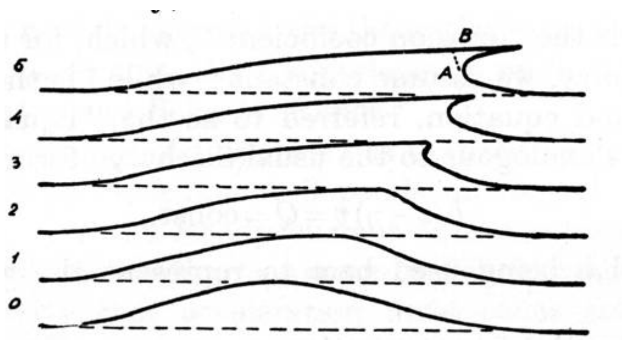
Conservation of mass and the birth of morphodynamics

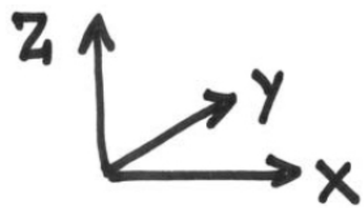


Professor Exner's mathematically derived dune p

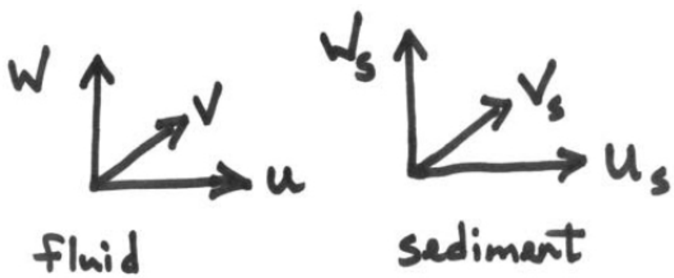
Credited with being the first in applying conservation of mass principles to sediment transport.

Exner realized that if flow velocity (and hence sediment flux) depends on elevation, then a nonlinear wave equation describes bed evolution, explaining why bedforms skew in the direction of flow

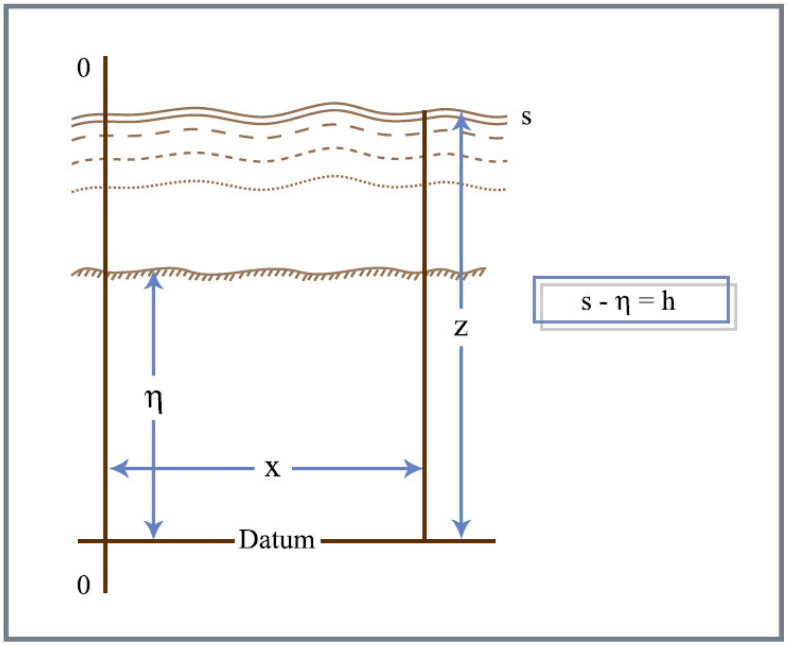
- All descriptions of fluid and sediment transport will be referenced to the following orthogonal coordinate system. In this system, ***x*** refers to the streawise or down-slope direction, ***y*** refers to the cross-stream or cross-slope direction, and ***z*** refers to the vertical direction.



The fluid velocity component in the x direction will be referred to as u, the fluid velocity component in the y direction will be referred to as v, and the component in the z direction is w. A subscript s will be added to each velocity component to distinguish the velocity of the sediment grains from the velocity of the fluid.



The following reference frame will be used for this discussion



- Prior to discussing fluid mechanics let's set the stage for why we care by looking at the Erosion Equation (aka the Exner Eq.). This is possibly the most important equation in all of sedimentology. We will come back to this at several points in the class.

Prior to getting to Exner, some other definitions:

Total Derivative: $\frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + \underset{\sim}{U} \cdot \nabla \rho$

Rewritten: $\frac{\partial \rho}{\partial t} + u_i \frac{\partial \rho}{\partial x_i} = \frac{d\rho}{dt}$

$A + B = C$

A: partial derivative, change in ρ with respect to *t* at a fixed point in space
B: space change
C: total derivative, total change in ρ with respect to *t*, and following a material element through space (i.e. point not fixed).

Change in density is due to:

1) Material diffusing away from that point
2) Material being compressible

If material is incompressible and there is negligible diffusion flow

$\frac{d\rho}{dt} = 0$, (but $\frac{\partial \rho}{\partial t}$ and $u_i \frac{\partial \rho}{\partial x_i}$) are not necessarily equal to zero or constant value

Frames of Reference

- Lagrangian – reference frame is attached to the parcel under observation (e.g., oceanographic buoys)

- Eulerian – reference fixed in space (e.g., stream gaging station)

Deriving an Expression for the conservation of Mass

Tool – Taylor Series: method of extrapolation: allows us to estimate what's going on somewhere else from a given or known point
Example – weather prediction

$$W_{later} = W_{now} + \left(\frac{dW}{dt}\right)_{now} \Delta t + \frac{1}{2!}\left(\frac{d^2W}{dt^2}\right)\left(\Delta t^2\right) + \ldots$$

$$\frac{1}{2!}\left(\frac{d^2W}{dt^2}\right)\left(\Delta t^2\right) = \frac{d}{dt}\left[\left(\frac{dW}{dt}\right)\Delta t\right]\Delta t$$

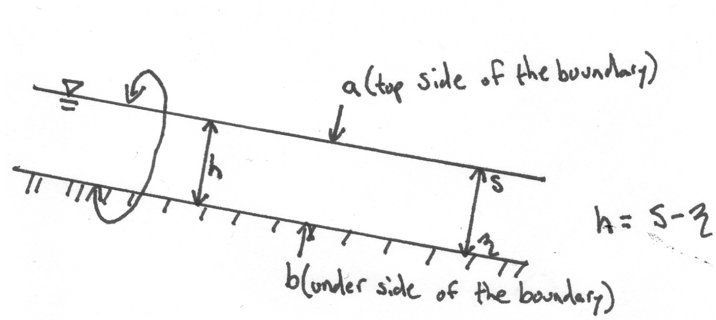
- Higher order terms get small so they are often dumped and expression written as:
- $N_t = N_0 + \left(\frac{\partial N}{\partial t}\right)_0 dt + O\left(\partial t^2\right)$

<u>Conservation of sediment mass</u> (assuming the sediment is incompressible)

$$\frac{d\varepsilon_{s_{total}}}{dt} = \left(\frac{\partial \varepsilon_{s_{total}}}{\partial t} + \nabla \cdot \varepsilon_{s_{total}} \underset{\sim}{U}_s\right) = \left(\frac{\partial \varepsilon_{s_{total}}}{\partial t} + \left(\frac{\partial \varepsilon_{s_{total}} u_s}{\partial x} + \frac{\partial \varepsilon_{s_{total}} v_s}{\partial y} + \frac{\partial \varepsilon_{s_{total}} w_s}{\partial z}\right)\right) = 0$$

$\varepsilon_{s_{total}}$ = total volume of concentration of sediment in the flow (bedload + suspended & washload)

The Erosion Equation is derived by integrating the expression for mass conservation through the entire depth of the flow and applying the following boundary conditions….



The erosion equation is derived by integrating the equation $\frac{\partial \varepsilon_s}{\partial t} + \nabla \cdot \varepsilon_s \underset{\sim}{U} = 0$ from above s (i.e. a) to below η (i.e. b). In doing this we cross two sediment concentration discontinuities. We want to get across these discontinuities because it allows us to have a desirable set of <u>boundary conditions</u>.

$(\varepsilon_s)_s \neq 0$, $\quad (\varepsilon_s)_a = 0$
$(u_s)_\eta \neq 0$, $\quad (u_s)_b = 0$ $\quad$ η = by definition the top of uppermost layer of nonmoving grains

We're going to end up with the equation: $\frac{\partial V}{\partial t} + \nabla \cdot \underset{\sim}{Q}_s + \frac{\partial \eta}{\partial t}\varepsilon_b = 0$

A + B + C

A: the rate of suspended sediment change
B: the divergence of the sediment discharge
C: The rate of elevation change of the bed (erosion, deposition)

Full derivation of Exner can be found in Paola & Voller 2005, and shown below

So let's start!, $\int_{\eta}^{s}\left(\frac{\partial \varepsilon}{\partial t}+\nabla \cdot \varepsilon_s \underset{\sim}{u_s}\right)dz = 0$

$$\int_{\eta}^{s}\frac{\partial \varepsilon_s}{\partial t}dz+\int_{\eta}^{s}\frac{\partial}{\partial x}(\varepsilon_s u_s)dz+\int_{\varepsilon}^{s}\frac{\partial}{\partial y}(\varepsilon_s v_s)dz+\int_{\eta}^{s}\frac{\partial}{\partial z}(\varepsilon_s w_s)dz = 0$$

Leibnitz Rule:

$$\frac{\partial}{\partial t}\int_{a(x,t)}^{b(x,t)} N(x,t)dx = \int_{a}^{b}\frac{\partial N}{\partial t}dx+\frac{\partial b}{\partial t}(N_b)-\frac{\partial a}{\partial t}(N_a)$$

N_b – integram evaluated at the upper limit

N_a – integram evaluated at the lower limit

or

$$\int_{a}^{b}\frac{\partial N}{\partial t}dx = \frac{\partial}{\partial t}\int_{a}^{b} Ndx-\frac{\partial b}{\partial t}(N_b)+\frac{\partial a}{\partial t}(N_a)$$

So,

$$\frac{\partial}{\partial t}\int_{\eta}^{s}\varepsilon_s dz-\frac{\partial s}{\partial t}(\varepsilon_s)_s+\frac{\partial \eta}{\partial t}(\varepsilon_s)_\eta+\frac{\partial}{\partial x}\int_{\eta}^{s}(\varepsilon_s u_s)dz-\frac{\partial s}{\partial x}(u_s\varepsilon_s)_s+\frac{\partial \eta}{\partial x}(u_s\varepsilon_s)_\eta+\frac{\partial}{\partial y}\int_{\eta}^{s}(\varepsilon_s v_s)dz$$
$$-\frac{\partial s}{\partial y}(\varepsilon_s v_s)_s+\frac{\partial \eta}{\partial y}(\varepsilon_s v_s)_\eta+(\varepsilon_s w_s)_s-(\varepsilon_s w_s)_\eta = 0$$

Back to some definitions:

$\int_{\eta}^{s}\varepsilon_s dz = V_s$; volume of sediment in suspension per unit area of bed

$\frac{\partial}{\partial t}\int_{\eta}^{s}\varepsilon_s dz$ = rate of increase (or decrease) of volume of sediment in the fluid per unit area of the bed

sed. discharge: $\underset{\sim}{Q_s} = \int_{\eta}^{s}\left(\varepsilon_s \underset{\sim}{U_s}\right)_H dz$; analogues to fluid discharge in a 2D system

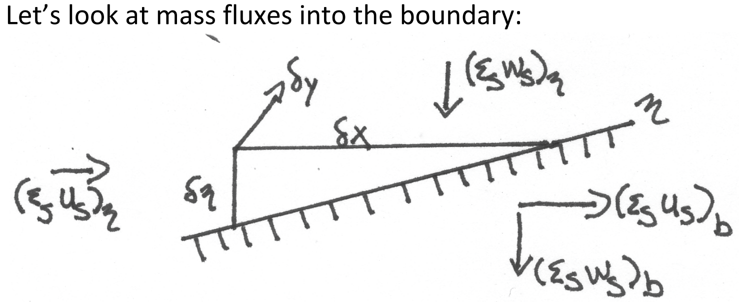
$\nabla \cdot \underset{\sim}{Q_s} = \frac{\partial}{\partial x}\int_{\eta}^{s}(\varepsilon_s u_s)dz+\frac{\partial}{\partial y}\int_{\eta}^{s}(\varepsilon_s v_s)dz$; (remember, no vertical component)

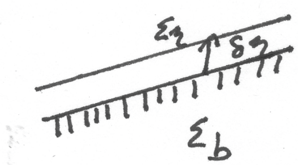
We can incorporate V_s and $\nabla \cdot \underset{\sim}{Q_s}$ into the erosion equation and get :

$$\frac{\partial V_s}{\partial t} + \nabla \cdot \underset{\sim}{Q_s} + (\varepsilon_s w_s)_s - (\varepsilon_s w_s)_\eta - \frac{\partial s}{\partial t}(\varepsilon_s)_s + \frac{\partial \eta}{\partial t}(\varepsilon_s)_\eta - \frac{\partial s}{\partial x}(\varepsilon_s u_s)_s + \frac{\partial \eta}{\partial x}(\varepsilon_s u_s)_\eta - \frac{\partial s}{\partial y}(\varepsilon_s v_s)_s$$
$$+ \frac{\partial \eta}{\partial y}(\varepsilon_s v_s)_\eta = 0$$

We want to replace η with b; s with a

Let's look at mass fluxes into the boundary:





$\delta\eta\delta x\delta y(\varepsilon_b - \varepsilon_\eta) =$ volume of sediment Added to bed

Using a Taylor series one can approximate $\delta\eta$:

$$\delta\eta = \eta_0 + \frac{\partial \eta}{\partial t}\delta t + O(\delta t^2) - \eta_0$$

Mass added to the boundary

$$= (\varepsilon_s u_s)_\eta \delta\eta\delta y\delta t + (\varepsilon_s v_s)_\eta \delta\eta\delta x\delta t - (\varepsilon_s w_s)_\eta \delta x\delta y\delta t - (\varepsilon_s u_s)_b \delta\eta\delta y\delta t - (\varepsilon_s v_s)_b \delta\eta\delta x\delta t$$
$$+ (\varepsilon_s w_s)_b \delta x\delta y\delta t + (\varepsilon_b - \varepsilon_\eta)\frac{\partial \eta}{\partial t}\delta x\delta y\delta t + o(\delta t)\delta x\delta y\delta t = 0$$

Now multiply each term by $\frac{1}{\delta x\delta y}$:

$$(\varepsilon_s u_s)_\eta \frac{\delta\eta}{\partial x} + (\varepsilon_s v_s)_\eta \frac{\delta\eta}{\partial y} - (\varepsilon_s w_s)_\eta - (\varepsilon_s u_s)_b \frac{\delta\eta}{\partial x} - (\varepsilon_s v_s)_b \frac{\delta\eta}{\partial y} + (\varepsilon_s w_s)_b + (\varepsilon_b - \varepsilon_\eta)\frac{\delta\eta}{\partial t} + O(\delta t) = 0$$

Now take lim as δx, δy, δz go to 0:

$$(\varepsilon_s u_s)_\eta \frac{\partial\eta}{\partial x} + (\varepsilon_s v_s)_\eta \frac{\partial\eta}{\partial y} - (w_s\varepsilon_s)_\eta - (\varepsilon_s u_s)_b \frac{\partial\eta}{\partial x} - (\varepsilon_s v_s)_b \frac{\partial\eta}{\partial y} + (\varepsilon_s w_s)_b + \varepsilon_b \frac{\partial\eta}{\partial t} - \varepsilon_\eta \frac{\partial\eta}{\partial t} = 0 \text{; so:}$$

$$(\varepsilon_s u_s)_b \frac{\partial\eta}{\partial x} + (\varepsilon_s v_s)_b \frac{\partial\eta}{\partial y} - (\varepsilon_s w_s)_b - \varepsilon_b \frac{\partial\eta}{\partial t} = (\varepsilon_s u_s)_\eta \frac{\partial\eta}{\partial x} + (\varepsilon_s v_s)_\eta \frac{\partial\eta}{\partial y} - (w_s\varepsilon_s)_\eta - \varepsilon_\eta \frac{\partial\eta}{\partial t}$$

(same surface in the limit so sums must be equal

The last expression on last page relates $(\quad)_b$ to $(\quad)_\eta$

You can perform a similar mass balance for the surface and relate $(\quad)_a$ to $(\quad)_s$.

Now that we have replaced η & s with b & a we can write the Erosion equation as:

$$\frac{\partial V_s}{\partial t}+\nabla\cdot \underset{\sim}{Q_s}+\left(\varepsilon_s u_s\right)_b\frac{\partial \eta}{\partial x}+\left(\varepsilon_s v_s\right)_b\frac{\partial \eta}{\partial y}-\left(\varepsilon_s w_s\right)_b+\varepsilon_b\frac{\partial \eta}{\partial t}+\left(\varepsilon_s u_s\right)_a\frac{\partial s}{\partial x}+\left(\varepsilon_s v_s\right)_a\frac{\partial s}{\partial y}$$
$$+\left(\varepsilon_s w_s\right)_a+\left(\varepsilon_s\right)_a\frac{\partial s}{\partial t}=0$$

If we assume $\left(\varepsilon_s\right)_a=0$ & $\left(\underset{\sim}{U_s}\right)_b=0$ than a lot of terms drop out and we are left with the Erosion Eq.

Relates erosion and deposition to sediment transport processes, a simple conservation of mass equation

$$\frac{\partial V_s}{\partial t}+\nabla\cdot \underset{\sim}{Q_s}+\varepsilon_b\frac{\partial \eta}{\partial t}=0$$

Or

$$\frac{\partial \eta}{\partial t}=-\frac{1}{\varepsilon_b}\left(\frac{\partial V_s}{\partial t}+\nabla\cdot \underset{\sim}{Q_s}\right)$$

A considerable portion of the rest of this course will deal with how we determine V_s and Q_s

$$\frac{\partial \eta}{\partial t} = -\frac{1}{\varepsilon_{bed}}\left(\nabla q_s + \frac{\partial V_s}{\partial t}\right) = -\frac{1}{\varepsilon_{bed}}\left(\left(\frac{\partial q_{sx}}{\partial x} + \frac{\partial q_{sy}}{\partial y} + \frac{\partial q_{sz}}{\partial z}\right) + \frac{\partial V_s}{\partial t}\right)$$

η = elevation of the uppermost layer of non-moving grains (units: m),
t = time (units: s),
ε_{bed} = concentration of sediment in the bed (1-porosity),
q_s = sediment flux or sediment discharge per unit width (units: m^2/s),
V_s = volume of sediment in motion per bed area (units: m)

The equation states that the rate of elevation change of the bed (i.e., erosion or deposition) is equal to the divergence or spatial change in the sediment flux plus the rate of change in suspended sediment (approximately V_s).

Useful Simplification of the Erosion Equation:

A. two-dimensional form.
B. Uniform sediment concentration in the vertical direction
C. V_s = small value, set to zero

$$\frac{\partial \eta}{\partial t} = -\frac{1}{\varepsilon_{bed}}\left(\frac{\partial q_s}{\partial x}\right)$$

What does this mean?

Next we will pull a trick and multiply the right side of this equation by 1 = $\frac{\partial \tau_b}{\partial \tau_b}$

$$\frac{\partial \eta}{\partial t} = -\frac{1}{\varepsilon_{bed}}\left(\frac{\partial q_s}{\partial x}\right) = -1\frac{1}{\varepsilon_{bed}}\left(\frac{\partial q_s}{\partial \tau_b}\right)\left(\frac{\partial \tau_b}{\partial x}\right)$$

$\left(\frac{\partial q_s}{\partial \tau_b}\right)$ is almost always positive, so the only way to go from erosion to deposition or vice versa is to change the sign of $\left(\frac{\partial \tau_b}{\partial x}\right)$.

- So **deposition** and **erosion** is primarily the consequence of a **spatial change** in **boundary shear stress**. Let's figure out exactly what boundary shear stress is…

An application of sediment conservation

Equilibrium shape: no shape change, migrating at constant speed C, 2-D case

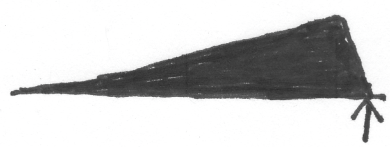
By definition: $\frac{d\eta}{dt} = \frac{\partial \eta}{\partial t} + C\frac{\partial \eta}{\partial x} = 0$

$\frac{\partial \eta}{\partial t} = -\frac{1}{\varepsilon_b}\frac{\partial q_s}{\partial x}$: Erosion equation (neglecting $\frac{\partial V_s}{\partial t}$ term)

$$-\frac{1}{\varepsilon_b}\frac{\partial q_s}{\partial x} + c\frac{\partial \eta}{\partial x} = 0$$

Integrate over x and rearrange terms

$\eta = \left(\frac{1}{\varepsilon_b C}\right) q_s + const$; typically imposed boundary condition: η = 0, q_s = 0, so const = 0



So if an equilibrium shape exists it has the same spatial structure as the sediment flux
<η> & <q_s> correlated

Crest height equals 2<η> therefore 2<q_s>

$$\langle q_s \rangle = \varepsilon_{bed} C \langle \eta \rangle = \varepsilon_{bed} C \frac{H}{2}$$

Simons, Richardson and Nordin (1965) Bedload equation for ripples and dunes

Yet another application of sediment conservation

Equilibrium or Graded Profile

Mackin (1948) described a graded stream as one where "… over a period of years, slope is delicately adjusted to provide, with available discharge and prevailing channel characteristics, just the velocity required for the transportation of the load supplied by the drainage basin".

In other words, the transport system is adjusted so that the sediment load moving through any particular reach is equal to the load entering it. Under ideal conditions this requirement leads to development of a steady state long profile for the transport system, and by extension, state state topography.

The connection between a graded system and a steady state profile is decribed by the erosion equation

$$\frac{\partial \eta}{\partial t} = -\frac{1}{\varepsilon_{bed}}\left(\nabla \cdot q_s + \frac{\partial V_s}{\partial t}\right)$$

Where η is elevation of the uppermost layer of non-moving grains, t is time, ε_{bed} is concentration of sediment in the bed (1-porosity), q_s is sediment flux or sediment discharge per unit width, and V_s is volume of sediment in motion per bed area. The above equation states that the rate of change in bed elevation by sedimentation or erosion is equal to the divergence or spatial change in the sediment flux plus the rate of change in suspended sediment concentration (approximately V_s).

Over time intervals of many days and longer, values for $\frac{\partial V_s}{\partial t}$ are typically much, much smaller than associated values for $\nabla \cdot q_s$ and the former term is often dropped from the balance to yield

$$\frac{\partial \eta}{\partial t} = -\frac{1}{\varepsilon_{bed}}\left(\frac{\partial q_{s_x}}{\partial x} + \frac{\partial q_{s_y}}{\partial y}\right)$$

Where x is the down-slope or dip direction, y is the cross-slope or strike direction, and

$$\frac{\partial q_{s_x}}{\partial x} + \frac{\partial q_{s_y}}{\partial y} = \nabla \cdot \underset{\sim}{q_s}$$

Mackin (1948) defined a graded system as one adjusted so that $\frac{\partial \eta}{\partial t} = 0$; this requires

$$\nabla \cdot \underset{\sim}{q_s} = 0$$

everywhere, the definition for an equilibrium profile and necessary condition for steady state topography.

This treatment breaks down at longer or 'geologic' time scales where relatively low rates of surface uplift or subsidence can add up to significantly deform the profile. Working at these scales a source/sink term must be added so that:

$$\frac{\partial \eta}{\partial t} = \sigma(x, y, t) - \frac{1}{\varepsilon_{bed}}\left(\frac{\partial q_{s_x}}{\partial x} + \frac{\partial q_{s_y}}{\partial y}\right)$$

Where σ is the source/sink term associated with deformation of the earth surface. In a sedimentary basin, σ is a subsidence rate, measured in mm/yr, varying as a function of *x*, *y*, and *t*.

Time-averaged, steady state topography requires $\frac{\partial \eta}{\partial t} = 0$

The necessary condition for development of an equilibrium profile then becomes

$$\sigma(x, y, t) = \frac{1}{\varepsilon_{bed}}\left(\frac{\partial q_{s_x}}{\partial x} + \frac{\partial q_{s_y}}{\partial y}\right)$$