NTNU
Norwegian University of
Science and Technology

Week 39: Lecture 2

The conditional distribution of the locations of events, and simulation of Poisson processes

Geir-Arne Fuglstad

September 23, 2020

Information

- **No lectures next week (September 28 and September 30).**
- Guidance with project next week:
  1. Smia and S21 on Monday 8:15–10:00. You need to **pre-register** using `https://s.ntnu.no/proj1mon`.
  2. R2 on Tuesday 8:15–10:00. No pre-registration, but we need to follow infection control rules such as maximum 4 people at each table.
  3. Digital guidance on Wednesday 12:15–14:00. More information later.
  4. Ask questions by e-mail to Mina (`mina.spremic@ntnu.no`) or Susan (`susan.anyosa@ntnu.no`)

Example

$\{X(t) : t \geq 0\}$ is a Poisson process with rate $\lambda > 0$. Determine the distribution of $W_1 | X(t) = 1$.

Theorem (Theorem 5.7)

Let $W_1, W_2, \ldots$ be the occurence times in a Poisson process $\{X(t) : t \geq 0\}$ with rate $\lambda > 0$. Then

$$\begin{aligned}(W_1, W_2, \ldots, W_n)|X(t) = n &\sim f(w_1, w_2, \ldots, w_n|X(t) = n) \\ &= \frac{n!}{t^n}, \quad 0 < w_1 < w_2 < \cdots < w_n \leq t.\end{aligned}$$

Example

Customers arrive according to a Poisson process with rate $\lambda > 0$ per hour. The store opens at 09:00. If 10 people have arrived at 11:00, what is the probability that exactly 5 of the 10 people arrived before 10:00?

Joint simulation

Input:

— Time interval: $(0, t]$

— Rate: $\lambda > 0$

Algorithm:

1. Simulate $n \sim \mathrm{Poisson}(\lambda t)$
2. Simulate $v_1, v_2, \ldots, v_n \overset{\text{iid}}{\sim} \mathcal{U}(0, t)$
3. Let $(w_1, w_2, \ldots, w_n) = \mathrm{sort}(v_1, v_2, \ldots, v_n)$

Output:

$$x(s) = \begin{cases} 0, & 0 \leq s < w_1 \\ 1, & w_1 \leq s < w_2 \\ \vdots & \\ n, & w_n \leq s < t. \end{cases}$$

Sequential simulation

Input:

— Number of jumps: N

— Rate: $\lambda > 0$

Algorithm:

1. $w_0 = 0$
2. for $n = 1 \ldots N$
3. Simulate $s_{n-1} \sim \mathrm{Exp}(\lambda)$
4. Set $w_n = w_{n-1} + s_{n-1}$
5. end

Output:

$$x(s) = \begin{cases} 0, & 0 \leq s < w_1 \\ 1, & w_1 \leq s < w_2 \\ \vdots & \\ n, & w_n \leq s < t. \end{cases}$$

Chapter 6

Definition

We call the stochastic process $\{X(t) : t \geq 0\}$ a **continuous-time Markov chain** with state space $\{0, 1, \ldots\}$ if it satisfies the **Markov property**

$$\Pr\{X(t+s) = j | X(s) = i, X(u), 0 \leq u < s\} = \Pr\{X(t+s) = j | X(s) = i\}$$

for $i, j = 0, 1, \ldots$ and for all $s \geq 0$ and $t > 0$.

Definition

Let $\{X(t) : t \geq 0\}$ be a continuous-time Markov chain with state space $\{0, 1, \ldots\}$ and stationary transition probabilities. We call

$$P_{ij}(t) = \Pr\{X(t) = j | X(0) = i\}, \quad i, j = 0, 1, \ldots,$$

the **transition probability functions.**