TMA 4190 Introduction to Topology
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Lecture 24[1]

24. Intersection Numbers and Euler Characteristics

Let us return to one of the initial motivations for the intersetion numbers and see what happens if both X and Z are submanifolds.

Intersection of submanifolds

Let X and Z be submanifolds of Y, with X compact and complementary dimensions $\dim X + \dim Z = \dim Y$, and all are oriented. Then we define the **intersection number of X and Z** in Y to be

$$I(X,Z) := I(i,Z)$$

where $i\colon X \hookrightarrow Y$ is the inclusion map.

Recall that calculating $I(X,Z)$ requires to bring X in transversal position to Z and then take the sum of the orientation numbers at the **finitely many intersection points** in $X \cap Z$.

A point $y \in X \cap Z$ has sign $+1$ if the orientation of $T_y(Y)$ induced by the direct sum decomposition

$$T_y(X) \oplus T_y(Z) = T_y(Y)$$

is the given orientation on $T_y(Y)$, and the sign is -1 if it is the opposite orientation.

Since the **order** of the summands in a direct sum **matters** for the orientation, it is clear that when both X and Z are compact we cannot expect $I(X,Z)$ to be equal $I(Z,X)$ in general.

All we should expect is $I(X,Z) = \pm I(Z,X)$. An example is given by intersecting the two circles on the torus. There we get $I(X,Z) = -I(Z,X)$.

Our next goal is to show that $I(X,Z)$ is homotopy invariant in both variables, and to determine the sign when we flip the factors.

Homotopy Invariance of intersection numbers revisited

[1]Following the books of Guillemin and Pollack: Differential Topology, and Milnor: Topology from the differentiable viewpoint.

Recall that a **deformation** of X in Y is a smooth homotopy from the embedding $i_0\colon X \hookrightarrow Y$ of X in Y to an embedding $i_1\colon X \hookrightarrow Y$ such that each i_t is an embedding.

We know that $I(X,Z)$ is **invariant under deformations of X**, since we calculate it point by point in $X \cap Z$ and a deformation of X is a homotopy of the inclusion. We need to prove that $I(X,Z)$ is **invariant under deformations of Z as well**. In order to show this we generalize our approach.

Let $f\colon X \to Y$ and $g\colon Z \to Y$ be two smooth maps with X and Z compact, all manifolds ae boundaryless and the dimensions satisfy $\dim X + \dim Z = \dim Y$. In particular, that the images of f and g are closed in Y and for g being the inclusion of Z into Y, we are back at the familiar situation.

As always we start with the case of transversal maps and then extend our defintion via homotopy.

In order to do so, we need to say what it means for **two maps** to be transversal:

Transversal maps

We say that $f\colon X \to Y$ and $g\colon Z \to Y$ are **transversal**, denoted $f \pitchfork g$, if

$$df_x(T_x(X)) + dg_z(T_z(Z)) = T_y(Y) \text{ whenever } f(x) = y = g(z).$$

In our situation, the assumption on dimensions implies that if $f \pitchfork g$ then the above **sum is direct**, i.e.

$$df_x(T_x(X)) \oplus dg_z(T_z(Z)) = T_y(Y) \text{ whenever } f(x) = y = g(z).$$

Moreover, the derivatives df_x and dg_z are both injective. Thus these derivatives map $T_x(X)$ and $T_z(Z)$ **isomorphically onto their images**. In particular, the **image spaces inherit an orientation** from X and Z, respectively.

Intersection numbers for maps

We define the **local intersection number** at (x,z) to be $+1$ if the direct sum orientation of $df_x(T_x(X)) \oplus dg_z(T_z(Z))$ equals the given orientation of $T_y(Y)$, and -1 otherwise.
Then $I(f,g)$ is defined as the **sum of the local intersection numbers** of all pairs (x,z) at which $f(x) = g(z)$.

When $g\colon Z \hookrightarrow Y$ is the inclusion map of a submanifold then $f \pitchfork g$ if and only if $f \pitchfork Z$, and if so $I(f,g) = I(f,Z)$. So everything remained consistent so far.

In the definition, we quietly assumed that the sum is **finite**. We should better check this! To do so, we are going to look at intersections from yet another angle. It will seem artificial at first glance, but it is actually a very useful perspective. For it can be generalized to many other situations, e.g. in Algebraic Geometry.

Let Δ denote the **diagonal** of $Y \times Y$, i.e. the set of points (y,y), and let

$$f \times g\colon X \times Z \to Y \times Y,\ (x,z) \mapsto (f(x),g(z))$$

be the product map. Then we have

$$f(x) = g(z) \iff (x,z) \in (f \times g)^{-1}(\Delta).$$

The dimension of $\dim(X \times Z)$ is $\dim X + \dim Z = \dim Y$, and the dimension of Δ is $\dim Y$. Thus $\dim(X \times Z) = \operatorname{codim}(\Delta)$ in $Y \times Y$. Hence if $f \times g \pitchfork \Delta$, then $(f \times g)^{-1}(\Delta)$ is a **compact zero-dimensional manifold**. Hence it is a **finite set**.

Transversality, $f \times g \pitchfork \Delta$, will follow from the following lemma from linear algebra:

Help from Linear Algebra

Let V be a finite dimensional vector space, and U and W be vector subspaces of V. Let Δ be the diagonal in $V \times V$. Then

$$U \oplus W = V \iff U \times W \oplus \Delta = V \times V.$$

Assume now that $U \oplus W = V$, and in addition that U and W are oriented, and give V the direct sum orientation. We assign Δ the orientation carried from V by the natural isomorphism $V \to \Delta$ which sends $v \mapsto (v,v)$. Then the **product orientation** on $V \times V$ **agrees** with the **direct sum orientation** induced from $U \times W \oplus \Delta$ if and only if **W is even dimensional**.

We skip the proof of the lemma which can be found in [GP], page 113+114. Instead we are ging to exploit its implications.

Transversality and diagonals

The maps f and g are transversal if and only if $f \times g$ is transversal to Δ, i.e.

$$f \pitchfork g \iff (f \times g) \pitchfork \Delta.$$

If $f \pitchfork g$, then

$$I(f,g) = (-1)^{\dim Z} I(f \times g, \Delta).$$

Proof: We apply the lemma to $U = df_x(T_x(X))$, $W = dg_z(T_z(Z))$, and $V = T_y(Y)$. Then the first part of the lemma yields the equivalence of transversality. The second part implies the formula on the signs, keeping in mind that we know $X \cap Z = (f \times g)^{-1}(\Delta)$. **QED**

The main point of the previous effort is that considering intersections as preimages of the diagonal allows us to extend our definition:

Intersection numbers via diagonals

For maps f and g as above which are not necessarily transversal, we **define** $I(f,g)$ to be

$$I(f,g) := (-1)^{\dim Z} I(f \times g, \Delta).$$

Moreover, the desired properties of $I(f,g)$ follow right away:

Homotopy Invariance

If f_0 and g_0 are **homotopic** to f_1 and g_1, respectively, i.e. $f_0 \sim f_1$ and $g_0 \sim g_1$, then

$$I(f_0,g_0) = I(f_1,g_1).$$

Proof: If F is a homotopy from f_0 to f_1 and G is a homotopy from g_0 to g_1, then $F \times G$ is a homotopy from $f_0 \times g_0$ to $f_1 \times g_1$. Then the homotopy invariance of $I(f \times g, \Delta)$ which we proved before implies the invariance of $I(f,g)$. **QED**

Recovering the previous definition

If Z is a **submanifold** of Y and $i\colon Z \to Y$ is its **inclusion map**, then $I(f,i) = I(f,Z)$ for any map $f\colon X \to Y$ (with the usual assumption that X is compact and complementary dimensions).

Proof: This follows just from the definition of $f \pitchfork Z$. If f is arbitrary, then we use the homotopy invariance of both $I(f,i)$ and $I(f,Z)$. **QED**

When we applied $I(f,Z)$ to the case $\dim X = \dim Y$ and $Z = \{y\}$, we obtained the degree of f. Let us check that this definition still works in the new setup.

Degrees are still well defined

If $\dim X = \dim Y$ and Y is connected, then $I(f,\{y\})$ is the same for every $y \in Y$. Thus $\deg(f) = I(f,\{y\})$ is well defined.

Proof: Since Y is connected and a smooth manifold, it is path-connected. Hence the inclusion maps i_0 and i_1 for any two points $y_0,y_1 \in Y$ are homotopic. Therefore

$$I(f,\{y_0\}) = I(f,i_0) = I(f,i_1) = I(f,\{y_1\}).$$

QED

How signs switch when we flip maps

When we flip the order of the maps, we get

$$I(f,g) = (-1)^{(\dim X)(\dim Z)} I(g,f).$$

Proof: We must compare the direct sum orientations of

$$T_y(Y) = df_x(T_x(X)) \oplus dg_z(T_z(Z)) \text{ and } T_y(Y) = dg_z(T_z(Z)) \oplus df_x(T_x(X)).$$

As we remarked in a previous lecture, switching the order of the summands reuqires to apply $\dim X) \cdot (\dim Z)$ many transpositions of the basis vectors. This gives the sign in the assertion. **QED**

Applying this result to the inclusions of two submanifolds yields the follwoing formula for signs when we switch the order of factors in intersection numbers:

How signs switch when we flip submanifolds

If X and Z are both compact submanifolds, then

$$I(X,Z) = (-1)^{(\dim X)(\dim Z)} I(Z,X).$$

Self-intersections and Euler Characteristic

As a special case, we can look at at the **self-intersection number** $I(X,X)$ when $\dim Y = 2\dim X$.

But the above sign formula implies that **if** $\dim X$ **is odd**, then

$$I(X,X) = (-1)^{(\dim X)^2} I(X,X) = -I(X,X) \text{ and hence } I(X,X) = 0.$$

As a consequence we also get $I_2(X,X) = I(X,X) \mod 2 = 0$.

This observation yields an insight into the nonorientability of some manifolds.

Obstruction for orientability

Let Y be any smooth manifold of **even dimension**. Then we can calculate the mod 2-self-intersection number $I_2(X,X)$ for any compact submanifold $X \subset Y$ of dimension $\dim X = \frac{1}{2}\dim Y$ as in the previous lecture without assuming orientability of Y.
If one of these self-intersection numbers **fails to vanish**, then Y is **not orientable**.

For example, the central circle in the Mobius strip has nonzero mod 2 self-intersection number, so the Möbius strip is nonorientable.

Self-intersection numbers can be used to define a very powerful and famous invariant. You will see different constructions for this invariant later in your mathematical life. Here is the first:

Euler Characterstics

Let Y be a compact, oriented manifold. Its **Euler characteristic**, denoted $\chi(Y)$, is defined to be the self-intersection number of the diagonal Δ in $Y \times Y$:

$$\chi(Y) := I(\Delta,\Delta).$$

Note: Our methods and contruction here makes it look like a differential invaiant. But note that the Euler characteristic is a **topological invariant** in the sense that it only depends on the topology of Y and not the differentiable structure.

As a first calculation of an Euler number, we deduce from the previous observations:

Euler characteristic in odd dimensions vanishes

The Euler characteristic of an odd-dimensional, compact, oriented manifold is zero.

Proof: If $\dim Y$ is odd, then $\dim \Delta = \dim Y$ is odd. Hence

$$\chi(Y) = I(\Delta,\Delta) = (-1)I(\Delta,\Delta) = 0$$

must be zero. **QED**

Lefschetz Fixed-Point Theorem

For a (smooth) map $f \to X \to X$ it is often desirable to know if the equation $f(x) = x$ has a solution, i.e., if f has a fixed point. In particular, we could ask how many fixed point does f have. On a compact oriented manifold X, intersection theory can help us answering that question.

Again it turns out to formulate the question first using diagonals. A point $x \in X$ is a fixed point of f if and only if $(x,f(x))$ is a point in the intersection of the graph $\Gamma(f)$ of f with the diagonal Δ of X in $X \times X$:

$$f(x) = x \iff (x,f(x)) \in \Delta \cap \Gamma(f).$$

Both Δ and $\Gamma(f)$ are submanifolds of X and their dimensions satisfy

$$\dim \Delta + \dim \Gamma(f) = \dim X + \dim X = \dim(X \times X).$$

Moreover, both receive an orientation from X via the natural diffeomorphism $X \to \Delta$ and $X \to \Gamma(f)$.

Thus we may use intersection theory to count their common points (if it is a finite number):

Global Lefschetz numbers

The **global Lefschetz number of f**, denoted by $L(f)$, is defined to be the intersection number

$$L(f) := I(\Delta, \Gamma(f))$$

Note: Again, our methods and contruction here makes it look like a differential invaiant. But the **Lefschetz number** is a **topological invariant** in the sense that it only depends on the topology of X and not the differentiable structure

Of course, f may have an infinite number of fixed points, as the identity map demonstrates. Thus the sense in which $L(f)$ measures the fixed-point set is somewhat subtle. However, we shall see that when the fixed points of f do happen to be finite, then $L(f)$ may be calculated directly in terms of the local behavior of f around its fixed points.

The significance of Lefschetz numbers may be illustrated by the following immediate consequences of the intersection theory approach. The following famous theorem in its many variations plays a crucial role in many branches in mathematics:

Smooth Lefschetz Fixed-Point Theorem

Let $f\colon X \to X$ be a smooth map on a compact orientable manifold. If $L(f) \neq 0$, then f has a **fixed point**.

Proof: If f has no fixed points, then Δ and $\Gamma(f)$ are disjoint, and hence trivially transversal. Consequently,

$$L(f) = I(\Delta, \Gamma(f)) = 0.$$

QED

Since $L(f)$ is an intersection number, we immediately get:

Lefschetz numbers are homotopy invariant

If $f_0 \sim f_1$, then $L(f_0) = L(f_1)$.

The graph of the identity map is just the diagonal itself. thus $L(\mathrm{Id}) = \chi(X)$ is just the Euler characteristic of X:

Lefschetz numbers and Euler characteristics

If f is homotopic to the identity, then $L(f)$ equals the Euler characteristic of X. In particular, if X admits any smooth map $f\colon X \to X$ that is **homotopic to the identity** and has **no fixed points**, then $\chi(X) = 0$.

Trnasversality is crucial for intersection theory. So let us call a smooth map $f\colon X \to X$ a **Lefschetz map** if $\Gamma(f) \pitchfork \Delta$.

Note that a Lefschetz map has only finitely many fixed points, since there are only finitely many points in the coplementary intersection $\Gamma(f) \cap \Delta$. Also not that the converse is false. Since Lefschetz maps are defined by a transversality condition, it should be plausible that most maps are Lefschetz.

Most maps are Lefschetz

Every smooth map $f\colon X \to X$ is homotopic to a Lefschetz map.

Proof: In the lecture on transversality we proved the following fact: Given $X \subset \mathbb{R}^N$ and $f\colon X \to X$, we can find an open ball S in $\mathbb{R}^N$ and a smooth map $F\colon X \times S \to X$ such that $F(x,0) = f(x)$ and $s \mapsto F(x,s)$ is a submersion for each $x \in X$.

Given this F, the map

$$G\colon X \times S \to X \times X,\ (x,s) \mapsto (x,F(x,s))$$

is also a submersion. For suppose that $G(x,s) = (x,y)$. Since G acts like the identity on the first X factor, the image of $dG_{(x,s)}$ contains a vector of the form (u,w) for every $u \in T_x(X)$. Since G restricted to $\{x\} \times S$ is a submersion to $\{x\} \times X$, the image also contains a vector of the form $(0,w)$ for every $w \in T_y(X)$. Therefore G is a submersion.

In particular, $G \pitchfork \Delta$. By the Transversality Theorem, for almost every s the map

$$X \to X \times X,\ \ x \mapsto G(x,s)$$

is transversal to Δ.

Now we observe that the image of this map is just the graph of the map $x \mapsto F(x,s)$. Hence, for any s, the map

$$X \to X, x \mapsto F(x,s)$$

is Lefschetz and homotopic to f. **QED**

Let us try to understand Lefschetz maps better. Suppose that x is a **fixed point** of f. As we showed in the exercises, the **tangent space of** $\Gamma(f)$ in $T_x(X \times X)$ is the graph of the derivative $df_x \colon T_x(X) \to T_x(X)$. Moreover, the **tangent space of the diagonal** Δ is the diagonal Δ_x in $T_x(X) \times T_x(X)$.

This implies

$$\Gamma(f) \pitchfork \Delta \text{ in } (x,x) \iff \Gamma(f) + \Delta_x = T_x(X) \times T_x(X).$$

As $\Gamma(df_x)$ and Δ_x are vector subspaces of $T_x(X)\times T_x(X)$ with **complementary dimension**, we have

$$\Gamma(f) + \Delta_x = T_x(X) \times T_x(X) \iff \Gamma(f) \cap \Delta_x = \{0\}.$$

But $\Gamma(f) \cap \Delta_x = \{0\}$ just means that df_x does **not have a fixed point**. In the language of linear algebra, this means that df_x **has no eigenvector of eigenvalue** $+1$.

Lefschetz fixed points

We call a fixed point x a **Lefschetz fixed point of f** if df_x has no nonzero fixed point, i.e., if the eigenvalues of df_x are all unequal to $+1$.
This shows that f is a **Lefschetz map** if and only if **all its fixed points are Lefschetz**.

Notice that the Lefschetz condition on x is simply the infinitesimal analog of the demand that x be an **isolated fixed point** of f. We have met Lefschetz fixed points on Exercise Set 6.

Local Lefschetz fixed points

If x is a Lefschetz fixed point, we denote the orientation number ± 1 of (x,x) in the intersection $\Delta\Gamma(f)$ by $L_x(f)$. It is called the **local Lelschetz number of f at x**.
For **Lefschetz maps**, we have

$$L(f) = \sum_{f(x)=x} L_x(f)$$

where the sum is taken over the finite number of fixed points of f.

Hence in order to calculate the global Lefschetz number $L(f)$, it suffices to calculate all the local Lefschetz numbers $L_x(f)$.

So let us have a closer look at the $L_x(f)$. First we observe that the condition for x to be a Lefschtz fixed point means that, for the identity map I on $T_x(X)$, $df_x - I$ is still an isomorphism on $T_x(X)$, since the kernel of $df_x - I$ is the space of fixed points of df_x. (We used that also to solve the exercise on Lefschetz fixed points and Lefschetz maps.) Now we observe:

Local Lefschetz numbers and orientations

Let x be a Lefschetz fixed point of f. Then $L_x(f)$ is $+1$ if the isomorphism $df_x - I$ preserves orientations on $T_x(X)$, and it is -1 if $df_x - I$ reverses orientations.
In other words,

$$L_x(f) = \mathrm{sign}(\det(df_x - I)).$$

Again, we skip the proof of this exercise in linear algebra ([GP] pages 121+122) and rther look at an important example.

The Euler characteristic of the two-sphere

As an example, we consider $X = S^2 \subset \mathbb{R}^3$. Let $g\colon \mathbb{R}^3 \to \mathbb{R}^3$ be the rotation by $\pi/2$ about the z-axis. The matrix representing g in the standard basis is

$$\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

In particular, g is a linear map and its own derivative dg_x at any point is just g.

Now let $f\colon S^2 \to S^2$ be the restriction of g to S^2. Then f has **exactly two fixed points**, the north pole $N = (0,0,+1)$ and the south pole $S = (0,0,-1)$. At both poles, $df_x\colon T_x(S^2) \to T_x(S^2)$ can be represented by the matrix $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Hence $\det(df_x - I) = 2$. In particular, f is a **Lefschetz map**, and the sign of the determininant is $+1$ at both poles. Thus $L(f) = L_N(f) + L_S(f) = 2$.

Any rotation with positive determinant is **homotopic to the identity** map of S^2. For a concrete homotopy from g to the identity map we can take

$$F(-,t) = \begin{pmatrix} t & t-1 & 0 \\ 1-t & t & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

By our previous discussion, this implies $L(f) = \chi(S^2)$. Hence we have proved the folowing important fact:

Euler characterstic of the two-sphere

The Euler characteristic of S^2 is 2: $\chi(S^2) = 2$.

As a consequence we get:

Self-maps on the two-sphere

Every map $S^2 \to S^2$ that is homotopic to the identity must possess a fixed point. In particular, the **antipodal map** $x \mapsto -x$ is **not** homotopic to the identity.