# MAT4400: Detailed Notes on Linear analysis

April 3, 2024

## 1  $\sigma$-Algebras (3, [Schilling(2017)])

**Definition 1.1** ($\sigma$-Algebra)**.** A family $\mathscr{A}$ of subsets of $X$ with:

(i) $X \in \mathscr{A}$,

(ii) $A \in \mathscr{A} \Rightarrow A^c \in \mathscr{A}$,

(iii) $(A_n)_{n\in\mathbb{N}} \in \mathscr{A} \Rightarrow \bigcup_{n\in\mathbb{N}}$

**Theorem 1.2** (and Definition)**.**

(i) *The intersection of arbitrarily many $\sigma$-algebras in $X$ is agaiang a $\sigma$-algebra in $X$.*

(ii) *For every system of sets $p \subset \mathscr{P}(X)$ there exists a smallest$\sigma$-algebra containing$p$. This is the $\sigma$-algebra generated by $p$, denoted $\sigma(p)$, and $\sigma(p)$ is called its generator.*

**Definition 1.3** (Borel)**.** The $\sigma$-algebra $\sigma(\mathcal{O})$ generated by the open sets $\mathcal{O} = \mathcal{O}_{\mathbb{R}^n}$ of $\mathbb{R}^n$ is called **Borel $\sigma$-algebra**, and its members are called **Borel sets** or **Borel measurable sets**.

## 3  Uniqueness of Measures (5, [Schilling(2017)])

**Lemma 3.1.** *A Dynkin system $D$ is a $\sigma$-algebra iff it is stable under finite intersections, i.e. $A, B \in D \Rightarrow A \cap B \in D$.*

**Theorem 3.2** (Dynkin)**.** *Assume $X$ is a set, $S$ is a collection of subsets of $X$ closed under finite intersections, that is, if $A, B \in S \Rightarrow A \cap B \in S$. Then $D(S) = \sigma(S)$.*

*Proof.* We clearly have that $D(S) \subset \sigma(S)$. If we can show that $D(S)$ is a $\sigma$-algebra, that is, that a Dynkin system generated by a subset $S \subset X$ (where $S$ is $\cap$-stable) is a $\sigma$-algebra, then the inverse conclusion $D(S) \supset \sigma(S)$ follows logically. This is the case because the $\sigma$-algebra $\sigma(S)$ is the smallest $\sigma$-algebra containing $S$, and so if $D(S)$ is a $\sigma$-algebra it must be a greater or equal (in some sense) than $\sigma(S)$.

Using Lemma 3.1 we only need to show that $D(S)$ is stable under finite intersections, to prove that $D(S)$ is a $\sigma$-algebra. Consider:

$$D_A := \{B \subset X : B \cap A \in D(S)\},$$

for some $A \in D(S)$. Notice that this set is $\cap$-stable, and so if we can show that $D_A = D(S)$ we must have that (by Lemma 3.1) $D(S)$ is a $\sigma$-algebra. Firstly, however, let us show that $D_A$ is a **Dynkin system**.

1. $\varnothing$ must be in $D_A$, since $\varnothing \cap A = \varnothing \in D(S)$.

2. Let $B \in D_A$. Then
$$A \cap B^c = A \setminus (A \cap B) = (A^c \cup (A \cap B))^c,$$
here $A \cap B$ and $A^c$ must be in $D(S)$. Furthermore, since disjoint unions of set from $D(S)$ are still in $D(S)$, we me must have $A^c \in D_A$.

3. Assume that $(B_n)_{n\in\mathbb{N}} \subset D_A$ is a pairwise disjoint sequence. Then

$$(B_n \cap A)_{n\in\mathbb{N}} \in D(S) \ \text{ (by def. of } D_A)$$
$$\Rightarrow \bigcup_{n\in\mathbb{N}} (B_n \cap A) = \left(\bigcup_{n\in\mathbb{N}} B_n\right) \cap A \in D(S)$$
$$\Rightarrow \bigcup_{n\in\mathbb{N}} B_n \in D_A.$$

So $D_A$ is indeed a Dynkin system.

We now want to show that $D(S)$ is $\cap$-stable, we have:

$$S \subset D_A \ \forall \, A \in S$$
$$\Rightarrow D(S) \subset D_A \ \forall \, A \in S$$
$$\Rightarrow B \cap A \in D(S) \ \forall \, B \in S, \ \forall \, A \in D(S)$$
$$\Rightarrow B \in D_A \ \forall \, B \in S, \ \forall A \in D(S)$$
$$\Rightarrow S \subset D_A \ \forall \, A \in D(S)$$
$$\Rightarrow D(S) \subset D_A \ \forall \, A \in D(S)$$
$$\Rightarrow A \cap B \in D(S) \ \forall \, A, B \in D(S),$$

and so $D(S)$ is $\cap$-stable and then $D(S) \supset \sigma(S) \Rightarrow D(S) = \sigma(S)$. $\qquad\square$

**Theorem 3.3** (uniqueness of measures)**.** *Let $(X, B)$ be a measurable space, and $S \subset P(X)$ be the generator of $B$, i.e. $B = \sigma(S)$. If $S$ satisfies the following conditions:*

1. *$S$ is stable under finite intersections ($\cap$-stable), i.e. $A, C \in S \Rightarrow A \cap C \in S$.*

2. *There exists an exhausting sequence $(G_n)_{N\in\mathbb{N}} \subset$ with $G_N \uparrow X$. Assume also that there are two measures $\mu, \nu$ satisfying:*

3. *$\mu(A) = \nu(A), \ \forall A \in S$.*

4. *$\mu(G_n) = \nu(G_n) < \infty$.*

*Then $\mu = \nu$.*

*Proof (outline).* Define

$$D_n := \{A \in B : \mu(G_n \cap A) = \nu(G_n \cap A) \ (< \infty)\},$$

and show that it is a Dynkin system. Then, use the fact that $S$ is $\cap$-stable and Theorem 3.2 to argue that $D(S) = \sigma(S)...\to...B = D_n$. $\qquad\square$

**Theorem 4.1** (Carathéodory). *Let $S \subset P(X)$ be a semi-ring and $\mu : S \to [0, \infty)$ a pre-measure. Then $\mu$ has an extension to a measure $\mu^*$ on $\sigma(S)$, i.e. that $\mu(s) = \mu^*(s)$, $\forall s \in \sigma(S)$.*

*Also, if $S$ contains an exhausting sequence, $S_n \uparrow X$, s.t. $\mu(S_n) < \infty$, then the extension is unique.*

*Outline of proof:* Firstly, let us define an outer measure.

**Definition 4.2** (Outer measure). An outer measure is a function $\mu^* : P(X) \to [0, \infty)$ with the following properties:

1. $\mu^*(\emptyset) = 0$,

2. $A \subset B \Rightarrow u^*(A) \leq \mu^*(B)$,

3. $\mu^* \left( \bigcup_{n \in \mathbb{N}} A_n \right) \leq \sum_{n \in \mathbb{N}} \mu^*(A_n)$,

and define for each $A \subset X$ the family of countable $S$-coverings:

$$C(A) := \left\{ (S_n)_{n \in \mathbb{N}} \subset S : \bigcup_{n \in \mathbb{N}} S_n \supset A \right\},$$

and the set function

$$\mu^*(A) := \inf \left\{ \sum_{n \in \mathbb{N}} \mu(S_n) : (S_n)_{n \in \mathbb{N}} \in C(A) \right\}.$$

**Step 1: Claim:** $\mu^*(A)$ *is an outer measure.*

*Proof.*

1. $C(\emptyset) = \{$any sequence in $S$ containing $\emptyset\}$
   $\Rightarrow \mu^*(\emptyset) = 0$.

2. Assume $A \subset B$. Then $C(A) \subset C(B)$
   $\Rightarrow \mu^*(A) \leq \mu^* B$.

3. If $\mu^*(A_n) = \infty$ for some $n$, then there is nothing to prove. Thus, assume $\mu^*(A_n) < \infty \, \forall n$. Fix $\epsilon > 0$, and for every $n$ choose $A_{n_k} \in S$ s.t.

$$A_n \subset \bigcup_{k \in \mathbb{N}} A_{n_k},$$

$$\sum_{k \in \mathbb{N}} \mu^*(A_{n_k}) < \mu^*(A_n) + \frac{\epsilon}{2^n}.$$

Then

$$\bigcup_{n \in \mathbb{N}} A_n \subset \bigcup_{k \in \mathbb{N}} \bigcup_{n \in \mathbb{N}} A_{n_k},$$

so

$$\mu^* \left( \bigcup_{n \in \mathbb{N}} A_n \right) \leq \sum_{n \in \mathbb{N}} \sum_{k \in \mathbb{N}} \mu(A_{n_k})$$
$$< \sum_{n \in \mathbb{N}} \left( \mu^*(A_n) + \frac{\epsilon}{2^n} \right)$$
$$= \sum_{n \in \mathbb{N}} \mu^*(A_n) + \epsilon.$$

As $\epsilon$ was arbitrarily, we get that

$$\mu^* \left( \bigcup_{n \in \mathbb{N}} A_n \right) \leq \sum_{n \in \mathbb{N}} \mu^*(A_n),$$

so $\mu^*$ fulfills all the conditions for being an outer measure. $\quad\square$

**Step 2:** *Showing that $\mu^*$ extends $\mu$, i.e. $\mu^*(s) = \mu(s) \, \forall s \in S$.*

**Step 3:** *Define $\mu^*$-measurable sets*

$$\Sigma^* := \left\{ A \subset X : \mu^*(Q) = \mu^*(Q \cap A) + \mu^*(Q \backslash A) \right.$$
$$\left. \forall \, Q \subset X \right\}$$

**Step 4:** *Show that $\mu|_{\Sigma^*}$ is a measure. In particular, $\mu|_{\sigma(S)}$ is a measure which extends $\mu$.*

We consider maps $T : X \to X'$ between two measurable spaces $(X, \mathcal{A})$ and $(X', \mathcal{A}')$ which respects the measurable structures, the $\sigma$-algbras on $X$ and $X'$. These maps are useful as we can transport a measure $\mu$, defined on $(X, \mathcal{A})$, to $(X', \mathcal{A}')$.

**Definition 5.1.** Let $(X, \mathcal{A})$, $(X', \mathcal{A}')$ b measurable spaces. A map $T : X \to X'$ is called $\mathcal{A}/\mathcal{A}'$-measurable if the pre-imag of every measurable set is a measurable set:

$$T^{-1}(A') \in \mathcal{A}, \quad \forall A' \in \mathcal{A}'.$$

- A $\mathcal{B}(\mathbb{R}^n)/\mathcal{B}(\mathbb{R}^m)$ measurable map is often called a Borel map.

- The notation $T : (X, \mathcal{A}) \to (X', \mathcal{A}')$ is often used to indicate measurability of the map $T$.

**Lemma 5.2.** *Let $(X, \mathcal{A})$, $(x', \mathcal{A}')$ be measurable spaces and let $\mathcal{A}' = \sigma(\mathcal{G}')$. Then $T : X \to X'$ is $\mathcal{A}/\mathcal{A}'$-measurable iff $T^{-1}(\mathcal{G}') \subset \mathcal{A}$, i.e. if*

$$T^{-1}(G') \in \mathcal{A}, \quad \forall G' \in \mathcal{G}'.$$

**Theorem 5.3.** *Let $(X_i, \mathcal{A}_i)$, $i = 1, 2, 3$, be measurable spaces and $T : X_1 \to X_2$, $S : X_2 \to X_3$ be $\mathcal{A}_1/\mathcal{A}_2$ and $\mathcal{A}_2/\mathcal{A}_3$-measurable maps respectivly. Then $S \circ T : X_1 \to X_3$ is $\mathcal{A}_1/\mathcal{A}_3$-measurable.*

**Corollary 5.4.** *Every continuous map between metric spaces is a Borel map.*

**Definition 5.5. (and lemma)** Let $(T_i)_{i \in I}$, $T_I : X \to X_i$, be arbitrarily many mappings from the same space X into measurable spaces $(X_i, \mathcal{A}_i)$. The smallest $\sigma$-algebra on $X$ that makes all $T_i$ simultanously measurable is

$$\sigma(T_i : i \in I) := \sigma \left( \bigcup_{i \in I} T_i^{-1}(\mathcal{A}_i) \right)$$

**Corollary 5.6.** *A function $f : (X, \mathcal{B}) \to \mathbb{R}$ is measurable if $f((a, +\infty)) \in \mathcal{B}$, $\forall a \in \mathbb{R}$.*

**Corollary 5.7.** *Assume $(X, \mathcal{B})$ is a measurable space, $(Y, d)$ is a metric space, and*
*$(f_n : (X, \mathcal{B}) \to Y)_{n=1}^{\infty}$ is a sequence of measurable maps. Assume this sequence of images $(f_n(x))_{n=1}^{\infty}$ is convergent in $Y$ $\forall x \in X$. Define*

$$f : X \to Y, \quad by \; f(x) = \lim_{n \to \infty} f_n(x).$$

*Then $f$ is measurable.*

**Theorem 5.8.** *Let $(X, \mathcal{A})$, $(X', \mathcal{A}')$ be measurable spaces and $T : X \to X'$ be an $\mathcal{A}/\mathcal{A}'$-measurable map. For every measurable $\mu$ on $(X, \mathcal{A})$,*

$$\mu'(A') := \mu(T^{-1}(A')), \quad A' \in \mathcal{A}',$$

*defines a measure on $(X', \mathcal{A}')$.*

**Definition 5.9.** The measure $\mu'(\cdot)$ in the above theorem is called the push forward or image measure of $\mu$ under $T$ and it is denoted as $T(\mu)(\cdot)$, $T_{*\mu}(\cdot)$ or $\mu \circ T^{-1}(\cdot)$.

**Theorem 5.10.** *If $T \in \mathbb{R}^{n \times n}$ is an orthogonal matrix, then $\lambda^n = T(\lambda^n)$.*

**Theorem 5.11.** *Let $S \in \mathbb{R}^{n \times n}$ be an invertible matrix. Then*

$$S(\lambda^n) = |\det s^{-1}| \lambda^n = |\det S|^{-1} \lambda^n.$$

**Corollary 5.12.** *Lebesgue measure is invariant under motions: $\lambda^n = M(\lambda^n)$ for all motions $M$ in $\mathbb{R}^n$. In particular, congruent sets have the same measure. Two sets of points are called congruent if, and only if, one can be transformed into the other by an isometry*

# Measurable Functions (8, [Schilling(2017)])

A *measurable function* is a measurable map $u : X \to \mathbb{R}$ from some measurable space $(X, \mathscr{A})$ to $(\mathbb{R}, \mathscr{B}(\mathbb{R}^1))$. They play central roles in the theory of integration.

We recall that $u : X \to \mathbb{R}$ is $\mathscr{A}/\mathscr{B}(\mathbb{R}^1)$-measurable if

$$u^{-1}(B) \in \mathscr{A}, \quad \forall B \in \mathscr{B}(\mathbb{R}^1).$$

Moreover from a lemma from chapter 7, we actually only need to show that

$$u^{-1}(G) \in \mathscr{A}, \quad \forall G \in \mathcal{G} \text{ where } \mathcal{G} \text{ generates } \mathscr{B}(\mathbb{R}^1).$$

**Proposition 5.13.**

1. *If $f, g : (X, \mathscr{B}) \to \mathbb{C}$ are measurable, then the function $f + g$, $f \cdot g$, $cf$, $(c \in \mathbb{C})$ are measurable.*

2. *If $b : \mathbb{C} \to \mathbb{C}$ is Borel and $b : (\mathbb{C}, \mathscr{B}) \to \mathbb{C}$ is measurable, then $b \circ f$ is measurable.*

3. *If $f(x) = \lim_{n \to \infty} f_n(x)$, $x \in X$ and $f_n$ are measurable, then $f$ is measurable.*

4. *If $X = \bigcup_{n=1}^{\infty} A_n$, $(A_n \in \mathscr{B})$, $f|_{A_n} : (A_n, \mathscr{B}_{A_n}) \to \mathbb{C}$ is measurable $\forall n$, then $f$ is measurable.*

**Definition 5.14.** Given a measurable space $(X, \mathscr{B})$, a measurable function $f : (X, \mathscr{B}) \to \mathbb{C}$ is called simple if

$$f(x) = \sum_{k=1}^{N} c_k \mathbb{1}_{A_k}(x),$$

for some $c_k \in \mathbb{C}$, $A_k \in \mathscr{B}$, where $\mathbb{1}$ is the characteristic function,

$$\mathbb{1}_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A. \end{cases}$$

The representation of simple function is **not** unique. We denote the standard representation of $f$ by

$$f(x) = \sum_{n=0}^{N} z_n \mathbb{1}_{B_n}(x),$$

for $N \in \mathbb{N}$, $z_n \in \mathbb{R}$, $B_n \in \mathscr{A}$, and

$$X = \bigcup_{n=1}^{N} B_n,$$

for $B_n \cap B_m = \emptyset$, $n \neq m$. The set of simple functions is denoted $\mathcal{E}(\mathscr{A})$ of $\mathcal{E}$.

**Definition 5.15.** Assume $\mu$ is a measure on $(X, \mathscr{B})$. Given a *positive* simple function

$$f = \sum_{k=1}^{N} c_k \mathbb{1}_{A_k}, \quad (c_k \geq 0).$$

We define

$$\int_X f d\mu = \sum_{k=1}^{n} c_k \mu(A_k) \in [0, +\infty].$$

We also denote this by $I_\mu(f)$.

**Lemma 5.16.** *This is well defined, that is, $\int_x f d\mu$ does not depend on the presentation of the simple function $f$.*

**Properties 5.17.** *For every positive simple function*

1. *$\int_X c f d\mu = c \int_X f d\mu$, for only $c \geq 0$*

2. *$\int_X (f + g) d\mu = \int_X f d\mu + \int_X g d\mu$.*

**Corollary 5.18.** *If $f \geq g \geq 0$ are simple functions, then*

$$\int_X f d\mu \geq \int_X g d\mu.$$

**Definition 5.19.** If $f : X \to [0, +\infty)$ is measurable, then we define

$$\int_X f d\mu = \sup \left\{ \int_X g d\mu : f \geq g \geq 0, \ g \text{ is simple} \right\}$$

**Remark.** *This means that any measurable function can be approximated by simple functions.*

**Properties 5.20.** *Measurable functions like this have the following properties*

1. *$\int_X c f d\mu = c \int_X f d\mu$, $\forall c \geq 0$.*

2. *If $f \geq g \geq 0$, then $\int_X f d\mu \geq \int_X g d\mu$ for any measurable $g, f$.*

3. *If $f \geq 0$ is simple, then $\int_X f d\mu$ is the same value as obtained before.*

To advance in measure theory we consider measurable functions

$$f : X \to [0, +\infty].$$

Measurability is understood w.r.t the $\sigma-$algebra $\mathscr{B}([0, +\infty])$ generated by $\mathscr{B}([0, +\infty))$ and $\{+\infty\}$. In other words, $A \subset [0, +\infty] \in B([0, +\infty])$ iff $A \cap [0, +\infty) \in \mathscr{B}([0, +\infty))$.

**Remark.** *Hence $f : X \to [0, +\infty]$ is measurable iff $f^{-1}(A)$ is measurable $\forall A \in \mathscr{B}([0, +\infty))$.*

**Definition 5.21.** For measurable functions $f_X \to [0, +\infty]$, we define

$$\int_X f d\mu = \sup \left\{ \int_x g d\mu : f \geq g \geq 0 : g \text{ is simple} \right\} \in [0, +\infty].$$

**Theorem 5.22.** ***Monotone convergence theorem*** *Assume* $(X, \mathcal{B}, \mu)$ *is a measure space,* $(f)_{n=1}^{\infty}$ *is an increasing sequence of measurable positive functions* $f_n : X \to [0, +\infty]$. *Define* $f(x) = \lim_{n \to \infty} f_n(x)$. *Then* $f$ *is measurable and*

$$\int_X f d\mu = \lim_{n \to \infty} \int_X f_n d\mu.$$

**Theorem 5.23.** *Assume* $(X, \mathcal{B})$ *is a measurable space and* $f : X \to [0, +\infty]$ *is measurable. Then there are simple functions* $g_n$, *s.t.*

$$0 \leq g_1 \leq g_2 \leq \dots \quad, \quad g_n(x) \to f(x), \quad \forall x \in X.$$

*Moreover, if* $f$ *is bounded, we can choose* $g_n$ *s.t. the convergence is uniform, that is,*

$$\lim_{n \to \infty} \sup_{x \in X} |g_n(x) - f(x)| = 0.$$

# 6 Integration of Measurable Functions

Through this chapter $(X, \mathcal{A}, \mu)$ will be some measure space. Recall that $\mathcal{M}^+(\mathcal{A})$ $[\mathcal{M}_{\mathbb{R}}^+(\mathcal{A})]$ are the $\mathcal{A}$-measurable positive functions and $\mathcal{E}(\mathcal{A})$ $[\mathcal{E}_{\mathbb{R}}^+(\mathcal{A})]$ are the positive and simple functions.

The fundamental idea of *Integration* is to measure the area between the graph of the function and the abscissa. For positive simple functions $f \in \mathcal{E}^+(\mathcal{A})$ in standard representation, this is done easily

$$\text{if } f = \sum_{i=0}^{M} y_i \mathbb{1}_{A_i} \in \mathcal{E}^+(\mathcal{A}) \quad \text{then} \quad \sum_{i=0}^{M} y_i \mu(A_i) \qquad (1)$$

would be the $\mu$-area enclosed by the graph and the abscissa. We note that the representation of $f$ should not impact the integral of $f$.

**Lemma 6.1.** *Let* $\sum_{i=0}^{M} y_i \mathbb{1}_{A_i} = \sum_{k=0}^{N} z_k \mathbb{1}_{B_k}$ *be two standard representations of the same function* $f \in \mathcal{E}^+(\mathcal{A})$. *Then*

$$\sum_{i=0}^{M} y_i \mu(A_i) = \sum_{k=0}^{N} z_k \mu(B_k). \qquad (2)$$

**Definition 6.2.** *Let* $f = \sum_{i=0}^{M} y_i \mathbb{1}_{A_i} \in \mathcal{E}^+(\mathcal{A})$ *be a simple function in standard representation. Then the number*

$$I_\mu(f) = \sum_{i=0}^{M} y_i \mu(A_i) \in [0, \infty] \qquad (3)$$

*(which is independent of the representation of* $f$*) is called the* $\mu$-*integral of* $f$.

**Proposition 6.3.** *Let* $f, g \in \mathcal{E}^+(\mathcal{A})$. *Then*

(i) $I_\mu(\mathbb{1}_A) = \mu(A) \quad \forall A \in \mathcal{A}$.

(ii) $I_\mu(\lambda f) = \lambda I_\mu(f) \quad \forall \lambda \geq 0$.

(iii) $I_\mu(f + g) = I_\mu(f) + I_\mu(g)$.

(iv) $f \leq g \Rightarrow I_\mu(f) \leq I_\mu(g)$.

In theorem 8.8 we saw that we could for every $u \in \mathcal{M}^+(\mathcal{A})$ write it as an increasing limit of simple functions. By corollary

8.10, the suprema of simple functions are again measurable, so that

$$u \in \mathcal{M}^+(\mathcal{A}) \Leftrightarrow u = \sup_{n \in \mathbb{N}} f_n, f \in \mathcal{E}^+(\mathcal{A}),$$

$$f_n \leq f_{n+1} \leq \dots.$$

We will use this to "inscribe" simple functions (which we know how to integrate) below the graph of a positive measurable function $u$ and exhaust the $\mu$-area below $u$.

**Definition 6.4.** *Let* $(X, \mathcal{A}, \mu)$ *be a measure space. The* $(\mu)$-*integral of a positive function* $u \in \mathcal{M}_{\mathbb{R}}^+(\mathcal{A})$ *is given by*

$$\int u d\mu = \sup \left\{ I_\mu(g) : g \leq u, \ g \in \mathcal{E}^+(\mathcal{A}) \right\}, \qquad (4)$$

*with* $\int u d\mu \in [0, +\infty]$. *If we need to emphasize the integration variable, we write* $\int u(x) \mu(dx)$. *The key observation is that the integral* $\int \dots d\mu$ *extends* $I_\mu$.

**Lemma 6.5.** *For all* $f \in \mathcal{E}^+(\mathcal{A})$ *we have* $\int f d\mu = I_\mu(f)$.

The next theorem is one of many convergence theorems. It shows that we could have defined 4 using any increasing sequence $f_n \uparrow u$ of simple functions $f_n \in \mathcal{E}^+(\mathcal{A})$.

**Theorem 6.6.** *(Beppo Levi) Let* $(X, \mathcal{A}, \mu)$ *be a measure space. For an increasing sequence of functions* $(u_n)_{n \in \mathbb{N}} \subset \mathcal{M}_{\mathbb{R}}^+(\mathcal{A})$, $0 \leq u_n \leq u_{n+1} \leq \dots$, *we have for the supremum* $u = \sup_{n \in \mathbb{N}} u_n \in \mathcal{M}_{\mathbb{R}}^+(\mathcal{A})$ *and*

$$\int \sup_{n \in \mathbb{N}} u_n d\mu = \sup_{n \in \mathbb{N}} \int u_n d\mu. \qquad (5)$$

*Note we can write* $\lim_{n \to \infty}$ *instead of* $\sup_{n \in \mathbb{N}}$ *as the supremum of an increasing sequence is its limit. Moreover, this theorem holds in* $[0, +\infty]$, *so the case* $+\infty = +\infty$ *is possible.*

**Corollary 6.7.** *Let* $u \in \mathcal{M}_{\mathbb{R}}^+(\mathcal{A})$. *Then*

$$\int u d\mu = \lim_{n \to \infty} \int f_n d\mu$$

*holds for every sequence* $(f_n)_{n \in \mathbb{N}} \subset \mathcal{E}^+(\mathcal{A})$ *with* $\lim_{n \to \infty} f_n = u$.

**Proposition 6.8.** *(of integral) Let* $u, v \in \mathcal{M}_{\mathbb{R}}^+(\mathcal{A})$. *Then*

(i) $\int \mathbb{1}_A d\mu = \mu(A) \quad \forall A \in \mathcal{A}$.

(ii) $\int \alpha u d\mu = \alpha \int u d\mu \quad \forall \alpha \geq 0$.

(iii) $\int u + v d\mu = \int u d\mu + \int v d\mu$.

(iv) $u \leq v \Rightarrow \int u d\mu \leq \int v d\mu$.

**Corollary 6.9.** *Let* $(u_n)_{n \in \mathbb{N}} \subset \mathcal{M}_{\mathbb{R}}^+(\mathcal{A})$. *Then* $\sum_{n=1}^{\infty} u_n$ *is measurable and we have*

$$\int \sum_{n=1}^{\infty} u_n d\mu = \sum_{n=1}^{\infty} \int u_n d\mu$$

*(including the possibility* $+\infty = +\infty$.)

**Theorem 6.10.** *(Fatou) Let* $(u_n)_{n \in \mathbb{N}} \subset \mathcal{M}_{\mathbb{R}}^+(\mathcal{A})$ *be a sequence of positive measurable functions. Then* $u = \liminf_{n \to \infty} u_n$ *is measurable and*

$$\int \liminf_{n \to \infty} u_n d\mu = \liminf_{n \to \infty} \int u_n d\mu \qquad (6)$$

# 7 Integrals of Measurable Functions

We have defined our integral for positive measurable functions, i.e. functions in $\mathcal{M}^+(\mathscr{A})$. To extend our integral to not only functions in $\mathcal{M}^+(\mathscr{A})$ we first notice that

$$u \in \mathcal{M}_{\overline{\mathbb{R}}}(\mathscr{A}) \Leftrightarrow u = u^+ - u^-, \ u^+, u^- \in \mathcal{M}_{\overline{\mathbb{R}}}^+,$$

i.e. that every measurable function can be written as a sum of **positive** measurable functions.

**Definition 7.1** ($\mu$-integrable)**.** A function $u : X \to \overline{\mathbb{R}}$ on $(X, \mathscr{A}, \mu)$ is $\mu$-integrable, if it is $\mathscr{A}/\mathscr{B}(\overline{\mathbb{R}})$-measurable and if $\int u^+ d\mu, \int u^- d\mu < \infty$ (recall the definition for the integral of positive measurable functions). Then

$$\int u d\mu := \int u^+ d\mu - \int u^- d\mu \in (-\infty, \infty)$$

is the ($\mu$-)integral of u. We write $\mathcal{L}^1(\mu)$ for the set of all real-valued $\mu$-integrable functions [1].

**Theorem 7.2.** *Let $u \in \mathcal{M}_{\overline{\mathbb{R}}}(\mathscr{A})$, then the following conditions are equivalent:*

*(i)* $u \in \mathcal{L}_{\mathbb{R}}^1(\mu)$.

*(ii)* $u^+, u^- \in \mathcal{L}_{\mathbb{R}}^1(\mu)$.

*(iii)* $|u| \in \mathcal{L}_{\mathbb{R}}^1(\mu)$.

*(iv)* $\exists w \in \mathcal{L}_{\mathbb{R}}^1(\mu)$ *with* $w \geq 0$ *s.t.* $|u| \leq w$.

**Theorem 7.3** (Properties of the $\mu$-integral)**.** *The $\mu$-integral is:* **homogeneous, additive**, *and:*

*(i)* $\min\{u, v\}, \max\{u, v\} \in \mathcal{L}_{\mathbb{R}}^1(\mu)$     (*lattice property*)

*(ii)* $u \leq v \Rightarrow \int u d\mu \leq \int v d\mu$     (*monotone*)

*(iii)* $\left| \int u d\mu \right| \leq \int |u| d\mu$     (*triangle inequality*)

**Remark.** *If $u(x) \pm v(x)$ is defined in $\overline{\mathbb{R}}$ for all $x \in X$ then we can exclude $\infty - \infty$ and the theorem above just says that the integral is linear:*

$$\int (au + bv) d\mu = a \int u d\mu + b \int v d\mu.$$

*This is always true for real-valued $u, v \in \mathcal{L}^1(\mu) = \mathcal{L}_{\mathbb{R}}^1(\mu)$, making $\mathcal{L}^1(\mu)$ a vector space with addition and scalar multiplication defined by*

$$(u + v)(x) := u(x) + v(x), \ (a \cdot u)(x) := a \cdot u(x),$$

*and*

$$\int ...d\mu : \mathcal{L}^1(\mu) \to \mathbb{R}, \ u \mapsto \int u d\mu,$$

*is a* **positive linear functional***.*

# 8 Null sets and the Almost Everywhere

**Definition 8.1.** A ($\mu$-)null set $N \in \mathcal{N}_\mu$ is a measurable set $N \in \mathscr{A}$ satisfying

$$N \in \mathcal{N}_\mu \Longleftrightarrow N \in \mathscr{A} \text{ and } \mu(N) = 0.$$

This can be used generally about a 'statement' or 'property', but we will be interested in questions like 'when is $u(x)$ equal to $v(x)$', and we answer this by saying

$$u = v \ a.e. \Leftrightarrow \{x : u(x) \neq v(x)\} \text{ is (contained in) a } \mu\text{-null set,}$$

i.e.

$$u = v \ \mu\text{-a.e.} \Leftrightarrow \mu(\{x : u(x) \neq v(x)\}) = 0.$$

The last phrasing should of course include that the set $\{x : u(x) \neq v(x)\}$ is in $\mathscr{A}$.

**Theorem 8.2.** *Let $u \in \mathcal{M}_{\overline{\mathbb{R}}}(\mathscr{A})$, then:*

*(i)* $\int |u| d\mu = 0 \Leftrightarrow |u| = 0 \ a.e. \ \Leftrightarrow \mu\{u \neq 0\} = 0$,

*(ii)* $\mathbb{1}_N u \in \mathcal{L}_{\mathbb{R}}^1(\mu) \ \forall \ N \in \mathcal{N}_\mu$,

*(iii)* $\int_N u d\mu = 0$.

*(i) is really useful, later we will define $\mathcal{L}^p$ and the $||\cdot||_p$-(semi)norm. Then (i) means that if we have a sequence $u_n$ converging to $u$ in the $|| \cdot ||_p$-norm then $u_n(x) = u(x) \ a.e.$*

**Corollary 8.3.** *Let $u = v \ \mu$-a.e. Then*

*(i)* $u, v \geq 0 \Rightarrow \int u d\mu = \int v d\mu$,

*(ii)* $u \in \mathcal{L}_{\mathbb{R}}^1(\mu) \Rightarrow v \in \mathcal{L}_{\mathbb{R}}^1(\mu)$ *and* $\int u d\mu = \int v d\mu$.

**Corollary 8.4.** *If $u \in \mathcal{M}_{\overline{\mathbb{R}}}(\mathscr{A})$, $v \in \mathcal{L}_{\mathbb{R}}^1(\mu)$ and $v \geq 0$ then*

$$|u| \leq v \ a.e. \ \Rightarrow u \in \mathcal{L}_{\mathbb{R}}^1(\mu).$$

**Proposition 8.5** (Markow inequality)**.** *For all $u \in \mathcal{L}_{\mathbb{R}}^1(\mu)$, $A \in \mathscr{A}$ and $c > 0$*

$$u(\{|u| \geq c\} \cap A) \leq \frac{1}{c} \int_A |u| d\mu,$$

*if $A = X$, then (obviosly)*

$$u\{|u| \geq c\} \leq \frac{1}{c} \int |u| d\mu.$$

**Corollary 8.6.** *If $u \in \mathcal{L}_{\overline{R}}^1(\mu)$, then $\mu$ is a.e. $\mathbb{R}$-vaued. In particular, we can find a version $\tilde{u} \in \mathcal{L}^1(\mu)$ s.t. $\tilde{u} = u$ a.e. and $\int \tilde{u} d\mu = \int u d\mu$*

---

[1] In words, we extend our integral to positive measurable functions by noticing that we can write every measurable function as a sum of positive measurable functions, something that we do know how to integrate. We don't want to run into the problem of $\infty - \infty$, thus we require the integral of the positive and negative parts to both (separately) be less than infinity.

# Completions of measure spaces

**Definition 8.7.** A measure space $(X, \mathscr{B}, \mu)$ is called **complete** if whenever $A \in \mathscr{B}$ and $\mu(A) = 0$, we have $B \in \mathscr{B} \; \forall B \subset A$.

**Remark.** *Any measure space can be completed as follows: Let $\bar{\mathscr{B}}$ be the $\sigma$-algebra generated by $\mathscr{B}$ and all sets $B \subset X$ s.t. there exists $A \in \mathscr{B}$ with $B \subset A$ and $\mu(A) = 0$.*

**Proposition 8.8.** *The $\sigma$-algebra $\bar{\mathscr{B}}$ can also be described as follows:*

$$\bar{\mathscr{B}} := \left\{ B \subset X : A_1 \subset B \subset A_2 \right.$$

$$\left. \text{for some } A_1, A_2 \in \mathscr{B} \text{ with } \mu(A_2 \backslash A_1) = 0 \right\},$$

*with $B, A_1, A_2$ as above, we define*

$$\bar{\mu} := \mu(A_1) = \mu(A_2)$$

*Then $\left(X, \bar{\mathscr{B}}, \bar{\mu}\right)$ is a complete measure space.*

**Definition 8.9.** *If $\mu$ is a Borel measure on a **metric** space $(X, d)$, then the completion $\bar{\mathscr{B}}(X)$ of the Borel $\sigma$-algebra with respect to $\mu$ is called the $\sigma$-algebra of $\mu$-measurable sets.*

**Remark.** *For $\mu = \lambda_n$ on $\mathbb{R}^n$ we talk about the $\sigma$-algebra of **Lebesgue measurable sets**. Instead of $\bar{\lambda_n}$ we still write $\lambda_n$ and call it the **Lebesgue measure**. A function $f : \mathbb{R}^n \to \mathbb{C}$, measurable w.r.t. the $\sigma$-algebra of Lebesgue measurable sets is called the **Lebesgue measurable**.*

*The following result shows that any Lebesgue measurable function coincides with a Borel function a.e.*

**Proposition 8.10.** *Assume $(X, \mathscr{B}, \mu)$ is a measure space and consider its completion $\left(X, \bar{\mathscr{B}}, \bar{\mu}\right)$. Assume $f : X \to \mathbb{C}$ is $\bar{\mathscr{B}}$-measurable. Then there is a $\mathscr{B}$-measurable function $g : X \to \mathbb{C}$ s.t. $f = g$ $\bar{\mu}$-a.e.*

# 9 Convergence Theorems and their Applications <span style="font-size:small">(12, [Schilling(2017)])</span>

- To interchange limits and integrals in **Riemann integrals** one typically has to assume uniform convergence. ¡- The set of Riemann integrable functions is somewhat limited, ¡- see theorem 9.7

**Theorem 9.1** (Generalization of Beppo Levi, monotone convergence)**.**

(i) *Let $(u_n)_{n \in \mathbb{N}} \subset \mathcal{L}^1(\mu)$ be s.t. $u_1 \le u_2 \le ...$ with limit $u := \sup_{n \in \mathbb{N}} u_n = \lim_{n \to \infty} u_n$. Then $u \in \mathcal{L}^1(\mu)$ **iff***

$$\sup_{n \in \mathbb{N}} \int u_n d\mu < +\infty,$$

*in which case*

$$\sup_{n \in \mathbb{N}} \int u_n d\mu = \int \sup_{n \in \mathbb{N}} u_n d\mu.$$

(ii) *Same thing only with a decreasing sequence $... > -\infty$ in which case*

$$\inf_{n \in \mathbb{N}} \int u_n d\mu = \int \inf_{n \in \mathbb{N}} u_n d\mu.$$

**Theorem 9.2** (Lebesgue; dominated convergence)**.** *Let $(u_n)_{n \in \mathbb{N}} \subset \mathcal{L}^1(\mu)$ s.t.*

(a) $|u_n|(x) \le w(x)$, $w \in \mathcal{L}^1(\mu)$,

(b) $u(x) = \lim_{n \to \infty} u_n(x)$ *exists in* $\bar{\mathbb{R}}$,

*then $u \in \mathcal{L}^1(\mu)$ and we have*

(i) $\lim_{n \to \infty} \int |u_n - u| d\mu = 0$;

(ii) $\lim_{n \to \infty} \int u_n d\mu = \int \lim_{n \to \infty} u_n d\mu = \int u d\mu$;

## Application 1: Parameter-Dependent Integrals

- We are interested in questions of the sort, when is

$$U(t) := \int u(t, x) \mu(dx), \; t \in (a, b),$$

again a smooth function of t? The answer involves interchange of limits and integration. Also, it turns out to better understand Riemann integrability, we need the Lebesgue integral.

**Theorem 9.3** (continuity lemma)**.** *Let $\emptyset \ne (a, b) \subset \mathbb{R}$ be a non-degenerate open interval and $u : (a, b) \times X \to \mathbb{R}$ satisfy*

(a) $x \mapsto u(t, x)$ *is in $\mathcal{L}^1(\mu)$ for every fixed $t \in (a, b)$;*

(b) $t \mapsto u(t, x)$ *is continuous for every fixed $x \in X$;*

(c) $|u(t, x)| \le w(x)$ *for all $(t, x) \in (a, b) \times X$ and some $w \in \mathcal{L}^1(\mu)$.*

*Then the function $U : (a, b) \to \mathbb{R}$ given by*

$$t \mapsto U(t) := \int u(t, x) \mu(dx) \tag{7}$$

*is continuous.*

**Theorem 9.4** (differentiability lemma)**.** *Let $\emptyset \le (a, b) \subset \mathbb{R}$ be a non-degenerate open interval and $u : (a, b) \times X \to \mathbb{R}$ satisfy*

(a) *Same*

(b) *Same*

(c) $|\partial_t u(t, x)| \le w(x)$ *for all $(t, x) \in (a, b) \times X$ and some $w \in \mathcal{L}^1(\mu)$.*

*Then the function in 7 is differentiable and its derivative is*

$$\frac{d}{dt} U(t) = \frac{d}{dt} \int u(t, x) \mu(dx) = \int \frac{\partial}{\partial t} u(t, x) \mu(dx). \tag{8}$$

## Application 2: Riemann vs Lebesgue Integration

Consider only $(X, \mathscr{A}, \mu) = (\mathbb{R}, \mathscr{B}(\mathbb{R}), \lambda)$.

**Definition 9.5** (The Riemann Inegral)**.** Consider on the finite interval $[a, b] \subset \mathbb{R}$ the partition

$$\Pi := \{a = t_0 < t_1 < ... < t_k < b\}, k = k(\Pi), \tag{9}$$

and introduce

$$S_\Pi[u] := \sum_{i=1}^{k(\Pi)} m_i (t_i - t_{i-1}), \qquad m_i := \inf_{x \in [t_{i-1}, t_i]} u(x), \tag{10}$$

$$S^\Pi[u] := \sum_{i=1}^{k(\Pi)} M_i (t_i - t_{i-1}), \qquad M_i := \sup_{x \in [t_{i-1}, t_i]} u(x). \tag{11}$$

$$\tag{12}$$

A bounded function $u : [a, b] \to \mathbb{R}$ is said to be **Riemann integrable** if the values

$$\underline{\int} u := \sup_\Pi S_\Pi[u] = \inf_\Pi S^\Pi[u] =: \overline{\int} u \qquad (13)$$

coincide and are finite. Their common value is called the **Riemann integral** of $u$ and denoted by $(R) \int_a^b u(x)dx$ or $\int_a^b u(x)dx$.

**Theorem 9.6.** *Let $u : [a, b] \to \mathbb{R}$ be a **measurable** and **Riemann integrable** function. Then*

$$u \in \mathcal{L}^1(\lambda) \ and \ \int_{[a,b]} u d\lambda = \int_a^b u(x)dx. \qquad (14)$$

**Theorem 9.7.** *Let $u : [a, b] \to \mathbb{R}$ be a bounded function, it is Riemann integrable **iff** the points in $(a, b)$ where $u$ is discontinuous are a (subset of) Borel measurable null set.*

## Improper Riemann Integrals

- The Lebesgue integral extends the (*proper*) Riemann integral. However, there is a further extension of the Riemann integral which cannot be captured by Lebesgue's theory. $u$ is Lebesgue integrable *iff* $|u|$ ha finite Lebesgue integral. ¡- The Lebesgue integral does not respect sign-changes and cancellations. However, the following *improper Riemann integral* does:

$$(R) \int_0^\infty u(x)dx := \lim_{n \to \infty} (R) \int_0^a u(x)dx. \qquad (15)$$

**Corollary 9.8.** *Let $u : [0, \infty) \to \mathbb{R}$ be a measurable, Riemann integrable function for every interval $[0, N]$, $N \in \mathbb{N}$. Then $u \in \mathcal{L}^1[0, \infty)$ **iff***

$$\lim_{N \to \infty} (R) \int_0^N |u(x)|dx < \infty. \qquad (16)$$

*In this case,* $(R) \int_0^\infty u(x)dx = \int_{[0,\infty)} u d\lambda$

**Example** of a function which is *improperly Riemann integrable* but **not** *Lebesgue integrable*:

$$f(x) = \frac{\sin(x)}{x}. \qquad (17)$$

**Proposition 9.9** (appearing as example 12.13 in Schilling). *Let $f_\alpha(x) := x^\alpha, x > 0$ and $\alpha \in \mathbb{R}$. Then*

*(i)* $f_{(\alpha)} \in \mathcal{L}^1(0, 1) \Leftrightarrow \alpha > -1.$

*(ii)* $f_{(\alpha)} \in \mathcal{L}^1[1, \infty) \Leftrightarrow \alpha < -1.$

## 10    Regularity of Measures (App. H, [Schilling(2017)])

We let $(X, d)$ be a metric space and denote by $\mathcal{O}$ the open, by $\mathcal{C}$ the closed and $\mathscr{B}(X) = \sigma(\mathcal{O})$ the Borel set of $X$.

**Definition 10.1.** A measure $\mu$ on $(X, d, \mathscr{B}(X))$ is called outer regular, if

$$\mu(B) = \inf \{\mu(U) \mid B \subset U, \ U \text{ open}\} \qquad (18)$$

and inner regular, if $\mu(K) < \infty$ for all compact sets $K \subset X$ and

$$\mu(U) = \sup \{\mu(K) \mid K \subset U, \ K \text{ compact}\}. \qquad (19)$$

A measure which is both inner and outer regular is called **regular**. We write $\mathfrak{m}_r^+(X)$ for the family of regular measures on $(X, \mathscr{B}(X))$.

**Remark.** *The space $X$ is called $\sigma$-compact if there is a sequence of compact sets $K_n \uparrow X$. A typical example of such a space is a locally compact, separable metric space.*

**Theorem 10.2.** *Let $(X, d)$ be a metric space. Every finite measure $\mu$ on $(X, \mathscr{B}(X))$ is outer regular. If $X$ is $\sigma$-compact, then $\mu$ is also inner regular, hence regular.*

**Theorem 10.3.** *Let $(X, d)$ be a metric space and $\mu$ be a measure on $(X, B(X))$ such that $\mu(K) < \infty$ for all compact sets $K \subset X$.*

1 If $X$ is $\sigma$-compact, then $\mu$ is inner regular.

2 If there exists a sequence $G_n \in \mathcal{O}$, $G_n \uparrow X$ such that $\mu(G_n) < \infty$, then $\mu$ is outer regular.

## 11    The Function Spaces $\mathcal{L}^p$ (13, [Schilling(2017)])

Assume $V$ is a vector space over $\mathbb{K} \in \{\mathbb{C}, \mathbb{R}\}$.

**Definition 11.1.** A seminorn on $V$ is a map $p : V \to [0, +\infty)$ s.t.

(1) $p(cx) = |c|p(x) \ \forall x \in V, \forall c \in \mathbb{K}.$

(2) $p(x + y) \leq p(x) + p(y) \ \forall x, y \in V.$   **triangle inequality**.

A seminorm is called a norm if we also have

$$p(x) = 0 \iff x = 0.$$

A norm is commonly denoted $||x||$, and a vectorspace equipped with a norm is called a **normed space**.

**Definition 11.2.** Assume $(X, d)$ is a measure space. Fix $1 \leq p \leq \infty$. For every measurable function $f : X \to \mathbb{C}$ we define the following

$$||f||_p = \left(\int_X |f|^p d\mu\right)^{1/p} \in [0, +\infty]. \qquad (20)$$

We can see that $||cf||_p = |c|\,||f||_p \ \forall c \in \mathbb{C}$.

Notice that by Theorem 8.2(i) we have that $||f||_p = 0 \Rightarrow f = 0$ a.e. Consider for example $\lim_{n\to\infty} ||f_n - f||_p = 0$, then we can find a subsequence s.t. $\lim_{k\to\infty} |f_{n(k)} - f| = 0$ a.e., i.e. $\lim_{k\to\infty} f_{n(k)} = f$ a.e.

**Lemma 11.3.**

$$||f + g||_p \leq ||f||_p + ||g||_p. \qquad (21)$$

**Definition 11.4.** We define

$$\mathcal{L}^p(X, d\mu) = \{f : X \to \mathbb{C} \mid f \text{ is measurable and } ||f||_p < \infty\}.$$

This is a vectorspace with seminorm $f \mapsto ||f||_p$. And in general this is not a normed space, since $||f||_p = 0 \iff f = 0$ a.e.

Generally, if $p$ is a seminorm on a vectorspace $V$, then

$$V_0 = \{x \in V \mid p(x) = 0\} \qquad (22)$$

which is a subspace of $V$. Then we consider the quotient/factor space $V/V_0$.

**Definition 11.5.** For $x, y \in V$, define

$$x \sim y \iff x - y \in V_0. \tag{23}$$

This is an equivalence relation on $V$. The representation class of $V$ is defined by $[x]$ or $x + V_0$.

Then $V/V_0$ is equals the set of equivalence classes. We can show that it is a normed space.

$$[x] + [y] = [x + y] \quad, \quad c[x] = [cx] \quad, \quad ||[x]|| = p(x).$$

Applying this to $\mathcal{L}^p(X, d\mu)$ we get the normed space

$$L^p(X, d\mu) := \mathcal{L}^p(X, d\mu)/\mathcal{N} = \mathcal{L}^p(X, d\mu)/\sim. \tag{24}$$

Where $\mathcal{N}$ is the space of measurable functions $f$ s.t. $f = 0$ a.e. The equivalence relation $\sim$ is defined by

$$u \sim v \iff \{u \neq v\} \in \mathcal{N}_\mu \iff \mu\{u \neq v\} = 0,$$

and so $L^p(X, d\mu)$ consists of *all equivalence classes* $[u]_p = \{v \in \mathcal{L}^p | u \sim v\}$. So for every $u \in L^p$ there is no $v \in L^p$ such that $\mu\{u \neq v\} \neq 0$.

We will further continue to denote the norm by $|| \cdot ||_p$, and we will normally **not** distinguish between $f \in \mathcal{L}^p(X, d\mu)$ and the vector in $L^p(X, d\mu)$ that $f$ defines.

**Definition 11.6.** A normed space $(X, || \cdot ||)$ is called a Banach space if $V$ is complete w.r.t the metric $d(x, y) = ||x - y||$.

**Theorem 11.7.** *If $(X, \mathscr{B}, \mu)$ is a measure space, $1 \leq p \leq \infty$, then $L^p(X, d\mu)$ is a Banach space.*

**Definition 11.8.** A measurable function $f : X \to \mathbb{C}$ is called **essentially bounded** if there is $c \geq 0$ s.t.

$$\mu(\{x \ : \ |f(x)| > c\}) = 0. \tag{25}$$

That is $|f| \leq c$ a.e. The smallest such $c$ is called the essential supremum of $f$ and is denoted by $||f||_\infty$.

**Definition 11.9.**

$$\mathcal{L}^\infty(X, d\mu) = \{f : X \to \mathbb{C} \ | \ f \text{ is measurable and } ||f||_\infty < \infty\}.$$

$$L^\infty(X, d\mu) = \mathcal{L}^\infty(X, d\mu)/\mathcal{N}.$$

Where by the previous definiton these spaces become the spaces of all essentially bounded functions.

**Theorem 11.10.** *If $(X, \mathscr{B}, \mu)$ is a $\sigma$-finite measure space, then $L^\infty(X, d\mu)$ is a Banach space.*

## Convergence in $\mathcal{L}^p$ and completeness

**Lemma 11.11.** *For any sequence $(u_n)_{n \in \mathbb{N}} \subset \mathcal{L}^p(\mu), p \in [1, \infty)$, of positive functions $u_n \geq 0$ we have*

$$\Big|\Big|\sum_{n=1}^\infty u_n\Big|\Big|_p \leq \sum_{n=1}^\infty ||u_n||_p.$$

**Theorem 11.12** (Riesz-Fischer). *The spaces $\mathcal{L}^p(\mu), p \in [1, \infty)$, are complete, i.e. every Cauchy sequence $(u_n)_{n \in \mathbb{N}} \subset \mathcal{L}^p(\mu)$ converges to some limit $u \in \mathcal{L}^p(\mu)$*

**Corollary 11.13.** *Let $(u_n)_{n \in \mathbb{N}} \subset \mathcal{L}^p(\mu), p \in [1, \infty)$ with $\mathcal{L}^p - \lim_{n \to \infty} u_n = u$. Then there exists a subsequence $(u_{n_k})_{k \in \mathbb{N}}$ s.t. $\lim_{k \to \infty} u_{n_k}(x) = u(x)$ holds for almost every $x \in X$.*

**Theorem 11.14.** *Let $(u_n)_{n \in \mathbb{N}} \subset \mathcal{L}^p(\mu), p \in [1, \infty)$, be a sequence of functions s.t. $|u_n| \leq w \ \forall n \in \mathbb{N}$ and some $w \in \mathcal{L}^p(\mu)$. If $u(x) = \lim_{n \to \infty} u_n(x)$ exists for (almost) every $x \in X$, then*

$$u \in \mathcal{L}^p \text{ and } \lim_{n \to \infty} ||u - u_n||_p = 0.$$

**Theorem 11.15** (F. Riesz (convergence theorem)). *Let $(u_n)_{n \in \mathbb{N}} \subset \mathcal{L}^p(\mu), p \in [1, \infty)$, be a sequence s.t. $\lim_{n \to \infty} u_n(x) = u(x)$ for almost every $x \in X$ and some $u \in \mathcal{L}^p(\mu)$. Then*

$$\lim_{n \to \infty} ||u_n - u||_p = 0 \iff \lim_{n \to \infty} ||u_n||_p = ||u||_p.$$

## 12 Dense and Determining Sets (17, [Schilling(2017)])

**Definition 12.1** (Dense Sets). A set $\mathcal{D} \subset \mathcal{L}^p(\mu), p \in [0, \infty]$, is called *dense* if for every $u \in \mathcal{L}^p(\mu)$ there exist a sequence $(f_n)_{n \in \mathbb{N}} \subset \mathcal{D}$ s.t. $\lim_{n \to \infty} ||u - f_n||_p = 0$.

**Theorem 12.2.** *Assume $X, d$ is a metric space and $\mu$ is a Borel measure that is finite on every ball $1 \leq p < \infty$. Then the space of bounded continuous functions with bounded support is dense in $\mathcal{L}^p(X, d\mu)$. Where bounded support means that $f$ vanishes outside some ball.*

*Proof.* We want to approximate $f \in \mathcal{L}^p(X, d\mu)$ by bounded continuous functions with bounded support. By considering separately $(\text{Re}(f))_I$ and $(\text{Im}(f))_I$ we may assume that $f \geq 0$. Then we can find simple functions $f_n$ s.t. $0 \leq f_n \leq f, f_n \to f$ pointwise. As $|f - f_n|^p \leq |f|^p$, by the dominated convergence theorem we have $f_n \to f \in \mathcal{L}^p(X, d\mu)$. Hence, it suffices to consider simple $f$, but then it suffices to approximate $f = \pi_A$. Note that $\pi_A \in \mathcal{L}^p(X, d\mu)$ iff $\mu(A) < \infty$.

Fix $x_0 \in X$. Then $\pi_{A \cap B_n(x_0)} \nearrow \pi_A$ pointwise, hence $\pi_{A \cap B_n(x_0)} \to \pi_A \in \mathcal{L}^p(X, d\mu)$, again by the dominated convergence theorem.

Therefor it suffices to consider $A \subset B_n(x_0)$. As $\mu$ is outer regular, we have

$$\mu(A) = \inf_{\substack{A \subset U \subset B_n(x_0) \\ U \text{ is open}}} \mu(U).$$

Note that $||\pi_U - \pi_A||_p = \mu(U \backslash A)^{1/p}$. Hence, we can choose $U_k \subset B_n(x_0)$ s.t. $A \subset U_k, U_k$ is open, $\pi_{U_k} \to \pi_A \in \mathcal{L}^p(X, d\mu)$.

Therefor it suffices to approximate $\pi_U$ for open $U \subset B_n(x_0)$. Consider the functions

$$f_k(x) = \frac{kd(x, U^c)}{1 + kd(x, U^c)}.$$

Then $0 \leq f_k \leq 1$, $f_k$ is continuous, supported on $\bar{U} \subset \bar{B}_n(x_0)$ and $f_k \nearrow \pi_U$ pointwise, hence $f_k \xrightarrow[k \to \infty]{} \pi_U \in \mathcal{L}^p(X, d\mu)$. $\square$

**Theorem 12.3.** *Assume $(X, d)$ is a separable locally compact metric space and $\mu$ is a Borel Measure on $X$ s.t. $\mu(K) < \infty \ \forall$ compact $K \subset K$. Then the space $C_c(X)$ of continuous compactly supported functions is dense in $\mathcal{L}^p(X, d\mu)$.*

Recall that the support of a function $f$ is $\text{supp}(f) = \{x \in X : f(x) \neq 0\}$, *closed support* is the closure of $\text{supp}(f)$ (i.e. boundary points are included), often just written as $\text{supp}(f)$, and a function is said to have *compact support* if $\text{supp}(f)$ is *compact*.

In particular, either theorem shows that if $\mu$ is a Borel measure on $\mathbb{R}^n$ s.t. the measure of every ball is finite, then $C_c(\mathbb{R}^n)$ is dense in $\mathcal{L}^p(\mathbb{R}^n, d\mu), 1 \leq p < \infty$. Later we will see that even $C_c^\infty(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n, d\mu)$.

**Remark.** *These results do not extend to $p = \infty$ in general. For $\mu = \lambda_n$ we write simply $\mathcal{L}^p(\mathbb{R}^n)$.*

**Remark.** *Theorem 17.8 in the book is WRONG. For example, $X = \mathbb{Q}$ with the usual metric is $\sigma$-compact, supports nonzero finite measure, but $C_c(\mathbb{Q}) = 0$.*

# Modes of Convergence

**Definition 12.4** (convergence in meassure). A sequence of measurable functions $u_n : X \to \bar{\mathbb{R}}$ *converges in measure* if

$$\forall \epsilon > 0 \forall A \in \mathscr{A}, \mu(A) < \infty : \lim_{n \to \infty} \mu\left(\{|u_n - u| > \epsilon\} \cap A\right) = 0$$

holds for some $u \in \mathcal{M}(\mathscr{A})$. We write $\mu\text{-}\lim_{n \to \infty} u_n = u$ or $u_n \xrightarrow{\mu} u$.

Assume $(X, \mathscr{B}, \mu)$ is a measure space. Given measurable functions $f_n, f : X \to \mathbb{C}$, recall that

$$f_n \to f \text{ a.e.}$$

means that $f_n(x) \xrightarrow[n \to \infty]{} f(x)$ for all x outside a set of measure zero.

**Theorem 12.5** (Egorov). *Assume $\mu(X) < \infty$ and $f_n \to f$ a.e. Then, $\forall \epsilon > 0$, there exists $X_\epsilon \in \mathscr{B}$ s.t. $\mu(X_\epsilon) < \epsilon$ and $f_n \to f$ uniformly on $X \backslash X_\epsilon$.*

In addition to pointwise and uniform convergence we also consider the following:
$f_n \to f$ *in the p-th mean* if $||f_n - f||_p \xrightarrow[n \to \infty]{} 0$. For $p = 1$ we say *in mean*, for $p = 2$ we say in quadratic mean.
$f_n \to f$ *in measure* if $\forall \epsilon > 0$ we have

$$\mu\left(\{x : |f_n(x) - f(x)| \geq \epsilon\}\right) \xrightarrow[n \to \infty]{} 0.$$

**Theorem 12.6** (Lemma 22.4 in the book?). *Assume $(X, \mathscr{B}, d\mu)$ is a measure space, $1 \leq p < \infty$, $f_n, f : X \to \mathbb{C}$ are measurable functions. Then*

*(i) If $f_n \to f$ in the p-th mean, then $f_n \to f$ in measure.*

*(ii) If $f_n \to f$ in measure, then there is a subsequence $(f_{n_k})_{k=1}^\infty$ s.t. $f_{n_k} \to f$ a.e.*

*(iii) If $f_n \to f$ a.e. and $\mu(X) < \infty$, then $f_n \to f$ in measure.*

*In particular, if $f_n \to f$ in the p-th mean, then $f_{n_k} \to f$ a.e. for a subsequence $(f_{n_k})_k$.*

# 13 Abstract Hilbert Spaces <span>(26, [Schilling(2017)])</span>

Assume $\mathcal{H}$ is a vector space over $\mathbb{C}$.

**Definition 13.1.** A *pre-inner product* on $\mathcal{H}$ is a map $(\cdot, \cdot) : H \times H \to \mathbb{C}$ which is

(i) Sesquilinear: linear in the first variable and antilinear in the second:

$$(\alpha u + \beta v, w) = \alpha(u, w) + \beta(v, w),$$
$$(w, \alpha u + \beta v) = \bar{\alpha}(w, u) + \bar{\beta}(w, v), \ u, v, w \in H \text{ and } \alpha, \beta \in \mathbb{C}.$$

(ii) Hermitian: $(u, v) = \overline{(v, u)}$.

(iii) Positive semidefinite: $(u, v) \geq 0$.

It is called an **inner product**, or a scalar product, if instead of (iii) the map is positive definite; $(u, v) > 0$. This definition also works for $\mathbb{R}$ instead of $\mathbb{C}$.

**Cauchy-Schwartz inequality** If $(\cdot, \cdot)$ is a pre-inner product, then $|(u, v)| \leq (u, u)^{1/2} (v, v)^{1/2}$.

**Corollary 13.2.** *Assume we have a seminorm $||u|| := (u, u)^{1/2}$. It is a norm iff $(\cdot, \cdot)$ is an inner product.*

**Definition 13.3** (Hilbert space). A Hilbert space is a complex vector space $\mathcal{H}$ with an inner product $(\cdot, \cdot)$ s.t. $\mathcal{H}$ is complete with respect to the norm $||u|| = (u, u)^{1/2}$.

1. The norm on a Hilbert space is determined by the inner product, but the inner product can also be recovered by the norm by the *polarization identity*: $(u, v) = \frac{||u+v||^2 - ||u-v||^2}{4} + i\frac{||u+iv||^2 - ||u-iv||^2}{4}$.

2. *Parallelogram law*: $||u + v||^2 + ||u - v||^2 = 2||u||^2 + 2||v||^2$.

3. A norm on a vector space is given by an inner product iff it satisfies the parallelogram law, and then the scalar product is uniquely determined by the polarization identity.

**Example 13.4.** Assume $(X, \mathscr{B}, \mu)$ is a measure space. Then $\mathcal{L}^2(X, d\mu)$ is a Hilbert space with inner product

$$(f, g) = \int_X f \bar{g} d\mu.$$

This is well-defined, as $|f\bar{g}| \leq \frac{1}{2}(|f|^2 + |g|^2)$.

In particular, if $\mathscr{B} = \mathcal{P}(X)$ and $\mu$ is the counting measure, then $L^2(X, d\mu)$ is denoted by $l^2(X)$; for $X = \mathbb{N}$ we write simply $l^2$. Note that in this case for $f : X \to [0, +\infty]$ we have

$$\int_X f d\mu = \sum_{x \in X} f(x) := \sup_{\substack{F \subset X \\ F \text{ is finite}}} \sum_{x \in F} f(x),$$

and if $\sum_{x \in X} f(x) < \infty$, then $\{x : f(x) > 0\}$ is at most countable, so $\sum_{x \in X} f(x) = \sum_{x : f(x) > 0} f(x)$ is the usual sum of a series.

Recall that a subset $\mathcal{C}$ of a vector space is called *convex* if

$$u, w \in \mathcal{C} \to tu + (1 - t)w \in \mathcal{C} \ \forall t \in (0, 1).$$

The following is one of the key properties of the Hilbert space

**Theorem 13.5** (projection theorem). *Assume $\mathcal{H}$ is a Hilbert space and $\mathcal{C} \subset H$ is a closed convex subset. Then for every $u \in H$ there is a unique $u_0 \in \mathcal{C}$ (minimizer) s.t.*

$$||u - u_0|| = d(u, \mathcal{C})(= \inf_{x \in \mathcal{C}} ||u - x||).$$

*Proof.* Let $d = d(u, \mathcal{C})$. Choose $u_n \in \mathcal{C}$ s.t. $||u - u_n|| \to d$. We claim that $(u_n)_{n=1}^\infty$ is a Cauchy sequence. As $\frac{u_n + u_2}{2} \in \mathcal{C}$, we have

$$d^2 \leq ||u - \frac{u_n + u_m}{2}||^2 = \frac{1}{4}||(u - u_m) + (u - u_m)||^2$$

$$\underset{\text{parallelogram law}}{=} \frac{1}{4}\left(2||u - u_n||^2 + 2||u - u_m||^2\right).$$

so

$$||u_n - u_m||^2 \leq 2\left(||u - u_n||^2 - d^2\right) + 2\left(||u - u_m||^2 - d^2\right).$$

Thus, $(u_n)_{n \in \mathbb{N}}$ is indeed Cauchy, hence $u_n \to u_0 \in \mathcal{C}$ for some $u_0$ and

$$||u - u_0|| = \lim_{n \to \infty} ||u - u_m|| = d = d(u, \mathcal{C}).$$

If $u_0'$ is another such point, we can take $u_{2n} = u_0, u_{2n+1} = u_0'$ and conclude that $u_0 = u_0'$. $\qquad \square$

# 14 Orthogonal Projections

For a Hilbert space $\mathcal{H}$ and a subset $A \subset H$, let

$$A^\perp := \{x \in H : x \perp y \ \forall y \in A\},$$

where $x \perp y$ means that $(x, y) = 0$. $A^\perp$ is a closed subspace of $\mathcal{H}$.

**Proposition 14.1.** *Assume $\mathcal{H}_0$ is a closed subspace of a Hilbert space $\mathcal{H}$. Then every $u \in H$ uniquely decomposes as*

$$u = u_0 + u_1, \ \text{with } u_0 \in H \ \text{and } u_1 \in \mathcal{H}_0^\perp.$$

*Moreover, $||u - u_0|| = d(u, \mathcal{H}_0)$ and $||u||^2 = ||u_0||^2 + ||u_1||^2$.*

For a closed subspace $\mathcal{H}_0 \subset \mathcal{H}$, consider the map $P : H \to \mathcal{H}_0$ s.t. $Pu \in \mathcal{H}_0$ is the unique element satisfying $u - Pu = H_0^\perp$. The operator $P$ is linear. It is also contractive, meaning that $||Pu|| \leq ||u||$, since $||u||^2 = ||Pu||^2 + ||u - Pu||^2$. It is called the orthogonal projection onto $\mathcal{H}_0$.

If $\mathcal{H}_0$ is finite dimensional with an orthonormal basis $u_1, ..., u_n$ then

$$Pu = \sum_{k=1}^n (u, u_k) u_k.$$

Orthonormal bases can be defined for arbitrary Hilbert spaces.

**Definition 14.2** (orthonormal system). An orthonormal system in $\mathcal{H}$ is a collection of vectors $u_i \in H$ $(i \in I)$ s.t.

$$(u_i, u_j) = \delta_{ij} \ \forall i, j \in I.$$

It is called an *orthonormal basis* if $\mathrm{span}\{u_i\}_{i \in I}$ denotes the linear span of $\{u_i\}_{i \in I}$, the space of finite linear combinations of the vectors $u_i$.

**Definition 14.3.** A Hilbert space $\mathcal{H}$ is said to be *separable* if $\mathcal{H}$ contains a countable dense subset $G \subset \mathcal{H}$.

**Theorem 14.4.** *Every Hilbert space $\mathcal{H}$ has an orthonormal basis. If $\mathcal{H}$ is separable, then there is a countable orthonormal basis.*

**Proposition 14.5.** *Assume $\{u_i\}_{i \in I}$ is an orthonormal system in a Hilbert space $H$. Take $u \in \mathcal{H}$. Then*

(i) Bessel's inequality: $\sum_{i \in I} |(u, u_i)|^2 \leq ||u||^2$, in particular, $\{i : (u, u_i) \neq 0\}$ is countable.

(ii) Parseval's identity: If $\{u_i\}_{i \in I}$ is an orthonormal basis, then $\sum_{i \in I} |(u, u_i)|^2 = ||u||^2$.

If $(u_i)_{i \in I}$ is an orthonormal basis, then the numbers $(u, u_i)$ are called the **Fourier coefficients** of $u$ with respect to $(u_i)_{i \in I}$. The Parseval identity then suggests that $u$ is determined by its Fourier coefficients. This is true, and even more, we have:

**Proposition 14.6.** *Assume $(u_i)_{i \in I}$ is an orthonormal basis in a Hilbert space $\mathcal{H}$. Then for every vector $(c_i)_{i \in I} \in l^2(I)$ there is a unique vector $u \in \mathcal{H}$ with Fourier coefficients $c_i$, and we write*

$$u = \sum_{i \in I} c_i u_i.$$

**Remark.** *Equivalently, the element $u = \sum_{i \in I} c_i u_i$ can be described as the unique element in $\mathcal{H}$ s.t. $\forall \epsilon > 0$ there is a finite $F_0 \subset I$ s.t. $||u - \sum_{i \in F} c_i u_i|| < \epsilon \ \forall$ finite $F \supset F_0$.*

**Corollary 14.7.** *We have a linear isomorphism $U : l^2(I) \xrightarrow{\sim} \mathcal{H}$, $U((c_i)_{i \in I}) = \sum_{i \in I} c_i u_i$. By Parseval's identity this isomorphism is isometric, that is, $||Ux|| = ||x|| \ \forall x \in l^2(I)$. By the polarization identity this is equivalent to*

$$(Ux, Uy) = (x, y) \ \forall x, y \in l^2(I).$$

*Therefor $U$ is unitary.*

**Corollary 14.8.** *Up to a unitary isomorphism, there is only one infinite dimensional separable Hilbert space, namely, $l^2$.*

# 15 Dual spaces

Given two orthonormal bases $(u_i)_{i \in I}$ and $(v_i)_{i \in I}$ in a Hilbert space $\mathcal{H}$, we can decompose

$$u_i = \sum_{j \in I} (u_i, v_j) v_j$$

and using that the sets $\{j : (u_i, v_j) \neq 0\}$ are countable proove the following:

*Claim*: Any two orthonormal bases in a Hilbert space have the same cardinality.

**Example 15.1** (classical Fourier series). Consider $\mathcal{H} = L^2(0, 2\bar{\mu}) = L^2((0, 2\bar{\mu}), d\lambda)$. For $n \in \mathbb{Z}$, define $e_n(t) = \frac{1}{\sqrt{2\bar{\mu}}} e^{int}$. By a version of Weierstrass' theorem it is known that $\mathrm{span}\{e_n\}_{n \in \mathbb{Z}}$ is dense in the supremum-norm in

$$\{f \in C[0, 2\bar{\mu}] : f(0) = f(2\bar{\mu})\}.$$

As $C[0, 2\bar{\mu}]$ is dense in $L^2(0, 2\bar{\mu})$, from this one can deduce that $\mathrm{span}\{e_n\}_{n \in \mathbb{Z}}$ is dense in $L^2(0, 2\bar{\mu})$. We then see that $(e_n)_{n \in \mathbb{Z}}$ is an orthonormal basis in $L^2(0, 2\bar{\mu})$. We therefor have a unitary isomorphism

$$l^2(\mathbb{Z}) \xrightarrow{\sim} L^2(0, 2\bar{\mu}), (c_n)_{n \in \mathbb{Z}} \mapsto \sum_{n \in \mathbb{Z}} c_n e_n.$$

The Fourier coefficients at $f \in L^2(0, 2\bar{\mu})$ are denoted by $\hat{f}(n)$, so

$$\hat{f}(n) = \frac{1}{\sqrt{2\bar{\mu}}} \int_0^{2\bar{\mu}} f(t) e^{-int} dt,$$

more practically

$$\hat{f}(n) = \frac{1}{\sqrt{2\bar{\mu}}} \int_{[0, 2\bar{\mu}]} f(t) e^{-int} d\lambda(t).$$

Therefor we have $f = \sum_{n \in \mathbb{Z}} \hat{f}(n) e_n$ in $L^2(0, 2\bar{\mu})$.

*Fact*: For every $f \in L^2(0, 2\bar{\mu})$, we have $\frac{1}{\sqrt{2\bar{\mu}}} \sum_{n=-N}^N \hat{f}(n) e^{int} \xrightarrow[N \to \infty]{} f(t)$ for a.e. t.

**Lemma 15.2.** *Assume $V$ is a normed space over $K = \mathbb{R}$ or $K = \mathbb{C}$. Consider a linear functional $f : V \to K$. The following are equivalent (TFAE):*

(i) $f$ is continuous;

(ii) $f$ is continuous at 0;

(iii) There is a $c \geq 0$ s.t. $|f(x)| \leq c||x|| \ \forall x \in V$.

If (i)-(iii) are satisfied, then $f$ is called a *bounded linear functional*. The constant $c$ in (iii) is denoted by $||f||$. We have $||f|| = \sup_{x \neq 0} \frac{|f(x)|}{||x||} = \sup_{||x||=1} |f(x)| = \sup_{||x|| \leq 1} |f(x)|$.

**Proposition 15.3.** *For every normed vector space $V$ over $K = \mathbb{R}$ or $K = \mathbb{C}$, the bounded linear functionals on $V$ form a Banach space $V^*$.*

**Remark.** *The sequence* $\{||f_n - f_m||\}_{m=1}^{\infty}$ *actually converges, since*

$$\left| ||f_n - f_m|| \right| \leq ||f_m - f_n||.$$

*When we study/use normed spaces, it is often important to understand the dual spaces. For Hilbert spaces this is particularly easy:*

**Theorem 15.4** (Riesz)**.** *Assume $\mathcal{H}$ is a Hilbert space. Then every $f \in \mathcal{H}^*$ has the form*

$$f(x) = (x, y),$$

*for a uniquely defined $y \in \mathcal{H}$. Moreover, we have $||f|| = ||y||$.*

For every Hilbert space $\mathcal{H}$ we can define the *conjugate Hilbert space* $\bar{\mathcal{H}}$, which has its elements as the symbols $\bar{x}$ for $x \in \mathcal{H}$, with the linear structure and inner product defined by

$$\bar{x} + \bar{y} = \overline{x + y}, c \cdot \bar{x} = \overline{\bar{c}x}, (\bar{x}, \bar{y}) = \overline{(x, y)} = (y, x).$$

**Corollary 15.5.** *For every Hilbert space $\mathcal{H}$, we have an isometric isomorphism $\bar{\mathcal{H}} \xrightarrow{\sim} \mathcal{H}^*$, $\bar{x} \mapsto (\cdot, x)$.*

# 16   Hahn-Banach Theorem   (4.2, [Teschl(2010)])

**Theorem 16.1** (Hahn-Banach)**.** *Assume $V$ is a real vector space, $V_0 \subset V$ a subspace, $e : V \to \mathbb{R}$ a convex function and $f : V_0 \to \mathbb{R}$ a linear functional s.t. $f \leq e$ on $V_0$. Then $f$ can be extended to a linear functional $F$ on $V$ s.t. $F \leq e$.*

Assume first that $V = V_0 + \mathbb{R}x$ for some $x \in V \backslash V_0$. To define $F$ we need to specify $F(x)$. The condition $F \leq e$ means that we need

$$F(y \pm tx) \leq e(y \pm tx) \ \forall y \in V_0, t \geq 0,$$

that is

$$\pm tF(x) + f(y) \leq e(y \pm tx).$$

Dividing by $t$ this is equivalent to

$$\begin{cases} F(x) \leq \frac{e(y+tx)-f(y)}{t}, \\ F(x) \geq \frac{f(y)-e(y-tx)}{t}, \end{cases}$$

$\forall y \in V_0, t \geq 0$. To show that there is a number satisfying these inequalities, we need to check that

$$\sup_{\substack{y \in V_0 \\ t > 0}} \frac{f(y) - e(y - tx)}{t} \leq \inf_{\substack{y \in V_0 \\ t > 0}} \frac{e(y + tx) - f(y)}{t}.$$

Then as $F(x)$ we can take any number in the interval [sup, inf]. In other words, we need to check that

$$\frac{f(y) - e(y - tx)}{t} \leq \frac{e(z + sx) - f(z)}{s},$$

$\forall y, z \in V_0, t, s \geq 0$. Equivalently,

$$sf(y) + tf(z) \leq se(y - tx) + te(z + sx).$$

We have

$$\frac{sf(y) + tf(z)}{s + t} = f\left(\frac{s}{s+t}y + \frac{t}{s+t}z\right)$$

$$\leq e\left(\frac{s}{s+t}y + \frac{t}{s+t}z\right)$$

$$\leq e\left(\frac{s}{s+t}(y - tx) + \frac{t}{s+t}(z + sx)\right)$$

$$\leq \frac{s}{s+t}e(y - tx) + \frac{t}{s+t}e(z + sx),$$

which is what we need.

The general case is deduced from this using transitive induction (Zorn's lemma), as follows. Consider the set $X$ of pairs $(W, F)$, where $w \subset V$ is a subspace, $F : W \to \mathbb{R}$ a linear functional, $V_0 \subset W$ and $F|_{v_0} = f$. Define a partial order on $X$ by $F \leq e$ on $W$.

$$(W_1, F_1) \leq (W_2, F_2) \text{ iff } W_1 \subset W_2 \text{ and } F_1 = F_2|_{W_1}.$$

If $\mathcal{C}$ is a chain in $X$, then it has an upper bound $(W, F)$ defined by

$$W = \bigcup_{(Y, \rho) \in \mathcal{C}} Y, F(y) = \rho(y) \text{ if } y \in Y, (Y, \rho) \in \mathcal{C}.$$

Hence, $X$ has a maximal element $(W, F)$ by Zorn's lemma. If $W \neq V$, then we can take $x \in V \backslash W$ and extend $F$ to $W + \mathbb{R}x$ preserving the inequality $F \leq e$. This contradicts maximality of $W, F$ in $X$. Hence, $W = V$.

**Theorem 16.2** (Hahn-Banach)**.** *Assume $V$ is a real or complex vector space, $p$ a seminorm on $V_0$, $V_0 \subset$, and $f$ a linear functional on $V_0$ s.t.*

$$|f(x)| \leq p(x) \ \forall x \in V_0.$$

*Then $f$ can be extended to a linear functional $F$ on $V$ s.t. $|F(x)| \leq p(x) \ \forall x \in V$.*

**Corollary 16.3.** *Assume $V$ is a normed real or complex vector space, $V_0 \subset V$ and $f \in V_0^*$. Then there is a $F \in V^*$ s.t.*

$$F|_{V_0}f \text{ and } ||F|| = ||f||.$$

*Proof.* We apply the previous theorem to $p(x) = ||f|| \cdot ||x||$. $\square$

**Corollary 16.4.** *Assume $V$ is a normed space and $x \in V, x \neq 0$. Then there is a $F \in V^*$ s.t. $||F|| = 1$ and $F(x) = ||x||$.*

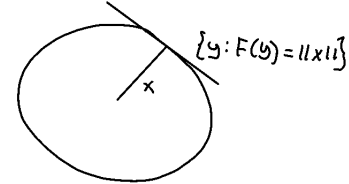Such an $F$ is called a *supporting functional at $x$.*



Figure 1: Tangent space?

If $V$ is a normed vector space, then every $x \in X$ defines a bounded linear functional on $V^*$ by

$$V^* \ni f \mapsto f(x).$$

As $|f(x)| \leq ||f|| \cdot ||x||$, this functional has norm $\leq ||x||$. By using a supporting functional at $x$, we actually see that we get norm $||x||$. Thus, we have an isometric embedding $V \subset V^{**} := (V^*)^*$. We can therefor see view $V$ as a subspace of $V^{**}$.

**Definition 16.5.** A normed space $V$ is called reflexive if $V^{**} = V$.

**Remark.** *This is stronger than requiring $V \simeq V^{**}$.*

**Remark.** *Every Hilbert space $\mathcal{H}$ is reflexive. Indeed, $\mathcal{H}^* = \bar{\mathcal{H}}$. By Riesz' theorem every bounded linear functional $f$ on $\bar{\mathcal{H}}$ has the form*

$$f(\bar{x}) = (\bar{x}, \bar{y}) = (y, x),$$

*for some $y \in \mathcal{H}$, which exactly means that $f = y$ in $\mathcal{H}^{**}$.*

*As we will see later, the spaces $\mathcal{L}^p(X, d\mu)$, with $\mu$ $\sigma$-finite and $1 < p < \infty$, are reflexive. The spaces $\mathcal{L}'(X, d\mu)$ and $\mathcal{L}^\infty(X, \mu)$ are usually not reflexive.*

## 17 Radon-Nikodym Theorem <span>(20, [Schilling(2017)])</span>

Assume $(X, \mathscr{B}, \mu)$ is a measure space. Are there other measures on $(X, \mathscr{B})$?

**Example 17.1.** Take a measurable function $f : X \to [0, +\infty]$ and define

$$\nu(A) = \int_A f d\mu \text{ for } A \in \mathscr{B}.$$

This is a measure by the monotone convergence theorem. We write $d\nu = f d\mu$.

**Proposition 17.2.** *Assume $(X, \mathscr{B})$ is a measurable space, $\mu$ and $\nu$ are $\sigma$-finite measures on $(X, \mathscr{B})$. Then there exist $N \in \mathscr{B}$ and a measurable $f : X \to [0, +\infty]$ s.t. $\mu(N) = 0$ and $\nu(A) = \nu(A \cap N) + \int_A f d\mu \; \forall A \in \mathscr{B}$.*

*Proof.* Assume first that the measure $\nu, \mu$ are finite. Consider the measure $\eta = \mu + \nu$ and define a linear functional $\rho$ on $L^2(x, d\eta)$ by

$$\rho(g) = \int_X g d\nu.$$

It is well-defined and bounded, since

$$|\rho(g)| \leq \underbrace{\int_X |\rho| d\nu}_{\|g\|_1} \underset{C.S}{\leq} \left( \int_X |\rho|^2 d\nu \right)^{\frac{1}{2}} \nu(X)^{\frac{1}{2}}$$

$$\leq \nu(x)^{\frac{1}{2}} \left( \int_X |\rho|^2 d\eta \right)^{\frac{1}{2}}.$$

By Riesz' theorem there exists $h \in L^2(x, d\eta)$ s.t. $\rho(g) = \int_X g h d\eta$ for all $g \in L^2(x, d\eta)$.

For $\rho = \mathbb{1}_A$ we get

$$\nu(A) = \int_A h d\eta \; \forall A \in \mathscr{B}.$$

In particular, $h \geq 0$ ($\eta$ − a.e.). As $\nu(A) \leq \eta(A)$, we also have $h \leq 1$ ($\eta$ − a.e.). From now on we view $h$ as a function on $X$ and assume $0 \leq h \leq 1$.

For $g = \mathbb{1}_A$ $(A \in \mathscr{B})$ we have

$$\int_X g d\nu = \rho(g) = \int_X g h d\eta = \int_X g h d\mu + \int_X g h d\nu,$$

hence

$$\int_X g(1 - h) d\nu = \int_X g h d\mu. \tag{26}$$

By extending eq. 26 to positive simple functions and then using the monotone convergence theorem, we conclude that eq. 26 holds for all measurable $g : X \to [0, +\infty]$.

We now let

$$N = \{x : h(x) = 1\} \text{ and } f = \frac{h}{1 - h} \mathbb{1}_{N^c}.$$

Letting $g = \mathbb{1}_N$ in eq. 26 we get $0 = \mu(N)$. For $A \in \mathscr{B}$, letting $g = \frac{\mathbb{1}_{A \cap N^c}}{1 - h}$ in eq. 26 we get

$$\nu(A \cap N^c) = \int_X \frac{\mathbb{1}_A \mathbb{1}_{N^c}}{1 - h} h d\mu = \int_X \mathbb{1}_A f d\mu = \int_A f d\mu.$$

Thus

$$\nu(A) = \nu(A \cap N) + \nu(A \cap N^c) = \nu(A \cap N) + \int_A f d\mu.$$

This finishes the proof for finite $\mu$ and $\nu$.

If $\mu$ and $\nu$ are $\sigma$-finite, we can write $X$ as disjoint unions $X = \cup_{n \in \mathbb{N}} X_n = \cup_{m \in \mathbb{N}} Y_m$ with $X_n, Y_m \in \mathscr{B}, \mu(X_n) < \infty, \nu(Y_m) < \infty$. Applying the first part of the proof to $X_n \cap Y_m$, we find $N_{nm} \in \mathscr{B}, N_{nm} \subset X_n \cap Y_m$, and measurable $f_{nm} : X_n \cap Y_m \to [0, +\infty]$ s.t. $\mu(N_{nm}) = 0$ and

$$\nu(A \cap X_n \cap Y_m) = \nu(A \cap N_{nm}) + \int_{A \cap X_n \cap Y_m} f_{nm} d\mu.$$

We then put $N = \cup_{n,m \in \mathbb{N}} N_{nm}$ and define $f : X \to [0, +\infty]$ by letting $f = f_{nm}$ on $X_n \cap Y_m$. $\square$

When can we discard the term $\nu(A \cap N)$?

**Definition 17.3.** Given measure $\mu$ and $\nu$ on $X, \mathscr{B}$, we say that $\nu$ is *absolutely continuous* with respect to $\mu$ and write $\nu << \mu$, if $\nu(A) = 0$ whenever $A \in \mathscr{B}, \mu(A) = 0$.

**Lemma 17.4.** *Assume $\mu$ and $\nu$ are measures on $(X, \mathscr{B}), \nu(X) < \infty$. Then $\nu << \mu$ iff $\forall \epsilon > 0 \exists \delta > 0$ s.t. if $A \in \mathscr{B}, \mu(A) < \delta$, then $\nu(A) < \epsilon$.*

*Proof.* "$\Rightarrow$": obvious. "$\Leftarrow$": Assume this is not true. Then, there is a $\epsilon > 0$ s.t. $\forall \delta > 0$ we can find $A \in \mathscr{B}$ satisfying $\mu(A) < \delta$, $\nu(A) \geq \epsilon$. Let $A_n$ be such a set $A$ for $\delta = 1/2^n$. Put $A = \cap_{n \in \mathbb{N}} \cup_{k=n} A_k$. Then

$$\mu(A) \leq \lim_{n \to \infty} \mu \left( \bigcup_{k=n}^{\infty} A_k \right) \leq \lim_{n \to \infty} \sum_{k=n}^{\infty} \mu(A_k)$$

$$\leq \lim_{n \to \infty} \sum_{k=n}^{\infty} \frac{1}{2^k} = \lim_{n \to} \frac{1}{2^{n-1}} = 0.$$

As $\nu(X) < \infty$, we also have

$$\nu(A) = \lim_{n \to \infty} \nu \left( \bigcup_{k=n}^{\infty} A_k \right) \geq \epsilon.$$

This contradicts the assumption $\nu << \mu$. $\square$

**Remark.** *The result is not true for infinite $\nu$.*

**Theorem 17.5** (Radon-Nikodym). *Assume $\mu$ and $\nu$ are $\sigma$-finite measures on a measurable space $(X, \mathscr{B})$, $\nu << \mu$. Then there is a measurable function $f : X \to [0, +\infty)$ s.t. $d\nu = f d\mu$ (that is, $\nu(A) \int_A f d\mu$). If $\tilde{f}$ is another function with the same properties, then $f = \tilde{f} \; \mu$ − a.e.*

*The function is called the* Radon-Nikodym *derivative at $\nu$ w.r.t. $\mu$ and is denoted by $\frac{d\nu}{d\mu}$.*

*Proof.* By the proposition above, we can find $N \in \mathscr{B}$ and $f : X \to [0, +\infty)$ s.t. $\mu(N) = 0$ and

$$\nu(A) = \nu(A \cap N) + \int_A f d\mu.$$

As $\nu(A \cap N) = 0$ by the assumption $\nu << \mu$, this proves the existence of $f$.

Assume we have another $\tilde{f}$. Then

$$\nu(A) = \int_A f d\mu = \int_A \tilde{f} d\mu \; \forall A \in \mathscr{B}.$$

If $B \in \mathscr{B}, \nu(B) < \infty$, then consider

$$A_1 = \left\{ x \in \mathscr{B} : f(x) > \tilde{f}(x) \right\}, A_2 = \left\{ x \in \mathscr{B} : f(x) < \tilde{f}(x) \right\}.$$

Then

$$\int_{A_1} (f - \tilde{f})d\mu = 0 \text{ and } \int_{A_2} (\tilde{f} - f)d\mu = 0,$$

hence $\mu(A_1) = \mu(A_2) = 0$. Therefore, $f = \tilde{f}$ $\mu$−a.e. on $\mathscr{B}$. As $\nu$ is $\sigma$-finite, we have $X = \cup_{n \in \mathbb{N}} B_n$, $\nu(B_n) < \infty$. Then $f = \tilde{f}$ $\mu$ − a.e. on $B_n$ for all $n$, hence $f = \tilde{f}$ $\mu$ − a.e. on $X$. $\qquad\square$

**Example 17.6.** Consider a real-valued function $f \in C'[a,b]$ s.t. $f'(t) > 0 \ \forall \ t \in [a,b]$. Let $c = f(a), d = f(b)$. We know that for every Riemann integrable function $g$ on $[c,d]$ we have

$$\int_c^d g(f)dt = \int_a^b g(f(t))f'(t)dt.$$

Equivalently,

$$\int_c^d g \circ g^{-1} dt = \int_a^b g f'(t)dt. \tag{27}$$

Denote by $\lambda_{[a,b]}, \lambda_{[c,d]}$ the Lebesgue measure restricted to the Borel subsets of $[a,b]$ and $[c,d]$, respectively. Then eq. 27 implies that

$$d\left((f^{-1})_* \lambda_{[c,d]}\right) = f' d\lambda_{[a,b]},$$

since the integration of $g = \mathbb{1}_{[\alpha,\beta]}$ gives the same results for any interval $[\alpha,\beta] \subset [a,b]$ and since a finite Borel measure on $[a,b]$ is

determined by its values on such intervals. Thus, $(f^{-1})_* \lambda_{[c,d]} <<$ $\lambda_{[a,b]}$ and

$$\frac{d\left((f^{-1})_* \lambda_{[c,d]}\right)}{d\lambda_{[a,b]}} = f'.$$

## Tips'n Tricks

- Assume we can write $X$ as a finite union: $X = \cup_{n \in I} A_n$, $i = 1, ..., N$. Then

$$\int fd\mu = \int_X fd\mu = \int_{A_1} fd\mu + \int_{A_2} fd\mu + ... + \int_{A_N} fd\mu.$$

## Questions

- In problem 26.18 we are supposed to show that $Y_n \perp Y_m = 0$, i.e. that $\langle y_n, y_m \rangle = 0$, $n \neq m$. I get ...

$$\langle y_n, y \rangle \subset \int_{A_m^c} |y_n|^2 |y_m|^2 d\mu,$$

and I want to argue that this is zero since $\int_{A_m^c} |y_m|^2 d\mu = 0$, but I don't see how. The solutions are not clear, and I think perhaps my setup is wrong. I am assuming $\langle f, g \rangle = \int_X f \bar{g} d\mu$, i.e. from $L^2$, but perhaps it is rather $\langle f, g \rangle = \int_{A_m^c \cup A_n^c} f \bar{g} d\mu$ or something?

## References

[Schilling(2017)] Schilling, R. 2017, Measures, Integrals and Martingales, Measures, Integrals and Martingales (Cambridge University Press). https://books.google.no/books?id=sdAoDwAAQBAJ

[Teschl(2010)] Teschl, G. 2010, Topics in Linear and Nonlinear Functional Analysis (Universität Wien). https://www.mat.univie.ac.at/~gerald/ftp/book-fa/fa.pdf