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Find the second order partial derivate of:

$$u = e^{x^2+xy+y^2}.$$

Solution:

Given function, $u = e^{x^2+xy+y^2}$ ...... (1)

Diff. (1) Partially w.r.t. x

$$\frac{\partial u}{\partial x} = u_x = e^{x^2+xy+y^2}.(2x+y+0) \;.\;.... (2)$$

Diff. (2) Partially w.r.t. x.

$$u_{xx} = e^{x^2+xy+y^2}.2 + (2x+y).e^{x^2+xy+y^2}.(2x+y).$$
$$= 2.e^{x^2+xy+y^2} + (2x+y)^2.e^{x^2+xy+y^2}.$$
$$= e^{x^2+xy+y^2}\{2+(2x+y)^2\}.$$

Diff. (2) Partially w.r.t y.

$$u_{yx} = e^{x^2+xy+y^2}.(0+1+0) + (2x+y).e^{x^2+xy+y^2}.(0+x+2y)$$
$$= e^{x^2+xy+y^2} + (2x+y)(x+2y).e^{x^2+xy+y^2}.$$
$$= e^{x^2+xy+y^2}\{1+(2x+y).(x+2y)\}.$$

Diff. (1) Partially. w.r.t. y.

$$u_y = e^{x^2+xy+y^2}.(x+2y) \;.... (3).$$

Diff. (3) Partially w.r.t. x.

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$$U_{xy} = e^{x^2+xy+y^2} \cdot 1 + (x+2y) \cdot e^{x^2+xy+y^2} \cdot (2x+y)$$

$$= e^{x^2+xy+y^2} \{1 + (2x+y)(x+2y)\}$$

Diff. (3) partially w.r.t. y.

$$U_{yy} = e^{x^2+xy+y^2} \cdot 2 + (x+2y) \cdot e^{x^2+xy+y^2} \cdot (0+x+2y)$$

$$= 2 \cdot e^{x^2+xy+y^2} + (x+2y)^2 \cdot e^{x^2+xy+y^2}$$

$$= e^{x^2+xy+y^2} \{2 + (x+2y)^2\}$$

Diff. (3) Partially w.r.t. y.

$U_{yy} = 6y.$

If $u = \log(x^3+y^3+z^3-3xyz)$ then show that.

(i) $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{3}{x+y+z}$.

ii) $\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = -\frac{9}{(x+y+z)^2}$

Here Given fxn : $u = \log(x^3+y^3+z^3-3xyz)$ .... (1).

Diff. (1) partially w.r.t. x,

$\frac{\partial u}{\partial x} = \frac{1}{x^3+y^3+z^3-3xyz} \cdot (3x^2-3yz)$

$\therefore \frac{\partial u}{\partial x} = 3\left[\frac{x^2-yz}{x^3+y^3+z^3-3xyz}\right]$ .... (a)

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Diff: (1). partially w.r.t. y, we get:

$$\frac{\partial u}{\partial y} = \frac{1}{x^3+y^3+z^3-3xyz} \cdot (3y^2-3xz)$$

$$\therefore \frac{\partial u}{\partial y} = 3\left[\frac{y^2-xz}{x^3+y^3+z^3-3xyz}\right] \quad \ldots (b)$$

Diff. (1) partially w.r.t. z, we get:

$$\frac{\partial u}{\partial z} = \frac{1}{x^3+y^3+z^3-3xyz} \cdot (3z^2-3xy)$$

$$\therefore \frac{\partial u}{\partial z} = 3\left[\frac{z^2-xy}{x^3+y^3+z^3-3xyz}\right] \quad \ldots (c)$$

So,

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 3\left[\frac{x^2-yz+y^2-xz+z^2-xy}{x^3+y^3+z^3-3xyz}\right]$$

we know that,
$x^3+y^3+z^3-3xyz$
$(x+y+z)(x^2+y^2+z^2-xy-yz-zx)$

$$= 3\left[\frac{(x^2-\cancel{yz+y^2-xz+z^2}-xy)}{(x^2-\cancel{yz+y^2-xz+z^2}-xy)(x+y+z)}\right]$$

$$= \frac{3}{x+y+z}.$$

$= R.H.S.$

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$$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u = \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\cdot\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)u$$

$$= \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\cdot\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right)$$

$$= \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\cdot\frac{3}{(x+y+z)}$$

$$= \frac{\partial}{\partial x}\left(\frac{3}{x+y+z}\right)+\frac{\partial}{\partial y}\left(\frac{3}{x+y+z}\right)+\frac{\partial}{\partial z}\left(\frac{3}{x+y+z}\right)$$

$$= -\frac{3}{(x+y+z)^2}\times 1-\frac{3}{(x+y+z)^2}\times 1-\frac{3}{(x+y+z)^2}\times 1$$

$$= \frac{-9}{(x+y+z)^2}$$

$= R.H.S.$

If $u = f(xyz)$ show that $x^2 u_{xx} = y^2 u_{yy} = z^2 u_{zz}$ (3)

Solution :

Given function $u = f(xyz)$ (1).

Diff. (1) partially w.r.t. x

$$\frac{\partial u}{\partial x} = \frac{\partial f(xyz)}{\partial x}$$

or, $$u_x = \frac{\partial f(xyz)}{\partial'(xyz)}\cdot\frac{\partial(xyz)}{\partial x}$$

$$= f'(xyz)\cdot yz$$

$\therefore U_x = yz \cdot f'(xyz)$

Diff. (2) Partially w.r.t. x we get,

$U_{xx} = yz \cdot f''(xyz) \cdot yz.$

$\therefore U_{xx} = (yz)^2 \cdot f''(xyz)$ .... (3)

Similarly;

$\therefore U_{yy} = (zx)^2 \cdot f''(xyz)$ ... (4)

$\therefore U_{zz} = (xy)^2 \cdot f''(xyz)$ .... (5).

Multiplying (3), (4), (5) by x^2, y^2, z^2 respectively we get:

$$x^2 U_{xx} = y^2 U_{yy} = z^2 U_{zz}.$$

If $x = r\cos\theta, y = r\sin\theta$. Prove that, (4)

$$\frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2} = \frac{1}{r}\left[\left(\frac{\partial r}{\partial x}\right)^2 + \left(\frac{\partial r}{\partial y}\right)^2\right]$$

Solution:

Given $x = r\cos\theta$, $y = r\sin\theta$.

Squaring and adding:

$r^2 = x^2 + y^2$ .... (1).

Diff. (1) partially w.r.t. x.

$$2r \cdot \frac{\partial r}{\partial x} = 2x + 0$$

$$\therefore \frac{\partial r}{\partial x} = \frac{x}{r} \quad \text{.... (2)}$$

Diff. (1) Partially w.r.t. y:

$$2r \cdot \frac{\partial r}{\partial y} = 0 + 2y.$$

$$\therefore \frac{\partial r}{\partial y} = \frac{y}{r} \quad \ldots (3).$$

Diff. (2) Partially w.r.t. x.

$$\frac{\partial^2 r}{\partial x^2} = \frac{r \cdot 1 - x \cdot \frac{\partial r}{\partial x}}{r^2}$$

$$= \frac{r - x \cdot \frac{x}{r}}{r^2} \quad \text{[using (2)].}$$

$$= \frac{r^2 - x^2}{r^3}$$

$$= \frac{x^2 + y^2 - x^2}{r^3} \quad \text{[using (1)]}$$

$$= \frac{y^2}{r^3} \quad \ldots (4)$$

Similarly, $\dfrac{\partial^2 r}{\partial y^2} = \dfrac{x^2}{r^3} \quad \ldots (5).$

Adding (4) and (5) we get,

$$\frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2} = \frac{y^2}{r^3} + \frac{x^2}{r^3}$$

$$= \frac{y^2 + x^2}{r^3}.$$

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$$= \frac{y^2}{r^3} + \frac{x^2}{r^3}$$

$$= \frac{1}{r}\left[\frac{y^2}{r^2} + \frac{x^2}{r^2}\right]$$

$$= \frac{1}{r}\left[\left(\frac{y}{r}\right)^2 + \left(\frac{x}{r}\right)^2\right]$$

$$= \frac{1}{r}\left[\left(\frac{\partial r}{\partial x}\right)^2 + \left(\frac{\partial r}{\partial y}\right)^2\right]$$ using (2) and (3)

$$\therefore \frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2} = \frac{1}{r}\left[\left(\frac{\partial r}{\partial x}\right)^2 + \left(\frac{\partial r}{\partial y}\right)^2\right]$$ proved.

3. If $u = \frac{1}{\sqrt{x^2+y^2+z^2}}$. prove that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$ 5

Solution:

Given $u = \frac{1}{\sqrt{x^2+y^2+z^2}}$ .... (1).

Diff. (1) partially w.r.t. x.

$$\frac{\partial u}{\partial x} = -\frac{1}{\cancel{2}}(x^2+y^2+z^2)^{-3/2} \cdot \cancel{2}x$$

$$\therefore \frac{\partial u}{\partial x} = \frac{-x}{(x^2+y^2+z^2)^{3/2}}$$ .... (2)

Diff. (2) partially w.r.t. x.

$$\frac{\partial^2 u}{\partial x^2} = -\left\{\frac{(x^2+y^2+z^2)^{3/2}\cdot 1 - x \cdot 3/\cancel{2} \cdot (x^2+y^2+z^2)^{1/2} \cdot \cancel{2}x}{(x^2+y^2+z^2)^3}\right\}$$

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3.

or, $\frac{\partial^2 u}{\partial x^2} = -\left\{ \frac{(x^2+y^2+z^2)^{3/2} - 3x^2(x^2+y^2+z^2)^{1/2}}{(x^2+y^2+z^2)^3} \right\}$

$\therefore \frac{\partial^2 u}{\partial x^2} = \frac{3x^2(x^2+y^2+z^2)^{1/2} - (x^2+y^2+z^2)^{3/2}}{(x^2+y^2+z^2)^3}$

Similarly,

$\frac{\partial^2 u}{\partial y^2} = \frac{3y^2(x^2+y^2+z^2)^{1/2} - (x^2+y^2+z^2)^{3/2}}{(x^2+y^2+z^2)^3}$

and,

$\frac{\partial^2 u}{\partial z^2} = \frac{3z^2(x^2+y^2+z^2)^{1/2} - (x^2+y^2+z^2)^{3/2}}{(x^2+y^2+z^2)^3}$

Now,

$L.H.S. = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$

$= \frac{3x^2(x^2+y^2+z^2)^{1/2} - (x^2+y^2+z^2)^{3/2} + 3y^2(x^2+y^2+z^2)^{1/2} - (x^2+y^2+z^2)^{3/2} + 3z^2(x^2+y^2+z^2)^{1/2} - (x^2+y^2+z^2)^{3/2}}{(x^2+y^2+z^2)^3}$

$= \frac{3(x^2+y^2+z^2)^{1/2}\{x^2+y^2+z^2\} - 3(x^2+y^2+z^2)^{3/2}}{(x^2+y^2+z^2)^3}$

$= \frac{3\cancel{(x^2+y^2+z^2)^{3/2}} - 3\cancel{(x^2+y^2+z^2)^{3/2}}}{(x^2+y^2+z^2)^3}$

$= 0$

$= R.H.S.$

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Euler Homogeneous theorem.

If $u = f(r)$, then prove that,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f''(r) + \frac{1}{r} f'(r)$$

Given, $u = f(r)$ ---(1)

Diff. (1) partially

$$\frac{\partial u}{\partial x} = f'(r) \cdot \frac{\partial r}{\partial x}.$$

IIly, $$\frac{\partial^2 u}{\partial x^2} = f''(r) \cdot \frac{\partial r}{\partial x} \quad \text{---(2)}$$

Again, Diff. (1) partially w.r.t. y.

$$\frac{\partial u}{\partial y} =$$

If $u = x^2+y^2+z^2$ then show that $xU_x + yU_y + zU_z = 2U$. (1)

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Given function:

$$u = x^2+y^2+z^2 \quad \text{---(1)}$$

Diff. (1) partially w.r.t. x we get,

$$\frac{\partial u}{\partial x} = u_x = 2x. \quad \text{...(a)}$$

Diff. (1) partially w.r.t. y we get,

$$\frac{\partial u}{\partial y} = u_y = 2y \quad \text{....(b)}$$

Diff. (1) partially w.r.t. z we get,

$$\frac{\partial u}{\partial z} = u_z = 2z. \quad \text{....(c)}$$

Now,

$$x.U_x + y.U_y + z.U_z$$

$= x.2x + y.2y + z.2z$ [from (a), (b), (c)]

$= 2x^2+2y^2+2z^2$

$= 2(x^2+y^2+z^2)$

$= 2u$

$=$ R.H.S.

Hence, $xU_x + y.U_y + z.U_z = 2U$.

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1. If $u = \log\sqrt{x^2+y^2+z^2}$, then show that:

$$(x^2+y^2+z^2)\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)=1.$$

Ans. Given function,

$$u = \log\sqrt{x^2+y^2+z^2}$$

or, $$u = \frac{1}{2}.\log(x^2+y^2+z^2) \quad \ldots (1).$$

Diff. (1) partially w.r.t. x,

$$\frac{\partial u}{\partial x} = \frac{1}{2(x^2+y^2+z^2)}\cdot 2x = \frac{x}{x^2+y^2+z^2} \quad \ldots (2)$$

Again, Diff. (2) partially w.r.t x

$$\frac{\partial^2 u}{\partial x^2} = \frac{(x^2+y^2+z^2).1-2x^2}{(x^2+y^2+z^2)^2} = \frac{-x^2+y^2+z^2}{(x^2+y^2+z^2)^2}$$

Similarly,

$$\cancel{\frac{\partial^2 u}{\partial y^2}} = \cancel{\frac{x^2-y^2+z^2}{(x^2+y^2+z^2)^2}}$$

~~and~~,

$$\cancel{\frac{\partial^2 u}{\partial z^2}} = \cancel{\frac{x^2+y^2-z^2}{(x^2+y^2+z^2)^2}}$$

Diff (1) partially w.r.t. y,

$$\frac{\partial u}{\partial y} = \frac{1\cdot 2y}{2(x^2+y^2+z^2)} = \frac{y}{x^2+y^2+z^2} \quad \ldots (3)$$

Diff. (3) partially w.r.t. y.

$$\frac{\partial^2 u}{\partial y^2} = \frac{(x^2+y^2+z^2).1-y.2y}{(x^2+y^2+z^2)^2}$$

$$= \frac{x^2-y^2+z^2}{(x^2+y^2+z^2)^2}.$$

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Ily we get,

$$\frac{\partial^2 u}{\partial z^2} = \frac{x^2+y^2-z^2}{(x^2+y^2+z^2)^2}$$

So,

$$(x^2+y^2+z^2)\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)$$

$$= (x^2+y^2+z^2)\left(\frac{-x^2+y^2+z^2}{(x^2+y^2+z^2)^2}+\frac{x^2-y^2+z^2}{(x^2+y^2+z^2)^2}+\frac{x^2+y^2-z^2}{(x^2+y^2+z^2)^2}\right)$$

$$= \cancel{(x^2+y^2+z^2)}\left(\frac{-\cancel{x^2}+\cancel{y^2}+\cancel{z^2}+\cancel{x^2}-\cancel{y^2}+z^2+x^2+y^2-\cancel{z^2}}{(x^2+y^2+z^2)^{\cancel{2}}}\right)$$

$$= \frac{x^2+y^2+z^2}{x^2+y^2+z^2}$$

$= 1. =$ R.H.S.

5. If $u = e^{xyz}$, then prove that $\frac{\partial^3 u}{\partial x\partial y\partial z} = (1+3xyz+x^2y^2z^2)e^{xyz}$. (9)

Given function: $u = e^{xyz}$ ..... (1).

Diff. (1) partially w.r.t. x.

$$\frac{\partial u}{\partial x} = e^{xyz}\cdot yz. \quad \ldots (2)$$

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Diff. (2) partially w.r.t. y on both sides,

$$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial y}\left(e^{xyz}.yz\right)$$

or, $$\frac{\partial^2 u}{\partial x.\partial y} = e^{xyz}.z + yz.e^{xyz}.xz$$

$$\therefore \frac{\partial^2 u}{\partial x.\partial y} = e^{xyz}.z + xyz^2.e^{xyz} \quad \text{...(3)}$$

Also, Again:

Diff. (3) partially w.r.t. z on both sides,

$$\frac{\partial}{\partial z}\left(\frac{\partial^2 u}{\partial x.\partial y}\right) = \frac{\partial}{\partial z}\left[e^{xyz}.z + xyz^2.e^{xyz}\right]$$

or, $$\frac{\partial^3 u}{\partial x.\partial y.\partial z} = e^{xyz}.1 + z.e^{xyz}.xy + xyz^2.e^{xyz}.xy + e^{xyz}.2xyz$$

$$= e^{xyz} + xyz.e^{xyz} + x^2y^2z^2.e^{xyz} + 2xyz.e^{xyz}$$

$$= e^{xyz} + 3xyz.e^{xyz} + x^2y^2z^2.e^{xyz}$$

$$= \left[1 + 3xyz + x^2y^2z^2\right]e^{xyz}$$

$$\therefore \frac{\partial^3 u}{\partial x\partial y\partial z} = \left\{1 + 3xyz + x^2y^2z^2\right\}e^{xyz}$$

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6. If $z = \phi(x+ay) + \psi(x-ay)$, then show that $\frac{\partial^2 z}{\partial x^2} = a^2 \cdot \frac{\partial^2 z}{\partial y^2}$. (10)

Ans. Here,

$$\frac{\partial z}{\partial x} = \phi'(x+ay).$$

$$\frac{\partial^2 z}{\partial x^2} = \phi''(x+ay).$$

And, $\frac{\partial z}{\partial y} = \phi'(x+ay) \cdot (0+a.1) = a \cdot \phi'(x+ay)$.

lly, $\frac{\partial^2 z}{\partial y^2} = a \cdot \phi''(x+ay) \cdot a$ ~~$= a^2$~~

or $\frac{\partial^2 z}{\partial y^2} = a^2 \cdot \phi''(x+ay)$

$$\therefore \frac{\partial^2 z}{\partial y^2} = a^2 \cdot \frac{\partial^2 z}{\partial x^2}.$$

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Homogenous function:

The function $u=f(x,y)$ be two independent variables x and y is said to be homogeneous function of two independent variables x and y of degree n if it can be expressed either of the form $x^n\phi\left(\frac{y}{x}\right)$ or $y^n\phi\left(\frac{x}{y}\right)$.

Example:

Let us consider the function,

$$u = f(x,y) = ax^2+2hxy+by^2$$

$$= x^2\left\{a+2h\left(\frac{y}{x}\right)+b\left(\frac{y^2}{x^2}\right)\right\}$$

$$\therefore u = x^2\phi\left(\frac{y}{x}\right).$$

Thus,

Theorem (Euler's theorem on homogeneous function). 11

State and prove Euler's theorem on homogeneous function of two independent variables.

Statement:

If $u=f(x,y)$ be homogeneous function of two indepent variables x and y degree n then. $x\cdot\frac{\partial u}{\partial x}+y\cdot\frac{\partial u}{\partial y}=nu$.

Proof:

Given that $u=f(x,y)$ be homogeneous function of two independent variables x and y degree n.

Then by definition.

It can be expressed either of the form.

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$$u = x^n . \phi . \left(\frac{y}{x}\right) \cdots (1)$$

Diff. (1) partially w.r.t. x and y respectively we get.

$$\frac{\partial u}{\partial x} = n.x^{n-1}. \phi\left(\frac{y}{x}\right) + x^n.\phi'\left(\frac{y}{x}\right).\left(\frac{-y}{x^2}\right).$$

or, $$\frac{\partial u}{\partial x} = n.x^{n-1} \phi\left(\frac{y}{x}\right) - x^{n-2}.y. \phi'\left(\frac{y}{x}\right) \cdots (2)$$

And,

$$\frac{\partial u}{\partial y} = x^n. \phi'\left(\frac{y}{x}\right).\frac{1}{x}.$$

or, $$\frac{\partial u}{\partial y} = x^{n-1}\phi'\left(\frac{y}{x}\right) \cdots (3)$$

Multiplying equation (2) and (3) by x and y respectively and adding we get,

$$x.\frac{\partial u}{\partial x} + y.\frac{\partial u}{\partial y} = nx^n\phi\left(\frac{y}{x}\right) - \cancel{x^{n-1}.y\,\phi'\left(\frac{y}{x}\right)} + \cancel{y.x^{n-1}\phi'}\left(\frac{y}{x}\right)$$

$$= n.x^n\phi\left(\frac{y}{x}\right)$$

$$\therefore x.\frac{\partial u}{\partial x} + y.\frac{\partial u}{\partial y} = nu. \quad [\text{using } (1)].$$

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Example:

1. Verify the Euler's theorem of the function. (12.)

$$u = \frac{x^{1/4} + y^{1/4}}{x^{1/5} + y^{1/5}}$$

Solution:

Given fxn: $u = \dfrac{x^{1/4} + y^{1/4}}{x^{1/5} + y^{1/5}}$

$$= \frac{x^{1/4}\left\{1 + \left(\frac{y}{x}\right)^{1/4}\right\}}{x^{1/5}\left\{1 + \left(\frac{y}{x}\right)^{1/5}\right\}}$$

$$= x^{1/4 - 1/5} \frac{\left\{1 + \left(\frac{y}{x}\right)^{1/4}\right\}}{\left\{1 + \left(\frac{y}{x}\right)^{1/5}\right\}}$$

$$= x^{1/20} \frac{\left\{1 + \left(\frac{y}{x}\right)^{1/4}\right\}}{\left\{1 + \left(\frac{y}{x}\right)^{1/5}\right\}}.$$

$\therefore$ $u = x^{1/20} \cdot \phi\left(\frac{y}{x}\right).$

Thus u is homogeneous function of degree $\frac{1}{20}$ then by Euler's theorem.

$$x \cdot \frac{\partial u}{\partial x} + y \cdot \frac{\partial u}{\partial y} = \frac{1}{20} u.$$

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Diff. (1) partially w.r.t. x and y respectively we get

$$u = \frac{x^{1/4}+y^{1/4}}{x^{1/5}+y^{1/5}}$$

$$\frac{\partial u}{\partial x} = \frac{(x^{1/5}+y^{1/5}).(\frac{1}{4}).x^{-3/4} - (x^{1/4}+y^{1/4}).(\frac{1}{5}).x^{-4/5}}{(x^{1/5}+y^{1/5})^2} \quad ---(2)$$

$$\text{And, } \frac{\partial u}{\partial y} = \frac{(x^{1/5}+y^{1/5})(\frac{1}{4}).y^{-3/4} - (x^{1/4}+y^{1/4}).(\frac{1}{5}).y^{-4/5}}{(x^{1/5}+y^{1/5})^2}.$$

Multiplying equation (2) and (3) by x and y respectively and adding we get,

$$x.\frac{\partial u}{\partial x} + y.\frac{\partial u}{\partial y}$$

$$= x.\left[\frac{(x^{1/5}+y^{1/5}).(\frac{1}{4}).x^{-3/4} - (x^{1/4}+y^{1/4}).(\frac{1}{5})x^{-4/5}}{(x^{1/5}+y^{1/5})^2}\right]$$

$$+ y.\left[\frac{(x^{1/5}+y^{1/5}).(\frac{1}{4}).y^{-3/4} - (x^{1/4}+y^{1/4})(\frac{1}{5})y^{-4/5}}{(x^{1/5}+y^{1/5})^2}\right]$$

$$= \frac{(x^{1/5}+y^{1/5}).(\frac{1}{4})x^{1/4} - (x^{1/4}+y^{1/4}).(\frac{1}{5}).x^{1/5}}{(x^{1/5}+y^{1/5})^2}$$

$$= \frac{(x^{1/5}+y^{1/5}).(\frac{1}{4}).x^{1/4} - (x^{1/4}+y^{1/4}).(\frac{1}{5}).x^{1/5}}{(x^{1/5}+y^{1/5})^2} + \frac{(x^{1/5}+y^{1/5}).(\frac{1}{4})y^{1/4} - (x^{1/4}+y^{1/4}).(\frac{1}{5})y^{1/5}}{}$$

$$= \frac{(\frac{1}{4})(x^{1/5}+y^{1/5})(x^{1/4}+y^{1/4}) - (\frac{1}{5})(x^{1/4}+y^{1/4})}{(x^{1/5}+y^{1/5})^2}$$

$$= \frac{1}{20}.\frac{(x^{1/4}+y^{1/4})}{(x^{1/5}+y^{1/5})}. = \frac{1}{20}u.$$

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$$\therefore\ x.\frac{\partial u}{\partial x} + y.\frac{\partial u}{\partial y} = nu.$$

2. Verify the Euler's theorem of the function (12. (ii))

$$u = x^n.\sin\left(\frac{y}{x}\right)$$

Ans. Given fxn, $u = x^n.\sin\left(\frac{y}{x}\right)$.

since, u is homogeneous function of degree n then by Euler's theorem.

$$x.\frac{\partial u}{\partial x} + y.\frac{\partial u}{\partial y} = nu.$$

Diff. (1) partially w.r.t. x and y respectively, we get.

$$\frac{\partial u}{\partial x} = x^n.\cos\left(\frac{y}{x}\right).\left(\frac{y}{-x^2}\right) + \sin\left(\frac{y}{x}\right).n x^{n-1}$$

OR,

$$\frac{\partial u}{\partial x} = -x^{n-2}y.\cos\left(\frac{y}{x}\right) + n.x^{n-1}.\sin\left(\frac{y}{x}\right) \quad \text{--(2)}$$

And, $$\frac{\partial u}{\partial y} = x^n.\cos\left(\frac{y}{x}\right).\left(\frac{1}{x}\right)$$

$$\therefore \quad = x^{n-1}.\cos\left(\frac{y}{x}\right) \quad \text{.....(3)}$$

Multiplying eq^n (2) and (3) by x and y respectively and adding we get,

$$x.\frac{\partial u}{\partial x} + y.\frac{\partial u}{\partial y} = x\left\{-x^{n-2}.y.\cos\left(\frac{y}{x}\right) + n.x^{n-1}.\sin\left(\frac{y}{x}\right)\right\} + y\left\{x^{n-1}.\cos\left(\frac{y}{x}\right)\right\}$$

$$= -\cancel{x^{n-1}y.\cos\left(\frac{y}{x}\right)} + n.x^n.\sin\left(\frac{y}{x}\right) + \cancel{x^{n-1}.y.\cos\left(\frac{y}{x}\right)}$$

$$= n.x^n.\sin\left(\frac{y}{x}\right)$$

$$= nu$$

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Homework:

Verify the Euler's theorem of the function.

1. $u = \frac{x^2y^2}{x^3+y^3}$ (12. iii.)

Ans. Solution:

Given fxn: $u = \frac{x^2y^2}{x^3+y^3}$ ... (1)

$$= \frac{x^4\left(\frac{y^2}{x^2}\right)}{x^3\left(1+\frac{y^3}{x^3}\right)}$$

$$= x \cdot \frac{\left(\frac{y^2}{x^2}\right)}{\left(1+\frac{y^3}{x^3}\right)}$$

$$\therefore u = x \cdot \phi\left(\frac{y}{x}\right).$$

Thus u is homogenous function of degree 1 then by Euler's theorem,

$$x \cdot \frac{\partial u}{\partial x} + y \cdot \frac{\partial u}{\partial y} = nu.$$

Now,

Diff. (1) partially w.r.t. x we get,

$$\frac{\partial u}{\partial x} = \frac{(x^3+y^3) \cdot y^2 \cdot 2x - x^2y^2 \cdot 3x^2}{(x^3+y^3)^2}$$

$$\therefore \frac{\partial u}{\partial x} = \frac{2xy^2(x^3+y^3) - 3x^4y^2}{(x^3+y^3)^2} \quad \text{.... (2)}$$

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||y,

Diff. (1) w.r.t. y we get,

$$\frac{\partial u}{\partial y} = \frac{(x^3+y^3).x^2.2y - x^2y^2.3y^2}{(x^3+y^3)^2}$$

$$\therefore \frac{\partial u}{\partial y} = \frac{2x^2y(x^3+y^3) - 3x^2y^4}{(x^3+y^3)^2} \quad \ldots (3)$$

Multiplying (2) and (3) by x and y respectively and adding we get,

$$x.\left\{\frac{2xy^2(x^3+y^3) - 3x^4y^2}{(x^3+y^3)^2}\right\} + y.\left\{\frac{2x^2y(x^3+y^3) - 3x^2y^4}{(x^3+y^3)^2}\right\}$$

$$= \frac{2x^2y^2(x^3+y^3) - 3x^5y^2}{(x^3+y^3)^2} + \frac{2x^2y^2(x^3+y^3) - 3x^2y^5}{(x^3+y^3)^2}$$

$$= \frac{2x^2y^2(x^3+y^3) - 3x^5y^2 + 2x^2y^2(x^3+y^3) - 3x^2y^5}{(x^3+y^3)^2}$$

$$= \frac{4x^2y^2(x^3+y^3) - 3x^2y^2(x^3+y^3)}{(x^3+y^3)^2}$$

$$= \frac{x^2y^2\cancel{(x^3+y^3)}}{(x^3+y^3)^{\cancel{2}}}$$

$$= \frac{x^2y^2}{(x^3+y^3)} = 1.\frac{x^2y^2}{x^3+y^3} = 1.u.$$

$$\therefore x.\frac{\partial u}{\partial x} + y.\frac{\partial u}{\partial y} = nu.$$

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2. $u = \frac{x^3y}{x^2+y^2}$. (12. (iv).)

Ans Solution:

Given fxn: $u = \frac{x^3y}{x^2+y^2}$ .... (1).

$$= \frac{x^4\left(\frac{y}{x}\right)}{x^2\left(1+\frac{y^2}{x^2}\right)}$$

$$= x^2 \cdot \left\{ \frac{\left(\frac{y}{x}\right)}{\left(1+\frac{y^2}{x^2}\right)} \right\}$$

$$\therefore u = x^2 \cdot \phi\left(\frac{y}{x}\right)$$

Thus u is homogenous function of degree 2 then by Euler's theorem,

$$x \cdot \frac{\partial u}{\partial x} + y \cdot \frac{\partial u}{\partial y} = nu.$$

Now,

Diff. (1) partially w.r.t. x we get,

$$\frac{\partial u}{\partial x} = \frac{(x^2+y^2) \cdot 3x^2y - x^3y \cdot 2x}{(x^2+y^2)^2}$$

$$\therefore \frac{\partial u}{\partial x} = \frac{3x^2y(x^2+y^2) - 2x^4y}{(x^2+y^2)^2}. \quad ....(2)$$

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Ily,

Diff. (1) partially w.r.t. y we get,

$$\frac{\partial u}{\partial y} = \frac{(x^2+y^2).x^3 - x^3y.(2y)}{(x^2+y^2)^2}$$

$$\therefore \frac{\partial u}{\partial y} = \frac{x^3(x^2+y^2) - 2x^3y^2}{(x^2+y^2)^2} \quad \ldots\ldots (3)$$

Multiplying (2) and (3) by x and y respectively & adding we get,

$$x.\frac{\partial u}{\partial x} + y.\frac{\partial u}{\partial y} = x.\left\{\frac{3x^2y(x^2+y^2) - 2x^4y}{(x^2+y^2)^2}\right\} + y.\left\{\frac{x^3(x^2+y^2) - 2x^3y^2}{(x^2+y^2)^2}\right\}$$

$$= \frac{3x^3y(x^2+y^2) - 2x^5y}{(x^2+y^2)^2} + \frac{x^3y(x^2+y^2) - 2x^3y^3}{(x^2+y^2)^2}$$

$$= \frac{3x^3y(x^2+y^2) - 2x^5y + x^3y(x^2+y^2) - 2x^3y^3}{(x^2+y^2)^2}$$

$$= \frac{4x^3y(x^2+y^2) - 2x^3y(x^2+y^2)}{(x^2+y^2)^2}$$

$$= \frac{2x^3y\cancel{(x^2+y^2)}}{(x^2+y^2)^{\cancel{2}}}$$

$$= 2.\frac{x^3y}{x^2+y^2} = nu.$$

$$\therefore x.\frac{\partial u}{\partial x} + y.\frac{\partial u}{\partial y} = nu$$

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(U.V)

3. $u = x^n . \tan^{-1}\left(\frac{y}{x}\right)$.

Ans. Solution:

Given fxn : $u = x^n . \tan^{-1}\left(\frac{y}{x}\right)$ ... (1)

since, u is homogenous fxn of degreen n then by Euler's theorem,

$$x . \frac{\partial u}{\partial x} + y . \frac{\partial u}{\partial y} = nu.$$

Now,

Diff. (1) partially w.r.t. x, we get,

$$\frac{\partial u}{\partial x} = x^n . \frac{1}{1+\left(\frac{y}{x}\right)^2} . \left(\frac{-y}{x^2}\right) + \tan^{-1}\left(\frac{y}{x}\right) . n . x^{n-1}$$

$$= \frac{-x^n . y}{x^2+y^2} . \frac{\cancel{x^2}}{\cancel{x^2}} + n . x^{n-1} . \tan^{-1}\left(\frac{y}{x}\right),$$

$$\therefore \frac{\partial u}{\partial x} = \frac{-x^n . y}{x^2+y^2} + n . x^{n-1} . \tan^{-1}\left(\frac{y}{x}\right) \quad \ldots (2)$$

Again,

Diff. (1) partially w.r.t. y we get,

$$\frac{\partial u}{\partial y} = x^n . \frac{1}{1+\left(\frac{y}{x}\right)^2} . \frac{1}{x}$$

$$= x^{n-1} . \frac{x^2}{x^2+y^2}. \qquad \therefore \frac{\partial u}{\partial y} = \frac{x^{n+1}}{x^2+y^2} \quad \ldots (3)$$

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Multiplying (1) and (2) by x and y respectively and adding them we get,

$$x.\frac{\partial u}{\partial x}+y.\frac{\partial u}{\partial y}=x.\left\{-\frac{x^n.y}{x^2+y^2}+n.x^{n-1}.\tan^{-1}\left(\frac{y}{x}\right)\right\}+ y.\left\{\frac{x^{n+1}}{x^2+y^2}\right\}$$

$$=-\cancel{\frac{x^{n+1}y}{x^2+y^2}}+n.x^n.\tan^{-1}\left(\frac{y}{x}\right)+\cancel{\frac{x^{n+1}.y}{x^2+y^2}}$$

$$=n.x^n.\tan^{-1}\left(\frac{y}{x}\right)$$

$$=nu.$$

$$\therefore x.\frac{\partial u}{\partial x}+y.\frac{\partial u}{\partial y}=nu.$$

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4. $u=x.f\left(\frac{y}{x}\right)$.

Ans. Here, $\frac{\partial u}{\partial x}=x.f'\left(\frac{y}{x}\right).\left(\frac{-y}{x^2}\right)+f\left(\frac{y}{x}\right)=f\left(\frac{y}{x}\right)-\frac{y}{x}.f'\left(\frac{y}{x}\right)$

Ily, $\frac{\partial u}{\partial y}=x.f'\left(\frac{y}{x}\right).\frac{1}{x}=f'\left(\frac{y}{x}\right)$.

$$\therefore x.\frac{\partial u}{\partial x}+y.\frac{\partial u}{\partial y}=x\left\{f\left(\frac{y}{x}\right)-\frac{y}{x}.f'\left(\frac{y}{x}\right)\right\}+y\left\{f'\left(\frac{y}{x}\right)\right\}$$

$$=x.f.\left(\frac{y}{x}\right)-\cancel{y.f'\left(\frac{y}{x}\right)}+\cancel{y.f'\left(\frac{y}{x}\right)}.$$

$$=x.f\left(\frac{y}{x}\right)$$

Proved.

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5. $u = \sin^{-1}\frac{x}{y} + \tan^{-1}\frac{x}{y}$.

Ans: Solution:

Given fxn: $u = \sin^{-1}\frac{x}{y} + \tan^{-1}\frac{x}{y}$ ....(1)

Diff. (1) partially w.r.t. x we get

$$\frac{\partial u}{\partial x} = \frac{1}{\sqrt{1-\frac{x^2}{y^2}}} \cdot \frac{1}{y} + \frac{1}{1+\frac{x^2}{y^2}} \cdot \frac{1}{y}.$$

$$\therefore \frac{\partial u}{\partial x} = \frac{1}{\sqrt{y^2-x^2}} + \frac{y}{x^2+y^2} \text{ ....(2)}$$

Also,

Diff. (1) partially w.r.t. y we get,

$$\frac{\partial u}{\partial y} = \frac{1}{\sqrt{1-\frac{x^2}{y^2}}} \cdot \left(\frac{-x}{y^2}\right) + \frac{1}{1+\frac{x^2}{y^2}} \cdot \left(\frac{-x}{y^2}\right).$$

$$\therefore \frac{\partial u}{\partial y} = \frac{-x}{\sqrt{y^2-x^2}\cdot y} - \frac{x}{x^2+y^2} \text{ ...(3)}$$

Multiplying (2) and (3) by x and y respectively and adding them, we get:

$$x \cdot \frac{\partial u}{\partial x} + y \cdot \frac{\partial u}{\partial y} = x\left\{\frac{1}{\sqrt{y^2-x^2}} + \frac{y}{x^2+y^2}\right\} + y \cdot \left\{\frac{-x}{y \cdot \sqrt{y^2-x^2}} - \frac{x}{x^2+y^2}\right\}$$

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$$= \cancel{\frac{x}{\sqrt{y^2-x^2}}} + \cancel{\frac{xy}{x^2+y^2}} - \cancel{\frac{x\cancel{y}}{\cancel{y}\sqrt{y^2-x^2}}} - \cancel{\frac{xy}{x^2+y^2}}$$

$$= 0.$$

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* If $u = \sin^{-1}\left(\frac{x^2y^2}{x+y}\right)$, show that $x.\frac{\partial u}{\partial x} + y.\frac{\partial u}{\partial y} = 3\tan u$.

Ans. Solution:

Given fxn: $u = \sin^{-1}\left(\frac{x^2y^2}{x+y}\right)$

$$\text{or, } \sin u = \frac{x^2y^2}{x+y}.$$

$$\text{or, } \sin u = \frac{x^4\left(\frac{y}{x}\right)^2}{x\left(1+\frac{y}{x}\right)} = \frac{x^3\left(\frac{y}{x}\right)^2}{\left(1+\frac{y}{x}\right)} = x^3\phi\left(\frac{y}{x}\right)$$

$$\therefore \sin u = x^3.\phi\left(\frac{y}{x}\right).$$

Thus, sinu is homogeneous function of degree three then by Euler's theorem,

$$x.\frac{\partial \sin u}{\partial x} + y.\frac{\partial \sin u}{\partial y} = 3\sin u.$$

$$\text{or, } x.\frac{\partial \sin u}{\partial u}.\frac{\partial u}{\partial x} + y.\frac{\partial \sin u}{\partial u}.\frac{\partial u}{\partial y} = 3\sin u.$$

$$\text{or, } x.\cos u.\frac{\partial u}{\partial x} + y.\cos u.\frac{\partial u}{\partial y} = 3\sin u.$$

$$\text{or, } \cos u\left(x.\frac{\partial u}{\partial x} + y.\frac{\partial u}{\partial y}\right) = 3\sin u$$

$$\text{or, } x.\frac{\partial u}{\partial x} + y.\frac{\partial u}{\partial y} = 3\tan u.$$

proved.

7. If $u = \tan^{-1}\left\{\frac{x^3+y^3}{x-y}\right\}$, show that $x.\frac{\partial u}{\partial x} + y.\frac{\partial u}{\partial y} = \sin 2u$.

Ans- Solution:

Given fxn: $u = \tan^{-1}\left\{\frac{x^3+y^3}{x-y}\right\}$

or, $\tan u = \frac{x^3+y^3}{x-y} = \frac{x^3\left(1+\frac{y^3}{x^3}\right)}{x\left(1-\frac{y}{x}\right)} = x^2.\frac{\left(1+\frac{y^3}{x^3}\right)}{\left(1-\frac{y}{x}\right)}$.

$\therefore \tan u = x^2.\phi\left(\frac{y}{x}\right)$.

Thus, tanu is homogenous function of degree two then by euler's law (theorem).

$$x.\frac{\partial \tan u}{\partial x} + y.\frac{\partial \tan u}{\partial y} = 2.\tan u.$$

or, $x.\frac{\partial \tan u}{\partial u}.\frac{\partial u}{\partial x} + y.\frac{\partial \tan u}{\partial u}.\frac{\partial u}{\partial y} = 2\tan u$

or, $x.\sec^2 u.\frac{\partial u}{\partial x} + y.\sec^2 u.\frac{\partial u}{\partial y} = 2\tan u$.

or, $\sec^2 u\left(x.\frac{\partial u}{\partial x} + y.\frac{\partial u}{\partial y}\right) = 2\tan u$.

or, $x.\frac{\partial u}{\partial x} + y.\frac{\partial u}{\partial y} = \frac{2\sin u/\cos u}{\sec^2 u}$

$\therefore x.\frac{\partial u}{\partial x} + y.\frac{\partial u}{\partial y} = 2\sin u.\cos u = \sin 2u$

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10. If $u = \log \frac{x^2+y^2}{x+y}$, show that $x \cdot \frac{\partial u}{\partial x} + y \cdot \frac{\partial u}{\partial y} = 1$.

Ans Here,

Given fxn: $u = \log \frac{x^2+y^2}{x+y}$

or, $e^u = \frac{x^2+y^2}{x+y} = \frac{x^2\left(1+\frac{y^2}{x^2}\right)}{x(1+y/x)} = \frac{x\left(1+\frac{y^2}{x^2}\right)}{1+y/x}$

or, $e^u = x\,\phi\left(\frac{y}{x}\right)$.

Thus, e^u is homogenous function of degree one then by Euler's theorem,

$$x \cdot \frac{\partial e^u}{\partial x} + y \cdot \frac{\partial e^u}{\partial y} = 1 \cdot e^u$$

or, $x \cdot \frac{\partial e^u}{\partial u} \cdot \frac{\partial u}{\partial x} + y \cdot \frac{\partial e^u}{\partial u} \cdot \frac{\partial u}{\partial y} = e^u$

or, $x \cdot e^u \cdot \frac{\partial u}{\partial x} + y \cdot e^u \cdot \frac{\partial u}{\partial y} = e^u$

or, $e^u \left[x \cdot \frac{\partial u}{\partial x} + y \cdot \frac{\partial u}{\partial y}\right] = e^u$.

$\therefore x \cdot \frac{\partial u}{\partial x} + y \cdot \frac{\partial u}{\partial y} = 1$

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* If $u = \log x$, show that $x \cdot \frac{\partial u}{\partial x} + y \cdot \frac{\partial u}{\partial y} = 1$

Ans Here, $u = \log x$ .... (1)

Diff. (1) partially w.r.t. x we get,

$$\frac{\partial u}{\partial x} = \frac{1}{x}.$$

Ily, Diff. (1) partially w.r.t. y we get,

$$\frac{\partial u}{\partial y} = 0$$

$$\therefore x \cdot \frac{\partial u}{\partial x} + y \cdot \frac{\partial u}{\partial y} = x \cdot \frac{1}{x} + y \times 0$$

$$= 1 + 0$$

$$= 0$$

$$= \text{R.H.S.}$$

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9. If $u = \cos^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$, show that. $x.\frac{\partial u}{\partial x} + y.\frac{\partial u}{\partial y} = -\frac{1}{2}\cot u$

Ans Here,

Given function, $u = \cos^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$

$$\text{or, } \cos u = \frac{x+y}{\sqrt{x}+\sqrt{y}} = \frac{x\,(1+y/x)}{x^{1/2}\left(1+\left(\frac{y}{x}\right)^{1/2}\right)}$$

$$\therefore \cos u = x^{+1/2}\left\{\frac{1+y/x}{1+(y/x)^{1/2}}\right\}$$

Thus $\cos u$ is homogenous function of degree $+1/2$ then by Euler's theorem,

$$x.\frac{\partial \cos u}{\partial x} + y.\frac{\partial \cos u}{\partial y} = +\frac{1}{2}.\cos u.$$

$$\text{or, } x.(-\sin u).\frac{\partial u}{\partial x} + y.(-\sin u).\frac{\partial u}{\partial y} = +\frac{1}{2}\cos u.$$

$$\text{or, } (-\sin u)\left(x.\frac{\partial u}{\partial x} + y.\frac{\partial u}{\partial y}\right) = +\frac{1}{2}\cos u$$

$$\text{or, } x.\frac{\partial u}{\partial x} + y.\frac{\partial u}{\partial y} = \frac{+1\cos u}{-2\sin u}$$

$$\therefore x.\frac{\partial u}{\partial x} + y.\frac{\partial u}{\partial y} = \frac{-1}{2}\cot u.$$

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8: If $u = \operatorname{cosec}^{-1}\left(\frac{x^{1/2}+y^{1/2}}{x^{1/3}+y^{1/3}}\right)$, show that $x.\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = -\frac{1}{6}\tan u$

Ans: Here,

Given function, $u = \operatorname{cosec}^{-1}\left(\frac{x^{1/2}+y^{1/2}}{x^{1/3}+y^{1/3}}\right)$

$$\text{or, } \operatorname{cosec} u = \frac{x^{1/2}+y^{1/2}}{x^{1/3}+y^{1/3}} = \frac{x^{1/2}\left(1+\frac{y^{1/2}}{x^{1/2}}\right)}{x^{1/3}\left(1+y^{1/3}/x^{1/3}\right)}$$

$$\text{or, } \operatorname{cosec} u = x^{1/2-1/3}.\left\{\frac{1+\frac{y^{1/2}}{x^{1/2}}}{1+y^{1/3}/x^{1/3}}\right\}$$

$$\therefore \operatorname{cosec} u = x^{1/6}\left(\frac{1+y^{1/2}/x^{1/2}}{1+y^{1/3}/x^{1/3}}\right)$$

Thus cosecu is homogenous function of degree 1/6 then by Euler's theorem,

$$x.\frac{\partial \operatorname{cosec} u}{\partial x} + y.\frac{\partial \operatorname{cosec} u}{\partial y} = \frac{1}{6}.\operatorname{cosec} u.$$

$$\text{or, } -x.\operatorname{cosec} u.\cot u.\frac{\partial u}{\partial x} + y.(-\operatorname{cosec} u.\cot u).\frac{\partial u}{\partial y} = \frac{1}{6}.\operatorname{cosec} u.$$

$$\text{or, } -\cancel{\operatorname{cosec} u}.\cot u\left(x.\frac{\partial u}{\partial x} + y.\frac{\partial u}{\partial y}\right) = \frac{1}{6}.\cancel{\operatorname{cosec} u}.$$

$$\text{or, } x.\frac{\partial u}{\partial x} + y.\frac{\partial u}{\partial y} = -\frac{1}{6\cot u}$$

$$\therefore x.\frac{\partial u}{\partial x} + y.\frac{\partial u}{\partial y} = -\frac{1}{6}.\tan u.$$

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6. If $\sin u = \dfrac{\sqrt{x} - \sqrt{y}}{\sqrt{x} + \sqrt{y}}$, show that $x \cdot \dfrac{\partial u}{\partial x} + y \cdot \dfrac{\partial u}{\partial y} = 0$.

Ans. Solution:

Given fxn: $\sin u = \dfrac{\sqrt{x} - \sqrt{y}}{\sqrt{x} + \sqrt{y}} = \dfrac{x^{1/2}\left(1 - \left(\frac{y}{x}\right)^{1/2}\right)}{x^{1/2}\left(1 + \left(\frac{y}{x}\right)^{1/2}\right)}$

$$\therefore \sin u = x^{0} \phi\left(\frac{y}{x}\right)$$

Thus, sinu is homogeneous function of degree 0. Then by Euler's theorem,

$$x \cdot \frac{\partial \sin u}{\partial x} + y \cdot \frac{\partial \sin u}{\partial y} = 0 \times \sin u$$

or, $x \cdot \cos u \cdot \dfrac{\partial u}{\partial x} + y \cdot \cos u \cdot \dfrac{\partial u}{\partial y} = 0$

or, $\cos u \left[x \cdot \dfrac{\partial u}{\partial x} + y \cdot \dfrac{\partial u}{\partial y} \right] = 0$.

$$\therefore x \cdot \frac{\partial u}{\partial x} + y \cdot \frac{\partial u}{\partial y} = 0$$

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4 Example:

Given that $u = \sin\left(\frac{x}{y}\right)$, $x = e^t$, $y = t^2$, find $\frac{du}{dt}$ (Total deriv)

Ans. Solution:

Let $u = f(x, y)$ where, $x = \phi(t)$ and $y = \psi(t)$ then u can be expressed as a function of t alone and the ordinary derivate $\frac{du}{dt}$ is called the Total Differential coefficients of u w.r.t. t.

$$\therefore \frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt}$$

Given that; $u = \sin\left(\frac{x}{y}\right)$ ....(1)

$$x = e^t \quad \text{....(2)}$$

$$y = t^2 \quad \text{....(3)}$$

Diff. (1) partially w.r.t. x and y respectively, we got

$$\frac{\partial u}{\partial x} = \frac{1}{y} \cdot \cos\left(\frac{x}{y}\right) \text{ and } \frac{\partial u}{\partial y} = -\frac{x}{y^2} \cdot \cos\left(\frac{x}{y}\right)$$

Diff. (2) and (3) w.r.t. t

$$\frac{dx}{dt} = e^t \text{ and } \frac{dy}{dt} = 2t.$$

$$\therefore \frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt}$$

$$= \frac{1}{y} \cdot \cos\left(\frac{x}{y}\right) \cdot e^t + \left(\frac{-x}{y^2}\right) \cdot \cos\left(\frac{x}{y}\right) \cdot 2t$$

$$= \frac{1}{t^2} \cdot \cos\left(\frac{e^t}{t^2}\right) \cdot e^t - \frac{e^t}{t^4} \cdot \cos\left(\frac{x}{y}\right) \cdot 2t$$

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$$= \frac{e^t}{t^2} \cdot \cos\left(\frac{e^t}{t^2}\right) - \frac{2 \cdot e^t}{t^3} \cdot \cos\left(\frac{e^t}{t^2}\right)$$

$$= \frac{(t-2)}{t^3} e^t \cdot \cos\left(\frac{e^t}{t^2}\right).$$

$$\therefore \frac{dy}{dt} = \frac{(t-2)}{t^3} \cdot e^t \cdot \cos\left(\frac{e^t}{t^2}\right)$$

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Example:

1. Examine and find the max^m and min^m value of $8-4x+4y-x^2-y^2$.

Ans. Solution:

Let $f(x,y) = 8-4x+4y-x^2-y^2$ ... (1)

Diff. (1) partially we get,

$f_x = -4-2x$

$\therefore f_{xx} = -2$

$f_{xy} = 0$ & $f_{yx} = 0$

and,

$f_y = 4-2y$

$f_{yy} = -2.$

Note: $f_{xy} = f_{yx}$.

For stationary points,

$f_x = 0$ and $f_y = 0$

or, $-4-2x = 0$ or, $4-2y = 0$

$\therefore x = -2$ $\therefore y = 2.$

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$\therefore$ The stationary point is $(-2, 2)$.

Now,

At $(-2,2)$; $f_{xx} f_{yy} - f(_{xy})^2 = (-2)(-2) - (0)^2$
$= 4 > 0$.

And, $f_{xx} = -2 < 0$.

Hence, the given function $f(x,y)$ is maximum value at $(-2, 2)$ and the maximum value is

$$f(-2,2) = 8 - 4(-2) + 4(2) - (-2)^2 - 2^2$$
$$= 8+8+8-4-4$$
$$= 16.$$

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2. Examine and find the maximum and minimum values of:
$$20 - x^2 - y^2 - z^2.$$

solution:

Let $f(x,y,z) = 20 - x^2 - y^2 - z^2$ ... (1)

Diff. (1) partially we get,

$f_x = -2x$ $\quad f_y = -2y$ $\quad f_z = -2z$.

$f_{xx} = -2$ $\quad f_{xy} = 0$ $\quad f_{xz} = 0$.

$\therefore f_{yx} = 0$ $\quad f_{yy} = -2$ $\quad f_{yz} = 0$

$f_{zx} = 0$ $\quad f_{zy} = 0$ $\quad f_{zz} = -2$

For stationary point,

$f_x = 0$, $\quad f_y = 0$ and $f_z = 0$

i.e. $-2x = 0$ $\quad -2y = 0$ $\quad -2z = 0$.

$\therefore$ solving these we get, $x = 0, y = 0$ and $z = 0$.

$\therefore$ The stationary point is $(0,0,0)$.

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Now,

At $(0,0,0)$: $f_{xx}f_{yy} - (f_{xy})^2 = (-2)(-2) - (0)^2 = 4 > 0$

$f_{xx} = -2 < 0$

And, $\begin{vmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{yz} & f_{yy} & f_{yz} \\ f_{zx} & f_{zy} & f_{zz} \end{vmatrix} = \begin{vmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{vmatrix} = -8 < 0.$

Hence the given function $f(x,y,z)$ is maximum value at $(0,0,0)$ and the maximum value is,

$f(0,0,0) = 20 - 0^2 - 0^2 - 0^2$

$= 20$

3. Obtain the maximum value of xyz such that $x+y+z = 24$.

Ans. Solution:

Let $f(x,y,z) = xyz$ ....(1)

such that, $x+y+z = 24$

or, $z = 24 - x - y$ ...(2)

From (1) and (2) we get,

$f(x,y) = xy(24 - x - y)$

$= 24xy - x^2y - xy^2$ ...(3)

Diff. (3) partially we get,

$f_x = 24y - 2xy - y^2$ $\quad$ $f_y = 24x - x^2 - 2xy$

$\therefore f_{xx} = -2y$ $\quad$ $f_{xy} = 24 - 2x - 2y$

$f_{yy} = -2x$

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For stationary points:

$f_x = 0$ and $f_y = 0$

i.e. $24y - 2xy - y^2 = 0$ and $24x - x^2 - 2xy = 0$

Solving these $x = 8$ and $y = 8$.

Putting the value of x and y in (2) we get $z = 8$.

$\therefore$ The stationary point is $(8,8,8)$

Now,

At $(8,8,8)$:

$f_{xx} f_{yy} - (f_{xy})^2 = [(-2) \times 8] \times [(-2) \times 8] - (24 - 2 \times 8 - 2 \times 8)^2$

$= 256 - 64$

$= 192 > 0.$

And $f_{xx} = -2 \times 8 = -16 < 0$

Hence the given function $f(x,y,z)$ is maximum value at $(8,8,8)$ and the maximum value is $f(8,8,8) = 8 \times 8 \times 8 = 512$.

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4. Find the extreme value of $x^2+y^2+z^2$ connected by the relation, $x+z=1$ and $2y+z=2$.

Ans. Solution :

Let $f(x,y,z) = x^2+y^2+z^2$ .... (1)

connected by the relation,

$x+z=1$ or, $z=1-x$ ... (2)

and, $2y+z=2$

or, $2y = 2-z$

$= 2-(1-x)$

$\therefore y = \dfrac{1+x}{2}$ .... (3)

Using (2) and (3) in (1) reduces to

$f(x) = x^2 + \left(\dfrac{1+x}{2}\right)^2 + (1-x)^2$ ... (4)

Diff. (4) w.r.t. x we get,

$f'(x) = 2x + \dfrac{1}{4}.2(1+x) + 2(1-x).(-1)$

$= 2x + \dfrac{1+x}{2} - 2(1-x)$

$\therefore f''(x) = 2 + \dfrac{1}{2} + 2 = \dfrac{9}{2}$.

For stationary points,

$f'(x) = 0$

i.e. $2x + \dfrac{1+x}{2} - 2(1-x) = 0$

or, $4x + 1 + x - 4 + 4x = 0$

or, $9x - 3 = 0$ $\therefore x = 1/3$.

... value is $f(8,8,8) = 8\times 8\times 8 = 512.$

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5. Find the minimum value of $x^2+xy+y^2+3z^2$ under the condition $x+2y+4z=60$.

Ans. Solution:

Let $f(x,y,z) = x^2+xy+y^2+3z^2 \quad \ldots(1)$

Under the condition, $x+2y+4z=60$

or, $4z = 60-x-2y$

or, $z = \dfrac{60-x-2y}{4} \quad \ldots(2)$

From (1) and (2)

$$f(x,y) = x^2+xy+y^2+3\left(\frac{60-x-2y}{4}\right)^2 \quad \ldots(3)$$

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Diff. (3) partially we get,

$$f_x = 2x + y + \frac{3}{16} \cdot 2(60 - x - 2y) \cdot (-1)$$

$$= \frac{16x + 8y - 180 + 3x + 6y}{8}$$

$$= \frac{19x + 14y - 180}{8}$$

$$f_y = x + 2y + \frac{3}{16} \cdot 2(60 - x - 2y) \cdot (-2)$$

$$= \frac{8x + 16y - 360 + 6x + 12y}{8}$$

$$= \frac{14x + 28y - 360}{8}.$$

$\therefore f_{xx} = \frac{19}{8}$, $f_{xy} = \frac{14}{8} = \frac{7}{4}$, and $f_{yy} = \frac{28}{8} = \frac{7}{2}$.

For stationary point,

$f_x = 0$ and $f_y = 0$

or, $\frac{19x + 14y - 180}{8} = 0$ $\quad$ $\frac{14x + 28y - 360}{8} = 0$

or, $19x + 14y - 180 = 0$ and, $14x + 28y - 360 = 0$.

On solving we get, $y = \frac{90}{7}$ and $x = 0$

Using this value in (2) we get $z = \frac{60}{7}$

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∴ The stationary point is $(0, 90/7, 60/7)$.

Now,

At $(0, 90/7, 60/7)$:

$$f_{xx}f_{yy} - (f_{xy})^2 = \frac{19}{8} \cdot \frac{7}{2} - \left(\frac{7}{4}\right)^2$$

$$= \frac{133}{16} - \frac{49}{16}$$

$$= \frac{84}{16} > 0$$

Also, $f_{xx} = \frac{19}{8} > 0.$

Hence, the given function $f(x,y,z)$ is min^m at $(0, 90/7, 60/7)$.

∴ The minimum ~~$\frac{90}{7}$~~ value of $x^2 + xy + y^2 + 3z^2$ at $(0, 90/7, 60/7)$

is, $(0)^2 + 0 \cdot \frac{19}{7} + \left(\frac{90}{7}\right)^2 + 3 \cdot \left(\frac{60}{7}\right)^2$.

$$= 0 + 0 + \frac{8100}{49} + 3 \cdot \frac{3600}{49}$$

$$= \frac{8100}{49} + 3 \cdot \frac{3600}{49}$$

$$= \frac{8100 + 10800}{49}$$

$$= \frac{2700}{7}$$

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6. Find the min^{m} value of $x^2+y^2+z^2$ connected by the relation $ax+by+cz=p$.

Ans: solution:

Let $f(x,y,z) = x^2+y^2+z^2 \dots (1)$

connected by the relation $ax+by+cz=p$.

$\therefore z = \frac{p-ax-by}{c} \dots (2)$

From (1) and (2) we get,

$f(x,y) = x^2+y^2+\left(\frac{p-ax-by}{c}\right)^2 \dots (3)$

Diff. (3) partially we get,

$f_x = 2x + 2\cdot\left(\frac{p-ax-by}{c}\right)\cdot\frac{1}{c}\cdot(-a)$

$= 2x - \frac{2a}{c^2}(p-ax-by)$

IIly, $f_y = 2y + \frac{2}{c^2}(p-ax-by)\cdot(-b)$

$= 2y - \frac{2b}{c^2}(p-ax-by)$

$\therefore f_{xx} = 2 - \frac{2a}{c^2}(-a) = 2 + \frac{2a^2}{c^2}$

$\therefore f_{xy} = -\frac{2a}{c^2}(-b) = \frac{2ab}{c^2}$.

and $f_{yy} = 2 - \frac{2b}{c^2}(-b)$

$= 2 + \frac{2b^2}{c^2}$.

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For stationary point,

$f_x = 0$ and $f_y = 0$

i.e. $2x - \frac{2a}{c^2}(P - ax - by) = 0$ i.e. $2y - \frac{2b}{c^2}(P - ax - by) = 0$

or, $2c^2x - 2a(P - ax - by) = 0$ or, $2c^2y - 2b(P - ax - by) = 0$

or, $(c^2+a^2)x + aby - ap$ ... (4) or, $abx + (c^2+b^2)y - bp = 0$ (5)

Solving (4) and (5) we get,

[By using Cross Multiplication Method]

$$\frac{x}{-apb^2 + ap(c^2+b^2)} = \frac{y}{-a^2bp + bp(c^2+a^2)} = \frac{1}{(c^2+a^2)(c^2+b^2) - a^2b^2}$$

or, $$\frac{x}{-\cancel{apb^2} + apc^2 + \cancel{apb^2}} = \frac{y}{-\cancel{bpa^2} + bpc^2 + \cancel{bpa^2}} = \frac{1}{c^4 + c^2b^2 + a^2c^2 + \cancel{a^2b^2} - \cancel{a^2b^2}}$$

or, $$\frac{x}{apc^2} = \frac{y}{bpc^2} = \frac{1}{c^2(a^2+b^2+c^2)}$$

or, $$\frac{x}{ap} = \frac{y}{bp} = \frac{1}{a^2+b^2+c^2}.$$

$\therefore x = \frac{ap}{a^2+b^2+c^2}$ and $y = \frac{bp}{a^2+b^2+c^2}$.

Using this value in (2) we get, $z = \frac{cP}{a^2+b^2+c^2}$

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$\therefore$ The stationary point is, $\left(\frac{ap}{a^2+b^2+c^2}, \frac{bp}{a^2+b^2+c^2}, \frac{cp}{a^2+b^2+c^2}\right)$.

Now, at stationary point,

$$f_{xx}f_{yy} - (f_{xy})^2 = \left(2+\frac{2a^2}{c^2}\right)\cdot\left(2+\frac{2b^2}{c^2}\right) - \left(\frac{2ab}{c^2}\right)^2$$

$$= 4\left(\frac{c^2+a^2}{c^2}\right)\cdot\left(\frac{c^2+b^2}{c^2}\right) - 4\,\frac{a^2b^2}{c^4}$$

$$= 4\left(\frac{c^4+c^2b^2+a^2c^2+a^2b^2}{c^4}\right) - 4\,\frac{a^2b^2}{c^4}$$

$$= 4\left(\frac{c^4+c^2b^2+a^2c^2+\cancel{a^2b^2}-\cancel{a^2b^2}}{c^4}\right)$$

$$= 4\left(\frac{c^2+b^2+a^2}{c^2}\right) > 0$$

And, $f_{xx} = \left(2+\frac{2a^2}{c^2}\right) > 0$.

Hence, the given function $f(x,y,z)$ is minimum value at $\left(\frac{ap}{a^2+b^2+c^2}, \frac{bp}{a^2+b^2+c^2}, \frac{cp}{a^2+b^2+c^2}\right)$.

and the ~~maximum~~ minimum value of $x^2+y^2+z^2$

$$= \left(\frac{ap}{a^2+b^2+c^2}\right)^2 + \left(\frac{bp}{a^2+b^2+c^2}\right)^2 + \left(\frac{cp}{a^2+b^2+c^2}\right)^2$$

$$= \frac{a^2p^2+b^2p^2+c^2p^2}{(a^2+b^2+c^2)^2}$$

$$= \frac{p^2}{a^2+b^2+c^2}.$$

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* Find the maximum value of xyz under the condition $x+y+z=8$.

Ans. Here, Given fxn : $f(x,y,z) = xyz \dots (1)$

under condition,

$x+y+z=8$

$\therefore z = 8-x-y \dots (2)$

Using value of z, then fxn (1) reduces to,

$f(x,y) = xy(8-x-y) \dots (3)$

Now, Diff.-(3) partially we get,

$f_x = y.(8-2x-y)$ & $f_y = x.(8-x-2y)$

$\therefore f_x = 8y-2xy-y^2$. $\therefore f_y = 8x-x^2-2xy$.

$\therefore f_{xx} = -2y$ $\therefore f_{yy} = -2x$

For stationary point,

$f_x = 0$ and $f_y = 0$

or, $8y-2xy-y^2=0$ or $8x-x^2-2xy=0$

On solving we get, $x = +8/3$ and $y = 8/3$

Putting value of x and y in z, (eq^n (2) we get,

$z = 8/3$.

$\therefore$ stationary point is $(+8/3, 8/3, 8/3)$.

And, At stationary point,

$f_{xx} = -2.\frac{8}{3} = \frac{-16}{3} < 0$

and, $f_{xx}.f_{yy} - (f_{xy})^2 = \frac{256}{3} + \frac{6}{9} > 0$

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Hence, the given fxn has max^m value of $= xyz$

$$= \frac{8}{3} \times \frac{8}{3} \times \frac{8}{3}$$

$$= \frac{512}{27}$$

* Obtain the minimum value of $x^2+y^2+z^2$ subject to the condition, $x+y+z-1=0$ and $xyz+1=0$.

Ans. Solution.

Let $f(x, y, z) = x^2+y^2+z^2$ .... (1)

connected by the relation,

$x+y+z-1=0$ and $xyz+1=0$

or, $y+z = 1-x$ ...(i) $\quad yz = \dfrac{-1}{x}$ ... (ii)

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From (1)

$f(x) = x^2+y^2+z^2$

$= x^2 + (y+z)^2 - 2yz$

$= x^2 + (1-x)^2 + \frac{2}{x}$ ......(2).

Diff. (2) ~~partially~~,

$\frac{d f(x)}{dx} = 2x + 2(1-x).(-1) - \frac{2}{x^2}$. $f_y =$

$= 2x - 2 + 2x - \frac{2}{x^2}$

$\therefore \frac{d f(x)}{dx} = 4x - 2 - \frac{2}{x^2}$.

$\therefore f_{xx} = 4 + \frac{4}{x^3}$.

For stationary point,

$f'(x) = 0$

i.e. $4x - 2 - \frac{2}{x^2} = 0$

or, $4x^3 - 2x^2 - 2 = 0$

or, $2x^3 - x^2 - 1 = 0$

$\Rightarrow x = 1 > 0$. ~~Hence~~ & $f''(x) = 4 + \frac{4}{(1)^3} = 8 > 0$ Hence the fxn is maximum

Using value of x in (1) and (ii) we get

$y+z=0$ and $yz=-1$.

On solving, $y = \pm 1$ and $z = \pm 1$.

Hence, the stationary points are $(1, \pm 1, \pm 1)$.

So the maximum value is, $x^2+y^2+z^2 = 3$.

* Find the extreme value of $x^2+y^2+z^2$ subject to the condition $x+y+z=1$.

Ans. Solution:

Let $f(x,y,z) = x^2+y^2+z^2$ ---- (1)

Under the condition, $x+y+z=1$

or, $z = 1-x-y$ .... (2)

From (1) and (2), the fxn reduces to.

$f(x,y) = x^2+y^2+(1-x-y)^2$ .....(3)

$\cancel{= x^2+y^2+}$

Diff. (3) partially we get,

$f_x = 2x + 2(1-x-y).(0-1)$ and, $f_y = 2y + 2(1-x-y).(-1)$

or $f_x = 2x - 2(1-x-y)$ $= 2y - 2(1-x-y)$

or $f_x = 2x - 2 + 2x + 2y$ $= 2y - 2 + 2x + 2y$

or, $f_x = 4x + 2y - 2$. $= 4y + 2x - 2$

$\therefore f_x = 2(2x+y-1)$ $\therefore f_y = 2(2y+x-1)$

$\therefore f_{xx} = 2(2) = 4$. & $f_{xy} = 2$. $\therefore f_{yy} = 2(2) = 4$

For stationary point,

$f_x = 0$ and, $f_y = 0$

or, $2(2x+y-1) = 0$ or, $2(2y+x-1) = 0$

or, $2x+y-1 = 0$ or, $2y+x-1 = 0$

On solving above equations we get, $x = 1/3$ and $y = 1/3$

Using this value in eq^n (2) we get, $z = 1/3$.

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$\therefore$ The stationary point is $(1/3, 1/3, 1/3)$.

Now,

At $(1/3, 1/3, 1/3)$:

$f_{xx} f_{yy} - (f_{xy})^2 = 4 \times 4 - (2)^2 = 16 - 4 = 12 > 0.$

Also, $f_{xx} = 4 > 0.$

Hence, the given fxn $f(x, y, z)$ is min^m at $f(1/3, 1/3, 1/3)$.

$\therefore$ The minimum value of $x^2 + y^2 + z^2$ at $(1/3, 1/3, 1/3)$ is, $\left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^2$

$= \frac{1}{9} + \frac{1}{9} + \frac{1}{9}$

$= \frac{3}{9}$

$= \frac{1}{3}.$

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* Find the minimum value of $x^2 + y^2 + z^2$ subject to the condition, $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 1.$

Ans Solution:

Let $f(x, y, z) = x^2 + y^2 + z^2$ ..... (1)

under the condition, $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 1.$

or, $\frac{1}{z} = 1 - \frac{1}{x} - \frac{1}{y}.$

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Or, $z = \dfrac{1}{1 - \frac{1}{x} - \frac{1}{y}}$

$\therefore z = \dfrac{xy}{xy - y - x}$ ... (2)

Using eqn (2) in (1), the function reduces to,

$f(x,y) = x^2 + y^2 + \left(\dfrac{xy}{xy-y-x}\right)^2$ ... (3)

Diff. (3) partially we get,

$f_x = 2x + 2.\left(\dfrac{xy}{xy-y-x}\right) . \dfrac{(xy-y-x).y - xy.(y-1)}{(xy-y-x)^2}$

$= 2x + \dfrac{2xy}{(xy-y-x)} \times \dfrac{\cancel{xy^2} - y^2 - \cancel{xy} - \cancel{xy^2} + \cancel{xy}}{(xy-y-x)^2}$

$= 2x - \dfrac{2xy^3}{(xy-y-x)^3}$

$\therefore f_{xx} = 2 - \dfrac{(xy-y-x)^3 . 2y^3 - 2xy^3 . 3.(xy-y-x)^2.}{(xy-y-x)^6}$

$= 2 - \dfrac{2y^3(xy-y-x)^3 - 6xy^3(y-1)(xy-y-x)^2}{(xy-y-x)^6}$

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$$f_y = 2y + 2\cdot\left(\frac{xy}{xy-y-x}\right)\cdot\left(\frac{(xy-y-x)\cdot x - xy\cdot(x-1)}{(xy-y-x)^2}\right)$$

$$= 2y + \frac{2xy}{xy-x-y} \times \frac{\cancel{x^2y} - \cancel{xy} - x^2 - \cancel{x^2y} + \cancel{xy}}{(xy-y-x)^2}$$

$$\therefore f_y = 2y - \frac{2x^3y}{(xy-y-x)^3}$$

$$\therefore f_{yy} = 2 - 2x^3\left[\frac{(xy-y-x)^3\cdot 1 - y\cdot 3\cdot(xy-y-x)^2\cdot(x-1)}{(xy-y-x)^6}\right]$$

$$= 2 - 2x^3\left[\frac{(xy-y-x^3) - 3y(x-1)(xy-y-x)^2}{(xy-y-x)^6}\right]$$

$$\& \quad f_{xy} = 0 - 2y\cdot\left[\frac{(xy-y-x)^3\cdot 3x^2 - x^3\cdot 3\cdot(xy-y-x)^2\cdot(y-1)}{(xy-y-x)^6}\right]$$

$$= -2y\cdot\left[\frac{3x^2(xy-y-x)^3 - 3x^3(xy-y-x)^2(y-1)}{(xy-y-x)^6}\right]$$

-1). For stationary point,

$f_x = 0$ and, $f_y = 0$

i.e. $2x - \dfrac{2xy^3}{(xy-y-x)^3} = 0$ i.e. $2y - \dfrac{2x^3y}{(xy-y-x)^3} = 0$

or, $(xy-y-x)^3 - y^3 = 0$ or, $(xy-y-x)^3 - x^3 = 0$