1.

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 3 & 0 \\ 2 & -4 & 2 \end{pmatrix}$$

(hint : find determinant of transpose)

finding eigenvalues :

$$0 = \det(A - \lambda I) = \det \begin{pmatrix} 1-\lambda & 2 & 0 \\ 0 & 3-\lambda & 0 \\ 2 & -4 & 2-\lambda \end{pmatrix} = (2-\lambda)(1-\lambda)(3-\lambda)$$

$\hookrightarrow$ eigenvalues are $\lambda = 1, 2, 3$.

finding eigenvectors : (vectors in $\ker(A - \lambda I)$)

$\underline{\lambda = 1}$

$$\begin{pmatrix} 0 & 2 & 0 \\ 0 & 2 & 0 \\ 2 & -4 & 1 \end{pmatrix} v_1 = \underline{0} \rightarrow v_1 = \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix} \text{ is an eigenvector}$$

$\underline{\lambda = 2}$

$$\begin{pmatrix} -1 & 2 & 0 \\ 0 & 1 & 0 \\ 2 & -4 & 0 \end{pmatrix} v_2 = \underline{0} \rightarrow v_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \text{ is an eigenvector}$$

$\underline{\lambda = 3}$

$$\begin{pmatrix} -2 & 2 & 0 \\ 0 & 0 & 0 \\ 2 & -4 & 1 \end{pmatrix} v_3 = \underline{0} \rightarrow v_3 = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} \text{ is an eigenvector}$$

diagonalization

And we find that setting $P = (v_1, v_2, v_3) = \begin{pmatrix} -1 & 0 & 1 \\ 0 & 0 & 1 \\ 2 & 1 & -2 \end{pmatrix}$,

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} = P^{-1} A P.$$

Therefore,

$$\exp(At) = \sum_{n=0}^{\infty} \frac{A^n t^n}{n!} = P \sum_{n=0}^{\infty} \frac{\operatorname{diag}(1,2,3)^n t^n}{n!} P^{-1}$$

$$= P \operatorname{diag}(e^t, e^{2t}, e^{3t}) P^{-1}$$

$$= \begin{pmatrix} e^t & e^{3t} - e^t & 0 \\ 0 & e^{3t} & 0 \\ 2(e^{2t} - e^t) & 2(e^t - e^{3t}) & e^{2t} \end{pmatrix}$$

4.

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 1 & 1 & 0 & 2 \end{pmatrix}$$

finding eigenvalues :

$0 = \det(A - \lambda I) = (1-\lambda)(2-\lambda)^3$.

$\rightarrow$ eigenvalues are $\lambda = 1$ and $\lambda = 2$ with algebraic multiplicity 3.

finding ~~find~~ eigenvectors :

$\lambda = 1$

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix} v_1 = \underline{0} \rightarrow v_1 = \begin{pmatrix} 1 \\ -1 \\ -1 \\ 0 \end{pmatrix}$$ is an eigenvector

$\lambda = 2$

$$\begin{pmatrix} -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix} v_2 = \underline{0} \rightarrow v_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \text{ and } v_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$ are eigenvectors.

however, these span the entire eigenspace $\ker(A - 2I)$.

so (the algebraic multiplicity of $\lambda = 2$) = 3 is greater than (the geometric multiplicity of $\lambda = 2$) = 2.

By the Jordan chain proceedure, we can find a final linearly independent vector v_4 in $\ker(A - 2I)^2$ by setting

$$(A - 2I)v_4 = v_2 \rightarrow v_4 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$ is a generalized eigenvector

diagonalization : Set $P = (v_1, v_2, v_4, v_3)$

↑ v_4 ~~next to~~ after v_2 because v_2 was used to make v_4.

we find that

$$J = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} = P^{-1} A P,$$ which is the Jordan normal form of A.