Linn - Anal - Lecture - Notes

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https://github.com/isakrukan/MAT4400-LinnAnaly

I. INTEGRATION OF COMPLEX FUNCTIONS (LEC. 7)

Assume $(X, \mathfrak{B}, \mu)$ is a measure space.

Definition I.1. A measurable function $f : X \to \mathbb{C}$ is called integrable (or μ-integrable) if

$$\int_X |f| d\mu < \infty.$$

Denote by $\mathcal{L}^1(X, \mathfrak{B}, d\mu)$, $\mathcal{L}^1(X, d\mu)$ or $\mathcal{L}^1_{\mathbb{C}}$ the set of integrable functions. This is also a vector space over $\mathbb{C}$, since

$$|f+g| \leq |f| + |g|, \; |cf| = |c||f| \; (c \in \mathbb{C}),$$

and the other axioms are should be easy.

This vector space is spanned by positive functions, since

$$f = \mathrm{Re}(f)_+ - \mathrm{Re}(f)_- + i\mathrm{Im}(f)_+ - i\mathrm{Im}(f)_-,$$

where for a function h we let

$$h_+ = \max\{h, 0\}, \; h_- = -\min\{h, 0\},$$

and if $f \in \mathcal{L}^1(X, d\mu)$, then

$$(\mathrm{Re}(f))_\pm, (\mathrm{Im}(f))_\pm \in \mathcal{L}^1(X, d\mu),$$

as

$$|(\mathrm{Re}(f))_\pm|, |(\mathrm{Im}(f))_\pm| \leq |f|.$$

Proposition 1. *The integral extends uniquely from the positive integrable functions to a linear function (functional?) $\mathcal{L}^1(X, d\mu) \to \mathbb{C}$, that is, to a map s.t.*

$$\int_X (f+g) d\mu = \int_X f d\mu + \int_X g d\mu,$$
$$\int_X cf d\mu = c \int_X f d\mu, \; c \in \mathbb{C}.$$

Proof. Uniqueness is clear, as positive functions in $\mathcal{L}^1(X, d\mu)$ spans the entire space. We first extend the integral to real integrable functions by letting

$$\int_X (g-h) d\mu := \int_X g d\mu - \int_X h d\mu,$$

for $g, h \in \mathcal{L}^1(X, d\mu), \; g, h \geq 0$.

This is well-defined, since if

$$g - h = g' - h',$$

then $g + h' = h + g'$ and hence $\int_X g d\mu + \int_X h' d\mu = \int_X g' d\mu + \int_X h' d\mu$. Now we extend the integral to the entire space $\mathcal{L}'(X, d\mu)$ by

$$\int_X f d\mu := \int_X (\mathrm{Re}(f)) d\mu + i \int_X (\mathrm{Im}(f)) d\mu.$$

We easily get that by definition:

$$\int_X (f_1 + f_2) d\mu = \int_X f_1 d\mu + \int_X f_2 d\mu, \; \forall f_1, f_2 \in \mathcal{L}^1(X, d\mu),$$

and

$$\int_X cf d\mu = c \int_X f d\mu \; \forall f \in \mathcal{L}^1(X, d\mu) \; \forall c \geq 0.$$

In order to prove the last property for all $c \in \mathbb{C}$, it remains to check it for $c = -1$ and $c = i$.

For $c = -1$ it follows, since if $g, h \geq 0$, then

$$\begin{aligned}\int_X (-(g-h)) \, d\mu &= \int_X (h-g) \, d\mu \\ &= \int_X h d\mu - \int_X g d\mu \\ &= -\int_X (g-h) \, d\mu.\end{aligned}$$

Similarly, for $c = i$ it is proved by a simple computation:

$$\begin{aligned}\int_X if d\mu &= \int_X \mathrm{Re}(if) d\mu + i \int_X \mathrm{Im}(if) d\mu \\ &= \int_X (-\mathrm{Im}(f)) \, d\mu + i \int_X (\mathrm{Re}(f)) \, d\mu \\ &= i \left(\int_X (\mathrm{Re}(f)) \, d\mu + i \int_X (\mathrm{Im}(f)) \, d\mu \right) \\ &= i \int_X f d\mu.\end{aligned}$$

$\square$

Proposition 2 (Triangle Inequality)**.** *For every $f \in \mathcal{L}^1(X, d\mu)$ we have*

$$\Big| \int_X f d\mu \Big| \leq \int_X |f| d\mu.$$

Proof. Choose $z \in \Pi := \{w \in \mathbb{C} : |w| = 1\}$ s.t.

$$z \int\limits_X f d\mu \geq 0.$$

Then

$$\begin{aligned}
\Big| \int\limits_X f d\mu \Big| &= \Big| z \int\limits_X \Big| \\
&= z \int\limits_X f d\mu \\
&= \int\limits_X z f d\mu \\
&= \int\limits_X \mathrm{Re}(zf) d\mu + \cancel{i \int\limits_X \mathrm{Im}(zf) d\mu} \\
&= \int\limits_X \left(\mathrm{Re}(zf)\right)_+ d\mu - \int\limits_X \left(\mathrm{Re}(zf)\right)_- d\mu \\
&\leq \int\limits_X \left(\mathrm{Re}(zf)\right)_+ d\mu \\
&\leq \int\limits_X |f| d\mu,
\end{aligned}$$

since $\left(\mathrm{Re}(zf)\right)_+ \leq |f|$. ∎