

# MAT4400: Notes on Linear analysis

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### 3 $\sigma$ -Algebras

**Definition 3.1** ( $\sigma$ -Algebra). A family  $\mathcal{A}$  of subsets of  $X$  with:

- (i)  $X \in \mathcal{A}$ ,
- (ii)  $A \in \mathcal{A} \Rightarrow A^c \in \mathcal{A}$ ,
- (iii)  $(A_n)_{n \in \mathbb{N}} \in \mathcal{A} \Rightarrow \bigcup_{n \in \mathbb{N}} A_n \in \mathcal{A}$

**Theorem 3.2** (and Definition).

(i) *The intersection of arbitrarily many  $\sigma$ -algebras in  $X$  is again a  $\sigma$ -algebra in  $X$ .*

(ii) *For every system of sets  $p \subset \mathcal{P}(X)$  there exists a smallest  $\sigma$ -algebra containing  $p$ .*

*This is the  $\sigma$ -algebra generated by  $p$ , denoted  $\sigma(p)$ , and  $p$  is called its generator.*

**Definition 3.3** (Borel). The  $\sigma$ -algebra  $\sigma(\mathcal{O})$  generated by the open sets  $\mathcal{O} = \mathcal{O}_{\mathbb{R}^n}$  of  $\mathbb{R}^n$  is called **Borel  $\sigma$ -algebra**, and its members are called **Borel sets** or **Borel measurable sets**.

## 5 Uniqueness of Measures

**Lemma 5.1.** *A Dynkin system  $D$  is a  $\sigma$ -algebra iff it is stable under finite intersections, i.e.  $A, B \in D \Rightarrow A \cap B \in D$ .*

**Theorem 5.2** (Dynkin). *Assume  $X$  is a set,  $S$  is a collection of subsets of  $X$  closed under finite intersections, that is, if  $A, B \in S \Rightarrow A \cap B \in S$ . Then  $D(S) = \sigma(S)$ .*

*Proof.* We clearly have that  $D(S) \subset \sigma(S)$ . If we can show that  $D(S)$  is a  $\sigma$ -algebra, that is, that a Dynkin system generated by a subset  $S \subset X$  (where  $S$  is  $\cap$ -stable) is a  $\sigma$ -algebra, then the inverse conclusion  $D(S) \supset \sigma(S)$  follows logically. This is the case because the  $\sigma$ -algebra  $\sigma(S)$  is the smallest  $\sigma$ -algebra containing  $S$ , and so if  $D(S)$  is a  $\sigma$ -algebra it must be a greater or equal (in some sense) than  $\sigma(S)$ .

Using Lemma 5.1 we only need to show that  $D(S)$  is stable under finite intersections, to prove that  $D(S)$  is a  $\sigma$ -algebra. Consider:

$$D_A := \{B \subset X : B \cap A \in D(S)\},$$

for some  $A \in D(S)$ . Notice that this set is  $\cap$ -stable, and so if we can show that  $D_A = D(S)$  we must have that (by Lemma 5.1)  $D(S)$  is a  $\sigma$ -algebra. Firstly, however, let us show that  $D_A$  is a **Dynkin system**.

1.  $\emptyset$  must be in  $D_A$ , since  $\emptyset \cap A = \emptyset \in D(S)$ .
2. Let  $B \in D_A$ . Then

$$A \cap B^c = A \setminus (A \cap B) = (A^c \cup (A \cap B))^c,$$

here  $A \cap B$  and  $A^c$  must be in  $D(S)$ . Furthermore, since disjoint unions of set from  $D(S)$  are still in  $D(S)$ , we must have  $A^c \in D_A$ .

3. Assume that  $(B_n)_{n \in \mathbb{N}} \subset D_A$  is a pairwise disjoint sequence. Then

$$\begin{aligned} (B_n \cap A)_{n \in \mathbb{N}} &\in D(S) \quad (\text{by def. of } D_A) \\ \Rightarrow \bigcup_{n \in \mathbb{N}} (B_n \cap A) &= \left( \bigcup_{n \in \mathbb{N}} B_n \right) \cap A \in D(S) \\ \Rightarrow \bigcup_{n \in \mathbb{N}} B_n &\in D_A. \end{aligned}$$

So  $D_A$  is indeed a Dynkin system.

We now want to show that  $D(S)$  is  $\cap$ -stable, we have:

$$\begin{aligned}
& S \subset D_A \quad \forall A \in S \\
& \Rightarrow D(S) \subset D_A \quad \forall A \in S \text{ (since } D_A \text{ is a Dynkin system)} \\
& \Rightarrow B \cap A \in D(S) \quad \forall B \in S, \forall A \in D(S) \text{ (by the definition of } D_A) \\
& \Rightarrow B \in D_A \quad \forall B \in S, \forall A \in D(S) \\
& \Rightarrow S \subset D_A \quad \forall A \in D(S) \\
& \Rightarrow D(S) \subset D_A \quad \forall A \in D(S) \text{ (since } D_A \text{ is a Dynkin system)} \\
& \Rightarrow A \cap B \in D(S) \quad \forall A, B \in D(S),
\end{aligned}$$

and so  $D(S)$  is  $\cap$ -stable and then  $D(S) \supset \sigma(S) \Rightarrow D(S) = \sigma(S)$ .  $\square$

**Theorem 5.3** (uniqueness of measures). *Let  $(X, B)$  be a measurable space, and  $S \subset P(X)$  be the generator of  $B$ , i.e.  $B = \sigma(S)$ . If  $S$  satisfies the following conditions:*

1.  *$S$  is stable under finite intersections ( $\cap$ -stable), i.e.  $A, C \in S \Rightarrow A \cap C \in S$ .*
2. *There exists an exhausting sequence  $(G_n)_{n \in \mathbb{N}} \subset B$  with  $G_n \uparrow X$ . Assume also that there are two measures  $\mu, \nu$  satisfying:*
3.  *$\mu(A) = \nu(A), \forall A \in S$ .*
4.  *$\mu(G_n) = \nu(G_n) < \infty$ .*

*Then  $\mu = \nu$ .*

*Proof (outline).* Define

$$D_n := \{A \in B : \mu(G_n \cap A) = \nu(G_n \cap A) (< \infty)\},$$

and show that it is a Dynkin system. Then, use the fact that  $S$  is  $\cap$ -stable and Theorem 5.2 to argue that  $D(S) = \sigma(S) \dots \rightarrow \dots B = D_n$ .  $\square$

## 6 Existence of Measures

**Theorem 6.1** (Carathéodory). *Let  $S \subset P(X)$  be a semi-ring and  $\mu : S \rightarrow [0, \infty)$  a pre-measure. Then  $\mu$  has an extension to a measure  $\mu^*$  on  $\sigma(S)$ , i.e. that  $\mu(s) = \mu^*(s)$ ,  $\forall s \in \sigma(S)$ .*

*Also, if  $S$  contains an exhausting sequence,  $S_n \uparrow X$ , s.t.  $\mu(S_n) < \infty$ , then the extension is unique.*

*Proof (outline).* Firstly, let us define an outer measure.

**Definition 6.2** (Outer measure). An outer measure is a function  $\mu^* : P(X) \rightarrow [0, \infty)$  with the following properties:

1.  $\mu^*(\emptyset) = 0$ ,
2.  $A \subset B \Rightarrow \mu^*(A) \leq \mu^*(B)$ ,
3.  $\mu^*\left(\bigcup_{n \in \mathbb{N}} A_n\right) \leq \sum_{n \in \mathbb{N}} \mu^*(A_n)$ ,

and define for each  $A \subset X$  the family of countable  $S$ -coverings:

$$C(A) := \left\{ (S_n)_{n \in \mathbb{N}} \subset S : \bigcup_{n \in \mathbb{N}} S_n \supset A \right\},$$

and the set function

$$\mu^*(A) := \inf \left\{ \sum_{n \in \mathbb{N}} \mu(S_n) : (S_n)_{n \in \mathbb{N}} \in C(A) \right\}.$$

**Step 1: Claim:**  $\mu^*(A)$  is an outer measure.

*Proof.*

1.  $C(\emptyset) = \{\text{any sequence in } S \text{ containing } \emptyset\} \Rightarrow \mu^*(\emptyset) = 0$ .
2. Assume  $A \subset B$ . Then  $C(A) \subset C(B) \Rightarrow \mu^*(A) \leq \mu^*(B)$ .
3. If  $\mu^*(A_n) = \infty$  for some  $n$ , then there is nothing to prove. Thus, assume  $\mu^*(A_n) < \infty \forall n$ . Fix  $\epsilon > 0$ , and for every  $n$  choose  $A_{n_k} \in S$  s.t.

$$A_n \subset \bigcup_{k \in \mathbb{N}} A_{n_k}, \quad \sum_{k \in \mathbb{N}} \mu^*(A_{n_k}) < \mu^*(A_n) + \frac{\epsilon}{2^n}.$$

Then

$$\bigcup_{n \in \mathbb{N}} A_n \subset \bigcup_{k \in \mathbb{N}} \bigcup_{n \in \mathbb{N}} A_{n_k},$$

so

$$\begin{aligned}
\mu^* \left( \bigcup_{n \in \mathbb{N}} A_n \right) &\leq \sum_{n \in \mathbb{N}} \sum_{k \in \mathbb{N}} \mu(A_{n_k}) \\
&< \sum_{n \in \mathbb{N}} \left( \mu^*(A_n) + \frac{\epsilon}{2^n} \right) \\
&= \sum_{n \in \mathbb{N}} \mu^*(A_n) + \epsilon.
\end{aligned}$$

As  $\epsilon$  was arbitrarily, we get that

$$\mu^* \left( \bigcup_{n \in \mathbb{N}} A_n \right) \leq \sum_{n \in \mathbb{N}} \mu^*(A_n),$$

so  $\mu^*$  fulfills all the conditions for being an outer measure.  $\square$

**Step 2:** Showing that  $\mu^*$  extends  $\mu$ , i.e.  $\mu^*(s) = \mu(s) \ \forall s \in S$ .

**Step 3:** Define  $\mu^*$ -measurable sets

$$\Sigma^* := \{A \subset X : \mu^*(Q) = \mu^*(Q \cap A) + \mu^*(Q \setminus A) \ \forall Q \subset X\}$$

**Step 4:** Show that  $\mu|_{\Sigma^*}$  is a measure. In particular,  $\mu|_{\sigma(S)}$  is a measure which extends  $\mu$ .  $\square$

## 7 Measurable Mappings

We consider maps  $T : X \rightarrow X'$  between two measurable spaces  $(X, \mathcal{A})$  and  $(X', \mathcal{A}')$  which respects the measurable structures, the  $\sigma$ -algebras on  $X$  and  $X'$ . These maps are useful as we can transport a measure  $\mu$ , defined on  $(X, \mathcal{A})$ , to  $(X', \mathcal{A}')$ .

**Definition 7.1.** Let  $(X, \mathcal{A})$ ,  $(X', \mathcal{A}')$  be measurable spaces. A map  $T : X \rightarrow X'$  is called  $\mathcal{A}/\mathcal{A}'$ -measurable if the pre-image of every measurable set is a measurable set:

$$T^{-1}(A') \in \mathcal{A}, \quad \forall A' \in \mathcal{A}'. \quad (1)$$

- A  $\mathcal{B}(\mathbb{R}^n)/\mathcal{B}(\mathbb{R}^m)$  measurable map is often called a Borel map.
- The notation  $T : (X, \mathcal{A}) \rightarrow (X', \mathcal{A}')$  is often used to indicate measurability of the map  $T$ .

**Lemma 7.2.** Let  $(X, \mathcal{A})$ ,  $(X', \mathcal{A}')$  be measurable spaces and let  $\mathcal{A}' = \sigma(\mathcal{G}')$ . Then  $T : X \rightarrow X'$  is  $\mathcal{A}/\mathcal{A}'$ -measurable iff  $T^{-1}(G') \in \mathcal{A}$ , i.e. if

$$T^{-1}(G') \in \mathcal{A}, \quad \forall G' \in \mathcal{G}'. \quad (2)$$

**Theorem 7.3.** Let  $(X_i, \mathcal{A}_i)$ ,  $i = 1, 2, 3$ , be measurable spaces and  $T : X_1 \rightarrow X_2$ ,  $S : X_2 \rightarrow X_3$  be  $\mathcal{A}_1/\mathcal{A}_2$  and  $\mathcal{A}_2/\mathcal{A}_3$ -measurable maps respectively. Then  $S \circ T : X_1 \rightarrow X_3$  is  $\mathcal{A}_1/\mathcal{A}_3$ -measurable.

**Corollary 7.4.** Every continuous map between metric spaces is a Borel map.

**Definition 7.5. (and lemma)** Let  $(T_i)_{i \in I}$ ,  $T_i : X \rightarrow X_i$ , be arbitrarily many mappings from the same space  $X$  into measurable spaces  $(X_i, \mathcal{A}_i)$ . The smallest  $\sigma$ -algebra on  $X$  that makes all  $T_i$  simultaneously measurable is

$$\sigma(T_i : i \in I) := \sigma \left( \bigcup_{i \in I} T_i^{-1}(\mathcal{A}_i) \right) \quad (3)$$

**Corollary 7.6.** A function  $f : (X, \mathcal{B}) \rightarrow \mathbb{R}$  is measurable if  $f((a, +\infty)) \in \mathcal{B}$ ,  $\forall a \in \mathbb{R}$ .

**Corollary 7.7.** Assume  $(X, \mathcal{B})$  is a measurable space,  $(Y, d)$  is a metric space,  $(f_n : (X, \mathcal{B}) \rightarrow Y)_{n=1}^{\infty}$  is a sequence of measurable maps. Assume this sequence of images  $(f_n(x))_{n=1}^{\infty}$  is convergent in  $Y \forall x \in X$ . Define

$$f : X \rightarrow Y, \quad \text{by } f(x) = \lim_{n \rightarrow \infty} f_n(x). \quad (4)$$

Then  $f$  is measurable.

**Theorem 7.8.** Let  $(X, \mathcal{A})$ ,  $(X', \mathcal{A}')$  be measurable spaces and  $T : X \rightarrow X'$  be an  $\mathcal{A}/\mathcal{A}'$ -measurable map. For every measurable  $\mu$  on  $(X, \mathcal{A})$ ,

$$\mu'(A') := \mu(T^{-1}(A')), \quad A' \in \mathcal{A}', \quad (5)$$

defines a measure on  $(X', \mathcal{A}')$ .

**Definition 7.9.** The measure  $\mu'(\cdot)$  in the above theorem is called the push forward or image measure of  $\mu$  under  $T$  and it is denoted as  $T(\mu)(\cdot)$ ,  $T_{*\mu}(\cdot)$  or  $\mu \circ T^{-1}(\cdot)$ .

**Theorem 7.10.** *If  $T \in \mathbb{R}^{n \times n}$  is an orthogonal matrix, then  $\lambda^n = T(\lambda^n)$ .*

**Theorem 7.11.** *Let  $S \in \mathbb{R}^{n \times n}$  be an invertible matrix. Then*

$$S(\lambda^n) = |\det S| \lambda^n = |\det S|^{-1} \lambda^n. \quad (6)$$

**Corollary 7.12.** *Lebesgue measure is invariant under motions:  $\lambda^n = M(\lambda^n)$  for all motions  $M$  in  $\mathbb{R}^n$ . In particular, congruent sets have the same measure. Two sets of points are called congruent if, and only if, one can be transformed into the other by an isometry*



## 8 Measurable Functions

A *measurable function* is a measurable map  $u : X \rightarrow \mathbb{R}$  from some measurable space  $(X, \mathcal{A})$  to  $(\mathbb{R}, \mathcal{B}(\mathbb{R}^1))$ . They play central roles in the theory of integration.

We recall that  $u : X \rightarrow \mathbb{R}$  is  $\mathcal{A}/\mathcal{B}(\mathbb{R}^1)$ -measurable if

$$u^{-1}(B) \in \mathcal{A}, \quad \forall B \in \mathcal{B}(\mathbb{R}^1). \quad (7)$$

Moreover from a lemma from chapter 7, we actually only need to show that

$$u^{-1}(G) \in \mathcal{A}, \quad \forall G \in \mathcal{G} \text{ where } \mathcal{G} \text{ generates } \mathcal{B}(\mathbb{R}^1). \quad (8)$$

**Proposition 8.1.**

- 1 If  $f, g : (X, \mathcal{B}) \rightarrow \mathbb{C}$  are measurable, then the function  $f+g, f \cdot g, cf, (c \in \mathbb{C})$  are measurable.
- 2 If  $b : \mathbb{C} \rightarrow \mathbb{C}$  is Borel and  $b : (\mathbb{C}, \mathcal{B}) \rightarrow \mathbb{C}$  is measurable, then  $b \circ f$  is measurable.
- 3 If  $f(x) = \lim_{n \rightarrow \infty} f_n(x)$ ,  $x \in X$  and  $f_n$  are measurable, then  $f$  is measurable.
- 4 If  $X = \bigcup_{n=1}^{\infty} A_n$ ,  $(A_n \in \mathcal{B})$ ,  $f|_{A_n} : (A_n, \mathcal{B}_{A_n}) \rightarrow \mathbb{C}$  is measurable  $\forall n$ , then  $f$  is measurable.

**Definition 8.2.** Given a measurable space  $(X, \mathcal{B})$ , a measurable function  $f : (X, \mathcal{B}) \rightarrow \mathbb{C}$  is called simple if

$$f(x) = \sum_{k=1}^N c_k \mathbb{1}_{A_k}(x), \quad (9)$$

for some  $c_k \in \mathbb{C}$ ,  $A_k \in \mathcal{B}$ , where  $\mathbb{1}$  is the characteristic function,

$$\mathbb{1}_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A. \end{cases} \quad (10)$$

The representation of simple function is **not** unique. We denote the standard representation of  $f$  by

$$f(x) = \sum_{n=0}^N z_n \mathbb{1}_{B_n}(x), \quad N \in \mathbb{N}, \quad z_n \in \mathbb{R}, \quad B_n \in \mathcal{A}, \quad X = \bigcup_{n=1}^N B_n, \quad \text{for } B_n \cap B_m = \emptyset, \quad n \neq m. \quad (11)$$

The set of simple functions is denoted  $\mathcal{E}(\mathcal{A})$  of  $\mathcal{E}$ .

**Definition 8.3.** Assume  $\mu$  is a measure on  $(X, \mathcal{B})$ . Given a *positive* simple function

$$f = \sum_{k=1}^N c_k \mathbf{1}_{A_k}, \quad (c_k \geq 0). \quad (12)$$

We define

$$\int_X f d\mu = \sum_{k=1}^n c_k \mu(A_k) \in [0, +\infty]. \quad (13)$$

We also denote this by  $I_\mu(f)$ .

**Lemma 8.4.** *This is well defined, that is,  $\int_X f d\mu$  does not depend on the presentation of the simple function  $f$ .*

**Properties 8.5.** *For every positive simple function*

$$1 \quad \int_X c f d\mu = c \int_X f d\mu, \quad \text{for only } c \geq 0$$

$$2 \quad \int_X (f + g) d\mu = \int_X f d\mu + \int_X g d\mu.$$

**Corollary 8.6.** *If  $f \geq g \geq 0$  are simple functions, then*

$$\int_X f d\mu \geq \int_X g d\mu. \quad (14)$$

**Definition 8.7.** If  $f : X \rightarrow [0, +\infty)$  is measurable, then we define

$$\int_X f d\mu = \sup \left\{ \int_X g d\mu : f \geq g \geq 0, \text{ } g \text{ is simple} \right\} \quad (15)$$

**Remark.** *This means that any measurable function can be approximated by simple functions.*

**Properties 8.8.** *Measurable functions like this have the following properties*

$$1 \quad \int_X c f d\mu = c \int_X f d\mu, \quad \forall c \geq 0.$$

$$2 \quad \text{If } f \geq g \geq 0, \text{ then } \int_X f d\mu \geq \int_X g d\mu \text{ for any measurable } g, f.$$

$$3 \quad \text{If } f \geq 0 \text{ is simple, then } \int_X f d\mu \text{ is the same value as obtained before.}$$

To advance in measure theory we consider measurable functions

$$f : X \rightarrow [0, +\infty].$$

Measurability is understood w.r.t the  $\sigma$ -algebra  $\mathcal{B}([0, +\infty])$  generated by  $\mathcal{B}([0, +\infty))$  and  $\{+\infty\}$ . In other words,  $A \subset [0, +\infty] \in \mathcal{B}([0, +\infty])$  iff  $A \cap [0, +\infty) \in \mathcal{B}([0, +\infty))$ .

**Remark.** Hence  $f : X \rightarrow [0, +\infty]$  is measurable iff  $f^{-1}(A)$  is measurable  $\forall A \in \mathcal{B}([0, +\infty))$ .

**Definition 8.9.** For measurable functions  $f_X \rightarrow [0, +\infty]$ , we define

$$\int_X f d\mu = \sup \left\{ \int_X g d\mu : f \geq g \geq 0 : g \text{ is simple} \right\} \in [0, +\infty]. \quad (16)$$

**Theorem 8.10. Monotone convergence theorem** Assume  $(X, \mathcal{B}, \mu)$  is a measure space,  $(f)_{n=1}^\infty$  is an increasing sequence of measurable positive functions  $f_n : X \rightarrow [0, +\infty]$ . Define  $f(x) = \lim_{n \rightarrow \infty} f_n(x)$ . Then  $f$  is measurable and

$$\int_X f d\mu = \lim_{n \rightarrow \infty} \int_X f_n d\mu. \quad (17)$$

**Theorem 8.11.** Assume  $(X, \mathcal{B})$  is a measurable space and  $f : X \rightarrow [0, +\infty]$  is measurable. Then there are simple functions  $g_n$ , s.t.

$$0 \leq g_1 \leq g_2 \leq \dots, \quad g_n(x) \rightarrow f(x), \quad \forall x \in X.$$

Moreover, if  $f$  is bounded, we can choose  $g_n$  s.t. the convergence is uniform, that is,

$$\lim_{n \rightarrow \infty} \sup_{x \in X} |g_n(x) - f(x)| = 0. \quad (18)$$

## 9 Integration of measurable functions

Through this chapter  $(X, \mathcal{A}, \mu)$  will be some measure space. Recall that  $\mathcal{M}^+(\mathcal{A})$   $[\mathcal{M}_{\mathbb{R}}^+(\mathcal{A})]$  are the  $\mathcal{A}$ -measurable positive functions and  $\mathcal{E}(\mathcal{A})$   $[\mathcal{E}_{\mathbb{R}}^+(\mathcal{A})]$  are the positive and simple functions.

The fundamental idea of *Integration* is to measure the area between the graph of the function and the abscissa. For positive simple functions  $f \in \mathcal{E}^+(\mathcal{A})$  in standard representation, this is done easily

$$\text{if } f = \sum_{i=0}^M y_i \mathbb{1}_{A_i} \in \mathcal{E}^+(\mathcal{A}) \quad \text{then} \quad \sum_{i=0}^M y_i \mu(A_i) \quad (19)$$

would be the  $\mu$ -area enclosed by the graph and the abscissa. We note that the representation of  $f$  should not impact the integral of  $f$ .

**Lemma 9.1.** *Let  $\sum_{i=0}^M y_i \mathbb{1}_{A_i} = \sum_{k=0}^N z_k \mathbb{1}_{B_k}$  be two standard representations of the same function  $f \in \mathcal{E}^+(\mathcal{A})$ . Then*

$$\sum_{i=0}^M y_i \mu(A_i) = \sum_{k=0}^N z_k \mu(B_k). \quad (20)$$

**Definition 9.2.** Let  $f = \sum_{i=0}^M y_i \mathbb{1}_{A_i} \in \mathcal{E}^+(\mathcal{A})$  be a simple function in standard representation. Then the number

$$I_\mu(f) = \sum_{i=0}^M y_i \mu(A_i) \in [0, \infty] \quad (21)$$

(which is independent of the representation of  $f$ ) is called the  $\mu$ -integral of  $f$ .

**Proposition 9.3.** *Let  $f, g \in \mathcal{E}^+(\mathcal{A})$ . Then*

- (i)  $I_\mu(\mathbb{1}_A) = \mu(A) \quad \forall A \in \mathcal{A}.$
- (ii)  $I_\mu(\lambda f) = \lambda I_\mu(f) \quad \forall \lambda \geq 0.$
- (iii)  $I_\mu(f + g) = I_\mu(f) + I_\mu(g).$
- (iv)  $f \leq g \Rightarrow I_\mu(f) \leq I_\mu(g).$

In theorem 8.8 we saw that we could for every  $u \in \mathcal{M}^+(\mathcal{A})$  write it as an increasing limit of simple functions. By corollary 8.10, the suprema of simple functions are again measurable, so that

$$u \in \mathcal{M}^+(\mathcal{A}) \Leftrightarrow u = \sup_{n \in \mathbb{N}} f_n, \quad f \in \mathcal{E}^+(\mathcal{A}), \quad f_n \leq f_{n+1} \leq \dots$$

We will use this to "inscribe" simple functions (which we know how to integrate) below the graph of a positive measurable function  $u$  and exhaust the  $\mu$ -area below  $u$ .

**Definition 9.4.** Let  $(X, \mathcal{A}, \mu)$  be a measure space. The  $(\mu)$ -integral of a positive function  $u \in \mathcal{M}_{\mathbb{R}}^+(\mathcal{A})$  is given by

$$\int u d\mu = \sup \{ I_{\mu}(g) : g \leq u, g \in \mathcal{E}^+(\mathcal{A}) \} \in [0, +\infty]. \quad (22)$$

If we need to emphasize the *integration variable*, we write  $\int u(x)\mu(dx)$ . The key observation is that the integral  $\int \dots d\mu$  extends  $I_{\mu}$ .

**Lemma 9.5.** For all  $f \in \mathcal{E}^+(\mathcal{A})$  we have  $\int f d\mu = I_{\mu}(f)$ .

The next theorem is one of many convergence theorems. It shows that we could have defined 22 using any increasing sequence  $f_n \uparrow u$  of simple functions  $f_n \in \mathcal{E}^+(\mathcal{A})$ .

**Theorem 9.6.** (*Beppo Levi*) Let  $(X, \mathcal{A}, \mu)$  be a measure space. For an increasing sequence of functions  $(u_n)_{n \in \mathbb{N}} \subset \mathcal{M}_{\mathbb{R}}^+(\mathcal{A})$ ,  $0 \leq u_n \leq u_{n+1} \leq \dots$ , we have for the supremum  $u = \sup_{n \in \mathbb{N}} u_n \in \mathcal{M}_{\mathbb{R}}^+(\mathcal{A})$  and

$$\int \sup_{n \in \mathbb{N}} u_n d\mu = \sup_{n \in \mathbb{N}} \int u_n d\mu. \quad (23)$$

Note we can write  $\lim_{n \rightarrow \infty}$  instead of  $\sup_{n \in \mathbb{N}}$  as the supremum of an increasing sequence is its limit. Moreover, this theorem holds in  $[0, +\infty]$ , so the case  $+\infty = +\infty$  is possible.

**Corollary 9.7.** Let  $u \in \mathcal{M}_{\mathbb{R}}^+(\mathcal{A})$ . Then

$$\int u d\mu = \lim_{n \rightarrow \infty} \int f_n d\mu$$

holds for every sequence  $(f_n)_{n \in \mathbb{N}} \subset \mathcal{E}^+(\mathcal{A})$  with  $\lim_{n \rightarrow \infty} f_n = u$ .

**Proposition 9.8.** (of integral) Let  $u, v \in \mathcal{M}_{\mathbb{R}}^+(\mathcal{A})$ . Then

$$(i) \int \mathbf{1}_A d\mu = \mu(A) \quad \forall A \in \mathcal{A}.$$

$$(ii) \int \alpha u d\mu = \alpha \int u d\mu \quad \forall \alpha \geq 0.$$

$$(iii) \int u + v d\mu = \int u d\mu + \int v d\mu.$$

$$(iv) u \leq v \Rightarrow \int u d\mu \leq \int v d\mu.$$

**Corollary 9.9.** Let  $(u_n)_{n \in \mathbb{N}} \subset \mathcal{M}_{\mathbb{R}}^+(\mathcal{A})$ . Then  $\sum_{n=1}^{\infty} u_n$  is measurable and we have

$$\int \sum_{n=1}^{\infty} u_n d\mu = \sum_{n=1}^{\infty} \int u_n d\mu$$

(including the possibility  $+\infty = +\infty$ .)

**Theorem 9.10.** (*Fatou*) Let  $(u_n)_{n \in \mathbb{N}} \subset \mathcal{M}_{\mathbb{R}}^+(\mathcal{A})$  be a sequence of positive measurable functions. Then  $u = \liminf_{n \rightarrow \infty} u_n$  is measurable and

$$\int \liminf_{n \rightarrow \infty} u_n d\mu = \liminf_{n \rightarrow \infty} \int u_n d\mu \quad (24)$$

## 10 Integrals of Measurable Functions

We have defined our integral for positive measurable functions, i.e. functions in  $\mathcal{M}^+(\mathcal{A})$ . To extend our integral to not only functions in  $\mathcal{M}^+(\mathcal{A})$  we first notice that

$$u \in \mathcal{M}_{\mathbb{R}}(\mathcal{A}) \Leftrightarrow u = u^+ - u^-, \quad u^+, u^- \in \mathcal{M}_{\mathbb{R}}^+, \quad (25)$$

i.e. that every measurable function can be written as a sum of **positive** measurable functions.

**Definition 10.1** ( $\mu$ -integrable). A function  $u : X \rightarrow \overline{\mathbb{R}}$  on  $(X, \mathcal{A}, \mu)$  is  $\mu$ -integrable, if it is  $\mathcal{A}/\mathcal{B}(\mathbb{R})$ -measurable and if  $\int u^+ d\mu, \int u^- d\mu < \infty$  (recall the definition for the integral of positive measurable functions). Then

$$\int u d\mu := \int u^+ d\mu - \int u^- d\mu \in (-\infty, \infty) \quad (26)$$

is the ( $\mu$ -)integral of  $u$ . We write  $\mathcal{L}^1(\mu)$  for the set of all real-valued  $\mu$ -integrable functions<sup>1</sup>.

**Theorem 10.2.** Let  $u \in \mathcal{M}_{\mathbb{R}}(\mathcal{A})$ , then the following conditions are equivalent:

- (i)  $u \in \mathcal{L}_{\mathbb{R}}^1(\mu)$ .
- (ii)  $u^+, u^- \in \mathcal{L}_{\mathbb{R}}^1(\mu)$ .
- (iii)  $|u| \in \mathcal{L}_{\mathbb{R}}^1(\mu)$ .
- (iv)  $\exists w \in \mathcal{L}_{\mathbb{R}}^1(\mu)$  with  $w \geq 0$  s.t.  $|u| \leq w$ .

**Theorem 10.3** (Properties of the  $\mu$ -integral). The  $\mu$ -integral is: **homogeneous, additive, and:**

- (i)  $\min\{u, v\}, \max\{u, v\} \in \mathcal{L}_{\mathbb{R}}^1(\mu)$  (lattice property)
- (ii)  $u \leq v \Rightarrow \int u d\mu \leq \int v d\mu$  (monotone)
- (iii)  $\left| \int u d\mu \right| \leq \int |u| d\mu$  (triangle inequality)

**Remark.** If  $u(x) \pm v(x)$  is defined in  $\overline{\mathbb{R}}$  for all  $x \in X$  then we can exclude  $\infty - \infty$  and the theorem above just says that the integral is linear:

$$\int (au + bv) d\mu = a \int u d\mu + b \int v d\mu. \quad (27)$$

---

<sup>1</sup>In words, we extend our integral to ~~positive~~ measurable functions by noticing that we can write every measurable function as a sum of positive measurable functions, something that we do know how to integrate. We don't want to run into the problem of  $\infty - \infty$ , thus we require the integral of the positive and negative parts to both (separately) be less than infinity.

*This is always true for real-valued  $u, v \in \mathcal{L}^1(\mu) = \mathcal{L}_{\mathbb{R}}^1(\mu)$ , making  $\mathcal{L}^1(\mu)$  a vector space with addition and scalar multiplication defined by*

$$(u + v)(x) := u(x) + v(x), \quad (a \cdot u)(x) := a \cdot u(x), \quad (28)$$

*and*

$$\int \dots d\mu : \mathcal{L}^1(\mu) \rightarrow \mathbb{R}, \quad u \mapsto \int u d\mu, \quad (29)$$

*is a **positive linear functional**.*



## 11 Null sets and the Almost Everywhere (lecture 08, 05. Feb.)

**Definition 11.1.** A  $(\mu)$ -null set  $N \in \mathcal{N}_\mu$  is a measurable set  $N \in \mathcal{A}$  satisfying

$$N \in \mu \Leftrightarrow N \in \mathcal{A} \text{ and } \mu(N) = 0. \quad (30)$$

This can be used generally about a ‘statement’ or ‘property’, but we will be interested in questions like ‘when is  $u(x)$  equal to  $v(x)$ ’, and we answer this by saying

$$u = v \text{ a.e.} \Leftrightarrow \{x : u(x) \neq v(x)\} \text{ is (contained in) a } \mu\text{-null set.}, \quad (31)$$

i.e.

$$u = v \text{ } \mu\text{-a.e.} \Leftrightarrow \mu(\{x : u(x) \neq v(x)\}) = 0. \quad (32)$$

The last phrasing should of course include that the set  $\{x : u(x) \neq v(x)\}$  is in  $\mathcal{A}$ .

**Theorem 11.2.** Let  $u \in \mathcal{M}_{\mathbb{R}}(\mathcal{A})$ , then:

- (i)  $\int |u| d\mu = 0 \Leftrightarrow |u| = 0 \text{ a.e.} \Leftrightarrow \mu\{u \neq 0\} = 0,$
- (ii)  $\mathbb{1}_N u \in \mathcal{L}_{\mathbb{R}}^1(\mu) \quad \forall N \in \mathcal{N}_\mu,$
- (iii)  $\int_N u d\mu = 0.$

**Corollary 11.3.** Let  $u = v \text{ } \mu\text{-a.e.}$  Then

- (i)  $u, v \geq 0 \Rightarrow \int u d\mu = \int v d\mu,$
- (ii)  $u \in \mathcal{L}_{\mathbb{R}}^1(\mu) \Rightarrow v \in \mathcal{L}_{\mathbb{R}}^1(\mu) \text{ and } \int u d\mu = \int v d\mu.$

**Corollary 11.4.** If  $u \in \mathcal{M}_{\mathbb{R}}(\mathcal{A})$ ,  $v \in \mathcal{L}_{\mathbb{R}}^1(\mu)$  and  $v \geq 0$  then

$$|u| \leq v \text{ a.e.} \Rightarrow u \in \mathcal{L}_{\mathbb{R}}^1(\mu). \quad (33)$$

**Proposition 11.5** (Markow inequality). For all  $u \in \mathcal{L}_{\mathbb{R}}^1(\mu)$ ,  $A \in \mathcal{A}$  and  $c > 0$

$$\mu(\{|u| \geq c\} \cap A) \leq \frac{1}{c} \int_A |u| d\mu, \quad (34)$$

if  $A = X$ , then (obviously)

$$\mu\{|u| \geq c\} \leq \frac{1}{c} \int |u| d\mu. \quad (35)$$

## Completions of measure spaces

**Definition 11.6.** A measure space  $(X, \mathcal{B}, \mu)$  is called **complete** if whenever  $A \in \mathcal{B}$  and  $\mu(A) = 0$ , we have  $B \in \mathcal{B} \ \forall B \subset A$ .

**Remark.** Any measure space can be completed as follows:

Let  $\bar{\mathcal{B}}$  be the  $\sigma$ -algebra generated by  $\mathcal{B}$  and all sets  $B \subset X$  s.t. there exists  $A \in \mathcal{B}$  with  $B \subset A$  and  $\mu(A) = 0$ .

**Proposition 11.7.** The  $\sigma$ -algebra  $\bar{\mathcal{B}}$  can also be described as follows:

$$\bar{\mathcal{B}} := \{B \subset X : A_1 \subset B \subset A_2 \text{ for some } A_1, A_2 \in \mathcal{B} \text{ with } \mu(A_2 \setminus A_1) = 0\}, \quad (36)$$

with  $B, A_1, A_2$  as above, we define

$$\bar{\mu} := \mu(A_1) = \mu(A_2) \quad (37)$$

Then  $(X, \bar{\mathcal{B}}, \bar{\mu})$  is a complete measure space.

**Definition 11.8.** If  $\mu$  is a Borel measure on a **metric** space  $(X, d)$ , then the completion  $\bar{\mathcal{B}}(X)$  of the Borel  $\sigma$ -algebra with respect to  $\mu$  is called the  $\sigma$ -algebra of  $\mu$ -measurable sets.

**Remark.** For  $\mu = \lambda_n$  on  $\mathbb{R}^n$  we talk about the  $\sigma$ -algebra of **Lebesgue measurable sets**. Instead of  $\bar{\lambda}_n$  we still write  $\lambda_n$  and call it the **Lebesgue measure**. A function  $f : \mathbb{R}^n \rightarrow \mathbb{C}$ , measurable w.r.t. the  $\sigma$ -algebra of Lebesgue measurable sets is called the **Lebesgue measurable**.

The following result shows that any Lebesgue measurable function coincides with a Borel function a.e.

**Proposition 11.9.** Assume  $(X, \mathcal{B}, \mu)$  is a measure space and consider its completion  $(X, \bar{\mathcal{B}}, \bar{\mu})$ . Assume  $f : X \rightarrow \mathbb{C}$  is  $\bar{\mathcal{B}}$ -measurable. Then there is a  $\mathcal{B}$ -measurable function  $g : X \rightarrow \mathbb{C}$  s.t.  $f = g$   $\bar{\mu}$ -a.e.

## 12 Convergence Theorems and Their Applications (lecture 9, 8. Feb.)

- To interchange limits and integrals in **Riemann integrals** one typically has to assume uniform convergence. - The set of Riemann integrable functions is somewhat limited, see theorem 12.4

**Theorem 12.1** (Generalization of Beppo Levi, monotone convergence).

(i) Let  $(u_n)_{n \in \mathbb{N}} \subset \mathcal{L}^1(\mu)$  be s.t.  $u_1 \leq u_2 \leq \dots$  with limit  $u := \sup_{n \in \mathbb{N}} u_n = \lim_{n \rightarrow \infty} u_n$ . Then  $u \in \mathcal{L}^1(\mu)$  **iff**

$$\sup_{n \in \mathbb{N}} \int u_n d\mu < +\infty,$$

in which case

$$\sup_{n \in \mathbb{N}} \int u_n d\mu = \int \sup_{n \in \mathbb{N}} u_n d\mu.$$

(ii) Same thing only with a decreasing sequence  $\dots > -\infty$  in which case

$$\inf_{n \in \mathbb{N}} \int u_n d\mu = \int \inf_{n \in \mathbb{N}} u_n d\mu.$$

**Theorem 12.2** (Lebesgue; dominated convergence). *Let  $(u_n)_{n \in \mathbb{N}} \subset \mathcal{L}^1(\mu)$  s.t.*

(a)  $|u_n|(x) \leq w(x)$ ,  $w \in \mathcal{L}^1(\mu)$ ,

(b)  $u(x) = \lim_{n \rightarrow \infty} u_n(x)$  exists in  $\bar{\mathbb{R}}$ ,

then  $u \in \mathcal{L}^1(\mu)$  and we have

(i)  $\lim_{n \rightarrow \infty} \int |u_n - u| d\mu = 0$ ;

(ii)  $\lim_{n \rightarrow \infty} \int u_n d\mu = \int \lim_{n \rightarrow \infty} u_n d\mu = \int u d\mu$ ;

## Application 2: Riemann vs Lebesgue Integration

Consider only  $(X, \mathcal{A}, \mu) = (\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda)$ .

**Theorem 12.3.** *Let  $u : [a, b] \rightarrow \mathbb{R}$  be a measurable and **Riemann integrable** function. Then*

$$u \in \mathcal{L}^1(\lambda) \text{ and } \int_{[a,b]} u d\lambda = \int_a^b u(x) dx. \quad (38)$$

**Theorem 12.4.** *Let  $u : [a, b] \rightarrow \mathbb{R}$  be a bounded function, it is Riemann integrable **iff** the points in  $(a, b)$  where  $u$  is discontinuous are a (subset of) Borel measurable null set.*

## Improper Riemann Integrals

- The Lebesgue integral extends the (*proper*) Riemann integral. However, there is a further extension of the Riemann integral which cannot be captured by Lebesgue's theory.  $u$  is Lebesgue integrable *iff*  $|u|$  has finite Lebesgue integral.  
 - The Lebesgue integral does not respect sign-changes and cancellations. However, the following *improper Riemann integral* does:

$$(R) \int_0^\infty u(x) dx := \lim_{n \rightarrow \infty} (R) \int_0^n u(x) dx. \quad (39)$$

**Corollary 12.5.** *Let  $u : [0, \infty) \rightarrow \mathbb{R}$  be a measurable, Riemann integrable function for every interval  $[0, N]$ ,  $N \in \mathbb{N}$ . Then  $u \in \mathcal{L}^1[0, \infty)$  **iff***

$$\lim_{N \rightarrow \infty} (R) \int_0^N |u(x)| dx < \infty. \quad (40)$$

*In this case,  $(R) \int_0^\infty u(x) dx = \int_{[0, \infty)} u d\lambda$*

**Proposition 12.6** (appearing as example 12.13 in Schilling). *Let  $f_\alpha(x) := x^\alpha$ ,  $x > 0$  and  $\alpha \in \mathbb{R}$ . Then*

$$(i) \quad f_\alpha \in \mathcal{L}^1(0, 1) \Leftrightarrow \alpha > -1.$$

$$(ii) \quad f_\alpha \in \mathcal{L}^1[1, \infty) \Leftrightarrow \alpha < -1.$$

### 13 The Function Spaces $\mathcal{L}^p$ (lecture 11, 15. Feb.)

Assume  $V$  is a vector space over  $\mathbb{K} \in \{\mathbb{C}, \mathbb{R}\}$ .

**Definition 13.1.** A seminorm on  $V$  is a map  $p : V \rightarrow [0, +\infty)$  s.t.

- (1)  $p(cx) = |c|p(x) \quad \forall x \in V, \forall c \in \mathbb{K}$ .
- (2)  $p(x + y) \leq p(x) + p(y) \quad \forall x, y \in V$ .    **triangle inequality**.

A seminorm is called a norm if we also have

$$p(x) = 0 \iff x = 0.$$

A norm is commonly denoted  $\|x\|$ , and a vectorspace equipped with a norm is called a **normed space**.

**Definition 13.2.** Assume  $(X, d)$  is a measure space. Fix  $1 \leq p \leq \infty$ . For every measurable function  $f : X \rightarrow \mathbb{C}$  we define the following

$$\|f\|_p = \left( \int_X |f|^p d\mu \right)^{1/p} \in [0, +\infty]. \quad (41)$$

We can see that  $\|cf\|_p = |c|\|f\|_p \quad \forall c \in \mathbb{C}$ .

**Lemma 13.3.**

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p. \quad (42)$$

**Definition 13.4.** We define

$$\mathcal{L}^p(X, d\mu) = \{f : X \rightarrow \mathbb{C} \mid f \text{ is measurable and } \|f\|_p < \infty\}. \quad (43)$$

This is a vectorspace with seminorm  $f \mapsto \|f\|_p$ . And in general this is not a normed space, since  $\|f\|_p = 0 \iff f = 0$  a.e.

Generally, if  $p$  is a seminorm on a vectorspace  $V$ , then

$$V_0 = \{x \in V \mid p(x) = 0\} \quad (44)$$

which is a subspace of  $V$ . Then we consider the quotient/factor space  $V/V_0$ .

**Definition 13.5.** For  $x, y \in V$ , define

$$x \sim y \iff x - y \in V_0. \quad (45)$$

This is an equivalence relation on  $V$ . The representation class of  $V$  is defined by  $[x]$  or  $x + V_0$ .

Then  $V/V_0$  is equals the set of equivalence classes. We can show that it is a normed space.

$$[x] + [y] = [x + y] \quad , \quad c[x] = [cx] \quad , \quad ||[x]|| = p(x).$$

Applying this to  $\mathcal{L}^p(X, d\mu)$  we get the normed space

$$L^p(X, d\mu) = \mathcal{L}^p(X, d\mu)/\mathcal{N}. \quad (46)$$

Where  $\mathcal{N}$  is the space of measurable functions  $f$  s.t.  $f = 0$  a.e. We will further continue to denote the norm by  $|| \cdot ||_p$ , and we will normally **not** distinguish between  $f \in \mathcal{L}^p(X, d\mu)$  and the vector in  $L^p(X, d\mu)$  that  $f$  defines.

**Definition 13.6.** A normed space  $(X, || \cdot ||)$  is called a Banach space if  $V$  is complete w.r.t the metric  $d(x, y) = ||x - y||$ .

**Theorem 13.7.** If  $(X, \mathcal{B}, \mu)$  is a measure space,  $1 \leq p \leq \infty$ , then  $L^p(X, d\mu)$  is a Banach space.

**Definition 13.8.** A measurable function  $f : X \rightarrow \mathbb{C}$  is called **essentially bounded** if there is  $c \geq 0$  s.t.

$$\mu(\{x : |f(x)| > c\}) = 0. \quad (47)$$

That is  $|f| \leq c$  a.e. The smallest such  $c$  is called the essential supremum of  $f$  and is denoted by  $||f||_\infty$ .

**Definition 13.9.**

$$\mathcal{L}^\infty(X, d\mu) = \{f : X \rightarrow \mathbb{C} \mid f \text{ is measurable and } ||f||_\infty < \infty\}.$$

$$L^\infty(X, d\mu) = \mathcal{L}^\infty(X, d\mu)/\mathcal{N}.$$

Where by the previous definiton these spaces become the spaces of all essentially bounded functions.

**Theorem 13.10.** If  $(X, \mathcal{B}, \mu)$  is a  $\sigma$ -finite measure space, then  $L^\infty(X, d\mu)$  is a Banach space.

## 17 Dense and Determining Sets (lecture 12, 19. Feb.)

**Definition 17.1** (Dense Sets). A set  $\mathcal{D} \subset \mathcal{L}^p(\mu), p \in [0, \infty]$ , is called *dense* if for every  $u \in \mathcal{L}^p(\mu)$  there exist a sequence  $(f_n)_{n \in \mathbb{N}} \subset \mathcal{D}$  s.t.  $\lim_{n \rightarrow \infty} ||u - f_n||_p = 0$ .

**Theorem 17.2.** Assume  $X, d$  is a metric space and  $\mu$  is a Borel measure that is finite on every ball  $1 \leq p < \infty$ . Then the space of bounded continuous functions with bounded support is dense in  $\mathcal{L}^p(X, d\mu)$ . Where bounded support means that  $f$  vanishes outside some ball.

**Theorem 17.3.** Assume  $(X, d)$  is a separable locally compact metric space and  $\mu$  is a Borel Measure on  $X$  s.t.  $\mu(K) < \infty \forall$  compact  $K \subset X$ . Then the space  $C_c(X)$  of continuous compactly supported functions is dense in  $\mathcal{L}^p(X, d\mu)$ .

In particular, either theorem shows that if  $\mu$  is a Borel measure on  $\mathbb{R}^n$  s.t. the measure of every ball is finite, then  $C_c(\mathbb{R}^n)$  is dense in  $\mathcal{L}^p(\mathbb{R}^n, d\mu)$ ,  $1 \leq p < \infty$ . Later we will see that even  $C_c^\infty(\mathbb{R}^n)$  is dense in  $L^p(\mathbb{R}^n, d\mu)$ .

**Remark.** These results do not extend to  $p = \infty$  in general.

For  $\mu = \lambda_n$  we write simply  $\mathcal{L}^p(\mathbb{R}^n)$ .

**Remark.** Theorem 17.8 in the book is *WRONG*. For example,  $X = \mathbb{Q}$  with the usual metric is  $\sigma$ -compact, supports nonzero finite measure, but  $C_c(\mathbb{Q}) = 0$ .

## Modes of Convergence (mixture of ex. 11.12 and ch. 22 p. 258-261.)

Assume  $(X, \mathcal{B}, \mu)$  is a measure space. Given measurable functions  $f_n, f : X \rightarrow \mathbb{C}$ , recall that

$$f_n \rightarrow f \text{ a.e.}$$

means that  $f_n(x) \xrightarrow{n \rightarrow \infty} f(x)$  for all  $x$  outside a set of measure zero.

**Theorem 17.4** (Egorov). Assume  $\mu(X) < \infty$  and  $f_n \rightarrow f$  a.e. Then,  $\forall \epsilon > 0$ , there exists  $X_\epsilon \in \mathcal{B}$  s.t.  $\mu(X_\epsilon) < \epsilon$  and  $f_n \rightarrow f$  uniformly on  $X \setminus X_\epsilon$ .

In addition to pointwise and uniform convergence we also consider the following:

$f_n \rightarrow f$  in the  $p$ -th mean if  $\|f_n - f\|_p \xrightarrow{n \rightarrow \infty} 0$ . For  $p = 1$  we say in mean, for  $p = 2$  we say in quadratic mean.

$f_n \rightarrow f$  in measure if  $\forall \epsilon > 0$  we have

$$\mu(\{x : |f_n(x) - f(x)| \geq \epsilon\}) \xrightarrow{n \rightarrow \infty} 0.$$

**Theorem 17.5** (Lemma 22.4 in the book?). Assume  $(X, \mathcal{B}, d\mu)$  is a measure space,  $1 \leq p < \infty$ ,  $f_n, f : X \rightarrow \mathbb{C}$  are measurable functions. Then

(i) If  $f_n \rightarrow f$  in the  $p$ -th mean, then  $f_n \rightarrow f$  in measure.

(ii) If  $f_n \rightarrow f$  in measure, then there is a subsequence  $(f_{n_k})_{k=1}^\infty$  s.t.  $f_{n_k} \rightarrow f$  a.e.

(iii) If  $f_n \rightarrow f$  a.e. and  $\mu(X) < \infty$ , then  $f_n \rightarrow f$  in measure.

In particular, if  $f_n \rightarrow f$  in the  $p$ -th mean, then  $f_{n_k} \rightarrow f$  a.e. for a subsequence  $(f_{n_k})_k$ .

## 26 Abstract Hilbert Spaces (lecture 13, 22. Feb)

Assume  $\mathcal{H}$  is a vector space over  $\mathbb{C}$ .

**Definition 26.1.** A pre-inner product on  $\mathcal{H}$  is a map  $(\cdot, \cdot) : H \times H \rightarrow \mathbb{C}$  which is

- (i) Sesquilinear: linear in the first variable and antilinear in the second:

$$\begin{aligned}(\alpha u + \beta v, w) &= \alpha(u, w) + \beta(v, w), \\(w, \alpha u + \beta v) &= \bar{\alpha}(u, w) + \bar{\beta}(v, w), \quad u, v, w \in H \text{ and } \alpha, \beta \in \mathbb{C}.\end{aligned}$$

- (ii) Hermitian:  $(u, v) = \overline{(v, u)}$ .

- (iii) Positive semidefinite:  $(u, u) \geq 0$ .

It is called an **inner product**, or a scalar product, if instead of (iii) the map is positive definite;  $(u, u) > 0$ . This definition also works for  $\mathbb{R}$  instead of  $\mathbb{C}$ .

**Cauchy-Schwartz inequality** If  $(\cdot, \cdot)$  is a pre-inner product, then  $|(u, v)| \leq (u, u)^{1/2}(v, v)^{1/2}$ .

**Corollary 26.2.** Assume we have a seminorm  $\|u\| := (u, u)^{1/2}$ . It is a norm iff  $(\cdot, \cdot)$  is an inner product.

**Definition 26.3** (Hilbert space). A Hilbert space is a complex vector space  $\mathcal{H}$  with an inner product  $(\cdot, \cdot)$  s.t.  $\mathcal{H}$  is complete with respect to the norm  $\|u\| = (u, u)^{1/2}$ .

1. The norm on a Hilbert space is determined by the inner product, but the inner product can also be recovered by the norm by the *polarization identity*:  $(u, v) = \frac{\|u+v\|^2 - \|u-v\|^2}{4} + i \frac{\|u+iv\|^2 - \|u-iv\|^2}{4}$ .
2. *Parallelogram law*:  $\|u+v\|^2 + \|u-v\|^2 = 2\|u\|^2 + 2\|v\|^2$ .
3. A norm on a vector space is given by an inner product iff it satisfies the parallelogram law, and then the scalar product is uniquely determined by the polarization identity.

Recall that a subset  $\mathcal{C}$  of a vector space is called *convex* if

$$u, w \in \mathcal{C} \rightarrow tu + (1-t)w \in \mathcal{C} \quad \forall t \in (0, 1).$$

The following is one of the key properties of the Hilbert space

**Theorem 26.4** (projection theorem). Assume  $\mathcal{H}$  is a Hilbert space and  $\mathcal{C} \subset H$  is a closed convex subset. Then for every  $u \in H$  there is a unique  $u_0 \in \mathcal{C}$  (minimizer) s.t.

$$\|u - u_0\| = d(u, \mathcal{C}) (= \inf_{x \in \mathcal{C}} \|u - x\|).$$



## Orthogonal Projections (lecture 14, 26. Feb.)

For a Hilbert space  $\mathcal{H}$  and a subset  $A \subset H$ , let

$$A^\perp := \{x \in H : x \perp y \ \forall y \in A\},$$

where  $x \perp y$  means that  $(x, y) = 0$ .  $A^\perp$  is a closed subspace of  $\mathcal{H}$ .

**Proposition 26.5.** *Assume  $\mathcal{H}_0$  is a closed subspace of a Hilbert space  $\mathcal{H}$ . Then every  $u \in H$  uniquely decomposes as*

$$u = u_0 + u_1, \text{ with } u_0 \in \mathcal{H}_0 \text{ and } u_1 \in \mathcal{H}_0^\perp.$$

Moreover,  $\|u - u_0\| = d(u, \mathcal{H}_0)$  and  $\|u\|^2 = \|u_0\|^2 + \|u_1\|^2$ .

For a closed subspace  $\mathcal{H}_0 \subset \mathcal{H}$ , consider the map  $P : H \rightarrow \mathcal{H}_0$  s.t.  $Pu \in \mathcal{H}_0$  is the unique element satisfying  $u - Pu \in \mathcal{H}_0^\perp$ . The operator  $P$  is linear. It is also contractive, meaning that  $\|Pu\| \leq \|u\|$ , since  $\|u\|^2 = \|Pu\|^2 + \|u - Pu\|^2$ . It is called the orthogonal projection onto  $\mathcal{H}_0$ .

If  $\mathcal{H}_0$  is finite dimensional with an orthonormal basis  $u_1, \dots, u_n$  then

$$Pu = \sum_{k=1}^n (u, u_k) u_k.$$

Orthonormal bases can be defined for arbitrary Hilbert spaces.

**Definition 26.6** (orthonormal system). An orthonormal system in  $\mathcal{H}$  is a collection of vectors  $u_i \in H$  ( $i \in I$ ) s.t.

$$(u_i, u_j) = \delta_{ij} \ \forall i, j \in I.$$

It is called an *orthonormal basis* if  $\text{span}\{u_i\}_{i \in I}$  denotes the linear span of  $\{u_i\}_{i \in I}$ , the space of finite linear combinations of the vectors  $u_i$ .

**Definition 26.7.** A Hilbert space  $\mathcal{H}$  is said to be *separable* if  $\mathcal{H}$  contains a countable dense subset  $G \subset \mathcal{H}$ .

**Theorem 26.8.** *Every Hilbert space  $\mathcal{H}$  has an orthonormal basis. If  $\mathcal{H}$  is separable, then there is a countable orthonormal basis.*

**Proposition 26.9.** *Assume  $\{u_i\}_{i \in I}$  is an orthonormal system in a Hilbert space  $H$ . Take  $u \in \mathcal{H}$ . Then*

(i) Bessel's inequality:  $\sum_{i \in I} |(u, u_i)|^2 \leq \|u\|^2$ , in particular,  $\{i : (u, u_i) \neq 0\}$  is countable.

(ii) Parseval's identity: *If  $\{u_i\}_{i \in I}$  is an orthonormal basis, then*  

$$\sum_{i \in I} |(u, u_i)|^2 = \|u\|^2.$$

If  $(u_i)_{i \in I}$  is an orthonormal basis, then the numbers  $(u, u_i)$  are called the **Fourier coefficients** of  $u$  with respect to  $(u_i)_{i \in I}$ . The Parseval identity then suggests that  $u$  is determined by its Fourier coefficients. This is true, and even more, we have:

**Proposition 26.10.** Assume  $(u_i)_{i \in I}$  is an orthonormal basis in a Hilbert space  $\mathcal{H}$ . Then for every vector  $(c_i)_{i \in I} \in l^2(I)$  there is a unique vector  $u \in \mathcal{H}$  with Fourier coefficients  $c_i$ , and we write

$$u = \sum_{i \in I} c_i u_i.$$

**Remark.** Equivalently, the element  $u = \sum_{i \in I} c_i u_i$  can be described as the unique element in  $\mathcal{H}$  s.t.  $\forall \epsilon > 0$  there is a finite  $F_0 \subset I$  s.t.  $\|u - \sum_{i \in F} c_i u_i\| < \epsilon \forall$  finite  $F \supset F_0$ .

**Corollary 26.11.** We have a linear isomorphism  $U : l^2(I) \xrightarrow{\sim} \mathcal{H}$ ,  $U((c_i)_{i \in I}) = \sum_{i \in I} c_i u_i$ . By Parseval's identity this isomorphism is isometric, that is,  $\|Ux\| = \|x\| \forall x \in l^2(I)$ . By the polarization identity this is equivalent to

$$(Ux, Uy) = (x, y) \forall x, y \in l^2(I).$$

Therefor  $U$  is unitary.

**Corollary 26.12.** Up to a unitary isomorphism, there is only one infinite dimensional separable Hilbert space, namely,  $l^2$ .

## Dual spaces (lecture 15, 29. Feb.)

**Lemma 26.13.** Assume  $V$  is a normed space over  $K = \mathbb{R}$  or  $K = \mathbb{C}$ . Consider a linear functional  $f : V \rightarrow K$ . The following are equivalent (TFAE):

- (i)  $f$  is continuous;
- (ii)  $f$  is continuous at 0;
- (iii) There is a  $c \geq 0$  s.t.  $|f(x)| \leq c\|x\| \forall x \in V$ .

If (i)-(iii) are satisfied, then  $f$  is called a *bounded linear functional*. The constant  $c$  in (iii) is denoted by  $\|f\|$ . We have  $\|f\| = \sup_{x \neq 0} \frac{|f(x)|}{\|x\|} = \sup_{\|x\|=1} |f(x)| = \sup_{\|x\| \leq 1} |f(x)|$ .

**Proposition 26.14.** For every normed vector space  $V$  over  $K = \mathbb{R}$  or  $K = \mathbb{C}$ , the bounded linear functionals on  $V$  form a Banach space  $V^*$ .

**Remark.** The sequence  $\{\|f_n - f_m\|\}_{m=1}^\infty$  actually converges, since

$$\left| \|f_n - f_m\| \right| \leq \|f_m - f_n\|.$$

When we study/use normed spaces, it is often important to understand the dual spaces. For Hilbert spaces this is particularly easy:

**Theorem 26.15 (Riesz).** Assume  $\mathcal{H}$  is a Hilbert space. Then every  $f \in \mathcal{H}^*$  has the form

$$f(x) = (x, y),$$

for a uniquely defined  $u \in \mathcal{H}$ . Moreover, we have  $\|f\| = \|y\|$ .

For every Hilbert space  $\mathcal{H}$  we can define the *conjugate Hilbert space*  $\bar{\mathcal{H}}$ , which has its elements as the symbols  $\bar{x}$  for  $x \in \mathcal{H}$ , with the linear structure and inner product defined by

$$\bar{x} + \bar{y} = \overline{x + y}, c \cdot \bar{x} = \overline{\bar{c}x}, (\bar{x}, \bar{y}) = \overline{(x, y)} = (y, x).$$

**Corollary 26.16.** *For every Hilbert space  $\mathcal{H}$ , we have an isometric isomorphism  $\bar{\mathcal{H}} \xrightarrow{\sim} \mathcal{H}^*$ ,  $\bar{x} \mapsto (\cdot, x)$ .*

## Appendix

### H Regularity of measures (lecture 10, 12. Feb.)

We let  $(X, d)$  be a metric space and denote by  $\mathcal{O}$  the open, by  $\mathcal{C}$  the closed and  $\mathcal{B}(X) = \sigma(\mathcal{O})$  the Borel set of  $X$ .

**Definition H.1.** A measure  $\mu$  on  $(X, d, \mathcal{B}(X))$  is called outer regular, if

$$\mu(B) = \inf \{ \mu(U) \mid B \subset U, U \text{ open} \} \quad (48)$$

and inner regular, if  $\mu(K) < \infty$  for all compact sets  $K \subset X$  and

$$\mu(U) = \sup \{ \mu(K) \mid K \subset U, K \text{ compact} \}. \quad (49)$$

A measure which is both inner and outer regular is called **regular**. We write  $\mathfrak{m}_r^+(X)$  for the family of regular measures on  $(X, \mathcal{B}(X))$ .

**Remark.** The space  $X$  is called  $\sigma$ -compact if there is a sequence of compact sets  $K_n \uparrow X$ . A typical example of such a space is a locally compact, separable metric space.

**Theorem H.2.** Let  $(X, d)$  be a metric space. Every finite measure  $\mu$  on  $(X, \mathcal{B}(X))$  is outer regular. If  $X$  is  $\sigma$ -compact, then  $\mu$  is also inner regular, hence regular.

**Theorem H.3.** Let  $(X, d)$  be a metric space and  $\mu$  be a measure on  $(X, \mathcal{B}(X))$  such that  $\mu(K) < \infty$  for all compact sets  $K \subset X$ .

- 1 If  $X$  is  $\sigma$ -compact, then  $\mu$  is inner regular.
- 2 If there exists a sequence  $G_n \in \mathcal{O}$ ,  $G_n \uparrow X$  such that  $\mu(G_n) < \infty$ , then  $\mu$  is outer regular.

### Hahn-Banach Theorem

**Theorem H.4** (Hahn-Banach). Assume  $V$  is a real vector space,  $V_0 \subset V$  a subspace,  $e : V \rightarrow \mathbb{R}$  a convex function and  $f : V_0 \rightarrow \mathbb{R}$  a linear functional s.t.  $f \leq e$  on  $V_0$ . Then  $f$  can be extended to a linear functional  $F$  on  $V$  s.t.  $F \leq e$ .

**Theorem H.5** (Hahn-Banach). Assume  $V$  is a real or complex vector space,  $p$  a seminorm on  $V_0$ ,  $V_0 \subset V$ , and  $f$  a linear functional on  $V_0$  s.t.

$$|f(x)| \leq p(x) \quad \forall x \in V_0.$$

Then  $f$  can be extended to a linear functional  $F$  on  $V$  s.t.  $|F(x)| \leq p(x) \quad \forall x \in V$ .

**Corollary H.6.** Assume  $V$  is a normed real or complex vector space,  $V_0 \subset V$  and  $f \in V_0^*$ . Then there is a  $F \in V^*$  s.t.

$$F|_{V_0} = f \text{ and } \|F\| = \|f\|.$$

**Corollary H.7.** Assume  $V$  is a normed space and  $x \in V, x \neq 0$ . Then there is a  $F \in V^*$  s.t.  $\|F\| = 1$  and  $F(x) = \|x\|$ .

Such an  $F$  is called a *supporting functional at  $x$* .

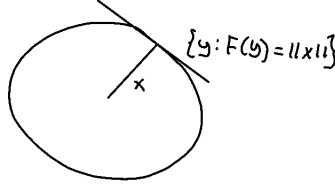


Figure 1: Tangent space?

If  $V$  is a normed vector space, then every  $x \in X$  defines a bounded linear functional on  $V^*$  by

$$V^* \ni f \mapsto f(x).$$

As  $|f(x)| \leq \|f\| \cdot \|x\|$ , this functional has norm  $\leq \|x\|$ . By using a supporting functional at  $x$ , we actually see that we get norm  $\|x\|$ . Thus, we have an isometric embedding  $V \subset V^{**} := (V^*)^*$ . We can therefore see  $V$  as a subspace of  $V^{**}$ .

**Definition H.8.** A normed space  $V$  is called reflexive if  $V^{**} = V$ .

**Remark.** This is stronger than requiring  $V \simeq V^{**}$ .

**Remark.** Every Hilbert space  $\mathcal{H}$  is reflexive. Indeed,  $\mathcal{H}^* = \bar{\mathcal{H}}$ . By Riesz' theorem every bounded linear functional  $f$  on  $\bar{\mathcal{H}}$  has the form

$$f(\bar{x}) = (\bar{x}, \bar{y}) = (y, x),$$

for some  $y \in \mathcal{H}$ , which exactly means that  $f = y$  in  $\mathcal{H}^{**}$ .

As we will see later, the spaces  $\mathcal{L}^p(X, d\mu)$ , with  $\mu$   $\sigma$ -finite and  $1 < p < \infty$ , are reflexive. The spaces  $\mathcal{L}'(X, d\mu)$  and  $\mathcal{L}^\infty(X, \mu)$  are usually not reflexive.