TMA4190 Introduction to Topology Spring 2018

Exercise set 3

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1 Let $A\colon \mathbb{R}^n \to \mathbb{R}^n$ be a linear map, and $b \in \mathbb{R}^n$. Show that the mapping

$$f\colon \mathbb{R}^n \to \mathbb{R}^n,\ x \mapsto Ax + b$$

is a diffeomorphism of $\mathbb{R}^n$ if and only if A is invertible.

2 Show that the map

$$f\colon \mathbb{R}^2 \to \mathbb{R}^3,\ (s,t) \mapsto ((2+\cos(2\pi s))\cos(2\pi t), (2+\cos(2\pi s))\sin(2\pi t), \sin(2\pi s))$$

is an immersion. Is it an embedding?

3 Let γ_α be the curve on the torus defined by

$$\gamma_\alpha\colon \mathbb{R} \to S^1 \times S^1,\ t \mapsto (e^{2\pi i t}, e^{2\pi i \alpha t})$$

where we consider S^1 as a subset of $\mathbb{C} \cong \mathbb{R}^2$. Show that γ_α factors through an embedding $S^1 \to S^1 \times S^1$ when α is rational, i.e. find a map $g_\alpha\colon S^1 \to S^1 \times S^1$ which is an embedding such that γ_α is the composite of the map $\mathbb{R} \to S^1,\ t \mapsto e^{2\pi i t}$, followed by g_α.

4 Consider the map $f\colon (0, 3\pi/4) \to \mathbb{R}^2,\ t \mapsto \sin(2t)(\cos t, \sin t)$.

a) Show that f is an immersion.

b) Let $\mathrm{Im}\,(f) = f((0,3\pi/4)) \subset \mathbb{R}^2$ be the image of f (considered as a subspace in $\mathbb{R}^2$). Show that $f\colon (0,3\pi/4) \to \mathrm{Im}\,(f)$ is not a homeomorphism. (Draw a picture of the image of f.)

c) To test your understanding answer the following questions (and give reasons for your answer):

- What is the difference between $\mathrm{Im}\,(f)$ and the graph $\Gamma(f)$?
- Is the map $F\colon (0,3\pi/4) \to (0,3\pi/4) \times \mathbb{R}^2,\ t \mapsto (t, f(t))$, an embedding?
- Would f be an embedding if it was defined on the closed interval $[0, 3\pi/4]$?
- Is the map $g\colon (0,3\pi/4) \to \mathbb{R}^3,\ t \mapsto \sin(2t)(\cos t, \sin t, t)$ an embedding?
- Is the map $h\colon [0,3\pi/4] \to \mathbb{R}^3,\ t \mapsto (\sin(2t)\cos t, \sin(2t)\sin t, 2t)$ an embedding?

$\boxed{5}$ Let X be an n-dimensional smooth manifold, Z be a k-dimensional smooth submanifold of X, and let $z \in Z$. Show that there exists a local coordinate system $(x_1, \ldots, x_n)$ defined in a neighborhood U of z in X such that $Z \cap U$ is defined by the equations $x_{k+1} = 0, \ldots, x_n = 0$, i.e. $Z \cap U$ is the subset of points in U for which the functions $x_{k+1}, \ldots, x_n$ all vanish.