TMA 4190 Introduction to Topology
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Lecture 17[1]

17. Manifolds with Boundary

In order to be able to analyze a wider class of phenomena we would like to enlarge the class of manifolds. A typical example which we would like to include is the domain of a homotopy $X \times [0,1]$ for a smooth k-dimensional manifold X. The points on $X \times \{0\}$ and $X \times \{1\}$ do not have an open neighborhhod which is diffeomorphic to $\mathbb{R}^k$. In fact, those subsets for the boundary of $X \times [0,1]$. Another example is the **closed** unit ball in $\mathbb{R}^k$. So far such guys do not qualify as a manifold. From now on, We would like to allow such subsets. We will see that most of the theorems we have proved so far are also valid for manifolds with boundary.

The idea for what a manifold with boundary should be is the same as before: it is a space which locally looks like some model space with boundary which we understand well. Hence we need to choose a good model space. But that is not hard to do.

In fact, the standard model of a Euclidean space with boundary is the half-plane

$$\mathbb{H}^k = \{(x_1, \dots, x_k) \in \mathbb{R}^k : x_k \geq 0\}$$

in $\mathbb{R}^k$. The **boundary of $\mathbb{H}^k$**, denoted $\partial\mathbb{H}^k$, is given by the points

$$\partial\mathbb{H}^k = \{(x_1, \dots, x_k) \in \mathbb{R}^k : x_k = 0\} = \mathbb{R}^{k-1} \times \{0\} \subset \mathbb{R}^k.$$

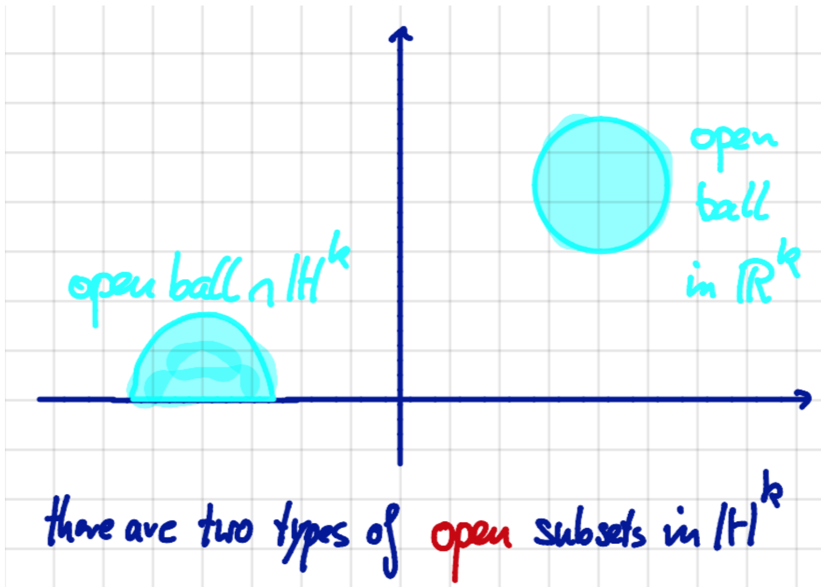
Now a manifold with boundary is a space which locally looks like $\mathbb{H}^k$:

Manifolds with boundary

A subset X of $\mathbb{R}^N$ is a **k-dimensional manifold with boundary** if every point x of X there is an open neighborhood $V \subset X$ containing x and an open neighborhood $U \subset \mathbb{H}^k$ together with a diffeomorphism $\phi \colon U \to V$. As before, any such a diffeomorphism is called a **local parametrization of X**.

[1] Following the books of Guillemin and Pollack: Differential Topology, and Milnor: Topology from the differentiable viewpoint.

The **boundary of X**, denoted ∂X, consists of those points that belong to the image of the boundary of $\mathbb{H}^k$ under some local parametrization. Its complement is called the **interior of X**, denoted $\mathrm{Int}(X) = X \setminus \partial X$.
A manifold X with $\partial X = \emptyset$ is just a smooth manifold in our initial terminology. In order to make the distinction clear if necessary, we call them also boundaryless manifolds or manifolds without a boundary.



Warning: The interior of $X \subset \mathbb{R}^N$ as a manifold is in general different from the interior of X as a subspace of $\mathbb{R}^N$. The interior of X as a manifold is the complement of the boundary, whereas the interior of the topologocal space X is the union of all its open subsets. But also every point in ∂X lies in some open neighborhood of X.

Let X be a manifold with boundary. We need to check that our definition of points in the interior and on the boudnary is independent of the choice of a local parametrization.

So let $x \in X$ be a point which is in the image of a local parametrization $\phi\colon U \to V \subset X$ such that $U \subset \mathbb{H}^k$ is an open set of $\mathbb{H}^k$ which is contained in the interior of $\mathbb{H}^k$. Then $\mathbb{R}^k$ is an open subset of $\mathbb{R}^k$. Now assume x is also in the image of another local parametrization $\phi'\colon U' \to V' \subset X$. Then $x \in W := V \cap V' \subset X$, and the composition $\phi' \circ \phi^{-1}\colon \phi^{-1}(W) \to (\phi')^{-1}(W)$ is a diffeomorphism. Hence, after possibly shrinking U', we see that U' is also an open subset in $\mathbb{R}^k$. Thus x is being an interior point is well-defined.

This shows in particular: if X is a manifold with boundary, then the interior of X, $\mathrm{Int}(X)$, is a boundaryless manifold of the same dimension as X.

It remains to show that being a boundary point is also well-defined. We show this by proving the following interest result:

Boundaries are manifolds

If X is a k-dimensional manifold with boundary, then ∂X is a $(k-1)$-dimensional manifold without boundary.

Proof: Let $x \in X$ and let ϕ and ψ be two local parametrizations around x. After possibly shrinking the domains and codomains, we can assume that $\phi\colon U \to V$ and $\psi\colon W \to V$ are both diffeomorphisms from open sets $U \subset \mathbb{H}^k$, $W \subset \mathbb{H}^k$ to the same open subset $V \subset X$.

We would like to show $\phi(\partial U) = \psi(\partial W)$. For then $\partial V = \phi(\partial U)$ is independent of our choice of local parametrization and therefore well-defined. Moreover, since $\partial U = U \cap \partial \mathbb{H}^k$ is an open subset of $\mathbb{R}^{k-1}$, we would get that every point $y \in \partial X$ is contained in a local parametrization $\phi_{|\partial U}\colon U \cap \partial \mathbb{H}^k \to \partial X$. This will show that ∂X is a manifold of dimension $k-1$.

By our assumption on ϕ and ψ, it suffices to show $\psi(\partial W) \subset \phi(\partial U)$. The other inclusion will follow by symmetry. Hence we woud like to show:

Claim: $\phi^{-1}(\psi(\partial W)) \subset \partial U$.

To simplify notation, we define the map $g = \phi^{-1} \circ \psi\colon W \to U$.

Suppose that the claim is false and there is a point $w \in \partial W$ which is mapped to an **interior point** $u = g(w)$ of U by g. Since both ϕ and ψ are diffeomorphisms, **g is a diffeomorphism** of W onto an open subset $g(W)$ of U. The chain rule implies that the derivative $d(g^{-1})_u$ of its inverse is bijective. But, since $u \in \mathrm{Int}(U)$, $g(W)$ contains a neighborhood of u that is **open in $\mathbb{R}^k$**. Thus the **Inverse Function Theorem**, applied to the map g^{-1} defined on this open subset of $\mathbb{R}^k$, implies that the image of g^{-1} contains a neighborhood of w that is **open in $\mathbb{R}^k$**. This contradicts the assumption $w \in \partial W$. **QED**

Tangent spaces and derivatives are still defined in the setting of manifolds with boundary.

Derivatives and tangent spaces vs boundaries

Derivatives of smooth maps can be defined as before. Since smoothness at a point requires a functions to be defined on open neighborhhod around that point, we need to be a bit more careful at boundary points:

Derivatives on $\mathbb{H}^k$:

Suppose that g is a smooth map of an open set U of $\mathbb{H}^k$ to R^l. If u is an interior point of U, then the derivative dg_u is defined as before.

If $u \in \partial U$ is a boundary poin, the smoothness of g means that it may be extended to a smooth map G defined in an open neighborhood of u in $\mathbb{R}^K$. We define dg_u to be the derivative $dG_u \colon \mathbb{R}^k \to \mathbb{R}^l$.

We must show that this definition is independent of the choice of G. So let G' be another local extension of g. We need to show $dG'_u = dG_u$.

The equality of the two derivatives is no problem at points in the interior $\mathrm{int}(U)$ of U, because then we have a small open neighborhood which is still in $\mathrm{int}(U)$. We are going to use this and approximate u be a sequence $\{u_i\}$ of interior points $u_i \in \mathrm{int}(U)$ which converge to u.

Since G and G' agree with g on $\mathrm{int}(U)$, we have

$$dG_{u_i} = dG'_{u_i} \text{ for all } i.$$

Since the derivative of a smooth map at a point depends continuously on the change of point, wthis implies that $dG_{u_i} \to dG_u$ and $dG'_{u_i} \to dG'_u$ when $u_i \to u$ and **both limits agree**. This shows that dg_u is also well-defined at boundary points.

One should note that, at all points, dg_u is still a linear map of all of $\mathbb{R}^k$ to $\mathbb{R}^l$. For we have defined dg_u as the derivative dG_u of an extension G to an **open subset of $\mathbb{R}^k$**.

Tangent spaces:

Let $X \subset \mathbb{R}^N$ be a smooth manifold with boundary, and $x\, inX$. Let $\phi \colon U \to X$ be a local parametrization with $U \subset \mathbb{H}^k$ open. Let $u \in U$ be the point with $\phi(u) = x$. Then we have just learned that we can form the derivative

$$d\phi_u \colon \mathbb{R}^k \to \mathbb{R}^N$$

no matter what kind of point x is. Thus, as before, we can define the **tangent space** to X at x, denoted $T_x(X)$, to be the image of $\mathbb{R}^k$ in $\mathbb{R}^N$ under the linear map $d\phi_u$. (One can check that $T_x(X)$ does not depend as a subspace of $\mathbb{R}^N$ on the choice of ϕ just as before using the chain rule.)

Derivatives on tangent spaces:

Now let $f \colon X \to Y$ be a smooth map between manifolds with boundaries with $X \subset \mathbb{R}^N$ and $Y \subset \mathbb{R}^M$. Given a point $x \in X$. Then after choosing local parametrizations $\phi \colon U \to X$ with $\phi(u) = x$ and $\psi \colon V \to Y$ with

$\psi(v) = f(x)$, then we **define**

$$df_x\colon T_x(X) \to T_{f(x)}(Y)$$

as the **linear map** which makes the following diagram commutative

$$\begin{tikzcd} T_x(X) \arrow[rr, dotted, "df_x"] & & T_y(Y) \\ \mathbb{R}^k \arrow[u, "d\phi_u"] \arrow[rr, "d\theta_u"'] & & \mathbb{R}^l. \arrow[u, "d\psi_v"'] \end{tikzcd}$$

where θ is the map $\psi^{-1} \circ f \circ \phi$ (note $v = \theta(u)$).

However, sometimes we do have to be careful when we apply our developed concepts to manifolds with boundaries. For example, the product of two manifolds with boundary may not be a manifold anymore. A simple example is the product $[0,1] \times [0,1]$.

But if only one manifold has a boundary we are ok:

Products and Boundaries

The product of a manifold without boundary X and a manifold with boundary Y is another manifold with boundary. Furthermore,

$$\partial(X \times Y) = X \times \partial Y,$$

and

$$\dim(X \times Y) = \dim X + \dim Y.$$

Proof: If $U \subset \mathbb{R}^k$ and $V \subset \mathbb{H}^l$ are open, then

$$U \times V \subset \mathbb{R}^k \times \mathbb{H}^l = \mathbb{H}^{k+l}$$

is open. Moreover, if $\phi\colon U \to X$ and $\psi\colon V \to Y$ are local parametrizations, so is $\phi \times \psi\colon U \times V \to X \times Y$. **QED**

Regular values and transversality

One of the most important concepts we have studied is transversality of smooth maps to submaifolds. We would like to extend this to manifolds with boundary. This is possible, but requires some care.

We start with the special case of regular values for functions on manifolds without boundary. This is a well-known case, but it turns out that it actually produces manifolds with boundary as follows:

Regular values for real-valued functions

Suppose that S is a manifold without boundary and that $\pi\colon S \to \mathbb{R}$ is a smooth function with regular value 0. Then the subset $\{s \in S : \pi(s) \geq 0\}$ is a manifold with boundary, and the boundary is $\pi^{-1}(0)$.

Proof: The set $\{x \in S : \pi(x) > 0\}$ is open in S, since it is the continuous preimage of the open subset $(0,\infty) \subset \mathbb{R}$. It is therefore a submanifold of the same dimension as S. Hence every point in $\{x \in S : \pi(x) > 0\}$ has an open neighborhood which is diffeomorphic to an open subset of $\mathbb{R}^k$, $k = \dim S$.

So suppose that $\pi(s) = 0$. By assumption, 0 is a regular value which means that s is a regular point of π. Hence π is locally near s equivalent to the canonical submersion. But for the canonical submersion

$$\pi\colon \mathbb{H}^k \to \mathbb{R},\ (x_1, \ldots, x_k) \to x_k$$

the lemma just states the definition of the boundary of $\mathbb{H}^k$:

$$\partial\mathbb{H}^k = \pi^{-1}(0) = \{(x_1, \ldots, x_k) \in \mathbb{R}^k : x_k = 0\}.$$

QED

An immediate consequence of this fact is:

Spheres are boundaries

Let π be the smooth function defined by

$$\pi\colon \mathbb{R}^k \to \mathbb{R},\ (x_1, \ldots, x_k) \mapsto 1 - \sum_i x_i^2.$$

Then 0 is a regular value of π, and the unit ball B^k in $\mathbb{R}^k$ can be described as

$$B^k = \{x \in \mathbb{R}^k : \pi(x) \geq 0\}.$$

The boundary of B^k is the $(k-1)$-sphere $S^{k-1} = \pi^{-1}(0)$.

Recall that transversality is formulated as a criterion on tangent spaces and derivatives. We would like to formulate a similar criterion for maps between manifolds with boundary.

As we learned above, the boundary ∂X of a k-manifold with boundary X is a manifold of dimension $k-1$ without boundary. Let $x \in \partial X$ be a point on the boundary. We have $\dim T_x(\partial X) = k-1$ and $\dim T_x(X) = k$. Moreover, since ∂X is a submanifold of X, we know that

$$T_x(\partial X) \subset T_x(X)$$

is a **vector subspace** of codimension 1 in $T_x(X)$.

For any smooth map $f\colon X \to Y$, we introduce the notation

$$\partial f = f_{|\partial X}$$

for the restriction of f to ∂X. The derivative of ∂f at x is just the restriction of df_x to the subspace $T_x(\partial X)$:

$$d(\partial f)_x = (df_x)_{|T_x(\partial X)}\colon T_x(\partial X) \to T_{f(x)}(Y).$$

Now let $f\colon X \to Y$ be a smooth map from a smooth manifold with boundary X to a boundaryless manifold Y, and let $Z \subset Y$ be a submanifold. We would like to know under which circumstances is $f^{-1}(Z)$ **a submanifold with boundary** of X (i.e. a subset of X which is itself a smooth manifold with boundary) with

$$\partial f^{-1}(Z) = f^{-1}(Z) \cap \partial X. \tag{1}$$

It turns out that it is **not enough** to ask that f is transversal to Z in the previous sense, i.e. $\mathrm{Im}\,(df_x) + T_{f(x)}(Z) = T_{f(x)}(Y)$.

A simple example

Even for the restriction of the canonical submersion

$$\pi\colon \mathbb{H}^2 \to \mathbb{R},\ (x_1,x_2) \mapsto x_2$$

this is not sufficient. For, $d\pi_{(x_1,x_2)}\colon \mathbb{R}^2 \to \mathbb{R}$ is just the projection onto the second factor. Hence it is surjective at every point (x_1,x_2). In particular, 0 is a regular value fo π. Let $Z := \{0\}$. Then

$$\pi^{-1}(Z) = \{(x_1,x_2) \in \mathbb{R}^2 : x_2 = 0\} = \partial\mathbb{H}^2.$$

Since 0 is regular value, we knew that $\pi^{-1}(Z)$ is a submanifold. The problem is that the boundary does not satisfy condition (1). For

$$\partial\pi^{-1}(Z) = \emptyset, \text{ whereas } \pi^{-1}(Z) \cap \partial X = \partial\mathbb{H}^2 \neq \emptyset.$$

In order to make sure that the boundary behaves well, we need to impose an additional transversality condition on ∂f.

We start again with regular values:

Preimages of regular values in manifolds with boundary

Let g be a smooth map of a k-manifold X with boundary onto a boundaryless n-manifold Y, and suppose that $y \in Y$ is a regular value for **both** $g\colon X \to Y$ and $\partial g\colon \partial X \to Y$. Then the preimage $g^{-1}(y)$ is a $(k-n)$-dimensional manifold with boundary

$$\partial(g^{-1}(y)) = g^{-1}(y) \cap \partial X.$$

Proof: To show that $g^{-1}(y)$ is a manifold with boundary is a local question, i.e. it suffices that each point in $g^{-1}(y)$ has an open neighborhood which is a manifold with boundary. So let $x \in X$ be a point with $g(x) = y$. After choosing local coordinates, we can assume that g is a map

$$g\colon \mathbb{H}^k \to \mathbb{R}^n.$$

If x is an interior point in X, then $g^{-1}(y)$ is a manifold without boundary in an open neighborhood around x by the Preimage Theorem for boundaryless manifolds.

So let us look at what happens if $x \in \partial X$. That g is smooth at x means by definition that there is an open subset $U \subset \mathbb{R}^k$ and a smooth map

$$G\colon U \to \mathbb{R}^n \text{ such that } G_{U\cap\mathbb{H}^k} = g_{U\cap\mathbb{H}^k}.$$

After possibly replacing U with a smaller subset, we can assume that **G has no critical points in U**. Then $G^{-1}(y)$ is a smooth manifold by the Preimage Theorem for boundaryless manifolds. We need to show that

$$g^{-1}(y) = G^{-1}(y) \cap \mathbb{H}^K \text{ is a manifold with boundary.}$$

In order to show this, we define a new smooth function π on the manifold $S := G^{-1}(y)$

$$\pi\colon S \to \mathbb{R},\ (x_1, \ldots, x_k) \mapsto x_k$$

as the projection to the last coordinate. Then

$$S \cap \mathbb{H}^k = \{s \in S : \pi(s) \geq 0\}.$$

Claim: 0 is a regular value of π.

If we can show the claim, then our previous lemma shows that $S \cap \mathbb{H}^k$ is a manifold with boundary and the boundary is $\pi^{-1}(0)$.

To show the claim, assume there was an $s \in S$ with both $\pi(s) = 0$, i.e. $s \in S \cap \partial\mathbb{H}^k$, and $d\pi_s = 0$. We want to show that the assumption $d\pi_s = 0$ leads to a contradiction.

To do so, first note that π is a **linear** map, and therefore $d\pi_s = \pi$. Thus,

$$d\pi_s = \pi \colon T_s(S) \to \mathbb{R}$$

being trivial, just means that the last coordinate of every vector in $T_s(X)$ is 0, i.e.

$$d\pi_s = 0 \Rightarrow T_s(S) \subset T_s(\partial\mathbb{H}^k) = \mathbb{R}^{k-1}.$$

Hence we want to show $T_s(S) \not\subset \mathbb{R}^{k-1}$.

The tangent space to $S = G^{-1}(y)$ at s is the kernel of dG_s:

$$T_s(S) = T_s(G^{-1}(y)) = \mathrm{Ker}\,(dG_s = dg_s \colon \mathbb{R}^k \to \mathbb{R}^n)$$

where $dg_s = dG_s$ by definition of dg_s.

We know that $d(\partial g)_s$ is the restriction of $dg_s \colon \mathbb{R}^k \to \mathbb{R}$ to $\mathbb{R}^{k-1}$:

$$d(\partial g)_s = (dg_s)_{|\mathbb{R}^{k-1}}.$$

Thus, **if** $T_s(S) = \mathrm{Ker}\,(dg_s) \subseteq \mathbb{R}^{k-1}$, then

$$\mathrm{Ker}\,(dg_s) = \mathrm{Ker}\,(d(\partial g)_s). \tag{2}$$

Now, finally, we apply the assumption of regularity of y. Since **y is a regular value of both g and ∂g**, we know that **both dg_s and $d(\partial g)_s$ are surjective**. This implies

$$\dim \mathrm{Ker}\,(dg_s) = k - n \text{ and } \dim \mathrm{Ker}\,(d(\partial g)_s) = k - 1 - n.$$

This **contradicts** assertion (2) about the kernels when $\mathrm{Ker}\,(dg_s) \subset \mathbb{R}^{k-1}$. Thus this assumption must be false, i.e.

$$T_s(S) = \mathrm{Ker}\,(dg_s) \not\subseteq \mathbb{R}^{k-1}$$

and hence $d\pi_s \neq 0$ and therefore $d\pi_s$ is surjective.

In other words, 0 is a regular value. **QED**

Preimages of manifolds with boundary

Let f be a smooth map of a manifold X with boundary onto a boundaryless manifold Y, and suppose that **both** $f\colon X \to Y$ and $\partial f\colon \partial X \to Y$ are **transversal** with respect to a boundaryless submanifold Z in Y. Then the preimage $f^{-1}(Z)$ is a manifold with boundary

$$\partial(f^{-1}(Z)) = f^{-1}(Z) \cap \partial X,$$

and the codimension of $f^{-1}(Z)$ in X equals the codimension of Z in Y.

Proof: The restriction of f to the boundaryless manifold $\mathrm{Int}(X)$ is transversal to Z. Hence, by the Preimage Theorem for boundaryless manifolds, $f^{-1}(Z) \cap \mathrm{Int}(X)$ **is a boundaryless manifold** of correct codimension. Thus it remains to examine $f^{-1}(Z)$ in a neighborhood of a point $x \in f^{-1}(Z) \cap \partial X$.

Let $l := \operatorname{codim} Z$ in Y. As in the boundaryless case, we can choose a submersion $h\colon W \to \mathbb{R}^l$ defined on an open neighborhood W of $f(x)$ in Y to $\mathbb{R}^l$ such that $Z \cap W = h^{-1}(0)$. Then $h \circ f$ is defined in a neighborhood V of x in X, and $f^{-1}(Z) \cap V = (h \circ f)^{-1}(0)$.

Now let $\phi\colon U \to X$ be a local parametrization around x, where U is an open subset of $\mathbb{H}^k$. Then define

$$g := h \circ f \circ \phi\colon U \to \mathbb{R}^l.$$

Since $\phi\colon V \to \phi(V)$ is a diffeomorphism, the set

$$\begin{aligned} & f^{-1}(Z) \text{ is a manifold with boundary in a neighhorhood of } x \\ \iff\ & (f \circ \phi)^{-1}(Z) = g^{-1}(0) \text{ is a manifold with boundary near } u = \phi^{-1}(x) \in \partial U. \end{aligned}$$

But the transversality assumptions of f and ∂f with respect to Z imply the 0 is a regular value of g. Hence we can apply the previous theorem and we are done. **QED**

Finally, also Sard's Theorem has a version with boundary.

Sard's Theorem with boundary

For any smooth map $f\colon X \to Y$ of a manifold X with boundary to a boundaryless manifold Y, almost every point of Y is a regular value of both f and ∂f.

Proof: For any point $x \in \partial X$ on the boundary of X,

$$d(\partial f)_x = (df_x)_{|T_x(\partial X)} \colon T_x(\partial X) \to T_{f(x)}(Y).$$

Hence if $d(\partial f)_x$ is surjective, then df_x is surjective. Hence if ∂f is regular at x, so is f.

Thus a point $y \in Y$ fails to be a regular value of both f and ∂f only when it is a critical value if both df_x fails to be surjective for all $x \in f^{-1}(y) \cap \mathrm{Int}(X)$ and $d(\partial f)_x$ fails to be surjective for all $x \in f^{-1}(y) \cap \partial X$.

But since $\mathrm{Int}(X)$ and ∂X are both boundaryless manifolds, both sets of critical values have measure zero by Sard's Theorem. Thus the complement of the set of common regular values for f and ∂f is the union of two sets of measure zero, and therefore itself a set of measure zero. **QED**