Convergence Theorems and Their Applications (lecture 9, 8. Feb.)

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- To interchange limits and integrals in **Riemann integrals** one typically has to assume uniform convergence. ¡- The set of Riemann integrable functions is somewhat limited, see theorem 12.19

Theorem 12.13 (Generalization of Beppo Levi, monotone convergence)**.**

(i) *Let* $(u_n)_{n\in\mathbb{N}} \subset \mathcal{L}^1(\mu)$ *be s.t.* $u_1 \leq u_2 \leq \ldots$ *with limit* $u := \sup_{n\in\mathbb{N}} u_n = \lim_{n\to\infty} u_n$*. Then* $u \in \mathcal{L}^1(\mu)$ ***iff***

$$\sup_{n\in\mathbb{N}} \int u_n d\mu < +\infty,$$

in which case

$$\sup_{n\in\mathbb{N}} \int u_n d\mu = \int \sup_{n\in\mathbb{N}} u_n d\mu.$$

(ii) *Same thing only with a decreasing sequence* $\ldots > -\infty$ *in which case*

$$\inf_{n\in\mathbb{N}} \int u_n d\mu = \int \inf_{n\in\mathbb{N}} u_n d\mu.$$

Theorem 12.14 (Lebesgue; dominated convergence)**.** *Let* $(u_n)_{n\in\mathbb{N}} \subset \mathcal{L}^1(\mu)$ *s.t.*

(a) $|u_n|(x) \leq w(x)$, $w \in \mathcal{L}^1(\mu)$,

(b) $u(x) = \lim_{n\to\infty} u_n(x)$ *exists in* $\bar{\mathbb{R}}$,

then $u \in \mathcal{L}^1(\mu)$ *and we have*

(i) $\lim\limits_{n\to\infty} \int |u_n - u| d\mu = 0$;

(ii) $\lim\limits_{n\to\infty} \int u_n d\mu = \int \lim\limits_{n\to\infty} u_n d\mu = \int u d\mu$;

Application 1: Parameter-Dependent Integrals

- We are interested in questions of the sort, when is

$$U(t) := \int u(t,x)\mu(dx),\ t \in (a,b),$$

again a smooth function of t? The answer involves interchange of limits and integration. Also, it turns out to better understand Riemann integrability, we need the Lebesgue integral.

Theorem 12.15 (continuity lemma)**.** *Let $\emptyset \neq (a,b) \subset \mathbb{R}$ be a non-degenerate open interval and $u : (a,b) \times X \to \mathbb{R}$ satisfy*

(a) $x \mapsto u(t,x)$ is in $\mathcal{L}^1(\mu)$ for every fixed $t \in (a,b)$;

(b) $t \mapsto u(t,x)$ is continuous for every fixed $x \in X$;

(c) $|u(t,x)| \leq w(x)$ for all $(t,x) \in (a,b) \times X$ and some $w \in \mathcal{L}^1(\mu)$.

Then the function $U : (a,b) \to \mathbb{R}$ given by

$$t \mapsto U(t) := \int u(t,x)\mu(dx) \tag{1}$$

is continuous.

Theorem 12.16 (differentiability lemma)**.** *Let $\emptyset \leq (a,b) \subset \mathbb{R}$ be a non-degenerate open interval and $u : (a,b) \times X \to \mathbb{R}$ satisfy*

(a) Same

(b) Same

(c) $|\partial_t u(t,x)| \leq w(x)$ for all $(t,x) \in (a,b) \times X$ and some $w \in \mathcal{L}^1(\mu)$.

Then the function in 1 is differentiable and its derivative is

$$\frac{d}{dt}U(t) = \frac{d}{dt}\int u(t,x)\mu(dx) = \int \frac{\partial}{\partial t}u(t,x)\mu(dx). \tag{2}$$

Application 2: Riemann vs Lebesgue Integration

Consider only $(X, \mathscr{A}, \mu) = (\mathbb{R}, \mathscr{B}(\mathbb{R}), \lambda)$.

Definition 12.17 (The Riemann Inegral)**.** Consider on the finite interval $[a,b] \subset \mathbb{R}$ the partition

$$\Pi := \{a = t_0 < t_1 < ... < t_k < b\}, k = k(\Pi), \tag{3}$$

and introduce

$$S_\Pi[u] := \sum_{i=1}^{k(\Pi)} m_i(t_i - t_{i-1}), \qquad m_i := \inf_{x\in[t_{i-1},t_i]} u(x), \tag{4}$$

$$S^\Pi[u] := \sum_{i=1}^{k(\Pi)} M_i(t_i - t_{i-1}), \qquad M_i := \sup_{x\in[t_{i-1},t_i]} u(x). \tag{5}$$

$$\tag{6}$$

A bounded function $u : [a,b] \to \mathbb{R}$ is said to be **Riemann integrable** if the values

$$\underline{\int} u := \sup_\Pi S_\Pi[u] = \inf_\Pi S^\Pi[u] =: \bar{\int} u \tag{7}$$

coincide and are finite. Their common value is called the **Riemann integral** of u and denoted by $(R)\int\limits_a^b u(x)dx$ or $\int\limits_a^b u(x)dx$.

Theorem 12.18. *Let $u : [a,b] \to \mathbb{R}$ be a* ***measurable*** *and* ***Riemann integrable*** *function. Then*

$$u \in \mathcal{L}^1(\lambda) \text{ and } \int\limits_{[a,b]} ud\lambda = \int\limits_a^b u(x)dx. \tag{8}$$

Theorem 12.19. *Let $u : [a,b] \to \mathbb{R}$ be a bounded function, it is Riemann integrable* ***iff*** *the points in (a,b) where u is discontinuous are a (subset of) Borel measurable null set.*

Improper Riemann Integrals

- The Lebesgue integral extends the (*proper*) Riemann integral. However, there is a further extension of the Riemann integral which cannot be captured by Lebesgue's theory. u is Lebesgue integrable *iff* $|u|$ ha finite Lebesgue integral. ¡- The Lebesgue integral does not respect sign-changes and cancellations. However, the following *improper Riemann integral* does:

$$(R)\int\limits_0^\infty u(x)dx := \lim_{n\to\infty} (R)\int\limits_0^a u(x)dx. \tag{9}$$

Corollary 12.20. *Let $u : [0,\infty) \to \mathbb{R}$ be a measurable, Riemann integrable function for every interval $[0,N]$, $N \in \mathbb{N}$. Then $u \in \mathcal{L}^1[0,\infty)$* ***iff***

$$\lim_{N\to\infty} (R)\int\limits_0^N |u(x)|dx < \infty. \tag{10}$$

In this case, $(R)\int_0^\infty u(x)dx = \int_{[0,\infty)} ud\lambda$

Example of a function which is *improperly Riemann integrable* but **not** *Lebesgue integrable*:

$$f(x) = \frac{\sin(x)}{x}. \tag{11}$$

Proposition 12.21 (appearing as example 12.13 in Schilling). *Let* $f_\alpha(x) := x^\alpha, x > 0$ *and* $\alpha \in \mathbb{R}$. *Then*

(i) $f_(\alpha) \in \mathcal{L}^1(0, 1) \Leftrightarrow \alpha > -1$.

(ii) $f_(\alpha) \in \mathcal{L}^1[1, \infty) \Leftrightarrow \alpha < -1$.