Having attained our goal it also bears remarking on a simple observation as to when periodic orbits can be excluded, again named after Dulac:

Theorem 17.8 (Dulac's Theorem)**.** *Let*

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x} = f(\mathbf{x})$$

be a C^1 planar system on the open set $U \subseteq \mathbb{R}^2$. If there exists a C^1 function $V : U \to \mathbb{R}$ such that $\nabla \cdot (Vf) \geq 0$ and not identically zero on U, then there are no periodic orbits lying entirely within U.

This is a result of the divergence theorem:

Proof. Suppose that $\Gamma \subseteq U$ is a periodic orbit enclosing the set E. By Green's/Stoke's/divergence theorem,

$$\int_E \nabla \cdot (Vf)\, \mathrm{d}\mathbf{x} = \int_\Gamma Vf \cdot \mathrm{d}\mathbf{s},$$

where $\mathbf{s}$ is aligned to the outer normal of E. Since f is tangent to the orbit all along Γ, the integral on the right is nought. Therefore Vf cannot be non-negative (or non-positive) without being identically zero.

$\square$

18. Lecture XVIII: Perturbation Theory I

Here we shall discuss a very practical tool in the analysis of dynamical systems. We shall step momentarily away from phase space and consider perturbative methods. In a way, perturbation methods is another facet of the same technique which in calculus is called "Taylor's expansion", and which we have repeatedly exploited to first order in another way.

Many physical systems have a characteristic scale or characteristic length, whether it be the diameter of a pipe in fluid flow, or the resonance frequency of beam in a structure under dynamic loading, or the frequency of the rotation of the earth in weather modelling, or something more apparently fundamental, like the Planck length, or the bond length of the hydrogen molecule.

Given a system $\dot{\mathbf{x}} = f(\mathbf{x}, \varepsilon)$, with a characteristic scale ε, we should like to propose an ansatz to bounded oscillatory solution of a form similar to

$$\mathbf{x}(t) = \mathbf{x}_0(t) + \varepsilon \mathbf{x}_1(t) + \varepsilon^2 \mathbf{x}_2(t) + \dots.$$

Inserting this into the equation we find that

$$\dot{\mathbf{x}}_0(t) + \varepsilon \dot{\mathbf{x}}_1(t) + \ldots = f(\mathbf{x}_0(t) + \varepsilon \mathbf{x}_1(t) + \ldots, \varepsilon)$$

Next we can collect the terms of the same order in ε, and what is hoped is that we should arrive at a collection of equations at each order that are simple not too coupled, each with suitably bounded solutions, so that $\mathbf{x}(t)$ can be well-approximated by short truncated expansion for small ε. We shall discover that there are many ways to do this badly.

18.1. **Examples.** We being with a few examples.

Example 18.1. First we consider a simple 1-dimensional equation with a small linear part, away from its fixed point:

$$\dot{x} = -x + \varepsilon x^2, \qquad x(0) = 1.$$

Writing $x(t) = x_0(t) + \varepsilon x_1(t) + \ldots$, we find

$$\begin{aligned}
&\dot{x}_0(t) + \varepsilon x_1(t) + \varepsilon^2 x_2(t) + O(\varepsilon^3) \\
&= -x_0(t) - \varepsilon x_1(t) - \varepsilon^2 x_2(t) + \varepsilon(x_0(t) + \varepsilon x_1(t) + \varepsilon^2 x_2(t))^2 + O(\varepsilon^3) \\
&= -x_0(t) + \varepsilon(-x_1(t) + x_0^2(t)) + \varepsilon^2(x_2(t) + 2x_0(t)x_1(t)) + O(\varepsilon^3).
\end{aligned}$$

Collecting the terms we find that if the equation above is to hold for every sufficiently small $\varepsilon > 0$, then

$$\begin{aligned}
\dot{x}_0(t) &= -x_0(t), \\
\dot{x}_1(t) &= -x_1(t) + x_0^2(t), \\
\dot{x}_2(t) &= -x_2(t) + 2x_0(t)x_1(t).
\end{aligned}$$

Since $x(0) = 1$ so that for every $\varepsilon > 0$, $1 = x_0(0) + \varepsilon x_1(0) + \ldots$, we find (by matching terms again), that $x_0(0) = 1$ and $x_n(0) = 0$ for $n > 0$.

These equations can be solved in sequence, which is not the general situation. They imply that

$$\begin{aligned}
x_0(t) &= e^{-t}, \\
x_1(t) &= e^{-t} - e^{-2t}, \\
x_2(t) &= e^{-t} - 2e^{-2t} + e^{-3t}.
\end{aligned}$$

We can in fact solve the full equation explicitly for comparison:

$$t = \int_0^{x(t)} \frac{1}{-y(1-\varepsilon y)}\,\mathrm{d}y = -\int_1^{x(t)} \frac{1}{y} + \frac{\varepsilon}{1-\varepsilon y}\,\mathrm{d}y = -\log(x(t)) + \varepsilon\log(1-\varepsilon x(t)) - \varepsilon\log(1-\varepsilon)$$

$$x(t) = \frac{e^{-t}}{1-\varepsilon+\varepsilon e^{-t}} = e^{-t}\sum_{n=0}^{\infty}\varepsilon^n(1-e^{-t})^n.$$

This verifies our pertubative method applied to this problem.

It is, however, not often the case that the perturbative expansion around $\varepsilon = 0$ (and, essentially, $t = 0$) holds well for all time $t > 0$.

Example 18.2. Let us consider another classical example where the function itself does not decay in time but oscillates indefinitely:

$$\ddot{x} + \varepsilon x + x = 0, \qquad x(0) = 1, \quad \dot{x}(0) = 0.$$

The solution is easily found to be $x(t) = \cos(\sqrt{1+\varepsilon}t)$.

Proceeding with the ansatz

$$x(t) = x_0(t) + \varepsilon x_1(t) + \varepsilon^2 x_2(t) + \ldots,$$

our perturbative method yields the following set of equations:

$$\begin{aligned}\ddot{x}_0(t) &= -x_0(t),\\ \ddot{x}_1(t) &= -x_0(t) - x_1(t),\\ \ddot{x}_2(t) &= -x_1(t) - x_2(t),\end{aligned}$$

to second order. The corresponding initial conditions are:

$$x_0(0) = 1, \qquad 0 = \dot{x}_0(0) = x_1(0) = \dot{x}_1(0) = \ldots.$$

These yield:

$$x_0(t) = \cos(t), \quad x_1(t) = -\frac{1}{2}t\sin(t), \quad x_2(t) = \frac{1}{8}(t\sin(t) - t^2\cos(t)). \tag{29}$$

The asymptotic expansion given by $x(t) = x_0(t) + \varepsilon x_1(t) + \varepsilon^2 x_2(t)$ is correct up to $O(\varepsilon^3)$ for finite times $t \in [0, T]$, $\varepsilon T \ll 1$, but if we allow $t \to \infty$, the higher order terms dominate the zeroth order term for $\varepsilon t \approx 1$.

Example 18.3. Finally let us consider an example in which the exact solution does decay to zero and yet a naive perturbative method yields a quickly diverging solution at any fixed order as $t \to \infty$:

$$\ddot{x} + \varepsilon\dot{x} + x = 0, \qquad x(0) = 0, \quad \dot{x}(0) = 1.$$

As can be verified, the exact solution is

$$x(t) = \frac{1}{\sqrt{1-\varepsilon^2/4}}e^{-\varepsilon t/2}\sin(\sqrt{1-\varepsilon^2/4}t).$$

Attempting a naive perturbative ansatz, we find to first order that:

$$\begin{aligned}0 &= \ddot{x}_0(t) + x_0(t),\\ 0 &= \ddot{x}_1(t) + \dot{x}_0(t) + x_1(t).\end{aligned}$$

The corresponding initial conditions are:

$$\dot{x}_0(0) = 1, \qquad 0 = x_0(0) = x_1(0) = \dot{x}_1(0) = \ldots.$$

These yield:

$$x_0(t) = \cos(t), \quad x_1(t) = -\frac{1}{2}t\sin(t),$$

and we again encounter the same problem of SECULAR TERMS — higher and higher order terms that come to dominate the expansion as $t \to \infty$.

We shall now consider a method to rectify our expansions. Observe that the difficulty here is primarily that the ocillatory functions x_k in the perturbative expansion do not quite match the exact solution in frequency. Over time there must needs be beating behaviour unless there are those polynomial factors present to cancel them out in what prima facie looks to be a rather divergent manner (we have not considered how the coefficients $\{-1/2, 1/8, \ldots\}$ decay), and in any case. we shall be needing far too many terms much too quickly as $t \to \infty$ to understand the behaviour of the solution as t grows.

18.2. **Poincaré-Lindstedt method.** Having made our observation as we have done, we see that one simple way to rectify this is to scale time properly as well so as to match the exact solution.

Let us introduce then a scaled time variable:

$$\tau(t, \varepsilon) = \omega(\varepsilon)t = (1 + \omega_1\varepsilon + \omega_2\varepsilon 2 + \ldots)t,$$

and consider the time-scaled ansatz

$$x(t, \varepsilon) = x_0(\tau(t, \varepsilon)) + \varepsilon x_1(\tau(t, \varepsilon)) + \varepsilon^2 x_2(\tau(t, x\varepsilon)) + \ldots.$$

This consideration is the basis of the POINCARÉ-LINDSTEDT METHOD.

Applying this expansion to Example 18.2, we can insert the ansatz into

$$\ddot{x} + (1 + \varepsilon)x = 0.$$

Now

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}(x \circ \tau)(t) = \frac{\mathrm{d}^2}{\mathrm{d}\tau^2}x(\tau)\dot{\tau}^2 + \frac{\mathrm{d}}{\mathrm{d}\tau}x(\tau)\ddot{\tau} = \ddot{x}(\tau)(1 + \omega_1\varepsilon + \omega_2\varepsilon^2 + \ldots)^2 + 0.$$

And so we obtain, to first order,

$$\frac{\mathrm{d}^2}{\mathrm{d}\tau^2}x_0(\tau) = -x_0(\tau),$$
$$\frac{\mathrm{d}^2}{\mathrm{d}\tau^2}x_1(\tau) = -2\omega_1\frac{\mathrm{d}^2}{\mathrm{d}\tau^2}x_0(\tau) - x_0(\tau) - x_1(\tau) = -x_1(\tau) + (2\omega_1 - 1)x_0(\tau).$$

The attendent intial conditions being unchanged from Example 18.2, we find that

$$x_0(\tau) = \cos(\tau).$$

Now observe that by choosing $\omega_1 = 1/2$, we have $x_1(t) \equiv 0$ since $x_1(0) = \dot{x_1}(0) = 0$.

Therefore to first order, we expect our solution to be

$$x(t) = \cos((1 + \varepsilon/2)t) + O(\varepsilon^2).$$

This is very much better an expansion than (29) — at least the first order is bounded in t. Moreover, on comparison with the exact solution, we see that we actually got the Taylor expansion around $t = 0$ of the correct frequency.