TMA 4190 Introduction to Topology
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Lecture 13[1]

13. Homotopy and Stability

Today we are going to introduce one of the most important concepts in topology. Actually, the idea of studying objects **up to homotopy** has turned out be extremely influential and successful in many areas in mathematics.

Homotopy

Let I denote the unit interval $[0,1]$ in $\mathbb{R}$. We say that two smooth maps f_0 and f_1 from X to Y are **homotopic**, denoted $f_0 \sim f_1$, if there exists a smooth map $F\colon X \times I \to Y$ such that

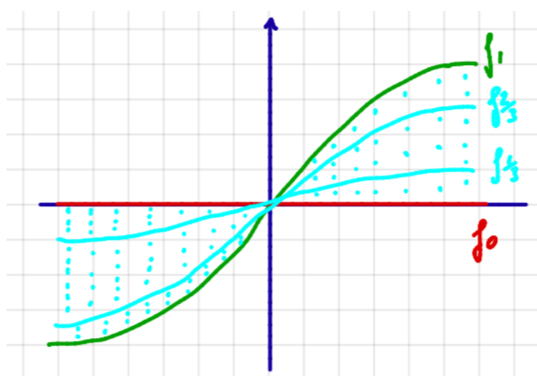
$$F(x,0) = f_0(x) \text{ and } F(x,1) = f_1(x).$$

F is called a **homotopy** between f_0 and f_1. We also write $f_t(x)$ for $F(x,t)$. In other words, a homotopy is a **family of smooth functions** f_t which smoothly interpolates between f_0 and f_1

To require that F is **smooth** is necessary because we are working with smooth manifolds. For general topological spaces, one just requires that F is **continuous**.

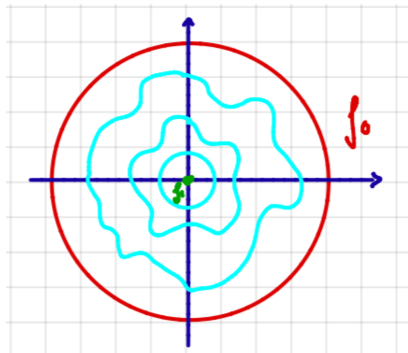
Some examples:

- $f_0\colon \mathbb{R} \to \mathbb{R}^2$, $x \mapsto (x,0)$ and $f_1\colon \mathbb{R} \to \mathbb{R}^2$, $x \mapsto (x, \sin x)$ with homotopy $F\colon \mathbb{R} \times [0,1] \to \mathbb{R}^2$, $(x,t) \mapsto (x, t\sin x)$.

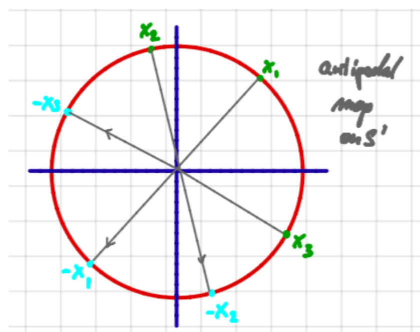


[1]Following the books of Guillemin and Pollack: Differential Topology, and Milnor: Topology from the differentiable viewpoint.

- Let $\gamma\colon S^1 \to \mathbb{R}^2$ be a smooth loop (a smooth path where start and end points agree). Then γ is homotopic to the constant map $S^1 \to \{0\} \subset \mathbb{R}^2$. In fact, this is true when we replace $\mathbb{R}^2$ with any $\mathbb{R}^k$, since $\mathbb{R}^k$ is contractible (see the exercises).



- In the exercises, we will show that the **antipodal map** on the k-sphere $S^k \to S^k$, $x \mapsto -x$ (which sends a point to the point on "the other side" of the sphere) is homotopic to the identity on S^k.



- An important example of two maps which are **not homotopic**: The constant map $f\colon S^1 \to \mathbb{R}^2 \setminus \{0\}$, $p \mapsto (1{,}0)$ and the map $g\colon S^1 \to \mathbb{R}^2 \setminus \{0\}$, $p \mapsto p$ are not homotopic. We will learn more about this later, and there are much better conceptual arguments in algebraic topology which explain this fact. Here is a first, hands-on argument:

  Assume there were a smooth homotopy $F\colon S^1 \times [0{,}1] \to \mathbb{R}^2 \setminus \{0\}$ from f to g. For every fixed point $p \in S^1$, $F(p{,}t)$ defines a path from p to $(1{,}0)$ in $\mathbb{R}^2 \setminus \{0\}$. Let Z be the subspace of S^1 of points with negative x-coordinate:

$$Z := \{p = (x{,}y) \in S^1 : x \leq 0\}.$$

  Then by the **Intermediate Value Theorem**, for every $p \in Z$, there is t such that the x-coordinate of $F(p{,}t)$ is 0. Since $[0{,}1]$ is **compact**, there is in fact a **minimal** such t for each $p \in Z$. We denote this minimum by

$t_0(p)$ and write

$$F(p,t_0(p)) = (0,y_0(p)).$$

As $(0,0)$ is not a point of $\mathbb{R}^2 \setminus \{0\}$, for each p, we have either $y_0(p) > 0$ or $y_0(p) < 0$.

Since F is smooth in both variables, $y_0(p)$ depends smoothly on p as well. Thus, if $y_0(p) > 0$ for some p, then there is an open neighborhood $U \subset S^1$ around p such that $y_0(q) > 0$ for all $q \in U$. In other words, the subset

$$U_{>0} := \{p = (x,y) \in Z : y_0(p) > 0\} \text{ is open in } Z.$$

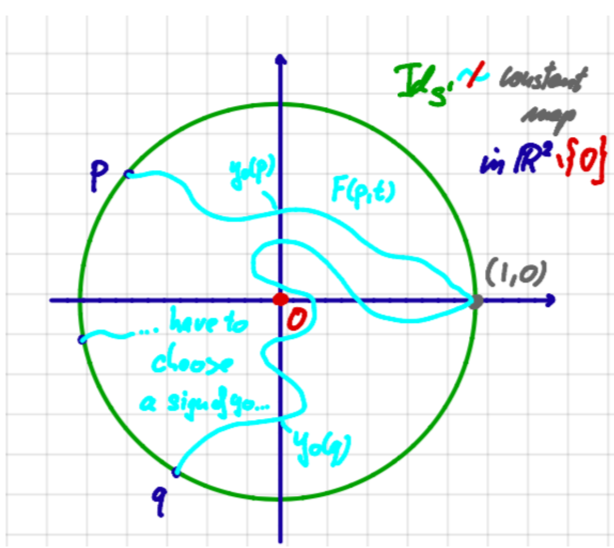
Similarly, the subset

$$U_{<0} := \{p = (x,y) \in Z : y_0(p) < 0\} \text{ is open in } Z.$$

Both spaces are nonempty, since $(0,1) \in U_{>0}$ and $(0,-1) \in U_{<0}$. Moreover, they are disjoint and mutual complements of each other in Z, i.e.

$$U_{>0} = Z \setminus U_{<0} \text{ and } U_{<0} = Z \setminus U_{>0}.$$

Thus, Z is the disjoint union of the two nonempty and both open and closed subsets $U_{>0}$ and $U_{<0}$. Since Z is connected (being the continuous image of a closed interval), this would imply either $Z = U_{>0}$ or $Z = U_{<0}$. But this is impossible. Thus the smooth homotopy F cannot exist.



Homotopy is an equivalence relation

Given two smooth manifolds X and Y, homotopy is an equivalence relation on smooth maps from X to Y. The equivalence class to which a mapping belongs is its **homotopy class**.

Proof:

We need to check that $\sim$ is reflexive, symmetric, and transitive:

Reflexivity is clear as every map is homotopic to itself via the homotopy $f_t = f$ for all t.

For symmetry, suppose $f \sim g$ and let F be a homotopy. Then the map defined by $(x,t) \mapsto F(x,1-t)$ is a homotopy from g to f. Hence $g \sim f$ as well.

For transitivity, we need to introduce a smart technique first:

Smooth bump functions

An extremely useful tool in differential topology are smooth bump functions which allow smooth transitions. We start with the function

$$f\colon \mathbb{R} \to \mathbb{R},\ f(x) = \begin{cases} e^{-1/x^2} & x > 0 \\ 0 & x \leq 0. \end{cases}$$

We observe that **f is smooth**: We only need to think about $x \geq 0$. Since the ith derivative has the form e^{-1/x^2} times a rational polynomial. Such a product is differentiable and

$$\lim_{x \to 0} f^{(i)}(x) = 0,$$

since e^{-1/x^2} goes to 0 faster than any rational polynomial can go to $\pm\infty$. Now, for any given real numbers $a < b$, we define a function

$$g(x) := f(x-a)f(b-x)$$

As a product of two smooth functions, g is smooth, and

$$\begin{cases} g(x) = 0 & x \leq a \text{ (since } f(x-a) = 0) \\ g(x) > 0 & a < x < b \\ g(x) = 0 & x \geq b \text{ (since } f(b-x) = 0) \end{cases}$$

Next we define yet another function

$$h\colon \mathbb{R} \to \mathbb{R},\ h(x) := \frac{\int_{-\infty}^{x} g(t)dt}{\int_{-\infty}^{\infty} g(t)dt}.$$

By the Fundamental Theorem of Calculus, h is smooth, nondecreasing, and

$$\begin{cases} h(x) = 0 & x \leq 0 \\ 0 < h(x) < 1 & a < x < b \\ h(x) = 1 & x \geq b \end{cases}$$

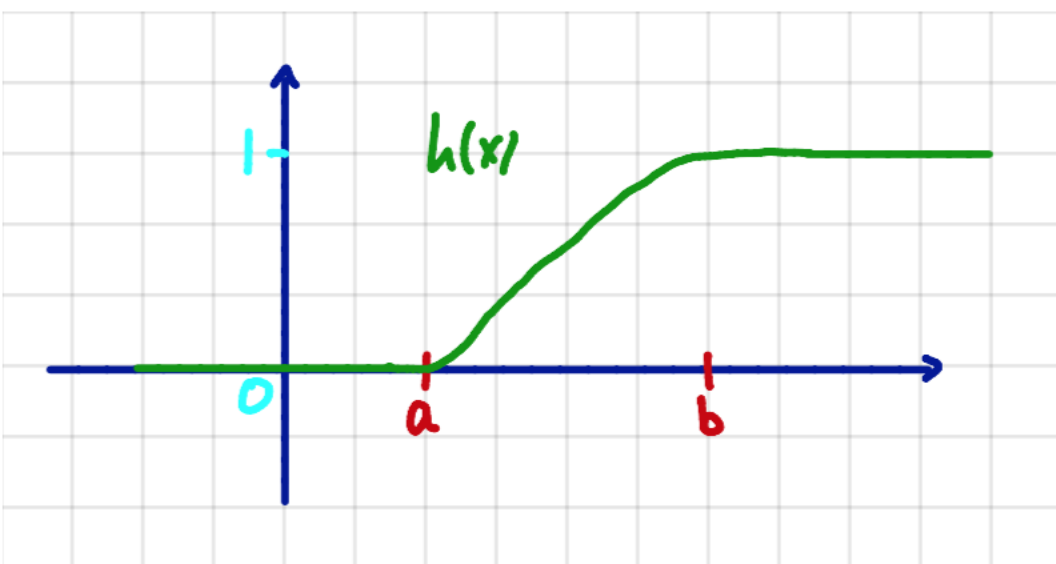
Then h is a **smooth bump function**.

Finally, we can also define higher dimensional smooth bump functions by setting

$$H\colon \mathbb{R}^k \to \mathbb{R},\ H(x) := 1 - h(|x|).$$

Then $H(x)$ is equal 1 on the closed ball around the origin with radius a, is 0 outside the open ball with radius b, and between 0 and 1 on the intermediate points:

$$\begin{cases} H(x) = 1 & x \in \bar{B}_a(0) \\ 0 < H(x) < 1 & a < |x| < b \\ H(x) = 0 & x \in \mathbb{R}^k \setminus B_b(0) \end{cases}$$



Back to the proof:

Suppose $f \sim g$ and $g \sim h$, and let F be a homotopy from f to g and G be a homotopy from g to h. We would like to compose F and G to get a homotopy from f to h. Since we require our homotopies to be **smooth**, we need to make sure that the transition from F to G is smooth.

In order to this, we need to manipulate F and G a bit. And here we are lucky that we have our smooth bump functions at our disposal. So let $\varphi\colon \mathbb{R} \to \mathbb{R}$ be a smooth function such that

$$\varphi(t) = \begin{cases} 0 & x \leq 1/4 \\ 1 & x \geq 3/4 \end{cases}$$

and define new homotopies $\tilde{F}$ from f to g and $\tilde{H}$ from g to h by

$$\tilde{F}(x,t) := F(x,\varphi(t)) \text{ and } \tilde{G}(x,t) := G(x,\varphi(t)).$$

Now we can define the map

$$H\colon X \times [0,1] \to Y,\ H(x,t) = \begin{cases} \tilde{F}(x,2t) & t \in [0,1/2] \\ \tilde{G}(x,2t-1) & t \in [1/2,1]. \end{cases}$$

This is map well-defined and smooth, since $\tilde{F}(x,2t) = \tilde{G}(x,2t-1)$ for $t \in [3/8,5/8]$. Thus H is a smooth homotopy from f to h. Hence $\sim$ is also transitive and an equivalence relation. **QED**

Homotopy is one of the most crucial notions in topology. In fact, a lot of properties in topology are **invariant under homotopy**. Therefore, they can be studied by considering maps only "up to homotopy". This led to the construction of the **homotopy category of spaces** in which morphisms are continuous maps modulo homotopy, i.e. $f \sim g$ if and only if f and g are homotopic. To be able to pass to the homotopy category is a very powerful method which has had great influences in many areas of mathematics. We will not be able to fully appreciate the homotopy category this semester.

However, we would like to start to exploit homotopy for our purposes. Despite the above remark, there also a lot of properties of maps which are not invariant under homotopy.

In fact, **many of the properties** we have studied so far are **not invariant**, i.e. if f_0 has a property P and f_t is a homotopy from f_0 to f_1, then it is often not true that f_1 has property P. For example, we could start with an embedding f_0 and end up with a constant map.

So let us ask **a more modest question**: given f_0 has property P, is there always a small $\epsilon > 0$ such that f_t has property P for all $t \in [0,\epsilon)$? For example, if f_0 is an embedding there is always a small $\epsilon > 0$ such that f_t remains an embedding for $0 \leq t < \epsilon$. In other words, embeddings are a so called stable class:

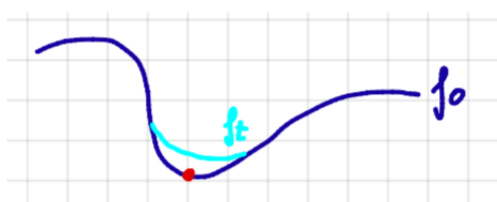
Stable properties

A property P is **stable** provided that whenever $f_0\colon X \to Y$ possesses the property and $f_t\colon X \to Y$ is a homotopy of f_0 then, for some $\epsilon > 0$, each f_t with $t < \epsilon$ also possesses the property.
We also call the maps which have a stable property, a **stable class**. Examples are the classes of embeddings, local diffeomorphisms, submersions,... as we will learn soon.

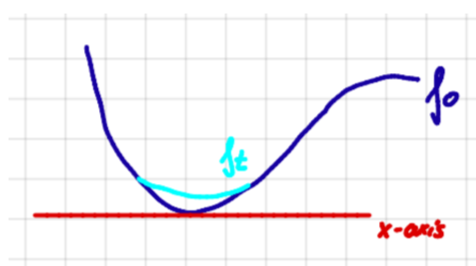
Note that stability is a very natural condition to ask for. For real-world measurements, only stable properties are interesting, since any tiny perturbation of the data would make an unstable property appear or disappear.

In order to get a better idea of stability, let us look at the difference between requiring that things merely intersect or that they intersect transversally:

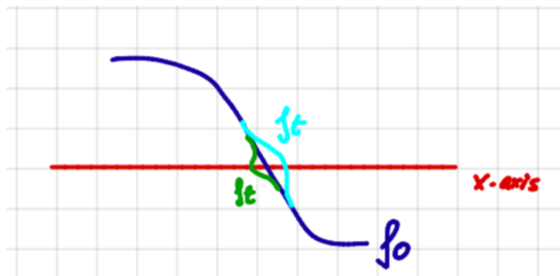
- That a smooth map $f_0\colon \mathbb{R} \to \mathbb{R}^2$ passes through a fixed point in $\mathbb{R}^2$ is **not** a stable property. It disappears immediately.



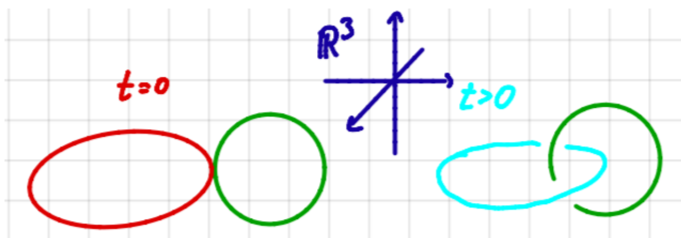
- That a smooth map $f_0\colon \mathbb{R} \to \mathbb{R}^2$ merely intersects the x-axis is **not** a stable property. It disappears immediately.



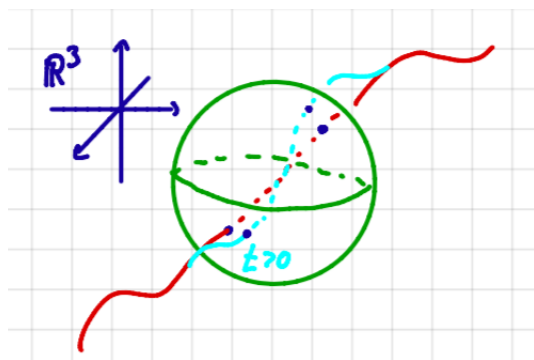
- However, that a smooth map $f_0\colon \mathbb{R} \to \mathbb{R}^2$ intersects the x-axis transversally **is** a stable property. It persists after a small perturbartion.



- That two smooth curves (connected 1-dimensional manifolds) meet in $\mathbb{R}^3$ is **not** a stable property. It disappears immediately.



- That a smooth curve and a smooth surface (2-dimensional manifold) intersect transversally in $\mathbb{R}^3$ **is** a stable property. It persists after a small perturbartion.



This reveals yet another very important feature of transversality. The following theorem tells us that the properties which turned out to be useful for us so far are all stable.

Stability Theorem

The following classes of smooth maps from a **compact** manifold X to a manifold Y are **stable classes**:
(a) local diffeomorphisms.
(b) immersions.

(c) submersions.
(d) maps which are transversal to any specified closed submanifold $Z \subset Y$.
(e) embeddings.
(f) diffeomorphisms.

Proof:

(a) First we note that **local diffeomorphisms** are just immersions in the special case when $\dim X = \dim Y$, so (a) follows from (b).

(b) Assume $f_0 \colon X \to Y$ is an **immersion** and $\dim X = m$. Letf_t be a homotopy of f_0. That f_0 is an immersion means that $d(f_0)_x$ is injective for all $x \in X$. We need to show that there is an $\epsilon > 0$ such that $d(f_t)_x$ is injective for all points (x,t) in $X \times [0,\epsilon) \subset X \times I$.

Given a point $x_0 \in X$, that $d(f_0)_{x_0}$ is injective implies that the matrix representing $d(f_0)_{x_0}$ (in local coordinates) has an $m \times m$-submatrix $A(x_0,0)$ with nonvanishing determinant. Since the determinant is continuous, this submatrix will have **nonvanishing determinant** in an open neighborhood of $(x_0,0)$ in $X \times [0,1]$. Since X **is compact**, **finitely many** such neighborhoods suffice to cover all of $X \times \{0\}$. Hence there is a small $\epsilon > 0$ (it is the minimum for the open intervals $[0,\epsilon_i)$ covering $\{0\}$) such that the intersection of these finitely many neighborhoods contains $X \times [0,\epsilon)$. This is what we needed.

(c) If f_0 is a **submersion**, almost the same argument works. We just need to choose an $n \times n$-submatrix of the surjective map $d(f_0)_x$ with $n = \dim Y$.

(d) Let $Z \subset Y$ be a **closed** submanifold, and assume that f_0 is a map which is **transversal to** Z. Then we have shown that, for every point $x \in X$, there is a smooth function g which sends a neighborhood of $f(x)$ to $0 \in \mathbb{R}^{\operatorname{codim} Z}$ and such that $g \circ f_0$ **is a submersion**. Since Z is closed in Y, $f^{-1}(Z)$ **is closed in** X and therefore **also compact**. Therefore, by (c), there is an $\epsilon > 0$ such that $g \circ f_t$ **is still a submersion** for all $t < \epsilon$. This is means that f_t is still transversal to Z for all $t < \epsilon$.

(e) Assume that f_0 is an **embedding**, and let f_t be a homotopy of f_0. Since X **is compact**, f_0 and each f_t are automaticlly proper maps. Hence we need to show that when f_0 is a one-to-one immersion, then so is f_t in a small neighborhood. We just checked that being an immersion is stable. Hence it remains to show that f_t **is still one-to-one if** t **is small enough**.

Therefor we define a smooth map

$$G\colon X \times I \to Y \times I,\; G(x,t) := (f_t(x),t).$$

Then **if (e) is false**, i.e. if f_t **not one-to-one** in some small neighborhood of 0, then, for every $\epsilon > 0$, we can find a t with $0 < t < \epsilon$ and $x,y \in X$ such that $f_t(x) = f_t(y)$. For example, for every $\epsilon_i = 1/i$, we could find such a t_i, x_i and y_i. Thus there is an **infinite sequence** $t_i \to 0$, and an infinite sequence of points $x_i \neq y_i \in X$ where f_{t_i} fails to be injective, i.e. such that

$$f_{t_i}(x_i) = G(x_i,t_i) = G(y_i,t_i) = f_{t_i}(y_i).$$

Since X **is compact**, we may pass to **subsequences which converges** $x_i \to x_0$ and $y_i \to y_0$. Then

$$G(x_0,0) = \lim_i G(x_i,t_i) = \lim_i G(y_i,t_i) = G(y_0,0).$$

But $G(x_0,0) = f_0(x_0)$ and $G(y_0,0) = f_0(x_0)$. By assumption, f_0 is injective, and **hence** $x_0 = y_0$.

Now, **after choosing local coordinates**, we can express the derivative of G at $(x_0,0)$ by the matrix

$$dG_{(x_0,0)} = \begin{pmatrix} & * \\ d(f_0)_{x_0} & \vdots \\ & * \\ 0\cdots 0 & 1 \end{pmatrix}$$

where the 0's in the lowest row arise from the fact that the first coordinates do not depend on t, and the 1 is the derivative of the function $t \mapsto t$.

Since f_0 is an **immersion**, $d(f_0)_{x_0}$ has $k = \dim X$ independent rows. Thus the matrix of $dG_{(x_0,0)}$ has $k+1$ independent rows, and hence $dG_{(x_0,0)}$ is an injective linear map. Thus, G is an **immersion around** $(x_0,0)$ and hence G must be **one-to-one on some neighborhood of** $(x_0,0)$. **But**, since the sequences (x_i,t_i) and (y_i,t_i) both **converge to** $(x_0,0)$, for large i, both (x_i,t_i) and (y_i,t_i) belong to this neighborhood. This contradicts the injectivity of G.

(f) Assume that $f_0\colon X \to Y$ is a diffeomorphism. Since X is compact, this implies that Y **is compact as well**. Let f_t be a homotopy of f_0. We need to show that there is an $\epsilon > 0$ such that f_t is diffeomorphism for all $t < \epsilon$.

Since X **is compact**, X has only finitely many connected components, and so does Y. Hence we can check the statement for each of these connected components separately. For, this gives us an ϵ_i for each component. Since there are finitely many components, we can just take the minimum of the ϵ_i's as the ϵ for all of X and Y.

Thus we may assume that **X and Y are connceted**. By (a) and (e), we know that being a local diffeomorphism and being an embedding is a stable property. Thus there is a $\epsilon > 0$ such that f_t is a local diffeomorphism and an embedding. For f_t being a diffeomorphism, it remains to show that f_t is surjective.

We fix a $t < \epsilon$. Since f_t is a local diffeomorphism, it is open and hence **$f_t(X)$ is open in Y**. But $f_t(X)$ is also closed, since it is compact being the image of a compact space. Since Y is connected, this implies $f_t(X) = Y$. **QED**

Note that the condition that Z is **closed in Y** in point (d) is **necessary**. For a simple example, in $Y = \mathbb{R}^2$ we consider the subspace

$$Z = \{(x,y) \in \mathbb{R}^2 : 0 < x < 1, y = 0\} \subset \mathbb{R}^2$$

(which is just image of an interval in $\mathbb{R}^2$). It a subspace which is neither open nor closed in $\mathbb{R}^2$. But Z is a one-dimensional submanifold of $\mathbb{R}^2$. Now, for $X = [-1,0]$, we define f_0 to be the smooth map

$$f_0 \colon [-1,0] \to \mathbb{R}^2,\ x \mapsto (x,0).$$

Since $f_0^{-1}(Z) = \emptyset$, **f_0 is transversal to Z**. But, for the homotopy f_t, given by

$$f_t \colon [-1,0] \times [0,1] \to \mathbb{R}^2,\ (x,t) \mapsto (x+t,0),$$

we have $f_t^{-1}(Z) \neq \emptyset$ **for every $t > 0$**. But both $\mathrm{Im}\,(df_x)$ and $T_{f(x)}(Z)$ are just $\mathbb{R}$ embedded as the x-axis in $\mathbb{R}^2 = T_{f(x)}(Y)$. Hence f_t is **not transversal** to Z **for any $t > 0$**. Note that this would not have happened if Z had been the closed submanifold $\{(x,0) : 0 \le x \le 1\}$.

An even more important assumption we made in the theorem is that X is **compact**. The next example will show that we cannot drop this assumption for any of the properties in theorem.

Compactness matters

The Stability Theorem fails when X is **not compact**. For a simple example, let $\rho \colon \mathbb{R} \to \mathbb{R}$ be a smooth function with $\rho(s) = 1$ for $|s| < 1$ and $\rho(s) = 0$ for $|s| > 2$. Then we define

$$f_t \colon \mathbb{R} \to \mathbb{R},\ f_t(x) = x\rho(tx).$$

For $t = 0$, $f_0(x) = x$ for all x, i.e. $f_0 = \mathrm{Id}$. Hence f_0 is a local diffeomorphism, an immersion, a submersion, an embedding, a diffeomorphism and transversal to every submanifold of $\mathbb{R}$.

But for **any fixed** $t > 0$, we have $|tx| > 2$ when $x > 2/|t|$. Hence, for this fixed t, $f_t(x) = 0$ **for all** $x > 2/|t|$.
Thus f_t **is neither** a local diffeomorphism, an immersion, a submersion, an embedding, nor a diffeomorphism, and is not transversal to $\{0\} \subset \mathbb{R}$.
We see what is going wrong when we replace the domain with a closed interval, i.e. a compact subspace of $\mathbb{R}$. Say $X = [a,b]$ with $b > 0$. Then we can choose $\epsilon > 0$ which is small enough such that $1/\epsilon > \max(|a|,|b|)$, and it would not be possible to choose x bigger than $1/|t|$. Then we had $f_t(x) = x$ for all x and all $t < \epsilon$.