Repetition

Definition

The stochastic variable X has a **Gaussian distribution** with **mean** $\mu \in \mathbb{R}$ and **variance** $\sigma^2 > 0$ if the probability density function is given by

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{1}{2\sigma^2}(x-\mu)^2\right), \quad x \in \mathbb{R}.$$

We will write $X \sim \mathcal{N}(\mu, \sigma^2)$.

Note: Also called a **normal distribution**.
Note 2: $\mathcal{N}(0,1)$ is called a **standard Gaussian distribution**.

Notation

We will use $\phi(\cdot)$ and $\Phi(\cdot)$ to denote the probability density function and cumulative distribution function, respectively, of $Z \sim \mathcal{N}(0,1)$. This means

$$\phi(z) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{z^2}{2}\right), \quad z \in \mathbb{R},$$
$$\Phi(z) = \int_{\infty}^{z} \phi(x)\mathrm{d}x, \quad z \in \mathbb{R}.$$

Definition

The stochastic vector $\boldsymbol{X} = (X_1, X_2, \ldots, X_n)$ has an n**-dimensional multivariate Gaussian distribution** with **mean vector** $\boldsymbol{\mu} = (\mu_1, \mu_2, \ldots, \mu_n)$ and $n \times n$ **covariance matrix** Σ if its probability density function is given by

$$f(\boldsymbol{x}) = \left(\frac{1}{\sqrt{2\pi}}\right)^n \frac{1}{|\Sigma|^{1/2}} \exp\left(-\frac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu})^{\mathrm{T}}\Sigma^{-1}(\boldsymbol{x}-\boldsymbol{\mu})\right), \quad \boldsymbol{x} \in \mathbb{R}^n.$$

We will write $\boldsymbol{X} \sim \mathcal{N}_n(\boldsymbol{\mu}, \Sigma)$.

Note: Also called a **multivariate normal distribution**.

Notation

For stochastic variables X and Y, $\mathrm{Cov}[X, Y] = \mathrm{E}[XY] - \mathrm{E}[X]\mathrm{E}[Y]$ denotes the covariance between X and Y. Note that

$$\mathrm{Cov}[X, X] = \mathrm{Var}[X].$$

For stochastic vectors $\boldsymbol{X}$ and $\boldsymbol{Y}$ of lengths m and n, respectively, $\Sigma = \mathrm{Cov}[\boldsymbol{X}, \boldsymbol{Y}] = \mathrm{E}[\boldsymbol{X}\boldsymbol{Y}^{\mathrm{T}}] - \mathrm{E}[\boldsymbol{X}]\mathrm{E}[\boldsymbol{Y}]^{\mathrm{T}}$ is an $m \times n$ matrix where

$$\Sigma_{ij} = \mathrm{Cov}[X_i, Y_j], \quad i = 1, 2, \ldots, m, \quad j = 1, 2, \ldots, n.$$

We **do not** write $\mathrm{Var}[\boldsymbol{X}]$ for a stochastic vector $\boldsymbol{X}$.

We often use the short-hand notation

$$\mathrm{Cov}[\boldsymbol{X}] = \mathrm{Cov}[\boldsymbol{X}, \boldsymbol{X}].$$

Theorem

Let $X_1, X_2, \ldots, X_n$ be stochastic variables, $a_1, a_2, \ldots, a_n \in \mathbb{R}$, and $b_1, b_2, \ldots, b_n \in \mathbb{R}$, then

$$\mathrm{Cov}\left[\sum_{i=1}^{n} a_i X_i, \sum_{j=1}^{n} b_j X_j\right] = \sum_{i=1}^{n}\sum_{j=1}^{n} a_i b_j \mathrm{Cov}[X_i, X_j].$$

This means that $\mathrm{Cov}[\cdot, \cdot]$ is bilinear.

In the most general form, one can write

$$\mathrm{Cov}[\mathbf{A}\boldsymbol{X}, \mathbf{B}\boldsymbol{Y}] = \mathbf{A}\mathrm{Cov}[\boldsymbol{X}, \mathbf{B}\boldsymbol{Y}] = \mathbf{A}\mathrm{Cov}[\boldsymbol{X}, \boldsymbol{Y}]\mathbf{B}^{\mathrm{T}},$$

where the matrices $\mathbf{A}, \mathbf{B}$ and the stochastic vectors $\boldsymbol{X}, \boldsymbol{Y}$ have compatible dimensions.

Theorem

Assume $\boldsymbol{X} \sim \mathcal{N}_n(\boldsymbol{\mu}, \Sigma)$, $\mathbf{A}$ is an $m \times n$ matrix, and $\boldsymbol{b}$ is a vector of length m. If $\boldsymbol{Y} = \mathbf{A}\boldsymbol{X} + \boldsymbol{b}$, then

$$\boldsymbol{Y} \sim \mathcal{N}_m(\mathbf{A}\boldsymbol{\mu} + \boldsymbol{b}, \mathbf{A}\Sigma\mathbf{A}^{\mathrm{T}}).$$