Repetition

Theorem

For a birth and death process, under suitable regularity conditions,

$$\begin{aligned}
&P'_{i0}(t) = -\lambda_0 P_{i0}(t) + \mu_1 P_{i1}(t), \quad t \geq 0, \\
&P'_{ij}(t) = \lambda_{j-1} P_{i,j-1}(t) - (\lambda_j + \mu_j) P_{i,j}(t) + \mu_{j+1} P_{i,j+1}(t), \quad t \geq 0, j > 0,
\end{aligned}$$

with initial conditions

$$P_{ij}(0) = \delta_{ij} = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases}$$

These are called the **forward Kolmogorov differential equations**.

Note: Suitable is basically referring to non-explosive behavior. E.g., for a pure birth process we must have $\sum_{i=0}^{\infty} \frac{1}{\lambda_i} = \infty$.

Theorem

For a birth and death process without absorbing states, the limiting probabilities

$$\pi_j = \lim_{t \to \infty} P_{ij}(t), \quad j = 0, 1, \ldots$$

1. exist
2. are not dependent on the state i.

Definition

If $\sum_{j=0}^{\infty} \pi_j = 1$, then $\boldsymbol{\pi} = (\pi_0, \pi_1, \ldots)$ is called the **limiting (probability) distribution**.

Note: The **limiting probabilities** fail to be a **limiting distribution** when $\pi_i = 0$, $i = 0, 1, \ldots$.

Theorem

Let π_j, $j = 0, 1, \ldots$, be the limiting probabilites of a birth and death process without absorbing states. If $\pi_j > 0$, $j = 0, 1, \ldots$, then $\boldsymbol{\pi} = (\pi_0, \pi_1, \ldots)$ is the unique solution of

$$\begin{aligned}
\lambda_0 \pi_0 &= \mu_1 \pi_1, \\
(\lambda_j + \mu_j)\pi_j &= \lambda_{j-1}\pi_{j-1} + \mu_{j+1}\pi_{j+1}, \quad j \geq 1, \\
\sum_{j=0}^{\infty} \pi_j &= 1.
\end{aligned}$$

Note: This is saying that if we have $\pi_j > 0$, $j = 0, 1, \ldots$, we automatically have $\sum_{i=0}^{\infty} \pi_i = 1$.
Note 2: Also works for finite state spaces $\{0, 1, \ldots, N\}$ with $\lambda_N = 0$ and $\mu_N > 0$.

Theorem

When

$$\begin{aligned}
\lambda_0 \pi_0 &= \mu_1 \pi_1, \\
(\lambda_j + \mu_j)\pi_j &= \lambda_{j-1}\pi_{j-1} + \mu_{j+1}\pi_{j+1}, \quad j \geq 1, \\
\sum_{j=0}^{\infty} \pi_j &= 1.
\end{aligned}$$

has a unique solution, it is given by

$$\pi_j = \frac{\theta_j}{\sum_{k=0}^{\infty} \theta_k}, \quad j = 0, 1, \ldots,$$

where

$$\theta_0 = 1, \quad \text{and} \quad \theta_k = \frac{\lambda_0 \lambda_1 \cdots \lambda_{k-1}}{\mu_1 \mu_2 \cdots \mu_k}, \quad k = 1, 2, \ldots.$$