Measurable Mappings

Morten Tryti Berg and Isak Cecil Onsager Rukan.

March 6, 2024

We consider maps $T : X \to X'$ between two measurable spaces $(X, \mathcal{A})$ and $(X', \mathcal{A}')$ which respects the measurable structurs, the σ-algbras on X and X'. These maps are useful as we can transport a measure μ, defined on $(X, \mathcal{A})$, to $(X', \mathcal{A}')$.

Definition 7.8. Let $(X, \mathcal{A})$, $(X', \mathcal{A}')$ b measurable spaces. A map $T : X \to X'$ is called $\mathcal{A}/\mathcal{A}'$-measurable if the pre-imag of every measurable set is a measurable set:

$$T^{-1}(A') \in \mathcal{A}, \quad \forall A' \in \mathcal{A}'. \tag{1}$$

- A $\mathcal{B}(\mathbb{R}^n)/\mathcal{B}(\mathbb{R}^m)$ measurable map is often called a Borel map.
- The notation $T : (X, \mathcal{A}) \to (X', \mathcal{A}')$ is often used to indicate measurability of the map T.

Lemma 7.9. *Let $(X, \mathcal{A})$, $(x', \mathcal{A}')$ be measurable spaces and let $\mathcal{A}' = \sigma(\mathcal{G}')$. Then $T : X \to X'$ is $\mathcal{A}/\mathcal{A}'$-measurable iff $T^{-1}(\mathcal{G}') \subset \mathcal{A}$, i.e. if*

$$T^{-1}(G') \in \mathcal{A}, \;\; \forall G' \in \mathcal{G}'. \tag{2}$$

Theorem 7.10. *Let $(X_i, \mathcal{A}_i)$, $\;i = 1, 2, 3$, be measurable spaces and $T : X_1 \to X_2$, $S : X_2 \to X_3$ be $\mathcal{A}_1/\mathcal{A}_2$ and $\mathcal{A}_2/\mathcal{A}_3$-measurable maps respectivly. Then $S \circ T : X_1 \to X_3$ is $\mathcal{A}_1/\mathcal{A}_3$-measurable.*

Corollary 7.11. *Every continuous map betwen metric spaces is a Borel map.*

Definition 7.12. (and lemma) Let $(T_i)_{i\in I}$, $T_I : X \to X_i$, be arbitrarily many mappings from the same space X into measurable spaces $(X_i, \mathcal{A}_i)$. The smallest σ-algebra on X that makes all T_i simultanously measurable is

$$\sigma(T_i : i \in I) := \sigma\left(\bigcup_{i\in I} T_i^{-1}(\mathcal{A}_i)\right) \tag{3}$$

Corollary 7.13. *A function $f : (X, \mathcal{B}) \to \mathbb{R}$ is measurable if $f((a, +\infty)) \in \mathcal{B}$, $\;\forall a \in \mathbb{R}$.*

Corollary 7.14. *Assume* $(X, \mathcal{B})$ *is a measurable space,* (Y, d) *is a metric space,* $(f_n : (X, \mathcal{B}) \to Y)_{n=1}^{\infty}$ *is a sequence of measurable maps. Assume this sequence of images* $(f_n(x))_{n=1}^{\infty}$ *is convergent in* Y $\forall x \in X$*. Define*

$$f : X \to Y, \quad \textit{by } f(x) = \lim_{n \to \infty} f_n(x). \tag{4}$$

Then f *is measurable.*

Theorem 7.15. *Let* $(X, \mathcal{A})$*,* $(X', \mathcal{A}')$ *be measurable spaces and* $T : X \to X'$ *be an* $\mathcal{A}/\mathcal{A}'$*-measurable map. For every measurable* μ *on* $(X, \mathcal{A})$*,*

$$\mu'(A') := \mu(T^{-1}(A')), \quad A' \in \mathcal{A}', \tag{5}$$

defines a measure on $(X', \mathcal{A}')$*.*

Definition 7.16. The measure $\mu'(\cdot)$ in the above theorem is called the push forward or image measure of μ under T and it is denoted as $T(\mu)(\cdot)$, $T_{*\mu}(\cdot)$ or $\mu \circ T^{-1}(\cdot)$.

Theorem 7.17. *If* $T \in \mathbb{R}^{n \times n}$ *is an orthogonal matrix, then* $\lambda^n = T(\lambda^n)$*.*

Theorem 7.18. *Let* $S \in \mathbb{R}^{n \times n}$ *be an invertible matrix. Then*

$$S(\lambda^n) = |\det s^{-1}|\lambda^n = |\det S|^{-1}\lambda^n. \tag{6}$$

Corollary 7.19. *Lebesgue measure is invariant under motions:* $\lambda^n = M(\lambda^n)$ *for all motions* M *in* $\mathbb{R}^n$*. In particular, congruent sets have the same measure. Two sets of points are called congruent if, and only if, one can be transformed into the other by an isometry*