NTNU
Norwegian University of
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Week 37: Lecture 1

Equivalence classes and classification of states in Markov chains

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Information

- You can use this Google form to provide feedback to the reference group:
  `https://s.ntnu.no/tma4265_2020_meeting1`
- Question last time: “How high powers do we need to check to exclude regularity?”
- Answer: For state space $\{0, 1, \ldots, N\}$, $m = N^2 + 1$ is guaranteed to be enough.

Section 4.3

We want to

- understand why regularity fails.
- extend regularity to infinite state spaces.

Introductory example

Let $\{X_n : n = 0, 1, \ldots\}$ be a Markov chain. It can fail to be regular due to, for example,

a) Reducibility

b) Periodic states

c) Transient states

Communication

Definition

Let $\{X_n : n = 0, 1, \ldots\}$ be a Markov chain with state space $\{0, 1, \ldots\}$ and transition probability matrix $\mathbf{P}$.

1) State j is **accessible** from state i if $\exists n \geq 0$ so that $P_{ij}^{(n)} > 0$.
2) If states i and j are accessible from each other, they are said to **communicate** and we write $i \sim j$. If states i and j do not communicate we write $i \nsim j$.

Equivalence relation

Theorem

Communication is an ***equivalence relation****, i.e.,*

1) ***reflexive****:* $i \sim i$
2) ***symmetric****:* $i \sim j \Rightarrow j \sim i$
3) ***transitive****:* $i \sim j$ *and* $j \sim k$ *implies* $i \sim k$

Equivalence relation

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Comment: An equivalence relation induces **equivalence classes** consisting of sets of states that communicate.

Irreducible

Definition

A Markov chain is **irreducible** if $\sim$ (communication) induces exactly one equivalence class. If not, it is called **reducible**.

Example

$$\mathbf{P} = \begin{bmatrix} 1/3 & 1/3 & 1/3 & 0 \\ 1/3 & 1/3 & 0 & 1/3 \\ 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 1/2 & 1/2 \end{bmatrix}$$

a) What are the equivalence classes?

b) Is the Markov chain irreducible?

Example

$$\mathbf{P} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

Is the Markov chain reducible?

Periodicity

Definition

The **period** of state i, written as $d(i)$, is

$$d(i) = \gcd\{n \geq 1 : P_{ii}^{(n)} > 0\}.$$

If $P_{ii}^{(n)} = 0$ for all $n \geq 1$, we define $d(i) = 0$.

If $d(i) = 1$, we say that state i is **aperiodic**.

Periodicity within an equivalence class

Theorem

If $i \sim j$, then $d(i) = d(j)$.

Notation

The state space may be infinite: $\{0, 1, \ldots\}$. We introduce:

1) The probability that the first return happens after exactly n steps:

$$f_{ii}^{(n)} = \Pr\{X_n = i, X_\nu \neq i, \nu = 1, 2, \ldots, n-1 | X_0 = i\}, \quad n > 0.$$

   We define $f_{ii}^{(0)} = 0$.

2) The probability of returning at some time:

$$f_{ii} = \sum_{k=0}^{\infty} f_{ii}^{(k)} = \lim_{n\to\infty} \sum_{k=0}^{n} f_{ii}^{(k)}.$$

Recurrent and transient states

Definition

State i is **recurrent** if the probability of returning to state i in a finite number of time steps is one, i.e., $f_{ii} = 1$. A state that is not recurrent, i.e., $f_{ii} < 1$, is called **transient**.

Theorem (Theorem 4.2)

A state i is recurrent if and only if

$$\sum_{n=1}^{\infty} P_{ii}^{(n)} = \infty.$$

Equivalently, state i is transient if and only if

$$\sum_{n=1}^{\infty} P_{ii}^{(n)} < \infty.$$