The Function Spaces $\mathcal{L}^p$

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March 6, 2024

Assume V is a vector space over $\mathbb{K} \in \{\mathbb{C}, \mathbb{R}\}$.

Definition 13.14. A seminorn on V is a map $p : V \to [0, +\infty)$ s.t.

(1) $p(cx) = |c|p(x) \;\; \forall x \in V, \forall c \in \mathbb{K}$.

(2) $p(x + y) \leq p(x) + p(y) \;\; \forall x, y \in V$. **triangle inequality**.

A seminorm is called a norm if we also have

$$p(x) = 0 \iff x = 0.$$

A norm is commonly denoted $||x||$, and a vectorspace equipped with a norm is called a **normed space**.

Definition 13.15. Assume (X, d) is a measure space. Fix $1 \leq p \leq \infty$. For every measurable function $f : X \to \mathbb{C}$ we define the following

$$||f||_p = \left(\int_X |f|^p d\mu \right)^{1/p} \in [0, +\infty]. \tag{1}$$

We can see that $||cf||_p = |c|||f||_p \;\; \forall c \in \mathbb{C}$.

Lemma 13.16.

$$||f + g||_p \leq ||f||_p + ||g||_p. \tag{2}$$

Definition 13.17. We define

$$\mathcal{L}^p(X, d\mu) = \{ f : X \to \mathbb{C} \;\mid\; f \text{ is measurable and } ||f||_p < \infty \}. \tag{3}$$

This is a vectorspace with seminorm $f \mapsto ||f||_p$. And in general this is not a normed space, since $||f||_p = 0 \iff f = 0$ a.e.

Generally, if p is a seminorm on a vectorspace V, then

$$V_0 = \{ x \in V \;\mid\; p(x) = 0 \} \tag{4}$$

which is a subspace of V. Then we consider the quotient/factor space V/V_0.

Definition 13.18. For $x, y \in V$, define

$$x \sim y \iff x - y \in V_0. \tag{5}$$

This is an equivalence relation on V. The representation class of V is defined by $[x]$ or $x + V_0$.

Then V/V_0 is equals the set of equivalence classes. We can show that it is a normed space.

$$[x] + [y] = [x + y] \ \ , \ \ c[x] = [cx] \ \ , \ \ ||[x]|| = p(x).$$

Applying this to $\mathcal{L}^p(X, d\mu)$ we get the normed space

$$L^p(X, d\mu) = \mathcal{L}^p(X, d\mu)/\mathcal{N}. \tag{6}$$

Where $\mathcal{N}$ is the space of measurable functions f s.t. $f = 0$ a.e. We will further continue to denote the norm by $|| \cdot ||_p$, and we will normally **not** distinguish between $f \in \mathcal{L}^p(X, d\mu)$ and the vector in $L^p(X, d\mu)$ that f defines.

Definition 13.19. A normed space $(X, || \cdot ||)$ is called a Banach space if V is complete w.r.t the metric $d(x, y) = ||x - y||$.

Theorem 13.20. *If $(X, \mathscr{B}, \mu)$ is a measure space, $1 \leq p \leq \infty$, then $L^p(X, d\mu)$ is a Banach space.*

Definition 13.21. A measurable function $f : X \to \mathbb{C}$ is called **essentially bounded** if there is $c \geq 0$ s.t.

$$\mu(\{x \ : \ |f(x)| > c\}) = 0. \tag{7}$$

That is $|f| \leq c$ a.e. The smallest such c is called the essential supremum of f and is denoted by $||f||_\infty$.

Definition 13.22.

$$\mathcal{L}^\infty(X, d\mu) = \{f : X \to \mathbb{C} \ | \ f \text{ is measurable and } ||f||_\infty < \infty\}.$$

$$L^\infty(X, d\mu) = \mathcal{L}^\infty(X, d\mu)/\mathcal{N}.$$

Where by the previous definiton these spaces become the spaces of all essentially bounded functions.

Theorem 13.23. *If $(X, \mathscr{B}, \mu)$ is a σ-finite measure space, then $L^\infty(X, d\mu)$ is a Banach space.*