NATCAR – Background Information

Feedback & Control Theory

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Outline

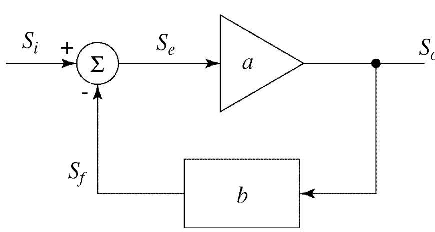
- Negative feedback
  - Basic principle
  - Control loop
- Stability of negative feedback loop
  - Loop gain and phase
  - Nyquist plots
  - Gain and phase margins
- Step responses (stable and unstable)
- Root-Locus Plots

Negative Feedback

- Negative feedback was invented by Harold Black in 1927, but is ubiquitous in biological and other natural systems as well as in electronics.
- Negative feedback amplifiers are covered in EEC110B, feedback control is covered in detail in EEC157A-B.
- Basic principle: If the output is supposed to be a scaled replica of the input, then you can take a fraction of the output and compare it with the input to generate an error. This error can be used to force the output to be closer to what it should be.

Negative Feedback Amplifier

Consider a negative feedback amplifier first:



$$S_o = aS_e = a\left(S_i - S_f\right) = a\left(S_i - bS_o\right)$$

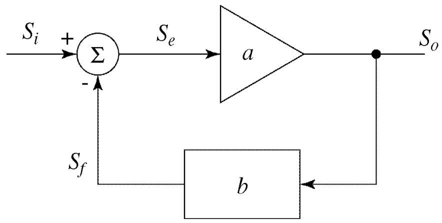
Solving yields, $A = \dfrac{S_o}{S_i} = \dfrac{a}{1+ab}$

If $ab >> 1$, then $A \approx \dfrac{1}{b}$

- For example; if $b = 0.1$, $A \approx 10$ so long as $a >> 10$. The exact value of a doesn't matter very much.
- If $ab = -1$ though, the transfer function blows up and the output can be finite with no input. We'll come back to this point in a moment.

Negative Feedback Amplifier

Examine the magnitude of the error term:



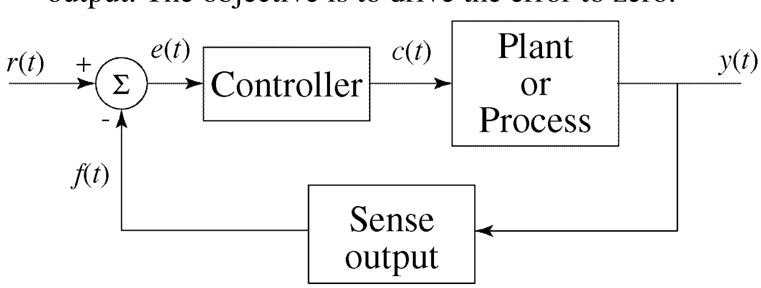
$$S_e = S_i - S_f = S_i - bS_o$$
$$= S_i - b(aS_e)$$
$$S_e(1+ab) = S_i$$
$$S_e = \frac{S_i}{1+ab}$$

- As $ab \rightarrow \infty$, $S_e \rightarrow 0$
- ab is called the loop gain as we saw before, so when the loop gain is large, the error is driven toward zero

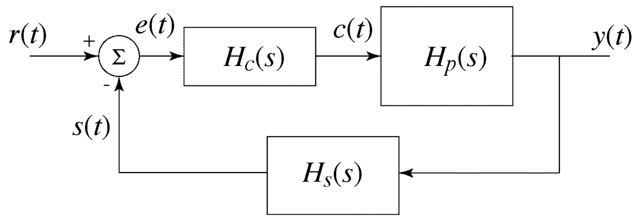
Feedback Control Loop

A feedback control loop uses different terminology than the amplifier just shown, but the principle is the same.

– Compare a signal fed back from the output with a reference input. Feed the resulting error into a controller that drives the plant or process to produce the desired output. The objective is to drive the error to zero.



Control Loop Transfer Functions



The overall transfer function is

$$H_T(s) = \frac{H_c(s)H_p(s)}{1 + H_c(s)H_p(s)H_S(s)}$$

denote the loop gain as $L(s) = H_c(s)H_p(s)H_S(s)$

Control Loop Stability

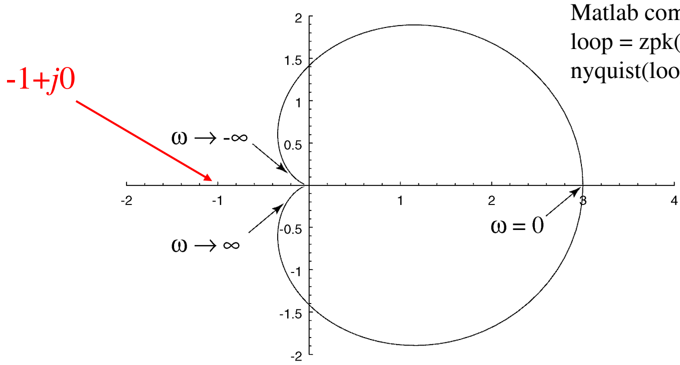
- The poles of $H_T(s)$ are given by the solutions of the characteristic equation $1 + L(s) = 0$.
- For the system to be stable, all of the poles of $H_T(s)$ must be in the left half of the complex plane.
- There are two common ways of checking whether or not all of the poles lie in the LHP:
  - By doing a Nyquist plot
  - By checking the gain and phase margins
- The Nyquist plot is the more general method, but gain and phase margins are more common for amplifiers.

Nyquist Plot

- A Nyquist plot is simply a plot of the complex values of $L(j\omega)$ as ω varies from –infinity to +infinity.
- The plot always has symmetry about the real axis.
- If the plot encircles the point $-1+j0$ *in the clockwise direction*, and $L(s)$ does not have poles in the RHP, then there are closed-loop poles in the RHP and the number of times the plot encircles the point $-1+j0$ is equal to the number of poles in the RHP.

Nyquist Plot Example

$$L(j\omega)=\frac{3}{\left(1+\frac{j\omega}{5}\right)\left(1+\frac{j\omega}{10}\right)}=\frac{150}{(5+j\omega)(10+j\omega)}$$



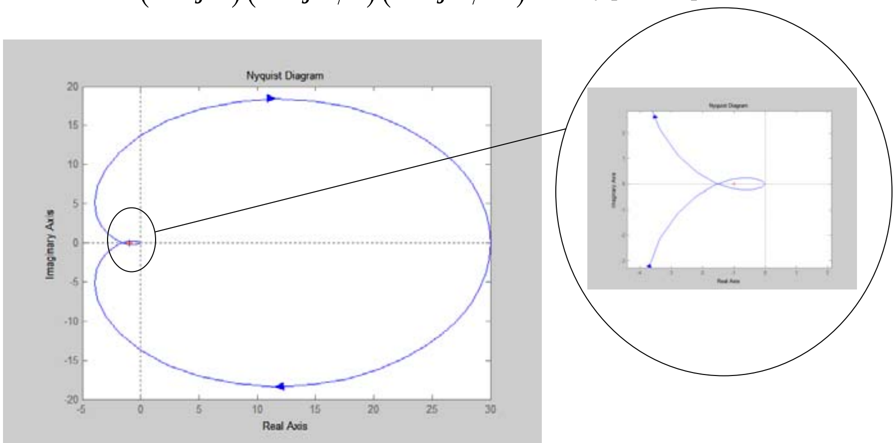
Matlab commands:
loop = zpk([], [-5 -10], 150)
nyquist(loop)

System is stable. All closed-loop poles are in the LHP

2^{nd} Nyquist Plot Example

$$L(j\omega)=\frac{30}{(1+j\omega)(1+j\omega/5)(1+j\omega/10)}$$

Matlab commands:
```
loop = zpk([], [-1 -5 -10], 1500)
nyquist(loop)
```



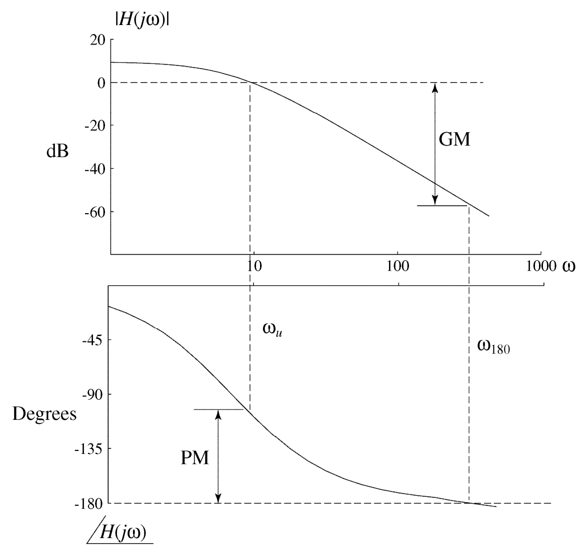
Encircles -1 twice CW ⇒ two poles in RHP, so unstable

Gain and Phase Margins

- A less general, but common, method for examining stability is to use a Bode plot of $L(j\omega)$ and look to see how close the phase is to -180° when the gain is 1 (0 dB) or more.
- The method works so long as the magnitude of the phase is monotonically non-decreasing with increasing frequency.
- A common textbook goal, which is easy to do graphically, is to set the phase margin to 45°; PM = 45°. A more typical number in real designs is PM = 60°.

Definitions of Gain and Phase Margins

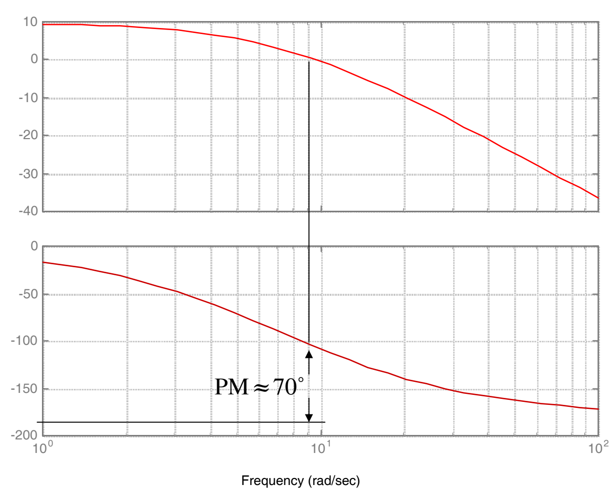
- The GM is the extra gain required to reach 0 dB when the phase is -180°
- The PM is the extra phase delay needed to reach -180° when the gain is 0 dB.



Gain and Phase Margin Example

$$L(j\omega) = \frac{3}{(1 + j\omega/5)(1 + j\omega/10)}$$

matlab commands:
loop = zpk([], [-5 -10], 150)
bode(loop)

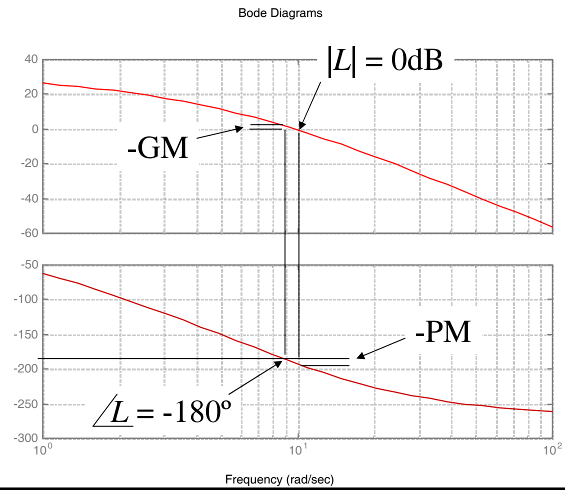


This system is stable since both the GM & PM are positive. In fact, the GM is infinite since the phase never reaches -180°.

2nd Gain and Phase Margin Example

$$L(j\omega) = \frac{30}{(1+j\omega)(1+j\omega/5)(1+j\omega/10)}$$

matlab commands:
loop = zpk([], [-1 -5 -10], 1500)
bode(loop)



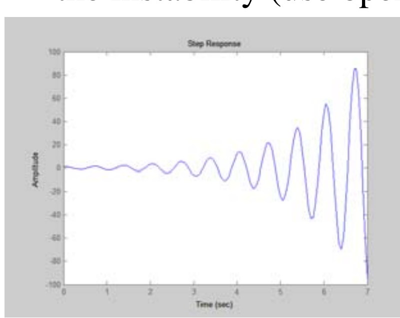
This system is unstable since the GM and PM are both negative.

Step Response of Unstable Loop

If the loop gain is given by

$$L(j\omega) = \frac{30}{(1+j\omega)(1+j\omega/5)(1+j\omega/10)}$$

then the closed-loop step response clearly shows the instability (use open-loop gain a = 1).



Matlab commands:

L = zpk([],[-1 -5 -10],1500)

HT = 1/(1+L)

step(HT,0:.05:7)

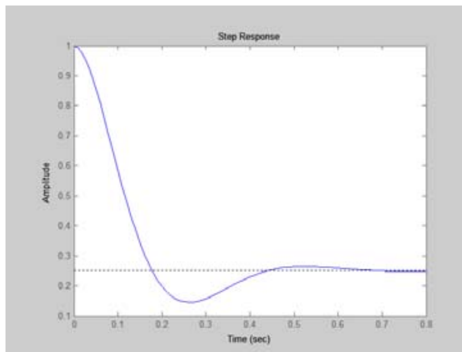
Step Response of Stable Loop

If the loop gain is given by

$$L(j\omega) = \frac{3}{(1 + j\omega/5)(1 + j\omega/10)}$$

then the closed-loop step response clearly shows the system is stable (use open-loop gain $a = 1$).

This is the step response of $H_T = 1/(1+L)$, so at $t = 0$ it starts at 1, and then the loop responds and starts to reduce the error. If we leave the step input forever, it is a DC term and $|H_T(j0)| = 1/4$



Root-Locus Plots

- We have seen that the poles of the closed-loop system are critically important in determining stability.
- We have also seen how to check stability if the loop gain is known (complete transfer function).
- One other very useful and intuitive technique allows us to examine how the closed-loop poles vary with the DC loop gain; this method is called a Root-Locus Plot.

Root-Locus Plots; Method

- Suppose the loop gain of our system is written in the form:

$$L(s) = K\frac{(s+z_1)(s+z_2)(s+z_3)}{(s+p_1)(s+p_2)(s+p_3)}$$

- The closed-loop transfer function is then of the form:

$$H(s) = \frac{N(s)}{1+K\dfrac{(s+z_1)(s+z_2)(s+z_3)}{(s+p_1)(s+p_2)(s+p_3)}}$$

- Now multiply the numerator and denominator by

$$(s+p_1)(s+p_2)(s+p_3)$$

Root-Locus Plots; Method cont.

- We obtain:

$$H(s) = \frac{N(s)(s+p_1)(s+p_2)(s+p_3)}{(s+p_1)(s+p_2)(s+p_3)+K(s+z_1)(s+z_2)(s+z_3)}$$

- The closed-loop poles are given by the solutions of the characteristic equation:

$$\underbrace{(s+p_1)(s+p_2)(s+p_3)}+K\underbrace{(s+z_1)(s+z_2)(s+z_3)}=0$$

NOTE: These are the poles of $L(s)$, and these are the zeros!

Root-Locus Plots; Method cont.

- The solutions of:

$$(s+p_1)(s+p_2)(s+p_3)+K(s+z_1)(s+z_2)(s+z_3)=0$$

- Are given by:
  - The poles of $L(s)$ when $K=0$
  - The zeros of $L(s)$ as K approaches infinity
- Therefore; the locus of the poles of $H(s)$ starts on the poles of $L(s)$ and ends on the zeros of $L(s)$.

Root-Locus Plots; Method cont.

- In addition to starting on the poles of $L(s)$ and ending on the zeros of $L(s)$, the locus of the poles of $H(s)$ must only include points that satisfy the characteristic equation:

$$1+L(s)=0$$

- In other words, these points must satisfy two conditions:

$$|L(s)|=1$$

and

$$\angle L(s)=\pm 180°$$

Root-Locus Plots; Method cont.

- Note that with

$$L(s) = K\frac{(s+z_1)(s+z_2)(s+z_3)}{(s+p_1)(s+p_2)(s+p_3)}$$

- The magnitude condition becomes (assume $K > 0$):

$$\left|K\frac{(s+z_1)(s+z_2)(s+z_3)}{(s+p_1)(s+p_2)(s+p_3)}\right| = K\left|\frac{(s+z_1)(s+z_2)(s+z_3)}{(s+p_1)(s+p_2)(s+p_3)}\right| = 1$$

which depends on the value of K. We will not use this condition to construct the locus, but it will allow us to find a specific point on the locus if we want to later.

Root-Locus Plots; Method cont.

- Also note that with

$$L(s) = K\frac{(s+z_1)(s+z_2)(s+z_3)}{(s+p_1)(s+p_2)(s+p_3)}$$

- The angle condition becomes (assume $K > 0$):

$$\angle\left(K\frac{(s+z_1)(s+z_2)(s+z_3)}{(s+p_1)(s+p_2)(s+p_3)}\right)$$

$$= \angle(s+z_1)(s+z_2)(s+z_3) - \angle(s+p_1)(s+p_2)(s+p_3) = \pm 180°$$

which is the condition we will use to find the locus.

Plotting the Root Locus

- We can now describe a procedure for plotting a root locus:
- The locus begins at the poles of the loop gain and ends on the zeros (we must include zeros at infinity)
- Any point in the plane that satisfies the angle condition is part of the locus; i.e., the sum of the angles of the vectors from the loop-gain zeros to the point, minus the sum of the angles of the vectors from the loop-gain poles to the point must be an odd integer multiple of 180º.

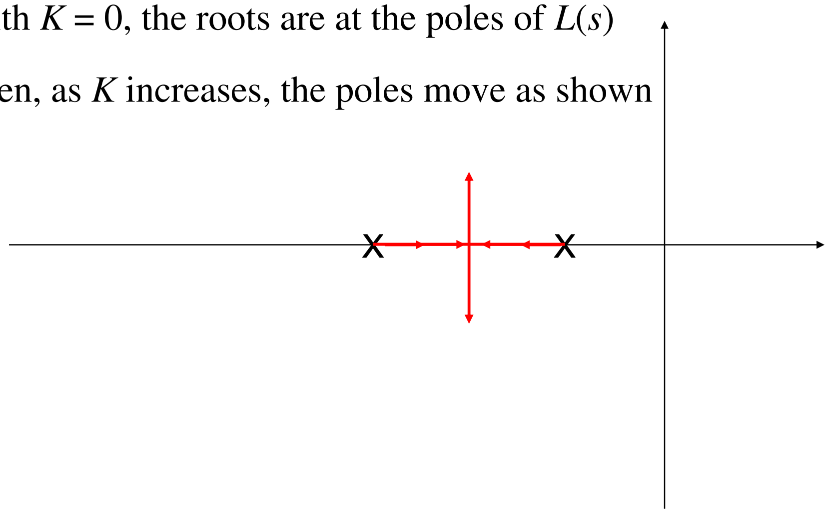
Root Locus Example

- Suppose we have: $L(s) = \dfrac{K}{(s+p_1)(s+p_2)}$
- This transfer function has two zeros at infinity. They will contribute zero phase to any finite point in the plane. Therefore, the locus will contain those points for which the sum of the angles of the two vectors from the poles to the point is ±180º.

Root Locus Example - continued

With $K = 0$, the roots are at the poles of $L(s)$

Then, as K increases, the poles move as shown



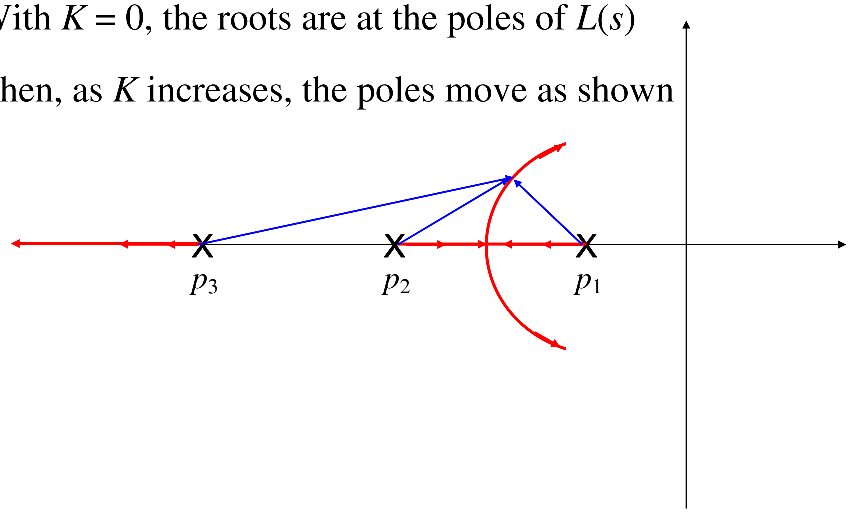
Root Locus Example

- Now suppose we have: $L(s) = \dfrac{K}{(s+p_1)(s+p_2)(s+p_3)}$
- This transfer function has three zeros at infinity. They will contribute zero phase to any finite point in the plane. Therefore, the locus will contain those points for which the sum of the angles of the three vectors from the poles to the point is 180º.
- Because of the added third pole, the complex part of the locus will bend to the right to keep the total phase of the three vectors equal to 180º.

Root Locus Example - continued

With $K = 0$, the roots are at the poles of $L(s)$

Then, as K increases, the poles move as shown



Root Locus with Matlab

$$L(j\omega) = \frac{30}{(1 + j\omega)(1 + j\omega/5)(1 + j\omega/10)}$$

Matlab commands:
loop = zpk([], [-1 -5 -10], 1500)
rlocus(loop)