Partial Differential Equations

isakhammer

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Contents

1 Lecture 1

1.1 Praktiske Ting

- Borthwick, Introduction to Partial Differential Equations - Springer Link
- Ingen obligatoriske øvinger.

1.2 Bevaring av Konserveringslov

- **Konserveringslov**

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x} f(u) = 0$$
$$u(t,x) \quad \text{ukjent}$$
$$f \quad \text{er oppgit}$$

- **Hamilton Jacobi**
$$\frac{\partial u}{\partial t} + f\left(\frac{\partial u}{\partial x}\right) = 0$$

- **Bølgelingingen**
$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0$$

- **Varmeligningen**
$$\frac{\partial u}{\partial t} - \mathbb{H} \frac{\partial^2}{\partial x^2} = f(t,x)$$

- **Possion lingingen**

$$-\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} = f(x,y)$$
$$-\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) u = f$$

- **Korteweg - de vries**

$$\frac{\partial u}{\partial t} + \frac{\partial^3}{\partial x^3} - 6u \frac{\partial u}{\partial x} = 0$$

- **Navier Stokes**

$$\rho\left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\nabla\mathbf{u}\right) = -\nabla p + \mu\Delta\mathbf{u} + \rho\mathbf{g}$$

1.3 Notation

En generell pde kan beskrives som

$$F\left(t, x, y, \ldots, u_t, u_t, \ldots, u_y, u_{xy}, \ldots\right) = 0$$

og blir beskrevet som en partiell diffligning.

$$u_t + f'\left(u\right) u_x = 0$$

En **klassisk løsning** til en PDE av order m er en C^m- funksjon som i ligningen.

Example. Bølgeligningen $u_{tt} - c^2 u_{xx} = 0$ har en klassisk løsning $u\left(t, u\right) = f\left(x \pm ct\right)$ med $f \in C^2$ der $\left[f, f', f''\right]$ er kontinuerlig.

$$\begin{aligned}
u_t &= \pm c f'\left(x \pm ct\right) \\
u_{tt} &= f''\left(x \pm ct\right) \\
u_{xx} &= c^2 f''\left(x \pm ct\right) \\
u_{tt} &= f''\left(x \pm ct\right) \\
u_{tt} &= c^2 u_{xx}.
\end{aligned}$$

Der av løsningen

$$u\left(t, x\right) = f_1\left(x + ct\right) + f_2\left(x - ct\right)$$

1.4 PDE-Teori

- Fine løsninger
- Analyse
  - Velstilt.
    - Løsninger eksisteres
    - De er entydige.
    - De avhenger kontiuerlig av data.
  - General opp oppførsel

$$\begin{aligned}
u_t - u_{xx} \quad t > 0 \\
u\left(0, x\right) = u_0\left(x\right)
\end{aligned}$$

  - Tilnærmede løsninger (numerikk)

1.5 Kap 3, Transportligningen

$$u_t + v u_x = 0 \quad \text{der} \quad v\left(t, x\right) = 0 \quad , u\left(0, x\right) = u_0\left(x\right)$$

Som kan omskrives til

$$\begin{aligned}
&\frac{du\left(t,x\left(t\right)\right)}{dt}=u_t+\dot{x}u_x=0\\
&\quad\text{dersom}\quad \dot{x}=v\left(t,x\right)\quad\text{har entydig løsning gitt}\quad x\left(0\right)=x_0\\
&\text{forutsatt}\quad v\in C^1
\end{aligned}$$

derfor er $u\left(t,x\left(t\right)\right)=u\left(0,x\left(0\right)\right)=u_0\left(x_0\right)$. La oss definere $X\left(t,x_0\right)=x\left(t\right)$

$$\text{dersom}\quad x\quad\text{løser}\quad\begin{cases}\dot{x} & =v\left(t,x\right)\\ x\left(0\right) & =x_0\end{cases}$$

La $u\left(t,X\left(t,x_0\right)\right)=u_0\left(x_0\right)$. For a finne $u\left(t,x\right)$, løs $\left(X\left(t,x_0\right)\right)$ med hennold pa x_0 og sett inn

Example.

$$u_t+\left(at+b\right)u_x=0\quad a,b\quad\text{er kont}$$

Da er ligningen $\dot{x}+at=b$ slik at

$$\begin{aligned}
x&=\frac{1}{2}at^2+bt+c\\
x_0&=x-\frac{1}{2}at^2-bt\\
u\left(t,x\right)&=u_0\left(x-\frac{1}{2}at^2-bt\right)\\
u_t&=-\left(at+b\right)
\end{aligned}$$

2 Lecture 26/08

2.1 ODE teori

Theorem 2.1.

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) \quad \begin{cases} \dot{\mathbf{x}}(t) & = \mathbf{f}(t, \mathbf{x}(t)) \\ \mathbf{x}(0) = \mathbf{x}_0 \end{cases}$$

Anta ligningen er et apoent interval $o \in I$ slik at $\Omega\mathbb{R}^n$ er et område slik at $\mathbf{f} \in C^1(I \times \Omega; \mathbb{R}^n)$. Da finnes et største intervall J $o \in J \subseteq I$ moden funcksjon $\mathbf{x} : I \to \Omega$ som løser problemet. Videre er løsningen gitt.

Proof. Ideen er eksistense. Pcard iterasjon

$$\mathbf{x}_{k+1}(t) = \mathbf{x}_0 + \int_0^t f(\tau, \mathbf{x}_k(\tau)) \, d\tau$$

Entydighet

Kontiuerlig /derivering avhenging av dote

□

Theorem 2.2. *Anta gitt*

$$\dot{\mathbf{x}} = f(t, \mathbf{x}, \hat{c})$$

der $\mathbf{x}(0) = \mathbf{a}$

Dersom f er C^{k+1} , så vil $\mathbf{f}$ vare en C^k funksjon av $(t, \mathbf{a}, \hat{c})$

Autonome system

$$\dot{\mathbf{x}} = f(\mathbf{x})$$

Løsningkurvene blander en en-dimensional foliering av Ω. Har en kurve gjennom hvert punkt med ingen krysninger.

Ikke autonomt system

$$\dot{\mathbf{x}} = f(t, \mathbf{x})$$

Ekvivalent

$$\begin{aligned}
\dot{\tau} &= 1 \\
\dot{\mathbf{x}} &= f(\tau,) \\
\tau(0) &= 0 \\
x^{(n)} &= f\left(t, x, \dot{x}, \ldots, x^{(n-1)}\right) \\
&.
\end{aligned}$$

Hvis vi setter

$$\begin{aligned}
\mathbf{w} &= \left(x, \dot{x}, \ldots, x^{(n-1)}\right) \\
\dot{w_1} &= w_2 \\
\dot{w_2} &= w_3 \\
&\vdots \\
\dot{w_n} &= f(t, \mathbf{w})
\end{aligned}$$

$$\dot{\mathbf{w}} = F(t, \mathbf{w}).$$

2.2 Kvasilinær Ligning

$$au_x + bu_y = c$$

a, b, c er alle funksjoner av $x, y, u(x, y)$

Grafen til u er
$$\{(x,y,z) \mid z = u(x,y)\}$$
Da antar vi en løsning u , en kurve γ i (x,y) - planet.
$$(x(\tau), y(\tau))$$
Git enn løsning $u(x,y)$ får vi en kurve i T i $\mathbb{R}^3$ i $(x(\tau), y(\tau), z(\tau))$. Da ender vi opp med
$$\begin{aligned} z(\tau) &= u(x(\tau), y(\tau)) \\ \dot{z} &= \dot{x}u_x(x,y) + \dot{y}u_y(x,y) \\ \text{anta} \quad &\begin{cases} \dot{x}(\tau) = & a(x,y,u(x,y)) = a(x,y,z) \\ \dot{y}(\tau) & = b(x,y,u(x,y)) = b(x,y,z) \end{cases} \end{aligned}$$
da blir
$$\dot{z} = au_x + bu_y = x(x,y,u(x,y)) = c(x,y,z)$$
Vi får da
$$\begin{cases} \dot{x} & = a(x,y,z) \\ \dot{y} & = b(x,y,z) \\ \dot{z} & = c(x,y,z) \end{cases}$$
Som er kaldt de karakteristiske ligningene til
$$au_x + bu_y = c$$
Grafen til en løsning u er an union av løsningkurven av de karakteristiske ligningene. (karakteristikk).

Ikke-karakteristiske data for ligningen har formen
$$u(x,y) = u_0(x,y) \quad \text{for} \quad (x,y) \in \gamma$$
der γ er en kurve i $\mathbb{R}^2$, slik at
$$\{(x,y,z) \mid (x,y) \in \gamma \quad \text{og} \quad z = k(x,y)\}$$
Blir en kurve Γ i $\mathbb{R}^3$ ned en tangent som stiller en parabola med
$$(a(x,y,z), b(x,y,z), c(x,y,z))$$

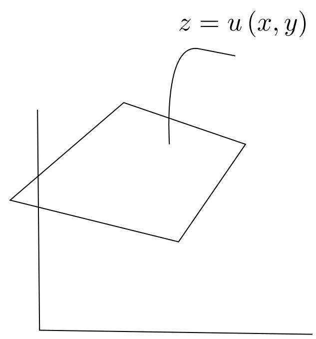

Figure 1: kurveiplan

Theorem 2.3. *Problemet har en entydig løsning $u(x,y)$ for (x,y) i et åpent område som inneholder γ .*

Konkret: Anta γ er gitt ved

$$(\chi(\sigma), \mu(\sigma), (\dot{\chi}, \dot{\mu}) \neq 0, 0)$$

Sett $\chi(\sigma) = u_0(\chi(\sigma), \mu(\sigma))$ og Γ gitt ved (χ, μ, ζ). Da løser vi k ?? med initialdata $(\chi(\sigma), \mu(\sigma), \zeta(\sigma))$. of kall resultatet

$$(x(\sigma, \tau), y(\sigma, \tau), z(\sigma, \tau))$$

Vi skal ha

$$u(x(\sigma, \tau), y(\sigma, \tau)) = z(\sigma, \tau)$$

Som er en implisitt løsning. Finn (σ, τ) skal være en funksjojn av (x, y).

Example.

$$\begin{aligned} u_t + a(t,u)\,u_x &= c(t,u) \\ u(0,x) &= u_0(x) \end{aligned}$$

Kontuerlighet slik at

$$\begin{aligned} &\dot{t} = 1, \quad t(0) = 0 \quad t(\tau) = \tau \\ &\dot{x} = a(t,x,z) \\ &\dot{z} = c(t,z) \end{aligned}$$

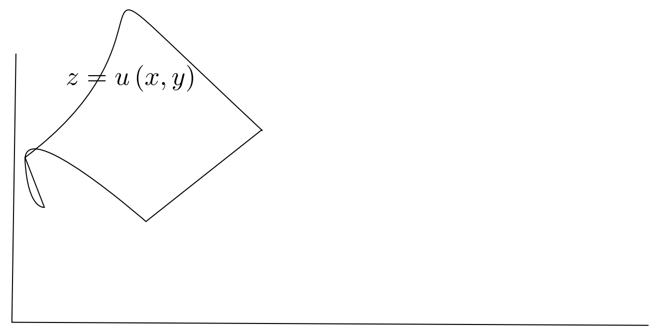

Figure 2: kurveiplan2

som kan forenkles til

$$.\begin{cases} \dot{x} & = a(t,x,z) \\ \dot{z} & = c(t,z) \end{cases}$$

Spesialtilfeller er

Tramert $a = (t,x)$

$$\text{så} \quad \begin{cases} \dot{x} & = a(t,x) \\ x(0) & = \zeta \end{cases}$$

Kan løses hver for seg of så løser vi

$$\begin{cases} \dot{z} = & c(t,x(t),z) \\ z(0) & = u_0(\zeta) \end{cases}$$

Slik at

$$\frac{Du}{Dt} = c(t,x,y) \implies u(0,\zeta) = u_0(\zeta)$$

Spesialtilfelle

$$u_t + a(u)\,u_x = 0$$
$$\dot{t} = 1, t(0) = 0, t = \tau$$
$$\dot{x} = a(z)$$
$$\dot{z} = 0$$

$x = \zeta + ta(u_0(\zeta))$ slik at $u(t,x) = u_0(\zeta)$

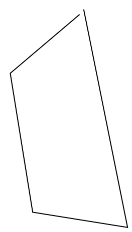

Figure 3: kdfkdkfd

Siste spesialtilfelle

$$
\begin{aligned}
&-yuu_x + xuu_y = 1, \quad u(x,0) = 0 \\
&\dot{x} = -yz \quad x(0) = \sigma \\
&\dot{y} = xz, \quad y(0) = 0 \\
&\dot{z} = 1 \quad z(0) = \sigma \quad z(\tau) = \sigma + \tau
\end{aligned}
$$

Et av resultatene er

$$
\frac{d}{d\tau}\left(x^2 + y^2\right) = 2x\dot{x} + 2y\dot{y} = 0
$$

Slik at $x^2 + y^2 = \text{konstant} = \sigma^2$.

La oss skrive

$$
\left.\begin{aligned}
x &= \sigma\cos(\phi(t)) \\
y &= \sigma\sin(\phi(t))
\end{aligned}\right\} \phi(0) = 0
$$

da er

$$
\begin{aligned}
&\dot{x} = -\sigma\sin(\phi(\tau)) \cdot \phi\dot{(\tau)} = -y\phi\dot{(\tau)} \\
&\dot{y} = \sigma\cos(\phi(\tau))\dot{\phi} = x\dot{\phi}(\tau)
\end{aligned}
$$

derfor er

$$
\dot{\phi}(\tau) = z = \sigma + \tau\phi = \sigma\tau + \frac{1}{2}\tau^2
$$

$$
\frac{1}{2}\tau^2 + \sigma\tau - \phi = 0, \quad \tau = \frac{1}{2}\left(-\sigma \pm \sqrt{\sigma^2 + 2\phi}\right)
$$

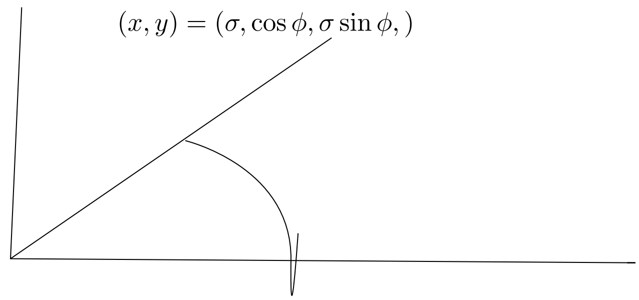

Figure 4: polarfigure

3 Lecture 02/09

3.1 Duhamds Prinsipp

$$
\begin{aligned}
u_t + c^2 u_{xx} &= f(t, x) \\
u(0, x) &= 0 \\
u_t(0, x) &= 0
\end{aligned}
$$

La $\eta_s(t, x)$ løse

$$
\begin{aligned}
\eta_{s,tt} + c^2 \eta_{s,xx} &= 0 \\
\eta_s(s, x) &= 0 \\
\eta_{s,j}(s, x) &= f(s, x)
\end{aligned}
$$

D. Alembtert

$$\eta_s(t, x) = \frac{1}{2c} \int_{x - x(t-s)} f(s, \chi)\, d\chi$$

Duhamed

$$u(t, x) = \int_0^t \eta_s(t, x)\, ds$$

Theorem 3.1. *Hvis $f \in C_1$ så vil dette løse probleme.*

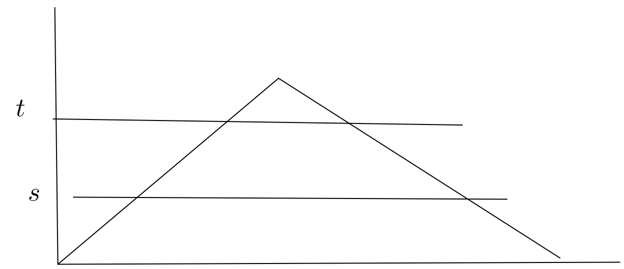

Figure 5: dkdkkd

Proof. la $u\left(0,x\right)=0$.

$$\begin{aligned} u_t &= \overbrace{\eta_t\left(t,x\right)}^{=0}+\int_0^t \eta_{s,t}\left(t,x\right)ds \\ u_t\left(0,x\right) &= 0 \\ u_{tt} &= \eta_{t,t}\left(t,x\right)+\int_0^t \eta_{s,tt}\left(t,x\right)ds \\ &= f\left(t,x\right)-c^2\int_0^t \eta_{s,xx}\left(t,x\right)ds \\ &= f\left(t,x\right)-c^2\partial_{xx}\int_0^t \eta_s\left(t,x\right)ds \\ &= f\left(t,x\right)-c^2+u_{xx}\left(t,x\right) \end{aligned}$$

We trenger $\eta\in C^2$, $\eta_{s,tt}$ og $\eta_{s,xx}$ kontinuerlig mhp. s,t,x .

$$\eta_{s,x}=\frac{1}{2c}\left(f\left(x+c\left(t-s\right)-f\left(x-c\left(t-s\right)\right)\right)\right)$$

$\eta_{s,xx},\eta_{s,tt}$ er kontinuerlig . Merk at

$$u\left(t,x\right)=\int_0^t \overbrace{\int_{x-c(t-s)}^{x+c(t-s)} f\left(s,\chi\right)d\chi}^{\eta_s(t,x)}\,ds$$

$\square$

Eksempel.

La $f(t,x) = \psi(t) \cdot h(x)$ og

$$\eta_s(t,x) = \frac{1}{2c} \int_{x-x(t-s)}^{x+c(t-s)} \psi(s) h(\chi) d\chi$$

$$=$$

$frac\psi(s)2c\left(H(x+c(t-s)) - H(x-c(t-s))\right)$

Der $\dot{H}(x) = h(x)$.

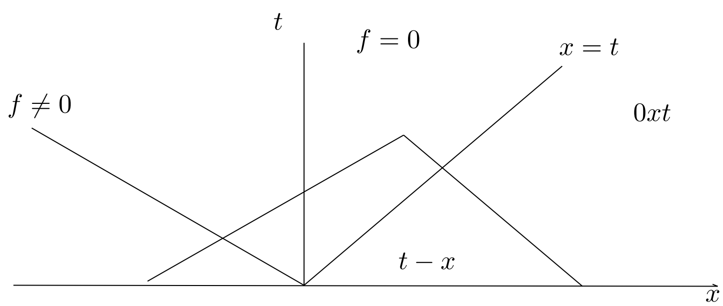

Figure 6: bølgeomraade

$$\begin{aligned}
\eta_s(t,x) &= 0 \quad \text{hvis} \quad s > t - x \\
\eta_s(t,s) &= \frac{\psi(s)}{2c}(x+t-s) \quad 0 < s < t-x \\
u(t,x) &= \int_0^{t-x} \frac{\cos s}{2}(x+t-s) ds \\
&= \frac{1}{2}\left[-s\sin s - \cos s + (t-x)\sin s\right]_{s=0}^{s=t-x} \\
&= \frac{1}{2}(1-\cos(t-x))
\end{aligned}$$

Her ble det brukt at

$$\int s\cos^2 s ds = s\sin s + \cos s$$

3.2 Randverdier

Vi kan ta bølgeligningen

$$\left.\begin{aligned} u_{tt} - c^2 u_{xx} &= 0 \\ u(0,x) &= g(x) \\ u_t(0,x) &= h(x) \\ u(t,0) &= 0 \end{aligned}\right\} x > 0$$

En forutsetning for en klassisk løsning er $g(0) = 0$ or $h(0) = 0$. Hvis ikke er ikke initalialbetingelsene konsistente.

D'Alembert

$$u(t,c) = \frac{g(x+ct) - g(x-ct)}{2} = \frac{1}{2c}\int_{x-ct}^{x+ct} h(\zeta)\, d\zeta$$

Løsning. Utvid g, h antisymmetrisk om 0

$$\begin{aligned} g(-x) &= -g(x) \\ h(-x) &= -h(x) \end{aligned}$$

Konsenvensen av antisymmentrien er at

$$u(t,-x) = -u(t,x)$$

Alternativt til randbetingelsen, også kjent som**Neumann betingelse** $u(t,0) = 0$ er

$$u_x(t,0) = 0$$

Analogien er at man har en trå festet på en ring i et stivt rør som ikke har friksjon. Da må tråen være horisontal.

Hvis vi ser på

$$\begin{aligned} &\left\{\frac{d}{dx}\left(g(x+ct) - g(x-ct)\right)\right\}_{x=0} \\ &= \dot{g}(ct) + \dot{g}(-ct) \\ &\vdots \end{aligned}$$

$$\begin{aligned} \dot{g}(ct) &= -\dot{g}(-ct) \\ g(x) &= g(-x) \\ g, h &\quad \text{symmetriske} \end{aligned}$$

$$\frac{d}{dx} h(\zeta)\, d\zeta \quad \text{for} \quad x = 0$$

Utvid g (og h) antisymmetrisk om $0, L$

$$\begin{aligned} g(-x) &= -g(x) \\ g(L-x) &= -g(L+x) \\ g(L+x) &= g(L-x) = g(x-L) \\ g(x+L) &= g(x-L) \quad x \to x+L \\ g(x+2L) &= g(x) \end{aligned}$$

Notater om kuler

$$\begin{aligned} B_r(\mathbf{a}) &= \{x \in \mathbb{R}^n \mid \|\mathbf{x}-\mathbf{a}\| < r\} \\ \overline{B}_r(\mathbf{a}) &= \{x \in \mathbb{R}^n \mid \|\mathbf{x}-\mathbf{a}\| \leq r\} \\ \partial B_r(\mathbf{a}) &= \{x \in \mathbb{R}^n \mid \|\mathbf{x}-\mathbf{a}\| = r\} \end{aligned}$$

For integratler

$$\int_{B_r(\mathbf{a})} f(\mathbf{x})\, d^n\mathbf{x} = \int_0^r \int_{\partial B_\rho(\mathbf{a})} f(\mathbf{x})\, dS dr$$

Vi kan skrive $\mathbf{x} = \mathbf{a} + \rho\mathbf{y}$ der $\|\mathbf{y}\| = 1$ og $\mathbf{y} \in S^{n-1}$ For $f = 1$

$$\begin{aligned} \int_{B_r(\mathbf{a})} d^n\mathbf{x} &= \int_0^r \underbrace{\int_{\partial B_\rho(s)} dS}_{A_n \rho^{r-n}} dr \\ &= A_n \int_0^r \rho^{n-1} d\rho = \frac{A_n}{n} r^n \end{aligned}$$

Der A_n er volumet av $S^{n-1} = \partial B_1(\mathbf{o}) \subseteq \mathbb{R}^n$

$$A_2 = 2\pi, \quad A_3 = 4\pi, \ldots$$

$$\int_{B_r(\mathbf{a})} f(\mathbf{x})\, d^n\mathbf{x} = \int_0^r, \int_{S^{n-1}} f(\mathbf{a} + \rho\mathbf{y})\, dS \rho^{n-1} d\rho$$

Integrasjon i generaliserte polarkoordinater.

$$\frac{d}{dr} \int_{B_r(\mathbf{a})} f(\mathbf{x})\, d^n\mathbf{x} = r^{n-1} \int_{S^{n-1}} f(\mathbf{a} + r\mathbf{y})\, dS$$

3.3 Darboux Formel

La oss ha en kule med sentrum $\mathbf{x}$ og radius ρ. Definer $\overline{f}(\mathbf{x};\rho)$ der

$$\begin{aligned}\overline{f(\mathbf{x};\rho)} &= \int_{\partial B_\rho(\mathbf{x})} f(\mathbf{y})\,dS(\mathbf{y}) := \frac{1}{A_n\rho^{n-1}}\int_{\partial B_\rho(\mathbf{x})} f(\mathbf{y})\,dS. \\ &= \int_{S^{n-1}} f(\mathbf{x}+\rho\mathbf{z})\,dS(\mathbf{z}) := \frac{1}{A_n}\int_{S^{n-1}}\cdots \\ \Delta\overline{f}(\mathbf{x};\rho) &= \overline{\Delta f}(\mathbf{x};\rho)\end{aligned}$$

Derbours Formel

$$\left(\rho^{n-1}\overline{f}_\rho\right)_\rho = \rho^{n-1}\Delta\overline{f}$$

4 Lecture 04/09/29

Bølgeligningen

$$u_{tt} - \nabla u = 0$$

$$u(0, \mathbf{x}) = g(\mathbf{x}), \quad u_t(0, \mathbf{x}) = h(\mathbf{x})$$

Darboux Formel. Anta $f : \mathbb{R}^n \to \mathbb{R}$

$$\begin{aligned}\overline{f}(\mathbf{x}, \rho) &= \int_{\partial B_\rho(\mathbf{x})} f(\mathbf{y})\, dS(\mathbf{y}) \\ &= \int_{f(\mathbf{x}+\rho\mathbf{z})} dS(\mathbf{z})\end{aligned}$$

$$\begin{aligned}\int_{B_\rho(\mathbf{x})} \Delta f(\mathbf{y})\, d^n\mathbf{y} &= \int_{\partial B_\rho(\mathbf{x})} \mathcal{V}\nabla f(\mathbf{y}) \\ &= A_n \rho^{n-1} \int_{S^{n-1}} \mathcal{V}\nabla f(\mathbf{x} + \rho\mathbf{z})\, dS(\xi) \\ &= A_n \rho^{n-1} \frac{\partial}{\partial \rho} f(\mathbf{x} - \rho\mathbf{z})\, dS(\mathbf{z}) \\ &= A_n \rho^{n-1} \overline{f}_\rho\end{aligned}$$

Theorem 4.1. *Darboux*

$$r^{n-1}\Delta\overline{f} = \left(r^{n-1}\overline{f}_\rho\right)_\rho.$$

Ide! Sett

$$\overline{u}(t, \mathbf{x}, \rho) = ⨍_{\partial B_\rho(\mathbf{x})} u(t, \mathbf{y})\, dS(\mathbf{y}) = ⨍_{S^{n-1}} u(t, \mathbf{x} + \rho\mathbf{z})$$

Merk at

$$\left.\begin{aligned}\overline{u}_{tt} - \Delta\overline{u} &= 0 \\ \Delta\overline{u} &= \tfrac{1}{\rho^{n-1}}\left(\rho^{n-1}u_\rho\right)_\rho\end{aligned}\right\} \underbrace{\overline{u}_{tt} - \frac{1}{\rho^{n-1}}\left(\rho^{n-1}\overline{u}_\rho\right)_\rho}_{\overline{u}_{\rho\rho} + \frac{n-1}{\rho}\overline{u}_\rho} = 0$$

$$\begin{aligned}\left(\rho^k\overline{u}\right)_{\rho\rho} &= \left(\rho^k\overline{u}_\rho + k\rho^{k-1}\overline{u}\right)_\rho \\ &= \rho^k\overline{u}_{\rho\rho} + 2k\rho^{k-1}\overline{u}_\rho + k(k-1)\rho^{k-2}\overline{u}\end{aligned}$$

Da ender vi opp med

$$(\rho\overline{u})_{\rho\rho} = \rho\overline{u}_{\rho\rho} + 2k\overline{u}_\rho$$

La oss definere

$$\hat{f}(\mathbf{x},\rho) = \rho\overline{f}(\mathbf{x})$$
$$\hat{u}(t,\mathbf{x},\rho) = \rho\overline{u}(t,\mathbf{x})$$

Som gjør vi ender opp i

$$\begin{aligned}
\hat{u}_{\rho\rho} = \rho\overline{u}_{\rho\rho} + 2\overline{u}_{\rho} &= \rho\left(\overline{u}_{\rho\rho} + (n-1)\,u_{p}\right) \\
\rho\overline{u}_{tt} &= \hat{u}_{tt} \\
\rho\overline{u}_{tt} - \hat{u_{\rho\rho}} &= 0
\end{aligned}$$

Andre interessante omskriving er

$$\hat{u}(t,\mathbf{x},\partial) = \rho\overline{u}(t,\mathbf{x},\rho) = \rho \fint_{S^{n-1}} u(t,\mathbf{x}+\rho\mathbf{z})\,dS(\mathbf{z})$$

NB! $\overline{u}(t,\mathbf{x},-\rho) = \overline{u}(t,\mathbf{x},\rho) \quad \Longrightarrow \quad \hat{u}(t,\mathbf{x},-\rho) = -\hat{u}(t,\mathbf{x},\rho)\,\hat{u}_{tt} - \Delta\hat{u} = 0$

$$\boxed{\begin{gathered}
\hat{u}_{tt} - \hat{u}_{\rho\rho} = 0 \\
\hat{u}(0,\mathbf{x},\rho) = \hat{g}(\mathbf{x},\rho) \\
\hat{u}_{t}(0,\mathbf{x},\rho) = \hat{h}(\mathbf{x},\rho)
\end{gathered}}$$

dAlembert

$$\begin{aligned}
\hat{u}(t,\mathbf{x},\rho) = {} & \frac{\hat{g}(\mathbf{x},\rho-t) + \hat{g}(\mathbf{x},\rho+t)}{2} \\
& + \frac{1}{2}\int_{\rho-t}^{\rho-t} \hat{h}(\mathbf{x},\sigma)\,d\sigma
\end{aligned}$$

Ved å ta gensene

$$\begin{aligned}
u(t,x) &= \lim_{\rho\to\infty} \overline{u}(t,\mathbf{x},\rho) \\
&= \lim_{\rho\to\infty} \frac{\hat{u}(t,\mathbf{x},\rho)}{\rho} \\
&= \lim_{\rho\to\infty} \left(\frac{\hat{g}(\mathbf{x},t+\rho)\,\hat{g}(\mathbf{x},t-\rho)}{2\rho} + \frac{1}{2\rho}\int_{t-\rho}^{t+\rho} \hat{h}(\mathbf{x},s)\,ds\right) \\
&= \hat{g}_{\rho}(\mathbf{x},t) + \hat{h}(\mathbf{x},t) \\
&= \partial_{t}\left(\hat{g}(\mathbf{x},t)\right) + \hat{h}(\mathbf{x},t)
\end{aligned}$$

Theorem 4.2. *Som er kjent som Kirchoffs Integralformel for $n = 3$*

$$u(t,x) = \hat{g}(\mathbf{x},t)_{t} + \hat{h}(\mathbf{x},t)$$

. Method of descent

Fra $n=3$ til $n=2$. Problem

$$u_{tt}-\Delta u=0,\quad t>0,x\in\mathbb{R}^2$$

$$\left.\begin{aligned} u(0,\mathbf{x}) &= g(\mathbf{x}) \\ u_t(0,\mathbf{x}) &= h(\mathbf{x}) \end{aligned}\right\} x\in\mathbb{R}^2$$

La

$$\begin{aligned} u(t,(x_1,x_2,x_3)) &= u(t,x_1,x_2) \\ g(x_1,x_2,x_3) &= g(x_1,x_2) \\ h(x_1,x_2,x_3) &= h(x_1,x_2) \end{aligned}$$

Resultat

$$u(t,x)=\overbrace{\hat{g}(x,t),\hat{h}(\mathbf{x},t)}^{\text{Regulert i }\mathbb{R}^3}$$

$$\begin{aligned} \hat{g}(\mathbf{x},t) &= t\fint_{S^2} g(\mathbf{x}+\rho\mathbf{z})\,dS(\mathbf{z}) \\ &= \frac{1}{4\pi}2t\int_{S^2} g(\mathbf{x}0t\mathbf{z})\,[z_3>0]\,dS(\mathbf{z}) \\ &= \frac{1}{4\pi}\int_D g(\mathbf{x}+\mathbf{tz})\,\frac{dz_1dz_2}{\sqrt{1-z_1^2-z_2^2}} \end{aligned}$$

Parametrisert med

$$\begin{aligned} z &= (z_1,z_2)\in D \\ dS &= \frac{dz_1dz_2}{\sqrt{1-z_1^2-z_2^2}} \\ dS &= \sqrt{1+\left(\frac{\partial z_3}{\partial z_1}\right)^2\left(\frac{\partial z_3}{\partial z_2}\right)^2}dz_1dz_2 \\ \implies z_3 &= \sqrt{1-z_1^2+z_2^2} \\ \left(\frac{\partial z_3}{\partial z_1}\right)^2 &= \left(\frac{-z_1}{\sqrt{1-z_1^2+z_2^2}}\right)^2=\frac{z_1^2}{1-z_1^2+z_2^2} \\ \implies dS &= \sqrt{1+} \end{aligned}$$

Poission Integralformel for n=2

$$u(t,x)=\frac{\partial}{\partial t}\left(\int_D g(\mathbf{x}+t\mathbf{z})\,\frac{1}{\sqrt{1-\|z\|^2}}dz_1dz_2\right)+\frac{t}{2\pi}\int_D\frac{h(\mathbf{x}+t\mathbf{z})}{\sqrt{1-\|z\|^2}}dz_1dz_2$$

5 References