MAT4400: Notes on Linear analysis (ONLY IMPORTANT STUFF)

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1 σ-Algebras (3, [Schilling(2017)])

Definition 1.1 (σ**-algebra**)**.** A family $\mathscr{A}$ of subsets of X with:

(i) $X \in \mathscr{A}$,

(ii) $A \in \mathscr{A} \Rightarrow A^c \in \mathscr{A}$,

(iii) $(A_n)_{n\in\mathbb{N}} \in \mathscr{A} \Rightarrow \bigcup\limits_{n\in\mathbb{N}}$

A set $A \in \mathscr{A}$ is said to be **measurable** or $\mathscr{A}$**-measurable**.

Theorem 1.2 (and Definition)**.**

(i) The intersection of arbitrarily many σ-algebras in X is againg a σ-algebra in X.

(ii) For every system of sets $p \subset \mathscr{P}(X)$ there exists a smallestσ-algebra containing$p$. This is the σ-algebra generated by p, denoted $\sigma(p)$, and $\sigma(p)$ is called its generator.

Definition 1.3 (Borel)**.** The σ-algebra $\sigma(\mathcal{O})$ generated by the open sets $\mathcal{O} = \mathcal{O}_{\mathbb{R}^n}$ of $\mathbb{R}^n$ is called **Borel σ-algebra**, and its members are called **Borel sets** or **Borel measurable sets**.

Definition 1.4 (**measure**)**.** A measure μ on X is a map $u : \mathcal{A} \to [0,\infty]$ satisfying

(i) $\mathcal{A}$ is a σ-algebra in X,

(ii) $\mu(\emptyset) = 0$,

(iii) $(A_n)_{n\in\mathbb{N}} \subset \mathcal{A}$ pairwise disjoint $\iff \mu(\cup_{n\in\mathbb{N}} A_n) = \sum_{n\in\mathbb{N}} \mu(A_n)$.

Definition 1.5 (σ-finite/sigma-finite)**.** A measure μ is said to be σ-finite and $(X, \mathcal{A}, \mu)$ is called a σ-finite measure space, if $\mathcal{A}$ contains a sequence $(A_n)_{n\in\mathbb{N}}$ s.t. $A_n \uparrow X$ and $\mu(A_n) < \infty$.

3 Uniqueness of Measures (5, [Schilling(2017)])

Lemma 3.1. *A Dynkin system D is a σ-algebra iff it is stable under finite intersections, i.e. $A, B \in D \Rightarrow A \cap B \in D$.*

Theorem 3.2 (Dynkin)**.** *Assume X is a set, S is a collection of subsets of X closed under finite intersections, that is, if $A, B \in S \Rightarrow A \cap B \in S$. Then $D(S) = \sigma(S)$.*

Theorem 3.3 (uniqueness of measures)**.** *Let (X, B) be a measurable space, and $S \subset P(X)$ be the generator of B, i.e. $B = \sigma(S)$. If S satisfies the following conditions:*

1. *S is stable under finite intersections ($\cap$-stable), i.e. $A, C \in S \Rightarrow A \cap C \in S$.*

2. *There exists an exhausting sequence $(G_n)_{N\in\mathbb{N}} \subset$ with $G_N \uparrow X$. Assume also that there are two measures μ, ν satisfying:*

3. *$\mu(A) = \nu(A)$, $\forall A \in S$.*

4. *$\mu(G_n) = \nu(G_n) < \infty$.*

Then $\mu = \nu$.

4 Existence of Measures (6, [Schilling(2017)])

Theorem 4.1 (**Carathéodory**)**.** *Let $S \subset P(X)$ be a semi-ring and $\mu : S \to [0,\infty)$ a pre-measure. Then μ has an extension to a measure μ^* on $\sigma(S)$, i.e. that $\mu(s) = \mu^*(s)$, $\forall\ s \in \sigma(S)$.*

Also, if S contains an exhausting sequence, $S_n \uparrow X$, s.t. $\mu(S_n) < \infty$, then the extension is unique.

Definition 4.2 (**Outer measure**)**.** An outer measure is a function $\mu^* : P(X) \to [0,\infty)$ with the following properties:

1. $\mu^*(\emptyset) = 0$,

2. $A \subset B \Rightarrow u^*(A) \le \mu^*(B)$,

3. $\mu^*\left(\bigcup\limits_{n\in\mathbb{N}} A_n\right) \le \sum\limits_{n\in\mathbb{N}} \mu^*(A_n)$,

5 Measurable Mappings (7, [Schilling(2017)])

We consider maps $T : X \to X'$ between two measurable spaces $(X, \mathcal{A})$ and $(X', \mathcal{A}')$ which respects the measurable structurs, the σ-algbras on X and X'. These maps are useful as we can transport a measure μ, defined on $(X, \mathcal{A})$, to $(X', \mathcal{A}')$.

Definition 5.1. Let $(X, \mathcal{A})$, $(X', \mathcal{A}')$ b measurable spaces. A map $T : X \to X'$ is called $\mathcal{A}/\mathcal{A}'$-measurable if the pre-imag of every measurable set is a measurable set:

$$T^{-1}(A') \in \mathcal{A}, \quad \forall A' \in \mathcal{A}'.$$

- $T^{-1}(A') := \{x \in X : f(x) \in A'\}$
- A $\mathcal{B}(\mathbb{R}^n)/\mathcal{B}(\mathbb{R}^m)$ measurable map is often called a Borel map.
- The notation $T : (X, \mathcal{A}) \to (X', \mathcal{A}')$ is often used to indicate measurability of the map T.

Lemma 5.2. *Let $(X, \mathcal{A})$, $(x', \mathcal{A}')$ be measurable spaces and let $\mathcal{A}' = \sigma(\mathcal{G}')$. Then $T : X \to X'$ is $\mathcal{A}/\mathcal{A}'$-measurable iff $T^{-1}(\mathcal{G}') \subset \mathcal{A}$, i.e. if*

$$T^{-1}(G') \in \mathcal{A}, \quad \forall G' \in \mathcal{G}'.$$

Theorem 5.3. *Let $(X_i, \mathcal{A}_i)$, $i = 1, 2, 3$, be measurable spaces and $T : X_1 \to X_2$, $S : X_2 \to X_3$ be $\mathcal{A}_1/\mathcal{A}_2$ and $\mathcal{A}_2/\mathcal{A}_3$-measurable maps respectivly. Then $S \circ T : X_1 \to X_3$ is $\mathcal{A}_1/\mathcal{A}_3$-measurable.*

Corollary 5.4. *Every continuous map betwen metric spaces is a Borel map.*

Definition 5.5. (and lemma) Let $(T_i)_{i \in I}, T_I : X \to X_i$, be arbitrarily many mappings from the same space X into measurable spaces $(X_i, \mathcal{A}_i)$. The smallest σ-algebra on X that makes all T_i simultanously measurable is

$$\sigma(T_i : i \in I) := \sigma\left(\bigcup_{i \in I} T_i^{-1}(\mathcal{A}_i)\right)$$

Corollary 5.6. *A function $f : (X, \mathcal{B}) \to \mathbb{R}$ is measurable if $f((a, +\infty)) \in \mathcal{B}$, $\forall a \in \mathbb{R}$.*

Corollary 5.7. *Assume $(X, \mathcal{B})$ is a measurable space, (Y, d) is a metric space, and*
$(f_n : (X, \mathcal{B}) \to Y)_{n=1}^{\infty}$ is a sequence of measurable maps. Assume this sequence of images $(f_n(x))_{n=1}^{\infty}$ is convergent in Y $\forall x \in X$. Define

$$f : X \to Y, \quad \text{by } f(x) = \lim_{n \to \infty} f_n(x).$$

Then f is measurable.

Theorem 5.8. *Let $(X, \mathcal{A})$, $(X', \mathcal{A}')$ be measurable spaces and $T : X \to X'$ be an $\mathcal{A}/\mathcal{A}'$-measurable map. For every measurable μ on $(X, \mathcal{A})$,*

$$\mu'(A') := \mu(T^{-1}(A')), \quad A' \in \mathcal{A}',$$

defines a measure on $(X', \mathcal{A}')$.

Definition 5.9. The measure $\mu'(\cdot)$ in the above theorem is called the **pushforward or image measure**of μ under T and it is denoted as $T(\mu)(\cdot)$, $T_{*\mu}(\cdot)$ or $\mu \circ T^{-1}(\cdot)$.

Theorem 5.10. *If $T \in \mathbb{R}^{n \times n}$ is an orthogonal matrix, then $\lambda^n = T(\lambda^n)$.*

Theorem 5.11. *Let $S \in \mathbb{R}^{n \times n}$ be an invertible matrix. Then*

$$S(\lambda^n) = |\det s^{-1}|\lambda^n = |\det S|^{-1}\lambda^n.$$

Corollary 5.12. *Lebesgue measure is invariant under motions: $\lambda^n = M(\lambda^n)$ for all motions M in $\mathbb{R}^n$. In particular, congruent sets have the same measure. Two sets of points are called **congruent** if, and only if, one can be transformed into the other by an isometry.*

Measurable Functions (8, [Schilling(2017)])

A *measurable function* is a measurable map $u : X \to \mathbb{R}$ from some measurable space $(X, \mathscr{A})$ to $(\mathbb{R}, \mathscr{B}(\mathbb{R}^1))$. They play central roles in the theory of integration.

We recall that $u : X \to \mathbb{R}$ is $\mathscr{A}/\mathscr{B}(\mathbb{R}^1)$-measurable if

$$u^{-1}(B) \in \mathscr{A}, \quad \forall B \in \mathscr{B}(\mathbb{R}^1).$$

Moreover from a lemma from chapter 7, we actually only need to show that

$$u^{-1}(G) \in \mathscr{A}, \quad \forall G \in \mathcal{G} \text{ where } \mathcal{G} \text{ generates } \mathscr{B}(\mathbb{R}^1).$$

Proposition 5.13.

1 If $f, g : (X, \mathscr{B}) \to \mathbb{C}$ are measurable, then the function $f + g$, $f \cdot g$, cf, $(c \in \mathbb{C})$ are measurable.

2 If $f : (\mathbb{C}, \mathscr{B}) \to \mathbb{C}$ is measurable and $h : \mathbb{C} \to \mathbb{C}$ is Borel measurable, then $h \circ f$ is measurable.

3 If $f(x) = \lim_{n \to \infty} f_n(x)$, $x \in X$ and f_n are measurable, then f is measurable.

4 If $X = \bigcup_{n=1}^{\infty} A_n$, $(A_n \in \mathscr{B})$, $f|_{A_n} : (A_n, \mathscr{B}_{A_n}) \to \mathbb{C}$ is measurable $\forall n$, then f is measurable.

Definition 5.14 (**simple function**). Given a measurable space $(X, \mathscr{B})$, a measurable function $f : (X, \mathscr{B}) \to \mathbb{C}$ is called simple if

$$f(x) = \sum_{k=1}^{N} c_k \mathbb{1}_{A_k}(x),$$

for some $c_k \in \mathbb{C}$, $A_k \in \mathscr{B}$, where $\mathbb{1}$ is the characteristic function,

$$\mathbb{1}_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A. \end{cases}$$

The representation of simple function is **not** unique. We denote the standard representation of f by

$$f(x) = \sum_{n=0}^{N} z_n \mathbb{1}_{B_n}(x),$$

for $N \in \mathbb{N}$, $z_n \in \mathbb{R}$, $B_n \in \mathscr{A}$, and

$$X = \bigcup_{n=1}^{N} B_n,$$

for $B_n \cap B_m = \emptyset$, $n \neq m$. The set of simple functions is denoted $\mathcal{E}(\mathscr{A})$ of $\mathcal{E}$.

Definition 5.15. Assume μ is a measure on $(X, \mathscr{B})$. Given a *positive* simple function

$$f = \sum_{k=1}^{N} c_k \mathbb{1}_{A_k}, \quad (c_k \geq 0).$$

We define

$$\int_X f d\mu = \sum_{k=1}^{n} c_k \mu(A_k) \in [0, +\infty].$$

We also denote this by $I_\mu(f)$.

Lemma 5.16. *This is well defined, that is, $\int_x f d\mu$ does not depend on the presentation of the simple function f.*

Properties 5.17. *For every positive simple function*

1 $\int_X cf d\mu = c\int_X f d\mu$, *for only* $c \geq 0$

2 $\int_X (f+g) d\mu = \int_X f d\mu + \int_X g d\mu$.

Corollary 5.18. *If $f \geq g \geq 0$ are simple functions, then*

$$\int_X f d\mu \geq \int_X g d\mu.$$

Remark. *This means that any measurable function can be approximated by simple functions.*

Properties 5.19. *Measurable functions like this have the following properties*

1 $\int_X cf d\mu = c\int_X f d\mu$, $\forall c \geq 0$.

2 *If $f \geq g \geq 0$, then $\int_X f d\mu \geq \int_X g d\mu$ for any measurable g, f.*

3 *If $f \geq 0$ is simple, then $\int_X f d\mu$ is the same value as obtained before.*

To advance in measure theory we consider measurable functions

$$f : X \longrightarrow [0, +\infty].$$

Measurability is understood w.r.t the $\sigma-$algebra $\mathscr{B}([0, +\infty])$ generated by $\mathscr{B}([0, +\infty))$ and $\{+\infty\}$. In other words, $A \subset [0, +\infty] \in B([0, +\infty])$ iff $A \cap [0, +\infty) \in \mathscr{B}([0, +\infty))$.

Remark. *Hence $f : X \longrightarrow [0, +\infty]$ is measurable iff $f^{-1}(A)$ is measurable $\forall A \in \mathscr{B}([0, +\infty))$.*

Definition 5.20 (**Lebesgue integral**)**.** For measurable functions $f : X \longrightarrow [0, +\infty]$, we define

$$\int_X f d\mu = \sup\left\{\int_x g d\mu \ : \ f \geq g \geq 0 \ : \ g \text{ is simple}\right\} \in [0, +\infty].$$

Theorem 5.21 (**Monotone convergence theorem**)**.** *Assume $(X, \mathscr{B}, \mu)$ is a measure space, $(f)_{n=1}^{\infty}$ is an increasing sequence of measurable positive functions $f_n : X \longrightarrow [0, +\infty]$. Define $f(x) = \lim_{n\to\infty} f_n(x)$. Then f is measurable and*

$$\int_X f d\mu = \lim_{n\to\infty} \int_X f_n d\mu.$$

Theorem 5.22. *Assume $(X, \mathscr{B})$ is a measurable space and $f : X \longrightarrow [0, +\infty]$ is measurable. Then there are simple functions g_n, s.t.*

$$0 \leq g_1 \leq g_2 \leq \dots \ , \ g_n(x) \longrightarrow f(x), \ \ \forall x \in X.$$

Moreover, if f is bounded, we can choose g_n s.t. the convergence is uniform, that is,

$$\lim_{n\to\infty} \sup_{x\in X} |g_n(x) - f(x)| = 0.$$

6 Integration of Measurable Functions

(9, [Schilling(2017)])

Through this chapter $(X, \mathscr{A}, \mu)$ will be some measure space. Recall that $\mathcal{M}^+(\mathscr{A})$ $[\mathcal{M}^+_{\mathbb{R}}(\mathscr{A})]$ are the $\mathscr{A}$-**measurable positive functions**and $\mathcal{E}(\mathscr{A})$ $[\mathcal{E}^+_{\mathbb{R}}(\mathscr{A})]$ are the **positive and simple functions**.

The fundamental idea of *Integration* is to measure the area between the graph of the function and the abscissa. For positive simple functions $f \in \mathcal{E}^+(\mathscr{A})$ in standard representation, this is done easily

$$\text{if } f = \sum_{i=0}^{M} y_i \mathbb{1}_{A_i} \in \mathcal{E}^+(\mathscr{A}) \quad \text{then } \sum_{i=0}^{M} y_i \mu(A_i) \tag{1}$$

would be the μ-area enclosed by the graph and the abscissa. We note that the representation of f should not impact the integral of f.

Lemma 6.1. *Let $\sum_{i=0}^{M} y_i \mathbb{1}_{A_i} = \sum_{k=0}^{N} z_k \mathbb{1}_{B_k}$ be two standard representations of the same function $f \in \mathcal{E}^+(\mathscr{A})$. Then*

$$\sum_{i=0}^{M} y_i \mu(A_i) = \sum_{k=0}^{N} z_k \mu(B_k). \tag{2}$$

Definition 6.2. Let $f = \sum_{i=0}^{M} y_i \mathbb{1}_{A_i} \in \mathcal{E}^+(\mathscr{A})$ be a simple function in standard representation. Then the number

$$I_\mu(f) = \sum_{i=0}^{M} y_i \mu(A_i) \in [0, \infty] \tag{3}$$

(which is independent of the representation of f) is called the μ-integral of f.

Proposition 6.3. *Let $f, g \in \mathcal{E}^+(\mathscr{A})$. Then*

(i) $I_\mu(\mathbb{1}_A) = \mu(A) \quad \forall A \in \mathscr{A}$.

(ii) $I_\mu(\lambda f) = \lambda I_\mu(f) \quad \forall \lambda \geq 0$.

(iii) $I_\mu(f+g) = I_\mu(f) + I_\mu(g)$.

(iv) $f \leq g \Rightarrow I_\mu(f) \leq I_\mu(g)$.

In theorem 8.8 we saw that we could for every $u \in \mathcal{M}^+(\mathscr{A})$ write it as an increasing limit of simple functions. By corollary 8.10, the suprema of simple functions are again measurable, so that

$$u \in \mathcal{M}^+(\mathscr{A}) \Leftrightarrow u = \sup_{n\in\mathbb{N}} f_n, f \in \mathcal{E}^+(\mathscr{A}),$$
$$f_n \leq f_{n+1} \leq \dots .$$

We will use this to "inscribe" simple functions (which we know how to integrate) below the graph of a positive measurable function u and exhaust the μ-area below u.

Definition 6.4. Let $(X, \mathscr{A}, \mu)$ be a measure space. The (μ)-integral of a positive function $u \in \mathcal{M}^+_{\mathbb{R}}(\mathscr{A})$ is given by

$$\int u d\mu = \sup\left\{I_\mu(g) : g \leq u, \ g \in \mathcal{E}^+(\mathscr{A})\right\}, \tag{4}$$

with $\int u d\mu \in [0, +\infty]$. If we need to emphasize the *integration variable*, we write $\int u(x)\mu(dx)$. The key observation is that the integral $\int \dots d\mu$ extends I_μ.

Lemma 6.5. *For all $f \in \mathcal{E}^+(\mathscr{A})$ we have $\int f d\mu = I_\mu(f)$.*

The next theorem is one of many convergence theorems. It shows that we could have defined 4 using any increasing sequence $f_n \uparrow u$ of simple functions $f_n \in \mathcal{E}^+(\mathscr{A})$.

Theorem 6.6 (**Beppo Levi**). *Let $(X, \mathscr{A}, \mu)$ be a measure space. For an increasing sequence of functions $(u_n)_{n\in\mathbb{N}} \subset \mathcal{M}^+_{\overline{\mathbb{R}}}(\mathscr{A})$, $0 \le u_n \le u_{n+1} \le \ldots$, we have for the supremum $u = \sup_{n\in\mathbb{N}} u_n \in \mathcal{M}^+_{\overline{\mathbb{R}}}(\mathscr{A})$ and*

$$\int \sup_{n\in\mathbb{N}} u_n d\mu = \sup_{n\in\mathbb{N}} \int u_n d\mu. \tag{5}$$

*Note we can write $\lim_{n\to\infty}$ **instead of** $\sup_{n\in\mathbb{N}}$ as the supremum of an increasing sequence is its limit. Moreover, this theorem holds in $[0, +\infty]$, so the case $+\infty = +\infty$ is possible.*

Corollary 6.7. *Let $u \in \mathcal{M}^+_{\overline{\mathbb{R}}}(\mathscr{A})$. Then*

$$\int u d\mu = \lim_{n\to\infty} \int f_n d\mu$$

holds for every sequence $(f_n)_{n\in\mathbb{N}} \subset \mathcal{E}^+(\mathscr{A})$ with $\lim_{n\to\infty} f_n = u$.

Proposition 6.8. *(of integral) Let $u, v \in \mathcal{M}^+_{\overline{\mathbb{R}}}(\mathscr{A})$. Then*

(i) $\int \mathbb{1}_A d\mu = \mu(A) \quad \forall A \in \mathscr{A}$.

(ii) $\int \alpha u d\mu = \alpha \int u d\mu \quad \forall \alpha \ge 0$.

(iii) $\int u + v d\mu = \int u d\mu + \int v d\mu$.

(iv) $u \le v \Rightarrow \int u d\mu \le \int v d\mu$.

Corollary 6.9 (**sum of positive functions**). *Let $(u_n)_{n\in\mathbb{N}} \subset \mathcal{M}^+_{\overline{\mathbb{R}}}(\mathscr{A})$. Then $\sum_{n=1}^\infty u_n$ is measurable and we have*

$$\int \sum_{n=1}^\infty u_n d\mu = \sum_{n=1}^\infty \int u_n d\mu$$

(including the possibility $+\infty = +\infty$.)

Theorem 6.10 (**Fatou**). *Let $(u_n)_{n\in\mathbb{N}} \subset \mathcal{M}^+_{\overline{\mathbb{R}}}(\mathscr{A})$ be a sequence of positive measurable functions. Then $u = \liminf_{n\to\infty} u_n$ is measurable and*

$$\int \liminf_{n\to\infty} u_n d\mu \le \liminf_{n\to\infty} \int u_n d\mu \tag{6}$$

7 Integrals of Measurable Functions

(10, [Schilling(2017)])

We have defined our integral for positive measurable functions, i.e. functions in $\mathcal{M}^+(\mathscr{A})$. To extend our integral to not only functions in $\mathcal{M}^+(\mathscr{A})$ we first notice that

$$u \in \mathcal{M}_{\overline{\mathbb{R}}}(\mathscr{A}) \Leftrightarrow u = u^+ - u^-, \; u^+, u^- \in \mathcal{M}^+_{\overline{\mathbb{R}}},$$

i.e. that every measurable function can be written as a sum of **positive** measurable functions.

Definition 7.1 (**μ-integrable**). A function $u : X \to \overline{\mathbb{R}}$ on $(X, \mathscr{A}, \mu)$ is *μ-integrable*, if it is $\mathscr{A}/\mathscr{B}(\overline{\mathbb{R}})$-measurable and if $\int u^+ d\mu, \int u^- d\mu < \infty$ (recall the definition for the integral of positive measurable functions). Then

$$\int u d\mu := \int u^+ d\mu - \int u^- d\mu \in (-\infty, \infty)$$

is the (μ-)*integral* of u. We write $\mathcal{L}^1(\mu)$ for the set of all real-valued μ-integrable functions [1].

Theorem 7.2. *Let $u \in \mathcal{M}_{\overline{\mathbb{R}}}(\mathscr{A})$, then the following conditions are equivalent:*

(i) $u \in \mathcal{L}^1_{\overline{\mathbb{R}}}(\mu)$.

(ii) $u^+, u^- \in \mathcal{L}^1_{\overline{\mathbb{R}}}(\mu)$.

(iii) $|u| \in \mathcal{L}^1_{\overline{\mathbb{R}}}(\mu)$.

(iv) $\exists w \in \mathcal{L}^1_{\overline{\mathbb{R}}}(\mu)$ *with* $w \ge 0$ *s.t.* $|u| \le w$.

Theorem 7.3 (Properties of the μ-integral). *The μ-integral is: **homogeneous, additive**, and:*

(i) $\min\{u, v\}, \max\{u, v\} \in \mathcal{L}^1_{\overline{\mathbb{R}}}(\mu)$ *(lattice property)*

(ii) $u \le v \Rightarrow \int u d\mu \le \int v d\mu$ *(monotone)*

(iii) $\left|\int u d\mu\right| \le \int |u| d\mu$ *(triangle inequality)*

Remark. *If $u(x) \pm v(x)$ is defined in $\overline{\mathbb{R}}$ for all $x \in X$ then we can exclude $\infty - \infty$ and the theorem above just says that the integral is linear:*

$$\int (au + bv) d\mu = a \int u d\mu + b \int v d\mu.$$

This is always true for real-valued $u, v \in \mathcal{L}^1(\mu) = \mathcal{L}^1_{\mathbb{R}}(\mu)$, making $\mathcal{L}^1(\mu)$ a vector space with addition and scalar multiplication defined by

$$(u + v)(x) := u(x) + v(x), \; (a \cdot u)(x) := a \cdot u(x),$$

and

$$\int ... d\mu : \mathcal{L}^1(\mu) \to \mathbb{R}, \; u \mapsto \int u d\mu,$$

*is a **positive linear functional**.*

8 Null sets and the Almost Everywhere

(11, [Schilling(2017)])

Definition 8.1 (**null set**). A (μ-)null set $N \in \mathcal{N}_\mu$ is a measurable set $N \in \mathscr{A}$ satisfying

$$N \in \mathcal{N}_\mu \Longleftrightarrow N \in \mathscr{A} \text{ and } \mu(N) = 0.$$

[1] In words, we extend our integral to ~~positive~~ measurable functions by noticing that we can write every measurable function as a sum of positive measurable functions, something that we do know how to integrate. We don't want to run into the problem of $\infty - \infty$, thus we require the integral of the positive and negative parts to both (separately) be less than infinity.

This can be used generally about a 'statement' or 'property', but we will be interested in questions like 'when is $u(x)$ equal to $v(x)$', and we answer this by saying

$$u = v \ a.e. \Leftrightarrow \{x : u(x) \neq v(x)\} \text{ is (contained in) a } \mu\text{-null set},$$

i.e.

$$u = v \quad \mu\text{-a.e.} \Leftrightarrow \mu\left(\{x : u(x) \neq v(x)\}\right) = 0.$$

The last phrasing should of course include that the set $\{x : u(x) \neq v(x)\}$ is in $\mathscr{A}$.

Theorem 8.2. *Let $u \in \mathcal{M}_{\overline{\mathbb{R}}}(\mathscr{A})$, then:*

(i) $\int |u| d\mu = 0 \Leftrightarrow |u| = 0$ ***a.e.*** $\Leftrightarrow \mu\{u \neq 0\} = 0$,

(ii) $\mathbb{1}_N u \in \mathcal{L}^1_{\overline{\mathbb{R}}}(\mu) \quad \forall\, N \in \mathcal{N}_\mu$,

(iii) $\int_N u d\mu = 0$.

(i) is really useful, later we will define $\mathcal{L}^p$ and the $||\cdot||_p$-(semi)norm. Then (i) means that if we have a sequence u_n converging to u in the $||\cdot||_p$-norm then $u_n(x) = u(x)$ a.e.

Corollary 8.3. *Let $u = v$ μ-a.e. Then*

(i) $u, v \geq 0 \Rightarrow \int u d\mu = \int v d\mu$,

(ii) $u \in \mathcal{L}^1_{\overline{\mathbb{R}}}(\mu) \Rightarrow v \in \mathcal{L}^1_{\overline{\mathbb{R}}}(\mu)$ *and* $\int u d\mu = \int v d\mu$.

Corollary 8.4. *If $u \in \mathcal{M}_{\overline{\mathbb{R}}}(\mathscr{A})$, $v \in \mathcal{L}^1_{\overline{\mathbb{R}}}(\mu)$ and $v \geq 0$ then*

$$|u| \leq v \ a.e. \quad \Rightarrow u \in \mathcal{L}^1_{\overline{\mathbb{R}}}(\mu).$$

Proposition 8.5 (Markow inequality)**.** *For all $u \in \mathcal{L}^1_{\overline{\mathbb{R}}}(\mu)$, $A \in \mathscr{A}$ and $c > 0$*

$$u\left(\{|u| \geq c\} \cap A\right) \leq \frac{1}{c} \int_A |u| d\mu,$$

if $A = X$, then (obviosly)

$$u\{|u| \geq c\} \leq \frac{1}{c} \int |u| d\mu.$$

Completions of measure spaces

Definition 8.6 (complete measure space)**.** A measure space $(X, \mathscr{B}, \mu)$ is called **complete** if whenever $A \in \mathscr{B}$ and $\mu(A) = 0$, we have $B \in \mathscr{B}\ \forall B \subset A$.

Remark. *Any measure space can be completed as follows:*
Let $\bar{\mathscr{B}}$ be the σ-algebra generated by $\mathscr{B}$ and all sets $B \subset X$ s.t. there exists $A \in \mathscr{B}$ with $B \subset A$ and $\mu(A) = 0$.

Proposition 8.7. *The σ-algebra $\bar{\mathscr{B}}$ can* ***also be described as follows****:*

$$\bar{\mathscr{B}} := \Bigg\{ B \subset X : A_1 \subset B \subset A_2$$

$$\text{for some } A_1, A_2 \in \mathscr{B} \text{ with } \mu(A_2 \backslash A_1) = 0 \Bigg\},$$

with B, A_1, A_2 as above, we define

$$\bar{\mu} := \mu(A_1) = \mu(A_2)$$

Then $\left(X, \bar{\mathscr{B}}, \bar{\mu}\right)$ is a complete measure space.

Definition 8.8. If μ is a Borel measure on a **metric** space (X, d), then the completion $\bar{\mathscr{B}}(X)$ of the Borel σ-algebra with respect to μ is called the σ-algebra of μ-measurable sets.

Remark. *For $\mu = \lambda_n$ on $\mathbb{R}^n$ we talk about the σ-algebra of* ***Lebesgue measurable sets****. Instead of $\bar{\lambda}_n$ we still write λ_n and call it the* ***Lebesgue measure****. A function $f : \mathbb{R}^n \to \mathbb{C}$, measurable w.r.t. the σ-algebra of Lebesgue measurable sets is called the* ***Lebesgue measurable****.*

The following result shows that any Lebesgue measurable function coincides with a Borel function a.e.

Proposition 8.9. *Assume $(X, \mathscr{B}, \mu)$ is a measure space and consider its completion $\left(X, \bar{\mathscr{B}}, \bar{\mu}\right)$. Assume $f : X \to \mathbb{C}$ is $\bar{\mathscr{B}}$-measurable. Then there is a $\mathscr{B}$-measurable function $g : X \to \mathbb{C}$ s.t. $f = g$ $\bar{\mu}$-a.e.*

9 Convergence Theorems and their Applications

(12, [Schilling(2017)])

- To interchange limits and integrals in **Riemann integrals** one typically has to assume uniform convergence. ¡- The set of Riemann integrable functions is somewhat limited, see theorem 9.5

Theorem 9.1 (Generalization of Beppo Levi, monotone convergence)**.**

(i) Let $(u_n)_{n\in\mathbb{N}} \subset \mathcal{L}^1(\mu)$ be s.t. $u_1 \leq u_2 \leq \ldots$ with limit $u := \sup_{n\in\mathbb{N}} u_n = \lim_{n\to\infty} u_n$. Then $u \in \mathcal{L}^1(\mu)$ ***iff***

$$\sup_{n\in\mathbb{N}} \int u_n d\mu < +\infty,$$

in which case

$$\sup_{n\in\mathbb{N}} \int u_n d\mu = \int \sup_{n\in\mathbb{N}} u_n d\mu.$$

(ii) Same thing only with a decreasing sequence ...> $-\infty$ in which case

$$\inf_{n\in\mathbb{N}} \int u_n d\mu = \int \inf_{n\in\mathbb{N}} u_n d\mu.$$

Theorem 9.2 (Lebesgue; dominated convergence)**.** *Let $(u_n)_{n\in\mathbb{N}} \subset \mathcal{L}^1(\mu)$ s.t.*

(a) $|u_n|(x) \leq w(x)$, $w \in \mathcal{L}^1(\mu)$,

(b) $u(x) = \lim_{n\to\infty} u_n(x)$ *exists in* $\bar{\mathbb{R}}$,

then $u \in \mathcal{L}^1(\mu)$ and we have

(i) $\lim\limits_{n\to\infty} \int |u_n - u| d\mu = 0$;

(ii) $\lim\limits_{n\to\infty} \int u_n d\mu = \int \lim\limits_{n\to\infty} u_n d\mu = \int u d\mu$;

Application 2: Riemann vs Lebesgue Integration

Consider only $(X, \mathscr{A}, \mu) = (\mathbb{R}, \mathscr{B}(\mathbb{R}), \lambda)$.

Definition 9.3 (The Riemann Inegral)**.** Consider on the finite interval $[a, b] \subset \mathbb{R}$ the partition

$$\Pi := \{a = t_0 < t_1 < \ldots < t_k < b\}, k = k(\Pi), \tag{7}$$

and introduce

$$S_\Pi[u] := \sum_{i=1}^{k(\Pi)} m_i(t_i - t_{i-1}), \qquad m_i := \inf_{x \in [t_{i-1}, t_i]} u(x), \tag{8}$$

$$S^\Pi[u] := \sum_{i=1}^{k(\Pi)} M_i(t_i - t_{i-1}), \qquad M_i := \sup_{x \in [t_{i-1}, t_i]} u(x). \tag{9}$$

$$\tag{10}$$

A bounded function $u : [a, b] \to \mathbb{R}$ is said to be **Riemann integrable** if the values

$$\underline{\int} u := \sup_\Pi S_\Pi[u] = \inf_\Pi S^\Pi[u] =: \bar{\int} u \tag{11}$$

coincide and are finite. Their common value is called the **Riemann integral** of u and denoted by $(R)\int_a^b u(x)dx$ or $\int_a^b u(x)dx$.

Theorem 9.4 (Lebesque → Riemann integrability)**.** *Let $u : [a, b] \to \mathbb{R}$ be a* ***measurable*** *and* ***Riemann integrable*** *function. Then*

$$u \in \mathcal{L}^1(\lambda) \text{ and } \int_{[a,b]} u d\lambda = \int_a^b u(x)dx. \tag{12}$$

Theorem 9.5 (Riemann integrability)**.** *Let $u : [a, b] \to \mathbb{R}$ be a bounded function, it is Riemann integrable* ***iff*** *the points in (a, b) where u is discontinuous are a (subset of) Borel measurable null set.*

Improper Riemann Integrals

- The Lebesgue integral extends the (*proper*) Riemann integral. However, there is a further extension of the Riemann integral which cannot be captured by Lebesgue's theory. u is Lebesgue integrable *iff* $|u|$ ha finite Lebesgue integral. ¡- The Lebesgue integral does not respect sign-changes and cancellations. However, the following *improper Riemann integral* does:

$$(R)\int_0^\infty u(x)dx := \lim_{n\to\infty} (R)\int_0^a u(x)dx. \tag{13}$$

Corollary 9.6. *Let $u : [0, \infty) \to \mathbb{R}$ be a measurable, Riemann integrable function for every interval $[0, N]$, $N \in \mathbb{N}$. Then $u \in \mathcal{L}^1[0, \infty)$* ***iff***

$$\lim_{N\to\infty} (R)\int_0^N |u(x)|dx < \infty. \tag{14}$$

In this case, $(R)\int_0^\infty u(x)dx = \int_{[0,\infty)} u d\lambda$

Proposition 9.7 (appearing as example 12.13 in Schilling)**.** *Let $f_\alpha(x) := x^\alpha, x > 0$ and $\alpha \in \mathbb{R}$. Then*

(i) $f_(\alpha) \in \mathcal{L}^1(0, 1) \Leftrightarrow \alpha > -1$.

(ii) $f_(\alpha) \in \mathcal{L}^1[1, \infty) \Leftrightarrow \alpha < -1$.

10 Regularity of Measures (App. H, [Schilling(2017)])

We let (X, d) be a metric space and denote by O the open, by C the closed and $\mathscr{B}(X) = \sigma(O)$ the Borel set of X.

Definition 10.1 (outer and inner regular measures)**.** A measure μ on $(X, d, \mathscr{B}(X))$ is called outer regular, if

$$\mu(B) = \inf\{\mu(U) \mid B \subset U, \ U \text{ open}\} \tag{15}$$

and inner regular, if $\mu(K) < \infty$ for all compact sets $K \subset X$ and

$$\mu(U) = \sup\{\mu(K) \mid K \subset U, \ K \text{ compact}\}. \tag{16}$$

A measure which is both inner and outer regular is called **regular**. We write $\mathfrak{m}_r^+(X)$ for the family of regular measures on $(X, \mathscr{B}(X))$.

Remark. *The space X is called σ-compact if there is a sequence of compact sets $K_n \uparrow X$. A typical example of such a space is a locally compact, separable metric space.*

Theorem 10.2. *Let (X, d) be a metric space. Every finite measure μ on $(X, \mathscr{B}(X))$ is outer regular. If X is σ-compact, then μ is also inner regular, hence regular.*

Theorem 10.3. *Let (X, d) be a metric space and μ be a measure on $(X, B(X))$ such that $\mu(K) < \infty$ for all compact sets $K \subset X$.*

1 If X is σ-compact, then μ is inner regular.

2 If there exists a sequence $G_n \in O$, $G_n \uparrow X$ such that $\mu(G_n) < \infty$, then μ is outer regular.

11 The Function Spaces $\mathcal{L}^p$ (13, [Schilling(2017)])

Assume V is a vector space over $\mathbb{K} \in \{\mathbb{C}, \mathbb{R}\}$.

Definition 11.1. A seminorm on V is a map $p : V \to [0, +\infty)$ s.t.

(1) $p(cx) = |c|p(x) \ \forall x \in V, \forall c \in \mathbb{K}$.

(2) $p(x + y) \le p(x) + p(y) \ \forall x, y \in V$. **triangle inequality**.

A seminorm is called a norm if we also have

$$p(x) = 0 \iff x = 0.$$

A norm is commonly denoted $||x||$, and a vectorspace equipped with a norm is called a **normed space**.

Definition 11.2 (p-norm)**.** Assume (X, d) is a measure space. Fix $1 \le p \le \infty$. For every measurable function $f : X \to \mathbb{C}$ we define the following

$$||f||_p = \left(\int_X |f|^p d\mu\right)^{1/p} \in [0, +\infty]. \tag{17}$$

We can see that $||cf||_p = |c|||f||_p \ \forall c \in \mathbb{C}$.

Notice that by Theorem 8.2(i) we have that $||f||_p = 0 \Rightarrow f = 0$ a.e. Consider for example $\lim_{n\to\infty} ||f_n - f||_p = 0$, then we can find a subsequence s.t. $\lim_{k\to\infty} |f_{n(k)} - f| = 0$ a.e., i.e. $\lim_{k\to\infty} f_{n(k)} = f$ a.e.

Theorem 11.3 (**Hölder's inequality**)**.** *Assume that* $u \in \mathcal{L}^p(\mu)$ *and* $v \in \mathcal{L}^q(\mu)$*, where* $1/p + 1/q = 1$ *and* $p, q \in [0, +\infty]$*. Then* $uv \in \mathcal{L}^1(\mu)$*, and the following inequality holds:*

$$\left| \int uv d\mu \right| \leq \int |uv| d\mu = ||uv||_1 \leq ||u||_p \cdot ||v||_q.$$

The generalized version reads:

$$\int |u_1 \cdot u_2 \cdots u_N| d\mu \leq ||u_1||_{p_1} \cdot ||u_2||_{p_2} \cdots ||u_N||_{p_N}.$$

Lemma 11.4.

$$||f + g||_p \leq ||f||_p + ||g||_p. \tag{18}$$

Definition 11.5 (**Lebesgue space**)**.** We define

$$\mathcal{L}^p(X, d\mu) = \left\{ f : X \to \mathbb{C} \ \mid \ f \text{ is measurable and } ||f||_p < \infty \right\}.$$

This is a vectorspace with seminorm $f \mapsto ||f||_p$. And in general this is not a normed space, since $||f||_p = 0 \iff f = 0$ a.e.

Generally, if p is a seminorm on a vectorspace V, then

$$V_0 = \{x \in V \ \mid \ p(x) = 0\} \tag{19}$$

which is a subspace of V. Then we consider the quotient/factor space V/V_0.

Definition 11.6. For $x, y \in V$, define

$$x \sim y \iff x - y \in V_0. \tag{20}$$

This is an equivalence relation on V. The representation class of V is defined by $[x]$ or $x + V_0$.

Then V/V_0 is equals the set of equivalence classes. We can show that it is a normed space.

$$[x] + [y] = [x + y] \ \ , \ \ c[x] = [cx] \ \ , \ \ ||[x]|| = p(x).$$

Applying this to $\mathcal{L}^p(X, d\mu)$ we get the normed space

$$L^p(X, d\mu) := \mathcal{L}^p(X, d\mu)/\mathcal{N} = \mathcal{L}^p(X, d\mu)/_\sim. \tag{21}$$

Where $\mathcal{N}$ is the space of measurable functions f s.t. $f = 0$ a.e. The equivalence relation $\sim$ is defined by

$$u \sim v \iff \{u \neq v\} \in \mathcal{N}_\mu \iff \mu\{u \neq v\} = 0,$$

and so $L^p(X, d\mu)$ consists of *all equivalence classes* $[u]_p = \{v \in \mathcal{L}^p | u \sim v\}$. So for every $u \in [u]_p$ there is no $v \in [u]_p$ such that $\mu\{u \neq v\} \neq 0$.

We will further continue to denote the norm by $|| \cdot ||_p$, and we will normally **not** distinguish between $f \in \mathcal{L}^p(X, d\mu)$ and the vector in $L^p(X, d\mu)$ that f defines.

Definition 11.7 (**Banach space**)**.** A normed space $(X, || \cdot ||)$ is called a Banach space if V is complete w.r.t the metric $d(x, y) = ||x - y||$.

Theorem 11.8. *If* $(X, \mathscr{B}, \mu)$ *is a measure space,* $1 \leq p \leq \infty$*, then* $L^p(X, d\mu)$ *is a Banach space.*

Definition 11.9. A measurable function $f : X \to \mathbb{C}$ is called **essentially bounded** if there is $c \geq 0$ s.t.

$$\mu(\{x \ : \ |f(x)| > c\}) = 0. \tag{22}$$

That is $|f| \leq c$ a.e. The smallest such c is called the essential supremum of f and is denoted by $||f||_\infty$. That is,

$$||u||_\infty := \inf \left\{ c > 0 : \mu\{|u| \geq c\} = 0 \right\},$$

and from problem 13.21 we have

$$\lim_{p \to \infty} || \cdot ||_p = || \cdot ||_\infty.$$

Definition 11.10 (**L^∞**)**.**

$$\mathcal{L}^\infty(X, d\mu) = \{f : X \to \mathbb{C} \ \mid \ f \text{ is measurable and } ||f||_\infty < \infty\}.$$

$$L^\infty(X, d\mu) = \mathcal{L}^\infty(X, d\mu)/\mathcal{N}.$$

Where by the previous definiton these spaces become the spaces of all essentially bounded functions.

Theorem 11.11. *If* $(X, \mathscr{B}, \mu)$ *is a* σ*-finite measure space, then* $L^\infty(X, d\mu)$ *is a Banach space.*

Convergence in $\mathcal{L}^p$ and completeness

Lemma 11.12. *For any sequence* $(u_n)_{n \in \mathbb{N}} \subset \mathcal{L}^p(\mu), p \in [1, \infty)$*, of positive functions* $u_n \geq 0$ *we have*

$$\left\| \sum_{n=1}^\infty u_n \right\|_p \leq \sum_{n=1}^\infty ||u_n||_p.$$

Theorem 11.13 (**Riesz-Fischer**)**.** *The spaces* $\mathcal{L}^p(\mu), p \in [1, \infty)$*, are complete, i.e. every Cauchy sequence* $(u_n)_{n \in \mathbb{N}} \subset \mathcal{L}^p(\mu)$ *converges to some limit* $u \in \mathcal{L}^p(\mu)$

Corollary 11.14. *Let* $(u_n)_{n \in \mathbb{N}} \subset \mathcal{L}^p(\mu), p \in [1, \infty)$ *with* $\mathcal{L}^p - \lim_{n \to \infty} u_n = u$*. Then there exists a subsequence* $(u_{n_k})_{k \in \mathbb{N}}$ *s.t.* $\lim_{k \to \infty} u_{n_k}(x) = u(x)$ *holds for almost every* $x \in X$*.*

Theorem 11.15. *Let* $(u_n)_{n \in \mathbb{N}} \subset \mathcal{L}^p(\mu), p \in [1, \infty)$*, be a sequence of functions s.t.* $|u_n| \leq w$ $\forall n \in \mathbb{N}$ *and some* $w \in \mathcal{L}^p(\mu)$*. If* $u(x) = \lim_{n \to \infty} u_n(x)$ *exists for (almost) every* $x \in X$*, then*

$$u \in \mathcal{L}^p \ \text{ and } \lim_{n \to \infty} ||u - u_n||_p = 0.$$

Theorem 11.16 (**F. Riesz (convergence theorem)**)**.** *Let* $(u_n)_{n \in \mathbb{N}} \subset \mathcal{L}^p(\mu), p \in [1, \infty)$*, be a sequence s.t.* $\lim_{n \to \infty} u_n(x) = u(x)$ *for almost every* $x \in X$ *and some* $u \in \mathcal{L}^p(\mu)$*. Then*

$$\lim_{n \to \infty} ||u_n - u||_p = 0 \iff \lim_{n \to \infty} ||u_n||_p = ||u||_p.$$

12 Dense and Determining Sets (17, [Schilling(2017)])

Definition 12.1 (dense sets). A set $\mathcal{D} \subset \mathcal{L}^p(\mu), p \in [0, \infty]$, is called *dense* if for every $u \in \mathcal{L}^p(\mu)$ there exist a sequence $(f_n)_{n\in\mathbb{N}} \subset \mathcal{D}$ s.t. $\lim_{n\to\infty} ||u - f_n||_p = 0$.

Definition 12.2 (support). The support of a function f is the set of points in X where f is non-zero:

$$\operatorname{supp}(f) := \{x \in X : f(x) \neq 0\}.$$

Dense subsets of $\mathcal{L}^p$:

Theorem 12.3. *Let μ be a finite measure on $X, d, \mathscr{B}(X)$. Then $C_b(X) \subset \mathcal{L}^p(\mu)$ is dense.*

Theorem 12.4. *Assume X, d is a metric space and μ is a Borel measure that is finite on every ball $1 \leq p < \infty$. Then the space of bounded continuous functions with bounded support is dense in $\mathcal{L}^p(X, d\mu)$. Where bounded support means that f vanishes outside some ball.*

Theorem 12.5. *Assume (X, d) is a separable locally compact metric space and μ is a Borel Measure on X s.t. $\mu(K) < \infty\ \forall$ compact $K \subset K$. Then the space $C_c(X)$ of continuous compactly supported functions is dense in $\mathcal{L}^p(X, d\mu)$.*

Recall that the support of a function f is $\operatorname{supp}(f) = \{x \in X : f(x) \neq 0\}$, *closed support* is the closure of $\operatorname{supp}(f)$ (i.e. boundary points are included), often just written as $\operatorname{supp}(f)$, and a function is said to have *compact support* if $\operatorname{supp}(f)$ is *compact*.

In particular, either theorem shows that if μ is a Borel measure on $\mathbb{R}^n$ s.t. the measure of every ball is finite, then $C_c(\mathbb{R}^n)$ is dense in $\mathcal{L}^p(\mathbb{R}^n, d\mu)$, $1 \leq p < \infty$. Later we will see that even $C_c^\infty(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n, d\mu)$.

Remark. *These results do not extend to $p = \infty$ in general.*

For $\mu = \lambda_n$ we write simply $\mathcal{L}^p(\mathbb{R}^n)$.

Remark. *Theorem 17.8 in the book is WRONG. For example, $X = \mathbb{Q}$ with the usual metric is σ-compact, supports nonzero finite measure, but $C_c(\mathbb{Q}) = 0$.*

Modes of Convergence

(mixture of ex. 11.12 and ch. 22 p. 258-261. in [Schilling(2017)])

Assume $(X, \mathscr{B}, \mu)$ is a measure space. Given measurable functions $f_n, f : X \to \mathbb{C}$, recall that

$$f_n \to f \text{ a.e.}$$

means that $f_n(x) \xrightarrow[n\to\infty]{} f(x)$ for all x outside a set of measure zero.

Theorem 12.6 (Egorov). *Assume $\mu(X) < \infty$ and $f_n \to f$ a.e. Then, $\forall \epsilon > 0$, there exists $X_\epsilon \in \mathscr{B}$ s.t. $\mu(X_\epsilon) < \epsilon$ and $f_n \to f$ uniformly on $X \backslash X_\epsilon$.*

In addition to pointwise and uniform convergence we also consider the following:

$f_n \to f$ in the *p-th mean* if $||f_n - f||_p \xrightarrow[n\to\infty]{} 0$. For $p = 1$ we say *in mean*, for $p = 2$ we say in quadratic mean.

$f_n \to f$ *in measure* if $\forall \epsilon > 0$ we have

$$\mu\left(\{x : |f_n(x) - f(x)| \geq \epsilon\}\right) \xrightarrow[n\to\infty]{} 0.$$

Theorem 12.7. *Assume $(X, \mathscr{B}, d\mu)$ is a measure space, $1 \leq p < \infty$, $f_n, f : X \to \mathbb{C}$ are measurable functions. Then*

(i) If $f_n \to f$ in the p-th mean, then $f_n \to f$ in measure.

(ii) If $f_n \to f$ in measure, then there is a subsequence $(f_{n_k})_{k=1}^\infty$ s.t. $f_{n_k} \to f$ a.e.

(iii) If $f_n \to f$ a.e. and $\mu(X) < \infty$, then $f_n \to f$ in measure.

In particular, if $f_n \to f$ in the p-th mean, then $f_{n_k} \to f$ a.e. for a subsequence $(f_{n_k})_k$.

13 Abstract Hilbert Spaces (26, [Schilling(2017)])

Assume $\mathcal{H}$ is a vector space over $\mathbb{C}$.

Definition 13.1. A pre-inner product on $\mathcal{H}$ is a map $(\cdot, \cdot) : H \times H \to \mathbb{C}$ which is

(i) Sesquilinear: linear in the first variable and antilinear in the second:

$$(\alpha u + \beta v, w) = \alpha(u, w) + \beta(v, w),$$
$$(w, \alpha u + \beta v) = \bar{\alpha}(w, u) + \bar{\beta}(w, v),\ \ u, v, w \in H \text{ and } \alpha, \beta \in \mathbb{C}.$$

(ii) Hermitian: $(u, v) = \overline{(u, v)}$.

(iii) Positive semidefinite: $(u, v) \geq 0$.

It is called an **inner product**, or a scalar product, if instead of (iii) the map is positive definite; $(u, v) > 0$. This definition also works for $\mathbb{R}$ instead of $\mathbb{C}$.

Definition 13.2 (adjoint operator). Assume $T : \mathcal{H}_1 \to \mathcal{H}_2$ is a linear operator. The adjoint operator T^* is a linear operator $T^* : \mathcal{H}_2 \to \mathcal{H}_1$ s.t. for all $h_1 \in \mathcal{H}_1$ and $h_2 \in \mathcal{H}_2$,

$$\langle Th_1, h_2\rangle_{\mathcal{H}_2} = \langle h_1, T^* h_2\rangle_{\mathcal{H}_1}.$$

Lemma 13.3 (Cauchy-Schwartz inequality). *If $(\cdot, \cdot)$ is a pre-inner product, then $|(u, v)| \leq (u, u)^{1/2}(v, v)^{1/2}$.*

Corollary 13.4. *Assume we have a seminorm $||u|| := (u, u)^{1/2}$. It is a norm iff $(\cdot, \cdot)$ is an inner product.*

Definition 13.5 (Hilbert space). A Hilbert space is a complex vector space $\mathcal{H}$ with an inner product $(\cdot, \cdot)$ s.t. $\mathcal{H}$ is complete with respect to the norm $||u|| = (u, u)^{1/2}$.

1. The norm on a Hilbert space is determined by the inner product, but the inner product can also be recovered by the norm by the *polarization identity*: $(u, v) = \frac{||u+v||^2 - ||u-v||^2}{4} + i\frac{||u+iv||^2 - ||u-iv||^2}{4}$.

2. *Parallelogram law*: $||u + v||^2 + ||u - v||^2 = 2||u||^2 + 2||v||^2$.

3. A norm on a vector space is given by an inner product iff it satisfies the parallelogram law, and then the scalar product is uniquely determined by the polarization identity.

Example 13.6. Assume $(X, \mathscr{B}, \mu)$ is a measure space. Then $\mathcal{L}^2(X, d\mu)$ is a Hilbert space with inner product

$$(f, g) = \int_X f\bar{g} d\mu.$$

This is well-defined, as $|f\bar{g}| \leq \frac{1}{2}(|f|^2 + |g|^2)$.

In particular, if $\mathscr{B} = \mathcal{P}(X)$ and μ is the counting measure, i.e.

$$\mu(A) = \begin{cases} \# \text{ if } A \text{ is finite,} \\ +\infty \text{ if } A \text{ is infinite,} \end{cases}$$

then $L^2(X, d\mu)$ is denoted by $l^2(X)$; for $X = \mathbb{N}$ we write simply l^2. Note that in this case for $f : X \to [0, +\infty]$ we have

$$\int_X f d\mu = \sum_{x \in X} f(x) := \sup_{\substack{F \subset X \\ F \text{ is finite}}} \sum_{x \in F} f(x),$$

and if $\sum_{x \in X} f(x) < \infty$, then $\{x : f(x) > 0\}$ is at most countable, so $\sum_{x \in X} f(x) = \sum_{x : f(x) > 0} f(x)$ is the usual sum of a series.

Recall that a subset C of a vector space is called *convex* if

$$u, w \in C \to tu + (1-t)w \in C \ \forall t \in (0, 1).$$

The following is one of the key properties of the Hilbert space

Theorem 13.7 (projection theorem)**.** *Assume $\mathcal{H}$ is a Hilbert space and $C \subset H$ is a closed convex subset. Then for every $u \in H$ there is a unique $u_0 \in C$ (*minimizer*) s.t.*

$$||u - u_0|| = d(u, C)(= \inf_{x \in C} ||u - x||).$$

This minimizer $u_0 = P_C u$ is called the orthogonal projection of u onto C.

14 Orthogonal Projections (26, [Schilling(2017)])

For a Hilbert space $\mathcal{H}$ and a subset $A \subset H$, the following is the orthogonal complement of A:

$$A^\perp := \{x \in H : x \perp y \ \forall y \in A\},$$

where $x \perp y$ means that $(x, y) = 0$. $A^\perp$ is a closed subspace of $\mathcal{H}$.

Proposition 14.1 (decomposition of Hilbert spaces)**.** *Assume $\mathcal{H}_0$ is a closed subspace of a Hilbert space $\mathcal{H}$. Then every $u \in H$ uniquely decomposes as*

$$u = u_0 + u_1, \text{ with } u_0 \in H \text{ and } u_1 \in \mathcal{H}_0^\perp.$$

Moreover, $||u - u_0|| = d(u, \mathcal{H}_0)$ and $||u||^2 = ||u_0||^2 + ||u_1||^2$.

For a closed subspace $\mathcal{H}_0 \subset \mathcal{H}$, consider the map $P : H \to \mathcal{H}_0$ s.t. $Pu \in \mathcal{H}_0$ is the unique element satisfying $u - Pu \in H_0^\perp$. The operator P is linear. It is also contractive, meaning that $||Pu|| \leq ||u||$, since $||u||^2 = ||Pu||^2 + ||u - Pu||^2$. It is called the orthogonal projection onto $\mathcal{H}_0$.

If $\mathcal{H}_0$ is finite dimensional with an orthonormal basis $u_1, \ldots, u_n$ then

$$Pu = \sum_{k=1}^n (u, u_k) u_k.$$

Orthonormal bases can be defined for arbitrary Hilbert spaces.

Definition 14.2 (orthonormal system)**.** An orthonormal system in $\mathcal{H}$ is a collection of vectors $u_i \in \mathcal{H}$ $(i \in I)$s.t.

$$(u_i, u_j) = \delta_{ij} \ \forall i, j \in I.$$

It is called an *orthonormal basis* if span$\{u_i\}_{i \in I}$ denotes the linear span of $\{u_i\}_{i \in I}$, the space of finite linear combinations of the vectors u_i.

Definition 14.3 (seperable Hilbert space)**.** A Hilbert space $\mathcal{H}$ is said to be *separable* if $\mathcal{H}$ contains a countable dense subset $G \subset \mathcal{H}$.

Theorem 14.4. ***Every Hilbert space $\mathcal{H}$ has an orthonormal basis.*** *If $\mathcal{H}$ is separable, then there is a countable orthonormal basis.*

Proposition 14.5. *Assume $\{u_i\}_{i \in I}$ is an orthonormal system in a Hilbert space $\mathcal{H}$ and let $u \in \mathcal{H}$. Then*

(i) Bessel's inequality*: $\sum_{i \in I} |(u, u_i)|^2 \leq ||u||^2$, in particular, $\{i : (u, u_i) \neq 0\}$ is countable.*

(ii) **Parseval's identity***: If $\{u_i\}_{i \in I}$ is an orthonormal basis, then $\sum_{i \in I} |(u, u_i)|^2 = ||u||^2$.*

*(iii) $\bigcup_{N=1}^{\infty} E(N)$ is **dense** in $\mathcal{H}$ where $E(N) = span\{e_1, \ldots, e_N\}$*

*(iv) $g = \sum_{n=1}^{\infty} \langle g, e_n \rangle e_n \ \forall g \in \mathcal{H}$. (**Fourier coefficients**)*

If $(u_i)_{i \in I}$ is an orthonormal basis, then the numbers (u, u_i) are called the **Fourier coefficients** of u with respect to $(u_i)_{i \in I}$. The Parseval identity then suggests that u is determined by its Fourier coefficients. This is true, and even more, we have:

Proposition 14.6. *Assume $(u_i)_{i \in I}$ is an orthonormal basis in a Hilbert space $\mathcal{H}$. Then for every vector $(c_i)_{i \in I} \in l^2(I)$ there is a unique vector $u \in \mathcal{H}$ with Fourier coefficients c_i, and we write*

$$u = \sum_{i \in I} c_i u_i.$$

Remark. *Equivalently, the element $u = \sum_{i \in I} c_i u_i$ can be described as the unique element in $\mathcal{H}$ s.t. $\forall \epsilon > 0$ there is a finite $F_0 \subset I$ s.t. $||u - \sum_{i \in F} c_i u_i|| < \epsilon \ \forall$ finite $F \supset F_0$.*

Corollary 14.7. *We have a linear isomorphism $U : l^2(I) \xrightarrow{\sim} \mathcal{H}$, $U((c_i)_{i \in I}) = \sum_{i \in I} c_i u_i$. By Parseval's identity this isomorphism is* isometric*, that is, $||Ux|| = ||x|| \ \forall x \in l^2(I)$. By the polarization identity this is equivalent to*

$$(Ux, Uy) = (x, y) \ \forall x, y \in l^2(I).$$

Therefor U is unitary.

Corollary 14.8. *Up to a unitary isomorphism, there is **only one infinite dimensional separable Hilbert space**, namely, l^2. Recall a unitary isomorphism is a bijective map between spaces, $U : H_1 \to H_2$ s.t. $\langle Ux, Uy \rangle_{H_2} = \langle x, y \rangle_{H_1}$.*

15 Dual spaces (26, [Schilling(2017)])

Lemma 15.1. *Assume V is a normed space over $K = \mathbb{R}$ or $K = \mathbb{C}$. Consider a linear functional $f : V \to K$. The following are equivalent (TFAE):*

(i) f is continuous;

(ii) f is continuous at 0;

(iii) There is a $c \geq 0$ s.t. $|f(x)| \leq c||x||$ $\forall x \in V$.

If (i)-(iii) are satisfied, then f is called a ***bounded linear functional***. The constant c in (iii) is denoted by $||f||$. We have $||f|| = \sup_{x\neq 0} \frac{|f(x)|}{||x||} = \sup_{||x||=1} |f(x)| = \sup_{||x||\leq 1} |f(x)|$. A bounded linear functional is a generalization of a bounded linear operator: $O : V \to V'$, where V' is $\mathbb{R}$ or $\mathbb{C}$.

Proposition 15.2. *For every normed vector space V over $K = \mathbb{R}$ or $K = \mathbb{C}$, the bounded linear functionals on V form a Banach space V^*.*

Remark. *The sequence $\{||f_n - f_m||\}_{m=1}^{\infty}$ actually converges, since*

$$\Big| ||f_n - f_m|| \Big| \leq ||f_m - f_n||.$$

When we study/use normed spaces, it is often important to understand the dual spaces. For Hilbert spaces this is particularly easy:

Theorem 15.3 (**Riesz**)**.** *Assume $\mathcal{H}$ is a Hilbert space. Then every $f \in \mathcal{H}^*$ has the form*

$$f(x) = (x, y),$$

for a uniquely defined $y \in \mathcal{H}$. Moreover, we have $||f|| = ||y||$.

For every Hilbert space $\mathcal{H}$ we can define the ***conjugate Hilbert space*** $\overline{\mathcal{H}}$, which has its elements as the symbols $\bar{x}$ for $x \in \mathcal{H}$, with the linear structure and inner product defined by

$$\bar{x} + \bar{y} = \overline{x+y}, c \cdot \bar{x} = \overline{\bar{c}x}, (\bar{x}, \bar{y}) = \overline{(x, y)} = (y, x).$$

Corollary 15.4. *For every Hilbert space $\mathcal{H}$, we have an isometric isomorphism (unitary isomorphism/transformation) $\bar{\mathcal{H}} \xrightarrow{\sim} \mathcal{H}^*$, $\bar{x} \mapsto (\cdot, x)$.*

16 Hahn-Banach Theorem (4.2, [Teschl(2010)])

Theorem 16.1 (**Hahn-Banach**)**.** *Assume V is a real vector space, $V_0 \subset V$ a subspace, $\phi : V \to \mathbb{R}$ a convex function (i.e., $\phi(\lambda x + (1 - \lambda)y) \leq \lambda\phi(x) + (1 - \lambda)\phi(y)$) and $f : V_0 \to \mathbb{R}$ a linear functional s.t. $f \leq \phi$ on V_0. Then f can be extended to a linear functional F on V s.t. $F \leq \phi$.*

Theorem 16.2 (**Hahn-Banach**)**.** *Assume V is a real or complex vector space, p a seminorm on V, $V_0 \subset V$, and f a linear functional on V_0 s.t.*

$$|f(x)| \leq p(x) \; \forall x \in V_0.$$

Then f can be extended to a linear functional F on V s.t. $|F(x)| \leq p(x)$ $\forall x \in V$.

Corollary 16.3. *Assume V is a normed real or complex vector space, $V_0 \subset V$ and $f \in V_0^*$. Then there is a $F \in V^*$ s.t.*

$$F|_{V_0} f \text{ and } ||F|| = ||f||.$$

Theorem 16.4 (**Hahn-Banach (dense subsets)**)**.** *Assume $\mathcal{H}_0$ is a dense subset of a normed vector space $\mathcal{H}$ and that $T : \mathcal{H}_0 \to Y$ is a bounded linear operator (where Y is a complete, normed vector space), then there is a unique extension of T to $T' : \mathcal{H} \to Y$.*

Corollary 16.5. *Assume V is a normed space and $x \in V, x \neq 0$. Then there is a $F \in V^*$ s.t. $||F|| = 1$ and $F(x) = ||x||$.*

Such an F is called a **supporting functional at x**.

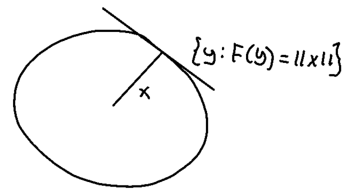

Figure 1: Tangent space?

If V is a normed vector space, then every $x \in X$ defines a bounded linear functional on V^* by

$$V^* \ni f \mapsto f(x).$$

As $|f(x)| \leq ||f|| \cdot ||x||$, this functional has norm $\leq ||x||$. By using a supporting functional at x, we actually see that we get norm $||x||$. Thus, we have an isometric embedding $V \subset V^{**} := (V^*)^*$. We can therefor see view V as a subspace of V^{**}.

Definition 16.6. A normed space V is called reflexive if $V^{**} = V$.

Remark. *This is stronger than requiring $V \simeq V^{**}$. And:*

(i) Every finite dimensional normed vector space V is reflexive for dimension reasons: $\dim V^{**} = \dim V^* = \dim V$.

(ii) Every Hilbert space $\mathcal{H}$ is reflexive. Indeed, $\mathcal{H}^* = \overline{\mathcal{H}}$. By Riesz' theorem every bounded linear functional f on $\overline{\mathcal{H}}$ has the form

$$f(\bar{x}) = (\bar{x}, \bar{y}) = (y, x),$$

for some $y \in \mathcal{H}$, which exactly means that $f = y$ in $\mathcal{H}^{**}$.

As we will see later, the spaces $\mathcal{L}^p(X, d\mu)$, with μ σ-finite and $1 < p < \infty$, are reflexive. The spaces $\mathcal{L}^1(X, d\mu)$ and $\mathcal{L}^\infty(X, \mu)$ are usually not reflexive.

17 Radon-Nikodym Theorem (20, [Schilling(2017)])

Assume $(X, \mathscr{B}, \mu)$ is a measure space. Are there other measures on $(X, \mathscr{B})$?

Example 17.1. Take a measurable function $f : X \to [0, +\infty]$ and define

$$\nu(A) := \int_A f d\mu \text{ for } A \in \mathscr{B}.$$

This is a measure by the monotone convergence theorem. We write $d\nu = f d\mu$. Furthermore, we say that f is the ***Radon-Nikodym* derivative**, and we denote it by $f = d\nu/d\mu$. If $\mu = \lambda^1$ we get $f(x) = d\nu(x)/dx$.

Proposition 17.2. *Assume $(X, \mathscr{B})$ is a measurable space, μ and ν are σ-finite measures on $(X, \mathscr{B})$. Then there exist $N \in \mathscr{B}$ and a measurable $f : X \to [0, +\infty]$ s.t. $\mu(N) = 0$ and $\nu(A) = \nu(A \cap N) + \int_A f d\mu$ $\forall A \in \mathscr{B}$.*

When can we discard the term $\nu(A \cap N)$?

Definition 17.3 (absolutely continuous measure)**.** Given measure μ and ν on $X, \mathscr{B}$, we say that ν is *absolutely continuous* with respect to μ and write $\nu << \mu$, if $\nu(A) = 0$ whenever $A \in \mathscr{B}, \mu(A) = 0$.

Lemma 17.4. *Assume μ and ν are measures on $(X, \mathscr{B})$, $\nu(X) < \infty$. Then $\nu << \mu$ iff $\forall \epsilon > 0 \exists \delta > 0$ s.t. if $A \in \mathscr{B}$, $\mu(A) < \delta$, then $\nu(A) < \epsilon$.*

Remark. *The result is not true for infinite ν.*

Theorem 17.5 (Radon-Nikodym)**.** *Assume μ and ν are σ-finite measures on a measurable space $(X, \mathscr{B})$, $\nu << \mu$. Then there is a measurable function $f : X \to [0, +\infty)$ s.t. $d\nu = f d\mu$ (that is, $\nu(A) = \int_A f d\mu$). If $\tilde{f}$ is another function with the same properties, then $f = \tilde{f}$ $\mu - a.e.$*

The function is called the Radon-Nikodym derivative *at ν w.r.t. μ and is denoted by $\frac{d\nu}{d\mu}$.*

Example 17.6. Consider a real-valued function $f \in C'[a, b]$ s.t. $f'(t) > 0$ $\forall$ $t \in [a, b]$. Let $c = f(a), d = f(b)$. We know that for every Riemann integrable function g on $[c, d]$ we have

$$\int_c^d g(f) dt = \int_a^b g(f(t)) f'(t) dt.$$

Equivalently,

$$\int_c^d g \circ g^{-1} dt = \int_a^b g f'(t) dt. \tag{23}$$

Denote by $\lambda_{[a,b]}, \lambda_{[c,d]}$ the Lebesgue measure restricted to the Borel subsets of $[a, b]$ and $[c, d]$, respectively. Then eq. 23 implies that

$$d\left((f^{-1})_* \lambda_{[c,d]}\right) = f' d\lambda_{[a,b]},$$

since the integration of $g = \mathbb{1}_{[\alpha,\beta]}$ gives the same results for any interval $[\alpha, \beta] \subset [a, b]$ and since a finite Borel measure on $[a, b]$ is determined by its values on such intervals. Thus, $(f^{-1})_* \lambda_{[c,d]} << \lambda_{[a,b]}$ and

$$\frac{d\left((f^{-1})_* \lambda_{[c,d]}\right)}{d\lambda_{[a,b]}} = f'.$$

18 Complex and Signed Measures (4.3, [Teschl(2010)])

Definition 18.1 (complex and fintie signed measure)**.** A **complex measure** on $(X, \mathscr{B})$ is a map $\nu : \mathscr{B} \to \mathbb{C}$ s.t. $\nu(\emptyset) = 0$ and

$$\nu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \nu(A_n)$$

for any disjoint $A_n \in \mathscr{B}$, where the series is assumed to be absolutely convergent. If ν takes values in $\mathbb{R}$ then ν is called a finite signed measure.

Remark. *More generally, a signed measure is allowed to take values in $\mathbb{R} \cup \{+\infty\}$ or $\mathbb{R} \cup \{-\infty\}$.*

Given a complex measure ν on $(X, \mathscr{B})$, its total variation is the map $|\nu| : \mathscr{B} \to [0, +\infty]$ defined by

$$|\nu|(A) = \sup\left\{\sum_{n=1}^{N} |\nu(A_n)| : A = \bigcup_{n=1}^{N} A_n, A_n \in \mathscr{B}, A_n \cap A_m = \emptyset\right\}.$$

Proposition 18.2. *$|\nu|$ is a finite measure on $(X, \mathscr{B})$.*

Example 18.3. Consider a measure space $(X, \mathscr{B}, \mu)$ and take $f \in L^1(X, d\mu)$. Define

$$\nu(A) = \int_A f d\mu.$$

Then ν is a complex measure on $(X, \mathscr{B})$, sinche this is true for $f \geq 0$ and a general f can be written as a linear combination of positive ones. We write $d\nu = f d\mu$.

We then have $d|\nu| = |f| d\mu$, that is,

$$|\nu|(A) = \int_A |f| d\mu.$$

Theorem 18.4 (Radon-Nikodym theorem for complex measures)**.** *Assume $(X, \mathscr{B}, \mu)$ is a measure space, ν is a complex measure on $(X, \mathscr{B})$, $\nu << \mu$. Then there is a unique $f \in L^1(X, d\mu)$ s.t. $d\nu = f d\mu$.*

19 Decomposition Theorems (20, [Schilling(2017)] and 4.3, [Teschl(2010)])

Definition 19.1 (mutually singular measurs)**.** Two measures ν and μ on a measurable space $(X, \mathscr{B})$ are called **mutually singular**, or we say that ν is **singular** w.r.t. μ, if there is a $N \in \mathscr{B}$ s.t. $\nu(N^c) = 0$, $\mu(N) = 0$. We then write $\nu \perp \mu$.

Theorem 19.2 (Lebesgue Decomposition Theorem)**.** *Assume ν, μ are σ-finite measures in $(X, \mathscr{B})$. Then there exist unique measures ν_a and ν_s s.t. $\nu = \nu_a + \nu_s$, $\nu_a << \mu$, $\nu_s \perp \mu$.*

Theorem 19.3 (Polar Decomposition of Complex Measure)**.** *Assume ν is a complex measure on $(X, \mathscr{B})$. Then there exist a finite measure μ on $(X, \mathscr{B})$ and a measurable function $f : X \to \Pi$ s.t. $d\nu = f d\mu$. If $(\tilde{\mu}, \tilde{f})$ is another such pair, then $\tilde{\mu} = \mu$ and $\tilde{f} = f$ μ-a.e.*

For signed measures this leads to the following.

Theorem 19.4 (Hahn Decomposition Theorem)**.** *Assume ν is a finite signed measure on $(X, \mathscr{B})$. Then there exist $P, N \in \mathscr{B}$ s.t.*

$$X = P \cup N, \; P \cap N = \emptyset,$$
$$\nu(A \cap P) \geq 0, \; \nu(A \cap N) \leq 0 \; \forall A \in \mathscr{B}.$$

Moreover, then $|\nu|(A) = \nu(A \cap P) - \nu(A \cap N)$, and if $X = \tilde{P} \cup \tilde{N}$ is another such decomposition, then

$$|\nu|\left(P \Delta \tilde{P}\right) = |\nu|\left(N \Delta \tilde{N}\right) = 0.$$

Corollary 19.5 (Jordan Decomposition Theorem)**.** *Assume ν is a finite signed measure on $(X, \mathscr{B})$. Then there exist unique finite measures ν_+, ν_- on $(X, \mathscr{B})$ s.t.*

$$\nu = \nu_+ - \nu_- \text{ and } \nu_+ \perp \nu_-.$$

Moreover, then $|\nu| = \nu_+ + \nu_-$, hence

$$\nu_+ = \frac{|\nu| + \nu}{2}, \nu_- = \frac{|\nu| - \nu}{2}.$$

20 More on Duals of L^p-spaces (21, p. 241, [Schilling(2017)])

- What is the dual of $L^p(X, d\mu)$? When does a measurable function $g : X \to \mathbb{C}$ define a bounded linear functional on $L^p(X, d\mu)$ by

$$\rho(f) = \int_X fg d\mu?$$

Theorem 20.1 (Young's inequality). *Assume $f : [0, a] \to [0, b]$ is a strictly increasing continuous function, $f(0) = 0$, $f(a) = b$. Then, for all $s \in [0, a]$ and $t \in [0, b]$, we have*

$$st \le \int_0^s f(x)dx + \int_0^t f^{-1}(y)dy,$$

and the equality holds iff $t = f(s)$.

With Holder's inequality it follows that every $g \in L^q(X, d\mu)$ defines a bounded linear functional

$$l_g : L^p(X, d\mu) \to \mathbb{C},\ l_g(f) = \int_X fg d\mu,$$

and $||l_q|| \le ||g||_q$.

The same makes sense for $p = 1, q = \infty$ and $p = \infty, q = 1$, when μ is σ-finite, as

$$\int_X |fg| d\mu \le \int_X |f| d\mu \cdot ||g||_\infty = ||f||_1 \cdot ||g||_\infty.$$

Lemma 20.2. *Assume $1 \le p \le \infty$, $1/p + 1/q = 1$, and μ is σ-finite if $p = 1$ or $p = \infty$. For $g \in L^q(X, d\mu)$ consider $l_q \in L^p(X, d\mu)^*$. Then*

$$||l_g|| = ||g||_q.$$

Therefor we can view $L^q(X, d\mu)$ as a subspace of $L^p(X, d\mu)^*$ using the isometric embedding

$$L^q(X, d\mu) \to L^p(X, d\mu)^*,\ g \mapsto l_g.$$

Theorem 20.3. *Assume $(X, \mathscr{B}, d\mu)$ is a σ-finite measure space, $1 \le p < \infty$, $1/p + 1/q = 1$. Then*

$$L^p(X, d\mu)^* = L^q(X, d\mu).$$

Remark. *This is usually not true for $p = \infty$.*

21 + 22 Riesz-Markow Theorem (21, p. [243-249], [Schilling(2017)])

Assume (X, d) is a locally compact metric space.

Definition 21.22 (positive linear functional). A linear functional $\rho : C_c(X) \to \mathbb{C}$ is called positive if $\rho(f) \ge 0$ for all $f \ge 0$. (Recall $C_c(X)$ is the space of continuous (C) compactly supported (c) functions.)

Theorem 21.23 (Riesz-Markov). *If $\rho : C_c(X) \to \mathbb{C}$ is a positive linear functional, where (X, d) is a locally compact metric space, then there exists a Borel measure μ on X s.t. $\mu(K) < \infty$ for every compact $K \subset X$ and*

$$\rho(f) = \int_X f d\mu\ \forall f \in C_c(X).$$

If X is separable, then such a measure μ is unique.

For the proof we need two auxiliary results.

Lemma 21.24 (Urysohn's Lemma). *Assume (X, d) is a metric space, $A, B \subset X$ are disjoint closed subsets. Then there exists a continuous function $f : X \to [0, 1]$ s.t. $f \equiv 1$ on A and $f \equiv 0$ on B.*

Lemma 21.25. *Assume (X, d) is a compact metric space, $U = (U_i)_{i=1}^n$ is a finite open cover of X (so U_i are open and $\cup_{i=1}^n U_i = X$). Then there exist functions $\rho_1, ..., \rho_n$ in $C(x)$ s.t.*

$$0 \le \rho_i \le 1,\ supp(\rho_i) \subset U_i,\ \sum_{i=1}^n \rho_i(x) = 1\ \forall x.$$

*Every such collection of functions is called a **partition of unity subordinate** to U.*

Remark. *Without separability, the uniqueness is not always true. It can be checked that the measure we constructed in the proof,*

$$\mu(U) := \sup \{\phi(f) : 0 \le f \le 1, supp(f) \subset U\},$$

has the following properties:

(i) $\mu(K) < \infty\ \forall$ *compact* $K \subset X$;

(ii) μ *is outer regular* ($\mu(A) = \inf_{\substack{A \subset U \\ U \text{ open}}} \mu(U)$);

(iii) μ *is inner regular on open sets (this is where we need the full strength of step 3):*

$$\mu(U) = \sup_{\substack{K \subset U \\ K \text{ compact}}}\ \forall \text{ open } U.$$

*Such measures are called **Radon measures**. It can be shown that the uniqueness holds within the class of Radon measures.*

Dual of $C(X)$

As an application of the Riesz-Markow Theorem we will describe $C(X)^*$ in terms of measures for compact metric spaces (X, d).

Denote by $M(X)$ the space of complex Borel measures on X. For every $\nu \in M(X)$ we want to make sense of $\int_X f d\nu$ for $f \in C(X)$. It is enough to consider finite signed measures, as then we can define

$$\int_X f d\nu = \int_X f d(\mathrm{Re}\nu) + i \int_X f d(\mathrm{Im}\nu).$$

So assume ν is a finite signed measure. Then, $\nu = \mu_1 - \mu_2$ for positive measures and we define

$$\int_X f d\nu = \int_X f d\mu_1 - \int_X f d\mu_2.$$

This is well-defined, since if

$$\nu = \mu_1 - \mu_2 = \omega_1 - \omega_2,$$

then $\mu_1 + \omega_2 = \mu_2 + \omega_1$ and

$$\int_X f d\mu_1 + \int_X f d\omega_2 = \int_X f d\mu_2 + \int_X f d\omega_1.$$

Thus, every $\nu \in M(X)$ defines a linear functional

$$\phi_\nu : C(X)\mathbb{C} \text{ by } \phi_\nu(f) = \int_x f d\nu,$$

and the map $\nu \mapsto \phi_\nu$ is linear.

Lemma 21.26. *If $\nu \in M(X)$ and $d\nu = gd|\nu|$ is its polar decomposition, then*

$$\int_X f d\nu = \int_X fgd|\nu| \ \forall f \in C(X).$$

Lemma 21.27. *For every $\nu \in M(X)$, the linear functional ϕ_ν is bounded and $||\phi_\nu|| = |\nu|(X)$. (Recall that the norm on $C(X)$ is $||f|| = \sup_{x\in X} |f(x)|$.)*

23 Product Measures and Fubini's Theorem

(14, [Schilling(2017)])

Throughout this chapter we assume that $(X, \mathscr{A}, \mu)$ and $(Y, \mathscr{B}, \nu)$ are σ-finite measure spaces.

Recall the Cartesian product of sets (assume $A \subset X, B \subset Y$):

$$A \times B = \{(a, b) | a \in A \text{ and } b \in B\}.$$

(there are hidden properties here.)

The Lebesgue measure on $\mathbb{R}^n$ has the following property for $n > d \geq 1$:

$$\lambda^n [a_1, b_1) \times ... \times [a_n, b_n) = \lambda^d [a_1, b_1) \times ... \times [a_d, b_d) \cdot \lambda^{n-d} [a_{d+1}, b_{d+1}) \times ... \times [a_n, b_n),$$

which means that

$$\lambda^n(E) = \int \mathbb{1}_E(x, y)\lambda^n(d(x, y)) = \int \left(\int \mathbb{1}_E(x_0, y)\lambda^{n-d}(dy)\right) \lambda^d(dx_0).$$

Goal: we want to define a measure ρ on rectangles on the form $A\times B$ s.t. $\rho(A \times B) = \mu(A)\nu(B)$.

Lemma 23.1. *Let $\mathscr{A}$ and $\mathscr{B}$ be two σ-algebras (or semi-rings), then $\mathscr{A} \times \mathscr{B}$ is a semi-ring.*

Definition 23.2 (**product σ-algebra**)**.** The σ-algebra $\mathscr{A} \otimes \mathscr{B} := \sigma(\mathscr{A} \times \mathscr{B})$ is called a **product σ-algebra**, and $(X \times Y, \mathscr{A} \otimes \mathscr{B})$ is the product of measurable spaces.

When considering $\mathscr{A}\otimes\mathscr{B}$, the following lemma shows that we can instead work with their generators.

Lemma 23.3. *If $\mathscr{A} = \sigma(\mathscr{F})$ and $\mathscr{B} = \sigma(\mathscr{G})$ and if $\mathscr{F}$ and $\mathscr{G}$ contain exhausting sequences $(F_n)_{n\in\mathbb{N}} \subset \mathscr{F}$, $F_n \uparrow X$ and $(G_n)_{n\in\mathbb{N}} \subset \mathscr{G}$, $G_n \uparrow X$, then*

$$\sigma(\mathscr{F} \times \mathscr{G}) = \sigma(\mathscr{A} \times \mathscr{B}) := \mathscr{A} \otimes \mathscr{B}.$$

Theorem 23.4 (uniqueness of product measures)**.** *Assume that $\mathscr{A} = \sigma(\mathscr{F})$ and $\mathscr{B} = \sigma(\mathscr{G})$. If*

- *$\mathscr{F}, \mathscr{G}$ is $\cap$-stable (stable under finite intersections),*
- *$\mathscr{F}, \mathscr{G}$ contain exhausting sequences $F_k \uparrow X$ and $G_k \uparrow Y$ with $\mu(F_k) < \infty$ and $\nu(G_n) < \infty$,*

then there is at most one measure ρ on $(X \times Y, \mathscr{A} \otimes \mathscr{B})$ satisfying

$$\rho(F \times G) = \mu(F)\nu(G) \ \forall F \in \mathscr{F}, G \in \mathscr{G}.$$

Theorem 23.5 (existence of product measures)**.** *The set function*

$$\rho : \mathscr{A} \times \mathscr{B} \to [0, \infty], \ \rho(A \times B) := \mu(A)\nu(B),$$

extends uniquely to a σ-finite measure on $(X \times Y, \mathscr{A} \otimes \mathscr{B})$ s.t.

$$\rho(E) = \int\int \mathbb{1}_E(x, y)\mu(dx)\nu(dy) = \int\int \mathbb{1}_E(x, y)\nu(dy)\mu(dx)$$

holds for all $E \in \mathscr{A} \otimes \mathscr{B}$ (the parenthesis in the expression above are left out). In particular, the functions

$$x \mapsto \mathbb{1}_E(x, y), x \mapsto \int \mathbb{1}_E(x, y)\nu(dy),$$

$$y \mapsto \mathbb{1}_E(x, y), y \mapsto \int \mathbb{1}_E(x, y)\mu(dx),$$

are $\mathscr{A}$, $\mathscr{B}$-measurable (respectively) for every fixed $y \in Y$, $x \in X$ (respectively).

Lecture 24

Definition 24.25 (**product measure $\mu\times\nu$**)**.** The unique measure ρ constructed in Theorem 23.5 is called the **product** of the measures μ and ν, denoted $\mu \times \nu$. $(X, Y, \mathscr{A} \otimes \mathscr{B}, \mu \times \nu)$ is called the **product measure space**

We can now finally construct the n-dimensional Lebesgue measure:

Corollary 24.26. *If $n > d \geq 1$,*

$$(\mathbb{R}^n, \mathscr{B}(\mathbb{R}^n), \lambda^n) = \left(\mathbb{R}^d \times \mathbb{R}^{n-d}, \mathscr{B}(\mathbb{R}^d) \otimes \mathscr{B}(\mathbb{R}^{n-d}), \lambda^d \times \lambda^{n-d}\right).$$

Great. The next step is to see how we can integrate w.r.t. to $\mu \times \nu$. The following two results are often stated together as the Fubini or Fubini-Tonelli theorem.

Theorem 24.27 (**Tonelli**)**.** *Let $(X, \mathscr{A}, \mu)$ $(Y, \mathscr{B}, \nu)$ be σ-finite measure spaces and let $u : X \times Y \to [0, \infty]$ be $\mathscr{A} \otimes \mathscr{B}$-measurable. Then*

(i) $x \mapsto u(x, y)$, $y \mapsto u(x, y)$ are $\mathscr{A}$-resp. $\mathscr{B}$-measurable for fixed y resp. x;

(ii) $x \mapsto \int_Y u(x, y)\nu(dy)$, $y \mapsto \int_X u(x, y)\mu(dx)$ are $\mathscr{A}$-resp. $\mathscr{B}$-measurable;

(iii) $\int\limits_{X\times Y} ud(\mu \times \nu) = \int\limits_Y\int\limits_X u(x, y)\mu(dx)\nu(dy = \int\limits_X\int\limits_Y u(x, y)\nu(dy)\mu(dx))$ which is in $[0, \infty]$.

The following corollary really extends Tonelli to not necessarily positive functions.

Corollary 24.28 (**Fubini's theorem**)**.** *Let $(X, \mathscr{A}, \mu)$ $(Y, \mathscr{B}, \nu)$ be σ-finite measure spaces and let $u : X \times Y \to \bar{\mathbb{R}}$ be $\mathscr{A} \otimes \mathscr{B}$-measurable. If at least one of the three integrals*

$$\int\limits_{X\times Y} |u| d(\mu \times \nu), \int_Y\int_X |u(x, y)|\mu(dx)\nu(dy), \int_X\int_Y |u(x, y)|\nu(dy)\mu(dx)$$

is finite, then all three integrals are finite, $u \in \mathcal{L}^1(\mu \times \nu)$, and

(i) $x \mapsto u(x, y)$ is in $\mathcal{L}^1(\mu)$ for ν-a.e. $y \in Y$;

(ii) $y \mapsto u(x, y)$ is in $\mathcal{L}^1(\nu)$ for μ-a.e. $x \in X$;

(iii) $y \mapsto \int_X u(x, y)\mu(dx)$ is in $\mathcal{L}^1(\nu)$;

(iv) $x \mapsto \int_Y u(x, y)\nu(dx)$ is in $\mathcal{L}^1(\mu)$;

(v) $\int\limits_{X\times Y} ud(\mu \times \nu) = \int\limits_Y\int\limits_X u(x, y)\mu(dx)\nu(dy) = \int\limits_X\int\limits_Y u(x, y)\nu(dy)\mu(dx)$.

25 Fourier Transform (§13 (pp. 125-128), §15 (pp. 157-158), §19 (pp. 214-217), [Schilling(2017)])

We write $L^1(\mathbb{R}^n)$ for $L^1(\mathbb{R}^n, d\lambda_n)$.

The Fourier transform of a function $f \in L^1(\mathbb{R}^n)$ is the function $\hat{f}$ on $\mathbb{R}^n$ defined by

$$\hat{f}(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} f(y) e^{-i\langle x,y\rangle} dy,$$

w where $\langle x, y\rangle = x_1y_1 + \ldots + x_n y_n$. More generally, given a finite Borel measure μ, is *Fourier transform* is the function $\hat{\mu}$ on $\mathbb{R}^n$ defined by

$$\hat{\mu}(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{-i\langle x,y\rangle} d\mu(y).$$

We can also define μ for complex Borel measures.

Warning. There are many different conventions for the Fourier transform: instead of $1/(2\pi^n)$ one also uses $1, 1/(2\pi)^{u/2}$; instead of $e^{-i\langle x,y\rangle}$ one also uses $e^{i\langle x,y\rangle}, e^{\pm 2\pi i\langle x,y\rangle}$.

Note that if μ_f for $f \in L^1(\mathbb{R}^n)$ is defined by $d\mu_f = f d\lambda_n$, then

$$\hat{\mu}_f = \hat{f}.$$

Lemma 25.1. *If μ is a complex Borel measure on $\mathbb{R}^n$, then $\hat{\mu}$ is a Bounded continuous function on $\mathbb{R}^n$, and $|\hat{\mu}(x)| \le \frac{|\mu|(\mathbb{R}^n)}{(2\pi)^n}$.*

In particular, if $f \in L^1(\mathbb{R}^n)$, then $\hat{f}$ is Bounded and continuous,

$$|\hat{f}(x)| \le \frac{||f||_1}{(2\pi)^n} \;\forall x.$$

Some properties:

(i) If $f_t(x) = f(x-t)$, then

$$\hat{f}_t(y) = e^{-i\langle t,y\rangle} \hat{f}(y).$$

(ii) If $e_t(x) = e^{i\langle t,x\rangle}$, then

$$\hat{e_t f}(y) = \hat{f}(y-t).$$

(iii) If $T \in GL_n(\mathbb{R})$ (invertible $n \times n$ matrices), then

$$f \hat{\circ} T = |\det T|^{-1} \hat{f} \circ (T^t)^{-1}.$$

(iv) $\hat{\hat{f}}(x) = \bar{\hat{f}}(-x)$.

Important example.

If $f(x) = e^{-\frac{|x|^2}{2}}$ ($|x| = \langle x, x\rangle^{1/2}$), then $\hat{f}(x) = 1/(2\pi)^n e^{-\frac{|x|^2}{2}}$. More generally, if $f(x) = e^{-\frac{c|x|^2}{2}}$, then

$$\hat{f}(x) = \frac{1}{(2\pi c)^{\frac{u}{2}}} e^{-\frac{|x|^2}{2c}} \;\forall c > 0.$$

This follows from property (iii).

For functions f, g on $\mathbb{R}^n$, their **convolution** is the function $f \times g$ defined by

$$(f * g)(x) := \int_{\mathbb{R}^n} f(y)g(x-y)dy = \int_{\mathbb{R}^n} f(x-y)g(y)dy$$

when is this well-defined?

Lemma 25.2. *If $f, g \in L^1(\mathbb{R}^n)$, then the function $y \mapsto f(y)g(x-y)$ is integrable for (λ_n)-a.e. x, $f*g \in L^1(\mathbb{R}^n)$ and $||f*g||_1 \le ||f||_1||g||_1$.*

Next let us show that $f * g$ is well-defined for $f \in L^1(\mathbb{R}^n)$, $g \in L^p(\mathbb{R}^n)$, $1 \le p \le \infty$.

Lemma 25.3. *Assume $\phi : (a, b) \to \mathbb{R}$ is a convex function. Then ϕ is continuous and $\phi(x) = \sup\{l(x) : \phi \ge l, l(s) = \alpha s + \beta\}$.*

Theorem 25.4 (**Jensen's inequality**). *Assume $(X, \mathscr{B}, \mu)$ is a probability measure space (so $\mu(x) = 1$), $\phi : [0, \infty) \to [0, \infty)$ is a convex function. Then, for every integrable function $f : X \to [0, \infty)$ we have*

$$\phi\left(\int_X f d\mu\right) \le \int_X \phi \circ f d\mu.$$

The same inequality holds for any measurable $f : X \to [0, \infty]$ if $\lim_{x\to\infty} \phi(x) = +\infty$ and we put $\phi(+\infty) = +\infty$.

26 Regularization (15 & 19, [Schilling(2017)])

Lemma 26.1. *Assume $f \in L^1(\mathbb{R}^n)$, $g \in L^p(\mathbb{R}^n)$ $(1 \le p \le \infty)$. Then the function $g \mapsto f(g)g(x-y)$ is integrable for a.e. x, $f * g \in L^p(\mathbb{R}^n)$ and $||f * g||_p \le ||f||_1||g||_p$.*

Note that

$$\int_{\mathbb{R}^n} f(y)g(x-y)dy = \int_{\mathbb{R}^n} f(x-y)g(y)dy,$$

so $f * g = g * f$.

Remark. *More generally, for $\mu \in M(\mathbb{R}^n)$ and $g \in L^p(\mathbb{R}^n)$, we can define $\mu * g = g * \mu \in L^p(\mathbb{R}^n)$ by*

$$(\mu * g)(x) = \int_{\mathbb{R}^n} g(x-y)d\mu(y).$$

Then $||\mu * g||_p \le |\mu|(\mathbb{R}^n)||g||_p$.

Proposition 26.2. *If $f, g \in L^1(\mathbb{R}^n)$, then $f \hat{*} g = (2\pi)^n \hat{f}\hat{g}$.*

What are convolutions good for?

Example 26.3. Consider

$$f = \frac{1}{\lambda_n(B_r(0))} \mathbb{1}_{B_r(0)}.$$

Then

$$\begin{aligned}(f * g)(x) &= \frac{1}{\lambda_n(B_r(0))} \int_{B_r(0)} g(x-y)dy \\ &= \frac{1}{\lambda_n(B_r(x))} \int_{B_r(x)} g(y)dy.\end{aligned}$$

For a multi-index $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}_+^n$, write ∂^α for

$$\frac{\partial^{\alpha_1+\ldots+\alpha_n}}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}}.$$

Denote by $L^1_{\text{loc}}(\mathbb{R}^n)$ the space of Lebesgue measurable functions that are independent on every ball. We identify functions that coincide a.e. (so, $L^1_{\text{loc}}(\mathbb{R}^n)$ is a space of equivalence classes of functions). We have $L^p(\mathbb{R}^n) \subset L^1_{\text{loc}}(\mathbb{R}^n)$ for all $1 \le p \le \infty$.

Lemma 26.4. *If $\phi \in C_c^\infty(\mathbb{R}^n)$ and $f \in L^1_{loc}(\mathbb{R}^n)$, then $\phi * f \in C^\infty(\mathbb{R}^n)$ and $\partial^\alpha(\phi * f) = (\partial^\alpha \phi) * f$.*

By choosing suitable ϕ we can make $\phi * f$ close to f, as we will see shortly.

Definition 26.5. A positive 'modifier' (?) is a function $\phi \in C_c(\mathbb{R}^n)$ s.t. $\phi \geq 0$ and $\int_{\mathbb{R}^n} \phi(x)dx = 1$.

For a function ϕ on $\mathbb{R}^n$ and $\epsilon > 0$, define $\phi^\epsilon(x) := \epsilon^{-n}\phi(x/\epsilon)$. Note that if $\phi \in L^1(\mathbb{R}^n)$, $\int_{\mathbb{R}^n} \phi dx = 1$, then $\int_{\mathbb{R}^n} \phi^\epsilon = 1$.

Example 26.6 (positive modifier)**.** Consider the function h in $\mathbb{R}$ defined by,

$$h(t) = \begin{cases} e^{-\frac{1}{1-t^2}}, & |t| < 1, \\ 0, & |t| \geq 1 \end{cases}.$$

Then $g \in C_c^\infty(\mathbb{R}^n)$. Hence, $\phi(x) = c_n h(|x|)$ is a modifier, where $c_n = \left(\int_{\mathbb{R}^n} h(|x|)dx\right)^{-1}$.

Proposition 26.7. *Let $\phi \in L^1(\mathbb{R}^n)$ be s.t. $\phi \geq 0$ and $\int_{\mathbb{R}^n} \phi dx = 1$. Then we have*

(i) If $f \in C_0(\mathbb{R}^n)$ (continuous functions vanishing at infinity (?)), then $\phi^\epsilon * f \in C_0(\mathbb{R}^n)$ and $||\phi^\epsilon * f|| \xrightarrow[\epsilon \downarrow 0]{} 0$ (uniform norm);

(ii) If $f \in L^p(\mathbb{R}^n)$, $1 \leq p \leq \infty$, then $||\phi^\epsilon * f - f||_p \xrightarrow[\epsilon \downarrow 0]{}$.

Corollary 26.8. *For any Radon measure μ on $\mathbb{R}^n$, $C_c^\infty(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n, d\mu)$ for $1 \leq p < \infty$.*

27 Fourier Inverse (19, [Schilling(2017)])

Corollary 27.1. *If $f \in L^1(\mathbb{R}^n)$, then $\hat{f} \in C_0(\mathbb{R}^n)$.*

Remark. *Another possibility is to approximate f by its linear combinations of $\mathbb{1}_{[a_1,b_1]\times\ldots\times[a_n,b_n]}$. Note that for $\mathbb{1}_{[a,b]} \in L^1(\mathbb{R})$ we have $\hat{\mathbb{1}}_{a,b}(x) = 1/(2\pi)\int_a^b e^{-ixy}dy \underset{x\neq 0}{=} \frac{e^{-iax}-e^{-ixb}}{2\pi i x} \xrightarrow[x\to\infty]{} 0$.*

Theorem 27.2 (Fourier Inversion Theorem)**.** *Assume $f \in L^1\mathbb{R}^n$ is s.t. $\hat{f} \in L^1(\mathbb{R}^n)$. Then, for a.e. x, we have*

$$f(x) = \int_{\mathbb{R}^n} \hat{f}(y)e^{i\langle x,y\rangle}dy.$$

Equivalently,

$$\hat{\hat{f}}(x) = \frac{1}{(2\pi)^n} f(-x).$$

Theorem 27.3 (convolution theorem)**.** *If μ, ν are finite measures on $(\mathbb{R}^n, \mathscr{B}(\mathbb{R}^n))$, then*

$$\widehat{\mu * \nu}(\xi) = (2\pi)^n\hat{\mu}(\xi)\hat{\nu}(\xi) \text{ and } \mathcal{F}^{-1}\mu * \nu(\xi) = \mathcal{F}^{-1}\mu(\xi)\mathcal{F}^{-1}\nu(\xi).$$

Lemma 27.4. *For any $f, g \in L^1(\mathbb{R}^n)$, we have*

$$\int_{\mathbb{R}^n} \hat{f} g dx = \int_{\mathbb{R}^n} f\hat{g}dx.$$

(Note that $\hat{f}g \in L^1(\mathbb{R}^n)$, as $\hat{f}$ is bounded.)

Corollary 27.5. *If $f \in L^1(\mathbb{R}^n)$ and $\hat{f} = 0$, then $f = 0$ (a.e.)*

Recall that a linear operator $U : H \to K$ between Hilbert spaces is called an **isometry** if

$$||Ux|| = ||x|| \ \forall x \in H.$$

Equivalently, by the polarization identity,

$$(Ux, Uy) = (x, y) \ \forall x, y \in H.$$

If U is in addition surjective, then it is called a **unitary**.

Theorem 27.6 (Plancherel)**.** *There is a unique unitary U : $L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ s.t. $Uf = (2\pi)^{n/2}\hat{f}$ for all $f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$.*
In the book: If $u \in L^2_{\mathbb{C}}(\lambda^n) \cap L^1_{\mathbb{C}}(\lambda^n)$, then

$$||\hat{u}||_2 = (2\pi)^{-n/2}||u||_2.$$

28 Schwartz Space 19, [Schilling(2017)]

Proposition 28.1. *Assume $f \in L^1(\mathbb{R}^n)$ and $x_j f \in L^1(\mathbb{R}^n)$ ($x_j f$ means $x \mapsto x_j f(x)$) for some $1 \leq j \leq n$. Then*

$$\partial_j \hat{f} = -i\widehat{x_j F}$$

Proposition 28.2. *Assume $f \in C(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$ is such that $\partial_j f \in L^1(\mathbb{R}^n)$. Then*

$$\widehat{\partial_j f} = ix_j f$$

Corollary 28.3. *If $f, \partial_j f \in L^1(\mathbb{R}^n)$, then*

$$x_j\hat{f}(x) \xrightarrow[x\to\infty]{} 0.$$

Corollary 28.4. *1. If $x^\alpha f \in L^1(\mathbb{R}^n)$ for all $|\alpha| \leq N$, then $\hat{f} \in C^N(\mathbb{R}^n)$ and $\partial^\alpha \hat{f} = (-i)^{|\alpha|}\widehat{x^\alpha f}$.*

2. If $f \in C^N(\mathbb{R}^n)$ and $\partial^\alpha f \in L^1(\mathbb{R}^n)$ for all $|\alpha| \leq N$, then $\widehat{\partial^\alpha f} = i^{|\alpha|}x^\alpha \hat{f}$ and hence $(1 + |x|)^N\hat{f}(x) \xrightarrow[x\to\infty]{} 0$.

(here $\alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{N}_0^n$: n-dim positive integers, and $|\alpha| = \alpha_1 + ... + \alpha_n$, $x^\alpha = x_1^{\alpha_1} \cdot ... \cdot x_n^{\alpha_n}$.)

Definition 28.5 (Schwartz function/space)**.** A function f on $\mathbb{R}^n$ is called a **Schwartz function** if $f \in C^\infty(\mathbb{R}^n)$ and $x^\alpha\partial^\beta f$ is bounded for all multi-indices α, β. The space $\mathcal{S}(\mathbb{R}^n)$ of Schwartz functions is called **Schwartz space**.

Note that for every $f \in C^\infty(\mathbb{R}^n)$ the following conditions are equivalent:

1. $x^\alpha\partial^\beta f$ is bounded for all α, β;
2. $x^\alpha(\partial^\beta f)(x) \xrightarrow[x\to\infty]{} 0$ for all α, β;
3. $(1 + |x|)^N\partial^\beta f$ is bounded for all $N \geq 1$ and all β.

Example 28.6. $C_c^\infty(\mathbb{R}^n) \subset \mathcal{S}(\mathbb{R}^n)$, $e^{-a|x|^2} \in \mathcal{S}(\mathbb{R}^n)$ for $a > 0$. If $f \in \mathcal{S}(\mathbb{R}^n)$, then $x^\alpha\partial^\beta f \in \mathcal{S}(\mathbb{R}^n)$. The product of two Schwartz functions is a Schwartz function.

Clearly, $\mathcal{S}(\mathbb{R}^n) \subset L^1(\mathbb{R}^n)$ for all $1 \leq p \leq \infty$. From the Corollary above we conclude that if $f \in \mathcal{S}(\mathbb{R}^n)$, then $\hat{f} \in \mathcal{S}(\mathbb{R}^n)$. By the Fourier inversion theorem we then get:

Theorem 28.7 (Fourier map in Schwartz space). *The Fourier transform maps $\mathcal{S}(\mathbb{R}^n)$ onto $\mathcal{S}(\mathbb{R}^n)$.*

Remark. *This gives another proof of the fact that the image of the Fourier transform $L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is dense, which we needed to prove Plancherel's theorem.*

Remark. *If $f \in C_c^\infty(\mathbb{R}^n)$, $f \neq 0$, then $\hat{f} \in \mathcal{S}(\mathbb{R}^n)$, but $\hat{f}$ is never compactly supported, since it extends to an analytic function on $\mathbb{C}^n : \hat{f}(z) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} f(y) e^{-i\langle z,y\rangle} dy$.*

29 Kolmogorov extension theorem

Assume X is a set and $(\mathcal{B}_n)_{n\in\mathbb{N}}$ is an increasing sequence of σ-algebras of subsets of X. Assume μ_n is a measure on $(X, \mathcal{B}_n)$ and

$$\mu_{n+1}|_{\mathcal{B}_n} = \mu_n \ \forall n.$$

Can we define a measure μ on $(X, \mathcal{B})$, where $\mathcal{B} = \sigma\left(\cup_{n\in\mathbb{N}} \mathcal{B}_n\right)$ s.t. $\mu|_{\mathcal{B}_n} = \mu_n \ \forall n$? - In general, no. But we have the following:

Theorem 29.1 (Kolmogorov extension theorem). *In the above settings, assume in addition that $\mu_n(X) = 1 \ \forall n$ and there is a collection of subesets $\mathcal{C} \subset \mathcal{B}$ s.t.:*

(i) $\mu_n(A) = \sup\{\mu_n(C) : C \subset A, C \in \mathcal{C} \cap \mathcal{B}_n\} \ \forall A \in \mathcal{B}_n$;

(ii) If $(C_n)_{n\in\mathbb{N}}$ is a sequence in $\mathcal{C}$ and $\cap_{n\in\mathbb{N}} C_n = \emptyset$, then $\cap_{n=1}^N C_n = \emptyset$ for some N.

Then there is a unique measure μ on $(X, \mathcal{B})$ s.t. $\mu|_{\mathcal{B}_n} = \mu$.

Assume now we have a collection $((X_i, \mathscr{B}_i))_{i\in I}$ of measurable spaces (I can be infinite and uncountable). Consider $X = \prod_{i\in I} X_i$. Denote by $\prod_{i\in I} \mathscr{B}_i$ the σ-algebra generated by all sets of the form

$$\prod_{i\in F} A_i \times \prod_{i\in F^c} X_j,$$

where $F \subset I$ is finite, $A_i \in \mathscr{B}_i$ $(i \in F)$.

Example 29.2. Consider a sequence $((X_n, d_n))_{n=1}^\infty$ of seperable metric spaces. Assume $d_n(x, y) \leq 1 \ \forall x, y$. (Any metric can be defined by such by defining $\tilde{d}(x, y) = \frac{d(x,y)}{1+d(x,y)}$.) Then $\prod_{n=1}^\infty X_n$ is a metric space with metric

$$d(\underline{x}, \underline{y}) = \sum_{n=1}^\infty \frac{1}{2^n} d_n(x_n, y_n),$$

where $\underline{x} = (x_n)_{n=1}^\infty \in X$. Given a sequence $(\underline{x}(n))_{n=1}^\infty$ in X, we have $\underline{x}(n) \xrightarrow[n\to\infty]{} \underline{x}$ iff

$$x(n)_k \xrightarrow[k\to\infty]{} x_k \ \forall k.$$

Consider the Borel σ-algebra $\mathscr{B}(X_n)$. Then $\prod_{n=1}^\infty \mathscr{B}(X_n) = \mathscr{B}(X)$. To see this, for every n, choose open sets $U_{n,k} \subset X_n$ $(k = 1, 2, \ldots)$ s.t. every open set in X_n is the union of some of $U'_{n,k}$s. This is possible by seperability: take a dense countable subset of X_n and then all balls of rational radii with centers at points of this subset. Then every open subset of X is the union of sets of the form

$$U_{1,k_1} \times U_{2,k_2} \times \ldots \times U_{n,k_n} \times \prod_{m=n+1}^\infty X_m.$$

Therefore such sets generate the σ-algebra $\mathscr{B}(X)$, and as $U_{n,k}$ $(k \in \mathbb{N})$ generate $\mathscr{B}(X_n)$, we conclude that $\prod_{n=1}^\infty \mathscr{B}(X_n) = \mathscr{B}(X)$.

In relation to this example, we will need the following:

Theorem 29.3 (Tikkonov, aso transcribed as Tychonoff). *Assume $((X_n, d_n))_{n=1}^\infty$ is a sequence of compact metric spaces. Then $\prod_{n=1}^\infty X_n$ (with metric as in the example) is compact.*

Return to a general collection $((X_i, \mathscr{B}_i))_{i\in I}$ of measurable spaces. Let us introduce the following notation: For $F \subset G \subset I$, write

$$X_F = \prod_{i\in F} X_i, \ X_G = \prod_{i\in G} X_i,$$
$\pi_{G,F} : X_G \to X_F$ for the projection map:
$$\pi_{G,F}((x_i)_{i\in G}) = (x_i)_{i\in F},$$

and recall the pushforward measure: given a measurable mapping $f : X_1 \to X_2$ and a measure $\mu : \mathscr{B} \to [0, +\infty]$, the pushforward of μ is the measure $f_*(\mu) : \mathscr{B}_2 \to [0, +\infty]$ given by

$$f_*(\mu)(B_2) = \mu\left(f^{-1}(B_2)\right) \text{ for } B_2 \in \mathscr{B}_2.$$

Theorem 29.4 (Kolmogorov extension theorem). *Assume $(X_i)_{i\in I}$ is a collection of metric spaces. Consider $X = \prod_{i\in I} X_i$, $\mathscr{B} = \prod_{i\in I} \mathscr{B}(X_i)$. Assume for every finite $F \subset I$ we are given a regular Borel probability measure μ_F on X_F s.t.*

$$(\pi_{G,F})_* \mu_G = \mu_F,$$

for all finite $F \subset G \subset I$. Then there is a unique probability measure μ on $(X, \mathscr{B})$ s.t.

$$(\pi_{I,F})_* \mu = \mu_F$$

for all finite $F \subset I$.

Remark. *If in addition the spaces X_i are seperable, then we can also conclude that for every $A \in \mathscr{B} = \prod_{i\in I} \mathscr{B}(X_i)$ we have*

$$\mu(A) = \sup \mu(C),$$

where the supremum is taken over all sets $C \subset A$ of the form

$$C = K \times \prod_{i\in I\setminus J} X_i,$$

where $J \subset I$ is countable and $K \subset X_J$ is compact.

30 Random variables and stochastic processes

Assume $(X, \mathscr{B}, \mathbb{P})$ is a proability measure space. If $Y, \mathscr{C}$ is a measurable space, a measurable map $f : X \to Y$ is called a random variable. For $A \in \mathscr{C}$, define

$$\mathbb{P}(f \in A) \stackrel{\text{def}}{=} \mathbb{P}(f^{-1}(A)) = (f_*\mathbb{P})(A),$$

the probability that f takes a value in A. The measure $f_*\mathbb{P}$ on $(Y, \mathscr{C})$ is called the probability distributionof f.

Definition 30.1. A stochastic processis a collection $(f_t : X \to Y)_{t\in T}$ of random variables.

T stands for "time" and is typically $\mathbb{Z}, \mathbb{Z}_+, \mathbb{R}$ or $\mathbb{R}_+$.

Given a different $t_1, ..., t_n \in T$, we can consider the **joint distribution** of $f_{t_1}, ..., f_{t_n}$, the measure

$$\mu_{t_1,...,t_n} = (f_{t_1} \times ... \times f_{t_n})_* \mathbb{P} \text{ on } (Y^n, \mathscr{C}^n).$$

When is a collection of measures defined by a sochastic process?

Theorem 30.2. *Assume T is a set and for all different elements $t_1, ..., t_n \in T$ we are given a Borel probability measure $\mu_{t_1}, ..., \mu_{t_n}$ on $\mathbb{R}^n$ s.t.*

(i) If $\sigma \in \delta_n$ and $A_1, ..., A_n \in \mathscr{B}(\mathbb{R}^n)$, then

$$\mu_{t_1,...,t_n}(A_1 \times ... \times A_n) = \mu_{t_{\sigma(a)},...,t_{\sigma(n)}}(A_{\sigma(1)} \times ... \times A_{\sigma(n)});$$

(ii) $\mu_{t_1}, ..., t_n, s_1, ..., s_m(A_1 \times ... \times A_n \times \underbrace{\mathbb{R} \times ... \times \mathbb{R}}_{m \text{ times}}) = \mu_{t_1,...,t_n}(A_1 \times ... \times A_n)$.

Then there is a probability measure space $(X, \mathscr{B}, \mathbb{P})$ and a stochastic process $(f_t : X \to \mathbb{R})_{t \in T}$ s.t. $\mu_{t_1,...,t_n}$ is the joint distribution of $f_{t_1}, ..., f_{t_n}$.

Remark. *Instead of $\mathbb{R}$ we could have taken any complete sperable metric space, as then the measure $\mu_{t_1,...,t_n}$ are regular. Or we could just require the measures $\mu_{t_1,...,t_n}$ to be regular.*

Random varaibles $f_1, ..., f_n : X \to \mathbb{R}$ are called **independent**if

$$\mathbb{P}(f_1 \in A_1, ..., f_n \in A_n) = \mathbb{P}(f_1 \in A_i)...\mathbb{P}(f_n \in A_n).$$

For all $A_1, ..., A_n \in \mathscr{B}(\mathbb{R})$. In other words, if μ_i is the probability distribution of f_i, then the joint distribution of $f_1, ..., f_n$ is $\mu_1 \times ... \times \mu_n$.

For such measures the above theorem gives the following result: if we are given a Borel probability measure μ_t on $\mathbb{R}$ for every $t \in T$, then we get a unique measure $\mu = \prod_{t \in T} \mu_t$ on $\mathbb{R}^T$ s.t. $(\overline{\mu}_{T,F})_* \mu = \prod_{t \in F} \mu_t$ $\forall$ finite $F \subset T$.

Example 30.3. Consider the process of tossing a coin. Write 0 for tail and 1 for head. We can model the process as follows

$$X = \prod_{n=0}^{\infty} \{0, 1\}, \ \mathbb{P} = \prod_{n=0}^{\infty} \nu,$$

where $\nu = \frac{1}{2} \int_0 + \frac{1}{2} \int_1$,

$$f_n : X \to \{0, 1\}, \ f_n(\underline{x}) = x_n,$$

f_n is the result of n-tosses.

While the Kolmogorov extension theorem requires some regularity, it turns out that infinite products of probability measures always exist:

Theorem 30.4. *Assume $((X_i, \mathscr{B}_i, \mu_i))_{i \in I}$ is a collection of probability measure spaces. Consider $X = \prod_{i \in I} X_i$, $\mathscr{B} = \prod_{i \in I} \mathscr{B}_i$. Then there exists a unique measaure μ on $(X, \mathscr{B})$ s.t.*

$$(\overline{\mu}_{I,F})_* \mu = \prod_{i \in F} \mu_i \ \forall \text{ finite } F \subset I.$$

Tips'n Tricks

- Assume we can write X as a finite union: $X = \cup_{n \in I} A_n$, $i = 1, ..., N$. Then

$$\int f d\mu = \int_X f d\mu = \int_{A_1} f d\mu + \int_{A_2} f d\mu + ... + \int_{A_N} f d\mu.$$

Questions

- In problem 26.18 we are supposed to show that $Y_n \perp Y_m = 0$, i.e. that $\langle y_n, y_m \rangle = 0$, $n \neq m$. I get ...

$$\langle y_n, y \rangle \subset \int_{A_m^c} |y_n|^2 |y_m|^2 d\mu,$$

and I want to argue that this is zero since $\int_{A_m^c} |y_m|^2 d\mu = 0$, but I don't see how. The solutions are not clear, and I think perhaps my setup is wrong. I am assuming $\langle f, g \rangle = \int_X f \bar{g} d\mu$, i.e. from L^2, but perhaps it is rather $\langle f, g \rangle = \int_{A_m^c \cup A_n^c} f \bar{g} d\mu$ or something?

References

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