## 2. Lecture II: Linear Systems on $\mathbb{R}^2$ II

2.1. **Autonomous linear systems on $\mathbb{R}^2$.** Part (v) of Lemma 1.1 allows us to decouple the equation (5) when $\mathbf{A}$ is diagonalizable. In fact an easier way to see it is as follows. Suppose that for a non-singular matrix $\mathbf{P}$,

$$\Big(\mathbf{P}^{-1}\mathbf{A}\mathbf{P}\Big)^j_i = \lambda^j \delta^j_i,$$

where $\delta^j_i$ is the Kronecker Delta that is zero everywhere only except at $i = j$, when it is unity, then the equation becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(\mathbf{P}\mathbf{x}\Big)^j = \lambda^j \delta^j_i \Big(\mathbf{P}\mathbf{x}\Big)^i.$$

These are $d$ decoupled one-dimensional equations for the new variable $\mathbf{y} = \mathbf{P}\mathbf{x} = (y^1, \ldots, y^d)^\top$:

$$\frac{\mathrm{d}y^j}{\mathrm{d}t} = \lambda^j y^j(t),$$

with solutions

$$y^j(t) = e^{\lambda^j t} y^j(0).$$

The matrix with entries $e^{\lambda^j t}\delta^j_i$ is sometimes written as $\operatorname{diag}(\exp(\lambda^j t))$. From the series representation it is clear that this is the same as $\exp(\operatorname{diag}(\lambda^j t))$.

In other words,

$$\mathbf{x}(t) = \mathbf{P}\exp(\mathbf{\Lambda}t)\mathbf{P}^{-1}\mathbf{x}(0) = \mathbf{P}\operatorname{diag}\big(\exp(\lambda^j t)\big)\mathbf{P}^{-1}\mathbf{x}(0). \tag{6}$$

We use the Einstein summation convention and sum up implicitly over repeated indices when one is an upper index and another is a lower index. Column vectors have upper indices and matrices have one upper and one lower index — we shall see why this is a good convention later. Context can usually inform us whether an upper index is just such or whether it is an exponent.

This derivation avoids the difficulties of having to re-derive some results when $\lambda^j$ are complex because we solve the equation after decoupling, and do not refer to the exponentiation of a complex matrix $\exp(\mathbf{\Lambda})$, where of course, $\big(\mathbf{\Lambda}\big)^j_i = \lambda^j\delta^j_i$, and the associated convergence issues in $\ell^2(\mathbb{C}) \to \ell^2(\mathbb{C})$. Naturally, it is also easy to show that all the previous derivations on convergence carry over.

Decoupling as shown does not happen when $\mathbf{A}$ is not diagonalizable, of course. In that instance it becomes helpful to resort to a Jordan normal form representation of $\mathbf{A}$, and we postpone that discussion in general to Lecture 3. For the remainder of this and the next lecture, we shall focus on the case $d = 2$.

When $d = 2$, the matrix $\mathbf{A}$ in (5) is

$$\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad a, b, c, d \in \mathbb{R}.$$

We can find its eigenvalues $\lambda$:

$$\begin{aligned} 0 &= \det\Big(\lambda\mathbf{I}_2 - \mathbf{A}\Big) \\ &= \det\begin{pmatrix} \lambda - a & -b \\ -c & \lambda - d \end{pmatrix} \\ &= \lambda^2 - (a+d)\lambda + (ad - bc). \end{aligned}$$

Therefore

$$\lambda_\pm = \frac{(a+d) \pm \sqrt{(a+d)^2 - 4(ad-bc)}}{2} = \frac{1}{2}\Big((a+d) \pm \sqrt{(a-d)^2 + 4bc}\Big).$$

This reduces to three cases:

(A) Two distinct real roots — $(a-d)^2 > -4bc$;
(B) Root with multiplicity — $(a-d)^2 = -4bc$ — this can only happen if $bc \leq 0$;

(C) Conjugate roots – $(a-d)^2 < -4bc$.

*The case (A).*

Consider now the foregoing discussion on decoupling. If there are two distinct real roots $\lambda_{\pm}$, there are two associated eigenvectors $\mathbf{v}_{\pm}$. Accordingly, we can write down the solution as

$$\mathbf{x}(t) = C_1 e^{\lambda_+ t}\mathbf{v}_+ + C_2 e^{\lambda_- t}\mathbf{v}_-, \tag{7}$$

where the constants are determined by the initial condition

$$\mathbf{x}(0) = C_1\mathbf{v}_+ + C_2\mathbf{v}_-.$$

Of course all this could have been derived using, e.g., the Laplace transform, for the second-order equation by reversing the proceedure for reducing equations to linear first-order systems described at the beginning of §1.3.

*The case (B).*

This case splits further into two: the first, where for the root $\lambda$ with multiplicity 2, there are associated eigenvectors that span $\mathbb{R}^2$ ("geometric multiplicity = algebraic multiplity), can be handled as in (A); and the second, where $\lambda$ does not have two linearly independent eigenvectors ("geometric multiplicity < algebraic multiplicity"), and the matrix is not in fact diagonalizable, we have to find an extra, generalized eigenvector.

From introductory linear algebra, we know that a Jordan chain gives us a second linearly independent vector so that if $\mathbf{v}_1$ is an eigenvector, then a generalized eigenvector $\mathbf{v}_2$ satisfies

$$(\mathbf{A} - \lambda\mathbf{I}_2)\mathbf{v}_2 = \mathbf{v}_1.$$

As $(\mathbf{A} - \lambda\mathbf{I}_2)$ is singular, $\mathbf{v}_2$ is not unique.

Writing

$$\mathbf{x}(t) = f(t)\mathbf{v}_1 + g(t)\mathbf{v}_2,$$

we can equate

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x} = f'(t)\mathbf{v}_1 + g'(t)\mathbf{v}_2$$

with

$$\mathbf{A}\mathbf{x} = f(t)\lambda\mathbf{v}_1 + g(t)\mathbf{v}_1 + g(t)\lambda\mathbf{v}_2$$

to arrive at the equations

$$\begin{aligned} f'(t) &= \lambda f(t) + g(t) \\ g'(t) &= \lambda g(t). \end{aligned}$$

This gives us the general solutions:

$$\mathbf{x}(t) = f(t)\mathbf{v}_1 + g(t)\mathbf{v}_2 = (C_1 + C_2 t)e^{\lambda t}\mathbf{v}_1 + C_2 e^{\lambda t}\mathbf{v}_2. \tag{8}$$

Again, the constants $C_1$ and $C_2$ are determined by the initial condition via

$$\mathbf{x}(0) = C_1\mathbf{v}_1 + C_2\mathbf{v}_2.$$

*The case (C).*

The case (C) is exactly as (A) as there are always two linearly independent (conjugate) eigenvectors:

$$\mathbf{x}(t) = C_1 e^{\lambda_+ t}\mathbf{v}_+ + C_2 e^{\lambda_- t}\mathbf{v}_-.$$

However, as our dynamics take place in $X = \mathbb{R}^2$, we have to exclude non-real solutions. We know that

$$\lambda_+ = \bar{\lambda}_-, \qquad \mathbf{v}_+ = \bar{\mathbf{v}}_-,$$

where the second conjugation is taken element-wise. This compels us to require

$$C_1 = \bar{C}_2.$$

in order that all solutions be real.

Now set

$$\sigma = \Re\lambda_\pm = \frac{a+d}{2}, \qquad \tau = \pm\Im\lambda_\pm = \frac{\sqrt{|(a-d)^2+4bc|}}{2},$$

so that

$$\lambda_\pm = \sigma \pm i\tau,$$

and

$$K_1 = \frac{C_1+C_2}{2}, \qquad K_2 = i\frac{C_1-C_2}{2}.$$

We can then write $\mathbf{x}$ as

$$\mathbf{x}(t) = e^{\sigma t}\big(K_1\cos(\tau t) + K_2\sin(\tau t)\big)\Re\mathbf{v}_+ + e^{\sigma t}\big(K_2\cos(\tau t) - K_1\sin(\tau t)\big)\Im\mathbf{v}_+, \tag{9}$$

where now we can determine the constants $K_1$ and $K_2$ by the initial condition:

$$\mathbf{x}(0) = K_1\Re\mathbf{v}_+ + K_2\Im\mathbf{v}_+.$$

2.2. **Asymptotic behaviour of solutions.** In this lecture we shall look at the qualitative behaviour of solutions by looking at their behaviour asymptotically. These include behaviour near fixed points, (quasi-)periodic behaviour, and escape to infinity.

Notice that if there is a fixed point at all, i.e., if $\mathrm{d}\mathbf{x}/\mathrm{d}t = 0$, then

$$\mathbf{A}\mathbf{x} = 0,$$

which only has a trivial solution if $\mathbf{A}$ is non-singular. If $\mathbf{A}$ is singular, then 0 is an eigenvalue, and any multiple of the eigenvector(s) is a fixed point. Therefore the fixed points of these systems are either $\mathbf{0}$, or a one-dimensional subspace, or the entire $\mathbb{R}^2$. Obviously the last possibility only occurs if $\mathbf{A}$ is the zero matrix itself.

We look again at the three cases we derived for the system:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x}(t) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\mathbf{x}(t).$$

These concerned the eigenvalues of the matrix foregoing,

$$\lambda_\pm = \frac{1}{2}\big((a+d) \pm \sqrt{(a-d)^2+4bc}\big),$$

and were:

(A) Two distinct real roots — $(a-d)^2 > -4bc$;
(B) Root with multiplicity — $(a-d)^2 = -4bc$ — this can only happen if $bc \le 0$;
(C) Conjugate roots – $(a-d)^2 < -4bc$.

Let us consider these cases in the asymptotic regime — when $t \to \infty$.

*The case (A).*

We already derived in (7) that solutions satisfy

$$\mathbf{x}(t) = C_1 e^{\lambda_+ t}\mathbf{v}_+ + C_2 e^{\lambda_- t}\mathbf{v}_-,$$

where the constants are determined by the initial condition

$$\mathbf{x}(0) = C_1\mathbf{v}_+ + C_2\mathbf{v}_-.$$

As defined, in this case, $\lambda_- < \lambda_+$.

We see that

(i) if $\lambda_- < \lambda_+ < 0$, then eventually $\mathbf{x}(t) \to \mathbf{0}$;

(ii) if $\lambda_- < 0 < \lambda_+$, then eventually $\mathbf{x}(t) \cdot \mathbf{v}_- \to 0$ ($\mathbf{x}(t)$ tends to zero in the direction of $\mathbf{v}_-$), and simultaneously, $\mathbf{x}(t) \cdot \mathbf{v}_+ \to \infty$;

(ii) if $0 < \lambda_- < \lambda_+$, then eventually $\mathbf{x}(t) \to \infty$ in the direction of $\mathbf{v}_+$.

*The case (B).*

When the geometric multiplicity of $\mathbf{A}$ equals its algebraic multiplicity, we may proceed as in the case above. The asymptotic behaviour exhibited shall either be exponential decay to the fixed point $\mathbf{0}$, or blow-up to infinity, according as the single eigenvalue-with-multiplicity satisfies $\lambda < 0$ or $\lambda > 0$.

As derived in (8), when $\mathbf{A}$ is non-diagonalizable, the solutions are

$$\mathbf{x}(t) = f(t)\mathbf{v}_1 + g(t)\mathbf{v}_2 = (C_1 + C_2 t)e^{\lambda t}\mathbf{v}_1 + C_2 e^{\lambda t}\mathbf{v}_2,$$

where $\mathbf{v}_1$ is one eigenvector associated with $\lambda$ and $(\mathbf{A} - \lambda\mathbf{I})\mathbf{v}_2 = \mathbf{v}_1$. Asymptotically, then, $\mathbf{x}$ tends to infinity or $\mathbf{0}$ in the direction of $\mathbf{v}_1$ according as $\lambda > 0$ or $\lambda < 0$.

*The case (C).*

Recall that we set $\sigma = \Re\lambda_\pm$ and $\tau = \pm\Im\lambda_-$. We derived in (9) that

$$\mathbf{x}(t) = e^{\sigma t}\big(K_1\cos(\tau t) + K_2\sin(\tau t)\big)\Re\mathbf{v}_+ + e^{\sigma t}\big(K_2\cos(\tau t) - K_1\sin(\tau t)\big)\Im\mathbf{v}_+,$$

where now we can determine the constants $K_1$ and $K_2$ by the initial condition:

$$\mathbf{x}(0) = K_1\Re\mathbf{v}_+ + K_2\Im\mathbf{v}_+.$$

These solutions are periodic/oscillatory with an attenuation/damping or amplification coefficient $e^{\sigma t}$. It is an attenuation factor if $\sigma < 0$ whereupon asymptotically, the solution tends to $\mathbf{0}$. It is an amplification factor if $\sigma > 0$, and asymptotically, the solution tends to infinity in an oscillatory manner. If $\sigma = 0$, then the oscillatory/periodic behaviour persists for all time.