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TMA4267 Linear statistical models Compulsory exercises 1



Problem 1 Bivariate normal distribution

Assume that X is a bivariate normal random variable with

$$\boldsymbol{\mu} = E \boldsymbol{X} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$$
 and $\Sigma = \operatorname{Cov} \boldsymbol{X} = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$.

Let

$$\mathbf{Y} = \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \mathbf{X}.$$

a) Find the mean vector and covariance matrix of Y. What is the distribution of Y? Are Y_1 and Y_2 independent random variables?

Let f be the pdf of X. Contours of f are the x satisfying f(x) = a for some constant a > 0, or equivalently $(x - \mu)^T \Sigma^{-1} (x - \mu) = b$ for a corresponding constant b > 0. In the figure below, the contour for b = 4.6 is shown.

b) Explain the connections between the covariance matrix Σ , the chosen value of b and features of the ellipse (e.g. principal axes and their half-lengths). Mark these features on the figure (make a drawing or use the printed figure). What is the probability that X falls within the given ellipse?

The following information might be useful:

> sigma=matrix(c(3,1,1,3),ncol=2)
> sigma

[,1] [,2] [1,] 3 1 [2,] 1 3

> eigen(sigma)

\$values

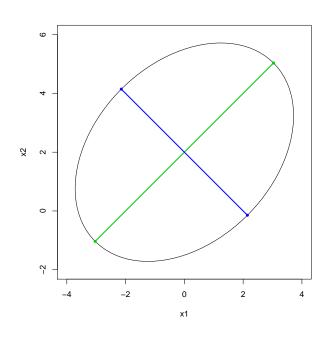
[1] 4 2

\$vectors

[,1] [,2] [1,] 0.7071068 -0.7071068 [2,] 0.7071068 0.7071068

> qchisq(0.9,2)

[1] 4.60517



Problem 2 Distributional results for \bar{X} and S^2 for a univariate normal sample

Let X_1, X_2, \ldots, X_n be a (univariate) random sample from some population. Define $\bar{X} = \frac{1}{n} \sum_{i=1}^{n} X_i$ and $S^2 = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X})^2$.

Futher, let **1** be an *n*-dimensional vector of 1s. Then $\mathbf{11}^{\mathrm{T}}$ is an $n \times n$ matrix of 1s. Let I be an $n \times n$ identity matrix. The matrix

$$C = I - \frac{1}{n} \mathbf{1} \mathbf{1}^{\mathrm{T}} = \begin{pmatrix} 1 - 1/n & -1/n & \cdots & -1/n \\ -1/n & 1 - 1/n & \cdots & -1/n \\ \vdots & \vdots & \ddots & \vdots \\ -1/n & -1/n & \cdots & 1 - 1/n \end{pmatrix}$$

is called the *centring matrix*. Let $\mathbf{X} = (X_1 \ X_2 \ \cdots \ X_n)^{\mathrm{T}}$.

a) Show that $\bar{X} = \frac{1}{n} \mathbf{1}^{\mathrm{T}} X$ and that $S^2 = \frac{1}{n-1} X^{\mathrm{T}} C X$.

Hint: Start by considering the *i*th component of CX, and observe that C is symmetric and idempotent.

Now, assume in addition that the random sample is taken from the univariate normal distribution with mean μ and variance σ^2 . In the notation of TMA4267, $\mathbf{X} \sim N(\mu \mathbf{1}, \sigma^2 I)$.

In your first statistics course, you were told that the T-statistic,

$$T = \frac{\bar{X} - \mu}{S/\sqrt{n}},$$

follows a t-distribution with n-1 degrees of freedom. This result follows from $(\bar{X}-\mu)/(\sigma/\sqrt{n}) \sim N(0,1)$ and $(n-1)S^2/\sigma^2 \sim \chi^2_{n-1}$, and independence of \bar{X} and S^2 .

With your new skills on the multivariate normal distribution, you can prove independence of \bar{X} and S^2 , and $(n-1)S^2/\sigma^2 \sim \chi^2_{n-1}$.

- **b)** Show that $\frac{1}{n}\mathbf{1}^{\mathrm{T}}C = \mathbf{0}^{\mathrm{T}}$. What does this imply about $\frac{1}{n}\mathbf{1}^{\mathrm{T}}X$ and CX? How can you use this to conclude that \bar{X} and S^2 are independent?
- c) Derive the distribution of $(n-1)S^2/\sigma^2$.

Hint: $S^2 = \frac{1}{n-1} \boldsymbol{X}^T C \boldsymbol{X}$, where C is symmetric and idempotent. In general, if R is a symmetric and idempotent matrix with rank r, and $\boldsymbol{Y} \sim N(\boldsymbol{0}, I)$, then $\boldsymbol{Y}^T R \boldsymbol{Y} \sim \chi_r^2$ (which we saw in a lecture and is stated by Fahrmeir, Kneib, Lang and Marx (2013) in Theorem B.8.2 on p. 651). Note, however, that \boldsymbol{X} is not assumed to be $N(\boldsymbol{0}, I)$.