NTNU
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Week 35: Lecture 2

Introduction to first step analysis

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Simulating Markov chains

Input:

- i_0: starting state
- $\mathbf{P}$: transition probability matrix
- T: number of time steps

Algorithm:

Output: One realization $x_0, x_1, \ldots, x_T$.

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Simulating Markov chains

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Algorithm:

1. Set $x_0 = i_0$
2. for $n = 1 \dots T$
3. Simulate x_n from $X_n | X_{n-1} = x_{n-1}$
4. end

Output: One realization $x_0, x_1, \dots, x_T$.

Section 3.3: Examples

Example: Ehrenfest urn model

We have $N = 100$ balls divided into two containers labelled A and B. At each time step n, one ball is selected at random and moved to the other container. Let Y_n denote the number of balls in container A at time n, and define $X_n = Y_n - 50$. Find the transition probabilities, and simulate and plot one realization of $\{X_n : n = 0, 1, \ldots, 500\}$.

Section 3.4: First step analysis

Motivation

Definition (Absorbing state)

For a Markov chain, a state i such that $\mathrm{P}_{ij} = 0 \; \forall j \neq i$ is called **absorbing**.

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Motivating example

Let $\{X_n\}$ be a Markov chain with transition probability matrix

$$\mathbf{P} = \begin{bmatrix} 1 & 0 & 0 \\ \alpha & \beta & \gamma \\ 0 & 0 & 1 \end{bmatrix},$$

where $\alpha, \beta, \gamma > 0$ and $\beta = 1 - \alpha - \gamma$. Assume $x_0 = 1$.

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Q1: What is the expected time until absorption?

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Q1: What is the expected time until absorption?
Q2: What is the probabibility to be absorbed in state 0?

Example 1

Let $\{X_n\}$ be a Markov chain with transition probability matrix

$$\mathbf{P} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0.4 & 0.3 & 0.2 & 0.1 \\ 0.1 & 0.3 & 0.3 & 0.3 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

The starting state is $x_0 = 1$. Calculate the probability to be absorbed in state 0.