Regularity of measures

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8 Regularity of measures

We let (X, d) be a metric space and denote by $\mathcal{O}$ the open, by $\mathcal{C}$ the closed and $\mathscr{B}(X) = \sigma(\mathcal{O})$ the Borel set of X.

Definition 8.1. A measure μ on $(X, d, \mathscr{B}(X))$ is called outer regular, if

$$\mu(B) = \inf \{\mu(U) \mid B \subset U,\ U \text{ open}\} \tag{1}$$

and inner regular, if $\mu(K) < \infty$ for all compact sets $K \subset X$ and

$$\mu(U) = \sup \{\mu(K) \mid K \subset U,\ K \text{ compact}\}\,. \tag{2}$$

A measure which is both inner and outer regular is called **regular**. We write $\mathfrak{m}_r^+(X)$ for the family of regular measures on $(X, \mathscr{B}(X))$.

Remark. *The space X is called σ-compact if there is a sequence of compact sets $K_n \uparrow X$. A typical example of such a space is a locally compact, separable metric space.*

Theorem 8.2. *Let (X, d) be a metric space. Every finite measure μ on $(X, \mathscr{B}(X))$ is outer regular. If X is σ-compact, then μ is also inner regular, hence regular.*

Theorem 8.3. *Let (X, d) be a metric space and μ be a measure on $(X, B(X))$ such that $\mu(K) < \infty$ for all compact sets $K \subset X$.*

1 If X is σ-compact, then μ is inner regular.

2 If there exists a sequence $G_n \in \mathcal{O},\ G_n \uparrow X$ such that $\mu(G_n) < \infty$, then μ is outer regular.