Integrals of Measurable Functions

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We have defined our integral for positive measurable functions, i.e. functions in $\mathcal{M}^+(\mathscr{A})$. To extend our integral to not only functions in $\mathcal{M}^+(\mathscr{A})$ we first notice that

$$u \in \mathcal{M}_{\overline{\mathbb{R}}}(\mathscr{A}) \Leftrightarrow u = u^+ - u^-,\ u^+, u^- \in \mathcal{M}^{\pm}_{\overline{\mathbb{R}}}, \tag{1}$$

i.e. that every measurable function can be written as a sum of **positive** measurable functions.

Definition 10.11 (μ-integrable)**.** A function $u : X \to \overline{\mathbb{R}}$ on $(X, \mathscr{A}, \mu)$ is μ-*integrable*, if it is $\mathscr{A}/\mathscr{B}(\overline{\mathbb{R}})$-measurable and if $\int u^+ d\mu, \int u^- d\mu < \infty$ (recall the definition for the integral of positive measurable functions). Then

$$\int u d\mu := \int u^+ d\mu - \int u^- d\mu \in (-\infty, \infty) \tag{2}$$

is the (μ-)*integral* of u. We write $\mathcal{L}^1(\mu)$ for the set of all real-valued μ-integrable functions [1].

Theorem 10.12. *Let $u \in \mathcal{M}_{\overline{\mathbb{R}}}(\mathscr{A})$, then the following conditions are equivalent:*

(i) $u \in \mathcal{L}^1_{\overline{\mathbb{R}}}(\mu)$.

(ii) $u^+, u^- \in \mathcal{L}^1_{\overline{\mathbb{R}}}(\mu)$.

(iii) $|u| \in \mathcal{L}^1_{\overline{\mathbb{R}}}(\mu)$.

(iv) $\exists w \in \mathcal{L}^1_{\overline{\mathbb{R}}}(\mu)$ *with* $w \geq 0$ *s.t.* $|u| \leq w$.

Theorem 10.13 (Properties of the μ-integral)**.** *The μ-integral is:* ***homogeneous, additive****, and:*

(i) $\min\{u, v\}, \max\{u, v\} \in \mathcal{L}^1_{\overline{\mathbb{R}}}(\mu)$ *(lattice property)*

(ii) $u \leq v \Rightarrow \int u d\mu \leq \int v d\mu$ *(monotone)*

[1] In words, we extend our integral to ~~positive~~ measurable functions by noticing that we can write every measurable function as a sum of positive measurable functions, something that we do know how to integrate. We don't want to run into the problem of $\infty - \infty$, thus we require the integral of the positive and negative parts to both (separately) be less than infinity.

(iii) $\left|\int u d\mu\right| \le \int |u| d\mu$ *(triangle inequality)*

Remark. *If $u(x) \pm v(x)$ is defined in $\overline{\mathbb{R}}$ for all $x \in X$ then we can exclude $\infty - \infty$ and the theorem above just says that the integral is linear:*

$$\int (au + bv) d\mu = a \int u d\mu + b \int v d\mu. \tag{3}$$

This is always true for real-valued $u, v \in \mathcal{L}^1(\mu) = \mathcal{L}^1_{\mathbb{R}}(\mu)$, making $\mathcal{L}^1(\mu)$ a vector space with addition and scalar multiplication defined by

$$(u + v)(x) := u(x) + v(x),\ \ (a \cdot u)(x) := a \cdot u(x), \tag{4}$$

and

$$\int ...d\mu : \mathcal{L}^1(\mu) \to \mathbb{R},\ \ u \mapsto \int u d\mu, \tag{5}$$

is a ***positive linear functional****.*