The next lemma will use this completeness to find a unique solution to a fixed-point problem on the Banach space.

Lemma 4.4 (Contraction Mapping Principle)**.** *Let $\mathfrak{T} : \mathfrak{X} \to \mathfrak{X}$ be a contraction map from a Banach space into itself. Then $\mathfrak{T}$ has a unique fixed point. that is, there exists a unique $\mathbf{x} \in \mathfrak{X}$ such that*

$$\mathfrak{T}(\mathbf{x}) = \mathbf{x}.$$

Proof. By assumption, $\mathfrak{T}$ is a contraction map. This means that for some $L < 1$,

$$\|\mathfrak{T}(\mathbf{x}) - \mathfrak{T}(\mathbf{y})\|_{\mathfrak{X}} \leq L\|\mathbf{x} - \mathbf{y}\|_{\mathfrak{X}}$$

for any two continuous functions $\mathbf{x}, \mathbf{y} \in \mathfrak{X}$.

Define the iteration $\mathbf{x}_{n+1} = \mathfrak{T}(\mathbf{x}_n)$.

Then we find that

$$\|\mathfrak{T}(\mathbf{x}_{n+1}) - \mathfrak{T}(\mathbf{x}_n)\|_{\mathfrak{X}} \leq L\|\mathfrak{T}(\mathbf{x}_n) - \mathfrak{T}(\mathbf{x}_{n-1})\|_{\mathfrak{X}} \leq L^{n+1}\|\mathfrak{T}(\mathbf{x}_0) - \mathbf{x}_0\|_{\mathfrak{X}}.$$

Since $L^n \to 0$,

$$\|\mathbf{x}_m - \mathbf{x}_n\|_{\mathfrak{X}} \leq \sum_{k=m+1}^{n+1} L^k\|\mathfrak{T}(\mathbf{x}_0) - \mathbf{x}_0\|_{\mathfrak{X}},$$

and $\{\mathbf{x}_n\}$ is a Cauchy sequence. By assumption the space $\mathfrak{X}$ is Banach and hence complete. Therefore, there is a point $\mathbf{x}$ to which the sequence converges in the norm of $\mathfrak{X}$. This must be a fixed point by construction.

Suppose there are two fixed points $\mathbf{x}$ and $\mathbf{y}$. Then

$$\mathfrak{T}(\mathbf{x}) = \mathbf{x}, \qquad \mathfrak{T}(\mathbf{y}) = \mathbf{y}, \qquad \|\mathfrak{T}(\mathbf{x}) - \mathfrak{T}(\mathbf{y})\|_{\mathfrak{X}} = \|\mathbf{x} - \mathbf{y}\|_{\mathfrak{X}}.$$

But this violates the contraction property as

$$\|\mathfrak{T}(\mathbf{x}) - \mathfrak{T}(\mathbf{y})\|_{\mathfrak{X}} \leq L\|\mathbf{x} - \mathbf{y}\|_{\mathfrak{X}} < \|\mathbf{x} - \mathbf{y}\|_{\mathfrak{X}}.$$

This contradiction establishes the theorem. □

Proof of Thm. 4.1. It remains to show that the map

$$\mathfrak{T}(\mathbf{x})(t) = \mathbf{x}(t_0) + \int_{t_0}^{t} f(s, \mathbf{x}(s))\, \mathrm{d}s$$

is a contraction map in the space $C(J)$.

To this end we simply take a difference. Let $\mathbf{x}, \mathbf{y} \in C(J)$.

$$\mathfrak{T}(\mathbf{x}) - \mathfrak{T}(\mathbf{y}) = \int_{t_0}^{t} f(s, \mathbf{x}(s)) - f(s, \mathbf{y}(s))\, \mathrm{d}s.$$

Since f is Lipschitz in its second argument, for some constant K (maybe even K_s),

$$\|\mathfrak{T}(\mathbf{x}) - \mathfrak{T}(\mathbf{y})\|_{C(J)} \leq \sup_{t\in J} \int_{t_0}^{t} |f(s, \mathbf{x}(s)) - f(s, \mathbf{y}(s))|\, \mathrm{d}s \leq \eta \sup_{s\in J} K_s\|\mathbf{x} - \mathbf{y}\|_{C(J)},$$

by the triangle inequality.

Now we simply require η to be small enough such that $\eta \sup_{s\in J} K_s < 1$. This will give us a contraction map, and by the previous lemma, a unique solution on $C(J)$ to the Cauchy problem in the theorem statement. In particular, the solution will be the limit under the $C(J)$ norm of the iterants in the following Picard iteration:

$$\mathbf{x}_{n+1}(t) := \mathbf{x}(t_0) + \int_{t_0}^{t} f(s, \mathbf{x}_n(s))\, \mathrm{d}s.$$

□

Remark 4.1 (Maximal time of existence and bootstrapping). Suppose $\sup_{s\in\mathbb{R}} K_s$ is bounded. Then when we reach $t_0+\eta$, we can extend it by another η, and then again, ad infinitum, and thus "bootstrap" our way to a globally unique solution. This would fail if the sequence of η defined by

$$t_n := \sup J_n, \quad J_n := J_{n-1} \cup (t_{n-1}+\eta_n), \quad \eta_n \sup_{s\in J_n} K_s < 1$$

sums to a convergent series.

■

4.1. **Some words on the Peano existence theorem.** [non-examinable]

It turns out that it is possible to weaken the condition of Lipschitz continuity and establish an existence theorem with continuity only. This is known as Peano's existence theorem:

Theorem 4.5. *Let $f : \mathbb{R}\times\mathbb{R}^d \to \mathbb{R}^d$ be a continuous function. Then there exists a non-empty interval $[t_0, t_0+\eta)$ on which Cauchy problem*

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x}(t) = f(t,\mathbf{x}(t)), \qquad \mathbf{x}(t_0) = \mathbf{b}$$

has a solution in $C([t_0, t_0+\eta))$.

But as we saw in Example 4.2, this solution may not be unique. Another way of looking at it is that an initial condition is not enough information to specify a unique solution in the space of continuous functions if f is just continuous in both of its arguments.

5. Lecture V: Local Well-Posedness II

5.1. **Gronwall's Inequality.** Also called Gronwall's lemma, this inequality is the archetypal bound arising from a differential inequality:

Theorem 5.1 (Gronwall's Inequality)**.** *Let $g : [0, T] \to \mathbb{R}$ be continuous and suppose that there are is a non-negative constant C and a non-negative $v : [0, T] \to \mathbb{R}$ such that*

$$g(t) \le C + \int_0^t v(s)g(s)\,\mathrm{d}s, \qquad t \in [0, T]. \tag{14}$$

Then

$$g(t) \le C \exp\left(\int_0^t v(s)\,\mathrm{d}s\right).$$

Remark 5.1. Note that this is slightly more general than the inequality found in *Schaeffer and Cain*. One can understand the inequality (14) as a differential inequality if $C = g(0)$, and a differential formulation from which we can deduce the integral formulation (14) is

$$\frac{\mathrm{d}}{\mathrm{d}t} g(t) \le v(t)g(t).$$

If v is bounded on $[0, T]$, then we also have

$$g(t) \le Ce^{\|v\|_{L^\infty([0,T])}t}, \qquad \left(\|v\|_{L^\infty([0,T])} := \sup_{t\in[0,T]} |v(t)|\right).$$

Proof. One simple way to prove the inequality is by iterating it and using the Taylor expansion for the exponential.

We can also set

$$G(t) = C + \int_0^t v(s)g(s)\,\mathrm{d}s,$$

from which we obtain

$$g(t) \le G(t), \qquad G'(t) = v(t)g(t).$$

Leibnitz's rule then shows that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{-\int_0^t v(s)\,\mathrm{d}s} G(t)\right) = e^{-\int_0^t v(s)\,\mathrm{d}s} v(t)\big(-G(t) + g(t)\big) \le 0,$$

because the exponential and v are both non-negative, and $g \le G$ pointwise.

Therefore we find that

$$e^{-\int_0^t v(s)\,\mathrm{d}s} G(t) \le G(0) = C,$$

which can be expanded to yield the inequality in the theorem statement.

$\square$

We shall apply Gronwall's inequality to derive continuous dependence on initial conditions.

Corollary 5.2. *Let $\mathbf{x}$ and $\mathbf{y}$ be solutions in $C([0, T])$ to the differential equation*

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{u} = f(t, \mathbf{u}(t)),$$

with initial conditions $\mathbf{u}(0) = \mathbf{x}(0)$ and $\mathbf{u}(0) = \mathbf{y}(0)$, respectively. Suppose f is continous $\mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$, and Lipschitz in its second argument:

$$|f(s, \boldsymbol{\xi}) - f(s, \boldsymbol{\zeta})| \le K_s|\boldsymbol{\xi} - \boldsymbol{\zeta}|, \qquad 0 \le K_s \in C([0, T]), \quad \boldsymbol{\xi}, \boldsymbol{\zeta} \in \mathbb{R}^d.$$

For $t \in [0, T]$, it holds that

$$|\mathbf{x}(t) - \mathbf{y}(t)| \le |\mathbf{x}(0) - \mathbf{y}(0)|e^{\int_0^t K_s\,\mathrm{d}s}.$$

Proof. This result follows from the previous theorem by application of the Lipschitz assumption of f. First integrate the differential equation over $[0, t]$, $t \in [0, T]$ with initial conditions $\mathbf{x}(0)$ and $\mathbf{y}(0)$, then take the difference. By the triangle inequality,

$$\begin{aligned} |\mathbf{x}(t) - \mathbf{y}(t)| &\leq |\mathbf{x}(0) - \mathbf{y}(0)| + \int_0^t |f(s, \mathbf{x}(s)) - f(s, \mathbf{y}(s))| \, \mathrm{d}s \\ &\leq |\mathbf{x}(0) - \mathbf{y}(0)| + \int_0^t K_s |\mathbf{x}(s) - \mathbf{y}(s)| \, \mathrm{d}s. \end{aligned}$$

Now we can apply the Gronwall inequality with $g(t) = |\mathbf{x}(t) - \mathbf{y}(t)|$ and $v(s) = K_s$ in (14). $\square$