UNIVERSITY OF OSLO

Faculty of Mathematics and Natural Sciences

Examination in MAT4410 — Advanced linear analysis.

Day of examination: Monday, December 3, 2018.

Examination hours: 9:00 – 13:00.

This problem set consists of 3 pages.

Appendices: None.

Permitted aids: None.

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Note: You must justify all your answers.

Problem 1

1a

(10 points) Let Ω be a nonempty set and $\mathcal{C}$ a countable set of pairwise disjoint subsets of Ω such that $\Omega = \bigcup_{E \in \mathcal{C}} E$. Form the σ-algebra $\sigma(\mathcal{C})$ generated by $\mathcal{C}$. Suppose that $f : \Omega \to \mathbb{R}$ is $\sigma(\mathcal{C})$-measurable. Prove that

$$f = \sum_{E \in \mathcal{C}} a_E \chi_E$$

for suitable real numbers a_E, $E \in \mathcal{C}$.

1b

(15 points) Assume now that $(\Omega, \mathcal{A}, P)$ is a probability space and $F_1, \ldots, F_n$ are pairwise disjoint sets in $\mathcal{A}$ such that

$$\Omega = \bigcup_{j=1}^{n} F_j$$

and $P(F_j) > 0$ for all $j = 1, \ldots, n$. Let $\mathcal{G} = \sigma(\{F_1, \ldots, F_n\})$ and let $f \in \mathcal{L}^1(\Omega, \mathcal{A}, P)$ be a real valued function. Prove that $\mathcal{E}(f|\mathcal{G})$, the conditional expectation of f given $\mathcal{G}$, has the form

$$\mathcal{E}(f|\mathcal{G}) = \sum_{j=1}^{n} a_j \chi_{F_j}$$

for suitable $a_j \in \mathbb{R}$, $j = 1, \ldots, n$. Show moreover that $a_j = P(F_j)^{-1} \int_{F_j} f \, dP$ for every $j = 1, \ldots, n$.

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Problem 2

Let $(\Omega, \mathcal{A}, \mu)$ be a σ-finite measure space and $(\mathbb{R}, \mathcal{M}, \lambda)$ be the Lebesgue measurable sets with the Lebesgue measure on $\mathbb{R}$. Let $f : \Omega \to [0, \infty)$ be $\mathcal{A}$-measurable and let $A \in \mathcal{A}$. Define a function $g : \Omega \times \mathbb{R} \to \mathbb{R}$ by $g(x,t) = f(x) - t$ and sets

$$B = \{(x,t) \in \Omega \times \mathbb{R} \mid 0 \le t \le f(x), x \in A\},$$
$$C = \{(x,t) \in \Omega \times \mathbb{R} \mid t = f(x), x \in A\}.$$

2a

(10 points) Show that g is measurable with respect to the product σ-algebra $\mathcal{A} \times \mathcal{M}$ on $\Omega \times \mathbb{R}$ and that $B, C \in \mathcal{A} \times \mathcal{M}$.

2b

(15 points) Show that $(\mu \times \lambda)(B) = \int_A f \, d\mu$ and $(\mu \times \lambda)(C) = 0$.

Problem 3

(10 points) Let $(\Omega, \mathcal{A}, \mu)$ be a σ-finite measure space and $1 < p < \infty$. Let $\mathcal{S}$ denote the space of $\mathcal{A}$-measurable simple complex valued functions s on Ω such that $\mu(\{x \in \Omega \mid s(x) \neq 0\}) < \infty$. Suppose that $\varphi : \mathcal{S} \to \mathbb{C}$ is a linear functional satisfying that $|\varphi(s)| \le M\|s\|_p$ for some positive constant M and all s in $\mathcal{S}$. Prove that there exists $g \in L^q(\Omega, \mathcal{A}, \mu)$, where $q = \frac{p}{p-1}$, such that

$$\varphi(s) = \int_\Omega sg \, d\mu$$

for all $s \in \mathcal{S}$.

Problem 4

4a

(5 points) Formulate the principle of uniform boundedness for a sequence of bounded linear operators between two Banach spaces.

4b

(10 points) Let $\{A_n\}_{n \in \mathbb{N}}$ be a sequence of bounded linear operators from the Banach space Ω to the Banach space Λ such that for each $y \in \Omega$ the sequence $\{A_n y\}_n$ converges in Λ. Define $A : \Omega \to \Lambda$ by $Ay = \lim_n (A_n y)$ for each $y \in \Omega$. Show that A is a bounded linear operator with $\|A\| \le \sup_{n \ge 1} \|A_n\|$.

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Let $c_0(\mathbb{N})$ denote the Banach space of complex sequences converging to zero, equipped with the supremum norm $\|\cdot\|_\infty$. Suppose that $x = \{x_j\}_{j\geq 1}$ is a sequence of complex numbers such that $\sum_{j=1}^{\infty} x_j y_j$ is a convergent series for every $\{y_j\}_{j\geq 1} \in c_0(\mathbb{N})$. For every $n \geq 1$, define $\phi_n : c_0(\mathbb{N}) \to \mathbb{C}$ by $\phi_n(y) = \sum_{j=1}^{n} x_j y_j$ for $y = \{y_j\}_{j\geq 1}$ in $c_0(\mathbb{N})$.

4c

(15 points) Show that ϕ_n is a bounded linear functional for every $n \geq 1$.

4d

(10 points) Conclude that $x \in l^1(\mathbb{N})$.

END