NTNU – Trondheim
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Department of Mathematical Sciences

Examination paper for **TMA4145 Linear Methods**

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Examination time (from–to): 09:00-13:00

Permitted examination support material: D: No written or handwritten material. Calculator Casio fx-82ES PLUS, Citizen SR-270X, Hewlett Packard HP30S

Other information:
The exam consists of twelve questions, and their order is not according to the level of difficulty. All solutions should be stated in a precise and rigorous way, with any assumptions written down and arguments justified. Each solution will be graded as *rudimentary* (F), *acceptable* (D), *good* (C) or *excellent* (A). Five acceptable solutions guarantee an E; seven acceptable with at least one good a D; seven acceptable with at least five good a C; nine good with at least two excellent a B; nine good with at least seven excellent an A. These are guaranteed limits. Beyond that, the grade is based on the total achievement.

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Problem 1 Let T be a bounded linear operator on a normed space $(X, \|.\|)$. Furthermore, T has a bounded inverse, i.e. T^{-1} exists and $\|T^{-1}\| < \infty$.

a) Show that $\|x\|_T := \|Tx\|$ is a norm on X.

b) Show that $\|.\|_T$ and $\|.\|$ are equivalent norms on X.

Problem 2

a) State (without proof) whether the assertion is true or false.

1. A uniformly continuous function on a metric space is Lipschitz continuous.
2. The kernel of a bounded linear operator T on a normed space X is always closed.
3. A Cauchy sequence $(x_i)_{i=1}^{\infty}$ in a normed space X is convergent.
4. $\mathbb{R}^n$ with $\|(x_1, x_2, ..., x_n)\|_1 = \sum_{i=1}^n |x_i|$ is not complete.
5. A linear mapping T on $\mathbb{C}^n$ has at least one eigenvalue.

b) Define the following notions.

1. Define the notion of a **Cauchy sequence** in a metric space (X, d).
2. Let f be a mapping between the normed spaces $(X, \|.\|_X)$ and $(Y, \|.\|_Y)$. Define the notion of **Lipschitz continuity** for f.
3. What does it mean for a subset S of a metric space (X, d) to be **dense**?
4. Let T be a linear operator between the normed spaces $(X, \|.\|_X)$ and $(Y, \|.\|_Y)$. Define the **operator norm** of T.
5. Define the **orthogonal complement** of a subspace M of a Hilbert space $\mathcal{H}$.

Problem 3 Let T be the linear operator on the space of polynomials $\mathcal{P}_2$ of degree at most 2 defined by $Tf(x) = f'(x)$, the derivative of f.

Find the matrix representation of T with respect to the basis $\{1, x, x^2\}$ of $\mathcal{P}_2$, its characteristic polynomial and show that T is nilpotent on $\mathcal{P}_2$.

Problem 4

Let the linear mapping $T : \mathbb{C}^3 \to \mathbb{C}^2$ be given by the matrix

$$A = \begin{pmatrix} 3 & 1 & 1 \\ -1 & 3 & 1 \end{pmatrix}$$

with respect to the standard bases. Compute the Singular Value Decomposition of A.

Problem 5 Give operators S and T on ℓ^2, where S is surjective but not injective and T is injective but not surjective.

Problem 6 Let M be a closed subspace of a Hilbert spae $(\mathcal{H}, \langle ., . \rangle)$. Define the orthogonal projection P_M of $\mathcal{H}$ onto M and show that P_M is a bounded, linear, selfadjoint operator on $\mathcal{H}$.

Problem 7 Let $\mathcal{H}$ be a real Hilbert space with respect to the inner product $\langle ., . \rangle$ and $\|.\|$ the associated norm.

a) Show that $2\|x\|^2 + 2\|y - z\|^2 = \|x + y - z\|^2 + \|x - y + z\|^2$ holds for any $x, y, z \in \mathcal{H}$.

Hint: Use the paralleogram identity for appropriate elements of $\mathcal{H}$.

b) Show that $\langle x, y \rangle = \frac{1}{4}\Big[\|x + y\|^2 - \|x - y\|^2\Big]$ for all $x, y \in \mathcal{H}$

Problem 8 Let $\mathcal{H}$ be a Hilbert space with respect to the inner product $\langle ., . \rangle$. Suppose $\{e_1, ..., e_n\}$ is a finite orthonormal system in $\mathcal{H}$. For $x \in \mathcal{H}$, show that the point in the closed linear span of $\{e_1, ..., e_n\}$ which is closest to x is given by:

$$\tilde{x} = \sum_{i=1}^{n} \langle x, e_i \rangle e_i$$

and that $\|\tilde{x} - x\| = \left(\|x\|^2 - \sum_{i=1}^{n} |\langle x, e_j \rangle|^2\right)^{1/2}$.

Problem 9 Let T be a linear operator from $\mathbb{R}^n$ to $\mathbb{R}^n$ and $(a_{ij})_{i,j=1}^n$ be the matrix represemtation with respect to a basis in $\mathbb{R}^n$. Determine the operator norm $\|T\|_{\text{op}}$ of $T : (\mathbb{R}^n, \|.\|_1) \to (\mathbb{R}^n, \|.\|_\infty)$ in terms of the matrix $(a_{ij})_{i,j=1}^n$, where one equips the domain of T with the norm $\|(x_1, ..., x_n)\|_1 = \sum_{i=1}^n |x_i|$ and the range space with $\|(x_1, ..., x_n)\|_\infty = \sup\{|x_i|\, i = 1, ..., n\}$.