

MAT4400: Detailed Notes on Linear analysis

Morten Tryti Berg and Isak Cecil Onsager Rukan.

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3 σ -Algebras

Definition 3.1 (σ -Algebra). A family \mathcal{A} of subsets of X with:

- (i) $X \in \mathcal{A}$,
- (ii) $A \in \mathcal{A} \Rightarrow A^c \in \mathcal{A}$,
- (iii) $(A_n)_{n \in \mathbb{N}} \in \mathcal{A} \Rightarrow \bigcup_{n \in \mathbb{N}} A_n \in \mathcal{A}$

Theorem 3.2 (and Definition).

- (i) *The intersection of arbitrarily many σ -algebras in X is again a σ -algebra in X .*
- (ii) *For every system of sets $p \subset \mathcal{P}(X)$ there exists a smallest σ -algebra containing p . This is the σ -algebra generated by p , denoted $\sigma(p)$, and p is called its generator.*

Definition 3.3 (Borel). The σ -algebra $\sigma(\mathcal{O})$ generated by the open sets $\mathcal{O} = \mathcal{O}_{\mathbb{R}^n}$ of \mathbb{R}^n is called **Borel σ -algebra**, and its members are called **Borel sets** or **Borel measurable sets**.

5 Uniqueness of Measures

Lemma 5.1. *A Dynkin system D is a σ -algebra iff it is stable under finite intersections, i.e. $A, B \in D \Rightarrow A \cap B \in D$.*

Theorem 5.2 (Dynkin). *Assume X is a set, S is a collection of subsets of X closed under finite intersections, that is, if $A, B \in S \Rightarrow A \cap B \in S$. Then $D(S) = \sigma(S)$.*

Proof. We clearly have that $D(S) \subset \sigma(S)$. If we can show that $D(S)$ is a σ -algebra, that is, that a Dynkin system generated by a subset $S \subset X$ (where S is \cap -stable) is a σ -algebra, then the inverse conclusion $D(S) \supset \sigma(S)$ follows logically. This is the case because the σ -algebra $\sigma(S)$ is the smallest σ -algebra containing S , and so if $D(S)$ is a σ -algebra it must be a greater or equal (in some sense) than $\sigma(S)$.

Using Lemma 5.1 we only need to show that $D(S)$ is stable under finite intersections, to prove that $D(S)$ is a σ -algebra. Consider:

$$D_A := \{B \subset X : B \cap A \in D(S)\},$$

for some $A \in D(S)$. Notice that this set is \cap -stable, and so if we can show that $D_A = D(S)$ we must have that (by Lemma 5.1) $D(S)$ is a σ -algebra. Firstly, however, let us show that D_A is a **Dynkin system**.

1. \emptyset must be in D_A , since $\emptyset \cap A = \emptyset \in D(S)$.
2. Let $B \in D_A$. Then

$$A \cap B^c = A \setminus (A \cap B) = (A^c \cup (A \cap B))^c,$$

here $A \cap B$ and A^c must be in $D(S)$. Furthermore, since disjoint unions of set from $D(S)$ are still in $D(S)$, we must have $A^c \in D_A$.

3. Assume that $(B_n)_{n \in \mathbb{N}} \subset D_A$ is a pairwise disjoint sequence. Then

$$\begin{aligned} (B_n \cap A)_{n \in \mathbb{N}} &\in D(S) \quad (\text{by def. of } D_A) \\ \Rightarrow \bigcup_{n \in \mathbb{N}} (B_n \cap A) &= \left(\bigcup_{n \in \mathbb{N}} B_n \right) \cap A \in D(S) \\ \Rightarrow \bigcup_{n \in \mathbb{N}} B_n &\in D_A. \end{aligned}$$

So D_A is indeed a Dynkin system.

We now want to show that $D(S)$ is \cap -stable, we have:

$$\begin{aligned}
& S \subset D_A \quad \forall A \in S \\
& \Rightarrow D(S) \subset D_A \quad \forall A \in S \text{ (since } D_A \text{ is a Dynkin system)} \\
& \Rightarrow B \cap A \in D(S) \quad \forall B \in S, \forall A \in D(S) \text{ (by the definition of } D_A) \\
& \Rightarrow B \in D_A \quad \forall B \in S, \forall A \in D(S) \\
& \Rightarrow S \subset D_A \quad \forall A \in D(S) \\
& \Rightarrow D(S) \subset D_A \quad \forall A \in D(S) \text{ (since } D_A \text{ is a Dynkin system)} \\
& \Rightarrow A \cap B \in D(S) \quad \forall A, B \in D(S),
\end{aligned}$$

and so $D(S)$ is \cap -stable and then $D(S) \supset \sigma(S) \Rightarrow D(S) = \sigma(S)$. \square

Theorem 5.3 (uniqueness of measures). *Let (X, B) be a measurable space, and $S \subset P(X)$ be the generator of B , i.e. $B = \sigma(S)$. If S satisfies the following conditions:*

1. *S is stable under finite intersections (\cap -stable), i.e. $A, C \in S \Rightarrow A \cap C \in S$.*
2. *There exists an exhausting sequence $(G_n)_{n \in \mathbb{N}} \subset B$ with $G_n \uparrow X$. Assume also that there are two measures μ, ν satisfying:*
3. *$\mu(A) = \nu(A), \forall A \in S$.*
4. *$\mu(G_n) = \nu(G_n) < \infty$.*

Then $\mu = \nu$.

Proof (outline). Define

$$D_n := \{A \in B : \mu(G_n \cap A) = \nu(G_n \cap A) (< \infty)\},$$

and show that it is a Dynkin system. Then, use the fact that S is \cap -stable and Theorem 5.2 to argue that $D(S) = \sigma(S) \dots \rightarrow \dots B = D_n$. \square

6 Existence of Measures

Theorem 6.1 (Carathéodory). *Let $S \subset P(X)$ be a semi-ring and $\mu : S \rightarrow [0, \infty)$ a pre-measure. Then μ has an extension to a measure μ^* on $\sigma(S)$, i.e. that $\mu(s) = \mu^*(s)$, $\forall s \in \sigma(S)$.*

Also, if S contains an exhausting sequence, $S_n \uparrow X$, s.t. $\mu(S_n) < \infty$, then the extension is unique.

Proof (outline). Firstly, let us define an outer measure.

Definition 6.2 (Outer measure). An outer measure is a function $\mu^* : P(X) \rightarrow [0, \infty)$ with the following properties:

1. $\mu^*(\emptyset) = 0$,
2. $A \subset B \Rightarrow \mu^*(A) \leq \mu^*(B)$,
3. $\mu^*\left(\bigcup_{n \in \mathbb{N}} A_n\right) \leq \sum_{n \in \mathbb{N}} \mu^*(A_n)$,

and define for each $A \subset X$ the family of countable S -coverings:

$$C(A) := \left\{ (S_n)_{n \in \mathbb{N}} \subset S : \bigcup_{n \in \mathbb{N}} S_n \supset A \right\},$$

and the set function

$$\mu^*(A) := \inf \left\{ \sum_{n \in \mathbb{N}} \mu(S_n) : (S_n)_{n \in \mathbb{N}} \in C(A) \right\}.$$

Step 1: Claim: $\mu^*(A)$ is an outer measure.

Proof.

1. $C(\emptyset) = \{\text{any sequence in } S \text{ containing } \emptyset\} \Rightarrow \mu^*(\emptyset) = 0$.
2. Assume $A \subset B$. Then $C(A) \subset C(B) \Rightarrow \mu^*(A) \leq \mu^*(B)$.
3. If $\mu^*(A_n) = \infty$ for some n , then there is nothing to prove. Thus, assume $\mu^*(A_n) < \infty \forall n$. Fix $\epsilon > 0$, and for every n choose $A_{n_k} \in S$ s.t.

$$A_n \subset \bigcup_{k \in \mathbb{N}} A_{n_k}, \quad \sum_{k \in \mathbb{N}} \mu^*(A_{n_k}) < \mu^*(A_n) + \frac{\epsilon}{2^n}.$$

Then

$$\bigcup_{n \in \mathbb{N}} A_n \subset \bigcup_{k \in \mathbb{N}} \bigcup_{n \in \mathbb{N}} A_{n_k},$$

so

$$\begin{aligned}
\mu^* \left(\bigcup_{n \in \mathbb{N}} A_n \right) &\leq \sum_{n \in \mathbb{N}} \sum_{k \in \mathbb{N}} \mu(A_{n_k}) \\
&< \sum_{n \in \mathbb{N}} \left(\mu^*(A_n) + \frac{\epsilon}{2^n} \right) \\
&= \sum_{n \in \mathbb{N}} \mu^*(A_n) + \epsilon.
\end{aligned}$$

As ϵ was arbitrarily, we get that

$$\mu^* \left(\bigcup_{n \in \mathbb{N}} A_n \right) \leq \sum_{n \in \mathbb{N}} \mu^*(A_n),$$

so μ^* fulfills all the conditions for being an outer measure. \square

Step 2: Showing that μ^* extends μ , i.e. $\mu^*(s) = \mu(s) \ \forall s \in S$.

Step 3: Define μ^* -measurable sets

$$\Sigma^* := \{A \subset X : \mu^*(Q) = \mu^*(Q \cap A) + \mu^*(Q \setminus A) \ \forall Q \subset X\}$$

Step 4: Show that $\mu|_{\Sigma^*}$ is a measure. In particular, $\mu|_{\sigma(S)}$ is a measure which extends μ . \square

7 Measurable Mappings

We consider maps $T : X \rightarrow X'$ between two measurable spaces (X, \mathcal{A}) and (X', \mathcal{A}') which respects the measurable structures, the σ -algebras on X and X' . These maps are useful as we can transport a measure μ , defined on (X, \mathcal{A}) , to (X', \mathcal{A}') .

Definition 7.1. Let (X, \mathcal{A}) , (X', \mathcal{A}') be measurable spaces. A map $T : X \rightarrow X'$ is called \mathcal{A}/\mathcal{A}' -measurable if the pre-image of every measurable set is a measurable set:

$$T^{-1}(A') \in \mathcal{A}, \quad \forall A' \in \mathcal{A}'. \quad (1)$$

- A $\mathcal{B}(\mathbb{R}^n)/\mathcal{B}(\mathbb{R}^m)$ measurable map is often called a Borel map.
- The notation $T : (X, \mathcal{A}) \rightarrow (X', \mathcal{A}')$ is often used to indicate measurability of the map T .

Lemma 7.2. Let (X, \mathcal{A}) , (X', \mathcal{A}') be measurable spaces and let $\mathcal{A}' = \sigma(\mathcal{G}')$. Then $T : X \rightarrow X'$ is \mathcal{A}/\mathcal{A}' -measurable iff $T^{-1}(G') \in \mathcal{A}$, i.e. if

$$T^{-1}(G') \in \mathcal{A}, \quad \forall G' \in \mathcal{G}'. \quad (2)$$

Theorem 7.3. Let (X_i, \mathcal{A}_i) , $i = 1, 2, 3$, be measurable spaces and $T : X_1 \rightarrow X_2$, $S : X_2 \rightarrow X_3$ be $\mathcal{A}_1/\mathcal{A}_2$ and $\mathcal{A}_2/\mathcal{A}_3$ -measurable maps respectively. Then $S \circ T : X_1 \rightarrow X_3$ is $\mathcal{A}_1/\mathcal{A}_3$ -measurable.

Corollary 7.4. Every continuous map between metric spaces is a Borel map.

Definition 7.5. (and lemma) Let $(T_i)_{i \in I}$, $T_i : X \rightarrow X_i$, be arbitrarily many mappings from the same space X into measurable spaces (X_i, \mathcal{A}_i) . The smallest σ -algebra on X that makes all T_i simultaneously measurable is

$$\sigma(T_i : i \in I) := \sigma \left(\bigcup_{i \in I} T_i^{-1}(\mathcal{A}_i) \right) \quad (3)$$

Corollary 7.6. A function $f : (X, \mathcal{B}) \rightarrow \mathbb{R}$ is measurable if $f((a, +\infty)) \in \mathcal{B}$, $\forall a \in \mathbb{R}$.

Corollary 7.7. Assume (X, \mathcal{B}) is a measurable space, (Y, d) is a metric space, $(f_n : (X, \mathcal{B}) \rightarrow Y)_{n=1}^{\infty}$ is a sequence of measurable maps. Assume this sequence of images $(f_n(x))_{n=1}^{\infty}$ is convergent in Y $\forall x \in X$. Define

$$f : X \rightarrow Y, \quad \text{by } f(x) = \lim_{n \rightarrow \infty} f_n(x). \quad (4)$$

Then f is measurable.

Theorem 7.8. Let (X, \mathcal{A}) , (X', \mathcal{A}') be measurable spaces and $T : X \rightarrow X'$ be an \mathcal{A}/\mathcal{A}' -measurable map. For every measurable μ on (X, \mathcal{A}) ,

$$\mu'(A') := \mu(T^{-1}(A')), \quad A' \in \mathcal{A}', \quad (5)$$

defines a measure on (X', \mathcal{A}') .

Definition 7.9. The measure $\mu'(\cdot)$ in the above theorem is called the push forward or image measure of μ under T and it is denoted as $T(\mu)(\cdot)$, $T_{*\mu}(\cdot)$ or $\mu \circ T^{-1}(\cdot)$.

Theorem 7.10. *If $T \in \mathbb{R}^{n \times n}$ is an orthogonal matrix, then $\lambda^n = T(\lambda^n)$.*

Theorem 7.11. *Let $S \in \mathbb{R}^{n \times n}$ be an invertible matrix. Then*

$$S(\lambda^n) = |\det S| \lambda^n = |\det S|^{-1} \lambda^n. \quad (6)$$

Corollary 7.12. *Lebesgue measure is invariant under motions: $\lambda^n = M(\lambda^n)$ for all motions M in \mathbb{R}^n . In particular, congruent sets have the same measure. Two sets of points are called congruent if, and only if, one can be transformed into the other by an isometry*

8 Measurable Functions

A *measurable function* is a measurable map $u : X \rightarrow \mathbb{R}$ from some measurable space (X, \mathcal{A}) to $(\mathbb{R}, \mathcal{B}(\mathbb{R}^1))$. They play central roles in the theory of integration.

We recall that $u : X \rightarrow \mathbb{R}$ is $\mathcal{A}/\mathcal{B}(\mathbb{R}^1)$ -measurable if

$$u^{-1}(B) \in \mathcal{A}, \quad \forall B \in \mathcal{B}(\mathbb{R}^1). \quad (7)$$

Moreover from a lemma from chapter 7, we actually only need to show that

$$u^{-1}(G) \in \mathcal{A}, \quad \forall G \in \mathcal{G} \text{ where } \mathcal{G} \text{ generates } \mathcal{B}(\mathbb{R}^1). \quad (8)$$

Proposition 8.1.

- 1 If $f, g : (X, \mathcal{B}) \rightarrow \mathbb{C}$ are measurable, then the function $f+g, f \cdot g, cf, (c \in \mathbb{C})$ are measurable.
- 2 If $b : \mathbb{C} \rightarrow \mathbb{C}$ is Borel and $b : (\mathbb{C}, \mathcal{B}) \rightarrow \mathbb{C}$ is measurable, then $b \circ f$ is measurable.
- 3 If $f(x) = \lim_{n \rightarrow \infty} f_n(x)$, $x \in X$ and f_n are measurable, then f is measurable.
- 4 If $X = \bigcup_{n=1}^{\infty} A_n$, $(A_n \in \mathcal{B})$, $f|_{A_n} : (A_n, \mathcal{B}_{A_n}) \rightarrow \mathbb{C}$ is measurable $\forall n$, then f is measurable.

Definition 8.2. Given a measurable space (X, \mathcal{B}) , a measurable function $f : (X, \mathcal{B}) \rightarrow \mathbb{C}$ is called simple if

$$f(x) = \sum_{k=1}^N c_k \mathbb{1}_{A_k}(x), \quad (9)$$

for some $c_k \in \mathbb{C}$, $A_k \in \mathcal{B}$, where $\mathbb{1}$ is the characteristic function,

$$\mathbb{1}_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A. \end{cases} \quad (10)$$

The representation of simple function is **not** unique. We denote the standard representation of f by

$$f(x) = \sum_{n=0}^N z_n \mathbb{1}_{B_n}(x), \quad N \in \mathbb{N}, \quad z_n \in \mathbb{R}, \quad B_n \in \mathcal{A}, \quad X = \bigcup_{n=1}^N B_n, \quad \text{for } B_n \cap B_m = \emptyset, \quad n \neq m. \quad (11)$$

The set of simple functions is denoted $\mathcal{E}(\mathcal{A})$ of \mathcal{E} .

Definition 8.3. Assume μ is a measure on (X, \mathcal{B}) . Given a *positive* simple function

$$f = \sum_{k=1}^N c_k \mathbf{1}_{A_k}, \quad (c_k \geq 0). \quad (12)$$

We define

$$\int_X f d\mu = \sum_{k=1}^n c_k \mu(A_k) \in [0, +\infty]. \quad (13)$$

We also denote this by $I_\mu(f)$.

Lemma 8.4. *This is well defined, that is, $\int_X f d\mu$ does not depend on the presentation of the simple function f .*

Properties 8.5. *For every positive simple function*

$$1 \quad \int_X c f d\mu = c \int_X f d\mu, \quad \text{for only } c \geq 0$$

$$2 \quad \int_X (f + g) d\mu = \int_X f d\mu + \int_X g d\mu.$$

Corollary 8.6. *If $f \geq g \geq 0$ are simple functions, then*

$$\int_X f d\mu \geq \int_X g d\mu. \quad (14)$$

Definition 8.7. If $f : X \rightarrow [0, +\infty)$ is measurable, then we define

$$\int_X f d\mu = \sup \left\{ \int_X g d\mu : f \geq g \geq 0, \text{ } g \text{ is simple} \right\} \quad (15)$$

Remark. *This means that any measurable function can be approximated by simple functions.*

Properties 8.8. *Measurable functions like this have the following properties*

$$1 \quad \int_X c f d\mu = c \int_X f d\mu, \quad \forall c \geq 0.$$

$$2 \quad \text{If } f \geq g \geq 0, \text{ then } \int_X f d\mu \geq \int_X g d\mu \text{ for any measurable } g, f.$$

$$3 \quad \text{If } f \geq 0 \text{ is simple, then } \int_X f d\mu \text{ is the same value as obtained before.}$$

To advance in measure theory we consider measurable functions

$$f : X \rightarrow [0, +\infty].$$

Measurability is understood w.r.t the σ -algebra $\mathcal{B}([0, +\infty])$ generated by $\mathcal{B}([0, +\infty))$ and $\{+\infty\}$. In other words, $A \subset [0, +\infty] \in \mathcal{B}([0, +\infty])$ iff $A \cap [0, +\infty) \in \mathcal{B}([0, +\infty))$.

Remark. Hence $f : X \rightarrow [0, +\infty]$ is measurable iff $f^{-1}(A)$ is measurable $\forall A \in \mathcal{B}([0, +\infty))$.

Definition 8.9. For measurable functions $f_X \rightarrow [0, +\infty]$, we define

$$\int_X f d\mu = \sup \left\{ \int_X g d\mu : f \geq g \geq 0 : g \text{ is simple} \right\} \in [0, +\infty]. \quad (16)$$

Theorem 8.10. Monotone convergence theorem Assume (X, \mathcal{B}, μ) is a measure space, $(f)_{n=1}^\infty$ is an increasing sequence of measurable positive functions $f_n : X \rightarrow [0, +\infty]$. Define $f(x) = \lim_{n \rightarrow \infty} f_n(x)$. Then f is measurable and

$$\int_X f d\mu = \lim_{n \rightarrow \infty} \int_X f_n d\mu. \quad (17)$$

Theorem 8.11. Assume (X, \mathcal{B}) is a measurable space and $f : X \rightarrow [0, +\infty]$ is measurable. Then there are simple functions g_n , s.t.

$$0 \leq g_1 \leq g_2 \leq \dots, \quad g_n(x) \rightarrow f(x), \quad \forall x \in X.$$

Moreover, if f is bounded, we can choose g_n s.t. the convergence is uniform, that is,

$$\lim_{n \rightarrow \infty} \sup_{x \in X} |g_n(x) - f(x)| = 0. \quad (18)$$

9 Integration of measurable functions

Through this chapter (X, \mathcal{A}, μ) will be some measure space. Recall that $\mathcal{M}^+(\mathcal{A})$ $[\mathcal{M}_{\mathbb{R}}^+(\mathcal{A})]$ are the \mathcal{A} -measurable positive functions and $\mathcal{E}(\mathcal{A})$ $[\mathcal{E}_{\mathbb{R}}^+(\mathcal{A})]$ are the positive and simple functions.

The fundamental idea of *Integration* is to measure the area between the graph of the function and the abscissa. For positive simple functions $f \in \mathcal{E}^+(\mathcal{A})$ in standard representation, this is done easily

$$\text{if } f = \sum_{i=0}^M y_i \mathbb{1}_{A_i} \in \mathcal{E}^+(\mathcal{A}) \quad \text{then} \quad \sum_{i=0}^M y_i \mu(A_i) \quad (19)$$

would be the μ -area enclosed by the graph and the abscissa. We note that the representation of f should not impact the integral of f .

Lemma 9.1. *Let $\sum_{i=0}^M y_i \mathbb{1}_{A_i} = \sum_{k=0}^N z_k \mathbb{1}_{B_k}$ be two standard representations of the same function $f \in \mathcal{E}^+(\mathcal{A})$. Then*

$$\sum_{i=0}^M y_i \mu(A_i) = \sum_{k=0}^N z_k \mu(B_k). \quad (20)$$

Definition 9.2. Let $f = \sum_{i=0}^M y_i \mathbb{1}_{A_i} \in \mathcal{E}^+(\mathcal{A})$ be a simple function in standard representation. Then the number

$$I_\mu(f) = \sum_{i=0}^M y_i \mu(A_i) \in [0, \infty] \quad (21)$$

(which is independent of the representation of f) is called the μ -integral of f .

Proposition 9.3. *Let $f, g \in \mathcal{E}^+(\mathcal{A})$. Then*

- (i) $I_\mu(\mathbb{1}_A) = \mu(A) \quad \forall A \in \mathcal{A}.$
- (ii) $I_\mu(\lambda f) = \lambda I_\mu(f) \quad \forall \lambda \geq 0.$
- (iii) $I_\mu(f + g) = I_\mu(f) + I_\mu(g).$
- (iv) $f \leq g \Rightarrow I_\mu(f) \leq I_\mu(g).$

In theorem 8.8 we saw that we could for every $u \in \mathcal{M}^+(\mathcal{A})$ write it as an increasing limit of simple functions. By corollary 8.10, the suprema of simple functions are again measurable, so that

$$u \in \mathcal{M}^+(\mathcal{A}) \Leftrightarrow u = \sup_{n \in \mathbb{N}} f_n, \quad f \in \mathcal{E}^+(\mathcal{A}), \quad f_n \leq f_{n+1} \leq \dots$$

We will use this to "inscribe" simple functions (which we know how to integrate) below the graph of a positive measurable function u and exhaust the μ -area below u .

Definition 9.4. Let (X, \mathcal{A}, μ) be a measure space. The (μ) -integral of a positive function $u \in \mathcal{M}_{\mathbb{R}}^+(\mathcal{A})$ is given by

$$\int u d\mu = \sup \{ I_{\mu}(g) : g \leq u, g \in \mathcal{E}^+(\mathcal{A}) \} \in [0, +\infty]. \quad (22)$$

If we need to emphasize the *integration variable*, we write $\int u(x)\mu(dx)$. The key observation is that the integral $\int \dots d\mu$ extends I_{μ} .

Lemma 9.5. For all $f \in \mathcal{E}^+(\mathcal{A})$ we have $\int f d\mu = I_{\mu}(f)$.

The next theorem is one of many convergence theorems. It shows that we could have defined 22 using any increasing sequence $f_n \uparrow u$ of simple functions $f_n \in \mathcal{E}^+(\mathcal{A})$.

Theorem 9.6. (*Beppo Levi*) Let (X, \mathcal{A}, μ) be a measure space. For an increasing sequence of functions $(u_n)_{n \in \mathbb{N}} \subset \mathcal{M}_{\mathbb{R}}^+(\mathcal{A})$, $0 \leq u_n \leq u_{n+1} \leq \dots$, we have for the supremum $u = \sup_{n \in \mathbb{N}} u_n \in \mathcal{M}_{\mathbb{R}}^+(\mathcal{A})$ and

$$\int \sup_{n \in \mathbb{N}} u_n d\mu = \sup_{n \in \mathbb{N}} \int u_n d\mu. \quad (23)$$

Note we can write $\lim_{n \rightarrow \infty}$ instead of $\sup_{n \in \mathbb{N}}$ as the supremum of an increasing sequence is its limit. Moreover, this theorem holds in $[0, +\infty]$, so the case $+\infty = +\infty$ is possible.

Corollary 9.7. Let $u \in \mathcal{M}_{\mathbb{R}}^+(\mathcal{A})$. Then

$$\int u d\mu = \lim_{n \rightarrow \infty} \int f_n d\mu$$

holds for every sequence $(f_n)_{n \in \mathbb{N}} \subset \mathcal{E}^+(\mathcal{A})$ with $\lim_{n \rightarrow \infty} f_n = u$.

Proposition 9.8. (of integral) Let $u, v \in \mathcal{M}_{\mathbb{R}}^+(\mathcal{A})$. Then

$$(i) \int \mathbf{1}_A d\mu = \mu(A) \quad \forall A \in \mathcal{A}.$$

$$(ii) \int \alpha u d\mu = \alpha \int u d\mu \quad \forall \alpha \geq 0.$$

$$(iii) \int u + v d\mu = \int u d\mu + \int v d\mu.$$

$$(iv) u \leq v \Rightarrow \int u d\mu \leq \int v d\mu.$$

Corollary 9.9. Let $(u_n)_{n \in \mathbb{N}} \subset \mathcal{M}_{\mathbb{R}}^+(\mathcal{A})$. Then $\sum_{n=1}^{\infty} u_n$ is measurable and we have

$$\int \sum_{n=1}^{\infty} u_n d\mu = \sum_{n=1}^{\infty} \int u_n d\mu$$

(including the possibility $+\infty = +\infty$.)

Theorem 9.10. (*Fatou*) Let $(u_n)_{n \in \mathbb{N}} \subset \mathcal{M}_{\mathbb{R}}^+(\mathcal{A})$ be a sequence of positive measurable functions. Then $u = \liminf_{n \rightarrow \infty} u_n$ is measurable and

$$\int \liminf_{n \rightarrow \infty} u_n d\mu = \liminf_{n \rightarrow \infty} \int u_n d\mu \quad (24)$$

10 Integrals of Measurable Functions

We have defined our integral for positive measurable functions, i.e. functions in $\mathcal{M}^+(\mathcal{A})$. To extend our integral to not only functions in $\mathcal{M}^+(\mathcal{A})$ we first notice that

$$u \in \mathcal{M}_{\mathbb{R}}(\mathcal{A}) \Leftrightarrow u = u^+ - u^-, \quad u^+, u^- \in \mathcal{M}_{\mathbb{R}}^+, \quad (25)$$

i.e. that every measurable function can be written as a sum of **positive** measurable functions.

Definition 10.1 (μ -integrable). A function $u : X \rightarrow \overline{\mathbb{R}}$ on (X, \mathcal{A}, μ) is μ -integrable, if it is $\mathcal{A}/\mathcal{B}(\mathbb{R})$ -measurable and if $\int u^+ d\mu, \int u^- d\mu < \infty$ (recall the definition for the integral of positive measurable functions). Then

$$\int u d\mu := \int u^+ d\mu - \int u^- d\mu \in (-\infty, \infty) \quad (26)$$

is the (μ -)integral of u . We write $\mathcal{L}^1(\mu)$ for the set of all real-valued μ -integrable functions¹.

Theorem 10.2. Let $u \in \mathcal{M}_{\mathbb{R}}(\mathcal{A})$, then the following conditions are equivalent:

- (i) $u \in \mathcal{L}_{\mathbb{R}}^1(\mu)$.
- (ii) $u^+, u^- \in \mathcal{L}_{\mathbb{R}}^1(\mu)$.
- (iii) $|u| \in \mathcal{L}_{\mathbb{R}}^1(\mu)$.
- (iv) $\exists w \in \mathcal{L}_{\mathbb{R}}^1(\mu)$ with $w \geq 0$ s.t. $|u| \leq w$.

Theorem 10.3 (Properties of the μ -integral). The μ -integral is: **homogeneous, additive, and:**

- (i) $\min\{u, v\}, \max\{u, v\} \in \mathcal{L}_{\mathbb{R}}^1(\mu)$ (lattice property)
- (ii) $u \leq v \Rightarrow \int u d\mu \leq \int v d\mu$ (monotone)
- (iii) $\left| \int u d\mu \right| \leq \int |u| d\mu$ (triangle inequality)

Remark. If $u(x) \pm v(x)$ is defined in $\overline{\mathbb{R}}$ for all $x \in X$ then we can exclude $\infty - \infty$ and the theorem above just says that the integral is linear:

$$\int (au + bv) d\mu = a \int u d\mu + b \int v d\mu. \quad (27)$$

¹In words, we extend our integral to ~~positive~~ measurable functions by noticing that we can write every measurable function as a sum of positive measurable functions, something that we do know how to integrate. We don't want to run into the problem of $\infty - \infty$, thus we require the integral of the positive and negative parts to both (separately) be less than infinity.

This is always true for real-valued $u, v \in \mathcal{L}^1(\mu) = \mathcal{L}_{\mathbb{R}}^1(\mu)$, making $\mathcal{L}^1(\mu)$ a vector space with addition and scalar multiplication defined by

$$(u + v)(x) := u(x) + v(x), \quad (a \cdot u)(x) := a \cdot u(x), \quad (28)$$

and

$$\int \dots d\mu : \mathcal{L}^1(\mu) \rightarrow \mathbb{R}, \quad u \mapsto \int u d\mu, \quad (29)$$

*is a **positive linear functional**.*

11 Null sets and the Almost Everywhere (lecture 08, 05. Feb.)

Definition 11.1. A (μ) -null set $N \in \mathcal{N}_\mu$ is a measurable set $N \in \mathcal{A}$ satisfying

$$N \in \mu \Leftrightarrow N \in \mathcal{A} \text{ and } \mu(N) = 0. \quad (30)$$

This can be used generally about a ‘statement’ or ‘property’, but we will be interested in questions like ‘when is $u(x)$ equal to $v(x)$ ’, and we answer this by saying

$$u = v \text{ a.e.} \Leftrightarrow \{x : u(x) \neq v(x)\} \text{ is (contained in) a } \mu\text{-null set.}, \quad (31)$$

i.e.

$$u = v \text{ } \mu\text{-a.e.} \Leftrightarrow \mu(\{x : u(x) \neq v(x)\}) = 0. \quad (32)$$

The last phrasing should of course include that the set $\{x : u(x) \neq v(x)\}$ is in \mathcal{A} .

Theorem 11.2. Let $u \in \mathcal{M}_{\mathbb{R}}(\mathcal{A})$, then:

$$(i) \quad \int |u| d\mu = 0 \Leftrightarrow |u| = 0 \text{ a.e.} \Leftrightarrow \mu\{u \neq 0\} = 0,$$

$$(ii) \quad \mathbb{1}_N u \in \mathcal{L}_{\mathbb{R}}^1(\mu) \quad \forall N \in \mathcal{N}_\mu,$$

$$(iii) \quad \int_N u d\mu = 0.$$

Corollary 11.3. Let $u = v \text{ } \mu\text{-a.e.}$ Then

$$(i) \quad u, v \geq 0 \Rightarrow \int u d\mu = \int v d\mu,$$

$$(ii) \quad u \in \mathcal{L}_{\mathbb{R}}^1(\mu) \Rightarrow v \in \mathcal{L}_{\mathbb{R}}^1(\mu) \text{ and } \int u d\mu = \int v d\mu.$$

Corollary 11.4. If $u \in \mathcal{M}_{\mathbb{R}}(\mathcal{A})$, $v \in \mathcal{L}_{\mathbb{R}}^1(\mu)$ and $v \geq 0$ then

$$|u| \leq v \text{ a.e.} \Rightarrow u \in \mathcal{L}_{\mathbb{R}}^1(\mu). \quad (33)$$

Proposition 11.5 (Markow inequality). For all $u \in \mathcal{L}_{\mathbb{R}}^1(\mu)$, $A \in \mathcal{A}$ and $c > 0$

$$\mu(\{|u| \geq c\} \cap A) \leq \frac{1}{c} \int_A |u| d\mu, \quad (34)$$

if $A = X$, then (obviously)

$$\mu\{|u| \geq c\} \leq \frac{1}{c} \int |u| d\mu. \quad (35)$$

Corollary 11.6. If $u \in \mathcal{L}_{\mathbb{R}}^1(\mu)$, then μ is a.e. \mathbb{R} -valued. In particular, we can find a version $\tilde{u} \in \mathcal{L}^1(\mu)$ s.t. $\tilde{u} = u$ a.e. and $\int \tilde{u} d\mu = \int u d\mu$

Completions of measure spaces

Definition 11.7. A measure space (X, \mathcal{B}, μ) is called **complete** if whenever $A \in \mathcal{B}$ and $\mu(A) = 0$, we have $B \in \mathcal{B} \forall B \subset A$.

Remark. Any measure space can be completed as follows:

Let $\bar{\mathcal{B}}$ be the σ -algebra generated by \mathcal{B} and all sets $B \subset X$ s.t. there exists $A \in \mathcal{B}$ with $B \subset A$ and $\mu(A) = 0$.

Proposition 11.8. The σ -algebra $\bar{\mathcal{B}}$ can also be described as follows:

$$\bar{\mathcal{B}} := \{B \subset X : A_1 \subset B \subset A_2 \text{ for some } A_1, A_2 \in \mathcal{B} \text{ with } \mu(A_2 \setminus A_1) = 0\}, \quad (36)$$

with B, A_1, A_2 as above, we define

$$\bar{\mu} := \mu(A_1) = \mu(A_2) \quad (37)$$

Then $(X, \bar{\mathcal{B}}, \bar{\mu})$ is a complete measure space.

Definition 11.9. If μ is a Borel measure on a **metric** space (X, d) , then the completion $\bar{\mathcal{B}}(X)$ of the Borel σ -algebra with respect to μ is called the σ -algebra of μ -measurable sets.

Remark. For $\mu = \lambda_n$ on \mathbb{R}^n we talk about the σ -algebra of **Lebesgue measurable sets**. Instead of $\bar{\lambda}_n$ we still write λ_n and call it the **Lebesgue measure**. A function $f : \mathbb{R}^n \rightarrow \mathbb{C}$, measurable w.r.t. the σ -algebra of Lebesgue measurable sets is called the **Lebesgue measurable**.

The following result shows that any Lebesgue measurable function coincides with a Borel function a.e.

Proposition 11.10. Assume (X, \mathcal{B}, μ) is a measure space and consider its completion $(X, \bar{\mathcal{B}}, \bar{\mu})$. Assume $f : X \rightarrow \mathbb{C}$ is $\bar{\mathcal{B}}$ -measurable. Then there is a \mathcal{B} -measurable function $g : X \rightarrow \mathbb{C}$ s.t. $f = g$ $\bar{\mu}$ -a.e.

12 Convergence Theorems and Their Applications (lecture 9, 8. Feb.)

- To interchange limits and integrals in **Riemann integrals** one typically has to assume uniform convergence. - The set of Riemann integrable functions is somewhat limited, see theorem 12.7

Theorem 12.1 (Generalization of Beppo Levi, monotone convergence).

(i) Let $(u_n)_{n \in \mathbb{N}} \subset \mathcal{L}^1(\mu)$ be s.t. $u_1 \leq u_2 \leq \dots$ with limit $u := \sup_{n \in \mathbb{N}} u_n = \lim_{n \rightarrow \infty} u_n$. Then $u \in \mathcal{L}^1(\mu)$ **iff**

$$\sup_{n \in \mathbb{N}} \int u_n d\mu < +\infty,$$

in which case

$$\sup_{n \in \mathbb{N}} \int u_n d\mu = \int \sup_{n \in \mathbb{N}} u_n d\mu.$$

(ii) Same thing only with a decreasing sequence $\dots > -\infty$ in which case

$$\inf_{n \in \mathbb{N}} \int u_n d\mu = \int \inf_{n \in \mathbb{N}} u_n d\mu.$$

Theorem 12.2 (Lebesgue; dominated convergence). *Let $(u_n)_{n \in \mathbb{N}} \subset \mathcal{L}^1(\mu)$ s.t.*

(a) $|u_n|(x) \leq w(x)$, $w \in \mathcal{L}^1(\mu)$,

(b) $u(x) = \lim_{n \rightarrow \infty} u_n(x)$ exists in $\bar{\mathbb{R}}$,

then $u \in \mathcal{L}^1(\mu)$ and we have

(i) $\lim_{n \rightarrow \infty} \int |u_n - u| d\mu = 0$;

(ii) $\lim_{n \rightarrow \infty} \int u_n d\mu = \int \lim_{n \rightarrow \infty} u_n d\mu = \int u d\mu$;

Application 1: Parameter-Dependent Integrals

- We are interested in questions of the sort, when is

$$U(t) := \int u(t, x) \mu(dx), \quad t \in (a, b),$$

again a smooth function of t ? The answer involves interchange of limits and integration. Also, it turns out to better understand Riemann integrability, we need the Lebesgue integral.

Theorem 12.3 (continuity lemma). *Let $\emptyset \neq (a, b) \subset \mathbb{R}$ be a non-degenerate open interval and $u : (a, b) \times X \rightarrow \mathbb{R}$ satisfy*

(a) $x \mapsto u(t, x)$ is in $\mathcal{L}^1(\mu)$ for every fixed $t \in (a, b)$;

(b) $t \mapsto u(t, x)$ is continuous for every fixed $x \in X$;

(c) $|u(t, x)| \leq w(x)$ for all $(t, x) \in (a, b) \times X$ and some $w \in \mathcal{L}^1(\mu)$.

Then the function $U : (a, b) \rightarrow \mathbb{R}$ given by

$$t \mapsto U(t) := \int u(t, x) \mu(dx) \tag{38}$$

is continuous.

Theorem 12.4 (differentiability lemma). *Let $\emptyset \neq (a, b) \subset \mathbb{R}$ be a non-degenerate open interval and $u : (a, b) \times X \rightarrow \mathbb{R}$ satisfy*

(a) Same

(b) Same

(c) $|\partial_t u(t, x)| \leq w(x)$ for all $(t, x) \in (a, b) \times X$ and some $w \in \mathcal{L}^1(\mu)$.

Then the function in 38 is differentiable and its derivative is

$$\frac{d}{dt} U(t) = \frac{d}{dt} \int u(t, x) \mu(dx) = \int \frac{\partial}{\partial t} u(t, x) \mu(dx). \quad (39)$$

Application 2: Riemann vs Lebesgue Integration

Consider only $(X, \mathcal{A}, \mu) = (\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda)$.

Definition 12.5 (The Riemann Inegral). Consider on the finite interval $[a, b] \subset \mathbb{R}$ the partition

$$\Pi := \{a = t_0 < t_1 < \dots < t_k < b\}, k = k(\Pi), \quad (40)$$

and introduce

$$S_\Pi[u] := \sum_{i=1}^{k(\Pi)} m_i(t_i - t_{i-1}), \quad m_i := \inf_{x \in [t_{i-1}, t_i]} u(x), \quad (41)$$

$$S^\Pi[u] := \sum_{i=1}^{k(\Pi)} M_i(t_i - t_{i-1}), \quad M_i := \sup_{x \in [t_{i-1}, t_i]} u(x). \quad (42)$$

$$(43)$$

A bounded function $u : [a, b] \rightarrow \mathbb{R}$ is said to be **Riemann integrable** if the values

$$\int u := \sup_{\Pi} S_\Pi[u] = \inf_{\Pi} S^\Pi[u] =: \int u \quad (44)$$

coincide and are finite. Their common value is called the **Riemann integral** of u and denoted by $(R) \int_a^b u(x) dx$ or $\int_a^b u(x) dx$.

Theorem 12.6. Let $u : [a, b] \rightarrow \mathbb{R}$ be a *measurable* and *Riemann integrable* function. Then

$$u \in \mathcal{L}^1(\lambda) \text{ and } \int_{[a,b]} u d\lambda = \int_a^b u(x) dx. \quad (45)$$

Theorem 12.7. Let $u : [a, b] \rightarrow \mathbb{R}$ be a bounded function, it is Riemann integrable **iff** the points in (a, b) where u is discontinuous are a (subset of) Borel measurable null set.

Improper Riemann Integrals

- The Lebesgue integral extends the (*proper*) Riemann integral. However, there is a further extension of the Riemann integral which cannot be captured by Lebesgue's theory. u is Lebesgue integrable *iff* $|u|$ has finite Lebesgue integral.
 - The Lebesgue integral does not respect sign-changes and cancellations. However, the following *improper Riemann integral* does:

$$(R) \int_0^\infty u(x) dx := \lim_{n \rightarrow \infty} (R) \int_0^n u(x) dx. \quad (46)$$

Corollary 12.8. *Let $u : [0, \infty) \rightarrow \mathbb{R}$ be a measurable, Riemann integrable function for every interval $[0, N]$, $N \in \mathbb{N}$. Then $u \in \mathcal{L}^1[0, \infty)$ **iff***

$$\lim_{N \rightarrow \infty} (R) \int_0^N |u(x)| dx < \infty. \quad (47)$$

In this case, $(R) \int_0^\infty u(x) dx = \int_{[0, \infty)} u d\lambda$

Example of a function which is *improperly Riemann integrable* but **not** Lebesgue integrable:

$$f(x) = \frac{\sin(x)}{x}. \quad (48)$$

Proposition 12.9 (appearing as example 12.13 in Schilling). *Let $f_\alpha(x) := x^\alpha$, $x > 0$ and $\alpha \in \mathbb{R}$. Then*

- (i) $f_\alpha \in \mathcal{L}^1(0, 1) \Leftrightarrow \alpha > -1$.
- (ii) $f_\alpha \in \mathcal{L}^1[1, \infty) \Leftrightarrow \alpha < -1$.

13 The Function Spaces \mathcal{L}^p (lecture 11, 15. Feb.)

Assume V is a vector space over $\mathbb{K} \in \{\mathbb{C}, \mathbb{R}\}$.

Definition 13.1. A seminorm on V is a map $p : V \rightarrow [0, +\infty)$ s.t.

- (1) $p(cx) = |c|p(x) \quad \forall x \in V, \forall c \in \mathbb{K}$.
- (2) $p(x + y) \leq p(x) + p(y) \quad \forall x, y \in V$. **triangle inequality**.

A seminorm is called a norm if we also have

$$p(x) = 0 \iff x = 0.$$

A norm is commonly denoted $\|x\|$, and a vectorspace equipped with a norm is called a **normed space**.

Definition 13.2. Assume (X, d) is a measure space. Fix $1 \leq p \leq \infty$. For every measurable function $f : X \rightarrow \mathbb{C}$ we define the following

$$\|f\|_p = \left(\int_X |f|^p d\mu \right)^{1/p} \in [0, +\infty]. \quad (49)$$

We can see that $\|cf\|_p = |c|\|f\|_p \quad \forall c \in \mathbb{C}$.

Lemma 13.3.

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p. \quad (50)$$

Definition 13.4. We define

$$\mathcal{L}^p(X, d\mu) = \{f : X \rightarrow \mathbb{C} \mid f \text{ is measurable and } \|f\|_p < \infty\}. \quad (51)$$

This is a vectorspace with seminorm $f \mapsto \|f\|_p$. And in general this is not a normed space, since $\|f\|_p = 0 \iff f = 0$ a.e.

Generally, if p is a seminorm on a vectorspace V , then

$$V_0 = \{x \in V \mid p(x) = 0\} \quad (52)$$

which is a subspace of V . Then we consider the quotient/factor space V/V_0 .

Definition 13.5. For $x, y \in V$, define

$$x \sim y \iff x - y \in V_0. \quad (53)$$

This is an equivalence relation on V . The representation class of V is defined by $[x]$ or $x + V_0$.

Then V/V_0 is equals the set of equivalence classes. We can show that it is a normed space.

$$[x] + [y] = [x + y] \quad , \quad c[x] = [cx] \quad , \quad ||[x]|| = p(x).$$

Applying this to $\mathcal{L}^p(X, d\mu)$ we get the normed space

$$L^p(X, d\mu) = \mathcal{L}^p(X, d\mu)/\mathcal{N}. \quad (54)$$

Where \mathcal{N} is the space of measurable functions f s.t. $f = 0$ a.e. We will further continue to denote the norm by $|| \cdot ||_p$, and we will normally **not** distinguish between $f \in \mathcal{L}^p(X, d\mu)$ and the vector in $L^p(X, d\mu)$ that f defines.

Definition 13.6. A normed space $(X, || \cdot ||)$ is called a Banach space if V is complete w.r.t the metric $d(x, y) = ||x - y||$.

Theorem 13.7. If (X, \mathcal{B}, μ) is a measure space, $1 \leq p \leq \infty$, then $L^p(X, d\mu)$ is a Banach space.

Definition 13.8. A measurable function $f : X \rightarrow \mathbb{C}$ is called **essentially bounded** if there is $c \geq 0$ s.t.

$$\mu(\{x : |f(x)| > c\}) = 0. \quad (55)$$

That is $|f| \leq c$ a.e. The smallest such c is called the essential supremum of f and is denoted by $||f||_\infty$.

Definition 13.9.

$$\mathcal{L}^\infty(X, d\mu) = \{f : X \rightarrow \mathbb{C} \mid f \text{ is measurable and } ||f||_\infty < \infty\}.$$

$$L^\infty(X, d\mu) = \mathcal{L}^\infty(X, d\mu)/\mathcal{N}.$$

Where by the previous definiton these spaces become the spaces of all essentially bounded functions.

Theorem 13.10. If (X, \mathcal{B}, μ) is a σ -finite measure space, then $L^\infty(X, d\mu)$ is a Banach space.

17 Dense and Determining Sets (lecture 12, 19. Feb.)

Definition 17.1 (Dense Sets). A set $\mathcal{D} \subset \mathcal{L}^p(\mu), p \in [0, \infty]$, is called *dense* if for every $u \in \mathcal{L}^p(\mu)$ there exist a sequence $(f_n)_{n \in \mathbb{N}} \subset \mathcal{D}$ s.t. $\lim_{n \rightarrow \infty} ||u - f_n||_p = 0$.

Theorem 17.2. Assume X, d is a metric space and μ is a Borel measure that is finite on every ball $1 \leq p < \infty$. Then the space of bounded continuous functions with bounded support is dense in $\mathcal{L}^p(X, d\mu)$. Where bounded support means that f vanishes outside some ball.

Proof. We want to approximate $f \in \mathcal{L}^p(X, d\mu)$ by bounded continuous functions with bounded support. By considering separately $(\operatorname{Re}(f))_I$ and $(\operatorname{Im}(f))_I$ we may assume that $f \geq 0$. Then we can find simple functions f_n s.t. $0 \leq f_n \leq f$, $f_n \rightarrow f$ pointwise. As $|f - f_n|^p \leq |f|^p$, by the dominated convergence theorem we have $f_n \rightarrow f \in \mathcal{L}^p(X, d\mu)$. Hence, it suffices to consider simple f , but then it suffices to approximate $f = \pi_A$. Note that $\pi_A \in \mathcal{L}^p(X, d\mu)$ iff $\mu(A) < \infty$.

Fix $x_0 \in X$. Then $\pi_{A \cap B_n(x_0)} \nearrow \pi_A$ pointwise, hence $\pi_{A \cap B_n(x_0)} \rightarrow \pi_A \in \mathcal{L}^p(X, d\mu)$, again by the dominated convergence theorem.

Therefor it suffices to consider $A \subset B_n(x_0)$. As μ is outer regular, we have

$$\mu(A) = \inf_{\substack{A \subset U \subset B_n(x_0) \\ U \text{ is open}}} \mu(U).$$

Note that $\|\pi_U - \pi_A\|_p = \mu(U \setminus A)^{1/p}$. Hence, we can choose $U_k \subset B_n(x_0)$ s.t. $A \subset U_k$, U_k is open, $\pi_{U_k} \rightarrow \pi_A \in \mathcal{L}^p(X, d\mu)$.

Therefor it suffices to approximate π_U for open $U \subset B_n(x_0)$. Consider the functions

$$f_k(x) = \frac{kd(x, U^c)}{1 + kd(x, U^c)}.$$

Then $0 \leq f_k \leq 1$, f_k is continuous, supported on $\bar{U} \subset \bar{B}_n(x_0)$ and $f_k \nearrow \pi_U$ pointwise, hence $f_k \xrightarrow[k \rightarrow \infty]{} \pi_U \in \mathcal{L}^p(X, d\mu)$. \square

Theorem 17.3. *Assume (X, d) is a separable locally compact metric space and μ is a Borel Measure on X s.t. $\mu(K) < \infty \forall$ compact $K \subset X$. Then the space $C_c(X)$ of continuous compactly supported functions is dense in $\mathcal{L}^p(X, d\mu)$.*

In particular, either theorem shows that if μ is a Borel measure on \mathbb{R}^n s.t. the measure of every ball is finite, then $C_c(\mathbb{R}^n)$ is dense in $\mathcal{L}^p(\mathbb{R}^n, d\mu)$, $1 \leq p < \infty$. Later we will see that even $C_c^\infty(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n, d\mu)$.

Remark. *These results do not extend to $p = \infty$ in general.*

For $\mu = \lambda_n$ we write simply $\mathcal{L}^p(\mathbb{R}^n)$.

Remark. *Theorem 17.8 in the book is WRONG. For example, $X = \mathbb{Q}$ with the usual metric is σ -compact, supports nonzero finite measure, but $C_c(\mathbb{Q}) = 0$.*

Modes of Convergence (mixture of ex. 11.12 and ch. 22 p. 258-261.)

Definition 17.4 (convergence in measure). A sequence of measurable functions $u_n : X \rightarrow \bar{\mathbb{R}}$ converges in measure if

$$\forall \epsilon > 0 \forall A \in \mathcal{A}, \mu(A) < \infty : \lim_{n \rightarrow \infty} \mu(\{|u_n - u| > \epsilon\} \cap A) = 0$$

holds for some $u \in \mathcal{M}(\mathcal{A})$. We write $\mu\text{-}\lim_{n \rightarrow \infty} u_n = u$ or $u_n \xrightarrow{\mu} u$.

Assume (X, \mathcal{B}, μ) is a measure space. Given measurable functions $f_n, f : X \rightarrow \mathbb{C}$, recall that

$$f_n \rightarrow f \text{ a.e.}$$

means that $f_n(x) \xrightarrow{n \rightarrow \infty} f(x)$ for all x outside a set of measure zero.

Theorem 17.5 (Egorov). Assume $\mu(X) < \infty$ and $f_n \rightarrow f$ a.e. Then, $\forall \epsilon > 0$, there exists $X_\epsilon \in \mathcal{B}$ s.t. $\mu(X_\epsilon) < \epsilon$ and $f_n \rightarrow f$ uniformly on $X \setminus X_\epsilon$.

In addition to pointwise and uniform convergence we also consider the following:

$f_n \rightarrow f$ in the p -th mean if $\|f_n - f\|_p \xrightarrow{n \rightarrow \infty} 0$. For $p = 1$ we say in mean, for $p = 2$ we say in quadratic mean.

$f_n \rightarrow f$ in measure if $\forall \epsilon > 0$ we have

$$\mu(\{x : |f_n(x) - f(x)| \geq \epsilon\}) \xrightarrow{n \rightarrow \infty} 0.$$

Theorem 17.6 (Lemma 22.4 in the book?). Assume $(X, \mathcal{B}, d\mu)$ is a measure space, $1 \leq p < \infty$, $f_n, f : X \rightarrow \mathbb{C}$ are measurable functions. Then

(i) If $f_n \rightarrow f$ in the p -th mean, then $f_n \rightarrow f$ in measure.

(ii) If $f_n \rightarrow f$ in measure, then there is a subsequence $(f_{n_k})_{k=1}^\infty$ s.t. $f_{n_k} \rightarrow f$ a.e.

(iii) If $f_n \rightarrow f$ a.e. and $\mu(X) < \infty$, then $f_n \rightarrow f$ in measure.

In particular, if $f_n \rightarrow f$ in the p -th mean, then $f_{n_k} \rightarrow f$ a.e. for a subsequence $(f_{n_k})_k$.

26 Abstract Hilbert Spaces (lecture 13, 22. Feb)

Assume \mathcal{H} is a vector space over \mathbb{C} .

Definition 26.1. A pre-inner product on \mathcal{H} is a map $(\cdot, \cdot) : H \times H \rightarrow \mathbb{C}$ which is

(i) Sesquilinear: linear in the first variable and antilinear in the second:

$$\begin{aligned} (\alpha u + \beta v, w) &= \alpha(u, w) + \beta(v, w), \\ (w, \alpha u + \beta v) &= \bar{\alpha}(u, w) + \bar{\beta}(v, w), \quad u, v, w \in H \text{ and } \alpha, \beta \in \mathbb{C}. \end{aligned}$$

(ii) Hermitian: $(u, v) = \overline{(v, u)}$.

(iii) Positive semidefinite: $(u, u) \geq 0$.

It is called an **inner product**, or a scalar product, if instead of (iii) the map is positive definite; $(u, u) > 0$. This definition also works for \mathbb{R} instead of \mathbb{C} .

Cauchy-Schwartz inequality If (\cdot, \cdot) is a pre-inner product, then $|(u, v)| \leq (u, u)^{1/2}(v, v)^{1/2}$.

Corollary 26.2. Assume we have a seminorm $\|u\| := (u, u)^{1/2}$. It is a norm iff (\cdot, \cdot) is an inner product.

Definition 26.3 (Hilbert space). A Hilbert space is a complex vector space \mathcal{H} with an inner product (\cdot, \cdot) s.t. \mathcal{H} is complete with respect to the norm $\|u\| = (u, u)^{1/2}$.

1. The norm on a Hilbert space is determined by the inner product, but the inner product can also be recovered by the norm by the *polarization identity*: $(u, v) = \frac{\|u+v\|^2 - \|u-v\|^2}{4} + i \frac{\|u+iv\|^2 - \|u-iv\|^2}{4}$.
2. *Parallelogram law*: $\|u+v\|^2 + \|u-v\|^2 = 2\|u\|^2 + 2\|v\|^2$.
3. A norm on a vector space is given by an inner product iff it satisfies the parallelogram law, and then the scalar product is uniquely determined by the polarization identity.

Example 26.4. Assume (X, \mathcal{B}, μ) is a measure space. Then $\mathcal{L}^2(X, d\mu)$ is a Hilbert space with inner product

$$(f, g) = \int_X f \bar{g} d\mu.$$

This is well-defined, as $|f\bar{g}| \leq \frac{1}{2}(|f|^2 + |g|^2)$.

In particular, if $\mathcal{B} = \mathcal{P}(X)$ and μ is the counting measure, then $L^2(X, d\mu)$ is denoted by $l^2(X)$; for $X = \mathbb{N}$ we write simply l^2 . Note that in this case for $f : X \rightarrow [0, +\infty]$ we have

$$\int_X f d\mu = \sum_{x \in X} f(x) := \sup_{\substack{F \subset X \\ F \text{ is finite}}} \sum_{x \in F} f(x),$$

and if $\sum_{x \in X} f(x) < \infty$, then $\{x : f(x) > 0\}$ is at most countable, so $\sum_{x \in X} f(x) = \sum_{x: f(x) > 0} f(x)$ is the usual sum of a series.

Recall that a subset \mathcal{C} of a vector space is called *convex* if

$$u, w \in \mathcal{C} \rightarrow tu + (1-t)w \in \mathcal{C} \quad \forall t \in (0, 1).$$

The following is one of the key properties of the Hilbert space

Theorem 26.5 (projection theorem). Assume \mathcal{H} is a Hilbert space and $\mathcal{C} \subset H$ is a closed convex subset. Then for every $u \in H$ there is a unique $u_0 \in \mathcal{C}$ (minimizer) s.t.

$$\|u - u_0\| = d(u, \mathcal{C}) (= \inf_{v \in \mathcal{C}} \|u - v\|).$$

Proof. Let $d = d(u, \mathcal{C})$. Choose $u_n \in \mathcal{C}$ s.t. $\|u - u_n\| \rightarrow d$. We claim that $(u_n)_{n=1}^\infty$ is a Cauchy sequence. As $\frac{u_n + u_m}{2} \in \mathcal{C}$, we have

$$\begin{aligned} d^2 &\leq \left\| u - \frac{u_n + u_m}{2} \right\|^2 = \frac{1}{4} \|(u - u_n) + (u - u_m)\|^2 \\ &\stackrel{\text{parallelogram law}}{=} \frac{1}{4} (2\|u - u_n\|^2 + 2\|u - u_m\|^2 - \|u_n - u_m\|^2), \end{aligned}$$

so

$$\|u_n - u_m\|^2 \leq 2(\|u - u_n\|^2 - d^2) + 2(\|u - u_m\|^2 - d^2).$$

Thus, $(u_n)_{n \in \mathbb{N}}$ is indeed Cauchy, hence $u_n \rightarrow u_0 \in \mathcal{C}$ for some u_0 and

$$\|u - u_0\| = \lim_{n \rightarrow \infty} \|u - u_n\| = d = d(u, \mathcal{C}).$$

If u'_0 is another such point, we can take $u_{2n} = u_0, u_{2n+1} = u'_0$ and conclude that $u_0 = u'_0$. \square

Orthogonal Projections (lecture 14, 26. Feb.)

For a Hilbert space \mathcal{H} and a subset $A \subset H$, let

$$A^\perp := \{x \in H : x \perp y \ \forall y \in A\},$$

where $x \perp y$ means that $(x, y) = 0$. A^\perp is a closed subspace of \mathcal{H} .

Proposition 26.6. Assume \mathcal{H}_0 is a closed subspace of a Hilbert space \mathcal{H} . Then every $u \in H$ uniquely decomposes as

$$u = u_0 + u_1, \text{ with } u_0 \in \mathcal{H}_0 \text{ and } u_1 \in \mathcal{H}_0^\perp.$$

Moreover, $\|u - u_0\| = d(u, \mathcal{H}_0)$ and $\|u\|^2 = \|u_0\|^2 + \|u_1\|^2$.

For a closed subspace $\mathcal{H}_0 \subset \mathcal{H}$, consider the map $P : H \rightarrow \mathcal{H}_0$ s.t. $Pu \in \mathcal{H}_0$ is the unique element satisfying $u - Pu = H_0^\perp$. The operator P is linear. It is also contractive, meaning that $\|Pu\| \leq \|u\|$, since $\|u\|^2 = \|Pu\|^2 + \|u - Pu\|^2$. It is called the orthogonal projection onto \mathcal{H}_0 .

If \mathcal{H}_0 is finite dimensional with an orthonormal basis u_1, \dots, u_n then

$$Pu = \sum_{k=1}^n (u, u_k) u_k.$$

Orthonormal bases can be defined for arbitrary Hilbert spaces.

Definition 26.7 (orthonormal system). An orthonormal system in \mathcal{H} is a collection of vectors $h_i \in H$ ($i \in I$) s.t.

$$(u_i, u_j) = \delta_{ij} \ \forall i, j \in I.$$

It is called an *orthonormal basis* if $\text{span}\{u_i\}_{i \in I}$ denotes the linear span of $\{u_i\}_{i \in I}$, the space of finite linear combinations of the vectors u_i .

Definition 26.8. A Hilbert space \mathcal{H} is said to be *separable* if \mathcal{H} contains a countable dense subset $G \subset \mathcal{H}$.

Theorem 26.9. Every Hilbert space \mathcal{H} has an orthonormal basis. If \mathcal{H} is separable, then there is a countable orthonormal basis.

Proposition 26.10. Assume $\{u_i\}_{i \in I}$ is an orthonormal system in a Hilbert space H . Take $u \in \mathcal{H}$. Then

- (i) Bessel's inequality: $\sum_{i \in I} |(u, u_i)|^2 \leq \|u\|^2$, in particular, $\{i : (u, u_i) \neq 0\}$ is countable.
- (ii) Parseval's identity: If $\{u_i\}_{i \in I}$ is an orthonormal basis, then $\sum_{i \in I} |(u, u_i)|^2 = \|u\|^2$.

If $(u_i)_{i \in I}$ is an orthonormal basis, then the numbers (u, u_i) are called the **Fourier coefficients** of u with respect to $(u_i)_{i \in I}$. The Parseval identity then suggests that u is determined by its Fourier coefficients. This is true, and even more, we have:

Proposition 26.11. Assume $(u_i)_{i \in I}$ is an orthonormal basis in a Hilbert space \mathcal{H} . Then for every vector $(c_i)_{i \in I} \in l^2(I)$ there is a unique vector $u \in \mathcal{H}$ with Fourier coefficients c_i , and we write

$$u = \sum_{i \in I} c_i u_i.$$

Remark. Equivalently, the element $u = \sum_{i \in I} c_i u_i$ can be described as the unique element in \mathcal{H} s.t. $\forall \epsilon > 0$ there is a finite $F_0 \subset I$ s.t. $\|u - \sum_{i \in F} c_i u_i\| < \epsilon \forall$ finite $F \supset F_0$.

Corollary 26.12. We have a linear isomorphism $U : l^2(I) \xrightarrow{\sim} \mathcal{H}$, $U((c_i)_{i \in I}) = \sum_{i \in I} c_i u_i$. By Parseval's identity this isomorphism is isometric, that is, $\|Ux\| = \|x\| \forall x \in l^2(I)$. By the polarization identity this is equivalent to

$$(Ux, Uy) = (x, y) \forall x, y \in l^2(I).$$

Therefore U is unitary.

Corollary 26.13. Up to a unitary isomorphism, there is only one infinite dimensional separable Hilbert space, namely, l^2 .

Dual spaces (lecture 15, 29. Feb.)

Given two orthonormal bases $(u_i)_{i \in I}$ and $(v_i)_{i \in I}$ in a Hilbert space \mathcal{H} , we can decompose

$$u_i = \sum_{j \in I} (u_i, v_j) v_j$$

and using that the sets $\{j : (u_i, v_j) \neq 0\}$ are countable prove the following:

Claim: Any two orthonormal bases in a Hilbert space have the same cardinality.

Example 26.14 (classical Fourier series). Consider $\mathcal{H} = L^2(0, 2\bar{\mu}) = L^2((0, 2\bar{\mu}), d\lambda)$. For $n \in \mathbb{Z}$, define $e_n(t) = \frac{1}{\sqrt{2\bar{\mu}}} e^{int}$. By a version of Weierstrass' theorem it is known that $\text{span}\{e_n\}_{n \in \mathbb{Z}}$ is dense in the supremum-norm in

$$\{f \in C[0, 2\bar{\mu}] : f(0) = f(2\bar{\mu})\}.$$

As $C[0, 2\bar{\mu}]$ is dense in $L^2(0, 2\bar{\mu})$, from this one can deduce that $\text{span}\{e_n\}_{n \in \mathbb{Z}}$ is dense in $L^2(0, 2\bar{\mu})$. We then see that $(e_n)_{n \in \mathbb{Z}}$ is an orthonormal basis in $L^2(0, 2\bar{\mu})$. We therefor have a unitary isomorphism

$$l^2(\mathbb{Z}) \xrightarrow{\sim} L^2(0, 2\bar{\mu}), (c_n)_{n \in \mathbb{Z}} \mapsto \sum_{n \in \mathbb{Z}} c_n e_n.$$

The Fourier coefficients at $f \in L^2(0, 2\bar{\mu})$ are denoted by $\hat{f}(n)$, so

$$\hat{f}(n) = \frac{1}{\sqrt{2\bar{\mu}}} \int_0^{2\bar{\mu}} f(t) e^{-int} dt,$$

more practically

$$\hat{f}(n) = \frac{1}{\sqrt{2\bar{\mu}}} \int_{[0, 2\bar{\mu})} f(t) e^{-int} d\lambda(t).$$

Therefor we have $f = \sum_{n \in \mathbb{Z}} \hat{f}(n) e_n$ in $L^2(0, 2\bar{\mu})$.

Fact: For every $f \in L^2(0, 2\bar{\mu})$, we have $\frac{1}{\sqrt{2\bar{\mu}}} \sum_{n=-N}^N \hat{f}(n) e^{int} \xrightarrow[N \rightarrow \infty]{} f(t)$ for a.e. t .

Lemma 26.15. Assume V is a normed space over $K = \mathbb{R}$ or $K = \mathbb{C}$. Consider a linear functional $f : V \rightarrow K$. The following are equivalent (TFAE):

- (i) f is continuous;
- (ii) f is continuous at 0;
- (iii) There is a $c \geq 0$ s.t. $|f(x)| \leq c\|x\| \ \forall x \in V$.

If (i)-(iii) are satisfied, then f is called a *bounded linear functional*. The constant c in (iii) is denoted by $\|f\|$. We have $\|f\| = \sup_{x \neq 0} \frac{|f(x)|}{\|x\|} = \sup_{\|x\|=1} |f(x)| = \sup_{\|x\| \leq 1} |f(x)|$.

Proposition 26.16. For every normed vector space V over $K = \mathbb{R}$ or $K = \mathbb{C}$, the bounded linear functionals on V form a Banach space V^* .

Remark. The sequence $\{\|f_n - f_m\|\}_{m=1}^\infty$ actually converges, since

$$\left| \|f_n - f_m\| \right| \leq \|f_m - f_n\|.$$

When we study/use normed spaces, it is often important to understand the dual spaces. For Hilbert spaces this is particularly easy:

Theorem 26.17 (Riesz). *Assume \mathcal{H} is a Hilbert space. Then every $f \in \mathcal{H}^*$ has the form*

$$f(x) = (x, y),$$

for a uniquely defined $y \in \mathcal{H}$. Moreover, we have $\|f\| = \|y\|$.

For every Hilbert space \mathcal{H} we can define the *conjugate Hilbert space* $\bar{\mathcal{H}}$, which has its elements as the symbols \bar{x} for $x \in \mathcal{H}$, with the linear structure and inner product defined by

$$\bar{x} + \bar{y} = \overline{x + y}, c \cdot \bar{x} = \overline{c x}, (\bar{x}, \bar{y}) = \overline{(x, y)} = (y, x).$$

Corollary 26.18. *For every Hilbert space \mathcal{H} , we have an isometric isomorphism $\bar{\mathcal{H}} \xrightarrow{\sim} \mathcal{H}^*$, $\bar{x} \mapsto (\cdot, x)$.*

Appendix

H Regularity of measures (lecture 10, 12. Feb.)

We let (X, d) be a metric space and denote by \mathcal{O} the open, by \mathcal{C} the closed and $\mathcal{B}(X) = \sigma(\mathcal{O})$ the Borel set of X .

Definition H.1. A measure μ on $(X, d, \mathcal{B}(X))$ is called outer regular, if

$$\mu(B) = \inf \{ \mu(U) \mid B \subset U, U \text{ open} \} \quad (56)$$

and inner regular, if $\mu(K) < \infty$ for all compact sets $K \subset X$ and

$$\mu(U) = \sup \{ \mu(K) \mid K \subset U, K \text{ compact} \}. \quad (57)$$

A measure which is both inner and outer regular is called **regular**. We write $\mathfrak{m}_r^+(X)$ for the family of regular measures on $(X, \mathcal{B}(X))$.

Remark. The space X is called σ -compact if there is a sequence of compact sets $K_n \uparrow X$. A typical example of such a space is a locally compact, separable metric space.

Theorem H.2. Let (X, d) be a metric space. Every finite measure μ on $(X, \mathcal{B}(X))$ is outer regular. If X is σ -compact, then μ is also inner regular, hence regular.

Theorem H.3. Let (X, d) be a metric space and μ be a measure on $(X, \mathcal{B}(X))$ such that $\mu(K) < \infty$ for all compact sets $K \subset X$.

- 1 If X is σ -compact, then μ is inner regular.
- 2 If there exists a sequence $G_n \in \mathcal{O}$, $G_n \uparrow X$ such that $\mu(G_n) < \infty$, then μ is outer regular.

Hahn-Banach Theorem

Theorem H.4 (Hahn-Banach). Assume V is a real vector space, $V_0 \subset V$ a subspace, $e : V \rightarrow \mathbb{R}$ a convex function and $f : V_0 \rightarrow \mathbb{R}$ a linear functional s.t. $f \leq e$ on V_0 . Then f can be extended to a linear functional F on V s.t. $F \leq e$.

Assume first that $V = V_0 + \mathbb{R}x$ for some $x \in V \setminus V_0$. To define F we need to specify $F(x)$. The condition $F \leq e$ means that we need

$$F(y \pm tx) \leq e(y \pm tx) \quad \forall y \in V_0, t \geq 0,$$

that is

$$\pm tF(x) + f(y) \leq e(y \pm tx).$$

Dividing by t this is equivalent to

$$\begin{cases} F(x) \leq \frac{e(y+tx)-f(y)}{t}, \\ F(x) \geq \frac{f(y)-e(y-tx)}{t}, \end{cases}$$

$\forall y \in V_0, t \geq 0$. To show that there is a number satisfying these inequalities, we need to check that

$$\sup_{\substack{y \in V_0 \\ t > 0}} \frac{f(y) - e(y - tx)}{t} \leq \inf_{\substack{y \in V_0 \\ t > 0}} \frac{e(y + tx) - f(y)}{t}.$$

Then as $F(x)$ we can take any number in the interval $[\sup, \inf]$. In other words, we need to check that

$$\frac{f(y) - e(y - tx)}{t} \leq \frac{e(z + sx) - f(z)}{s},$$

$\forall y, z \in V_0, t, s \geq 0$. Equivalently,

$$sf(y) + tf(z) \leq se(y - tx) + te(z + sx).$$

We have

$$\begin{aligned} \frac{sf(y) + tf(z)}{s + t} &= f\left(\frac{s}{s + t}y + \frac{t}{s + t}z\right) \\ &\leq e\left(\frac{s}{s + t}y + \frac{t}{s + t}z\right) \\ &\leq e\left(\frac{s}{s + t}(y - tx) + \frac{t}{s + t}(z + sx)\right) \\ &\leq \frac{s}{s + t}e(y - tx) + \frac{t}{s + t}e(z + sx), \end{aligned}$$

which is what we need.

The general case is deduced from this using transitive induction (Zorn's lemma), as follows. Consider the set X of pairs (W, F) , where $w \subset V$ is a subspace, $F : W \rightarrow \mathbb{R}$ a linear functional, $V_0 \subset W$ and $F|_{V_0} = f$. Define a partial order on X by $F \leq e$ on W .

$$(W_1, F_1) \leq (W_2, F_2) \text{ iff } W_1 \subset W_2 \text{ and } F_1 = F_2|_{W_1}.$$

If \mathcal{C} is a chain in X , then it has an upper bound (W, F) defined by

$$W = \bigcup_{(Y, \rho) \in \mathcal{C}} Y, F(y) = \rho(y) \text{ if } y \in Y, (Y, \rho) \in \mathcal{C}.$$

Hence, X has a maximal element (W, F) by Zorn's lemma. If $W \neq V$, then we can take $x \in V \setminus W$ and extend F to $W + \mathbb{R}x$ preserving the inequality $F \leq e$. This contradicts maximality of W, F in X . Hence, $W = V$.

Theorem H.5 (Hahn-Banach). Assume V is a real or complex vector space, p a seminorm on V_0 , $V_0 \subset V$, and f a linear functional on V_0 s.t.

$$|f(x)| \leq p(x) \quad \forall x \in V_0.$$

Then f can be extended to a linear functional F on V s.t. $|F(x)| \leq p(x) \quad \forall x \in V$.

Corollary H.6. Assume V is a normed real or complex vector space, $V_0 \subset V$ and $f \in V_0^*$. Then there is a $F \in V^*$ s.t.

$$F|_{V_0} = f \quad \text{and} \quad \|F\| = \|f\|.$$

Proof. We apply the previous theorem to $p(x) = \|f\| \cdot \|x\|$. □

Corollary H.7. Assume V is a normed space and $x \in V, x \neq 0$. Then there is a $F \in V^*$ s.t. $\|F\| = 1$ and $F(x) = \|x\|$.

Such an F is called a *supporting functional at x* .

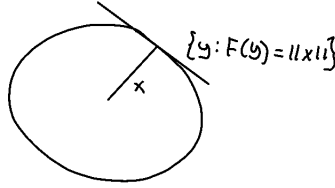


Figure 1: Tangent space?

If V is a normed vector space, then every $x \in X$ defines a bounded linear functional on V^* by

$$V^* \ni f \mapsto f(x).$$

As $|f(x)| \leq \|f\| \cdot \|x\|$, this functional has norm $\leq \|x\|$. By using a supporting functional at x , we actually see that we get norm $\|x\|$. Thus, we have an isometric embedding $V \subset V^{**} := (V^*)^*$. We can therefore see V as a subspace of V^{**} .

Definition H.8. A normed space V is called reflexive if $V^{**} = V$.

Remark. This is stronger than requiring $V \simeq V^{**}$.

Remark. Every Hilbert space \mathcal{H} is reflexive. Indeed, $\mathcal{H}^* = \bar{\mathcal{H}}$. By Riesz' theorem every bounded linear functional f on $\bar{\mathcal{H}}$ has the form

$$f(\bar{x}) = (\bar{x}, \bar{y}) = (y, x),$$

for some $y \in \mathcal{H}$, which exactly means that $f = y$ in \mathcal{H}^{**} .

As we will see later, the spaces $\mathcal{L}^p(X, d\mu)$, with μ σ -finite and $1 < p < \infty$, are reflexive. The spaces $\mathcal{L}^1(X, d\mu)$ and $\mathcal{L}^\infty(X, \mu)$ are usually not reflexive.