MAT3400/4400 – H13 – Solution outline

Problem 1

1a) Obviously, since we are considering the σ-algebra $\mathcal{P}(X)$, every function from X into $\mathbb{K}$ is measurable, sowe don't have to bother about measurability.

Let $f \in \ell^\infty(X)$. Since $|f| \leq \|f\|_u \, 1_X$ where 1_X denote the function on X constantly equal to 1, using monotonicity and linearity of the integral w.r.t. a measure, we get

$$\int |f| \, d\mu \;\leq\; \int \|f\|_u \, 1_X \, d\mu = \|f\|_u \int 1_X \, d\mu = \mu(X) \cdot \|f\|_u \; < \infty \, .$$

Hence, f is integrable w.r.t. μ. Moreover, this gives that

$$|\, I_\mu(f) \,| \;\leq\; \int |f| \, d\mu \leq \mu(X) \cdot \|f\|_u$$

for every $f \in \ell^\infty(X)$, so I_μ is bounded, with $\|I_\mu\| \leq \mu(X)$. As $\|1_X\|_u = 1$ and $I_\mu(1_X) = \mu(X)$, we also have $\|I_\mu\| \geq \mu(X)$. Hence, $\|I_\mu\| = \mu(X)$.

1b)

i) $\nu(\emptyset) = I(\chi_\emptyset) = I(\mathcal{O}) = 0$ (since I is linear).

ii) Let $n \in \mathbb{N}$ and $A_1, \ldots, A_n \subset X$ be (pairwise) disjoint. Set $A = \cup_{j=1}^n A_j$.

Then $\chi_A = \sum_{j=1}^n \chi_{A_j}$, so, using the linearity of I, we get

$$\nu(A) = I(\chi_A) = I\big(\sum_{j=1}^n \chi_{A_j}\big) = \sum_{j=1}^n I(\chi_{A_j}) = \sum_{j=1}^n \nu(A_j)\, .$$

iii) Since I is bounded, $\nu(X) = I(\chi_X) = I(1_X) = |I(1_X)| \leq \|I\| \, \|1_X\|_u = \|I\| < \infty$.

A suitable additional condition is as follows.

Assume that I also satisfies that $I(f_n) \to I(f)$ as $n \to \infty$ whenever $\{f_n\}$ is a nondecreasing sequence of nonnegative functions in $\ell^\infty(X)$ converging pointwise to some $f \in \ell^\infty(X)$.

Then ν becomes a measure on $\mathcal{P}(X)$ such that $I = I_\nu$.

[For interested readers, we sketch a proof. First, one checks that ν becomes a measure on $\mathcal{P}(X)$ such that $I(f) = \int f \, d\nu$ for all $f \in \ell^\infty(X)$ such that $f \geq 0$: this may be proven essentially in the same way as Exercise 1 in the compulsory assignment. By writing a given $f \in \ell^\infty(X)$ as $f = f_1^+ - f_1^- + i\,(f_2^+ - f_2^-)$ where $f_1 = \mathrm{Re}(f)$, $f_2 = \mathrm{Im}(f)$, and using that both I and the integral are linear, one gets that $I(f) = \int f \, d\nu$. Hence, $I = I_\nu$.]

1c)

Consider g as in the hint. Then we have $\|g\|_u = \max\{|\lambda_1|, \ldots, |\lambda_n|\}$. Moreover, $X = \cup_{j=1}^n A_j$ (disjoint union), by definition of the standard representation of g, so $\nu(X) = \sum_{j=1}^n \nu(A_j)$. Thus, using the triangle inequality, we get

$$|I_0(g)| \leq \sum_{j=1}^n |\lambda_j|\, \nu(A_j) \leq \|g\|_u \sum_{j=1}^n \nu(A_j) = \|g\|_u\, \nu(X)\,.$$

This shows that I_0 is bounded, with $\|I_0\| \leq \nu(X)$. As $\|1_X\|_u = 1$ and $|I_0(1_X)| = \nu(X)$, we also have $\|I_0\| \geq \nu(X)$, so $\|I_0\| = \nu(X)$.

Now, we know that $\mathcal{E}$ is dense in $\ell^\infty(X)$ (since $\ell^\infty(X) = \mathcal{L}^\infty(X, \mathcal{P}(X), \mu_c)$, where μ_c denotes the counting measure on $\mathcal{P}(X)$, and $\|g\|_u = \|g\|_\infty$ for $g \in \ell^\infty(X)$). Since $\mathbb{K}$ is a Banach space, we may extend I_0 (in a unique way) to a linear, bounded map $I : \ell^\infty(X) \to \mathbb{K}$, satisfying $\|I\| = \|I_0\|$. Hence, we have $\|I\| = \|I_0\| = \nu(X)$. Moreover,

$$I(\chi_A) = I_0(\chi_A) = I_0\big(1 \cdot \chi_A + 0 \cdot \chi_{A^c}\big) = \nu(A) + 0 \cdot \nu(A^c) = \nu(A)$$

for every $A \subset X$, as desired.

Problem 2

2a) For each $n \in \mathbb{N}$, set $g_n = \sum_{j=1}^n |h_n|$. Then $\{g_n\}_{n\in\mathbb{N}}$ is a nondecreasing sequence in $\overline{\mathcal{M}}^+$ converging pointwise to g. Hence the MCT, combined with linearity of the integral, gives that

$$\int |g|\, d\mu = \int g\, d\mu = \lim_{n\to\infty} \int g_n\, d\mu = \lim_{n\to\infty} \sum_{j=1}^n \Big(\int |h_j|\, d\mu\Big) = \sum_{j=1}^\infty \Big(\int |h_j|\, d\mu\Big) < \infty\,.$$

2b) For each $n \in \mathbb{N}$, set $f_n = \sum_{j=1}^n h_j$, which belongs to $\mathcal{L}^1$ since each h_j belongs to $\mathcal{L}^1$ and $\mathcal{L}^1$ is closed under addition. By the triangle inequality, we have $|f_n| \leq \sum_{j=1}^n |h_j| \leq g$ for every n. Moreover, $i)$ says that $\lim_{n\to\infty} f_n = f$ (pointwise) μ-a.e., and we have seen in $a)$ that g is integrable. Hence, it follows from the LDCT that f is integrable and

$$\int f\, d\mu = \lim_{n\to\infty} \int f_n\, d\mu = \lim_{n\to\infty} \sum_{j=1}^n \Big(\int h_j\, d\mu\Big) = \sum_{j=1}^\infty \Big(\int h_j\, d\mu\Big)\,.$$

2c) For each $k \in \mathbb{N}$, define $h_k : [-1,1] \to \mathbb{R}$ by $h_k(x) = \frac{1}{k}\, x^k$. As each h_k is continuous, each h_k is (Riemann-integrable and) Lebesgue-integrable on $[-1,1]$. As $f(x) = \sum_{k=1}^\infty h_k(x)$ for all $x \in [-1,1)$, by definition of f, and $\mu(\{1\}) = 0$, we see that property $i)$ is satisfied.

Moreover, we have

$$\int_{[-1,1]} |h_k|\, d\mu = \int_{-1}^1 \frac{1}{k}\, |x|^k\, dx = \frac{2}{k} \int_0^1 x^k\, dx = \frac{2}{k\,(k+1)}\,.$$

Hence,

$$\sum_{k=1}^{\infty}\Big(\int_{[-1,1]} |h_k|\, d\mu\Big) = \sum_{k=1}^{\infty} \frac{2}{k\,(k+1)} \le 2\sum_{k=1}^{\infty}\frac{1}{k^2} < \infty\,.$$

Hence, property $ii)$ is also satisfied and we can conclude from $b)$ that $f \in \mathcal{L}^1$.

Using the last assertion in $b)$, we get

$$\int_{[-1,1]} f\, d\mu = \sum_{k=1}^{\infty}\Big(\int_{[-1,1]} \frac{1}{k}\, x^k\, d\mu\Big) = \sum_{m=1}^{\infty} \frac{1}{m\,(2m+1)}$$

since

$$\int_{[-1,1]} \frac{1}{k}\, x^k\, d\mu = \frac{1}{k}\int_{-1}^{1} x^k\, dx = \frac{1}{k}\Big[\frac{x^{k+1}}{k+1}\Big]_{x=-1}^{x=1} = \begin{cases} 0 & \text{if } k \text{ is odd} \\ \frac{2}{k\,(k+1)} & \text{if } k \text{ is even} \end{cases}.$$

[Remark for interested readers: we know from elementary calculus that $f(x) = -\ln(1-x)$ when $x \in [-1, 1)$. One can show that

$$\int_{[-1,1]} f\, d\mu = -\int_{-1}^{1^-} \ln(1-x)\, dx$$

and this improper Riemann integral is easily evaluated to be equal to $2(1-\ln 2)$. It follows that $\sum_{m=1}^{\infty} \frac{1}{m\,(2m+1)} = 2(1-\ln 2)$.]

Problem 3

3a) Assume first that M is invariant under T. Let $v \in M^\perp$. For all $u \in M$, we have $T(u) \in M$, so

$$\langle u,\, T^*(v)\rangle = \langle T(u),\, v\rangle = 0\,.$$

This shows that $T^*(v) \in M^\perp$. Hence, $M^\perp$ is invariant under T^*.

Conversely, assume that $M^\perp$ is invariant under T^*. Using what we just have shown, we get that $(M^\perp)^\perp = M$ is invariant under $(T^*)^* = T$.

Finally, let $v \in E_\lambda^T$. Since $TS = ST$, we have

$$T(S(v)) = S(T(v)) = S(\lambda v) = \lambda S(v)\,.$$

This shows that $S(v) \in E_\lambda^T$. Hence, E_λ^T is invariant under S.

3b) Set $H' = E_\lambda^T$. Then H' is finite-dimensional and non-zero (since λ is an eigenvalue for T). As H' is invariant under S (see a)), we may restrict S to H'; letting $S' : H' \to H'$ being defined by $S'(v) = S(v)$, we get a map $S' \in B(H')$, which is self-adjoint since S is self-adjoint: indeed, we have

$$\langle S'(u),\, v\rangle = \langle S(u),\, v\rangle = \langle u,\, S(v)\rangle = \langle u,\, S'(v)\rangle$$

for all $u, v \in H'$.

Moreover, S' is compact (since $\dim H' < \infty$). So the spectral theorem tells us that H' has an orthonormal basis consisting of eigenvectors for S'. All vectors in this basis are then eigenvectors for S, and also for T (since $H' = E^T_\lambda$).

3c) Since $(ST)^* = T^*S^* = TS = ST$, ST is self-adjoint. As T is compact, ST is also compact. Hence, the first assertion follows from the spectral theorem for compact, self-adjoint operators on a separable Hilbert space.

Assume now that T is also one-to-one, i.e. $\mathrm{Ker}(T) = \{0\}$. As T is compact and self-adjoint, we then know that $H = \overline{\mathrm{Im}(T)}$ has an o.n.b., say $\mathcal{B}'$, consisting of eigenvectors for T. Moreover, T has only non-zero eigenvalues, and the associated eigenspaces are finite-dimensional and orthogonal to each other.

Let $\{\alpha_k\}_{k\in K}$ be a countable list of all the *different* eigenvalues of T, and set $E_k = E^T_{\alpha_k}$ for each $k \in K$. Applying b) to each E_k, we get that for each $k \in K$, there exists an o.n.b. $\mathcal{B}_k$ for E_k that consists of vectors that are also eigenvectors for S. Then $\mathcal{B} = \cup_{k\in K}\mathcal{B}_k$ is clearly orthonormal and consists of vectors that are eigenvectors for both S and T (and therefore also eigenvectors for ST).

Since each $b' \in \mathcal{B}'$ lies in one of the E_k's, we have $\mathcal{B}' \subset \mathrm{Span}\, \cup_{k\in K}\mathcal{B}_k = \mathrm{Span}\, \mathcal{B}$.

Hence, $\mathrm{Span}\, \mathcal{B}' \subset \mathrm{Span}\, \mathcal{B}$, so $H = \overline{\mathrm{Span}\, \mathcal{B}'} \subset \overline{\mathrm{Span}\, \mathcal{B}} \subset H$. Thus, $H = \overline{\mathrm{Span}\, \mathcal{B}}$.

This shows that $\mathcal{B}$ is an o.n.b. for H, so the final assertion is proved.