Norwegian University of Science and Technology
Department of Mathematical Sciences

Exercise set 2

$\boxed{1}$ Given the problem

$$-u_{xx} + u_x = f(x), \qquad 0 < x < 1,$$
$$u_x(0) + u(0) = g_0, \qquad u(1) = 0.$$

Modify your code from Exercise 1, Problem 4, to handle this problem (only the boundary condition in $x = 0$ has changed.). Check the order of the approximation. You may try both first and second order approximations.

$\boxed{2}$ Consider the boundary value problem

$$-u_{xx} + u_x + u = f, \qquad \text{on } [0,1], \qquad u_x(0) = u_x(1) = 0.$$

a) Set up the weak formulation of the problem.

b) Use Lax-Milgram theorem to prove that the problem has a unique solution.

c) Given an equidistributed mesh $x_i = ih$, $h = 1/M$ on the interval $[0,1]$ and the linear finite element space

$$X_h^1 = \{v \in C^0(0,1);\ v|_{x_{i-1},x_i} \in \mathbb{P}_1\}.$$

that is the space of continuous functions, that are also linear on each element $K_i = (x_i, x_{i+1})$. Notice that $X_h^1 \subset H^1(0,1)$.

Now use the Galerkin method to find an approximation to the problem.

- Set up the stiffness matrix A_h.
- Find the load vector F_h.
- Implement the method.
- Test it on a freely chosen test problem.

d) Fint an error bound for the numerical solution,~~and verify it numerically.~~ (This topic will be lectured on Tuesday 21.01).

3 Before doing this exercise, read section 2.3 in the note by Charles Curry.

Consider the problem

$$-u_{xx} = f, \qquad u(0) = u(1) = 0.$$

This exercise is about how to solve this problem with a finite element method on the same equidistributed mesh as in in the previous problem, but now with a finite element space

$$X_h^2 = \{v \in C^0(\Omega);\ v|_{(x_{i-1},x_i)} \in \mathbb{P}_2\}.$$

The space X_h^2 can be constructed as follows: At each element (x_{i-1}, x_i) choose the points x_{i-1}, x_i and the midpoint $(x_{i-1} + x_i)/2$, and use the corresponding cardinal functions as basis functions.

a) Let $M = 2$ and make a figure of the basis functions in this case.

b) Set up the element stiffness matrix and the element load vector.

c) Set up the complete stiffness matrix and load vector (by hand) for some arbitrary M. Use $f = 1$.

Solve the problem numerically for $M = 5$ (for example). If your calculations are correct, the scheme solves this particular problem exactly (why?)

Reminder: Given the gridpoints $(x_i)_{i=0}^n$, the corresponding cardinal-functions are given by

$$\ell_i(x) = \prod_{j=0, j\neq i}^{n} \frac{x - x_j}{x_i - x_j} \in \mathbb{P}_2$$

satisfying

$$\ell_i(x_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}.$$