UNIVERSITY OF OSLO

Faculty of Mathematics and Natural Sciences

Examination in: MAT3400/4400 — Linear analysis with applications

Day of examination: Thursday, June 14, 2018

Examination hours: 14.30 – 18.30

This problem set consists of 6 pages.

Appendices: None.

Permitted aids: None.

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

With solutions !

Problem 1 (weight 15 points)

Let $(\Omega, \mathcal{A}, \mu)$ be a measure space. For each integer $k \in \mathbb{Z}$, let $A_k \in \mathcal{A}$ and assume that $A_k \cap A_l = \varnothing$ whenever $k \neq l$. Set $A := \bigcup_{k\in\mathbb{Z}} A_k \in \mathcal{A}$.

Assume $f : \Omega \to \mathbb{C}$ is $\mathcal{A}$-measurable and integrable over A with respect to μ.

1a (weight 5 points)

Explain why f is integrable over A_k with respect to μ for each $k \in \mathbb{Z}$.

Solution: The assumption implies that $\int |f|\chi_A \, d\mu < \infty$. For each $k \in \mathbb{Z}$ we have $A_k \subset A$, so $|f|\chi_{A_k} \leq |f|\chi_A$, and we get

$$\int |f|\chi_{A_k} \, d\mu \leq \int |f|\chi_A \, d\mu < \infty \,,$$

thus showing that f is integrable over A_k with respect to μ.

1b (weight 10 points)

Show that

$$\int_A f \, d\mu = \lim_{n\to\infty} \sum_{k=-n}^{n} \int_{A_k} f \, d\mu \,.$$

Solution: For each $n \in \mathbb{N}$, set $f_n = \sum_{k=-n}^{n} f\chi_{A_k}$ and $g = |f|\chi_A$. Then all f_n's, and also g, are $\mathcal{A}$-measurable. Moreover, since A is the disjoint union of the A_k's, we have that $\sum_{k=-n}^{n} \chi_{A_k} \leq \chi_A$ and $|f_n| \leq g$ for all $n \in \mathbb{N}$, and also that $\lim_{n\to\infty} f_n = f\chi_A$ (pointwise). As g is μ-integrable (from the

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assumption), Lebesgue's Dominated Convergence Theorem gives that each f_n is μ-integrable, and

$$\int_A f\,d\mu = \int f\chi_A\,d\mu = \lim_{n\to\infty}\int f_n\,d\mu = \lim_{n\to\infty}\sum_{k=-n}^{n}\int f\chi_{A_k}\,d\mu\,.$$

Problem 2 (weight 40 points)

Let $\mathcal{M}$ denote the σ-algebra of all Lebesgue measurable subsets of $\mathbb{R}$ and let λ denote the Lebesgue measure on $(\mathbb{R},\mathcal{M})$. For any $E\subset\mathbb{R}$ and $a\in\mathbb{R}$, set

$$E-a := \big\{x-a \mid x\in E\big\}\,.$$

We recall that for all $E\in\mathcal{M}$ and $a\in\mathbb{R}$, we have $E-a\in\mathcal{M}$ and $\lambda(E-a)=\lambda(E)$.

2a (weight 15 points)

Let $a\in\mathbb{R}$ and $E\in\mathcal{M}$. Show that if $f:\mathbb{R}\to[0,\infty)$ is $\mathcal{M}$-measurable, then

$$\int_E f(x)\,d\lambda(x) = \int_{E-a} f(x+a)\,d\lambda(x)\,.$$

Hint: Check this first when $f=\chi_A$ for some $A\in\mathcal{M}$.

Solution: If $f=\chi_A$ for some $A\in\mathcal{A}$, we get

$$\int_E f(x)\,d\lambda(x) = \int \chi_{E\cap A}\,d\lambda = \lambda(E\cap A), \quad \text{while}$$

$$\begin{aligned}\int_{E-a} f(x+a)\,d\lambda(x) &= \int_{E-a}\chi_{A-a}\,d\lambda = \lambda\big((E-a)\cap(A-a)\big)\\ &= \lambda\big((E\cap A)-a\big) = \lambda(E\cap A)\,,\end{aligned}$$

which shows that the desired inequality holds in this case. By linearity of the integral, we get that it also holds for all elements in the set $\mathcal{S}$ of all nonnegative $\mathcal{A}$-measurable simple functions on Ω.

Now, if $f:\mathbb{R}\to[0,\infty)$ is $\mathcal{M}$-measurable, we can pick a nondecreasing sequence $\{g_n\}_{n\in\mathbb{N}}$ in $\mathcal{S}$ such that $g_n\to f$ pointwise as $n\to\infty$. We may then apply the Monoton Convergence Theorem, first to the sequence $\{g_n\chi_E\}_{n\in\mathbb{N}}$, then also to the sequence $\{h_n\chi_{E-a}\}_{n\in\mathbb{N}}$, where $h_n(x):=g_n(x+a)$, to get that

$$\begin{aligned}\int_E f(x)\,d\lambda(x) &= \int f\chi_E\,d\mu = \lim_{n\to\infty}\int g_n\chi_E\,d\mu = \lim_{n\to\infty}\int_E g_n\,d\mu\\ &= \lim_{n\to\infty}\int_{E-a} g_n(x+a)\,d\lambda(x) = \lim_{n\to\infty}\int h_n(x)\chi_{E-a}(x)\,d\lambda(x)\\ &= \int f(x+a)\chi_{E-a}(x)\,d\lambda(x) = \int_{E-a} f(x+a)\,d\lambda(x)\end{aligned}$$

as desired.

Next, we recall that a function $g:\mathbb{R}\to\mathbb{C}$ is called *periodic* if there exists some $a>0$ such that $g(x+a)=g(x)$ for all $x\in\mathbb{R}$.

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2b (weight 10 points)

Assume $g : \mathbb{R} \to \mathbb{C}$ is $\mathcal{M}$-measurable and periodic.

Prove that g is Lebesgue integrable if and only if $g = 0$ λ-a-e.

Solution: Let $a > 0$ denote a period for g and assume g is Lebesgue integrable, i.e.,

$$M := \int_{\mathbb{R}} |g| \, d\lambda < \infty \,.$$

For each $k \in \mathbb{Z}$, set $A_k := \big[ka, (k+1)a\big)$. Then each A_k is Lebesgue measurable, the A_k's are pairwise disjoint and $\mathbb{R} = \bigcup_{k\in\mathbb{Z}} A_k$. Moreover, using 2a and that $g(x + a) = g(x)$ for all $x \in \mathbb{R}$, we get that

$$\int_{A_k} |g| \, d\lambda = \int_{A_k - a} |g(x + a)| \, d\lambda(x) = \int_{A_{k-1}} |g| \, d\lambda$$

for every $k \in \mathbb{Z}$. This implies that

$$\int_{A_k} |g| \, d\lambda = \int_{A_0} |g| \, d\lambda = \int_{[0,a)} |g| \, d\lambda$$

for every $k \in \mathbb{Z}$. Applying now 1b, we get that

$$M = \int_{\mathbb{R}} |g| \, d\lambda = \lim_{n\to\infty} \sum_{k=-n}^{n} \int_{A_k} |g| \, d\mu = \lim_{n\to\infty} \; (2n + 1) \int_{[0,a)} |g| \, d\lambda \,.$$

Since $M < \infty$, this is only possible when $\int_{[0,a)} |g| \, d\lambda = 0$, in which case we get that $M = 0$. This implies that $|g| = 0$ λ-a.e., i.e. $g = 0$ λ-a.e. The converse implication is trivial.

2c (weight 15 points)

Let $h : \mathbb{R} \to \mathbb{R}$ be given by

$$h(x) = \frac{2 + \sin(x)}{\sqrt{x^2 + 1}} \quad \text{for all } x \in \mathbb{R}.$$

Determine for which $p \in [1, \infty)$ we have that $h \in \mathcal{L}^p(\mathbb{R}, \mathcal{M}, \lambda)$.

Solution: Assume $1 < p < \infty$. Clearly, if $x \neq 0$, then $|h(x)| \leq 3\,|x|^{-1}$. Thus we get

$$\int_{\mathbb{R}} |h|^p \, d\lambda \;\leq\; 3^p \int_{(-\infty,-1]} \frac{1}{|x|^p} \, d\lambda(x) \;+\; \int_{[-1,1]} |h|^p \, d\lambda \;+\; 3^p \int_{[1,\infty)} \frac{1}{|x|^p} \, d\lambda(x) \,.$$

Since $\int_{[-1,1]} |h|^p \, d\lambda = \int_{-1}^{1} |h(x)|^p \, dx < \infty$ (as $|h|^p$ is continuous),

$$\int_{(-\infty,-1]} \frac{1}{|x|^p} \, d\lambda(x) = \lim_{n\to\infty} \int_{-n}^{-1} \frac{1}{(-x)^p} \, dx = \lim_{n\to\infty} \int_{1}^{n} \frac{1}{x^p} \, dx = \frac{1}{p-1} \,,$$

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and, similarly, $\int_{[1,\infty)} |x|^{-p}\, d\lambda(x) = 1/(p-1)$, we get that $\int_{\mathbb{R}} |h|^p\, d\lambda < \infty$. Hence, $h \in \mathcal{L}^p(\mathbb{R}, \mathcal{M}, \lambda)$ in this case.

On the other hand, we have $2 + \sin(x) \geq 1$ for all x and $\sqrt{x^2+1} \leq \sqrt{2}\, x$ when $x \geq 1$. Thus $h(x) \geq (\sqrt{2}\, x)^{-1}$ when $x \geq 1$, so

$$\begin{aligned}\int_{\mathbb{R}} |h|\, d\lambda \geq \int_{[1,\infty)} |h|\, d\lambda &\geq \frac{1}{\sqrt{2}} \int_{[1,\infty)} \frac{1}{x}\, d\lambda(x) \\ &= \frac{1}{\sqrt{2}} \lim_{n\to\infty} \int_1^n \frac{1}{x}\, dx = \frac{1}{\sqrt{2}} \lim_{n\to\infty} \ln(n) = \infty .\end{aligned}$$

Hence, $h \notin \mathcal{L}^1(\mathbb{R}, \mathcal{M}, \lambda)$.

Problem 3 (weight 10 points)

Let $\mathbb{F}$ denote $\mathbb{R}$ or $\mathbb{C}$, and let X be a normed space over $\mathbb{F}$. Let $\varphi \in \mathcal{B}(X, \mathbb{F})$ and assume that $\varphi \neq 0$.

Show that there exists some $y \in X \setminus \{0\}$ such that

$$X = \ker(\varphi) \oplus N ,$$

where $N := \mathrm{Span}\{y\}$.

Solution: Since $\varphi \neq 0$, there exists $y \in X \setminus \{0\}$ such that $\varphi(y) \neq 0$. Let $x \in X$. Then, by linearity of φ, we have

$$\varphi\Big(x - \frac{\varphi(x)}{\varphi(y)}\, y\Big) = \varphi(x) - \frac{\varphi(x)}{\varphi(y)}\, \varphi(y) = 0 .$$

Thus, $m := x - \frac{\varphi(x)}{\varphi(y)}\, y \in \ker(\varphi)$, $z := \frac{\varphi(x)}{\varphi(y)}\, y \in \mathrm{Span}\{y\}$, and $x = m + z$. Hence,

$$X = \ker(\varphi) + \mathrm{Span}\{y\} .$$

Moreover, if $v \in \ker(\varphi) \cap \mathrm{Span}\{y\}$, then we have $\varphi(v) = 0$ and $v = \lambda\, y$ for some $\lambda \in \mathbb{F}$, and this gives that $\lambda\, \varphi(y) = \varphi(\lambda y) = \varphi(v) = 0$, hence $\lambda = 0$ (since $\varphi(y) \neq 0$), so $v = 0$. Thus,

$$\ker(\varphi) \cap \mathrm{Span}\{y\} = \{0\}.$$

This shows that X is the algebraic direct sum of $\ker(\varphi)$ and $N := \mathrm{Span}\{y\}$. As $\ker(\varphi)$ is closed (since φ is continuous, by assumption) and N is closed too (being finite-dimensional), we get that $X = \ker(\varphi) \oplus N$, as desired.

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Problem 4 (weight 25 points)

Let H be an infinite-dimensional Hilbert space (over $\mathbb{R}$ or $\mathbb{C}$) having a countable orthonormal basis $\mathcal{B} = \{u_n \mid n \in \mathbb{N}\}$.

4a (weight 5 points)

Let $T \in \mathcal{B}(H)$. Justify that for every $k \in \mathbb{N}$ we have

$$\big\langle T(u_k), T(u_k)\big\rangle = \sum_{n=1}^{\infty} \big|\big\langle T(u_k), u_n\big\rangle\big|^2 .$$

Solution: Let $k \in \mathbb{N}$. Parseval's identity says that for any $x \in H$ we have

$$\big\langle x, x\big\rangle = \|x\|^2 = \sum_{n=1}^{\infty} \big|\big\langle x, u_n\big\rangle\big|^2 .$$

Applying this with $x = T(u_k)$ gives the assertion.

4b (weight 10 points)

Let $P \in \mathcal{B}(H)$ be a self-adjoint projection (i.e., we have $P^* = P = P^2$).

Assume that $\big\langle P(u_k), u_k\big\rangle \in \{0, 1\}$ for some $k \in \mathbb{N}$.

Show that $\big\langle P(u_k), u_n\big\rangle = 0$ for all $n \in \mathbb{N}$ such that $n \neq k$.

Solution: We first note that the assumptions give that

$$\big\langle P(u_k), P(u_k)\big\rangle = \big\langle (P^*P)(u_k), u_k\big\rangle = \big\langle P(u_k), u_k\big\rangle \in \{0, 1\} .$$

We can now apply 4a (with $T = P$):

If $\big\langle P(u_k), P(u_k)\big\rangle = 0$, then we get that $\sum_{n=1}^{\infty} \big|\big\langle P(u_k), u_n\big\rangle\big|^2 = 0$, so $\big\langle P(u_k), u_n\big\rangle = 0$ for all n.

On the other hand, if $\big\langle P(u_k), P(u_k)\big\rangle = 1$, then we also have that $\big\langle P(u_k), u_k\big\rangle = 1$, so get that

$$1 = 1 + \sum_{n \in \mathbb{N}, n \neq k} \big|\big\langle P(u_k), u_n\big\rangle\big|^2 ,$$

so $\big\langle P(u_k), u_n\big\rangle = 0$ for all $n \neq k$.

4c (weight 10 points)

Let $P \in \mathcal{B}(H)$ be a self-adjoint projection and let $D_P \in \mathcal{B}(H)$ denote the diagonal operator w.r.t. $\mathcal{B}$ which satisfies that

$$D_P(u_n) = \big\langle P(u_n), u_n\big\rangle\, u_n \quad \text{for all } n \in \mathbb{N}.$$

Show that if D_P is a projection, then we have $P = D_P$.

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Solution: Assume that D_P is a projection, i.e., we have $(D_P)^2 = D_P$, and set $\lambda_k := \big\langle P(u_k), u_k \big\rangle \in \mathbb{F}$ for each $k \in \mathbb{N}$. Then we have that

$$\lambda_k\, u_k = D_P(u_k) = D_P\big(D_P(u_k)\big) = D_P(\lambda_k\, u_k) = \lambda_k\, D_P(u_k) = \lambda_k^2\; u_k\,,$$

hence that $\lambda_k = \lambda_k^2$ for all $k \in \mathbb{N}$. This implies that

$$\big\langle P(u_k), u_k \big\rangle = \lambda_k \;\in\; \{0, 1\}$$

for every $k \in \mathbb{N}$. Hence, using 4b, we deduce that $\big\langle P(u_k), u_n \big\rangle = 0$ for all $k, n \in \mathbb{N}$ such that $n \neq k$. Since $\mathcal{B}$ is an orthonormal basis, we get that

$$P(u_k) = \sum_{n=1}^{\infty} \big\langle P(u_k), u_n \big\rangle\, u_n = \big\langle P(u_k), u_k \big\rangle\, u_k = D_P(u_k)$$

for every $k \in \mathbb{N}$. Since a bounded operator is determined by its values on an orthonormal basis, we can conclude that $P = D_P$, as desired.

Problem 5 (weight 10 points)

Let H be an infinite-dimensional Hilbert space (over $\mathbb{R}$ or $\mathbb{C}$) and let $T \in \mathcal{B}(H)$ be compact. Explain what it means that T is compact. Then show that $\big\langle T(u_n), u_n \big\rangle \to 0$ as $n \to \infty$ whenever $\{u_n\}_{n\in\mathbb{N}}$ is an orthonormal sequence in H.

Solution: That T is compact means that $\{T(x_n)\}_{n\in\mathbb{N}}$ has a convergent subsequence whenever $\{x_n\}_{n\in\mathbb{N}}$ is a bounded sequence in H. Equivalently, that T maps bounded sets into relatively compact sets.

Next, let $\{u_n\}_{n\in\mathbb{N}}$ be an orthonormal sequence in H and assume (for contradiction) that $\big\langle T(u_n), u_n \big\rangle \not\to 0$ as $n \to \infty$. Then there exists $\varepsilon > 0$ and natural numbers $n_1 < n_2 < \cdots < n_j < \cdots$ such that

$$\big|\big\langle T(u_{n_j}), u_{n_j} \big\rangle\big| \geq \varepsilon \quad \text{for all } j \in \mathbb{N}.$$

Set $x_j := u_{n_j}$ for each $j \in \mathbb{N}$. Since $\{x_j\}_{j\in\,\mathbb{N}}$ is bounded, we can find a subsequence $\{y_k\}_{k\in\mathbb{N}} := \{x_{j_k}\}_{k\in\mathbb{N}}$ of $\{x_j\}_{j\in\mathbb{N}}$ such that $T(y_k) \to z$ for some $z \in H$. We can then find $K \in \mathbb{N}$ such that $\|z - T(y_k)\| < \varepsilon/2$ for all $k \geq K$, which gives that

$$|\big\langle z - T(y_k), y_k \big\rangle| \leq \|z - T(y_k)\|\, \|y_k\| < \varepsilon/2$$

for all $k \geq K$. Since $\big|\big\langle T(y_k), y_k \big\rangle\big| \geq \varepsilon$ for all $k \in \mathbb{N}$, we obtain that

$$\begin{aligned}
\big|\big\langle z, y_k \big\rangle\big| &= \big|\big\langle T(y_k), y_k \big\rangle \,+\, \big\langle z - T(y_k), y_k \big\rangle\big| \\
&\geq \big|\big\langle T(y_k), y_k \big\rangle\big| \,-\, \big|\big\langle z - T(y_k), y_k \big\rangle\big| \\
&> \varepsilon - \varepsilon/2 \\
&= \varepsilon/2
\end{aligned}$$

for all $k \geq K$. But this implies that $\sum_{k=1}^{\infty} |\big\langle z, y_k \big\rangle|^2 = \infty$, which contradicts Bessel's inequality. Thus, we must have $\big\langle T(u_n), u_n \big\rangle \to 0$ as $n \to \infty$.