Solution to exam in MAT3400/4400, Linear analysis with applications.
Exam date Monday, December 6, 2010.

Problem 1. The n-th Fourier coefficient of $f(x)=x^2$ on $[-\pi,\pi]$ is (for $n\neq 0$)

$$\begin{aligned}
c_n(f) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} x^2 e^{-inx}dx \\
&= \frac{1}{2\pi}\Big[-\frac{x^2e^{-inx}}{in}\Big]_{-\pi}^{\pi} + \frac{1}{\pi in}\int_{-\pi}^{\pi} xe^{-inx}dx = \frac{1}{\pi in}\int_{-\pi}^{\pi} xe^{-inx}dx \\
&= \frac{1}{\pi in}\Big[-\frac{x}{in}e^{-inx}\Big]_{-\pi}^{\pi} + \frac{1}{\pi(in)^2}\int_{-\pi}^{\pi} e^{-inx}dx \\
&= \frac{1}{\pi n^2}(\pi e^{-in\pi}+\pi e^{in\pi}) = 2\frac{(-1)^n}{n^2}.
\end{aligned}$$

We have $c_0(f)=\frac{1}{2\pi}\int_{-\pi}^{\pi}x^2dx=\frac{\pi^2}{3}$. The Fourier series is

$$\begin{aligned}
\sum_{n\in\mathbb{Z}} c_n(f)e^{inx} &= c_0(f)+\sum_{n=1}^{\infty}c_n(f)e^{inx}+\sum_{n=1}^{\infty}c_{-n}(f)e^{-inx} \\
&= \frac{\pi^2}{3}+\sum_{n=1}^{\infty}\big(2\frac{(-1)^n}{n^2}e^{inx}+2\frac{(-1)^n}{n^2}e^{-inx}\big) \\
&= \frac{\pi^2}{3}+4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx.
\end{aligned}$$

If we let $\bar{f}$ be the extension of f to a 2π periodic function on $\mathbb{R}$, then at every $x_0\in(2k+1)\mathbb{Z}$ for $k\in\mathbb{Z}$ we have $\bar{f}(x_0+)=\bar{f}(x_0-)=\pi^2$, so $\bar{f}$ is continuous on $\mathbb{R}$. Note that f has a derivative at all points in $(-\pi,\pi)$. At $x_0=\pi$ we compute that $f'(x_0+)=2\pi$ and $f'(x_0-)=-2\pi$. We conclude from the corollary about pointwise convergence that the Fourier series is pointwise convergent at every $x\in\mathbb{R}$ with sum $f(x)$. In fact, since f' is piecewise continuous on $(-\pi,\pi)$, it follows from the theorem on uniform convergence that the Fourier series of f is uniformly convergent on $[-\pi,\pi]$ with limit f.

For (c), let $x=0$ in (b). Since $f(0)=0$, the pointwise convergence of the Fourier series of f implies that $\frac{\pi^2}{3}+4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}=0$. Then

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n^2}=\frac{\pi^2}{12}.$$

Problem 2.

Let $a<b$ be real numbers and κ a continuous function on $[a,b]\times[a,b]$ with values in $\mathbb{C}$ such that $\kappa(x,y)=\overline{\kappa(y,x)}$ for all x,y. Let H be the

Hilbert space $L^2([a,b],d\lambda)$, where λ is Lebesgue measure and the inner product is given by $\langle f,g\rangle = \int_a^b \overline{f(y)}g(y)dy$. Let K denote the self-adjoint, compact integral operator

$$(Kf)(x) = \int_a^b \kappa(x,y)f(y)dy.$$

We know from the spectral theorem for compact self-adjoint operators that the eigenvalues of K are real and (if they form an infinite set) form a sequence $\{\lambda_j\}_{j\geq 1}$ converging to zero. Moreover, there is an orthonormal sequence $\{e_j\}_{j\geq 1}$ where e_j is eigenvector corresponding to λ_j, and $Kf = \sum_j \lambda_j\langle e_j, f\rangle e_j$ for all $f\in H$. Let $M = \sup\{|\kappa(x,y)| \mid x,y\in(a,b)\}$.

For (a), note that for any $x\in[a,b]$ we have

$$(Ke_j)(x) = \int_a^b \kappa(x,y)e_j(y)dy = \int_a^b \overline{\kappa(y,x)}e_j(y)dy = \langle \kappa(\cdot,x), e_j\rangle.$$

But $\kappa(\cdot,x)\in L^2[a,b]$ because $\|\kappa(\cdot,x)\|^2 = \int_a^b |\kappa(y,x)|^2dy \leq M^2(b-a)$. Bessel's inequality says that $\sum_{j\geq 1}|\langle\kappa(\cdot,x),e_j\rangle|^2 \leq \|\kappa(\cdot,x)\|^2$. Therefore

$$\sum_{j=1}^{\infty}|(Ke_j)(x)|^2 \leq \|\kappa(\cdot,x)\|^2 \leq M^2(b-a).$$

Let $f_n(x) = \sum_{j=1}^n |(Ke_j)(x)|^2$. Then $\{f_n\}$ is a non-decreasing sequence of non-negative measurable functions. By the monotone convergence theorem,

$$\lim_{n\to\infty}\int_a^b f_n(x)dx = \int_a^b (\lim_n f_n(x))dx.$$

Now $\int_a^b f_n(x)dx = \sum_{j=1}^n \int_a^b |\lambda_j|^2|e_j(x)|^2dx = \sum_{j=1}^n \lambda_j^2\|e_j\|^2 = \sum_{j=1}^n \lambda_j^2$. Since $\lim_n f_n(x) \leq M^2(b-a)$, we conclude that $\sum_{j=1}^{\infty}\lambda_j^2 \leq M^2(b-a)^2$.

Problem 3. We have $Lu = -u^{''}$ on $\mathcal{D}(L) = \{f\in C^2[0,1] \mid f(0) = 0, f'(1) = 0\} \subset L^2[0,1]$. A number α is eigenvalue for L if $Lu = \alpha u$, equivalently $u^{''} + \alpha u = 0$. The characteristic equation is $r^2+\alpha = 0$. The discriminant is -4α.

Case **1**: $\alpha < 0$. Write $a = \sqrt{-\alpha} > 0$. Then there are two real solutions a and $-a$. The equation $u^{''}+\alpha u = 0$ has solutions of the general form $u(x) = Ae^{ax}+Be^{-ax}$. To have $u\in\mathcal{D}(L)$ implies $A+B = 0$ and $Ae^a - Be^{-a} = 0$, from which we get $A = B = 0$, so u can't be an eigenvector. This shows no $\alpha < 0$ is an eigenvalue.

Case **2**: $\alpha = 0$. Then $r = 0$, so the solutions are of form $u(x) = A+Bx$, and $u\in\mathcal{D}(L)$ is only possible for $u = 0$.

Case **3**: $\alpha > 0$. Now the solutions of $r^2+\alpha = 0$ are $i\sqrt{\alpha}$ and $-i\sqrt{\alpha}$, so $u(x) = A\cos\sqrt{\alpha}x + B\sin\sqrt{\alpha}x$. Then $u\in\mathcal{D}(L)$ gives $A = 0$, so $u(x) = B\sin\sqrt{\alpha}x$. We look for $u(x)$ non-trivial, so $u'(1) = 0$ gives $\cos\sqrt{\alpha} = 0$. Thus $\sqrt{\alpha} = \frac{\pi}{2}, \frac{3\pi}{2}, \ldots$. We obtain $\alpha_n = (n-\frac{1}{2})^2\pi^2$,

$n = 1, 2, \dots$ for the eigenvalues of L. A normalized eigenvector for α_n is $u_n(x) = \sqrt{2}\sin((n - \frac{1}{2})\pi x)$.

Since 0 is not an eigenvalue, it follows that $\ker L = 0$. So L is injective.

Problem 4. Suppose K is a compact, self-adjoint operator on H such that $\ker(K) = \{0\}$.

We know from the spectral theorem that if $\{\lambda_n\}_{n\geq 1}$ are the eigenvalues of K, there is an orthonormal sequence $\{e_n\}_{n\geq 1}$ where e_n is eigenvector corresponding to λ_n, and $x = \sum_j \langle e_j, x\rangle e_j$ for all $x \in H$ (since $\ker(K) = \{0\}$).

Since $\ker(K) = \{0\}$, the value 0 is not an eigenvalue, so $\lambda_n \neq 0$ for all n. Then we can let $A_n x = \sum_{m=1}^{n} \lambda_m^{-1}\langle e_m, x\rangle e_m$. Then A_n is of finite rank because it has range equal to $\operatorname{span}\{e_m \mid m = 1, \dots, n\}$. We have

$$A_n K x = \sum_{m=1}^{n} \lambda_m^{-1}\langle e_m, Kx\rangle e_m = \sum_{m=1}^{n} \lambda_m^{-1}\lambda_m\langle e_m, x\rangle e_m,$$

which converges to x when $n \to \infty$. Similarly,

$$KA_n x = \sum_{j=1}^{\infty} \lambda_j \langle e_j, A_n x\rangle e_j = \sum_{m=1}^{n} \lambda_m \lambda_m^{-1}\langle e_m, x\rangle e_m,$$

which converges to x as $n \to \infty$.