Linn - Anal - Lecture - Notes

(Dated: February 8, 2024)

https://github.com/isakrukan/MAT4400-LinnAnaly

I. INTEGRATION OF COMPLEX FUNCTIONS (LEC. 7)

Assume $(X, \mathfrak{B}, \mu)$ is a measure space.

Definition I.1. A measurable function $f : X \to \mathbb{C}$ is called integrable (or μ-integrable) if

$$\int_X |f| d\mu < \infty.$$

Denote by $\mathcal{L}^1(X, \mathfrak{B}, d\mu)$, $\mathcal{L}^1(X, d\mu)$ or $\mathcal{L}^1_{\mathbb{C}}$ the set of integrable functions. This is also a vector space over $\mathbb{C}$, since

$$|f+g| \leq |f| + |g|, \ |cf| = |c||f| \ (c \in \mathbb{C}),$$

and the other axioms are should be easy.

This vector space is spanned by positive functions, since

$$f = \mathrm{Re}(f)_+ - \mathrm{Re}(f)_- + i\mathrm{Im}(f)_+ - i\mathrm{Im}(f)_-,$$

where for a function h we let

$$h_+ = \max\{h, 0\}, \ h_- = -\min\{h, 0\},$$

and if $f \in \mathcal{L}^1(X, d\mu)$, then

$$(\mathrm{Re}(f))_\pm, (\mathrm{Im}(f))_\pm \in \mathcal{L}^1(X, d\mu),$$

as

$$|(\mathrm{Re}(f))_\pm|, |(\mathrm{Im}(f))_\pm| \leq |f|.$$

Proposition 1. *The integral extends uniquely from the positive integrable functions to a linear function (functional?) $\mathcal{L}^1(X, d\mu) \to \mathbb{C}$, that is, to a map s.t.*

$$\int_X (f+g) d\mu = \int_X f d\mu + \int_X g d\mu,$$
$$\int_X cf d\mu = c \int_X f d\mu, \ c \in \mathbb{C}.$$

Proof. Uniqueness is clear, as positive functions in $\mathcal{L}^1(X, d\mu)$ spans the entire space. We first extend the integral to real integrable functions by letting

$$\int_X (g-h) d\mu := \int_X g d\mu - \int_X h d\mu,$$

for $g, h \in \mathcal{L}^1(X, d\mu), \ g, h \geq 0$.

This is well-defined, since if

$$g - h = g' - h',$$

then $g + h' = h + g'$ and hence $\int_X g d\mu + \int_X h' d\mu = \int_X g' d\mu + \int_X h' d\mu$. Now we extend the integral to the entire space $\mathcal{L}'(X, d\mu)$ by

$$\int_X f d\mu := \int_X (\mathrm{Re}(f)) d\mu + i \int_X (\mathrm{Im}(f)) d\mu.$$

We easily get that by definition:

$$\int_X (f_1 + f_2) d\mu = \int_X f_1 d\mu + \int_X f_2 d\mu, \ \forall f_1, f_2 \in \mathcal{L}^1(X, d\mu),$$

and

$$\int_X cf d\mu = c \int_X f d\mu \ \forall f \in \mathcal{L}^1(X, d\mu) \ \forall c \geq 0.$$

In order to prove the last property for all $c \in \mathbb{C}$, it remains to check it for $c = -1$ and $c = i$.

For $c = -1$ it follows, since if $g, h \geq 0$, then

$$\begin{aligned}\int_X (-(g-h)) \, d\mu &= \int_X (h-g) \, d\mu \\ &= \int_X h d\mu - \int_X g d\mu \\ &= -\int_X (g-h) \, d\mu.\end{aligned}$$

Similarly, for $c = i$ it is proved by a simple computation:

$$\begin{aligned}\int_X if d\mu &= \int_X \mathrm{Re}(if) d\mu + i \int_X \mathrm{Im}(if) d\mu \\ &= \int_X (-\mathrm{Im}(f)) \, d\mu + i \int_X (\mathrm{Re}(f)) \, d\mu \\ &= i \left(\int_X (\mathrm{Re}(f)) \, d\mu + i \int_X (\mathrm{Im}(f)) \, d\mu \right) \\ &= i \int_X f d\mu.\end{aligned}$$

□

Proposition 2 (Triangle Inequality)**.** *For every $f \in \mathcal{L}^1(X, d\mu)$ we have*

$$\left| \int_X f d\mu \right| \leq \int_X |f| d\mu.$$

Proof. Choose $z \in \Pi := \{w \in \mathbb{C} : |w| = 1\}$ s.t.

$$z \int\limits_X f d\mu \geq 0.$$

Then

$$\begin{aligned}
\Big| \int\limits_X f d\mu \Big| &= \Big| z \int\limits_X \Big| \\
&= z \int\limits_X f d\mu \\
&= \int\limits_X z f d\mu \\
&= \int\limits_X \mathrm{Re}(zf) d\mu + i \cancel{\int\limits_X \mathrm{Im}(zf) d\mu} \\
&= \int\limits_X \left(\mathrm{Re}(zf)\right)_+ d\mu - \int\limits_X \left(\mathrm{Re}(zf)\right)_- d\mu \\
&\leq \int\limits_X \left(\mathrm{Re}(zf)\right)_+ d\mu \\
&\leq \int\limits_X |f| d\mu,
\end{aligned}$$

since $\left(\mathrm{Re}(zf)\right)_+ \leq |f|$. $\square$

II. COMPLETIONS OF MEASURE SPACES (LEC. 8)

Definition II.1. A measure space $(X, \mathfrak{B}, \mu)$ is called **complete** if whenever $A \in \mathfrak{B}, \mu(A) = 0$, we have $B \in \mathfrak{B}$ for all $B \subset A$.

Any measure space $(X, \mathfrak{B}, \mu)$ can be completed as follows:

Let $\bar{\mathfrak{B}}$ be the σ-algebra generated by $\mathfrak{B}$ and all sets $B \subset X$ s.t. $\exists A \in \mathfrak{B}$ with $B \subset A$ and $\mu(A) = 0$.

Proposition 3. *The σ-algebra $\bar{\mathfrak{B}}$ can also be described as follows:*

$$\bar{\mathfrak{B}} := \left\{B \subset X : A_1 \subset B \subset A_2 \text{ for some } A_1, A_2 \in \mathfrak{B} \text{ with } \mu(A_2 \backslash A_1)\right\},$$

with B, A_1, A_2 as above, we define

$$\bar{\mu}(B) := \mu(A_1) = \mu(A_2).$$

Then $(X, \bar{\mathfrak{B}}, \bar{\mu})$ is a complete measure space.

Proof. Consider the collection of sets

$$\tilde{\mathfrak{B}} := \left\{B \subset X : A_1 \subset B \subset A_2 \text{ for some } A_1, A_2 \in \mathfrak{B} \text{ with } \mu(A_2 \backslash A_1)\right\},$$

this is a σ-algebra; $\emptyset \in \tilde{\mathfrak{B}}$ (take $A_1 = A_2 = \emptyset$), if $B \in \tilde{\mathfrak{B}}$ and A_1, A_2 are as above, then $A_2^c \subset B^c \subset A_1^c$ and $\mu(A_1 \backslash A_2) = \mu(A_2 \backslash A_1) = 0$, so $B^c \in \tilde{\mathfrak{B}}$. Finally, assume $(B_n)_{n \in \mathbb{N}} \subset \tilde{\mathfrak{B}}$ and let $A_n', A_n'' \in \mathfrak{B}$ be s.t.

$$A_n' \subset B_n \subset A_n'', \ \mu(A_n'' \backslash A_n') = 0.$$

Put $A_1 = \bigcup\limits_{n=1}^{\infty} A_n'$, $A_2 = \bigcup\limits_{n=1}^{\infty} A_n''$. Then

$$A_1 \subset \bigcup_{n=1}^{\infty} B_n \subset A_2, \ A_2 \backslash A_1 \subset \bigcup_{n=1}^{\infty} \left(A_n'' \backslash A_n'\right),$$

so

$$\mu(A_2 \backslash A_1) \leq \sum_{n=1}^{\infty} \mu(A_n'' \backslash A_n'),$$

hence

$$\bigcup_{n=1}^{\infty} B_n \in \tilde{\mathfrak{B}}.$$

Thus, $\tilde{\mathfrak{B}}$ is a σ-algebra. It contains $\mathfrak{B}$ and all sets $B \subset X$ s.t. $B \subset A$ and $\mu(A) = 0$ for some $A \in \mathfrak{B}$. Hence, $\bar{\mathfrak{B}} \subset \tilde{\mathfrak{B}}$. On the other hand, if $B \in \tilde{\mathfrak{text}}$, $A_1 \subset B \subset A_2$, $A_1, A_2 \in \mathfrak{B}$ and $\mu(A_2 \backslash A_1) = 0$, then

$$B = A_1 \cup (B \backslash A_1) \text{ and } B \backslash A_1 \subset A_2 \backslash A_1,$$

hence $B \in \bar{\mathfrak{B}}$. Thus $\bar{\mathfrak{B}} = \tilde{\mathfrak{B}}$.

Next, we need to show that $\bar{\mu}$ is well-defined. Assume

$$A_1 \subset B \subset A_2, \ A_1' \subset B \subset A_2',$$

with $A_1, A_2, A_1', A_2' \in \mathfrak{B}$, $\mu(A_2 \backslash A_1) = \mu(A_2' \backslash A_1') = 0$. Then

$$A_1 \backslash A_1' \subset A_2' \backslash A_1',$$

hence

$$\mu(A_1 \backslash A_1') = 0.$$

It follows that

$$\mu(A_1) = \mu(A_1 \cap A_1'),$$

and for the same reason that

$$\mu(A_1') = \mu(A_1 \cap A_1').$$

Therefor $\mu(A_1) = \mu(A_1')$, so $\bar{\mu}$ is well-defined.

Finally, we have to check that μ is a measure. Assume $(B_n)_{n \in \mathbb{N}}$ is a sequence of disjoint sets in $\bar{\mathfrak{B}}$. As above, choose $A_n', A_n'' \in \mathfrak{B}$ s.t.

$$A_n' \subset B_n \subset A_n'' \text{ and } \mu(A_n'' \backslash A_n') = 0.$$

Then for $A_1 = \bigcup\limits_{n=1}^{\infty} A_n'$ and $A_2 = \bigcup\limits_{n=1}^{\infty} A_n''$ we have

$$A_1 \subset \bigcup_{n=1}^{\infty} B_n \subset A_2 \text{ and } \mu(A_2 \backslash A_1) = 0.$$

Hence,

$$\bar{\mu}\left(\bigcup_{n=1}^{\infty} B_n\right) = \mu(A_1) = \mu\left(\bigcup_{n=1}^{\infty} A'_n\right) = \sum_{n=1}^{\infty} \mu(A'_n) = \sum_{n=1}^{\infty} \bar{\mu}(B_n).$$

We also have $\bar{\mu}(\emptyset) = \mu(\emptyset) = 0$. □

Definition II.2. If μ is a Borel measure on a **metric** space (X, d), then the completion $\mathfrak{B}(X)$ of the Borel σ-algebra with respect to μ is called the σ-algebra of μ-measurable sets.

For $\mu = \lambda_n$ on $\mathbb{R}^n$ we talk about the σ-algebra of **Lebesque measurable sets**. Instead of $\bar{\lambda}_n$ we still write λ_n and call it the **Lebesgue measure**. A function $f : \mathbb{R}^n \to \mathbb{C}$, measurable with respect to the σ-algebra of Lebesgue measurable sets is called **Lebesgue measurable**.

Proposition 4. *Assume $(X, \mathfrak{B}, \mu)$ is a measure space and consider its completion $(X, \bar{\mathfrak{B}}, \bar{\mu})$. Assume $f : X \to \mathbb{C}$ is $\bar{\mathfrak{B}}$-measurable. Then there is a $\mathfrak{B}$-measurable function $g : X \to \mathbb{C}$ s.t. $f = g$ $\bar{\mu}$-a.e (almost everywhere).*

Proof. By considering separately $(\mathrm{Re}(f))_{\pm}$, $(\mathrm{Im}(f))_{\pm}$ we may assume that $f \geq 0$. Assume first that f is simple,

$$f = \sum_{k=1}^{\infty} c_k 1_{B_k},\ c_k \geq 0,\ B_k \in \bar{\mathfrak{B}}.$$

Let $A_k \in \mathfrak{B}$ be s.t. $A_K \subset B_k$, $\bar{\mu}(B_k \backslash A_k) = 0$. Then define

$$g = \sum_{k=1}^{\infty} c_k 1_{A_k}.$$

We have $0 \leq g \leq f$ and $f = g$ $\bar{\mu}$-a.e., namely,

$$\{x : f(x) \neq g(x)\} \subset \bigcup_{n=1}^{n} (B_k \backslash A_k).$$

For general $f \geq 0$ choose simple $\bar{\mathfrak{B}}$-measurable functions f_n s.t.

$$0 \leq f_1 \leq f_2 \leq ...,\ f_n \uparrow f \text{ pointwise.}$$

Then we can find simple $\mathfrak{B}$-measurable functions g_n s.t. $0 \leq g_n \leq f_n$ and the set

$$A_n := \{x : f_n(x) \neq g_n(x)\}$$

has measure zero. Consider

$$\tilde{g}_n := \max\{g_1, ..., g_n\}.$$

Then

$$0 \leq \tilde{g}_1 \leq \tilde{g}_2 \leq ...,\ \tilde{g}_n \leq f_n,$$

and

$$\{x : \tilde{g}_n(x) < f_n(x)\} \subset A_n.$$

Define $g(x) := \lim_{n\to\infty} \tilde{g_n}(x)$. Then $g(x)$ is $\mathfrak{B}$-measurable, $g \leq f$ and

$$\{x : g(x) < f(x)\} \subset \bigcup_{n=1}^{\infty} A_n,$$

so $g = f$ $\bar{\mu}$-a.e. □