

# Integration of measurable functions

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Through this chapter  $(X, \mathcal{A}, \mu)$  will be some measure space. Recall that  $\mathcal{M}^+(\mathcal{A})$  [ $\mathcal{M}_{\mathbb{R}}^+(\mathcal{A})$ ] are the  $\mathcal{A}$ -measurable positive functions and  $\mathcal{E}(\mathcal{A})$  [ $\mathcal{E}_{\mathbb{R}}^+(\mathcal{A})$ ] are the positive and simple functions.

The fundamental idea of *Integration* is to measure the area between the graph of the function and the abscissa. For positive simple functions  $f \in \mathcal{E}^+(\mathcal{A})$  in standard representation, this is done easily

$$\text{if } f = \sum_{i=0}^M y_i \mathbb{1}_{A_i} \in \mathcal{E}^+(\mathcal{A}) \quad \text{then} \quad \sum_{i=0}^M y_i \mu(A_i) \quad (1)$$

would be the  $\mu$ -area enclosed by the graph and the abscissa. We note that the representation of  $f$  should not impact the integral of  $f$ .

**Lemma 9.10.** *Let  $\sum_{i=0}^M y_i \mathbb{1}_{A_i} = \sum_{k=0}^N z_k \mathbb{1}_{B_k}$  be two standard representations of the same function  $f \in \mathcal{E}^+(\mathcal{A})$ . Then*

$$\sum_{i=0}^M y_i \mu(A_i) = \sum_{k=0}^N z_k \mu(B_k). \quad (2)$$

**Definition 9.11.** Let  $f = \sum_{i=0}^M y_i \mathbb{1}_{A_i} \in \mathcal{E}^+(\mathcal{A})$  be a simple function in standard representation. Then the number

$$I_{\mu}(f) = \sum_{i=0}^M y_i \mu(A_i) \in [0, \infty] \quad (3)$$

(which is independent of the representation of  $f$ ) is called the  $\mu$ -integral of  $f$ .

**Proposition 9.12.** *Let  $f, g \in \mathcal{E}^+(\mathcal{A})$ . Then*

$$(i) \quad I_{\mu}(\mathbb{1}_A) = \mu(A) \quad \forall A \in \mathcal{A}.$$

$$(ii) \quad I_{\mu}(\lambda f) = \lambda I_{\mu}(f) \quad \forall \lambda \geq 0.$$

$$(iii) \quad I_{\mu}(f + g) = I_{\mu}(f) + I_{\mu}(g).$$

$$(iv) \quad f \leq g \Rightarrow I_{\mu}(f) \leq I_{\mu}(g).$$

In theorem 8.8 we saw that we could for every  $u \in \mathcal{M}^+(\mathcal{A})$  write it as an increasing limit of simple functions. By corollary 8.10, the suprema of simple functions are again measurable, so that

$$u \in \mathcal{M}^+(\mathcal{A}) \Leftrightarrow u = \sup_{n \in \mathbb{N}} f_n, \quad f \in \mathcal{E}^+(\mathcal{A}), \quad f_n \leq f_{n+1} \leq \dots$$

We will use this to "inscribe" simple functions (which we know how to integrate) below the graph of a positive measurable function  $u$  and exhaust the  $\mu$ -area below  $u$ .

**Definition 9.13.** Let  $(X, \mathcal{A}, \mu)$  be a measure space. The  $(\mu)$ -integral of a positive function  $u \in \mathcal{M}_{\mathbb{R}}^+(\mathcal{A})$  is given by

$$\int u d\mu = \sup \{ I_\mu(g) : g \leq u, g \in \mathcal{E}^+(\mathcal{A}) \} \in [0, +\infty]. \quad (4)$$

If we need to emphasize the *integration variable*, we write  $\int u(x)\mu(dx)$ . The key observation is that the integral  $\int \dots d\mu$  extends  $I_\mu$ .

**Lemma 9.14.** For all  $f \in \mathcal{E}^+(\mathcal{A})$  we have  $\int f d\mu = I_\mu(f)$ .

The next theorem is one of many convergence theorems. It shows that we could have defined 4 using any increasing sequence  $f_n \uparrow u$  of simple functions  $f_n \in \mathcal{E}^+(\mathcal{A})$ .

**Theorem 9.15.** (*Beppo Levi*) Let  $(X, \mathcal{A}, \mu)$  be a measure space. For an increasing sequence of functions  $(u_n)_{n \in \mathbb{N}} \subset \mathcal{M}_{\mathbb{R}}^+(\mathcal{A})$ ,  $0 \leq u_n \leq u_{n+1} \leq \dots$ , we have for the supremum  $u = \sup_{n \in \mathbb{N}} u_n \in \mathcal{M}_{\mathbb{R}}^+(\mathcal{A})$  and

$$\int \sup_{n \in \mathbb{N}} u_n d\mu = \sup_{n \in \mathbb{N}} \int u_n d\mu. \quad (5)$$

Note we can write  $\lim_{n \rightarrow \infty}$  instead of  $\sup_{n \in \mathbb{N}}$  as the supremum of an increasing sequence is its limit. Moreover, this theorem holds in  $[0, +\infty]$ , so the case  $+\infty = +\infty$  is possible.

**Corollary 9.16.** Let  $u \in \mathcal{M}_{\mathbb{R}}^+(\mathcal{A})$ . Then

$$\int u d\mu = \lim_{n \rightarrow \infty} \int f_n d\mu$$

holds for every sequence  $(f_n)_{n \in \mathbb{N}} \subset \mathcal{E}^+(\mathcal{A})$  with  $\lim_{n \rightarrow \infty} f_n = u$ .

**Proposition 9.17.** (of integral) Let  $u, v \in \mathcal{M}_{\mathbb{R}}^+(\mathcal{A})$ . Then

- (i)  $\int \mathbb{1}_A d\mu = \mu(A) \quad \forall A \in \mathcal{A}$ .
- (ii)  $\int \alpha u d\mu = \alpha \int u d\mu \quad \forall \alpha \geq 0$ .
- (iii)  $\int u + v d\mu = \int u d\mu + \int v d\mu$ .

(iv)  $u \leq v \Rightarrow \int u d\mu \leq \int v d\mu$ .

**Corollary 9.18.** *Let  $(u_n)_{n \in \mathbb{N}} \subset \mathcal{M}_{\mathbb{R}}^+(\mathcal{A})$ . Then  $\sum_{n=1}^{\infty} u_n$  is measurable and we have*

$$\int \sum_{n=1}^{\infty} u_n d\mu = \sum_{n=1}^{\infty} \int u_n d\mu$$

(including the possibility  $+\infty = +\infty$ .)

**Theorem 9.19.** (*Fatou*) *Let  $(u_n)_{n \in \mathbb{N}} \subset \mathcal{M}_{\mathbb{R}}^+(\mathcal{A})$  be a sequence of positive measurable functions. Then  $u = \liminf_{n \rightarrow \infty} u_n$  is measurable and*

$$\int \liminf_{n \rightarrow \infty} u_n d\mu = \liminf_{n \rightarrow \infty} \int u_n d\mu \tag{6}$$