

Mathematical Modelling

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1 Lecture 1

1.1 Practical Information

You need to know

- Separable 1. order equations.
- Linear 1. order equations.
- 2. order linear equations with constant coefficients.

1.2 Dimensional Analysis

Basic facts

- Any physical relation has to make sense dimensionally.
- Any physical relation must be valid for any choice of fundamental units.

Remark.

- **Forbidden** $3m + 2kg = ?$
- $m = f(x, t)$ is legal
- e^{-t} and $s = 5t^2$, is nonsense
- **Dimension** is length, mass, energy, etc.
- **Unit** is meter, feet, year, etc

Make sure
remark looks
better

Given a variable R , we write $R = \overbrace{v(R)}^{\text{numerical value}} \underbrace{[R]}_{\text{unit}}.$

If we have a physical relation that is dimensionally correct that

$$f(R_1, R_2, \dots, R_n) = 0 \rightarrow f(v(R_1), v(R_2), \dots, v(R_n)) = 0$$

1.3 Fundamental Units

Given units F_1, F_2, \dots, F_m for fundamental if

$$F_1^{\alpha_1}, F_2^{\alpha_2}, \dots, F_m^{\alpha_m} = 0 \rightarrow \alpha_1 = \alpha_2 = \dots = 0$$

This units are then independent. **Example.** The units kg, m, s are independent.

Example. In a right angle triangle with angle α and hypotenuse c . We know the area A is uniquely determined by α and c

$$A = f(c, \alpha)$$

α is dimensionless since $\alpha = \frac{s}{r}$. Since A scales as the square of the length, then is

$$\begin{aligned} f(ac, \alpha) &= a^2 f(c, \alpha) \\ c = 1 &\rightarrow f(a, \alpha) = a^2 f(1, \alpha) = a^2 h(\alpha) \end{aligned}$$

Which then ends up with the relation

$$A = a^2 h(\alpha)$$

Make corollary environmet

Lets derive $A = a^2 h(\alpha)$ somewhat differently. We know there is a relation $f(A, c, \alpha) = 0$. We want to introduce new variables.

$$\Pi_1 = \frac{A}{c^2}, \quad c = c_1, \quad \alpha = \alpha_1$$

which means $f(c^2 \Pi_1, c, \alpha) = 0$ and $h(\Pi_1, \alpha, c) = 0$. h must be dimensionally consistent $\rightarrow h$ must be independent of c .

$$\begin{aligned} h(\Pi_1, \alpha) &= 0 \leftrightarrow \Pi_1 = k(\alpha) \\ &\rightarrow \frac{A}{c^2} = k(\alpha) \quad \leftrightarrow \quad A = c^2 k(\alpha) \end{aligned}$$

1.4 Trinity of the first atomic blast

We assume there is a relation

$$f(E, \rho, r, t) = 0$$

- Energy: $E, [E] = kgm^2s^{-2}$
- Mass density of air: $\rho, [\rho] = kg^{-3}$
- Radius: $r, [r] = m$
- Time: $t, [t] = s$

We choose 3 independent variables, say r, t, ρ . Also we call r, t, ρ **core variables**. Let us define a dimensionless number Π_1 such that

$$[\Pi_1] = 0$$

The relation is now given by $h(\Pi, t, r, \rho) = 0$, where h is independent of t, r and ρ . Which in fact is $h(\Pi) = 0$, where $\Pi_1 = c$ s.t. $[c] = 1$.

Given by the definition is

$$\frac{Et^2}{\rho r^5} = c \quad \rightarrow \quad E = \frac{c \rho r^5}{t^2}$$

Using $\rho = 12kgm^{-3}$, $r = 110m$, $t = 6 \cdot 10^{-3}$ do we end up with the relation

$$E = c \cdot 7.5 \cdot 10^{13} J$$

1.5 Steady-state single phase flow in a uniform straight pipeline

Figure of a pipe

Pipe with flow u , length L and pressure drop Δp Then there is a relation between

- L : length, $[L] = m$
- D : diameter $[D] = m$
- u : flow rate $[u] = ms^{-1}$
- Δp : Pressure drop, $[\Delta p] = kgm^{-1}s^{-2}$
- μ : (Shear) viscosity $[\mu] = kgm^{-1}s^{-1}$
- ρ : mass density: $[\rho] = kgm^{-3}$
- E : Wall roughness: $[E] = m$

We have to choose 3 core variables and they are not unique. Since we have 3 independent units ρ, u, D are independent such that it can be a core variable:

$$\Pi_1 = \frac{L}{D} \quad , \quad \Pi_2 = \frac{\Delta p}{\rho u^2} \quad , \quad \Pi_3 = \frac{\rho}{\mu} \quad , \quad \Pi_4 = \frac{E}{D}$$

Then the relation is

$$\begin{aligned} f(\Pi_1, \Pi_2, \Pi_3, \Pi_4, \rho, D, u) &= 0 \quad \Pi_2 = h(\Pi_1, \Pi_3, \Pi_4) \leftrightarrow \frac{\Delta p}{\rho u^2} = h(\Pi_1, \Pi_3, \Pi_4) \\ &\rightarrow \frac{\Delta p}{u^2 \rho} = \Pi_1 k(\Pi_3, \Pi_4) \\ \Delta p &= u^2 \rho \frac{L}{D} k\left(\frac{\rho D u}{\mu}, \frac{E}{D}\right) \\ \text{measure } \frac{\rho D \mu}{\mu} \quad , \quad k &= \frac{\Delta p D}{u^2 \rho} \end{aligned}$$

2 Lecture 2

2.1 Practical Information

Ask for zoom meeting. ola.mahlen@ntnu.no, wednesday 13-14.

2.2 Recall

Last time did we consider steady-state single phase in a flow in a pipe.

- Assuming $f(L, \Delta p, u, \mu, D, E, \rho) = 0$ we arrive with this formula

$$\frac{\Delta p D}{u^2 \rho L} = k \left(\underbrace{\frac{\rho u D}{\mu}}_{\text{Reynhold number}}, \underbrace{\frac{E}{D}}_{\text{Relative wall roughness}} \right)$$

- Dimensionless numbers are often called **dimensionless groups**. Such numbers are independent of choice of fundamental units. They have real physical meaning. **Reynholds number** R_e essentially define what type of flow. Usually $R_e < 2000$ is it laminar flow and $R_e > 4000$ turbulent flow.

2.3 Scaling

Let a pipe have diameter D and flow rate u such that $t_v = \frac{D}{u}$. Then can we describe

$$t_\alpha = \frac{D^2}{\frac{\mu}{\rho}}$$

where μ is the kinematic viscosity. Then is R_e defined such that

$$R_e = \frac{t_\alpha}{t_v}$$

Assume we have the relation

$$R_1 = f(R_2, \dots, R_m)$$

Such that it exist an

$$\Pi_1 = g(\Pi_2, \Pi_2, \dots, \Pi_{m-k}).$$

2.4 Buckingham's Pi-Theorem

Assume we have a dimensionally valid relation $f(R_1, \dots, R_m) = 0$ and a set of fundamental units F_1, F_2, \dots, F_n such that

$$[R_j] = F_1^{a_{j1}} F_2^{a_{j2}} \dots F_n^{a_{jn}} \quad j = 1, 2, \dots, m$$

This then defines the dimension matrix A given by

	Table 1:			
	F_1	F_2	\dots	F_n
R_1	a_{11}	a_{11}		a_{1n}
R_2	a_{21}	a_{21}		a_{2n}
\vdots		\ddots		
R_n	a_{m1}	\dots		a_{mn}

Fix better table environment.

Let $\text{rank}(A) = \dim(\text{row}(A)) = k$. This translates to that we have k dimensionally independent variables. Choosing k linearly independent row vectors, corresponds to choosing core variables. Let this basis be $\mathbf{a}_{i1}, \mathbf{a}_{i2}, \dots, \mathbf{a}_{ik}$. Let the rest of the row vectors be

$$\mathbf{a}_{j_1}, \mathbf{a}_{j_2}, \dots, \mathbf{a}_{j_{m-k}}$$

Then is $\mathbf{a}_{j_r} = \sum_{s=1}^k C_{j_r,s} \mathbf{a}_{i_s}$ where $r = 1, \dots, m-k$. We end up with the equation

$$\Pi_r = \frac{R_{j_r}}{R_{i_1}^{r_{j_r,1}} R_{j_2}^{a_{j_r,2}} \dots R_{j_k}^{a_{j_r,k}}}$$

Are dimensionally numbers.

Our relation becomes

$$g(\Pi_1, \dots, \Pi_{m-k}) = 0, \quad \begin{cases} i_1, i_2, \dots, i_k \\ j_1, \dots, j_{m-k} \end{cases}$$

Example. Swinging pendulum

Assume there is a relation

$$f(w, \alpha_0, L, M, g) = 0$$

where w is the frequency, g gravitational acceleration, M mass, α_0 the swinging angle. We can set L, M, g as core variables such that

$$\left[\frac{L}{g} \right] = s^2 \quad \rightarrow \quad \left[\frac{L}{g} w^2 \right] = 1$$

$$f(w, \alpha_0, L, M, g) = 0 \implies g \left(\alpha_0, \frac{Lw^2}{g} \right) = 0$$

2.5 Scaling

We have a problem at hand, usually differential equations. Then we try to find representative scales for the various variables, and then write the equation on so-called dimensionless form. This has several advantages

- Our dimensionless variables are of order 1 .
- We get rid of a lot of physical constants.
- It makes us able to see what terms are "small" in the equation. The idea is to introduce dimensionless variables by introducing appropriate scales. If we have a stick of length L , we choose L as length scale i.e

$$x^* = Lx \quad \text{Where } x \text{ is dimensionless}$$

Example. Heat flow in a rod with length L . Let $u^*(x^*, t^*)$ be the temperature with the boundary conditions

$$u^*(0, t^*) = 0 \quad u^*(L, t^*) = 0$$

If we let the model be

$$\frac{\partial u^*}{\partial t^*} = D \cdot \frac{\partial^2 u^*}{\partial x^{*2}}, \quad u^*(0, t^*) = 0 \quad u^*(L, t^*) = 0$$

$$u^*(x^*, 0) = u_0 \sin\left(\pi \frac{x^*}{L}\right)$$

We find the time scale T by scales **balancing the equation** .

Let $x^* = Lx$, and $t^* = Tt$, where T is to be determined $u^* = u_0 u$. If we find $u(x, t)$, then the physical temperature is given by

$$u^*(x^*, t^*) = u_0 u\left(\frac{x^*}{L}, \frac{t^*}{T}\right)$$

We have $u(0, t) = u(1, t) = 0$

$$\frac{\partial u^*}{\partial t^*} = D \frac{\partial^2 u^*}{\partial x^{*2}} \implies \frac{u_0}{T} \frac{\partial u}{\partial t} = \frac{u_0}{L^2} D \frac{\partial^2 u}{\partial x^2}$$

$$\leftrightarrow \frac{\partial u}{\partial t} = \left(\frac{TD}{L^2}\right) \frac{\partial^2 u}{\partial x^2} \quad \text{Balancing the equation}$$

$$\frac{TD}{L^2} = 1 \implies T = \frac{L^2}{D}$$

$$u^*(x^*, 0) = u_0 \sin\left(\pi \frac{x^*}{L}\right)$$

$$u(x, 0) = \sin(\pi x)$$

which fulfills the condition

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad u(0, t) = u(1, t) = 0$$

3 Lecture 3

3.1 Recall

$$\begin{aligned}\frac{\partial u^*}{\partial t^*} &= D \frac{\partial^2 u^*}{\partial x^{*2}} \\ 0 &\leq x^* \leq L \\ x^* &= Lx \\ t^* &= Tt \\ u^* &= u_0\end{aligned}$$

We can also recall

$$\begin{aligned}u^*(x^*, t^*) &= u_0 u\left(\frac{x^*}{L}, \frac{t^*}{T}\right) \\ \frac{u_0}{T} \frac{\partial u}{\partial t} &= D \frac{u_0}{L^2} \implies \frac{\partial u}{\partial t} = \frac{TD}{L^2} \frac{\partial^2 u}{\partial x^2} \\ \text{Require } \frac{TD}{L^2} &= 1 \implies T = \frac{L^2}{D}\end{aligned}$$

This can be generalized to

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad 0 \leq x \leq 1$$

3.2 Sinking Ball

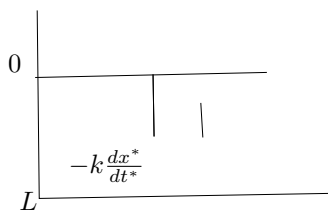


Figure 1: sinkingball

Let

- ρ_b e mass density of ball
- ρ_f mass density of fluid
- V Volume of ball

Then is the equation

$$\begin{aligned}\rho_b V g - \rho_f V g &= V g \rho_b \left(1 - \frac{\rho_f}{\rho_b}\right) \\ &= m \hat{g} \implies \hat{g} = g \left(1 - \frac{\rho_f}{\rho_b}\right)\end{aligned}$$

And we then end up with the newtons law

$$m \frac{dx^{*2}}{dt^{*2}} = m \hat{g} - k \frac{dx^*}{dt^*}, \quad \text{Friction coefficient } k$$

where

$$x^*(0) = 0, \quad \frac{dx^*}{dt^*}(0) = V$$

The cases can be described as follows

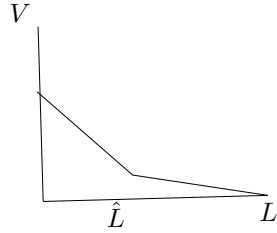


Figure 2: highV

1. High friction, not so high V . Ball will sink at constant speed most of the time.
2. Friction is low, and C not "too high". ("Free fall with $V=0$ ")
3. High V , and high friction $m \frac{d^2 x^*}{dt^{*2}} = m \hat{g} - k \frac{dx^*}{dt^*}$

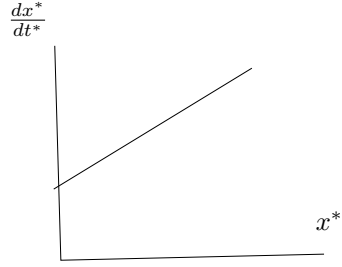


Figure 3: frefall

For this problem there is 3 characteristic speeds

1. V : initial velocity
2. v_0 : equilibrium speed in case A $v_0 = \frac{m\hat{g}}{k}$
3. v_f : free fall $v_f = \sqrt{2\hat{g}L}$

Let us put

$$\begin{aligned} \frac{d^2 x^*}{dt^{*2}} = 0 &\implies k \frac{dx^*}{dt} = \hat{g}m \\ &\implies \frac{dx^*}{dt^*} = \hat{g} \frac{m}{k} = v_0 \end{aligned}$$

and put

$$\begin{aligned} x^*(0) = \frac{dx^*}{dt^*}(0) &= 0 \\ k &= 0 \end{aligned}$$

3.2.1 Scaling

1. Case A: The ball sinks at constant speed "most" of the time.
 - (a) Length scale L : $x^* = Lx$. Since the ball falls with speed most of the time, a timescale would be $T = \frac{L}{v_0}$. v is not much larger than v_0

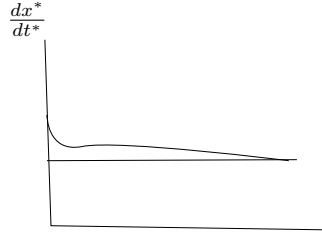


Figure 4: sinking

\Rightarrow it is not so that $v \gg v_0$

$$\begin{aligned}
 m \frac{L}{T^2} x'' &= m \hat{g} - k \frac{L}{T} x' \quad \text{Divide by } L \\
 \Rightarrow m \frac{1}{kT} x'' &= \frac{Tm\hat{g}}{KL} - x' \\
 \frac{m}{k \frac{L}{v_0}} x'' &= \frac{\frac{k}{v_0} m \hat{g}}{kL} - x' \\
 \Rightarrow \frac{mv_0}{Lk} x'' &= \frac{Lm\hat{g}}{KLv_0} - x'
 \end{aligned}$$

We can then derive

$$\begin{aligned}
 \frac{m \frac{m\hat{g}}{k}}{Lk} x'' &= 1 - x' \\
 \Rightarrow \frac{m^2 \hat{g}}{Lk^2} x'' &= 1 - x' \\
 \Rightarrow \frac{m^2 \hat{g}^2}{\hat{g} L k^2} x'' &= 1 - x' \\
 \epsilon x'' &= 1 - x' \quad \text{Where } \epsilon = 2 \left(\frac{v_0}{v_f} \right)^2
 \end{aligned}$$

The condition are $x(0) = 0$, $\frac{L}{T} x'(0) = V$ which can be rewritten to

$$x'(0) = \frac{TV}{L} \frac{\frac{L}{v_0 V}}{L} = \frac{V}{v_0} = \mu$$

3.3 Let Analyze The equation

In case A is the

$$\epsilon \ddot{x} = 1 - \dot{x}$$

An approximation we can do is to put $\epsilon = 0$ such that

$$0 = 1 - \dot{x} \quad x(0) = 0, \quad \dot{x}(0) = \mu \quad \ddot{x} = 0$$

unless $\mu = 1$, we cant find a solution.

3.3.1 Case B

Small friction, V is not too high. Let the lengthscale be L .

$$\begin{aligned} \frac{d^2}{dt^{*2}} x^{*2} &= \hat{g}, \quad x^*(0) = \frac{dx^*}{dt^*}(0) = 0 \\ x^*(t^*) &= \frac{1}{2} \hat{g} (t^*)^2 \end{aligned}$$

Hit the bottom with speed V_f . We can choose time scale T such that

$$T = \frac{L}{v_f}$$

So gain

$$\frac{mL}{T^2} \ddot{x} = m\hat{g} - \frac{kL}{T} \dot{x}$$

What you can observe is that gravity dominates so we modify the equation to be

$$\begin{aligned} \frac{L}{\hat{g}T^2} \ddot{x} &= 1 - \frac{kL}{gmT} \dot{x} \\ \implies 2\ddot{x} &= 1 - \left(\frac{v_F}{v_0} \right), \quad \frac{K}{T} \dot{x}(0) = 0 \\ 2\ddot{x} &= 1 - \epsilon \dot{x} \quad \dot{x}(0) = \frac{V}{v_f} = \mu \end{aligned}$$

3.3.2 Case C: High V and high friction

Let us consider

$$m \frac{d^2 x^*}{dt^{*2}} = -kV \quad \frac{dx^*}{dt^*} = V - \frac{kV}{m} t^* = 0$$

Where we choose the scales $t^* = \frac{m}{k} = T$, $L = \frac{Vm}{k}$, where $TV = L$.

$$\implies \ddot{x} = \epsilon - \dot{x}, \quad x(0) = 1, \quad \dot{x} = 1, \quad \epsilon = \frac{v_0}{V}$$

Example. Let

$$a \frac{d^2 x^*}{dt^{*2}} + b \frac{dx^*}{dt^*} + cx^* = 0$$

$$x^*(0) = x_0, \quad \frac{dx^*}{dt^*}(0) = 0$$

Three ways to scale by balancing the equation. Last term "small"

$$x^* = x_0 x, \quad t^* = Tt$$

Where T is to be determined.

$$a \frac{x_0}{T^2} \ddot{x} + b \frac{x_0}{T} \dot{x} + cx_0 = 0$$

$$\ddot{x} + \frac{bT}{a} \dot{x} + \frac{cT^2}{a} = 0$$

If we are smart can we choose the timescale $T = \frac{a}{b}$ then we get

$$\ddot{x} + \dot{x} + \frac{ca^2}{b^2 a} = 0.$$

$$\implies \ddot{x} + \dot{x} + \left(\frac{ca}{b^2}\right) x = 0$$

3.4 Turbulence

Reynold number

$$Re = \frac{u\rho L}{\mu} = \frac{uL}{\frac{\mu u}{\rho}} = \frac{uL}{\nu}$$

Then we have

$$\frac{\partial v}{\partial t} = \nu \frac{\partial^2 v}{\partial x^2}$$

4 Lecture 31/08

4.1 Turbulence

Kolmogorov's Microscales .

$$\rho \frac{du}{dt} = \mu \frac{\partial^2 u}{\partial x^2}$$

Time scale for convective flow over a distance L

$$t_c = \frac{L}{U}, \quad U \text{ is velocity.}$$

This can be rearranged such that

$$\frac{\partial u}{\partial t} = \left(\frac{\mu}{\rho} \frac{\partial^2 u}{\partial x^2} \right).$$

We also define $\mathcal{V} = \frac{\mu}{\rho}$ where $[\mathcal{V}] = m^2 s^{-1}$, which is the time for dispersion of velocity.

Let $t_d = \frac{L^2}{\mathcal{V}}$ such that the Reynolds number can be written

$$Re = \frac{v\rho L}{\mu} = \frac{UL}{\left(\frac{\mu}{\rho}\right)} = \frac{UL}{\mathcal{V}} = \frac{t_d}{t_0}$$

For water is $\mathcal{V} = 10^{-6} m^2 s^{-1}$. So for a river, put $L = 100m$ with $U = 1m s^{-1}$

$$Re = \frac{1m s^{-1} \cdot 100m}{10^{-6} m^2 s^{-1}} = 10^8$$

Assume the generation of new whirls stops when $t_d \approx t_c \rightarrow Re \approx 1$. Let

$$E = \frac{\text{Energy}}{\text{time per unit mass}}$$
$$[E] = kg m^2 s^{-2} s^{-1} kg$$

Let l be the scale of the smallest whirls and u the unit velocity then is

$$E = E(l, u, \mathcal{V}).$$

We assume that E is proportional to u^2 .

$$f\left(\frac{E}{u^2}, l, \mathcal{V}\right) = 0$$

Table 2:

	m	s
$\frac{E}{n^2}$	1	0
l	1	0
v	2	-2

$$\left[\begin{array}{c} \frac{E}{u^2} \\ \mathcal{V} \end{array} \right] = m^{-2}$$

$$\Pi = \frac{\frac{E}{u^2}}{\mathcal{V}} l^2$$

choose $\Pi = 1$

$$\rightarrow E = \mathcal{V} \left(\frac{u^2}{l} \right)^2$$

$$ul = \mathcal{V}$$

$$\Rightarrow k = \left(\mathcal{V}^3 \frac{1}{E} \right)^{\frac{1}{4}}, \quad u = (VE)^{\frac{1}{4}}$$

Example . Let us have $1kg$ what in a mixmaster and apply $100W$ power.
then is

$$l = \left(\frac{(10^{-6} m^2 s^{-1})^3}{100 m^2 s^{-3}} \right)^{\frac{1}{4}} = 0.01 mm$$

4.2 Regular Perturbation Theory

Assume we have an equation s.t.

$$D(x, \varepsilon) = 0 \quad \text{where} \quad \varepsilon \ll 1$$

meaning that ε is small.

We have a solution $x(\varepsilon)$ to the problem $D(x, \varepsilon)$. The perturbation problem is regular if $\lim_{\varepsilon \rightarrow 0} x(\varepsilon)$ is a solution to $D(x, 0) = 0$. The idea is

1. Put $x(\varepsilon) = x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \dots$

$$x(\varepsilon) \approx x_0 \quad \text{in 0. order}$$

$$x(\varepsilon) \approx x_0 + \varepsilon x_1 \quad \text{to 1. order}$$

2. Insert $x(\varepsilon) = x_0 + \varepsilon x_1 + \dots$ into $D(x, \varepsilon)$.

3. Collect all terms of order 0, all terms of order 1 so that

$$D(x, \varepsilon) = 0 \Leftrightarrow \underbrace{(\quad)}_{=0} + \underbrace{(\quad)}_{=0} \varepsilon^2 + \dots = 0$$

Example. Let

$$x^3 + x^2 + \varepsilon x - 2 = 0, \quad \varepsilon \ll 1$$

For $\varepsilon = 0$ we have $x = 1$ as a solution. To find a solution "close to" 1 when $\varepsilon \neq 0$ we put

$$x = 1 + \varepsilon x_1 + \varepsilon^2 x_2 + O(\varepsilon)$$

Want an approximation to 2. order. We get

$$\begin{aligned} (1 + \varepsilon x_1 + \varepsilon^2 x_2)^3 + (1 + \varepsilon x_1 + \varepsilon^2 x_2)^2 + \varepsilon(1 + \varepsilon x_1 + \varepsilon^2 x_2) - 2 &= 0 \\ \implies \varepsilon(5x_1 + 1) + \varepsilon^2(\dots) &= 0 \end{aligned}$$

$$x(\varepsilon) \approx 1 - \frac{\varepsilon}{5} + \frac{\varepsilon^2}{125}$$

4.3 The Projectile Problem

Let v_0 be the vertical velocity and v_e be escape velocity such that $v_0 \ll v_e$.

Newton gravitational law

$$\mathbf{F} = -m \frac{R^2 g}{(R + x^*)^2}$$

Where g is the gravitational constant at $x^* = 0$.

Energy to move to $x^* = \infty$

$$\begin{aligned}
-\int_0^\infty \mathbf{F} dx^* &= mgR^2 \int_0^\infty \frac{dx^*}{(R+x^*)^2} \\
&= mgR^2 \left[-\frac{1}{(R+x^*)} \right]_0^\infty \\
&= mgR = \frac{1}{2}mv_e^2 \\
\implies v_e &= \sqrt{2gR}
\end{aligned}$$

We have

$$m \frac{d^2 x^*}{dt^{*2}} = -m \frac{gR^2}{(R+x^*)^2}$$

Such that

$$\frac{d^2}{dt^{*2}} = -\frac{R^2 g}{(R+x^*)^2}, \quad x^*(0) = 0, \quad \frac{dx^*}{dt^*}(0) = v_0$$

and $v_0 \ll v_e$, when $x^* \ll R$ (a consequence of $v_0 \ll v_e$)

$$\frac{d^2 x^*}{dt^{*2}} \approx -g \quad \frac{dx^*}{dt^*} = v_0 - t^* g = 0 \quad \leftrightarrow t^* = \frac{v_0}{g} = T = \text{timescale}$$

$$X^* = v_0 t^* - \frac{1}{2} t^{*2} g \quad x^*(T) = \frac{v_0^2}{g} - \frac{1}{2} \frac{v_0^2}{g} = \frac{1}{2} \frac{v_0^2}{g}$$

Let $L = \frac{v_0^2}{g}$ and scale the equation $\left(\frac{L}{T}\right) = v_0$ and $x^* = Lx$.

$$\begin{aligned}
\frac{L}{T^2} \ddot{x} &= \frac{-gR^2}{(R+Lx)^2} \leftrightarrow \frac{L}{T^2} \ddot{x} = -\frac{gR^2}{R^2 \left(1 + \frac{L}{R}x\right)^2} \\
\rightarrow \ddot{x} &= \frac{-T^2 \frac{g}{L}}{\left(1 + \frac{L}{R}x\right)^2} \rightarrow \ddot{x} = \frac{-1}{(1 + \varepsilon x)^2}
\end{aligned}$$

Where

$$\varepsilon = \frac{L}{R} = \frac{v_0^2}{Rg} = 2 \frac{2v_0^2}{v_e^2}$$

Following problem

$$\ddot{x} = \frac{-1}{(1 + \varepsilon x)^2}, \quad x(0) = 0, \quad \dot{x}(0) = 1$$

Recall that

$$\begin{aligned}
f(u) &= \frac{1}{(1+u)^2} \rightarrow \int f(u) = \frac{1}{1+u} + C \\
&= C - (1 - u + u^2 - u^3 + \dots) \\
\implies f(u) &= 1 - 2u + 3u^2 + O(u^3)
\end{aligned}$$

Then to second order

$$\ddot{x} = -(1 - 2\varepsilon x + 3\varepsilon x^2), \quad x(0) = 0, \quad \dot{x}(0) = q$$

Next et

$$x(t) = x_0(t) + \varepsilon x_1(t) + \varepsilon^2 x_2(t) + O(\varepsilon^3)$$

So let

$$\begin{aligned} x_j(0) &= 0 \quad \text{for } j = 0, 1, 2 \\ \ddot{x}_0(0) &= 1, \quad \dot{x}_1(0) = \dot{x}_2(0) = 0 \\ &\rightarrow \ddot{x}_0 + \varepsilon \ddot{x}_1 + \varepsilon^2 \ddot{x}_2 = -1 + 2\varepsilon(x_0 + \varepsilon x_1) - 3\varepsilon^2 x_0^2 \\ &\rightarrow (\ddot{x}_0 + 1) + \varepsilon(\ddot{x}_1 - 2x_0) + \varepsilon^2(\ddot{x}_2 + 2x_1 + 3x_0^2) = 0 \\ \ddot{x}_0 &= -1 \quad x_0(0) = 0, \quad \dot{x}_0 = 1 \\ \ddot{x}_1 &= 2x_0, \quad \dot{x}_1(0) = \dot{x}_2(0) = 0 \\ \ddot{x}_2 &= 2x_1 - 3x_0^2, \quad x_2(0) = \dot{x}_2(0) = 0 \end{aligned}$$

$$\begin{aligned} &\rightarrow x_0(t) = t - \frac{1}{2}t^2 \\ \ddot{x}_1(t) &= 2t - t^2 \\ \dot{x}_1(t) &= t^2 - \frac{1}{3}t^3 \\ x_1(t) &= \frac{1}{3}t^3 - \frac{1}{12}t^4 \end{aligned}$$

Where

$$\begin{aligned} \ddot{x}_2 &= \frac{2}{3}t^3 - \frac{1}{6}t^4 - 3\left(t^2 - t^3 + \frac{1}{4}t^4\right) \\ x_2 &= -\frac{1}{4}t^4 + \frac{11}{60}t^5 - \frac{11}{360}t^6 \end{aligned}$$

Which end up with

$$x(t) = t - \frac{1}{2}t^2 + \varepsilon\left(\frac{1}{3}t^3 - \frac{1}{12}t^4\right) + \varepsilon^2\left(-\frac{1}{4}t^4 + \frac{11}{60}t^5 - \frac{11}{360}t^6\right)$$

Gives the idea of how to approx the time to the maximum height. $\dot{x}(t) = 0$ is a 5. degree equation containing ε .

Lets put

$$t = 1 + \varepsilon t_2 + \varepsilon^2 t_3$$

Into the 5. degree equation and to regular perturbation

$$\rightarrow t = 1 + \frac{2}{3}\varepsilon + \frac{2}{5}\varepsilon^2 + O(\varepsilon^3)$$

such that

$$\begin{aligned}\ddot{x} &= \frac{-1}{(1+\varepsilon x)^2} \rightarrow \ddot{x}\dot{x} = \frac{\dot{x}}{(1+\varepsilon x)^2} \\ \rightarrow \frac{d}{dt} \left(\frac{1}{2} \dot{x}^2 \right) &= \frac{d}{dt} \left(\frac{-1}{\varepsilon} \frac{1}{1+\varepsilon x} \right) \\ \frac{1}{2} \dot{x}^2 &= \frac{-1}{\varepsilon} \frac{1}{1+\varepsilon x} + C \\ \frac{1}{2} &= \frac{-1}{\varepsilon}\end{aligned}$$

$$C = \frac{1}{2} + \frac{1}{\varepsilon}$$

where

$$\frac{1}{2} \dot{x}^2 = \frac{-1}{\varepsilon} \frac{1}{1+\varepsilon x} + \frac{1}{2} + \frac{1}{\varepsilon}$$

At maximum height $\dot{x} = 0$

$$0 = -\frac{1}{\varepsilon}.$$

5 Lecture 02/09

Let Newtons Law be

$$\frac{d^2 s^*}{dt^{*2}} = g \sin(\alpha^*) \implies \frac{d^2 \alpha^*}{dt^{*2}} = -\frac{g}{L} \sin(\alpha^*)$$

scaling:

$$\begin{aligned} \alpha^* &= \varepsilon \alpha, \quad t^* = Tt \\ \frac{\varepsilon}{T^2} \ddot{\alpha} &= \frac{-g}{L} \sin(\varepsilon \alpha) \implies \ddot{\alpha} = -\left(T^2 g \frac{1}{L}\right) \frac{\sin(\varepsilon \alpha)}{\varepsilon} \\ T &= \sqrt{\frac{L}{g}} \implies \ddot{\alpha} = -\frac{\sin(\varepsilon \alpha)}{\varepsilon} \\ \alpha(0) &= 1 \quad \dot{\alpha}(0) = 0 \end{aligned}$$

Let put $\alpha = \alpha_0(t) + \varepsilon^2 \alpha_2(t) + O(\varepsilon^4)$. where $\alpha(t)$ is an even function of ε due to symmetry.

$$\alpha_0(0) = 1, \quad \dot{\alpha}_0(0) = 0, \quad \alpha_2(0) = \dot{\alpha}_2(0) = 0$$

Inserted into the equation

$$\begin{aligned} \ddot{\alpha}_0 + \varepsilon^2 \ddot{\alpha}_2 &= -\frac{\sin(\varepsilon(\alpha_0 + \varepsilon^2 \alpha_2))}{\varepsilon} \implies \ddot{\alpha}_0 + \varepsilon^2 \ddot{\alpha}_2 \\ &= \frac{-1}{3} \left(\varepsilon \underbrace{(\alpha_0 + \varepsilon^2 \alpha_2)}_u \frac{\varepsilon^2}{6} (\alpha_0 + \alpha \varepsilon^2) \right) \end{aligned}$$

Let

$$\begin{aligned} \alpha_0(t) &= A \cos t + B \sin t \\ \alpha_0(0) &= 1, \quad \dot{\alpha}_0(0) = 0 \implies \alpha_0(t) = \cos t \\ \alpha_2(t) &= A \cos t + B \sin t + \alpha_{2,f}(t) \\ \cos^3 t &= \left(\frac{1}{2} (e^{it} - e^{-it}) \right)^3 = \frac{1}{8} (e^{i3t} + 3e^{it} - 3e^{-it} - e^{-i3t}) \\ &= \frac{1}{4} (\cos 3t + 3 \cos t) \\ \alpha_{20}(t) &= A \cos 3t + B \sin 3t + Ct \cos t + Dt \sin t \\ \alpha_2(t) &= \frac{1}{192} (\cos t + \cos 3t) + \frac{1}{16} t \sin t \\ \alpha(t) &= \alpha_0(t) + \varepsilon^2 \alpha_2(t) \quad \text{is not periodic} \end{aligned}$$

Poincare-Lin Stel Method . Instead let

$$\alpha(t) = \alpha_0(\omega(\varepsilon)t) + \alpha_2(\omega(\varepsilon)t)\varepsilon^2 + O(\varepsilon^4)$$

Where $\omega(\varepsilon) = 1 + \omega_2\varepsilon^2 + O(\varepsilon^4)$. See exercise.

5.1 Modelling how the kidney disposes salt and water.

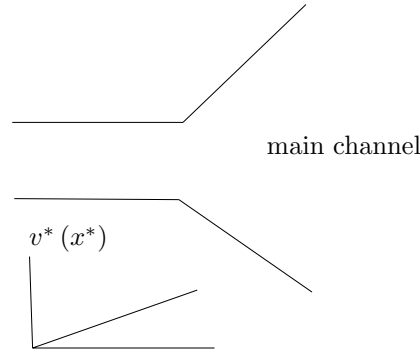


Figure 5: watermodell

Assumptions

1. Secondary channel is fed water by osmosis from the surrounding tissue.
2. Ions are transported down the channel by convection and diffusion.
3. Ions are fed into the channel by a chemical pump.

We want the steady-state profiles of ion concentration $C^*(x^*)$ and the velocity $v^*(x^*)$ of the ion water solution.

The ion concentration is written as

$$[C^*] = \frac{\text{ions}}{m^3} = \frac{\text{osmol}}{m^3}$$

One mole salt gives two moles ions

Osmosis :

$$J^* = P(c^* - c_0)$$

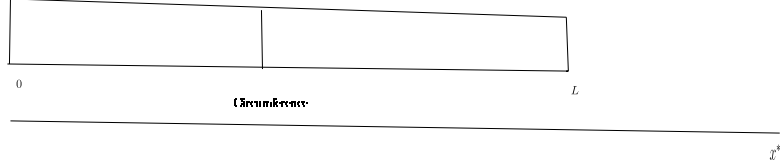


Figure 6: molefig

Is flux density of water entering the secondary channel. J^* is volume water in per area per time. c_0 ion concentration is tissue and main channel. P is called membrane permeability.

$$[P] = \frac{[J^*]}{[c^*]} = \frac{ms^{-1}}{osmol \cdot m^{-3}} = \frac{m^4}{s \cdot osmol}$$

Ion flux density

$$N^* = \begin{cases} N_0, & 0 \leq x^* \leq \delta \\ 0, & \delta \leq x^* \leq L \end{cases}$$

Where $[N_0] = \frac{osmol}{m^2 \cdot s}$. The total rate of salt entering the channel

$$N_0 \cdot c \cdot \delta$$

Where c is the area of pump.

- The flux density of ions in the secondary channel

$$F^* = F_c^* + F_\alpha^*$$

$$[F^*] = \frac{osmol}{m^2 \cdot s}$$

- Convective flow

$$F_c^* = c^* v^*$$

- Diffusion: **Ficus law**

$$F_1^* = -D \frac{dc^*}{dx^*}.$$

where D is the diffusion of salt in water.

Conservation of water

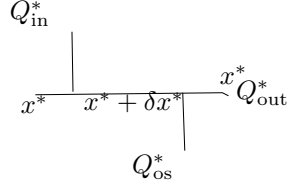


Figure 7: conssswater

$$\begin{aligned}
 Q_{\text{out}}^* &= Q_{\text{in}}^* + Q_{\text{os}}^* \\
 v^*(x^* + \Delta x^*) &= v^* A + P(c^*(\hat{x}) - c_0) c \Delta x^*, \\
 &\text{where } \hat{x}^* \in \langle x^*, x^* + \Delta x^* \rangle \\
 \implies \frac{v^*(x^* + \Delta x^*) - v^*(x^*)}{\Delta x^*} &= \frac{c}{A} P(c^*(\hat{x}^*) - c_0) \\
 \Delta x^* \rightarrow 0 \implies \frac{dv^*}{dx^*} &= \left(\frac{cP}{A} \right) (c^* - c_0)
 \end{aligned}$$

Conservation of salt

$$F^*(x^* + \Delta x^*) A = F^*(x^*) A + N^*(\hat{x}^*) c \Delta x^*$$

This ends up with

$$\begin{aligned}
&\implies \frac{dF^*}{dx^*} = \frac{c}{A} N^*(x^*) \\
&\text{or } \frac{dF^*}{dx^*} = \frac{c}{A} \cdot \begin{cases} N_0, & 0 < x^* < \delta \\ 0, & \delta < x^* < L \end{cases} \\
&F^*(0) = 0 \implies F(x^*) = \begin{cases} \frac{N_0 c}{A} x^*, & 0 < x^* < \delta \\ \frac{N_0 \delta c}{A}, & \delta < x^* < L \end{cases} \\
&\implies v^* c^* - D \frac{dc^*}{dx^*} = F^*(x^*) \\
&\frac{dv^*}{dt^*} = \frac{cP}{A} (c^* - c_0) \\
&v^*(0) = 0 \\
&c^*(L) = c_0
\end{aligned}$$

Also same that v^* and c^* are continious at $x^* = \delta$.

5.1.1 Scaling the model

Two length scales δ and L . Choose δ as length scale. Natural to use c_0 as scale for c^* . The rate salt supplied is

$$N_0 \delta c = c_0 U A$$

Ions supplied is convectiv flux with c^* such that $U = \frac{N_0 \delta c}{c_0 A}$.

$$\begin{aligned}
x^* &= \delta, \\
c^* &= c_0 c \\
v^* &= U v
\end{aligned}$$

1. $(U c_0) c v - \frac{D c_0}{\delta} \dot{c} = F^*$ such that

$$\implies v c - \frac{D c}{\delta U c_0} \dot{c} = \frac{1}{U c} \cdot \begin{cases} \frac{N_0 c \delta x}{A U c_0}, & 0 < x \delta < \delta \\ \frac{N_0 c \delta}{A u c_0}, & \delta < x \delta < L \end{cases}$$

$$v c - \varepsilon \dot{c} = \begin{cases} x & 0 < x < 1 \\ 1 & 1 < x < \lambda \end{cases}$$

where $\varepsilon = \frac{D}{\delta u}$, and $\lambda = \frac{L}{\delta}$

$$\implies U = \frac{N_0 \delta c}{c_0 A}$$

2. $\frac{U}{\delta} \dot{v} = \frac{cP}{A} c_0 (c - 1)$

6 Lecture 07/09

6.1 Emergent Osmotic Concentration

- (i) Total rate of salt pumped per second $\delta c N_0$
- (ii) Water out per second $v^*(L) A = U v(\lambda) A$, where $\lambda = \frac{L}{\delta}$

$$\begin{aligned}\delta c N_0 &= C_0 U \\ &\approx \text{Flow out of salt per sec} \\ \implies U &= \frac{\delta c N_0}{C_0 A}\end{aligned}$$

Measure of the efficiency

$$\begin{aligned}\frac{\text{Salt out}}{\text{Water out}} &= O s^* \\ &= \frac{\delta c N_0}{U v(\lambda) A} = \frac{C_0}{v(\lambda)}\end{aligned}$$

Thus $v(\lambda) > \frac{1}{4}$

6.2 Boundary Value Problem

We know that

$$\begin{aligned}\sum v'(x) &= C(x) - 1 \\ v(x) C(x) - \mu C'(x) &= f(x) = \begin{cases} x, & 0 \leq x \leq 1 \\ 1, & 1 \leq x \leq \lambda \end{cases}\end{aligned}$$

Where $v(0) = 0$, $C(\lambda) = 1$. In addition v and C must be continuous.

Let assume $0 < \varepsilon \ll 1$. Put $C = c_0 + \varepsilon C_1 + O(\varepsilon^2)$ and $v = v_0 + \varepsilon v_1 + O(\varepsilon^2)$. Inserted into the equation

$$\begin{aligned}\varepsilon (v'_0) &= C_0 + \varepsilon C_1 - 1 + O(\varepsilon^2) \\ (v_0 + \varepsilon v_1) (1 + \varepsilon C_1)^2 - \mu (\varepsilon C'_1) &= f(x) + O(\varepsilon^2) \\ C_0 - 1 &= 0 \iff C_0 = 1 \\ C_1 - v'_0 &= 0 \implies C_1 = v'_0 \implies C_1 = f(x), \quad C_1 \text{ is discontinuity} \\ v_0 - f(x) &= 0, \quad v_0 = f(x) \\ v_1 + v_0 C_1 - \mu \varepsilon C'_1 &= 0\end{aligned}$$

Something is wrong.

$$\begin{aligned} \varepsilon v' &= C - 1 \\ \varepsilon v C - \underbrace{(\varepsilon \mu)}_{\text{not small}} &= \varepsilon f(x) \end{aligned}$$

For notation convenience let

$$\begin{aligned} (\varepsilon \mu) &= \omega^{-1} \\ \varepsilon v' &= C - 1 \\ \varepsilon v C - \frac{1}{\omega^2} C' &= \varepsilon f(x) \\ \implies \varepsilon (\omega^2 v C) - C' &= \varepsilon \omega^2 f(x) \end{aligned} .$$

We then get

$$\begin{aligned} v &= v_0 + \varepsilon v_1 \\ C &= C_0 + \varepsilon C_1 \\ \varepsilon v'_0 &= C_0 + \varepsilon C_1 \implies C_0 = 1, \quad v'_0 = 1 \\ \varepsilon (\omega^2 (v_0 C_0)) - C'_0 - \varepsilon C'_1 &= \omega^2 \varepsilon f(\varepsilon) \\ \omega^2 v_0 - v''_0 &= \omega^2 f(x) \\ v''_0 - \omega^2 v_0 &= -\omega^2 f(x) \\ v(0) = 0 &\implies v_0(0) = 0 \end{aligned}$$

Also

$$\begin{aligned} C(\lambda) &= 1 = 1 + \varepsilon C_1(\lambda) + O(\varepsilon) \\ \implies C_1(\lambda) &= 0 \implies v'_0(\lambda) = 0 \end{aligned}$$

v and C is continuous . v_0 and v'_0 continuous.

For $0 \leq x \leq 1$ we have

$$v''_0 + \omega^2 = -\omega^2 x$$

A solution to $v''_0 + \omega = 0$

$$E e^{\omega x} + E e^{-\omega x} = A \cosh(\omega x) + B \sinh(\omega x)$$

Identities.

$$\cosh u = \frac{1}{2} (e^u + e^{-u})$$

$$\sinh u = \frac{1}{2} (e^u - e^{-u})$$

$$\cosh' u = \sinh u$$

$$\sinh' u = \cosh u$$

$$\cosh u - v = \cosh u \cosh v - \sinh u \sinh v$$

$$\cosh 0 = 1$$

$$\sinh 0 = 0$$

The solution is for $0 \leq x \leq 1$

$$v_0(x) = x + A \cosh \omega x + B \sinh \omega x$$

In the same manner

$$v_0^+(x) = \overbrace{1 + C \cosh \omega x + D \sinh \omega x}^{0 \leq x \leq \lambda = \frac{l}{8}}$$

$$v_0(0) = 0 \implies v_0^- = 0$$

$$\implies v_0^-(x) = x + B \sinh \omega x$$

$$\frac{dv_0^+}{dx}(\lambda) = 0$$

$$C\omega \sinh \omega \lambda + D\omega \cosh \omega \lambda = 0$$

The solution is

$$v_0(x) = E \cosh \varepsilon (x - \lambda)$$

Require continuity at $x = 1$ of $v_0(x)$ and $C_1(x) = \frac{dv_0}{dx}(x)$

$$v_0^-(1) = v_0^+(1)$$

$$\frac{dv_0^-}{dx} = \frac{dv_0^+}{dx}$$

We get

$$v_0^-(x) = x - \frac{\cosh(\omega(\lambda - 1))}{\omega \cosh(\omega \lambda)} \sinh \omega \lambda \quad 0 \leq x \leq 1$$

$$v_0^+ = 1 - \frac{\sinh(h\omega)}{\omega \cosh(\omega \lambda)} \cosh \omega (x - \lambda)$$

$$\begin{aligned} Os^* &= \frac{C_0}{v(\lambda)} \approx \frac{C_0}{v_0(\lambda)} \\ &= \frac{C_0}{\left(1 - \frac{\sinh \omega}{\omega} \frac{1}{\cosh \omega \lambda}\right)} \end{aligned}$$

$\varepsilon \ll 1$, Os^* depends on ω and $\lambda\omega = k$.

If ω is small then is

$$\frac{\sinh \omega}{\omega} \approx 1 + \frac{1}{6}\omega^2 + \dots$$

Let

$$\begin{aligned} Os^* &\approx \frac{C_0}{1 - \frac{1}{\cosh k}} = C_0 \left(\frac{\cosh k}{\cosh k - 1} \right) = C_0 \left(\frac{1 + \frac{1}{2}k^2 + O(k^4)}{\frac{1}{2}k^2 + O(k^4)} \right) \\ &\approx \left(1 + \frac{2}{k^2} \right) C^* \end{aligned}$$

Argue that

$$\frac{2}{k^2} \approx \frac{F_{\text{Diffusion}}^*}{F_{\text{Convection}}^*}$$

We can finally conclude that

$$Os^* \approx C_0 \left(1 + \frac{F_{\text{diff}}^*}{F_{\text{conv}}^*} \right)$$

7 Singular Perturbation

$$\varepsilon m^2 + 2m + 1 = 0, \quad 0 < \varepsilon \ll 1, \quad m = \frac{1}{2}$$

If εm^2 and 1 are important

$$m \pm e\varepsilon^{\frac{1}{2}} \implies \varepsilon m^2 + 2m \approx 0$$

$$\leftrightarrow m(\varepsilon m + 2) = 0$$

$$m \approx -\frac{2}{\varepsilon}$$

$$\varepsilon m^2 \approx -\frac{2}{\varepsilon}$$

$$2m \approx \frac{4}{3}$$

$$\varepsilon m^2 + 2m + 1 = 0$$

$$m = -\frac{1}{2} + \varepsilon m_1$$

$$m = -\frac{2}{3}\widetilde{m}_1\varepsilon$$

7.1 Singular perturbation applied to differential equations

$$\varepsilon y'' + 2y' + y = 0$$

$$y(0) = 0, \quad y(1) = 1$$

$$0 \leq x \leq 1$$

Let $\varepsilon = 0$ then is

$$\begin{aligned} 2y' + y = 0 &\implies y = ke^{-\frac{x}{2}}, k \in \mathbb{R} \\ y(0) = 0 &\implies y := 0 \\ y(1) = 1 &\implies y(x) = e^{\frac{1}{2}}e^{-\frac{x}{2}} \end{aligned}$$

The characteristic equation for

$$\begin{aligned} \varepsilon y'' + y' + y &= 0 \\ \varepsilon r^2 + 2r + 1 &= 0, \quad r_1 \approx -\frac{1}{2}, r_2 \approx -\frac{2}{3} \\ y(x) &\approx Ae^{-\frac{x}{2}}Be^{-\frac{2x}{\varepsilon}} \end{aligned}$$

For $y(0) = 0$

$$y(x) = A \left(e^{-\frac{x}{2}} - e^{-\frac{2x}{\varepsilon}} \right)$$

And for $y(1) = 1$

$$y(x) \approx e^{-\frac{1}{2}} \left(e^{-\frac{x}{2}} - e^{-\frac{2x}{\varepsilon}} \right)$$

7.2 Further look at Singular Perturbation

Our main equation

$$\varepsilon y'' + 2y' + y = 0, \quad y(0) = 0, \quad y(1) = 1$$

- (i) Find outer solution y_o by setting $\varepsilon = 0$. Since the solution $\varepsilon y_0(x) \approx y(x)$ for

$$\begin{aligned} x &> \delta(\varepsilon), \quad \text{where } \delta(\varepsilon) \rightarrow 0 \text{ when } \varepsilon \rightarrow 0 \\ y_0(x) &= e^{\frac{1}{2}}e^{-\frac{x}{2}} \end{aligned}$$

Characteristic equation for

$$\begin{aligned} Y \left(\frac{x}{\delta(\varepsilon)} \right) &= y(x) \quad \text{is} \\ \zeta &= \frac{x}{\delta(\varepsilon)}, \quad Y(\zeta) = \frac{x}{y(\zeta\delta(\varepsilon))} \end{aligned}$$

$$\varepsilon Y'' + 2Y' + Y = 0, \quad \implies \quad \varepsilon \frac{1}{\delta^2} Y'' + \frac{2}{\delta} Y' + Y = 0$$

Are of order of 1.

$$\implies \quad \varepsilon \frac{1}{\delta^2}, \frac{1}{\delta}, 1 \quad \text{are}$$

the "size" of the terms.

Choosing $\delta = \varepsilon$ gives

$$\frac{1}{3}Y'' + \frac{2}{3}Y' = 0 \implies Y'' + 2Y' + \varepsilon Y = 0$$

Let

$$Y\left(\frac{x}{\delta(\varepsilon)}\right) = y(x), \quad y(0) = 0 \implies Y(0) =$$

Which is called the **inner equation**. Putting $\varepsilon = 0$ and $Y'' + 2Y' = 0$ where

$$\implies Y(\zeta) = D + Ee^{2\zeta}$$

We see that

$$\begin{aligned} Y(0) = 0 &\implies E = -D \\ Y(\zeta) &= E(1 - e^{-2\zeta}) \end{aligned}$$

Let us match it with this equation

$$y_0(x) = e^{\frac{1}{2}}e^{-\frac{x}{2}}$$

We can try to match the solution at $x = \theta(\varepsilon)$. Then we need to require that

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0^+} \theta(\varepsilon) &= 0 \\ \lim_{\varepsilon \rightarrow 0^+} \frac{\theta(\varepsilon)}{\delta(\varepsilon)} &= \infty \end{aligned}$$

Example.

$$\delta = \varepsilon, \quad \theta = \varepsilon^{\frac{1}{2}}$$

We know that

$$Y\left(\frac{x}{\delta(\varepsilon)}\right) = y_I(x)$$

Then can we start matching such that

$$y_I(\theta(\varepsilon) \approx y_0(\theta(\varepsilon))) \implies Y\left(\frac{\theta(\varepsilon)}{\delta(\varepsilon)}\right) = y_0(\theta(\varepsilon))$$

Let $\varepsilon \rightarrow 0$ and require equality since $\frac{\theta(\varepsilon)}{\delta(\varepsilon)} \rightarrow \infty$, $\theta(\varepsilon) = 0$. Then we obtain

$$\lim_{\zeta \rightarrow \infty} Y(\zeta) = \lim_{x \rightarrow 0} y_0(x)$$

the matching condition

$$\lim_{\zeta \rightarrow \infty} E (1 - e^{-2\zeta}) = \lim_{x \rightarrow 0} e^{\frac{1}{2}} e^{-\frac{x}{2}}, \quad \implies E = e^{\frac{1}{2}}$$

$$y_0(x) + Y\left(\frac{x}{\varepsilon}\right) - \lim_{x \rightarrow 0} y_0(x) = y_u(x)$$

The uniform solution

$$\begin{aligned} y_u(x) &= e^{\frac{1}{2}} e^{-\frac{x}{2}} + e^{\frac{1}{2}} \left(1 - e^{-\frac{2x}{\varepsilon}}\right) - e^{\frac{1}{2}} \\ &= e^{\frac{1}{2}} \left(e^{-\frac{x}{2}} - e^{-\frac{2x}{\varepsilon}}\right) \end{aligned}$$

7.3 Biochemical reaction kinetics

Let the differential equation be

$$\frac{df^*(t^*)}{dt^*} = ka^*(t^*)b^*(t^*)$$

Where s^*, e^*, c^*, p^* be molar concentrations of S, E, C and P at time t^* .

$$\frac{ds^*}{dt^*} = -k, \quad s^*, e^* + k_{-1}c^* \quad (1)$$

$$\frac{de^*}{dt^*} = -k_1s^*e^*0k_2 \quad (2)$$

$$\frac{dc^*}{dt^*} = k_1s^*e^* - (k_{-1} + k_2) \quad (3)$$

$$\frac{dp^*}{dt^*} = k_2c^*. \quad (4)$$

Add 2) and 3) we get

$$\begin{aligned} \frac{d}{dt^*} (e^* + c^*) &= 0 \\ e^*(t^*) + c^*(t^*) &= k \end{aligned}$$

Initial conditions

$$\begin{aligned} s^*(0) &= \bar{s}, \quad e^*(0) = \bar{e} \\ c^*(0) &= p^*(0) = 0 \end{aligned}$$

We have that

$$e^*(t) = \bar{e} - c^*(t^*)$$

1) gives out

$$\begin{aligned}\frac{ds^*}{dt^*} &= -k_1 s^* (\bar{e} - c^*) + k_1 c^* \\ \frac{ds^*}{dt^*} &= -(k_1 \bar{e}) s^* + (k_1) s^* c^* + k_{-1} c^* \\ \implies \frac{ds^*}{dt^*} &= -(k_1 \bar{e}) s^* + [k_1 s^* + k_{-1}] c^*\end{aligned}$$

$$\begin{aligned}\frac{dc^*}{dt^*} &= k_1 s^* (\bar{e} - c^*) (k_{-1} + k_2) c^* \\ \implies \frac{dc^*}{dt^*} &= (k_1 \bar{e}) - [k_1 s^* + k_{-1} + k_2] c^*\end{aligned}$$

Let the scalars be $s^* = \bar{s}s$, $c^* = \bar{e}c$, $t^* = Tt$.

$$\frac{\bar{s}}{T} s' = -(k_1 \bar{e}) \bar{s}s + [k_1 \bar{s} + k_{-1}] \bar{e}c$$

$$s' = -(Tk_1 \bar{e}) s + \left[Tk_1 \bar{e}s + k_{-1} \frac{\bar{e}T}{\bar{s}} \right] c$$

$$\text{Let } Tk_1 \bar{e} = 1 \implies T = \frac{1}{\bar{e}k_1}$$

$$s' = -s + \left[s0 \left(\frac{k_{-1}}{k_1 \bar{s}} \right) \right]$$

8 Lecture 2020-09-14

8.1 Biochemical example, kinetics

We start with the initial conditions

$$s^*(0) = \bar{s}$$

$$e^*(0) = \bar{e}$$

$$c^*(0) = 0$$

$$p^*(0) = 0$$

With the reaction equations

$$\frac{ds^*}{dt^*} = -k_1 e^* s^* + k_{-1} c^* \quad (5)$$

$$\frac{de^*}{dt^*} = -k_1 e^* s^* + k_{-1} c^* + k_2 c^* \quad (6)$$

$$\frac{dc^*}{dt^*} = k_1 e^* s^* - k_{-1} c^* - k_2 c^* \quad (7)$$

$$\frac{dp^*}{dt^*} = -k_2 c^* \quad (8)$$

We can eliminate

$$e^* + c^* = \bar{e}$$

$$e^* = \bar{e} - c^*$$

Insert into (1) .

$$\begin{aligned} \frac{ds^*}{dt^*} &= -k_1 s^* (\bar{e} - c^*) + k_{-1} c^* \\ \frac{dc^*}{dt^*} &= +k_1 s^* (\bar{e} - c^*) - (k_{-1} + k_2) c^* \end{aligned}$$

Which can be transformed to

$$\frac{ds^*}{dt^*} = -(k_1 \bar{e}) s^* + [k_1 s^* + k_{-1}] c^* \quad (9)$$

$$\frac{dc^*}{dt^*} = (k_1 \bar{e}) s^* - [k_1 s^* - (k_{-1} + k_2)] c^*. \quad (10)$$

We can then scale such that

$$s^* = \bar{s} s, \quad c^* = \bar{e} c, \quad t^* = T t$$

Using (9),

$$\begin{aligned}\frac{\bar{s}}{T}s' &= -T(k_1\bar{e})\bar{s}s + \left[Tk_1\bar{s}s + \frac{k_{-1}T}{\bar{s}}\right]\bar{e}c \\ s' &= -(T\bar{e}k_1)s + \left[(Tk_1\bar{e})s + \frac{k_{-1}T\bar{e}}{\bar{s}}\right]c\end{aligned}$$

Put $T = \frac{1}{\bar{e}k_1}$ we find

$$\implies s' = -s + \left[s + \frac{k_{-1}}{k_1\bar{s}}\right]c$$

Now seeing (10) we get

$$\begin{aligned}\bar{e}\frac{c'}{\bar{s}} &= (Tk_1\bar{e})\bar{s}s - \left[k_1\bar{s}sT + \frac{(k_{-1} + k_2)T}{\bar{s}}\right]\bar{e}c \\ \implies \left(\frac{\bar{e}}{\bar{s}}\right)c' &= s - \left[s + \frac{(k_{-1} + k_2)}{\bar{s}k_1}\right]c\end{aligned}$$

$$\frac{\bar{e}}{\bar{s}} = \varepsilon, \quad \frac{k_{-1} + k_2}{\bar{s}k_1} = k, \quad \frac{k_2}{\bar{s}k_1} = \lambda$$

We then end up with

$$\begin{aligned}s' &= -s + [s + k - \lambda]c \\ \varepsilon c' &= s - [s + k]c \\ , s(0) &= 1, c(0) = 0\end{aligned}$$

Assume that

$$\frac{\bar{e}}{\bar{s}} = \varepsilon \ll 1$$

Outer solution :

$$\begin{aligned}s &= s_0 + \varepsilon s_1 + \dots \\ c &= c_0 + \varepsilon c_1 + \dots\end{aligned}$$

Put $\varepsilon = 0$. This gives

$$\begin{aligned}0 &= s - [s + k]c \\ c &= \frac{s}{s + k}\end{aligned}$$

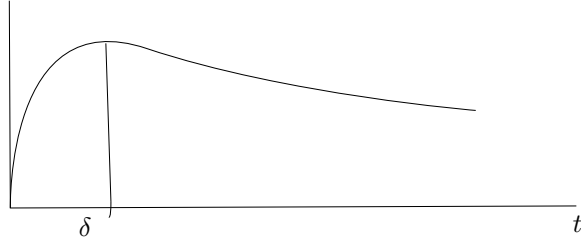


Figure 8: whines good

C is then

$$\begin{aligned}
 s' &= -s + [(s+k) - \lambda] \frac{s}{s+k} \\
 \implies s' &= -\frac{\lambda s}{s+k} \\
 \implies \left(\frac{s+k}{s} \right) ds &= -\lambda dt \\
 &\downarrow \text{Integration} \\
 \text{Outer solution } \begin{cases} s + k \ln s &= -\lambda t + K, \quad K \text{ is constant.} \\ c &= \frac{s}{s+k} \end{cases}
 \end{aligned}$$

Let us introduce

$$\begin{aligned}
 S\left(\frac{t}{\delta}\right) &= s(t), \quad \tau = \frac{t}{\delta} \\
 C\left(\frac{t}{\delta}\right) &= c(t)
 \end{aligned}$$

For the inner solution (now capital) .

$$\begin{aligned}
 \frac{1}{\delta} S' &= -S + [S+k-\lambda] C \\
 \frac{\varepsilon}{\delta} C' &= S - [S+k] C
 \end{aligned}$$

To retain $(\frac{\varepsilon}{\delta}C')$ we choose $\delta = \varepsilon$. This gives

$$S' = \varepsilon (-S + [S + k - \lambda] C)$$

$$C' = S - [S + k] C$$

If we let $\varepsilon = 0$: $S' = 0$. So we have that $S(\tau) = L$, but $s(0) = 1$, means $S(0) = 1$

$$S(\tau) = 1$$

This gives $C' = 1 - [1 + k] C$, with the solution

$$C(\tau) = \frac{1}{1+k} + M e^{-(1+k)\tau}$$

$$C_I(0) = 0, \quad \implies \quad C(\tau) = \frac{1}{1+k} \left[1 - e^{-(k+1)\tau} \right]$$

$$S_I(\tau) = 1$$

$$C_0(t) = \frac{S_0(t)}{S_0(t) + k}$$

$$S_0(t) + k \ln S_0(t) = -\lambda t K$$

Matching.

$$\theta(\delta) \rightarrow 0, \quad \text{when } \delta \rightarrow 0$$

$$\frac{\theta(\delta)}{\delta} \rightarrow \infty, \quad \text{when } \delta \rightarrow 0$$

$$\begin{aligned} \lim_{\delta \rightarrow 0} \begin{bmatrix} S^I \left(\theta(\delta) \frac{1}{\delta} \right) \\ C^I \left(\frac{\theta(\delta)}{\delta} \right) \end{bmatrix} &= \lim_{\delta \rightarrow 0} \begin{bmatrix} S_0 \\ C_0(\theta(\delta)) \end{bmatrix} \\ &\implies \lim_{\tau \rightarrow 0} \begin{bmatrix} S_I(\tau) \\ C_I(\tau) \end{bmatrix} \end{aligned}$$

$$\lim_{t \rightarrow 0} \begin{bmatrix} S_0(t) \\ C_0(t) \end{bmatrix} = \lim_{\tau \rightarrow \infty} \begin{bmatrix} 1 \\ \frac{1}{1+k} (1 - e^{-(1+k)\tau}) \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{1}{1+k} \end{bmatrix}$$

Uniform solution

$$\begin{aligned} \begin{bmatrix} S_u \\ C_u \end{bmatrix} &= \begin{bmatrix} S_0(t) \\ C_0(t) \end{bmatrix} + \begin{bmatrix} S_I \left(\frac{t}{\varepsilon} \right) \\ C_I \left(\frac{t}{\varepsilon} \right) \end{bmatrix} - \begin{bmatrix} 1 \\ \frac{1}{1+k} \end{bmatrix} \\ &= \begin{bmatrix} S_0(t) \\ C_0(t) - \frac{1}{1+k} e^{-(1+k)\frac{t}{\varepsilon}} \end{bmatrix} \\ &= \begin{bmatrix} S_0(t) \\ S_0(t) \frac{1}{S_0(t)+k} - \frac{1}{1+k} e^{(1+k)\frac{t}{\varepsilon}} \end{bmatrix} \end{aligned}$$

$$S'_0 + k + k \frac{S'_0}{S_0} = -\lambda$$

$$S_0(0) = 1 \implies S'_0(0) = \frac{-\lambda}{1+k} S_0(t) = 1 - \frac{\lambda}{1+k} t + O(t^2)$$

For large $\lambda t : k \ln S_0(t) \approx -\lambda t$

$$S_0(t) \approx e^{\frac{\lambda}{k}t}$$

8.2 Stability

8.2.1 Dynamical Systems

Let

$$\begin{aligned} x' &= f_1(x_1, x_2, \dots, x_n) \\ x'_2 &= f_2(x_1, x_2, \dots, x_n) \\ &\vdots \\ x'_n &= f_n(x_1, x_2, \dots, x_n) \end{aligned}$$

Where $x_j(0) = x_j^{(0)}$ are given. Write this as

$$\mathbf{x}' = \mathbf{f}(\mathbf{x}), \quad \mathbf{x}(0) = \mathbf{x}^{(0)}, \quad \mathbf{x}(t) \in \mathbb{R}$$

Example.

$$\begin{aligned} x'_1 &= -x_2, & x_1(0) &= 1 \\ x'_2 &= x_1, & x_2(0) &= 0 \end{aligned}$$

An equilibrium point for

$$\mathbf{x}' = \mathbf{f}(\mathbf{x})$$

is a constant solution. I.e. \mathbf{x}_e is an equilibrium point

$$\implies \mathbf{f}(\mathbf{x}_e) = 0$$

Definition 8.1. An equilibrium point \mathbf{x}_e is **stable** if for any $\varepsilon > 0$, there exist a $\delta > 0$ such that if

$$\|\mathbf{x}(0) - \mathbf{x}_e\| < \delta \implies \|\mathbf{x}(t) - \mathbf{x}_e\| < \varepsilon, \quad \text{for } t > 0$$

Definition 8.2. If \mathbf{x}_e is stable and, there exists a $\delta > 0$ such that always

$$\|\mathbf{x}(0) - \mathbf{x}_e\| < \delta$$

Implies

$$\lim_{t \rightarrow \infty} \mathbf{x}(t) = \mathbf{x}_e$$

Then \mathbf{x}_e is an asymptotically stable equilibrium point.

If \mathbf{x}_e is not stable, it is **unstable**.

Example.

$$x' = -x, \quad x_e = 0 \text{ is an equilibrium point.}$$

Where the solution is

$$x = Ce^{-t} \rightarrow \text{for any } C$$

8.2.2 Linearization

$$x'_j = f_j(x_1, x_2, \dots, x_n), \quad j = 1, 2, \dots, n$$

Assume \mathbf{x}_e is an equilibrium point. If f_j is differentiable we can write

$$\frac{f_j(\mathbf{x}_0 + \delta\mathbf{x}) - f_j(\mathbf{x}) + \sum_{i=1}^n \frac{\partial f_j}{\partial x_i} \Delta x_i}{\|\Delta x\|} \xrightarrow{\|\Delta x \rightarrow 0\|} 0$$

From matrix notation

$$\mathbf{f}(\mathbf{x}_1 + \Delta\mathbf{x}) = \mathbf{f}(\mathbf{x}_1) + J(\mathbf{x}_1) \Delta\mathbf{x}_1$$

Where the $n \times n$ matrix $J(\mathbf{x}_0)$ is given by

$$(J(\mathbf{x}_0))_{ij} = \frac{\partial f_j}{\partial x_i}(\mathbf{x}_0)$$

And is called the jacobian matrix of $\mathbf{f}(\mathbf{x})$ at $\mathbf{x} = \mathbf{x}_0$

9 Lecture 2020-09-16

9.1 Stability

Dynamic system

$$\mathbf{x}' = f(\mathbf{x}(t))$$

Where $\mathbf{x}(t) \in \mathbb{R}^n$. The equilibrium point $\mathbf{x}_e \implies f(\mathbf{x}_e) = 0$.

$$\mathbf{x}_e \text{ is either } \begin{cases} \text{stable} \\ \text{asymptotically stable} \\ \text{unstable} \end{cases}$$

We can determine the linear approximation if

$$\frac{\partial f_i}{\partial x_i \partial x_q}.$$

is continuous at \mathbf{x}_e . Then we have

$$\begin{aligned} f(\mathbf{x}_e + \delta \mathbf{x}) &= f(\mathbf{x}_e) + J(\mathbf{x}_e) \Delta \mathbf{x} + O(\|\Delta \mathbf{x}\|^2) \\ \implies f(\mathbf{x}_e + \Delta \mathbf{x}) &\approx J(\mathbf{x}_e) \Delta \mathbf{x}, \quad \mathbf{x}_e + \Delta \mathbf{x} = \mathbf{x} \\ \Delta \mathbf{x}' &= J(\mathbf{x}_e) \Delta \mathbf{x}, \quad \Delta \mathbf{x}(0) = 0 \end{aligned}$$

If $J(\mathbf{x}_e)$ has n linearly independent eigenvectors v_1, v_2, \dots, v_n with corresponding eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$. Then the solution to

$$\Delta \mathbf{x}' = J(\mathbf{x}_0) \mathbf{x}_0$$

Is

$$\Delta \mathbf{x}(t) = c_1 \mathbf{x}_1 e^{\lambda_1 t} + c_2 \mathbf{x}_2 e^{\lambda_2 t} + \dots + c_n \mathbf{x}_n e^{\lambda_n t}$$

Where c_1, \dots, c_n is determined by $\Delta \mathbf{x}(0)$. \mathbf{x}_e is an asymptotically stable eq. Point if $\operatorname{Re} \lambda_j < 0$ for $j = 1, 2, \dots, n$ for the system $\mathbf{x}' = f(\mathbf{x})$. If $\operatorname{Re} \lambda_k > 0$ for one k , then \mathbf{x}_e is unstable.

Example.

$$\begin{aligned} x_1 &= x_2 \\ x_2' &= -2x_1 - 2x_2 \\ \mathbf{x} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \mathbf{x}' = \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix} \\ &= A\mathbf{x}, \quad \|A - \lambda I\| = 0 \\ \implies \begin{vmatrix} -\lambda & 1 \\ -2 & -2 - \lambda \end{vmatrix} &= 0 \end{aligned}$$

Example.

$$\begin{aligned}x_1' &= x_1^2 - x_2' \\x_2' &= 2x_1 + x_2 + 3\end{aligned}$$

Solve

$$\begin{aligned}x_1^2 + x_2^2 &= 0 \\2x_1 + x_2 + 3 &= 0 \\ \implies x_2 &= \pm x_1\end{aligned}$$

when we get

- $x_2 = x_1$:

$$3x_1 + 3 = 0 \implies x_1 = -1$$

Which means $(-1, -1)$ is a eq. point.

- $x_2 = -x_1$: $2x_1 - x_1 + 3 = 0$. Which means $x_1 = -3$ and $(-3, 3)$ is a equ. point.

Let

$$\begin{aligned}A &= \begin{bmatrix} a & b \\ x & d \end{bmatrix}, \quad |A - \lambda I| = (a - \lambda)(d - \lambda) - bc \\ &= \lambda^2 - (tr A)\lambda + det A = (\lambda - \lambda_1)(\lambda - \lambda_2) \\ tr A &= a + d, \quad \lambda = a + b \\ det A &< 0, \quad \text{unstable} \\ det A &> 0, \quad tr A < 0 \implies \text{asymptotic stable}\end{aligned}$$

Back to the example

$$\begin{aligned}J(\mathbf{x}) &= \begin{bmatrix} 2x_1 & -2x_2 \\ 2 & 1 \end{bmatrix} \\ J(-1, -1) &= \begin{bmatrix} -2 & 2 \\ 2 & 2 \end{bmatrix} \implies |J| = -6, \quad \mathbf{x}_e \text{ is unstable} \\ J(-3, -3) &= \begin{bmatrix} -6 & -6 \\ 2 & 1 \end{bmatrix}, |J| = 6, \quad tr A = -5\end{aligned}$$

$$Re\lambda_1 < 0, \quad Re\lambda_2 < 0, \implies (-3, -3) \text{ asymptotically stable .}$$

9.2 Amoebae and chemotaxis

Let $\phi(x, t)$ be the amoebae concentration at position x at time t . Let A be the cross-sectional area of the tube. Then

$$\left(\int_{x_1}^{x_2} \phi(x, t) dA \right) A = \text{nr amoebae in } [x_1, x_2]$$

$$\frac{d}{dt} \left(\int_{x_1}^{x_2} \phi(x, t) dx \right) A = J(x_1, t) A - J(x_2, t) A$$

- Flux density: $J = -M \frac{\partial \phi}{\partial x} + E \frac{\partial c}{\partial x}$, where $M > 0$ motility and $E > 0$ strength of chemotaxis.
- $c(x_1 t)$ concentration of signaling substance.

$$x_2 - x_1 = \Delta x, \quad \tilde{x} \in \langle x_1, x_2 \rangle$$

$$\Rightarrow \frac{\partial}{\partial t} (\phi(\tilde{x}, t)) \Delta x + J(x_2, t) - J(x_1, t) = 0$$

$$\Delta x \rightarrow 0$$

$$\frac{\partial \phi}{\partial t} + \frac{\partial J}{\partial x} = 0$$

We can rewrite such that

$$\Rightarrow \frac{\partial \phi}{\partial t} + \frac{\partial}{\partial x} \left(-M \frac{\partial \phi}{\partial x} + E \phi \frac{\partial c}{\partial x} \right) = 0$$

$$\Rightarrow \frac{\partial \phi}{\partial t} = \frac{\partial}{\partial x} \left(M \frac{\partial \phi}{\partial x} - A \phi \frac{\partial c}{\partial x} \right)$$

Flux density for c : $J_c = -D \frac{\partial c}{\partial x}$

$$\frac{\partial c}{\partial x} + \frac{\partial J_c}{\partial x} = q_1 \phi - q_2 c$$

- q_1 strength of secretion.
- q_2 decay rate for c_1

$$\frac{\partial c}{\partial t} + \left(-D \frac{\partial^2 c}{\partial x^2} \right) = q_1 \phi - q_2 c$$

$$\frac{\partial \phi}{\partial t} = M \frac{\partial^2 \phi}{\partial x^2} - E \frac{\partial}{\partial x} \left(\phi \frac{\partial c}{\partial x} \right)$$

$$\frac{\partial c}{\partial t} = D \frac{\partial^2 c}{\partial x^2} + q_1 \phi - q_2 c$$

$$\left. \begin{array}{l} \phi(x, t) = \phi_0 \\ c(x, t) = c_0 \end{array} \right\} \text{a solution as long as: } q_1 u_0 - q_2 c_0 = 0$$

Let write

$$\begin{aligned}
\phi(x, t) &= \phi_0 + \tilde{\phi}(x, t) \\
c(x, t) &= c_0 + \tilde{c}(x, t) \\
\Rightarrow \frac{\partial \tilde{\phi}}{\partial t} &= M \frac{\partial^2 \tilde{\phi}}{\partial x^2} - E \left(\left(\phi_0 + \tilde{\phi} \right) \frac{\partial \tilde{c}}{\partial x} \right)_x \\
\frac{\partial \tilde{c}}{\partial t} &= -D \frac{\partial^2 \tilde{c}}{\partial x^2} + q_1 \tilde{c} - q_2 \tilde{c} \\
\Rightarrow \frac{\partial \tilde{\phi}}{\partial t} &= M \frac{\partial^2 \tilde{\phi}}{\partial x^2} - E \phi_0 \frac{\partial^2 \tilde{c}}{\partial x^2} - E \left(\tilde{\phi} \frac{\partial \tilde{c}}{\partial x} \right)_x
\end{aligned}$$

Linearization

$$\begin{aligned}
\frac{\partial \tilde{\phi}}{\partial t} &= M \frac{\partial^2 \tilde{\phi}}{\partial x^2} - E \phi_0 \frac{\partial^2 \tilde{c}}{\partial x^2} \\
\frac{\partial \tilde{c}}{\partial t} &= D \frac{\partial^2 \tilde{c}}{\partial x^2} + q_1 \tilde{\phi} - q_2 \tilde{c}, \quad \tilde{c}(x, 0) \text{ and } \tilde{\phi}(x, 0) \text{ is given}
\end{aligned}$$

Let

$$\tilde{\phi}(x, t) = \sum \alpha_n(t) e^{i\beta_n x}$$

$$\tilde{c}(x, t) = \sum \gamma_n(t) e^{\beta_n x n}$$

(i)

$$\sum \alpha'_n(t) e^{i\beta_n x} = \sum \left(-M \beta_n^2 \alpha_n(t) e^{i\beta_n x} + E \phi_0 \gamma_n(t) e^{i\beta_n x} \right)$$

$$\alpha'_n(t) = -M \beta_n^2 \alpha_n(t) + E \phi_0 \gamma_n(t) \beta_n^2$$

$$\gamma'_n(t) = -D \beta_n^2 \gamma_n(t) + q_1 \alpha_n(t) - q_2 \gamma_n(t)$$

$$\Rightarrow \begin{bmatrix} \alpha_n \\ \gamma_n \end{bmatrix}_t = \begin{bmatrix} M \beta_n^2 & E \phi_0 \beta_n^2 \\ q_1 & -D \beta_n^2 - q_1 \end{bmatrix} \begin{bmatrix} \alpha_n \\ \gamma_n \end{bmatrix}$$

$$tr A < 0, \quad det A = M \beta_n^2 (D \beta_n^2 + q_2) - q_1 E \phi_0 \beta_n \begin{cases} < 0, & \text{stable} \\ < 0, & \text{unstable} \end{cases}$$

Unstable when

$$det A < 0$$

$$\Rightarrow M (D \beta_n^2) + q_1 \phi_0 E < 0$$

$$q_1 > \frac{M (D \beta_n^2 + q_2)}{\phi_0 E}$$

$$\tilde{\phi} = \sum \alpha_n(0) e^{i\beta_n x}, \quad \beta_n^2 \text{ is increasing.}$$

10 Lecture 2020-09-21

10.1 Bifurcation

First, we will only consider 1-D dynamical systems in general. Given

$$\frac{du}{dt} = f(\mu, u)$$

where $\mu \in \mathbb{R}$ is a parameter. For given μ , we have equilibrium points when

$$f(\mu, u) = 0$$

If $u_e = u(\mu)$ is an equilibrium point, u_e is asymptotically stable if

$$\frac{\partial f}{\partial u}(\mu, u_e) < 0$$

and unstable if

$$\frac{\partial f}{\partial u}(\mu, u_e) > 0$$

Example. Let the problem be formulated as

$$\begin{aligned} u' &= (u-1)(\mu-u^2) = f(\mu, u) \\ f(\mu, u) &= 0, \implies u = 1 \text{ or } u^2 = \mu \end{aligned}$$

Example.

$$\frac{du}{dt} = u\mu - u^2 = \underbrace{u(\mu - u)}_{f(\mu, u)}$$

$$\frac{\partial f}{\partial u} = \mu - 2u,$$

$$\frac{\partial f}{\partial u}(\mu, u=0) = \mu$$

$$\frac{\partial f}{\partial u}(\mu, u=\mu) = -\mu$$

Definition 10.1. Implicit function theorem. Let (μ, u) have continuous derivatives around (μ_0, u_0) , where $f(\mu_0, u_0) = 0$. Then there are constants $a > 0, b > 0$ such that $f(\mu, u) = 0$ has a unique solution $u(\mu)$ for

$$\|\mu - \mu_0\| > a, \quad \|u - u_0\| > b$$

if $\frac{\partial f}{\partial u}(\mu_0, u_0) \neq 0$. Then

$$\frac{du}{d\mu} = -\frac{\frac{\partial f}{\partial \mu}}{\frac{\partial f}{\partial u}}, \quad \frac{\partial f}{\partial \mu}(\mu_0, u_0) \neq 0$$

We can find

$$\mu(u)$$

Remark. If

$$\begin{aligned} f(\mu_0, u_0) &= \frac{\partial f}{\partial \mu}(\mu_0, u_0) \\ &= \frac{\partial f}{\partial u}(\mu_0, u_0) \end{aligned}$$

Then (μ_0, u_0) is a singular point. Assume second derivatives are continuous, and not all zero. Then

$$f(\mu_0 + \Delta\mu, u_0 + \Delta u) \approx \frac{1}{2}f_{\mu\mu}(\mu_0, u_0)\Delta\mu^2 + f_{\mu u}(\mu_0, u_0)\Delta\mu\Delta u + \frac{1}{2}f_{uu}(\mu_0, u_0)\Delta u^2 = 0$$

Assume $f_{uu}(\mu_0, u_0) \neq 0$.

$$\Rightarrow \underbrace{f_{\mu\mu}}_c + 2\underbrace{f_{\mu u}}_b \left(\frac{\Delta u}{\Delta \mu}\right) + \underbrace{f_{uu}}_a \left(\frac{\Delta u}{\Delta \mu}\right)^2 = 0$$

$$4f_{\mu u}^2 - 4f_{\mu\mu}f_{uu} > 0, \quad \text{two solutions for } \frac{\Delta u}{\Delta \mu}$$

Example. Legislate:

$$f(\mu, u) = (\mu^2 + u^2)^2 - 2(\mu^2 - u^2)$$

Where $(0, 0)$ is a solution.

$$\left. \begin{aligned} \frac{\partial f}{\partial \mu} &= 2(\mu^2 + u^2) \cdot 2\mu - 2(2\mu) \\ \frac{\partial f}{\partial u} &= 2(\mu^2 + u^2) \cdot 2u + 4u \end{aligned} \right\} = 0 \text{ at } (0, 0)$$

Where $(0, 0)$ is singular.

$$f_{\mu\mu} = 12\mu^2 + 4u^2 - 4$$

$$f_{\mu u} = 8u\mu$$

$$f_{uu} = 12u^2 + 4$$

$$4 \cdot 0^2 - 4(4)(-4) \geq 0 \Rightarrow \text{????}$$

Example. Tank Reactor. Let q be inflow rate, $[q] = m^3 s^{-1}$. With a concentration c_{in} and temperature θ_{in} and out c^*, θ^* .

- c^* reactor concentration.
- θ^* Temperature.
- $c^*(0) = c_{in}$
- $\theta^*(0) = \theta_{in}$

Assume that c^* disappears at rate

$$kc^* e^{-\frac{A}{\theta^*}}.$$

The reaction generates heat given by

$$h \left(kc^* e^{\frac{A}{\theta^*}} \right)$$

where h is a constant, $[k] = s^{-1}$, $[A] = Kelvin = K$ and $[h] = Jmole^{-1}$. The concentration of reactant is as follows

$$\begin{aligned} \frac{d}{dt^*} (Vc^*) &= qc_{in} - qc^* - Vkc^* e^{-\frac{A}{\theta^*}} \\ \frac{d}{dt} (VC_v \theta^*) &= qC_v \theta_{in} - qC_v \theta^* + Vhlc^* e^{\frac{-A}{\theta^*}} \end{aligned}$$

Where C_v is heat capacity. Let start with scaling

$$\begin{aligned} c^* &= c_{in} c, & \theta^* &= \theta_{in} \theta \\ t^* &= T t \end{aligned}$$

Then is the equations

$$\begin{aligned} \frac{V}{T} c_{in} \frac{dc}{dt} &= qc_{in} - qc_{in} C - kV c_{in} e^{-\frac{A}{\theta_{in} \theta}} \\ \implies \frac{dc}{dt} &= T \left(\frac{q}{v} \right) - T \left(\frac{q}{v} \right) C - T k c \cdot e^{-\frac{A}{\theta_{in} \theta}} \end{aligned}$$

If we choose $T = \frac{V}{q}$ then is

$$\implies c' = 1 - c + \left(\frac{Vk}{q} \right) c e^{-\frac{A}{\theta_{in} \theta}}$$

To simplify, let us define $\mu = \frac{q}{kv}$ and $\gamma = \frac{A}{\theta_{in} \theta}$. We then end up with the problem

$$c' = 1 - c \frac{1}{\mu} e^{\gamma \frac{1}{\theta}}, \quad c(0) = 1$$

For the other problem is

$$\begin{aligned}\frac{1}{T}vC_v\theta_{in}\theta'(t) &= qC_v\theta_{in}\frac{1}{V} - \frac{qC_v\theta_{in}\theta}{V} + hkVc_{in}ce^{\frac{\gamma}{\theta}} \\ \implies \theta' &= 1 - \theta - \left(\frac{Vhk c_{in}}{qC_v\theta_{in}}\right)ce^{\frac{\gamma}{\theta}} \\ \theta &= 1 - \theta + \nu\frac{1}{\mu}ce^{\gamma\frac{1}{\theta}}\end{aligned}$$

Where $\nu = \frac{hc_{in}}{c_v\theta_{in}}$. We then have the equations

$$c' = 1 - c - \frac{1}{\mu}ce^{\gamma\frac{1}{\theta}} \quad (11)$$

$$\theta' = 1 - \theta + \frac{\nu}{\mu}ce^{\frac{\gamma}{\theta}}. \quad (12)$$

Multiply (11) by μ and add

$$\begin{aligned}(\mu c + \theta)' &= \mu - \mu c + 1 - \theta \\ \implies (\nu c + \theta)' &= \nu + 1 \\ h' + h &= \nu + 1\nu c + \theta = \nu + 1 \implies \nu c = \nu + 1 - \theta\end{aligned}$$

Inserted into (12) given

$$\begin{aligned}\theta' = 1 - \theta + \frac{(\nu + 1 - \theta)}{\mu}e^{-\frac{\gamma}{\theta}} &\implies \theta' = 1 - \theta + \frac{\nu + 1 - \theta}{\mu}e^{-\frac{\gamma}{\theta}} \\ \theta &= u + 1 \\ \implies u' &= -u + \frac{\nu - u}{\mu}e^{-\frac{\gamma}{u+1}}\end{aligned}$$

Where $u(0) = 0$ since $\theta(0) = 1$. Assume that γ, ν are constant while we vary $\mu = \frac{q}{V^k}$ physically

$$f(\mu, u) = -u + \frac{(\nu - u)}{\mu}e^{-\frac{\gamma}{u+1}}$$

Define $h(u) = (\nu - u)e^{-\frac{\gamma}{u+1}}$.

$$f = 0 \implies -u + \frac{h(u)}{\mu} = 0 \implies \mu u = h(u)$$

Lets check the stability

$$\begin{aligned}f(\mu, u) &= -uh(u) \cdot \frac{1}{\mu} \\ \frac{\partial f}{\partial u} &= -1 + \frac{h'(u)}{\mu} < 0 \\ \implies h'(u) &< \mu \implies \text{stability}\end{aligned}$$

11 Lecture 2020-09-23

11.1 Dynamic population models

Natural growth: $\frac{dv^*}{dt^*} = rv^*$. Ok as long as there is food and space. But in reality there are limitation to growth, so a model

$$\frac{dv^*}{dt^*} = rN^* \left(1 - \left(\frac{N^*}{K} \right)^\alpha \right), \quad K > 0$$

when $N^* \ll K$

$$\frac{dN^*}{dt^*} \approx rN^*$$

Scaling. $[r] = \frac{1}{s}$. One time scale is $T = \frac{1}{r}$. K natural scale for N^*

$$\begin{aligned} t^* &= \frac{1}{r}t, \quad N^* = KN \\ rkN' &= KrN \left(1 - \frac{KN}{K} \right) \\ \implies N' &= N(1 - N) \\ f(N) &= 0, \quad \implies N = 0, N = 1 \end{aligned}$$

Stability.

$$\begin{aligned} f'(N) &= 1 - 2N \\ f'(0) &= 1 > 0, \quad \text{unstable} \\ f'(1) &= 1 - 2 = -1 < 0, \quad \text{totally stable} \end{aligned}$$

The solution is

$$N(t) = \frac{N(0)}{N(0) + (1 - N(0))e^{-t}}$$

11.1.1 Whales and krill

Let $N^*(t^*)$ and $H^*(t^*)$ be the number of krill and whales respectively.

$$\frac{dN^*}{dt^*} = rN^* \left(1 - \frac{N^*}{K_N} \right) - \alpha_2 N^* H^* - u_N F_N N^* \quad (13)$$

$$\frac{dH^*}{dt^*} = qH^* \left(1 - \frac{H^*}{\alpha N^*} \right) - u_H F_H H^*, \quad N^* \neq 0 \quad (14)$$

$$r \ll q \implies T_N = \frac{1}{r} \ll T_H = \frac{1}{q}. \quad (15)$$

Scaling. Choose $T = T_H$ as time scale. K_N scale for N^* , αK_N scale for H^* .

$$t^* = \frac{1}{q}t, \quad N^* = K_N N, \quad H^* = \alpha K_N H$$

Using (13)

$$\begin{aligned} qN' &= rK_N \left(1 - \frac{K_N N}{K_N}\right) - \frac{\alpha_2 K_N K_r N H}{r} - u_N F_N \frac{K_N N}{r} \\ \implies \left(\frac{q}{r}\right) N' &= N(1 - N) - \left(\frac{\alpha \alpha_2 K_N}{r}\right) N H - \left(u_N F_N \frac{1}{r}\right) N \end{aligned}$$

Lets put

$$\varepsilon = \frac{q}{r}, \quad \gamma = \frac{\alpha \alpha_2 K_N}{r}, \quad f_N = \frac{u_N F_N}{r}$$

We then end up with

$$\varepsilon N' = N(1 - N) - \gamma H N - f_N N.$$

For equation (14)

$$\alpha K_N q H' = \alpha K_N q H \left(1 - \frac{\alpha K_N H}{\alpha K_N N}\right) - u_H F_H K_N \alpha H$$

Let $f_H = \frac{u_H F_H}{q}$ be

$$H' = H \left(1 - \frac{H}{N}\right) - f_H H$$

such that

$$\begin{aligned} \varepsilon N' &= N(1 - N) - \gamma H N - f_N N \\ H' &= H \left(1 - \frac{H}{N}\right) - f_H H \end{aligned}$$

Stability Properties.

$$\mathbf{f}(N, H) = \begin{bmatrix} N(1 - N) - \gamma H N - f_N N \\ H \left(1 - \frac{H}{N}\right) - f_H H \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad N \neq 0$$

Fromt he first equation

$$\begin{aligned} N(1 - N - \gamma H - f_N) &= 0 \\ H \left(1 - \frac{H}{N} - f_H\right) &= 0 \end{aligned}$$

If we llok at $H = 0$ it implies that

$$\begin{aligned} 1 - N - f_N &= 0 \\ N &= 1 - f_N \end{aligned}$$

We have now one eq. point such that

$$(1 - f_N, 0) = (N_e, H_e)$$

which can be considered as an unstable point if we introduce perturb by introducing , let say, one whale. For $H \neq 0$

$$\begin{aligned} 1 - N - \gamma H f_N &= 0 \\ 1 - \frac{H}{N} f_H &= 0 \end{aligned}$$

$$1 - N - \gamma H f_N = 0 \quad (16)$$

$$N - H - f_H N = 0. \quad (17)$$

Insert (17) we get $H = (1 - f_H) N$. And inserting (16)

$$\begin{aligned} 1 - N - \gamma (1 - f_H) N - f_N &= 0 \\ \implies N (1 + \gamma (1 - f_H)) &= 1 - f_N \\ \implies N &= \frac{1 - f_N}{1 + \gamma (1 - f_H)} \end{aligned}$$

In summary is the eq. point

$$(N_e, H_e) = \left(\frac{1 - f_N}{1 + \gamma (1 - f_H)}, \frac{(1 - f_H)(1 - f_N)}{1 + \gamma (1 - f_H)} \right)$$

Rate of krill being caught is

$$N_e f_N = \frac{f_N (1 - f_N)}{1 + \gamma (1 - f_H)} = P_N$$

P_N has maximum when $f_N = \frac{1}{2}$.

$$\mathbf{f}(N, H) = \begin{bmatrix} N(1 - N) - \gamma NH - f_N N \\ H(1 - \frac{H}{N}) - f_H H \end{bmatrix}$$

The jacobian matrix

$$J(N, H) = \begin{bmatrix} 1 - 2N - \gamma & -\gamma N \\ \frac{H^2}{N^2} & 1 - \frac{2H}{N} - f_H \end{bmatrix}$$

The first eq. point we obtained $(N_e, H_e) = (1 - f_N, 0)$

$$J(N_e, H_e) = \begin{bmatrix} \overbrace{1 - f_N - 2(1 - f_N)}^{-(1 - f_N)} & -\gamma(1 - f_N) \\ 0 & 1 - f_H \end{bmatrix}$$

Observe that

$$\begin{aligned} \det(J) &= -(1 - f_N)(1 - f_H) < 0 \\ \implies (1 - f_N, 0), &\text{ is in fact unstable.} \end{aligned}$$

However,

$$\begin{aligned} \begin{bmatrix} \tilde{N} \\ \tilde{H} \end{bmatrix}' &= J(N_e, 0) \begin{bmatrix} \tilde{N} \\ \tilde{H} \end{bmatrix} \\ \begin{bmatrix} \tilde{N}(t) \\ \tilde{H}(t) \end{bmatrix} &= c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{-(1-f_N)t} + c_2 \begin{bmatrix} 1 \\ \frac{-(1-f_N)-(1-f_H)}{\gamma(1-f_H)} e^{(1-f_H)t} \end{bmatrix} \end{aligned}$$

In the same manner we find

$$J(N_e, H_e)$$

for

$$(N_e, H_e) = \left(\frac{1-f_N}{1+\gamma(1-f_H)}, \frac{(1-f_N)(1-f_N)}{1+\gamma(1-f_H)} 1 \right)$$

And discover that $\det(J) > 0$ and $\text{trace}(J) < 0$

- Both eigenvalues have negative part
- (N_e, H_e) is asymptotic stable

Rate of whale catching

$$H_e f_H = P_H$$

p_N and p_H is the return value of each N and H

$$f(f_N, f_H) = p_N P_N + p_H P_H$$

11.2 Conservation Laws

In general we can formulate everything in n - dimensions. But we stick to $n = 3$. Three main players .

- Conserved entity: ϕ
- Flux density: \mathbf{J}
- Sinks/sources : Q

12 References