Suggested solution to the exam in MAT3400/4400, December 7, 2015.

Problem 1. Solution: Note that $e^{-(x+y+\frac{x^2y^2}{n})} \to e^{-(x+y)}$ as $n \to \infty$ pointwise on $X \times X$. By Tonelli's theorem, which applies since all the functions involved are measurable and non-negative, it follows that

$$\int_{X\times X} e^{-(x+y)} d(\lambda \otimes \lambda)(x, y) = (\int_X e^{-x} d\lambda(x))(\int_X e^{-y} d\lambda(y)),$$

and since $\int_X e^{-x} d\lambda(x) = \lim_n \int_{[0,n]} e^{-x} dx = 1$ e.g. by Monotone Convergence theorem, it follows that $e^{-(x+y)} \in L^1(X \times X)$. Then

$$\lim_{n\to\infty} \int_{X\times X} e^{-(\frac{x^2y^2}{n}+x+y)} d(\lambda \otimes \lambda)(x, y) = 1.$$

Problem 2. Solution for 2a: The assumptions imply that $|f| \leq g$ μ-a.e. on Ω. (You need to provide details for this claim.) Then

$$|f_n - f|^p \leq (|f_n| + |f|)^p \leq 2^p g^p.$$

Since $g \in \mathcal{L}^p(\mu)$, it follows that $g^p \in \mathcal{L}^1(\mu)$, and therefore $2^p g^p \in \mathcal{L}^1(\mu)$. Now apply the Dominated Convergence Theorem.

Solution for 2b: Note that for all $n \geq 1$ and all $x \in [1, \infty)$,

$$|f_n(x)| = \frac{n}{n\sqrt{x} + 1} \leq \frac{1}{\sqrt{x}} = v_{1/2}(x).$$

Then $|f_n|^p \leq v_{p/2}$. For $p > 2$, the function $v_{p/2}$ is in $\mathcal{L}^1(\lambda)$, so $v_{1/2}$ is in $\mathcal{L}^p(\lambda)$. By the properties of the integral (it is monotone), also $f_n \in \mathcal{L}^p(\lambda)$ when $2 < p < \infty$.

Solution for 2c: Since f_n converges pointwise to $v_{1/2}$, part a implies that the convergence is in $\mathcal{L}^p(\lambda)$ when $2 < p < \infty$.

Problem 3. Denote $X = [0, 1]$. Assume $|u(x)| = 1$ λ-a.e. on X. Let $N \in \mathcal{B}$ such that $\lambda(N) = 0$ and $N = \{x \mid |u(x)| \neq 1\}$. Then for $f \in L^2(\lambda)$,

$$\|U(f)\|_2^2 - \|f\|_2^2 = \int_X (|u(x)|^2 - 1)|f(x)|^2\, d\lambda(x) = \int_{N^c} (|u(x)|^2 - 1)|f(x)|^2\, d\lambda(x) = 0,$$

so $\|U(f)\|_2 = \|f\|_2$ for every $f \in L^2(\lambda)$. This shows that U is an isometry. For the converse implication, assume that $\|U(f)\|_2 = \|f\|_2$ for every $f \in L^2(\lambda)$. Then

$$\int_X (|u(x)|^2 - 1)|f(x)|^2\, d\lambda(x) = 0$$

for arbitrary $f \in L^2(\lambda)$. If there is $A \in \mathcal{B}$ such that $\lambda(A) > 0$ and $|u(x)| > 1$ for all $x \in A$, then with $f = \chi_A$ we have

$$0 < \lambda(A) \leq \int_X (|u(x)|^2 - 1)|f(x)|^2 \, d\lambda(x),$$

a contradiction. A similar argument applies in case that there is $A \in \mathcal{B}$ such that $\mu(A) > 0$ and $|u(x)| < 1$ for all $x \in A$. For the last claim, show that $U^*(f)(x) = \overline{u(x)}f(x)$ for $f \in L^2(\lambda)$ (similar to what was done in class), and verify directly that $UU^* = U^*U = I$.

Problem 4. For 4a, if λ is an eigenvalue of S with eigenvector $u \neq 0$, then $0 \leq (S(u) \mid u) = (\lambda u \mid u) = \lambda\|u\|^2$, so $\lambda \geq 0$.

Solution 4b: The operator B^*B is self adjoint because $(B^*B)^* = B^*(B^*)^* = B^B*$, and it is compact because B is and the product of a compact operator with a bounded operator in $B(H)$ is again compact. Furthermore, it is positive because $(B^*B(x) \mid x) = (B(x) \mid B(x)) \geq 0$ for all $x \in H$. The spectral theorem for compact self-adjoint operators gives the required existence of the sequence $\{\lambda_n\}$ of real eigenvalues with corresponding eigenvectors $\{e_n\}$ that form an orthonormal basis for the Hilbert space $\overline{\mathrm{Im}(B^*B)}$. By 4a, $\lambda_n \geq 0$ for all n.

Solution 4c: Note that $(B(e_n) \mid B(e_m)) = (e_n \mid B^*B(e_m)) = \lambda_m(e_n \mid e_m)$, so the vectors $\{B(e_n)\}_n$ are pairwise orthogonal. Then $f_n = \frac{1}{\sqrt{\lambda_n}}B(e_n)$ are orthonormal. Let M denote the closed subspace $\overline{\mathrm{Im}(B^*B)}$ of H (equal to $\overline{\mathrm{Im}(B)}$ by assumption). Then $M^\perp = \ker(B^*B) = \ker(B)$. Let P_M be the orthogonal projection corresponding to M. Given $x \in H$,

$$B(x) = B(P_M(x) + (I - P_M)(x)) = B(P_M(x)) + B(P_{M^\perp}(x)) = B(P_M(x)).$$

But $P_M(x) = \sum_n (x \mid e_n)e_n$ because $\{e_n\}_n$ is an onb for M, so applying B to this series and using the continuity of B implies the claimed equality.

Solution 4d. To show that B is compact follow the proof given in the class in case of the sequence $\alpha_n = 1/n$ (example 7.11). Here we have $\alpha_n = \frac{i^n}{n}$, which converges to zero. Compute B^*. Let $\{e_n\}_n$ be the canonical orthonormal basis in H with e_n having all entries equal to zero except the place n where the entry is 1. A computation shows that $B^*B(e_n) = \frac{1}{n^2}e_n$. Thus $\lambda_n = \frac{1}{n^2}$ is an eigenvalue with corresponding eigenvector e_n for B^*B. Since $B(e_n) = \frac{i^n}{n}e_n$, it follows that $f_n = nB(e_n) = i^n e_n$.