UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Exam in: MAT4410 — Advanced Linear Analysis

Day of examination: Monday, December 7, 2020

Examination hours: 15.00 – 19.00

This problem set consists of 4 pages.

Appendices: Ingen

Permitted aids: Ingen

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

We make our convention as follows:

- $C_c(\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{R} \mid \text{continuous with compact support}\}$
- $\mathbb{N} = \{1, 2, \dots\}$

We also use maps $\xi : \mathbb{N} \to \mathbb{R}$ to represent real sequences $(a_n)_{n=1}^{\infty}$ up to the correspondence $\xi(n) = a_n$.

Problems 1–4 are the common ones that everyone should solve to get a passing grade. To score better, answer one or more of Problems 5–7.

Problem 1

Write $S = [0,1] \times [0,1]$, and consider the subsets $A_n^{(m)}, B_p^{(m)} \subset S$ for $1 \leq n \leq 8^{m-1}$ and $1 \leq p \leq 8^m$, with $m = 1, 2, \dots$, inductively as follows:

- $A_1^{(1)} = [\frac{1}{3}, \frac{2}{3}] \times [\frac{1}{3}, \frac{2}{3}]$
- $B_1^{(1)}, \dots, B_8^{(1)}$: the squares of the form $[\frac{i}{3}, \frac{i+1}{3}] \times [\frac{j}{3}, \frac{j+1}{3}] \subset S$ for $i, j = 0, 1, 2$, other than $A_1^{(1)}$
- $A_n^{(2)} = [\frac{i_n}{3} + \frac{1}{9}, \frac{i_n}{3} + \frac{2}{9}] \times [\frac{j_n}{3} + \frac{1}{9}, \frac{j_n}{3} + \frac{2}{9}]$ for $B_n^{(1)} = [\frac{i_n}{3}, \frac{i_n+1}{3}] \times [\frac{j_n}{3}, \frac{j_n+1}{3}]$, $n = 1, \dots, 8$
- $B_1^{(2)}, \dots, B_{64}^{(2)}$: the squares of the form $[\frac{i}{9}, \frac{i+1}{9}] \times [\frac{j}{9}, \frac{j+1}{9}] \subset B_n^{(1)}$, other than $A_n^{(2)}$, for some n
- repeat this procedure.

We put $C = \bigcup_{m=1}^{\infty} \bigcup_{n=1}^{8^{m-1}} A_n^{(m)}$. ($S \setminus C$ is called the *Sierpinski carpet*.)

a

Explain how to make sense of the area of C using concepts of measure theory.

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b

Compute the area of C following the previous explanation.

Problem 2

a

Explain the measure theoretic framework for the manipulation

$$\sum_{n=1}^{\infty} \int_0^{\infty} (-1)^n e^{-n^2 x} dx = \int_0^{\infty} \sum_{n=1}^{\infty} (-1)^n e^{-n^2 x} dx.$$

(As part of the explanation, you should give the relevant measure spaces.)

b

Consider the double sequence $(a_{i,j})_{i,j=1}^{\infty}$ defined as

$$a_{1,1} = 1, \qquad a_{i,i+1} = 1, \qquad a_{i+1,i} = -1, \qquad a_{i,j} = 0 \qquad \text{(otherwise)}$$

Does $\sum_i \sum_j a_{i,j} = \sum_j \sum_i a_{i,j}$ hold? If yes, give a measure theoretic justification. If no, explain the difference with Part **a**.

Problem 3

Consider the sequence of real functions $(g_n)_{n=1}^{\infty}$ given by

$$g_n(x) = \begin{cases} n & (0 < x < \frac{1}{n}) \\ 0 & \text{(otherwise)}, \end{cases}$$

and put $\phi_n(f) = \int_{-\infty}^{\infty} f(x) g_n(x) dx$ for $f \in C_c(\mathbb{R})$.

a

Consider $C_c(\mathbb{R})$ as a normed vector space with the norm $\|f\|_\infty = \sup_x |f(x)|$. What is the norm of ϕ_n as a functional on $C_c(\mathbb{R})$?

b

Describe the limit functional $\phi(f) = \lim_n \phi_n(f)$.

c

We have $\lim_n g_n(x) = 0$ for each x. Explain why this does not contradict with Part **b**.

(Continued on page 3.)

Problem 4

Consider the measurable space $[0,\infty)$ with $\mathcal{B}_{[0,\infty)} = \{B \subset [0,\infty) \mid \text{Borel set}\}$ as its associated σ-algebra. Let m denote the restriction of Lebesgue measure on $[0,\infty)$, and δ_n be the Dirac measure at n, for $n \in \mathbb{N}$.

Consider the signed measure μ on $[0,\infty)$ given by

$$\mu(A) = \int_{[0,\infty)} 1_A(x) e^{-x} \sin x \, dm(x) + \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \delta_n(A)$$

for $A \in \mathcal{B}_{[0,\infty)}$.

a

Describe $\mu^{\pm}$ appearing in the Jordan decomposition $\mu = \mu^+ - \mu^-$.

b

Describe μ_a^+ and μ_s^+ appearing in the Lebesgue decomposition $\mu^+ = \mu_a^+ + \mu_s^+$ relative to m.

Problem 5

Let X be a nonnegative random variable. Show that its expectation $\mathbb{E}[X]$ satisfies

$$\mathbb{E}[X] = \int_0^{\infty} \mathbb{P}[X \geq t] dt,$$

where $\mathbb{P}[X \geq t]$ is the probability of $X \geq t$.

Problem 6

Consider the Banach spaces

$$\ell^p(\mathbb{N}) = \{(a_n)_{n=1}^{\infty} \mid a_n \in \mathbb{R}, \|(a_n)_n\|_p = (\sum_n |a_n|^p)^{\frac{1}{p}} < \infty\}$$

for $p = 1, 2$. For $n \in \mathbb{N}$, consider the vector $\delta_n \in \ell^1(\mathbb{N}) \cap \ell^2(\mathbb{N})$ defined by $\delta_n(k) = \delta_{n,k}$ (the Kronecker delta).

a

Show that $\delta_n \to 0$ $(n \to \infty)$ holds in the weak topology of $\ell^2(\mathbb{N})$.

b

Show that $\delta_n \to 0$ $(n \to \infty)$ does not hold in the weak topology of $\ell^1(\mathbb{N})$.

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Problem 7

In this problem we work over complex coefficient. Consider the Hilbert space

$$\ell^2(\mathbb{Z}) = \left\{ (a_n)_{n=-\infty}^{\infty} \;\middle|\; a_n \in \mathbb{C}, \sum_n |a_n|^2 < \infty \right\}$$

with respect to the standard Hermitian inner product $\langle \xi, \eta \rangle = \sum_{n=-\infty}^{\infty} \xi(n)\overline{\eta(n)}$.

Let U be the operator $(U\xi)(n) = \xi(n-1)$ on $\ell^2(\mathbb{Z})$. (In other words, U satisfies $U\delta_n = \delta_{n+1}$ for $n \in \mathbb{Z}$.) Fix a vector $\eta \in \ell^2(\mathbb{Z})$. For a Laurent polynomial $p(z) = \sum_{k=-M}^{N} c_k z^k$ with $M, N \in \mathbb{N}$ and $c_k \in \mathbb{C}$, put

$$\phi(p) = \langle p(U)\eta, \eta \rangle = \langle c_{-M} U^{-M} \eta + \cdots + c_N U^N \eta, \eta \rangle.$$

Show that there is a measure μ on the unit circle $\mathbb{T} = \{w \in \mathbb{C} \mid |w| = 1\}$ such that $\phi(p) = \int p(w) d\mu(w)$.

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