TMA4183
Optimisation II
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Exercise set 1

$\boxed{1}$ The *Cantor set* is constructed by the following approach: We start with the interval $[0,1]$ and remove from it the central open interval $(1/3, 2/3)$. This results in the set $[0,1/3] \cup [2/3,1]$, which is the union of two disjoint closed intervals. From each of those intervals, we then remove again their central parts, that is, the intervals $(1/9, 2/9)$ and $(7/9, 8/9)$, and end up with the union of four disjoint intervals of length $1/9$. Again, we remove the central part of each of the subintervals and obtain a union of eight disjoint intervals of length $1/27$. This process of always removing the central part of each subinterval is then continued *ad infinitum* and the resulting set is called the *Cantor set*, denoted in the following by C.

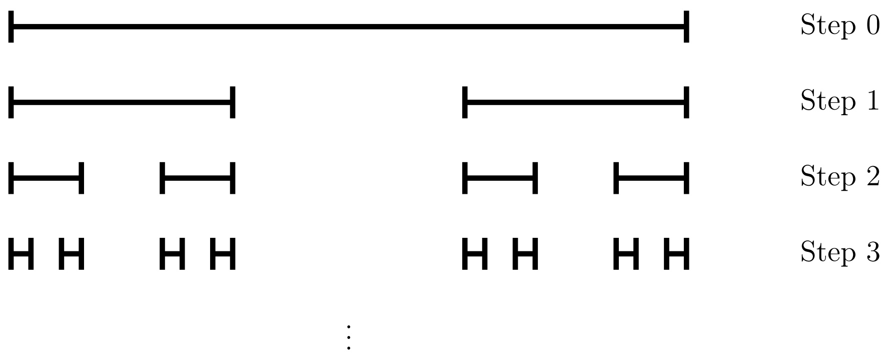

Figure 1: Sketch of the construction of the Cantor set.

a) Show that C is a closed set and that $\mathcal{L}^1(C) = 0$.

b) Show that the set C consists precisely of the reals in the unit interval that have an expansion in base 3, where none of the digits is equal to 1. In other words, show that

$$C = \Big\{ x \in [0,1] : x = \sum_{k=1}^{\infty} a_k 3^{-k} \text{ with } a_k \in \{0,2\} \text{ for all } k \Big\}.$$

c) Show that the mapping $f\colon C \to [0,1]$ defined by

$$f\Big(\sum_{k=1}^{\infty} a_k 3^{-k}\Big) = \sum_{k=1}^{\infty} \frac{a_k}{2} 2^{-k}$$

is surjective, and conclude that the cardinality of C is the same as the cardinality $\mathfrak{c}$ of the reals.

In particular, this shows that there exists uncountable sets of measure zero.

$\boxed{2}$ *(Reverse Hölder inequality)*

Assume that $0 < p < 1$ and denote by q the Hölder conjugate exponent of p, that is, $q = p/(p-1)$ (note that $q < 0!$). Let moreover u, $v\colon E \to \mathbb{R}$ be measurable functions such that $u(x) \geq 0$ and $v(x) > 0$ for almost every $x \in E$. Show that

$$\int_E uv\,dx \geq \Big(\int_E u^p\,dx\Big)^{1/p}\Big(\int_E v^q\,dx\Big)^{1/q}.$$

Hint: Write $u^p = (uv)^p v^{-p}$ and apply the Hölder inequality to $\int_E (uv)^p v^{-p}\,dx$.

$\boxed{3}$ Assume that $1 \leq p < q \leq +\infty$. Show that $L^p([0,1]) \not\subseteq L^q([0,1])$.

$\boxed{4}$ Assume that $1 \leq p < q \leq +\infty$. Show that $L^q(\mathbb{R}) \not\subseteq L^p(\mathbb{R})$.