Fluid Mechanics I

Conservation of linear momentum:

An expression of Newton's 2nd Law for a continuous medium. m x a added up over the entire volume

$$\int_V \rho \frac{d\underset{\sim}{U}}{dt} \partial V = \sum_{forces} = \text{surface forces + body forces}$$

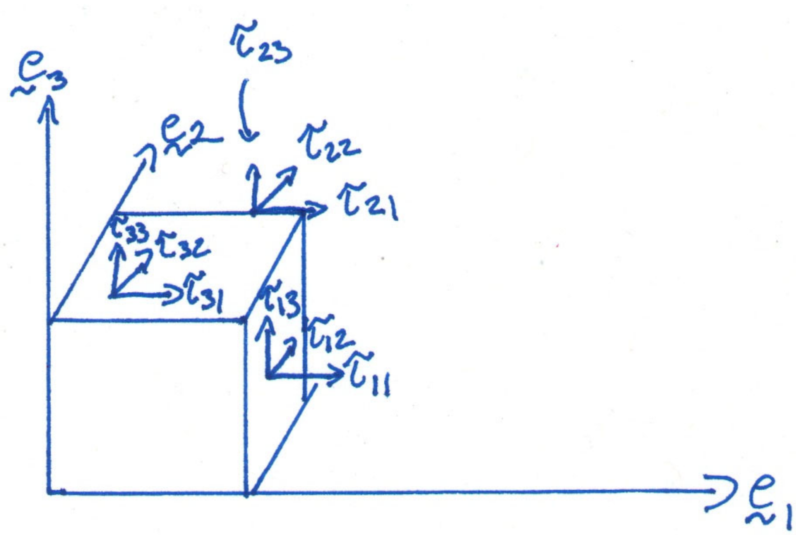
$$\rho \partial V \frac{d\underset{\sim}{U}}{dt} = f_B \partial V + \left(\nabla \cdot \tilde{\tau} \right) \partial V$$

Remember this is nothing more than Newton's Second law. This is also called Cauchy's first law of mechanics

$$\rho \frac{d\underset{\sim}{U}}{dt} = f_B + \nabla \cdot \tilde{\tau}_{total}$$

$$\rho \frac{d\underset{\sim}{U}}{dt} = -\rho g + \nabla \cdot \tilde{\tau}_{total}$$

Stress = force/area (need geometric data and force info)



$$\tilde{\tau} = \underset{\sim}{n}\,\underset{\sim}{F}$$

In fluid mechanics it is often convenient to break the total stress into two components

$$\tilde{\tau}_{total} = -\tilde{p} + \tilde{\tau}$$

$\underset{\sim}{p}$; Pressure (defined in the outward direction)

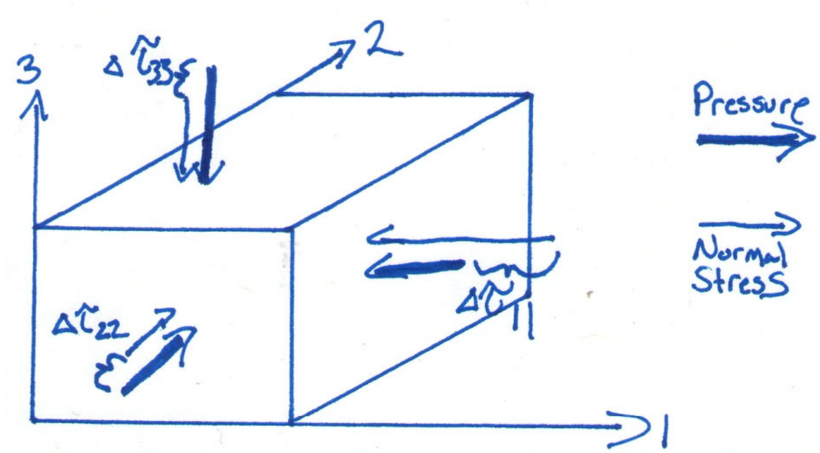
Isotropic portion (at a point the property doesn't vary with direction)

$\underset{\sim}{\tau}$; Deviatoric stress

Deviates from the pressure

Non-isotropic

$p(pressure) \overset{def}{=} -\left(\frac{\tau_{11}+\tau_{22}+\tau_{33}}{3}\right) = -\frac{\tau_{ii}}{3}$ = the average of normal stresses



$\Delta\tau_{ij}$ = deviatoric stress

$$\rho\frac{d\underset{\sim}{U}}{dt} = -\rho g + \nabla\cdot\underset{\sim}{\tau}_{total}$$

$$\nabla\cdot\underset{\sim}{\tau}_{total} = \nabla\cdot\left(-\underset{\sim}{p} + \underset{\sim}{\tau}_{deviatoric}\right)$$

Define $p = -\frac{\tau_{11}+\tau_{22}+\tau_{33}}{3} = -\frac{\tau_{ii}}{3}$

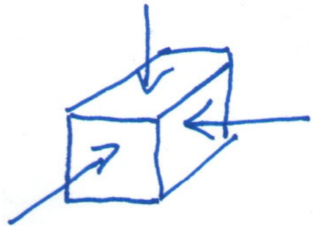
Can be shown $\nabla\cdot -\underset{\sim}{p} = -\nabla p$

So: $\nabla\cdot\underset{\sim}{\tau}_{total} = -\nabla p + \nabla\cdot\underset{\sim}{\tau}_{dev} = -\frac{\partial p}{\partial x} - \frac{\partial p}{\partial y} - \frac{\partial p}{\partial z} + \frac{\tau_{xj}}{\partial x} + \frac{\partial\tau_{yj}}{\partial y} + \frac{\partial\tau_{zj}}{\partial z}$

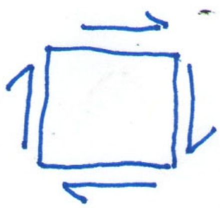
So: $\rho\frac{d\underset{\sim}{U}}{dt} = \underset{\sim}{f}_{B} + \nabla\cdot\underset{\sim}{\tau}_{total} = \underset{\sim}{f}_{B} - \nabla p + \nabla\cdot\underset{\sim}{\tau}_{dev}$

I) Translation: occurs just by applying forces (push or pull) 

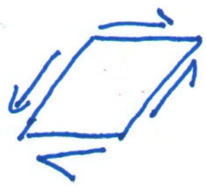
II) Deformation:



a. Dilation/compression:



b. Rotation:



c. Pure Shear (no rotation)

Balance forces so there is no rotation, just shear

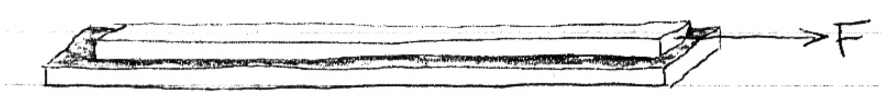
Viscous forces

$$\vec{\nabla} \cdot \underset{\sim}{\tau} = \frac{\partial \tau_{ij}}{\partial x_i} = \frac{\partial \tau_{1j}}{\partial x_i} + \frac{\partial \tau_{2j}}{\partial x_2} + \frac{\partial \tau_{3j}}{\partial x_3}$$

Pressure is a force per unit area. Pressure at base of a river is equal to: $P = \rho g h$
Stress is a measure of Force on the inside of a continuum
Strain is the response of material particles to stress

<u>Constitutive Equations</u> express the relationship between **stress** and **strain (rate)**.

Constitutive Equation for a Newtonian fluid



(stress on z face in x direction) $\tau_{zx} = \dfrac{F}{A_p}$

(strain rate) $\dfrac{\partial u}{\partial z} = \dfrac{u_p}{h}$; h is depth of flow, No slip B.C. - $\partial u = u_{plate} - u_{lower\ boundary}$

$u_p - 0 = u_p$

so, $\dfrac{\tau_{zx}}{\frac{\partial u}{\partial z}} = \dfrac{F / A_p}{\frac{u_p}{h}} = \mu$; μ = dynamic viscosity(constant for a given material at given temp.)

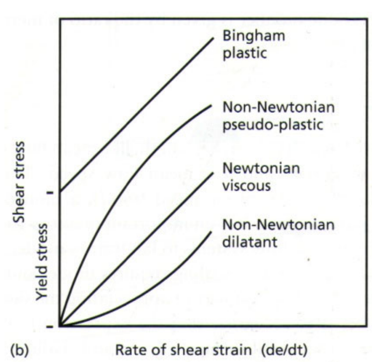
Newton's viscous law: $\tau_{zx} = \mu \dfrac{\partial u}{\partial z}$; ($\upsilon = \dfrac{\mu}{\rho}$, kinematic viscosity)

μ has units of stress/strain rate = Pa/(1/t) = N s /m^2 = Pa s; in primary units M/(LT)

Water (20^{o}C) $\mu = 10^{-3}$ Pa s
Asthenosphere $\mu = 10^{20}$ Pa s

$$\upsilon = \frac{\mu}{\rho} = \frac{[M / LT]}{[M / L^3]} = \frac{L^2}{T}$$; kinematic viscosity

Note, this is only applicable to Newtonian fluids where strain is linearly related to stress:



We now want a generalized tensor expression for Newton's Viscous Law:

$\tau_{ij} = \mu_{ijkl} \dfrac{\partial u_k}{\partial x_l}$ (i = face pulling on, j = direction of pull, k = velocity component [u, v, w], l = direction of change [x, y, z])

Remember: viscous forces

$$\vec{\nabla} \cdot \underset{\sim}{\tau} = \frac{\partial \tau_{ij}}{\partial x_i} = \frac{\partial \tau_{1j}}{\partial x_i} + \frac{\partial \tau_{2j}}{\partial x_2} + \frac{\partial \tau_{3j}}{\partial x_3}$$

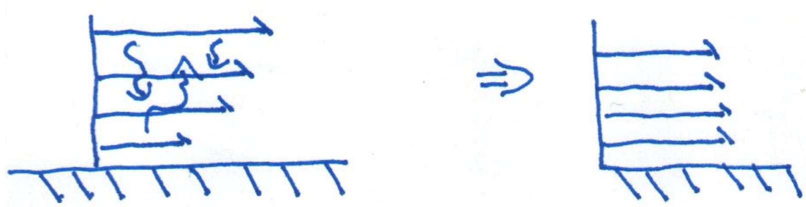
Apply constitutive relationship above and after some math end up with…

Assuming constant temperature and incompressible fluid to get simple stress-strain relation

$$\vec{\nabla} \cdot \tilde{\tau}_{dev} = \mu \nabla^2 \vec{u}$$

Right side of equation is essentially the friction in the system

Left side of equation: diffusion of velocity or momentum (because ρ = constant): random motion of molecules lead to destruction of velocity gradient $\vec{\nabla} u \to 0$



Viscous force that diffuses momentum.

Navier – Stokes equation (Early 1800's) Navier – French engineer, Stokes – English Mathematician

A = B – C + D

$\rho \frac{d\underset{\sim}{U}}{dt} = -\rho g - \nabla p + \mu \nabla^2 \underset{\sim}{U}$; For Isotropic, incompressible, viscous Newtonian fluid, at constant temp.

A = acceleration of fluid

B = body force

C = pressure gradient

D = diffusion of velocity

Navier – Stokes Equation is really nothing more than a modified expression of conservation of linear momentum.

In this class we will encounter many non dimensional numbers (numbers that do not have any units associated with them). These numbers are rather useful in comparing dynamics of systems that have different scales. Many of the non-dimensional numbers we will focus on are actually ‘hidden’ in the NS Eq.

Example of non-dimensional numbers

$$\rho \frac{\partial \underset{\sim}{U}}{\partial t} + \rho \underset{\sim}{U} \cdot \nabla \underset{\sim}{U} = -\rho g - \nabla p + \mu \nabla^2 \underset{\sim}{U}$$

(incompressible situation)

Discrete form of NS Eq.

$$\rho \frac{\underset{\sim}{U}}{T} + \rho \frac{\underset{\sim}{U}^2}{d} = -\frac{p}{d} + \mu \frac{\underset{\sim}{U}}{d^2} - \rho g\,;$$

now we will divide each term by $\rho \frac{\underset{\sim}{U}^2}{d}$

You then get: $$\frac{d}{\underset{\sim}{U} T} + 1 = -\frac{p}{\rho \left(\frac{\underset{\sim}{U}^2}{2} \right)} + \frac{\upsilon}{\underset{\sim}{U} d} - \frac{gd}{\underset{\sim}{U}^2}$$

A +1 = - B + C - D

A = Strouhal #

B = Euler # (if $\underset{\sim}{U}^2$ is divided by 2)

C = inverse of Reynolds #

D = (Froude #)$^{-1/2}$

Remember, for channel flow problems: u = <u>, characteristic length = h = depth (for uniform channel flow the only characteristic length is the depth)

$$\mathrm{Re} = \frac{<u> h}{\upsilon}, \qquad Fr = \frac{<u>}{\sqrt{gh}}$$

It is important to remember that Re and Fr are defined independently of one another.

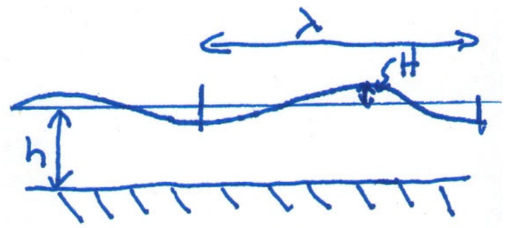
FROUNDE NUMBER

Froude numbers (Fr). Fr is a nondimensional number that is a ratio of inertial forces in a flow to gravitation forces. It is useful for many problems we will face later in class.

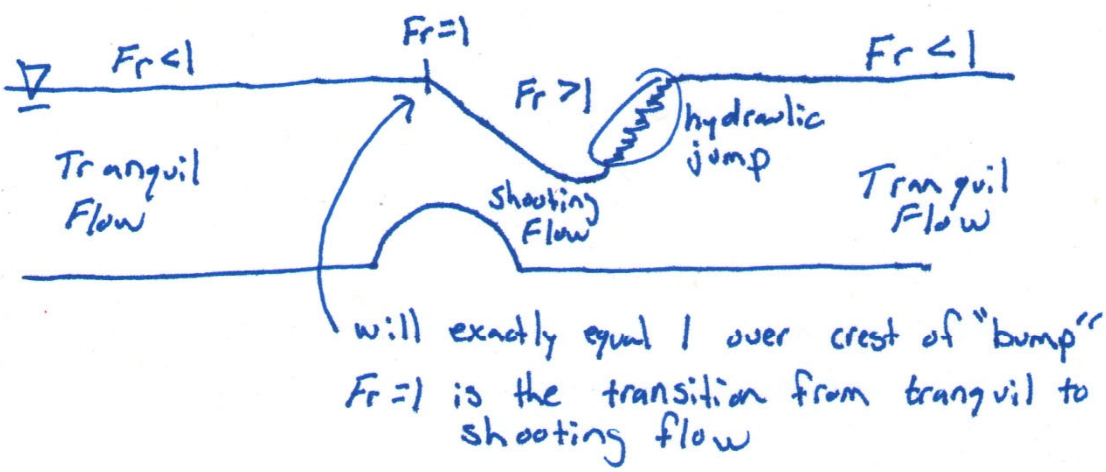
$$Fr = \frac{u}{\sqrt{gh}}$$

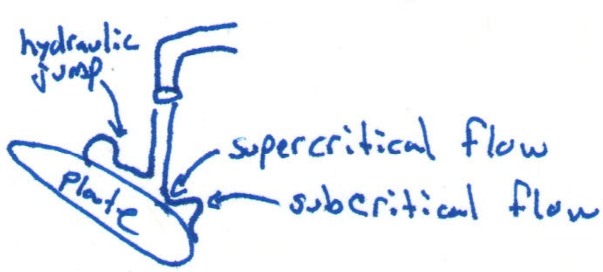
Froude number: applicable to laminar and turbulent flows having a free surface OR interface such that gravity forces play an important role in causing the flow.

$\sqrt{gh}$ = c = phase speed for shallow water surface waves (wavelength >>water depth) and (H<<h)



Fr < 1 waves can propogate both upstream and downstream, Tranquil flow.
Fr > 1 wave cannot propogate upstream, Shooting flow
Fr = 1 hydraulic jump, all upstream propagating waves are “stuck” here.





Reynolds Number: by looking at the size of the number you can see if inertia or friction is more important in the system.

Re < 1 Laminar flow: stable to small disturbances (reversible deformation)
Re >> 1 Turbulent flow: unstable to small disturbances, stretching and twisting

In nature you always have disturbances, questions is when do they decay versus grow?

Re < 500 Laminar
Re > 500 Turbulent (import natural flow, could through away laminar conditions, if not for boundary layers, length scale gets very small)

If Re > 1000, one can neglect the viscous forces & drop the $\mu\nabla^2 \underset{\sim}{U}$ term out of NS Eq.

If Re ~ 10^{-3}, one can neglect convective accelerations & drop the $\rho \underset{\sim}{U}\cdot\nabla \underset{\sim}{U}$ term out of the NS Eq.

If Re ~ 1, one needs to keep both terms in the NS Eq.

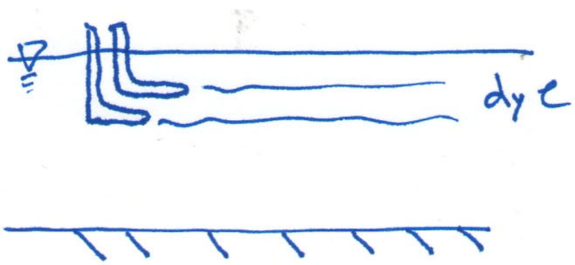
This "scaling" allows one to get at the "essence" of the NS Eq for a particular problem without having to solve for the entire equation

Aside:

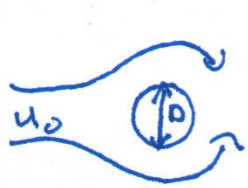
If Re is small then laminar flow (a very organized fow) – A Poiseuille flow (if steady and horizontally uniform).

Perturbations of the flow decay with time – the stable condition is complete organization (eg an introduced wave would decay with time)

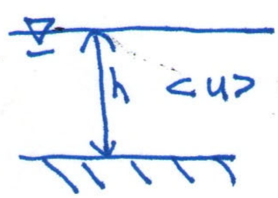
For laminar flows you can completely reverse this process (you can put all the particles back where they started).



Samples of scale velocity & length



$R_D = \frac{u_0 D}{\nu}$



$\Rightarrow R = \frac{\langle u\rangle h}{\nu}$

NS equation describes instantaneous velocity field. What happens if flow field is time dependent?

Velocity data can be deconvolved into three potential signals; mean, periodic, and deviatoric variations. (e.g. Lyn et al., 1995)

Time-dependent, periodic variations in velocity associated with the shedding of vortices.

Strouhal #: When a flow becomes time dependant, the structure of the flow characterized by the Strouhal and Reynolds #'s

$St = \frac{L}{uT}$ here *T* will be characteristic time of vortex shedding

Experimentally observed that $S \approx 0.2$

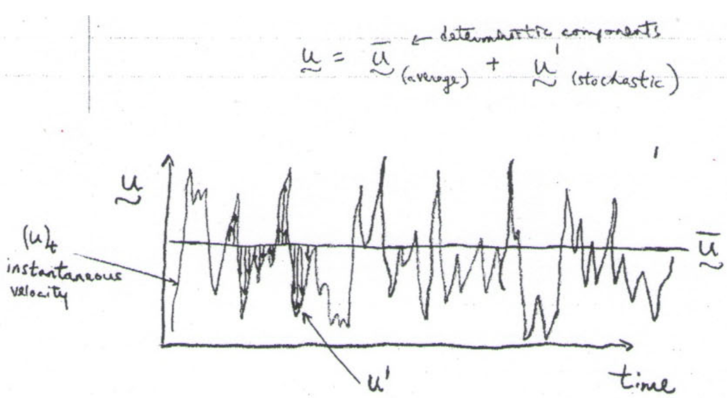
2 m/s wind | 1 mm wire | $T = 2.5 \times 10^{-3}$ sec | 2(0.001)

1 cm/s river | 20 cm pipe | Would shed vortices at a time scale of 100 sec

Euler number

$Eu = \frac{p}{\rho \underset{\sim}{U}^2}$; basically ratio of pressure forces to inertia forces

Turbulent flows: Non averaged NS Eq. describes the instantaneous velocity field. If a turbulent flow field is of a deterministic plus completely ramdom part it is not possible to solve for this instantaneous field. Therefore the NS Eq. must be averaged in orger to produce a conservative relationship for the deterministic components of the flow.



U = instantaneous vel
U_bar = averaged over time
U' = deviations of one from other

In turbulent flows velocity varies with time. Therefore we will characterize turbulent flow in terms of a mean or average velocity, and its deterministic component.

$$\underset{\sim}{u} = \underset{\sim}{\overline{u}} + \underset{\sim}{u'}$$

$\underset{\sim}{\overline{u}}$ = deterministic portion of the turbulent flows velocity (can be calculated)

$\underset{\sim}{u'}$ = stochastic (random) portion of the velocity term (fluctuates)

It is clear that if one is interested in describing turbulent flows one needs to have expressions possessing averaged variables.

So let’s Reynolds average the Navier-Stokes Equation

$$\rho \frac{d\underset{\sim}{\overline{U}}}{dt} = -\rho g - \nabla \overline{p} + \overline{\mu} \nabla^2 \underset{\sim}{\overline{U}}$$

So how do you calculate $\underset{\sim}{\overline{u}}$?

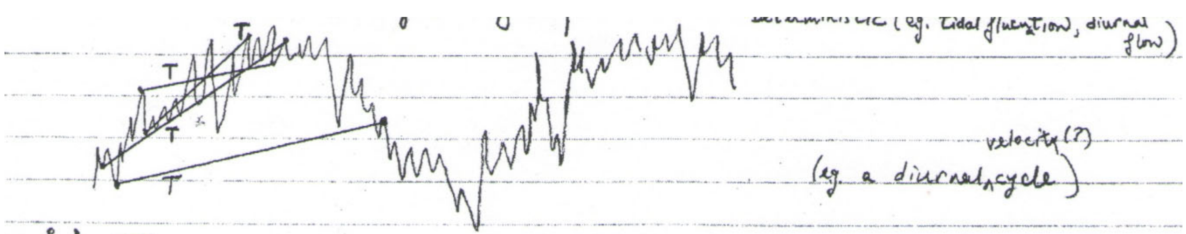
You would like to have an Ensemble Average (a collection of average measurements). Unfortunately you can’t expect to measure the velocity in a stream reach a 1000 different times and get the same $\underset{\sim}{\overline{u}}$.

Because of this $\underset{\sim}{\overline{u}}$ is often determined by making an approximation.

Approximation for $\underset{\sim}{\overline{u}}$:

$$\underset{\sim}{\overline{u}} = \lim_{T \to \infty} \frac{1}{T} \int_{t}^{t+T} \underset{\sim}{u}\, dt \qquad \text{approximation: } \frac{1}{T} \int_{t}^{t+T} \underset{\sim}{u}\, dt$$

Where $T_{deterministic} >> T >> T_{turbulence}$; This is the deterministic period (i.e. the period which doesn’t allow T to go to infinity: e.g. tidal fluctuations, diurnal fluctuations).
Remember that T can not go to infinity: $T << T_{deterministic}$ (e.g. tidal fluctuations, diurnal flow)



[] - T's choosen which are too long or too short lead to errors in the characterizing of the velocity profile
[] - Best fit

How do you determine the length of the period associated with the turbulence?
Taylor Hypothesis: Most of the energy in a flow is in waves that have lengths on the scale of the depth of the flow.

$T_{turb} = \dfrac{h}{<u>}$; Example: $\dfrac{10m}{1m/s} = 10s$; so the peak in the turbulent spectrum occurs every 10s. (i.e. major turbulent fluctuations have 10s periods).

Largest scale eddy is most effective in distributing any property.

So, it is now clear that if one is interested in turbulent flows one needs relationships that have averaged variables. (We want an averaged Navier – Stokes Equation, Continuity Equation).

$$\frac{\partial \underset{\sim}{u}}{\partial t} + \rho \underset{\sim}{u} \nabla \underset{\sim}{u} + \nabla p - \mu \nabla^2 \underset{\sim}{u} + \rho g = 0$$

1 + 2 + 3 - 4 + 5 = 0

1: $\overline{\dfrac{\partial \underset{\sim}{u}}{\partial t}} = \dfrac{1}{T}\displaystyle\int_t^{t+T} \frac{\partial \underset{\sim}{u}}{\partial t}\partial t = \frac{\underset{\sim}{u}(t+T) - \underset{\sim}{u}(t)}{T} = \frac{\Delta \underset{\sim}{u}}{\Delta T} = \frac{\partial \underset{\sim}{u}}{\partial t}$; we can make the last estimate if $\dfrac{\partial \underset{\sim}{u}}{\partial t}$ is large and occurs over a long period (i.e. large T)

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Going to time average one term of the N-S equation (ρ, P, u: all averaged)

$$\rho \vec{u} \cdot \vec{\nabla} \vec{u} = \rho \vec{\nabla} \cdot \vec{u}\vec{u} - \rho \vec{u} \vec{\nabla} \cdot \vec{u}$$

$$\rho \vec{\nabla} \cdot [(\overline{u} + u')(\overline{u} + u')] = \rho \vec{\nabla} \cdot [\overline{u}\,\overline{u} + 2\overline{u}u' + u'u']$$

$$\frac{\overline{\rho \vec{\nabla}}}{T}\int_0^T \overline{u}\,\overline{u}\,\partial t = \frac{\overline{\rho \vec{\nabla}}}{T}(\overline{u}\,\overline{u}T - 0) = \overline{\rho} \vec{\nabla} \cdot \overline{u}\,\overline{u}$$ Already averaged with time, therefore constant wrt time

$\displaystyle\int_0^t 2\overline{u}u' \Rightarrow 0$ because $\overline{u'} = 0$ Average of fluctuations = 0

$$\frac{\overline{\rho \vec{\nabla}}}{T}\int_0^T u'u'\,\partial T = \rho \vec{\nabla} \cdot \overline{u'u'}$$

$$\rho\left(\overline{\frac{\partial \underset{\sim}{u}}{\partial t}} + \overline{\underset{\sim}{u} \cdot \nabla \underset{\sim}{u}}\right) = -\overline{\nabla p} + \overline{\mu \nabla^2 \underset{\sim}{u}} - \overline{\rho g}$$
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After a number of steps you end up with the averaged N-S Equation, or Reynolds Equation

$$\rho \frac{\partial \bar{\underset{\sim}{u}}}{\partial t} + \rho \bar{\underset{\sim}{u}} \cdot \nabla \bar{\underset{\sim}{u}} = -\nabla \bar{p} + \mu \nabla^2 \bar{\underset{\sim}{u}} - \rho g + \nabla \cdot \rho \overline{\underset{\sim}{u}'\underset{\sim}{u}'}$$

Reynolds stress or turbulent momentum flux

$\rho \overline{u_i u_j}$ = flux of x_i momentum in the x_j direction

A momentum flux can be thought of as a stress

Reynolds Stress = a turbulent momentum flux

Mass flux (gm/s) through a surface = $\rho\left(\frac{gm}{cm^3}\right) u\left(\frac{cm}{s}\right) \cdot A\left(cm^2\right)$

Momentum flux $\left(\frac{gmcm}{s^2}\right)$ through a surface = $\rho\left(\frac{gm}{cm^3}\right) u\left(\frac{cm}{s}\right) u\left(\frac{cm}{s}\right) A\left(cm^2\right)$; notice units of force

Stress = force/area = $\left(\frac{gm \cdot cm}{s^2}\right) / \left(cm^2\right) = \frac{gm}{cms^2}$

$\tau_{zx} = \mu \frac{\partial u}{\partial z} \Rightarrow \left(\frac{gm}{scm}\right)\left(\frac{cm}{s} / cm\right) = \frac{gm}{scm}$; units of stress

$\tau_R = \rho \overline{\underset{\sim}{u}'\underset{\sim}{u}'} \Rightarrow \left(\frac{gm}{cm^3}\right)\left(\frac{cm}{s}\right)\left(\frac{cm}{s}\right) = \frac{gm}{cms^2}$; units of stress