Uniqueness of Measures

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Lemma 5.6. *A Dynkin system D is a σ-algebra iff it is stable under finite intersections, i.e. $A, B \in D \Rightarrow A \cap B \in D$.*

Theorem 5.7 (Dynkin)**.** *Assume X is a set, S is a collection of subsets of X closed under finite intersections, that is, if $A, B \in S \Rightarrow A \cap B \in S$. Then $D(S) = \sigma(S)$.*

Proof. We clearly have that $D(S) \subset \sigma(S)$. If we can show that $D(S)$ is a σ-algebra, that is, that a Dynkin system generated by a subset $S \subset X$ (where S is $\cap$-stable) is a σ-algebra, then the inverse conclusion $D(S) \supset \sigma(S)$ follows logically. This is the case because the σ-algebra $\sigma(S)$ is the smallest σ-algebra containing S, and so if $D(S)$ is a σ-algebra it must be a greater or equal (in some sense) than $\sigma(S)$.

Using Lemma 5.6 we only need to show that $D(S)$ is stable under finite intersections, to prove that $D(S)$ is a σ-algebra. Consider:

$$D_A := \{B \subset X : B \cap A \in D(S)\},$$

for some $A \in D(S)$. Notice that this set is $\cap$-stable, and so if we can show that $D_A = D(S)$ we must have that (by Lemma 5.6) $D(S)$ is a σ-algebra. Firstly, however, let us show that D_A is a **Dynkin system**.

1. $\varnothing$ must be in D_A, since $\varnothing \cap A = \varnothing \in D(S)$.

2. Let $B \in D_A$. Then

   $$A \cap B^c = A \setminus (A \cap B) = (A^c \cup (A \cap B))^c,$$

   here $A \cap B$ and A^c must be in $D(S)$. Furthermore, since disjoint unions of set from $D(S)$ are still in $D(S)$, we me must have $A^c \in D_A$.

3. Assume that $(B_n)_{n\in\mathbb{N}} \subset D_A$ is a pairwise disjoint sequence. Then

   $$\begin{aligned} &(B_n \cap A)_{n\in\mathbb{N}} \in D(S) \quad \text{(by def. of } D_A) \\ \Rightarrow &\bigcup_{n\in\mathbb{N}} (B_n \cap A) = \left(\bigcup_{n\in\mathbb{N}} B_n\right) \cap A \in D(S) \\ \Rightarrow &\bigcup_{n\in\mathbb{N}} B_n \in D_A. \end{aligned}$$

So D_A is indeed a Dynkin system.

We now want to show that $D(S)$ is $\cap$-stable, we have:

$$\begin{aligned}
S \subset D_A &\ \ \forall\ A \in S \\
\Rightarrow D(S) \subset D_A &\ \ \forall\ A \in S \text{ (since } D_A \text{ is a Dynkin system)} \\
\Rightarrow B \cap A \in D(S) &\ \ \forall\ B \in S,\ \forall\ A \in D(S) \text{ (by the definition of } D_A) \\
\Rightarrow B \in D_A &\ \ \forall\ B \in S,\ \forall A \in D(S) \\
\Rightarrow S \subset D_A &\ \ \forall\ A \in D(S) \\
\Rightarrow D(S) \subset D_A &\ \ \forall\ A \in D(S) \text{ (since } D_A \text{ is a Dynkin system)} \\
\Rightarrow A \cap B \in D(S) &\ \ \forall\ A, B \in D(S),
\end{aligned}$$

and so $D(S)$ is $\cap$-stable and then $D(S) \supset \sigma(S) \Rightarrow D(S) = \sigma(S)$. □

Theorem 5.8 (uniqueness of measures)**.** *Let (X, B) be a measurable space, and $S \subset P(X)$ be the generator of B, i.e. $B = \sigma(S)$. If S satisfies the following conditions:*

1. *S is stable under finite intersections ($\cap$-stable), i.e. $A, C \in S \Rightarrow A \cap C \in S$.*
2. *There exists an exhausting sequence $(G_n)_{N \in \mathbb{N}} \subset$ with $G_N \uparrow X$. Assume also that there are two measures μ, ν satisfying:*
3. *$\mu(A) = \nu(A),\ \forall A \in S$.*
4. *$\mu(G_n) = \nu(G_n) < \infty$.*

Then $\mu = \nu$.

Proof (outline). Define

$$D_n := \{A \in B : \mu(G_n \cap A) = \nu(G_n \cap A)\ (< \infty)\},$$

and show that it is a Dynkin system. Then, use the fact that S is $\cap$-stable and Theorem 5.7 to argue that $D(S) = \sigma(S) ... \rightarrow ... B = D_n$. □