# Chapter 10

February 14, 2024

## 10.1 Integration of Complex Functions

Assume $(X, \mathfrak{B}, \mu)$ is a measure space.

**Definition 10.10.1.** A measurable function $f : X \to \mathbb{C}$ is called integrable (or $\mu$-integrable) if

$$\int_X |f| d\mu < \infty.$$

Denote by $\mathcal{L}^1(X, \mathfrak{B}, d\mu)$, $\mathcal{L}^1(X, d\mu)$ or $\mathcal{L}^1_{\mathbb{C}}$ the set of integrable functions. This is also a vector space over $\mathbb{C}$, since

$$|f + g| \leq |f| + |g|, \; |cf| = |c||f| \; (c \in \mathbb{C}),$$
$$\text{the other axioms are trivial.}$$

This vector space is spanned by positive functions, since

$$f = \mathrm{Re}(f)_+ - \mathrm{Re}(f)_- + i\mathrm{Im}(f)_+ - i\mathrm{Im}(f)_-,$$

where for a function $h$ we let

$$h_+ = \max\{h, 0\}, \; h_- = -\min\{h, 0\},$$

and if $f \in \mathcal{L}^1(X, d\mu)$, then

$$(\mathrm{Re}(f))_\pm, (\mathrm{Im}(f))_\pm \in \mathcal{L}^1(X, d\mu),$$

as

$$|(\mathrm{Re}(f))_\pm|, |(\mathrm{Im}(f))_\pm| \leq |f|.$$

**Proposition 1.** *The integral extends uniquely from the positive integrable functions to a linear function (functional?) $\mathcal{L}^1(X, d\mu) \to \mathbb{C}$, that is, to a map s.t.*

$$\int_X (f + g) d\mu = \int_X f d\mu + \int_X g d\mu,$$
$$\int_X cf d\mu = c \int_X f d\mu, \; c \in \mathbb{C}.$$

*Proof.* Uniqueness is clear, as positive functions in $\mathcal{L}^1(X, d\mu)$ spans the entire space. We first extend the integral to real integrable functions by letting

$$\int\limits_X (g-h)d\mu \equiv \int\limits_X g d\mu - \int\limits_X h d\mu,$$

for $g, h \in \mathcal{L}^1(X, d\mu),\ g, h \geq 0$.

This is well-defined, since if

$$g - h = g' - h',$$

then $g + h' = h + g'$ and hence $\int\limits_X g d\mu + \int\limits_X h' d\mu$ $\square$