Integration of measurable functions

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Through this chapter $(X, \mathscr{A}, \mu)$ will be some measure space. Recall that $\mathcal{M}^+(\mathscr{A})$ $[\mathcal{M}^+_{\bar{\mathbb{R}}}(\mathscr{A})]$ are the $\mathscr{A}$-measurable positive functions and $\mathcal{E}(\mathscr{A})$ $[\mathcal{E}^+_{\bar{\mathbb{R}}}(\mathscr{A})]$ are the positive and simple functions.

The fundamental idea of *Integration* is to measure the area between the graph of the function and the abscissa. For positive simple functions $f \in \mathcal{E}^+(\mathscr{A})$ in standard representation, this is done easily

$$\text{if } f = \sum_{i=0}^{M} y_i \mathbb{1}_{A_i} \in \mathcal{E}^+(\mathscr{A}) \quad \text{then } \sum_{i=0}^{M} y_i \mu(A_i) \tag{1}$$

would be the μ-area enclosed by the graph and the abscissa. We note that the representation of f should not impact the integral of f.

Lemma 9.10. *Let $\sum_{i=0}^{M} y_i \mathbb{1}_{A_i} = \sum_{k=0}^{N} z_k \mathbb{1}_{B_k}$ be two standard representations of the same function $f \in \mathcal{E}^+(\mathscr{A})$. Then*

$$\sum_{i=0}^{M} y_i \mu(A_i) = \sum_{k=0}^{N} z_k \mu(B_k). \tag{2}$$

Definition 9.11. Let $f = \sum_{i=0}^{M} y_i \mathbb{1}_{A_i} \in \mathcal{E}^+(\mathscr{A})$ be a simple function in standard representation. Then the number

$$I_\mu(f) = \sum_{i=0}^{M} y_i \mu(A_i) \in [0, \infty] \tag{3}$$

(which is independent of the representation of f) is called the μ-integral of f.

Proposition 9.12. *Let $f, g \in \mathcal{E}^+(\mathscr{A})$. Then*

(i) $I_\mu(\mathbb{1}_A) = \mu(A) \quad \forall A \in \mathscr{A}$.

(ii) $I_\mu(\lambda f) = \lambda I_\mu(f) \quad \forall \lambda \geq 0$.

(iii) $I_\mu(f + g) = I_\mu(f) + I_\mu(g)$.

(iv) $f \leq g \Rightarrow I_\mu(f) \leq I_\mu(g)$.

In theorem 8.8 we saw that we could for every $u \in \mathcal{M}^+(\mathscr{A})$ write it as an increasing limit of simple functions. By corollary 8.10, the suprema of simple functions are again measurable, so that

$$u \in \mathcal{M}^+(\mathscr{A}) \Leftrightarrow u = \sup_{n \in \mathbb{N}} f_n, \quad f \in \mathcal{E}^+(\mathscr{A}), \quad f_n \leq f_{n+1} \leq \dots.$$

We will use this to "inscribe" simple functions (which we know how to integrate) below the graph of a positive measurable function u and exhaust the μ-area below u.

Definition 9.13. Let $(X, \mathscr{A}, \mu)$ be a measure space. The (μ)-integral of a positive function $u \in \mathcal{M}^+_{\bar{\mathbb{R}}}(\mathscr{A})$ is given by

$$\int u d\mu = \sup \left\{ I_\mu(g) : g \leq u,\ g \in \mathcal{E}^+(\mathscr{A}) \right\} \in [0, +\infty]. \tag{4}$$

If we need to emphasize the *integration variable*, we write $\int u(x)\mu(dx)$. The key observation is that the integral $\int \dots d\mu$ extends I_μ.

Lemma 9.14. *For all $f \in \mathcal{E}^+(\mathscr{A})$ we have $\int f d\mu = I_\mu(f)$.*

The next theorem is one of many convergence theorems. It shows that we could have defined 4 using any increasing sequence $f_n \uparrow u$ of simple functions $f_n \in \mathcal{E}^+(\mathscr{A})$.

Theorem 9.15. *(Beppo Levi) Let $(X, \mathscr{A}, \mu)$ be a measure space. For an increasing sequence of functions $(u_n)_{n \in \mathbb{N}} \subset \mathcal{M}^+_{\bar{\mathbb{R}}}(\mathscr{A})$, $0 \leq u_n \leq u_{n+1} \leq \dots$, we have for the supremum $u = \sup_{n \in \mathbb{N}} u_n \in \mathcal{M}^+_{\bar{\mathbb{R}}}(\mathscr{A})$ and*

$$\int \sup_{n \in \mathbb{N}} u_n d\mu = \sup_{n \in \mathbb{N}} \int u_n d\mu. \tag{5}$$

Note we can write $\lim_{n \to \infty}$ instead of $\sup_{n \in \mathbb{N}}$ as the supremum of an increasing sequence is its limit. Moreover, this theorem holds in $[0, +\infty]$, so the case $+\infty = +\infty$ is possible.

Corollary 9.16. *Let $u \in \mathcal{M}^+_{\bar{\mathbb{R}}}(\mathscr{A})$. Then*

$$\int u d\mu = \lim_{n \to \infty} \int f_n d\mu$$

holds for every sequence $(f_n)_{n \in \mathbb{N}} \subset \mathcal{E}^+(\mathscr{A})$ with $\lim_{n \to \infty} f_n = u$.

Proposition 9.17. *(of integral) Let $u, v \in \mathcal{M}^+_{\bar{\mathbb{R}}}(\mathscr{A})$. Then*

(i) $\int \mathbb{1}_A d\mu = \mu(A) \quad \forall A \in \mathscr{A}$.

(ii) $\int \alpha u d\mu = \alpha \int u d\mu \quad \forall \alpha \geq 0$.

(iii) $\int u + v d\mu = \int u d\mu + \int v d\mu$.

(iv) $u \leq v \Rightarrow \int u d\mu \leq \int v d\mu.$

Corollary 9.18. *Let* $(u_n)_{n\in\mathbb{N}} \subset \mathcal{M}^+_{\bar{\mathbb{R}}}(\mathscr{A})$*. Then* $\sum_{n=1}^{\infty} u_n$ *is measurable and we have*

$$\int \sum_{n=1}^{\infty} u_n d\mu = \sum_{n=1}^{\infty} \int u_n d\mu$$

(including the possibility $+\infty = +\infty$*.)*

Theorem 9.19. *(Fatou) Let* $(u_n)_{n\in\mathbb{N}} \subset \mathcal{M}^+_{\bar{\mathbb{R}}}(\mathscr{A})$ *be a sequence of positive measurable functions. Then* $u = \liminf_{n\to\infty} u_n$ *is measurable and*

$$\int \liminf_{n\to\infty} u_n d\mu = \liminf_{n\to\infty} \int u_n d\mu \tag{6}$$