6

Polynomial interpolation

6.1 Introduction

It is time to take a break from solving equations. In this chapter we consider the problem of polynomial interpolation; it involves finding a polynomial that agrees exactly with some information that we have about a real-valued function f of a single real variable x. This information may be in the form of values $f(x_0), \dots, f(x_n)$ of the function f at some finite set of points $\{x_0, \dots, x_n\}$ on the real line, and the corresponding polynomial is then called the **Lagrange interpolation polynomial**[1] or, provided that f is differentiable, it may include values of the derivative of f at these points, in which case the associated polynomial is referred to as a **Hermite interpolation polynomial**.[2]

Why should we be interested in constructing Lagrange or Hermite interpolation polynomials? If the function values $f(x)$ are known for all x in a closed interval of the real line, then the aim of polynomial

[1] Joseph-Louis Lagrange (25 January 1736, Turin, Sardinia–Piedmont (now in Italy) – 10 April 1813, Paris, France) made fundamental contributions to the calculus of variations. He succeeded Euler as Director of Mathematics at the Berlin Academy of Sciences in 1766. During his stay in Berlin Lagrange worked on astronomy, the stability of the solar system, mechanics, dynamics, fluid mechanics, probability, number theory, and the foundations of calculus. In 1787 he moved to Paris and became a member of the Académie des Sciences. Napoleon named Lagrange to the Legion of Honour and as a Count of the Empire in 1808, and on 3 April 1813, a week before his death, he received the Grand Croix of the Ordre Impérial de la Réunion.

[2] Charles Hermite (24 December 1822, Dieuze, Lorraine, France – 14 January 1901, Paris, France). Hermite did not enjoy formal examinations and had to spend five years to complete his undergraduate degree. He contributed to the theory of elliptic functions and their application to the general polynomial equation of the fifth degree. In 1873 he published the first proof that e is a transcendental number. Using methods similar to those of Hermite, Lindemann established in 1882 that π was also transcendental. A number of mathematical entities bear Hermite's name: Hermite orthogonal polynomials, Hermite's differential equation, Hermite's formula of interpolation and Hermitian matrices.

interpolation is to approximate the function f by a polynomial over this interval. Given that any polynomial can be completely specified by its (finitely many) coefficients, storing the interpolation polynomial for f in a computer will be, generally, more economical than storing f itself.

Frequently, it is the case, though, that the function values $f(x)$ are only known at a finite set of points $x_0, \ldots, x_n$, perhaps as the results of some measurements. The aim of polynomial interpolation is then to attempt to reconstruct the unknown function f by seeking a polynomial p_n whose graph in the (x, y)-plane passes through the points with coordinates $(x_i, f(x_i))$, $i = 0, \ldots, n$. Of course, in general, the resulting polynomial p_n will differ from f (unless f itself is a polynomial of the same degree as p_n), so an error will be incurred. In this chapter we shall also establish results which provide bounds on the size of this error.

6.2 Lagrange interpolation

Given that n is a nonnegative integer, let $\mathcal{P}_n$ denote the set of all (real-valued) polynomials of degree $\leq n$ defined over the set $\mathbb{R}$ of real numbers. The simplest interpolation problem can be stated as follows: given x_0 and y_0 in $\mathbb{R}$, find a polynomial $p_0 \in \mathcal{P}_0$ such that $p_0(x_0) = y_0$. The solution to this is, trivially, $p_0(x) \equiv y_0$. The purpose of this section is to explore the following more general problem.

> Let $n \geq 1$, and suppose that $x_i, i = 0, 1, \ldots, n$, are *distinct* real numbers (*i.e.*, $x_i \neq x_j$ for $i \neq j$) and $y_i, i = 0, 1, \ldots, n$, are real numbers; we wish to find $p_n \in \mathcal{P}_n$ such that $p_n(x_i) = y_i$, $i = 0, 1, \ldots, n$.

To prove that this problem has a unique solution, we begin with a useful lemma.

Lemma 6.1 *Suppose that $n \geq 1$. There exist polynomials $L_k \in \mathcal{P}_n$, $k = 0, 1, \ldots, n$, such that*

$$L_k(x_i) = \begin{cases} 1\,, & i = k\,, \\ 0\,, & i \neq k\,, \end{cases} \tag{6.1}$$

for all $i, k = 0, 1, \ldots, n$. Moreover,

$$p_n(x) = \sum_{k=0}^{n} L_k(x) y_k \tag{6.2}$$

satisfies the above interpolation conditions; in other words, $p_n \in \mathcal{P}_n$ and $p_n(x_i) = y_i,\ i = 0, 1, \ldots, n$.

Proof For each fixed k, $0 \le k \le n$, L_k is required to have n zeros – x_i, $i = 0, 1, \ldots, n$, $i \neq k$; thus, $L_k(x)$ is of the form

$$L_k(x) = C_k \prod_{\substack{i=0 \\ i \neq k}}^{n} (x - x_i)\,, \tag{6.3}$$

where $C_k \in \mathbb{R}$ is a constant to be determined. It is easy to find the value of C_k by recalling that $L_k(x_k) = 1$; using this in (6.3) yields

$$C_k = \prod_{\substack{i=0 \\ i \neq k}}^{n} \frac{1}{x_k - x_i}\,.$$

On inserting this expression for C_k into (6.3) we get

$$L_k(x) = \prod_{\substack{i=0 \\ i \neq k}}^{n} \frac{x - x_i}{x_k - x_i}\,. \tag{6.4}$$

As the function p_n defined by (6.2) is a linear combination of the polynomials $L_k \in \mathcal{P}_n$, $k = 0, 1, \ldots, n$, also $p_n \in \mathcal{P}_n$. Finally, $p_n(x_i) = y_i$ for $i = 0, 1, \ldots, n$ is a trivial consequence of using (6.1) in (6.2). $\square$

Remark 6.1 *Although the statement of Lemma 6.1 required that $n \ge 1$, the trivial case of $n = 0$ mentioned at the beginning of the section can also be included by defining, for $n = 0$, $L_0(x) \equiv 1$, and observing that the function p_0 defined by*

$$p_0(x) = L_0(x) y_0 \ (\equiv y_0\,)$$

is the unique polynomial in $\mathcal{P}_0$ that satisfies $p_0(x_0) = y_0$.

We note that, implicitly, the polynomials L_k, $k = 0, 1, \ldots, n$, depend on the polynomial degree n, $n \ge 0$. To highlight this fact, a more accurate but cumbersome notation would have involved writing, for example, $L_k^n(x)$ instead of $L_k(x)$; this would have made it clear that $L_k^n(x)$ differs from $L_k^m(x)$ when the polynomial degrees n and m differ. For the sake of notational simplicity, we have chosen to write $L_k(x)$; the implied value of n will always be clear from the context.

Theorem 6.1 (Lagrange's Interpolation Theorem) *Assume that $n \ge 0$. Let x_i, $i = 0, \ldots, n$, be* distinct *real numbers and suppose that $y_i, i = 0, \ldots, n$, are real numbers. Then, there exists a unique polynomial $p_n \in \mathcal{P}_n$ such that*

$$p_n(x_i) = y_i\,, \quad i = 0, \ldots, n\,. \tag{6.5}$$

Proof In view of Remark 6.1, for $n = 0$ the proof is trivial. Let us therefore suppose that $n \geq 1$. It follows immediately from Lemma 6.1 that the polynomial $p_n \in \mathcal{P}_n$ defined by

$$p_n(x) = \sum_{k=0}^{n} L_k(x) y_k$$

satisfies the conditions (6.5), thus showing the *existence* of the required polynomial. It remains to show that p_n is the *unique* polynomial in $\mathcal{P}_n$ satisfying the interpolation property

$$p_n(x_i) = y_i\,, \qquad i = 0, 1, \ldots, n\,.$$

Suppose, otherwise, that there exists $q_n \in \mathcal{P}_n$, different from p_n, such that $q_n(x_i) = y_i$, $i = 0, 1, \ldots, n$. Then, $p_n - q_n \in \mathcal{P}_n$ and $p_n - q_n$ has $n+1$ distinct roots, x_i, $i = 0, 1, \ldots, n$; since a polynomial of degree n cannot have more than n distinct roots, unless it is identically 0, it follows that

$$p_n(x) - q_n(x) \equiv 0\,,$$

which contradicts our assumption that p_n and q_n are distinct. Hence, there exists only one polynomial $p_n \in \mathcal{P}_n$ which satisfies (6.5). $\square$

Definition 6.1 *Suppose that $n \geq 0$. Let x_i, $i = 0, \ldots, n$, be distinct real numbers, and y_i, $i = 0, \ldots, n$, real numbers. The polynomial p_n defined by*

$$\boxed{p_n(x) = \sum_{k=0}^{n} L_k(x) y_k\,, \qquad (6.6)}$$

with $L_k(x)$, $k = 0, 1, \ldots, n$, defined by (6.4) when $n \geq 1$, and $L_0(x) \equiv 1$ when $n = 0$, is called the **Lagrange interpolation polynomial** *of degree n for the set of points $\{(x_i, y_i)\colon i = 0, \ldots, n\}$. The numbers x_i, $i = 0, \ldots, n$, are called the* **interpolation points**.

Frequently, the real numbers y_i are given as the values of a real-valued function f, defined on a closed real interval $[a, b]$, at the (distinct) interpolation points $x_i \in [a, b]$, $i = 0, \ldots, n$.

Definition 6.2 *Let $n \geq 0$. Given the real-valued function f, defined and continuous on a closed real interval $[a, b]$, and the (distinct) interpolation points $x_i \in [a, b]$, $i = 0, \ldots, n$, the polynomial p_n defined by*

$$p_n(x) = \sum_{k=0}^{n} L_k(x) f(x_k) \tag{6.7}$$

is the **Lagrange interpolation polynomial of degree** n **(with interpolation points** x_i**,** $i = 0, \ldots, n$**) for the function** f.

Example 6.1 *We shall construct the Lagrange interpolation polynomial of degree* 2 *for the function* $f \colon x \mapsto \mathrm{e}^x$ *on the interval* $[-1, 1]$, *with interpolation points* $x_0 = -1$, $x_1 = 0$, $x_2 = 1$.

As $n = 2$, we have that

$$L_0(x) = \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} = \tfrac{1}{2} x(x-1)\,.$$

Similarly, $L_1(x) = 1 - x^2$ and $L_2(x) = \frac{1}{2}x(x+1)$. Therefore,

$$p_2(x) = \tfrac{1}{2}x(x-1)\,\mathrm{e}^{-1} + (1 - x^2)\,\mathrm{e}^0 + \tfrac{1}{2}x(x+1)\,\mathrm{e}^1\,.$$

Thus, after some simplification, $p_2(x) = 1 + x \sinh 1 + x^2(\cosh 1 - 1)$. $\diamond$

Although the values of the function f and those of its Lagrange interpolation polynomial coincide at the interpolation points, $f(x)$ may be quite different from $p_n(x)$ when x is *not* an interpolation point. Thus, it is natural to ask just how large the difference $f(x) - p_n(x)$ is when $x \neq x_i$, $i = 0, \ldots, n$. Assuming that the function f is sufficiently smooth, an estimate of the size of the **interpolation error** $f(x) - p_n(x)$ is given in the next theorem.

Theorem 6.2 *Suppose that* $n \geq 0$, *and that* f *is a real-valued function, defined and continuous on the closed real interval* $[a, b]$, *such that the derivative of* f *of order* $n+1$ *exists and is continuous on* $[a, b]$. *Then, given that* $x \in [a, b]$, *there exists* $\xi = \xi(x)$ *in* (a, b) *such that*

$$f(x) - p_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}\pi_{n+1}(x)\,, \tag{6.8}$$

where

$$\pi_{n+1}(x) = (x - x_0)\ldots(x - x_n)\,. \tag{6.9}$$

Moreover

$$|f(x) - p_n(x)| \leq \frac{M_{n+1}}{(n+1)!}|\pi_{n+1}(x)|\,, \tag{6.10}$$

where

$$M_{n+1} = \max_{\zeta \in [a,b]} |f^{(n+1)}(\zeta)|\,.$$

Proof When $x = x_i$ for some i, $i = 0, 1, \dots, n$, both sides of (6.8) are zero, and the equality is trivially satisfied. Suppose then that $x \in [a, b]$ and $x \neq x_i$, $i = 0, 1, \dots, n$. For such a value of x, let us consider the auxiliary function $t \mapsto \varphi(t)$, defined on the interval $[a, b]$ by

$$\varphi(t) = f(t) - p_n(t) - \frac{f(x) - p_n(x)}{\pi_{n+1}(x)} \pi_{n+1}(t)\,. \tag{6.11}$$

Clearly $\varphi(x_i) = 0$, $i = 0, 1, \dots, n$, and $\varphi(x) = 0$. Thus, φ vanishes at $n + 2$ points which are all distinct in $[a, b]$. Consequently, by Rolle's Theorem, Theorem A.2, $\varphi'(t)$, the first derivative of φ with respect to t, vanishes at $n + 1$ points in (a, b), one between each pair of consecutive points at which φ vanishes.

In particular, if $n = 0$, we then deduce the existence of $\xi = \xi(x)$ in the interval (a, b) such that $\varphi'(\xi) = 0$. Since $p_0(x) \equiv f(x_0)$ and $\pi_1(t) = t - x_0$, it follows from (6.11) that

$$0 = \varphi'(\xi) = f'(\xi) - \frac{f(x) - p_0(x)}{\pi_1(x)}\,,$$

and hence (6.8) in the case of $n = 0$.

Now suppose that $n \geq 1$. As $\varphi'(t)$ vanishes at $n+1$ points in (a, b), one between each pair of consecutive points at which φ vanishes, applying Rolle's Theorem again, we see that φ'' vanishes at n distinct points. Our assumptions about f are sufficient to apply Rolle's Theorem $n + 1$ times in succession, showing that $\varphi^{(n+1)}$ vanishes at some point $\xi \in (a, b)$, the exact value of ξ being dependent on the value of x. By differentiating $n + 1$ times the function φ with respect to t, and noting that p_n is a polynomial of degree n or less, it follows that

$$0 = \varphi^{(n+1)}(\xi) = f^{(n+1)}(\xi) - \frac{f(x) - p_n(x)}{\pi_{n+1}(x)} (n + 1)!$$

Hence

$$f(x) - p_n(x) = \frac{f^{(n+1)}(\xi)}{(n + 1)!} \pi_{n+1}(x)\,.$$

In order to prove (6.10), we note that as $f^{(n+1)}$ is a continuous function on $[a, b]$ the same is true of $|f^{(n+1)}|$. Therefore, the function $x \mapsto |f^{(n+1)}(x)|$ is bounded on $[a, b]$ and achieves its maximum there; so (6.10) follows from (6.8). □

It is perhaps worth noting that since the location of ξ in the interval $[a, b]$ is unknown (to the extent that the exact dependence of ξ on x is not revealed by the proof of Theorem 6.2), (6.8) is of little practical value; on the other hand, given the function f, an upper bound on the maximum value of $f^{(n+1)}$ over $[a, b]$ is, at least in principle, possible to obtain, and thereby we can provide an upper bound on the size of the interpolation error by means of inequality (6.10).

6.3 Convergence

An important theoretical question is whether or not a sequence (p_n) of interpolation polynomials for a continuous function f converges to f as $n \to \infty$. This question needs to be made more specific, as p_n depends on the distribution of the interpolation points x_j, $j = 0, 1, \ldots, n$, not just on the value of n. Suppose, for example, that we agree to choose equally spaced points, with

$$x_j = a + j(b-a)/n\,, \qquad j = 0, 1, \ldots, n\,, \quad n \geq 1\,.$$

The question of convergence then clearly depends on the behaviour of M_{n+1} as n increases. In particular, if

$$\lim_{n\to\infty} \frac{M_{n+1}}{(n+1)!} \max_{x\in[a,b]} |\pi_{n+1}(x)| = 0\,,$$

then, by (6.10),

$$\lim_{n\to\infty} \max_{x\in[a,b]} |f(x) - p_n(x)| = 0\,, \tag{6.12}$$

and we say that the sequence of interpolation polynomials (p_n), with equally spaced points on $[a, b]$, converges to f as $n \to \infty$, uniformly on the interval $[a, b]$.

You may now think that if all derivatives of f exist and are continuous on $[a, b]$, then (6.12) will hold. Unfortunately, this is not so, since the sequence

$$\left(M_{n+1} \max_{x\in[a,b]} |\pi_{n+1}(x)| \right)$$

may tend to ∞, as $n \to \infty$, faster than the sequence $(1/(n+1)!)$ tends to 0.

In order to convince you of the existence of such 'pathological' functions, we consider the sequence of Lagrange interpolation polynomials

Table 6.1. *Runge phenomenon: n denotes the degree of the interpolation polynomial p_n to f, with equally spaced points on $[-5,5]$.* 'Max error' *signifies* $\max_{x\in[-5,5]} |f(x) - p_n(x)|$.

Degree n	Max error
2	0.65
4	0.44
6	0.61
8	1.04
10	1.92
12	3.66
14	7.15
16	14.25
18	28.74
20	58.59
22	121.02
24	252.78

p_n, $n = 0, 1, 2, \ldots$, with equally spaced interpolation points on the interval $[-5,5]$, to

$$f(x) = \frac{1}{1+x^2}\,, \qquad x \in [-5,5]\,.$$

This example is due to Runge,[1] and the characteristic behaviour exhibited by the sequence of interpolation polynomials p_n in Table 6.1 is referred to as the **Runge phenomenon**: Table 6.1 shows the maximum difference between $f(x)$ and $p_n(x)$ for $-5 \le x \le 5$, for values of n from 2 up to 24. The numbers indicate clearly that the maximum error increases exponentially as n increases. Figure 6.1 shows the interpolation polynomial p_{10}, using the equally spaced interpolation points $x_j = -5 + j$, $j = 0, 1, \ldots, 10$. The sizes of the local maxima near ± 5 grow exponentially as the degree n increases.

Note that, in many ways, the function f is well behaved; all its deriva-

[1] Carle David Tolmé Runge (30 August 1856, Bremen, Germany – 3 January 1927, Göttingen, Germany) studied mathematics and physics at the University of Munich. His doctoral dissertation in 1880 was in the area of differential geometry. Gradually, his research interests shifted to more applied topics: he devised a numerical procedure for the solution of algebraic equations where the roots were expressed as infinite series of rational functions of the coefficients, and in 1887 he started to work on the wavelengths of the spectral lines of elements. In 1904 Runge became Professor of Applied Mathematics in Göttingen. He was a fit and active man: on his 70th birthday he entertained his grandchildren by performing handstands. A few months later he suffered a fatal heart attack.

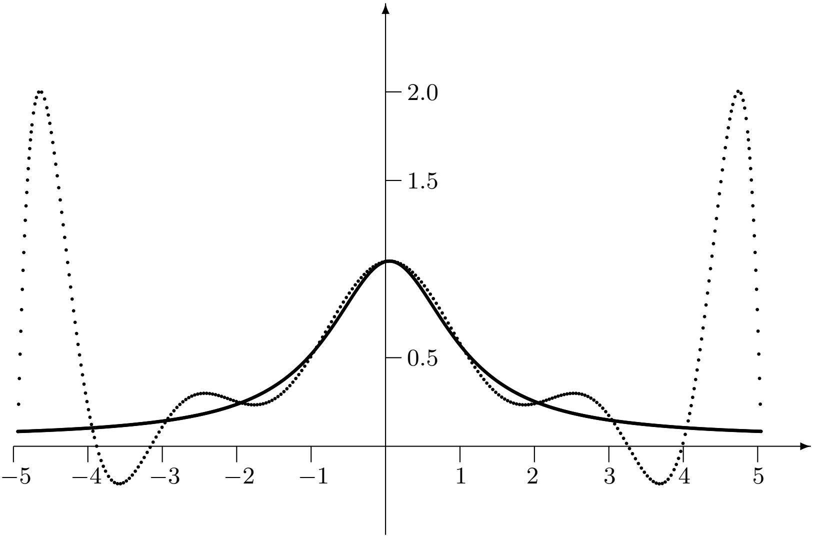

Fig. 6.1. Polynomial interpolation of $f\colon x \mapsto 1/(1+x^2)$ for $x \in [-5,5]$. The continuous curve is f; the dotted curve is the associated Lagrange interpolation polynomial p_{10} of degree 10, using equally spaced interpolation points.

tives are continuous and bounded for all $x \in [-5,5]$. The apparent divergence of the sequence of Lagrange interpolation polynomials (p_n) is related to the fact that, when extended to the complex plane, the Taylor series of the complex-valued function $f\colon z \mapsto 1/(1+z^2)$ converges in the open unit disc of radius 1 but not in any disc of larger radius centred at $z=0$, given that f has poles on the imaginary axis at $z = \pm\imath$. Some further insight into this problem is given in Exercise 11, and a similar difficulty in numerical integration is discussed in Section 7.4.

6.4 Hermite interpolation

The idea of Lagrange interpolation can be generalised in various ways; we shall consider here one simple extension where a polynomial p is required to take given values and derivative values at the interpolation points. Given the distinct interpolation points $x_i, i = 0,\dots,n$, and two sets of real numbers y_i, $i = 0,\dots,n$, and z_i, $i = 0,\dots,n$, with $n \ge 0$, we need to find a polynomial $p_{2n+1} \in \mathcal{P}_{2n+1}$ such that

$$p_{2n+1}(x_i) = y_i\,, \qquad p'_{2n+1}(x_i) = z_i\,, \qquad i = 0,\dots,n\,.$$