Radon-Nikodym Theorem (lecture 17. 11. March)

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Assume $(X, \mathscr{B}, \mu)$ is a measure space. Are there other measures on $(X, \mathscr{B})$?

Example 20.21. Take a measurable function $f : X \to [0, +\infty]$ and define

$$\nu(A) = \int_A f d\mu \text{ for } A \in \mathscr{B}.$$

This is a measure by the monotone convergence theorem. We write $d\nu = f d\mu$.

Proposition 20.22. *Assume $(X, \mathscr{B})$ is a measurable space, μ and ν are σ-finite measures on $(X, \mathscr{B})$. Then there exist $N \in \mathscr{B}$ and a measurable $f : X \to [0, +\infty]$ s.t. $\mu(N) = 0$ and $\nu(A) = \nu(A \cap N) + \int_A f d\mu$ $\forall A \in \mathscr{B}$.*

Proof. Assume first that the measure ν, μ are finite. Consider the measure $\eta = \mu + \nu$ and define a linear functional ρ on $L^2(x, d\eta)$ by

$$\rho(g) = \int_X g d\nu.$$

It is well-defined and bounded, since

$$\begin{aligned} |\rho(g)| &\leq \underbrace{\int_X |\rho| d\nu}_{||g||_1} \underset{C.S}{\leq} (\int_X |\rho|^2 d\nu)^{\frac{1}{2}} \nu(X)^{\frac{1}{2}} \\ &\leq \nu(x)^{\frac{1}{2}} (\int_X |\rho|^2 d\eta)^{\frac{1}{2}}. \end{aligned}$$

By Riesz' theorem there exists $h \in L^2(x, d\eta)$ s.t. $\rho(g) = \int_X g h d\eta$ for all $g \in L^2(x, d\eta)$.

For $\rho = \mathbb{1}_A$ we get

$$\nu(A) = \int_A h d\eta \ \forall A \in \mathscr{B}.$$

In particular, $h \geq 0$ ($\eta -$ a.e.). As $\nu(A) \leq \eta(A)$, we also have $h \leq 1$ ($\eta -$ a.e.). From now on we view h as a function on X and assume $0 \leq h \leq 1$.

For $g = \mathbb{1}_A$ ($A \in \mathscr{B}$) we have

$$\int_X g d\nu = \rho(g) = \int_X g h d\eta = \int_X g h d\mu + \int_X g h d\nu,$$

hence

$$\int_X g(1-h) d\nu = \int_X g h d\mu. \tag{1}$$

By extending eq. 1 to positive simple functions and then using the monotone convergence theorem, we conclude that eq. 1 holds for all measurable $g : X \to [0, +\infty]$.

We now let

$$N = \{x : h(x) = 1\} \text{ and } f = \frac{h}{1-h} \mathbb{1}_{N^c}.$$

Letting $g = \mathbb{1}_N$ in eq. 1 we get $0 = \mu(N)$. For $A \in \mathscr{B}$, letting $g = \frac{\mathbb{1}_{A \cap N^c}}{1-h}$ in eq. 1 we get

$$\nu(A \cap N^c) = \int_X \frac{\mathbb{1}_A \mathbb{1}_{N^c}}{1-h} h d\mu = \int_X \mathbb{1}_A f d\mu = \int_A f d\mu.$$

Thus

$$\nu(A) = \nu(A \cap N) + \nu(A \cap N^c) = \nu(A \cap N) + \int_A f d\mu.$$

This finishes the proof for finite μ and ν.

If μ and ν are σ-finite, we can write X as disjoint unions $X = \cup_{n \in \mathbb{N}} X_n = \cup_{m \in \mathbb{N}} Y_m$ with $X_n, Y_m \in \mathscr{B}, \mu(X_n) < \infty, \nu(Y_m) < \infty$. Applying the first part of the proof to $X_n \cap Y_m$, we find $N_{nm} \in \mathscr{B}, N_{nm} \subset X_n \cap Y_m$, and measurable $f_{nm} : X_n \cap Y_m \to [0, +\infty]$ s.t. $\mu(N_{nm}) = 0$ and

$$\nu(A \cap X_n \cap Y_m) = \nu(A \cap N_{nm}) + \int_{A \cap X_n \cap Y_m} f_{nm} d\mu.$$

We then put $N = \cup_{n,m \in \mathbb{N}} N_{nm}$ and define $f : X \to [0, +\infty]$ by letting $f = f_{nm}$ on $X_n \cap Y_m$. $\square$

When can we discard the term $\nu(A \cap N)$?

Definition 20.23. Given measure μ and ν on $X, \mathscr{B}$, we say that ν is *absolutely continuous* with respect to μ and write $\nu << \mu$, if $\nu(A) = 0$ whenever $A \in \mathscr{B}, \mu(A) = 0$.

Lemma 20.24. *Assume μ and ν are measures on $(X, \mathscr{B})$, $\nu(X) < \infty$. Then $\nu << \mu$ iff $\forall \epsilon > 0 \exists \delta > 0$ s.t. if $A \in \mathscr{B}$, $\mu(A) < \delta$, then $\nu(A) < \epsilon$.*