MAT3400/4400 SPRING 2021: SOLUTIONS TO THE EXAM

Problem 1.

(a) Consider a subset $A \subseteq \mathbb{R}$ and a covering $\mathcal{C} = \{I_n\}_{n\geq 1}$ of A comprised of open intervals. There are two cases.

(i) If $0 \notin A$, we choose the covering $I_1 = (0, \infty)$, $I_n = (-\infty, 1/n)$ for $n \geq 2$. Note that $\varrho(I_1) = 1 - 1 = 0$ and $\varrho(I_n) = 0 - 0 = 0$ for $n \geq 2$. Moreover,

$$\bigcup_{n=1}^{\infty} I_n = (-\infty, 0) \cup (0, \infty).$$

Since $0 \notin A$ we find that $A \subseteq \bigcup_{n\geq 1} I_n$ and so

$$\mu^*(A) \leq \sum_{n=1}^{\infty} \varrho(I_n) = 0.$$

Since $\mu^*(A) \geq 0$ for every outer measure and every set, we conclude that $\mu^*(A) = 0$.

(ii) If $0 \in A$, we can choose the covering defined by $I_1 = \mathbb{R}$ and $I_n = \emptyset$ for $n \geq 2$ and conclude, as in (i), that $\mu^*(A) \leq 1$. However, any covering $\mathcal{C}$ of A comprised of open intervals must contain an interval which contains 0. Hence $\mu^*(A) \geq 1$. In conclusion we have $\mu^*(A) = 1$.

Note. The open interval $I = (-\infty, 0)$ exemplifies the possibility that $\mu^*(I) < \varrho(I)$. Another example is in *Spaces* Exercise 8.1.2.

(b) Let E be any subset of $\mathbb{R}$. We need to check that

$$\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \setminus E)$$

holds for every $A \subseteq \mathbb{R}$. If $0 \notin A$, this trivially holds since

$$\mu^*(A) = \mu^*(A \cap E) = \mu^*(A \setminus E) = 0.$$

If $0 \in A \cap E$, then $\mu^*(A) = \mu^*(A \cap E) = 1$ and $\mu^*(A \setminus E) = 0$. The case $0 \in A \setminus E$ is similar. Hence E is μ^*-measurable.

Problem 2.

(a) Clearly, $f = \mathbf{1}_A$ is in N since $\mu(A) < \infty$ and $f \not\equiv 0$ since $\mu(A) > 0$. However, $-f$ is not in N and hence N is not a subspace of H.

(b) To prove the claim, take any function $f \in H$ and write $f = f_1 - f_2$, where

$$f_1(x) = \begin{cases} f(x), & f(x) \geq 0, \\ 0, & f(x) < 0, \end{cases} \qquad \text{and} \qquad f_2(x) = \begin{cases} -f(x), & f(x) \leq 0, \\ 0, & f(x) > 0. \end{cases}$$

Since f_1 and f_2 are measurable and $\|f\|^2 = \|f_1\|^2 + \|f_2\|^2$, we conclude that $f_1, f_2 \in N$. If K is a subspace of H which contains N, then every linear combination of f_1 and f_2 must be in K. In particular, $f = f_1 - f_2$ is in K and hence $K = H$.

(c) Suppose that $\{f_n\}_{n\geq 1}$ is a Cauchy sequence in N. Since the elements of H are defined up to sets of measure 0, we may assume that $f_n(x) \geq 0$ for every x. Since H is a Hilbert space, the sequence $\{f_n\}_{n\geq 1}$ converges to some function f in $H = L^2(X)$. Hence by Corollary 44 (*Spaces* 7.8.3), there is a subsequence $\{f_{n_k}\}_{k\geq 1}$ such that $f_{n_k} \to f$ pointwise almost everywhere. A convergent limit of non-negative numbers is non-negative, so we find that $f(x) \geq 0$ for almost every x. Hence $f \in N$ and consequently N is closed.
(It is also possible to argue more directly as follows. Suppose that $f_n \to f$ in L^2 for a sequence $(f_n)_{n\geq 1}$ in N and a function $f \in L^2$. Consider the measurable set $F = \{x \in X : f(x) < 0\}$. Then

$$0 \leq \int_X \mathbf{1}_F(x)|f(x)|^2\, d\mu(x) = \langle \mathbf{1}_F f, f\rangle = \lim_{n\to\infty} \langle \mathbf{1}_F f, f_n\rangle \leq 0$$

where the final equality follows from the fact that $f_n \to f$ in L^2. However, this is only possible when $\mu(F) = 0$. Hence $f \in N$ and hence N is closed.)

(d) If f_1 and f_2 are in N, then $(1-t)f_1 + tf_2$ is also in N for every $0 \leq t \leq 1$. Hence N is a convex subset of N. Since N is closed by (c), we can appeal to *ELA* Theorem 4.1.1 to conclude that for every fixed $g \in H$, there is a unique $f \in H$ such that

$$d_H(g, N) = \|f - g\|.$$

(It is also possible to directly show that the unique minimizer is

$$f = \mathbf{1}_{\{x\in X\,:\,g(x)\geq 0\}}\, g$$

by arguing as in (b), but we were not asked to find f.)

Problem 3. Note that $\ker(T) \subseteq \ker(T^*T)$ since if $T(x) = 0$, then certainly

$$T^*T(x) = T^*(T(x)) = T^*(0) = 0.$$

For the other inclusion, here are two possible arguments.

(i) Using the properties of the adjoint operator, we get

$$\langle T^*T(x), x\rangle = \langle T(x), T(x)\rangle = \|T(x)\|^2.$$

If $x \in \ker(T^*T)$ then this shows that $\|T(x)\|^2 = 0$, so clearly $T(x) = 0$. This means that $x \in \ker(T)$ and consequently $\ker(T^*T) \subseteq \ker(T)$.

(ii) Recall from *ELA* Proposition 4.3.8 that

$$\ker(T^*) = (T(H))^\perp.$$

Hence if $T(x) \neq 0$, then $T^*(T(x)) \neq 0$ since $T(x) \in T(H) \setminus \{0\}$. By the contrapositive this demonstrates that $\ker(T^*T) \subseteq \ker(T)$.

Problem 4.

(a) Clearly, $X = \operatorname{span}(\{e_1, e_2, \ldots, e_n\})$ is a subspace of H with $\dim(X) = n$. Define $\widetilde{S}$ as the restriction of S to X, i.e. the operator $\widetilde{S}\colon X \to H$ defined by $\widetilde{S}(x) = S(x)$. Evidently, $\operatorname{rank}(\widetilde{S}) \leq \operatorname{rank}(S) = n-1$. Since X is finite-dimensional we have

$$\dim(X) = \dim(\widetilde{S}(X)) + \dim(\ker(\widetilde{S})) = \operatorname{rank}(\widetilde{S}) + \dim(\ker(\widetilde{S})),$$

so $\dim(\ker(\widetilde{S})) \geq 1$. Since $\ker(\widetilde{S}) \subseteq X \cap \ker(S)$ we conclude that the latter intersection cannot be equal to $\{0\}$.

(b) Using the spectral theorem for compact self-adjoint operators, we may write

$$T(x) = \sum_{k=1}^{\infty} \lambda_k \langle x, e_k \rangle e_k$$

where $\mathcal{E} = \{e_k\}_{k\geq 1}$ are the eigenvectors corresponding to $\{\lambda_k\}_{k\geq 1}$. By the spectral theorem we also know that $\mathcal{E}$ is an orthonormal set. The operator

$$S(x) = \sum_{k=1}^{n-1} \lambda_k \langle x, e_k \rangle e_k$$

satisfies $\operatorname{rank}(S) = n - 1$ since $|\lambda_k| \neq 0$ for every $k \geq 1$. Fix any $x \in H$. By Parseval's formula (*ELA* Theorem 4.2.8 (c)), we find that

$$\|(T - S)(x)\|^2 = \left\| \sum_{k=n}^{\infty} \lambda_k \langle x, e_k \rangle e_k \right\|^2 = \sum_{k=n}^{\infty} |\lambda_k|^2 |\langle x, e_k \rangle|^2$$

and since $|\lambda_k| \leq |\lambda_n|$ for every $k \geq n$ we estimate

$$\begin{aligned} &\leq |\lambda_n|^2 \sum_{k=n}^{\infty} |\langle x, e_k \rangle|^2 \\ &\leq |\lambda_n|^2 \|x\|^2 \end{aligned}$$

where the final estimate is Bessel's inequality (*ELA* Proposition 1.2.7). Hence $\|T - S\| \leq |\lambda_n|$ and so $a_n(T) \leq |\lambda_n|$.

(c) Let $S \in \mathcal{B}(H)$ be any operator with $\operatorname{rank}(S) = n - 1$. By (a) we can find

$$x = \sum_{k=1}^{n} c_k e_k$$

in $\ker(S)$ with $\|x\| = 1$, where $\{e_k\}_{k\geq 1}$ are the eigenvectors corresponding to the eigenvalues $\{\lambda_k\}_{k\geq 1}$ of T. Since $\|x\| = 1$ and $x \in \ker(S)$, we get

$$\|T - S\| \geq \|(T - S)(x)\| = \|T(x)\|.$$

Using Parseval's formula again

$$\|T(x)\|^2 = \left\| \sum_{k=1}^{n} \lambda_k c_k e_k \right\|^2 = \sum_{k=1}^{n} |\lambda_k|^2 |c_k|^2$$

and since $|\lambda_n| \leq |\lambda_k|$ for $1 \leq k \leq n$ we get

$$\geq |\lambda_n|^2 \sum_{k=1}^{n} |c_k|^2 = |\lambda_n|^2$$

where the final equality follows from Parseval's formula yet again and the fact that $\|x\| = 1$. Hence $\|T - S\| \geq |\lambda_n|$ and so $a_n(T) \geq |\lambda_n|$.