NTNU
Norwegian University of
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Week 38: Lecture 1

Stationary distributions, and introduction to Poisson processes

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Information

— The minutes from the reference group meeting will be put on Blackboard. If you disagree with any thing there or would like to add something, let us know.

Examples: stationary distributions

Find the stationary distributions.

a)

$$\mathbf{P} = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$$

b)

$$\mathbf{P} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

c)

$$\mathbf{P} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Section 4.5

Read this yourselves.

Why do we care so much about Markov chains?

— Importance goes far beyond statistical modelling of physical phenomena.

— In the end of the 80s and start of the 90s, computation power was growing stronger.

— We realized that we could sample from difficult distributions by constructing Markov chains whose stationary distribution matched a desired target distribution.

— The theory we have discussed is a part of the theory developed to show that these methods worked.

Markov chain Monte Carlo

I will show you a simple demonstration. Markov chain Monte Carlo (MCMC) itself is not part of the curriculum in this course, but can be found in TMA4300 Computational Statistics.

The main purpose is to demonstrate the potential of discrete-time Markov chains and motivate their importance.

Section 5.1.1

Poisson distribution

Definition

The stochastic variable X has a **Poisson distribution** with (mean) parameter $\mu > 0$ if

$$p(x) = \frac{\mu^x}{x!}\mathrm{e}^{-\mu}, \quad x = 0, 1, \ldots .$$

We write $X \sim \mathrm{Poisson}(\mu)$.

Theorem

If $X \sim \text{Poisson}(\mu)$*, then*

— $\mathrm{E}[X] = \mu$.

— $\mathrm{Var}[X] = \mu$.

Important!

Theorem (Theorem 5.1)

If $X \sim \mathrm{Poisson}(\mu)$, $Y \sim \mathrm{Poisson}(\nu)$, and X and Y are independent, then

$$X + Y \sim \mathrm{Poisson}(\mu + \nu).$$

We saw this in week 34

Theorem (Theorem 5.2)

If $N \sim \mathrm{Poisson}(\mu)$ *and* $M|N \sim \mathrm{Binomial}(N, p)$, *then*

$$M \sim \mathrm{Poisson}(\mu p).$$

Section 5.1.2

Definition

A **Poisson process** with **rate (intensity)** $\lambda > 0$ is an integer-valued stochastic process $\{X(t) : t \geq 0\}$ for which

1. for any $n > 0$ and any time points $0 = t_0 < t_1 < \cdots < t_n$, the increments

$$X(t_1) - X(t_0), X(t_2) - X(t_1), \ldots, X(t_n) - X(t_{n-1})$$

   are independent.

2. for $s \geq 0$ and $t > 0$,

$$X(s+t) - X(s) \sim \mathrm{Poisson}(\lambda t).$$

3. $X(0) = 0$.

Example

We assume the arrival of customers to a store follows a Poisson process with rate $\lambda = 4$ customers per hour. The store opens at 09:00. What is the probability that exactly one customer has arrived by 09:30 and exactly five customers have arrived by 11:30?

Section 5.1.3

Definition

An **inhomogeneous Poisson process** with **rate (intensity)** $\lambda(t) \geq 0$, $t \geq 0$, is an integer-valued stochastic process $\{X(t) : t \geq 0\}$ for which

1. for any $n > 0$ and any time points $0 = t_0 < t_1 < \cdots < t_n$, the increments

$$X(t_1) - X(t_0), X(t_2) - X(t_1), \ldots, X(t_n) - X(t_{n-1})$$

   are independent.

2. for $s \geq 0$ and $t > 0$,

$$X(s+t) - X(s) \sim \mathrm{Poisson}\left(\int_s^{s+t} \lambda(y)\,\mathrm{d}y\right).$$

3. $X(0) = 0$.

Example

Assume the arrival of customers to a store follows an inhomogeneous Poisson process with rate $\lambda(t) = t$ [customers per hour], $t \geq 0$. Assume the store opens at 09:00. What is the probability that no-one has arrived at 10:00?