TMA 4190 Introduction to Topology
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Lecture 15[1]

15. Embedding Manifolds in Euclidean Space

We have two objectives today. The first one is to study how manifolds can be embedded into Euclidean space. In particular, given a k-dimensional manifold, what is the minimal N such that we can be sure that there is an embedding $X \subset \mathbb{R}^N$? The second one is to give an intrinsic definition of manifolds. In the next lecture, we are going to relate these two objectives and show that every abstract smooth manifold can be embedded into some Euclidean space.

To address the first question we need a useful new device, the tangent bundle.

The Tangent Bundle

Let $X \subset \mathbb{R}^N$ be a smooth manifold. For every $x \in X$, the tangent space $T_x(X)$ to X at x is a vector subspace of $\mathbb{R}^N$. If we let x vary, these tangent space will in general overlap in $\mathbb{R}^N$. (For example, if X is a vector space, they will all be equal.)

Hence, in order to be able to keep track of the information contained in all the different tangent spaces, we need a smart device that keeps those spaces apart:

Tangent bundles

The **tangent bundle** of X, denoted $T(X)$, is the subset of $X \times \mathbb{R}^N$ defined by
$$T(X) := \{(x,v) \in X \times \mathbb{R}^N : v \in T_x(X)\}.$$
In particular, $T(X)$ contains a natural copy X_0 of X, consisting of the points $(x,0)$. In the direction perpendicular to X_0, it contains copies of each tangent space $Tx(X)$ embedded as the sets
$$\{(x,v) \in T(X) : \text{ for a fixed } x\}.$$
There is a natural projection map
$$\pi \colon T(X) \to X,\ (x,v) \mapsto x.$$

[1]Following the books of Guillemin and Pollack: Differential Topology, and Milnor: Topology from the differentiable viewpoint.

Any smooth map $f\colon X \to Y$ induces a **global derivative map**

$$df\colon T(X) \to T(Y),\ (x,v) \mapsto (f(x), dfx(v)).$$

Note that, since $X \subset \mathbb{R}^N$ and $T_x(X) \subset \mathbb{R}^N$ for every x, $T(X)$ is also a subset of Euclidean space:

$$T(X) \subset \mathbb{R}^N \times \mathbb{R}^N.$$

Therefore, if $Y \subset \mathbb{R}^M$, then df maps a subset of R^{2N} into $\mathbb{R}^{2M}$.

We claim that ***df* is smooth**. For since $f\colon X \to \mathbb{R}^M$ is smooth, it extends by definition around any point $x \in X$ to a smooth map $F\colon U \to \mathbb{R}^M$, where U is an open set of R^N. Then $dF\colon T(U) \to \mathbb{R}^{2M}$ **locally extends *df***. But, since $U \subset \mathbb{R}^N$ is open and hence $T_u(U) = \mathbb{R}^N$ for every $u \in U$, $T(U)$ is all of $U \times \mathbb{R}^N$. Since $U \times \mathbb{R}^N$ is an open set in $\mathbb{R}^{2N}$, dF is a linear and hence smooth map defined on an open subset of $\mathbb{R}^{2N}$. This shows that $df\colon T(X) \to \mathbb{R}^{2M}$ may be locally extended to a smooth map on an open subset of R^{2N}, meaning that df is smooth.

Given smooth maps $f\colon X \to Y$ and $g\colon Y \to Z$, the global derivative of the composite is equal to the composite of global derivatives:

$$d(g \circ f) = dg \circ df\colon T(X) \to T(Z).$$

For, the chain rule implies that, for any $(x,v) \in T(X)$,

$$\begin{aligned}
d(g \circ f)(x,v) &= ((g \circ f)(x), d(g \circ f)_x(v) \\
&= ((g(f(x)), (dg_{f(x)} \circ df_x)(v)) \\
&= dg(df(x,v)) \\
&= dg \circ df(x,v).
\end{aligned}$$

As a consequence we get:

Tangent bundles are intrinsic

If $f\colon X \to Y$ is a diffeomorphism, so is $df\colon T(X) \to T(Y)$. For the chain rule implies that $df^{-1} \circ df$ is the identity map of $T(X)$ and $df \circ df^{-1}$ is the identity map of $T(Y)$. Thus diffeomorphic manifolds have diffeomorphic tangent bundles. As a result, $T(X)$ is an object intrinsically associated to X.

Finally, we are going to show that $T(X)$ is in fact itself a smooth manifold. Let W be an open set of X. In particular, W is also a manifold, and we can consider its tangent bundle $T(W)$. Since $T_x(W) = T_x(X)$ for every $x \in W$, $T(W)$ is by definition

$$T(W) = \{(x,v) \in T(X) : x \in W\} = T(X) \cap (W \times \mathbb{R}^N) \subset T(X).$$

Since $W \times \mathbb{R}^N$ is open in $X \times \mathbb{R}^N$, **$T(W)$ is open in $T(X)$**.

Now suppose that W is the image of a **local parametrization** $\phi\colon U \to W$, where U is an open set in $\mathbb{R}^k$. Then the global derivative $d\phi\colon T(U) \to T(W)$ is a diffeomorphism. But $T(U) = U \times \mathbb{R}^k$ is an open subset of R^{2k}, so **$d\phi$ is a parametrization** of the open set $T(W)$ in $T(X)$. Since every point of $T(X)$ sits in such a neighborhood, we have proved the following useful result:

Tangent bundles are manifolds

The tangent bundle of a manifold X is a smooth manifold of dimension $\dim T(X) = 2\dim X$.

Whitney's Theorem

Whitney's Theorem

Every k-dimensional manifold admits a one-to-one immersion in $\mathbb{R}^{2k+1}$.

Proof: Let $X \subset \mathbb{R}^N$ be k-dimensional manifold which is a subset in $\mathbb{R}^N$ for some $N > 2k+1$. In particular, we are given an **injective immersion** $X \to \mathbb{R}^N$. Our goal is to show that we can choose N to be $2k+1$ and still have an injective immerison. Therefor we are going to construct a linear projection $\mathbb{R}^N \to \mathbb{R}^{2k+1}$ that restricts to a one-to-one immersion $X \to \mathbb{R}^{2k+1}$ on X.

The construction works by induction: Whenever we are given an injective immersion $f\colon X \to \mathbb{R}^M$ with $M > 2k+1$, then there exists a unit vector $a \in \mathbb{R}^M$ such that the composition of f with the projection map carrying $\mathbb{R}^M$ onto the orthogonal complement of a is still an injective immersion. The complement $H := \{b \in \mathbb{R}^M : b \perp a\}$ is an $M-1$-dimensional vector subspace of $\mathbb{R}^M$, hence isomorphic to R^{M-1}. Thus, after choosing a basis for H, we obtain an injective immersion into $\mathbb{R}^{M-1}$.

Continuing this procedure yields a chain of linear maps

$$\mathbb{R}^N \to \mathbb{R}^{N-1} \to \cdots \to \mathbb{R}^{2k+1}$$

such that the composition $X \to \mathbb{R}^N \to \mathbb{R}^{2k+1}$ is still an injective immersion.

So let us **assume we have an injective immersion**

$$f\colon X \to \mathbb{R}^M \text{ with } M > 2k+1.$$

We define two smooth maps

$$\begin{tikzcd} & X \times X \times \mathbb{R} \arrow[d, "h"] \\ T(X) \arrow[r, "g"'] & \mathbb{R}^M \end{tikzcd}$$

by

$$h\colon X \times X \times \mathbb{R} \to \mathbb{R}^M,\ (x,y,t) \mapsto t(f(x) - f(y)).$$

and

$$g\colon T(X) \to \mathbb{R}^M,\ (x,v) \mapsto df_x(v).$$

By **Sard's theorem**, the sets S_g and $_h$ of critical values of g and h, respectively, have measrure zero in $\mathbb{R}^M$. Hence the union of S_g and S_h still has measure zero in $\mathbb{R}^M$. Thus the intersection of the sets of regular values of g and h, which is the complement of $S_g \cup S_h$, is nonempty.

Since $\dim T(X) = 2k$, $\dim X \times X \times \mathbb{R} = 2k+1$, but $M > 2k+1$, the only regular values of g and h are the points in $\mathbb{R}^M$ which are not in the image of g or h. Hence there exists a point $a \in \mathbb{R}^M$ which is neither in the image of g nor in the image of h. Note that, since 0 belongs to both images, we must have $a \neq 0$.

Let π be the projection of $\mathbb{R}^M$ onto the orthogonal complement H of a.

First claim: $\pi \circ f\colon X \to H$ is **injective**.

For suppose that $\pi \circ f(x) = \pi \circ f(y)$. Then, since π is **linear**, we have $\pi(f(x) - f(y)) = 0$, i.e.

$$\begin{aligned} f(x) - f(y) \in \operatorname{Ker}(\pi) &= \operatorname{span}(a) \text{ in } \mathbb{R}^M \\ &= \{w \in \mathbb{R}^M : w = t \cdot a \text{ for some } t \in \mathbb{R}\}. \end{aligned}$$

Thus there is a $t \in \mathbb{R}$ with $f(x) - f(y) = ta$. If $x \neq y$ then $t \neq 0$, since f **is injective**. But then

$$a = 1/t(f(x) - f(y)) = h(x,y,1/t)$$

which contradicts the choice of a.

Second claim: $\pi \circ f\colon X \to H$ is an **immersion**.

For suppose there was a nonzero vector v in $T_x(X)$ for which $d(\pi \circ f)_x = 0$. Because π is **linear**, the chain rule yields

$$d(\pi \circ f)_x = \pi \circ df_x.$$

Thus $\pi(df_x(v)) = 0$, so $df_x(v) = ta$ for some $t \in \mathbb{R}$. Because **f is an immersion**, we must have $ta \neq 0$. But since we know $a \neq 0$, this implies $t \neq 0$. Thus, since df_x is linear,

$$a = \frac{1}{t} df_x(v) = df_x(\frac{1}{t}v) = g(x, \frac{1}{t}v)$$

which again contradicts the choice of a. **QED**

For compact manifolds, one-to-one immersions are the same as embeddings. So we have just proved the embedding theorem in the compact case.

Whitney's Embedding for compact manifolds

Every compact k-dimensional manifold admits an embedding in $\mathbb{R}^{2k+1}$.

Note that Whitney's result does **not** give us the minimal N for an individual manifold. For example, we know that S^n is embedded in $\mathbb{R}^{n+1}$ for every n. The result tells us that, in general, **$N = 2k + 1$ will always work**. In fact, Whitney showed that $N = 2k$ always works. But the proof is much harder, and we will not discuss it in this course.

In order to extend Whitney's theorem (for $N = 2k + 1$) to noncompact manifolds, we must modify the immersion to make it proper. This is a topological, not a differential problem.

Before we develop the necessary tools to address this problem, we are going to contemplate a bit on a way to define manifolds without referring to a given embedding into some $\mathbb{R}^N$. The key idea that should be preserved in any new definition should be that **a manifold is a space which locally looks like Euclidean space**.

Abstract smooth manifolds

Hausdorff spaces

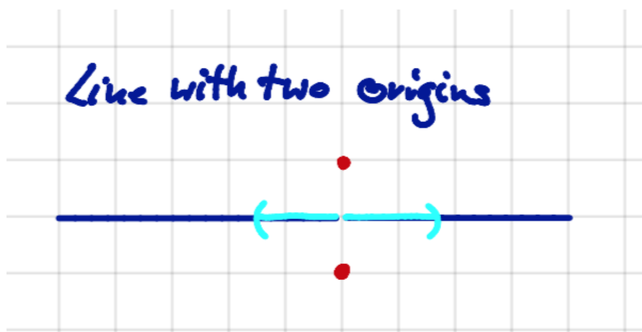
A topological space X is called **Hausdorff** if, for any two distinct points x,$y \in X$, there are two **disjoint** open subsets U,$V \subset X$ such that $x \in U$ and $y \in V$.

In other words, in a Hausdorff space we can separate points by open neighborhoods.

Every subspace of $\mathbb{R}^N$ (with the relative topology) is a Hausdorff space. However, there are spaces which are not Hausdorff.

For a typical example, consider two copies of the real line $Y_1 := \mathbb{R} \times \{1\}$ and $Y_2 := \mathbb{R} \times \{2\}$ as subspaces of $\mathbb{R}^2$. On $Y_1 \cup Y_2$, we define the equivalence realtion $(x,1) \sim (x,2)$ for all $x \neq 0$.

Let X be the set of equivalence classes. The topology on X is the quotient topology defined as follows: a subset $W \subset X$ is open in X if and only if both its preimages in $\mathbb{R} \times \{1\}$ and $\mathbb{R} \times \{2\}$ are open.



Then X looks like the real line except that the origin is replaced with two different copies of the origin. Away from the double origin, X looks perfectly nice like a one-dimensional manfiold. But every neighborhood of one of the origins contains the other. Hence we cannot separate the two origins by open subsets, and X is **not Hausdorff**.

For our definition of an abstract manfiold, we want to avoid such pathological spaces.

Abstract manifolds

Let X be a topological space.
A **chart** on X is a pair (V,ϕ) where $V \subset X$ is an open subset and $\phi\colon V \to U$ is a homeomorphism from V to an open subset $U \subset \mathbb{R}^k$ of $\mathbb{R}^k$.
An **abstract smooth k-manifold** is a Hausdorff space X together with a (countable) collection of charts (V_α,ϕ_α) on X such that

(1) every point in X is in the domain of some chart, and

(2) for every pair of overlapping charts ϕ_α and ϕ_β, i.e.

$$V_{\alpha\beta} := V_\alpha \cap V_\beta \neq \emptyset,$$

the **change-of-coordinates map**

$$\phi_\beta \circ \phi_\alpha^{-1} \colon \phi_\alpha(V_{\alpha\beta}) \to \phi_\beta(V_{\alpha\beta})$$

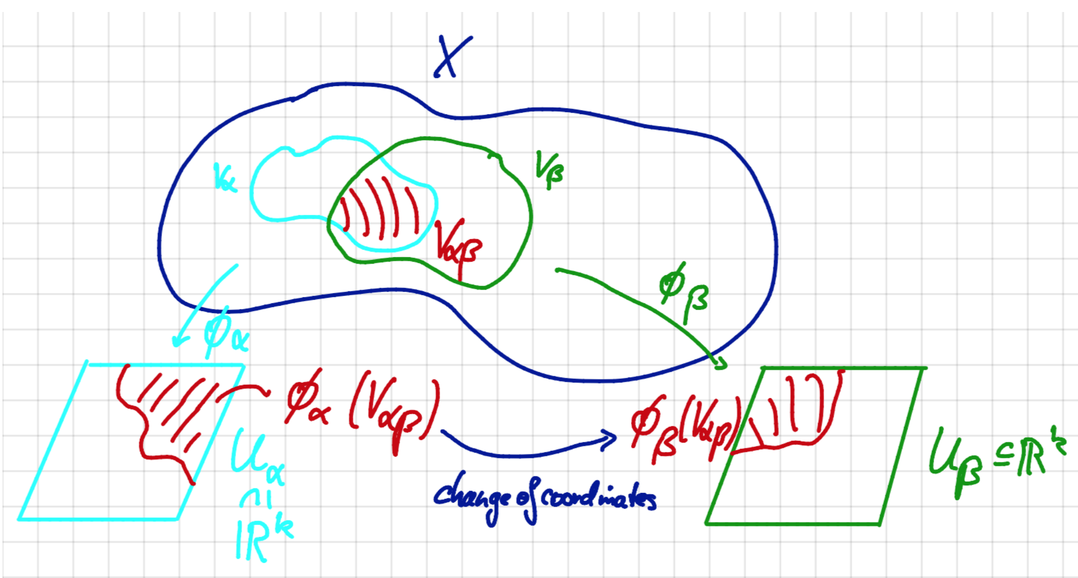
is **smooth** (as a map between open subsets of $\mathbb{R}^k$). In fact, this means that the change-of-coordinates maps are diffeomorphisms, since they are mutual smooth inverses to each other.

Let X be an abstract smooth k-manifold and $f \colon X \to \mathbb{R}^n$ be a continuous map. Then f is smooth if for every chart $\phi_\alpha \colon V_\alpha \to U_\alpha$, the composition $f \circ \phi_\alpha^{-1} \colon U_\alpha \to \mathbb{R}^n$ is smooth.

More generally, let X be an abstract smooth k-manifold and Y an abstract smooth m-manifold and $f \colon X \to Y$ a continuous map. Then **f is smooth at $x \in X$** if, for every chart $\phi \colon V \to U$ on X around x and every chart $\psi \colon V' \to U'$ on Y around $f(x)$, the map

$$\psi \circ f_{|V \cap f^{-1}(V')} \circ \phi^{-1}_{|U \cap \phi(f^{-1}(V'))} \colon U \cap \phi(f^{-1}(V')) \to U'$$

is a smooth map as a map from an open subset of $\mathbb{R}^k$ to an open subset of $\mathbb{R}^m$. We call f smooth if it is smooth at every $x \in X$.



Note that the smooth k-dimensional manifolds $X \subset \mathbb{R}^N$ we have been studying so far are examples of abstract smooth k-manifolds:

- The **Hausdorff** property is satisfied in $\mathbb{R}^N$ and therefore also for every subspace of $\mathbb{R}^N$ (with relative topology we have been using).
- Moreover, every open cover $\{U_\alpha\}$ of $\mathbb{R}^N$ has a **countable refinement**. For, we can take the collection of all open balls which are contained in some U_α, which have rational radii, and which are centered at points having only rational coordinates.
- For an open cover $\{V_\alpha\}$ of a subset $X \subset \mathbb{R}^N$, we can write $V_\alpha = U_\alpha \cap X$ for some open subsets U_α of $\mathbb{R}^N$. Then let $\{\tilde{U}_i\}$ be a **countable refinement** of $\{U_\alpha\}$ in R^N, and define $\tilde{V}_i = \tilde{U}_i \cap X$.
- The **charts** are just what we called **local coordinates** and the inverses of charts are what we called **local parametrizations**. One difference is that we required local parametrizations to be diffeomorphisms. For an abstract manifold X, we need the charts to **define what smoothness means** for a map on X. Hence a priori it makes only sense to talk about the smoothness of the change of coordinate maps. A posteriori we can then check that charts are in fact diffeomorphisms.
- Similarly for smooth maps between manifolds. We only know what smoothness of maps between Euclidean spaces means. Hence we need to use the charts to first translate the maps into maps between Euclidean spaces.
- In the abstract definition, we take care of the fact that the images of the charts/local parametrizations overlap. In fact, we use the overlap to define the smooth structure.
- Finally, a chosen collection of charts is called an **atlas** on the manifold. One can show that every manifold has a maximal atlas, i.e. the images of the charts are as “big as possible”.

Here is an important example which we can easily be described with the new definition of an abstract manifold, but for which it is not obvious how we can embedd it into $\mathbb{R}^N$.

(Actually, it is a difficult question how to embedd these guys into $\mathbb{R}^N$ with N as small as possible. In fact, if $n = 2^m$ for some m and if there is an immersion $\mathbb{R}\mathrm{P}^n \to \mathbb{R}^N$, then N must be at least $2^m - 1$. You will learn about the techniques to show this in the Algebraic Topology course.)

Real Projective Space

The **real projective n-space $\mathbb{R}\mathrm{P}^n$** is the set of all straight lines through the origin in R^{n+1}. As a topological space, $\mathbb{R}\mathrm{P}^n$ is the quotient space

$$\mathbb{R}\mathrm{P}^n = (\mathbb{R}^{n+1} \setminus \{0\})/ \sim$$

where the equivalence relation is given by $x \sim y$ if there is a nonzero real number λ such that $x = \lambda y$. This means that a subset V is open in $\mathbb{R}\mathrm{P}^n$ if and only if its preimage $U = \{x \in \mathbb{R}^{n+1} \setminus \{0\} : [x] \in V\}$ is open in $\mathbb{R}^{n+1} \setminus \{0\}$.
Note that since each line through the origin intersects the unit sphere in two (antipodal) points, $\mathbb{R}\mathrm{P}^n$ can alternatively be described as

$$S^n/\sim$$

where the equivalence relation is $x \sim -x$. As a quotient of S^n, we see that $\mathbb{R}\mathrm{P}^n$ is **compact**.
We claim that $\mathbb{R}\mathrm{P}^n$ is an **abstract n-dimensional smooth manifold**.
If $x = (x_0, \ldots, x_n) \in \mathbb{R}^{n+1} \setminus \{0\}$, we write $[x]$ for its equivalence class considered as a point in $\mathbb{R}\mathrm{P}^n$. One also often writes $[x] = [x_0 : \ldots : x_n]$.
For $0 \leq i \leq n$, let

$$V_i := \{[x] \in \mathbb{R}P^n : x_i \neq 0\}.$$

The preimage of V_i in $\mathbb{R}^{n+1}$ is the open subset $\{x \in \mathbb{R}^{n+1} : x_i \neq 0\}$. Hence each V_i is open in $\mathbb{R}\mathrm{P}^n$. By varying i, this gives an open cover of $\mathbb{R}\mathrm{P}^n$ because every representative $(x_0, \ldots, x_n)$ of a point $[x] \in \mathbb{R}\mathrm{P}^n$ must have at least one coordinate $\neq 0$ (otherwise it would be the origin which is excluded).
For each i, we have the maps $\phi_i \colon \mathbb{R}^n \to V_i$

$$(x_0, \ldots, \widehat{x_i}, \ldots, x_n) \mapsto [x_0 : \ldots : x_{i-1} : 1 : x_{i+1} : \ldots : x_n].$$

and $\phi_i^{-1} \colon V_i \to \mathbb{R}^n$

$$[x_0 : \ldots : x_i : \ldots : x_n] \mapsto \frac{1}{x_i}(x_0, \ldots, \widehat{x_i}, \ldots, x_n)$$

where $\widehat{x_i}$ means that x_i is omitted.
Since we use a representative of an equivalence class for the definition of ϕ_i^{-1}, we need to check that the definition is independent of the chosen representative. But if $[x_0 : \ldots : x_i : \ldots : x_n] = [\lambda x_0 : \ldots : \lambda x_i : \ldots : \lambda x_n]$ for some $\lambda \neq 0$, then

$$\begin{aligned}
\phi_i^{-1}([\lambda x]) &= \frac{1}{\lambda x_i}(\lambda x_0, \ldots, \lambda x_{i-1}, \lambda x_{i+1}, \ldots, \lambda x_n) \\
&= \frac{1}{x_i}(x_0, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) = \phi_k^{-1}([x]).
\end{aligned}$$

It is easy to see that ϕ and ϕ_i^{-1} are mutual inverses which are both continuous.
Finally, the change-of-coordinate maps are smooth: For

$$\phi_i^{-1}(V_i \cap V_j) \xrightarrow{\phi_i} V_i \cap V_j \xrightarrow{\phi_j^{-1}} \phi_j^{-1}(V_i \cap V_j)$$

is just

$$(x_0,\ldots,\widehat{x_i},\ldots,x_n) \mapsto \frac{1}{x_j}(x_0,\ldots,x_{i-1},1,x_{i+1},\ldots,\widehat{x_j},\ldots,x_n)$$

which is **smooth** whenever $x_j \neq 0$.

To have such an intrinsic definition of a manifold is important and nice. However, the definition is quite abstract indeed. And, in fact, we are going to show that every abstract smooth manifold can be embedded into Euclidean space and is therefore a manifold for our previous definition. Hence all the machinery we have developed can be applied to abstract manifolds.