

Linn - Anal - Lecture - Notes

(Dated: February 8, 2024)

<https://github.com/isakrukan/MAT4400-LinnAnaly>

I. INTEGRATION OF COMPLEX FUNCTIONS (LEC. 7)

Assume (X, \mathfrak{B}, μ) is a measure space.

Definition I.1. A measurable function $f : X \rightarrow \mathbb{C}$ is called integrable (or μ -integrable) if

$$\int_X |f| d\mu < \infty.$$

Denote by $\mathcal{L}^1(X, \mathfrak{B}, d\mu)$, $\mathcal{L}^1(X, d\mu)$ or $\mathcal{L}_{\mathbb{C}}^1$ the set of integrable functions. This is also a vector space over \mathbb{C} , since

$$|f + g| \leq |f| + |g|, \quad |cf| = |c||f| \quad (c \in \mathbb{C}),$$

the other axioms are trivial.

This vector space is spanned by positive functions, since

$$f = \operatorname{Re}(f)_+ - \operatorname{Re}(f)_- + i\operatorname{Im}(f)_+ - i\operatorname{Im}(f)_-,$$

where for a function h we let

$$h_+ = \max\{h, 0\}, \quad h_- = -\min\{h, 0\},$$

and if $f \in \mathcal{L}^1(X, d\mu)$, then

$$(\operatorname{Re}(f))_{\pm}, (\operatorname{Im}(f))_{\pm} \in \mathcal{L}^1(X, d\mu),$$

as

$$|(\operatorname{Re}(f))_{\pm}|, |(\operatorname{Im}(f))_{\pm}| \leq |f|.$$

Proposition 1. *The integral extends uniquely from the positive integrable functions to a linear function (functional?) $\mathcal{L}^1(X, d\mu) \rightarrow \mathbb{C}$, that is, to a map s.t.*

$$\begin{aligned} \int_X (f + g) d\mu &= \int_X f d\mu + \int_X g d\mu, \\ \int_X cf d\mu &= c \int_X f d\mu, \quad c \in \mathbb{C}. \end{aligned}$$

Proof. Uniqueness is clear, as positive functions in $\mathcal{L}^1(X, d\mu)$ spans the entire space. We first extend the integral to real integrable functions by letting

$$\int_X (g - h) d\mu := \int_X g d\mu - \int_X h d\mu,$$

for $g, h \in \mathcal{L}^1(X, d\mu)$, $g, h \geq 0$.

This is well-defined, since if

$$g - h = g' - h',$$

then $g + h' = h + g'$ and hence $\int_X g d\mu + \int_X h' d\mu = \int_X g' d\mu + \int_X h d\mu$. Now we extend the integral to the entire space $\mathcal{L}^1(X, d\mu)$ by

$$\int_X f d\mu := \int_X (\operatorname{Re}(f)) d\mu + i \int_X (\operatorname{Im}(f)) d\mu.$$

We easily get that by definition (?):

$$\int_X (f_1 + f_2) d\mu = \int_X f_1 d\mu + \int_X f_2 d\mu, \quad \forall f_1, f_2 \in \mathcal{L}^1(X, d\mu),$$

and

$$\int_X cf d\mu = c \int_X f d\mu \quad \forall f \in \mathcal{L}^1(X, d\mu) \quad \forall c \geq 0.$$

In order to prove the last property for all $c \in \mathbb{C}$, it remains to check it for $c = -1$ and $c = i$.

For $c = -1$ it follows, since if $g, h \geq 0$, then

$$\begin{aligned} \int_X -(g - h) d\mu &= \int_X (h - g) d\mu \\ &= \int_X h d\mu - \int_X g d\mu \\ &= - \int_X (g - h) d\mu. \end{aligned}$$

Similarly, for $c = i$ it is proved by a simple computation:

$$\begin{aligned} \int_X if d\mu &= \int_X \operatorname{Re}(if) d\mu + i \int_X \operatorname{Im}(if) d\mu \\ &= \int_X (-\operatorname{Im}(f)) d\mu + i \int_X (\operatorname{Re}(f)) d\mu \\ &= i \left(\int_X (\operatorname{Re}(f)) d\mu + \int_X (\operatorname{Im}(f)) d\mu \right) \\ &= i \int_X f d\mu. \end{aligned}$$

□

Proposition 2 (Triangle Inequality). *For every $f \in \mathcal{L}^1(X, d\mu)$ we have*

$$\left| \int_X f d\mu \right| \leq \int_X |f| d\mu.$$

Proof. Choose $z \in \Pi := \{w \in \mathbb{C} : |w| = 1\}$ s.t.

since $(\operatorname{Re}(zf))_+ \leq |f|$.

□

$$z \int_X f d\mu \geq 0.$$

Then

$$\begin{aligned} \left| \int_X f d\mu \right| &= \left| z \int_X f d\mu \right| \\ &= z \int_X f d\mu \\ &= \int_X z f d\mu \\ &= \int_X \operatorname{Re}(zf) d\mu + i \int_X \operatorname{Im}(zf) d\mu \quad (\text{since } \left| \int_X f d\mu \right| \in \mathbb{R}) \\ &= \int_X (\operatorname{Re}(zf))_+ d\mu - \int_X (\operatorname{Re}(zf))_- d\mu \\ &\leq \int_X (\operatorname{Re}(zf))_+ d\mu \\ &\leq \int_X |f| d\mu, \end{aligned}$$