

Measurable Mappings

February 14, 2024

We consider maps $T : X \rightarrow X'$ between two measurable spaces (X, \mathcal{A}) and (X', \mathcal{A}') which respects the measurable structures, the σ -algebras on X and X' . These maps are useful as we can transport a measure μ , defined on (X, \mathcal{A}) , to (X', \mathcal{A}') .

Definition 7.7.1. Let (X, \mathcal{A}) , (X', \mathcal{A}') be measurable spaces. A map $T : X \rightarrow X'$ is called \mathcal{A}/\mathcal{A}' -measurable if the pre-image of every measurable set is a measurable set:

$$T^{-1}(A') \in \mathcal{A}, \quad \forall A' \in \mathcal{A}'. \quad (1)$$

- A $\mathcal{B}(\mathbb{R}^n)/\mathcal{B}(\mathbb{R}^m)$ measurable map is often called a Borel map.
- The notation $T : (X, \mathcal{A}) \rightarrow (X', \mathcal{A}')$ is often used to indicate measurability of the map T .

Lemma 7.8. Let (X, \mathcal{A}) , (X', \mathcal{A}') be measurable spaces and let $\mathcal{A}' = \sigma(\mathcal{G}')$. Then $T : X \rightarrow X'$ is \mathcal{A}/\mathcal{A}' -measurable iff $T^{-1}(G') \in \mathcal{A}$, i.e. if

$$T^{-1}(G') \in \mathcal{A}, \quad \forall G' \in \mathcal{G}'. \quad (2)$$

Theorem 7.9. Let (X_i, \mathcal{A}_i) , $i = 1, 2, 3$, be measurable spaces and $T : X_1 \rightarrow X_2$, $S : X_2 \rightarrow X_3$ be $\mathcal{A}_1/\mathcal{A}_2$ and $\mathcal{A}_2/\mathcal{A}_3$ -measurable maps respectively. Then $S \circ T : X_1 \rightarrow X_3$ is $\mathcal{A}_1/\mathcal{A}_3$ -measurable.

Definition 7.9.1. (and lemma) Let $(T_i)_{i \in I}$, $T_i : X \rightarrow X_i$, be arbitrarily many mappings from the same space X into measurable spaces (X_i, \mathcal{A}_i) . The smallest σ -algebra on X that makes all T_i simultaneously measurable is

$$\sigma(T_i : i \in I) := \sigma \left(\bigcup_{i \in I} T_i^{-1}(\mathcal{A}_i) \right) \quad (3)$$

Theorem 7.10. Let (X, \mathcal{A}) , (X', \mathcal{A}') be measurable spaces and $T : X \rightarrow X'$ be an \mathcal{A}/\mathcal{A}' -measurable map. For every measurable μ on (X, \mathcal{A}) ,

$$\mu'(A') := \mu(T^{-1}(A')), \quad A' \in \mathcal{A}', \quad (4)$$

defines a measure on (X', \mathcal{A}') .

Definition 7.10.1. The measure $\mu'(\cdot)$ in the above theorem is called the push forward or image measure of μ under T and it is denoted as $T(\mu)(\cdot)$, $T_{*\mu}(\cdot)$ or $\mu \circ T^{-1}(\cdot)$.

Theorem 7.11. *If $T \in \mathbb{R}^{n \times n}$ is an orthogonal matrix, then $\lambda^n = T(\lambda^n)$.*

Theorem 7.12. *Let $S \in \mathbb{R}^{n \times n}$ be an invertible matrix. Then*

$$S(\lambda^n) = |\det S| \lambda^n = |\det S|^{-1} \lambda^n. \quad (5)$$

Corollary 7.12.1. *Lebesgue measure is invariant under motions: $\lambda^n = M(\lambda^n)$ for all motions M in \mathbb{R}^n . In particular, congruent sets have the same measure. Two sets of points are called congruent if, and only if, one can be transformed into the other by an isometry*