

MAT4400: Detailed Notes on Linear analysis

May 1, 2024

1 σ -Algebras (3, [Schilling(2017)])

Definition 1.1 (σ -Algebra). A family \mathcal{A} of subsets of X with:

- (i) $X \in \mathcal{A}$,
- (ii) $A \in \mathcal{A} \Rightarrow A^c \in \mathcal{A}$,
- (iii) $(A_n)_{n \in \mathbb{N}} \in \mathcal{A} \Rightarrow \bigcup_{n \in \mathbb{N}} A_n \in \mathcal{A}$

A set $A \in \mathcal{A}$ is said to be **measurable** or **\mathcal{A} -measurable**.

Theorem 1.2 (and Definition).

- (i) The intersection of arbitrarily many σ -algebras in X is again a σ -algebra in X .
- (ii) For every system of sets $p \subset \mathcal{P}(X)$ there exists a smallest σ -algebra containing p . This is the σ -algebra generated by p , denoted $\sigma(p)$, and $\sigma(p)$ is called its generator.

Definition 1.3 (Borel). The σ -algebra $\sigma(\mathcal{O})$ generated by the open sets $\mathcal{O} = \mathcal{O}_{\mathbb{R}^n}$ of \mathbb{R}^n is called **Borel σ -algebra**, and its members are called **Borel sets** or **Borel measurable sets**.

Definition 1.4 (σ -finite/sigma-finite). A measure μ is said to be σ -finite and (X, \mathcal{A}, μ) is called a σ -finite measure space, if \mathcal{A} contains a sequence $(A_n)_{n \in \mathbb{N}}$ s.t. $A_n \uparrow X$ and $\mu(A_n) < \infty$.

3 Uniqueness of Measures (5, [Schilling(2017)])

Lemma 3.1. A Dynkin system D is a σ -algebra iff it is stable under finite intersections, i.e. $A, B \in D \Rightarrow A \cap B \in D$.

Theorem 3.2 (Dynkin). Assume X is a set, S is a collection of subsets of X closed under finite intersections, that is, if $A, B \in S \Rightarrow A \cap B \in S$. Then $D(S) = \sigma(S)$.

Proof. We clearly have that $D(S) \subset \sigma(S)$. If we can show that $D(S)$ is a σ -algebra, that is, that a Dynkin system generated by a subset $S \subset X$ (where S is \cap -stable) is a σ -algebra, then the inverse conclusion $D(S) \supset \sigma(S)$ follows logically. This is the case because the σ -algebra $\sigma(S)$ is the smallest σ -algebra containing S , and so if $D(S)$ is a σ -algebra it must be a greater or equal (in some sense) than $\sigma(S)$.

Using Lemma 3.1 we only need to show that $D(S)$ is stable under finite intersections, to prove that $D(S)$ is a σ -algebra. Consider:

$$D_A := \{B \subset X : B \cap A \in D(S)\},$$

for some $A \in D(S)$. Notice that this set is \cap -stable, and so if we can show that $D_A = D(S)$ we must have that (by Lemma 3.1) $D(S)$ is a σ -algebra. Firstly, however, let us show that D_A is a **Dynkin system**.

1. \emptyset must be in D_A , since $\emptyset \cap A = \emptyset \in D(S)$.

2. Let $B \in D_A$. Then

$$A \cap B^c = A \setminus (A \cap B) = (A^c \cup (A \cap B))^c,$$

here $A \cap B$ and A^c must be in $D(S)$. Furthermore, since disjoint unions of set from $D(S)$ are still in $D(S)$, we must have $A^c \in D_A$.

3. Assume that $(B_n)_{n \in \mathbb{N}} \subset D_A$ is a pairwise disjoint sequence. Then

$$\begin{aligned} (B_n \cap A)_{n \in \mathbb{N}} &\in D(S) \quad (\text{by def. of } D_A) \\ \Rightarrow \bigcup_{n \in \mathbb{N}} (B_n \cap A) &= \left(\bigcup_{n \in \mathbb{N}} B_n \right) \cap A \in D(S) \\ \Rightarrow \bigcup_{n \in \mathbb{N}} B_n &\in D_A. \end{aligned}$$

So D_A is indeed a Dynkin system.

We now want to show that $D(S)$ is \cap -stable, we have:

$$\begin{aligned} S &\subset D_A \quad \forall A \in S \\ \Rightarrow D(S) &\subset D_A \quad \forall A \in S \\ \Rightarrow B \cap A &\in D(S) \quad \forall B \in S, \forall A \in D(S) \\ \Rightarrow B &\in D_A \quad \forall B \in S, \forall A \in D(S) \\ \Rightarrow S &\subset D_A \quad \forall A \in D(S) \\ \Rightarrow D(S) &\subset D_A \quad \forall A \in D(S) \\ \Rightarrow A \cap B &\in D(S) \quad \forall A, B \in D(S), \end{aligned}$$

and so $D(S)$ is \cap -stable and then $D(S) \supset \sigma(S) \Rightarrow D(S) = \sigma(S)$. \square

Theorem 3.3 (uniqueness of measures). Let (X, \mathcal{B}) be a measurable space, and $S \subset \mathcal{P}(X)$ be the generator of \mathcal{B} , i.e. $\mathcal{B} = \sigma(S)$. If S satisfies the following conditions:

1. S is stable under finite intersections (\cap -stable), i.e. $A, C \in S \Rightarrow A \cap C \in S$.
2. There exists an exhausting sequence $(G_n)_{n \in \mathbb{N}} \subset S$ with $G_n \uparrow X$. Assume also that there are two measures μ, ν satisfying:
3. $\mu(A) = \nu(A)$, $\forall A \in S$.
4. $\mu(G_n) = \nu(G_n) < \infty$.

Then $\mu = \nu$.

Proof (outline). Define

$$D_n := \{A \in \mathcal{B} : \mu(G_n \cap A) = \nu(G_n \cap A) (< \infty)\},$$

and show that it is a Dynkin system. Then, use the fact that S is \cap -stable and Theorem 3.2 to argue that $D(S) = \sigma(S) \dots \rightarrow \dots \mathcal{B} = D_n$. \square

4 Existence of Measures (6, [Schilling(2017)])

Theorem 4.1 (Carathéodory). *Let $S \subset P(X)$ be a semi-ring and $\mu : S \rightarrow [0, \infty)$ a pre-measure. Then μ has an extension to a measure μ^* on $\sigma(S)$, i.e. that $\mu(s) = \mu^*(s)$, $\forall s \in \sigma(S)$.*

Also, if S contains an exhausting sequence, $S_n \uparrow X$, s.t. $\mu(S_n) < \infty$, then the extension is unique.

Outline of proof: Firstly, let us define an outer measure.

Definition 4.2 (Outer measure). An outer measure is a function $\mu^* : P(X) \rightarrow [0, \infty)$ with the following properties:

1. $\mu^*(\emptyset) = 0$,
2. $A \subset B \Rightarrow \mu^*(A) \leq \mu^*(B)$,
3. $\mu^*\left(\bigcup_{n \in \mathbb{N}} A_n\right) \leq \sum_{n \in \mathbb{N}} \mu^*(A_n)$,

and define for each $A \subset X$ the family of countable S -coverings:

$$C(A) := \left\{ (S_n)_{n \in \mathbb{N}} \subset S : \bigcup_{n \in \mathbb{N}} S_n \supset A \right\},$$

and the set function

$$\mu^*(A) := \inf \left\{ \sum_{n \in \mathbb{N}} \mu(S_n) : (S_n)_{n \in \mathbb{N}} \in C(A) \right\}.$$

Step 1: Claim: $\mu^*(A)$ is an outer measure.

Proof.

1. $C(\emptyset) = \{\text{any sequence in } S \text{ containing } \emptyset\} \Rightarrow \mu^*(\emptyset) = 0$.
2. Assume $A \subset B$. Then $C(A) \subset C(B) \Rightarrow \mu^*(A) \leq \mu^*(B)$.
3. If $\mu^*(A_n) = \infty$ for some n , then there is nothing to prove. Thus, assume $\mu^*(A_n) < \infty \forall n$. Fix $\epsilon > 0$, and for every n choose $A_{n_k} \in S$ s.t.

$$\begin{aligned} A_n &\subset \bigcup_{k \in \mathbb{N}} A_{n_k}, \\ \sum_{k \in \mathbb{N}} \mu^*(A_{n_k}) &< \mu^*(A_n) + \frac{\epsilon}{2^n}. \end{aligned}$$

Then

$$\bigcup_{n \in \mathbb{N}} A_n \subset \bigcup_{k \in \mathbb{N}} \bigcup_{n \in \mathbb{N}} A_{n_k},$$

so

$$\begin{aligned} \mu^*\left(\bigcup_{n \in \mathbb{N}} A_n\right) &\leq \sum_{n \in \mathbb{N}} \sum_{k \in \mathbb{N}} \mu(A_{n_k}) \\ &< \sum_{n \in \mathbb{N}} \left(\mu^*(A_n) + \frac{\epsilon}{2^n}\right) \\ &= \sum_{n \in \mathbb{N}} \mu^*(A_n) + \epsilon. \end{aligned}$$

As ϵ was arbitrarily, we get that

$$\mu^*\left(\bigcup_{n \in \mathbb{N}} A_n\right) \leq \sum_{n \in \mathbb{N}} \mu^*(A_n),$$

so μ^* fulfills all the conditions for being an outer measure. \square

Step 2: Showing that μ^* extends μ , i.e. $\mu^*(s) = \mu(s) \forall s \in S$.

Step 3: Define μ^* -measurable sets

$$\Sigma^* := \left\{ A \subset X : \mu^*(Q) = \mu^*(Q \cap A) + \mu^*(Q \setminus A) \right. \\ \left. \forall Q \subset X \right\}$$

Step 4: Show that $\mu|_{\Sigma^*}$ is a measure. In particular, $\mu|_{\sigma(S)}$ is a measure which extends μ .

5 Measurable Mappings (7, [Schilling(2017)])

We consider maps $T : X \rightarrow X'$ between two measurable spaces (X, \mathcal{A}) and (X', \mathcal{A}') which respects the measurable structures, the σ -algebras on X and X' . These maps are useful as we can transport a measure μ , defined on (X, \mathcal{A}) , to (X', \mathcal{A}') .

Definition 5.1. Let (X, \mathcal{A}) , (X', \mathcal{A}') be measurable spaces. A map $T : X \rightarrow X'$ is called \mathcal{A}/\mathcal{A}' -measurable if the pre-image of every measurable set is a measurable set:

$$T^{-1}(A') \in \mathcal{A}, \quad \forall A' \in \mathcal{A}'.$$

- A $\mathcal{B}(\mathbb{R}^n)/\mathcal{B}(\mathbb{R}^m)$ measurable map is often called a Borel map.
- The notation $T : (X, \mathcal{A}) \rightarrow (X', \mathcal{A}')$ is often used to indicate measurability of the map T .

Lemma 5.2. Let (X, \mathcal{A}) , (X', \mathcal{A}') be measurable spaces and let $\mathcal{A}' = \sigma(\mathcal{G}')$. Then $T : X \rightarrow X'$ is \mathcal{A}/\mathcal{A}' -measurable iff $T^{-1}(\mathcal{G}') \subset \mathcal{A}$, i.e. if

$$T^{-1}(G') \in \mathcal{A}, \quad \forall G' \in \mathcal{G}'.$$

Theorem 5.3. Let (X_i, \mathcal{A}_i) , $i = 1, 2, 3$, be measurable spaces and $T : X_1 \rightarrow X_2$, $S : X_2 \rightarrow X_3$ be $\mathcal{A}_1/\mathcal{A}_2$ and $\mathcal{A}_2/\mathcal{A}_3$ -measurable maps respectively. Then $S \circ T : X_1 \rightarrow X_3$ is $\mathcal{A}_1/\mathcal{A}_3$ -measurable.

Corollary 5.4. Every continuous map between metric spaces is a Borel map.

Definition 5.5. (and lemma) Let $(T_i)_{i \in I}$, $T_i : X \rightarrow X_i$, be arbitrarily many mappings from the same space X into measurable spaces (X_i, \mathcal{A}_i) . The smallest σ -algebra on X that makes all T_i simultaneously measurable is

$$\sigma(T_i : i \in I) := \sigma\left(\bigcup_{i \in I} T_i^{-1}(\mathcal{A}_i)\right)$$

Corollary 5.6. A function $f : (X, \mathcal{B}) \rightarrow \mathbb{R}$ is measurable if $f((a, +\infty)) \in \mathcal{B}$, $\forall a \in \mathbb{R}$.

Corollary 5.7. Assume (X, \mathcal{B}) is a measurable space, (Y, d) is a metric space, and $(f_n : (X, \mathcal{B}) \rightarrow Y)_{n=1}^\infty$ is a sequence of measurable maps. Assume this sequence of images $(f_n(x))_{n=1}^\infty$ is convergent in $Y \forall x \in X$. Define

$$f : X \rightarrow Y, \text{ by } f(x) = \lim_{n \rightarrow \infty} f_n(x).$$

Then f is measurable.

Theorem 5.8. Let (X, \mathcal{A}) , (X', \mathcal{A}') be measurable spaces and $T : X \rightarrow X'$ be an \mathcal{A}/\mathcal{A}' -measurable map. For every measurable μ on (X, \mathcal{A}) ,

$$\mu'(A') := \mu(T^{-1}(A')), \quad A' \in \mathcal{A}',$$

defines a measure on (X', \mathcal{A}') .

Definition 5.9. The measure $\mu'(\cdot)$ in the above theorem is called the push forward or image measure of μ under T and it is denoted as $T(\mu)(\cdot)$, $T_*\mu(\cdot)$ or $\mu \circ T^{-1}(\cdot)$.

Theorem 5.10. If $T \in \mathbb{R}^{n \times n}$ is an orthogonal matrix, then $\lambda^n = T(\lambda^n)$.

Theorem 5.11. Let $S \in \mathbb{R}^{n \times n}$ be an invertible matrix. Then

$$S(\lambda^n) = |\det S|^{-1} \lambda^n = |\det S|^{-1} \lambda^n.$$

Corollary 5.12. Lebesgue measure is invariant under motions: $\lambda^n = M(\lambda^n)$ for all motions M in \mathbb{R}^n . In particular, congruent sets have the same measure. Two sets of points are called congruent if, and only if, one can be transformed into the other by an isometry

Measurable Functions (8, [Schilling(2017)])

A measurable function is a measurable map $u : X \rightarrow \mathbb{R}$ from some measurable space (X, \mathcal{A}) to $(\mathbb{R}, \mathcal{B}(\mathbb{R}^1))$. They play central roles in the theory of integration.

We recall that $u : X \rightarrow \mathbb{R}$ is $\mathcal{A}/\mathcal{B}(\mathbb{R}^1)$ -measurable if

$$u^{-1}(B) \in \mathcal{A}, \quad \forall B \in \mathcal{B}(\mathbb{R}^1).$$

Moreover from a lemma from chapter 7, we actually only need to show that

$$u^{-1}(G) \in \mathcal{A}, \quad \forall G \in \mathcal{G} \text{ where } \mathcal{G} \text{ generates } \mathcal{B}(\mathbb{R}^1).$$

Proposition 5.13.

- 1 If $f, g : (X, \mathcal{B}) \rightarrow \mathbb{C}$ are measurable, then the function $f + g$, $f \cdot g$, cf , ($c \in \mathbb{C}$) are measurable.
- 2 If $b : \mathbb{C} \rightarrow \mathbb{C}$ is Borel and $b : (\mathbb{C}, \mathcal{B}) \rightarrow \mathbb{C}$ is measurable, then $b \circ f$ is measurable.
- 3 If $f(x) = \lim_{n \rightarrow \infty} f_n(x)$, $x \in X$ and f_n are measurable, then f is measurable.
- 4 If $X = \bigcup_{n=1}^\infty A_n$, ($A_n \in \mathcal{B}$), $f|_{A_n} : (A_n, \mathcal{B}_{A_n}) \rightarrow \mathbb{C}$ is measurable $\forall n$, then f is measurable.

Definition 5.14. Given a measurable space (X, \mathcal{B}) , a measurable function $f : (X, \mathcal{B}) \rightarrow \mathbb{C}$ is called simple if

$$f(x) = \sum_{k=1}^N c_k \mathbb{1}_{A_k}(x),$$

for some $c_k \in \mathbb{C}$, $A_k \in \mathcal{B}$, where $\mathbb{1}$ is the characteristic function,

$$\mathbb{1}_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A. \end{cases}$$

The representation of simple function is **not** unique. We denote the standard representation of f by

$$f(x) = \sum_{n=0}^N z_n \mathbb{1}_{B_n}(x),$$

for $N \in \mathbb{N}$, $z_n \in \mathbb{R}$, $B_n \in \mathcal{A}$, and

$$X = \bigcup_{n=1}^N B_n,$$

for $B_n \cap B_m = \emptyset$, $n \neq m$. The set of simple functions is denoted $\mathcal{E}(\mathcal{A})$ of \mathcal{E} .

Definition 5.15. Assume μ is a measure on (X, \mathcal{B}) . Given a positive simple function

$$f = \sum_{k=1}^N c_k \mathbb{1}_{A_k}, \quad (c_k \geq 0).$$

We define

$$\int_X f d\mu = \sum_{k=1}^n c_k \mu(A_k) \in [0, +\infty].$$

We also denote this by $I_\mu(f)$.

Lemma 5.16. This is well defined, that is, $\int_X f d\mu$ does not depend on the presentation of the simple function f .

Properties 5.17. For every positive simple function

$$1 \int_X cf d\mu = c \int_X f d\mu, \quad \text{for only } c \geq 0$$

$$2 \int_X (f + g) d\mu = \int_X f d\mu + \int_X g d\mu.$$

Corollary 5.18. If $f \geq g \geq 0$ are simple functions, then

$$\int_X f d\mu \geq \int_X g d\mu.$$

Definition 5.19. If $f : X \rightarrow [0, +\infty)$ is measurable, then we define

$$\int_X f d\mu = \sup \left\{ \int_X g d\mu : f \geq g \geq 0, \text{ } g \text{ is simple} \right\}$$

Remark. This means that any measurable function can be approximated by simple functions.

Properties 5.20. Measurable functions like this have the following properties

$$1 \int_X cf d\mu = c \int_X f d\mu, \quad \forall c \geq 0.$$

$$2 \text{ If } f \geq g \geq 0, \text{ then } \int_X f d\mu \geq \int_X g d\mu \text{ for any measurable } g, f.$$

3 If $f \geq 0$ is simple, then $\int_X f d\mu$ is the same value as obtained before.

To advance in measure theory we consider measurable functions

$$f : X \rightarrow [0, +\infty].$$

Measurability is understood w.r.t the σ -algebra $\mathcal{B}([0, +\infty])$ generated by $\mathcal{B}([0, +\infty))$ and $\{+\infty\}$. In other words, $A \subset [0, +\infty] \in \mathcal{B}([0, +\infty])$ iff $A \cap [0, +\infty) \in \mathcal{B}([0, +\infty))$.

Remark. Hence $f : X \rightarrow [0, +\infty]$ is measurable iff $f^{-1}(A)$ is measurable $\forall A \in \mathcal{B}([0, +\infty))$.

Definition 5.21. For measurable functions $f_X \rightarrow [0, +\infty]$, we define

$$\int_X f d\mu = \sup \left\{ \int_X g d\mu : f \geq g \geq 0 : g \text{ is simple} \right\} \in [0, +\infty].$$

Theorem 5.22. Monotone convergence theorem Assume (X, \mathcal{B}, μ) is a measure space, $(f)_{n=1}^\infty$ is an increasing sequence of measurable positive functions $f_n : X \rightarrow [0, +\infty]$. Define $f(x) = \lim_{n \rightarrow \infty} f_n(x)$. Then f is measurable and

$$\int_X f d\mu = \lim_{n \rightarrow \infty} \int_X f_n d\mu.$$

Theorem 5.23. Assume (X, \mathcal{B}) is a measurable space and $f : X \rightarrow [0, +\infty]$ is measurable. Then there are simple functions g_n , s.t.

$$0 \leq g_1 \leq g_2 \leq \dots, \quad g_n(x) \rightarrow f(x), \quad \forall x \in X.$$

Moreover, if f is bounded, we can choose g_n s.t. the convergence is uniform, that is,

$$\lim_{n \rightarrow \infty} \sup_{x \in X} |g_n(x) - f(x)| = 0.$$

6 Integration of Measurable Functions

(9, [Schilling(2017)])

Through this chapter (X, \mathcal{A}, μ) will be some measure space. Recall that $\mathcal{M}^+(\mathcal{A})$ [$\mathcal{M}_{\mathbb{R}}^+(\mathcal{A})$] are the \mathcal{A} -measurable positive functions and $\mathcal{E}(\mathcal{A})$ [$\mathcal{E}_{\mathbb{R}}^+(\mathcal{A})$] are the positive and simple functions.

The fundamental idea of *Integration* is to measure the area between the graph of the function and the abscissa. For positive simple functions $f \in \mathcal{E}^+(\mathcal{A})$ in standard representation, this is done easily

$$\text{if } f = \sum_{i=0}^M y_i \mathbb{1}_{A_i} \in \mathcal{E}^+(\mathcal{A}) \quad \text{then} \quad \sum_{i=0}^M y_i \mu(A_i) \quad (1)$$

would be the μ -area enclosed by the graph and the abscissa. We note that the representation of f should not impact the integral of f .

Lemma 6.1. Let $\sum_{i=0}^M y_i \mathbb{1}_{A_i} = \sum_{k=0}^N z_k \mathbb{1}_{B_k}$ be two standard representations of the same function $f \in \mathcal{E}^+(\mathcal{A})$. Then

$$\sum_{i=0}^M y_i \mu(A_i) = \sum_{k=0}^N z_k \mu(B_k). \quad (2)$$

Definition 6.2. Let $f = \sum_{i=0}^M y_i \mathbb{1}_{A_i} \in \mathcal{E}^+(\mathcal{A})$ be a simple function in standard representation. Then the number

$$I_\mu(f) = \sum_{i=0}^M y_i \mu(A_i) \in [0, \infty] \quad (3)$$

(which is independent of the representation of f) is called the μ -integral of f .

Proposition 6.3. Let $f, g \in \mathcal{E}^+(\mathcal{A})$. Then

- (i) $I_\mu(\mathbb{1}_A) = \mu(A) \quad \forall A \in \mathcal{A}.$
- (ii) $I_\mu(\lambda f) = \lambda I_\mu(f) \quad \forall \lambda \geq 0.$
- (iii) $I_\mu(f + g) = I_\mu(f) + I_\mu(g).$
- (iv) $f \leq g \Rightarrow I_\mu(f) \leq I_\mu(g).$

In theorem 8.8 we saw that we could for every $u \in \mathcal{M}^+(\mathcal{A})$ write it as an increasing limit of simple functions. By corollary 8.10, the suprema of simple functions are again measurable, so that

$$u \in \mathcal{M}^+(\mathcal{A}) \Leftrightarrow u = \sup_{n \in \mathbb{N}} f_n, f \in \mathcal{E}^+(\mathcal{A}), \\ f_n \leq f_{n+1} \leq \dots$$

We will use this to "inscribe" simple functions (which we know how to integrate) below the graph of a positive measurable function u and exhaust the μ -area below u .

Definition 6.4. Let (X, \mathcal{A}, μ) be a measure space. The (μ) -integral of a positive function $u \in \mathcal{M}_{\mathbb{R}}^+(\mathcal{A})$ is given by

$$\int u d\mu = \sup \{ I_\mu(g) : g \leq u, g \in \mathcal{E}^+(\mathcal{A}) \}, \quad (4)$$

with $\int u d\mu \in [0, +\infty]$. If we need to emphasize the *integration variable*, we write $\int u(x) \mu(dx)$. The key observation is that the integral $\int \dots d\mu$ extends I_μ .

Lemma 6.5. For all $f \in \mathcal{E}^+(\mathcal{A})$ we have $\int f d\mu = I_\mu(f)$.

The next theorem is one of many convergence theorems. It shows that we could have defined 4 using any increasing sequence $f_n \uparrow u$ of simple functions $f_n \in \mathcal{E}^+(\mathcal{A})$.

Theorem 6.6. (Beppo Levi) Let (X, \mathcal{A}, μ) be a measure space. For an increasing sequence of functions $(u_n)_{n \in \mathbb{N}} \subset \mathcal{M}_{\mathbb{R}}^+(\mathcal{A})$, $0 \leq u_n \leq u_{n+1} \leq \dots$, we have for the supremum $u = \sup_{n \in \mathbb{N}} u_n \in \mathcal{M}_{\mathbb{R}}^+(\mathcal{A})$ and

$$\int \sup_{n \in \mathbb{N}} u_n d\mu = \sup_{n \in \mathbb{N}} \int u_n d\mu. \quad (5)$$

Note we can write $\lim_{n \rightarrow \infty}$ instead of $\sup_{n \in \mathbb{N}}$ as the supremum of an increasing sequence is its limit. Moreover, this theorem holds in $[0, +\infty]$, so the case $+\infty = +\infty$ is possible.

Corollary 6.7. Let $u \in \mathcal{M}_{\mathbb{R}}^+(\mathcal{A})$. Then

$$\int u d\mu = \lim_{n \rightarrow \infty} \int f_n d\mu$$

holds for every sequence $(f_n)_{n \in \mathbb{N}} \subset \mathcal{E}^+(\mathcal{A})$ with $\lim_{n \rightarrow \infty} f_n = u$.

Proposition 6.8. (of integral) Let $u, v \in \mathcal{M}_{\mathbb{R}}^+(\mathcal{A})$. Then

- (i) $\int \mathbb{1}_A d\mu = \mu(A) \quad \forall A \in \mathcal{A}.$
- (ii) $\int \alpha u d\mu = \alpha \int u d\mu \quad \forall \alpha \geq 0.$
- (iii) $\int u + v d\mu = \int u d\mu + \int v d\mu.$

(iv) $u \leq v \Rightarrow \int u d\mu \leq \int v d\mu$.

Corollary 6.9. Let $(u_n)_{n \in \mathbb{N}} \subset \mathcal{M}_{\mathbb{R}}^+(\mathcal{A})$. Then $\sum_{n=1}^{\infty} u_n$ is measurable and we have

$$\int \sum_{n=1}^{\infty} u_n d\mu = \sum_{n=1}^{\infty} \int u_n d\mu$$

(including the possibility $+\infty = +\infty$.)

Theorem 6.10. (*Fatou*) Let $(u_n)_{n \in \mathbb{N}} \subset \mathcal{M}_{\mathbb{R}}^+(\mathcal{A})$ be a sequence of positive measurable functions. Then $u = \liminf_{n \rightarrow \infty} u_n$ is measurable and

$$\int \liminf_{n \rightarrow \infty} u_n d\mu \leq \liminf_{n \rightarrow \infty} \int u_n d\mu \quad (6)$$

7 Integrals of Measurable Functions

(10, [Schilling(2017)])

We have defined our integral for positive measurable functions, i.e. functions in $\mathcal{M}^+(\mathcal{A})$. To extend our integral to not only functions in $\mathcal{M}^+(\mathcal{A})$ we first notice that

$$u \in \mathcal{M}_{\mathbb{R}}(\mathcal{A}) \Leftrightarrow u = u^+ - u^-, \quad u^+, u^- \in \mathcal{M}_{\mathbb{R}}^+,$$

i.e. that every measurable function can be written as a sum of **positive** measurable functions.

Definition 7.1 (μ -integrable). A function $u : X \rightarrow \overline{\mathbb{R}}$ on (X, \mathcal{A}, μ) is μ -integrable, if it is $\mathcal{A}/\mathcal{B}(\overline{\mathbb{R}})$ -measurable and if $\int u^+ d\mu, \int u^- d\mu < \infty$ (recall the definition for the integral of positive measurable functions). Then

$$\int u d\mu := \int u^+ d\mu - \int u^- d\mu \in (-\infty, \infty)$$

is the (μ -)integral of u . We write $\mathcal{L}^1(\mu)$ for the set of all real-valued μ -integrable functions¹.

Theorem 7.2. Let $u \in \mathcal{M}_{\mathbb{R}}(\mathcal{A})$, then the following conditions are equivalent:

$$(i) \quad u \in \mathcal{L}_{\mathbb{R}}^1(\mu).$$

$$(ii) \quad u^+, u^- \in \mathcal{L}_{\mathbb{R}}^1(\mu).$$

$$(iii) \quad |u| \in \mathcal{L}_{\mathbb{R}}^1(\mu).$$

$$(iv) \quad \exists w \in \mathcal{L}_{\mathbb{R}}^1(\mu) \text{ with } w \geq 0 \text{ s.t. } |u| \leq w.$$

Theorem 7.3 (Properties of the μ -integral). The μ -integral is: **homogeneous, additive, and:**

$$(i) \quad \min\{u, v\}, \max\{u, v\} \in \mathcal{L}_{\mathbb{R}}^1(\mu) \quad (\text{lattice property})$$

$$(ii) \quad u \leq v \Rightarrow \int u d\mu \leq \int v d\mu \quad (\text{monotone})$$

$$(iii) \quad \left| \int u d\mu \right| \leq \int |u| d\mu \quad (\text{triangle inequality})$$

Remark. If $u(x) \pm v(x)$ is defined in $\overline{\mathbb{R}}$ for all $x \in X$ then we can exclude $\infty - \infty$ and the theorem above just says that the integral is linear:

$$\int (au + bv) d\mu = a \int u d\mu + b \int v d\mu.$$

This is always true for real-valued $u, v \in \mathcal{L}^1(\mu) = \mathcal{L}_{\mathbb{R}}^1(\mu)$, making $\mathcal{L}^1(\mu)$ a vector space with addition and scalar multiplication defined by

$$(u + v)(x) := u(x) + v(x), \quad (a \cdot u)(x) := a \cdot u(x),$$

and

$$\int \dots d\mu : \mathcal{L}^1(\mu) \rightarrow \mathbb{R}, \quad u \mapsto \int u d\mu,$$

is a **positive linear functional**.

8 Null sets and the Almost Everywhere

(11, [Schilling(2017)])

Definition 8.1. A (μ -)null set $N \in \mathcal{N}_{\mu}$ is a measurable set $N \in \mathcal{A}$ satisfying

$$N \in \mathcal{N}_{\mu} \iff N \in \mathcal{A} \text{ and } \mu(N) = 0.$$

This can be used generally about a ‘statement’ or ‘property’, but we will be interested in questions like ‘when is $u(x)$ equal to $v(x)$ ’, and we answer this by saying

$$u = v \text{ a.e.} \Leftrightarrow \{x : u(x) \neq v(x)\} \text{ is (contained in) a } \mu\text{-null set,}$$

i.e.

$$u = v \quad \mu\text{-a.e.} \Leftrightarrow \mu(\{x : u(x) \neq v(x)\}) = 0.$$

The last phrasing should of course include that the set $\{x : u(x) \neq v(x)\}$ is in \mathcal{A} .

Theorem 8.2. Let $u \in \mathcal{M}_{\mathbb{R}}(\mathcal{A})$, then:

$$(i) \quad \int |u| d\mu = 0 \Leftrightarrow |u| = 0 \text{ a.e.} \Leftrightarrow \mu\{u \neq 0\} = 0,$$

$$(ii) \quad \mathbb{1}_N u \in \mathcal{L}_{\mathbb{R}}^1(\mu) \quad \forall N \in \mathcal{N}_{\mu},$$

$$(iii) \quad \int_N u d\mu = 0.$$

(i) is really useful, later we will define \mathcal{L}^p and the $\|\cdot\|_p$ -(semi)norm. Then (i) means that if we have a sequence u_n converging to u in the $\|\cdot\|_p$ -norm then $u_n(x) = u(x)$ a.e.

Corollary 8.3. Let $u = v$ μ -a.e. Then

$$(i) \quad u, v \geq 0 \Rightarrow \int u d\mu = \int v d\mu,$$

$$(ii) \quad u \in \mathcal{L}_{\mathbb{R}}^1(\mu) \Rightarrow v \in \mathcal{L}_{\mathbb{R}}^1(\mu) \text{ and } \int u d\mu = \int v d\mu.$$

Corollary 8.4. If $u \in \mathcal{M}_{\mathbb{R}}(\mathcal{A})$, $v \in \mathcal{L}_{\mathbb{R}}^1(\mu)$ and $v \geq 0$ then

$$|u| \leq v \text{ a.e.} \Rightarrow u \in \mathcal{L}_{\mathbb{R}}^1(\mu).$$

¹In words, we extend our integral to **positive** measurable functions by noticing that we can write every measurable function as a sum of positive measurable functions, something that we do know how to integrate. We don’t want to run into the problem of $\infty - \infty$, thus we require the integral of the positive and negative parts to both (separately) be less than infinity.

Proposition 8.5 (Markow inequality). For all $u \in \mathcal{L}^1_{\mathbb{R}}(\mu)$, $A \in \mathcal{A}$ and $c > 0$

$$u(\{|u| \geq c\} \cap A) \leq \frac{1}{c} \int_A |u| d\mu,$$

if $A = X$, then (obviously)

$$u\{|u| \geq c\} \leq \frac{1}{c} \int |u| d\mu.$$

Corollary 8.6. If $u \in \mathcal{L}^1_{\mathbb{R}}(\mu)$, then μ is a.e. \mathbb{R} -valued. In particular, we can find a version $\tilde{u} \in \mathcal{L}^1(\mu)$ s.t. $\tilde{u} = u$ a.e. and $\int \tilde{u} d\mu = \int u d\mu$

Completions of measure spaces

Definition 8.7. A measure space (X, \mathcal{B}, μ) is called **complete** if whenever $A \in \mathcal{B}$ and $\mu(A) = 0$, we have $B \in \mathcal{B} \forall B \subset A$.

Remark. Any measure space can be completed as follows: Let $\bar{\mathcal{B}}$ be the σ -algebra generated by \mathcal{B} and all sets $B \subset X$ s.t. there exists $A \in \mathcal{B}$ with $B \subset A$ and $\mu(A) = 0$.

Proposition 8.8. The σ -algebra $\bar{\mathcal{B}}$ can also be described as follows:

$$\bar{\mathcal{B}} := \left\{ B \subset X : A_1 \subset B \subset A_2 \text{ for some } A_1, A_2 \in \mathcal{B} \text{ with } \mu(A_2 \setminus A_1) = 0 \right\},$$

with B, A_1, A_2 as above, we define

$$\bar{\mu} := \mu(A_1) = \mu(A_2)$$

Then $(X, \bar{\mathcal{B}}, \bar{\mu})$ is a complete measure space.

Definition 8.9. If μ is a Borel measure on a **metric** space (X, d) , then the completion $\bar{\mathcal{B}}(X)$ of the Borel σ -algebra with respect to μ is called the σ -algebra of μ -measurable sets.

Remark. For $\mu = \lambda_n$ on \mathbb{R}^n we talk about the σ -algebra of **Lebesgue measurable sets**. Instead of $\bar{\lambda}_n$ we still write λ_n and call it the **Lebesgue measure**. A function $f : \mathbb{R}^n \rightarrow \mathbb{C}$, measurable w.r.t. the σ -algebra of Lebesgue measurable sets is called the **Lebesgue measurable**.

The following result shows that any Lebesgue measurable function coincides with a Borel function a.e.

Proposition 8.10. Assume (X, \mathcal{B}, μ) is a measure space and consider its completion $(X, \bar{\mathcal{B}}, \bar{\mu})$. Assume $f : X \rightarrow \mathbb{C}$ is $\bar{\mathcal{B}}$ -measurable. Then there is a \mathcal{B} -measurable function $g : X \rightarrow \mathbb{C}$ s.t. $f = g$ $\bar{\mu}$ -a.e.

9 Convergence Theorems and their Applications

(12, [Schilling(2017)])

- To interchange limits and integrals in **Riemann integrals** one typically has to assume uniform convergence. - The set of Riemann integrable functions is somewhat limited, see theorem 9.7

Theorem 9.1 (Generalization of Beppo Levi, monotone convergence).

(i) Let $(u_n)_{n \in \mathbb{N}} \subset \mathcal{L}^1(\mu)$ be s.t. $u_1 \leq u_2 \leq \dots$ with limit $u := \sup_{n \in \mathbb{N}} u_n = \lim_{n \rightarrow \infty} u_n$. Then $u \in \mathcal{L}^1(\mu)$ iff

$$\sup_{n \in \mathbb{N}} \int u_n d\mu < +\infty,$$

in which case

$$\sup_{n \in \mathbb{N}} \int u_n d\mu = \int \sup_{n \in \mathbb{N}} u_n d\mu.$$

(ii) Same thing only with a decreasing sequence $\dots > -\infty$ in which case

$$\inf_{n \in \mathbb{N}} \int u_n d\mu = \int \inf_{n \in \mathbb{N}} u_n d\mu.$$

Theorem 9.2 (Lebesgue; dominated convergence). Let $(u_n)_{n \in \mathbb{N}} \subset \mathcal{L}^1(\mu)$ s.t.

(a) $|u_n|(x) \leq w(x)$, $w \in \mathcal{L}^1(\mu)$,

(b) $u(x) = \lim_{n \rightarrow \infty} u_n(x)$ exists in $\bar{\mathbb{R}}$,

then $u \in \mathcal{L}^1(\mu)$ and we have

(i) $\lim_{n \rightarrow \infty} \int |u_n - u| d\mu = 0$;

(ii) $\lim_{n \rightarrow \infty} \int u_n d\mu = \int \lim_{n \rightarrow \infty} u_n d\mu = \int u d\mu$;

Application 1: Parameter-Dependent Integrals

- We are interested in questions of the sort, when is

$$U(t) := \int u(t, x) \mu(dx), \quad t \in (a, b),$$

again a smooth function of t ? The answer involves interchange of limits and integration. Also, it turns out to better understand Riemann integrability, we need the Lebesgue integral.

Theorem 9.3 (continuity lemma). Let $\emptyset \neq (a, b) \subset \mathbb{R}$ be a non-degenerate open interval and $u : (a, b) \times X \rightarrow \mathbb{R}$ satisfy

(a) $x \mapsto u(t, x)$ is in $\mathcal{L}^1(\mu)$ for every fixed $t \in (a, b)$;

(b) $t \mapsto u(t, x)$ is continuous for every fixed $x \in X$;

(c) $|u(t, x)| \leq w(x)$ for all $(t, x) \in (a, b) \times X$ and some $w \in \mathcal{L}^1(\mu)$.

Then the function $U : (a, b) \rightarrow \mathbb{R}$ given by

$$t \mapsto U(t) := \int u(t, x) \mu(dx) \quad (7)$$

is continuous.

Theorem 9.4 (differentiability lemma). Let $\emptyset \leq (a, b) \subset \mathbb{R}$ be a non-degenerate open interval and $u : (a, b) \times X \rightarrow \mathbb{R}$ satisfy

(a) Same

(b) Same

(c) $|\partial_t u(t, x)| \leq w(x)$ for all $(t, x) \in (a, b) \times X$ and some $w \in \mathcal{L}^1(\mu)$.

Then the function in 7 is differentiable and its derivative is

$$\frac{d}{dt} U(t) = \frac{d}{dt} \int u(t, x) \mu(dx) = \int \frac{\partial}{\partial t} u(t, x) \mu(dx). \quad (8)$$

Application 2: Riemann vs Lebesgue Integration

Consider only $(X, \mathcal{A}, \mu) = (\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda)$.

Definition 9.5 (The Riemann Inegral). Consider on the finite interval $[a, b] \subset \mathbb{R}$ the partition

$$\Pi := \{a = t_0 < t_1 < \dots < t_k < b\}, k = k(\Pi), \quad (9)$$

and introduce

$$S_\Pi[u] := \sum_{i=1}^{k(\Pi)} m_i(t_i - t_{i-1}), \quad m_i := \inf_{x \in [t_{i-1}, t_i]} u(x), \quad (10)$$

$$S^\Pi[u] := \sum_{i=1}^{k(\Pi)} M_i(t_i - t_{i-1}), \quad M_i := \sup_{x \in [t_{i-1}, t_i]} u(x). \quad (11)$$

$$(12)$$

A bounded function $u : [a, b] \rightarrow \mathbb{R}$ is said to be **Riemann integrable** if the values

$$\int u := \sup_{\Pi} S_\Pi[u] = \inf_{\Pi} S^\Pi[u] =: \int u \quad (13)$$

coincide and are finite. Their common value is called the **Riemann integral** of u and denoted by $(R) \int_a^b u(x) dx$ or $\int_a^b u(x) dx$.

Theorem 9.6. Let $u : [a, b] \rightarrow \mathbb{R}$ be a *measurable* and *Riemann integrable* function. Then

$$u \in \mathcal{L}^1(\lambda) \text{ and } \int_{[a,b]} u d\lambda = \int_a^b u(x) dx. \quad (14)$$

Theorem 9.7. Let $u : [a, b] \rightarrow \mathbb{R}$ be a bounded function, it is Riemann integrable *iff* the points in (a, b) where u is discontinuous are a (subset of) Borel measurable null set.

Improper Riemann Integrals

- The Lebesgue integral extends the (*proper*) Riemann integral. However, there is a further extension of the Riemann integral which cannot be captured by Lebesgue's theory. u is Lebesgue integrable *iff* $|u|$ has finite Lebesgue integral. - The Lebesgue integral does not respect sign-changes and cancellations. However, the following *improper Riemann integral* does:

$$(R) \int_0^\infty u(x) dx := \lim_{n \rightarrow \infty} (R) \int_0^n u(x) dx. \quad (15)$$

Corollary 9.8. Let $u : [0, \infty) \rightarrow \mathbb{R}$ be a measurable, Riemann integrable function for every interval $[0, N]$, $N \in \mathbb{N}$. Then $u \in \mathcal{L}^1[0, \infty)$ *iff*

$$\lim_{N \rightarrow \infty} (R) \int_0^N |u(x)| dx < \infty. \quad (16)$$

In this case, $(R) \int_0^\infty u(x) dx = \int_{[0, \infty)} u d\lambda$

Example of a function which is *improperly Riemann integrable* but **not** *Lebesgue integrable*:

$$f(x) = \frac{\sin(x)}{x}. \quad (17)$$

Proposition 9.9 (appearing as example 12.13 in Schilling). Let $f_\alpha(x) := x^\alpha, x > 0$ and $\alpha \in \mathbb{R}$. Then

$$(i) f_\alpha \in \mathcal{L}^1(0, 1) \Leftrightarrow \alpha > -1.$$

$$(ii) f_\alpha \in \mathcal{L}^1[1, \infty) \Leftrightarrow \alpha < -1.$$

10 Regularity of Measures (App. H, [Schilling(2017)])

We let (X, d) be a metric space and denote by \mathcal{O} the open, by \mathcal{C} the closed and $\mathcal{B}(X) = \sigma(\mathcal{O})$ the Borel set of X .

Definition 10.1. A measure μ on $(X, d, \mathcal{B}(X))$ is called *outer regular*, if

$$\mu(B) = \inf \{ \mu(U) \mid B \subset U, U \text{ open} \} \quad (18)$$

and *inner regular*, if $\mu(K) < \infty$ for all compact sets $K \subset X$ and

$$\mu(U) = \sup \{ \mu(K) \mid K \subset U, K \text{ compact} \}. \quad (19)$$

A measure which is both inner and outer regular is called **regular**. We write $\mathfrak{m}_r^+(X)$ for the family of regular measures on $(X, \mathcal{B}(X))$.

Remark. The space X is called *σ -compact* if there is a sequence of compact sets $K_n \uparrow X$. A typical example of such a space is a locally compact, separable metric space.

Theorem 10.2. Let (X, d) be a metric space. Every finite measure μ on $(X, \mathcal{B}(X))$ is outer regular. If X is *σ -compact*, then μ is also inner regular, hence regular.

Theorem 10.3. Let (X, d) be a metric space and μ be a measure on $(X, \mathcal{B}(X))$ such that $\mu(K) < \infty$ for all compact sets $K \subset X$.

- 1 If X is *σ -compact*, then μ is inner regular.
- 2 If there exists a sequence $G_n \in \mathcal{O}$, $G_n \uparrow X$ such that $\mu(G_n) < \infty$, then μ is outer regular.

11 The Function Spaces \mathcal{L}^p (13, [Schilling(2017)])

Assume V is a vector space over $\mathbb{K} \in \{\mathbb{C}, \mathbb{R}\}$.

Definition 11.1. A seminorm on V is a map $p : V \rightarrow [0, +\infty)$ s.t.

- (1) $p(cx) = |c|p(x) \quad \forall x \in V, \forall c \in \mathbb{K}$.
- (2) $p(x + y) \leq p(x) + p(y) \quad \forall x, y \in V$. **triangle inequality**.

A seminorm is called a norm if we also have

$$p(x) = 0 \iff x = 0.$$

A norm is commonly denoted $\|x\|$, and a vectorspace equipped with a norm is called a **normed space**.

Definition 11.2. Assume (X, d) is a measure space. Fix $1 \leq p \leq \infty$. For every measurable function $f : X \rightarrow \mathbb{C}$ we define the following

$$\|f\|_p = \left(\int_X |f|^p d\mu \right)^{1/p} \in [0, +\infty]. \quad (20)$$

We can see that $\|cf\|_p = |c| \|f\|_p \quad \forall c \in \mathbb{C}$.

Notice that by Theorem 8.2(i) we have that $\|f\|_p = 0 \Rightarrow f = 0$ a.e. Consider for example $\lim_{n \rightarrow \infty} \|f_n - f\|_p = 0$, then we can find a subsequence s.t. $\lim_{k \rightarrow \infty} |f_{n(k)} - f| = 0$ a.e., i.e. $\lim_{k \rightarrow \infty} f_{n(k)} = f$ a.e.

Theorem 11.3 (Hölder's inequality). Assume that $u \in \mathcal{L}^p(\mu)$ and $v \in \mathcal{L}^q(\mu)$, where $1/p + 1/q = 1$ and $p, q \in [0, +\infty]$. Then $uv \in \mathcal{L}^1(\mu)$, and the following inequality holds:

$$\left| \int uv d\mu \right| \leq \int |uv| d\mu = \|uv\|_1 \leq \|u\|_p \cdot \|v\|_q.$$

The generalized version reads:

$$\int |u_1 \cdot u_2 \cdots u_N| d\mu \leq \|u_1\|_{p_1} \cdot \|u_2\|_{p_2} \cdots \|u_N\|_{p_N}.$$

Proof. Assume $\|f\|_p > 0$, $\|g\|_q > 0$, otherwise $fg = 0$ (μ -a.e.). We have

$$\frac{|fg|}{\|f\|_p \|g\|_q} \leq \frac{1}{p} \frac{|f|^p}{\|f\|_p^p} + \frac{1}{q} \frac{|g|^q}{\|g\|_q^q}.$$

Integrating over X we get

$$\begin{aligned} \frac{\|fg\|_1}{\|f\|_p \|g\|_q} &\leq \frac{1}{p} \int_X \frac{|f|^p}{\|f\|_p^p} d\mu + \frac{1}{q} \int_X \frac{|g|^q}{\|g\|_q^q} d\mu \\ &= \frac{1}{p} \frac{\|f\|_p^p}{\|f\|_p^p} + \frac{1}{q} \frac{\|g\|_q^q}{\|g\|_q^q} \\ &= \frac{1}{p} + \frac{1}{q} = 1, \end{aligned}$$

so $\|gf\|_1 \leq \|f\|_p \|g\|_q$ □

Lemma 11.4.

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p. \quad (21)$$

Definition 11.5. We define

$$\mathcal{L}^p(X, d\mu) = \{f : X \rightarrow \mathbb{C} \mid f \text{ is measurable and } \|f\|_p < \infty\}.$$

This is a vectorspace with seminorm $f \mapsto \|f\|_p$. And in general this is not a normed space, since $\|f\|_p = 0 \iff f = 0$ a.e.

Generally, if p is a seminorm on a vectorspace V , then

$$V_0 = \{x \in V \mid p(x) = 0\} \quad (22)$$

which is a subspace of V . Then we consider the quotient/factor space V/V_0 .

Definition 11.6. For $x, y \in V$, define

$$x \sim y \iff x - y \in V_0. \quad (23)$$

This is an equivalence relation on V . The representation class of V is defined by $[x]$ or $x + V_0$.

Then V/V_0 is equals the set of equivalence classes. We can show that it is a normed space.

$$[x] + [y] = [x + y] \quad , \quad c[x] = [cx] \quad , \quad \|[x]\| = p(x).$$

Applying this to $\mathcal{L}^p(X, d\mu)$ we get the normed space

$$L^p(X, d\mu) := \mathcal{L}^p(X, d\mu)/\mathcal{N} = \mathcal{L}^p(X, d\mu)/\sim. \quad (24)$$

Where \mathcal{N} is the space of measurable functions f s.t. $f = 0$ a.e. The equivalence relation \sim is defined by

$$u \sim v \iff \{u \neq v\} \in \mathcal{N}_\mu \iff \mu\{u \neq v\} = 0,$$

and so $L^p(X, d\mu)$ consists of all equivalence classes $[u]_p = \{v \in \mathcal{L}^p \mid u \sim v\}$. So for every $u \in [u]_p$ there is no $v \in [u]_p$ such that $\mu\{u \neq v\} \neq 0$.

We will further continue to denote the norm by $\|\cdot\|_p$, and we will normally **not** distinguish between $f \in \mathcal{L}^p(X, d\mu)$ and the vector in $L^p(X, d\mu)$ that f defines.

Definition 11.7. A normed space $(X, \|\cdot\|)$ is called a Banach space if V is complete w.r.t the metric $d(x, y) = \|x - y\|$.

Theorem 11.8. If (X, \mathcal{B}, μ) is a measure space, $1 \leq p \leq \infty$, then $L^p(X, d\mu)$ is a Banach space.

Definition 11.9. A measurable function $f : X \rightarrow \mathbb{C}$ is called **essentially bounded** if there is $c \geq 0$ s.t.

$$\mu(\{x : |f(x)| > c\}) = 0. \quad (25)$$

That is $|f| \leq c$ a.e. The smallest such c is called the essential supremum of f and is denoted by $\|f\|_\infty$. That is,

$$\|u\|_\infty := \inf \{c > 0 : \mu\{|u| \geq c\} = 0\},$$

and from problem 13.21 we have

$$\lim_{p \rightarrow \infty} \|\cdot\|_p = \|\cdot\|_\infty.$$

Definition 11.10.

$$\mathcal{L}^\infty(X, d\mu) = \{f : X \rightarrow \mathbb{C} \mid f \text{ is measurable and } \|f\|_\infty < \infty\}.$$

$$L^\infty(X, d\mu) = \mathcal{L}^\infty(X, d\mu)/\mathcal{N}.$$

Where by the previous definiton these spaces become the spaces of all essentially bounded functions.

Theorem 11.11. If (X, \mathcal{B}, μ) is a σ -finite measure space, then $L^\infty(X, d\mu)$ is a Banach space.

Convergence in \mathcal{L}^p and completeness

Lemma 11.12. For any sequence $(u_n)_{n \in \mathbb{N}} \subset \mathcal{L}^p(\mu)$, $p \in [1, \infty)$, of positive functions $u_n \geq 0$ we have

$$\left\| \sum_{n=1}^{\infty} u_n \right\|_p \leq \sum_{n=1}^{\infty} \|u_n\|_p.$$

Theorem 11.13 (Riesz-Fischer). The spaces $\mathcal{L}^p(\mu)$, $p \in [1, \infty)$, are complete, i.e. every Cauchy sequence $(u_n)_{n \in \mathbb{N}} \subset \mathcal{L}^p(\mu)$ converges to some limit $u \in \mathcal{L}^p(\mu)$

Corollary 11.14. Let $(u_n)_{n \in \mathbb{N}} \subset \mathcal{L}^p(\mu)$, $p \in [1, \infty)$ with $\mathcal{L}^p - \lim_{n \rightarrow \infty} u_n = u$. Then there exists a subsequence $(u_{n_k})_{k \in \mathbb{N}}$ s.t. $\lim_{k \rightarrow \infty} u_{n_k}(x) = u(x)$ holds for almost every $x \in X$.

Theorem 11.15. Let $(u_n)_{n \in \mathbb{N}} \subset \mathcal{L}^p(\mu)$, $p \in [1, \infty)$, be a sequence of functions s.t. $|u_n| \leq w \ \forall n \in \mathbb{N}$ and some $w \in \mathcal{L}^p(\mu)$. If $u(x) = \lim_{n \rightarrow \infty} u_n(x)$ exists for (almost) every $x \in X$, then

$$u \in \mathcal{L}^p \text{ and } \lim_{n \rightarrow \infty} \|u - u_n\|_p = 0.$$

Theorem 11.16 (F. Riesz (convergence theorem)). Let $(u_n)_{n \in \mathbb{N}} \subset \mathcal{L}^p(\mu)$, $p \in [1, \infty)$, be a sequence s.t. $\lim_{n \rightarrow \infty} u_n(x) = u(x)$ for almost every $x \in X$ and some $u \in \mathcal{L}^p(\mu)$. Then

$$\lim_{n \rightarrow \infty} \|u_n - u\|_p = 0 \iff \lim_{n \rightarrow \infty} \|u_n\|_p = \|u\|_p.$$

12 Dense and Determining Sets (17, [Schilling(2017)])

Definition 12.1 (Dense Sets). A set $\mathcal{D} \subset \mathcal{L}^p(\mu)$, $p \in [0, \infty]$, is called *dense* if for every $u \in \mathcal{L}^p(\mu)$ there exist a sequence $(f_n)_{n \in \mathbb{N}} \subset \mathcal{D}$ s.t. $\lim_{n \rightarrow \infty} \|u - f_n\|_p = 0$.

Definition 12.2 (Support). The support of a function f is the set of points in X where f is non-zero:

$$\text{supp}(f) := \{x \in X : f(x) \neq 0\}.$$

Theorem 12.3. Assume X, d is a metric space and μ is a Borel measure that is finite on every ball $1 \leq p < \infty$. Then the space of bounded continuous functions with bounded support is dense in $\mathcal{L}^p(X, d\mu)$. Where bounded support means that f vanishes outside some ball.

Proof. We want to approximate $f \in \mathcal{L}^p(X, d\mu)$ by bounded continuous functions with bounded support. By considering separately $(\text{Re}(f))_I$ and $(\text{Im}(f))_I$ we may assume that $f \geq 0$. Then we can find simple functions f_n s.t. $0 \leq f_n \leq f$, $f_n \rightarrow f$ pointwise. As $|f - f_n|^p \leq |f|^p$, by the dominated convergence theorem we have $f_n \rightarrow f \in \mathcal{L}^p(X, d\mu)$. Hence, it suffices to consider simple f , but then it suffices to approximate $f = \pi_A$. Note that $\pi_A \in \mathcal{L}^p(X, d\mu)$ iff $\mu(A) < \infty$.

Fix $x_0 \in X$. Then $\pi_{A \cap B_n(x_0)} \nearrow \pi_A$ pointwise, hence $\pi_{A \cap B_n(x_0)} \rightarrow \pi_A \in \mathcal{L}^p(X, d\mu)$, again by the dominated convergence theorem.

Therefor it suffices to consider $A \subset B_n(x_0)$. As μ is outer regular, we have

$$\mu(A) = \inf_{\substack{A \subset U \subset B_n(x_0) \\ U \text{ is open}}} \mu(U).$$

Note that $\|\pi_U - \pi_A\|_p = \mu(U \setminus A)^{1/p}$. Hence, we can choose $U_k \subset B_n(x_0)$ s.t. $A \subset U_k$, U_k is open, $\pi_{U_k} \rightarrow \pi_A \in \mathcal{L}^p(X, d\mu)$.

Therefor it suffices to approximate π_U for open $U \subset B_n(x_0)$. Consider the functions

$$f_k(x) = \frac{kd(x, U^c)}{1 + kd(x, U^c)}.$$

Then $0 \leq f_k \leq 1$, f_k is continuous, supported on $\bar{U} \subset \bar{B}_n(x_0)$ and $f_k \nearrow \pi_U$ pointwise, hence $f_k \xrightarrow[k \rightarrow \infty]{} \pi_U \in \mathcal{L}^p(X, d\mu)$. \square

Theorem 12.4. Assume (X, d) is a separable locally compact metric space and μ is a Borel Measure on X s.t. $\mu(K) < \infty \ \forall$ compact $K \subset X$. Then the space $C_c(X)$ of continuous compactly supported functions is dense in $\mathcal{L}^p(X, d\mu)$.

Recall that the support of a function f is $\text{supp}(f) = \{x \in X : f(x) \neq 0\}$, *closed support* is the closure of $\text{supp}(f)$ (i.e. boundary points are included), often just written as $\text{supp}(f)$, and a function is said to have *compact support* if $\text{supp}(f)$ is *compact*.

In particular, either theorem shows that if μ is a Borel measure on \mathbb{R}^n s.t. the measure of every ball is finite, then $C_c(\mathbb{R}^n)$ is dense in $\mathcal{L}^p(\mathbb{R}^n, d\mu)$, $1 \leq p < \infty$. Later we will see that even $C_c^\infty(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n, d\mu)$.

Remark. These results do not extend to $p = \infty$ in general.

For $\mu = \lambda_n$ we write simply $\mathcal{L}^p(\mathbb{R}^n)$.

Remark. Theorem 17.8 in the book is *WRONG*. For example, $X = \mathbb{Q}$ with the usual metric is σ -compact, supports nonzero finite measure, but $C_c(\mathbb{Q}) = 0$.

Modes of Convergence

(mixture of ex. 11.12 and ch. 22 p. 258-261. in [Schilling(2017)])

Definition 12.5 (convergence in measure). A sequence of measurable functions $u_n : X \rightarrow \mathbb{R}$ *converges in measure* if

$$\forall \epsilon > 0 \forall A \in \mathcal{A}, \mu(A) < \infty : \lim_{n \rightarrow \infty} \mu(\{|u_n - u| > \epsilon\} \cap A) = 0$$

holds for some $u \in \mathcal{M}(\mathcal{A})$. We write $\mu\text{-}\lim_{n \rightarrow \infty} u_n = u$ or $u_n \xrightarrow{\mu} u$.

Assume (X, \mathcal{B}, μ) is a measure space. Given measurable functions $f_n, f : X \rightarrow \mathbb{C}$, recall that

$$f_n \rightarrow f \text{ a.e.}$$

means that $f_n(x) \xrightarrow[n \rightarrow \infty]{} f(x)$ for all x outside a set of measure zero.

Theorem 12.6 (Egorov). Assume $\mu(X) < \infty$ and $f_n \rightarrow f$ a.e. Then, $\forall \epsilon > 0$, there exists $X_\epsilon \in \mathcal{B}$ s.t. $\mu(X_\epsilon) < \epsilon$ and $f_n \rightarrow f$ uniformly on $X \setminus X_\epsilon$.

In addition to pointwise and uniform convergence we also consider the following:

$f_n \rightarrow f$ in the p -th mean if $\|f_n - f\|_p \xrightarrow[n \rightarrow \infty]{} 0$. For $p = 1$ we say in mean, for $p = 2$ we say in quadratic mean.

$f_n \rightarrow f$ in measure if $\forall \epsilon > 0$ we have

$$\mu(\{x : |f_n(x) - f(x)| \geq \epsilon\}) \xrightarrow[n \rightarrow \infty]{} 0.$$

Theorem 12.7 (Lemma 22.4 in the book?). Assume $(X, \mathcal{B}, d\mu)$ is a measure space, $1 \leq p < \infty$, $f_n, f : X \rightarrow \mathbb{C}$ are measurable functions. Then

(i) If $f_n \rightarrow f$ in the p -th mean, then $f_n \rightarrow f$ in measure.

(ii) If $f_n \rightarrow f$ in measure, then there is a subsequence $(f_{n_k})_{k=1}^\infty$ s.t. $f_{n_k} \rightarrow f$ a.e.

(iii) If $f_n \rightarrow f$ a.e. and $\mu(X) < \infty$, then $f_n \rightarrow f$ in measure.

In particular, if $f_n \rightarrow f$ in the p -th mean, then $f_{n_k} \rightarrow f$ a.e. for a subsequence $(f_{n_k})_k$.

13 Abstract Hilbert Spaces (26, [Schilling(2017)])

Assume \mathcal{H} is a vector space over \mathbb{C} .

Definition 13.1. A pre-inner product on \mathcal{H} is a map $(\cdot, \cdot) : H \times H \rightarrow \mathbb{C}$ which is

- (i) Sesquilinear: linear in the first variable and antilinear in the second:

$$\begin{aligned} (\alpha u + \beta v, w) &= \alpha(u, w) + \beta(v, w), \\ (w, \alpha u + \beta v) &= \bar{\alpha}(w, u) + \bar{\beta}(w, v), \quad u, v, w \in H \text{ and } \alpha, \beta \in \mathbb{C}. \end{aligned}$$

- (ii) Hermitian: $(u, v) = \overline{(v, u)}$.

- (iii) Positive semidefinite: $(u, u) \geq 0$.

It is called an **inner product**, or a scalar product, if instead of (iii) the map is positive definite; $(u, u) > 0$. This definition also works for \mathbb{R} instead of \mathbb{C} .

Cauchy-Schwartz inequality If (\cdot, \cdot) is a pre-inner product, then $|(u, v)| \leq (u, u)^{1/2} (v, v)^{1/2}$.

Corollary 13.2. Assume we have a seminorm $\|u\| := (u, u)^{1/2}$. It is a norm iff (\cdot, \cdot) is an inner product.

Definition 13.3 (Hilbert space). A Hilbert space is a complex vector space \mathcal{H} with an inner product (\cdot, \cdot) s.t. \mathcal{H} is complete with respect to the norm $\|u\| = (u, u)^{1/2}$.

- The norm on a Hilbert space is determined by the inner product, but the inner product can also be recovered by the norm by the *polarization identity*: $(u, v) = \frac{\|u+v\|^2 - \|u-v\|^2}{4} + i \frac{\|u+iv\|^2 - \|u-iv\|^2}{4}$.
- Parallelogram law*: $\|u+v\|^2 + \|u-v\|^2 = 2\|u\|^2 + 2\|v\|^2$.
- A norm on a vector space is given by an inner product iff it satisfies the parallelogram law, and then the scalar product is uniquely determined by the polarization identity.

Example 13.4. Assume (X, \mathcal{B}, μ) is a measure space. Then $\mathcal{L}^2(X, d\mu)$ is a Hilbert space with inner product

$$(f, g) = \int_X f \bar{g} d\mu.$$

This is well-defined, as $|f \bar{g}| \leq \frac{1}{2}(|f|^2 + |g|^2)$.

In particular, if $\mathcal{B} = \mathcal{P}(X)$ and μ is the counting measure, then $L^2(X, d\mu)$ is denoted by $l^2(X)$; for $X = \mathbb{N}$ we write simply l^2 . Note that in this case for $f : X \rightarrow [0, +\infty]$ we have

$$\int_X f d\mu = \sum_{x \in X} f(x) := \sup_{\substack{F \subset X \\ F \text{ is finite}}} \sum_{x \in F} f(x),$$

and if $\sum_{x \in X} f(x) < \infty$, then $\{x : f(x) > 0\}$ is at most countable, so $\sum_{x \in X} f(x) = \sum_{x: f(x) > 0} f(x)$ is the usual sum of a series.

Recall that a subset C of a vector space is called *convex* if

$$u, w \in C \rightarrow tu + (1-t)w \in C \quad \forall t \in (0, 1).$$

The following is one of the key properties of the Hilbert space

Theorem 13.5 (projection theorem). Assume \mathcal{H} is a Hilbert space and $C \subset H$ is a closed convex subset. Then for every $u \in H$ there is a unique $u_0 \in C$ (minimizer) s.t.

$$\|u - u_0\| = d(u, C) (= \inf_{x \in C} \|u - x\|).$$

Proof. Let $d = d(u, C)$. Choose $u_n \in C$ s.t. $\|u - u_n\| \rightarrow d$. We claim that $(u_n)_{n=1}^\infty$ is a Cauchy sequence. As $\frac{u_n + u_m}{2} \in C$, we have

$$\begin{aligned} d^2 &\leq \|u - \frac{u_n + u_m}{2}\|^2 = \frac{1}{4} \|(u - u_m) + (u - u_n)\|^2 \\ &= \frac{1}{4} (2\|u - u_n\|^2 + 2\|u - u_m\|^2 - \|u_n - u_m\|^2), \end{aligned}$$

parallelogram law

so

$$\|u_n - u_m\|^2 \leq 2(\|u - u_n\|^2 - d^2) + 2(\|u - u_m\|^2 - d^2).$$

Thus, $(u_n)_{n \in \mathbb{N}}$ is indeed Cauchy, hence $u_n \rightarrow u_0 \in C$ for some u_0 and

$$\|u - u_0\| = \lim_{n \rightarrow \infty} \|u - u_n\| = d = d(u, C).$$

If u'_0 is another such point, we can take $u_{2n} = u_0, u_{2n+1} = u'_0$ and conclude that $u_0 = u'_0$. \square

14 Orthogonal Projections (26, [Schilling(2017)])

For a Hilbert space \mathcal{H} and a subset $A \subset H$, let

$$A^\perp := \{x \in H : x \perp y \quad \forall y \in A\},$$

where $x \perp y$ means that $(x, y) = 0$. A^\perp is a closed subspace of \mathcal{H} .

Proposition 14.1. Assume \mathcal{H}_0 is a closed subspace of a Hilbert space \mathcal{H} . Then every $u \in H$ uniquely decomposes as

$$u = u_0 + u_1, \text{ with } u_0 \in \mathcal{H}_0 \text{ and } u_1 \in \mathcal{H}_0^\perp.$$

Moreover, $\|u - u_0\| = d(u, \mathcal{H}_0)$ and $\|u\|^2 = \|u_0\|^2 + \|u_1\|^2$.

For a closed subspace $\mathcal{H}_0 \subset \mathcal{H}$, consider the map $P : H \rightarrow \mathcal{H}_0$ s.t. $Pu \in \mathcal{H}_0$ is the unique element satisfying $u - Pu = H_0^\perp$. The operator P is linear. It is also contractive, meaning that $\|Pu\| \leq \|u\|$, since $\|u\|^2 = \|Pu\|^2 + \|u - Pu\|^2$. It is called the orthogonal projection onto \mathcal{H}_0 .

If \mathcal{H}_0 is finite dimensional with an orthonormal basis u_1, \dots, u_n then

$$Pu = \sum_{k=1}^n (u, u_k) u_k.$$

Orthonormal bases can be defined for arbitrary Hilbert spaces.

Definition 14.2 (orthonormal system). An orthonormal system in \mathcal{H} is a collection of vectors $u_i \in H$ ($i \in I$) s.t.

$$(u_i, u_j) = \delta_{ij} \quad \forall i, j \in I.$$

It is called an *orthonormal basis* if $\text{span}\{u_i\}_{i \in I}$ denotes the linear span of $\{u_i\}_{i \in I}$, the space of finite linear combinations of the vectors u_i .

Definition 14.3. A Hilbert space \mathcal{H} is said to be *separable* if \mathcal{H} contains a countable dense subset $G \subset \mathcal{H}$.

Theorem 14.4. Every Hilbert space \mathcal{H} has an orthonormal basis. If \mathcal{H} is separable, then there is a countable orthonormal basis.

Proposition 14.5. Assume $\{u_i\}_{i \in I}$ is an orthonormal system in a Hilbert space H . Take $u \in \mathcal{H}$. Then

(i) Bessel's inequality: $\sum_{i \in I} |(u, u_i)|^2 \leq \|u\|^2$, in particular, $\{i : (u, u_i) \neq 0\}$ is countable.

(ii) Parseval's identity: If $\{u_i\}_{i \in I}$ is an orthonormal basis, then $\sum_{i \in I} |(u, u_i)|^2 = \|u\|^2$.

If $(u_i)_{i \in I}$ is an orthonormal basis, then the numbers (u, u_i) are called the **Fourier coefficients** of u with respect to $(u_i)_{i \in I}$. The Parseval identity then suggests that u is determined by its Fourier coefficients. This is true, and even more, we have:

Proposition 14.6. Assume $(u_i)_{i \in I}$ is an orthonormal basis in a Hilbert space \mathcal{H} . Then for every vector $(c_i)_{i \in I} \in l^2(I)$ there is a unique vector $u \in \mathcal{H}$ with Fourier coefficients c_i , and we write

$$u = \sum_{i \in I} c_i u_i.$$

Remark. Equivalently, the element $u = \sum_{i \in I} c_i u_i$ can be described as the unique element in \mathcal{H} s.t. $\forall \epsilon > 0$ there is a finite $F_0 \subset I$ s.t. $\|u - \sum_{i \in F} c_i u_i\| < \epsilon \forall$ finite $F \supset F_0$.

Corollary 14.7. We have a linear isomorphism $U : l^2(I) \xrightarrow{\sim} \mathcal{H}$, $U((c_i)_{i \in I}) = \sum_{i \in I} c_i u_i$. By Parseval's identity this isomorphism is isometric, that is, $\|Ux\| = \|x\| \forall x \in l^2(I)$. By the polarization identity this is equivalent to

$$(Ux, Uy) = (x, y) \forall x, y \in l^2(I).$$

Therefor U is unitary.

Corollary 14.8. Up to a unitary isomorphism, there is only one infinite dimensional separable Hilbert space, namely, l^2 .

15 Dual spaces (26, [Schilling(2017)])

Given two orthonormal bases $(u_i)_{i \in I}$ and $(v_i)_{i \in I}$ in a Hilbert space \mathcal{H} , we can decompose

$$u_i = \sum_{j \in I} (u_i, v_j) v_j$$

and using that the sets $\{j : (u_i, v_j) \neq 0\}$ are countable prove the following:

Claim: Any two orthonormal bases in a Hilbert space have the same cardinality.

Example 15.1 (classical Fourier series). Consider $\mathcal{H} = L^2(0, 2\bar{\mu}) = L^2((0, 2\bar{\mu}), d\lambda)$. For $n \in \mathbb{Z}$, define $e_n(t) = \frac{1}{\sqrt{2\bar{\mu}}} e^{int}$. By a version of Weierstrass' theorem it is known that $\text{span}\{e_n\}_{n \in \mathbb{Z}}$ is dense in the supremum-norm in

$$\{f \in C[0, 2\bar{\mu}] : f(0) = f(2\bar{\mu})\}.$$

As $C[0, 2\bar{\mu}]$ is dense in $L^2(0, 2\bar{\mu})$, from this one can deduce that $\text{span}\{e_n\}_{n \in \mathbb{Z}}$ is dense in $L^2(0, 2\bar{\mu})$. We then see that $(e_n)_{n \in \mathbb{Z}}$ is an orthonormal basis in $L^2(0, 2\bar{\mu})$. We therefor have a unitary isomorphism

$$l^2(\mathbb{Z}) \xrightarrow{\sim} L^2(0, 2\bar{\mu}), (c_n)_{n \in \mathbb{Z}} \mapsto \sum_{n \in \mathbb{Z}} c_n e_n.$$

The Fourier coefficients at $f \in L^2(0, 2\bar{\mu})$ are denoted by $\hat{f}(n)$, so

$$\hat{f}(n) = \frac{1}{\sqrt{2\bar{\mu}}} \int_0^{2\bar{\mu}} f(t) e^{-int} dt,$$

more practically

$$\hat{f}(n) = \frac{1}{\sqrt{2\bar{\mu}}} \int_{[0, 2\bar{\mu})} f(t) e^{-int} d\lambda(t).$$

Therefor we have $f = \sum_{n \in \mathbb{Z}} \hat{f}(n) e_n$ in $L^2(0, 2\bar{\mu})$.

Fact: For every $f \in L^2(0, 2\bar{\mu})$, we have $\frac{1}{\sqrt{2\bar{\mu}}} \sum_{n=-N}^N \hat{f}(n) e^{int} \xrightarrow[N \rightarrow \infty]{} f(t)$ for a.e. t .

Lemma 15.2. Assume V is a normed space over $K = \mathbb{R}$ or $K = \mathbb{C}$. Consider a linear functional $f : V \rightarrow K$. The following are equivalent (TFAE):

(i) f is continuous;

(ii) f is continuous at 0;

(iii) There is a $c \geq 0$ s.t. $|f(x)| \leq c\|x\| \forall x \in V$.

If (i)-(iii) are satisfied, then f is called a *bounded linear functional*. The constant c in (iii) is denoted by $\|f\|$. We have $\|f\| = \sup_{x \neq 0} \frac{|f(x)|}{\|x\|} = \sup_{\|x\|=1} |f(x)| = \sup_{\|x\| \leq 1} |f(x)|$.

Proposition 15.3. For every normed vector space V over $K = \mathbb{R}$ or $K = \mathbb{C}$, the bounded linear functionals on V form a Banach space V^* .

Remark. The sequence $\{\|f_n - f_m\|\}_{m=1}^\infty$ actually converges, since

$$\left| \|f_n - f_m\| \right| \leq \|f_m - f_n\|.$$

When we study/use normed spaces, it is often important to understand the dual spaces. For Hilbert spaces this is particularly easy:

Theorem 15.4 (Riesz). Assume \mathcal{H} is a Hilbert space. Then every $f \in \mathcal{H}^*$ has the form

$$f(x) = (x, y),$$

for a uniquely defined $y \in \mathcal{H}$. Moreover, we have $\|f\| = \|y\|$.

For every Hilbert space \mathcal{H} we can define the *conjugate Hilbert space* $\bar{\mathcal{H}}$, which has its elements as the symbols \bar{x} for $x \in \mathcal{H}$, with the linear structure and inner product defined by

$$\bar{x} + \bar{y} = \overline{x + y}, c \cdot \bar{x} = \overline{c x}, (\bar{x}, \bar{y}) = \overline{(x, y)} = (y, x).$$

Corollary 15.5. For every Hilbert space \mathcal{H} , we have an isometric isomorphism $\bar{\mathcal{H}} \xrightarrow{\sim} \mathcal{H}^*$, $\bar{x} \mapsto (\cdot, x)$.

16 Hahn-Banach Theorem (4.2, [Teschl(2010)])

Theorem 16.1 (Hahn-Banach). Assume V is a real vector space, $V_0 \subset V$ a subspace, $e : V \rightarrow \mathbb{R}$ a convex function and $f : V_0 \rightarrow \mathbb{R}$ a linear functional s.t. $f \leq e$ on V_0 . Then f can be extended to a linear functional F on V s.t. $F \leq e$.

Assume first that $V = V_0 + \mathbb{R}x$ for some $x \in V \setminus V_0$. To define F we need to specify $F(x)$. The condition $F \leq e$ means that we need

$$F(y \pm tx) \leq e(y \pm tx) \quad \forall y \in V_0, t \geq 0,$$

that is

$$\pm tF(x) + f(y) \leq e(y \pm tx).$$

Dividing by t this is equivalent to

$$\begin{cases} F(x) \leq \frac{e(y+tx) - f(y)}{t}, \\ F(x) \geq \frac{f(y) - e(y-tx)}{t}, \end{cases}$$

$\forall y \in V_0, t \geq 0$. To show that there is a number satisfying these inequalities, we need to check that

$$\sup_{\substack{y \in V_0 \\ t > 0}} \frac{f(y) - e(y - tx)}{t} \leq \inf_{\substack{y \in V_0 \\ t > 0}} \frac{e(y + tx) - f(y)}{t}.$$

Then as $F(x)$ we can take any number in the interval $[\sup, \inf]$. In other words, we need to check that

$$\frac{f(y) - e(y - tx)}{t} \leq \frac{e(z + sx) - f(z)}{s},$$

$\forall y, z \in V_0, t, s \geq 0$. Equivalently,

$$sf(y) + tf(z) \leq se(y - tx) + te(z + sx).$$

We have

$$\begin{aligned} \frac{sf(y) + tf(z)}{s+t} &= f\left(\frac{s}{s+t}y + \frac{t}{s+t}z\right) \\ &\leq e\left(\frac{s}{s+t}y + \frac{t}{s+t}z\right) \\ &\leq e\left(\frac{s}{s+t}(y - tx) + \frac{t}{s+t}(z + sx)\right) \\ &\leq \frac{s}{s+t}e(y - tx) + \frac{t}{s+t}e(z + sx), \end{aligned}$$

which is what we need.

The general case is deduced from this using transitive induction (Zorn's lemma), as follows. Consider the set X of pairs (W, F) , where $W \subset V$ is a subspace, $F : W \rightarrow \mathbb{R}$ a linear functional, $V_0 \subset W$ and $F|_{V_0} = f$. Define a partial order on X by $F \leq e$ on W .

$$(W_1, F_1) \leq (W_2, F_2) \text{ iff } W_1 \subset W_2 \text{ and } F_1 = F_2|_{W_1}.$$

If C is a chain in X , then it has an upper bound (W, F) defined by

$$W = \bigcup_{(Y, \rho) \in C} Y, F(y) = \rho(y) \text{ if } y \in Y, (Y, \rho) \in C.$$

Hence, X has a maximal element (W, F) by Zorn's lemma. If $W \neq V$, then we can take $x \in V \setminus W$ and extend F to $W + \mathbb{R}x$ preserving the inequality $F \leq e$. This contradicts maximality of W, F in X . Hence, $W = V$.

Theorem 16.2 (Hahn-Banach). Assume V is a real or complex vector space, p a seminorm on V_0 , $V_0 \subset V$, and f a linear functional on V_0 s.t.

$$|f(x)| \leq p(x) \quad \forall x \in V_0.$$

Then f can be extended to a linear functional F on V s.t. $|F(x)| \leq p(x) \quad \forall x \in V$.

Corollary 16.3. Assume V is a normed real or complex vector space, $V_0 \subset V$ and $f \in V_0^*$. Then there is a $F \in V^*$ s.t.

$$F|_{V_0} = f \text{ and } \|F\| = \|f\|.$$

Proof. We apply the previous theorem to $p(x) = \|f\| \cdot \|x\|$. \square

Corollary 16.4. Assume V is a normed space and $x \in V, x \neq 0$. Then there is a $F \in V^*$ s.t. $\|F\| = 1$ and $F(x) = \|x\|$.

Such an F is called a *supporting functional* at x .

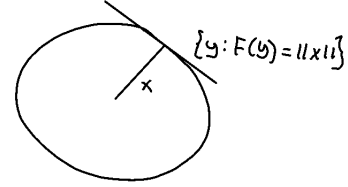


Figure 1: Tangent space?

If V is a normed vector space, then every $x \in X$ defines a bounded linear functional on V^* by

$$V^* \ni f \mapsto f(x).$$

As $|f(x)| \leq \|f\| \cdot \|x\|$, this functional has norm $\leq \|x\|$. By using a supporting functional at x , we actually see that we get norm $\|x\|$. Thus, we have an isometric embedding $V \subset V^{**} := (V^*)^*$. We can therefore see V as a subspace of V^{**} .

Definition 16.5. A normed space V is called reflexive if $V^{**} = V$.

Remark. This is stronger than requiring $V \simeq V^{**}$.

Remark. Every Hilbert space \mathcal{H} is reflexive. Indeed, $\mathcal{H}^* = \bar{\mathcal{H}}$. By Riesz' theorem every bounded linear functional f on $\bar{\mathcal{H}}$ has the form

$$f(\bar{x}) = (\bar{x}, \bar{y}) = (y, x),$$

for some $y \in \mathcal{H}$, which exactly means that $f = y$ in \mathcal{H}^{**} .

As we will see later, the spaces $\mathcal{L}^p(X, d\mu)$, with μ σ -finite and $1 < p < \infty$, are reflexive. The spaces $\mathcal{L}^1(X, d\mu)$ and $\mathcal{L}^\infty(X, \mu)$ are usually not reflexive.

17 Radon-Nikodym Theorem (20, [Schilling(2017)])

Assume (X, \mathcal{B}, μ) is a measure space. Are there other measures on (X, \mathcal{B}) ?

Example 17.1. Take a measurable function $f : X \rightarrow [0, +\infty]$ and define

$$\nu(A) := \int_A f d\mu \text{ for } A \in \mathcal{B}.$$

This is a measure by the monotone convergence theorem. We write $d\nu = f d\mu$. Furthermore, we say that f is the **Radon-Nikodym derivative**, and we denote it by $f = d\nu/d\mu$. If $\mu = \lambda^1$ we get $f(x) = d\nu(x)/dx$.

Proposition 17.2. Assume (X, \mathcal{B}) is a measurable space, μ and ν are σ -finite measures on (X, \mathcal{B}) . Then there exist $N \in \mathcal{B}$ and a measurable $f : X \rightarrow [0, +\infty]$ s.t. $\mu(N) = 0$ and $\nu(A) = \nu(A \cap N) + \int_A f d\mu \forall A \in \mathcal{B}$.

Proof. Assume first that the measure ν, μ are finite. Consider the measure $\eta = \mu + \nu$ and define a linear functional ρ on $L^2(X, d\eta)$ by

$$\rho(g) = \int_X g d\nu.$$

It is well-defined and bounded, since

$$\begin{aligned} |\rho(g)| &\leq \underbrace{\int_X |g| d\nu}_{\|g\|_1} \stackrel{C.S.}{\leq} \left(\int_X |g|^2 d\nu \right)^{\frac{1}{2}} \nu(X)^{\frac{1}{2}} \\ &\leq \nu(X)^{\frac{1}{2}} \left(\int_X |g|^2 d\eta \right)^{\frac{1}{2}}. \end{aligned}$$

By Riesz' theorem there exists $h \in L^2(X, d\eta)$ s.t. $\rho(g) = \int_X g h d\eta$ for all $g \in L^2(X, d\eta)$.

For $g = \mathbb{1}_A$ we get

$$\nu(A) = \int_A h d\eta \forall A \in \mathcal{B}.$$

In particular, $h \geq 0$ (η -a.e.). As $\nu(A) \leq \eta(A)$, we also have $h \leq 1$ (η -a.e.). From now on we view h as a function on X and assume $0 \leq h \leq 1$.

For $g = \mathbb{1}_A$ ($A \in \mathcal{B}$) we have

$$\int_X g d\nu = \rho(g) = \int_X g h d\eta = \int_X g h d\mu + \int_X g h d\nu,$$

hence

$$\int_X g(1-h) d\nu = \int_X g h d\mu. \quad (26)$$

By extending eq. 26 to positive simple functions and then using the monotone convergence theorem, we conclude that eq. 26 holds for all measurable $g : X \rightarrow [0, +\infty]$.

We now let

$$N = \{x : h(x) = 1\} \text{ and } f = \frac{h}{1-h} \mathbb{1}_{N^c}.$$

Letting $g = \mathbb{1}_N$ in eq. 26 we get $0 = \mu(N)$. For $A \in \mathcal{B}$, letting $g = \frac{\mathbb{1}_{A \cap N^c}}{1-h}$ in eq. 26 we get

$$\nu(A \cap N^c) = \int_X \frac{\mathbb{1}_A \mathbb{1}_{N^c}}{1-h} h d\mu = \int_X \mathbb{1}_A f d\mu = \int_A f d\mu.$$

Thus

$$\nu(A) = \nu(A \cap N) + \nu(A \cap N^c) = \nu(A \cap N) + \int_A f d\mu.$$

This finishes the proof for finite μ and ν .

If μ and ν are σ -finite, we can write X as disjoint unions $X = \cup_{n \in \mathbb{N}} X_n = \cup_{m \in \mathbb{N}} Y_m$ with $X_n, Y_m \in \mathcal{B}, \mu(X_n) < \infty, \nu(Y_m) < \infty$. Applying the first part of the proof to $X_n \cap Y_m$, we find $N_{nm} \in \mathcal{B}, N_{nm} \subset X_n \cap Y_m$, and measurable $f_{nm} : X_n \cap Y_m \rightarrow [0, +\infty]$ s.t. $\mu(N_{nm}) = 0$ and

$$\nu(A \cap X_n \cap Y_m) = \nu(A \cap N_{nm}) + \int_{A \cap X_n \cap Y_m} f_{nm} d\mu.$$

We then put $N = \cup_{n,m \in \mathbb{N}} N_{nm}$ and define $f : X \rightarrow [0, +\infty]$ by letting $f = f_{nm}$ on $X_n \cap Y_m$. \square

When can we discard the term $\nu(A \cap N)$?

Definition 17.3. Given measure μ and ν on X, \mathcal{B} , we say that ν is *absolutely continuous* with respect to μ and write $\nu \ll \mu$, if $\nu(A) = 0$ whenever $A \in \mathcal{B}, \mu(A) = 0$.

Lemma 17.4. Assume μ and ν are measures on (X, \mathcal{B}) , $\nu(X) < \infty$. Then $\nu \ll \mu$ iff $\forall \epsilon > 0 \exists \delta > 0$ s.t. if $A \in \mathcal{B}, \mu(A) < \delta$, then $\nu(A) < \epsilon$.

Proof. " \Rightarrow ": obvious. " \Leftarrow ": Assume this is not true. Then, there is a $\epsilon > 0$ s.t. $\forall \delta > 0$ we can find $A \in \mathcal{B}$ satisfying $\mu(A) < \delta, \nu(A) \geq \epsilon$. Let A_n be such a set A for $\delta = 1/2^n$. Put $A = \cap_{n \in \mathbb{N}} \cup_{k=n} A_k$. Then

$$\begin{aligned} \mu(A) &\leq \lim_{n \rightarrow \infty} \mu \left(\bigcup_{k=n}^{\infty} A_k \right) \leq \lim_{n \rightarrow \infty} \sum_{k=n}^{\infty} \mu(A_k) \\ &\leq \lim_{n \rightarrow \infty} \sum_{k=n}^{\infty} \frac{1}{2^k} = \lim_{n \rightarrow \infty} \frac{1}{2^{n-1}} = 0. \end{aligned}$$

As $\nu(X) < \infty$, we also have

$$\nu(A) = \lim_{n \rightarrow \infty} \nu \left(\bigcup_{k=n}^{\infty} A_k \right) \geq \epsilon.$$

This contradicts the assumption $\nu \ll \mu$. \square

Remark. The result is not true for infinite ν .

Theorem 17.5 (Radon-Nikodym). Assume μ and ν are σ -finite measures on a measurable space (X, \mathcal{B}) , $\nu \ll \mu$. Then there is a measurable function $f : X \rightarrow [0, +\infty]$ s.t. $d\nu = f d\mu$ (that is, $\nu(A) = \int_A f d\mu$). If \tilde{f} is another function with the same properties, then $f = \tilde{f}$ μ -a.e.

The function is called the Radon-Nikodym derivative at ν w.r.t. μ and is denoted by $\frac{d\nu}{d\mu}$.

Proof. By the proposition above, we can find $N \in \mathcal{B}$ and $f : X \rightarrow [0, +\infty]$ s.t. $\mu(N) = 0$ and

$$\nu(A) = \nu(A \cap N) + \int_A f d\mu.$$

As $\nu(A \cap N) = 0$ by the assumption $\nu \ll \mu$, this proves the existence of f .

Assume we have another \tilde{f} . Then

$$\nu(A) = \int_A f d\mu = \int_A \tilde{f} d\mu \quad \forall A \in \mathcal{B}.$$

If $B \in \mathcal{B}$, $\nu(B) < \infty$, then consider

$$A_1 = \{x \in \mathcal{B} : f(x) > \tilde{f}(x)\}, A_2 = \{x \in \mathcal{B} : f(x) < \tilde{f}(x)\}.$$

Then

$$\int_{A_1} (f - \tilde{f}) d\mu = 0 \text{ and } \int_{A_2} (\tilde{f} - f) d\mu = 0,$$

hence $\mu(A_1) = \mu(A_2) = 0$. Therefore, $f = \tilde{f} \mu - \text{a.e. on } \mathcal{B}$. As ν is σ -finite, we have $X = \cup_{n \in \mathbb{N}} B_n$, $\nu(B_n) < \infty$. Then $f = \tilde{f} \mu - \text{a.e. on } B_n$ for all n , hence $f = \tilde{f} \mu - \text{a.e. on } X$. \square

Example 17.6. Consider a real-valued function $f \in C'[a, b]$ s.t. $f'(t) > 0 \quad \forall t \in [a, b]$. Let $c = f(a), d = f(b)$. We know that for every Riemann integrable function g on $[c, d]$ we have

$$\int_c^d g(f) dt = \int_a^b g(f(t)) f'(t) dt.$$

Equivalently,

$$\int_c^d g \circ g^{-1} dt = \int_a^b g f'(t) dt. \quad (27)$$

Denote by $\lambda_{[a,b]}, \lambda_{[c,d]}$ the Lebesgue measure restricted to the Borel subsets of $[a, b]$ and $[c, d]$, respectively. Then eq. 27 implies that

$$d((f^{-1})_* \lambda_{[c,d]}) = f' d\lambda_{[a,b]},$$

since the integration of $g = \mathbf{1}_{[\alpha, \beta]}$ gives the same results for any interval $[\alpha, \beta] \subset [a, b]$ and since a finite Borel measure on $[a, b]$ is determined by its values on such intervals. Thus, $(f^{-1})_* \lambda_{[c,d]} \ll \lambda_{[a,b]}$ and

$$\frac{d((f^{-1})_* \lambda_{[c,d]})}{d\lambda_{[a,b]}} = f'.$$

18 Complex and Signed Measures (4.3, [Teschl(2010)])

Definition 18.1. A **complex measure** on (X, \mathcal{B}) is a map $\nu : \mathcal{B} \rightarrow \mathbb{C}$ s.t. $\nu(\emptyset) = 0$ and

$$\nu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \nu(A_n)$$

for any disjoint $A_n \in \mathcal{B}$, where the series is assumed to be absolutely convergent. If ν takes values in \mathbb{R} then ν is called a **finite signed measure**.

Remark. More generally, a signed measure is allowed to take values in $\mathbb{R} \cup \{+\infty\}$ or $\mathbb{R} \cup \{-\infty\}$.

Given a complex measure ν on (X, \mathcal{B}) , its **total variation** is the map $|\nu| : \mathcal{B} \rightarrow [0, +\infty]$ defined by

$$|\nu|(A) = \sup \left\{ \sum_{n=1}^N |\nu(A_n)| : A = \bigcup_{n=1}^N A_n, A_n \in \mathcal{B}, A_n \cap A_m = \emptyset \right\}.$$

Proposition 18.2. $|\nu|$ is a finite measure on (X, \mathcal{B}) .

Proof. Let us first show that $|\nu|$ is a measure. As $\nu(\emptyset) = 0$, we have $|\nu|(\emptyset) = 0$. For σ -additivity, take $A = \cup_{n \in \mathbb{N}} A_n$, $A_n \in \mathcal{B}$, $A_n \cap A_m = \emptyset$. If $A = \cup_{k=1}^N B_n$, $B_k \in \mathcal{B}$, $B_k \cap B_l = \emptyset$ then

$$\begin{aligned} \sum_{k=1}^N |\nu(B_k)| &= \sum_{k=1}^N \left| \sum_{n=1}^{\infty} \nu(B_k \cap A_n) \right| \\ &\leq \sum_{n=1}^{\infty} \sum_{k=1}^N |\nu(B_k \cap A_n)| \\ &\leq \sum_{n=1}^{\infty} |\nu|(A_n). \end{aligned}$$

Taking the supremum over all such decompositions $A_n \in \mathcal{B}$ we get

$$|\nu|(A) \leq \sum_{n=1}^{\infty} |\nu|(A_n).$$

To prove the opposite inequality we may assume $|\nu|(A) < \infty$. Then $|\nu|(A_n) \leq |\nu|(A) < \infty$. It suffices to show that $\sum_{n=1}^N |\nu|(A_n) \leq |\nu|(A)$.

Fix $\epsilon > 0$ and choose decompositions $A_n = \cup_{k=1}^{M_n} B_{nk}$, $B_{nk} \in \mathcal{B}$, $B_{nk} \cap B_{nl} = \emptyset$ s.t. $\sum_{k=1}^{M_n} |\nu(B_{nk})| > |\nu|(A_n) - \epsilon$. Then

$$\begin{aligned} |\nu|(A) &\geq \sum_{n=1}^N \sum_{k=1}^{M_n} |\nu(B_{nk})| + \left| \nu \left(A \setminus \bigcup_{n=1}^N A_n \right) \right| \\ &> \sum_{n=1}^N (|\nu|(A_n) - \epsilon) = \sum_{n=1}^N |\nu|(A_n) - N\epsilon. \end{aligned}$$

As $\epsilon > 0$ was arbitrary, we conclude that $|\nu|(A) \geq \sum_{n=1}^N |\nu|(A_n)$.

It remains to check that $|\nu|(X) < \infty$. As $|\nu|(X) \leq |\text{Re } \nu|(X) + |\text{Im } \nu|(X)$, we can consider $\text{Re } \nu$ and $\text{Im } \nu$ separately. In other words, it is enough to consider finite signed measures.

Assume $|\nu|(X) = \pm\infty$. We have that there exists $A, B \in \mathcal{B}$ s.t.

$$X = A \cup B, \quad A \cap B = \emptyset, \quad |\nu|(A) = +\infty, \quad |\nu|(B) \geq 1.$$

To see this, find a decomposition

$$X = \bigcup_{n=1}^N A_n, \quad A_n \in \mathcal{B}, \quad A_n \cap A_m = \emptyset$$

s.t.

$$\sum_{n=1}^N |\nu(A_n)| \geq |\nu(X)| + 2 = \left| \sum_{n=1}^N \nu(A_n) \right| + 2.$$

Consider $I := \{i : 1 \leq i \leq N, \nu(A_i) \geq 0\}$ and $J := \{i : 1 \leq i \leq N, \nu(A_i) < 0\}$. We then have

$$\sum_{k \in I} \nu(A_k) - \sum_{k \in J} \nu(A_k) \geq \left| \sum_{k \in I} \nu(A_k) - \sum_{k \in J} \nu(A_k) \right| + 2,$$

which implies that

$$\sum_{k \in I} \nu(A_k) \geq 1 \text{ and } \sum_{k \in J} \nu(A_k) \leq -1.$$

Let $B_0 = \cup_{k \in I} A_k$, $B_1 = \cup_{k \in J} A_k$. Then

$$X = B_0 \cup B_1, \quad B_0 \cap B_1 = \emptyset, \quad |\nu(B_0)| \geq 1, \quad |\nu(B_1)| \geq 1.$$

As $|\nu|(X) = |\nu|(B_0) + |\nu|(B_1)$, we must have $|\nu|(B_0) = +\infty$ or $|\nu|(B_1) = +\infty$. If $|\nu|(B_0) = +\infty$, we let $A = B_0$, $B = B_1$. If $|\nu|(B_0) < \infty$, we let $A = B_1$, $B = B_0$. This proves the claim.

We can apply the claim to the restriction of ν to A , and so on. Therefor by induction we can construct disjoint measurable sets B_1, B_2, \dots s.t.

$$|\nu|(B_n) \geq 1 \quad \forall n \text{ and } |\nu|\left(X \setminus \bigcup_{n=1}^N B_n\right) = +\infty \quad \forall N.$$

But then $\nu(\cup_{n \in \mathbb{N}} B_n) = \sum_{n \in \mathbb{N}} \nu(B_n)$, which does not make sense, as the series is not absolutely convergent. Hence $|\nu|(X) < \infty$. \square

Example 18.3. Consider a measure space (X, \mathcal{B}, μ) and take $f \in L^1(X, d\mu)$. Define

$$\nu(A) = \int_A f d\mu.$$

Then ν is a complex measure on (X, \mathcal{B}) , sinche this is true for $f \geq 0$ and a general f can be written as a linear combination of positive ones. We write $d\nu = f d\mu$.

We then have $d|\nu| = |f| d\mu$, that is,

$$|\nu|(A) = \int_A |f| d\mu.$$

Proof. To see this, consider the measure ω defined by $d\omega = |f| d\mu$. We want to show that $|\nu| = \omega$.

If $A = \cup_{n=1}^N A_n$, $A_n \cap A_m = \emptyset$, then

$$\begin{aligned} \sum_{n=1}^N |\nu(A_n)| &= \sum_{n=1}^N \left| \int_{A_n} f d\mu \right| \leq \sum_{n=1}^N \int_{A_n} |f| d\mu \\ &= \sum_{n=1}^N \omega(A_n) = \omega(A). \end{aligned}$$

Therefore, $|\nu| \leq \omega$.

To prove the equality, assume first that f is simple, so $f = \sum_{n=1}^N c_n \mathbb{1}_{A_n}$, $c_n \in \mathbb{C}$, $A_n \in \mathcal{B}$, $A_n \cap A_m = \emptyset$. Then, for every $A \in \mathcal{B}$,

$$\begin{aligned} |\nu|(A) &\geq \sum_{n=1}^N |\nu(A \cap A_n)| = \sum_{n=1}^N |c_n| \mu(A \cap A_n) \\ &= \sum_{n=1}^N \int_{A \cap A_n} |f| d\mu = \int_A |f| d\mu = \omega(A). \end{aligned}$$

Thus, $|\nu| \geq \omega$, hence $|\nu| = \omega$.

For general f , choose simple functions $f_n \in L^1(X, d\mu)$ s.t. $\|f - f_n\|_1 \xrightarrow{n \rightarrow \infty} 0$. Consider the corresponding complex measures ν_n , $d\nu_n = f_n d\mu$. Fix $A \in \mathcal{B}$. For every $B \in \mathcal{B}$, we have

$$|\nu(B)| \leq |(\nu - \nu_n)(B)| + |\nu_n(B)|.$$

This implies that

$$|\nu(A)| \leq |\nu - \nu_n|(A) + |\nu_n(A)|.$$

Similarly, $|\nu_n(A)| \leq |\nu - \nu_n|(A) + |\nu|(A)$. Therefore,

$$\begin{aligned} \left| |\nu|(A) - |\nu_n|(A) \right| &\leq |\nu - \nu_n|(A) \leq \int_A |f - f_n| d\mu \\ &\leq \|f - f_n\|_1 \xrightarrow{n \rightarrow \infty} 0. \end{aligned}$$

It follows that

$$|\nu|(A) = \lim_{n \rightarrow \infty} |\nu_n|(A) = \lim_{n \rightarrow \infty} \int_A |f_n| d\mu = \int_A |f| d\mu$$

(since $\|f\| - \|f_n\| \leq \|f - f_n\|$.) This completes the proof. \square

Definition 18.4. If (X, \mathcal{B}, μ) is a measure space, ν is a complex measure on (X, \mathcal{B}) , then we say that ν is **absolutely continuous** w.r.t. μ and write $\nu \ll \mu$, if $\nu(A) = 0$ whenever $A \in \mathcal{B}, \mu(A) = 0$. Equivalently, $|\nu| \ll \mu$.

Theorem 18.5 (Radon-Nikodym theorem for complex measures). Assume (X, \mathcal{B}, μ) is a measure space, ν is a complex measure on (X, \mathcal{B}) , $\nu \ll \mu$. Then there is a unique $f \in L^1(X, d\mu)$ s.t. $d\nu = f d\mu$.

Proof. Existence: By considering separately $\operatorname{Re} \nu$ and $\operatorname{Im} \nu$, we may assume that ν is a finite signed measure. Then

$$\nu = \nu_+ - \nu_-, \text{ where } \nu_{\pm} = \frac{|\nu| \pm \nu}{2}$$

are positive measures, since $|\nu(A)| \leq |\nu|(A)$. Clearly, $\nu_{\pm} \ll \mu$. Therefor the proof reduces to the case when ν is positive, in which case we already know the result: we take $f = d\nu/d\mu$; note that $\int_X f d\mu = \nu(X) < \infty$, so $f \in L^1(X, d\mu)$.

Uniqueness: It suffices to show that if $\nu = 0$ and $d\nu = f d\mu$, then $f = 0$ (μ -a.e.). This is true, for example, because $\int_X |f| d\mu = |\nu|(X) = 0$. \square

19 Decomposition Theorems (20, [Schilling(2017)] and 4.3, [Teschl(2010)])

Definition 19.1. Two measures ν and μ on a measurable space (X, \mathcal{B}) are called **mutually singular**, or we say that ν is **singular** w.r.t. μ , if there is a $N \in \mathcal{B}$ s.t. $\nu(N^c) = 0$, $\mu(N) = 0$. We then write $\nu \perp \mu$.

Theorem 19.2 (Lebesgue Decomposition Theorem). Assume ν, μ are σ -finite measures in (X, \mathcal{B}) . Then there exist unique measures ν_a and ν_s s.t. $\nu = \nu_a + \nu_s$, $\nu_a \ll \mu$, $\nu_s \perp \mu$.

Proof. As we showed earlier, there exist $N \in \mathcal{B}$ and measurable $f : X \rightarrow [0, +\infty]$ s.t. $\mu(N) = 0$ and

$$\nu(A) = \nu(A \cap N) + \int_A f d\mu \quad \forall A \in \mathcal{B}.$$

We then define

$$\nu_a(A) := \int_A f d\mu, \quad \nu_s(A) = \nu(A \cap N).$$

Assume we have another such decomposition $\nu = \tilde{\nu}_a + \tilde{\nu}_s$. Let $\tilde{N} \in \mathcal{B}$ be s.t. $\mu(\tilde{N}) = 0$ and $\tilde{\nu}_s(\tilde{N}^c) = 0$. Then $\mu(N \cup \tilde{N}) = 0$ and $\nu_s((N \cup \tilde{N})^c) = 0$, $\tilde{\nu}_s((N \cup \tilde{N})^c) = 0$. Hence, for every $A \in \mathcal{B}$,

$$\begin{aligned} \nu_s(A) &= \nu_s\left(A \cap \left(N \cup \tilde{N}\right)\right) \\ &= \nu_s\left(A \cap \left(N \cup \tilde{N}\right)\right) + \nu_a\left(A \cap \left(N \cup \tilde{N}\right)\right) \\ &= \nu\left(A \cap \left(N \cup \tilde{N}\right)\right), \end{aligned}$$

and for the same reasons

$$\tilde{\nu}_s(A) = \nu\left(A \cap \left(N \cup \tilde{N}\right)\right).$$

Thus, $\nu_s = \tilde{\nu}_s$.

In a similar way,

$$\begin{aligned}\nu_a(A) &= \nu_a\left(A \cap \left(N \cup \tilde{N}\right)^c\right) \\ &= \nu_a\left(A \cap \left(N \cup \tilde{N}\right)^c\right) + \nu_s\left(A \cap \left(N \cup \tilde{N}\right)^c\right) \\ &= \nu\left(A \cap \left(N \cup \tilde{N}\right)^c\right), \\ \tilde{\nu}_a(A) &= \nu\left(A \cap \left(N \cup \tilde{N}\right)^c\right).\end{aligned}$$

Hence $\nu_a = \tilde{\nu}_a$. \square

Theorem 19.3 (Polar Decomposition of Complex Measure). *Assume ν is a complex measure on (X, \mathcal{B}) . Then there exist a finite measure μ on (X, \mathcal{B}) and a measurable function $f : X \rightarrow \Pi$ s.t. $d\nu = f d\mu$. If $(\tilde{\mu}, \tilde{f})$ is another such pair, then $\tilde{\mu} = \mu$ and $\tilde{f} = f$ μ -a.e.*

Proof. Take $\mu = |\nu|$. Then $\nu \ll |\nu|$. Hence, by the Radon-Nikodym theorem for complex measures we have $d\nu = f d\mu$ for a unique $f \in L^1(X, d\mu)$. Then $d|\nu| = |f| d\mu = |f| d|\nu|$. Hence, $|f| = 1$ (μ -a.e.). By viewing f as a function, we can therefore assume that $f : X \rightarrow \Pi$.

Assume we have another such decomposition $d\nu = \tilde{f} d\tilde{\mu}$. Then $d|\nu| = |\tilde{f}| d\tilde{\mu} = d\tilde{\mu}$, that is, $\tilde{\mu} = \mu = |\nu|$. Then $d\nu = f d\mu = \tilde{f} d\tilde{\mu}$, hence $f = \tilde{f}$ μ -a.e. \square

For signed measures this leads to the following.

Theorem 19.4 (Hahn Decomposition Theorem). *Assume ν is a finite signed measure on (X, \mathcal{B}) . Then there exist $P, N \in \mathcal{B}$ s.t.*

$$\begin{aligned}X &= P \cup N, \quad P \cap N = \emptyset, \\ \nu(A \cap P) &\geq 0, \quad \nu(A \cap N) \leq 0 \quad \forall A \in \mathcal{B}.\end{aligned}$$

Moreover, then $|\nu|(A) = \nu(A \cap P) - \nu(A \cap N)$, and if $X = \tilde{P} \cup \tilde{N}$ is another such decomposition, then

$$|\nu|(P \Delta \tilde{P}) = |\nu|(N \Delta \tilde{N}) = 0.$$

Proof. Consider the polar decomposition $d\nu = f d|\nu|$, $f : X \rightarrow \Pi$. Since ν is real-valued, we have $\text{Im} f = 0$ $|\nu|$ -a.e. We may therefore assume that f takes values in $\Pi \cap \mathbb{R} = \{-1, 1\}$. Let $P := \{x : f(x) = 1\}$, $N := \{x : f(x) = -1\}$. Then

$$\begin{aligned}\nu(A \cap P) &= \int_{A \cap P} f d|\nu| = \int_{A \cap P} 1 d|\nu| = |\nu|(A \cap P) \geq 0, \\ \nu(A \cap N) &= \int_{A \cap N} f d|\nu| = \int_{A \cap N} (-1) d|\nu| = -|\nu|(A \cap N) \leq 0.\end{aligned}$$

This gives the required decomposition. Moreover, we see that

$$\begin{aligned}\nu(A \cap P) - \nu(A \cap N) &= |\nu|(A \cap P) + |\nu|(A \cap N) \\ &= |\nu|(A).\end{aligned}$$

Assume $X = \tilde{P} \cup \tilde{N}$ another decomposition as in the statement of the theorem. For $A \subset P \setminus \tilde{P}$ ($A \in \mathcal{B}$) we have

$$\begin{aligned}\nu(A) &= \nu(A \cap P) \geq 0, \\ \nu(A) &= \nu(A \cap \tilde{N}) \leq 0.\end{aligned}$$

It follows that $\nu(A) = 0$. Hence, $|\nu|(P \setminus \tilde{P}) = 0$. Similarly, $|\nu|(\tilde{P} \setminus P) = 0$. Hence, $|\nu|(P \Delta \tilde{P}) = |\nu|(P \setminus \tilde{P}) + |\nu|(\tilde{P} \setminus P) = 0$, $|\nu|(N \Delta \tilde{N}) = |\nu|(N \setminus \tilde{N}) + |\nu|(\tilde{N} \setminus N) = 0$. \square

Corollary 19.5 (Jordan Decomposition Theorem). *Assume ν is a finite signed measure on (X, \mathcal{B}) . Then there exist unique finite measures ν_+, ν_- on (X, \mathcal{B}) s.t.*

$$\nu = \nu_+ - \nu_- \quad \text{and} \quad \nu_+ \perp \nu_-.$$

Moreover, then $|\nu| = \nu_+ + \nu_-$, hence

$$\nu_+ = \frac{|\nu| + \nu}{2}, \quad \nu_- = \frac{|\nu| - \nu}{2}.$$

Proof. Consider the Hahn decompositions $X = P \cup N$. Define

$$\nu_+(A) = \nu(A \cap P), \quad \nu_-(A) = -\nu(A \cap N).$$

Then by the properties of the Hahn decomposition we have

$$\nu = \nu_+ - \nu_-, \quad \nu_+, \nu_- \geq 0, \quad |\nu| = \nu_+ + \nu_-,$$

and $\nu_+ \perp \nu_-$ by definition.

Assume we have another such decomposition $\nu = \tilde{\nu}_+ - \tilde{\nu}_-$, $\tilde{\nu}_+ \perp \tilde{\nu}_-$. Let $\tilde{N} \in \mathcal{B}$ be such that $\tilde{\nu}_+(\tilde{N}) = 0$, $\tilde{\nu}_-(\tilde{N}^c) = 0$. Put $\tilde{P} = X \setminus \tilde{N}$. Then $X = \tilde{P} \cup \tilde{N}$ is again a Hahn decomposition. In particular,

$$\begin{aligned}|\nu|(A) &= \nu(A \cap \tilde{P}) - \nu(A \cap \tilde{N}) \\ &= \tilde{\nu}_+(A \cap \tilde{P}) + \tilde{\nu}_-(A \cap \tilde{N}) = \tilde{\nu}_+(A) - \tilde{\nu}_-(A).\end{aligned}$$

Thus $|\nu| = \tilde{\nu}_+ - \tilde{\nu}_-$. It follows that

$$\tilde{\nu}_+ = \frac{|\nu| + \nu}{2} = \nu_+, \quad \tilde{\nu}_- = \frac{|\nu| - \nu}{2} = \nu_-.$$

\square

20 More on Duals of L^p -spaces (21, p. 241, [Schilling(2017)])

- What is the dual of $L^p(X, d\mu)$? When does a measurable function $g : X \rightarrow \mathbb{C}$ define a bounded linear functional on $L^p(X, d\mu)$ by

$$\rho(f) = \int_X f g d\mu?$$

Theorem 20.1 (Young's inequality). *Assume $f : [0, a] \rightarrow [0, b]$ is a strictly increasing continuous function, $f(0) = 0$, $f(a) = b$. Then, for all $s \in [0, a]$ and $t \in [0, b]$, we have*

$$st \leq \int_0^s f(x) dx + \int_0^t f^{-1}(y) dy,$$

and the equality holds iff $t = f(s)$.

With Holder's inequality it follows that every $g \in L^q(X, d\mu)$ defines a bounded linear functional

$$l_g : L^p(X, d\mu) \rightarrow \mathbb{C}, \quad l_g(f) = \int_X f g d\mu,$$

and $\|l_g\| \leq \|g\|_q$.

The same makes sense for $p = 1, q = \infty$ and $p = \infty, q = 1$, when μ is σ -finite, as

$$\int_X |f g| d\mu \leq \int_X |f| d\mu \cdot \|g\|_\infty = \|f\|_1 \cdot \|g\|_\infty.$$

Lemma 20.2. Assume $1 \leq p \leq \infty$, $1/p + 1/q = 1$, and μ is σ -finite if $p = 1$ or $p = \infty$. For $g \in L^q(X, d\mu)$ consider $l_g \in L^p(X, d\mu)^*$. Then

$$\|l_g\| = \|g\|_q.$$

Proof. We may assume $\|g\|_q > 0$, otherwise $l_g = 0$. Consider three cases:

(i) $1 < p < \infty$, so $1 < q < \infty$. We already know that $\|l_g\| \leq \|g\|_q$. To prove the opposite inequality, consider

$$f = \frac{\bar{g}}{|g|} |g|^{q-1}.$$

Then

$$|f|^p = |g|^{(q-1)p} = |g|^{q(1-\frac{1}{q})p} = |g|^q,$$

so $\|f\|_p^p = \|g\|_q^p$. We have

$$l_q(f) = \int_X |g|^q d\mu = \|g\|_q^q.$$

Hence

$$\begin{aligned} \|l_g\| &\geq \frac{|l_g(f)|}{\|f\|_p} = \frac{\|g\|_q^q}{\|g\|_q^{q/p}} = \|g\|_q^{q-q/p} \\ &= \|g\|_q^{q(1-1/p)} = \|g\|_q. \end{aligned}$$

(ii) $p = 1$, so $q = \infty$. Take $0 < c < \|g\|_\infty$. Then $\mu\{x : |g(x)| > c\} > 0$. As μ is σ -finite, so $X = \cup_{n \in \mathbb{N}} X_n$, $\mu(X_n) < \infty$, we can find $A = X_n \cap \{x : |g(x)| > c\}$ s.t. $\mu(A) > 0$ and $\mu(A) < \infty$. Consider $f = \bar{g}/|g| \mathbb{1}_A$. Then $\|f\|_1 = \mu(A)$,

$$l_g(f) = \int_X |g| \mathbb{1}_A d\mu = \int_A |g| d\mu > c\mu(A).$$

Hence, $\|l_g\| \geq |l_g(f)|/\|f\|_1 > c$. It follows that $\|l_g\| \geq \|g\|_\infty$.

(iii) $p = \infty$, so $q = 1$. Define

$$f(x) = \begin{cases} \frac{\bar{g}(x)}{|g(x)|}, & \text{if } g(x) \neq 0, \\ 0, & \text{otherwise.} \end{cases}$$

Then $\|f\|_\infty$ and

$$l_g(f) = \int_X |g| d\mu = \|g\|_1.$$

Hence, $\|l_g\| \geq \|g\|_1$. \square

Therefor we can view $L^q(X, d\mu)$ as a subspace of $L^p(X, d\mu)^*$ using the isometric embedding

$$L^q(X, d\mu) \rightarrow L^p(X, d\mu)^*, \quad g \mapsto l_g.$$

Theorem 20.3. Assume $(X, \mathcal{B}, d\mu)$ is a σ -finite measure space, $1 \leq p < \infty$, $1/p + 1/q = 1$. Then

$$L^p(X, d\mu)^* = L^q(X, d\mu).$$

Remark. This is usually not true for $p = \infty$.

Proof. Proof can be found in lec. notes. \square

21 + 22 Riesz-Markow Theorem (21, p. [243-249],

[Schilling(2017)])

Assume (X, d) is a locally compact metric space.

Definition 21.22 (positive linear functional). A linear functional $\rho : C_c(X) \rightarrow \mathbb{C}$ is called positive if $\rho(f) \geq 0$ for all $f \geq 0$. (Recall $C_c(X)$ is the space of continuous (C) compactly supported (c) functions.)

Theorem 21.23 (Riesz-Markov). If $\rho : C_c(X) \rightarrow \mathbb{C}$ is a positive linear functional, where (X, d) is a locally compact metric space, then there exists a Borel measure μ on X s.t. $\mu(K) < \infty$ for every compact $K \subset X$ and

$$\rho(f) = \int_X f d\mu \quad \forall f \in C_c(X).$$

If X is separable, then such a measure μ is unique.

For the proof we need two auxiliary results.

Lemma 21.24 (Urysohn's Lemma). Assume (X, d) is a metric space, $A, B \subset X$ are disjoint closed subsets. Then there exists a continuous function $f : X \rightarrow [0, 1]$ s.t. $f \equiv 1$ on A and $f \equiv 0$ on B .

Proof. Define,

$$f(x) = \frac{d(x, B)}{d(x, A) + d(x, B)}.$$

Note that this is well-defined, since $d(x, A) + d(x, B) > 0$ for all x , as $\{x : d(x, A) = d(x, B) = 0\} = A \cap B = \emptyset$. \square

Lemma 21.25. Assume (X, d) is a compact metric space, $U = (U_i)_{i=1}^n$ is a finite open cover of X (so U_i are open and $\cup_{i=1}^n U_i = X$). Then there exist functions ρ_1, \dots, ρ_n in $C(X)$ s.t.

$$0 \leq \rho_i \leq 1, \quad \text{supp}(\rho_i) \subset U_i, \quad \sum_{i=1}^n \rho_i(x) = 1 \quad \forall x.$$

Every such collection of functions is called a **partition of unity subordinate** to U .

Proof. Let us show first that there exists an open set V_1 s.t. $\bar{V}_1 \subset U_1$ and $X = V_1 \cup U_2 \cup \dots \cup U_n$. For this, consider the closed set $K = X \setminus (U_2 \cup \dots \cup U_n) \subset U_1$. For every $x \in K$ we can find a ball $B_{r_x}(x)$ ($r_x > 0$) s.t. $B_{r_x}(x) \subset U_1$. As K is compact and $(B_{r_x}(x))_{x \in K}$ is an open cover of K , we can find a finite subcover $(B_{r_{x_k}}(x_k))_{k=1}^m$. Then put $V_1 = \cup_{k=1}^m B_{r_{x_k}}(x_k)$. Using this construction $2n$ times we can find open sets V'_1, \dots, V'_n and V''_1, \dots, V''_n s.t. $\bar{V}'_i \subset V''_i$, $\bar{V}''_i \subset U_i$, $\cup_{i=1}^n V'_i = X$. By Urysohn's lemma, for every i , we can find $f_i \in C(X)$ s.t. $0 \leq f_i \leq 1$, $f_i \equiv 1$ on \bar{V}'_i , $f_i \equiv 0$ on $X \setminus V''_i$. Then $\text{supp}(f_i) \subset \bar{V}'_i \subset U_i$. Define

$$\rho_i(x) = \frac{f_i(x)}{\sum_{j=1}^n f_j(x)}.$$

Step 4. $\phi(f) = \int_X f d\mu \forall f \in C_c(X)$.

It suffices to consider $0 \leq f \leq 1$, as such functions span $C_c(F)$.

Consider $K = \text{supp}(f)$ and define for some N ,

$$K_n = \left\{ x \in K : f(x) \geq \frac{n}{N} \right\} \quad (n = 0, \dots, N), \quad (28)$$

$$f_n(x) = \min \left\{ f(x), \frac{n}{N} \right\}, \quad (n = 0, \dots, N), \quad (29)$$

$$g_n(f) = f_n(x) - f_{n-1}(x), \quad (n = 1, \dots, N). \quad (30)$$

Then $f_0 = 0$, $f_N = f$, so $f = \sum_{n=1}^N g_n$.

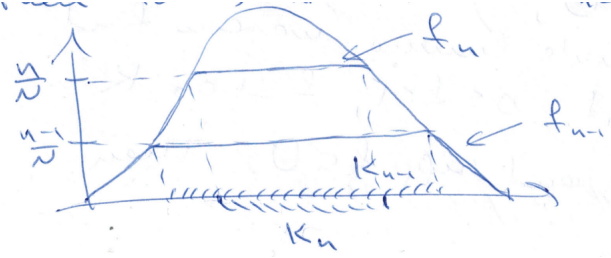


Figure 2: Drawing of the f_n 's defined in eq. (29).

We have

$$\frac{1}{N} \mathbb{1}_{K_n} \leq g_n \leq \frac{1}{N} \mathbb{1}_{K_{n-1}} \quad (n = 1, \dots, N).$$

By integrating we get

$$\frac{1}{N} \mu(K_n) \leq \int_X g_n d\mu \leq \frac{1}{N} \mu(K_{n-1}).$$

On the other hand, by step 3 we have

$$\frac{1}{N} \mu(K_n) \leq \phi(g_n) \leq \frac{1}{N} \mu(K_{n-1}).$$

It follows that both $\int_X f d\mu$ and $\phi(f)$ lie in the interval

$$\left[\frac{1}{N} \sum_{n=1}^N \mu(K_n), \frac{1}{N} \sum_{n=1}^N \mu(K_{n-1}) \right].$$

The length of this interval is

$$\frac{1}{N} (\mu(K_0) - \mu(K_N)) \leq \frac{1}{N} \mu(K_0) = \frac{\mu(K)}{N}.$$

Therefore,

$$\left| \int_X f d\mu - \phi(f) \right| \leq \frac{\mu(K)}{N}.$$

Letting $N \rightarrow \infty$ we conclude that

$$\phi(f) = \int_X f d\mu.$$

Step 5. If X is separable, then μ is uniquely determined by ϕ .

Assume μ is a Borel measure s.t. $\mu(K) < \infty \forall$ compact $K \subset X$ and

$$\phi(f) = \int_X f d\mu \quad \forall f \in C_c(X).$$

Recall from the lecture on regularity of measures that then μ is regular (as X is separable, locally compact). In particular,

$$\mu(A) = \inf_{\substack{A \subset U \\ U \text{ open}}} \mu(U) \quad \forall A \in \mathcal{B}(X).$$

Therefore it suffices to show that $\mu(U)$ is determined by ϕ .

If \bar{U} is compact, then $f_n \nearrow \mathbb{1}_U$, where

$$f_n(x) = \frac{nd(x, U^c)}{1 + nd(x, U^c)}.$$

Hence,

$$\mu(U) = \lim_{n \rightarrow \infty} \int_X f_n d\mu = \lim_{n \rightarrow \infty} \phi(f_n).$$

For general U , we can find open U_n s.t. \bar{U}_n are compact and $U_n \uparrow U$ (again, recall the lecture on regularity). Then $\mu(U_n) \nearrow \mu(U)$, so $\mu(U)$ is determined by ϕ . \square

Remark. Without separability, the uniqueness is not always true. It can be checked that the measure we constructed in the proof has the following properties:

(i) $\mu(K) < \infty \forall$ compact $K \subset X$;

(ii) μ is outer regular ($\mu(A) = \inf_{U \text{ open}} \mu(U)$);

(iii) μ is inner regular on open sets (this is where we need the full strength of step 3):

$$\mu(U) = \sup_{\substack{K \subset U \\ K \text{ compact}}} \mu(K) \quad \forall \text{ open } U.$$

Such measures are called **Radon measures**. It can be shown that the uniqueness holds within the class of Radon measures.

Dual of $C(X)$

As an application of the Riesz-Markow Theorem we will describe $C(X)^*$ in terms of measures for compact metric spaces (X, d) .

Denote by $M(X)$ the space of complex Borel measures on X . For every $\nu \in M(X)$ we want to make sense of $\int_X f d\nu$ for $f \in C(X)$. It is enough to consider finite signed measures, as then we can define

$$\int_X f d\nu = \int_X f d(\text{Re} \nu) + i \int_X f d(\text{Im} \nu).$$

So assume ν is a finite signed measure. Then, $\nu = \mu_1 - \mu_2$ for positive measures and we define

$$\int_X f d\nu = \int_X f d\mu_1 - \int_X f d\mu_2.$$

This is well-defined, since if

$$\nu = \mu_1 - \mu_2 = \omega_1 - \omega_2,$$

then $\mu_1 + \omega_2 = \mu_2 + \omega_1$ and

$$\int_X f d\mu_1 + \int_X f d\omega_2 = \int_X f d\mu_2 + \int_X f d\omega_1.$$

Thus, every $\nu \in M(X)$ defines a linear functional

$$\phi_\nu : C(X) \rightarrow \mathbb{C} \text{ by } \phi_\nu(f) = \int_X f d\nu,$$

and the map $\nu \mapsto \phi_\nu$ is linear.

Lemma 22.23. If $\nu \in M(X)$ and $d\nu = g d|\nu|$ is its polar decomposition, then

$$\int_X f d\nu = \int_X f g d|\nu| \quad \forall f \in C(X).$$

Proof. More generally, let us show that if μ is a finite Borel measure, $g \in L^1(X, d\mu)$, and $d\nu = g d\mu$, then

$$\int_X f d\nu = \int_X f g d\mu \quad \forall f \in C(X).$$

It suffices to consider $g \geq 0$. Then ν is positive, so $\int_X f d\nu$ has the usual meaning.

We will then prove an even more general statement:

$$\int_X f d\nu = \int_X f g d\mu \quad \forall f \in L^\infty(X, \mu).$$

As simple functions are dense in $L^\infty(X, \mu)$, it suffices to consider simple f , hence it suffices to consider $f = \mathbb{1}_A$. Then

$$\int_X \mathbb{1}_A d\nu = \nu(A) = \int_A d\mu = \int_X \mathbb{1}_A g d\mu,$$

where the second equality comes from the definition of $d\nu = g d\mu$. \square

Lemma 22.24. For every $\nu \in M(X)$, the linear functional ϕ_ν is bounded and $\|\phi_\nu\| = |\nu|(X)$. (Recall that the norm on $C(X)$ is $\|f\| = \sup_{x \in X} |f(x)|$.)

Proof. We have, with $d\nu = g d|\nu|$ being the polar decomposition,

$$\begin{aligned} |\phi_\nu(f)| &= \left| \int_X f d\nu \right| = \left| \int_X f g d|\nu| \right| \\ &\leq \int_X |f| g d|\nu| \leq \|f\| \int_X g d|\nu| \\ &\leq \|f\| |\nu|(X). \end{aligned}$$

Therefore,

$$\|\phi_\nu\| \leq |\nu|(X).$$

For the opposite inequality, recall that $|\nu|$ is regular and $X(X)$ is dense in $L^1(X, d|\nu|)$. Hence, we can find $f_n \in C(X)$ s.t. $f_n \rightarrow g$ in $L^1(X, d|\nu|)$. Define

$$g_n(x) = \begin{cases} f_n(x), & \text{if } |f_n(x)| \leq 1, \\ \frac{f_n(x)}{|f_n(x)|}, & \text{if } |f_n(x)| > 1. \end{cases}$$

Then $\|g_n\| \leq 1$ and $|g - g_n| \leq |f - f_n|$. Therefore, $\|g - g_n\| \leq \|g - f_n\|_1 \xrightarrow{n \rightarrow \infty} 0$. We have

$$\phi_\nu(\bar{f}_n) = \int_X \bar{g}_n g d|\nu| \rightarrow \int_X d|\nu| = |\nu|(X),$$

since

$$\begin{aligned} \left| \int_X (\bar{g}_n g - 1) d|\nu| \right| &\leq \int_X |\bar{g}_n g - 1| d|\nu| \\ &= \int_X |\bar{g}_n - \bar{g}| d|\nu| = \|g_n - g\|_1 \xrightarrow{n \rightarrow \infty} 0, \end{aligned}$$

where the first equality is because $|g(x)| = 1$. Therefore, $\|\phi_\nu\| \geq \sup_n |\phi_\nu(\bar{g}_n)| \geq |\nu|(X)$. \square

23 Product Measures and Fubini's Theorem

(14, [Schilling(2017)])

Throughout this chapter we assume that (X, \mathcal{A}, μ) and (Y, \mathcal{B}, ν) are σ -finite measure spaces.

Recall the Cartesian product of sets (assume $A \subset X, B \subset Y$):

$$A \times B = \{(a, b) | a \in A \text{ and } b \in B\}.$$

$$\left(\bigcup_i A_i \right) \times B = \bigcup_i (A_i \times B),$$

$$\left(\bigcap_i A_i \right) \times B = \bigcap_i (A_i \times B),$$

$$(A \times B) \cap (A' \times B') = (A \cap A') \times (B \cap B'),$$

$$A^c \times B = (X \times B) \setminus (A \times B),$$

$$A \times B \subset A' \times B' \iff A \subset A' \text{ and } B \subset B'.$$

The Lebesgue measure on \mathbb{R}^n has the following property for $n > d \geq 1$:

$$\lambda^n[a_1, b_1) \times \dots \times [a_n, b_n) = \lambda^d[a_1, b_1) \times \dots \times [a_d, b_d) \cdot \lambda^{n-d}[a_{d+1}, b_{d+1}) \times \dots \times [a_n, b_n)$$

which means that

$$\lambda^n(E) = \int \mathbb{1}_E(x, y) \lambda^n(d(x, y)) = \int \left(\int \mathbb{1}_E(x_0, y) \lambda^{n-d}(dy) \right) \lambda^d(dx_0).$$

Goal: we want to define a measure ρ on rectangles on the form $A \times B$ s.t. $\rho(A \times B) = \mu(A)\nu(B)$.

Lemma 23.1. Let \mathcal{A} and \mathcal{B} be two σ -algebras (or semi-rings), then $\mathcal{A} \times \mathcal{B}$ is a semi-ring.

Proof. The proof is straight forwards. Recall that a semi-ring \mathcal{S} has $\emptyset \in \mathcal{S}$ (S_1), $S, T \in \mathcal{S} \rightarrow S \cap T \in \mathcal{S}$ (S_2), and for $S, T \in \mathcal{S}$ there exist finitely many disjoint $S_1, S_2, \dots, S_M \in \mathcal{S}$ s.t. $S \setminus T = \bigcup_{i=1}^M S_i$. \square

Definition 23.2 (product σ -algebra). The σ -algebra $\mathcal{A} \otimes \mathcal{B} := \sigma(A \times B)$ is called a **product σ -algebra**, and $(X \times Y, \mathcal{A} \otimes \mathcal{B})$ is the product of measurable spaces.

When considering $\mathcal{A} \otimes \mathcal{B}$, the following lemma shows that we can instead work with their generators.

Lemma 23.3. If $\mathcal{A} = \sigma(\mathcal{F})$ and $\mathcal{B} = \sigma(\mathcal{G})$ and if \mathcal{F} and \mathcal{G} contain exhausting sequences $(F_n)_{n \in \mathbb{N}} \subset \mathcal{F}$, $F_n \uparrow X$ and $(G_n)_{n \in \mathbb{N}} \subset \mathcal{G}$, $G_n \uparrow Y$, then

$$\sigma(\mathcal{F} \times \mathcal{G}) = \sigma(\mathcal{A} \times \mathcal{B}) := \mathcal{A} \otimes \mathcal{B}.$$

Proof. Since $\mathcal{F} \times \mathcal{G} \subset \mathcal{A} \times \mathcal{B}$, we have $\sigma(\mathcal{F} \times \mathcal{G}) \subset \mathcal{A} \otimes \mathcal{B}$. On the other hand, the system

$$\Sigma := \{A \in \mathcal{A} : A \times G \in \sigma(\mathcal{F} \times \mathcal{G}) \forall G \in \mathcal{G}\}$$

is a σ -algebra. This can be shown, but is more or less straightforward. Furthermore, clearly, $\mathcal{F} \subset \Sigma \subset \mathcal{A}$, but $\mathcal{A} \subset \Sigma$ by construction, and so $\Sigma = \mathcal{A}$. We conclude that $\mathcal{A} \times \mathcal{G} \subset \sigma(\mathcal{F} \times \mathcal{G})$. This means that for all $A \in \mathcal{A}$ and $B \in \mathcal{B}$

$$A \times B = (A \times Y) \cap (X \times B) = \bigcup_{k, n \in \mathbb{N}} \underbrace{(A \times G_k)}_{\in \sigma(\mathcal{F} \times \mathcal{G})} \cap \underbrace{(F_n \times B)}_{\in \sigma(\mathcal{F} \times \mathcal{G})},$$

so that $\mathcal{A} \times \mathcal{B} \subset \sigma(\mathcal{F} \times \mathcal{G})$ and thus $\mathcal{A} \otimes \mathcal{B} \subset \sigma(\mathcal{F} \times \mathcal{G})$. \square

Theorem 23.4 (uniqueness of product measures). *Assume that $\mathcal{A} = \sigma(\mathcal{F})$ and $\mathcal{B} = \sigma(\mathcal{G})$. If*

- \mathcal{F}, \mathcal{G} is \cap -stable (stable under finite intersections),
- \mathcal{F}, \mathcal{G} contain exhausting sequences $F_k \uparrow X$ and $G_k \uparrow Y$ with $\mu(F_k) < \infty$ and $\nu(G_k) < \infty$,

then there is at most one measure ρ on $(X \times Y, \mathcal{A} \otimes \mathcal{B})$ satisfying

$$\rho(F \times G) = \mu(F)\nu(G) \quad \forall F \in \mathcal{F}, G \in \mathcal{G}.$$

By Lemma 23.3 $\mathcal{F} \times \mathcal{G}$ generates $\mathcal{A} \otimes \mathcal{B}$. Moreover, $\mathcal{F} \times \mathcal{G}$ inherits the \cap -stability of \mathcal{F} and \mathcal{G} (not obvious), the sequence $F_n \times G_n$ increases towards $X \times Y$ and $\rho(F_n \times G_n) = \mu(F_n)\nu(G_n) < \infty$. These are the assumptions of the uniqueness theorem for measures (Theorem 3.3).

Theorem 23.5 (existence of product measures). *The set function*

$$\rho : \mathcal{A} \times \mathcal{B} \rightarrow [0, \infty], \quad \rho(A \times B) := \mu(A)\nu(B),$$

extends uniquely to a σ -finite measure on $(X \times Y, \mathcal{A} \otimes \mathcal{B})$ s.t.

$$\rho(E) = \int \int \mathbb{1}_E(x, y) \mu(dx) \nu(dy) = \int \int \mathbb{1}_E(x, y) \nu(dy) \mu(dx)$$

holds for all $E \in \mathcal{A} \otimes \mathcal{B}$ (the parenthesis in the expression above are left out). In particular, the functions

$$\begin{aligned} x &\mapsto \mathbb{1}_E(x, y), \quad x \mapsto \int \mathbb{1}_E(x, y) \nu(dy), \\ y &\mapsto \mathbb{1}_E(x, y), \quad y \mapsto \int \mathbb{1}_E(x, y) \mu(dx), \end{aligned}$$

are \mathcal{A} , \mathcal{B} -measurable (respectively) for every fixed $y \in Y$, $x \in X$ (respectively).

Lecture 24

Definition 24.25 (product measure $\mu \times \nu$). The unique measure ρ constructed in Theorem 23.5 is called the **product** of the measures μ and ν , denoted $\mu \times \nu$. $(X, Y, \mathcal{A} \otimes \mathcal{B}, \mu \times \nu)$ is called the **product measure space**.

We can now finally construct the n -dimensional Lebesgue measure:

Corollary 24.26. *If $n > d \geq 1$,*

$$(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n), \lambda^n) = \left(\mathbb{R}^d \times \mathbb{R}^{n-d}, \mathcal{B}(\mathbb{R}^d) \otimes \mathcal{B}(\mathbb{R}^{n-d}), \lambda^d \times \lambda^{n-d} \right).$$

Great. The next step is to see how we can integrate w.r.t. to $\mu \times \nu$. The following two results are often stated together as the Fubini or Fubini-Tonelli theorem.

Theorem 24.27 (Tonelli). *Let (X, \mathcal{A}, μ) (Y, \mathcal{B}, ν) be σ -finite measure spaces and let $u : X \times Y \rightarrow [0, \infty]$ be $\mathcal{A} \otimes \mathcal{B}$ -measurable. Then*

- (i) $x \mapsto u(x, y)$, $y \mapsto u(x, y)$ are \mathcal{A} -resp. \mathcal{B} -measurable for fixed y resp. x ;
- (ii) $x \mapsto \int_Y u(x, y) \nu(dy)$, $y \mapsto \int_X u(x, y) \mu(dx)$ are \mathcal{A} -resp. \mathcal{B} -measurable;

$$\begin{aligned} \int_{X \times Y} u d(\mu \times \nu) &= \int_Y \int_X u(x, y) \mu(dx) \nu(dy) = \int_X \int_Y u(x, y) \nu(dy) \mu(dx) \\ &\text{which is in } [0, \infty]. \end{aligned}$$

The following corollary really extends Tonelli to not necessarily positive functions.

Corollary 24.28 (Fubini's theorem). *Let (X, \mathcal{A}, μ) (Y, \mathcal{B}, ν) be σ -finite measure spaces and let $u : X \times Y \rightarrow \mathbb{R}$ be $\mathcal{A} \otimes \mathcal{B}$ -measurable. If at least one of the three integrals*

$$\int_{X \times Y} |u| d(\mu \times \nu), \quad \int_Y \int_X |u(x, y)| \mu(dx) \nu(dy), \quad \int_X \int_Y |u(x, y)| \nu(dy) \mu(dx)$$

is finite, then all three integrals are finite, $u \in L^1(\mu \times \nu)$, and

- (i) $x \mapsto u(x, y)$ is in $L^1(\mu)$ for ν -a.e. $y \in Y$;
- (ii) $y \mapsto u(x, y)$ is in $L^1(\nu)$ for μ -a.e. $x \in X$;
- (iii) $y \mapsto \int_X u(x, y) \mu(dx)$ is in $L^1(\nu)$;
- (iv) $x \mapsto \int_Y u(x, y) \nu(dy)$ is in $L^1(\mu)$;
- (v) $\int_{X \times Y} u d(\mu \times \nu) = \int_Y \int_X u(x, y) \mu(dx) \nu(dy) = \int_X \int_Y u(x, y) \nu(dy) \mu(dx)$.

25 Fourier Transform (§13 (pp. 125-128), §15 (pp. 157-158), §19 (pp. 214-217), [Schilling(2017)])

We write $L^1(\mathbb{R}^n)$ for $L^1(\mathbb{R}^n, d\lambda_n)$.

The Fourier transform of a function $f \in L^1(\mathbb{R}^n)$ is the function \hat{f} on \mathbb{R}^n defined by

$$\hat{f}(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} f(y) e^{-i\langle x, y \rangle} dy,$$

where $\langle x, y \rangle = x_1 y_1 + \dots + x_n y_n$. More generally, given a finite Borel measure μ , is *Fourier transform* is the function $\hat{\mu}$ on \mathbb{R}^n defined by

$$\hat{\mu}(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{-i\langle x, y \rangle} d\mu(y).$$

We can also define μ for complex Borel measures.

Warning. There are many different conventions for the Fourier transform: instead of $1/(2\pi)^n$ one also uses $1, 1/(2\pi)^{n/2}$; instead of $e^{-i\langle x, y \rangle}$ one also uses $e^{i\langle x, y \rangle}, e^{\pm 2\pi i \langle x, y \rangle}$.

Note that if μ_f for $f \in L^1(\mathbb{R}^n)$ is defined by $d\mu_f = f d\lambda_n$, then

$$\hat{\mu}_f = \hat{f}.$$

Lemma 25.1. *If μ is a complex Borel measure on \mathbb{R}^n , then $\hat{\mu}$ is a Bounded continuous function on \mathbb{R}^n , and $|\hat{\mu}(x)| \leq \frac{|\mu|(\mathbb{R}^n)}{(2\pi)^n}$.*

In particular, if $f \in L^1(\mathbb{R}^n)$, then \hat{f} is Bounded and continuous,

$$|\hat{f}(x)| \leq \frac{\|f\|_1}{(2\pi)^n} \quad \forall x.$$

Proof. We have

$$\begin{aligned} |\hat{\mu}(x)| &= \left| \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{-i\langle x, y \rangle} d\mu(y) \right| \\ &\leq \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} 1 d|\mu|(y) = \frac{1}{(2\pi)^n} |\mu|(\mathbb{R}^n). \end{aligned}$$

Continuity follows from the dominated convergence theorem. \square

Some properties:

(i) If $f_t(x) = f(x - t)$, then

$$\hat{f}_t(y) = e^{-i\langle t, y \rangle} \hat{f}(y).$$

(ii) If $e_t(x) = e^{i\langle t, x \rangle}$, then

$$e_t \hat{f}(y) = \hat{f}(y - t).$$

(iii) If $T \in GL_n(\mathbb{R})$ (invertible $n \times n$ matrices), then

$$f \circ T = |\det T|^{-1} \hat{f} \circ (T^t)^{-1}.$$

Proof.

$$\begin{aligned} f \circ T(x) &= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} f(Ty) e^{-i\langle x, y \rangle} dy \\ &= \frac{1}{|\det T|} \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} f(y) e^{-i\langle x, T^{-1}y \rangle} dy \\ &= \frac{1}{|\det T|} \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} f(y) e^{-i\langle (T^t)^{-1}x, y \rangle} dy \\ &= \frac{1}{|\det T|} \hat{f}((T^t)^{-1}x). \end{aligned}$$

(iv) $\hat{\hat{f}}(x) = \hat{f}(-x)$.

Important example.

If $f(x) = e^{-\frac{|x|^2}{2}}$ ($|x| = \langle x, x \rangle^{1/2}$), then $\hat{f}(x) = 1/(2\pi)^n e^{-\frac{|x|^2}{2}}$.

Proof. We need to show that

$$\frac{1}{2\pi} \int_{\mathbb{R}^n} e^{-\frac{y^2}{2} - ixy} dy = \frac{e^{-\frac{x^2}{2}}}{(2\pi)^{1/2}}.$$

We have $e^{-\frac{y^2}{2} - ixy} = e^{-\frac{1}{2}(y+ix)^2 - \frac{x^2}{2}}$, so we need to check that

$$\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-\frac{1}{2}(y+ix)^2} dy = 1$$



Let c_n be as in the figure above, and consider $\oint e^{-\frac{z^2}{2}} dz = 0$. Letting $n \rightarrow \infty$, we conclude that

$$\int_{\mathbb{R}^n} e^{-\frac{1}{2}(y+ix)^2} dy = \int_{\mathbb{R}^n} e^{-\frac{y^2}{2}} dy.$$

It remains to show that

$$\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-\frac{y^2}{2}} dy = 1.$$

Let $a = \int_0^\infty e^{-\frac{y^2}{2}} dy$, so $\int_{\mathbb{R}} e^{-\frac{y^2}{2}} dy = 2a$. Then $a^2 = \int_0^\infty \int_0^\infty e^{-\frac{x^2+y^2}{2}} dx dy$. Passing to spherical coordinates we get $a^2 = \int_0^\infty dr \int_0^{\frac{\pi}{2}} d\phi e^{-\frac{r^2}{2}} r = \frac{\pi}{2} \int_0^\infty dr e^{-\frac{r^2}{2}} r = \frac{\pi}{2}$. Therefore $2a = \sqrt{2\pi}$. \square

More generally, if $f(x) = e^{-\frac{c|x|^2}{2}}$, then

$$\hat{f}(x) = \frac{1}{(2\pi c)^{\frac{n}{2}}} e^{-\frac{|x|^2}{2c}} \quad \forall c > 0.$$

This follows from property (iii).

For functions f, g on \mathbb{R}^n , their **convolution** is the function $f \times g$ defined by

$$(f * g)(x) := \int_{\mathbb{R}^n} f(y) g(x - y) dy = \int_{\mathbb{R}^n} f(x - y) g(y) dy$$

when is this well-defined?

Lemma 25.2. If $f, g \in L^1(\mathbb{R}^n)$, then the function $y \mapsto f(y)g(x - y)$ is integrable for (λ_n) -a.e. x , $f * g \in L^1(\mathbb{R}^n)$ and $\|f * g\|_1 \leq \|f\|_1 \|g\|_1$.

Proof. For everything except the last inequality we may assume that $f, g \geq 0$. Consider $F : \mathbb{R}^n \times \mathbb{R}^n \rightarrow [0, \infty)$,

$$F(x, y) = f(x - y)g(y).$$

\square Then

$$\begin{aligned} \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} F(x, y) dx \right) dy &= \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} f(x - y) dx \right) g(y) dy \\ &= \|f\|_1 \int_{\mathbb{R}^n} g(y) dy = \|f\|_1 \|g\|_1. \end{aligned}$$

Therefore, by Tonelli's theorem, $F \in L^1(\mathbb{R}^n \times \mathbb{R}^n)$. Then, by the same theorem,

$$\|F\|_1 = \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} F(x, y) dy \right) dx = \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} f(y) g(x - y) dy \right) dx.$$

It follows that $\int_{\mathbb{R}^n} f(y) g(x - y) dy < \infty$ for a.e. x , the function

$$x \mapsto (f * g)(x) = \int_{\mathbb{R}^n} F(x, y) dy$$

is measurable and $\|f * g\|_1 = \|F\|_1 = \|f\|_1 \|g\|_1$.

Finally, for general $f, g \in L^1(\mathbb{R}^n)$,

$$\begin{aligned} \|f * g\|_1 &= \int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} f(y) g(x - y) dy \right| dx \\ &\leq \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} |f(y)| \cdot |g(x - y)| dy \right) dx \\ &= \| |f| * |g| \|_1 = \| |f| \|_1 \cdot \| |g| \|_1 = \|f\|_1 \cdot \|g\|_1. \end{aligned}$$

\square

Next let us show that $f * g$ is well-defined for $f \in L^1(\mathbb{R}^n)$, $g \in L^p(\mathbb{R}^n)$, $1 \leq p \leq \infty$.

Lemma 25.3. Assume $\phi : (a, b) \rightarrow \mathbb{R}$ is a convex function. Then ϕ is continuous and $\phi(x) = \sup\{\langle l, x \rangle : \phi \geq l, l(s) = \alpha s + \beta\}$.

Proof. We claim that the function $t \mapsto \frac{\phi(t) - \phi(x)}{t - x}$, $t \neq x$, is increasing for all x . This is equivalent to the inequalities:

$$\frac{\phi(t) - \phi(x)}{t - x} \leq \frac{\phi(y) - \phi(x)}{y - x} \leq \frac{\phi(y) - \phi(t)}{y - t},$$

for $a < x < t < y < b$. To prove the inequalities write $t = \lambda x + (1 - \lambda)y$, $0 < \lambda < 1$. Then

$$\begin{aligned} \frac{\phi(t) - \phi(x)}{t - x} &\leq \frac{\lambda\phi(x) + (1 - \lambda)\phi(y) - \phi(x)}{t - x} = \frac{\phi(y) - \phi(x)}{y - x}, \\ \frac{\phi(y) - \phi(t)}{y - t} &\geq \frac{\phi(y) - \lambda\phi(x) - (1 - \lambda)\phi(y)}{y - t} = \frac{\phi(y) - \phi(x)}{y - x}, \end{aligned}$$

prooving the claim.

The claim implies that the left and right derivatives ϕ'_- and ϕ'_+ exist and

$$\frac{\phi(t_1) - \phi(x)}{t_1 - x} \leq \phi'_-(x) \leq \phi'_+(x) \leq \frac{\phi(t_2) - \phi(x)}{t_2 - x},$$

for $t_1 < x < t_2$. From the existence of ϕ'_- , ϕ'_+ we conclude that ϕ is continuous. We see also that for $l(s) = \phi'_+(x)(s - x) + \phi(x)$ we have $\phi \geq l$ and $\phi(x) = l(x)$. \square

Theorem 25.4 (Jensen's inequality). *Assume (X, \mathcal{B}, μ) is a probability measure space (so $\mu(X) = 1$), $\phi : [0, \infty) \rightarrow [0, \infty)$ is a convex function. Then, for every integrable function $f : X \rightarrow [0, \infty)$ we have*

$$\phi\left(\int_X f d\mu\right) \leq \int_X \phi \circ f d\mu.$$

The same inequality holds for any measurable $f : X \rightarrow [0, \infty]$ if $\lim_{x \rightarrow \infty} \phi(x) = +\infty$ and we put $\phi(+\infty) = +\infty$.

Proof. Note that as ϕ is continuous on $(0, \infty)$ the function $\phi \circ f$ is measurable. If $\int_X f d\mu = 0$, so $f = 0$ a.e., then there is nothing to prove, as then

$$\phi\left(\int_X f d\mu\right) = \phi(0), \quad \int_X \phi \circ f d\mu = \phi(0)\mu(X) = \phi(0).$$

Assume therefore that $f : X \rightarrow [0, \infty)$ and $t = \int_X f d\mu$ satisfies $0 < t < \infty$. Take a linear function $l(s) = \alpha s + \beta$ s.t. $\phi \geq l$ on $(0, \infty)$. Then also $\phi(0) \geq l(0) = \beta$. (Indeed, for $s > 0$ we have $l(s) \leq \phi(s) \leq (1 - s)\phi(0) + s\phi(1)$, and letting $s \downarrow 0$ we get $\phi(0) \geq l(0)$.) Then

$$\begin{aligned} l\left(\int_X f d\mu\right) &= \alpha \int_X f d\mu + \beta \\ &= \int_X l \circ f d\mu \leq \int_X \phi \circ f d\mu. \end{aligned}$$

Taking the supremum over l as above, by the previous lemma, we get

$$\phi\left(\int_X f d\mu\right) \leq \int_X \phi \circ f d\mu.$$

Assume now that $\lim_{x \rightarrow \infty} \phi(x) = \infty$ and put $\phi(\infty) = \infty$.

If $f : X \rightarrow [0, \infty]$ and $\int_X f d\mu < \infty$ (so $f < \infty$ a.e.), then the same proof as above gives the required inequality. It remains to show that if

$$\int_X f d\mu = \infty, \text{ then also } \int_X \phi \circ f d\mu = \infty.$$

For this it suffices to show that there is $l(s) = \alpha s + \beta$ s.t. $\phi \geq l$ and $\alpha > 0$, as then

$$\int_X \phi \circ f d\mu \geq \alpha \int_X f d\mu + \beta = \infty.$$

Such l exists by the proof of the previous lemma: take $t > 0$ and find x s.t. $\phi(x) > \phi(t)$. Then

$$\phi'_+(x) \geq \frac{\phi(t) - \phi(x)}{t - x} > 0$$

and we can take $l(s) = \phi'_+(x)(s - x) + \phi(x)$. \square

26 Regularization (15 & 19, [Schilling(2017)])

Lemma 26.1. *Assume $f \in L^1(\mathbb{R}^n)$, $g \in L^p(\mathbb{R}^n)$ ($1 \leq p \leq \infty$). Then the function $g \mapsto f(g)g(x - y)$ is integrable for a.e. x , $f * g \in L^p(\mathbb{R}^n)$ and $\|f * g\|_p \leq \|f\|_1 \|g\|_p$.*

Proof. For $p = \infty$ the proof is easy, since for every $x \in \mathbb{R}^n$,

$$|f(y)g(x - y)| \leq |f(y)| \|g\|_\infty \text{ for a.e. } y.$$

Assume $1 < p < \infty$ (for $p = 1$ we already know the lemma). For everything except the last inequality we may assume that $f, g \geq 0$. If $f = 0$ a.e., there is nothing to prove, so we may assume that $\|f\|_1 > 0$ and then replacing f by $\frac{f}{\|f\|_1}$, that $\|f\|_1 = 1$. By Tensen's inequality,

$$\begin{aligned} \left(\int_{\mathbb{R}^n} f(y)g(x - y)dy\right)^p &\leq \int_{\mathbb{R}^n} f(y)g(x - y)^p dy \\ &= (f * g^p)(x). \end{aligned}$$

As $g^p \in L^1(\mathbb{R}^n)$, it follows that $\int_{\mathbb{R}^n} f(y)g(x - y)dy < \infty$ for a.e. x and

$$\begin{aligned} \|f * g\|_p &\leq \left(\int_{\mathbb{R}^n} (f * g^p)(x)dx\right)^{1/p} \\ &= (\|f\|_1 \|g^p\|_1)^{1/p} = \|g^p\|_1^{1/p} = \|g\|_p. \end{aligned}$$

Finally, for general $f \in L^1(\mathbb{R}^n)$, $g \in L^p(\mathbb{R}^n)$ we have

$$|f * g| \leq |f| * |g|,$$

so

$$\begin{aligned} \|f * g\|_p &\leq \| |f| * |g| \|_p \leq \| |f| \|_1 \cdot \| |g| \|_p \\ &= \|f\|_1 \cdot \|g\|_p. \end{aligned}$$

\square

Note that

$$\int_{\mathbb{R}^n} f(y)g(x - y)dy = \int_{\mathbb{R}^n} f(x - y)g(y)dy,$$

so $f * g = g * f$.

Remark. *More generally, for $\mu \in M(\mathbb{R}^n)$ and $g \in L^p(\mathbb{R}^n)$, we can define $\mu * g = g * \mu \in L^p(\mathbb{R}^n)$ by*

$$(\mu * g)(x) = \int_{\mathbb{R}^n} g(x - y)d\mu(y).$$

*Then $\|\mu * g\|_p \leq |\mu|(\mathbb{R}^n) \|g\|_p$.*

Proposition 26.2. If $f, g \in L^1(\mathbb{R}^n)$, then $f \hat{*} g = (2\pi)^n \hat{f} \hat{g}$.

Proof.

$$\begin{aligned} f \hat{*} g(x) &= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} (f * g)(y) e^{-i\langle x, y \rangle} dy \\ &= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} f(z) g(y-z) e^{-i\langle x, y \rangle} dz \right) dy \\ &= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} f(z) e^{-i\langle x, z \rangle} g(y-z) e^{-i\langle x, y-z \rangle} dy \right) dz \\ &= (2\pi)^n \hat{f} \hat{g}. \end{aligned}$$

What are convolutions good for?

Example 26.3. Consider

$$f = \frac{1}{\lambda_n(B_r(0))} \mathbb{1}_{B_r(0)}.$$

Then

$$\begin{aligned} (f * g)(x) &= \frac{1}{\lambda_n(B_r(0))} \int_{B_r(0)} g(x-y) dy \\ &= \frac{1}{\lambda_n(B_r(x))} \int_{B_r(x)} g(y) dy. \end{aligned}$$

For a multi-index $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}_+^n$, write ∂^α for

$$\frac{\partial^{\alpha_1 + \dots + \alpha_n}}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}}.$$

Denote by $L_{\text{loc}}^1(\mathbb{R}^n)$ the space of Lebesgue measurable functions that are independent on every ball. We identify functions that coincide a.e. (so, $L_{\text{loc}}^1(\mathbb{R}^n)$ is a space of equivalence classes of functions). We have $L^p(\mathbb{R}^n) \subset L_{\text{loc}}^1(\mathbb{R}^n)$ for all $1 \leq p \leq \infty$.

Lemma 26.4. If $\phi \in C_c^\infty(\mathbb{R}^n)$ and $f \in L_{\text{loc}}^1(\mathbb{R}^n)$, then $\phi * f \in C^\infty(\mathbb{R}^n)$ and $\partial^\alpha(\phi * f) = (\partial^\alpha \phi) * f$.

Proof. It suffices to show that $\frac{\partial}{\partial x_i}(\phi * f) = \frac{\partial \phi}{\partial x_i} * f$. Let $e_i = (0, \dots, 0, 1, 0, \dots, 0) \in \mathbb{R}^n$. Then

$$\begin{aligned} \frac{\partial(\phi * f)}{\partial x_i}(x) &= \frac{d}{df}(\phi * f)(x + te_i) \\ &= \frac{d}{df} \int_{\mathbb{R}^n} \phi(x + te_i - y) f(y) dy \\ &= \int_{\mathbb{R}^n} \frac{\partial \phi}{\partial x_i}(x - y) f(y) dy = \left(\frac{\partial \phi}{\partial x_i} * f \right)(x), \end{aligned}$$

where we used theorem on differentiability of integrals depending on parameters. \square

By choosing suitable ϕ we can make $\phi * f$ close to f , as we will see shortly.

Definition 26.5. A positive ‘modifier’ (?) is a function $\phi \in C_c(\mathbb{R}^n)$ s.t. $\phi \geq 0$ and $\int_{\mathbb{R}^n} \phi(x) dx = 1$.

For a function ϕ on \mathbb{R}^n and $\epsilon > 0$, define $\phi^\epsilon(x) := \epsilon^{-n} \phi(x/\epsilon)$. Note that if $\phi \in L^1(\mathbb{R}^n)$, $\int_{\mathbb{R}^n} \phi dx = 1$, then $\int_{\mathbb{R}^n} \phi^\epsilon = 1$.

Example 26.6 (positive modifier). Consider the function h in \mathbb{R} defined by,

$$h(t) = \begin{cases} e^{-\frac{1}{1-t^2}}, & |t| < 1, \\ 0, & |t| \geq 1 \end{cases}.$$

Then $g \in C_c^\infty(\mathbb{R}^n)$. Hence, $\phi(x) = c_n h(|x|)$ is a modifier, where $c_n = \left(\int_{\mathbb{R}^n} h(|x|) dx \right)^{-1}$.

Proposition 26.7. Let $\phi \in L^1(\mathbb{R}^n)$ be s.t. $\phi \geq 0$ and $\int_{\mathbb{R}^n} \phi dx = 1$. Then we have

\square (i) If $f \in C_0(\mathbb{R}^n)$ (continuous functions vanishing at infinity (?)), then $\phi^\epsilon * f \in C_0(\mathbb{R}^n)$ and $\|\phi^\epsilon * f\| \xrightarrow{\epsilon \downarrow 0} 0$ (uniform norm);

(ii) If $f \in L^p(\mathbb{R}^n)$, $1 \leq p \leq \infty$, then $\|\phi^\epsilon * f - f\|_p \xrightarrow{\epsilon \downarrow 0} 0$.

Proof. Can be found in lecture. \square

Corollary 26.8. For any Radon measure μ on \mathbb{R}^n , $C_c^\infty(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n, d\mu)$ for $1 \leq p < \infty$.

Proof. Can be found in lecture. \square

27 Fourier Inverse (19, [Schilling(2017)])

Corollary 27.1. If $f \in L^1(\mathbb{R}^n)$, then $\hat{f} \in C_0(\mathbb{R}^n)$.

Proof. We already know that $\hat{f} \in C_b(\mathbb{R}^n)$ (the space of bounded continuous functions) and $\|\hat{f}\|_\infty \leq \|f\|_1$. Consider $\phi(x) = 1/(2\pi)^{n/2} e^{-\frac{|x|^2}{2}}$, so $\int_{\mathbb{R}^n} \phi(x) dx = 1$. We know that $\hat{\phi} = 1/(2\pi)^{n/2} \phi \in C_0(\mathbb{R}^n)$. Then $\widehat{\phi^\epsilon} = \hat{\phi}(\epsilon x) = 1/(2\pi)^{n/2} \phi(\epsilon x)$, $\hat{\phi}^\epsilon \in C_0(\mathbb{R}^n)$. As $\phi^\epsilon * f \xrightarrow{\epsilon \downarrow 0} f$ in $L^1(\mathbb{R}^n)$, we have

$$\|\widehat{\phi^\epsilon * f} - \hat{f}\|_\infty \xrightarrow{\epsilon \downarrow 0} 0.$$

But $\widehat{\phi^\epsilon * f} = (2\pi)^n \widehat{\phi^\epsilon} \hat{f} \in C_0(\mathbb{R}^n)$, hence $\hat{f} \in C_0(\mathbb{R}^n)$. \square

Remark. Another possibility is to approximate f by its linear combinations of $\mathbb{1}_{[a_1, b_1] \times \dots \times [a_n, b_n]}$. Note that for $\mathbb{1}_{[a, b]} \in L^1(\mathbb{R})$ we have

$$\hat{\mathbb{1}}_{a, b}(x) = 1/(2\pi) \int_a^b e^{-ixy} dy \underset{x \neq 0}{=} \frac{e^{-iax} - e^{-ibx}}{2\pi ix} \xrightarrow{x \rightarrow \infty} 0.$$

Theorem 27.2 (Fourier Inversion Theorem). Assume $f \in L^1 \mathbb{R}^n$ is s.t. $\hat{f} \in L^1(\mathbb{R}^n)$. Then, for a.e. x , we have

$$f(x) = \int_{\mathbb{R}^n} \hat{f}(y) e^{i\langle x, y \rangle} dy.$$

Equivalently,

$$\hat{\hat{f}}(x) = \frac{1}{(2\pi)^n} f(-x).$$

Lemma 27.3. For any $f, g \in L^1(\mathbb{R}^n)$, we have

$$\int_{\mathbb{R}^n} \hat{f} g dx = \int_{\mathbb{R}^n} f \hat{g} dx.$$

(Note that $\hat{f} g \in L^1(\mathbb{R}^n)$, as \hat{f} is bounded.)

Proof. This follows by Fubini's theorem:

$$\begin{aligned}\int_{\mathbb{R}^n} \hat{f}(x)g(x)dx &= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} f(y)g(x)e^{-i\langle x,y \rangle} dy \right) dx \\ &= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} f(y)g(x)e^{-i\langle x,y \rangle} dx \right) dy \\ &= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} f(y)\hat{g}(y)dy.\end{aligned}$$

□

proof of Fourier Inversion Theorem. We already know that the theorem is true for $\phi(x) = 1/(2\pi)^{n/2}e^{-\frac{|x|^2}{2}}$, hence also for ϕ^ϵ (so $\phi^\epsilon(x) = \epsilon^{-n}\phi(x/\epsilon)$). Consider the function I_ϵ defined by

$$\begin{aligned}I_\epsilon(x) &= \int_{\mathbb{R}^n} \widehat{\phi^\epsilon * f}(y)e^{i\langle x,y \rangle} dy \\ &= (2\pi)^n \int_{\mathbb{R}^n} \widehat{\phi^\epsilon}(y)\hat{f}(y)e^{i\langle x,y \rangle} dy.\end{aligned}$$

On the other hand,

$$(2\pi)^n \widehat{\phi^\epsilon}(y) = (2\pi)^n \hat{\phi}(\epsilon y) = e^{-\frac{\epsilon^2 |y|^2}{2}}.$$

As $\hat{f} \in L^1(\mathbb{R}^n)$ by assumption, by the dominated convergence theorem we conclude that

$$I_\epsilon(x) \rightarrow \int_{\mathbb{R}^n} \hat{f}(y)e^{i\langle x,y \rangle} dy \quad \forall x \in \mathbb{R}^n.$$

On the other hand, applying the previous lemma to $g(y) = \widehat{\phi^\epsilon}e^{i\langle x,y \rangle}$ (with x fixed) we have

$$\begin{aligned}I_\epsilon(x) &= (2\pi)^n \int_{\mathbb{R}^n} g(y)\hat{f}(y)dy \\ &= (2\pi)^n \int_{\mathbb{R}^n} \hat{g}(y)f(y)dy \\ &= (2\pi)^n \int_{\mathbb{R}^n} \widehat{\phi^\epsilon}(y-x)f(y)dy \\ &= \int_{\mathbb{R}^n} \phi^\epsilon(y-x)f(y)dy \\ &= (\phi^\epsilon * f)(x).\end{aligned}$$

Therefore $I_\epsilon = \phi^\epsilon * f \xrightarrow[\epsilon \downarrow 0]{} f$ in $L^1(\mathbb{R}^n)$. In particular $I_{1/n} \xrightarrow[n \rightarrow \infty]{} f$ in $L^1(\mathbb{R}^n)$, hence, $I_{1/n_k} \xrightarrow[k \rightarrow \infty]{} f$ a.e. for some subsequence. It follows that for a.e. x we have $f(x) = \int_{\mathbb{R}^n} \hat{f}(y)e^{i\langle x,y \rangle} dy$. □

Corollary 27.4. *If $f \in L^1(\mathbb{R}^n)$ and $\hat{f} = 0$, then $f = 0$ (a.e.)*

Recall that a linear operator $U : H \rightarrow K$ between Hilbert spaces is called an **isometry** if

$$\|Ux\| = \|x\| \quad \forall x \in H.$$

Equivalently, by the polarization identity,

$$(Ux, Uy) = (x, y) \quad \forall x, y \in H.$$

If U is in addition surjective, then it is called a **unitary**.

Theorem 27.5 (Plancherel). *There is a unique unitary $U : L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n)$ s.t. $Uf = (2\pi)^{n/2}\hat{f}$ for all $f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$.*

Can be found in lecture.

Tips'n Tricks

- Assume we can write X as a finite union: $X = \cup_{n \in I} A_n$, $i = 1, \dots, N$. Then

$$\int f d\mu = \int_X f d\mu = \int_{A_1} f d\mu + \int_{A_2} f d\mu + \dots + \int_{A_N} f d\mu.$$

Questions

- In problem 26.18 we are supposed to show that $Y_n \perp Y_m = 0$, i.e. that $\langle y_n, y_m \rangle = 0$, $n \neq m$. I get ...

$$\langle y_n, y \rangle \subset \int_{A_m^c} |y_n|^2 |y_m|^2 d\mu,$$

and I want to argue that this is zero since $\int_{A_m^c} |y_m|^2 d\mu = 0$, but I don't see how. The solutions are not clear, and I think perhaps my setup is wrong. I am assuming $\langle f, g \rangle = \int_X f \bar{g} d\mu$, i.e. from L^2 , but perhaps it is rather $\langle f, g \rangle = \int_{A_m^c \cup A_n^c} f \bar{g} d\mu$ or something?

References

- [Schilling(2017)] Schilling, R. 2017, Measures, Integrals and Martingales, Measures, Integrals and Martingales (Cambridge University Press). <https://books.google.no/books?id=sdAoDwAAQBAJ>
- [Teschl(2010)] Teschl, G. 2010, Topics in Linear and Nonlinear Functional Analysis (Universität Wien). <https://www.mat.univie.ac.at/~gerald/ftp/book-fa/fa.pdf>