Norwegian University of Science and Technology
Deptartment of Mathematical Sciences

TMA4190 Introduction to Topology Spring 2018

Suggestions for solutions
Exam May 31, 2018

1 **a)** By definition of embeddings, we need to show that f is an injective, proper immersion.

- f is injective: If $f(t) = f(s)$, then $\frac{e^t+e^{-t}}{2} = \frac{e^s+e^{-s}}{2}$ and $\frac{e^t-e^{-t}}{2} = \frac{e^s-e^{-s}}{2}$. Adding these two equations, implies $e^t = e^s$. Since the exponential function is injective, this shows $t = s$.
- f is proper: Let K be a compact subset of $\mathbb{R}^2$. That means that K is both closed and bounded in $\mathbb{R}^2$. Since f is continuous, $f^{-1}(K)$ is closed in $\mathbb{R}$. Since both coordinates of $f(t)$ are unbounded when t varies in all of $\mathbb{R}$, $f^{-1}(K)$ must be bounded as well. Thus $f^{-1}(K)$ is both closed and bounded in $\mathbb{R}$ and therefore compact.
- f is an immersion: The derivative of f at any $t \in \mathbb{R}$ is given in the standard basis by the 2×1-matrix

$$df_t = \begin{pmatrix} \frac{e^t-e^{-t}}{2} \\ \frac{e^t+e^{-t}}{2} \end{pmatrix}.$$

For each t, df_t is a linear map $\mathbb{R} \to \mathbb{R}^2$. Since $\mathrm{Ker}\,(df_t)$ is a vector subspace of $\mathbb{R}$, it is either $\{0\}$ or $\mathbb{R}$ itself. Since df_t is not the zero matrix for any t, df_t must be injective for all $t \in \mathbb{R}$.

b) The derivative of g at a point (x, y) is given by the 1×2-matrix

$$dg_{(x,y)} = \begin{pmatrix} 2x & -2y \end{pmatrix}.$$

As a linear map from $\mathbb{R}^2$ to $\mathbb{R}$, $dg_{(x,y)}$ is surjective whenever it is not the zero map. Hence $dg_{(x,y)}$ is surjective for all $(x, y) \neq (0, 0)$.

Thus the set of regular values of g is the subset $\mathbb{R} \setminus \{0\}$. Since $g(0, 0) = 0$, the only critical value is 0.

Since the derivative of g is not surjective at all points, g is not a submersion.

c) The image of f is a submanifold of $\mathbb{R}^2$. This follows, for example, from the fact that f is an embedding. We could also obeserve that $\mathrm{Im}\,(f) = g^{-1}(1)$ and remark that 1 is a regular value of g. The composition $g \circ f$ is the constant map $\mathbb{R} \to \mathbb{R}$ with value 1. Hence $(g \circ f)^{-1}(1) = \mathbb{R}$ is a manifold.

$\boxed{2}$ **a)** We define the map

$$f\colon \mathbb{R}^4 \to \mathbb{R},$$
$$(x_1, , x_2, x_3, x_4) \mapsto x_1 + x_2^2 + x_3^3 + x_4^4.$$

Then Z is the preimage of 0 under f, i.e., $Z = f^{-1}(0)$. In order to show that Z is a manifold, we just need to show that 0 is a regular value of f. To check this, we calculate the derivative of f at any $z = (x_1, x_2, x_3, x_4)$ in $f^{-1}(0)$. The derivative df_z is a linear map $\mathbb{R}^4 \to \mathbb{R}$ given in the standard basis by the 1×4-matrix

$$df_z = \begin{pmatrix} 1 & 2x_2 & 3x_3^2 & 4x_4^3 \end{pmatrix}.$$

We need to show that df_z is surjective. Since df_z is a map with values in $\mathbb{R}$, it suffices to observe that df_z is not the zero map. Thus df_z is surjective for all $z \in f^{-1}(0)$, and 0 is a regular value of f. By the Preimage Theorem, $Z = f^{-1}(0)$ is a manifold of dimension $3 = \dim \mathbb{R}^4 - \dim \mathbb{R}$.

b) We show that Z and S^3 meet transversally in $\mathbb{R}^4$. To do this we need to check that $T_z(Z) + T_z(S^3) = T_z(\mathbb{R}^4) = \mathbb{R}^4$ for all $z \in Z \cap S^3$. Since $T_z(Z)$ and $T_z(S^3)$ are both three-dimensional subspaces of $\mathbb{R}^4$, it suffices to show that, for every $z \in Z \cap S^3$, there is at least one vector v in $T_z(Z)$ which is not contained in $T_z(S^3)$.

The tangent space to Z in a point $z \in Z$ is the subspace in $\mathbb{R}^4$ given by the kernel of the derivative df_z. Let $z = (x_1, x_2, x_3, x_4)$ be any point in $Z \cap S^3$. Then the vector $v := (12x_1, 6x_2, 4x_3, 2x_4)$ lies in $T_z(Z)$, since

$$\begin{aligned} df_z(v) &= \begin{pmatrix} 1 & 2x_2 & 3x_3^2 & 4x_4^3 \end{pmatrix} \begin{pmatrix} 12x_1 \\ 6x_2 \\ 4x_3 \\ 3x_4 \end{pmatrix} \\ &= 12x_1 + 12x_2^2 + 12x_3^3 + 12x_4^3 \\ &= 12f(z) \\ &= 0. \end{aligned}$$

But v is not an element in $T_z(S^3)$. For, recall that $T_z(S^3)$ is the subspace in $\mathbb{R}^4$ which is orthogonal to the vector z, i.e.,

$$T_z(S^3) = \{w \in \mathbb{R}^4 : z \perp w = 0\}.$$

We can check orthogonality via the scalar product in $\mathbb{R}^4$:

$$z \perp w \iff z \cdot w = 0.$$

For v we calculate

$$z \cdot v = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 \end{pmatrix} \begin{pmatrix} 12x_2 \\ 6x_2 \\ 4x_3 \\ 3x_4 \end{pmatrix} = 12x_1^2 + 6x_2^2 + 4x_3^2 + 3x_4^2 > 0.$$

Thus v is not an element in $T_z(S^3)$. Hence Z and S^3 meet transversally in $\mathbb{R}^4$. By the Preimage Theorem, the codimension of $Z \cap S^3$ in S^3 equals the codimension of Z in $\mathbb{R}^4$. Thus $\dim Z \cap S^3 = 2$.

$\boxed{3}$ Let $X = \{(x,y) \in \mathbb{R}^2 : x \geq -1\}$, $Y = \mathbb{R}$ and

$$f\colon X \to Y,\ (x,y) \mapsto x^2 + y^2.$$

a) The boundary of X is $\partial X = \{(x,y) \in \mathbb{R}^2 : x = -1\}$. The derivative of f is given by the 1×2-matrix $df_{(x,y)} = \begin{pmatrix} 2x & 2y \end{pmatrix}$. Hence $df_{(x,y)}$ is a surjective linear map for all $(x,y) \neq (0,0)$. Since $f(0,0) = 0 \neq 1$, $df_{(x,y)}$ is surjective for all $(x,y) \in f^{-1}(1)$ and 1 is a regular value of f.
The restriction of f to the boundary of X is

$$\partial f\colon \partial X \to Y,\ (-1,y) \mapsto 1 + y^2.$$

Hence the derivative of ∂f is given by the 1×1-matrix $(\partial f)_{(-1,y)} = 2y$. This is a linear map which is surjective if and only if $y \neq 0$. Since $(-1,0) \in \partial X$ and $\partial f(-1,0) = 1$, we see that 1 is not a regular value of ∂f.

b) The preimage $f^{-1}(1)$ is just the unit sphere S^1. Hence the boundary $\partial(f^{-1}(1))$ is empty. However,

$$f^{-1}(1)\cap\partial X = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 = 1\}\cap\{(x,y) \in \mathbb{R}^2 : x = -1\} = \{(-1,0)\} \neq \emptyset.$$

In particular, $\partial(f^{-1}(1)) \neq f^{-1}(1) \cap \partial X$.
This is not a contradiction to the Preimage Theorem for manifolds with boundary, since the conclusion of the theorem required that 1 was a regular of both f and ∂f. But we showed in the first part that 1 is not a regular value of ∂f.

$\boxed{4}$ **a)** Since $\deg_2(f) \neq 0$, we must have $\#f^{-1}(y) \neq 0 \mod 2$ for some $y \in Y$. But, since Y is connected, the function

$$\#f^{-1}(-) \colon Y \to \mathbb{Z}/2, y \mapsto \#f^{-1}(y) \mod 2$$

is constant. Thus we must have $\#f^{-1}(y) \neq 0 \mod 2$ for all $y \in Y$. Hence $f^{-1}(y) \neq \emptyset$ for all $y \in Y$ and f is surjective.

b) Let us assume $\deg_2(f) \neq 0$ and derive a contradiction. By the previous point, if $\deg_2(f) \neq 0$, then f is surjective. But that means $Y = f(X)$. Since f is, in particular, continuous and X is compact, the image of X under f is compact. Hence Y would be compact as the continuous image of a compact space. This contradicts the assumption. Hence we must have $\deg_2(f) = 0$.

c) Let $f \colon S^1 \to S^1$ be a smooth map without fixed points. We define the map

$$G(x,t) \colon S^1 \times [0,1] \to \mathbb{R}^2, (x,t) \mapsto f(x)(1-t) - tx.$$

We would like to turn G into a homotopy between f and α. Hence we need to manipulate G such that its image is contained in $S^1 \subset \mathbb{R}^2$. We can arrange this if $G(x,t) \neq 0$. For then $\frac{G(x,t)}{|G(x,t)|}$ is in S^1. Hence we need to check $G(x,t) \neq 0$ for all $(x,t) \in S^1 \times [0,1]$.

For a fixed x and varying t, $f(x)(1-t) - tx$ describes the line segment in $\mathbb{R}^2$ between the two points $f(x)$ and $-x$ on S^1. The only way, this line segment can pass $0 \in \mathbb{R}^2$, is when $f(x) = x$ is the antipodal point to $-x$. But, by the assumption on f, $f(x) \neq x$ for all $x \in S^1$.

Thus the smooth map

$$F(x,t) \colon S^1 \times [0,1] \to S^1, (x,t) \mapsto \frac{f(x)(1-t) - tx}{|f(x)(1-t) - tx|}$$

is a homotopy between f and α.

Since $\alpha^{-1}(x) = -x$ for all $x \in S^1$, there is exactly one preimage point for each x. Hence $\deg_2(\alpha) = 1$. Since f and α are homtopic, the invariance of $\deg_2$ under homotopy implies $\deg_2(f) = 1$. By the first point, $\deg_2(f) = 1$ implies that f is surjective.