MAT 4410 Advanced Linear Analysis.
December 13, 2016
SOLUTIONS.

PROBLEM 1. Let $F : \mathbb{R} \to \mathbb{R}$ be the right continuous function defined by

$$F(t) = \begin{cases} 2 - \frac{1}{(1+t)^2}\,, & \text{if } t \geq 0 \\ 0\,, & \text{if } t < 0 \end{cases} \tag{1}$$

and let μ_F be the Borel measure on $\mathbb{R}$ associated to F.

(**a**)

$$\mu_F(a,b] = F(b) - F(a) = \begin{cases} \frac{1}{(1+a)^2} - \frac{1}{(1+b)^2}\,, & b > a \geq 0 \\ 2 - \frac{1}{(1+b)^2}\,, & b \geq 0, a < 0 \\ 0, & a < 0, b < 0 \end{cases}$$

Here the Lebesgue decomposition of $\mu = \mu_F$ can be written as

$$\mu = \mu_1 + \mu_2, \qquad \mu_1 \perp \lambda,\ \mu_2 << \lambda$$

Now, for each $E \in \mathcal{B}(\mathbb{R})$,

$$\mu_F(E \cap (-\infty, 0)) = 0,\ \mu_F\{0\} = \lim_{n\to\infty} \mu_F(0, 1/n] = \lim_{n\to\infty} [-\frac{1}{(1+n)^2} + 1] = 1,$$

and

$$\mu_F(E \cap (0,\infty)) = \int_{E\cap(0,\infty)} F' \, \mathrm{d}\lambda = \int_E \chi_{[0,\infty)}(t) \frac{1}{(1+t)^3} \, \mathrm{d}\lambda(t).$$

(As an alternative, $\mu_F\{0\} = 1$ follows from the second formula for $\mu_F(a,b]$.) Moreover, $\mu_F(E) = \mu_F(E \cap \{0\}) + \mu_F(\{0\}^c)$. By uniqueness of the Lebesgue decomposition it follows that $\mu_1 = \delta_0$ is the point mass at 0 and

$$\mu_2(E) = \int_E \chi_{[0,\infty)}(t) \frac{1}{(1+t)^3} \, \mathrm{d}\lambda(t) \quad (E \in \mathcal{B}(\mathbb{R})).$$

(**b**) Hence

$$\int_{\mathbb{R}} g \, \mathrm{d}\mu_F = \int_{\mathbb{R}} g \, \mathrm{d}\mu_1 + \int_{\mathbb{R}} g \, \mathrm{d}\mu_2 = g(0) + \int_0^\infty g(t) \frac{2}{(1+t)^3} \, \mathrm{d}\lambda(t)$$

for all g such that $t \mapsto g(t)\frac{1}{(1+t)^3}$ is integrable.

PROBLEM 2. Let λ denote Lebesgue measure on the Borel σ-algebraen $\mathcal{B}(\mathbb{R})$ on $\mathbb{R}$ and let $f \in \mathcal{L}^1(\lambda)$. We define

$$f^*(x) = \sup_{r>0} \frac{1}{2r} \int_{x-r}^{x+r} |f(t)| \, \mathrm{d}\lambda(t), \quad x \in \mathbb{R}$$

and

$$U_t = \{x \in \mathbb{R} : f^*(x) > t\}, \quad t > 0.$$

For every $r > 0$ and every $x \in \mathbb{R}$ we denote by $I_r(x)$ the open interval $I_r(x) = (x - r, x + r)$.

(**a**) Let $z \in I_{r_2 - r_1}(x)$. Then

$$-(r_2 - r_1) < x - z < r_2 - r_1, \quad x + r_1 < z + r_2 \text{ and } x - r_1 > z - r_2,$$

and hence

$$\int_{x-r_1}^{x+r_1} |f| \, \mathrm{d}\lambda \leq \int_{z-r_2}^{z+r_2} |f| \, \mathrm{d}\lambda.$$

(**b**) Let $x \in U_t$. Then there is an $r_1 > 0$ such that

$$J_1 = \frac{1}{2r_1} \int_{x-r_1}^{x+r_1} |f| \, \mathrm{d}\lambda > t.$$

Hence we can find an $r_2 > r_1$ such that

$$J_1 > \frac{1}{2r_2} \int_{x-r_1}^{x+r_1} |f| \, \mathrm{d}\lambda > t$$

Now let $z \in I_{r_2 - r_1}(x)$. Then by (**a**)

$$t < \frac{1}{2r_2} \int_{x-r_1}^{x+r_1} |f| \, \mathrm{d}\lambda \leq \frac{1}{2r_2} \int_{z-r_2}^{z+r_2} |f| \, \mathrm{d}\lambda$$

Consequently, $z \in U_t$. We have shown that $I_{r_2 - r_1}(x) \subset U_t$. Therefore, all points of U_t are interior points, hence U_t is open.

(**c**) Assume that $\lambda(K) \leq \frac{3}{t}||f||_1$ for all compact sets K contained in U_t. Let $\epsilon > 0$ be arbitrary. Since λ is (inner) regular we can find a compact $K \subset U_t$ such that $\lambda(U_t) \leq \lambda(K) + \epsilon$. Hence

$$\lambda(U_t) < \lambda(K) + \epsilon \leq \frac{3}{t}||f||_1 + \epsilon$$

As ϵ was arbitrary we conclude that $\lambda(U_t) \leq \frac{3}{t}||f||_1$

(**d**) Let K be compact and $K \subset U_t$. For each $x \in K$ there is an interval $I_r(x)$ such that

$$(*) \qquad \frac{1}{2r}\int_{I_r(x)} |f|\,\mathrm{d}\lambda > t, \qquad \text{that is,} \;\; \lambda(I_r(x)) = 2r < \frac{1}{t}\int_{I_r(x)} |f|\,\mathrm{d}\lambda$$

Now the family

$$\{I_r(x) \,:\, r > 0,\, x \in K, \text{ and } (*) \text{ holds}\}$$

forms an open cover of K in U_t. Hence, as K is compact, we can find a finite subcover which we denote by $\{I_1, \ldots, I_N\}$. As is given above in this problem, we can choose a finite subfamily $\{I'_1, \ldots, I'_M\}$ of pairwise disjoint intervals such that

$$\lambda(\bigcup_1^N I_j) \leq 3\sum_1^M \lambda(I'_k)$$

Therefore,

$$\begin{aligned}\lambda(K) \leq \lambda(\bigcup_1^N I_j) &\leq 3\sum_1^M \lambda(I'_k) \leq \frac{3}{t}\sum_1^M \int_{I'_k} |f|\,\mathrm{d}\lambda \\ &= \frac{3}{t}\int_{\bigcup_1^M I'_k} |f|\,\mathrm{d}\lambda \leq \frac{3}{t}\int_{\mathbb{R}} |f|\,\mathrm{d}\lambda\end{aligned}$$

Hence the proof is complete.

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PROBLEM 3.

(**a**) Let X be a normed linear space, X^* the dual of X.
For all $x \in X$, there is a natural linear functional $l_x : X^* \to \mathbb{C}$ such that $l_x(g) = g(x),\; g \in X^*$. Then $|l_x(g)| \leq ||g|| \cdot ||x||$ and hence $||l_x|| \leq ||x||$. To prove the reverse inequality it suffices to find $g \in X^*$ such that $||g|| \leq 1$ and $|l_x(g)| = ||x||$. On the subspace $\mathbb{C}x$ we define a linear functional g_0 of norm less than or equal to 1 by $g_0(\alpha x) = \alpha||x||$. By the Hahn-Banach Extension Theorem there is an extension g of g_0 to all of X with $||g|| = ||g_0||$. Now $l_x(g) = g(x) = ||x||$ and the argument is complete. Thus $||l_x|| = ||x||$.

(b) Suppose $\mathcal{F} \subset \mathcal{L}^p(\mu)$, and $||f||_p \leq M$, for all $f \in \mathcal{F}$. Let $f \in \mathcal{F}$. For all $g \in \mathfrak{L}^q$ we have from the hypothesis:

$$|\int gf \, \mathrm{d}\mu| \leq ||g||_q ||f||_p \leq M||g||_q$$

Hence, for all $g \in \mathcal{L}^q$, $\sup_{f \in \mathcal{F}} | \int gf \, \mathrm{d}\mu| \leq M||g||_q < \infty$.

Conversely, assume that for all $g \in \mathcal{L}^q$,

$$\sup_{f \in \mathcal{F}} | \int gf \, \mathrm{d}\mu| = \sup_{f \in \mathcal{F}} |l_f(g)| \leq M||g||_q$$

Then $\{l_f\}_{f \in \mathcal{F}}$ is a pointwise bounded family of continuous linear functionals defined on the Banach space $\mathfrak{L}^q$. By the Principle of Uniform Boundedness for linear maps there is an $M < \infty$ such that $||f||_p = ||l_f|| \leq M$, for all f in $\mathcal{F}$. Hence the argument is complete.

PROBLEM 4.

For all $n \in \mathbb{N}$, $|f_n'| \leq F \in \mathcal{L}^1(\lambda)$. Therefore, the Dominated Convergence Theorem yields

$$\lim_{n \to \infty} \int_0^x f_n' \, \mathrm{d}\lambda = \int_0^x g \, \mathrm{d}\lambda$$

In particular, $g \in \mathcal{L}^1(\lambda)$. Let $E = \{t : \lim_{n \to \infty} f_n(t) = g(t)\}$. Then $\lambda(E^c) = 0$. Since each f_n is absolutely continuous (ac),

$$f_n(x) - f_n(0) = \int_0^x f_n'(t) \, \mathrm{d}\lambda(t) \underset{n \to \infty}{\longrightarrow} \int_0^x g \, \mathrm{d}\lambda.$$

Hence $\lim_{n \to \infty} [f_n(x) - f_n(0)]$ exists. If $x \in E$ then $\lim_{n \to \infty} f_n(x) = g(x)$. Hence there is a c_0 such that $\lim_{n \to \infty} f_n(0) = c_0$. Accordingly,

$$f(x) = c_0 + \int_0^x g \, \mathrm{d}\lambda, \qquad \text{for all } x \in E.$$

Set $h(x) = c_0 + \int_0^x g \, \mathrm{d}\lambda$, for all $x \in [0,1]$. As $g \in \mathcal{L}^1$, the First Fundamental Theorem yields that h is differentiable ae and $h' = g$ ae. In addition, h is ac by its definition. Hence f is equal to the ac function h on E thus $f = h$ ae. Moreover, $f' = h' = g$ ae on $[0,1]$.

[Actually $f = h$ on all of $[0,1]$, hence f is ac on $[0,1]$. To see this, let $k = f - h$ and suppose $k(x) \neq 0$ for some $x \in [0,1]$. Notice that k is continuous, hence there is an open interval I_x containing x such that $k(y) \neq 0$ for all $y \in I_x$. This contradicts that $k = 0$ ae. Thus $k(x) = 0$ for all $x \in [0,1]$, as claimed.]

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