MAT4400: Notes on Linear analysis

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3 σ-Algebras

Definition 3.1 (σ-Algebra)**.** A family $\mathscr{A}$ of subsets of X with:

(i) $X \in \mathscr{A}$,

(ii) $A \in \mathscr{A} \Rightarrow A^c \in \mathscr{A}$,

(iii) $(A_n)_{n\in\mathbb{N}} \in \mathscr{A} \Rightarrow \bigcup\limits_{n\in\mathbb{N}}$

Theorem 3.2 (and Definition)**.**

(i) The intersection of arbitrarily many σ-algebras in X is againg a σ-algebra in X.

(ii) For every system of sets $p \subset \mathscr{P}(X)$ there exists a smallestσ-algebra containing$p$. This is the σ-algebra generated by p, denoted $\sigma(p)$, and $\sigma(p)$ is called its generator.

Definition 3.3 (Borel)**.** The σ-algebra $\sigma(\mathcal{O})$ generated by the open sets $\mathcal{O} = \mathcal{O}_{\mathbb{R}^n}$ of $\mathbb{R}^n$ is called **Borel σ-algebra**, and its members are called **Borel sets** or **Borel measurable sets**.

5 Uniqueness of Measures

Lemma 5.1. *A Dynkin system D is a σ-algebra iff it is stable under finite intersections, i.e. $A, B \in D \Rightarrow A \cap B \in D$.*

Theorem 5.2 (Dynkin)**.** *Assume X is a set, S is a collection of subsets of X closed under finite intersections, that is, if $A, B \in S \Rightarrow A \cap B \in S$. Then $D(S) = \sigma(S)$.*

Theorem 5.3 (uniqueness of measures)**.** *Let (X, B) be a measurable space, and $S \subset P(X)$ be the generator of B, i.e. $B = \sigma(S)$. If S satisfies the following conditions:*

1. *S is stable under finite intersections ($\cap$-stable), i.e. $A, C \in S \Rightarrow A \cap C \in S$.*

2. *There exists an exhausting sequence* $(G_n)_{N\in\mathbb{N}} \subset$ *with* $G_N \uparrow X$*. Assume also that there are two measures* μ, ν *satisfying:*

3. $\mu(A) = \nu(A),\ \forall A \in S.$

4. $\mu(G_n) = \nu(G_n) < \infty.$

Then $\mu = \nu$*.*

6 Existence of Measures

Theorem 6.1 (Carathéodory)**.** *Let* $S \subset P(X)$ *be a semi-ring and* $\mu : S \to [0, \infty)$ *a pre-measure. Then* μ *has an extension to a measure* μ^* *on* $\sigma(S)$*, i.e. that* $\mu(s) = \mu^*(s),\ \forall\ s \in \sigma(S)$*.*

Also, if S *contains an exhausting sequence,* $S_n \uparrow X$*, s.t.* $\mu(S_n) < \infty$*, then the extension is unique.*

7 Measurable Mappings

We consider maps $T : X \to X'$ between two measurable spaces $(X, \mathcal{A})$ and $(X', \mathcal{A}')$ which respects the measurable structurs, the σ-algbras on X and X'. These maps are useful as we can transport a measure μ, defined on $(X, \mathcal{A})$, to $(X', \mathcal{A}')$.

Definition 7.1. Let $(X, \mathcal{A})$, $(X', \mathcal{A}')$ b measurable spaces. A map $T : X \to X'$ is called $\mathcal{A}/\mathcal{A}'$-measurable if the pre-imag of every measurable set is a measurable set:

$$T^{-1}(A') \in \mathcal{A}, \quad \forall A' \in \mathcal{A}'. \tag{1}$$

- A $\mathcal{B}(\mathbb{R}^n)/\mathcal{B}(\mathbb{R}^m)$ measurable map is often called a Borel map.
- The notation $T : (X, \mathcal{A}) \to (X', \mathcal{A}')$ is often used to indicate measurability of the map T.

Lemma 7.2. *Let* $(X, \mathcal{A})$*,* $(x', \mathcal{A}')$ *be measurable spaces and let* $\mathcal{A}' = \sigma(\mathcal{G}')$*. Then* $T : X \to X'$ *is* $\mathcal{A}/\mathcal{A}'$*-measurable iff* $T^{-1}(\mathcal{G}') \subset \mathcal{A}$*, i.e. if*

$$T^{-1}(G') \in \mathcal{A},\ \ \forall G' \in \mathcal{G}'. \tag{2}$$

Theorem 7.3. *Let* $(X_i, \mathcal{A}_i),\ \ i = 1, 2, 3,$ *be measurable spaces and* $T : X_1 \to X_2$*,* $S : X_2 \to X_3$ *be* $\mathcal{A}_1/\mathcal{A}_2$ *and* $\mathcal{A}_2/\mathcal{A}_3$*-measurable maps respectivly. Then* $S \circ T : X_1 \to X_3$ *is* $\mathcal{A}_1/\mathcal{A}_3$*-measurable.*

Corollary 7.4. *Every continuous map betwen metric spaces is a Borel map.*

Definition 7.5. (and lemma) Let $(T_i)_{i\in I}$, $T_I : X \to X_i$, be arbitrarily many mappings from the same space X into measurable spaces $(X_i, \mathcal{A}_i)$. The smallest σ-algebra on X that makes all T_i simultanously measurable is

$$\sigma(T_i : i \in I) := \sigma\left(\bigcup_{i\in I} T_i^{-1}(\mathcal{A}_i)\right) \tag{3}$$

Corollary 7.6. *A function $f : (X, \mathcal{B}) \to \mathbb{R}$ is measurable if $f((a, +\infty)) \in \mathcal{B},\ \forall a \in \mathbb{R}$.*

Corollary 7.7. *Assume $(X, \mathcal{B})$ is a measurable space, (Y, d) is a metric space, $(f_n : (X, \mathcal{B}) \to Y)_{n=1}^{\infty}$ is a sequence of measurable maps. Assume this sequence of images $(f_n(x))_{n=1}^{\infty}$ is convergent in Y $\forall x \in X$. Define*

$$f : X \to Y, \ \ by\ f(x) = \lim_{n\to\infty} f_n(x). \tag{4}$$

Then f is measurable.

Theorem 7.8. *Let $(X, \mathcal{A})$, $(X', \mathcal{A}')$ be measurable spaces and $T : X \to X'$ be an $\mathcal{A}/\mathcal{A}'$-measurable map. For every measurable μ on $(X, \mathcal{A})$,*

$$\mu'(A') := \mu(T^{-1}(A')), \quad A' \in \mathcal{A}', \tag{5}$$

defines a measure on $(X', \mathcal{A}')$.

Definition 7.9. The measure $\mu'(\cdot)$ in the above theorem is called the push forward or image measure of μ under T and it is denoted as $T(\mu)(\cdot)$, $T_{*\mu}(\cdot)$ or $\mu \circ T^{-1}(\cdot)$.

Theorem 7.10. *If $T \in \mathbb{R}^{n\times n}$ is an orthogonal matrix, then $\lambda^n = T(\lambda^n)$.*

Theorem 7.11. *Let $S \in \mathbb{R}^{n\times n}$ be an invertible matrix. Then*

$$S(\lambda^n) = |\det s^{-1}|\lambda^n = |\det S|^{-1}\lambda^n. \tag{6}$$

Corollary 7.12. *Lebesgue measure is invariant under motions: $\lambda^n = M(\lambda^n)$ for all motions M in $\mathbb{R}^n$. In particular, congruent sets have the same measure. Two sets of points are called congruent if, and only if, one can be transformed into the other by an isometry*

8 Measurable Functions

A *measurable function* is a measurable map $u : X \to \mathbb{R}$ from some measurable space $(X, \mathscr{A})$ to $(\mathbb{R}, \mathscr{B}(\mathbb{R}^1))$. They play central roles in the theory of integration.

We recall that $u : X \to \mathbb{R}$ is $\mathscr{A}/\mathscr{B}(\mathbb{R}^1)$-measurable if

$$u^{-1}(B) \in \mathscr{A}, \ \ \forall B \in \mathscr{B}(\mathbb{R}^1). \tag{7}$$

Moreover from a lemma from chapter 7, we actually only need to show that

$$u^{-1}(G) \in \mathscr{A}, \ \ \forall G \in \mathcal{G} \text{ where } \mathcal{G} \text{ generates } \mathscr{B}(\mathbb{R}^1). \tag{8}$$

Proposition 8.1.

1 *If $f, g : (X, \mathscr{B}) \to \mathbb{C}$ are measurable, then the function $f+g$, $f \cdot g$, cf, $(c \in \mathbb{C})$ are measurable.*

2 *If $b : \mathbb{C} \to \mathbb{C}$ is Borel and $b : (\mathbb{C}, \mathscr{B}) \to \mathbb{C}$ is measurable, then $b \circ f$ is measurable.*

3 *If $f(x) = \lim_{n\to\infty} f_n(x)$, $x \in X$ and f_n are measurable, then f is measurable.*

4 *If $X = \bigcup_{n=1}^{\infty} A_n$, $(A_n \in \mathscr{B})$, $f|_{A_n} : (A_n, \mathscr{B}_{A_n}) \to \mathbb{C}$ is measurable $\forall n$, then f is measurable.*

Definition 8.2. Given a measurable space $(X, \mathscr{B})$, a measurable function $f : (X, \mathscr{B}) \to \mathbb{C}$ is called simple if

$$f(x) = \sum_{k=1}^{N} c_k \mathbb{1}_{A_k}(x), \tag{9}$$

for some $c_k \in \mathbb{C}$, $A_k \in \mathscr{B}$, where $\mathbb{1}$ is the characteristic function,

$$\mathbb{1}_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A. \end{cases} \tag{10}$$

The representation of simple function is **not** unique. We denote the standard representation of f by

$$f(x) = \sum_{n=0}^{N} z_n \mathbb{1}_{B_n}(x), \quad N \in \mathbb{N},\ z_n \in \mathbb{R},\ B_n \in \mathscr{A},\ X = \bigcup_{n=1}^{N} B_n,\ \text{ for } B_n \cap B_m = \emptyset,\ \ n \neq m. \tag{11}$$

The set of simple functions is denoted $\mathcal{E}(\mathscr{A})$ of $\mathcal{E}$.

Definition 8.3. Assume μ is a measure on $(X, \mathscr{B})$. Given a *positive* simple function

$$f = \sum_{k=1}^{N} c_k \mathbb{1}_{A_k}, \quad (c_k \geq 0). \tag{12}$$

We define

$$\int_X f d\mu = \sum_{k=1}^{n} c_k \mu(A_k) \in [0, +\infty]. \tag{13}$$

We also denote this by $I_\mu(f)$.

Lemma 8.4. *This is well defined, that is,* $\int_x f d\mu$ *does not depend on the presentation of the simple function* f*.*

Properties 8.5. *For every positive simple function*

1 $\int_X cf d\mu = c \int_X f d\mu, \quad$ *for only* $c \geq 0$

2 $\int_X (f+g) d\mu = \int_X f d\mu + \int_X g d\mu.$

Corollary 8.6. *If* $f \geq g \geq 0$ *are simple functions, then*

$$\int_X f d\mu \geq \int_X g d\mu. \tag{14}$$

Definition 8.7. If $f : X \to [0, +\infty)$ is measurable, then we define

$$\int_X f d\mu = \sup \left\{ \int_X g d\mu \; : \; f \geq g \geq 0, \; \; g \text{ is simple} \right\} \tag{15}$$

Remark. *This means that any measurable function can be approximated by simple functions.*

Properties 8.8. *Measurable functions like this have the following properties*

1 $\int_X cf d\mu = c \int_X f d\mu, \quad \forall c \geq 0.$

2 *If* $f \geq g \geq 0$*, then* $\int_X f d\mu \geq \int_X g d\mu$ *for any measurable* g, f*.*

3 *If* $f \geq 0$ *is simple, then* $\int_X f d\mu$ *is the same value as obtained before.*

To advance in measure theory we consider measurable functions

$$f : X \to [0, +\infty].$$

Measurability is understood w.r.t the $\sigma-$algebra $\mathscr{B}([0, +\infty])$ generated by $\mathscr{B}([0, +\infty))$ and $\{+\infty\}$. In other words, $A \subset [0, +\infty] \in B([0, +\infty])$ iff $A \cap [0, +\infty) \in \mathscr{B}([0, +\infty)$.

Remark. *Hence* $f : X \to [0, +\infty]$ *is measurable iff* $f^{-1}(A)$ *is measurable* $\forall A \in \mathscr{B}([0, +\infty))$*.*

Definition 8.9. For measurable functions $f_X \to [0, +\infty]$, we define

$$\int_X f d\mu = \sup \left\{ \int_x g d\mu \; : \; f \geq g \geq 0 \; : \; g \text{ is simple} \right\} \in [0, +\infty]. \tag{16}$$

Theorem 8.10. ***<u>Monotone convergence theorem</u>*** *Assume* $(X, \mathscr{B}, \mu)$ *is a measure space,* $(f)_{n=1}^{\infty}$ *is an increasing sequence of measurable positive functions* $f_n : X \to [0, +\infty]$*. Define* $f(x) = \lim_{n\to\infty} f_n(x)$*. Then* f *is measurable and*

$$\int_X f d\mu = \lim_{n \to \infty} \int_X f_n d\mu. \tag{17}$$

Theorem 8.11. *Assume* $(X, \mathscr{B})$ *is a measurable space and* $f : X \to [0, +\infty]$ *is measurable. Then there are simple functions* g_n*, s.t.*

$$0 \leq g_1 \leq g_2 \leq \dots \quad , \quad g_n(x) \to f(x), \quad \forall x \in X.$$

Moreover, if f *is bounded, we can choose* g_n *s.t. the convergence is uniform, that is,*

$$\lim_{n\to\infty} \sup_{x\in X} |g_n(x) - f(x)| = 0. \tag{18}$$

10 Integrals of Measurable Functions

We have defined our integral for positive measurable functions, i.e. functions in $\mathcal{M}^+(\mathscr{A})$. To extend our integral to not only functions in $\mathcal{M}^+(\mathscr{A})$ we first notice that

$$u \in \mathcal{M}_{\overline{\mathbb{R}}}(\mathscr{A}) \Leftrightarrow u = u^+ - u^-, \; u^+, u^- \in \mathcal{M}^+_{\overline{\mathbb{R}}}, \tag{19}$$

i.e. that every measurable function can be written as a sum of **positive** measurable functions.

Definition 10.1 (μ-integrable)**.** A function $u : X \to \overline{\mathbb{R}}$ on $(X, \mathscr{A}, \mu)$ is μ-*integrable*, if it is $\mathscr{A}/\mathscr{B}(\overline{\mathbb{R}})$-measurable and if $\int u^+ d\mu, \int u^- d\mu < \infty$ (recall the definition for the integral of positive measurable functions). Then

$$\int u d\mu := \int u^+ d\mu - \int u^- d\mu \in (-\infty, \infty) \tag{20}$$

is the (μ-)*integral* of u. We write $\mathcal{L}^1(\mu)$ for the set of all real-valued μ-integrable functions [1].

Theorem 10.2. *Let* $u \in \mathcal{M}_{\overline{\mathbb{R}}}(\mathscr{A})$*, then the following conditions are equivalent:*

(i) $u \in \mathcal{L}^1_{\overline{\mathbb{R}}}(\mu)$.

(ii) $u^+, u^- \in \mathcal{L}^1_{\overline{\mathbb{R}}}(\mu)$.

(iii) $|u| \in \mathcal{L}^1_{\overline{\mathbb{R}}}(\mu)$.

(iv) $\exists w \in \mathcal{L}^1_{\overline{\mathbb{R}}}(\mu)$ *with* $w \geq 0$ *s.t.* $|u| \leq w$.

Theorem 10.3 (Properties the μ-integral)**.** *The* μ*-integral has the following properties:* ***homogeneous, additive****, and:*

(i) $\min\{u, v\}, \max\{u, v\} \in \mathcal{L}^1_{\overline{\mathbb{R}}}(\mu)$ *(lattice property)*

[1] In words, we extend our integral to ~~positive~~ measurable functions by noticing that we can write every measurable function as a sum of positive measurable functions, something that we do know how to integrate. We don't want to run into the problem of $\infty - \infty$, thus we require the integral of the positive and negative parts to both (separately) be less than infinity.

(ii) $u \le v \Rightarrow \int u d\mu \le \int v d\mu$ *(monotone)*

(iii) $\left| \int u d\mu \right| \le \int |u| d\mu$ *(triangle inequality)*

Remark. *If $u(x) \pm v(x)$ is defined in $\overline{\mathbb{R}}$ for all $x \in X$ then we can exclude $\infty - \infty$ and the theorem above just says that the integral is linear:*

$$\int (au + bv) d\mu = a \int u d\mu + b \int v d\mu. \tag{21}$$

This is always true for real-valued $u, v \in \mathcal{L}^1(\mu) = \mathcal{L}^1_{\mathbb{R}}(\mu)$, making $\mathcal{L}^1(\mu)$ a vector space with addition and scalar multiplication defined by

$$(u + v)(x) := u(x) + v(x), \;\; (a \cdot u)(x) := a \cdot u(x), \tag{22}$$

and

$$\int ...d\mu : \mathcal{L}^1(\mu) \to \mathbb{R}, \;\; u \mapsto \int u d\mu, \tag{23}$$

is a ***positive linear functional****.*

11 Null sets and the "Almost Everywhere"

Definition 11.1. A (μ-)null set $N \in \mathcal{N}_\mu$ is a measurable set $N \in \mathscr{A}$ satisfying

$$N \in \mu \Leftrightarrow N \in \mathscr{A} \text{ and } \mu(N) = 0. \tag{24}$$

This can be used generally about a 'statement' or 'property', but we will be interested in questions like 'when is $u(x)$ equal to $v(x)$', and we answer this by saying

$$u = v \; a.e. \Leftrightarrow \{x : u(x) \neq v(x)\} \text{ is (contained in) a } \mu\text{-null set.}, \tag{25}$$

i.e.

$$u = v \;\; \mu\text{-a.e.} \Leftrightarrow \mu\left(\{x : u(x) \neq v(x)\}\right) = 0 . \tag{26}$$

The last phrasing should of course include that the set $\{x : u(x) \neq v(x)\}$ is in $\mathscr{A}$, but this can be trivially seen.

Theorem 11.2. *Let $u \in \mathcal{M}_{\overline{\mathbb{R}}}(\mathscr{A})$, then:*

(i) $\int |u| d\mu = 0 \Leftrightarrow |u| = 0 \;\; a.e. \;\; \Leftrightarrow \mu\{u \neq 0\} = 0$,

(ii) $\mathbb{1}_N u \in \mathcal{L}^1_{\overline{\mathbb{R}}}(\mu) \;\; \forall \; N \in \mathcal{N}_\mu$,

(iii) $\int_N u d\mu = 0$.

Corollary 11.3. *Let $u = v$ μ-a.e. Then*

(i) $u, v \geq 0 \Rightarrow \int u d\mu = \int v d\mu$,

(ii) $u \in \mathcal{L}^1_{\mathbb{R}}(\mu) \Rightarrow v \in \mathcal{L}^1_{\mathbb{R}}(\mu)$ *and* $\int u d\mu = \int v d\mu$.

Corollary 11.4. *If* $u \in \mathcal{M}_{\overline{\mathbb{R}}}(\mathscr{A})$, $v \in \mathcal{L}^1_{\overline{\mathbb{R}}}(\mu)$ *and* $v \geq 0$ *then*

$$|u| \leq v \text{ a.e. } \Rightarrow u \in \mathcal{L}^1_{\overline{\mathbb{R}}}(\mu). \tag{27}$$

Proposition 11.5 (Markow inequality)**.** *For all* $u \in \mathcal{L}^1_{\overline{\mathbb{R}}}(\mu)$, $A \in \mathscr{A}$ *and* $c > 0$

$$u\left(\{|u| \geq c\} \cap A\right) \leq \frac{1}{c} \int_A |u| d\mu, \tag{28}$$

if $A = X$, *then (obviosly)*

$$u\left\{|u| \geq c\right\} \leq \frac{1}{c} \int |u| d\mu. \tag{29}$$

Corollary 11.6. *If* $u \in \mathcal{L}^1_{\overline{R}}(\mu)$, *then* μ *is a.e.* $\mathbb{R}$*-vaued. In particular, we can find a version* $\tilde{u} \in \mathcal{L}^1(\mu)$ *s.t.* $\tilde{u} = u$ *a.e. and* $\int \tilde{u} d\mu = \int u d\mu$

Appendix

H Regularity of measures

We let (X,d) be a metric space and denote by $\mathcal{O}$ the open, by $\mathcal{C}$ the closed and $\mathscr{B}(X) = \sigma(\mathcal{O})$ the Borel set of X.

Definition H.1. A measure μ on $(X, d, \mathscr{B}(X))$ is called outer regular, if

$$\mu(B) = \inf \{\mu(U) \mid B \subset U,\ U \text{ open}\} \tag{30}$$

and inner regular, if $\mu(K) < \infty$ for all compact sets $K \subset X$ and

$$\mu(U) = \sup \{\mu(K) \mid K \subset U,\ K \text{ compact}\}. \tag{31}$$

A measure which is both inner and outer regular is called **regular**. We write $\mathfrak{m}_r^+(X)$ for the family of regular measures on $(X, \mathscr{B}(X))$.

Remark. *The space X is called σ-compact if there is a sequence of compact sets $K_n \uparrow X$. A typical example of such a space is a locally compact, separable metric space.*

Theorem H.2. *Let (X,d) be a metric space. Every finite measure μ on $(X, \mathscr{B}(X))$ is outer regular. If X is σ-compact, then μ is also inner regular, hence regular.*

Theorem H.3. *Let (X,d) be a metric space and μ be a measure on $(X, B(X))$ such that $\mu(K) < \infty$ for all compact sets $K \subset X$.*

1 If X is σ-compact, then μ is inner regular.

2 If there exists a sequence $G_n \in \mathcal{O},\ G_n \uparrow X$ such that $\mu(G_n) < \infty$, then μ is outer regular.