NTNU – Trondheim
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Examination paper for **TMA4145 Linear methods**

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Examination time (from–to): 9:00-13:00

Permitted examination support material: D: No written or handwritten material are allowed. Calculators Casio fx-82ES PLUS, Citizen SR-270X or Citizen SR-270X College, Hewlett Packard HP30S are allowed

Other information:
The exam consists of twelve questions, the order is according to the topics in the course not to the level of difficulty. All solutions should be stated in a precise and rigorous way, with any assumptions written down and arguments justified. Each solution will be graded as *rudimentary* (F), *acceptable* (E), *good* (C) or *excellent* (A). Five acceptable solutions guarantee an E; seven acceptable with at least one good a D; seven acceptable with at least five good a C; nine good with at least two excellent a B; nine good with at least seven excellent an A. These are guaranteed limits. Beyond that, the grade is based on the total achievement.

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Problem 1 Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be norms on a vector space V.

a) Show that $\|x\| = \|x\|_1 + \|x\|_2$ is also a norm and if $\{x_n\}$ is a Cauchy sequence in $(V, \|\cdot\|)$ then $\{x_n\}$ is a Cauchy sequence in $(V, \|\cdot\|_1)$.

b) Give an example of a vector space V, two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on V, and a sequence $\{x_n\}$ such that $\{x_n\}$ is a Cauchy sequence in $(V, \|\cdot\|_1)$ but not in $(V, \|\cdot\|)$, where $\|\cdot\|$ was defined in **a)**. Prove that the dimension of V has to be infinite for such an example.

Solution

a) To show that $\|\cdot\|$ is a norm we should check that it (i) is non-negative and is zero only for the zero vector, (ii) is positive homogeneous (iii) satisfies triangle inequality.
(i) We have $\|x\| = \|x\|_1 + \|x_2 \geq 0$ since $\|\cdot\|_1$ and $\|\cdot\|_2$ are norms. If $\|x\| = 0$ then $\|x\|_1 = 0$ and then $x = 0$. Also if $x = 0$ then $\|x\|_1 = \|x_2\| = 0$ and therefore $\|x\| = 0$.
(ii) For $x \in V$ and $\lambda \in \mathbb{R}(\mathbb{C})$ we have $\|\lambda x\| = \|\lambda x\|_1 + \|\lambda x\|_2 = |\lambda|\|x\|_1 + |\lambda|\|x\|_2 = |\lambda|\|x\|$.
(iii) For any $x, y \in V$, $\|x+y\| = \|x+y\|_1 + \|x+y\|_2 \leq \|x\|_1 + \|y\|_1 + \|x\|_2 + \|y\|_2 = \|x\| + \|y\|$.

Suppose now that $\{x_n\}$ is a Cauchy sequence in $(V, \|\cdot\|)$. We want to check that it is a Cauchy sequence in $(V, \|\cdot\|_1$. Note that $\|x\|_1 \leq \|x\|$ for any $x \in V$. For any $\epsilon > 0$ there exists N such that $\|x_n - x_m\| < \epsilon$ for $n, m > N$ (since $\{x_n\}$ is a Cauchy sequence in $(V, \|\cdot\|)$). Then we have also $\|x_n - x_m\|_1 \leq \|x_n - x_m\| < \epsilon$ and thus $\{x_n\}$ is a Cauchy sequence in $(V, \|\cdot\|_1)$.

b) Let V be the space of all polynomials,

$$V = \{p(t) = a_0 + a_1 t + ... + a_k t^k, a_1, ..., a_k \in \mathbb{C}\}.$$

We consider $\|p\|_1 = \max_j |a_j|$ and $\|p\|_2 = \sum_j |a_j|$. Now let $p_n(t) = \sum_{j=1}^n t^j/n$. We have

$$\|p_n - p_m\|_1 \leq \|p_n\|_1 + \|p_m\|_1 \leq \frac{1}{n} + \frac{1}{m}.$$

Then $\|p_n - p_m\|_1 \leq 2/N$ when $n, m > N$. Clearly, $\{p_n\}$ is a Cauchy sequence in $(V, \|\cdot\|_1)$. However for the norm $\|\cdot\| = \|\cdot\|_1 + \|\cdot\|_2$ we have when $n < m$

$$\|p_n - p_m\| \geq \|p_n - p_m\|_2 = n\left|\frac{1}{n} - \frac{1}{m}\right| + \frac{m-n}{m}.$$

In particular $\|p_n - p_{2n}\| \geq 1$. Thus $\{p_n\}$ is not a Cauchy sequence in $(V, \|\cdot\|)$.

If the dimension of V is finite and $\|\cdot\|_1$ and $\|\cdot\|$ are too norms on V then these norms are equivalent. It implies that there exists a constant C such that $\|x-y\| \leq C\|x-y\|_1$. Therefore any Cauchy sequence in $(V, \|\cdot\|_1)$ is also a Cauchy sequence in $(V, \|\cdot\|)$.

Problem 2 Let

$$A = \begin{bmatrix} 8 & 0 & -1 \\ -2 & 5 & 0 \\ 0 & -4 & 7 \end{bmatrix}, \quad b = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$$

a) Find an LU-decomposition of A and solve the linear system $Ax = b$.

b) Rewrite the system $Ax = b$ in the form $x = Bx + c$ such that $B : \mathbb{R}^3 \to \mathbb{R}^3$ is a contraction in the norm $\|x\|_\infty = \max\{|x_1|, |x_2|, |x_3|\}$, $x = (x_1, x_2, x_3) \in \mathbb{R}^3$. Show how the new system may be solved by iteration starting from any $x_0 \in \mathbb{R}^3$.

Solution

a) We perform the Gauss elimination on A

$$A = \begin{bmatrix} 8 & 0 & -1 \\ -2 & 5 & 0 \\ 0 & -4 & 7 \end{bmatrix} \to \begin{bmatrix} 8 & 0 & -1 \\ 0 & 5 & -0.25 \\ 0 & -4 & 7 \end{bmatrix} \to \begin{bmatrix} 8 & 0 & -1 \\ 0 & 5 & -0.25 \\ 0 & 0 & 6.8 \end{bmatrix} = U$$

The row operations we used were: (1) add 1/4th of the first row to the second and (2) 4/5th of the second row to the third. Thus the L-matrix in LU decomposition is

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -0.25 & 1 & 0 \\ 0 & -0.8 & 1 \end{bmatrix}$$

Now we can solve the system $Ax = b$ by solving first $Ly = b$ and then $Ux = y$. We have

$$\begin{bmatrix} 1 & 0 & 0 \\ -0.25 & 1 & 0 \\ 0 & -0.8 & 1 \end{bmatrix} y = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \quad \Rightarrow \quad y = \begin{bmatrix} 2 \\ 3.5 \\ 6.8 \end{bmatrix}$$

Finally,

$$\begin{bmatrix} 8 & 0 & -1 \\ 0 & 5 & -0.25 \\ 0 & 0 & 6.8 \end{bmatrix} x = \begin{bmatrix} 2 \\ 3.5 \\ 6.8 \end{bmatrix} \quad \Rightarrow \quad x = \begin{bmatrix} 0.375 \\ 0.75 \\ 1 \end{bmatrix}$$

b) We rewrite the system $Ax = b$ in the form

$$\begin{aligned} 8x_1 - x_3 &= 2 \\ -2x_1 + 5x_2 &= 3 \\ -4x_2 + 7x_3 &= 4 \end{aligned}$$

It is equivalent to

$$\begin{aligned} x_1 &= 1/8x_3 + 1/4 \\ x_2 &= 2/5x_1 + 3/5 \\ x_3 &= 4/7x_2 + 4/7 \end{aligned}$$

The last system has the form $x = Bx + c$, where

$$B = \begin{bmatrix} 0 & 0 & 1/8 \\ 2/5 & 0 & 0 \\ 0 & 4/7 & 0 \end{bmatrix}, \quad c = \begin{bmatrix} 1/4 \\ 3/5 \\ 4/7 \end{bmatrix}$$

We have $B(x_1, x_2, x_3)^T = (x_3/8, 2x_1/5, 4x_2/7)$ and

$$\|Bx\|_\infty \le \max\{1/8, 2/5, 4/7\}\|x\|_\infty = 4/7\|x\|_\infty.$$

Therefore $B : \mathbb{R}^3 \to \mathbb{R}^3$ is a contraction in the norm $\|\cdot\|_\infty$. Further $x \mapsto Bx + c$ is also a contraction since $\|Bx + c - (By + c)\| = \|Bx - By\| \le 4/7\|x - y\|$. The space $\mathbb{R}^3$ with the norm $\|\cdot\|_\infty$ is complete, thus by the Banach fixed point theorem there exists a unique solution to the equation $x = Bx + c$. It could be found as the limit of the sequence x_n, where $x_0 \in \mathbb{R}^3$ is arbitrary and $x_{n+1} = Bx_n + c$ for $n \ge 0$.

Problem 3

a) Let $C([0, 2] \times [0, 2], \mathbb{R})$ be an inner-product space with

$$\langle f, g\rangle = \int_0^2 \int_0^2 f(x, y)g(x, y)dxdy.$$

Find an orthogonal basis for $\operatorname{span}\{1, x, y\}$ in this space.

b) Find $a, b, c \in \mathbb{R}$ such that $\int_0^2 \int_0^2 |xy - a - bx - cy|^2 dxdy$ is minimal.

Solution

a) We apply the Gram-Schmidt algorithm to find an orthogonal basis for the subspace $W = \text{span}\{1, x, y\}$. We have $v_1 = 1$,

$$\langle x, 1\rangle = \int_0^2 \int_0^2 x dx dy = 2\int_0^2 x dx = 4, \quad \langle 1, 1\rangle = \int_0^2 \int_0^2 1 dx dy = 4.$$

Then $v_2 = x - \langle x, 1\rangle(\langle 1, 1\rangle)^{-2}1 = x - 1$ and

$$v_3 = y - \langle y, 1\rangle(\langle 1, 1\rangle)^{-1}1 - \langle y, x-1\rangle(\langle x-1, x-1\rangle)^{-1}x - 1 = y - 1.$$

Therefore $\{1, x-1, y-1\}$ is an orthogonal basis for $\text{span}\{1, x, y\}$.

b) We want to find the orthogonal projection of the function $f(x, y) = xy$ onto the subspace W generated by $\{1, x, y\}$. This orthogonal projection is of the form $a + bx + cy$ and provides the minimal to

$$\|f - a - bx - cy\|_2 = \left(\int_0^2 \int_0^2 |f(x, y) - a - bx - cy|^2 dx dy\right)^{1/2}.$$

We have the orthogonal basis for W, $\{1, x-1, y-1\}$. Then the orthogonal projection satisfies

$$\text{Pr}_W(xy) = \frac{\langle xy, 1\rangle}{\langle 1, 1\rangle}1 + \frac{\langle xy, x-1\rangle}{\langle x-1, x-1\rangle}(x-1) + \frac{\langle xy, y-1\rangle}{\langle y-1, y-1\rangle}(y-1).$$

Computing the intergals,

$$\int_0^2 \int_0^2 xy dx dy = \int_0^2 x dx \int_0^2 y dy = 4,$$

$$\int_0^2 \int_0^2 y^2 dx dy = \int_0^2 \int_0^2 x^2 dx dy = 2\int_0^2 x^2 dx = 16/3,$$

$$\int_0^2 \int_0^2 xy^2 dx dy = \int_0^2 \int_0^2 x^2 y dx dy = \int_0^2 x^2 dx \int_0^2 y dy = 16/3,$$

we obtain $\langle xy, 1\rangle = 4$, $\langle xy, x-1\rangle = \langle xy, y-1\rangle = 16/3 - 4 = 4/3$ and $\langle x-1, x-1\rangle = \langle y-1, y-1\rangle = 16/3 - 8 + 4 = 4/3$. Finally,

$$\text{Pr}_W(xy) = 1 + (x-1) + (y-1) = x + y - 1.$$

The answer is $a = -1, b = 1, c = 1$. (It is easy to check that $xy + 1 - x - y$ is orthogonal to W.)

Problem 4

a) Let M be a closed subspace of a Hilbert space H. For each $x \in H$ denote by $P_M(x)$ the orthogonal projection of x onto M. Prove that $P_M^2 = P_M$, $P_M^* = P_M$ and $\|P_M\| = 1$.

b) Let H be a Hilbert space and $P : H \to H$ be a bounded linear transformation that satisfy $P = P^*$ and $P^2 = P$. Prove that P is the orthogonal projection on some closed subspace M of H.

Solution

a) First, if $v \in M$ then $v = v + \mathbf{0}$ and $P_M(v) = v$ by the projection theorem ($v \in M$, $\mathbf{0} \in M^\perp$). By the definition of the projection $P_M(x) = v \in M$, then $P_M(v) = v$ and $P_M(P_M(x)) = P_M(x)$.

For any $x, y \in H$ let $x = P_M x + u$ and $y = P_m y + w$, where $u, w \in M^\perp$. Then

$$\langle P_M x, y\rangle = \langle P_M x, P_M y + w\rangle = \langle P_M x, P_M y\rangle = \langle P_M x + u, P_M y\rangle = \langle x, P_M y\langle.$$

Thus $P_M^* = P_M$.

By the Pythagoras theorem $\|x\|^2 = \|P_M x\|^2 + \|x - P_M x\|^2$ since $P_M x$ and $x - P_M x$ are orthogonal. Thus $\|P_M x\| \le \|x\|$ and $\|P_M\| \le 1$. If $M \neq \{0\}$ then there exists $v \in M$, $v \neq 0$ such that $P_M v = v$ and therefore $\|P_M\| = 1$.

b) Suppose that $P : H \to H$ is bounded linear and $P^2 = P$. Let $M = P(H)$ be the image of P. Then M is a subspace of H, $P(y) = y$ for any $y \in M$. Further, since $P^* = P$, we have

$$\|Px\|^2 = \langle P(x), P(x)\rangle = \langle x, P(P(x))\rangle = \langle x, P(x)\rangle \le \|x\|\|Px\|$$

by the Cauchy-Schwarz inequality. Thus $\|Px\| \le \|x\|$ and $\{y : P(y) = y\}$ is a closed subspace ($y_n \to y$ and $P(y_n) = y_n$ implies $P(y) = y$).

Further for any $y \in M$ we have $\langle Px, y\rangle = \langle x, Py\rangle = \langle x, y\rangle$. Thus $P(x) - x \in M^\perp$. We get $x = P(x) + (x - P(x))$, $P(x) \in M$ and $x - P(x) \in M^\perp$. Thus by the orthogonal projection theorem $P(x) = P_M(x)$.

Problem 5 Let X, Y be Banach spaces and $T : X \to Y$ be a bounded linear transformation.

a) Prove that the kernel of T is a closed subspace of X.

b) Give an example of two Banach spaces X and Y and a bounded linear transformation T for which the range of T is not closed.

Solution

a) Let $W = \ker(\mathrm{T}) = \{\mathrm{x} \in \mathrm{X} : \mathrm{Tx} = \mathbf{0}\}$. Then W is a subspace of X, if $x, y \in W$ then $T(\alpha x + \beta y) = \alpha T(x) + \beta T(y) = \mathbf{0}$ since T is linear. To show that W is closed assume that $x_n \in W$ and $x_n \to x$ in X. Since T is a bounded operator and $Tx_n = \mathbf{0}$, we get

$$\|Tx\| = \|Tx - Tx_n\| \leq \|T\|\|x - x_n\|.$$

But $\|x - x_n\|$ tends to zero as n tends to infinity. Thus $\|Tx\| = 0$, the definition of a norm implies that then $Tx = 0$ and $x \in \ker(\mathrm{T})$. Thus W is a closed subspace of X.

b) Consider $X = Y = l_\infty$ and define $Tx(j) = j^{-1}x(j)$ when $j = 1, 2, ...,$ where $x = \{x(j)\}_{j=1}^{\infty} \in l_\infty$. Then T is a linear operator from l_∞ to l_∞. A bounded sequence is mapped to a bounded sequence and T is linear, $T(\alpha x + \beta y) = \alpha T(x) + \beta T(y)$. Further, T is bounded, $\|Tx\|_\infty = \sup_j |j^{-1}x(j)| \leq \sup_j |x(j)| = \|x\|_\infty$.

We want to show that $T(X) = \mathrm{ran}(\mathrm{T})$ is not closed. Let $x_0(j) = j^{-1/2}$, $j = 1, 2, ...$ Clearly $x_0 \in l_\infty$, $\|x_0\|_\infty = 1$ and $x_0 \notin T(X)$ since the sequence $\{j^{1/2}\}$ is not bounded. Further let $x_n(j) = j^{-1/2}$ if $j \leq n$ and $x_n(j) = 0$ if $j > n$, $n = 1, 2, ...$ Then $x_n \to x_0$ in l_∞, we have $\|x_n - x_0\|_\infty = \sup_{j>n} |j^{-1/2}| = (n+1)^{-1/2} \to 0$ when $n \to \infty$. Also, $x_n = T(y_n)$ where $y_n(j) = j^{1/2}$ if $j \leq n$ and $y_n(j) = 0$ if $j > n$, $y_n \in l_\infty$. We have constructed a sequence $\{x_n\}$ such that $x_n \in T(X)$, $x_n \to x_0$ and $x_0 \notin T(X)$. Thus $T(X)$ is not closed.

Problem 6 Let

$$A = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 1 & -1 & 3 \end{bmatrix}, \quad x_0 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$$

a) Show that A has two eigenvalues $\lambda_1 = 2$ and $\lambda_2 = 3$ and find the Jordan normal form of A, determine both the matrix J and the change-of-basis matrix T in $A = TJT^{-1}$.

b) Solve the initial-value problem $\dot{x} = Ax,\ x(0) = x_0$.

Solution

a) The characteristic polynomial of A is

$$p_A(\lambda) = \det \begin{bmatrix} 2-\lambda & 1 & 0 & 0 \\ 0 & 2-\lambda & 1 & 0 \\ 0 & 0 & 3-\lambda & 0 \\ 0 & 1 & -1 & 3-\lambda \end{bmatrix} = (3-\lambda) \det \begin{bmatrix} 2-\lambda & 1 & 0 \\ 0 & 2-\lambda & 1 \\ 0 & 0 & 3-\lambda \end{bmatrix}$$

$$= (3-\lambda)^2 \det \begin{bmatrix} 2-\lambda & 1 \\ 0 & 2-\lambda \end{bmatrix} = (3-\lambda)^2(2-\lambda)^2.$$

Thus A has two eigenvalues $\lambda_1 = 3$ and $\lambda_2 = 2$ both of algebraic multiplicity two.

To find the Jordan normal form of A we first look at its eigenvectors.

$$A - 3I = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \end{bmatrix} \sim \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

thus there are two linearly independent eigenvectors corresponding to $\lambda_1 = 3$, we may choose

$$v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

For the second eigenvalue $\lambda_2 = 2$, we have

$$A - 2I = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

and there is only one eigenvector corresponding to λ_2 (all others are multiples of this one),

$$v_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

We already know that Jordan form of A is (up to the order of the blocks)

$$J = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

To find the change-of-basis matrix T it is enough to find a generalized eigenvector corresponding to $\lambda_2 = 2$, we look for v_4 such that $(A-2I)v_4 = v_3$. Applying the Gauss elimination, we get

$$[A-2I|v_3] = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

thus we may choose

$$v_4 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}.$$

Now, using v_1, v_2, v_3, v_4 we see that $Av_1 = 3v_1$, $Av_2 = 3v_2$, $Av_3 = 2v_3$ and $Av_4 = v_3 + 2v_4$. It means that the matrix of A in the basis $\{v_1, v_2, v_3, v_4\}$ is J. Thus the change-of-basis matrix is

$$T = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix}.$$

(Note that the answer here is not unique.)

b) We know that the solution to the initial-value problem $\dot{x} = Ax$, $x(0) = x_0$ is given by $x(t) = \exp(tA)(x_0)$ and $\exp(tA) = T\exp(tJ)T^{-1}$. Now, to find $\exp(tJ)$ we write $J = D + N$, where D is the diagonal matrix with values $3, 3, 2, 2$ on the main diagonal and N is a nilpotent matrix,

$$N = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

A simple calculation shows that $N^2 = 0$, then $\exp(tN) = I + tN$ and since N and D satisfy $DN = ND$ we get

$$\exp(tA) = \exp(tD + tN) = \exp(tD)\exp(tN) =$$
$$\begin{bmatrix} e^{3t} & 0 & 0 & 0 \\ 0 & e^{3t} & 0 & 0 \\ 0 & 0 & e^{2t} & 0 \\ 0 & 0 & 0 & e^{2t} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & t \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} e^{3t} & 0 & 0 & 0 \\ 0 & e^{3t} & 0 & 0 \\ 0 & 0 & e^{2t} & te^{2t} \\ 0 & 0 & 0 & e^{2t} \end{bmatrix}$$

Therefore

$$x(t) = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} e^{3t} & 0 & 0 & 0 \\ 0 & e^{3t} & 0 & 0 \\ 0 & 0 & e^{2t} & te^{2t} \\ 0 & 0 & 0 & e^{2t} \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix}^{-1} x_0.$$

We have $x(t) = T\exp(tA)T^{-1}$. Now, if we solve the system $Tc = x$ then we get $x_0 = c_1v_1 + c_2v_2 + c_3v_3 + c_4v_4$, i.e., $T^{-1}x_0 = (c_1\ c_2\ c_3\ c_4)^t$, and

$$x(t) = T \begin{bmatrix} c_1e^{3t} \\ c_2e^{3t} \\ c_3e^{2t} + c_4te^{2t} \\ c_4e^{2t} \end{bmatrix} = c_1e^{3t}v_1 + c_2e^{3t}v_2 + (c_3 + tc_4)e^{2t}v_3 + c_4e^{2t}v_4.$$

We have $c = (0, 1, 1, 1)^t$ and $x(t) = [(1 + t)e^{2t}, e^{2t}, 0, e^{3t} - e^{2t}]^t$.