NTNU
Norwegian University of
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Week 42: Lecture 2

General continuous-time Markov chains, and introduction to queue theory

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Information

- Reference group meeting will be on October 22.
- Let me or the reference group know if you have feedback.

Section 6.5

Example

Calculate the expected time to reach state 0 when starting in state 2.

Theorem

Let $\{X(t) : t \geq 0\}$ be a birth and death process. Assume that state 0 is the only absorbing state, and that the probability of absorption in state 0 is 1 for all starting states. Then we can find

$$v_i = \mathrm{E}[\min\{t \geq 0 : X(t) = 0\} | X(0) = i], \quad i = 0, 1, \ldots,$$

by solving

$$\begin{aligned} v_0 &= 0, \\ v_i &= \frac{1}{\lambda_i + \mu_i} + \sum_{j \neq i} \mathrm{Pr}\{i \to j\} v_j, \quad j \neq i. \end{aligned}$$

Theorem

The calculation of the probability to be absorbed in state i for a continuous-time Markov chain works exactly like for a discrete-time Markov chain with one-step transition probabilties given by $P_{ij} = \mathrm{Pr}\{i \to j\}$, where $P_{ii} = 0$ for $i = 0, 1, \ldots$.

Section 6.6

Definition (Infinitesimal)

A **continuous-time Markov chain** $\{X(t) : t \geq 0\}$ with state space $\{0, 1, \ldots, N\}$ and stationary transition probabilities is defined through **(transition) rates** $q_{ij} \geq 0$ for $j \neq i$.

Let $q_i = \sum_{j \neq i} q_{ij}$, $i = 0, 1, \ldots, N$, then $\{X(t) : t \geq 0\}$ is defined through

1. $P_{ij}(h) = q_{ij}h + o(h) \quad (\text{as } h \to 0^+)$ for $i \neq j$.
2. $P_{ii}(h) = 1 - q_i h + o(h) \quad (\text{as } h \to 0^+)$
3. $$P_{ij}(0) = \delta_{ij} = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases}$$

Definition (Constructive)

A **continuous-time Markov chain** $\{X(t) : t \geq 0\}$ with state space $\{0, 1, \ldots, N\}$ and stationary transition probabilities is defined through **(transition) rates** $q_{ij} \geq 0$ for $j \neq i$.

Let $q_i = \sum_{j \neq i} q_{ij}$, $i = 0, 1, \ldots, N$, then each time $\{X(t) : t \geq 0\}$ jumps to a new state state i

1. the sojourn time is $\mathrm{Exp}(q_i)$
2. after the sojourn time ends, the jump probabilities to the next state are $\mathrm{Pr}\{i \to j\} = \frac{q_{ij}}{q_i}$ for $j \neq i$.

Notation

We collect all the probability transition functions in a matrix

$$\mathbf{P}(t) = \begin{bmatrix} P_{0,0}(t) & P_{0,1}(t) & \cdots & P_{0,N}(t) \\ P_{1,0}(t) & P_{1,1}(t) & & \vdots \\ \vdots & & \ddots & \\ P_{N,0}(t) & \cdots & & P_{N,N}(t) \end{bmatrix}$$

and we define the **infinitesimal matrix** as

$$\mathbf{A} = \begin{bmatrix} -q_0 & q_{0,1} & \cdots & q_{0,N} \\ q_{1,0}(t) & -q_1 & & \vdots \\ \vdots & & \ddots & \\ q_{N,0} & \cdots & & -q_N \end{bmatrix}$$

Theorem

The stationary distributions of a continuous-time Markov chain with state space $\{0, 1, \ldots, N\}$ and stationary transition probabilities are found by solving

$$\pi_i q_i = \sum_{k \neq i} \pi_k q_{ki}, \quad i = 0, 1, \ldots, N,$$

$$\sum_{k=0}^{N} \pi_k = 1.$$

Example

What is the long-run mean fractions of time spent in state 0, in state 1, and in state 2?

Section 9.1

Theorem (Little's law)

$$L = \lambda W$$

L: *Average number of customers in the system.*

λ: *Rate of arrival to the system.*

W: *Average time spent by a customer in the system.*