

Regularity of measures

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We let (X, d) be a metric space and denote by \mathcal{O} the open, by \mathcal{C} the closed and $\mathcal{B}(X) = \sigma(\mathcal{O})$ the Borel set of X .

Definition 0.1. A measure μ on $(X, d, \mathcal{B}(X))$ is called outer regular, if

$$\mu(B) = \inf \{ \mu(U) \mid B \subset U, U \text{ open} \} \quad (1)$$

and inner regular, if $\mu(K) < \infty$ for all compact sets $K \subset X$ and

$$\mu(U) = \sup \{ \mu(K) \mid K \subset U, K \text{ compact} \}. \quad (2)$$

A measure which is both inner and outer regular is called **regular**. We write $\mathbf{m}_r^+(X)$ for the family of regular measures on $(X, \mathcal{B}(X))$.

Remark. The space X is called σ -compact if there is a sequence of compact sets $K_n \uparrow X$. A typical example of such a space is a locally compact, separable metric space.

Theorem 0.2. Let (X, d) be a metric space. Every finite measure μ on $(X, \mathcal{B}(X))$ is outer regular. If X is σ -compact, then μ is also inner regular, hence regular.

Theorem 0.3. Let (X, d) be a metric space and μ be a measure on $(X, \mathcal{B}(X))$ such that $\mu(K) < \infty$ for all compact sets $K \subset X$.

- 1 If X is σ -compact, then μ is inner regular.
- 2 If there exists a sequence $G_n \in \mathcal{O}$, $G_n \uparrow X$ such that $\mu(G_n) < \infty$, then μ is outer regular.