Repetition

Theorem

Let $W_1, W_2, \dots$ be the occurence times in a Poisson process $\{X(t) : t \geq 0\}$ with rate $\lambda > 0$. Then

$$\begin{aligned}(W_1, W_2, \dots, W_n)|X(t) = n &\sim f(w_1, w_2, \dots, w_n|X(t) = n)\\ &= \frac{n!}{t^n}, \quad 0 < w_1 < w_2 < \cdots < w_n \leq t.\end{aligned}$$

Note: The theorem states that conditional on n events occuring in $(0, t]$, the (unsorted) times of the events follow independent uniform distributions on $(0, t]$.

Joint simulation

Input:

- Time interval: $(0, t]$
- Rate: $\lambda > 0$

Algorithm:

1. Simulate $n \sim \text{Poisson}(\lambda t)$
2. Simulate $v_1, v_2, \dots, v_n \overset{\text{iid}}{\sim} \mathcal{U}(0, t)$
3. Let $(w_1, w_2, \dots, w_n) = \text{sort}(v_1, v_2, \dots, v_n)$

Output:

$$x(s) = \begin{cases} 0, & 0 \leq s < w_1 \\ 1, & w_1 \leq s < w_2 \\ \vdots & \\ n, & w_n \leq s < t. \end{cases}$$

Sequential simulation

Input:

- Number of jumps: N
- Rate: $\lambda > 0$

Algorithm:

1. $w_0 = 0$
2. for $n = 1 \dots N$
3. Simulate $s_{n-1} \sim \text{Exp}(\lambda)$
4. Set $w_n = w_{n-1} + s_{n-1}$
5. end

Output:

$$x(s) = \begin{cases} 0, & 0 \leq s < w_1 \\ 1, & w_1 \leq s < w_2 \\ \vdots & \\ n, & w_n \leq s < t. \end{cases}$$

Definition

We call the stochastic process $\{X(t) : t \geq 0\}$ a **continuous-time Markov chain** with state space $\{0, 1, \dots\}$ if it satisfies the **Markov property**

$$\Pr\{X(t+s) = j | X(s) = i, X(u), 0 \leq u < s\} = \Pr\{X(t+s) = j | X(s) = i\}$$

for $i, j = 0, 1, \dots$ and for all $s \geq 0$ and $t > 0$.

Note: The idea is the same as for discrete-time Markov chains: the entire history for time $t \in [0, s]$ does not provide any more information than the current state in time $t = s$.

Definition

Let $\{X(t) : t \geq 0\}$ be a continuous-time Markov chain with state space $\{0, 1, \ldots\}$ and stationary transition probabilities. We call

$$P_{ij}(t) = \Pr\{X(t) = j | X(0) = i\}, \quad i, j = 0, 1, \ldots,$$

the **transition probability functions**.

Note: We will only consider continuous-time Markov chains with stationary transition probabilities in this course.