NTNU
Norwegian University of
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Week 44: Lecture 1

Conditional multivariate Gaussian distributions

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Information

- Minutes from reference group meeting 2 available under "course information".
- We aim to conduct a physical exam with letter grades, but faculty/NTNU will assess the infection situation in the next two weeks.
- The backup plan is a digital home exam with pass/fail.
- Physical guidance on October 27 and November 3 in R2, and November 2 in S21/Smia.
- Looking into possibility for guidance also on November 5 or 6.
- Online intro course to R available here: `https://digit.ntnu.no/courses/course-v1:NTNU+IMF001+2020/course/`

Section 2.3 (Note)

Example 1

$$\boldsymbol{X} \sim \mathcal{N}_2\left(\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}1 & 0.5\\0.5 & 2\end{bmatrix}\right).$$

a) Determine the distribution of X_2.

b) Determine the distribution of $\bar{X} = 0.5(X_1 + X_2)$.

Theorem (Theorem 1)

Assume $\boldsymbol{X} \sim \mathcal{N}_n(\boldsymbol{\mu}, \Sigma)$ *and* $\mathbf{L}$ *is the Cholesky factor of* Σ *(i.e.,* $\Sigma = \mathbf{L}\mathbf{L}^{\mathrm{T}}$*). Then*

1) $\boldsymbol{X} \sim \mathcal{N}_n(\boldsymbol{\mu}, \Sigma)$ *implies* $\boldsymbol{Z} = \mathbf{L}^{-1}(\boldsymbol{X} - \boldsymbol{\mu}) \sim \mathcal{N}_n(\mathbf{0}, \mathbf{I})$.
2) $\boldsymbol{Z} \sim \mathcal{N}_n(\mathbf{0}, \mathbf{I})$ *implies* $\boldsymbol{X} = \mathbf{L}\boldsymbol{Z} + \boldsymbol{\mu} \sim \mathcal{N}_n(\boldsymbol{\mu}, \Sigma)$.

Section 2.4 (Note)

Theorem (Theorem 2)

If

$$\boldsymbol{X} = (\boldsymbol{X}_\mathrm{A}, \boldsymbol{X}_\mathrm{B}) \sim \mathcal{N}_{n_\mathrm{A}+n_\mathrm{B}}\left(\begin{bmatrix}\boldsymbol{\mu}_\mathrm{A}\\ \boldsymbol{\mu}_\mathrm{B}\end{bmatrix}, \begin{bmatrix}\Sigma_\mathrm{AA} & \Sigma_\mathrm{AB}\\ \Sigma_\mathrm{BA} & \Sigma_\mathrm{BB}\end{bmatrix}\right),$$

where $\boldsymbol{X}_\mathrm{A}$ *is* n_A*-dimensional and* $\boldsymbol{X}_\mathrm{B}$ *is* n_B*-dimensional, then*

$$\boldsymbol{X}_\mathrm{A}|\boldsymbol{X}_\mathrm{B} = \boldsymbol{x}_\mathrm{B} \sim \mathcal{N}_{n_\mathrm{A}}(\boldsymbol{\mu}_\mathrm{C}, \Sigma_\mathrm{C}),$$

where

$$\begin{aligned}\boldsymbol{\mu}_\mathrm{C} &= \boldsymbol{\mu}_\mathrm{A} + \Sigma_\mathrm{AB}\Sigma_\mathrm{BB}^{-1}(\boldsymbol{x}_\mathrm{B} - \boldsymbol{\mu}_\mathrm{B})\\ \Sigma_\mathrm{C} &= \Sigma_\mathrm{AA} - \Sigma_\mathrm{AB}\Sigma_\mathrm{BB}^{-1}\Sigma_\mathrm{BA}.\end{aligned}$$

Example 2

Assume $-1 < \rho < 1$ and $\sigma^2 > 0$, and let

$$(X_1, X_2, X_3) \sim \mathcal{N}_3\left(\begin{bmatrix}0\\0\\0\end{bmatrix}, \sigma^2\begin{bmatrix}1 & \rho & \rho^2\\ \rho & 1 & \rho\\ \rho^2 & \rho & 1\end{bmatrix}\right).$$

Determine the distribution of $X_1 | X_3 = x_3$.

Example 3

Assume $-1 < \rho < 1$ and $\sigma^2 > 0$, and let

$$(X_1, X_2, X_3) \sim \mathcal{N}_3\left(\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \sigma^2 \begin{bmatrix} 1 & \rho & \rho^2 \\ \rho & 1 & \rho \\ \rho^2 & \rho & 1 \end{bmatrix}\right).$$

Determine the distribution of $X_1 | X_2 = x_2, X_3 = x_3$.

Section 2.5 (Note)

Simulation from $\mathcal{N}_n(\boldsymbol{\mu}, \Sigma)$

Input:

n: dimension

$\boldsymbol{\mu}$: mean vector

Σ: covariance matrix

Algorithm:

1. calculate Choleksy factorization $\Sigma = \mathbf{L}\mathbf{L}^{\mathrm{T}}$.
2. for $i = 1 \ldots n$
3. draw $z_i \sim \mathcal{N}(0, 1)$
4. end
5. set $\boldsymbol{x} = \mathbf{L}\boldsymbol{z} + \boldsymbol{\mu}$

Output: $\boldsymbol{x}$ is a simulation from $\mathcal{N}_n(\boldsymbol{\mu}, \Sigma)$.

Section 3 (Note)

Definition (Def. 1)

The stochastic process $\{B(t) : t \geq 0\}$ with state space $\mathbb{R}$ is called **Brownian motion** with **variance parameter** $\sigma^2 > 0$ if

1) $B(s+t) - B(s) \sim \mathcal{N}(0, t\sigma^2)$ for $s \geq 0$ and $t > 0$.
2) for $0 \leq t_1 < t_2 \leq t_3 < t_4$,

$$B(t_2) - B(t_1) \quad \text{and} \quad B(t_4) - B(t_3)$$

   are independent.
3) $B(0) = 0$ (and the realizations are continuous).

Example 4

We consider simulations from Brownian motions with

1. $\sigma^2 = 1$ and $t \in [0, 1]$
2. $\sigma^2 = 1$ and $t \in [0, 10]$
3. $\sigma^2 = 1/10$ and $t \in [0, 100]$
4. $\sigma^2 = 1/100$ and $t \in [0, 1000]$

Example 5

Let $\{B(t) : t \geq 0\}$ be Brownian motion with variance parameter $\sigma^2 > 0$. Derive an expression for the function $C(t, s) = \mathrm{Cov}[B(t), B(s)]$.