Suggested solution, exam TMA4265, Stochastic Modeling, Aug 5, 2019

Task 1

a)

$$P(X_2 = 1|X_0 = 1) = \sum_{k=1}^{3} P(X_1 = k, X_2 = 1|X_0 = 1) = \sum_{k=1}^{3} P(X_1 = k|X_0 = 1)P(X_2 = 1|X_1 = k)$$

Where the Markov property is used;
$P(X_2 = 1|X_1 = k, X_0 = 1) = P(X_2 = 1|X_1 = k)$. This gives

$$P(X_2 = 1|X_0 = 1) = 0.6 \cdot 0.6 + 0.4 \cdot 0.3 + 0 \cdot 0 = 0.48$$

$$P(X_1 = 1|X_0 = 1, X_2 = 1) = \frac{P(X_1 = 1, X_2 = 1|X_0 = 1)}{P(X_2 = 1|X_0 = 1)} = \frac{0.6 \cdot 0.6}{0.48} = 0.75$$

b)

Two possible realizations of the chains are visualized in Figure 1. The chain moves between state 1 and 3 before it gets absorbed in state 2, always from state 3.

The expected number of time steps to absorption $T = \min\{t; X_t = 2\}$, starting in state i is denoted $v_i = E(T|X_0 = i)$. By a first step analysis we get a system of equations:

$$\begin{aligned} v_1 &= 1 + v_1 0.3 + v_3 0.7 \\ v_2 &= 0 \\ v_3 &= 1 + v_1 0.5 + v_2 0.1 + v_3 0.4 \end{aligned}$$

This simplifies to

$$\begin{aligned} v_1 &= 1/0.7 + v_3 \\ v_3 &= 1 + (1/0.7 + v_3)0.5 + v_3 0.4 \end{aligned}$$

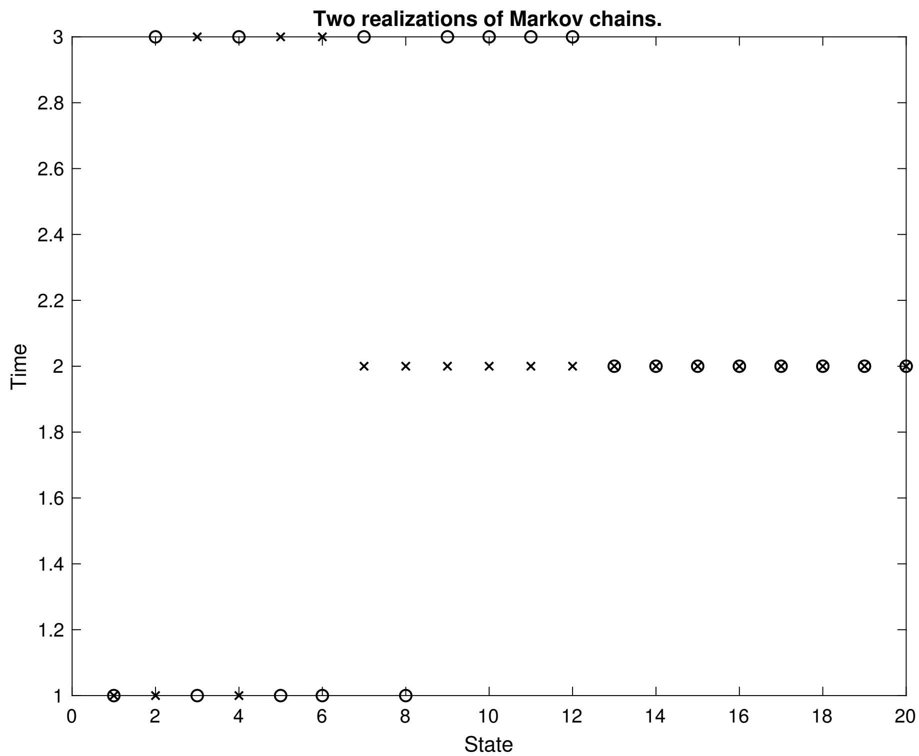

Figure 1: Markov chain realizations of the Markov chain, plotted as a function of time.

With solution $v_3 = 17.1$ and $v_1 = 18.6$.

Task 2

This task is a random walk (gambler's ruin problem). See Pinsky and Karlin, Sect 3.5.3.

a)

Player A can move one up (Prob p) or one down (Prob q) at each time.

$$P(X_2 = i | X_0 = i) = P(X_2 = i, X_1 = i{+}1 | X_0 = i) + P(X_2 = i, X_1 = i{-}1 | X_0 = i)$$

$$P(X_2 = i | X_0 = i) = pq + qp = 2qp = 0.48$$

In 4 time steps there are 6 combinations of moves that brings player A

back to state i. All have two down and two up (prob p^2q^2). This gives:

$$P(X_4 = i|X_0 = i) = 6p^2q^2 = 0.34$$

In 10 steps the player must take 5 steps up and 5 steps down. The combinatorial numbers of possible up/down moves getting back to i are $\binom{10}{5}$ $= 252$, giving

$$P(X_{10} = i|X_0 = i) = 252p^5q^5 = 0.20$$

The return probability is connected with calculations used for recurrence results. (See Pinsky and Karlin, Sect 4.3.3.)

b)

Define $\eta = q/p = 0.66$. $u_i = P(\text{Absorption in state 0})$, starting in $X_0 = i$. By a first step analysis:

$$\begin{aligned} u_0 &= 1 \\ u_i &= pu_{i+1} + qu_{i-1} \\ u_N &= 0 \end{aligned}$$

This means that $p(u_{i+1} - u_i) = q(u_i - u_{i-1})$. By summing all elements from 1 to i we have

$$u_i = 1 + (1 + \frac{q}{p} + \ldots + \frac{q^{i-1}}{p^{i-1}})(u_1 - 1)$$

and the formula for geometric series can be used, with kvotient $\eta = q/p$.

But $u_N = 0$, and then we can solve for u_1 and subsequently for u_i getting

$$u_i = \frac{\eta^i - \eta^N}{1 - \eta^N}$$

(See also derivations in Sect 3.6 of Pinsky and Karlin book.)

c)

By a fair game, we have chance of winning equal to $u_i = 1/2$.

Solving for i we get

$$i = \frac{\log(\eta^{10} + 0.5(1 - \eta^{10}))}{\log(\eta)} = 1.63$$

Here $u_1 = 0.65$, $u_2 = 0.45$, so starting at $i = 1$ would favour player B, while starting at $i = 2$ or higher would favour player A.

Task 3

a)

Let N be the number of birds he sees; $N \sim \text{Poisson}(\mu)$. Let X be the number of birds he hits; $X|N \sim \text{Binomial}(N, p)$. Marginalizing over N gives

$$P(X = x) = \sum_n \frac{\mu^n}{n!} e^{-\mu} \binom{n}{x} p^x (1-p)^{n-x}$$

$$P(X = x) = \frac{(\mu p)^x}{x!} e^{-\mu} \sum_i \frac{(\mu(1-p))^i}{i!} = \frac{(\mu p)^x}{x!} e^{-\mu p}$$

which is a Poisson distribution with parameter μp. This means that the original number with intensity μ is thinned at random with probability p.

(See Example in Sect 2.1 of Pinsky and Karlin.)

b)

$$P(X(t) = 0) = \frac{(\lambda t)^0}{0!} e^{-\lambda t} = e^{-\lambda t}$$

We get $P(X(4) = 0) = e^{-0.75 \cdot 4} = 0.05$.

There are 4 hours from 8 to 12 and 2 hours from 8 to 12. The time intervals of the Poisson process are independent and intervals of equal length are equally likely to contain a single event. This means that one single event has chance 0.5 of happening in the first 2 hours. With two events this gives $P(X(2) = 2|X(4) = 2) = 0.5^2 = 0.25$.

The probability can also be derived from the formula of the Poisson distribution and assumption of independent increments in time intervals (see next point).

c)

The intensity is now inhomogeneous. $X(2)$ is Poisson with parameter $\Lambda(2) = \int_0^2 (0.8 - 0.1t)dt = 0.8 \cdot 2 - 0.05 \cdot 2^2 = 1.4$, $X(4)$ is Poisson with parameter $\Lambda(4) = \int_0^4 (0.8 - 0.1t)dt = 0.8 \cdot 4 - 0.05 \cdot 4^2 = 2.4$, and $X(4) - X(2)$ is Poisson parameter $\Lambda(4) - \Lambda(2) = 1$ and independent of $X(2)$.

From basic principles, using independent increments:

$$P(X(2)=2|X(4)=2)=\frac{P(X(2)=2)P(X(4)-X(2)=0)}{P(X(4)=2)}$$

The probabilities are $P(X(2)=2)=\frac{1.4^2}{2}e^{-1.4}=0.24$, $P(X(4)=2)=\frac{2.4^2}{2}e^{-2.4}=0.26$ and $P(X(4)-X(2)=0)=\frac{1^0}{0!}e^{-1}=0.37$. Then,

$$P(X(2)=2|X(4)=2)=\frac{0.24\cdot 0.37}{0.26}=0.34$$

Task 4

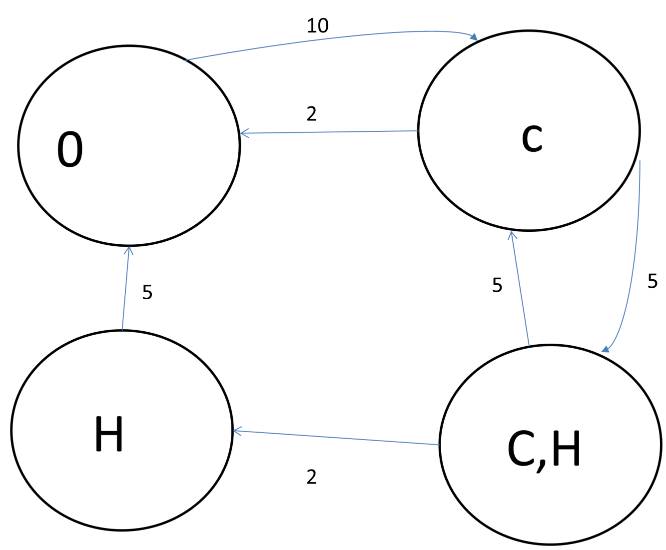

Figure 2: Transition diagram between the states, with rates indicated.

The long-term distribution, defined by probabilities π_0, π_C, π_H, π_{CH} are determined by setting long-term rates in and out of states equal:

$$\begin{aligned}
\pi_0 10 &= \pi_C 2 + \pi_H 5 \\
\pi_C(2+5) &= \pi_0 10 + \pi_{CH} 5 \\
\pi_{CH}(2+5) &= \pi_C 5 \\
\pi_H 5 &= \pi_{CH} 2 \\
1 &= \pi_0 + \pi_C + \pi_H + \pi_{CH}
\end{aligned}$$

Here, $5\pi_H = 2\pi_{CH} = (10/7)\pi_C$, and the system is reduced to only two unknowns in 2 equations:

$$\begin{aligned}
\pi_0 10 &= \pi_C 2 + \pi_C(10/7) \\
1 &= \pi_0 + \pi_C + (2/7)\pi_C + (5/7)\pi_C
\end{aligned}$$

The solution is

$$\pi_C = 1/((24/70) + 1 + (2/7) + (5/7)) = 0.43$$

and for the others $\pi_0 = 0.15$, $\pi_H = 0.12$ and $\pi_{CH} = 0.30$.

Task 5

Without knowing the price at $t = 50$, the joint distribution of $X(25)$ and $X(50)$ is Gaussian distributed with means equal to 9, $\text{var}(X(25)) = 0.05^2 25 = 0.0625$, $\text{var}(X(50)) = 0.05^2 50 = 0.125$ and $\text{cov}(X(25), X(50)) = 0.0625$. The conditional distribution is then Gaussian with mean

$$E(X(25)|X(50)) = 9 + (0.0625/0.125)(9.5 - 9) = 9.25$$

$$Var(X(25)|X(50)) = 0.0625 - (0.0625/0.125)0.0625 = 0.312 = 0.176^2$$

$$P(X(25) > 9|X(50) = 9.5) = 1 - \Phi((9 - 9.25)/0.176) = 0.922$$