Solutions to the exam problems in MAT3400/4400, Fall 2012

Problem 1. Using Fourier series show that if $f\colon [0,2\pi] \to \mathbb{C}$ is a C^1-function such that $f(0) = f(2\pi)$ and $\int_0^{2\pi} f(t)dt = 0$, then

$$\int_0^{2\pi} |f(t)|^2 dt \le \int_0^{2\pi} |f'(t)|^2 dt.$$

Describe all functions f as above such that the equality holds.

Solution. If $c_n(f)$, $n \in \mathbb{Z}$, denote the Fourier coefficients of f (with respect to the orthonormal basis $e_n(t) = e^{int}/\sqrt{2\pi}$), then we know that $c_n(f') = inc_n(f)$ for any $f \in C^1[0,2\pi]$ such that $f(0) = f(2\pi)$. By assumption we also have $c_0(f) = 0$. Therefore by Parseval's identity we get

$$\int_0^{2\pi} |f(t)|^2 dt = \sum_{n\ne 0} |c_n(f)|^2 \quad \text{and} \quad \int_0^{2\pi} |f'(t)|^2 dt = \sum_{n\ne 0} n^2 |c_n(f)|^2.$$

This immediately gives

$$\int_0^{2\pi} |f(t)|^2 dt \le \int_0^{2\pi} |f'(t)|^2 dt.$$

Furthermore, the equality holds if and only if $c_n(f) = 0$ for all $|n| > 1$. In other words, the equality holds exactly for the functions $f(t) = ae^{it} + be^{-it}$, with $a, b \in \mathbb{C}$.

Problem 2. Assume $(a_{ij})_{i,j=1}^\infty$ is an infinite matrix with complex coefficients such that $\sum_{i,j=1}^\infty |a_{ij}|^2 < \infty$.

(a) Consider the Hilbert space ℓ_2. Show that the following formula defines a bounded linear operator T from ℓ_2 into itself:

$$T(x_1, x_2, \dots) = \left(\sum_{j=1}^\infty a_{1j} x_j, \sum_{j=1}^\infty a_{2j} x_j, \dots \right).$$

(b) Show that T is a Hilbert-Schmidt operator and compute its Hilbert-Schmidt norm.

Solution. (a) By the Cauchy-Schwarz inequality we have

$$\left| \sum_{j=1}^\infty a_{ij} x_j \right| \le \left(\sum_{j=1}^\infty |a_{ij}|^2 \right)^{1/2} \left(\sum_{j=1}^\infty |x_j|^2 \right)^{1/2} = \left(\sum_{j=1}^\infty |a_{ij}|^2 \right)^{1/2} \|x\|_2.$$

Therefore

$$\|Tx\|_2 = \left(\sum_{i=1}^\infty \left| \sum_{j=1}^\infty a_{ij} x_j \right|^2 \right)^{1/2} \le \left(\sum_{i=1}^\infty \sum_{j=1}^\infty |a_{ij}|^2 \right)^{1/2} \|x\|_2.$$

Thus T indeed maps ℓ_2 into itself. It is clear that the operator T is linear. The above inequality shows that T is bounded, of norm not larger than $\left(\sum_{i,j=1}^\infty |a_{ij}|^2 \right)^{1/2}$.

(b) Consider the standard orthonormal basis $\{e_n\}_{n=1}^\infty$ in ℓ_2. We have

$$Te_n = (a_{1n}, a_{2n}, \dots).$$

Therefore

$$\sum_{j=1}^{\infty} \|Te_j\|_2^2 = \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} |a_{ij}|^2 < \infty.$$

Hence T is Hilbert-Schmidt and its Hilbert-Schmidt norm is

$$\|T\|_2 = \left(\sum_{i,j=1}^{\infty} |a_{ij}|^2\right)^{1/2}.$$

Problem 3. Consider the operator $T\colon L^2[0,1] \to L^2[0,1]$ defined by

$$(Tf)(s) = \int_0^s f(t)dt, \quad s \in [0,1].$$

It is assumed to be known that T is a compact operator and that the image of T is contained in the subspace $L^2_{cont}[0,1] \subset L^2[0,1]$ of continuous functions.

(a) Show that the adjoint operator is given by

$$(T^*f)(s) = \int_s^1 f(t)dt, \quad s \in [0,1].$$

(b) Show that a function $f \in L^2[0,1]$ is an eigenvector of T^*T with eigenvalue $\lambda > 0$ if and only if it is smooth and satisfies the differential equation

$$\begin{cases} \lambda f'' + f = 0, \\ f(1) = 0,\ f'(0) = 0. \end{cases}$$

(c) Find the singular values of T. What is the operator norm of T? Is T a trace-class operator?

Solution. (a) Consider the operator $S\colon L^2[0,1] \to L^2[0,1]$ defined by

$$(Sf)(s) = \int_s^1 f(t)dt, \quad s \in [0,1].$$

Then for $f, g \in L^2_{cont}[0,1]$ we have

$$(Tf, g) = \int_0^1 (Tf)(s)\overline{g(s)}ds = \int_0^1 ds \int_0^s dt\, f(t)\overline{g(s)}$$

and

$$(f, Sg) = \int_0^1 f(t)\overline{(Sg)(t)}dt = \int_0^1 dt \int_t^1 ds\, f(t)\overline{g(s)}.$$

We see that $(Tf, g) = (f, Sg)$, since both double integrals above coincide with the integral of the function $(t,s) \mapsto f(t)\overline{g(s)}$ over the region consisting of points (t,s) such that $0 \le t \le 1$, $t \le s \le 1$. Hence $S = T^*$.

In fact, essentially the same computation was done in greater generality in the class. The operator T is the integral operator $(Tf)(s) = \int_0^1 K(s,t)f(t)dt$ with K given by

$$K(s,t) = \begin{cases} 1, & \text{if } t \le s, \\ 0, & \text{otherwise.} \end{cases}$$

It follows that the adjoint operator is the integral operator $(T^*f)(s) = \int_0^1 \tilde{K}(s,t)f(t)dt$ with $\tilde{K}$ given by

$$\tilde{K}(s,t) = \overline{K(t,s)} = \begin{cases} 1, & \text{if } s \le t, \\ 0, & \text{otherwise,} \end{cases}$$

so $(T^*f) = \int_s^1 f(t)dt$. Formally, though, we discussed only the case when K is continuous.

(b) Assume $f \in L^2[0,1]$ is an eigenvector of T^*T with eigenvalue $\lambda > 0$. Since T maps $L^2[0,1]$ into the space of continuous functions and the operators T and T^* map $C^n[0,1]$ into $C^{n+1}[0,1]$ for any $n \ge 0$, we conclude that $f = \lambda^{-1}T^*Tf$ is contained in the subspace of C^∞-functions. From the integral formulas for T and T^* we see that for any continuous function g we have

$$(Tg)(0) = 0, \quad (Tg)' = g, \quad (T^*g)(1) = 0, \quad (T^*g)' = -g.$$

Therefore

$$\lambda f(1) = (T^*Tf)(1) = 0, \quad \lambda f'(0) = (T^*Tf)'(0) = -(Tf)(0) = 0,$$
$$\lambda f'' = (T^*Tf)'' = -(Tf)' = -f.$$

Conversely, assume f is a smooth function such that $f(1) = 0$, $f'(0) = 0$, $\lambda f'' + f = 0$. Then the function $g = T^*Tf - \lambda f$ has the properties

$$g(1) = 0, \quad g'(0) = -(Tf)(0) - \lambda f'(0) = 0, \quad g'' = (T^*Tf)'' - \lambda f'' = -f - \lambda f'' = 0.$$

From $g'' = 0$ we conclude that g is linear, so $g(t) = a + bt$, and then the conditions $g'(0) = 0$ and $g(1) = 0$ imply that $a = b = 0$. Therefore $g = 0$, that is, $T^*Tf = \lambda f$.

(c) By part (b) the eigenvalues $\lambda > 0$ of T^*T are the numbers for which the differential equation

$$\begin{cases} \lambda f'' + f = 0, \\ f(1) = 0, \; f'(0) = 0 \end{cases}$$

has a non-zero solution. The general solution of $\lambda f'' + f = 0$ has the form

$$f(t) = a\cos(t/\sqrt{\lambda}) + b\sin(t/\sqrt{\lambda}).$$

The condition $f'(0) = 0$ means that $b = 0$. Then the condition $f(1) = 0$ is satisfied for $a \ne 0$ if and only if $1/\sqrt{\lambda} = \pi/2 + n\pi$ for some $n \ge 0$. Therefore the nonzero eigenvalues $\lambda_n(T^*T)$ of T^*T, counted with multplicities and ordered in the decreasing order, are given by

$$\lambda_n(T^*T) = \frac{1}{(\pi/2 + (n-1)\pi)^2}, \quad n = 1, 2, \dots.$$

Hence the singular values of T are given by

$$s_n(T) = \lambda_n(T^*T)^{1/2} = \frac{1}{\pi/2 + (n-1)\pi}, \quad n = 1, 2, \dots.$$

We have

$$\|T\| = s_1(T) = \frac{2}{\pi}.$$

The operator T is not of trace class, since $\sum_{n=1}^\infty s_n(T) = \infty$.

Problem 4.

(a) Assume that the Fourier coefficients of a function $f \in L^1[0, 2\pi]$ are all zero. Show that $f = 0$ a.e.

(b) Assume $f \in L^1[0,2\pi]$ is such that its Fourier coefficients $c_n(f)$ satisfy $\sum_{n\in\mathbb{Z}} |c_n(f)|^2 < \infty$. Show that $f \in L^2[0,2\pi]$.

Solution. (a) The proof is motivated by the proof of the same result for $L^2[0,2\pi]$. But since for $f \in L^1[0,2\pi]$ and $g \in L^2[0,2\pi]$ the function $f\bar{g}$ is not necessarily integrable, we have to be more careful.

Consider the set $\mathcal{A}$ of bounded Borel functions g on $[0,2\pi]$ such that $\int_0^{2\pi} fg\,dt = 0$. This is a vector space containing the functions e^{int}, $n \in \mathbb{Z}$. This space is closed under pointwise limits of bounded sequences. Indeed, if $\{g_n\}_{n=1}^\infty$ is a bounded sequence in $\mathcal{A}$, so $|g_n(t)| \le M$ for all $t \in [0,2\pi]$ and $n \ge 1$, and $g_n(t) \to g(t)$ as $n \to \infty$ for all $t \in [0,2\pi]$, then $|fg_n| \le M|f|$ and therefore

$$\int_0^{2\pi} fg\,dt = \lim_{n\to\infty} \int_0^{2\pi} fg_n dt = 0$$

by the dominated convergence theorem. By the Weierstrass theorem the functions e^{int} span a dense subspace of $C[0,2\pi]$ (with respect to the supremum-norm). Hence $C[0,2\pi] \subset \mathcal{A}$. Using the hint we conclude that $\mathcal{A}$ coincides with the space of all bounded Borel functions.

Consider now the bounded function $g(t) = \overline{f(t)}/|f(t)|$, with the convention that $g(t) = 0$ if $f(t) = 0$. Since g is Lebesgue measurable, it coincides a.e. with a Borel function[1], so

$$0 = \int_0^{2\pi} fg\,dt = \int_0^{2\pi} |f| dt.$$

Hence $f = 0$ a.e.

(b) Since $\sum_{n\in\mathbb{Z}} |c_n(f)|^2 < \infty$, the series $\sum_{n\in\mathbb{Z}} c_n(f)e_n$ in $L^2[0,2\pi]$, where $e_n(t) = e^{int}/\sqrt{2\pi}$, converges to an element $\tilde{f} \in L^2[0,2\pi]$. Then the function $f - \tilde{f}$ is integrable and has all the Fourier coefficients equal to zero. Hence $f - \tilde{f} = 0$ a.e. by part (a), so $f \in L^2[0,2\pi]$.

[1] Depending on the conventions this is either obvious, as in Teschl, who defines $L^1[0,2\pi]$ using only Borel functions, or a bit non-trivial, but was anyway proved in the class.