

# MAT4400: Notes on Linear analysis

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## 3 $\sigma$ -Algebras

**Definition 3.1** ( $\sigma$ -Algebra). A family  $\mathcal{A}$  of subsets of  $X$  with:

- (i)  $X \in \mathcal{A}$ ,
- (ii)  $A \in \mathcal{A} \Rightarrow A^c \in \mathcal{A}$ ,
- (iii)  $(A_n)_{n \in \mathbb{N}} \in \mathcal{A} \Rightarrow \bigcup_{n \in \mathbb{N}} A_n \in \mathcal{A}$

**Theorem 3.2** (and Definition).

- (i) *The intersection of arbitrarily many  $\sigma$ -algebras in  $X$  is again a  $\sigma$ -algebra in  $X$ .*
- (ii) *For every system of sets  $p \subset \mathcal{P}(X)$  there exists a smallest  $\sigma$ -algebra containing  $p$ . This is the  $\sigma$ -algebra generated by  $p$ , denoted  $\sigma(p)$ , and  $p$  is called its generator.*

**Definition 3.3** (Borel). The  $\sigma$ -algebra  $\sigma(\mathcal{O})$  generated by the open sets  $\mathcal{O} = \mathcal{O}_{\mathbb{R}^n}$  of  $\mathbb{R}^n$  is called **Borel  $\sigma$ -algebra**, and its members are called **Borel sets** or **Borel measurable sets**.

## 5 Uniqueness of Measures

**Lemma 5.1.** *A Dynkin system  $D$  is a  $\sigma$ -algebra iff it is stable under finite intersections, i.e.  $A, B \in D \Rightarrow A \cap B \in D$ .*

**Theorem 5.2** (Dynkin). *Assume  $X$  is a set,  $S$  is a collection of subsets of  $X$  closed under finite intersections, that is, if  $A, B \in S \Rightarrow A \cap B \in S$ . Then  $D(S) = \sigma(S)$ .*

**Theorem 5.3** (uniqueness of measures). *Let  $(X, B)$  be a measurable space, and  $S \subset P(X)$  be the generator of  $B$ , i.e.  $B = \sigma(S)$ . If  $S$  satisfies the following conditions:*

1.  *$S$  is stable under finite intersections ( $\cap$ -stable), i.e.  $A, C \in S \Rightarrow A \cap C \in S$ .*

2. There exists an exhausting sequence  $(G_n)_{n \in \mathbb{N}} \subset X$  with  $G_N \uparrow X$ . Assume also that there are two measures  $\mu, \nu$  satisfying:
3.  $\mu(A) = \nu(A), \forall A \in S$ .
4.  $\mu(G_n) = \nu(G_n) < \infty$ .

Then  $\mu = \nu$ .

## 6 Existence of Measures

**Theorem 6.1** (Carathéodory). *Let  $S \subset P(X)$  be a semi-ring and  $\mu : S \rightarrow [0, \infty)$  a pre-measure. Then  $\mu$  has an extension to a measure  $\mu^*$  on  $\sigma(S)$ , i.e. that  $\mu(s) = \mu^*(s), \forall s \in \sigma(S)$ .*

*Also, if  $S$  contains an exhausting sequence,  $S_n \uparrow X$ , s.t.  $\mu(S_n) < \infty$ , then the extension is unique.*

## 7 Measurable Mappings

We consider maps  $T : X \rightarrow X'$  between two measurable spaces  $(X, \mathcal{A})$  and  $(X', \mathcal{A}')$  which respects the measurable structures, the  $\sigma$ -algebras on  $X$  and  $X'$ . These maps are useful as we can transport a measure  $\mu$ , defined on  $(X, \mathcal{A})$ , to  $(X', \mathcal{A}')$ .

**Definition 7.1.** Let  $(X, \mathcal{A}), (X', \mathcal{A}')$  be measurable spaces. A map  $T : X \rightarrow X'$  is called  $\mathcal{A}/\mathcal{A}'$ -measurable if the pre-image of every measurable set is a measurable set:

$$T^{-1}(A') \in \mathcal{A}, \quad \forall A' \in \mathcal{A}'. \quad (1)$$

- A  $\mathcal{B}(\mathbb{R}^n)/\mathcal{B}(\mathbb{R}^m)$  measurable map is often called a Borel map.
- The notation  $T : (X, \mathcal{A}) \rightarrow (X', \mathcal{A}')$  is often used to indicate measurability of the map  $T$ .

**Lemma 7.2.** *Let  $(X, \mathcal{A}), (X', \mathcal{A}')$  be measurable spaces and let  $\mathcal{A}' = \sigma(\mathcal{G}')$ . Then  $T : X \rightarrow X'$  is  $\mathcal{A}/\mathcal{A}'$ -measurable iff  $T^{-1}(G') \in \mathcal{A}$ , i.e. if*

$$T^{-1}(G') \in \mathcal{A}, \quad \forall G' \in \mathcal{G}'. \quad (2)$$

**Theorem 7.3.** *Let  $(X_i, \mathcal{A}_i), i = 1, 2, 3$ , be measurable spaces and  $T : X_1 \rightarrow X_2, S : X_2 \rightarrow X_3$  be  $\mathcal{A}_1/\mathcal{A}_2$  and  $\mathcal{A}_2/\mathcal{A}_3$ -measurable maps respectively. Then  $S \circ T : X_1 \rightarrow X_3$  is  $\mathcal{A}_1/\mathcal{A}_3$ -measurable.*

**Corollary 7.4.** *Every continuous map between metric spaces is a Borel map.*

**Definition 7.5. (and lemma)** Let  $(T_i)_{i \in I}$ ,  $T_i : X \rightarrow X_i$ , be arbitrarily many mappings from the same space  $X$  into measurable spaces  $(X_i, \mathcal{A}_i)$ . The smallest  $\sigma$ -algebra on  $X$  that makes all  $T_i$  simultaneously measurable is

$$\sigma(T_i : i \in I) := \sigma \left( \bigcup_{i \in I} T_i^{-1}(\mathcal{A}_i) \right) \quad (3)$$

**Corollary 7.6.** A function  $f : (X, \mathcal{B}) \rightarrow \mathbb{R}$  is measurable if  $f((a, +\infty)) \in \mathcal{B}$ ,  $\forall a \in \mathbb{R}$ .

**Corollary 7.7.** Assume  $(X, \mathcal{B})$  is a measurable space,  $(Y, d)$  is a metric space,  $(f_n : (X, \mathcal{B}) \rightarrow Y)_{n=1}^\infty$  is a sequence of measurable maps. Assume this sequence of images  $(f_n(x))_{n=1}^\infty$  is convergent in  $Y \forall x \in X$ . Define

$$f : X \rightarrow Y, \text{ by } f(x) = \lim_{n \rightarrow \infty} f_n(x). \quad (4)$$

Then  $f$  is measurable.

**Theorem 7.8.** Let  $(X, \mathcal{A})$ ,  $(X', \mathcal{A}')$  be measurable spaces and  $T : X \rightarrow X'$  be an  $\mathcal{A}/\mathcal{A}'$ -measurable map. For every measurable  $\mu$  on  $(X, \mathcal{A})$ ,

$$\mu'(A') := \mu(T^{-1}(A')), \quad A' \in \mathcal{A}', \quad (5)$$

defines a measure on  $(X', \mathcal{A}')$ .

**Definition 7.9.** The measure  $\mu'(\cdot)$  in the above theorem is called the push forward or image measure of  $\mu$  under  $T$  and it is denoted as  $T(\mu)(\cdot)$ ,  $T_{*\mu}(\cdot)$  or  $\mu \circ T^{-1}(\cdot)$ .

**Theorem 7.10.** If  $T \in \mathbb{R}^{n \times n}$  is an orthogonal matrix, then  $\lambda^n = T(\lambda^n)$ .

**Theorem 7.11.** Let  $S \in \mathbb{R}^{n \times n}$  be an invertible matrix. Then

$$S(\lambda^n) = |\det S|^{-1} \lambda^n = |\det S|^{-1} \lambda^n. \quad (6)$$

**Corollary 7.12.** Lebesgue measure is invariant under motions:  $\lambda^n = M(\lambda^n)$  for all motions  $M$  in  $\mathbb{R}^n$ . In particular, congruent sets have the same measure. Two sets of points are called congruent if, and only if, one can be transformed into the other by an isometry

## 8 Measurable Functions

A *measurable function* is a measurable map  $u : X \rightarrow \mathbb{R}$  from some measurable space  $(X, \mathcal{A})$  to  $(\mathbb{R}, \mathcal{B}(\mathbb{R}^1))$ . They play central roles in the theory of integration.

We recall that  $u : X \rightarrow \mathbb{R}$  is  $\mathcal{A}/\mathcal{B}(\mathbb{R}^1)$ -measurable if

$$u^{-1}(B) \in \mathcal{A}, \quad \forall B \in \mathcal{B}(\mathbb{R}^1). \quad (7)$$

Moreover from a lemma from chapter 7, we actually only need to show that

$$u^{-1}(G) \in \mathcal{A}, \quad \forall G \in \mathcal{G} \text{ where } \mathcal{G} \text{ generates } \mathcal{B}(\mathbb{R}^1). \quad (8)$$

**Proposition 8.1.**

- 1 If  $f, g : (X, \mathcal{B}) \rightarrow \mathbb{C}$  are measurable, then the function  $f+g, f \cdot g, cf, (c \in \mathbb{C})$  are measurable.
- 2 If  $b : \mathbb{C} \rightarrow \mathbb{C}$  is Borel and  $b : (\mathbb{C}, \mathcal{B}) \rightarrow \mathbb{C}$  is measurable, then  $b \circ f$  is measurable.
- 3 If  $f(x) = \lim_{n \rightarrow \infty} f_n(x), x \in X$  and  $f_n$  are measurable, then  $f$  is measurable.
- 4 If  $X = \bigcup_{n=1}^{\infty} A_n, (A_n \in \mathcal{B}), f|_{A_n} : (A_n, \mathcal{B}_{A_n}) \rightarrow \mathbb{C}$  is measurable  $\forall n$ , then  $f$  is measurable.

**Definition 8.2.** Given a measurable space  $(X, \mathcal{B})$ , a measurable function  $f : (X, \mathcal{B}) \rightarrow \mathbb{C}$  is called simple if

$$f(x) = \sum_{k=1}^N c_k \mathbb{1}_{A_k}(x), \quad (9)$$

for some  $c_k \in \mathbb{C}, A_k \in \mathcal{B}$ , where  $\mathbb{1}$  is the characteristic function,

$$\mathbb{1}_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A. \end{cases} \quad (10)$$

The representation of simple function is **not** unique. We denote the standard representation of  $f$  by

$$f(x) = \sum_{n=0}^N z_n \mathbb{1}_{B_n}(x), \quad N \in \mathbb{N}, z_n \in \mathbb{R}, B_n \in \mathcal{A}, X = \bigcup_{n=1}^N B_n, \text{ for } B_n \cap B_m = \emptyset, n \neq m. \quad (11)$$

The set of simple functions is denoted  $\mathcal{E}(\mathcal{A})$  of  $\mathcal{E}$ .

**Definition 8.3.** Assume  $\mu$  is a measure on  $(X, \mathcal{B})$ . Given a *positive* simple function

$$f = \sum_{k=1}^N c_k \mathbb{1}_{A_k}, \quad (c_k \geq 0). \quad (12)$$

We define

$$\int_X f d\mu = \sum_{k=1}^n c_k \mu(A_k) \in [0, +\infty]. \quad (13)$$

We also denote this by  $I_\mu(f)$ .

**Lemma 8.4.** *This is well defined, that is,  $\int_x f d\mu$  does not depend on the presentation of the simple function  $f$ .*

**Properties 8.5.** *For every positive simple function*

$$1 \int_X c f d\mu = c \int_X f d\mu, \quad \text{for only } c \geq 0$$

$$2 \int_X (f + g) d\mu = \int_X f d\mu + \int_X g d\mu.$$

**Corollary 8.6.** *If  $f \geq g \geq 0$  are simple functions, then*

$$\int_X f d\mu \geq \int_X g d\mu. \quad (14)$$

**Definition 8.7.** *If  $f : X \rightarrow [0, +\infty)$  is measurable, then we define*

$$\int_X f d\mu = \sup \left\{ \int_X g d\mu : f \geq g \geq 0, \text{ } g \text{ is simple} \right\} \quad (15)$$

**Remark.** *This means that any measurable function can be approximated by simple functions.*

**Properties 8.8.** *Measurable functions like this have the following properties*

$$1 \int_X c f d\mu = c \int_X f d\mu, \quad \forall c \geq 0.$$

$$2 \text{ If } f \geq g \geq 0, \text{ then } \int_X f d\mu \geq \int_X g d\mu \text{ for any measurable } g, f.$$

$$3 \text{ If } f \geq 0 \text{ is simple, then } \int_X f d\mu \text{ is the same value as obtained before.}$$

To advance in measure theory we consider measurable functions

$$f : X \rightarrow [0, +\infty].$$

Measurability is understood w.r.t the  $\sigma$ -algebra  $\mathcal{B}([0, +\infty])$  generated by  $\mathcal{B}([0, +\infty))$  and  $\{+\infty\}$ . In other words,  $A \subset [0, +\infty] \in \mathcal{B}([0, +\infty])$  iff  $A \cap [0, +\infty) \in \mathcal{B}([0, +\infty))$ .

**Remark.** *Hence  $f : X \rightarrow [0, +\infty]$  is measurable iff  $f^{-1}(A)$  is measurable  $\forall A \in \mathcal{B}([0, +\infty])$ .*

**Definition 8.9.** *For measurable functions  $f : X \rightarrow [0, +\infty]$ , we define*

$$\int_X f d\mu = \sup \left\{ \int_X g d\mu : f \geq g \geq 0 : g \text{ is simple} \right\} \in [0, +\infty]. \quad (16)$$

**Theorem 8.10. Monotone convergence theorem** *Assume  $(X, \mathcal{B}, \mu)$  is a measure space,  $(f)_{n=1}^\infty$  is an increasing sequence of measurable positive functions  $f_n : X \rightarrow [0, +\infty]$ . Define  $f(x) = \lim_{n \rightarrow \infty} f_n(x)$ . Then  $f$  is measurable and*

$$\int_X f d\mu = \lim_{n \rightarrow \infty} \int_X f_n d\mu. \quad (17)$$

**Theorem 8.11.** Assume  $(X, \mathcal{B})$  is a measurable space and  $f : X \rightarrow [0, +\infty]$  is measurable. Then there are simple functions  $g_n$ , s.t.

$$0 \leq g_1 \leq g_2 \leq \dots, \quad g_n(x) \rightarrow f(x), \quad \forall x \in X.$$

Moreover, if  $f$  is bounded, we can choose  $g_n$  s.t. the convergence is uniform, that is,

$$\lim_{n \rightarrow \infty} \sup_{x \in X} |g_n(x) - f(x)| = 0. \quad (18)$$

## 10 Integrals of Measurable Functions

We have defined our integral for positive measurable functions, i.e. functions in  $\mathcal{M}^+(\mathcal{A})$ . To extend our integral to not only functions in  $\mathcal{M}^+(\mathcal{A})$  we first notice that

$$u \in \mathcal{M}_{\mathbb{R}}(\mathcal{A}) \Leftrightarrow u = u^+ - u^-, \quad u^+, u^- \in \mathcal{M}_{\mathbb{R}}^+, \quad (19)$$

i.e. that every measurable function can be written as a sum of **positive** measurable functions.

**Definition 10.1** ( $\mu$ -integrable). A function  $u : X \rightarrow \overline{\mathbb{R}}$  on  $(X, \mathcal{A}, \mu)$  is  $\mu$ -integrable, if it is  $\mathcal{A}/\mathcal{B}(\overline{\mathbb{R}})$ -measurable and if  $\int u^+ d\mu, \int u^- d\mu < \infty$  (recall the definition for the integral of positive measurable functions). Then

$$\int u d\mu := \int u^+ d\mu - \int u^- d\mu \in (-\infty, \infty) \quad (20)$$

is the ( $\mu$ -)integral of  $u$ . We write  $\mathcal{L}^1(\mu)$  for the set of all real-valued  $\mu$ -integrable functions<sup>1</sup>.

**Theorem 10.2.** Let  $u \in \mathcal{M}_{\mathbb{R}}(\mathcal{A})$ , then the following conditions are equivalent:

- (i)  $u \in \mathcal{L}_{\mathbb{R}}^1(\mu)$ .
- (ii)  $u^+, u^- \in \mathcal{L}_{\mathbb{R}}^1(\mu)$ .
- (iii)  $|u| \in \mathcal{L}_{\mathbb{R}}^1(\mu)$ .
- (iv)  $\exists w \in \mathcal{L}_{\mathbb{R}}^1(\mu)$  with  $w \geq 0$  s.t.  $|u| \leq w$ .

**Theorem 10.3** (Properties of the  $\mu$ -integral). The  $\mu$ -integral is: **homogeneous, additive, and:**

$$(i) \quad \min \{u, v\}, \max \{u, v\} \in \mathcal{L}_{\mathbb{R}}^1(\mu) \quad (\text{lattice property})$$

<sup>1</sup>In words, we extend our integral to ~~positive~~ measurable functions by noticing that we can write every measurable function as a sum of positive measurable functions, something that we do know how to integrate. We don't want to run into the problem of  $\infty - \infty$ , thus we require the integral of the positive and negative parts to both (separately) be less than infinity.

$$(ii) \quad u \leq v \Rightarrow \int u d\mu \leq \int v d\mu \quad (\text{monotone})$$

$$(iii) \quad \left| \int u d\mu \right| \leq \int |u| d\mu \quad (\text{triangle inequality})$$

**Remark.** If  $u(x) \pm v(x)$  is defined in  $\overline{\mathbb{R}}$  for all  $x \in X$  then we can exclude  $\infty - \infty$  and the theorem above just says that the integral is linear:

$$\int (au + bv) d\mu = a \int u d\mu + b \int v d\mu. \quad (21)$$

This is always true for real-valued  $u, v \in \mathcal{L}^1(\mu) = \mathcal{L}_{\mathbb{R}}^1(\mu)$ , making  $\mathcal{L}^1(\mu)$  a vector space with addition and scalar multiplication defined by

$$(u + v)(x) := u(x) + v(x), \quad (a \cdot u)(x) := a \cdot u(x), \quad (22)$$

and

$$\int \dots d\mu : \mathcal{L}^1(\mu) \rightarrow \mathbb{R}, \quad u \mapsto \int u d\mu, \quad (23)$$

is a **positive linear functional**.

## 11 Null sets and the "Almost Everywhere"

**Definition 11.1.** A  $(\mu)$ -null set  $N \in \mathcal{N}_{\mu}$  is a measurable set  $N \in \mathcal{A}$  satisfying

$$N \in \mu \Leftrightarrow N \in \mathcal{A} \text{ and } \mu(N) = 0. \quad (24)$$

This can be used generally about a 'statement' or 'property', but we will be interested in questions like 'when is  $u(x)$  equal to  $v(x)$ ', and we answer this by saying

$$u = v \text{ a.e.} \Leftrightarrow \{x : u(x) \neq v(x)\} \text{ is (contained in) a } \mu\text{-null set.}, \quad (25)$$

i.e.

$$u = v \quad \mu\text{-a.e.} \Leftrightarrow \mu(\{x : u(x) \neq v(x)\}) = 0. \quad (26)$$

The last phrasing should of course include that the set  $\{x : u(x) \neq v(x)\}$  is in  $\mathcal{A}$ .

**Theorem 11.2.** Let  $u \in \mathcal{M}_{\overline{\mathbb{R}}}(\mathcal{A})$ , then:

$$(i) \quad \int |u| d\mu = 0 \Leftrightarrow |u| = 0 \text{ a.e.} \Leftrightarrow \mu\{u \neq 0\} = 0,$$

$$(ii) \quad \mathbb{1}_N u \in \mathcal{L}_{\overline{\mathbb{R}}}^1(\mu) \quad \forall N \in \mathcal{N}_{\mu},$$

$$(iii) \quad \int_N u d\mu = 0.$$

**Corollary 11.3.** Let  $u = v$   $\mu$ -a.e. Then

$$(i) \quad u, v \geq 0 \Rightarrow \int u d\mu = \int v d\mu,$$

$$(ii) \quad u \in \mathcal{L}^1_{\mathbb{R}}(\mu) \Rightarrow v \in \mathcal{L}^1_{\mathbb{R}}(\mu) \text{ and } \int u d\mu = \int v d\mu.$$

**Corollary 11.4.** If  $u \in \mathcal{M}_{\mathbb{R}}(\mathcal{A})$ ,  $v \in \mathcal{L}^1_{\mathbb{R}}(\mu)$  and  $v \geq 0$  then

$$|u| \leq v \text{ a.e.} \Rightarrow u \in \mathcal{L}^1_{\mathbb{R}}(\mu). \quad (27)$$

**Proposition 11.5** (Markow inequality). For all  $u \in \mathcal{L}^1_{\mathbb{R}}(\mu)$ ,  $A \in \mathcal{A}$  and  $c > 0$

$$\mu(\{|u| \geq c\} \cap A) \leq \frac{1}{c} \int_A |u| d\mu, \quad (28)$$

if  $A = X$ , then (obviously)

$$\mu\{|u| \geq c\} \leq \frac{1}{c} \int |u| d\mu. \quad (29)$$

**Corollary 11.6.** If  $u \in \mathcal{L}^1_{\mathbb{R}}(\mu)$ , then  $\mu$  is a.e.  $\mathbb{R}$ -valued. In particular, we can find a version  $\tilde{u} \in \mathcal{L}^1(\mu)$  s.t.  $\tilde{u} = u$  a.e. and  $\int \tilde{u} d\mu = \int u d\mu$

## Completions of measure spaces (from lecture notes)

**Definition 11.7.** A measure space is called **complete** if whenever  $A \in \mathcal{B}$  and  $\mu(A) = 0$ , we have  $B \in \mathcal{B} \forall B \subset A$ .

**Remark.** Any measure space can be completed as follows:

Let  $\bar{\mathcal{B}}$  be the  $\sigma$ -algebra generated by  $\mathcal{B}$  and all sets  $B \subset X$  s.t. there exists  $A \in \mathcal{B}$  with  $B \subset A$  and  $\mu(A) = 0$ .

**Proposition 11.8.** The  $\sigma$ -algebra  $\bar{\mathcal{B}}$  can also be described as follows:

$$\bar{\mathcal{B}} := \{B \subset X : A_1 \subset B \subset A_2 \text{ for some } A_1, A_2 \in \mathcal{B} \text{ with } \mu(A_2 \setminus A_1) = 0\}, \quad (30)$$

with  $B, A_1, A_2$  as above, we define

$$\bar{\mu} := \mu(A_1) = \mu(A_2) \quad (31)$$

Then  $(X, \bar{\mathcal{B}}, \bar{\mu})$  is a complete measure space.

**Definition 11.9.** If  $\mu$  is a Borel measure on a **metric** space  $(X, d)$ , then the completion  $\bar{\mathcal{B}}(X)$  of the Borel  $\sigma$ -algebra with respect to  $\mu$  is called the  $\sigma$ -algebra of  $\mu$ -measurable sets.

**Remark.** For  $\mu = \lambda_n$  on  $\mathbb{R}^n$  we talk about the  $\sigma$ -algebra of **Lebesgue measurable sets**. Instead of  $\bar{\lambda}_n$  we still write  $\lambda_n$  and call it the **Lebesgue measure**. A function  $f : \mathbb{R}^n \rightarrow \mathbb{C}$ , measurable w.r.t. the  $\sigma$ -algebra of Lebesgue measurable sets is called the **Lebesgue measurable**.



The following result shows that any Lebesgue measurable function coincides with a Borel function a.e.

**Proposition 11.10.** Assume  $(X, \mathcal{B}, \mu)$  is a measure space and consider its completion  $(X, \bar{\mathcal{B}}, \bar{\mu})$ . Assume  $f : X \rightarrow \mathbb{C}$  is  $\bar{\mathcal{B}}$ -measurable. Then there is a  $\mathcal{B}$ -measurable function  $g : X \rightarrow \mathbb{C}$  s.t.  $f = g$   $\bar{\mu}$ -a.e.

## 12 Convergence Theorems and Their Applications

- To interchange limits and integrals in **Riemann integrals** one typically has to assume uniform convergence. - The set of Riemann integrable functions is somewhat limited, see theorem 12.7

**Theorem 12.1** (Generalization of Beppo Levi, monotone convergence).

(i) Let  $(u_n)_{n \in \mathbb{N}} \subset \mathcal{L}^1(\mu)$  be s.t.  $u_1 \leq u_2 \leq \dots$  with limit  $u := \sup_{n \in \mathbb{N}} u_n = \lim_{n \rightarrow \infty} u_n$ . Then  $u \in \mathcal{L}^1(\mu)$  **iff**

$$\sup_{n \in \mathbb{N}} \int u_n d\mu < +\infty,$$

in which case

$$\sup_{n \in \mathbb{N}} \int u_n d\mu = \int \sup_{n \in \mathbb{N}} u_n d\mu.$$

(ii) Same thing only with a decreasing sequence  $\dots > -\infty$  in which case

$$\inf_{n \in \mathbb{N}} \int u_n d\mu = \int \inf_{n \in \mathbb{N}} u_n d\mu.$$

**Theorem 12.2** (Lebesgue; dominated convergence). Let  $(u_n)_{n \in \mathbb{N}} \subset \mathcal{L}^1(\mu)$  s.t.

(a)  $|u_n|(x) \leq w(x)$ ,  $w \in \mathcal{L}^1(\mu)$ ,

(b)  $u(x) = \lim_{n \rightarrow \infty} u_n(x)$  exists in  $\bar{\mathbb{R}}$ ,

then  $u \in \mathcal{L}^1(\mu)$  and we have

(i)  $\lim_{n \rightarrow \infty} \int |u_n - u| d\mu = 0$ ;

(ii)  $\lim_{n \rightarrow \infty} \int u_n d\mu = \int \lim_{n \rightarrow \infty} u_n d\mu = \int u d\mu$ ;

## Application 1: Parameter-Dependent Integrals

- We are interested in questions of the sort, when is

$$U(t) := \int u(t, x) \mu(dx), \quad t \in (a, b),$$

again a smooth function of  $t$ ? The answer involves interchange of limits and integration. Also, it turns out to better understand Riemann integrability, we need the Lebesgue integral.

**Theorem 12.3** (continuity lemma). *Let  $\emptyset \neq (a, b) \subset \mathbb{R}$  be a non-degenerate open interval and  $u : (a, b) \times X \rightarrow \mathbb{R}$  satisfy*

- (a)  $x \mapsto u(t, x)$  is in  $\mathcal{L}^1(\mu)$  for every fixed  $t \in (a, b)$ ;
- (b)  $t \mapsto u(t, x)$  is continuous for every fixed  $x \in X$ ;
- (c)  $|u(t, x)| \leq w(x)$  for all  $(t, x) \in (a, b) \times X$  and some  $w \in \mathcal{L}^1(\mu)$ .

*Then the function  $U : (a, b) \rightarrow \mathbb{R}$  given by*

$$t \mapsto U(t) := \int u(t, x) \mu(dx) \tag{32}$$

*is continuous.*

**Theorem 12.4** (differentiability lemma). *Let  $\emptyset \leq (a, b) \subset \mathbb{R}$  be a non-degenerate open interval and  $u : (a, b) \times X \rightarrow \mathbb{R}$  satisfy*

- (a) *Same*
- (b) *Same*
- (c)  $|\partial_t u(t, x)| \leq w(x)$  for all  $(t, x) \in (a, b) \times X$  and some  $w \in \mathcal{L}^1(\mu)$ .

*Then the function in 32 is differentiable and its derivative is*

$$\frac{d}{dt} U(t) = \frac{d}{dt} \int u(t, x) \mu(dx) = \int \frac{\partial}{\partial t} u(t, x) \mu(dx). \tag{33}$$

## Application 2: Riemann vs Lebesgue Integration

Consider only  $(X, \mathcal{A}, \mu) = (\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda)$ .

**Definition 12.5** (The Riemann Inegral). Consider on the finite interval  $[a, b] \subset \mathbb{R}$  the partition

$$\Pi := \{a = t_0 < t_1 < \dots < t_k < b\}, k = k(\Pi), \tag{34}$$

and introduce

$$S_{\Pi}[u] := \sum_{i=1}^{k(\Pi)} m_i(t_i - t_{i-1}), \quad m_i := \inf_{x \in [t_{i-1}, t_i]} u(x), \quad (35)$$

$$S^{\Pi}[u] := \sum_{i=1}^{k(\Pi)} M_i(t_i - t_{i-1}), \quad M_i := \sup_{x \in [t_{i-1}, t_i]} u(x). \quad (36)$$

$$(37)$$

A bounded function  $u : [a, b] \rightarrow \mathbb{R}$  is said to be **Riemann integrable** if the values

$$\int u := \sup_{\Pi} S_{\Pi}[u] = \inf_{\Pi} S^{\Pi}[u] =: \int u \quad (38)$$

coincide and are finite. Their common value is called the **Riemann integral** of  $u$  and denoted by  $(R) \int_a^b u(x)dx$  or  $\int_a^b u(x)dx$ .

**Theorem 12.6.** *Let  $u : [a, b] \rightarrow \mathbb{R}$  be a **measurable** and **Riemann integrable** function. Then*

$$u \in \mathcal{L}^1(\lambda) \text{ and } \int_{[a,b]} u d\lambda = \int_a^b u(x)dx. \quad (39)$$

**Theorem 12.7.** *Let  $u : [a, b] \rightarrow \mathbb{R}$  be a bounded function, it is Riemann integrable **iff** the points in  $(a, b)$  where  $u$  is discontinuous are a (subset of) Borel measurable null set.*

## Improper Riemann Integrals

- The Lebesgue integral extends the (*proper*) Riemann integral. However, there is a further extension of the Riemann integral which cannot be captured by Lebesgue's theory.  $u$  is Lebesgue integrable *iff*  $|u|$  has finite Lebesgue integral.  
 - The Lebesgue integral does not respect sign-changes and cancellations. However, the following *improper Riemann integral* does:

$$(R) \int_0^{\infty} u(x)dx := \lim_{n \rightarrow \infty} (R) \int_0^n u(x)dx. \quad (40)$$

**Corollary 12.8.** *Let  $u : [0, \infty) \rightarrow \mathbb{R}$  be a measurable, Riemann integrable function for every interval  $[0, N]$ ,  $N \in \mathbb{N}$ . Then  $u \in \mathcal{L}^1[0, \infty)$  **iff***

$$\lim_{N \rightarrow \infty} (R) \int_0^N |u(x)|dx < \infty. \quad (41)$$

In this case,  $(R) \int_0^{\infty} u(x)dx = \int_{[0, \infty)} u d\lambda$

**Example** of a function which is *improperly Riemann integrable* but **not Lebesgue integrable**:

$$f(x) = \frac{\sin(x)}{x}. \quad (42)$$

**Proposition 12.9** (appearing as example 12.13 in Schilling). *Let  $f_\alpha(x) := x^\alpha, x > 0$  and  $\alpha \in \mathbb{R}$ . Then*

$$(i) \quad f_\alpha \in \mathcal{L}^1(0, 1) \Leftrightarrow \alpha > -1.$$

$$(ii) \quad f_\alpha \in \mathcal{L}^1[1, \infty) \Leftrightarrow \alpha < -1.$$

### 13 The Function Spaces $\mathcal{L}^p$

Assume  $V$  is a vector space over  $\mathbb{K} \in \{\mathbb{C}, \mathbb{R}\}$ .

**Definition 13.1.** A seminorm on  $V$  is a map  $p : V \rightarrow [0, +\infty)$  s.t.

$$(1) \quad p(cx) = |c|p(x) \quad \forall x \in V, \forall c \in \mathbb{K}.$$

$$(2) \quad p(x+y) \leq p(x) + p(y) \quad \forall x, y \in V. \quad \textbf{triangle inequality.}$$

A seminorm is called a norm if we also have

$$p(x) = 0 \quad \Longleftrightarrow \quad x = 0.$$

A norm is commonly denoted  $\|x\|$ , and a vectorspace equipped with a norm is called a **normed space**.

**Definition 13.2.** Assume  $(X, d)$  is a measure space. Fix  $1 \leq p \leq \infty$ . For every measurable function  $f : X \rightarrow \mathbb{C}$  we define the following

$$\|f\|_p = \left( \int_X |f|^p d\mu \right)^{1/p} \in [0, +\infty]. \quad (43)$$

We can see that  $\|cf\|_p = |c|\|f\|_p \quad \forall c \in \mathbb{C}$ .

**Lemma 13.3.**

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p. \quad (44)$$

**Definition 13.4.** We define

$$\mathcal{L}^p(X, d\mu) = \{f : X \rightarrow \mathbb{C} \mid f \text{ is measurable and } \|f\|_p < \infty\}. \quad (45)$$

This is a vectorspace with seminorm  $f \mapsto \|f\|_p$ . And in general this is not a normed space, since  $\|f\|_p = 0 \Longleftrightarrow f = 0$  a.e.

Generally, if  $p$  is a seminorm on a vectorspace  $V$ , then

$$V_0 = \{x \in V \mid p(x) = 0\} \quad (46)$$

which is a subspace of  $V$ . Then we consider the quotient/factor space  $V/V_0$ .

**Definition 13.5.** For  $x, y \in V$ , define

$$x \sim y \iff x - y \in V_0. \quad (47)$$

This is an equivalence relation on  $V$ . The representation class of  $V$  is defined by  $[x]$  or  $x + V_0$ .

Then  $V/V_0$  is equals the set of equivalence classes. We can show that it is a normed space.

$$[x] + [y] = [x + y] \quad , \quad c[x] = [cx] \quad , \quad \|[x]\| = p(x).$$

Applying this to  $\mathcal{L}^p(X, d\mu)$  we get the normed space

$$L^p(X, d\mu) = \mathcal{L}^p(X, d\mu)/\mathcal{N}. \quad (48)$$

Where  $\mathcal{N}$  is the space of measurable functions  $f$  s.t.  $f = 0$  a.e. We will further continue to denote the norm by  $\|\cdot\|_p$ , and we will normally **not** distinguish between  $f \in \mathcal{L}^p(X, d\mu)$  and the vector in  $L^p(X, d\mu)$  that  $f$  defines.

**Definition 13.6.** A normed space  $(X, \|\cdot\|)$  is called a Banach space if  $V$  is complete w.r.t the metric  $d(x, y) = \|x - y\|$ .

**Theorem 13.7.** If  $(X, \mathcal{B}, \mu)$  is a measure space,  $1 \leq p \leq \infty$ , then  $L^p(X, d\mu)$  is a Banach space.

**Definition 13.8.** A measurable function  $f : X \rightarrow \mathbb{C}$  is called **essentially bounded** if there is  $c \geq 0$  s.t.

$$\mu(\{x : |f(x)| > c\}) = 0. \quad (49)$$

That is  $|f| \leq c$  a.e. The smallest such  $c$  is called the essential supremum of  $f$  and is denoted by  $\|f\|_\infty$ .

**Definition 13.9.**

$$\mathcal{L}^\infty(X, d\mu) = \{f : X \rightarrow \mathbb{C} \mid f \text{ is measurable and } \|f\|_\infty < \infty\}.$$

$$L^\infty(X, d\mu) = \mathcal{L}^\infty(X, d\mu)/\mathcal{N}.$$

Where by the previous definiton these spaces become the spaces of all essentially bounded functions.

**Theorem 13.10.** If  $(X, \mathcal{B}, \mu)$  is a  $\sigma$ -finite measure space, then  $L^\infty(X, d\mu)$  is a Banach space.

## Appendix

### H Regularity of measures

We let  $(X, d)$  be a metric space and denote by  $\mathcal{O}$  the open, by  $\mathcal{C}$  the closed and  $\mathcal{B}(X) = \sigma(\mathcal{O})$  the Borel set of  $X$ .

**Definition H.1.** A measure  $\mu$  on  $(X, d, \mathcal{B}(X))$  is called outer regular, if

$$\mu(B) = \inf \{ \mu(U) \mid B \subset U, U \text{ open} \} \quad (50)$$

and inner regular, if  $\mu(K) < \infty$  for all compact sets  $K \subset X$  and

$$\mu(U) = \sup \{ \mu(K) \mid K \subset U, K \text{ compact} \}. \quad (51)$$

A measure which is both inner and outer regular is called **regular**. We write  $\mathfrak{m}_r^+(X)$  for the family of regular measures on  $(X, \mathcal{B}(X))$ .

**Remark.** The space  $X$  is called  $\sigma$ -compact if there is a sequence of compact sets  $K_n \uparrow X$ . A typical example of such a space is a locally compact, separable metric space.

**Theorem H.2.** Let  $(X, d)$  be a metric space. Every finite measure  $\mu$  on  $(X, \mathcal{B}(X))$  is outer regular. If  $X$  is  $\sigma$ -compact, then  $\mu$  is also inner regular, hence regular.

**Theorem H.3.** Let  $(X, d)$  be a metric space and  $\mu$  be a measure on  $(X, \mathcal{B}(X))$  such that  $\mu(K) < \infty$  for all compact sets  $K \subset X$ .

- 1 If  $X$  is  $\sigma$ -compact, then  $\mu$  is inner regular.
- 2 If there exists a sequence  $G_n \in \mathcal{O}$ ,  $G_n \uparrow X$  such that  $\mu(G_n) < \infty$ , then  $\mu$  is outer regular.