map $f\colon X \to S^k$ as in the theorem is defined as

$$\deg(f) = \sum_{x \in f^{-1}(y)} \operatorname{sign}(df_x)$$

where y is a regular value of f and $\operatorname{sign}(df_x)$ is $+1$ if df_x preserves orientations and -1 if df_x reverses orientations. We will refer to this sign rule as **our usual orientation convention**.

[1]Following the books of Guillemin and Pollack: Differential Topology, and Milnor: Topology from the differentiable viewpoint.

Some Remarks on Hopf's Theorem

- We can think of the degree as a map

$$\mathrm{Hom}(X,S^k)_{/\sim} =: [X,S^k] \to \mathbb{Z}.$$

Hopf's Theorem tells us that this map is **injective**, where $\sim$ denotes the homotopy relation. One can show that it is also surjective, i.e., there is exactly one homotopy class of maps $X \to S^k$ for every integer $n \in Z$.

- For $X = S^k$, one usually rephrases this result by saying that the kth homotopy group of S^k is $\mathbb{Z}$, i.e.,

$$\pi_k(S^k) =: [S^k,S^k] := \mathrm{Hom}(X,S^k)_{/\sim} = \mathbb{Z}.$$

- Note that the situation is different for nonorientable manifolds: Two maps of a compact, connected, **nonorientable**, boundaryless k-manifold X to S^k are **homotopic** if and only if they have the **same degree modulo 2**.

Now we start our march towards a proof Hopf's theorem. We will follow the guideline of Guillemin-Pollack as usual. But it is worth noting that there are many different ways to prove this theorem. In particular, there is Pontryagin's proof as presented in Milnor's book which introduces an extremely important and interesting concept, called cobordism. We recommend to have a look at that proof as well.

Strategy for the proof of Hopf's Theorem

Assume given two maps f_0 and f_1 from X to S^k.

- Set $W := X \times [0,1]$, define $f\colon \partial W \to S^k$ by $f := f_0$ on $X \times \{0\}$ and $f := f_1$ on $X \times \{1\}$. Then $\deg(f) = \deg(f_1) - \deg(f_0) = 0$. Moreover, a homotopy between f_0 and f_1 is a **global extension of f to W**.
- Show the **Extension Theorem**: $f\colon \partial W \to S^k$ has a **global extension** $W \to S^k$ if and only if $\deg(f) = 0$, for any compact, connected, oriented $k+1$-manifold W. (We knew already: existence of global extensions $\Rightarrow \deg(f) = 0$.)
- To show the Extension Theorem, use the **Isotopy Lemma** to move W inside some ball $B \subset \mathbb{R}^{k+1}$ with $\mathrm{Int}(W) \subset B$. This reduces to checking an extension statement on balls and spheres.
- Use **winding numbers** to show that a map which is homotopic to a **constant map** on the boundary of a ball B extends to all of B.
- Show the **Special Case**: For $f\colon S^k \to S^k$,

$$\deg(f) = 0 \Rightarrow f \sim \text{constant map}.$$

This follows by **induction on the dimension** k of S^k. We have shown previously that $f,g\colon S^1 \to S^1$ are homotopic if and only if $\deg(f) = \deg(g)$. The induction step is actually a zigzag argument using **winding numbers**. The Isotopy Lemma is frequently used to move points into appropriate open neighborhoods and balls.

In order to make this strategy work, we need to prove a series of technical results. This will occupy the rest of the lecture. Two main technical ingredients are isotopies which allow to move points, and winding numbers which help us calculating degrees.

Isotopies and the Isotopy Lemma

We will need an important special type of homotopy which preserves more information than homotopies in general:

Isotopies

An **isotopy** is a homotopy h_t in which **each map h_t is a diffeomorphism**, and two diffeomorphisms are isotopic if they can be joined by an isotopy. An isotopy is **compactly supported** if the maps h_t are all equal to the identity map outside some fixed compact set.

A particular case of isotopies are linear isotopies.

Linear Isotopy Lemma

Suppose that E is a linear isomorphism of $\mathbb{R}^k$ that preserves orientations. Then there exists a homotopy E_t consisting of linear isomorphisms, such that $E_0 = E$ and E_1 is the identity. If E reverses orientation, then there exists such a homotopy with E_1 equal to the reflection map

$$r_1(x_1,\dots,x_k) = (-x_1,x_2,\dots,x_k).$$

Proof: First we remark that it suffices to deal with the case that E preserves orientations. For if E is orientation reversing, then $r_1 \circ E$ preserves orientations. Then if there is a homotopy F between $r_1 \circ E$ and Id, then, after composing all maps with r_1, $r_1 \circ F$ is a homotopy between $E = r_1 \circ r_1 \circ E$ and r_1.

So let E be a linear isomorphism of $\mathbb{R}^k$ that preserves orientations. The proof is by induction on the dimension k. We need to check two initial cases.

First, let $k = 1$. Then $E\colon \mathbb{R} to \mathbb{R}$ is given by multiplication by a real number $\lambda > 0$. Then $E_t = t \cdot 1 + (1-t)\cdot\lambda$ is a homotopy between $E = \lambda$ and $\mathrm{Id} = 1$. Note that each E_t is nonzero and therefore a linear isomorphsm.

Now let $k = 2$ and assume that E has only complex eigenvalues. Then $E_t = tE + (1-t)\mathrm{Id}$ is a linear homotopy between Id and E. Moreover, each E_t is a linear isomorphism. To show this we show that $\det(E_t) \neq 0$ for all $t \in [0,1]$. If $E = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then we get

$$\begin{aligned}\det(E_t) &= (t(a-1)+1)(t(d-1)+1) - t^2bc \\ &= t^2(a-1)(d-1) + t(a+d-2) + 1 - t^2bc \\ &= t^2(ad-bc-a-d+1) + t(a+d-2) + 1.\end{aligned}$$

The discriminant of this quadratic equation in t is

$$\begin{aligned}&(a+d-2)^2 - 4(ad-bc-a-d+1) \\ =&(a+d)^2 - 4(a+d) + 4 - 4(ad-bc) + 4(a+d) - 4 \\ =&(a+d)^2 - 4(ad-bc).\end{aligned}$$

But this is exactly the discriminant of the equation

$$t^2 + t(a+d) - (ad-bc) = 0$$

which is the characteristic polynomial (in t) of E. By assumption, this polynomial has only complex roots, i.e. its discriminant is negative. Hence there is no real t such that $\det(E_t) = 0$.

Now we show the induction step. So assume $k \geq 2$ and the assertion to be true in all dimensions $< k$. Then E has either at least one real eigenvalue or at least one complex eigenvalue. Let $V \subset \mathbb{R}^k$ be the corresponding eigenspace, which is either one- or two-dimensional. Then E maps V into itself. Hence $\mathbb{R}^k$ splits into a direct sum $\mathbb{R}^k = V \oplus W$. By choosing a basis of $\mathbb{R}^k$ consisting of a basis of V and one for W, we can represent E as a matrix of the form

$$E = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix}.$$

(Here A is either a 1×1- or a 2×2-matrix given by the eigenvalue.)

Then we can define a linear homotopy E_t by

$$E_t = \begin{pmatrix} A & tB \\ 0 & C \end{pmatrix}.$$

Since E is a linear isomorphism and the determinant is multiplicative, we have

$$0 \neq \det(E) = \det(A)\det(C) = \det(E_t).$$

Thus E_t is also a linear isomorphism for every t.

For $t = 0$, we see that E_0 maps V to V by A and W to W by C. Since $\dim W$ is strictly less than k, we can apply the induction hypothesis to C and W and the initial cases to A and V, respectively. Hence we have a homotopyC_t consisting of linear isomorphisms between C and the identity and a homotopy A_t between A and the identity. Then

$$\begin{pmatrix} A_t & tB \\ 0 & C_t \end{pmatrix}$$

is a homotopy consisting of linear homotopies between E and the identity of $\mathbb{R}^k$. **QED**

The following theorem will allow us to move points on connected manifolds via a family of diffeomorphisms. The fact that every map in the homotopy family is a diffeomorphism makes it much easier to keep track of the orientation numbers at preimages.

The Isotopy Lemma

Given any two points y and z in the **connected** manifold Y, there exists a diffeomorphism $h\colon Y \to Y$ such that $h(y) = z$ and h is isotopic to the identity. Moreover, the isotopy may be taken to be compactly supported.

Today we are lazy and skip the proof of this reuslt (it is in [GP] on pages 142, 143). Instead we look at a consequence which we will actually use later.

Corollary to the Isotopy Lemma

Suppose that Y is a **connected** manifold of dimension greater than 1, and let $\{y_1, \dots, y_n\}$ and $\{z_1, \dots, z_n\}$ be two sets of distinct points in Y. Then there exists a diffeomorphism $h\colon Y \to Y$ which is isotopic to the identity with

$$h(y_1) = z_1, \dots, h(y_n) = z_n.$$

Moreover, the isotopy may be taken to be compactly supported.

Proof of the Corollary: The proof works by induction. The Isotopy Lemma is the case $n = 1$. Now we assume the corollary being true for $n - 1$. Then we have a compactly supported isotopy $h'_t\colon Y \setminus \{y_n, z_n\} \to Y \setminus \{y_n, z_n\}$ such that $h'_1(y_i) = z_i$ for all $i < n$ and $h'_0 = \mathrm{Id}$.

Since $\dim Y > 1$, the punctured manifold $Y \setminus \{y_n, z_n\}$ is connected. Since the isotopy h'_t has compact support, there are open neighborhoods around y_n and z_n in Y on which the h'_t are all equal to the identity. Hence we can extend the family h'_t to a family of diffeomorphisms of Y that fix those two points.

Now we apply the induction hypothesis again to the punctured manifold

$$Y \setminus \{y_1, \dots, y_{n-1}, z_1, \dots, z_{n-1}\} \text{ and the points } y_n, z_n.$$

Then we get a compectly supported isotopy h''_t with $h''_1(y_n) = z_n$ and $h''_0 = \mathrm{Id}$. By the same argument as for h'_t, we can extend h''_t to an isotopy on all of Y such that all h''_t satisfy $h''_t(y_i) = z_i$ for all $i < n$. Then

$$h_t := h''_t \circ h'_t$$

is the desired isotopy. **QED**

Winding numbers revisited

As for many results on maps between spheres, the winding number is useful concept. We used it before with values modulo 2. Today, we need an integral version:

Integer winding numbers

Let X be a compact oriented k-dimensional smooth manifold, and let $f\colon X \to \mathbb{R}^{k+1}$ be a smooth map. The **winding number** of f, denoted $W(f,z)$, around any point $z \in \mathbb{R}^{k+1} \setminus f(X)$ is defined as the degree of the map

$$u\colon X \to S^k,\ x \mapsto \frac{f(x)-z}{|f(x)-z|}.$$

As a formula:

$$W(f,z) = \deg(u).$$

The winding number will be the main tool in the proof of Hopf's theorem. In order to exploit it effectively, we investigate some of its properties:

Step 1

Let $f\colon U \to \mathbb{R}^k$ be a smooth map defined on an open subset U of $\mathbb{R}^k$, and let x be a regular point, with $f(x) = z$. Let B be a sufficiently small closed ball centered at x, and define $\partial f\colon \partial B \to \mathbb{R}^k$ to be the restriction of f to the boundary of B. Then we have

$$W(\partial f,z) = \begin{cases} +1 \text{ if } f \text{ preserves orientation at } x, \\ -1 \text{ if } f \text{ reerses orientation at } x. \end{cases}$$

Proof: After possibly translating things, we can assume $x = 0 = z$, which keeps the notation simpler. We set $A = df_0$. We are going to show that $W(A,0)$ can be used to calculate $W(\partial f,0)$. This will follow if we show that we can choose B small enough such that there is a homotopy $F_t\colon \partial B \times [0,1] \to S^{k-1}$ between $Ax/|Ax|$ and $\partial f(x)/|\partial f(x)|$. For then

$$W(\partial f,0) = \deg\left(\frac{\partial f(x)}{|\partial f(x)|}\right) = \deg\left(\frac{Ax}{|Ax|}\right) = W(A,0).$$

Now we are going to construct the homotopy F_t. By Taylor theory, we can write

$$f(x) = Ax + \epsilon(x), \text{ where } \epsilon(x)/|x| \to 0 \text{ when } x \to 0. \tag{1}$$

We define

$$f_t(x) = Ax + t\epsilon(x) \text{ for } t \in [0{,}1].$$

Then, f_t is a homotopy from $f_0(x) = Ax$ to $f_1(x) = f(x)$.

Since $x = 0$ is a regularit point, we know that A is an isomorphism. Hence the image of the unit ball in $\mathbb{R}^k$ under A strictly contains a closed ball of some radius $r > 0$. Since every linear isomorphism is a diffeomorphism, we also know that A maps boundaries to boundaries, i.e., S^{k-1} to the boundary of the closed ball of radius r. Hence

$$|Ax| > r \text{ for all } x \in S^{k-1}.$$

As a consequence,

$$|A\frac{x}{|x|}| > c \text{ and thus } |Ax| > |rx| \text{ for all } x \in \mathbb{R}^k \setminus \{0\}.$$

Now we use (1). Since $\epsilon(x)/|x| \to 0$ as $x \to 0$, we can choose a ball B small enough such that

$$\epsilon(x)/|x| < \frac{r}{2} \text{ for all } x \in \partial B.$$

Then we have

$$|f_t(x)| = |Ax| - t|\epsilon(x)| > r|x| - \frac{r}{2}|x| = \frac{r}{2}|x|,$$
$$i.e., |f_t(x)| > 0 \text{ for all } x \in \partial B.$$

Hence we can define the desired homotopy F_t by

$$F_t \colon \partial B \times [0{,}1] \to S^k,\ x \mapsto \frac{f_t(x)}{|f_t(x)|}.$$

Now we compute $W(A{,}0)$. Therefor we apply the Linear Isotopy Lemma and get that A is homotopic to the identity if it preserves orientions, and homotopic to the reflection map $(x_1, \ldots, x_k) \mapsto (-x_1, x_2, \ldots, x_k)$ if it reverses orientations. In the former case, we have $W(A{,}0) = +1$, and in the latter case $W(A{,}0) = -1$. **QED**

This result determines how local diffeomorphisms can wind. Now we are going to use this information to count preimages.

Step 2

Let $f\colon B \to \mathbb{R}^k$ be a smooth map defined on some closed ball B in $\mathbb{R}^k$. Suppose that z is a regular value of f that has no preimages on the boundary sphere ∂B, and let $\partial f\colon \partial B \to \mathbb{R}^k$ be its restriction to the boundary. Then the number of preimages of z, counted with our usual orientation convention, equals the winding number $W(\partial f, z)$.

Proof: By the Stack of Records Theorem, we know that $f^1(z)$ is a finite set $\{x_1, \ldots, x_n\}$, and we can choose disjoint balls B_i around each x_i. Since $f^1(z)$ is disjoint from ∂B by assumption, we can shrink these balls such that $B_i \cap \partial B = \emptyset$ and so that each B_i is sufficiently small so that Step 1 can be applied.

Let $\partial f_i = f_{|\partial B_i}$. Then **Step 1** implies that the **number of preimage points**, counted with our usual orientation convention, equals $\sum_{i=1}^n W(\partial f_i, z)$.

Let $B' := B \setminus \cup_i B_i$ and consider the map

$$u\colon \partial B \to S^{k-1},\ x \mapsto \frac{f(x) - z}{|f(x) - z|}.$$

Since $f(x) \neq z$ on B', this map extends to all of B'. This implies

$$W(f_{|\partial B'}, z) = \deg(u) = 0.$$

The orientations of the boundaries are related by

$$\partial B' = \partial B \cup_{i=1}^n (-\partial B_i).$$

This implies

$$W(f_{|\partial B'}, z) = W(\partial f, z) - \sum_{i=1}^n W(\partial f_i, z).$$

Hence in total we get $W(\partial f, z) = \sum_{i=1}^n W(\partial f_i, z)$. **QED**

Step 3

Let B be a closed ball in $\mathbb{R}^k$, and let $f\colon \mathbb{R}^k \setminus \mathrm{Int}(B) \to Y$ be a smooth map defined outside the open ball $\mathrm{Int}(B)$. Let $\partial f\colon \partial B \to Y$ be the restriction to the boundary. Assume that ∂f is homotopic to a constant map. Then f extends to a smooth map defined on all of $\mathbb{R}^k$ into Y.

Proof: For simplicity, we assume that B is centered at 0. Then we can write every non-zero point $x \in B$ uniquely as $x = ty$ for some $y \in \partial B$ and some

$t \in [0,1]$. By assumption, there is a homotopy $g_t\colon \partial B \to Y$ with $g_1 = \partial f$ and g_0 being a constant map.

Now we define the map $F\colon \mathbb{R}^k \to Y$ by setting

$$F(x) = \begin{cases} f(x) & \text{if } x \in \mathbb{R}^k \setminus \mathrm{Int}(B) \\ g_t(x) & \text{if } x \in B \text{ and } x = ty \text{ for some } y \in \partial B \text{ and } t \in [0,1]. \end{cases}$$

Note that F is well-defined on $R^k \setminus \mathrm{Int}(B)$, since f and g_t agree on $\partial B = B \cap (\mathbb{R}^k \setminus \mathrm{Int}(B))$ where we have $f = \partial f = g_1$. Note also that $F(0)$ is well-defined as the constant value of g_0.

Now it remains to use smooth bump function to turn F into a smooth homotopy (it is already smooth except, possibly, on ∂B). **QED**

The Special Case

Special case

Any smooth map $f\colon S^k \to S^k$ having **degree zero** is homotopic to a **constant map**.

The special case implies:

Corollary

Any smooth map $f\colon S^k \to \mathbb{R}^{k+1} \setminus \{0\}$ having winding number zero with respect to the origin is homotopic to a constant map.

Proof of the Corollary: By assumption, the degree of the map $\frac{f}{|f|}$ is zero. By the special case, this implies that $\frac{f}{|f|}$ is homotopic to a constant map. But $\frac{f}{|f|}$ and f are homotopic via the homotopy

$$F\colon S^k \times [0,1] \to \mathbb{R}^{k+1} \setminus \{0\},\ (x,t) \mapsto tf(x) + (1-t)\frac{f}{|f|}$$

Since homotopy is a transitive relation, f is also homotopic to a constant map. **QED**

Proof of the special case:

The proof is by induction on the dimension k. We have established the case $k = 1$ in a previous lecture. So we assume the special case being true for $k - 1$ and want to deduce it for k.

We need to prove a lemma first:

A lemma

Let $f\colon \mathbb{R}^k \to \mathbb{R}^k$ be a smooth map with 0 as a regular value. Suppose that $f^{-1}(0)$ is finite and that the number of preimage points in $f^{-1}(0)$ is zero when counted with the usual orientation convention. Assuming the special case in dimension $k - 1$. Then there exists a map $g\colon \mathbb{R}^k \to \mathbb{R}^k \setminus \{0\}$ such that $g = f$ outside a compact set.
In particular, the homotopy $tf + (1t)g$ from g to f is constant outside this compact set.

Proof: Since $f^{-1}(0)$ is a finite, we can choose a ball B centered at the origin with $f^{-1}(0) \subset \mathrm{Int}(B)$. By assumption, the number of preimages is zero when counted with the usual orientation convention. By Step 2, the map $\partial f\colon \partial B \to \mathbb{R}^k \setminus \{0\}$ has winding number zero. Since ∂B is diffeomorphic to S^{k-1}, so ∂f is a map from ${}^{k-1}$ to $\mathbb{R}^k \setminus \{0\}$.

Since we are assuming the special case being true in dimension $k - 1$, we can apply its corollary in that dimension. Thus, ∂f is homotopic to a constant map. Hence

$$f_{|\mathbb{R}^k \setminus \mathrm{Int}(B)}\colon \mathbb{R}^k \setminus \mathrm{Int}(B) \to \mathbb{R}^k \setminus \{0\}$$

is a map to which we can apply Step 3. This implies that f extends to a smooth map $g\colon \mathbb{R}^k \to \mathbb{R}^k \setminus \{0\}$ with $f = g$ outside the compact space B. **QED**

Now we get back to the proof of the special case, and we are given a smooth map $f\colon S^k \to S^k$ with $\deg(f) = 0$.

The **idea of the proof** is to show that f is homotopic to a map $h\colon S^k \to S^k \setminus \{b\}$, where b is some point in S^k. But $S^k \setminus \{b\}$ is diffeomorphic to $\mathbb{R}^k$ via stereographic projection (from b). Since $\mathbb{R}^k$ is **contractible**, this implies h is homotopic to a constant map. Then f is also homotopic to a constant map.

So we need to show:

Claim: f is homotopic to a smooth map $g\colon S^k \to S^k \setminus \{b\}$.

By Sards Theorem, we can choose distinct regular values a and b of f. By the Stack of Records Theorem, the preimage sets are finite, say $f^{-1}(a) = \{a_1, \ldots, a_n\}$ and $f^{-1}(b) = \{b_1, \ldots, b_m\}$.

Moreover, we can find an open neighborhood U of a_1 such that U is diffeomorphic to $\mathbb{R}^k$ via a diffeomorphism $\alpha\colon \mathbb{R}^k \to U$ and such that $b_i \notin U$ for all $i = 1, \ldots, m$.

Since $k > 1$, we can apply the corollary of the Isotopy Lemma to the points $\{a_2, \ldots, a_n\}$ in $Y := S^k \setminus \{b\}$ to get a diffeomorphism which is isotopic to the identity, compactly supported, and moves the points a_i into U.

Since homotopy is a transitive relation, we can therefore assume that U is an open neighborhood of $f^{-1}(a)$ with $b \notin f(U)$.

Now let $\beta\colon S^k \setminus \{b\} \to \mathbb{R}^k$ be a diffeomorphism with $\beta(a) = 0$. Then

$$\beta \circ f \circ \alpha\colon \mathbb{R}^k \xrightarrow{\alpha} U \xrightarrow{f} S^k \setminus \{b\} \xrightarrow{\beta} \mathbb{R}^k$$

is a smooth map from $\mathbb{R}^k$ to $\mathbb{R}^k$. Since a is a regular value of f, 0 is a regular value of $\beta \circ f \circ \alpha$. Moreover, since $f^{-1}(a)$ is finite, $(\beta \circ f \circ \alpha)^{-1}(0)$ is finite as well.

Now we use the **assumption** $\deg(f) = 0$. For this means that the **number of preimages of a under f is zero** when counted with our usual orientation convention. Hence the number of **preimages of 0 under $\beta \circ f \circ \alpha$ is zero** when counted with the usual orientation convention.

Thus, we can **apply the lemma** to $\beta \circ f \circ \alpha\colon \mathbb{R}^k \to \mathbb{R}^k$ and get a map $g\colon \mathbb{R}^k \to \mathbb{R}^k \setminus \{0\}$ such that $g = \beta \circ f \circ \alpha$ outside a compact set B and g is homotopic to $\beta \circ f \circ \alpha$ on $\mathbb{R}^k$.

Since α and β are **diffeomorphisms**, this implies that f is homotopic to $\beta^{-1} \circ g \circ \alpha^{-1}$ as a map from U to $S^k \setminus \{b\}$.

Since $g = \beta \circ f \circ \alpha$ outside B, we have

$$\beta^{-1} \circ g \circ \alpha^{-1} = f \text{ on } U \setminus \alpha^{-1}(B).$$

Thus, the map

$$h\colon S^k \to S^k \setminus \{b\}$$

defined by setting

$$h = \begin{cases} f & \text{on } S^k \setminus \alpha^{-1}(B) \\ \beta^{-1} \circ g \circ \alpha^{-1} & \text{on } \alpha^{-1}(B) \end{cases}$$

is smooth, and h is the desired map homotopic to f. This proves the special case. **QED**

Towards proof of Hopf's theorem

Now we are almost ready to prove Hopf's result.

Extending maps to Euclidean spaces

Let W be a compact smooth manifold with boundary, and let $f\colon \partial W \to \mathbb{R}^k$ be a smooth map. Then f can be extended to a globally defined map $F\colon W \to \mathbb{R}^k$.

Proof: As always we assume that W is a subset of some $\mathbb{R}^N$. Since W is compact, it is a closed subset of $\mathbb{R}^N$, and so is ∂W. Since f is a smooth map defined on a closed subset of $\mathbb{R}^N$, it may be locally extended to a smooth map on open sets. Since ∂W is compact and boundaryless, we can apply the **ϵ-Neighborhood Theorem** to extend f to a map F defined on a neighborhood U of ∂W in $\mathbb{R}^N$.

Now we choose a smooth bump function ρ that is constant 1 on ∂W and 0 outside some compact subset of U.

Then we can extend f to all of W by letting it be

$$\rho \cdot F \text{ on } U, \text{ and } 0 \text{ outside of } U.$$

This is a smooth function defined on all of $\mathbb{R}^N$ with values in $\mathbb{R}^k$ and being $f = 1 \cdot F$ on ∂W. **QED**

Now we apply this lemma to maps with values in spheres:

Extension Theorem

Let W be a compact, connected, oriented $k{+}1$-dimensional smooth manifold with boundary, and let $f\colon \partial W \to S^k$ be a smooth map. Then f **extends** to a **globally** defined map $F\colon W \to S^k$ with $\partial F = f$ if and only if $\deg(f) = 0$.

Proof: We already know that if f can be extended to all of W, then $\deg(f) = 0$. It remains to show the opposite direction.

So let f be as in the thereom, and assume $\deg(f) = 0$. By the previous lemma, we can extend f to a smooth map $F\colon W \to \mathbb{R}^{k+1}$. By the **Transversality Extension Theorem**, we can assume that 0 is a regular value of F. Since W

is compact of dimension $k+1$, we know that $F^{-1}(0)$ is a finite set. Hence we can apply the corollary to the Isotopy Lemma to this finite set, and move $F^{-1}(0)$ inside $\mathrm{Int}(B)$ where B is a closed ball contained $\mathrm{Int}(W)$.

In particular, since $F^{-1}(0) \subset \mathrm{Int}(B)$, the map $\frac{F}{|F|}$ extends to $W' := W \setminus \mathrm{Int}(B)$. Hence

$$W\left(\frac{F}{|F|},0\right) = \deg\left(\frac{F}{|F|}\right) = 0.$$

On the other hand, we know by our assumtpion that

$$W(F_{|\partial W},0) = W(f,0) = \deg(f) = 0,$$

where we use $f = f/|f|$, since f has values in S^k.

Now let

$$\partial F = F_{\partial B} \colon \partial B \to \mathbb{R}^{k+1} \setminus \{0\}$$

be the restriction to the boundary. By the definition of W' and boundary orientations, we have

$$\partial W' = (\partial W) \cup (-\partial B).$$

Hence we get

$$W(F_{|\partial W'},0) = W(F_{|\partial W},0) - W(F_{|\partial B},0)$$

and therefore $W(F_{|\partial B},0)$ by our previous observations.

Now the corollary to the special case implies that ∂F is homotopic to a constant map. By **Step 3**, this implies that ∂F extends to a map $G \colon W \to \mathbb{R}^{k+1} \setminus \{0\}$. Then the map $\frac{G}{|G|} \colon W \to S^k$ is the global extension of f. **QED**

And, finally, the last step:

Proof the Hopf Degree Theorem: Let f_0 and f_1 be two maps $X \to S^k$ and let $W := X \times [0,1]$. We define a map $f \colon \partial W \to S^k$ by setting

$$f = \begin{cases} f_0 & \text{on } X \times \{0\} \\ f_1 & \text{on } X \times \{1\}. \end{cases}$$

By the **Extension Theorem**, f extends to a map on all of W if and only if $\deg(f) = 0$. By definition, such an extension would be a homotopy between f_0 and f_1. Thus we have

$$f_0 \sim f_1 \iff \deg(f) = 0.$$

It remains to relate $\deg(f)$ to $\deg(f_0)$ and $\deg(f_1)$. But, since $\partial W = (X \times \{1\}) \cup (X \times \{0\})$ with the opposite orientation on $X \times \{0\}$, it follows that

$$\deg(f) = \deg(f_1) - \deg(f_0).$$

Thus

$$f_0 \sim f_1 \iff \deg(f_1) = \deg(f_0).$$

QED