MAT3400/4400 Spring 2020: Solutions.

Problem 1 a) We have

$$\int \mathbf{1}_A \mathbf{1}_B \, d\mu = \int \mathbf{1}_{A\cap B} \, d\mu = \mu(A \cap B) = \mu(A)\mu(B) = \left(\int \mathbf{1}_A \, d\mu\right)\left(\int \mathbf{1}_B \, d\mu\right).$$

b) Using a), we get

$$\int fg \, d\mu = \int \left(\sum_{i=1}^n a_i \mathbf{1}_{A_i}\right)\left(\sum_{j=1}^m b_j \mathbf{1}_{B_j}\right) d\mu = \sum_{i=1}^n \sum_{j=1}^m a_i b_j \int \mathbf{1}_{A_i} \mathbf{1}_{B_j} \, d\mu$$

$$= \sum_{i=1}^n \sum_{j=1}^m a_i b_j \left(\int \mathbf{1}_{A_i} \, d\mu\right)\left(\int \mathbf{1}_{B_j} \, d\mu\right)$$

$$= \left(\sum_{i=1}^n \int a_i \mathbf{1}_{A_i} \, d\mu\right)\left(\sum_{j=1}^m \int b_j \mathbf{1}_{B_j} \, d\mu\right) = \left(\int f \, d\mu\right)\left(\int g \, d\mu\right).$$

c) Let $\{f_n\}$ be an increasing sequence of nonnegative, $\mathcal{A}$-measurable simple functions converging pointwise to f, and let $\{g_n\}$ be an increasing sequence of nonnegative, $\mathcal{B}$-measurable simple functions converging pointwise to g (such sequences exist by Proposition 7.5.3 in *Spaces*). Then $\{f_n g_n\}$ is an increasing sequence of simple functions converging pointwise to fg, and by the Monotone Convergence Theorem and part b)

$$\int fg \, d\mu = \lim_{n\to\infty} \int f_n g_n \, d\mu = \lim_{n\to\infty} \left(\int f_n \, d\mu\right)\left(\int g_n \, d\mu\right)$$

$$= \left(\lim_{n\to\infty} \int f_n \, d\mu\right)\left(\lim_{n\to\infty} \int g_n \, d\mu\right) = \left(\int f \, d\mu\right)\left(\int g \, d\mu\right).$$

d) We split f and g in positive and negative parts, $f = f_+ - f_-$, $g = g_+ - g_-$, and observe that $fg = f_+g_+ - f_+g_- - f_-g_+ + f_-g_-$. Note that f_+ and f_- are $\mathcal{A}$-measurable, while g_+ and g_- are $\mathcal{B}$-measurable. Note also that by c), all the products on the right hand side are integrable. Hence fg is integrable and

$$\int fg \, d\mu = \int f_+g_+ \, d\mu - \int f_+g_- \, d\mu - \int f_-g_+ \, d\mu + \int f_-g_- \, d\mu$$

$$= \left(\int f_+ \, d\mu\right)\left(\int g_+ \, d\mu\right) - \left(\int f_+ \, d\mu\right)\left(\int g_- \, d\mu\right)$$

$$- \left(\int f_- \, d\mu\right)\left(\int g_+ \, d\mu\right) + \left(\int f_- \, d\mu\right)\left(\int g_- \, d\mu\right)$$

$$= \left(\int f_+ \, d\mu - \int f_- \, d\mu\right)\left(\int g_+ \, d\mu - \int g_- \, d\mu\right) = \left(\int f \, d\mu\right)\left(\int g \, d\mu\right).$$

Problem 2 a) To see that T is linear, note that

$$\begin{aligned}&T(\alpha f+\beta g)\\&=2(\alpha f(n)+\beta g(n))-(\alpha f(n+1)+\beta g(n+1))-(\alpha f(n-1)+\beta g(n-1))\\&=\alpha\left(2f(n)-f(n+1)-f(n-1)\right)+\beta\left(2g(n)-g(n+1)-g(n-1)\right)\\&=\alpha Tf(n)+\beta Tg(n).\end{aligned}$$

To estimate the norm, define f^+ and f^- by

$$f^+(n)=f(n+1)\qquad\text{and}\qquad f^-(n)=f(n-1).$$

Clearly, $\|f^+\|=\|f\|$ and $\|f^-\|=\|f\|$, and hence by the triangle inequality

$$\|Tf\|=\|2f-f^+-f^-\|\le 2\|f\|+\|f^+\|+\|f^-\|=4\|f\|,$$

which shows that $\|T\|\le 4$.

b) We shall use the functions from the hint:

$$f_N(n)=\begin{cases}1 & \text{if } |n|\le N \text{ and } n \text{ is even}\\ -1 & \text{if } |n|\le N \text{ and } n \text{ is odd}\\ 0 & \text{if } |n|>N\end{cases}$$

Note that $\|f_N\|=\sqrt{2N+1}$ and observe also that

$$Tf_N(n)=\begin{cases}4 & \text{if } |n|<N \text{ and } n \text{ is even}\\ -4 & \text{if } |n|<N \text{ and } n \text{ is odd}\end{cases}$$

Hence $\|Tf_N\|\ge 4\sqrt{2N-1}$, and we get

$$\frac{\|Tf_N\|}{\|f_N\|}\ge 4\sqrt{\frac{2N-1}{2N+1}}=4\sqrt{\frac{1-\frac{1}{2N}}{1+\frac{1}{2N}}}\ \to\ 4.$$

This means that we can get $\frac{\|Tf_N\|}{\|f_N\|}$ as close to 4 as we want, and hence $\|T\|=4$.

c) It is easier to work backwards:

$$\sum_{n=-\infty}^{\infty}(f(n+1)-f(n))\overline{(g(n+1)-g(n))}$$

$$=\sum_{n=-\infty}^{\infty}\left(f(n+1)\overline{g(n+1)}-f(n+1)\overline{g(n)}-f(n)\overline{g(n+1)}+f(n)\overline{g(n)}\right)$$

$$\begin{aligned}&=\sum_{n=-\infty}^{\infty}f(n+1)\overline{g(n+1)}-\sum_{n=-\infty}^{\infty}f(n+1)\overline{g(n)}\\&\quad-\sum_{n=-\infty}^{\infty}f(n)\overline{g(n+1)}+\sum_{n=-\infty}^{\infty}f(n)\overline{g(n)}.\end{aligned}$$

If we change summation variable in the first and third term, we see that

$$\sum_{n=-\infty}^{\infty}f(n+1)\overline{g(n+1)}=\sum_{n=-\infty}^{\infty}f(n)\overline{g(n)}$$

and
$$\sum_{n=-\infty}^{\infty} f(n)\overline{g(n+1)} = \sum_{n=-\infty}^{\infty} f(n-1)\overline{g(n)}.$$
Substituting this into the expression above, we get
$$\sum_{n=-\infty}^{\infty} (f(n+1)-f(n))\overline{(g(n+1)-g(n))}$$
$$= 2\sum_{n=-\infty}^{\infty} f(n)\overline{g(n)} - \sum_{n=-\infty}^{\infty} f(n+1)\overline{g(n)} - \sum_{n=-\infty}^{\infty} f(n-1)\overline{g(n)} = \langle Tf, g\rangle.$$
To show that T is self-adjoint, note that
$$\langle f, Tg\rangle = \overline{\langle Tg, f\rangle} = \overline{\sum_{n=-\infty}^{\infty} (g(n+1)-g(n))\overline{(f(n+1)-f(n))}}$$
$$= \sum_{n=-\infty}^{\infty} \overline{(g(n+1)-g(n))}(f(n+1)-f(n)) = \langle Tf, g\rangle.$$
d) From b) we have
$$\langle Tf, f\rangle = \sum_{n=-\infty}^{\infty} (f(n+1)-f(n)\overline{(f(n+1)-f(n))}$$
$$= \sum_{n=-\infty}^{\infty} |f(n+1)-f(n)|^2 \geq 0.$$
Observe next that in order to have $\langle Tf, f\rangle = 0$, we must have $f(n+1) = f(n)$ for all $n \in \mathbb{N}$, i.e., f has to be constant. As the only constant function in $l^2(\mathbb{Z})$ is the one that is constant zero, we see that T is positive definite.

Since T is self-adjoint and $\|T\| = 4$, we know from the *Notes* that all eigenvalues λ must be real and satisfy $|\lambda| \leq 4$. To show that $\lambda > 0$, note that if f is a corresponding eigenvector, we have
$$\langle Tf, f\rangle = \langle \lambda f, f\rangle = \lambda\|f\|^2,$$
and hence
$$\lambda = \frac{\langle Tf, f\rangle}{\|f\|^2} > 0.$$
e) Assume for contradiction that f is an eigenvector with eigenvalue 4. Then $Tf(n) = 4f(n)$ for all $n \in \mathbb{Z}$, i.e.
$$2f(n) - f(n+1) - f(n-1) = 4f(n).$$
If we rearrange the terms, we get
$$f(n+1) + 2f(n) + f(n-1) = 0.$$
This means that the sequence $\{f(n)\}$ is a solution of the difference equation
$$x_{n+1} + 2x_n + x_{n-1} = 0,$$

and must be of the form $f(n) = (Cn + D)(-1)^n$. However, for f to be in $H = l^2(\mathbb{Z})$, we need to have $\sum_{n=-\infty}^{\infty} |f(n)|^2 < \infty$, and hence $\lim_{n\to\infty} f(n) = 0$. This is only the case if $C = D = 0$, i.e., if $f = 0$. As $f = 0$ can't be an eigenvector, we have our contradiction.

Problem 3: a) Since $y_n \in (T - \lambda I)H$, there are elements $z_n \in H$ such that $y_n = (T - \lambda I)z_n$. Using the orthogonal decomposition

$$H = \ker(T - \lambda I) \oplus \ker(T - \lambda I)^\perp,$$

we see that

$$z_n = u_n + x_n$$

where $u_n \in \ker(T - \lambda I)$ and $x_n \in \ker(T - \lambda I)^\perp$. Hence

$$y_n = (T - \lambda I)z_n = (T - \lambda I)u_n + (T - \lambda I)x_n = 0 + (T - \lambda I)x_n = (T - \lambda I)x_n.$$

b) If $\{x_n\}$ is unbounded, we can choose the subsequence $\{x_{n_k}\}$ in this way: Let $n_1 = 1$, and if n_k is chosen, let n_{k+1} be the first number larger than n_k such that $\|x_{n_{k+1}}\| \geq k + 1$.

If $v_k = \frac{x_{n_k}}{\|x_{n_k}\|}$, we now have

$$\lim_{k\to\infty} (T - \lambda I)v_k = \lim_{k\to\infty} \frac{(T - \lambda I)x_{n_k}}{\|x_{n_k}\|} = 0$$

since $\lim_{k\to\infty}(T - \lambda I)x_{n_k} = y$ and $\|x_{n_k}\| \to \infty$.

c) Since $\{v_k\}$ is bounded and T is compact, there is a subsequence $\{v_{k_m}\}$ of $\{v_k\}$ such that $\{Tv_{k_m}\}$ converges to an element $x \in H$. Since (using b))

$$\lim_{m\to\infty} \lambda v_{k_m} = \lim_{m\to\infty} \big((Tv_{k_m} - (T - \lambda I)v_{k_m})\big)$$

$$= \lim_{m\to\infty} Tv_{k_m} - \lim_{m\to\infty} (T - \lambda I)v_{k_m} = x - 0 = x,$$

we see that $v_{k_m} \to \frac{x}{\lambda}$.

d) Since the elements x_n are in $\ker(T - \lambda I)^\perp$, so are the elements v_{k_m}, and since $x = \lambda \lim_{m\to\infty} v_{k_m}$ and $\ker(T - \lambda I)^\perp$ is closed, x must be in $\ker(T - \lambda I)^\perp$. On the other hand, since $(T - \lambda I)v_{k_m} \to 0$, we have

$$(T - \lambda I)x = \lim_{m\to\infty} (T - \lambda I)(\lambda v_{k_m}) = 0,$$

which shows that $x \in \ker(T - \lambda)$. Hence

$$x \in \ker(T - \lambda I) \cap \ker(T - \lambda I)^\perp = \{0\}$$

which means that $x = 0$. But as $x = \lambda \lim_{m\to\infty} v_{k_m}$ where $\|v_{k_m}\| = 1$ and $\lambda \neq 0$, this is impossible, and hence our assumption that $\{x_n\}$ is unbounded leads to a contradiction.

e) As we now know that $\{x_n\}$ is bounded, the compactness of T implies that there is a subsequence $\{x_{n_\ell}\}$ such that $\{Tx_{n_\ell}\}$ converges to some point a. Since $\{(T-\lambda I)x_{n_\ell}\}$ converges to y by assumption, we get

$$\lim_{\ell\to\infty} \lambda x_{n_\ell} = \lim_{\ell\to\infty} (Tx_{n_\ell} - (T-\lambda I)x_{n_\ell}) = a - y,$$

and hence $\lim_{\ell\to\infty} x_{n_\ell} = \frac{a-y}{\lambda}$. Thus if we let $z = \frac{a-y}{\lambda}$, we have

$$(T-\lambda I)z = \lim_{\ell\to\infty} (T-\lambda I)x_{n_\ell} = y,$$

which shows that $y \in (T-\lambda I)H$.

We have now proved that if y is the limit of a sequence $\{y_n\}$ of elements in $(T-\lambda I)H$, then $y \in (T-\lambda I)H$, and hence $(T-\lambda I)H$ is closed (see *Spaces*, Proposition 3.3.7).