

# MAT4400: Notes on Linear analysis (ONLY IMPORTANT STUFF)

May 16, 2024

## 1 $\sigma$ -Algebras (3, [Schilling(2017)])

**Definition 1.1** ( $\sigma$ -Algebra). A family  $\mathcal{A}$  of subsets of  $X$  with:

- (i)  $X \in \mathcal{A}$ ,
- (ii)  $A \in \mathcal{A} \Rightarrow A^c \in \mathcal{A}$ ,
- (iii)  $(A_n)_{n \in \mathbb{N}} \in \mathcal{A} \Rightarrow \bigcup_{n \in \mathbb{N}} A_n \in \mathcal{A}$

A set  $A \in \mathcal{A}$  is said to be **measurable** or  **$\mathcal{A}$ -measurable**.

**Theorem 1.2** (and Definition).

- (i) The intersection of arbitrarily many  $\sigma$ -algebras in  $X$  is again a  $\sigma$ -algebra in  $X$ .
- (ii) For every system of sets  $p \subset \mathcal{P}(X)$  there exists a smallest  $\sigma$ -algebra containing  $p$ . This is the  $\sigma$ -algebra generated by  $p$ , denoted  $\sigma(p)$ , and  $\sigma(p)$  is called its generator.

**Definition 1.3** (Borel). The  $\sigma$ -algebra  $\sigma(\mathcal{O})$  generated by the open sets  $\mathcal{O} = \mathcal{O}_{\mathbb{R}^n}$  of  $\mathbb{R}^n$  is called **Borel  $\sigma$ -algebra**, and its members are called **Borel sets** or **Borel measurable sets**.

**Definition 1.4** ( $\sigma$ -algebra). A measure  $\mu$  on  $X$  is a map  $u : \mathcal{A} \rightarrow [0, \infty]$  satisfying

- (i)  $\mathcal{A}$  is a  $\sigma$ -algebra in  $X$ ,
- (ii)  $\mu(\emptyset) = 0$ ,
- (iii)  $(A_n)_{n \in \mathbb{N}} \subset \mathcal{A}$  pairwise disjoint  $\iff \mu(\bigcup_{n \in \mathbb{N}} A_n) = \sum_{n \in \mathbb{N}} \mu(A_n)$ .

**Definition 1.5** ( $\sigma$ -finite/sigma-finite). A measure  $\mu$  is said to be  $\sigma$ -finite and  $(X, \mathcal{A}, \mu)$  is called a  $\sigma$ -finite measure space, if  $\mathcal{A}$  contains a sequence  $(A_n)_{n \in \mathbb{N}}$  s.t.  $A_n \uparrow X$  and  $\mu(A_n) < \infty$ .

## 3 Uniqueness of Measures (5, [Schilling(2017)])

**Lemma 3.1.** A Dynkin system  $D$  is a  $\sigma$ -algebra iff it is stable under finite intersections, i.e.  $A, B \in D \Rightarrow A \cap B \in D$ .

**Theorem 3.2** (Dynkin). Assume  $X$  is a set,  $S$  is a collection of subsets of  $X$  closed under finite intersections, that is, if  $A, B \in S \Rightarrow A \cap B \in S$ . Then  $D(S) = \sigma(S)$ .

**Theorem 3.3** (uniqueness of measures). Let  $(X, \mathcal{B})$  be a measurable space, and  $S \subset \mathcal{P}(X)$  be the generator of  $\mathcal{B}$ , i.e.  $\mathcal{B} = \sigma(S)$ . If  $S$  satisfies the following conditions:

1.  $S$  is stable under finite intersections ( $\cap$ -stable), i.e.  $A, C \in S \Rightarrow A \cap C \in S$ .
2. There exists an exhausting sequence  $(G_n)_{n \in \mathbb{N}} \subset S$  with  $G_n \uparrow X$ . Assume also that there are two measures  $\mu, \nu$  satisfying:
3.  $\mu(A) = \nu(A)$ ,  $\forall A \in S$ .
4.  $\mu(G_n) = \nu(G_n) < \infty$ .

Then  $\mu = \nu$ .

*Proof (outline).* Define

$$D_n := \{A \in \mathcal{B} : \mu(G_n \cap A) = \nu(G_n \cap A) (< \infty)\},$$

and show that it is a Dynkin system. Then, use the fact that  $S$  is  $\cap$ -stable and Theorem 3.2 to argue that  $D(S) = \sigma(S) \dots \rightarrow \dots \mathcal{B} = D_n$ .  $\square$

## 4 Existence of Measures (6, [Schilling(2017)])

**Theorem 4.1** (Carathéodory). Let  $S \subset \mathcal{P}(X)$  be a semi-ring and  $\mu : S \rightarrow [0, \infty)$  a pre-measure. Then  $\mu$  has an extension to a measure  $\mu^*$  on  $\sigma(S)$ , i.e. that  $\mu(s) = \mu^*(s)$ ,  $\forall s \in \sigma(S)$ .

Also, if  $S$  contains an exhausting sequence,  $S_n \uparrow X$ , s.t.  $\mu(S_n) < \infty$ , then the extension is unique.

**Definition 4.2** (Outer measure). An outer measure is a function  $\mu^* : \mathcal{P}(X) \rightarrow [0, \infty)$  with the following properties:

1.  $\mu^*(\emptyset) = 0$ ,
2.  $A \subset B \Rightarrow \mu^*(A) \leq \mu^*(B)$ ,
3.  $\mu^*\left(\bigcup_{n \in \mathbb{N}} A_n\right) \leq \sum_{n \in \mathbb{N}} \mu^*(A_n)$ ,

## 5 Measurable Mappings (7, [Schilling(2017)])

We consider maps  $T : X \rightarrow X'$  between two measurable spaces  $(X, \mathcal{A})$  and  $(X', \mathcal{A}')$  which respects the measurable structures, the  $\sigma$ -algebras on  $X$  and  $X'$ . These maps are useful as we can transport a measure  $\mu$ , defined on  $(X, \mathcal{A})$ , to  $(X', \mathcal{A}')$ .

**Definition 5.1.** Let  $(X, \mathcal{A})$ ,  $(X', \mathcal{A}')$  be measurable spaces. A map  $T : X \rightarrow X'$  is called  $\mathcal{A}/\mathcal{A}'$ -measurable if the pre-image of every measurable set is a measurable set:

$$T^{-1}(A') \in \mathcal{A}, \quad \forall A' \in \mathcal{A}'.$$

- $T^{-1}(A') := \{x \in X : f(x) \in A'\}$
- A  $\mathcal{B}(\mathbb{R}^n)/\mathcal{B}(\mathbb{R}^m)$  measurable map is often called a Borel map.

- The notation  $T : (X, \mathcal{A}) \rightarrow (X', \mathcal{A}')$  is often used to indicate measurability of the map  $T$ .

**Lemma 5.2.** *Let  $(X, \mathcal{A})$ ,  $(X', \mathcal{A}')$  be measurable spaces and let  $\mathcal{A}' = \sigma(\mathcal{G}')$ . Then  $T : X \rightarrow X'$  is  $\mathcal{A}/\mathcal{A}'$ -measurable iff  $T^{-1}(\mathcal{G}') \subset \mathcal{A}$ , i.e. if*

$$T^{-1}(G') \in \mathcal{A}, \quad \forall G' \in \mathcal{G}'.$$

**Theorem 5.3.** *Let  $(X_i, \mathcal{A}_i)$ ,  $i = 1, 2, 3$ , be measurable spaces and  $T : X_1 \rightarrow X_2$ ,  $S : X_2 \rightarrow X_3$  be  $\mathcal{A}_1/\mathcal{A}_2$  and  $\mathcal{A}_2/\mathcal{A}_3$ -measurable maps respectively. Then  $S \circ T : X_1 \rightarrow X_3$  is  $\mathcal{A}_1/\mathcal{A}_3$ -measurable.*

**Corollary 5.4.** *Every continuous map between metric spaces is a Borel map.*

**Definition 5.5. (and lemma)** Let  $(T_i)_{i \in I}$ ,  $T_i : X \rightarrow X_i$ , be arbitrarily many mappings from the same space  $X$  into measurable spaces  $(X_i, \mathcal{A}_i)$ . The smallest  $\sigma$ -algebra on  $X$  that makes all  $T_i$  simultaneously measurable is

$$\sigma(T_i : i \in I) := \sigma\left(\bigcup_{i \in I} T_i^{-1}(\mathcal{A}_i)\right)$$

**Corollary 5.6.** *A function  $f : (X, \mathcal{B}) \rightarrow \mathbb{R}$  is measurable if  $f((a, +\infty)) \in \mathcal{B}$ ,  $\forall a \in \mathbb{R}$ .*

**Corollary 5.7.** *Assume  $(X, \mathcal{B})$  is a measurable space,  $(Y, d)$  is a metric space, and  $(f_n : (X, \mathcal{B}) \rightarrow Y)_{n=1}^{\infty}$  is a sequence of measurable maps. Assume this sequence of images  $(f_n(x))_{n=1}^{\infty}$  is convergent in  $Y \forall x \in X$ . Define*

$$f : X \rightarrow Y, \quad \text{by } f(x) = \lim_{n \rightarrow \infty} f_n(x).$$

*Then  $f$  is measurable.*

**Theorem 5.8.** *Let  $(X, \mathcal{A})$ ,  $(X', \mathcal{A}')$  be measurable spaces and  $T : X \rightarrow X'$  be an  $\mathcal{A}/\mathcal{A}'$ -measurable map. For every measurable  $\mu$  on  $(X, \mathcal{A})$ ,*

$$\mu'(A') := \mu(T^{-1}(A')), \quad A' \in \mathcal{A}',$$

*defines a measure on  $(X', \mathcal{A}')$ .*

**Definition 5.9.** The measure  $\mu'(\cdot)$  in the above theorem is called the push forward or image measure of  $\mu$  under  $T$  and it is denoted as  $T(\mu)(\cdot)$ ,  $T_*\mu(\cdot)$  or  $\mu \circ T^{-1}(\cdot)$ .

**Theorem 5.10.** *If  $T \in \mathbb{R}^{n \times n}$  is an orthogonal matrix, then  $\lambda^n = T(\lambda^n)$ .*

**Theorem 5.11.** *Let  $S \in \mathbb{R}^{n \times n}$  be an invertible matrix. Then*

$$S(\lambda^n) = |\det S|^{-1} \lambda^n = |\det S|^{-1} \lambda^n.$$

**Corollary 5.12.** *Lebesgue measure is invariant under motions:  $\lambda^n = M(\lambda^n)$  for all motions  $M$  in  $\mathbb{R}^n$ . In particular, congruent sets have the same measure. Two sets of points are called congruent if, and only if, one can be transformed into the other by an isometry*

## Measurable Functions (8, [Schilling(2017)])

A *measurable function* is a measurable map  $u : X \rightarrow \mathbb{R}$  from some measurable space  $(X, \mathcal{A})$  to  $(\mathbb{R}, \mathcal{B}(\mathbb{R}^1))$ . They play central roles in the theory of integration.

We recall that  $u : X \rightarrow \mathbb{R}$  is  $\mathcal{A}/\mathcal{B}(\mathbb{R}^1)$ -measurable if

$$u^{-1}(B) \in \mathcal{A}, \quad \forall B \in \mathcal{B}(\mathbb{R}^1).$$

Moreover from a lemma from chapter 7, we actually only need to show that

$$u^{-1}(G) \in \mathcal{A}, \quad \forall G \in \mathcal{G} \text{ where } \mathcal{G} \text{ generates } \mathcal{B}(\mathbb{R}^1).$$

**Proposition 5.13.**

- 1 If  $f, g : (X, \mathcal{B}) \rightarrow \mathbb{C}$  are measurable, then the function  $f + g$ ,  $f \cdot g$ ,  $cf$ , ( $c \in \mathbb{C}$ ) are measurable.
- 2 If  $b : \mathbb{C} \rightarrow \mathbb{C}$  is Borel and  $b : (\mathbb{C}, \mathcal{B}) \rightarrow \mathbb{C}$  is measurable, then  $b \circ f$  is measurable.
- 3 If  $f(x) = \lim_{n \rightarrow \infty} f_n(x)$ ,  $x \in X$  and  $f_n$  are measurable, then  $f$  is measurable.
- 4 If  $X = \bigcup_{n=1}^{\infty} A_n$ , ( $A_n \in \mathcal{B}$ ),  $f|_{A_n} : (A_n, \mathcal{B}_{A_n}) \rightarrow \mathbb{C}$  is measurable  $\forall n$ , then  $f$  is measurable.

**Definition 5.14.** Given a measurable space  $(X, \mathcal{B})$ , a measurable function  $f : (X, \mathcal{B}) \rightarrow \mathbb{C}$  is called simple if

$$f(x) = \sum_{k=1}^N c_k \mathbb{1}_{A_k}(x),$$

for some  $c_k \in \mathbb{C}$ ,  $A_k \in \mathcal{B}$ , where  $\mathbb{1}$  is the characteristic function,

$$\mathbb{1}_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A. \end{cases}$$

The representation of simple function is **not** unique. We denote the standard representation of  $f$  by

$$f(x) = \sum_{n=0}^N z_n \mathbb{1}_{B_n}(x),$$

for  $N \in \mathbb{N}$ ,  $z_n \in \mathbb{R}$ ,  $B_n \in \mathcal{A}$ , and

$$X = \bigcup_{n=1}^N B_n,$$

for  $B_n \cap B_m = \emptyset$ ,  $n \neq m$ . The set of simple functions is denoted  $\mathcal{E}(\mathcal{A})$  of  $\mathcal{E}$ .

**Definition 5.15.** Assume  $\mu$  is a measure on  $(X, \mathcal{B})$ . Given a *positive* simple function

$$f = \sum_{k=1}^N c_k \mathbb{1}_{A_k}, \quad (c_k \geq 0).$$

We define

$$\int_X f d\mu = \sum_{k=1}^n c_k \mu(A_k) \in [0, +\infty].$$

We also denote this by  $I_\mu(f)$ .

**Lemma 5.16.** *This is well defined, that is,  $\int_X f d\mu$  does not depend on the presentation of the simple function  $f$ .*

**Properties 5.17.** For every positive simple function

$$1 \int_X c f d\mu = c \int_X f d\mu, \quad \text{for only } c \geq 0$$

$$2 \int_X (f + g) d\mu = \int_X f d\mu + \int_X g d\mu.$$

**Corollary 5.18.** If  $f \geq g \geq 0$  are simple functions, then

$$\int_X f d\mu \geq \int_X g d\mu.$$

**Definition 5.19.** If  $f : X \rightarrow [0, +\infty)$  is measurable, then we define

$$\int_X f d\mu = \sup \left\{ \int_X g d\mu : f \geq g \geq 0, \quad g \text{ is simple} \right\}$$

**Remark.** This means that any measurable function can be approximated by simple functions.

**Properties 5.20.** Measurable functions like this have the following properties

$$1 \int_X c f d\mu = c \int_X f d\mu, \quad \forall c \geq 0.$$

$$2 \text{ If } f \geq g \geq 0, \text{ then } \int_X f d\mu \geq \int_X g d\mu \text{ for any measurable } g, f.$$

$$3 \text{ If } f \geq 0 \text{ is simple, then } \int_X f d\mu \text{ is the same value as obtained before.}$$

To advance in measure theory we consider measurable functions

$$f : X \rightarrow [0, +\infty].$$

Measurability is understood w.r.t the  $\sigma$ -algebra  $\mathcal{B}([0, +\infty])$  generated by  $\mathcal{B}([0, +\infty))$  and  $\{+\infty\}$ . In other words,  $A \subset [0, +\infty] \in \mathcal{B}([0, +\infty])$  iff  $A \cap [0, +\infty) \in \mathcal{B}([0, +\infty))$ .

**Remark.** Hence  $f : X \rightarrow [0, +\infty]$  is measurable iff  $f^{-1}(A)$  is measurable  $\forall A \in \mathcal{B}([0, +\infty])$ .

**Definition 5.21.** For measurable functions  $f_X \rightarrow [0, +\infty]$ , we define

$$\int_X f d\mu = \sup \left\{ \int_X g d\mu : f \geq g \geq 0 : g \text{ is simple} \right\} \in [0, +\infty].$$

**Theorem 5.22. Monotone convergence theorem** Assume  $(X, \mathcal{B}, \mu)$  is a measure space,  $(f)_{n=1}^\infty$  is an increasing sequence of measurable positive functions  $f_n : X \rightarrow [0, +\infty]$ . Define  $f(x) = \lim_{n \rightarrow \infty} f_n(x)$ . Then  $f$  is measurable and

$$\int_X f d\mu = \lim_{n \rightarrow \infty} \int_X f_n d\mu.$$

**Theorem 5.23.** Assume  $(X, \mathcal{B})$  is a measurable space and  $f : X \rightarrow [0, +\infty]$  is measurable. Then there are simple functions  $g_n$ , s.t.

$$0 \leq g_1 \leq g_2 \leq \dots, \quad g_n(x) \rightarrow f(x), \quad \forall x \in X.$$

Moreover, if  $f$  is bounded, we can choose  $g_n$  s.t. the convergence is uniform, that is,

$$\lim_{n \rightarrow \infty} \sup_{x \in X} |g_n(x) - f(x)| = 0.$$

## 6 Integration of Measurable Functions

(9, [Schilling(2017)])

Through this chapter  $(X, \mathcal{A}, \mu)$  will be some measure space. Recall that  $\mathcal{M}^+(\mathcal{A})$  [ $\mathcal{M}_{\mathbb{R}}^+(\mathcal{A})$ ] are the  $\mathcal{A}$ -measurable positive functions and  $\mathcal{E}(\mathcal{A})$  [ $\mathcal{E}_{\mathbb{R}}^+(\mathcal{A})$ ] are the positive and simple functions.

The fundamental idea of *Integration* is to measure the area between the graph of the function and the abscissa. For positive simple functions  $f \in \mathcal{E}^+(\mathcal{A})$  in standard representation, this is done easily

$$\text{if } f = \sum_{i=0}^M y_i \mathbb{1}_{A_i} \in \mathcal{E}^+(\mathcal{A}) \quad \text{then} \quad \sum_{i=0}^M y_i \mu(A_i) \quad (1)$$

would be the  $\mu$ -area enclosed by the graph and the abscissa. We note that the representation of  $f$  should not impact the integral of  $f$ .

**Lemma 6.1.** Let  $\sum_{i=0}^M y_i \mathbb{1}_{A_i} = \sum_{k=0}^N z_k \mathbb{1}_{B_k}$  be two standard representations of the same function  $f \in \mathcal{E}^+(\mathcal{A})$ . Then

$$\sum_{i=0}^M y_i \mu(A_i) = \sum_{k=0}^N z_k \mu(B_k). \quad (2)$$

**Definition 6.2.** Let  $f = \sum_{i=0}^M y_i \mathbb{1}_{A_i} \in \mathcal{E}^+(\mathcal{A})$  be a simple function in standard representation. Then the number

$$I_\mu(f) = \sum_{i=0}^M y_i \mu(A_i) \in [0, \infty] \quad (3)$$

(which is independent of the representation of  $f$ ) is called the  $\mu$ -integral of  $f$ .

**Proposition 6.3.** Let  $f, g \in \mathcal{E}^+(\mathcal{A})$ . Then

$$(i) \quad I_\mu(\mathbb{1}_A) = \mu(A) \quad \forall A \in \mathcal{A}.$$

$$(ii) \quad I_\mu(\lambda f) = \lambda I_\mu(f) \quad \forall \lambda \geq 0.$$

$$(iii) \quad I_\mu(f + g) = I_\mu(f) + I_\mu(g).$$

$$(iv) \quad f \leq g \Rightarrow I_\mu(f) \leq I_\mu(g).$$

In theorem 8.8 we saw that we could for every  $u \in \mathcal{M}^+(\mathcal{A})$  write it as an increasing limit of simple functions. By corollary 8.10, the suprema of simple functions are again measurable, so that

$$u \in \mathcal{M}^+(\mathcal{A}) \Leftrightarrow u = \sup_{n \in \mathbb{N}} f_n, \quad f \in \mathcal{E}^+(\mathcal{A}), \\ f_n \leq f_{n+1} \leq \dots$$

We will use this to "inscribe" simple functions (which we know how to integrate) below the graph of a positive measurable function  $u$  and exhaust the  $\mu$ -area below  $u$ .

**Definition 6.4.** Let  $(X, \mathcal{A}, \mu)$  be a measure space. The  $(\mu)$ -integral of a positive function  $u \in \mathcal{M}_{\mathbb{R}}^+(\mathcal{A})$  is given by

$$\int u d\mu = \sup \{ I_\mu(g) : g \leq u, \quad g \in \mathcal{E}^+(\mathcal{A}) \}, \quad (4)$$

with  $\int u d\mu \in [0, +\infty]$ . If we need to emphasize the *integration variable*, we write  $\int u(x) \mu(dx)$ . The key observation is that the integral  $\int \dots d\mu$  extends  $I_\mu$ .

**Lemma 6.5.** For all  $f \in \mathcal{E}^+(\mathcal{A})$  we have  $\int f d\mu = I_\mu(f)$ .

The next theorem is one of many convergence theorems. It shows that we could have defined 4 using any increasing sequence  $f_n \uparrow u$  of simple functions  $f_n \in \mathcal{E}^+(\mathcal{A})$ .

**Theorem 6.6.** (*Beppo Levi*) Let  $(X, \mathcal{A}, \mu)$  be a measure space. For an increasing sequence of functions  $(u_n)_{n \in \mathbb{N}} \subset \mathcal{M}_{\mathbb{R}}^+(\mathcal{A})$ ,  $0 \leq u_n \leq u_{n+1} \leq \dots$ , we have for the supremum  $u = \sup_{n \in \mathbb{N}} u_n \in \mathcal{M}_{\mathbb{R}}^+(\mathcal{A})$  and

$$\int \sup_{n \in \mathbb{N}} u_n d\mu = \sup_{n \in \mathbb{N}} \int u_n d\mu. \quad (5)$$

Note we can write  $\lim_{n \rightarrow \infty}$  instead of  $\sup_{n \in \mathbb{N}}$  as the supremum of an increasing sequence is its limit. Moreover, this theorem holds in  $[0, +\infty]$ , so the case  $+\infty = +\infty$  is possible.

**Corollary 6.7.** Let  $u \in \mathcal{M}_{\mathbb{R}}^+(\mathcal{A})$ . Then

$$\int u d\mu = \lim_{n \rightarrow \infty} \int f_n d\mu$$

holds for every sequence  $(f_n)_{n \in \mathbb{N}} \subset \mathcal{E}^+(\mathcal{A})$  with  $\lim_{n \rightarrow \infty} f_n = u$ .

**Proposition 6.8.** (of integral) Let  $u, v \in \mathcal{M}_{\mathbb{R}}^+(\mathcal{A})$ . Then

$$(i) \int \mathbb{1}_A d\mu = \mu(A) \quad \forall A \in \mathcal{A}.$$

$$(ii) \int \alpha u d\mu = \alpha \int u d\mu \quad \forall \alpha \geq 0.$$

$$(iii) \int u + v d\mu = \int u d\mu + \int v d\mu.$$

$$(iv) u \leq v \Rightarrow \int u d\mu \leq \int v d\mu.$$

**Corollary 6.9.** Let  $(u_n)_{n \in \mathbb{N}} \subset \mathcal{M}_{\mathbb{R}}^+(\mathcal{A})$ . Then  $\sum_{n=1}^{\infty} u_n$  is measurable and we have

$$\int \sum_{n=1}^{\infty} u_n d\mu = \sum_{n=1}^{\infty} \int u_n d\mu$$

(including the possibility  $+\infty = +\infty$ .)

**Theorem 6.10.** (*Fatou*) Let  $(u_n)_{n \in \mathbb{N}} \subset \mathcal{M}_{\mathbb{R}}^+(\mathcal{A})$  be a sequence of positive measurable functions. Then  $u = \liminf_{n \rightarrow \infty} u_n$  is measurable and

$$\int \liminf_{n \rightarrow \infty} u_n d\mu \leq \liminf_{n \rightarrow \infty} \int u_n d\mu \quad (6)$$

## 7 Integrals of Measurable Functions

(10, [Schilling(2017)])

We have defined our integral for positive measurable functions, i.e. functions in  $\mathcal{M}^+(\mathcal{A})$ . To extend our integral to not only functions in  $\mathcal{M}^+(\mathcal{A})$  we first notice that

$$u \in \mathcal{M}_{\mathbb{R}}(\mathcal{A}) \Leftrightarrow u = u^+ - u^-, \quad u^+, u^- \in \mathcal{M}_{\mathbb{R}}^+(\mathcal{A}),$$

i.e. that every measurable function can be written as a sum of **positive** measurable functions.

**Definition 7.1** ( $\mu$ -integrable). A function  $u : X \rightarrow \overline{\mathbb{R}}$  on  $(X, \mathcal{A}, \mu)$  is  $\mu$ -integrable, if it is  $\mathcal{A}/\mathcal{B}(\mathbb{R})$ -measurable and if  $\int u^+ d\mu, \int u^- d\mu < \infty$  (recall the definition for the integral of positive measurable functions). Then

$$\int u d\mu := \int u^+ d\mu - \int u^- d\mu \in (-\infty, \infty)$$

is the ( $\mu$ -)integral of  $u$ . We write  $\mathcal{L}^1(\mu)$  for the set of all real-valued  $\mu$ -integrable functions<sup>1</sup>.

<sup>1</sup>In words, we extend our integral to **positive** measurable functions by noticing that we can write every measurable function as a sum of positive measurable functions, something that we do know how to integrate. We don't want to run into the problem of  $\infty - \infty$ , thus we require the integral of the positive and negative parts to both (separately) be less than infinity.

**Theorem 7.2.** Let  $u \in \mathcal{M}_{\mathbb{R}}(\mathcal{A})$ , then the following conditions are equivalent:

$$(i) \quad u \in \mathcal{L}^1(\mu).$$

$$(ii) \quad u^+, u^- \in \mathcal{L}^1(\mu).$$

$$(iii) \quad |u| \in \mathcal{L}^1(\mu).$$

$$(iv) \quad \exists w \in \mathcal{L}_{\mathbb{R}}^1(\mu) \text{ with } w \geq 0 \text{ s.t. } |u| \leq w.$$

**Theorem 7.3** (Properties of the  $\mu$ -integral). The  $\mu$ -integral is: **homogeneous, additive**, and:

$$(i) \quad \min\{u, v\}, \max\{u, v\} \in \mathcal{L}_{\mathbb{R}}^1(\mu) \quad (\text{lattice property})$$

$$(ii) \quad u \leq v \Rightarrow \int u d\mu \leq \int v d\mu \quad (\text{monotone})$$

$$(iii) \quad \left| \int u d\mu \right| \leq \int |u| d\mu \quad (\text{triangle inequality})$$

**Remark.** If  $u(x) \pm v(x)$  is defined in  $\overline{\mathbb{R}}$  for all  $x \in X$  then we can exclude  $\infty - \infty$  and the theorem above just says that the integral is linear:

$$\int (au + bv) d\mu = a \int u d\mu + b \int v d\mu.$$

This is always true for real-valued  $u, v \in \mathcal{L}^1(\mu) = \mathcal{L}_{\mathbb{R}}^1(\mu)$ , making  $\mathcal{L}^1(\mu)$  a vector space with addition and scalar multiplication defined by

$$(u + v)(x) := u(x) + v(x), \quad (a \cdot u)(x) := a \cdot u(x),$$

and

$$\int \dots d\mu : \mathcal{L}^1(\mu) \rightarrow \mathbb{R}, \quad u \mapsto \int u d\mu,$$

is a **positive linear functional**.

## 8 Null sets and the Almost Everywhere

(11, [Schilling(2017)])

**Definition 8.1.** A ( $\mu$ -)null set  $N \in \mathcal{N}_{\mu}$  is a measurable set  $N \in \mathcal{A}$  satisfying

$$N \in \mathcal{N}_{\mu} \iff N \in \mathcal{A} \text{ and } \mu(N) = 0.$$

This can be used generally about a 'statement' or 'property', but we will be interested in questions like 'when is  $u(x)$  equal to  $v(x)$ ', and we answer this by saying

$$u = v \text{ a.e.} \Leftrightarrow \{x : u(x) \neq v(x)\} \text{ is (contained in) a } \mu\text{-null set,}$$

i.e.

$$u = v \quad \mu\text{-a.e.} \Leftrightarrow \mu(\{x : u(x) \neq v(x)\}) = 0.$$

The last phrasing should of course include that the set  $\{x : u(x) \neq v(x)\}$  is in  $\mathcal{A}$ .

**Theorem 8.2.** Let  $u \in \mathcal{M}_{\bar{\mathbb{R}}}(\mathcal{A})$ , then:

$$(i) \int |u| d\mu = 0 \Leftrightarrow |u| = 0 \text{ a.e.} \Leftrightarrow \mu\{u \neq 0\} = 0,$$

$$(ii) \mathbb{1}_N u \in \mathcal{L}_{\bar{\mathbb{R}}}^1(\mu) \quad \forall N \in \mathcal{N}_\mu,$$

$$(iii) \int_N u d\mu = 0.$$

(i) is really useful, later we will define  $\mathcal{L}^p$  and the  $\|\cdot\|_p$ -(semi)norm. Then (i) means that if we have a sequence  $u_n$  converging to  $u$  in the  $\|\cdot\|_p$ -norm then  $u_n(x) = u(x)$  a.e.

**Corollary 8.3.** Let  $u = v$   $\mu$ -a.e. Then

$$(i) u, v \geq 0 \Rightarrow \int u d\mu = \int v d\mu,$$

$$(ii) u \in \mathcal{L}_{\bar{\mathbb{R}}}^1(\mu) \Rightarrow v \in \mathcal{L}_{\bar{\mathbb{R}}}^1(\mu) \text{ and } \int u d\mu = \int v d\mu.$$

**Corollary 8.4.** If  $u \in \mathcal{M}_{\bar{\mathbb{R}}}(\mathcal{A})$ ,  $v \in \mathcal{L}_{\bar{\mathbb{R}}}^1(\mu)$  and  $v \geq 0$  then

$$|u| \leq v \text{ a.e.} \Rightarrow u \in \mathcal{L}_{\bar{\mathbb{R}}}^1(\mu).$$

**Proposition 8.5** (Markow inequality). For all  $u \in \mathcal{L}_{\bar{\mathbb{R}}}^1(\mu)$ ,  $A \in \mathcal{A}$  and  $c > 0$

$$\mu(\{|u| \geq c\} \cap A) \leq \frac{1}{c} \int_A |u| d\mu,$$

if  $A = X$ , then (obviously)

$$\mu\{|u| \geq c\} \leq \frac{1}{c} \int |u| d\mu.$$

## Completions of measure spaces

**Definition 8.6.** A measure space  $(X, \mathcal{B}, \mu)$  is called **complete** if whenever  $A \in \mathcal{B}$  and  $\mu(A) = 0$ , we have  $B \in \mathcal{B} \quad \forall B \subset A$ .

**Remark.** Any measure space can be completed as follows:

Let  $\bar{\mathcal{B}}$  be the  $\sigma$ -algebra generated by  $\mathcal{B}$  and all sets  $B \subset X$  s.t. there exists  $A \in \mathcal{B}$  with  $B \subset A$  and  $\mu(A) = 0$ .

**Proposition 8.7.** The  $\sigma$ -algebra  $\bar{\mathcal{B}}$  can also be described as follows:

$$\bar{\mathcal{B}} := \left\{ B \subset X : A_1 \subset B \subset A_2 \right.$$

$$\left. \text{for some } A_1, A_2 \in \mathcal{B} \text{ with } \mu(A_2 \setminus A_1) = 0 \right\},$$

with  $B, A_1, A_2$  as above, we define

$$\bar{\mu} := \mu(A_1) = \mu(A_2)$$

Then  $(X, \bar{\mathcal{B}}, \bar{\mu})$  is a complete measure space.

**Definition 8.8.** If  $\mu$  is a Borel measure on a **metric** space  $(X, d)$ , then the completion  $\bar{\mathcal{B}}(X)$  of the Borel  $\sigma$ -algebra with respect to  $\mu$  is called the  $\sigma$ -algebra of  $\mu$ -measurable sets.

**Remark.** For  $\mu = \lambda_n$  on  $\mathbb{R}^n$  we talk about the  $\sigma$ -algebra of **Lebesgue measurable sets**. Instead of  $\bar{\lambda}_n$  we still write  $\lambda_n$  and call it the **Lebesgue measure**. A function  $f : \mathbb{R}^n \rightarrow \mathbb{C}$ , measurable w.r.t. the  $\sigma$ -algebra of Lebesgue measurable sets is called the **Lebesgue measurable**.

The following result shows that any Lebesgue measurable function coincides with a Borel function a.e.

**Proposition 8.9.** Assume  $(X, \mathcal{B}, \mu)$  is a measure space and consider its completion  $(X, \bar{\mathcal{B}}, \bar{\mu})$ . Assume  $f : X \rightarrow \mathbb{C}$  is  $\bar{\mathcal{B}}$ -measurable. Then there is a  $\mathcal{B}$ -measurable function  $g : X \rightarrow \mathbb{C}$  s.t.  $f = g$   $\bar{\mu}$ -a.e.

## 9 Convergence Theorems and their Applications

(12, [Schilling(2017)])

- To interchange limits and integrals in **Riemann integrals** one typically has to assume uniform convergence. - The set of Riemann integrable functions is somewhat limited, see theorem 9.5

**Theorem 9.1** (Generalization of Beppo Levi, monotone convergence).

(i) Let  $(u_n)_{n \in \mathbb{N}} \subset \mathcal{L}^1(\mu)$  be s.t.  $u_1 \leq u_2 \leq \dots$  with limit  $u := \sup_{n \in \mathbb{N}} u_n = \lim_{n \rightarrow \infty} u_n$ . Then  $u \in \mathcal{L}^1(\mu)$  iff

$$\sup_{n \in \mathbb{N}} \int u_n d\mu < +\infty,$$

in which case

$$\sup_{n \in \mathbb{N}} \int u_n d\mu = \int \sup_{n \in \mathbb{N}} u_n d\mu.$$

(ii) Same thing only with a decreasing sequence  $\dots > -\infty$  in which case

$$\inf_{n \in \mathbb{N}} \int u_n d\mu = \int \inf_{n \in \mathbb{N}} u_n d\mu.$$

**Theorem 9.2** (Lebesgue; dominated convergence). Let  $(u_n)_{n \in \mathbb{N}} \subset \mathcal{L}^1(\mu)$  s.t.

$$(a) |u_n|(x) \leq w(x), \quad w \in \mathcal{L}^1(\mu),$$

$$(b) u(x) = \lim_{n \rightarrow \infty} u_n(x) \text{ exists in } \bar{\mathbb{R}},$$

then  $u \in \mathcal{L}^1(\mu)$  and we have

$$(i) \lim_{n \rightarrow \infty} \int |u_n - u| d\mu = 0;$$

$$(ii) \lim_{n \rightarrow \infty} \int u_n d\mu = \int \lim_{n \rightarrow \infty} u_n d\mu = \int u d\mu;$$

### Application 2: Riemann vs Lebesgue Integration

Consider only  $(X, \mathcal{A}, \mu) = (\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda)$ .

**Definition 9.3** (The Riemann Integral). Consider on the finite interval  $[a, b] \subset \mathbb{R}$  the partition

$$\Pi := \{a = t_0 < t_1 < \dots < t_k < b\}, k = k(\Pi), \quad (7)$$

and introduce

$$S_\Pi[u] := \sum_{i=1}^{k(\Pi)} m_i(t_i - t_{i-1}), \quad m_i := \inf_{x \in [t_{i-1}, t_i]} u(x), \quad (8)$$

$$S^\Pi[u] := \sum_{i=1}^{k(\Pi)} M_i(t_i - t_{i-1}), \quad M_i := \sup_{x \in [t_{i-1}, t_i]} u(x). \quad (9)$$

(10)

A bounded function  $u : [a, b] \rightarrow \mathbb{R}$  is said to be **Riemann integrable** if the values

$$\int u := \sup_\Pi S_\Pi[u] = \inf_\Pi S^\Pi[u] =: \int^\sim u \quad (11)$$

coincide and are finite. Their common value is called the **Riemann integral** of  $u$  and denoted by  $(R) \int_a^b u(x) dx$  or  $\int_a^b u(x) dx$ .



**Theorem 9.4.** Let  $u : [a, b] \rightarrow \mathbb{R}$  be a *measurable* and *Riemann integrable* function. Then

$$u \in \mathcal{L}^1(\lambda) \text{ and } \int_{[a,b]} u d\lambda = \int_a^b u(x) dx. \quad (12)$$

**Theorem 9.5.** Let  $u : [a, b] \rightarrow \mathbb{R}$  be a bounded function, it is Riemann integrable *iff* the points in  $(a, b)$  where  $u$  is discontinuous are a (subset of) Borel measurable null set.

## Improper Riemann Integrals

- The Lebesgue integral extends the (*proper*) Riemann integral. However, there is a further extension of the Riemann integral which cannot be captured by Lebesgue's theory.  $u$  is Lebesgue integrable *iff*  $|u|$  has finite Lebesgue integral. - The Lebesgue integral does not respect sign-changes and cancellations. However, the following *improper Riemann integral* does:

$$(R) \int_0^\infty u(x) dx := \lim_{n \rightarrow \infty} (R) \int_0^n u(x) dx. \quad (13)$$

**Corollary 9.6.** Let  $u : [0, \infty) \rightarrow \mathbb{R}$  be a measurable, Riemann integrable for every interval  $[0, N]$ ,  $N \in \mathbb{N}$ . Then  $u \in \mathcal{L}^1[0, \infty)$  *iff*

$$\lim_{N \rightarrow \infty} (R) \int_0^N |u(x)| dx < \infty. \quad (14)$$

In this case,  $(R) \int_0^\infty u(x) dx = \int_{[0, \infty)} u d\lambda$

**Proposition 9.7** (appearing as example 12.13 in Schilling). Let  $f_\alpha(x) := x^\alpha, x > 0$  and  $\alpha \in \mathbb{R}$ . Then

$$(i) f_\alpha \in \mathcal{L}^1(0, 1) \Leftrightarrow \alpha > -1.$$

$$(ii) f_\alpha \in \mathcal{L}^1[1, \infty) \Leftrightarrow \alpha < -1.$$

## 10 Regularity of Measures (App. H, [Schilling(2017)])

We let  $(X, d)$  be a metric space and denote by  $\mathcal{O}$  the open, by  $\mathcal{C}$  the closed and  $\mathcal{B}(X) = \sigma(\mathcal{O})$  the Borel set of  $X$ .

**Definition 10.1.** A measure  $\mu$  on  $(X, d, \mathcal{B}(X))$  is called outer regular, if

$$\mu(B) = \inf \{ \mu(U) \mid B \subset U, U \text{ open} \} \quad (15)$$

and inner regular, if  $\mu(K) < \infty$  for all compact sets  $K \subset X$  and

$$\mu(U) = \sup \{ \mu(K) \mid K \subset U, K \text{ compact} \}. \quad (16)$$

A measure which is both inner and outer regular is called **regular**. We write  $\mathbf{m}_r^+(X)$  for the family of regular measures on  $(X, \mathcal{B}(X))$ .

**Remark.** The space  $X$  is called  $\sigma$ -compact if there is a sequence of compact sets  $K_n \uparrow X$ . A typical example of such a space is a locally compact, separable metric space.

**Theorem 10.2.** Let  $(X, d)$  be a metric space. Every finite measure  $\mu$  on  $(X, \mathcal{B}(X))$  is outer regular. If  $X$  is  $\sigma$ -compact, then  $\mu$  is also inner regular, hence regular.

**Theorem 10.3.** Let  $(X, d)$  be a metric space and  $\mu$  be a measure on  $(X, \mathcal{B}(X))$  such that  $\mu(K) < \infty$  for all compact sets  $K \subset X$ .

1 If  $X$  is  $\sigma$ -compact, then  $\mu$  is inner regular.

2 If there exists a sequence  $G_n \in \mathcal{O}$ ,  $G_n \uparrow X$  such that  $\mu(G_n) < \infty$ , then  $\mu$  is outer regular.

## 11 The Function Spaces $\mathcal{L}^p$ (13, [Schilling(2017)])

Assume  $V$  is a vector space over  $\mathbb{K} \in \{\mathbb{C}, \mathbb{R}\}$ .

**Definition 11.1.** A seminorm on  $V$  is a map  $p : V \rightarrow [0, +\infty)$  s.t.

$$(1) p(cx) = |c|p(x) \quad \forall x \in V, \forall c \in \mathbb{K}.$$

$$(2) p(x+y) \leq p(x) + p(y) \quad \forall x, y \in V. \quad \text{triangle inequality.}$$

A seminorm is called a norm if we also have

$$p(x) = 0 \iff x = 0.$$

A norm is commonly denoted  $\|x\|$ , and a vectorspace equipped with a norm is called a **normed space**.

**Definition 11.2.** Assume  $(X, d)$  is a measure space. Fix  $1 \leq p \leq \infty$ . For every measurable function  $f : X \rightarrow \mathbb{C}$  we define the following

$$\|f\|_p = \left( \int_X |f|^p d\mu \right)^{1/p} \in [0, +\infty]. \quad (17)$$

We can see that  $\|cf\|_p = |c|\|f\|_p \quad \forall c \in \mathbb{C}$ .

Notice that by Theorem 8.2(i) we have that  $\|f\|_p = 0 \Rightarrow f = 0$  a.e. Consider for example  $\lim_{n \rightarrow \infty} \|f_n - f\|_p = 0$ , then we can find a subsequence s.t.  $\lim_{k \rightarrow \infty} |f_{n(k)} - f| = 0$  a.e., i.e.  $\lim_{k \rightarrow \infty} f_{n(k)} = f$  a.e.

**Theorem 11.3** (Hölder's inequality). Assume that  $u \in \mathcal{L}^p(\mu)$  and  $v \in \mathcal{L}^q(\mu)$ , where  $1/p + 1/q = 1$  and  $p, q \in [0, +\infty]$ . Then  $uv \in \mathcal{L}^1(\mu)$ , and the following inequality holds:

$$\left| \int uv d\mu \right| \leq \int |uv| d\mu = \|uv\|_1 \leq \|u\|_p \cdot \|v\|_q.$$

The generalized version reads:

$$\int |u_1 \cdot u_2 \cdots u_N| d\mu \leq \|u_1\|_{p_1} \cdot \|u_2\|_{p_2} \cdots \|u_N\|_{p_N}.$$

**Lemma 11.4.**

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p. \quad (18)$$

**Definition 11.5.** We define

$$\mathcal{L}^p(X, d\mu) = \{ f : X \rightarrow \mathbb{C} \mid f \text{ is measurable and } \|f\|_p < \infty \}.$$

This is a vectorspace with seminorm  $f \mapsto \|f\|_p$ . And in general this is not a normed space, since  $\|f\|_p = 0 \iff f = 0$  a.e.

Generally, if  $p$  is a seminorm on a vectorspace  $V$ , then

$$V_0 = \{ x \in V \mid p(x) = 0 \} \quad (19)$$

which is a subspace of  $V$ . Then we consider the quotient/factor space  $V/V_0$ .

**Definition 11.6.** For  $x, y \in V$ , define

$$x \sim y \iff x - y \in V_0. \quad (20)$$

This is an equivalence relation on  $V$ . The representation class of  $V$  is defined by  $[x]$  or  $x + V_0$ .

Then  $V/V_0$  is equals the set of equivalence classes. We can show that it is a normed space.

$$[x] + [y] = [x + y], \quad c[x] = [cx], \quad ||[x]|| = p(x).$$

Applying this to  $\mathcal{L}^p(X, d\mu)$  we get the normed space

$$L^p(X, d\mu) := \mathcal{L}^p(X, d\mu)/\mathcal{N} = \mathcal{L}^p(X, d\mu)/\sim. \quad (21)$$

Where  $\mathcal{N}$  is the space of measurable functions  $f$  s.t.  $f = 0$  a.e. The equivalence relation  $\sim$  is defined by

$$u \sim v \iff \{u \neq v\} \in \mathcal{N}_\mu \iff \mu\{u \neq v\} = 0,$$

and so  $L^p(X, d\mu)$  consists of all equivalence classes  $[u]_p = \{v \in \mathcal{L}^p | u \sim v\}$ . So for every  $u \in [u]_p$  there is no  $v \in [u]_p$  such that  $\mu\{u \neq v\} \neq 0$ .

We will further continue to denote the norm by  $|| \cdot ||_p$ , and we will normally **not** distinguish between  $f \in \mathcal{L}^p(X, d\mu)$  and the vector in  $L^p(X, d\mu)$  that  $f$  defines.

**Definition 11.7.** A normed space  $(X, || \cdot ||)$  is called a Banach space if  $V$  is complete w.r.t the metric  $d(x, y) = ||x - y||$ .

**Theorem 11.8.** If  $(X, \mathcal{B}, \mu)$  is a measure space,  $1 \leq p \leq \infty$ , then  $L^p(X, d\mu)$  is a Banach space.

**Definition 11.9.** A measurable function  $f : X \rightarrow \mathbb{C}$  is called **essentially bounded** if there is  $c \geq 0$  s.t.

$$\mu(\{x : |f(x)| > c\}) = 0. \quad (22)$$

That is  $|f| \leq c$  a.e. The smallest such  $c$  is called the essential supremum of  $f$  and is denoted by  $||f||_\infty$ . That is,

$$||u||_\infty := \inf \{c > 0 : \mu\{|u| \geq c\} = 0\},$$

and from problem 13.21 we have

$$\lim_{p \rightarrow \infty} || \cdot ||_p = || \cdot ||_\infty.$$

**Definition 11.10.**

$$\mathcal{L}^\infty(X, d\mu) = \{f : X \rightarrow \mathbb{C} \mid f \text{ is measurable and } ||f||_\infty < \infty\}.$$

$$L^\infty(X, d\mu) = \mathcal{L}^\infty(X, d\mu)/\mathcal{N}.$$

Where by the previous definiton these spaces become the spaces of all essentially bounded functions.

**Theorem 11.11.** If  $(X, \mathcal{B}, \mu)$  is a  $\sigma$ -finite measure space, then  $L^\infty(X, d\mu)$  is a Banach space.

## Convergence in $\mathcal{L}^p$ and completeness

**Lemma 11.12.** For any sequence  $(u_n)_{n \in \mathbb{N}} \subset \mathcal{L}^p(\mu)$ ,  $p \in [1, \infty)$ , of positive functions  $u_n \geq 0$  we have

$$\left\| \sum_{n=1}^{\infty} u_n \right\|_p \leq \sum_{n=1}^{\infty} ||u_n||_p.$$

**Theorem 11.13** (Riesz-Fischer). The spaces  $\mathcal{L}^p(\mu)$ ,  $p \in [1, \infty)$ , are complete, i.e. every Cauchy sequence  $(u_n)_{n \in \mathbb{N}} \subset \mathcal{L}^p(\mu)$  converges to some limit  $u \in \mathcal{L}^p(\mu)$

**Corollary 11.14.** Let  $(u_n)_{n \in \mathbb{N}} \subset \mathcal{L}^p(\mu)$ ,  $p \in [1, \infty)$  with  $\mathcal{L}^p - \lim_{n \rightarrow \infty} u_n = u$ . Then there exists a subsequence  $(u_{n_k})_{k \in \mathbb{N}}$  s.t.  $\lim_{k \rightarrow \infty} u_{n_k}(x) = u(x)$  holds for almost every  $x \in X$ .

**Theorem 11.15.** Let  $(u_n)_{n \in \mathbb{N}} \subset \mathcal{L}^p(\mu)$ ,  $p \in [1, \infty)$ , be a sequence of functions s.t.  $|u_n| \leq w \forall n \in \mathbb{N}$  and some  $w \in \mathcal{L}^p(\mu)$ . If  $u(x) = \lim_{n \rightarrow \infty} u_n(x)$  exists for (almost) every  $x \in X$ , then

$$u \in \mathcal{L}^p \text{ and } \lim_{n \rightarrow \infty} ||u - u_n||_p = 0.$$

**Theorem 11.16** (F. Riesz (convergence theorem)). Let  $(u_n)_{n \in \mathbb{N}} \subset \mathcal{L}^p(\mu)$ ,  $p \in [1, \infty)$ , be a sequence s.t.  $\lim_{n \rightarrow \infty} u_n(x) = u(x)$  for almost every  $x \in X$  and some  $u \in \mathcal{L}^p(\mu)$ . Then

$$\lim_{n \rightarrow \infty} ||u_n - u||_p = 0 \iff \lim_{n \rightarrow \infty} ||u_n||_p = ||u||_p.$$

## 12 Dense and Determining Sets (17, [Schilling(2017)])

**Definition 12.1** (Dense Sets). A set  $\mathcal{D} \subset \mathcal{L}^p(\mu)$ ,  $p \in [0, \infty]$ , is called *dense* if for every  $u \in \mathcal{L}^p(\mu)$  there exist a sequence  $(f_n)_{n \in \mathbb{N}} \subset \mathcal{D}$  s.t.  $\lim_{n \rightarrow \infty} ||u - f_n||_p = 0$ .

**Definition 12.2** (Support). The support of a function  $f$  is the set of points in  $X$  where  $f$  is non-zero:

$$\text{supp}(f) := \{x \in X : f(x) \neq 0\}.$$

**Theorem 12.3.** Assume  $X, d$  is a metric space and  $\mu$  is a Borel measure that is finite on every ball  $1 \leq p < \infty$ . Then the space of bounded continuous functions with bounded support is dense in  $\mathcal{L}^p(X, d\mu)$ . Where bounded support means that  $f$  vanishes outside some ball.

**Theorem 12.4.** Assume  $(X, d)$  is a separable locally compact metric space and  $\mu$  is a Borel Measure on  $X$  s.t.  $\mu(K) < \infty \forall$  compact  $K \subset X$ . Then the space  $C_c(X)$  of continuous compactly supported functions is dense in  $\mathcal{L}^p(X, d\mu)$ .

Recall that the support of a function  $f$  is  $\text{supp}(f) = \{x \in X : f(x) \neq 0\}$ , *closed support* is the closure of  $\text{supp}(f)$  (i.e. boundary points are included), often just written as  $\text{supp}(f)$ , and a function is said to have *compact support* if  $\text{supp}(f)$  is compact.

In particular, either theorem shows that if  $\mu$  is a Borel measure on  $\mathbb{R}^n$  s.t. the measure of every ball is finite, then  $C_c(\mathbb{R}^n)$  is dense in  $\mathcal{L}^p(\mathbb{R}^n, d\mu)$ ,  $1 \leq p < \infty$ . Later we will see that even  $C_c^\infty(\mathbb{R}^n)$  is dense in  $L^p(\mathbb{R}^n, d\mu)$ .

**Remark.** These results do not extend to  $p = \infty$  in general.

For  $\mu = \lambda_n$  we write simply  $\mathcal{L}^p(\mathbb{R}^n)$ .

**Remark.** Theorem 17.8 in the book is *WRONG*. For example,  $X = \mathbb{Q}$  with the usual metric is  $\sigma$ -compact, supports nonzero finite measure, but  $C_c(\mathbb{Q}) = 0$ .

## Modes of Convergence

(mixture of ex. 11.12 and ch. 22 p. 258-261. in [Schilling(2017)])

Assume  $(X, \mathcal{B}, \mu)$  is a measure space. Given measurable functions  $f_n, f : X \rightarrow \mathbb{C}$ , recall that

$$f_n \rightarrow f \text{ a.e.}$$

means that  $f_n(x) \xrightarrow{n \rightarrow \infty} f(x)$  for all  $x$  outside a set of measure zero.

**Theorem 12.5** (Egorov). Assume  $\mu(X) < \infty$  and  $f_n \rightarrow f$  a.e. Then,  $\forall \epsilon > 0$ , there exists  $X_\epsilon \in \mathcal{B}$  s.t.  $\mu(X_\epsilon) < \epsilon$  and  $f_n \rightarrow f$  uniformly on  $X \setminus X_\epsilon$ .

In addition to pointwise and uniform convergence we also consider the following:

$f_n \rightarrow f$  in the  $p$ -th mean if  $\|f_n - f\|_p \xrightarrow{n \rightarrow \infty} 0$ . For  $p = 1$  we say in mean, for  $p = 2$  we say in quadratic mean.

$f_n \rightarrow f$  in measure if  $\forall \epsilon > 0$  we have

$$\mu(\{x : |f_n(x) - f(x)| \geq \epsilon\}) \xrightarrow{n \rightarrow \infty} 0.$$

**Theorem 12.6** (Lemma 22.4 in the book?). Assume  $(X, \mathcal{B}, d\mu)$  is a measure space,  $1 \leq p < \infty$ ,  $f_n, f : X \rightarrow \mathbb{C}$  are measurable functions. Then

- (i) If  $f_n \rightarrow f$  in the  $p$ -th mean, then  $f_n \rightarrow f$  in measure.
- (ii) If  $f_n \rightarrow f$  in measure, then there is a subsequence  $(f_{n_k})_{k=1}^\infty$  s.t.  $f_{n_k} \rightarrow f$  a.e.
- (iii) If  $f_n \rightarrow f$  a.e. and  $\mu(X) < \infty$ , then  $f_n \rightarrow f$  in measure.

In particular, if  $f_n \rightarrow f$  in the  $p$ -th mean, then  $f_{n_k} \rightarrow f$  a.e. for a subsequence  $(f_{n_k})_k$ .

## 13 Abstract Hilbert Spaces (26, [Schilling(2017)])

Assume  $\mathcal{H}$  is a vector space over  $\mathbb{C}$ .

**Definition 13.1.** A pre-inner product on  $\mathcal{H}$  is a map  $(\cdot, \cdot) : \mathcal{H} \times \mathcal{H} \rightarrow \mathbb{C}$  which is

- (i) Sesquilinear: linear in the first variable and antilinear in the second:

$$\begin{aligned} (\alpha u + \beta v, w) &= \alpha(u, w) + \beta(v, w), \\ (w, \alpha u + \beta v) &= \bar{\alpha}(w, u) + \bar{\beta}(w, v), \quad u, v, w \in \mathcal{H} \text{ and } \alpha, \beta \in \mathbb{C}. \end{aligned}$$

- (ii) Hermitian:  $(u, v) = \overline{(v, u)}$ .

- (iii) Positive semidefinite:  $(u, v) \geq 0$ .

It is called an **inner product**, or a scalar product, if instead of (iii) the map is positive definite;  $(u, v) > 0$ . This definition also works for  $\mathbb{R}$  instead of  $\mathbb{C}$ .

**Cauchy-Schwartz inequality** If  $(\cdot, \cdot)$  is a pre-inner product, then  $|(u, v)| \leq (u, u)^{1/2}(v, v)^{1/2}$ .

**Corollary 13.2.** Assume we have a seminorm  $\|u\| := (u, u)^{1/2}$ . It is a norm iff  $(\cdot, \cdot)$  is an inner product.

**Definition 13.3** (Hilbert space). A Hilbert space is a complex vector space  $\mathcal{H}$  with an inner product  $(\cdot, \cdot)$  s.t.  $\mathcal{H}$  is complete with respect to the norm  $\|u\| = (u, u)^{1/2}$ .

1. The norm on a Hilbert space is determined by the inner product, but the inner product can also be recovered by the norm by the *polarization identity*:  $(u, v) = \frac{\|u+v\|^2 - \|u-v\|^2}{4} + i \frac{\|u+iv\|^2 - \|u-iv\|^2}{4}$ .
2. *Parallelogram law*:  $\|u+v\|^2 + \|u-v\|^2 = 2\|u\|^2 + 2\|v\|^2$ .
3. A norm on a vector space is given by an inner product iff it satisfies the parallelogram law, and then the scalar product is uniquely determined by the polarization identity.

**Example 13.4.** Assume  $(X, \mathcal{B}, \mu)$  is a measure space. Then  $\mathcal{L}^2(X, d\mu)$  is a Hilbert space with inner product

$$(f, g) = \int_X f \bar{g} d\mu.$$

This is well-defined, as  $|f \bar{g}| \leq \frac{1}{2}(|f|^2 + |g|^2)$ .

In particular, if  $\mathcal{B} = \mathcal{P}(X)$  and  $\mu$  is the counting measure, i.e.

$$\mu(A) = \begin{cases} \# & \text{if } A \text{ is finite,} \\ +\infty & \text{if } A \text{ is infinite,} \end{cases}$$

then  $\mathcal{L}^2(X, d\mu)$  is denoted by  $l^2(X)$ ; for  $X = \mathbb{N}$  we write simply  $l^2$ . Note that in this case for  $f : X \rightarrow [0, +\infty]$  we have

$$\int_X f d\mu = \sum_{x \in X} f(x) := \sup_{\substack{F \subset X \\ F \text{ is finite}}} \sum_{x \in F} f(x),$$

and if  $\sum_{x \in X} f(x) < \infty$ , then  $\{x : f(x) > 0\}$  is at most countable, so  $\sum_{x \in X} f(x) = \sum_{x: f(x) > 0} f(x)$  is the usual sum of a series.

Recall that a subset  $C$  of a vector space is called *convex* if

$$u, w \in C \rightarrow tu + (1-t)w \in C \quad \forall t \in (0, 1).$$

The following is one of the key properties of the Hilbert space

**Theorem 13.5** (projection theorem). Assume  $\mathcal{H}$  is a Hilbert space and  $C \subset \mathcal{H}$  is a closed convex subset. Then for every  $u \in \mathcal{H}$  there is a unique  $u_0 \in C$  (minimizer) s.t.

$$\|u - u_0\| = d(u, C) = \inf_{x \in C} \|u - x\|.$$

## 14 Orthogonal Projections (26, [Schilling(2017)])

For a Hilbert space  $\mathcal{H}$  and a subset  $A \subset \mathcal{H}$ , let

$$A^\perp := \{x \in \mathcal{H} : x \perp y \quad \forall y \in A\},$$

where  $x \perp y$  means that  $(x, y) = 0$ .  $A^\perp$  is a closed subspace of  $\mathcal{H}$ .

**Proposition 14.1.** Assume  $\mathcal{H}_0$  is a closed subspace of a Hilbert space  $\mathcal{H}$ . Then every  $u \in \mathcal{H}$  uniquely decomposes as

$$u = u_0 + u_1, \quad \text{with } u_0 \in \mathcal{H}_0 \text{ and } u_1 \in \mathcal{H}_0^\perp.$$

Moreover,  $\|u - u_0\| = d(u, \mathcal{H}_0)$  and  $\|u\|^2 = \|u_0\|^2 + \|u_1\|^2$ .



For a closed subspace  $\mathcal{H}_0 \subset \mathcal{H}$ , consider the map  $P : H \rightarrow \mathcal{H}_0$  s.t.  $Pu \in \mathcal{H}_0$  is the unique element satisfying  $u - Pu = H_0^\perp$ . The operator  $P$  is linear. It is also contractive, meaning that  $\|Pu\| \leq \|u\|$ , since  $\|u\|^2 = \|Pu\|^2 + \|u - Pu\|^2$ . It is called the orthogonal projection onto  $\mathcal{H}_0$ .

If  $\mathcal{H}_0$  is finite dimensional with an orthonormal basis  $u_1, \dots, u_n$  then

$$Pu = \sum_{k=1}^n (u, u_k) u_k.$$

Orthonormal bases can be defined for arbitrary Hilbert spaces.

**Definition 14.2** (orthonormal system). An orthonormal system in  $\mathcal{H}$  is a collection of vectors  $u_i \in H$  ( $i \in I$ ) s.t.

$$(u_i, u_j) = \delta_{ij} \quad \forall i, j \in I.$$

It is called an *orthonormal basis* if  $\text{span}\{u_i\}_{i \in I}$  denotes the linear span of  $\{u_i\}_{i \in I}$ , the space of finite linear combinations of the vectors  $u_i$ .

**Definition 14.3.** A Hilbert space  $\mathcal{H}$  is said to be *separable* if  $\mathcal{H}$  contains a countable dense subset  $G \subset \mathcal{H}$ .

**Theorem 14.4.** Every Hilbert space  $\mathcal{H}$  has an orthonormal basis. If  $\mathcal{H}$  is separable, then there is a countable orthonormal basis.

**Proposition 14.5.** Assume  $\{u_i\}_{i \in I}$  is an orthonormal system in a Hilbert space  $H$ . Take  $u \in \mathcal{H}$ . Then

- (i) Bessel's inequality:  $\sum_{i \in I} |(u, u_i)|^2 \leq \|u\|^2$ , in particular,  $\{i : (u, u_i) \neq 0\}$  is countable.
- (ii) Parseval's identity: If  $\{u_i\}_{i \in I}$  is an orthonormal basis, then  $\sum_{i \in I} |(u, u_i)|^2 = \|u\|^2$ .

If  $(u_i)_{i \in I}$  is an orthonormal basis, then the numbers  $(u, u_i)$  are called the **Fourier coefficients** of  $u$  with respect to  $(u_i)_{i \in I}$ . The Parseval identity then suggests that  $u$  is determined by its Fourier coefficients. This is true, and even more, we have:

**Proposition 14.6.** Assume  $(u_i)_{i \in I}$  is an orthonormal basis in a Hilbert space  $\mathcal{H}$ . Then for every vector  $(c_i)_{i \in I} \in l^2(I)$  there is a unique vector  $u \in \mathcal{H}$  with Fourier coefficients  $c_i$ , and we write

$$u = \sum_{i \in I} c_i u_i.$$

**Remark.** Equivalently, the element  $u = \sum_{i \in I} c_i u_i$  can be described as the unique element in  $\mathcal{H}$  s.t.  $\forall \epsilon > 0$  there is a finite  $F_0 \subset I$  s.t.  $\|u - \sum_{i \in F} c_i u_i\| < \epsilon \quad \forall \text{ finite } F \supset F_0$ .

**Corollary 14.7.** We have a linear isomorphism  $U : l^2(I) \xrightarrow{\sim} \mathcal{H}$ ,  $U((c_i)_{i \in I}) = \sum_{i \in I} c_i u_i$ . By Parseval's identity this isomorphism is isometric, that is,  $\|Ux\| = \|x\| \quad \forall x \in l^2(I)$ . By the polarization identity this is equivalent to

$$(Ux, Uy) = (x, y) \quad \forall x, y \in l^2(I).$$

Therefor  $U$  is unitary.

**Corollary 14.8.** Up to a unitary isomorphism, there is only one infinite dimensional separable Hilbert space, namely,  $l^2$ .

## 15 Dual spaces (26, [Schilling(2017)])

**Lemma 15.1.** Assume  $V$  is a normed space over  $K = \mathbb{R}$  or  $K = \mathbb{C}$ . Consider a linear functional  $f : V \rightarrow K$ . The following are equivalent (TFAE):

- (i)  $f$  is continuous;
- (ii)  $f$  is continuous at 0;
- (iii) There is a  $c \geq 0$  s.t.  $|f(x)| \leq c\|x\| \quad \forall x \in V$ .

If (i)-(iii) are satisfied, then  $f$  is called a *bounded linear functional*. The constant  $c$  in (iii) is denoted by  $\|f\|$ . We have  $\|f\| = \sup_{x \neq 0} \frac{|f(x)|}{\|x\|} = \sup_{\|x\|=1} |f(x)| = \sup_{\|x\| \leq 1} |f(x)|$ .

**Proposition 15.2.** For every normed vector space  $V$  over  $K = \mathbb{R}$  or  $K = \mathbb{C}$ , the bounded linear functionals on  $V$  form a Banach space  $V^*$ .

**Remark.** The sequence  $\{\|f_n - f_m\|\}_{m=1}^\infty$  actually converges, since

$$\|f_n - f_m\| \leq \|f_m - f_n\|.$$

When we study/use normed spaces, it is often important to understand the dual spaces. For Hilbert spaces this is particularly easy:

**Theorem 15.3** (Riesz). Assume  $\mathcal{H}$  is a Hilbert space. Then every  $f \in \mathcal{H}^*$  has the form

$$f(x) = (x, y),$$

for a uniquely defined  $y \in \mathcal{H}$ . Moreover, we have  $\|f\| = \|y\|$ .

For every Hilbert space  $\mathcal{H}$  we can define the *conjugate Hilbert space*  $\bar{\mathcal{H}}$ , which has its elements as the symbols  $\bar{x}$  for  $x \in \mathcal{H}$ , with the linear structure and inner product defined by

$$\bar{x} + \bar{y} = \overline{x + y}, \quad c \cdot \bar{x} = \overline{cx}, \quad (\bar{x}, \bar{y}) = \overline{(x, y)} = (y, x).$$

**Corollary 15.4.** For every Hilbert space  $\mathcal{H}$ , we have an isometric isomorphism  $\bar{\mathcal{H}} \xrightarrow{\sim} \mathcal{H}^*$ ,  $\bar{x} \mapsto (\cdot, x)$ .

## 16 Hahn-Banach Theorem (4.2, [Teschl(2010)])

**Theorem 16.1** (Hahn-Banach). Assume  $V$  is a real vector space,  $V_0 \subset V$  a subspace,  $e : V \rightarrow \mathbb{R}$  a convex function and  $f : V_0 \rightarrow \mathbb{R}$  a linear functional s.t.  $f \leq e$  on  $V_0$ . Then  $f$  can be extended to a linear functional  $F$  on  $V$  s.t.  $F \leq e$ .

**Theorem 16.2** (Hahn-Banach). Assume  $V$  is a real or complex vector space,  $p$  a seminorm on  $V_0$ ,  $V_0 \subset V$ , and  $f$  a linear functional on  $V_0$  s.t.

$$|f(x)| \leq p(x) \quad \forall x \in V_0.$$

Then  $f$  can be extended to a linear functional  $F$  on  $V$  s.t.  $|F(x)| \leq p(x) \quad \forall x \in V$ .

**Corollary 16.3.** Assume  $V$  is a normed real or complex vector space,  $V_0 \subset V$  and  $f \in V_0^*$ . Then there is a  $F \in V^*$  s.t.

$$F|_{V_0} = f \quad \text{and} \quad \|F\| = \|f\|.$$

**Corollary 16.4.** Assume  $V$  is a normed space and  $x \in V, x \neq 0$ . Then there is a  $F \in V^*$  s.t.  $\|F\| = 1$  and  $F(x) = \|x\|$ .

Such an  $F$  is called a *supporting functional* at  $x$ .

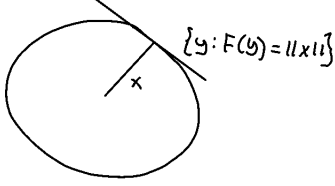


Figure 1: Tangent space?

If  $V$  is a normed vector space, then every  $x \in X$  defines a bounded linear functional on  $V^*$  by

$$V^* \ni f \mapsto f(x).$$

As  $|f(x)| \leq \|f\| \cdot \|x\|$ , this functional has norm  $\leq \|x\|$ . By using a supporting functional at  $x$ , we actually see that we get norm  $\|x\|$ . Thus, we have an isometric embedding  $V \subset V^{**} := (V^*)^*$ . We can therefor see view  $V$  as a subspace of  $V^{**}$ .

**Definition 16.5.** A normed space  $V$  is called reflexive if  $V^{**} = V$ .

**Remark.** This is stronger than requiring  $V \simeq V^{**}$ .

**Remark.** Every Hilbert space  $\mathcal{H}$  is reflexive. Indeed,  $\mathcal{H}^* = \bar{\mathcal{H}}$ . By Riesz' theorem every bounded linear functional  $f$  on  $\mathcal{H}$  has the form

$$f(\bar{x}) = (\bar{x}, \bar{y}) = (y, x),$$

for some  $y \in \mathcal{H}$ , which exactly means that  $f = y$  in  $\mathcal{H}^{**}$ .

As we will see later, the spaces  $\mathcal{L}^p(X, d\mu)$ , with  $\mu$   $\sigma$ -finite and  $1 < p < \infty$ , are reflexive. The spaces  $\mathcal{L}'(X, d\mu)$  and  $\mathcal{L}^\infty(X, \mu)$  are usually not reflexive.

## 17 Radon-Nikodym Theorem (20, [Schilling(2017)])

Assume  $(X, \mathcal{B}, \mu)$  is a measure space. Are there other measures on  $(X, \mathcal{B})$ ?

**Example 17.1.** Take a measurable function  $f : X \rightarrow [0, +\infty]$  and define

$$\nu(A) := \int_A f d\mu \text{ for } A \in \mathcal{B}.$$

This is a measure by the monotone convergence theorem. We write  $d\nu = f d\mu$ . Furthermore, we say that  $f$  is the **Radon-Nikodym** derivative, and we denote it by  $f = d\nu/d\mu$ . If  $\mu = \lambda^1$  we get  $f(x) = d\nu(x)/dx$ .

**Proposition 17.2.** Assume  $(X, \mathcal{B})$  is a measurable space,  $\mu$  and  $\nu$  are  $\sigma$ -finite measures on  $(X, \mathcal{B})$ . Then there exist  $N \in \mathcal{B}$  and a measurable  $f : X \rightarrow [0, +\infty]$  s.t.  $\mu(N) = 0$  and  $\nu(A) = \nu(A \cap N) + \int_A f d\mu \forall A \in \mathcal{B}$ .

When can we discard the term  $\nu(A \cap N)$ ?

**Definition 17.3.** Given measure  $\mu$  and  $\nu$  on  $X, \mathcal{B}$ , we say that  $\nu$  is *absolutely continuous* with respect to  $\mu$  and write  $\nu \ll \mu$ , if  $\nu(A) = 0$  whenever  $A \in \mathcal{B}, \mu(A) = 0$ .

**Lemma 17.4.** Assume  $\mu$  and  $\nu$  are measures on  $(X, \mathcal{B})$ ,  $\nu(X) < \infty$ . Then  $\nu \ll \mu$  iff  $\forall \epsilon > 0 \exists \delta > 0$  s.t. if  $A \in \mathcal{B}, \mu(A) < \delta$ , then  $\nu(A) < \epsilon$ .

**Remark.** The result is not true for infinite  $\nu$ .

**Theorem 17.5** (Radon-Nikodym). Assume  $\mu$  and  $\nu$  are  $\sigma$ -finite measures on a measurable space  $(X, \mathcal{B})$ ,  $\nu \ll \mu$ . Then there is a measurable function  $f : X \rightarrow [0, +\infty]$  s.t.  $d\nu = f d\mu$  (that is,  $\nu(A) = \int_A f d\mu$ ). If  $\tilde{f}$  is another function with the same properties, then  $f = \tilde{f}$   $\mu$ -a.e.

The function is called the Radon-Nikodym derivative at  $\nu$  w.r.t.  $\mu$  and is denoted by  $\frac{d\nu}{d\mu}$ .

**Example 17.6.** Consider a real-valued function  $f \in C'[a, b]$  s.t.  $f'(t) > 0 \forall t \in [a, b]$ . Let  $c = f(a), d = f(b)$ . We know that for every Riemann integrable function  $g$  on  $[c, d]$  we have

$$\int_c^d g(f) dt = \int_a^b g(f(t)) f'(t) dt.$$

Equivalently,

$$\int_c^d g \circ f^{-1} dt = \int_a^b g f'(t) dt. \quad (23)$$

Denote by  $\lambda_{[a,b]}, \lambda_{[c,d]}$  the Lebesgue measure restricted to the Borel subsets of  $[a, b]$  and  $[c, d]$ , respectively. Then eq. 23 implies that

$$d((f^{-1})_* \lambda_{[c,d]}) = f' d\lambda_{[a,b]},$$

since the integration of  $g = \mathbb{1}_{[\alpha, \beta]}$  gives the same results for any interval  $[\alpha, \beta] \subset [a, b]$  and since a finite Borel measure on  $[a, b]$  is determined by its values on such intervals. Thus,  $(f^{-1})_* \lambda_{[c,d]} \ll \lambda_{[a,b]}$  and

$$\frac{d((f^{-1})_* \lambda_{[c,d]})}{d\lambda_{[a,b]}} = f'.$$

## 18 Complex and Signed Measures (4.3, [Teschl(2010)])

**Definition 18.1.** A **complex measure** on  $(X, \mathcal{B})$  is a map  $\nu : \mathcal{B} \rightarrow \mathbb{C}$  s.t.  $\nu(\emptyset) = 0$  and

$$\nu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \nu(A_n)$$

for any disjoint  $A_n \in \mathcal{B}$ , where the series is assumed to be absolutely convergent. If  $\nu$  takes values in  $\mathbb{R}$  then  $\nu$  is called a **finite signed measure**.

**Remark.** More generally, a signed measure is allowed to take values in  $\mathbb{R} \cup \{+\infty\}$  or  $\mathbb{R} \cup \{-\infty\}$ .

Given a complex measure  $\nu$  on  $(X, \mathcal{B})$ , its **total variation** is the map  $|\nu| : \mathcal{B} \rightarrow [0, +\infty]$  defined by

$$|\nu|(A) = \sup \left\{ \sum_{n=1}^N |\nu(A_n)| : A = \bigcup_{n=1}^N A_n, A_n \in \mathcal{B}, A_n \cap A_m = \emptyset \right\}.$$

**Proposition 18.2.**  $|\nu|$  is a finite measure on  $(X, \mathcal{B})$ .

**Example 18.3.** Consider a measure space  $(X, \mathcal{B}, \mu)$  and take  $f \in L^1(X, d\mu)$ . Define

$$\nu(A) = \int_A f d\mu.$$

Then  $\nu$  is a complex measure on  $(X, \mathcal{B})$ , since this is true for  $f \geq 0$  and a general  $f$  can be written as a linear combination of positive ones. We write  $d\nu = f d\mu$ .

We then have  $d|\nu| = |f| d\mu$ , that is,

$$|\nu|(A) = \int_A |f| d\mu.$$

**Definition 18.4.** If  $(X, \mathcal{B}, \mu)$  is a measure space,  $\nu$  is a complex measure on  $(X, \mathcal{B})$ , then we say that  $\nu$  is **absolutely continuous** w.r.t.  $\mu$  and write  $\nu \ll \mu$ , if  $\nu(A) = 0$  whenever  $A \in \mathcal{B}, \mu(A) = 0$ . Equivalently,  $|\nu| \ll \mu$ .

**Theorem 18.5** (Radon-Nikodym theorem for complex measures). Assume  $(X, \mathcal{B}, \mu)$  is a measure space,  $\nu$  is a complex measure on  $(X, \mathcal{B})$ ,  $\nu \ll \mu$ . Then there is a unique  $f \in L^1(X, d\mu)$  s.t.  $d\nu = f d\mu$ .

## 19 Decomposition Theorems (20, [Schilling(2017)] and 4.3, [Teschl(2010)])

**Definition 19.1.** Two measures  $\nu$  and  $\mu$  on a measurable space  $(X, \mathcal{B})$  are called **mutually singular**, or we say that  $\nu$  is **singular** w.r.t.  $\mu$ , if there is a  $N \in \mathcal{B}$  s.t.  $\nu(N^c) = 0$ ,  $\mu(N) = 0$ . We then write  $\nu \perp \mu$ .

**Theorem 19.2** (Lebesgue Decomposition Theorem). Assume  $\nu, \mu$  are  $\sigma$ -finite measures in  $(X, \mathcal{B})$ . Then there exist unique measures  $\nu_a$  and  $\nu_s$  s.t.  $\nu = \nu_a + \nu_s$ ,  $\nu_a \ll \mu$ ,  $\nu_s \perp \mu$ .

**Theorem 19.3** (Polar Decomposition of Complex Measure). Assume  $\nu$  is a complex measure on  $(X, \mathcal{B})$ . Then there exist a finite measure  $\mu$  on  $(X, \mathcal{B})$  and a measurable function  $f : X \rightarrow \mathbb{C}$  s.t.  $d\nu = f d\mu$ . If  $(\tilde{\mu}, \tilde{f})$  is another such pair, then  $\tilde{\mu} = \mu$  and  $\tilde{f} = f$   $\mu$ -a.e.

For signed measures this leads to the following.

**Theorem 19.4** (Hahn Decomposition Theorem). Assume  $\nu$  is a finite signed measure on  $(X, \mathcal{B})$ . Then there exist  $P, N \in \mathcal{B}$  s.t.

$$\begin{aligned} X &= P \cup N, \quad P \cap N = \emptyset, \\ \nu(A \cap P) &\geq 0, \quad \nu(A \cap N) \leq 0 \quad \forall A \in \mathcal{B}. \end{aligned}$$

Moreover, then  $|\nu|(A) = \nu(A \cap P) - \nu(A \cap N)$ , and if  $X = \tilde{P} \cup \tilde{N}$  is another such decomposition, then

$$|\nu|(P \Delta \tilde{P}) = |\nu|(N \Delta \tilde{N}) = 0.$$

**Corollary 19.5** (Jordan Decomposition Theorem). Assume  $\nu$  is a finite signed measure on  $(X, \mathcal{B})$ . Then there exist unique finite measures  $\nu_+, \nu_-$  on  $(X, \mathcal{B})$  s.t.

$$\nu = \nu_+ - \nu_- \quad \text{and} \quad \nu_+ \perp \nu_-.$$

Moreover, then  $|\nu| = \nu_+ + \nu_-$ , hence

$$\nu_+ = \frac{|\nu| + \nu}{2}, \quad \nu_- = \frac{|\nu| - \nu}{2}.$$

## 20 More on Duals of $L^p$ -spaces (21, p. 241, [Schilling(2017)])

- What is the dual of  $L^p(X, d\mu)$ ? When does a measurable function  $g : X \rightarrow \mathbb{C}$  define a bounded linear functional on  $L^p(X, d\mu)$  by

$$\rho(f) = \int_X f g d\mu?$$

**Theorem 20.1** (Young's inequality). Assume  $f : [0, a] \rightarrow [0, b]$  is a strictly increasing continuous function,  $f(0) = 0$ ,  $f(a) = b$ . Then, for all  $s \in [0, a]$  and  $t \in [0, b]$ , we have

$$st \leq \int_0^s f(x) dx + \int_0^t f^{-1}(y) dy,$$

and the equality holds iff  $t = f(s)$ .

With Holder's inequality it follows that every  $g \in L^q(X, d\mu)$  defines a bounded linear functional

$$l_g : L^p(X, d\mu) \rightarrow \mathbb{C}, l_g(f) = \int_X f g d\mu,$$

and  $\|l_g\| \leq \|g\|_q$ .

The same makes sense for  $p = 1, q = \infty$  and  $p = \infty, q = 1$ , when  $\mu$  is  $\sigma$ -finite, as

$$\int_X |f g| d\mu \leq \int_X |f| d\mu \cdot \|g\|_\infty = \|f\|_1 \cdot \|g\|_\infty.$$

**Lemma 20.2.** Assume  $1 \leq p \leq \infty$ ,  $1/p + 1/q = 1$ , and  $\mu$  is  $\sigma$ -finite if  $p = 1$  or  $p = \infty$ . For  $g \in L^q(X, d\mu)$  consider  $l_g \in L^p(X, d\mu)^*$ . Then

$$\|l_g\| = \|g\|_q.$$

Therefor we can view  $L^q(X, d\mu)$  as a subspace of  $L^p(X, d\mu)^*$  using the isometric embedding

$$L^q(X, d\mu) \rightarrow L^p(X, d\mu)^*, \quad g \mapsto l_g.$$

**Theorem 20.3.** Assume  $(X, \mathcal{B}, d\mu)$  is a  $\sigma$ -finite measure space,  $1 \leq p < \infty$ ,  $1/p + 1/q = 1$ . Then

$$L^p(X, d\mu)^* = L^q(X, d\mu).$$

**Remark.** This is usually not true for  $p = \infty$ .

## 21 + 22 Riesz-Markow Theorem (21, p. [243-249], [Schilling(2017)])

Assume  $(X, d)$  is a locally compact metric space.

**Definition 21.22** (positive linear functional). A linear functional  $\rho : C_c(X) \rightarrow \mathbb{C}$  is called positive if  $\rho(f) \geq 0$  for all  $f \geq 0$ . (Recall  $C_c(X)$  is the space of continuous (C) compactly supported (c) functions.)

**Theorem 21.23** (Riesz-Markov). If  $\rho : C_c(X) \rightarrow \mathbb{C}$  is a positive linear functional, where  $(X, d)$  is a locally compact metric space, then there exists a Borel measure  $\mu$  on  $X$  s.t.  $\mu(K) < \infty$  for every compact  $K \subset X$  and

$$\rho(f) = \int_X f d\mu \quad \forall f \in C_c(X).$$

If  $X$  is separable, then such a measure  $\mu$  is unique.

For the proof we need two auxiliary results.

**Lemma 21.24** (Urysohn's Lemma). Assume  $(X, d)$  is a metric space,  $A, B \subset X$  are disjoint closed subsets. Then there exists a continuous function  $f : X \rightarrow [0, 1]$  s.t.  $f \equiv 1$  on  $A$  and  $f \equiv 0$  on  $B$ .

**Lemma 21.25.** Assume  $(X, d)$  is a compact metric space,  $U = (U_i)_{i=1}^n$  is a finite open cover of  $X$  (so  $U_i$  are open and  $\cup_{i=1}^n U_i = X$ ). Then there exist functions  $\rho_1, \dots, \rho_n$  in  $C(X)$  s.t.

$$0 \leq \rho_i \leq 1, \text{ supp}(\rho_i) \subset U_i, \sum_{i=1}^n \rho_i(x) = 1 \quad \forall x.$$

Every such collection of functions is called a **partition of unity subordinate** to  $U$ .

**Remark.** Without separability, the uniqueness is not always true. It can be checked that the measure we constructed in the proof has the following properties:

- (i)  $\mu(K) < \infty \quad \forall \text{ compact } K \subset X$ ;
- (ii)  $\mu$  is outer regular ( $\mu(A) = \inf_{U \text{ open}, A \subset U} \mu(U)$ );
- (iii)  $\mu$  is inner regular on open sets (this is where we need the full strength of step 3):

$$\mu(U) = \sup_{\substack{K \subset U \\ K \text{ compact}}} \mu(K) \quad \forall \text{ open } U.$$

Such measures are called **Radon measures**. It can be shown that the uniqueness holds within the class of Radon measures.

## Dual of $C(X)$

As an application of the Riesz-Markow Theorem we will describe  $C(X)^*$  in terms of measures for compact metric spaces  $(X, d)$ .

Denote by  $M(X)$  the space of complex Borel measures on  $X$ . For every  $\nu \in M(X)$  we want to make sense of  $\int_X f d\nu$  for  $f \in C(X)$ . It is enough to consider finite signed measures, as then we can define

$$\int_X f d\nu = \int_X f d(\text{Re}\nu) + i \int_X f d(\text{Im}\nu).$$

So assume  $\nu$  is a finite signed measure. Then,  $\nu = \mu_1 - \mu_2$  for positive measures and we define

$$\int_X f d\nu = \int_X f d\mu_1 - \int_X f d\mu_2.$$

This is well-defined, since if

$$\nu = \mu_1 - \mu_2 = \omega_1 - \omega_2,$$

then  $\mu_1 + \omega_2 = \mu_2 + \omega_1$  and

$$\int_X f d\mu_1 + \int_X f d\omega_2 = \int_X f d\mu_2 + \int_X f d\omega_1.$$

Thus, every  $\nu \in M(X)$  defines a linear functional

$$\phi_\nu : C(X)\mathbb{C} \text{ by } \phi_\nu(f) = \int_X f d\nu,$$

and the map  $\nu \mapsto \phi_\nu$  is linear.

**Lemma 21.26.** If  $\nu \in M(X)$  and  $d\nu = g d|\nu|$  is its polar decomposition, then

$$\int_X f d\nu = \int_X f g d|\nu| \quad \forall f \in C(X).$$

**Lemma 21.27.** For every  $\nu \in M(X)$ , the linear functional  $\phi_\nu$  is bounded and  $\|\phi_\nu\| = |\nu|(X)$ . (Recall that the norm on  $C(X)$  is  $\|f\| = \sup_{x \in X} |f(x)|$ .)

## 23 Product Measures and Fubini's Theorem

(14, [Schilling(2017)])

Throughout this chapter we assume that  $(X, \mathcal{A}, \mu)$  and  $(Y, \mathcal{B}, \nu)$  are  $\sigma$ -finite measure spaces.

Recall the Cartesian product of sets (assume  $A \subset X, B \subset Y$ ):

$$A \times B = \{(a, b) | a \in A \text{ and } b \in B\}.$$

(there are hidden properties here.)

The Lebesgue measure on  $\mathbb{R}^n$  has the following property for  $n > d \geq 1$ :

$$\lambda^n[a_1, b_1] \times \dots \times [a_n, b_n] = \lambda^d[a_1, b_1] \times \dots \times [a_d, b_d] \cdot \lambda^{n-d}[a_{d+1}, b_{d+1}]$$

which means that

$$\lambda^n(E) = \int \mathbb{1}_E(x, y) \lambda^n(d(x, y)) = \int \left( \int \mathbb{1}_E(x_0, y) \lambda^{n-d}(dy) \right) \lambda^d(dx_0).$$

**Goal:** we want to define a measure  $\rho$  on rectangles on the form  $A \times B$  s.t.  $\rho(A \times B) = \mu(A)\nu(B)$ .

**Lemma 23.1.** Let  $\mathcal{A}$  and  $\mathcal{B}$  be two  $\sigma$ -algebras (or semi-rings), then  $\mathcal{A} \times \mathcal{B}$  is a semi-ring.

**Definition 23.2** (product  $\sigma$ -algebra). The  $\sigma$ -algebra  $\mathcal{A} \otimes \mathcal{B} := \sigma(\mathcal{A} \times \mathcal{B})$  is called a **product  $\sigma$ -algebra**, and  $(X \times Y, \mathcal{A} \otimes \mathcal{B})$  is the product of measurable spaces.

When considering  $\mathcal{A} \otimes \mathcal{B}$ , the following lemma shows that we can instead work with their generators.

**Lemma 23.3.** If  $\mathcal{A} = \sigma(\mathcal{F})$  and  $\mathcal{B} = \sigma(\mathcal{G})$  and if  $\mathcal{F}$  and  $\mathcal{G}$  contain exhausting sequences  $(F_n)_{n \in \mathbb{N}} \subset \mathcal{F}$ ,  $F_n \uparrow X$  and  $(G_n)_{n \in \mathbb{N}} \subset \mathcal{G}$ ,  $G_n \uparrow Y$ , then

$$\sigma(\mathcal{F} \times \mathcal{G}) = \sigma(\mathcal{A} \times \mathcal{B}) := \mathcal{A} \otimes \mathcal{B}.$$

**Theorem 23.4** (uniqueness of product measures). Assume that  $\mathcal{A} = \sigma(\mathcal{F})$  and  $\mathcal{B} = \sigma(\mathcal{G})$ . If

- $\mathcal{F}, \mathcal{G}$  is  $\cap$ -stable (stable under finite intersections),
- $\mathcal{F}, \mathcal{G}$  contain exhausting sequences  $F_k \uparrow X$  and  $G_k \uparrow Y$  with  $\mu(F_k) < \infty$  and  $\nu(G_k) < \infty$ ,

then there is at most one measure  $\rho$  on  $(X \times Y, \mathcal{A} \otimes \mathcal{B})$  satisfying

$$\rho(F \times G) = \mu(F)\nu(G) \quad \forall F \in \mathcal{F}, G \in \mathcal{G}.$$

**Theorem 23.5** (existence of product measures). The set function

$$\rho : \mathcal{A} \times \mathcal{B} \rightarrow [0, \infty], \quad \rho(A \times B) := \mu(A)\nu(B),$$

extends uniquely to a  $\sigma$ -finite measure on  $(X \times Y, \mathcal{A} \otimes \mathcal{B})$  s.t.

$$\rho(E) = \int \int \mathbb{1}_E(x, y) \mu(dx) \nu(dy) = \int \int \mathbb{1}_E(x, y) \nu(dy) \mu(dx)$$

holds for all  $E \in \mathcal{A} \otimes \mathcal{B}$  (the parenthesis in the expression above are left out). In particular, the functions

$$\begin{aligned} x &\mapsto \mathbb{1}_E(x, y), x \mapsto \int \mathbb{1}_E(x, y) \nu(dy), \\ y &\mapsto \mathbb{1}_E(x, y), y \mapsto \int \mathbb{1}_E(x, y) \mu(dx), \end{aligned}$$

are  $\mathcal{A}$ ,  $\mathcal{B}$ -measurable (respectively) for every fixed  $y \in Y$ ,  $x \in X$  (respectively).

## Lecture 24

**Definition 24.25** (product measure  $\mu \times \nu$ ). The unique measure  $\rho$  constructed in Theorem 23.5 is called the **product** of the measures  $\mu$  and  $\nu$ , denoted  $\mu \times \nu$ .  $(X, Y, \mathcal{A} \otimes \mathcal{B}, \mu \times \nu)$  is called the **product measure space**

We can now finally construct the  $n$ -dimensional Lebesgue measure:

**Corollary 24.26.** If  $n > d \geq 1$ ,

$$(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n), \lambda^n) = \left( \mathbb{R}^d \times \mathbb{R}^{n-d}, \mathcal{B}(\mathbb{R}^d) \otimes \mathcal{B}(\mathbb{R}^{n-d}), \lambda^d \times \lambda^{n-d} \right).$$

Great. The next step is to see how we can integrate w.r.t. to  $\mu \times \nu$ . The following two results are often stated together as the Fubini or Fubini-Tonelli theorem.

**Theorem 24.27** (Tonelli). Let  $(X, \mathcal{A}, \mu)$   $(Y, \mathcal{B}, \nu)$  be  $\sigma$ -finite measure spaces and let  $u : X \times Y \rightarrow [0, \infty]$  be  $\mathcal{A} \otimes \mathcal{B}$ -measurable. Then

- (i)  $x \mapsto u(x, y)$ ,  $y \mapsto u(x, y)$  are  $\mathcal{A}$ -resp.  $\mathcal{B}$ -measurable for fixed  $y$  resp.  $x$ ;
- (ii)  $x \mapsto \int_Y u(x, y) \nu(dy)$ ,  $y \mapsto \int_X u(x, y) \mu(dx)$  are  $\mathcal{A}$ -resp.  $\mathcal{B}$ -measurable;
- (iii)  $\int_{X \times Y} u d(\mu \times \nu) = \int_Y \int_X u(x, y) \mu(dx) \nu(dy) = \int_X \int_Y u(x, y) \nu(dy) \mu(dx)$  which is in  $[0, \infty]$ .

The following corollary really extends Tonelli to not necessarily positive functions.

**Corollary 24.28** (Fubini's theorem). Let  $(X, \mathcal{A}, \mu)$   $(Y, \mathcal{B}, \nu)$  be  $\sigma$ -finite measure spaces and let  $u : X \times Y \rightarrow \mathbb{R}$  be  $\mathcal{A} \otimes \mathcal{B}$ -measurable. If at least one of the three integrals

$$\int_{X \times Y} |u| d(\mu \times \nu), \int_Y \int_X |u(x, y)| \mu(dx) \nu(dy), \int_X \int_Y |u(x, y)| \nu(dy) \mu(dx)$$

is finite, then all three integrals are finite,  $u \in \mathcal{L}^1(\mu \times \nu)$ , and

- (i)  $x \mapsto u(x, y)$  is in  $\mathcal{L}^1(\mu)$  for  $\nu$ -a.e.  $y \in Y$ ;
- (ii)  $y \mapsto u(x, y)$  is in  $\mathcal{L}^1(\nu)$  for  $\mu$ -a.e.  $x \in X$ ;
- (iii)  $y \mapsto \int_X u(x, y) \mu(dx)$  is in  $\mathcal{L}^1(\nu)$ ;
- (iv)  $x \mapsto \int_Y u(x, y) \nu(dy)$  is in  $\mathcal{L}^1(\mu)$ ;
- (v)  $\int_{X \times Y} u d(\mu \times \nu) = \int_Y \int_X u(x, y) \mu(dx) \nu(dy) = \int_X \int_Y u(x, y) \nu(dy) \mu(dx)$ .

## 25 Fourier Transform (§13 (pp. 125-128), §15 (pp. 157-158), §19 (pp. 214-217), [Schilling(2017)])

We write  $L^1(\mathbb{R}^n)$  for  $L^1(\mathbb{R}^n, d\lambda_n)$ .

The Fourier transform of a function  $f \in L^1(\mathbb{R}^n)$  is the function  $\hat{f}$  on  $\mathbb{R}^n$  defined by

$$\hat{f}(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} f(y) e^{-i\langle x, y \rangle} dy,$$

where  $\langle x, y \rangle = x_1 y_1 + \dots + x_n y_n$ . More generally, given a finite Borel measure  $\mu$ , its *Fourier transform* is the function  $\hat{\mu}$  on  $\mathbb{R}^n$  defined by

$$\hat{\mu}(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{-i\langle x, y \rangle} d\mu(y).$$

We can also define  $\mu$  for complex Borel measures.

**Warning.** There are many different conventions for the Fourier transform: instead of  $1/(2\pi)^n$  one also uses  $1/(2\pi)^{n/2}$ ; instead of  $e^{-i\langle x, y \rangle}$  one also uses  $e^{i\langle x, y \rangle}$ ,  $e^{\pm 2\pi i \langle x, y \rangle}$ .

Note that if  $\mu_f$  for  $f \in L^1(\mathbb{R}^n)$  is defined by  $d\mu_f = f d\lambda_n$ , then

$$\hat{\mu}_f = \hat{f}.$$

**Lemma 25.1.** If  $\mu$  is a complex Borel measure on  $\mathbb{R}^n$ , then  $\hat{\mu}$  is a Bounded continuous function on  $\mathbb{R}^n$ , and  $|\hat{\mu}(x)| \leq \frac{|\mu|(\mathbb{R}^n)}{(2\pi)^n}$ .

In particular, if  $f \in L^1(\mathbb{R}^n)$ , then  $\hat{f}$  is Bounded and continuous,

$$|\hat{f}(x)| \leq \frac{\|f\|_1}{(2\pi)^n} \quad \forall x.$$

Some properties:

- (i) If  $f_t(x) = f(x - t)$ , then

$$\hat{f}_t(y) = e^{-i\langle t, y \rangle} \hat{f}(y).$$

- (ii) If  $e_t(x) = e^{i\langle t, x \rangle}$ , then

$$e_t \hat{f}(y) = \hat{f}(y - t).$$

- (iii) If  $T \in GL_n(\mathbb{R})$  (invertible  $n \times n$  matrices), then

$$f \circ T = |\det T|^{-1} \hat{f} \circ (T^t)^{-1}.$$

- (iv)  $\hat{\hat{f}}(x) = \tilde{f}(-x)$ .

**Important example.**

If  $f(x) = e^{-\frac{|x|^2}{2}}$  ( $|x| = \langle x, x \rangle^{1/2}$ ), then  $\hat{f}(x) = 1/(2\pi)^n e^{-\frac{|x|^2}{2}}$ . More generally, if  $f(x) = e^{-\frac{c|x|^2}{2}}$ , then

$$\hat{f}(x) = \frac{1}{(2\pi c)^{\frac{n}{2}}} e^{-\frac{|x|^2}{2c}} \quad \forall c > 0.$$

This follows from property (iii).

For functions  $f, g$  on  $\mathbb{R}^n$ , their **convolution** is the function  $f \times g$  defined by

$$(f * g)(x) := \int_{\mathbb{R}^n} f(y) g(x - y) dy = \int_{\mathbb{R}^n} f(x - y) g(y) dy$$

when is this well-defined?



**Lemma 25.2.** If  $f, g \in L^1(\mathbb{R}^n)$ , then the function  $y \mapsto f(y)g(x-y)$  is integrable for  $(\lambda_n)$ -a.e.  $x$ ,  $f * g \in L^1(\mathbb{R}^n)$  and  $\|f * g\|_1 \leq \|f\|_1 \|g\|_1$ .

Next let us show that  $f * g$  is well-defined for  $f \in L^1(\mathbb{R}^n)$ ,  $g \in L^p(\mathbb{R}^n)$ ,  $1 \leq p \leq \infty$ .

**Lemma 25.3.** Assume  $\phi : (a, b) \rightarrow \mathbb{R}$  is a convex function. Then  $\phi$  is continuous and  $\phi(x) = \sup\{l(x) : \phi \geq l, l(s) = \alpha s + \beta\}$ .

**Theorem 25.4** (Jensen's inequality). Assume  $(X, \mathcal{B}, \mu)$  is a probability measure space (so  $\mu(X) = 1$ ),  $\phi : [0, \infty) \rightarrow [0, \infty)$  is a convex function. Then, for every integrable function  $f : X \rightarrow [0, \infty)$  we have

$$\phi\left(\int_X f d\mu\right) \leq \int_X \phi \circ f d\mu.$$

The same inequality holds for any measurable  $f : X \rightarrow [0, \infty]$  if  $\lim_{x \rightarrow \infty} \phi(x) = +\infty$  and we put  $\phi(+\infty) = +\infty$ .

## 26 Regularization (15 & 19, [Schilling(2017)])

**Lemma 26.1.** Assume  $f \in L^1(\mathbb{R}^n)$ ,  $g \in L^p(\mathbb{R}^n)$  ( $1 \leq p \leq \infty$ ). Then the function  $g \mapsto f(g)g(x-y)$  is integrable for a.e.  $x$ ,  $f * g \in L^p(\mathbb{R}^n)$  and  $\|f * g\|_p \leq \|f\|_1 \|g\|_p$ .

Note that

$$\int_{\mathbb{R}^n} f(y)g(x-y)dy = \int_{\mathbb{R}^n} f(x-y)g(y)dy,$$

so  $f * g = g * f$ .

**Remark.** More generally, for  $\mu \in M(\mathbb{R}^n)$  and  $g \in L^p(\mathbb{R}^n)$ , we can define  $\mu * g = g * \mu \in L^p(\mathbb{R}^n)$  by

$$(\mu * g)(x) = \int_{\mathbb{R}^n} g(x-y)d\mu(y).$$

Then  $\|\mu * g\|_p \leq |\mu|(\mathbb{R}^n)\|g\|_p$ .

**Proposition 26.2.** If  $f, g \in L^1(\mathbb{R}^n)$ , then  $f \hat{*} g = (2\pi)^n \hat{f} \hat{g}$ .

What are convolutions good for?

**Example 26.3.** Consider

$$f = \frac{1}{\lambda_n(B_r(0))} \mathbb{1}_{B_r(0)}.$$

Then

$$\begin{aligned} (f * g)(x) &= \frac{1}{\lambda_n(B_r(0))} \int_{B_r(0)} g(x-y)dy \\ &= \frac{1}{\lambda_n(B_r(x))} \int_{B_r(x)} g(y)dy. \end{aligned}$$

For a multi-index  $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}_+^n$ , write  $\partial^\alpha$  for

$$\frac{\partial^{\alpha_1 + \dots + \alpha_n}}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}}.$$

Denote by  $L_{\text{loc}}^1(\mathbb{R}^n)$  the space of Lebesgue measurable functions that are independent on every ball. We identify functions that coincide a.e. (so,  $L_{\text{loc}}^1(\mathbb{R}^n)$  is a space of equivalence classes of functions). We have  $L^p(\mathbb{R}^n) \subset L_{\text{loc}}^1(\mathbb{R}^n)$  for all  $1 \leq p \leq \infty$ .

**Lemma 26.4.** If  $\phi \in C_c^\infty(\mathbb{R}^n)$  and  $f \in L_{\text{loc}}^1(\mathbb{R}^n)$ , then  $\phi * f \in C^\infty(\mathbb{R}^n)$  and  $\partial^\alpha(\phi * f) = (\partial^\alpha \phi) * f$ .

By choosing suitable  $\phi$  we can make  $\phi * f$  close to  $f$ , as we will see shortly.

**Definition 26.5.** A positive ‘modifier’ (?) is a function  $\phi \in C_c(\mathbb{R}^n)$  s.t.  $\phi \geq 0$  and  $\int_{\mathbb{R}^n} \phi(x)dx = 1$ .

For a function  $\phi$  on  $\mathbb{R}^n$  and  $\epsilon > 0$ , define  $\phi^\epsilon(x) := \epsilon^{-n} \phi(x/\epsilon)$ . Note that if  $\phi \in L^1(\mathbb{R}^n)$ ,  $\int_{\mathbb{R}^n} \phi dx = 1$ , then  $\int_{\mathbb{R}^n} \phi^\epsilon = 1$ .

**Example 26.6** (positive modifier). Consider the function  $h$  in  $\mathbb{R}$  defined by,

$$h(t) = \begin{cases} e^{-\frac{1}{1-t^2}}, & |t| < 1, \\ 0, & |t| \geq 1 \end{cases}.$$

Then  $g \in C_c^\infty(\mathbb{R}^n)$ . Hence,  $\phi(x) = c_n h(|x|)$  is a modifier, where  $c_n = \left(\int_{\mathbb{R}^n} h(|x|)dx\right)^{-1}$ .

**Proposition 26.7.** Let  $\phi \in L^1(\mathbb{R}^n)$  be s.t.  $\phi \geq 0$  and  $\int_{\mathbb{R}^n} \phi dx = 1$ . Then we have

(i) If  $f \in C_0(\mathbb{R}^n)$  (continuous functions vanishing at infinity (?)), then  $\phi^\epsilon * f \in C_0(\mathbb{R}^n)$  and  $\|\phi^\epsilon * f\| \xrightarrow{\epsilon \downarrow 0} 0$  (uniform norm);

(ii) If  $f \in L^p(\mathbb{R}^n)$ ,  $1 \leq p \leq \infty$ , then  $\|\phi^\epsilon * f - f\|_p \xrightarrow{\epsilon \downarrow 0} 0$ .

**Corollary 26.8.** For any Radon measure  $\mu$  on  $\mathbb{R}^n$ ,  $C_c^\infty(\mathbb{R}^n)$  is dense in  $L^p(\mathbb{R}^n, d\mu)$  for  $1 \leq p < \infty$ .

## 27 Fourier Inverse (19, [Schilling(2017)])

**Corollary 27.1.** If  $f \in L^1(\mathbb{R}^n)$ , then  $\hat{f} \in C_0(\mathbb{R}^n)$ .

**Remark.** Another possibility is to approximate  $f$  by its linear combinations of  $\mathbb{1}_{[a_1, b_1] \times \dots \times [a_n, b_n]}$ . Note that for  $\mathbb{1}_{[a, b]} \in L^1(\mathbb{R})$  we have  $\hat{\mathbb{1}}_{a, b}(x) = 1/(2\pi) \int_a^b e^{-ixy} dy = \frac{e^{-iax} - e^{-ibx}}{2\pi ix} \xrightarrow{x \rightarrow \infty} 0$ .

**Theorem 27.2** (Fourier Inversion Theorem). Assume  $f \in L^1 \mathbb{R}^n$  is s.t.  $\hat{f} \in L^1(\mathbb{R}^n)$ . Then, for a.e.  $x$ , we have

$$f(x) = \int_{\mathbb{R}^n} \hat{f}(y) e^{i\langle x, y \rangle} dy.$$

Equivalently,

$$\hat{\hat{f}}(x) = \frac{1}{(2\pi)^n} f(-x).$$

**Theorem 27.3** (convolution theorem). If  $\mu, \nu$  are finite measures on  $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$ , then

$$\widehat{\mu * \nu}(\xi) = (2\pi)^n \hat{\mu}(\xi) \hat{\nu}(\xi) \text{ and } \mathcal{F}^{-1} \mu * \nu(\xi) = \mathcal{F}^{-1} \mu(\xi) \mathcal{F}^{-1} \nu(\xi).$$

**Lemma 27.4.** For any  $f, g \in L^1(\mathbb{R}^n)$ , we have

$$\int_{\mathbb{R}^n} \hat{f} g dx = \int_{\mathbb{R}^n} f \hat{g} dx.$$

(Note that  $\hat{f} g \in L^1(\mathbb{R}^n)$ , as  $\hat{f}$  is bounded.)

**Corollary 27.5.** If  $f \in L^1(\mathbb{R}^n)$  and  $\hat{f} = 0$ , then  $f = 0$  (a.e.)

Recall that a linear operator  $U : H \rightarrow K$  between Hilbert spaces is called an **isometry** if

$$\|Ux\| = \|x\| \quad \forall x \in H.$$

Equivalently, by the polarization identity,

$$(Ux, Uy) = (x, y) \quad \forall x, y \in H.$$

If  $U$  is in addition surjective, then it is called a **unitary**.

**Theorem 27.6** (Plancherel). There is a unique unitary  $U : L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n)$  s.t.  $Uf = (2\pi)^{n/2} \hat{f}$  for all  $f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ .

In the book: If  $u \in L^2_{\mathbb{C}}(\lambda^n) \cap L^1_{\mathbb{C}}(\lambda^n)$ , then

$$\|\hat{u}\|_2 = (2\pi)^{-n/2} \|u\|_2.$$

## 28 Schwartz Space 19, [Schilling(2017)]

**Proposition 28.1.** Assume  $f \in L^1(\mathbb{R}^n)$  and  $x_j f \in L^1(\mathbb{R}^n)$  ( $x_j f$  means  $x \mapsto x_j f(x)$ ) for some  $1 \leq j \leq n$ . Then

$$\partial_j \hat{f} = -ix_j \hat{f}$$

**Proposition 28.2.** Assume  $f \in C(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$  is such that  $\partial_j f \in L^1(\mathbb{R}^n)$ . Then

$$\widehat{\partial_j f} = ix_j \hat{f}$$

**Corollary 28.3.** If  $f, \partial_j f \in L^1(\mathbb{R}^n)$ , then

$$x_j \hat{f}(x) \xrightarrow{x \rightarrow \infty} 0.$$

**Corollary 28.4.** 1. If  $x^\alpha f \in L^1(\mathbb{R}^n)$  for all  $|\alpha| \leq N$ , then  $\hat{f} \in C^N(\mathbb{R}^n)$  and  $\partial^\alpha \hat{f} = (-i)^{|\alpha|} \widehat{x^\alpha f}$ .

2. If  $f \in C^N(\mathbb{R}^n)$  and  $\partial^\alpha f \in L^1(\mathbb{R}^n)$  for all  $|\alpha| \leq N$ , then  $\widehat{\partial^\alpha f} = i^{|\alpha|} \widehat{x^\alpha f}$  and hence  $(1 + |x|)^N \hat{f}(x) \xrightarrow{x \rightarrow \infty} 0$ .

(here  $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n$ :  $n$ -dim positive integers, and  $|\alpha| = \alpha_1 + \dots + \alpha_n$ ,  $x^\alpha = x_1^{\alpha_1} \cdot \dots \cdot x_n^{\alpha_n}$ .)

**Definition 28.5.** A function  $f$  on  $\mathbb{R}^n$  is called a **Schwartz function** if  $f \in C^\infty(\mathbb{R}^n)$  and  $x^\alpha \partial^\beta f$  is bounded for all multi-indices  $\alpha, \beta$ . The space  $\mathcal{S}(\mathbb{R}^n)$  of Schwartz functions is called **Schwartz space**.

Note that for every  $f \in C^\infty(\mathbb{R}^n)$  the following conditions are equivalent:

1.  $x^\alpha \partial^\beta f$  is bounded for all  $\alpha, \beta$ ;
2.  $x^\alpha (\partial^\beta f)(x) \xrightarrow{x \rightarrow \infty} 0$  for all  $\alpha, \beta$ ;
3.  $(1 + |x|)^N \partial^\beta f$  is bounded for all  $N \geq 1$  and all  $\beta$ .

**Example 28.6.**  $C_c^\infty(\mathbb{R}^n) \subset \mathcal{S}(\mathbb{R}^n)$ ,  $e^{-a|x|^2} \in \mathcal{S}(\mathbb{R}^n)$  for  $a > 0$ . If  $f \in \mathcal{S}(\mathbb{R}^n)$ , then  $x^\alpha \partial^\beta f \in \mathcal{S}(\mathbb{R}^n)$ . The product of two Schwartz functions is a Schwartz function.

## References

[Schilling(2017)] Schilling, R. 2017, Measures, Integrals and Martingales, Measures, Integrals and Martingales (Cambridge University Press). <https://books.google.no/books?id=sdAoDwAAQBAJ>

Clearly,  $\mathcal{S}(\mathbb{R}^n) \subset L^1(\mathbb{R}^n)$  for all  $1 \leq p \leq \infty$ . From the Corollary above we conclude that if  $f \in \mathcal{S}(\mathbb{R}^n)$ , then  $\hat{f} \in \mathcal{S}(\mathbb{R}^n)$ . By the Fourier inversion theorem we then get:

**Theorem 28.7.** The Fourier transform maps  $\mathcal{S}(\mathbb{R}^n)$  onto  $\mathcal{S}(\mathbb{R}^n)$ .

**Remark.** This gives another proof of the fact that the image of the Fourier transform  $L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n)$  is dense, which we needed to prove Plancherel's theorem.

**Remark.** If  $f \in C_c^\infty(\mathbb{R}^n)$ ,  $f \neq 0$ , then  $\hat{f} \in \mathcal{S}(\mathbb{R}^n)$ , but  $\hat{f}$  is never compactly supported, since it extends to an analytic function on  $\mathbb{C}^n$ :  $\hat{f}(z) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} f(y) e^{-i\langle z, y \rangle} dy$ .

**Theorem 28.8** (Kolmogorov extension theorem). Assume  $X$  is a set and  $(\mathcal{B}_n)_{n \in \mathbb{N}}$  is an increasing sequence of  $\sigma$ -algebras of subsets of  $X$ . Assume  $\mu_n$  if a measure on  $(X, \mathcal{B}_n)$  and

$$\mu_{n+1}|_{\text{vert}_{\mathcal{B}_n}} = \mu_n \quad \forall n.$$

Can we define a measure  $\mu$  on  $(X, \mathcal{B})$ , where  $\mathcal{B} = \sigma(\cup_{n \in \mathbb{N}} \mathcal{B}_n)$  s.t.  $\mu|_{\mathcal{B}_n} = \mu_n \quad \forall n$ ? - In general, no. But we have the following:

**Theorem 28.9.** In the above settings, assume in addition that  $\mu_n(X) = 1 \quad \forall n$  and there is a collection of subsets  $C \subset \mathcal{B}$  s.t.:

- (i)  $\mu_n(A) = \sup \{ \mu_n(C) : C \subset A, C \in \mathcal{C} \cap \mathcal{B}_n \} \quad \forall A \in \mathcal{B}_n$ ;
- (ii) If  $(C_n)_{n \in \mathbb{N}}$  is a sequence in  $\mathcal{C}$  and  $\cap_{n \in \mathbb{N}} C_n = \emptyset$ , then  $\cap_{n=1}^N C_n = \emptyset$  for some  $N$ .

Then there is a unique measure  $\mu$  on  $(X, \mathcal{B})$  s.t.  $\mu|_{\mathcal{B}_n} = \mu_n$ .

## Tips'n Tricks

- Assume we can write  $X$  as a finite union:  $X = \cup_{n \in I} A_n$ ,  $i = 1, \dots, N$ . Then

$$\int f d\mu = \int_X f d\mu = \int_{A_1} f d\mu + \int_{A_2} f d\mu + \dots + \int_{A_N} f d\mu.$$

## Questions

- In problem 26.18 we are supposed to show that  $Y_n \perp Y_m = 0$ , i.e. that  $\langle y_n, y_m \rangle = 0$ ,  $n \neq m$ . I get ...

$$\langle y_n, y \rangle \subset \int_{A_m^c} |y_n|^2 |y_m|^2 d\mu,$$

and I want to argue that this is zero since  $\int_{A_m^c} |y_m|^2 d\mu = 0$ , but I don't see how. The solutions are not clear, and I think perhaps my setup is wrong. I am assuming  $\langle f, g \rangle = \int_X f \bar{g} d\mu$ , i.e. from  $L^2$ , but perhaps it is rather  $\langle f, g \rangle = \int_{A_m^c \cup A_n^c} f \bar{g} d\mu$  or something?

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[Teschl(2010)] Teschl, G. 2010, Topics in Linear and Nonlinear Functional Analysis (Universität Wien). <https://www.mat.univie.ac.at/~gerald/ftp/book-fa/fa.pdf>