Pease Exercises

November 2, 2022

Recall that the iterative Cooley-Tukey factorization for a DFT of size 2^t is

$$F_{2^t} = \left(\prod_{c=1}^{t} (\mathrm{I}_{2^{c-1}} \otimes \mathrm{F}_2 \otimes \mathrm{I}_{2^{t-c}})(\mathrm{I}_{2^{c-1}} \otimes \mathrm{T}^{2^{t-c+1}}_{2^{t-c}}) \right) \mathrm{R}_{2^t}$$

where

$$\mathrm{T}^{mn}_{n}(e^m_i \otimes e^n_j) = \omega^{ij}(e^m_i \otimes e^n_j)$$

and

$$\mathrm{L}^{mn}_{n}(e^m_i \otimes e^n_j) = (e^n_j \otimes e^m_i).$$

From this we can derive the Pease Factorization

$$\mathrm{F}_{2^t} = \left(\prod_{c=1}^{t} \mathrm{L}^{2^t}_{2} (\mathrm{I}_{2^{t-1}} \otimes \mathrm{F}_2) T_c \right) \mathrm{R}_{2^t}$$

where

$$T_c = \mathrm{L}^{2^t}_{2^{t-c}} (\mathrm{I}_{2^{c-1}} \otimes \mathrm{T}^{2^{t-c+1}}_{2^{t-c}}) \mathrm{L}^{2^t}_{2^c}$$

by manipulating each factor to be of the form $(\mathrm{I}_{2^{t-1}} \otimes \mathrm{F}_2)$.

Finally, recall the Commutation Theorem,

$$A^{m\times m} \otimes B^{n\times n} = \mathrm{L}^{mn}_{m} \left(B^{n\times n} \otimes A^{m\times m} \right) \mathrm{L}^{mn}_{n}.$$

1. Show $\mathrm{T}^{mn}_{n} = \bigoplus_{i=0}^{m-1} W^i_n(\omega)$ where

$$W_n(\omega) = \begin{bmatrix} 1 & & & & \\ & \omega & & & \\ & & \omega^2 & & \\ & & & \ddots & \\ & & & & \omega^{n-1} \end{bmatrix},$$

   and ω is a primitive mn^{th} root of unity.

2. Show $\mathrm{L}^{mn}_{n} \mathrm{T}^{mn}_{n} \mathrm{L}^{mn}_{m} = \mathrm{T}^{mn}_{m}$

3. Show $T_c = \bigoplus_{i=0}^{2^{t-c}-1} \left(\mathrm{I}_{2^{c-1}} \otimes W_2^{i2^{c-1}}(\omega) \right)$. It may be useful to first show how T_c acts on a standard basis.

4. Starting from the iterative Cooley-Tukey factorization, derive the Korn-Lambiotte factorization,

$$\mathrm{F}_{2^t} = \left(\prod_{c=1}^{t} (\mathrm{F}_2 \otimes \mathrm{I}_{2^{t-1}})(\mathrm{T}_{2^{t-c}}^{2^{t-c+1}} \otimes \mathrm{I}_{2^{c-1}})\mathrm{L}_2^{2^t} \right) \mathrm{R}_{2^t}.$$

5. Derive the same factorization starting from the Pease factorization.