TMA 4190 Introduction to Topology
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Lecture 21[1]

21. Winding Numbers and the Borsuk-Ulam Theorem

Today we are going to exploit intersection numbers and degree modulo 2 a bit further and prove a famous theorem. As a starter, we introduce a useful new invariant.

Let X be a **compact**, **connected** smooth manifold, and let

$$f\colon X \to \mathbb{R}^n$$

be a smooth map. We assume $\dim X = n - 1$.

Let z be a point of $\mathbb{R}^n$ **not** lying in the image $f(X)$. We would like to understand how $f(x)$ **winds around** z. To do this, we look at the unit vector

$$u(x) = \frac{f(x) - z}{|f(x) - z|}.$$

It points in the direction from z to $f(x)$ and has length one.

With z **fixed** and x varying, we can consider u as a map

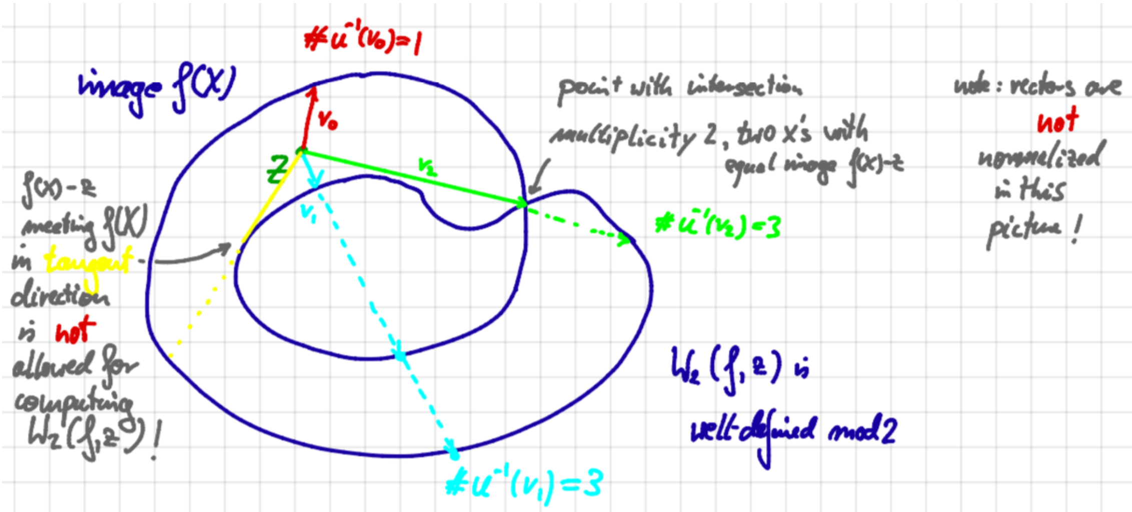
$$u\colon X \to S^{n-1}.$$

We would like to know how often this vector points in a given direction, i.e. how often $u(x)$ has a given value. We learned from the previous lecture, that the degree of u is an invariant that encodes this information. For, we know that, modulo 2, $\#u^{-1}(y)$ is constant for **regular values** y of u, i.e. where $y - z$ hits $f(X)$ transversally, and is equal $\deg_2(u)$ by definition of the latter. (We will see in the proof of our main theorem today, that y being a regular value of u means that the line through z and y must be tranversal to $f(X)$.)

We give this number a name and call it the **winding number of f around z**. We denote it by

$$W_2(f,z) := \deg_2(u).$$

[1]Following the books of Guillemin and Pollack: Differential Topology, and Milnor: Topology from the differentiable viewpoint.



The goal for today is to prove the following famous result:

Borsuk-Ulam Theorem

Let $f\colon S^k \to \mathbb{R}^{k+1} \setminus \{0\}$ be a smooth map, and suppose that f is odd, i.e. satisfies the symmetry condition

$$f(-x) = -f(x) \text{ for all } x \in S^k. \tag{1}$$

Then $W_2(f,0) = 1$.

In other words, any map that is odd, i.e. symmetric around the origin, must wind around the origin an **odd number of times**.

A we will see below, there is a nice interpretation of this result for the meteorologists among us: At any given time, there are **two antipodal points** on the **Earth** that have the **same temperature and pressure**. (Assuming temperature and pressure vary smoothly on the Earth.)

Before we approach the proof, we observe:

Equivalent formulation of BUT

The Borsuk-Ulam theorem is equivalent to the following assertion:
If $f\colon S^k \to S^k$ is a map which sends antipodal points to antipodal points, i.e. $f(-x) = -f(x)$, then $\deg_2(f) = 1$.

Proof: Assume BUT is true: given a smooth map $f\colon S^k \to S^k$ with $f(-x) = -f(x)$, we can consider it as a map $f\colon S^k \to S^k \subset \mathbb{R}^{k+1}$. Then we have $1 = W_2(f,0) = \deg_2(f/|f|) = \deg_2(f)$.

Assume the assertion is true: given a smooth map $f\colon S^k \to \mathbb{R}^{k+1} \setminus \{0\}$ with $f(-x) = -f(x)$, then $f/|f|$ is a well-defined smooth map $f/|f|\colon S^k \to S^k$. Hence $1 = \deg_2(f/|f|) = W_2(f,0)$ by definition of winding number. **QED**

As a slogan, we can remember the Borsuk-Ulam Theorem for a smooth map $f\colon S^k \to S^k$ as follows:

BUT in a nutshell

If f is odd, its degree is odd.

In order to prove the theorem, we first need to investigate the relationship of winding numbers and boundaries:

Winding numbers and boundaries

Suppose that X is the **boundary** ∂D of a compact manifold D of dimension n with boundary, and let $F\colon D \to \mathbb{R}^n$ be a smooth map extending $f\colon X \to \mathbb{R}^n$, i.e. $\partial F = f$. Suppose that z is a **regular value** of F that does **not** belong to the image of f.
Then $F^{-1}(z)$ is a **finite set**, and

$$W_2(f,z) = \#F^{-1}(z) \mod 2.$$

In other words, f winds X around z as often as F hits z, at least modulo 2.

Proof:

First case: $F^{-1}(z) = \emptyset$, i.e. $\#F^{-1}(z) = 0$.

In this case, the map

$$u\colon X = \partial D \to S^{n-1},\ x \mapsto \frac{f(x) - z}{|f(x) - z|}$$

can be extended to a map

$$D \to S^{n-1},\ x \mapsto \frac{F(x) - z}{|F(x) - z|}$$

since $F(x) - z$ is never 0. Hence by the **Boundary Theorem**,

$$W_2(f,z) = \deg_2(u) = 0 \mod 2.$$

Second case: $F^{-1}(z) \neq \emptyset$.

Since D is **compact** and of dimension n, $F^{-1}(z)$ is a zero-dimensional closed submanifold of D, and hence compact and hence a **finite set**. Suppose

$$F^{-1}(z) = \{y_1, \ldots, y_m\}.$$

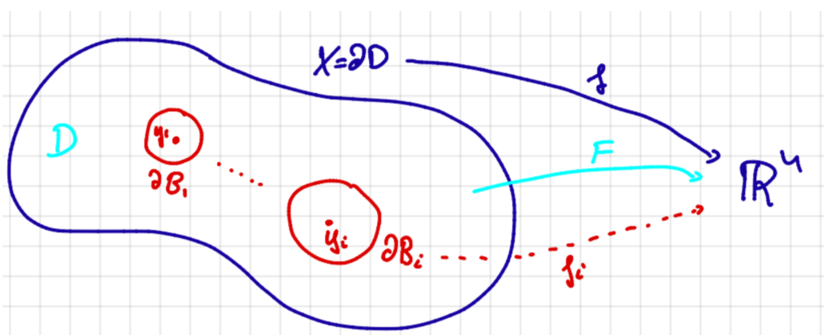
Then we can choose local parametrizations around each y_i in D and let B_i be the image of a closed ball in $\mathbb{R}^n$ around y_i. Since z is a **regular value**, the **Stack of Records Theorem** shows that $F^{-1}(z)$ is discrete and disjoint to $X = \partial D$. Thus we can choose the radii of these balls small enough such that

$$B_i \cap B_j = \emptyset \text{ and } B_i \cap X = \emptyset \text{ for all } i \neq j, \text{ and } i = 1, \ldots, m.$$

We define

$$f_i := F_{|\partial B_i} \colon \partial B_i \to \mathbb{R}^n.$$

to be the restriction of F to ∂B_i.



Now we observe that the subset

$$\tilde{D} := D \setminus (\cup_i \mathrm{Int}(B_i))$$

is a closed submanifold of D with boundary

$$\partial \tilde{D} = \partial D \,\dot{\cup}\, \partial B_1 \dot{\cup} \,\cdots\, \dot{\cup}\, \partial B_m$$

the disjoint union of the boundaries of D and the B_i's.

By the choice of the B_i's, we have $F^{-1}(z) \cap \tilde{D} = \emptyset$. Hence

$$F^{-1}(z) \cap \tilde{D} = (F_{|\tilde{D}})^{-1}(z) = \emptyset.$$

Hence the **winding number of $\partial F_{|\tilde{D}}$ at z is zero**.

Since degrees and hence winding numbers are **additive with respect to connected components** this yields

$$0 = W_2(\partial F_{|\tilde{D}}, z) = W_2(f, z) + W(f_1, z) + \cdots + W_2(f_m, z) \quad \mod 2.$$

Since we are working **modulo** 2, this implies

$$W_2(f, z) = W(f_1, z) + \cdots + W_2(f_m, z) \quad \mod 2.$$

Now it **remains to show** $W_2(f_i, z) = 1$ for each $i = 1, \dots, m$. For then

$$\#F^{-1}(z) = m = \sum_i W_2(f_i, z) = W(f, z) \quad \mod 2.$$

Since z is a **regular value**, dF_{y_i} is an isomorphism (remember $\dim D = n$). Thus, by the **Inverse Function Theorem**, we can choose the radius of B_i small enough such that $F_{|B_i}$ is a **diffeomorphism onto its image** (which contains z). By continuity, this implies also that $f_i = \partial F_{|B_i}$ is one-to-one onto the boundary of $F(B_i)$.

By possibly rescaling and translating, we are **reduced to showing**:

Let B be the closed unit ball in $\mathbb{R}^n$ and $F\colon B \to B$ be a diffeomorphism. Let $f = \partial F\colon S^{n-1} \to S^{n-1}$. Then

$$\#F^{-1}(0) = W(f, 0) = 1 \mod 2.$$

But this is obvious, since $W(f, 0) = \deg_2(f) = \#f^{-1}(v) = 1$ for any $v \in S^{n-1}$. **QED**

Now we are ready to attack the proof of BUT.

Proof of the Borsuk-Ulam Theorem: The proof is by induction.

The case $k = 1$:

By the previous remark, to show that theorem is equivalent to showing that a map $f\colon S^1 \to S^1$ with $f(-x) = -f(x)$ has $\deg_2(f) = 1$.

The **idea** is that, given any smooth map $f\colon S^1 \to S^1$, we can **lift f locally** using the Stack of Records Theorem and then **patch the pieces together** to get a smooth map

$$g\colon \mathbb{R} \to \mathbb{R} \text{ such that } p(g(t)) = f(p(t))$$

where p is the (covering) map

$$p\colon \mathbb{R} \to S^1,\ t \mapsto e^{2\pi i t}.$$

To make g compatible with f in the above sense, we must have

$$p(g(t+1)) = f(p(t+1)) = f(p(t)) = p(g(t)) \Rightarrow p(g(t+1) - g(t) = 1.$$

Since $p(t) = 1$ if and only if $t \in \mathbb{Z}$, we must have $g(t+1) - g(t) \in \mathbb{Z}$. Since the function $t \mapsto g(t+1) - g(t)$ takes only values in the discrete space $\mathbb{Z}$, it is **locally constant**. Since $\mathbb{R}$ is **connected**, it must be **constant**. Hence q is a **fixed integer** depending only on f. In other words, for all $t \in \mathbb{R}$, we have

$$g(t+1) = g(t) + q \text{ for some } \textbf{fixed } q \in \mathbb{Z}.$$

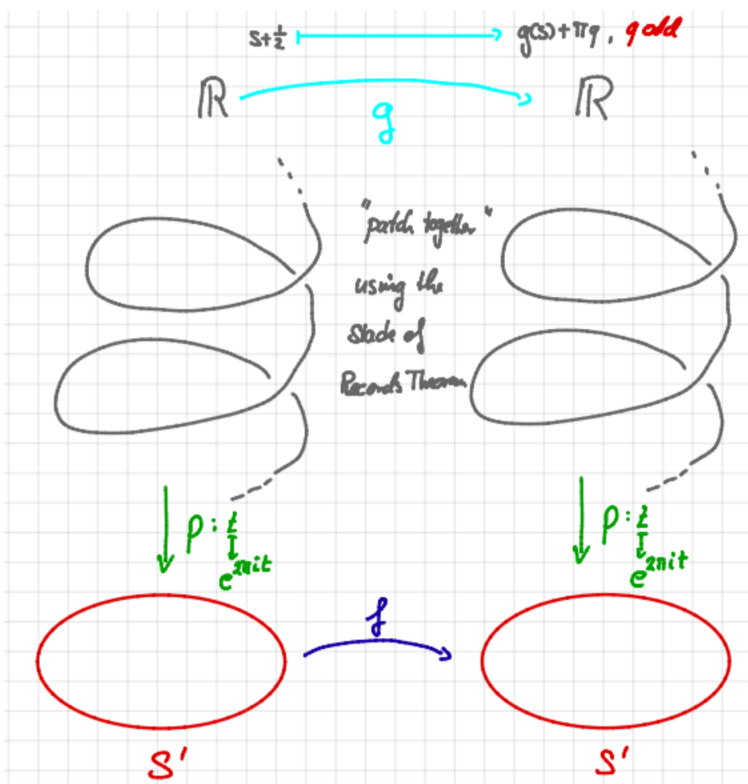
Then we have $\deg_2(f) = q$, since q tells us **how often** f hits the same point when t moves from 0 to 1, or vary t around S^1 once.

When **f is odd**, then

$$\begin{aligned} p(g(t+1/2)) &= f(p(t+1/2)) = f(-p(t)) = -f(p(t)) \\ &= -p(g(t)) = p(g(t) + q/2) \text{ for some fixed } \textbf{odd } q \in \mathbb{Z}. \end{aligned}$$

(For $p(s_1) = -p(s_2) \iff e^{2\pi i s_1} = -e^{2\pi i s_2} = e^{2\pi i s_2} e^{q\pi i}$ for some odd $q \in \mathbb{Z}$, and hence $p(s_1) = -p(s_2) \iff s_1 = s_2 + q/2$ for this odd q.)

Hence $\deg_2(f) = q = 1 \mod 2$.



Aside: There is a deeper general reason why this works. For $\mathbb{R}$ is a **(universal) covering space** of S^1, and continuous paths can always be lifted to a cavering space. You will learn more about this phenomenon later.

Induction step: Assume the theorem is true for $k-1$ and $k \geq 2$. Let $f\colon S^k \to \mathbb{R}^{k+1} \setminus \{0\}$ satisfy the symmetry condition (1). We consider $^{k-1}$ to be the equator of S^k, embedded by

$$(x_1, \ldots, x_k) \mapsto (x_1, \ldots, x_k, 0).$$

The idea is to compute $W_2(f,0)$ by **counting how often f intersects a line L** in $\mathbb{R}^{k+1}$. By choosing L disjoint from the image of the equator, we can use the inductive hypothesis to show that the equator winds around L an odd number of times. Finally, it is easy to calculate the intersection of f with L once we know the behavior of f on the equator.

Let $g\colon S^{k-1} \to \mathbb{R}^{k+1} \setminus \{0\}$ be the restriction of f to the equator. By Sard's Theorem, we can choose a value $y \in S^k$ which is **regular for both** smooth maps

$$\frac{g}{|g|}\colon S^{k-1} \to S^k, \text{ and } \frac{f}{|f|}\colon S^k \to S^k.$$

The **symmetry condition** implies that **y is regular for both** these maps if and only if **$-y$ is regular for both** maps , since the derivatives at preimages of y and $-y$ just differ by multpying with (-1).

Since $\dim S^{k-1} < \dim S^k$, the only way y can be a **regular value of $\frac{g}{|g|}$** is when y is **not** in the image. Hence neither y nor $-y$ are in the image of $\frac{g}{|g|}$.

Thus, for the **line** $L := \mathbb{R} \cdot y = \operatorname{span}(y)$, we have

$$y \text{ is a regular value of } g \iff \operatorname{Im}(g) \cap L = \emptyset.$$

That y is **regular for $\frac{f}{|f|}$** means by definition

$$\operatorname{Im}\left(d\left(\frac{f}{|f|}\right)_x\right) = T_y(S^k).$$

The tangent space to S^k at y is the orthogonal complement of the line pointing in direction of y. The map $x \mapsto \frac{f(x)}{|f(x)|}$ is the composite of f and $x \mapsto x/|x|$ (which is smooth in dimensions $k \geq 2$).

The derivative of the latter map satisfies

$$\operatorname{Im}(d(x/|x|)_x) = (\operatorname{span}(x))^\perp \subset \mathbb{R}^{k+1}, \text{ i.e. } \operatorname{Ker}(d(x/|x|)_x) = Span(x).$$

For $f/|f|$, this means

$$\operatorname{Ker}\left(d\left(\frac{f}{|f|}\right)_x\right) = \operatorname{span}(f(x)) \cap \operatorname{Im}(df_x).$$

Thus

$$\begin{aligned}
\operatorname{Im}\left(d\left(\frac{f}{|f|}\right)_x\right) = T_y(S^k) &\iff \operatorname{Ker}\left(d\left(\frac{f}{|f|}\right)_x\right) = \{0\} \\
&\iff \operatorname{span}(f(x)) \cap \operatorname{Im}(df_x) = \{0\} \\
&\iff \operatorname{span}(f(x)) \not\subset \operatorname{Im}(df_x) \\
&\iff L + \operatorname{Im}(df_x) = \mathbb{R}^{k+1} \\
&\iff f \pitchfork L.
\end{aligned}$$

Summarzing the argument, we have obtained

$$y \text{ is a regular value of } \frac{f}{|f|} \iff f \pitchfork L. \tag{2}$$

Now we are going to exploit these two observations for calculating $W_2(f,0)$. By definition, we have

$$W_2(f,0) = \deg_2\left(\frac{f-0}{|f-0|}\right) = \deg_2\left(\frac{f}{|f|}\right) = \#\left(\frac{f}{|f|}\right)^{-1}(y) \quad \bmod 2.$$

By **symmetry**, we have

$$\#\left(\frac{f}{|f|}\right)^{-1}(y) = \#\left(\frac{f}{|f|}\right)^{-1}(-y).$$

From (2) we know

$$\begin{aligned}
f^{-1}(L) &= \{x \in S^k : f(x) \in L\} \\
&= \{x \in S^k : \frac{f(x)}{|f(x)|} = \pm y\} \\
&= \left(\frac{f}{|f|}\right)^{-1}(y) \cup \left(\frac{f}{|f|}\right)^{-1}(-y).
\end{aligned}$$

Thus

$$\#\left(\frac{f}{|f|}\right)^{-1}(y) = \frac{1}{2}\#f^{-1}(L).$$

Hence we need to **calculate the number** $\frac{1}{2}\#f^{-1}(L)$, at least modulo 2.

By **symmetry**, we can do this on the **upper hemisphere** S^k_+ of S^k, i.e. the points on S^k with $x_{k+1} \geq 0$. Let f_+ be the restriction of f to S^k_+. By the choice of y, L does not meet the equator, and hence no point on the equator is in $f^{-1}(L)$. This implies

$$\frac{1}{2}\#f^{-1}(L) = \#f_+^{-1}(L).$$

The upper hemisphere is a manifold with **boundary**

$$\partial S^k_+ = \{x = (x_1, \ldots, x_{k+1}) : \sum_i x_i^2 = 1 \text{ and } x_{k+1} = 0\} = S^{k-1}$$

being the equator.

Now we would like to apply the previous theorem to the f_+ and $g = \partial f_+$ and use the induction hypothesis. But the target of f_+ has dimension $k+1$, whereas for both the theorem and the induction hypothesis we need as target a Euclidean space of dimension k. So we need to fix this.

The key is that the **orthogonal complement** of L in $\mathbb{R}^{k+1}$, denoted by V, is a vector space of dimension k. By choosing a basis of V, we can identify it with $\mathbb{R}^k$.

To complete the argument, let $\pi\colon \mathbb{R}^{k+1} \to V$ be the orthogonal projection onto V. Since g is symmetric and π is linear,

$$\pi \circ g\colon S^{k-1} \to V \text{ is } \textbf{symmetric} : \pi(g(-x)) = \pi(-g(x)) = -\pi(g(x)).$$

Moreover, we have

$$\pi(g(x)) = 0 \iff g(x) \in L, \text{ hence } \pi(\mathbf{g}(\mathbf{x})) \neq \mathbf{0} \text{ for all } x \in S^{k-1}$$

by the definition of π and the choice of L.

Thus, after choosing a basis for V, we can consider $\pi \circ g$ as a map

$$\pi \circ g\colon S^{k-1} \to \mathbb{R}^k \setminus \{0\}.$$

Now we **apply the induction hypothesis** to $\pi \circ g$ and get $W_2(\pi \circ g, 0) = 1$.

To finish, **recall** $f_+ \pitchfork L$ and hence for

$$\pi \circ f_+\colon S^k \to V, \ (\pi \circ f_+) \pitchfork \{0\}.$$

In other words, 0 **is a regular value** of $\pi \circ f_+$.

Hence we can **apply the previous theorem** to $\pi \circ f_+$ and its boundary map $\partial(\pi \circ f_+) = \pi \circ g$ to get

$$W_2(\pi \circ g, 0) = \#(\pi \circ f_+)^{-1}(0).$$

But, by the choice of L, we have

$$\pi(f_+(x)) = 0 \iff f_+(x) \in L, \text{ and hence } (\pi \circ f_+)^{-1}(0) = f_+^{-1}(L).$$

Thus

$$W_2(f, 0) = \# f_+^{-1}(L) = W_2(\pi \circ g, 0) = 1.$$

QED

Remark: Going back to the definition of $W_2(f, z)$ and the picture at the beginning, we learn from the proof, in particular, that lines **tangential** to $f(X)$ are **not allowed** for calculating $W_2(f, z)$.

Let us look at some of the consequences of this theorem.

Corollary 1 of BUT

If $f \colon S^k \to \mathbb{R}^{k+1} \setminus \{0\}$ is symmetric about the origin, i.e. $f(-x) = -f(x)$, then f intersects every line through 0 at least once.

Proof: Let L be a line in $\mathbb{R}^{k+1}$ through the origin. If f never hits L, then $\# f^{-1}(L) = 0$ and $f \pitchfork L$. By repeating the above proof using this f and L for calculating $W_2(f, 0)$, we would get the contradiction

$$W_2(f, 0) = \# f^{-1}(L) = 0.$$

QED

Corollary 2 of BUT

Any k smooth odd real-valued functions $f_1, \ldots, f_k$ on S^k must have a common zero.

Proof: If they did not have a common zero, then we can form the smooth odd map

$$f := (f_1, \ldots, f_k, 0) \colon S^k \to \mathbb{R}^{k+1} \setminus \{0\}.$$

Then we can apply Corollary 1 of BUT to f and L being the x_{k+1}-axis. Hence f intersects L at least once. But x with $f(x) \in L$ is a common zero of the $f_1, \ldots, f_k$. Contradiction. **QED**

Corollary 3 of BUT

For any k smooth real-valued functions $g_1, \ldots, g_k$ on S^k, there exists a point $p \in S^k$ such that

$$g_1(p) = g_1(-p), \ldots, g_k(p) = g_k(-p).$$

Proof: We define functions $f_1, \ldots, f_k$ on S^k by

$$f_i(x) := g_i(x) - g_i(-x).$$

Then each f_i is smooth and odd. Hence there is a common zero which is the desired point $p \in S^k$. **QED**

In order to get the meteorologic interpretation, take g_1 measuring temperature and g_2 measuring pressure.