Norwegian University of Science and Technology
Deptartment of Mathematical Sciences

TMA4190 Introduction to Topology Spring 2018

Solutions to exercise set 4

$\boxed{1}$ We can assume that $U = X$, since we just need to consider the map $U \subseteq X \xrightarrow{f} Y$. Let y be any point in $f(X)$. We need to show that there is an open nieghborhood W around y which is contained in $f(X)$. Let x be a point in X with $f(x) = y$ which exists since $y \in f(X)$. By the Local Submersion Theorem, we can choose local parametrizations $\phi\colon V \to X$ around x with $V \subset \mathbb{R}^n$ open and $\psi\colon V' \to Y$ around y with $V' \subset \mathbb{R}^m$ open such that the induced map $V \to V'$ is the canonical submersion:

$$\begin{tikzcd} X \arrow[r, "f"] & Y \\ V \arrow[u, "\phi"] \arrow[r, "\text{canonical}", "\text{submersion}"'] & V' \arrow[u, "\psi"'] \end{tikzcd}$$

By possibly shrinking V and V', we can assume that $V = B_\epsilon(0) \subset \mathbb{R}^n$ and $V' = B_\epsilon(0) \subset \mathbb{R}^m$. Then the canonical submersion maps V **onto** V'. Since the above diagram commutes, we see that $W := \psi(V')$ is contained in $f(X)$. Since ψ is a diffeomorphism onto its image in Y, W is open in Y.

$\boxed{2}$ **a)** Let $f\colon X \to Y$ be a submersion. We obviously have $Y = f(X) \cup (Y \setminus f(X))$. Since X is compact and f is continuous, we know $f(X)$ is compact and therefore closed in Y. Since X is open in X and f is a submersion, the previous exercise shows that $f(X)$ is open in Y. Hence $f(X)$ is both open and closed in Y. Hence $f(X)$ must be either Y or $\emptyset$. Assuming f is nontrivial, $f(X)$ must be all of Y.

b) Given a compact smooth manifold X. Assume we had a submersion $f\colon X \to \mathbb{R}^n$ with for some n. By the previous point, we would have $f(X) = \mathbb{R}^n$. But since X is compact, $f(X)$ is compact too. But $\mathbb{R}^n$ is not compact. Hence such a submersion cannot exist.

$\boxed{3}$ Given $A = (a_{ij}) \in O(n)$. Unfolding the matrix-multiplication, we see that $\sum_j a_{ij}^2$ is the ith diagonal entry in AA^t. Moreover, $\sum_j a_{ij}^2$ is also the square of the norm of the ith row vector of A. Since $A \in O(n)$, we have $AA^t = I$ and the ith diagonal entry in AA^t is equal 1. This shows that $O(n)$ is contained in the product of n spheres $\prod S^{n-1}$ in $\prod \mathbb{R}^n = \mathbb{R}^{n^2} = M(n)$. Hence $O(n)$ is bounded. But $O(n)$ is also closed in $\mathbb{R}^{n^2}$, since we can define it as the inverse image of the closed point $I \in S(n)$ under the map $M(n) \to S(n)$ sending A to AA^t. Thus $O(n)$ is closed and bounded in $\mathbb{R}^{n^2}$ and therefore compact.

$\boxed{4}$ We consider $O(n)$ as a subspace in $M(n) = \mathbb{R}^{n^2}$. We defined $O(n)$ as $f^{-1}(I)$ under the map $f\colon M(n) \to S(n)$, $f(A) = AA^t$. We have checked in the lecture that I is a regular value for f. By the first propostion of Lecture 8, this implies that $T_I(O(n))$ equals the kernel of $df_I\colon M(n) = T_I(M(n)) \to T_I(S(n)) = S(n)$. We calculated the derivative df_A for any $A \in O(n)$ in the lecture. It is given by $df_A(B) = BA^t + AB^t$. For $A = I$, this gives $df_I(B) = B + B^t$. Hence the kernel of df_I is the space of matrices satisfying $B - B^t = 0$, i.e. $B^t = -B$.

$\boxed{5}$ As an open subspace of $M(n)$, the space of nonzero 2×2-matrices $M(2) \setminus \{0\}$ is a manifold of dimension 4. A 2×2-matrix A has rank 0 if and only if it is the zero matrix. Thus $A \in M(2) \setminus \{0\}$ has rank 1 if and only if it does not have rank 2, i.e. if and only if it is not invertible. Hence $A \in M(2) \setminus \{0\}$ has rank 1 if and only if $\det A = 0$. Thus it suffices to show that the determinant function is a submersion $M(2) \setminus \{0\} \to \mathbb{R}$. For then $R_1 = \det^{-1}(0)$ is a submanifold of dimension $4 - 1 = 3$.

We need to show that the derivative of the determinant at a matrix $A \in R_1$ is surjective. Since $d(\det)_A$ is a linear map $\mathbb{R}^4 \to \mathbb{R}$, it suffices to show $d(\det)_A$ is nonzero. Therefore, it suffices to show that $d(\det)_A(B) \neq 0$ for some matrix B.

Since $A \neq 0$, there is at least one entry in A which is nonzero. Assume that $a_{11} \neq 0$ (for the other cases the argument is similar). Then we take the matrix $E_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$:

$$\begin{aligned}
d(\det)_A(E_{22}) &= \lim_{s\to 0} \frac{\det(A + sE_{22}) - \det A}{s} \\
&= \lim_{s\to 0} \frac{\det(A + sE_{22}) - 0}{s} \\
&= \lim_{s\to 0} \frac{(a_{11})(a_{22} + s) - a_{12}a_{21}}{s} \\
&= \lim_{s\to 0} \frac{a_{11}a_{22} + a_{11}s - a_{12}a_{21}}{s} \\
&= \lim_{s\to 0} \frac{a_{11}s + \det A}{s} \\
&= \lim_{s\to 0} \frac{a_{11}s}{s} = a_{11} \neq 0.
\end{aligned}$$