Curve Signatures in Theory and Applications

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January 10, 2023

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***Abstract*—What are Curve signatures? How can they be used and lastly**

I. Introduction

A. Context

B. Outline of this report

Signatures has proved to be fundamental for several applications. THe theory is quite applicable in the algebraic theory of rough paths [1]. This has showed to be a very interesting way to represent data because of the special properties of signatures [2]. In fact, in a 2013 competition focused on recognizing handwritten Chinese characters, the winner represented the characters as arrays based on a "signature" from rough path theory, then classified them using a convolutional neural network [3].

We will divide the report into two parts. One part for the fundamental properties of signatures and the second pat for the recent applications of this method.

II. Background Theory

The concept of the signature approach is to extract characteristic features from a data, that is a function or data points in a non parametric way. First we want to define the so-called path integral. Consider a two parameterized one dimensional paths $Y_t : [a, b] \to \mathbb{R}^d$ and $X_t : [a, b] \to \mathbb{R}$, then we say the path integral of Y_t against X_t is

$$\int_a^b Y_t dX_t = \int_a^b Y_t \dot{X}_t dt, \tag{1}$$

where we defined $\dot{X}_t = \frac{d}{dt} X(t)$.

A fundamental piece in the definition of a signature is the so-called path integral. Lets consider the parametrized smooth path of d dimensions be $X_t : [a, b] \to \mathbb{R}^d$ such that $X_t = \left\{ X_t^1, X_t^2, \dots, X_t^d \right\}$. Now since each path is $X_t^i : [a, b] \to \mathbb{R}$ for $i \in \{1, \dots, d\}$, we say define the integral

$$S(X)^i_{a,t} = \int_{a<s<t} dX^i = X^i_t - X^i_0 \tag{2}$$

Similarly we define the double-iterated double integral

$$S(X)^{i,j}_{a,t} = \int_{a<s<t} S(X)^i_{a,s} dX^j_s = \int_{\substack{a<r<t \\ a<s<t}} dX^i_r dX^j_s$$

Continuing recursively we obtain the definition

$$S(X)^{i_1,\dots,i_k}_{a,t} = \int_{a<s<t} S(X)^{i_1,\dots,i_{k-1}} dX^{i_k}_s$$

where $i_1, \dots, i_k \in \{1, \dots, d\}$. Notice that we still obtain the mapping $S(X)^{i_1,\dots,i_k} : [a, b] \to \mathbb{R}$. Finally we have the tools rquired to define a signature.

> **Definition II.1** (Signature)**.** *We say a signature of a path $X : [a, b] \to \mathbb{R}^d$, denoted by $S(X)_{a,b}$ is the collection of all the iterated integrals of X. Thus we, nor have the sequence of numbers*
>
> $$\begin{aligned} S(X)_{a,b} =& (1, S(X)^1_{a,b}, \dots, S(X)^d_{a,b}, S(X)^{1,1}_{a,b}, \\ & S(X)^{1,2}_{a,b}, S(X)^{2,1}_{a,b}, \dots). \end{aligned} \tag{3}$$
>
> *Here the first term is defined as* 1*. Keep in mind that we iterate over all multi-indexes, that is the set*
>
> $$W = \left\{ \begin{matrix} (i_1, \dots, i_k) \text{ where } k \geq 1, \\ \text{for all } i_1, \dots, i_k \in \{1, \dots, d\}\,. \end{matrix} \right\} \tag{4}$$
>
> *We denote the set W as words and $A = \{1, \dots, d\}$ as the alphabet of d letters .*

One of the most fundamental properties of the signature is its invariance under time reparameterization. This can easily be demonstrated using the definitions of the path integral. Consider two paths $X, Y : [a, b] \to \mathbb{R}$, which are real-valued. Now, consider two corresponding reparameterized paths $\widetilde{X}, \widetilde{Y} : [a, b] \to \mathbb{R}$, where $\widetilde{X}_t = X_{\psi(t)}$ and $\widetilde{Y} = Y_{\psi(t)}$, with some smooth reparameterization $\psi : [a, b] \to [a, b]$. From the chain rule it is clear that

$$\frac{d}{dt} \widetilde{X}_t = \dot{\widetilde{X}}_t \dot{\psi}(t)$$

, thus it follows that

$$\int_a^b \widetilde{Y}_t d\widetilde{X}_t = \int_a^b Y_{\psi(t)} \dot{X}_{\psi(t)} \dot{\psi}(t) = \int_a^b Y_u dX_u \tag{5}$$

Here we used the subsitution $u = \psi(t)$. This is of course applicable in the multidimensional case in the case of a signature. Let $\widetilde{X}, X : [a, b] \to \mathbb{R}d$ where $\widetilde{X}_t = X_t$.Then we see that

$$S(\widetilde{X})^{i_1,\dots,i_k}_{a,b} = S(X)^{i_1,\dots,i_k}_{a,b} \tag{6}$$

for any $i_1, \dots, i_k \in \{1, \dots, d\}$. Thus we see that the signature is, in fact, invariant under time re parameterization.

A fundamental property of the signature, first shown by Ree, is that the product of two signature terms $S(X)^{i_1,\dots,i_k,b}_a$ and $S(X)^{j_1,\dots,j_m,b}_a$ can be expressed as a sum of terms depending on shuffled multi-indexes.

To formalize this, we define the shuffle product for two multi-indexes. A permutation σ of the set $\{1, \dots, k+m\}$ is called a (k, m)-shuffle if $\sigma^{-1}(1) < \cdots < \sigma^{-1}(k)$ and $\sigma^{-1}(k+1) < \cdots < \sigma^{-1}(k+m)$. The list $(\sigma(1), \dots, \sigma(k+m))$ forms a shuffle of $(1, \dots, k)$ and $(k+1, \dots, k+m)$. Let $\mathrm{Shuffles}(k, m)$ denote the collection of all such shuffles.

The shuffle product $I \sqcup\!\sqcup J$ is the set of multi-indexes of length $k+m$.

Theorem (Shuffle product identity). For a path $X : [a, b] \to \mathbb{R}^d$ and multi-indexes $I = (i_1, \dots, i_k)$ and $J = (j_1, \dots, j_m)$, it holds that

$$S(X)_a^b S(X)_a^b = \sum_{K \in I \sqcup\!\sqcup J} S(X)_a^{K,b}.$$

III. Rough paths

References

[1] Xi Geng. "An Introduction to the Theory of Rough Paths". In: *Lecture Notes* (2021), p. 9.

[2] Ilya Chevyrev and Andrey Kormilitzin. "A primer on the signature method in machine learning". In: *arXiv preprint arXiv:1603.03788* (2016).

[3] Fei Yin et al. "Chinese handwriting recognition competition". In: *International Journal on Document Analysis and Recognition (ICDAR)* 12.4 (2013), pp. 1464–1470.