Repetition

Theorem: The Law of rare events

Let $p_1, p_2, \ldots \in [0,1]$ be a sequence such that $\lim_{n\to\infty} np_n = \lambda < \infty$, then

$$\lim_{n\to\infty} \binom{n}{k} p_n^k (1-p_n)^{n-k} = \frac{\lambda^k}{k!} \mathrm{e}^{-\lambda}, \quad k = 0, 1, \ldots.$$

Note: We are saying that Binomial(n, p_n) converges to Poisson(λ) when $n \to \infty$ if $\lim_{n\to} np_n = \lambda < 0$.
Note 2: If you are counting the number of successes among many independent trials and success is rare, then the Poisson distribution is a reasonable model.

Definition

Let f and g be real functions. We use **little-oh notation** in the two following ways

$$f(n) = o(g(n)) \quad (\text{as } n \to \infty) \quad \iff \quad \lim_{n\to\infty} \frac{f(n)}{g(n)} = 0,$$
$$f(h) = o(g(h) \quad (\text{as } h \to 0^+) \quad \iff \quad \lim_{h\to 0^+} \frac{f(h)}{g(h)} = 0.$$

Note: Our main focus are statements such as $f(h) = \lambda h + o(h)$ (as $h \to 0^+$). The interpretation is that f decays linearly as h goes to zero with an error that is very small compared to h.

Definition

A **counting process** is a stochastic process $\{N(t) : t \geq 0\}$ so that

1. $N(t)$ is integer for $t \geq 0$.
2. $N(t) \geq 0$ for $t \geq 0$.
3. If $s \leq t$, then $N(s) \leq N(t)$.

Note: We sometimes write $N((a, b]) = N(b) - N(a) =$ "Number of events in $(a, b]$", $0 \leq a < b$. This notation is not a focus for us.

Definition

Let $\{N(t) : t \geq 0\}$ be a counting process. Then $\{N(t) : t \geq 0\}$ is a **Poisson process** with **rate (intensity)** $\lambda > 0$ if

1. For every integer $m > 1$, for any time points $0 = t_0 < t_1 < \cdots < t_m$,
$$N(t_1) - N(t_0), N(t_2) - N(t_1), \ldots, N(t_m) - N(t_{m-1})$$
are independent.
2. For $t \geq 0$ and $h > 0$, the distribution of $N(t + h) - N(t)$ only depends on h and not t.
3. $\Pr\{N(t + h) - N(t) = 1\} = \lambda h + o(h) \quad (\text{as } h \to 0^+), \quad \forall t \geq 0.$
4. $\Pr\{N(t + h) - N(t) = 0\} = 1 - \lambda h + o(h) \quad (\text{as } h \to 0^+), \quad \forall t \geq 0.$
5. $N(0) = 0$.

Note: 1. is often shortened to "independent increments" and 2. is often shortened to "stationary increments".

Theorem

The above definition is equivalent to the counting process satisfying

1. increments are independent.
2. for $s \geq 0$ and $t > 0$,
$$N(s + t) - N(s) \sim \text{Poisson}(\lambda t).$$
3. $N(0) = 0$.