MAT4400: Notes on Linear analysis
(Proofs excluded)

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1 σ-Algebras (Ch. 3 in [Schilling(2017)])

Definition 1.1 (σ-Algebra)**.** A family $\mathscr{A}$ of subsets of X with:

(i) $X \in \mathscr{A}$,

(ii) $A \in \mathscr{A} \Rightarrow A^c \in \mathscr{A}$,

(iii) $(A_n)_{n\in\mathbb{N}} \in \mathscr{A} \Rightarrow \bigcup\limits_{n\in\mathbb{N}}$

Theorem 1.2 (and Definition)**.**

(i) The intersection of arbitrarily many σ-algebras in X is againg a σ-algebra in X.

(ii) For every system of sets $p \subset \mathscr{P}(X)$ there exists a smallestσ-algebra containing$p$. This is the σ-algebra generated by p, denoted $\sigma(p)$, and $\sigma(p)$ is called its generator.

Definition 1.3 (Borel)**.** The σ-algebra $\sigma(\mathcal{O})$ generated by the open sets $\mathcal{O} = \mathcal{O}_{\mathbb{R}^n}$ of $\mathbb{R}^n$ is called **Borel σ-algebra**, and its members are called **Borel sets** or **Borel measurable sets**.

3 Uniqueness of Measures (Ch. 5 in [Schilling(2017)])

Lemma 3.1. *A Dynkin system D is a σ-algebra iff it is stable under finite intersections, i.e. $A, B \in D \Rightarrow A \cap B \in D$.*

Theorem 3.2 (Dynkin)**.** *Assume X is a set, S is a collection of subsets of X closed under finite intersections, that is, if $A, B \in S \Rightarrow A \cap B \in S$. Then $D(S) = \sigma(S)$.*

Theorem 3.3 (uniqueness of measures)**.** *Let (X, B) be a measurable space, and $S \subset P(X)$ be the generator of B, i.e. $B = \sigma(S)$. If S satisfies the following conditions:*

1. S is stable under finite intersections ($\cap$-stable), i.e. $A, C \in S \Rightarrow A \cap C \in S$.

2. There exists an exhausting sequence $(G_n)_{N\in\mathbb{N}} \subset$ with $G_N \uparrow X$. Assume also that there are two measures μ, ν satisfying:

3. $\mu(A) = \nu(A),\ \forall A \in S$.

4. $\mu(G_n) = \nu(G_n) < \infty$.

Then $\mu = \nu$.

Proof (outline). Define

$$D_n := \{A \in B : \mu(G_n \cap A) = \nu(G_n \cap A)\ (< \infty)\},$$

and show that it is a Dynkin system. Then, use the fact that S is $\cap$-stable and Theorem 3.2 to argue that $D(S) = \sigma(S) ... \to ... B = D_n$. ☐

4 Existence of Measures (Ch. 6 in [Schilling(2017)])

Theorem 4.1 (Carathéodory)**.** *Let $S \subset P(X)$ be a semi-ring and $\mu : S \to [0, \infty)$ a pre-measure. Then μ has an extension to a measure μ^* on $\sigma(S)$, i.e. that $\mu(s) = \mu^*(s),\ \forall\ s \in \sigma(S)$.*

Also, if S contains an exhausting sequence, $S_n \uparrow X$, s.t. $\mu(S_n) < \infty$, then the extension is unique.

Outline of proof: Firstly, let us define an outer measure.

Definition 4.2 (Outer measure)**.** An outer measure is a function $\mu^* : P(X) \to [0, \infty)$ with the following properties:

1. $\mu^*(\emptyset) = 0$,

2. $A \subset B \Rightarrow u^*(A) \le \mu^*(B)$,

3. $\mu^*\left(\bigcup\limits_{n\in\mathbb{N}} A_n\right) \le \sum\limits_{n\in\mathbb{N}} \mu^*(A_n)$,

and define for each $A \subset X$ the family of countable S-*coverings*:

$$C(A) := \left\{ (S_n)_{n\in\mathbb{N}} \subset S : \bigcup_{n\in\mathbb{N}} S_n \supset A \right\},$$

and the set function

$$\mu^*(A) := \inf \left\{ \sum_{n\in\mathbb{N}} \mu(S_n) : (S_n)_{n\in\mathbb{N}} \in C(A) \right\}.$$

Step 1: Claim: *$\mu^*(A)$ is an outer measure.*

Proof.

1. $C(\emptyset) = \{$any sequence in S containing $\emptyset\}$ $\Rightarrow \mu^*(\emptyset) = 0$.

2. Assume $A \subset B$. Then $C(A) \subset C(B)$ $\Rightarrow \mu^*(A) \leq \mu^* B$.

3. If $\mu^*(A_n) = \infty$ for some n, then there is nothing to prove. Thus, assume $\mu^*(A_n) < \infty\ \forall n$. Fix $\epsilon > 0$, and for every n choose $A_{n_k} \in S$ s.t.

$$A_n \subset \bigcup_{k\in\mathbb{N}} A_{n_k},$$
$$\sum_{k\in\mathbb{N}} \mu^*(A_{n_k}) < \mu^*(A_n) + \frac{\epsilon}{2^n}.$$

Then

$$\bigcup_{n\in\mathbb{N}} A_n \subset \bigcup_{k\in\mathbb{N}} \bigcup_{n\in\mathbb{N}} A_{n_k},$$

so

$$\begin{aligned} \mu^* \left(\bigcup_{n\in\mathbb{N}} A_n \right) &\leq \sum_{n\in\mathbb{N}} \sum_{k\in\mathbb{N}} \mu\left(A_{n_k}\right) \\ &< \sum_{n\in\mathbb{N}} \left(\mu^*(A_n) + \frac{\epsilon}{2^n} \right) \\ &= \sum_{n\in\mathbb{N}} \mu^*(A_n) + \epsilon. \end{aligned}$$

As ϵ was arbitrarily, we get that

$$\mu^* \left(\bigcup_{n\in\mathbb{N}} A_n \right) \leq \sum_{n\in\mathbb{N}} \mu^*(A_n),$$

so μ^* fulfills all the conditions for being an outer measure. ◻

Step 2: *Showing that μ^* extends μ, i.e. $\mu^*(s) = \mu(s)\ \forall s \in S$.*

Step 3: *Define μ^*-measurable sets*

$$\Sigma^* := \left\{ A \subset X : \mu^*(Q) = \mu^*(Q \cap A) + \mu^*(Q \backslash A) \quad \forall\, Q \subset X \right\}$$

Step 4: *Show that $\mu|_{\Sigma^*}$ is a measure. In particular, $\mu|_{\sigma(S)}$ is a measure which extends μ.*

5 Measurable Mappings (Ch. 7 in [Schilling(2017)])

We consider maps $T : X \to X'$ between two measurable spaces $(X, \mathcal{A})$ and $(X', \mathcal{A}')$ which respects the measurable structurs, the σ-algbras on X and X'. These maps are useful as we can transport a measure μ, defined on $(X, \mathcal{A})$, to $(X', \mathcal{A}')$.

Definition 5.1. Let $(X, \mathcal{A})$, $(X', \mathcal{A}')$ b measurable spaces. A map $T : X \to X'$ is called $\mathcal{A}/\mathcal{A}'$-measurable if the pre-imag of every measurable set is a measurable set:

$$T^{-1}(A') \in \mathcal{A}, \quad \forall A' \in \mathcal{A}'.$$

- A $\mathcal{B}(\mathbb{R}^n)/\mathcal{B}(\mathbb{R}^m)$ measurable map is often called a Borel map.
- The notation $T : (X, \mathcal{A}) \to (X', \mathcal{A}')$ is often used to indicate measurability of the map T.

Lemma 5.2. *Let $(X, \mathcal{A})$, $(x', \mathcal{A}')$ be measurable spaces and let $\mathcal{A}' = \sigma(\mathcal{G}')$. Then $T : X \to X'$ is $\mathcal{A}/\mathcal{A}'$-measurable iff $T^{-1}(\mathcal{G}') \subset \mathcal{A}$, i.e. if*

$$T^{-1}(G') \in \mathcal{A}, \ \ \forall G' \in \mathcal{G}'.$$

Theorem 5.3. *Let $(X_i, \mathcal{A}_i)$, $i = 1, 2, 3$, be measurable spaces and $T : X_1 \to X_2$, $S : X_2 \to X_3$ be $\mathcal{A}_1/\mathcal{A}_2$ and $\mathcal{A}_2/\mathcal{A}_3$-measurable maps respectivly. Then $S \circ T : X_1 \to X_3$ is $\mathcal{A}_1/\mathcal{A}_3$-measurable.*

Corollary 5.4. *Every continuous map betwen metric spaces is a Borel map.*

Definition 5.5. (and lemma) Let $(T_i)_{i\in I}$, $T_I : X \to X_i$, be arbitrarily many mappings from the same space X into measurable spaces

$(X_i, \mathcal{A}_i)$. The smallest σ-algebra on X that makes all T_i simultanously measurable is

$$\sigma(T_i : i \in I) := \sigma\left(\bigcup_{i\in I} T_i^{-1}(\mathcal{A}_i)\right)$$

Corollary 5.6. *A function $f : (X, \mathcal{B}) \to \mathbb{R}$ is measurable if $f((a, +\infty)) \in \mathcal{B}, \ \forall a \in \mathbb{R}$.*

Corollary 5.7. *Assume $(X, \mathcal{B})$ is a measurable space, (Y, d) is a metric space, and $(f_n : (X, \mathcal{B}) \to Y)_{n=1}^{\infty}$ is a sequence of measurable maps. Assume this sequence of images $(f_n(x))_{n=1}^{\infty}$ is convergent in Y $\forall x \in X$. Define*

$$f : X \to Y, \ \text{by } f(x) = \lim_{n\to\infty} f_n(x).$$

Then f is measurable.

Theorem 5.8. *Let $(X, \mathcal{A})$, $(X', \mathcal{A}')$ be measurable spaces and $T : X \to X'$ be an $\mathcal{A}/\mathcal{A}'$-measurable map. For every measurable μ on $(X, \mathcal{A})$,*

$$\mu'(A') := \mu(T^{-1}(A')), \quad A' \in \mathcal{A}',$$

defines a measure on $(X', \mathcal{A}')$.

Definition 5.9. The measure $\mu'(\cdot)$ in the above theorem is called the push forward or image measure of μ under T and it is denoted as $T(\mu)(\cdot)$, $T_{*\mu}(\cdot)$ or $\mu \circ T^{-1}(\cdot)$.

Theorem 5.10. *If $T \in \mathbb{R}^{n\times n}$ is an orthogonal matrix, then $\lambda^n = T(\lambda^n)$.*

Theorem 5.11. *Let $S \in \mathbb{R}^{n\times n}$ be an invertible matrix. Then*

$$S(\lambda^n) = |\det s^{-1}|\lambda^n = |\det S|^{-1}\lambda^n.$$

Corollary 5.12. *Lebesgue measure is invariant under motions: $\lambda^n = M(\lambda^n)$ for all motions M in $\mathbb{R}^n$. In particular, congruent sets have the same measure. Two sets of points are called congruent if, and only if, one can be transformed into the other by an isometry*

Measurable Functions (Ch. 8 in [Schilling(2017)])

A *measurable function* is a measurable map $u : X \to \mathbb{R}$ from some measurable space $(X, \mathscr{A})$ to $(\mathbb{R}, \mathscr{B}(\mathbb{R}^1))$. They play central roles in the theory of integration.

We recall that $u : X \to \mathbb{R}$ is $\mathscr{A}/\mathscr{B}(\mathbb{R}^1)$-measurable if

$$u^{-1}(B) \in \mathscr{A}, \ \ \forall B \in \mathscr{B}(\mathbb{R}^1).$$

Moreover from a lemma from chapter 7, we actually only need to show that

$$u^{-1}(G) \in \mathscr{A}, \ \ \forall G \in \mathcal{G} \text{ where } \mathcal{G} \text{ generates } \mathscr{B}(\mathbb{R}^1).$$

Proposition 5.13.

1 If $f, g : (X, \mathscr{B}) \to \mathbb{C}$ are measurable, then the function $f + g$, $f \cdot g$, cf, $(c \in \mathbb{C})$ are measurable.

2 If $b : \mathbb{C} \to \mathbb{C}$ is Borel and $b : (\mathbb{C}, \mathscr{B}) \to \mathbb{C}$ is measurable, then $b \circ f$ is measurable.

3 If $f(x) = \lim_{n\to\infty} f_n(x)$, $x \in X$ and f_n are measurable, then f is measurable.

4 If $X = \bigcup_{n=1}^{\infty} A_n$, $(A_n \in \mathscr{B})$, $f|_{A_n} : (A_n, \mathscr{B}_{A_n}) \to \mathbb{C}$ is measurable $\forall n$, then f is measurable.

Definition 5.14. Given a measurable space $(X, \mathscr{B})$, a measurable function $f : (X, \mathscr{B}) \to \mathbb{C}$ is called simple if

$$f(x) = \sum_{k=1}^{N} c_k \mathbb{1}_{A_k}(x),$$

for some $c_k \in \mathbb{C}$, $A_k \in \mathscr{B}$, where $\mathbb{1}$ is the characteristic function,

$$\mathbb{1}_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A. \end{cases}$$

The representation of simple function is **not** unique. We denote the standard representation of f by

$$f(x) = \sum_{n=0}^{N} z_n \mathbb{1}_{B_n}(x),$$

for $N \in \mathbb{N}$, $z_n \in \mathbb{R}$, $B_n \in \mathscr{A}$, and

$$X = \bigcup_{n=1}^{N} B_n,$$

for $B_n \cap B_m = \emptyset$, $n \neq m$. The set of simple functions is denoted $\mathcal{E}(\mathscr{A})$ of $\mathcal{E}$.

Definition 5.15. Assume μ is a measure on $(X, \mathscr{B})$. Given a *positive* simple function

$$f = \sum_{k=1}^{N} c_k \mathbb{1}_{A_k}, \quad (c_k \geq 0).$$

We define

$$\int_X f d\mu = \sum_{k=1}^{n} c_k \mu(A_k) \in [0, +\infty].$$

We also denote this by $I_\mu(f)$.

Lemma 5.16. *This is well defined, that is, $\int_x f d\mu$ does not depend on the presentation of the simple function f.*

Properties 5.17. *For every positive simple function*

1 $\int_X cf d\mu = c \int_X f d\mu, \quad$ *for only* $c \geq 0$

2 $\int_X (f+g) d\mu = \int_X f d\mu + \int_X g d\mu.$

Corollary 5.18. *If $f \geq g \geq 0$ are simple functions, then*

$$\int_X f d\mu \geq \int_X g d\mu.$$

Definition 5.19. If $f : X \to [0, +\infty)$ is measurable, then we define

$$\int_X f d\mu = \sup \left\{ \int_X g d\mu : f \geq g \geq 0, \ \ g \text{ is simple} \right\}$$

Remark. *This means that any measurable function can be approximated by simple functions.*

Properties 5.20. *Measurable functions like this have the following properties*

1 $\int_X cf d\mu = c \int_X f d\mu, \quad \forall c \geq 0.$

2 *If $f \geq g \geq 0$, then $\int_X f d\mu \geq \int_X g d\mu$ for any measurable g, f.*

3 *If $f \geq 0$ is simple, then $\int_X f d\mu$ is the same value as obtained before.*

To advance in measure theory we consider measurable functions

$$f : X \to [0, +\infty].$$

Measurability is understood w.r.t the $\sigma-$algebra $\mathscr{B}([0, +\infty])$ generated by $\mathscr{B}([0, +\infty))$ and $\{+\infty\}$. In other words, $A \subset [0, +\infty] \in B([0, +\infty])$ iff $A \cap [0, +\infty) \in \mathscr{B}([0, +\infty)$.

Remark. *Hence $f : X \to [0, +\infty]$ is measurable iff $f^{-1}(A)$ is measurable $\forall A \in \mathscr{B}([0, +\infty))$.*

Definition 5.21. For measurable functions $f_X \to [0, +\infty]$, we define

$$\int_X f d\mu = \sup \left\{ \int_x g d\mu \ : \ f \geq g \geq 0 \ : \ g \text{ is simple} \right\} \in [0, +\infty].$$

Theorem 5.22. ***Monotone convergence theorem*** *Assume $(X, \mathscr{B}, \mu)$ is a measure space, $(f)_{n=1}^{\infty}$ is an increasing sequence of measurable positive functions $f_n : X \to [0, +\infty]$. Define $f(x) = \lim_{n\to\infty} f_n(x)$. Then f is measurable and*

$$\int_X f d\mu = \lim_{n\to\infty} \int_X f_n d\mu.$$

Theorem 5.23. *Assume $(X, \mathscr{B})$ is a measurable space and $f : X \to [0, +\infty]$ is measurable. Then there are simple functions g_n, s.t.*

$$0 \leq g_1 \leq g_2 \leq \dots \ \ , \ \ g_n(x) \to f(x), \ \ \forall x \in X.$$

Moreover, if f is bounded, we can choose g_n s.t. the convergence is uniform, that is,

$$\lim_{n\to\infty} \sup_{x \in X} |g_n(x) - f(x)| = 0.$$

6 Integration of measurable functions (Ch. 9 in [Schilling(2017)])

Through this chapter $(X, \mathscr{A}, \mu)$ will be some measure space. Recall that $\mathcal{M}^+(\mathscr{A})$ $[\mathcal{M}^+_{\mathbb{R}}(\mathscr{A})]$ are the $\mathscr{A}$-measurable positive functions and $\mathcal{E}(\mathscr{A})$ $[\mathcal{E}^+_{\mathbb{R}}(\mathscr{A})]$ are the positive and simple functions.

The fundamental idea of *Integration* is to measure the area between the graph of the function and the abscissa. For positive simple functions $f \in \mathcal{E}^+(\mathscr{A})$ in standard representation, this is done easily

$$\text{if } f = \sum_{i=0}^{M} y_i \mathbb{1}_{A_i} \in \mathcal{E}^+(\mathscr{A}) \quad \text{then } \sum_{i=0}^{M} y_i \mu(A_i) \tag{1}$$

would be the μ-area enclosed by the graph and the abscissa. We note that the representation of f should not impact the integral of f.

Lemma 6.1. *Let $\sum_{i=0}^{M} y_i \mathbb{1}_{A_i} = \sum_{k=0}^{N} z_k \mathbb{1}_{B_k}$ be two standard representations of the same function $f \in \mathcal{E}^+(\mathscr{A})$. Then*

$$\sum_{i=0}^{M} y_i \mu(A_i) = \sum_{k=0}^{N} z_k \mu(B_k). \tag{2}$$

Definition 6.2. Let $f = \sum_{i=0}^{M} y_i \mathbb{1}_{A_i} \in \mathcal{E}^+(\mathscr{A})$ be a simple function in standard representation. Then the number

$$I_\mu(f) = \sum_{i=0}^{M} y_i \mu(A_i) \in [0, \infty] \tag{3}$$

(which is independent of the representation of f) is called the μ-integral of f.

Proposition 6.3. *Let $f, g \in \mathcal{E}^+(\mathscr{A})$. Then*

(i) $I_\mu(\mathbb{1}_A) = \mu(A) \quad \forall A \in \mathscr{A}$.

(ii) $I_\mu(\lambda f) = \lambda I_\mu(f) \quad \forall \lambda \geq 0$.

(iii) $I_\mu(f + g) = I_\mu(f) + I_\mu(g)$.

(iv) $f \leq g \Rightarrow I_\mu(f) \leq I_\mu(g)$.

In theorem 8.8 we saw that we could for every $u \in \mathcal{M}^+(\mathscr{A})$ write it as an increasing limit of simple functions. By corollary 8.10, the suprema of simple functions are again measurable, so that

$$u \in \mathcal{M}^+(\mathscr{A}) \Leftrightarrow u = \sup_{n \in \mathbb{N}} f_n, f \in \mathcal{E}^+(\mathscr{A}), \\ f_n \leq f_{n+1} \leq \dots .$$

We will use this to "inscribe" simple functions (which we know how to integrate) below the graph of a positive measurable function u and exhaust the μ-area below u.

Definition 6.4. Let $(X, \mathscr{A}, \mu)$ be a measure space. The (μ)-integral of a positive function $u \in \mathcal{M}^+_{\bar{\mathbb{R}}}(\mathscr{A})$ is given by

$$\int u d\mu = \sup \left\{ I_\mu(g) : g \leq u, \ g \in \mathcal{E}^+(\mathscr{A}) \right\}, \tag{4}$$

with $\int u d\mu \in [0, +\infty]$. If we need to emphasize the *integration variable*, we write $\int u(x) \mu(dx)$. The key observation is that the integral $\int \dots d\mu$ extends I_μ.

Lemma 6.5. *For all $f \in \mathcal{E}^+(\mathscr{A})$ we have $\int f d\mu = I_\mu(f)$.*

The next theorem is one of many convergence theorems. It shows that we could have defined 4 using any increasing sequence $f_n \uparrow u$ of simple functions $f_n \in \mathcal{E}^+(\mathscr{A})$.

Theorem 6.6. *(Beppo Levi) Let $(X, \mathscr{A}, \mu)$ be a measure space. For an increasing sequence of functions $(u_n)_{n \in \mathbb{N}} \subset \mathcal{M}^+_{\bar{\mathbb{R}}}(\mathscr{A})$, $0 \leq u_n \leq u_{n+1} \leq \dots$, we have for the supremum $u = \sup_{n \in \mathbb{N}} u_n \in \mathcal{M}^+_{\bar{\mathbb{R}}}(\mathscr{A})$ and*

$$\int \sup_{n \in \mathbb{N}} u_n d\mu = \sup_{n \in \mathbb{N}} \int u_n d\mu. \tag{5}$$

Note we can write $\lim_{n \to \infty}$ instead of $\sup_{n \in \mathbb{N}}$ as the supremum of an increasing sequence is its limit. Moreover, this theorem holds in $[0, +\infty]$, so the case $+\infty = +\infty$ is possible.

Corollary 6.7. *Let $u \in \mathcal{M}^+_{\bar{\mathbb{R}}}(\mathscr{A})$. Then*

$$\int u d\mu = \lim_{n \to \infty} \int f_n d\mu$$

holds for every sequence $(f_n)_{n \in \mathbb{N}} \subset \mathcal{E}^+(\mathscr{A})$ with $\lim_{n \to \infty} f_n = u$.

Proposition 6.8. *(of integral) Let $u, v \in \mathcal{M}^+_{\bar{\mathbb{R}}}(\mathscr{A})$. Then*

(i) $\int \mathbb{1}_A d\mu = \mu(A) \quad \forall A \in \mathscr{A}$.

(ii) $\int \alpha u d\mu = \alpha \int u d\mu \quad \forall \alpha \geq 0$.

(iii) $\int u + v d\mu = \int u d\mu + \int v d\mu$.

(iv) $u \leq v \Rightarrow \int u d\mu \leq \int v d\mu$.

Corollary 6.9. *Let $(u_n)_{n \in \mathbb{N}} \subset \mathcal{M}^+_{\bar{\mathbb{R}}}(\mathscr{A})$. Then $\sum_{n=1}^{\infty} u_n$ is measurable and we have*

$$\int \sum_{n=1}^{\infty} u_n d\mu = \sum_{n=1}^{\infty} \int u_n d\mu$$

(including the possibility $+\infty = +\infty$.)

Theorem 6.10. *(Fatou) Let $(u_n)_{n \in \mathbb{N}} \subset \mathcal{M}^+_{\bar{\mathbb{R}}}(\mathscr{A})$ be a sequence of positive measurable functions. Then $u = \liminf_{n \to \infty} u_n$ is measurable and*

$$\int \liminf_{n \to \infty} u_n d\mu = \liminf_{n \to \infty} \int u_n d\mu \tag{6}$$

7 Integrals of Measurable Functions (Ch. 10 in [Schilling(2017)])

We have defined our integral for positive measurable functions, i.e. functions in $\mathcal{M}^+(\mathscr{A})$. To extend our integral to not only functions in $\mathcal{M}^+(\mathscr{A})$ we first notice that

$$u \in \mathcal{M}_{\overline{\mathbb{R}}}(\mathscr{A}) \Leftrightarrow u = u^+ - u^-, \ u^+, u^- \in \mathcal{M}^+_{\overline{\mathbb{R}}},$$

i.e. that every measurable function can be written as a sum of **positive** measurable functions.

Definition 7.1 (μ-integrable)**.** A function $u : X \to \overline{\mathbb{R}}$ on $(X, \mathscr{A}, \mu)$ is *μ-integrable*, if it is $\mathscr{A}/\mathscr{B}(\overline{\mathbb{R}})$-measurable and if $\int u^+ d\mu, \int u^- d\mu < \infty$ (recall the definition for the integral of positive measurable functions). Then

$$\int u d\mu := \int u^+ d\mu - \int u^- d\mu \in (-\infty, \infty)$$

is the (μ-)*integral* of u. We write $\mathcal{L}^1(\mu)$ for the set of all real-valued μ-integrable functions [1].

Theorem 7.2. *Let $u \in \mathcal{M}_{\overline{\mathbb{R}}}(\mathscr{A})$, then the following conditions are equivalent:*

(i) $u \in \mathcal{L}^1_{\overline{\mathbb{R}}}(\mu)$.

(ii) $u^+, u^- \in \mathcal{L}^1_{\overline{\mathbb{R}}}(\mu)$.

(iii) $|u| \in \mathcal{L}^1_{\overline{\mathbb{R}}}(\mu)$.

(iv) $\exists w \in \mathcal{L}^1_{\overline{\mathbb{R}}}(\mu)$ *with* $w \geq 0$ *s.t.* $|u| \leq w$.

Theorem 7.3 (Properties of the μ-integral)**.** *The μ-integral is:* ***homogeneous, additive****, and:*

(i) $\min\{u, v\}, \max\{u, v\} \in \mathcal{L}^1_{\overline{\mathbb{R}}}(\mu)$ *(lattice property)*

(ii) $u \leq v \Rightarrow \int u d\mu \leq \int v d\mu$ *(monotone)*

(iii) $\left|\int u d\mu\right| \leq \int |u| d\mu$ *(triangle inequality)*

Remark. *If $u(x) \pm v(x)$ is defined in $\overline{\mathbb{R}}$ for all $x \in X$ then we can exclude $\infty - \infty$ and the theorem above just says that the integral is linear:*

$$\int (au + bv) d\mu = a \int u d\mu + b \int v d\mu.$$

This is always true for real-valued $u, v \in \mathcal{L}^1(\mu) = \mathcal{L}^1_{\mathbb{R}}(\mu)$, making $\mathcal{L}^1(\mu)$ a vector space with addition and scalar multiplication defined by

$$(u+v)(x) := u(x) + v(x), \ (a \cdot u)(x) := a \cdot u(x),$$

and

$$\int ... d\mu : \mathcal{L}^1(\mu) \to \mathbb{R}, \ u \mapsto \int u d\mu,$$

is a ***positive linear functional****.*

8 Null sets and the Almost Everywhere (Ch. 11 in [Schilling(2017)])

Definition 8.1. A (μ-)null set $N \in \mathcal{N}_\mu$ is a measurable set $N \in \mathscr{A}$ satisfying

$$N \in \mathcal{N}_\mu \iff N \in \mathscr{A} \text{ and } \mu(N) = 0.$$

This can be used generally about a 'statement' or 'property', but we will be interested in questions like 'when is $u(x)$ equal to $v(x)$', and we answer this by saying

$$u = v \ a.e. \Leftrightarrow \{x : u(x) \neq v(x)\} \text{ is (contained in) a } \mu\text{-null set},$$

i.e.

$$u = v \quad \mu\text{-a.e.} \Leftrightarrow \mu\left(\{x : u(x) \neq v(x)\}\right) = 0.$$

The last phrasing should of course include that the set $\{x : u(x) \neq v(x)\}$ is in $\mathscr{A}$.

Theorem 8.2. *Let $u \in \mathcal{M}_{\overline{\mathbb{R}}}(\mathscr{A})$, then:*

(i) $\int |u| d\mu = 0 \Leftrightarrow |u| = 0$ *a.e.* $\Leftrightarrow \mu\{u \neq 0\} = 0$,

(ii) $\mathbb{1}_N u \in \mathcal{L}^1_{\overline{\mathbb{R}}}(\mu) \ \ \forall \ N \in \mathcal{N}_\mu$,

(iii) $\int_N u d\mu = 0$.

(i) is really useful, later we will define $\mathcal{L}^p$ and the $||\cdot||_p$-(semi)norm. Then (i) means that if we have a sequence u_n converging to u in the $||\cdot||_p$-norm then $u_n(x) = u(x)$ a.e.

Corollary 8.3. *Let $u = v$ μ-a.e. Then*

(i) $u, v \geq 0 \Rightarrow \int u d\mu = \int v d\mu$,

[1] In words, we extend our integral to ~~positive~~ measurable functions by noticing that we can write every measurable function as a sum of positive measurable functions, something that we do know how to integrate. We don't want to run into the problem of $\infty - \infty$, thus we require the integral of the positive and negative parts to both (separately) be less than infinity.

(ii) $u \in \mathcal{L}^1_{\bar{\mathbb{R}}}(\mu) \Rightarrow v \in \mathcal{L}^1_{\bar{\mathbb{R}}}(\mu)$ *and* $\int u d\mu = \int v d\mu$.

Corollary 8.4. *If* $u \in \mathcal{M}_{\bar{\mathbb{R}}}(\mathscr{A})$, $v \in \mathcal{L}^1_{\bar{\mathbb{R}}}(\mu)$ *and* $v \geq 0$ *then*

$$|u| \leq v \ a.e. \ \Rightarrow u \in \mathcal{L}^1_{\bar{\mathbb{R}}}(\mu).$$

Proposition 8.5 (Markow inequality)**.** *For all* $u \in \mathcal{L}^1_{\bar{\mathbb{R}}}(\mu)$, $A \in \mathscr{A}$ *and* $c > 0$

$$u\left(\{|u| \geq c\} \cap A\right) \leq \frac{1}{c} \int_A |u| d\mu,$$

if $A = X$*, then (obviosly)*

$$u\left\{|u| \geq c\right\} \leq \frac{1}{c} \int |u| d\mu.$$

Completions of measure spaces

Definition 8.6. A measure space $(X, \mathscr{B}, \mu)$ is called **complete** if whenever $A \in \mathscr{B}$ and $\mu(A) = 0$, we have $B \in \mathscr{B}\ \forall B \subset A$.

Remark. *Any measure space can be completed as follows:*
Let $\bar{\mathscr{B}}$ *be the* σ*-algebra generated by* $\mathscr{B}$ *and all sets* $B \subset X$ *s.t. there exists* $A \in \mathscr{B}$ *with* $B \subset A$ *and* $\mu(A) = 0$.

Proposition 8.7. *The* σ*-algebra* $\bar{\mathscr{B}}$ *can also be described as follows:*

$$\bar{\mathscr{B}} := \Bigg\{ B \subset X : A_1 \subset B \subset A_2$$
$$\text{for some } A_1, A_2 \in \mathscr{B} \text{ with } \mu(A_2 \backslash A_1) = 0 \Bigg\},$$

with B, A_1, A_2 *as above, we define*

$$\bar{\mu} := \mu(A_1) = \mu(A_2)$$

Then $\left(X, \bar{\mathscr{B}}, \bar{\mu}\right)$ *is a complete measure space.*

Definition 8.8. If μ is a Borel measure on a **metric** space (X, d), then the completion $\bar{\mathscr{B}}(X)$ of the Borel σ-algebra with respect to μ is called the σ-algebra of μ-measurable sets.

Remark. *For* $\mu = \lambda_n$ *on* $\mathbb{R}^n$ *we talk about the* σ*-algebra of* ***Lebesgue measurable sets****. Instead of* $\bar{\lambda}_n$ *we still write* λ_n *and call it the* ***Lebesgue measure****. A function* $f : \mathbb{R}^n \to \mathbb{C}$*, measurable w.r.t. the* σ*-algebra of Lebesgue measurable sets is called the* ***Lebesgue measurable****.*

The following result shows that any Lebesgue measurable function coincides with a Borel function a.e.

Proposition 8.9. *Assume* $(X, \mathscr{B}, \mu)$ *is a measure space and consider its completion* $\left(X, \bar{\mathscr{B}}, \bar{\mu}\right)$*. Assume* $f : X \to \mathbb{C}$ *is* $\bar{\mathscr{B}}$*-measurable. Then there is a* $\mathscr{B}$*-measurable function* $g : X \to \mathbb{C}$ *s.t.* $f = g\ \bar{\mu}$*-a.e.*

9 Convergence Theorems and Their Applications (Ch. 12 in [Schilling(2017)])

- To interchange limits and integrals in **Riemann integrals** one typically has to assume uniform convergence. ¡- The set of Riemann integrable functions is somewhat limited, see theorem 9.4

Theorem 9.1 (Generalization of Beppo Levi, monotone convergence)**.**

(i) *Let* $(u_n)_{n\in\mathbb{N}} \subset \mathcal{L}^1(\mu)$ *be s.t.* $u_1 \leq u_2 \leq ...$ *with limit* $u := \sup_{n\in\mathbb{N}} u_n = \lim_{n\to\infty} u_n$*. Then* $u \in \mathcal{L}^1(\mu)$ ***iff***

$$\sup_{n\in\mathbb{N}} \int u_n d\mu < +\infty,$$

in which case

$$\sup_{n\in\mathbb{N}} \int u_n d\mu = \int \sup_{n\in\mathbb{N}} u_n d\mu.$$

(ii) *Same thing only with a decreasing sequence ...*$> -\infty$ *in which case*

$$\inf_{n\in\mathbb{N}} \int u_n d\mu = \int \inf_{n\in\mathbb{N}} u_n d\mu.$$

Theorem 9.2 (Lebesgue; dominated convergence)**.** *Let* $(u_n)_{n\in\mathbb{N}} \subset \mathcal{L}^1(\mu)$ *s.t.*

(a) $|u_n|(x) \leq w(x)$, $w \in \mathcal{L}^1(\mu)$,

(b) $u(x) = \lim_{n\to\infty} u_n(x)$ *exists in* $\bar{\mathbb{R}}$,

then $u \in \mathcal{L}^1(\mu)$ *and we have*

(i) $\lim\limits_{n\to\infty} \int |u_n - u| d\mu = 0$;

(ii) $\lim\limits_{n\to\infty} \int u_n d\mu = \int \lim\limits_{n\to\infty} u_n d\mu = \int u d\mu$;

Application 2: Riemann vs Lebesgue Integration

Consider only $(X, \mathscr{A}, \mu) = (\mathbb{R}, \mathscr{B}(\mathbb{R}), \lambda)$.

Theorem 9.3. *Let $u : [a, b] \to \mathbb{R}$ be a **measurable** and **Riemann integrable** function. Then*

$$u \in \mathcal{L}^1(\lambda) \text{ and } \int_{[a,b]} u d\lambda = \int_a^b u(x)dx. \tag{7}$$

Theorem 9.4. *Let $u : [a, b] \to \mathbb{R}$ be a bounded function, it is Riemann integrable **iff** the points in (a, b) where u is discontinuous are a (subset of) Borel measurable null set.*

Improper Riemann Integrals

- The Lebesgue integral extends the (*proper*) Riemann integral. However, there is a further extension of the Riemann integral which cannot be captured by Lebesgue's theory. u is Lebesgue integrable *iff* $|u|$ ha finite Lebesgue integral. ¡- The Lebesgue integral does not respect sign-changes and cancellations. However, the following *improper Riemann integral* does:

$$(R) \int_0^\infty u(x)dx := \lim_{n\to\infty} (R) \int_0^a u(x)dx. \tag{8}$$

Corollary 9.5. *Let $u : [0, \infty) \to \mathbb{R}$ be a measurable, Riemann integrable function for every interval $[0, N]$, $N \in \mathbb{N}$. Then $u \in \mathcal{L}^1[0, \infty)$ **iff***

$$\lim_{N\to\infty} (R) \int_0^N |u(x)|dx < \infty. \tag{9}$$

In this case, $(R) \int_0^\infty u(x)dx = \int_{[0,\infty)} ud\lambda$

Proposition 9.6 (appearing as example 12.13 in Schilling)**.** *Let $f_\alpha(x) := x^\alpha, x > 0$ and $\alpha \in \mathbb{R}$. Then*

(i) $f_(\alpha) \in \mathcal{L}^1(0, 1) \Leftrightarrow \alpha > -1$.

(ii) $f_(\alpha) \in \mathcal{L}^1[1, \infty) \Leftrightarrow \alpha < -1$.

10 Regularity of measures (Append H in [Schilling(2017)])

We let (X, d) be a metric space and denote by $\mathcal{O}$ the open, by $\mathcal{C}$ the closed and $\mathscr{B}(X) = \sigma(\mathcal{O})$ the Borel set of X.

Definition 10.1. A measure μ on $(X, d, \mathscr{B}(X))$ is called outer regular, if

$$\mu(B) = \inf \{\mu(U) \mid B \subset U,\ U \text{ open}\} \tag{10}$$

and inner regular, if $\mu(K) < \infty$ for all compact sets $K \subset X$ and

$$\mu(U) = \sup \{\mu(K) \mid K \subset U,\ K \text{ compact}\}. \tag{11}$$

A measure which is both inner and outer regular is called **regular**. We write $\mathfrak{m}_r^+(X)$ for the family of regular measures on $(X, \mathscr{B}(X))$.

Remark. *The space X is called σ-compact if there is a sequence of compact sets $K_n \uparrow X$. A typical example of such a space is a locally compact, separable metric space.*

Theorem 10.2. *Let (X, d) be a metric space. Every finite measure μ on $(X, \mathscr{B}(X))$ is outer regular. If X is σ-compact, then μ is also inner regular, hence regular.*

Theorem 10.3. *Let (X, d) be a metric space and μ be a measure on $(X, B(X))$ such that $\mu(K) < \infty$ for all compact sets $K \subset X$.*

1 If X is σ-compact, then μ is inner regular.

2 If there exists a sequence $G_n \in \mathcal{O},\ G_n \uparrow X$ such that $\mu(G_n) < \infty$, then μ is outer regular.

11 The Function Spaces $\mathcal{L}^p$ (Ch. 13 in [Schilling(2017)])

Assume V is a vector space over $\mathbb{K} \in \{\mathbb{C}, \mathbb{R}\}$.

Definition 11.1. A seminorn on V is a map $p : V \to [0, +\infty)$ s.t.

(1) $p(cx) = |c|p(x) \ \ \forall x \in V, \forall c \in \mathbb{K}$.

(2) $p(x + y) \le p(x) + p(y) \ \forall x, y \in V$. **triangle inequality**.

A seminorm is called a norm if we also have

$$p(x) = 0 \iff x = 0.$$

A norm is commonly denoted $||x||$, and a vectorspace equipped with a norm is called a **normed space**.

Definition 11.2. Assume (X, d) is a measure space. Fix $1 \leq p \leq \infty$. For every measurable function $f : X \to \mathbb{C}$ we define the following

$$||f||_p = \left(\int_X |f|^p d\mu \right)^{1/p} \in [0, +\infty]. \tag{12}$$

We can see that $||cf||_p = |c|||f||_p \ \forall c \in \mathbb{C}$.

Notice that by Theorem 8.2(i) we have that $||f||_p = 0 \Rightarrow f = 0$ a.e. Consider for example $\lim_{n\to\infty} ||f_n - f||_p = 0$, then we can find a subsequence s.t. $\lim_{k\to\infty} |f_{n(k)} - f| = 0$ a.e., i.e. $\lim_{k\to\infty} f_{n(k)} = f$ a.e.

Lemma 11.3.

$$||f + g||_p \leq ||f||_p + ||g||_p. \tag{13}$$

Definition 11.4. We define

$$\mathcal{L}^p(X, d\mu) = \{f : X \to \mathbb{C} \ | \ f \text{ is measurable and } ||f||_p < \infty\}.$$

This is a vectorspace with seminorm $f \mapsto ||f||_p$. And in general this is not a normed space, since $||f||_p = 0 \iff f = 0$ a.e.

Generally, if p is a seminorm on a vectorspace V, then

$$V_0 = \{x \in V \ | \ p(x) = 0\} \tag{14}$$

which is a subspace of V. Then we consider the quotient/factor space V/V_0.

Definition 11.5. For $x, y \in V$, define

$$x \sim y \iff x - y \in V_0. \tag{15}$$

This is an equivalence relation on V. The representation class of V is defined by $[x]$ or $x + V_0$.

Then V/V_0 is equals the set of equivalence classes. We can show that it is a normed space.

$$[x] + [y] = [x + y] \ \ , \ \ c[x] = [cx] \ \ , \ \ ||[x]|| = p(x).$$

Applying this to $\mathcal{L}^p(X, d\mu)$ we get the normed space

$$L^p(X, d\mu) := \mathcal{L}^p(X, d\mu)/\mathcal{N} = \mathcal{L}^p(X, d\mu)/_\sim. \tag{16}$$

Where $\mathcal{N}$ is the space of measurable functions f s.t. $f = 0$ a.e. The equivalence relation $\sim$ is defined by

$$u \sim v \iff \{u \neq v\} \in \mathcal{N}_\mu \iff \mu\{u \neq v\} = 0,$$

and so $L^p(X, d\mu)$ consists of *all equivalence classes* $[u]_p = \{v \in \mathcal{L}^p | u \sim v\}$. So for every $u \in L^p$ there is no $v \in L^p$ such that $\mu\{u \neq v\} \neq 0$.

We will further continue to denote the norm by $|| \cdot ||_p$, and we will normally **not** distinguish between $f \in \mathcal{L}^p(X, d\mu)$ and the vector in $L^p(X, d\mu)$ that f defines.

Definition 11.6. A normed space $(X, || \cdot ||)$ is called a Banach space if V is complete w.r.t the metric $d(x, y) = ||x - y||$.

Theorem 11.7. *If $(X, \mathscr{B}, \mu)$ is a measure space, $1 \leq p \leq \infty$, then $L^p(X, d\mu)$ is a Banach space.*

Definition 11.8. A measurable function $f : X \to \mathbb{C}$ is called **essentially bounded** if there is $c \geq 0$ s.t.

$$\mu(\{x \ : \ |f(x)| > c\}) = 0. \tag{17}$$

That is $|f| \leq c$ a.e. The smallest such c is called the essential supremum of f and is denoted by $||f||_\infty$.

Definition 11.9.

$$\mathcal{L}^\infty(X, d\mu) = \{f : X \to \mathbb{C} \ | \ f \text{ is measurable and } ||f||_\infty < \infty\}.$$

$$L^\infty(X, d\mu) = \mathcal{L}^\infty(X, d\mu)/\mathcal{N}.$$

Where by the previous definiton these spaces become the spaces of all essentially bounded functions.

Theorem 11.10. *If $(X, \mathscr{B}, \mu)$ is a σ-finite measure space, then $L^\infty(X, d\mu)$ is a Banach space.*

Convergence in $\mathcal{L}^p$ and completeness

Lemma 11.11. *For any sequence $(u_n)_{n\in\mathbb{N}} \subset \mathcal{L}^p(\mu), p \in [1, \infty)$, of positive functions $u_n \geq 0$ we have*

$$\Big|\Big| \sum_{n=1}^\infty u_n \Big|\Big|_p \leq \sum_{n=1}^\infty ||u_n||_p.$$

Theorem 11.12 (Riesz-Fischer)**.** *The spaces $\mathcal{L}^p(\mu), p \in [1, \infty)$, are complete, i.e. every Cauchy sequence $(u_n)_{n\in\mathbb{N}} \subset \mathcal{L}^p(\mu)$ converges to some limit $u \in \mathcal{L}^p(\mu)$*

Corollary 11.13. *Let $(u_n)_{n\in\mathbb{N}} \subset \mathcal{L}^p(\mu), p \in [1, \infty)$ with $\mathcal{L}^p - \lim_{n\to\infty} u_n = u$. Then there exists a subsequence $(u_{n_k})_{k\in\mathbb{N}}$ s.t. $\lim_{k\to\infty} u_{n_k}(x) = u(x)$ holds for almost every $x \in X$.*

Theorem 11.14. *Let $(u_n)_{n\in\mathbb{N}} \subset \mathcal{L}^p(\mu), p \in [1,\infty)$, be a sequence of functions s.t. $|u_n| \le w$ $\forall n \in \mathbb{N}$ and some $w \in \mathcal{L}^p(\mu)$. If $u(x) = \lim_{n\to\infty} u_n(x)$ exists for (almost) every $x \in X$, then*

$$u \in \mathcal{L}^p \text{ and } \lim_{n\to\infty} ||u - u_n||_p = 0.$$

Theorem 11.15 (F. Riesz (convergence theorem)). *Let $(u_n)_{n\in\mathbb{N}} \subset \mathcal{L}^p(\mu), p \in [1,\infty)$, be a sequence s.t. $\lim_{n\to\infty} u_n(x) = u(x)$ for almost every $x \in X$ and some $u \in \mathcal{L}^p(\mu)$. Then*

$$\lim_{n\to\infty} ||u_n - u||_p = 0 \iff \lim_{n\to\infty} ||u_n||_p = ||u||_p.$$

12 Dense and Determining Sets (Ch. 17 in [Schilling(2017)])

Definition 12.1 (Dense Sets)**.** A set $\mathcal{D} \subset \mathcal{L}^p(\mu), p \in [0,\infty]$, is called *dense* if for every $u \in \mathcal{L}^p(\mu)$ there exist a sequence $(f_n)_{n\in\mathbb{N}} \subset \mathcal{D}$ s.t. $\lim_{n\to\infty} ||u - f_n||_p = 0$.

Theorem 12.2. *Assume X, d is a metric space and μ is a Borel measure that is finite on every ball $1 \le p < \infty$. Then the space of bounded continuous functions with bounded support is dense in $\mathcal{L}^p(X, d\mu)$. Where bounded support means that f vanishes outside some ball.*

Theorem 12.3. *Assume (X, d) is a separable locally compact metric space and μ is a Borel Measure on X s.t. $\mu(K) < \infty$ $\forall$ compact $K \subset K$. Then the space $C_c(X)$ of continuous compactly supported functions is dense in $\mathcal{L}^p(X, d\mu)$.*

Recall that the support of a function f is $\mathrm{supp}(f) = \{x \in X : f(x) \neq 0\}$, *closed support* is the closure of $\mathrm{supp}(f)$ (i.e. boundary points are included), often just written as $\mathrm{supp}(f)$, and a function is said to have *compact support* if $\mathrm{supp}(f)$ is *compact.*

In particular, either theorem shows that if μ is a Borel measure on $\mathbb{R}^n$ s.t. the measure of every ball is finite, then $C_c(\mathbb{R}^n)$ is dense in $\mathcal{L}^p(\mathbb{R}^n, d\mu)$, $1 \le p < \infty$. Later we will see that even $C_c^\infty(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n, d\mu)$.

Remark. *These results do not extend to $p = \infty$ in general.*

For $\mu = \lambda_n$ we write simply $\mathcal{L}^p(\mathbb{R}^n)$.

Remark. *Theorem 17.8 in the book is WRONG. For example, $X = \mathbb{Q}$ with the usual metric is σ-compact, supports nonzero finite measure, but $C_c(\mathbb{Q}) = 0$.*

Modes of Convergence (mixture of ex. 11.12 and ch. 22 p. 258-261.)

Assume $(X, \mathscr{B}, \mu)$ is a measure space. Given measurable functions $f_n, f : X \to \mathbb{C}$, recall that

$$f_n \to f \text{ a.e.}$$

means that $f_n(x) \xrightarrow[n\to\infty]{} f(x)$ for all x outside a set of measure zero.

Theorem 12.4 (Egorov). *Assume $\mu(X) < \infty$ and $f_n \to f$ a.e. Then, $\forall \epsilon > 0$, there exists $X_\epsilon \in \mathscr{B}$ s.t. $\mu(X_\epsilon) < \epsilon$ and $f_n \to f$ uniformly on $X \backslash X_\epsilon$.*

In addition to pointwise and uniform convergence we also consider the following:

$f_n \to f$ in the *p-th mean* if $||f_n - f||_p \xrightarrow[n\to\infty]{} 0$. For $p = 1$ we say *in mean*, for $p = 2$ we say in quadratic mean.

$f_n \to f$ *in measure* if $\forall \epsilon > 0$ we have

$$\mu(\{x : |f_n(x) - f(x)| \ge \epsilon\}) \xrightarrow[n\to\infty]{} 0.$$

Theorem 12.5 (Lemma 22.4 in the book?)**.** *Assume $(X, \mathscr{B}, d\mu)$ is a measure space, $1 \le p < \infty$, $f_n, f : X \to \mathbb{C}$ are measurable functions. Then*

(i) If $f_n \to f$ in the p-th mean, then $f_n \to f$ in measure.

(ii) If $f_n \to f$ in measure, then there is a subsequence $(f_{n_k})_{k=1}^\infty$ s.t. $f_{n_k} \to f$ a.e.

(iii) If $f_n \to f$ a.e. and $\mu(X) < \infty$, then $f_n \to f$ in measure.

In particular, if $f_n \to f$ in the p-th mean, then $f_{n_k} \to f$ a.e. for a subsequence $(f_{n_k})_k$.

13 Abstract Hilbert Spaces (Ch. 26 in [Schilling(2017)])

Assume $\mathcal{H}$ is a vector space over $\mathbb{C}$.

Definition 13.1. A pre-inner product on $\mathcal{H}$ is a map $(\cdot,\cdot) : H \times H \to \mathbb{C}$ which is

(i) Sesquilinear: linear in the first variable and antilinear in the second:

$$(\alpha u + \beta v, w) = \alpha(u, w) + \beta(v, w),$$
$$(w, \alpha u + \beta v) = \bar{\alpha}(w, u) + \bar{\beta}(w, v), \; u, v, w \in H \text{ and } \alpha, \beta \in \mathbb{C}.$$

(ii) Hermitian: $(u,v) = \overline{(u,v)}$.

(iii) Positive semidefinite: $(u,v) \geq 0$.

It is called an **inner product**, or a scalar product, if instead of (iii) the map is positive definite; $(u,v) > 0$. This definition also works for $\mathbb{R}$ instead of $\mathbb{C}$.

Cauchy-Schwartz inequality If $(\cdot,\cdot)$ is a pre-inner product, then $|(u,v)| \leq (u,u)^{1/2}(v,v)^{1/2}$.

Corollary 13.2. *Assume we have a seminorm* $||u|| := (u,u)^{1/2}$. *It is a norm iff* $(\cdot,\cdot)$ *is an inner product.*

Definition 13.3 (Hilbert space)**.** A Hilbert space is a complex vector space $\mathcal{H}$ with an inner product $(\cdot,\cdot)$ s.t. $\mathcal{H}$ is complete with respect to the norm $||u|| = (u,u)^{1/2}$.

1. The norm on a Hilbert space is determined by the inner product, but the inner product can also be recovered by the norm by the *polarization identity*: $(u,v) = \frac{||u+v||^2 - ||u-v||^2}{4} + i\frac{||u+iv||^2 - ||u-iv||^2}{4}$.
2. *Parallelogram law*: $||u+v||^2 + ||u-v||^2 = 2||u||^2 + 2||v||^2$.
3. A norm on a vector space is given by an inner product iff it satisfies the parallelogram law, and then the scalar product is uniquely determined by the polarization identity.

Recall that a subset $\mathcal{C}$ of a vector space is called *convex* if

$$u, w \in \mathcal{C} \to tu + (1-t)w \in \mathcal{C} \ \forall t \in (0,1).$$

The following is one of the key properties of the Hilbert space

Theorem 13.4 (projection theorem)**.** *Assume* $\mathcal{H}$ *is a Hilbert space and* $\mathcal{C} \subset H$ *is a closed convex subset. Then for every* $u \in H$ *there is a unique* $u_0 \in \mathcal{C}$ *(*minimizer*) s.t.*

$$||u - u_0|| = d(u, \mathcal{C}) (= \inf_{x \in \mathcal{C}} ||u - x||).$$

14 Orthogonal Projections (Ch. 26 in [Schilling(2017)])

For a Hilbert space $\mathcal{H}$ and a subset $A \subset H$, let

$$A^\perp := \{x \in H : x \perp y \ \forall y \in A\},$$

where $x \perp y$ means that $(x,y) = 0$. $A^\perp$ is a closed subspace of $\mathcal{H}$.

Proposition 14.1. *Assume* $\mathcal{H}_0$ *is a closed subspace of a Hilbert space* $\mathcal{H}$. *Then every* $u \in H$ *uniquely decomposes as*

$$u = u_0 + u_1, \text{ with } u_0 \in H \text{ and } u_1 \in \mathcal{H}_0^\perp.$$

Moreover, $||u - u_0|| = d(u, \mathcal{H}_0)$ *and* $||u||^2 = ||u_0||^2 + ||u_1||^2$.

For a closed subspace $\mathcal{H}_0 \subset \mathcal{H}$, consider the map $P : H \to \mathcal{H}_0$ s.t. $Pu \in \mathcal{H}_0$ is the unique element satisfying $u - Pu = H_0^\perp$. The operator P is linear. It is also contractive, meaning that $||Pu|| \leq ||u||$, since $||u||^2 = ||Pu||^2 + ||u - Pu||^2$. It is called the orthogonal projection onto $\mathcal{H}_0$.

If $\mathcal{H}_0$ is finite dimensional with an orthonormal basis $u_1, ..., u_n$ then

$$Pu = \sum_{k=1}^{n} (u, u_k) u_k.$$

Orthonormal bases can be defined for arbitrary Hilbert spaces.

Definition 14.2 (orthonormal system)**.** An orthonormal system in $\mathcal{H}$ is a collection of vectors $u_i \in H$ $(i \in I)$s.t.

$$(u_i, u_j) = \delta_{ij} \ \forall i, j \in I.$$

It is called an *orthonormal basis* if span$\{u_i\}_{i \in I}$ denotes the linear span of $\{u_i\}_{i \in I}$, the space of finite linear combinations of the vectors u_i.

Definition 14.3. A Hilbert space $\mathcal{H}$ is said to be *separable* if $\mathcal{H}$ contains a countable dense subset $G \subset \mathcal{H}$.

Theorem 14.4. *Every Hilbert space* $\mathcal{H}$ *has an orthonormal basis. If* $\mathcal{H}$ *is separable, then there is a countable orthonormal basis.*

Proposition 14.5. *Assume* $\{u_i\}_{i \in I}$ *is an orthonormal system in a Hilbert space* H. *Take* $u \in \mathcal{H}$. *Then*

(i) Bessel's inequality*:* $\sum_{i \in I} |(u, u_i)|^2 \leq ||u||^2$, *in particular,* $\{i : (u, u_i) \neq 0\}$ *is countable.*

(ii) Parseval's identity*: If* $\{u_i\}_{i \in I}$ *is an orthonormal basis, then* $\sum_{i \in I} |(u, u_i)|^2 = ||u||^2$.

If $(u_i)_{i\in I}$ is an orthonormal basis, then the numbers (u, u_i) are called the **Fourier coefficients** of u with respect to $(u_i)_{i\in I}$. The Parseval identity then suggests that u is determined by its Fourier coefficients. This is true, and even more, we have:

Proposition 14.6. *Assume $(u_i)_{i\in I}$ is an orthonormal basis in a Hilbert space $\mathcal{H}$. Then for every vector $(c_i)_{i\in I} \in l^2(I)$ there is a unique vector $u \in \mathcal{H}$ with Fourier coefficients c_i, and we write*

$$u = \sum_{i\in I} c_i u_i.$$

Remark. *Equivalently, the element $u = \sum_{i\in I} c_i u_i$ can be described as the unique element in $\mathcal{H}$ s.t. $\forall \epsilon > 0$ there is a finite $F_0 \subset I$ s.t. $||u - \sum_{i\in F} c_i u_i|| < \epsilon$ $\forall$ finite $F \supset F_0$.*

Corollary 14.7. *We have a linear isomorphism $U : l^2(I) \xrightarrow{\sim} \mathcal{H}$, $U((c_i)_{i\in I}) = \sum_{i\in I} c_i u_i$. By Parseval's identity this isomorphism is* isometric*, that is, $||Ux|| = ||x||$ $\forall x \in l^2(I)$. By the polarization identity this is equivalent to*

$$(Ux, Uy) = (x, y)\ \forall x, y \in l^2(I).$$

Therefor U is unitary.

Corollary 14.8. *Up to a unitary isomorphism, there is only one infinite dimensional separable Hilbert space, namely, l^2.*

15 Dual spaces (Ch. 26 in [Schilling(2017)])

Lemma 15.1. *Assume V is a normed space over $K = \mathbb{R}$ or $K = \mathbb{C}$. Consider a linear functional $f : V \to K$. The following are equivalent (TFAE):*

(i) f is continuous;

(ii) f is continuous at 0;

(iii) There is a $c \geq 0$ s.t. $|f(x)| \leq c||x||$ $\forall x \in V$.

If (i)-(iii) are satisfied, then f is called a *bounded linear functional*. The constant c in (iii) is denoted by $||f||$. We have $||f|| = \sup_{x\neq 0} \frac{|f(x)|}{||x||} = \sup_{||x||=1} |f(x)| = \sup_{||x||\leq 1} |f(x)|$.

Proposition 15.2. *For every normed vector space V over $K = \mathbb{R}$ or $K = \mathbb{C}$, the bounded linear functionals on V form a Banach space V^*.*

Remark. *The sequence $\{||f_n - f_m||\}_{m=1}^{\infty}$ actually converges, since*

$$\Big| ||f_n - f_m|| \Big| \leq ||f_m - f_n||.$$

When we study/use normed spaces, it is often important to understand the dual spaces. For Hilbert spaces this is particularly easy:

Theorem 15.3 (Riesz)**.** *Assume $\mathcal{H}$ is a Hilbert space. Then every $f \in \mathcal{H}^*$ has the form*

$$f(x) = (x, y),$$

for a uniquely defined $y \in \mathcal{H}$. Moreover, we have $||f|| = ||y||$.

For every Hilbert space $\mathcal{H}$ we can define the *conjugate Hilbert space* $\bar{\mathcal{H}}$, which has its elements as the symbols $\bar{x}$ for $x \in \mathcal{H}$, with the linear structure and inner product defined by

$$\bar{x} + \bar{y} = \overline{x + y}, c \cdot \bar{x} = \overline{\bar{c}x}, (\bar{x}, \bar{y}) = \overline{(x, y)} = (y, x).$$

Corollary 15.4. *For every Hilbert space $\mathcal{H}$, we have an isometric isomorphism $\bar{\mathcal{H}} \xrightarrow{\sim} \mathcal{H}^*$, $\bar{x} \mapsto (\cdot, x)$.*

16 Hahn-Banach Theorem (Ch. 4.2 in [Teschl(2010)])

Theorem 16.1 (Hahn-Banach)**.** *Assume V is a real vector space, $V_0 \subset V$ a subspace, $e : V \to \mathbb{R}$ a convex function and $f : V_0 \to \mathbb{R}$ a linear functional s.t. $f \leq e$ on V_0. Then f can be extended to a linear functional F on V s.t. $F \leq e$.*

Theorem 16.2 (Hahn-Banach)**.** *Assume V is a real or complex vector space, p a seminorm on V_0, $V_0 \subset$, and f a linear functional on V_0 s.t.*

$$|f(x)| \leq p(x)\ \forall x \in V_0.$$

Then f can be extended to a linear functional F on V s.t. $|F(x)| \leq p(x)$ $\forall x \in V$.

Corollary 16.3. *Assume V is a normed real or complex vector space, $V_0 \subset V$ and $f \in V_0^*$. Then there is a $F \in V^*$ s.t.*

$$F|_{V_0} f \text{ and } ||F|| = ||f||.$$

Corollary 16.4. *Assume V is a normed space and $x \in V, x \neq 0$. Then there is a $F \in V^*$ s.t. $||F|| = 1$ and $F(x) = ||x||$.*

Such an F is called a *supporting functional at x.*

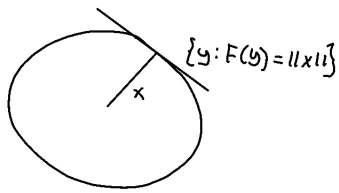

Figure 1: Tangent space?

If V is a normed vector space, then every $x \in X$ defines a bounded linear functional on V^* by

$$V^* \ni f \mapsto f(x).$$

As $|f(x)| \leq ||f|| \cdot ||x||$, this functional has norm $\leq ||x||$. By using a supporting functional at x, we actually see that we get norm $||x||$. Thus, we have an isometric embedding $V \subset V^{**} := (V^*)^*$. We can therefor see view V as a subspace of V^{**}.

Definition 16.5. A normed space V is called reflexive if $V^{**} = V$.

Remark. *This is stronger than requiring $V \simeq V^{**}$.*

Remark. *Every Hilbert space $\mathcal{H}$ is reflexive. Indeed, $\mathcal{H}^* = \bar{\mathcal{H}}$. By Riesz' theorem every bounded linear functional f on $\bar{\mathcal{H}}$ has the form*

$$f(\bar{x}) = (\bar{x}, \bar{y}) = (y, x),$$

for some $y \in \mathcal{H}$, which exactly means that $f = y$ in $\mathcal{H}^{**}$.

As we will see later, the spaces $\mathcal{L}^p(X, d\mu)$, with μ σ-finite and $1 < p < \infty$, are reflexive. The spaces $\mathcal{L}'(X, d\mu)$ and $\mathcal{L}^\infty(X, \mu)$ are usually not reflexive.

17 Radon-Nikodym Theorem (Ch. 20 in [Schilling(2017)])

Assume $(X, \mathscr{B}, \mu)$ is a measure space. Are there other measures on $(X, \mathscr{B})$?

Example 17.1. Take a measurable function $f : X \to [0, +\infty]$ and define

$$\nu(A) = \int_A f d\mu \text{ for } A \in \mathscr{B}.$$

This is a measure by the monotone convergence theorem. We write $d\nu = f d\mu$.

Proposition 17.2. *Assume $(X, \mathscr{B})$ is a measurable space, μ and ν are σ-finite measures on $(X, \mathscr{B})$. Then there exist $N \in \mathscr{B}$ and a measurable $f : X \to [0, +\infty]$ s.t. $\mu(N) = 0$ and $\nu(A) = \nu(A \cap N) + \int_A f d\mu \; \forall A \in \mathscr{B}$.*

When can we discard the term $\nu(A \cap N)$?

Definition 17.3. Given measure μ and ν on $X, \mathscr{B}$, we say that ν is *absolutely continuous* with respect to μ and write $\nu << \mu$, if $\nu(A) = 0$ whenever $A \in \mathscr{B}, \mu(A) = 0$.

Lemma 17.4. *Assume μ and ν are measures on $(X, \mathscr{B})$, $\nu(X) < \infty$. Then $\nu << \mu$ iff $\forall \epsilon > 0 \exists \delta > 0$ s.t. if $A \in \mathscr{B}$, $\mu(A) < \delta$, then $\nu(A) < \epsilon$.*

Proof. "$\Rightarrow$": obvious. "$\Leftarrow$": Assume this is not true. Then, there is a $\epsilon > 0$ s.t. $\forall \delta > 0$ we can find $A \in \mathscr{B}$ satisfying $\mu(A) < \delta$, $\nu(A) \geq \epsilon$. Let A_n be such a set A for $\delta = 1/2^n$. Put $A = \cap_{n \in \mathbb{N}} \cup_{k=n} A_k$. Then

$$\mu(A) \leq \lim_{n\to\infty} \mu\left(\bigcup_{k=n}^{\infty} A_k\right) \leq \lim_{n\to\infty} \sum_{k=n}^{\infty} \mu(A_k)$$
$$\leq \lim_{n\to\infty} \sum_{k=n}^{\infty} \frac{1}{2^k} = \lim_{n\to} \frac{1}{2^{n-1}} = 0.$$

As $\nu(X) < \infty$, we also have

$$\nu(A) = \lim_{n\to\infty} \nu\left(\bigcup_{k=n}^{\infty} A_k\right) \geq \epsilon.$$

This contradicts the assumption $\nu << \mu$. ☐

Remark. *The result is not true for infinite ν.*

Theorem 17.5 (Radon-Nikodym)**.** *Assume μ and ν are σ-finite measures on a measurable space $(X, \mathscr{B})$, $\nu << \mu$. Then there is a measurable function $f : X \to [0, +\infty)$ s.t. $d\nu = f d\mu$ (that is, $\nu(A) \int_A f d\mu$). If $\tilde{f}$ is another function with the same properties, then $f = \tilde{f}$ $\mu - a.e.$*

The function is called the Radon-Nikodym derivative *at ν w.r.t. μ and is denoted by $\frac{d\nu}{d\mu}$.*

Example 17.6. Consider a real-valued function $f \in C'[a, b]$ s.t. $f'(t) > 0 \; \forall \; t \in [a, b]$. Let $c = f(a), d = f(b)$. We know that for every Riemann integrable function g on $[c, d]$ we have

$$\int_c^d g(f) dt = \int_a^b g(f(t)) f'(t) dt.$$

Equivalently,

$$\int_c^d g \circ g^{-1} dt = \int_a^b g f'(t) dt. \tag{18}$$

Denote by $\lambda_{[a,b]}$, $\lambda_{[c,d]}$ the Lebesgue measure restricted to the Borel subsets of $[a,b]$ and $[c,d]$, respectively. Then eq. 18 implies that

$$d\left((f^{-1})_* \lambda_{[c,d]}\right) = f' d\lambda_{[a,b]},$$

since the integration of $g = \mathbb{1}_{[\alpha,\beta]}$ gives the same results for any interval $[\alpha,\beta] \subset [a,b]$ and since a finite Borel measure on $[a,b]$ is determined by its values on such intervals. Thus, $(f^{-1})_* \lambda_{[c,d]} << \lambda_{[a,b]}$ and

$$\frac{d\left((f^{-1})_* \lambda_{[c,d]}\right)}{d\lambda_{[a,b]}} = f'.$$

References

[Schilling(2017)] Schilling, R. 2017, Measures, Integrals and Martingales, Measures, Integrals and Martingales (Cambridge University Press). https://books.google.no/books?id=sdAoDwAAQBAJ

[Teschl(2010)] Teschl, G. 2010, Topics in Linear and Nonlinear Functional Analysis (Universität Wien). https://www.mat.univie.ac.at/~gerald/ftp/book-fa/fa.pdf