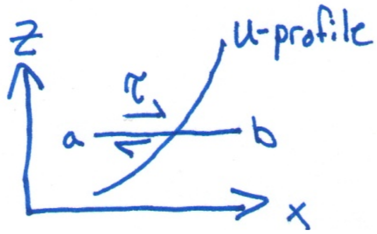
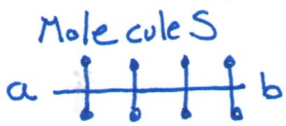
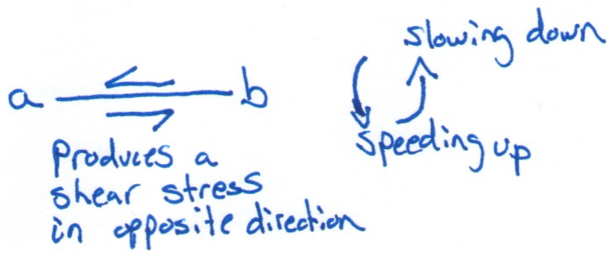
Fluid Mechanics II

Laminar flow



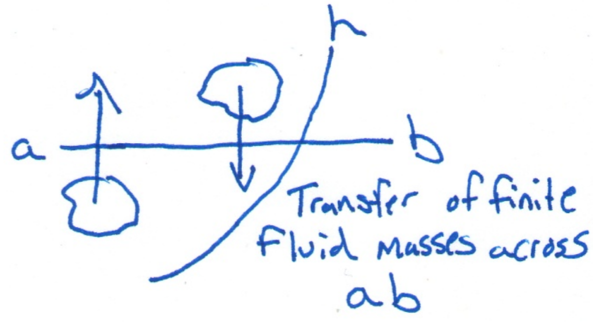
Even though the parcels of fluid are moving horizontally, molecular motion will transport momentum across surface a-b. Molecules above and below surface a-b are traveling at higher and lower velocities, respectively. The vertical motion of these molecules produces a resisting shear stress.





Turbulent flow

Momentum transfer by eddy transport:



Transfer of finite volumes of fluid across surface a-b redistributes fluid momentum, producing a resisting stress. This stress is commonly referred to as the **Reynolds Stress**, τ_r.

$$\rho \frac{\partial \bar{\underset{\sim}{u}}}{\partial t} + \rho \bar{\underset{\sim}{u}} \cdot \nabla \bar{\underset{\sim}{u}} = -\nabla \bar{p} + \mu \nabla^2 \bar{\underset{\sim}{u}} - \rho g + \nabla \cdot \rho \overline{\underset{\sim}{u}' \underset{\sim}{u}'}$$

A **B**

Where term A is $\nabla \cdot \tilde{\tau}_{v(dev.)}$

And term B is $\nabla \cdot \tilde{\tau}_{R(\approx deviatoric)}$ in other words $\rho \overline{u'u'}$, $\rho \overline{v'v'}$, $\rho \overline{w'w'}$ terms are very small

And $\nabla \cdot \tilde{\tau}_{total(dev)} = \nabla \cdot \tilde{\tau}_v + \nabla \cdot \tilde{\tau}_R$

Near the boundary the Reynold's Stress is >> the Viscous Stress

so, $\bar{\rho} \frac{\partial \bar{\underset{\sim}{u}}}{\partial t} + \bar{\rho} \bar{\underset{\sim}{u}} \cdot \nabla \bar{\underset{\sim}{u}} = -\nabla \bar{p} + \nabla \cdot \tilde{\tau}_{total}$

where $\nabla \cdot \tilde{\tau}_{total} \Rightarrow \tau_{viscous} + \tau_{\mathrm{Re}ynolds} \cong \tau_R$ because $|\tau_R| >> |\tau_V|$

Thus it is clear that we need to find a new constitutive Equation, one in which:

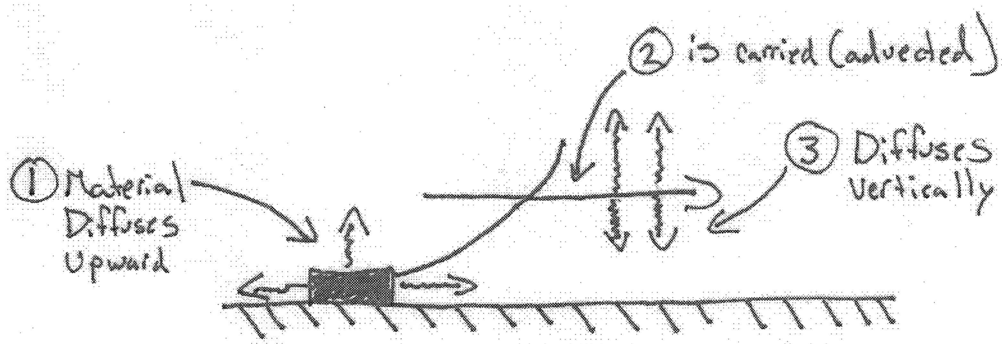
$$\tilde{\tau}_R = f\left(\frac{\partial \bar{\underset{\sim}{u}}}{\partial z}\right)$$

Near the boundary $\tilde{\tau}_R >> \tilde{\tau}_V$ so we need a new constitutive equation relating strain rate

In terms of $\tilde{\tau}_R$. You therefore can see why it is very important that we be able to close the Reynolds Equation (i.e. express the Reynolds stress in terms of a deterministic variable)

The N-S equation picked up an extra term during Reynolds Averaging because of the nonlinearity of the $\rho \underset{\sim}{u} \cdot \nabla \underset{\sim}{u}$ term. In order to <u>close</u> the Reynolds Equation one has to find a way to express the newly added stochastic variable in terms of average variables already in the equation.

We are always near boundary in sed. Transport. This is the appropriate simplifying assumption.



The vertical diffusion + lateral advection effect is >> then lateral diffusion therefore we are going to close the Eq. only in the vertical.

Near the boundary we will assume that the dominant direction of turbulent diffusion is in the vertical ($\therefore$ the dominant terms of the Reynolds stress will be in the vertical (z))

[vertical diffusion + lateral advection >> lateral diffusion effects}

Because of this we are going to close the Reynolds Equation only in the vertical

$$\nabla\cdot\left(\rho\overline{\underset{\sim}{u}\,\underset{\sim}{u}'}\right)\approx\frac{\partial}{\partial z}\rho\left(\overline{u'w'}+\underbrace{\cancel{\overline{v'w'}}}_{0\text{ if Flow is 2D}}+\underbrace{\cancel{\overline{w'w'}}}_{0;\text{ small}}\right)$$

Can approximate for 2D flow as

$$\nabla\cdot\left(\rho\overline{\underset{\sim}{u}\,\underset{\sim}{u}'}\right)\approx\frac{\partial}{\partial z}\rho\left(\overline{u'w'}\right)$$

For a horizontally uniform, 2D Flow

$$\frac{\partial}{\partial z}\left(\rho\overline{u'w'}\right)=\frac{\partial}{\partial z}\rho K\frac{\partial\bar{u}}{\partial z}$$

Early researchers on turbulence hypothesized turbulent eddies would have the same diffusive or mixing effect as molecular diffusion, although MUCH STRONGER. Based on analogy to a laminar flow, they proposed an **Eddy Viscosity** closure for the description of the Reynolds stress/momentum flux.

$$\rho\overline{u'w'}\approx\left(\tau_{zx}\right)_R=-\rho\upsilon_{turb}\frac{\partial\bar{u}}{\partial z}$$

The closure: $\left(\tau_{zx}\right)_R = -\rho K \frac{\partial \bar{u}}{\partial z}$

Expressed only in terms of mean quantities

$$\rho \frac{\partial \bar{\underset{\sim}{u}}}{\partial t} + \rho \bar{\underset{\sim}{u}} \cdot \underset{\sim}{\nabla} \bar{\underset{\sim}{u}} = -\underset{\sim}{\nabla} \bar{p} + \mu \nabla^2 \bar{\underset{\sim}{u}} - \rho g + \frac{\partial}{\partial z} K \frac{\partial \bar{u}}{\partial z}$$

K (eddy viscosity) = $k u_* z f\left(\frac{z}{h}\right)$

$K = k u_* z\left(1 - \frac{z}{h}\right)$ for $\frac{z}{h} \le 0.2$

$K = k u_* h / 6.24$ for $\frac{z}{h} > 0.2$

k = 0.407, von Karman's constant of proportionality

u_* = shear velocity

Cauchy's 1st Law:

A) $$\rho\frac{d\underset{\sim}{u}}{dt} = -\nabla p - \rho \underset{\sim}{g} + \nabla\cdot\tau = \rho\frac{\partial \underset{\sim}{u}}{\partial t} + \rho\underset{\sim}{u}\cdot\nabla\underset{\sim}{u}$$

A1) $$\underset{\text{II}}{\cancel{\rho\frac{\partial u}{\partial t}}} + \underset{\text{I+III}}{\cancel{\rho u_i\frac{\partial u}{\partial x_i}}} = -\frac{\partial p}{\partial x} + \frac{\partial \tau_{ix}}{\partial x_i} - \rho g_x$$

A2) $$\underset{\text{II}}{\cancel{\rho\frac{\partial v}{\partial t}}} + \underset{\text{I+III}}{\cancel{\rho u_i\frac{\partial v}{\partial x_i}}} = -\frac{\partial p}{\partial y} + \frac{\partial \tau_{iy}}{\partial x_i} - \rho g_y$$

A3) $$\underset{\text{II}}{\cancel{\rho\frac{\partial w}{\partial t}}} + \underset{\text{I+III}}{\cancel{\rho u_i\frac{\partial w}{\partial x_i}}} = -\frac{\partial p}{\partial z} + \frac{\partial \tau_{iz}}{\partial x_i} - \rho g_z$$

Let's start making assumptions & dropping out terms

I) Assume fluid is incompressible: $\underbrace{\cancel{\frac{1}{\rho}\frac{\partial\rho}{\partial t}}}_{0} + \nabla\cdot\underset{\sim}{u} = 0$ (Continuity Eq.)

$$\nabla\cdot\underset{\sim}{u} = 0$$

$$\frac{d\underset{\sim}{u}}{dt} = \frac{\partial\underset{\sim}{u}}{\partial t} + \underset{\sim}{u}\cdot\cancel{\nabla\underset{\sim}{u}} \qquad \underset{\sim}{u}\cdot\nabla\underset{\sim}{u} = u_i\frac{\partial\underset{\sim}{u}}{\partial x_i}$$

spatial accelerations ← (equal zero by continuity eq)

II) Assume <u>steady flow</u>: $\frac{\partial}{\partial t} \overset{def}{=} 0$

III) Assume flow is <u>horizontally uniform</u>: $\frac{\partial}{\partial x}, \frac{\partial}{\partial y} \overset{def}{=} 0$

Flow doesn't vary in the down-stream & cross-stream directions

The continuity equation says that $\frac{\partial w}{\partial z} = 0$, $\therefore$ w = constant wrt z

Remember no slip condition: $(w)_b = (w)_B = 0$

$\therefore$ if velocity is zero at boundary & constant everywhere then the vertical velocity is zero everywhere

Steady, horizontally uniform flows $\Rightarrow$ are unaccelerated flows

$$\frac{\partial \tau_{ij}}{\partial x_i} = \underbrace{\cancel{\frac{\partial \tau_{xj}}{\partial x}}}_{0\ (\mathrm{III})} + \underbrace{\cancel{\frac{\partial \tau_{yj}}{\partial y}}}_{0\ (\mathrm{III})} + \frac{\partial \tau_{zj}}{\partial z}$$

So here's what we're left with:

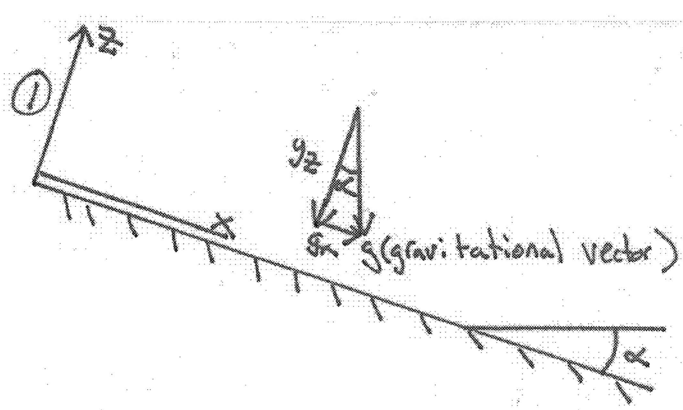
Component Equations for Cauchy's 1st

Law for steady, uniform flows

$$\cancel{\frac{\partial p}{\partial x}} = \frac{\partial \tau_{zx}}{\partial z} - \rho g_x$$

$$\cancel{\frac{\partial p}{\partial y}} = \frac{\partial \tau_{zy}}{\partial z} - \rho g_y$$

$$\frac{\partial p}{\partial z} = \frac{\partial \tau_{zz}}{\partial z} - \rho g_z$$



$g_x = |g| \sin\alpha$
$g_z = |g| \cos\alpha$

Case 1)

If the flow is able to move from plus to minus infinity with no barriers (no pressure)

$$\underbrace{\cancel{\frac{\partial p}{\partial x}}}_{0} = \frac{\partial \tau_{zx}}{\partial z} - \rho g \sin\alpha$$

$$\cancel{\frac{\partial p}{\partial y} = \frac{\partial \tau_{zy}}{\partial z}}$$

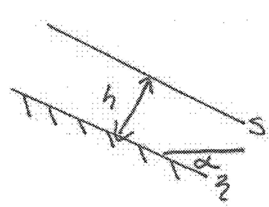
$$\frac{\partial p}{\partial z} = \frac{\partial \tau_{zz}}{\partial z} - \rho g \cos\alpha$$

Component equations of Cauchy's

1st Law for steady, uniform, isotropic

viscous flows

By assuming that the flow is two dimensional & positioning the x-axis exactly in the direction of flow you can make the problem two dimensional and drop out the expression for change along the y-axis (e.g. eliminates one equation of motion).



$$\underline{h \overset{\mathrm{def}}{=} s - z}$$

So, $$\boxed{\begin{aligned}\frac{\partial \tau_{zx}}{\partial z} &= \rho g \sin\alpha \\ \frac{\partial p}{\partial z} &= \frac{\partial \tau_{zz}}{\partial z} - \rho g \cos\alpha\end{aligned}}$$

Equation shows that the pressure is hydrostatic (hydrostatic Equation in the derivative form). Hydrostatic pressure (exp. Struct. Geol.) = stress that is uniform in all directions (e.g. beneath a homogeneous fluid & causes dilation rather than distortion)

$$\frac{\partial \tau_{zx}}{\partial z} = \rho g \sin\alpha$$

(a force balance, balancing the body force (gravitational) against the change in stress at that level)

$$\frac{\partial \tau_{zx}}{\partial z} dz = \rho g \sin\alpha dz$$

$$\int \frac{\partial \tau_{zx}}{\partial z} dz = \int \rho g \sin\alpha dz$$

$$\tau_{zx} = (\upsilon g \sin\alpha) z + const.$$

Now in order to determine the value of the constant let's invoke a boundary condition

1) $P_{surface} = P_{atmosphere}$
2) $\tau_{zx\ surface} = (\tau_{zx})_{atmosphere} = 0$
   For now we'll say there is no stress on the surface of this fluid. An unstressed surface is called a <u>free surface</u>
3) $z = s = h$

$$0 - (\rho g \sin\alpha) h = const.$$

So, $$\tau_{zx} = (\rho g \sin\alpha) z - (\rho g \sin\alpha) h$$
$$\tau_{zx} = (\rho g \sin\alpha)(z - h)$$

$$(\tau_{zx})_b \overset{def}{=} \tau_b$$ (boundary shear stress)

$$\tau_b = (\rho g \sin\alpha)(0 - h) = -(\rho g \sin\alpha) h = -\rho g h \sin\alpha$$

$$\tau_{zx} = (\rho g \sin\alpha)(z - h)$$

$$\tau_{zx} = \tau_b \left(1 - \frac{z}{h}\right)$$

These equations show that the stress must vary linearly wrt z.

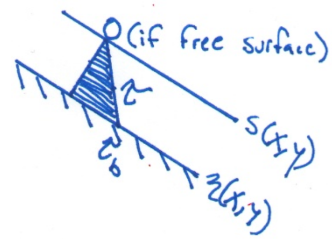
$$\tau_b = -\rho g h \sin\alpha$$

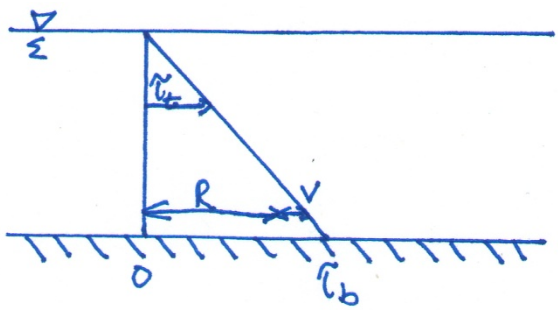
$$\tau_{zx} = \tau_b\left(1-\frac{z}{h}\right)$$

Expressions which are valid for any Newtonian-viscous material as long as the flow is unaccelerated

The pull downstream is gravitational and is resisted by the boundary shear stress. The friction at the boundary is transmitted through the fluid column linearly (i.e. the shear stress profile is linear)

The stress profile for a horizontally uniform, steady flow is linear whether the flow is laminar or turbulent.





$$\left|\tau_{\mathrm{Re}ynolds}\right| >> \left|\tau_{viscous}\right|$$

Back to a constitutive relationship between Reynolds stress and mean strain rate

$$\tau_{zx(total)} = \rho\upsilon\frac{\partial \bar{u}}{\partial z} - \rho\overline{u'w'} = \tau_b\left(1-\frac{z}{h}\right) = \rho u_*^2\left(1-\frac{z}{h}\right)$$

Assumptions:

1) Steady and horizontally uniform flow
2) In a turbulent flow near a wall $\tau_R \cong \tau_b = \rho u_*^2$

LAW OF THE WALL

Next we replace K with (ku_*z): kz is chosen as it represents a mixing length. $k = 0.407$, von Karman's constant of proportionality. This mixing length assumes that the dimension of turbulent eddies in the lower flow scale with distance from the boundary. Small eddies near the bed, larger eddies further away from the bed. This leaves a term with units of m/s (our shear velocity: u_*)

$$\tau_{zx} = \rho(ku_* z)\frac{\partial u}{\partial z}$$

$$\tau_b\left(1-\frac{z}{h}\right) = \rho(ku_* z)\frac{\partial u}{\partial z} \qquad \rightarrow \qquad \rho u_*^2\left(1-\frac{z}{h}\right) = \rho k u_* z\frac{\partial u}{\partial z}$$

$$\cancel{\rho} u_*^{\cancel{2}}\left(1-\cancel{\frac{z}{h}}\right) = \cancel{\rho} k \cancel{u_*} z\frac{\partial u}{\partial z}$$

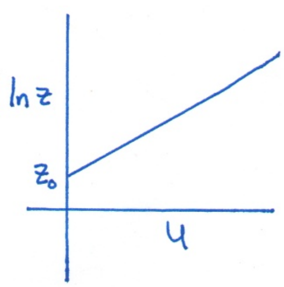
Small near bed

$$u_* = kz\frac{\partial u}{\partial z}$$

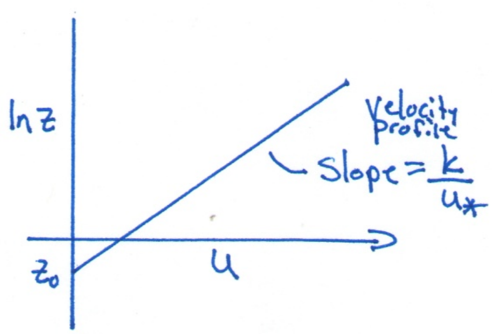
$$\int\frac{\partial u}{\partial z}dz = \int\frac{u_*}{kz} \qquad \rightarrow \qquad u = \frac{u_*}{k}\int\frac{1}{z} = \frac{u_*}{k}\ln z + const$$

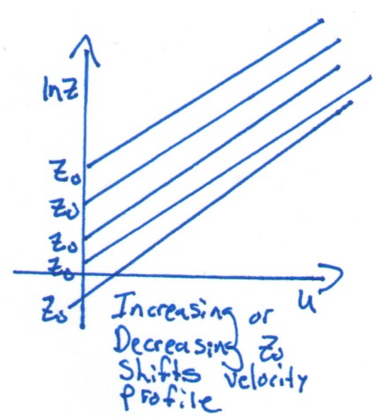
Apply boundary condition at z_0, $u = 0$ $\rightarrow$ $0 = \frac{u_*}{k}\ln z_0 + const$ $\rightarrow$ $-\frac{u_*}{k}\ln z_0 = const$

So, $u = \frac{u_*}{k}(\ln z - \ln z_0) = \frac{u_*}{k}\left(\ln\frac{z}{z_0}\right)$



z_0 = roughness parameter (level at which $u = 0$)

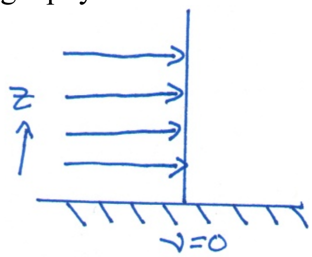




Adjusting z_0 changes the value of u for a flow of constant u_*.

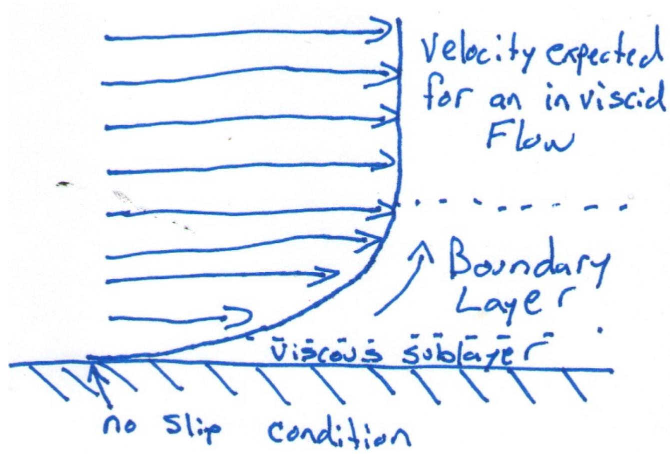
The law of the wall strictly applies to the flow near the bed ($z < 0.2h$). Empirically it provides a reasonable approximation for the entire velocity profile in most rivers.
Remember that the viscous sublayer seperates the turbulent flow from the bed. It is therefore not valid to extrapolate the logarithmic profile to $z = 0$. The level z_0 is defined as the distance above the bed at which $u = 0$ if the turbulent velocity profile was extended downward to that position in the flow.

BOUNDARY LAYERS
At high ***Re*** (***Re*** > 500) the term describing the redistribution and dissipation of momentum by molecular viscosity is typically dropped from the equations describing the flow field for a geophysical fluid. Such a flow is said to be **inviscid** (having no viscosity).



While an inviscid approximation for the flow may be appropriate away from the rigid boundary (the bed), viscous effects can never be neglected near the solid boundary. This thin layer where viscosity is important is called the **Viscous Sublayer**. (An estimate for its thickness can be calculated taking the distance from the bed as the representative length scale for the Reynolds number).

<u>Boundary Layer Flow</u>:



Boundary Layer = Portion of flow substantially affected by the presence of a solid boundary.

A boundary layer matches the 'inviscid' portion of a flow with its viscous boundary condition (no slip at the solid boundary).

The concept of a boundary layer has meaning only at high *Re*. At low *Re*, molecular viscosity acts on the entire flow and there is no local part of the flow that is uniquely identifiable as the boundary layer.

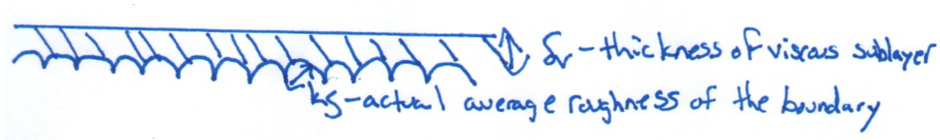
Measuring z_0: Boundary Roughness

Key to determining the appropriate roughness parameter is selecting the appropriate characteristic length scale for the bed roughness.

Selection of this scale is trivial in cases where the bed is composed of a single grain size. In this case the nominal diameter = $\boldsymbol{k_s}$, the roughness length scale.

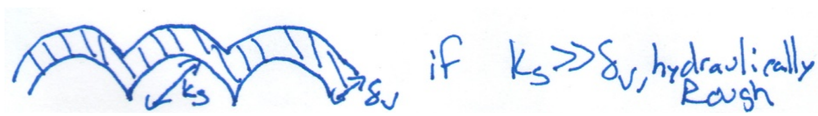
The trivial cases (One grain size, no sediment transport):

1. **Hydraulically Smooth Flow** [$\boldsymbol{k_s} < \delta_v$, the average thickness of the viscous sublayer]



2. **Hydraulically Transitional Flow** [$\boldsymbol{k_s} \approx \delta_v$]

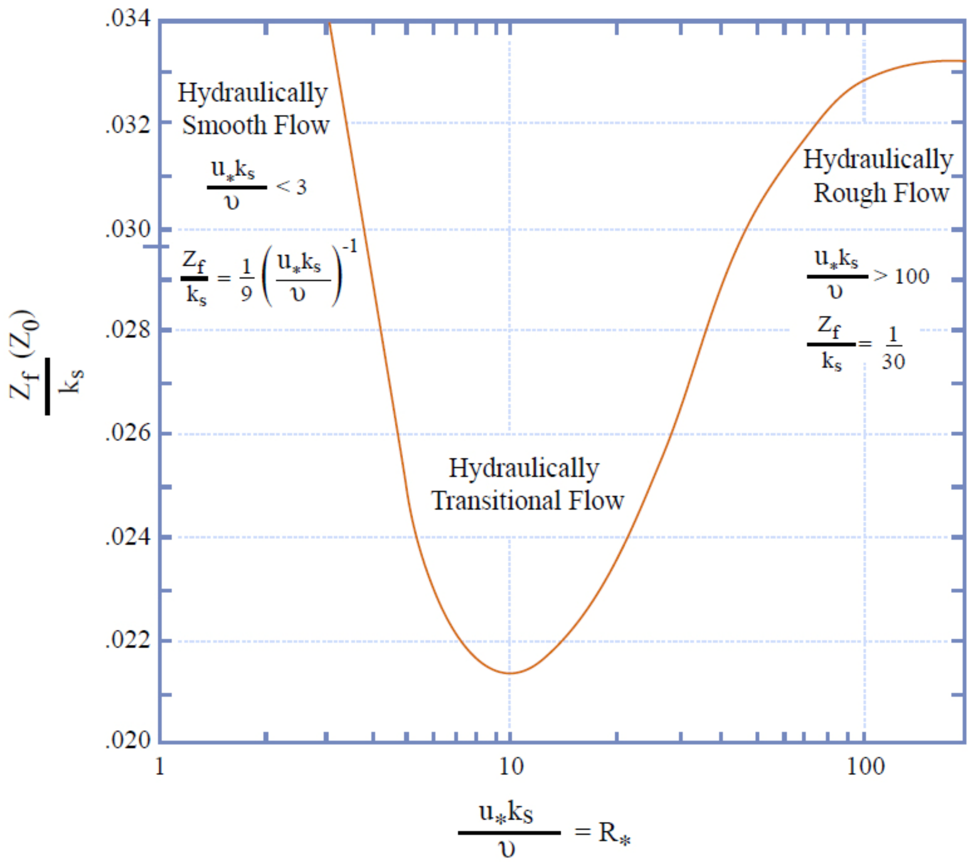
3. Hydraulically Rough Flow [$\boldsymbol{k_s} > \delta_v$]



Note: The flow directly above δ_v for the HRF case is accelerating and decelerating over the roughness elements. This interval of the flow does not satisfy the requirement of horizontally uniform flow assumed in derivation of the law of the wall. Quasi-uniform flow is set up at a distance about $3k_s$ above the bed.

Nikuradse Diagram: $z_0 = k_s f(R_*)$ where $R_* = [u_* \times k_s]/\nu$
Nikuradse experimentally measured values for R_* as a function of z_0/k_s.

He did this by gluing well-sorted sand to the interiors of pipes and measuring pipe-flow velocity profiles.



Necessary first step. To attack most geological problems the relationship needs to be expanded to handle 1) poorly sorted sediment (multiple potential roughness scales), and 2) bed irregularities (e.g. ripples, dunes, bars).

Comments on simple hydraulic drag coefficients for unidirectional flow

Assuming $K = ku_* z(1 - z/h)$

In this case $\quad \tau_{zx} = \cancel{\rho} u_*^{\cancel{2}}(1 - \cancel{z/h}) = \cancel{\rho}(k\cancel{u_*}z(1 - \cancel{z/h}))\dfrac{\partial \bar{u}}{\partial z}$

$$\int \frac{\partial \bar{u}}{\partial z} dz = \int \frac{u_*}{kz} dz$$

$$\int \ln z dz = z(\ln z - 1)$$

$$u = \frac{u_*}{k} \ln \frac{z}{z_0}$$

Now, $Q_{(discharge)} = \int\limits_{\eta=0.}^{s=h} u dz \approx \dfrac{u_*}{k} \int\limits_{\eta=z_0}^{s=h} \ln \dfrac{z}{z_0} dz$

$$Q = \frac{u_*}{k}\left(\ln\frac{z}{z_0} - 1\right)z\Bigg|_{z_0}^{h} = \frac{u_*}{k}\left[\left(\ln\frac{h}{z_0} - 1\right)h - \underbrace{\left(\ln\cancel{\frac{z_0}{z_0}} - 1\right)}_{=0}\underbrace{\cancel{z_0}}_{\approx 0}\right]$$

$$\boxed{Q \approx \frac{u_*}{k}\left(\ln\frac{h}{z_0} - 1\right)h}$$

$$\boxed{\langle u \rangle = \frac{Q}{h} = \frac{u_*}{k}\left(\ln\frac{h}{z_0} - 1\right)}$$

$$\rho\langle u \rangle^2 = \rho u_*^2\left(\frac{1}{k}\left(\ln\frac{h}{z_0 - 1}\right)\right)^2$$

Where: $\quad C_f^{-1} = \left(\dfrac{1}{k}\left(\ln\dfrac{h}{z_0} - 1\right)\right)^2$ $\quad C_f$ = surface drag coefficient or the friction coefficient (the coefficient $\neq$ constant, $C_f = C_f\left(\dfrac{h}{z_0}\right)$).

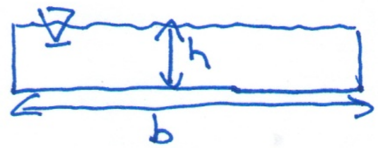
$\boxed{\rho u_*^2 = \tau_b = \rho C_f \langle u \rangle^2}$ ← τ_b can be calculated from the $\langle u \rangle$ and a C_f.

The Manning Equation

$$\langle u \rangle = \frac{1}{M} R^{2/3} S^{1/2}$$ where M = Manning coef., R = hydraulic radius, S = Slope

$$\langle u \rangle = \frac{\left(g^{1/2} R^{1/2} S^{1/2}\right) R^{1/6}}{g^{1/2} M} = \frac{u_* R^{1/6}}{g^{1/2} M}$$

Hydraulic perimeter Radius (R) = $\frac{x\text{−}sectional\ area\ of\ flow}{wetted\ perimeter}$ $\frac{hb}{b+2h}$



$\approx h$ if $b >> h$