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Department of Mathematical Sciences

Examination paper for
TMA4145 Continuation exam - solutions

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Other information:
There are 5 problems on the exam and each problem counts for 20 points. All solutions should be stated in a precise and rigorous way, with any assumptions written down and arguments justified.

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Problem 1

a) (5 points)
Let A be an $m \times n$ matrix. State the singular value decomposition of A and describe all its building blocks.

b) (15 points)
Determine the singular value decomposition of

$$A = \begin{pmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

and express the inverse of A in terms of its singular value decomposition.

Solution.
a) Given an $m \times n$-matrix A with rank r, the singular value decomposition states that we can find a unitary $m \times m$ matrix U, a unitary $n \times n$ matrix V and a diagonal $m \times n$ matrix Σ such that

$$A = U\Sigma V^*.$$

Σ has the singular values $\sigma_1, \sigma_2, ..., \sigma_r$ of A (i.e. the square roots of the eigenvalues of either A^*A or AA^*) in the first r entries of the diagonal and zeros elsewhere.

b) Recall that the singular values of A are the eigenvalues of A^*A, or equivalently of AA^*. We will use A^*A. First note that

$$A^* = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ -1 & 0 & 1 \end{pmatrix},$$

and then calculate that

$$A^*A = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & 1 \\ -1 & 1 & 2 \end{pmatrix}.$$

Finding Σ
To find the eigenvalues $\lambda_i = \sigma_i^2$ of A^*A we need to solve $\det(A^*A - \lambda I) = 0$ for λ. Written out, this equation is

$$\lambda^3 - 6\lambda^2 + 9\lambda - 4 = 0.$$

The roots of this polynomial are $4, 1$ and 1: by trial and error, one finds that 1 is a root. By dividing the polynomial $\lambda^3 - 6\lambda^2 + 9\lambda - 4$ by $\lambda - 1$ we get

$$\frac{\lambda^3 - 6\lambda^2 + 9\lambda - 4}{\lambda - 1} = \lambda^2 - 5\lambda + 4,$$

and this quadratic polynomial has roots 1 and 4, which one can find either by inspection or by using the quadratic formula. Therefore the singular values of A are $\sqrt{4} = 2, 1$ and 1, so

$$\Sigma = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Finding V: To find V, we need the eigenvectors of A^*A. We start with $\lambda = 4$. The eigenvectors form the solution space for $A - 4I = 0$, and solving this equation by row reduction gives us the normalized eigenvector

$$v_1 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}.$$

A similar calculation for $\lambda = 1$ leads to the orthonormal eigenvectors

$$v_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \qquad v_3 = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}.$$

The two eigenvectors you first find for $\lambda = 1$ might not be orthogonal. In that case you will need to use the Gram-Schmidt method to produce an orthogonal pair of eigenvectors. In conclusion:

$$V = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \end{pmatrix}.$$

Finding U: The columns u_1, u_2, u_3 of U are obtained by $u_i = \frac{1}{\sigma_i} A v_i$, where σ_i are the singular values of A. By calculating these matrix products we find that

$$u_1 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} \qquad u_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \qquad u_3 = \frac{1}{\sqrt{6}} \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix}.$$

Hence

$$U = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{3}} & 0 & -\frac{2}{\sqrt{6}} \\ -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{pmatrix}.$$

The singular value decomposition of A is

$$A = U\Sigma V^* = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{3}} & 0 & -\frac{2}{\sqrt{6}} \\ -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} \end{pmatrix}.$$

Finally, we use this to express the inverse of A. We know that $A = U\Sigma V^*$, hence $A^{-1} = (V^*)^{-1}\Sigma^{-1}U^{-1}$. Luckily, the inverses of the the building blocks of the SVD are easy to find. U and V^* are unitary, so their inverses are U^* and $(V^*)^* = V$. Σ is diagonal, so its inverse is obtained by taking the inverse of the diagonal elements. This means that

$$A^{-1} = (V^*)^{-1}\Sigma^{-1}U^{-1} = V\Sigma^{-1}U^* = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \end{pmatrix} \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{pmatrix}.$$

Problem 2 Let T be a bounded linear operator on a Hilbert space X and we denote the operator norm of T by $\|T\|$.

a) (10 points) Show that $\|T\| = \|T^*\|$, where T^* is the adjoint of T.

b) (10 points)
Show that $\|T^*T\| = \|T\|^2$.

Solution.
a) We start by showing that $\|T\| \leq \|T^*\|$. By the definition of the operator norm it will suffice to show that $\|Tx\| \leq \|T^*\|\|x\|$ for any $x \in X$. Since we want to use the adjoint operator, we write the norm of Tx as an inner product, and find that

$$\begin{aligned} \|Tx\|^2 &= \langle Tx, Tx\rangle \\ &= \langle T^*Tx, x\rangle \\ &\leq \|T^*Tx\|\|x\| \\ &\leq \|T^*\|\|Tx\|\|x\|. \end{aligned}$$

The first inequality is Cauchy-Schwarz, and the second inequality ($\|T^*x\| \leq \|T^*\|\|x\|$) follows from the definition of the operator norm of T^*. By dividing both sides of the inequality by $\|Tx\|$, we obtain $\|Tx\| \leq \|T^*\|\|x\|$. Hence $\|T\| \leq \|T^*\|$. Since $(T^*)^* = T$, we also have that $\|T^*\| \leq \|(T^*)^*\| = \|T\|$, so $\|T^*\| = \|T\|$.

b) We know that $\|ST\| \leq \|S\|\|T\|$ for two operators S and T. In particular, $\|T^*T\| \leq \|T^*\|\|T\| = \|T\|^2$, where the last equality is part a). To show the opposite inequality we proceed as we did in part a):

$$\begin{aligned} \|Tx\|^2 &= \langle Tx, Tx\rangle \\ &= \langle T^*Tx, x\rangle \\ &\leq \|T^*Tx\|\|x\| \\ &\leq \|T^*T\|\|x\|^2, \end{aligned}$$

where only the last step differs from part a). This shows that $\|T\|^2 \leq \|T^*T\|$.

Problem 3

a) (10 points)

(1) Suppose $(X, \|.\|_X)$ and $(Y, \|.\|_Y)$ are normed spaces. Define for a linear transformation $T : X \to Y$ the *operator norm* of T.

(2) Let X be a vector space and let $\|.\|_a$ and $\|.\|_b$ be two norms on X. Define when the norms $\|.\|_a$ and $\|.\|_b$ are *equivalent* on X.

(3) Suppose $(x_n)_{n\in\mathbb{N}}$ is a sequence in a normed space $(X, \|.\|_X)$. Define the *series* $\sum_{n=1}^{\infty} x_n$ of $(x_n)_{n\in\mathbb{N}}$.

(4) Let M be a subset of an innerproduct space $(X, \langle ., .\rangle)$. Define the *orthogonal complement* of M.

(5) Let T be a linear transformation on a finite-dimensional vector space X. Define the *characteristic polynomial* and the *minimal polynomial* of T.

b) (10 points)
Determine if the following statements are true or false and if the statement is not true, give a counterexample.

(1) A linear transformation T between the normed spaces $(X, \|.\|_X)$ and $(Y, \|.\|_Y)$ is continuous if and only if T is a bounded operator.

(2) Any linear transformation on a finite-dimensional vector space is unitarily equivalent to an upper-triangular matrix.

(3) Any Cauchy sequence in a normed space $(X, \|.\|_X)$ converges to an element in X.

(4) Let X be an infinite-dimensional Hilbert space. Then any isometric linear operator on X is a unitary operator on X.

(5) The kernel of any bounded linear map on an infinite-dimensional normed space $(X, \|.\|_X)$ is closed.

Solution.
a)

(1) $\|T\| = \sup\left\{\frac{\|Tx\|_Y}{\|x\|_X} : x \neq 0\right\}$. As we have seen in the lectures, there are some other, equivalent expressions too. These will of course also be acceptable answers.

(2) The norms are equivalent if there exist two positive constants C_1, C_2 such that $C_1\|x\|_a \leq \|x\|_b \leq C_2\|x\|_a$ for any $x \in X$.

(3) The series $\sum_{n=1}^{\infty} x_n$ is the limit of the sequence of partial sums $(s_N)_{N\in\mathbb{N}}$, where $s_N = \sum_{n=1}^{N} x_n$, when this limit exists.

(4) The orthogonal complement of M is the set $M^\perp = \{x \in X : \langle x, y\rangle = 0 \text{ for any } y \in M\}$.

(5) Let A be a matrix representing T. The characteristic polynomial of T is the polynomial $p_A(z) = \det(zI - A)$, where I is the identity matrix[1]. The minimal polynomial m of T is the monic polynomial of smallest degree such that $m(T) = 0$.

b)

(1) True.

(2) True.

(3) False. This is only true for Banach spaces. Let the normed space be the set $\mathbb{R} \setminus \{0\}$ of all the real numbers except the origin, with norm given by the absolute value. The sequence $(1/n)_{n\in\mathbb{N}}$ is Cauchy, but does not converge to an element in our space (in $\mathbb{R}$ it converges to 0, which does not belong to our space).

[1] Of course, the characteristic polynomial is independent of which matrix representation we choose.

(4) False. We have seen that a linear operator $T : X \to X$ is an isometry if and only if $T^*T = I$, and by definition T is unitary if $T^*T = TT^* = I$. So we need to find an operator satisfying $T^*T = I$ and $TT^* \neq I$. Let T be the left-shift operator on the Hilbert space $\ell^2(\mathbb{N})$, i.e.

$$T(x_1, x_2, \dots) = (0, x_1, x_2, \dots).$$

In the lectures we have seen that the adjoint of T is the right-shift operator

$$T^*(x_1, x_2, \dots) = (x_2, x_3, \dots).$$

Clearly we have that $T^*T = I$, but

$$TT^*(x_1, x_2, \dots) = (0, x_2, x_3, \dots),$$

so $TT^* \neq I$.
Another approach would be to note that the left-shift operator clearly is an isometry, but not surjective. Since a unitary operator is invertible, it must in particular be surjective. Therefore the left-shift operator cannot be unitary.

(5) True.

Problem 4

a) (10 points)
Let $T : (C[1,3], \|.\|_\infty) \to (C[1,3], \|.\|_\infty)$ be given by $Tf(x) = \int_1^3 \alpha e^{-(x-y)} f(y)dy$ for some positive real number α.

(1) Show that T is a bounded operator.

(2) Determine the operator norm of T.

(3) Determine the set of α's for which T is a contraction.

b) (10 points)

(1) Give an example of a linear operator on a normed space that is not bounded.

(2) Let T be a linear operator on $(X, \|.\|)$ that is not bounded. Show that then X has to be infinite-dimensional.

Solution.
a) Let us first note that we can simplify the expression for T a bit, since $Tf(x) = \alpha e^{-x} \int_1^3 e^y f(y)\, dy$.

(1) We find for $f \in C[1,3]$ that

$$\begin{aligned}|Tf(x)| &= \alpha e^{-x} \left| \int_1^3 e^y f(y) dy \right| \\ &\leq \alpha e^{-x} \int_1^3 |e^y f(y)|\, dy \\ &\leq \alpha e^{-x} \|f\|_\infty \int_1^3 e^y dy \\ &= \alpha e^{-x} \|f\|_\infty (e^3 - e).\end{aligned}$$

Since $e^{-x} \leq e^{-1}$ for $x \in [1,3]$, we get that

$$\|Tf\|_\infty \leq \alpha(e^2 - 1)\|f\|_\infty.$$

Thus T is bounded, with $\|T\| \leq \alpha(e^2 - 1)$.

(2) We will show that the operator norm of T is $\alpha(e^2 - 1)$. Let f be the constant function $f(x) = 1$ for $x \in [1,3]$, and note that $\|f\|_\infty = 1$. Then

$$Tf(x) = \alpha e^{-x} \int_1^3 e^y\ dy = \alpha e^{-x}(e^3 - e),$$

hence $\|Tf\|_\infty = \alpha(e^2 - 1)$. This shows[2] that $\|T\| \geq \alpha(e^2 - 1)$, and combining this with part a) we get that $\|T\| = \alpha(e^2 - 1)$.

(3) T is a contraction when $\|T\| < 1$. This happens when $\alpha(e^2 - 1) < 1$, in other words when $\alpha \in (0, \frac{1}{e^2-1})$.

b)

(1) One example is the differentiation operator $\frac{d}{dx}$ on the normed space $(C^\infty(0,1), \|\cdot\|_\infty)$ where $C^\infty(0,1)$ is the space of functions on $(0,1)$ that are differentiable infinitely many times. $\frac{d}{dx}$ is a linear operator on $C^\infty(0,1)$, but is not bounded. For instance, if $f_n(x) = e^{inx}$ for $n \in \mathbb{N}$, then

$$\|f_n\|_\infty = 1, \qquad \frac{d}{dx} f_n(x) = in f_n(x) \qquad \|\frac{d}{dx} f_n\|_\infty = n.$$

[2]By definition, $\|T\|$ is an upper bound for $\left\{ \frac{\|Tf\|_\infty}{\|f\|_\infty} : f \neq 0 \right\}$. Since we have found an f with $\|f\|_\infty = 1$ and $\|Tf\|_\infty = \alpha(e^2 - 1)$, we must in particular have that $\|T\|$ is greater than $\frac{\alpha(e^2-1)}{1}$.

Hence

$$\|\frac{d}{dx}\| = \sup_{f\neq 0} \frac{\|\frac{df}{dx}\|_\infty}{\|f\|_\infty} \geq n$$

for any $n \in \mathbb{N}$, showing that the operator is unbounded.

(2) We will prove the contrapositive, which in this case states that any linear operator on a finite-dimensional normed space is bounded. Assume therefore that $(X, \|\cdot\|)$ is a finite-dimensional normed space, and let $\{e_1, e_2, \ldots, e_n\}$ be a basis for X. If $x \in X$, then x has a unique basis expansion $x = \sum_{i=1}^n a_i e_i$ for a sequence $(a_n)_{n\in\mathbb{N}}$ of scalars. We define a new norm on X by

$$\|x\|_1 = \sum_{i=1}^n |a_i|.$$

Note that $\|e_i\|_1 = 1$ for $i = 1, 2, \ldots n$. Now define the matrix $(b_{i,j})_{i,j=1}^n$ by the basis expansion $Te_i = \sum_{j=1}^n b_{i,j} e_j$. We will show that T is bounded with respect to the norm $\|\cdot\|_1$.

$$\begin{aligned}
\|Tx\|_1 &= \|T\sum_{i=1}^n a_i e_i\|_1 \\
&= \|\sum_{i=1}^n a_i T e_i\|_1 \\
&= \|\sum_{i,j=1}^n a_i b_{i,j} e_j\|_1 \\
&\leq \sum_{i,j=1}^n |a_i||b_{i,j}|\|e_j\|_1 \\
&\leq \sup_{i,j} |b_{i,j}| \sum_{i=1}^n |a_i| \\
&= \sup_{i,j} |b_{i,j}|\|x\|_1.
\end{aligned}$$

Hence T is bounded with respect to the norm $\|\cdot\|_1$, with norm $\leq \sup_{i,j} |b_{i,j}|$. Since all norms on a finite-dimensional space are equivalent, we can find constants $C_1, C_2 > 0$ such that

$$C_1\|y\| \leq \|y\|_1 \leq C_2\|y\| \quad \text{for any } y \in X.$$

From this we get that

$$C_1\|Tx\| \leq \|Tx\|_1 \leq \sup_{i,j} |b_{i,j}|\|x\|_1 \leq C_2 \sup_{i,j} |b_{i,j}\|x\|,$$

and if we divide both sides of this inequality by C_1 we have $\|Tx\| \leq \sup_{i,j} |b_{i,j}| \frac{C_2}{C_1}\|x\|$, hence T is bounded with respect to the norm $\|\cdot\|$.

Problem 5 (20 points)
Let $M_e = \{f \in L^2[-2,2] : f(-x) = f(x)\}$ be the subspace of even functions of $L^2[-2,2]$ and $M_o = \{f \in L^2[-2,2] : f(-x) = -f(x)\}$ be the subspace of odd functions of $L^2[-2,2]$.

(1) Show that M_e is closed.

(2) Determine the orthogonal complement of M_o.

(3) Find the projection onto $M_o^\perp$.

(4) Show that $M_o \cap M_o^\perp = \{0\}$.

Solution. Note: there are some minor technicalities related to this problem. Elements of $L^2[-2,2]$ are actually not functions, but equivalence classes of functions. Since this was not covered in the course, the students' solutions were not required to comment on this.

(1) The easiest way to prove this, is to use the result from (2), namely that the orthogonal complement of M_o is M_e. We know that orthogonal complements are always closed.

(2) We claim the $M_e^\perp = M_o$. If $f \in M_e$ and $g \in M_o$, then

$$\langle f, g \rangle = \int_{-2}^{2} f(t)\overline{g(t)}\, dt = 0,$$

since the integrand $f\overline{g}$ is an odd function and we integrate from -2 to 2. This shows that $M_o \subset M_e^\perp$. Now assume that $f \in M_e^\perp$, and recall that any function can be written as a sum of an odd and an even function:

$$f(x) = \frac{f(x) + f(-x)}{2} + \frac{f(x) - f(-x)}{2} := f_e(x) + f_o(x).$$

Since $f_e \in M_e$ and $f \in M_e^\perp$, we find, using the fact that $\langle f_e, f_o \rangle = 0$ by $M_o \subset M_e^\perp$, that

$$0 = \langle f, f_e \rangle = \langle f_e + f_o, f_e \rangle = \langle f_e, f_e \rangle = \|f_e\|^2,$$

hence $f_e = 0$. Thus $f = f_e + f_o = f_o$, and we see that $f \in M_o$. This proves that $M_e^\perp \subset M_o$, so $M_e^\perp = M_o$.

(3) The projection Pf of a function f onto the closed subspace $M_e = M_o^\perp$ is given by writing f as a sum $f = f_e + f_o$ with $f_e \in M_e$ and $f_o \in M_e^\perp$, and selecting $Pf = f_e$. We saw in part (2) that we can write

$$f(x) = \frac{f(x) + f(-x)}{2} + \frac{f(x) - f(-x)}{2} := f_e(x) + f_o(x),$$

hence $Pf(x) = f_e(x) = \frac{f(x)+f(-x)}{2}$.

(4) Assume that $f \in M_o \cap M_o^\perp$. Since f belongs to both M_o and the orthogonal complement of M_o, $\|f\| = \langle f, f\rangle = 0$, hence $f = 0$. We have used that $f \in M_o \cap M_o^\perp$ to get that $\langle f, f\rangle = 0$.