

MAT4400: Notes on Linear analysis

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3 σ -Algebras

Definition 3.1 (σ -Algebra). A family \mathcal{A} of subsets of X with:

- (i) $X \in \mathcal{A}$,
- (ii) $A \in \mathcal{A} \Rightarrow A^c \in \mathcal{A}$,
- (iii) $(A_n)_{n \in \mathbb{N}} \in \mathcal{A} \Rightarrow \bigcup_{n \in \mathbb{N}} A_n \in \mathcal{A}$

Theorem 3.2 (and Definition).

(i) *The intersection of arbitrarily many σ -algebras in X is again a σ -algebra in X .*

(ii) *For every system of sets $p \subset \mathcal{P}(X)$ there exists a smallest σ -algebra containing p .*

This is the σ -algebra generated by p , denoted $\sigma(p)$, and p is called its generator.

Definition 3.3 (Borel). The σ -algebra $\sigma(\mathcal{O})$ generated by the open sets $\mathcal{O} = \mathcal{O}_{\mathbb{R}^n}$ of \mathbb{R}^n is called **Borel σ -algebra**, and its members are called **Borel sets** or **Borel measurable sets**.

5 Uniqueness of Measures

Lemma 5.1. *A Dynkin system D is a σ -algebra iff it is stable under finite intersections, i.e. $A, B \in D \Rightarrow A \cap B \in D$.*

Theorem 5.2 (Dynkin). *Assume X is a set, S is a collection of subsets of X closed under finite intersections, that is, if $A, B \in S \Rightarrow A \cap B \in S$. Then $D(S) = \sigma(S)$.*

Proof. We clearly have that $D(S) \subset \sigma(S)$. If we can show that $D(S)$ is a σ -algebra, that is, that a Dynkin system generated by a subset $S \subset X$ (where S is \cap -stable) is a σ -algebra, then the inverse conclusion $D(S) \supset \sigma(S)$ follows logically. This is the case because the σ -algebra $\sigma(S)$ is the smallest σ -algebra containing S , and so if $D(S)$ is a σ -algebra it must be a greater or equal (in some sense) than $\sigma(S)$.

Using Lemma 5.1 we only need to show that $D(S)$ is stable under finite intersections, to prove that $D(S)$ is a σ -algebra. Consider:

$$D_A := \{B \subset X : B \cap A \in D(S)\},$$

for some $A \in D(S)$. Notice that this set is \cap -stable, and so if we can show that $D_A = D(S)$ we must have that (by Lemma 5.1) $D(S)$ is a σ -algebra. Firstly, however, let us show that D_A is a **Dynkin system**.

1. \emptyset must be in D_A , since $\emptyset \cap A = \emptyset \in D(S)$.
2. Let $B \in D_A$. Then

$$A \cap B^c = A \setminus (A \cap B) = (A^c \cup (A \cap B))^c,$$

here $A \cap B$ and A^c must be in $D(S)$. Furthermore, since disjoint unions of set from $D(S)$ are still in $D(S)$, we must have $A^c \in D_A$.

3. Assume that $(B_n)_{n \in \mathbb{N}} \subset D_A$ is a pairwise disjoint sequence. Then

$$\begin{aligned} (B_n \cap A)_{n \in \mathbb{N}} &\in D(S) \quad (\text{by def. of } D_A) \\ \Rightarrow \bigcup_{n \in \mathbb{N}} (B_n \cap A) &= \left(\bigcup_{n \in \mathbb{N}} B_n \right) \cap A \in D(S) \\ \Rightarrow \bigcup_{n \in \mathbb{N}} B_n &\in D_A. \end{aligned}$$

So D_A is indeed a Dynkin system.

We now want to show that $D(S)$ is \cap -stable, we have:

$$\begin{aligned}
& S \subset D_A \quad \forall A \in S \\
& \Rightarrow D(S) \subset D_A \quad \forall A \in S \text{ (since } D_A \text{ is a Dynkin system)} \\
& \Rightarrow B \cap A \in D(S) \quad \forall B \in S, \forall A \in D(S) \text{ (by the definition of } D_A) \\
& \Rightarrow B \in D_A \quad \forall B \in S, \forall A \in D(S) \\
& \Rightarrow S \subset D_A \quad \forall A \in D(S) \\
& \Rightarrow D(S) \subset D_A \quad \forall A \in D(S) \text{ (since } D_A \text{ is a Dynkin system)} \\
& \Rightarrow A \cap B \in D(S) \quad \forall A, B \in D(S),
\end{aligned}$$

and so $D(S)$ is \cap -stable and then $D(S) \supset \sigma(S) \Rightarrow D(S) = \sigma(S)$. \square

Theorem 5.3 (uniqueness of measures). *Let (X, B) be a measurable space, and $S \subset P(X)$ be the generator of B , i.e. $B = \sigma(S)$. If S satisfies the following conditions:*

1. *S is stable under finite intersections (\cap -stable), i.e. $A, C \in S \Rightarrow A \cap C \in S$.*
2. *There exists an exhausting sequence $(G_n)_{n \in \mathbb{N}} \subset B$ with $G_n \uparrow X$. Assume also that there are two measures μ, ν satisfying:*
3. *$\mu(A) = \nu(A), \forall A \in S$.*
4. *$\mu(G_n) = \nu(G_n) < \infty$.*

Then $\mu = \nu$.

Proof (outline). Define

$$D_n := \{A \in B : \mu(G_n \cap A) = \nu(G_n \cap A) (< \infty)\},$$

and show that it is a Dynkin system. Then, use the fact that S is \cap -stable and Theorem 5.2 to argue that $D(S) = \sigma(S) \dots \rightarrow \dots B = D_n$. \square

6 Existence of Measures

Theorem 6.1 (Carathéodory). *Let $S \subset P(X)$ be a semi-ring and $\mu : S \rightarrow [0, \infty)$ a pre-measure. Then μ has an extension to a measure μ^* on $\sigma(S)$, i.e. that $\mu(s) = \mu^*(s)$, $\forall s \in \sigma(S)$.*

Also, if S contains an exhausting sequence, $S_n \uparrow X$, s.t. $\mu(S_n) < \infty$, then the extension is unique.

Proof (outline). Firstly, let us define an outer measure.

Definition 6.2 (Outer measure). An outer measure is a function $\mu^* : P(X) \rightarrow [0, \infty)$ with the following properties:

1. $\mu^*(\emptyset) = 0$,
2. $A \subset B \Rightarrow \mu^*(A) \leq \mu^*(B)$,
3. $\mu^*\left(\bigcup_{n \in \mathbb{N}} A_n\right) \leq \sum_{n \in \mathbb{N}} \mu^*(A_n)$,

and define for each $A \subset X$ the family of countable S -coverings:

$$C(A) := \left\{ (S_n)_{n \in \mathbb{N}} \subset S : \bigcup_{n \in \mathbb{N}} S_n \supset A \right\},$$

and the set function

$$\mu^*(A) := \inf \left\{ \sum_{n \in \mathbb{N}} \mu(S_n) : (S_n)_{n \in \mathbb{N}} \in C(A) \right\}.$$

Step 1: Claim: $\mu^*(A)$ is an outer measure.

Proof.

1. $C(\emptyset) = \{\text{any sequence in } S \text{ containing } \emptyset\} \Rightarrow \mu^*(\emptyset) = 0$.
2. Assume $A \subset B$. Then $C(A) \subset C(B) \Rightarrow \mu^*(A) \leq \mu^*(B)$.
3. If $\mu^*(A_n) = \infty$ for some n , then there is nothing to prove. Thus, assume $\mu^*(A_n) < \infty \forall n$. Fix $\epsilon > 0$, and for every n choose $A_{n_k} \in S$ s.t.

$$A_n \subset \bigcup_{k \in \mathbb{N}} A_{n_k}, \quad \sum_{k \in \mathbb{N}} \mu^*(A_{n_k}) < \mu^*(A_n) + \frac{\epsilon}{2^n}.$$

Then

$$\bigcup_{n \in \mathbb{N}} A_n \subset \bigcup_{k \in \mathbb{N}} \bigcup_{n \in \mathbb{N}} A_{n_k},$$

so

$$\begin{aligned}
\mu^* \left(\bigcup_{n \in \mathbb{N}} A_n \right) &\leq \sum_{n \in \mathbb{N}} \sum_{k \in \mathbb{N}} \mu(A_{n_k}) \\
&< \sum_{n \in \mathbb{N}} \left(\mu^*(A_n) + \frac{\epsilon}{2^n} \right) \\
&= \sum_{n \in \mathbb{N}} \mu^*(A_n) + \epsilon.
\end{aligned}$$

As ϵ was arbitrarily, we get that

$$\mu^* \left(\bigcup_{n \in \mathbb{N}} A_n \right) \leq \sum_{n \in \mathbb{N}} \mu^*(A_n),$$

so μ^* fulfills all the conditions for being an outer measure. \square

Step 2: Showing that μ^* extends μ , i.e. $\mu^*(s) = \mu(s) \ \forall s \in S$.

Step 3: Define μ^* -measurable sets

$$\Sigma^* := \{A \subset X : \mu^*(Q) = \mu^*(Q \cap A) + \mu^*(Q \setminus A) \ \forall Q \subset X\}$$

Step 4: Show that $\mu|_{\Sigma^*}$ is a measure. In particular, $\mu|_{\sigma(S)}$ is a measure which extends μ . \square

7 Measurable Mappings

We consider maps $T : X \rightarrow X'$ between two measurable spaces (X, \mathcal{A}) and (X', \mathcal{A}') which respects the measurable structures, the σ -algebras on X and X' . These maps are useful as we can transport a measure μ , defined on (X, \mathcal{A}) , to (X', \mathcal{A}') .

Definition 7.1. Let (X, \mathcal{A}) , (X', \mathcal{A}') be measurable spaces. A map $T : X \rightarrow X'$ is called \mathcal{A}/\mathcal{A}' -measurable if the pre-image of every measurable set is a measurable set:

$$T^{-1}(A') \in \mathcal{A}, \quad \forall A' \in \mathcal{A}'. \quad (1)$$

- A $\mathcal{B}(\mathbb{R}^n)/\mathcal{B}(\mathbb{R}^m)$ measurable map is often called a Borel map.
- The notation $T : (X, \mathcal{A}) \rightarrow (X', \mathcal{A}')$ is often used to indicate measurability of the map T .

Lemma 7.2. Let (X, \mathcal{A}) , (X', \mathcal{A}') be measurable spaces and let $\mathcal{A}' = \sigma(\mathcal{G}')$. Then $T : X \rightarrow X'$ is \mathcal{A}/\mathcal{A}' -measurable iff $T^{-1}(G') \in \mathcal{A}$, i.e. if

$$T^{-1}(G') \in \mathcal{A}, \quad \forall G' \in \mathcal{G}'. \quad (2)$$

Theorem 7.3. Let (X_i, \mathcal{A}_i) , $i = 1, 2, 3$, be measurable spaces and $T : X_1 \rightarrow X_2$, $S : X_2 \rightarrow X_3$ be $\mathcal{A}_1/\mathcal{A}_2$ and $\mathcal{A}_2/\mathcal{A}_3$ -measurable maps respectively. Then $S \circ T : X_1 \rightarrow X_3$ is $\mathcal{A}_1/\mathcal{A}_3$ -measurable.

Corollary 7.4. Every continuous map between metric spaces is a Borel map.

Definition 7.5. (and lemma) Let $(T_i)_{i \in I}$, $T_i : X \rightarrow X_i$, be arbitrarily many mappings from the same space X into measurable spaces (X_i, \mathcal{A}_i) . The smallest σ -algebra on X that makes all T_i simultaneously measurable is

$$\sigma(T_i : i \in I) := \sigma \left(\bigcup_{i \in I} T_i^{-1}(\mathcal{A}_i) \right) \quad (3)$$

Corollary 7.6. A function $f : (X, \mathcal{B}) \rightarrow \mathbb{R}$ is measurable if $f((a, +\infty)) \in \mathcal{B}$, $\forall a \in \mathbb{R}$.

Corollary 7.7. Assume (X, \mathcal{B}) is a measurable space, (Y, d) is a metric space, $(f_n : (X, \mathcal{B}) \rightarrow Y)_{n=1}^{\infty}$ is a sequence of measurable maps. Assume this sequence of images $(f_n(x))_{n=1}^{\infty}$ is convergent in $Y \forall x \in X$. Define

$$f : X \rightarrow Y, \quad \text{by } f(x) = \lim_{n \rightarrow \infty} f_n(x). \quad (4)$$

Then f is measurable.

Theorem 7.8. Let (X, \mathcal{A}) , (X', \mathcal{A}') be measurable spaces and $T : X \rightarrow X'$ be an \mathcal{A}/\mathcal{A}' -measurable map. For every measurable μ on (X, \mathcal{A}) ,

$$\mu'(A') := \mu(T^{-1}(A')), \quad A' \in \mathcal{A}', \quad (5)$$

defines a measure on (X', \mathcal{A}') .

Definition 7.9. The measure $\mu'(\cdot)$ in the above theorem is called the push forward or image measure of μ under T and it is denoted as $T(\mu)(\cdot)$, $T_{*\mu}(\cdot)$ or $\mu \circ T^{-1}(\cdot)$.

Theorem 7.10. *If $T \in \mathbb{R}^{n \times n}$ is an orthogonal matrix, then $\lambda^n = T(\lambda^n)$.*

Theorem 7.11. *Let $S \in \mathbb{R}^{n \times n}$ be an invertible matrix. Then*

$$S(\lambda^n) = |\det S| \lambda^n = |\det S|^{-1} \lambda^n. \quad (6)$$

Corollary 7.12. *Lebesgue measure is invariant under motions: $\lambda^n = M(\lambda^n)$ for all motions M in \mathbb{R}^n . In particular, congruent sets have the same measure. Two sets of points are called congruent if, and only if, one can be transformed into the other by an isometry*

8 Measurable Functions

A *measurable function* is a measurable map $u : X \rightarrow \mathbb{R}$ from some measurable space (X, \mathcal{A}) to $(\mathbb{R}, \mathcal{B}(\mathbb{R}^1))$. They play central roles in the theory of integration.

We recall that $u : X \rightarrow \mathbb{R}$ is $\mathcal{A}/\mathcal{B}(\mathbb{R}^1)$ -measurable if

$$u^{-1}(B) \in \mathcal{A}, \quad \forall B \in \mathcal{B}(\mathbb{R}^1). \quad (7)$$

Moreover from a lemma from chapter 7, we actually only need to show that

$$u^{-1}(G) \in \mathcal{A}, \quad \forall G \in \mathcal{G} \text{ where } \mathcal{G} \text{ generates } \mathcal{B}(\mathbb{R}^1). \quad (8)$$

Proposition 8.1.

- 1 If $f, g : (X, \mathcal{B}) \rightarrow \mathbb{C}$ are measurable, then the function $f+g, f \cdot g, cf, (c \in \mathbb{C})$ are measurable.
- 2 If $b : \mathbb{C} \rightarrow \mathbb{C}$ is Borel and $b : (\mathbb{C}, \mathcal{B}) \rightarrow \mathbb{C}$ is measurable, then $b \circ f$ is measurable.
- 3 If $f(x) = \lim_{n \rightarrow \infty} f_n(x)$, $x \in X$ and f_n are measurable, then f is measurable.
- 4 If $X = \bigcup_{n=1}^{\infty} A_n$, $(A_n \in \mathcal{B})$, $f|_{A_n} : (A_n, \mathcal{B}_{A_n}) \rightarrow \mathbb{C}$ is measurable $\forall n$, then f is measurable.

Definition 8.2. Given a measurable space (X, \mathcal{B}) , a measurable function $f : (X, \mathcal{B}) \rightarrow \mathbb{C}$ is called simple if

$$f(x) = \sum_{k=1}^N c_k \mathbb{1}_{A_k}(x), \quad (9)$$

for some $c_k \in \mathbb{C}$, $A_k \in \mathcal{B}$, where $\mathbb{1}$ is the characteristic function,

$$\mathbb{1}_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A. \end{cases} \quad (10)$$

The representation of simple function is **not** unique. We denote the standard representation of f by

$$f(x) = \sum_{n=0}^N z_n \mathbb{1}_{B_n}(x), \quad N \in \mathbb{N}, \quad z_n \in \mathbb{R}, \quad B_n \in \mathcal{A}, \quad X = \bigcup_{n=1}^N B_n, \quad \text{for } B_n \cap B_m = \emptyset, \quad n \neq m. \quad (11)$$

The set of simple functions is denoted $\mathcal{E}(\mathcal{A})$ of \mathcal{E} .

Definition 8.3. Assume μ is a measure on (X, \mathcal{B}) . Given a *positive* simple function

$$f = \sum_{k=1}^N c_k \mathbf{1}_{A_k}, \quad (c_k \geq 0). \quad (12)$$

We define

$$\int_X f d\mu = \sum_{k=1}^n c_k \mu(A_k) \in [0, +\infty]. \quad (13)$$

We also denote this by $I_\mu(f)$.

Lemma 8.4. *This is well defined, that is, $\int_X f d\mu$ does not depend on the presentation of the simple function f .*

Properties 8.5. *For every positive simple function*

$$1 \quad \int_X c f d\mu = c \int_X f d\mu, \quad \text{for only } c \geq 0$$

$$2 \quad \int_X (f + g) d\mu = \int_X f d\mu + \int_X g d\mu.$$

Corollary 8.6. *If $f \geq g \geq 0$ are simple functions, then*

$$\int_X f d\mu \geq \int_X g d\mu. \quad (14)$$

Definition 8.7. If $f : X \rightarrow [0, +\infty)$ is measurable, then we define

$$\int_X f d\mu = \sup \left\{ \int_X g d\mu : f \geq g \geq 0, \text{ } g \text{ is simple} \right\} \quad (15)$$

Remark. *This means that any measurable function can be approximated by simple functions.*

Properties 8.8. *Measurable functions like this have the following properties*

$$1 \quad \int_X c f d\mu = c \int_X f d\mu, \quad \forall c \geq 0.$$

$$2 \quad \text{If } f \geq g \geq 0, \text{ then } \int_X f d\mu \geq \int_X g d\mu \text{ for any measurable } g, f.$$

$$3 \quad \text{If } f \geq 0 \text{ is simple, then } \int_X f d\mu \text{ is the same value as obtained before.}$$

To advance in measure theory we consider measurable functions

$$f : X \rightarrow [0, +\infty].$$

Measurability is understood w.r.t the σ -algebra $\mathcal{B}([0, +\infty])$ generated by $\mathcal{B}([0, +\infty))$ and $\{+\infty\}$. In other words, $A \subset [0, +\infty] \in \mathcal{B}([0, +\infty])$ iff $A \cap [0, +\infty) \in \mathcal{B}([0, +\infty))$.

Remark. Hence $f : X \rightarrow [0, +\infty]$ is measurable iff $f^{-1}(A)$ is measurable $\forall A \in \mathcal{B}([0, +\infty))$.

Definition 8.9. For measurable functions $f_X \rightarrow [0, +\infty]$, we define

$$\int_X f d\mu = \sup \left\{ \int_X g d\mu : f \geq g \geq 0 : g \text{ is simple} \right\} \in [0, +\infty]. \quad (16)$$

Theorem 8.10. Monotone convergence theorem Assume (X, \mathcal{B}, μ) is a measure space, $(f)_{n=1}^\infty$ is an increasing sequence of measurable positive functions $f_n : X \rightarrow [0, +\infty]$. Define $f(x) = \lim_{n \rightarrow \infty} f_n(x)$. Then f is measurable and

$$\int_X f d\mu = \lim_{n \rightarrow \infty} \int_X f_n d\mu. \quad (17)$$

Theorem 8.11. Assume (X, \mathcal{B}) is a measurable space and $f : X \rightarrow [0, +\infty]$ is measurable. Then there are simple functions g_n , s.t.

$$0 \leq g_1 \leq g_2 \leq \dots, \quad g_n(x) \rightarrow f(x), \quad \forall x \in X.$$

Moreover, if f is bounded, we can choose g_n s.t. the convergence is uniform, that is,

$$\lim_{n \rightarrow \infty} \sup_{x \in X} |g_n(x) - f(x)| = 0. \quad (18)$$

9 Integration of measurable functions

Through this chapter (X, \mathcal{A}, μ) will be some measure space. Recall that $\mathcal{M}^+(\mathcal{A})$ $[\mathcal{M}_{\mathbb{R}}^+(\mathcal{A})]$ are the \mathcal{A} -measurable positive functions and $\mathcal{E}(\mathcal{A})$ $[\mathcal{E}_{\mathbb{R}}^+(\mathcal{A})]$ are the positive and simple functions.

The fundamental idea of *Integration* is to measure the area between the graph of the function and the abscissa. For positive simple functions $f \in \mathcal{E}^+(\mathcal{A})$ in standard representation, this is done easily

$$\text{if } f = \sum_{i=0}^M y_i \mathbb{1}_{A_i} \in \mathcal{E}^+(\mathcal{A}) \quad \text{then} \quad \sum_{i=0}^M y_i \mu(A_i) \quad (19)$$

would be the μ -area enclosed by the graph and the abscissa. We note that the representation of f should not impact the integral of f .

Lemma 9.1. *Let $\sum_{i=0}^M y_i \mathbb{1}_{A_i} = \sum_{k=0}^N z_k \mathbb{1}_{B_k}$ be two standard representations of the same function $f \in \mathcal{E}^+(\mathcal{A})$. Then*

$$\sum_{i=0}^M y_i \mu(A_i) = \sum_{k=0}^N z_k \mu(B_k). \quad (20)$$

Definition 9.2. Let $f = \sum_{i=0}^M y_i \mathbb{1}_{A_i} \in \mathcal{E}^+(\mathcal{A})$ be a simple function in standard representation. Then the number

$$I_\mu(f) = \sum_{i=0}^M y_i \mu(A_i) \in [0, \infty] \quad (21)$$

(which is independent of the representation of f) is called the μ -integral of f .

Proposition 9.3. *Let $f, g \in \mathcal{E}^+(\mathcal{A})$. Then*

- (i) $I_\mu(\mathbb{1}_A) = \mu(A) \quad \forall A \in \mathcal{A}.$
- (ii) $I_\mu(\lambda f) = \lambda I_\mu(f) \quad \forall \lambda \geq 0.$
- (iii) $I_\mu(f + g) = I_\mu(f) + I_\mu(g).$
- (iv) $f \leq g \Rightarrow I_\mu(f) \leq I_\mu(g).$

In theorem 8.8 we saw that we could for every $u \in \mathcal{M}^+(\mathcal{A})$ write it as an increasing limit of simple functions. By corollary 8.10, the suprema of simple functions are again measurable, so that

$$u \in \mathcal{M}^+(\mathcal{A}) \Leftrightarrow u = \sup_{n \in \mathbb{N}} f_n, \quad f \in \mathcal{E}^+(\mathcal{A}), \quad f_n \leq f_{n+1} \leq \dots$$

We will use this to "inscribe" simple functions (which we know how to integrate) below the graph of a positive measurable function u and exhaust the μ -area below u .

Definition 9.4. Let (X, \mathcal{A}, μ) be a measure space. The (μ) -integral of a positive function $u \in \mathcal{M}_{\mathbb{R}}^+(\mathcal{A})$ is given by

$$\int u d\mu = \sup \{ I_{\mu}(g) : g \leq u, g \in \mathcal{E}^+(\mathcal{A}) \} \in [0, +\infty]. \quad (22)$$

If we need to emphasize the *integration variable*, we write $\int u(x) \mu(dx)$. The key observation is that the integral $\int \dots d\mu$ extends I_{μ} .

Lemma 9.5. For all $f \in \mathcal{E}^+(\mathcal{A})$ we have $\int f d\mu = I_{\mu}(f)$.

The next theorem is one of many convergence theorems. It shows that we could have defined 22 using any increasing sequence $f_n \uparrow u$ of simple functions $f_n \in \mathcal{E}^+(\mathcal{A})$.

Theorem 9.6. (*Beppo Levi*) Let (X, \mathcal{A}, μ) be a measure space. For an increasing sequence of functions $(u_n)_{n \in \mathbb{N}} \subset \mathcal{M}_{\mathbb{R}}^+(\mathcal{A})$, $0 \leq u_n \leq u_{n+1} \leq \dots$, we have for the supremum $u = \sup_{n \in \mathbb{N}} u_n \in \mathcal{M}_{\mathbb{R}}^+(\mathcal{A})$ and

$$\int \sup_{n \in \mathbb{N}} u_n d\mu = \sup_{n \in \mathbb{N}} \int u_n d\mu. \quad (23)$$

Note we can write $\lim_{n \rightarrow \infty}$ instead of $\sup_{n \in \mathbb{N}}$ as the supremum of an increasing sequence is its limit. Moreover, this theorem holds in $[0, +\infty]$, so the case $+\infty = +\infty$ is possible.

Corollary 9.7. Let $u \in \mathcal{M}_{\mathbb{R}}^+(\mathcal{A})$. Then

$$\int u d\mu = \lim_{n \rightarrow \infty} \int f_n d\mu$$

holds for every sequence $(f_n)_{n \in \mathbb{N}} \subset \mathcal{E}^+(\mathcal{A})$ with $\lim_{n \rightarrow \infty} f_n = u$.

Proposition 9.8. (of integral) Let $u, v \in \mathcal{M}_{\mathbb{R}}^+(\mathcal{A})$. Then

$$(i) \int \mathbf{1}_A d\mu = \mu(A) \quad \forall A \in \mathcal{A}.$$

$$(ii) \int \alpha u d\mu = \alpha \int u d\mu \quad \forall \alpha \geq 0.$$

$$(iii) \int u + v d\mu = \int u d\mu + \int v d\mu.$$

$$(iv) u \leq v \Rightarrow \int u d\mu \leq \int v d\mu.$$

Corollary 9.9. Let $(u_n)_{n \in \mathbb{N}} \subset \mathcal{M}_{\mathbb{R}}^+(\mathcal{A})$. Then $\sum_{n=1}^{\infty} u_n$ is measurable and we have

$$\int \sum_{n=1}^{\infty} u_n d\mu = \sum_{n=1}^{\infty} \int u_n d\mu$$

(including the possibility $+\infty = +\infty$.)

Theorem 9.10. (*Fatou*) Let $(u_n)_{n \in \mathbb{N}} \subset \mathcal{M}_{\mathbb{R}}^+(\mathcal{A})$ be a sequence of positive measurable functions. Then $u = \liminf_{n \rightarrow \infty} u_n$ is measurable and

$$\int \liminf_{n \rightarrow \infty} u_n d\mu = \liminf_{n \rightarrow \infty} \int u_n d\mu \quad (24)$$

10 Integrals of Measurable Functions

We have defined our integral for positive measurable functions, i.e. functions in $\mathcal{M}^+(\mathcal{A})$. To extend our integral to not only functions in $\mathcal{M}^+(\mathcal{A})$ we first notice that

$$u \in \mathcal{M}_{\mathbb{R}}(\mathcal{A}) \Leftrightarrow u = u^+ - u^-, \quad u^+, u^- \in \mathcal{M}_{\mathbb{R}}^+, \quad (25)$$

i.e. that every measurable function can be written as a sum of **positive** measurable functions.

Definition 10.1 (μ -integrable). A function $u : X \rightarrow \overline{\mathbb{R}}$ on (X, \mathcal{A}, μ) is μ -integrable, if it is $\mathcal{A}/\mathcal{B}(\mathbb{R})$ -measurable and if $\int u^+ d\mu, \int u^- d\mu < \infty$ (recall the definition for the integral of positive measurable functions). Then

$$\int u d\mu := \int u^+ d\mu - \int u^- d\mu \in (-\infty, \infty) \quad (26)$$

is the (μ -)integral of u . We write $\mathcal{L}^1(\mu)$ for the set of all real-valued μ -integrable functions¹.

Theorem 10.2. Let $u \in \mathcal{M}_{\mathbb{R}}(\mathcal{A})$, then the following conditions are equivalent:

- (i) $u \in \mathcal{L}_{\mathbb{R}}^1(\mu)$.
- (ii) $u^+, u^- \in \mathcal{L}_{\mathbb{R}}^1(\mu)$.
- (iii) $|u| \in \mathcal{L}_{\mathbb{R}}^1(\mu)$.
- (iv) $\exists w \in \mathcal{L}_{\mathbb{R}}^1(\mu)$ with $w \geq 0$ s.t. $|u| \leq w$.

Theorem 10.3 (Properties of the μ -integral). The μ -integral is: **homogeneous, additive, and:**

- (i) $\min\{u, v\}, \max\{u, v\} \in \mathcal{L}_{\mathbb{R}}^1(\mu)$ (lattice property)
- (ii) $u \leq v \Rightarrow \int u d\mu \leq \int v d\mu$ (monotone)
- (iii) $\left| \int u d\mu \right| \leq \int |u| d\mu$ (triangle inequality)

Remark. If $u(x) \pm v(x)$ is defined in $\overline{\mathbb{R}}$ for all $x \in X$ then we can exclude $\infty - \infty$ and the theorem above just says that the integral is linear:

$$\int (au + bv) d\mu = a \int u d\mu + b \int v d\mu. \quad (27)$$

¹In words, we extend our integral to ~~positive~~ measurable functions by noticing that we can write every measurable function as a sum of positive measurable functions, something that we do know how to integrate. We don't want to run into the problem of $\infty - \infty$, thus we require the integral of the positive and negative parts to both (separately) be less than infinity.

This is always true for real-valued $u, v \in \mathcal{L}^1(\mu) = \mathcal{L}_{\mathbb{R}}^1(\mu)$, making $\mathcal{L}^1(\mu)$ a vector space with addition and scalar multiplication defined by

$$(u + v)(x) := u(x) + v(x), \quad (a \cdot u)(x) := a \cdot u(x), \quad (28)$$

and

$$\int \dots d\mu : \mathcal{L}^1(\mu) \rightarrow \mathbb{R}, \quad u \mapsto \int u d\mu, \quad (29)$$

*is a **positive linear functional**.*

11 Null sets and the Almost Everywhere (lecture 08, 05. Feb.)

Definition 11.1. A (μ) -null set $N \in \mathcal{N}_\mu$ is a measurable set $N \in \mathcal{A}$ satisfying

$$N \in \mu \Leftrightarrow N \in \mathcal{A} \text{ and } \mu(N) = 0. \quad (30)$$

This can be used generally about a ‘statement’ or ‘property’, but we will be interested in questions like ‘when is $u(x)$ equal to $v(x)$ ’, and we answer this by saying

$$u = v \text{ a.e.} \Leftrightarrow \{x : u(x) \neq v(x)\} \text{ is (contained in) a } \mu\text{-null set.}, \quad (31)$$

i.e.

$$u = v \text{ } \mu\text{-a.e.} \Leftrightarrow \mu(\{x : u(x) \neq v(x)\}) = 0. \quad (32)$$

The last phrasing should of course include that the set $\{x : u(x) \neq v(x)\}$ is in \mathcal{A} .

Theorem 11.2. Let $u \in \mathcal{M}_{\mathbb{R}}(\mathcal{A})$, then:

$$(i) \int |u| d\mu = 0 \Leftrightarrow |u| = 0 \text{ a.e.} \Leftrightarrow \mu\{u \neq 0\} = 0,$$

$$(ii) \mathbb{1}_N u \in \mathcal{L}_{\mathbb{R}}^1(\mu) \quad \forall N \in \mathcal{N}_\mu,$$

$$(iii) \int_N u d\mu = 0.$$

(i) is really useful, later we will define \mathcal{L}^p and the $\|\cdot\|_p$ -(semi)norm. Then (i) means that if we have a sequence u_n converging to u in the $\|\cdot\|_p$ -norm then $u_n(x) = u(x)$ a.e.

Corollary 11.3. Let $u = v$ μ -a.e. Then

$$(i) u, v \geq 0 \Rightarrow \int u d\mu = \int v d\mu,$$

$$(ii) u \in \mathcal{L}_{\mathbb{R}}^1(\mu) \Rightarrow v \in \mathcal{L}_{\mathbb{R}}^1(\mu) \text{ and } \int u d\mu = \int v d\mu.$$

Corollary 11.4. If $u \in \mathcal{M}_{\mathbb{R}}(\mathcal{A})$, $v \in \mathcal{L}_{\mathbb{R}}^1(\mu)$ and $v \geq 0$ then

$$|u| \leq v \text{ a.e.} \Rightarrow u \in \mathcal{L}_{\mathbb{R}}^1(\mu). \quad (33)$$

Proposition 11.5 (Markow inequality). For all $u \in \mathcal{L}_{\mathbb{R}}^1(\mu)$, $A \in \mathcal{A}$ and $c > 0$

$$\mu(\{|u| \geq c\} \cap A) \leq \frac{1}{c} \int_A |u| d\mu, \quad (34)$$

if $A = X$, then (obviously)

$$\mu\{|u| \geq c\} \leq \frac{1}{c} \int |u| d\mu. \quad (35)$$

Completions of measure spaces

Definition 11.6. A measure space (X, \mathcal{B}, μ) is called **complete** if whenever $A \in \mathcal{B}$ and $\mu(A) = 0$, we have $B \in \mathcal{B} \forall B \subset A$.

Remark. Any measure space can be completed as follows:

Let $\bar{\mathcal{B}}$ be the σ -algebra generated by \mathcal{B} and all sets $B \subset X$ s.t. there exists $A \in \mathcal{B}$ with $B \subset A$ and $\mu(A) = 0$.

Proposition 11.7. The σ -algebra $\bar{\mathcal{B}}$ can also be described as follows:

$$\bar{\mathcal{B}} := \{B \subset X : A_1 \subset B \subset A_2 \text{ for some } A_1, A_2 \in \mathcal{B} \text{ with } \mu(A_2 \setminus A_1) = 0\}, \quad (36)$$

with B, A_1, A_2 as above, we define

$$\bar{\mu} := \mu(A_1) = \mu(A_2) \quad (37)$$

Then $(X, \bar{\mathcal{B}}, \bar{\mu})$ is a complete measure space.

Definition 11.8. If μ is a Borel measure on a **metric** space (X, d) , then the completion $\bar{\mathcal{B}}(X)$ of the Borel σ -algebra with respect to μ is called the σ -algebra of μ -measurable sets.

Remark. For $\mu = \lambda_n$ on \mathbb{R}^n we talk about the σ -algebra of **Lebesgue measurable sets**. Instead of $\bar{\lambda}_n$ we still write λ_n and call it the **Lebesgue measure**. A function $f : \mathbb{R}^n \rightarrow \mathbb{C}$, measurable w.r.t. the σ -algebra of Lebesgue measurable sets is called the **Lebesgue measurable**.

The following result shows that any Lebesgue measurable function coincides with a Borel function a.e.

Proposition 11.9. Assume (X, \mathcal{B}, μ) is a measure space and consider its completion $(X, \bar{\mathcal{B}}, \bar{\mu})$. Assume $f : X \rightarrow \mathbb{C}$ is $\bar{\mathcal{B}}$ -measurable. Then there is a \mathcal{B} -measurable function $g : X \rightarrow \mathbb{C}$ s.t. $f = g$ $\bar{\mu}$ -a.e.

12 Convergence Theorems and Their Applications (lecture 9, 8. Feb.)

- To interchange limits and integrals in **Riemann integrals** one typically has to assume uniform convergence. - The set of Riemann integrable functions is somewhat limited, see theorem 12.4

Theorem 12.1 (Generalization of Beppo Levi, monotone convergence).

(i) Let $(u_n)_{n \in \mathbb{N}} \subset \mathcal{L}^1(\mu)$ be s.t. $u_1 \leq u_2 \leq \dots$ with limit $u := \sup_{n \in \mathbb{N}} u_n = \lim_{n \rightarrow \infty} u_n$. Then $u \in \mathcal{L}^1(\mu)$ **iff**

$$\sup_{n \in \mathbb{N}} \int u_n d\mu < +\infty,$$

in which case

$$\sup_{n \in \mathbb{N}} \int u_n d\mu = \int \sup_{n \in \mathbb{N}} u_n d\mu.$$

(ii) Same thing only with a decreasing sequence $\dots > -\infty$ in which case

$$\inf_{n \in \mathbb{N}} \int u_n d\mu = \int \inf_{n \in \mathbb{N}} u_n d\mu.$$

Theorem 12.2 (Lebesgue; dominated convergence). *Let $(u_n)_{n \in \mathbb{N}} \subset \mathcal{L}^1(\mu)$ s.t.*

(a) $|u_n|(x) \leq w(x)$, $w \in \mathcal{L}^1(\mu)$,

(b) $u(x) = \lim_{n \rightarrow \infty} u_n(x)$ exists in $\bar{\mathbb{R}}$,

then $u \in \mathcal{L}^1(\mu)$ and we have

(i) $\lim_{n \rightarrow \infty} \int |u_n - u| d\mu = 0$;

(ii) $\lim_{n \rightarrow \infty} \int u_n d\mu = \int \lim_{n \rightarrow \infty} u_n d\mu = \int u d\mu$;

Application 2: Riemann vs Lebesgue Integration

Consider only $(X, \mathcal{A}, \mu) = (\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda)$.

Theorem 12.3. *Let $u : [a, b] \rightarrow \mathbb{R}$ be a measurable and **Riemann integrable** function. Then*

$$u \in \mathcal{L}^1(\lambda) \text{ and } \int_{[a,b]} u d\lambda = \int_a^b u(x) dx. \quad (38)$$

Theorem 12.4. *Let $u : [a, b] \rightarrow \mathbb{R}$ be a bounded function, it is Riemann integrable **iff** the points in (a, b) where u is discontinuous are a (subset of) Borel measurable null set.*

Improper Riemann Integrals

- The Lebesgue integral extends the (*proper*) Riemann integral. However, there is a further extension of the Riemann integral which cannot be captured by Lebesgue's theory. u is Lebesgue integrable *iff* $|u|$ has finite Lebesgue integral.
 - The Lebesgue integral does not respect sign-changes and cancellations. However, the following *improper Riemann integral* does:

$$(R) \int_0^\infty u(x) dx := \lim_{n \rightarrow \infty} (R) \int_0^n u(x) dx. \quad (39)$$

Corollary 12.5. *Let $u : [0, \infty) \rightarrow \mathbb{R}$ be a measurable, Riemann integrable function for every interval $[0, N]$, $N \in \mathbb{N}$. Then $u \in \mathcal{L}^1[0, \infty)$ **iff***

$$\lim_{N \rightarrow \infty} (R) \int_0^N |u(x)| dx < \infty. \quad (40)$$

In this case, $(R) \int_0^\infty u(x) dx = \int_{[0, \infty)} u d\lambda$

Proposition 12.6 (appearing as example 12.13 in Schilling). *Let $f_\alpha(x) := x^\alpha$, $x > 0$ and $\alpha \in \mathbb{R}$. Then*

$$(i) \quad f_\alpha \in \mathcal{L}^1(0, 1) \Leftrightarrow \alpha > -1.$$

$$(ii) \quad f_\alpha \in \mathcal{L}^1[1, \infty) \Leftrightarrow \alpha < -1.$$

13 The Function Spaces \mathcal{L}^p (lecture 11, 15. Feb.)

Assume V is a vector space over $\mathbb{K} \in \{\mathbb{C}, \mathbb{R}\}$.

Definition 13.1. A seminorm on V is a map $p : V \rightarrow [0, +\infty)$ s.t.

- (1) $p(cx) = |c|p(x) \quad \forall x \in V, \forall c \in \mathbb{K}$.
- (2) $p(x + y) \leq p(x) + p(y) \quad \forall x, y \in V$. **triangle inequality.**

A seminorm is called a norm if we also have

$$p(x) = 0 \iff x = 0.$$

A norm is commonly denoted $\|x\|$, and a vectorspace equipped with a norm is called a **normed space**.

Definition 13.2. Assume (X, d) is a measure space. Fix $1 \leq p \leq \infty$. For every measurable function $f : X \rightarrow \mathbb{C}$ we define the following

$$\|f\|_p = \left(\int_X |f|^p d\mu \right)^{1/p} \in [0, +\infty]. \quad (41)$$

We can see that $\|cf\|_p = |c|\|f\|_p \quad \forall c \in \mathbb{C}$.

Notice that by Theorem 11.2(i) we have that $\|f\|_p = 0 \Rightarrow f = 0$ a.e. Consider for example $\lim_{n \rightarrow \infty} \|f_n - f\|_p = 0$, then we can find a subsequence s.t. $\lim_{k \rightarrow \infty} |f_{n(k)} - f| = 0$ a.e., i.e. $\lim_{k \rightarrow \infty} f_{n(k)} = f$ a.e.

Lemma 13.3.

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p. \quad (42)$$

Definition 13.4. We define

$$\mathcal{L}^p(X, d\mu) = \{f : X \rightarrow \mathbb{C} \mid f \text{ is measurable and } \|f\|_p < \infty\}.$$

This is a vectorspace with seminorm $f \mapsto \|f\|_p$. And in general this is not a normed space, since $\|f\|_p = 0 \iff f = 0$ a.e.

Generally, if p is a seminorm on a vectorspace V , then

$$V_0 = \{x \in V \mid p(x) = 0\} \quad (43)$$

which is a subspace of V . Then we consider the quotient/factor space V/V_0 .

Definition 13.5. For $x, y \in V$, define

$$x \sim y \iff x - y \in V_0. \quad (44)$$

This is an equivalence relation on V . The representation class of V is defined by $[x]$ or $x + V_0$.

Then V/V_0 is equals the set of equivalence classes. We can show that it is a normed space.

$$[x] + [y] = [x + y] \quad , \quad c[x] = [cx] \quad , \quad ||[x]|| = p(x).$$

Applying this to $\mathcal{L}^p(X, d\mu)$ we get the normed space

$$L^p(X, d\mu) = \mathcal{L}^p(X, d\mu)/\mathcal{N}. \quad (45)$$

Where \mathcal{N} is the space of measurable functions f s.t. $f = 0$ a.e. We will further continue to denote the norm by $|| \cdot ||_p$, and we will normally **not** distinguish between $f \in \mathcal{L}^p(X, d\mu)$ and the vector in $L^p(X, d\mu)$ that f defines.

Definition 13.6. A normed space $(X, || \cdot ||)$ is called a Banach space if V is complete w.r.t the metric $d(x, y) = ||x - y||$.

Theorem 13.7. If (X, \mathcal{B}, μ) is a measure space, $1 \leq p \leq \infty$, then $L^p(X, d\mu)$ is a Banach space.

Definition 13.8. A measurable function $f : X \rightarrow \mathbb{C}$ is called **essentially bounded** if there is $c \geq 0$ s.t.

$$\mu(\{x : |f(x)| > c\}) = 0. \quad (46)$$

That is $|f| \leq c$ a.e. The smallest such c is called the essential supremum of f and is denoted by $||f||_\infty$.

Definition 13.9.

$$\mathcal{L}^\infty(X, d\mu) = \{f : X \rightarrow \mathbb{C} \mid f \text{ is measurable and } ||f||_\infty < \infty\}.$$

$$L^\infty(X, d\mu) = \mathcal{L}^\infty(X, d\mu)/\mathcal{N}.$$

Where by the previous definiton these spaces become the spaces of all essentially bounded functions.

Theorem 13.10. If (X, \mathcal{B}, μ) is a σ -finite measure space, then $L^\infty(X, d\mu)$ is a Banach space.

Convergence in \mathcal{L}^p and completeness

Lemma 13.11. For any sequence $(u_n)_{n \in \mathbb{N}} \subset \mathcal{L}^p(\mu), p \in [1, \infty)$, of positive functions $u_n \geq 0$ we have

$$\left\| \sum_{n=1}^{\infty} u_n \right\|_p \leq \sum_{n=1}^{\infty} ||u_n||_p.$$

Theorem 13.12 (Riesz-Fischer). The spaces $\mathcal{L}^p(\mu), p \in [1, \infty)$, are complete, i.e. every Cauchy sequence $(u_n)_{n \in \mathbb{N}} \subset \mathcal{L}^p(\mu)$ converges to some limit $u \in \mathcal{L}^p(\mu)$

Corollary 13.13. Let $(u_n)_{n \in \mathbb{N}} \subset \mathcal{L}^p(\mu), p \in [1, \infty)$ with $\mathcal{L}^p - \lim_{n \rightarrow \infty} u_n = u$. Then there exists a subsequence $(u_{n_k})_{k \in \mathbb{N}}$ s.t. $\lim_{k \rightarrow \infty} u_{n_k}(x) = u(x)$ holds for almost every $x \in X$.

Theorem 13.14. Let $(u_n)_{n \in \mathbb{N}} \subset \mathcal{L}^p(\mu), p \in [1, \infty)$, be a sequence of functions s.t. $|u_n| \leq w \ \forall n \in \mathbb{N}$ and some $w \in \mathcal{L}^p(\mu)$. If $u(x) = \lim_{n \rightarrow \infty} u_n(x)$ exists for (almost) every $x \in X$, then

$$u \in \mathcal{L}^p \text{ and } \lim_{n \rightarrow \infty} \|u - u_n\|_p = 0.$$

Theorem 13.15 (F. Riesz (convergence theorem)). Let $(u_n)_{n \in \mathbb{N}} \subset \mathcal{L}^p(\mu), p \in [1, \infty)$, be a sequence s.t. $\lim_{n \rightarrow \infty} u_n(x) = u(x)$ for almost every $x \in X$ and some $u \in \mathcal{L}^p(\mu)$. Then

$$\lim_{n \rightarrow \infty} \|u_n - u\|_p = 0 \iff \lim_{n \rightarrow \infty} \|u_n\|_p = \|u\|_p.$$

17 Dense and Determining Sets (lecture 12, 19. Feb.)

Definition 17.1 (Dense Sets). A set $\mathcal{D} \subset \mathcal{L}^p(\mu), p \in [0, \infty]$, is called *dense* if for every $u \in \mathcal{L}^p(\mu)$ there exist a sequence $(f_n)_{n \in \mathbb{N}} \subset \mathcal{D}$ s.t. $\lim_{n \rightarrow \infty} \|u - f_n\|_p = 0$.

Theorem 17.2. Assume X, d is a metric space and μ is a Borel measure that is finite on every ball $1 \leq p < \infty$. Then the space of bounded continuous functions with bounded support is dense in $\mathcal{L}^p(X, d\mu)$. Where bounded support means that f vanishes outside some ball.

Theorem 17.3. Assume (X, d) is a separable locally compact metric space and μ is a Borel Measure on X s.t. $\mu(K) < \infty \ \forall$ compact $K \subset X$. Then the space $C_c(X)$ of continuous compactly supported functions is dense in $\mathcal{L}^p(X, d\mu)$.

In particular, either theorem shows that if μ is a Borel measure on \mathbb{R}^n s.t. the measure of every ball is finite, then $C_c(\mathbb{R}^n)$ is dense in $\mathcal{L}^p(\mathbb{R}^n, d\mu), 1 \leq p < \infty$. Later we will see that even $C_c^\infty(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n, d\mu)$.

Remark. These results do not extend to $p = \infty$ in general.

For $\mu = \lambda_n$ we write simply $\mathcal{L}^p(\mathbb{R}^n)$.

Remark. Theorem 17.8 in the book is *WRONG*. For example, $X = \mathbb{Q}$ with the usual metric is σ -compact, supports nonzero finite measure, but $C_c(\mathbb{Q}) = 0$.

Modes of Convergence (mixture of ex. 11.12 and ch. 22 p. 258-261.)

Assume (X, \mathcal{B}, μ) is a measure space. Given measurable functions $f_n, f : X \rightarrow \mathbb{C}$, recall that

$$f_n \rightarrow f \text{ a.e.}$$

means that $f_n(x) \xrightarrow{n \rightarrow \infty} f(x)$ for all x outside a set of measure zero.

Theorem 17.4 (Egorov). Assume $\mu(X) < \infty$ and $f_n \rightarrow f$ a.e. Then, $\forall \epsilon > 0$, there exists $X_\epsilon \in \mathcal{B}$ s.t. $\mu(X_\epsilon) < \epsilon$ and $f_n \rightarrow f$ uniformly on $X \setminus X_\epsilon$.

In addition to pointwise and uniform convergence we also consider the following:

$f_n \rightarrow f$ in the p -th mean if $\|f_n - f\|_p \xrightarrow{n \rightarrow \infty} 0$. For $p = 1$ we say in mean, for $p = 2$ we say in quadratic mean.

$f_n \rightarrow f$ in measure if $\forall \epsilon > 0$ we have

$$\mu(\{x : |f_n(x) - f(x)| \geq \epsilon\}) \xrightarrow{n \rightarrow \infty} 0.$$

Theorem 17.5 (Lemma 22.4 in the book?). Assume $(X, \mathcal{B}, d\mu)$ is a measure space, $1 \leq p < \infty$, $f_n, f : X \rightarrow \mathbb{C}$ are measurable functions. Then

(i) If $f_n \rightarrow f$ in the p -th mean, then $f_n \rightarrow f$ in measure.

(ii) If $f_n \rightarrow f$ in measure, then there is a subsequence $(f_{n_k})_{k=1}^\infty$ s.t. $f_{n_k} \rightarrow f$ a.e.

(iii) If $f_n \rightarrow f$ a.e. and $\mu(X) < \infty$, then $f_n \rightarrow f$ in measure.

In particular, if $f_n \rightarrow f$ in the p -th mean, then $f_{n_k} \rightarrow f$ a.e. for a subsequence $(f_{n_k})_k$.

26 Abstract Hilbert Spaces (lecture 13, 22. Feb)

Assume \mathcal{H} is a vector space over \mathbb{C} .

Definition 26.1. A pre-inner product on \mathcal{H} is a map $(\cdot, \cdot) : H \times H \rightarrow \mathbb{C}$ which is

(i) Sesquilinear: linear in the first variable and antilinear in the second:

$$\begin{aligned} (\alpha u + \beta v, w) &= \alpha(u, w) + \beta(v, w), \\ (w, \alpha u + \beta v) &= \bar{\alpha}(u, w) + \bar{\beta}(v, w), \quad u, v, w \in H \text{ and } \alpha, \beta \in \mathbb{C}. \end{aligned}$$

(ii) Hermitian: $(u, v) = \overline{(v, u)}$.

(iii) Positive semidefinite: $(u, u) \geq 0$.

It is called an **inner product**, or a scalar product, if instead of (iii) the map is positive definite; $(u, u) > 0$. This definition also works for \mathbb{R} instead of \mathbb{C} .

Cauchy-Schwartz inequality If (\cdot, \cdot) is a pre-inner product, then $|(u, v)| \leq (u, u)^{1/2}(v, v)^{1/2}$.

Corollary 26.2. Assume we have a seminorm $\|u\| := (u, u)^{1/2}$. It is a norm iff (\cdot, \cdot) is an inner product.

Definition 26.3 (Hilbert space). A Hilbert space is a complex vector space \mathcal{H} with an inner product (\cdot, \cdot) s.t. \mathcal{H} is complete with respect to the norm $\|u\| = (u, u)^{1/2}$.

1. The norm on a Hilbert space is determined by the inner product, but the inner product can also be recovered by the norm by the *polarization identity*: $(u, v) = \frac{\|u+v\|^2 - \|u-v\|^2}{4} + i \frac{\|u+iv\|^2 - \|u-iv\|^2}{4}$.
2. *Parallelogram law*: $\|u+v\|^2 + \|u-v\|^2 = 2\|u\|^2 + 2\|v\|^2$.
3. A norm on a vector space is given by an inner product iff it satisfies the parallelogram law, and then the scalar product is uniquely determined by the polarization identity.

Recall that a subset \mathcal{C} of a vector space is called *convex* if

$$u, w \in \mathcal{C} \rightarrow tu + (1-t)w \in \mathcal{C} \quad \forall t \in (0, 1).$$

The following is one of the key properties of the Hilbert space

Theorem 26.4 (projection theorem). Assume \mathcal{H} is a Hilbert space and $\mathcal{C} \subset H$ is a closed convex subset. Then for every $u \in H$ there is a unique $u_0 \in \mathcal{C}$ (minimizer) s.t.

$$\|u - u_0\| = d(u, \mathcal{C}) (= \inf_{x \in \mathcal{C}} \|u - x\|).$$

Orthogonal Projections (lecture 14, 26. Feb.)

For a Hilbert space \mathcal{H} and a subset $A \subset H$, let

$$A^\perp := \{x \in H : x \perp y \quad \forall y \in A\},$$

where $x \perp y$ means that $(x, y) = 0$. A^\perp is a closed subspace of \mathcal{H} .

Proposition 26.5. Assume \mathcal{H}_0 is a closed subspace of a Hilbert space \mathcal{H} . Then every $u \in H$ uniquely decomposes as

$$u = u_0 + u_1, \quad \text{with } u_0 \in \mathcal{H}_0 \text{ and } u_1 \in \mathcal{H}_0^\perp.$$

Moreover, $\|u - u_0\| = d(u, \mathcal{H}_0)$ and $\|u\|^2 = \|u_0\|^2 + \|u_1\|^2$.

For a closed subspace $\mathcal{H}_0 \subset \mathcal{H}$, consider the map $P : H \rightarrow \mathcal{H}_0$ s.t. $Pu \in \mathcal{H}_0$ is the unique element satisfying $u - Pu \in \mathcal{H}_0^\perp$. The operator P is linear. It is also contractive, meaning that $\|Pu\| \leq \|u\|$, since $\|u\|^2 = \|Pu\|^2 + \|u - Pu\|^2$. It is called the orthogonal projection onto \mathcal{H}_0 .

If \mathcal{H}_0 is finite dimensional with an orthonormal basis u_1, \dots, u_n then

$$Pu = \sum_{k=1}^n (u, u_k) u_k.$$

Orthonormal bases can be defined for arbitrary Hilbert spaces.

Definition 26.6 (orthonormal system). An orthonormal system in \mathcal{H} is a collection of vectors $u_i \in H$ ($i \in I$) s.t.

$$(u_i, u_j) = \delta_{ij} \quad \forall i, j \in I.$$

It is called an *orthonormal basis* if $\text{span}\{u_i\}_{i \in I}$ denotes the linear span of $\{u_i\}_{i \in I}$, the space of finite linear combinations of the vectors u_i .

Definition 26.7. A Hilbert space \mathcal{H} is said to be *separable* if \mathcal{H} contains a countable dense subset $G \subset \mathcal{H}$.

Theorem 26.8. Every Hilbert space \mathcal{H} has an orthonormal basis. If \mathcal{H} is separable, then there is a countable orthonormal basis.

Proposition 26.9. Assume $\{u_i\}_{i \in I}$ is an orthonormal system in a Hilbert space H . Take $u \in \mathcal{H}$. Then

- (i) Bessel's inequality: $\sum_{i \in I} |(u, u_i)|^2 \leq \|u\|^2$, in particular, $\{i : (u, u_i) \neq 0\}$ is countable.
- (ii) Parseval's identity: If $\{u_i\}_{i \in I}$ is an orthonormal basis, then $\sum_{i \in I} |(u, u_i)|^2 = \|u\|^2$.

If $(u_i)_{i \in I}$ is an orthonormal basis, then the numbers (u, u_i) are called the **Fourier coefficients** of u with respect to $(u_i)_{i \in I}$. The Parseval identity then suggests that u is determined by its Fourier coefficients. This is true, and even more, we have:

Proposition 26.10. Assume $(u_i)_{i \in I}$ is an orthonormal basis in a Hilbert space \mathcal{H} . Then for every vector $(c_i)_{i \in I} \in l^2(I)$ there is a unique vector $u \in \mathcal{H}$ with Fourier coefficients c_i , and we write

$$u = \sum_{i \in I} c_i u_i.$$

Remark. Equivalently, the element $u = \sum_{i \in I} c_i u_i$ can be described as the unique element in \mathcal{H} s.t. $\forall \epsilon > 0$ there is a finite $F_0 \subset I$ s.t. $\|u - \sum_{i \in F} c_i u_i\| < \epsilon \quad \forall \text{ finite } F \supset F_0$.

Corollary 26.11. We have a linear isomorphism $U : l^2(I) \xrightarrow{\sim} \mathcal{H}$, $U((c_i)_{i \in I}) = \sum_{i \in I} c_i u_i$. By Parseval's identity this isomorphism is isometric, that is, $\|Ux\| = \|x\| \quad \forall x \in l^2(I)$. By the polarization identity this is equivalent to

$$(Ux, Uy) = (x, y) \quad \forall x, y \in l^2(I).$$

Therefore U is unitary.

Corollary 26.12. Up to a unitary isomorphism, there is only one infinite dimensional separable Hilbert space, namely, l^2 .

Dual spaces (lecture 15, 29. Feb.)

Lemma 26.13. Assume V is a normed space over $K = \mathbb{R}$ or $K = \mathbb{C}$. Consider a linear functional $f : V \rightarrow K$. The following are equivalent (TFAE):

- (i) f is continuous;
- (ii) f is continuous at 0;
- (iii) There is a $c \geq 0$ s.t. $|f(x)| \leq c\|x\| \ \forall x \in V$.

If (i)-(iii) are satisfied, then f is called a *bounded linear functional*. The constant c in (iii) is denoted by $\|f\|$. We have $\|f\| = \sup_{x \neq 0} \frac{|f(x)|}{\|x\|} = \sup_{\|x\|=1} |f(x)| = \sup_{\|x\| \leq 1} |f(x)|$.

Proposition 26.14. For every normed vector space V over $K = \mathbb{R}$ or $K = \mathbb{C}$, the bounded linear functionals on V form a Banach space V^* .

Remark. The sequence $\{\|f_n - f_m\|\}_{m=1}^\infty$ actually converges, since

$$\left| \|f_n - f_m\| \right| \leq \|f_m - f_n\|.$$

When we study/use normed spaces, it is often important to understand the dual spaces. For Hilbert spaces this is particularly easy:

Theorem 26.15 (Riesz). Assume \mathcal{H} is a Hilbert space. Then every $f \in \mathcal{H}^*$ has the form

$$f(x) = (x, y),$$

for a uniquely defined $y \in \mathcal{H}$. Moreover, we have $\|f\| = \|y\|$.

For every Hilbert space \mathcal{H} we can define the *conjugate Hilbert space* $\bar{\mathcal{H}}$, which has its elements as the symbols \bar{x} for $x \in \mathcal{H}$, with the linear structure and inner product defined by

$$\bar{x} + \bar{y} = \overline{x + y}, c \cdot \bar{x} = \overline{c x}, (\bar{x}, \bar{y}) = \overline{(x, y)} = (y, x).$$

Corollary 26.16. For every Hilbert space \mathcal{H} , we have an isometric isomorphism $\bar{\mathcal{H}} \xrightarrow{\sim} \mathcal{H}^*$, $\bar{x} \mapsto (\cdot, x)$.

Appendix

H Regularity of measures (lecture 10, 12. Feb.)

We let (X, d) be a metric space and denote by \mathcal{O} the open, by \mathcal{C} the closed and $\mathcal{B}(X) = \sigma(\mathcal{O})$ the Borel set of X .

Definition H.1. A measure μ on $(X, d, \mathcal{B}(X))$ is called outer regular, if

$$\mu(B) = \inf \{ \mu(U) \mid B \subset U, U \text{ open} \} \quad (47)$$

and inner regular, if $\mu(K) < \infty$ for all compact sets $K \subset X$ and

$$\mu(U) = \sup \{ \mu(K) \mid K \subset U, K \text{ compact} \}. \quad (48)$$

A measure which is both inner and outer regular is called **regular**. We write $\mathfrak{m}_r^+(X)$ for the family of regular measures on $(X, \mathcal{B}(X))$.

Remark. The space X is called σ -compact if there is a sequence of compact sets $K_n \uparrow X$. A typical example of such a space is a locally compact, separable metric space.

Theorem H.2. Let (X, d) be a metric space. Every finite measure μ on $(X, \mathcal{B}(X))$ is outer regular. If X is σ -compact, then μ is also inner regular, hence regular.

Theorem H.3. Let (X, d) be a metric space and μ be a measure on $(X, \mathcal{B}(X))$ such that $\mu(K) < \infty$ for all compact sets $K \subset X$.

- 1 If X is σ -compact, then μ is inner regular.
- 2 If there exists a sequence $G_n \in \mathcal{O}$, $G_n \uparrow X$ such that $\mu(G_n) < \infty$, then μ is outer regular.

Hahn-Banach Theorem

Theorem H.4 (Hahn-Banach). Assume V is a real vector space, $V_0 \subset V$ a subspace, $e : V \rightarrow \mathbb{R}$ a convex function and $f : V_0 \rightarrow \mathbb{R}$ a linear functional s.t. $f \leq e$ on V_0 . Then f can be extended to a linear functional F on V s.t. $F \leq e$.

Theorem H.5 (Hahn-Banach). Assume V is a real or complex vector space, p a seminorm on V_0 , $V_0 \subset V$, and f a linear functional on V_0 s.t.

$$|f(x)| \leq p(x) \quad \forall x \in V_0.$$

Then f can be extended to a linear functional F on V s.t. $|F(x)| \leq p(x) \quad \forall x \in V$.

Corollary H.6. Assume V is a normed real or complex vector space, $V_0 \subset V$ and $f \in V_0^*$. Then there is a $F \in V^*$ s.t.

$$F|_{V_0} = f \text{ and } \|F\| = \|f\|.$$

Corollary H.7. Assume V is a normed space and $x \in V, x \neq 0$. Then there is a $F \in V^*$ s.t. $\|F\| = 1$ and $F(x) = \|x\|$.

Such an F is called a *supporting functional at x* .

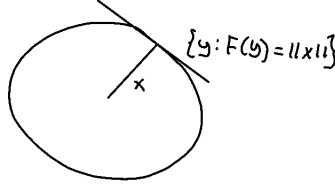


Figure 1: Tangent space?

If V is a normed vector space, then every $x \in X$ defines a bounded linear functional on V^* by

$$V^* \ni f \mapsto f(x).$$

As $|f(x)| \leq \|f\| \cdot \|x\|$, this functional has norm $\leq \|x\|$. By using a supporting functional at x , we actually see that we get norm $\|x\|$. Thus, we have an isometric embedding $V \subset V^{**} := (V^*)^*$. We can therefore see V as a subspace of V^{**} .

Definition H.8. A normed space V is called reflexive if $V^{**} = V$.

Remark. This is stronger than requiring $V \simeq V^{**}$.

Remark. Every Hilbert space \mathcal{H} is reflexive. Indeed, $\mathcal{H}^* = \bar{\mathcal{H}}$. By Riesz' theorem every bounded linear functional f on $\bar{\mathcal{H}}$ has the form

$$f(\bar{x}) = (\bar{x}, \bar{y}) = (y, x),$$

for some $y \in \mathcal{H}$, which exactly means that $f = y$ in \mathcal{H}^{**} .

As we will see later, the spaces $\mathcal{L}^p(X, d\mu)$, with μ σ -finite and $1 < p < \infty$, are reflexive. The spaces $\mathcal{L}'(X, d\mu)$ and $\mathcal{L}^\infty(X, \mu)$ are usually not reflexive.