Formulas: TMA4265 Stochastic Modeling:

The law of total probability

Let $B_1, B_2, \ldots$ be pairwise disjoint events with $P(\cup_{i=1}^{\infty} B_i) = 1$. Then

$$P(A|C) = \sum_{i=1}^{\infty} P(A|B_i \cap C)P(B_i|C),$$

$$E[X|C] = \sum_{i=1}^{\infty} E[X|B_i \cap C]P(B_i|C).$$

Discrete time Markov chains

Chapman-Kolmogorov equations

$$P_{ij}^{(m+n)} = \sum_{k=0}^{\infty} P_{ik}^{(m)} P_{kj}^{(n)}.$$

For an irreducible and ergodic Markov chain, $\pi_j = \lim_{n\to\infty} P_{ij}^{(n)}$ exist and is given by the equations

$$\pi_j = \sum_i \pi_i P_{ij} \quad \text{and} \quad \sum_i \pi_i = 1.$$

For transient states i, j and k, the mean passage time from i to $j \neq i$, M_{ij}, is

$$M_{ij} = 1 + \sum_k P_{ik} M_{kj}.$$

The Poisson process

The waiting time to the n-th event (the n-th arrival time), X_n, has probability density

$$f_{X_n}(t) = \frac{\lambda^n t^{n-1}}{(n-1)!} e^{-\lambda t} \quad \text{for} \quad t \geq 0.$$

Given that the number of events $N(t) = n$, the arrival times $X_1, X_2, \ldots, X_n$ have the uniform joint probability density

$$f_{X_1,X_2,\ldots,X_n|N(t)}(x_1, x_2, \ldots, x_n|n) = \frac{n!}{t^n} \quad \text{for} \quad 0 < x_1 < x_2 < \ldots < x_n \leq t.$$

Markov processes in continuous time

A (homogeneous) Markov process $X(t)$, $0 \le t \le \infty$, with state space $\Omega \subseteq \mathbf{Z}^+ = \{0, 1, 2, \ldots\}$, is called a birth and death process if

$$P_{i,i+1}(h) = \lambda_i h + o(h)$$

$$P_{i,i-1}(h) = \mu_i h + o(h)$$

$$P_{i,i}(h) = 1 - (\lambda_i + \mu_i)h + o(h)$$

$$P_{ij}(h) = o(h) \quad \text{for } |j - i| \ge 2$$

where $P_{ij}(s) = P(X(t+s) = j | X(t) = i)$, $i, j \in \mathbf{Z}^+$, $\lambda_i \ge 0$ are birth rates, $\mu_i \ge 0$ are death rates.

The Chapman-Kolmogorov equations

$$P_{ij}(t+s) = \sum_{k=0}^{\infty} P_{ik}(t) P_{kj}(s).$$

Limit relations

$$\lim_{h \to 0} \frac{1 - P_{ii}(h)}{h} = v_i \, , \quad \lim_{h \to 0} \frac{P_{ij}(h)}{h} = q_{ij} \, , \;\; i \ne j$$

Kolmogorov's forward equations

$$P'_{ij}(t) = \sum_{k \ne j} q_{kj} P_{ik}(t) - v_j P_{ij}(t).$$

Kolmogorov's backward equations

$$P'_{ij}(t) = \sum_{k \ne i} q_{ik} P_{kj}(t) - v_i P_{ij}(t).$$

If $P_j = \lim_{t \to \infty} P_{ij}(t)$ exist, P_j are given by

$$v_j P_j = \sum_{k \ne j} q_{kj} P_k \quad \text{and} \quad \sum_j P_j = 1.$$

In particular, for birth and death processes

$$P_0 = \frac{1}{\sum_{k=0}^{\infty} \theta_k} \quad \text{and} \quad P_k = \theta_k P_0 \quad \text{for} \quad k = 1, 2, \ldots$$

where

$$\theta_0 = 1 \quad \text{and} \quad \theta_k = \frac{\lambda_0 \lambda_1 \cdot \ldots \cdot \lambda_{k-1}}{\mu_1 \mu_2 \cdot \ldots \cdot \mu_k} \quad \text{for} \quad k = 1, 2, \ldots$$

Queueing theory

For the average number of customers in the system L, in the queue L_Q; the average amount of time a customer spends in the system W, in the queue W_Q; the service time S; the average remaining time (or work) in the system V, and the arrival rate λ_a, the following relations obtain

$$L = \lambda_a W.$$
$$L_Q = \lambda_a W_Q.$$
$$Z = \lambda_a E[S].$$
$$V = \lambda_a E[SW_Q^*] + \lambda_a E[S^2]/2.$$

Gaussian processes

The multivariate Gaussian density for $n \times 1$ random vector $\boldsymbol{x} = (x_1, \ldots, x_n)$ is

$$p(\boldsymbol{x}) = \frac{1}{(2\pi)^{n/2}|\boldsymbol{\Sigma}|^{1/2}} \exp\left(-\frac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\boldsymbol{x}-\boldsymbol{\mu})\right), \quad \boldsymbol{x} \in \mathbb{R}^n,$$

where size $n \times 1$ mean vector $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_n)$, $E(x_i) = \mu_i$, and

$$\boldsymbol{\Sigma} = \begin{bmatrix} \Sigma_{1,1} & \ldots & \Sigma_{1,n} \\ \ldots & \ldots & \ldots \\ \Sigma_{n,1} & \ldots & \Sigma_{n,n} \end{bmatrix}, \quad \Sigma_{i,j} = \mathrm{Cov}(x_i, x_j).$$

Let $\boldsymbol{x}_A = (x_{A,1}, \ldots, x_{A,n_A})$ and $\boldsymbol{x}_B = (x_{B,1}, \ldots, x_{B,n_B})$, be two subsets of variables, with block mean and covariance structure

$$\boldsymbol{\mu} = (\boldsymbol{\mu}_A, \boldsymbol{\mu}_B), \quad \boldsymbol{\Sigma} = \begin{bmatrix} \boldsymbol{\Sigma}_A & \boldsymbol{\Sigma}_{A,B} \\ \boldsymbol{\Sigma}_{B,A} & \boldsymbol{\Sigma}_B \end{bmatrix}.$$

The conditional density of $\boldsymbol{x}_A$, given $\boldsymbol{x}_B$, is Gaussian with

$$\begin{aligned} E(\boldsymbol{x}_A|\boldsymbol{x}_B) &= \boldsymbol{\mu}_A + \boldsymbol{\Sigma}_{A,B}\boldsymbol{\Sigma}_B^{-1}(\boldsymbol{x}_B - \boldsymbol{\mu}_B), \\ \mathrm{Var}(\boldsymbol{x}_A|\boldsymbol{x}_B) &= \boldsymbol{\Sigma}_A - \boldsymbol{\Sigma}_{A,B}\boldsymbol{\Sigma}_B^{-1}\boldsymbol{\Sigma}_{B,A}. \end{aligned}$$

The Brownian motion has increments $x(t_i) - x(t_{i-1})$ with the following properties, for any configuration of times $t_0 = 0 < t_1 < t_2 < \ldots$:

- $x(t_i) - x(t_{i-1})$ and $x(t_j) - x(t_{j-1})$ are independent for all $i \neq j$.
- the distribution of $x(t_i) - x(t_{i-1})$ is identical to that of $x(t_i + s) - x(t_{i-1} + s)$, for any s.
- $x(t_i) - x(t_{i-1})$ is Gaussian distributed with 0 mean and variance $\sigma^2(t_i - t_{i-1})$.

Unless otherwise stated, $x(0) = 0$.

Some mathematical series

$$\sum_{k=0}^{n} a^k = \frac{1 - a^{n+1}}{1 - a} \quad , \qquad \sum_{k=0}^{\infty} k a^k = \frac{a}{(1 - a)^2} \quad .$$