

Measurable Mappings

Morten Tryti Berg and Isak Cecil Onsager Rukan.

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We consider maps $T : X \rightarrow X'$ between two measurable spaces (X, \mathcal{A}) and (X', \mathcal{A}') which respects the measurable structures, the σ -algebras on X and X' . These maps are useful as we can transport a measure μ , defined on (X, \mathcal{A}) , to (X', \mathcal{A}') .

Definition 7.8. Let (X, \mathcal{A}) , (X', \mathcal{A}') be measurable spaces. A map $T : X \rightarrow X'$ is called \mathcal{A}/\mathcal{A}' -measurable if the pre-image of every measurable set is a measurable set:

$$T^{-1}(A') \in \mathcal{A}, \quad \forall A' \in \mathcal{A}'. \quad (1)$$

- A $\mathcal{B}(\mathbb{R}^n)/\mathcal{B}(\mathbb{R}^m)$ measurable map is often called a Borel map.
- The notation $T : (X, \mathcal{A}) \rightarrow (X', \mathcal{A}')$ is often used to indicate measurability of the map T .

Lemma 7.9. Let (X, \mathcal{A}) , (X', \mathcal{A}') be measurable spaces and let $\mathcal{A}' = \sigma(\mathcal{G}')$. Then $T : X \rightarrow X'$ is \mathcal{A}/\mathcal{A}' -measurable iff $T^{-1}(G') \in \mathcal{A}$, i.e. if

$$T^{-1}(G') \in \mathcal{A}, \quad \forall G' \in \mathcal{G}'. \quad (2)$$

Theorem 7.10. Let (X_i, \mathcal{A}_i) , $i = 1, 2, 3$, be measurable spaces and $T : X_1 \rightarrow X_2$, $S : X_2 \rightarrow X_3$ be $\mathcal{A}_1/\mathcal{A}_2$ and $\mathcal{A}_2/\mathcal{A}_3$ -measurable maps respectively. Then $S \circ T : X_1 \rightarrow X_3$ is $\mathcal{A}_1/\mathcal{A}_3$ -measurable.

Corollary 7.11. Every continuous map between metric spaces is a Borel map.

Definition 7.12. (and lemma) Let $(T_i)_{i \in I}$, $T_i : X \rightarrow X_i$, be arbitrarily many mappings from the same space X into measurable spaces (X_i, \mathcal{A}_i) . The smallest σ -algebra on X that makes all T_i simultaneously measurable is

$$\sigma(T_i : i \in I) := \sigma \left(\bigcup_{i \in I} T_i^{-1}(\mathcal{A}_i) \right) \quad (3)$$

Corollary 7.13. A function $f : (X, \mathcal{B}) \rightarrow \mathbb{R}$ is measurable if $f((a, +\infty)) \in \mathcal{B}$, $\forall a \in \mathbb{R}$.

Corollary 7.14. Assume (X, \mathcal{B}) is a measurable space, (Y, d) is a metric space, $(f_n : (X, \mathcal{B}) \rightarrow Y)_{n=1}^{\infty}$ is a sequence of measurable maps. Assume this sequence of images $(f_n(x))_{n=1}^{\infty}$ is convergent in $Y \ \forall x \in X$. Define

$$f : X \rightarrow Y, \text{ by } f(x) = \lim_{n \rightarrow \infty} f_n(x). \quad (4)$$

Then f is measurable.

Theorem 7.15. Let (X, \mathcal{A}) , (X', \mathcal{A}') be measurable spaces and $T : X \rightarrow X'$ be an \mathcal{A}/\mathcal{A}' -measurable map. For every measurable μ on (X, \mathcal{A}) ,

$$\mu'(A') := \mu(T^{-1}(A')), \quad A' \in \mathcal{A}', \quad (5)$$

defines a measure on (X', \mathcal{A}') .

Definition 7.16. The measure $\mu'(\cdot)$ in the above theorem is called the push forward or image measure of μ under T and it is denoted as $T(\mu)(\cdot)$, $T_{*\mu}(\cdot)$ or $\mu \circ T^{-1}(\cdot)$.

Theorem 7.17. If $T \in \mathbb{R}^{n \times n}$ is an orthogonal matrix, then $\lambda^n = T(\lambda^n)$.

Theorem 7.18. Let $S \in \mathbb{R}^{n \times n}$ be an invertible matrix. Then

$$S(\lambda^n) = |\det S|^{-1} \lambda^n = |\det S|^{-1} \lambda^n. \quad (6)$$

Corollary 7.19. Lebesgue measure is invariant under motions: $\lambda^n = M(\lambda^n)$ for all motions M in \mathbb{R}^n . In particular, congruent sets have the same measure. Two sets of points are called congruent if, and only if, one can be transformed into the other by an isometry