Problem 1.

(a) Let $x \in \Omega$ be given. Since $f(x) \in \mathbb{C}$, there is some $N \in \mathbb{N}$ such that $|f(x)| < N$. Hence for $n \geq N$ we have $|f(x)| < N \leq n$ so $|f(x)| \not\geq n$. In other words $x \notin A_n$ so $\chi_{A_n}(x) = 0$. Therefore $\lim_n \chi_{A_n}(x) = 0$.

(b) Since f is measurable so is $|f|$ and hence $A_n = |f|^{-1}([n, \infty))$ is $\mathcal{A}$-measurable. Therefore we have, by monotinicity of the integral,

$$\int_\Omega |f| \, d\mu \geq \int_{A_n} |f| \, d\mu \geq \int_{A_n} n \, d\mu = n\mu(A_n).$$

for all $n \in \mathbb{N}$.

(c) Define for each $n \in \mathbb{N}$ an $\mathcal{A}$-measurable function $f_n : \Omega \to \mathbb{C}$ by $f_n = |f|\chi_{A_n}$. It follows from problem **1a** that for all $x \in \Omega$

$$f_n(x) = |f(x)|\chi_{A_n}(x) \to |f(x)| \cdot 0 = 0, \text{ as } n \to \infty.$$

That is, f_n converges to 0 pointwise. Furthermore, for each $n \in \mathbb{N}$ we have $|f_n| \leq |f|$, and by assumption $|f|$ is integrable. Therefore we can apply the Dominated Convergence Theorem to see that

$$\lim_n \int_\Omega f_n \, d\mu = \int_\Omega \lim_n f_n \, d\mu = \int_\Omega 0 \, d\mu = 0.$$

By problem **1b** we have

$$0 \leq n\mu(A_n) \leq \int_{A_n} |f| \, d\mu = \int_\Omega f_n \, d\mu.$$

Since we just showed that the right hand side goes to zero, we must have

$$\lim_n n\mu(A_n) = 0.$$

Problem 2.
Define a continuous function $g\colon [1, \infty) \to \mathbb{R}$ by $g(x) = \frac{1}{x^2}$. For any $n \in \mathbb{N}$ and any $x \in [1, \infty)$ we have

$$|f_n(x)| = \frac{|\sin(x)|^n}{x^2} \leq \frac{1}{x^2} = g(x).$$

We know, from exercises from the course, that g is integrable over $[1, \infty)$, so by monotinicity of the integral

$$\int_{[1,\infty)} |f_n| \, d\lambda \leq \int_{[1,\infty)} g \, d\lambda < \infty.$$

Hence f_n is integrable for all $n \in \mathbb{N}$.

Let now

$$A = \left\{ \frac{n\pi}{2} \;\middle|\; n \in \mathbb{N} \right\}.$$

Since A is countable it is measurable and has Lebesgue measure 0. If $x \in [1, \infty) \setminus A$ then $-1 < \sin(x) < 1$ so $\sin(x)^n \to 0$ as $n \to 0$. Therefore we have that

$$f_n(x) = \frac{(\sin(x))^n}{x^2} \to \frac{0}{x^2} = 0, \text{ as } n \to \infty,$$

for all $x \in [1, \infty) \setminus A$. That is f_n converges pointwise to 0 λ-almost everywhere.

We showed before $|f_n| \le g$ and g is integrable, so by the Dominated Convergence Theorem we get

$$\lim_n \int_{[1,\infty)} f_n \, d\lambda = \int_{[1,\infty)} 0 \, d\lambda = 0.$$

Problem 3.
Let $x_1, x_2 \in \mathbb{R}$ and $a_1, a_2 \in [0, 1]$ with $a_1 + a_2 = 1$ be given. Define a simple function $s \colon [0, 1] \to \mathbb{R}$ by

$$s = x_1 \chi_{[0,a_1)} + x_2 \chi_{[a_1,1]}.$$

Then s is λ-integrable and we have that

$$\begin{aligned}\int_{[0,1]} s \, d\lambda &= x_1 \lambda([0, a_1)) + x_2 \lambda([a_1, 1]) \\ &= x_1(a_1 - 0) + x_2(1 - a_1) = x_1 a_1 + x_2 a_2.\end{aligned}$$

Whence

$$\psi(x_1 a_1 + x_2 a_2) = \psi\left(\int_{[0,1]} s \, d\lambda\right). \tag{†}$$

Suppose now that $t \in [0, 1]$. Then either $t \in [0, a_1)$ or $t \in [a_1, 1]$. In the first case we have

$$(\psi \circ s)(t) = \psi\,(s(t)) = \psi(x_1),$$

and in the second case

$$(\psi \circ s)(t) = \psi\,(s(t)) = \psi(x_2).$$

Hence

$$\psi \circ s = \psi(x_1) \chi_{[0,a_1)} + \psi(x_2) \chi([a_1, 1]),$$

for all $t \in [0, 1]$. Computing as before we therefore get

$$\begin{aligned}\int_{[0,1]} \psi \circ s \, d\lambda &= \int_{[0,1]} \psi(x_1) \chi_{[0,a_1)} + \psi(x_2) \chi([a_1, 1]) \, d\lambda \\ &= \psi(x_1) \lambda([0, a_1)) + \psi(x_2) \lambda([a_1, 1]) = \psi(x_1) a_1 + \psi(x_2) a_2.\end{aligned}$$

Hence

$$\int_{[0,1]} \psi \circ s \, d\lambda = a_1 \psi(x_1) + a_2 \psi(x_2). \tag{‡}$$

Combining (†), (‡), and the inequality from the problem formulation we get

$$\psi(x_1 a_1 + x_2 a_2) = \psi\left(\int_{[0,1]} s \, d\lambda\right) \le \int_{[0,1]} \psi \circ s \, d\lambda = a_1 \psi(x_1) + a_2 \psi(x_2),$$

as desired.

Problem 4.

(a) For any $f \in L^2(\Omega)$ we have

$$P_E(P_E f) = P_E(f\chi_E) = f\chi_E\chi_E = f\chi_E = P_E f,$$

since $\chi_E^2 = \chi_E$. Hence $P_E^2 = P_E$. For any $f, g \in L^2(\Omega)$ we see, using that χ_E is equal to its own complex conjugate at each point in Ω, that

$$\begin{aligned}\langle P_E f, g\rangle = \langle f\chi_E, g\rangle &= \int_\Omega f\chi_E\overline{g}\,d\mu = \int_\Omega f\overline{g\chi_E}\,d\mu \\ &= \langle f, g\chi_E\rangle = \langle f, P_E g\rangle.\end{aligned}$$

Since P_E^* is defined as the unique operator that such that

$$\langle P_E f, g\rangle = \langle f, P_E^* g\rangle,$$

for all $f, g \in L^2(\Omega)$, we must have that $P_E = P_E^*$. Since $P_E = P_E^* = P_E^2$, P_E is a projection.

(b) Let $f \in L^2(\Omega)$ be given. (Formally we are picking a representative for an equivalence class). We observe that for any $n \in \mathbb{N}$

$$f - f\chi_{E_n} = f \cdot (1 - \chi_{E_n}) = f\chi_{E_n^c}. \tag{$\star$}$$

From the equations defining the E_n's we obtain the following by taking complements

$$E_1^c \supseteq E_2^c \supseteq E_3^c \supseteq \cdots \quad \text{and} \quad \bigcap_n E_n^c = \emptyset.$$

Hence, for any $x \in \Omega$ we can find an $N \in \mathbb{N}$ such that if $n \geq N$ then $\chi_{E_n^c}(x) = 0$. Hence we have

$$f - f\chi_{E_n} = f\chi_{E_n^c} \to 0 \text{ pointwise as } n \to \infty.$$

Since $f \in L^2(\Omega)$ we have that $|f^2|$ is integrable and for any $n \in \mathbb{N}$ we have $|f^2\chi_{E_n^c}| \leq |f^2|$, so by the Dominated Convergence Theorem we get

$$\begin{aligned}\lim_n \|f - P_{E_n} f\|_2^2 = \lim_n \|f\chi_{E_n^c}\|_2^2 &= \lim_n \int_\Omega |f\chi_{E_n^c}|^2\,d\mu \\ &= \int_\Omega \lim_n |f\chi_{E_n^c}|^2\,d\mu = \int_\Omega 0\,d\mu = 0.\end{aligned}$$

Therefore $\lim_n \|f - P_{E_n} f\|_2 = 0$.

(c) From ($\star$) we see that $I - P_{E_n} = P_{E_n^c}$. We have

$$0 < \mu(E_n^c) \leq \mu(\Omega) < \infty,$$

so $\chi_{E_n^c}$ is a non-zero function in $L^2(\Omega)$. And since

$$P_{E_n^c}\chi_{E_n^c} = \chi_{E_n^c}\chi_{E_n^c} = \chi_{E_n^c},$$

the operator $P_{E_n^c}$ is a non-zero projection. In particular it has norm 1. Therefore

$$\|I - P_{E_n}\| = \|P_{E_n^c}\| = 1 \not\to 0.$$

Problem 5.

(a) Let $\{x_n\} \subseteq H$ be a bounded sequence. To show that S is compact, we will show that there is a subsequence of $\{x_n\}$ such that S applied to it converges. Since T is compact there is a subsequence $\{x_{n_k}\}$ such that $\{Tx_{n_k}\}$ is a convergent sequence. For $k, l \in \mathbb{N}$ we have

$$\|Sx_{n_k} - Sx_{n_l}\| = \|S(x_{n_k} - x_{n_l})\| \leq \|T(x_{n_k} - x_{n_l})\| = \|Tx_{n_k} - Tx_{n_l}\|.$$

Since $\{Tx_{n_k}\}$ is convergent it is Cauchy so the right hand side above tends to 0 when k, l tend to infinity. Hence $\{Sx_{n_k}\}$ is a Cauchy sequence, and therefore $\{Sx_{n_k}\}$ is convergent as H is complete.

(b) Let $x \in H$ be given. We have

$$0 \leq \langle (S - I)x, x \rangle = \langle Sx, x \rangle - \langle Ix, x \rangle = \langle Sx, x \rangle - \langle x, x \rangle.$$

So
$$\|x\|^2 = \langle x, x \rangle \leq \langle Sx, x \rangle.$$

By the Cauchy-Schwarz inequality we have

$$\langle Sx, x \rangle \leq \|Sx\|\|x\|$$

Hence we see that
$$\|x\|^2 \leq \|Sx\|\|x\|,$$

so
$$\|Ix\| = \|x\| \leq \|Sx\|.$$

It now follows from problem **5a** that if S was compact then I would be compact, but I is not compact since H is infinite dimensional. Hence S cannot be compact.