# UNIVERSITY OF OSLO

## Faculty of Mathematics and Natural Sciences

Examination in MAT4410 — Advanced linear analysis.

Day of examination: Friday, December 1, 2017.

Examination hours: 14:30 – 18:30.

This problem set consists of 3 pages.

Appendices: None.

Permitted aids: None.

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

**Note: You must justify all your answers!**

## Problem 1

Let $(\Omega, \mathcal{A}, P)$ be a probability space, thus $P(\Omega) = 1$. Let $\mathcal{G}$ be a $\sigma$-algebra of subsets of $\Omega$ such that $\mathcal{G} \subset \mathcal{A}$ and denote by $P|_{\mathcal{G}}$ the restriction of $P$ to a finite measure on $\mathcal{G}$. Recall that for $f \in L^1(\Omega, \mathcal{A}, P)$, the conditional expectation of $f$ given $\mathcal{G}$ is a function $\mathcal{E}(f \mid \mathcal{G})$ in $L^1(\Omega, \mathcal{G}, P|_{\mathcal{G}})$ defined by the identity

$$\int_G f\,dP = \int_G \mathcal{E}(f \mid \mathcal{G})\,dP|_{\mathcal{G}} \text{ for every } G \in \mathcal{G}, \tag{1}$$

where $\mathcal{E}(f \mid \mathcal{G})$ is unique $P$-a.e. among $\mathcal{G}$-measurable functions satisfying (1). It is assumed known that $\mathcal{E}(\cdot \mid \mathcal{G}) : L^1(\Omega, \mathcal{A}, P) \to L^1(\Omega, \mathcal{G}, P|_{\mathcal{G}})$ is linear and satisfies $\mathcal{E}(f \mid \mathcal{G}) \geq 0$ whenever $f \geq 0$.

### 1a

(15 points) Show that $|\mathcal{E}(f \mid \mathcal{G})| \leq \mathcal{E}(|f| \mid \mathcal{G})$ $P$-a.e for every $f \in L^1(\Omega, \mathcal{A}, P)$.

### 1b

(15 points) Prove that $\|\mathcal{E}(f \mid \mathcal{G})\|_1 \leq \|f\|_1$ for every $f \in L^1(\Omega, \mathcal{A}, P)$.

### 1c

(5 points) If $\mathcal{H}$ is a $\sigma$-algebra of subsets of $\Omega$ such that $\mathcal{H} \subset \mathcal{G} \subset \mathcal{A}$, prove that

$$\mathcal{E}\big(\mathcal{E}(f \mid \mathcal{H}) \mid \mathcal{G}\big) = \mathcal{E}(f \mid \mathcal{H})\, P\text{-a.e.}$$

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### 1d

(5 points) If $\mathcal{H}$ is a $\sigma$-algebra of subsets of $\Omega$ such that $\mathcal{H} \subset \mathcal{G} \subset \mathcal{A}$, prove that

$$\mathcal{E}\big(\mathcal{E}(f \mid \mathcal{G}) \mid \mathcal{H}\big) = \mathcal{E}(f \mid \mathcal{H}) \; P\text{-a.e.}$$

## Problem 2

Let $(\Omega, \mathcal{A}, \mu)$ be a measure space such that $\mu$ is $\sigma$-finite. For every $1 < p < \infty$, consider the space

$$L^p(\mu) = \{f : \Omega \to \mathbb{C} \mid f \text{ is } \mathcal{A} - \text{measurable }, \|f\|_p^p = \int_\Omega |f|^p \, d\mu < \infty\};$$

it is assumed known that this is a Banach space with the $\|\cdot\|_p$-norm. Let $1 < q < \infty$ such that $\frac{1}{p} + \frac{1}{q} = 1$.

### 2a

(15 points) Suppose that $\{f_n\}_{n \geq 1}$ is a sequence in $L^p(\mu)$ such that

$$\lim_{n\to\infty} \int_\Omega f_n g \, d\mu = 0 \tag{2}$$

for every $g \in L^q(\mu)$. Prove that $\sup\{\|f_n\|_p \mid n \geq 1\} < \infty$ and for every $E \in \mathcal{A}$ with $\mu(E) < \infty$ it holds that

$$\lim_{n\to\infty} \int_E f_n \, d\mu = 0.$$

### 2b

(15 points) Prove conversely that if $\sup\{\|f_n\|_p \mid n \geq 1\} < \infty$ and for every $E \in \mathcal{A}$ with $\mu(E) < \infty$ we have $\lim_{n\to\infty} \int_E f_n \, d\mu = 0$, then (2) holds for every $g \in L^q(\mu)$. (You may use without proof that the space of simple $\mathcal{A}$-measurable functions $s$ such that $\mu(\{x \in \Omega \mid s(x) \neq 0\}) < \infty$ is dense in $L^q(\mu)$ for the $\|\cdot\|_q$-norm.)

## Problem 3

Let $X$ be a normed space over $\mathbb{C}$.

### 3a

(10 points) Prove that for each $x \in X$ there exists $\ell_x \in X^*$ such that $\|\ell_x\| \leq 1$ and $\|\ell_x(x)\| = \|x\|$.

Let $B = \{x \in X \mid \|x\| \leq 1\}$ be the closed unit ball in $X$ and let $S = \{x \in X \mid \|x\| = 1\}$ be the closed unit sphere. Assume that $B$ is compact.

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## 3b

(10 points) Prove that there exist bounded linear functionals $\ell_1, \ldots, \ell_n$ in $X^*$ for some $n \in \mathbb{N}$ such that

$$S \subset \bigcup_{j=1}^{n} \{x \in X \mid |\ell_n(x)| > \frac{1}{2}\}.$$

## 3c

(10 points) Prove that there exists a linear injective map $F : X \to \mathbb{C}^n$ given by

$$F(x) = \big(\ell_1(x), \ldots, \ell_n(x)\big).$$

What can you conclude about the dimension of $X$ as a vector space?

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