TMA4190 Introduction to Topology Spring 2018

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Exercise set 6

1 As a first test of our understanding of transversality, answer the following questions:

a) Let $z = (a, b) \in S^1 \subseteq \mathbb{R}^2$ and let $N_z = \{(a, y) : y \in \mathbb{R}\}$ be the vertical line intersecting the circle at z. When is $S^1 \subseteq \mathbb{R}^2$ transverse to $N_z \subseteq \mathbb{R}^2$?

b) Which of the following linear spaces intersect transversally?

- The plane spanned by $\{(1, 0, 0), (2, 1, 1)\}$ and the y-axis in $\mathbb{R}^3$.
- $\mathbb{R}^k \times \{0\}$ and $\{0\} \times \mathbb{R}^l$ in $\mathbb{R}^n$. (The answer depends on k, l, and n.)
- $V \times \{0\}$ and the diagonal in $V \times V$, for a real vector space V.
- The spaces of symmetric ($A^t = A$) and skew symmetric ($A^t = -A$) matrices in $M(n)$.

c) Do $SL(n)$ and $O(n)$ meet transversally in $M(n)$?

2 **a)** Let $f \colon X \to Y$ be a map transversal to a submanifold Z in Y. Then we know that $W = f^{-1}(Z)$ is a submanifold of X. Prove that $T_x(W)$ is the preimage of $T_{f(x)}(Z)$ under the linear map $df_x \colon T_x(X) \to T_{f(x)}(Y)$.

b) Let X and Z be transversal submanifolds of Y. Deduce from the previous point that, for every $y \in X \cap Z$,

$$T_y(X \cap Z) = T_y(X) \cap T_y(Z).$$

3 Let V be a vector space, and let Δ be the diagonal of $V \times V$. For a linear map $A \colon V \to V$, consider the graph $\Gamma(A) = \{(v, Av) : v \in V\}$. Show that $\Gamma(A) \pitchfork \Delta$ if and only if $+1$ is not an eigenvalue of A.

4 Let $f \colon X \to X$ be a map, and let x be a fixed point of f, i.e. $f(x) = x$. If $+1$ is not an eigenvalue of $df_x \colon T_x(X) \to T_x(X)$, then x is called a *Lefschetz fixed point* of f. The map f is called a *Lefschetz map* if all its fixed points are Lefschetz. Prove that if X is compact and f is Lefschetz, then f has only finitely many fixed points.

(Hint: Show that the intersection of the graph of f and the diagonal of X is a 0-dimensional submanifold of $X \times X$.

5 Consider the following intersections in $\mathbb{C}^5 \setminus \{0\}$:

$$S^7_k = \{z_1^2 + z_2^2 + z_3^2 + z_4^3 + z_5^{6k-1} = 0\} \cap \{|z_1|^2 + |z_2|^2 + |z_3|^2 + |z_4|^2 + |z_5|^2 = 1\}.$$

Show S^7_k is a 7-dimensional submanifold by showing that the intersection is transversal in $\mathbb{C}^5 \setminus \{0\}$.

(Hint: At some point you may want to show that, at a point $z = (z_1, \ldots, z_5)$, the vector $w := (\frac{m}{2}z_1, \frac{m}{2}z_2, \frac{m}{2}z_3, \frac{m}{3}z_4, \frac{m}{6k-1}z_5)$, with $m := 2 \cdot 3 \cdot (6k-1)$, lies in one of the tangent spaces but not in the other.)