Exercise set 2: Solutions

Please justify your answers! Note that *how* you arrive at an answer is more important than the answer itself.

1 **a)** Determine the following numbers and decide in each case whether "supremum" can be replaced by "maximum":

1. $\sup_{x\in(0,\infty)} \frac{1}{x^2}$;
2. $\sup_{x\in\mathbb{R}} e^{-2|x|}$;
3. $\sup_{n\in\mathbb{N}} \frac{n^2+3}{n^2+1}$;
4. $\sup_{n\in\mathbb{N}}(-1)^n \frac{n+3}{n^2+1}$.

b) Determine the following numbers and decide in each case whether "infimum" can be replaced by "minimum":

1. $\inf_{x\in(0,\infty)} \frac{1}{x^2}$;
2. $\inf_{x\in\mathbb{R}} e^{-2|x|}$;
3. $\inf_{n\in\mathbb{N}} \frac{n^2+3}{n^2+1}$;
4. $\inf_{n\in\mathbb{N}}(-1)^n \frac{n+3}{n^2+1}$.

Solution. a) In general, to show that a is the supremum of a set, we need to show that *a is an upper bound for the set* and that *there is no smaller upper bound for the set.*

1. This supremum is clearly ∞, since the supremum is an upper bound and $\frac{1}{x^2}$ grows arbitrarily large[1] when x approaches zero. There is clearly no $x \in (0,\infty)$ such that $1/x^2 = \infty$, so we cannot replace supremum with a maximum.

2. The function $e^{-2|x|}$ decreases when $|x|$ increases - one might for instance consider the derivative or graph of the function to see this. Hence $e^{-2|x|}$ is never larger than its value in $x = 0$, which is 1. Therefore 1 is an upper bound, and since $e^0 = 1$ it is clearly the *least* upper bound. This supremum could be replaced by a maximum, since $1 = e^0$.

[1] Read: “as large as you like”. Norsk: vilkårlig stor

3. The expression $\frac{n^2+3}{n^2+1}$ decreases as n increases, this is perhaps most apparent after writing $\frac{n^2+3}{n^2+1} = 1 + \frac{2}{n^2+1}$. Hence it is never larger than its value at $n = 1$, which is $\frac{4}{2} = 2$. Hence the supremum is 2, and could be replaced by a maximum as it is obtained for the value $n = 1$.

4. The expression $\frac{n+3}{n^2+1}$ also clearly decreases when n increases, since n^2 grows faster than n. Hence the supremum of $(-1)^n \frac{n+3}{n^2+1}$ is the first positive term in the sequence (the greatest absolute value is obtained for $n = 1$, but then the expression is negative!). The first positive term is $n = 2$, hence the supremum (which can be replaced by a maximum) is $\frac{2+3}{5} = 1$

b) In general, to show that b is the infinum of a set, we need to show that b *is a lower bound for the set* and that *there is no greater lower bound for the set.*

1. $\frac{1}{x^2}$ is always greater than 0, hence 0 is a lower bound. On the other hand $\frac{1}{x^2}$ becomes arbitrarily small when $x \to \infty$, so 0 is the *greatest* lower bound. The supremum cannot be replaced by a minimum, since there is no $x \in (0, \infty)$ such that $1/x^2 = 0$.

2. The same reasoning as above shows that 0 is the infinum, and that it cannot be replaced by a minimum: $e^{-2|x|}$ is always greater than zero, but becomes arbitrarily small as $|x| \to \infty$.

3. As we discussed in part 3. of a), the expression $\frac{n^2+3}{n^2+1}$ is decreasing when n increases. It is also easy to see that the limit $\lim_{n\to\infty} \frac{n^2+3}{n^2+1} = 1$ (just apply the usual techniques!). Since $\frac{n^2+3}{n^2+1}$ is a decreasing sequence converging to 1, we must conclude that the infinum is 1. It cannot be replaced by a minimum, as there is no value $n \in \mathbb{N}$ such that $\frac{n^2+3}{n^2+1} = 1$ (just try to solve this equation if you do not believe me!).

   This solution was rather short - for the benefit of the reader we show how to make it painfully precise. To show that 1 is the infinum, we need to show that 1 is a lower bound for the set, and that it is the *greatest* lower bound. Clearly 1 is a lower bound, since $\frac{n^2+3}{n^2+1} > 1$ for every $n \in \mathbb{N}$. After all, the nominator is clearly greater than the denominator! Could there possibly exist a greater upper bound than 1 – i.e. a $b \in \mathbb{R}$ such that $b > 1$ and $\frac{n^2+3}{n^2+1} \geq b$ for any $n \in \mathbb{N}$? To show that this is not possible, we will use that $\lim_{n\to\infty} \frac{n^2+3}{n^2+1} = 1$. By the definition of the limit, this means that for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $\frac{n^2+3}{n^2+1} - 1 < \epsilon$ [2]. If we now let $\epsilon = b - 1$, this says that we can find a $N_b \in \mathbb{N}$ such that $\frac{n^2+3}{n^2+1} - 1 < b - 1$ for $n \geq N_b$. Now move the -1 to the right side:

$$\frac{n^2+3}{n^2+1} < 1 + b - 1 = b.$$

[2]Strictly speaking we should have absolute values on the left side here, but we know that the fraction always exceeds 1

It follows that b cannot be a lower bound, since $\frac{N_b^2+3}{N_b^2+1} < b$. Hence 1 is the infinum.

4. As we observed in a), the expression $\frac{n+3}{n^2+1}$ is decreasing. Since the sign of $(-1)^n$ is alternatively positive and negative, the smallest value of $(-1)^n \frac{n+3}{n^2+1}$ must be an n where $(-1)^n$ is negative, and it must be the smallest such n since $\frac{n+3}{n^2+1}$ is decreasing. In this case, we clearly have that the infinum, and the minimum, is the value obtained for $n = 1$, namely $(-1)\frac{1+3}{1+1} = -2$.

$\boxed{2}$ Let A be bounded above. Show that the supremum of A is unique.

Solution: First a general hint: to show that there is a unique object satisfying some property, you often assume that you have two such objects, and prove that they must be equal.
Assume that the supremum of A is not unique, in other words we have two numbers m and m' such that both m and m' are least upper bounds for A and $m \neq m'$. In particular m is an upper bound, and since m' is a *least* upper bound, we must have $m' \leq m$. Similarly m' is an upper bound and m is a *least* upper bound, hence $m \leq m'$. Of course, these two inequalities together imply that $m = m'$, hence our assumption $m \neq m'$ is contradicted – in words, there cannot be two different supremums, showing that the supremum is unique.

$\boxed{3}$ Let A be a bounded subset of $\mathbb{R}$, and let cA be its dilate by a positive constant $c > 0$. Show that

$$\inf cA = c \inf A.$$

Solution. We need to show two things: $c \inf A$ is a lower bound for cA, and $c \inf A$ is the *greatest lower bound* for cA. For any $x \in A$ we have that $cx \geq c \inf A$, since $\inf A$ is a lower bound for A by definition. Since cA consists exactly of elements of the form cx, this shows that $c \inf A$ is a lower bound for cA.
To show that $c \inf A$ is the greatest lower bound, we pick any $\epsilon > 0$ and show that $c \inf A + \epsilon$ cannot be a lower bound. By definition [3] of $\inf A$, we can find $x \in A$ such that

$$x < \inf A + \frac{\epsilon}{c}.$$

By multiplying this with c (which is assumed to be positive) we find

$$cx < c \inf A + \epsilon,$$

hence $\inf A + \epsilon$ cannot be a lower bound for cA.

[3] Since $\inf A$ is the greatest lower bound, the larger number $\inf A + \epsilon$ cannot be a lower bound.

$\boxed{4}$ Let X be a vector space.

1. Prove that the additive inverse is unique (meaning for any $x \in X$ there exists a unique vector $y \in X$ such that $x + y = 0$; we denote the additive inverse of x by $-x$.)
2. Show that for every $x \in X$ we have $(-1)x = -x$. In words multiplication by the scalar -1 gives the additive inverse of a vector.

Solution. 1) Once again we need to prove that some object is unique, and as before we start by assuming that we have two objects satisfying the property. In this case we assume that there are two elements y and y' in X such that $x + y = 0$ and $x + y' = 0$ – we want to show that $y = y'$.

Start with the equation

$$x + y = 0.$$

Then add y' to both sides of the equation to obtain

$$y' + x + y = y' + 0 = y'.$$

By the assumption on y', $y' + x = 0$. Hence

$$y' = y' + x + y = (y' + x) + y = 0 + y = y.$$

2) $-x$ is by definition the unique vector such that $x + (-x) = 0$, so we need to show that $x + (-1)x = 0$. By an axiom for vector spaces, $x = 1x$, and another axiom (distributivity) then allows us to show that

$$x + (-1)x = 1x + (-1)x = (1 - 1)x = 0x.$$

Hence it only remains to show that $0x = 0$ (note that the 0 one the left side is a scalar, and 0 on the right side a vector). This seems rather obvious, but still requires a proof. In fact,

$$x + 0x = 1x + 0x = (1 + 0)x = x,$$

and if we add $-x$ to both sides of this equality we find that

$$-x + x + 0x = -x + x = 0,$$

hence $0x = 0$.

$\boxed{5}$ A function $f : \mathbb{R} \to \mathbb{R}$ is called *odd* if $f(t) = -f(-t)$ for all $t \in \mathbb{R}$.

Prove or disprove that the set of odd functions $\mathbb{R} \to \mathbb{R}$ with component-wise addition

$$(f + g)(t) = f(t) + g(t),$$

and scalar multiplication

$$(\lambda f)(t) = \lambda f(t), \quad \forall\, \lambda \in \mathbb{R},$$

form a real vector space.

Hint: you need to find a zero vector, an additive inverse of each element f, and check the axioms of a real vector space. Most importantly, check that the operations of addition and scalar multiplication does not "lead out of space", i.e., that they are indeed operations $V \times V \to V$ and $\mathbb{R} \times V \to V$, respectively.

Solution. We will need to check that axioms (1) to (6) in definition 2.1.1 are true, but as the exercise suggests it is important to also check that the operations do not lead out of the space – this is the part of the definition of vector spaces stated before the list of axioms, and can be easy to miss. In our case, we need to show that if f and g are odd functions and $\lambda \in \mathbb{R}$, then

$$f + g \text{ is an odd function}$$

and

$$\lambda f \text{ is an odd function.}$$

We find that

$$\begin{aligned}
(f+g)(-t) &= f(-t) + g(-t) && \text{by our definition of addition of functions.}\\
&= -f(t) - g(t) && f, g \text{ are odd functions.}\\
&= -(f(t) + g(t)) && \\
&= -(f+g)(t) && \text{by our definition of addition of functions.}
\end{aligned}$$

This shows that $f + g$ is odd. Similarly, λf is odd because

$$\begin{aligned}
(\lambda f)(-t) &= \lambda f(-t) && \text{our definition of scalar multiplication.}\\
&= -\lambda f(t) && \text{since } f \text{ is odd.}\\
&= -(\lambda f)(t) && \text{our definition of scalar multiplication..}
\end{aligned}$$

We can then move to the list of axioms. The calculations may look long and boring, but the trick is to always reduce the statements to the corresponding statements for real numbers, where we know that the statements are true.

1. If f, g are odd functions and $\lambda, \mu \in \mathbb{R}$, then

   $$(f+g)(x) = f(x) + g(x) = g(x) + f(x) = (g+f)(x).$$

   The point here is that the expression $f(x) + g(x)$ is simply the addition of real numbers, so we can safely switch the order. Similarly

   $$((\lambda\mu)f)(x) = (\lambda\mu)f(x) = \lambda(\mu f(x)) = (\lambda(\mu f))(x).$$

2. If f, g, h are odd functions, then

$$((f+g)+h)(x) = (f+g)(x)+h(x) = f(x)+g(x)+h(x) = f(x)+(g(x)+h(x)) = (f+(g+h))(x),$$

which shows that $(f+g)+h = f+(g+h)$. The point is that $f(x)+g(x)+h(x)$ is simply the addition of real numbers, so we know that we can safely put parentheses wherever we want.

3. Consider the function f defined by $f(t) = 0$ for any t. Then f is odd since $f(-t) = 0 = -0 = -f(t)$, and for any odd function g we have that $(f+g)(t) = f(t)+g(t) = 0+g(t) = g(t)$, hence $f+g = g$. It follows that f is an additive identity, so we will denote f by 0 from now on.

4. Given an odd function f, we consider the function defined by $g(t) = -f(t)$. Then g is odd, since $g(-t) = -f(-t) = f(t) = -g(t)$, and $(f+g)(t) = f(t)+g(t) = 0$. Therefore g is an additive inverse to f.

5. Let f be an odd function. Then $(1f)(t) = 1f(t) = f(t)$ by our definition of scalar multiplication, hence $1f = f$.

6. If $\lambda \in \mathbb{R}$ and f, g are odd functions, then

$$(\lambda(f+g))(t) = \lambda(f+g)(t) = \lambda(f(t)+g(t)) = \lambda f(t)+\lambda g(t) = (\lambda f)(t)+(\lambda g)(t) = (\lambda f+\lambda g)(t),$$

hence $\lambda(f+g) = \lambda f + \lambda g$. The main point is that the relation $\lambda(f(t)+g(t)) = \lambda f(t) + \lambda g(t)$ holds since this is just the distributive law for multiplication of real numbers, which we know is true. The proof that $(\mu+\lambda)f = \mu f + \lambda f$ is similar.

$\boxed{6}$ Let X be a vector space and T a linear mapping $T : X \to X$.

1. Show that the range of T is a subspace of X.
2. Let D be the differentiation operator $Df(x) = f'(x)$. Determine the kernel and the range of $Tf = f' - 3f$ for $f \in C^{(1)}(\mathbb{R})$, the space of continuously differentiable functions on $\mathbb{R}$.

Solution. 1) Assume that $y, y' \in X$ are elements of the range of T, which by definition means that there are $x, x' \in X$ with $Tx = y$ and $Tx' = y'$. To show that the range of T is a subspace, we need to show that $y + \lambda y'$ is in the range of T for any scalar λ. This is straightforward to show using the linearity of T:

$$\begin{aligned} y + \lambda y' &= T(x) + \lambda T(x') \\ &= T(x) + T(\lambda)x') \\ &= T(x + \lambda x'). \end{aligned}$$

Since this shows that $y + \lambda y'$ is the image of $x + \lambda x'$ under T, $y + \lambda y'$ belongs to the range of T.

2) The kernel of T is the set of functions f such that $Tf = f' - 3f = 0$. This is a simple, separable differential equation for f, which one easily solves to find that $f = Ce^{3t}$ where $C \in \mathbb{R}$ is some constant. Thus the kernel of T is the subspace

$\{Ce^{3t} : C \in \mathbb{R}\}$, in other words the subspace spanned by e^{3t}. The range of T is the set of functions of the form $f' - 3f$ for $f \in C^{(1)}(\mathbb{R})$. We claim that T is surjective, and to prove this we need, for every $g \in C(\mathbb{R})$, to find $f \in C^{(1)}(\mathbb{R})$ that solves $f' - 3f = g$. Now, this is a first order linear differential equation, and you know how to solve such an equation from earlier courses. If one uses the theory of integrating factors, one finds that the integrating factor is $-3x$, and the solution to the equation is

$$f(x) = e^{3x} \int_0^x e^{-3t} g(t)\ dt.$$

This is f is such that $Tf = f' - 3f = g$, hence T is surjective.