UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Exam in: MAT3400/4400 — Linear Analysis with Applications

Day of examination: Tuesday, 31 May 2022

Examination hours: 15.00 – 19.00

This problem set consists of 6 pages.

Appendices: None.

Permitted aids: All.

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

All subproblems count equally. If there is a subproblem you cannot solve, you may still use the result in the sequel. All answers have to be substantiated.

Problem 1 (weight 10%)

Let $(X, \mathcal{A}, \mu)$ be a complete measure space. Suppose that f and g are functions from X to $\overline{\mathbb{R}}$ and set

$$N = \big\{x \in X \,:\, f(x) \neq g(x)\big\}.$$

Prove that if f is measurable and N is a null set, then g is also measurable.

Solution. Consider the interval $I = [-\infty, r)$ for a real number r. We need to prove that $g^{-1}(I)$ is in $\mathcal{A}$. We begin by decomposing

$$g^{-1}(I) = \big(g^{-1}(I) \cap N\big) \cup \big(g^{-1}(I) \cap N^{\mathrm{c}}\big) = \big(g^{-1}(I) \cap N\big) \cup \big(f^{-1}(I) \cap N^{\mathrm{c}}\big)$$

where the second equality holds since $f(x) = g(x)$ for x in N^{c}. Since f is measurable, it follows that $f^{-1}(I)$ is in $\mathcal{A}$. Since N is a null set and $(X, \mathcal{A}, \mu)$ is complete, we know that N is in $\mathcal{A}$. Since $g^{-1}(I) \cap N \subseteq N$ we see that $g^{-1}(I) \cap N$ is also a null set and hence included in $\mathcal{A}$ by the same reasoning as above. It follows that $g^{-1}(I)$ is in $\mathcal{A}$ and hence g is measurable.

Problem 2 (weight 20%)

(a) Suppose that $\{a_{j,k}\}_{k \geq 1}$ is an increasing sequence of nonnegative extended real numbers for each integer $j \geq 1$. Explain why

$$\lim_{k\to\infty} \sum_{j=1}^{\infty} a_{j,k} = \sum_{j=1}^{\infty} \lim_{k\to\infty} a_{j,k}.$$

Hint. Counting measure!

(Continued on page 2.)

Solution. Consider the measure space $(\mathbb{N}, 2^{\mathbb{N}}, \nu)$, where $2^{\mathbb{N}}$ is the σ-algebra of all subsets of $\mathbb{N}$ and where ν is the counting measure. Since $2^{\mathbb{N}}$ contains all subsets of $\mathbb{N}$, every function $f\colon \mathbb{N} \to \overline{\mathbb{R}}$ is measurable. If f is nonnegative, then

$$\int_{\mathbb{N}} f\,d\nu = \sum_{j=1}^{\infty} f(j).$$

For $k \geq 1$, define $f_k(j) = a_{j,k}$. Then $\{f_k\}_{k\geq 1}$ is an increasing sequence of nonnegative, measurable functions on $(\mathbb{N}, 2^{\mathbb{N}}, \nu)$. By the Monotone Convergence Theorem, we find that

$$\lim_{k\to\infty} \sum_{j=1}^{\infty} a_{j,k} = \lim_{k\to\infty} \int_{\mathbb{N}} f_k\,d\nu = \int_{\mathbb{N}} \lim_{k\to\infty} f_k\,d\nu = \sum_{j=1}^{\infty} \lim_{k\to\infty} a_{j,k}.$$

(b) Let $(X, \mathcal{A})$ be a measurable space and let $\{\mu_k\}_{k\geq 1}$ be a sequence of measures on $\mathcal{A}$ which enjoy the property that

$$\mu_1(A) \leq \mu_2(A) \leq \mu_3(A) \leq \cdots$$

for every A in $\mathcal{A}$. Show that the limit

$$\mu(A) = \lim_{k\to\infty} \mu_k(A)$$

defines a measure on $\mathcal{A}$.

Solution. Since the limit is increasing for each fixed A in $\mathcal{A}$, it is clear it will either converge to a finite nonnegative number or it will diverge to $+\infty$. In particular, the function $\mu\colon \mathcal{A} \to \overline{\mathbb{R}_+}$ is well-defined. Since $\mu_k(\emptyset) = 0$ for all $k \geq 1$, it follows that $\mu(\emptyset) = 0$. It remains to check countable additivity. Let $\{A_j\}_{j\geq 1}$ be a disjoint sequence of sets from $\mathcal{A}$. Then

$$\mu\left(\bigcup_{j=1}^{\infty} A_j\right) = \lim_{k\to\infty} \mu_k\left(\bigcup_{j=1}^{\infty} A_j\right) = \lim_{k\to\infty} \sum_{j=1}^{\infty} \mu_k(A_j).$$

Since the sequences defined by $a_{j,k} = \mu_k(A_j)$ satisfy the assumptions of Problem 1 (a), we conclude from this that

$$\mu\left(\bigcup_{j=1}^{\infty} A_j\right) = \sum_{j=1}^{\infty} \lim_{k\to\infty} \mu_k(A_j) = \sum_{j=1}^{\infty} \mu(A_j).$$

Problem 3 (weight 20%)

Let μ denote the Lebesgue measure on $\mathbb{R}$. For subsets A and B of $\mathbb{R}$, consider

$$A + B = \{a + b \,:\, a \in A \text{ and } b \in B\}.$$

(a) Suppose that A and B are subsets of $[0, 1]$ and that $A + B$ is Lebesgue measurable. Prove that $\mu(A + B) \leq 2$.

(Continued on page 3.)

Solution. Since A and B are subsets of $[0,1]$, it follows that $A+B$ is a subset of $[0,1]+[0,1]$. Since $[0,1]+[0,1]=[0,2]$, we find by monotonicity of measure that $\mu(A+B)\leq\mu([0,2])=2$.

(b) Find a subset C of $[0,1]$ such that $\mu(C)=0$ and $\mu(C+C)=2$.

Solution. Consider the Cantor set

$$C=\left\{x=\sum_{j=1}^{\infty}\frac{c_j}{3^j}\;:\;c_j\in\{0,2\}\right\}.$$

Recall from *AMoLM* Theorem 1.3.3 that $\mu(C)=0$. Since C is a subset of $[0,1]$, it follows that $C+C\subseteq[0,2]$ from the same argument as in Problem 2 (a). We will be done if we can establish that any z in $[0,2]$ can be written as $x+y$ for x and y in C. This would imply that $C+C=[0,2]$ and hence $\mu(C+C)=2$. It is equivalent to establish that any z in $[0,1]$ can be written as $(x+y)/2$ for x and y in C. If $\{c_j\}_{j\geq1}$ and $\{d_j\}_{j\geq1}$ are the ternary expansions of x and y, then

$$\frac{x+y}{2}=\frac{1}{2}\left(\sum_{j=1}^{\infty}\frac{c_j}{3^j}+\sum_{j=1}^{\infty}\frac{d_j}{3^j}\right)=\sum_{j=1}^{\infty}\frac{(c_j+d_j)/2}{3^j}.$$

It remains only to observe that the map $(c,d)\mapsto(c+d)/2$ is surjective from $\{0,2\}\times\{0,2\}$ to $\{0,1,2\}$.

Problem 4 (weight 10%)

Let H be a Hilbert space and let T be a linear operator on H with $\|T\|=1$. Prove that if $T(x)=x$ for some vector x in H, then $T^*(x)=x$ as well.

Solution. Exploiting the connection between the norm and the inner product and using the definition of the adjoint twice, we find that

$$\begin{aligned}\|T^*(x)-x\|^2&=\|T^*(x)\|^2-\langle T^*(x),x\rangle-\langle x,T^*(x)\rangle+\|x\|^2\\&=\|T^*(x)\|^2-\langle x,T(x)\rangle-\langle T(x),x\rangle+\|x\|^2\\&=\|T^*(x)\|^2-\|x\|^2.\end{aligned}$$

In the final equality we used twice the assumption that $T(x)=x$. From *ELA* Theorem 4.3.2 (iv) and the assumption that $\|T\|=1$, we find that $\|T^*\|=\|T\|=1$. Hence we conclude that

$$\|T^*(x)\|^2-\|x\|^2\leq\|T^*\|^2\|x\|^2-\|x\|^2=0$$

which shows that $\|T^*(x)-x\|^2=0$ and consequently that $T^*(x)=x$.

Problem 5 (weight 30%)

Consider the Hilbert space $H=L^2([0,1],\mu)$, where μ is the Lebesgue measure.

(Continued on page 4.)

(a) For $0 \le a \le 1$, consider the bounded linear functional on H defined by

$$\varphi_a(f) = \int_{[0,a)} f \, d\mu.$$

Find an element g_a in H such that $\varphi_a(f) = \langle f, g_a \rangle$ and compute $\|\varphi_a\|$.

> **Solution.** Writing
>
> $$\int_{[0,a)} f \, d\mu = \int_{[0,1]} \mathbf{1}_{[0,a)} f \, d\mu = \int_{[0,1]} f \, \overline{\mathbf{1}_{[0,a)}} \, d\mu = \langle f, \mathbf{1}_{[0,a)} \rangle,$$
>
> we see that $g_a = \mathbf{1}_{[0,a)}$. By the Riesz Representation Theorem (*ELA* Theorem 3.4.1), we then compute $\|\varphi_a\| = \|g_a\| = \sqrt{a}$.

(b) Let n be a fixed positive integer. Set $\mathcal{U}_n = \{u_{n,k}\}_{k=1}^n$, where

$$u_{n,k}(x) = \begin{cases} 1, & \text{if } \frac{k-1}{n} \le x < \frac{k}{n}, \\ 0, & \text{else.} \end{cases}$$

Consider the bounded linear operator $T_n \colon H \to H$ defined by

$$T_n f(x) = \varphi_{\frac{k}{n}}(f) \quad \text{for} \quad \tfrac{k-1}{n} \le x < \tfrac{k}{n} \quad \text{and} \quad T_n f(1) = 0.$$

Prove that $T_n(H) = \mathrm{Span}(\mathcal{U}_n)$. What is $\mathrm{rank}(T_n)$?

> **Solution.** Since $\mathcal{U}_n$ is a orthogonal set of n elements none of which are the 0 vector, it follows that $\mathrm{Span}(\mathcal{U}_n)$ is n-dimensional. If we could prove that $T_n(H) = \mathrm{Span}(\mathcal{U}_n)$, then it would follow that $\mathrm{rank}(T_n) = n$. Using Problem 5 (a) we write
>
> $$T_n f(x) = \sum_{j=1}^{n} \varphi_{\frac{k}{n}}(f) u_{n,k}(x) = \sum_{j=1}^{n} \langle f, g_{\frac{k}{n}} \rangle u_{n,k},$$
>
> from which it is clear that $T_n(H) \subseteq \mathrm{Span}(\mathcal{U}_n)$. It remains to establish the other inclusion. One way to prove this is to show that $\mathcal{U}_n$ is a subset $T_n(H)$. Since $T_n(H)$ is a subspace of H, this would imply that $\mathrm{Span}(\mathcal{U}_n) \subseteq T_n(H)$. Set $f_j = u_{n,j} - u_{n,j+1}$ for $1 \le j < n$ and $f_n = u_{n,n}$. Then
>
> $$\langle f_j, g_{\frac{k}{n}} \rangle = \begin{cases} \frac{1}{n}, & j = k, \\ 0, & j \ne k, \end{cases}$$
>
> which shows that $T_n(f_k) = \frac{1}{n} u_{n,k}$ and we are done.

(c) Let T be the linear operator on H defined by

$$Tf(x) = \int_{[0,x)} f \, d\mu.$$

Prove that T is compact.

(Continued on page 5.)

Solution. By *ELA* Corollary 4.2.3 and Problem 5 (b), it is sufficient to establish that $\|T_n - T\| \to 0$ as $n \to \infty$ to conclude that T is compact. If $(k-1)/n \leq x < k/n$, then

$$T_n f(x) = \varphi_{\frac{k}{n}}(f) = \int_{[0,k/n)} f \, d\mu.$$

For x in the same interval a small computation reveals that

$$T_n f(x) - Tf(x) = \int_{[0,k/n)} f \, d\mu - \int_{[0,x)} f \, d\mu = \int_{[0,1]} \mathbf{1}_{[x,k/n)} f \, d\mu.$$

Using the Cauchy–Schwarz inequality, we find that

$$|T_n f(x) - Tf(x)| = \left| \int_{[0,1]} \mathbf{1}_{[x,k/n)} f \, d\mu \right| \leq \sqrt{\frac{k}{n} - x} \, \|f\|$$

for every x on the interval $(k-1)/n \leq x < k/n$. Consequently,

$$\begin{aligned} \|(T_n - T)f\|^2 &= \sum_{k=1}^{n} \int_{[(k-1)/n,k/n)} |T_n f(x) - Tf(x)|^2 \, d\mu(x) \\ &\leq \sum_{k=1}^{n} \int_{[(k-1)/n,k/n)} \left(\frac{k}{n} - x\right) \|f\|^2 \, d\mu(x) = \frac{\|f\|^2}{2n} \end{aligned}$$

which implies that $\|T_n - T\| \leq \frac{1}{\sqrt{2n}}$ for $n \geq 1$. Hence T is compact.

It is also possible to prove directly that T is Hilbert–Schmidt and then appeal to *ELA* Proposition 4.2.8 to see that T is compact. To do this, we recall that $e_n(x) = \exp(2\pi i n x)$ for $n \in \mathbb{Z}$ is an orthonormal basis for $L^2([0,1])$. We then compute $Te_0(x) = x$ and

$$Te_n(x) = \frac{e_n(x) - 1}{2\pi i n}$$

for $n \neq 0$. It follows that

$$\|T\|_2^2 = \sum_{n \in \mathbb{Z}} \|Te_n\|^2 = \frac{1}{3} + 2 \sum_{n=1}^{\infty} \frac{2}{4\pi^2 n^2} = \frac{1}{2}.$$

Problem 6 (weight 10%)

Let T be a compact self-adjoint operator on a Hilbert space H and assume that $\ker T = 0$. Let $\{\lambda_j\}_{j \geq 1}$ denote the sequence of eigenvalues of T repeated according to their multiplicity. Define

$$a = \inf_{j \geq 1} \lambda_j \qquad \text{and} \qquad b = \sup_{j \geq 1} \lambda_j.$$

Let W_T denote the numerical range of T. Prove that $W_T \subseteq [a, b]$.

(Continued on page 6.)

Solution. Let $\mathcal{U} = \{u_j\}_{j\geq 1}$ be the orthonormal sequence of eigenvectors corresponding to the eigenvalues $\{\lambda_j\}_{j\geq 1}$. Since $\ker T = 0$ it follows from the spectral theorem that $\mathcal{U}$ is an orthonormal basis for H. The numerical range is defined as

$$W_T = \big\{\langle T(x), x\rangle \,:\, \|x\| = 1\big\}.$$

Since T is self-adjoint, the numerical range is a subset of the real line. By the spectral theorem and the fact that $\mathcal{U}$ is a basis for H, we write

$$T(x) = \sum_{j=1}^{\infty} \lambda_j \langle x, u_j\rangle u_j \qquad \text{and} \qquad x = \sum_{j=1}^{\infty} \langle x, u_j\rangle.$$

By orthogonality, we conclude that

$$\langle T(x), x\rangle = \sum_{j=1}^{\infty} \lambda_j |\langle x, u_j\rangle|^2.$$

It now follows from Parseval's formula (*ELA* Theorem 3.3.8 (c)) that

$$\langle T(x), x\rangle \leq b \sum_{j=1}^{\infty} |\langle x, u_j\rangle|^2 = b\|x\|^2 = b$$

if $\|x\| = 1$. That $\langle T(x), x\rangle \geq a$ if $\|x\| = 1$ is established analogously. From the definition of W_T it is now clear that $W_T \subseteq [a, b]$.

THE END