TMA 4215 Numerical Mathematics: Lecture 03 Appendix

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The goal of this appendix is to provide you will a complete and clean proof of the LU factorization using the pivotal column strategy.

Theorem 1. Let $A \in \mathbb{R}^{n,n}$ for $n \geqslant n$. Then there exists a permutation matrix P, a unit lower triangular matix L and an upper triangular matrix U, all in $\mathbb{R}^{n,n}$ such that

$$PA = LU. \tag{1}$$

Proof. As usual, we proof this by induction. *Step 1:* We consider the base case for $n = 2$ to illustrate the main idea. The case $n = 1$ would equally well work as base case, but it is just so trivial. We start from the 2×2 matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

If $a \neq 0$, the proof follows immediately from Thm.2 in Lecture 2 (in fact, the base case in the induction step there treated exactly the case $a \neq 0$). If $a = 0$ but $c \neq 0$, we set

$$P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

and write

$$\underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}_{=:P}\underbrace{\begin{pmatrix} 0 & b \\ c & d \end{pmatrix}}_{A} = \begin{pmatrix} c & d \\ 0 & b \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{=:L}\underbrace{\begin{pmatrix} c & d \\ 0 & b \end{pmatrix}}_{=:U}$$

Finally, if both $a = c = 0$, we simply rewrite A twice is the most trivial way

$$\underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{=:P}\underbrace{\begin{pmatrix} 0 & b \\ 0 & d \end{pmatrix}}_{A} = \begin{pmatrix} 0 & b \\ 0 & d \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{=:L}\underbrace{\begin{pmatrix} 0 & b \\ 0 & d \end{pmatrix}}_{=:U}$$

This conclude the base case.

Step 2 (induction step $n \mapsto n+1$): Assume that the theorem is valid for all $k \leqslant n$ and let $A \in \mathbb{R}^{n+1,n+1}$. Locate in the first column the element α with the largest absolute value, if there is several with the same absolute value, pick one of them. Assume it is in row r, swap row 1 and r by multiplying with the permutation matrix P_{1r} from the left. Then we make the ansatz

$$P_{1r}A = \begin{pmatrix} \alpha & \boldsymbol{w}^T \\ \boldsymbol{p} & B \end{pmatrix} = \begin{pmatrix} 1 & \mathbf{0}^T \\ \boldsymbol{m} & \mathrm{Id} \end{pmatrix} \begin{pmatrix} \alpha & \boldsymbol{v}^T \\ \mathbf{0} & C \end{pmatrix} \tag{2}$$

where $\boldsymbol{w}, \boldsymbol{p} \in \mathbb{R}^n$ and $B \in \mathbb{R}^{n,n}$. We need to determine $\boldsymbol{m}, \boldsymbol{v} \in \mathbb{R}^n$ and $C \in \mathbb{R}^{n,n}$. Multiplying out the matrices and setting the block elements equal to $P_{1r}A$ we obtain the system of equations

$$\boldsymbol{v}^T = \boldsymbol{w}^T, \quad \alpha\boldsymbol{m} = \boldsymbol{p}, \quad C = B - \boldsymbol{m}\boldsymbol{v}^T.$$

First equation determines $\boldsymbol{w}$. To solve the second, we distinguish two cases. If $\alpha = 0$, this implies that the entire vector $\boldsymbol{p} = \mathbf{0}$ since α was supposed to be the one with the largest absolut value. Then we can simply set $\boldsymbol{m} = \mathbf{0}$ and $C = B$. If $\alpha \neq 0$, we set $\boldsymbol{m} = 1/\alpha\boldsymbol{p}$ and by the choice of α, all the elements of $\boldsymbol{m}$ have absolute values ≤ 1.

Next, by induction there is a permuation matrix P^*, a unit lower triangular matrix L^*, and an upper triangular matrix U^*, all in $\mathbb{R}^{n,n}$ such

$$P^*C = L^*U^*$$

Replacing C with $(P^*)^{-1}L^*U^*$ we find that

$$\begin{aligned} P_{1r}A = \begin{pmatrix} \alpha & \boldsymbol{w}^T \\ \boldsymbol{p} & B \end{pmatrix} &= \begin{pmatrix} 1 & \mathbf{0}^T \\ \boldsymbol{m} & \mathrm{Id} \end{pmatrix} \begin{pmatrix} \alpha & \boldsymbol{v}^T \\ \mathbf{0} & (P^*)^{-1}L^*U^* \end{pmatrix} \\ &= \begin{pmatrix} 1 & \mathbf{0}^T \\ \mathbf{0} & (P^*)^{-1} \end{pmatrix} \begin{pmatrix} 1 & \mathbf{0}^T \\ P^*\boldsymbol{m} & L^* \end{pmatrix} \begin{pmatrix} \alpha & \boldsymbol{v}^T \\ \mathbf{0} & U^* \end{pmatrix} \end{aligned}$$

Inverting the permutation matrix on the right-hand side, we arrive at

$$\underbrace{\begin{pmatrix} 1 & \mathbf{0}^T \\ \mathbf{0} & P^* \end{pmatrix} P_{1r}}_{=:P} A = \begin{pmatrix} \alpha & \boldsymbol{w}^T \\ \boldsymbol{p} & B \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & \mathbf{0}^T \\ P^*\boldsymbol{m} & L^* \end{pmatrix}}_{=:L} \underbrace{\begin{pmatrix} \alpha & \boldsymbol{v}^T \\ \mathbf{0} & U^* \end{pmatrix}}_{=:U}$$

which concludes the induction step.