Existence of Measures

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Theorem 6.7 (Carathéodory)**.** *Let $S \subset P(X)$ be a semi-ring and $\mu : S \to [0, \infty)$ a pre-measure. Then μ has an extension to a measure μ^* on $\sigma(S)$, i.e. that $\mu(s) = \mu^*(s),\ \forall\ s \in \sigma(S)$.*

Also, if S contains an exhausting sequence, $S_n \uparrow X$, s.t. $\mu(S_n) < \infty$, then the extension is unique.

Proof (outline). Firstly, let us define an outer measure.

Definition 6.8 (Outer measure)**.** An outer measure is a function $\mu^* : P(X) \to [0, \infty)$ with the following properties:

1. $\mu^*(\emptyset) = 0$,
2. $A \subset B \Rightarrow u^*(A) \leq \mu^*(B)$,
3. $\mu^*\left(\bigcup\limits_{n\in\mathbb{N}} A_n\right) \leq \sum\limits_{n\in\mathbb{N}} \mu^*(A_n)$,

and define for each $A \subset X$ the family of countable S-*coverings*:

$$C(A) := \left\{ (S_n)_{n\in\mathbb{N}} \subset S : \bigcup_{n\in\mathbb{N}} S_n \supset A \right\},$$

and the set function

$$\mu^*(A) := \inf \left\{ \sum_{n\in\mathbb{N}} \mu(S_n) : (S_n)_{n\in\mathbb{N}} \in C(A) \right\}.$$

Step 1: Claim: $\mu^*(A)$ *is an outer measure.*

Proof.

1. $C(\emptyset) = \{\text{any sequence in } S \text{ containing } \emptyset\} \Rightarrow \mu^*(\emptyset) = 0$.
2. Assume $A \subset B$. Then $C(A) \subset C(B) \Rightarrow \mu^*(A) \leq \mu^* B$.

3. If $\mu^*(A_n) = \infty$ for some n, then there is nothing to prove. Thus, assume $\mu^*(A_n) < \infty$ $\forall n$. Fix $\epsilon > 0$, and for every n choose $A_{n_k} \in S$ s.t.

$$A_n \subset \bigcup_{k\in\mathbb{N}} A_{n_k}, \; \sum_{k\in\mathbb{N}} \mu^*(A_{n_k}) < \mu^*(A_n) + \frac{\epsilon}{2^n}.$$

Then

$$\bigcup_{n\in\mathbb{N}} A_n \subset \bigcup_{k\in\mathbb{N}} \bigcup_{n\in\mathbb{N}} A_{n_k},$$

so

$$\begin{aligned}\mu^*\left(\bigcup_{n\in\mathbb{N}} A_n\right) &\leq \sum_{n\in\mathbb{N}} \sum_{k\in\mathbb{N}} \mu\left(A_{n_k}\right) \\ &< \sum_{n\in\mathbb{N}} \left(\mu^*(A_n) + \frac{\epsilon}{2^n}\right) \\ &= \sum_{n\in\mathbb{N}} \mu^*(A_n) + \epsilon.\end{aligned}$$

As ϵ was arbitrarily, we get that

$$\mu^*\left(\bigcup_{n\in\mathbb{N}} A_n\right) \leq \sum_{n\in\mathbb{N}} \mu^*(A_n),$$

so μ^* fulfills all the conditions for being an outer measure. □

Step 2: *Showing that μ^* extends μ, i.e. $\mu^*(s) = \mu(s)$ $\forall s \in S$.*
Step 3: *Define μ^*-measurable sets*

$$\Sigma^* := \{A \subset X : \mu^*(Q) = \mu^*(Q \cap A) + \mu^*(Q \backslash A) \; \forall \; Q \subset X\}$$

Step 4: *Show that $\mu|_{\Sigma^*}$ is a measure. In particular, $\mu|_{\sigma(S)}$ is a measure which extends μ.*

□