NTNU
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Department of Mathematical Sciences

# Examination paper for **TMA4145 Linear Methods**

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**Examination date:**

**Examination time (from–to):**

**Permitted examination support material:** D:No written or handwritten material. Calculator Casio fx-82ES PLUS, Citizen SR-270X, Hewlett Packard HP30S

**Other information:**
There are 5 problems on the exam and each problem counts for 20 points. All solutions should be stated in a precise and rigorous way, with any assumptions written down and arguments justified.

**Language:** English

**Number of pages:** 3

**Number of pages enclosed:** 0

**Checked by:**

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| **Informasjon om trykking av eksamensoppgave** |
| **Originalen er:** |
| **1-sidig** ☐ **2-sidig** ☐ |
| **sort/hvit** ☐ **farger** ☐ |
| **skal ha flervalgskjema** ☐ |

Date Signature

**Problem 1**

**a)** (5 points)
Let $A$ be an $m \times n$ matrix. State the singular value decomposition of $A$ and describe all its building blocks.

**b)** (15 points)
Determine the singular value decomposition of

$$A = \begin{pmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

and express the inverse of $A$ in terms of its singular value decomposition.

**Problem 2** Let $T$ be a bounded linear operator on a Hilbert space $X$ and we denote the operator norm of $T$ by $\|T\|$.

**a)** (10 points) Show that $\|T\| = \|T^*\|$, where $T^*$ is the adjoint of $T$.

**b)** (10 points)
Show that $\|T^*T\| = \|T\|^2$.

**Problem 3**

**a)** (10 points)

(1) Suppose $(X, \|.\|_X)$ and $(Y, \|.\|_Y)$ are normed spaces. Define for a linear transformation $T : X \to Y$ the *operator norm* of T.

(2) Let $X$ be a vector space and let $\|.\|_a$ and $\|.\|_b$ be two norms on $X$. Define when the norms $\|.\|_a$ and $\|.\|_b$ are *equivalent* on $X$.

(3) Suppose $(x_n)_{n\in\mathbb{N}}$ is a sequence in a normed space $(X, \|.\|_X)$. Define the *series* $\sum_{n=1}^{\infty} x_n$ of $(x_n)_{n\in\mathbb{N}}$.

(4) Let $M$ be a subset of an innerproduct space $(X, \langle ., . \rangle)$. Define the *orthogonal complement* of $M$.

(5) Let $T$ be a linear transformation on a finite-dimensional vector space $X$. Define the *characteristic polynomial* and the *minimal polynomial* of $T$.

**b)** (10 points)
Determine if the following statements are true or false and if the statement is not true, give a counterexample.

(1) A linear transformation $T$ between the normed spaces $(X, \|.\|_X)$ and $(Y, \|.\|_Y)$ is continuous if and only if $T$ is a bounded operator.

(2) Any linear transformation on a finite-dimensional vector space is unitarily equivalent to an upper-triangular matrix.

(3) Any Cauchy sequence in a normed space $(X, \|.\|_X)$ converges to an element in $X$.

(4) Let $X$ be an infinite-dimensional Hilbert space. Then any isometric linear operator on $X$ is a unitary operator on $X$.

(5) The kernel of any bounded linear map on an infinite-dimensional normed space $(X, \|.\|_X)$ is closed.

## Problem 4

**a)** (10 points)
Let $T : (C[1,3], \|.\|_\infty) \to (C[1,3], \|.\|_\infty)$ be given by $Tf(x) = \int_1^3 \alpha e^{-(x-y)} f(y)dy$ for some positive real number $\alpha$.

(1) Show that $T$ is a bounded operator.

(2) Determine the operator norm of $T$.

(3) Determine the set of $\alpha$'s for which $T$ is a contraction.

**b)** (10 points)

(1) Give an example of a linear operator on a normed space that is not bounded.

(2) Let $T$ be a linear operator on $(X, \|.\|)$ that is not bounded. Show that then $X$ has to be infinite-dimensional.

## Problem 5 (20 points)

Let $M_e = \{f \in L^2[-2,2] : f(-x) = f(x)\}$ be the subspace of even functions of $L^2[-2,2]$ and $M_o = \{f \in L^2[-2,2] : f(-x) = -f(x)\}$ be the subspace of odd functions of $L^2[-2,2]$.

(1) Show that $M_e$ is closed.

(2) Determine the orthogonal complement of $M_o$.

(3) Find the projection onto $M_o^{\perp}$.

(4) Show that $M_o \cap M_o^{\perp} = \{0\}$.