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Exercise set 3:Solutions

1 **a)** Determine if the following expressions are norms for $\mathbb{R}^3$.

1. $f(x_1, x_2, x_3) = |x_1| + |x_2|$;
2. $f(x_1, x_2, x_3) = |x_1| + (|x_2|^2 + |x_3|^2)^{1/2}$;
3. $f(x_1, x_2, x_3) = (w_1|x_1|^3 + w_2|x_2|^2 + w_3|x_3|)^{1/2}$ for some positive real numbers w_1, w_2, w_3.

b) Determine $\|z\|_1$, $\|z\|_2$ and $\|z\|_\infty$ for $z = (1+i, 1-i)$ and $z = (e^{i\pi/2}, e^{3i\pi/2})$ in $\mathbb{C}^2$.

Solution. a)

1. This is not a norm, since $f(0, 0, a) = 0$ for any $a \in \mathbb{R}$. Hence we do not have $\|x\| = 0$ only if $x = 0$, and positivity is not satisfied.
2. This function defines a norm. Let us check the axioms.
   - Positivity: Clearly $f(x_1, x_2, x_3) \geq 0$ for any $(x_1, x_2, x_3) \in \mathbb{R}^3$ and $f(0) = 0$. If $f(x_1, x_2, x_3) = 0$, then $|x_1| + (|x_2|^2 + |x_3|^2)^{1/2} = 0$, and since this is a sum of positive numbers each summand must be zero. Hence $x_1 = 0$ and $(|x_2|^2 + |x_3|^2)^{1/2} = 0$, and by a similar argument we then get $x_2 = x_3 = 0$.
   - Homogeneity: If $\lambda \in \mathbb{R}$, we have
   $$\begin{aligned} f(\lambda x_1, \lambda x_2, \lambda x_3) &= |\lambda x_1| + (|\lambda x_2|^2 + |\lambda x_3|^2)^{1/2} \\ &= |\lambda||x_1| + (|\lambda|^2|x_2|^2 + |\lambda|^2|x_3|^2)^{1/2} \\ &= |\lambda| \left(|x_1| + (|x_2|^2 + |x_3|^2)^{1/2}\right). \end{aligned}$$
   - Triangle inequality: We know from earlier courses and the lecture notes that $|x_1 + y_1| \leq |x_1| + |y_1|$ and $(|x_2 + y_2|^2 + |x_3 + y_3|^2)^{1/2} \leq (|x_2|^2 + |x_3|^2)^{1/2} + (|y_2|^2 + |y_3|^2)^{1/2}$. These are just two instances of the usual triangle inequality in $\mathbb{R}^n$ for $n = 1, 2$. Combining these, we find for $(x_1, x_2, x_3), (y_1, y_2, y_3) \in \mathbb{R}^3$:
   $$\begin{aligned} f(x_1 + y_1, x_2 + y_2, x_3 + y_3) &= |x_1 + y_1| + (|x_2 + y_2|^2 + |x_3 + y_3|^2)^{1/2} \\ &\leq |x_1| + |y_1| + (|x_2|^2 + |x_3|^2)^{1/2} + (|y_2|^2 + |y_3|^2)^{1/2} \\ &= f(x_1, x_2, x_3) + f(y_1, y_2, y_3). \end{aligned}$$

3. This f fails to satisfy the homogeneity axiom. For instance, we have that $f(0,0,3\cdot 2) = \sqrt{w_3|3\cdot 1|} = \sqrt{3}\sqrt{w_3|1|} = \sqrt{3}\sqrt{\omega_3}$. This is not equal to $3f(0,0,1) = 3\sqrt{\omega_3}$.

b)

1. If $z = (1+i, 1-i)$, then $\|z\|_1 = |1+i|+|1-i| = 2\sqrt{2}$ (I trust that you are able to calculate $|1+i|$ and similar expressions). Similarly $\|z\|_2 = \sqrt{|1+i|^2 + |1-i|^2} = \sqrt{2+2} = 2$. Finally $\|z\|_\infty = \sup\{|i+1|, |1-i|\} = \sup\{\sqrt{2}, \sqrt{2}\} = \sqrt{2}$.
2. If $z = (e^{i\pi/2}, e^{3i\pi/2})$, then $\|z\|_1 = |e^{i\pi/2}| + |e^{3i\pi/2}| = 2$. Similarly $\|z\|_2 = \sqrt{|e^{i\pi/2}|^2 + |e^{3i\pi/2}|^2} = \sqrt{2} = \sqrt{2}$. Finally $\|z\|_\infty = \sup\{|e^{i\pi/2}|, |e^{3i\pi/2}|\} = \sup\{1,1\} = 1$.

$\boxed{2}$

1. What is an *open* ball $B_1(x_0)$ in $\mathbb{C}$ with the metric induced by the norm $\|x\| = \sqrt{\mathrm{Re}(x)^2 + \mathrm{Im}(x)^2}$?
2. What is a *closed* ball $\overline{B_1}(f_0)$ in the space of bounded and continuous real-valued functions on $[a,b]$, denoted $BC([a,b],\mathbb{R})$, with the metric induced by the supremum norm?

Solution. Recall that the metric induced by a norm $\|\cdot\|$ on a normed space X is given by $d(x,y) = \|x-y\|$. Hence $B_1(x_0) = \{x \in X : \|x - x_0\| < 1\}$ and $\overline{B_1}(x_0) = \{x \in X : \|x-x_0\| \leq 1\}$.

1. The open unit ball $B_1(x_0)$ is by definition the set

$$\begin{aligned} B_1(x_0) &= \{x \in \mathbb{C} : \|x - x_0\| < 1\} \\ &= \{x \in \mathbb{C} : \sqrt{\mathrm{Re}(x-x_0)^2 + \mathrm{Im}(x-x_0)^2} < 1\} \\ &= \{x \in \mathbb{C} : \sqrt{(\mathrm{Re}(x) - \mathrm{Re}(x_0))^2 + (\mathrm{Im}(x) - \mathrm{Im}(x_0))^2} < 1\}. \end{aligned}$$

As we know, this equation describes an open disk of radius 1 in the complex plane centered at the point $x_0 = \mathrm{Re}(x_0) + i\mathrm{Im}(x_0)$.

2. By definition, where $\|\cdot\|_\infty$ denotes the supremum norm as usual,

$$\begin{aligned} \overline{B_1}(f_0) &= \{f \in BC([a,b],\mathbb{R}) : \|f - f_0\|_\infty \leq 1\} \\ &= \{f \in BC([a,b],\mathbb{R}) : \sup_{t\in[a,b]} |f(t) - f_0(t)| \leq 1\}. \end{aligned}$$

The condition $\sup_{t\in[a,b]} |f(t) - f_0(t)| \leq 1$ is fulfilled if and only if $|f(t)-f_0(t)| \leq 1$ for any $t \in [a,b]$. Hence the closed ball $\overline{B_1}(f_0)$ consists of those functions f such that the distance between $f(t)$ and $f_0(t)$ is less than 1 at each point $t \in [a,b]$. See also the plot.

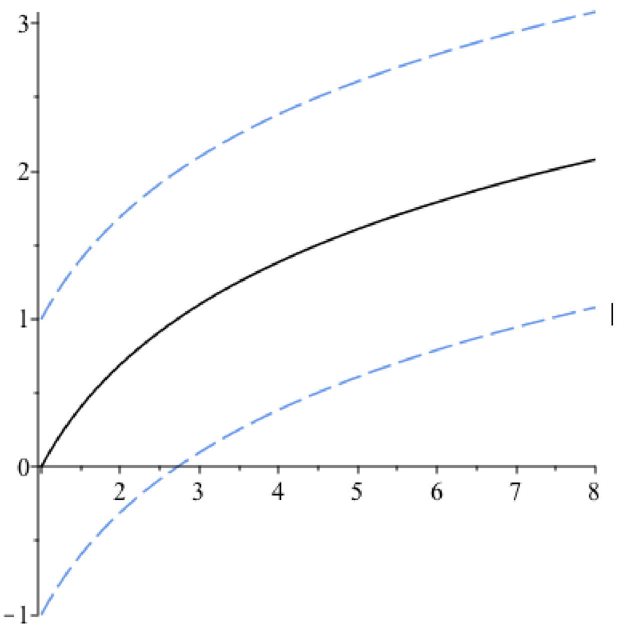

Figure 1: Illustration of problem 2.2. In this example we are working on the interval $I = [1, 8]$, and the function f_0 is the natural logarithm $f_0(t) = \ln(t)$, drawn in black. The dashed blue lines are the function $f_0(t) + 1$ and $f_0(t) - 1$. By our solution of problem 2.2, the closed ball of radius 1 centered at f_0 consists of those continuous functions whose graph stays between the two dashed lines at every point $t \in [1, 8]$

$\boxed{3}$ Recall that any set X can be endowed with the discrete metric

$$d(x, y) := \begin{cases} 1, & x \neq y, \\ 0, & x = y. \end{cases}$$

Show that in a discrete metric space, every subset is both open and closed.

Solution. Let U be any subset of X. We start by showing that U is open, i.e. that every point $x_0 \in U$ is an interior point of U. By definition, this means that we must find an $\epsilon > 0$ such that $B_\epsilon(x_0) \subset U$. Let us pick $\epsilon = \frac{1}{2}$. Then

$$\begin{aligned} B_{1/2}(x_0) &= \{x \in X : d(x, x_0) < 1/2\} \\ &= \{x_0\} \subset U. \end{aligned}$$

In words, $B_{1/2}(x_0)$ is a subset of U as it only consists of the point x_0 by the definition of the metric d. This shows that U is open.
To show that U is closed, we need to show that U contains all its boundary points. So assume that x_0 is a boundary point of U. This means that for every $\epsilon > 0$, the ball $B_\epsilon(x_0)$ contains a point $x \in U$ and a point $y \in X \setminus U$. In particular this must be true for $\epsilon = 1/2$, but we have seen that $B_{1/2}(x_0) = \{x_0\}$. For this ball to contain a point $x \in U$ and a point $y \in X \setminus U$, we must have $x = y = x_0$. But this implies that x_0 lies in both U and $X \setminus U$, which is not possible. We conclude that U *has*

no boundary points, so it trivially contains all its boundary points and is therefore closed.

$\boxed{4}$ Let X be a vector space and $\|.\|_a$ and $\|.\|_b$ norms on x. Show that $\|x\| := (\|x\|_a^2 + \|x\|_b^2)^{1/2}$ defines a norm on X.

Try to define a variant of this norm for $p \neq 2$ and contemplate about a possible proof of this statement.

Solution.
The case p=2

$\|x\| := (\|x\|_a^2 + \|x\|_b^2)^{1/2}$ is a norm, because

1. Positivity: $\|x\| = 0$ if and only if $\|x\|_a = 0$ and $\|x\|_b = 0$, which is the case only if $x = 0$.
2. Homogeneity: $\|\lambda x\| = (\|\lambda x\|_a^2 + \|\|ax\|_b^2)^{1/2} = (\lambda^2\|x\|_a^2 + \lambda^2\|x\|_b^2)^{1/2} = |\lambda|(\|x\|_a^2 + \|x\|_b^2)^{1/2} = |\lambda|\|x\|$.
3. Triangle inequaliy: By definition and the triangle inequalities for $\|\cdot\|_a$ and $\|\cdot\|_b$
$$\begin{aligned}\|x+y\| &= (\|x+y\|_a^2 + \|x+y\|_b^2)^{1/2}\\ &\leq \left[(\|x\|_a + \|y\|_a)^2 + (\|x\|_b + \|y\|_b)^2\right]^{1/2}.\end{aligned}$$
By the Minkowski inequality for $p = 2$ (also known as the triangle inequality for $\mathbb{R}^2$ with the p-norm), we then find that
$$\begin{aligned}\left[(\|x\|_a + \|y\|_a)^2 + (\|x\|_b + \|y\|_b)^2\right]^{1/2} &\leq (\|x\|_a^2 + \|x\|_b^2)^{1/2} + (\|y\|_a^2 + \|y\|_b^2)^{1/2}\\ &= \|x\| + \|y\|.\end{aligned}$$

Extension to $p \neq 2$
The natural extension of this norm for $p \neq 2$ is
$$\|x\|_p := (\|x\|_a^p + \|x\|_b^p)^{1/p}$$
for $p < \infty$, and
$$\|x\|_\infty := \max\{\|x\|_a, \|x\|_b\}$$
for $p = \infty$. The reason why this is the natural extension, is that we define our norms to be the ℓ^p-norms of the pair $(\|x\|_a, \|x\|_b) \in \mathbb{R}^2$. Hopefully we will then be able to use the triangle inequality for the ℓ^p-spaces to deduce the triangle inequality for our new norms.

We need to check that $\|\cdot\|_p$ and $\|\cdot\|_\infty$ satisfy the three axioms for being a norm. Let us start with $\|\cdot\|_p$.

1. Positivity: Clearly $\|x\|_p$ is positive, since $\|x\|_a$ and $\|x\|_b$ are positive numbers, and similarly $\|0\|_p = 0$ since $\|0\|_a = 0 = \|0\|_b$. If $\|x\|_p = 0$, then $\|x\|_a^p + \|x\|_b^p = 0$, and since this is a sum of positive numbers each summand must be 0. In particular $\|x\|_a = 0$, which implies that $x = 0$ since $\|\cdot\|_a$ is a norm.

2. Homegeneity: For λ a scalar, we have $\|\lambda x\|_p = (\|\lambda x\|_a^p + \|\lambda x\|_b^p)^{1/p} = (|\lambda|^p \|x\|_a^p + |\lambda|^p \|x\|_b^p)^{1/p} = |\lambda| \|x\|_p$, where the main step is to use that $\|\cdot\|_a$ and $\|\cdot\|_b$ are homogeneous by assumption.

3. Triangle inequality: Let $x, y \in X$. We find that

$$\begin{aligned} \|x+y\|_p &= (\|x+y\|_a^p + \|x+y\|_b^p)^{1/p} \\ &\leq [(\|x\|_a + \|y\|_a)^p + (\|x\|_b + \|y\|_b)^p]^{1/p} \end{aligned}$$

by using the triangle inequalities for $\|\cdot\|_a$ and $\|\cdot\|_b$. Now recall the Minkowski inequality for $\mathbb{R}^2$ (which is also the triangle inequality on $\mathbb{R}^2$ with the ℓ^p norm). It says that if $(a, b), (c, d) \in \mathbb{R}^2$, then

$$\begin{aligned} \|(a, b) + (c, d)\|_{\ell^p} &= [(|a+c|)^p + (|b+d|)^p]^{1/p} \\ &\leq (|a|^p + |b|^p)^{1/p} + (|c|^p + |d|^p)^{1/p}. \end{aligned}$$

Let us now pick $a = \|x\|_a$, $b = \|y\|_a$, $c = \|x\|_b$ and $d = \|y\|_b$. The Minkowski inequality then says that

$$\begin{aligned} [(\|x\|_a + \|y\|_a)^p + (\|x\|_b + \|y\|_b)^p]^{1/p} &\leq (\|x\|_a^p + \|x\|_b^p)^{1/p} + (\|y\|_a^p + \|y\|_b^p)^{1/p} \\ &= \|x\|_p + \|y\|_p. \end{aligned}$$

Now consider $\|\cdot\|_\infty$.

1. Positivity: Since $\|x\|_\infty$ is the maximum of two positive numbers, it must be positive. Also, $\|0\|_\infty = \max\{0, 0\} = 0$. Finally, if $\|x\|_\infty = 0$, then in particular $\|x\|_a = 0$, and since $\|\cdot\|_a$ is a norm we must have $x = 0$.

2. Homogeneity: If λ is a scalar, we have that

$$\begin{aligned} \|\lambda x\|_\infty &= \max\{\|\lambda x\|_a, \|\lambda x\|_b\} \\ &= \max\{|\lambda| \|x\|_a, |\lambda| \|x\|_b\} \\ &= |\lambda| \max\{\|x\|_a, \|x\|_b\} = \lambda \|x\|_\infty. \end{aligned}$$

3. Triangle inequality:

$$\begin{aligned} \|x+y\|_\infty &= \max\{\|x+y\|_a, \|x+y\|_b\} \\ &\leq \max\{\|x\|_a + \|y\|_a, \|x\|_b + \|y\|_b\} \\ &\leq \max\{\|x\|_a, \|x\|_b\} + \max\{\|y\|_a, \|y\|_b\} = \|x\|_\infty + \|y\|_\infty. \end{aligned}$$

The first inequality is a result of the triangle inequality for the norms $\|\cdot\|_a$ and $\|\cdot\|_b$. The second inequality is a property of taking the maximum of sums of real number - make sure that you see why we need an inequality rather than an equality.

$\boxed{5}$ Let $M_n(\mathbb{R})$ be the vector space of $n \times n$ matrices. Define for $A \in M_n(\mathbb{R})$ the function $\|A\|_2 = (\sum_{i,j=1}^n |a_{ij}|^2)^{1/2}$. Show that $\|.\|_2$ is a norm on $M_n(\mathbb{R})$. The trace of a matrix $A \in M_n(\mathbb{R})$ is defined as the sum of its diagonal elements, $\mathrm{tr}(A) = a_{11} + \cdots + a_{nn}$. Prove that $\|A\|_2^2 = \mathrm{tr}(A^T A)$. If the general case is too difficult, try to do it for $n = 3$.

Solution. We first show that $\|\cdot\|_2$ defines a norm on $M_n(\mathbb{R})$. Intuitively, this space is exactly the same as $\mathbb{R}^{n^2}$ with the ℓ^2-norm. Let us check the axioms:

1. Positivity: $\|A\|_2$ is obviously positive for any matrix A. Also, the zero element of $M_n(\mathbb{R})$ is the zero matrix, i.e. the matrix 0 where all entries are zero. Clearly $\|0\|_2 = (\sum_{i,j=1}^n |0|^2)^{1/2} = 0$. If $\|A\|_2 = 0$ for some matrix A, we have by the definition that $\sum_{i,j=1}^n |a_{ij}|^2 = 0$. Since this is a sum of positive numbers $|a_{ij}|$ whose sum is 0, we must have that every $a_{ij} = 0$, hence $A = 0$ - the zero matrix.

2. Homogeneity: Let $\lambda \in \mathbb{R}$, $A \in M_n(\mathbb{R})$. Then λA is defined by multiplying every entry of A by λ, so that the entries of λA are λa_{ij}. We then find that

$$\begin{aligned}
\|\lambda A\|_2 &= (\sum_{i,j=1}^n |\lambda a_{ij}|^2)^{1/2} \\
&= (\sum_{i,j=1}^n |\lambda|^2 |a_{ij}|^2)^{1/2} \\
&= |\lambda| (\sum_{i,j=1}^n |a_{ij}|^2)^{1/2} = |\lambda| \|A\|_2.
\end{aligned}$$

3. Triangle inequality: Let $A, B \in M_n(\mathbb{R})$, and let a_{ij} be the entries of A and b_{ij} the entries of B. Then the entries of $A + B$ are $a_{ij} + b_{ij}$ (adding two matrices does of course correspond to adding each entry, as you know!). Hence, by using the triangle inequality for $\mathbb{R}^{n^2}$ in the ℓ^2-norm (Minkowski's inequality):

$$\begin{aligned}
\|A + B\|_2 &= (\sum_{i,j=1}^n |a_{ij} + b_{ij}|^2)^{1/2} \\
&\leq (\sum_{i,j=1}^n |a_{ij}|^2)^{1/2} + (\sum_{i,j=1}^n |b_{ij}|^2)^{1/2} = \|A\|_2 + \|B\|_2.
\end{aligned}$$

We now move on to show that $\|A\|_2^2 = \mathrm{tr}(A^T A)$. Since $\mathrm{tr}(A^T A)$ is the sum of the diagonal entries of $A^T A$, we start by studying these diagonal entries. Let us consider the multiplication of two arbitrary matrices A and B. If we write $C = BA$, we know that the entry c_{ij} is the dot product of row i of B with column j of A [1] - $c_{ij} = \sum_{k=1}^n b_{ik} a_{k,j}$. In our case we have $B = A^T$, and we are interested in the diagonal entries c_{ii} of the product $A^T A$. As we discussed, c_{ii} is the dot product of row i of

[1] There is nothing mysterious about this, it is the way you were taught to multiply matrices.

A^T with column i of A. But by definition row i of A^T is column i of A - hence c_{ii} is actually just the dot product of column i of A with itself! In detail, we have $c_{ii} = \sum_{j=1}^n a_{ji}^2$. If we now sum all the diagonal entries c_{ii}, we get that

$$\begin{aligned}\operatorname{tr}(A^T A) &= \sum_{i=1}^n c_{ii} \\ &= \sum_{i,j=1}^n a_{ji}^2 \\ &= \|A\|_2^2.\end{aligned}$$

Hence $\sqrt{\operatorname{tr}(A^T A)} = \|A\|_2$.
If this seemed a bit too abstract, let us show the reasoning on 3×3-matrices. Consider

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}.$$

Then $\|A\|_2 = \sqrt{\sum_{i,j=1}^3 a_{ij}^2}$. Clearly

$$A^T = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}.$$

If we multiply these matrices (focusing on the diagonal), we get that

$$A^T A = \begin{bmatrix} a_{11}^2 + a_{21}^2 + a_{31}^2 & & \\ & a_{12}^2 + a_{22}^2 + a_{32}^2 & \\ & & a_{13}^2 + a_{23}^2 + a_{33}^2 \end{bmatrix}.$$

Clearly, if we sum over all the diagonal entries of $A^T A$, we will obtain the sum $\sum_{i,j=1}^3 a_{ij}^2$, which is exactly $\|A\|_2^2$.

6 Let $(X, \|.\|)$ be a normed vector space. Show that for any $x, y \in X$ we have

$$|\|x\| - \|y\|| \le \|x - y\|.$$

This is known as the *reversed triangle inequality.*

Solution. An inequality with absolute values, such as this one, can equivalently be written as a set of *two* equations with no absolute values[2]:

$$\begin{aligned} \|x\| - \|y\| &\le \|x - y\| \\ -\|x - y\| &\le \|x\| - \|y\|. \end{aligned}$$

[2]This should be known from Matte 1 or equivalent courses, but make sure that you understand it.

The first of these inequalities follows from writing $x = y + (x - y)$ and using the triangle inequality:

$$\|x\| = \|y + (x - y)\| \leq \|y\| + \|x - y\|.$$

If we subtract $\|y\|$ from both sides, we end up with $\|x\| - \|y\| \leq \|x - y\|$. The second inequality is proved similarly, by writing $y = x + (y - x)$ and using the triangle inequality:

$$\|y\| = \|x + (y - x)\| \leq \|x\| + \|x - y\|.$$

Here we use that $\|y - x\| = \|x - y\|$. Subtracting $\|y\|$ and $\|x - y\|$ from both sides, we have that $-\|x - y\| \leq \|x\| - \|y\|$. We have proved both inequalities, and therefore proved the original inequality with the absolute value.