Suggested solution to the exam in MAT3400/4400, 2014

Problem 1 Part 1a. This is a standard result proved during the course: let $g_n = \sum_{k=1}^{n} f_k$ for $n \in \mathbb{N}$, and show that the sequence $\{g_n\}_n$ satisfies the hypotheses of the MCT. Then apply the theorem.

Part 1b. An example of a measure space her is $\Omega = [0, \infty)$ with the Lebesgue measure, where $A_n = [n, n+1)$ for $n \geq 1$. Let $f_n = 1/n\mu(A_n)^{-1}\chi_{A_n}$ for $n \in \mathbb{N}$. Applying part 1a gives

$$\int_\Omega f\, d\mu = \int_\Omega (\sum_{n=1}^{\infty} f_n)\, d\mu = \sum_{n=1}^{\infty} \int_\Omega f_n\, d\mu = \sum_{n=1}^{\infty} \frac{1}{n}.$$

Since the series $\sum_{n=1}^{\infty} 1/n$ is divergent and $f \geq 0$, it follows that $f \notin \mathcal{L}^1(\Omega, \mathcal{A}, \mu)$.

The other claim is that $\int_\Omega |f|^2\, d\mu < \infty$. Since the sets A_n are pairwise disjoint, it follows that $f^2 = (\sum_n f_n)^2 = \sum_n f_n^2$. Now apply part 1a to the series $\sum_n f_n^2$ and use that $\sum_n 1/n^2$ is convergent.

Part 1c. Since $\mu(\Omega) < \infty$, we have $1 \in \mathcal{L}^2(\Omega, \mathcal{A}, \mu)$. Let $f \in \mathcal{L}^2(\Omega, \mathcal{A}, \mu)$. By Hölder's inequality (for $p = q = 2$) it follows that

$$\int_\Omega |f|\, d\mu = \int_\Omega |f|\cdot 1\, d\mu \leq \big(\int_\Omega |f|^2\, d\mu\big)^{\frac{1}{2}} \big(\int_\Omega 1\, d\mu\big)^{\frac{1}{2}} = \mu(\Omega)^{\frac{1}{2}} \big(\int_\Omega |f|^2\, d\mu\big)^{\frac{1}{2}} < \infty,$$

which implies $f \in \mathcal{L}^1(\Omega, \mathcal{A}, \mu)$.

Problem 2. The following are given: the measure space $[0, 1]$ with the Borel σ-algebra $\mathcal{B}_{[0,1]}$ and the Lebesgue measure λ restricted to $\mathcal{B}_{[0,1]}$; moreover, $\{u_m\}_{m\in\mathbb{N}}$ and $\{v_n\}_{n\in\mathbb{N}}$ are two orthonormal bases in $L^2(\lambda)$.

Part 2a. The claim that $\{w_{n,m}\}_{n,m\in\mathbb{N}}$ defined by $w_{n,m}(x, y) = u_m(x)v_n(y)$ for $(x, y) \in [0, 1]\times[0, 1]$ form an orthonormal family in $L^2(\lambda\times\lambda)$ follows by applying Fubini's theorem (applying the theorem is justified since $L^2(\lambda) \subset L^1(\lambda)$ by problem 1c):

$$\begin{aligned}\int_{[0,1]^2} w_{(m,n)}\overline{w_{m',n'}} d(\lambda \times \lambda) &= \int_{[0,1]} u_m(x)\overline{u_{m'}(x)} d\lambda(x) \int_{[0,1]} v_n(y)\overline{v_{n'}(y)} d\lambda(y) \\ &= \delta_{m,m'}\delta_{n,n'}.\end{aligned}$$

Part 2b. Let $f \in L^2(\lambda \times \lambda)$. For each $n \in \mathbb{N}$, let F_n be the function

$$F_n(x) = \int_{[0,1]} f(x, y)\overline{v_n(y)}\, d\lambda(y).$$

The first claim is that F_n defined for λ-almost all $x \in [0, 1]$ is in $L^2(\lambda)$. The definition of F_n suggests that $F_n(x)$ may be considered as the inner-product of the functions $f(x, \cdot)$ and v_n, both views as functions of the variable y, and this would allow applying the Cauchy-Schwarz inequality. It is given that $v_n \in L^2(\lambda)$, so it remains to justify that $f(x, \cdot) \in L^2(\lambda)$ for λ-almost all $x \in [0, 1]$. Now, $f \in L^2(\lambda \times \lambda)$ implies that $|f|^2 \in L^1(\lambda \times \lambda)$. Applying Fubini's theorem gives

$$\int_{[0,1]^2} |f|^2 d(\lambda \times \lambda) = \int_{[0,1]} \big(\int_{[0,1]} |f(x,y)|^2 d\lambda(y)\big) d\lambda(x) < \infty.$$

Thus, $\int_{[0,1]} |f(x,y)|^2 d\lambda(y) < \infty$ for λ-almost all $x \in [0,1]$, as wanted. Now apply the Cauchy-Schwarz inequality to get

$$\begin{aligned}\int_{[0,1]} |F_n|^2 d\lambda &= \int_{[0,1]} |\int_{[0,1]} f(x,y)\overline{v_n(y)}\, d\lambda(y)|^2 d\lambda(x) \\ &\leq \int_{[0,1]} \Big(\big(\int_{[0,1]} |f(x,y)|^2 d\lambda(y)\big)\big(\int_{[0,1]} |v_n(y)|^2 d\lambda(y)\big) \Big) d\lambda(x)\end{aligned}$$

Since v_n is an element of an orthonormal basis, $\|v_n\|_2 = 1$, so the last expression above is finite and therefore $F \in L^2(\lambda)$.

The equality of the inner product $\big(f, w_{n,m}\big)$ with (F_n, u_m) is now an immediate application of Fubini's theorem, because $w_{n,m}$ is a product of the function u_m in the variable x and of the function v_n in the variable y.

Part 2c. To conclude that $\{w_{n,m}\}_{n,m\in\mathbb{N}}$ forms an orthonormal basis for $L^2(\lambda\times\lambda)$, let $f \in L^2(\lambda\times\lambda)$. Compute $\|f\|_2^2$ using Fubini's theorem, Parseval's identity for $f(x,\cdot)$ and for each F_n, and also using problem 1a and problem 2b:

$$\begin{aligned}\int_{[0,1]^2} |f|^2 d(\lambda \times \lambda) &= \int_{[0,1]} \big(\int_{[0,1]} |f(x,y)|^2 d\lambda(y)\big) d\lambda(x) \\ &= \int_{[0,1]} \|f(x,\cdot)\|_2^2 d\lambda(x) \\ &= \int_{[0,1]} \sum_{n=1}^{\infty} |\big(f(x,\cdot), v_n\big)|^2 d\lambda(x) \\ &= \sum_{n=1}^{\infty} \int_{[0,1]} |\big(f(x,\cdot), v_n\big)|^2 d\lambda(x) \\ &= \sum_{n=1}^{\infty} \int_{[0,1]} |F_n(x)|^2 d\lambda(x) \\ &= \sum_{n=1}^{\infty} \|F_n\|_2^2 \\ &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} |\big(F_n, u_m\big)|^2 \\ &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} |\big(f, w_{n,m}\big)|^2.\end{aligned}$$

This shows that Parseval's identity is satisfied by $\{w_{m,n}\}$, so this is an orthonormal basis as claimed.

Problem 3.

An operator $R \in B(H)$ is called an orthogonal reflection if there exists a closed subspace M of H such that

$$x + R(x) \in M \text{ and } x - R(x) \in M^{\perp} \text{ for every } x \in H.$$

Part 3a. Assume that R is an orthogonal reflection. Need to show that $\frac{1}{2}(R + I)$ is an orthogonal projection. Note that for every $x \in H$ we can write

$$x = \frac{1}{2}(x + R(x)) + \frac{1}{2}(x - R(x)),$$

where $\frac{1}{2}(x + R(x)) \in M$ and $\frac{1}{2}(x - R(x)) \in M^{\perp}$. On the other hand, since M is a closed subspace, there is an associated orthogonal projection P_M in $B(H)$ such that if $x = y + z$ is the unique decomposition of x with respect to M, with $y \in M$ and $z \in M^{\perp}$, then $P_M(x) = y$. By uniqueness of the orthogonal decomposition, it follows that

$$P_M(x) = y = \frac{1}{2}(x + R(x)).$$

Since this is true for arbitrary x, part 1 is proved.

Part 3b. Assume this time that $R \in B(H)$ is such that $P = \frac{1}{2}(R+I)$ is an orthogonal projection in $B(H)$. Let M be the closed subspace equal to $P(H)$. For $x \in H$ we have $x + R(x) = 2P(x) \in M$. Since $I - P$ is the orthogonal projection onto $M^{\perp}$, it follows that $I - P = I - 1/2(R + I) = 1/2(I - R)$, so $x - R(x) = 2(I - P)(x) \in M^{\perp}$.

Part 3c. Assume first that R is an orthogonal reflection corresponding to a closed subspace M. By part 3a, $R = 2P_M - I$, so $R^* = 2P_M^* - I = 2P_M - I = R$ because P_M is a projection. Also, $R^2 = (2P_M - I)^2 = 4P_M^2 - 4P_M + I = I$, again because P_M is a projection. If conversely $R = R^*$ and $R^2 = I$, then verify directly that the operator $1/2(I + R)$ is self-adjoint and has square power equal to the operator itself, thus it is a projection.

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