Mathemathical Modelling

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Contents

1 Lecture 1

1.1 Practical Information

You need to know

- Separable 1. order equations.
- Linear 1. order equations.
- 2. order linear equations with constant coefficients.

1.2 Dimensional Analysis

Basic facts

- Any physical relation has to make sense dimensionally.
- Any physical relation must be valid for any choice of fundamental units.

Remark.

Make sure remark looks better

- **Forbidden** $3m + 2kg = ?$
- $m = f(x, t)$ is legal
- e^{-t} and $s = 5t^2$, is nonsense
- **Dimension** is length, mass , energy, etc.
- **Unit** is meter, feet, year, etc

Given a variable R, we write $R = \overbrace{v(R)}^{\text{numerical value}} \underbrace{[R]}_{\text{unit}}$.

If we have a physical relation that is dimensionall correct that

$$f(R_1, R_2, \ldots, R_n) = 0 \quad \rightarrow \quad f(v(R_1), v(R_2), \ldots, v(R_n)) = 0$$

1.3 Fundamental Units

Given units $F_1, F_2, \ldots, F_m$ for fundamental if

$$F_1^{\alpha_1}, F_2^{\alpha_2}, \ldots, F_m^{\alpha m} = 0 \quad \rightarrow \quad \alpha_1 = \alpha_2 = \ldots = 0$$

This units are then independent. **Example.** The units kg, m, s are independent.

Example. In a right angle triangle with angle α and hypothenus c. We know the area A is uniquely determined by α and c

$$A = f(c, \alpha)$$

α is dimensialless since $\alpha = \frac{s}{r}$. Since A scales as the square of the length, then is

$$f(ac, \alpha) = a^2 f(c, \alpha)$$
$$c = 1 \to f(a, \alpha) = a^2 f(1, \alpha) = a^2 h(\alpha)$$

Which then ends up with the relation

$$A = a^2 h(\alpha)$$

Make corollary environmet

Lets derive $A = a^2 h(\alpha)$ somwhat differently. We know there is a relation $f(A, c, \alpha) = 0$. We want to introduce new variables.

$$\Pi_1 = \frac{A}{c^2}, \quad c = c_1, \quad \alpha = \alpha_1$$

which means $f\left(c^2\Pi_1, c, \alpha\right) = 0$ and $h(\Pi_1, \alpha, c) = 0$. h must be dimensially consistent $\to$ h must be independent of c.

$$h(\Pi_1, \alpha) = 0 \leftrightarrow \Pi_1 = k(\alpha)$$
$$\to \frac{A}{c^2} = k(\alpha) \quad \leftrightarrow \quad A = c^2 k(\alpha)^{.}$$

1.4 Trinity of the first atomic blast

We assume there is a relation

$$f(E, \rho, r, t) = 0$$

- Energy: E, $[E] = kgm^2s^{-2}$
- Mass density of air: ρ, $[\rho] = kg^{-3}$
- Radius: r , $[r] = m$
- Time: t, $[t] = s$

We choose 3 independent variables, say r, t, ρ. Also we call r, t, ρ **core variables**. Let is define a dimensionalless number Π_1 such that

$$[\Pi_1] = 0$$

The relation is now given by $h(\Pi, t, r, \rho) = 0$, where h is independent of t , r and ρ. Which in fact is $h(\Pi) = 0$, where $\Pi_1 = c$ s.t. $[c] = 1$.

Given by the definitnion is

$$\frac{Et^2}{\rho r^5} = c \quad \to \quad E = \frac{c\rho r^5}{t^2}$$

Using $\rho = 12kgm^{-3}$, $r = 110m$, $t = 6 \cdot 10^{-3}$ do we end up with the relation

$$E = c \cdot 7.5 \cdot 10^{13} J$$

1.5 Steady-state single phase flow in a uniform straight pipeline

Figure of a pipe

Pipe with flow u, length L and pressure drop Δp Then there is a relation between

- L : length, $[L] = m$
- D: diameter $[D] = m$
- u: flow rate $[u] = ms^{-1}$
- Δp: Pressure drop, $\left[\Delta kgm^{-1}s^{-2}\right]$
- μ: (Shear) viscousity $[\mu] = kgm^{-1}s^{-1}$
- ρ: mass density: $[\rho] = kgm^{-3}$
- E: Wall roughness: $[E] = m$

We have to choose 3 core variables and they are not unique. Since we have 3 independent units ρ, u, D are independent such that it can be a core variable:

$$\Pi_1 = \frac{L}{D} \quad , \quad \Pi_2 = \frac{\Delta p}{\rho u^2} \quad , \quad \Pi_3 = \frac{\rho}{\mu} \quad , \quad \Pi_4 = \frac{E}{D}$$

Then the relation is

$$f\left(\Pi_1, \Pi_2, \Pi_3, \Pi^4, \rho, D, u\right) = 0 \quad \Pi_2 = h\left(\Pi_1, \Pi_3, \Pi_4\right) \leftrightarrow \frac{\Delta p}{\rho u^2} = h\left(\Pi_1, \Pi_3, \Pi_4\right)$$

$$\rightarrow \frac{\Delta p}{u^2 \rho} = \Pi_1 k\left(\Pi_3, \Pi_4\right)$$

$$\Delta p = u^2 \rho \frac{L}{D} k\left(\frac{\rho D u}{\mu}, \frac{E}{D}\right)$$

$$\text{measure} \quad \frac{\rho D \mu}{\mu} \quad , \quad k = \frac{\Delta p D}{u^2 \rho}$$

2 Lecture 2

2.1 Practical Information

Ask for zoom meeting. ola.mahlen@ntnu.no, wednesday 13-14.

2.2 Recall

Last time did we consider steady-state single phase in a flow in a pipe.

- Assuming $f(L, \Delta p, u, \mu, D, E, \rho) = 0$ we arrive with this formula

$$\frac{\Delta p D}{u^2 \rho L} = k \left(\overbrace{\frac{\rho u D}{\mu}}^{\text{Reynhold number}}, \underbrace{\frac{E}{D}}_{\text{Relative wall roughness}} \right)$$

- Dimensionless numbers are often called **dimensionless groups**. Such numbers are independent of choice of fundamental units. They have real physical meaning. **Reynholds number** R_e essentially define what type of flow. Usually $R_e < 2000$ is it laminar flow and $R_e > 4000$ turbulent flow.

2.3 Scaling

Let a pipe have diameter D and flow rate u such that $t_v = \frac{D}{u}$. Then can we describe

$$t_\alpha = \frac{D^2}{\frac{\mu}{e}}$$

where μ is the kinematic viscosity. Then is R_e defined such that

$$R_e = \frac{t_\alpha}{t_v}$$

Assume we have the relation

$$R_1 = f(R_2, \ldots, R_m)$$

Such that it exist an

$$\Pi_1 = g(\Pi_2, \Pi_2, \ldots, \Pi_{m-k}).$$

2.4 Buckinghams Pi-Theorem

Assume we have a dimensionally valid relation $f(R_1, \ldots, R_m) = 0$ and a set of fundemental units $F_1, F_2, \ldots, F_n$ such that

$$[R_j] = F_1^{a_{j1}} F_2^{a_{j2}} \ldots F_n^{a_{jn}} \quad j = 1, 2, \ldots, m$$

This then defines the dimension matrix A given by

Table 1:

	F_1	F_2	$\dots$	F_n
R_1	a_{11}	a_{11}		a_{1n}
R_2	a_{21}	a_{21}		a_{2n}
$\vdots$		$\ddots$		
R_n	a_{m1}	$\dots$		a_{mn}

Fix better table environment.

Let $rank\left(A\right) = dim\left(row\left(A\right)\right) = k$. This translates to that we have k dimensionally independent variables. Choosing k linearly independent row vectors, corresponds to choosing core variables. Let this basis be $\mathbf{a}_{i1}, \mathbf{a}_{i2}, \dots, \mathbf{a}_{ik}$. Let the rest of the row vectors be

$$\mathbf{a}_{j_1}, \mathbf{a}_{j_2}, \dots, \mathbf{a}_{j_{m-k}}$$

Then is $\mathbf{a}_{j_r} = \sum_{s=1}^{k} C_{j_r,s} \mathbf{a}_{\mathbf{i_s}}$ where $r = 1, \dots, m-k$. We end up with the equation

$$\Pi_r = \frac{R_{j_r}}{R_{i_1}^{r_{j_r,1}} R_{j_2}^{a_{j_r,2}} \dots R_{j_k}^{a_{j_r,k}}}$$

Are dimensionally numbers.

Our relation becomes

$$g\left(\Pi_1, \dots, \Pi_{m-k}\right) = 0, \qquad \begin{cases} i_1, i_2, \dots, i_k \\ j_1, \dots, j_{m-k} \end{cases}$$

Example. Swinging pendulum
Assume there is a relation

$$f\left(w, \alpha_0, L, M, g, \right) = 0$$

where w is the frequenxy, g gravitational acceleration, M mass, α_0 the swinging angle. We can set L, M, g as core variables such that

$$\left[\frac{L}{g}\right] = s^2 \quad \rightarrow \quad \left[\frac{L}{g} w^2\right] = 1$$

$$f\left(w, \alpha_0, L, M, g\right) = 0 \implies \quad g\left(\alpha_0, \frac{Lw^2}{g}\right) = 0$$

2.5 Scaling

We have a problem at hand, usually differential equations. Then we tru to find representative scales for the various variables, and then write the equation on so-called fimensionless form. This has several advantages

- Our dimensionless variables are of order 1 .
- We get rid of a lot of physical constants.
- It makes us able to see what terms are "small" in the equation. The idea is to introduce dimensionless variables by introducing appropiate scales. If we have a stick of lenght L, we choose L as length scale i.e

$$x^* = Lx \quad \text{Where} \quad x \quad \text{is dimenionless}$$

Example. Heat flow in a rod with length L. Let $u^*(x^*, t^*)$ be the temperatur with the boundary conditions

$$u^*(0, t^*) = 0 \quad u^*(L, t^*) = 0$$

If we let the model be

$$\frac{\partial u^*}{\partial t^*} = D \cdot \frac{\partial^2 u^*}{\partial x^{*2}}, \quad u^*(0, t^*) = 0 \quad u^*(L, t^*) = 0$$
$$u^*(x^*, 0) = u_0 \sin\left(\pi \frac{x^*}{L}\right)$$

We fund the tune scale T by scales **balancing the equation** .
Let $x^* = Lx$, and $t^* = Tt$, where T is to be determined $u^* = u_0 u$. If we find $u(x, t)$, then the physical temperature is given by

$$u^*(x^*, t^*) = u_0 u\left(\frac{x^*}{L}, \frac{t^*}{T}\right)$$

We have $u(0, t) = u(1, t) = 0$

$$\frac{\partial u^*}{\partial t^*} = D\frac{\partial^2 u^*}{\partial x^{*2}} \quad \Longrightarrow \quad \frac{u_0}{T}\frac{\partial u}{\partial t} = \frac{u_0}{L^2} D \frac{\partial^2}{\partial x^2}$$
$$\leftrightarrow \frac{\partial u}{\partial t} = \left(\frac{TD}{L^2}\right)\frac{\partial^2 u}{\partial x^2} \quad \text{Balancing the equation}$$
$$\frac{TD}{L^2} = 1 \quad \Longrightarrow T = \frac{L^2}{D}$$
$$u^*(x^*, 0) = u_0 \sin\left(\pi \frac{x^*}{L}\right)$$
$$u(x, 0) = \sin(\pi x)$$

which fulfills the condition

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \quad , u(0, t) = u(1, t) = 0$$

3 Lecture 3

3.1 Recall

$$\begin{aligned}\frac{\partial u^*}{\partial t^*} &= D\frac{\partial^2 u^*}{\partial x^{*2}}\\ 0 \leq x^* &\leq L\\ x^* &= Lx\\ t^* &= Tt\\ u^* &= u_0\end{aligned}$$

We can also recall

$$u^*\left(x^*, t^*\right) = u_0 u\left(\frac{x^*}{L}, \frac{t^*}{T}\right)$$

$$\frac{u_0}{T}\frac{\partial u}{\partial t} = D\frac{u_0}{L^2} \implies \frac{\partial u}{\partial t} = \frac{TD}{L^2}\frac{\partial^2 u}{\partial x^2}$$

$$\text{Require} \quad \frac{TD}{L^2} = 1 \implies T = \frac{L^2}{D}$$

This can be generelized to

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad 0 \leq x \leq 1$$

3.2 Sinking Ball

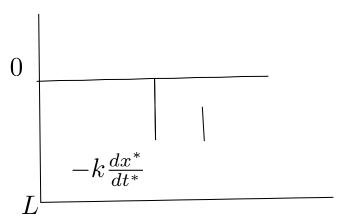

Figure 1: sinkingball

Let

- ρ_b e mass density of ball
- $\rho_f$mass density of fluid
- V Volume of ball

Then is the equation

$$\begin{aligned}\rho_b V g - \rho_f V g &= V g \rho_b \left(1 - \frac{\rho_f}{\rho_b}\right) \\ &= m\hat{g} \implies \hat{g} = g\left(1 - \frac{\rho_f}{\rho_b}\right)\end{aligned}$$

And we then end up with the newtions law

$$m\frac{dx^{*2}}{dt^{*2}} = m\hat{g} - k\frac{dx^*}{dt}, \quad \text{Friction coefficient} \quad k$$

where

$$x^*(0) = 0, \quad \frac{dx^*}{dt^*}(0) = V$$

The cases can be described as follows

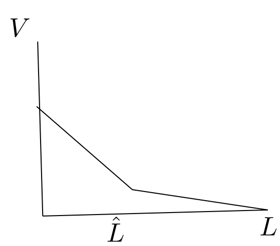

Figure 2: highV

1. High friction, not so high V. Ball will sink at constant speed most of the time.
2. Friction is low, andC not "too high". ("Free fall with V=0")
3. High V, and high friction $m\frac{d^2x^*}{dt^{*2}} = m\hat{g} - k\frac{dx^*}{dt^*}$

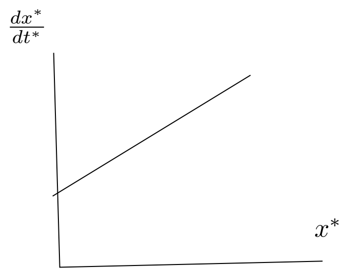

Figure 3: frefall

For this problem there is 3 characteristic speeds

1. V : initial velocity
2. v_0 : equilibrium speed in case A $v_0 = \frac{m\hat{g}}{k}$
3. v_f : free fall $v_f = \sqrt{2\hat{g}L}$

Let us put

$$\frac{d^2x^*}{dt^{*2}} = 0 \implies k\frac{dx^*}{dt} = \hat{g}m$$
$$\implies \frac{dx^*}{dt^*} = \hat{g}\frac{m}{k} = v_0$$

and put

$$x^*(0) = \frac{dx^*}{dt^*}(0) = 0$$
$$k = 0$$

3.2.1 Scaling

1. Case A: The ball sinks at constant speed "most" of the time.
   (a) Length scale L : $x^* = Lx$. Since the ball falls with speed most of the time, a timescale would be $T = \frac{L}{v_0}$. v is not much larger than v_0

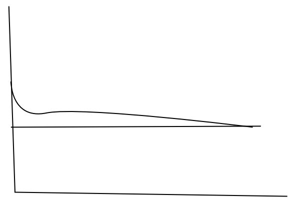

Figure 4: sinking

$\implies$ it is not so that $v \gg v_0$

$$
\begin{aligned}
m\frac{L}{T^2}x^{''} &= m\hat{g} - k\frac{L}{T}x^{'} \quad \text{Divide by } L \\
\implies m\frac{1}{kT}x^{''} &= \frac{Tm\hat{g}}{KL} - x^{'} \\
\frac{m}{k\frac{L}{v_0}}x^{''} &= \frac{\frac{k}{v_0}m\hat{g}}{kL} - x^{'} \\
\implies \frac{mv_0}{Lk}x^{''} &= \frac{Lm\hat{g}}{KLv_0} - x^{'}
\end{aligned}
$$

We can then derive

$$
\begin{aligned}
\frac{m\frac{m\hat{g}}{k}}{Lk}x^{''} &= 1 - x^{'} \\
\implies \frac{m^2\hat{g}}{Lk^2}x^{''} &= 1 - x^{'} \\
\implies \frac{m^2\hat{g}^2}{\hat{g}Lk^2}x^{''} &= 1 - x^{'} \\
\epsilon x^{''} &= 1 - x^{'} \quad \text{Where} \quad \epsilon = 2\left(\frac{v_0}{v_f}\right)^2
\end{aligned}
$$

The condition are $x(0) = 0$, $\frac{L}{T}x^{'}(0) = V$ which can be rewritten to

$$
x^{'}(0) = \frac{TV}{L})\frac{\frac{L}{v_0V}}{L} = \frac{V}{v_0} = \mu
$$

3.3 Let Analyze The equation

In case A is the

$$\epsilon\ddot{x} = 1 - \dot{x}$$

An approximation we can do is to put $\epsilon = 0$ such that

$$0 = 1 - \dot{x} \quad x(0) = 0, \quad \dot{x}(0) = \mu \quad \dot{x} = 0$$

unless $\mu = 1$, we cant find a solution.

3.3.1 Case B

Small friction, V is not too high. Let the lengthscale be L.

$$\begin{aligned} \frac{d^2}{dt^{*2}}x^{*2} &= \hat{g}, \quad x^*(0) = \frac{dx^*}{dt^*}(0) = 0 \\ x^*(t^*) &= \frac{1}{2}\hat{g}(t^*)^2 \end{aligned}$$

Hit the bottom with speed V_f . We can choose time scale T such that

$$T = \frac{L}{v_f}$$

So gain

$$\frac{mL}{T^2}\ddot{x} = m\hat{g} - \frac{kL}{T}\dot{x}$$

What you can observe is that gravity dominates so we modify the equation to be

$$\begin{aligned} \frac{L}{\hat{g}T^2}\ddot{x} &= 1 - \frac{kL}{gmT}\dot{x} \\ \implies 2\ddot{x} &= 1 - \left(\frac{v_F}{v_0}\right), \quad \frac{K}{T}\dot{x}(0) = 0 \\ 2\ddot{x} &= 1 - \epsilon\dot{x} \quad \dot{x}(0) = \frac{V}{v_f} = \mu \end{aligned}$$

3.3.2 Case C: High V and high friction

Let us consider

$$m\frac{d^2x^*}{dt^{*2}} = -kV \quad \frac{dx^*}{dt^*} = V - \frac{kV}{m}t^* = 0$$

Where we choose the scales $t^* = \frac{m}{k} = T$, $L = \frac{Vm}{k}$, where $TV = L$.

$$\implies \ddot{x} = \epsilon - \dot{x}, \quad x(0) = 1, \quad \dot{x} = 1, \quad \epsilon = \frac{v_0}{V}$$

Example. Let

$$a\frac{d^2x^*}{dt^{*2}} + b\frac{dx^*}{dt^*} + cx^* = 0$$
$$x^*(0) = x_0, \quad \frac{dx^*}{dt^*}(0) = 0$$

Three waus to scale by balancing the equation. Last term "small"

$$x^* = x_0 x, \quad t^* = Tt$$

Where T is to be determined.

$$a\frac{x_0}{T^2}\ddot{x} + b\frac{x_0}{T}\dot{x} + cx_0 = 0$$

$$\ddot{x} + \frac{bT}{a}\dot{x} + \frac{cT^2}{a} = 0$$

If we are smart can we choose the timescale $T = \frac{a}{b}$ then we get

$$\ddot{x} + \dot{x} + \frac{ca^2}{b^2a} = 0.$$
$$\implies \ddot{x} + \dot{x} + \left(\frac{ca}{b^2}\right)x = 0$$

3.4 Turbulence

Reynold number

$$R_e = \frac{u\rho L}{\mu} = \frac{uL}{\frac{mu}{\rho}} = \frac{uL}{\mathcal{V}}$$

Then we have

$$\frac{\partial v}{\partial t} = \mathcal{V}\frac{\partial^2 v}{\partial x^2}$$

4 Lecture 31/08

4.1 Turbulence

Kolmogorvs Microscales .

$$\rho \frac{du}{dt} = \mu \frac{\partial^2 u}{\partial x^2}$$

Time svale for convitive flow over a distance L

$$t_c = \frac{L}{U}, \quad U \quad \text{is velocity.}$$

This can be rearranged such that

$$\frac{\partial u}{\partial t} = \left(\frac{\mu}{\rho} \frac{\partial^2 u}{\partial x^2} \right).$$

We also define $\mathcal{V} = \frac{\mu}{\rho}$ where $[\mathcal{V}] = m^2 s^{-1}$, which is the time for dispersion of velocity.

Let $t_d = \frac{L^2}{\mathcal{V}}$ such that the Reynolds number can be written

$$R_e = \frac{v \rho L}{\mu} = \frac{UL}{\left(\frac{\mu}{\rho}\right)} = \frac{UL}{\mathcal{V}} = \frac{t_d}{t_0}$$

For water is $\mathcal{V} = 10^{-6} m^2 s^{-1}$. So for a river , put $L = 100m$ with $U = 1ms^{-1}$

$$R_e = \frac{1ms^{-1} \cdot 100m}{10^{-6} m^2 s^{-1}} = 10^8$$

Assume the generation of new whrils stops when $t_d \approx t_c \rightarrow R_e \approx 1$. Let

$$E = \frac{\text{Energy}}{\text{time per unit mass}}$$
$$[E] = kgm^2 s^{-2} s^{-1} kg$$

Let l be bthe scale of the smallest whirls and u the unit velocity then is

$$E = E\left(l, u, \mathcal{V}\right).$$

We assume that E is proportional to u^2.

$$f\left(\frac{E}{u^2}, l, \mathcal{V}\right) = 0$$

Table 2:

	m	s
$\frac{E}{n^2}$	1	0
l	1	0
v	2	-2

$$\left[\frac{\frac{E}{u^2}}{\mathcal{V}}\right] = m^{-2}$$

$$\Pi = \frac{\frac{E}{u^2}}{\mathcal{V}} l^2$$

$$\text{choose } \Pi = 1$$

$$\to E = \mathcal{V}(\frac{u^2}{l})^2$$

$$ul = \mathcal{V}$$

$$\implies k = \left(\mathcal{V}^3 \frac{1}{E}\right)^{\frac{1}{4}}, \quad u = (VE)^{\frac{1}{4}}$$

Example . Let us have $1kg$ what in a mixmaster and apply $100W$ power. then is

$$l = \left(\frac{\left(10^{-6} m^2 s^{-1}\right)^3}{100 m^2 s^{-3}}\right)^{\frac{1}{4}} = 0.01mm$$

4.2 Regular Perturbation Theory

Assume we have an equation s.t.

$$D(x,\varepsilon) = 0 \quad \text{where} \quad \varepsilon \ll 1$$

meaning that ε is small.

We have a solution $x(\varepsilon)$ to the problem $D(x,\varepsilon)$. The perturbation problem is regular if $\lim_{\varepsilon\to 0} x(\varepsilon)$ is a solution to $D(x,0)=0$. The idea is

1. Put $x(\varepsilon) = x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \ldots$

$$\begin{aligned} x(\varepsilon) &\approx x_0 \quad \text{in 0. order} \\ x(\varepsilon) &\approx x_0 + \varepsilon x_1 \quad \text{to 1. order} \end{aligned}$$

2. Insert $x(\varepsilon) = x_0 + \varepsilon x_1 + \ldots$ into $D(x,\varepsilon)$.

3. Collect all terms of order 0, all terms of order 1 so that

$$D(x,\varepsilon) = 0 \leftrightarrow \overbrace{()}^{=0} + \overbrace{()\,\varepsilon^2}^{=0} + \ldots = 0$$

Example. Let

$$x^3 + x^2 + \varepsilon x - 2 = 0, \quad \varepsilon \ll 1$$

For $\varepsilon = 0$ we have $x = 1$ as a solution. To find a solution "close to" 1 when $\varepsilon \neq 0$ we put

$$x = 1 + \varepsilon x_1 + \varepsilon^2 x_2 + O(\varepsilon)$$

Want an approximation to 2. order. We get

$$\begin{aligned} \left(1+\varepsilon x_1 + \varepsilon^2 x_2\right)^3 + \left(1+\varepsilon x_1 + \varepsilon^2 x_2\right)^2 + \varepsilon\left(1+\varepsilon x_1 + \varepsilon^2 x_2\right) - 2 &= 0 \\ \implies \varepsilon(5x_1+1) + \varepsilon^2(\ldots) &= 0 \end{aligned}$$

$$x(\varepsilon) \approx 1 - \frac{\varepsilon}{5} + \frac{\varepsilon^2}{125}$$

4.3 The Projectile Problem

Let v_0 be the vertical velocity and v_e be escape velocity such that $v_0 \ll v_e$.

Newton gravitational law

$$\mathbf{F} = -m\frac{R^2 g}{(R+x^*)^2}$$

Where g is the gravitational constand at $x^* = 0$.

Energy to move to $x^* = \infty$

$$
\begin{aligned}
-\int_0^\infty \mathbf{F} dx^* &= mgR^2 \int_0^\infty \frac{dx^*}{(R+x^*)^2} \\
&= mgR^2 \left[-\frac{1}{(R+x^2)} \right]_0^\infty \\
&= mgR = \frac{1}{2} m v_e^2 \\
\implies v_e &= \sqrt{2gR}
\end{aligned}
$$

We have

$$
m\frac{d^2x^*}{dt^{*2}} = -m\frac{gR^2}{(R+x^*)^2}
$$

Such that

$$
\frac{d^2}{dt^{*2}} = -\frac{R^2 g}{(R\ x^*)^2}, \quad x^*(0) = 0, \quad \frac{dx^*}{dt^*}(0) = v_0
$$

and $v_0 \ll v_e$, when $x^* \ll R$ (a consequence of $v_0 \ll v_e$)

$$
\frac{d^2x^*}{dt^{*2}} \approx -g \quad \frac{dx^*}{dt^*} = v_0 - t^* g = 0 \quad \leftrightarrow t^* = \frac{v_0}{g} = T = \text{timescale}
$$

$$
X^* = v_0 t^* - \frac{1}{2} t^* g \quad x^*(T) = \frac{v_0^2}{g} - \frac{1}{2}\frac{v_0^2}{g} = \frac{1}{2}\frac{v_0^2}{g}
$$

Let $L = \frac{v_0^2}{g}$ and scale the equation $\left(\frac{L}{T}\right) = v_0$ and $x^* = Lx$.

$$
\begin{aligned}
\frac{L}{T^2}\ddot{x} = \frac{-gR^2}{(R+Lx)^2} &\leftrightarrow \frac{L}{T^2}\ddot{x} = -\frac{gR^2}{R^2\left(1+\frac{L}{R}x\right)^2} \\
\rightarrow \ddot{x} = \frac{-T^2\frac{g}{L}}{\left(1+\frac{L}{R}x^2\right)} &\rightarrow \ddot{x} = \frac{-1}{(1+\varepsilon x)^2}
\end{aligned}
$$

Where

$$
\varepsilon = \frac{L}{R} = \frac{v_0^2}{Rg} = 2\frac{2v_0^2}{v_e^2}
$$

Following problem

$$
\ddot{x} = \frac{-1}{(1+\varepsilon x)^2}, \quad x(0) = 0, \quad \dot{x}(0) = 1
$$

Recall that

$$
\begin{aligned}
f(u) = \frac{1}{(1+u)^2} \rightarrow \int f(u) &= \frac{1}{1+u} + C \\
&= C - \left(1 - u + u^2 - u^3 + \ldots\right) \\
\implies f(u) &= 1 - 2u03u^2 + O\left(u^3\right)
\end{aligned}
$$

Then to second order

$$\ddot{x} = -\left(1 - 2\varepsilon x + 3\varepsilon x^2\right), \quad x(0) = 0, \quad \dot{x}(0) = q$$

Next et

$$x(t) = x_0(t) + \varepsilon x_1(t) + \varepsilon x_2(t) + O(\varepsilon)$$

So let

$$x_j(0) = 0 \quad \text{for} \quad j = 0, 1, 2$$
$$\ddot{x_0}(0) = 1, \quad \dot{x_1}(0) = \dot{x_2}(0) = 0$$
$$\rightarrow \ddot{x_0} + \varepsilon \ddot{x_1} + \varepsilon^2 \ddot{x_2} = -1 + 2\varepsilon(x_0 0 \varepsilon x_1) - 3\varepsilon^2 x_0^2$$
$$\rightarrow (\ddot{x_0} + 1) + \varepsilon(\ddot{x_1} - 2x_0) + \varepsilon^2\left(\ddot{x_2} + 2x_1 + 3x_0^2\right) = 0$$
$$\ddot{x_0} = -1 \quad x_0(0) = 0, \quad \dot{x_0} = 1$$
$$\ddot{x_1} = 2x_0, \quad x\dot{1}(0) = \dot{x_i}(0) = 0$$
$$\ddot{x_2} = 2x_1 - 3x_0^2, \quad x_2(0) = \dot{x_2}(0) = 0$$

$$\rightarrow x_0(t) = t - \frac{1}{2}tst$$
$$\ddot{x_1}(t) = 2t - t^2$$
$$\dot{x_1}(t) = t^2 - \frac{1}{3}t^3$$
$$x_1(t) = \frac{1}{3}t^3 - \frac{1}{12}t^4$$

Where

$$\ddot{x_2} = \frac{2}{3}t^3 - \frac{1}{6}t^4 - 3\left(t^2 - t^3 + \frac{1}{4}t^4\right)$$
$$x_2 = -\frac{1}{4}t^4 + \frac{11}{60}t^5 - \frac{11}{360}t^6$$

Which end up with

$$x(t) = t - \frac{1}{2}t^2 0\varepsilon\left(\frac{1}{3}t^3 - \frac{1}{12}\right) + \varepsilon^2\left(-\frac{t^4}{4}0\frac{11}{60}t^5 - \frac{11}{360}t^6\right)$$

Gives the diea of how to approx the time to the macimum height. $\dot{x}(t) = 0$ is a 5. degree equation containing ε .

Lets put

$$t = 1 + \varepsilon t_2 \varepsilon^2 t_2$$

Into the 5. degree edition and to regular perturabation

$$\rightarrow t = 1 + \frac{2}{3}\varepsilon + 2/5\varepsilon^2 + O(\varepsilon)$$

such that

$$\begin{aligned}
\ddot{x} &= \frac{-1}{(1+\varepsilon x)^2} \rightarrow \ddot{x}\dot{x} = \frac{\dot{x}}{(1+\varepsilon x)^2} \\
\rightarrow \frac{d}{dt}\left(\frac{1}{2}\dot{x}^2\right) &= \frac{d}{dt}\left(\frac{-1}{\varepsilon}\frac{1}{1+\varepsilon x}\right) \\
\frac{1}{2}\dot{x}^2 &= \frac{-1}{\varepsilon}\frac{1}{1+\varepsilon x} + C \\
\frac{1}{2} &= \frac{-1}{\varepsilon}
\end{aligned}$$

$$C = \frac{1}{2} + \frac{1}{\varepsilon}$$

where

$$\frac{1}{2}\dot{x}^2 = \frac{-1}{\varepsilon}\frac{1}{1+\varepsilon x} + \frac{1}{2} + \frac{1}{\varepsilon}$$

At maximum height $\dot{x} = 0$

$$0 = -\frac{1}{\varepsilon}.$$

5 Lecture 02/09

Let Newtons Law be

$$\frac{d^2 s^*}{dt^{*2}} = g \sin(\alpha^*) \implies \frac{d^2 \alpha^*}{dt^{*2}} = -\frac{g}{L} \sin(\alpha^*)$$

scaling:

$$\begin{aligned}
\alpha^* &= \varepsilon \alpha, \quad t^* = Tt \\
\frac{\varepsilon}{T^2} \ddot{\alpha} &= \frac{-g}{L} \sin(\varepsilon \alpha) \implies \ddot{\alpha} = -\left(T^2 g \frac{1}{L}\right) \frac{\sin(\varepsilon \alpha)}{\varepsilon} \\
T &= \sqrt{\frac{L}{g}} \implies \ddot{\alpha} = -\frac{\sin(\varepsilon \alpha)}{\varepsilon} \\
\alpha(0) &= 1 \quad \dot{\alpha}(0) = 0
\end{aligned}$$

Let put $\alpha = \alpha_0(t) + \varepsilon^2 \alpha_2(t) + O(\varepsilon^4)$. where $\alpha(t)$ is an even function of ε due to symmetry.

$$\alpha_0(0) = 1, \quad \dot{\alpha}_0(0) = 0, \quad \alpha_2(0) = \dot{\alpha_2}(0) = 0$$

Inserted into the equation

$$\begin{aligned}
\ddot{\alpha_0} + \varepsilon^2 \ddot{\alpha_2} = -\frac{\sin\left(\varepsilon\left(\alpha_0 + \varepsilon^2 \alpha_2\right)\right)}{\varepsilon} &\implies \quad \ddot{\alpha_0} + \varepsilon^2 \ddot{\alpha_2} \\
&= \frac{-1}{3}\left(\varepsilon \underbrace{\left(\alpha_0 + \varepsilon^2 \alpha_2\right)}_{u} \frac{\varepsilon^2}{6}\left(\alpha_0 + \alpha \varepsilon^2\right)\right)
\end{aligned}$$

Let

$$\begin{aligned}
\alpha_0(t) &= A \cos t + B \sin t \\
\alpha_0(0) &= 1, \quad \dot{\alpha}(0) = 0 \quad \implies \alpha_0(t) = \cos t \\
\alpha_2(t) &= A \cos t + B \sin t + \alpha_{2,f}(t) \\
\cos^3 t &= \left(\frac{1}{2}\left(e^{it} - e^{it}\right)\right)^3 = \frac{1}{8}\left(e^{i3t} + 3e^{it} 0 3 e^{-i3t}\right) \\
&= \frac{1}{4}(\cos 3t + 3 \cos t) \\
\alpha_{20}(t) &= A \cos 3t + B \sin 3t + Ct \cos t + Dt \sin t \\
\alpha_2(t) &= \frac{1}{192}(\cos t + \cos 3t) + \frac{1}{16} t \sin t \\
\alpha(t) &= \alpha_0(t) + \varepsilon_2^2(t) \quad \text{is not periodic}
\end{aligned}$$

Poincare-Lin Stel Method . Instead let

$$\alpha(t) = \alpha_0(\omega(\varepsilon)t) + \alpha_2(\omega(\varepsilon)t)\varepsilon^2 + O(\varepsilon^4)$$

Where $\omega(\varepsilon) = 1 + \omega_2\varepsilon^2\ O(\varepsilon^4)$. See exercise.

5.1 Modelling how the kidney disposes salt and water.

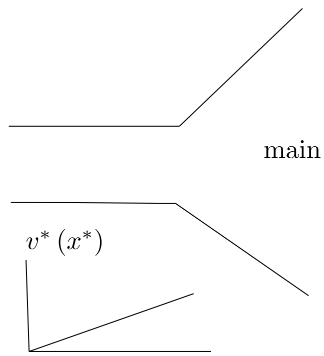

Figure 5: watermodell

Assumptions

1. Secondary channel is fed water by osmosis from the sorrouinding tissue.
2. Ions are transported down the channel by connection and diffusion.
3. Ions are fed into the channel be a chemical ppump-

We want the steady-state profiles of ion concenstration $C^*(x^*)$ and the velocity $v^*(x^*)$ of the ion water solution.

The ion concentration is written as

$$[C^*] = \frac{ions}{m^3} = \frac{osmol}{m^3}$$

One mole salt give two moles ions
Osomosis :

$$J^* = P(c^* - c_0)$$

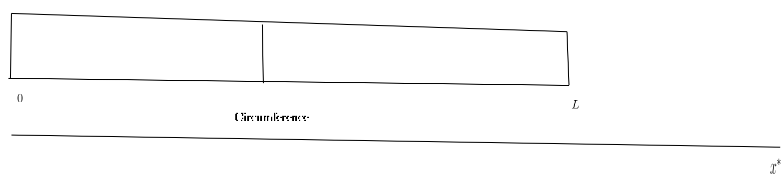

Figure 6: molefig

Is flux density of water entering the secondary channel. J^* is volume water in per area per time. c_0 ion concentration is tissue and main channel. P is called membrance permeability.

$$[P] = \frac{[J^*]}{[c^*]} = \frac{ms^{-1}}{osmol \cdot m^{-3}} = \frac{m^4}{s \cdot osmol}$$

Ion flux density

$$N^* = \begin{cases} N_0, & 0 \leq x^* \leq \delta \\ 0, & \delta \leq x^* \leq L \end{cases}$$

Where $[N_0] = \frac{osmol}{m^2 \cdot s}$. The toal rate of salt entering the channel

$$N_0 \cdot c \cdot \delta$$

Where c is the area of pump.

- The flux density of ions in the secondary channel

$$F^* = F_c^* + F_\alpha^*$$

$$[F^*] = \frac{osmol}{m^2 \cdot s}$$

- Convective flow

$$F_c^* = c^* v^*$$

- Diffusion: **Ficus law**

$$F_1^* = -D\frac{dc^*}{dx^*}.$$

where D is the diffusion of salt in water.

Conservation of water

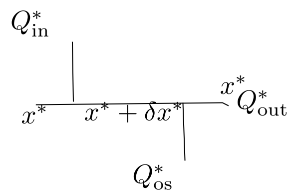

Figure 7: conssswater

$$
\begin{aligned}
Q^*_{\mathrm{out}} &= Q^*_{\mathrm{in}} + Q^*_{\mathrm{os}} \\
v^*\left(x^* + \Delta x^*\right) &= v^* A + P\left(c^*\left(\hat{x}\right) - c_0\right) c \Delta x^*, \\
&\text{where } \hat{x^*} \in \langle x^*, x^* + \Delta x^* \rangle \\
\implies \frac{v^*\left(x^* + \Delta x^*\right) - v^*\left(x^*\right)}{\Delta x^*} &= \frac{c}{A} P\left(c^*\left(\hat{x^*}\right) - c_0\right) \\
\Delta x^* \to 0 \qquad \implies \frac{dv^*}{dx^*} &= \left(\frac{cP}{A}\right)\left(c^* - c_0\right)
\end{aligned}
$$

COnservation of salt

$$
F^*\left(x^* + \Delta x^*\right) A = F^*\left(x^*\right) A + N^*\left(\hat{x^*}\right) c \Delta x^*
$$

This ends up with

$$\begin{aligned}
\implies \frac{dF^*}{dx^*} &= \frac{c}{A}N^*\left(x^*\right) \\
\text{or} \quad \frac{dF^*}{dx^*} &= \frac{c}{A}\cdot\begin{cases} N_0, & 0<x^*<\delta \\ 0, & \delta<x^*<L \end{cases} \\
F^*\left(0\right)=0 \qquad \implies F\left(x^*\right) &= \begin{cases} \frac{N_0 c}{A}x^*, & 0<x^*<\delta \\ \frac{N_0 \delta c}{A}, & \delta<x^*<L \end{cases} \\
\implies v^*c^* - D\frac{dc^*}{dx^*} &= F^*\left(x^*\right) \\
\frac{dv^*}{dt^*} &= \frac{cP}{A}\left(c^*-c_0\right) \\
v^*\left(0\right) &= 0 \\
c^*\left(L\right) &= c_0
\end{aligned}$$

Also same that v^* and c^* are continious at $x^*=\delta$.

5.1.1 Scaling the model

Two length scales δ and L. Choose δ as length svale. Natural to use c_0 as scale for c^*. The rate salt supplied is

$$N_0\delta c = c_0 U A$$

Ions supplied is convectiv flux with c^* such that $U=\frac{N_0\delta c}{c_0 A}$.

$$\begin{aligned}
x^* &= \delta, \\
c^* &= c_0 c \\
v^* &= Uv
\end{aligned}$$

1. $\left(Uc_0\right)cv - \frac{Dc_0}{\delta}\dot{c} = F^*$ such that

$$\implies vc - \frac{Dc}{\delta U c_0}\dot{c} = \frac{1}{Uc}\cdot\begin{cases} \frac{N_0 c\delta x}{AUc_0}, & 0<x\delta<\delta \\ \frac{N_0 c\delta}{Auc_0}, & \delta<x\delta<L \end{cases}$$

$$vc-\varepsilon\dot{c} = \begin{cases} x & 0<x<1 \\ 1 & 1<x<\lambda \end{cases}$$

where $\varepsilon=\frac{D}{\delta u}$, and $\lambda=\frac{L}{\delta}$

$$\implies U = \frac{N_0\delta c}{c_0 A}$$

2. $\frac{U}{\delta}\dot{v} = \frac{cP}{A}c_0\left(c-1\right)$

6 Lecture 07/09

6.1 Emergent Osmotic Concentration

(i) Total rate of salgt pumped per second $\delta c N_0$

(ii) Water out per second $v^*(L) A = Uv(\lambda) A$, where $\lambda = \frac{L}{\delta}$

$$\begin{aligned}\delta c N_0 &= C_0 U \\ &\approx \text{Flow out of salt per sec} \\ \implies U &= \frac{\delta c N_0}{C_0 A}\end{aligned}$$

Measure of the efficiency

$$\begin{aligned}\frac{\text{Salt out}}{\text{Water out}} &= Os^* \\ &= \frac{\delta c N_0}{Uv(\lambda) A} = \frac{C_0}{v(\lambda)}\end{aligned}$$

Thus $v(\lambda) > \frac{1}{4}$

6.2 Boundary Value Problem

We know that

$$\sum v'(x) = C(x) - 1$$

$$v(x) C(x) - \mu C'(x) = f(x) = \begin{cases} x, & 0 \leq x \leq \\ 1, & 1 \leq \leq \lambda \end{cases}$$

Where $v(0) = 0, \quad C(\lambda) = 1$. In addition v and C must be continuous.

Let assume $0 < \varepsilon \ll 1$. Put $C = c_0 + \varepsilon C_1 + O(\varepsilon^2)$ and $v = v_0 + \varepsilon v_1 + \emptyset(\varepsilon^2)$. Inserted into the equation

$$\begin{aligned}\varepsilon(v_0') &= C_0 + \varepsilon C_1 - 1 + O(\varepsilon^2) \\ (v_0 + \varepsilon v_1)(1 + C_1 \varepsilon)^2 - \mu(\varepsilon C_1') &= f(x) + O(\varepsilon^2) \\ C_0 - 1 = 0 &\leftrightarrow \quad C_0 = 1 \\ C_1 - v_0' = 0 \implies C_1 = v_0' &\implies \quad C_1 = f(x), \quad C_1 \text{ is discontinuity} \\ v_0 - f(x) = 0, &\quad v_0 = f(x) \\ v_1 + v_0 C_1 - \mu \varepsilon C_1' &= 0\end{aligned}$$

Something is wrong.

$$\begin{aligned} \varepsilon v' &= C - 1 \\ \varepsilon v C - \underbrace{(\varepsilon \mu)}_{\text{not small}} &= \varepsilon f(x) \end{aligned}$$

For notation convenience let

$$\begin{aligned} (\varepsilon \mu) &= \omega^{-1} \\ \varepsilon v' &= C - 1 \\ \varepsilon v C - \frac{1}{\omega^2} C' &= \varepsilon f(x) \\ &\implies \varepsilon \left(\omega^2 v C\right) - C' = \varepsilon \omega^2 f(x) \end{aligned}.$$

We then get

$$\begin{aligned} v &= v_0 + \varepsilon v_1 \\ C &= C_0 + \varepsilon C_1 \\ \varepsilon v_0' &= C_0 + \varepsilon C_1 \implies C_0 = 1, \quad v_0' = 1 \\ \varepsilon \left(\omega^2 (v_0 C_0)\right) - C_0' - \varepsilon C_1' &= \omega^2 \varepsilon f(\varepsilon) \\ \omega^2 v_0 - v_0'' &= \omega^2 f(x) \\ v_0'' - \omega^2 v_0 &= -\omega^2 f(x) \\ v(0) = 0 &\implies v_0(0) = 0 \end{aligned}$$

Also

$$\begin{aligned} C(\lambda) &= 1 = 1 + \varepsilon C_1(\lambda) + O(\varepsilon) \\ &\implies C_1(\lambda) = 0 \implies v_0'(\lambda) = 0 \end{aligned}$$

v and C is continuous . v_0 and v_0' continuous.

For $0 \le x \le 1$ we have

$$v_0'' + \omega^2 = -\omega^2 x$$

A solution to $v_0'' + \omega = 0$

$$E e^{\omega x} + E e^{-\omega x} = A \cosh(\omega x) + B \sinh(\omega x)$$

Identities.

$$\begin{aligned}
\cosh u &= \frac{1}{2}\left(e^{u}+e^{-u}\right)\\
\sinh u &= \frac{1}{2}\left(e^{u}-e^{-u}\right)\\
\cosh' u &= \sinh u\\
\sinh' u &= \cosh u\\
\cosh u - v &= \cosh u \cosh u - \sinh u \sinh v\\
\cosh 0 &= 1\\
\sinh 0 &= 0
\end{aligned}$$

The solution is for $0 \leq x \leq 1$

$$v_0\left(x\right) = x + A\cosh \omega x + B \sinh \omega x$$

In the same manner

$$\begin{aligned}
v_0^+\left(x\right) &= \overbrace{1 + C\cosh \omega x + D\sinh \omega x}^{0\leq x\leq \lambda = \frac{L}{\delta}}\\
v_0\left(0\right) = 0 &\implies v_0^- = 0\\
&\implies v_0^-\left(x\right) = x + B\sinh \omega x\\
\frac{dv_0^+}{dx}\left(\lambda\right) &= 0\\
C\omega \sinh \omega\lambda + D\omega\cosh\omega\lambda &= 0
\end{aligned}$$

The soution is

$$v_0\left(x\right) = E\cosh \varepsilon\left(x-\lambda\right)$$

Require continuity at $x = 1$ of $v_0\left(x\right)$ and $C_1\left(x\right) = \frac{dv_0}{dx}\left(x\right)$

$$\begin{aligned}
v_0^-\left(1\right) &= v_0^+\left(1\right)\\
\frac{dv_0^-}{dx} &= \frac{dv_0^+}{dx}
\end{aligned}$$

We get

$$\begin{aligned}
v_0^-\left(x\right) &= x - \frac{\cosh\left(\omega\left(\lambda-1\right)\right)}{\omega\cosh\left(\omega\lambda\right)}\sinh\omega\lambda \quad 0\leq x\leq 1\\
v_0^+ &= 1 - \frac{\sinh\left(h\omega\right)}{\omega\cosh\left(\omega\lambda\right)}\cosh\omega\left(x-\lambda\right)\\
Os^* &= \frac{C_0}{v\left(\lambda\right)} \approx \frac{C_0}{v_0\left(\lambda\right)}\\
&= \frac{C_0}{\left(1-\frac{\sinh\omega}{\omega}\frac{1}{\cosh\omega\lambda}\right)}
\end{aligned}$$

$\varepsilon \ll 1$, Os^* depends on ω and $\lambda\omega = k$.

If ω is smak then is

$$\frac{\sinh \omega}{\omega} \approx 1 + \frac{1}{6}\omega^2 + \dots$$

Let

$$\begin{aligned} Os^* &\approx \frac{C_0}{1 - \frac{1}{\cosh k}} = C_0 \left(\frac{\cosh k}{\cosh k - 1} \right) = C_0 \left(\frac{1 + \frac{1}{2}k^2 + O\left(k^4\right)}{\frac{1}{2}k^2 + O\left(k^4\right)} \right) \\ &\approx \left(1 + \frac{2}{k^2} \right) C^* \end{aligned}$$

Argue that

$$\frac{2}{k^2} \approx \frac{F^*_{\text{Diffusion}}}{F^*_{\text{Convection}}}$$

We can finally conclude that

$$Os^* \approx C_0 \left(1 + \frac{F^*_{\text{diff}}}{F^*_{\text{conv}}} \right)$$

7 Singular Perturbation

$$\varepsilon m^2 + 2m + 1 = 0, \quad 0 < \varepsilon \ll 1, \quad m = .\frac{1}{2}$$

If εm^2 and 1 are important

$$\begin{aligned} m \pm e\varepsilon^{\frac{1}{2}} \quad &\implies \varepsilon m^2 + 2m \approx 0 \\ \leftrightarrow m\left(\varepsilon m + 2\right) &= 0 \\ m &\approx -\frac{2}{\varepsilon} \\ \varepsilon m^2 &\approx -\frac{2}{\varepsilon} \\ 2m &\approx \frac{4}{3} \\ \varepsilon m^2 + 2m + 1 &= 0 \\ m &= -\frac{1}{2} + \varepsilon m_1 \\ m &= -\frac{2}{3}\widetilde{m_1}\varepsilon \end{aligned}$$

7.1 Singular perturbation applied to differential equations

$$\begin{aligned} \varepsilon y'' + 2y' + y &= 0 \\ y\left(0\right) = 0, \quad y\left(1\right) &= 1 \\ 0 \leq x &\leq 1 \end{aligned}$$

Let $\varepsilon = 0$ then is

$$\begin{aligned} 2y' + y = 0 &\implies y = ke^{-\frac{x}{2}}, k \in \mathbb{R} \\ y(0) = 0 &\implies y := 0 \\ y(1) = 1 &\implies y(x) = e^{\frac{1}{2}} e^{-\frac{x}{2}} \end{aligned}$$

Ther characteristic equation for

$$\varepsilon y'' + y' + y = 0$$
$$\varepsilon r^2 + 2r + 1 = 0, \quad r_1 \approx -\frac{1}{2}, r_2 \approx -\frac{2}{3}$$
$$y(x) \approx Ae^{-\frac{x}{2}} Be^{-\frac{2x}{\varepsilon}}$$

For $y(0) = 0$

$$y(x) = A\left(e^{-\frac{x}{2}} - e^{-\frac{2x}{\varepsilon}}\right)$$

And for $y(1) = 1$

$$y(x) \approx e^{-\frac{1}{2}}\left(e^{-\frac{x}{2}} - e^{-\frac{2x}{\varepsilon}}\right)$$

7.2 Further look at Singular Perturbation

Our main equation

$$\varepsilon y'' + 2y' + y = 0, \quad y(0) = 0, \quad y(1) = 1$$

(i) Find outer solution y_o by setting $\varepsilon = 0$. Since the solution $\varepsilon y_0(x) \approx y(x)$ for

$$x > \delta(\varepsilon), \quad \text{where} \quad \delta(\varepsilon) \to 0 \text{ when } \varepsilon \to 0$$
$$y_0(x) = e^{\frac{1}{2}} e^{-\frac{x}{2}}$$

Characteristic equation for

$$Y\left(\frac{x}{\delta(\varepsilon)}\right) = y(x) \quad \text{is}$$
$$\zeta = \frac{x}{\delta(\varepsilon)}, \quad Y(\zeta) = \frac{x}{y(\zeta\delta(\varepsilon))}$$

$$\varepsilon Y'' + 2Y' + Y = 0, \quad \implies \quad \varepsilon\frac{1}{\delta^2}Y'' + \frac{2}{\delta}Y' + Y = 0$$

Are of order of 1.

$$\implies \varepsilon\frac{1}{\delta^2}, \frac{1}{\delta}, 1 \quad \text{are}$$

the "size" of the terms.

CHossing $\delta = \varepsilon$ gives

$$\frac{1}{3}Y'' + \frac{2}{3}Y' = 0 \implies \quad Y'' + 2Y' + \varepsilon Y = 0$$

Let

$$Y\left(\frac{x}{\delta(\varepsilon)}\right) = y(x), \quad y(0) = 0 \implies Y(0) =$$

Which is called the **inner equation.** Putting $\varepsilon = 0$ and $Y'' + 2Y' = 0$ where

$$\implies Y(\zeta) = D + Ee^{2\zeta}$$

We see that

$$Y(0) = 0 \implies E = -D$$
$$Y(\zeta) = E\left(1 - e^{-2\zeta}\right)$$

Let us match it with this equation

$$y_0(x) = e^{\frac{1}{2}}e^{-\frac{x}{2}}$$

We can try to match the solution at $x = \theta(\varepsilon)$. Then we need to require that

$$\lim_{\varepsilon \to 0^+} \theta(\varepsilon) = 0$$
$$\lim_{\varepsilon \to 0^+} \frac{\theta(\varepsilon)}{\delta(\varepsilon)} = \infty$$

Example.

$$\delta = \varepsilon, \quad \theta = \varepsilon^{\frac{1}{2}}$$

We know that

$$Y\left(\frac{x}{\delta(\varepsilon)}\right) = y_I(x)$$

Then can we start matching such that

$$y_I(\theta(\varepsilon) \approx y_0(\theta(\varepsilon))) \implies Y\left(\frac{\theta(\varepsilon)}{\delta(\varepsilon)}\right) = y_0(\theta(\varepsilon))$$

Let $\varepsilon \to 0$ and require equality since $\frac{\theta(\varepsilon)}{\delta(\varepsilon)} \to \infty$, $\theta(\varepsilon) = 0$. THen we obtain

$$\lim_{\zeta \to \infty} Y(\zeta) = \lim_{x \to 0} y_0(x)$$

the matching condition

$$\lim_{\zeta\to\infty} E\left(1-e^{-2\zeta}\right) = \lim_{x\to 0} e^{\frac{1}{2}} e^{-\frac{x}{2}}, \quad \implies E = e^{\frac{1}{2}}$$

$$y_0(x) + Y\left(\frac{x}{\varepsilon}\right) - \lim_{x\to 0} y_0(x) = y_u(x)$$

The uniform solution

$$\begin{aligned} y_u(x) &= e^{\frac{1}{2}} e^{-\frac{x}{2}} + e^{\frac{1}{2}}\left(1 - e^{-\frac{2x}{\varepsilon}}\right) - e^{\frac{1}{2}} \\ &= e^{\frac{1}{2}}\left(e^{-\frac{x}{2}} - e^{-\frac{2x}{\varepsilon}}\right) \end{aligned}$$

7.3 Biochemical reaction kinetics

Let the differential equation be

$$\frac{df^*(t^*)}{dt^*} = ka^*(t^*)\, b^*(t^*)$$

Where s^*, e^*, c^*, p^* be molar concentrations of S, E, C and P at time t^*.

$$\frac{ds^*}{dt^*} = -k, \quad s^*, e^* + k_{-1}c^* \tag{1}$$

$$\frac{de^*}{dt^*} = -k_1 s^* e^* 0 k_2 \tag{2}$$

$$\frac{dc^*}{dt^*} = k_1 s^* e^* - (k_{-1} + k_2) \tag{3}$$

$$\frac{dp^*}{dt^*} = k_2 c^*. \tag{4}$$

Add 2) and 3) we get

$$\begin{aligned} \frac{d}{dt^*}(e^* + c^*) &= 0 \\ e^*(t^*) + c^*(t^*) &= k \end{aligned}$$

Inital conditions

$$\begin{aligned} s^*(0) = \overline{s}, \quad e^*(0) &= \overline{e} \\ c^*(0) = p^*(0) &= 0 \end{aligned}$$

We have that

$$e^*(t) = \overline{e} - c^*(t^*)$$

1) gives out

$$\frac{ds^*}{dt^*} = -k_1 s^* (\overline{e} - c^*) + k_1 c^*$$
$$\frac{ds^*}{dt^*} = -(k_1\overline{e}) s^* + (k_1) s^* c^* + k_{-1} c^*$$
$$\implies \frac{ds^*}{dt^*} = -(k_1\overline{e}) s^* + [k_1 s^* + k_{-1}] c^*$$

$$\frac{dc^*}{dt^*} = k_1 s^* (\overline{e} - c^*) (k_{-1} + k_2) c^*$$
$$\implies \frac{dc^*}{dt^*} = (k_1\overline{e}) - [k_1 s^* + k_{-1} + k2] c^*$$

Let the scalars be $s^* = \overline{s}s$, $c^* = \overline{e}c$, $t^* = Tt$.

$$\frac{\overline{s}}{T} s' = -(k_1\overline{e}) \overline{s}s + [k_1\overline{s} + k_{-1}] \overline{e}c$$

$$s' = -(Tk_1\overline{e}) s + \left[Tk_1\overline{e}s + k_{-1}\frac{\overline{e}T}{\overline{s}} \right] c$$

Let $Tk_1e = 1 \implies T = \dfrac{1}{\overline{e}k_1}$

$$s' = -s + \left[s0 \left(\frac{k_{-1}}{k_1\overline{s}} \right) \right]$$

8 Lecture 2020-09-14

8.1 Biochemical example, kinetics

We start with the inital conditions

$$\begin{aligned}
s^*(0) &= \overline{s} \\
e^*(0) &= \overline{e} \\
c^*(0) &= 0 \\
p^*(0) &= 0
\end{aligned}$$

With the reaction equations

$$\frac{ds^*}{dt^*} = -k_1 e^* s^* + k_{-1} c^* \tag{5}$$
$$\frac{de^*}{dt^*} = -k_1 e^* s^* + k_{-1} c^* + k_2 c^* \tag{6}$$
$$\frac{dc^*}{dt^*} = k_1 e^* s^* - k_1 c^* - k_2 c^* \tag{7}$$
$$\frac{dp^*}{dt^*} = -k_2 c^* \tag{8}$$

We can eliminate

$$\begin{aligned}
e^* + c^* &= \overline{e} \\
e^* &= \overline{e} - c^*
\end{aligned}$$

Inserter into (1) .

$$\begin{aligned}
\frac{ds^*}{dt^*} &= -k_1 s^* \left(\overline{e} - c^*\right) + k_{-1} c^* \\
\frac{dc^*}{dt^*} &= +k_1 s^* \left(\overline{e} - c^*\right) - \left(k_{-1} + k_2\right) c^*
\end{aligned}$$

Which can be transformed to

$$\frac{ds^*}{dt^*} = -\left(k_1 \overline{e}\right) s^* + \left[k_1 s^* + k_{-1}\right] c^* \tag{9}$$
$$\frac{dc^*}{dt^*} = \left(k_1 \overline{e}\right) s^* - \left[k_1 s^* - \left(k_{-1} + k_1\right)\right] c^*. \tag{10}$$

We can then scale such that

$$s^* = \overline{s} s, \quad c^* = \overline{e} c, \quad t^* = T t$$

Using (**??**),

$$\frac{\overline{s}}{T}s' = -T\left(k_1\overline{e}\right)\overline{s}s + \left[Tk_1\overline{s}s + \frac{k_{-1}T}{\overline{s}}\right]\overline{e}c$$
$$s' = -\left(T\overline{e}k_1\right)s + \left[\left(Tk_1\overline{e}\right)s + \frac{k_{-}T\overline{e}}{\overline{s}}\right]c$$

Put $T = \frac{1}{\overline{e}k_1}$ we find

$$\Longrightarrow\ s' = -s + \left[s + \frac{k_{-1}}{k_1\overline{s}}\right]c$$

Now seeing (**??**) we get

$$\overline{e}\frac{c'}{\overline{s}} = \left(Tk_1\overline{e}\right)\overline{s}s - \left[k_1\overline{s}sT + \frac{\left(k_{-1}+k_2\right)T}{\overline{s}}\right]\overline{e}c$$
$$\Longrightarrow\ \left(\frac{\overline{e}}{\overline{s}}\right)c' = s - \left[s + \frac{\left(k_{-1}+k_2\right)}{\overline{s}k_1}\right]c$$

$$\frac{\overline{e}}{\overline{s}} = \varepsilon, \quad \frac{k_{-1}+k_2}{\overline{s}k_1} = k, \quad \frac{k_2}{\overline{s}k_1} = \lambda$$

We then end up with

$$s' = -s + \left[s + k - \lambda\right]c$$
$$\varepsilon c' = s - \left[s + k\right]c$$
$$, s(0) = 1, c\left(0\right) = 0$$

Assume that

$$\frac{\overline{e}}{\overline{s}} = \varepsilon \ll 1$$

Outer solution :

$$s = s_0 + \varepsilon s_1 + \ldots$$
$$c = c_0 + \varepsilon c_1 + \ldots$$

Put $\varepsilon = 0$. This gives

$$0 = s - \left[s + k\right]c$$
$$c = \frac{s}{s+k}$$

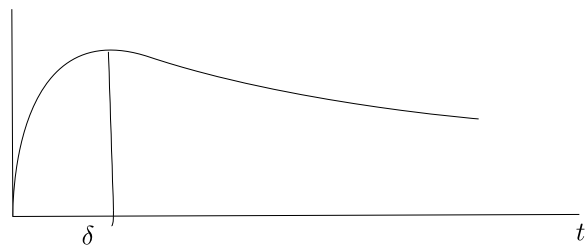

Figure 8: whines good

C is then

$$
\begin{aligned}
s' &= -s + \left[(s+k) - \lambda\right] \frac{s}{s+k} \\
\implies s' &= -\frac{\lambda s}{s+k} \\
\implies \left(\frac{s+k}{s}\right) ds &= -\lambda dt
\end{aligned}
$$

$$\downarrow \text{Integration}$$

$$
\text{Outer solution} \begin{cases} s + k \ln s &= -\lambda t + K, \quad K \text{ is constant.} \\ c &= \frac{s}{s+k} \end{cases}
$$

Let us introduce

$$
S\left(\frac{t}{\delta}\right) = s\left(t\right), \quad \tau = \frac{t}{\delta}
$$

$$
C\left(\frac{t}{\delta}\right) = c\left(t\right)
$$

For the inner solution (now capital) .

$$
\begin{aligned}
\frac{1}{\delta} S' &= -S + \left[S + k - \lambda\right] C \\
\frac{\varepsilon}{\delta} C' &= S - \left[S + k\right] C
\end{aligned}
$$

To retain $\left(\frac{\varepsilon}{\delta}C'\right)$ we choose $\delta = \varepsilon$. This gives

$$S' = \varepsilon\left(-S + [S + k - \lambda]\, C\right)$$
$$C' = S - [S + k]\, C$$

If we let $\varepsilon = 0$: $S' = 0$. So we have that $S(\tau) = L$, but $s(0) = 1$, means $S(0) = 1$

$$S(\tau) = 1$$

This gives $C' = 1 - [1 + k]\, C$, with the solution

$$C(\tau) = \frac{1}{1+k} + Me^{-(1+k)\tau}$$
$$C_I(0) = 0, \quad \Longrightarrow \quad C(\tau) = \frac{1}{1+k}\left[1 - e^{-(k+1)\tau}\right]$$
$$S_I(\tau) = 1$$
$$C_0(t) = \frac{S_0(t)}{S_0(t) + k}$$
$$S_0(t) + k\ln S_0(t) = -\lambda t K$$

Matching.

$$\theta(\delta) \to 0, \quad \text{when} \quad \delta \to 0$$
$$\frac{\theta(\delta)}{\delta} \to \infty, \quad \text{when} \quad \delta \to 0$$
$$\lim_{\delta\to 0}\begin{bmatrix} S^I\left(\theta(\delta)\frac{1}{\delta}\right) \\ C^I\left(\frac{\theta(\delta)}{\delta}\right)\end{bmatrix} = \lim_{\delta\to 0}\begin{bmatrix} S_0 \\ C_0(\theta(\delta))\end{bmatrix}$$
$$\Longrightarrow \lim_{\tau\to 0}\begin{bmatrix} S_I(\tau) \\ C_I(\tau)\end{bmatrix}$$

$$\lim_{t\to 0}\begin{bmatrix} S_0(t) \\ C_0(t)\end{bmatrix} = \lim_{\tau\to\infty}\begin{bmatrix} 1 \\ \frac{1}{1+k}\left(1 - e^{-(1+k)\tau}\right)\end{bmatrix} = \begin{bmatrix} 1 \\ \frac{1}{1+k}\end{bmatrix}$$

Uniform solution

$$\begin{bmatrix} S_u \\ C_u\end{bmatrix} = \begin{bmatrix} S_0(t) \\ C_0(t)\end{bmatrix} + \begin{bmatrix} S_I\left(\frac{t}{\varepsilon}\right) \\ C_I\left(\frac{t}{\varepsilon}\right)\end{bmatrix} - \begin{bmatrix} 1 \\ \frac{1}{1+k}\end{bmatrix}$$
$$= \begin{bmatrix} S_0(t) \\ C_0(t) - \frac{1}{1+k}e^{-(1+k)\frac{t}{\varepsilon}}\end{bmatrix}$$
$$= \begin{bmatrix} S_0(t) \\ S_0(t)\,\frac{1}{S_0(t)+k} - \frac{1}{1+k}e^{(1+k)\frac{t}{\varepsilon}}\end{bmatrix}$$

$$S_0' + k + k\frac{S_0'}{S_0} = -\lambda$$
$$S_0(0) = 1 \implies \quad S_0'(0) = \frac{-\lambda}{1+k}S_0(t) = 1 - \frac{\lambda}{1+k}t + O\left(t^2\right)$$

For large $\lambda t : k \ln S_0(t) \approx -\lambda t$

$$S_0(t) \approx e^{\frac{\lambda}{k} t}$$

8.2 Stability

8.2.1 Dynamical Systems

Let

$$\begin{aligned} x' &= f_1(x_1, x_2, \ldots, x_n) \\ x_2' &= f_2(x_1, x_2, \ldots, x_n) \\ &\vdots \\ x_n' &= f_n(x_1, x_2, \ldots, x_n) \end{aligned}$$

Where $x_j(0) = x_j^{(0)}$ are given. Write this as

$$\mathbf{x}' = \mathbf{f}(\mathbf{x}), \quad \mathbf{x}(0) = \mathbf{x}^{(0)}, \quad \mathbf{x}(t) \in \mathbb{R}$$

Example.

$$\begin{aligned} x_1' &= -x_2, \quad x_1(0) = 1 \\ x_2' &= x_1, \quad x_2(0) = 0 \end{aligned}$$

An equilibrium point for

$$\mathbf{x}' = \mathbf{f}(\mathbf{x})$$

is a constant solution. I.e. $\mathbf{x}_e$ is an equilibrium point

$$\implies \mathbf{f}(\mathbf{x}_e) = 0$$

Definition 8.1. *An equilibrium point $\mathbf{x}_e$ is **stable** if for any $\varepsilon > 0$, there exist a $\delta > 0$ such that if*

$$\|\mathbf{x}(0) - \mathbf{x_e}\| < \delta \implies \|x(t) - \mathbf{x_e}\| < \varepsilon, \quad \text{for } t > 0$$

Definition 8.2. *If $\mathbf{x_e}$ is stable and, there exists a $\delta > 0$ such that always*

$$\|x(0) - \mathbf{x_e}\| < \delta$$

Implies

$$\lim_{t \to \infty} \mathbf{x}(t_1) = \mathbf{x_e}$$

Then $\mathbf{x_e}$ is an asymptotically stable equilibrium point.

If $\mathbf{x_e}$ *is not stable, it is* ***unstable*** *.*

Example.

$$x' = -x, \quad x_e = 0 \text{ is a equilibrium point.}$$

Where the solution is

$$x = Ce^{-t} \quad \rightarrow \quad \text{for any } C$$

8.2.2 Linearization

$$x_j' = f_j\left(x_1, x_2, \ldots, x_n\right), \quad j = 1, 2, \ldots, n$$

Assume $\mathbf{x_e}$ is an equilibrium point. If f_j is differentiable we can write

$$.\frac{f_j\left(\mathbf{x_0} + \delta\delta\mathbf{x}\right) - f_j\left(\mathbf{x}\right) + \sum_{i=1}^{n} \frac{\partial f_j}{\partial x_i} \Delta x_i}{\|\Delta x\|} \overbrace{\rightarrow}^{\|\Delta x \rightarrow 0\|} 0$$

From matrix notation

$$\mathbf{f}\left(\mathbf{x_1} \Delta \mathbf{x}\right) = \mathbf{f}\left(\mathbf{x_2}\right) + J\left(\mathbf{x_0}\right) \Delta \mathbf{x_1}$$

Where the $n \times n$ matrix $J\left(\mathbf{x_0}\right)$ is given by

$$\left(J\left(\mathbf{x_0}\right)\right)_{ij} = \frac{\partial f_j}{\partial x_i}\left(\mathbf{x_0}\right)$$

And is called the jacobian matrix of $\mathbf{f}\left(\mathbf{x}\right)$ at $\mathbf{x} = \mathbf{x_0}$

9 Lecture 2020-09-16

9.1 Stability

Dynamic system

$$\mathbf{x}' = f(\mathbf{x}(t))$$

Where $\mathbf{x}(t) \in \mathbb{R}^n$. The equilibrium point $\mathbf{x}_e \implies f(\mathbf{x}_e) = 0$.

$$\mathbf{x}_e \text{ is either } \begin{cases} \text{stable} \\ \text{asymptotically stable} \\ \text{unstable} \end{cases}$$

We can determinde the linear approximation if

$$\frac{\partial f_i}{\partial x_i \partial x_q}.$$

is contiionus at $\mathbf{x_e}$. Then we have

$$\begin{aligned} f(\mathbf{x_e} + \delta\mathbf{x}) &= f(\mathbf{x}_e) + J(\mathbf{x_e})\Delta x + O\left(\|\Delta x\|^2\right) \\ \implies f(\mathbf{x_e} + \Delta\mathbf{x}) &\approx J(\mathbf{x_e})\Delta\mathbf{x}, \quad \mathbf{x_e} + \Delta\mathbf{x} = \mathbf{x} \\ \Delta\mathbf{x}' &= J(\mathbf{x_e})\Delta\mathbf{x}, \quad \Delta\mathbf{x}(0) = 0 \end{aligned}$$

If $J(\mathbf{x}_e)$ has n linearly independent eigenvectors $v_1, v_2, \ldots, v_n$ with corresonding eigen values $\lambda_1, \lambda_2, \ldots, \lambda_n$. Then the solution to

$$\Delta\mathbf{x}' = J(\mathbf{x_0})\mathbf{x}_0$$

Is

$$\Delta\mathbf{x}(t) = c_1\mathbf{x_1}e^{\lambda_1 t} + c_2\mathbf{x_2}e^{\lambda_2 t} + \ldots + c_n\mathbf{x_n}e^{\lambda_n t}$$

Where $c_1, \ldots, c_n$ is determined by $\Delta\mathbf{x}(0)$. $\mathbf{x_e}$ is an asymptotically stable eq. Point if $Re\lambda_j < 0$ for $j = 1, 2, \ldots, n$ for the system $\mathbf{x}' = f(\mathbf{x})$. If $Re\lambda_k > 0$ for one k , then $\mathbf{x_e}$ is unstable.

Example.

$$\begin{aligned} x_1 &= x_2 \\ x_2' &= -2x_1 - 2x_2 \end{aligned}$$

$$\begin{aligned} \mathbf{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, &\quad \mathbf{x}' = \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix} \\ &= A\mathbf{x}, \quad \|A - \lambda I\| = 0 \\ \implies &\begin{vmatrix} -\lambda & 1 \\ -2 & -2-\lambda \end{vmatrix} = 0 \end{aligned}$$

Example.

$$x_1' = x_1^2 - x_2'$$
$$x_2' = 2x_1 + x_2 + 3$$

Solve

$$\begin{aligned} x_1^2 + x_2^2 &= 0 \\ 2x_1 + x_2 + 3 &= 0 \\ \implies x_2 &= \pm x_1 \end{aligned}$$

when we get

- $x_2 = x_1$:
$$3x_1 + 3 = 0 \qquad \implies x_1 = -1$$
Which means $(-1,-1)$ is a eq. point.
- $x_2 = -x_1$: $2x_1 - x_1 + 3 = 0$. Which means $x_1 = -3$ and $(-3,3)$ is a equ. point.

Let

$$\begin{aligned} A = \begin{bmatrix} a & b \\ x & d \end{bmatrix}, &\quad |A - \lambda I| = (a-\lambda)(d-\lambda) - bc \\ &= \lambda^2 - (trA)\lambda + detA = (\lambda - \lambda_1)\, l\, (\lambda - \lambda_2) \\ trA = a + d, &\quad \lambda = a + b \\ detA < 0, &\quad \text{unstable} \\ detA > 0, &\quad trA < 0 \implies \text{asymptotic stable} \end{aligned}$$

Back to the example

$$\begin{aligned} J(\mathbf{x}) &= \begin{bmatrix} 2x_1 & -2x_2 \\ 2 & 1 \end{bmatrix} \\ J(-1,-1) &= \begin{bmatrix} -2 & 2 \\ 2 & 2 \end{bmatrix} \implies |J| = -6, \quad \mathbf{x_e} \text{ is unstable} \\ J(-3,-3) &= \begin{bmatrix} -6 & -6 \\ 2 & 1 \end{bmatrix}, |J| = 6, \quad trA = -5 \end{aligned}$$

$$Re\lambda_1 < 0, \quad Re\lambda_2 < 0, \implies \quad (-3,-3) \text{ asymptotically stable .}$$

9.2 Amoebae and chemotaxis

Let $\phi(x,t)$ be the amoebae concentration at position x at time t. Let A be the cross-sectional area of the tube. Then

$$\left(\int_{x_1}^{x_2} \phi(x,t)\, dA\right) A = \text{ nr amoebae in } [x_1, x_2]$$

$$\frac{d}{dt}\left(\int_{x_1}^{x_2} \phi(x,t)\, dx\right) A = J(x_1,t)\, A - J(x_2,t)\, A$$

- Flux density: $J = -M\dfrac{\partial \phi}{\partial x} + E\dfrac{\partial c}{\partial x}$, where $M > 0$ motility and $E > 0$ strength of chemotaxis.
- $c(x_1 t)$ concentration of signaling substance.

$$x_2 - x_1 = \Delta x, \quad \widetilde{x} \in \langle x_1, x_2\rangle$$

$$\Longrightarrow \frac{\partial}{\partial t}\left(\phi(\widetilde{x}, t)\right)\Delta x + J(x_2,t) - J(x_1,t) = 0$$

$$\Delta x \to 0$$

$$\frac{\partial \phi}{\partial t} + \frac{\partial J}{\partial x} = 0$$

We can rewrite such that

$$\Longrightarrow \frac{\partial \phi}{\partial t} + \frac{\partial}{\partial x}\left(-M\frac{\partial \phi}{\partial x} + E\phi\frac{\partial c}{\partial x}\right) = 0$$

$$\Longrightarrow \frac{\partial \phi}{\partial t} = \frac{\partial}{\partial x}\left(M\frac{\partial \phi}{\partial x} - A\phi\frac{\partial c}{\partial x}\right)$$

Flux density for c : $\quad J_c = -D\frac{\partial c}{\partial x}$

$$\frac{\partial c}{\partial x} + \frac{\partial J_c}{\partial x} = q_1\phi - q_2 c$$

- q_1 strength of secretion.
- q_2 decay rate for c_1

$$\frac{\partial c}{\partial t} + \left(-D\frac{\partial^2 c}{\partial x^2}\right) = q_1\phi - q_2 c$$

$$\frac{\partial \phi}{\partial t} = M\frac{\partial^2 \phi}{\partial x^2} - E\frac{\partial}{\partial x}\left(\phi\frac{\partial c}{\partial x}\right)$$

$$\frac{\partial c}{\partial t} = D\frac{\partial^2}{\partial x^2}c + q_1\phi - q_2 c$$

$$\left.\begin{aligned}\phi(x,t) &= \phi_0\\ c(x,t) &= c_0\end{aligned}\right\}\text{a solution as long as: } q_1 u_0 - q_2 c_0 = 0$$

Let write

$$\begin{aligned}
\phi(x,t) &= \phi_0 + \widetilde{\phi}(x,t) \\
c(x,t) &= c_0 + \widetilde{c}(x,t) \\
\implies \frac{\partial \widetilde{\phi}}{\partial t} &= M \frac{\partial^2 \widetilde{\phi}}{\partial \widetilde{x}} - E\left(\left(\phi_0 + \widetilde{\phi}\right)\frac{\partial \widetilde{c}}{\partial x}\right)_x \\
\frac{\partial \widetilde{c}}{\partial t} &= -D\frac{\partial \widetilde{c}}{\partial x^2} + q_1\widetilde{c} - q_2\widetilde{c} \\
\implies \frac{\partial \widetilde{\phi}}{\partial t} &= M\frac{\partial^2 \widetilde{\phi}}{\partial x^2} - E\phi_0\frac{\partial^2 \widetilde{c}}{\partial x^2} - E\left(\widetilde{\phi}\frac{\partial \widetilde{c}}{\partial x}\right)_x
\end{aligned}$$

Linearization

$$\begin{aligned}
\frac{\partial \widetilde{\phi}}{\partial t} &= M\frac{\partial^2 \widetilde{\phi}}{\partial x^2} - E\phi_0\frac{\partial^2 \widetilde{c}}{\partial x^2} \\
\frac{\partial \widetilde{c}}{\partial t} &= D\frac{\partial^2 c\widetilde{c}}{\partial x^2} + q_1\widetilde{\phi} - q_2\widetilde{c}, \quad \widetilde{c}(x,0) \text{ and } \widetilde{\phi}(x,0) \text{ is geiven}
\end{aligned}$$

Let

$$\widetilde{\phi}(x,t) = \sum \alpha_n(t)\, e^{i\beta_n x}$$

$$\widetilde{c}(x,t) = \sum \gamma_n(t)\, e^{\beta_n x n}$$

(i)

$$\sum \alpha_n'(t)\, e^{i\beta_n x} = \sum\left(-M\beta_n^2\alpha_n(t)\, e^{i\beta_n x} + E\phi_0\gamma_n(t)\, e^{i\beta_n x}\right)$$

$$\begin{aligned}
\alpha_n'(t) &= -M\beta_n^2\alpha_n(t) + E\phi_0\gamma_n(t)\,\beta_n^2 \\
\gamma_n'(t) &= -D\beta_n^2\gamma_n(t) + q_1\alpha_n(t) - q_2\gamma_n(t) \\
\implies \begin{bmatrix}\alpha_n \\ \gamma_n\end{bmatrix}_t &= \begin{bmatrix} M\beta_n^2 & E\phi_0\beta_n^2 \\ q_1 & -D\beta_n^2 - q_1\end{bmatrix}\begin{bmatrix}\alpha_n \\ \beta_n\end{bmatrix}
\end{aligned}$$

$$trA < 0, \quad detA = M\beta_n^2\left(D\beta_n^2 + q_2\right) - q_1E\phi_0\beta_n \begin{cases} < 0, & \text{stable} \\ < 0, & textunstable \end{cases}$$

Unstable when

$$\begin{aligned}
detA &< 0 \\
\implies M\left(D\beta_n^2\right) + q_1\phi_0E &< 0 \\
q_1 &> \frac{M\left(D\beta_n^2 + q_2\right)}{\phi_0 E} \\
\widetilde{\phi} &= \sum \alpha_n(0)\, e^{i\beta_n x}, \quad \beta_n^2 \text{ is increasing.}
\end{aligned}$$

10 Lecture 2020-09-21

10.1 Bifurcation

First, wee will only consider 1-D dynamical systems in general. Given

$$\frac{du}{dt} = f(\mu, u)$$

where $\mu \in \mathbb{R}$ is a parameter. For given μ, we have equilibrium points when

$$f(\mu, u) = 0$$

If $u_e = u(\mu)$ is an equilibrium point, u_e is asymptotically stable if

$$\frac{\partial f}{\partial u}(\mu, u_e) < 0$$

and unstable if

$$\frac{\partial f}{\partial u}(\mu, u_e) > 0$$

Example. Let the problem be formulated as

$$\begin{aligned} u' = (u-1)\left(\mu - u^2\right) &= f(\mu, u) \\ f(\mu, u) = 0, \implies u &= 1 \text{ or } u^2 = \mu \end{aligned}$$

Example.

$$\frac{du}{dt} = u\mu - u^2 = \underbrace{u(\mu - u)}_{f(\mu,u)}$$

$$\begin{aligned} \frac{\partial f}{\partial u} &= \mu - 2u, \\ \frac{\partial f}{\partial u}(\mu, u = 0) &= \mu \\ \frac{\partial f}{\partial u}(\mu, u = \mu) &= -\mu \end{aligned}$$

Definition 10.1. ***Implicit function theorem.*** *Let (μ, u) have continious derivatives around (μ_0, u_0), where $f(\mu_0, u_0) = 0$. Then there are constants $a > 0, b > 0$ such that $f(\mu, u) = 0$ has a unique solution $u(\mu)$ for*

$$\|\mu - \mu_0\| > a, \quad \|u - u_0\| > b$$

if $\frac{\partial f}{\partial u}(\mu_0, u_0) \neq 0$. Then

$$\frac{du}{d\mu} = -\frac{\frac{\partial f}{\partial \mu}}{\frac{\partial f}{\partial u}}, \quad \frac{\partial f}{\partial \mu}(\mu_0, u_0) \neq 0$$

We can find

$$\mu(u)$$

Remark. If

$$\begin{aligned} f(\mu_0, u_0) &= \frac{\partial f}{\partial \mu}(\mu_0, u_0) \\ &= \frac{\partial f}{\partial u}(\mu_0, u_0) \end{aligned}.$$

Then (μ_0, u_0) is a singular point. Assume ass second derivatives are continuous, and not all zero. Then

$$\begin{aligned} f(\mu_0 + \Delta\mu_1, u_0 + \Delta u) \approx &\frac{1}{2} f_{\mu\mu}(\mu_0, u_0)\Delta\mu^2 \\ &+ f_{\mu u}(\mu_0, u_0)\Delta\mu\Delta u + \frac{1}{2} f_{uu}(\mu_0, u_0)\Delta u^2 = 0 \end{aligned}$$

Assume $f_{uu}(\mu_0, u_0) \neq 0$.

$$\implies \underbrace{f_{\mu\mu}}_{c} + \underbrace{2f_{\mu u}}_{b}\left(\frac{\Delta u}{\Delta\mu}\right) + \underbrace{f_{uu}}_{a}\left(\frac{\Delta u}{\Delta\mu}\right)^2 = 0$$

$$4f_{\mu u}^2 - 4f_{\mu\mu}f_{uu} > 0, \quad \text{two solutions for } \frac{\Delta u}{\Delta\mu}$$

Example. Legislate:

$$f(\mu, u) = \left(\mu^2 + u^2\right)^2 - 2\left(\mu^2 - u^2\right)$$

Where $(0, 0)$ is a solution.

$$\left.\begin{aligned} \tfrac{\partial f}{\partial \mu} &= 2\left(\mu^2 + u^2\right) \cdot 2\mu - 2(2\mu) \\ \tfrac{\partial f}{\partial u} &= 2\left(\mu^2 + u^2\right) 2u + 4u \end{aligned}\right\} = 0 \text{ at } (0, 0)$$

Where $(0,0)$ is singular.

$$\begin{aligned}
f_{\mu\mu} &= 12\mu^2 + 4u^2 - 4 \\
f_{\mu\mu} &= 8u\mu \\
f_{uu} &= 12u^2 + 4 \\
4 * 0^2 - 4\left(4\right)\left(-4\right) \geq 0 &\implies ?????
\end{aligned}$$

Example. Tank Reactor. Let q be inflow rate, $[q] = m^3 s^{-1}$. With a concentration c_{in} and temperature θ_{in} and out c^*, θ^* .

- c^* reactor concentration.
- θ^* Temperature.
- $c^*\left(0\right) = c_{\text{in}}$
- $\theta^*\left(0\right) = \theta_{in}$

Assume that c^* disappears at rate

$$kc^* e^{-\frac{A}{\theta^*}}.$$

The reaction generates heat given by

$$h\left(kc^* e^{\frac{A}{\theta^*}}\right)$$

where h is a constant, $[k] = s^{-1}$, $[A] = Kelvin = K$ and $[h] = Jmole^{-1}$. The consentration of reactant is as follows

$$\begin{aligned}
\frac{d}{dt^*}\left(Vc^*\right) &= qc_{in} - qc^* - Vkc^* e^{-\frac{A}{\theta^*}} \\
\frac{d}{dt}\left(VC_v\theta^*\right) &= qC_v\theta_{in} - qC_v\theta^* + Vhlc^* e^{\frac{-A}{\theta^*}}
\end{aligned}$$

Where C_v is heat capacity . Let start with scaling

$$\begin{aligned}
&c^* = c_{in}c, \quad \theta^* = \theta_{in}\theta \\
&t^* = Tt
\end{aligned}$$

Then is the equations

$$\begin{aligned}
\frac{V}{T}c_{in}\frac{dc}{dt} &= qc_{in} - qc_{in}C - kVc_{in}e^{-\frac{A}{\theta_{in}\theta}} \\
\implies \frac{dc}{dt} &= T\left(\frac{q}{v}\right) - T\left(\frac{q}{v}\right) - Tkc \cdot e^{-\frac{A}{\theta_{in}\theta}}
\end{aligned}$$

If we choose $T = \frac{V}{q}$ then is

$$\implies c' = 1 - c + \left(\frac{Vk}{q}\right) ce^{-\frac{A}{\theta_{in}\theta}}$$

To simplify, let us define $\mu = \frac{q}{kv}$ and $\gamma = \frac{A}{\theta_{in}\theta}$. We then end up with the problem

$$c' = 1 - c\frac{1}{\mu}e^{\gamma\frac{1}{\theta}}, \quad c(0) = 1$$

For the other problem is

$$\begin{aligned}
\frac{1}{T}vC_v\theta_{in}\theta'(t) &= qC_v\theta_{in}\frac{1}{V} - \frac{qC_v\theta_{in}\theta}{V} + hkVc_{in}ce^{\frac{\gamma}{\theta}} \\
\implies \theta' &= 1 - \theta - + \left(\frac{Vhkc_{in}}{qc_v\theta_{in}}\right)ce^{\frac{\gamma}{\theta}} \\
\theta &= 1 - \theta + \nu\frac{1}{\mu}ce^{\gamma\frac{1}{\theta}}
\end{aligned}$$

Where $\nu = \frac{hc_1n}{c_v\theta_{in}}$. We then have the equations

$$c' = 1 - c - \frac{1}{\mu}ce^{\gamma\frac{1}{\theta}} \tag{11}$$

$$\theta' = 1 - \theta + \frac{\nu}{\mu}ce^{\frac{\gamma}{\theta}}. \tag{12}$$

Multiply (**??**) by μ and add

$$\begin{aligned}
(\mu c + \theta)' &= \mu - \mu c + 1 - \theta \\
\implies (\nu c + \theta)' &= \nu + 1 \\
h' + h = \nu + 1\nu c + \theta &= \nu + 1 \implies \nu c = \nu + 1 - \theta
\end{aligned}$$

Inserted into (**??**) given

$$\begin{aligned}
\theta' = 1 - \theta + \frac{(\nu + 1 - \theta)}{\mu}e^{\frac{\gamma}{-\theta}} \implies \theta' &= 1 - \theta + \frac{\nu + 1 - \theta}{\mu}e^{-\frac{\gamma}{\theta}} \\
\theta &= u + 1 \\
\implies u' &= -u + \frac{\nu - u}{\mu}e^{\frac{\gamma}{u+1}}
\end{aligned}$$

Where $u(0) = 0$ since $\theta(0) = 1$. Assume that γ, ν are constant while we vary $\mu = \frac{q}{Vk}$ physically

$$f(\mu, u) = -u + \frac{(\nu - u)}{\mu}e^{-\frac{\gamma}{u+1}}$$

Define $h(u) = (\nu - u)e^{-\frac{\gamma}{u+1}}$.

$$f = 0 \implies -u + \frac{h(u)}{\mu} = 0 \implies \mu u = h(u)$$

Lets check the stability

$$\begin{aligned}
f(\mu, u) &= -uh(u) \cdot \frac{1}{\mu} \\
\frac{\partial f}{\partial u} &= -1 + \frac{h'(u)}{\mu} < 0 \\
\implies h'(u) < \mu &\quad \implies \text{stability}
\end{aligned}$$

11 Lecture 2020-09-23

11.1 Dynamic population models

Natural growth: $\frac{dv^*}{dt^*} = rv^*$. Ok as long as there is food and space. But in reality there are limitiation to growth, so a model

$$\frac{dv^*}{dt^*} = rN^*\left(1 - \left(\frac{N^*}{K}\right)^\alpha\right), \quad K > 0$$

when $N^* \ll K$

$$\frac{dN^*}{dt^*} \approx rN^*$$

Scaling. $[r] = \frac{1}{s}$. One time scale is $T = \frac{1}{r}$. K natural scale for N^*

$$\begin{aligned} t^* &= \frac{1}{r}t, \quad N^* = KN \\ rkN' &= KrN\left(1 - \frac{KN}{K}\right) \\ \implies N' &= N(1-N) \\ f(N) &= 0, \quad \implies N = 0, N = 1 \end{aligned}$$

Stability.

$$\begin{aligned} f'(N) &= 1 - 2N \\ f'(0) &= 1 > 0, \quad \text{unstable} \\ f'(1) = 1 - 2 = -1 < 0, &\text{totally stable} \end{aligned}$$

The solution is

$$N(t) = \frac{N(0)}{N(0) + (1 - N(0)e^{-t})}$$

11.1.1 Whales and krill

Let $N^*(t^*)$ and $H^*(t^*)$ be the number of krill and whales respectively.

$$\frac{dN^*}{dt^*} = rN^*\left(1 - \frac{N^*}{K_N}\right) - \alpha_2 N^* H^* - u_N F_N N^* \tag{13}$$

$$\frac{dH^*}{dt^*} = qH^*\left(1 - \frac{H^*}{\alpha N^*}\right) - u_H F_H H^*, \quad N^* \neq 0 \tag{14}$$

$$r \ll q \implies T_N = \frac{1}{r} \ll T_H = \frac{1}{q}. \tag{15}$$

Scaling. Choose $T = T_H$ as time scale. K_N scale for N^*, αK_N scale for H^*.

$$t^* = \frac{1}{q}t, \quad N^* = K_N N, \quad H^* = \alpha K_N H$$

Using (**??**)

$$qN' = rK_N\left(1 - \frac{K_N N}{K_N}\right) - \frac{\alpha_2 K_N K_r N H}{r} - u_N F_N \frac{K_N N}{r}$$

$$\implies \left(\frac{q}{r}\right) N' = N\left(1 - N\right) - \left(\frac{\alpha\alpha_2 K_N}{r}\right) NH - \left(u_N F_N \frac{1}{r}\right) N$$

Lets put

$$\varepsilon = \frac{q}{r}, \quad \gamma = \frac{\alpha\alpha_2 K_N}{r}, \quad f_N = \frac{u_N F_N}{r}$$

We then end up with

$$\varepsilon N' = N\left(1 - N\right) - \gamma HN - f_N N.$$

For equation (**??**)

$$\alpha K_N q H' = \alpha K_N q H\left(1 - \frac{\alpha K_N H}{\alpha K_N N}\right) - u_H F_H K_N \alpha H$$

Let $f_H = \frac{u_N F_N}{q}$ be

$$H' = H\left(1 - \frac{H}{N}\right) - f_H H$$

such that

$$\begin{aligned} \varepsilon N' &= N\left(1 - N\right) - \gamma HN - f_N N \\ H' &= H\left(1 - \frac{H}{N}\right) - f_H H \end{aligned}$$

Stability Properties.

$$\mathbf{f}\left(N, H\right) = \begin{bmatrix} N\left(1 - N\right) - \gamma HN f_N N \\ H\left(1 - \frac{H}{N}\right) - f_H H \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad N \neq 0$$

Fromt he first equation

$$\begin{aligned} N\left(1 - N - \gamma H - f_N\right) &= 0 \\ H\left(1 - \frac{H}{N} - f_H\right) &= 0 \end{aligned}$$

If we llok at $H = 0$ it implies that

$$\begin{aligned} 1 - N - f_N &= 0 \\ N &= 1 - f_N \end{aligned}$$

We have now one eq. point such that

$$\left(1 - f_N, 0\right) = \left(N_e, H_e\right)$$

which can be considered as an unstable point if we introduce perturb by introducing , let say, one whale. For $H \neq 0$

$$\begin{aligned} 1 - N - \gamma H f_N &= 0 \\ 1 - \frac{H}{N} f_H &= 0 \end{aligned}$$

$$1 - N - \gamma H f_N = 0 \tag{16}$$
$$N - H - f_H N = 0. \tag{17}$$

Insert (**??**) we get $H = (1 - f_H) N$. And inserting (**??**)

$$\begin{aligned} 1 - N - \gamma (1 - f_H) N - f_N &= 0 \\ \implies N (1 + \gamma (1 - f_H)) &= 1 - f_N \\ \implies N &= \frac{1 - f_N}{1 + \gamma (1 - f_H)} \end{aligned}$$

In summary is the eq. point

$$(N_e, H_e) = \left(\frac{1 - f_N}{1 + \gamma (1 - f_H)} \right), \frac{(1 - f_H)(1 - f_N)}{1 + \gamma (1 - f_H)}$$

Rate of krill being caught is

$$N_e f_N = \frac{f_N (1 - f_N)}{1 + \gamma (1 - f_H)} = P_N$$

P_N has maximum when $f_N = \frac{1}{2}$.

$$\mathbf{f}(N, H) = \begin{bmatrix} N (1 - N) - \gamma N H - f_N N \\ H \left(1 - \frac{H}{N}\right) - f_H H \end{bmatrix}$$

The jacobian matrix

$$J(N, H) = \begin{bmatrix} 1 - 2N - \gamma & -\gamma N \\ \frac{H^2}{N^2} & 1 - \frac{2H}{N} - f_H \end{bmatrix}$$

The first eq. point we obtained $(N_e, H_e) = (1 - f_N, 0)$

$$J(N_e, H_e) = \begin{bmatrix} \overbrace{1 - f_N - 2(1 - f_N)}^{-(1 - f_N)} & -\gamma (1 - f_N) \\ 0 & 1 - f_H \end{bmatrix}$$

Observe that

$$\begin{aligned} det(J) &= -(1 - f_N)(1 - f_H) < 0 \\ \implies (1 - f_N, 0), &\text{ is in fact unstable.} \end{aligned}$$

However,

$$\begin{bmatrix}\widetilde{N}\\ \widetilde{H}\end{bmatrix}' = J\left(N_e,0\right)\begin{bmatrix}\widetilde{N}\\ \widetilde{H}\end{bmatrix}$$

$$\begin{bmatrix}\widetilde{N}\left(t\right)\\ \widetilde{H}\left(t\right)\end{bmatrix} = c_1\begin{bmatrix}1\\ 0\end{bmatrix}e^{-(1-f_N)t} + c_2\begin{bmatrix}1\\ \frac{-(1-f_N)-(1-f_H)}{\gamma(1-f_H)}e^{(1-f_H)t}\end{bmatrix}$$

In the same manner we find

$$J\left(N_e,H_e\right)$$

for

$$\left(N_e,H_e\right) = \left(\frac{1-f_N}{1+\gamma\left(1-f_H\right)},\frac{\left(1-f_N\right)\left(1-f_N\right)}{1+\gamma\left(1-f_H\right)}1\right)$$

And discover that $det\left(J\right) > 0$ and $trace\left(J\right) < 0$

- Both eigenvalues have negative part
- $\left(N_e,H_e\right)$ is asymptotic stable

Rate of whale catching

$$H_e f_H = P_H$$

p_N and p_H is the return value of each N and H

$$f\left(f_N,f_H\right) = p_N P_N + p_H P_H$$

11.2 Conservation Laws

In general we can formulate everything in n- dimensions. But we stick to $n = 3$. Three main players .

- Conserved entity: ϕ
- Flux density: $\mathbf{J}$
- Sinks/sources : Q

12 Lecture 2020-09-28

12.1 Universial conservation law

Main criterias

(i) **Converved entity** φ.

(ii) **Flux density y**

(iii) **Source/sink term** Q

12.1.1 Conserved entity

φ could be a number off different entities, such as, nass, energy, momentum etc. Let φ be the number of gas molecules . Density of gas. Let r_m be the typical distance betweeen molecules. We need a moving average to describe the density ρ as smoothly varying quantity. Let

$$\rho(\mathbf{x}, t) = \frac{\text{Number molecules in } B_{\mathbf{x}'}(r_a)}{\text{volume}(B_{\mathbf{x}}(r_a))}$$

Where $B_{\mathbf{x_0}}(r_a) = \left\{\mathbf{x} \in \mathbb{R}^3 : \|\mathbf{x} - \mathbf{x_0}\| < r_a\right\}$. we want $r_a \gg r_m$. If r_ρ is the length scala where ρ changes then we want

$$r_\rho \gg r_a \gg r_m$$

which is called the separation .

12.1.2 Flux density

Let $\mathbf{J}(\mathbf{x}, t)$ be a vector field for each t. Let $\mathbf{n}$ be a unit vector, then is the flux described as

$$|\mathbf{J}| \cos(\alpha) = \mathbf{J} \cdot \mathbf{n}, \quad |\mathbf{n}| = 1$$

THen is the flux through a area ΔA computed as

$$\Delta A = \mathbf{J} \cdot \mathbf{n} \Delta S$$

Where ΔS is the surface area of ΔA. S is an oriented surface. Flux through S is

$$\sum \mathbf{J} \cdot \mathbf{n} \Delta S \overset{\Delta S \to 0}{\Longrightarrow} \int_S \int \mathbf{J} \cdot \mathbf{n} dS$$

$[\mathbf{J}] = [\varphi] \, m^{-2} s^{-1}$

Example. Let S_R be the sphere surface

$$S_R = \{\mathbf{x} : \|\mathbf{x}\| = R\}$$

and let $\mathbf{r} = [x, y, z]$. We define $\mathbf{J} = a\frac{\mathbf{r}}{\|r\|^3}$, a constant.

$$\begin{aligned}
\int_{S_R}\int \mathbf{J}\cdot\mathbf{n}\cdot dS &= \int_{S_R}\int a\frac{\mathbf{r}}{R^3}\frac{\mathbf{r}}{R}dS \\
&= \int_{S_R}\int a\frac{\|\mathbf{r}\|^2}{R^4}dS = \int_{S_R}\int \frac{a}{R^3}dS \\
&= \frac{a}{R^2}4\pi R^2 = 4\pi a
\end{aligned}$$

12.2 Divergence Theorem

Let $V \subseteq \mathbb{R}^3$ be a domain with a piecewise smooth surface ∂V, and let $\mathbf{J}$ be a vectorfield with components that are continiously differential in V. Then

$$\iint_{\partial V} \mathbf{F}\cdot\mathbf{n}dS = \iiint_V \nabla\cdot\mathbf{J}dV$$

Where $\nabla\cdot\mathbf{J} = \frac{\partial J_1}{\partial x} + \frac{\partial J_2}{\partial y} + \frac{\partial J_3}{\partial z}$. Check $\nabla\cdot\frac{\mathbf{r}}{\|\mathbf{r}\|^3} = 0$

12.2.1 Source/Sink

If the system we are considering is not closed, and we have a source for φ from the outside we let

$$\iiint_V Q\cdot dV = \text{rate } Q$$

flows into V . Q is a source density function

$$[Q] = [\varphi]\, m^{-3}s^{-1}$$

It does exists several kind of sources

- Point source
- Line source
- Surface source.

To include these types of sources in our formalism Q need to be a so-called distribution. Let us define $Q = a\delta_{\mathbf{x}_0}(\mathbf{x})$ such that

$$\iint_V QdV = \begin{cases} a, & x_0 \in V \\ 0, & x_0 \notin V \end{cases}$$

Let us define

$$\iiint_V f(\mathbf{x})\,\delta_{x_0} dV = f(\mathbf{x_0})$$

For the line source define

$$Q = f(\mathbf{x})\,\delta_c(\mathbf{x})$$

by

$$\iiint_V f(\mathbf{x})\,\delta_c(x)\,dV = \int_{C\cap V} f(x)\,ds$$

Surface source density

$$Q = f(x)\,\delta_s(\mathbf{x})$$

where

$$\iiint_{V\cap S} Q dV = \iint_{V\cap S} f(x)\,dS$$

12.3 Conservation law

Let our solution domain be $\Omega \subseteq \mathbb{R}^3$ and let $V \subseteq \Omega$ have piecewise smooth surface . Accounting for φ in V

$$\frac{d}{dt}\iiint_V \rho(\mathbf{x},t)\,dV = -\iint_{\partial V} \mathbf{J}\cdot\mathbf{n}dS + \iiint_V QdV$$

Often written like this

$$\frac{d}{dt}\iiint_V \rho dV + \iint_{\partial V} \mathbf{J}\cdot\mathbf{n}dS = \iiint_V QdV$$

$$\downarrow \text{Subject to } V \text{ conditions.}$$

$$\Longrightarrow \iiint_V \frac{\partial\rho}{\partial t}dV + \iiint_V \nabla\cdot\mathbf{J}dV = \iiint_V QdV$$

$$\Longrightarrow \iiint_V \left(\frac{\partial\rho}{\partial t} + \nabla\cdot\mathbf{J} - Q\right) = 0$$

Valid for any region in Ω. Thus

$$\frac{\partial\rho}{\partial t} + \nabla\cdot\mathbf{J} = Q$$

in Ω . Conservation law of differentiable form.

Example. Heat conduction.

$$k(\mathbf{x}) = \{k_{ij}(x)\}$$

is a symmetric non-negative 3×3 matrix, the conductivity. Fouriers law states that

$$\mathbf{J} = -k\nabla T$$

where J is the thermal energy flux density and T the temperature. Thermal energy in $V \subseteq \Omega$ is

$$\iiint_V C_V \rho T dV$$

where C_V is the specific heat capacity which is assumed to be constant, ρ the mass density. Then can this be further described as

$$C_V \rho \frac{d}{dt} \iiint_V T dV + \iint_{\partial v} (-k\nabla T) \cdot \mathbf{n} dS = \iiint_V Q dV$$

If k is continuously differentiable

$$C_V \rho \frac{\partial \rho}{\partial t} - \nabla \cdot (k \nabla T) = Q$$

Assume $k = k_0 I$ we get

$$\frac{\partial T}{\partial t} = \left(\frac{k_0}{C_v \rho}\right) \nabla^2 T + Q$$

12.4 Method of Characteristics

We consider equations on the form

$$u_t + a(u,x,t) u_x = b(u,x,t)$$

If we put $z(t) = u(x(t), t)$. If $x'(t) = a$, then $z' = u_x x' + u_t = b$. Assume $u(x,0) = u_0(x)$ we can solve for each x_0

$$\begin{aligned} x' &= a(z,x,t), \quad x(0) = x_0 \\ z' &= b(z,x,t), \quad z(0) = u_0(x_0) \end{aligned}$$

To obtain $x(t, x_0)$ and $z(t, x_0)$ we solve for $x = x(t, x_0)$ for x_0 , i.e

$$x_0 = x_0(x,t).$$

We get the solution

$$u(x,t) = z(t, x_0(x,t))$$

Example. Let $u_t + 3u = u^2$ and $u(x,0) = u_0(x)$

$$x' = 3, \quad x(0) = x_0$$
$$z' = z^2, \quad z(0) = u_0(x_0)$$

Solve for x_0 :

$$x = 3t + x_0, \quad x_0 = x - 3t$$

And the second one

$$z' = z^2 \implies \frac{dz}{z^2} = dt, \implies -\frac{1}{z} = t - Cz$$

$$= \frac{1}{C - t}$$

$$z(0) = u_0(x_0), \quad \implies C = \frac{1}{u_0(x_0)}$$

$$z = \frac{1}{\frac{1}{u_0(x_0)} - t} - \frac{u_0(x_0)}{1 - tu_0(x_0)}$$

$$\implies u(x,t) = \frac{u_0(x - 3t)}{1 - tu_0(x - 3t)}$$

13 References