UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Examination in MAT4410 — Videregående Linear Analyse

Day of examination: 13. desember 2016

Examination hours: 09.00 – 13.00

This problem set consists of 3 pages.

Appendices: Ingen

Permitted aids: Ingen

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Problem 1 (25p)

Let $F : \mathbb{R} \to \mathbb{R}$ be the right continuous function defined by

$$F(t) = \begin{cases} 2 - \frac{1}{(1+t)^2}\,, & \text{if } t \geq 0 \\ 0\,, & \text{if } t < 0 \end{cases} \tag{1}$$

and let μ_F be the Borel measure on $\mathbb{R}$ associated to the (distribution) function F.

(**a**) What is $\mu_F(a, b]$, $a < b$, $(a, b \in \mathbb{R})$? Give detailed formulas. Find the Lebesgue decomposition of μ_F with respect to Lebesgue measure λ.

(**b**) Explain why

$$\int_{\mathbb{R}} g \,\mathrm{d}\mu_F = g(0) + \int_0^\infty g(t) \frac{2}{(1+t)^3} \,\mathrm{d}\lambda(t)$$

for all Borel measurable $g : \mathbb{R} \to \mathbb{R}$ for which the function $t \mapsto \frac{g(t)}{(1+t)^3}$ is λ-integrable on $[0, \infty)$.

Problem 2 (35p)

Let λ be Lebesgue measure on the Borel σ-algebraen $\mathcal{B}(\mathbb{R})$ on $\mathbb{R}$ and let $f \in \mathcal{L}^1(\lambda)$. We define

$$f^*(x) = \sup_{r>0} \frac{1}{2r} \int_{x-r}^{x+r} |f(t)| \,\mathrm{d}\lambda(t), \quad x \in \mathbb{R}$$

and

$$U_t = \{x \in \mathbb{R} : f^*(x) > t\}, \quad t > 0.$$

Let, for all $r > 0$ and all $x \in \mathbb{R}$, $I_r(x)$ denote the open interval $I_r(x) = (x - r, x + r)$.

(Continued on page 2.)

(**a**) Suppose that $r_2 > r_1 > 0$. Show that

$$\int_{x-r_1}^{x+r_1} |f| \, d\lambda \leq \int_{z-r_2}^{z+r_2} |f| \, d\lambda.$$

for all $z \in I_{r_2 - r_1}(x)$.

(**b**) Prove that U_t is open (and hence Borel measurable) for every $z \in I_{r_2-r_1}(x)$.

Hint: For all $x \in U_t$ there are r_1 and r_2 such that $r_2 > r_1 > 0$ and

$$\frac{1}{2r_1} \int_{x-r_1}^{x+r_1} |f| \, d\lambda > t \,, \quad \frac{1}{2r_2} \int_{x-r_1}^{x+r_1} |f| \, d\lambda > t$$

Consider $I_{r_2-r_1}(x)$.

(**c**) Next we will prove that

$$(2) \qquad \lambda(U_t) \leq \frac{3}{t} ||f||_1$$

Explain that it suffices to prove (2) for all compact subsets K of U_t (hence that $\lambda(K) \leq (3/t)||f||_1$ for all such K).

(**d**) In what follows you can take for granted that for every finite set $\{I_1, I_2, ..., I_N\}$ of open intervals, there is a subset $\{I'_1, I'_2, ..., I'_M\}$ of pairwise disjoint intervals ($I'_j \cap I'_k = \emptyset, \ 1 \leq j < k \leq M$) such that

$$\lambda(\bigcup_{k=1}^{N} I_k) \leq 3 \sum_{j=1}^{M} \lambda(I'_j).$$

Show that

$$\lambda(K) \leq \frac{3}{t} ||f||_1$$

for all compact $K \subset U_t$.

Hint: For all $x \in K$, choose $I_r(x) = (x - r, x + r)$ such that

$$\frac{1}{\lambda(I_r(x))} \int_{I_r(x)} |f| \, d\lambda > t.$$

Problem 3 (25p)

(**a**) Let X be a linear normed space, X^* the dual space of X.
Justify that at every $x \in X$ induces a bounded linear fuctional l_x defined on X^*.

Show that we also have $||l_x|| = ||x||$. **Hint:** Hahn-Banach.

(**b**) Assume that $(\Omega, \mathcal{A}, \mu)$ is a σ-finite measure space ($\mu \geq 0$), $1 \leq p < +\infty, \frac{1}{p} + \frac{1}{q} = 1$.

Let $\mathcal{F} \subset \mathcal{L}^p(\mu)$. Show that there is a finite positive M such that $||f||_p \leq M$ for all $f \in \mathcal{F}$ if and only if

$$(3) \qquad \sup_{f \in \mathcal{F}} | \int_{\Omega} g f \, d\mu \, | < \infty, \ \text{for all } g \in \mathcal{L}^q(\mu).$$

(Continued on page 3.)

Problem 4 (15p)

Consider Lebesgue measure λ on the Borel σ-algebra $\mathcal{B}$ on $[0,1]$. Assume that (f_n) is a sequence of absolutely continuous functions on $[0,1]$ enjoying the following properties:

(1) There is a continuous function f such that $f = \lim_{n\to\infty} f_n$ pointwise ae on $[0,1]$.

(2) There is a function g such that $g = \lim_{n\to\infty} f_n'$ pointwise ae on $[0,1]$.

(3) There is an integrable function F such that

$$|f_n'(x)| \leq F(x) \text{ for alle } x \in [0,1] \text{ and for all } n \in \mathbb{N}.$$

Show that f is equal to an absolutely continuous function ae and that $f' = g$ ae on $[0,1]$.

THE END