

MAT4400: Notes on Linear analysis (ONLY IMPORTANT STUFF)

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1 σ -Algebras (3, [Schilling(2017)])

Definition 1.1 (σ -algebra). A family \mathcal{A} of subsets of X with:

- (i) $X \in \mathcal{A}$,
- (ii) $A \in \mathcal{A} \Rightarrow A^c \in \mathcal{A}$,
- (iii) $(A_n)_{n \in \mathbb{N}} \in \mathcal{A} \Rightarrow \bigcup_{n \in \mathbb{N}} A_n \in \mathcal{A}$

A set $A \in \mathcal{A}$ is said to be **measurable** or \mathcal{A} -**measurable**.

Theorem 1.2 (and Definition).

- (i) The intersection of arbitrarily many σ -algebras in X is again a σ -algebra in X .
- (ii) For every system of sets $p \subset \mathcal{P}(X)$ there exists a smallest σ -algebra containing p . This is the σ -algebra generated by p , denoted $\sigma(p)$, and $\sigma(p)$ is called its generator.

Definition 1.3 (Borel). The σ -algebra $\sigma(\mathcal{O})$ generated by the open sets $\mathcal{O} = \mathcal{O}_{\mathbb{R}^n}$ of \mathbb{R}^n is called **Borel σ -algebra**, and its members are called **Borel sets** or **Borel measurable sets**.

Definition 1.4 (measure). A measure μ on X is a map $\mu : \mathcal{A} \rightarrow [0, \infty]$ satisfying

- (i) \mathcal{A} is a σ -algebra in X ,
- (ii) $\mu(\emptyset) = 0$,
- (iii) $(A_n)_{n \in \mathbb{N}} \subset \mathcal{A}$ pairwise disjoint $\iff \mu(\bigcup_{n \in \mathbb{N}} A_n) = \sum_{n \in \mathbb{N}} \mu(A_n)$.

Definition 1.5 (σ -finite/sigma-finite). A measure μ is said to be σ -finite and (X, \mathcal{A}, μ) is called a σ -finite measure space, if \mathcal{A} contains a sequence $(A_n)_{n \in \mathbb{N}}$ s.t. $A_n \uparrow X$ and $\mu(A_n) < \infty$.

3 Uniqueness of Measures (5, [Schilling(2017)])

Lemma 3.1. A Dynkin system D is a σ -algebra iff it is stable under finite intersections, i.e. $A, B \in D \Rightarrow A \cap B \in D$.

Theorem 3.2 (Dynkin). Assume X is a set, S is a collection of subsets of X closed under finite intersections, that is, if $A, B \in S \Rightarrow A \cap B \in S$. Then $D(S) = \sigma(S)$.

Theorem 3.3 (uniqueness of measures). Let (X, \mathcal{B}) be a measurable space, and $S \subset \mathcal{P}(X)$ be the generator of \mathcal{B} , i.e. $\mathcal{B} = \sigma(S)$. If S satisfies the following conditions:

1. S is stable under finite intersections (\cap -stable), i.e. $A, C \in S \Rightarrow A \cap C \in S$.
2. There exists an exhausting sequence $(G_n)_{n \in \mathbb{N}} \subset S$ with $G_n \uparrow X$. Assume also that there are two measures μ, ν satisfying:
3. $\mu(A) = \nu(A), \forall A \in S$.
4. $\mu(G_n) = \nu(G_n) < \infty$.

Then $\mu = \nu$.

Theorem 3.4 (**Carathéodory extension theorem**). Assume X is a set and $\mu^* : \mathcal{P}(X) \rightarrow [0, +\infty]$ is an outer measure. Let $\mathcal{B} \subset \mathcal{P}(X)$ be the collection of sets A s.t.

$$\mu^*(B) = \mu^*(A \cap B) + \mu^*(A^c \cap B) \quad \forall B \subset X,$$

(such sets are called **Carathéodory measurable** with respect to μ^* .) then \mathcal{B} is a σ -algebra and $\mu := \mu^*|_{\mathcal{B}}$ is a measure on (X, \mathcal{B}) .

4 Existence of Measures (6, [Schilling(2017)])

Theorem 4.1 (**Carathéodory**). Let $S \subset \mathcal{P}(X)$ be a semi-ring and $\mu : S \rightarrow [0, \infty)$ a pre-measure. Then μ has an extension to a measure μ^* on $\sigma(S)$, i.e. that $\mu(s) = \mu^*(s), \forall s \in \sigma(S)$.

Also, if S contains an exhausting sequence, $S_n \uparrow X$, s.t. $\mu(S_n) < \infty$, then the extension is unique.

Definition 4.2 (**Outer measure**). An outer measure is a function $\mu^* : \mathcal{P}(X) \rightarrow [0, \infty)$ with the following properties:

1. $\mu^*(\emptyset) = 0$,
2. $A \subset B \Rightarrow \mu^*(A) \leq \mu^*(B)$,
3. $\mu^*\left(\bigcup_{n \in \mathbb{N}} A_n\right) \leq \sum_{n \in \mathbb{N}} \mu^*(A_n)$,

5 Measurable Mappings (7, [Schilling(2017)])

We consider maps $T : X \rightarrow X'$ between two measurable spaces (X, \mathcal{A}) and (X', \mathcal{A}') which respects the measurable structures, the σ -algebras on X and X' . These maps are useful as we can transport a measure μ , defined on (X, \mathcal{A}) , to (X', \mathcal{A}') .

Definition 5.1. Let $(X, \mathcal{A}), (X', \mathcal{A}')$ be measurable spaces. A map $T : X \rightarrow X'$ is called \mathcal{A}/\mathcal{A}' -measurable if the pre-image of every measurable set is a measurable set:

$$T^{-1}(A') \in \mathcal{A}, \quad \forall A' \in \mathcal{A}'.$$

- $T^{-1}(A') := \{x \in X : f(x) \in A'\}$

- A $\mathcal{B}(\mathbb{R}^n)/\mathcal{B}(\mathbb{R}^m)$ measurable map is often called a Borel map.
- The notation $T : (X, \mathcal{A}) \rightarrow (X', \mathcal{A}')$ is often used to indicate measurability of the map T .

Lemma 5.2. *Let (X, \mathcal{A}) , (X', \mathcal{A}') be measurable spaces and let $\mathcal{A}' = \sigma(\mathcal{G}')$. Then $T : X \rightarrow X'$ is \mathcal{A}/\mathcal{A}' -measurable iff $T^{-1}(\mathcal{G}') \subset \mathcal{A}$, i.e. if*

$$T^{-1}(G') \in \mathcal{A}, \quad \forall G' \in \mathcal{G}'.$$

Theorem 5.3. *Let (X_i, \mathcal{A}_i) , $i = 1, 2, 3$, be measurable spaces and $T : X_1 \rightarrow X_2$, $S : X_2 \rightarrow X_3$ be $\mathcal{A}_1/\mathcal{A}_2$ and $\mathcal{A}_2/\mathcal{A}_3$ -measurable maps respectively. Then $S \circ T : X_1 \rightarrow X_3$ is $\mathcal{A}_1/\mathcal{A}_3$ -measurable.*

Corollary 5.4. *Every continuous map between metric spaces is a Borel map.*

Definition 5.5. (and lemma) Let $(T_i)_{i \in I}$, $T_i : X \rightarrow X_i$, be arbitrarily many mappings from the same space X into measurable spaces (X_i, \mathcal{A}_i) . The smallest σ -algebra on X that makes all T_i simultaneously measurable is

$$\sigma(T_i : i \in I) := \sigma\left(\bigcup_{i \in I} T_i^{-1}(\mathcal{A}_i)\right)$$

Corollary 5.6. *A function $f : (X, \mathcal{B}) \rightarrow \mathbb{R}$ is measurable if $f^{-1}((a, +\infty)) \in \mathcal{B}$, $\forall a \in \mathbb{R}$.*

Corollary 5.7. *Assume (X, \mathcal{B}) is a measurable space, (Y, d) is a metric space, and $(f_n : (X, \mathcal{B}) \rightarrow Y)_{n=1}^{\infty}$ is a sequence of measurable maps. Assume this sequence of images $(f_n(x))_{n=1}^{\infty}$ is convergent in $Y \forall x \in X$. Define*

$$f : X \rightarrow Y, \quad \text{by } f(x) = \lim_{n \rightarrow \infty} f_n(x).$$

Then f is measurable.

Theorem 5.8. *Let (X, \mathcal{A}) , (X', \mathcal{A}') be measurable spaces and $T : X \rightarrow X'$ be an \mathcal{A}/\mathcal{A}' -measurable map. For every measurable μ on (X, \mathcal{A}) ,*

$$\mu'(A') := \mu(T^{-1}(A')), \quad A' \in \mathcal{A}',$$

defines a measure on (X', \mathcal{A}') .

Definition 5.9. The measure $\mu'(\cdot)$ in the above theorem is called the **pushforward or image measure** of μ under T and it is denoted as $T(\mu)(\cdot)$, $T_*\mu(\cdot)$ or $\mu \circ T^{-1}(\cdot)$.

Theorem 5.10. *If $T \in \mathbb{R}^{n \times n}$ is an orthogonal matrix, then $\lambda^n = T(\lambda^n)$.*

Theorem 5.11. *Let $S \in \mathbb{R}^{n \times n}$ be an invertible matrix. Then*

$$S(\lambda^n) = |\det S|^{-1} \lambda^n = |\det S|^{-1} \lambda^n.$$

Corollary 5.12. *Lebesgue measure is invariant under motions: $\lambda^n = M(\lambda^n)$ for all motions M in \mathbb{R}^n . In particular, congruent sets have the same measure. Two sets of points are called **congruent** if, and only if, one can be transformed into the other by an isometry.*

Measurable Functions (8, [Schilling(2017)])

A *measurable function* is a measurable map $u : X \rightarrow \mathbb{R}$ from some measurable space (X, \mathcal{A}) to $(\mathbb{R}, \mathcal{B}(\mathbb{R}^1))$. They play central roles in the theory of integration.

We recall that $u : X \rightarrow \mathbb{R}$ is $\mathcal{A}/\mathcal{B}(\mathbb{R}^1)$ -measurable if

$$u^{-1}(B) \in \mathcal{A}, \quad \forall B \in \mathcal{B}(\mathbb{R}^1).$$

Moreover from a lemma from chapter 7, we actually only need to show that

$$u^{-1}(G) \in \mathcal{A}, \quad \forall G \in \mathcal{G} \text{ where } \mathcal{G} \text{ generates } \mathcal{B}(\mathbb{R}^1).$$

Proposition 5.13.

- 1 If $f, g : (X, \mathcal{B}) \rightarrow \mathbb{C}$ are measurable, then the function $f + g$, $f \cdot g$, cf , ($c \in \mathbb{C}$) are measurable.
- 2 If $f : (\mathbb{C}, \mathcal{B}) \rightarrow \mathbb{C}$ is measurable and $h : \mathbb{C} \rightarrow \mathbb{C}$ is Borel measurable, then $h \circ f$ is measurable.
- 3 If $f(x) = \lim_{n \rightarrow \infty} f_n(x)$, $x \in X$ and f_n are measurable, then f is measurable.
- 4 If $X = \bigcup_{n=1}^{\infty} A_n$, ($A_n \in \mathcal{B}$), $f|_{A_n} : (A_n, \mathcal{B}_{A_n}) \rightarrow \mathbb{C}$ is measurable $\forall n$, then f is measurable.

Definition 5.14 (simple function). Given a measurable space (X, \mathcal{B}) , a measurable function $f : (X, \mathcal{B}) \rightarrow \mathbb{C}$ is called simple if

$$f(x) = \sum_{k=1}^N c_k \mathbb{1}_{A_k}(x),$$

for some $c_k \in \mathbb{C}$, $A_k \in \mathcal{B}$, where $\mathbb{1}$ is the characteristic function,

$$\mathbb{1}_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A. \end{cases}$$

The representation of simple function is **not** unique. We denote the standard representation of f by

$$f(x) = \sum_{n=0}^N z_n \mathbb{1}_{B_n}(x),$$

for $N \in \mathbb{N}$, $z_n \in \mathbb{R}$, $B_n \in \mathcal{A}$, and

$$X = \bigcup_{n=1}^N B_n,$$

for $B_n \cap B_m = \emptyset$, $n \neq m$. The set of simple functions is denoted $\mathcal{E}(\mathcal{A})$ of \mathcal{E} .

Definition 5.15. Assume μ is a measure on (X, \mathcal{B}) . Given a *positive* simple function

$$f = \sum_{k=1}^N c_k \mathbb{1}_{A_k}, \quad (c_k \geq 0).$$

We define

$$\int_X f d\mu = \sum_{k=1}^N c_k \mu(A_k) \in [0, +\infty].$$

We also denote this by $I_\mu(f)$.

Lemma 5.16. *This is well defined, that is, $\int_X f d\mu$ does not depend on the presentation of the simple function f .*

Properties 5.17. *For every positive simple function*

$$1 \quad \int_X c f d\mu = c \int_X f d\mu, \quad \text{for only } c \geq 0$$

$$2 \quad \int_X (f + g) d\mu = \int_X f d\mu + \int_X g d\mu.$$

Corollary 5.18. *If $f \geq g \geq 0$ are simple functions, then*

$$\int_X f d\mu \geq \int_X g d\mu.$$

Remark. *This means that any measurable function can be approximated by simple functions.*

Properties 5.19. *Measurable functions like this have the following properties*

$$1 \quad \int_X c f d\mu = c \int_X f d\mu, \quad \forall c \geq 0.$$

$$2 \quad \text{If } f \geq g \geq 0, \text{ then } \int_X f d\mu \geq \int_X g d\mu \text{ for any measurable } g, f.$$

$$3 \quad \text{If } f \geq 0 \text{ is simple, then } \int_X f d\mu \text{ is the same value as obtained before.}$$

To advance in measure theory we consider measurable functions

$$f : X \rightarrow [0, +\infty].$$

Measurability is understood w.r.t the σ -algebra $\mathcal{B}([0, +\infty])$ generated by $\mathcal{B}([0, +\infty])$ and $\{+\infty\}$. In other words, $A \subset [0, +\infty] \in \mathcal{B}([0, +\infty])$ iff $A \cap [0, +\infty) \in \mathcal{B}([0, +\infty))$.

Remark. *Hence $f : X \rightarrow [0, +\infty]$ is measurable iff $f^{-1}(A)$ is measurable $\forall A \in \mathcal{B}([0, +\infty])$.*

Definition 5.20 (Lebesgue integral). For measurable functions $f : X \rightarrow [0, +\infty]$, we define

$$\int_X f d\mu = \sup \left\{ \int_X g d\mu : f \geq g \geq 0 : g \text{ is simple} \right\} \in [0, +\infty].$$

Theorem 5.21 (Monotone convergence theorem). *Assume (X, \mathcal{B}, μ) is a measure space, $(f_n)_{n=1}^\infty$ is an increasing sequence of measurable positive functions $f_n : X \rightarrow [0, +\infty]$. Define $f(x) = \lim_{n \rightarrow \infty} f_n(x)$. Then f is measurable and*

$$\int_X f d\mu = \lim_{n \rightarrow \infty} \int_X f_n d\mu.$$

Theorem 5.22. *Assume (X, \mathcal{B}) is a measurable space and $f : X \rightarrow [0, +\infty]$ is measurable. Then there are simple functions g_n , s.t.*

$$0 \leq g_1 \leq g_2 \leq \dots, \quad g_n(x) \rightarrow f(x), \quad \forall x \in X.$$

Moreover, if f is bounded, we can choose g_n s.t. the convergence is uniform, that is,

$$\lim_{n \rightarrow \infty} \sup_{x \in X} |g_n(x) - f(x)| = 0.$$

6 Integration of Measurable Functions

(9, [Schilling(2017)])

Through this chapter (X, \mathcal{A}, μ) will be some measure space. Recall that $\mathcal{M}^+(\mathcal{A})$ [$\mathcal{M}_{\mathbb{R}}^+(\mathcal{A})$] are the \mathcal{A} -**measurable positive functions** and $\mathcal{E}(\mathcal{A})$ [$\mathcal{E}_{\mathbb{R}}^+(\mathcal{A})$] are the **positive and simple functions**.

The fundamental idea of *Integration* is to measure the area between the graph of the function and the abscissa. For positive simple functions $f \in \mathcal{E}^+(\mathcal{A})$ in standard representation, this is done easily

$$\text{if } f = \sum_{i=0}^M y_i \mathbb{1}_{A_i} \in \mathcal{E}^+(\mathcal{A}) \quad \text{then} \quad \sum_{i=0}^M y_i \mu(A_i) \quad (1)$$

would be the μ -area enclosed by the graph and the abscissa. We note that the representation of f should not impact the integral of f .

Lemma 6.1. *Let $\sum_{i=0}^M y_i \mathbb{1}_{A_i} = \sum_{k=0}^N z_k \mathbb{1}_{B_k}$ be two standard representations of the same function $f \in \mathcal{E}^+(\mathcal{A})$. Then*

$$\sum_{i=0}^M y_i \mu(A_i) = \sum_{k=0}^N z_k \mu(B_k). \quad (2)$$

Definition 6.2. Let $f = \sum_{i=0}^M y_i \mathbb{1}_{A_i} \in \mathcal{E}^+(\mathcal{A})$ be a simple function in standard representation. Then the number

$$I_\mu(f) = \sum_{i=0}^M y_i \mu(A_i) \in [0, \infty] \quad (3)$$

(which is independent of the representation of f) is called the μ -integral of f .

Proposition 6.3. *Let $f, g \in \mathcal{E}^+(\mathcal{A})$. Then*

$$(i) \quad I_\mu(\mathbb{1}_A) = \mu(A) \quad \forall A \in \mathcal{A}.$$

$$(ii) \quad I_\mu(\lambda f) = \lambda I_\mu(f) \quad \forall \lambda \geq 0.$$

$$(iii) \quad I_\mu(f + g) = I_\mu(f) + I_\mu(g).$$

$$(iv) \quad f \leq g \Rightarrow I_\mu(f) \leq I_\mu(g).$$

In theorem 8.8 we saw that we could for every $u \in \mathcal{M}^+(\mathcal{A})$ write it as an increasing limit of simple functions. By corollary 8.10, the suprema of simple functions are again measurable, so that

$$u \in \mathcal{M}^+(\mathcal{A}) \Leftrightarrow u = \sup_{n \in \mathbb{N}} f_n, f \in \mathcal{E}^+(\mathcal{A}), \\ f_n \leq f_{n+1} \leq \dots$$

We will use this to "inscribe" simple functions (which we know how to integrate) below the graph of a positive measurable function u and exhaust the μ -area below u .

Definition 6.4. Let (X, \mathcal{A}, μ) be a measure space. The (μ) -integral of a positive function $u \in \mathcal{M}_{\mathbb{R}}^+(\mathcal{A})$ is given by

$$\int u d\mu = \sup \{ I_\mu(g) : g \leq u, g \in \mathcal{E}^+(\mathcal{A}) \}, \quad (4)$$

with $\int u d\mu \in [0, +\infty]$. If we need to emphasize the *integration variable*, we write $\int u(x) \mu(dx)$. The key observation is that the integral $\int \dots d\mu$ extends I_μ .

Lemma 6.5. *For all $f \in \mathcal{E}^+(\mathcal{A})$ we have $\int f d\mu = I_\mu(f)$.*

The next theorem is one of many convergence theorems. It shows that we could have defined 4 using any increasing sequence $f_n \uparrow u$ of simple functions $f_n \in \mathcal{E}^+(\mathcal{A})$.

Theorem 6.6 (Beppo Levi). Let (X, \mathcal{A}, μ) be a measure space. For an increasing sequence of functions $(u_n)_{n \in \mathbb{N}} \subset \mathcal{M}_{\mathbb{R}}^+(\mathcal{A})$, $0 \leq u_n \leq u_{n+1} \leq \dots$, we have for the supremum $u = \sup_{n \in \mathbb{N}} u_n \in \mathcal{M}_{\mathbb{R}}^+(\mathcal{A})$ and

$$\int \sup_{n \in \mathbb{N}} u_n d\mu = \sup_{n \in \mathbb{N}} \int u_n d\mu. \quad (5)$$

Note we can write $\lim_{n \rightarrow \infty}$ **instead of** $\sup_{n \in \mathbb{N}}$ as the supremum of an increasing sequence is its limit. Moreover, this theorem holds in $[0, +\infty]$, so the case $+\infty = +\infty$ is possible.

Corollary 6.7. Let $u \in \mathcal{M}_{\mathbb{R}}^+(\mathcal{A})$. Then

$$\int u d\mu = \lim_{n \rightarrow \infty} \int f_n d\mu$$

holds for every sequence $(f_n)_{n \in \mathbb{N}} \subset \mathcal{E}^+(\mathcal{A})$ with $\lim_{n \rightarrow \infty} f_n = u$.

Proposition 6.8. (of integral) Let $u, v \in \mathcal{M}_{\mathbb{R}}^+(\mathcal{A})$. Then

- (i) $\int \mathbb{1}_A d\mu = \mu(A) \quad \forall A \in \mathcal{A}$.
- (ii) $\int \alpha u d\mu = \alpha \int u d\mu \quad \forall \alpha \geq 0$.
- (iii) $\int u + v d\mu = \int u d\mu + \int v d\mu$.
- (iv) $u \leq v \Rightarrow \int u d\mu \leq \int v d\mu$.

Corollary 6.9 (sum of positive functions). Let $(u_n)_{n \in \mathbb{N}} \subset \mathcal{M}_{\mathbb{R}}^+(\mathcal{A})$. Then $\sum_{n=1}^{\infty} u_n$ is measurable and we have

$$\int \sum_{n=1}^{\infty} u_n d\mu = \sum_{n=1}^{\infty} \int u_n d\mu$$

(including the possibility $+\infty = +\infty$.)

Theorem 6.10 (Fatou). Let $(u_n)_{n \in \mathbb{N}} \subset \mathcal{M}_{\mathbb{R}}^+(\mathcal{A})$ be a sequence of positive measurable functions. Then $u = \liminf_{n \rightarrow \infty} u_n$ is measurable and

$$\int \liminf_{n \rightarrow \infty} u_n d\mu \leq \liminf_{n \rightarrow \infty} \int u_n d\mu \quad (6)$$

7 Integrals of Measurable Functions

(10, [Schilling(2017)])

We have defined our integral for positive measurable functions, i.e. functions in $\mathcal{M}^+(\mathcal{A})$. To extend our integral to not only functions in $\mathcal{M}^+(\mathcal{A})$ we first notice that

$$u \in \mathcal{M}_{\mathbb{R}}(\mathcal{A}) \Leftrightarrow u = u^+ - u^-, \quad u^+, u^- \in \mathcal{M}_{\mathbb{R}}^+,$$

i.e. that every measurable function can be written as a sum of **positive** measurable functions.

¹In words, we extend our integral to **positive** measurable functions by noticing that we can write every measurable function as a sum of positive measurable functions, something that we do know how to integrate. We don't want to run into the problem of $\infty - \infty$, thus we require the integral of the positive and negative parts to both (separately) be less than infinity.

Definition 7.1 (μ -integrable). A function $u : X \rightarrow \overline{\mathbb{R}}$ on (X, \mathcal{A}, μ) is μ -integrable, if it is $\mathcal{A}/\mathcal{B}(\overline{\mathbb{R}})$ -measurable and if $\int u^+ d\mu, \int u^- d\mu < \infty$ (recall the definition for the integral of positive measurable functions). Then

$$\int u d\mu := \int u^+ d\mu - \int u^- d\mu \in (-\infty, \infty)$$

is the (μ) -integral of u . We write $\mathcal{L}^1(\mu)$ for the set of all real-valued μ -integrable functions ¹.

Theorem 7.2. Let $u \in \mathcal{M}_{\mathbb{R}}(\mathcal{A})$, then the following conditions are equivalent:

- (i) $u \in \mathcal{L}_{\mathbb{R}}^1(\mu)$.
- (ii) $u^+, u^- \in \mathcal{L}_{\mathbb{R}}^1(\mu)$.
- (iii) $|u| \in \mathcal{L}_{\mathbb{R}}^1(\mu)$.
- (iv) $\exists w \in \mathcal{L}_{\mathbb{R}}^1(\mu)$ with $w \geq 0$ s.t. $|u| \leq w$.

Theorem 7.3 (Properties of the μ -integral). The μ -integral is: **homogeneous, additive, and:**

- (i) $\min\{u, v\}, \max\{u, v\} \in \mathcal{L}_{\mathbb{R}}^1(\mu)$ (lattice property)
- (ii) $u \leq v \Rightarrow \int u d\mu \leq \int v d\mu$ (monotone)
- (iii) $\left| \int u d\mu \right| \leq \int |u| d\mu$ (triangle inequality)

Remark. If $u(x) \pm v(x)$ is defined in $\overline{\mathbb{R}}$ for all $x \in X$ then we can exclude $\infty - \infty$ and the theorem above just says that the integral is linear:

$$\int (au + bv) d\mu = a \int u d\mu + b \int v d\mu.$$

This is always true for real-valued $u, v \in \mathcal{L}^1(\mu) = \mathcal{L}_{\mathbb{R}}^1(\mu)$, making $\mathcal{L}^1(\mu)$ a vector space with addition and scalar multiplication defined by

$$(u + v)(x) := u(x) + v(x), \quad (a \cdot u)(x) := a \cdot u(x),$$

and

$$\int \dots d\mu : \mathcal{L}^1(\mu) \rightarrow \mathbb{R}, \quad u \mapsto \int u d\mu,$$

is a **positive linear functional**.

8 Null sets and the Almost Everywhere

(11, [Schilling(2017)])

Definition 8.1 (null set). A (μ) -null set $N \in \mathcal{N}_{\mu}$ is a measurable set $N \in \mathcal{A}$ satisfying

$$N \in \mathcal{N}_{\mu} \iff N \in \mathcal{A} \text{ and } \mu(N) = 0.$$

This can be used generally about a ‘statement’ or ‘property’, but we will be interested in questions like ‘when is $u(x)$ equal to $v(x)$ ’, and we answer this by saying

$$u = v \text{ a.e.} \Leftrightarrow \{x : u(x) \neq v(x)\} \text{ is (contained in) a } \mu\text{-null set,}$$

i.e.

$$u = v \text{ } \mu\text{-a.e.} \Leftrightarrow \mu(\{x : u(x) \neq v(x)\}) = 0.$$

The last phrasing should of course include that the set $\{x : u(x) \neq v(x)\}$ is in \mathcal{A} .

Theorem 8.2. Let $u \in \mathcal{M}_{\overline{\mathbb{R}}}(\mathcal{A})$, then:

- (i) $\int |u| d\mu = 0 \Leftrightarrow |u| = 0 \text{ a.e.} \Leftrightarrow \mu\{u \neq 0\} = 0$,
 - (ii) $\mathbb{1}_N u \in \mathcal{L}_{\overline{\mathbb{R}}}^1(\mu) \quad \forall N \in \mathcal{N}_{\mu}$,
 - (iii) $\int_N u d\mu = 0$.
- (i) is really useful, later we will define \mathcal{L}^p and the $\|\cdot\|_p$ -(semi)norm. Then (i) means that if we have a sequence u_n converging to u in the $\|\cdot\|_p$ -norm then $u_n(x) = u(x)$ a.e.

Corollary 8.3. Let $u = v$ μ -a.e. Then

- (i) $u, v \geq 0 \Rightarrow \int u d\mu = \int v d\mu$,
- (ii) $u \in \mathcal{L}_{\overline{\mathbb{R}}}^1(\mu) \Rightarrow v \in \mathcal{L}_{\overline{\mathbb{R}}}^1(\mu)$ and $\int u d\mu = \int v d\mu$.

Corollary 8.4. If $u \in \mathcal{M}_{\overline{\mathbb{R}}}(\mathcal{A})$, $v \in \mathcal{L}_{\overline{\mathbb{R}}}^1(\mu)$ and $v \geq 0$ then

$$|u| \leq v \text{ a.e.} \Rightarrow u \in \mathcal{L}_{\overline{\mathbb{R}}}^1(\mu).$$

Proposition 8.5 (Markow inequality). For all $u \in \mathcal{L}_{\overline{\mathbb{R}}}^1(\mu)$, $A \in \mathcal{A}$ and $c > 0$

$$\mu(\{|u| \geq c\} \cap A) \leq \frac{1}{c} \int_A |u| d\mu,$$

if $A = X$, then (obviously)

$$\mu\{|u| \geq c\} \leq \frac{1}{c} \int |u| d\mu.$$

Completions of measure spaces

Definition 8.6 (**complete measure space**). A measure space (X, \mathcal{B}, μ) is called **complete** if whenever $A \in \mathcal{B}$ and $\mu(A) = 0$, we have $B \in \mathcal{B} \quad \forall B \subset A$.

Remark. Any measure space can be completed as follows: Let $\bar{\mathcal{B}}$ be the σ -algebra generated by \mathcal{B} and all sets $B \subset X$ s.t. there exists $A \in \mathcal{B}$ with $B \subset A$ and $\mu(A) = 0$.

Proposition 8.7. The σ -algebra $\bar{\mathcal{B}}$ can **also be described as follows**:

$$\bar{\mathcal{B}} := \left\{ B \subset X : A_1 \subset B \subset A_2 \right.$$

$$\left. \text{for some } A_1, A_2 \in \mathcal{B} \text{ with } \mu(A_2 \setminus A_1) = 0 \right\},$$

with B, A_1, A_2 as above, we define

$$\bar{\mu} := \mu(A_1) = \mu(A_2)$$

Then $(X, \bar{\mathcal{B}}, \bar{\mu})$ is a complete measure space.

Definition 8.8. If μ is a Borel measure on a **metric** space (X, d) , then the completion $\bar{\mathcal{B}}(X)$ of the Borel σ -algebra with respect to μ is called the σ -algebra of μ -measurable sets.

Remark. For $\mu = \lambda_n$ on \mathbb{R}^n we talk about the σ -algebra of **Lebesgue measurable sets**. Instead of $\bar{\lambda}_n$ we still write λ_n and call it the **Lebesgue measure**. A function $f : \mathbb{R}^n \rightarrow \mathbb{C}$, measurable w.r.t. the σ -algebra of Lebesgue measurable sets is called the **Lebesgue measurable**.

The following result shows that any Lebesgue measurable function coincides with a Borel function a.e.

Proposition 8.9. Assume (X, \mathcal{B}, μ) is a measure space and consider its completion $(X, \bar{\mathcal{B}}, \bar{\mu})$. Assume $f : X \rightarrow \mathbb{C}$ is $\bar{\mathcal{B}}$ -measurable. Then there is a \mathcal{B} -measurable function $g : X \rightarrow \mathbb{C}$ s.t. $f = g$ $\bar{\mu}$ -a.e.

9 Convergence Theorems and their Applications

(12, [Schilling(2017)])

- To interchange limits and integrals in **Riemann integrals** one typically has to assume uniform convergence. - The set of Riemann integrable functions is somewhat limited, see theorem 9.5

Theorem 9.1 (**Generalization of Beppo Levi, monotone convergence**).

- (i) Let $(u_n)_{n \in \mathbb{N}} \subset \mathcal{L}^1(\mu)$ be s.t. $u_1 \leq u_2 \leq \dots$ with limit $u := \sup_{n \in \mathbb{N}} u_n = \lim_{n \rightarrow \infty} u_n$. Then $u \in \mathcal{L}^1(\mu)$ iff

$$\sup_{n \in \mathbb{N}} \int u_n d\mu < +\infty,$$

in which case

$$\sup_{n \in \mathbb{N}} \int u_n d\mu = \int \sup_{n \in \mathbb{N}} u_n d\mu.$$

- (ii) Same thing only with a decreasing sequence $\dots > -\infty$ in which case

$$\inf_{n \in \mathbb{N}} \int u_n d\mu = \int \inf_{n \in \mathbb{N}} u_n d\mu.$$

Theorem 9.2 (**Lebesgue; dominated convergence**). Let $(u_n)_{n \in \mathbb{N}} \subset \mathcal{L}^1(\mu)$ s.t.

- (a) $|u_n|(x) \leq w(x)$, $w \in \mathcal{L}^1(\mu)$ and $\forall n$,
- (b) $u(x) = \lim_{n \rightarrow \infty} u_n(x)$ exists in $\bar{\mathbb{R}}$,

then $u \in \mathcal{L}^1(\mu)$ and we have

- (i) $\lim_{n \rightarrow \infty} \int |u_n - u| d\mu = 0$;
- (ii) $\lim_{n \rightarrow \infty} \int u_n d\mu = \int \lim_{n \rightarrow \infty} u_n d\mu = \int u d\mu$;

Application 2: Riemann vs Lebesgue Integration

Consider only $(X, \mathcal{A}, \mu) = (\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda)$.

Definition 9.3 (The Riemann Integral). Consider on the finite interval $[a, b] \subset \mathbb{R}$ the partition

$$\Pi := \{a = t_0 < t_1 < \dots < t_k < b\}, k = k(\Pi), \quad (7)$$

and introduce

$$S_\Pi[u] := \sum_{i=1}^{k(\Pi)} m_i(t_i - t_{i-1}), \quad m_i := \inf_{x \in [t_{i-1}, t_i]} u(x), \quad (8)$$

$$S^\Pi[u] := \sum_{i=1}^{k(\Pi)} M_i(t_i - t_{i-1}), \quad M_i := \sup_{x \in [t_{i-1}, t_i]} u(x). \quad (9)$$

(10)

A bounded function $u : [a, b] \rightarrow \mathbb{R}$ is said to be **Riemann integrable** if the values

$$\int u := \sup_{\Pi} S_\Pi[u] = \inf_{\Pi} S^\Pi[u] =: \int u \quad (11)$$

coincide and are finite. Their common value is called the **Riemann integral** of u and denoted by $(R) \int_a^b u(x) dx$ or $\int_a^b u(x) dx$.

Theorem 9.4 (**Lebesgue \rightarrow Riemann integrability**). Let $u : [a, b] \rightarrow \mathbb{R}$ be a **measurable** and **Riemann integrable** function. Then

$$u \in \mathcal{L}^1(\lambda) \text{ and } \int_{[a,b]} u d\lambda = \int_a^b u(x) dx. \quad (12)$$

Theorem 9.5 (**Riemann integrability**). Let $u : [a, b] \rightarrow \mathbb{R}$ be a bounded function, it is Riemann integrable **iff** the points in (a, b) where u is discontinuous are a (subset of) Borel measurable null set.

Improper Riemann Integrals

- The Lebesgue integral extends the (*proper*) Riemann integral. However, there is a further extension of the Riemann integral which cannot be captured by Lebesgue's theory. u is Lebesgue integrable *iff* $|u|$ has finite Lebesgue integral. - The Lebesgue integral does not respect sign-changes and cancellations. However, the following *improper Riemann integral* does:

$$(R) \int_0^\infty u(x) dx := \lim_{n \rightarrow \infty} (R) \int_0^n u(x) dx. \quad (13)$$

Corollary 9.6. Let $u : [0, \infty) \rightarrow \mathbb{R}$ be a measurable, Riemann integrable function for every interval $[0, N]$, $N \in \mathbb{N}$. Then $u \in \mathcal{L}^1[0, \infty)$ **iff**

$$\lim_{N \rightarrow \infty} (R) \int_0^N |u(x)| dx < \infty. \quad (14)$$

In this case, $(R) \int_0^\infty u(x) dx = \int_{[0, \infty)} u d\lambda$

Proposition 9.7 (appearing as example 12.13 in Schilling). Let $f_\alpha(x) := x^\alpha$, $x > 0$ and $\alpha \in \mathbb{R}$. Then

$$(i) f_\alpha \in \mathcal{L}^1(0, 1) \Leftrightarrow \alpha > -1.$$

$$(ii) f_\alpha \in \mathcal{L}^1[1, \infty) \Leftrightarrow \alpha < -1.$$

10 Regularity of Measures (App. H, [Schilling(2017)])

We let (X, d) be a metric space and denote by \mathcal{O} the open, by \mathcal{C} the closed and $\mathcal{B}(X) = \sigma(\mathcal{O})$ the Borel set of X .

Definition 10.1 (**outer and inner regular measures**). A measure μ on $(X, d, \mathcal{B}(X))$ is called outer regular, if

$$\mu(B) = \inf \{ \mu(U) \mid B \subset U, U \text{ open} \} \quad (15)$$

and inner regular, if $\mu(K) < \infty$ for all compact sets $K \subset X$ and

$$\mu(U) = \sup \{ \mu(K) \mid K \subset U, K \text{ compact} \}. \quad (16)$$

A measure which is both inner and outer regular is called **regular**. We write $\mathfrak{m}_r^+(X)$ for the family of regular measures on $(X, \mathcal{B}(X))$.

Remark. The space X is called σ -compact if there is a sequence of compact sets $K_n \uparrow X$. A typical example of such a space is a locally compact, separable metric space.

Theorem 10.2. Let (X, d) be a metric space. Every finite measure μ on $(X, \mathcal{B}(X))$ is outer regular. If X is σ -compact, then μ is also inner regular, hence regular.

Theorem 10.3. Let (X, d) be a metric space and μ be a measure on $(X, \mathcal{B}(X))$ such that $\mu(K) < \infty$ for all compact sets $K \subset X$.

- 1 If X is σ -compact, then μ is inner regular.
- 2 If there exists a sequence $G_n \in \mathcal{O}$, $G_n \uparrow X$ such that $\mu(G_n) < \infty$, then μ is outer regular.

11 The Function Spaces \mathcal{L}^p (13, [Schilling(2017)])

Assume V is a vector space over $\mathbb{K} \in \{\mathbb{C}, \mathbb{R}\}$.

Definition 11.1. A seminorm on V is a map $p : V \rightarrow [0, +\infty)$ s.t.

- (1) $p(cx) = |c|p(x) \quad \forall x \in V, \forall c \in \mathbb{K}$.
- (2) $p(x + y) \leq p(x) + p(y) \quad \forall x, y \in V$. **triangle inequality**.

A seminorm is called a norm if we also have

$$p(x) = 0 \iff x = 0.$$

A norm is commonly denoted $\|x\|$, and a vectorspace equipped with a norm is called a **normed space**.

Definition 11.2 (**p-norm**). Assume (X, d) is a measure space. Fix $1 \leq p \leq \infty$. For every measurable function $f : X \rightarrow \mathbb{C}$ we define the following

$$\|f\|_p = \left(\int_X |f|^p d\mu \right)^{1/p} \in [0, +\infty]. \quad (17)$$

We can see that $\|cf\|_p = |c| \|f\|_p \quad \forall c \in \mathbb{C}$.

Notice that by Theorem 8.2(i) we have that $\|f\|_p = 0 \Rightarrow f = 0$ a.e. Consider for example $\lim_{n \rightarrow \infty} \|f_n - f\|_p = 0$, then we can find a subsequence s.t. $\lim_{k \rightarrow \infty} |f_{n(k)} - f| = 0$ a.e., i.e. $\lim_{k \rightarrow \infty} f_{n(k)} = f$ a.e.

Theorem 11.3 (Hölder's inequality). Assume that $u \in \mathcal{L}^p(\mu)$ and $v \in \mathcal{L}^q(\mu)$, where $1/p + 1/q = 1$ and $p, q \in [0, +\infty]$. Then $uv \in \mathcal{L}^1(\mu)$, and the following inequality holds:

$$\left| \int uv d\mu \right| \leq \int |uv| d\mu = \|uv\|_1 \leq \|u\|_p \cdot \|v\|_q.$$

The generalized version reads:

$$\int |u_1 \cdot u_2 \cdots u_N| d\mu \leq \|u_1\|_{p_1} \cdot \|u_2\|_{p_2} \cdots \|u_N\|_{p_N}.$$

Lemma 11.4.

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p. \quad (18)$$

Definition 11.5 (Lebesgue space). We define

$$\mathcal{L}^p(X, d\mu) = \{f : X \rightarrow \mathbb{C} \mid f \text{ is measurable and } \|f\|_p < \infty\}.$$

This is a vectorspace with seminorm $f \mapsto \|f\|_p$. And in general this is not a normed space, since $\|f\|_p = 0 \iff f = 0$ a.e.

Generally, if p is a seminorm on a vectorspace V , then

$$V_0 = \{x \in V \mid p(x) = 0\} \quad (19)$$

which is a subspace of V . Then we consider the quotient/factor space V/V_0 .

Definition 11.6. For $x, y \in V$, define

$$x \sim y \iff x - y \in V_0. \quad (20)$$

This is an equivalence relation on V . The representation class of V is defined by $[x]$ or $x + V_0$.

Then V/V_0 equals the set of equivalence classes. We can show that it is a normed space.

$$[x] + [y] = [x + y], \quad c[x] = [cx], \quad \|[x]\| = p(x).$$

Applying this to $\mathcal{L}^p(X, d\mu)$ we get the normed space

$$L^p(X, d\mu) := \mathcal{L}^p(X, d\mu)/\mathcal{N} = \mathcal{L}^p(X, d\mu)/\sim. \quad (21)$$

Where \mathcal{N} is the space of measurable functions f s.t. $f = 0$ a.e. The equivalence relation \sim is defined by

$$u \sim v \iff \{u \neq v\} \in \mathcal{N}_\mu \iff \mu\{u \neq v\} = 0,$$

and so $L^p(X, d\mu)$ consists of all equivalence classes $[u]_p = \{v \in \mathcal{L}^p \mid u \sim v\}$. So for every $u \in [u]_p$ there is no $v \in [u]_p$ such that $\mu\{u \neq v\} \neq 0$.

We will further continue to denote the norm by $\|\cdot\|_p$, and we will normally **not** distinguish between $f \in \mathcal{L}^p(X, d\mu)$ and the vector in $L^p(X, d\mu)$ that f defines.

Definition 11.7 (Banach space). A normed space $(X, \|\cdot\|)$ is called a Banach space if V is complete w.r.t the metric $d(x, y) = \|x - y\|$.

Theorem 11.8. If (X, \mathcal{B}, μ) is a measure space, $1 \leq p \leq \infty$, then $L^p(X, d\mu)$ is a Banach space.

Definition 11.9. A measurable function $f : X \rightarrow \mathbb{C}$ is called **essentially bounded** if there is $c \geq 0$ s.t.

$$\mu(\{x : |f(x)| > c\}) = 0. \quad (22)$$

That is $|f| \leq c$ a.e. The smallest such c is called the essential supremum of f and is denoted by $\|f\|_\infty$. That is,

$$\|u\|_\infty := \inf \{c > 0 : \mu\{|u| \geq c\} = 0\},$$

and from problem 13.21 we have

$$\lim_{p \rightarrow \infty} \|\cdot\|_p = \|\cdot\|_\infty.$$

Definition 11.10 (L^∞).

$$\mathcal{L}^\infty(X, d\mu) = \{f : X \rightarrow \mathbb{C} \mid f \text{ is measurable and } \|f\|_\infty < \infty\}.$$

$$L^\infty(X, d\mu) = \mathcal{L}^\infty(X, d\mu)/\mathcal{N}.$$

Where by the previous definition these spaces become the spaces of all essentially bounded functions.

Theorem 11.11. If (X, \mathcal{B}, μ) is a σ -finite measure space, then $L^\infty(X, d\mu)$ is a Banach space.

Convergence in \mathcal{L}^p and completeness

Lemma 11.12. For any sequence $(u_n)_{n \in \mathbb{N}} \subset \mathcal{L}^p(\mu)$, $p \in [1, \infty)$, of positive functions $u_n \geq 0$ we have

$$\left\| \sum_{n=1}^{\infty} u_n \right\|_p \leq \sum_{n=1}^{\infty} \|u_n\|_p.$$

Theorem 11.13 (Riesz-Fischer). The spaces $\mathcal{L}^p(\mu)$, $p \in [1, \infty)$, are complete, i.e. every Cauchy sequence $(u_n)_{n \in \mathbb{N}} \subset \mathcal{L}^p(\mu)$ converges to some limit $u \in \mathcal{L}^p(\mu)$

Corollary 11.14. Let $(u_n)_{n \in \mathbb{N}} \subset \mathcal{L}^p(\mu)$, $p \in [1, \infty)$ with $\mathcal{L}^p - \lim_{n \rightarrow \infty} u_n = u$. Then there exists a subsequence $(u_{n_k})_{k \in \mathbb{N}}$ s.t. $\lim_{k \rightarrow \infty} u_{n_k}(x) = u(x)$ holds for almost every $x \in X$.

Theorem 11.15. Let $(u_n)_{n \in \mathbb{N}} \subset \mathcal{L}^p(\mu)$, $p \in [1, \infty)$, be a sequence of functions s.t. $|u_n| \leq w \forall n \in \mathbb{N}$ and some $w \in \mathcal{L}^p(\mu)$. If $u(x) = \lim_{n \rightarrow \infty} u_n(x)$ exists for (almost) every $x \in X$, then

$$u \in \mathcal{L}^p \text{ and } \lim_{n \rightarrow \infty} \|u - u_n\|_p = 0.$$

Theorem 11.16 (F. Riesz (convergence theorem)). Let $(u_n)_{n \in \mathbb{N}} \subset \mathcal{L}^p(\mu)$, $p \in [1, \infty)$, be a sequence s.t. $\lim_{n \rightarrow \infty} u_n(x) = u(x)$ for almost every $x \in X$ and some $u \in \mathcal{L}^p(\mu)$. Then

$$\lim_{n \rightarrow \infty} \|u_n - u\|_p = 0 \iff \lim_{n \rightarrow \infty} \|u_n\|_p = \|u\|_p.$$

12 Dense and Determining Sets (17, [Schilling(2017)])

Definition 12.1 (dense sets). A set $\mathcal{D} \subset \mathcal{L}^p(\mu)$, $p \in [0, \infty]$, is called *dense* if for every $u \in \mathcal{L}^p(\mu)$ there exist a sequence $(f_n)_{n \in \mathbb{N}} \subset \mathcal{D}$ s.t. $\lim_{n \rightarrow \infty} \|u - f_n\|_p = 0$.

Definition 12.2 (support). The support of a function f is the set of points in X where f is non-zero:

$$\text{supp}(f) := \{x \in X : f(x) \neq 0\}.$$

Dense subsets of \mathcal{L}^p :

Theorem 12.3. Let μ be a finite measure on $X, d, \mathcal{B}(X)$. Then $C_b(X) \subset \mathcal{L}^p(\mu)$ is dense.

Theorem 12.4. Assume X, d is a metric space and μ is a Borel measure that is finite on every ball $1 \leq p < \infty$. Then the space of bounded continuous functions with bounded support is dense in $\mathcal{L}^p(X, d\mu)$. Where bounded support means that f vanishes outside some ball.

Theorem 12.5. Assume (X, d) is a separable locally compact metric space and μ is a Borel Measure on X s.t. $\mu(K) < \infty \forall$ compact $K \subset X$. Then the space $C_c(X)$ of continuous compactly supported functions is dense in $\mathcal{L}^p(X, d\mu)$.

Recall that the support of a function f is $\text{supp}(f) = \{x \in X : f(x) \neq 0\}$, *closed support* is the closure of $\text{supp}(f)$ (i.e. boundary points are included), often just written as $\text{supp}(f)$, and a function is said to have *compact support* if $\text{supp}(f)$ is compact.

In particular, either theorem shows that if μ is a Borel measure on \mathbb{R}^n s.t. the measure of every ball is finite, then $C_c(\mathbb{R}^n)$ is dense in $\mathcal{L}^p(\mathbb{R}^n, d\mu)$, $1 \leq p < \infty$. Later we will see that **even $C_c^\infty(\mathbb{R}^n)$ is dense in $\mathcal{L}^p(\mathbb{R}^n, d\mu)$** .

Remark. These results do not extend to $p = \infty$ in general.

For $\mu = \lambda_n$ we write simply $\mathcal{L}^p(\mathbb{R}^n)$.

Remark. Theorem 17.8 in the book is *WRONG*. For example, $X = \mathbb{Q}$ with the usual metric is σ -compact, supports nonzero finite measure, but $C_c(\mathbb{Q}) = 0$.

Modes of Convergence

(mixture of ex. 11.12 and ch. 22 p. 258-261. in [Schilling(2017)])

Assume (X, \mathcal{B}, μ) is a measure space. Given measurable functions $f_n, f : X \rightarrow \mathbb{C}$, recall that

$$f_n \rightarrow f \text{ a.e.}$$

means that $f_n(x) \xrightarrow[n \rightarrow \infty]{} f(x)$ for all x outside a set of measure zero.

Theorem 12.6 (Egorov). Assume $\mu(X) < \infty$ and $f_n \rightarrow f$ a.e. Then, $\forall \epsilon > 0$, there exists $X_\epsilon \in \mathcal{B}$ s.t. $\mu(X_\epsilon) < \epsilon$ and $f_n \rightarrow f$ uniformly on $X \setminus X_\epsilon$.

In addition to pointwise and uniform convergence we also consider the following:

$f_n \rightarrow f$ in the p -th mean if $\|f_n - f\|_p \xrightarrow[n \rightarrow \infty]{} 0$. For $p = 1$ we say *in mean*, for $p = 2$ we say *quadratic mean*.

$f_n \rightarrow f$ in measure if $\forall \epsilon > 0$ we have

$$\mu(\{x : |f_n(x) - f(x)| \geq \epsilon\}) \xrightarrow[n \rightarrow \infty]{} 0.$$

Theorem 12.7. Assume $(X, \mathcal{B}, d\mu)$ is a measure space, $1 \leq p < \infty$, $f_n, f : X \rightarrow \mathbb{C}$ are measurable functions. Then

- (i) If $f_n \rightarrow f$ in the p -th mean, then $f_n \rightarrow f$ in measure.
- (ii) If $f_n \rightarrow f$ in measure, then there is a subsequence $(f_{n_k})_{k=1}^\infty$ s.t. $f_{n_k} \rightarrow f$ a.e.
- (iii) If $f_n \rightarrow f$ a.e. and $\mu(X) < \infty$, then $f_n \rightarrow f$ in measure.

In particular, if $f_n \rightarrow f$ in the p -th mean, then $f_{n_k} \rightarrow f$ a.e. for a subsequence $(f_{n_k})_k$.

13 Abstract Hilbert Spaces (26, [Schilling(2017)])

Assume \mathcal{H} is a vector space over \mathbb{C} .

Definition 13.1. A pre-inner product on \mathcal{H} is a map $(\cdot, \cdot) : \mathcal{H} \times \mathcal{H} \rightarrow \mathbb{C}$ which is

- (i) Sesquilinear: linear in the first variable and antilinear in the second:

$$\begin{aligned} (\alpha u + \beta v, w) &= \alpha(u, w) + \beta(v, w), \\ (w, \alpha u + \beta v) &= \bar{\alpha}(w, u) + \bar{\beta}(w, v), \quad u, v, w \in \mathcal{H} \text{ and } \alpha, \beta \in \mathbb{C}. \end{aligned}$$

- (ii) Hermitian: $(u, v) = \overline{(v, u)}$.

- (iii) Positive semidefinite: $(u, u) \geq 0$.

It is called an **inner product**, or a scalar product, if instead of (iii) the map is positive definite; $(u, u) > 0$. This definition also works for \mathbb{R} instead of \mathbb{C} .

Definition 13.2 (adjoint operator). Assume $T : \mathcal{H}_1 \rightarrow \mathcal{H}_2$ is a linear operator. The adjoint operator T^* is a linear operator $T^* : \mathcal{H}_2 \rightarrow \mathcal{H}_1$ s.t. for all $h_1 \in \mathcal{H}_1$ and $h_2 \in \mathcal{H}_2$,

$$\langle Th_1, h_2 \rangle_{\mathcal{H}_2} = \langle h_1, T^*h_2 \rangle_{\mathcal{H}_1}.$$

Lemma 13.3 (Cauchy-Schwartz inequality). If (\cdot, \cdot) is a pre-inner product, then $|(u, v)| \leq (u, u)^{1/2}(v, v)^{1/2}$.

Corollary 13.4. Assume we have a seminorm $\|u\| := (u, u)^{1/2}$. It is a norm iff (\cdot, \cdot) is an inner product.

Definition 13.5 (Hilbert space). A Hilbert space is a complex vector space \mathcal{H} with an inner product (\cdot, \cdot) s.t. \mathcal{H} is complete with respect to the norm $\|u\| = (u, u)^{1/2}$.

1. The norm on a Hilbert space is determined by the inner product, but the inner product can also be recovered by the norm by the *polarization identity*: $(u, v) = \frac{\|u+v\|^2 - \|u-v\|^2}{4} + i \frac{\|u+iv\|^2 - \|u-iv\|^2}{4}$.
2. *Parallelogram law*: $\|u+v\|^2 + \|u-v\|^2 = 2\|u\|^2 + 2\|v\|^2$.
3. A norm on a vector space is given by an inner product iff it satisfies the parallelogram law, and then the scalar product is uniquely determined by the polarization identity.

Example 13.6. Assume (X, \mathcal{B}, μ) is a measure space. Then $\mathcal{L}^2(X, d\mu)$ is a Hilbert space with inner product

$$(f, g) = \int_X f \bar{g} d\mu.$$

This is well-defined, as $|f\bar{g}| \leq \frac{1}{2}(|f|^2 + |g|^2)$.

In particular, if $\mathcal{B} = \mathcal{P}(X)$ and μ is the counting measure, i.e.

$$\mu(A) = \begin{cases} \# & \text{if } A \text{ is finite,} \\ +\infty & \text{if } A \text{ is infinite,} \end{cases}$$

then $L^2(X, d\mu)$ is denoted by $l^2(X)$; for $X = \mathbb{N}$ we write simply l^2 . Note that in this case for $f : X \rightarrow [0, +\infty]$ we have

$$\int_X f d\mu = \sum_{x \in X} f(x) := \sup_{\substack{F \subset X \\ F \text{ is finite}}} \sum_{x \in F} f(x),$$

and if $\sum_{x \in X} f(x) < \infty$, then $\{x : f(x) > 0\}$ is at most countable, so $\sum_{x \in X} f(x) = \sum_{x: f(x) > 0} f(x)$ is the usual sum of a series.

Recall that a subset C of a vector space is called *convex* if

$$u, w \in C \rightarrow tu + (1-t)w \in C \quad \forall t \in (0, 1).$$

The following is one of the key properties of the Hilbert space

Theorem 13.7 (projection theorem). Assume \mathcal{H} is a Hilbert space and $C \subset H$ is a closed convex subset. Then for every $u \in H$ there is a unique $u_0 \in C$ (minimizer) s.t.

$$\|u - u_0\| = d(u, C) = \inf_{x \in C} \|u - x\|.$$

This minimizer $u_0 = P_C u$ is called the **orthogonal projection** of u onto C .

14 Orthogonal Projections (26, [Schilling(2017)])

For a Hilbert space \mathcal{H} and a subset $A \subset H$, the following is the **orthogonal complement** of A :

$$A^\perp := \{x \in H : x \perp y \quad \forall y \in A\},$$

where $x \perp y$ means that $(x, y) = 0$. A^\perp is a closed subspace of \mathcal{H} .

Proposition 14.1 (decomposition of Hilbert spaces). Assume \mathcal{H}_0 is a closed subspace of a Hilbert space \mathcal{H} . Then every $u \in H$ uniquely decomposes as

$$u = u_0 + u_1, \quad \text{with } u_0 \in \mathcal{H}_0 \text{ and } u_1 \in \mathcal{H}_0^\perp.$$

Moreover, $\|u - u_0\| = d(u, \mathcal{H}_0)$ and $\|u\|^2 = \|u_0\|^2 + \|u_1\|^2$.

For a closed subspace $\mathcal{H}_0 \subset \mathcal{H}$, consider the map $P : H \rightarrow \mathcal{H}_0$ s.t. $Pu \in \mathcal{H}_0$ is the unique element satisfying $u - Pu \in \mathcal{H}_0^\perp$. The operator P is linear. It is also contractive, meaning that $\|Pu\| \leq \|u\|$, since $\|u\|^2 = \|Pu\|^2 + \|u - Pu\|^2$. It is called the **orthogonal projection** onto \mathcal{H}_0 .

If \mathcal{H}_0 is **finite dimensional with an orthonormal basis** u_1, \dots, u_n then

$$Pu = \sum_{k=1}^n (u, u_k) u_k.$$

Orthonormal bases can be defined for arbitrary Hilbert spaces.

Definition 14.2 (orthonormal system). An orthonormal system in \mathcal{H} is a collection of vectors $u_i \in \mathcal{H}$ ($i \in I$) s.t.

$$(u_i, u_j) = \delta_{ij} \quad \forall i, j \in I.$$

It is called an *orthonormal basis* if $\text{span}\{u_i\}_{i \in I}$ denotes the linear span of $\{u_i\}_{i \in I}$, the space of finite linear combinations of the vectors u_i .

Definition 14.3 (separable Hilbert space). A Hilbert space \mathcal{H} is said to be *separable* if \mathcal{H} contains a countable dense subset $G \subset \mathcal{H}$.

Theorem 14.4. Every Hilbert space \mathcal{H} has an orthonormal basis. If \mathcal{H} is separable, then there is a countable orthonormal basis.

Proposition 14.5. Assume $\{u_i\}_{i \in I}$ is an orthonormal system in a Hilbert space \mathcal{H} and let $u \in \mathcal{H}$. Then

(i) Bessel's inequality: $\sum_{i \in I} |(u, u_i)|^2 \leq \|u\|^2$, in particular, $\{i : (u, u_i) \neq 0\}$ is countable.

(ii) Parseval's identity: If $\{u_i\}_{i \in I}$ is an orthonormal basis, then $\sum_{i \in I} |(u, u_i)|^2 = \|u\|^2$.

(iii) $\bigcup_{N=1}^{\infty} E(N)$ is dense in \mathcal{H} where $E(N) = \text{span}\{e_1, \dots, e_N\}$

(iv) $g = \sum_{n=1}^{\infty} \langle g, e_n \rangle e_n \quad \forall g \in \mathcal{H}$. (Fourier coefficients)

If $(u_i)_{i \in I}$ is an orthonormal basis, then the numbers (u, u_i) are called the **Fourier coefficients** of u with respect to $(u_i)_{i \in I}$. The Parseval identity then suggests that u is determined by its Fourier coefficients. This is true, and even more, we have:

Proposition 14.6. Assume $(u_i)_{i \in I}$ is an orthonormal basis in a Hilbert space \mathcal{H} . Then for every vector $(c_i)_{i \in I} \in l^2(I)$ there is a unique vector $u \in \mathcal{H}$ with Fourier coefficients c_i , and we write

$$u = \sum_{i \in I} c_i u_i.$$

Remark. Equivalently, the element $u = \sum_{i \in I} c_i u_i$ can be described as the unique element in \mathcal{H} s.t. $\forall \epsilon > 0$ there is a finite $F_0 \subset I$ s.t. $\|u - \sum_{i \in F} c_i u_i\| < \epsilon \quad \forall \text{ finite } F \supset F_0$.

Corollary 14.7. We have a linear isomorphism $U : l^2(I) \xrightarrow{\sim} \mathcal{H}$, $U((c_i)_{i \in I}) = \sum_{i \in I} c_i u_i$. By Parseval's identity this isomorphism is isometric, that is, $\|Ux\| = \|x\| \quad \forall x \in l^2(I)$. By the polarization identity this is equivalent to

$$(Ux, Uy) = (x, y) \quad \forall x, y \in l^2(I).$$

Therefore U is unitary.

Corollary 14.8. Up to a unitary isomorphism, there is **only one infinite dimensional separable Hilbert space**, namely, l^2 . Recall a unitary isomorphism is a bijective map between spaces, $U : H_1 \rightarrow H_2$ s.t. $\langle Ux, Uy \rangle_{H_2} = \langle x, y \rangle_{H_1}$.

15 Dual spaces (26, [Schilling(2017)])

Given two orthonormal bases $(u_i)_{i \in I}$ and $(v_i)_{i \in I}$ in a Hilbert space \mathcal{H} , we can decompose

$$u_i = \sum_{j \in I} (u_i, v_j) v_j$$

and using that the sets $\{j : (u_i, v_j) \neq 0\}$ are countable prove the following:

Fact: Any two orthonormal bases in a Hilbert space have the same cardinality.

Example 15.1 (classical Fourier series). Consider $\mathcal{H} = L^2(0, 2\pi) = L^2((0, 2\pi), d\lambda)$. For $n \in \mathbb{Z}$, define $e_n(t) = \frac{1}{\sqrt{2\pi}} e^{int}$. By a version of Weierstrass' theorem it is known that $\text{span}\{e_n\}_{n \in \mathbb{Z}}$ is dense in the supremum-norm in

$$\{f \in C[0, 2\pi] : f(0) = f(2\pi)\}.$$

As $C[0, 2\pi]$ is dense in $L^2(0, 2\pi)$, from this one can deduce that $\text{span}\{e_n\}_{n \in \mathbb{Z}}$ is dense in $L^2(0, 2\pi)$. We then see that $(e_n)_{n \in \mathbb{Z}}$ is an orthonormal basis in $L^2(0, 2\pi)$. We therefor have a unitary isomorphism

$$l^2(\mathbb{Z}) \xrightarrow{\sim} L^2(0, 2\pi), (c_n)_{n \in \mathbb{Z}} \mapsto \sum_{n \in \mathbb{Z}} c_n e_n.$$

The Fourier coefficients at $f \in L^2(0, 2\pi)$ are denoted by $\hat{f}(n)$, so

$$\hat{f}(n) = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} f(t) e^{-int} dt,$$

more practically

$$\hat{f}(n) = \frac{1}{\sqrt{2\pi}} \int_{[0, 2\pi)} f(t) e^{-int} d\lambda(t).$$

Therefor we have $f = \sum_{n \in \mathbb{Z}} \hat{f}(n) e_n$ in $L^2(0, 2\pi)$.

Fact: For every $f \in L^2(0, 2\pi)$, we have $\frac{1}{\sqrt{2\pi}} \sum_{n=-N}^N \hat{f}(n) e^{int} \xrightarrow{N \rightarrow \infty} f(t)$ for a.e. t .

Lemma 15.2. Assume V is a normed space over $K = \mathbb{R}$ or $K = \mathbb{C}$. Consider a linear functional $f : V \rightarrow K$. The following are equivalent (TFAE):

- (i) f is continuous;
- (ii) f is continuous at 0;
- (iii) There is a $c \geq 0$ s.t. $|f(x)| \leq c \|x\| \forall x \in V$.

If (i)-(iii) are satisfied, then f is called a **bounded linear functional**. The constant c in (iii) is denoted by $\|f\|$. We have $\|f\| = \sup_{x \neq 0} \frac{|f(x)|}{\|x\|} = \sup_{\|x\|=1} |f(x)| = \sup_{\|x\| \leq 1} |f(x)|$. A bounded linear functional is a generalization of a bounded linear operator: $O : V \rightarrow V'$, where V' is \mathbb{R} or \mathbb{C} .

Proposition 15.3. For every normed vector space V over $K = \mathbb{R}$ or $K = \mathbb{C}$, the bounded linear functionals on V form a Banach space V^* .

Remark. The sequence $\{\|f_n - f_m\|\}_{m=1}^\infty$ actually converges, since

$$\|\|f_n - f_m\|\| \leq \|f_m - f_n\|.$$

When we study/use normed spaces, it is often important to understand the dual spaces. For Hilbert spaces this is particularly easy:

Theorem 15.4 (Riesz). Assume \mathcal{H} is a Hilbert space. Then every $f \in \mathcal{H}^*$ has the form

$$f(x) = (x, y),$$

for a uniquely defined $y \in \mathcal{H}$. Moreover, we have $\|f\| = \|y\|$.

For every Hilbert space \mathcal{H} we can define the **conjugate Hilbert space** $\overline{\mathcal{H}}$, which has its elements as the symbols \bar{x} for $x \in \mathcal{H}$, with the linear structure and inner product defined by

$$\bar{x} + \bar{y} = \overline{x + y}, c \cdot \bar{x} = \overline{cx}, (\bar{x}, \bar{y}) = \overline{(x, y)} = (y, x).$$

Corollary 15.5. For every Hilbert space \mathcal{H} , we have an isometric isomorphism (unitary isomorphism/transformation) $\overline{\mathcal{H}} \xrightarrow{\sim} \mathcal{H}^*$, $\bar{x} \mapsto (\cdot, x)$.

16 Hahn-Banach Theorem (4.2, [Teschl(2010)])

Theorem 16.1 (Hahn-Banach). Assume V is a real vector space, $V_0 \subset V$ a subspace, $\phi : V \rightarrow \mathbb{R}$ a convex function (i.e., $\phi(\lambda x + (1 - \lambda)y) \leq \lambda\phi(x) + (1 - \lambda)\phi(y)$) and $f : V_0 \rightarrow \mathbb{R}$ a linear functional s.t. $f \leq \phi$ on V_0 . Then f can be extended to a linear functional F on V s.t. $F \leq \phi$.

Theorem 16.2 (Hahn-Banach). Assume V is a real or complex vector space, p a seminorm on V , $V_0 \subset V$, and f a linear functional on V_0 s.t.

$$|f(x)| \leq p(x) \forall x \in V_0.$$

Then f can be extended to a linear functional F on V s.t. $|F(x)| \leq p(x) \forall x \in V$.

Corollary 16.3. Assume V is a normed real or complex vector space, $V_0 \subset V$ is a subspace and $f \in V_0^*$. Then there is a $F \in V^*$ s.t.

$$F|_{V_0} = f \text{ and } \|F\| = \|f\|.$$

Corollary 16.4. Assume V is a normed space and $x \in V, x \neq 0$. Then there is a $F \in V^*$ s.t. $\|F\| = 1$ and $F(x) = \|x\|$.

Such an F is called a **supporting functional at x** .

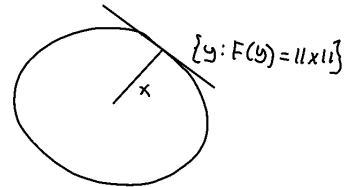


Figure 1: Tangent space?

If V is a normed vector space, then every $x \in X$ defines a bounded linear functional on V^* by

$$V^* \ni f \mapsto f(x).$$

As $|f(x)| \leq \|f\| \cdot \|x\|$, this functional has norm $\leq \|x\|$. By using a supporting functional at x , we actually see that we get norm $\|x\|$. Thus, we have an isometric embedding $V \subset V^{**} := (V^*)^*$. We can therefor see view V as a subspace of V^{**} .

Definition 16.5. A normed space V is called reflexive if $V^{**} = V$.

Remark. This is stronger than requiring $V \simeq V^{**}$. And:

- (i) Every finite dimensional normed vector space V is reflexive for dimension reasons: $\dim V^{**} = \dim V^* = \dim V$.
- (ii) Every Hilbert space \mathcal{H} is reflexive. Indeed, $\mathcal{H}^* = \overline{\mathcal{H}}$. By Riesz' theorem every bounded linear functional f on $\overline{\mathcal{H}}$ has the form

$$f(\bar{x}) = (\bar{x}, \bar{y}) = (y, x),$$

for some $y \in \mathcal{H}$, which exactly means that $f = y$ in \mathcal{H}^{**} .

As we will see later, the spaces $\mathcal{L}^p(X, d\mu)$, with μ σ -finite and $1 < p < \infty$, are reflexive. The spaces $\mathcal{L}^1(X, d\mu)$ and $\mathcal{L}^\infty(X, \mu)$ are usually not reflexive.

17 Radon-Nikodym Theorem (20, [Schilling(2017)])

Assume (X, \mathcal{B}, μ) is a measure space. Are there other measures on (X, \mathcal{B}) ?

Example 17.1. Take a measurable function $f : X \rightarrow [0, +\infty]$ and define

$$\nu(A) := \int_A f d\mu \text{ for } A \in \mathcal{B}.$$

This is a measure by the monotone convergence theorem. We write $d\nu = f d\mu$. Furthermore, we say that f is the **Radon-Nikodym derivative**, and we denote it by $f = d\nu/d\mu$. If $\mu = \lambda^1$ we get $f(x) = d\nu(x)/dx$.

Proposition 17.2. Assume (X, \mathcal{B}) is a measurable space, μ and ν are σ -finite measures on (X, \mathcal{B}) . Then there exist $N \in \mathcal{B}$ and a measurable $f : X \rightarrow [0, +\infty]$ s.t. $\mu(N) = 0$ and $\nu(A) = \nu(A \cap N) + \int_A f d\mu \forall A \in \mathcal{B}$.

When can we discard the term $\nu(A \cap N)$?

Definition 17.3 (absolutely continuous measure). Given measure μ and ν on X, \mathcal{B} , we say that ν is *absolutely continuous* with respect to μ and write $\nu \ll \mu$, if $\nu(A) = 0$ whenever $A \in \mathcal{B}, \mu(A) = 0$.

Lemma 17.4. Assume μ and ν are measures on (X, \mathcal{B}) , $\nu(X) < \infty$. Then $\nu \ll \mu$ iff $\forall \epsilon > 0 \exists \delta > 0$ s.t. if $A \in \mathcal{B}, \mu(A) < \delta$, then $\nu(A) < \epsilon$.

Remark. The result is not true for infinite ν .

Theorem 17.5 (Radon-Nikodym). Assume μ and ν are σ -finite measures on a measurable space (X, \mathcal{B}) , $\nu \ll \mu$. Then there is a measurable function $f : X \rightarrow [0, +\infty)$ s.t. $d\nu = f d\mu$ (that is, $\nu(A) = \int_A f d\mu$). If \tilde{f} is another function with the same properties, then $f = \tilde{f}$ μ -a.e.

The function is called the Radon-Nikodym derivative at ν w.r.t. μ and is denoted by $\frac{d\nu}{d\mu}$.

Example 17.6. Consider a real-valued function $f \in C'[a, b]$ s.t. $f'(t) > 0 \forall t \in [a, b]$. Let $c = f(a), d = f(b)$. We know that for every Riemann integrable function g on $[c, d]$ we have

$$\int_c^d g(f) dt = \int_a^b g(f(t)) f'(t) dt.$$

Equivalently,

$$\int_c^d g \circ f^{-1} dt = \int_a^b g f'(t) dt. \quad (23)$$

Denote by $\lambda_{[a,b]}, \lambda_{[c,d]}$ the Lebesgue measure restricted to the Borel subsets of $[a, b]$ and $[c, d]$, respectively. Then eq. 23 implies that

$$d((f^{-1})_* \lambda_{[c,d]}) = f' d\lambda_{[a,b]}$$

(recall that

$$(f^{-1})_* \lambda_{[c,d]}(A) := \lambda_{[c,d]}(f^{-1}(A)) := \lambda_{[c,d]}(\{x \in X : f(x) \in A\}),$$

since the integration of $g = \mathbf{1}_{[\alpha, \beta]}$ gives the same results for any interval $[\alpha, \beta] \subset [a, b]$ and since a finite Borel measure on $[a, b]$ is determined by its values on such intervals. Thus, $(f^{-1})_* \lambda_{[c,d]} \ll \lambda_{[a,b]}$ and

$$\frac{d((f^{-1})_* \lambda_{[c,d]})}{d\lambda_{[a,b]}} = f'.$$

18 Complex and Signed Measures (4.3, [Teschl(2010)])

Definition 18.1 (complex and finite signed measure). A complex measure on (X, \mathcal{B}) is a map $\nu : \mathcal{B} \rightarrow \mathbb{C}$ s.t. $\nu(\emptyset) = 0$ and

$$\nu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \nu(A_n)$$

for any disjoint $A_n \in \mathcal{B}$, where the series is assumed to be absolutely convergent. If ν takes values in \mathbb{R} then ν is called a **finite signed measure**.

Remark. More generally, a signed measure is allowed to take values in $\mathbb{R} \cup \{+\infty\}$ or $\mathbb{R} \cup \{-\infty\}$.

Given a complex measure ν on (X, \mathcal{B}) , its **total variation** is the map $|\nu| : \mathcal{B} \rightarrow [0, +\infty]$ defined by

$$|\nu|(A) = \sup \left\{ \sum_{n=1}^N |\nu(A_n)| : A = \bigcup_{n=1}^N A_n, A_n \in \mathcal{B}, A_n \cap A_m = \emptyset \right\}.$$

Proposition 18.2. $|\nu|$ is a finite measure on (X, \mathcal{B}) .

Example 18.3. Consider a measure space (X, \mathcal{B}, μ) and take $f \in L^1(X, d\mu)$. Define

$$\nu(A) = \int_A f d\mu.$$

Then ν is a complex measure on (X, \mathcal{B}) , since this is true for $f \geq 0$ and a general f can be written as a linear combination of positive ones. We write $d\nu = f d\mu$.

We then have $d|\nu| = |f| d\mu$, that is,

$$|\nu|(A) = \int_A |f| d\mu.$$

Theorem 18.4 (Radon-Nikodym theorem for complex measures). Assume (X, \mathcal{B}, μ) is a measure space, ν is a complex measure on (X, \mathcal{B}) , $\nu \ll \mu$. Then there is a unique $f \in L^1(X, d\mu)$ s.t. $d\nu = f d\mu$.

19 Decomposition Theorems (20, [Schilling(2017)] and 4.3,

[Teschl(2010)]

Definition 19.1 (mutually singular measures). Two measures ν and μ on a measurable space (X, \mathcal{B}) are called **mutually singular**, or we say that ν is **singular** w.r.t. μ , if there is a $N \in \mathcal{B}$ s.t. $\nu(N^c) = 0$, $\mu(N) = 0$. We then write $\nu \perp \mu$.

Theorem 19.2 (Lebesgue Decomposition Theorem). Assume ν, μ are σ -finite measures in (X, \mathcal{B}) . Then there exist unique measures ν_a and ν_s s.t. $\nu = \nu_a + \nu_s$, $\nu_a \ll \mu$, $\nu_s \perp \mu$.

Theorem 19.3 (Polar Decomposition of Complex Measure). Assume ν is a complex measure on (X, \mathcal{B}) . Then there exist a finite measure μ on (X, \mathcal{B}) and a measurable function $f : X \rightarrow \mathbb{C}$ s.t. $d\nu = f d\mu$. If $(\tilde{\mu}, \tilde{f})$ is another such pair, then $\tilde{\mu} = \mu$ and $\tilde{f} = f \mu$ -a.e.

For signed measures this leads to the following.

Theorem 19.4 (Hahn Decomposition Theorem). Assume ν is a finite signed measure on (X, \mathcal{B}) . Then there exist $P, N \in \mathcal{B}$ s.t.

$$X = P \cup N, \quad P \cap N = \emptyset, \\ \nu(A \cap P) \geq 0, \quad \nu(A \cap N) \leq 0 \quad \forall A \in \mathcal{B}.$$

Moreover, then $|\nu|(A) = \nu(A \cap P) - \nu(A \cap N)$, and if $X = \tilde{P} \cup \tilde{N}$ is another such decomposition, then

$$|\nu|(P \Delta \tilde{P}) = |\nu|(N \Delta \tilde{N}) = 0.$$

Corollary 19.5 (Jordan Decomposition Theorem). Assume ν is a finite signed measure on (X, \mathcal{B}) . Then there exist unique finite measures ν_+, ν_- on (X, \mathcal{B}) s.t.

$$\nu = \nu_+ - \nu_- \quad \text{and} \quad \nu_+ \perp \nu_-.$$

Moreover, then $|\nu| = \nu_+ + \nu_-$, hence

$$\nu_+ = \frac{|\nu| + \nu}{2}, \quad \nu_- = \frac{|\nu| - \nu}{2}.$$

20 More on Duals of L^p -spaces (21, p. 241, [Schilling(2017)])

- What is the dual of $L^p(X, d\mu)$? When does a measurable function $g : X \rightarrow \mathbb{C}$ define a bounded linear functional on $L^p(X, d\mu)$ by

$$\rho(f) = \int_X f g d\mu?$$

Theorem 20.1 (Young's inequality). Assume $f : [0, a] \rightarrow [0, b]$ is a strictly increasing continuous function, $f(0) = 0$, $f(a) = b$. Then, for all $s \in [0, a]$ and $t \in [0, b]$, we have

$$st \leq \int_0^s f(x) dx + \int_0^t f^{-1}(y) dy,$$

and the equality holds iff $t = f(s)$.

With Holder's inequality it follows that every $g \in L^q(X, d\mu)$ defines a bounded linear functional

$$l_g : L^p(X, d\mu) \rightarrow \mathbb{C}, \quad l_g(f) = \int_X f g d\mu,$$

and $\|l_g\| \leq \|g\|_q$.

The same makes sense for $p = 1, q = \infty$ and $p = \infty, q = 1$, when μ is σ -finite, as

$$\int_X |f g| d\mu \leq \int_X |f| d\mu \cdot \|g\|_\infty = \|f\|_1 \cdot \|g\|_\infty.$$

Lemma 20.2. Assume $1 \leq p \leq \infty$, $1/p + 1/q = 1$, and μ is σ -finite if $p = 1$ or $p = \infty$. For $g \in L^q(X, d\mu)$ consider $l_g \in L^p(X, d\mu)^*$. Then

$$\|l_g\| = \|g\|_q.$$

Therefor we can view $L^q(X, d\mu)$ as a subspace of $L^p(X, d\mu)^*$ using the isometric embedding

$$L^q(X, d\mu) \rightarrow L^p(X, d\mu)^*, \quad g \mapsto l_g.$$

Theorem 20.3. Assume $(X, \mathcal{B}, d\mu)$ is a σ -finite measure space, $1 \leq p < \infty$, $1/p + 1/q = 1$. Then

$$L^p(X, d\mu)^* = L^q(X, d\mu).$$

Remark. This is usually not true for $p = \infty$.

21 + 22 Riesz-Markow Theorem (21, p. [243-249], [Schilling(2017)])

Assume (X, d) is a locally compact metric space.

Definition 21.22 (positive linear functional). A linear functional $\rho : C_c(X) \rightarrow \mathbb{C}$ is called positive if $\rho(f) \geq 0$ for all $f \geq 0$. (Recall $C_c(X)$ is the space of continuous (C) compactly supported (c) functions.)

Theorem 21.23 (Riesz-Markov). If $\rho : C_c(X) \rightarrow \mathbb{C}$ is a positive linear functional, where (X, d) is a locally compact metric space, then there exists a Borel measure μ on X s.t. $\mu(K) < \infty$ for every compact $K \subset X$ and

$$\rho(f) = \int_X f d\mu \quad \forall f \in C_c(X).$$

If X is separable, then such a measure μ is unique.

For the proof we need two auxiliary results.

Lemma 21.24 (Urysohn's Lemma). Assume (X, d) is a metric space, $A, B \subset X$ are disjoint closed subsets. Then there exists a continuous function $f : X \rightarrow [0, 1]$ s.t. $f \equiv 1$ on A and $f \equiv 0$ on B .

Lemma 21.25. Assume (X, d) is a compact metric space, $U = (U_i)_{i=1}^n$ is a finite open cover of X (so U_i are open and $\cup_{i=1}^n U_i = X$). Then there exist functions ρ_1, \dots, ρ_n in $C(X)$ s.t.

$$0 \leq \rho_i \leq 1, \quad \text{supp}(\rho_i) \subset U_i, \quad \sum_{i=1}^n \rho_i(x) = 1 \quad \forall x.$$

Every such collection of functions is called a **partition of unity subordinate** to U .

Remark. Without separability, the uniqueness is not always true. It can be checked that the measure we constructed in the proof,

$$\mu(U) := \sup \{ \rho(f) : 0 \leq f \leq 1, \text{supp}(f) \subset U \},$$

has the following properties:

(i) $\mu(K) < \infty \quad \forall \text{ compact } K \subset X$;

(ii) μ is outer regular ($\mu(A) = \inf_{U \subset U \text{ open}} \mu(U)$);

(iii) μ is inner regular on open sets (this is where we need the full strength of step 3):

$$\mu(U) = \sup_{\substack{K \subset U \\ K \text{ compact}}} \mu(K) \quad \forall \text{ open } U.$$

Such measures are called **Radon measures**. It can be shown that the uniqueness holds within the class of Radon measures.

Dual of $C(X)$

As an application of the Riesz-Markow Theorem we will describe $C(X)^*$ in terms of measures for compact metric spaces (X, d) .

Denote by $M(X)$ the space of complex Borel measures on X . For every $\nu \in M(X)$ we want to make sense of $\int_X f d\nu$ for $f \in C(X)$. It is enough to consider finite signed measures, as then we can define

$$\int_X f d\nu = \int_X f d(\text{Re}\nu) + i \int_X f d(\text{Im}\nu).$$

So assume ν is a finite signed measure. Then, $\nu = \mu_1 - \mu_2$ for positive measures and we define

$$\int_X f d\nu = \int_X f d\mu_1 - \int_X f d\mu_2.$$

This is well-defined, since if

$$\nu = \mu_1 - \mu_2 = \omega_1 - \omega_2,$$

then $\mu_1 + \omega_2 = \mu_2 + \omega_1$ and

$$\int_X f d\mu_1 + \int_X f d\omega_2 = \int_X f d\mu_2 + \int_X f d\omega_1.$$

Thus, every $\nu \in M(X)$ defines a linear functional

$$\phi_\nu : C(X) \rightarrow \mathbb{C} \text{ by } \phi_\nu(f) = \int_X f d\nu,$$

and the map $\nu \mapsto \phi_\nu$ is linear.

Lemma 21.26. If $\nu \in M(X)$ and $d\nu = g d|\nu|$ is its polar decomposition, then

$$\int_X f d\nu = \int_X f g d|\nu| \quad \forall f \in C(X).$$

Lemma 21.27. For every $\nu \in M(X)$, the linear functional ϕ_ν is bounded and $\|\phi_\nu\| = |\nu|(X)$. (Recall that the norm on $C(X)$ is $\|f\| = \sup_{x \in X} |f(x)|$.)

23 Product Measures and Fubini's Theorem

(14, [Schilling(2017)])

Throughout this chapter we assume that (X, \mathcal{A}, μ) and (Y, \mathcal{B}, ν) are σ -finite measure spaces.

Recall the Cartesian product of sets (assume $A \subset X, B \subset Y$):

$$A \times B = \{(a, b) | a \in A \text{ and } b \in B\}.$$

(there are hidden properties here.)

The Lebesgue measure on \mathbb{R}^n has the following property for $n > d \geq 1$:

$$\lambda^n[a_1, b_1) \times \dots \times [a_n, b_n) = \lambda^d[a_1, b_1) \times \dots \times [a_d, b_d) \cdot \lambda^{n-d}[a_{d+1}, b_{d+1}) \times \dots \times [a_n, b_n)$$

which means that

$$\lambda^n(E) = \int \mathbb{1}_E(x, y) \lambda^n(d(x, y)) = \int \left(\int \mathbb{1}_E(x_0, y) \lambda^{n-d}(dy) \right) \lambda^d(dx_0).$$

Goal: we want to define a measure ρ on rectangles on the form $A \times B$ s.t. $\rho(A \times B) = \mu(A)\nu(B)$.

Lemma 23.1. Let \mathcal{A} and \mathcal{B} be two σ -algebras (or semi-rings), then $\mathcal{A} \times \mathcal{B}$ is a semi-ring.

Definition 23.2 (product σ -algebra). The σ -algebra $\mathcal{A} \otimes \mathcal{B} := \sigma(\mathcal{A} \times \mathcal{B})$ is called a **product σ -algebra**, and $(X \times Y, \mathcal{A} \otimes \mathcal{B})$ is the product of measurable spaces.

When considering $\mathcal{A} \otimes \mathcal{B}$, the following lemma shows that we can instead work with their generators.

Lemma 23.3. If $\mathcal{A} = \sigma(\mathcal{F})$ and $\mathcal{B} = \sigma(\mathcal{G})$ and if \mathcal{F} and \mathcal{G} contain exhausting sequences $(F_n)_{n \in \mathbb{N}} \subset \mathcal{F}$, $F_n \uparrow X$ and $(G_n)_{n \in \mathbb{N}} \subset \mathcal{G}$, $G_n \uparrow Y$, then

$$\sigma(\mathcal{F} \times \mathcal{G}) = \sigma(\mathcal{A} \times \mathcal{B}) := \mathcal{A} \otimes \mathcal{B}.$$

Theorem 23.4 (uniqueness of product measures). Assume that $\mathcal{A} = \sigma(\mathcal{F})$ and $\mathcal{B} = \sigma(\mathcal{G})$. If

- \mathcal{F}, \mathcal{G} is \cap -stable (stable under finite intersections),
- \mathcal{F}, \mathcal{G} contain exhausting sequences $F_k \uparrow X$ and $G_k \uparrow Y$ with $\mu(F_k) < \infty$ and $\nu(G_k) < \infty$,

then there is at most one measure ρ on $(X \times Y, \mathcal{A} \otimes \mathcal{B})$ satisfying

$$\rho(F \times G) = \mu(F)\nu(G) \quad \forall F \in \mathcal{F}, G \in \mathcal{G}.$$

Theorem 23.5 (existence of product measures). The set function

$$\rho : \mathcal{A} \times \mathcal{B} \rightarrow [0, \infty], \quad \rho(A \times B) := \mu(A)\nu(B),$$

extends uniquely to a σ -finite measure on $(X \times Y, \mathcal{A} \otimes \mathcal{B})$ s.t.

$$\rho(E) = \int \int \mathbb{1}_E(x, y) \mu(dx) \nu(dy) = \int \int \mathbb{1}_E(x, y) \nu(dy) \mu(dx)$$

holds for all $E \in \mathcal{A} \otimes \mathcal{B}$ (the parenthesis in the expression above are left out). In particular, the functions

$$x \mapsto \mathbb{1}_E(x, y), \quad x \mapsto \int \mathbb{1}_E(x, y) \nu(dy),$$

$$y \mapsto \mathbb{1}_E(x, y), \quad y \mapsto \int \mathbb{1}_E(x, y) \mu(dx),$$

are \mathcal{A} , \mathcal{B} -measurable (respectively) for every fixed $y \in Y$, $x \in X$ (respectively).

Lecture 24

Definition 24.25 (product measure $\mu \times \nu$). The unique measure ρ constructed in Theorem 23.5 is called the **product** of the measures μ and ν , denoted $\mu \times \nu$. $(X, Y, \mathcal{A} \otimes \mathcal{B}, \mu \times \nu)$ is called the **product measure space**

We can now finally construct the n -dimensional Lebesgue measure:

Corollary 24.26. *If $n > d \geq 1$,*

$$(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n), \lambda^n) = \left(\mathbb{R}^d \times \mathbb{R}^{n-d}, \mathcal{B}(\mathbb{R}^d) \otimes \mathcal{B}(\mathbb{R}^{n-d}), \lambda^d \times \lambda^{n-d} \right).$$

Great. The next step is to see how we can integrate w.r.t. to $\mu \times \nu$. The following two results are often stated together as the Fubini or Fubini-Tonelli theorem.

Theorem 24.27 (Tonelli). *Let (X, \mathcal{A}, μ) (Y, \mathcal{B}, ν) be σ -finite measure spaces and let $u : X \times Y \rightarrow [0, \infty]$ be $\mathcal{A} \otimes \mathcal{B}$ -measurable. Then*

- (i) $x \mapsto u(x, y)$, $y \mapsto u(x, y)$ are \mathcal{A} -resp. \mathcal{B} -measurable for fixed y resp. x ;
- (ii) $x \mapsto \int_Y u(x, y) \nu(dy)$, $y \mapsto \int_X u(x, y) \mu(dx)$ are \mathcal{A} -resp. \mathcal{B} -measurable;
- (iii) $\int_{X \times Y} u d(\mu \times \nu) = \int_Y \int_X u(x, y) \mu(dx) \nu(dy) = \int_X \int_Y u(x, y) \nu(dy) \mu(dx)$ which is in $[0, \infty]$.

The following corollary really extends Tonelli to not necessarily positive functions.

Corollary 24.28 (Fubini's theorem). *Let (X, \mathcal{A}, μ) (Y, \mathcal{B}, ν) be σ -finite measure spaces and let $u : X \times Y \rightarrow \mathbb{R}$ be $\mathcal{A} \otimes \mathcal{B}$ -measurable. If at least one of the three integrals*

$$\int_{X \times Y} |u| d(\mu \times \nu), \int_Y \int_X |u(x, y)| \mu(dx) \nu(dy), \int_X \int_Y |u(x, y)| \nu(dy) \mu(dx)$$

is finite, then all three integrals are finite, $u \in \mathcal{L}^1(\mu \times \nu)$, and

- (i) $x \mapsto u(x, y)$ is in $\mathcal{L}^1(\mu)$ for ν -a.e. $y \in Y$;
- (ii) $y \mapsto u(x, y)$ is in $\mathcal{L}^1(\nu)$ for μ -a.e. $x \in X$;
- (iii) $y \mapsto \int_X u(x, y) \mu(dx)$ is in $\mathcal{L}^1(\nu)$;
- (iv) $x \mapsto \int_Y u(x, y) \nu(dy)$ is in $\mathcal{L}^1(\mu)$;
- (v) $\int_{X \times Y} u d(\mu \times \nu) = \int_Y \int_X u(x, y) \mu(dx) \nu(dy) = \int_X \int_Y u(x, y) \nu(dy) \mu(dx)$.

25 Fourier Transform (§13 (pp. 125-128), §15 (pp. 157-158), §19 (pp. 214-217), [Schilling(2017)])

We write $L^1(\mathbb{R}^n)$ for $L^1(\mathbb{R}^n, d\lambda_n)$.

The Fourier transform of a function $f \in L^1(\mathbb{R}^n)$ is the function \hat{f} on \mathbb{R}^n defined by

$$\hat{f}(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} f(y) e^{-i\langle x, y \rangle} dy,$$

w where $\langle x, y \rangle = x_1 y_1 + \dots + x_n y_n$. More generally, given a finite Borel measure μ , is *Fourier transform* is the function $\hat{\mu}$ on \mathbb{R}^n defined by

$$\hat{\mu}(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{-i\langle x, y \rangle} d\mu(y).$$

We can also define μ for complex Borel measures.

Warning. There are many different conventions for the Fourier transform: instead of $1/(2\pi)^n$ one also uses $1, 1/(2\pi)^{n/2}$; instead of $e^{-i\langle x, y \rangle}$ one also uses $e^{i\langle x, y \rangle}, e^{\pm 2\pi i \langle x, y \rangle}$.

Note that if μ_f for $f \in L^1(\mathbb{R}^n)$ is defined by $d\mu_f = f d\lambda_n$, then

$$\hat{\mu}_f = \hat{f}.$$

Lemma 25.1. *If μ is a complex Borel measure on \mathbb{R}^n , then $\hat{\mu}$ is a Bounded continuous function on \mathbb{R}^n , and $|\hat{\mu}(x)| \leq \frac{|\mu|(\mathbb{R}^n)}{(2\pi)^n}$.*

In particular, if $f \in L^1(\mathbb{R}^n)$, then \hat{f} is Bounded and continuous,

$$|\hat{f}(x)| \leq \frac{\|f\|_1}{(2\pi)^n} \quad \forall x.$$

Some properties:

- (i) If $f_t(x) = f(x - t)$, then

$$\hat{f}_t(y) = e^{-i\langle t, y \rangle} \hat{f}(y).$$

- (ii) If $e_t(x) = e^{i\langle t, x \rangle}$, then

$$e_t \hat{f}(y) = \hat{f}(y - t).$$

- (iii) If $T \in GL_n(\mathbb{R})$ (invertible $n \times n$ matrices), then

$$f \circ T = |\det T|^{-1} \hat{f} \circ (T^t)^{-1}.$$

- (iv) $\hat{\hat{f}}(x) = \hat{f}(-x)$.

Important example.

If $f(x) = e^{-\frac{|x|^2}{2}}$ ($|x| = \langle x, x \rangle^{1/2}$), then $\hat{f}(x) = 1/(2\pi)^n e^{-\frac{|x|^2}{2}}$. More generally, if $f(x) = e^{-\frac{c|x|^2}{2}}$, then

$$\hat{f}(x) = \frac{1}{(2\pi c)^{\frac{n}{2}}} e^{-\frac{|x|^2}{2c}} \quad \forall c > 0.$$

This follows from property (iii).

For functions f, g on \mathbb{R}^n , their **convolution** is the function $f \times g$ defined by

$$(f * g)(x) := \int_{\mathbb{R}^n} f(y) g(x - y) dy = \int_{\mathbb{R}^n} f(x - y) g(y) dy$$

when is this well-defined?

Lemma 25.2. *If $f, g \in L^1(\mathbb{R}^n)$, then the function $y \mapsto f(y)g(x - y)$ is integrable for (λ_n) -a.e. x , $f * g \in L^1(\mathbb{R}^n)$ and $\|f * g\|_1 \leq \|f\|_1 \|g\|_1$.*

Next let us show that $f * g$ is well-defined for $f \in L^1(\mathbb{R}^n)$, $g \in L^p(\mathbb{R}^n)$, $1 \leq p \leq \infty$.

Lemma 25.3. *Assume $\phi : (a, b) \rightarrow \mathbb{R}$ is a convex function. Then ϕ is continuous and $\phi(x) = \sup\{l(x) : \phi \geq l, l(s) = \alpha s + \beta\}$.*

Theorem 25.4 (Jensen's inequality). *Assume (X, \mathcal{B}, μ) is a probability measure space (so $\mu(X) = 1$), $\phi : [0, \infty) \rightarrow [0, \infty)$ is a convex function. Then, for every integrable function $f : X \rightarrow [0, \infty)$ we have*

$$\phi\left(\int_X f d\mu\right) \leq \int_X \phi \circ f d\mu.$$

The same inequality holds for any measurable $f : X \rightarrow [0, \infty]$ if $\lim_{x \rightarrow \infty} \phi(x) = +\infty$ and we put $\phi(+\infty) = +\infty$.

26 Regularization (15 & 19, [Schilling(2017)])

Lemma 26.1. Assume $f \in L^1(\mathbb{R}^n)$, $g \in L^p(\mathbb{R}^n)$ ($1 \leq p \leq \infty$). Then the function $g \mapsto f(g)g(x-y)$ is integrable for a.e. x , $f * g \in L^p(\mathbb{R}^n)$ and $\|f * g\|_p \leq \|f\|_1 \|g\|_p$.

Note that

$$\int_{\mathbb{R}^n} f(y)g(x-y)dy = \int_{\mathbb{R}^n} f(x-y)g(y)dy,$$

so $f * g = g * f$.

Remark. More generally, for $\mu \in M(\mathbb{R}^n)$ and $g \in L^p(\mathbb{R}^n)$, we can define $\mu * g = g * \mu \in L^p(\mathbb{R}^n)$ by

$$(\mu * g)(x) = \int_{\mathbb{R}^n} g(x-y)d\mu(y).$$

Then $\|\mu * g\|_p \leq |\mu|(\mathbb{R}^n)\|g\|_p$.

Proposition 26.2. If $f, g \in L^1(\mathbb{R}^n)$, then $f * \hat{g} = (2\pi)^n \hat{f} \hat{g}$.

What are convolutions good for?

Example 26.3. Consider

$$f = \frac{1}{\lambda_n(B_r(0))} \mathbb{1}_{B_r(0)}.$$

Then

$$\begin{aligned} (f * g)(x) &= \frac{1}{\lambda_n(B_r(0))} \int_{B_r(0)} g(x-y)dy \\ &= \frac{1}{\lambda_n(B_r(x))} \int_{B_r(x)} g(y)dy. \end{aligned}$$

For a multi-index $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}_+^n$, write ∂^α for

$$\frac{\partial^{\alpha_1 + \dots + \alpha_n}}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}}.$$

Denote by $L_{\text{loc}}^1(\mathbb{R}^n)$ the space of Lebesgue measurable functions that are independent on every ball. We identify functions that coincide a.e. (so, $L_{\text{loc}}^1(\mathbb{R}^n)$ is a space of equivalence classes of functions). We have $L^p(\mathbb{R}^n) \subset L_{\text{loc}}^1(\mathbb{R}^n)$ for all $1 \leq p \leq \infty$.

Lemma 26.4. If $\phi \in C_c^\infty(\mathbb{R}^n)$ and $f \in L_{\text{loc}}^1(\mathbb{R}^n)$, then $\phi * f \in C^\infty(\mathbb{R}^n)$ and $\partial^\alpha(\phi * f) = (\partial^\alpha \phi) * f$.

By choosing suitable ϕ we can make $\phi * f$ close to f , as we will see shortly.

Definition 26.5. A positive mollifier is a function $\phi \in C_c(\mathbb{R}^n)$ s.t. $\phi \geq 0$ and $\int_{\mathbb{R}^n} \phi(x)dx = 1$.

Example 26.6 (positive mollifier). Consider the function h in \mathbb{R} defined by,

$$h(t) = \begin{cases} e^{-\frac{1}{1-t^2}}, & |t| < 1, \\ 0, & |t| \geq 1 \end{cases}.$$

Then $h \in C_c^\infty(\mathbb{R}^n)$. Hence, $\phi(x) = c_n h(|x|)$ is a mollifier, where $c_n = \left(\int_{\mathbb{R}^n} h(|x|)dx \right)^{-1}$.

A mollifier also has the property that if we define $\phi^\epsilon(x) := \epsilon^{-n} \phi(x/\epsilon)$, then $\lim_{\epsilon \rightarrow 0} \phi^\epsilon = \delta(x)$. **Which has the usage:** $\phi^\epsilon f \xrightarrow{\epsilon \rightarrow 0} f$. Note that if $\phi \in L^1(\mathbb{R}^n)$, $\int_{\mathbb{R}^n} \phi dx = 1$, then $\int_{\mathbb{R}^n} \phi^\epsilon = 1$.

Proposition 26.7. Let $\phi \in L^1(\mathbb{R}^n)$ be s.t. $\phi \geq 0$ and $\int_{\mathbb{R}^n} \phi dx = 1$. Then we have

(i) If $f \in C_0(\mathbb{R}^n)$ (continuous functions vanishing at infinity (?)), then $\phi^\epsilon * f \in C_0(\mathbb{R}^n)$ and $\|\phi^\epsilon * f\| \xrightarrow{\epsilon \downarrow 0} 0$ (uniform norm);

(ii) If $f \in L^p(\mathbb{R}^n)$, $1 \leq p \leq \infty$, then $\|\phi^\epsilon * f - f\|_p \xrightarrow{\epsilon \downarrow 0} 0$.

Corollary 26.8. For any Radon measure μ on \mathbb{R}^n , $C_c^\infty(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n, d\mu)$ for $1 \leq p < \infty$.

27 Fourier Inverse (19, [Schilling(2017)])

Corollary 27.1. If $f \in L^1(\mathbb{R}^n)$, then $\hat{f} \in C_0(\mathbb{R}^n)$.

Remark. Another possibility is to approximate f by its linear combinations of $\mathbb{1}_{[a_1, b_1] \times \dots \times [a_n, b_n]}$. Note that for $\mathbb{1}_{[a, b]} \in L^1(\mathbb{R})$ we have $\hat{\mathbb{1}}_{a, b}(x) = 1/(2\pi) \int_a^b e^{-ixy} dy \underset{x \neq 0}{=} \frac{e^{-iax} - e^{-ibx}}{2\pi ix} \xrightarrow{x \rightarrow \infty} 0$.

Theorem 27.2 (Fourier Inversion Theorem). Assume $f \in L^1 \mathbb{R}^n$ is s.t. $\hat{f} \in L^1(\mathbb{R}^n)$. Then, for a.e. x , we have

$$f(x) = \int_{\mathbb{R}^n} \hat{f}(y) e^{i\langle x, y \rangle} dy.$$

Equivalently,

$$\hat{\hat{f}}(x) = \frac{1}{(2\pi)^n} f(-x).$$

Theorem 27.3 (convolution theorem). If μ, ν are finite measures on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$, then

$$\widehat{\mu * \nu}(\xi) = (2\pi)^n \hat{\mu}(\xi) \hat{\nu}(\xi) \text{ and } \mathcal{F}^{-1} \mu * \nu(\xi) = \mathcal{F}^{-1} \mu(\xi) \mathcal{F}^{-1} \nu(\xi).$$

Lemma 27.4. For any $f, g \in L^1(\mathbb{R}^n)$, we have

$$\int_{\mathbb{R}^n} \hat{f} g dx = \int_{\mathbb{R}^n} f \hat{g} dx.$$

(Note that $\hat{f} g \in L^1(\mathbb{R}^n)$, as \hat{f} is bounded.)

Corollary 27.5. If $f \in L^1(\mathbb{R}^n)$ and $\hat{f} = 0$, then $f = 0$ (a.e.)

Recall that a linear operator $U : H \rightarrow K$ between Hilbert spaces is called an **isometry** if

$$\|Ux\| = \|x\| \quad \forall x \in H.$$

Equivalently, by the polarization identity,

$$(Ux, Uy) = (x, y) \quad \forall x, y \in H.$$

If U is in addition surjective, then it is called a **unitary**.

Theorem 27.6 (Plancherel). There is a unique unitary $U : L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n)$ s.t. $Uf = (2\pi)^{n/2} \hat{f}$ for all $f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$.

In the book: If $u \in L_C^2(\lambda^n) \cap L_C^1(\lambda^n)$, then

$$\|\hat{u}\|_2 = (2\pi)^{-n/2} \|u\|_2.$$

28 Schwartz Space 19, [Schilling(2017)]

Proposition 28.1. Assume $f \in L^1(\mathbb{R}^n)$ and $x_j f \in L^1(\mathbb{R}^n)$ ($x_j f$ means $x \mapsto x_j f(x)$) for some $1 \leq j \leq n$. Then

$$\partial_j \hat{f} = -i \widehat{x_j f}$$

Proposition 28.2. Assume $f \in C(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$ is such that $\partial_j f \in L^1(\mathbb{R}^n)$. Then

$$\widehat{\partial_j f} = i x_j \hat{f}$$

Corollary 28.3. If $f, \partial_j f \in L^1(\mathbb{R}^n)$, then

$$x_j \hat{f}(x) \xrightarrow{x \rightarrow \infty} 0.$$

Corollary 28.4. 1. If $x^\alpha f \in L^1(\mathbb{R}^n)$ for all $|\alpha| \leq N$, then $\hat{f} \in C^N(\mathbb{R}^n)$ and $\partial^\alpha \hat{f} = (-i)^{|\alpha|} \widehat{x^\alpha f}$.

2. If $f \in C^N(\mathbb{R}^n)$ and $\partial^\alpha f \in L^1(\mathbb{R}^n)$ for all $|\alpha| \leq N$, then $\widehat{\partial^\alpha f} = i^{|\alpha|} x^\alpha \hat{f}$ and hence $(1 + |x|)^N \hat{f}(x) \xrightarrow{x \rightarrow \infty} 0$.

(here $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n$: n -dim positive integers, and $|\alpha| = \alpha_1 + \dots + \alpha_n$, $x^\alpha = x_1^{\alpha_1} \cdot \dots \cdot x_n^{\alpha_n}$.)

Definition 28.5 (Schwartz function/space). A function f on \mathbb{R}^n is called a **Schwartz function** if $f \in C^\infty(\mathbb{R}^n)$ and $x^\alpha \partial^\beta f$ is bounded for all multi-indices α, β . The space $\mathcal{S}(\mathbb{R}^n)$ of Schwartz functions is called **Schwartz space**.

Note that for every $f \in C^\infty(\mathbb{R}^n)$ the following conditions are equivalent:

1. $x^\alpha \partial^\beta f$ is bounded for all α, β ;
2. $x^\alpha (\partial^\beta f)(x) \xrightarrow{x \rightarrow \infty} 0$ for all α, β ;
3. $(1 + |x|)^N \partial^\beta f$ is bounded for all $N \geq 1$ and all β .

Example 28.6. $C_c^\infty(\mathbb{R}^n) \subset \mathcal{S}(\mathbb{R}^n)$, $e^{-a|x|^2} \in \mathcal{S}(\mathbb{R}^n)$ for $a > 0$. If $f \in \mathcal{S}(\mathbb{R}^n)$, then $x^\alpha \partial^\beta f \in \mathcal{S}(\mathbb{R}^n)$. The product of two Schwartz functions is a Schwartz function.

Clearly, $\mathcal{S}(\mathbb{R}^n) \subset L^1(\mathbb{R}^n)$ for all $1 \leq p \leq \infty$. From the Corollary above we conclude that if $f \in \mathcal{S}(\mathbb{R}^n)$, then $\hat{f} \in \mathcal{S}(\mathbb{R}^n)$. By the Fourier inversion theorem we then get:

Theorem 28.7 (Fourier map in Schwartz space). The Fourier transform maps $\mathcal{S}(\mathbb{R}^n)$ onto $\mathcal{S}(\mathbb{R}^n)$.

Remark. This gives another proof of the fact that the image of the Fourier transform $L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n)$ is dense, which we needed to prove Plancherel's theorem.

Remark. If $f \in C_c^\infty(\mathbb{R}^n)$, $f \neq 0$, then $\hat{f} \in \mathcal{S}(\mathbb{R}^n)$, but \hat{f} is never compactly supported, since it extends to an analytic function on \mathbb{C}^n : $\hat{f}(z) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} f(y) e^{-i\langle z, y \rangle} dy$.

29 Kolmogorov extension theorem

Assume X is a set and $(\mathcal{B}_n)_{n \in \mathbb{N}}$ is an increasing sequence of σ -algebras of subsets of X . Assume μ_n is a measure on (X, \mathcal{B}_n) and

$$\mu_{n+1}|_{\mathcal{B}_n} = \mu_n \quad \forall n.$$

Can we define a measure μ on (X, \mathcal{B}) , where $\mathcal{B} = \sigma(\cup_{n \in \mathbb{N}} \mathcal{B}_n)$ s.t. $\mu|_{\mathcal{B}_n} = \mu_n \quad \forall n$? - In general, no. But we have the following:

Theorem 29.1 (Kolmogorov extension theorem). In the above settings, assume in addition that $\mu_n(X) = 1 \quad \forall n$ and there is a collection of subsets $C \subset \mathcal{B}$ s.t.:

- (i) $\mu_n(A) = \sup \{ \mu_n(C) : C \subset A, C \in \mathcal{C} \cap \mathcal{B}_n \} \quad \forall A \in \mathcal{B}_n$;
- (ii) If $(C_n)_{n \in \mathbb{N}}$ is a sequence in \mathcal{C} and $\cap_{n \in \mathbb{N}} C_n = \emptyset$, then $\cap_{n=1}^N C_n = \emptyset$ for some N .

Then there is a unique measure μ on (X, \mathcal{B}) s.t. $\mu|_{\mathcal{B}_n} = \mu_n$.

Assume now we have a collection $((X_i, \mathcal{B}_i))_{i \in I}$ of measurable spaces (I can be infinite and uncountable). Consider $X = \prod_{i \in I} X_i$. Denote by $\prod_{i \in I} \mathcal{B}_i$ the σ -algebra generated by all sets of the form

$$\prod_{i \in F} A_i \times \prod_{i \in F^c} X_i,$$

where $F \subset I$ is finite, $A_i \in \mathcal{B}_i$ ($i \in F$).

Example 29.2. Consider a sequence $((X_n, d_n))_{n=1}^\infty$ of separable metric spaces. Assume $d_n(x, y) \leq 1 \quad \forall x, y$. (Any metric can be defined by such by defining $\tilde{d}(x, y) = \frac{d(x, y)}{1+d(x, y)}$.) Then $\prod_{n=1}^\infty X_n$ is a metric space with metric

$$d(\underline{x}, \underline{y}) = \sum_{n=1}^\infty \frac{1}{2^n} d_n(x_n, y_n),$$

where $\underline{x} = (x_n)_{n=1}^\infty \in X$. Given a sequence $(\underline{x}(n))_{n=1}^\infty$ in X , we have $\underline{x}(n) \xrightarrow{n \rightarrow \infty} \underline{x}$ iff

$$x(n)_k \xrightarrow{k \rightarrow \infty} x_k \quad \forall k.$$

Consider the Borel σ -algebra $\mathcal{B}(X_n)$. Then $\prod_{n=1}^\infty \mathcal{B}(X_n) = \mathcal{B}(X)$. To see this, for every n , choose open sets $U_{n,k} \subset X_n$ ($k = 1, 2, \dots$) s.t. every open set in X_n is the union of some of $U'_{n,k}$ s. This is possible by separability: take a dense countable subset of X_n and then all balls of rational radii with centers at points of this subset. Then every open subset of X is the union of sets of the form

$$U_{1,k_1} \times U_{2,k_2} \times \dots \times U_{n,k_n} \times \prod_{m=n+1}^\infty X_m.$$

Therefore such sets generate the σ -algebra $\mathcal{B}(X)$, and as $U_{n,k}$ ($k \in \mathbb{N}$) generate $\mathcal{B}(X_n)$, we conclude that $\prod_{n=1}^\infty \mathcal{B}(X_n) = \mathcal{B}(X)$.

In relation to this example, we will need the following:

Theorem 29.3 (Tikkonov, also transcribed as Tychonoff). Assume $((X_n, d_n))_{n=1}^\infty$ is a sequence of compact metric spaces. Then $\prod_{n=1}^\infty X_n$ (with metric as in the example) is compact.

Return to a general collection $((X_i, \mathcal{B}_i))_{i \in I}$ of measurable spaces. Let us introduce the following notation: For $F \subset G \subset I$, write

$$X_F = \prod_{i \in F} X_i, \quad X_G = \prod_{i \in G} X_i,$$

$\pi_{G,F} : X_G \rightarrow X_F$ for the projection map:

$$\pi_{G,F}((x_i)_{i \in G}) = (x_i)_{i \in F},$$

and recall the **pushforward measure**: given a measurable mapping $f : X_1 \rightarrow X_2$ and a measure $\mu : \mathcal{B} \rightarrow [0, +\infty]$, the pushforward of μ is the measure $f_*(\mu) : \mathcal{B}_2 \rightarrow [0, +\infty]$ given by

$$f_*(\mu)(B_2) = \mu(f^{-1}(B_2)) \quad \text{for } B_2 \in \mathcal{B}_2.$$

Theorem 29.4 (Kolmogorov extension theorem). Assume $(X_i)_{i \in I}$ is a collection of metric spaces. Consider $X = \prod_{i \in I} X_i$, $\mathcal{B} = \prod_{i \in I} \mathcal{B}(X_i)$. Assume for every finite $F \subset I$ we are given a regular Borel probability measure μ_F on X_F s.t.

$$(\pi_{G,F})_* \mu_G = \mu_F,$$

for all finite $F \subset G \subset I$. Then there is a unique probability measure μ on (X, \mathcal{B}) s.t.

$$(\pi_{I,F})_* \mu = \mu_F$$

for all finite $F \subset I$.

Remark. If in addition the spaces X_i are separable, then we can also conclude that for every $A \in \mathcal{B} = \prod_{i \in I} \mathcal{B}(X_i)$ we have

$$\mu(A) = \sup \mu(C),$$

where the supremum is taken over all sets $C \subset A$ of the form

$$C = K \times \prod_{i \in I \setminus J} X_i,$$

where $J \subset I$ is countable and $K \subset X_J$ is compact.

30 Random variables and stochastic processes

Assume $(X, \mathcal{B}, \mathbb{P})$ is a probability measure space. If Y, \mathcal{C} is a measurable space, a measurable map $f : X \rightarrow Y$ is called a **random variable**. For $A \in \mathcal{C}$, define

$$\mathbb{P}(f \in A) \stackrel{\text{def}}{=} \mathbb{P}(f^{-1}(A)) = (f_* \mathbb{P})(A),$$

the probability that f takes a value in A . The measure $f_* \mathbb{P}$ on (Y, \mathcal{C}) is called the **probability distribution** of f .

Definition 30.1. A **stochastic process** is a collection $(f_t : X \rightarrow Y)_{t \in T}$ of random variables.

T stands for "time" and is typically $\mathbb{Z}, \mathbb{Z}_+, \mathbb{R}$ or \mathbb{R}_+ .

Given a different $t_1, \dots, t_n \in T$, we can consider the **joint distribution** of f_{t_1}, \dots, f_{t_n} , the measure

$$\mu_{t_1, \dots, t_n} = (f_{t_1} \times \dots \times f_{t_n})_* \mathbb{P} \text{ on } (Y^n, \mathcal{C}^n).$$

When is a collection of measures defined by a stochastic process?

Theorem 30.2. Assume T is a set and for all different elements $t_1, \dots, t_n \in T$ we are given a Borel probability measure $\mu_{t_1}, \dots, \mu_{t_n}$ on \mathbb{R}^n s.t.

(i) If $\sigma \in \delta_n$ and $A_1, \dots, A_n \in \mathcal{B}(\mathbb{R}^n)$, then

$$\mu_{t_1, \dots, t_n}(A_1 \times \dots \times A_n) = \mu_{t_{\sigma(1)}, \dots, t_{\sigma(n)}}(A_{\sigma(1)} \times \dots \times A_{\sigma(n)});$$

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$$(ii) \mu_{t_1, \dots, t_n, s_1, \dots, s_m}(A_1 \times \dots \times A_n \times \underbrace{\mathbb{R} \times \dots \times \mathbb{R}}_{m \text{ times}}) = \mu_{t_1, \dots, t_n}(A_1 \times \dots \times A_n).$$

Then there is a probability measure space $(X, \mathcal{B}, \mathbb{P})$ and a stochastic process $(f_t : X \rightarrow \mathbb{R})_{t \in T}$ s.t. μ_{t_1, \dots, t_n} is the joint distribution of f_{t_1}, \dots, f_{t_n} .

Remark. Instead of \mathbb{R} we could have taken any complete separable metric space, as then the measure μ_{t_1, \dots, t_n} are regular. Or we could just require the measures μ_{t_1, \dots, t_n} to be regular.

Random variables $f_1, \dots, f_n : X \rightarrow \mathbb{R}$ are called **independent** if

$$\mathbb{P}(f_1 \in A_1, \dots, f_n \in A_n) = \mathbb{P}(f_1 \in A_1) \dots \mathbb{P}(f_n \in A_n).$$

For all $A_1, \dots, A_n \in \mathcal{B}(\mathbb{R})$. In other words, if μ_i is the probability distribution of f_i , then the joint distribution of f_1, \dots, f_n is $\mu_1 \times \dots \times \mu_n$.

For such measures the above theorem gives the following result: if we are given a Borel probability measure μ_t on \mathbb{R} for every $t \in T$, then we get a unique measure $\mu = \prod_{t \in T} \mu_t$ on \mathbb{R}^T s.t. $(\bar{\mu}_{T,F})_* \mu = \prod_{t \in F} \mu_t \forall$ finite $F \subset T$.

Example 30.3. Consider the process of tossing a coin. Write 0 for tail and 1 for head. We can model the process as follows

$$X = \prod_{n=0}^{\infty} \{0, 1\}, \quad \mathbb{P} = \prod_{n=0}^{\infty} \nu,$$

where $\nu = \frac{1}{2} \delta_0 + \frac{1}{2} \delta_1$,

$$f_n : X \rightarrow \{0, 1\}, \quad f_n(x) = x_n,$$

f_n is the result of n -tosses.

While the Kolmogorov extension theorem requires some regularity, it turns out that infinite products of probability measures always exist:

Theorem 30.4. Assume $((X_i, \mathcal{B}_i, \mu_i))_{i \in I}$ is a collection of probability measure spaces. Consider $X = \prod_{i \in I} X_i$, $\mathcal{B} = \prod_{i \in I} \mathcal{B}_i$. Then there exists a unique measure μ on (X, \mathcal{B}) s.t.

$$(\bar{\mu}_{I,F})_* \mu = \prod_{i \in F} \mu_i \quad \forall \text{ finite } F \subset I.$$

Recall

(i) Dirac measure:

$$\delta_x(A) = \mathbb{1}_A(x) = \begin{cases} 0, & x \notin A, \\ 1, & x \in A. \end{cases}$$