19.3. **The WKB method.** What if β were large? Scaling time by $t/\beta \mapsto t$ (the slow time scale), we can consider the scaled equation

$$\frac{1}{\beta^2}\ddot{x} + (x^2-1)\dot{x} + x = 0.$$

The limit $\beta \to \infty$ is singular because it doesn't have oscillatory behaviour and we do not expect its solutions to approximate limit cycles of systems with large but finite β except for very short times. And indeed, for a short enough time, there is a solution to the truncated, integrable equation

$$(x^2-1)\dot{x} + x = 0,$$

which breaks down in finite time.

There is a fairly popular perturbation method that handles vanishing highest-order terms known as the WKB method, or the Carlini-Liouville-Green method. However, it is generally restricted to equations of the form

$$\varepsilon^2\ddot{x}(t) + V(t)x(t) = 0.$$

Let us step back from the van der Pol system and consider simpler systems of the above form, where V has isolated roots.

Suppose V were constant. Then we expect

$$x(t) = C_1 e^{-\sqrt{V}t/\varepsilon} + C_2 e^{\sqrt{V}t/\varepsilon}.$$

Comparing this to Example 18.3, it is evident that a vanishing highest order term gives dynamics that are very different from a vanishing lower-order term. The particular feature of the WKB method is the assumption of decay in the fast time variable $\tau_1 = t/\varepsilon$, so that ansatz are taken of the form:

$$x(t) = e^{\vartheta(t)/\varepsilon^\alpha}\big(x_0(t) + \varepsilon^\alpha x_1(t) + \cdots\big).$$

Taking derivatives of this ansatz, we find that

$$\begin{aligned}\dot{x} &= e^{\vartheta/\varepsilon^\alpha}\big(\varepsilon^{-\alpha}\dot{\vartheta}x_0 + \dot{x}_0 + \dot{\vartheta}x_1 + \varepsilon^\alpha\dot{x}_1 + \cdots\big)\\ \ddot{x} &= e^{\vartheta/\varepsilon^\alpha}\big(\varepsilon^{-2\alpha}\dot{\vartheta}^2x_0 + \varepsilon^{-\alpha}(\ddot{\vartheta}x_0 + 2\dot{\vartheta}\dot{x}_0 + \dot{\vartheta}^2x_1) + \cdots\big),\end{aligned}$$

etc., and upon substitution into the equation we have

$$\varepsilon^2\Big[\varepsilon^{-2\alpha}\dot{\vartheta}^2x_0 + \varepsilon^{-\alpha}\big(\ddot{\vartheta}x_0 + 2\dot{\vartheta}\dot{x}_0 + \dot{\vartheta}^2x_1\big) + \cdots\Big] + V(t)(x_0 + \varepsilon^\alpha x_1 + \cdots) = 0.$$

The linearity of the equation allowed us to multiply through with $\exp(-\vartheta(t)\varepsilon^{-\alpha})$.

Setting $\alpha = 1$ allows us to collect terms of like powers. The zeroth order equation is

$$\dot{\vartheta}^2 = V.$$

This can be integrated directly.

The first order equation is

$$\ddot{\vartheta}x_0 + 2\dot{\vartheta}\dot{x}_0 + \dot{\vartheta}^2x_1 - V(t)x_1 = 0.$$

Using the first equation, we find that we need only solve

$$\ddot{\vartheta}x_0 + 2\dot{\vartheta}\dot{x}_0 = 0,$$

which has the solution

$$x_0(t) = \frac{c}{\sqrt{\dot{\vartheta}}}.$$

These give a first order approximation of

$$x(t) = V^{-1/4}(t)\Bigg[C_0\exp\bigg(\frac{-1}{\varepsilon}\int^x \sqrt{V}(s)\,\mathrm{d}s\bigg) + C_1\exp\bigg(\frac{1}{\varepsilon}\int^x \sqrt{V}(s)\,\mathrm{d}s\bigg)\Bigg].$$

20. Lecture XX: Index Theory I

20.1. **Notion and Properties of the Index.** Index theory in dynamical systems is another manifestation of a phenomenon you may have encountered before in, say, residue theory/winding number/monodromy in complex analysis, or Euler-Poincaré indices of compact 2-D (or higher dimensional) manifolds or graphs, or degree theory of differentiable maps, or indeed, index theory of linear maps in Fourier analysis (Toeplitz operators).

The basic idea is that it is often possible to assign an integer invariant to maps, motivated by algebraic, geometric, or topological considerations, that proves remarkably useful, for example, in distinguishing one type of (nonlinear) dynamics from another.

Let us begin with some definitions. The INDEX of a Jordan curve $C \subseteq \mathbb{R}^2$ on a vector field $f : \mathbb{R}^2 \to \mathbb{R}^2$ given by $f(x,y) = (P(x,y), Q(x,y))^\top$, with no critical point on C is defined as

$$I_f(C) := \frac{1}{2\pi} \oint_C \mathrm{d}\Big(\arctan(\frac{\mathrm{d}y}{\mathrm{d}x}) \Big) = \frac{1}{2\pi} \oint_C \frac{P\mathrm{d}Q - Q\mathrm{d}P}{P^2 + Q^2},$$

the second equality following from L'Hospital's rule. This is the averaged curl around the curve C, the amount of circulation picked up along C, reminiscent of Green's theorem as well as of Thm.13.1. From arctangent definition, and from the fact that f is C^1, with none of its critical points on the curve C, it can be seen that $I_f(C)$ must needs be an integer (because f varies smoothly along C).

We say that a(n oriented) Jordan curve C is the sum of two (oriented) Jordan curves C_1 and C_2, written as $C = C_1 + C_2$, if C_1 and C_2 (are compatibly oriented) and share a line segment. In a theorem reminscent of the Cauchy-Goursat integral theorem,

Theorem 20.1. 1. *Let a Jordan curve C be the sum of Jordan curves C_1 and C_2. Then with respect to any C^1 vector field f with no critical point on C_1 or C_2,*

$$I_f(C) = I_f(C_1) + I_f(C_2).$$

2. *If neither C nor its interior contain a critical point of f, then $I_f(C) = 0$.*

Proof. 1.

This is a direct consequence of the linearity of integrals.

2.

Suppose $P^2 + Q^2 > \sigma > 0$ is bounded away from nought on an open set E by the lack of critical points of the system/vector field f on E. Then for any $\varepsilon > 0$, a piecewise-C^1 Jordan curve C entirely within E of length δ may be chosen so that the quantity

$$I_f(C) = \left| \frac{1}{2\pi} \oint_C \frac{P\mathrm{d}Q - Q\mathrm{d}P}{P^2 + Q^2} \right| < \frac{\delta}{2\pi\sigma} \sup_{\mathbf{x} \in E} \left| P\frac{\mathrm{d}Q}{\mathrm{d}r} - Q\frac{\mathrm{d}P}{\mathrm{d}r} \right|, \tag{31}$$

where $\mathrm{d}/\mathrm{d}r$ is the directional derivative along C in the counter-clockwise direction, is bounded by ε by taking δ to be sufficiently small. This is because $P, Q \in (C^1(\mathbb{R}^2))^2$.

Since the integral $I_f(C)$ is an integer, with $\varepsilon < 1$, we can conclude that $I_f(C) = 0$.

Suppose now that $C \subseteq E$ is a bounded Jordan curve of any finite length. Denote its interior by U. We can divide $C \cup U$ by a finite grid of side length $\delta/4$ and apply the estimate (31) to every square lying entirely within U. Part 1. of this theorem then suggests that for the curve $\tilde{C}$, the sum of the perimeter of these squares, $I_f(\tilde{C}) = 0$.

Each of the remaining squares contain a portion C. Since C is of finite length we can subdivide these squares into smaller and smaller squares only finitely many times until either a square is inside U entirely or a square S contains a portion of C shorter than δ. Then the reduced square $S\backslash(C \cup U)^c$ has a boundary that is piecewise C^1 and of length 2δ. By choosing $\varepsilon < 1/2$, we can still use (31) to conlude that the remaining fintely many squares and reduced squares have index zero. $\square$

Corollary 20.2. *Let C_1 and C_2 be Jordan curves for which C_2 lies in the interior of C_1 and there exists only one critical point of the C^1-vector field f on the union of C_1 and its interior, which also lies in C_2. Then $I_f(C_1) = I_f(C_2)$.*

Proof. Let p_1, p_2 be distinct points on C_1 and q_1 and q_2 be distinct points on C_2. Let the counter clockwise arcs on C_1 and C_2 be defined:

$$A_1 = \wideparen{p_1p_2}, \quad A_2 = \wideparen{p_2p_1}, \quad B_1 = \wideparen{q_1q_2}, \quad B_2 = \wideparen{q_2q_1}.$$

Let $M_1 = \overline{p_1q_1}$ and $M_2 = \overline{p_2q_2}$ be two line segments that do not intersect one another or the interior of C_2.

The corollary statement is immediate as we can consider the sum of three Jordan curves whose enclosed areas do not intersect:

$$\Gamma_1 = A_1M_2\bar{B}_1\bar{M}_1, \qquad \Gamma_2 = A_2M_1\bar{B}_2\bar{M}_2,$$

and C_2, defined counter clockwise. Here we use the over bar to denote the same arc in the reverse sense. We see that $\Gamma_1 + C_2 + \Gamma_2 = C_1$.

The theorem then tells us that

$$I_f(\Gamma_1) + I_f(C_2) + I_f(\Gamma_2) = I_f(C_1),$$

and since Γ_1 does not enclose or include a critical point of f, $I_f(\Gamma_1) = 0$. Likewise, $I_f(\Gamma_2) = 0$. $\square$

This implies that any Jordan curve enclosing a unique critical point $\mathbf{x}_0$ of a vector field $f \in (C^1(\mathbb{R}^2))^2$ has the same index. Therefore it makes sense to define

$$I_f(\mathbf{x}_0) := I_f(C),$$

where C is a C^1-Jordan curve enclosing the unique critical point $\mathbf{x}_0$.

Using a similar construction to the above, we also have

Corollary 20.3. *Let C be a Jordan curve enclosing n isolated critical points $\{\mathbf{x}_i\}_{i=1}^n$ of $f \in (C^1(\mathbb{R}^2))^2$ and only these critical points. Then*

$$I_f(C) = \sum_{i=1}^{n} I_f(\mathbf{x}_i).$$

One simply considers C in place of C_1 above and a collection $\{J_i\}_{i=1}^n$ of n mutually disjoint Jordan curves, each enclosed in C with J_i enclosing $\mathbf{x}_i$, so that $I_f(C_i) = I_f(\mathbf{x}_i)$, in place of C_2 above.

20.2. **Applications of the Index.** Till now we have yet to apply the index to describe actual systems.

Since the previous results suggest that there is some robustness to the value of the index, it is no surprise that

Theorem 20.4. *If a C^1-first order planar system is governed by*

$$f(\mathbf{x}) = \mathrm{D}f|_{\mathbf{x}_0} + g(\mathbf{x} - \mathbf{x}_0),$$

where $\mathbf{x}_0$ is an isolated critical point, and $|g(\mathbf{x})| = o(|\mathbf{x}|)$ as $|\mathbf{x}| \to 0$, then

$$I_f(\mathbf{x}_0) = I_v(\mathbf{x}_0),$$

where v is the vector field given by the linearisation, $v(\mathbf{y}) = \mathrm{D}f|_{\mathbf{x}_0}\mathbf{y}$ for $\mathbf{y} \in \mathbb{R}^2$.

Next we look at some results on calculation of indices and what the values of indices imply about the (local) vector fields they are defined over.

Theorem 20.5. *Let $f \in (C^1(E))^2$ be a vector field on an open subset $E \subseteq \mathbb{R}^2$. Let Γ be a periodic orbit of the system $\dot{\mathbf{x}} = f(\mathbf{x})$ lying entirely in E. Then $I_f(E) = 1$.*

Proof. This is a very simple calculation in polar coordinates. Let Γ be parameterized by (r, ϑ), and have period T:

$$\begin{aligned}
I_f(\Gamma) &= \frac{1}{2\pi} \oint_\Gamma \mathrm{d}\Big(\arctan(\frac{\mathrm{d}y}{\mathrm{d}x})\Big) \\
&= \frac{1}{2\pi} \int_0^T \frac{x\dot{y} - y\dot{x}}{x^2 + y^2}\,\mathrm{d}t \\
&= \frac{1}{2\pi} \int_0^{2\pi} \frac{r\cos(\theta)\mathrm{d}\big(r\sin(\theta)\big) - r\sin(\theta)\mathrm{d}\big(r\cos(\theta)\big)}{r^2} \\
&= \frac{1}{2\pi} \int_0^{2\pi} \cos(\theta)\mathrm{d}\sin(\theta) + \frac{1}{r}\cos(\theta)\sin(\theta)\mathrm{d}r - \sin(\theta)\mathrm{d}\cos(\theta) - \frac{1}{r}\sin(\theta)\mathrm{d}\cos(\theta) \\
&= \frac{1}{2\pi} \int_0^{2\pi} (\cos^2(\theta) + \sin^2(\theta))\mathrm{d}\theta \\
&= 1.
\end{aligned}$$

□

The theorem above, along with part 2 of Thm.20.1 clearly implies

Corollary 20.6. *Let $f \in (C^1(E))^2$ be a vector field on an open subset $E \subseteq \mathbb{R}^2$. Let Γ be a periodic orbit of the system $\dot{\mathbf{x}} = f(\mathbf{x})$ lying entirely in E. Then there is at least one critical point of f in the interior of Γ.*