MATHEMATICAL CONTROL AND
RELATED FIELDS

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STOCHASTIC OPTIMAL CONTROL — A CONCISE INTRODUCTION

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Abstract. This is a concise introduction to stochastic optimal control theory. We assume that the readers have basic knowledge of real analysis, functional analysis, elementary probability, ordinary differential equations and partial differential equations. We will present the following topics: (i) A brief presentation of relevant results on stochastic analysis; (ii) Formulation of stochastic optimal control problems; (iii) Variational method and Pontryagin's maximum principle, together with a brief introduction of backward stochastic differential equations; (iv) Dynamic programming method and viscosity solutions to Hamilton-Jacobi-Bellman equation; (v) Linear-quadratic optimal control problems, including a careful discussion on open-loop optimal controls and closed-loop optimal strategies, linear forward-backward stochastic differential equations, and Riccati equations.

Contents

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1. **Some Preliminaries.** In this chapter, we make some minimum preparations for our later presentations. We do not have an intention to be self-contained or exhaustive. It is assumed that the readers have the basic knowledge of real analysis, functional analysis, probability theory, ordinary and partial differential equations.

1.1. **Elements in probability.** In this section, we will recall some basic definitions together with some basic results that will be necessary in sequel. Let us first recall the following definitions.

Definition 1.1. (i) A pair $(\Omega, \mathcal{F})$ is called a *measurable space* if Ω is a nonempty set, $\mathcal{F}$ is a σ-*field* of Ω, i.e., $\mathcal{F}$ is a set of some subsets of Ω such that the following hold:

$$\begin{cases} \Omega \in \mathcal{F}, \\ A, B \in \mathcal{F} \quad \Rightarrow \quad A \setminus B \in \mathcal{F}, \\ A_i \in \mathcal{F}, \quad i = 1, 2, \cdots \quad \Rightarrow \quad \displaystyle\bigcup_{i=1}^{\infty} A_i \in \mathcal{F}. \end{cases}$$

In the above case, any $A \in \mathcal{F}$ is called an *event* (which is also called a *measurable set* in real analysis).

(ii) A map $\mathbb{P} : \mathcal{F} \to [0, 1]$ is called a *probability measure* on a given measurable space $(\Omega, \mathcal{F})$ if the following hold:

$$\begin{cases} \mathbb{P}(\emptyset) = 0, \qquad \mathbb{P}(\Omega) = 1, \\ A_i \in \mathcal{F},\ A_i \cap A_j = \emptyset,\ i, j \geq 1,\ i \neq j \quad \Rightarrow \quad \mathbb{P}\Big(\displaystyle\bigcup_{i=1}^{\infty} A_i\Big) = \sum_{i=1}^{\infty} \mathbb{P}(A_i). \end{cases}$$

In the above case, $\mathbb{P}(A)$ is called the *probability* of the given event $A \in \mathcal{F}$. Any $A \in \mathcal{F}$ with $\mathbb{P}(A) = 0$ is called a $\mathbb{P}$-*null set*.

(iii) If $(\Omega, \mathcal{F})$ is a measurable space and $\mathbb{P}$ is a probability measure on $(\Omega, \mathcal{F})$, then $(\Omega, \mathcal{F}, \mathbb{P})$ is called a *probability space*. Such a probability space is said to be *complete* if $\mathcal{F}$ contains all subsets of $\mathbb{P}$-null sets, i.e.,

$$A \in \mathcal{F},\ \mathbb{P}(A) = 0,\ B \subseteq A \qquad \Rightarrow \qquad B \in \mathcal{F}.$$

(iv) In a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, if an event $A \in \mathcal{F}$ satisfies $\mathbb{P}(A) = 1$, we say that A holds almost surely, denoted by

$$A \text{ holds}, \quad \mathbb{P}\text{-a.s.}$$

Example 1.2. (i) Let $\mathcal{B}_0(\mathbb{R}^n)$ be the set of all open sets in $\mathbb{R}^n$. Let $\mathcal{B}(\mathbb{R}^n)$ be the smallest σ-field containing $\mathcal{B}_0(\mathbb{R}^n)$. We call $\mathcal{B}(\mathbb{R}^n)$ the *Borel σ-field* of $\mathbb{R}^n$. In this case, $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$ is a measurable space.

(ii) Let $\Omega = [0,1]$ and $\mathcal{F} = \mathcal{B}([0,1])$ with

$$\mathcal{B}([0,1]) = \mathcal{B}(\mathbb{R}) \bigcap [0,1] \equiv \big\{ A \cap [0,1] \bigm| A \in \mathcal{B}(\mathbb{R}) \big\}.$$

Then $(\Omega, \mathcal{F})$ is a measurable space. Further, let $\mathbb{P}$ be the Lebesgue measure on $[0,1]$. Then $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space. Now, let $\mathcal{F}$ be the *completion* of $\mathcal{B}([0,1])$ with respect to the Lebesgue measure, i.e., $\mathcal{F}$ is the smallest σ-field containing

$$\mathcal{B}([0,1]) \bigcup \big\{ A \subseteq [0,1] \bigm| \exists B \in \mathcal{B}([0,1]),\ A \subseteq B,\ \mathbb{P}(B) = 0 \big\}.$$

Then $(\Omega, \mathcal{F}, \mathbb{P})$ is a complete probability space. Actually, in this last case, $\mathcal{F} = \mathcal{L}([0,1])$, the set of all Lebesgue measurable sets in $[0,1]$.

(iii) Let Ω be a finite or a countably infinite set. In this case, we also call Ω a *discrete set*. Let $\mathcal{F} = 2^{\Omega}$, the set of all subsets of Ω. Then $(\Omega, \mathcal{F})$ is a measurable space. If $\mathbb{P}$ is a probability on $(\Omega, \mathcal{F})$, then $(\Omega, \mathcal{F}, \mathbb{P})$ is (always) a complete probability space.

Definition 1.3. (i) Let $(\Omega, \mathcal{F})$ be a measurable space. A map $\xi \equiv (\xi_1, \cdots, \xi_n)^{\top} : \Omega \to \mathbb{R}^n$ is called an $\mathbb{R}^n$-valued *random variable* if

$$\{\xi \le c\} \equiv \{\omega \in \Omega \bigm| \xi_i(\omega) \le c_i,\ 1 \le i \le n\} \in \mathcal{F}, \quad \forall c \equiv (c_1, \cdots, c_n) \in \mathbb{R}^n.$$

This is also equivalent to the following:

$$\xi^{-1}(A) \in \mathcal{F}, \qquad \forall A \in \mathcal{B}(\mathbb{R}^n).$$

In this case, we also say that ξ is $\mathcal{F}$-measurable. The set of all ($\mathcal{F}$-measurable) random variables is denoted by $L^0_{\mathcal{F}}(\Omega; \mathbb{R}^n)$.

(ii) Let Ω be a nonempty set and $\xi : \Omega \to \mathbb{R}^n$ be a given map. Let $\mathcal{F}^{\xi}$ be the smallest σ-field in Ω such that ξ is $\mathcal{F}^{\xi}$-measurable. We call $\mathcal{F}^{\xi}$ the σ-field generated by ξ. One actually has

$$\mathcal{F}^{\xi} = \xi^{-1}\big(\mathcal{B}(\mathbb{R}^n)\big).$$

Example 1.4. (i) Let $(\Omega, \mathcal{F})$ be a measurable space with $\mathcal{F} = 2^{\Omega}$. Then any map $\xi : \Omega \to \mathbb{R}^n$ is a random variable.

(ii) Let $(\Omega, \mathcal{F})$ be a measurable space such that $\mathcal{F}$ is a finite set. Then $\xi : \Omega \to \mathbb{R}^n$ is a random variable if and only if ξ is a piecewise constant map. In particular, if $\mathcal{F} = \{\Omega, \emptyset\}$. Then $\xi : \Omega \to \mathbb{R}^n$ is a random variable if and only if ξ is a constant map.

Definition 1.5. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space.

(i) For any $\xi \in L^0_{\mathcal{F}}(\Omega; \mathbb{R}^n)$, define its *distribution function* $F^{\xi} : \mathbb{R}^n \to [0,1]$ by the following:

$$F^{\xi}(x) = \mathbb{P}\big(\{\xi \le x\}\big), \qquad \forall x \in \mathbb{R}^n.$$

If $x \mapsto F^{\xi}(x)$ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^n$, i.e., there exists an integrable function $p : \mathbb{R}^n \to \mathbb{R}_+ \equiv [0, \infty)$ such that

$$F^{\xi}(x) = \int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_n} p(y_1, \cdots, y_n) dy_1 \cdots dy_n, \qquad \forall x \in \mathbb{R}^n,$$

then $p(\cdot)$ is called the *density function* of ξ.

(ii) For any $\xi \in L^0_{\mathcal{F}}(\Omega; \mathbb{R}^n)$, define its *mean* (or *mathematical expectation*) as follows

$$\mathbb{E}\xi \overset{\Delta}{=} \int_{\Omega} \xi(\omega) d\mathbb{P}(\omega) = \int_{\mathbb{R}^n} y dF^{\xi}(y),$$

if it exists.

(iii) For any $p \in [1,\infty)$, define $L^p_{\mathcal{F}}(\Omega;\mathbb{R}^n)$ to be the set of all $\xi \in L^0_{\mathcal{F}}(\Omega;\mathbb{R}^n)$ such that

$$\|\xi\|_p \overset{\Delta}{=} \Big(\mathbb{E}|\xi|^p\Big)^{\frac{1}{p}} = \Big(\int_\Omega |\xi(\omega)|^p d\mathbb{P}(\omega)\Big)^{\frac{1}{p}} = \Big(\int_{\mathbb{R}^n} |y|^p dF^\xi(y)\Big)^{\frac{1}{p}} < \infty.$$

Also, $L^\infty_{\mathcal{F}}(\Omega;\mathbb{R}^n)$ is defined to be the set of all random variables $\xi : \Omega \to \mathbb{R}^n$ such that

$$\|\xi\|_\infty \overset{\Delta}{=} \inf_{A\in\mathcal{F},\,\mathbb{P}(A)=0} \sup_{\omega\in\Omega\setminus A} |\xi(\omega)| \equiv \operatorname*{esssup}_{\omega\in\Omega} |\xi(\omega)| < \infty.$$

For $p \in [1,\infty]$, $\|\cdot\|_p$ is a norm under which $L^p_{\mathcal{F}}(\Omega;\mathbb{R}^n)$ is a Banach space. Further, $L^2_{\mathcal{F}}(\Omega;\mathbb{R}^n)$ is a Hilbert space with the inner product

$$\langle\, \xi, \eta \,\rangle = \int_\Omega \xi(\omega)^\top \eta(\omega) d\mathbb{P}(\omega), \qquad \forall \xi,\eta \in L^2_{\mathcal{F}}(\Omega;\mathbb{R}^n).$$

The following result is very important and interesting.

Proposition 1.6. *Let $\xi \in L^1_{\mathcal{F}}(\Omega;\mathbb{R}^n)$, and $\mathcal{G}$ be a sub-σ-field of $\mathcal{F}$. Then there exists a unique $f \in L^1_{\mathcal{G}}(\Omega;\mathbb{R}^n)$ such that*

$$\int_A \xi(\omega)d\mathbb{P}(\omega) = \int_A f(\omega)d\mathbb{P}(\omega), \qquad \forall A \in \mathcal{G}. \tag{1.1}$$

Proof. Define $\mu : \mathcal{G} \to \mathbb{R}^n$ as follows:

$$\mu(A) = \int_A \xi(\omega)d\mathbb{P}(\omega), \qquad \forall A \in \mathcal{G}. \tag{1.2}$$

Then $\mu(\cdot)$ is a *vector-valued measure* on $\mathcal{G}$ with a bounded total variation

$$\|\mu\| \overset{\Delta}{=} \int_\Omega |\xi(\omega)|d\mathbb{P}(\omega) \equiv \mathbb{E}|\xi|,$$

and it is absolutely continuous with respect to $\mathbb{P}|_{\mathcal{G}}$, the restriction of $\mathbb{P}$ on $\mathcal{G}$. Thus, by the Radon-Nikodým theorem, there exists a unique $f \in L^1_{\mathcal{G}}(\Omega;\mathbb{R}^n)$ (called the *Radon-Nikodým derivative* of μ with respect to $\mathbb{P}|_{\mathcal{G}}$) such that

$$\mu(A) = \int_A f(\omega)d\mathbb{P}(\omega), \qquad \forall A \in \mathcal{G}.$$

Then (1.1) follows. □

The unique f (depending on ξ) in the above proposition is called the *conditional expectation* of ξ given $\mathcal{G}$, denoted by $\mathbb{E}(\xi|\,\mathcal{G})$. Thus, one has

$$\int_A \xi(\omega)d\mathbb{P}(\omega) = \int_A \mathbb{E}[\xi|\,\mathcal{G}](\omega)d\mathbb{P}(\omega), \quad \forall A \in \mathcal{G}. \tag{1.3}$$

The following collects some useful properties of conditional expectations.

Proposition 1.7. *Let $\mathcal{G},\mathcal{G}_1,\mathcal{G}_2 \subseteq \mathcal{F}$ be σ-fields with $\mathcal{G}_1 \subseteq \mathcal{G}_2$.*

(i) *Let $\xi \in L^1_{\mathcal{G}}(\Omega;\mathbb{R}^n)$, $\eta,\zeta \in L^1_{\mathcal{F}}(\Omega;\mathbb{R})$ with $\eta \le \zeta$. Then*

$$\mathbb{E}\big(\eta|\,\mathcal{G}\big) \le \mathbb{E}\big(\zeta|\,\mathcal{G}\big),$$

and

$$\mathbb{E}\big(\xi\eta|\,\mathcal{G}\big) = \xi\mathbb{E}\big(\eta|\,\mathcal{G}\big).$$

In particular,

$$\mathbb{E}\big(\xi|\,\mathcal{G}\big) = \xi.$$

(ii) *Let* $\xi \in L^1_{\mathcal{F}}(\Omega;\mathbb{R}^n)$ *be independent of* $\mathcal{G}$, *i.e.,*

$$\mathbb{P}\big(A\cap B\big)=\mathbb{P}(A)\mathbb{P}(B),\qquad \forall A\in\mathcal{F}^{\xi},\ B\in\mathcal{G}.$$

Then

$$\mathbb{E}[\xi\big|\mathcal{G}]=\mathbb{E}\xi.$$

(iii) **(Tower property)** *For any* $\eta\in L^1_{\mathcal{F}}(\Omega;\mathbb{R}^n)$,

$$\mathbb{E}\big(\mathbb{E}(\eta|\,\mathcal{G}_2)\big|\mathcal{G}_1\big)=\mathbb{E}\big(\mathbb{E}(\eta|\,\mathcal{G}_1)\big|\,\mathcal{G}_2\big)=\mathbb{E}(\eta|\,\mathcal{G}_1\big).$$

(iv) **(Jensen's inequality)** *If* $\varphi:\mathbb{R}^n\to\mathbb{R}$ *is convex such that* $\varphi(\xi)\in L^1_{\mathcal{F}}(\Omega;\mathbb{R})$, *then*

$$\varphi\big(\mathbb{E}(\xi|\,\mathcal{G})\big)\le\mathbb{E}\big(\varphi(\xi)\big|\,\mathcal{G}\big).$$

In particular, for any $p\ge1$, *and* $\xi\in L^p_{\mathcal{F}}(\Omega;\mathbb{R}^n)$,

$$\big|\mathbb{E}(\xi|\,\mathcal{G})\big|^p\le\mathbb{E}\big(|\xi|^p\big|\,\mathcal{G}\big).$$

(v) **(Hölder's inequality)** *Let* $p,q\in(1,\infty)$ *with* $\frac{1}{p}+\frac{1}{q}=1$. *Suppose* $\xi\in L^p_{\mathcal{F}}(\Omega;\mathbb{R})$ *and* $\eta\in L^q_{\mathcal{F}}(\Omega;\mathbb{R})$. *Then*

$$\mathbb{E}\Big(|\xi\eta|\ \big|\ \mathcal{G}\Big)\le\Big(\mathbb{E}\big[|\xi|^p\ \big|\ \mathcal{G}\big]\Big)^{\frac{1}{p}}\Big(\mathbb{E}\big[|\eta|^q\ \big|\ \mathcal{G}\big]\Big)^{\frac{1}{q}},\qquad \text{a.s.}$$

(vi) **(Minkowski's inequality)** *Let* $1\le p<\infty$ *and* $\xi,\eta\in L^p_{\mathcal{F}}(\Omega;\mathbb{R})$. *Then*

$$\Big(\mathbb{E}\big[\,|\xi+\eta|^p\ \big|\ \mathcal{G}\big]\Big)^{\frac{1}{p}}\le\Big(\mathbb{E}\big[|\xi|^p\ \big|\ \mathcal{G}\big]\Big)^{\frac{1}{p}}+\Big(\mathbb{E}\big[|\eta|^p\ \big|\ \mathcal{G}\big]\Big)^{\frac{1}{p}}.$$

Now, in a probability space $(\Omega,\mathcal{F},\mathbb{P})$, a family $\mathbb{F}\equiv\{\mathcal{F}_t\ \big|\ t\in[0,T]\}$ of sub-σ-fields of $\mathcal{F}$ is called a *filtration* if

$$\mathcal{F}_s\subseteq\mathcal{F}_t,\qquad 0\le s\le t\le T.$$

We call $(\Omega,\mathcal{F},\mathbb{F},\mathbb{P})$ a *filtered probability space.* Further, $(\Omega,\mathcal{F},\mathbb{F},\mathbb{P})$ is said to satisfy the *usual condition* if $(\Omega,\mathcal{F},\mathbb{P})$ is complete, $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets in $\mathcal{F}$, and $\mathbb{F}$ is right-continuous, i.e.,

$$\mathcal{F}_{t+}\stackrel{\Delta}{=}\bigcap_{s>t}\mathcal{F}_s=\mathcal{F}_t.$$

From now on, we always assume that the filtered probability space $(\Omega,\mathcal{F},\ \mathbb{F},\mathbb{P})$ is given and satisfies the usual condition.

Definition 1.8. A map $X:[0,T]\times\Omega\to\mathbb{R}^n$ is called a *stochastic process* if for each $t\in[0,T]$, $\omega\mapsto X(t,\omega)$ is a random variable. In this case, for any $\omega\in\Omega$, map $t\mapsto X(t,\omega)$ is called a *sample path.* Further,

(i) $X(\cdot)$ is said to be *measurable* if $(t,\omega)\mapsto X(t,\omega)$ is $\mathcal{B}[0,T]\otimes\mathcal{F}$-measurable;

(ii) $X(\cdot)$ is said to be $\mathbb{F}$-*adapted* if for all $t\in[0,T]$, $\omega\mapsto X(t,\omega)$ is $\mathcal{F}_t$-measurable

(iii) $X(\cdot)$ is said to be $\mathbb{F}$-*progressively measurable* if for all $t\in[0,T]$, $(s,\omega)\mapsto X(s,\omega)$ is $\mathcal{B}[0,t]\otimes\mathcal{F}_t$-measurable.

(iv) $X(\cdot)$ is said to be (left, right) continuous on $[0,T]$ if there exists a $\mathbb{P}$-null set N such that for any $\omega\in\Omega\setminus N$, $t\mapsto X(t,\omega)$ is (left, right) continuous.

It is clear that if $X(\cdot)$ is $\mathbb{F}$-progressively measurable, it must be measurable and $\mathbb{F}$-adapted. Conversely, we have the following result.

Theorem 1.9. (Chung–Doob–Meyer) *Let $X(\cdot)$ be a measurable and $\mathbb{F}$-adapted process. Then it admits an $\mathbb{F}$-progressively measurable modification $\bar X(\cdot)$, i.e., $\bar X(\cdot)$ is $\mathbb{F}$-progressively measurable such that*

$$X(t)=\bar X(t),\qquad \text{a.s. },\quad \forall t\in[0,T].$$

Further, if $X(\cdot)$ is left or right continuous, then $X(\cdot)$ itself is $\mathbb{F}$-progressively measurable.

Next, for any $p,q\in[1,\infty)$, and $0\le S<T$, we let

$$\begin{aligned}L^0_{\mathbb{F}}(\Omega;L^q(S,T;\mathbb{R}^n))=\Big\{X:[S,T]\times\Omega\to\mathbb{R}^n\;\big|\;&X(\cdot)\text{ is }\mathbb{F}\text{-adapted},\\ &\int_S^T|X(t)|^qdt<\infty,\ \text{a.s.}\ \Big\},\end{aligned}\tag{1.4}$$

and

$$\begin{aligned}L^p_{\mathbb{F}}(\Omega;L^q(S,T;\mathbb{R}^n))=\Big\{X:[S,T]\times\Omega\to\mathbb{R}^n\;\big|\;X(\cdot)\text{ is }\mathbb{F}\text{-adapted},\\ \|X(\cdot)\|_{L^p_{\mathbb{F}}(\Omega;L^q(S,T;\mathbb{R}^n))}\stackrel{\Delta}{=}\Big[\mathbb{E}\Big(\int_S^T|X(t)|^qdt\Big)^{\frac{p}{q}}\Big]^{\frac1p}<\infty\Big\}.\end{aligned}\tag{1.5}$$

We denote

$$L^p_{\mathbb{F}}(S,T;\mathbb{R}^n)=L^p_{\mathbb{F}}(\Omega;L^p(S,T;\mathbb{R}^n)).$$

Note that $L^2_{\mathbb{F}}(S,T;\mathbb{R}^n)$ is a Hilbert space with the inner product defined by the following:

$$\langle\, X(\cdot),Y(\cdot)\,\rangle=\mathbb{E}\int_S^T\langle\, X(t),Y(t)\,\rangle\, dt.$$

Let $L^p_{\mathbb{F}}(\Omega;C([S,T];\mathbb{R}^n))$ be the Banach space consisting of all continuous processes $X(\cdot)$ with the norm

$$\|X(\cdot)\|_{L^p_{\mathbb{F}}(\Omega;C([S,T];\mathbb{R}^n))}\stackrel{\Delta}{=}\Big\{\mathbb{E}\Big(\sup_{t\in[S,T]}|X(t)|^p\Big)\Big\}^{\frac1p}<\infty.$$

The following gives a sufficient condition for processes to have continuous paths.

Theorem 1.10. (Kolmogorov Continuity Criterion) *Let $X(\cdot)$ be a stochastic process. Suppose there are constants $\alpha,\beta,K>0$ such that*

$$\mathbb{E}|X(t)-X(s)|^\alpha\le K|t-s|^{1+\beta},\qquad t,s\in[0,T].$$

Then $X(\cdot)$ has a γ-Hölder continuous modification for all $\gamma\in(0,\frac{\beta}{\alpha})$. Namely, there exists a modification $\bar X(\cdot)$ of $X(\cdot)$ such that for some $\gamma\in(0,\frac{\beta}{\alpha})$ and $\delta>0$,

$$\mathbb{P}\Big\{\omega\in\Omega\;\big|\sup_{\substack{0<t-s<h(\omega)\\ t,s\in[0,T]}}\frac{|\bar X(t)-\bar X(s)|}{|t-s|^\gamma}\le\delta\Big\}=1,$$

where $h(\cdot)$ is a positive-valued random variable.

Note that in the above, if $\alpha<\beta$, then $X(\cdot)$ must be a constant process. Thus, the case $\alpha\ge\beta$ is the real interesting case.

Definition 1.11. A map $\tau:\Omega\to[0,\infty]$ is called an $\mathbb{F}$-stopping time if

$$(\tau\le t)\stackrel{\Delta}{=}\{\omega\in\Omega\;\big|\;\tau(\omega)\le t\}\in\mathcal{F}_t,\qquad\forall t\ge0.$$

For any $\mathbb{F}$-stopping time τ, define

$$\mathcal{F}_\tau \overset{\Delta}{=} \big\{A \in \mathcal{F} \bigm| A \cap (\tau \le t) \in \mathcal{F}_t,\ \forall t \ge 0\big\}.$$

The following result is very useful.

Theorem 1.12. (Debut Theorem) *Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a filtered probability space satisfying the usual condition. If $X(\cdot)$ is $\mathbb{F}$-progressively measurable and $G \in \mathcal{B}(\mathbb{R}^n)$, then*

$$\tau \equiv \inf\{t \ge 0 \bigm| X(t) \in G\}$$

is a stopping time.

The following are the usual forms one frequently encounters.

Example 1.13. Let $X(\cdot)$ be $\mathbb{F}$-adapted and continuous. Let $E \subseteq \mathbb{R}^n$ be an open set. Define the *first hitting time* $\sigma_E(\cdot)$ of $X(\cdot)$ to E and the *first exit time* $\tau_E(\cdot)$ of $X(\cdot)$ from E, respectively, by

$$\begin{cases} \sigma_E(\omega) \overset{\Delta}{=} \inf\{t \ge 0 \bigm| X(t,\omega) \in E\}, \\ \tau_E(\omega) \overset{\Delta}{=} \inf\{t \ge 0 \bigm| X(t,\omega) \notin E\}. \end{cases}$$

Then both $\sigma_E(\cdot)$ and $\tau_E(\cdot)$ are $\mathbb{F}$-stopping times.

Definition 1.14. A process $X : [0,T] \times \Omega \to \mathbb{R}^n$ is called an $\mathbb{F}$-*martingale* if it is $\mathbb{F}$-adapted and for any $t \ge 0$, $\mathbb{E}X(t)$ exists with the property that

$$\mathbb{E}\big[X(t) \,|\, \mathcal{F}_s\big] = X(s), \qquad \mathbb{P}\text{-a.s. },\ 0 \le s \le t.$$

Example 1.15. For any $\xi \in L^p_{\mathcal{F}_T}(\Omega; \mathbb{R}^n)$ with $p \ge 1$, the process defined by $M(t) = \mathbb{E}(\xi \,|\, \mathcal{F}_t)$ $(t \in [0,T])$ is an $\mathbb{F}$-martingale.

Proposition 1.16. (Doob) *Let $p > 1$ and $X(\cdot)$ be an $\mathbb{R}$-valued right-continuous $\mathbb{F}$-martingale such that for all $t \ge 0$, $\mathbb{E}|X(t)|^p < \infty$. Then*

$$\mathbb{E}\Big(\sup_{t\in[0,T]} |X(t)|^p\Big) \le \Big(\frac{p}{p-1}\Big)^p \mathbb{E}|X(T)|^p, \qquad \forall T > 0.$$

1.2. **Brownian motion and Itô's integral.** We first introduce the following.

Definition 1.17. Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a filtered probability space. An $\mathbb{F}$-adapted $\mathbb{R}^d$-valued continuous process $W(\cdot)$ is called a d-dimensional standard Brownian motion if $\mathbb{P}(W(0) = 0) = 1$, for any $0 \le s \le t$, almost surely,

$$\mathbb{E}\Big[W(t) - W(s) \Bigm| \mathcal{F}_s\Big] = 0, \tag{1.6}$$

and $W(t) - W(s) \sim N\big(0, (t-s)I\big)$, the standard normal distribution.

Next, for a standard d-dimensional Brownian motion on the filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, for any $t \ge 0$, we let $\mathcal{F}_t^W$ be the smallest σ-field such that all the random variables $W(s)$, $s \in [0,t]$ are measurable, denoted by

$$\mathcal{F}_t^W = \sigma\Big\{W(s), 0 \le s \le t\Big\} \subseteq \mathcal{F}_t.$$

It can be shown that the augmentation $\widehat{\mathbb{F}}^W \equiv \{\widehat{\mathcal{F}}_t^W\}_{t\ge0}$ of $\mathbb{F}^W \equiv \{\mathcal{F}_t^W\}_{t\ge0}$ by all the $\mathbb{P}$-null sets is continuous (in t). We call $\widehat{\mathbb{F}}^W$ the *natural filtration* of $W(\cdot)$. In what follows, we will take $\mathbb{F} \equiv \{\mathcal{F}_t\}_{t\ge0}$ to be the natural filtration of $W(\cdot)$. We point out that if $W(\cdot)$ is a d-dimensional standard Brownian motion under $\mathbb{F}$, then it stays under $\widehat{\mathbb{F}}^W$. For simplicity of presentation, hereafter, we let $d = 1$, unless we explicitly otherwise state.

One can show that the above-defined standard Brownian motions exist. We now would like to define the integral of form

$$\int_0^T f(t)dW(t), \tag{1.7}$$

where f is some stochastic process and $W(\cdot)$ is a (one-dimensional) standard Brownian motion. Such an integral will play an essential role in the sequel. Note that if for almost all $\omega \in \Omega$, the map $t \mapsto W(t, \omega)$ were of *bounded variation*, a natural definition of the above integral would be a Lebesgue-Stieltjes type, with ω being a parameter. Unfortunately, we have the following result.

Proposition 1.18. *Let $W(\cdot)$ be a standard Brownian motion. Let $\gamma > 1/2$ and set*

$$\mathcal{S}_\gamma \stackrel{\Delta}{=} \Big\{\omega \in \Omega \bigm| W(\cdot\,, \omega) \text{ is } \gamma\text{-Hölder continuous at some } t \in [0, \infty)\Big\}.$$

Then $\mathbb{P}(\mathcal{S}_\gamma) = 0$. In particular, for almost all $\omega \in \Omega$, the map $t \mapsto W(t, \omega)$ is nowhere differentiable.

From the above result, we see that integral (3.3) cannot be defined as a Lebesgue-Stieltjes integral. To properly define such an integral, let us first introduce the function space consisting of all possible integrands. Let $T > 0$ and denote

$$\|f\|_T \stackrel{\Delta}{=} \|f\|_{L^2_{\mathbb{F}}(0,T;\mathbb{R})}. \tag{1.8}$$

Next, we introduce the following sets:

$$\begin{cases} \mathcal{M}^2[0, T] = \Big\{X \in L^2_{\mathbb{F}}(0, T; \mathbb{R}) \bigm| X \text{ is a right-continuous} \\ \qquad\qquad\qquad\qquad\qquad \mathbb{F}\text{-martingale with } X(0) = 0,\ \mathbb{P}\text{-a.s. } \Big\}, \\ \mathcal{M}^2_c[0, T] = \Big\{X \in \mathcal{M}^2[0, T] \bigm| t \mapsto X(t) \text{ is continuous, } \mathbb{P}\text{-a.s. } \Big\}. \end{cases}$$

We identify $X, Y \in \mathcal{M}^2[0, T]$ if there exists a $\mathbb{P}$-null set $N \in \mathcal{F}$, such that $X(t, \omega) = Y(t, \omega)$, for all $t \geq 0$ and $\omega \notin N$. Define

$$|X|_T = \big(\mathbb{E}X(T)^2\big)^{1/2}, \qquad \forall X \in \mathcal{M}^2[0, T]. \tag{1.9}$$

We can show by the martingale property that (1.9) is a norm under which $\mathcal{M}^2[0, T]$ is a Hilbert space. Moreover, $\mathcal{M}^2_c[0, T]$ is a closed subspace of $\mathcal{M}^2[0, T]$. We should distinguish the norms $\|\cdot\|_T$ (defined by (1.8)) and $|\cdot|_T$ (defined by (1.9)). It is important to note that any Brownian motion $W(\cdot)$ is in $\mathcal{M}^2_c[0, T]$ with $|W|^2_T = T$ (see Definition 1.17).

Now, let $f(\cdot) \in L^2_{\mathbb{F}}(0, T; \mathbb{R})$ be fixed. For any $\varepsilon > 0$, one can find a *simple process* $f^\varepsilon(\cdot)$ of form:

$$f^\varepsilon(t, \omega) = f^\varepsilon_0(\omega) I_{\{t=0\}}(t) + \sum_{i\geq 0} f^\varepsilon_i(\omega) I_{(t^\varepsilon_i, t^\varepsilon_{i+1}]}(t), \qquad t \in [0, T],$$

where $0 = t^\varepsilon_0 < t^\varepsilon_1 < \cdots,\ \ t^\varepsilon_i \leq T$ and $f^\varepsilon_i(\cdot)$ is $\mathcal{F}_{t^\varepsilon_i}$-measurable, such that

$$\|f(\cdot) - f^\varepsilon(\cdot)\|_T < \varepsilon. \tag{1.10}$$

We let $\mathcal{L}_0[0,T]$ be the set of all simple processes. Define

$$\mathbb{I}(f^\varepsilon)(t,\omega)\stackrel{\Delta}{=}\sum_{i\ge0}f_i^\varepsilon(\omega)\big[W(t\wedge t_{i+1}^\varepsilon,\omega)-W(t\wedge t_i^\varepsilon,\omega)\big],\qquad t\in[0,T].$$

Clearly, $\mathbb{I}:\mathcal{L}_0[0,T]\to\mathcal{M}_c^2[0,T]$ is linear. Moreover,

$$|\mathbb{I}(f^\varepsilon-f^\delta)|_T=\|f^\varepsilon-f^\delta\|_T\to0,\qquad\text{as }\varepsilon,\delta\to0.$$

Namely, $\{\mathbb{I}(f^\varepsilon),\varepsilon>0\}$ is Cauchy in $\mathcal{M}_c^2[0,T]$. Thus, there exists a unique limit in $\mathcal{M}_c^2[0,T]$, denoted by $\mathbb{I}(f)$. It is seen that this limit only depends on f and is independent of the choice of the sequence f^ε. Hence $\mathbb{I}(f)$ is well-defined on $L^2(0,T;\mathbb{R})$ and it is called the *Itô integral* of $f(\cdot)$, denoted by

$$\int_0^t f(s)dW(s)\stackrel{\Delta}{=}\mathbb{I}(f)(t),\qquad 0\le t\le T.$$

Further, for any $f\in L^2_{\mathbb{F}}(0,T;\mathbb{R})$ and any two $\mathbb{F}$-stopping times σ and τ with $0\le\sigma\le\tau\le T$, $\mathbb{P}$-a.s. , we denote

$$\int_\sigma^\tau f(s)dW(s)\stackrel{\Delta}{=}\mathbb{I}(f)(\tau)-\mathbb{I}(f)(\sigma).$$

Now, let us collect some fundamental properties of the Itô integral.

Proposition 1.19. (i) *For any $f,g\in L^2_{\mathbb{F}}(0,T;\mathbb{R})$ and $\mathbb{F}$-stopping times σ and τ with $\sigma\le\tau$ ($\mathbb{P}$-a.s.), it holds, almost surely,*

$$\mathbb{E}\Big\{\int_{t\wedge\sigma}^{t\wedge\tau}f(r)dW(r)\big|\,\mathcal{F}_\sigma\Big\}=0,$$

$$\mathbb{E}\Big\{\Big[\int_{t\wedge\sigma}^{t\wedge\tau}f(r)dW(r)\Big]\Big[\int_{t\wedge\sigma}^{t\wedge\tau}g(r)dW(r)\Big]\big|\,\mathcal{F}_\sigma\Big\}=\mathbb{E}\Big\{\int_{t\wedge\sigma}^{t\wedge\tau}f(r)g(r)dr\big|\,\mathcal{F}_\sigma\Big\}.$$

In particular, for any $0\le s<t\le T$,

$$\mathbb{E}\Big\{\int_s^t f(r)dW(r)\big|\,\mathcal{F}_s\Big\}=0,$$

$$\mathbb{E}\Big\{\Big[\int_s^t f(r)dW(r)\Big]\Big[\int_s^t g(r)dW(r)\Big]\big|\,\mathcal{F}_s\Big\}=\mathbb{E}\Big\{\int_s^t f(r)g(r)dr\big|\,\mathcal{F}_s\Big\}.$$

(ii) *For any $\mathbb{F}$-topping time τ and $f\in L^2_{\mathbb{F}}(0,T;\mathbb{R})$, let $\widetilde f(t)=f(t)I_{(\tau\ge t)}$. Then*

$$\int_0^{t\wedge\tau}f(s)dW(s)=\int_0^t\widetilde f(s)dW(s).\tag{1.11}$$

We now extend the Itô integral to a bigger class of integrands than $L^2_{\mathbb{F}}(0,T;\mathbb{R})$. To this end, we recall $L^0_{\mathbb{F}}(\Omega;L^2(0,T;\mathbb{R}))$ (see (1.4)), and introduce the set of *local martingales*:

$$\begin{cases}\mathcal{M}^{2,loc}[0,T]=\Big\{X:[0,T]\times\Omega\to\mathbb{R}\bigm|\exists\text{ nondecreasing}\\ \qquad\qquad\qquad\text{stopping times }\tau_j\text{ with }\mathbb{P}(\lim\limits_{j\to\infty}\tau_j\ge T)=1,\\ \qquad\qquad\qquad\text{and }X(\cdot\wedge\tau_j)\in\mathcal{M}^2[0,T],\quad\forall j=1,2,\cdots\Big\},\\ \mathcal{M}_c^{2,loc}[0,T]=\Big\{X\in\mathcal{M}^{2,loc}[0,T]\bigm|t\mapsto X(\cdot)\text{ continuous, a.s. }\Big\}.\end{cases}$$

For any $f(\cdot) \in L^0_{\mathbb{F}}(\Omega; L^2(0,T;\mathbb{R}))$, define

$$\tau_j(\omega) \overset{\Delta}{=} \inf \Big\{ t \in [0,T] \mid \int_0^t |f(s,\omega)|^2 ds \geq j \Big\}, \quad j = 1, 2, \cdots .$$

In the above, we define $\inf \emptyset \overset{\Delta}{=} T$. Clearly, $\{\tau_j\}_{j\geq 1}$ is a sequence of nondecreasing $\mathbb{F}$-stopping times. Set $f_j(t) \overset{\Delta}{=} f(t) I_{\{\tau_j \geq t\}}$. Then

$$\int_0^T |f_j(s)|^2 ds = \int_0^{\tau_j} |f(s)|^2 ds \leq j, \qquad \text{a.s. },$$

which implies $f_j(\cdot) \in L^2_{\mathbb{F}}(0,T;\mathbb{R})$. By (1.11), we have

$$\begin{aligned} &\int_0^{t\wedge\tau_i} f_j(s) dW(s) = \int_0^t f_j(s) I_{\{\tau_i \geq s\}} dW(s) \\ &= \int_0^t f(s) I_{\{\tau_j \geq s\}} I_{\{\tau_i \geq s\}} dW(s) = \int_0^t f_i(s) dW(s), \quad \forall i < j. \end{aligned}$$

Hence, the following is well-defined:

$$\int_0^t f(s) dW(s) \overset{\Delta}{=} \int_0^t f_j(s) dW(s), \qquad \forall t \in [0,\tau_j], \; j = 1, 2, \cdots .$$

This is called the *Itô integral* of $f(\cdot) \in L^0_{\mathbb{F}}(\Omega; L^2(0,T;\mathbb{R}))$. It is easy to see that $\int_0^t f(s) dW(s) \in \mathcal{M}^{2,loc}_c[0,T]$ for any $f(\cdot) \in L^0_{\mathbb{F}}(\Omega; L^2(0,T;\mathbb{R}))$.

Theorem 1.20. *Let $f(\cdot) \in L^0_{\mathbb{F}}(\Omega; L^2(0,T;\mathbb{R}^n))$. Then, for any $q > 0$, there exist constants $0 < \underline{K}_q < \overline{K}_q$ such that for any $\mathbb{F}$-stopping time τ,*

$$\begin{aligned} \underline{K}_q \mathbb{E}\Big(\int_0^\tau |f(s)|^2 ds \Big)^{\frac{q}{2}} &\leq \mathbb{E}\Big(\sup_{0\leq t\leq\tau} \Big| \int_0^\tau f(s) dW(s) \Big|^q \Big) \\ &\leq \overline{K}_q \mathbb{E}\Big(\int_0^\tau |f(s)|^2 ds \Big)^{\frac{q}{2}}. \end{aligned} \tag{1.12}$$

The above is called *Burkholder–Davis–Gundy inequalities*. By such a result, one can obtain a sufficient condition under which $t \mapsto \int_0^t f(s) dW(s)$ is in $L^q_{\mathbb{F}}(\Omega; C([0,T]; \mathbb{R}))$. Note that the constants $\underline{K}_q$ and $\overline{K}_q$ are independent of the $\mathbb{F}$-stopping time τ, only depends on q. For $q > 1$, one interesting choice is the following: (see [5], p.285)

$$\underline{K}_q = q^{-(q+1)} (7 + 4\sqrt{2})^{-q}, \qquad \overline{K}_q = q^{q+1} (2\sqrt{6})^q.$$

Also, we have

$$\underline{K}_2 = 1, \qquad \overline{K}_2 = 4.$$

Further, $[0,T]$ can be replaced by any $[a,b]$ with $0 < a < b < \infty$.

In applications, Itô's integral $\int_a^b f(s) dW(s)$ is constantly encountered and one expects that it has zero mean. In order to have this, it is sufficient to have

$$\mathbb{E}\Big(\int_a^b |f(s)|^2 ds \Big)^{\frac{1}{2}} < \infty.$$

In fact, we may let $f^N(s)$ be the truncation of $f(s)$ by N:

$$f^N(s) = f(s) \mathbf{1}_{(|f(s)|\leq N)} + N \frac{f(s)}{|f(s)|} \mathbf{1}_{(|f(s)|>N)}.$$

Then

$$\begin{aligned}&\Big|\mathbb{E}\int_a^b f(s)dW(s)\Big|=\Big|\mathbb{E}\int_a^b[f(s)-f^N(s)]dW(s)\Big|\\&\le\mathbb{E}\Big[\sup_{t\in[a,b]}\Big|\int_a^t[f(s)-f^N(s)]dW(s)\Big|\Big]\\&\le\overline{K}_1\mathbb{E}\Big(\int_a^b|f(s)-f^N(s)|^2ds\Big)^{\frac12}\to 0,\qquad N\to\infty.\end{aligned}$$

Hence, the left-hand side is zero.

Theorem 1.21. *Let $(\Omega,\mathcal{F},\mathbb{F},\mathbb{P})$ be a complete filtered probability space on which a one-dimensional standard Brownian motion $W(\cdot)$ is defined with $\mathbb{F}$ being its natural filtration. Let $X(\cdot)\in\mathcal{M}_c^2[0,T]$ (resp. $\mathcal{M}_c^{2,loc}[0,T]$). Then there exists a unique $f(\cdot)\in L^2_{\mathbb{F}}(0,T;\mathbb{R})$ (resp. $L^0_{\mathbb{F}}(\Omega;L^2(0,T;\mathbb{R}))$) such that*

$$X(t)=X(0)+\int_0^t f(s)dW(s),\qquad t\in[0,T],\ \text{a.s.}$$

The above is called the *Martingale Representation Theorem*. This result roughly says that being an $\mathbb{F}$-martingale (with $\mathbb{F}$ being the natural filtration of a standard Brownian motion $W(\cdot)$), $X(\cdot)$ is "differentiable" with respect to $W(\cdot)$.

Next, recall the chain rule for deterministic functions. Suppose

$$X(t)=X(0)+\int_0^t b(s)ds,\qquad t\in[0,T],$$

for some $b(\cdot)\in L^1(0,T;\mathbb{R}^n)$. Then for any $F\in C^1([0,T]\times\mathbb{R}^n)$,

$$F(t,X(t))=F(0,X(0))+\int_0^t\big[F_t(s,X(s))+F_x(s,X(s))b(s)\big]ds,\quad\forall t\in[0,T].$$

For the stochastic case, we have a similar formula which is significantly different from the above. Consider the following:

$$X(t)=X(0)+\int_0^t b(s)ds+\int_0^t\sigma(s)dW(s),\qquad t\in[0,T],\tag{1.13}$$

with $b(\cdot)\in L^0_{\mathbb{F}}(\Omega;L^1(0,T;\mathbb{R}^n))$ and $\sigma(\cdot)\in L^0_{\mathbb{F}}(\Omega;L^2(0,T;\mathbb{R}^{n\times d}))$, where $W(\cdot)$ is a d-dimensional standard Brownian motion. Any process $X(\cdot)$ of form (1.13) is called an *Itô process*. We have the following *Itô's formula*.

Theorem 1.22. *Let $b(\cdot)\in L^0_{\mathbb{F}}(\Omega;L^1(0,T;\mathbb{R}^n))$, $\sigma(\cdot)\in L^0_{\mathbb{F}}(\Omega;L^2(0,T;\mathbb{R}^{n\times d}))$, and $X(\cdot)$ be given by (1.13). Let $F(t,x):[0,\infty)\times\mathbb{R}^n\to\mathbb{R}$ be C^1 in t and C^2 in x with F_t, F_x and F_{xx} being continuous such that*

$$\begin{cases}F_t(\cdot\,,X(\cdot)),\ F_x(\cdot\,,X(\cdot))b(\cdot)\in L^0_{\mathbb{F}}(\Omega;L^1(0,T;\mathbb{R})),\\ \sigma(\cdot)^\top F_{xx}(\cdot\,,X(\cdot))\sigma(\cdot)\in L^0_{\mathbb{F}}(\Omega;L^1(0,T;\mathbb{R}^{d\times d})),\\ F_x(\cdot\,,X(\cdot))\sigma(\cdot)\in L^0_{\mathbb{F}}(\Omega;L^2(0,T;\mathbb{R}^d)).\end{cases}$$

Then for any $t\ge 0$, almost surely,

$$\begin{aligned}F(t,X(t))=F(0,X(0))&+\int_0^t\Big\{F_s(s,X(s))+F_x(s,X(s))b(s)\\&+\frac12\mathrm{tr}\,\big[\sigma(s)^\top F_{xx}(s,X(s))\sigma(s)\big]\Big\}ds+\int_0^t F_x(s,X(s))\sigma(s)dW(s).\end{aligned}$$

Let us make an observation. Take $\sigma(\cdot)\in L^2_{\mathbb F}(0,T;\mathbb R)$ and consider

$$X(t)=\int_0^t\sigma(s)dW(s),\quad t\in[0,T].$$

Then $X(\cdot)\in\mathcal M^2_c[0,T]$. By Itô's formula above, we have

$$X(t)^2=\int_0^t\sigma(s)^2ds+2\int_0^t X(s)\sigma(s)dW(s),\quad t\in[0,T].$$

Note that $X(\cdot)\sigma(\cdot)$ is in $L^0_{\mathbb F}(\Omega;L^2(0,T;\mathbb R))$, but not necessarily in $L^2_{\mathbb F}(0,T;\mathbb R)$. Hence, the introduction of Itô's integral for integrands in $L^0_{\mathbb F}(\Omega;L^2(0,T;\mathbb R))$ is not just some routine generalization. It is really necessary even for as simple calculus as the above.

As an application of the Itô's formula, we present the following result which will be frequently used in this paper.

Corollary 1.23. *Let Z and $\widehat Z$ be $\mathbb R^n$-valued continuous process satisfying*

$$\begin{cases}dZ(t)=b(t)dt+\sigma(t)dW(t),\\ d\widehat Z(t)=\widehat b(t)dt+\widehat\sigma(t)dW(t),\end{cases}$$

where $b,\widehat b,\sigma,\widehat\sigma$ are proper functions valued in $\mathbb R^n$ and $W(\cdot)$ is a one-dimensional standard Brownian motion. Then

$$\begin{aligned}d\langle Z(t),\widehat Z(t)\rangle=&\big[\langle Z(t),\widehat b(t)\rangle+\langle b(t),\widehat Z(t)\rangle+\langle\sigma(t),\widehat\sigma(t)\rangle\big]dt\\&+\big[\langle\sigma(t),\widehat Z(t)\rangle+\langle Z(t),\widehat\sigma(t)\rangle\big]dW(t).\end{aligned}$$

This corollary can be easily proved by Itô's formula with $F(x,y)=\langle x,y\rangle$ for $(x,y)\in\mathbb R^{2n}$.

1.3. **Stochastic differential equations.** For $0\le S<T$, we consider the following equation

$$\begin{cases}dX(t)=b(t,X(t))dt+\sigma(t,X(t))dW(t),\quad t\in[S,T],\\ X(S)=\xi\,,\end{cases}\tag{1.14}$$

where ξ is $\mathcal F_S$-measurable and b and σ are given maps. In the above, $X(\cdot)$ is regarded as an unknown. Such an equation is called a *stochastic differential equation.*

Definition 1.24. An $\mathbb F$-adapted continuous process $X(\cdot)$ is called a *strong solution* of (1.14) if

$$\int_S^t\big\{|b(r,X(r))|+|\sigma(r,X(r))|^2\big\}dr<\infty,\quad\forall t\in[S,T],\ \text{a.s.}$$

and

$$X(t)=\xi+\int_S^t b(r,X(r))dr+\int_S^t\sigma(r,X(r))dW(r),\quad\forall t\in[S,T],\ \text{a.s.}$$

In the above, the first integral on the right is a usual Lebesgue integral (regarding $\omega\in\Omega$ as a parameter) and the second is the Itô integral defined in the previous section. From our assumption, these two integrals are well-defined. We refer to $\int_S^t b(s,X(s))ds$ as the *drift* term, and $\int_S^t\sigma(s,X(s))dW(s)$ as the *diffusion* term. Next we introduce the following hypothesis.

(H) The maps $b, \sigma : [0,T] \times \mathbb{R}^n \times \Omega \to \mathbb{R}^n$ are measurable such that for every $x \in \mathbb{R}^n$, $t \mapsto (b(t,x), \sigma(t,x))$ is $\mathbb{F}$-progressively measurable, there exists a constant $L > 0$ such that

$$|b(t,x) - b(t,x')| + |\sigma(t,x) - \sigma(t,x')| \le L|x - x'|, \qquad \forall t \in [0,T],\ x, x' \in \mathbb{R}^n,\ \text{a.s.}\ ,$$

and for some $p \in [1,\infty)$,

$$\mathbb{E}\Big(\int_0^T |b(s,0)|ds\Big)^p + \mathbb{E}\Big(\int_0^T |\sigma(s,0)|^2 ds\Big)^{\frac{p}{2}} < \infty.$$

We have the following result.

Theorem 1.25. *Let* (H) *hold. Then, for any $\xi \in L^p_{\mathcal{F}_S}(\Omega;\mathbb{R}^n)$, (1.14) admits a unique strong solution $X(\cdot)$, such that*

$$\mathbb{E}\Big[\max_{S\le r\le T}|X(r)|^p\Big] \le K\mathbb{E}\Big[|\xi|^p + \Big(\int_S^T |b(t,0)|dt\Big)^p + \Big(\int_S^T |\sigma(t,0)|^2 dt\Big)^{\frac{p}{2}}\Big], \tag{1.15}$$

and

$$\begin{aligned}\mathbb{E}|X(t)-X(s)|^p \le K\Big\{&\mathbb{E}\Big(\int_s^t |b(r,0)|dr\Big)^p + \mathbb{E}\Big(\int_s^t |\sigma(r,0)|^2 dr\Big)^{\frac{p}{2}} \\ &+ (t-s)^{\frac{p}{2}}\mathbb{E}\Big[|\xi|^p + \Big(\int_S^T |b(r,0)|dr\Big)^p + \Big(\int_S^T |\sigma(r,0)|^2 dr\Big)^{\frac{p}{2}}\Big]\Big\}, \\ &\forall S \le s < t \le T.\end{aligned} \tag{1.16}$$

Hereafter $K > 0$ represents a generic constant which could be different from line to line. Moreover, if $\widehat{\xi} \in L^p_{\mathcal{F}_S}(\Omega;\mathbb{R}^n)$ and $\widehat{X}(\cdot)$ is the strong solution of (1.14) corresponding to $\widehat{\xi}$, then

$$\mathbb{E}\Big[\max_{S\le r\le T}|X(r) - \widehat{X}(r)|^p\Big] \le K\mathbb{E}|\xi - \widehat{\xi}|^p. \tag{1.17}$$

Proof. For any $S \le \tau \le T$, set

$$\mathcal{X}_p[S,\tau] \triangleq L^p_{\mathbb{F}}(\Omega; C([S,\tau];\mathbb{R}^n)) \triangleq \Big\{X : [S,\tau]\times\Omega \to \mathbb{R}^n \;\Big|\; X(\cdot) \text{ is } \mathbb{F}\text{-adapted, continuous, } \mathbb{E}\Big[\sup_{S\le t\le\tau}|X(t)|^p\Big] < \infty\Big\}.$$

Clearly $\mathcal{X}_p[S,\tau]$ is a Banach space with the norm

$$\|X(\cdot)\|_{\mathcal{X}_p[S,\tau]} \triangleq \Big\{\mathbb{E}\Big[\sup_{S\le t\le\tau}|X(t)|^p\Big]\Big\}^{\frac{1}{p}}.$$

For any $x(\cdot), \bar{x}(\cdot) \in \mathcal{X}_p[S,\tau]$, define

$$\begin{cases} X(t) = \xi + \displaystyle\int_S^t b(s,x(s))ds + \int_S^t \sigma(s,x(s))dW(s), \\ \bar{X}(t) = \xi + \displaystyle\int_S^t b(s,\bar{x}(s))ds + \int_S^t \sigma(s,\bar{x}(s))dW(s), \end{cases} \qquad t \in [S,\tau].$$

Then by Burkholder–Davis–Gundy inequalities, we have

$$
\begin{aligned}
\|X(\cdot)\|_{\mathcal{X}_p[S,\tau]} &\le \|\xi\|_p + \Big[\mathbb{E}\Big(\int_S^\tau |b(s,x(s))|ds\Big)^p\Big]^{\frac{1}{p}} \\
&\qquad + \Big[\mathbb{E}\Big(\sup_{S\le t\le\tau}\Big|\int_S^t \sigma(s,x(s))dW(s)\Big|^p\Big)\Big]^{\frac{1}{p}} \\
&\le \|\xi\|_p + \Big[\mathbb{E}\Big(\int_S^\tau |b(s,0)|ds\Big)^p\Big]^{\frac{1}{p}} + L\Big[\mathbb{E}\Big(\int_S^\tau |x(s)|ds\Big)^p\Big]^{\frac{1}{p}} \\
&\quad + \overline{K}_p^{\frac{1}{p}}\Big[\mathbb{E}\Big(\int_S^\tau |\sigma(s,x(s))|^2 ds\Big)^{\frac{p}{2}}\Big]^{\frac{1}{p}} \\
&\le \|\xi\|_p + \Big[\mathbb{E}\Big(\int_S^\tau |b(s,0)|ds\Big)^p\Big]^{\frac{1}{p}} + L\Big[\mathbb{E}\Big(\int_S^\tau |x(s)|ds\Big)^p\Big]^{\frac{1}{p}} \\
&\quad + \overline{K}_p^{\frac{1}{p}}\Big[\mathbb{E}\Big(\int_S^\tau |\sigma(s,0)|^2 ds\Big)^{\frac{p}{2}}\Big]^{\frac{1}{p}} + \overline{K}_p^{\frac{1}{p}} L\Big[\mathbb{E}\Big(\int_S^\tau |x(s)|^2 ds\Big)^{\frac{p}{2}}\Big]^{\frac{1}{p}} \\
&\le \|\xi\|_p + \Big[\mathbb{E}\Big(\int_S^\tau |b(s,0)|ds\Big)^p\Big]^{\frac{1}{p}} + \overline{K}_p^{\frac{1}{p}}\Big[\mathbb{E}\Big(\int_S^\tau |\sigma(s,0)|^2 ds\Big)^{\frac{p}{2}}\Big]^{\frac{1}{p}} \\
&\qquad + \bar{K}(\tau-S)^{\frac{1}{2}}\|x(\cdot)\|_{\mathcal{X}_p[S,\tau]},
\end{aligned}
$$

with $\bar{K}>0$ being an absolute constant. Now, let $\tau = S + (2\bar{K})^{-2}$, and for given $\xi \in L^p_{\mathcal{F}_S}(\Omega;\mathbb{R}^n)$, denote

$$
M = 2\Big\{\|\xi\|_p + \Big[\mathbb{E}\Big(\int_S^\tau |b(s,0)|ds\Big)^p\Big]^{\frac{1}{p}} + \overline{K}_p^{\frac{1}{p}}\Big[\mathbb{E}\Big(\int_S^\tau |\sigma(s,0)|^2 ds\Big)^{\frac{p}{2}}\Big]^{\frac{1}{p}}\Big\}.
$$

Then the above estimate can be written as

$$
\|X(\cdot)\|_{\mathcal{X}_p[S,\tau]} \le \frac{M}{2} + \frac{1}{2}\|x(\cdot)\|_{\mathcal{X}_p[S,\tau]}.
$$

Hence,

$$
\|X(\cdot)\|_{\mathcal{X}_p[S,\tau]} \le M, \qquad \forall \|x(\cdot)\|_{\mathcal{X}_p[S,\tau]} \le M.
$$

On the other hand, similar to the above, we have

$$
\begin{aligned}
\|X(\cdot)-\bar{X}(\cdot)\|_{\mathcal{X}_p[S,\tau]} &\le \bar{K}(\tau-S)^{\frac{1}{2}}\|x(\cdot)-\bar{x}(\cdot)\|_{\mathcal{X}_p[S,\tau]} \\
&= \frac{1}{2}\|x(\cdot)-\bar{x}(\cdot)\|_{\mathcal{X}_p[S,\tau]}.
\end{aligned}
$$

Hence, the map $x(\cdot)\mapsto X(\cdot)$ is from a closed ball centered at 0 with radius M in $\mathcal{X}_p[S,\tau]$ to itself and is a contraction. By contraction mapping theorem, there exists a unique fixed point, which gives the strong solution $X(\cdot)$ to (1.14) on $[S,\tau]$. Next, repeating the procedure on $[S+(2\bar{K})^{-2}, S+2(2\bar{K})^{-2}]$, etc., we are able to get the unique strong solution on $[S,T]$. Clearly, estimate (1.15) and (1.17) can be obtained following the above procedure. Now, for (1.16), let $S\le s<t\le T$, one has

$$
\begin{aligned}
&\mathbb{E}|X(t)-X(s)|^p = \mathbb{E}\Big|\int_s^t b(r,X(t))dr + \int_s^t \sigma(r,X(r))dW(r)\Big|^p \\
&\le K\Big[\mathbb{E}\Big(\int_s^t |b(r,0)|dr\Big)^p + \mathbb{E}\Big(\int_s^t |\sigma(r,0)|^2 dr\Big)^{\frac{p}{2}} + (t-s)^{\frac{p}{2}}\mathbb{E}\Big(\sup_{r\in[s,t]}|X(r)|^p\Big)\Big] \\
&\le K\Big\{\mathbb{E}\Big(\int_s^t |b(r,0)|dr\Big)^p + \mathbb{E}\Big(\int_s^t |\sigma(r,0)|^2 dr\Big)^{\frac{p}{2}}
\end{aligned}
$$

$$+(t-s)^{\frac{p}{2}}\mathbb{E}\Big[|\xi|^p+\mathbb{E}\Big(\int_S^T|b(r,0)|ds\Big)^p+\mathbb{E}\Big(\int_S^T|\sigma(r,0)|^2dr\Big)^{\frac{p}{2}}\Big]\Big\}.$$

Then (1.16) follows. $\square$

1.4. **Feynman-Kac formula and its consequences.** In this section, we look at some interesting relations between SDEs and PDEs. We let $(\Omega,\mathcal{F},\mathbb{F},\mathbb{P})$ be a filtered complete probability space on which a d-dimensional standard Brownian motion $W(\cdot)$ is defined, with $\mathbb{F}=\{\mathcal{F}_t\}_{t\ge0}$ being its natural filtration augmented by all the $\mathbb{P}$-null sets. We have the following result.

Theorem 1.26. (Feynman-Kac Formula) *Let b,σ,c,g,h be proper deterministic functions such that the following Cauchy problem of linear partial differential equation:*

$$\begin{cases}V_t(t,x)+\dfrac12\mathrm{tr}\Big[V_{xx}(t,x)\sigma(t,x)\sigma(t,x)^\top\Big]+V_x(t,x)b(t,x)\\ \qquad\qquad+c(t,x)V(t,x)+g(t,x)=0,\qquad (t,x)\in[0,T]\times\mathbb{R}^n,\\ V(T,x)=h(x),\qquad x\in\mathbb{R}^n,\end{cases}\tag{1.18}$$

admits a unique classical solution $V(\cdot\,,\cdot)$. Suppose for any $(t,x)\in[0,T)\times\mathbb{R}^n$, the following SDE:

$$\begin{cases}dX(s)=b(s,X(s))ds+\sigma(s,X(s))dW(s),\qquad s\in[t,T],\\ X(t)=x,\end{cases}\tag{1.19}$$

admits a unique strong solution $X(\cdot)\equiv X(\cdot\,;t,x)$ such that

$$\mathbb{E}\Big(\int_t^T e^{\int_t^s 2c(r,X(r))dr}|V_x(s,X(s))\sigma(s,X(s))|^2ds\Big)^{\frac12}<\infty.\tag{1.20}$$

Then $V(\cdot\,,\cdot)$ admits the following representation:

$$V(t,x)=\mathbb{E}\Big[\int_t^T e^{\int_t^s c(r,X(r))dr}g(s,X(s))ds+e^{\int_t^T c(r,X(r))dr}h(X(T))\Big],\tag{1.21}$$

where $X(\cdot)=X(\cdot\,;t,x)$.

Proof. Applying Itô's formula to $s\mapsto V(s,X(s))e^{\int_t^s c(r,X(r))dr}$, we obtain

$$\begin{aligned}&V(T,X(T))e^{\int_t^T c(r,X(r))dr}-V(t,x)\\&=\int_t^T e^{\int_t^s c(r,X(r))dr}\Big[V_s(s,X(s))+c(s,X(s))V(s,X(s))\\&\qquad+V_x(s,X(s))b(s,X(s))+\frac12\mathrm{tr}\Big[V_{xx}(s,X(s))\sigma(s,X(s))\sigma(s,X(s))^\top\Big]ds\\&\qquad+\int_t^T e^{\int_t^s c(r,X(r))dr}V_x(s,X(s))\sigma(s,X(s))dW(s)\\&=-\int_t^T e^{\int_t^s c(r,X(r))dr}g(s,X(s))ds\\&\qquad+\int_t^T e^{\int_t^s c(r,X(r))dr}V_x(s,X(s))\sigma(s,X(s))dW(s).\end{aligned}$$

Consequently,

$$\begin{aligned}V(t,x)&=e^{\int_t^T c(r,X(r))dr}h(X(T))+\int_t^T e^{\int_t^s c(r,X(r))dr}g(s,X(s))ds\\&\quad-\int_t^T e^{\int_t^s c(r,X(r))dr}V_x(s,X(s))\sigma(s,X(s))dW(s).\end{aligned}$$

By condition (1.20), we have

$$V(t,x)=\mathbb{E}\Big[e^{\int_t^T c(r,X(r))dr}h(X(T))+\int_t^T e^{\int_t^s c(r,X(r))dr}g(s,X(s))ds\ \Big|\ \mathcal{F}_t\Big]$$

On the other hand, from

$$X(s)=x+\int_t^s b(r,X(r))dr+\int_t^s \sigma(r,X(r))dW(r),\qquad s\in[t,T],$$

we see that for deterministic $(t,x)\in[0,T)\times\mathbb{R}^n$, $X(s;t,x)$ is $\mathcal{F}_{t,s}$-measurable, with

$$\mathcal{F}_{t,s}=\sigma\Big\{W(r)-W(t)\ \big|\ r\in[t,s]\Big\},$$

which is independent of $\mathcal{F}_t$. Hence, in the above $\mathbb{E}_t$ can be replaced by $\mathbb{E}$, which completes our proof. $\square$

In the case $c(\cdot\,,\cdot)=0$, (1.18) becomes

$$\begin{cases}V_t(t,x)+\frac{1}{2}\mathrm{tr}\,\Big[\sigma(t,x)\sigma(t,x)^\top V_{xx}(t,x)\Big]+V_x(t,x)b(t,x)+g(t,x)=0,\\ \hspace{7cm}(t,x)\in[0,T]\times\mathbb{R}^n,\\ V(T,x)=h(x),\qquad x\in\mathbb{R}^n,\end{cases}\tag{1.22}$$

and (1.21) becomes

$$V(t,x)=\mathbb{E}\Big[\int_t^T g(s,X(s))ds+h(X(T))\Big].\tag{1.23}$$

Let us now formally look at a couple of interesting consequences of the above Feynman-Kac formula.

Note that for given $(t,x)\in[0,T)\times\mathbb{R}^n$ and $s\in[t,T]$, the distribution function of the random variable $X(s;t,x)$ is given by

$$F(s,y;t,x)=\mathbb{P}(X(s;t,x)\le y)=\mathbb{E}\Big[I_{(-\infty,y]}\big(X(s;t,x)\big)\Big],\quad y\in\mathbb{R}^n,$$

where $(-\infty,y]=(-\infty,y_1]\times\cdots\times(-\infty,y_n]$. Suppose $y\mapsto F(s,y;t,x)$ is absolutely continuous (with respect to the Lebesgue measure). Then the random variable $X(s;t,x)$ has a density $y\mapsto p(s,y;t,x)$. Thus, for any $h(\cdot)\in C(\mathbb{R}^n;\mathbb{R})$,

$$v(s;t,x,h(\cdot))\equiv\mathbb{E}\Big[h\big(X(s;t,x)\big)\Big]=\int_{\mathbb{R}^n}h(y)p(s,y;t,x)dy,\tag{1.24}$$

and by Feynman-Kac formula, with $c(t,x)=0$, $g(t,x)=0$, and $T=s$, we see that $v(t,x)\equiv v(s;t,x,h(\cdot))$ is the unique solution to the following:

$$\begin{cases}v_t(t,x)+\frac{1}{2}\mathrm{tr}\,\Big[v_{xx}(t,x)\sigma(t,x)\sigma(t,x)^\top\Big]+v_x(t,x)b(t,x)=0,\\ \hspace{6cm}(t,x)\in[0,s)\times\mathbb{R}^n,\\ v(s,x)=h(x),\qquad x\in\mathbb{R}^n.\end{cases}\tag{1.25}$$

Then for $0 \le t < s$, using the integral representation (1.24), we obtain

$$\begin{aligned} 0 &= v_t(t,x) + \frac{1}{2}\mathrm{tr}\left[v_{xx}(t,x)\sigma(t,x)\sigma(t,x)^\top\right] + v_x(t,x)b(t,x) \\ &= \int_{\mathbb{R}^n} h(y)\Big[p_t(s,y;t,x) + \frac{1}{2}\mathrm{tr}\left(p_{xx}(s,y;t,x)\sigma(t,x)\sigma(t,x)^\top\right) \\ &\qquad\qquad + p_x(s,y;t,x)b(t,x)\Big]dy. \end{aligned}$$

Since the above holds for any $h(\cdot) \in C(\mathbb{R}^n;\mathbb{R})$, we have the following:

$$\begin{cases} p_t(s,y;t,x) + \frac{1}{2}\mathrm{tr}\left[p_{xx}(s,y;t,x)\sigma(t,x)\sigma(t,x)^\top\right] + p_x(s,y;t,x)b(t,x) = 0, \\ \qquad\qquad\qquad\qquad\qquad 0 \le t < s \le T, \quad x,y \in \mathbb{R}^n, \\ p(s,y;s,x) = \delta(y-x), \qquad s \in [0,T], \quad x,y \in \mathbb{R}^n. \end{cases} \tag{1.26}$$

This is called the *Kolmogorov backward equation.*

Next, for any $\tau \in (t,s)$, by the uniqueness of the solution to (1.25) (or, the semigroup property of the solution), we have

$$\begin{aligned} &\int_{\mathbb{R}^n} v(\tau,y)p(\tau,y;t,x)dy = \mathbb{E}\Big[v(\tau,X(\tau;t,x))\Big] = \mathbb{E}\Big[v(s,X(s;t,x))\Big] \\ &= \mathbb{E}\Big[h(X(s;t,x))\Big] = v(t,x). \end{aligned}$$

Thus, differentiating in τ, we obtain

$$\begin{aligned} 0 = \partial_\tau v(t,x) &= \int_{\mathbb{R}^n} \Big[v_\tau(\tau,y)p(\tau,y;t,x) + v(\tau,y)p_\tau(\tau,y;t,x)\Big]dy \\ &= \int_{\mathbb{R}^n} \Big\{\Big[-v_y(\tau,y)b(\tau,y) - \frac{1}{2}\mathrm{tr}\left(v_{yy}(\tau,y)\sigma(\tau,y)\sigma(\tau,y)^\top\right)\Big]p(\tau,y;t,x) \\ &\qquad + v(\tau,y)p_\tau(\tau,y;t,x)\Big\}dy \\ &\equiv \int_{\mathbb{R}^n} \Big[v(\tau,y)p_\tau(\tau,y;t,x) - \Big(\mathcal{L}_{(\tau,y)}v(\tau,y)\Big)p(\tau,y;t,x)\Big]dy \\ &= \int_{\mathbb{R}^n} v(\tau,y)\Big[p_\tau(\tau,y;t,x) - \mathcal{L}^*_{(\tau,y)}p(\tau,y;t,x)\Big]dy, \end{aligned}$$

where

$$\mathcal{L}_{(\tau,y)}\varphi(y) = \frac{1}{2}\mathrm{tr}\left[\varphi_{yy}(y)\sigma(\tau,y)\sigma(\tau,y)^\top\right] + \varphi_y(y)b(\tau,y),$$

and

$$\mathcal{L}^*_{(\tau,y)}\varphi(y) = -\nabla\cdot\Big[b(\tau,y)\varphi(y)\Big] + \sum_{i,j=1}^n \Big(a_{ij}(\tau,y)\varphi(y)\Big)_{y_iy_j},$$

with

$$\Big(a_{ij}(\tau,y)\Big) = \frac{1}{2}\sigma(\tau,y)\sigma(\tau,y)^\top.$$

Hence, we obtain

$$\begin{cases} p_s(s,y;t,x) - \sum_{i,j=1}^n \big(a_{ij}(s,y)p(s,y;t,x)\big)_{y_iy_j} + \sum_{i=1}^n \big(b(s,y)p(s,y;t,x)\big)_{y_i} = 0, \\ \qquad\qquad\qquad\qquad\qquad 0 \le t < s \le T, \ x,y \in \mathbb{R}^n, \\ p(t,y;t,x) = \delta(y-x), \quad t \in [0,T], \quad x,y \in \mathbb{R}^n. \end{cases} \tag{1.27}$$

The above is called the *Kolmogorov Forward Equation* which is also known as the *Fokker–Planck Equation*.

2. Stochastic Optimal Control Problems.

2.1. Some problems from real life.

In this subsection, we are presenting some typical optimal control problems.

2.1.1. Investment-consumption problem We consider a financial market in which there are $n+1$ assets continuously traded. The 0-th asset is a *bond*, and the last n are *stocks*. The price process of the i-th asset is denoted by $P_i(\cdot)$ and they satisfy the following system:

$$\begin{cases} dP_0(t) = r(t)P_0(t)dt, \\ dP_i(t) = b_i(t)P_i(t)dt + P_i(t)\displaystyle\sum_{j=1}^{d}\sigma_{ij}(t)dW_j(t), \\ P_i(0) = p_i, \qquad 0 \le i \le n, \end{cases} \tag{2.1}$$

where $r(\cdot)$, $b_i(\cdot)$ and $\sigma_{ij}(\cdot)$ are the *interest rate*, the *appreciation rate*, and the *volatility*, respectively; $W(\cdot) \equiv (W_1(\cdot), \cdots, W_d(\cdot))$ is a d-dimensional standard Brownian motion defined on some complete filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ such that $\mathbb{F} = \{\mathcal{F}_t\}_{t\ge0}$ is the natural filtration of $W(\cdot)$ augmented by all the $\mathbb{P}$-null sets. We denote $b(\cdot) = (b_1(\cdot), \cdots, b_n(\cdot))^\top$ and $\sigma(\cdot) = (\sigma_{ij}(\cdot))_{n\times d}$.

Let an investor have an initial wealth $x \in \mathbb{R}$. She invests this amount in the market described above and the wealth process $X(\cdot)$ satisfies the following SDE:

$$\begin{cases} dX(t) = \big\{r(t)X(t) + \langle\, b(t) - r(t)\mathbf{1}, \pi(t)\,\rangle - c(t)\big\}dt + \langle\, \pi(t), \sigma(t)dW(t)\,\rangle, \\ \hfill t \in [0,T], \\ X(0) = x, \end{cases} \tag{2.2}$$

where $\pi(\cdot) = (\pi_1(\cdot), \cdots, \pi_n(\cdot))^\top$ is called the *portfolio*, with $\pi_i(t)$ being the amount (at t) invested in the i-th stock, $c(\cdot)$ is the *consumption rate*, and $\mathbf{1} = (1, 1, \cdots, 1)^\top \in \mathbb{R}^n$. For this particular investor, she has her own *utility* (a satisfaction level) of consuming at rate $c(t)$ at time $t \in [0,T]$, represented by $g(t, c(t))$ and her own attitude to the risk versus the gain at the final time T, described by $h(X(T))$. Functions $c \mapsto g(t,c)$ and $x \mapsto h(x)$ are strictly increasing and concave. The investor would like to maximize the following payoff functional

$$J(\pi(\cdot), c(\cdot)) = \mathbb{E}\Big[h(X(T)) + \int_0^T g(t, c(t))dt\Big], \tag{2.3}$$

by choosing a suitable *control* $(\pi(\cdot), c(\cdot))$ from certain class. Sometimes, we also have various constraints on the wealth and the strategy. For example, if the investor is regarded *bankruptcy* and has to leave the market if the wealth ever reaches some specified level ν. Thus, she would like to keep the wealth above that level all the time:

$$X(t) \ge \nu > 0, \quad \forall t \in [0,T], \ \text{a.s.} \tag{2.4}$$

This is a constraint for the wealth. Suppose the *short-selling* is prohibited for all the securities (including the bond). Then we have the constraints on the portfolio:

$$\pi_i(t) \ge 0, \quad X(t) - \sum_{i=1}^{n}\pi_i(t) \ge 0, \quad \forall t \in [0,T], \ \text{a.s.} \tag{2.5}$$

If the investor needs a certain consumption rate to survive, then we need

$$c(t) \geq \delta > 0, \qquad t \in [0,T].$$

Many other constraints are possible. In general, we might have some constraints on the wealth and controls.

2.1.2. Optimal inventory Suppose a supermarket is selling n products. The inventory of these n products at time t is denoted by $X(t) \equiv (X_1(t), \cdots, X_n(t))^\top$ with $X_i(t)$ being the inventory of the i-th product. We assume that $X(\cdot)$ satisfies the following SDE:

$$dX(t) = \Big[-b(t, X(t), p(t)) + u(t) \Big] dt + \sigma(t, X(t), p(t)) dW(t),$$

where $p(t) \equiv (p_1(t), \cdots, p_n(t))^\top$ is the price vector with $p_i(t)$ being the price of the i-th product. Map $b : [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n_+$ represents the rate of selling, which gives a decrease of inventory (thus it has a negative sign in front). Normally, $b_i(t,x,p)$ is decreasing in p_i and increasing in x_i. The coupling between $X_i(\cdot)$ and $X_j(\cdot)$ means that when people buy the i-product, certain amount of the j-th product will also be bought (for example, bread and butter). Also, the coupling of $X_i(\cdot)$ with $p_j(\cdot)$ means that the change of the price for j-th product will affect the selling of the i-th product. For example, an increase of the price for type A bread might lead to a decrease of selling this type of bread which might lead to an increase of selling for type B bread. In the above, $u(\cdot) \equiv (u_1(\cdot), \cdots, u_n(\cdot))^\top$ stands for the order rate made by the store, with $u_i(\cdot)$ being the order rate for the i-th product. The diffusion term in the state equation represents the possible disturbances of the demand. For example, the temperature going down might lead to the demand of ice cream decreasing.

As the manager of the supermarket, he would like to minimize the cost for the inventory. Thus, the following cost functional can be introduced:

$$J(u(\cdot), p(\cdot)) = \mathbb{E}\Big[\int_0^T g(t, X(t)) dt + h(X(T)) \Big].$$

Here, the map $x_i \mapsto g(t,x)$ is super-linear since the cost of keeping the i-product is not of linear growth. Typically, it could be

$$g(t,x) = \sum_{i=1}^n \alpha_i(t) (x_i^+)^{1+\beta_i}, \qquad \beta_i > 0.$$

In practice, the store orders of the product are not continuously. Thus, the problem should be described as an *impulse control* problem, by which we mean that the drift term $\int_0^t u(s) ds$ in the state equation is replaced by a step function, and the cost functional will also be changed properly. We prefer not to get into the details here.

2.1.3. Epidemic problems Consider a control of epidemic system problem. Suppose N is the total population under consideration and assume it stays constant (for a certain period of time). Let the total population be divided into three groups: *susceptible* (who could be infected), *infected* (who has been infected), and *recovered* (or *removed*, who has recovered and is immuned, will not become a susceptible anymore). Let $S(t)$, $I(t)$, and $R(t)$ be the total population of the susceptible, infected, and removed, at time t, respectively. Then $N = S(t) + I(t) + R(t)$ is assumed to be a constant. Here is a so-called deterministic SIR model:

$$\begin{cases} \dot{S}(t) = -\beta I(t)S(t), \\ \dot{I}(t) = \beta I(t)S(t) - \gamma I(t), \\ \dot{R}(t) = \gamma I(t). \end{cases}$$

In the above, the rate of new infected occur is $\beta I(t)S(t)$ (due to the interaction of the susceptible and the infected with $\beta > 0$ being a parameter). On the other hand, the (naturally) recovered (removed) rate of infected is γ.

Now, suppose the dynamics of the processes $S(\cdot)$ and $I(\cdot)$ contain some random disturbances affecting the interaction of susceptible and infected. Then our model will become the following:

$$\begin{cases} dS(t) = -\beta I(t)S(t)dt - \sigma I(t)S(t)dW(t), \\ dI(t) = \big[\beta I(t)S(t) - \gamma I(t)\big]dt + \sigma I(t)S(t)dW(t), \\ dR(t) = \gamma I(t)dt. \end{cases}$$

This is called a stochastic differential equation SIR model. One can further add birth rate and death rate. For simplicity, we do not consider those.

Note that in the above, no control is applied, and the recovers is just by the nature. We now introduce the control process into the equation for $I(\cdot)$. Then, the above will become

$$\begin{cases} dS(t) = -\beta I(t)S(t)dt - \sigma I(t)S(t)dW(t), \\ dI(t) = \big\{\beta I(t)S(t) - [\gamma + u(t)]I(t)\big\}dt + \sigma I(t)S(t)dW(t), \\ dR(t) = [\gamma + u(t)]I(t)dt, \end{cases}$$

meaning that the treatment will speed up the recovery. Then we may introduce the cost functional to measure the performance of the control; an optimal control problem is formulated. More comprehensive control model is possible.

2.2. **General formulation of problems.** We first give a mathematical formulation of the problem.

Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a complete filtered probability space on which a one-dimensional standard Brownian motion $W(\cdot)$ is defined with $\mathbb{F} \overset{\Delta}{=} \{\mathcal{F}_t\}_{t \geq 0}$ being its natural filtration augmented by all the $\mathbb{P}$-null sets. Consider the following stochastic differential equation:

$$\begin{cases} dX(t) = b(t, X(t), u(t))dt + \sigma(t, X(t), u(t))dW(t), \qquad t \in [0, T], \\ X(0) = x, \end{cases} \tag{2.6}$$

where $b, \sigma : [0, T] \times \mathbb{R}^n \times U \times \Omega \to \mathbb{R}^n$ are given maps, and U is a metric space. In the above, $X(\cdot)$ is called a *state process* and $u(\cdot)$ is called a *control process* which is taken from the following set:

$$\mathcal{U}[0, T] = \Big\{ u : [0, T] \to U \;\Big|\; u(\cdot) \text{ is } \mathbb{F}\text{-progressively measurable} \Big\}.$$

We refer to (2.6) as the *state equation*. From the previous chapter, we know that under some mild conditions, for any $x \in \mathbb{R}^n$ and $u(\cdot) \in \mathcal{U}[0, T]$, state equation (2.6) admits a unique solution $X(\cdot) \equiv X(\cdot\,; x, u(\cdot))$. To measure the performance of the control process $u(\cdot)$, we introduce the *cost functional* as follows

$$J(u(\cdot)) = \mathbb{E}\Big[\int_0^T g(t, X(t), u(t))dt + h(X(T))\Big]. \tag{2.7}$$

Our optimal control problem can be stated as follows:

Problem (C). *For given $x \in \mathbb{R}^n$, find a $\bar{u}(\cdot) \in \mathcal{U}[0,T]$ such that*

$$J(\bar{u}(\cdot)) = \inf_{u(\cdot)\in\mathcal{U}[0,T]} J(u(\cdot)). \tag{2.8}$$

Any $\bar{u}(\cdot) \in \mathcal{U}[0,T]$ satisfying the above is called an *open-loop optimal control*, the corresponding state process $\bar{X}(\cdot)$ is called an *open-loop optimal state process*, and $(\bar{X}(\cdot), \bar{u}(\cdot))$ is called an *open-loop optimal pair*.

Let us look at several special cases.

2.2.1. Linear-quadratic problems

Suppose the state equation reads

$$\begin{cases} dX(t) = \big[A(t)X(t) + B(t)u(t)\big]dt + \big[C(t)X(t) + D(t)u(t)\big]dW(t), \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad t \in [0,T], \\ X(0) = x, \end{cases}$$

and the cost functional reads:

$$J(u(\cdot)) = \mathbb{E}\Big[\int_0^T \Big(\langle\, Q(t)X(t), X(t)\,\rangle + \langle\, R(t)u(t), u(t)\,\rangle\Big)dt + \langle\, HX(T), X(T)\,\rangle\Big].$$

Here, $A(\cdot), B(\cdot), C(\cdot), D(\cdot), Q(\cdot), R(\cdot)$ are suitable bounded matrix valued functions, and H is a matrix. Thus, the state equation is linear and the cost functional is quadratic in $(X(\cdot), u(\cdot))$. In the current case, $U \subseteq \mathbb{R}^m$. The corresponding optimal control is referred to as a linear-quadratic (LQ, for short) problem. Note that if U is unbounded, some integrability condition has to be introduced when $\mathcal{U}[0,T]$ is defined so that for any $x \in \mathbb{R}^n$, $u(\cdot) \in \mathcal{U}[0,T]$, the cost functional is well-defined.

2.2.2. Linear-convex problems When the state equation is linear and the cost functional is of the following form:

$$J(u(\cdot)) = \mathbb{E}\Big[\int_0^T \Big(q(t, X(t)) + \rho(t, u(t))\Big)dt + h(X(T))\Big],$$

where

$$x \mapsto q(t,x), \quad u \mapsto \rho(t,u), \quad x \mapsto h(x) \tag{2.9}$$

are convex, we call the corresponding optimal control problem a linear-convex problem.

2.2.3. Linear-semiconvex problems In the above, if instead of the convexity conditions imposed for the maps (2.9), there is a constant $K \geq 0$ such that

$$x \mapsto q(t,x) + K|x|^2, \quad u \mapsto \rho(t,u) + K|u|^2, \quad x \mapsto h(x) + K|x|^2$$

are convex. Functions satisfying the above condition are called semi-convex functions. Thus, the corresponding optimal control problem is referred to as a linear-semiconvex problem. It is clear that when $K = 0$, the current problem is reduced to the linear-convex problem.

2.2.4. Affine-quadratic problems Suppose the state equation takes the following form:

$$\begin{cases} dX(t) = \big[A(t, X(t)) + B(t, X(t))u(t)\big]dt \\ \qquad\qquad\quad + \big[C(t, X(t)) + D(t, X(t))u(t)\big]dW(t), \\ X(0) = x, \end{cases}$$

and the cost functional takes the following form:

$$J(u(\cdot))=\mathbb{E}\Big[\int_0^T\Big(q(t,X(t))+\langle\, R(t,X(t))u(t),u(t)\,\rangle\Big)dt+h(X(T))\Big].$$

The main feature of the above is that the drift and the diffusion of the state equation are affine in the control process and the integrand in the cost functional is quadratic in the control process. Due to this, we refer to the corresponding optimal control problem as an *affine-quadratic* (AQ, for short) problem. Similar to the LQ problems, when U is unbounded, we need to have some integrability for any $u(\cdot)\in\mathcal{U}[0,T]$ to guarantee the cost functional being well-defined.

2.3. **An existence result.** We now look at a special case for which optimal control exists. Consider the following stochastic linear controlled system:

$$\begin{cases} dX(t)=\big[AX(t)+Bu(t)\big]dt+\big[CX(t)+Du(t)\big]dW(t), \quad t\in[0,T],\\ X(0)=X_0,\end{cases} \tag{2.10}$$

where A,B,C,D are matrices of suitable sizes. The state $X(\cdot)$ takes values in $\mathbb{R}^n$ and the control $u(\cdot)$ is in

$$\mathcal{U}[0,T]\overset{\Delta}{=}\Big\{u(\cdot)\in L^2_{\mathbb{F}}(0,T;\mathbb{R}^m)\bigm| u(t)\in U,\ \text{a.e. } t\in[0,T],\ \mathbb{P}\text{-a.s.}\ \Big\},$$

with $U\subseteq\mathbb{R}^m$. Note that we have an additional constraint that a control must be square integrable just to ensure the existence of solutions of (2.10) under $u(\cdot)$. If U is bounded, then this restriction is not necessary. The cost functional is

$$J(u(\cdot))=\mathbb{E}\Big[\int_0^T g(X(t),u(t))dt+h(X(T))\Big], \tag{2.11}$$

with $g:\mathbb{R}^n\times U\to\mathbb{R}$ and $h:\mathbb{R}^n\to\mathbb{R}$. The optimal control problem can be stated as follows.

Problem (LC). *Minimize (2.11) subject to the state equation (2.10) with* $u(\cdot)\in\mathcal{U}[0,T]$.

Now we introduce the following assumptions:

(L1) The set $U\subseteq\mathbb{R}^m$ is convex and closed, and the functions g and h are convex and for some $\delta,K>0$,

$$g(x,u)\ge\delta|u|^2-K,\quad h(x)\ge -K,\quad \forall (x,u)\in\mathbb{R}^n\times U. \tag{2.12}$$

(L2) The set $U\subseteq\mathbb{R}^m$ is convex and compact, and the functions g and h are convex.

When either (L1) or (L2) is assumed, Problem (LC) is also called a *stochastic linear-convex control problem* which is a little more general than that introduced in Subsection 2.2.2. When both g and h are quadratic functions, Problem (LC) is reduced to a stochastic linear quadratic problem. The main result of this section is the following:

Proposition 2.1. *Let either* (L1) *or* (L2) *hold. Then Problem* (LC) *admits an optimal control.*

Proof. First suppose (L1) holds. Let $(X_j(\cdot),u_j(\cdot))$ be a minimizing sequence. By (2.12), we have

$$\mathbb{E}\int_0^T|u_j(t)|^2dt\le K,\qquad \forall j\ge1, \tag{2.13}$$

for some constant $K > 0$. Thus, there is a subsequence, still labeling by $u_j(\cdot)$, such that

$$u_j(\cdot) \to \bar{u}(\cdot), \qquad \text{weakly in } L^2_{\mathbb{F}}(0,T;\mathbb{R}^m).$$

By Mazur's Theorem, we have a sequence of convex combinations

$$\widetilde{u}_j(\cdot) \stackrel{\Delta}{=} \sum_{i\geq 1} \alpha_{ij} u_{i+j}(\cdot), \qquad \text{with } \alpha_{ij} \geq 0,\ \sum_{i\geq 1} \alpha_{ij} = 1$$

so that

$$\widetilde{u}_j(\cdot) \to \bar{u}(\cdot), \qquad \text{strongly in } L^2_{\mathbb{F}}(0,T;\mathbb{R}^m).$$

Since the set $U \subseteq \mathbb{R}^k$ is convex and closed, it follows that $\bar{u}(\cdot) \in \mathcal{U}[0,T]$. On the other hand, if $\widetilde{X}_j(\cdot)$ is the state under the control $\widetilde{u}_j(\cdot)$, then we have the convergence

$$\widetilde{X}_j(\cdot) \to \bar{X}(\cdot), \qquad \text{strongly in } L^2_{\mathbb{F}}(\Omega; C([0,T],\mathbb{R}^n)).$$

Clearly, $(\bar{X}(\cdot), \bar{u}(\cdot))$ is admissible, and the convexity of g and h implies

$$J(\bar{u}(\cdot)) = \lim_{j\to\infty} J(\widetilde{u}_j(\cdot)) \leq \lim_{j\to\infty} \sum_{i\geq 1} \alpha_{ij} J(u_{i+j}(\cdot)) = \inf_{u(\cdot)\in\mathcal{U}[0,T]} J(u(\cdot)).$$

Hence, $(\bar{X}(\cdot), \bar{u}(\cdot))$ is optimal.

In the case when (L2) holds, we automatically have (2.13). Then the above proof applies. □

3. **Variational Method and Pontryagin Maximum Principle.**

In this chapter, we discuss some necessary conditions for optimal controls via variational techniques.

3.1. **Variation along an optimal pair.**

We recall Problem (C) stated in Section 2.2. To get some feeling, in this section, we will derive a variational inequality along the optimal pair, under some suitable conditions. More precisely, we assume the following.

(H1) Let $U \subseteq \mathbb{R}^m$ be convex and bounded, $b, \sigma : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$, $g : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$, and $h : \mathbb{R}^n \to \mathbb{R}$ be continuously differentiable in (x,u), with uniformly bounded first order derivatives.

For convenience, we only consider the case that functions b, σ, g are independent of t. We have the following result.

Proposition 3.1. *Let* (H1) *hold. Suppose* $(\bar{X}(\cdot), \bar{u}(\cdot))$ *is an optimal pair of Problem (C). Then for any* $u(\cdot) \in \mathcal{U}[0,T]$, *the following variational inequality holds:*

$$0 \leq \mathbb{E}\Big[\int_0^T \Big(g_x(t)X_1(t) + g_u(t)[u(t) - \bar{u}(t)]\Big)dt + h_x(\bar{X}(T))X_1(T)\Big], \tag{3.1}$$

where

$$g_x(\cdot) = g_x(\bar{X}(\cdot), \bar{u}(\cdot)), \quad g_u(\cdot) = g_u(\bar{X}(\cdot), \bar{u}(\cdot)), \tag{3.2}$$

and $X_1(\cdot)$ *is the solution to the following variational equation:*

$$\begin{cases} dX_1(t) = \big\{b_x(t)X_1(t) + b_u(t)[u(t) - \bar{u}(t)]\big\}dt \\ \qquad\qquad +\big\{\sigma_x(t)X_1(t) + \sigma_u(t)[u(t) - \bar{u}(t)]\big\}dW(t), \quad t \in [0,T], \\ X_1(0) = 0, \end{cases} \tag{3.3}$$

with

$$\begin{cases} b_x(\cdot) = b_x(\bar{X}(\cdot), \bar{u}(\cdot)), \quad b_u(\cdot) = b_u(\bar{X}(\cdot), \bar{u}(\cdot)), \\ \sigma_x(\cdot) = \sigma_x(\bar{X}(\cdot), \bar{u}(\cdot)), \quad \sigma_u(\cdot) = \sigma_u(\bar{X}(\cdot), \bar{u}(\cdot)). \end{cases} \tag{3.4}$$

Proof. Let $(\bar{X}(\cdot), \bar{u}(\cdot))$ be an optimal pair. Let $u(\cdot) \in \mathcal{U}[0,T]$ be fixed. For any $\varepsilon > 0$, by the convexity of U, we have

$$u^\varepsilon(\cdot) = \bar{u}(\cdot) + \varepsilon[u(\cdot) - \bar{u}(\cdot)] = (1-\varepsilon)\bar{u}(\cdot) + \varepsilon u(\cdot) \in \mathcal{U}[0,T]. \tag{3.5}$$

The above is called a *convex perturbation* of $\bar{u}(\cdot)$ associated with $u(\cdot)$. Let $X^\varepsilon(\cdot)$ be the state process corresponding to $u^\varepsilon(\cdot)$, and denote

$$\xi^\varepsilon(t) = \frac{X^\varepsilon(t) - \bar{X}(t)}{\varepsilon}, \qquad \Delta u(t) = u(\cdot) - \bar{u}(\cdot).$$

Then $\xi^\varepsilon(\cdot)$ satisfies the following:

$$\begin{aligned} d\xi^\varepsilon(t) &= \frac{1}{\varepsilon}\big[b(X^\varepsilon(t), u^\varepsilon(t)) - b(\bar{X}(t), \bar{u}(t))\big]dt \\ &\qquad + \frac{1}{\varepsilon}\big[\sigma(X^\varepsilon(t), u^\varepsilon(t)) - \sigma(\bar{X}(t), \bar{u}(t))\big]dW(t) \\ &= \big[b_x^\varepsilon(t)\xi^\varepsilon(t) + b_u^\varepsilon(t)\Delta u(t)\big]dt + \big[\sigma_x^\varepsilon(t)\xi^\varepsilon(t) + \sigma_u^\varepsilon(t)\Delta u(t)\big]dW(t), \end{aligned} \tag{3.6}$$

with $\xi^\varepsilon(0) = 0$, where

$$\begin{cases} b_x^\varepsilon(t) = \displaystyle\int_0^1 b_x(\bar{X}(t) + \alpha\varepsilon\xi^\varepsilon(t), \bar{u}(t) + \alpha\varepsilon\Delta u(t))d\alpha, \\ b_u^\varepsilon(t) = \displaystyle\int_0^1 b_u(\bar{X}(t) + \alpha\varepsilon\xi^\varepsilon(t), \bar{u}(t) + \alpha\varepsilon\Delta u(t))d\alpha, \\ \sigma_x^\varepsilon(t) = \displaystyle\int_0^1 \sigma_x(\bar{X}(t) + \alpha\varepsilon\xi^\varepsilon(t), \bar{u}(t) + \alpha\varepsilon\Delta u(t))d\alpha, \\ \sigma_u^\varepsilon(t) = \displaystyle\int_0^1 \sigma_u(\bar{X}(t) + \alpha\varepsilon\xi^\varepsilon(t), \bar{u}(t) + \alpha\varepsilon\Delta u(t))d\alpha. \end{cases}$$

Thus, for $p \geq 2$,

$$\mathbb{E}\Big[\sup_{t\in[0,T]} |\xi^\varepsilon(t)|^p\Big] \equiv \mathbb{E}\Big[\sup_{t\in[0,T]} \Big|\frac{X^\varepsilon(t) - \bar{X}(t)}{\varepsilon}\Big|^p\Big] \leq K\Big[\int_0^T |\Delta u(s)|^2 ds\Big]^{\frac{p}{2}}. \tag{3.7}$$

Now, let $X_1(\cdot)$ be the solution to the variational equation (3.3). Similar to (3.7), we have

$$\mathbb{E}\Big[\sup_{t\in[0,T]} |X_1(t)|^p\Big] \leq K\Big[\int_0^T |\Delta u(s)|^2 ds\Big]^{\frac{p}{2}} \leq K. \tag{3.8}$$

Next, set $\xi_1^\varepsilon(\cdot) \equiv \xi^\varepsilon(\cdot) - X_1(\cdot)$ which satisfies

$$\begin{cases} d\xi_1^\varepsilon(t) = \big\{b_x^\varepsilon(t)\xi_1^\varepsilon(t) + \big[b_x^\varepsilon(t) - b_x(t)\big]X_1(t) + \big[b_u^\varepsilon(t) - b_u(t)\big]\Delta u(t)\big\}dt \\ \qquad\qquad + \big\{\sigma_x^\varepsilon(t)\xi_1^\varepsilon(t) + \big[\sigma_x^\varepsilon(t) - \sigma_x(t)\big]X_1(t) + \big[\sigma_u^\varepsilon(t) - \sigma_u(t)\big]\Delta u(t)\big\}dW(t), \\ \xi_1^\varepsilon(0) = 0. \end{cases}$$

Then

$$
\begin{aligned}
&\mathbb{E}\Big[\sup_{t\in[0,T]}|\xi_1^\varepsilon(t)|^p\Big]\\
&\le K\Big\{\mathbb{E}\Big[\int_0^T\Big|\big[b_x^\varepsilon(t)-b_x(t)\big]X_1(t)+\big[b_u^\varepsilon(t)-b_u(t)\big]\Delta u(t)\Big|dt\Big]^p\\
&\qquad+\mathbb{E}\Big[\int_0^T\Big|\big[\sigma_x^\varepsilon(t)-\sigma_x(t)\big]X_1(t)+\big[\sigma_u^\varepsilon(t)-\sigma_u(t)\big]\Delta u(t)\Big|^2dt\Big]^{\frac{p}{2}}\Big\}\\
&\le K\Big\{\Big[\mathbb{E}\Big(\sup_{t\in[0,T]}|X_1(t)|^p\Big)\Big]\Big[\mathbb{E}\Big(\int_0^T|b_x^\varepsilon(t)-b_x(t)|dt\Big)^{\frac{p}{p-1}}\Big]^{p-1}\\
&\qquad+\mathbb{E}\Big[\int_0^T|b_u^\varepsilon(t)-b_u(t)|dt\Big]^p\\
&\qquad+\Big[\mathbb{E}\Big(\sup_{t\in[0,T]}|X_1(t)|^p\Big)\Big]\Big[\mathbb{E}\Big(\int_0^T|\sigma_x^\varepsilon(t)-\sigma_x(t)|^2dt\Big)^{\frac{p}{p-2}}\Big]^{p-2}\\
&\qquad+\mathbb{E}\Big[\int_0^T|\sigma_u^\varepsilon(t)-\sigma_u(t)|^2dt\Big]^{\frac{p}{2}}\Big\}\\
&\le K\Big\{\Big[\mathbb{E}\Big(\int_0^T|b_x^\varepsilon(t)-b_x(t)|dt\Big)^{\frac{p}{p-1}}\Big]^{p-1}+\mathbb{E}\Big[\int_0^T|b_u^\varepsilon(t)-b_u(t)|dt\Big]^p\\
&\qquad+\Big[\mathbb{E}\Big(\int_0^T|\sigma_x^\varepsilon(t)-\sigma_x(t)|^2dt\Big)^{\frac{p}{p-2}}\Big]^{p-2}+\mathbb{E}\Big[\int_0^T|\sigma_u^\varepsilon(t)-\sigma_u(t)|^2dt\Big]^{\frac{p}{2}}\Big\}\\
&=o(1).
\end{aligned}
\tag{3.9}
$$

The above means that

$$
\|X^\varepsilon(\cdot)-\bar X(\cdot)-\varepsilon X_1(\cdot)\|_{L^p_{\mathbb{F}}(\Omega;C([0,T];\mathbb{R}^n))}=o(\varepsilon). \tag{3.10}
$$

This is a Taylor expansion of the map $\varepsilon\mapsto X^\varepsilon(\cdot)$ near $\varepsilon=0$, in the space $L^p_{\mathbb{F}}(\Omega;C([0,T];\mathbb{R}^n))$. In particular, it implies that

$$
\|X^\varepsilon(\cdot)-\bar X(\cdot)\|_{L^p_{\mathbb{F}}(\Omega;C([0,T];\mathbb{R}^n))}=O(\varepsilon). \tag{3.11}
$$

Now, by the optimality of $(\bar X(\cdot),\bar u(\cdot))$,

$$
\begin{aligned}
0&\le\lim_{\varepsilon\to0}\frac{J(u^\varepsilon(\cdot))-J(\bar u(\cdot))}{\varepsilon}\\
&=\mathbb{E}\Big\{\int_0^T\Big(\langle g_x(\bar X(t),\bar u(t)),X_1(t)\rangle+\langle g_u(\bar X(t),\bar u(t)),\Delta u(t)\rangle\Big)dt\\
&\qquad\qquad+\langle h_x(\bar X(T)),X_1(T)\rangle\Big\}\\
&\equiv\mathbb{E}\Big\{\int_0^T\Big(g_x(t)X_1(t)+\langle g_u(t),\Delta u(t)\rangle\Big)dt+h_x(\bar X(T))X_1(T)\Big\},
\end{aligned}
$$

which proves (3.1). $\square$

Note that (3.1) is a necessary condition for the optimal pair $(\bar X(\cdot),\bar u(\cdot))$. Such a condition is not very useful, because process $X_1(\cdot)$ depends on $u(\cdot)$. Hence, we need to introduce a device to get rid of this process. This is the purpose of the following section.

3.2. **Linear BSDEs and duality relations.** We consider the following linear stochastic differential equation:

$$\begin{cases} dY(t)=-\big[B(t)Y(t)+C(t)Z(t)+g(t)\big]dt+Z(t)dW(t),\\ Y(T)=\eta,\end{cases}\tag{3.12}$$

where $B(\cdot)$, $C(\cdot)$ are $\mathbb{R}^{n\times n}$-valued $\mathbb{F}$-adapted processes, $g(\cdot)$ is an $\mathbb{R}^n$-valued $\mathbb{F}$-adapted process, and η is an $\mathbb{R}^n$-valued $\mathcal{F}_T$-measurable random variable. This is a terminal value problem for SDE. We call it a *backward stochastic differential equation* (BSDE, for short). The unknown of BSDE (3.12) that we are looking for is a pair $(Y(\cdot),Z(\cdot))$, called adapted solution, of $\mathbb{F}$-adapted processes that satisfy the following integral equation:

$$Y(t)=\eta+\int_t^T\big[B(s)Y(s)+C(s)Z(s)+g(s)\big]ds-\int_t^T Z(s)dW(s),\qquad t\in[0,T].$$

Note that η and the first integral in the above are merely $\mathcal{F}_T$-measurable. Hence, without the last term, it is not possible to have the $\mathbb{F}$-adaptiveness for $Y(\cdot)$. The last term is therefore called the *adjustment term*. The following result gives the well-posedness for BSDE (3.12).

Proposition 3.2. *Let $B(\cdot),C(\cdot)\in L^\infty_{\mathbb{F}}(0,T;\mathbb{R}^{n\times n})$. Then for any $g(\cdot)\in L^p_{\mathbb{F}}(\Omega;L^1(0,T;\mathbb{R}^n))$ and $\eta\in L^p_{\mathcal{F}_T}(\Omega;\mathbb{R}^n)$, with $p\in(1,\infty)$, there exists a unique adapted solution $(Y(\cdot),Z(\cdot))\in L^p_{\mathbb{F}}(\Omega;C([0,T];\mathbb{R}^n))\times L^p_{\mathbb{F}}(\Omega;L^2(0,T;\mathbb{R}^n))$ such that*

$$\mathbb{E}\Big[\sup_{t\in[0,T]}|Y(t)|^p+\Big(\int_0^T|Z(t)|^2dt\Big)^{\frac{p}{2}}\Big]\le KE\Big[|\eta|^p+\Big(\int_0^T|g(t)|dt\Big)^p\Big].$$

Proof. We only prove the case $n=1$ (The general case is left to the interested readers). Let $S\in[0,T)$ be undetermined, and

$$\mathcal{H}^p[S,T]=L^p_{\mathbb{F}}(\Omega;C([S,T];\mathbb{R}))\times L^p_{\mathbb{F}}(\Omega;L^2(S,T;\mathbb{R})).$$

For any $(y(\cdot),z(\cdot))\in\mathcal{H}^p[S,T]$, let

$$f(\cdot)=B(\cdot)y(\cdot)+C(\cdot)z(\cdot)+g(\cdot)\in L^p_{\mathbb{F}}(\Omega;L^1(S,T;\mathbb{R})),$$

and consider the following BSDE:

$$\begin{cases} dY(t)=-f(t)dt+Z(t)dW(t), & t\in[S,T],\\ Y(T)=\eta.\end{cases}\tag{3.13}$$

Define

$$\begin{cases} M(t)=\mathbb{E}\Big[\eta+\int_S^T f(s)ds\big|\,\mathcal{F}_t\Big],\\ Y(t)=\mathbb{E}\Big[\eta+\int_t^T f(s)ds\big|\,\mathcal{F}_t\Big]=M(t)-\int_S^t f(s)ds,\end{cases}\qquad t\in[S,T].\tag{3.14}$$

Clearly, $M(\cdot)$ is a continuous $\mathbb{F}$-martingale on $[S,T]$ with $M(S)=Y(S)$ and

$$\begin{aligned}&\mathbb{E}|M(t)|^p=\mathbb{E}\Big|\mathbb{E}\Big[\eta+\int_S^T f(s)ds\big|\,\mathcal{F}_t\Big]\Big|^p\\ &\le\mathbb{E}\Big\{\mathbb{E}\Big[\big|\eta+\int_S^T f(s)ds\big|^p\big|\,\mathcal{F}_t\Big]\Big\}=\mathbb{E}\big|\eta+\int_S^T f(s)ds\big|^p\\ &\le 2^{p-1}\Big[\mathbb{E}|\eta|^p+\mathbb{E}\Big(\int_S^T|f(s)|ds\Big)^p\Big].\end{aligned}$$

Thus, by martingale representation theorem, there exists a unique $Z(\cdot)\in L^p_{\mathbb{F}}(\Omega; L^2(S,T;\mathbb{R}))$ such that

$$M(t)=Y(S)+\int_S^t Z(s)dW(s),\qquad t\in[S,T]. \tag{3.15}$$

Since η is $\mathcal{F}_T$-measurable, we further have (from (3.14))

$$\eta+\int_S^T f(s)ds=M(T)=Y(S)+\int_S^T Z(s)dW(s).$$

Hence,

$$\begin{aligned}Y(t)&=M(t)-\int_S^t f(s)ds=Y(S)+\int_S^t Z(s)dW(s)-\int_S^t f(s)ds\\&=\eta+\int_S^T f(s)ds-\int_S^T Z(s)dW(s)+\int_S^t Z(s)dW(s)-\int_S^t f(s)ds\\&=\eta+\int_t^T f(s)ds-\int_t^T Z(s)dW(s),\qquad t\in[S,T].\end{aligned}\tag{3.16}$$

This means that $(Y(\cdot),Z(\cdot))$ is an adapted solution to BSDE (3.13). We now establish the estimate. First of all, we have

$$\begin{aligned}&\mathbb{E}\Big[\sup_{t\in[S,T]}|Y(t)|^p\Big]=\mathbb{E}\Big\{\sup_{t\in[S,T]}\Big|\mathbb{E}\Big[\eta+\int_t^T f(s)ds\mid\mathcal{F}_t\Big]\Big|^p\Big\}\\&\le\mathbb{E}\Big(|\eta|+\int_S^T|f(s)|ds\Big)^p\le 2^{p-1}\Big[\mathbb{E}|\eta|^p+\mathbb{E}\Big(\int_S^T|f(s)|ds\Big)^p\Big].\end{aligned}$$

Then by Burkholder–Davis–Gundy's inequalities, we have

$$\begin{aligned}&\mathbb{E}\Big(\int_S^T|Z(s)|^2ds\Big)^{\frac{p}{2}}\le K\mathbb{E}\Big[\sup_{t\in[S,T]}\Big|\int_S^t Z(s)dW(s)\Big|^p\Big]\\&\le K\mathbb{E}\Big[\sup_{t\in[S,T]}\Big|\int_S^T Z(s)dW(s)-\int_t^T Z(s)dW(s)\Big|^p\Big]\\&\le K\mathbb{E}\Big[\sup_{t\in[S,T]}\Big|\int_t^T Z(s)dW(s)\Big|^p\Big]\\&\le K\mathbb{E}\Big[\sup_{t\in[S,T]}\Big|\eta+\int_t^T f(s)ds-Y(t)\Big|^p\Big]\\&\le K\mathbb{E}\Big[|\eta|^p+\Big(\int_S^T|f(s)|ds\Big)^p\Big].\end{aligned}$$

Hence,

$$\mathbb{E}\Big[\sup_{t\in[S,T]}|Y(t)|^p+\Big(\int_S^T|Z(s)|^2ds\Big)^{\frac{p}{2}}\le K\mathbb{E}\Big[|\eta|^p+\Big(\int_S^T|f(s)|ds\Big)^p\Big].$$

Since BSDE (3.13) is linear, the above estimate also gives the uniqueness of adapted solution to (3.13). It is seen that a map $\Phi:\mathcal{H}^p[S,T]\to\mathcal{H}^p[S,T]$, $(y(\cdot),z(\cdot))\mapsto(Y(\cdot),Z(\cdot))$ is well-defined. Now, let $(\bar y(\cdot),\bar z(\cdot))\in\mathcal{H}^p[S,T]$ and let $(\bar Y(\cdot),\bar Z(\cdot))$ be the corresponding adapted solution to (3.13) with $f(\cdot)$ replaced by $B(\cdot)\bar y(\cdot)+C(\cdot)\bar z(\cdot)+g(\cdot)$. Then

$$\begin{aligned}&\mathbb{E}\Big[\sup_{t\in[S,T]}|Y(t)-\bar Y(t)|^p\Big]+\mathbb{E}\Big(\int_S^T|Z(t)-\bar Z(t)|^2dt\Big)^{\frac p2}\\&\le K\mathbb{E}\Big[\int_S^T\Big(|B(t)|\,|y(t)-\bar y(t)|+|C(t)|\,|z(t)-\bar z(t)|\Big)dt\Big]^p\\&\le K_0\Big[\Big(\int_S^T\|B(t)\|_\infty dt\Big)^p+\int_S^T\|C(t)\|^2_\infty dt\Big)^{\frac p2}\Big]\\&\qquad\cdot\Big\{\mathbb{E}\Big[\sup_{t\in[S,T]}|y(t)-\bar y(t)|^p\Big]+\mathbb{E}\Big(\int_S^T|z(t)-\bar z(t)|^2dt\Big)^{\frac p2}\Big\}.\end{aligned}$$

In the above, the constant $K_0>0$ is uniform (independent of $(y(\cdot),z(\cdot))$, $(\bar y(\cdot),\bar z(\cdot))$, $B(\cdot)$, and $C(\cdot)$). Hence, by choosing $S=T-\delta$ with $\delta>0$ small, we see that Φ is a contraction on $\mathcal{H}^p[T-\delta,T]$. By contraction mapping theorem, there exists a unique fixed point which is the unique adapted solution to (3.12) over $[T-\delta,T]$. Further,

$$\begin{aligned}&\mathbb{E}\Big[\sup_{t\in[S,T]}|Y(t)|^p+\Big(\int_S^T|Z(s)|^2ds\Big)^{\frac p2}\Big]\\&\le K\mathbb{E}\Big\{|\eta|^p+\Big[\int_S^T\Big(|B(s)|\,|Y(s)|+|C(s)|\,|Z(s)|+|g(s)|\Big)ds\Big]^p\Big\}\\&\le K\mathbb{E}\Big[|\eta|^p+\Big(\int_S^T|g(s)|ds\Big)^p+(T-S)\|B(\cdot)\|^p_\infty\sup_{t\in[S,T]}|Y(t)|^p\\&\qquad+(T-S)^{\frac p2}\|C(\cdot)\|^p_\infty\Big(\int_S^T|Z(s)|^2ds\Big)^{\frac p2}\Big]\\&\le K\mathbb{E}\Big[|\eta|^p+\Big(\int_S^T|g(s)|ds\Big)^p\Big]\\&\qquad+K_0(T-S)^{\frac p2}\mathbb{E}\Big[\sup_{t\in[S,T]}|Y(t)|^p+\Big(\int_S^T|Z(s)|^2ds\Big)^{\frac p2}\Big],\end{aligned}$$

with $K,K_0>0$ being absolute constants. Hence, by shrinking $T-S$, and iteration, we obtain

$$\mathbb{E}\Big[\sup_{t\in[S,T]}|Y(t)|^p+\Big(\int_S^T|Z(s)|^2ds\Big)^{\frac p2}\Big]\le K\mathbb{E}\Big[|\eta|^p+\Big(\int_S^T|g(s)|ds\Big)^p\Big].$$

Then use the same argument we can obtain the solvability of (3.12) over $[T-2\delta,T-\delta]$, and so on, and eventually obtain the unique solvability of (3.12) over $[0,T]$, with the desired estimate. □

Now, we introduce the following linear BSDE:

$$\begin{cases}dY(t)=-\Big[b_x(t)^\top Y(t)+\sigma_x(t)^\top Z(t)-g_x(t)\Big]dt+Z(t)dW(t),\\ Y(T)=-h_x(\bar X(T)),\end{cases}\tag{3.17}$$

where $b_x(\cdot)$, $\sigma_x(\cdot)$, and $g_x(\cdot)$ are defined by (3.4) and (3.2). The above is called the *adjoint equation* of (3.3). According to our condition, the above linear BSDE admits a unique adapted solution $(Y(\cdot), Z(\cdot))$. Next, let $X_1(\cdot)$ be the solution to (3.3). Applying Itô's formula to $\langle X_1(\cdot), Y(\cdot)\rangle$, one has

$$\begin{aligned} d\,\langle X_1(t), Y(t)\rangle &= \Big\{ \langle b_x(t)X_1(t) + b_u(t)\Delta u(t), Y(t)\rangle \\ &\quad - \langle X_1(t), b_x(t)^\top Y(t) + \sigma_x(t)^\top Z(t) - g_x(t)\rangle \\ &\quad + \langle \sigma_x(t)X_1(t) + \sigma_u(t)\Delta u(t), Z(t)\rangle \Big\} dt \\ &\quad + \Big\{ \langle \sigma_x(t)X_1(t) + \sigma_u(t)\Delta u(t), Y(t)\rangle + \langle X_1(t), Z(t)\rangle \Big\} dW(t) \\ &= \Big[\langle b_u(t)^\top Y(t) + \sigma_u(t)^\top Z(t), \Delta u(t)\rangle + \langle g_x(t), X_1(t)\rangle \Big] dt + \{\cdots\} dW(t). \end{aligned}$$

Consequently,

$$\begin{aligned} -\mathbb{E}\,\langle h_x(X(T)), X_1(T)\rangle &= \mathbb{E}\,\langle Y(T), X_1(T)\rangle - \langle Y(0), X_1(0)\rangle \\ &= \mathbb{E}\int_0^T \Big[\langle b_u(t)^\top Y(t) + \sigma_u(t)^\top Z(t), \Delta u(t)\rangle + \langle g_x(t), X_1(t)\rangle \Big] dt. \end{aligned}$$

The above is called a *duality relation* between $X_1(\cdot)$ and $Y(\cdot)$. By (3.1), we have

$$\begin{aligned} 0 &\le \mathbb{E}\Big\{ \int_0^T [\langle g_x(t), X_1(t)\rangle + \langle g_u(t), \Delta u(t)\rangle]\, dt + \langle h_x(\bar X(T)), X_1(T)\rangle \Big\} \\ &= -\mathbb{E}\int_0^T \langle b_u(t)^\top Y(t) + \sigma_u(t)^\top Z(t) - g_u(t), \Delta u(t)\rangle\, dt. \end{aligned} \tag{3.18}$$

To proceed further, we need the following lemma.

Lemma 3.3. *If $F(\cdot) \in L^2_{\mathbb{F}}(0,T;\mathbb{R}^m)$ such that*

$$\mathbb{E}\int_0^T \langle F(t), u(t) - \bar u(t)\rangle\, dt \le 0, \qquad \forall u(\cdot) \in \mathcal{U}[0,T], \tag{3.19}$$

then

$$\langle F(t), u - \bar u(t)\rangle \le 0, \qquad \forall u \in U, \ \text{a.e.} \ \ t \in [0,T], \ \text{a.s.} \tag{3.20}$$

Proof. Suppose (3.20) fails. Then for some $u_0 \in U$, and some $\varepsilon > 0$, by defining

$$\Lambda_\varepsilon \overset{\Delta}{=} \{(t,\omega) \in [0,T]\times\Omega \;\big|\; \langle F(t,\omega), u_0 - \bar u(t,\omega)\rangle \ge \varepsilon\},$$

one has

$$\int_\Omega \int_0^T I_{\Lambda_\varepsilon}(t,\omega)\, dt\, dP(\omega) > 0.$$

Since $\langle F(\cdot), u_0 - \bar u(\cdot)\rangle$ is $\mathbb{F}$-progressively measurable, so is the process $t \mapsto I_{\Lambda_\varepsilon}(t,\omega)$. Now, we define

$$\widehat u(t,\omega) = u_0 I_{\Lambda_\varepsilon}(t,\omega) + \bar u(t,\omega) I_{\Lambda_\varepsilon^c}(t,\omega), \quad (t,\omega) \in [0,T]\times\Omega.$$

Clearly, $\widehat u(\cdot) \in \mathcal{U}[0,T]$. By taking $u(\cdot) = \widehat u(\cdot)$ in (3.19), we obtain

$$\begin{aligned} &\mathbb{E}\int_0^T \langle F(t), u(t) - \bar u(t)\rangle\, dt = \int_\Omega\int_0^T I_{\Lambda_\varepsilon}(t,\omega)\,\langle F(t,\omega), u_0 - \bar u(t,\omega)\rangle\, dt\, dP(\omega) \\ &\ge \varepsilon \int_\Omega \int_0^T I_{\Lambda_\varepsilon}(t,\omega)\, dt\, dP(\omega) > 0, \end{aligned}$$

which contradicts (3.19). $\square$

Now, applying the above lemma to (3.18), we have

$$\begin{aligned}&\langle\, b_u(\bar X(t),\bar u(t))^\top Y(t)+\sigma_u(\bar X(t),\bar u(t))^\top Z(t)-g_u(\bar X(t),\bar u(t)),u-\bar u(t)\,\rangle\le 0,\\ &\qquad\qquad\qquad\qquad\qquad\forall u\in U,\ \text{a.e.}\ \ t\in[0,T],\ \text{a.s.}\end{aligned}\tag{3.21}$$

To summarize the above, we can state the following result.

Theorem 3.4. *Let* (H1) *hold. Let* $(\bar X(\cdot),\bar u(\cdot))$ *be an optimal pair of Problem (C). Then there exists an adapted solution* $(Y(\cdot),Z(\cdot))$ *to* (3.17) *such that* (3.21) *holds.*

The above result is called *Pontryagin's maximum principle*. If we define

$$H(x,y,z,u)=\langle\, y,b(x,u)\,\rangle+\langle\, z,\sigma(x,u)\,\rangle-g(x,u),\tag{3.22}$$

which is called a *Hamiltonian*, then condition (3.21) reads

$$\langle\, H_u(\bar X(t),Y(t),Z(t),\bar u(t)),u-\bar u(t)\,\rangle\le 0,\quad \forall u\in U,\ \text{a.e.}\ \ t\in[0,T],\ \text{a.s.}$$

Now, if $U=\mathbb{R}^m$, condition (3.21) becomes

$$b_u(\bar X(t),\bar u(t))^\top Y(t)+\sigma_u(\bar X(t),\bar u(t))^\top Z(t)-g_u(\bar X(t),\bar u(t))=0.$$

Then, by dropping bars in $(\bar X(\cdot),\bar u(\cdot))$, we obtain the following *optimality system*:

$$\begin{cases} dX(t)=b(X(t),u(t))dt+\sigma(X(t),u(t))dW(t),\\ dY(t)=-\Big[b_x(X(t),u(t))^\top Y(t)+\sigma_x(X(t),u(t))^\top Z(t)\\ \qquad\qquad\qquad -g_x(X(t),u(t))\Big]dt+Z(t)dW(t),\\ X(0)=X_0,\qquad Y(T)=h_x(X(T)),\\ b_u(X(t),u(t))^\top Y(t)+\sigma_u(X(t),u(t))^\top Z(t)-g_u(X(t),u(t))=0.\end{cases}\tag{3.23}$$

This is called a (*coupled*) *forward-backward stochastic differential equation* (FBSDE, for short). Note that the coupling appears through the last equality which is also called the *stationarity condition*.

If one can show through a different method that Problem (C) admits a unique optimal control and can show that the above FBSDE admits a unique adapted solution $(X(\cdot),Y(\cdot),Z(\cdot))$, then $X(\cdot)$ is the optimal state process and the corresponding $u(\cdot)$ is the optimal control.

3.3. **General stochastic maximum principle.** Note that if the control set U is just a metric space (not necessarily to have an algebraic structure; for example, $U=\{0,1\}$), then convex perturbation of form (3.5) is not allowed. For such a case, one needs to adopt *spike variations*. To get some feeling, let us look at the following example.

Example 3.5. Consider the following control system ($n=m=1$):

$$\begin{cases} dX(t)=u(t)dW(t),\qquad t\in[0,T],\\ X(0)=0.\end{cases}\tag{3.24}$$

with the control set being $U=\{0,1\}$ and the cost functional being

$$J(u)=\mathbb{E}X(T)^2.$$

Then the optimal pair is clearly given by $(\bar{X}(t), \bar{u}(t)) \equiv (0,0)$. Since U is not convex, convex perturbation is not allowed. To make an admissible perturbation, let $u(t) \equiv 1$, $E_\varepsilon = [s, s+\varepsilon] \subseteq [0,T]$, and set

$$u^\varepsilon(t) = \begin{cases} 0, & \text{if } t \in [0,T] \setminus E_\varepsilon, \\ 1, & \text{if } t \in E_\varepsilon. \end{cases}$$

Clearly, $u^\varepsilon(\cdot) \in \mathcal{U}[0,T]$. Such a control is called a spike variation of the optimal control $\bar{u}(\cdot) = 0$ associated with the control $u(t) \equiv 1$. Let $X^\varepsilon(\cdot)$ be the solution of the state equation (3.24) corresponding to $u^\varepsilon(\cdot)$. Then, by Itô's formula, for $t \geq s + \varepsilon$,

$$\mathbb{E}|X^\varepsilon(t) - \bar{X}(t)|^2 = \mathbb{E}|X^\varepsilon(t)|^2 = \mathbb{E}\int_0^t |u^\varepsilon(r)|^2 dr = \varepsilon.$$

This means that when the spike variations are used, we do not have (3.11). Hence, one could not expect the existence of the limit

$$\lim_{\varepsilon \to 0} \frac{X^\varepsilon(\cdot) - \bar{X}(\cdot)}{\varepsilon}.$$

Consequently, the approach used for the case of convex control set U does not apply here.

This section is devoted to the introduction of a new method. For simplicity of presentation, we again use the same framework of the previous section. Instead of assumption (H1), we introduce the following assumption.

(H2) Let (U,d) be a separable metric space and $T > 0$. The maps b, σ, g and h are measurable and C^2 in x. Moreover, there exist a constant $L > 0$ and a modulus of continuity $\overline{\omega} : [0,\infty) \to [0,\infty)$, such that for $\varphi(x,u) = b(x,u), \sigma(x,u), g(x,u), h(x)$,

$$\begin{cases} |\varphi(0,u)| \leq L, \qquad \forall u \in U, \\ |\varphi(x,u) - \varphi(\widehat{x}, \widehat{u})| \leq L|x - \widehat{x}| + \overline{\omega}\big(d(u,\widehat{u})\big), \\ |\varphi_x(x,u) - \varphi_x(\widehat{x}, \widehat{u})| \leq L|x - \widehat{x}| + \overline{\omega}(d(u,\widehat{u})), \\ |\varphi_{xx}(x,u) - \varphi_{xx}(\widehat{x}, \widehat{u})| \leq \overline{\omega}(|x - \widehat{x}| + d(u,\widehat{u})), \end{cases} \qquad \forall x, \widehat{x} \in \mathbb{R}^n,\ u, \widehat{u} \in U.$$

Our main result of this section is the following theorem which is called the general stochastic Pontryagin maximum principle.

Theorem 3.6. *Let* (H2) *hold. Let* $(\bar{X}(\cdot), \bar{u}(\cdot))$ *be an optimal pair of Problem (C). Then there are pairs of processes*

$$\begin{cases} (Y(\cdot), Z(\cdot)) \in L^2_{\mathbb{F}}(\Omega; C([0,T]; \mathbb{R}^n)) \times L^2_{\mathbb{F}}(0,T;\mathbb{R}^n), \\ (\Gamma(\cdot), \Lambda(\cdot)) \in L^2_{\mathbb{F}}(\Omega; C([0,T; \mathbb{S}^n)) \times L^2_{\mathbb{F}}(0,T;\mathbb{S}^n), \end{cases}$$

such that

$$\begin{aligned} &H(\bar{X}(t), \bar{u}(t), Y(t), Z(t)) - H(\bar{X}(t), u, Y(t), Z(t)) \\ &-\frac{1}{2}\mathrm{tr}\Big(\big[\sigma(\bar{X}(t), \bar{u}(t)) - \sigma(\bar{X}(t), u)\big]^\top \Gamma(t) \big[\sigma(\bar{X}(t), \bar{u}(t)) - \sigma(\bar{X}(t), u)\big]\Big) \geq 0, \\ &\qquad\qquad \forall u \in U, \quad \text{a.e. } t \in [0,T], \quad \text{a.s. }, \end{aligned} \tag{3.25}$$

where $(Y(\cdot),Z(\cdot))$ and $(\Gamma(\cdot),\Lambda(\cdot))$ are adapted solutions to the following first-order and second-order adjoint equations, respectively:

$$\begin{cases} dY(t)=-H_x(\bar X(t),\bar u(t),Y(t),Z(t))dt+Z(t)dW(t), \quad t\in[0,T], \\ Y(T)=-h_x(\bar X(T)), \end{cases} \tag{3.26}$$

and

$$\begin{cases} d\Gamma(t)=-\big[b_x(\bar X(t),\bar u(t))^\top\Gamma(t)+\Gamma(t)b_x(\bar X(t),\bar u(t)) \\ \qquad\qquad +\sigma_x(\bar X(t),\bar u(t))^\top\Gamma(t)\sigma_x(\bar X(t),\bar u(t)) \\ \qquad\qquad +\sigma_x(\bar X(t),\bar u(t))^\top\Lambda(t)+\Lambda(t)\sigma_x(\bar X(t),\bar u(t)) \\ \qquad\qquad +H_{xx}(\bar X(t),\bar u(t),Y(t),Z(t))\big]dt+\Lambda(t)dW(t), \\ \Gamma(T)=-h_{xx}(\bar X(T)), \end{cases} \tag{3.27}$$

with the Hamiltonian H given by

$$\begin{aligned} H(x,u,y,z)=b(x,u)^\top y+\sigma(x,u)^\top z-g(x,u),& \\ (x,u,y,z)\in\mathbb{R}^n\times U\times\mathbb{R}^n\times\mathbb{R}^n.& \end{aligned}$$

Proof. Let $(\bar X(\cdot),\bar u(\cdot))$ be a given optimal pair. Fix any $u(\cdot)\in\mathcal{U}[0,T]$ and $\varepsilon>0$. Define

$$u^\varepsilon(t)=\begin{cases} \bar u(t), & t\in[0,T]\setminus E_\varepsilon, \\ u(t), & t\in E_\varepsilon, \end{cases}$$

where $E_\varepsilon\subseteq[0,T]$ is a measurable set with $|E_\varepsilon|=\varepsilon$. Let $X^\varepsilon(\cdot)$ be the state process corresponding to $u^\varepsilon(\cdot)$. Next, for $\varphi=b^i,\sigma^i,g$ $(1\le i\le n)$, we define

$$\begin{cases} \varphi_x(t)=\varphi_x(\bar X(t),\bar u(t)), \quad \varphi_{xx}(t)=\varphi_{xx}(\bar X(t),\bar u(t)), \\ \delta\varphi(t)=\varphi(\bar X(t),u(t))-\varphi(\bar X(t),\bar u(t)), \\ \delta\varphi_x(t)=\varphi_x(\bar X(t),u(t))-\varphi_x(\bar X(t),\bar u(t)), \\ \delta\varphi_{xx}(t)=\varphi_{xx}(\bar x(t),u(t))-\varphi_{xx}(\bar X(t),\bar u(t)). \end{cases}$$

Let $X_1^\varepsilon(\cdot)$ and $X_2^\varepsilon(\cdot)$ be the solution of the following equations, respectively:

$$\begin{cases} dX_1^\varepsilon(t)=b_x(t)X_1^\varepsilon(t)dt+\big[\sigma_x(t)X_1^\varepsilon(t)+\delta\sigma(t)\chi_{E_\varepsilon}(t)\big]dW(t), \quad t\in[0,T], \\ X_1^\varepsilon(0)=0, \end{cases}$$

and

$$\begin{cases} dX_2^\varepsilon(t)=\big[b_x(t)X_2^\varepsilon(t)+\delta b(t)\chi_{E_\varepsilon}(t)+\frac{1}{2}b_{xx}(t)[X_1^\varepsilon(t)]^2\big]dt \\ \qquad\qquad +\big[\sigma_x(t)X_2^\varepsilon(t)+\delta\sigma_x(t)X_1^\varepsilon(t)\chi_{E_\varepsilon}(t) \\ \qquad\qquad +\frac{1}{2}\sigma_{xx}(t)[X_1^\varepsilon(t)]^2\big]dW(t), \qquad t\in[0,T], \\ X_2^\varepsilon(0)=0, \end{cases}$$

where

$$b_{xx}(t)[X_1^\varepsilon(t)]^2 \triangleq \begin{pmatrix} \mathrm{tr}\left[b_{xx}^1(t)X_1^\varepsilon(t)X_1^\varepsilon(t)^\top\right] \\ \vdots \\ \mathrm{tr}\left[b_{xx}^n(t)X_1^\varepsilon(t)X_1^\varepsilon(t)^\top\right] \end{pmatrix},$$

$$\sigma_{xx}(t)[X_1^\varepsilon(t)]^2 \triangleq \begin{pmatrix} \mathrm{tr}\left[\sigma_{xx}^1(t)X_1^\varepsilon(t)X_1^\varepsilon(t)^\top\right] \\ \vdots \\ \mathrm{tr}\left[\sigma_{xx}^n(t)X_1^\varepsilon(t)X_1^\varepsilon(t)^\top\right] \end{pmatrix}.$$

Then we have the following *Taylor expansion* of the state with respect to the spike variation of the control: For any $p \geq 1$, it holds that

$$\begin{aligned}
&\mathbb{E}\Big[\sup_{t\in[0,T]}|X^\varepsilon(t)-\bar{X}(t)|^p\Big] = O(\varepsilon^{\frac{p}{2}}), \quad \mathbb{E}\Big[\sup_{t\in[0,T]}|X_1^\varepsilon(t)|^p\Big] = O(\varepsilon^{\frac{p}{2}}),\\
&\mathbb{E}\Big[\sup_{t\in[0,T]}|X_2^\varepsilon(t)|^p\Big] = O(\varepsilon^{p}), \quad \mathbb{E}\Big[\sup_{t\in[0,T]}|X^\varepsilon(t)-\bar{X}(t)-X_1^\varepsilon(t)|^p\Big] = O(\varepsilon^{p}),\\
&\mathbb{E}\Big[\sup_{t\in[0,T]}|X^\varepsilon(t)-\bar{X}(t)-X_1^\varepsilon(t)-X_2^\varepsilon(t)|^p\Big] = o(\varepsilon^{p}).
\end{aligned}$$

Moreover, the following expansion holds for the cost functional:

$$\begin{aligned}
J(u^\varepsilon(\cdot)) = J(\bar{u}(\cdot)) &+ \mathbb{E}\big\{h_x(\bar{X}(T))[X_1^\varepsilon(T)+X_2^\varepsilon(T)]\big\}\\
&+\frac{1}{2}\mathbb{E}\,\langle\, h_{xx}(\bar{X}(T))X_1^\varepsilon(T), X_1^\varepsilon(T)\,\rangle\\
&+\mathbb{E}\int_0^T \big\{g_x(t)[X_1^\varepsilon(t)+X_2^\varepsilon(t)] + \frac{1}{2}\,\langle\, g_{xx}(t)X_1^\varepsilon(t), X_1^\varepsilon(t)\,\rangle\\
&\qquad +\delta g(t)\chi_{E_\varepsilon}(t)\big\}dt + o(\varepsilon).
\end{aligned}$$

Then by the optimality of $(\bar{X}(\cdot),\bar{u}(\cdot))$, we have the following:

$$\begin{aligned}
0 \leq\ &\mathbb{E}\big\{h_x(\bar{X}(T))[X_1^\varepsilon(T)+X_2^\varepsilon(T)] + \frac{1}{2}\,\langle\, h_{xx}(\bar{X}(T))X_1^\varepsilon(T), X_1^\varepsilon(T)\,\rangle\,\big\}\\
&+\mathbb{E}\int_0^T \Big\{g_x(t)[X_1^\varepsilon(t)+X_2^\varepsilon(t)] + \frac{1}{2}\,\langle\, g_{xx}(t)X_1^\varepsilon(t), X_1^\varepsilon(t)\,\rangle\\
&\qquad +\delta g(t)\chi_{E_\varepsilon}(t)\Big\}dt + o(\varepsilon), \quad \forall u(\cdot)\in\mathcal{U}[0,T], \forall\varepsilon>0.
\end{aligned}$$

The following two equalities are dualities between $X_1^\varepsilon(\cdot)$ and $Y(\cdot)$, and between $X_2^\varepsilon(\cdot)$ and $Y(\cdot)$.

$$\mathbb{E}\,\langle\, Y(T), X_1^\varepsilon(T)\,\rangle = \mathbb{E}\int_0^T \Big[g_x(t)X_1^\varepsilon(t) + \mathrm{tr}\left[Z(t)^\top\delta\sigma(t)\right]\chi_{E_\varepsilon}(t)\Big]dt,$$

and

$$\begin{aligned}
\mathbb{E}\,\langle\, Y(T), X_2^\varepsilon(T)\,\rangle = \mathbb{E}&\int_0^T \Big\{g_x(t)X_2^\varepsilon(t)\\
&+\frac{1}{2}\Big(\,\langle\, Y(t), b_{xx}(t)[X_1^\varepsilon(t)]^2\,\rangle + \langle\, Z(t), \sigma_{xx}(t)[X_1^\varepsilon(t)]^2\,\rangle\,\Big)\\
&+\big[\,\langle\, Y(t), \delta b(t)\,\rangle + \langle\, Z(t), \delta\sigma_x(t)X_1^\varepsilon(t)\,\rangle\,\big]\chi_{E_\varepsilon}(t)\Big\}dt.
\end{aligned}$$

Consequently,

$$
\begin{aligned}
0 \geq J(\bar{u}(\cdot)) - J(u^\varepsilon(\cdot)) &= -\frac{1}{2}\mathbb{E}\,\langle\, h_{xx}(\bar{X}(T))X_1^\varepsilon(T), X_1^\varepsilon(T)\,\rangle \\
&\quad + \frac{1}{2}\mathbb{E}\int_0^T \Big(-\langle\, g_{xx}(t)X_1^\varepsilon(t), X_1^\varepsilon(t)\,\rangle + \langle\, Y(t), b_{xx}(t)[X_1^\varepsilon(t)]^2\,\rangle \\
&\qquad\qquad + \langle\, Z(t), \sigma_{xx}(t)[X_1^\varepsilon(t)]^2\,\rangle \Big)dt \\
&\quad + \mathbb{E}\int_0^T \Big(-\delta g(t) + \langle\, Y(t), \delta b(t)\,\rangle + \langle\, Z(t), \delta\sigma(t)\,\rangle \Big)\chi_{E_\varepsilon}(t)dt + o(\varepsilon) \\
&= \frac{1}{2}\mathbb{E}\Big[\mathrm{tr}\,\big\{\Gamma(T)[X_1^\varepsilon(T)X_1^\varepsilon(T)^\top]\big\}\Big] \\
&\quad + \mathbb{E}\Big[\int_0^T \Big(\frac{1}{2}\mathrm{tr}\,\big\{H_{xx}(t)[X_1^\varepsilon(t)X_1^\varepsilon(t)^\top]\big\} + \delta H(t)\chi_{E_\varepsilon}(t)\Big)dt\Big] + o(\varepsilon),
\end{aligned}
$$

where

$$
\begin{cases}
H_{xx}(t) = H_{xx}(t, \bar{X}(t), \bar{u}(t), Y(t), Z(t)), \\
\delta H(t) = H(t, \bar{X}(t), u(t), Y(t), Z(t)) - H(t, \bar{X}(t), \bar{u}(t), Y(t), Z(t)).
\end{cases}
$$

Next, we have the duality between $[X_1^\varepsilon(\cdot)X_1^\varepsilon(\cdot)^\top]$ and $\Gamma(\cdot)$:

$$
\begin{aligned}
&\mathbb{E}\Big(\mathrm{tr}\,\big[\Gamma(T)X_1^\varepsilon(T)X_1^\varepsilon(T)^\top\big]\Big) \\
&= \mathbb{E}\int_0^T \mathrm{tr}\,\big[\delta\sigma(t)^\top\Gamma(t)\delta\sigma(t)\chi_{E_\varepsilon}(t) - H_{xx}(t)X_1^\varepsilon(t)X_1^\varepsilon(t)^\top\big]dt + o(\varepsilon).
\end{aligned}
$$

Hence,

$$
\begin{aligned}
0 &\geq J(\bar{u}(\cdot)) - J(u^\varepsilon(\cdot)) \\
&= \mathbb{E}\int_0^T \Big(\delta H(t) + \frac{1}{2}\mathrm{tr}\,\big[\delta\sigma(t)^\top\Gamma(t)\delta\sigma(t)\big]\Big)\chi_{E_\varepsilon}(t)dt + o(\varepsilon).
\end{aligned}
$$

Then we can obtain the variational inequality (3.25). $\square$

We now present an illustrative example.

Example 3.7. Consider the following state equation:

$$
\begin{cases}
dX(t) = u(t)dW(t), \qquad t \in [0,T], \\
X(0) = x,
\end{cases}
$$

with the cost functional

$$
J(u(\cdot)) = \mathbb{E}\Big[\int_0^T u(t)dt + \frac{1}{2}|X(T)|^2\Big],
$$

and $U = \{\pm 1\}$. The Hamiltonian is given by

$$
H(x,u,y,z) = (z-1)u.
$$

Hence, if $(\bar{X}(\cdot), \bar{u}(\cdot))$ is an optimal pair, then the first order adjoint equation reads

$$
\begin{cases}
dY(t) = Z(t)dW(t), \qquad t \in [0,T], \\
Y(T) = -\bar{X}(T),
\end{cases}
$$

whose unique adapted solution is given by

$$Y(t) = -\bar{X}(t), \quad Z(t) = -\bar{u}(t), \qquad t \in [0,T].$$

The second order adjoint equation reads

$$\begin{cases} d\Gamma(t) = \Lambda(t)dW(t), & t \in [0,T], \\ \Gamma(T) = -1, \end{cases} \tag{3.28}$$

whose unique adapted solution is given by

$$\Gamma(t) = -1, \quad \Lambda(t) = 0, \qquad t \in [0,T],$$

Then the maximum condition implies

$$\begin{aligned} 0 \le\, & H(\bar{X}(t),\bar{u}(t),Y(t),Z(t)) - H(\bar{X}(t),u,Y(t),Z(t)) \\ & -\frac{1}{2}\Gamma(t)\big[\sigma(\bar{X}(t),\bar{u}(t)) - \sigma(\bar{X}(t),u)\big]^2 \\ =\, & \big[Z(t)-1\big]\big[\bar{u}(t)-u\big] + \frac{1}{2}\big[\bar{u}(t)-u\big]^2 \\ =\, & -[\bar{u}(t)+1][\bar{u}(t)-u] + \big[1-u\bar{u}(t)\big] = u - \bar{u}, \qquad u = \pm 1. \end{aligned}$$

Therefore, by letting $u=-1$, we see that $\bar{u}(t) \equiv -1$.

As a matter of fact, since

$$\begin{aligned} J(u(\cdot)) &= \mathbb{E}\Big[\int_0^T u(t)dt + \frac{1}{2}\Big(x + \int_0^T u(t)dW(t)\Big)^2\Big] \\ &= \mathbb{E}\int_0^T u(t)dt + \frac{x^2+T}{2}, \end{aligned}$$

one sees that the optimal control is

$$\bar{u}(t) = -1, \qquad t \in [0,T],$$

with optimal state process

$$\bar{X}(t) = x - W(t), \qquad t \in [0,T].$$

4. **Dynamic Programming Method and HJB Equations.**

4.1. **The value function.** Let $\mathcal{T}[0,T]$ be the set of all $\mathbb{F}$-stopping times valued in $[0,T]$, and denote $\mathcal{X}_\tau = L^p_{\mathcal{F}_\tau}(\Omega;\mathbb{R}^n)$, for any $\tau \in \mathcal{T}[0,T]$. Next, let $\mathcal{D}$ be the set of all *admissible initial pairs*:

$$\mathcal{D} = \big\{(\tau,\xi) \bigm| \tau \in \mathcal{T}[0,T],\ \xi \in \mathcal{X}_\tau\big\}.$$

Also, let

$$\mathcal{U}[\tau,T] = \big\{u : [\tau,T] \to U \bigm| u(\cdot) \text{ is } \mathbb{F}\text{-progressively measurable }\big\}.$$

Although the control set U could be any separable metric space, for simplicity of presentation, we assume that $U \subseteq \mathbb{R}^m$ is a compact set. Moreover, by shifting U, we assume that $0 \in U$. Now, for any $(\tau,\xi) \in \mathcal{D}$. Consider the following state equation on $[\tau,T]$:

$$\begin{cases} dX(s) = b(s,X(s),u(s))ds + \sigma(s,X(s),u(s))dW(s), & s \in [\tau,T], \\ X(\tau) = \xi, \end{cases} \tag{4.1}$$

and cost functional

$$J(\tau,\xi;u(\cdot))=\mathbb{E}_\tau\Big[\int_\tau^T g(s,X(s),u(s))ds+h(X(T))\Big],\tag{4.2}$$

where $\mathbb{E}_\tau[\,\cdot\,]=\mathbb{E}[\,\cdot\,|\mathcal{F}_\tau]$. We introduce the following assumption.

(D1) Let $b:[0,T]\times\mathbb{R}^n\times U\to\mathbb{R}^n$, $\sigma:[0,T]\times\mathbb{R}^n\times U\to\mathbb{R}^{n\times d}$, $g:[0,T]\times\mathbb{R}^n\times U\to\mathbb{R}$, and $h:\mathbb{R}^n\to\mathbb{R}$ be continuous. There exists a constant $L>0$ and a modulus of continuity $\rho:[0,\infty)\to[0,\infty)$ such that

$$\begin{aligned}&|b(t,x,u)-b(t',x',u')|+|\sigma(t,x,u)-\sigma(t',x',u')|\\&\qquad+|g(t,x,u)-g(t',x',u')|+|h(x)-h(x')|\\&\le L|x-x'|+\rho\big(|t-t'|+|u-u'|\big),\\&\qquad\qquad\qquad\qquad(t,x,u),(t',x',u')\in[0,T]\times\mathbb{R}^n\times U.\end{aligned}\tag{4.3}$$

The following is standard.

Proposition 4.1. *Let* (D1) *hold. Then for any $(\tau,\xi)\in\mathcal{D}$ and $u(\cdot)\in\mathcal{U}[\tau,T]$, the state equation* (4.1) *admits a unique solution $X(\cdot)=X(\cdot\,;\tau,\xi,u(\cdot))$ such that for all $(\tau,\xi),(\tau,\xi')\in\mathcal{D}$ and $\tau'\in\mathcal{T}[\tau,T]$,*

$$\mathbb{E}_\tau\Big[\sup_{s\in[\tau,T]}|X(s;\tau,\xi,u(\cdot))|^p\Big]\le K\mathbb{E}_\tau\big(1+|\xi|^p\big),\tag{4.4}$$

$$\begin{aligned}&\mathbb{E}_\tau\Big[\sup_{s\in[\tau,T]}|X(s;\tau,\xi,u(\cdot))-X(s;\tau,\xi',u'(\cdot))|^p\Big]\\&\le K\mathbb{E}_\tau\Big[|\xi-\xi'|^p+\Big(\int_\tau^T\rho(u(s),u'(s))^2ds\Big)^{\frac p2}\Big],\end{aligned}\tag{4.5}$$

$$\mathbb{E}_\tau\Big[\sup_{s\in[\tau,\tau']}|X(s;\tau,\xi,u(\cdot))-\xi|^p\Big]\le K(1+|\xi|^p)\mathbb{E}_\tau(\tau'-\tau)^{\frac p2}.\tag{4.6}$$

Note that for any $(\tau,\xi)\in\mathcal{D}$, $J(\tau,\xi;\cdot):\mathcal{U}[\tau,T]\to L^0_{\mathcal{F}_\tau}(\Omega;\mathbb{R})$. Our optimal control problem can be stated as follows.

Problem (C). *For any given $(\tau,\xi)\in\mathcal{D}$, find a $\bar u(\cdot)\in\mathcal{U}[\tau,T]$ such that*

$$J(\tau,\xi;\bar u(\cdot))=\operatorname*{essinf}_{u(\cdot)\in\mathcal{U}[\tau,T]}J(\tau,\xi;u(\cdot))\equiv\mathbb{V}(\tau,\xi).\tag{4.7}$$

We call $\mathbb{V}:\mathcal{D}\times\Omega\to\mathbb{R}$ the *value function* of Problem (C). Before going further, the meaning of essinf in (4.7) needs to be clarified. We therefore introduce the following definition.

Definition 4.2. Let $\tau\in\mathcal{T}[0,T]$ be fixed and let $\mathbb{J}\subseteq L^0_{\mathcal{F}_\tau}(\Omega;\mathbb{R})$. A random variable $\bar J\in L^0_{\mathcal{F}_\tau}(\Omega;\mathbb{R})$ is called the *essential infimum* of $\mathbb{J}$ if the following hold:

$$\bar J(\omega)\le J(\omega),\qquad\text{a.s. }\ \omega\in\Omega,\quad\forall J\in\mathbb{J},\tag{4.8}$$

and if $\widehat J\in L^0_{\mathcal{F}_\tau}(\Omega;\mathbb{R})$ with the property that

$$\widehat J(\omega)\le J(\omega),\qquad\text{a.s. }\ \omega\in\Omega,\ \forall J\in\mathbb{J},$$

then

$$\widehat J(\omega)\le\bar J(\omega),\quad\text{a.s. }\ \omega\in\Omega.$$

We denote

$$\bar J=\operatorname{essinf}\mathbb{J}.$$

For any given $(\tau,\xi)\in\mathcal{D}$, let

$$\mathbb{J}(\tau,\xi)\equiv\big\{J(\tau,\xi;u(\cdot))\bigm| u(\cdot)\in\mathcal{U}[\tau,T]\big\}\subseteq L^0_{\mathcal{F}_\tau}(\Omega;\mathbb{R}).$$

It is natural to ask: Does $\operatorname{essinf}\mathbb{J}(\tau,\xi)$ exist? In order to answer this question positively, we present the following lemma.

Lemma 4.3. *Let $(\tau,\xi)\in\mathcal{D}$, and $u_1(\cdot),u_2(\cdot)\in\mathcal{U}[\tau,T]$. Define*

$$A=\big\{\omega\in\Omega\bigm| J(\tau,\xi;u_1(\cdot))\le J(\tau,\xi;u_2(\cdot))\big\},$$

and denote

$$u_1(\cdot)\sqcap u_2(\cdot)=u_1(\cdot)\mathbf{1}_A+u_2(\cdot)\mathbf{1}_{A^c}.$$

Then

$$\begin{aligned}J(\tau,\xi;u_1(\cdot)\sqcap u_2(\cdot))(\omega)=\big[J(\tau,\xi;u_1(\cdot))\wedge J(\tau,\xi;u_2(\cdot))\big](\omega),&\\ \text{a.s. }\ \omega\in\Omega,&\end{aligned}\tag{4.9}$$

where $\alpha\wedge\beta=\min\{\alpha,\beta\}$ for any $\alpha,\beta\in\mathbb{R}$.

Proof. Let

$$X_1(\cdot)=X(\cdot\,;\tau,\xi,u_1(\cdot)),\quad X_2(\cdot)=X(\cdot\,;\tau,\xi,u_2(\cdot)).$$

Since $A\in\mathcal{F}_\tau$, we have

$$\begin{aligned}&X_1(s)\mathbf{1}_A+X_2(\cdot)\mathbf{1}_{A^c}\\&=\xi+\int_\tau^s b(r,X_1(r),u_1(r))\mathbf{1}_A+b(r,X_2(r),u_2(r))\mathbf{1}_{A^c}\big]dr\\&\quad+\int_\tau^s\sigma(r,X_1(r),u_1(r))\mathbf{1}_A+\sigma(r,X_2(r),u_2(r))\mathbf{1}_{A^c}\big]dW(r)\\&=\xi+\int_\tau^s b\big(r,X_1(r)\mathbf{1}_A+X_2(r)\mathbf{1}_{A^c},u_1(r)\mathbf{1}_A+u_2(r)\mathbf{1}_{A^c}\big)dr\\&\quad+\int_\tau^s\sigma\big(r,X_1(r)\mathbf{1}_A+X_2(r)\mathbf{1}_{A^c},u_1(r)\mathbf{1}_A+u_2(r)\mathbf{1}_{A^c}\big)dW(r).\end{aligned}$$

Hence,

$$X\big(\cdot\,;\tau,\xi,u_1(\cdot)\sqcap u_2(\cdot)\big)=X(\cdot\,;\tau,\xi,u_1(\cdot))\mathbf{1}_A+X(\cdot\,;\tau,\xi,u_2(\cdot))\mathbf{1}_{A^c}.$$

Then

$$\begin{aligned}&J(\tau,\xi;u_1(\cdot))\wedge J(\tau,\xi;u_2(\cdot))=J(\tau,\xi;u_1(\cdot))\mathbf{1}_A+J(\tau,\xi;u_2(\cdot))\mathbf{1}_{A^c}\\&=\mathbb{E}_\tau\Big[\int_\tau^T\big[g(r,X_1(r),u_1(r))\mathbf{1}_A+g(r,X_2(r),u_2(r))\mathbf{1}_{A^c}\big]dr\\&\qquad\quad+h(X_1(T))\mathbf{1}_A+h(X_2(T))\mathbf{1}_{A^c}\Big]\\&=\mathbb{E}_\tau\Big[\int_\tau^T g\big(r,X_1(r)\mathbf{1}_A+X_2(r)\mathbf{1}_{A^c},u_1(r)\mathbf{1}_A+u_2(r)\mathbf{1}_{A^c}\big)dr\\&\qquad\quad+h\big(X_1(T))\mathbf{1}_A+X_2(T))\mathbf{1}_{A^c}\big)\Big]\\&=J\big(\tau,\xi;u_1(\cdot)\mathbf{1}_A+u_2(\cdot)\mathbf{1}_{A^c}\big)=J\big(\tau,\xi;u_1(\cdot)\sqcap u_2(\cdot)\big),\end{aligned}$$

proving our claim. $\square$

The following gives the existence of the essential infimum appeared in (4.7), and a *minimizing sequence.*

Lemma 4.4. *For any $(\tau,\xi)\in\mathcal{D}$, there exists a sequence $u_k(\cdot)\in\mathcal{U}[\tau,T]$ such that*

$$J(\tau,\xi;u_k(\cdot))(\omega)\searrow \operatorname*{essinf}_{u(\cdot)\in\mathcal{U}[\tau,T]} J(\tau,\xi;u_k(\cdot))=\mathbb{V}(\tau,\xi),\qquad \text{a.s.}\tag{4.10}$$

Proof. For given $(\tau,\xi)\in\mathcal{D}$, we let

$$\mathbb{J}(\tau,\xi)=\big\{J(\tau,\xi;u(\cdot))\bigm| u(\cdot)\in\mathcal{U}[\tau,T]\big\}\subseteq L^0_{\mathcal{F}_\tau}(\Omega;\mathbb{R}).$$

Without loss of generality, we assume that $\mathbb{J}(\tau,\xi)\subseteq L^\infty_{\mathcal{F}_\tau}(\Omega;\mathbb{R})$, otherwise, we may consider

$$\arctan\mathbb{J}(\tau,\xi)\equiv\{\arctan J\bigm| J\in\mathbb{J}(\tau,\xi)\}$$

instead. Next, by looking at the set

$$\mathbb{E}\big[\mathbb{J}(\tau,\xi)\big]\equiv\big\{\mathbb{E}[J]\bigm| J\in\mathbb{J}(\tau,\xi)\big\}\subseteq\mathbb{R},$$

its infimum exists and there exists a sequence $u_k(\cdot)\in\mathcal{U}[\tau,T]$ such that

$$\lim_{k\to\infty}\mathbb{E}[J_k(\tau,\xi;u_k(\cdot))]=\inf_{u(\cdot)\in\mathcal{U}[\tau,T]}\mathbb{E}[J(\tau,\xi;u(\cdot))].$$

By Lemma 4.3, if we let

$$\widehat u_k(\cdot)=u_1(\cdot)\sqcap u_2(\cdot)\sqcap\cdots\sqcap u_k(\cdot),\qquad k\ge1,$$

then $\widehat u_k(\cdot)\in\mathcal{U}[\tau,T]$ and

$$J(\tau,\xi;\widehat u_k(\cdot))=\min_{1\le i\le k}J(\tau,\xi;u_i(\cdot)).$$

Thus, we may assume that $J(\tau,\xi;u_k(\cdot))$ is nonincreaing, otherwise, we may replace $u_k(\cdot)$ by $\widehat u_k(\cdot)$. Now, we define

$$\bar J(\omega)=\inf_{k\ge1}J(\tau,\xi;u_k(\cdot)),\qquad\omega\in\Omega,$$

which is $\mathcal{F}_\tau$-measurable. We claim that

$$\bar J=\operatorname*{essinf}_{u(\cdot)\in\mathcal{U}[\tau,T]}J(\tau,\xi;u(\cdot)).$$

In fact, for any $\mathcal{F}_\tau$-measurable random variable $\widehat J$ with the property

$$\widehat J(\omega)\le J(\tau,\xi;u(\cdot)),\qquad\forall u(\cdot)\in\mathcal{U}[\tau,T],$$

one has

$$\widehat J(\omega)\le J(\tau,\xi;u_k(\cdot)),\qquad\forall k\ge1,$$

which leads to

$$\widehat J(\omega)\le\bar J(\omega),\qquad\text{a.s. }\ \omega\in\Omega.$$

On the other hand, for any $v(\cdot)\in\mathcal{U}[\tau,T]$, by Lemma 4.3 again,

$$J\big(\tau,\xi;v(\cdot)\sqcap u_k(\cdot)\big)=J(\tau,\xi;v(\cdot))\wedge J(\tau,\xi;u_k(\cdot))\searrow J(\tau,\xi;v(\cdot))\wedge\bar J(\omega).$$

Thus, by monotone convergence theorem, one has

$$\begin{aligned}&\mathbb{E}\big[J(\tau,\xi;v(\cdot))\wedge\bar J\big]=\lim_{k\to\infty}\mathbb{E}\big[J(\tau,\xi;v(\cdot))\wedge J(\tau,\xi;u_k(\cdot))\big]\\&\ge\inf_{u(\cdot)\in\mathcal{U}[\tau,T]}\mathbb{E}\big[J(\tau,\xi;u(\cdot))\big]=\mathbb{E}[\bar J].\end{aligned}$$

Hence, the equality must hold and

$$J(\tau,\xi;v(\cdot))\ge\bar J,\qquad\text{a.s.}$$

This completes the proof. $\square$

Let us point out that the existence of the essential infimum of the cost functional $J(\tau,\xi;u(\cdot))$ does not mean that an optimal control exists, i.e., the essential infimum might not be achieved.

From the definition of the value function, we see that for any $(\tau,\xi)\in\mathcal{D}$, $\mathbb{V}(\tau,\xi)$ is an $\mathcal{F}_\tau$-measurable random variable, and it is not necessarily a non-random constant in general. Let us now look at some properties of $\mathbb{V}(\cdot\,,\cdot)$.

Proposition 4.5. *For any $(\tau,\xi),(\tau,\widetilde{\xi}\,)\in\mathcal{D}$,*

$$\begin{cases} |\mathbb{V}(\tau,\xi)|\le K(1+|\xi|),\\ |\mathbb{V}(\tau,\xi)-\mathbb{V}(\tau,\widetilde{\xi}\,)|\le K|\xi-\widetilde{\xi}|.\end{cases}\tag{4.11}$$

Proof. For any $(\tau,\xi)\in\mathcal{D}$ and $u(\cdot)\in\mathcal{U}[\tau,T]$,

$$\begin{aligned}&|J(\tau,\xi;u(\cdot))|\le\mathbb{E}_\tau\Big(\int_\tau^T|g(r,X(r),u(r))|dr+|h(X(T)|\Big)\\&\le\mathbb{E}_\tau\Big(\int_\tau^T L(1+|X(r)|)dr+L(1+|X(T)|)\Big)\le K(1+|\xi|).\end{aligned}$$

This proves the first one.

Now, let $u_k(\cdot),\widetilde{u}_k(\cdot)\in\mathcal{U}[\tau,T]$ such that

$$J(\tau,\xi;u_k(\cdot))\searrow\mathbb{V}(\tau,\xi),\qquad J(\tau,\widetilde{\xi};\widetilde{u}_k(\cdot))\searrow\mathbb{V}(\tau,\widetilde{\xi}\,).$$

Note that

$$|J(\tau,\xi;u_k(\cdot))-J(\tau,\widetilde{\xi};u_k(\cdot))|+|J(\tau,\xi;\widetilde{u}_k(\cdot))-J(\tau,\widetilde{\xi};\widetilde{u}_k(\cdot))|\le K|\xi-\widetilde{\xi}|.$$

On the other hand,

$$\begin{aligned}&J\big(\tau,\xi;\widetilde{u}_k(\cdot)\big)\ge\mathbb{V}(\tau,\xi)\ge J\big(\tau,\xi;u_k(\cdot)\big)-\delta_k(\omega),\\&J\big(\tau,\widetilde{\xi};u_k(\cdot)\big)\ge\mathbb{V}(\tau,\widetilde{\xi})\ge J\big(\tau,\widetilde{\xi};\widetilde{u}_k(\cdot)\big)-\widetilde{\delta}_k(\omega),\end{aligned}$$

with $\delta_k,\widetilde{\delta}_k$ being uniformly bounded and

$$\delta_k,\ \widetilde{\delta}_k\searrow 0,\qquad\text{a.s. },$$

we obtain

$$\begin{aligned}&-K|\xi-\widetilde{\xi}|-\delta_k\le J\big(\tau,\xi;u_k(\cdot)\big)-J\big(\tau,\widetilde{\xi};u_k(\cdot)\big)-\delta_k\le\mathbb{V}(\tau,\xi)-\mathbb{V}(\tau,\widetilde{\xi})\\&\le J\big(\tau,\xi;\widetilde{u}_k(\cdot)\big)-J\big(\tau,\widetilde{\xi};\widetilde{u}_k(\cdot)\big)+\widetilde{\delta}_k\le K|\xi-\widetilde{\xi}|+\widetilde{\delta}_k.\end{aligned}$$

This proves the second one. □

The following is called the *Bellman's principle of optimality*:

Theorem 4.6. *For any $(\tau,\xi)\in\mathcal{D}$ and $\bar{\tau}\in\mathcal{T}(\tau,T]$, the following holds:*

$$\mathbb{V}(\tau,\xi)=\inf_{u(\cdot)\in\mathcal{U}[\tau,\,\bar{\tau}]}\mathbb{E}_\tau\Big\{\int_\tau^{\bar{\tau}}g(s,X(s),u(s))ds+\mathbb{V}(\bar{\tau},X(\bar{\tau}))\Big\},\tag{4.12}$$

where $X(\cdot)$ is the state process corresponding to the control $u(\cdot)$, with the initial pair (τ,ξ).

Proof. Let $(\tau,\xi)\in\mathcal{D}$ be given. For any $u(\cdot)\in\mathcal{U}[\tau,T]$, by the definition of the value function, we have

$$\begin{aligned}\mathbb{V}(\tau,\xi)\le J(\tau,\xi;u(\cdot))&=\mathbb{E}_\tau\Big[\int_\tau^{\bar\tau}g(s,X(s),u(s))ds\\&\qquad+\int_{\bar\tau}^T g(s,X(s)),u(s))ds+h(X(T))\Big]\\&=\mathbb{E}_\tau\Big[\int_\tau^{\bar\tau}g(s,X(s),u(s))ds+J\big(\bar\tau,X(\bar\tau);u(\cdot)\big|_{[\bar\tau,T]}\big)\Big].\end{aligned}$$

Taking infimum for $u(\cdot)\big|_{[\bar\tau,T]}\in\mathcal{U}[\bar\tau,T]$, we obtain

$$\mathbb{V}(\tau,\xi)\le\mathbb{E}_\tau\Big[\int_\tau^{\bar\tau}g(s,X(s),u(s))ds+\mathbb{V}(\bar\tau,X(\bar\tau))\Big].$$

Hence,

$$\mathbb{V}(\tau,\xi)\le\inf_{u(\cdot)\in\mathcal{U}[\tau,\bar\tau]}\mathbb{E}_\tau\Big[\int_\tau^{\bar\tau}g(s,X(s),u(s))ds+\mathbb{V}(\bar\tau,X(\bar\tau))\Big].$$

Next, for any $\varepsilon>0$, there exists a $u^\varepsilon(\cdot)\in\mathcal{U}[\tau,T]$ such that

$$\begin{aligned}&\mathbb{V}(\tau,\xi)+\varepsilon>J(\tau,\xi;u^\varepsilon(\cdot))\\&=\mathbb{E}_\tau\Big[\int_\tau^{\bar\tau}g(s,X^\varepsilon(s),u^\varepsilon(s))ds+\int_{\bar\tau}^T g(s,X^\varepsilon(s)),u^\varepsilon(s))ds+h(X^\varepsilon(T))\Big]\\&=\mathbb{E}_\tau\Big[\int_\tau^{\bar\tau}g(s,X^\varepsilon(s),u^\varepsilon(s))ds+J\big(\bar\tau,X^\varepsilon(\bar\tau);u^\varepsilon(\cdot)\big|_{[\bar\tau,T]}\big)\Big]\\&\ge\mathbb{E}_\tau\Big[\int_\tau^{\bar\tau}g(s,X^\varepsilon(s),u^\varepsilon(s))ds+\mathbb{V}(\bar\tau,X^\varepsilon(\bar\tau))\Big]\\&\ge\inf_{u(\cdot)\in\mathcal{U}[\tau,\bar\tau]}\mathbb{E}_\tau\Big[\int_\tau^{\bar\tau}g(s,X(s),u(s))ds+\mathbb{V}(\bar\tau,X(\bar\tau))\Big].\end{aligned}$$

This proves (4.12). ∎

We have the following interesting corollary.

Corollary 4.7. *For any $(\tau,\xi)\in\mathcal{D}$ and $\tau'\in\mathcal{T}(\tau,T]$,*

$$|\mathbb{V}(\tau,\xi)-\mathbb{E}_\tau\mathbb{V}(\tau',\xi)|\le K(1+|\xi|)\mathbb{E}_\tau|\tau-\tau'|^{\frac12}.\tag{4.13}$$

Proof. From (4.12), we see that for any $u(\cdot)\in\mathcal{U}[\tau,\tau']$,

$$\begin{aligned}&\mathbb{V}(\tau,\xi)-\mathbb{E}_\tau\mathbb{V}(\tau',\xi)\le\mathbb{E}_\tau\Big[\mathbb{V}(\tau',X(\tau'))-\mathbb{V}(\tau',\xi)\Big]+\mathbb{E}_\tau\int_\tau^{\tau'}g(s,X(s),u(s))ds\\&\le K\mathbb{E}_\tau|X(\tau')-\xi|+\mathbb{E}_\tau\int_\tau^{\tau'}K\big(1+|X(s)|\big)ds\le K(1+|\xi|)\mathbb{E}_\tau|\tau-\tau'|^{\frac12}.\end{aligned}$$

On the other hand, for any $\varepsilon > 0$, there exists a $u^\varepsilon(\cdot) \in \mathcal{U}[\tau, \tau']$ such that (with $X^\varepsilon(\cdot)$ being the corresponding state process)

$$\begin{aligned}
&V(\tau,\xi) - \mathbb{E}_\tau V(\tau',\xi) + \varepsilon \\
&\geq \mathbb{E}_\tau\Big[\mathbb{V}(\tau', X^\varepsilon(\tau')) - \mathbb{V}(\tau',\xi)\Big] + \mathbb{E}_\tau \int_\tau^{\tau'} g(s, X^\varepsilon(s), u^\varepsilon(s))ds \\
&\geq -K\mathbb{E}_\tau|X^\varepsilon(\tau') - \xi| - \mathbb{E}_\tau \int_\tau^{\tau'} K\big(1 + |X^\varepsilon(s)|\big)ds \\
&\geq -K(1+|\xi|)\mathbb{E}_\tau|\tau - \tau'|^{\frac{1}{2}}.
\end{aligned}$$

Then our conclusion follows. $\square$

Let $V(\cdot\,,\cdot)$ be the restriction of $\mathbb{V}$ on $[0,T]\times\mathbb{R}^n$:

$$V(t,x) = \mathbb{V}(t,x), \qquad \forall (t,x) \in [0,T]\times\mathbb{R}^n. \tag{4.14}$$

We call $V(\cdot\,,\cdot)$ the *restricted value function*, or still simply the value function if the specific meaning is clear. In general, $V : [0,T]\times\mathbb{R}^n \to L^0_{\mathcal{F}_t}(\Omega;\mathbb{R})$. In another word, although (t,x) is deterministic, $V(t,x)$ might still be random. It is natural to ask if the following holds:

$$\mathbb{V}(t,\xi)(\omega) = V(t,\xi(\omega)), \qquad \text{a.s. ?} \tag{4.15}$$

Note that the right-hand side is the value of plugging $x = \xi(\omega)$ into $V(t,x)$. This might not be true for a general function $\varphi : \mathcal{D} \to L^0_{\mathcal{F}}(\Omega;\mathbb{R}^n)$. For example, if $n = 1$ and

$$\varphi(\tau,\xi) = \xi\mathbb{E}[\xi],$$

then the restriction is given by

$$\bar\varphi(t,x) = x^2.$$

Hence, when ξ is not deterministic, one has

$$\bar\varphi(t,\xi(\omega)) = \xi(\omega)^2 \neq \xi(\omega)\mathbb{E}[\xi] = \varphi(t,\xi(\omega)),$$

i.e., (4.15) fails. However, for the value function, we have the following result.

Proposition 4.8. *Let $t \in [0,T)$. Then for any $\xi \in L^p_{\mathcal{F}_t}(\Omega;\mathbb{R}^n)$, (4.15) holds.*

Proof. Let

$$\xi = \sum_{i=1}^N x_i \mathbf{1}_{A_i},$$

with $\{A_i \bigm| 1 \leq i \leq N\}$ being a partition of Ω in $\mathcal{F}_t$, i.e.,

$$A_i \in \mathcal{F}_t, \quad A_i \cap A_j = \emptyset,\ i \neq j, \quad \bigcup_{i=1}^N A_i = \Omega,$$

and $x_1, x_2, \cdots, x_N \in \mathbb{R}^n$. For any $u(\cdot) \in \mathcal{U}[t,T]$, let

$$u_i(\cdot) = u(\cdot)\mathbf{1}_{A_i} \in \mathcal{U}[t,T].$$

Recall that $0 \in U$. Similar to the proof of Lemma 4.3, we obtain

$$J(t,\xi;u(\cdot)) = \sum_{i=1}^N J\big(t,x_i;u_i(\cdot)\big)\mathbf{1}_{A_i} \geq \sum_{i=1}^N V(t,x_i)\mathbf{1}_{A_i} = V(t,\xi).$$

Hence,

$$\mathbb{V}(t,\xi)\geq V(t,\xi).$$

On the other hand, let $u_{ik}(\cdot)\in\mathcal{U}[t,T]$ such that

$$J(t,x_i;u_{ik}(\cdot))\searrow\mathbb{V}(t,x_i),\qquad k\to\infty.$$

Then, similar to the proof of Lemma 4.3, one has

$$\begin{aligned}\mathbb{V}(t,\xi)&\leq J\big(t,\sum_{i=1}^N x_i\mathbf{1}_{A_i};\sum_{i=1}^N u_{ik}(\cdot)\mathbf{1}_{A_i}\big)\\&=\sum_{i=1}^N J(t,x_i;u_{ik}(\cdot))\mathbf{1}_{A_i}\searrow\sum_{i=1}^N V(t,x_i)\mathbf{1}_{A_i}=V(t,\xi).\end{aligned}$$

This completes the proof of (4.15). $\square$

Because of the above result, (4.12) implies

$$V(t,\xi)=\inf_{u(\cdot)\in\mathcal{U}[t,\bar t]}\mathbb{E}_t\Big\{\int_t^{\bar t}g(s,X(s),u(s))ds+V(\bar t,X(\bar t))\Big\}.\tag{4.16}$$

The following is a local form of Bellman's principle of optimality.

Theorem 4.9. (HJB Equation) *Suppose that the restriction $V(\cdot\,,\cdot)$ of $\mathbb{V}(\cdot\,,\cdot)$ is a deterministic function. Moreover, $V_t(\cdot\,,\cdot)$, $V_x(\cdot\,,\cdot)$ and $V_{xx}(\cdot\,,\cdot)$ are continuous. Then $V(\cdot\,,\cdot)$ is a solution to the following Hamilton-Jacobi-Bellman (HJB, for short) equation:*

$$\begin{cases}V_t+H(t,x,V_x,V_{xx})=0,\qquad(t,x)\in[0,T]\times\mathbb{R}^n,\\V(T,x)=h(x),\qquad x\in\mathbb{R}^n,\end{cases}\tag{4.17}$$

where

$$\begin{aligned}&H(t,x,\mathbf{p}^\top,\mathbf{P})=\inf_{u\in U}\mathbb{H}(t,x,u,\mathbf{p}^\top,\mathbf{P}),\\&\mathbb{H}(t,x,u,\mathbf{p}^\top,\mathbf{P})=\mathbf{p}^\top b(t,x,u)+\frac{1}{2}\mathrm{tr}\,[\mathbf{P}\sigma(t,x,u)\sigma(t,x,u)^\top]+g(t,x,u),\\&\qquad\qquad(t,x,u,\mathbf{p},\mathbf{P})\in[0,T]\times\mathbb{R}^n\times U\times\mathbb{R}^n\times\mathbb{S}^n.\end{aligned}$$

which is called a Hamiltonian.

Proof. Let $(t,x)\in[0,T]\times\mathbb{R}^n$ be given and for any $u\in U$, taking the constant control $u(\cdot)=u$, we have (noting (4.16))

$$\begin{aligned}0&\leq\mathbb{E}_t\Big[\int_t^{t+\varepsilon}g(s,X(s),u)ds+V(t+\varepsilon,X(t+\varepsilon))-V(t,x)\Big]\\&=\mathbb{E}_t\Big[\int_t^{t+\varepsilon}\Big(g(s,X(s),u)+V_s(s,X(s))+V_x(s,X(s))b(s,X(s),u)\\&\qquad+\frac{1}{2}\mathrm{tr}\,\big(V_{xx}(s,X(s))\sigma(s,X(s),u)\sigma(s,X(s),u)\big)^\top\Big)ds\Big].\end{aligned}$$

Taking expectation, dividing ε and sending $\varepsilon\to0$, one can obtain that

$$0\leq g(t,x,u)+V_t(t,x)+V_x(t,x)b(t,x,u)+\frac{1}{2}\mathrm{tr}\,\big[V_{xx}(t,x)\sigma(t,x,u)\sigma(t,x,u)^\top\big].$$

Thus,

$$V_t+\inf_{u\in U}\mathbb{H}(t,x,u,V_x,V_{xx})\geq0.$$

Next, for any $\delta,\varepsilon>0$, there exists a $u^{\delta,\varepsilon}(\cdot)\in\mathcal{U}[t,T]$ such that

$$\begin{aligned}
&\delta\varepsilon>\mathbb{E}_t\Big[\int_t^{t+\varepsilon}g(s,X^{\delta,\varepsilon}(s),u^{\delta,\varepsilon}(s))ds+V(t+\varepsilon,X^{\delta,\varepsilon}(t+\varepsilon))-V(t,x)\Big]\\
&=\mathbb{E}_t\Big[\int_t^{t+\varepsilon}\Big(g(s,X^{\delta,\varepsilon}(s),u^{\delta,\varepsilon}(s))\\
&\qquad+V_s(s,X^{\delta,\varepsilon}(s))+V_x(s,X^{\delta,\varepsilon}(s))b(s,X^{\delta,\varepsilon}(s),u^{\delta,\varepsilon}(s))\\
&\qquad+\frac{1}{2}\mathrm{tr}\left[V_{xx}(s,X^{\delta,\varepsilon}(s))\sigma(s,X^{\delta,\varepsilon}(s),u^{\delta,\varepsilon}(s))\sigma(s,X^{\delta,\varepsilon}(s),u^{\delta,\varepsilon}(s))^\top\right]\Big)ds\Big]\\
&\geq\mathbb{E}_t\int_t^{t+\varepsilon}\Big[V_s(s,X^{\delta,\varepsilon}(s))\\
&\qquad\qquad+\inf_{u\in U}\mathbb{H}\big(s,X^{\delta,\varepsilon}(s),u,V_x(s,X^{\delta,\varepsilon}(s)),V_{xx}(s,X^{\delta,\varepsilon}(s))\big)\Big]ds\\
&\geq\varepsilon\Big[V_t(t,x)+\inf_{u\in U}\mathbb{H}(t,x,u,V_x(t,x),V_{xx}(t,x))\Big]-K\mathbb{E}_t\int_t^{t+\varepsilon}|X^{\delta,\varepsilon}(s)-x|ds.
\end{aligned}$$

Then dividing $\varepsilon>0$ and sending $\varepsilon\to 0$, we obtain

$$\delta\geq V_t(t,x)+\inf_{u\in U}\mathbb{H}(t,x,u,V_x(t,x),V_{xx}(t,x)).$$

Since $\delta>0$ is arbitrary, we obtain (4.17). $\square$

We see that in the above, the condition that $V(\cdot\,,\cdot)$ is a deterministic function plays an essential role. Otherwise, we will need a much more complicated Itô type formula (for stochastic functionals) to derive a corresponding (much more complicated) HJB equation. The following is a major motivation of the dynamic programming method, since formally, an optimal control can be constructed through the value function.

Theorem 4.10. (Verification Theorem) *Let $V(\cdot\,,\cdot)$ be a classical solution of HJB equation and let function $\Phi:[0,T]\times\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{S}^n\to U$ satisfy*

$$\mathbb{H}(t,x,\Phi(t,x,V_x,V_{xx}),V_x,V_{xx})=\inf_{u\in U}\mathbb{H}(t,x,u,V_x,V_{xx}).$$

Further, under the control

$$u(t)=\Phi(t,X(t),V_x(t,X(t)),V_{xx}(t,X(t))),\qquad t\in[0,T],\tag{4.18}$$

the state equation admits a unique solution. Then $u(\cdot)$ defined by the above is an optimal control.

Proof. Let $(t,x)\in[0,T)\times\mathbb{R}^n$ be given and let $u(\cdot)$ be constructed as in the theorem. Let $X(\cdot)=X(\cdot\,;t,x,u(\cdot))$. Then by Itô's formula,

$$\begin{aligned}
&\mathbb{E}_t[h(X(T))]-V(t,x)=\mathbb{E}_t[V(T,X(T))]-V(t,x)\\
&=\mathbb{E}_t\int_t^T\Big(V_s(s,X(s))+V_x(s,X(s))b(s,X(s),u(s))\\
&\qquad\qquad+\frac{1}{2}\mathrm{tr}\left[V_{xx}(s,X(s))\sigma(s,X(s),u(s))\sigma(s,X(s),u(s))^\top\right]\Big)ds\\
&=-\mathbb{E}_t\int_t^T g(s,X(s),u(s))ds.
\end{aligned}$$

Hence,

$$V(t,x)=J(t,x;u(\cdot)),$$

proving the optimality of $u(\cdot)$. $\square$

To conclude this section, we make some observations.

(i) Suppose we consider a case with no control, i.e.,

$$\begin{aligned} b(t,x,u)\equiv b(t,x),\quad \sigma(t,x,u)\equiv\sigma(t,x),\quad g(t,x,u)\equiv g(t,x),\\ \forall(t,x,u)\in[0,T]\times\mathbb{R}^n\times U.\end{aligned}$$

Then the above verification theorem is reduced to the Feynman-Kac formula with $c(\cdot\,,\cdot)=0$ (see Theorem 1.26). Therefore, the above is a generalization of the Feynman-Kac formula.

(ii) It is seen that in HJB equation, we need $V:[0,T]\times\mathbb{R}^n\to\mathbb{R}$, i.e, the restriction $V(\cdot\,,\cdot)$ of the value function $\mathbb{V}(\cdot\,,\cdot)$ is a deterministic function. But, this is not obvious. On the other hand, one may instead consider the cost functional

$$\bar J(\tau,\xi;u(\cdot))=\mathbb{E}[J(\tau,\xi;u(\cdot))]=\mathbb{E}\Big[\int_t^T g(s,X(s),u(s))ds+h(X(T))\Big]. \tag{4.19}$$

With such a cost functional, the corresponding value function, denoted by $\overline{\mathbb{V}}(\cdot\,,\cdot)$ which is real valued. We then may further define the restriction $\overline{V}(\cdot\,,\cdot)$ of $\overline{\mathbb{V}}(\cdot\,,\cdot)$:

$$\overline{V}(t,x)=\overline{\mathbb{V}}(t,x),\qquad (t,x)\in[0,T]\times\mathbb{R}^n,$$

and $\overline{V}:[0,T]\times\mathbb{R}^n\to\mathbb{R}$ is a real valued function now. However, the following is not true:

$$\overline{\mathbb{V}}(\tau,\xi)=\overline{V}(\tau,\xi)\overset{\Delta}{=}\overline{V}(t,x)\big|_{(t,x)=(\tau,\xi)}, \tag{4.20}$$

since the left-hand side is deterministic and the right-hand side is random in general. Further, we do not expect to have the following:

$$\begin{aligned}&\inf_{u(\cdot)\in\mathcal{U}[t,T]}\int_\Omega J(t,\xi(\omega);u(\cdot))d\mathbb{P}(\omega)\overset{\Delta}{=}\overline{\mathbb{V}}(t,\xi)\\ &=\mathbb{E}\overline{V}(t,\xi)\overset{\Delta}{=}\int_\Omega\inf_{u(\cdot)\in\mathcal{U}[t,T]}J(t,\xi(\omega);u(\cdot))d\mathbb{P}(\omega).\end{aligned}$$

To illustrate this, we need only to look at the following simple example: Let $\zeta\sim N(0,1)$ and $\Gamma=\{\zeta,-\zeta\}$. Then

$$\mathbb{E}\Big[\inf_{\xi\in\Gamma}\xi\Big]=-\mathbb{E}|\zeta|<0=\inf_{\xi\in\Gamma}\mathbb{E}\xi.$$

In quite a few literatures, the cost functional of form (4.19) has been used under the weak solution framework, with some further technicalities. We prefer to use strong solution framework. Hence the determinicity of $V(\cdot\,,\cdot)$ is an issue that we have to face.

(iii) Even if we obtain that, under certain conditions, $V(\cdot\,,\cdot)$ is a deterministic function, it is not necessarily true that $V_t(\cdot\,,\cdot)$, $V_x(\cdot\,,\cdot)$ and $V_{xx}(\cdot\,,\cdot)$ exist and continuous. Therefore, the beautiful HJB equation so far is actually formal, and is lack of rigorous foundation.

In the rest of this chapter, we will try to deal with the above issues (ii) and (iii).

4.2. **Determinicity of restricted value function.** In this section, we will show that when all the coefficients are deterministic, the restricted value function $V(\cdot\,,\cdot)$ is a deterministic function. To this end, we first make some preparations, which are interesting by themselves. We introduce the following additional hypothesis.

(D2) The control set $U \subseteq \mathbb{R}^m$ is convex and compact.

The results in this subsection relies on the convexity of U. It is not clear if our approach goes through without convexity of U.

For any $\tau \in \mathcal{T}[0,T]$, we introduce

$$\mathcal{F}_s^\tau = \sigma\Big(W(r) - W(\tau), \quad r \in [\tau, s]\Big), \quad \mathbb{F}^\tau = \{\mathcal{F}_s^\tau\}_{s \geq \tau}.$$

Then $\mathcal{F}_\tau$ and $\mathcal{F}_s^\tau$ are independent, and

$$\mathcal{F}_s = \mathcal{F}_\tau \vee \mathcal{F}_s^\tau, \qquad 0 \leq \tau < s \leq T. \tag{4.21}$$

Let

$$\widehat{\mathcal{U}}[\tau,T] = \{\widehat{u}(\cdot) \in \mathcal{U}[\tau,T] \bigm| \widehat{u}(\cdot) \text{ is } \mathbb{F}^\tau\text{-adapted}\} \subset \mathcal{U}[\tau,T].$$

Note that any $u(\cdot) \in \mathcal{U}[\tau,T]$ is $\mathbb{F}$-adapted, in particular, for such a $u(\cdot)$, if $u(\tau)$ is defined, then $u(\tau)$ is $\mathcal{F}_\tau$-measurable and is not necessarily deterministic, whereas, if $\widehat{u}(\cdot) \in \widehat{\mathcal{U}}[\tau,T]$ and $\widehat{u}(\tau)$ is defined, then $\widehat{u}(\tau)$ is deterministic. We define

$$\widehat{\mathbb{V}}(\tau,\xi) = \operatorname*{essinf}_{\widehat{u}(\cdot) \in \widehat{\mathcal{U}}[\tau,T]} J(\tau,\xi;\widehat{u}(\cdot)). \tag{4.22}$$

For such a value function, we have the following result.

Proposition 4.11. *Let* (D1) *hold. Then* $\widehat{\mathbb{V}}\big|_{[0,T]\times\mathbb{R}^n} : [0,T]\times\mathbb{R}^n \to \mathbb{R}$. *Namely, the restriction of* $\widehat{\mathbb{V}}(\cdot\,,\cdot)$ *on* $[0,T]\times\mathbb{R}^n$ *is a deterministic function.*

Proof. For any $(t,x) \in [0,T]\times\mathbb{R}^n$, and $u(\cdot) \in \widehat{\mathcal{U}}[t,T]$, let $X(\cdot) = X(\cdot\,;t,x,u(\cdot))$ be the corresponding state process. Then

$$X(s) = x + \int_t^s b(r,X(r),u(r))dr + \int_t^s \sigma(r,X(r),u(r))dW(r), \quad s \in [t,T].$$

We see that $X(\cdot)$ is $\mathbb{F}^t$-adapted. Now, consider the following BSDE:

$$Y(s) = h(X(T)) + \int_s^T g(r,X(r),u(r))dr - \int_s^T Z(r)dW(r), \quad s \in [t,T].$$

The adapted solution $(Y(\cdot),Z(\cdot))$ is $\mathbb{F}^t$-adapted. In particular, $Y(t)$ is $\mathcal{F}_t^t$-measurable, which means that $Y(t)$ is a (deterministic) real number. On the other hand, we know that

$$J(t,x;u(\cdot)) = Y(t).$$

Hence, $\widehat{\mathbb{V}}(t,x)$ is deterministic for each $(t,x) \in [0,T]\times\mathbb{R}^n$. $\square$

Since $\widehat{\mathcal{U}}[\tau,T] \subseteq \mathcal{U}[\tau,T]$, in general, we have

$$\widehat{\mathbb{V}}(\tau,\xi) \geq \mathbb{V}(\tau,\xi), \qquad \forall (\tau,\xi) \in \mathcal{D}. \tag{4.23}$$

A natural question is whether the above two are the same? To answer this question, let us first present the following result.

Lemma 4.12. *Let* (D2) *hold. Then for any* $u(\cdot)\in\mathcal{U}[\tau,T]$, *and any* $\varepsilon>0$, *there exists a* $u^\varepsilon(\cdot)$ *of the following form:*

$$u^\varepsilon(\cdot)=\sum_{i=1}^N\mathbf{1}_{A_i}\widehat{u}_i^\varepsilon(\cdot),\tag{4.24}$$

with $\{A_i\bigm|1\le i\le N\}$ *being a partition of* Ω *in* $\mathcal{F}_\tau$ *and* $\widehat{u}_i^\varepsilon(\cdot)\in\widehat{\mathcal{U}}[\tau,T]$ *for each* $i=1,2,\cdots,N$, *such that*

$$\mathbb{E}\int_\tau^T|u(r)-u^\varepsilon(r)|dr<\varepsilon.\tag{4.25}$$

Proof. The proof is split into two steps.

Step 1. Filtration approximation. For any $s\in\mathcal{T}[\tau,T]$, let

$$\mathcal{G}=\Big\{\bigcup_{k=1}^N(A^k\cap\widehat{A}^k)\bigm|A^k\in\mathcal{F}_\tau,\ \widehat{A}^k\in\mathcal{F}_s^\tau,\ 1\le k\le N,\ N\ge1\Big\}.$$

It is easy to see that

$$\mathcal{F}_\tau\cup\mathcal{F}_s^\tau\subseteq\mathcal{G}\subseteq\mathcal{F}_\tau\vee\mathcal{F}_s^\tau=\mathcal{F}_s,$$

which leads to

$$\mathcal{F}_s=\mathcal{F}_\tau\vee\mathcal{F}_s^\tau=\sigma(\mathcal{G}).$$

This means that $\mathcal{G}$ generates $\mathcal{F}_s$. Next, it is clear that $\mathcal{G}$ is closed under finite union (which is obvious), finite intersection, and complement. In fact, let

$$\bigcup_{k=1}^N(A^k\cap\widehat{A}^k),\ \bigcup_{\ell=1}^M(B^\ell\cap\widehat{B}^\ell)\in\mathcal{G}.$$

Then

$$\Big[\bigcup_{k=1}^N(A^k\cap\widehat{A}^k)\Big]\cap\Big[\bigcup_{\ell=1}^M(B^\ell\cap\widehat{B}^\ell)\Big]=\bigcup_{k=1}^N\bigcup_{\ell=1}^M\Big[(A^k\cap B^\ell)\cap(\widehat{A}^k\cap\widehat{B}^\ell)\Big]\in\mathcal{G}.$$

This gives that $\mathcal{G}$ is closed under finite intersection. Also, for $A,B\in\mathcal{F}_\tau$ and $\widehat{A},\widehat{B}\in\mathcal{F}_s^\tau$, let us look at

$$\begin{aligned}&\Big[(A\cap\widehat{A})\cup(B\cap\widehat{B})\Big]^c=(A^c\cup\widehat{A}^c)\cap(B^c\cup\widehat{B}^c)\\&=(A^c\cap B^c)\cup(B^c\cap\widehat{A}^c)\cup(A^c\cap\widehat{B}^c)\cup(\widehat{A}^c\cap\widehat{B}^c)\\&=\Big[(A^c\cap B^c)\cap\Omega\Big]\cup[B^c\cap\widehat{A}^c]\cup[A^c\cap\widehat{B}^c]\cup\Big[\Omega\cap(\widehat{A}^c\cap\widehat{B}^c)\Big]\in\mathcal{G}.\end{aligned}$$

This gives $\mathcal{G}$ is closed under complementary. Now, let $\widetilde{\mathcal{G}}$ be defined by the following:

$$\mathcal{G}\subseteq\widetilde{\mathcal{G}}\triangleq\Big\{F\in\mathcal{F}_s\bigm|\forall\varepsilon>0,\ \exists F_\varepsilon\in\mathcal{G},\ \mathbb{P}(F\Delta F_\varepsilon)<\varepsilon\Big\}\subseteq\mathcal{F}_s,$$

where $F\Delta F_\varepsilon=(F\cap F_\varepsilon^c)\cup(F_\varepsilon\cap F^c)=F^c\Delta F_\varepsilon^c$ which is called the *symmetric difference* of F and F_ε. We now show that $\widetilde{\mathcal{G}}$ is a σ-field. To this end, take any $F\in\widetilde{\mathcal{G}}\subseteq\mathcal{F}_s$, let $F_\varepsilon\in\mathcal{G}$ such that

$$\varepsilon>\mathbb{P}(F\Delta F_\varepsilon)=\mathbb{P}(F^c\Delta F_\varepsilon^c),$$

which means that $F^c \in \widetilde{\mathcal{G}}$, since $F^c_\varepsilon \in \mathcal{G}$. Thus, $\widetilde{\mathcal{G}}$ is closed under complement. Next, we let $F^k \in \widetilde{\mathcal{G}}$. Choose $F^k_\varepsilon \in \mathcal{G}$ such that

$$\mathbb{P}(F^k \Delta F^k_\varepsilon) < \frac{\varepsilon}{2^{k+1}}, \qquad k \geq 1.$$

Then

$$\widetilde{F}^k_\varepsilon = \bigcup_{i=1}^k F^k_\varepsilon \in \mathcal{G}, \quad \widetilde{F}^k = \bigcup_{i=1}^k F^i, \; \widetilde{F} = \bigcup_{i=1}^\infty F^i \in \mathcal{F}_s,$$

and

$$\begin{aligned}
\widetilde{F}^k \Delta \widetilde{F}^k_\varepsilon &= \Big(\bigcup_{i=1}^k F^i\Big) \Delta \Big(\bigcup_{j=1}^k F^j_\varepsilon\Big) \\
&= \Big[\Big(\bigcup_{i=1}^k F^i\Big) \cap \Big(\bigcup_{j=1}^k F^j_\varepsilon\Big)^c\Big] \cup \Big[\Big(\bigcup_{i'=1}^k F^{i'}_\varepsilon\Big) \cap \Big(\bigcup_{j'=1}^k F^{j'}\Big)^c\Big] \\
&= \Big\{\bigcup_{i=1}^k \Big[F^i \cap \Big(\bigcap_{j=1}^k (F^j_\varepsilon)^c\Big)\Big]\Big\} \cup \Big\{\bigcup_{i'=1}^k \Big[F^{i'}_\varepsilon \cap \Big(\bigcap_{j'=1}^k (F^{j'})^c\Big)\Big]\Big\} \\
&\subseteq \Big\{\bigcup_{i=1}^k \Big[F^i \cap (F^i_\varepsilon)^c\Big]\Big\} \cup \Big\{\bigcup_{i'=1}^k \Big[F^{i'}_\varepsilon \cap (F^{i'})^c\Big]\Big\} \\
&\subseteq \Big[\bigcup_{i=1}^k (F^i \Delta F^i_\varepsilon)\Big] \cup \Big[\bigcup_{i'=1}^k (F^{i'} \Delta F^{i'}_\varepsilon)\Big] = \bigcup_{i=1}^k (F^i \Delta F^i_\varepsilon).
\end{aligned}$$

Hence,

$$\mathbb{P}(\widetilde{F}^k \Delta \widetilde{F}^k_\varepsilon) \leq \sum_{i=1}^k \mathbb{P}(F^i \Delta F^i_\varepsilon) < \sum_{i=1}^k \frac{\varepsilon}{2^{i+1}} < \frac{\varepsilon}{2}.$$

Further, since

$$\lim_{k\to\infty} \mathbb{P}(\widetilde{F} \setminus \widetilde{F}^k) = 0,$$

there exists an $N > 0$ such that $\mathbb{P}(\widetilde{F} \setminus \widetilde{F}^N) < \frac{\varepsilon}{2}$. Consequently, noting $\widetilde{F} = \widetilde{F}^N \cup (\widetilde{F} \setminus \widetilde{F}^N)$, we have

$$\begin{aligned}
\mathbb{P}(\widetilde{F} \Delta \widetilde{F}^N_\varepsilon) &= \mathbb{P}\Big([\widetilde{F} \cap (\widetilde{F}^N_\varepsilon)^c] \cup [\widetilde{F}^N_\varepsilon \cap \widetilde{F}^c]\Big) \\
&= \mathbb{P}\Big([(\widetilde{F} \setminus \widetilde{F}^N) \cap (\widetilde{F}^N_\varepsilon)^c] \cup [\widetilde{F}^N \cap (\widetilde{F}^N_\varepsilon)^c] \cup [\widetilde{F}^N_\varepsilon \cap \widetilde{F}^c]\Big) \\
&\leq \mathbb{P}(\widetilde{F} \setminus \widetilde{F}^N) + \mathbb{P}(\widetilde{F}^N \Delta \widetilde{F}^N_\varepsilon) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

This implies that $\widetilde{F} \in \widetilde{\mathcal{G}}$, proving that $\widetilde{\mathcal{G}}$ is a σ-field. Hence, combining $\mathcal{G} \subseteq \widetilde{\mathcal{G}} \subseteq \mathcal{F}_s$, we obtain

$$\mathcal{F}_s = \widetilde{\mathcal{G}}.$$

Step 2. Control process approximation. Let $u(\cdot) \in \mathcal{U}[\tau, T]$. Fix any $u_0 \in U$, and extend $u(\cdot)$ to $[-1, \tau]$ by letting $u(r) = u_0$, $r \in [-1, \tau)$. For any $\delta \in (0, 1]$, define

$$u_\delta(s, \omega) = \frac{1}{\delta} \int_{s-\delta}^s u(r, \omega) dr, \qquad s \geq \tau, \; \omega \in \Omega.$$

Then $u_\delta(\cdot)$ is continuous, uniformly bounded, U-valued (since U is compact and convex), and $\mathbb{F}$-adapted. Moreover, for almost all $\omega\in\Omega$, almost every $s\in[\tau,T]$ is a Lebesgue point of $s\mapsto u(s,\omega)$, which implies

$$\lim_{\delta\to0}u_\delta(s)=u(s),\qquad \text{a.e.}\quad s\in[\tau,T],\ \text{a.s.}$$

Then by the Dominated Convergence Theorem,

$$\lim_{\delta\to0}\int_\tau^T|u_\delta(r)-u(r)|dr=0,\quad \text{a.s.}$$

Hence, for $\varepsilon>0$, we may let $u_\delta(\cdot)$ (with δ depending on ε) be constructed as above such that

$$\mathbb{E}\int_\tau^T|u_\delta(s)-u(s)|ds<\varepsilon.$$

For such a $u_\delta(\cdot)$, we have

$$|u_\delta(s)-u_\delta(t)|=\frac{1}{\delta}\Big|\int_{s-\delta}^s u(r)dr-\int_{t-\delta}^t u(r)dr\Big|\le\frac{2M}{\delta}|s-t|,$$

with M being the bound for U. Then we choose a partition Π of $[\tau,T]$ as

$$\tau=\tau^0<\tau^1<\cdots<\tau^{N_1+1}=T,\quad \|\Pi\|\equiv\max_{1\le i\le N_1+1}|\tau^i-\tau^{i-1}|<\frac{\delta\varepsilon}{2M},$$

and define

$$\widetilde u_\delta(r)=\sum_{i=0}^{N_1}u_\delta(\tau^i)\mathbf{1}_{[\tau^i,\tau^{i+1})}(r),\quad r\in[\tau,T].$$

Clearly,

$$|\widetilde u_\delta(r)-u_\delta(r)|<\varepsilon,\qquad r\in[\tau,T],\ \text{a.s.}$$

Since U is compact, for any $\widehat\varepsilon>0$, we may choose a partition of U as follows:

$$U=\bigcup_{j=1}^{N_2}U_j,\qquad \operatorname{diam}(U^j)<\widehat\varepsilon,\quad 1\le j\le N_2.$$

Let

$$B_\delta^{ij}=\big\{\omega\in\Omega\bigm| u_\delta(\tau^i)\in U_j\big\}\in\mathcal{F}_{\tau^i},\quad 0\le i\le N_1,\ 1\le j\le N_2.$$

Then $\{B_\delta^{ij}\bigm|1\le j\le N_2\}$ is a partition of Ω in $\mathcal{F}_{\tau^i}$. We may assume that

$$\min_{0\le i\le N_1,1\le j\le N_2}\mathbb{P}(B_\delta^{ij})=\varepsilon_0>0. \tag{4.26}$$

Define

$$\widehat u_\delta(r)=\sum_{i=0}^{N_1}\sum_{j=1}^{N_2}u^{ij}\mathbf{1}_{B_\delta^{ij}}\mathbf{1}_{[\tau^i,\tau^{i+1})}(r),\quad r\in[\tau,T], \tag{4.27}$$

where u^{ij} is picked from U_j. One has $\widehat u_\delta(\cdot)\in\mathcal{U}[\tau,T]$ and

$$|\widetilde u_\delta(r)-\widehat u_\delta(r)|\le\widehat\varepsilon,\quad r\in[\tau,T],\ \text{a.s.} \tag{4.28}$$

By *Step 1*, for each $B_\delta^{ij}\in\mathcal{F}_{\tau^i}$, for any $\widetilde\varepsilon\in(0,\varepsilon_0)$, there exists an $A_\delta^{ij}\in\mathcal{G}$ such that

$$\mathbb{P}(B_\delta^{ij}\Delta A_\delta^{ij})<\widetilde\varepsilon. \tag{4.29}$$

Then

$$\mathbb{P}(B_\delta^{ij}\cap A_\delta^{ij})=\mathbb{P}(B_\delta^{ij}\cup A_\delta^{ij})-\mathbb{P}(B_\delta^{ij}\Delta A_\delta^{ij})\ge\mathbb{P}(B_\delta^{ij})-\widetilde\varepsilon. \tag{4.30}$$

From

$$\bigcup_{j=1}^{N_2} A_\delta^{ij} = \bigcup_{j=1}^{N_2} \Big[\Big(A_\delta^{ij} \cap B_\delta^{ij}\Big) \cup \Big(A_\delta^{ij} \setminus B_\delta^{ij}\Big)\Big],$$

together with (4.30), one obtains, for each $1 \le i \le N_1$,

$$\mathbb{P}\Big(\bigcup_{j=1}^{N_2} A_\delta^{ij}\Big) \ge \sum_{j=1}^{N_2} \mathbb{P}\Big(A_\delta^{ij} \cap B_\delta^{ij}\Big) \ge \sum_{j=1}^{N_2} \Big[\mathbb{P}(B_\delta^{ij}) - \widetilde{\varepsilon}\Big] = 1 - N_2\widetilde{\varepsilon}. \tag{4.31}$$

We let

$$A_\delta^{i0} = \Big(\bigcup_{j=1}^{N_2} A_\delta^{ij}\Big)^c, \quad 0 \le i \le N_1.$$

Then

$$\mathbb{P}(A_\delta^{i0}) \le N_2\widetilde{\varepsilon}, \qquad 0 \le i \le N_1.$$

Next, since

$$B_\delta^{ij} \cap B_\delta^{ij'} = \emptyset, \quad j \ne j',$$

for $j \ne j'$, we have

$$\begin{aligned}
&A_\delta^{ij} \cap A_\delta^{ij'} = \big[\big(A_\delta^{ij} \cap B_\delta^{ij}\big) \cup \big(A_\delta^{ij} \setminus B_\delta^{ij}\big)\big] \cap \big[\big(A_\delta^{ij'} \cap B_\delta^{ij'}\big) \cup \big(A_\delta^{ij'} \setminus B_\delta^{ij'}\big)\big] \\
&= \big[(A_\delta^{ij} \cap B_\delta^{ij}) \cap (A_\delta^{ij'} \cap B_\delta^{ij'})\big] \cup \big[(A_\delta^{ij} \cap B_\delta^{ij}) \cap (A_\delta^{ij'} \setminus B_\delta^{ij'})\big] \\
&\quad \cup \big[(A_\delta^{ij} \setminus B_\delta^{ij}) \cap (A_\delta^{ij'} \cap B^{ij'})\big] \cup \big[(A_\delta^{ij} \setminus B_\delta^{ij}) \cap (A_\delta^{ij'} \setminus B_\delta^{ij'})\big] \\
&\subseteq \big(A_\delta^{ij'} \setminus B_\delta^{ij'}\big) \cup \big(A_\delta^{ij} \setminus B_\delta^{ij}\big) \subseteq \big(A_\delta^{ij} \Delta B_\delta^{ij}\big) \cup \big(A_\delta^{ij'} \Delta B_\delta^{ij'}\big),
\end{aligned}$$

which leads to

$$\mathbb{P}\big(A_\delta^{ij} \cap A_\delta^{ij'}\big) < 2\widetilde{\varepsilon}, \qquad \forall j \ne j'. \tag{4.32}$$

Note that the following is a partition of Ω in $\mathcal{F}_{\tau^i}$:

$$\Big\{A_\delta^{i0}, A_\delta^{ij} \setminus \bigcup_{j'=1}^{j-1} A_\delta^{ij'}, \quad 1 \le j \le N_2\Big\}.$$

We now define $\bar{u}_\delta(\cdot)$ as follows:

$$\bar{u}_\delta(r,\omega) = \left\{\begin{array}{ll} u^{ij}, \quad \omega \in A_\delta^{ij} \setminus \displaystyle\bigcup_{j'=1}^{j-1} A_\delta^{ij'}, \quad 1 \le j \le N_2, \\ u^0, \quad \omega \in A_\delta^{i0}, \end{array}\right. \qquad r \in [\tau_i, \tau_{i+1}),$$

for some fixed $u^0 \in U$. Thus, $\bar{u}_\delta(\cdot)$ is well-defined, and for each $0 \le i \le N_1$, $1 \le j \le N_2$,

$$\bar{u}_\delta(r,\omega) = u^{ij} = \widehat{u}_\delta(r,\omega), \quad \forall \omega \in B_\delta^{ij} \cap \Big(A_\delta^{ij} \setminus \bigcup_{j'=1}^{j-1} A_\delta^{ij'}\Big)$$

Therefore, for each $0 \le i \le N_1$ and $r \in [\tau^i, \tau^{i+1})$,

$$\begin{aligned}
&\{\omega \in \Omega \bigm| \bar{u}_\delta(r,\omega) \ne \widehat{u}_\delta(r,\omega)\} \\
&\subseteq \bigcup_{j=1}^{N_2} \Big\{A_\delta^{ij} \setminus \Big[B_\delta^{ij} \cap \Big(A_\delta^{ij} \setminus \bigcup_{j'=1}^{j-1} A_\delta^{ij'}\Big)\Big]\Big\} \cup A_\delta^{i0}
\end{aligned}$$

$$
\begin{aligned}
&= \bigcup_{j=1}^{N_2} \Big\{ A_\delta^{ij} \cap \Big[(B_\delta^{ij})^c \cup \Big((A_\delta^{ij})^c \cup \bigcup_{j'=1}^{j-1} A_\delta^{ij'} \Big) \Big] \Big\} \cup \Big(\bigcup_{j=1}^{N_2} A_\delta^{ij} \Big)^c \\
&= \bigcup_{j=1}^{N_2} \Big[\Big(A_\delta^{ij} \setminus B_\delta^{ij} \Big) \cup \Big(\bigcup_{j'=1}^{j-1} (A_\delta^{ij} \cap A_\delta^{ij'}) \Big) \Big] \cup \Big(\bigcup_{j=1}^{N_2} A_\delta^{ij} \Big)^c .
\end{aligned}
$$

Hence, by (4.29), (4.32), and (4.31),

$$
\begin{aligned}
&\mathbb{P}\Big(\big\{ \omega \in \Omega \bigm| \bar u_\delta(t,\omega) \neq \widehat u_\delta(r,\omega) \big\} \Big) \\
&\le \sum_{j=1}^{N_2} \Big[\mathbb{P}(A_\delta^{ij} \setminus B_\delta^{ij}) + \sum_{j'=1}^{j-1} \mathbb{P}\big(A_\delta^{ij} \cap A_\delta^{ij'} \big) \Big] + 1 - \mathbb{P}\Big(\bigcup_{j=1}^{N_2} A_\delta^{ij} \Big) \\
&\le N_2 \widetilde\varepsilon + \sum_{j=1}^{N_2} 2(j-1)\widetilde\varepsilon + N_2\widetilde\varepsilon = (2N_2 + N_2^2)\widetilde\varepsilon, \qquad r \in [\tau_i, \tau_{i+1}).
\end{aligned}
$$

By the boundedness of U, we have

$$
\mathbb{E} \int_\tau^T |\bar u_\delta(r) - \widehat u_\delta(r)| dr \le 2MT(2N_2 + N_2^2)\widetilde\varepsilon .
$$

Now, since $A_\delta^{ij} \in \mathcal{G}$, we may let

$$
A_\delta^{ij} = \bigcup_{k=1}^{N_{ij}} \big(A_\delta^{ijk} \cap \widehat A_\delta^{ijk} \big), \qquad 1 \le i \le N_1,\ 1 \le j \le N_2,
$$

with $A_\delta^{ijk} \in \mathcal{F}_\tau$, $\widehat A_\delta^{ijk} \in \mathcal{F}^\tau_{\tau^i}$ and $N_{ij} \ge 1$. Note that

$$
\{ A_\delta^{ijk} \bigm| 0 \le i \le N_1,\ 0 \le j \le N_2,\ 1 \le k \le N_{ij} \}
$$

is a finite subset of $\mathcal{F}_\tau$. Let $\mathcal{F}_\tau^{\{\delta\}}$ be the σ-field generated by the above set, which is still a finite subset of $\mathcal{F}_\tau$. Then we can obtain a (finite) partition of Ω in $\mathcal{F}_\tau$, denoted by the following:

$$
\mathbb{A}_\delta = \{ A_\delta^\ell \bigm| 1 \le \ell \le N_3 \},
$$

such that $\mathcal{F}_\tau^{\{\delta\}} = \sigma(\mathbb{A}_\delta)$. Likewise,

$$
\{ \widehat A_\delta^{ijk} \bigm| 0 \le i \le N_1,\ 1 \le j \le N_2,\ 1 \le k \le N_{ij} \}
$$

is a finite subset of $\mathcal{F}^\tau_{\tau^i}$, which generates a (finite) σ-field $\widehat{\mathcal{F}}^{\tau,\{\delta\}}_{\tau^i}$ and we can obtain a (finite) partition of Ω in $\mathcal{F}^\tau_{\tau^i}$, denoted by

$$
\widehat{\mathbb{A}}_\delta = \{ \widehat A_\delta^\ell \bigm| 1 \le \ell \le \widehat N_3 \},
$$

such that $\sigma(\widehat{\mathbb{A}}_\delta) = \widehat{\mathcal{F}}^{\tau,\{\delta\}}_{t^i}$. Hence, we have the following:

$$
\begin{aligned}
A_\delta^{ij} \setminus \bigcup_{j'=1}^{j-1} A_\delta^{ij'} &= \Big(\bigcup_{k=1}^{N_{ij}} A_\delta^{ijk} \cap \widehat A_\delta^{ijk} \Big) \setminus \bigcup_{j'=1}^{j-1} \Big(\bigcup_{k=1}^{N_{ij'}} A_\delta^{ij'k} \cap \widehat A_\delta^{ij'k} \Big) \\
&= \bigcup_{\ell \in \Lambda(i,j)} A_\delta^\ell \cap \Big(\bigcup_{\ell' \in \Lambda'(i,j,\ell)} \widehat A_\delta^{\ell'} \Big), \qquad 0 \le i \le N_1,\ 1 \le j \le N_2,
\end{aligned}
$$

$$
\begin{aligned}
A_\delta^{i0} &= \Big(\bigcup_{j=1}^{N_2} A_\delta^{ij}\Big)^c = \Big(\bigcup_{j=1}^{N_2}\bigcup_{k=1}^{N_{ij}} A_\delta^{ijk}\cap \widehat A_\delta^{ijk}\Big)^c \\
&= \Big(\bigcap_{j=1}^{N_2}\bigcap_{k=1}^{N_{ij}} (A_\delta^{ijk})^c\cup(\widehat A_\delta^{ijk})^c\Big) = \bigcup_{\ell\in\Lambda(i,0)} A_\delta^\ell \cap \Big(\bigcup_{\ell'\in\Lambda'(i,0,\ell)} \widehat A_\delta^{\ell'}\Big), \quad 0\le i\le N_1,
\end{aligned}
$$

where $\Lambda(i,j)$ and $\Lambda'(i,j,\ell)$ are suitable index sets.

Now, for any $(r,\omega)\in[\tau,T)\times\Omega$, there exists a unique $i\in\{0,1,\cdots,N_1\}$ such that $r\in[\tau^i,\tau^{i+1})$. Next, from the above, we see that either

$$
\omega\in\bigcup_{\ell\in\Lambda(i,j)} A_\delta^\ell\cap\Big(\bigcup_{\ell'\in\Lambda'(i,j,\ell)}\widehat A_\delta^{\ell'}\Big),
$$

leading to $\bar u_\delta(r,\omega)=u^{ij}$ $(1\le j\le N_2)$, or

$$
\omega\in\bigcup_{\ell\in\Lambda(i,0)} A_\delta^\ell\cap\Big(\bigcup_{\ell'\in\Lambda'(i,0,\ell)}\widehat A_\delta^{\ell'}\Big),
$$

leading to $\bar u_\delta(r,\omega)=u^0$. Thus, we have the following representation:

$$
\bar u_\delta(r,\omega)=\sum_{\ell=1}^{N_3}\mathbf 1_{A_\delta^\ell}\sum_{i=0}^{N_1}\sum_{\ell'\in\widehat\Lambda(i,\ell)} u_\ell^i\mathbf 1_{\widehat A_\delta^{\ell'}}(\omega)\mathbf 1_{[\tau^i,\tau^{i+1})}(r)\equiv\sum_{\ell=1}^{N_1}\mathbf 1_{A_\delta^\ell}(\omega)\widehat u_\delta^\ell(r,\omega),
$$

with $u_\ell^i\in\{u^0,u^{ij},1\le j\le N_2\}$, and $\widehat u_\delta^\ell(\cdot)\in\widehat{\mathcal U}[\tau,T]$. This proves the lemma. $\square$

To summarize the above, we have constructed the following functions from $u(\cdot)$:

$$
\begin{aligned}
&u_\delta(s,\omega)=\frac1\delta\int_{s-\delta}^s u(r,\omega)dr, \qquad s\ge\tau,\ \omega\in\Omega,\\
&\widetilde u_\delta(r)=\sum_{i=0}^{N_1} u_\delta(\tau^i)\mathbf 1_{[\tau^i,\tau^{i+1})}(r), \qquad r\in[\tau,T],\\
&\widehat u_\delta(r)=\sum_{i=0}^{N_1}\sum_{j=1}^{N_2} u^{ij}\mathbf 1_{B_\delta^{ij}}\mathbf 1_{[\tau^i,\tau^{i+1})}(r), \quad r\in[\tau,T],\\
&\bar u_\delta(r,\omega)=\sum_{\ell=1}^{N_3}\mathbf 1_{A_\delta^\ell}\sum_{i=0}^{N_1}\sum_{\ell'\in\widehat\Lambda(i,\ell)} u_\ell^i\mathbf 1_{\widehat A_\delta^{\ell'}}(\omega)\mathbf 1_{[\tau^i,\tau^{i+1})}(r)\equiv\sum_{\ell=1}^{N_1}\mathbf 1_{A_\delta^\ell}(\omega)\widehat u_\delta^\ell(r,\omega).
\end{aligned}
$$

With the aid of above result, we can prove the following theorem.

Theorem 4.13. *Let* (D1)–(D2) *hold. Then*

$$
\mathbb V(\tau,\xi)=\widehat{\mathbb V}(\tau,\xi), \qquad \forall(\tau,\xi)\in\mathcal D. \tag{4.33}
$$

Consequently, the restricted value function $V(\cdot\,,\cdot)$ is a deterministic function.

Proof. By (4.23), it suffices to show

$$
\widehat{\mathbb V}(\tau,\xi)\le\mathbb V(\tau,\xi), \qquad \forall(\tau,\xi)\in\mathcal D. \tag{4.34}
$$

To this end, let $u_k(\cdot)\in\mathcal U[\tau,T]$ such that

$$
J(\tau,\xi;u_k(\cdot))\searrow\mathbb V(\tau,\xi).
$$

By Lemma 4.12, we have

$$u_{k\ell}(\cdot)=\sum_{i\ge1}\mathbf{1}_{A^i_{k\ell}}\widehat u^i_{k\ell}(\cdot),$$

with $A^i_{k\ell}\in\mathcal{F}_\tau$ and $\widehat u^i_{k\ell}(\cdot)\in\widehat{\mathcal{U}}[\tau,T]$ and

$$\mathbb{E}\int_\tau^T|u_k(r)-u_{k\ell}(r)|dr<\frac{1}{\ell},\qquad \ell\ge1.$$

Then

$$J(\tau,\xi;u_{k\ell}(\cdot))=\sum_{i\ge1}\mathbf{1}_{A^i_{k\ell}}J(\tau,\xi;\widehat u^i_{k\ell}(\cdot))\ge\sum_{i\ge1}\mathbf{1}_{A^i_{k\ell}}\widehat{\mathbb{V}}(\tau,\xi)=\widehat{\mathbb{V}}(\tau,\xi).$$

Hence, letting $\ell\to\infty$, we see that (4.33) holds. □

Thanks to the above result, we see that the (restricted) value function $V(\cdot\,,\cdot)$ satisfies the following: For any $(t,x),(t',x')\in[0,T]\times\mathbb{R}^n$,

$$\begin{aligned}&|V(t,x)|\le K(1+|x|),\\&|V(t,x)-V(t',x')|\le K\big[(1+|x|\vee|x'|)|t-t'|^{\frac12}+|x-x'|\big],\end{aligned}\tag{4.35}$$

and for $\bar t\in(t,T]$ (comparing with (4.12) and (4.16))

$$V(t,x)=\inf_{u(\cdot)\in\mathcal{U}[t,\,\bar t]}\mathbb{E}_t\Big\{\int_t^{\bar t}g(s,X(s),u(s))ds+V(\bar t,X(\bar t))\Big\}.\tag{4.36}$$

4.3. **Some examples.** In this section, we present some examples for which the HJB equations can be solved explicitly.

4.3.1. Merton type problems Consider a market with one bond and one stock whose price processes $S_0(\cdot)$ and $S(\cdot)$ are descried by the following classical Merton model:

$$\begin{cases}dS_0(t)=rS_0(t)dt,\\dS(t)=\mu S(t)dt+\sigma S(t)dW(t).\end{cases}\tag{4.37}$$

An investor is trading in the market, whose wealth process $X(\cdot)$ satisfies the following state equation:

$$\begin{cases}dX(s)=\big[rX(s)+(\mu-r)\pi(s)\big]ds+\sigma\pi(s)dW(s),\qquad s\in[t,T],\\X(t)=x,\end{cases}\tag{4.38}$$

where $x>0$ is the initial wealth at t and $\pi(t)$ is the dollar amount in the stock. To measure the performance of the trading process $\pi(\cdot)$, we introduce the following payoff functional:

$$J(t,x;\pi(\cdot))=\mathbb{E}\big[h(X(T))\big],\tag{4.39}$$

for some concave function $h:\mathbb{R}\to\mathbb{R}$. The classical Merton's problem is the following:

Problem (M). *For given $(t,x)\in[0,T)\times(0,\infty)$, find a $\bar\pi(\cdot)$ such that*

$$J(t,x;\bar\pi(\cdot))=\sup_{\pi(\cdot)}J(t,x;\pi(\cdot))=V(t,x).\tag{4.40}$$

Let

$$
\begin{aligned}
H(t,x,p,P,\pi) &= \frac{1}{2}P\sigma^2\pi^2 + p\big[rx + (\mu - r)\pi\big] \\
&= \frac{P\sigma^2}{2}\Big(\pi^2 + \frac{2p(\mu - r)}{P\sigma^2}\pi\Big) + prx \\
&= \frac{P\sigma^2}{2}\Big(\pi + \frac{p(\mu - r)}{P\sigma^2}\Big)^2 - \frac{p^2(\mu - r)^2}{2P\sigma^2} + prx.
\end{aligned}
$$

Then the HJB equation reads

$$
\begin{cases}
V_t(t,x) - \dfrac{\theta^2 V_x(t,x)^2}{2V_{xx}(t,x)} + rxV_x(t,x) = 0, \quad (t,x) \in [0,T] \times [0,\infty), \\
V(T,x) = h(x), \qquad x \in [0,\infty), \\
V(t,0) = 0, \qquad t \in [0,T], \\
V_{xx}(t,x) < 0, \qquad (t,x) \in [0,T] \times [0,\infty),
\end{cases}
\tag{4.41}
$$

where $\theta = \frac{\mu - r}{\sigma}$ is called the *risk premium.* If $V(\cdot\,,\cdot)$ is the classical solution to the above HJB equation, then the optimal control is given by

$$
\bar{\pi}(t) = -\frac{(\mu - r)V_x(t,X(t))}{\sigma^2 V_{xx}(t,X(t))}, \qquad t \in [0,T].
$$

Now, we look at three cases.

(i) The classical Merton problem. In this case, one has

$$
h(x) = \frac{x^\beta}{\beta}, \qquad x \geq 0,
$$

for some $\beta < 1$. In this case, we let

$$
V(t,x) = \varphi(t)x^\beta.
$$

Then,

$$
\begin{aligned}
0 &= V_t(t,x) - \frac{\theta^2 V_x(t,x)^2}{2V_{xx}(t,x)} + rxV_x(t,x) \\
&= \varphi'(t)x^\beta - \frac{\theta^2}{2}\frac{\varphi(t)^2\beta^2 x^{2(\beta-1)}}{\varphi(t)\beta(\beta-1)x^{\beta-2}} + rx\varphi(t)\beta x^{\beta-1} \\
&= \Big[\varphi'(t) + \Big(\frac{\theta^2\beta}{2(1-\beta)} + r\beta\Big)\varphi(t)\Big]x^\beta.
\end{aligned}
$$

Hence, $\varphi(\cdot)$ satisfies the following linear equation:

$$
\begin{cases}
\varphi'(t) + \Big(\dfrac{\theta^2\beta}{2(1-\beta)} + r\beta\Big)\varphi(t) = 0, \qquad t \in [0,T], \\
\varphi(T) = \dfrac{1}{\beta}.
\end{cases}
$$

Consequently,

$$
\varphi(t) = \frac{e^{\lambda(T-t)}}{\beta}, \qquad \lambda = \frac{\theta^2\beta}{2(1-\beta)} + r\beta,
$$

and

$$
V(t,x) = \varphi(t)x^\beta = \frac{e^{\lambda(T-t)}x^\beta}{\beta}.
$$

Note that

$$\frac{V_x(t,x)(\mu-r)}{V_{xx}(t,x)\sigma^2}=\frac{\varphi(t)\beta(\mu-r)x^{\beta-1}}{\sigma^2\varphi(t)\beta(\beta-1)x^{\beta-2}}=\frac{\mu-r}{\sigma^2(\beta-1)}x.$$

The optimal portfolio is given by

$$\bar\pi(t)=\frac{\mu-r}{\sigma^2(1-\beta)}X(t),\qquad t\in[0,T].$$

(ii) The case of exponential utility. In this case, we have

$$h(x)=-e^{-\beta x}.$$

For this case, we are solving the following HJB equation:

$$\begin{cases} V_t(t,x)-\dfrac{\theta^2V_x(t,x)^2}{2V_{xx}(t,x)}+rxV_x(t,x)=0, & (t,x)\in[0,T]\times[0,\infty),\\ V(T,x)=-e^{-\beta x}, \qquad x\in\mathbb{R}, \\ V_{xx}(t,x)<0, \qquad (t,x)\in[0,T]\times\mathbb{R}. \end{cases}\tag{4.42}$$

Let

$$V(t,x)=-\varphi(t)e^{-\psi(t)x}.$$

Then

$$\begin{aligned}0&=V_t(t,x)-\frac{\theta^2V_x(t,x)^2}{2V_{xx}(t,x)}+rxV_x(t,x)\\ &=-\varphi'(t)e^{-\psi(t)x}+\varphi(t)\psi'(t)xe^{-\psi(t)x}\\ &\qquad-\frac{\theta^2\varphi(t)^2\psi(t)^2e^{-2\psi(t)x}}{2\varphi(t)\psi(t)^2e^{-\psi(t)x}}+rx\varphi(t)\psi(t)e^{-\psi(t)x}\\ &=-\big[\varphi'(t)+\frac{\theta^2}{2}\varphi(t)\big]e^{-\psi(t)x}+\big[\psi'(t)+r\psi(t)\big]\varphi(t)xe^{-\psi(t)x}.\end{aligned}$$

Thus,

$$\varphi'(t)=-\frac{\theta^2}{2}\varphi(t),\qquad \varphi(T)=1,$$

and

$$\psi'(t)=-r\psi(t),\qquad \psi(T)=\beta.$$

Consequently,

$$\varphi(t)=e^{\frac{\theta^2}{2}(T-t)},\qquad \psi(t)=e^{r(T-t)}\beta,\qquad t\in[0,T].$$

This leads to

$$V(t,x)=-e^{\frac{\theta^2}{2}(T-t)-e^{r(T-t)}\beta x},\qquad (t,x)\in[0,T]\times\mathbb{R}.$$

In this case, we have

$$\frac{V_x(t,x)(\mu-r)}{V_{xx}(t,x)\sigma^2}=\frac{\varphi(t)\psi(t)e^{-\psi(t)x}(\mu-r)}{-\sigma^2\varphi(t)\psi(t)^2e^{-\psi(t)x}}=-\frac{\mu-r}{\sigma^2\psi(t)}=-\frac{\mu-r}{\sigma^2\beta}e^{-r(T-t)}.$$

Hence, the optimal portfolio is given by the following:

$$\bar\pi(t)=\frac{\mu-r}{\sigma^2\beta}e^{-r(T-t)}.$$

(iii) The case of logarithmic utility. We consider the following utility function:

$$h(x)=\ln x.$$

For this case, we are solving the following HJB equation:

$$\begin{cases} V_t(t,x) - \dfrac{\theta^2 V_x(t,x)^2}{2V_{xx}(t,x)} + rxV_x(t,x) = 0, \quad (t,x) \in [0,T] \times (0,\infty), \\ V(T,x) = \ln x, \qquad x \in (0,\infty), \\ V(t,0) = -\infty, \qquad t \in [0,T], \\ V_{xx}(t,x) < 0, \qquad (t,x) \in [0,T] \times (0,\infty). \end{cases} \tag{4.43}$$

Let

$$V(t,x) = \varphi(t) + \ln x.$$

Then

$$\begin{aligned} 0 &= V_t(t,x) - \frac{\theta^2 V_x(t,x)^2}{2V_{xx}(t,x)} + rxV_x(t,x) \\ &= \varphi'(t) - \frac{\theta^2 x^{-2}}{-2x^{-2}} + rxx^{-1} = \varphi'(t) + \frac{\theta^2}{2} + r. \end{aligned}$$

Hence,

$$\varphi(t) = \Big(\frac{\theta^2}{2} + r\Big)(T-t).$$

Then

$$V(t,x) = \Big(\frac{\theta^2}{2} + r\Big)(T-t) + \ln x, \qquad (t,x) \in [0,T] \times (0,\infty).$$

For such a case,

$$\frac{V_x(t,x)(\mu - r)}{V_{xx}(t,x)\sigma^2} = \frac{x^{-1}(\mu - r)}{-x^{-2}\sigma^2} = -\frac{(\mu - r)x}{\sigma^2}.$$

Hence, the optimal portfolio is given by

$$\bar\pi(t) = \frac{(\mu - r)X(t)}{\sigma^2}, \qquad t \in [0,T].$$

4.3.2. Linear-quadratic problems

Consider the following controlled linear SDE:

$$\begin{cases} dX(s) = \big[AX(s) + Bu(s)\big]ds + \big[CX(s) + Du(s)\big]dW(s), \quad s \in [t,T], \\ X(t) = x. \end{cases}$$

The cost functional reads

$$\begin{aligned} J(t,x;u(\cdot)) = \frac{1}{2}\mathbb{E}_t\Big[\int_t^T \Big(\langle QX(s), X(s)\rangle + \langle Ru(s), u(s)\rangle\Big)ds \\ + \langle GX(T), X(T)\rangle\Big]. \end{aligned}$$

For such a problem, the Hamiltonian takes the following form:

$$\begin{aligned} \mathbb{H}(t,x,u,\mathbf{p}^\top,\mathbf{P}) &= \mathbf{p}^\top(Ax+Bu) + \frac{1}{2}(Cx+Du)^\top \mathbf{P}(Cx+Du) \\ &\quad + \frac{1}{2}\big(x^\top Qx + u^\top Ru\big) \\ &= \frac{1}{2}u^\top(R + D^\top \mathbf{P}D)u + (\mathbf{p}^\top B + x^\top C^\top \mathbf{P}D)u \\ &\quad + \frac{1}{2}x^\top Qx + \mathbf{p}^\top Ax + \frac{1}{2}x^\top C^\top \mathbf{P}Cx \end{aligned}$$

$$\begin{aligned}&=\frac{1}{2}\Big|(R+D^\top\mathbf{P}D)^{\frac12}\big[u+(R+D^\top\mathbf{P}D)^{-1}(B^\top\mathbf{p}+D^\top\mathbf{P}Cx)\big]\Big|^2\\&\quad+\frac{1}{2}\Big[x^\top Qx+2\mathbf{p}^\top Ax+x^\top C^\top\mathbf{P}Cx\\&\quad-(\mathbf{p}^\top B+x^\top C^\top\mathbf{P}D)(R+D^\top\mathbf{P}D)^{-1}(B^\top\mathbf{p}+D^\top\mathbf{P}Cx)\Big].\end{aligned}$$

Here, we assume that

$$R+D^\top\mathbf{P}D>0.$$

Then the corresponding HJB equation reads

$$\begin{cases}V_t(t,x)+\dfrac{1}{2}\Big[x^\top Qx+2V_x(t,x)Ax+x^\top C^\top V_{xx}(t,x)Cx\\\quad-\big[V_x(t,x)B+x^\top C^\top V_{xx}(t,x)D\big]\big[R+D^\top V_{xx}(t,x)D\big]^{-1}\\\qquad\cdot\big[B^\top V_x(t,x)^\top+D^\top V_{xx}(t,x)Cx\big]\Big]=0,\\V(T,x)=\dfrac{1}{2}x^\top Gx,\end{cases}$$

and the optimal control is given by

$$\bar u(t)=\big[R+D^\top V_{xx}(t,X(t))D\big]^{-1}\Big[B^\top V_x(t,X(t))^\top+D^\top V_{xx}(t,X(t))CX(t)\Big].$$

We now take the following ansatz:

$$V(t,x)=\frac{1}{2}x^\top P(t)x.$$

Then

$$V_t(t,x)=\frac{1}{2}x^\top\dot P(t)x,\quad V_x(t,x)=x^\top P(t),\quad V_{xx}(t,x)=P(t).$$

Consequently,

$$\begin{aligned}0=x^\top\Big[&\dot P+PA+A^\top P+C^\top PC+Q\\&-(PB+C^\top PD)(R+D^\top PD)^{-1}(B^\top P+D^\top PC)\Big]x.\end{aligned}$$

Thus, it suffices to have $P(\cdot)$ being the solution to the following equation:

$$\begin{cases}\dot P(t)+P(t)A+A^\top P(t)+C^\top P(t)C+Q-[P(t)B+C^\top P(t)D]\\\qquad\cdot[R+D^\top P(t)D]^{-1}\big[B^\top P(t)+D^\top P(t)C\big]=0,\quad t\in[0,T],\\P(T)=G.\end{cases}\tag{4.44}$$

The above is called a *Riccati equation* which admits a solution $P(\cdot)$, under certain conditions. We will come back later.

4.4.3. Affine-quadratic problems We would like to look at a further extension of the above linear-quadratic problem. More precisely, we consider the following state equation:

$$\begin{cases}dX(s)=\big[A(s,X(s))+B(s,X(s))u(s)\big]ds\\\qquad+\big[C(s,X(s))+D(s,X(s))u(s)\big]dW(s),\quad s\in[\tau,T],\\X(\tau)=\xi,\end{cases}\tag{4.45}$$

with the cost functional

$$
\begin{aligned}
J(\tau,\xi;u(\cdot)) = \mathbb{E}_\tau\Big[\int_\tau^T \big[q(s,X(s)) + \frac{1}{2}u(s)^\top R(s,X(s))u(s)\big]ds \\
+h(X(T))\Big].
\end{aligned}
\tag{4.46}
$$

Under proper conditions, we could get the well-posedness of the state equation for any initial pair (τ,ξ) and control $u(\cdot)\in\mathcal{U}[\tau,T]$. Also, under suitable conditions, the cost functional will be meaningful. We see that both drift and diffusion of the state equation are affine in the control variable $u(\cdot)$ and the cost is quadratic in $u(\cdot)$. Therefore, we call the corresponding optimal control an *affine-quadratic* problem. In the current case, the Hamiltonian is given by the following:

$$
\begin{aligned}
&H(t,x,u,\mathbf{p}^\top,\mathbf{P}) = \mathbf{p}^\top[A(t,x)+B(t,x)u]\\
&\quad+\frac{1}{2}\big(C(t,x)+D(t,x)u\big)^\top\mathbf{P}\big(C(t,x)+D(t,x)u\big)+q(t,x)+\frac{1}{2}u^\top R(t,x)u\\
&=\mathbf{p}^\top A(t,x)+\frac{1}{2}C(t,x)^\top\mathbf{P}C(t,x)+q(t,x)\\
&\quad+\frac{1}{2}u^\top\big[R(t,x)+D(t,x)^\top\mathbf{P}D(t,x)\big]u+\big[\mathbf{p}^\top B(t,x)+C(t,x)^\top\mathbf{P}D(t,x)\big]u\\
&=\frac{1}{2}\Big|\big[R(t,x)+D(t,x)^\top\mathbf{P}D(t,x)\big]^{\frac{1}{2}}\\
&\quad\cdot\Big(u+\big[R(t,x)+D(t,x)^\top\mathbf{P}D(t,x)\big]^{-1}\big[B(t,x)^\top\mathbf{p}+D(t,x)^\top\mathbf{P}C(t,x)\big]\Big)\Big|^2\\
&\quad+\mathbf{p}^\top A(t,x)+\frac{1}{2}C(t,x)^\top\mathbf{P}C(t,x)+q(t,x)\\
&\quad-\big[\mathbf{p}^\top B(t,x)+C(t,x)^\top\mathbf{P}D(t,x)\big]\big[R(t,x)+D(t,x)^\top\mathbf{P}D(t,x)\big]^{-1}\\
&\quad\cdot\big[B(t,x)^\top\mathbf{p}+D(t,x)^\top\mathbf{P}C(t,x)\big].
\end{aligned}
$$

Hence, the corresponding HJB equation is the following:

$$
\begin{cases}
V_t(t,x)+V_x(t,x)A(t,x)+\frac{1}{2}C(t,x)^\top V_{xx}(t,x)C(t,x)+q(t,x)\\
\qquad-\big[V_x(t,x)B(t,x)+C(t,x)^\top V_{xx}(t,x)D(t,x)\big]\\
\qquad\cdot\big[R(t,x)+D(t,x)^\top V_{xx}(t,x)D(t,x)\big]^{-1}\\
\qquad\cdot\big[B(t,x)^\top V_x(t,x)^\top+D(t,x)^\top V_{xx}(t,x)C(t,x)\big]=0,\\
V(T,x)=h(x).
\end{cases}
\tag{4.47}
$$

The optimal control is given by

$$
\begin{aligned}
\bar u(t) = -\big[R(t,\bar X(t))+D(t,\bar X(t))^\top V_{xx}(t,\bar X(t))D(t,\bar X(t))\big]^{-1}\\
\cdot\big[B(t,\bar X(t))^\top V_x(t,\bar X(t))^\top+D(t,\bar X(t))^\top V_{xx}(t,\bar X(t))C(t,\bar X(t))\big].
\end{aligned}
$$

We see that the linear-quadratic problem is a special case of the above.

It is open that under what suitable conditions, HJB equation (4.47) admits a classical solution.

4.4.4. Convexification of control set Let U be an arbitrary closed set and $\mathrm{co}\,U$ be its convex hull. Then we could pose two optimal control problems, associated with U and $\operatorname{co} U$, respectively. We denote the value function associated with $\operatorname{co} U$ by $V^{\mathrm{co}}(\cdot\,,\cdot)$. Then, in general,

$$V^{\mathrm{co}}(t,x) \le V(t,x). \tag{4.48}$$

The following example shows that the inequality could be strict.

Example 4.14. Consider

$$\begin{cases} dX(t) = \sigma u(s) dW(s), \qquad & s \in [t,T], \\ X(t) = x, \end{cases}$$

with the control set $U = \{\pm 1\}$ and the cost functional

$$J(t,x;u(\cdot)) = \frac{1}{2}\mathbb{E}\Big[\int_t^T |u(s)|^2 ds + |X(T)|^2\Big],$$

Then, noting $|u(s)| = 1$, we have

$$\begin{aligned} J(t,x;u(\cdot)) &= \frac{1}{2}\mathbb{E}\Big[\int_t^T |u(s)|^2 ds + \Big|x + \int_t^T \sigma u(s) dW(s)\Big|^2\Big] \\ &= \frac{1}{2}\Big[(1+\sigma^2)(T-t) + x^2\Big] = V(t,x). \end{aligned} \tag{4.49}$$

Now, for convexified $\operatorname{co} U = [-1,1]$, we have

$$\begin{aligned} J(t,x;u(\cdot)) &= \frac{1}{2}\mathbb{E}\Big[\int_t^T |u(s)|^2 ds + \Big|x + \int_t^T \sigma u(s) dW(s)\Big|^2\Big] \\ &= \frac{1}{2}\Big[x^2 + (1+\sigma^2)\int_t^T |u(s)|^2 ds\Big] \ge \frac{x^2}{2} = V^{\mathrm{co}}(t,x). \end{aligned} \tag{4.50}$$

Hence, the strict inequality in (4.48) holds. In the current case, the HJB equations for $V(\cdot\,,\cdot)$ and $V^{\mathrm{co}}(\cdot\,,\cdot)$ are, respectively,

$$\begin{cases} V_t(t,x) + \dfrac{1+\sigma^2}{2} V_{xx}(t,x) = 0, \qquad & (t,x) \in [0,T] \times \mathbb{R}, \\ V(T,x) = \dfrac{1}{2}x^2, \qquad x \in \mathbb{R}, \end{cases} \tag{4.51}$$

and

$$\begin{cases} V^{\mathrm{co}}_t(t,x) + \dfrac{1+\sigma^2}{2} \min\Big\{V^{\mathrm{co}}_{xx}(t,x), 0\Big\} = 0, \qquad & (t,x) \in [0,T] \times \mathbb{R}, \\ V^{\mathrm{co}}(T,x) = \dfrac{1}{2}x^2, \qquad x \in \mathbb{R}. \end{cases} \tag{4.52}$$

Clearly, $V(\cdot\,,\cdot)$ and $V^{\mathrm{co}}(\cdot\,,\cdot)$ are classical solutions to HJB equations (4.51) and (4.52), respectively.

The above example shows that for the problem with general control set U, the properties of the value function cannot be obtained by considering the problem with convexified control set $\operatorname{co} U$.

4.4. **Viscosity solution of HJB equations.** In this section we will handle the issue that the value function $V(\cdot\,,\cdot)$ is not differentiable. First of all, we present an example that the value function is really not differentiable.

Example 4.15. Consider the state equation

$$\begin{cases} dX(s)=u(s)ds+\mathbf{1}_{[0,a)}(s)dW(s), & s\in[t,T],\\ X(t)=x, \end{cases}$$

with $0<a<T$. The cost functional is

$$J(t,x;u(\cdot))=\mathbb{E}_t|X(T)|^2.$$

The control set is $U=[-1,1]$. For any $(t,x)\in[a,T]\times\mathbb{R}$,

$$J(t,x;u(\cdot))=\mathbb{E}_t\Big(x+\int_t^T u(s)ds\Big)^2.$$

Thus, it is clear that

$$V(t,x)=\begin{cases} \Big(\big[x-(T-t)\big]^+\Big)^2, & x\geq 0,\\ \Big(\big[x+(T-t)\big]^+\Big)^2, & x\leq 0. \end{cases}$$

Clearly, $x\mapsto V(t,x)$ is not C^2. The Hamiltonian is given by

$$\mathbb{H}(t,x,u,\mathbf{p},\mathbf{P})=\mathbf{p}u+\frac{1}{2}\mathbf{1}_{[0,a]}(t)\mathbf{P}.$$

Hence,

$$H(t,x,\mathbf{p},\mathbf{P})=\inf_{u\in[-1,1]}\mathbb{H}(t,x,u,\mathbf{p},\mathbf{P})=-|\mathbf{p}|+\frac{1}{2}\mathbf{1}_{[0,a]}(t)\mathbf{P}.$$

Consequently, HJB equation reads

$$\begin{cases} V_t(t,x)+\frac{1}{2}\mathbf{1}_{[0,a]}(t)V_{xx}(t,x)-|V_x(t,x)|=0, & (t,x)\in[0,a)\times\mathbb{R},\\ V(T,x)=x^2. \end{cases}$$

4.4.1. Viscosity solution

Let us first introduce the following definition.

Definition 4.16. (i) A continuous function $v(\cdot\,,\cdot)$ is called a *viscosity sub-solution* of (4.17) if

$$\varphi(T,x)\leq h(x), \qquad x\in\mathbb{R}^n,$$

and for any $\varphi(\cdot\,,\cdot)\in C^{1,2}([0,T]\times\mathbb{R}^n)$, whenever $v(\cdot\,,\cdot)-\varphi(\cdot\,,\cdot)$ attains a local maximum at $(t_0,x_0)\in[0,T)\times\mathbb{R}^n$, the following holds:

$$\varphi_t(t_0,x_0)+H(t_0,x_0,\varphi_x(t_0,x_0),\varphi_{xx}(t_0,x_0))\geq 0. \tag{4.53}$$

(ii) A continuous function $v(\cdot\,,\cdot)$ is called a *viscosity super-solution* of (4.17) if

$$\varphi(T,x)\geq h(x), \qquad x\in\mathbb{R}^n,$$

and for any $\varphi(\cdot\,,\cdot)\in C^{1,2}([0,T]\times\mathbb{R}^n)$, whenever $v(\cdot\,,\cdot)-\varphi(\cdot\,,\cdot)$ attains a local minimum at $(t_0,x_0)\in[0,T)\times\mathbb{R}^n$, the following holds:

$$\varphi_t(t_0,x_0)+H(t_0,x_0,\varphi_x(t_0,x_0),\varphi_{xx}(t_0,x_0))\leq 0. \tag{4.54}$$

(iii) A continuous function $v(\cdot\,,\cdot)$ is called a *viscosity solution* of (4.17) if it is both viscosity sub- and siper-solution of (4.17).

The following proposition shows that viscosity solution is a good generalization of classical solution to the HJB equation.

Proposition 4.17. (i) *If $v(\cdot\,,\cdot)$ is a classical solution to* (4.17)*, then it is a viscosity solution to* (4.17).

(ii) *If $v(\cdot\,,\cdot)$ is a viscosity solution to* (4.17) *and $v(\cdot\,,\cdot)\in C^{1,2}([0,T]\times\mathbb{R}^n)$, then it is a classical solution to* (4.17).

Proof. (i) Let $v(\cdot\,,\cdot)$ be a classical solution to (4.17). Then $v(T,x)=h(x)$, and for any $\varphi(\cdot\,,\cdot)\in C^{1,2}([0,T]\times\mathbb{R}^n)$, if $v(\cdot\,,\cdot)-\varphi(\cdot\,,\cdot)$ attains a local maximum at $(t_0,x_0)\in(0,T)\times\mathbb{R}^n$, then

$$v_t(t_0,x_0)=\varphi_t(t_0,x_0),\quad v_x(t_0,x_0)=\varphi_x(t_0,x_0),\quad v_{xx}(t_0,x_0)\le\varphi_{xx}(t_0,x_0).$$

Thus,

$$\begin{aligned}&\varphi_t(t_0,x_0)+H(t_0,x_0,\varphi_x(t_0,x_0),\varphi_{xx}(t_0,x_0))\\&\ge v_t(t_0,x_0)+H(t_0,x_0,v_x(t_0,x_0),v_{xx}(t_0,x_0))=0.\end{aligned}$$

This means that $v(\cdot\,,\cdot)$ is a viscosity sub-solution of (4.17). Likewise, $v(\cdot\,,\cdot)$ is a viscosity super-solution of (4.17). Hence, $v(\cdot\,,\cdot)$ is a viscosity solution to (4.17).

(ii) Let $v(\cdot\,,\cdot)$ be a viscosity solution of (4.17) and $v(\cdot\,,\cdot)\in C^{1,2}([0,T]\times\mathbb{R}^n)$. Then, by picking $\varphi(\cdot\,,\cdot)=v(\cdot\,,\cdot)$, we have $v(\cdot\,,\cdot)-\varphi(\cdot\,,\cdot)$ attains a local maximum and a local minimum at every point (t_0,x_0). Thus,

$$\begin{aligned}&v_t(t_0,x_0)+H(t_0,x_0,v_x(t_0,x_0),v_{xx}(t_0,x_0))\\&=\varphi_t(t_0,x_0)+H(t_0,x_0,\varphi_x(t_0,x_0),\varphi_{xx}(t_0,x_0))\ge0 \text{ and } \le 0.\end{aligned}$$

Hence, $v(\cdot\,,\cdot)$ is a classical solution of (4.17). □

Proposition 4.18. *Let* (D1)–(D2) *hold, Then the value function $V(\cdot\,,\cdot)$ is a viscosity solution of* (4.17).

Proof. Let $V(\cdot\,,\cdot)-\varphi(\cdot\,,\cdot)$ attains a local maximum at (t_0,x_0). Then for (t,x) near (t_0,x_0), we have

$$V(t,x)-\varphi(t,x)\le V(t_0,x_0)-\varphi(t_0,x_0).$$

Then

$$V(t,X(t))-V(t_0,x_0)\le\varphi(t,X(t))-\varphi(t_0,x_0).$$

Since

$$\begin{aligned}V(t_0,x_0)&=\inf_{u(\cdot)}\mathbb{E}_{t_0}\Big[\int_{t_0}^t g(s,X(s),u(s))ds+V(t,X(t))\Big]\\&\le\mathbb{E}_{t_0}\Big[\int_{t_0}^t g(s,X(s),u)ds+V(t,X(t))\Big],\qquad\forall u\in U,\end{aligned}$$

one has the following:

$$\begin{aligned}0 &\le \mathbb{E}_{t_0}\Big[\int_{t_0}^t g(s,X(s),u)ds + V(t,X(t)) - V(t_0,x_0)\Big]\\ &\le \mathbb{E}_{t_0}\Big[\int_{t_0}^t g(s,X(s),u)ds + \varphi(t,X(t)) - \varphi(t_0,x_0)\Big]\\ &= \mathbb{E}_{t_0}\Big[\int_{t_0}^t \Big(g(s,X(s),u) + \varphi_s(s,X(s)) + \varphi_x(s,X(s))b(s,X(s),u)\\ &\qquad\qquad + \frac{1}{2}\mathrm{tr}\,\big[\varphi_{xx}(s,X(s))\sigma(s,X(s),u)\sigma(s,X(s),u)^\top\big]\Big)ds\Big].\end{aligned}$$

Dividing $t-t_0$ and sending $t\to t_0$, we get

$$0 \le \varphi_t(t_0,x_0) + \mathbb{H}(t_0,x_0,u,\varphi_x(t_0,x_0),\varphi_{xx}(t_0,x_0)).$$

This leads to

$$0 \le \varphi_t(t_0,x_0) + H(t_0,x_0,\varphi_x(t_0,x_0),\varphi_{xx}(t_0,x_0).$$

Thus, $V(\cdot\,,\cdot)$ is a viscosity sub-solution.

Next, let $V(\cdot\,,\cdot)-\varphi(\cdot\,,\cdot)$ attains a local minimum at (t_0,x_0). For any $\varepsilon>0$, there exists a $u^\varepsilon(\cdot)$ such that

$$\begin{aligned}&\varepsilon(t-t_0) > \mathbb{E}_{t_0}\Big[\int_{t_0}^t g(s,X^\varepsilon(s),u^\varepsilon(s))ds + V(t,X^\varepsilon(t)) - V(t_0,x_0)\Big]\\ &\ge \mathbb{E}_{t_0}\Big[\int_{t_0}^t g(s,X^\varepsilon(s),u^\varepsilon(s))ds + \varphi(t,X^\varepsilon(t)) - \varphi(t_0,x_0)\Big]\\ &= \mathbb{E}_{t_0}\Big[\int_{t_0}^t \Big(\varphi_t(s,X^\varepsilon(s)) + \mathbb{H}(s,X^\varepsilon(s),u^\varepsilon(s),\varphi_x(s,X^\varepsilon(s)),\varphi_{xx}(s,X^\varepsilon(s))ds\Big]\\ &\ge \mathbb{E}_{t_0}\Big[\int_{t_0}^t \Big(\varphi_t(s,X^\varepsilon(s)) + H(s,X^\varepsilon(s),\varphi_x(s,X^\varepsilon(s)),\varphi_{xx}(s,X^\varepsilon(s))ds\Big].\end{aligned}$$

Then, dividing $(t-t_0)$ and sending $t\to t_0$, we get

$$0 \ge \varphi_t(t_0,x_0) + H(t_0,x_0,\varphi_x(t_0,x_0),\varphi_{xx}(t_0,x_0)).$$

This means that $V(\cdot\,,\cdot)$ is a viscosity super-solution of (4.17).

Combining the above two parts, we see that the value function $V(\cdot\,,\cdot)$ is a viscosity solution to (4.17). □

4.4.2. Uniqueness

In this subsection, we will prove the uniqueness of viscosity solution of HJB equation (4.17).

Theorem 4.19. *Let* (D1) *hold. Then the HJB equation admits at most one viscosity solution* $v(\cdot\,,\cdot)$ *in the class of functions satisfying* (4.35).

To prove the above theorem, we need to make some preparations.

Lemma 4.20. *Let* $v\in C([0,T]\times\mathbb{R}^n)$ *satisfying* (4.35). *For any* $\gamma>0$, *define*

$$v^\gamma(t,x) \triangleq \sup_{(s,y)\in[0,T]\times\mathbb{R}^n}\Big\{v(s,y) - \frac{1}{2\gamma^2}\big[|t-s|^2+|x-y|^2\big]\Big\},\qquad (t,x)\in[0,T]\times\mathbb{R}^n. \tag{4.55}$$

Then $v^\gamma(\cdot\,,\cdot)$ is semiconvex, by which we mean that there exists a constant $K_0>0$ such that $(t,x)\mapsto v^\gamma(t,x)+K_0(|t|^2+|x|^2)$ is convex. Moreover, the following is satisfied:

$$|v^\gamma(t,x)|\le K(1+|x|),\quad \forall (t,x)\in[0,T]\times\mathbb{R}^n,\ \gamma>0, \tag{4.56}$$

and

$$\begin{aligned}|v^\gamma(t,x)-v^\gamma(t',x')|\le \frac{K}{\gamma^2}\big[|t-t'|+(1+|x|^{\frac12}\vee|x'|^{\frac12})|x-x'|\big],\\ \forall (t,x),(t',x')\in[0,T]\times\mathbb{R}^n,\ \gamma>0.\end{aligned} \tag{4.57}$$

Further, for any $(t,x)\in[0,T]\times\mathbb{R}^n$, there exists a $(\hat t,\hat x)\in[0,T]\times\mathbb{R}^n$ such that

$$v^\gamma(t,x)=v(\hat t,\hat x)-\frac{1}{2\gamma^2}\big[|t-\hat t|^2+|x-\hat x|^2\big], \tag{4.58}$$

and for some absolute constant K (independent of (t,x) and γ)

$$\left\{\begin{aligned}&|t-\hat t|\le K\gamma^{\frac43}\big(1+|x|^{\frac23}\big),\\ &|x-\hat x|\le K\gamma(1+|x|^{\frac12}),\end{aligned}\right. \tag{4.59}$$

and

$$0\le v^\gamma(t,x)-v(t,x)\le K\gamma^{\frac23}(1+|x|^{\frac43}). \tag{4.60}$$

Proof. First of all, from the following

$$\begin{aligned}&(t,x)\mapsto v^\gamma(t,x)+\frac{1}{2\gamma^2}\big(|t|^2+|x|^2\big)\\ &=\sup_{(s,y)\in[0,T]\times\mathbb{R}^n}\Big\{v(s,y)-\frac{1}{2\gamma^2}\Big(|t-s|^2+|x-y|^2\Big)+\frac{1}{2\gamma^2}\Big(|t|^2+|x|^2\Big)\Big\}\\ &=\sup_{(s,y)\in[0,T]\times\mathbb{R}^n}\Big\{v(s,y)-\frac{1}{2\gamma^2}\Big(|s|^2+|y|^2\Big)+\frac{1}{\gamma^2}\Big(ts+x^\top y\Big)\Big\},\end{aligned}$$

which is the supremum of a family of linear functions, we see that the map $(t,x)\mapsto v^\gamma(t,x)+\frac{1}{2\gamma^2}(|t|^2+|x|^2)$ is convex, which means that $v^\gamma(t,x)$ is semiconvex. Next,

$$v^\gamma(t,x)\ge v(t,x)\ge -K(1+|x|),$$

and

$$\begin{aligned}v^\gamma(t,x)&\le \sup_{(s,y)\in[0,T]\times\mathbb{R}^n}\Big[K(1+|y|)-\frac{1}{2\gamma^2}(|x|^2-2x^\top y+|y|^2)\Big]\\ &\le \sup_{(s,y)\in[0,T]\times\mathbb{R}^n}\Big[-\frac{1}{2\gamma^2}|y|^2+(K+\frac{1}{\gamma^2}|x|)|y|+K-\frac{1}{2\gamma^2}|x|^2\Big]\\ &=\sup_{(s,y)\in[0,T]\times\mathbb{R}^n}\Big[-\frac{1}{2\gamma^2}\Big(|y|^2-2(K\gamma^2+|x|)|y|\Big)+K-\frac{1}{2\gamma^2}|x|^2\Big]\\ &\le\frac{1}{2\gamma^2}(K\gamma^2+|x|)^2+K-\frac{1}{2\gamma^2}|x|^2=K+\frac{\gamma^2K^2}{2}+K|x|.\end{aligned}$$

Hence, (4.56) holds.

Next, since $(s,y)\mapsto v(s,y)$ grows at most linearly, for any $(t,x)\in[0,T]\times\mathbb{R}^n$, there exists a $(\hat t,\hat x)\in[0,T]\times\mathbb{R}^n$ such that (4.58) holds. We now establish (4.59). From (4.58), one has

$$\frac{1}{2\gamma^2}\big(|t-\hat t|^2+|x-\hat x|^2\big)=v(\hat t,\hat x)-v^\gamma(t,x)\le v(\hat t,\hat x)-v(t,x)$$

$$
\begin{aligned}
&\le K\big[|x-\hat x| + (1+|x|\vee|\hat x|)|t-\hat t|^{\frac12}\big]\\
&\le K\big[|x-\hat x| + (1+|x|+|x-\hat x|)|t-\hat t|^{\frac12}\big]\\
&\le K\big[|x-\hat x| + (1+|x|)|t-\hat t|^{\frac12}\big].
\end{aligned}\tag{4.61}
$$

Dropping $|t-\hat t|^2$ on the left-hand side of (4.61), and noting $|t-\hat t|\le T$, we have

$$
\begin{aligned}
|x-\hat x|^2 &\le 2\gamma^2K(1+|x|) + 2\gamma^2K|x-\hat x|\\
&\le 2\gamma^2K(1+|x|) + 2\gamma^4K^2 + \frac{|x-\hat x|^2}{2}.
\end{aligned}
$$

Hence, the second inequality of (4.59) holds. Next, dropping $|x-\hat x|^2$ on the left of (4.61), making use of the second inequality in (4.59) and Young's inequality, we get

$$
\begin{aligned}
|t-\hat t|^2 &\le 2\gamma^2K\big[\gamma K(1+|x|^{\frac12}) + (1+|x|)|t-\hat t|^{\frac12}\big]\\
&= 2\gamma^3K^2(1+|x|^{\frac12}) + 2\gamma^2K(1+|x|)|t-\hat t|^{\frac12}\\
&\le 2\gamma^3K^2(1+|x|^{\frac12}) + \frac34\big[2\gamma^2K(1+|x|)\big]^{\frac43} + \frac{|t-\hat t|^2}{4},
\end{aligned}
$$

which implies the first inequality in (4.59).

Now, for any $(t,x),(t',x')\in[0,T]\times\mathbb{R}^n$, let

$$
G = \Big\{(s,y)\in[0,T]\times\mathbb{R}^n \;\Big|\; |x'-y|+|x-y|\le K\gamma(1+|x|^{\frac12}\vee|x'|^{\frac12})\Big\}.
$$

Then for any $(s,y)\in G$,

$$
\begin{aligned}
&\Big|\Big[v(s,y)-\frac{1}{2\gamma^2}\Big(|t-s|^2+|x-y|^2\Big)\Big] - \Big[v(s,y)-\frac{1}{2\gamma^2}\Big(|t'-s|^2+|x'-y|^2\Big)\Big|\\
&= \frac{1}{2\gamma^2}\big|(t+t'-2s)(t-t') + (x+x'-2y)(x-x')\big|\\
&\le \frac{1}{2\gamma^2}\Big(2T|t-t'| + 2K\gamma(1+|x|^{\frac12}\vee|x'|^{\frac12})|x-x'|\Big)\\
&\le \frac{K}{2\gamma^2}\Big(|t-t'| + (1+|x|^{\frac12}\vee|x'|^{\frac12})|x-x'|\Big).
\end{aligned}
$$

Hence, taking supremum over (s,y) ovewr G, we obtain

$$
|v^\gamma(t,x)-v^\gamma(t',x')| \le \frac{K}{\gamma^2}\Big(|t-t'| + (1+|x|^{\frac12}\vee|x'|^{\frac12})|x-x'|\Big).
$$

Finally,

$$
\begin{aligned}
0 &\le v^\gamma(t,x)-v(t,x)\\
&= \sup_{(s,y)\in[0,T]\times\mathbb{R}^n}\Big[v(s,y)-v(t,x)-\frac{1}{2\gamma^2}\Big(|t-s|^2+|x-y|^2\Big)\Big]\\
&\le \sup_{(s,y)\in[0,T]\times\mathbb{R}^n}\Big[K\Big(|x-y| + (1+|x|\vee|y|)|t-s|^{\frac12}\Big)\\
&\qquad\qquad -\frac{1}{2\gamma^2}\Big(|t-s|^2+|x-y|^2\Big)\Big]
\end{aligned}
$$

$$\begin{aligned}
&\leq \sup_{(s,y)\in[0,T]\times\mathbb{R}^n}\Big[K\Big(|x-y|+(1+|x|+|x-y|)|t-s|^{\frac12}\Big)\\
&\qquad\qquad -\frac{1}{2\gamma^2}\Big(|t-s|^2+|x-y|^2\Big)\Big]\\
&\leq \sup_{(s,y)\in[0,T]\times\mathbb{R}^n}\Big[K\Big(|x-y|+(1+|x|)|t-s|^{\frac12}\Big)-\frac{1}{2\gamma^2}\Big(|t-s|^2+|x-y|^2\Big)\Big].
\end{aligned}$$

Note that

$$K|x-y|-\frac{1}{2\gamma^2}|x-y|^2=-\frac{1}{2\gamma^2}\Big(|x-y|-K\gamma^2\Big)^2+\frac{K^2\gamma^2}{2}\leq K\gamma^2,$$

and by Young's inequality,

$$\begin{aligned}
&K(1+|x|)|t-s|^{\frac12}-\frac{1}{2\gamma^2}|t-s|^2\\
&=\Big(\frac{K\gamma^{\frac12}}{2^{\frac14}}(1+|x|)\Big)\Big(\frac{2|t-s|^2}{\gamma^2}\Big)^{\frac14}-\frac{1}{2\gamma^2}|t-s|^2\leq K\gamma^{\frac23}(1+|x|)^{\frac43}.
\end{aligned}$$

Hence,

$$0\leq v^\gamma(t,x)-v(t,x)\leq K\gamma^2+K\gamma^{\frac32}\big(1+|x|^{\frac43}\big)\leq K\gamma^{\frac23}\big(1+|x|^{\frac43}\big),$$

leading to (4.60). □

From the above, we see that $v^\gamma(\cdot\,,\cdot)$ is semiconvex and

$$\lim_{\gamma\to0}v^\gamma(t,x)=v(t,x),$$

uniformly for x in any compact sets. We call $v^\gamma(\cdot\,,\cdot)$ the *semiconvex approximation* of $v(\cdot\,,\cdot)$. Likewise, we can define the *semiconcave approximation* of $v(\cdot\,,\cdot)$ as follows:

$$\begin{aligned}
v_\gamma(t,x)\stackrel{\Delta}{=}\inf_{(s,y)\in[0,T]\times\mathbb{R}^n}\Big\{v(s,y)+\frac{1}{2\gamma^2}\big[|t-s|^2+|x-y|^2\big]\Big\},\\
(t,x)\in[0,T]\times\mathbb{R}^n.
\end{aligned}\tag{4.62}$$

Similar to Lemma 4.20, we have

Lemma 4.21. *Let $v\in C([0,T]\times\mathbb{R}^n)$ satisfying (4.35). Then $v_\gamma(\cdot\,,\cdot)$ is semiconcave, by which we mean that there exists a constant $K_0>0$ such that $(t,x)\mapsto v^\gamma(t,x)-K_0(|t|^2+|x|^2)$ is concave. Moreover, the following is satisfied:*

$$|v_\gamma(t,x)|\leq K(1+|x|),\quad\forall(t,x)\in[0,T]\times\mathbb{R}^n,\ \gamma>0,\tag{4.63}$$

and

$$\begin{aligned}
|v_\gamma(t,x)-v_\gamma(t',x')|\leq\frac{K}{\gamma^2}\Big[|t-t'|+(1+|x|^{\frac12}\vee|x'|^{\frac12})|x-x'|\Big],\\
\forall(t,x),(t',x')\in[0,T]\times\mathbb{R}^n,\ \gamma>0.
\end{aligned}\tag{4.64}$$

Further, for any $(t,x)\in[0,T]\times\mathbb{R}^n$, there exists a $(\hat t,\hat x)\in[0,T]\times\mathbb{R}^n$ such that

$$v_\gamma(t,x)=v(\hat t,\hat x)+\frac{1}{2\gamma^2}\big[|t-\hat t|^2+|x-\hat x|^2\big],\tag{4.65}$$

and for some absolute constant K (independent of (t,x) and γ)

$$\begin{cases}|t-\hat t|\leq K\gamma^{\frac43}\big(1+|x|^{\frac23}\big),\\ |x-\hat x|\leq K\gamma(1+|x|^{\frac12}),\end{cases}\tag{4.66}$$

and

$$0 \le v(t,x) - v_\gamma(t,x) \le K\gamma^{\frac{2}{3}}(1+|x|^{\frac{4}{3}}). \tag{4.67}$$

Since any convex/concave functions are Lipschitz continuous in any compact set, so are any semiconvex/semiconcave functions. The following result exhibits an even nicer property of semiconvex/semiconcave functions.

Lemma 4.22. (Alexandrov's Theorem) *Let $Q \subseteq \mathbb{R}^n$ be a convex domain and $\varphi : Q \to \mathbb{R}$ be a semiconvex (or semiconcave) function. Then there exists a set $N \subseteq Q$ with $|N| = 0$ such that at any $x \in Q \setminus N$, φ is twice differentiable, i.e., there are $(\mathbf{p}, \mathbf{P}) \in \mathbb{R}^n \times \mathbb{S}^n$ such that*

$$\varphi(x+y) = \varphi(x) + \mathbf{p}^\top y + \frac{1}{2} y^\top \mathbf{P} y + o(|y|^2), \quad \textit{for all } |y| \textit{ small enough.}$$

Further, we have the following result.

Lemma 4.23. (Jensen's Lemma) *Let $Q \subseteq \mathbb{R}^n$ be a convex domain, $\varphi : Q \to \mathbb{R}$ be a semiconvex function, and $\bar{x} \in \operatorname{Int} Q$ be a local strict maximum of φ. Then, for any small $r, \delta > 0$, the set*

$$\begin{aligned}\mathcal{K} \stackrel{\Delta}{=} \big\{ x \in B_r(\bar{x}) \bigm| \exists |\mathbf{p}| \le \delta, \text{ such that } x \mapsto \varphi(x) + \mathbf{p}^\top x \\ \text{attains a strict local maximum at } x \ \big\}\end{aligned}$$

has a positive Lebesgue measure.

Note that in Jensen's Lemma the condition that φ attains a local *strict* maximum at $\bar{x}$ is crucial. It excludes the case where $\varphi(x)$ is a constant near $\bar{x}$, in such a case $\mathcal{K} = \partial B_r(\bar{x})$, and the lemma does not hold.

Note that for a viscosity solution $v(\cdot\,,\cdot)$ of (4.17), although $v^\gamma(\cdot\,,\cdot)$ and $v_\gamma(\cdot\,,\cdot)$ are nice semiconvex/semiconcave approximations of $v(\cdot\,,\cdot)$, they might be neither viscosity sub- nor super-solution of the same HJB equation (4.17). In order to make use of these approximations, we need to get the HJB equations which are nice approximation of the original HJB equation, and to which $v^\gamma(\cdot\,,\cdot)$ and $v_\gamma(\cdot\,,\cdot)$ will be either viscosity sub- or super-solution. To this end, for any $(t, x, \mathbf{p}, \mathbf{P}) \in [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{S}^n$, we define for some $K > 0$,

$$\begin{cases} H^\gamma(t,x,\mathbf{p}^\top,\mathbf{P}) \stackrel{\Delta}{=} \displaystyle\sup_{(s,y)\in[0,T]\times\mathbb{R}^n} \Big\{ H(s,y,\mathbf{p}^\top,\mathbf{P}) \bigm| |t-s| \le K\gamma^{\frac{4}{3}}(1+|x|^{\frac{2}{3}}), \\ \qquad\qquad\qquad\qquad\qquad\qquad |x-y| \le K\gamma\big[1+|x|^{\frac{1}{2}}\big] \Big\}, \\ H_\gamma(t,x,\mathbf{p},\mathbf{P}) \stackrel{\Delta}{=} \displaystyle\inf_{(s,y)\in[0,T]\times\mathbb{R}^n} \Big\{ H(s,y,\mathbf{p}^\top,\mathbf{P}) \bigm| |t-s| \le K\gamma^{\frac{4}{3}}(1+|x|^{\frac{2}{3}}), \\ \qquad\qquad\qquad\qquad\qquad\qquad |x-y| \le K\gamma\big[1+|x|^{\frac{1}{2}}\big] \Big\}. \end{cases} \tag{4.68}$$

Clearly,

$$\lim_{\gamma\to 0} H^\gamma(t,x,\mathbf{p}^\top,\mathbf{P}) = \lim_{\gamma\to 0} H_\gamma(t,x,\mathbf{p}^\top,\mathbf{P}) = H(t,x,\mathbf{p}^\top,\mathbf{P}), \tag{4.69}$$

uniformly for $(t, x, \mathbf{p}, \mathbf{P})$ in compact sets. We now present the following result.

Lemma 4.24. *Let (D1) hold and $v(\cdot\,,\cdot) \in C([0,T] \times \mathbb{R}^n)$ be a viscosity subsolution of (4.17). Then, for each $\gamma > 0$, $v^\gamma(\cdot\,,\cdot)$ is a viscosity subsolution of the following:*

$$\begin{cases} v_t + H^\gamma(t,x,v_x,v_{xx}) = 0, & (t,x) \in [0,T) \times \mathbb{R}^n, \\ v\big|_{t=T} = v^\gamma(T,x), & x \in \mathbb{R}^n. \end{cases} \tag{4.70}$$

Likewise, if $v(\cdot\,,\cdot) \in C([0,T]\times\mathbb{R}^n)$ is a viscosity supersolution of (4.17), then, for each $\gamma>0$, $v_\gamma(\cdot\,,\cdot)$ is a viscosity supersolution of the following:

$$\begin{cases} v_t + H_\gamma(t,x,v_x,v_{xx}) = 0, & (t,x)\in[0,T)\times\mathbb{R}^n,\\ v\big|_{t=T} = v_\gamma(T,x), & x\in\mathbb{R}^n. \end{cases} \tag{4.71}$$

Proof. Let $\varphi\in C^{1,2}([0,T]\times\mathbb{R}^n)$ such that $v^\gamma-\varphi$ attains a local maximum at (t,x). Suppose $(\hat t,\hat x)\in[0,T]\times\mathbb{R}^n$ satisfies (4.58). Then, for any $(s,y)\in[0,T]\times\mathbb{R}^n$, one has (noting the definition of v^γ)

$$\begin{aligned} &v(\hat t,\hat x)-\varphi(t,x) = v^\gamma(t,x)+\frac{1}{2\gamma^2}\big(|t-\hat t|^2+|x-\hat x|^2\big)-\varphi(t,x)\\ &\geq v^\gamma(s,y)-\varphi(s,y)+\frac{1}{2\gamma^2}\big(|t-\hat t|^2+|x-\hat x|^2\big)\\ &=\sup_{(r,z)\in[0,T]\times\mathbb{R}^n}\Big\{v(r,z)-\frac{1}{2\gamma^2}\big(|r-s|^2+|y-z|^2\big)\Big\}\\ &\qquad\qquad\qquad +\frac{1}{2\gamma^2}\big(|t-\hat t|^2+|x-\hat x|^2\big)-\varphi(s,y)\\ &\geq v(s-t+\hat t,y-x+\hat x)-\varphi(s,y). \end{aligned}$$

Consequently, for any $(\tau,\zeta)\in[0,T]\times\mathbb{R}^n$, near $(\hat t,\hat x)$, by taking $s=\tau+t-\hat t$ and $y=\zeta+x-\hat x$, which are close to t and x, respectively, we can rewrite the above as

$$v(\hat t,\hat x)-\varphi(t,x)\geq v(\tau,\zeta)-\varphi(\tau+t-\hat t,\zeta+x-\hat x).$$

This means that the function $(\tau,\zeta)\mapsto v(\tau,\zeta)-\varphi(\tau+t-\hat t,\zeta+x-\hat x)$ attains a local maximum at $(\tau,\zeta)=(\hat t,\hat x)$. Thus, by the definition of viscosity subsolution for (4.17) and (4.68), we obtain

$$\begin{aligned} &\varphi_t(t,x)+H^\gamma(t,x,\varphi_x(t,x),\varphi_{xx}(t,x))\\ &\geq \varphi_t(t,x)+H(\hat t,\hat x,\varphi_x(t,x),\varphi_{xx}(t,x))\geq 0. \end{aligned}$$

Here, we should note that

$$\begin{aligned} &\varphi_x(\tau+t-\hat t,\zeta+x-\hat x)\Big|_{(\tau,\zeta)=(\hat t,\hat x)}=\varphi_x(t,x),\\ &\varphi_{xx}(\tau+t-\hat t,\zeta+x-\hat x)\Big|_{(\tau,\zeta)=(\hat t,\hat x)}=\varphi_{xx}(t,x). \end{aligned}$$

This proves that $v^\gamma(\cdot\,,\cdot)$ is a viscosity subsolution of (4.70). In a similar manner, we can prove that $v_\gamma(\cdot\,,\cdot)$ is a viscosity supersolution of (4.71). $\square$

Note that we only have

$$v^\gamma(T,x)\geq v(T,x)=h(x)\geq v_\gamma(T,x),\qquad x\in\mathbb{R}^n. \tag{4.72}$$

Thus, in (4.70) and (4.71), $v^\gamma(T,x)$ and $v_\gamma(T,x)$ are used, respectively, instead of $h(x)$ as the terminal condition. However, we will see that this will not matter due to the relations (4.60) and (4.67).

Now, we are ready to present a proof of Theorem 4.19 which is technical and lengthy.

Proof of Theorem 4.19. Let $v(\cdot\,,\cdot)$ and $\widehat v(\cdot\,,\cdot)$ be two viscosity solutions of (4.17) satisfying (4.35). We claim that

$$v(t,x)\leq\widehat v(t,x),\qquad\forall(t,x)\in[T-\mu,T]\times\mathbb{R}^n, \tag{4.73}$$

with $0<\mu\le[2L(4+L)]^{-1}$ (L is the constant in Assumption (D1)). If this is proved, by the symmetry, the reversed inequality must also hold, and thus $v(\cdot\,,\cdot)$ and $\widehat v(\cdot\,,\cdot)$ are equal on $[T-\mu,T]\times\mathbb{R}^n$. Repeating this argument on $[T-2\mu,T-\mu]\times\mathbb{R}^n$, etc., we can obtain the uniqueness of the viscosity solution on $[0,T]\times\mathbb{R}^n$.

Now we prove (4.73) by contradiction. Suppose (4.73) fails. Then there exists a point $(\bar t,\bar x)\in(T-\mu,T]\times\mathbb{R}^n$ such that

$$\eta\overset{\Delta}{=}v(\bar t,\bar x)-\widehat v(\bar t,\bar x)>0. \tag{4.74}$$

Let $v^\gamma(\cdot\,,\cdot)$ and $\widehat v_\gamma(\cdot\,,\cdot)$ be the semiconvex and semiconcave approximations of $v(\cdot\,,\cdot)$ and $\widehat v(\cdot\,,\cdot)$, respectively. By (4.60) and (4.67), for all $\gamma>0$,

$$\begin{aligned}&v^\gamma(\bar t,\bar x)-\widehat v_\gamma(\bar t,\bar x)=v^\gamma(\bar t,\bar x)-v(\bar t,\bar x)+v(\bar t,\bar x)-\widehat v(\bar t,\bar x)+\widehat v(\bar t,\bar x)-\widehat v_\gamma(\bar t,\bar x)\\&\ge v(\bar t,\bar x)-\widehat v(\bar t,\bar x)=\eta>0.\end{aligned} \tag{4.75}$$

Step 1. Construction of an auxiliary function and properties of its maximum. For any $\alpha,\beta,\varepsilon,\delta,\lambda,\gamma\in(0,1)$, define

$$\begin{cases}\varphi(t,x,s,y)=\alpha\Big(\dfrac{T-t+\mu}{T+\mu}\Big)|x|^2+\alpha\Big(\dfrac{T-s+\mu}{T+\mu}\Big)|y|^2-\beta(t+s)\\ \qquad\qquad +\dfrac{1}{2\varepsilon}|t-s|^2+\dfrac{1}{2\delta}|x-y|^2+\dfrac{\lambda}{t-T+\mu}+\dfrac{\lambda}{s-T+\mu},\\ \Phi(t,x,s,y)=v^\gamma(t,x)-\widehat v_\gamma(s,y)-\varphi(t,x,s,y),\\ \qquad\qquad \forall(t,x),(s,y)\in(T-\mu,T]\times\mathbb{R}^n.\end{cases}$$

We see that $(t,x,s,y)\mapsto\Phi(t,x,s,y)$ is semiconvex. By (4.56) and (4.63), we have

$$\begin{cases}\displaystyle\lim_{|x|+|y|\to\infty}\Phi(t,x,s,y)=-\infty,\quad\text{uniformly in } t,s\in(T-\mu,T],\\ \displaystyle\lim_{t\wedge s\downarrow T-\mu}\Phi(t,s,x,y)=-\infty,\quad\text{uniformly in } x,y\in\mathbb{R}^n.\end{cases}$$

Thus, $\Phi(\cdot\,,\cdot\,,\cdot\,,\cdot)$ attains its maximum at some $(t_0,x_0,s_0,y_0)\in\{(T-\mu,T]\times\mathbb{R}^n\}^2$ (depending on the parameters $\alpha,\beta,\varepsilon,\delta,\lambda$, and γ). By (4.60) and (4.67), one has

$$0\le v^\gamma(T,0)-v(T,0)+\widehat v(T,0)-\widehat v_\gamma(T,0)=v^\gamma(T,0)-\widehat v_\gamma(T,0)\le 2K\gamma^{\frac23}.$$

Then

$$\begin{aligned}&2\beta T-\frac{2\lambda}{\mu}\le v^\gamma(T,0)-\widehat v_\gamma(T,0)+2\beta T-\frac{2\lambda}{\mu}=\Phi(T,0,T,0)\\&\le\max_{\{(T-\mu,T]\times\mathbb{R}^n\}^2}\Phi(t,x,s,y)=\Phi(t_0,x_0,s_0,y_0)\\&=v^\gamma(t_0,x_0)-\widehat v_\gamma(s_0,y_0)-\alpha\Big(\frac{T-t_0+\mu}{T+\mu}\Big)|x_0|^2-\alpha\Big(\frac{T-s_0+\mu}{T+\mu}\Big)|y_0|^2\\&\quad+\beta(t_0+s_0)-\frac{1}{2\varepsilon}|t_0-s_0|^2-\frac{1}{2\delta}|x_0-y_0|^2-\frac{\lambda}{t_0-T+\mu}-\frac{\lambda}{s_0-T+\mu}\\&\le v^\gamma(t_0,x_0)-\widehat v_\gamma(s_0,y_0)-\frac{\alpha\mu}{T+\mu}\big(|x_0|^2+|y_0|^2\big)\\&\quad+2\beta T-\frac{1}{2\varepsilon}|t_0-s_0|^2-\frac{1}{2\delta}|x_0-y_0|^2-\frac{\lambda}{t_0-T+\mu}-\frac{\lambda}{s_0-T+\mu}.\end{aligned}$$

This, together with (4.56) and (4.63), yields the following

$$
\begin{aligned}
&\frac{\alpha\mu}{T+\mu}\big(|x_0|^2+|y_0|^2\big)+\frac{1}{2\varepsilon}|t_0-s_0|^2+\frac{1}{2\delta}|x_0-y_0|^2\\
&+\frac{\lambda(T-t_0)}{\mu(t_0-T+\mu)}+\frac{\lambda(T-s_0)}{\mu(s_0-T+\mu)}\\
&\le v^\gamma(t_0,x_0)-\widehat v_\gamma(s_0,y_0)\le K(1+|x_0|+|y_0|),
\end{aligned}
\tag{4.76}
$$

for some $K>0$, independent of $\alpha,\beta,\varepsilon,\delta,\lambda$, and γ. Note that

$$
\begin{aligned}
&\frac{\alpha\mu}{T+\mu}|x_0|^2-K|x_0|\\
&=\frac{\alpha\mu}{T+\mu}\Big(|x_0|^2-\frac{K(T+\mu)}{\alpha\mu}|x_0|+\frac{K^2(T+\mu)^2}{4\alpha^2\mu^2}\Big)-\frac{\alpha\mu}{T+\mu}\frac{K^2(T+\mu)^2}{4\alpha^2\mu^2}\\
&=\frac{\alpha\mu}{T+\mu}\Big(|x_0|-\frac{K(T+\mu)}{2\alpha\mu}\Big)^2-\frac{K^2(T+\mu)}{4\alpha\mu}.
\end{aligned}
$$

Thus, (4.76) implies

$$
\begin{aligned}
&\frac{\alpha\mu}{T+\mu}\Big[\Big(|x_0|-\frac{K(T+\mu)}{2\alpha\mu}\Big)^2+\Big(|y_0|-\frac{K(T+\mu)}{2\alpha\mu}\Big)^2\Big]+\frac{1}{2\varepsilon}|t_0-s_0|^2\\
&+\frac{1}{2\delta}|x_0-y_0|^2+\frac{\lambda(T-t_0)}{\mu(t_0-T+\mu)}+\frac{\lambda(T-s_0)}{\mu(s_0-T+\mu)}\le\frac{K^2(T+\mu)}{2\alpha\mu},
\end{aligned}
$$

which leads

$$
\Big(|x_0|-\frac{K(T+\mu)}{2\alpha\mu}\Big)^2+\Big(|y_0|-\frac{K(T+\mu)}{2\alpha\mu}\Big)^2\le\frac{K^2(T+\mu)^2}{2\alpha^2\mu^2}.
$$

Hence,

$$
|x_0|+|y_0|\le\frac{K(T+\mu)}{\alpha\mu},
\tag{4.77}
$$

for some $K>0$. Consequently, there is a constant $K>0$ (independent of α,β,ε,δ, λ,γ) such that

$$
\begin{aligned}
&|x_0|+|y_0|+\frac{1}{2\varepsilon}|t_0-s_0|^2+\frac{1}{2\delta}|x_0-y_0|^2\\
&\qquad+\frac{\lambda(T-t_0)}{\mu(t_0-T+\mu)}+\frac{\lambda(T-s_0)}{\mu(s_0-T+\mu)}\le\frac{K(T+\mu)}{\alpha\mu}.
\end{aligned}
\tag{4.78}
$$

This implies

$$
\alpha\lambda(T-t_0)\le K(T+\mu)(t_0-T+\mu).
$$

Then

$$
[\alpha\lambda+K(T+\mu)](T-t_0)\le K\mu(T+\mu),
$$

leading to

$$
T-t_0\le\frac{K(T+\mu)}{\alpha\lambda+K(T+\mu)}\mu\equiv\widehat\mu_{\alpha\lambda}<\mu,\quad\text{if }\alpha\lambda>0.
\tag{4.79}
$$

Consequently (the same for s_0)

$$
T-\mu<T-\widehat\mu_{\alpha\lambda}\le t_0,s_0\le T.
\tag{4.80}
$$

Next, noting

$$\varphi(t_0,x_0,t_0,x_0)=2\alpha\Big(\frac{T-t_0+\mu}{T+\mu}\Big)|x_0|^2-2\beta t_0+\frac{2\lambda}{t_0-T+\mu},$$
$$\varphi(s_0,y_0,s_0,y_0)=2\alpha\Big(\frac{T-s_0+\mu}{T+\mu}\Big)|y_0|^2-2\beta s_0+\frac{2\lambda}{s_0-T+\mu},$$

and by the inequality

$$\Phi(t_0,x_0,t_0,x_0)+\Phi(s_0,y_0,s_0,y_0)\le 2\Phi(t_0,x_0,s_0,y_0),$$

we have

$$\begin{aligned}
&v^\gamma(t_0,x_0)-\widehat v_\gamma(t_0,x_0)+v^\gamma(s_0,y_0)-\widehat v_\gamma(s_0,y_0)-2\alpha\Big(\frac{T-t_0+\mu}{T+\mu}\Big)|x_0|^2\\
&\quad-2\alpha\Big(\frac{T-s_0+\mu}{T+\mu}\Big)|y_0|^2+2\beta(t_0+s_0)-\frac{2\lambda}{t_0-T+\mu}-\frac{2\lambda}{s_0-T+\mu}\\
&=v^\gamma(t_0,x_0)-\widehat v_\gamma(t_0,x_0)-\varphi(t_0,x_0,t_0,x_0)\\
&\quad+v^\gamma(s_0,y_0)-\widehat v_\gamma(s_0,y_0)-\varphi(s_0,y_0,s_0,y_0)\\
&\le 2v^\gamma(t_0,x_0)-2\widehat v_\gamma(s_0,y_0)-2\varphi(t_0,x_0,s_0,y_0)\\
&=2v^\gamma(t_0,x_0)-2\widehat v_\gamma(s_0,y_0)-2\alpha\Big(\frac{T-t_0+\mu}{T+\mu}\Big)|x_0|^2-2\alpha\Big(\frac{T-s_0+\mu}{T+\mu}\Big)|y_0|^2\\
&\quad+2\beta(t_0+s_0)-\frac{1}{\varepsilon}|t_0-s_0|^2-\frac{1}{\delta}|x_0-y_0|^2-\frac{2\lambda}{t_0-T+\mu}-\frac{2\lambda}{s_0-T+\mu}.
\end{aligned}$$

This leads to (by (4.57), (4.64), and (4.77))

$$\begin{aligned}
&\frac{1}{\varepsilon}|t_0-s_0|^2+\frac{1}{\delta}|x_0-y_0|^2\\
&\le v^\gamma(t_0,x_0)-v^\gamma(s_0,y_0)+\widehat v_\gamma(t_0,x_0)-\widehat v_\gamma(s_0,y_0)\\
&\le |v^\gamma(t_0,x_0)-v^\gamma(s_0,y_0)|+|\widehat v_\gamma(t_0,x_0)-\widehat v_\gamma(s_0,y_0)|\\
&\le \frac{K}{\gamma^2}\{|t_0-s_0|+(1+|x_0|^{\frac12}\vee|y_0|^{\frac12})|x_0-y_0|\}\\
&\le \frac{K}{\gamma^2}\Big\{|t_0-s_0|+\Big(1+\frac{\sqrt{K(T+\mu)}}{\sqrt{\alpha\mu}}\Big)|x_0-y_0|\Big\}\to 0,\qquad \text{as } \varepsilon,\delta\to 0.
\end{aligned}\tag{4.81}$$

Step 2. Discussion of two mutually exclusive cases. We keep in mind that (t_0,x_0,s_0,y_0) depends on the parameters $\alpha,\beta,\varepsilon,\delta,\lambda$ and γ.

Case 1. Along some sequence $(\beta,\varepsilon,\delta,\lambda,\gamma)\to 0$, the corresponding t_0,s_0 satisfy

$$t_0\vee s_0=T.\tag{4.82}$$

For this case, we observe the following: (recalling that $(\bar t,\bar x)\in(T-\mu,T]\times\mathbb{R}^n$ satisfying (4.75).

$$\begin{aligned}
&v^\gamma(\bar t,\bar x)-\widehat v_\gamma(\bar t,\bar x)-2\alpha\frac{T-\bar t+\mu}{T+\mu}|\bar x|^2+2\beta\bar t-\frac{2\lambda}{\bar t-T+\mu}\\
&=\Phi(\bar t,\bar x,\bar t,\bar x)\le\Phi(t_0,x_0,s_0,y_0)\le v^\gamma(t_0,x_0)-\widehat v_\gamma(s_0,y_0)+\beta(t_0+s_0).
\end{aligned}\tag{4.83}$$

Now we send $\varepsilon,\delta\to 0$. By (4.78), some subsequence of (t_0,x_0,s_0,y_0), still denoted by itself, converges. By (4.81)–(4.82), the limit has to be of the form $(T,\bar x_0,T,\bar x_0)$.

Then (4.83) becomes

$$v^\gamma(\bar t,\bar x)-\widehat v_\gamma(\bar t,\bar x)-2\alpha\frac{T-\bar t+\mu}{T+\mu}|\bar x|^2+2\beta\bar t-\frac{2\lambda}{\bar t-T+\mu}$$
$$\le v^\gamma(T,\bar x_0)-\widehat v_\gamma(T,\bar x_0)+2\beta T.$$

Next, (fixing $\alpha>0$ so that x_0 stays bounded, see (4.78)) by sending $\gamma\to 0$ and using (4.60) and (4.67), we obtain

$$v(\bar t,\bar x)-\widehat v(\bar t,\bar x)-2\alpha\frac{T-\bar t+\mu}{T+\mu}|\bar x|^2+2\beta\bar t-\frac{2\lambda}{\bar t-T+\mu}\le 2\beta T.$$

Finally, by sending $\alpha,\beta,\lambda\to 0$, we obtain a contradiction to (4.74).

Case 2. For any $\alpha,\beta,\varepsilon,\delta,\lambda,\gamma\in(0,1)$, the corresponding t_0,s_0 satisfy (see (4.80))

$$T-\mu<T-\widehat\mu_{\alpha\lambda}\le t_0,s_0<T, \tag{4.84}$$

recalling $\widehat\mu_{\alpha\lambda}=\frac{K(T+\mu)}{\alpha\lambda+K(T+\mu)}\mu<\mu$. For fixed $\alpha,\lambda\in(0,1)$, recalling (4.78), let

$$Q\overset{\Delta}{=}\Big\{(t,x,s,y)\in\{[0,T]\times\mathbb{R}^n\}^2\;\Big|\;t,s\in[T-\widehat\mu_{\alpha\lambda},T],\;|x|,|y|\le\frac{2K(T+\mu)}{\alpha\mu}\Big\},$$

Thus, by restricting (t,x,s,y) on Q, the function $\varphi(t,x,s,y)$ is smooth with bounded derivatives, which implies its semiconcavity. Consequently, $\Phi(t,x,s,y)$ is semiconvex and attains its maximum at (t_0,x_0,s_0,y_0) in the interior of Q (noting (4.78)). Hence, for any small $r>0$,

$$\widehat\Phi(t,x,s,y)\overset{\Delta}{=}\Phi(t,x,s,y)-r\big(|t-t_0|^2+|s-s_0|^2+|x-x_0|^2+|y-y_0|^2\big)$$

is semiconvex on Q, attaining a strict maximum at (t_0,x_0,s_0,y_0). Note that $\widehat\Phi$ might not be twice differentiable at this point. Thanks to Lemmas 4.22 and 4.23, for the above $r>0$, there exist $q,\hat q\in\mathbb{R}$ and $p,\hat p\in\mathbb{R}^n$ with

$$|q|+|\hat q|+|p|+|\hat p|\le r, \tag{4.85}$$

and $(\hat t_0,\hat x_0,\hat s_0,\hat y_0)\in Q$ with

$$|\hat t_0-t_0|+|\hat x_0-x_0|+|\hat s_0-s_0|+|\hat y_0-y_0|\le r, \tag{4.86}$$

such that

$$\begin{aligned}&\widehat\Phi(t,x,s,y)+qt+\hat qs+\langle\,p,x\,\rangle+\langle\,\hat p,y\,\rangle\\&\equiv v^\gamma(t,x)-\widehat v_\gamma(s,y)-\varphi(t,x,s,y)-r\big(|t-t_0|^2+|s-s_0|^2\\&\qquad+|x-x_0|^2+|y-y_0|^2\big)+qt+\hat qs+\langle\,p,x\,\rangle+\langle\,\hat p,y\,\rangle\end{aligned}$$

attains a maximum at $(\hat t_0,\hat x_0,\hat s_0,\hat y_0)$, and crucially, at which $v^\gamma(t,x)-\widehat v_\gamma(s,y)$ is twice differentiable. For notational simplicity, we now drop γ in $v^\gamma(t,x)$ and $\widehat v_\gamma(s,y)$. Then, by the first- and second-order necessary conditions for a maximum point, at the point $(\hat t_0,\hat x_0,\hat s_0,\hat y_0)$, we must have

$$\begin{cases}v_t=\varphi_t+2r(\hat t_0-t_0)-q,\\ \widehat v_s=-\varphi_s-2r(\hat s_0-s_0)+\hat q,\\ v_x=\varphi_x+2r(\hat x_0-x_0)-p,\\ \widehat v_y=-\varphi_y-2r(\hat y_0-y_0)+\hat p,\\ \begin{pmatrix}v_{xx}&0\\0&-\widehat v_{yy}\end{pmatrix}\le\begin{pmatrix}\varphi_{xx}&\varphi_{xy}\\ \varphi_{xy}^\top&\varphi_{yy}\end{pmatrix}+2rI_{2n},\end{cases} \tag{4.87}$$

where I_{2n} is the $2n\times 2n$ identity matrix. Now, at $(\hat t_0,\hat x_0,\hat s_0,\hat y_0)$, we calculate the following:

$$\begin{cases}\varphi_t=-\beta-\dfrac{\lambda}{(\hat t_0-T+\mu)^2}-\dfrac{\alpha}{T+\mu}|\hat x_0|^2+\dfrac{1}{\varepsilon}(\hat t_0-\hat s_0),\\ \varphi_s=-\beta-\dfrac{\lambda}{(\hat s_0-T+\mu)^2}-\dfrac{\alpha}{T+\mu}|\hat y_0|^2+\dfrac{1}{\varepsilon}(\hat s_0-\hat t_0),\\ \varphi_x=\dfrac{2\alpha(T-\hat t_0+\mu)}{T+\mu}\hat x_0+\dfrac{\hat x_0-\hat y_0}{\delta},\\ \varphi_y=\dfrac{2\alpha(T-\hat s_0+\mu)}{T+\mu}\hat y_0+\dfrac{\hat y_0-\hat x_0}{\delta},\\ A\equiv\begin{pmatrix}\varphi_{xx}&\varphi_{xy}\\ \varphi_{xy}^\top&\varphi_{yy}\end{pmatrix}=\dfrac{1}{\delta}\begin{pmatrix}I_n&-I_n\\-I_n&I_n\end{pmatrix}-\dfrac{2\alpha}{T+\mu}\begin{pmatrix}\hat t_0I_n&0\\0&\hat s_0I_n\end{pmatrix}+2\alpha I_{2n}.\end{cases}\tag{4.88}$$

On the other hand, by Lemma 4.24 and the definition of viscosity sub- and super-solutions, we have (note that γ has been dropped in $v^\gamma(\cdot\,,\cdot)$ and $\widehat v_\gamma(\cdot\,,\cdot)$)

$$\begin{cases}v_t(\hat t_0,\hat x_0)+H^\gamma(\hat t_0,\hat x_0,v_x(\hat t_0,\hat x_0),v_{xx}(\hat t_0,\hat x_0))\ge 0,\\ \widehat v_s(\hat s_0,\hat y_0)+H_\gamma(\hat s_0,\hat y_0,\widehat v_y(\hat s_0,\hat y_0),\widehat v_{yy}(\hat s_0,\hat y_0))\le 0.\end{cases}$$

By (4.68) (the definition of H^γ and H_γ), one can find a $(\bar t_0,\bar x_0,\bar s_0,\bar y_0)$ with

$$|\bar t_0-\hat t_0|+|\bar x_0-\hat x_0|+|\bar s_0-\hat s_0|+|\bar y_0-\hat y_0|\le\bar K_\alpha\gamma,\tag{4.89}$$

for some $\bar K_\alpha>0$ (depending only on α), such that

$$\begin{aligned}&\widehat v_s(\hat s_0,\hat y_0)-v_t(\hat t_0,\hat x_0)\\ &\le H^\gamma(\hat t_0,\hat x_0,v_x(\hat t_0,\hat x_0),v_{xx}(\hat t_0,\hat x_0))-H_\gamma(\hat s_0,\hat y_0,\widehat v_y(\hat s_0,\hat y_0),\widehat v_{yy}(\hat s_0,\hat y_0))\\ &=H(\bar t_0,\bar x_0,u,v_x(\hat t_0,\hat x_0),v_{xx}(\hat t_0,\hat x_0))-H(\bar s_0,\bar y_0,u,\widehat v_y(\hat s_0,\hat y_0),\widehat v_{yy}(\hat s_0,\hat y_0))\\ &\le\sup_{u\in U}\Big\{\mathbb H(\bar t_0,\bar x_0,u,v_x(\hat t_0,\hat x_0),v_{xx}(\hat t_0,\hat x_0))\\ &\qquad\qquad-\mathbb H(\bar s_0,\bar y_0,u,\widehat v_y(\hat s_0,\hat y_0),\widehat v_{yy}(\hat s_0,\hat y_0))\Big\}.\end{aligned}$$

Hence,

$$\begin{aligned}&\widehat v_s(\hat s_0,\hat y_0)-v_t(\hat t_0,\hat x_0)\\ &\le\sup_{u\in U}\Big\{\frac12\mathrm{tr}\Big[v_{xx}(\hat t_0,\hat x_0)\sigma(\bar t_0,\bar x_0,u)\sigma(\bar t_0,\bar x_0,u)^\top\\ &\qquad\qquad-\widehat v_{yy}(\hat s_0,\hat y_0)\sigma(\bar s_0,\bar y_0,u)\sigma(\bar s_0,\bar y_0,u)^\top\Big]\\ &\qquad+\Big[v_x(\hat t_0,\hat x_0)b(\bar t_0,\bar x_0,u)-\widehat v_y(\hat s_0,\hat y_0)b(\bar s_0,\bar y_0,u)\Big]\\ &\qquad+\Big[g(\bar t_0,\bar x_0,u)-g(\bar s_0,\bar y_0,u)\Big]\Big\}\\ &\equiv\sup_{u\in U}\big\{(I)+(II)+(III)\big\}.\end{aligned}\tag{4.90}$$

By (4.87)–(4.88) and (4.85)–(4.86), we have

$$\begin{aligned}&\widehat v_s(\hat s_0,\hat y_0)-v_t(\hat t_0,\hat x_0)\\ &=-\varphi_s(\hat s_0,\hat y_0)-2r(\hat s_0-s_0)+\hat q-\varphi_t(\hat t_0,\hat x_0)-2r(\hat t_0-t_0)+q\end{aligned}$$

$$
\begin{aligned}
&= 2\beta + \frac{2\alpha}{T+\mu}\big(|\hat{x}_0|^2 + |\hat{y}_0|^2\big) + \frac{\lambda}{(\hat{t}_0 - T+\mu)^2} + \frac{\lambda}{(\hat{s}_0 - T+\mu)^2} \\
&\qquad -2r(\hat{t}_0 - t_0 + \hat{s}_0 - s_0) + q + \hat{q} \\
&\geq 2\beta + \frac{2\alpha}{T+\mu}\big(|\hat{x}_0|^2 + |\hat{y}_0|^2\big) - Kr,
\end{aligned}
$$

for some absolute constant $K > 0$. By (4.81) and (4.86), we see that one may assume that as $\varepsilon, \delta, r \to 0$, $(\hat{t}_0, \hat{x}_0)$ and $(\hat{s}_0, \hat{y}_0)$ converge to the same limit, denoted by (t_α, x_α), to emphasize the dependence on α. Thus, letting $\varepsilon, \delta, r \to 0$ in the above leads to

$$
\widehat{v}_s(t_\alpha, x_\alpha) - v_t(t_\alpha, x_\alpha) \geq 2\beta + \frac{4\alpha}{T+\mu}|x_\alpha|^2. \tag{4.91}
$$

This gives an estimate for the left-hand side of (4.90). We now estimate the terms (I), (II), and (III) on the right-hand side of (4.90) one by one. First of all, from (4.3), (4.81), (4.86), and (4.89), one obtains an estimate for (III):

$$
(III) \triangleq g(\bar{t}_0, \bar{x}_0, u) - g(\bar{s}_0, \bar{y}_0, u) \to 0, \qquad \text{as } \varepsilon, \delta, \gamma, r \to 0, \tag{4.92}
$$

uniformly in $u \in U$. Next,

$$
\begin{aligned}
(II) &\triangleq v_x(\hat{t}_0, \hat{x}_0) b(\bar{t}_0, \bar{x}_0, u) - \widehat{v}_y(\hat{s}_0, \hat{y}_0) b(\bar{s}_0, \bar{y}_0, u) \\
&= \Big(\frac{2\alpha(T - \hat{t}_0 + \mu)}{T+\mu}\hat{x}_0 + \frac{\hat{x}_0 - \hat{y}_0}{\delta} + 2r(\hat{x}_0 - x_0) - p\Big)^\top b(\bar{t}_0, \bar{x}_0, u) \\
&\quad + \Big(\frac{2\alpha(T - \hat{s}_0 + \mu)}{T+\mu}\hat{y}_0 + \frac{\hat{y}_0 - \hat{x}}{\delta} + 2r(\hat{y}_0 - y_0) - \hat{p}\Big)^\top b(\bar{s}_0, \bar{y}_0, u) \\
&\leq \Big(\frac{\hat{x}_0 - \hat{y}_0}{\delta}\Big)^\top \Big(b(\bar{t}_0, \bar{x}_0, u) - b(\bar{s}_0, \bar{y}_0, u)\Big) \\
&\quad + \frac{2\alpha(T - \hat{t}_0 + \mu)}{T+\mu} L|\hat{x}_0|(1 + |\bar{x}_0|) + \frac{2\alpha(T - \hat{s}_0 + \mu)}{T+\mu} L|\hat{y}_0|(1 + |\bar{y}_0|) \\
&\quad + rK(1 + |\bar{x}_0| + |\bar{y}_0|).
\end{aligned} \tag{4.93}
$$

Letting $\varepsilon, \gamma, r \to 0$ (fixing $\alpha, \delta > 0$), we may assume that $(\hat{t}_0, \hat{x}_0, \hat{s}_0, \hat{y}_0)$ and $(\bar{t}_0, \bar{x}_0, \bar{s}_0, \bar{y}_0)$ converge. Clearly, the limits of these two sequences have to be the same (see (4.59), (4.81), (4.86), and (4.89)), which is denoted by (t_0, x_0, t_0, y_0). Thus,

$$
\lim_{\varepsilon, \gamma, r \to 0} (II) \leq L\frac{|x_0 - y_0|^2}{\delta} + \frac{4\alpha(T - t_0 + \mu)L}{T+\mu}\big(|x_0| + |y_0| + |x_0|^2 + |y_0|^2\big).
$$

Then letting $\delta \to 0$, one concludes that (t_0, x_0) and (t_0, y_0) approach a common limit (see (4.81)), called (t_α, x_α). Consequently,

$$
\lim_{\delta \to 0} \lim_{\varepsilon, \gamma, r \to 0} (II) \leq \frac{8\alpha(T - t_\alpha + \mu)}{T+\mu} L(|x_\alpha| + |x_\alpha|^2). \tag{4.94}
$$

Now, we treat (I) in (4.90). By the inequality in (4.87),

$$
\begin{aligned}
(I) \triangleq \frac{1}{2}\mathrm{tr}\Big[&v_{xx}(\hat{t}_0, \hat{x}_0)\sigma(\bar{t}_0, \bar{x}_0, u)\sigma(\bar{t}_0, \bar{x}_0, u)^\top \\
&-\widehat{v}_{yy}(\hat{s}_0, \hat{y}_0)\sigma(\bar{s}_0, \bar{y}_0, u)\sigma(\bar{s}_0, \bar{y}_0, u)^\top\Big]
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{2}\mathrm{tr}\Big[\begin{pmatrix}\sigma(\bar t_0,\bar x_0,u)\\ \sigma(\bar s_0,\bar y_0,u)\end{pmatrix}^\top \begin{pmatrix} v_{xx}(\hat t_0,\hat x_0) & 0\\ 0 & -\widehat v_{yy}(\hat s_0,\hat y_0)\end{pmatrix}\begin{pmatrix}\sigma(\bar t_0,\bar x_0,u)\\ \sigma(\bar s_0,\bar y_0,u)\end{pmatrix}\Big]\\
&\le \frac{1}{2}\mathrm{tr}\Big[\begin{pmatrix}\sigma(\bar t_0,\bar x_0,u)\\ \sigma(\bar s_0,\bar x_0,u)\end{pmatrix}^\top (A+2rI_{2n})\begin{pmatrix}\sigma(\bar t_0,\bar x_0,u)\\ \sigma(\bar s_0,\bar y_0,u)\end{pmatrix}\Big]\\
&\le \frac{1}{2}\Big\{\frac{1}{\delta}|\sigma(\bar t_0,\bar x_0,u)-\sigma(\bar s_0,\bar y_0,u)|^2\\
&\qquad +\Big(2(\alpha+r)-\frac{2\alpha\hat t_0}{T+\mu}\Big)|\sigma(\bar t_0,\bar x_0,u)|^2+\Big(2(\alpha+r)-\frac{2\alpha\hat s_0}{T+\mu}\Big)|\sigma(\bar s_0,\bar y_0,u)|^2\Big\}.
\end{aligned}
$$

As above, we first let $\varepsilon,\gamma,r\to 0$ and then let $\delta\to 0$ to get

$$
\lim_{\delta\to 0}\lim_{\varepsilon,\gamma,r\to 0}(I)\le \frac{2\alpha(T-t_\alpha+\mu)}{T+\mu}L^2(1+|x_\alpha|)^2. \tag{4.95}
$$

Combining (4.90)–(4.95), we obtain

$$
2\beta+\frac{4\alpha}{T+\mu}|x_\alpha|^2\le \frac{2\alpha(T-t_\alpha+\mu)}{T+\mu}L\Big[4(|x_\alpha|+|x_\alpha|^2)+L(1+|x_\alpha|)^2\Big],
$$

which leads to

$$
\begin{aligned}
\beta &\le -\frac{2\alpha}{T+\mu}|x_\alpha|^2+\frac{\alpha(T-t_\alpha+\mu)}{T+\mu}L\Big[(4+L)|x_\alpha|^2+(4+2L)|x_\alpha|+L\Big]\\
&= -\frac{\alpha}{T+\mu}\Big\{\Big[2-L(4+L)(T-t_\alpha+\mu)\Big]|x_\alpha|^2\\
&\qquad\qquad -(T-t_\alpha+\mu)(4+2L)L|x_\alpha|-(T-t_\alpha+\mu)L^2\Big\},
\end{aligned}
$$

for some absolute constant $K>0$. Noting $t_\alpha\in[T-\mu,T]$ and $0<\mu\le[2L(4+L)]^{-1}$, one has

$$
2-L(4+L)(T-t_\alpha+\mu)\ge 2-L(4+L)2\mu=1.
$$

Hence,

$$
\beta\le -\frac{\alpha}{T+\mu}\Big\{|x_\alpha|^2-K\big(1+|x_\alpha|\big)\Big\}, \tag{4.96}
$$

for some constant $K>0$ independent of $\alpha>0$. Clearly $\{\cdots\}$ on the right-hand side of the above is bounded from below uniformly in α. Thus, by sending $\alpha\to 0$, we obtain $\beta\le 0$, which contradicts our assumption $\beta>0$. This proves (4.73). $\square$

Let us make some comments on the auxiliary function $\varphi(t,x,s,y)$ which is recalled here

$$
\begin{aligned}
\varphi(t,x,s,y) &= \alpha\Big(\frac{T-t+\mu}{T+\mu}\Big)|x|^2+\alpha\Big(\frac{T-s+\mu}{T+\mu}\Big)|y|^2-\beta(t+s)\\
&\qquad +\frac{1}{2\varepsilon}|t-s|^2+\frac{1}{2\delta}|x-y|^2+\frac{\lambda}{t-T+\mu}+\frac{\lambda}{s-T+\mu}.
\end{aligned}
$$

First of all, the term $\beta(t+s)$ with $\beta>0$ plays a role in obtaining the contradiction in (4.96). Second, the terms involving $\alpha>0$ play a role in (4.96) so that the term $\{\cdots\}$ on the right-hand side is bounded from below. Third, the terms involving ε and δ make the maximum point (t_0,x_0,s_0,y_0) having property (4.81) which leads to the comparison of $v(\cdot\,,\cdot)$ and $\widehat v(\cdot\,,\cdot)$ at the same points. Finally, the terms involving λ makes t_0,s_0 away from $T-\mu$.

5. **Linear Quadratic Optimal Control Problems.**

5.1. **Formulation of the problem.** Consider the following linear controlled stochastic differential equation:

$$\begin{cases} dX(s) = [A(s)X(s) + B(s)u(s) + b(s)]ds \\ \qquad\qquad +[C(s)X(s) + D(s)u(s) + \sigma(s)]dW(s), \quad s \in [t,T], \\ X(t) = x. \end{cases} \tag{5.1}$$

In the above, $X(\cdot)$ is the *state process* with the initial state x at initial time t, and $u(\cdot)$ is the *control process* belonging to $\mathcal{U}[t,T] \equiv L^2_{\mathbb{F}}(t,T;\mathbb{R}^m)$ of admissible controls; *coefficients* $A(\cdot), B(\cdot), C(\cdot), D(\cdot)$ are given matrix-valued deterministic maps, and *nonhomogeneous terms* $b(\cdot)$ and $\sigma(\cdot)$ are given stochastic processes. Under certain conditions, for any *initial pair* $(t,x) \in [0,T) \times \mathbb{R}^n$ and control $u(\cdot) \in \mathcal{U}[t,T]$, state equation (5.1) admits a unique (strong) solution $X(\cdot) \equiv X(\cdot\,;t,x,u(\cdot)) \in L^2_{\mathbb{F}}(\Omega;C([t,T];\mathbb{R}^n))$. We introduce the following *cost functional* used to measure the performance of the control process:

$$\begin{aligned} J(t,x;u(\cdot)) = \mathbb{E}\Big\{ \int_t^T & \big[\,\langle\, Q(s)X(s),X(s)\,\rangle + 2\,\langle\, S(s)X(s),u(s)\,\rangle \\ & + \langle\, R(s)u(s),u(s)\,\rangle + 2\,\langle\, q(s),X(s)\,\rangle + 2\,\langle\, \rho(s),u(s)\,\rangle\,\big]ds \\ & + \langle\, HX(T),X(T)\,\rangle + 2\,\langle\, h,X(T)\,\rangle\,\Big\}, \end{aligned} \tag{5.2}$$

where $Q(\cdot)$, $S(\cdot)$ and $R(\cdot)$ are matrix-valued deterministic functions and $q(\cdot)$ and $\rho(\cdot)$ are some stochastic processes, H is a matrix and h is a random variable. Note that $J(t,x;u(\cdot))$ actually depends on the processes $b(\cdot),\sigma(\cdot),q(\cdot),\rho(\cdot)$ and random variable h. Therefore, we may write

$$J(t,x;u(\cdot)) = J(t,x;u(\cdot);b(\cdot),\sigma(\cdot),q(\cdot),\rho(\cdot),h).$$

We denote

$$J_0(t,x;u(\cdot)) = J(t,x;u(\cdot);0,0,0,0,0).$$

Namely, $J_0(t,x;u(\cdot))$ is the cost functional corresponding to the case that $b(\cdot),\sigma(\cdot)$, $q(\cdot),\rho(\cdot)$ and h are all zero.

Before going further, let us introduce the following hypotheses.

(L1) The coefficients of the state equation and the nonhomogeneous terms of the state equation satisfy the following:

$$\begin{cases} A(\cdot), C(\cdot) \in L^\infty(0,T;\mathbb{R}^{n\times n}), \\ B(\cdot) \in L^2(0,T;\mathbb{R}^{n\times m}), \quad D(\cdot) \in L^\infty(0,T;\mathbb{R}^{n\times m}), \\ b(\cdot) \in L^2_{\mathbb{F}}(\Omega;L^1(0,T;\mathbb{R}^n)), \quad \sigma(\cdot) \in L^2_{\mathbb{F}}(0,T;\mathbb{R}^n). \end{cases}$$

(L2) The weighting functions in the cost functional satisfy the following:

$$\begin{cases} Q(\cdot) \in L^\infty(0,T;\mathbb{S}^n), \quad S(\cdot) \in L^2(0,T;\mathbb{R}^{m\times n}), \quad R(\cdot) \in L^\infty(0,T;\mathbb{S}^m), \\ q(\cdot) \in L^2_{\mathbb{F}}(\Omega;L^1(0,T;\mathbb{R}^n)), \quad \rho(\cdot) \in L^2_{\mathbb{F}}(0,T;\mathbb{R}^m), \quad H \in \mathbb{S}^n, \quad h \in \mathbb{R}^n, \end{cases}$$

recalling $\mathbb{S}^n$ is the set of all $(n\times n)$ symmetric matrices.

The following result is standard.

Proposition 5.1. *Let* (L1) *hold. Then for any* $(t,x)\in[0,T)\times\mathbb{R}^n$ *and* $u(\cdot)\in\mathcal{U}[t,T]$, (5.1) *admits a unique solution* $X(\cdot)$, *and the following holds:*

$$\begin{aligned}\mathbb{E}\Big[\sup_{s\in[t,T]}|X(s)|^2\Big]\le K\mathbb{E}\Big[|x|^2+\Big(\int_t^T|b(s)|ds\Big)^2\\+\int_t^T|\sigma(s)|^2ds+\int_t^T|u(s)|^2ds\Big].\end{aligned}\tag{5.3}$$

Due to the above proposition, under (L1)–(L2), for any $(t,x)\in[0,T)\times\mathbb{R}^n$ and $u(\cdot)\in\mathcal{U}[t,T]$, the cost functional $J(t,x;u(\cdot))$ is well-defined. Hence, we can state our stochastic linear quadratic optimal control problem as follows:

Problem (SLQ). *For each* $(t,x)\in[0,T]\times\mathbb{R}^n$, *find a* $\bar u(\cdot)\in\mathcal{U}[t,T]$ *such that*

$$J(t,x;\bar u(\cdot))=\inf_{u(\cdot)\in\mathcal{U}[t,T]}J(t,x;u(\cdot))\overset{\Delta}{=}V(t,x).\tag{5.4}$$

Next, we introduce the following definition.

Definition 5.2. (i) Problem (SLQ) is said to be *finite* at $(t,x)\in[0,T]\times\mathbb{R}^n$ if

$$V(t,x)>-\infty.$$

If the above holds for all $(t,x)\in[0,T]\times\mathbb{R}^n$, we simply say that Problem (SLQ) is *finite*.

(ii) Problem (SLQ) is said to be (*uniquely*) *open-loop solvable* at $(t,x)\in[0,T]\times\mathbb{R}^n$ if there exists a (unique) control $\bar u(\cdot)\in\mathcal{U}[t,T]$, such that

$$J(t,x;\bar u(\cdot))=V(t,x).$$

In this case, $\bar u(\cdot)$ is called an (the) *open-loop optimal control*, the corresponding $\bar X(\cdot)\equiv X(\cdot\,;t,x,\bar u(\cdot))$ is called an (the) *open-loop optimal state process* and $(\bar X(\cdot),\bar u(\cdot))$ is called an (the) *open-loop optimal pair*. If for any $(t,x)\in[0,T]\times\mathbb{R}^n$, Problem (SLQ) is (uniquely) open-loop solvable at (t,x), we simply say that Problem (SLQ) is (*uniquely*) *open-loop solvable*.

Let us look at some examples.

Example 5.3. Consider the following controlled ODE system (where both the state and control are one-dimensional)

$$\begin{cases}\dot X(s)=u(s), & s\in[t,T],\\ X(t)=x,\end{cases}\tag{5.5}$$

with the cost functional

$$J_1^D(t,x;u(\cdot))=\int_t^T X(s)^2ds.$$

It is clear that this problem is finite at any $t\in[0,T)$ since

$$\inf_{u(\cdot)\in\mathcal{U}[t,T]}J_1^D(t,x;u(\cdot))\ge0.$$

But, we claim that the problem is not open-loop solvable at any (t,x) with $x\ne0$. In fact, by letting

$$u_k(s)=-kxI_{[t,t+\frac1k]}(s),\qquad s\in[t,T],$$

we have

$$X(s;t,x,u_k(\cdot))=\begin{cases} x-kx(s-t), & s\in[t,t+\frac{1}{k}],\\ 0, & s\in[t+\frac{1}{k},T].\end{cases}$$

Thus,

$$\begin{aligned} J_1^D(t,x;u_k(\cdot))&=\int_t^{t+\frac{1}{k}} x^2[1-k(s-t)]^2ds\\ &=x^2\int_0^{\frac{1}{k}}(1-ks)^2ds=\frac{x^2}{3k}\to 0,\qquad k\to\infty.\end{aligned}$$

Consequently,

$$\inf_{u(\cdot)\in\mathcal{U}[t,T]} J_1^D(t,x;u(\cdot))=0<J_1^D(t,x;u(\cdot)),\qquad \forall u(\cdot)\in\mathcal{U}[t,T].$$

This shows that the problem is not open-loop solvable and the value function is given by

$$V_1^D(t,x)=0,\qquad \forall(t,x)\in[0,T]\times\mathbb{R}.$$

Now, we consider the following controlled SDE system:

$$\begin{cases} dX(s)=u(s)ds+\delta u(s)dW(s), & s\in[t,T],\\ X(t)=x,\end{cases}\tag{5.6}$$

for some $\delta\neq 0$, with the cost functionals

$$J_1^S(t,x;u(\cdot))=\mathbb{E}\Big[\int_t^T X(s)^2ds\Big].$$

Define

$$P(t)=\delta^2\big(1-e^{\frac{t-T}{\delta^2}}\big),\qquad t\in[0,T].\tag{5.7}$$

Then $P(\cdot)$ solves

$$\begin{cases} \dot P(t)-\dfrac{P(t)}{\delta^2}+1=0, & t\in[0,T],\\ P(T)=0.\end{cases}$$

By Itô's formula, we have

$$\begin{aligned} 0=\mathbb{E}\Big[P(T)x(T)^2\Big]&=P(t)x^2+\mathbb{E}\Big\{\int_t^T\Big[\Big(\frac{P(s)}{\delta^2}-1\Big)X(s)^2\\ &\qquad +2P(s)X(s)u(s)+\delta^2P(s)u(s)^2\Big]ds\Big\}.\end{aligned}$$

Thus, for any $u(\cdot)\in\mathcal{U}[t,T]$,

$$\begin{aligned} J_1^S(t,x;u(\cdot))&=\mathbb{E}\Big[\int_t^T X(s)^2ds\Big]\\ &=P(t)x^2+\mathbb{E}\Big[\int_t^T\delta^2P(s)\big[u(s)+\frac{X(s)}{\delta^2}\big]^2ds\Big]\ge P(t)x^2.\end{aligned}$$

This suggests that we first solve the following SDE:

$$\begin{cases} d\bar X(s)=-\dfrac{\bar X(s)}{\delta^2}ds-\dfrac{\bar X(s)}{\delta}dW(s), & s\in[t,T],\\ \bar X(t)=x,\end{cases}\tag{5.8}$$

and then define

$$\bar{u}(s) = -\frac{\bar{X}(s)}{\delta^2}, \qquad s \in [t, T]. \tag{5.9}$$

By the above calculation, we see that

$$\begin{aligned} J_1^S(t, x; \bar{u}(\cdot)) = P(t)x^2 &= \inf_{u(\cdot)\in\mathcal{U}[t,T]} J_1^S(t, x; u(\cdot)) \\ &= V_1^S(t, x) = \delta^2\big(1 - e^{\frac{t-T}{\delta^2}}\big)x^2, \quad \forall (t, x) \in [0, T] \times \mathbb{R}. \end{aligned}$$

This example shows that a deterministic LQ problem with $R = 0$ may be finite but not open-loop solvable. Whereas, a stochastic LQ problem with $R = 0$ could still be open-loop solvable. As a matter of fact, the following example tells us that we can go even further.

Example 5.4. Consider the controlled ODE system (5.5) with the cost functional

$$J_2^D(t, x; u(\cdot)) = -\int_t^T u(s)^2 ds + X(T)^2.$$

We see that the above cost functional has a negative weight $R = -1$ on the control term. If we take

$$u_\varepsilon(s) = \frac{1}{\varepsilon} I_{[T-\varepsilon,T]}(s), \qquad s \in [t, T],$$

with $0 < \varepsilon < T - t$, one has

$$X(s) = \begin{cases} x, & s \in [t, T - \varepsilon], \\ x + \dfrac{s - T + \varepsilon}{\varepsilon}, & s \in [T - \varepsilon, T]. \end{cases}$$

Consequently,

$$J_2^D(t, x; u_\varepsilon(\cdot)) = -\frac{1}{\varepsilon} + (x + 1)^2 \to -\infty, \qquad \varepsilon \to 0.$$

Thus, the corresponding value function

$$V_2^D(t, x) = -\infty, \qquad \forall (t, x) \in [0, T) \times \mathbb{R},$$

the corresponding LQ problem is not finite, and, of course, no optimal control exists.

Now, we consider the controlled SDE system (5.6) with the cost functional

$$J_2^S(t, x; u(\cdot)) = \mathbb{E}\Big[-\int_t^T u(s)^2 ds + X(T)^2\Big].$$

It is seen from (5.6) that the control affects the size of the noise in the system. We assume that $|\delta| > 1$ with

$$\delta^2\big(2\ln|\delta| - 1\big) > T - 1, \tag{5.10}$$

and introduce the following differential equation:

$$\begin{cases} \dot{P}(t) = \dfrac{P(t)^2}{\delta^2 P(t) - 1}, & t \in [0, T], \\ P(T) = 1. \end{cases} \tag{5.11}$$

Solving the above by separation of variables, we can get

$$\delta^2 \ln P(t) + \frac{1}{P(t)} = t + 1 - T, \qquad t \in (\bar{t}, T],$$

where $\bar{t} < T$ is such that $\delta^2 P(\bar{t}) = 1$. Thus

$$\delta^2\big(1 - 2\ln|\delta|\big) = \bar{t} + 1 - T,$$

or

$$\bar{t} = T - 1 - \delta^2\big(2\ln|\delta| - 1\big) < 0.$$

Hence, under (5.10), equation (5.11) is uniquely solvable on $[0,T]$, and

$$P(t) > \frac{1}{\delta^2}, \qquad t \in [0,T]. \tag{5.12}$$

Now, for any $u(\cdot) \in \mathcal{U}[t,T]$, let $X(\cdot)$ be the corresponding state process. Then

$$\begin{aligned}\mathbb{E}|X(T)|^2 &= \mathbb{E}\big[P(T)X(T)^2\big]\\ &= P(t)x^2 + \int_t^T \Big[\dot{P}(s)X(s)^2 + 2P(s)X(s)u(s) + \delta^2 P(s)u(s)^2\Big]ds.\end{aligned}$$

Hence, noting (5.12),

$$\begin{aligned}J_2^S(t,x;u(\cdot)) &= \mathbb{E}\Big[-\int_t^T u(s)^2 ds + X(T)^2\Big]\\ &= P(t)x^2 + \mathbb{E}\int_t^T \Big[\dot{P}(s)X(s)^2 + 2P(s)X(s)u(s) + (\delta^2 P(s) - 1)u(s)^2\Big]ds\\ &= P(t)x^2 + \mathbb{E}\int_t^T \Big[\Big(\dot{P}(s) - \frac{P(s)^2}{\delta^2 P(s) - 1}\Big)X(s)^2\\ &\qquad + (\delta^2 P(s) - 1)\Big(u(s) + \frac{P(s)X(s)}{\delta^2 P(s) - 1}\Big)^2\Big]ds\\ &= P(t)x^2 + \mathbb{E}\int_t^T (\delta^2 P(s) - 1)\Big(u(s) + \frac{P(s)X(s)}{\delta^2 P(s) - 1}\Big)^2 ds \geq P(t)x^2.\end{aligned}$$

Similar to the previous example, the optimal control exists and is given by

$$\bar{u}(s) = -\frac{P(s)\bar{X}(s)}{\delta^2 P(s) - 1}, \qquad s \in [t,T], \tag{5.13}$$

with $\bar{X}(\cdot)$ being the solution to the following:

$$\begin{cases} d\bar{X}(s) = -\dfrac{P(s)\bar{X}(s)}{\delta^2 P(s) - 1}ds - \dfrac{\delta P(s)\bar{X}(s)}{\delta^2 P(s) - 1}dW(s), \quad s \in [t,T],\\ \bar{X}(t) = x. \end{cases} \tag{5.14}$$

This example shows that a stochastic LQ problem with a *negative* control weighting matrix in the cost may still be open-loop solvable. The following example shows another interesting point.

Example 5.5. Consider controlled ODE system (5.5) with the cost functional

$$J_3^D(t,x;u(\cdot)) = \int_t^T u(s)^2 ds - X(T)^2.$$

This is an LQ problem with the weight on the square of the terminal state being negative. We assume that $T < 1$. Let

$$P(t) = -\frac{1}{(t + 1 - T)}, \qquad t \in [0,T].$$

Then $P(\cdot)$ satisfies

$$\begin{cases} \dot{P}(t) = P(t)^2, \qquad t \in [0,T],\\ P(T) = -1. \end{cases}$$

Consequently,

$$
\begin{aligned}
J_3^D(t,x;u(\cdot)) &= \int_t^T u(s)^2 ds + P(T)X(T)^2 \\
&= P(t)x^2 + \int_t^T \Big(\dot P(s)X(s)^2 + 2P(s)X(s)u(s) + u(s)^2\Big) ds \\
&= P(t)x^2 + \int_t^T \Big(\big[\dot P(s) - P(s)^2\big]X(s)^2 + \big[u(s) + P(s)X(s)\big]^2\Big) ds \\
&= P(t)x^2 + \int_t^T \big[u(s) + P(s)X(s)\big]^2 ds \geq P(t)x^2.
\end{aligned}
$$

Hence, an optimal control is given by

$$
\bar u(s) = \frac{\bar X(s)}{s+1-T}, \qquad s \in [t,T], \tag{5.15}
$$

with $\bar X(\cdot)$ being the solution to the following:

$$
\begin{cases}
\dot{\bar X}(s) = \dfrac{\bar X(s)}{s+1-T}, \qquad s \in [t,T], \\
\bar X(t) = x.
\end{cases}
$$

For this case,

$$
\bar X(s) = \frac{s+1-T}{t+1-T}x, \quad \bar u(s) = \frac{x}{t+1-T}, \qquad s \in [t,T],
$$

and the corresponding value function is

$$
V_3^D(t,x) = \frac{-x^2}{t+1-T}, \qquad \forall (t,x) \in [0,T] \times \mathbb{R}.
$$

Now, we consider controlled SDE system (5.6) with $|\delta| > 1$, and with the cost functionals:

$$
J(t,x;u(\cdot)) = E\Big\{ \int_t^T u(s)^2 ds - X(T)^2 \Big\}.
$$

For any $\ell > 0$, we take

$$
u_\ell(s) = \ell \chi_{[T-\frac{1}{\ell},T]}(s), \quad s \in [t,T].
$$

Let $X_\ell(\cdot)$ be the corresponding trajectory. Then

$$
E|X_\ell(T)|^2 = E\big|x + 1 + \delta\ell\big[W(T) - W(T - \frac{1}{\ell})\big]\big|^2 = (x+1)^2 + \delta^2\ell.
$$

Thus,

$$
J(t,x;u_\ell(\cdot)) \leq -(1+x)^2 - (\delta^2 - 1)\ell \to -\infty, \qquad \text{as } \ell \to +\infty,
$$

which yields that $V(t,x)$ is not finite if $|\delta| > 1$ and $T < 1$.

This example shows that an open-loop solvable deterministic LQ problem may become not finite if the noise gets into the control system.

5.2. **Representations of the cost functional.** In this section, we will give some proper representations of the cost functional, which will be useful for studying the finiteness and open-loop solvability of Problem (SLQ). The method is a combination of functional analysis and BSDEs.

Let us first introduce the following SDE for matrix-valued process:

$$\begin{cases} d\Phi(t)=A(t)\Phi(t)dt+C(t)\Phi(t)dW(t), \qquad t\ge 0,\\ \Phi(0)=I. \end{cases} \tag{5.16}$$

We know that $\Phi(t)^{-1}$ exists for all $t\ge 0$ and the following holds:

$$\begin{cases} d\Phi(t)^{-1}=-\Phi(t)^{-1}[A(t)-C(t)^2]dt-\Phi(t)^{-1}C(t)dW(t), \quad t\ge 0,\\ \Phi(0)^{-1}=I. \end{cases} \tag{5.17}$$

The solution $X(\cdot)$ of (5.1) can be written as follows:

$$\begin{aligned} X(s)&=\Phi(s)\Phi(t)^{-1}x\\ &+\Phi(s)\int_t^s\Phi(r)^{-1}\big\{\big[B(r)-C(r)D(r)\big]u(r)+b(r)-C(r)\sigma(r)\big\}dr\\ &+\Phi(s)\int_t^s\Phi(r)^{-1}\big[D(r)u(r)+\sigma(r)\big]dW(r), \qquad s\in[t,T]. \end{aligned} \tag{5.18}$$

Also, we have

$$\begin{aligned} \mathbb{E}\Big\{\sup_{t\le r\le s}|X(r)|^2\Big\}\le K\mathbb{E}\Big\{|x|^2+\Big(\int_t^s|b(r)|dr\Big)^2&\\ +\int_t^s\big[|u(r)|^2+|\sigma(r)|^2\big]dr\Big\},& \quad \forall s\in[t,T]. \end{aligned}$$

Now, we define the following operators: For any $(t,x)\in[0,T)\times\mathbb{R}^n$, and $u(\cdot)\in\mathcal{U}[t,T]$,

$$\begin{cases} (L_tu(\cdot))(\cdot)=\Phi(\cdot)\Big\{\displaystyle\int_t^\cdot\Phi(r)^{-1}\big[B(r)-C(r)D(r)\big]u(r)dr\\ \qquad\qquad\qquad\qquad\qquad+\displaystyle\int_t^\cdot\Phi(r)^{-1}D(r)u(r)dW(r)\Big\},\\ (\Gamma_tx)(\cdot)=\Phi(\cdot)\Phi(t)^{-1}x,\\ f_t(\cdot)=\Phi(\cdot)\Big\{\displaystyle\int_t^\cdot\Phi(r)^{-1}[b(r)-C(r)\sigma(r)]dr+\int_t^\cdot\Phi(r)^{-1}\sigma(r)dW(r)\Big\},\\ \widehat L_tu(\cdot)=(L_tu(\cdot))(T), \quad \widehat\Gamma_tx=(\Gamma_tx)(T), \quad \widehat f_t=f_t(T). \end{cases}$$

Then, for any $(t,x)\in[0,T)\times\mathbb{R}^n$ and $u(\cdot)\in\mathcal{U}[t,T]$, the corresponding state process $X(\cdot)$ and its terminal value $X(T)$ are given by

$$\begin{cases} X(\cdot)=(\Gamma_tx)(\cdot)+(L_tu(\cdot))(\cdot)+f_t(\cdot),\\ X(T)=\widehat\Gamma_tx+\widehat L_tu(\cdot)+\widehat f_t. \end{cases} \tag{5.19}$$

Our next goal is to find a representation of the cost functional in terms of the control. To this end, we first note that, for any $t\in[0,T]$,

$$\begin{cases} L_t:\mathcal{U}[t,T]\to\mathcal{X}[t,T], \qquad \widehat L_t:\mathcal{U}[t,T]\to\mathcal{X}_T,\\ \Gamma_t:\mathbb{R}^n\to\mathcal{X}[t,T], \qquad\quad \widehat\Gamma_t:\mathbb{R}^n\to\mathcal{X}_T, \end{cases} \tag{5.20}$$

are all bounded linear operators, where $\mathcal{X}[t,T]\equiv L^2_{\mathbb{F}}(t,T;\mathbb{R}^n)$ and $\mathcal{X}_T\equiv L^2_{\mathcal{F}_T}(\Omega;\mathbb{R}^n)$ are Hilbert spaces. Therefore, their adjoint operators exist:

$$\begin{cases} L_t^*:\mathcal{X}[t,T]\to\mathcal{U}[s,T], \qquad \widehat{L}_t^*:\mathcal{X}_T\to\mathcal{U}[t,T],\\ \Gamma_t^*:\mathcal{X}[t,T]\to\mathbb{R}^n, \qquad \widehat{\Gamma}_t^*:\mathcal{X}_T\to\mathbb{R}^n.\end{cases}\tag{5.21}$$

We want to find representations of the above operators. By definition, the above operators satisfy the following conditions:

$$\begin{cases} \mathbb{E}\int_t^T\langle (L_tu(\cdot))(s),\xi(s)\,\rangle\,ds=\mathbb{E}\int_t^T\langle\, u(s),(L_t^*\xi(\cdot))(s)\,\rangle\,ds,\\ \mathbb{E}\int_t^T\langle(\Gamma_tx)(s),\xi(s)\,\rangle\,ds=\mathbb{E}\,\langle\, x,\Gamma_t^*\xi(\cdot)\,\rangle,\\ \qquad\qquad\qquad\forall u(\cdot)\in\mathcal{U}[t,T],\ \xi(\cdot)\in\mathcal{X}[t,T],\ x\in\mathbb{R}^n,\end{cases}\tag{5.22}$$

and

$$\begin{cases} \mathbb{E}\,\langle\,\widehat{L}_tu(\cdot),\eta\,\rangle=\mathbb{E}\int_t^T\langle\, u(s),(\widehat{L}_t^*\eta)(s)\,\rangle\,ds,\\ \mathbb{E}\,\langle\,\widehat{\Gamma}_tx,\eta\,\rangle=\mathbb{E}\,\langle\, x,\widehat{\Gamma}_t^*\eta\,\rangle,\\ \qquad\qquad\qquad\forall u(\cdot)\in\mathcal{U}[t,T],\ x\in\mathbb{R}^n,\ \eta\in\mathcal{X}_T.\end{cases}\tag{5.23}$$

In the above, we have used $\langle\,\cdot\,,\cdot\,\rangle$ as inner products in different Euclidean spaces, which can be identified from the context.

We note that the operators in (5.21) satisfying (5.22)–(5.23) are *not* formal transpose of those in (5.20). For example, the formal transpose of Γ_t is $[\Phi(T)\Phi(t)^{-1}]^\top$ which maps from $\mathcal{X}_T$ to $\mathcal{X}_T$ (*not* to $\mathbb{R}^n$) in general. Similar situation happens for the other three operators in (5.20). To find the operators (5.21) satisfying (5.22)–(5.23), let us introduce the following BSDE:

$$\begin{cases} dY(s)=-\big[A(s)^\top Y(s)+C(s)^\top Z(s)+\xi(s)\big]dt+Z(s)dW(s),\\ \qquad\qquad\qquad\qquad\qquad\qquad s\in[t,T],\\ Y(T)=\eta\in\mathcal{X}_T.\end{cases}\tag{5.24}$$

For any $\xi(\cdot)\in\mathcal{X}[t,T]$ and $\eta\in\mathcal{X}_T$, the above BSDE admits a unique adapted solution

$$(Y(\cdot),Z(\cdot))\equiv(Y(\cdot\,;\xi(\cdot),\eta),Z(\cdot\,;\xi(\cdot),\eta)).$$

We have the following result.

Proposition 5.6. *Let* (L1)–(L2) *hold. Then the following representations hold:*

$$\begin{cases} (L_t^*\xi)(s)=B(s)^\top Y(s;\xi(\cdot),0)+D(s)^\top Z(s;\xi(\cdot),0), \quad s\in[t,T],\\ \Gamma_t^*\xi(\cdot)=Y(t;\xi(\cdot),0),\end{cases}\tag{5.25}$$

and

$$\begin{cases} (\widehat{L}_t^*\eta)(s)=B(s)^\top Y(s;0,\eta)+D(s)^\top Z(s;0,\eta), \qquad s\in[t,T],\\ \widehat{\Gamma}_t^*\eta=Y(t;0,\eta).\end{cases}\tag{5.26}$$

Proof. For any $\eta\in\mathcal{X}_T$ and $\xi(\cdot)\in\mathcal{X}[t,T]$, there exists a unique adapted solution $(Y(\cdot),Z(\cdot))\in\mathcal{X}[t,T]\times\mathcal{X}[t,T]$ to BSDE (5.24) satisfying

$$E\Big(\sup_{s\le t\le T}|Y(t)|^2+\int_s^T|Z(t)|^2dt\Big)\le KE\Big(|\eta|^2+\int_s^T|\xi(t)|^2dt\Big),$$

for some constant $K > 0$. It follows that all the operators defined in (5.25) and (5.26) are bounded. Next, for any $x \in \mathbb{R}^n$ and $u(\cdot) \in \mathcal{U}[t,T]$, let $X(\cdot) = X(\cdot\,;t,x,u(\cdot))$ be the solution of (5.1). Applying Itô's formula to $\langle X(\cdot), Y(\cdot)\rangle$, we obtain

$$
\begin{aligned}
&\langle X(T), \eta\rangle - \langle x, Y(t)\rangle \\
&= \int_t^T \big[\,\langle u(s), B(s)^\top Y(s) + D(s)^\top Z(s)\rangle - \langle X(s), \xi(s)\rangle\,\big] ds + \int_t^T [\cdots] dW(s).
\end{aligned}
$$

Consequently, using (5.19), we have

$$
\begin{aligned}
&\mathbb{E}\big[\,\langle \widehat{\Gamma}_t x + \widehat{L}_t u(\cdot), \eta\rangle - \langle x, Y(t)\rangle \\
&= \mathbb{E}\int_t^T \big\{\,\langle u(s), B(s)^\top Y(s) + D(s)^\top Z(s)\rangle \\
&\qquad\qquad - \langle (\Gamma_t x)(s) + (L_t u(\cdot))(s), \xi(s)\rangle\,\big\} ds.
\end{aligned}
\tag{5.27}
$$

Then, in (5.27), by taking $x = 0$ and $\eta = 0$, we obtain the first relations in (5.25); by taking $u(\cdot) = 0$ and $\eta = 0$, we obtain the second relations in (5.25); by taking $x = 0$ and $\xi(\cdot) = 0$, we obtain the first relations in (5.26); and by taking $u(\cdot) = 0$ and $\xi(\cdot) = 0$, we obtain the second relations in (5.26). $\square$

Having the above result, we can obtain the following.

Proposition 5.7. *Let* (L1)–(L2) *hold. For any $t \in [0,T)$, there exists a bounded self-adjoint linear operator $M_2(t) : \mathcal{U}[t,T] \to \mathcal{U}[t,T]$, a bounded linear operator $M_1(t) : \mathbb{R}^n \to \mathcal{U}[t,T]$, an $M_0(t) \in \mathbb{S}^n$ and $\nu_t(\cdot) \in \mathcal{U}[t,T], y_t \in \mathbb{R}^n, c_t \in \mathbb{R}$ such that*

$$
\begin{aligned}
J(t,x;u(\cdot)) &= \langle M_2(t)u, u\rangle + 2\langle M_1(t)x, u\rangle + \langle M_0(t)x, x\rangle \\
&\quad + 2\langle \nu_t, u\rangle + 2\langle y_t, x\rangle + c_t, \\
J_0(t,x;u(\cdot)) &= \langle M_2(t)u, u\rangle + 2\langle M_1(t)x, u\rangle + \langle M_0(t)x, x\rangle, \\
&\qquad\qquad \forall (x, u(\cdot)) \in \mathbb{R}^n \times \mathcal{U}[t,T].
\end{aligned}
\tag{5.28}
$$

Moreover, denote

$$
\varphi(\cdot) = (b(\cdot), \sigma(\cdot), q(\cdot), g),
$$

and let

$$
(X(\cdot), Y(\cdot), Z(\cdot)) \equiv \big(X(\cdot\,;x,u(\cdot),\varphi(\cdot)), Y(\cdot\,;x,u(\cdot),\varphi(\cdot)), Z(\cdot\,;x,u(\cdot),\varphi(\cdot))\big)
$$

be the adapted solution of the following (decoupled) linear FBSDE:

$$
\begin{cases}
dX(s) = \big[A(s)X(s) + B(s)u(s) + b(s)\big] ds \\
\qquad\qquad + \big[C(s)X(s) + D(s)u(s) + \sigma(s)\big] dW(s), \quad s \in [t,T], \\
dY(s) = -\big[A(s)^\top Y(s) + C(s)^\top Z(s) + Q(s)X(s) \\
\qquad\qquad + S(s)^\top u(s) + q(s)\big] ds + Z_0(s) dW(s), \quad s \in [t,T], \\
X(t) = x, \qquad Y(T) = HX(T) + h.
\end{cases}
\tag{5.29}
$$

Then

$$\left\{\begin{aligned}&(M_2(t)u(\cdot))(s)=B(s)^\top Y(s;0,u(\cdot),0)+D(s)^\top Z(s;0,u(\cdot),0)\\&\qquad\qquad\qquad +S(s)X(s;0,u(\cdot),0)+R(s)u(s),\quad s\in[t,T],\\&(M_1(t)x)(s)=B(s)^\top Y(s;x,0,0)+D(s)^\top Z(s;x,0,0)\\&\qquad\qquad\qquad +S(s)X(s;x,0,0),\qquad s\in[t,T],\\&M_0(t)=\mathbb{E}\big[Y(t;x,0,0)\big].\end{aligned}\right.\tag{5.30}$$

Finally, $M_0(\cdot)$ solves the following Lyapunov equation:

$$\left\{\begin{aligned}&\dot M_0(t)+M_0(t)A(t)+A(t)^\top M_0(t)\\&\qquad\qquad +C(t)^\top M_0(t)C(t)+Q(t)=0,\qquad t\in[0,T],\\&M_0(T)=H,\end{aligned}\right.\tag{5.31}$$

and it admits the following representation:

$$\begin{aligned}M_0(t)=\mathbb{E}\Big\{&\big[\Phi(T)\Phi(t)^{-1}\big]^\top H\big[\Phi(T)\Phi(t)^{-1}\big]\\&+\int_t^T\big[\Phi(s)\Phi(t)^{-1}\big]^\top Q(s)\big[\Phi(s)\Phi(t)^{-1}\big]ds\Big\},\end{aligned}\tag{5.32}$$

where $\Phi(\cdot)$ is the solution to (5.16).

Proof. Note that

$$\begin{aligned}J(t,x;u(\cdot))&=\langle QX,X\rangle+2\langle SX,u\rangle+\langle Ru,u\rangle+2\langle q,X\rangle+2\langle\rho,u\rangle\\&\quad+\langle HX(T),X(T)\rangle+2\langle h,X(T)\rangle\\&=\langle Q(\Gamma x+Lu+f),\Gamma x+Lu+f\rangle+2\langle S(\Gamma x+Lu+f),u\rangle\\&\quad+\langle Ru,u\rangle+2\langle q,\Gamma x+Lu+f\rangle+2\langle\rho,u\rangle\\&\quad+\langle H(\widehat\Gamma x+\widehat Lu+\widehat f),\widehat\Gamma x+\widehat Lu+\widehat f\rangle+2\langle h,\widehat\Gamma x+\widehat Lu+\widehat f\rangle\\&=\langle(\widehat L^*H\widehat L+L^*QL+SL+L^*S^\top+R)u,u\rangle\\&\quad+2\langle(\widehat L^*H\widehat\Gamma+L^*Q\Gamma+S\Gamma)x,u\rangle\\&\quad+2\langle\widehat L^*H\widehat f+\widehat L^*h+L^*Qf+L^*q+Sf+\rho,u\rangle\\&\quad+\langle(\widehat\Gamma^*H\widehat\Gamma+\Gamma^*Q\Gamma)x,x\rangle+2\langle\widehat\Gamma^*H\widehat f+\widehat\Gamma^*h+\Gamma^*Qf+\Gamma^*q,x\rangle\\&\quad+\langle Qf,f\rangle+2\langle q,f\rangle+\langle H\widehat f,\widehat f\rangle+2\langle h,\widehat f\rangle\end{aligned}$$

Denote

$$\left\{\begin{aligned}&M_2(t)=\widehat L_t^*H\widehat L_t+L_t^*QL_t+SL_t+L_t^*S^\top+R,\\&M_1(t)=\widehat L_t^*H\widehat\Gamma_t+(L_t^*Q+S)\Gamma_t,\\&M_0(t)=\widehat\Gamma_t^*H\widehat\Gamma_t+\Gamma_t^*Q\Gamma_t,\\&\nu_t=\widehat L_t^*H\widehat f_t+\widehat L_t^*h+L_t^*Qf_t+L_t^*q+Sf_t+\rho,\\&y_t=\widehat\Gamma_t^*H\widehat f_t+\widehat\Gamma_t^*h+\Gamma_t^*Qf_t+\Gamma_t^*q,\\&c_t=\langle H\widehat f_t,\widehat f_t\rangle+2\langle h,\widehat f_t\rangle+\langle Qf_t,f_t\rangle+2\langle q,f_t\rangle.\end{aligned}\right.\tag{5.33}$$

Then (5.28) follows. The rest conclusions can be proved. □

From the representation of the cost functional, we have the following simple corollary.

Corollary 5.8. *Let* (L1)–(L2) *hold and* $t \in [0,T)$ *be given. For any* $x \in \mathbb{R}^n$, $\lambda \in \mathbb{R}$ *and* $u(\cdot), v(\cdot) \in \mathcal{U}[t,T]$, *the following holds:*

$$\begin{aligned}&J(t,x;u(\cdot)+\lambda v(\cdot))\\&=J(t,x;u(\cdot))+\lambda^2 J_0(t,0;v(\cdot))\\&\quad+2\lambda\mathbb{E}\int_t^T\langle\, B(s)^\top Y(s)+D(s)^\top Z(s)+S(s)X(s)\\&\qquad\qquad\qquad\qquad\qquad+R(s)u(s)+\rho(s),v(s)\,\rangle\,ds,\end{aligned}\tag{5.34}$$

where $(X(\cdot),Y(\cdot),Z(\cdot))$ *is the adapted solution to the following (decoupled) linear FBSDE:*

$$\begin{cases}dX(s)=\big[A(s)X(s)+B(s)u(s)+b(s)\big]ds\\ \qquad\qquad+\big[C(s)X(s)+D(s)u(s)+\sigma(s)\big]dW(s),\quad s\in[t,T],\\ dY(s)=-\big[A(s)^\top Y(s)+C(s)^\top Z(s)+Q(s)X(s)\\ \qquad\qquad+S(s)^\top u(s)+q(s)\big]ds+Z(s)dW(s),\quad s\in[t,T],\\ X(t)=x,\qquad Y(T)=HX(T)+h.\end{cases}\tag{5.35}$$

Consequently, the map $u(\cdot)\mapsto J(t,x;u(\cdot))$ *is Fréchet differentiable with the Fréchet derivative given by*

$$\begin{aligned}\mathcal{D}_{u(\cdot)}J(t,x;u(\cdot))(s)=2\big[&B(s)^\top Y(s)+D(s)^\top Z(s)+S(s)X(s)\\&+R(s)u(s)+\rho(s)\big],\qquad s\in[t,T],\end{aligned}\tag{5.36}$$

and (5.34) *can also be written as*

$$\begin{aligned}&J(t,x;u(\cdot)+\lambda v(\cdot))\\&=J(t,x;u(\cdot))+\lambda^2J_0(t,0;v(\cdot))+\lambda\mathbb{E}\int_t^T\langle\,\mathcal{D}J(t,x;u(\cdot))(s),v(s)\,\rangle\,ds.\end{aligned}\tag{5.37}$$

Proof. From Proposition 5.7, we have

$$\begin{aligned}&J(t,x;u(\cdot)+\lambda v(\cdot))\\&=\langle\, M_2(t)(u+\lambda v),u+\lambda v\,\rangle+2\,\langle\, M_1(t)x,u+\lambda v\,\rangle\\&\qquad+\langle\, M_0(t)x,x\,\rangle+2\,\langle\, u+\lambda v,\nu_t\,\rangle+2\,\langle\, x,y_t\,\rangle+c_t\\&=J(t,x;u(\cdot))+\lambda^2J_0(t,0;v(\cdot))+2\lambda\,\langle\, M_2(t)u+M_1(t)x+\nu_t,v\,\rangle\,.\end{aligned}$$

From the representation of $M_1(t)$, $M_2(t)$ and ν_t in Proposition 5.7, we see that

$$\begin{aligned}&(M_2(t)u)(s)+(M_1(t)x)(s)+\nu_t(s)\\&=B(s)^\top Y(s)+D(s)^\top Z(s)+S(s)X(s)+R(s)u(s)+\rho(s),\quad s\in[t,T],\end{aligned}$$

with $(X(\cdot),Y(\cdot),Z(\cdot))$ being the adapted solution to the FBSDE (5.35). The rest of the proof is clear. $\square$

The following is concerned with the convexity of the cost functional, whose proof is straightforward, by making use of the representation (5.28) of the cost functional.

Corollary 5.9. *Let* (L1)–(L2) *hold and let* $t \in [0,T)$ *be given. Then the following are equivalent:*

(i) $u(\cdot)\mapsto J(t,x;u(\cdot))$ *is convex, for some* $x\in\mathbb{R}^n$.

(ii) $u(\cdot)\mapsto J(t,x;u(\cdot))$ *is convex, for any* $x\in\mathbb{R}^n$.

(iii) $u(\cdot)\mapsto J_0(t,x;u(\cdot))$ *is convex, for some* $x\in\mathbb{R}^n$.
(iv) $u(\cdot)\mapsto J_0(t,x;u(\cdot))$ *is convex, for any* $x\in\mathbb{R}^n$.
(v) $J_0(t,0;u(\cdot))\ge 0$, *for all* $u(\cdot)\in\mathcal{U}[t,T]$.
(vi) $M_2(t)\ge 0$.

Similar to the above, we have that $u(\cdot)\mapsto J(t,x;u(\cdot))$ is uniformly convex if and only if

$$J_0(t,0;u(\cdot))\ge\lambda\mathbb{E}\int_t^T|u(s)|^2ds,\qquad\forall u(\cdot)\in\mathcal{U}[t,T],\tag{5.38}$$

for some $\lambda>0$. This is also equivalent to the following:

$$M_2(t)\ge\lambda I,\tag{5.39}$$

for some $\lambda>0$. Further, if Problem (SLQ) is a *standard stochastic LQ problem*, by which we mean following conditions hold:

$$R(\cdot)\gg 0,\quad Q(\cdot)-S(\cdot)R(\cdot)^{-1}S(\cdot)^\top\ge 0,\quad H\ge 0,\tag{5.40}$$

then

$$\begin{aligned}M_2(t)&=\widehat L_t^*H\widehat L_t+L_t^*(Q-S^\top R^{-1}S)L_t\\&\quad+(L_t^*S^\top R^{-\frac12}+R^{\frac12})(R^{-\frac12}SL_t+R^{\frac12})\ge 0,\end{aligned}\tag{5.41}$$

which means that the functional $u(\cdot)\mapsto J_0(t,0,u(\cdot))$ is convex. The following result tells us that under (5.40), one actually has the uniform convexity of the cost functional.

Proposition 5.10. *Let* (L1)–(L2) *and* (5.40) *hold. Then for any* $t\in[0,T)$, *the map* $u(\cdot)\mapsto J_0(t,0;u(\cdot))$ *is uniformly convex.*

Proof. For any $u(\cdot)\in\mathcal{U}[t,T]$, let $X^{(u)}(\cdot)$ be the solution of

$$\begin{cases}dX^{(u)}(s)=\big[A(s)X^{(u)}(s)+B(s)u(s)\big]ds\\ \qquad\qquad\qquad+\big[C(s)X^{(u)}(s)+D(s)u(s)\big]dW(s),\qquad s\in[t,T],\\ X^{(u)}(t)=0.\end{cases}$$

Since (5.40) holds, we have

$$\begin{aligned}&J^0(t,0;u(\cdot))\ge\mathbb{E}\int_t^T\Big[\langle QX^{(u)},X^{(u)}\rangle+2\langle SX^{(u)},u\rangle+\langle Ru,u\rangle\Big]ds\\&=\mathbb{E}\int_t^T\Big[\langle\big(Q-S^\top R^{-1}S\big)X^{(u)},X^{(u)}\rangle\\&\qquad+\langle R\big(u+R^{-1}SX^{(u)}\big),u+R^{-1}SX^{(u)}\rangle\Big]ds\\&\ge\delta\mathbb{E}\int_t^T\big|u+R^{-1}SX^{(u)}\big|^2ds.\end{aligned}\tag{5.42}$$

Now we define a bounded linear operator $\mathfrak{L}:\mathcal{U}[t,T]\to\mathcal{U}[t,T]$ by

$$\mathfrak{L}u=u+R^{-1}SX^{(u)}.$$

It is easy to see that $\mathfrak{L}$ is bijective, and that its inverse $\mathfrak{L}^{-1}$ is given by

$$\mathfrak{L}^{-1}u=u-R^{-1}S\widetilde X^{(u)},$$

where $\widetilde{X}^{(u)}(\cdot)$ is the solution of

$$\begin{cases} d\widetilde{X}^{(u)}(s)=\big[(A-BR^{-1}S)\widetilde{X}^{(u)}+Bu\big]ds\\ \qquad\qquad +\big[(C-DR^{-1}S)\widetilde{X}^{(u)}+Du\big]dW(s), \quad s\in[t,T],\\ \widetilde{X}^{(u)}(t)=0.\end{cases}$$

By the bounded inverse theorem, $\mathfrak{L}^{-1}$ is also bounded with norm $\|\mathfrak{L}^{-1}\|>0$. Thus,

$$\mathbb{E}\int_t^T|u(s)|^2ds=\mathbb{E}\int_t^T|(\mathfrak{L}^{-1}\mathfrak{L}u)(s)|^2ds\le\|\mathfrak{L}^{-1}\|^2\mathbb{E}\int_t^T|(\mathfrak{L}u)(s)|^2ds. \tag{5.43}$$

Combining (5.42) and (5.43), we obtain

$$J^0(t,0;u(\cdot))\ge\delta\mathbb{E}\int_t^T|(\mathfrak{L}u)(s)|^2ds\ge\frac{\delta}{\|\mathfrak{L}^{-1}\|^2}\mathbb{E}\int_t^T|u(s)|^2ds.$$

Since $u(\cdot)\in\mathcal{U}[t,T]$ is arbitrary, the desired conclusion follows. □

5.3. **Open-loop solvability.** In this section, we study the *open-loop solvability* of Problem (SLQ).

We have the following result for the finiteness and solvability of our Problem (SLQ).

Theorem 5.11. *Let* (L1)–(L2) *hold.*

(i) *If Problem* (SLQ) *is finite at some* $(t,x)\in[0,T]\times\mathbb{R}^n$, *then*

$$M_2(t)\ge0. \tag{5.44}$$

(ii) *Problem* (SLQ) *is (uniquely) open-loop solvable at* $(t,x)\in[0,T]\times\mathbb{R}^n$ *if and only if* (5.44) *holds and there exists a (unique)* $\bar u(\cdot)\in\mathcal{U}[t,T]$ *such that*

$$M_2(t)\bar u(\cdot)+M_1(t)x+\nu_t=0. \tag{5.45}$$

In this case, $\bar u(\cdot)$ *is an (the) optimal control.*

(iii) *If* $M_2(t)\gg0$ *for some* $t\in[0,T)$, *then for any* $x\in\mathbb{R}^n$, $J(t,x;\cdot)$ *admits a unique minimizer* $\bar u(\cdot)$ *given by*

$$\bar u(\cdot)=-M_2(t)^{-1}\big(M_1(t)x+\nu_t\big). \tag{5.46}$$

In this case, it holds

$$\begin{aligned}&V(t,x)\overset{\Delta}{=}\inf_{u(\cdot)\in\mathcal{U}[t,T]}J(t,x;u(\cdot))=J(t,x;\bar u(\cdot))\\&=\langle M_0(t)x,x\rangle+2\langle y_t,x\rangle+c_t\\&\qquad-\langle M_2(t)^{-1}\big(M_1(t)x+\nu_t\big),M_1(t)x+\nu_t\rangle,\ \forall(t,x)\in[0,T]\times\mathbb{R}^n.\end{aligned} \tag{5.47}$$

The proof is pretty straightforward and we leave it to the readers. Note that the above result is in an abstract framework. The advantage is that the general picture of the LQ problem is pretty clear. Whereas, the conditions expressed by some complicated operators which are determined by some FBSDE. Therefore, one expects that the open-loop solvability might be characterized directly in terms of that of FBSDE. The following result gives such a result.

Theorem 5.12. *Let* (L1)–(L2) *hold. For a given initial pair* $(t,x)\in[0,T)\times\mathbb{R}^n$, *a state-control pair* $(\bar X(\cdot),\bar u(\cdot))$ *is an open-loop optimal pair of Problem* (SLQ) *if and only if the following stationarity condition holds:*

$$\begin{aligned}&B(s)^\top\bar Y(s)+D(s)^\top\bar Z(s)+S(s)\bar X(s)+R(s)\bar u(s)+\rho(s)=0,\\&\qquad\qquad\qquad\qquad\qquad\qquad\text{a.e. } s\in[t,T],\ \text{a.s.}\end{aligned}\tag{5.48}$$

where $(\bar Y(\cdot),\bar Z(\cdot))$ *is the adapted solution to the following BSDE:*

$$\begin{cases}d\bar Y(s)=-\big[A(s)^\top\bar Y(s)+C(s)^\top\bar Z(s)+Q(s)\bar X(s)\\\qquad\qquad+S(s)^\top\bar u(s)+q(s)\big]+\bar Z(s)dW(s),\quad s\in[t,T],\\\bar Y(T)=H\bar X(T)+h,\end{cases}\tag{5.49}$$

and the following convexity condition holds: For any $u(\cdot)\in\mathcal{U}[t,T]$,

$$\begin{aligned}&\mathbb{E}\Big\{\langle HX_0(T),X_0(T)\rangle+\int_t^T\big[\langle Q(s)X_0(s),X_0(s)\rangle\\&\qquad+2\langle S(s)X_0(s),u(s)\rangle+\langle R(s)u(s),u(s)\rangle\big]ds\Big\}\ge0,\end{aligned}\tag{5.50}$$

where $X_0(\cdot)$ *is the solution to the following:*

$$\begin{cases}dX_0(s)=\big[A(s)X_0(s)+B(s)u(s)\big]ds\\\qquad\qquad+\big[C(s)X_0(s)+D(s)u(s)\big]dW(s),\quad s\in[t,T],\\X_0(t)=0.\end{cases}\tag{5.51}$$

Proof. Suppose $(\bar X(\cdot),\bar u(\cdot))$ is a state-control pair corresponding to the given initial pair $(t,x)\in[0,T)\times\mathbb{R}^n$. For any $u(\cdot)\in\mathcal{U}[t,T]$ and $\varepsilon\in\mathbb{R}$, let $X^\varepsilon(\cdot)=X(\cdot\,;t,x,\bar u(\cdot)+\varepsilon u(\cdot))$. Then

$$\begin{cases}dX^\varepsilon(s)=\big\{A(s)X^\varepsilon(s)+B(s)\big[\bar u(s)+\varepsilon u(s)\big]+b(s)\big\}ds\\\qquad\qquad+\big\{C(s)X^\varepsilon(s)+D(s)\big[\bar u(s)+\varepsilon u(s)\big]+\sigma(s)\big\}dW(s),\\\qquad\qquad\qquad\qquad\qquad\qquad s\in[t,T],\\X^\varepsilon(t)=x.\end{cases}$$

Thus, $X_0(\cdot)\equiv\frac{X^\varepsilon(\cdot)-\bar X(\cdot)}{\varepsilon}$ is independent of ε and satisfies (5.51). Then

$$\begin{aligned}&J(t,x;\bar u(\cdot)+\varepsilon u(\cdot))-J(t,x;\bar u(\cdot))\\&=\varepsilon\mathbb{E}\Big\{\langle H\big[2\bar X(T)+\varepsilon X_0(T)\big],X_0(T)\rangle+2\langle h,X_0(T)\rangle\\&\quad+\int_t^T\Big[\langle\begin{pmatrix}Q(s)&S(s)^T\\S(s)&R(s)\end{pmatrix}\begin{pmatrix}2\bar X(s)+\varepsilon X_0(s)\\2\bar u(s)+\varepsilon u(s)\end{pmatrix},\begin{pmatrix}X_0(s)\\u(s)\end{pmatrix}\rangle\\&\qquad\qquad+2\langle\begin{pmatrix}q(s)\\\rho(s)\end{pmatrix},\begin{pmatrix}X_0(s)\\u(s)\end{pmatrix}\rangle\Big]ds\Big\}\end{aligned}$$

$$
\begin{aligned}
&= 2\varepsilon\mathbb{E}\Big\{ \langle H\bar X(T), X_0(T)\rangle + \langle h, X_0(T)\rangle \\
&\qquad + \int_t^T \Big[\langle Q\bar X, X_0\rangle + \langle S\bar X, u\rangle + \langle SX_0, \bar u\rangle + \langle R\bar u, u\rangle \\
&\qquad\qquad\qquad + \langle q, X_0\rangle + \langle \rho, u\rangle \Big] ds \Big\} \\
&\qquad + \varepsilon^2\mathbb{E}\Big\{ \langle HX_0(T), X_0(T)\rangle + \int_t^T \Big[\langle Q(s)X_0(s), X_0(s)\rangle \\
&\qquad\qquad + 2\langle S(s)X_0(s), u(s)\rangle + \langle R(s)u(s), u(s)\rangle \Big] ds \Big\}.
\end{aligned}
$$

Now, let $(\bar Y(\cdot), \bar Z(\cdot))$ be the adapted solution to the BSDE (5.49). Then

$$
\begin{aligned}
&\mathbb{E}\Big\{ \langle H\bar X(T) + h, X_0(T)\rangle + \int_t^T \Big[\langle Q\bar X + S^\top \bar u + q, X_0\rangle \\
&\qquad\qquad + \langle S\bar X + R\bar u + \rho, u\rangle \Big] ds \Big\} \\
&= \mathbb{E}\Big\{ \int_t^T \Big[\langle -(A^\top \bar Y + C^\top \bar Z + Q\bar X + S^\top \bar u + q), X_0\rangle \\
&\qquad + \langle \bar Y, AX_0 + Bu\rangle + \langle \bar Z, CX_0 + Du\rangle \\
&\qquad + \langle Q\bar X + S^\top \bar u + q, X_0\rangle + \langle S\bar X + R\bar u + \rho, u\rangle \Big] ds \Big\} \\
&= \mathbb{E}\int_t^T \langle B^\top \bar Y + D^\top \bar Z + S\bar X + R\bar u + \rho, u\rangle\, ds.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&J(t,x;\bar u(\cdot) + \varepsilon u(\cdot)) \\
&= J(t,x;\bar u(\cdot)) + 2\varepsilon\mathbb{E}\Big\{ \int_t^T \langle B^\top \bar Y + D^\top \bar Z + S\bar X + R\bar u + \rho, u\rangle\, ds \Big\} \\
&\quad + \varepsilon^2\mathbb{E}\Big\{ \langle HX_0(T), X_0(T)\rangle + \int_t^T \Big[\langle Q(s)X_0(s), X_0(s)\rangle \\
&\qquad + 2\langle S(s)X_0(s), u(s)\rangle + \langle R(s)u(s), u(s)\rangle \Big] ds \Big\}.
\end{aligned}
$$

Therefore, $(\bar X(\cdot), \bar u(\cdot))$ is an open-loop optimal control of Problem (SLQ) for (t,x) if and only if (5.48) and (5.50) hold. $\square$

Putting the state equation, the adjoint equation and the stationarity condition together, we obtain the following *optimality system*:

$$
\begin{cases}
d\bar X(s) = \big[A(s)\bar X(s) + B(s)\bar u(s) + b(s)\big] ds \\
\qquad\qquad + \big[C(s)\bar X(s) + D(s)\bar u(s) + \sigma(s)\big] dW(s), \\
d\bar Y(s) = -\big[A(s)^\top \bar Y(s) + C(s)^\top \bar Z(s) + Q(s)\bar X(s) \\
\qquad\qquad\qquad + S(s)^\top \bar u(s) + q(s)\big] + \bar Z(s) dW(s), \\
B(s)^\top \bar Y(s) + D(s)^\top \bar Z(s) + S(s)\bar X(s) + R(s)\bar u(s) + \rho(s) = 0, \\
\qquad\qquad\qquad\qquad\qquad\qquad \text{a.e. } s \in [t,T], \text{ a.s.} \\
\bar X(t) = x, \qquad \bar Y(T) = H\bar X(T) + h.
\end{cases}
\tag{5.52}
$$

This is a coupled linear FBSDE for $(\bar X(\cdot),\bar Y(\cdot),\bar Z(\cdot))$ with the coupling through the stationarity condition. We also call the above the *Hamiltonian system* for Problem (SLQ). Therefore, Theorem 5.12 can also be stated as follows.

Theorem 5.13. *Let* (L1)–(L2) *hold. Then Problem (SLQ) is open-loop solvable if and only if the convexity condition holds and the Hamiltonian system* (5.52) *admits a solution* $(\bar X(\cdot),\bar Y(\cdot),\bar Z(\cdot),\bar u(\cdot))$.

5.4. **Closed-loop solvability.** In this section, we study the *closed-loop solvability* of Problem (SLQ).

For any $t\in[0,T)$, take $\Theta(\cdot)\in L^2(t,T;\mathbb{R}^{m\times n})\equiv\mathcal{Q}[t,T]$, and $v(\cdot)\in\mathcal{U}[t,T]$. For any $x\in\mathbb{R}^n$, we consider the following equation:

$$\begin{cases} dX(s)=\big\{\big[A(s)+B(s)\Theta(s)\big]X(s)+B(s)v(s)+b(s)\big\}ds\\ \qquad\qquad+\big\{\big[C(s)+D(s)\Theta(s)\big]X(s)+D(s)v(s)+\sigma(s)\big\}dW(s),\\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad s\in[t,T],\\ X(t)=x,\end{cases}\tag{5.53}$$

which admits a unique solution $X(\cdot)\equiv X(\cdot\,;t,x,\Theta(\cdot),v(\cdot))$, depending on $\Theta(\cdot)$ and $v(\cdot)$. The above is called a *closed-loop system* of the original state equation (5.1) under *closed-loop strategy* $(\Theta(\cdot),v(\cdot))$. We point out that $(\Theta(\cdot),v(\cdot))$ is independent of the initial state x. With the above solution $X(\cdot)$, we define

$$\begin{aligned}J(t,x;\Theta(\cdot)X(\cdot)+v(\cdot))&=\frac12\mathbb{E}\Big\{\langle HX(T),X(T)\rangle+2\langle h,X(T)\rangle\\&+\int_t^T\Big[\langle\begin{pmatrix}Q(s)&S(s)^T\\S(s)&R(s)\end{pmatrix}\begin{pmatrix}X(s)\\\Theta(s)X(s)+v(s)\end{pmatrix},\begin{pmatrix}X(s)\\\Theta(s)X(s)+v(s)\end{pmatrix}\rangle\\&\qquad+2\langle\begin{pmatrix}q(s)\\\rho(s)\end{pmatrix},\begin{pmatrix}X(s)\\\Theta(s)X(s)+v(s)\end{pmatrix}\rangle\Big]ds\Big\}.\end{aligned}$$

We now introduce the following definition.

Definition 5.14. A pair $(\bar\Theta(\cdot),\bar v(\cdot))\in\mathcal{Q}[t,T]\times\mathcal{U}[t,T]$ is called a *closed-loop optimal strategy* of Problem (SLQ) on $[t,T]$ if

$$\begin{aligned}&J(t,x;\bar\Theta(\cdot)\bar X(\cdot)+\bar v(\cdot))\le J(t,x;\Theta(\cdot)X(\cdot)+v(\cdot)),\\&\qquad\qquad\forall x\in\mathbb{R}^n,\ \Theta(\cdot)\in\mathcal{Q}[t,T],\ v(\cdot)\in\mathcal{U}[t,T],\end{aligned}\tag{5.54}$$

where $\bar X(\cdot)=X(\cdot\,;t,x,\bar\Theta(\cdot),\bar v(\cdot))$ on the left and $X(\cdot)=X(\cdot\,;t,x,\Theta(\cdot),v(\cdot))$ on the right. In the above case, we say that Problem (SLQ) is *closed-loop solvable* on $[t,T]$.

In the above, both $\bar\Theta(\cdot)$ and $\bar v(\cdot)$ are required to be independent of the initial state $x\in\mathbb{R}^n$. Note that in (5.4), for given initial pair (t,x), the inequality holds. Therefore, open-loop optimal control $\bar u(\cdot)$ is depending on (t,x). Whereas, in the above (5.54), the inequality holds for all $x\in\mathbb{R}^n$. We have the following equivalence theorem.

Theorem 5.15. *Let* (L1)–(L2) *hold. Then the following are equivalent:*

(i) $(\bar\Theta(\cdot),\bar v(\cdot))\in\mathcal{Q}[t,T]\times\mathcal{U}[t,T]$ *is a closed-loop optimal strategy of Problem* (SLQ).

(ii) *The following holds:*

$$\begin{aligned}&J(t,x;\bar\Theta(\cdot)\bar X(\cdot)+\bar v(\cdot))\le J(t,x;\bar\Theta(\cdot)\widetilde X(\cdot)+v(\cdot)),\\&\qquad\qquad\forall x\in\mathbb{R}^n,\ v(\cdot)\in\mathcal{U}[t,T],\end{aligned}\tag{5.55}$$

where $\widetilde X(\cdot)=X(\cdot\,;t,x,\bar\Theta(\cdot),v(\cdot))$.

(iii) *The following holds:*

$$J(t,x;\bar\Theta(\cdot)\bar X(\cdot)+\bar v(\cdot))\le J(t,x;u(\cdot)),\quad \forall x\in\mathbb{R}^n,\ u(\cdot)\in\mathcal{U}[t,T]. \tag{5.56}$$

Proof. (i) $\Rightarrow$ (ii) is trivial.

(ii) $\Rightarrow$ (iii). For any $u(\cdot)\in\mathcal{U}[t,T]$, let $X(\cdot)=X(\cdot\,;t,x,u(\cdot))$. Define

$$v(\cdot)=u(\cdot)-\bar\Theta(\cdot)X(\cdot),$$

Then

$$X(\cdot)\equiv X(\cdot\,;t,x,u(\cdot))=X(\cdot\,;t,x,\bar\Theta(\cdot),v(\cdot))\equiv\widetilde X(\cdot).$$

Therefore, by (ii), we have

$$J(t,x;\bar\Theta(\cdot)\bar X(\cdot)+\bar v(\cdot))\le J(t,x;\bar\Theta(\cdot)\widetilde X(\cdot)+v(\cdot))=J(t,x;u(\cdot)),$$

proving (iii).

(iii) $\Rightarrow$ (i). For any $\Theta(\cdot)\in\mathcal{Q}[t,T]$ and $v(\cdot)\in\mathcal{U}[t,T]$, let $X(\cdot)=X(\cdot\,;t,x,\Theta(\cdot),v(\cdot))$. Define

$$u(\cdot)=\Theta(\cdot)X(\cdot)+v(\cdot).$$

Then

$$X(\cdot\,;t,x,\Theta(\cdot),v(\cdot))=X(\cdot\,;t,x,u(\cdot)).$$

Therefore, by (iii), one has

$$J(t,x;\bar\Theta(\cdot)\bar X(\cdot)+\bar v(\cdot))\le J(t,x;u(\cdot))=J(t,x;\Theta(\cdot)X(\cdot)+v(\cdot)).$$

This completes the proof. □

From (5.56), we see that for a fixed initial pair $(t,x)\in[0,T)\times\mathbb{R}^n$, the *outcome*

$$\bar u(\cdot)\equiv\bar\Theta(\cdot)\bar X(\cdot)+\bar v(\cdot)\in\mathcal{U}[t,T]$$

of the closed-loop optimal strategy $(\bar\Theta(\cdot),\bar v(\cdot))$ is an open-loop optimal control of Problem (SLQ) for (t,x). Therefore, Problem (SLQ) is closed-loop solvable implies that Problem (SLQ) is open-loop solvable.

The following simple example shows that Problem (SLQ) could be open-loop solvable, but could be not closed-loop solvable.

Example 5.16. Consider the following controlled ODE:

$$\begin{cases}\dot X(s)=u(s), & s\in[t,T],\\ X(t)=x.\end{cases}$$

The cost functional is defined to be

$$J(t,x;u(\cdot))=|X(T)|^2.$$

Then for any $\tau\in(t,T]$, one could define

$$u_\tau(s)=-\frac{x}{\tau-t}I_{[t,\tau]}(s),\qquad s\in[t,T],$$

which leads to

$$X(T)=0.$$

Therefore, each $u_\tau(\cdot)$ is an open-loop optimal control, and the corresponding LQ problem is open-loop solvable. However, such a problem is not closed-loop solvable. In fact, if it were closed-loop solvable, then we may let $(\bar\Theta(\cdot),\bar v(\cdot))$ be a closed-loop optimal strategy. Therefore, it is necessary that

$$0=\bar X(T)=e^{\int_t^T\bar\Theta(r)dr}x+\int_t^T e^{\int_s^T\bar\Theta(r)dr}v(s)ds,\qquad\forall x\in\mathbb{R}.$$

This is impossible.

The above example shows that for Problem (LQ), the open-loop solvability is weaker than the closed-loop solvability. For closed-loop optimal strategies, we have the following characterization.

Theorem 5.17. *Let* (L1)–(L2) *hold. Then Problem* (SLQ) *admits a closed-loop optimal strategy if and only if the following Riccati equation admits a solution* $P(\cdot) \in C([t,T];\mathbb{S}^n)$*:*

$$\begin{cases} \dot P + PA + A^\top P + C^\top PC + Q - (PB + C^\top PD + S^\top) \\ \qquad\qquad \cdot (R + D^\top PD)^\dagger (B^\top P + D^\top PC + S) = 0, \\ \mathcal{R}\big(B^\top P + D^\top PC + S\big) \subseteq \mathcal{R}\big(R + D^\top PD\big), \quad R + D^\top PD \geq 0, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{a.e. } \ s \in [t,T], \\ P(T) = H, \end{cases} \tag{5.57}$$

such that

$$\big[R + D^\top PD\big]^\dagger \big[B^\top P + D^\top PC + S\big] \in L^2(t,T;\mathbb{R}^{m\times n}), \tag{5.58}$$

and the adapted solution $(\eta(\cdot),\zeta(\cdot))$ *of the following BSDE:*

$$\begin{cases} d\eta = -\Big\{ \big[A^\top - (PB + C^\top PD + S^\top)(R + D^\top PD)^\dagger B^\top\big]\eta \\ \qquad + \big[C^\top - (PB + C^\top PD + S^\top)(R + D^\top PD)^\dagger D^\top\big]\zeta \\ \qquad + \big[C^\top - (PB + C^\top PD + S^\top)(R + D^\top PD)^\dagger D^\top\big]P\sigma \\ \quad -(PB + C^\top PD + S^\top)(R + D^\top PD)^\dagger \rho + Pb + q\Big\} ds + \zeta dW(s), \\ B^\top\eta + D^\top\zeta + D^\top P\sigma + \rho \in \mathcal{R}(R + D^\top PD), \quad \text{a.e. } s \in [t,T], \text{ a.s.} \\ \eta(T) = h, \end{cases} \tag{5.59}$$

satisfies

$$\big[R + D^\top PD\big]^\dagger \big[B^\top\eta + D^\top\zeta + D^\top P\sigma + \rho\big] \in L^2_{\mathbb{F}}(t,T;\mathbb{R}^m). \tag{5.60}$$

In this case, any closed-loop optimal strategy $(\bar\Theta(\cdot),\bar v(\cdot))$ *of Problem* (SLQ) *admits the following representation:*

$$\begin{cases} \bar\Theta = -(R + D^\top PD)^\dagger (B^\top P + D^\top PC + S) \\ \qquad\quad + \big[I - (R + D^\top PD)^\dagger (R + D^\top PD)\big]\theta, \\ \bar v = -(R + D^\top PD)^\dagger (B^\top\eta + D^\top\zeta + D^\top P\sigma + \rho) \\ \qquad\quad + \big[I - (R + D^\top PD)^\dagger (R + D^\top PD)\big]\nu, \end{cases} \tag{5.61}$$

for some $\theta(\cdot) \in L^2(t,T;\mathbb{R}^{m\times n})$ *and* $\nu(\cdot) \in L^2_{\mathbb{F}}(t,T;\mathbb{R}^m)$*. Further, the value function admits the following representation:*

$$\begin{aligned} V(t,x) &\equiv \inf_{u(\cdot)\in\mathcal{U}[t,T]} J(t,x;u(\cdot)) \\ &= \mathbb{E}\Big\{ \langle P(t)x, x\rangle + 2\langle \eta(t), x\rangle + \int_t^T \big[\langle P\sigma,\sigma\rangle + 2\langle \eta, b\rangle + 2\langle \zeta,\sigma\rangle \\ &\qquad - \big|[(R + D^\top PD)^\dagger]^{\frac12} (B^\top\eta + D^\top\zeta + D^\top P\sigma + \rho)\big|^2 \big] ds \Big\}. \end{aligned} \tag{5.62}$$

Proof. Necessity. Let $(\bar\Theta(\cdot),\bar v(\cdot))\in\mathcal{Q}[t,T]\times\mathcal{U}[t,T]$ be a closed-loop optimal control Problem (SLQ) over $[t,T]$. Then, by Theorem 5.15, $\bar v(\cdot)$ is an open-loop optimal control of the LQ problem with $A(\cdot),C(\cdot),Q(\cdot),S(\cdot),q(\cdot)$ replaced by the following:

$$\begin{aligned}&A(\cdot)+B(\cdot)\bar\Theta(\cdot),\qquad C(\cdot)+D(\cdot)\bar\Theta(\cdot),\\&Q(\cdot)+\bar\Theta(\cdot)^\top S(\cdot)+S(\cdot)^\top\bar\Theta(\cdot)+\bar\Theta(\cdot)^\top R(\cdot)\bar\Theta(\cdot),\\&S(\cdot)+R(\cdot)\bar\Theta(\cdot),\qquad q(\cdot)+\bar\Theta(\cdot)^\top\rho(\cdot).\end{aligned}$$

Hence, by Theorem 5.13, for any $x\in\mathbb{R}^n$, the following FBSDE admits an adapted solution $(\bar X(\cdot),\bar Y(\cdot),\bar Z(\cdot))$:

$$\begin{cases}d\bar X=\big\{(A+B\bar\Theta)\bar X+B\bar v+b\big\}ds+\big\{(C+D\bar\Theta)\bar X+D\bar v+\sigma\big\}dW(s),\\ d\bar Y=-\big\{(A+B\bar\Theta)^\top\bar Y+(C+D\bar\Theta)^\top\bar Z+(Q+\bar\Theta^\top S+S^\top\bar\Theta+\bar\Theta^\top R\bar\Theta)\bar X\\ \qquad\qquad+(S+R\bar\Theta)^\top\bar v+q+\bar\Theta^\top\rho\big\}ds+\bar ZdW(s),\\ \bar X(t)=x,\qquad \bar Y(T)=H\bar X(T)+h,\end{cases}\tag{5.63}$$

and the following stationarity condition holds:

$$B^\top\bar Y+D^\top\bar Z+(S+R\bar\Theta)\bar X+R\bar v+\rho=0,\quad\text{a.e.}\quad\text{a.s.}\tag{5.64}$$

Making use of (5.64), we may rewrite the BSDE in (5.63) as follows:

$$\begin{aligned}d\bar Y&=-\big\{A^\top\bar Y+C^\top\bar Z+(Q+S^\top\bar\Theta)\bar X\\&\qquad\qquad+\bar\Theta^\top[B^\top\bar Y+D^\top\bar Z+(S+R\bar\Theta)\bar X]\\&\qquad\qquad+(S^\top+\bar\Theta^\top R)\bar v+q+\bar\Theta^\top\rho\big\}ds+\bar ZdW(s)\\&=-\big\{A^\top\bar Y+C^\top\bar Z+(Q+S^\top\bar\Theta)\bar X-\bar\Theta^\top(R\bar v+\rho)\\&\qquad\qquad+(S^\top+\bar\Theta^\top R)\bar v+q+\bar\Theta^\top\rho\big\}ds+\bar ZdW(s)\\&=-\big\{A^\top\bar Y+C^\top\bar Z+(Q+S^\top\bar\Theta)\bar X+S^\top\bar v+q\big\}ds+\bar ZdW(s).\end{aligned}$$

Thus, we obtain

$$\begin{cases}d\bar X=\big\{(A+B\bar\Theta)\bar X+B\bar v+b\big\}ds+\big\{(C+D\bar\Theta)\bar X+D\bar v+\sigma\big\}dW(s),\\ d\bar Y=-\big\{A^\top\bar Y+C^\top\bar Z+(Q+S^\top\bar\Theta)\bar X+S^\top\bar v+q\big\}ds+\bar ZdW(s),\\ \bar X(t)=x,\qquad \bar Y(T)=H\bar X(T)+h,\\ B^\top\bar Y+D^\top\bar Z+(S+R\bar\Theta)\bar X+R\bar v+\rho=0,\quad\text{a.e.}\quad\text{a.s.}\end{cases}\tag{5.65}$$

Since the above admits a solution for each $x\in\mathbb{R}^n$, and $(\bar\Theta(\cdot),\bar v(\cdot))$ is independent of x, by subtracting solutions corresponding x and 0, the later from the former, we see that for any $x\in\mathbb{R}^n$, the following FBSDE admits an adapted solution $(X(\cdot),Y(\cdot),Z(\cdot))$:

$$\begin{cases}dX=(A+B\bar\Theta)Xds+(C+D\bar\Theta)XdW(s),\ s\in[t,T],\\ dY=-\big[A^\top Y+C^\top Z+(Q+S^\top\bar\Theta)X\big]ds+ZdW(s),\ s\in[t,T],\\ X(t)=x,\qquad Y(T)=HX(T),\\ B^\top Y+C^\top Z+(S+R\bar\Theta)X=0,\quad\text{a.e. } s\in[t,T],\ \text{a.s.}\end{cases}\tag{5.66}$$

Now, we let

$$\begin{cases} d\mathbb{X}(s) = \big[A(s) + B(s)\bar{\Theta}(s)\big]\mathbb{X}(s)ds \\ \qquad\qquad +\big[C(s) + D(s)\bar{\Theta}(s)\big]\mathbb{X}(s)dW(s), \quad s \in [t,T], \\ \mathbb{X}(t) = I, \end{cases}$$

and let

$$\begin{cases} d\mathbb{Y}(s) = \big\{-A(s)^\top\mathbb{Y}(s) - C(s)^\top\mathbb{Z}(s) - \big[Q(s) + S(s)^\top\bar{\Theta}(s)\big]\mathbb{X}(s)\big\}ds \\ \qquad\qquad\qquad\qquad\qquad\qquad +\mathbb{Z}(s)dW(s), \quad s \in [t,T], \\ \mathbb{Y}(T) = H\mathbb{X}(T). \end{cases}$$

Clearly, $\mathbb{X}(\cdot)$, $\mathbb{Y}(\cdot)$, and $\mathbb{Z}(\cdot)$ are all well-defined square matrix valued processes. Further,

$$\begin{aligned} &B(s)^\top\mathbb{Y}(s) + D(s)^\top\mathbb{Z}(s) + \big[S(s) + R(s)\bar{\Theta}(s)\big]\mathbb{X}(s) = 0, \\ &\qquad\qquad\qquad\qquad\qquad\qquad \text{a.e. } s \in [t,T], \text{ a.s.} \end{aligned} \tag{5.67}$$

Clearly, $\mathbb{X}(\cdot)^{-1}$ exists and satisfies the following:

$$\begin{cases} d\big[\mathbb{X}(s)^{-1}\big] = \mathbb{X}(s)^{-1}\big\{\big[C(s)+D(s)\bar{\Theta}(s)\big]^2 - A(s) - B(s)\bar{\Theta}(s)\big\}ds \\ \qquad\qquad -\mathbb{X}(s)^{-1}\big[C(s) + D(s)\bar{\Theta}(s)\big]dW(s), \quad s \in [t,T], \\ \mathbb{X}(t)^{-1} = I. \end{cases} \tag{5.68}$$

We define

$$P(\cdot) = \mathbb{Y}(\cdot)\mathbb{X}(\cdot)^{-1}, \qquad \Pi(\cdot) = \mathbb{Z}(\cdot)\mathbb{X}(\cdot)^{-1}.$$

Then (5.67) implies

$$B^\top P + D^\top\Pi + (S + R\bar{\Theta}) = 0, \quad \text{a.e.} \tag{5.69}$$

Also, by Itô's formula,

$$\begin{aligned} dP = &\Big\{-\big[A^\top\mathbb{Y} + C^\top\mathbb{Z} + (Q + S^\top\bar{\Theta})\mathbb{X}\big]\mathbb{X}^{-1} \\ &\quad +\mathbb{Y}\mathbb{X}^{-1}\big[(C + D\bar{\Theta})^2 - A - B\bar{\Theta}\big] - \mathbb{Z}\mathbb{X}^{-1}(C + D\bar{\Theta})\Big\}ds \\ &\quad +\Big\{\mathbb{Z}\mathbb{X}^{-1} - \mathbb{Y}\mathbb{X}^{-1}(C + D\bar{\Theta})\Big\}dW(s) \\ = &\Big\{ -A^\top P - C^\top\Pi - Q - S^\top\bar{\Theta} + P\big[(C + D\bar{\Theta})^2 - A - B\bar{\Theta}\big] \\ &\quad -\Pi(C + D\bar{\Theta})\Big\}ds + \Big\{\Pi - P(C + D\bar{\Theta})\Big\}dW(s). \end{aligned}$$

Let

$$\Lambda = \Pi - P(C + D\bar{\Theta}).$$

Then

$$\begin{aligned} dP = &\Big\{ -A^\top P - C^\top[\Lambda + P(C + D\bar{\Theta})] - Q - S^\top\bar{\Theta} \\ &\quad +P\big[(C + D\bar{\Theta})^2 - A - B\bar{\Theta}\big] \\ &\quad -[\Lambda + P(C + D\bar{\Theta})](C + D\bar{\Theta})\Big\}ds + \Lambda dW(s) \\ = &\Big\{-PA - A^\top P - \Lambda C - C^\top\Lambda - C^\top PC \\ &\quad -(PB + C^\top PD + S^\top + \Lambda D)\bar{\Theta} - Q\Big\}ds + \Lambda dW(s), \end{aligned}$$

and $P(T)=H$. Thus, $(P(\cdot),\Lambda(\cdot))$ is the adapted solution of a BSDE with deterministic coefficients. Hence, $P(\cdot)$ is deterministic and $\Lambda(\cdot)=0$ which means

$$\Pi=\mathbb{Z}\mathbb{X}^{-1}=P(C+D\Theta^*). \tag{5.70}$$

Therefore,

$$\dot P+PA+A^\top P+C^\top PC+(PB+C^\top PD+S^\top)\bar\Theta+Q=0,\quad \text{a.e.} \tag{5.71}$$

and (5.69) becomes

$$\begin{aligned}0&=B^\top P+D^\top P(C+D\bar\Theta)+S+R\bar\Theta\\&=B^\top P+D^\top PC+S+(R+D^\top PD)\bar\Theta,\quad \text{a.e.}\end{aligned} \tag{5.72}$$

This implies

$$\mathcal{R}\big(B^\top P+D^\top PC+S\big)\subseteq\mathcal{R}\big(R+D^\top PD\big),\quad \text{a.e.}$$

Using (5.72), (5.71) can be written as

$$\begin{aligned}0=\dot P+P(A+B\bar\Theta)+(A+B\bar\Theta)^\top P+(C+D\bar\Theta)^\top P(C+D\bar\Theta)\\+\bar\Theta^\top R\bar\Theta+S^\top\bar\Theta+\bar\Theta^\top S+Q,\quad \text{a.e.}\end{aligned}$$

Since $P(T)=H\in\mathbb{S}^n$ and $Q(\cdot),R(\cdot)$ are symmetric, by uniqueness, we must have $P(\cdot)\in C([t,T];\mathbb{S}^n)$. Denoting $\widehat R=R+D^\top PD$, since

$$\widehat R^\dagger(B^\top P+D^\top PC+S)=-\widehat R^\dagger\widehat R\bar\Theta,$$

and $\widehat R^\dagger\widehat R$ is an orthogonal projection, we see that (5.58) holds and

$$\bar\Theta=-\widehat R^\dagger(B^\top P+D^\top PC+S)+\big(I-\widehat R^\dagger\widehat R\big)\theta$$

for some $\theta(\cdot)\in L^2(t,T;\mathbb{R}^{m\times n})$. Consequently,

$$\begin{aligned}&(PB+C^\top PD+S^\top)\bar\Theta=\bar\Theta^\top\widehat R\widehat R^\dagger(B^\top P+D^\top PC+S)\\&=-(PB+C^\top PD+S^\top)\widehat R^\dagger(B^\top P+D^\top PC+S).\end{aligned} \tag{5.73}$$

Plug the above into (5.71), we obtain Riccati equation in (5.57). To determine $\bar v(\cdot)$, we define

$$\begin{cases}\eta=\bar Y-P\bar X,\\ \zeta=\bar Z-P(C+D\bar\Theta)\bar X-PD\bar v-P\sigma.\end{cases}\qquad s\in[t,T].$$

Then

$$\begin{aligned}d\eta&=d\bar Y-\dot P\bar Xds-Pd\bar X\\&=-\big[A^\top\bar Y+C^\top\bar Z+(Q+S^\top\bar\Theta)\bar X+S^\top\bar v+q\big]ds+\bar ZdW\\&\qquad+\Big\{\big[PA+A^\top P+C^\top PC+Q\\&\qquad-(PB+C^\top PD+S^\top)(R+D^\top PD)^\dagger(B^\top P+D^\top PC+S)\big]\bar X\\&\qquad-P[(A+B\bar\Theta)\bar X+B\bar v+b]\Big\}ds-P\big[(C+D\bar\Theta)\bar X+D\bar v+\sigma]dW\end{aligned}$$

$$
\begin{aligned}
&= -\Big\{A^\top(\eta + P\bar X) + C^\top\big[\zeta + P(C+D\bar\Theta)\bar X + PD\bar v + P\sigma\big]\\
&\qquad +(Q+S^\top\bar\Theta)\bar X + S^\top\bar v + q - \big[PA{+}A^\top P{+}C^\top PC\\
&\qquad +Q{-}(PB{+}C^\top PD{+}S^\top)(R{+}D^\top PD)^\dagger(B^\top P{+}D^\top PC{+}S)\big]\bar X\\
&\qquad +P[(A+B\bar\Theta)\bar X + B\bar v + b]\Big\}ds + \zeta dW\\
&= \Big\{-A^\top\eta - C^\top\zeta{-}(PB{+}C^\top PD{+}S^\top)\bar\Theta\bar X\\
&\quad -(PB{+}C^\top PD{+}S^\top)\bar v{-}C^\top P\sigma{-}Pb{-}q\\
&\quad -\big[(PB{+}C^\top PD{+}S^\top)(R{+}D^\top PD)^\dagger(B^\top P{+}D^\top PC{+}S)\big]\bar X\Big\}ds{+}\zeta dW\\
&= -\big[A^\top\eta{+}C^\top\zeta{+}(PB+C^\top PD{+}S^\top)\bar v{+}C^\top P\sigma{+}Pb{+}q\big]ds{+}\zeta dW.
\end{aligned}
$$

According to (5.64), we have

$$
\begin{aligned}
0 &= B^\top\bar Y + D^\top\bar Z + (S+R\bar\Theta)\bar X + R\bar v + \rho\\
&= B^\top(\eta + P\bar X) + D^\top[\zeta + P(C+D\bar\Theta)\bar X + PD\bar v + P\sigma]\\
&\qquad +(S+R\bar\Theta)\bar X + R\bar v + \rho\\
&= [B^\top P + D^\top PC + S + (R+D^\top PD)\bar\Theta]\bar X\\
&\qquad +B^\top\eta + D^\top\zeta + D^\top P\sigma + \rho + (R+D^\top PD)\bar v\\
&= B^\top\eta + D^\top\zeta + D^\top P\sigma + \rho + (R+D^\top PD)\bar v.
\end{aligned}
$$

Hence,

$$
B^\top\eta + D^\top\zeta + D^\top P\sigma + \rho \in \mathcal R(R+D^\top PD), \quad \text{a.e. a.s.}
$$

Since

$$
\widehat R^\dagger(B^\top\eta + D^\top\zeta + D^\top P\sigma + \rho) = -\widehat R^\dagger\widehat R\bar v,
$$

and $\widehat R^\dagger\widehat R$ is an orthogonal projection, we see that (5.60) holds and

$$
\bar v = -\widehat R^\dagger(B^\top\eta + D^\top\zeta + D^\top P\sigma + \rho) + \big[I - \widehat R^\dagger\widehat R\big]\nu
$$

for some $\nu(\cdot)\in L^2_{\mathbb F}(t,T;\mathbb R^m)$. Consequently,

$$
\begin{aligned}
&(PB + C^\top PD + S^\top)\bar v\\
&= -(PB + C^\top PD + S^\top)(R+D^\top PD)^\dagger(B^\top\eta + D^\top\zeta + D^\top P\sigma + \rho)\\
&\qquad +(PB + C^\top PD + S^\top)\big[I - (R+D^\top PD)^\dagger(R+D^\top PD)\big]\nu\\
&= -(PB + C^\top PD + S^\top)(R+D^\top PD)^\dagger(B^\top\eta + D^\top\zeta + D^\top P\sigma + \rho).
\end{aligned}
$$

Then

$$
\begin{aligned}
&A^\top\eta + C^\top\zeta + (PB + C^\top PD + S^\top)v^* + C^\top P\sigma + Pb + q\\
&= A^\top\eta{+}C^\top\zeta{-}(PB{+}C^\top PD{+}S^\top)(R{+}D^\top PD)^\dagger\\
&\qquad \cdot(B^\top\eta{+}D^\top\zeta{+}D^\top P\sigma{+}\rho){+}C^\top P\sigma{+}Pb{+}q\\
&= \big[A^\top{-}(PB{+}C^\top PD+S^\top)(R{+}D^\top PD)^\dagger B^\top\big]\eta\\
&\qquad +\big[C^\top{-}\,(PB{+}\,C^\top PD{+}S^\top)(R{+}D^\top PD)^\dagger D^\top\big]\zeta\\
&\qquad +\big[C^\top - (PB + C^\top PD + S^\top)(R+D^\top PD)^\dagger D^\top\big]P\sigma\\
&\qquad -(PB + C^\top PD + S^\top)(R+D^\top PD)^\dagger\rho + Pb + q.
\end{aligned}
$$

Therefore, (η,ζ) is the adapted solution to the following BSDE:

$$\begin{cases} d\eta=-\Big\{\big[A^\top-(PB+C^\top PD+S^\top)(R+D^\top PD)^\dagger B^\top\big]\eta \\ \qquad\qquad +\big[C^\top-(PB+C^\top PD+S^\top)(R+D^\top PD)^\dagger D^\top\big]\zeta \\ \qquad\qquad +\big[C^\top-(PB+C^\top PD+S^\top)(R+D^\top PD)^\dagger D^\top\big]P\sigma \\ \qquad\qquad -(PB+C^\top PD+S^\top)(R+D^\top PD)^\dagger\rho+Pb+q\Big\}ds+\zeta dW(s), \\ \eta(T)=h. \end{cases}$$

To prove $R+D^\top PD\geq 0$, as well as the sufficiency, we take any $u(\cdot)\in\mathcal{U}[t,T]$, let $X(\cdot)\equiv X(\cdot\,;t,x,u(\cdot))$ be the corresponding state process. Then

$$\begin{aligned} J(t,x;u(\cdot)) &= \mathbb{E}\Big\{\langle HX(T),X(T)\rangle+2\langle h,X(T)\rangle \\ &\qquad +\int_t^T\big[\langle QX,X\rangle+2\langle SX,u\rangle+\langle Ru,u\rangle+2\langle q,X\rangle+2\langle \rho,u\rangle\big]ds\Big\} \\ &= \mathbb{E}\Big\{\langle P(t)x,x\rangle+2\langle \eta(t),x\rangle \\ &\quad +\int_t^T\big\{\langle\big[-PA-A^\top P-C^\top PC-Q \\ &\quad +(PB+C^\top PD+S^\top)(R+D^\top PD)^\dagger(B^\top P+D^\top PC+S)\big]X,X\rangle \\ &\quad +\langle P(AX+Bu+b),X\rangle+\langle PX,AX+Bu+b\rangle \\ &\quad +\langle P(CX+Du+\sigma),CX+Du+\sigma\rangle \\ &\quad +2\langle\big[-A^\top+(PB+C^\top PD+S^\top)(R+D^\top PD)^\dagger B^\top\big]\eta,X\rangle \\ &\quad +2\langle\big[-C^\top+(PB+C^\top PD+S^\top)(R+D^\top PD)^\dagger D^\top\big]P\sigma,X\rangle \\ &\quad +2\langle\big[-C^\top+(PB+C^\top PD+S^\top)(R+D^\top PD)^\dagger D^\top\big]\zeta,X\rangle \\ &\quad +2\langle\big(PB+C^\top PD+S^\top)(R+D^\top PD)^\dagger\rho-Pb-q,X\rangle \\ &\quad +2\langle\zeta,CX+Du+\sigma\rangle+2\langle\eta,AX+Bu+b\rangle \\ &\quad +\langle QX,X\rangle+2\langle SX,u\rangle+\langle Ru,u\rangle+2\langle q,X\rangle+2\langle\rho,u\rangle\big\}ds\Big\} \\ &= \mathbb{E}\Big\{\langle P(t)x,x\rangle+2\langle\eta(t),x\rangle+\int_t^T\Big[\langle P\sigma,\sigma\rangle+2\langle\eta,b\rangle+2\langle\zeta,\sigma\rangle \\ &\quad +\langle(PB+C^\top PD+S^\top)(R+D^\top PD)^\dagger(B^\top P+D^\top PC+S)X,X\rangle \\ &\quad +2\langle(B^\top P+D^\top PC+S)X+B^\top\eta+D^\top\zeta+D^\top P\sigma+\rho,u\rangle \\ &\quad +\langle(R+D^\top PD)u,u\rangle \\ &\quad +2\langle(PB+C^\top PD+S^\top)(R+D^\top PD)^\dagger(B^\top\eta+D^\top\zeta+D^\top P\sigma+\rho),X\rangle\Big]ds\Big\}. \end{aligned}$$

Note that

$$\begin{cases} B^\top P+D^\top PC+S=-(R+D^\top PD)\bar\Theta\equiv-\widehat R\bar\Theta, \\ B^\top\eta+D^\top\zeta+D^\top P\sigma+\rho=-(R+D^\top PD)\bar v\equiv-\widehat R\bar v. \end{cases}$$

Also, one has

$$\begin{aligned}
&\langle (R+D^\top PD)\bar v,\bar v\,\rangle\\
&=\langle\,\widehat R\widehat R^\dagger(B^\top\eta+D^\top\zeta+D^\top P\sigma+\rho),\widehat R^\dagger(B^\top\eta+D^\top\zeta+D^\top P\sigma+\rho)\,\rangle\\
&=\langle (R+D^\top PD)^\dagger(B^\top\eta+D^\top\zeta+D^\top P\sigma+\rho),B^\top\eta+D^\top\zeta+D^\top P\sigma+\rho\,\rangle\,.
\end{aligned}$$

Thus,

$$\begin{aligned}
&J(t,x;u(\cdot))=\mathbb{E}\Big\{\,\langle\,P(t)x,x\,\rangle+2\,\langle\,\eta(t),x\,\rangle+\int_t^T\Big[\,\langle\,P\sigma,\sigma\,\rangle+2\,\langle\,\eta,b\,\rangle+2\,\langle\,\zeta,\sigma\,\rangle\\
&\quad+\langle (PB+C^\top PD+S^\top)(R+D^\top PD)^\dagger(B^\top P+D^\top PC+S)X,X\,\rangle\\
&\quad+2\,\langle (B^\top P+D^\top PC+S)X+B^\top\eta+D^\top\zeta+D^\top P\sigma+\rho,u\,\rangle\\
&\quad+\langle (R+D^\top PD)u,u\,\rangle\\
&\quad+2\,\langle (PB+C^\top PD+S^\top)(R+D^\top PD)^\dagger(B^\top\eta+D^\top\zeta+D^\top P\sigma+\rho),X\,\rangle\Big]ds\Big\}\\
&=\mathbb{E}\Big\{\,\langle\,P(t)x,x\,\rangle+2\,\langle\,\eta(t),x\,\rangle+\int_t^T\Big[\,\langle\,P\sigma,\sigma\,\rangle+2\,\langle\,\eta,b\,\rangle+2\,\langle\,\zeta,\sigma\,\rangle\\
&\quad+\langle\,\bar\Theta^\top\widehat R\widehat R^\dagger\widehat R\bar\Theta X,X\,\rangle-2\,\langle\,\widehat R(\bar\Theta X+\bar v),u\,\rangle+\langle\,\widehat Ru,u\,\rangle+2\,\langle\,\bar\Theta^\top\widehat R\widehat R^\dagger\widehat R\bar v,X\,\rangle\Big]ds\Big\}\\
&=\mathbb{E}\Big\{\,\langle\,P(t)x,x\,\rangle+2\,\langle\,\eta(t),x\,\rangle+\int_t^T\Big[\,\langle\,P\sigma,\sigma\,\rangle+2\,\langle\,\eta,b\,\rangle+2\,\langle\,\zeta,\sigma\,\rangle\\
&\quad+\langle\,\widehat R\bar\Theta X,\bar\Theta X\,\rangle-2\,\langle\,\widehat R(\bar\Theta X+\bar v),u\,\rangle+\langle\,\widehat Ru,u\,\rangle+2\,\langle\,\widehat R\bar\Theta X,\bar v\,\rangle\Big]ds\Big\}\\
&=\mathbb{E}\Big\{\,\langle\,P(t)x,x\,\rangle+2\,\langle\,\eta(t),x\,\rangle+\int_t^T\Big[\,\langle\,P\sigma,\sigma\,\rangle+2\,\langle\,\eta,b\,\rangle+2\,\langle\,\zeta,\sigma\,\rangle\\
&\quad-\langle (R+D^\top PD)^\dagger(B^\top\eta+D^\top\zeta+D^\top P\sigma+\rho),B^\top\eta+D^\top\zeta+D^\top P\sigma+\rho\,\rangle\\
&\quad+\langle (R+D^\top PD)(u-\bar\Theta X-\bar v),u-\bar\Theta X-\bar v\,\rangle\Big]ds\Big\}\\
&=J\big(t,x;\bar\Theta(\cdot)\bar X(\cdot)+\bar v(\cdot)\big)+\mathbb{E}\int_t^T\langle (R+D^\top PD)(u-\bar\Theta X-\bar v),u-\bar\Theta X-\bar v\,\rangle\,ds.
\end{aligned}$$

Consequently,

$$\begin{aligned}
&J(t,x;\bar\Theta(\cdot)X(\cdot)+v(\cdot))\\
&=J(t,x;\bar\Theta(\cdot)\bar X(\cdot)+\bar v(\cdot))+\mathbb{E}\int_t^T\langle (R+D^\top PD)(v-\bar v),v-\bar v\,\rangle\,ds.
\end{aligned}$$

Hence,

$$J(t,x;\bar\Theta(\cdot)\bar X(\cdot)+\bar v(\cdot))\le J(t,x;\bar\Theta(\cdot)X(\cdot)+v(\cdot)),\quad\forall v(\cdot)\in\mathcal{U}[t,T],$$

if and only if

$$R+D^\top PD\ge 0,\quad \text{a.e. } s\in[t,T].$$

This completes the proof. $\square$

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REFERENCES

[1] M. G. Crandall and P. L. Lions, Viscosity solutions of Hamilton-Jacobi equations, *Trans. Amer. Math. Soc.*, **277** (1983), 1–42.

[2] L. C. Evans and R. F. Gariepy, *Measure Theory and Fine Properties of Functions*, CRC Press, Boca Raton, FL, 1992.

[3] W. H. Fleming and H. M. Soner, *Controlled Markov Processes and Viscosity Solutions*, Springer-Verlag, New York, 1993.

[4] A. Gary, D. Greenhalgh, L. Hu, X. Mao and J. Pan, A stochastic differential equation SIS epidemic model, *SIAM J. Appl. Math.*, **71** (2011), 876–902.

[5] S. He, J. Wang and J. Yan, *Semimartingale Theory and Stochastic Calculus*, Science Press and CRC Press, Beijing, 1992.

[6] I. Karatzas and S. Shreve, *Brownian Motion and Stochastic Calculus*, Springer-Verlag, New York, 1988, 47–127.

[7] E. Pardoux and S. Peng, Adapted solution of backward stochastic differential equations, *Systems Control Lett.*, **14** (1990), 55–61.

[8] S. Peng, A general stochastic maximum principle for optimal control problems, *SIAM J. Control Optim.*, **28** (1990), 966–979.

[9] J. Sun, X. Li and J. Yong, Open-loop and closed-loop solvabilities for stochastic linear quadratic optimal control problems, *SIAM J. Control Optim.*, **54** (2016), 2274–2308.

[10] J. Sun and J. Yong, Linear quadratic stochastic differential games: Open-loop and closed-loop saddle points, *SIAM J. Control Optim.*, **52** (2014), 4082–4121.

[11] E. Tornatore, S. M. Buccellato and P. Vetro, Stability of a stochastic SIR system, *Physica A*, **354** (2005), 111–126.

[12] J. Yong and X. Y. Zhou, *Stochastic Controls: Hamiltonian Systems and HJB Equations*, Springer-Verlag, New York, 1999.

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