Abstract Hilbert Spaces (lecture 13, 22. Feb)

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Assume $\mathcal{H}$ is a vector space over $\mathbb{C}$.

Definition 26.27. A pre-inner product on $\mathcal{H}$ is a map $(\cdot,\cdot) : H \times H \to \mathbb{C}$ which is

(i) Sesquilinear: linear in the first variable and antilinear in the second:

$$(\alpha u + \beta v, w) = \alpha(u,w) + \beta(v,w),$$
$$(w, \alpha u + \beta v) = \bar{\alpha}(u,w) + \bar{\beta}(v,w), \; u,v,w \in H \text{ and } \alpha,\beta \in \mathbb{C}.$$

(ii) Hermitian: $(u,v) = \overline{(u,v)}$.

(iii) Positive semidefinite: $(u,v) \geq 0$.

It is called an **inner product**, or a scalar product, if instead of (iii) the map is positive definite; $(u,v) > 0$. This definition also works for $\mathbb{R}$ instead of $\mathbb{C}$.

Cauchy-Schwartz inequality If $(\cdot,\cdot)$ is a pre-inner product, then $|(u,v)| \leq (u,u)^{1/2}(v,v)^{1/2}$.

Corollary 26.28. *Assume we have a seminorm* $||u|| := (u,u)^{1/2}$. *It is a norm iff* $(\cdot,\cdot)$ *is an inner product.*

Definition 26.29 (Hilbert space)**.** A Hilbert space is a complex vector space $\mathcal{H}$ with an inner product $(\cdot,\cdot)$ s.t. $\mathcal{H}$ is complete with respect to the norm $||u|| = (u,u)^{1/2}$.

1. The norm on a Hilbert space is determined by the inner product, but the inner product can also be recovered by the norm by the *polarization identity*: $(u,v) = \frac{||u+v||^2 - ||u-v||^2}{4} + i\frac{||u+iv||^2 - ||u-iv||^2}{4}$.
2. *Parallelogram law*: $||u+v||^2 + ||u-v||^2 = 2||u||^2 + 2||v||^2$.
3. A norm on a vector space is given by an inner product iff it satisfies the parallelogram law, and then the scalar product is uniquely determined by the polarization identity.

Example 26.30. Assume $(X, \mathscr{B}, \mu)$ is a measure space. Then $\mathcal{L}^2(X, d\mu)$ is a Hilbert space with inner product

$$(f, g) = \int_X f\bar{g} d\mu.$$

This is well-defined, as $|f\bar{g}| \leq \frac{1}{2}(|f|^2 + |g|^2)$.

In particular, if $\mathscr{B} = \mathcal{P}(X)$ and μ is the counting measure, then $L^2(X, d\mu)$ is denoted by $l^2(X)$; for $X = \mathbb{N}$ we write simply l^2. Note that in this case for $f : X \to [0, +\infty]$ we have

$$\int_X f d\mu = \sum_{x \in X} f(x) := \sup_{\substack{F \subset X \\ F \text{ is finite}}} \sum_{x \in F} f(x),$$

and if $\sum_{x \in X} f(x) < \infty$, then $\{x : f(x) > 0\}$ is at most countable, so $\sum_{x \in X} f(x) = \sum_{x : f(x) > 0} f(x)$ is the usual sum of a series.

Recall that a subset $\mathcal{C}$ of a vector space is called *convex* if

$$u, w \in \mathcal{C} \to tu + (1-t)w \in \mathcal{C} \ \forall t \in (0, 1).$$

The following is one of the key properties of the Hilbert space

Theorem 26.31 (projection theorem)**.** *Assume $\mathcal{H}$ is a Hilbert space and $\mathcal{C} \subset H$ is a closed convex subset. Then for every $u \in H$ there is a unique $u_0 \in \mathcal{C}$ (*minimizer*) s.t.*

$$||u - u_0|| = d(u, \mathcal{C})(= \inf_{v \in \mathcal{C}} ||u - x||).$$

Proof. Let $d = d(u, \mathcal{C})$. Choose $u_n \in \mathcal{C}$ s.t. $||u - u_n|| \to d$. We claim that $(u_n)_{n=1}^{\infty}$ is a Cauchy sequence. As $\frac{u_n + u_2}{2} \in \mathcal{C}$, we have

$$\begin{aligned} d^2 \leq ||u - \frac{u_n + u_m}{2}||^2 = \frac{1}{4}||(u - u_m) + (u - u_m)||^2 \\ \underset{\text{parallelogram law}}{=} = \frac{1}{4}\left(2||u - u_n||^2 + 2||u - u_m||^2 - ||u_n - u_m||^2\right), \end{aligned}$$

so

$$||u_n - u_m||^2 \leq 2\left(||u - u_n||^2 - d^2\right) + 2\left(||u - u_m||^2 - d^2\right).$$

Thus, $(u_n)_{n \in \mathbb{N}}$ is indeed Cauchy, hence $u_n \to u_0 \in \mathcal{C}$ for some u_0 and

$$||u - u_0|| = \lim_{n \to \infty} ||u - u_m|| = d = d(u, \mathcal{C}).$$

If u_0' is another such point, we can take $u_{2n} = u_0, u_{2n+1} = u_0'$ and conclude that $u_0 = u_0'$. $\square$

Orthogonal Projections (lecture 14, 26. Feb.)

For a Hilbert space $\mathcal{H}$ and a subset $A \subset H$, let

$$A^{\perp} := \{x \in H : x \perp y \ \forall y \in A\},$$

where $x \perp y$ means that $(x, y) = 0$. $A^{\perp}$ is a closed subspace of $\mathcal{H}$.

Proposition 26.32. *Assume $\mathcal{H}_0$ is a closed subspace of a Hilbert space $\mathcal{H}$. Then every $u \in H$ uniquely decomposes as*

$$u = u_0 + u_1, \ \textit{with } u_0 \in H \textit{ and } u_1 \in \mathcal{H}_0^{\perp}.$$

Moreover, $||u - u_0|| = d(u, \mathcal{H}_0)$ *and* $||u||^2 = ||u_0||^2 + ||u_1||^2$.

For a closed subspace $\mathcal{H}_0 \subset \mathcal{H}$, consider the map $P : H \to \mathcal{H}_0$ s.t. $Pu \in \mathcal{H}_0$ is the unique element satisfying $u - Pu = H_0^{\perp}$. The operator P is linear. It is also contractive, meaning that $||Pu|| \leq ||u||$, since $||u||^2 = ||Pu||^2 + ||u - Pu||^2$. It is called the orthogonal projection onto $\mathcal{H}_0$.

If $\mathcal{H}_0$ is finite dimensional with an orthonormal basis $u_1, ..., u_n$ then

$$Pu = \sum_{k=1}^{n} (u, u_k) u_k.$$

Orthonormal bases can be defined for arbitrary Hilbert spaces.

Definition 26.33 (orthonormal system)**.** An orthonormal system in $\mathcal{H}$ is a collection of vectors $h_i \in H$ $(i \in I)$s.t.

$$(u_i, u_j) = \delta_{ij} \ \forall i, j \in I.$$

It is called an *orthonormal basis* if $\operatorname{span}\{u_i\}_{i \in I}$ denotes the linear span of $\{u_i\}_{i \in I}$, the space of finite linear combinations of the vectors u_i.

Definition 26.34. A Hilbert space $\mathcal{H}$ is said to be *separable* if $\mathcal{H}$ contains a countable dense subset $G \subset \mathcal{H}$.

Theorem 26.35. *Every Hilbert space $\mathcal{H}$ has an orthonormal basis. If $\mathcal{H}$ is separable, then there is a countable orthonormal basis.*

Proposition 26.36. *Assume $\{u_i\}_{i \in I}$ is an orthonormal system in a Hilbert space H. Take $u\mathcal{H}$. Then*

(i) Bessel's inequality*:* $\sum_{i \in I} |(u, u_i)|^2 \leq ||u||^2$*, in particular,* $\{i : (u, u_i) \neq 0\}$ *is countable.*

(ii) Parseval's identity*: If $\{u_i\}_{i \in I}$ is an orthonormal basis, then* $\sum_{i \in I} |(u, u_i)|^2 = ||u||^2$.

If $(u_i)_{i \in I}$ is an orthonormal basis, then the numbers (u, u_i) are called the **Fourier coefficients** of u with respect to $(u_i)_{i \in I}$. The Parseval identity then suggests that u is determined by its Fourier coefficients. This is true, and even more, we have:

Proposition 26.37. *Assume $(u_i)_{i\in I}$ is an orthonormal basis in a Hilbert space $\mathcal{H}$. Then for every vector $(c_i)_{i\in I} \in l^2(I)$ there is a unique vector $u \in \mathcal{H}$ with Fourier coefficients c_i, and we write*

$$u = \sum_{i\in I} c_i u_i.$$

Remark. *Equivalently, the element $u = \sum_{i\in I} c_i u_i$ can be described as the unique element in $\mathcal{H}$ s.t. $\forall \epsilon > 0$ there is a finite $F_0 \subset I$ s.t. $||u - \sum_{i\in F} c_i u_i|| < \epsilon$ $\forall$ finite $F \supset F_0$.*

Corollary 26.38. *We have a linear isomorphism $U : l^2(I) \xrightarrow{\sim} \mathcal{H}$, $U\left((c_i)_{i\in I}\right) = \sum_{i\in I} c_i u_i$. By Parseval's identity this isomorphism is* isometric, *that is, $||Ux|| = ||x||$ $\forall x \in l^2(I)$. By the polarization identity this is equivalent to*

$$(Ux, Uy) = (x, y) \ \forall x, y \in l^2(I).$$

Therefor U is unitary.

Corollary 26.39. *Up to a unitary isomorphism, there is only one infinite dimensional separable Hilbert space, namely, l^2.*

Dual spaces (lecture 15, 29. Feb.)

Given two orthonormal bases $(u_i)_{i\in I}$ and $(v_i)_{i\in I}$ in a Hilbert space $\mathcal{H}$, we can decompose

$$u_i = \sum_{j\in I} (u_i, v_j) v_j$$

and using that the sets $\{j : (u_i, v_j) \neq 0\}$ are countable proove the following:

Claim: Any two orthonormal bases in a Hilbert space have the same cardinality.

Example 26.40 (classical Fourier series)**.** Consider $\mathcal{H} = L^2(0, 2\bar{\mu}) = L^2((0, 2\bar{\mu}), d\lambda)$. For $n \in \mathbb{Z}$, define $e_n(t) = \frac{1}{\sqrt{2\bar{\mu}}} e^{int}$. By a version of Weierstrass' theorem it is known that $\text{span}\{e_n\}_{n\in\mathbb{Z}}$ is dense in the supremum-norm in

$$\{f \in C[0, 2\bar{\mu}] : f(0) = f(2\bar{\mu})\}.$$

As $C[0, 2\bar{\mu}]$ is dense in $L^2(0, 2\bar{\mu})$, from this one can deduce that $\text{span}\{e_n\}_{n\in\mathbb{Z}}$ is dense in $L^2(0, 2\bar{\mu})$. We then see that $(e_n)_{n\in\mathbb{Z}}$ is an orthonormal basis in $L^2(0, 2\bar{\mu})$. We therefor have a unitary isomorphism

$$l^2(\mathbb{Z}) \xrightarrow{\sim} L^2(0, 2\bar{\mu}), (c_n)_{n\in\mathbb{Z}} \mapsto \sum_{n\in\mathbb{Z}} c_n e_n.$$

The Fourier coefficients at $f \in L^2(0, 2\bar{\mu})$ are denoted by $\hat{f}(n)$, so

$$\hat{f}(n) = \frac{1}{\sqrt{2\bar{\mu}}} \int_0^{2\bar{\mu}} f(t) e^{-int} dt,$$

more practically

$$\hat{f}(n) = \frac{1}{\sqrt{2\bar{\mu}}} \int_{[0,2\bar{\mu})} f(t) e^{-int} d\lambda(t).$$

Therefor we have $f = \sum_{n\in\mathbb{Z}} \hat{f}(n) e_n$ in $L^2(0, 2\bar{\mu})$.

Fact: For every $f \in L^2(0, 2\bar{\mu})$, we have $\frac{1}{\sqrt{2\bar{\mu}}} \sum_{n=-N}^{N} \hat{f}(n) e^{int} \xrightarrow[N\to\infty]{} f(t)$ for a.e. t.

Lemma 26.41. *Assume V is a normed space over $K = \mathbb{R}$ or $K = \mathbb{C}$. Consider a linear functional $f : V \to K$. The following are equivalent (TFAE):*

(i) f is continuous;

(ii) f is continuous at 0;

(iii) There is a $c \geq 0$ s.t. $|f(x)| \leq c||x||$ $\forall x \in V$.

If (i)-(iii) are satisfied, then f is called a *bounded linear functional*. The constant c in (iii) is denoted by $||f||$. We have $||f|| = \sup_{x\neq 0} \frac{|f(x)|}{||x||} = \sup_{||x||=1} |f(x)| = \sup_{||x||\leq 1} |f(x)|$.

Proposition 26.42. *For every normed vector space V over $K = \mathbb{R}$ or $K = \mathbb{C}$, the bounded linear functionals on V form a Banach space V^*.*

Remark. *The sequence $\{||f_n - f_m||\}_{m=1}^{\infty}$ actually converges, since*

$$\Big| ||f_n - f_m|| \Big| \leq ||f_m - f_n||.$$

When we study/use normed spaces, it is often important to understand the dual spaces. For Hilbert spaces this is particularly easy:

Theorem 26.43 (Riesz)**.** *Assume $\mathcal{H}$ is a Hilbert space. Then every $f \in \mathcal{H}^*$ has the form*

$$f(x) = (x, y),$$

for a uniquely defined $u \in \mathcal{H}$. Moreover, we have $||f|| = ||y||$.

For every Hilbert space $\mathcal{H}$ we can define the *conjugate Hilbert space* $\bar{\mathcal{H}}$, which has its elements as the symbols $\bar{x}$ for $x \in \mathcal{H}$, with the linear structure and inner product defined by

$$\bar{x} + \bar{y} = \overline{x + y}, c \cdot \bar{x} = \overline{\bar{c}x}, (\bar{x}, \bar{y}) = \overline{(x, y)} = (y, x).$$

Corollary 26.44. *For every Hilbert space $\mathcal{H}$, we have an isometric isomorphism $\bar{\mathcal{H}} \xrightarrow{\sim} \mathcal{H}^*$, $\bar{x} \mapsto (\cdot, x)$.*