

MAT4400: Notes on Linear analysis

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3 σ -Algebras

Definition 3.0.1 (Borel). The σ -algebra $\sigma(\mathcal{O})$ generated by the open sets $\mathcal{O} = \mathcal{O}_{\mathbb{R}^n}$ of \mathbb{R}^n is called **Borel σ -algebra**, and its members are called **Borel sets** or **Borel measurable sets**.

5 Uniqueness of Measures

Lemma 5.1. A Dynkin system D is a σ -algebra iff it is stable under finite intersections, i.e. $A, B \in D \Rightarrow A \cap B \in D$.

Theorem 5.2 (Dynkin). Assume X is a set, S is a collection of subsets of X closed under finite intersections, that is, if $A, B \in S \Rightarrow A \cap B \in S$. Then $D(S) = \sigma(S)$.

Theorem 5.3 (uniqueness of measures). Let (X, B) be a measurable space, and $S \subset P(X)$ be the generator of B , i.e. $B = \sigma(S)$. If S satisfies the following conditions:

1. S is stable under finite intersections (\cap -stable), i.e. $A, C \in S \Rightarrow A \cap C \in S$.
2. There exists an exhausting sequence $(G_n)_{n \in \mathbb{N}} \subset S$ with $G_n \uparrow X$. Assume also that there are two measures μ, ν satisfying:
3. $\mu(A) = \nu(A), \forall A \in S$.
4. $\mu(G_n) = \nu(G_n) < \infty$.

Then $\mu = \nu$.

6 Existence of Measures

Theorem 6.1 (Carathéodory). Let $S \subset P(X)$ be a semi-ring and $\mu : S \rightarrow [0, \infty)$ a pre-measure. Then μ has an extension to a measure μ^* on $\sigma(S)$, i.e. that $\mu(s) = \mu^*(s), \forall s \in \sigma(S)$.

Also, if S contains an exhausting sequence, $S_n \uparrow X$, s.t. $\mu(S_n) < \infty$, then the extension is unique.

7 Measurable Mappings

We consider maps $T : X \rightarrow X'$ between two measurable spaces (X, \mathcal{A}) and (X', \mathcal{A}') which respects the measurable structures, the σ -algebras on X and X' . These maps are useful as we can transport a measure μ , defined on (X, \mathcal{A}) , to (X', \mathcal{A}') .

Definition 7.0.1. Let (X, \mathcal{A}) , (X', \mathcal{A}') be measurable spaces. A map $T : X \rightarrow X'$ is called \mathcal{A}/\mathcal{A}' -measurable if the pre-image of every measurable set is a measurable set:

$$T^{-1}(A') \in \mathcal{A}, \quad \forall A' \in \mathcal{A}'. \quad (1)$$

- A $\mathcal{B}(\mathbb{R}^n)/\mathcal{B}(\mathbb{R}^m)$ measurable map is often called a Borel map.
- The notation $T : (X, \mathcal{A}) \rightarrow (X', \mathcal{A}')$ is often used to indicate measurability of the map T .

Lemma 7.1. Let (X, \mathcal{A}) , (X', \mathcal{A}') be measurable spaces and let $\mathcal{A}' = \sigma(\mathcal{G}')$. Then $T : X \rightarrow X'$ is \mathcal{A}/\mathcal{A}' -measurable iff $T^{-1}(\mathcal{G}') \subset \mathcal{A}$, i.e. if

$$T^{-1}(G') \in \mathcal{A}, \quad \forall G' \in \mathcal{G}'. \quad (2)$$

Theorem 7.2. Let (X_i, \mathcal{A}_i) , $i = 1, 2, 3$, be measurable spaces and $T : X_1 \rightarrow X_2$, $S : X_2 \rightarrow X_3$ be $\mathcal{A}_1/\mathcal{A}_2$ and $\mathcal{A}_2/\mathcal{A}_3$ -measurable maps respectively. Then $S \circ T : X_1 \rightarrow X_3$ is $\mathcal{A}_1/\mathcal{A}_3$ -measurable.

Definition 7.2.1. (and lemma) Let $(T_i)_{i \in I}$, $T_i : X \rightarrow X_i$, be arbitrarily many mappings from the same space X into measurable spaces (X_i, \mathcal{A}_i) . The smallest σ -algebra on X that makes all T_i simultaneously measurable is

$$\sigma(T_i : i \in I) := \sigma \left(\bigcup_{i \in I} T_i^{-1}(\mathcal{A}_i) \right) \quad (3)$$

Theorem 7.3. Let (X, \mathcal{A}) , (X', \mathcal{A}') be measurable spaces and $T : X \rightarrow X'$ be an \mathcal{A}/\mathcal{A}' -measurable map. For every measurable μ on (X, \mathcal{A}) ,

$$\mu'(A') := \mu(T^{-1}(A')), \quad A' \in \mathcal{A}', \quad (4)$$

defines a measure on (X', \mathcal{A}') .

Definition 7.3.1. The measure $\mu'(\cdot)$ in the above theorem is called the push forward or image measure of μ under T and it is denoted as $T(\mu)(\cdot)$, $T_{*\mu}(\cdot)$ or $\mu \circ T^{-1}(\cdot)$.

Theorem 7.4. If $T \in \mathbb{R}^{n \times n}$ is an orthogonal matrix, then $\lambda^n = T(\lambda^n)$.

Theorem 7.5. Let $S \in \mathbb{R}^{n \times n}$ be an invertible matrix. Then

$$S(\lambda^n) = |\det S|^{-1} \lambda^n = |\det S|^{-1} \lambda^n. \quad (5)$$

Corollary 7.5.1. Lebesgue measure is invariant under motions: $\lambda^n = M(\lambda^n)$ for all motions M in \mathbb{R}^n . In particular, congruent sets have the same measure. Two sets of points are called congruent if, and only if, one can be transformed into the other by an isometry

8 Measurable Functions

A *measurable function* is a measurable map $u : X \rightarrow \mathbb{R}$ from some measurable space (X, \mathcal{A}) to $(\mathbb{R}, \mathcal{B}(\mathbb{R}^1))$. They play central roles in the theory of integration.

We recall that $u : X \rightarrow \mathbb{R}$ is $\mathcal{A}/\mathcal{B}(\mathbb{R}^1)$ -measurable if

$$u^{-1}(B) \in \mathcal{A}, \quad \forall B \in \mathcal{B}(\mathbb{R}^1). \quad (6)$$

Moreover from a lemma from chapter 7, we actually only need to show that

$$u^{-1}(G) \in \mathcal{A}, \quad \forall G \in \mathcal{G} \text{ where } \mathcal{G} \text{ generates } \mathcal{B}(\mathbb{R}^1). \quad (7)$$