

Existence of Measures

February 14, 2024

Theorem 6.7 (Carathéodory). *Let $S \subset P(X)$ be a semi-ring and $\mu : S \rightarrow [0, \infty)$ a pre-measure. Then μ has an extension to a measure μ^* on $\sigma(S)$, i.e. that $\mu(s) = \mu^*(s)$, $\forall s \in \sigma(S)$.*

Also, if S contains an exhausting sequence, $S_n \uparrow X$, s.t. $\mu(S_n) < \infty$, then the extension is unique.

Proof (outline). Firstly, let us define an outer measure.

Definition 6.7.1 (Outer measure). An outer measure is a function $\mu^* : P(X) \rightarrow [0, \infty)$ with the following properties:

1. $\mu^*(\emptyset) = 0$,
2. $A \subset B \Rightarrow \mu^*(A) \leq \mu^*(B)$,
3. $\mu^*\left(\bigcup_{n \in \mathbb{N}} A_n\right) \leq \sum_{n \in \mathbb{N}} \mu^*(A_n)$,

and define for each $A \subset X$ the family of countable S -coverings:

$$C(A) := \left\{ (S_n)_{n \in \mathbb{N}} \subset S : \bigcup_{n \in \mathbb{N}} S_n \supset A \right\},$$

and the set function

$$\mu^*(A) := \inf \left\{ \sum_{n \in \mathbb{N}} \mu(S_n) : (S_n)_{n \in \mathbb{N}} \in C(A) \right\}.$$

Step 1: Claim: $\mu^*(A)$ is an outer measure.

Proof.

1. $C(\emptyset) = \{\text{any sequence in } S \text{ containing } \emptyset\} \Rightarrow \mu^*(\emptyset) = 0$.
2. Assume $A \subset B$. Then $C(A) \subset C(B) \Rightarrow \mu^*(A) \leq \mu^*(B)$.

3. If $\mu^*(A_n) = \infty$ for some n , then there is nothing to prove. Thus, assume $\mu^*(A_n) < \infty \forall n$. Fix $\epsilon > 0$, and for every n choose $A_{n_k} \in S$ s.t.

$$A_n \subset \bigcup_{k \in \mathbb{N}} A_{n_k}, \quad \sum_{k \in \mathbb{N}} \mu^*(A_{n_k}) < \mu^*(A_n) + \frac{\epsilon}{2^n}.$$

Then

$$\bigcup_{n \in \mathbb{N}} A_n \subset \bigcup_{k \in \mathbb{N}} \bigcup_{n \in \mathbb{N}} A_{n_k},$$

so

$$\begin{aligned} \mu^* \left(\bigcup_{n \in \mathbb{N}} A_n \right) &\leq \sum_{n \in \mathbb{N}} \sum_{k \in \mathbb{N}} \mu(A_{n_k}) \\ &< \sum_{n \in \mathbb{N}} \left(\mu^*(A_n) + \frac{\epsilon}{2^n} \right) \\ &= \sum_{n \in \mathbb{N}} \mu^*(A_n) + \epsilon. \end{aligned}$$

As ϵ was arbitrarily, we get that

$$\mu^* \left(\bigcup_{n \in \mathbb{N}} A_n \right) \leq \sum_{n \in \mathbb{N}} \mu^*(A_n),$$

so μ^* fulfills all the conditions for being an outer measure. □

Step 2: Showing that μ^* extends μ , i.e. $\mu^*(s) = \mu(s) \forall s \in S$.

Step 3: Define μ^* -measurable sets

$$\Sigma^* := \{A \subset X : \mu^*(Q) = \mu^*(Q \cap A) + \mu^*(Q \setminus A) \forall Q \subset X\}$$

Step 4: Show that $\mu|_{\Sigma^*}$ is a measure. In particular, $\mu|_{\sigma(S)}$ is a measure which extends μ . □