Cooley-Tukey Exercises

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October 14, 2022

1. Construct F_4 in $\mathbb{Z}_p$ with $p = 17$.

2. Show that the Cooley-Tukey Factorization works mod p for $p = 17$.

3. Let e_i^m with $0 \le i < m$ denote the i^{th} standard basis vector of size m where $(e_i^m)_j = \begin{cases} 1 & i = j \\ 0 & i \ne j \end{cases}$. Any vector x can be expressed as a linear combination of standard basis vectors.

   (a) Let $x = \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix}$. Express x as a linear combination of standard basis vectors.

   (b) Evaluation on a standard basis vector can be used to show that two matrices are the same. Prove this fact — i.e. show that if $Ae_i^n = Be_i^n$ for all $0 \le i < n$, then $A = B$.

   (c) The tensor product of vectors is defined as $(x \otimes y) = [x_i y]$. Use this definition to show $e_i^m \otimes e_j^n = e_{in+j}^{mn}$.

   (d) Use the above to show that the tensor product is associative,

   (e) Compute the result of $e_i^2 \otimes e_j^2 \otimes e_k^2$. Note that this tensor product corresponds to the binary representation of index at which the standard basis vector is 1.

4. Defining the stride permutation and bit-reversal permutation by how they act on a standard basis,

$$\mathrm{L}_n^{mn}(e_i^m \otimes e_j^n) = (e_j^n \otimes e_i^m)$$

   and

$$\mathrm{R}_{2^k}(e_{i_0}^2 \otimes e_{i_1}^2 \otimes \cdots \otimes e_{i_{k-1}}^2) = (e_{i_{k-1}}^2 \otimes e_{i_{k-2}}^2 \otimes \cdots \otimes e_{i_0}^2).$$

   Use these definitions to show by evaluation on a standard basis that $\mathrm{R}_{2^n} = (\mathrm{R}_{2^k} \otimes \mathrm{R}_{2^{n-k}})\mathrm{L}_{2^k}^{2^n}$.

5. Given $(A \otimes B)(C \otimes D) = AC \otimes BD$, prove that $\mathrm{I}_m \otimes \prod_{i=0}^k A_i = \prod_{i=0}^k (\mathrm{I}_m \otimes A_i)$.

6. (a) Show that F_n is symmetric, i.e. $\mathrm{F}_n^T = \mathrm{F}_n$

   (b) Given

$$(AB)^T = B^T A^T,$$
$$(A \otimes B)^T = (A^T \otimes B^T)$$

and

$$(\mathrm{L}_m^{mn})^T = (\mathrm{L}_m^{mn})^{-1} = \mathrm{L}_n^{mn},$$

show $\mathrm{F}_n = \mathrm{L}_m^n (\mathrm{I}_2 \otimes \mathrm{F}_m) \mathrm{T}_m^n (\mathrm{F}_2 \otimes \mathrm{I}_m)$ where $n = 2m$.