UNIVERSITY OF OSLO

Faculty of Mathematics and Natural Sciences

Examination in: MAT3400/4400 — Linear analysis with applications

Day of examination: Thursday, June 6, 2019

Examination hours: 09.00 – 13.00

This problem set consists of 7 pages.

Appendices: None.

Permitted aids: None.

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

The points in parentheses indicate the maximum score for each problem or subproblem. The maximum score is granted for a correct and complete solution to the respective question. If you are unable to solve a subproblem, you may assume the result of that problem when solving later problems. E.g., if you cannot solve problem 3a, you may assume the result of 3a when solving 3b.

Note: You must justify all your answers!

Problem 1 (weight 25 points)

Let $(X, \mathcal{A}, \mu)$ be a measure space and set

$$\widetilde{\mathcal{A}} = \big\{ B \subseteq X \mid B \cap E \in \mathcal{A} \text{ for all } E \in \mathcal{A} \text{ such that } \mu(E) < \infty \big\}.$$

1a (weight 10 points)

Show that $\widetilde{\mathcal{A}}$ is a σ-algebra on X such that $\mathcal{A} \subseteq \widetilde{\mathcal{A}}$.

Solution: Let $B \in \widetilde{\mathcal{A}}$ and $\{B_n\}_{n\in\mathbb{N}}$ be a sequence in $\widetilde{\mathcal{A}}$. Assume that $E \in \mathcal{A}$ and $\mu(E) < \infty$. Then, using that $\mathcal{A}$, being a σ-algebra, is closed under complementation, under countable (hence finite) intersections, and countable unions, we get

- $\emptyset \cap E = \emptyset \in \mathcal{A}$,
- $B^c \cap E = (B^c \cup E^c) \cap E = (B \cap E)^c \cap E \in \mathcal{A}$ (since $B \cap E \in \mathcal{A}$),
- $\big(\cup_{n\in\mathbb{N}} B_n\big) \cap E = \cup_{n\in\mathbb{N}} (B_n \cap E) \in \mathcal{A}$ (since $B_n \cap E \in \mathcal{A}$ for every n).

This shows that $\emptyset \in \widetilde{\mathcal{A}}$, $B^c \in \widetilde{\mathcal{A}}$, and $\cup_{n\in\mathbb{N}} B_n \in \widetilde{\mathcal{A}}$, hence that $\widetilde{\mathcal{A}}$ is a σ-algebra.

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Further, let $A \in \mathcal{A}$. For any $E \in \mathcal{A}$ such that $\mu(E) < \infty$, we have that $A \cap E \in \mathcal{A}$ (since $\mathcal{A}$ is closed under finite intersections). Thus, $A \in \mathcal{A}$. Hence, we have shown that $\mathcal{A} \subseteq \widetilde{\mathcal{A}}$, as desired.

1b (weight 10 points)

Define $\tilde{\mu} : \widetilde{\mathcal{A}} \to [0, \infty]$ by

$$\tilde{\mu}(B) = \begin{cases} \mu(B) & \text{if } B \in \mathcal{A}\,, \\ \\ \infty & \text{if } B \in \widetilde{\mathcal{A}} \setminus \mathcal{A}\,. \end{cases}$$

Show that $\tilde{\mu}$ is a measure on $\widetilde{\mathcal{A}}$.

Solution: Since $\emptyset \in \mathcal{A}$, we have that $\tilde{\mu}(\emptyset) = \mu(\emptyset) = 0$. Next, let $\{B_n\}$ be a sequence of disjoints sets in $\widetilde{\mathcal{A}}$ and set $B := \cup_{n\in\mathbb{N}} B_n \in \widetilde{\mathcal{A}}$. We have to show that

$$\tilde{\mu}(B) = \sum_{n=1}^{\infty} \tilde{\mu}(B_n). \tag{1}$$

Assume first that $B_n \in \mathcal{A}$ for every $n \in \mathbb{N}$. Then we get that

$$\tilde{\mu}(B) = \tilde{\mu}(\cup_{n\in\mathbb{N}} B_n) = \mu(\cup_{n\in\mathbb{N}} B_n) = \sum_{n=1}^{\infty} \mu(B_n) = \sum_{n=1}^{\infty} \tilde{\mu}(B_n).$$

Next, assume that there exists some $k \in \mathbb{N}$ such that $B_k \notin \mathcal{A}$. Then we get that

$$\sum_{n=1}^{\infty} \tilde{\mu}(B_n) \ \geq\ \tilde{\mu}(B_k) = \infty,$$

hence that $\sum_{n=1}^{\infty} \tilde{\mu}(B_n) = \infty$. On the other hand,

- if $B \notin \mathcal{A}$, then $\tilde{\mu}(B) = \infty$;
- if $B \in \mathcal{A}$ and $\mu(B) = \infty$, then $\tilde{\mu}(B) = \mu(B) = \infty$;
- If $B \in \mathcal{A}$ and $\mu(B) < \infty$, then we get that $B_k = B_k \cap B \in \mathcal{A}$ (since $B_k \in \widetilde{\mathcal{A}}$), which contradicts that $B_k \notin \mathcal{A}$; so this eventuality can not occur.

This means that we have $\tilde{\mu}(B) = \infty = \sum_{n=1}^{\infty} \tilde{\mu}(B_n)$ in this case. Thus, altogether, we have shown that (1) holds, so $\tilde{\mu}$ is a measure on $\widetilde{\mathcal{A}}$.

1c (weight 5 points)

Assume now that $X = \mathbb{R}$, $\mathcal{A}$ consists of all Lebesgue measurable subsets of $\mathbb{R}$, and μ is the Lebesgue measure on $\mathcal{A}$.

Show that $\widetilde{\mathcal{A}} = \mathcal{A}$.

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Solution: Let $B \in \widetilde{\mathcal{A}}$. For each $n \in \mathbb{N}$, set $E_n = [-n, n] \in \mathcal{A}$. As $\mathbb{R} = \cup_{n\in\mathbb{N}} E_n$, we have that

$$B = B \cap \mathbb{R} = \bigcup_{n\in\mathbb{N}} B \cap E_n.$$

Since $\mu(E_n) = 2n < \infty$, we also have that $B \cap E_n \in \mathcal{A}$ for each $n \in \mathbb{N}$. Since $\mathcal{A}$ is closed under countable unions, we get that $B = \cup_{n\in\mathbb{N}} B \cap E_n \in \mathcal{A}$.

This shows that $\widetilde{\mathcal{A}} \subseteq \mathcal{A}$. As $\mathcal{A} \subseteq \widetilde{\mathcal{A}}$, the desired conclusion follows.

Problem 2 (weight 15 points)

Let $\mathcal{A}$ denote the σ-algebra consisting of all Lebesgue measurable subsets of $[1, \infty)$ and let μ denote the Lebesgue measure on $\mathcal{A}$. For each $n \in \mathbb{N}$, let $f_n : [1, \infty) \to \mathbb{R}$ be defined by

$$f_n(x) = \frac{n\,\sin(\frac{x}{n})}{x(1 + x^3)}\,.$$

Show that each f_n is integrable w.r.t. μ and decide whether the limit

$$\lim_{n\to\infty} \int_{[1,\infty)} f_n\,d\mu$$

exists.

Hint: You may use that $|\sin(t)| \leq t$ for all $t \in (0, \infty)$.

Solution: We first note that each f_n, being continuous on $[1, \infty)$, is measurable w.r.t. $\mathcal{A}$. Next, using the hint, we get that

$$|f_n(x)| \leq \frac{1}{1 + x^3} \leq \frac{1}{x^3} \quad \text{for all } x \in [1, \infty),$$

i.e., $|f_n| \leq g$, where the function $g : [1, \infty) \to [0, \infty)$ is defined by $g(x) = 1/x^3$. As is well-known, g is integrable w.r.t. μ. Moreover, as $\sin(t)/t \to 1$ when $t \to 0$, we get that

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{\sin(x/n)}{x/n}\,\frac{1}{(1 + x^3)} = \frac{1}{1 + x^3}$$

for every $x \in [1, \infty)$. Thus, the sequence $\{f_n\}_{n\in\mathbb{N}}$ converges pointwise to f on $[1, \infty)$, where f is the (continuous) function given by $f(x) = \frac{1}{1+x^3}, x \geq 1$.

Hence, applying the LDCT, we get that all f_n's, and f, are integrable w.r.t. μ, and

$$\lim_{n\to\infty} \int_{[1,\infty)} f_n\,d\mu = \int_{[1,\infty)} f\,d\mu < \infty.$$

(Continued on page 4.)

Problem 3 (weight 25 points)

Let $(X, \mathcal{A}, \mu)$ be a measure space and $M \in (0, \infty)$.

Suppose that $\Phi : X \to X$ is a map satisfying that

$$\Phi^{-1}(A) \in \mathcal{A} \quad \text{and} \quad \mu\big(\Phi^{-1}(A)\big) \leq M\,\mu(A)$$

for all $A \in \mathcal{A}$.

3a (weight 15 points)

Show that if $f : X \to [0, \infty]$ is measurable w.r.t. $\mathcal{A}$, then $f \circ \Phi : X \to [0, \infty]$ is measurable w.r.t. $\mathcal{A}$. Moreover, show that

$$\int_X f \circ \Phi \, \mathrm{d}\mu \leq M \int_X f \, \mathrm{d}\mu. \tag{2}$$

Hint: For the second assertion, consider first the case where $f = \mathbf{1}_A$ for some $A \in \mathcal{A}$.

Solution: Assume $f : X \to [0, \infty]$ is measurable w.r.t. $\mathcal{A}$. Let $t \in \mathbb{R}$. Then $f^{-1}\big((-\infty, t]\big) \in \mathcal{A}$, so

$$(f \circ \Phi)^{-1}\big((\infty, t]\big) = \Phi^{-1}\big(f^{-1}\big((-\infty, t]\big)\big) \in \mathcal{A}.$$

As this holds for every $t \in \mathbb{R}$, we have shown that $f \circ \Phi : X \to [0, \infty]$ is measurable w.r.t. $\mathcal{A}$.

To show the second assertion, we consider first the case where $f = \mathbf{1}_A$ for some $A \in \mathcal{A}$. As $\mathbf{1}_A \circ \Phi = \mathbf{1}_{\Phi^{-1}(A)}$, we get

$$\begin{aligned}\int_X \mathbf{1}_A \circ \Phi \, \mathrm{d}\mu = \int_X \mathbf{1}_{\Phi^{-1}(A)} \, \mathrm{d}\mu &= \mu\big(\Phi^{-1}(A)\big) \\ &\leq M\,\mu(A) = M \int_X \mathbf{1}_A \, \mathrm{d}\mu,\end{aligned}$$

showing that (2) holds in this case. As $\big(\sum_{j=1}^n \mathbf{1}_{A_j}\big) \circ \Phi = \sum_{j=1}^n \big(\mathbf{1}_{A_j} \circ \Phi\big)$, it follows then readily, using linearity of the integral, that (2) also holds for every nonnegative simple $\mathcal{A}$-measurable function f. Finally, if f is a function as given at the outset, we can pick an increasing sequence $\{f_n\}$ of nonnegative simple $\mathcal{A}$-measurable functions converging pointwise to f on X. Then $\{f_n \circ \Phi\}$ is a sequence of nonnegative simple $\mathcal{A}$-measurable functions converging pointwise to $f \circ \Phi$ on X, so using the MCT (twice), we get that

$$\int_X f \circ \Phi \, \mathrm{d}\mu = \lim_{n\to\infty} \int_X f_n \circ \Phi \, \mathrm{d}\mu \leq M \lim_{n\to\infty} \int_X f_n \, \mathrm{d}\mu = M \int_X f \, \mathrm{d}\mu\,,$$

as desired.

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3b (weight 10 points)

Let $p \in [1, \infty)$. Show that if $f \in \mathcal{L}^p(X, \mathcal{A}, \mu)$, then $f \circ \Phi \in \mathcal{L}^p(X, \mathcal{A}, \mu)$. Then show that the map $T : L^p(X, \mathcal{A}, \mu) \to L^p(X, \mathcal{A}, \mu)$ given by

$$T\big([f]\big) = \big[f \circ \Phi\big] \quad \text{for all} \quad f \in \mathcal{L}^p(X, \mathcal{A}, \mu)$$

is a well-defined, bounded linear operator on $L^p(X, \mathcal{A}, \mu)$.

(We recall that if $f \in \mathcal{L}^p(X, \mathcal{A}, \mu)$, then $[f]$ denotes the class of all functions in $\mathcal{L}^p(X, \mathcal{A}, \mu)$ which agree μ-a.e. with f.)

Solution: Let $f \in \mathcal{L}^p(X, \mathcal{A}, \mu)$. Then, using the first part of a), and the fact that linear combinations of measurable real-valued function are measurable, we get that

$$f \circ \Phi = \big((\mathrm{Re}f)^+ \circ \Phi - (\mathrm{Re}f)^- \circ \Phi\big) + i\,\big((\mathrm{Im}f)^+ \circ \Phi - (\mathrm{Im}f)^- \circ \Phi\big)$$

is measurable w.r.t. $\mathcal{A}$. Moreover, as $|f|^p$ is nonnegative and integrable w.r.t. $\mathcal{A}$, we can use the second assertion in a) and get that

$$\int_X |f \circ \Phi|^p \, \mathrm{d}\mu = \int_X |f|^p \circ \Phi \, \mathrm{d}\mu \le M \int_X |f|^p \, \mathrm{d}\mu < \infty.$$

Hence, $f \circ \Phi \in \mathcal{L}^p(X, \mathcal{A}, \mu)$.

To prove the second assertion, we first note that if $f, g \in \mathcal{L}^p(X, \mathcal{A}, \mu)$ and $f = g$ μ-a.e., then $f \circ \Phi = g \circ \Phi$ μ-a.e.

Indeed, if $\mu(A) = 0$, where $A := \{x \in X : f(x) \neq g(x)\}$, then we get

$$0 \le \mu\big(\{x \in X : (f \circ \Phi)(x) \neq (g \circ \Phi)(x)\}\big) = \mu\big(\Phi^{-1}(A)\big) \le M\,\mu(A) = 0\,,$$

and the claim follows. This means that the map T is well-defined.

Moreover, T is linear (this is straightforward, so we skip the proof here). Finally, T is bounded: given $f \in \mathcal{L}^p(X, \mathcal{A}, \mu)$, we get that

$$\|\, T\big([f]\big)\|_p^p = \|\,[f \circ \Phi]\,\|_p^p = \int_X |f \circ \Phi|^p \, \mathrm{d}\mu \le M \int_X |f|^p \, \mathrm{d}\mu = M\,\|\,[f]\,\|_p^p\,,$$

which clearly implies that T is bounded, with $\|T\| \le M^{1/p}$.

Problem 4 (weight 15 points)

Let H be a Hilbert space (over $\mathbb{R}$ or $\mathbb{C}$), $H \neq \{0\}$, and let $I : H \to H$ denote the identity operator.

4a (weight 5 points)

Assume M is a closed subspace of H such that $\{0\} \neq M \neq H$. Let $\mathcal{B}$ denote an orthonormal basis for M and $\mathcal{B}'$ denote an orthonormal basis for $M^\perp$.

Show that $\mathcal{B} \cup \mathcal{B}'$ is an orthonormal basis for H.

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Solution: One easily checks that the set $\mathcal{B} \cup \mathcal{B}'$ is orthonormal (so we skip the proof here). So it remains to show that $\mathrm{Span}(\mathcal{B} \cup \mathcal{B}')$ is dense in H:

Let $x \in H$ and $\varepsilon > 0$. Then we may write

$$x = x_M + x_{M^\perp},$$

where $x_M \in M$ and $x_{M\perp} \in M^\perp$, and pick $y \in \mathrm{Span}(\mathcal{B})$, $z \in \mathrm{Span}(\mathcal{B}')$ such that $\|x_M - y\| < \varepsilon/2$, $\|x_{M^\perp} - z\| < \varepsilon/2$. With $w := y + z \in \mathrm{Span}(\mathcal{B} \cup \mathcal{B}')$, we get that

$$\| x - w\| = \|x_M - y + x_{M^\perp} - z\| \le \|x_M - y\| + \|x_{M^\perp} - z\| < \varepsilon/2 + \varepsilon/2 = \varepsilon\,.$$

Since $\varepsilon > 0$ was arbitrary, this shows that $x \in \overline{\mathrm{Span}(\mathcal{B} \cup \mathcal{B}')}$, as desired.

4b (weight 10 points)

Assume $S \in \mathcal{B}(H)$ is self-adjoint and let M denote the closed subspace of H given by $M = \ker(I - S) = \{y \in H : S(y) = y\}$.

Show that S is an isometry if and only if S is unitary, in which case we have $S = 2\,P_M - I$, where P_M denotes the orthogonal projection of H onto M.

Solution: As a unitary operator is always isometric, we assume that S is an isometry. This is equivalent to $S^*S = I$. As $S^* = S$ by assumption, this gives that $I = S^*S = S^2 = SS^*$. Thus, S is unitary.

Moreover, since $(I - S)^* = I - S$, we have that

$$M^\perp = \big(\ker((I - S)^*)\big)^\perp = \overline{(I - S)(H)}.$$

Now, for $x \in H$, we have that

$$x = \frac{1}{2}(I + S)(x) + \frac{1}{2}(I - S)(x). \tag{3}$$

Set $y := \frac{1}{2}(I + S)(x)$. Then we have that

$$S(y) = \frac{1}{2}(S + S^2)(x) = \frac{1}{2}(I + S)(x) = y,$$

i.e., $y \in M$. On the other hand,

$$\frac{1}{2}(I - S)(x) \in \overline{(I - S)(H)} = M^\perp.$$

So we see from (3) that $P_M(x) = \frac{1}{2}(I+S)(x)$. This shows that $P_M = \frac{1}{2}(I+S)$, i.e., $S = 2\,P_M - I$, as desired.

Problem 5 (weight 20 points)

Let H be a separable infinite-dimensional Hilbert space over $\mathbb{C}$.

(Continued on page 7.)

5a (weight 10 points)

Let $T \in \mathcal{B}(H)$ be diagonalizable, in the sense that there exists an orthonormal basis $\mathcal{B} = \{u_j\}_{j\in\mathbb{N}}$ for H consisting of eigenvectors for T.

Show that there exists some $S \in \mathcal{B}(H)$ such that $S^2 = T$.

Solution: For each $j \in \mathbb{N}$, let $\mu_j \in \mathbb{C}$ denote the eigenvalue of T corresponding to u_j. We may then choose $\lambda_j \in \mathbb{C}$ such that $\lambda_j^2 = \mu_j$ for each $j \in \mathbb{N}$. Since $\{\mu_j\}_{n\in\mathbb{N}}$ is a bounded sequence in $\mathbb{C}$ (with $\sup_j |\mu_j| = \|T\|$), the sequence $\{\lambda_j\}_{n\in\mathbb{N}}$ is also bounded, so we can let $S \in \mathcal{B}(H)$ be the diagonal operator associated with $\{\lambda_j\}_{n\in\mathbb{N}}$ (w.r.t. $\mathcal{B}$). Since

$$S^2(u_j) = \lambda_j^2\, u_j = \mu_j\, u_j = T(u_j) \quad \text{for all } j \in \mathbb{N},$$

and the values of a bounded operator are determined by its values on an orthonormal basis, we get that $S^2 = T$, as desired.

5b (weight 10 points)

Let $T \in \mathcal{K}(H)$ be selfadjoint. Show that there exists some $S \in \mathcal{K}(H)$ such that $S^2 = T$. Can S always be chosen to be self-adjoint ?

Solution: Since $T \in \mathcal{K}(H)$ is selfadjoint, the spectral theorem says that T is diagonalizable. Applying part a), we get that there exists $S \in \mathcal{B}(H)$ such that $S^2 = T$. Using the notation from the proof of a), we know that the compactness of T implies that $\mu_j \to 0$ as $j \to \infty$. But then $|\lambda_j| = |\mu_j|^{1/2} \to 0$, so it follows that S is compact (as we saw in an exercise during the course).

Assume we can find a self-adjoint $S \in \mathcal{K}(H)$ satisfying that $S^2 = T$. Then, letting $\{v_k\}_{j\in\mathbb{N}}$ be an o.n.b. for H consisting of eigenvectors for S, each with associated eigenvalue $\lambda_k \in \mathbb{R}$, we get (as above) that each v_k is an eigenvector for T with associated eigenvalue $\lambda_k^2 \geq 0$. This implies that T is positive (i.e., $\langle T(x), x\rangle \geq 0$ for all $x \in H$). Thus, S can not be chosen to be self-adjoint when T is not positive (i.e., T has at least one negative eigenvalue).