MAT4400: Notes on Linear analysis

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3 σ-Algebras

Definition 3.0.1 (Borel)**.** The σ-algebra $\sigma(\mathcal{O})$ generated by the open sets $\mathcal{O} = \mathcal{O}_{\mathbb{R}^n}$ of $\mathbb{R}^n$ is called **Borel σ-algebra**, and its members are called **Borel sets** or **Borel measurable sets**.

5 Uniqueness of Measures

Lemma 5.1. *A Dynkin system D is a σ-algebra iff it is stable under finite intersections, i.e. $A, B \in D \Rightarrow A \cap B \in D$.*

Theorem 5.2 (Dynkin)**.** *Assume X is a set, S is a collection of subsets of X closed under finite intersections, that is, if $A, B \in S \Rightarrow A \cap B \in S$. Then $D(S) = \sigma(S)$.*

Theorem 5.3 (uniqueness of measures)**.** *Let (X, B) be a measurable space, and $S \subset P(X)$ be the generator of B, i.e. $B = \sigma(S)$. If S satisfies the following conditions:*

1. *S is stable under finite intersections ($\cap$-stable), i.e. $A, C \in S \Rightarrow A \cap C \in S$.*
2. *There exists an exhausting sequence $(G_n)_{N \in \mathbb{N}} \subset$ with $G_N \uparrow X$. Assume also that there are two measures μ, ν satisfying:*
3. *$\mu(A) = \nu(A),\ \forall A \in S$.*
4. *$\mu(G_n) = \nu(G_n) < \infty$.*

Then $\mu = \nu$.

6 Existence of Measures

Theorem 6.1 (Carathéodory)**.** *Let $S \subset P(X)$ be a semi-ring and $\mu : S \to [0, \infty)$ a pre-measure. Then μ has an extension to a measure μ^* on $\sigma(S)$, i.e. that $\mu(s) = \mu^*(s),\ \forall\ s \in \sigma(S)$.*

Also, if S contains an exhausting sequence, $S_n \uparrow X$, s.t. $\mu(S_n) < \infty$, then the extension is unique.

7 Measurable Mappings

We consider maps $T : X \to X'$ between two measurable spaces $(X, \mathcal{A})$ and $(X', \mathcal{A}')$ which respects the measurable structurs, the σ-algbras on X and X'. These maps are useful as we can transport a measure μ, defined on $(X, \mathcal{A})$, to $(X', \mathcal{A}')$.

Definition 7.0.1. Let $(X, \mathcal{A})$, $(X', \mathcal{A}')$ b measurable spaces. A map $T : X \to X'$ is called $\mathcal{A}/\mathcal{A}'$-measurable if the pre-imag of every measurable set is a measurable set:

$$T^{-1}(A') \in \mathcal{A}, \quad \forall A' \in \mathcal{A}'. \tag{1}$$

- A $\mathcal{B}(\mathbb{R}^n)/\mathcal{B}(\mathbb{R}^m)$ measurable map is often called a Borel map.
- The notation $T : (X, \mathcal{A}) \to (X', \mathcal{A}')$ is often used to indicate measurability of the map T.

Lemma 7.1. *Let $(X, \mathcal{A})$, $(x', \mathcal{A}')$ be measurable spaces and let $\mathcal{A}' = \sigma(\mathcal{G}')$. Then $T : X \to X'$ is $\mathcal{A}/\mathcal{A}'$-measurable iff $T^{-1}(\mathcal{G}') \subset \mathcal{A}$, i.e. if*

$$T^{-1}(G') \in \mathcal{A}, \ \forall G' \in \mathcal{G}'. \tag{2}$$

Theorem 7.2. *Let $(X_i, \mathcal{A}_i)$, $i = 1, 2, 3$, be measurable spaces and $T : X_1 \to X_2$, $S : X_2 \to X_3$ be $\mathcal{A}_1/\mathcal{A}_2$ and $\mathcal{A}_2/\mathcal{A}_3$-measurable maps respectivly. Then $S \circ T : X_1 \to X_3$ is $\mathcal{A}_1/\mathcal{A}_3$-measurable.*

Definition 7.2.1. (and lemma) Let $(T_i)_{i\in I}$, $T_I : X \to X_i$, be arbitrarily many mappings from the same space X into measurable spaces $(X_i, \mathcal{A}_i)$. The smallest σ-algebra on X that makes all T_i simultanously measurable is

$$\sigma(T_i : i \in I) := \sigma\left(\bigcup_{i\in I} T_i^{-1}(\mathcal{A}_i)\right) \tag{3}$$

Theorem 7.3. *Let $(X, \mathcal{A})$, $(X', \mathcal{A}')$ be measurable spaces and $T : X \to X'$ be an $\mathcal{A}/\mathcal{A}'$-measurable map. For every measurable μ on $(X, \mathcal{A})$,*

$$\mu'(A') := \mu(T^{-1}(A')), \quad A' \in \mathcal{A}', \tag{4}$$

defines a measure on $(X', \mathcal{A}')$.

Definition 7.3.1. The measure $\mu'(\cdot)$ in the above theorem is called the push forward or image measure of μ under T and it is denoted as $T(\mu)(\cdot)$, $T_{*\mu}(\cdot)$ or $\mu \circ T^{-1}(\cdot)$.

Theorem 7.4. *If $T \in \mathbb{R}^{n\times n}$ is an orthogonal matrix, then $\lambda^n = T(\lambda^n)$.*

Theorem 7.5. *Let $S \in \mathbb{R}^{n\times n}$ be an invertible matrix. Then*

$$S(\lambda^n) = |\det s^{-1}|\lambda^n = |\det S|^{-1}\lambda^n. \tag{5}$$

Corollary 7.5.1. *Lebesgue measure is invariant under motions: $\lambda^n = M(\lambda^n)$ for all motions M in $\mathbb{R}^n$. In particular, congruent sets have the same measure. Two sets of points are called congruent if, and only if, one can be transformed into the other by an isometry*

8 Measurable Functions

A *measurable function* is a measurable map $u : X \to \mathbb{R}$ from some measurable space $(X, \mathscr{A})$ to $(\mathbb{R}, \mathscr{B}(\mathbb{R}^1))$. They play central roles in the theory of integration.

We recall that $u : X \to \mathbb{R}$ is $\mathscr{A}/\mathscr{B}(\mathbb{R}^1)$-measurable if

$$u^{-1}(B) \in \mathscr{A}, \ \ \forall B \in \mathscr{B}(\mathbb{R}^1). \tag{6}$$

Moreover from a lemma from chapter 7, we actually only need to show that

$$u^{-1}(G) \in \mathscr{A}, \ \ \forall G \in \mathcal{G} \text{ where } \mathcal{G} \text{ generates } \mathscr{B}(\mathbb{R}^1). \tag{7}$$

10 Chapter 10

10.1 Integration of Complex Functions

Assume $(X, \mathfrak{B}, \mu)$ is a measure space.

Definition 10.0.1. A measurable function $f : X \to \mathbb{C}$ is called integrable (or μ-integrable) if

$$\int_X |f| d\mu < \infty.$$

Denote by $\mathcal{L}^1(X, \mathfrak{B}, d\mu)$, $\mathcal{L}^1(X, d\mu)$ or $\mathcal{L}^1_{\mathbb{C}}$ the set of integrable functions. This is also a vector space over $\mathbb{C}$, since

$$|f + g| \leq |f| + |g|, \ |cf| = |c||f| \ (c \in \mathbb{C}),$$
$$\text{the other axioms are trivial.}$$

This vector space is spanned by positive functions, since

$$f = \mathrm{Re}(f)_+ - \mathrm{Re}(f)_- + i\mathrm{Im}(f)_+ - i\mathrm{Im}(f)_-,$$

where for a function h we let

$$h_+ = \max\{h, 0\}, \ h_- = -\min\{h, 0\},$$

and if $f \in \mathcal{L}^1(X, d\mu)$, then

$$(\mathrm{Re}(f))_\pm, (\mathrm{Im}(f))_\pm \in \mathcal{L}^1(X, d\mu),$$

as

$$|(\mathrm{Re}(f))_\pm|, |(\mathrm{Im}(f))_\pm| \leq |f|.$$

Proposition 1. *The integral extends uniquely from the positive integrable functions to a linear function (functional?) $\mathcal{L}^1(X, d\mu) \to \mathbb{C}$, that is, to a map s.t.*

$$\int_X (f+g)d\mu = \int_X f d\mu + \int_X g d\mu,$$

$$\int_X cf d\mu = c \int_X f d\mu, \ c \in \mathbb{C}.$$