11.3. **Gradient Systems.** Since Hamiltonian systems arise naturally in modelling potentials, it is of interest also to consider the orthogonal system. Given the Hamiltonian system

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x} = \frac{\partial H}{\partial \mathbf{y}}, \qquad \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{y} = -\frac{\partial H}{\partial \mathbf{x}},$$

the ORTHOGONAL SYSTEM is

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x} = \frac{\partial H}{\partial \mathbf{x}}, \qquad \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{y} = \frac{\partial H}{\partial \mathbf{y}},$$

which we can also write as

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\mathbf{x}\\ \mathbf{y}\end{pmatrix} = \nabla H. \tag{20}$$

Since

$$\nabla H \cdot \Big(\frac{\partial H}{\partial \mathbf{y}}, -\frac{\partial H}{\partial \mathbf{x}}\Big)^{\top} = 0,$$

the trajectories of the orthogonal system are orthogonal to the trajectories of the Hamiltonian system. Where the trajectories of the Hamiltonian system are iso-energetic lines, the orthogonal trajectories run along the direction of the steepest change of H, that is, along ∇H. If H were of a form giving a Newton system, the $\mathbf{x}$ trajectories would be the field lines of the "force" under the energy potential given by H. Systems of the form (20) are known as GRADIENT SYSTEMS.

Since trajectories of the gradient system are always orthogonal to the level sets of H, we can use H (or $-H$) as a Lyapunov function and formulate and show the "dual" theorem to Thm.11.4:

Theorem 11.5. *Let $\mathbf{x}_0$ be a nondegenerate critical point of a planar analytic gradient system with potential H. Then $\mathbf{x}_0$ is a topolgocial saddle for the gradient system if it is a saddle for H and it is a stable or unstable node of the system according as it is a strict local maximum or strict local minimum for H.*

Example 11.1 (Simple pendulum)**.** The simple one-dimensional (i.e., one degree of freedom) pendulum with a massless string under the effect of gravity only can be modelled as

$$\ddot{\vartheta} + \sin(\vartheta) = 0.$$

This system is usually linearised by considering the small angle approximation $\sin(\vartheta) \approx \vartheta$ when ϑ is small.

The full system can be readily analysed however, as it is a Newton system:

$$\dot{\vartheta} = \psi, \qquad \dot{\psi} = \sin(\vartheta).$$

There are fixed points at $\psi = 0$ and $\vartheta \in \pi\mathbb{Z}$.

The potential energy/Hamiltonian in this case is

$$H(\vartheta) = \int_0^{\vartheta} \sin(\eta)\,\mathrm{d}\eta = 1 - \cos(\vartheta).$$

As can be computed with the Hessian of H, via Thm. 11.4, the fixed points at $2\pi\mathbb{Z}$ are centres and the fixed points at $(2n+1)\pi\mathbb{Z}$ are saddles.

12. Lecture XII: Critical Points of Planar Systems I

Let us continue our discussion on critical points of planar systems in greater generality than we have done in the previous lecture. In this lecture we are still interested in the four types of nondegenerate behaviours that are manifested in planar linear systems — namely, centres, foci, nodes, and saddles, and perturbations of them by nonlinearities. Recall that we have also defined the centre-focus which does not occur in linear or even analytic systems. We shall see in the next lecture that nonlinear systems can manifest behaviours vastly different from these five.

With reference to the theorems that shall follow, by an ISOLATED CRITICAL POINT, we mean a critical point $\mathbf{x}_0$ for which there exists a $\delta > 0$ such that $B_\delta(\mathbf{x}_0)$ does not contain any other critical points. Before we begin let us also review and expand the list of fixed points of nonlinear systems we considered last time:

(i) A critical point $\mathbf{x}_0 \in \mathbb{R}^2$ for an autonomous system is a NODE if there exists a $\delta > 0$ such that for $\mathbf{y}_0$ with $0 < |\mathbf{x}_0 - \mathbf{y}_0| < \delta$, $|\phi_t(\mathbf{y}_0) - \mathbf{x}_0| \to 0$ and $\arg(\phi_t(\mathbf{y}_0) - \mathbf{x}_0)$ tends to a finite limit as $t \to \infty$ (STABLE) or as $t \to -\infty$ (UNSTABLE). It is a PROPER NODE if each ray from $\mathbf{x}_0$ is tangent to some trajectory.

(ii) A critical point $\mathbf{x}_0 \in \mathbb{R}^2$ for an autonomous system is a FOCUS (or SPIRAL) if there exists a $\delta > 0$ such that for $\mathbf{y}_0$ with $0 < |\mathbf{x}_0 - \mathbf{y}_0| < \delta$, $|\phi_t(\mathbf{y}_0) - \mathbf{x}_0| \to 0$ and $|\arg(\phi_t(\mathbf{y}_0) - \mathbf{x}_0)| \to \infty$ as $t \to \infty$ (STABLE) or as $t \to -\infty$ (UNSTABLE).

(iii) A critical point $\mathbf{x}_0 \in \mathbb{R}^2$ for an autonomous system is a CENTRE if there exists a $\delta > 0$ such that every trajectory in $B_\delta(\mathbf{x}_0)\backslash\{\mathbf{x}_0\}$ is a closed curve.

(iv) A critical point $\mathbf{x}_0 \in \mathbb{R}^2$ for an autnomous system is a CENTRE-FOCUS if there exists an sequence of closed curves $\{\Gamma_n\}$ and a sequence of number $\delta_n \to 0$ such that Γ_{n+1} is in the open set enclosed by Γ_n and $\Gamma_n \subseteq B_{\delta_n}(\mathbf{x}_0)$, and every trajectory between Γ_n and Γ_{n+1} tends to one closed curve or the other as $t \to \pm\infty$. These closed curves Γ_n are known as LIMIT CYCLES.

(v) A critical point $\mathbf{x}_0 \in \mathbb{R}^2$ for an autonomous system is a TOPOLOGICAL SADDLE if there exist two trajectories which approach $\mathbf{x}_0$ as $t \to \infty$, and two trajectories that approach $\mathbf{x}_0$ as $t \to -\infty$, and if there exists a $\delta > 0$ such that all other trajectories in $B_\delta(\mathbf{x}_0)\backslash\{\mathbf{x}_0\}$ leave $B_\delta(\mathbf{x}_0)$ as $t \to \pm\infty$. We call the four special trajectories SEPARATRICES.

First, we present a general theorem due to Bendixon, whose work shall be the topic of further lectures in this module:

Theorem 12.1 (Bendixson). *Let $(x_0, y_0) \in \mathbb{R}^2$ be an isolated critical point of a C^1-first order autonomous system, then either (i) every neighbourhood of the critical point contains a closed trajectory, or (ii) there exists a trajectory that tends to (x_0, y_0) as $t \to \pm\infty$.*

This may seem a rather vacuous theorem, and indeed it can be verified quite readily if one attempts to draw trajectories near an isolated fixed point. We shall not spend time proving it.

Recall that from the Hartman-Grobman theorem, around a hyperbolic critical point $\mathbf{x}_0 \in \mathbb{R}^d$ of a C^1-autonomous system, we can always find neighbourhoods U and V and a homeomorphism $\psi : U \to V$ such that the flow was homeomorphic to the linear flow:

$$\psi \circ \phi_t = \exp(\mathrm{D}f(\mathbf{x}_0))\psi.$$

It turns out that this simply continuous homeomorphism is capable of turning a node into a focus. But if ψ is a C^1-diffeomorphism, this is not possible. Happily, as we also mentioned in remarks following the Hartman-Grobman Theorem, Hartman also showed that if the system is a C^2-first order autonomous system, a homeomorphism that is additionally a C^1-diffeomorphism exists. To see what is happening, consider the topologist's sine curve, $y = \sin(1/x)$, which is continuous but has unbounded derivative closer and closer to the origin.

From this we can conclude that

Theorem 12.2. *Let $(x_0, y_0) \in \mathbb{R}^2$ be a hyperbolic fixed point of a C^2-first order autonomous system. Then (x_0, y_0) is a (topological) saddle, a stable or unstable focus, or a stable or unstable node, according as (x_0, y_0) is a saddle, a stable or unstable focus, or a stable or unstable node of the linearized system, and conversely.*

The correspondence between the full system and the linearized system in the saddle case can be had for C^1-first order autonomous systems as well. Let us look at one example where the system is only C^1 and the conclusion of the theorem above does not hold.

Example 12.1. Consider the system with f defined by the following for $(x, y) \neq (0, 0)$:

$$\dot{x} = -x - \frac{y}{\log(\sqrt{x^2 + y^2})},$$
$$\dot{y} = -y + \frac{x}{\log(\sqrt{x^2 + y^2})}.$$

We complete this definition by setting $f((0, 0)) = (0, 0)$, so that the origin is a fixed point by fiat. In polar coordinates, the system becomes

$$\dot{r} = -r, \qquad \dot{\vartheta} = 1/\log(r).$$

Recall that a change of coordinates is really a local homeomorphism.

This nonlinear system can be readily integrated:

$$r(t) = r_0 e^{-t}, \qquad \vartheta(t) = -\log\left(1 - t/\log(r_0)\right) + \vartheta_0.$$

This shows that starting in a small enough neighbourhood of the origin, i.e., $r_0 < 1$ we have $|r| \to 0$ and $|\vartheta| \to \infty$ as $t \to \infty$, and the origin is a stable node.

The linearized system is determined by

$$\mathrm{D}f = \begin{pmatrix} -1 & -(\log(\sqrt{x^2+y^2}))^{-1} \\ (\log(\sqrt{x^2+y^2}))^{-1} & -1 \end{pmatrix} + (\log(\sqrt{x^2+y^2}))^{-2}(x^2+y^2)^{-1}\begin{pmatrix} xy & y^2 \\ -x^2 & -xy \end{pmatrix},$$

and

$$\mathrm{D}f((0,0)) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix},$$

which yields a stable node.

Changing to polar coordinates in planar systems may be the main technique introduced in this lecture.

The aim of this lecture is to show the following theorem which underpinned our deductions last time:

Theorem 12.3. *Let $U \subseteq \mathbb{R}^2$ be a neighbourhood of a critical point $\mathbf{x}_0$ of an autonomous system $\dot{\mathbf{x}} = f(\mathbf{x})$, $f \in C^1(U)$. Suppose $\mathbf{x}_0$ is a centre for the linearized dynamics. Then $\mathbf{x}_0$ is wither a centre, a centre-focus, or a focus for the original autonomous system.*

Proof. By a translation if necessary, let the critical point $\mathbf{x}_0$ be the origin. If the origin is a centre, we know that by a change of variables, we can write $\mathrm{D}f$ as

$$\mathrm{D}f = \begin{pmatrix} 0 & -b \\ b & 0 \end{pmatrix},$$

which has eigenvalues $\lambda_\pm = \pm\imath b$.

The full planar system is then

$$\dot{x} = -bx + p(x, y)$$
$$\dot{y} = by + q(x, y).$$

Since $(0, 0)$ is a fixed point, $p(0, 0) = q(0, 0) = 0$.

Recall our decomposition of a general C^1 system in our discussion of the Stable Manifold Theorem. For every $\delta > 0$, there is an $\varepsilon > 0$ such that if $|(x, y)^\top| < \delta$,

$$|(p(x, y), q(x, y))^\top| < \varepsilon |(x, y)^\top|.$$

This means that $p = o(r)$ and $q = o(r)$.

Now

$$\begin{aligned} 2r\dot{r} &= \frac{\mathrm{d}}{\mathrm{d}t} r^2 = 2x\dot{x} + 2y\dot{y} = -2xyb + 2xyb + 2xp + 2yq = o(r^2), \\ \dot{\vartheta} &= \frac{\mathrm{d}}{\mathrm{d}t} \arctan(y/x) = \frac{1}{1 + (y/x)^2} \left(\frac{1}{x}\dot{y} - \frac{y}{x^2}\dot{x} \right) = b + o(1), \end{aligned}$$

as $r \to 0$, and we have

$$\dot{r} = o(r), \qquad \dot{\vartheta} = b + o(1).$$

This means that for sufficiently small r_0, say $r_0 < \delta$, we find

$$\dot{\vartheta} \geq b/2, \qquad \vartheta \geq bt/2 + \theta_0,$$

as $t \to \infty$, and ϑ is a monotonically increasing function of t. This means ϑ is invertible for small enough r, and so it makes sense to write the radius $\tilde{r} = r \circ \vartheta^{-1}$ as a function of the angle/argument.

Suppose the origin is neither a centre nor a centre focus. Then for $\delta > 0$ small enough, there are no closed trajectories in $B_\delta(\mathbf{0}) \backslash \{\mathbf{0}\}$. Without loss of generality (otherwise use time reversal $t \mapsto -t$) we can take

$$\tilde{r}(\vartheta_0 + 2\pi) < \tilde{r}(\vartheta_0).$$

This argument can be iterated and in order to avoid trajectories that cross (which cannot happen in autonomous systems), we have

$$\tilde{r}(\vartheta_0 + 2k\pi) < \tilde{r}(\vartheta_0 + 2(k-1)\pi)$$

for every $k \in \mathbb{N}$.

Therefore the sequence of numbers in k is monotonically decreasing and lower bounded by 0, which means there is a number ϱ such that

$$\varrho = \lim_{k \to \infty} \tilde{r}(\vartheta_0 + 2k\pi).$$

Now we are going to consider the convergence of the functions $r_k(\vartheta) = \tilde{r}(\vartheta_0 + 2k\pi + \vartheta)$ on $[0, 2\pi]$. We shall show that it is uniformly bounded and equicontinuous, whereby the Arzela-Ascoli Theorem will allow us to conclude that $r_k(\vartheta)$ converges to a continuous function $R(\vartheta)$ on $[0, 2\pi]$ in the continuous/uniform norm that we have seen before. Since $R(\vartheta)$ will then be arbitrarily cloes to $R(\vartheta + 2\pi)$ for any $\vartheta \in [0, 2\pi]$, R will be non-zero periodic function, and a closed trajectory, a contradiction.

We already know that the functions r_k are uniformly bounded :

$$\varrho \leq r_k(\vartheta) \leq \sup_{\theta \in [0, 2\pi]} \tilde{r}(\vartheta_0 + \theta).$$

We also know that

$$\begin{aligned} \frac{\mathrm{d}\tilde{r}}{\mathrm{d}\vartheta} = \frac{\dot{\tilde{r}}}{\dot{\vartheta}} &= \frac{(x\dot{x} + y\dot{y})/\tilde{r}}{(x\dot{y} - y\dot{x})/\tilde{r}^2} \\ &= \frac{p(x, y)\cos(\vartheta) + q(x, y)\sin(\vartheta)}{(\cos(\vartheta)q(x, y) - \sin(\vartheta)p(x, y))/\tilde{r}} \\ &\leq \frac{M}{b/2}. \end{aligned}$$

This shows that r_k are equicontinuous. And thus is our theorem proven.

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