# UNIVERSITY OF OSLO

## Faculty of mathematics and natural sciences

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| Exam in: | MAT4410 — Advanced linear analysis |
| Day of examination: | Wednesday, December 11, 2019 |
| Examination hours: | 14:30 – 18:30 |

This problem set consists of 2 pages.

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| Appendices: | None |
| Permitted aids: | None |

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

All subproblems (1a, 1b,...) carry the same weight.

## Problem 1

### 1a

Formulate the Uniform Boundedness Principle.

### 1b

Let $X$ be a locally compact space. Assume we are given functions $f_n$ $(n \in \mathbb{N})$ and $f$ in $C_0(X)$ such that

$$\int_X f_n \, d\mu \underset{n}{\to} \int_X f \, d\mu \tag{1}$$

for all regular complex Borel measures $\mu$ on $X$. Show that $\sup_n \|f_n\| < \infty$ (where $\|f_n\|$ denotes the supremum-norm of $f_n$) and $f_n \to f$ pointwise.

### 1c

Show that conversely, if $f_n, f \in C_0(X)$ are such that $\sup_n \|f_n\| < \infty$ and $f_n \to f$ pointwise, then we have (1) for all complex Borel measures $\mu$ on $X$.

## Problem 2

### 2a

Formulate the Fubini–Tonelli theorem.

### 2b

Consider the interval $[0, 1]$ and the $\sigma$-algebra $\mathcal{B}$ of Borel subsets of $[0, 1]$. Let $\lambda$ and $\mu$ be the Lebesgue and counting measures, respectively, on $([0, 1], \mathcal{B})$.

*(Continued on page 2.)*

(Thus $\mu(A)$ equals the number of elements of $A$.) Denote by $D$ the diagonal $\{(x,x)|x \in [0,1]\}$ in $[0,1]\times[0,1]$ and consider the characteristic function $\chi_D$ of $D$.

Compute the integrals

$$\int_{[0,1]}\left(\int_{[0,1]}\chi_D(x,y)d\lambda(x)\right)d\mu(y) \quad\text{and}\quad \int_{[0,1]}\left(\int_{[0,1]}\chi_D(x,y)d\mu(y)\right)d\lambda(x).$$

Why the Fubini–Tonelli theorem does not apply in this case?

## Problem 3

### 3a

Formulate the Radon–Nikodym theorem.

### 3b

Consider the measures $\lambda$ and $\mu$ from Problem 2b. Show that $\lambda \ll \mu$, but there is no Radon–Nikodym derivative $\frac{d\lambda}{d\mu}$. What goes wrong with the Radon–Nikodym theorem here?

### 3c

Show that there is no Lebesgue decomposition of $\mu$ with respect to $\lambda$, that is, we cannot write $\mu = \mu_a + \mu_s$, with $\mu_a \ll \lambda$ and $\mu_s \perp \lambda$.

## Problem 4

Assume $f\colon \mathbb{R}^n \to \mathbb{C}$ is an integrable function and consider its Fourier transform

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x)e^{-2\pi i\xi x}dx,$$

where $\xi x$ denotes the scalar product $\xi_1 x_1 + \cdots + \xi_n x_n$. For $k \in \mathbb{N}$, consider also the function $\phi_k$ on $\mathbb{R}^n$ defined by

$$\phi_k(x) = k^n e^{-\pi k^2 x^2}.$$

### 4a

Show that the functions $\phi_k * f$ are continuous and for every point of continuity $x$ of $f$ we have

$$(\phi_k * f)(x) \underset{k}{\to} f(x).$$

### 4b

Show that if $f$ is continuous at 0 and $\hat{f} \geq 0$, then $\hat{f}$ is integrable and

$$\int_{\mathbb{R}^n} \hat{f}(\xi)d\xi = f(0).$$

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