Linear Methods Lecture

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1 Lecture 1

1.1 Set Theory

Definition 1.1. *A* ***set*** *is a collection of distinct objects, its elements.*

$$x \in X \quad x \text{ is a element of the set } X$$

and similary

$$x \notin X \quad x \text{ is not an element of } X$$

Two sets are identical $X = Y$ *, if*

$$x \in X \leftrightarrow x \in Y$$

for any element x *.*

Definition 1.2. Y *is a subset of* X*,* $Y \mathbb{C} X$ *if for all* $y \in X$*. If* $Y \subset X$ *and* $Y \neq X$*, we write* $y \subset X$ *(or* $Y \not\subset X$*).* Y *is then a proper subset of* X *. Showing to sets are equal,*

- $x \in X \leftrightarrow x \in Y$
- $x \subset Y$ *and* $y \subset X$

The empty set are denoted by null.

Example 1.

- $\mathbb{N} = \{1, 2, 3, 4, 5, \ldots\}$
- $\mathbb{Z} = \{\ldots, -1, 0, 1, \ldots\}$
- $\mathbb{Q} = \left\{\frac{p}{q} : p, q \in \mathbb{Z}, q \neq 0\right\}$
- $\mathbb{R} = \text{reals}$
- $\mathbb{C}$: Complex numbers $\quad a + ib$
- Finite set $\{3, 4, 5, 6\}$
- Intervals in $\mathbb{R}$ For real numbers $a < b < \infty$

$$(a, b)$$
$$[a, b]$$
$$(a, b], \quad [a, b).$$

Definition 1.3. *Let X and Y be two sets then*

- *Union.* $X \cup Y = \{z \mid z \in X \quad or \quad z \in Y\}$

$$\bigcup_{i \in I} X_i = \{z \mid z \in X_i \quad \text{for some} \quad i \in I\}$$

- *Intersection if* $\bigcap_{i \in I} = \{z \mid z \in X_i \quad \text{For every} \quad i \in I\}$
- *Complement if S is a subset of X , then the complement of S is*

$$X \setminus S = S^c = \{x \in X : x \notin S\}.$$

- *Cartesian product*

$$X \times Y = \{(x, y) : x \in X, \quad y \in Y\}$$

Lemma 1.1.

- $x \cap (Y \cup Z) = (X \cap Y) \cup (X \cap Z)$ *and*

$$X \cup (Y \cap Z) = (X \cup Y) \cap (X \cup Z)$$

- $(X \cup Y)^c = X^c \cap Y^c$
- $(X \cap Y)^c = X^c \cup Y^c$
- *Demo organs law*

$$X \setminus (Y \cup Z) = (X \setminus Y) \cap (X \setminus Z)$$

.

- $(X^c)^c = X$

Proof. Proof of $(X \cup Y)^c) = X^c \cap Y^c$

$$\begin{aligned} x \in (X \cup Y)^c \to x \in X \cup U \\ x \notin X \quad \text{and} \quad x \notin Y \\ x \in X^c \quad \text{and} \quad x \in Y^{\cdot} \\ x \in X^c \cap Y^c \end{aligned}$$

□

1.2 Functions

Let X, Y be sets. A function f from X to Y, denoted $f : X \to Y$, is defined by a set G of ordered pairs (x, y), where $x \in X, \quad y \in Y$ and with the property that;

For each set is there a unique $y \in Y \quad$ s.t. $\quad (x, y) \in G$. We write $f(x) = y$.

- We say that X is the domain and Y is the codomain.
- The (direct) image of a set $A \subset X$ under f is

$$f(A) = \{f(t) : t \in A\} \subset Y$$

- The **inverse image** of a set $B \subset Y$ under f is

$$f^{-1}(B) = \{x \in X \mid f(x) \in B\} \subset X$$

- The **range** if f is the image of its domain X is

$$ran(f) = f(X) = \{f(t) : t \in X\}$$

Example 2. Let $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = max\{x, 0\} = x^+$$

Then is the $ran(f) = [0, \infty)$. The inverse is $f^{-1}(\{y\}) = \{y\}$ and $f^{-1}(\{0\}) = (-\infty, 0]$ and

$$f^{-1}(\{y\}) = \text{NULL} \quad \text{if} \quad y < 0$$

Definition 1.4. *Let $f : X \to Y$ be a function*

- *f is* ***injective*** *or* ***one-to-one*** *if $f(x_1) \to x_1 = x_1$*
- *f is* ***surjective*** *or* ***onto*** *if $ran(f) = y$*
- *f is* ***bijective*** *if it is both surjective and injective.*

Example 3. Lets continue the example.

- Let $f : \mathbb{R} \to \mathbb{R}$, $f(x) = max\{x, 0\}$. Injective? No; $f(x_1) = \underbrace{f(x_2)}_{=0}$ for any two $x_1, x_1 < 0$.
- A **bijection** $f : X \in Y$ has a **inverse** function $f^{-1} : Y \to X$, defined by $f^{-1}(y) = x$ if $f(x) = y$.

  THe inverse function f^{-1} is also a bijection.

Remark. Not to be confused with the inverse image of a set $f^{-1}(B)$ introduced earlier.

2 Lecture 2

2.1 Recall

Let $f : X \to Y$ then is

i) Inective: $f(x_1) = f(x_2) \to \quad x_1 = x_2$

ii) Surjective: For all y in Y there is a x in X s.t. $f(x) = y$.

iii) Bijective if i) and ii) holds.

- If $F : X \to Y$ is a bijective then there is an inverse

$$f^{-1} : Y \to X$$

  Given by

$$f^{-1}(y) = x \quad \text{if} \quad f(x) = y$$

- Identify function/map
  - id: $X \to X$
  - $id_x(x) = x$ for all $x \in X$
- The composition of a function

$$g : Y \to Z \quad \text{with} \quad f : X \to X$$

  is the function $g \cdot f : X \to Y$ defined by

$$(g \cdot f)(x) = g(f(x)) \quad \text{for} \quad x \in X$$

Definition 2.1. *Anternative version. Given a bijection $f : X \to Y$ the inverse function $f^{-1} : Y \to X$ is the unique function satisftying $f^{-1} \cdot f = id_x$ and $f \cdot f^{-1} = id_y$*

Example 4. $\frac{d}{dx} : C^1(\mathbb{R}, \mathbb{R}) \to C(\mathbb{R}, \mathbb{R})$. Inverse? no.
Let $g \in C^1(\mathbb{R}, \mathbb{R})$. Then is

$$\frac{d(g+c)}{dx} = \frac{dg}{dx} \quad \text{where c is the constant.}$$

It is surjective because given any $f \in C(\mathbb{R}, \mathbb{R})$ we can define $F \in C^1(\mathbb{R}, \mathbb{R})$ by

$$F : X \to \int_0^x f(t)\, dt$$

and

$$\frac{dF}{dx} = f \quad \text{fundamental theorem of calculus.}$$

2.2 Cardinality

Cardinality is a tool for comparing the sizes of sets.

Definition 2.2. *We say that two sets A and B has the same cardinality if there exist a bijection between A and B.*

Example.

i) The two inervals $[0,2]$ and $[0,1]$ have the same cardinality.

$$f:[0,2]\to[0,1]$$
$$f(t)=\frac{t}{2}$$

ii) Let $\mathbb{N}=\{1,2,3,4,\ldots\}$ and $\mathbb{N}\setminus\{1\}=\{2,3,4,5,\ldots\}$ have the same cardinality

$$f(n)=n+1$$

iii) n is finite integer. Then there is no bijection

$$f:\{1,2,3,\ldots,n\}\to\mathbb{N}$$

These two sets **do not** have the same cardinality.

Definition 2.3. *Let X be a set. We say X is **finite** if either $X = NULL$ or there exist $n\in\mathbb{N}$ s. T. X has the same cardinality as $\{1,2,3,4,\ldots,n\}$ if*

$$\textit{There exist}\quad f:\{1,2,3,\ldots mb\}\to X\quad \textit{for some}\quad n$$

*X is **infinite** if it is not finite.*

Definition 2.4. *A set X is*

- *Countable infinite if it has the same cardinality as $\mathbb{N}$.*

$$\exists \textit{bijection} \quad f : X \to \mathbb{N}$$

- *Countable if it is either countably infinite or finite. or equivalently*
  - *if $\exists$ injection $f : X \to \mathbb{N}$*
  - *$\exists$ surjection $f : \mathbb{N} \to X$*
- *Uncountable if it is not countable.*

Example.

- Any finitie set is, e.g. $\{2, 5, 9\}$
- $X = \{1, 4, 9, 16, \ldots, n^2, \ldots\}$ such that

$$f : \mathbb{N} \to X, \quad f(n) = n^2$$

- $\mathbb{N} \times \mathbb{N}$ is countable ;

  We arrange $N \times N$ in a table.

$$\begin{gathered} f : \mathbb{N} \to \mathbb{N} \times \mathbb{N} \\ f(1) = (1, 1) \\ f(2) = (2, 1) \\ f(3) = (1, 2) \\ f(4) = (3, 1) \\ \vdots \end{gathered}$$

- $\mathbb{Z}$ and $\mathbb{Q}$ are countable (Prob set 1).
- If X and Y are countable, then so is $X \cup Y$.

2.3 Schroeder Bernstein Theorem

Let X and Y by two be two sets. Suppose there are injective maps $f : X \to Y$ and $g : Y \to X$. Then there exists a bijection between X and Y.

Example. The interval $(0,1) \subseteq \mathbb{R}$. Claim it is uncountable.

Proof. The Cantor diagonalization argument. Suppose that $(0,1)$ is countable.

$$(0,1) = \{x_1, x_2, x_3, x_4, \ldots\}$$
$$f(1), f(2), f(3), \ldots$$

$$f : \mathbb{N} \to (0,1)$$
$$x_i = 0{,}x_{i1}, x_{i2}, x_{i3}, \ldots$$

Now let

$$a = 0, a_1, a_2, a_3, a_4, a_5, \ldots$$

where

$$a_i = \begin{cases} 3 & \text{if} \quad x_{ii} \neq 3 \\ 1 & \text{if} \quad x_{ii} = 3 \end{cases}$$

Then $a_i \neq x_{ii}$, so by construction $a \neq x_i$ for all i. Moreover, we must have $a \in (0,1)$. This is a contradiction, so $(0,1)$ cannot be countable. □

Example. The set of all binary sequences $X = \{(x_1, x_2, x_3, \ldots)\}$: $x_i \in \{0,1\}$ is uncountable .

Proof. Problem set 2. □

Lemma 2.1. *Let X and Y be sets. Then*

- *If X is countable and $Y \subseteq X$, then Y is also countable.*

$$\{1, 2, 3, 4, 5, \ldots\} \to \{x_1, x_2, x_3, x_4, \ldots\}$$

- *If X is uncountable and $X \subseteq Y$, then Y is uncountable.*
- *If X is countable and there is an injection*

$$f : Y \to X$$

  then Y is countable.

- *If X is uncountable and*

$$\exists \quad \textit{injective} \quad f : X \to Y,$$

  then Y is uncountable.

Example. Have proved formally that $(0, 1) \subseteq \mathbb{R}$ is countable $\overbrace{\to}^{\text{ii)}} \mathbb{R}$ must be uncountable

$$R \subset \mathbb{C} \overbrace{\longrightarrow}^{\text{ii)}} \mathbb{C} \quad \text{is uncountable}$$

Example. $R = \mathbb{Q} \cup \mathbb{I}$. Know: $\mathbb{Q}$ countable.
Assume $\mathbb{I}$ countable. Then $R \cup \mathbb{I}$ which is a contradiction. So $\mathbb{I}$ is uncountable

3 Lecture 3

3.1 Sequences

Fixed set J and set X with elements $x_j \in X$ for $j \in J$. J is a **index set** , x_j is the j-th component of the sequence $\{x_{j\in J}\}_j$.

Remark. (x_j) is equivalent to $(x_j)_j$. More technically $x : J \to X$ s.t. $x_{(j)} = x_j$.

3.2 Infima and Suprema

Definition 3.1. *Suppose $A \subseteq \mathbb{R}$ is nonempty.*

1. *A is **bounded** if*

$$\exists \quad M \in \mathbb{R} \quad s.t. \quad a \leq M \quad for\ all \quad a \in A$$

2. *A is **bounded below** if*

$$\exists \quad m \in \mathbb{R} \quad s.t. a \geq m \quad for\ all\ a \in A$$

3. *A is **bounded** is* 1., 2.

4. *v is a **maximal element** of A if $v \in A$ and $a \leq v$ for every $a \in A$. We write $v = \max(A)$*

5. *v is a **minimal element** of A if $v \in A$ and $a \geq v$ for every $a \in A$. We write $v = \min(A)$*

Definition 3.2 (Infimum and supremum)**.** *Suppose $A \subseteq \mathbb{R}$ is nonemtpy.*

1. *We say that $M \in \mathbb{R}$ is the **supreme** or **least upper bound** of A if*
   - (a) *M is a upper bound of A , i.e. $a \leq M$ for every $a \in A$.*
   - (b) *All other upper bounds $M^{'}$ of A satisfied $M^{'} \geq M$. We write $M = sup(A)$ (and if it exists a max element $u \in A$, then $u = sup(A) = max(A)$*

2. *$m \in \mathbb{R}$ is the **infimum** or the **greated lower bound** of A if*
   - (a) *It is a lower bound, $a \geq m \forall a \in A$*
   - (b) *All other lower bounds $m^{'}$ are smaller $m^{'} < m$*

Example.

$$A = \left(0 \quad 1\right) \to \begin{cases} inf\left(A\right) = 0 \\ sup\left(A\right) = 1 \end{cases}$$

Remark.

- If $A \subset \mathbb{R}$ is not bounded from above, we write $sup\left(A\right) = \infty$
- If $A \subset \mathbb{R}$ is not bounded below, we write $inf\left(A\right) = -\infty$

Lemma 3.1. *$A \subseteq \mathbb{R}$ is nonemtpy.*

1. *Say A is bounded above. Then $M \in \mathbb{R}$ is the sup of A if*
   (a) $a \ll M \quad \forall \quad a \in A$
   (b) $\forall \epsilon > 0 \quad \exists \quad a \in A \quad s.t. \quad a > M - \epsilon$
2. *Say A is bounded from below. Then $m \in \mathbb{R}$ is the inf of A if*
   (a) $a \geq m \quad \forall \quad a \in A$
   (b) $\forall \epsilon > 0 \quad \exists a \in A \quad s.t. \quad a < m + \epsilon$

Example. Let $A = \left\{\frac{1}{n} : n \in \mathbb{N}\right\} = \left\{1, \frac{1}{2}, \frac{1}{3}, \dots\right\}$ then is

1. $\inf\left(A\right) = 0$, since $\frac{1}{n} \geq 0 \quad \forall n \in \mathbb{N}$, and for any $\epsilon > 0$ we can find N s.t. $\frac{1}{N} < \epsilon$
2. $sup\left(A\right) = 1$, since $\frac{1}{n} \leq 1$ for all $n \in \mathbb{N}$ and for any $\epsilon > 0$ we have $1 > 1 - \epsilon$ which concludes

$$\max\left(A\right) = sup\left(A\right) = 1$$

- From our definitnion, it follows that

$$\inf\left(A\right) \geq sup\left(A\right)$$

- If $A = \left(a_n\right)_n$ then we usually write

$$sup_{\cap}\left(a_n\right)$$

- If we have a function $f : X \to Y$ then

$$sup_x f = sup\left\{f\left(x\right) : x \in X\right\}$$

Definition 3.3 (Dilate Set)**.** *We define the* ***dilate*** *by* $c \in \mathbb{R}$ *of a set* $A \subseteq \mathbb{R}$ *by*

$$cA = \{b \in \mathbb{R} : \quad b = ca, \quad a \in A\}$$

Lemma 3.2 (Properties of dilates, subsets, sums)**.** *Let* $A, B \subseteq \mathbb{R}$ *be nonempty and bounded.*

1. *if* $c > 0$ *, then* $\sup cA = c \sup A$ *and* $\inf cA = c \inf A$
2. *If* $c < 0$ *, then* $\sup cA = c \inf A$ *and* $\inf cA = c \sup A$
3. $\sup (A + B) = \sup A + \sup B$ *and* $\inf (A + B) = \inf A + \inf B$
4. *If* $B \subset A$ *, then is* $\inf B \geq \inf A$ *and* $\sup B \leq \sup A$

Proof. We want to show that $\sup cA = c \sup A$ for $c > 0$. Let $\sup A = M$. Then is $\forall a \in A$, $a \leq M \implies ca \geq cM$ and $\sup cA \leq cM$. Moroever, for every $\epsilon > 0$ does exist $a \in A$ s.t. $a \geq M - \frac{\epsilon}{c}$. This can be rewritten such that

$$ca \geq cM - \epsilon \implies \sup cA = cM$$

$\square$

Example. Let $X = \{g \in C\,[0, 2] : |g| < M\}$ and

$$\begin{aligned} f : X &\to \mathbb{R} \\ g &:\to \int_0^2 g\,(x)\,dx \\ \sup_x f &= \sup \{f\,(g) : g \in X\} \\ &= \sup \left\{ \int_0^2 g\,(x)\,dx : g \in X \right\} \end{aligned}$$

We can show that

$$\int_0^2 g\,(x)\,dx \leq \overbrace{\sup_{x \in [0,2]} g\,(x)}^{<M} \underbrace{\int_0^2 dx}_{=2} \leq 2M$$

Claim that $\sup_x f = 2M$. And then is the task: For any $\epsilon > 0$, find $g \in X$ s.t.

$$\int_0^2 g\,(x)\,dx > 2M - \epsilon$$

3.3 Known material (self-study)

- 1.7 : Convergent sequenxes of numbers.
  - Say $(x_n)_{n\in\mathbb{N}}$ sequenc of real/complex numbers. (x_n) converges if $\exists$ some x in $\frac{\mathbb{R}}{\mathbb{C}}$ s.t.

    $$\forall\epsilon > 0 \exists N \in \mathbb{N} \quad \text{s.t.} n \geq N \qquad \implies \|x_n - x\| < \epsilon$$

    We write $x_n \to x$, $\lim_{n\to\infty} x_n = x$, $\lim x_n = x$
  - If (x_n) sequence of real numbers, then we say that (x_n) diverges to ∞ if

    $$\forall R > 0 \exists N > 0 \quad \text{s.t.} \quad x_n > R \forall n > N$$

    We write $\lim_{n\to\infty} = \infty$, $\lim x_n \to \infty$, $x_n \to \infty$
- 1.8: Infinite Series of numbers
  - $\displaystyle\sum_{n=1}^{\infty} c_n$ series of real/complex numbers converges if the sequence of partial sums

    $$s_n = \sum_{n=1}^{N} c_n$$

    Converges as $N \to \infty$. Say $S_N \to S$. We then write $\sum_{i=1}^{\infty} c_i = s$.
  - Recall if $\sum_{i=1}^{\infty} c_i$ converges, then $\lim_{i\to\infty} c_i = 0$
  - Recall if $\sum_{i=1}^{\infty} c_i$ converges, then $\lim_{N\to\infty} \left(\sum_{i=N} c_i\right) = 0$
- Concerning 1.9 $\to$ read it!