Suggested solution, exam TMA4265, Stochastic Modeling, Nov 28, 2018

Task 1

a)

Markov chain 1 is the only one who allows direct transitions from state 1 to 3 and from state 3 to 1. This occurs only in Display c), hence chain 1 must be related to c).

Markov chain 2 has $P_{11} = 0.5$ while chain 3 has $P_{11} = 0.9$. There is clearly a tendency of longer stays in state 1 for Display a). Also, when in state 2, chain 2 has $P_{22} = 0.8$ with equal probability 0.1 of going to state 1 and 3, while chain 3 has $P_{22} = 0.5$ with larger probability 0.4 of going to 1 than state 3. Display a) appears to stay longer in state 2, and have more equal moves up and down, we hence argue that Markov chain 2 is visualized in display a) and Markov chain 3 in Display b).

Possible realizations of the chains are visualized in Figure 1. Chain 1 moves between state 1 and 2 before it gets absorbed in state 3 from 2. Chain 2 has a tendency of moving up from 1 to 2, then from 2 to 3, and back to state 1.

b)

Because of the Markov property:

$$P(X_3 = 2|X_1 = 2) = \sum_{j=1}^{3} P(X_3 = 2|X_2 = j, X_1 = 2)P(X_2 = j|X_1 = 2) = \sum_{j=1}^{3} P_{j2}P_{2j}$$

$$P(X_3 = 2|X_1 = 2) = 0.8^2 + 0.1 \cdot 0.5 + 0.1^2 = 0.7$$

$$P(X_2 = 2|X_3 = 2, X_1 = 2) = \frac{P(X_3 = 2, X_2 = 2|X_1 = 2)}{P(X_3 = 2|X_1 = 2)} = \frac{0.8^2}{0.7} = 0.91$$

The long-run probabilities are determined by

$$\pi_i = \sum_{j=1}^{3} \pi_j P(j, i), \quad \sum_{j=1}^{3} \pi_j = 1$$

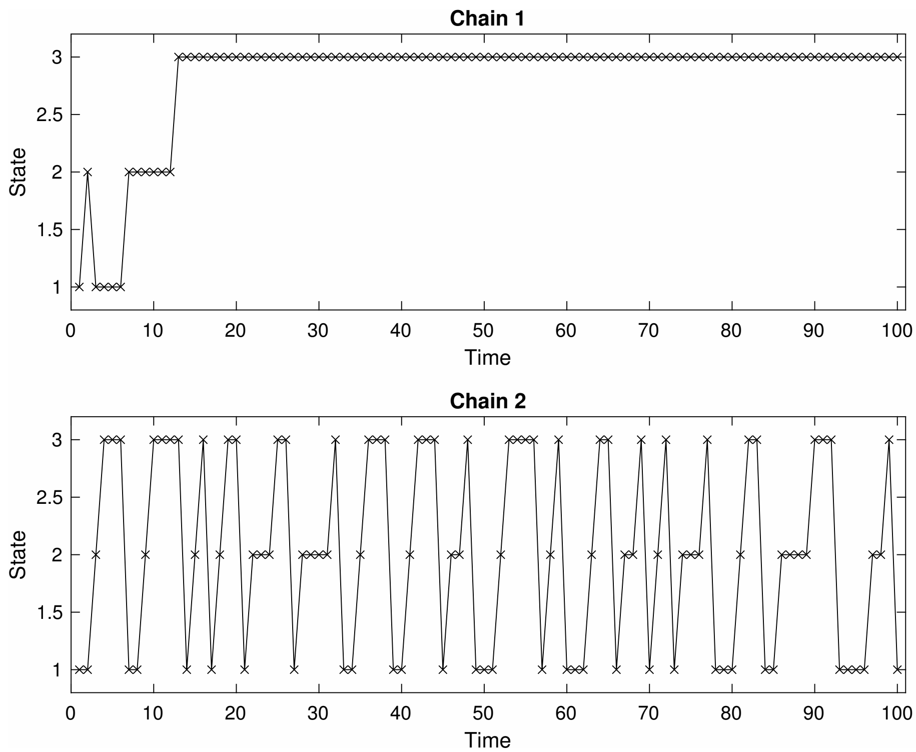

Figure 1: Markov chain realizations of the two different Markov transition probabilities, plotted as a function of time.

From the last two columns of the transition matrix, we get:

$$\begin{aligned}
\pi_1 &= 0.5\pi_1 + 0.1\pi_2, \;\; \pi_1 = \pi_2/5 \\
\pi_3 &= 0.1\pi_2 + 0.9\pi_3, \;\; \pi_3 = \pi_2 \\
\pi_1 &+ \pi_2 + \pi_3 = \pi_2/5 + \pi_2 + \pi_2 = 1,
\end{aligned}$$

$\pi_2 = 1/(1/5+2) = 0.445$
$\pi_1 = 0.445/5 = 0.09,$
$\pi_3 = 0.445.$

Task 2

a)

X_t cannot be larger than 9, and for $t < 9$ it cannot be larger than t (only drawing Ingrid's cards first).

X_t cannot be smaller than 0, and for $t > 9$ it cannot be smaller than $t-9$ (only drawing Sverre's cards first).

Hence,

$$X_t \in \{\max(0, t-9), \ldots, \min(t, 9)\}, \;\; t = 1, \ldots, 18.$$

The transition probabilities are:

$$P(X_{t+1} = k+1 | X_t = k) = \frac{9-k}{18-t}, \quad t = 0, \ldots, 17.$$

$$P(X_{t+1} = k | X_t = k) = 1 - \frac{9-k}{18-t}, \quad t = 0, \ldots, 17.$$

b)

Define $u_{t,i} = P(\text{Ingrid wins}|X_t = i)$.

When $X_{15} = 8$, there are two of Sverre's card on the table, and one of Ingrid's: By a first-step analysis:

$$\begin{aligned} u_{15,8} &= u_{16,9}(1/3) + u_{16,8}(2/3) = 1(1/3) + (1/2)(2/3) = 2/3 \\ u_{14,8} &= u_{15,9}(1/4) + u_{15,8}(3/4) = 1(1/4) + (2/3)(3/4) = 3/4 \end{aligned}$$

$$\begin{aligned} u_{15,7} &= u_{16,8}(2/3) + u_{16,7}0 = 1/2(2/3) = 1/3 \\ u_{14,7} &= u_{15,8}(2/4) + u_{15,7}(2/4) = (2/3)(2/4) + (1/3)(2/4) = 1/2 \\ u_{13,7} &= u_{14,8}(2/5) + u_{14,7}(3/5) = 3/4(2/5) + (1/2)(3/5) = 3/5 \end{aligned}$$

Which are all intuitive considering the cards left on the table.

Task 3

a)

$$
\begin{aligned}
& P(N_{\text{Minor}}(7)=0, N_{\text{Major}}(7)=0) = P(N_{\text{Minor}}(7)=0)P(N_{\text{Minor}}(7)=0) \\
= \quad & \exp(-\mu_1 7)\exp(-\mu_2 7) = \exp(-7(0.2+0.4)) = 0.015
\end{aligned}
$$

Let S_{Minor} and S_{Major} be the waiting times for the first minor failure and major failure, respectively.

$$P(S_{\text{Minor}} < S_{\text{Major}}) = \int_0^\infty P(s < S_{\text{Major}} | S_{\text{Minor}} = s) f(s) ds$$

where $f(s) = \mu_1 \exp(-\mu_1 s)$ is the density function of S_{Minor}.

$$P(S_{\text{Minor}} < S_{\text{Major}}) = \int_0^\infty \exp(-\mu_2 s)\mu_1 \exp(-\mu_1 s) ds = \mu_1/(\mu_1+\mu_2)$$

b)

Let $N(7) = N_{\text{Minor}}(7) + N_{\text{Major}}(7)$ be the total number of failures. The sum of Poisson variables is also Poisson, and $N(7)$ has parameter $\mu_1+\mu_2$.

The probabilty of having less major than minor failures is defined by either having $N_{\text{Minor}}(7) = 4$, $N_{\text{Major}}(7) = 0$ or $N_{\text{Minor}}(7) = 3$, $N_{\text{Major}}(7) = 1$. We have:

$$
\begin{aligned}
& P(N_{\text{Minor}} = 4, N_{\text{Major}} = 0 | N(7) = 4) = \frac{P(N_{\text{Minor}}(7)=4)P(N_{\text{Major}}(7)=0)}{P(N(7)=4)} \\
= \quad & \frac{(\mu_1 7)^4/24)\exp(-\mu_1 7)\exp(-\mu_2 7)}{((\mu_1+\mu_2)7)^4/24)\exp(-(\mu_1+\mu_2)7)} \\
= \quad & \frac{\mu_1^4}{(\mu_1+\mu_2)^4} = \frac{0.4^4}{0.6^4} = 0.198
\end{aligned}
$$

$$
\begin{aligned}
& P(N_{\text{Minor}} = 3, N_{\text{Major}} = 1 | N(7) = 4) = \frac{P(N_{\text{Minor}}(7)=3)P(N_{\text{Major}}(7)=1)}{P(N(7)=4)} \\
= \quad & \frac{(\mu_1 7)^3/6)\exp(-\mu_1 7)(\mu_2 7)/1)\exp(-\mu_2 7)}{((\mu_1+\mu_2)7)^4/24)\exp(-(\mu_1+\mu_2)7)} \\
= \quad & 4\frac{(\mu_1^3\mu_2)}{(\mu_1+\mu_2)^4} = 4\frac{0.4^3 0.2}{0.6^4} = 0.395
\end{aligned}
$$

In total the probability is then $0.198 + 0.395 = 0.59$

c)

Define the total number of repair costs as $C = C_{\text{minor}} + C_{\text{major}}$. The costs are independent by the assumptions of the model. For minor repairs; $C_{\text{minor}} = \sum_{i=1}^{N_{\text{minor}}(7)} C_{\text{minor},i}$, where $C_{\text{minor},i}$ are the individual independent repair costs for minor failures: $E(C_{\text{minor},i}) = 10000$, $Var(C_{\text{minor},i}) = 1000^2$. For major repairs; $C_{\text{major}} = \sum_{i=1}^{N_{\text{major}}(7)} C_{\text{major},i}$, where $C_{\text{major},i}$ are the individual independent costs for major failures: $E(C_{\text{major},i}) = 50000$, $Var(C_{\text{major},i}) = 10000^2$.

By double mean we have:

$$E(C_{\text{minor}}) = E(N_{\text{minor}}(7))10000 = 0.4 \cdot 7 \cdot 10000 = 28000$$

$$E(C_{\text{major}}) = E(N_{\text{major}}(7))50000 = 0.2 \cdot 7 \cdot 50000 = 70000$$

$E(C) = 28000 + 70000 = 98000$

By double variance we have:

$$\begin{aligned}
Var(C_{\text{minor}}) &= E(Var(C_{\text{minor}}|N_{\text{minor}}(7))) + Var(E(C_{\text{minor}}|N_{\text{minor}}(7))) \\
&= E(N_{\text{minor}}(7))1000^2 + Var(N_{\text{minor}}(7))10000^2 \\
&= 0.4 \cdot 7 \cdot 1000^2 + 0.4 \cdot 7 \cdot 10000^2 = 16800^2
\end{aligned}$$

$$\begin{aligned}
Var(C_{\text{major}}) &= E(Var(C_{\text{major}}|N_{\text{major}}(7))) + Var(E(C_{\text{major}}|N_{\text{major}}(7))) \\
&= E(N_{\text{major}}(7))10000^2 + Var(N_{\text{major}}(7))50000^2 \\
&= 0.2 \cdot 7 \cdot 10000^2 + 0.2 \cdot 7 \cdot 50000^2 = 60330^2
\end{aligned}$$

$Var(C) = 16800^2 + 60330^2 = 62600^2$

d)

There appears to be inhomogeneous rate in both processes. This inhomogeneity seems to be coupled in the two processes - there is a (delayed) correlation between the two failure type occurences, where the major failures cluster in time right after several minor failures.

A possible improvement of the model could be an inhomogeneous Poisson process with larger rates near 10-20 days and 35-45 days, for both processes. The rate for the minor failures process is about 10 per day in these time periods, while it is about 7 per day for the major failures process. Outside the two time intervals there are very few failures.

Task 4

a)

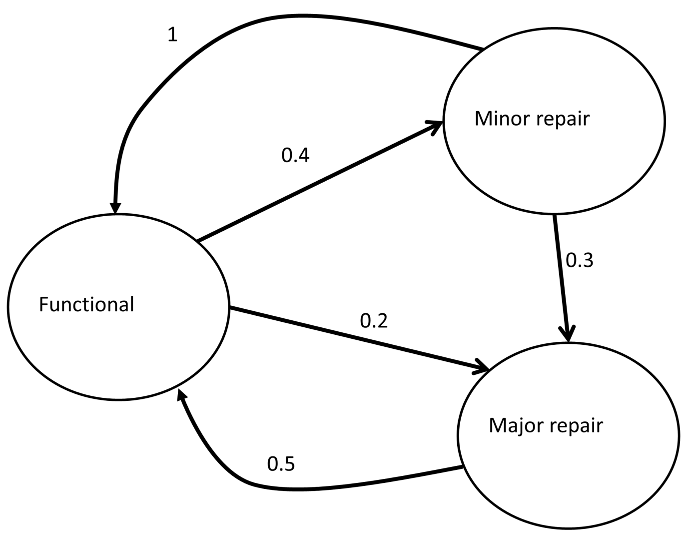

Figure 2: Transition diagram between the three states, with rates indicated.

The long-term distribution is determined by setting long-term rates in and out of states equal:

$$\begin{aligned}
\pi_0(\mu_1 + \mu_2) &= \pi_2\lambda_2 + \pi_1\lambda_1 \\
\pi_1(\mu_{12} + \lambda_1) &= \pi_0\mu_1 \\
\pi_2\lambda_2 &= \pi_0\mu_1 + \pi_1\mu_{12} \\
1 &= \pi_0 + \pi_1 + \pi_2
\end{aligned}$$

Inserting the rates, the first two equations give:

$$\begin{aligned}\pi_0(0.4+0.2) &= \pi_2 0.5 + \pi_1 1 \\ \pi_1(0.3+1) &= \pi_0 0.4\end{aligned}$$

$$\pi_0+\pi_1+\pi_2 = \pi_0(1+(0.4/1.3)+(0.6-(0.4/1.3))/0.5) = 1$$

This means that $\pi_0 = 1/1.89 = 0.53$, $\pi_1 = 0.53(0.4/1.3) = 0.16$, and $\pi_2 = 0.53(0.6-(0.4/1.3))/0.5 = 0.31$.

Task 5

a)

The mean in the process is $\mu_t = E(x(t)) = 1 + 0.05t$. The variance is $\sigma_t^2 = 0.02^2 t$. That is, $X(5) \sim N(1.25, 5 \cdot 0.02^2)$.

$$P(X(5) > 1.2) = P(Z > \frac{1.2-1.25}{0.02\sqrt{5}}) = P(Z > -1.12) = 0.87$$

Because of the increments of the Brownian motion are independent, $X(5)-\mu_5 = X(4) - \mu_4 + B(1)$. This means that $X(5) - X(4) \sim N(0.05, 0.02^2)$.

$$P(X(5) - X(4) > 0) = P(Z > \frac{0-0.05}{0.02}) = P(Z > -2.5) = 0.994$$