Norwegian University of Science and Technology
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Exercise set 3

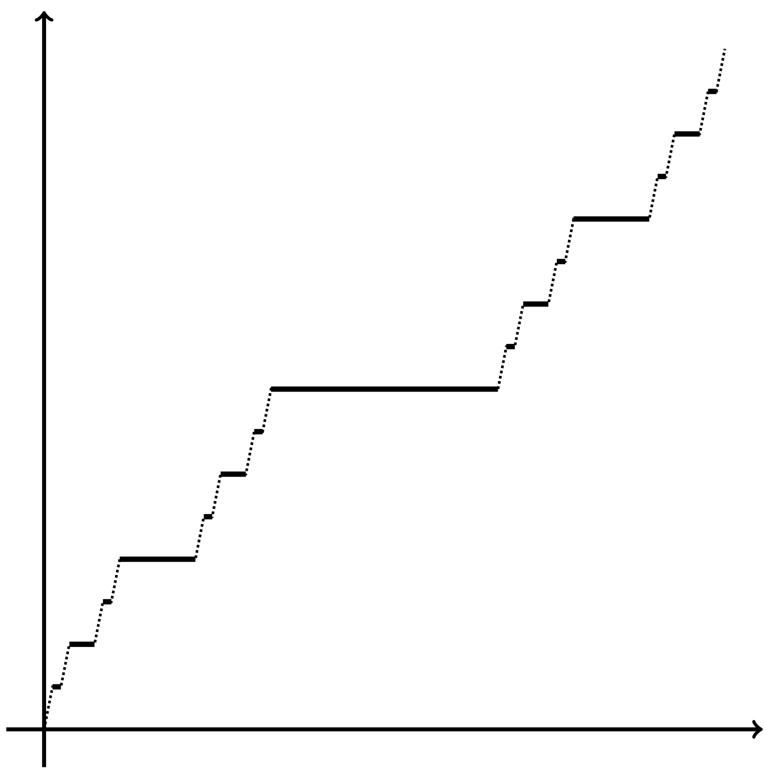
$\boxed{1}$ Recall the Cantor set from exercise sheet 1, which is constructed from the unit interval by removing first the central interval $(1/3, 2/3)$, then the two intervals $(1/9, 2/9)$ and $(7/9, 8/9)$, then four (central) intervals of length $1/27$, and so on.

We now define *Lebesgue's singular function* (sometimes called *Devil's staircase*) by setting

$$f(x) = \begin{cases} 1/2 & \text{for } 1/3 \le x \le 2/3, \\ 1/4 & \text{for } 1/9 \le x \le 2/9, \\ 3/4 & \text{for } 7/9 \le x \le 8/9, \\ \vdots & \end{cases}$$

Or, using again ternary expansions,[1] we have

$$f(x) = \sum_{k=1}^{\infty} \lfloor a_k/2 \rfloor 2^{-k} \text{ for } x = \sum_{k=1}^{\infty} a_k 3^{-k}.$$



[1] Here we have to use finite expansions whenever possible.

a) Show that f is continuous.
Hint: either show that f is the uniform limit of a sequence of continuous functions, or use the fact that the function f is a monotone continuation of the function defined in exercise 1c) on exercise sheet 1.

b) Show that f is differentiable with $f'(x) = 0$ in every point $x \notin C$.

c) Conclude that $f \notin H^1(0,1)$.

$\boxed{2}$ The *brachistochrone problem*[2] is one of the classical problems in the calculus of variations, and was posed (and solved) by Bernoulli in 1696. Given two points A and B, it asks for the path from A to B along which an object would slide in the shortest possible time, if it is originally at rest and is only accelerated by gravity (we disregard friction).

We can assume without loss of generality that $A = (0,0)$. In order to simplify the notation, we will assume moreover that the y-axis points downwards and that the point B is situated at $B = (a,b)$ with $a > 0$ and $b > 0$. We can then write the path as the graph of a function $y \colon [0,a] \to \mathbb{R}$ with $y(0) = 0$ and $y(a) = b$.[3]

In the following, we will derive an explicit formula for the travel time along the path y. To that end, we will denote by v the speed of the object.

a) Using the principle of conservation of energy, show that v and y satisfy the relation

$$v(x) = \sqrt{2gy(x)},$$

where g is the gravitational acceleration at the earth's surface. Show moreover that $y(x) \geq 0$ for all x.

b) The length of the path between 0 and x is given by $s(x) = \int_0^x \sqrt{1 + y'(\hat{x})^2}\, d\hat{x}$. Use this and the formula for the speed in order to show that the travel time along the path y is given by

$$T(y) = \int_0^a \frac{\sqrt{1 + y'(x)^2}}{\sqrt{2gy(x)}}\, dx.$$

As a consequence, we have obtained a reformulation of the brachistochrone problem as a typical variational problem of the form: "Minimise $T(y)$ subject to the constraints $y(0) = 0$ and $y(a) = b$."

c) Formulate (and simplify) the Euler–Lagrange equation for the brachistochrone problem and show that the solution satisfies

$$y + y(y')^2 = C$$

for some constant $C > 0$.

d) Show that the solution can be parameterised as

$$t \mapsto \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = C \begin{pmatrix} t - \frac{1}{2}\sin 2t \\ \frac{1}{2} - \frac{1}{2}\cos 2t \end{pmatrix}.$$

[2] The term is composed of the greek words *brachistos* meaning shortest and *chronos* meaning time. Thus it literally translates to *shortest time problem*.

[3] Try to argue why this must be the case!