Solution of exam MAT3400/4400 — Fall 2016

December 2, 2016

Problem 1

Part (a): We check that ν satisfies the axioms of a measure ([MW99, Definition 5.1]).

For any $A \in \mathcal{A}$ we have

$$\nu(A) = \mu_1(A) + \mu_2(A) \geq 0 + 0 = 0,$$

since both μ_1 and μ_2 are measures.

We also have

$$\nu(\emptyset) = \mu_1(\emptyset) + \mu_2(\emptyset) = 0 + 0 = 0.$$

Finally, if $\{A_n\}_{n=1}^{\infty} \subseteq \mathcal{A}$ is a sequence of pairwise disjoint sets, then

$$\begin{aligned}\nu\left(\cup_n A_n\right) = \mu_1\left(\cup_n A_n\right) + \mu_2\left(\cup_n A_n\right) &= \sum_n \mu_1(\cup_n A_n) + \sum_n \mu_2(A_n) \\ &= \sum_n \left(\mu_1(A_n) + \mu_2(A_n)\right) = \sum_n \nu(A_n).\end{aligned}$$

Thus ν is a measure on $(\Omega, \mathcal{A})$.

Part (b): We will use bootstrapping to prove that

$$\int_\Omega f\, d\nu = \int_\Omega f\, d\mu_1 + \int_\Omega f\, d\mu_2, \tag{1}$$

for all non-negative $\mathcal{A}$-measurable functions f. Hence we will first show (1) for non-negative simple functions, and then for general non-negative functions. Let a simple $\mathcal{A}$-measurable function $s\colon \Omega \to [0, \infty]$ be given, with canonical presentation

$$s = \sum_{n=1}^{k} a_n \chi_{A_n},$$

where the $a_n \in [0,\infty]$ are distinct and the $A_n \in \mathcal{A}$ are pairwise disjoint. Then

$$\begin{aligned}\int_\Omega s\,d\nu &= \sum_{n=1}^k a_n \nu(A_n) = \sum_{n=1}^k a_n \left(\mu_1(A_n) + \mu_2(A_n)\right) \\ &= \sum_{n=1}^k a_n \mu_1(A_n) + \sum_{n=1}^k a_n \mu_2(A_n) = \int_\Omega s\,d\mu_1 + \int_\Omega s\,d\mu_2.\end{aligned}$$

Hence (1) holds for non-negative simple functions.

Let now a non-negative $\mathcal{A}$-measurable function f be given. By [MW99, Proposition 5.7] we can find a non-decreasing sequence $\{s_n\}_n$ of non-negative simple $\mathcal{A}$-measurable functions such that s_n converges pointwise towards f. Using the Monotone Convergence Theorem ([MW99, Theorem 5.6]) three times we then get

$$\begin{aligned}\int_\Omega f\,d\nu &= \int_\Omega \lim_n s_n\,d\nu = \lim_n \int_\Omega s_n\,d\nu = \lim_n \left(\int_\Omega s_n\,d\mu_1 + \int_\Omega s_n\,d\mu_2\right) \\ &= \lim_n \int_\Omega s_n\,d\mu_1 + \lim_n \int_\Omega s_n\,d\mu_2 = \int_\Omega \lim_n s_n\,d\mu_1 + \int_\Omega \lim_n s_n\,d\mu_2 \\ &= \int_\Omega f\,d\mu_1 + \int_\Omega f\,d\mu_2.\end{aligned}$$

Hence (1) holds for all non-negative $\mathcal{A}$-measurable functions.

Problem 2

Part (a): Let $p \in [1,\infty)$ and $a \in \left(0, \frac{1}{p}\right)$ be given. We first observe that

$$0 < ap < \frac{1}{p}p = 1.$$

The function g_a is continuous so it measurable. Since the fucntion g_a is positive, we can use the Monotone Convergence Theorem ([MW99, Theorem 5.6]) to see that

$$\int_{(0,1]} |g_a|^p\,d\lambda = \int_{(0,1]} x^{-ap}\,d\lambda(x) = \lim_n \int_{[\frac{1}{n},1]} x^{-ap}\,d\lambda(x).$$

To compute the last integral, we use that the Lebesgue and Riemann integrals agree for continuous functions on closed and bounded intervals ([MW99, Theorem 4.9]). Since $-ap \neq -1$ we then get

$$\begin{aligned}\int_{[\frac{1}{n},1]} x^{-ap}\,d\lambda(x) &= \int_{\frac{1}{n}}^1 x^{-ap}\,dx = \left[\frac{1}{-ap+1}x^{-ap+1}\right]_{\frac{1}{n}}^1 \\ &= \frac{1}{1-ap} - \frac{1}{1-ap}\left(\frac{1}{n}\right)^{1-ap}.\end{aligned}$$

Since $1 - ap > 0$ we have that $\lim_n \left(\frac{1}{n}\right)^{1-ap} = 0$. Hence we get that

$$\int_{(0,1]} |g_a|^p \, d\lambda = \lim_n \left(\frac{1}{1-ap} - \frac{1}{1-ap} \left(\frac{1}{n}\right)^{1-ap} \right) = \frac{1}{1-ap} < \infty.$$

Therefore $g \in \mathcal{L}^p$.

Part (b): Let $p \in [1, \infty)$ be given. Note first that h is continuous and therefore measurable. Pick $a \in \left(0, \frac{1}{p}\right)$. By the hint we have

$$|h| < \frac{1}{ax^a} = \frac{1}{a} g_a.$$

Using monotonicity and linearity of the integral ([MW99,]) and the fact that $g_a \in \mathcal{L}^p$ we get

$$\int_{(0,1]} |h| \, d\lambda \leq \frac{1}{a} \int_{(0,1]} g_a \, d\lambda < \infty.$$

Hence $h \in \mathcal{L}^p$.

Part (c): Let $K \in \mathbb{R}$ be given. Since $|h|$ is strictly decreasing on $(0,1]$ and $|h(x)|$ tends to ∞ as x tends to 0, we can find some $x_0 \in (0,1]$ such that $|h(x)| > K$ for all $x \in (0, x_0)$. Thus

$$\lambda\left(\{x \in (0,1] \mid |h(x)| > K\}\right) \geq \lambda(0, x_0) = x_0 > 0.$$

Hence $|h|$ is not less than K λ-a.e. As this was for an arbitrary K, we deduce that h is not in $\mathcal{L}^\infty$.

Problem 3

Part (a): Consider the function $g \colon [0, \infty) \to [0, \infty)$ given by

$$g(x) = e^{-x}.$$

Since g is continuous it is measurable.

For all $n \in \mathbb{N}$ and all $x \in [0, \infty)$ we have

$$|f_n(x)| = e^{-x} \left| \cos\left(\frac{x}{n}\right) \right| \leq e^{-x} = g(x).$$

Since all the f_n are continuous and therefore measurable, it suffices to show that g is integrable in order to conclude that all the f_n are integrable.

Using the Monotone Convergence Theorem ([MW99, Theorem 5.6]) and the fact that for continuous functions on closed bounded the Lebesgue and Riemann integrals agree ([MW99, Theorem 4.9]), we see that

$$\int_{[0,\infty)} g \, d\lambda = \lim_n \int_{[0,n]} g \, d\lambda = \lim_n \int_0^n e^{-x} \, dx = \lim_n [-e^{-x}]_0^n = 1 - \lim_n e^{-n} = 1.$$

Hence g is integrable and therefore all the f_n are integrable.

Part (b): We observe that for each $x \in [0, \infty)$ we have

$$\lim_n f_n(x) = \lim_n e^{-x} \cos\left(\frac{x}{n}\right) = e^{-x} \cos\left(\lim_n \frac{x}{n}\right) = e^{-x} \cos(0) = e^{-x} = g(x).$$

So $\{f_n\}$ converges pointwise to g. Since g is integrable and dominates all the f_n, we get from the Dominated Convergence Theorem ([MW99, Theorem 5.8]) that

$$\lim_n \int_{[0,\infty)} f_n \, d\lambda = \int_{[0,\infty)} \lim_n f_n \, d\lambda = \int_{[0,\infty)} g \, d\lambda = 1.$$

Problem 4

Part (a): Let $x, y \in W$ be given, say

$$x = \sum_{n=1}^{k} \alpha_n e_n, \quad \text{and} \quad y = \sum_{n=1}^{l} \beta_n e_n.$$

By appending the sums with zero terms if necessary, we may assume that $l = k$. We have

$$\begin{aligned} T_p(x+y) &= T_p\left(\sum_{n=1}^{k} (\alpha_n + \beta_n) e_n\right) = \sum_{n=1}^{k} (\alpha_n + \beta_n) e_{p^n} = \sum_{n=1}^{k} \alpha_n e_{p^n} + \sum_{n=1}^{k} \beta_n e_{p^n} \\ &= T_p\left(\sum_{n=1}^{k} \alpha_n e_n\right) + T_p\left(\sum_{n=1}^{k} \beta_n e_{p^n}\right) = T_p x + T_p y. \end{aligned}$$

If $\beta \in \mathbb{C}$ then

$$T_p(\beta x) = T_p\left(\sum_{n=1}^{k} \beta \alpha_n e_n\right) = \sum_{n=1}^{k} \beta \alpha_n e_{p^n} = \beta \left(\sum_{n=1}^{k} \alpha_n e_{p^n}\right) = \beta T_p x.$$

Hence T_p is linear.

To see that T is bounded, note that since $\{e_n\}$ is a basis we have ([RY08, Theorem 3.47])

$$\|x\|^2 = \left\|\sum_{n=1}^{k} \alpha_n e_n\right\|^2 = \sum_{n=1}^{k} |\alpha_n|^2 = \left\|\sum_{n=1}^{k} \alpha_n e_{p^n}\right\|^2 = \|T_p x\|^2.$$

Thus T_p is not only bounded it is an isometry.

Since T_p is a bounded linear operator and W is dense in H, we can extend T by continuity ([RY08, Theorem 4.19]) to a an operator S_p such that $\|S_p\| = 1$ and $S_p e_n = T e_n = e_{p^n}$.

Part (b): Let $p, q \in \mathbb{P}$. For any $n, m \in \mathbb{N}$ we have

$$\langle S_p^* S_q e_n, e_m \rangle = \langle S_q e_n, S_p e_m \rangle = \langle e_{q^n}, e_{p^m} \rangle = \begin{cases} 1, & \text{if } q = p \text{ and } n = m \\ 0, & \text{otherwise} \end{cases}.$$

The last equality follows since p, q are prime. In particular if $p \neq q$ then

$$\langle S_p^* S_q e_n, e_m \rangle = 0,$$

for all $n, m \in \mathbb{N}$. Hence, for any $m \in \mathbb{N}$ we have, since $\{e_n\}$ is a basis,

$$S_p^* S_q e_m = \sum_{n=1}^{\infty} \langle S_p^* S_q e_m, e_n \rangle e_n = 0.$$

And so $S_p^* S_q$ and O agree on basis elements, and therefore $S_p^* S_q = O$.

The above computation also shows that

$$S_p^* S_p e_m = \sum_{n=1}^{\infty} \langle S_p^* S_p e_m, e_n \rangle e_n = e_m.$$

Hence $S_p^* S_p e_m = I e_m$ for all m, so $S_p^* S_p = I$.

Problem 5

The fist step to applying the hint, is to show that if $f \in \mathcal{L}^2(\mu)$ then $f \in \mathcal{L}^1(\nu)$. So let $f \in \mathcal{L}^2(\mu)$ be given. Since $\mu(\Omega) < \infty$ the constant function 1 is an $\mathcal{L}^2(\mu)$ function, so we get from Hölder's inequality ([MW99, Theorem 13.9]) that

$$\int_\Omega |f| \, d\mu = \int_\Omega |f 1| \, d\mu \leq \|f\|_2 \|1\|_2 = \|f\|_2 \sqrt{\mu(\Omega)}.$$

Where the $\|\cdot\|_2$ norms are computed in $\mathcal{L}^2(\mu)$. Hence we see that $f \in \mathcal{L}^1(\mu)$. Now since $\nu \leq \mu$ we get for any non-negative $\mathcal{A}$-measurable function g that

$$\begin{aligned}
\int_\Omega g \, d\nu &= \sup \left\{ \int_\Omega s \, d\nu \;\middle|\; s \text{ simple non-negative } \mathcal{A}\text{-measurable, } s \leq g \right\} \\
&= \sup \left\{ \sum_{n=1}^{k} a_k \nu(A_k) \;\middle|\; \begin{array}{c} s \text{ simple non-negative } \mathcal{A}\text{-measurable, } s \leq g \\ s = \sum_{n=1}^{k} a_k \chi_{A_k} \text{ is the canonical presentation of } s \end{array} \right\} \\
&\leq \sup \left\{ \sum_{n=1}^{k} a_k \mu(A_k) \;\middle|\; \begin{array}{c} s \text{ simple non-negative } \mathcal{A}\text{-measurable, } s \leq g \\ s = \sum_{n=1}^{k} a_k \chi_{A_k} \text{ is the canonical presentation of } s \end{array} \right\} \\
&= \sup \left\{ \int_\Omega s \, d\mu \;\middle|\; s \text{ simple non-negative } \mathcal{A}\text{-measurable, } s \leq g \right\} \\
&= \int_\Omega g \, d\mu.
\end{aligned}$$

Putting it all together, we have that

$$\int_\Omega |f|\,d\nu \le \int_\Omega |f|\,d\mu \le \|f\|_2\sqrt{\mu(\Omega)} < \infty. \tag{2}$$

So that $f \in \mathcal{L}^1(\nu)$.

Suppose that $h_1, h_2 \in \mathcal{L}^2(\mu)$ are equal μ-a.e. Then

$$0 = \mu(\{x \in \Omega \mid h_1(x) \ne h_2(x)\}) \ge \nu(\{x \in \Omega \mid h_1(x) \ne h_2(x)\}).$$

So h_1, h_2 are equal ν-a.e. Combining this with the above, we have that the map $\phi\colon L^2(\mu) \to \mathbb{C}$ given by

$$\phi([h]) = \int_\Omega h\,d\nu,$$

is well-defined (i.e. it doesn't depend on the representative h of the class $[h]$, and the integral on the right always makes sense). The map ϕ is linear since integration is a linear operation, and it is bounded, since by (2) and the triangle inequality for integrals

$$|\phi([h])| = \left|\int_\Omega h\,d\nu\right| \le \int_\Omega |h|\,d\nu \le \sqrt{\mu(\Omega)}\|h\|_2,$$

so $\|\phi\| \le \sqrt{\mu(\Omega)}$.

The Riesz-Fréchet Theorem ([RY08, Theorem 5.2]) now tells us that there exists $g \in \mathcal{L}^2(\mu)$ such that for all $h \in \mathcal{L}^2(\mu)$ we have

$$\phi([h]) = \langle [h], [g] \rangle = \int_\Omega h\overline{g}\,d\mu.$$

For any $A \in \mathcal{A}$ we have $\mu(A) < \infty$ so the indicator function χ_A is in $\mathcal{L}^2(\mu)$, hence we have

$$\nu(A) = \int_A d\nu = \int_\Omega \chi_A\,d\nu = \phi([\chi_A]) = \int_\Omega \chi_A\overline{g}\,d\mu = \int_A \overline{g}\,d\mu,$$

as desired.

References

[MW99] John N. McDonald and Neil A. Weiss, *A course in real analysis*, Academic Press, Inc., San Diego, CA, 1999, Biographies by Carol A. Weiss. MR 1680810

[RY08] Bryan P. Rynne and Martin A. Youngson, *Linear functional analysis*, second ed., Springer Undergraduate Mathematics Series, Springer-Verlag London, Ltd., London, 2008. MR 2370216