MAT 4410 Solutions

December 21, 2015

Problem 1.

(**a**) Let λ be Lebesgue measure on the σ-algebra $\mathcal{B}$ of all Borel measurable sets on the half open interval $(0,1]$. We define a signed measure α on $\mathcal{B}$ by

$$\alpha(E) = \int_E \cos(2\pi x)\,\mathrm{d}\lambda(x), \quad E \in \mathcal{B}.$$

Find a Hahn-decomposition for α in $(0,1]$ and determine the positive and negative variations α^+ and α^- of α.

Solution:

Set $f(x) = \cos(2\pi x), \quad x \in (0,1]$. Let $A = (0, \frac{1}{4}] \cup [\frac{3}{4}, 1], \quad B = (\frac{1}{4}, \frac{3}{4})$. Then $A = \{x : f(x) \geq 0\}$ and $B = \{x : f(x) < 0\}$. Hence A, B forms a Hahn-decomposition for the signed measure α and

$$\begin{aligned} \alpha^+(E) &= \alpha(E \cap A) = \int_{A\cap E} \cos(2\pi x)\,\mathrm{d}\lambda(x), \\ \alpha^-(E) &= -\alpha(E \cap B) = -\int_{B\cap E} \cos(2\pi x)\,\mathrm{d}\lambda(x), \quad E \in \mathcal{B}. \end{aligned}$$

A measure β is given by

$$\beta(E) = \int_E \frac{1}{x}\,\mathrm{d}\lambda(x), \quad E \in \mathcal{B}.$$

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(**b**) Explain that $\lambda << \beta$, $|\alpha| << \beta$, and find the Radon-Nikodym derivatives

$$\frac{\mathrm{d}\lambda}{\mathrm{d}\beta} \quad \text{and} \quad \frac{\mathrm{d}|\alpha|}{\mathrm{d}\beta}.$$

Solution:

$\beta(E) = \int_E \frac{1}{x}\,\mathrm{d}\lambda(x), \quad E \in \mathcal{B},$ yields $\lambda(E) = \int_E x \cdot \frac{1}{x}\,\mathrm{d}\lambda(x) = \int_E x\,\mathrm{d}\beta(x)$, hence $\lambda << \beta$ with $\frac{\mathrm{d}\lambda}{\mathrm{d}\beta}(x) = x$ ae. Furthermore,

$$|\alpha|(E) = \int_E |\cos(2\pi x)|\,\mathrm{d}\lambda(x),$$

so that $|\alpha| << \lambda$ and $\frac{\mathrm{d}|\alpha|}{\mathrm{d}\lambda} = |\cos(2\pi x)|$. Since $|\alpha| << \lambda$ and $\lambda << \beta$, we have $|\alpha| << \beta$ and

$$|\alpha|(E) = \int_E |\cos(2\pi x)| \,\mathrm{d}\lambda(x) = \int_E |\cos(2\pi x)| x \,\mathrm{d}\beta(x)$$

Hence $\frac{\mathrm{d}|\alpha|}{\mathrm{d}\beta} = |\cos(2\pi x)| x \qquad (= \frac{|\alpha|}{\mathrm{d}\lambda}\frac{\mathrm{d}\lambda}{\mathrm{d}\beta})$.

Above we have used that if $\nu << \mu$ and $\frac{\mathrm{d}\nu}{\mathrm{d}\mu} = f$, then $\int g \,\mathrm{d}\nu = \int gf \,\mathrm{d}\mu$. (This should be known from the course, and can readily be verified for simple functions g. Then one may apply the MCT.) ■

(**c**) Let μ be counting measure on $\mathcal{B}$ (that is, $\mu(E) =$ the number of elements of E if E is finite, $\mu(E) = +\infty$ if E is infinite). Show that $\lambda << \mu$. Prove that there is no $\mathcal{B}$-measurable function f on $(0, 1]$ such that

$$\lambda(E) = \int_E f \,\mathrm{d}\mu, \quad E \in \mathcal{B}.$$

Explain why this does not contradict the Radon-Nikodym Theorem.

Solution:

If $E \in \mathcal{B}$, $\mu(E) = 0 \Leftrightarrow E = \emptyset \Rightarrow$ any measure on $\mathcal{B}$ is absolutely continuous with respect μ.

Suppose next f is $\mathcal{B}$-measurable on $(0, 1]$ and $\lambda(E) = \int_E f \,\mathrm{d}\mu, \quad E \in \mathcal{B}$. Then $f \neq 0$ so there is $x \in (0, 1]$ such that $f(x) \neq 0$. Hence

$$\lambda\{x\} = 0 \neq f(x) = \int_{\{x\}} f \,\mathrm{d}\mu,$$

a contradiction.

Since$(0, 1]$ is uncountable, μ is not σ-finite. Therefore, the hypothesis of the Radon-Nikodym Theorem is not satisfied and the theorem does not apply. ■

Problem 2

Let $(\Omega, \mathcal{A}, \mu)$ be a finite measure space (that is, $\mu(\Omega) < \infty$). Suppose that

(*i*) V is a closed subspace of $\mathcal{L}^p(\mu)$ for some $p,\ 1 \leq p < \infty$,

and

(*ii*) V is contained in $\mathcal{L}^\infty(\mu)$.

In other words, V is a subspace of $\mathcal{L}^p(\mu) \cap \mathcal{L}^\infty(\mu)$, for some p, $1 \leq p < \infty$, that is closed in the p-norm.

(**a**) Show that $V \subset \mathcal{L}^2(\mu)$ and that there is a constant C such that

$$(1) \qquad ||f||_2 \leq C||f||_\infty \quad \text{whenever } f \in V.$$

Solution:
Let $f \in V$. Then $f \in \mathcal{L}^\infty(\mu)$ by (ii). Hence

$$||f||_2^2 = \int_\Omega |f|^2 \, \mathrm{d}\mu \leq \int_\Omega ||f||_\infty^2 \, \mathrm{d}\mu = ||f||_\infty^2 \mu(\Omega)$$

Thus

$$||f||_2 \leq C||f||_\infty, \quad C = \mu(\Omega)^{\frac{1}{2}}$$

In particular, $f \in \mathcal{L}^2(\mu)$. ■

Below we shall see that the inequality in (1) can be reversed.

(**b**) What do we mean by a closed linear map between two linear normed spaces? Show that the identity map

$$I \,:\, f \mapsto f, \; (V, ||\cdot||_p) \to (V, ||\cdot||_\infty)$$

is a closed map. Explain that there is an $M > 0$ such that

$$(2) \qquad ||f||_\infty \leq M||f||_p\,, \quad \text{for all } f \in V.$$

Solution:
Let $T : V \to W$ be a linear map between two normed linear spaces V and W. We say that T is closed if

$$x_n \underset{n\to\infty}{\longrightarrow} x \text{ in } V \text{ and } Tx_n \underset{n\to\infty}{\longrightarrow} y \text{ in } W \Longrightarrow y = tx.$$

Suppose $f_n \underset{n\to\infty}{\longrightarrow} f$ in p-norm and $I(f_n) \underset{n\to\infty}{\longrightarrow} g$ in ∞-norm. There is a subsequence (f_{n_k}) of (f_n) that converges pointwise ae to f on Ω. In addition,

$$||f_{n_k} - g||_\infty \underset{n\to\infty}{\longrightarrow} 0, \text{ so } f_{n_k} \underset{n\to\infty}{\longrightarrow} g \text{ pointwise ae on } \Omega.$$

Hence $f = g$ ae. This means that I is a closed map. Observe that V is also closed in $\mathcal{L}^\infty(\mu)$ since $||f||_p \leq \mu(\Omega)^{\frac{1}{p}}||f||_\infty$. This follows as in (a). Hence $(V, ||\cdot||_\infty)$ is a complete normed space. By the Closed Graph Theorem the map $I \,:\, (V, ||\cdot||_p) \to \mathcal{L}^\infty(\mu)$ is continuous. Hence there is a constant $M > 0$ such that

$$||f||_\infty \leq M||f||_p \qquad \text{for all } f \in V.$$

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(**c**) Assume next that $2 < p < \infty$ in (i) and (ii). Show that there is an $A > 0$ so that

$$||f||_\infty \leq A||f||_2 \quad \text{whenever } f \in V.$$

Hint: $|f(x)|^p \le ||f||_\infty^{p-2}|f(x)|^2, \quad f \in V.$
Solution:
Suppose $2 < p < \infty$ in (i) and (ii). Let $f \in V$. Then $f \in \mathcal{L}^\infty(\mu)$ so that

$$|f(x)|^p \le ||f||_\infty^{p-2}|f(x)|^2 \quad \text{for almost all } x \in \Omega.$$

Integration of the last inequality yields

$$(*) \qquad ||f||_p^p \le ||f||_\infty^{p-2}\int |f(x)|^2 \, \mathrm{d}\mu = ||f||_\infty^{p-2}||f||_2^2$$

We may assume that $||f||_\infty > 0$, otherwise the inequality is obvious. By $(*)$ and (2) in (b),

$$||f||_\infty^p \le M^p||f||_p^p \le M^p||f||_\infty^{p-2}||f||_2^2$$

Dividing the above inequalities by $||f||_\infty^{p-2}$, we deduce $||f||_\infty^2 \le M^p||f||_2^2$ which yields the desired result. ■

(**d**) Suppose that $1 \le p \le 2$ in (i) and (ii). Show that there exists $B > 0$ such that

$$||f||_\infty \le B||f||_2$$

Hint: Consider $r = \frac{2}{p}$ and $s = \frac{2}{2-p}$. Then apply Hölder's inequality.
Solution:
Suppose $1 \le p \le 2$ in (i) and (ii). Now $r = 2/p\,(\ge 1)$ and $s = 2/(2-p)\,(\le \infty)$ are conjugate exponents, $\frac{1}{r} + \frac{1}{s} = 1$. By Hölder's inequality, using that $|f|^p \in \mathcal{L}^{2/p}(\mu)$, $1 \in \mathcal{L}^s(\mu)$, we deduce that

$$\int_\Omega |f|^p \cdot 1 \, \mathrm{d}\mu \le (\int_\Omega |f|^2 \, \mathrm{d}\mu)^{p/2} \cdot (\int_\Omega 1^s \, \mathrm{d}\mu)^{\frac{1}{s}} = ||f||_2^p M^p, \quad M^p = \mu(\Omega)^{\frac{1}{s}}.$$

By (b) , $||f||_\infty \le B||f||_2 \qquad (B = MC)$. ■

Problem 3. Let $\mathcal{M}$ be the σ-algebra of all Lebesgue measurable subsets of $\mathbb{R}$, λ be Lebesgue measure on $\mathcal{M}$. Assume that f is a continuous map from $[0,1]$ into $\mathbb{R}$ and consider the condition

$$(N) \qquad \lambda(f(E)) = 0 \text{ whenever } \lambda(E) = 0 \text{ and } E \subset [0,1].$$

(**a**) Suppose f satisfies (N). Show that $f(E) \in \mathcal{M}$ whenever $E \in \mathcal{M}$ and $E \subset [0,1]$.
Solution:
Let $E \subset (0,1)$ be Lebesgue measurable. Lebesgue measure λ is regular so there are sequences of compact sets $\{K_n\}$ and of open sets $\{O_n\}$ such that

$$K_n \subset E \subset O_n \subset (0,1) \text{ and } \lambda(O_n \setminus K_n) < \frac{1}{n}, \quad n = 1, 2, \ldots$$

Hence $\lambda^*(\bigcap_{n=1}^{\infty} O_n \setminus \bigcup_{n=1}^{\infty} K_n) = 0$. Set $N = E \setminus \bigcup_{n=1}^{\infty} K_n$. Then

$$N \subset \bigcup_n O_n \setminus \bigcup_n K_n = B, \text{ where } \lambda(B) = 0.$$

Therefore, as λ is a complete measure, N is Lebesgue measurable and $\lambda(N) = 0$. Clearly $F = \bigcup_n K_n$ is measurable (it is even a Borel set) and $E = N \cup F$ where $E \cup N = \emptyset$. Since f is continuous and satisfies (N), it follows that $f(K_n)$ is compact ($n \in \mathbb{N}$) and

$$f(E) = f(N) \cup f(F) = f(N) \cup (\bigcup_{n=1}^{\infty} f(K_n))$$

is Lebesgue measurable. ∎

(**b**) Show that if $f(E) \in \mathcal{M}$ whenever $E \in \mathcal{M}$ and $E \subset [0, 1]$, then f satisfies (N). You can use (without proof) the fact that every $A \in \mathcal{M}$ for which $\lambda(A) > 0$ contains a nonmeasurable subset D with positive outer Lebesgue measure ($\lambda^*(D) > 0$).

Solution:

Assume that

$$E \subset [0, 1] \text{ is Lebesgue measurable } \Rightarrow f(E) \text{ is Lebesgue measurable.}$$

Further assume that $\lambda(N) = 0, \quad N \subset [0, 1]$. If $\lambda(f(N)) > 0$, there is a nonmeasurable subset D of $f(N)$ with $\lambda^*(D) > 0$. Then $D = f(A)$ for some $A \subset N$. Hence, λ being a complete measure on $\mathcal{M}$, we have $\lambda(A) = 0$ so that A is Lebesgue measurable and $D = f(A)$ is Lebesgue measurable, a contradiction. Consequently, $\lambda(f(N)) = 0$, and the condition (N) follows. ∎

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