Measurable Functions

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8 Measurable Functions

A *measurable function* is a measurable map $u : X \to \mathbb{R}$ from some measurable space $(X, \mathscr{A})$ to $(\mathbb{R}, \mathscr{B}(\mathbb{R}^1))$. They play central roles in the theory of integration.

We recall that $u : X \to \mathbb{R}$ is $\mathscr{A}/\mathscr{B}(\mathbb{R}^1)$-measurable if

$$u^{-1}(B) \in \mathscr{A}, \ \ \forall B \in \mathscr{B}(\mathbb{R}^1). \tag{1}$$

Moreover from a lemma from chapter 7, we actually only need to show that

$$u^{-1}(G) \in \mathscr{A}, \ \ \forall G \in \mathcal{G} \text{ where } \mathcal{G} \text{ generates } \mathscr{B}(\mathbb{R}^1). \tag{2}$$

Proposition 8.1.

1. *If $f, g : (X, \mathscr{B}) \to \mathbb{C}$ are measurable, then the function $f+g$, $f \cdot g$, cf, $(c \in \mathbb{C})$ are measurable.*

2. *If $b : \mathbb{C} \to \mathbb{C}$ is Borel and $b : (\mathbb{C}, \mathscr{B}) \to \mathbb{C}$ is measurable, then $b \circ f$ is measurable.*

3. *If $f(x) = \lim_{n\to\infty} f_n(x)$, $x \in X$ and f_n are measurable, then f is measurable.*

4. *If $X = \bigcup_{n=1}^{\infty} A_n$, $(A_n \in \mathscr{B})$, $f|_{A_n} : (A_n, \mathscr{B}_{A_n}) \to \mathbb{C}$ is measurable $\forall n$, then f is measurable.*

Definition 8.2. Given a measurable space $(X, \mathscr{B})$, a measurable function $f : (X, \mathscr{B}) \to \mathbb{C}$ is called simple if

$$f(x) = \sum_{k=1}^{N} c_k \mathbb{1}_{A_k}(x), \tag{3}$$

for some $c_k \in \mathbb{C}$, $A_k \in \mathscr{B}$, where $\mathbb{1}$ is the characteristic function,

$$\mathbb{1}_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A. \end{cases} \tag{4}$$

The representation of simple function is **not** unique. We denote the standard representation of f by

$$f(x) = \sum_{n=0}^{N} z_n \mathbb{1}_{B_n}(x), \quad N \in \mathbb{N},\ z_n \in \mathbb{R},\ B_n \in \mathscr{A},\ X = \bigcup_{n=1}^{N} B_n,\ \text{ for } B_n \cap B_m = \emptyset,\ \ n \neq m. \tag{5}$$

The set of simple functions is denoted $\mathcal{E}(\mathscr{A})$ of $\mathcal{E}$.

Definition 8.3. Assume μ is a measure on $(X, \mathscr{B})$. Given a *positive* simple function

$$f = \sum_{k=1}^{N} c_k \mathbb{1}_{A_k}, \quad (c_k \geq 0). \tag{6}$$

We define

$$\int_X f d\mu = \sum_{k=1}^{n} c_k \mu(A_k) \in [0, +\infty]. \tag{7}$$

We also denote this by $I_\mu(f)$.

Lemma 8.4. *This is well defined, that is, $\int_x f d\mu$ does not depend on the presentation of the simple function f.*

Properties 8.5. *For every positive simple function*

1 $\int_X cf d\mu = c \int_X f d\mu,$ *for only* $c \geq 0$

2 $\int_X (f+g) d\mu = \int_X f d\mu + \int_X g d\mu.$

Corollary 8.6. *If $f \geq g \geq 0$ are simple functions, then*

$$\int_X f d\mu \geq \int_X g d\mu. \tag{8}$$

Definition 8.7. If $f : X \to [0, +\infty)$ is measurable, then we define

$$\int_X f d\mu = \sup \left\{ \int_X g d\mu \ :\ f \geq g \geq 0,\ \ g \text{ is simple} \right\} \tag{9}$$

Remark. *This means that any measurable function can be approximated by simple functions.*

Properties 8.8. *Measurable functions like this have the following properties*

1 $\int_X cf d\mu = c \int_X f d\mu, \quad \forall c \geq 0.$

2 If $f \geq g \geq 0$, then $\int_X f d\mu \geq \int_X g d\mu$ for any measurable g, f.

3 If $f \geq 0$ is simple, then $\int_X f d\mu$ is the same value as obtained before.

To advance in measure theory we consider measurable functions

$$f : X \to [0, +\infty].$$

Measurability is understood w.r.t the $\sigma-$algebra $\mathscr{B}([0, +\infty])$ generated by $\mathscr{B}([0, +\infty))$ and $\{+\infty\}$. In other words, $A \subset [0, +\infty] \in B([0, +\infty])$ iff $A \cap [0, +\infty) \in \mathscr{B}([0, +\infty)$.

Remark. *Hence $f : X \to [0, +\infty]$ is measurable iff $f^{-1}(A)$ is measurable $\forall A \in \mathscr{B}([0, +\infty))$.*

Definition 8.9. For measurable functions $f_X \to [0, +\infty]$, we define

$$\int_X f d\mu = \sup \left\{ \int_x g d\mu \ : \ f \geq g \geq 0 \ : \ g \text{ is simple} \right\} \in [0, +\infty]. \tag{10}$$

Theorem 8.10. ***Monotone convergence theorem*** *Assume $(X, \mathscr{B}, \mu)$ is a measure space, $(f)_{n=1}^{\infty}$ is an increasing sequence of measurable positive functions $f_n : X \to [0, +\infty]$. Define $f(x) = \lim_{n\to\infty} f_n(x)$. Then f is measurable and*

$$\int_X f d\mu = \lim_{n\to\infty} \int_X f_n d\mu. \tag{11}$$

Theorem 8.11. *Assume $(X, \mathscr{B})$ is a measurable space and $f : X \to [0, +\infty]$ is measurable. Then there are simple functions g_n, s.t.*

$$0 \leq g_1 \leq g_2 \leq \dots \ , \ g_n(x) \to f(x), \ \ \forall x \in X.$$

Moreover, if f is bounded, we can choose g_n s.t. the convergence is uniform, that is,

$$\lim_{n\to\infty} \sup_{x\in X} |g_n(x) - f(x)| = 0. \tag{12}$$