# Euler characteristic and surfaces

How can we prove that the two surfaces


are not homeomorphic? One way is to use the Euler characteristic.

## The 'classical' definition

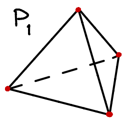
$P$ polyhedron in $\mathbb{R}^n$ with $a_0$ vertices, $a_1$ edges, $a_2$ faces (2-dim 'sides'), ..., $a_n$ n-dim 'sides'.

$$\chi(P) := \sum_{i=0}^{n} (-1)^i a_i .$$

Note: $\chi(P) \in \mathbb{Z}$.

$X \subseteq \mathbb{R}^n$ with $X \cong P$ then $\chi(X) = \chi(P)$, and is independent of $P$ as long as $P \cong X$. (Poincaré - Alexander)

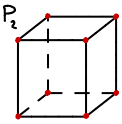
Examples: (1) $P_1$



$a_0 = 4$ (V)
$a_1 = 6$ (E)
$a_2 = 4$ (F)

$\chi(P_1) = a_0 - a_1 + a_2 = 2.$

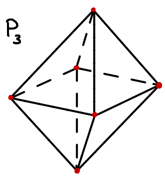
(2) $P_2$



$a_0 = 8$
$a_1 = 12$
$a_2 = 6$

$\chi(P_2) = a_0 - a_1 + a_2 = 2.$

(3) $P_3$



$a_0 = 6$
$a_1 = 12$
$a_2 = 8$

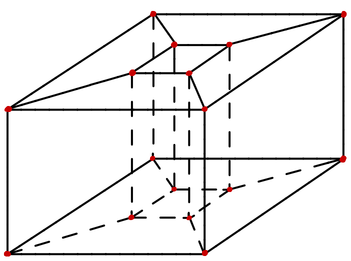
$\chi(P_3) = a_0 - a_1 + a_2 = 2.$

(See also dodecahedron and icosahedron.)

Note: $P_1 \cong P_2 \cong P_3 \cong S^2$, $\chi(S^2) = 2$ (Euler).

The Euler characteristic is a topological invariant: if $X \cong Y$ then $\chi(X) = \chi(Y)$. In other words: if $\chi(X) \neq \chi(Y)$ then $X \ncong Y$.

Example: $P$

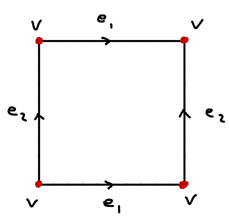


$a_0 = 16$
$a_1 = 32$
$a_2 = 16$

$\chi(P) = a_0 - a_1 + a_2 = 16 - 32 + 16 = 0.$

$P \cong T^2$. $\chi(T^2) = 0$.

As $\chi(T^2) = 0 \neq \chi(S^2)$, $T^2$ and $S^2$ can't be homeomorphic, i.e. $T^2 \ncong S^2$.

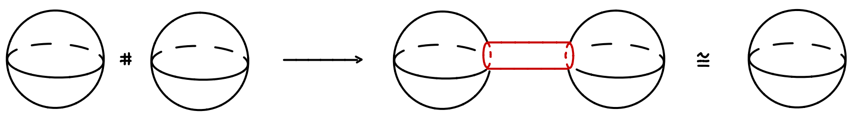



$a_0 = 1$
$a_1 = 2$
$a_2 = 1$

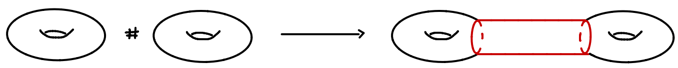
$\Rightarrow \chi = 0.$

$X, Y$ surfaces. The connected sum $X \# Y$ is (roughly) obtained by removing a (small) disk from each of $X$ and $Y$ and connecting the resulting holes with a cylinder.

<u>Examples</u>: 1) $S^2 \# S^2 \cong S^2$ (For an arbitrary surface $X$, $S^2 \# X \cong X$.)

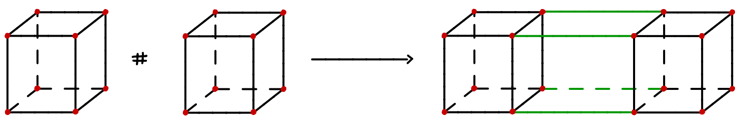


2) $T^2 \# \overset{g}{\cdots} \# T^2 = \Sigma_g$ (genus $g$)



$\chi(X \# Y) = ?$

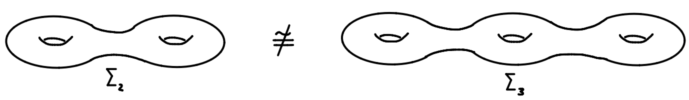
<u>Example</u>: As $S^2 \# S^2 \cong S^2$, $\chi(S^2 \# S^2) = 2$. Let $P$ be a cube. Then we remove 2 faces and add 4, add 4 edges and no vertices



when constructing $P \# P$ as above. Hence, $\chi(P \# P) = \chi(P) + \chi(P) + 0 - 4 + (4-2) = \chi(P) + \chi(P) - 2 = 2$.

<u>Theorem</u>: $X, Y$ surfaces. Then $\chi(X \# Y) = \chi(X) + \chi(Y) - 2$.

Thus $\chi(\Sigma_g) = 2(1-g)$, and hence it follows that $\Sigma_2$ and $\Sigma_3$ are not homeomorphic as $\chi(\Sigma_2) = 2(1-2) = -2 \neq \chi(\Sigma_3) = 2(1-3) = -4$.




<u>Theorem</u> (Classification of surfaces): Two connected compact surfaces are homeomorphic if and only if they have the same Euler characteristic and the same number of boundary components, and both are orientable or both are non-orientable.

By a deep theorem in differential topology any pair of homeomorphic smooth surfaces are diffeomorphic. (Holds for dim $\leq 3$.) The first example of homeomorphic but not diffeomorphic was given by Milnor where he constructed a smooth 7-manifold homeomorphic but not diffeomorphic to the standard $S^7$.

A proof of the classification of surfaces (as stated above) is given by Hirsch (GTM 33, SpringerLink). Another proof (and statement) is given by Lawson (OUP, GTinM 9).

How do we relate the 'classical' and the 'intersection number' definition of the Euler charateristic (when they both make sense)?

<u>The Poincaré–Hopf theorem</u>

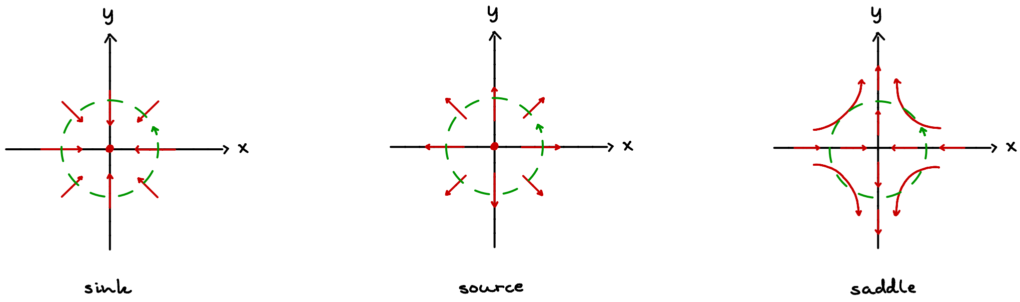
The Poincaré–Hopf theorem provides a way of computing the Euler characteristic by relating it to the indices of vector fields.

A smooth manifold $M^n$ is parallelizable if the tangent bundle $TM$ (Lecture 15) is trivial: $TM \cong M \times \mathbb{R}^n$, $T_pM \longrightarrow \{p\} \times \mathbb{R}^n$.

Using the Poincaré–Hopf theorem we can compute the Euler characteristic for every parallelizable manifold $M$: $\chi(M) = 0$. Thus, $\chi(M) = 0$ for all Lie groups $M$, as all Lie groups are parallelizable

Consider the following three (smooth) vector fields in $\mathbb{R}^2$:

(1) $F_1(x,y) = (-x,-y)$ (2) $F_2(x,y) = (x,y)$ (3) $F_3(x,y) = (-x,y)$




The index of $F_i$ at $(0,0)$ counts the number of times $F_i$ rotates completely while traversing the (small) circle centered at $(0,0)$ with rotation of $F_i$ counterclockwise gives $+1$ and rotation of $F_i$ clockwise gives $-1$.

Hence, $\mathrm{ind}_0 F_1 = +1$, $\mathrm{ind}_0 F_2 = +1$ and $\mathrm{ind}_0 F_3 = -1$.

A vector field on a manifold $M$ in $\mathbb{R}^N$ is a smooth map $F: M \longrightarrow \mathbb{R}^N$ such that $F(x) \in T_x M$ for every $x \in M$.

$F$ vector field in $\mathbb{R}^k$ with an isolated zero at $0$. We define the index of $F$ at $0$ as

$$\mathrm{ind}_0(F) := \deg(u) \quad , \quad u: S_\varepsilon \longrightarrow S^{k-1}, \quad x \longmapsto F(x)/\|F(x)\|.$$

Note that $F_1$ corresponds to the antipodal map on $S^1$, hence $\mathrm{ind}_0(F_1) = \deg(F_1) = (-1)^2 = 1$. $F_2$ corresponds to the identity map, hence $\mathrm{ind}_0(F_2) = \deg(F_2) = 1$. Finally, $F_3$ corresponds to the map

$$\begin{pmatrix} x \\ y \end{pmatrix} \longmapsto A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -x \\ y \end{pmatrix} \quad , \quad \det(A) = -1$$

with $\mathrm{ind}_0(F_3) = \deg(F_3) = -1$.

To define the index of vector fields at isolated zeros on arbitrary manifolds, use local parametrization or charts. The index does not depend on the choice of local parametrization or chart.

$\varphi: \mathcal{U} \longrightarrow M$ local parametrization, $\varphi(0) = x$, $0 \in \mathcal{U} \subseteq \mathbb{R}^k$. The pullback vector field $\varphi^* F$ on $\mathcal{U}$ is defined by

$$\varphi^* F(u) = d\varphi_u^{-1} F(\varphi(u)) \quad , \quad u \in \mathcal{U}. \qquad (d\varphi_u : T_{\varphi(u)} M \xrightarrow{\cong} \mathbb{R}^k)$$

If $F$ has an isolated zero at $x$, $\varphi^* F$ has an isolated zero at $0$. Hence,

$$\mathrm{ind}_x(F) := \mathrm{ind}_0(\varphi^* F).$$

<u>Theorem</u> (Poincaré – Hopf): If $F$ is a smooth vector field on a compact oriented manifold $M$ with only finitely many zeros. Then

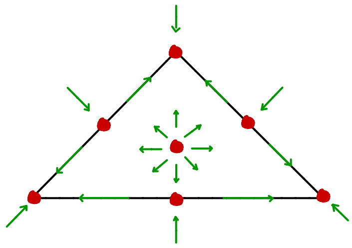
$$\sum_x \mathrm{ind}_x(F) = \chi(M).$$

A proof using (local) Lefschetz numbers is presented in Guillemin and Pollack, pp. 134 – 137.

As a consequence, we have the following:

Theorem: For a smooth oriented compact 2-manifold, the 'classical' and the 'intersection number' definition of the Euler characteristic agree.

Proof (sketch): Triangulate the manifold (can always be done; see Cairns (1935)). Define a vector field $F$ on $M$ with a source on each face, a saddle on each edge and a sink at each vertex:



For each source there is a zero of $F$ of index $1$, and similarly each saddle has a zero of index $-1$ and each sink has a zero of index $1$.

By Poincaré–Hopf, $\sum_x \operatorname{ind}_x(F) = \chi(M) = I(\Delta, \Delta)$ [$\Delta$ : diagonal in $M \times M$].

But this sum is precisely $a_0 - a_1 + a_2$ with $a_0 = \#$vertices, $a_1 = \#$ edges and $a_2 = \#$ faces. $\square$

The theorem also holds for higher dimensions.

The Euler characteristic can be defined in many ways. One way that uses homology is as follows: For a space $X$ the ith Betti number of $X$, $b_i(X)$, is the rank of $H_i(X)$ (rank of an abelian group is somewhat like the dimension of a vector space).

$b_0(X)$ is the number of path components in $X$. $b_i(X)$ measure a form of higher-dimensional connectivity of $X$.

The Euler characteristic of $X$ is then given by

$$\chi(X) = \sum_i (-1)^i b_i(X).$$