NTNU – Trondheim
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Department of Mathematical Sciences

Examination paper for **TMA4145 Linear Methods**

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Examination date: 17.12.2015

Examination time (from–to): 09:00–13:00

Permitted examination support material: D: No written or handwritten material. Calculator Casio fx-82ES PLUS, Citizen SR-270X, Hewlett Packard HP30S

Other information:
The exam consists of twelve questions, and their order is not according to the level of difficulty. All solutions should be stated in a precise and rigorous way, with any assumptions written down and arguments justified. Each solution will be graded as *rudimentary* (F), *acceptable* (D), *good* (C) or *excellent* (A). Five acceptable solutions guarantee an E; seven acceptable with at least one good a D; seven acceptable with at least five good a C; nine good with at least two excellent a B; nine good with at least seven excellent an A. These are guaranteed limits. Beyond that, the grade is based on the total achievement.

Language: English

Number of pages: 3

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Checked by:

Date Signature

Problem 1

a) State (without proof) whether the assertion is true or false.

1. A Lipschitz continuous function is uniformly continuous.
2. The range of a linear operator T on a normed space X is always closed.
3. Suppose f is a function on $\mathbb{R}$. Then $d(x, y) = |f(x) - f(y)|$ defines a metric on $\mathbb{R}$.
4. $\mathbb{R}^n$ with $\|(x_1, x_2, ..., x_n)\|_\infty = \max_i |x_i|$ is complete.
5. A contraction on a non-zero metric space has a unique fixed point.

b) Define the following notions.

1. Define the **orthogonal complement** of a subspace M of a Hilbert space $\mathcal{H}$.
2. Let T be a linear operator between two normed spaces $(X, \|.\|_X)$ and $(Y, \|.\|_Y)$. Define the **operator norm** of T.
3. Let T be a linear mapping on $\mathbb{C}^n$. Define the notion of a **generalized eigenvector**.
4. Suppose f is a function between two metric spaces (X, d_X) and (Y, d_Y). Define the notion of **uniform continuity** for f.
5. Define the notion of a **nilpotent** operator $T : V \to V$ for a finite-dimensional vector space V.

Problem 2 Let T be the linear operator on the space of polynomials $\mathcal{P}_2$ of degree at most 2 defined by $Tf(x) = -f(x) - f'(x)$.

a) Find the matrix representation of T with respect to the basis $1, x, x^2$ of $\mathcal{P}_2$ and its characteristic polynomial.

b) Find the generalized eigenvectors and eigenvalues of T. Determine a basis for the space of generalized eigenvectors.

Problem 3 We consider the matrix $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$.

a) Determine the eigenvalues and eigenvectors of A^*A.

b) Find the Singular Value Decomposition of A.

Problem 4 Let T be the linear operator defined on ℓ^2 by

$$T(x_1, x_2, ...) = (0, 2x_1, x_2, 2x_3, ...).$$

a) Show that T is a bounded operator on ℓ^2 and determine the adjoint of T. (The operator may be viewed as the composition of a multiplication operator and the left shift operator on ℓ^2.)

b) Determine the kernel and the range of T. Use these results to find the kernel and the range of T^*.

Problem 5 Let $(X, \langle ., . \rangle)$ be an inner product space.

a) Define the linear functional $\varphi_y(x) := \langle x, y \rangle$ for a fixed $y \in X$. Show that $\varphi_y : X \to \mathbb{C}$ is bounded.

b) Suppose that (x_n) and (y_n) are convergent sequences, with $\lim_n x_n = x$ and $\lim_n y_n = y$. Show that $\lim_n \langle x_n, y_n \rangle = \langle x, y \rangle$.

Problem 6 We define the following matrix and vector:

$$A = \begin{pmatrix} 5 & 1 & 0 \\ 2 & 8 & 0 \\ 0 & 1 & 3 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}.$$

a) Use Banach's fixed point theorem to solve $Ax = b$ for the normed space $(\mathbb{R}^3, \|.\|_\infty)$. In other words, write $Ax = b$ in the form $x = Bx + c$ such that B is a contraction with a constant K on $\mathbb{R}^3$ with respect to the norm $\|y\|_\infty = \max\{|y_1|, |y_2|, |y_3|\}$. Then show that one can solve this problem by iteration starting from any $x_0 \in \mathbb{R}^3$.

b) We denote the fixed point of the problem by $\tilde{x}$ and by (x_n) the sequence of iterations. Show that

$$d_\infty(x_n, \tilde{x}) \leq \frac{K^n}{1-K} d_\infty(x_0, x_1),$$

where K is the contraction constant K from part a).