titleLinear Methods Exams

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Contents

1 Exam 18h

1.1 Problem 1

Determine whether the followeing statements are true or false. if the statement are true, no further explanation is required. If the statement is false, give a counter example.

1. The Kerner of a bounded linear operator $T : X \mapsto Y$ between normed spaces X and Y is closed.

   Answer. True

2. The range of a bounded linear operator $T : X \rightarrow Y$ between normed spaces X and Y is closed.

   Answer. False. Lets assume that X and Y is closed. Then is this true.

3. The dual space $X^{'}$ of a normed space is a Banach Space.

   Answer. True.

4. A closed subspace of a Banach Space is itself a Banish Space.

   Answer. True

1.2 Problem 2

Let $(x_k)_{k\in\mathbb{N}}$ be a sequence in a normed space $(X, \|.\|)$.

a) Prove that $(x_k)_{k\in\mathbb{N}}$ is a Cauchy sequence, then $(x_k)_{k\in\mathbb{N}}$ is bounded.

   Answer. We need to show that it exist $d(x_m, x_n) < \epsilon$. First let $x_n \mapsto x$, then does is this true $d(x_n, x) < \frac{\epsilon}{2}$ for an $n \geq N$. Using the triangle equality can we determine

   $$d(x_n, x_m) = d(x, x_m) + d(x, x_n) < \epsilon$$

   This is then true.

b) Let $\|.\|_a$ and $\|.\|_b$ be equivalent norms on X and let $x \in X$. Prove that $(x_k)_{k\in\mathbb{N}}$ converges to x in $(X, \|.\|_a)$ if and only if $(x_k)_{k\in\mathbb{N}}$ converges to x in $(X, \|.\|_b)$.

Answer.

Proof. Let $x_n \mapsto x$ and $x_m \mapsto x$. Then is $\|x_n - x\|_a < \frac{\epsilon}{2}$ for an $n > N_a$. This also holds for x_m such that $\|x_m - x\|_b < \frac{\epsilon}{2}$ for an $m > N_b$. If we let $m, n > \max(N_a, N_b)$ then can we conclude that

$$\|x_n - x\|_a + \|x_m - x\|_b < \epsilon.$$

Which proves that if $\|.\|_b$ is converging does this hold for $\|.\|_a$ for all $(x_n)_{n\in\mathbb{N}}$ $\square$

1.3 Problem 3

Let $\left(\ell^2, \langle ., . \rangle\right)$ be the inner product space of complex-valued sequences $x \in (x_k)_{k\in\mathbb{N}}$ equipped with the standard inner product

$$\langle x, y \rangle = \sum_{k=1}^{\infty} x_k \overline{y_k} \quad \text{for} \quad x, y \in \ell^2. \tag{1}$$

and let $T : \ell^2 \mapsto \ell^2$ be the multiplication operator given by

$$Tx = \left(i^k x_k / k\right)_{k\in\mathbb{N}}$$

where $i = \sqrt{-1}$.

a) Show that T is a bounded linear operator on ℓ^2, and determine the operator norm $\|T\|$.

Answer. We want to show that T is Cauchy. Let $(x_n)_{n\in\mathbb{N}}$ be a Cauchy sequence and let $\epsilon > 0$ such that $\|x_n - x\| < \frac{\epsilon}{2}$ for a N. By observing that

$$Tx_m \mapsto Tx$$

can we use the argument such that $\|Tx_m - Tx\| = \|T(x_m - x)\| < \frac{\epsilon}{2}$ if $m > M$. Applying the triangle in equality can it be shown that

$$\|Tx_m - Tx_n\| \leq \|Tx_m - Tx\| + \|Tx_n - Tx\| < \epsilon \quad n, m = \max(N, M)$$

And then shows that T is bounded.
The operator norm of T is

$$\|T\| = \sup_{\substack{x_k \in X \\ \|x_k\|=1}} \frac{\|Tx_k\|}{\|x_k\|} = \|\frac{i^k}{k}\| = \frac{1}{k}$$

b) Determine the adjoint operator T^*. State what it means for an operator to be normal, and determine whether or not T is normal.

> **Answer.** The adjoint operator should have this property, $\langle T^* y, y\rangle = \langle y, Tx\rangle$.

c) Show that the range of T is dense in ℓ^2.

1.4 Problem 4

Let

$$A = \begin{bmatrix} 2 & 2 \\ 2 & 2 \\ -1 & -1 \end{bmatrix}$$

a) Find a singular value decomposition of A.

b) Find the pseudoinverse A^+ of A and use it to find the best approximation for a solution of the inconsistent system.

$$\begin{aligned} 2x_1 + 2x_2 &= 3 \\ 2x_1 + 2x_2 &= 4 \\ -x_1 - x_2 &= -4. \end{aligned}$$

1.5 Problem 5

Find $a, b \in \mathbb{C}$ such that

$$\int_0^{2\pi} |t - a\sin(t) - b\sin(2t)|^2 \, dt$$

Tip: You might find the formula $(\sin(t))^2 = \frac{1-\cos(2t)}{2}$ useful.

1.6 Problem 6

a) Show that if $X \neq \emptyset$ is a complete metric space, and $T : X \to X$ is a mapping such that

$$T^k = T \cdot T \cdot \ldots \cdot T$$

Is a contraction for some natural number $k > 1$, then T has a unique fixed point.

b) Consider the space of continuous functionsa $C[0,1]$ equppised with the metric induced by the supremenum norm

$$d(f,g) = \|f - g\|_\infty \sup_{0 \le t \le 1} |f(t) - g(t)|$$

and let $T : C[0,1] \to C[0,1]$ be given by

$$(Tf)(t) = 1 - \int_0^t f(s)\,ds, \quad 0 \le t \le 1.$$

Show that T has a unique fixed point, and use iteration to find it starting with $f_0(t) = 1$

Tip: You can use the results from a) even if you did not solve this problem.

2 Appendix

2.1 Sequences in metric spaces and normed spaces

Definition 2.1 (Norm). *Criterias for norms*

(i) $\|cx\| = c\|x\|$

(ii) $\|xy\| \leq \|x\|\|y\|$

(iii) $\|x+y\| \leq \|x\| + \|y\|$

(iv) $\|x\| = 0 \quad \textit{only if} \quad x = 0$

Theorem 2.1 (Inequalities). *This inequliaties hold*

- *Holder Inequality*

$$\sum_{n=1}^{\infty} |\chi_n \mu_n| \leq \left(\sum_{k=1}^{\infty} |\chi_k|^p \right)^{\frac{1}{p}} \cdot \left(\sum_{m=1}^{\infty} |\mu_m|^q \right)^{\frac{1}{p}}$$

- *Cauchy Schwarts Inequality*

$$\sum_{n=1}^{\infty} |\chi_n \mu_n| = \sqrt{\sum_{k=1}^{\infty} |\chi_k|^2} + \sqrt{\sum_{j=1}^{\infty} |\mu_k|^2}$$

- *Minowsky Inequality*

$$\left(\sum_{n=1}^{\infty} |\chi_n + \mu_n|^p \right)^{\frac{1}{p}} \leq \left(\sum_{k=1}^{\infty} |\chi_k| \right)^{\frac{1}{p}} + \left(\sum_{i=1}^{\infty} |\mu_i|^p \right)^{\frac{1}{p}}$$

Definition 2.2 (Sequence). *Let (X, d) be a metric space. A sequence $(x_n)_{n \in \mathbb{N}}$ in X is said to **converge to** $x \in X$ for every $\epsilon > 0$ one can find $N = N(\epsilon) \in \mathbb{N}$ such that*

$$d(x_n, x) < \epsilon.$$

*whenever $b \geq N$. The element x is called the **limit** of the sequence $(x_n)_{n \in \mathbb{N}}$. In particular, in $(X, \|.\|)$ is a normed space. then $(x_n)_{n \in \mathbb{N}}$ converge to $x \in X$ for every $\epsilon > 0$ one can find $N = N(\epsilon) \in \mathbb{N}$ such that*

$$\|x - x_n\| < \epsilon.$$

whenever $n \geq N$..

Definition 2.3. *Given a point* $x_0 \in X$ *and a real number* $r > 0$*, we define three types of sets:*

(i) $B(x_0; r) = \{x \in X \mid d(x, x_0) < r\}$ ***(Open ball)***

(ii) $\hat{B}(x_0; r) = \{x \in X \mid d(x, x_0) \leq r\}$ ***(Closed ball)***

(iii) $S(x_0; r) = \{x \in X \mid d(x, x_0) = r\}$ ***(Sphere)***

Here is x_0 *called the center and* r *the radius. Remark that* $S(x_0, r) = \hat{B}(x_0, r) - B(x_0, r)$.

Definition 2.4 (Open and Closed Set)**.** *A subset* M *of a metric space* X *is said to be open if it contains a ball around each of its points. A subset* K *of* X *is said to be closed if its complement (in* X *) is open, that is,* $K^c = X - K$ *is open.*

Remark. A complement set is defined such that $A^c = U \setminus A$ or more formally $A^c = \{x \in U \mid x \notin A\}$

Lemma 2.1. *A convergent sequence in a metric space* (X, d) *is bounded.*

Definition 2.5 (Dense Set)**.** *Formally,* $S \subset X$ *is dense in* X *if, for any* $\epsilon > 0$ *and* $x \in X$*, there is some* $s \in S$ *such that* $\|x - s\| < \epsilon$*. An equivalent definition is that* S *is dense in* X *if, for any* $x \in X$*, there is a sequence* $\{x_n\} \subset S$ *such that*

$$\lim_{n \to \infty} x_n = x$$

Definition 2.6. *The* ***completeness*** *axiom says that every nonempty subset of* $\mathbb{R}$ *that is bounded above has a supremum. Equivalently is that nonempty subset that is bounded below as a infimum ("greated lower bound").*

Definition 2.7. ***Lots of definitions.*** *Given a metric space X , we make the following definitions*

- *If $x \in X$ and $r > 0$ then the* ***open ball*** *in X is centered at x with radius r is the set*

$$B_r(x) = \{y \in X : d(x, y) < r\}$$

- *A set $U \subseteq X$ is* ***open*** *if for each $x \in U$ there exist a radius $r > 0$ such that $B_r \subseteq U$.*

- *The* ***topoplogy*** *of X is the collection of all open subsets of X .*

- *The* ***interior*** *of a set $E \subseteq X$ is the largest open set E° that is contained in E. Explicity, $E^{\circ} = \cup \{U : U \quad \text{is open and} \quad U \subseteq E\}$*

- *A set $E \subseteq X$ is* ***closed*** *if uts complement $X \setminus E$ is open.*

- *A set $E \subseteq X$ is* ***bounded*** *if it is contained in some open ball, i.e, there exists some $x \in X$ and some $r > 0$ such that $E \subseteq B_r(x)$*

- *A point $x \in X$ is a* ***accumulation point*** *of a set $E \subseteq X$ if there exist points $x_n \in E$ with all $x_n \neq x$ such that $x_n \to x$*

- *A point $x \in X$ is a* ***boundary point*** *of a set $E \subseteq X$ if for every $r > 0$ we have both $B_r(x) \cap E \neq \emptyset$. The set of all boundary point of E is called the* ***boundary*** *if E, and it is denoted by ∂E .*

- *The* ***closure*** *if a set $E \subseteq X$ is the smallest closed set $\overline{E}$ that containst E. Explicity, $\overline{E} = \cap \{F : F \text{ is closed and } E \subseteq F\}$*

- *$E \subseteq X$ is* ***dense*** *in X if $\overline{E} = X$*

- *We say that X is* ***separable*** *if there exists a countable sebset E that is dense in X.*

2.2 Linear Operator

Definition 2.8. *A linear operator T is an operator such that*

1. *the domain $\mathbb{D}(T)$ of T is a vector space and the range $R(T)$ lies in a vector space over the same field.*

2. *$\forall x, y \in \mathbb{D}(T)$ and scalars α*

$$T(x + y) = Tx + Ty \quad \text{and} \quad T(\alpha x) = \alpha Tx. \tag{2}$$

Definition 2.9 (Bounded Linear Operator)**.** *An linear operator $T : X \mapsto Y$ is bounded if $\forall x \in X$ and $c > 0$ such that $\|Tx\| = \|T\|\|x\| \leq c\|x\|$*

Remark. What is the smallest possible c such that $\|Tx\| \leq c\|x\|$ still hold for all non-zero $x \in \mathbb{D}(T)$? (We can leave out $x = 0$ since $Tx = 0$ for $x = 0$) By division,

$$\frac{\|Tx\|}{\|x\|} \leq c.$$

and this shows that c must be at least as big as the supremum of the expression on the left taken over the range $\mathbb{D}(T) - \{0\}$. Hense the answer to our question is that the smallest possible c is that supremum. This quantity denoted by $\|T\|$, thus

$$\|T\| = \sup_{\substack{x \in \mathbb{D}(T) \\ x \neq 0}} \frac{\|Tx\|}{\|x\|}$$

$\|T\|$ is called the **norm** of the operator T. If the range $\mathbb{D}(T) = \{0\}$, we define $\|T\| = 0$. Note that with $c = \|T\|$ is

$$\|Tx\| \leq \|T\|\|x\|$$

which is a quite frequently used formula.

Lemma 2.2. *Let T be a bounded linear operator. Then is this true,*

(i)

$$.\|T\| = \sup_{\substack{x \in \mathbb{D}(T) \\ x \neq 0}} \frac{\|Tx\|}{\|x\|} = \sup_{\substack{x \in \mathbb{D}(T) \\ x = 1}} \|Tx\|$$

(ii) The norm satisfy general norm aksioms.

Proof. (i) Let $\|x\| = a$ and define $y = \frac{x}{a}$. Using this definition can we see that $\|y\| = 1$. Hense can we rewrite the definition.

$$\sup_{\substack{x \in \mathbb{D}(T) \\ x \neq 0}} \frac{\|Tx\|}{\|x\|} = \sup_{\substack{x \in \mathbb{D}(T) \\ x \neq 0}} \frac{\|Tx\|}{a} = \sup_{\substack{x \in \mathbb{D}(T) \\ x \neq 0}} \|\frac{Tx}{a}\| = \sup_{\substack{y \in \mathbb{D}(T) \\ y=1}} \|Ty\|$$

(ii) We need to prove that it satisfy the criteria $\|cT\| = c\|T\|$ and $\|T_1 + T_2\| \leq \|T_1\| + \|T_2\|$.

$$\begin{aligned} \|cT\| &= \sup_{\substack{y \in \mathbb{D}(T) \\ \|y\|=1}} \|Tcy\| = \sup_{\substack{y \in \mathbb{D}(T) \\ \|y\|=1}} c\|Ty\| \\ &= c\|T\|. \end{aligned}$$

$$\begin{aligned} \|T_1 + T_2\| &= \sup_{x \in \mathbb{D}(T), \|x\|=1} \|\left(T_1 x + T_2 x\right)\| \leq \sup_{x \in \mathbb{D}(T), \|x\|=1} \|T_1 x\| + \|T_2 x\| \\ &= \|T_1\| + \|T_2\|. \end{aligned}$$

□

Theorem 2.2. *Let $T : \mathbb{D} \mapsto Y$ be a linear operator where $\mathbb{D} \subset X$ and X, Y are normed spaces, then*

1. *T is continous if and only if T is bounded.*
2. *If T is continous at a single point, T is continious.*

Proof. 1. For $T = 0$ the statement is trivial. Let $T \neq 0$. Then $\|T\| \neq 0$. We Assume T To be bounded and consider any $x_0 \in \mathbb{D}(T)$. Let any $\epsilon > 0$. Then, since T is linear, for every $x \in \mathbb{D}(T)$ such that

$$\|x - x_0\| < \delta \quad where \quad \delta = \frac{\epsilon}{\|T\|}$$

we obtain

$$\|Tx - Tx_0\| = \|T\left(x - x_0\right)\| \leq \|T\|\|x - x_0\| < \|T\|\delta = \epsilon$$

. Since $x_0 \in \mathbb{D}(T)$ was arbitary, this shows that T is continous.

Conversely, assume that T is continous at an arbitary $x_0 \in \mathbb{D}(T)$ then, given any $\epsilon > 0$, there is a $\delta > 0$ such that

$$\|Tx - Tx_0\| \leq \epsilon \quad \text{for all } x \in \mathbb{D}(T) \text{ satisfying} \quad \|x - x_0\| \leq \delta.. \qquad (3)$$

We now take any $y \neq 0$ in $\mathbb{D}(T)$ and set

$$x = x_0 + \frac{\delta}{\|y\|} y. \quad \text{then} \quad x - x_0 = \frac{\delta}{\|y\|} y.$$

Hence $\|x - x_0\| = \delta$, so that we may use the result in (3) . Since T is linear we have

$$\|Tx_0 - Tx\| = \|T(x - x_0)\| = \|T\left(\frac{\delta}{\|y\|} y\right)\| = \frac{\delta}{\|y\|}\|Ty\|$$

and this implies

$$\frac{\delta}{\|y\|}\|Ty\| \leq \epsilon. \quad \text{Thus} \quad \|Ty \leq \frac{\epsilon}{\delta}\|\|y\|.$$

This can be written $\|Ty\| \leq \|y\|$, where $c = \frac{\epsilon}{\delta}$ and shows that T is bounded.

2. Continuity of T at a point implies boundedness of T by the second part of the proof of (a), which in turn implies boundedness of T by (a).

□

2.3 Banach Spaces

Definition 2.10 (Cauchy Sequence)**.** *Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in the metric space (X, d). We say that $(x_n)_{n\in\mathbb{N}}$ is **Cauchy Sequence** if for any $\epsilon > 0$ there exist an $N \in \mathbb{N}$ such that*

$$d(x_n, x_m) < \epsilon.$$

In particular if $(x_n)_{n\in\mathbb{N}}$ is a sequence in the normed space $(X, \|.\|)$, then $(x_n)_{n\in\mathbb{N}}$ is Cauchy if for any $\epsilon > 0$ there exist an $N \in \mathbb{N}$ such that

$$\|x_n - x_m\| < \epsilon, \quad s.t. \quad n, m \geq N.$$

In an inner product space $(X, \langle ., . \rangle)$, we say that a sequenxe $(x_n)_{n\in\mathbb{N}}$ is Cauchy if the sequence is Cauchy with respect to the indeuced norn $\|x\| := \langle x, x \rangle^{\frac{1}{2}}$.

Lemma 2.3. *Any Cauchy sequence in (X, d) is bounded.*

Proof. Let $(x_n)_{n\in\mathbb{N}}$ be a Cauchy sequence. Then there exist $N \in \mathbb{N}$ such that for all $m, n \geq N$ we have

$$d(x_m, x_n) < 1.$$

In particular, we have

$$d(x_N, x_m) < 1 \quad \forall \quad m \geq N.$$

Or equivalently $x_m \in B_1(x_N)$ for all $m \geq N$. Now let

$$r = max\{1, d(x_1, x_N), d(x_2, x_N), \dots, d(x_{N-1}, x_N)\}.$$

Then for any $n \in \mathbb{N}$ we have $x_n \in B_{r+1}(x_N)$ so $(x_n)_{n\in\mathbb{N}}$ is bounded.

□

Remark. A set is **closed** if the set contains all of its boundary points (the closure of the set is equal to the set). There are some other definitions for closed also. A set is **bounded** if the distance between any two points in the set is less then some finite constant. A set in $\mathbb{R}^n$ is bounded if all of the points are contained within a disc of finite radius.

Definition 2.11 (Completeness)**.** *A sequence $(x_n)_{n\in\mathbb{N}}$ in a metric space $X = (X, d)$ is said to be Cauchy (or fundemental) if for every $\epsilon > 0$ there is an $N = N(\epsilon)$ such that $d(x_m, x_n) < \epsilon$ for every $m, n \geq N$. The space X is said to be complete if every Cauchy sequence in X converges (that is, has a limit which is an element of X).*

Remark (Procedure for Completeness proofs). To prove completeness do we choose an arbitary Cauchy sequence $(x_n)_{n\in\mathbb{N}}$ in X and show that it does converge in X. They often have the same pattern.

1. Contruct an element x (to be used as an limit).
2. Prove that x is in the space considered.
3. Prove convergence $x_n \mapsto x$

Theorem 2.3 (Convergent sequences)**.** *Every convergent sequences in a metric space is a Cauchy Sequence.*

Proof. Let $x_n \mapsto x$ for $x \in X$, then is for an $N = N(\epsilon)$

$$d(x_n, x) < \frac{\epsilon}{2} \quad \text{for any} \quad n > N.$$

To prove that this is Cauchy can we use the triangulation theorem such that

$$d(x_n, x_m) \leq d(x, x_n) + d(x, x_m) < \epsilon \quad \text{such that} \quad m, n \geq N(\epsilon)$$

This proves that $(x_n)_{n\in\mathbb{N}}$ is Cauchy.

□

Definition 2.12 (Banach Space and Hilbert Space). *A metric space (X, d) is said to be complete if every Cauchy sequence $(x_n)_{x_n \in \mathbb{N}} \in X$ converges to a limit $x \in X$. A complete normed space $(X, \|.\|)$ is classed a Banach Space. Similary, a complete inner product space $(X, \langle ., . \rangle)$ is called a Hilbert space.*

Theorem 2.4. *Let (f_n) be a sequence of continious functions on $[a, b]$ which converges uniformly to a limit function f. Then f is continious on $[a, b]$.*

Proof. We want to show that for any fixed $y \in [a, b]$ and $\epsilon > 0$ we can find a $\delta > 0$ such that

$$\|x - y\| < \delta \qquad \Longrightarrow \qquad \|f(x) - f(y)\| < \epsilon$$

By the uniformly convergence (f_n) to f, there exist an N such that

$$\|f_n(x) - f(x)\| < \epsilon \quad \text{for all} \quad x \in [a, b], n \geq N.$$

Moreover, the function f_n is continious, so there exist a $\delta > 0$ such that

$$\|x - y\| < \delta \qquad \Longrightarrow \qquad \|f_N(x) - f_N(y)\| < \frac{\epsilon}{3}.$$

It follow that

$$\|f(x) - f(y)\| \leq \|f(x) - f_N(x)\| - \|f_N(x) - f_N(y)\| + \|f_N(y) - f(y)\| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.$$

whenever $\|x - y\| < \delta$ ∎

Theorem 2.5. *$(C[a, b], \|.\|_\infty)$ is a Banach Space*

Proof. (i) **Find a candidate for the limit**

Fix $x \in [a, b]$ and note that

$$\|f_n(x) - f(x)\| \leq \|f_n - f_m\|_\infty = \max_{a \leq x \leq b} \|f_n(x) - f_m(x)\|.$$

This if (f_n) is a Cauchy sequence in $(C[a, b], \|.\|_\infty)$, then $(f_n(x))_{n \in \mathbb{N}}$ is a Cauchy Sequence in $(\mathbb{R}, \|.\|)$. Since $(\mathbb{R}, \|.\|)$ is complete, there exist a point $f(x) \in \mathbb{R}$ such that $f_n(x) \mapsto f(x)$. A reasonable candidate for the limit is the function f given by the pointwise limits.

(ii) **Show that** $f \in C[a,b]$

We observe that the convergence of f_n to f is not onlypointwise, but in fact uniform; Since (f_n) is Cauchy, there is for every $\epsilon > 0$ an integer N such that

$$\|f_n - f\|_\infty = \max_{a \leq x \leq b} \|f_n(x) - f_m(x)\| < \frac{\epsilon}{2}, \quad n, m \geq N$$

In particular, this hold as $m \mapsto \infty$, and we get

$$\max_{a \leq x \leq b} \|f_n(x) - f(x)\| \leq \frac{\epsilon}{2} < \epsilon, \quad n \geq N \quad . \tag{4}$$

Thus, f_n converges uniformly to f on the interval $[a,b]$, and it follows by Theorem 3.13 (linear method lecture notes) that $f \in C[a,b]$.

(iii) **Show that** $f_n \mapsto f$

Follows from (4)

□

2.4 Banach Fixed Point

Definition 2.13 (Contraction)**.** *Let $X = (X,d)$ be a metric space. A mapping $T : X \mapsto X$ is called a* ***contraction*** *on X if there is a positive real number $\alpha < 1$ such that for all $x, y \in X$*

$$d(Tx, Ty) \leq \alpha d(x,y) \quad , \alpha < 1$$

Geometrically this means that any point x and y have images that are closer together than those points x and y; more precisely, the ratio

$$\frac{d(Tx, Ty)}{d(x,y)}$$

does not exceed a constant α which is strictly less than 1.

Theorem 2.6 (Banach Fixed Point Theorem)**.** *Consider a metric space $X = (X,d)$, where $X \neq \emptyset$. Suppose that X is complete and let $T : X \mapsto X$ be a contraction on X. Then T has precisely one fixed point.*

Proof. We constrict a sequence (x_n) and show that it is Cauchy so that it converges in the complete space X, and then we prove that its limit x is a fixed point on T and T has no further fixed points. This is the idea of the proof.

We choose any $x_0 \in X$ and define the "iterative sequence " (x_n) by

$$x_0, \quad x_1 = Tx_0, \quad x_2 = Tx_1 = T^2x_0 \quad \ldots \quad x_n = T^n x_0, \quad \ldots . \tag{5}$$

Clearly, this is the sequence of the image of x_0 under repeated application of T. We show that (x_n) is Cauchy by the contraction definition and (5) ,

$$
\begin{aligned}
d\left(x_{m+1}, x_m\right) &= d\left(Tx_m, Tx_{m-1}\right) && (6)\\
&\leq \alpha d\left(x_m, x_{m-1}\right) && (7)\\
&= \alpha d\left(Tx_{m-1} . Tx_{m-2}\right) && (8)\\
&\leq \alpha^2 d\left(x_{m-1}, x_{m-2}\right) && (9)\\
\ldots &= \alpha^m d\left(x_1, x_0\right) && (10)\\
&. && (11)
\end{aligned}
$$

Hense by the triangle inequality and the formula for the sum of a geometric progression we obtain for $n \geq m$.

$$
\begin{aligned}
\left(x_m, x_n\right) &\leq d\left(x_m, x_{m+1}\right) d\left(x_{m+1}, x_{m+2}\right) + \ldots + d\left(x_{n-1}, x_n\right)\\
&\leq \left(\alpha^m + \alpha^{m+1} + \ldots + \alpha^{n-1}\right) d\left(x_0, x_1\right)\\
&= \alpha^m \frac{1-\alpha^{n-m}}{1-\alpha} d\left(x_0, x_1\right)
\end{aligned}
$$

.

Since $0 < \alpha < 1$, in the numerator we have $1 - \alpha^{n-m} < 1$. Consequently

$$
d\left(x_m, x_n\right) \leq \frac{\alpha^m}{1-\alpha} d\left(x_0, x_1\right), \quad n > m.. \tag{12}
$$

On the right is $0 < \alpha < 1$and $d\left(x_0, x_1\right)$ is fixed, so that we can make the right-hand side as small as we please by taking m sufficiently large (and $n > m$). This proves that (x_m) is Cauchy. Since X is complete, (x_m) converges, say, $x_m \mapsto x$. We show hat this limit x is a fixed point of the mapping T.

From the triangle inequality and the contraction theorem we have

$$
\begin{aligned}
d\left(x, Tx\right) &= d\left(x, x_m\right) + d\left(x_m, Tx\right) && (13)\\
&\leq d\left(x, x_m\right) + \alpha d\left(x_{m-1}, x\right). && (14)
\end{aligned}
$$

and can make the sum in the second line smaller than any preassigned $\epsilon > 0$ because $x_m \mapsto x$. We conclude that $d\left(x, Tx\right) = 0$, so that $x = Tx$. This shows that x is a fixed point of T .

x is the only fixed point of T because from $Tx = x$ and $T\hat{x} = \hat{x}$ we obtain by

$$
d\left(\hat{x}, x\right) = d\left(T\hat{x}, Tx\right) \leq \alpha d\left(\hat{x}, x\right)
$$

Which implies $d\left(\hat{x}, x, \right) = 0$ since $\alpha < 1$. Hense $x = \hat{x}$ and the theorem is proved.

□

2.5 Hilber Spaces

Definition 2.14 (Separable). *A metric space is said to be **separable** if it contains a countable dense set*

$$X \quad \textit{separable} \leftrightarrow (x_n)_{n\in\mathbb{N}}\,\mathbb{C} \quad \textit{such that} \quad \overline{(x_n)_{n\in\mathbb{N}}} = X.$$

Definition 2.15 (Inner product space, Hilbert space). *An inner product space (or pre-Hilbert space) is a vector space X with an inner product defined on X. A Hilbert space is a complete inner product space. Here, an **inner product** on X is a mapping from $X \times X$ into the scalar field K of X ; that is, with every pair of vectors x and y there is assiciated a scalar which is written*

$$\langle x, y\rangle$$

and is called the inner product of x and y such that for all vectors x, y, z and scalars α we have

IP1) $\langle x+y, z\rangle = \langle x, z\rangle + \langle y, z\rangle$

IP2) $\langle \alpha x, y\rangle = \alpha \langle x, y\rangle$

IP3) $\langle x, y\rangle = \overline{\langle y, x\rangle}$

IP4) $\langle x, x\rangle \geq 0 \quad \textit{and} \quad \langle x, x\rangle = 0 \quad \implies \quad x = 0$

Definition 2.16 (Hilbert-adjoint operator). *Let $T : H_1 \mapsto H_2$ be a bounded linear operator, where H_1 and H_2 then the Hilbert adjoint operator T^* of T is the operator*

$$T^* : H_2 \mapsto H_1.$$

Such that for all $x \in H_1$ and $y \in H_2$

$$\langle Tx, y\rangle = \langle x, T^* y\rangle . \tag{15}$$

Theorem 2.7 (Properties of Hilber adjoint operators). *Let H_1 and H_2 be hilbert spaces, $S : H_1 \mapsto H_2$ and $T : H_1 \mapsto H_2$ bounded linear operators* *and α any scalar. Then we have*

$$\langle T^* y, x \rangle = \langle y, Tx \rangle \tag{16}$$
$$(S+T)^* = S^* + T^* \tag{17}$$
$$(\alpha T)^* = \hat{\alpha} T^* \tag{18}$$
$$(T^*)^* = T^* \tag{19}$$
$$\|TT^*\| = \|T^*T\| = \|T\|^2 \tag{20}$$
$$T^*T = 0 \quad \implies \quad T = 0 \tag{21}$$
$$(ST)^* = T^*S^*. \tag{22}$$

Definition 2.17 (Self, Adjoint, unitary and normal operators)**.** *A bounded linear operator $T : H \mapsto H$ on a Hilbert space H is said to be*

$$\begin{aligned} \textit{Self adjoint or Hermition} \quad & T^* = T, \\ \textit{Unitary if } T \textit{ is bijective and} \quad & T^* = T^{-1}, \\ \textit{Normal if} \quad & TT^* = T^*T. \end{aligned}$$

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2.6 Series and Normes

Definition 2.18 (Hamel Basis)**.** *We call a linearly independent set S of a vector space X a* ***Hamel basis*** *if S spans X, i.e. if any $x \in X$ has a unique and finite representation.*

$$x = a_1 x_1 + \ldots + a_n x_n, \quad x_j \in S, a_j \in \mathbb{F}$$

Theorem 2.8 (Finite-dimensional norm equivalence)**.** *On a finite-dimensional vector space X, all norms are equivalent. For instance, all norms are quivalent on $\mathbb{R}^n$*

2.7 Common

Definition 2.19 (Range)**.** *A range of a function $f : X \mapsto Y$, is denoted by range (f) or $f(X)$, is the set of all $y \in Y$ that are the image of some $x \in X$. More compact can this be written.*

$$range\,(f) = \{y \in Y \mid \textit{there exist } x \in X \textit{ such that } f(x) = y\}$$

Definition 2.20. *Let $f : X \mapsto Y$ be a function.*

1. *We call f injective or one-to-one if $f(x_1) = f(x_2)$ implies $x_1 = x_2$, i.e, no two elements of the domain have tha same image. Equivalently, if $x \neq x_2$ then $f(x_1) \neq f(x_2)$.*
2. *We call f surjective or onto if range $(f) = Y$, i.e each $y \in Y$ is the image of at least one $x \in X$.*
3. *We call f bijective if f is both injective and surjective.*

Definition 2.21 (Testing)**.** *I am a big test*

Definition 2.22 (Closed Set)**.** *Let X be a subset of a set Y. If X is closed is this true.*

(i) *The compliment X^c is an open set.*

(ii) *X is it own set closure.*

(iii) *Sequences/nets/filters in X that converge do so in X.*

(iv) *Every point outside X has a neightbourhood disjoint from X*

3 References