MAT4400: Detailed Notes on Linear analysis

April 24, 2024

1 σ-Algebras (3, [Schilling(2017)])

Definition 1.1 (σ-Algebra)**.** A family $\mathscr{A}$ of subsets of X with:

(i) $X \in \mathscr{A}$,

(ii) $A \in \mathscr{A} \Rightarrow A^c \in \mathscr{A}$,

(iii) $(A_n)_{n\in\mathbb{N}} \in \mathscr{A} \Rightarrow \bigcup_{n\in\mathbb{N}}$

A set $A \in \mathscr{A}$ is said to be **measurable** or **$\mathscr{A}$-measurable**.

Theorem 1.2 (and Definition)**.**

(i) The intersection of arbitrarily many σ-algebras in X is againg a σ-algebra in X.

(ii) For every system of sets $p \subset \mathscr{P}(X)$ there exists a smallestσ-algebra containing$p$. This is the σ-algebra generated by p, denoted $\sigma(p)$, and $\sigma(p)$ is called its generator.

Definition 1.3 (Borel)**.** The σ-algebra $\sigma(\mathcal{O})$ generated by the open sets $\mathcal{O} = \mathcal{O}_{\mathbb{R}^n}$ of $\mathbb{R}^n$ is called **Borel σ-algebra**, and its members are called **Borel sets** or **Borel measurable sets**.

Definition 1.4 (σ-finite/sigma-finite)**.** A measure μ is said to be σ-finite and $(X, \mathcal{A}, \mu)$ is called a σ-finite measure space, if $\mathcal{A}$ contains a sequence $(A_n)_{n\in\mathbb{N}}$ s.t. $A_n \uparrow X$ and $\mu(A_n) < \infty$.

3 Uniqueness of Measures (5, [Schilling(2017)])

Lemma 3.1. *A Dynkin system D is a σ-algebra iff it is stable under finite intersections, i.e. $A, B \in D \Rightarrow A \cap B \in D$.*

Theorem 3.2 (Dynkin)**.** *Assume X is a set, S is a collection of subsets of X closed under finite intersections, that is, if $A, B \in S \Rightarrow A \cap B \in S$. Then $D(S) = \sigma(S)$.*

Proof. We clearly have that $D(S) \subset \sigma(S)$. If we can show that $D(S)$ is a σ-algebra, that is, that a Dynkin system generated by a subset $S \subset X$ (where S is $\cap$-stable) is a σ-algebra, then the inverse conclusion $D(S) \supset \sigma(S)$ follows logically. This is the case because the σ-algebra $\sigma(S)$ is the smallest σ-algebra containing S, and so if $D(S)$ is a σ-algebra it must be a greater or equal (in some sense) than $\sigma(S)$.

Using Lemma 3.1 we only need to show that $D(S)$ is stable under finite intersections, to prove that $D(S)$ is a σ-algebra. Consider:

$$D_A := \{B \subset X : B \cap A \in D(S)\},$$

for some $A \in D(S)$. Notice that this set is $\cap$-stable, and so if we can show that $D_A = D(S)$ we must have that (by Lemma 3.1) $D(S)$ is a σ-algebra. Firstly, however, let us show that D_A is a **Dynkin system**.

1. $\varnothing$ must be in D_A, since $\varnothing \cap A = \varnothing \in D(S)$.
2. Let $B \in D_A$. Then
$$A \cap B^c = A \setminus (A \cap B) = (A^c \cup (A \cap B))^c,$$
here $A \cap B$ and A^c must be in $D(S)$. Furthermore, since disjoint unions of set from $D(S)$ are still in $D(S)$, we me must have $A^c \in D_A$.
3. Assume that $(B_n)_{n\in\mathbb{N}} \subset D_A$ is a pairwise disjoint sequence. Then
$$\begin{aligned} &(B_n \cap A)_{n\in\mathbb{N}} \in D(S) \quad \text{(by def. of } D_A) \\ \Rightarrow &\bigcup_{n\in\mathbb{N}} (B_n \cap A) = \left(\bigcup_{n\in\mathbb{N}} B_n\right) \cap A \in D(S) \\ \Rightarrow &\bigcup_{n\in\mathbb{N}} B_n \in D_A. \end{aligned}$$

So D_A is indeed a Dynkin system.

We now want to show that $D(S)$ is $\cap$-stable, we have:

$$\begin{aligned} S \subset D_A \quad &\forall\ A \in S \\ \Rightarrow D(S) \subset D_A \quad &\forall\ A \in S \\ \Rightarrow B \cap A \in D(S) \quad &\forall\ B \in S,\ \forall\ A \in D(S) \\ \Rightarrow B \in D_A \quad &\forall\ B \in S,\ \forall A \in D(S) \\ \Rightarrow S \subset D_A \quad &\forall\ A \in D(S) \\ \Rightarrow D(S) \subset D_A \quad &\forall\ A \in D(S) \\ \Rightarrow A \cap B \in D(S) \quad &\forall\ A, B \in D(S), \end{aligned}$$

and so $D(S)$ is $\cap$-stable and then $D(S) \supset \sigma(S) \Rightarrow D(S) = \sigma(S)$. □

Theorem 3.3 (uniqueness of measures)**.** *Let (X, B) be a measurable space, and $S \subset P(X)$ be the generator of B, i.e. $B = \sigma(S)$. If S satisfies the following conditions:*

1. *S is stable under finite intersections ($\cap$-stable), i.e. $A, C \in S \Rightarrow A \cap C \in S$.*
2. *There exists an exhausting sequence $(G_n)_{N\in\mathbb{N}} \subset$ with $G_N \uparrow X$. Assume also that there are two measures μ, ν satisfying:*
3. *$\mu(A) = \nu(A),\ \forall A \in S$.*
4. *$\mu(G_n) = \nu(G_n) < \infty$.*

Then $\mu = \nu$.

Proof (outline). Define

$$D_n := \{A \in B : \mu(G_n \cap A) = \nu(G_n \cap A)\ (< \infty)\},$$

and show that it is a Dynkin system. Then, use the fact that S is $\cap$-stable and Theorem 3.2 to argue that $D(S) = \sigma(S) ... \rightarrow ... B = D_n$. □

4 Existence of Measures (6, [Schilling(2017)])

Theorem 4.1 (Carathéodory)**.** *Let $S \subset P(X)$ be a semi-ring and $\mu : S \to [0, \infty)$ a pre-measure. Then μ has an extension to a measure μ^* on $\sigma(S)$, i.e. that $\mu(s) = \mu^*(s)$, $\forall\ s \in \sigma(S)$.*

Also, if S contains an exhausting sequence, $S_n \uparrow X$, s.t. $\mu(S_n) < \infty$, then the extension is unique.

Outline of proof: Firstly, let us define an outer measure.

Definition 4.2 (Outer measure)**.** An outer measure is a function $\mu^* : P(X) \to [0, \infty)$ with the following properties:

1. $\mu^*(\emptyset) = 0$,
2. $A \subset B \Rightarrow u^*(A) \leq \mu^*(B)$,
3. $\mu^*\left(\bigcup\limits_{n\in\mathbb{N}} A_n\right) \leq \sum\limits_{n\in\mathbb{N}} \mu^*(A_n)$,

and define for each $A \subset X$ the family of countable S-*coverings*:

$$C(A) := \left\{(S_n)_{n\in\mathbb{N}} \subset S : \bigcup_{n\in\mathbb{N}} S_n \supset A\right\},$$

and the set function

$$\mu^*(A) := \inf\left\{\sum_{n\in\mathbb{N}} \mu(S_n) : (S_n)_{n\in\mathbb{N}} \in C(A)\right\}.$$

Step 1: Claim: *$\mu^*(A)$ is an outer measure.*

Proof.

1. $C(\emptyset) = \{$any sequence in S containing $\emptyset\}$
   $\Rightarrow \mu^*(\emptyset) = 0$.
2. Assume $A \subset B$. Then $C(A) \subset C(B)$
   $\Rightarrow \mu^*(A) \leq \mu^* B$.
3. If $\mu^*(A_n) = \infty$ for some n, then there is nothing to prove. Thus, assume
   $\mu^*(A_n) < \infty\ \forall n$. Fix $\epsilon > 0$, and for every n choose $A_{n_k} \in S$ s.t.
   $$A_n \subset \bigcup_{k\in\mathbb{N}} A_{n_k},$$
   $$\sum_{k\in\mathbb{N}} \mu^*(A_{n_k}) < \mu^*(A_n) + \frac{\epsilon}{2^n}.$$
   Then
   $$\bigcup_{n\in\mathbb{N}} A_n \subset \bigcup_{k\in\mathbb{N}} \bigcup_{n\in\mathbb{N}} A_{n_k},$$
   so
   $$\mu^*\left(\bigcup_{n\in\mathbb{N}} A_n\right) \leq \sum_{n\in\mathbb{N}} \sum_{k\in\mathbb{N}} \mu\left(A_{n_k}\right) < \sum_{n\in\mathbb{N}} \left(\mu^*(A_n) + \frac{\epsilon}{2^n}\right) = \sum_{n\in\mathbb{N}} \mu^*(A_n) + \epsilon.$$
   As ϵ was arbitrarily, we get that
   $$\mu^*\left(\bigcup_{n\in\mathbb{N}} A_n\right) \leq \sum_{n\in\mathbb{N}} \mu^*(A_n),$$
   so μ^* fulfills all the conditions for being an outer measure. □

Step 2: *Showing that μ^* extends μ, i.e. $\mu^*(s) = \mu(s)\ \forall s \in S$.*
Step 3: *Define μ^*-measurable sets*

$$\Sigma^* := \left\{A \subset X : \mu^*(Q) = \mu^*(Q \cap A) + \mu^*(Q \backslash A)\right.$$
$$\left.\forall\ Q \subset X\right\}$$

Step 4: *Show that $\mu|_{\Sigma^*}$ is a measure. In particular, $\mu|_{\sigma(S)}$ is a measure which extends μ.*

5 Measurable Mappings (7, [Schilling(2017)])

We consider maps $T : X \to X'$ between two measurable spaces $(X, \mathcal{A})$ and $(X', \mathcal{A}')$ which respects the measurable structurs, the σ-algbras on X and X'. These maps are useful as we can transport a measure μ, defined on $(X, \mathcal{A})$, to $(X', \mathcal{A}')$.

Definition 5.1. Let $(X, \mathcal{A})$, $(X', \mathcal{A}')$ b measurable spaces. A map $T : X \to X'$ is called $\mathcal{A}/\mathcal{A}'$-measurable if the pre-imag of every measurable set is a measurable set:

$$T^{-1}(A') \in \mathcal{A}, \quad \forall A' \in \mathcal{A}'.$$

- A $\mathcal{B}(\mathbb{R}^n)/\mathcal{B}(\mathbb{R}^m)$ measurable map is often called a Borel map.
- The notation $T : (X, \mathcal{A}) \to (X', \mathcal{A}')$ is often used to indicate measurability of the map T.

Lemma 5.2. *Let $(X, \mathcal{A})$, $(x', \mathcal{A}')$ be measurable spaces and let $\mathcal{A}' = \sigma(\mathcal{G}')$. Then $T : X \to X'$ is $\mathcal{A}/\mathcal{A}'$-measurable iff $T^{-1}(\mathcal{G}') \subset \mathcal{A}$, i.e. if*

$$T^{-1}(G') \in \mathcal{A}, \ \ \forall G' \in \mathcal{G}'.$$

Theorem 5.3. *Let $(X_i, \mathcal{A}_i)$, $i = 1, 2, 3$, be measurable spaces and $T : X_1 \to X_2$, $S : X_2 \to X_3$ be $\mathcal{A}_1/\mathcal{A}_2$ and $\mathcal{A}_2/\mathcal{A}_3$-measurable maps respectivly. Then $S \circ T : X_1 \to X_3$ is $\mathcal{A}_1/\mathcal{A}_3$-measurable.*

Corollary 5.4. *Every continuous map betwen metric spaces is a Borel map.*

Definition 5.5. (and lemma) Let $(T_i)_{i\in I}$, $T_I : X \to X_i$, be arbitrarily many mappings from the same space X into measurable spaces $(X_i, \mathcal{A}_i)$. The smallest σ-algebra on X that makes all T_i simultanously measurable is

$$\sigma(T_i : i \in I) := \sigma\left(\bigcup_{i\in I} T_i^{-1}(\mathcal{A}_i)\right)$$

Corollary 5.6. *A function $f : (X, \mathcal{B}) \to \mathbb{R}$ is measurable if $f((a, +\infty)) \in \mathcal{B}$, $\forall a \in \mathbb{R}$.*

Corollary 5.7. *Assume* $(X, \mathcal{B})$ *is a measurable space,* (Y, d) *is a metric space, and* $(f_n : (X, \mathcal{B}) \to Y)_{n=1}^{\infty}$ *is a sequence of measurable maps. Assume this sequence of images* $(f_n(x))_{n=1}^{\infty}$ *is convergent in* Y $\forall x \in X$*. Define*

$$f : X \to Y, \quad \text{by } f(x) = \lim_{n\to\infty} f_n(x).$$

Then f is measurable.

Theorem 5.8. *Let* $(X, \mathcal{A})$, $(X', \mathcal{A}')$ *be measurable spaces and* $T : X \to X'$ *be an* $\mathcal{A}/\mathcal{A}'$*-measurable map. For every measurable* μ *on* $(X, \mathcal{A})$,

$$\mu'(A') := \mu(T^{-1}(A')), \quad A' \in \mathcal{A}',$$

defines a measure on $(X', \mathcal{A}')$.

Definition 5.9. The measure $\mu'(\cdot)$ in the above theorem is called the push forward or image measure of μ under T and it is denoted as $T(\mu)(\cdot)$, $T_{*\mu}(\cdot)$ or $\mu \circ T^{-1}(\cdot)$.

Theorem 5.10. *If* $T \in \mathbb{R}^{n\times n}$ *is an orthogonal matrix, then* $\lambda^n = T(\lambda^n)$.

Theorem 5.11. *Let* $S \in \mathbb{R}^{n\times n}$ *be an invertible matrix. Then*

$$S(\lambda^n) = |\det s^{-1}|\lambda^n = |\det S|^{-1}\lambda^n.$$

Corollary 5.12. *Lebesgue measure is invariant under motions:* $\lambda^n = M(\lambda^n)$ *for all motions* M *in* $\mathbb{R}^n$*. In particular, congruent sets have the same measure. Two sets of points are called congruent if, and only if, one can be transformed into the other by an isometry*

Measurable Functions (8, [Schilling(2017)])

A *measurable function* is a measurable map $u : X \to \mathbb{R}$ from some measurable space $(X, \mathscr{A})$ to $(\mathbb{R}, \mathscr{B}(\mathbb{R}^1))$. They play central roles in the theory of integration.

We recall that $u : X \to \mathbb{R}$ is $\mathscr{A}/\mathscr{B}(\mathbb{R}^1)$-measurable if

$$u^{-1}(B) \in \mathscr{A}, \quad \forall B \in \mathscr{B}(\mathbb{R}^1).$$

Moreover from a lemma from chapter 7, we actually only need to show that

$$u^{-1}(G) \in \mathscr{A}, \quad \forall G \in \mathcal{G} \text{ where } \mathcal{G} \text{ generates } \mathscr{B}(\mathbb{R}^1).$$

Proposition 5.13.

1 *If* $f, g : (X, \mathscr{B}) \to \mathbb{C}$ *are measurable, then the function* $f + g$, $f \cdot g$, cf, $(c \in \mathbb{C})$ *are measurable.*

2 *If* $b : \mathbb{C} \to \mathbb{C}$ *is Borel and* $b : (\mathbb{C}, \mathscr{B}) \to \mathbb{C}$ *is measurable, then* $b \circ f$ *is measurable.*

3 *If* $f(x) = \lim_{n\to\infty} f_n(x)$, $x \in X$ *and* f_n *are measurable, then* f *is measurable.*

4 *If* $X = \bigcup_{n=1}^{\infty} A_n$, $(A_n \in \mathscr{B})$, $f|_{A_n} : (A_n, \mathscr{B}_{A_n}) \to \mathbb{C}$ *is measurable* $\forall n$*, then* f *is measurable.*

Definition 5.14. Given a measurable space $(X, \mathscr{B})$, a measurable function $f : (X, \mathscr{B}) \to \mathbb{C}$ is called simple if

$$f(x) = \sum_{k=1}^{N} c_k \mathbb{1}_{A_k}(x),$$

for some $c_k \in \mathbb{C}$, $A_k \in \mathscr{B}$, where $\mathbb{1}$ is the characteristic function,

$$\mathbb{1}_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A. \end{cases}$$

The representation of simple function is **not** unique. We denote the standard representation of f by

$$f(x) = \sum_{n=0}^{N} z_n \mathbb{1}_{B_n}(x),$$

for $N \in \mathbb{N}$, $z_n \in \mathbb{R}$, $B_n \in \mathscr{A}$, and

$$X = \bigcup_{n=1}^{N} B_n,$$

for $B_n \cap B_m = \emptyset$, $n \neq m$. The set of simple functions is denoted $\mathcal{E}(\mathscr{A})$ of $\mathcal{E}$.

Definition 5.15. Assume μ is a measure on $(X, \mathscr{B})$. Given a *positive* simple function

$$f = \sum_{k=1}^{N} c_k \mathbb{1}_{A_k}, \quad (c_k \geq 0).$$

We define

$$\int_X f d\mu = \sum_{k=1}^{n} c_k \mu(A_k) \in [0, +\infty].$$

We also denote this by $I_\mu(f)$.

Lemma 5.16. *This is well defined, that is,* $\int_X f d\mu$ *does not depend on the presentation of the simple function* f.

Properties 5.17. *For every positive simple function*

1 $\int_X cf d\mu = c\int_X f d\mu$, *for only* $c \geq 0$

2 $\int_X (f+g) d\mu = \int_X f d\mu + \int_X g d\mu$.

Corollary 5.18. *If* $f \geq g \geq 0$ *are simple functions, then*

$$\int_X f d\mu \geq \int_X g d\mu.$$

Definition 5.19. If $f : X \to [0, +\infty)$ is measurable, then we define

$$\int_X f d\mu = \sup\left\{\int_X g d\mu : f \geq g \geq 0, \ \ g \text{ is simple}\right\}$$

Remark. *This means that any measurable function can be approximated by simple functions.*

Properties 5.20. *Measurable functions like this have the following properties*

1 $\int_X cf d\mu = c\int_X f d\mu$, $\forall c \geq 0$.

2 *If* $f \geq g \geq 0$*, then* $\int_X f d\mu \geq \int_X g d\mu$ *for any measurable* g, f.

3 If $f \geq 0$ is simple, then $\int_X f d\mu$ is the same value as obtained before.

To advance in measure theory we consider measurable functions

$$f : X \to [0, +\infty].$$

Measurability is understood w.r.t the $\sigma-$algebra $\mathscr{B}([0, +\infty])$ generated by $\mathscr{B}([0, +\infty))$ and $\{+\infty\}$. In other words, $A \subset [0, +\infty] \in B([0, +\infty])$ iff $A \cap [0, +\infty) \in \mathscr{B}([0, +\infty)$.

Remark. *Hence $f : X \to [0, +\infty]$ is measurable iff $f^{-1}(A)$ is measurable $\forall A \in \mathscr{B}([0, +\infty))$.*

Definition 5.21. For measurable functions $f_X \to [0, +\infty]$, we define

$$\int_X f d\mu = \sup \left\{ \int_x g d\mu \ : \ f \geq g \geq 0 \ : \ g \text{ is simple} \right\} \in [0, +\infty].$$

Theorem 5.22. *Monotone convergence theorem* *Assume $(X, \mathscr{B}, \mu)$ is a measure space, $(f)_{n=1}^{\infty}$ is an increasing sequence of measurable positive functions $f_n : X \to [0, +\infty]$. Define $f(x) = \lim_{n\to\infty} f_n(x)$. Then f is measurable and*

$$\int_X f d\mu = \lim_{n\to\infty} \int_X f_n d\mu.$$

Theorem 5.23. *Assume $(X, \mathscr{B})$ is a measurable space and $f : X \to [0, +\infty]$ is measurable. Then there are simple functions g_n, s.t.*

$$0 \leq g_1 \leq g_2 \leq \dots \ , \ g_n(x) \to f(x), \ \forall x \in X.$$

Moreover, if f is bounded, we can choose g_n s.t. the convergence is uniform, that is,

$$\lim_{n\to\infty} \sup_{x\in X} |g_n(x) - f(x)| = 0.$$

6 Integration of Measurable Functions

(9, [Schilling(2017)])

Through this chapter $(X, \mathscr{A}, \mu)$ will be some measure space. Recall that $\mathcal{M}^+(\mathscr{A})$ $[\mathcal{M}^+_{\overline{\mathbb{R}}}(\mathscr{A})]$ are the $\mathscr{A}$-measurable positive functions and $\mathcal{E}(\mathscr{A})$ $[\mathcal{E}^+_{\overline{\mathbb{R}}}(\mathscr{A})]$ are the positive and simple functions.

The fundamental idea of *Integration* is to measure the area between the graph of the function and the abscissa. For positive simple functions $f \in \mathcal{E}^+(\mathscr{A})$ in standard representation, this is done easily

$$\text{if } f = \sum_{i=0}^{M} y_i \mathbb{1}_{A_i} \in \mathcal{E}^+(\mathscr{A}) \quad \text{then } \sum_{i=0}^{M} y_i \mu(A_i) \tag{1}$$

would be the μ-area enclosed by the graph and the abscissa. We note that the representation of f should not impact the integral of f.

Lemma 6.1. *Let $\sum_{i=0}^{M} y_i \mathbb{1}_{A_i} = \sum_{k=0}^{N} z_k \mathbb{1}_{B_k}$ be two standard representations of the same function $f \in \mathcal{E}^+(\mathscr{A})$. Then*

$$\sum_{i=0}^{M} y_i \mu(A_i) = \sum_{k=0}^{N} z_k \mu(B_k). \tag{2}$$

Definition 6.2. Let $f = \sum_{i=0}^{M} y_i \mathbb{1}_{A_i} \in \mathcal{E}^+(\mathscr{A})$ be a simple function in standard representation. Then the number

$$I_\mu(f) = \sum_{i=0}^{M} y_i \mu(A_i) \in [0, \infty] \tag{3}$$

(which is independent of the representation of f) is called the μ-integral of f.

Proposition 6.3. *Let $f, g \in \mathcal{E}^+(\mathscr{A})$. Then*

(i) $I_\mu(\mathbb{1}_A) = \mu(A) \quad \forall A \in \mathscr{A}$.

(ii) $I_\mu(\lambda f) = \lambda I_\mu(f) \quad \forall \lambda \geq 0$.

(iii) $I_\mu(f + g) = I_\mu(f) + I_\mu(g)$.

(iv) $f \leq g \Rightarrow I_\mu(f) \leq I_\mu(g)$.

In theorem 8.8 we saw that we could for every $u \in \mathcal{M}^+(\mathscr{A})$ write it as an increasing limit of simple functions. By corollary 8.10, the suprema of simple functions are again measurable, so that

$$u \in \mathcal{M}^+(\mathscr{A}) \Leftrightarrow u = \sup_{n\in\mathbb{N}} f_n, f \in \mathcal{E}^+(\mathscr{A}),$$
$$f_n \leq f_{n+1} \leq \dots .$$

We will use this to "inscribe" simple functions (which we know how to integrate) below the graph of a positive measurable function u and exhaust the μ-area below u.

Definition 6.4. Let $(X, \mathscr{A}, \mu)$ be a measure space. The (μ)-integral of a positive function $u \in \mathcal{M}^+_{\overline{\mathbb{R}}}(\mathscr{A})$ is given by

$$\int u d\mu = \sup \left\{ I_\mu(g) : g \leq u, \ g \in \mathcal{E}^+(\mathscr{A}) \right\}, \tag{4}$$

with $\int u d\mu \in [0, +\infty]$. If we need to emphasize the *integration variable*, we write $\int u(x)\mu(dx)$. The key observation is that the integral $\int \dots d\mu$ extends I_μ.

Lemma 6.5. *For all $f \in \mathcal{E}^+(\mathscr{A})$ we have $\int f d\mu = I_\mu(f)$.*

The next theorem is one of many convergence theorems. It shows that we could have defined 4 using any increasing sequence $f_n \uparrow u$ of simple functions $f_n \in \mathcal{E}^+(\mathscr{A})$.

Theorem 6.6. *(Beppo Levi) Let $(X, \mathscr{A}, \mu)$ be a measure space. For an increasing sequence of functions $(u_n)_{n\in\mathbb{N}} \subset \mathcal{M}^+_{\overline{\mathbb{R}}}(\mathscr{A})$, $0 \leq u_n \leq u_{n+1} \leq \dots$, we have for the supremum $u = \sup_{n\in\mathbb{N}} u_n \in \mathcal{M}^+_{\overline{\mathbb{R}}}(\mathscr{A})$ and*

$$\int \sup_{n\in\mathbb{N}} u_n d\mu = \sup_{n\in\mathbb{N}} \int u_n d\mu. \tag{5}$$

Note we can write $\lim_{n\to\infty}$ instead of $\sup_{n\in\mathbb{N}}$ as the supremum of an increasing sequence is its limit. Moreover, this theorem holds in $[0, +\infty]$, so the case $+\infty = +\infty$ is possible.

Corollary 6.7. *Let $u \in \mathcal{M}^+_{\overline{\mathbb{R}}}(\mathscr{A})$. Then*

$$\int u d\mu = \lim_{n\to\infty} \int f_n d\mu$$

holds for every sequence $(f_n)_{n\in\mathbb{N}} \subset \mathcal{E}^+(\mathscr{A})$ with $\lim_{n\to\infty} f_n = u$.

Proposition 6.8. *(of integral) Let $u, v \in \mathcal{M}^+_{\overline{\mathbb{R}}}(\mathscr{A})$. Then*

(i) $\int \mathbb{1}_A d\mu = \mu(A) \quad \forall A \in \mathscr{A}$.

(ii) $\int \alpha u d\mu = \alpha \int u d\mu \quad \forall \alpha \geq 0$.

(iii) $\int u + v d\mu = \int u d\mu + \int v d\mu$.

(iv) $u \le v \Rightarrow \int u d\mu \le \int v d\mu$.

Corollary 6.9. *Let $(u_n)_{n\in\mathbb{N}} \subset \mathcal{M}^+_{\overline{\mathbb{R}}}(\mathscr{A})$. Then $\sum_{n=1}^{\infty} u_n$ is measurable and we have*

$$\int \sum_{n=1}^{\infty} u_n d\mu = \sum_{n=1}^{\infty} \int u_n d\mu$$

(including the possibility $+\infty = +\infty$.)

Theorem 6.10. *(Fatou) Let $(u_n)_{n\in\mathbb{N}} \subset \mathcal{M}^+_{\overline{\mathbb{R}}}(\mathscr{A})$ be a sequence of positive measurable functions. Then $u = \liminf_{n\to\infty} u_n$ is measurable and*

$$\int \liminf_{n\to\infty} u_n d\mu \le \liminf_{n\to\infty} \int u_n d\mu \tag{6}$$

7 Integrals of Measurable Functions

(10, [Schilling(2017)])

We have defined our integral for positive measurable functions, i.e. functions in $\mathcal{M}^+(\mathscr{A})$. To extend our integral to not only functions in $\mathcal{M}^+(\mathscr{A})$ we first notice that

$$u \in \mathcal{M}_{\overline{\mathbb{R}}}(\mathscr{A}) \Leftrightarrow u = u^+ - u^-,\ u^+, u^- \in \mathcal{M}^+_{\overline{\mathbb{R}}},$$

i.e. that every measurable function can be written as a sum of **positive** measurable functions.

Definition 7.1 (μ-integrable)**.** A function $u : X \to \overline{\mathbb{R}}$ on $(X, \mathscr{A}, \mu)$ is μ-*integrable*, if it is $\mathscr{A}/\mathscr{B}(\overline{\mathbb{R}})$-measurable and if $\int u^+ d\mu, \int u^- d\mu < \infty$ (recall the definition for the integral of positive measurable functions). Then

$$\int u d\mu := \int u^+ d\mu - \int u^- d\mu \in (-\infty, \infty)$$

is the (μ-)*integral* of u. We write $\mathcal{L}^1(\mu)$ for the set of all real-valued μ-integrable functions [1].

Theorem 7.2. *Let $u \in \mathcal{M}_{\overline{\mathbb{R}}}(\mathscr{A})$, then the following conditions are equivalent:*

(i) $u \in \mathcal{L}^1_{\overline{\mathbb{R}}}(\mu)$.

(ii) $u^+, u^- \in \mathcal{L}^1_{\overline{\mathbb{R}}}(\mu)$.

(iii) $|u| \in \mathcal{L}^1_{\overline{\mathbb{R}}}(\mu)$.

(iv) $\exists w \in \mathcal{L}^1_{\overline{\mathbb{R}}}(\mu)$ *with* $w \ge 0$ *s.t.* $|u| \le w$.

Theorem 7.3 (Properties of the μ-integral)**.** *The μ-integral is: **homogeneous, additive**, and:*

(i) $\min\{u, v\}, \max\{u, v\} \in \mathcal{L}^1_{\overline{\mathbb{R}}}(\mu)$ *(lattice property)*

(ii) $u \le v \Rightarrow \int u d\mu \le \int v d\mu$ *(monotone)*

(iii) $\left|\int u d\mu\right| \le \int |u| d\mu$ *(triangle inequality)*

Remark. *If $u(x) \pm v(x)$ is defined in $\overline{\mathbb{R}}$ for all $x \in X$ then we can exclude $\infty - \infty$ and the theorem above just says that the integral is linear:*

$$\int (au + bv) d\mu = a \int u d\mu + b \int v d\mu.$$

This is always true for real-valued $u, v \in \mathcal{L}^1(\mu) = \mathcal{L}^1_{\mathbb{R}}(\mu)$, making $\mathcal{L}^1(\mu)$ a vector space with addition and scalar multiplication defined by

$$(u + v)(x) := u(x) + v(x),\ \ (a \cdot u)(x) := a \cdot u(x),$$

and

$$\int ...d\mu : \mathcal{L}^1(\mu) \to \mathbb{R},\ u \mapsto \int u d\mu,$$

*is a **positive linear functional**.*

8 Null sets and the Almost Everywhere

(11, [Schilling(2017)])

Definition 8.1. A (μ-)null set $N \in \mathcal{N}_\mu$ is a measurable set $N \in \mathscr{A}$ satisfying

$$N \in \mathcal{N}_\mu \Longleftrightarrow N \in \mathscr{A} \text{ and } \mu(N) = 0.$$

This can be used generally about a 'statement' or 'property', but we will be interested in questions like 'when is $u(x)$ equal to $v(x)$', and we answer this by saying

$$u = v\ a.e. \Leftrightarrow \{x : u(x) \ne v(x)\} \text{ is (contained in) a } \mu\text{-null set},$$

i.e.

$$u = v\ \ \mu\text{-a.e.} \Leftrightarrow \mu(\{x : u(x) \ne v(x)\}) = 0.$$

The last phrasing should of course include that the set $\{x : u(x) \ne v(x)\}$ is in $\mathscr{A}$.

Theorem 8.2. *Let $u \in \mathcal{M}_{\overline{\mathbb{R}}}(\mathscr{A})$, then:*

(i) $\int |u| d\mu = 0 \Leftrightarrow |u| = 0$ *a.e.* $\Leftrightarrow \mu\{u \ne 0\} = 0$,

(ii) $\mathbb{1}_N u \in \mathcal{L}^1_{\overline{\mathbb{R}}}(\mu)\ \ \forall\ N \in \mathcal{N}_\mu$,

(iii) $\int_N u d\mu = 0$.

(i) is really useful, later we will define $\mathcal{L}^p$ and the $||\cdot||_p$-(semi)norm. Then (i) means that if we have a sequence u_n converging to u in the $||\cdot||_p$-norm then $u_n(x) = u(x)$ a.e.

Corollary 8.3. *Let $u = v$ μ-a.e. Then*

(i) $u, v \ge 0 \Rightarrow \int u d\mu = \int v d\mu$,

(ii) $u \in \mathcal{L}^1_{\overline{\mathbb{R}}}(\mu) \Rightarrow v \in \mathcal{L}^1_{\overline{\mathbb{R}}}(\mu)$ *and* $\int u d\mu = \int v d\mu$.

Corollary 8.4. *If $u \in \mathcal{M}_{\overline{\mathbb{R}}}(\mathscr{A})$, $v \in \mathcal{L}^1_{\overline{\mathbb{R}}}(\mu)$ and $v \ge 0$ then*

$$|u| \le v\ \ a.e.\ \Rightarrow u \in \mathcal{L}^1_{\overline{\mathbb{R}}}(\mu).$$

[1] In words, we extend our integral to ~~positive~~ measurable functions by noticing that we can write every measurable function as a sum of positive measurable functions, something that we do know how to integrate. We don't want to run into the problem of $\infty - \infty$, thus we require the integral of the positive and negative parts to both (separately) be less than infinity.

Proposition 8.5 (Markow inequality)**.** *For all $u \in \mathcal{L}^1_{\bar{\mathbb{R}}}(\mu)$, $A \in \mathscr{A}$ and $c > 0$*

$$u(\{|u| \geq c\} \cap A) \leq \frac{1}{c}\int_A |u| d\mu,$$

if $A = X$, then (obviosly)

$$u\{|u| \geq c\} \leq \frac{1}{c}\int |u| d\mu.$$

Corollary 8.6. *If $u \in \mathcal{L}^1_{\bar{R}}(\mu)$, then μ is a.e. $\mathbb{R}$-vaued. In particular, we can find a version $\tilde{u} \in \mathcal{L}^1(\mu)$ s.t. $\tilde{u} = u$ a.e. and $\int \tilde{u} d\mu = \int u d\mu$*

Completions of measure spaces

Definition 8.7. A measure space $(X, \mathscr{B}, \mu)$ is called **complete** if whenever $A \in \mathscr{B}$ and $\mu(A) = 0$, we have $B \in \mathscr{B}$ $\forall B \subset A$.

Remark. *Any measure space can be completed as follows:*
Let $\bar{\mathscr{B}}$ be the σ-algebra generated by $\mathscr{B}$ and all sets $B \subset X$ s.t. there exists $A \in \mathscr{B}$ with $B \subset A$ and $\mu(A) = 0$.

Proposition 8.8. *The σ-algebra $\bar{\mathscr{B}}$ can also be described as follows:*

$$\bar{\mathscr{B}} := \Big\{ B \subset X : A_1 \subset B \subset A_2$$

$$\text{for some } A_1, A_2 \in \mathscr{B} \text{ with } \mu(A_2 \backslash A_1) = 0 \Big\},$$

with B, A_1, A_2 as above, we define

$$\bar{\mu} := \mu(A_1) = \mu(A_2)$$

Then $(X, \bar{\mathscr{B}}, \bar{\mu})$ is a complete measure space.

Definition 8.9. If μ is a Borel measure on a **metric** space (X, d), then the completion $\bar{\mathscr{B}}(X)$ of the Borel σ-algebra with respect to μ is called the σ-algebra of μ-measurable sets.

Remark. *For $\mu = \lambda_n$ on $\mathbb{R}^n$ we talk about the σ-algebra of* ***Lebesgue measurable sets****. Instead of $\bar{\lambda_n}$ we still write λ_n and call it the* ***Lebesgue measure****. A function $f : \mathbb{R}^n \to \mathbb{C}$, measurable w.r.t. the σ-algebra of Lebesgue measurable sets is called the* ***Lebesgue measurable****.*

The following result shows that any Lebesgue measurable function coincides with a Borel function a.e.

Proposition 8.10. *Assume $(X, \mathscr{B}, \mu)$ is a measure space and consider its completion $(X, \bar{\mathscr{B}}, \bar{\mu})$. Assume $f : X \to \mathbb{C}$ is $\bar{\mathscr{B}}$-measurable. Then there is a $\mathscr{B}$-measurable function $g : X \to \mathbb{C}$ s.t. $f = g$ $\bar{\mu}$-a.e.*

9 Convergence Theorems and their Applications

(12, [Schilling(2017)])

- To interchange limits and integrals in **Riemann integrals** one typically has to assume uniform convergence. ¡- The set of Riemann integrable functions is somewhat limited, see theorem 9.7

Theorem 9.1 (Generalization of Beppo Levi, monotone convergence)**.**

(i) *Let $(u_n)_{n\in\mathbb{N}} \subset \mathcal{L}^1(\mu)$ be s.t. $u_1 \leq u_2 \leq \ldots$ with limit $u := \sup_{n\in\mathbb{N}} u_n = \lim_{n\to\infty} u_n$. Then $u \in \mathcal{L}^1(\mu)$* ***iff***

$$\sup_{n\in\mathbb{N}} \int u_n d\mu < +\infty,$$

in which case

$$\sup_{n\in\mathbb{N}} \int u_n d\mu = \int \sup_{n\in\mathbb{N}} u_n d\mu.$$

(ii) *Same thing only with a decreasing sequence ...> $-\infty$ in which case*

$$\inf_{n\in\mathbb{N}} \int u_n d\mu = \int \inf_{n\in\mathbb{N}} u_n d\mu.$$

Theorem 9.2 (Lebesgue; dominated convergence)**.** *Let $(u_n)_{n\in\mathbb{N}} \subset \mathcal{L}^1(\mu)$ s.t.*

(a) *$|u_n|(x) \leq w(x)$, $w \in \mathcal{L}^1(\mu)$,*

(b) *$u(x) = \lim_{n\to\infty} u_n(x)$ exists in $\bar{\mathbb{R}}$,*

then $u \in \mathcal{L}^1(\mu)$ and we have

(i) *$\lim_{n\to\infty} \int |u_n - u| d\mu = 0$;*

(ii) *$\lim_{n\to\infty} \int u_n d\mu = \int \lim_{n\to\infty} u_n d\mu = \int u d\mu$;*

Application 1: Parameter-Dependent Integrals

- We are interested in questions of the sort, when is

$$U(t) := \int u(t, x)\mu(dx), \ t \in (a, b),$$

again a smooth function of t? The answer involves interchange of limits and integration. Also, it turns out to better understand Riemann integrability, we need the Lebesgue integral.

Theorem 9.3 (continuity lemma)**.** *Let $\emptyset \neq (a, b) \subset \mathbb{R}$ be a non-degenerate open interval and $u : (a, b) \times X \to \mathbb{R}$ satisfy*

(a) *$x \mapsto u(t, x)$ is in $\mathcal{L}^1(\mu)$ for every fixed $t \in (a, b)$;*

(b) *$t \mapsto u(t, x)$ is continuous for every fixed $x \in X$;*

(c) *$|u(t, x)| \leq w(x)$ for all $(t, x) \in (a, b) \times X$ and some $w \in \mathcal{L}^1(\mu)$.*

Then the function $U : (a, b) \to \mathbb{R}$ given by

$$t \mapsto U(t) := \int u(t, x)\mu(dx) \tag{7}$$

is continuous.

Theorem 9.4 (differentiability lemma)**.** *Let $\emptyset \leq (a, b) \subset \mathbb{R}$ be a non-degenerate open interval and $u : (a, b) \times X \to \mathbb{R}$ satisfy*

(a) *Same*

(b) *Same*

(c) *$|\partial_t u(t, x)| \leq w(x)$ for all $(t, x) \in (a, b) \times X$ and some $w \in \mathcal{L}^1(\mu)$.*

Then the function in 7 is differentiable and its derivative is

$$\frac{d}{dt}U(t) = \frac{d}{dt}\int u(t, x)\mu(dx) = \int \frac{\partial}{\partial t}u(t, x)\mu(dx). \tag{8}$$

Application 2: Riemann vs Lebesgue Integration

Consider only $(X, \mathscr{A}, \mu) = (\mathbb{R}, \mathscr{B}(\mathbb{R}), \lambda)$.

Definition 9.5 (The Riemann Inegral)**.** Consider on the finite interval $[a, b] \subset \mathbb{R}$ the partition

$$\Pi := \{a = t_0 < t_1 < \ldots < t_k < b\}, k = k(\Pi), \tag{9}$$

and introduce

$$S_\Pi[u] := \sum_{i=1}^{k(\Pi)} m_i(t_i - t_{i-1}), \qquad m_i := \inf_{x \in [t_{i-1}, t_i]} u(x), \tag{10}$$

$$S^\Pi[u] := \sum_{i=1}^{k(\Pi)} M_i(t_i - t_{i-1}), \qquad M_i := \sup_{x \in [t_{i-1}, t_i]} u(x). \tag{11}$$

$$\tag{12}$$

A bounded function $u : [a, b] \to \mathbb{R}$ is said to be **Riemann integrable** if the values

$$\underline{\int} u := \sup_\Pi S_\Pi[u] = \inf_\Pi S^\Pi[u] =: \bar{\int} u \tag{13}$$

coincide and are finite. Their common value is called the **Riemann integral** of u and denoted by $(R)\int_a^b u(x)dx$ or $\int_a^b u(x)dx$.

Theorem 9.6. *Let $u : [a, b] \to \mathbb{R}$ be a **measurable** and **Riemann integrable** function. Then*

$$u \in \mathcal{L}^1(\lambda) \text{ and } \int_{[a,b]} u d\lambda = \int_a^b u(x)dx. \tag{14}$$

Theorem 9.7. *Let $u : [a, b] \to \mathbb{R}$ be a bounded function, it is Riemann integrable **iff** the points in (a, b) where u is discontinuous are a (subset of) Borel measurable null set.*

Improper Riemann Integrals

- The Lebesgue integral extends the (*proper*) Riemann integral. However, there is a further extension of the Riemann integral which cannot be captured by Lebesgue's theory. u is Lebesgue integrable *iff* $|u|$ ha finite Lebesgue integral. ¡- The Lebesgue integral does not respect sign-changes and cancellations. However, the following *improper Riemann integral* does:

$$(R)\int_0^\infty u(x)dx := \lim_{n\to\infty}(R)\int_0^a u(x)dx. \tag{15}$$

Corollary 9.8. *Let $u : [0, \infty) \to \mathbb{R}$ be a measurable, Riemann integrable function for every interval $[0, N]$, $N \in \mathbb{N}$. Then $u \in \mathcal{L}^1[0, \infty)$ **iff***

$$\lim_{N\to\infty}(R)\int_0^N |u(x)|dx < \infty. \tag{16}$$

In this case, $(R)\int_0^\infty u(x)dx = \int_{[0,\infty)} u d\lambda$

Example of a function which is *improperly Riemann integrable* but **not** *Lebesgue integrable*:

$$f(x) = \frac{\sin(x)}{x}. \tag{17}$$

Proposition 9.9 (appearing as example 12.13 in Schilling)**.** *Let $f_\alpha(x) := x^\alpha, x > 0$ and $\alpha \in \mathbb{R}$. Then*

(i) $f_(\alpha) \in \mathcal{L}^1(0, 1) \Leftrightarrow \alpha > -1$.

(ii) $f_(\alpha) \in \mathcal{L}^1[1, \infty) \Leftrightarrow \alpha < -1$.

10 Regularity of Measures (App. H, [Schilling(2017)])

We let (X, d) be a metric space and denote by $\mathcal{O}$ the open, by $\mathcal{C}$ the closed and $\mathscr{B}(X) = \sigma(\mathcal{O})$ the Borel set of X.

Definition 10.1. A measure μ on $(X, d, \mathscr{B}(X))$ is called outer regular, if

$$\mu(B) = \inf\{\mu(U) \mid B \subset U,\ U \text{ open}\} \tag{18}$$

and inner regular, if $\mu(K) < \infty$ for all compact sets $K \subset X$ and

$$\mu(U) = \sup\{\mu(K) \mid K \subset U,\ K \text{ compact}\}. \tag{19}$$

A measure which is both inner and outer regular is called **regular**. We write $\mathfrak{m}_r^+(X)$ for the family of regular measures on $(X, \mathscr{B}(X))$.

Remark. *The space X is called σ-compact if there is a sequence of compact sets $K_n \uparrow X$. A typical example of such a space is a locally compact, separable metric space.*

Theorem 10.2. *Let (X, d) be a metric space. Every finite measure μ on $(X, \mathscr{B}(X))$ is outer regular. If X is σ-compact, then μ is also inner regular, hence regular.*

Theorem 10.3. *Let (X, d) be a metric space and μ be a measure on $(X, B(X))$ such that $\mu(K) < \infty$ for all compact sets $K \subset X$.*

1 If X is σ-compact, then μ is inner regular.

2 If there exists a sequence $G_n \in \mathcal{O}$, $G_n \uparrow X$ such that $\mu(G_n) < \infty$, then μ is outer regular.

11 The Function Spaces $\mathcal{L}^p$ (13, [Schilling(2017)])

Assume V is a vector space over $\mathbb{K} \in \{\mathbb{C}, \mathbb{R}\}$.

Definition 11.1. A seminorn on V is a map $p : V \to [0, +\infty)$ s.t.

(1) $p(cx) = |c|p(x) \quad \forall x \in V, \forall c \in \mathbb{K}$.

(2) $p(x + y) \leq p(x) + p(y) \quad \forall x, y \in V$. **triangle inequality**.

A seminorm is called a norm if we also have

$$p(x) = 0 \iff x = 0.$$

A norm is commonly denoted $||x||$, and a vectorspace equipped with a norm is called a **normed space**.

Definition 11.2. Assume (X,d) is a measure space. Fix $1 \le p \le \infty$. For every measurable function $f : X \to \mathbb{C}$ we define the following

$$||f||_p = \left(\int_X |f|^p d\mu\right)^{1/p} \in [0,+\infty]. \tag{20}$$

We can see that $||cf||_p = |c|||f||_p \;\; \forall c \in \mathbb{C}$.

Notice that by Theorem 8.2(i) we have that $||f||_p = 0 \Rightarrow f = 0$ a.e. Consider for example $\lim_{n\to\infty} ||f_n - f||_p = 0$, then we can find a subsequence s.t. $\lim_{k\to\infty} |f_{n(k)} - f| = 0$ a.e., i.e. $\lim_{k\to\infty} f_{n(k)} = f$ a.e.

Theorem 11.3 (Hölder's inequality)**.** *Assume that $u \in \mathcal{L}^p(\mu)$ and $v \in \mathcal{L}^q(\mu)$, where $1/p+1/q = 1$ and $p, q \in [0,+\infty]$. Then $uv \in \mathcal{L}^1(\mu)$, and the following inequality holds:*

$$\left|\int uv d\mu\right| \le \int |uv| d\mu = ||uv||_1 \le ||u||_p \cdot ||v||_q.$$

The generalized version reads:

$$\int |u_1 \cdot u_2 \cdots u_N| d\mu \le ||u_1||_{p_1} \cdot ||u_2||_{p_2} \cdots ||u_N||_{p_N}.$$

Proof. Assume $||f||_p > 0$, $||g||_q > 0$, otherwise $fg = 0$ (μ-a.e.). We have

$$\frac{|fg|}{||f||_p||g||_q} \le \frac{1}{p}\frac{|f|^p}{||f||_p^p} + \frac{1}{q}\frac{|g|^1}{||g||_q^q}.$$

Integrating over X we get

$$\begin{aligned}\frac{||fg||_1}{||f||_p||g||_q} &\le \frac{1}{p}\int_X \frac{|f|^p}{||f||_p^p} d\mu + \frac{1}{q}\int_X \frac{|g|^p}{||g||_p^p} d\mu \\ &= \frac{1}{p}\frac{||f||_p^p}{||f||_p^p} + \frac{1}{q}\frac{||g||_q^q}{||g||_q^q} \\ &= \frac{1}{p} + \frac{1}{q} = 1,\end{aligned}$$

so $||gf||_1 \le ||f||_p||g||_q$ □

Lemma 11.4.

$$||f+g||_p \le ||f||_p + ||g||_p. \tag{21}$$

Definition 11.5. We define

$$\mathcal{L}^p(X, d\mu) = \left\{f : X \to \mathbb{C} \;\mid\; f \text{ is measurable and } ||f||_p < \infty\right\}.$$

This is a vectorspace with seminorm $f \mapsto ||f||_p$. And in general this is not a normed space, since $||f||_p = 0 \iff f = 0$ a.e.

Generally, if p is a seminorm on a vectorspace V, then

$$V_0 = \{x \in V \;\mid\; p(x) = 0\} \tag{22}$$

which is a subspace of V. Then we consider the quotient/factor space V/V_0.

Definition 11.6. For $x, y \in V$, define

$$x \sim y \iff x - y \in V_0. \tag{23}$$

This is an equivalence relation on V. The representation class of V is defined by $[x]$ or $x + V_0$.

Then V/V_0 is equals the set of equivalence classes. We can show that it is a normed space.

$$[x] + [y] = [x+y] \;\;,\;\; c[x] = [cx] \;\;,\;\; ||[x]|| = p(x).$$

Applying this to $\mathcal{L}^p(X, d\mu)$ we get the normed space

$$L^p(X, d\mu) := \mathcal{L}^p(X, d\mu)/\mathcal{N} = \mathcal{L}^p(X, d\mu)/_\sim. \tag{24}$$

Where $\mathcal{N}$ is the space of measurable functions f s.t. $f = 0$ a.e. The equivalence relation $\sim$ is defined by

$$u \sim v \iff \{u \ne v\} \in \mathcal{N}_\mu \iff \mu\{u \ne v\} = 0,$$

and so $L^p(X, d\mu)$ consists of *all equivalence classes* $[u]_p = \{v \in \mathcal{L}^p | u \sim v\}$. So for every $u \in [u]_p$ there is no $v \in [u]_p$ such that $\mu\{u \ne v\} \ne 0$.

We will further continue to denote the norm by $||\cdot||_p$, and we will normally **not** distinguish between $f \in \mathcal{L}^p(X, d\mu)$ and the vector in $L^p(X, d\mu)$ that f defines.

Definition 11.7. A normed space $(X, ||\cdot||)$ is called a Banach space if V is complete w.r.t the metric $d(x,y) = ||x - y||$.

Theorem 11.8. *If $(X, \mathscr{B}, \mu)$ is a measure space, $1 \le p \le \infty$, then $L^p(X, d\mu)$ is a Banach space.*

Definition 11.9. A measurable function $f : X \to \mathbb{C}$ is called **essentially bounded** if there is $c \ge 0$ s.t.

$$\mu(\{x \;:\; |f(x)| > c\}) = 0. \tag{25}$$

That is $|f| \le c$ a.e. The smallest such c is called the essential supremum of f and is denoted by $||f||_\infty$. That is,

$$||u||_\infty := \inf\{c > 0 : \mu\{|u| \ge c\} = 0\},$$

and from problem 13.21 we have

$$\lim_{p\to\infty} ||\cdot||_p = ||\cdot||_\infty.$$

Definition 11.10.

$$\mathcal{L}^\infty(X, d\mu) = \{f : X \to \mathbb{C} \;\mid\; f \text{ is measurable and } ||f||_\infty < \infty\}.$$

$$L^\infty(X, d\mu) = \mathcal{L}^\infty(X, d\mu)/\mathcal{N}.$$

Where by the previous definiton these spaces become the spaces of all essentially bounded functions.

Theorem 11.11. *If $(X, \mathscr{B}, \mu)$ is a σ-finite measure space, then $L^\infty(X, d\mu)$ is a Banach space.*

Convergence in $\mathcal{L}^p$ and completeness

Lemma 11.12. *For any sequence $(u_n)_{n\in\mathbb{N}} \subset \mathcal{L}^p(\mu)$, $p \in [1,\infty)$, of positive functions $u_n \ge 0$ we have*

$$\left\|\sum_{n=1}^\infty u_n\right\|_p \le \sum_{n=1}^\infty ||u_n||_p.$$

Theorem 11.13 (Riesz-Fischer)**.** *The spaces $\mathcal{L}^p(\mu)$, $p \in [1,\infty)$, are complete, i.e. every Cauchy sequence $(u_n)_{n\in\mathbb{N}} \subset \mathcal{L}^p(\mu)$ converges to some limit $u \in \mathcal{L}^p(\mu)$*

Corollary 11.14. *Let $(u_n)_{n\in\mathbb{N}} \subset \mathcal{L}^p(\mu), p \in [1,\infty)$ with $\mathcal{L}^p - \lim_{n\to\infty} u_n = u$. Then there exists a subsequence $(u_{n_k})_{k\in\mathbb{N}}$ s.t. $\lim_{k\to\infty} u_{n_k}(x) = u(x)$ holds for almost every $x \in X$.*

Theorem 11.15. *Let $(u_n)_{n\in\mathbb{N}} \subset \mathcal{L}^p(\mu), p \in [1,\infty)$, be a sequence of functions s.t. $|u_n| \le w \; \forall n \in \mathbb{N}$ and some $w \in \mathcal{L}^p(\mu)$. If $u(x) = \lim_{n\to\infty} u_n(x)$ exists for (almost) every $x \in X$, then*

$$u \in \mathcal{L}^p \text{ and } \lim_{n\to\infty} ||u - u_n||_p = 0.$$

Theorem 11.16 (F. Riesz (convergence theorem))**.** *Let $(u_n)_{n\in\mathbb{N}} \subset \mathcal{L}^p(\mu), p \in [1,\infty)$, be a sequence s.t. $\lim_{n\to\infty} u_n(x) = u(x)$ for almost every $x \in X$ and some $u \in \mathcal{L}^p(\mu)$. Then*

$$\lim_{n\to\infty} ||u_n - u||_p = 0 \Longleftrightarrow \lim_{n\to\infty} ||u_n||_p = ||u||_p.$$

12 Dense and Determining Sets (17, [Schilling(2017)])

Definition 12.1 (Dense Sets)**.** A set $\mathcal{D} \subset \mathcal{L}^p(\mu), p \in [0,\infty]$, is called *dense* if for every $u \in \mathcal{L}^p(\mu)$ there exist a sequence $(f_n)_{n\in\mathbb{N}} \subset \mathcal{D}$ s.t. $\lim_{n\to\infty} ||u - f_n||_p = 0$.

Definition 12.2 (Support)**.** The support of a function f is the set of points in X where f is non-zero:

$$\operatorname{supp}(f) := \{x \in X : f(x) \neq 0\}.$$

Theorem 12.3. *Assume X, d is a metric space and μ is a Borel measure that is finite on every ball $1 \le p < \infty$. Then the space of bounded continuous functions with bounded support is dense in $\mathcal{L}^p(X, d\mu)$. Where bounded support means that f vanishes outside some ball.*

Proof. We want to approximate $f \in \mathcal{L}^p(X, d\mu)$ by bounded continuous functions with bounded support. By considering separately $(\mathrm{Re}(f))_I$ and $(\mathrm{Im}(f))_I$ we may assume that $f \ge 0$. Then we can find simple functions f_n s.t. $0 \le f_n \le f, f_n \to f$ pointwise. As $|f - f_n|^p \le |f|^p$, by the dominated convergence theorem we have $f_n \to f \in \mathcal{L}^p(X, d\mu)$. Hence, it suffices to consider simple f, but then it suffices to approximate $f = \pi_A$. Note that $\pi_A \in \mathcal{L}^p(X, d\mu)$ iff $\mu(A) < \infty$.

Fix $x_0 \in X$. Then $\pi_{A\cap B_n(x_0)} \nearrow \pi_A$ pointwise, hence $\pi_{A\cap B_n(x_0)} \to \pi_A \in \mathcal{L}^p(X, d\mu)$, again by the dominated convergence theorem.

Therefor it suffices to consider $A \subset B_n(x_0)$. As μ is outer regular, we have

$$\mu(A) = \inf_{\substack{A\subset U\subset B_n(x_0)\\ U \text{ is open}}} \mu(U).$$

Note that $||\pi_U - \pi_A||_p = \mu(U\backslash A)^{1/p}$. Hence, we can choose $U_k \subset B_n(x_0)$ s.t. $A \subset U_k$, U_k is open, $\pi_{U_k} \to \pi_A \in \mathcal{L}^p(X, d\mu)$.

Therefor it suffices to approximate π_U for open $U \subset B_n(x_0)$. Consider the functions

$$f_k(x) = \frac{kd(x, U^c)}{1 + kd(x, U^c)}.$$

Then $0 \le f_k \le 1$, f_k is continuous, supported on $\bar{U} \subset \bar{B}_n(x_0)$ and $f_k \nearrow \pi_U$ pointwise, hence $f_k \xrightarrow[k\to\infty]{} \pi_U \in \mathcal{L}^p(X, d\mu)$. □

Theorem 12.4. *Assume (X, d) is a separable locally compact metric space and μ is a Borel Measure on X s.t. $\mu(K) < \infty \; \forall$ compact $K \subset K$. Then the space $C_c(X)$ of continuous compactly supported functions is dense in $\mathcal{L}^p(X, d\mu)$.*

Recall that the support of a function f is $\operatorname{supp}(f) = \{x \in X : f(x) \neq 0\}$, *closed support* is the closure of $\operatorname{supp}(f)$ (i.e. boundary points are included), often just written as $\operatorname{supp}(f)$, and a function is said to have *compact support* if $\operatorname{supp}(f)$ is *compact.*

In particular, either theorem shows that if μ is a Borel measure on $\mathbb{R}^n$ s.t. the measure of every ball is finite, then $C_c(\mathbb{R}^n)$ is dense in $\mathcal{L}^p(\mathbb{R}^n, d\mu)$, $1 \le p < \infty$. Later we will see that even $C_c^\infty(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n, d\mu)$.

Remark. *These results do not extend to $p = \infty$ in general.*

For $\mu = \lambda_n$ we write simply $\mathcal{L}^p(\mathbb{R}^n)$.

Remark. *Theorem 17.8 in the book is WRONG. For example, $X = \mathbb{Q}$ with the usual metric is σ-compact, supports nonzero finite measure, but $C_c(\mathbb{Q}) = 0$.*

Modes of Convergence

(mixture of ex. 11.12 and ch. 22 p. 258-261. in [Schilling(2017)])

Definition 12.5 (convergence in meassure)**.** A sequence of measurable functions $u_n : X \to \bar{\mathbb{R}}$ *converges in measure* if

$$\forall\epsilon > 0 \forall A \in \mathscr{A}, \mu(A) < \infty : \lim_{n\to\infty} \mu(\{|u_n - u| > \epsilon\} \cap A) = 0$$

holds for some $u \in \mathcal{M}(\mathscr{A})$. We write $\mu\text{-}\lim_{n\to\infty u_n = u}$ or $u_n \xrightarrow{\mu} u$.

Assume $(X, \mathscr{B}, \mu)$ is a measure space. Given measurable functions $f_n, f : X \to \mathbb{C}$, recall that

$$f_n \to f \text{ a.e.}$$

means that $f_n(x) \xrightarrow[n\to\infty]{} f(x)$ for all x outside a set of measure zero.

Theorem 12.6 (Egorov)**.** *Assume $\mu(X) < \infty$ and $f_n \to f$ a.e. Then, $\forall\epsilon > 0$, there exists $X_\epsilon \in \mathscr{B}$ s.t. $\mu(X_\epsilon) < \epsilon$ and $f_n \to f$ uniformly on $X\backslash X_\epsilon$.*

In addition to pointwise and uniform convergence we also consider the following:

$f_n \to f$ in the *p-th mean* if $||f_n - f||_p \xrightarrow[n\to\infty]{} 0$. For $p = 1$ we say *in mean*, for $p = 2$ we say in quadratic mean.

$f_n \to f$ *in measure* if $\forall\epsilon > 0$ we have

$$\mu(\{x : |f_n(x) - f(x)| \ge \epsilon\}) \xrightarrow[n\to\infty]{} 0.$$

Theorem 12.7 (Lemma 22.4 in the book?)**.** *Assume $(X, \mathscr{B}, d\mu)$ is a measure space, $1 \le p < \infty$, $f_n, f : X \to \mathbb{C}$ are measurable functions. Then*

(i) If $f_n \to f$ in the p-th mean, then $f_n \to f$ in measure.

(ii) If $f_n \to f$ in measure, then there is a subsequence $(f_{n_k})_{k=1}^\infty$ s.t. $f_{n_k} \to f$ a.e.

(iii) If $f_n \to f$ a.e. and $\mu(X) < \infty$, then $f_n \to f$ in measure.

In particular, if $f_n \to f$ in the p-th mean, then $f_{n_k} \to f$ a.e. for a subsequence $(f_{n_k})_k$.

13 Abstract Hilbert Spaces (26, [Schilling(2017)])

Assume $\mathcal{H}$ is a vector space over $\mathbb{C}$.

Definition 13.1. A pre-inner product on $\mathcal{H}$ is a map $(\cdot,\cdot) : H\times H \to \mathbb{C}$ which is

(i) Sesquilinear: linear in the first variable and antilinear in the second:

$$(\alpha u+\beta v, w) = \alpha(u,w)+\beta(v,w),$$
$$(w, \alpha u+\beta v) = \bar{\alpha}(w,u)+\bar{\beta}(w,v),\ u,v,w\in H \text{ and } \alpha,\beta\in\mathbb{C}.$$

(ii) Hermitian: $(u,v) = \overline{(u,v)}$.

(iii) Positive semidefinite: $(u,v) \geq 0$.

It is called an **inner product**, or a scalar product, if instead of (iii) the map is positive definite; $(u,v) > 0$. This definition also works for $\mathbb{R}$ instead of $\mathbb{C}$.

Cauchy-Schwartz inequality If $(\cdot,\cdot)$ is a pre-inner product, then $|(u,v)| \leq (u,u)^{1/2}(v,v)^{1/2}$.

Corollary 13.2. *Assume we have a seminorm $||u|| := (u,u)^{1/2}$. It is a norm iff $(\cdot,\cdot)$ is an inner product.*

Definition 13.3 (Hilbert space)**.** A Hilbert space is a complex vector space $\mathcal{H}$ with an inner product $(\cdot,\cdot)$ s.t. $\mathcal{H}$ is complete with respect to the norm $||u|| = (u,u)^{1/2}$.

1. The norm on a Hilbert space is determined by the inner product, but the inner product can also be recovered by the norm by the *polarization identity*: $(u,v) = \frac{||u+v||^2-||u-v||^2}{4} + i\frac{||u+iv||^2-||u-iv||^2}{4}$.
2. *Parallelogram law*: $||u+v||^2+||u-v||^2 = 2||u||^2+2||v||^2$.
3. A norm on a vector space is given by an inner product iff it satisfies the parallelogram law, and then the scalar product is uniquely determined by the polarization identity.

Example 13.4. Assume $(X,\mathscr{B},\mu)$ is a measure space. Then $\mathcal{L}^2(X,d\mu)$ is a Hilbert space with inner product

$$(f,g) = \int_X f\bar{g}d\mu.$$

This is well-defined, as $|f\bar{g}| \leq \frac{1}{2}(|f|^2+|g|^2)$.

In particular, if $\mathscr{B} = \mathcal{P}(X)$ and μ is the counting measure, then $L^2(X,d\mu)$ is denoted by $l^2(X)$; for $X=\mathbb{N}$ we write simply l^2. Note that in this case for $f : X \to [0,+\infty]$ we have

$$\int_X f d\mu = \sum_{x\in X} f(x) := \sup_{\substack{F\subset X\\ F \text{ is finite}}} \sum_{x\in F} f(x),$$

and if $\sum_{x\in X} f(x) < \infty$, then $\{x : f(x) > 0\}$ is at most countable, so $\sum_{x\in X} f(x) = \sum_{x:f(x)>0} f(x)$ is the usual sum of a series.

Recall that a subset C of a vector space is called *convex* if

$$u,w\in C \to tu+(1-t)w\in C\ \forall t\in(0,1).$$

The following is one of the key properties of the Hilbert space

Theorem 13.5 (projection theorem)**.** *Assume $\mathcal{H}$ is a Hilbert space and $C\subset H$ is a closed convex subset. Then for every $u\in H$ there is a unique $u_0\in C$ (*minimizer*) s.t.*

$$||u-u_0|| = d(u,C)(= \inf_{x\in C} ||u-x||).$$

Proof. Let $d = d(u,C)$. Choose $u_n\in C$ s.t. $||u-u_n|| \to d$. We claim that $(u_n)_{n=1}^{\infty}$ is a Cauchy sequence. As $\frac{u_n+u_2}{2}\in C$, we have

$$d^2 \leq ||u-\frac{u_n+u_m}{2}||^2 = \frac{1}{4}||(u-u_m)+(u-u_m)||^2$$
$$\underset{\text{parallelogram law}}{=} \frac{1}{4}\left(2||u-u_n||^2+2||u-u_m||^2-||u_n-u_m||^2\right),$$

so

$$||u_n-u_m||^2 \leq 2\left(||u-u_n||^2-d^2\right)+2\left(||u-u_m||^2-d^2\right).$$

Thus, $(u_n)_{n\in\mathbb{N}}$ is indeed Cauchy, hence $u_n\to u_0\in C$ for some u_0 and

$$||u-u_0|| = \lim_{n\to\infty} ||u-u_m|| = d = d(u,C).$$

If u_0' is another such point, we can take $u_{2n} = u_0, u_{2n+1} = u_0'$ and conclude that $u_0 = u_0'$. □

14 Orthogonal Projections (26, [Schilling(2017)])

For a Hilbert space $\mathcal{H}$ and a subset $A\subset H$, let

$$A^\perp := \{x\in H : x\perp y\ \forall y\in A\},$$

where $x\perp y$ means that $(x,y)=0$. $A^\perp$ is a closed subspace of $\mathcal{H}$.

Proposition 14.1. *Assume $\mathcal{H}_0$ is a closed subspace of a Hilbert space $\mathcal{H}$. Then every $u\in H$ uniquely decomposes as*

$$u = u_0+u_1,\ \textit{with } u_0\in H \textit{ and } u_1\in\mathcal{H}_0^\perp.$$

Moreover, $||u-u_0|| = d(u,\mathcal{H}_0)$ and $||u||^2 = ||u_0||^2+||u_1||^2$.

For a closed subspace $\mathcal{H}_0\subset\mathcal{H}$, consider the map $P : H\to\mathcal{H}_0$ s.t. $Pu\in\mathcal{H}_0$ is the unique element satisfying $u-Pu = H_0^\perp$. The operator P is linear. It is also contractive, meaning that $||Pu|| \leq ||u||$, since $||u||^2 = ||Pu||^2+||u-Pu||^2$. It is called the orthogonal projection onto $\mathcal{H}_0$.

If $\mathcal{H}_0$ is finite dimensional with an orthonormal basis $u_1,...,u_n$ then

$$Pu = \sum_{k=1}^{n}(u,u_k)u_k.$$

Orthonormal bases can be defined for arbitrary Hilbert spaces.

Definition 14.2 (orthonormal system)**.** An orthonormal system in $\mathcal{H}$ is a collection of vectors $u_i\in H$ $(i\in I)$s.t.

$$(u_i,u_j) = \delta_{ij}\ \forall i,j\in I.$$

It is called an *orthonormal basis* if span$\{u_i\}_{i\in I}$ denotes the linear span of $\{u_i\}_{i\in I}$, the space of finite linear combinations of the vectors u_i.

Definition 14.3. A Hilbert space $\mathcal{H}$ is said to be *separable* if $\mathcal{H}$ contains a countable dense subset $G\subset\mathcal{H}$.

Theorem 14.4. *Every Hilbert space $\mathcal{H}$ has an orthonormal basis. If $\mathcal{H}$ is separable, then there is a countable orthonormal basis.*

Proposition 14.5. *Assume $\{u_i\}_{i\in I}$ is an orthonormal system in a Hilbert space H. Take $u \in \mathcal{H}$. Then*

(i) Bessel's inequality*: $\sum_{i\in I} |(u,u_i)|^2 \le ||u||^2$, in particular, $\{i : (u,u_i) \neq 0\}$ is countable.*

(ii) Parseval's identity*: If $\{u_i\}_{i\in I}$ is an orthonormal basis, then $\sum_{i\in I} |(u,u_i)|^2 = ||u||^2$.*

If $(u_i)_{i\in I}$ is an orthonormal basis, then the numbers (u,u_i) are called the **Fourier coefficients** of u with respect to $(u_i)_{i\in I}$. The Parseval identity then suggests that u is determined by its Fourier coefficients. This is true, and even more, we have:

Proposition 14.6. *Assume $(u_i)_{i\in I}$ is an orthonormal basis in a Hilbert space $\mathcal{H}$. Then for every vector $(c_i)_{i\in I} \in l^2(I)$ there is a unique vector $u \in \mathcal{H}$ with Fourier coefficients c_i, and we write*

$$u = \sum_{i\in I} c_i u_i.$$

Remark. *Equivalently, the element $u = \sum_{i\in I} c_i u_i$ can be described as the unique element in $\mathcal{H}$ s.t. $\forall \epsilon > 0$ there is a finite $F_0 \subset I$ s.t. $||u - \sum_{i\in F} c_i u_i|| < \epsilon$ $\forall$ finite $F \supset F_0$.*

Corollary 14.7. *We have a linear isomorphism $U : l^2(I) \xrightarrow{\sim} \mathcal{H}$, $U((c_i)_{i\in I}) = \sum_{i\in I} c_i u_i$. By Parseval's identity this isomorphism is* isometric*, that is, $||Ux|| = ||x||$ $\forall x \in l^2(I)$. By the polarization identity this is equivalent to*

$$(Ux, Uy) = (x, y)\ \forall x, y \in l^2(I).$$

Therefor U is unitary.

Corollary 14.8. *Up to a unitary isomorphism, there is only one infinite dimensional separable Hilbert space, namely, l^2.*

15 Dual spaces (26, [Schilling(2017)])

Given two orthonormal bases $(u_i)_{i\in I}$ and $(v_i)_{i\in I}$ in a Hilbert space $\mathcal{H}$, we can decompose

$$u_i = \sum_{j\in I} (u_i, v_j) v_j$$

and using that the sets $\{j : (u_i, v_j) \neq 0\}$ are countable proove the following:

Claim: Any two orthonormal bases in a Hilbert space have the same cardinality.

Example 15.1 (classical Fourier series)**.** Consider $\mathcal{H} = L^2(0, 2\bar{\mu}) = L^2((0,2\bar{\mu}), d\lambda)$. For $n \in \mathbb{Z}$, define $e_n(t) = \frac{1}{\sqrt{2\bar{\mu}}} e^{int}$. By a version of Weierstrass' theorem it is known that span$\{e_n\}_{n\in\mathbb{Z}}$ is dense in the supremum-norm in

$$\{f \in C[0, 2\bar{\mu}] : f(0) = f(2\bar{\mu})\}.$$

As $C[0, 2\bar{\mu}]$ is dense in $L^2(0, 2\bar{\mu})$, from this one can deduce that span$\{e_n\}_{n\in\mathbb{Z}}$ is dense in $L^2(0, 2\bar{\mu})$. We then see that $(e_n)_{n\in\mathbb{Z}}$ is an orthonormal basis in $L^2(0, 2\bar{\mu})$. We therefor have a unitary isomorphism

$$l^2(\mathbb{Z}) \xrightarrow{\sim} L^2(0, 2\bar{\mu}), (c_n)_{n\in\mathbb{Z}} \mapsto \sum_{n\in\mathbb{Z}} c_n e_n.$$

The Fourier coefficients at $f \in L^2(0, 2\bar{\mu})$ are denoted by $\hat{f}(n)$, so

$$\hat{f}(n) = \frac{1}{\sqrt{2\bar{\mu}}} \int_0^{2\bar{\mu}} f(t) e^{-int} dt,$$

more practically

$$\hat{f}(n) = \frac{1}{\sqrt{2\bar{\mu}}} \int_{[0,2\bar{\mu})} f(t) e^{-int} d\lambda(t).$$

Therefor we have $f = \sum_{n\in\mathbb{Z}} \hat{f}(n) e_n$ in $L^2(0, 2\bar{\mu})$.

Fact: For every $f \in L^2(0, 2\bar{\mu})$, we have $\frac{1}{\sqrt{2\bar{\mu}}} \sum_{n=-N}^{N} \hat{f}(n) e^{int} \xrightarrow[N\to\infty]{} f(t)$ for a.e. t.

Lemma 15.2. *Assume V is a normed space over $K = \mathbb{R}$ or $K = \mathbb{C}$. Consider a linear functional $f : V \to K$. The following are equivalent (TFAE):*

(i) f is continuous;

(ii) f is continuous at 0;

(iii) There is a $c \ge 0$ s.t. $|f(x)| \le c||x||$ $\forall x \in V$.

If (i)-(iii) are satisfied, then f is called a *bounded linear functional.* The constant c in (iii) is denoted by $||f||$. We have $||f|| = \sup_{x\neq 0} \frac{|f(x)|}{||x||} = \sup_{||x||=1} |f(x)| = \sup_{||x||\le 1} |f(x)|$.

Proposition 15.3. *For every normed vector space V over $K = \mathbb{R}$ or $K = \mathbb{C}$, the bounded linear functionals on V form a Banach space V^*.*

Remark. *The sequence $\{||f_n - f_m||\}_{m=1}^{\infty}$ actually converges, since*

$$\Big| ||f_n - f_m|| \Big| \le ||f_m - f_n||.$$

When we study/use normed spaces, it is often important to understand the dual spaces. For Hilbert spaces this is particularly easy:

Theorem 15.4 (Riesz)**.** *Assume $\mathcal{H}$ is a Hilbert space. Then every $f \in \mathcal{H}^*$ has the form*

$$f(x) = (x, y),$$

for a uniquely defined $y \in \mathcal{H}$. Moreover, we have $||f|| = ||y||$.

For every Hilbert space $\mathcal{H}$ we can define the *conjugate Hilbert space* $\bar{\mathcal{H}}$, which has its elements as the symbols $\bar{x}$ for $x \in \mathcal{H}$, with the linear structure and inner product defined by

$$\bar{x} + \bar{y} = \overline{x + y}, c \cdot \bar{x} = \overline{\bar{c}x}, (\bar{x}, \bar{y}) = \overline{(x, y)} = (y, x).$$

Corollary 15.5. *For every Hilbert space $\mathcal{H}$, we have an isometric isomorphism $\bar{\mathcal{H}} \xrightarrow{\sim} \mathcal{H}^*$, $\bar{x} \mapsto (\cdot, x)$.*

16 Hahn-Banach Theorem (4.2, [Teschl(2010)])

Theorem 16.1 (Hahn-Banach)**.** *Assume V is a real vector space, $V_0 \subset V$ a subspace, $e : V \to \mathbb{R}$ a convex function and $f : V_0 \to \mathbb{R}$ a linear functional s.t. $f \le e$ on V_0. Then f can be extended to a linear functional F on V s.t. $F \le e$.*

Assume first that $V = V_0 + \mathbb{R}x$ for some $x \in V \backslash V_0$. To define F we need to specify $F(x)$. The condition $F \le e$ means that we need

$$F(y \pm tx) \le e(y \pm tx) \ \forall y \in V_0, t \ge 0,$$

that is

$$\pm tF(x) + f(y) \le e(y \pm tx).$$

Dividing by t this is equivalent to

$$\begin{cases} F(x) \le \frac{e(y+tx)-f(y)}{t}, \\ F(x) \ge \frac{f(y)-e(y-tx)}{t}, \end{cases}$$

$\forall y \in V_0, t \ge 0$. To show that there is a number satisfying these inequalities, we need to check that

$$\sup_{\substack{y \in V_0 \\ t > 0}} \frac{f(y) - e(y - tx)}{t} \le \inf_{\substack{y \in V_0 \\ t > 0}} \frac{e(y + tx) - f(y)}{t}.$$

Then as $F(x)$ we can take any number in the interval $[\sup, \inf]$. In other words, we need to check that

$$\frac{f(y) - e(y - tx)}{t} \le \frac{e(z + sx) - f(z)}{s},$$

$\forall y, z \in V_0, t, s \ge 0$. Equivalently,

$$sf(y) + tf(z) \le se(y - tx) + te(z + sx).$$

We have

$$\begin{aligned} \frac{sf(y) + tf(z)}{s+t} &= f\left(\frac{s}{s+t}y + \frac{t}{s+t}z\right) \\ &\le e\left(\frac{s}{s+t}y + \frac{t}{s+t}z\right) \\ &\le e\left(\frac{s}{s+t}(y - tx) + \frac{t}{s+t}(z + sx)\right) \\ &\le \frac{s}{s+t}e\left(y - tx\right) + \frac{t}{s+t}e\left(z + sx\right), \end{aligned}$$

which is what we need.

The general case is deduced from this using transitive induction (Zorn's lemma), as follows. Consider the set X of pairs (W, F), where $w \subset V$ is a subspace, $F : W \to \mathbb{R}$ a linear functional, $V_0 \subset W$ and $F|_{v_0} = f$. Define a partial order on X by $F \le e$ on W.

$$(W_1, F_1) \le (W_2, F_2) \text{ iff } W_1 \subset W_2 \text{ and } F_1 = F_2|_{W_1}.$$

If C is a chain in X, then it has an upper bound (W, F) defined by

$$W = \bigcup_{(Y,\rho) \in C} Y, F(y) = \rho(y) \text{ if } y \in Y, (Y, \rho) \in C.$$

Hence, X has a maximal element (W, F) by Zorn's lemma. If $W \neq V$, then we can take $x \in V \backslash W$ and extend F to $W + \mathbb{R}x$ preserving the inequality $F \le e$. This contradicts maximality of W, F in X. Hence, $W = V$.

Theorem 16.2 (Hahn-Banach)**.** *Assume V is a real or complex vector space, p a seminorm on V_0, $V_0 \subset$, and f a linear functional on V_0 s.t.*

$$|f(x)| \le p(x) \ \forall x \in V_0.$$

Then f can be extended to a linear functional F on V s.t. $|F(x)| \le p(x) \ \forall x \in V$.

Corollary 16.3. *Assume V is a normed real or complex vector space, $V_0 \subset V$ and $f \in V_0^*$. Then there is a $F \in V^*$ s.t.*

$$F|_{V_0} f \text{ and } ||F|| = ||f||.$$

Proof. We apply the previous theorem to $p(x) = ||f|| \cdot ||x||$. □

Corollary 16.4. *Assume V is a normed space and $x \in V, x \neq 0$. Then there is a $F \in V^*$ s.t. $||F|| = 1$ and $F(x) = ||x||$.*

Such an F is called a *supporting functional at x.*

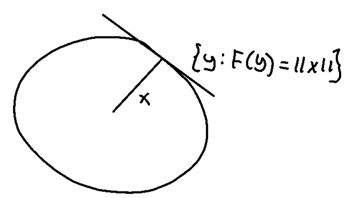

Figure 1: Tangent space?

If V is a normed vector space, then every $x \in X$ defines a bounded linear functional on V^* by

$$V^* \ni f \mapsto f(x).$$

As $|f(x)| \le ||f|| \cdot ||x||$, this functional has norm $\le ||x||$. By using a supporting functional at x, we actually see that we get norm $||x||$. Thus, we have an isometric embedding $V \subset V^{**} := (V^*)^*$. We can therefor see view V as a subspace of V^{**}.

Definition 16.5. A normed space V is called reflexive if $V^{**} = V$.

Remark. *This is stronger than requiring $V \simeq V^{**}$.*

Remark. *Every Hilbert space $\mathcal{H}$ is reflexive. Indeed, $\mathcal{H}^* = \bar{\mathcal{H}}$. By Riesz' theorem every bounded linear functional f on $\bar{\mathcal{H}}$ has the form*

$$f(\bar{x}) = (\bar{x}, \bar{y}) = (y, x),$$

for some $y \in \mathcal{H}$, which exactly means that $f = y$ in $\mathcal{H}^{**}$.

As we will see later, the spaces $\mathcal{L}^p(X, d\mu)$, with μ σ-finite and $1 < p < \infty$, are reflexive. The spaces $\mathcal{L}'(X, d\mu)$ and $\mathcal{L}^\infty(X, \mu)$ are usually not reflexive.

17 Radon-Nikodym Theorem (20, [Schilling(2017)])

Assume $(X,\mathscr{B},\mu)$ is a measure space. Are there other measures on $(X,\mathscr{B})$?

Example 17.1. Take a measurable function $f : X \to [0,+\infty]$ and define

$$\nu(A) := \int_A f d\mu \text{ for } A \in \mathscr{B}.$$

This is a measure by the monotone convergence theorem. We write $d\nu = f d\mu$. Furthermore, we say that f is the ***Radon-Nikodym*** derivative, and we denote it by $f = d\nu/d\mu$. If $\mu = \lambda^1$ we get $f(x) = d\nu(x)/dx$.

Proposition 17.2. *Assume $(X,\mathscr{B})$ is a measurable space, μ and ν are σ-finite measures on $(X,\mathscr{B})$. Then there exist $N \in \mathscr{B}$ and a measurable $f : X \to [0,+\infty]$ s.t. $\mu(N) = 0$ and $\nu(A) = \nu(A\cap N) + \int_A f d\mu$ $\forall A \in \mathscr{B}$.*

Proof. Assume first that the measure ν,μ are finite. Consider the measure $\eta = \mu + \nu$ and define a linear functional ρ on $L^2(x,d\eta)$ by

$$\rho(g) = \int_X g d\nu.$$

It is well-defined and bounded, since

$$|\rho(g)| \le \underbrace{\int_X |g| d\nu}_{||g||_1} \underset{C.S}{\le} (\int_X |g|^2 d\nu)^{\frac{1}{2}} \nu(X)^{\frac{1}{2}}$$
$$\le \nu(x)^{\frac{1}{2}} (\int_X |g|^2 d\eta)^{\frac{1}{2}}.$$

By Riesz' theorem there exists $h \in L^2(x,d\eta)$ s.t. $\rho(g) = \int_X g h d\eta$ for all $g \in L^2(x,d\eta)$.

For $g = \mathbb{1}_A$ we get

$$\nu(A) = \int_A h d\eta \ \forall A \in \mathscr{B}.$$

In particular, $h \ge 0$ $(\eta -$ a.e.). As $\nu(A) \le \eta(A)$, we also have $h \le 1$ $(\eta -$ a.e.). From now on we view h as a function on X and assume $0 \le h \le 1$.

For $g = \mathbb{1}_A$ $(A \in \mathscr{B})$ we have

$$\int_X g d\nu = \rho(g) = \int_X g h d\eta = \int_X g h d\mu + \int_X g h d\nu,$$

hence

$$\int_X g(1-h) d\nu = \int_X g h d\mu. \tag{26}$$

By extending eq. 26 to positive simple functions and then using the monotone convergence theorem, we conclude that eq. 26 holds for all measurable $g : X \to [0,+\infty]$.

We now let

$$N = \{x : h(x) = 1\} \text{ and } f = \frac{h}{1-h}\mathbb{1}_{N^c}.$$

Letting $g = \mathbb{1}_N$ in eq. 26 we get $0 = \mu(N)$. For $A \in \mathscr{B}$, letting $g = \frac{\mathbb{1}_{A\cap N^c}}{1-h}$ in eq. 26 we get

$$\nu(A\cap N^c) = \int_X \frac{\mathbb{1}_A \mathbb{1}_{N^c}}{1-h} h d\mu = \int_X \mathbb{1}_A f d\mu = \int_A f d\mu.$$

Thus

$$\nu(A) = \nu(A\cap N) + \nu(A \cap N^c) = \nu(A\cap N) + \int_A f d\mu.$$

This finishes the proof for finite μ and ν.

If μ and ν are σ-finite, we can write X as disjoint unions $X = \cup_{n\in\mathbb{N}} X_n = \cup_{m\in\mathbb{N}} Y_m$ with $X_n, Y_m \in \mathscr{B}, \mu(X_n) < \infty, \nu(Y_m) < \infty$. Applying the first part of the proof to $X_n \cap Y_m$, we find $N_{nm} \in \mathscr{B}, N_{nm} \subset X_n \cap Y_m$, and measurable $f_{nm} : X_n \cap Y_m \to [0,+\infty]$ s.t. $\mu(N_{nm}) = 0$ and

$$\nu(A\cap X_n \cap Y_m) = \nu(A \cap N_{nm}) + \int_{A\cap X_n\cap Y_m} f_{nm} d\mu.$$

We then put $N = \cup_{n,m\in\mathbb{N}} N_{nm}$ and define $f : X \to [0,+\infty]$ by letting $f = f_{nm}$ on $X_n \cap Y_m$. □

When can we discard the term $\nu(A\cap N)$?

Definition 17.3. Given measure μ and ν on $X,\mathscr{B}$, we say that ν is *absolutely continuous* with respect to μ and write $\nu << \mu$, if $\nu(A) = 0$ whenever $A \in \mathscr{B}, \mu(A) = 0$.

Lemma 17.4. *Assume μ and ν are measures on $(X,\mathscr{B})$, $\nu(X) < \infty$. Then $\nu << \mu$ iff $\forall \epsilon > 0 \exists \delta > 0$ s.t. if $A \in \mathscr{B}$, $\mu(A) < \delta$, then $\nu(A) < \epsilon$.*

Proof. "⇒": obvious. "⇐": Assume this is not true. Then, there is a $\epsilon > 0$ s.t. $\forall \delta > 0$ we can find $A \in \mathscr{B}$ satisfying $\mu(A) < \delta$, $\nu(A) \ge \epsilon$. Let A_n be such a set A for $\delta = 1/2^n$. Put $A = \cap_{n\in\mathbb{N}} \cup_{k=n} A_k$. Then

$$\mu(A) \le \lim_{n\to\infty} \mu\left(\bigcup_{k=n}^{\infty} A_k\right) \le \lim_{n\to\infty} \sum_{k=n}^{\infty} \mu(A_k)$$
$$\le \lim_{n\to\infty} \sum_{k=n}^{\infty} \frac{1}{2^k} = \lim_{n\to} \frac{1}{2^{n-1}} = 0.$$

As $\nu(X) < \infty$, we also have

$$\nu(A) = \lim_{n\to\infty} \nu\left(\bigcup_{k=n}^{\infty} A_k\right) \ge \epsilon.$$

This contradicts the assumption $\nu << \mu$. □

Remark. *The result is not true for infinite ν.*

Theorem 17.5 (Radon-Nikodym)**.** *Assume μ and ν are σ-finite measures on a measurable space $(X,\mathscr{B})$, $\nu << \mu$. Then there is a measurable function $f : X \to [0,+\infty)$ s.t. $d\nu = f d\mu$ (that is, $\nu(A) = \int_A f d\mu$). If $\tilde{f}$ is another function with the same properties, then $f = \tilde{f}$ $\mu -$ a.e.*

The function is called the Radon-Nikodym derivative *at ν w.r.t. μ and is denoted by $\frac{d\nu}{d\mu}$.*

Proof. By the proposition above, we can find $N \in \mathscr{B}$ and $f : X \to [0,+\infty)$ s.t. $\mu(N) = 0$ and

$$\nu(A) = \nu(A\cap N) + \int_A f d\mu.$$

As $\nu(A \cap N) = 0$ by the assumption $\nu << \mu$, this proves the existence of f.

Assume we have another $\tilde{f}$. Then

$$\nu(A) = \int_A f d\mu = \int_A \tilde{f} d\mu \ \forall A \in \mathscr{B}.$$

If $B \in \mathscr{B}$, $\nu(B) < \infty$, then consider

$$A_1 = \left\{x \in \mathscr{B} : f(x) > \tilde{f}(x)\right\}, A_2 = \left\{x \in \mathscr{B} : f(x) < \tilde{f}(x)\right\}.$$

Then

$$\int_{A_1} (f - \tilde{f}) d\mu = 0 \text{ and } \int_{A_2} (\tilde{f} - f) d\mu = 0,$$

hence $\mu(A_1) = \mu(A_2) = 0$. Therefore, $f = \tilde{f}$ $\mu -$ a.e. on $\mathscr{B}$. As ν is σ-finite, we have $X = \cup_{n\in\mathbb{N}} B_n$, $\nu(B_n) < \infty$. Then $f = \tilde{f}$ $\mu -$ a.e. on B_n for all n, hence $f = \tilde{f}$ $\mu -$ a.e. on X. □

Example 17.6. Consider a real-valued function $f \in C'[a, b]$ s.t. $f'(t) > 0 \ \forall \ t \in [a, b]$. Let $c = f(a), d = f(b)$. We know that for every Riemann integrable function g on $[c, d]$ we have

$$\int_c^d g(f) dt = \int_a^b g(f(t)) f'(t) dt.$$

Equivalently,

$$\int_c^d g \circ g^{-1} dt = \int_a^b g f'(t) dt. \tag{27}$$

Denote by $\lambda_{[a,b]}$, $\lambda_{[c,d]}$ the Lebesgue measure restricted to the Borel subsets of $[a, b]$ and $[c, d]$, respectively. Then eq. 27 implies that

$$d\left((f^{-1})_* \lambda_{[c,d]}\right) = f' d\lambda_{[a,b]},$$

since the integration of $g = \mathbb{1}_{[\alpha,\beta]}$ gives the same results for any interval $[\alpha, \beta] \subset [a, b]$ and since a finite Borel measure on $[a, b]$ is determined by its values on such intervals. Thus, $(f^{-1})_* \lambda_{[c,d]} << \lambda_{[a,b]}$ and

$$\frac{d\left((f^{-1})_* \lambda_{[c,d]}\right)}{d\lambda_{[a,b]}} = f'.$$

18 Complex and Signed Measures (4.3, [Teschl(2010)])

Definition 18.1. A **complex measure** on $(X, \mathscr{B})$ is a map $\nu : \mathscr{B} \to \mathbb{C}$ s.t. $\nu(\emptyset) = 0$ and

$$\nu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \nu(A_n)$$

for any disjoint $A_n \in \mathscr{B}$, where the series is assumed to be absolutely convergent. If ν takes values in $\mathbb{R}$ then ν is called a **finite signed measure**.

Remark. *More generally, a signed measure is allowed to take values in* $\mathbb{R} \cup \{+\infty\}$ *or* $\mathbb{R} \cup \{-\infty\}$.

Given a complex measure ν on $(X, \mathscr{B})$, its **total variation** is the map $|\nu| : \mathscr{B} \to [0, +\infty]$ defined by

$$|\nu|(A) = \sup\left\{\sum_{n=1}^{N} |\nu(A_n)| : A = \bigcup_{n=1}^{N} A_n, A_n \in \mathscr{B}, A_n \cap A_m = \emptyset\right\}.$$

Proposition 18.2. *$|\nu|$ is a finite measure on $(X, \mathscr{B})$.*

Proof. Let us first show that $|\nu|$ is a measure. As $\nu(\emptyset) = 0$, we have $|\nu|(\emptyset) = 0$. For σ-addiativity, take $A = \cup_{n\in\mathbb{N}} A_n$, $A_n \in \mathscr{B}$, $A_n \cap A_m = \emptyset$. If $A = \cup_{k=1}^{N} B_n$, $B_k \in \mathscr{B}$, $B_k \cap B_l = \emptyset$ then

$$\begin{aligned} \sum_{k=1}^{N} |\nu(B_k)| &= \sum_{k=1}^{N} \left| \sum_{n=1}^{\infty} \nu\left(B_k \cap A_n\right) \right| \\ &\le \sum_{n=1}^{\infty} \sum_{k=1}^{N} \left|\nu\left(B_k \cap A_n\right)\right| \\ &\le \sum_{n=1}^{\infty} |\nu|(A_n). \end{aligned}$$

Taking the supremum over all such decompositions $A_n \in \mathscr{B}$ we get

$$|\nu|(A) \le \sum_{n=1}^{\infty} |\nu|(A_n).$$

To prove the opposite inequality we may assume $|\nu|(A) < \infty$. Then $|\nu|(A_n) \le |\nu|(A) < \infty$. It suffices to show that $\sum_{n=1}^{N} |\nu|(A_n) \le |\nu|(A)$.

Fix $\epsilon > 0$ and choose decompositions $A_n = \cup_{k=1}^{M_n} B_{n_k}$, $B_{n_k} \in \mathscr{B}$, $B_{n_k} \cap B_{n_l} = \emptyset$ s.t. $\sum_{k=1}^{M_n} |\nu(B_{n_k})| > |\nu|(A_n) - \epsilon$. Then

$$\begin{aligned} |\nu|(A) &\ge \sum_{n=1}^{N} \sum_{k=1}^{M_n} |\nu(B_{n_k})| + \left|\nu\left(A \setminus \bigcup_{n=1}^{N} A_n\right)\right| \\ &> \sum_{n=1}^{N} (|\nu|(A_n) - \epsilon) = \sum_{n=1}^{N} |\nu|(A_n) - N\epsilon. \end{aligned}$$

As $\epsilon > 0$ was arbitrary, we conclude that $|\nu|(A) \ge \sum_{n=1}^{N} |\nu|(A_n)$.

It remains to check that $|\nu|(X) < \infty$. As $|\nu|(X) \le |\mathrm{Re}\nu|(X) + |\mathrm{Im}\nu|(X)$, we can consider $\mathrm{Re}\nu$ and $\mathrm{Im}\nu$ separately. In other words, it is enough to consider finite signed measures.

Assume $|\nu|(X) = \pm\infty$. We have that there exists $A, B \in \mathscr{B}$ s.t.

$$X = A \cup B,\ A \cap B = \emptyset,\ |\nu|(A) = +\infty,\ |\nu|(B) \ge 1.$$

To see this, find a decomposition

$$X = \bigcup_{n=1}^{N} A_n,\ A_n \in \mathscr{B},\ A_n \cap A_m = \emptyset$$

s.t.

$$\sum_{n=1}^{N} |\nu(A_n)| \ge |\nu(X)| + 2 = \left|\sum_{n=1}^{N} \nu(A_n)\right| + 2.$$

Consider $I := \{i : 1 \le i \le N, \nu(A_i) \ge 0\}$ and $J := \{i : 1 \le i \le N, \nu(A_i) < 0\}$. We then have

$$\sum_{k\in I} \nu(A_k) - \sum_{k\in J} \nu(A_k) \ge \left|\sum_{k\in I} \nu(A_k) - \sum_{k\in J} \nu(A_k)\right| + 2,$$

which implies that

$$\sum_{k\in I} \nu(A_k) \ge 1 \text{ and } \sum_{k\in J} \nu(A_k) \le -1.$$

Let $B_0 = \cup_{k\in I} A_k$, $B_1 = \cup_{k\in J} A_k$. Then

$$X = B_0 \cup B_1,\ B_0 \cap B_1 = \emptyset,\ |\nu(B_0)| \ge 1,\ |\nu(B_1)| \ge 1.$$

As $|\nu|(X) = |\nu|(B_0) + |\nu|(B_1)$, we must have $|\nu|(B_0) = +\infty$ or $|\nu|(B_1) = +\infty$. If $|\nu|(B_0) = +\infty$, we let $A = B_0,\ B = B_1$. If $|\nu|(B_0) < \infty$, we let $A = B_1,\ B = B_0$. This proves the claim.

We can apply the claim to the restriction of ν to A, and so on. Therefor by induction we can construct disjoint measurable sets $B_1, B_2, \ldots$ s.t.

$$|\nu|(B_n) \geq 1\ \forall n \text{ and } |\nu|\left(X \setminus \bigcup_{n=1}^{N} B_n\right) = +\infty\ \forall N.$$

But then $\nu(\cup_{n\in\mathbb{N}} B_n) = \sum_{n\in\mathbb{N}}$, which does not make sense, as the series is not absolutely convergent. Hence $|\nu|(X) < \infty$. □

Example 18.3. Consider a measure space $(X, \mathscr{B}, \mu)$ and take $f \in L^1(X, d\mu)$. Define

$$\nu(A) = \int_A f d\mu.$$

Then ν is a complex measure on $(X, \mathscr{B})$, sinche this is true for $f \geq 0$ and a general f can be written as a linear combination of positive ones. We write $d\nu = f d\mu$.

We then have $d|\nu| = |f|d\mu$, that is,

$$|\nu|(A) = \int_A |f| d\mu.$$

Proof. To see this, consider the measure ω defined by $d\omega = |f|d\mu$. We want to show that $|\nu| = \omega$.

If $A = \cup_{n=1}^{N} A_n,\ A_n \cap A_m = \emptyset$, then

$$\begin{aligned}\sum_{n=1}^{N} |\nu(A_n)| &= \sum_{n=1}^{N} \left|\int_{A_n} f d\mu\right| \leq \sum_{n=1}^{N} \int_{A_n} |f| d\mu \\ &= \sum_{n=1}^{N} \omega(A_n) = \omega(A).\end{aligned}$$

Therefore, $|\nu| \leq \omega$.

To prove the equality, assume first that f is simple, so $f = \sum_{n=1}^{N} c_n \mathbb{1}_{A_n},\ c_n \in \mathbb{C},\ A_n \in \mathscr{B},\ A_n \cap A_m = \emptyset$. Then, for every $A \in \mathscr{B}$,

$$\begin{aligned}|\nu|(A) &\geq \sum_{n=1}^{N} |\nu\,(A \cap A_n)\,| = \sum_{n=1}^{N} |c_n| \mu\,(A \cap A_n) \\ &= \sum_{n=1}^{N} \int_{A \cap A_n} |f| d\mu = \int_A |f| d\mu = \omega(A).\end{aligned}$$

Thus, $|\nu| \geq \omega$, hence $|\nu| = \omega$.

For general f, choose simple functions $f_n \in L^1(X, d\mu)$ s.t. $||f - f_n||_1 \xrightarrow[n\to\infty]{} 0$. Consider the corresponding complex measures $\nu_n,\ d\nu_n = f_n d\mu$. Fix $A \in \mathscr{B}$. For every $B \in \mathscr{B}$, we have

$$|\nu(B)| \leq |(\nu - \nu_n)(B)| + |\nu_n(B)|.$$

This implies that

$$|\nu(A)| \leq |\nu - \nu_n|(A) + |\nu_n|(A).$$

Similarly, $|\nu_n|(A) \leq |\nu - \nu_n|(A) + |\nu|(A)$. Therefore,

$$\begin{aligned}\Big||\nu|(A) - |\nu_n|(A)\Big| &\leq |\nu - \nu_n|(A) \leq \int_A |f - f_n| d\mu \\ &\leq ||f - f_n||_1 \xrightarrow[n\to\infty]{} \infty.\end{aligned}$$

It follows that

$$|\nu|(A) = \lim_{n\to\infty} |\nu_n|(A) = \lim_{n\to\infty} \int_A |f_n| d\mu = \int_A |f| d\mu$$

(since $||f| - |f_n|| \leq |f - f_n|$.) This completes the proof. □

Definition 18.4. If $(X, \mathscr{B}, \mu)$ is a measure space, ν is a complex measure on $(X, \mathscr{B})$, then we say that ν is **absolutely continuous** w.r.t. μ and write $\nu << \mu$, if $\nu(A) = 0$ whenever $A \in \mathscr{B}, \mu(A) = 0$. Equivalently, $|\nu| << \mu$.

Theorem 18.5 (Radon-Nikodym theorem for complex measures)**.** *Assume $(X, \mathscr{B}, \mu)$ is a measure space, ν is a complex measure on $(X, \mathscr{B})$, $\nu << \mu$. Then there is a unique $f \in L^1(X, d\mu)$ s.t. $d\nu = f d\mu$.*

Proof. **Existence**: By considering separately $\mathrm{Re}\nu$ and $\mathrm{Im}\nu$, we may assume that ν is a finite signed measure. Then

$$\nu = \nu_+ - \nu_-, \text{ where } \nu_\pm = \frac{|\nu| \pm \nu}{2}$$

are positive measures, since $|\nu(A)| \leq |\nu|(A)$. Clearly, $\nu_\pm << \mu$. Therefor the proof reduces to the case when ν is positive, in which case we already know the result: we take $f = d\nu/d\mu$; note that $\int_x f d\mu = \nu(X) < \infty$, so $f \in L^1(X, d\mu)$.

Uniqueness: It suffices to show that if $\nu = 0$ and $d\nu = f d\mu$, then $f = 0$ (μ–a.e.). This is true, for example, because $\int_X |f| d\mu = |\nu|(X) = 0$. □

19 Decomposition Theorems (20, [Schilling(2017)] and 4.3, [Teschl(2010)])

Definition 19.1. Two measures ν and μ on a measurable space $(X, \mathscr{B})$ are called **mutually singular**, or we say that ν is **singular** w.r.t. μ, if there is a $N \in \mathscr{B}$ s.t. $\nu(N^c) = 0,\ \mu(N) = 0$. We then write $\nu \perp \mu$.

Theorem 19.2 (Lebesgue Decomposition Theorem)**.** *Assume ν, μ are σ-finite measures in $(X, \mathscr{B})$. Then there exist unique measures ν_a and ν_s s.t. $\nu = \nu_a + \nu_s$, $\nu_a << \mu$, $\nu_s \perp \mu$.*

Proof. As we showed earlier, there exist $N \in \mathscr{B}$ and measurable $f : X \longrightarrow [0, +\infty]$ s.t. $\mu(N) = 0$ and

$$\nu(A) = \nu(A \cap N) + \int_A f d\mu\ \forall\ A \in \mathscr{B}.$$

We then define

$$\nu_a(A) := \int_A f d\mu,\ \ \nu_s(A) = \nu(A \cap N).$$

Assume we have another such decomposition $\nu = \tilde{\nu}_a + \tilde{\nu}_s$. Let $\tilde{N} \in \mathscr{B}$ be s.t. $\mu(\tilde{N}) = 0$ and $\tilde{\nu}_s(\tilde{N}^c) = 0$. Then $\mu(N \cup \tilde{N}) = 0$ and $\nu_s((N \cup \tilde{N})^c) = 0$, $\tilde{\nu}_s((N \cup \tilde{N})^c) = 0$. Hence, for every $A \in \mathscr{B}$,

$$\begin{aligned}\nu_s(A) &= \nu_s\left(A \cap \left(N \cup \tilde{N}\right)\right) \\ &= \nu_s\left(A \cap \left(N \cup \tilde{N}\right)\right) + \nu_a\left(a \cap \left(N \cup \tilde{N}\right)\right) \\ &= \nu\left(A \cap \left(N \cup \tilde{N}\right)\right),\end{aligned}$$

and for the same reasons

$$\tilde{\nu}_s(A) = \nu\left(A \cap \left(N \cup \tilde{N}\right)\right).$$

Thus, $\nu_s = \tilde{\nu}_s$.

In a similar way,

$$\begin{aligned}\nu_a(A) &= \nu_a\left(A \cap \left(N \cup \tilde{N}\right)^c\right)\\ &= \nu_a\left(A \cap \left(N \cup \tilde{N}\right)^c\right) + \nu_s\left(A \cap \left(N \cup \tilde{N}\right)^c\right)\\ &= \nu\left(A \cap \left(N \cup \tilde{N}\right)^c\right),\\ \tilde{\nu}_a(A) &= \nu\left(A \cap \left(N \cup \tilde{N}\right)^c\right).\end{aligned}$$

Hence $\nu_a = \tilde{\nu}_a$. □

Theorem 19.3 (Polar Decomposition of Complex Measure)**.** *Assume ν is a complex measure on $(X, \mathscr{B})$. Then there exist a finite measure μ on $(X, \mathscr{B})$ and a measurable function $f : X \to \Pi$ s.t. $d\nu = f d\mu$. If $(\tilde{\mu}, \tilde{f})$ is another such pair, then $\tilde{\mu} = \mu$ and $\tilde{f} = f$ μ-a.e.*

Proof. Take $\mu = |\nu|$. Then $\nu << |\nu|$. Hence, by the Radon-Nikodyn theorem for complex measures we have $d\nu = f d\mu$ for a unique $f \in L^1(X, d\mu)$. Then $d|\nu| = |f|d\mu = |f|d|\nu|$. Hence, $|f| = 1$ (μ-a.e.). By viewing f as a function, we can therefore assume that $f : X \to \Pi$.

Assume we have another such decomposition $d\nu = \tilde{f}d\tilde{\mu}$. Then $d|\nu| = |\tilde{f}|d\tilde{\mu} = d\tilde{\mu}$, that is, $\tilde{\mu} = \mu = |\nu|$. Then $d\nu = f d\mu = \tilde{f}d\mu$, hence $f = \tilde{f}$ μ-a.e. □

For signed measures this leads to the following.

Theorem 19.4 (Hahn Decomposition Theorem)**.** *Assume ν is a finite signed measure on $(X, \mathscr{B})$. Then there exist $P, N \in \mathscr{B}$ s.t.*

$$\begin{aligned}X = P \cup N,\ \ P \cap N = \emptyset,\\ \nu\left(A \cap P\right) \geq 0,\ \ \nu\left(A \cap N\right) \leq 0\ \forall A \in \mathscr{B}.\end{aligned}$$

Moreover, then $|\nu|(A) = \nu\left(A \cap P\right) - \nu\left(A \cap N\right)$, and if $X = \tilde{P} \cup \tilde{N}$ is another such decomposition, then

$$|\nu|\left(P \Delta \tilde{P}\right) = |\nu|\left(N \Delta \tilde{N}\right) = 0.$$

Proof. Consider the polar decomposition $d\nu = f d|\nu|$, $f : X \to \Pi$. Since ν is real-valued, we have $\mathrm{Im} f = 0$ $|\nu|$-a.e. We may therefore assume that f takes values in $\Pi \cap \mathbb{R} = \{-1, 1\}$. Let $P := \{x : f(x) = 1\}$, $N := \{x : f(x) = -1\}$. Then

$$\begin{aligned}\nu\left(A \cap P\right) &= \int_{A\cap P} f d|\nu| = \int_{A\cap P} 1 d|\nu| = |\nu|\left(A \cap P\right) \geq 0,\\ \nu\left(A \cap N\right) &= \int_{A\cap N} f d|\nu| = \int_{A\cap N} (-1) d|\nu| = -|\nu|\left(A \cap N\right) \leq 0.\end{aligned}$$

This gives the required decomposition. Moreover, we see that

$$\begin{aligned}\nu\left(A \cap P\right) - \nu\left(A \cap N\right) &= |\nu|\left(A \cap P\right) + |\nu|\left(A \cap N\right)\\ &= |\nu|(A).\end{aligned}$$

Assume $X = \tilde{P} \cup \tilde{N}$ another decomposition as in the statement of the theorem. For $A \subset P \backslash \tilde{P}$ ($A \in \mathscr{B}$) we have

$$\begin{aligned}\nu(A) &= \nu\left(A \cap P\right) \geq 0,\\ \nu(A) &= \nu\left(A \cap \tilde{N}\right) \leq 0.\end{aligned}$$

It follows that $\nu(A) = 0$. Hence, $|\nu|(P\backslash\tilde{P}) = 0$. Similarly, $|\nu|(\tilde{P}\backslash P) = 0$. Hence, $|\nu|(P\Delta\tilde{P}) = |\nu|(P\backslash\tilde{P}) + |\nu|(\tilde{P}\backslash P) = 0$, $|\nu|(N\Delta\tilde{N}) = |\nu|(N^c\Delta\tilde{N}^c) = |\nu|(P\Delta\tilde{P}) = 0$. □

Corollary 19.5 (Jordan Decomposition Theorem)**.** *Assume ν is a finite signed measure on $(X, \mathscr{B})$. Then there exist unique finite measures ν_+, ν_- on $(X, \mathscr{B})$ s.t.*

$$\nu = \nu_+ - \nu_- \text{ and } \nu_+ \perp \nu_-.$$

Moreover, then $|\nu| = \nu_+ + \nu_-$, hence

$$\nu_+ = \frac{|\nu| + \nu}{2}, \nu_- = \frac{|\nu| - \nu}{2}.$$

Proof. Consider the Hahn decompositions $X = P \cup N$. Define

$$\nu_+(A) = \nu(A \cap P),\ \ \nu_-(A) = -\nu(A \cap N).$$

Then by the properties of the Hahn decomposition we have

$$\nu = \nu_+ - \nu_-,\ \ \nu_+, \nu_- \geq 0,\ \ |\nu| = \nu_+ + \nu_-,$$

and $\nu_+ \perp \nu_-$ by definition.

Assume we have another such decomposition $\nu = \tilde{\nu}_+ - \tilde{\nu}_-$, $\tilde{\nu}_+ \perp \tilde{\nu}_-$. Let $\tilde{N} \in \mathscr{B}$ be such that $\tilde{\nu}_+(\tilde{N}) = 0$, $\tilde{\nu}_-(\tilde{N}^c) = 0$. Put $\tilde{P} = X\backslash\tilde{N}$. Then $X = \tilde{P} \cup \tilde{N}$ is again a Hahn decomposition. In particular,

$$\begin{aligned}|\nu|(A) &= \nu(A \cap \tilde{P}) - \nu(A \cap \tilde{N})\\ &= \tilde{\nu}_+(A \cap \tilde{P}) + \tilde{\nu}_-(A \cap \tilde{N}) = \tilde{\nu}_+(A) - \tilde{\nu}_-(A).\end{aligned}$$

Thus $|\nu| = \tilde{\nu}_+ - \tilde{\nu}_-$. It follows that

$$\tilde{\nu}_+ = \frac{|\nu| + \nu}{2} = \nu_+,\ \ \tilde{\nu}_- = \frac{|\nu| - \nu}{2} = \nu_-.$$

□

20 More on Duals of L^p-spaces (21, p. 241, [Schilling(2017)])

- What is the dual of $L^p(X, d\mu)$? When does a measurable function $g : X \to \mathbb{C}$ define a bounded linear functional on $L^p(X, d\mu)$ by

$$\rho(f) = \int_X f g d\mu?$$

Theorem 20.1 (Young's inequality)**.** *Assume $f : [0, a] \to [0, b]$ is a strictly increasing continuous function, $f(0) = 0$, $f(a) = b$. Then, for all $s \in [0, a]$ and $t \in [0, b]$, we have*

$$st \leq \int_0^s f(x)dx + \int_0^t f^{-1}(y)dy,$$

and the equality holds iff $t = f(s)$.

With Holder's inequality it follows that every $g \in L^q(X, d\mu)$ defines a bounded linear functional

$$l_g : L^p(X, d\mu) \to \mathbb{C}, l_q(f) = \int_X f g d\mu,$$

and $||l_q|| \leq ||g||_q$.

The same makes sense for $p = 1, q = \infty$ and $p = \infty, q = 1$, when μ is σ-finite, as

$$\int_X |fg|d\mu \leq \int_X |f|d\mu \cdot ||g||_\infty = ||f||_1 \cdot ||g||_\infty.$$

Lemma 20.2. *Assume $1 \le p \le \infty$, $1/p+1/q=1$, and μ is σ-finite if $p=1$ or $p=\infty$. For $g \in L^q(X,d\mu)$ consider $l_q \in L^p(X,d\mu)^*$. Then*

$$||l_g|| = ||g||_q.$$

Proof. We may assume $||g||_q > 0$, otherwise $l_q = 0$. Consider three cases:

(i) $1 < p < \infty$, so $1 < q < \infty$. We already know that $||l_g|| \le ||g||_q$. To prove the opposite inequality, consider

$$f = \frac{\bar{g}}{|g|}|g|^{q-1}.$$

Then

$$|f|^p = |g|^{(q-1)p} = |g|^{q(1-\frac{1}{q})p} = |g|^q,$$

so $||f||_p^p = ||g||_p^p$. We have

$$l_q(f) = \int_X |g|^q d\mu = ||g||_q^q.$$

Hence

$$\begin{aligned} ||l_g|| \ge \frac{|l_g(f)|}{||f||_p} &= \frac{||g||_q^q}{||g||_q^{q/p}} = ||g||_q^{q-q/p} \\ &= ||g||_q^{q(1-1/p)} = ||g||_q. \end{aligned}$$

(ii) $p=1$, so $q=\infty$. Take $0 < c < ||g||_\infty$. Then $\mu\{x : |g(x)| > c\} > 0$. As μ is σ-finite, so $X = \cup_{n\in\mathbb{N}} X_n$, $\mu(X_n) < \infty$, we can find $A = X_n \cap \{x : |g(x)| > c\}$ s.t. $\mu(A) > 0$ and $\mu(A) < \infty$. Consider $f = \bar{g}/|g|\mathbb{1}_A$. Then $||f||_1 = \mu(A)$,

$$l_g(f) = \int_X |g|\mathbb{1}_A d\mu = \int_A |g| d\mu > c\mu(A).$$

Hence, $||l_g|| \ge |l_g(f)|/||f||_1 > c$. It follows that $||l_g|| \ge ||g||_\infty$.

(iii) $p=\infty$, so $q=1$. Define

$$f(x) = \begin{cases} \frac{\bar{g}(x)}{|g(x)|}, & \text{if } g(x) \ne 0, \\ 0, & \text{otherwise.} \end{cases}$$

Then $||f||_\infty$ and

$$l_g(f) = \int_X |g| d\mu = ||g||_1.$$

Hence, $||l_g|| \ge ||g||_1$. □

Therefor we can view $L^q(X,d\mu)$ as a subspace of $L^p(X,d\mu)^*$ using the isometric embedding

$$L^q(X,d\mu) \to L^p(X,d\mu)^*, \; g \mapsto l_g.$$

Theorem 20.3. *Assume $(X,\mathscr{B},d\mu)$ is a σ-finite measure space, $1 \le p < \infty$, $1/p+1/q=1$. Then*

$$L^p(X,d\mu)^* = L^q(X,d\mu).$$

Remark. *This is usually not true for $p=\infty$.*

Proof. Proof can be found in lec. notes. □

21 + 22 Riesz-Markow Theorem (21, p. [243-249], [Schilling(2017)])

Assume (X,d) is a locally compact metric space.

Definition 21.22 (positive linear functional)**.** A linear functional $\rho : C_c(X) \to \mathbb{C}$ is called positive if $\rho(f) \ge 0$ for all $f \ge 0$. (Recall $C_c(X)$ is the space of continuous (C) compactly supported (c) functions.)

Theorem 21.23 (Riesz-Markov)**.** *If $\rho : C_c(X) \to \mathbb{C}$ is a positive linear functional, where (X,d) is a locally compact metric space, then there exists a Borel measure μ on X s.t. $\mu(K) < \infty$ for every compact $K \subset X$ and*

$$\rho(f) = \int_X f d\mu \; \forall f \in C_c(X).$$

If X is separable, then such a measure μ is unique.

For the proof we need two auxiliary results.

Lemma 21.24 (Urysohn's Lemma)**.** *Assume (X,d) is a metric space, $A, B \subset X$ are disjoint closed subsets. Then there exists a continuous function $f : X \to [0,1]$ s.t. $f \equiv 1$ on A and $f \equiv 0$ on B.*

Proof. Define,

$$f(x) = \frac{d(x,B)}{d(x,A)+d(x,B)}.$$

Note that this is well-defined, since $d(x,A)+d(x,B) > 0$ for all x, as $\{x : d(x,A) = d(x,B) = 0\} = A \cap B = \emptyset$. □

Lemma 21.25. *Assume (X,d) is a compact metric space, $U = (U_i)_{i=1}^n$ is a finite open cover of X (so U_i are open and $\cup_{i=1}^n U_i = X$). Then there exist functions $\rho_1, ..., \rho_n$ in $C(x)$ s.t.*

$$0 \le \rho_i \le 1, \; supp(\rho_i) \subset U_i, \; \sum_{i=1}^n \rho_i(x) = 1 \; \forall x.$$

Every such collection of functions is called ***a partition of unity subordinate*** *to U.*

Proof. Let us show first that there exists an open set V_1 s.t. $\bar{V}_1 \subset U_1$ and $X = V_1 \cup U_2 \cup ... \cup U_n$. For this, consider the closed set $K = X \setminus (U_2 \cup ... \cup U_n) \subset U_1$. For every $x \in K$ we can find a ball $B_{r_x}(X)$ $(r_x > 0)$ s.t. $B_{r_x}(x) \subset U_1$. As K is compact and $(B_{r_x}(x))_{x\in K}$ is an open covert of K, we can find a finite subcover $(B_{r_x}(x_k))_{k=1}^m$. Then put $V_1 = \cup_{k=1}^m B_{r_{x_k}}(x_k)$. Using this construction $2n$ times we can find open sets $V_1', ..., V_n'$ and $V_1'', ..., V_n''$ s.t. $\bar{V}_i' \subset V_i'', \bar{V}_i'' \subset U_i$, $\cup_{i=1}^n V_i' = X$. By Urysohn's lemma, for every i, we can find $f_i \in C(X)$ s.t. $0 \le f_i \le 1$, $f_i \equiv 1$ on $\bar{V}_i'$, $f_i \equiv 0$ on $X \setminus V_i''$. Then $\text{supp}(f_i) \subset \bar{V}_j'' \subset U_j$. Define

$$\rho_i(x) = \frac{f_i(x)}{\sum_{j=1}^n f_j(x)}.$$

□

Proof of the Riesz-Markov Theorem. Assume $\phi : C_c(X) \to \mathbb{C}$ is positive for open $U \subset X$, define

$$\mu(U) := \sup \{\phi(f) : 0 \le f \le 1, \operatorname{supp}(f) \subset Y\}.$$

For a set $A \subset X$, define then

$$\mu^*(A) := \inf_{\substack{A \subset U \\ U \text{ open}}}.$$

Step 1. μ^* is an outer measure on X and $\mu^*(U) = \mu(U)$ for open U. Observe first that if $V \subset U$ is open, then $\mu(V) \le \mu(U)$. This implies that $\mu^*(V) = V$ for all open V. Obviously, $\mu^*(\emptyset) = \mu(\emptyset) = 0$. We need to show that

$$\mu^*\left(\bigcup_{n=1}^{\infty} A_n\right) \le \sum_{n=1}^{\infty} \mu^*(A_n).$$

We may assume that $\mu^*(A_n) < \infty$ for all n, as otherwise there is nothing to prove.

Assume first that $A_n = U_n$ are open. Take $f \in C_c(X)$ with $0 \le f \le 1$ s.t. $\operatorname{supp}(f) \subset \cup_{n\in\mathbb{N}} U_n$. As $\operatorname{supp}(f)$ is compact, there is a N s.t. $\operatorname{supp}(f) \subset \cup_{n=1}^N U_n$. Let $\rho_1, \ldots, \rho_N$ be a partition of unity in $C(\operatorname{supp}(f))$ subordinate to $(U_n \cap \operatorname{supp}(f))_{n=1}^N$. Define

$$f_n(x) = \begin{cases} \rho_n(x) f(x), & \text{if } x \in \operatorname{supp}(f), \\ 0, & \text{otherwise} \end{cases}.$$

Note that the functions f_n are continuous.

$\Big($Indeed, if $x \notin \operatorname{supp}(f)$ or $x \in \inf(\operatorname{supp}(f))$, then $f(y) \to f(x)$ as $y \to x$, as 0 is continuous on $X \backslash \operatorname{supp}(f)$ and ρ_n, f are continuous on $\inf(\operatorname{supp}(f))$. Assume now that $x \in (\operatorname{supp}(f)) \backslash \inf(\operatorname{supp}(f))$, so $x \in \operatorname{supp}(f)$, but $B_r(x) \backslash \operatorname{supp}(f) \ne \emptyset\ \forall r > 0$. Then $f(x) = 0$ and $f(y) \to 0$ as $y \to x$, since

$$\lim_{\substack{y \to x \\ y \in \operatorname{supp}(f)}} \rho_n(y) f(y) = \rho_n(x) f(x) = 0.\Big)$$

Note also that

$$\operatorname{supp}(f_n) < \operatorname{supp}(\rho_n) \subset U_n.$$

Hence, $\phi(f_n) \le \mu(U_n)$. Therefore

$$\phi(f) = \sum_{n=1}^{N} \phi(f_n) \le \sum_{n=1}^{N} \mu(U_n) \le \sum_{n=1}^{\infty} \mu(U_n).$$

Taking the supremum over all f s.t. $0 \le f \le 1$, supp(f) $\subset \cup_{n\in\mathbb{N}} U_n$, we set

$$\mu\left(\bigcup_{n=1}^{\infty} U_n\right) \le \sum_{n=1}^{\infty} \mu(U_n).$$

For general A_n, fix $\epsilon > 0$ and choose open U_n s.t.

$$A_n \subset U_n \text{ and } \mu(U_n) < \mu^*(A_n) + \frac{\epsilon}{2^n}.$$

Then

$$\mu^*\left(\bigcup_{n=1}^{\infty} A_n\right) \le \sum_{n=1}^{\infty} \mu^*(A_n).$$

This completes step 1.

Step 2. All Borel sets are Caratheodory measurable (with respect to μ^*). It suffices to show that every open set U is Caratheodory measurable. We need to check that

$$\mu^*(A) \ge \mu^*(A \cap U) + \mu^*(A \cap U^c)\ \ \forall A \subset X.$$

Consider first an open set $A = V$. We may assume that $\mu(V) < \infty$, as otherwise there is nothing to prove.

Fix $\epsilon > 0$ and choose f s.t.

$$0 \le f \le 1, \operatorname{supp}(f) \subset V \cap U, \phi(f) > \mu(V \cap U) - \epsilon.$$

Consider the open set $W = V \backslash \operatorname{supp}(f)$.

Choose $0 \le g \le 1$, $\operatorname{supp}(g) \subset W$, $\phi(g) > \mu(W) - \epsilon$. Then

$$\begin{aligned} \mu(V \cap U) + \mu^*(V \cap U^c) &\le \mu(V \cap U) + \mu(W) \\ &< \phi(f) + \phi(g) + 2\epsilon = \phi(f+g) + 2\epsilon \\ &\le \mu(V) + 2\epsilon, \end{aligned}$$

since $0 \le f + g \le 1$, $\operatorname{supp}(f+g) \subset V$. As $\epsilon > 0$ was arbitrary, we get the required inequality.

For general A, take $V \supset A$, V open. Then

$$\mu(V) \ge \mu(V \cap U) + \mu^*(V \cap U^c) \ge \mu^*(A \cap U) + \mu^*(A \cap U^c).$$

Taking the infimum over all V we get

$$\mu^*(A) \ge \mu^*(A \cap U) + \mu^*(A \cap U^c).$$

Lecture 22

Step 3. If $K \subset X$ is compact, then $\mu(K) < \infty$ and $\mu(K) = \inf\{\phi(f) : \mathbb{1}_(K \le f)\}$. Also, if $0 \le f \le \mathbb{1}_K$, then $\phi(f) \le \mu(K)$.

Assume $\mathbb{1}_K \le f$. Fix $\epsilon > 0$ and consider the open set $U = \{x : f(x) > 1 - \epsilon\}$. Assume $0 \le g \le 1$, $\operatorname{supp}(g) \subset U$. Then $g \le f/(1-\epsilon)$, hence $\phi(g) \le \phi(f)/(1-\epsilon)$. Taking the supremum over all such g, we get $\mu(U) \le \phi(f)/(1-\epsilon)$. Hence

$$\mu(K) \le \mu(U) \le \phi(f)/(1-\epsilon).$$

As $\epsilon > 0$ was arbitrary, we get $\mu(K) \le \phi(f)$. In particular, $\mu(K) < \infty$. (Note that $f \in C_c(X)$ s.t. $\mathbb{1}_K \le f$ exists by Usysohn's lemma.) We see also that

$$\mu(K) \le \int \{\phi(f) : \mathbb{1}_K \le f\}.$$

In order to prove the equality, fix $\epsilon > 0$ and choose an open U s.t.

$$K \subset U,\ \mu(K) > \mu(U) - \epsilon.$$

By Urysohn's lemma we can find $f \in C_c(X)$ s.t. $0 \le f \le 1$, $f \equiv 1$ on K (so $\mathbb{1}_K \le f$) and $\operatorname{supp}(f) \subset U$. Then

$$\phi(f) \le \mu(U) < \mu(K) + \epsilon.$$

Therefore,

$$\inf\{\phi(f) : \mathbb{1}_K \le f\}.$$

Finally, assume $0 \le f \le \mathbb{1}_K$. Then for every open $U \supset K$ we have $\operatorname{supp}(f) \subset U$, so $\phi(f) \le \mu(U)$. Hence

$$\phi(f) \le \inf_{\substack{K \subset U \\ U \text{ open}}} \mu(U) = \mu(K).$$

Step 4. $\phi(f) = \int_X f d\mu \ \forall f \in C_c(X)$.

It suffices to consider $0 \le f \le 1$, as such functions span $C_c(F)$. Consider $K = \mathrm{supp}(f)$ and define for some N,

$$K_n = \left\{x \in K : f(x) \ge \frac{n}{N}\right\} \ (n = 0, ..., N), \tag{28}$$

$$f_n(x) = \min\left\{f(x), \frac{n}{N}\right\}, \ (n = 0, ..., N), \tag{29}$$

$$g_n(f) = f_n(x) - f_{n-1}(x), \ (n = 1, ..., N). \tag{30}$$

Then $f_0 = 0$, $f_N = f$, so $f = \sum_{n=1}^N g_n$.

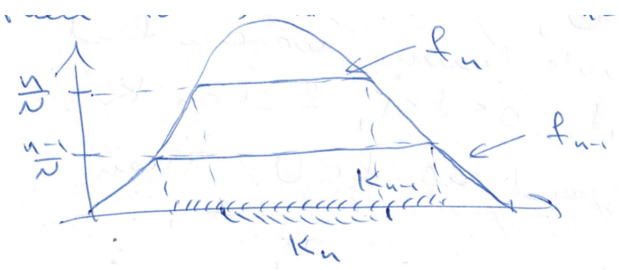

Figure 2: Drawing of the f_n's defined in eq. (29).

We have

$$\frac{1}{N}\mathbb{1}_{K_n} \le g_n \le \frac{1}{N}\mathbb{1}_{K_{n-1}} \ (n = 1, ..., N).$$

By integrating we get

$$\frac{1}{N}\mu(K_n) \le \int_X g_n d\mu \le \frac{1}{N}\mu(K_{n-1}).$$

On the other hand, by step 3 we have

$$\frac{1}{N}\mu(K_n) \le \phi(g_n) \le \frac{1}{N}\mu(K_{n-1}).$$

It follows that both $\int_X f d\mu$ and $\phi(f)$ lie in the interval

$$\left[\frac{1}{N}\sum_{n=1}^N \mu(K_n), \frac{1}{N}\sum_{n=1}^N \mu(K_{n-1})\right].$$

The length of this interval is

$$\frac{1}{N}\left(\mu(K_0) - \mu(K_n)\right) \le \frac{1}{N}\mu(K_0) = \frac{\mu(K)}{N}.$$

Therefore,

$$\left|\int_X f d\mu - \phi(f)\right| \le \frac{\mu(K)}{N}.$$

Letting $N \to \infty$ we conclude that

$$\phi(f) = \int_X f d\mu.$$

Step 5. If X is separable, then μ is uniquely determined by ϕ.

Assume μ is a Borel measure s.t. $\mu(K) < \infty \ \forall$ compact $K \subset X$ and

$$\phi(f) = \int_X f d\mu \ \forall f \in C_c(X).$$

Recall from the lecture on regularity of measures that then μ is regular (as X is separable, locally compact). In particular,

$$\mu(A) = \inf_{\substack{A \subset U \\ U \text{ open}}} \mu(U) \ \forall A \in \mathscr{B}(X).$$

Therefore it suffices to show that $\mu(U)$ is determined by ϕ.

If $\bar{U}$ is compact, then $f_n \nearrow \mathbb{1}_U$, where

$$f_n(x) = \frac{nd(x, U^c)}{1 + nd(x, U^c)}.$$

Hence,

$$\mu(U) = \lim_{n\to\infty} \int_X f_n d\mu = \lim_{n\to\infty} \phi(f_n).$$

For general U, we can find open U_n s.t. $\bar{U}_n$ are compact and $U_n \uparrow U$ (again, recall the lecture on regularity). Then $\mu(U_n) \nearrow \mu(U)$, so $\mu(U)$ is determined by ϕ. □

Remark. *Without separability, the uniqueness is not always true. It can be checked that the measure we constructed in the proof has the following properties:*

(i) $\mu(K) < \infty \ \forall$ *compact* $K \subset X$*;*

(ii) μ *is outer regular* ($\mu(A) = \inf_{\substack{A\subset U \\ U \text{ open}}} \mu(U)$)*;*

(iii) μ *is inner regular on open sets (this is where we need the full strength of step 3):*

$$\mu(U) = \sup_{\substack{K \subset U \\ K \text{ compact}}} \quad \forall \text{ open } U.$$

Such measures are called ***Radon measures****. It can be shown that the uniqueness holds within the class of Radon measures.*

Dual of $C(X)$

As an application of the Riesz-Markow Theorem we will describe $C(X)^*$ in terms of measures for compact metric spaces (X, d).

Denote by $M(X)$ the space of complex Borel measures on X. For every $\nu \in M(X)$ we want to make sense of $\int_X f d\nu$ for $f \in C(X)$. It is enough to consider finite signes measures, as then we can define

$$\int_X f d\nu = \int_X f d(\mathrm{Re}\nu) + i\int_X f d(\mathrm{Im}\nu).$$

So assume ν is a finite signed measure. Then, $\nu = \mu_1 - \mu_2$ for positive measures and we define

$$\int_X f d\nu = \int_X f d\mu_1 - \int_X f d\mu_2.$$

This is well-defined, since if

$$\nu = \mu_1 - \mu_2 = \omega_1 - \omega_2,$$

then $\mu_1 + \omega_2 = \mu_2 + \omega_1$ and

$$\int_X f d\mu_1 + \int_X f d\omega_2 = \int_X f d\mu_2 + \int_X f d\omega_1.$$

Thus, every $\nu \in M(X)$ defines a linear functional

$$\phi_\nu : C(X)\mathbb{C} \text{ by } \phi_\nu(f) = \int_x f d\nu,$$

and the map $\nu \mapsto \phi_\nu$ is linear.

Lemma 22.23. *If $\nu \in M(X)$ and $d\nu = gd|\nu|$ is its polar decomposition, then*

$$\int_X f d\nu = \int_X fgd|\nu| \;\forall f \in C(X).$$

Proof. More generally, let us show that if μ is a finite Borel measure, $g \in L^1(X, d\mu)$, and $d\nu = gd\mu$, then

$$\int_X f d\nu = \int_X fgd\mu \;\forall f \in C(X).$$

It suffices to consider $g \geq 0$. Then ν is positive, so $\int_X f d\nu$ has the usual meaning.

We will then prove an even more general statement:

$$\int_X f d\nu = \int_X fgd\mu \;\forall f \in L^\infty(X, \mu).$$

As simple functions are dense in $L^\infty(X, \mu)$, it suffices to consider simple f, hence it suffices to consider $f = \mathbb{1}_A$. Then

$$\int_X \mathbb{1}_A d\nu = \nu(A) \int_A f d\mu = \int_X \mathbb{1} gd\mu,$$

where the second equality comes from the definition of $d\nu = gd\mu$. □

Lemma 22.24. *For every $\nu \in M(X)$, the linear functional ϕ_ν is bounded and $||\phi_\nu|| = |\nu|(X)$. (Recall that the norm on $C(X)$ is $||f|| = \sup_{x\in X} |f(x)|$.)*

Proof. We have, with $d\nu = gd|\nu|$ being the polar decomposition,

$$\begin{aligned} |\phi_\nu(f)| &= \left| \int_X f d\nu \right| = \left| \int_X fgd|\nu| \right| \\ &\leq \int_X |f|d|\nu| \leq ||f|| \int_X d|\nu| \\ &\leq ||f|| |\nu|(X). \end{aligned}$$

Therefore,

$$||\phi_\nu|| \leq |\nu|(X).$$

For the opposite inequality, recall that $|\nu|$ is regular and $X(X)$ is dense in $L^1(X, d|\nu|)$. Hence, we can find $f_n \in C(X)$ s.t. $f_n \to g$ in $L^1(X, d|\nu|)$. Define

$$g_n(x) = \begin{cases} f_n(X), & \text{if } |f_n(x)| \leq 1, \\ \frac{f_n(x)}{|f_n(x)|}, & \text{if } |f_n(x)| > 1. \end{cases}$$

Then $||g_n|| \leq 1$ and $|g - g_n| \leq |f - f_n|$. Therefore, $||g - g_n|| \leq ||g - f_n||_1 \xrightarrow[n\to\infty]{} 0$. We have

$$\phi_\nu(\bar{f_n}) = \int_X \bar{g}_n g d|\nu| \to \int_X d|\nu| = |\nu|(X),$$

since

$$\begin{aligned} \left| \int_X (\bar{g}_n g - 1)\, d|\nu| \right| &\leq \int_X |\bar{g}_n g - 1| d|\nu| \\ &= \int_X |\bar{g}_n - \bar{g}| d|\nu| = ||g_n - g||_1 \xrightarrow[n\to\infty]{} 0, \end{aligned}$$

where the first equality is because $|g(x)| = 1$. Therefore, $||\phi_\nu|| \geq \sup_n |\phi_\nu(\bar{g}_n)| \geq |\nu|(X)$. □

23 Product Measures and Fubini's Theorem

(14, [Schilling(2017)])

Throughout this chapter we assume that $(X, \mathscr{A}, \mu)$ and $(Y, \mathscr{B}, \nu)$ are σ-finite measure spaces.

Recall the Cartesian product of sets (assume $A \subset X, B \subset Y$):

$$A \times B = \{(a, b) | a \in A \text{ and } b \in B\}.$$

$$\begin{aligned} (\bigcup_i A_i) \times B &= \bigcup_i (A_i \times B), \\ (\bigcap_i A_i) \times B &= \bigcap_i (A_i \times B), \\ (A \times B) \cap (A' \times B') &= (A \cap A') \times (B \cap B'), \\ A^c \times B &= (X \times B) \setminus (A \times B), \\ A \times B \subset A' \times B' &\iff A \subset A' \text{ and } B \subset B'. \end{aligned}$$

The Lebesgue measure on $\mathbb{R}^n$ has the following property for $n > d \geq 1$:

$$\lambda^n [a_1, b_1) \times ... \times [a_n, b_n) = \lambda^d [a_1, b_1) \times ... \times [a_d, b_d) \cdot \lambda^{n-d} [a_{d+1}, b_{d+1})$$

which means that

$$\lambda^n(E) = \int \mathbb{1}_E(x, y) \lambda^n(d(x, y)) = \int \left(\int \mathbb{1}_E(x_0, y) \lambda^{n-d}(dy) \right) \lambda^d(dx_0).$$

Goal: we want to define a measure ρ on rectangles on the form $A \times B$ s.t. $\rho(A \times B) = \mu(A)\nu(B)$.

Lemma 23.1. *Let $\mathscr{A}$ and $\mathscr{B}$ be two σ-algebras (or semi-rings), then $\mathscr{A} \times \mathscr{B}$ is a semi-ring.*

Proof. The proof is straight forwards. Recall that a semi-ring $\mathcal{S}$ has $\emptyset \in \mathcal{S}$ (S_1), $S, T \in \mathcal{S} \to S \cap T \in \mathcal{S}$ (S_2), and for $S, T \in \mathcal{S}$ there exist finitely many disjoint $S_1, S_2, ..., S_M \in \mathcal{S}$ s.t. $S \backslash T = \cup_{i=1}^M S_i$. □

Definition 23.2 (product σ-algebra)**.** The σ-algebra $\mathscr{A} \otimes \mathscr{B} := \sigma(A \times B)$ is called a **product σ-algebra**, and $(X \times Y, \mathscr{A} \otimes \mathscr{B})$ is the product of measurable spaces.

When considering $\mathscr{A} \otimes \mathscr{B}$, the following lemma shows that we can instead work with their generators.

Lemma 23.3. *If $\mathscr{A} = \sigma(\mathscr{F})$ and $\mathscr{B} = \sigma(\mathscr{G})$ and if $\mathscr{F}$ and $\mathscr{G}$ contain exhausting sequences $(F_n)_{n\in\mathbb{N}} \subset \mathscr{F}$, $F_n \uparrow X$ and $(G_n)_{n\in\mathbb{N}} \subset \mathscr{G}$, $G_n \uparrow X$, then*

$$\sigma(\mathscr{F} \times \mathscr{G}) = \sigma(\mathscr{A} \times \mathscr{B}) := \mathscr{A} \otimes \mathscr{B}.$$

Proof. Since $\mathscr{F} \times \mathscr{G} \subset \mathscr{A} \times \mathscr{B}$, we have $\sigma(\mathscr{F} \times \mathscr{G}) \subset \mathscr{A} \otimes \mathscr{B}$, On the other hand, the system

$$\Sigma := \{A \in \mathscr{A} : A \times G \in \sigma(\mathscr{F} \times \mathscr{G}) \forall G \in \mathscr{G}\}$$

is a σ-algebra. This can be shown, but is more or less straightforward. Furthermore, clearly, $\mathscr{F} \subset \Sigma \subset \mathscr{A}$, but $\mathscr{A} \subset \Sigma$ by construction, and so $\Sigma = \mathscr{A}$. We conclude that $\mathscr{A} \times \mathscr{G} \subset \sigma(\mathscr{F} \times \mathscr{G})$. This means that for all $A \in \mathscr{A}$ and $B \in \mathscr{B}$

$$A \times B = (A \times Y) \cap (X \times B) = \bigcup_{k,n\in\mathbb{N}} \underbrace{(A \times G_k)}_{\in\sigma(\mathscr{F}\times\mathscr{G})} \cap \underbrace{(F_n \times B)}_{\in\sigma(\mathscr{F}\times\mathscr{G})},$$

so that $\mathscr{A} \times \mathscr{B} \subset \sigma(\mathscr{F} \times \mathscr{G})$ and thus $\mathscr{A} \otimes \mathscr{B} \subset \sigma(\mathscr{F} \times \mathscr{G})$. □

Theorem 23.4 (uniqueness of product measures). *Assume that $\mathscr{A} = \sigma(\mathscr{F})$ and $\mathscr{B} = \sigma(\mathscr{G})$. If*

- *$\mathscr{F}, \mathscr{G}$ is $\cap$-stable (stable under finite intersections),*
- *$\mathscr{F}, \mathscr{G}$ contain exhausting sequences $F_k \uparrow X$ and $G_k \uparrow Y$ with $\mu(F_k) < \infty$ and $\nu(G_n) < \infty$,*

then there is at most one measure ρ on $(X \times Y, \mathscr{A} \otimes \mathscr{B})$ satisfying

$$\rho(F \times G) = \mu(F)\nu(G)\ \forall F \in \mathscr{F}, G \in \mathscr{G}.$$

By Lemma 23.3 $\mathscr{F} \times \mathscr{G}$ generates $\mathscr{A} \otimes \mathscr{B}$. Moreover, $\mathscr{F} \times \mathscr{G}$ inherits the $\cap$-stability of $\mathscr{F}$ and $\mathscr{G}$ (not obvious), the sequence $F_n \times G_n$ increases towards $X \times Y$ and $\rho(F_n \times G_n) = \mu(F_n)\nu(G_n) < \infty$. These are the assumptions of the uniqueness theorem for measures (Theorem 3.3).

Theorem 23.5 (existence of product measures). *The set function*

$$\rho : \mathscr{A} \times \mathscr{B} \to [0, \infty],\ \rho(A \times B) := \mu(A)\nu(B),$$

extends uniquely to a σ-finite measure on $(X \times Y, \mathscr{A} \otimes \mathscr{B})$ s.t.

$$\rho(E) = \int \int \mathbb{1}_E(x, y)\mu(dx)\nu(dy) = \int \int \mathbb{1}_E(x, y)\nu(dy)\mu(dx)$$

holds for all $E \in \mathscr{A} \otimes \mathscr{B}$ (the parenthesis in the expression above are left out). In particular, the functions

$$x \mapsto \mathbb{1}_E(x, y), x \mapsto \int \mathbb{1}_E(x, y)\nu(dy),$$
$$y \mapsto \mathbb{1}_E(x, y), y \mapsto \int \mathbb{1}_E(x, y)\mu(dx),$$

are $\mathscr{A}$, $\mathscr{B}$-measurable (respectively) for every fixed $y \in Y$, $x \in X$ (respectively).

Lecture 24

Definition 24.25 (product measure $\mu \times \nu$). The unique measure ρ constructed in Theorem 23.5 is called the **product** of the measures μ and ν, denoted $\mu \times \nu$. $(X, Y, \mathscr{A} \otimes \mathscr{B}, \mu \times \nu)$ is called the **product measure space**

We can now finally construct the n-dimensional Lebesgue measure:

Corollary 24.26. *If $n > d \geq 1$,*

$$(\mathbb{R}^n, \mathscr{B}(\mathbb{R}^n), \lambda^n) = \left(\mathbb{R}^d \times \mathbb{R}^{n-d}, \mathscr{B}(\mathbb{R}^d) \otimes \mathscr{B}(\mathbb{R}^{n-d}), \lambda^d \times \lambda^{n-d}\right).$$

Great. The next step is to see how we can integrate w.r.t. to $\mu \times \nu$. The following two results are often stated together as the Fubini or Fubini-Tonelli theorem.

Theorem 24.27 (Tonelli). *Let $(X, \mathscr{A}, \mu)$ $(Y, \mathscr{B}, \nu)$ be σ-finite measure spaces and let $u : X \times Y \to [0, \infty]$ be $\mathscr{A} \otimes \mathscr{B}$-measurable. Then*

(i) $x \mapsto u(x, y)$, $y \mapsto u(x, y)$ *are $\mathscr{A}$-resp. $\mathscr{B}$-measurable for fixed y resp. x;*

(ii) $x \mapsto \int_Y u(x, y)\nu(dy)$, $y \mapsto \int_X u(x, y)\mu(dx)$ *are $\mathscr{A}$-resp. $\mathscr{B}$-measurable;*

(iii) $\int\limits_{X \times Y} u d(\mu \times \nu) = \int\limits_Y \int\limits_X u(x, y)\mu(dx)\nu(dy = \int\limits_X \int\limits_Y u(x, y)\nu(dy)\mu(dx))$ *which is in $[0, \infty]$.*

The following corollary really extends Tonelli to not necessarily positive functions.

Corollary 24.28 (Fubini's theorem). *Let $(X, \mathscr{A}, \mu)$ $(Y, \mathscr{B}, \nu)$ be σ-finite measure spaces and let $u : X \times Y \to \bar{\mathbb{R}}$ be $\mathscr{A} \otimes \mathscr{B}$-measurable. If at least one of the three integrals*

$$\int\limits_{X \times Y} |u| d(\mu \times \nu), \int_Y \int_X |u(x, y)|\mu(dx)\nu(dy), \int_X \int_Y |u(x, y)|\nu(dy)\mu(dx)$$

is finite, then all three integrals are finite, $u \in \mathcal{L}^1(\mu \times \nu)$, and

(i) $x \mapsto u(x, y)$ *is in $\mathcal{L}^1(\mu)$ for ν-a.e. $y \in Y$;*

(ii) $y \mapsto u(x, y)$ *is in $\mathcal{L}^1(\nu)$ for μ-a.e. $x \in X$;*

(iii) $y \mapsto \int_X u(x, y)\mu(dx)$ *is in $\mathcal{L}^1(\nu)$;*

(iv) $x \mapsto \int_Y u(x, y)\nu(dx)$ *is in $\mathcal{L}^1(\mu)$;*

(v) $\int\limits_{X \times Y} u d(\mu \times \nu) = \int\limits_Y \int\limits_X u(x, y)\mu(dx)\nu(dy) = \int\limits_X \int\limits_Y u(x, y)\nu(dy)\mu(dx).$

Tips'n Tricks

- Assume we can write X as a finite union: $X = \cup_{n \in I} A_n$, $i = 1, ..., N$. Then

$$\int f d\mu = \int_X f d\mu = \int_{A_1} f d\mu + \int_{A_2} f d\mu + ... + \int_{A_N} f d\mu.$$

Questions

- In problem 26.18 we are supposed to show that $Y_n \perp Y_m = 0$, i.e. that $\langle y_n, y_m \rangle = 0$, $n \neq m$. I get ...

$$\langle y_n, y \rangle \subset \int_{A_m^c} |y_n|^2 |y_m|^2 d\mu,$$

  and I want to argue that this is zero since $\int_{A_m^c} |y_m|^2 d\mu = 0$, but I don't see how. The solutions are not clear, and I think perhaps my setup is wrong. I am assuming $\langle f, g \rangle = \int_X f \bar{g} d\mu$, i.e. from L^2, but perhaps it is rather $\langle f, g \rangle = \int_{A_m^c \cup A_n^c} f \bar{g} d\mu$ or something?

References

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