• 2.3 Locate and classify the equilibrium points of the following systems. Sketch the phase diagrams: it will often be helpful to obtain isoclines and path directions at other points in the plane.

(i) $\dot{x} = x - y$, $\dot{y} = x + y - 2xy$;
(ii) $\dot{x} = ye^y$, $\dot{y} = 1 - x^2$;
(iii) $\dot{x} = 1 - xy$, $\dot{y} = (x - 1)y$;
(iv) $\dot{x} = (1 + x - 2y)x$, $\dot{y} = (x - 1)y$;
(v) $\dot{x} = x - y$, $\dot{y} = x^2 - 1$;
(vi) $\dot{x} = -6y + 2xy - 8$, $\dot{y} = y^2 - x^2$;
(vii) $\dot{x} = 4 - 4x^2 - y^2$, $\dot{y} = 3xy$;
(viii) $\dot{x} = -y\sqrt{(1 - x^2)}$, $\dot{y} = x\sqrt{(1 - x^2)}$ for $|x| \le 1$;
(ix) $\dot{x} = \sin y$, $\dot{y} = -\sin x$;
(x) $\dot{x} = \sin x \cos y$, $\dot{y} = \sin y \cos x$.

2.3. For the system $\dot{x} = X(x, y)$, $\dot{y} = Y(x, y)$, the equilibrium points are given by solutions of $X(x, y) = 0$, $Y(x, y) = 0$. The linear approximations (Section 2.3) near each equilibrium point are classified using the table in Section 2.5, or Figure 2.10 (both in NODE). Curve sketching can be helped by plotting the isoclines $Y(x, y) = 0$ (phase paths locally parallel to the x axis) and $X(x, y) = 0$ (phase paths locally parallel to the y axis). Since these problems are nonlinear, scales along the axes are now significant.

(i) $\dot{x} = x - y$, $\dot{y} = x + y - 2xy$. The equilibrium points are given by

$$x - y = 0, \quad x + y - 2xy = 0.$$

There are two equilibrium points, at $(0, 0)$ and $(1, 1)$.

(a) $(0, 0)$. The linear approximation is

$$\dot{x} = x - y, \quad \dot{y} \approx x + y.$$

Hence the parameters are

$$p = 2 > 0, \quad q = 1 + 1 = 2 > 0, \quad \Delta = 4 - 8 = -4 < 0,$$

which means that the origin is locally an unstable spiral.

(b) $(1, 1)$. Put $x = 1 + \xi$ and $y = 1 + \eta$. The linear approximation is

$$\dot{\xi} = \xi - \eta, \quad \dot{\eta} \approx -\xi - \eta.$$

For this linear approximation the parameters are

$$p = 0, \quad q = -2 < 0, \quad \Delta = 0 + 8 = 4 > 0,$$

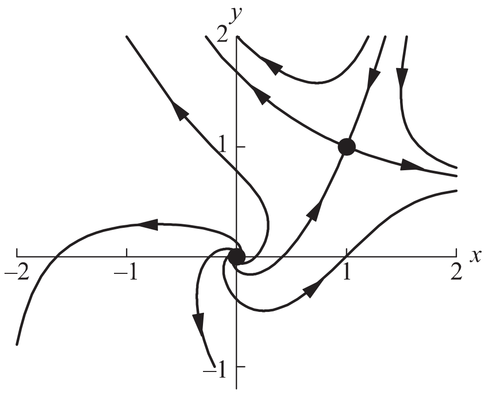

Figure 2.7 Problem 2.3(i): $\dot{x} = x - y$, $\dot{y} = x + y - 2xy$.

which means that $(1, 1)$ is locally a saddle point with asymptotes in the directions of the slopes $1 \pm \sqrt{2}$.

The zero-slope isocline is the curve $x + y - 2xy = 0$ and the infinite-slope isocline is the line $y = x$. A computed phase diagram is shown in Figure 2.7.

(ii) $\dot{x} = ye^y$, $\dot{y} = 1 - x^2$. The equilibrium points are given by

$$ye^y = 0, \quad 1 - x^2 = 0.$$

Therefore there are two equilibrium points, at $(1, 0)$ and $(-1, 0)$.

(a) $(1, 0)$. Put $x = 1 + \xi$. The linear approximation is

$$\dot{\xi} \approx y, \quad \dot{y} \approx -2\xi.$$

The parameters are

$$p = 0, \quad q = 2 > 0, \quad \Delta = -8 < 0,$$

from which we infer that the $(1, 0)$ is a centre.

(b) $(-1, 0)$. Put $x = -1 + \xi$. The linear approximation is

$$\dot{\xi} \approx y, \quad \dot{y} \approx 2\xi.$$

The parameters are

$$p = 0, \quad q = -2 < 0, \quad \Delta = 8 > 0,$$

which implies that $(-1, 0)$ is a saddle. The phase diagram is shown in Figure 2.8. Note that the isoclines of zero slope are the straight lines $x = \pm 1$.

(iii) $\dot{x} = 1 - xy$, $\dot{y} = (x - 1)y$. The equilibrium points are given by

$$1 - xy = 0, \quad (x - 1)y = 0,$$

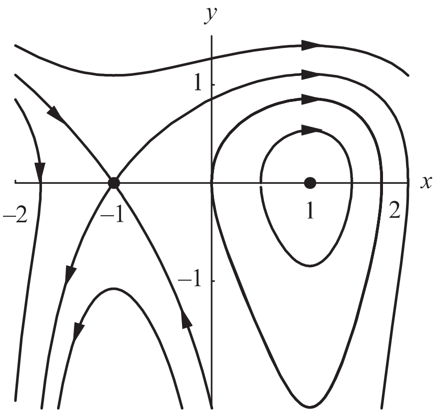

Figure 2.8 Problem 2.3(ii) :$\dot{x} = ye^y$, $\dot{y} = 1 - x^2$.

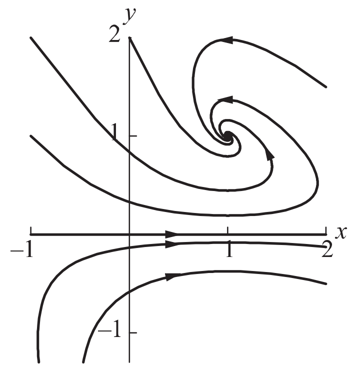

Figure 2.9 Problem 2.3(iii): $\dot{x} = 1 - xy$, $\dot{y} = (x - 1)y$.

which has the single solution $(1, 1)$. Let $x = 1 + \xi$ and $y = 1 + \eta$. Then the linear approximation is

$$\dot{\xi} \approx -\xi - \eta, \quad \dot{\eta} \approx \xi.$$

The parameters are

$$p = -1 < 0, \quad q = 1 > 0, \quad \Delta = 1 - 4 = -3 < 0,$$

which means that $(1, 1)$ is a stable spiral. Note that $y = 0$ is a phase path. The phase diagram is shown in Figure 2.9.

(iv) $\dot{x} = (1 + x - 2y)x$, $\dot{y} = (x - 1)y$. The equilibrium points are given by

$$(1 + x - 2y)x = 0, \quad (x - 1)y = 0.$$

There are three equilibrium points: at $(0, 0)$, $(1, 1)$ and $(-1, 0)$. Note that the axes $x = 0$ and $y = 0$ are phase paths. The straight line $x = 1$ is an isocline of zero slope.

(a) $(0, 0)$. The linear approximation is

$$\dot{x} \approx x, \quad \dot{y} = -y.$$

The parameters are

$$p = 0, \quad q = -1 < 0, \quad \Delta = -4 < 0,$$

which implies that $(0, 0)$ is a saddle.

(b) $(1, 1)$. Let $x = 1 + \xi$ and $y = 1 + \eta$. The linear approximation is

$$\dot{\xi} = (2 + \xi - 2 - 2\eta)(1 + \xi) \approx \xi - 2\eta, \quad \dot{\eta} = \xi.$$

The parameters are

$$p = 1 > 0, \quad q = 2 > 0, \quad \Delta = 1 - 4 = -3 < 0.$$

Hence $(1, 1)$ is an unstable spiral

(c) $(-1, 0)$. Let $x = -1 + \xi$. Then the linear approximation is

$$\dot{\xi} \approx -\xi + 2y, \quad \dot{y} \approx -2y.$$

Hence the parameters are

$$p = -3 < 0, \quad q = 2 > 0, \quad \Delta = 9 - 8 = 1 > 0,$$

which means that $(-1, 0)$ is a stable node.

The phase diagram is shown in Figure 2.10.

(v) $\dot{x} = x - y$, $\dot{y} = x^2 - 1$. The equilibrium points are given by

$$x - y = 0, \quad x^2 - 1 = 0.$$

Therefore the equilibrium points occur at $(1, 1)$ and $(-1, -1)$. The isoclines of zero slope are the lines $x = \pm 1$, and the isocline of infinite slope is the line $y = x$.

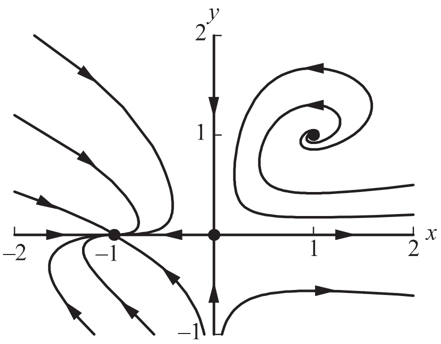

Figure 2.10 Problem 2.3(iv): $\dot{x} = (1 + x - 2y)x$, $\dot{y} = (x - 1)y$.

(a) $(1, 1)$. Let $x = 1 + \xi$ and $y = 1 + \eta$. The linear approximation is

$$\dot{\xi} = \xi - \eta, \quad \dot{\eta} \approx 2\xi,$$

which has the parameters

$$p = 1 > 0, \quad q = 2 > 0, \quad \Delta = 1 - 8 = -7 < 0.$$

Hence $(1, 1)$ is an unstable spiral.
(b) $(-1, -1)$. Let $x = -1 + \xi$ and $y = -1 + \eta$. The linear approximation is

$$\dot{\xi} = \xi - \eta, \quad \dot{\eta} \approx -2\xi,$$

which has the parameters

$$p = 1 > 0, \quad q = -2 < 0, \quad \Delta = 1 + 8 = 9 > 0.$$

Therefore $(-1, -1)$ is a saddle point. The phase diagram is shown in Figure 2.11.

(vi) $\dot{x} = -6y + 2xy - 8$, $\dot{y} = y^2 - x^2$. The equilibrium points are given by

$$-3y + xy - 4 = 0, \quad y^2 - x^2 = (y - x)(y + x) = 0.$$

If $y = -x$, the first equation has no real solutions, whilst for $y = x$, there are two solutions, leading to equilibrium points at $(-1, -1)$ and $(4, 4)$.

(a) $(-1, -1)$. Let $x = -1 + \xi$ and $y = -1 + \eta$. The linear approximation is

$$\dot{\xi} \approx -2\xi - 8\eta, \quad \dot{\eta} \approx 2\xi - 2\eta,$$

which has the parameters

$$p = -4 < 0, \quad q = 20 > 0, \quad \Delta = 16 - 80 = -64 < 0.$$

Hence $(-1, -1)$ is a stable spiral.

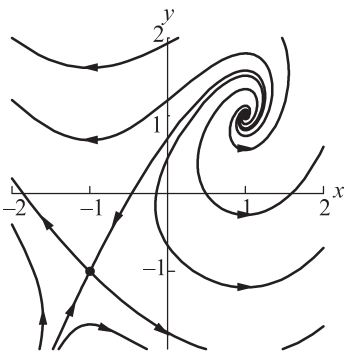

Figure 2.11 Problem 2.3(v): $\dot{x} = x - y$, $\dot{y} = x^2 - 1$.

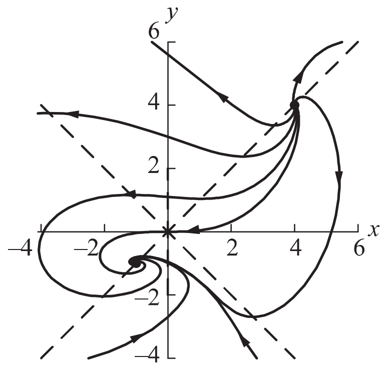

Figure 2.12 Problem 2.3(vi): $\dot{x} = -6x + 2xy - 8$, $\dot{y} = y^2 - x^2$: the dashed lines are the isoclines of zero slope.

(b) $(4, 4)$. Let $x = 4 + \xi$ and $y = 4 + \eta$. The linear approximation is

$$\dot{\xi} \approx 8\xi + 2\eta, \quad \dot{\eta} \approx -8\xi + 8\eta,$$

which has the parameters

$$p = 16 > 0, \quad q = 80 > 0, \quad \Delta = 256 - 320 = -64 < 0.$$

Hence $(4, 4)$ is an unstable spiral.

The phase diagram is shown in Figure 2.12.

(vii) $\dot{x} = 4 - 4x^2 - y^2$, $\dot{y} = 3xy$. The equilibrium points are solutions of

$$4 - 4x^2 - y^2 = 0, \quad 3xy = 0.$$

The complete set of solutions is $(0, 2)$, $(0, -2)$, $(1, 0)$ and $(-1, 0)$. The x axis is a phase path, and the y axis is a zero-slope isocline.

(a) $(0, 2)$. Let $y = 2 + \eta$. The linear approximation is

$$\dot{x} \approx -4\eta, \quad \dot{\eta} \approx 6x,$$

which has the parameters

$$p = 0, \quad q = 24 > 0, \quad \Delta = -96 < 0.$$

Hence $(0, 2)$ is a centre.

(b) $(0, -2)$. Let $y = -2 + \eta$. The linear approximation is

$$\dot{x} \approx 4\eta, \quad \dot{\eta} \approx -6x,$$

which has the parameter values

$$p = 0, \quad q = 24 > 0, \quad \Delta = -96 < 0.$$

Therefore $(0, -2)$ is also a centre.

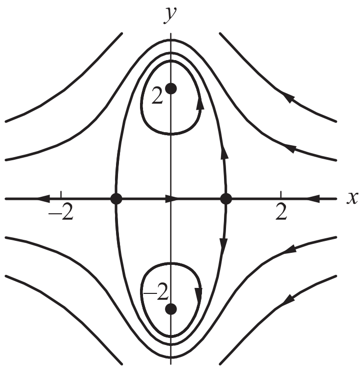

Figure 2.13 Problem 2.3(vii): $\dot{x} = 4 - 4x^2 - y^2$, $\dot{y} = 3xy$.

(c) $(1, 0)$. Let $x = 1 + \xi$. Then the linear approximation is

$$\dot{\xi} \approx -8\xi, \quad \dot{y} \approx 3y,$$

which has the parameter values

$$p = -8 + 3 = -5 < 0, \quad q = -24 < 0, \Delta = 25 + 96 = 121 > 0.$$

This equilibrium point is a saddle.

(d) $(-1, 0)$. Let $x = -1 + \xi$. The linear approximation is

$$\dot{\xi} \approx 8\xi, \quad \dot{y} \approx -3y,$$

which has the parameter values

$$p = 8 - 3 = 5 > 0, \quad q = -24 < 0, \quad \Delta = 25 + 96 = 121 > 0.$$

The equilibrium point is also a saddle.

The phase diagram is shown in Figure 2.13.

(viii) $\dot{x} = -y\surd(1 - x^2)$, $\dot{y} = x\surd(1 - x^2)$, for $|x| \leq 1$. The equilibrium points include the origin $(0, 0)$ and all points on the lines $x = \pm 1$. The equations are real only in the strip $|x| \leq 1$. The phase paths are given by

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{x}{y},$$

which has the general solution $x^2 + y^2 = C$. All phase paths in the strip $|x| < 1$ are circles which means that the origin is a centre. The phase diagram is shown in Figure 2.14.

(ix) $\dot{x} = \sin y$, $\dot{y} = -\sin x$. Equilibrium points occur where both $\sin y = 0$ and $\sin x = 0$. Hence there is an infinite set of such points at $(m\pi, n\pi)$ where $m = 0, \pm 1, \pm 2, \ldots$ and $n = 0 \pm 1, \pm 2, \ldots$. Since the equations are unchanged by the transformations $x \to x + 2m\pi$,

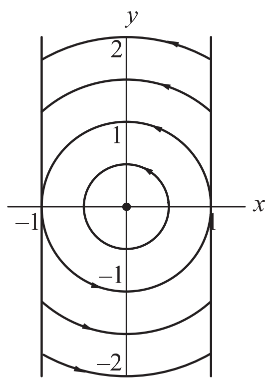

Figure 2.14 Problem 2.3(viii): $\dot{x} = -y\sqrt{(1-x^2)}$, $\dot{y} = x\sqrt{(1-x^2)}$, for $|x| \le 1$.

$y \to y + 2n\pi$, the phase diagram is periodic with period 2π in both the x and y directions. The equations of the phase paths can be found from

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{\sin x}{\sin y}.$$

This is a separable equation with general solution $\cos x + \cos y = C$. Note that this system is Hamiltonian (Section 2.8) from which we infer that any simple equilibrium points will be centres or saddle points. Near the origin

$$\dot{x} \approx y, \quad \dot{y} \approx -x,$$

which indicates a centre. Near $(\pi, 0)$, let $x = \pi + \xi$. Then the linear approximation is

$$\dot{\xi} \approx y, \quad \dot{y} \approx \xi,$$

which indicates a saddle. In fact the centres and saddles alternate in both the x and y directions.
The phase diagram is shown in Figure 2.15.

(x) $\dot{x} = \sin x \cos y$, $\dot{y} = \sin y \cos x$. The consistent pairings of $\dot{x} = 0$ and $\dot{y} = 0$ are

$$\sin x = 0, \quad \sin y = 0, \text{ and } \cos y = 0, \quad \cos x = 0.$$

Therefore there are equilibrium points at

$$x = m\pi, \quad y = n\pi, \text{ and at } x = \tfrac{1}{2}(2p+1)\pi, \quad y = \tfrac{1}{2}(2q+1)\pi,$$

where $m, n, p, q = 0, \pm 1, \pm 2, \ldots$. There are the obvious singular solutions given by the straight lines $x = r\pi$ and $y = s\pi$, where $r, s = 0, \pm 1, \pm 2, \ldots$. Near the origin the linear approximation is

$$\dot{x} \approx x, \quad \dot{y} \approx y$$

Locally the phase paths are given by

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y}{x} \Rightarrow y = Cx.$$

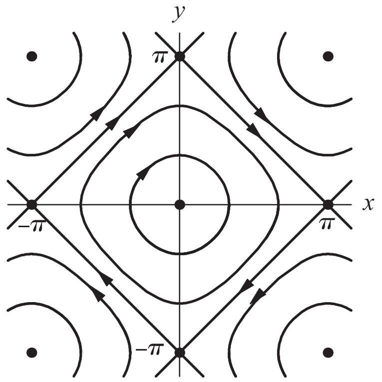

Figure 2.15 Problem 2.3(ix): $\dot{x} = \sin y$, $\dot{y} = -\sin x$.

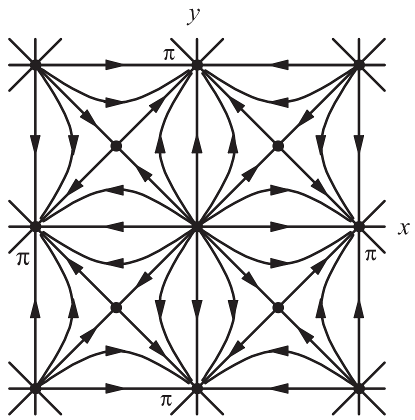

Figure 2.16 Problem 2.3(x): $\dot{x} = \sin x \cos y$, $\dot{y} = \sin y \cos x$.

Hence the origin (and similarly all other grid points) have locally star-shaped phase diagrams. It can also be verified that the lines $y = x + p\pi$ and $y = -x + p\pi$ for $p = 0, \pm 1, \pm 2, \ldots$ are also phase paths (separatrices) and that these equilibrium points are saddle points. The phase diagram, which is periodic with period 2π in both the x and y directions, is shown in Figure 2.16.

• 2.4 Construct phase diagrams for the following differential equations, using the phase plane in which $y = \dot{x}$.

(i) $\ddot{x} + x - x^3 = 0$;

(ii) $\ddot{x} + x + x^3 = 0$;

(iii) $\ddot{x} + \dot{x} + x - x^3 = 0$;

(iv) $\ddot{x} + \dot{x} + x + x^3 = 0$;

(v) $\ddot{x} = (2\cos x - 1)\sin x$.