Null sets and the Almost Everywhere (lecture 08, 05. Feb.)

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April 2, 2024

Definition 11.12. A (μ-)null set $N \in \mathcal{N}_\mu$ is a measurable set $N \in \mathscr{A}$ satisfying

$$N \in \mathcal{N}_\mu \iff N \in \mathscr{A} \text{ and } \mu(N) = 0. \tag{1}$$

This can be used generally about a 'statement' or 'property', but we will be interested in questions like 'when is $u(x)$ equal to $v(x)$', and we answer this by saying

$$u = v \ a.e. \Leftrightarrow \{x : u(x) \neq v(x)\} \text{ is (contained in) a } \mu\text{-null set.}, \tag{2}$$

i.e.

$$u = v \quad \mu\text{-a.e.} \Leftrightarrow \mu\left(\{x : u(x) \neq v(x)\}\right) = 0. \tag{3}$$

The last phrasing should of course include that the set $\{x : u(x) \neq v(x)\}$ is in $\mathscr{A}$.

Theorem 11.13. *Let $u \in \mathcal{M}_{\overline{\mathbb{R}}}(\mathscr{A})$, then:*

(i) $\int |u| d\mu = 0 \Leftrightarrow |u| = 0 \ a.e. \ \Leftrightarrow \mu\{u \neq 0\} = 0$,

(ii) $\mathbb{1}_N u \in \mathcal{L}^1_{\mathbb{R}}(\mu) \ \ \forall\ N \in \mathcal{N}_\mu$,

(iii) $\int_N u d\mu = 0$.

(i) is really useful, later we will define $\mathcal{L}^p$ and the $||\cdot||_p$-(semi)norm. Then (i) means that if we have a sequence u_n converging to u in the $||\cdot||_p$-norm then $u_n(x) = u(x)$ a.e.

Corollary 11.14. *Let $u = v$ μ-a.e. Then*

(i) $u, v \geq 0 \Rightarrow \int u d\mu = \int v d\mu$,

(ii) $u \in \mathcal{L}^1_{\overline{\mathbb{R}}}(\mu) \Rightarrow v \in \mathcal{L}^1_{\overline{\mathbb{R}}}(\mu)$ *and* $\int u d\mu = \int v d\mu$.

Corollary 11.15. *If $u \in \mathcal{M}_{\overline{\mathbb{R}}}(\mathscr{A})$, $v \in \mathcal{L}^1_{\overline{\mathbb{R}}}(\mu)$ and $v \geq 0$ then*

$$|u| \leq v \ a.e. \ \Rightarrow u \in \mathcal{L}^1_{\overline{\mathbb{R}}}(\mu). \tag{4}$$

Proposition 11.16 (Markow inequality)**.** *For all $u \in \mathcal{L}^1_{\mathbb{R}}(\mu)$, $A \in \mathscr{A}$ and $c > 0$*

$$u\left(\{|u| \geq c\} \cap A\right) \leq \frac{1}{c} \int_A |u| d\mu, \tag{5}$$

if $A = X$, then (obviosly)

$$u\left\{|u| \geq c\right\} \leq \frac{1}{c} \int |u| d\mu. \tag{6}$$

Corollary 11.17. *If $u \in \mathcal{L}^1_{\overline{R}}(\mu)$, then μ is a.e. $\mathbb{R}$-vaued. In particular, we can find a version $\tilde{u} \in \mathcal{L}^1(\mu)$ s.t. $\tilde{u} = u$ a.e. and $\int \tilde{u} d\mu = \int u d\mu$*

Completions of measure spaces

Definition 11.18. A measure space $(X, \mathscr{B}, \mu)$ is called **complete** if whenever $A \in \mathscr{B}$ and $\mu(A) = 0$, we have $B \in \mathscr{B}$ $\forall B \subset A$.

Remark. *Any measure space can be completed as follows:*
Let $\bar{\mathscr{B}}$ be the σ-algebra generated by $\mathscr{B}$ and all sets $B \subset X$ s.t. there exists $A \in \mathscr{B}$ with $B \subset A$ and $\mu(A) = 0$.

Proposition 11.19. *The σ-algebra $\bar{\mathscr{B}}$ can also be described as follows:*

$$\bar{\mathscr{B}} := \left\{B \subset X : A_1 \subset B \subset A_2 \text{ for some } A_1, A_2 \in \mathscr{B} \text{ with } \mu(A_2 \backslash A_1) = 0\right\}, \tag{7}$$

with B, A_1, A_2 as above, we define

$$\bar{\mu} := \mu(A_1) = \mu(A_2) \tag{8}$$

Then $\left(X, \bar{\mathscr{B}}, \bar{\mu}\right)$ is a complete measure space.

Definition 11.20. If μ is a Borel measure on a **metric** space (X, d), then the completion $\bar{\mathscr{B}}(X)$ of the Borel σ-algebra with respect to μ is called the σ-algebra of μ-measurable sets.

Remark. *For $\mu = \lambda_n$ on $\mathbb{R}^n$ we talk about the σ-algebra of* ***Lebesgue measurable sets****. Instead of $\bar{\lambda_n}$ we still write λ_n and call it the* ***Lebesgue measure****. A function $f : \mathbb{R}^n \to \mathbb{C}$, measurable w.r.t. the σ-algebra of Lebesgue measurable sets is called the* ***Lebesgue measurable****.*

The following result shows that any Lebesgue measurable function coincides with a Borel function a.e.

Proposition 11.21. *Assume $(X, \mathscr{B}, \mu)$ is a measure space and consider its completion $\left(X, \bar{\mathscr{B}}, \bar{\mu}\right)$. Assume $f : X \to \mathbb{C}$ is $\bar{\mathscr{B}}$-measurable. Then there is a $\mathscr{B}$-measurable function $g : X \to \mathbb{C}$ s.t. $f = g$ $\bar{\mu}$-a.e.*