Q1)

mean = θ_1

variance = θ_2 standard deviation = $(\theta_2)^{1/2}$

The likelyhood function of sample x_1, x_2, x_m can be written as

$$L = \sum_{i=1}^{m} p(x_1, \mu, \sigma^2) = \prod_{i=1}^{m} \frac{1}{\sqrt{2\pi\theta_2}} e^{-1/2 \frac{(x_i-\theta_1)^2}{\theta_2}}$$

$$\Rightarrow \frac{1}{\sqrt{2\pi\theta_2}} e^{-\frac{(x_1-\theta_1)^2}{2\theta_2}} * \frac{1}{\sqrt{2\pi\theta_2}} e^{-\frac{(x_2-\theta_1)^2}{2\theta_2}} \cdots \frac{1}{\sqrt{2\pi\theta_2}} e^{-\frac{(x_m-\theta_1)^2}{2\theta_2}}$$

$$= \left(\frac{1}{\sqrt{2\pi\theta_2}}\right)^m e^{-\frac{1}{2}\sum_{i=1}^{m} \frac{(x_2-\theta_1)^2}{\theta_2}}$$

$$= \left(\frac{1}{\sqrt{2\pi\theta_2}}\right)^m e^{-\frac{1}{2\theta_2}\sum_{i=1}^{m} (x_i-\theta_1)^2}$$

taking log on both sides

$$\log L = m \log \frac{1}{\sqrt{2\pi\theta_2}} + \log e^{-\frac{1}{2}\sum_{i=1}^{m}(x_i-\mu)^2}$$

$$= m(\cancel{\log 1}^{0} - \log\sqrt{2\pi\theta_2}) - \frac{1}{2\theta_2}\sum_{i=1}^{m}(x_i-\mu)^2$$

$$= -m \log(2\pi\theta_2)^{1/2} - \frac{1}{2\theta_2}\sum_{i=1}^{m}(x_i-\theta_1)^2$$

$$= -\frac{m}{2}\log(2\pi\theta_2) - \frac{1}{2\theta_2}\sum_{i=1}^{m}(x_i-\theta_1)^2$$

$$= -\frac{m}{2}(\log\theta_2 + \log 2\pi) - \frac{1}{2\theta_2}\sum_{i=1}^{m}(x_i-\theta_1)^2$$

differentiate partially wrt to μ and σ^2, we get

$$\frac{\partial}{\partial\mu}\log L = -\frac{m}{2}(0+0) - \frac{1}{2\sigma^2}\sum_{i=1}^{m} 2(x_i-\theta_1)(-1)$$

$$= \frac{1}{\theta_2}\sum_{i=1}^{m}(x_i-\mu)$$

and, $$\frac{\partial}{\partial\sigma^2}(\log L) = -\frac{m}{2}\left(\frac{1}{\theta_2}+0\right) + \frac{1}{2\theta_2^2}\sum_{i=1}^{m}(x_i-\theta_1)^2$$

Equating ① to zero and solving for μ, we get

$$\frac{\partial}{\partial \mu}\log L = 0 \rightarrow \frac{1}{\theta_2}\sum_{i=1}^{m}(x_i-\theta_1) \Rightarrow \sum_{i=1}^{m} x_i - m\theta_1$$

$$\Rightarrow \theta_1 = \frac{1}{m}\sum_{i=1}^{m} x_i \rightarrow \hat{\theta}_1 = \bar{x}$$

Equating ② to zero and solving for σ^2 and after substituting $\mu = \bar{x}$, we get

$$\frac{\partial}{\partial \sigma^2}\log L = 0 \Rightarrow -\frac{m}{2\theta_2} + \frac{1}{2\theta_2^2}\sum_{i=1}^{m}(x_i-\theta_1)^2 = 0$$

$$\sum_{i=1}^{m}(x_i-\bar{x})^2 = m\theta_2$$

$$\hat{\theta}_2 = \frac{1}{m}\sum_{i=1}^{m}(x_i-\bar{x})^2$$, the sample variance S

Again partial derivate w.r.t μ

$$\frac{\partial^2}{\partial \theta_1^2} - \frac{1}{\theta_2}\sum_{i=1}^{m}(1)(-\theta_1)^{1-1}(-1)$$

$$= -\frac{m}{\theta_2} < 0$$

$\theta_1 = \bar{x}$ is mle for θ_1

Similarly differentiate w.r.t σ^2, we get

$$\frac{\partial^2}{\partial \theta_2^2} = \frac{m}{2}(\theta_2)^{-2} + (-1)(\theta_2)^{-3}\sum_{i=1}^{m}(x_i-\theta_1)^2$$

$$= \frac{m}{2\theta_2^2} - \frac{1}{\theta_2^3}(x_i-\theta_1)^2$$

$$\frac{\partial^2}{\partial(\theta_2)^2}(\log L) \text{ at } \hat{\theta}_2 = S^2$$

$$= -\left(-\frac{m^2}{2S^4} + \frac{1}{S^6}\sum_{i=1}^{m}(x_i-\bar{x})^2\right) < 0$$

$\hat{\theta}_2 = S^2$ is MLE of σ^2

Q2)

The pmf of X_i is given by

$$P(X_i | m, p) = \binom{m}{x_i} p^{x_i} (1-p)^{m-x_i} \quad , x \in \{0,1,2 \dots, m\}$$

the likelihood function is -

$$L(p) = \prod_{i=1}^{n} P(x_i | m, p) = \prod_{i=1}^{n} \binom{m}{x_i} p^{x_i} (1-p)^{m-x_i}$$

$$\Rightarrow \prod_{i=1}^{n} \binom{m}{x_i} \prod_{i=1}^{n} p^{x_i} \prod_{i=1}^{n} (1-p)^{m-x_i}$$

$$= \prod_{i=1}^{n} \binom{m}{x_i} p^{\sum_{i=1}^{n} x_i} (1-p)^{\sum_{i=1}^{n} m-x_i}$$

$$\Rightarrow \prod_{i=1}^{n} \binom{m}{x_i} p^{\sum_{i=1}^{n} x_i} (1-p)^{n \cdot m - \sum_{i=1}^{n} x_i}$$

$\Rightarrow$ The log likelihood function

$$l(p) = \log L(p)$$

$$= \log\left(\prod_{i=1}^{n} \binom{m}{x_i} p^{\sum_{i=1}^{n} x_i} (1-p)^{nm - \sum_{i=1}^{n} x_i}\right)$$

$$= \log\left(\prod_{i=1}^{n} \binom{m}{x_i}\right) + \log\left(p^{\sum_{i=1}^{n} x_i}\right) + \log\left((1-p)^{nm - \sum_{i=1}^{n} x_i}\right)$$

$$\Rightarrow \log\left[\prod_{i=1}^{n} \binom{m}{x_i}\right] + \log(p)\left(\sum_{i=1}^{n} x_i\right) + \log(1-p)\left(nm - \sum_{i=1}^{n} x_i\right)$$

differentiate wrt p

$$\frac{d\, l(p)}{\partial p} = \frac{1}{p} \sum_{i=1}^{n} x_i + \frac{1}{1-p}\left(nm - \sum_{i=1}^{n} x_i\right)(-1)$$

$$\frac{\partial^2 l(p)}{\partial p^2} = -\frac{1}{p^2} \sum_{i=1}^{n} x_i - \frac{1}{(1-p)^2}\left(nm - \sum_{i=1}^{n} x_i\right)$$

To find MLE of p, $\hat{p}$, we solve the following -

$$\frac{\partial l(p)}{\partial p}\Big|_{p=\hat{p}} = 0 \rightarrow \frac{1}{p}\sum_{i=1}^{n} x_i - \frac{1}{1-p}\left(mn - \sum_{i=1}^{n} x_i\right)$$

$$\rightarrow \frac{1}{1-\hat{p}}\left(mn - \sum_{i=1}^{n} x_i\right) = \frac{1}{\hat{p}}\sum_{i=1}^{n} x_i$$

$$\Rightarrow \frac{mn - \sum_{i=1}^{n} x_i}{\sum_{i=1}^{n} x_i} = \frac{1-\hat{p}}{\hat{p}}$$

$$\frac{mn}{\sum_{i=1}^{n} x_i} \cancel{-1} = \frac{1}{\hat{p}} \cancel{-1}$$

$$\hat{p} = \frac{1}{mn}\sum_{i=1}^{n} x_i \Rightarrow \frac{\bar{X}_n}{m}$$

To prove MLE is maximize $p = \hat{p}$ we must show

$$\frac{\partial^2 l(p)}{\partial p^2}\Big|_{p=\hat{p}} < 0$$

NOTE:- $$\frac{\partial^2 l(p)}{\partial p^2} = -\left[\frac{1}{p^2}\sum_{i=1}^{n} x_i + \frac{1}{(1-p)^2}\left(mn - \sum_{i=1}^{n} x_i\right)\right]$$

$\forall\ p\ (0,1)$