Dense and Determining Sets (lecture 12, 19. Feb.)

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Definition 17.18 (Dense Sets)**.** A set $\mathcal{D} \subset \mathcal{L}^p(\mu), p \in [0, \infty]$, is called *dense* if for every $u \in \mathcal{L}^p(\mu)$ there exist a sequence $(f_n)_{n\in\mathbb{N}} \subset \mathcal{D}$ s.t. $\lim_{n\to\infty} ||u - f_n||_p = 0$.

Theorem 17.19. *Assume X, d is a metric space and μ is a Borel measure that is finite on every ball $1 \leq p < \infty$. Then the space of bounded continuous functions with bounded support is dense in $\mathcal{L}^p(X, d\mu)$. Where bounded support means that f vanishes outside some ball.*

Proof. We want to approximate $f \in \mathcal{L}^p(X, d\mu)$ by bounded continuous functions with bounded support. By considering separately $(\mathrm{Re}(f))_I$ and $(\mathrm{Im}(f))_I$ we may assume that $f \geq 0$. Then we can find simple functions f_n s.t. $0 \leq f_n \leq f, f_n \to f$ pointwise. As $|f - f_n|^p \leq |f|^p$, by the dominated convergence theorem we have $f_n \to f \in \mathcal{L}^p(X, d\mu)$. Hence, it suffices to consider simple f, but then it suffices to approximate $f = \pi_A$. Note that $\pi_A \in \mathcal{L}^p(X, d\mu)$ iff $\mu(A) < \infty$.

Fix $x_0 \in X$. Then $\pi_{A\cap B_n(x_0)} \nearrow \pi_A$ pointwise, hence $\pi_{A\cap B_n(x_0)} \to \pi_A \in \mathcal{L}^p(X, d\mu)$, again by the dominated convergence theorem.

Therefor it suffices to consider $A \subset B_n(x_0)$. As μ is outer regular, we have

$$\mu(A) = \inf_{\substack{A\subset U\subset B_n(x_0)\\ U \text{ is open}}} \mu(U).$$

Note that $||\pi_U - \pi_A||_p = \mu(U\setminus A)^{1/p}$. Hence, we can choose $U_k \subset B_n(x_0)$ s.t. $A \subset U_k$, U_k is open, $\pi_{U_k} \to \pi_A \in \mathcal{L}^p(X, d\mu)$.

Therefor it suffices to approximate π_U for open $U \subset B_n(x_0)$. Consider the functions

$$f_k(x) = \frac{kd(x, U^c)}{1 + kd(x, U^c)}.$$

Then $0 \leq f_k \leq 1$, f_k is continuous, supported on $\bar{U} \subset \bar{B}_n(x_0)$ and $f_k \nearrow \pi_U$ pointwise, hence $f_k \xrightarrow[k\to\infty]{} \pi_U \in \mathcal{L}^p(X, d\mu)$. ☐

Theorem 17.20. *Assume (X, d) is a separable locally compact metric space and μ is a Borel Measure on X s.t. $\mu(K) < \infty\ \forall$ compact $K \subset K$. Then the space $C_c(X)$ of continuous compactly supported functions is dense in $\mathcal{L}^p(X, d\mu)$.*

Recall that the support of a function f is $\text{supp}(f) = \{x \in X : f(x) \neq 0\}$, *closed support* is the closure of $\text{supp}(f)$ (i.e. boundary points are included), often just written as $\text{supp}(f)$, and a function is said to have *compact support* if $\text{supp}(f)$ is *compact*.

In particular, either theorem shows that if μ is a Borel measure on $\mathbb{R}^n$ s.t. the measure of every ball is finite, then $C_c(\mathbb{R}^n)$ is dense in $\mathcal{L}^p(\mathbb{R}^n, d\mu)$, $1 \leq p < \infty$. Later we will see that even $C_c^\infty(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n, d\mu)$.

Remark. *These results do not extend to $p = \infty$ in general.*
For $\mu = \lambda_n$ we write simply $\mathcal{L}^p(\mathbb{R}^n)$.

Remark. *Theorem 17.8 in the book is WRONG. For example, $X = \mathbb{Q}$ with the usual metric is σ-compact, supports nonzero finite measure, but $C_c(\mathbb{Q}) = 0$.*

Modes of Convergence (mixture of ex. 11.12 and ch. 22 p. 258-261.)

Definition 17.21 (convergence in meassure)**.** A sequence of measurable functions $u_n : X \to \bar{\mathbb{R}}$ *converges in measure* if

$$\forall \epsilon > 0 \forall A \in \mathscr{A}, \mu(A) < \infty : \lim_{n \to \infty} \mu\left(\{|u_n - u| > \epsilon\} \cap A\right) = 0$$

holds for some $u \in \mathcal{M}(\mathscr{A})$. We write $\mu\text{-}\lim_{n\to\infty u_n = u}$ or $u_n \xrightarrow{\mu} u$.

Assume $(X, \mathscr{B}, \mu)$ is a measure space. Given measurable functions $f_n, f : X \to \mathbb{C}$, recall that

$$f_n \to f \text{ a.e.}$$

means that $f_n(x) \xrightarrow[n\to\infty]{} f(x)$ for all x outside a set of measure zero.

Theorem 17.22 (Egorov)**.** *Assume $\mu(X) < \infty$ and $f_n \to f$ a.e. Then, $\forall \epsilon > 0$, there exists $X_\epsilon \in \mathscr{B}$ s.t. $\mu(X_\epsilon) < \epsilon$ and $f_n \to f$ uniformly on $X \backslash X_\epsilon$.*

In addition to pointwise and uniform convergence we also consider the following:

$f_n \to f$ in the *p-th mean* if $||f_n - f||_p \xrightarrow[n\to\infty]{} 0$. For $p = 1$ we say *in mean*, for $p = 2$ we say in quadratic mean.

$f_n \to f$ *in measure* if $\forall \epsilon > 0$ we have

$$\mu\left(\{x : |f_n(x) - f(x)| \geq \epsilon\}\right) \xrightarrow[n\to\infty]{} 0.$$

Theorem 17.23 (Lemma 22.4 in the book?)**.** *Assume $(X, \mathscr{B}, d\mu)$ is a measure space, $1 \leq p < \infty$, $f_n, f : X \to \mathbb{C}$ are measurable functions. Then*

(i) If $f_n \to f$ in the p-th mean, then $f_n \to f$ in measure.

(ii) *If* $f_n \to f$ *in measure, then there is a subsequence* $(f_{n_k})_{k=1}^{\infty}$ *s.t.* $f_{n_k} \to f$ *a.e.*

(iii) *If* $f_n \to f$ *a.e. and* $\mu(X) < \infty$*, then* $f_n \to f$ *in measure.*

In particular, if $f_n \to f$ *in the p-th mean, then* $f_{n_k} \to f$ *a.e. for a subsequence* $(f_{n_k})_k$.