Given
$$\frac{\mathbb{C}[x]}{x^{rs} - 1} \tag{1}$$

We can represent this as a coefficient vector of size rs like so

$$\begin{pmatrix} f_0 \\ f_1 \\ ... \\ f_{rs-1} \end{pmatrix} \tag{2}$$

for the polynomial

$$f_0 + f_1 x + f_2 x^2 .... f_{rs-1} x^{rs-1} \tag{3}$$

Now given this, we want to do a discrete fourier transform. To do this, we can write the above equation as

$$\prod_{i=0}^{r-1} \frac{\mathbb{C}[x]}{x^s - \omega_r^i} \tag{4}$$

This is equivalent to factoring

$$(x^{rs} - 1) = (x^s - 1)(x^s - \omega_r)(x^s - \omega_r^2).... \tag{5}$$

The reason this works is because if $x = \omega_r$, x to the rth power is always 1 since $\omega_r^r = 1$. Now, let us examine each of

$$\frac{\mathbb{C}[x]}{x^s - \omega_r^i} \tag{6}$$

Given the polynomial

$$f_0 + f_1 x + f_2 x^2 .... f_{rs-1} x^{rs-1} \tag{7}$$

If we take this mod $x^s - \omega_r^i$, we replace every x^s with ω_r^i. So, we have

$$(f_0 + f_s \omega_r^i + f_{2s} \omega_r^{2i} ....) + (f_1 + f_{s+1} \omega_r^i + f_{2s+1} \omega_r^{2i} ....) x + (f_2 + f_{s+2} \omega_r^i + f_{2s+2} \omega_r^{2i} ....) x^2 \tag{8}$$

Now, if we have $i = 0$, we can have

$$(f_0 + f_s + f_{2s}....) + (f_1 + f_{s+1} + f_{2s+1}....)x + (f_2 + f_{s+2} + f_{2s+2}....)x^2 \tag{9}$$

In the previous coefficient vector format, we can write this as

$$\begin{pmatrix} f_0 + f_s + f_{2s}.... \\ f_1 + f_{s+1} + f_{2s+1}.... \\ f_2 + f_{s+2} + f_{2s+2}.... \\ ... \\ f_{s-1} + f_{2s-1}.... \end{pmatrix} \tag{10}$$

1

So we have s rows of coefficients. This can be computed as a matrix multiplication from the initial coefficient vector as

$$
\begin{pmatrix} I_s & I_s & I_s & \dots & I_s \end{pmatrix} \begin{pmatrix} f_0 \\ f_1 \\ \dots \\ f_{rs-1} \end{pmatrix} \tag{11}
$$

This would get stride s.

Now, in the case of ω^i, we have

$$
\begin{pmatrix} I_s & \omega^i I_s & \omega^{2i} I_s & \dots & \omega^{-i} I_s \end{pmatrix} \begin{pmatrix} f_0 \\ f_1 \\ \dots \\ f_{rs-1} \end{pmatrix} \tag{12}
$$

Now, the ith row of $F_r \otimes I_s$ is exactly this. Also, we can transpose so we can write it as

$$
\begin{pmatrix} f_0 & f_1 & \dots & f_{rs-1} \end{pmatrix} \begin{pmatrix} I_s \\ \omega^i I_s \\ \omega^{2i} I_s \\ \dots \\ \omega^{-i} I_s \end{pmatrix} \tag{13}
$$

will be a one times s matrix of all the coefs. Now if we do

$$
\begin{pmatrix} f_0 & f_1 & \dots & f_{rs-1} \end{pmatrix} F_r \otimes I_s \tag{14}
$$

We would have a one times rs matrix which goes from the coefficient mod $x^r - 1$, $x^r - \omega$ and onwards concatenated in a row.

Now, our goal for this whole thing is to make a matrix F_{sm} which when we multiply by our coefficient vector, we get the function evaluated at $1, \omega, \omega^2$ and so on. For this, we want to use F_s to do this. Now so far, we transformed our original coefficient vector into a concatenated vector given mod $x^2 \equiv \omega_r^i$ which is

$$
\frac{\mathbb{C}[x]}{x^s - \omega_r^i} \tag{15}
$$

Now we want to do an operation where we can just use F_s, or in other words we can just calculate

$$
\frac{\mathbb{C}[x]}{x^s - 1} \tag{16}
$$

and convert it to the case of

$$
\frac{\mathbb{C}[x]}{x^s - \omega_r^i} \tag{17}
$$

Now, if $i = 0$, then the matrix that ties this together is just the identity. Now, at i, we can see that it becomes W_s^i as we multiply the original $\dfrac{\mathbb{C}[x]}{x^s - 1}$ coefficient

2

vector by

$$\begin{pmatrix} 1 & 0 & .... \\ 0 & \omega^i & .... \\ .... & & \end{pmatrix} \tag{18}$$

and so on. Now, if we do this for all the cases, we notice that this is just the twiddle matrix T_s^{rs} which is the identity at the top left then W, W^2 and so on. Then, finally, we multiply by $I_r \otimes F_s$ to get the initial coefficients.

Now, while this can get confusing from the direction of thinking multiplying the coefficient vector to this but the idea is different. The idea is we get the coefficient vector from

$$\frac{\mathbb{C}[x]}{x^s - 1} \tag{19}$$

And concatenate it together like so

$$\begin{pmatrix} f_0 & f_s & f_{2s} & .... & f_1 & f_{s+1} & .... \end{pmatrix} \tag{20}$$

This is the same as the $I_r \otimes F_s$ because for each block of f_0 to one before f_1, if we multiply by $I_r \otimes F_s$ we get the fourier/evaluation matrix for that block.

Now, we can multiply by the twiddle/triangle matrix and we get

$$(f_0 + f_s \omega_r^i + f_{2s} \omega_r^{2i}....) + (f_1 + f_{s+1} \omega_r^i + f_{2s+1} \omega_r^{2i}....)x + (f_2 + f_{s+2} \omega_r^i + f_{2s+2} \omega_r^{2i}....)x^2 \tag{21}$$

for each block in equation form.

Now, this is the same as the initial coefficient vector times $F_r \otimes I_s$. So I'm guessing we need an inverse to get back to the original?