Mathemathical Modelling

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1 Lecture 1

1.1 Practical Information

You need to know

- Separable 1. order equations.
- Linear 1. order equations.
- 2. order linear equations with constant coefficients.

1.2 Dimensional Analysis

Basic facts

- Any physical relation has to make sense dimensionally.
- Any physical relation must be valid for any choice of fundamental units.

Remark.

Make sure remark looks better

- **Forbidden** $3m + 2kg = ?$
- $m = f(x, t)$ is legal
- e^{-t} and $s = 5t^2$, is nonsense
- **Dimension** is length, mass , energy, etc.
- **Unit** is meter, feet, year, etc

Given a variable R, we write $R = \overbrace{v(R)}^{\text{numerical value}} \underbrace{[R]}_{\text{unit}}$.

If we have a physical relation that is dimensionall correct that

$$f(R_1, R_2, \ldots, R_n) = 0 \quad \to \quad f(v(R_1), v(R_2), \ldots, v(R_n)) = 0$$

1.3 Fundamental Units

Given units $F_1, F_2, \ldots, F_m$ for fundamental if

$$F_1^{\alpha_1}, F_2^{\alpha_2}, \ldots, F_m^{\alpha m} = 0 \quad \to \quad \alpha_1 = \alpha_2 = \ldots = 0$$

This units are then independent.

Example 1. The units kg, m, s are independent.

Example 2. In a right angle triangle with angle α and hypothenus c. We know the area A is uniquely determined by α and c

$$A = f(c, \alpha)$$

α is dimensialless since $\alpha = \frac{s}{r}$. Since A scales as the square of the length, then is

$$\begin{aligned} f(ac, \alpha) &= a^2 f(c, \alpha) \\ c = 1 \to f(a, \alpha) &= a^2 f(1, \alpha) = a^2 h(\alpha) \end{aligned}$$

Which then ends up with the relation

$$A = a^2 h(\alpha)$$

Make corollary environmet

Lets derive $A = a^2 h(\alpha)$ somwhat differently. We know there is a relation $f(A, c, \alpha) = 0$. We want to introduce new variables.

$$\Pi_1 = \frac{A}{c^2}, \quad c = c_1, \quad \alpha = \alpha_1$$

which means $f\left(c^2\Pi_1, c, \alpha\right) = 0$ and $h(\Pi_1, \alpha, c) = 0$. h must be dimensially consistent $\to$ h must be independent of c.

$$\begin{aligned} h(\Pi_1, \alpha) = 0 \leftrightarrow \Pi_1 = k(\alpha) \\ \to \frac{A}{c^2} = k(\alpha) \quad \leftrightarrow \quad A = c^2 k(\alpha) \end{aligned}.$$

1.4 Trinity of the first atomic blast

We assume there is a relation

$$f(E, \rho, r, t) = 0$$

- Energy: E, $[E] = kgm^2s^{-2}$
- Mass density of air: ρ, $[\rho] = kg^{-3}$
- Radius: r , $[r] = m$
- Time: t, $[t] = s$

We choose 3 independent variables, say r, t, ρ. Also we call r, t, ρ **core variables**. Let is define a dimensionalless number Π_1 such that

$$[\Pi_1] = 0$$

The relation is now given by $h(\Pi, t, r, \rho) = 0$, where h is independent of t , r and ρ. Which in fact is $h(\Pi) = 0$, where $\Pi_1 = c$ s.t. $[c] = 1$.

Given by the definitnion is

$$\frac{Et^2}{\rho r^5} = c \quad \rightarrow \quad E = \frac{c\rho r^5}{t^2}$$

Using $\rho = 12kgm^{-3}$, $r = 110m$, $t = 6 \cdot 10^{-3}$ do we end up with the relation

$$E = c \cdot 7.5 \cdot 10^{13} J$$

1.5 Steady-state single phase flow in a uniform straight pipeline

Figure of a pipe

Pipe with flow u, length L and pressure drop Δp Then there is a relation between

- L : length, $[L] = m$
- D: diameter $[D] = m$
- u: flow rate $[u] = ms^{-1}$
- Δp: Pressure drop, $\left[\Delta kgm^{-1}s^{-2}\right]$
- μ: (Shear) viscousity $[\mu] = kgm^{-1}s^{-1}$
- ρ: mass density: $[\rho] = kgm^{-3}$
- E: Wall roughness: $[E] = m$

We have to choose 3 core variables and they are not unique. Since we have 3 independent units ρ, u, D are independent such that it can be a core variable:

$$\Pi_1 = \frac{L}{D} \quad , \quad \Pi_2 = \frac{\Delta p}{\rho u^2} \quad , \quad \Pi_3 = \frac{\rho}{\mu} \quad , \quad \Pi_4 = \frac{E}{D}$$

Then the relation is

$$f\left(\Pi_1, \Pi_2, \Pi_3, \Pi^4, \rho, D, u\right) = 0 \quad \Pi_2 = h\left(\Pi_1, \Pi_3, \Pi_4\right) \leftrightarrow \frac{\Delta p}{\rho u^2} = h\left(\Pi_1, \Pi_3, \Pi_4\right)$$

$$\rightarrow \frac{\Delta p}{u^2 \rho} = \Pi_1 k\left(\Pi_3, \Pi_4\right)$$

$$\Delta p = u^2 \rho \frac{L}{D} k\left(\frac{\rho D u}{\mu}, \frac{E}{D}\right)$$

$$\text{measure} \quad \frac{\rho D \mu}{\mu} \quad , \quad k = \frac{\Delta p D}{u^2 \rho}$$