3.3. **A class of nonautonomous linear systems.** Finally we take a look at a simple class of *non*-autonomous systems, which is the autnomous linear system, with an additional nonhomogeneity:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{g}(t). \tag{11}$$

From a first course in differential equations, it can be guessed that the solution is the matrix reformulation of Duhamel's formula, by obtained by convolving a fundamental solution against the inhomogeneity. This is in fact the case.

Define a FUNDAMENTAL MATRIX SOLUTION to

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x}(t) = \mathbf{A}\mathbf{x}(t)$$

as any continuously differentiable matrix-valued function $\Phi : t \to \mathbb{R}^{d\times d}$ satisfying

$$\Phi'(t) = \mathbf{A}\Phi(t) \qquad \forall t \in \mathbb{R}.$$

In particular, $\Phi(t) = \exp(\mathbf{A}t)$ is a fundamental matrix solution of the first-order system with $\Phi(0) = \mathbf{I}_d$. We have the theorem:

Theorem 3.4. *Let Φ be any fundamental matrix solution to*

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x}(t) = \mathbf{A}\mathbf{x}(t).$$

The solution of (11) *with intitial condition* $\mathbf{x}(0) = \mathbf{b}$ *can be written as*

$$\mathbf{x}(t) = \Phi(t)\Phi^{-1}(0)\mathbf{b} + \int_0^t \Phi(t)\Phi^{-1}(s)\mathbf{g}(s)\,\mathrm{d}s,$$

and is unique.

This result can be verified readily by differentiation.

This also means that taking $\Phi(t) = \exp(\mathbf{A}t)$, we have

$$\mathbf{x}(t) = \exp(\mathbf{A}t)\mathbf{b} + \int_0^t \exp(\mathbf{A}(t-s))\mathbf{g}(s)\,\mathrm{d}s.$$

It bears mentioning that, in fact, we can always turn a non-autonmous system into an autonmous one by increasing our dimension. Suppose we have some non-autonomous system

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x}(t) = f(t, \mathbf{x}(t)).$$

Now introduce a new variable $\mathbf{y}(t) = (\tau(t), \mathbf{x}^\top(t))^\top$, where $\tau(t) = t$. Then the system becomes

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\tau &= 1, \\ \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x} &= f(\mathbf{y}(t)), \end{aligned}$$

which is obviously autonomous. However, even if f were linear in $\mathbf{x}(t)$, it is usually very far from being linear in $\mathbf{y}(t)$.

4. Lecture IV: Local Well-Posedness I

The study of differential equations is usually concerned with four main problems. Given a differential equation, we should like to know:

(i) Existence: given an appropriate set of initial/boundary conditions, do solutions exist (in a certain function space, in an appropriate sense)?
(ii) Uniqueness: given an appropriate set of initial/boundary conditions, are solutions unique if they exist (in an appropriate sense)?
(iii) Continuous dependence: if initial/boundary/other conditions are perturbed slightly (in a given topology), does the corresponding solution also only change slightly (in some topology)? (Continuous dependence implies uniqueness in appropriate spaces.)
(iv) Asymptotic/long-time behaviour: concerning evolution equations — how do solutions develope eventually ?

Along the way, we are interested also in dependence on other parameters, transient behaviour of solutions, etc.. Affirmative answers to the first three questions characterize a phenomenon known as WELL-POSEDNESS. We say the equation with its accompanying conditions constitute then a "well-posed problem".

In previous lectures, we have answered all of the questions above as they concern linear autonomous systems of two first-order equations by constructing solutions explicitly. Direct construction is not generally possible. We should like to provide some general conditions for well-posedness of first-order systems of the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x}(t) = f(t, \mathbf{x}(t)).$$

To this end we shall establish the Picard-Lindelöf Theorem in this lecture about existence and uniqueness, and we shall discuss continuous dependence in Lecture 5. Before we do so we shall recall an important definition: A function $g : \mathbb{R}^m \to \mathbb{R}^d$ is LIPSHITZ in the open set $U \subseteq \mathbb{R}^m$ if there is a constant L, known as the LIPSCHITZ CONSTANT, such that for every $\mathbf{x}, \mathbf{y} \in U$,

$$|g(\mathbf{x}) - g(\mathbf{y})| \leq L|\mathbf{x} - \mathbf{y}|.$$

Clearly, Lipschitz functions are continuous, and in fact, they are differentiable except on a Lebesgue null set. Where a Lipschitz function is differentiable, its derivative is readily seen to be bounded by the local Lipschitz constant. Therefore Lpischitzness is a smoothness condition. When $L < 1$ globally, we call the Lipschitz map a CONTRACTION MAP.

This condition is important in ways we shall appreciate later. First we take a look at two classic examples of non-(global) existence and non-uniqueness where the Lipschitz condition is violated:

Example 4.1. On the space $X = \mathbb{R}$ consider the equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = x^2, \qquad x(0) = 1.$$

It can be checked that $x \mapsto x^2$ fails to be globally Lipschitz, though it is locally so on every bounded open set. This equation is separable, and we can solve it to find

$$\frac{\mathrm{d}x}{x^2} = \mathrm{d}t, \qquad \frac{-1}{x} = t + C.$$

The constant is -1 from the initial condition and we find that the solution is

$$x(t) = \frac{1}{1-t}.$$

There is no existence beyond $[0, 1)$. We call this mode of non-existence FINITE-TIME BLOW-UP.

Example 4.2. Again consider a one-dimensional system this time given by the equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \sqrt{|x|}, \qquad x(0) = 0.$$

It can be checked that $x \mapsto \sqrt{|x|}$ fails to be Lipschitz in any neighbourhood of 0.

The equation is again separable, and we find that

$$x(t) = t^2/4$$

solves the problem with the given initial condition. But this initial condition is not enough to enforce uniqueness because $x(t) \equiv 0$ also solves the equation with the given initial condition.

We are now ready to state the main theorem of this lecture.

Theorem 4.1 (Picard-Lindelöf Theorem)**.** *Let $f(t, \mathbf{x}(t))$ be Lipschitz in its second argument over the open set $U \subseteq \mathbb{R}^d$ and continuous in t. Then for each $\mathbf{b} \in \mathbb{R}^d$, there exists an $\eta > 0$, and a C^1 map $\mathbf{x} : (t_0 - \eta, t_0 + \eta) \to U$ solving the Cauchy problem*

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x}(t) = f(t, \mathbf{x}(t)), \qquad \mathbf{x}(t_0) = \mathbf{b}. \tag{12}$$

Furthermore, $\mathbf{x}$ is unique on its interval of definition.

Note that this theorem extends the theorem in *Cain and Schaeffer* to the non-autonomous case.

The proof depends on several lemmas and is, along with the proofs of these lemmas, examinable. The proof is bipartite.

(1) First we shall show that the Cauchy problem for the differential equation can be re-formulated as a Cauchy problem for an integral equation. Simultaneously, we shall show that if a continuous $\mathbf{x}(t)$ exists at all, it is in fact C^1. This shall re-cast our problem as a FIXED-POINT PROBLEM.
(2) Secondly, we shall show that $\mathbf{x}(t)$ exists and is continuous and unique.

Part 1.

Lemma 4.2. *Let $U \subseteq \mathbb{R}^d$ be an open set. Let $f : \mathbb{R} \times U \to \mathbb{R}^d$ be continuous. Write J for the interval $(t_0 - \eta, t_0 + \eta)$. If $\mathbf{x} \in C^1(J)$ solves the Cauchy problem* (12) *then it also solves the integral equation*

$$\mathbf{x}(t_0 + t) = \mathbf{x}(t_0) + \int_{t_0}^{t_0+t} f(s, \mathbf{x}(s))\, \mathrm{d}s, \qquad 0 < t < \eta. \tag{13}$$

Conversely, if $\mathbf{x} \in C(J)$ and satisfies the foregoing integral equation, then it is in fact in $C^1(J)$ and solves the Cauchy problem (12)*.*

This is a direct result of an application of the fundamental theorem of calculus. Let it be pointed out that $\mathbf{x}(t)$ needs only be continuous for the integral equation to make sense. But once it does make sense, it is immediate that $\mathbf{x}(t)$ must also be once continuously differentiable if f is continuous in both its arguments as a composition of two continuous functions is continuous.

This transforms our Cauchy problem into one of finding a fixed point $\mathbf{x} \in C(J)$ (the "point" is a continuous function — a point on the function space) in the following way. Let $\mathfrak{T} : C(J) \to C(J)$ via

$$\mathfrak{T}(\mathbf{x})(t) = \mathbf{x}(t_0) + \int_{t_0}^{t} f(s, \mathbf{x}(s))\, \mathrm{d}s.$$

Then the Cauchy problem is equivalent to finding a fixed point of the map $\mathfrak{T}$.

Part 2.

Establishing existence usually turns out to be a topological result of compactness, or on a metric space, completeness. We know that a complete space is one for which every Cauchy sequence converges. That is, roughly if a sequence does not escape to infinity — that a subsequence is Cauchy — then the space ensures that it also does not eventually fall through a hole in the space, and so a limit point exists. Therefore existence reduces to setting up an approximate sequence and showing that the elements of the sequence gets closer and closer to one another.

We shall require then three ingredients:

(i) finding a complete metric space,
(ii) constructing a sequence,
(iii) showing that the sequence is Cauchy,

or satisfy some other compactness structure analogous to the ones here described.

Looking back at Part 1., we see that we have an essentially free upgrade to the smaller space of continuously differentiable functions, and we need only to show that our solution exists as a continuous function over some interval J. Therefore our candidate for a complete metric space is $C(J)$, with η a free constant to be later determined. Obviously we should like to make η (in $J = (t_0 - \eta, t_0 + \eta)$) as big as we can.

Lemma 4.3. *The continuous functions $C(J; \mathbb{R}^d)$ form a complete metric space under the norm-induced metric*

$$\|\mathbf{x} - \mathbf{y}\|_{C(J)} = \sup_{t \in J} |\mathbf{x}(t) - \mathbf{y}(t)|.$$

It is also relatively transparent that $C(J)$ is a vector space. Recall that a complete normed vector space is known as a Banach space.

Proof. To show completeness, we postulate a Cauchy sequence of continuous functions and show that it must converge to a continuous function. Let $\mathbf{x}_n(t)$ be a sequence of functions in $C(J)$ such that for any ε, there is an N such that for $m, n > N$,

$$\|\mathbf{x}_n - \mathbf{x}_m\|_{C(J)} = \sup_{t \in J} |\mathbf{x}_n(t) - \mathbf{x}_m(t)| < \varepsilon.$$

For each fixed $t \in J$, given any ε, we can choose the same N and say that if $n, m > N$,

$$|\mathbf{x}_n(t) - \mathbf{x}_m(t)| \leq \|\mathbf{x}_n - \mathbf{x}_m\|_{C(J)} < \varepsilon.$$

Therefore by the completeness of the reals, for each fixed t, $\mathbf{x}_n(t)$ converges pointwise and defines a point $\mathbf{x}(t)$. Now we shall show that $t \mapsto \mathbf{x}(t)$ is continuous:

For two points $s, t \in (t_0 - \eta, t_0 + \eta)$, by the triangle inequality,

$$|\mathbf{x}(t) - \mathbf{x}(s)| \leq |\mathbf{x}(t) - \mathbf{x}_n(t)| + |\mathbf{x}_n(t) - \mathbf{x}_n(s)| + |\mathbf{x}_n(s) - \mathbf{x}(s)|.$$

Let us choose N large enough that $\sup_{t \in J} \mathbf{x}(t) - \mathbf{x}_n(t)| < \varepsilon/3$. Since $\mathbf{x}_n$ is continuous, for any $\varepsilon > 0$, there exists a δ such that if $|t - s| < \delta$, $|\mathbf{x}_n(t) - \mathbf{x}_n(s)| < \varepsilon/3$. The calculation above then implies that for every $\varepsilon > 0$, we can use the same δ and find that if $|t - s| < \delta$, $|\mathbf{x}(t) - \mathbf{x}(s)| < \varepsilon$. Therefore $\mathbf{x} \in C(J)$, and the space is complete.

□