MAT4400: Notes on Linear analysis

Morten Tryti Berg and Isak Cecil Onsager Rukan.

March 6, 2024

3 σ-Algebras

Definition 3.1 (σ-Algebra)**.** A family $\mathscr{A}$ of subsets of X with:

(i) $X \in \mathscr{A}$,

(ii) $A \in \mathscr{A} \Rightarrow A^c \in \mathscr{A}$,

(iii) $(A_n)_{n \in \mathbb{N}} \in \mathscr{A} \Rightarrow \bigcup\limits_{n \in \mathbb{N}}$

Theorem 3.2 (and Definition)**.**

(i) The intersection of arbitrarily many σ-algebras in X is againg a σ-algebra in X.

(ii) For every system of sets $p \subset \mathscr{P}(X)$ there exists a smallestσ-algebra containingp. This is the σ-algebra generated by p, denoted $\sigma(p)$, and $\sigma(p)$ is called its generator.

Definition 3.3 (Borel)**.** The σ-algebra $\sigma(\mathcal{O})$ generated by the open sets $\mathcal{O} = \mathcal{O}_{\mathbb{R}^n}$ of $\mathbb{R}^n$ is called **Borel σ-algebra**, and its members are called **Borel sets** or **Borel measurable sets**.

5 Uniqueness of Measures

Lemma 5.1. *A Dynkin system D is a σ-algebra iff it is stable under finite intersections, i.e. $A, B \in D \Rightarrow A \cap B \in D$.*

Theorem 5.2 (Dynkin)**.** *Assume X is a set, S is a collection of subsets of X closed under finite intersections, that is, if $A, B \in S \Rightarrow A \cap B \in S$. Then $D(S) = \sigma(S)$.*

Theorem 5.3 (uniqueness of measures)**.** *Let (X, B) be a measurable space, and $S \subset P(X)$ be the generator of B, i.e. $B = \sigma(S)$. If S satisfies the following conditions:*

1. *S is stable under finite intersections ($\cap$-stable), i.e. $A, C \in S \Rightarrow A \cap C \in S$.*
2. *There exists an exhausting sequence $(G_n)_{N \in \mathbb{N}} \subset$ with $G_N \uparrow X$. Assume also that there are two measures μ, ν satisfying:*
3. *$\mu(A) = \nu(A),\ \forall A \in S$.*
4. *$\mu(G_n) = \nu(G_n) < \infty$.*

Then $\mu = \nu$.

6 Existence of Measures

Theorem 6.1 (Carathéodory)**.** *Let* $S \subset P(X)$ *be a semi-ring and* $\mu : S \to [0, \infty)$ *a pre-measure. Then* μ *has an extension to a measure* μ^* *on* $\sigma(S)$*, i.e. that* $\mu(s) = \mu^*(s),\ \forall\ s \in \sigma(S)$*.*

Also, if S *contains an exhausting sequence,* $S_n \uparrow X$*, s.t.* $\mu(S_n) < \infty$*, then the extension is unique.*

7 Measurable Mappings

We consider maps $T : X \to X'$ between two measurable spaces $(X, \mathcal{A})$ and $(X', \mathcal{A}')$ which respects the measurable structurs, the σ-algbras on X and X'. These maps are useful as we can transport a measure μ, defined on $(X, \mathcal{A})$, to $(X', \mathcal{A}')$.

Definition 7.1. Let $(X, \mathcal{A})$, $(X', \mathcal{A}')$ b measurable spaces. A map $T : X \to X'$ is called $\mathcal{A}/\mathcal{A}'$-measurable if the pre-imag of every measurable set is a measurable set:

$$T^{-1}(A') \in \mathcal{A}, \quad \forall A' \in \mathcal{A}'. \tag{1}$$

- A $\mathcal{B}(\mathbb{R}^n)/\mathcal{B}(\mathbb{R}^m)$ measurable map is often called a Borel map.
- The notation $T : (X, \mathcal{A}) \to (X', \mathcal{A}')$ is often used to indicate measurability of the map T.

Lemma 7.2. *Let $(X, \mathcal{A})$, $(x', \mathcal{A}')$ be measurable spaces and let $\mathcal{A}' = \sigma(\mathcal{G}')$. Then $T : X \to X'$ is $\mathcal{A}/\mathcal{A}'$-measurable iff $T^{-1}(\mathcal{G}') \subset \mathcal{A}$, i.e. if*

$$T^{-1}(G') \in \mathcal{A}, \ \ \forall G' \in \mathcal{G}'. \tag{2}$$

Theorem 7.3. *Let $(X_i, \mathcal{A}_i)$, $\ i = 1, 2, 3$, be measurable spaces and $T : X_1 \to X_2$, $S : X_2 \to X_3$ be $\mathcal{A}_1/\mathcal{A}_2$ and $\mathcal{A}_2/\mathcal{A}_3$-measurable maps respectivly. Then $S \circ T : X_1 \to X_3$ is $\mathcal{A}_1/\mathcal{A}_3$-measurable.*

Corollary 7.4. *Every continuous map betwen metric spaces is a Borel map.*

Definition 7.5. (and lemma) Let $(T_i)_{i\in I}$, $T_I : X \to X_i$, be arbitrarily many mappings from the same space X into measurable spaces $(X_i, \mathcal{A}_i)$. The smallest σ-algebra on X that makes all T_i simultanously measurable is

$$\sigma(T_i : i \in I) := \sigma\left(\bigcup_{i\in I} T_i^{-1}(\mathcal{A}_i)\right) \tag{3}$$

Corollary 7.6. *A function $f : (X, \mathcal{B}) \to \mathbb{R}$ is measurable if $f((a, +\infty)) \in \mathcal{B}$, $\ \forall a \in \mathbb{R}$.*

Corollary 7.7. *Assume $(X, \mathcal{B})$ is a measurable space, (Y, d) is a metric space, $(f_n : (X, \mathcal{B}) \to Y)_{n=1}^{\infty}$ is a sequence of measurable maps. Assume this sequence of images $(f_n(x))_{n=1}^{\infty}$ is convergent in Y $\forall x \in X$. Define*

$$f : X \to Y, \ \ \text{by } f(x) = \lim_{n\to\infty} f_n(x). \tag{4}$$

Then f is measurable.

Theorem 7.8. *Let $(X, \mathcal{A})$, $(X', \mathcal{A}')$ be measurable spaces and $T : X \to X'$ be an $\mathcal{A}/\mathcal{A}'$-measurable map. For every measurable μ on $(X, \mathcal{A})$,*

$$\mu'(A') := \mu(T^{-1}(A')), \quad A' \in \mathcal{A}', \tag{5}$$

defines a measure on $(X', \mathcal{A}')$.

Definition 7.9. The measure $\mu'(\cdot)$ in the above theorem is called the push forward or image measure of μ under T and it is denoted as $T(\mu)(\cdot)$, $T_{*\mu}(\cdot)$ or $\mu \circ T^{-1}(\cdot)$.

Theorem 7.10. *If $T \in \mathbb{R}^{n\times n}$ is an orthogonal matrix, then $\lambda^n = T(\lambda^n)$.*

Theorem 7.11. *Let $S \in \mathbb{R}^{n\times n}$ be an invertible matrix. Then*

$$S(\lambda^n) = |\det s^{-1}|\lambda^n = |\det S|^{-1}\lambda^n. \tag{6}$$

Corollary 7.12. *Lebesgue measure is invariant under motions: $\lambda^n = M(\lambda^n)$ for all motions M in $\mathbb{R}^n$. In particular, congruent sets have the same measure. Two sets of points are called congruent if, and only if, one can be transformed into the other by an isometry*

8 Measurable Functions

A *measurable function* is a measurable map $u : X \to \mathbb{R}$ from some measurable space $(X, \mathscr{A})$ to $(\mathbb{R}, \mathscr{B}(\mathbb{R}^1))$. They play central roles in the theory of integration.

We recall that $u : X \to \mathbb{R}$ is $\mathscr{A}/\mathscr{B}(\mathbb{R}^1)$-measurable if

$$u^{-1}(B) \in \mathscr{A}, \ \ \forall B \in \mathscr{B}(\mathbb{R}^1). \tag{7}$$

Moreover from a lemma from chapter 7, we actually only need to show that

$$u^{-1}(G) \in \mathscr{A}, \ \ \forall G \in \mathcal{G} \text{ where } \mathcal{G} \text{ generates } \mathscr{B}(\mathbb{R}^1). \tag{8}$$

Proposition 8.1.

1 If $f, g : (X, \mathscr{B}) \to \mathbb{C}$ are measurable, then the function $f+g$, $f \cdot g$, cf, $(c \in \mathbb{C})$ are measurable.

2 If $b : \mathbb{C} \to \mathbb{C}$ is Borel and $b : (\mathbb{C}, \mathscr{B}) \to \mathbb{C}$ is measurable, then $b \circ f$ is measurable.

3 If $f(x) = \lim_{n\to\infty} f_n(x)$, $x \in X$ and f_n are measurable, then f is measurable.

4 If $X = \bigcup_{n=1}^{\infty} A_n$, $(A_n \in \mathscr{B})$, $f|_{A_n} : (A_n, \mathscr{B}_{A_n}) \to \mathbb{C}$ is measurable $\forall n$, then f is measurable.

Definition 8.2. Given a measurable space $(X, \mathscr{B})$, a measurable function $f : (X, \mathscr{B}) \to \mathbb{C}$ is called simple if

$$f(x) = \sum_{k=1}^{N} c_k \mathbb{1}_{A_k}(x), \tag{9}$$

for some $c_k \in \mathbb{C}$, $A_k \in \mathscr{B}$, where $\mathbb{1}$ is the characteristic function,

$$\mathbb{1}_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A. \end{cases} \tag{10}$$

The representation of simple function is **not** unique. We denote the standard representation of f by

$$f(x) = \sum_{n=0}^{N} z_n \mathbb{1}_{B_n}(x), \quad N \in \mathbb{N},\ z_n \in \mathbb{R},\ B_n \in \mathscr{A},\ X = \bigcup_{n=1}^{N} B_n, \ \text{ for } B_n \cap B_m = \emptyset, \ \ n \neq m. \tag{11}$$

The set of simple functions is denoted $\mathcal{E}(\mathscr{A})$ of $\mathcal{E}$.

Definition 8.3. Assume μ is a measure on $(X, \mathscr{B})$. Given a *positive* simple function

$$f = \sum_{k=1}^{N} c_k \mathbb{1}_{A_k}, \quad (c_k \geq 0). \tag{12}$$

We define

$$\int_X f d\mu = \sum_{k=1}^{n} c_k \mu(A_k) \in [0, +\infty]. \tag{13}$$

We also denote this by $I_\mu(f)$.

Lemma 8.4. *This is well defined, that is, $\int_x f d\mu$ does not depend on the presentation of the simple function f.*

Properties 8.5. *For every positive simple function*

1 $\int_X cf d\mu = c \int_X f d\mu, \quad$ *for only* $c \geq 0$

2 $\int_X (f + g) d\mu = \int_X f d\mu + \int_X g d\mu$.

Corollary 8.6. *If $f \geq g \geq 0$ are simple functions, then*

$$\int_X f d\mu \geq \int_X g d\mu. \tag{14}$$

Definition 8.7. If $f : X \to [0, +\infty)$ is measurable, then we define

$$\int_X f d\mu = \sup \left\{ \int_X g d\mu \ : \ f \geq g \geq 0, \ \ g \text{ is simple} \right\} \tag{15}$$

Remark. *This means that any measurable function can be approximated by simple functions.*

Properties 8.8. *Measurable functions like this have the following properties*

1 $\int_X cf d\mu = c \int_X f d\mu, \quad \forall c \geq 0.$

2 If $f \geq g \geq 0$, then $\int_X f d\mu \geq \int_X g d\mu$ for any measurable g, f.

3 If $f \geq 0$ is simple, then $\int_X f d\mu$ is the same value as obtained before.

To advance in measure theory we consider measurable functions

$$f : X \to [0, +\infty].$$

Measurability is understood w.r.t the $\sigma-$algebra $\mathscr{B}([0, +\infty])$ generated by $\mathscr{B}([0, +\infty))$ and $\{+\infty\}$. In other words, $A \subset [0, +\infty] \in B([0, +\infty])$ iff $A \cap [0, +\infty) \in \mathscr{B}([0, +\infty)$.

Remark. *Hence* $f : X \to [0, +\infty]$ *is measurable iff* $f^{-1}(A)$ *is measurable* $\forall A \in \mathscr{B}([0, +\infty))$.

Definition 8.9. For measurable functions $f_X \to [0, +\infty]$, we define

$$\int_X f d\mu = \sup \left\{ \int_x g d\mu \ : \ f \geq g \geq 0 \ : \ g \text{ is simple} \right\} \in [0, +\infty]. \tag{16}$$

Theorem 8.10. ***Monotone convergence theorem*** *Assume* $(X, \mathscr{B}, \mu)$ *is a measure space,* $(f)_{n=1}^{\infty}$ *is an increasing sequence of measurable positive functions* $f_n : X \to [0, +\infty]$. *Define* $f(x) = \lim_{n\to\infty} f_n(x)$. *Then* f *is measurable and*

$$\int_X f d\mu = \lim_{n\to\infty} \int_X f_n d\mu. \tag{17}$$

Theorem 8.11. *Assume* $(X, \mathscr{B})$ *is a measurable space and* $f : X \to [0, +\infty]$ *is measurable. Then there are simple functions* g_n, *s.t.*

$$0 \leq g_1 \leq g_2 \leq \dots \ , \ \ g_n(x) \to f(x), \ \ \forall x \in X.$$

Moreover, if f *is bounded, we can choose* g_n *s.t. the convergence is uniform, that is,*

$$\lim_{n\to\infty} \sup_{x\in X} |g_n(x) - f(x)| = 0. \tag{18}$$

9 Integration of measurable functions

Through this chapter $(X, \mathscr{A}, \mu)$ will be some measure space. Recall that $\mathcal{M}^+(\mathscr{A})$ $[\mathcal{M}^+_{\overline{\mathbb{R}}}(\mathscr{A})]$ are the $\mathscr{A}$-measurable positive functions and $\mathcal{E}(\mathscr{A})$ $[\mathcal{E}^+_{\overline{\mathbb{R}}}(\mathscr{A})]$ are the positive and simple functions.

The fundamental idea of *Integration* is to measure the area between the graph of the function and the abscissa. For positive simple functions $f \in \mathcal{E}^+(\mathscr{A})$ in standard representation, this is done easily

$$\text{if } f = \sum_{i=0}^{M} y_i \mathbb{1}_{A_i} \in \mathcal{E}^+(\mathscr{A}) \quad \text{then } \sum_{i=0}^{M} y_i \mu(A_i) \tag{19}$$

would be the μ-area enclosed by the graph and the abscissa. We note that the representation of f should not impact the integral of f.

Lemma 9.1. *Let $\sum_{i=0}^{M} y_i \mathbb{1}_{A_i} = \sum_{k=0}^{N} z_k \mathbb{1}_{B_k}$ be two standard representations of the same function $f \in \mathcal{E}^+(\mathscr{A})$. Then*

$$\sum_{i=0}^{M} y_i \mu(A_i) = \sum_{k=0}^{N} z_k \mu(B_k). \tag{20}$$

Definition 9.2. Let $f = \sum_{i=0}^{M} y_i \mathbb{1}_{A_i} \in \mathcal{E}^+(\mathscr{A})$ be a simple function in standard representation. Then the number

$$I_\mu(f) = \sum_{i=0}^{M} y_i \mu(A_i) \in [0, \infty] \tag{21}$$

(which is independent of the representation of f) is called the μ-integral of f.

Proposition 9.3. *Let $f, g \in \mathcal{E}^+(\mathscr{A})$. Then*

(i) $I_\mu(\mathbb{1}_A) = \mu(A) \quad \forall A \in \mathscr{A}$.

(ii) $I_\mu(\lambda f) = \lambda I_\mu(f) \quad \forall \lambda \geq 0$.

(iii) $I_\mu(f + g) = I_\mu(f) + I_\mu(g)$.

(iv) $f \leq g \Rightarrow I_\mu(f) \leq I_\mu(g)$.

In theorem 8.8 we saw that we could for every $u \in \mathcal{M}^+(\mathscr{A})$ write it as an increasing limit of simple functions. By corollary 8.10, the suprema of simple functions are again measurable, so that

$$u \in \mathcal{M}^+(\mathscr{A}) \Leftrightarrow u = \sup_{n \in \mathbb{N}} f_n, \quad f \in \mathcal{E}^+(\mathscr{A}), \quad f_n \leq f_{n+1} \leq \dots.$$

We will use this to "inscribe" simple functions (which we know how to integrate) below the graph of a positive measurable function u and exhaust the μ-area below u.

Definition 9.4. Let $(X, \mathscr{A}, \mu)$ be a measure space. The (μ)-integral of a positive function $u \in \mathcal{M}^+_{\bar{\mathbb{R}}}(\mathscr{A})$ is given by

$$\int u d\mu = \sup \left\{ I_\mu(g) : g \leq u,\ g \in \mathcal{E}^+(\mathscr{A}) \right\} \in [0, +\infty]. \tag{22}$$

If we need to emphasize the *integration variable*, we write $\int u(x)\mu(dx)$. The key observation is that the integral $\int \ldots d\mu$ extends I_μ.

Lemma 9.5. *For all $f \in \mathcal{E}^+(\mathscr{A})$ we have $\int f d\mu = I_\mu(f)$.*

The next theorem is one of many convergence theorems. It shows that we could have defined 22 using any increasing sequence $f_n \uparrow u$ of simple functions $f_n \in \mathcal{E}^+(\mathscr{A})$.

Theorem 9.6. *(Beppo Levi) Let $(X, \mathscr{A}, \mu)$ be a measure space. For an increasing sequence of functions $(u_n)_{n\in\mathbb{N}} \subset \mathcal{M}^+_{\bar{\mathbb{R}}}(\mathscr{A})$, $0 \leq u_n \leq u_{n+1} \leq \ldots$, we have for the supremum $u = \sup_{n\in\mathbb{N}} u_n \in \mathcal{M}^+_{\bar{\mathbb{R}}}(\mathscr{A})$ and*

$$\int \sup_{n\in\mathbb{N}} u_n d\mu = \sup_{n\in\mathbb{N}} \int u_n d\mu. \tag{23}$$

Note we can write $\lim_{n\to\infty}$ instead of $\sup_{n\in\mathbb{N}}$ as the supremum of an increasing sequence is its limit. Moreover, this theorem holds in $[0, +\infty]$, so the case $+\infty = +\infty$ is possible.

Corollary 9.7. *Let $u \in \mathcal{M}^+_{\bar{\mathbb{R}}}(\mathscr{A})$. Then*

$$\int u d\mu = \lim_{n\to\infty} \int f_n d\mu$$

holds for every sequence $(f_n)_{n\in\mathbb{N}} \subset \mathcal{E}^+(\mathscr{A})$ with $\lim_{n\to\infty} f_n = u$.

Proposition 9.8. *(of integral) Let $u, v \in \mathcal{M}^+_{\bar{\mathbb{R}}}(\mathscr{A})$. Then*

(i) $\int \mathbb{1}_A d\mu = \mu(A) \quad \forall A \in \mathscr{A}$.

(ii) $\int \alpha u d\mu = \alpha \int u d\mu \quad \forall \alpha \geq 0$.

(iii) $\int u + v d\mu = \int u d\mu + \int v d\mu$.

(iv) $u \leq v \Rightarrow \int u d\mu \leq \int v d\mu$.

Corollary 9.9. *Let $(u_n)_{n\in\mathbb{N}} \subset \mathcal{M}^+_{\bar{\mathbb{R}}}(\mathscr{A})$. Then $\sum_{n=1}^\infty u_n$ is measurable and we have*

$$\int \sum_{n=1}^\infty u_n d\mu = \sum_{n=1}^\infty \int u_n d\mu$$

(including the possibility $+\infty = +\infty$.)

Theorem 9.10. *(<u>Fatou</u>) Let $(u_n)_{n\in\mathbb{N}} \subset \mathcal{M}^+_{\overline{\mathbb{R}}}(\mathscr{A})$ be a sequence of positive measurable functions. Then $u = \liminf_{n\to\infty} u_n$ is measurable and*

$$\int \liminf_{n\to\infty} u_n d\mu = \liminf_{n\to\infty} \int u_n d\mu \tag{24}$$

10 Integrals of Measurable Functions

We have defined our integral for positive measurable functions, i.e. functions in $\mathcal{M}^+(\mathscr{A})$. To extend our integral to not only functions in $\mathcal{M}^+(\mathscr{A})$ we first notice that

$$u \in \mathcal{M}_{\overline{\mathbb{R}}}(\mathscr{A}) \Leftrightarrow u = u^+ - u^-,\ u^+, u^- \in \mathcal{M}^+_{\overline{\mathbb{R}}}, \tag{25}$$

i.e. that every measurable function can be written as a sum of **positive** measurable functions.

Definition 10.1 (μ-integrable)**.** A function $u : X \to \overline{\mathbb{R}}$ on $(X, \mathscr{A}, \mu)$ is μ-*integrable*, if it is $\mathscr{A}/\mathscr{B}(\overline{\mathbb{R}})$-measurable and if $\int u^+ d\mu, \int u^- d\mu < \infty$ (recall the definition for the integral of positive measurable functions). Then

$$\int u d\mu := \int u^+ d\mu - \int u^- d\mu \in (-\infty, \infty) \tag{26}$$

is the (μ-)*integral* of u. We write $\mathcal{L}^1(\mu)$ for the set of all real-valued μ-integrable functions [1].

Theorem 10.2. *Let $u \in \mathcal{M}_{\overline{\mathbb{R}}}(\mathscr{A})$, then the following conditions are equivalent:*

(i) $u \in \mathcal{L}^1_{\overline{\mathbb{R}}}(\mu)$.

(ii) $u^+, u^- \in \mathcal{L}^1_{\overline{\mathbb{R}}}(\mu)$.

(iii) $|u| \in \mathcal{L}^1_{\overline{\mathbb{R}}}(\mu)$.

(iv) $\exists w \in \mathcal{L}^1_{\overline{\mathbb{R}}}(\mu)$ *with* $w \geq 0$ *s.t.* $|u| \leq w$.

Theorem 10.3 (Properties of the μ-integral)**.** *The μ-integral is:* ***homogeneous, additive****, and:*

(i) $\min\{u, v\}, \max\{u, v\} \in \mathcal{L}^1_{\overline{\mathbb{R}}}(\mu)$ *(lattice property)*

(ii) $u \leq v \Rightarrow \int u d\mu \leq \int v d\mu$ *(monotone)*

(iii) $\left|\int u d\mu\right| \leq \int |u| d\mu$ *(triangle inequality)*

Remark. *If $u(x) \pm v(x)$ is defined in $\overline{\mathbb{R}}$ for all $x \in X$ then we can exclude $\infty - \infty$ and the theorem above just says that the integral is linear:*

$$\int (au + bv) d\mu = a \int u d\mu + b \int v d\mu. \tag{27}$$

[1] In words, we extend our integral to ~~positive~~ measurable functions by noticing that we can write every measurable function as a sum of positive measurable functions, something that we do know how to integrate. We don't want to run into the problem of $\infty - \infty$, thus we require the integral of the positive and negative parts to both (separately) be less than infinity.

This is always true for real-valued $u, v \in \mathcal{L}^1(\mu) = \mathcal{L}^1_{\mathbb{R}}(\mu)$, making $\mathcal{L}^1(\mu)$ a vector space with addition and scalar multiplication defined by

$$(u+v)(x) := u(x) + v(x), \ \ (a \cdot u)(x) := a \cdot u(x), \tag{28}$$

and

$$\int ...d\mu : \mathcal{L}^1(\mu) \to \mathbb{R}, \ \ u \mapsto \int u d\mu, \tag{29}$$

is a ***positive linear functional****.*

11 Null sets and the "Almost Everywhere"

Definition 11.1. A (μ-)null set $N \in \mathcal{N}_\mu$ is a measurable set $N \in \mathscr{A}$ satisfying

$$N \in \mu \Leftrightarrow N \in \mathscr{A} \text{ and } \mu(N) = 0. \tag{30}$$

This can be used generally about a 'statement' or 'property', but we will be interested in questions like 'when is $u(x)$ equal to $v(x)$', and we answer this by saying

$$u = v \ a.e. \Leftrightarrow \{x : u(x) \neq v(x)\} \text{ is (contained in) a } \mu\text{-null set.}, \tag{31}$$

i.e.

$$u = v \quad \mu\text{-a.e.} \Leftrightarrow \mu\left(\{x : u(x) \neq v(x)\}\right) = 0. \tag{32}$$

The last phrasing should of course include that the set $\{x : u(x) \neq v(x)\}$ is in $\mathscr{A}$.

Theorem 11.2. *Let $u \in \mathcal{M}_{\overline{\mathbb{R}}}(\mathscr{A})$, then:*

(i) $\int |u| d\mu = 0 \Leftrightarrow |u| = 0 \ a.e. \quad \Leftrightarrow \mu\{u \neq 0\} = 0$,

(ii) $\mathbb{1}_N u \in \mathcal{L}^1_{\overline{\mathbb{R}}}(\mu) \quad \forall \ N \in \mathcal{N}_\mu$,

(iii) $\int_N u d\mu = 0$.

Corollary 11.3. *Let $u = v$ μ-a.e. Then*

(i) $u, v \geq 0 \Rightarrow \int u d\mu = \int v d\mu$,

(ii) $u \in \mathcal{L}^1_{\overline{\mathbb{R}}}(\mu) \Rightarrow v \in \mathcal{L}^1_{\overline{\mathbb{R}}}(\mu)$ *and* $\int u d\mu = \int v d\mu$.

Corollary 11.4. *If $u \in \mathcal{M}_{\overline{\mathbb{R}}}(\mathscr{A})$, $v \in \mathcal{L}^1_{\overline{\mathbb{R}}}(\mu)$ and $v \geq 0$ then*

$$|u| \leq v \ a.e. \quad \Rightarrow u \in \mathcal{L}^1_{\overline{\mathbb{R}}}(\mu). \tag{33}$$

Proposition 11.5 (Markow inequality)**.** *For all $u \in \mathcal{L}^1_{\overline{\mathbb{R}}}(\mu)$, $A \in \mathscr{A}$ and $c > 0$*

$$u\left(\{|u| \geq c\} \cap A\right) \leq \frac{1}{c} \int_A |u| d\mu, \tag{34}$$

if $A = X$, then (obviosly)

$$u\{|u| \geq c\} \leq \frac{1}{c} \int |u| d\mu. \tag{35}$$

Corollary 11.6. *If $u \in \mathcal{L}^1_{\overline{R}}(\mu)$, then μ is a.e. $\mathbb{R}$-vaued. In particular, we can find a version $\tilde{u} \in \mathcal{L}^1(\mu)$ s.t. $\tilde{u} = u$ a.e. and $\int \tilde{u} d\mu = \int u d\mu$*

Completions of measure spaces (from lecture notes 8, 05. february)

Definition 11.7. A measure space $(X, \mathscr{B}, \mu)$ is called **complete** if whenever $A \in \mathscr{B}$ and $\mu(A) = 0$, we have $B \in \mathscr{B}\ \forall B \subset A$.

Remark. *Any measure space can be completed as follows:*
Let $\bar{\mathscr{B}}$ be the σ-algebra generated by $\mathscr{B}$ and all sets $B \subset X$ s.t. there exists $A \in \mathscr{B}$ with $B \subset A$ and $\mu(A) = 0$.

Proposition 11.8. *The σ-algebra $\bar{\mathscr{B}}$ can also be described as follows:*

$$\bar{\mathscr{B}} := \{B \subset X : A_1 \subset B \subset A_2 \text{ for some } A_1, A_2 \in \mathscr{B} \text{ with } \mu(A_2 \backslash A_1) = 0\}, \tag{36}$$

with B, A_1, A_2 as above, we define

$$\bar{\mu} := \mu(A_1) = \mu(A_2) \tag{37}$$

Then $\left(X, \bar{\mathscr{B}}, \bar{\mu}\right)$ is a complete measure space.

Definition 11.9. If μ is a Borel measure on a **metric** space (X, d), then the completion $\bar{\mathscr{B}}(X)$ of the Borel σ-algebra with respect to μ is called the σ-algebra of μ-measurable sets.

Remark. *For $\mu = \lambda_n$ on $\mathbb{R}^n$ we talk about the σ-algebra of* ***Lebesgue measurable sets****. Instead of $\bar{\lambda_n}$ we still write λ_n and call it the* ***Lebesgue measure****. A function $f : \mathbb{R}^n \to \mathbb{C}$, measurable w.r.t. the σ-algebra of Lebesgue measurable sets is called the* ***Lebesgue measurable****.*

The following result shows that any Lebesgue measurable function coincides with a Borel function a.e.

Proposition 11.10. *Assume $(X, \mathscr{B}, \mu)$ is a measure space and consider its completion $\left(X, \bar{\mathscr{B}}, \bar{\mu}\right)$. Assume $f : X \to \mathbb{C}$ is $\bar{\mathscr{B}}$-measurable. Then there is a $\mathscr{B}$-measurable function $g : X \to \mathbb{C}$ s.t. $f = g$ $\bar{\mu}$-a.e.*

12 Convergence Theorems and Their Applications

- To interchange limits and integrals in **Riemann integrals** one typically has to assume uniform convergence. ¡- The set of Riemann integrable functions is somewhat limited, see theorem 12.7

Theorem 12.1 (Generalization of Beppo Levi, monotone convergence)**.**

(i) Let $(u_n)_{n\in\mathbb{N}} \subset \mathcal{L}^1(\mu)$ be s.t. $u_1 \leq u_2 \leq \ldots$ with limit $u := \sup_{n\in\mathbb{N}} u_n = \lim_{n\to\infty} u_n$. Then $u \in \mathcal{L}^1(\mu)$ ***iff***

$$\sup_{n\in\mathbb{N}} \int u_n d\mu < +\infty,$$

in which case

$$\sup_{n\in\mathbb{N}}\int u_n d\mu = \int \sup_{n\in\mathbb{N}} u_n d\mu.$$

(ii) Same thing only with a decreasing sequence ...$> -\infty$ in which case

$$\inf_{n\in\mathbb{N}}\int u_n d\mu = \int \inf_{n\in\mathbb{N}} u_n d\mu.$$

Theorem 12.2 (Lebesgue; dominated convergence). *Let $(u_n)_{n\in\mathbb{N}} \subset \mathcal{L}^1(\mu)$ s.t.*

(a) $|u_n|(x) \leq w(x)$, $w \in \mathcal{L}^1(\mu)$,

(b) $u(x) = \lim_{n\to\infty}$ exists in $\bar{\mathbb{R}}$,

then $u \in \mathcal{L}^1(\mu)$ and we have

(i) $\lim\limits_{n\to\infty} \int |u_n - u| d\mu = 0$;

(ii) $\lim\limits_{n\to\infty} \int u_n d\mu = \int \lim\limits_{n\to\infty} u_n d\mu = \int u d\mu$;

Application 1: Parameter-Dependent Integrals

- We are interested in questions of the sort, when is

$$U(t) := \int u(t,x)\mu(dx),\ t \in (a,b),$$

again a smooth function of t? The answer involves interchange of limits and integration. Also, it turns out to better understand Riemann integrability, we need the Lebesgue integral.

Theorem 12.3 (continuity lemma). *Let $\emptyset \neq (a,b) \subset \mathbb{R}$ be a non-degenerate open interval and $u : (a,b)\times X \to \mathbb{R}$ satisfy*

(a) $x \mapsto u(t,x)$ is in $\mathcal{L}^1(\mu)$ for every fixed $t \in (a,b)$;

(b) $t \mapsto u(t,x)$ is continuous for every fixed $x \in X$;

(c) $|u(t,x)| \leq w(x)$ for all $(t,x) \in (a,b)\times X$ and some $w \in \mathcal{L}^1(\mu)$.

Then the function $U : (a,b) \to \mathbb{R}$ given by

$$t \mapsto U(t) := \int u(t,x)\mu(dx) \tag{38}$$

is continuous.

Theorem 12.4 (differentiability lemma). *Let $\emptyset \leq (a,b) \subset \mathbb{R}$ be a non-degenerate open interval and $u : (a,b)\times X \to \mathbb{R}$ satisfy* *(a) Same*

(b) Same

(c) $|\partial_t u(t,x)| \leq w(x)$ *for all* $(t,x) \in (a,b) \times X$ *and some* $w \in \mathcal{L}^1(\mu)$.

Then the function in 38 is differentiable and its derivative is

$$\frac{d}{dt}U(t) = \frac{d}{dt}\int u(t,x)\mu(dx) = \int \frac{\partial}{\partial t}u(t,x)\mu(dx). \tag{39}$$

Application 2: Riemann vs Lebesgue Integration

Consider only $(X, \mathscr{A}, \mu) = (\mathbb{R}, \mathscr{B}(\mathbb{R}), \lambda)$.

Definition 12.5 (The Riemann Inegral)**.** Consider on the finite interval $[a,b] \subset \mathbb{R}$ the partition

$$\Pi := \{a = t_0 < t_1 < ... < t_k < b\}, k = k(\Pi), \tag{40}$$

and introduce

$$S_\Pi[u] := \sum_{i=1}^{k(\Pi)} m_i(t_i - t_{i-1}), \qquad m_i := \inf_{x \in [t_{i-1}, t_i]} u(x), \tag{41}$$

$$S^\Pi[u] := \sum_{i=1}^{k(\Pi)} M_i(t_i - t_{i-1}), \qquad M_i := \sup_{x \in [t_{i-1}, t_i]} u(x). \tag{42}$$

$$\tag{43}$$

A bounded function $u : [a,b] \to \mathbb{R}$ is said to be **Riemann integrable** if the values

$$\underline{\int} u := \sup_\Pi S_\Pi[u] = \inf_\Pi S^\Pi[u] =: \overline{\int} u \tag{44}$$

coincide and are finite. Their common value is called the **Riemann integral** of u and denoted by $(R)\int_a^b u(x)dx$ or $\int_a^b u(x)dx$.

Theorem 12.6. *Let* $u : [a,b] \to \mathbb{R}$ *be a* ***measurable*** *and* ***Riemann integrable*** *function. Then*

$$u \in \mathcal{L}^1(\lambda) \text{ and } \int_{[a,b]} u d\lambda = \int_a^b u(x)dx. \tag{45}$$

Theorem 12.7. *Let* $u : [a,b] \to \mathbb{R}$ *be a bounded function, it is Riemann integrable* ***iff*** *the points in* (a,b) *where* u *is discontinuous are a (subset of) Borel measurable null set.*

Improper Riemann Integrals

- The Lebesgue integral extends the (*proper*) Riemann integral. However, there is a further extension of the Riemann integral which cannot be captured by Lebesgue's theory. u is Lebesgue integrable *iff* $|u|$ ha finite Lebesgue integral. ¡- The Lebesgue integral does not respect sign-changes and cancellations. However, the following *improper Riemann integral* does:

$$(R)\int_0^\infty u(x)dx := \lim_{n\to\infty}(R)\int_0^a u(x)dx. \tag{46}$$

Corollary 12.8. *Let $u : [0,\infty) \to \mathbb{R}$ be a measurable, Riemann integrable function for every interval $[0,N],\ N \in \mathbb{N}$. Then $u \in \mathcal{L}^1[0,\infty)$* ***iff***

$$\lim_{N\to\infty}(R)\int_0^N |u(x)|dx < \infty. \tag{47}$$

In this case, $(R)\int_0^\infty u(x)dx = \int_{[0,\infty)} ud\lambda$

Example of a function which is *improperly Riemann integrable* but **not** *Lebesgue integrable*:

$$f(x) = \frac{\sin(x)}{x}. \tag{48}$$

Proposition 12.9 (appearing as example 12.13 in Schilling)**.** *Let $f_\alpha(x) := x^\alpha, x > 0$ and $\alpha \in \mathbb{R}$. Then*

(i) $f_(\alpha) \in \mathcal{L}^1(0,1) \Leftrightarrow \alpha > -1.$

(ii) $f_(\alpha) \in \mathcal{L}^1[1,\infty) \Leftrightarrow \alpha < -1.$

13 The Function Spaces $\mathcal{L}^p$

Assume V is a vector space over $\mathbb{K} \in \{\mathbb{C}, \mathbb{R}\}$.

Definition 13.1. A seminorn on V is a map $p : V \to [0, +\infty)$ s.t.

(1) $p(cx) = |c|p(x) \;\; \forall x \in V, \forall c \in \mathbb{K}$.

(2) $p(x + y) \leq p(x) + p(y) \;\; \forall x, y \in V$. **triangle inequality**.

A seminorm is called a norm if we also have

$$p(x) = 0 \iff x = 0.$$

A norm is commonly denoted $||x||$, and a vectorspace equipped with a norm is called a **normed space**.

Definition 13.2. Assume (X, d) is a measure space. Fix $1 \leq p \leq \infty$. For every measurable function $f : X \to \mathbb{C}$ we define the following

$$||f||_p = \left(\int_X |f|^p d\mu \right)^{1/p} \in [0, +\infty]. \tag{49}$$

We can see that $||cf||_p = |c| ||f||_p \;\; \forall c \in \mathbb{C}$.

Lemma 13.3.

$$||f + g||_p \leq ||f||_p + ||g||_p. \tag{50}$$

Definition 13.4. We define

$$\mathcal{L}^p(X, d\mu) = \{ f : X \to \mathbb{C} \;\mid\; f \text{ is measurable and } ||f||_p < \infty \}. \tag{51}$$

This is a vectorspace with seminorm $f \mapsto ||f||_p$. And in general this is not a normed space, since $||f||_p = 0 \iff f = 0$ a.e.

Generally, if p is a seminorm on a vectorspace V, then

$$V_0 = \{ x \in V \;\mid\; p(x) = 0 \} \tag{52}$$

which is a subspace of V. Then we consider the quotient/factor space V/V_0.

Definition 13.5. For $x, y \in V$, define

$$x \sim y \iff x - y \in V_0. \tag{53}$$

This is an equivalence relation on V. The representation class of V is defined by $[x]$ or $x + V_0$.

Then V/V_0 is equals the set of equivalence classes. We can show that it is a normed space.

$$[x] + [y] = [x + y] \;\; , \;\; c[x] = [cx] \;\; , \;\; ||[x]|| = p(x).$$

Applying this to $\mathcal{L}^p(X, d\mu)$ we get the normed space

$$L^p(X, d\mu) = \mathcal{L}^p(X, d\mu)/\mathcal{N}. \tag{54}$$

Where $\mathcal{N}$ is the space of measurable functions f s.t. $f = 0$ a.e. We will further continue to denote the norm by $|| \cdot ||_p$, and we will normally **not** distinguish between $f \in \mathcal{L}^p(X, d\mu)$ and the vector in $L^p(X, d\mu)$ that f defines.

Definition 13.6. A normed space $(X, || \cdot ||)$ is called a Banach space if V is complete w.r.t the metric $d(x, y) = ||x - y||$.

Theorem 13.7. *If $(X, \mathscr{B}, \mu)$ is a measure space, $1 \leq p \leq \infty$, then $L^p(X, d\mu)$ is a Banach space.*

Definition 13.8. A measurable function $f : X \to \mathbb{C}$ is called **essentially bounded** if there is $c \geq 0$ s.t.

$$\mu(\{x \; : \; |f(x)| > c\}) = 0. \tag{55}$$

That is $|f| \leq c$ a.e. The smallest such c is called the essential supremum of f and is denoted by $||f||_\infty$.

Definition 13.9.

$$\mathcal{L}^\infty(X, d\mu) = \{f : X \to \mathbb{C} \; | \; f \text{ is measurable and } ||f||_\infty < \infty\}.$$

$$L^\infty(X, d\mu) = \mathcal{L}^\infty(X, d\mu)/\mathcal{N}.$$

Where by the previous definiton these spaces become the spaces of all essentially bounded functions.

Theorem 13.10. *If $(X, \mathscr{B}, \mu)$ is a σ-finite measure space, then $L^\infty(X, d\mu)$ is a Banach space.*

Appendix

H Regularity of measures

We let (X, d) be a metric space and denote by $\mathcal{O}$ the open, by $\mathcal{C}$ the closed and $\mathscr{B}(X) = \sigma(\mathcal{O})$ the Borel set of X.

Definition H.1. A measure μ on $(X, d, \mathscr{B}(X))$ is called outer regular, if

$$\mu(B) = \inf \{\mu(U) \mid B \subset U,\ U \text{ open}\} \tag{56}$$

and inner regular, if $\mu(K) < \infty$ for all compact sets $K \subset X$ and

$$\mu(U) = \sup \{\mu(K) \mid K \subset U,\ K \text{ compact}\} . \tag{57}$$

A measure which is both inner and outer regular is called **regular**. We write $\mathfrak{m}_r^+(X)$ for the family of regular measures on $(X, \mathscr{B}(X))$.

Remark. *The space X is called σ-compact if there is a sequence of compact sets $K_n \uparrow X$. A typical example of such a space is a locally compact, separable metric space.*

Theorem H.2. *Let (X, d) be a metric space. Every finite measure μ on $(X, \mathscr{B}(X))$ is outer regular. If X is σ-compact, then μ is also inner regular, hence regular.*

Theorem H.3. *Let (X, d) be a metric space and μ be a measure on $(X, B(X))$ such that $\mu(K) < \infty$ for all compact sets $K \subset X$.*

1 If X is σ-compact, then μ is inner regular.

2 If there exists a sequence $G_n \in \mathcal{O},\ G_n \uparrow X$ such that $\mu(G_n) < \infty$, then μ is outer regular.