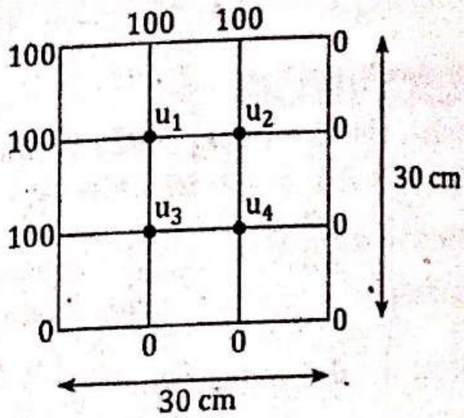
1. **The steady state two dimensional heat flow in a metal plate of size 30×30 cm is defined by $\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$. Two adjacent sides are placed at $100°$ C and other side at $0°$ C. Find the temperature at inner points, assuming the grid size of 10×10 cm.** **[2013/Fall]**

Solution:

The metal plate can be drawn as,



Given that;

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$$

Let the inner points be defined as u_1, u_2, u_3 and u_4. Now using standard five point formula. We have,

$$u_1 = \frac{1}{4}(100 + 100 + u_2 + u_3)$$
$$= \frac{1}{4}(200 + u_2 + u_3)$$
$$u_2 = \frac{1}{4}(0 + 100 + u_1 + u_4) = \frac{1}{4}(100 + u_1 + u_4)$$
$$u_3 = \frac{1}{4}(0 + 100 + u_1 + u_4) = \frac{1}{4}(100 + u_1 + u_4)$$
$$u_4 = \frac{1}{4}(0 + 0 + u_2 + u_3) = \frac{1}{4}(u_2 + u_3)$$

To obtain the values let initial values of

$u_1 = 0, u_2 = 0, u_3 = 0$ and $u_4 = 0$ then

Using gauss Siedal method of iteration in tabular form

Itn.	$u_1 = \frac{1}{4}(200 + u_2 + u_3)$	$u_2 = \frac{1}{4}(100 + u_1 + u_4)$	$u_3 = \frac{1}{4}(100 + u_1 + u_4)$	$u_4 = \frac{1}{4}(u_2 + u_3)$
1	$\frac{1}{4}(200 + 0 + 0) = 50$	$\frac{1}{4}(100 + 50 + 0) = 37.5$	$\frac{1}{4}(100 + 50 + 0) = 37.5$	$\frac{1}{4}(37.5 + 37.5) = 18.75$
2	68.75	46.875	46.875	23.437
3	73.437	49.218	49.218	24.409
4	74.609	49.804	49.804	24.902
5	74.902	49.951	49.951	24.975
6	74.975	49.987	49.987	24.993
7	74.993	49.996	49.996	24.998
8	74.998	49.999	49.999	24.9995
9	74.9995	49.9997	49.9997	24.9998

Here the values of u_1, u_2, u_3 and u_4 are correct up to 3 decimal places

Hence, the required temperature at inner points are;

$u_1 = 74.9995 \approx 75°C$

$u_2 = 49.9997 \approx 50°C$

$u_3 = 49.9997 \approx 50°C$

and, $u_4 = 24.9998 \approx 25°C$

NOTE:

Procedure to iterate in programmable calculator:

Let, $A = u_1$, $B = u_2$, $C = u_3$, $D = u_4$

Step 1: Set the following in calculator;

$$A = \frac{200 + B + C}{4} : B = \frac{100 + A + D}{4} : C = \frac{100 + A + D}{4} : D = \frac{B + C}{4}$$

Step 2: Press CALC then,

enter the value of B? then press =

enter the value of C? then press =

enter the value of D? then press =

Step 3: Now press = only, again and again to get the values for the respective row for each column.

Step 4: The values are updated automatically so continue pressing = till the required number of iterations.

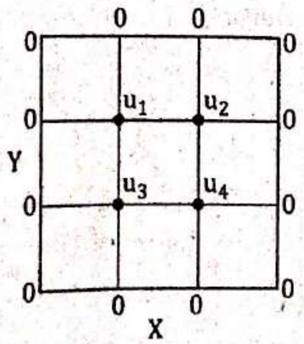
2. **Solve the Poisson equation $\nabla^2 f = 2x^2 y^2$ over the square domain $0 \le x \le 3$ and $0 \le y \le 3$ with $f = 0$ on the boundary and $h = 1$.**

[2013/Spring, 2014/Spring, 2018/Spring]

Solution:

Given that;

$$\nabla^2 f = 2x^2 y^2 \qquad \text{..... (1)}$$



Also the square domain of $0 \le x \le 3$ and $0 \le y \le 3$ with $f = 0$ on the boundary.

It is illustrated in figure as,

Here, let u_1, u_2, u_3 and u_4 be the initial nodes of Poisson equation and replacing $\nabla^2 f$ by difference equation with $x = ih$, $y = jk$ where, $(h = k = 1)$

Then, $u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1} - 4u_{i,j} = (2i^2 j^2)(1)^2$

For node u_1, put $i = 1, j = 2$

Or, $u_{0,2} + u_{2,2} + u_{1,1} + u_{1,3} - 4u_{1,2} = 2(1)^2(2)^2$

Or, $0 + u_2 + u_3 + 0 - 4u_1 = 8$

Or, $u_2 + u_3 - 4u_1 = 8$

$u_1 = \frac{1}{4}(u_2 + u_3 - 8)$

Likewise,

For node u_2, put $i = 2, j = 2$

$$u_{1,2} + u_{3,2} + u_{2,1} + u_{2,3} - 4u_{2,2} = 2\,(2)^2(2)^2$$

or, $u_1 + 0 + u_4 + 0 - 4u_2 = 32$

$$u_2 = \frac{1}{4}(u_1 + u_4 - 32) \qquad \text{---- (2)}$$

For node, u_4 put $i = 2, j = 1$

$$u_{1,1} + u_{3,1} + u_{2,0} + u_{2,2} - 4u_{2,1} = 2\,(2)^2(1)^2$$

or, $u_3 + 0 + 0 + u_2 - 4u_4 = 8$

or, $u_3 + u_2 - 4u_4 = 8$

or, $u_4 = \frac{1}{4}(u_3 + u_2 - 8)$

or, $u_4 = u_1$

Equation (2) becomes

$$u_2 = \frac{1}{4}(2u_1 - 32) = \frac{1}{2}(u_1 - 16)$$

For node u_3, put $i = 1, j = 1$

$$u_{0,1} + u_{2,1} + u_{1,0} + u_{1,2} - 4u_{1,1} = 2(1)^2(1)^2$$

or, $0 + u_4 + 0 + u_1 - 4u_3 = 2$

or, $u_3 = \frac{1}{4}(u_1 + u_4 - 2)$

or, $u_3 = \frac{1}{4}(2u_1 - 2)$

or, $u_3 = \frac{1}{2}(u_1 - 1)$

Now, let initial guess for u_1, u_2, u_3 and u_4 be 0. Then using Gauss Seidel method of iteration in tabular form.

Iteration	$u_1 = \frac{1}{4}(u_2 + u_3 - 8) = u_4$	$u_2 = \frac{1}{2}(u_1 - 16)$	$u_3 = \frac{1}{2}(u_1 - 1)$
1	$\frac{1}{4}(0 + 0 - 8) = -2$	$\frac{1}{2}(-2 - 16) = -9$	$\frac{1}{2}(-2 - 1) = -1.5$
2	-4.625	-10.312	-2.812
3	-5.281	-10.640	-3.140
4	-5.445	-10.722	-3.222
5	-5.486	-10.743	-3.243
6	-5.496	-10.748	-3.248
7	-5.499	-10.749	-3.249
8	-5.499	-10.749	-3.249

Hence the required values of nodes are

$$u_1 = u_4 = -5.499 \approx -5.5$$

$$u_2 = -10.749 \approx -10.75$$

$$u_3 = -3.249 \approx -3.25$$

NOTE:

Procedure to iterate in programmable calculator:

Let, $A = u_1 = u_4$, $B = u_2$, $C = u_3$

Set the following in calculator;

$$A = \frac{B + C - 8}{4} : B = \frac{A - 16}{2} : C = \frac{A - 1}{2}$$

Now press CALC and enter the initial value of B and C and continue pressing = only for the required number of iterations.

3. **Torsion on a square bar of size 9 cm × 9 cm subject to twisting is governed by $\nabla^2 u = -4$ with Dirichlet boundary condition of $u(x, y) = 0$ and $h = 1$. Calculate the steady state temperatures at interior points. Assume a grid size of 3 cm × 3 cm. Iterate until the minimum difference at any point is correct to two decimal places by applying Gauss point is correct to two decimal places by applying gauss Seidel method.** **[2014/Fall]**

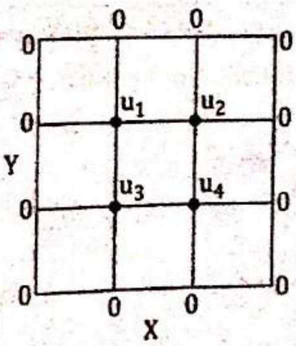
Solution:

Given that;

$$\nabla^2 u = -4 \qquad \text{..... (1)}$$

with $u(x, y) = 0$ and $h = 1$

Torsion on a square bar of size 9 cm × 9 cm with grid size of 3 cm × 3 cm It is illustrated in figure as:



Let u_1, u_2, u_3 and u_4 be the internal points of Poisson equation and replacing $\nabla^2 u$ by difference equation with $x = ih$, $y = jk$ where $(h = k = 1)$

Then, $u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1} - 4u_{i,j} = -4\,(1)^2$

For node u_1, put $i = 1, j = 2$

$$u_{0,2} + u_{2,2} + u_{1,1} + u_{1,3} - 4\,u_{1,2} = -4$$

or, $0 + u_2 + u_3 + 0 - 4u_1 = -4$

or, $u_2 + u_3 - 4u_1 = -4$

or, $u_1 = \frac{1}{4}(u_2 + u_3 + 4)$

For node u_4, put $i = 2, j = 1$

or, $u_{1,1} + u_{3,1} + u_{2,0} + u_{2,2} - 4\,u_{2,1} = -4$

or, $u_3 + 0 + 0 + u_2 - 4\,u_4 = -4$

or, $u_3 + u_2 - 4u_4 = -4$

or, $u_4 = \frac{1}{4}(u_3 + u_2 + 4)$

$u_4 = u_1$

For node u_2, put $i = 2, j = 2$

or, $u_{1,2} + u_{3,2} + u_{2,1} + u_{2,3} - 4u_{2,2} = -4$

or, $u_1 + 0 + u_4 + 0 - 4u_2 = -4$

$u_2 = \frac{1}{4}(u_1 + u_4 + 4)$

or, $u_2 = \frac{1}{2}(u_1 + 2)$ [$\because u_1 = u_4$]

For node u_3, put $i = 1$ $j = 1$

$$u_{0,1} + u_{2,1} + u_{1,0} + u_{1,2} - 4u_{1,1} = -4$$

or, $0 + u_4 + 0 + u_1 - 4u_3 = -4$

or, $u_3 = \frac{1}{4}(u_1 + u_4 + 4)$

or, $u_3 = \frac{1}{2}(u_1 + 2)$

or, $u_3 = u_2$

Now, let the initial guess for u_1, u_2, u_3 and u_4 be 0.

Then using Gauss Seidel method of iteration in tabular form,

Iteration	$u_1 = u_4 = \frac{1}{2}(u_2 + 2)$	$u_2 = u_3 = \frac{1}{2}(u_1 + 2)$
1	$\frac{1}{2}(0 + 2) = 1$	1.5
2	1.75	1.875
3	1.9375	1.9688
4	1.9844	1.9922
5	1.9961	1.9980

Here, the obtained values are correct up to two decimal places.

Hence the required steady state temperatures at interior points are

$$u_1 = u_2 = u_3 = u_4 = 1.99 \approx 2.$$

NOTE:

Procedure to iterate in programmable calculator:

Let, $A = u_1 = u_4$, $B = u_2 = u_3$

Set the following in calculator;

$$A = \frac{B+2}{2} : B = \frac{A+2}{2}$$

Now press CALC and enter the initial value of B and C and continue pressing = only for the required number of iterations.

4. Solve the Poisson equation $\nabla^2 f = (2 + x^2 y)$, over the square domain of $0 \le x \le 3$ and $0 \le y \le 3$ with $f = 0$ on the boundary and $h = 1$.

[2015/Fall]

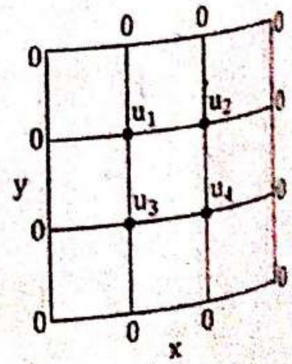
Solution:

Given that;

$$\nabla^2 f = 2 + x^2 y$$

Over the square domain of $0 \le x \le 3$ and $0 \le y \le 3$ with $f = 0$ on the boundary.

It is illustrated on the figure as:



Let u_1, u_2, u_3 and u_4 be the interior points and using Poisson formula with $x = ih$, $y = jk$ where $(h = k = 1)$

$$u_{i-1,j} + u_{i+1,j} + u_{i,j-1} - 4u_{i,j} = (2 + i^2 j)(1)^2$$

Now, for interior point u_1, put $i = 1, j = 2$

$$u_{0,2} + u_{2,2} + u_{1,1} + u_{1,3} - 4u_{1,2} = 2 + (1)^2 2$$

or, $0 + u_2 + u_3 + 0 - 4u_1 = 4$

or, $u_1 = \frac{1}{4}(u_2 + u_3 - 4)$

For interior point u_4, put $i = 2, j = 1$

$$u_{1,1} + u_{3,1} + u_{2,0} + u_{2,2} - 4u_{2,1} = 2 + (2)^2 \cdot 1$$

or, $u_3 + 0 + 0 + u_2 - 4u_4 = 6$

or, $u_4 = \frac{1}{4}(u_2 + u_3 - 6)$

For interior point u_2, put $i = 2, j = 2$

$$u_{1,2} + u_{3,2} + u_{2,1} + u_{2,3} - 4u_{2,2} = 2 + (2)^2 \cdot 2$$

or, $u_1 + 0 + u_4 + 0 - 4u_2 = 10$

or, $u_2 = \frac{1}{4}(u_1 + u_4 - 10)$

For interior point u_3, put $i = 1, j = 1$

$$u_{0,1} + u_{2,1} + u_{1,0} + u_{1,2} - 4u_{1,1} = 2 + (1)^2 \cdot 1$$

or, $0 + u_4 + 0 + u_1 - 4u_3 = 3$

or, $u_3 = \frac{1}{4}(u_1 + u_4 - 3)$

Now, let the initial guess for u_1, u_2, u_3 and u_4 be 0.

Then using Gauss Seidel method of iteration in tabular form,

Iteration	$u_1 = \frac{1}{4}(u_2 + u_3 - 4)$	$u_2 = \frac{1}{4}(u_1 + u_4 - 10)$	$u_3 = \frac{1}{4}(u_1 + u_4 - 3)$	$u_4 = \frac{1}{4}(u_2 + u_3 - 6)$
1	$\frac{1}{4}(0 - 4) = -1$	$\frac{1}{4}(-1 + 0 - 10) = -2.75$	$\frac{1}{4}(-1 + 0 - 3) = -1$	$\frac{1}{4}(-2.75 - 1 - 6) = -2.437$
2	-1.9375	-3.5936	-1.8436	-2.8593
3	-2.3593	-3.8047	-2.0547	-2.9648
4	-2.4649	-3.8574	-2.1074	-2.9912
5	-2.4912	-3.8706	-2.1206	-2.9978
6	-2.4978	-3.8739	-2.1239	-2.9995
7	-2.4995	-3.8747	-2.1248	-2.9999
8	-2.4999	-3.8749	-2.1250	-3.0000
9	-2.5000	-3.8750	-2.1250	-3.0000
10	-2.5000	-3.8750	-2.1250	-3.0000

Here, the obtained values are values are correct up to 4 decimal places.

Hence the required interior points are,

$u_1 = -2.5$

$u_2 = -3.875$

$u_3 = -2.125$

and, $u_4 = -3$

NOTE:

Procedure to iterate in programmable calculator:

Let, $A = u_1$, $B = u_2$, $C = u_3$, $D = u_4$

Set the following in calculator;

$$A = \frac{B + C - 4}{4} : B = \frac{A + D - 10}{4} : C = \frac{A + D - 3}{4} : D = \frac{B + C - 6}{4}$$

Now press CALC and enter the initial value of B, C and D and continue pressing = only for the required number of iterations.

5. Solve the Poisson equation $\nabla^2 f = 2x^2 + y$, over the square domain $1 \le x \le 3$, $1 \le, \le 3$ with $f = 1$ on the boundary. Take $h = k = 1$. [2015/Spring]

Solution:

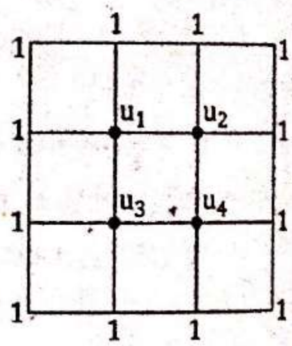
Given that;

$$\nabla^2 f = 2x^2 + y$$

Over the square domain $1 \le x \le 3$, $1 \le y \le 3$

With $f = 1$ on the boundary.

It is illustrated in figure as:



Let u_1, u_2, u_3 and u_4 be the interior points and using Poisson formula with $x = ih$, $y = jk$ where, $(h = k = 1)$

$$u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1} - 4u_{i,j} = (2i^2 + j)(1)^2$$

Now for interior point u_1, put $i = 1, j = 2$

$$u_{0,2} + u_{2,2} + u_{1,1} + u_{1,3} - 4u_{1,2} = 2(1)^2 + 2$$

or, $1 + u_2 + u_3 + 1 - 4u_1 = 4$

or, $u_2 + u_3 - 4u_1 = 2$

or, $u_1 = \frac{1}{2}(u_2 + u_3 - 2)$

For interior point u_2, put $i = 2, j = 2$

$$u_{1,2} + u_{3,2} + u_{2,1} + u_{2,3} - 4u_{2,2} = 2(2)^2 + 2$$

or, $u_1 + 1 + u_4 + 1 - 4u_2 = 10$

or, $u_2 = \frac{1}{4}(u_1 + u_4 - 8)$

For interior point u_3, put $i = 1, j = 1$

$$u_{0,1} + u_{2,1} + u_{1,0} + u_{1,2} - 4u_{1,1} = 2(1)^2 + 1$$

or, $1 + u_4 + 1 + u_1 - 4u_3 = 3$

or, $u_3 = \frac{1}{4}(u_1 + u_4 - 1)$

For interior point u_4, put $i = 2, j = 1$

$$u_{1,1} + u_{3,1} + u_{2,0} + u_{2,2} - 4u_{2,1} = 2(2)^2 + 1$$

or, $u_3 + 1 + 1 + u_2 - 4u_4 = 9$

or, $u_4 = \frac{1}{4}(u_2 + u_3 - 7)$

Now let the initial guess for u_1, u_2, u_3 and u_4 be 0.

Then using Gauss Seidel method of iteration in tabular form

Itn.	$u_1 = \frac{1}{4}(u_2+u_3-2)$	$u_2 = \frac{1}{4}(u_1+u_4-8)$	$u_3 = \frac{1}{4}(u_1+u_4-1)$	$u_4 = \frac{1}{4}(u_2+u_3-7)$
1	−0.5	−2.1250	−0.3750	−2.3750
2	−1.1250	−2.8750	−1.1250	−2.7500
3	−1.5000	−3.0625	−1.3125	−2.8438
4	−1.5938	−3.1094	−1.3594	−2.8672
5	−1.6172	−3.1211	−1.3711	−2.8731
6	−1.6231	−3.1240	−1.3741	−2.8745
7	−1.6245	−3.1248	−1.3748	−2.849
8	−1.6249	−3.1250	−1.3750	−2.8750
9	−1.6250	−3.1250	−1.3750	−2.8750

Here, the obtained values are correct up to 4 decimal places

Hence the required interior points are

$u_1 = -1.6250$

$u_2 = -3.1250$

$u_3 = -1.3750$

and, $u_4 = -2.8750$

NOTE:

Procedure to iterate in programmable calculator:

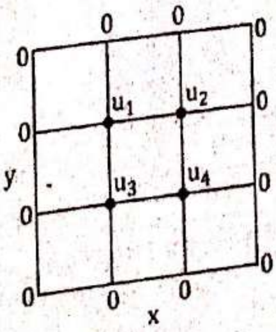
Let, $A = u_1$, $B = u_2$, $C = u_3$, $D = u_4$

Set the following in calculator;

$$A = \frac{B+C-2}{4} : B = \frac{A+D-8}{4} : C = \frac{A+D-1}{4} ; D = \frac{B+C-7}{4}$$

Now press CALC and enter the initial value of B and C and continue pressing = only for the required number of iterations.

6. **Given the Poisson's equation. $\Delta^2 f = -10(x^2 + y^2 + 10)$ over the square domain $0 \le x \le 3$ and $0 \le y \le 3$ with Dirichlet boundary condition of $f(x, y) = 0$ and $h = 1$. Calculate the steady state temperatures at the interior nodes by using Gauss Seidel method.** [2016/Fall, 2018/Fall]



Solution:

Given that;

$$\Delta^2 f = -10(x^2 + y^2 + 10)$$

Over the square domain $0 \le x \le 3$ and $0 \le y \le 3$

With Dirichlet boundary condition of $f(x, y) = 0$

It is illustrated in figure as:

Let, u_1, u_2, u_3, u_4 be the interior nodes and using Poisson formula with $x = ih$, $y = jk$ where $(h = k = 1)$

$$u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1} - 4u_{i,j} = -10(i^2 + j^2 + 10) \cdot (1)^2$$

Now, for interior node u_1, put i = 1, j = 2

$$u_{0,2} + u_{2,2} + u_{1,1} + u_{1,3} - 4\,u_{1,2} = -10\,((1)^2 + (2)^2 + 10)$$

or, $0 + u_2 + u_3 + 0 - 4u_1 = -150$

or, $u_1 = \frac{1}{4}(u_2 + u_3 + 150)$

For interior node u_2, put i = 2, j = 2

$$u_{1,2} + u_{3,2} + u_{2,1} + u_{2,3} - 4u_{2,2} = -10\,((2)^2 + (2)^2 + 10)$$

or, $u_1 + 0 + u_4 + 0 - 4u_2 = -180$

or, $u_1 = \frac{1}{4}(u_1 + u_4 + 180)$

For interior node u_3, put i = 1, j = 1

$$u_{0,1} + u_{2,1} + u_{1,0} + u_{1,2} - 4u_{1,1} = -10[(1)^2 + (1)^2 + 10]$$

or, $0 + u_4 + 0 + u_1 - 4u_3 = -120$

or, $u_3 = \frac{1}{4}(u_1 + u_4 + 120)$

For interior node u_4, put i = 2, j = 1

$$u_{1,1} + u_{3,1} + u_{2,0} + u_{2,2} - 4u_{2,1} = -10\,[(2)^2 + (1)^2 + 10]$$

or, $u_3 + 0 + 0 + u_2 - 4\,u_4 = -150$

or, $u_4 = \frac{1}{4}(u_2 + u_3 + 150)$

Here, $u_1 = u_4 = \frac{1}{4}(u_2 + u_3 + 150)$

so, $u_2 = \frac{1}{2}(u_1 + 90)$ and $u_3 = \frac{1}{2}(u_1 + 60)$

Now, let initial Guess for u_1, u_2, u_3 and u_4 be 0.

Now, solving the equations by the Gauss Seidel method,

Iteration	$u_1 = u_4 = \frac{1}{4}(u_2 + u_3 + 150)$	$u_2 = \frac{1}{2}(u_1 + 90)$	$u_3 = \frac{1}{2}(u_1 + 60)$
1	37.5	63.75	48.75
2	65.625	77.8125	62.8125
3	72.6563	81.3281	66.3282
4	74.4141	82.2070	67.2071
5	74.8535	82.4268	67.4268
6	74.9634	82.4817	67.4817
7	74.9909	82.4954	67.4955
8	74.9977	82.4989	67.4989
9	74.9994	82.4997	67.4997
10	74.9999	82.4999	67.4999

Hence the required steady state temperatures at the interior nodes are

$u_1 = u_4 = 75$

$u_2 = 82.5$

and, $u_3 = 67.5$

NOTE:

Procedure to iterate in programmable calculator:

Let, $A = u_1 = u_4$, $B = u_2$, $C = u_3$

Set the following in calculator;

$$A = \frac{B + C + 150}{4} : B = \frac{A + 90}{2} : C = \frac{A + 60}{2}$$

Now press CALC and enter the initial value of B and C and continue pressing = only for the required number of iterations.

7. **Solve the parabolic equation $2f_{xx}(x, t) = f_t(x, t)$, $0 \le t \le 1.5$ and given initial condition $f(x, 0) = 50(4 - x)$, $0 \le x \le 4$ with boundary condition $f(0, t) = 0 = f(4, t)$ $0 \le t \le 1.5$.** **[2017/Fall]**

Solution:

Given that;

$$f_t(x, t) = 2f_{xx}(x, t)$$

We have the parabolic equation,

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$

where, c^2 is the diffusivity of the substance

$$c^2 = 2$$

Let, $h = 1 \rightarrow$ Spacing along x-direction, $0 \le x \le 4$

Let, $k = 0.5 \rightarrow$ Spacing along time, t-direction, $0 \le t \le 1.5$

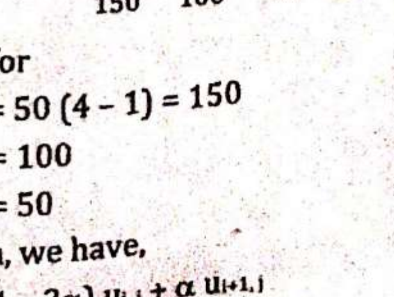
Now, solving the parabolic equation using Schmidt method.

We have,

$$\alpha = \frac{kc^2}{h^2} = \frac{0.5 \times 2}{1^2} = 1$$

Here, α lies between $0 < \alpha \le 12$ which satisfies the condition

The figure is illustrated as shown for $f(0, t) = 0 = f(4, t)$



Here, boundary values for

$$u_{1,0} = 50(4 - x) = 50(4 - 1) = 150$$

$$u_{2,0} = 50(4 - 2) = 100$$

$$u_{3,0} = 50(4 - 3) = 50$$

From Schmidt's formula, we have,

$$u_{i,j+1} = \alpha\, u_{i-1,j} + (1 - 2\alpha)\, u_{i,j} + \alpha\, u_{i+1,j}$$

Substituting the value of $\alpha = 1$

$$u_{i,j+1} = u_{i-1,j} - u_{i,j} + u_{i+1,j}$$

Now, for i = 1, 2, 3 and j = 0

$$u_{1,1} = [u_{0,0} - u_{1,0} + u_{2,0}] = 0 - 150 + 100 = -50$$
$$u_{2,1} = [u_{1,0} - u_{2,0} + u_{3,0}] = 150 - 100 + 50 = 100$$
$$u_{3,1} = [u_{2,0} - u_{3,0} + u_{4,0}] = 100 - 50 + 0 = 50$$

For i = 1, 2, 3 and j = 1

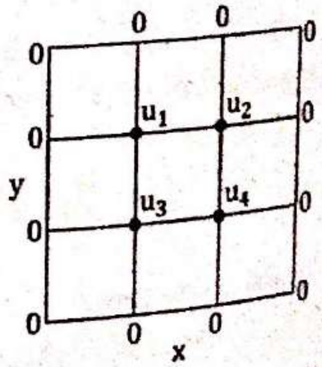
$$u_{1,2} = [u_{0,1} - u_{1,1} + u_{2,1}] = 0 + 50 + 100 = 150$$
$$u_{2,2} = [u_{1,1} - u_{2,1} + u_{3,1}] = -50 - 100 + 50 = -100$$
$$u_{3,2} = [u_{2,1} - u_{3,1} + u_{4,1}] = 100 - 50 + 0 = 50$$

For i = 1, 2, 3 and j = 2

$$u_{1,3} = [u_{0,2} - u_{1,2} + u_{2,2}] = 0 - 150 + (-100) = -250$$
$$u_{2,3} = [u_{1,2} - u_{2,2} + u_{3,2}] = 150 + 100 + 50 = 300$$
$$u_{3,3} = [u_{2,2} - u_{3,2} + u_{4,2}] = -100 - 50 + 0 = -150$$

8. Given the Poisson's equation $\nabla^2 u = -10\,(x^2 + y^2 + 10)$ over the square domain such that $0 \le x \le 3$ and $0 \le y \le 3$ with Dirichlet boundary condition of $u(x, y) = 0$. Calculate the steady state temperature at interior points by suing successive over relaxation method up to 5th iteration. Assume h = k = 1. **[2017/Spring]**

Solution:



Given that;

$$\nabla^2 u = -10\,(x^2 + y^2 + 10)$$

Over the square domain; $0 \le x \le 3$ and $0 \le y \le 3$

With Dirichlet boundary condition of $u(x, y) = 0$

It is illustrated in figure as:

Let u_1, u_2, u_3 and u_4 be the interior points and using Poisson formula with x = ih, y = jk where (h = k = 1)

$$u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1} - 4u_{i,j} = -10\,(i^2 + j^2 + 10)\cdot(1)^2$$

Now, for interior point u_1, put i = 1, j = 2

$$u_{0,2} + u_{2,2} + u_{1,1} + u_{1,3} - 4u_{1,2} = -10\,[(1)^2 + (2)^2 + 10]$$

or, $\quad 0 + u_2 + u_3 + 0 - 4u_1 = -150$

or, $\quad u_1 = \frac{1}{4}(u_2 + u_3 + 150)$

For interior node u_2, put i = 2, j = 2

$$u_{1,2} + u_{3,2} + u_{2,1} + u_{2,3} - 4u_{2,2} = -10\,[(2)^2 + (2)^2 + 10]$$

or, $\quad u_1 + 0 + u_4 + 0 - 4u_2 = -180$

or, $\quad u_2 = \frac{1}{4}(u_1 + u_4 + 180)$

For interior node u_3, put i = 1, j = 1

$$u_{0,1} + u_{2,1} + u_{1,0} + u_{1,2} - 4\,u_{1,1} = -10\,[(1)^2 + (1)^2 + 10]$$

$0 + u_4 + 0 + u_1 - 4u_3 = -120$

or, $u_3 = \frac{1}{4}(u_1 + u_4 + 120)$

For interior node u_4, put $i = 2, j = 1$

$u_{1,1} + u_{3,1} + u_{2,0} + u_{2,2} - 4u_{2,1} = -10[(2)^2 + (1)^2 + 10]$

or, $u_3 + 0 + 0 + u_2 - 4u_4 = -150$

or, $u_4 = \frac{1}{4}(u_2 + u_3 + 150)$

Here, $u_1 = u_4 = \frac{1}{4}(u_2 + u_3 + 150)$

so, $u_2 = \frac{1}{2}(u_1 + 90)$ and $u_3 = \frac{1}{2}(u_1 + 60)$

Now, using successive over relaxation method

We have,

$x_i^{n+1} = (1 - w)x_i^n + w$ [Gauss Seidel iteration]

Here, w is relaxation parameter which value lies from $0 < w < 2$ for convergence reason.

Lets choose $w = 1.25$

$x_i^{n+1} = -0.25x_i^n + 1.25$ [Gauss Seidel iteration]

Now, the equations are formed as

$$u_1^{n+1} = -0.25\,u_1^n + \frac{1.25}{4}(u_2^n + u_3^n + 150)$$

$$u_2^{n+1} = -0.25\,u_2^n + \frac{1.25}{4}(u_1^{n+1} + u_4^n + 180)$$

$$u_3^{n+1} = -0.25\,u_3^n + \frac{1.25}{4}(u_1^{n+1} + u_4^n + 120)$$

$$u_4^{n+1} = -0.25\,u_4^n + \frac{1.25}{4}(u_2^{n+1} + u_3^{n+1} + 150)$$

Here, $u_1^{n+1} = u_4^{n+1}$

Then, $u_1^{n+1} = -0.25\,u_1^n + 0.3125\,(u_2^n + u_3^n + 150) = u_4^{n+1}$

$u_2^{n+1} = -0.25\,u_2^n + 0.3125\,(u_1^{n+1} + u_4^n + 180)$

$u_3^{n+1} = -0.25\,u_3^n + 0.3125\,(u_1^{n+1} + u_4^n + 120)$

Let the initial guess for u_1, u_2, u_3 and u_4 be 0.

Now, 1st iteration,

For $n = 0$

$u_4^1 = u_1^{(1)} = -0.25\,u_1^0 + 0.3125\,(u_2^0 + u_3^0 + 150)$

$= -0.25 \times 0 + 0.3125\,(0 + 0 + 150)$

$= 46.875$

$u_2^1 = -0.25\,u_2^0 + 0.3125\,(u_1^1 + u_4^0 + 180)$

$= 0 + 0.3125\,(46.875 + 0 + 180)$

$= 70.8984$

$$u_3^1 = -0.25\, u_3^0 + 0.3125\, (u_1^1 + u_4^0 + 120)$$
$$= 0 + 0.3125\, (46.875 + 0 + 120)$$
$$= 52.1484$$

Likewise,

2nd iteration, n = 1

$$u_1^2 = -0.25\, u_1^1 + 0.3125\, (u_1^2 + u_3^1 + 150)$$
$$= 73.6084$$
$$u_2^2 = -0.25\, u_2^1 + 0.3125\, (u_1^2 + u_4^1 + 180)$$
$$= 76.1765$$
$$u_3^2 = -0.25\, u_3^1 + 0.3125\, (u_1^2 + u_4^1 + 120)$$
$$= 62.1140$$
$$u_4^2 = u_1^2 = 73.6084$$

3rd iteration, n = 2

$$u_1^3 = -0.25\, u_1^2 + 0.3125\, (u_2^2 + u_3^2 + 150)$$
$$= 71.6887$$
$$u_2^3 = -0.25\, u_2^2 + 0.3125\, (u_1^3 + u_4^2 ++ 180)$$
$$= 82.6112$$
$$u_3^3 = -0.25\, u_3^2 + 0.3125\, (u_1^3 + u_4^2 + 120)$$
$$= 67.3768$$
$$u_4^3 = u_1^3 = 71.6887$$

4th iteration

n = 3 then

$$u_1^4 = -0.25\, u_1^3 + 0.3125\, (u_2^3 + u_3^3 + 150)$$
$$= 75.8241$$
$$u_2^4 = -0.25\, u_2^3 + 0.3125\, (u_1^4 + u_4^3 + 180)$$
$$= 81.6950$$
$$u_3^4 = -0.25\, u_3^3 + 0.3125\, (u_1^4 + u_4^3 + 120)$$
$$= 66.7536$$
$$u_4^4 = u_1^4 = 75.8241$$

5th iteration, n = 4

$$u_1^5 = -0.25\, u_1^4 + 0.3125\, (u_2^4 + u_3^4 + 150)$$
$$= 74.3092$$
$$u_2^5 = -0.25\, u_2^4 + 0.3125\, (u_1^5 + u_4^4 + 180)$$
$$= 82.7429$$
$$u_3^5 = -0.25\, u_3^4 + 0.3125\, (u_1^5 + u_4^4 + 120)$$
$$= 67.7283$$
$$u_4^5 = u_1^5 = 74.3092$$

Hence the required steady state temperature at interior points are

$u_1 = u_4 = 74.3092$

$u_2 = 82.7429$

$u_3 = 67.7283$

NOTE:
Procedure to iterate in programmable calculator:
Let, $A = u_1 = u_4$, $B = u_2$, $C = u_3$
Set the following in calculator;

$X = -0.25A + 0.3125(150 + B + C) : Y = -0.25B + 0.3125(180 + X + A) :$
$M = -0.25C + 0.3125(120 + X + A)$

Press CALC and enter the initial value of A, B and C and continue pressing = only for the required row for each column.
Update the values of A?, B? and C? when asked again.

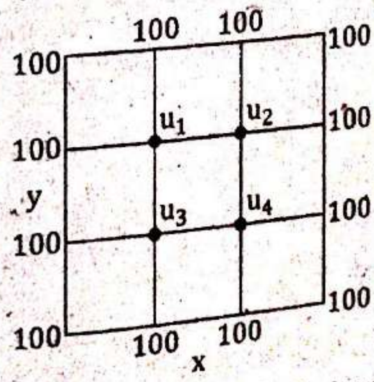
9. **Given the Poisson's equation $\Delta^2 f = 4x^2 y^2$ over the square domain $0 \leq x \leq 3$ and $0 \leq y \leq 3$ with Dirichlet boundary condition of $f(x, y) = 100$ and $h = k = 1$. Calculate the steady state temperatures at the interior nodes by using Gauss Seidel method. Iterate until the successive values at any point is correct to two decimal places. [2019/Fall]**

Solution:

Given that;

$$\Delta^2 f = 4x^2 y^2$$

Over the square domain $0 \leq x \leq 3$ and $0 \leq y \leq 3$
With Dirichlet boundary condition of $f(x, y) = 100$
It is illustrated in figure as,



Let u_1, u_2, u_3 and u_4 be the interior nodes of Poisson's equation with $x = ih$, $y = jk$ where $h = k = 1$

Then, $u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1} - 4u_{i,j} = (4\, i^2 j^2)\,(1)^2$

For node u_1, put $i = 1, j = 2$

Or, $u_{0,2} + u_{2,2} + u_{1,1} + u_{1,3} - 4u_{1,2} = 4(1)^2\,(2)^2$

Or, $100 + u_2 + u_3 + 100 - 4u_1 = 16$

$$u_1 = \frac{1}{4}(u_2 + u_3 + 184)$$

For node u_2, put $i = 2, j = 2$

$u_{1,2} + u_{3,2} + u_{2,1} + u_{2,3} - 4u_{2,2} = 4\,(2)^2\,(2)^2$

or, $u_1 + 100 + u_4 + 100 - 4u_2 = 64$

or, $u_2 = \frac{1}{4}(u_1 + u_4 + 136)$

For node u_3, put $i = 1, j = 1$

$u_{0,1} + u_{2,1} + u_{1,0} + u_{1,2} - 4u_{1,1} = 4(1)^2(1)^2$

or, $100 + u_4 + 100 + u_1 - 4u_3 = 4$

or, $u_3 = \frac{1}{4}(u_1 + u_4 + 196)$

For node u_4, put $i = 2, j = 1$

$u_{1,1} + u_{3,1} + u_{2,0} + u_{2,2} - 4u_{2,1} = 4(2)^2(1)^2$

or, $u_3 + 100 + 100 + u_2 - 4u_4 = 16$

or, $u_4 = \frac{1}{4}(u_2 + u_3 + 184)$

Here, $u_1 = u_4 = \frac{1}{4}(u_2 + u_3 + 184)$ then,

$u_2 = \frac{1}{2}(u_1 + 68)$

$u_3 = \frac{1}{2}(u_1 + 98)$

Let the initial guess for u_1, u_2, u_3, u_4 be zero.

Now, using Gauss Seidel method in tabular form,

Iteration	$u_1 = u_4 = \frac{1}{4}(u_2 + u_3 + 184)$	$u_2 = \frac{1}{2}(u_1 + 68)$	$u_3 = \frac{1}{2}(u_1 + 98)$
1	46	57	72
2	78.25	73.125	88.125
3	86.3125	77.1563	92.1563
4	88.3281	78.1641	93.1641
5	88.8320	78.4160	934160
6	88.9580	78.4790	93.4790
7	88.9895	78.4948	93.4948
8	88.9974	78.4987	93.4987

Hence the required values of temperatures at interior nodes are

$u_1 = u_4 = 88.9974$

$u_2 = 78.4987$

and, $u_3 = 93.4987$

NOTE:

Procedure to iterate in programmable calculator:

Let, $A = u_1 = u_4$, $B = u_2$, $C = u_3$

Set the following in calculator;

$$A = \frac{B + C + 184}{4} : B = \frac{A + 68}{2} : C = \frac{A + 98}{2}$$

Now press CALC and enter the initial value of B and C and continue pressing = only for the required number of iterations.

Solve the Poisson's equation $u_{xx} + u_{yy} = 243(x^2 + y^2)$ over a square domain $0 \le x \le 1, 0 \le y \le 1$ with step size $h = \frac{1}{3}$ with $u = 100$ on the boundary.

[20*?/Spring]

Solution:

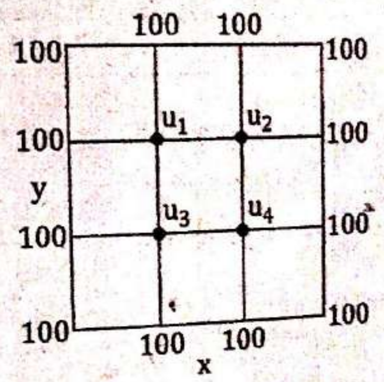
Given that;

$$u_{xx} + u_{yy} = 243(x^2 + y^2)$$

Over a square domain $0 \le x \le 1, 0 \le y \le 1$

With $u = 100$ on the boundary.

It is illustrated in the figure as,



Let u_1, u_2, u_3 and u_4 be the interior nodes of Poisson's equation and replacing $u_{xx} + u_{yy}$ by difference equation with $x = ih, y = jk$ where $h = k = \frac{1}{3}$.

Then, $u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1} - 4u_{i,j} = h^2 f(ih, jk)$

or, $u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1} - 4u_{i,j} = \frac{1}{9} \times 243\left(\frac{i^2}{9} + \frac{j^2}{9}\right)$

or, $u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1} - 4u_{i,j} = 3(i^2 + j^2)$

Now, for node u_1, put $i = 1, j = 2$

$$u_{0,2} + u_{2,2} + u_{1,1} + u_{1,3} - 4u_{1,2} = 3(1^2 + 2^2)$$

or, $100 + u_2 + u_3 + 100 - 4u_1 = 15$

or, $u_1 = \frac{1}{4}(u_2 + u_3 + 185)$

For node u_2, put $i = 2, j = 2$

$$u_{1,2} + u_{3,2} + u_{2,1} + u_{2,3} - 4u_{2,2} = 3[(2)^2 + (2)^2]$$

or, $u_1 + 100 + u_4 + 100 - 4u_2 = 24$

or, $u_2 = \frac{1}{4}(u_1 + u_4 + 176)$

For node u_3, put $i = 1, j = 1$

$$u_{0,1}\ u_{2,1} + u_{1,0} + u_{1,2} - 4u_{1,1} = 3[(1)^2 + (1)^2]$$

or, $100 + u_4 + 100 + u_1 - 4u_3 = 6$

or, $u_3 = \frac{1}{4}(u_1 + u_4 + 194)$

For node u_4, put $i = 2, j = 1$

$$u_{1,1} + u_{3,1} + u_{2,0} + u_{2,2} - 4u_{2,1} = 3[(2)^2 + (1)^2]$$

or, $u_2 + 100 + 100 + u_3 - 4u_4 = 15$

or, $u_4 = \frac{1}{4}(u_2 + u_3 + 185)$

Here, $u_1 = u_4 = \frac{1}{4}(u_2 + u_3 + 185)$ and

$$u_2 = \frac{1}{2}(u_1 + 88),\ u_3 = \frac{1}{2}(u_1 + 97)$$

Let the initial guess for u_1, u_2, u_3 and u_4 be zero.

Now, using Gauss Seidel method in tabular form,

Iteration	$u_1 = u_4 = \frac{1}{4}(u_2 + u_3 + 185)$	$u_2 = \frac{1}{2}(u_1 + 88)$	$u_3 = \frac{1}{2}(u_1 + 97)$
1	46.250	67.125	71.625
2	80.938	84.469	88.969
3	89.610	88.805	93.305
4	91.778	89.889	94.389
5	92.320	90.160	94.660
6	92.455	90.228	94.728
7	92.489	90.244	94.745
8	92.497	90.249	94.749
9	92.499	90.250	94.750
10	92.500	90.250	94.750

Hence the required values of interior points are,

$$u_1 = u_4 = 92.5,\ u_2 = 90.25 \text{ and } u_3 = 94.75$$

NOTE:

Procedure to iterate in programmable calculator:

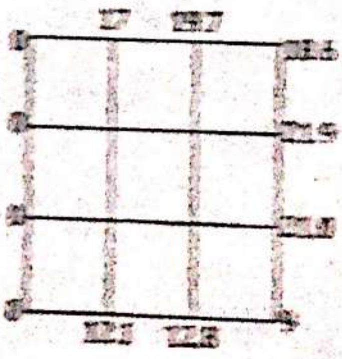
Let, $A = u_1 = u_4$, $B = u_2$, $C = u_3$

Set the following in calculator,

$$A = \frac{B + C + 185}{4} : B = \frac{A + 88}{2} : C = \frac{A + 97}{2}$$

Now press CALC and enter the initial value of B and C and continue pressing = only for the required number of iterations.

11. Solve the Poisson equation $\nabla^2 f = 4x^2 y + 2xy^2$, over the square domain $x \le 3$, $1 \le y \le 3$, with f on the boundary is given in figure below. Take $h = k = 1$. [2070 Fall]

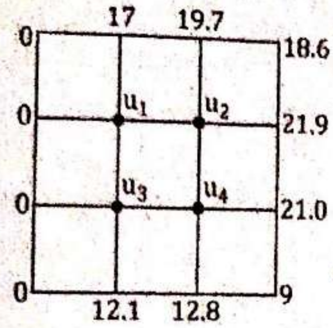


Solution:

Given that;

$$\nabla^2 f = 4x^2y + 3xy^2$$

Over the square domain $x \le 3$, $1 \le y \le 3$ with f on the boundary



Let u_1, u_2, u_3 and u_4 be the interior nodes of Poisson's equation and replacing $\nabla^2 f$ by difference equation with $x = ih$, $y = jk$ where, $(h = k = 1)$

Then,

$$u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1} - 4u_{i,j} = (1)^2\,(4i^2j + 3ij^2)$$

or, $\quad u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1} - 4u_{i,j} = 4i^2j + 3ij^2$

Now, for node, u_1, put $i = 1, j = 2$

$$u_{0,2} + u_{2,2} + u_{1,1} + u_{1,3} - 4u_{1,2} = 4(1)^2(2) + 3(1)(2)^2$$

or, $\quad 0 + u_2 + u_3 + 17 - 4u_1 = 20$

or, $\quad u_1 = \frac{1}{4}(u_2 + u_3 - 3)$

Now for node u_2, put $i = 2, j = 2$

$$u_{1,2} + u_{3,2} + u_{2,1} + u_{2,3} - 4u_{2,2} = 4(2)^2(2) + 3(2)(2)^2$$

or, $\quad u_1 + 21.9 + u_4 + 19.7 - 4u_2 = 56$

or, $\quad u_2 = \frac{1}{4}(u_1 + u_4 - 14.4)$

For node u_3, put $i = 1, j = 1$

$$u_{0,1} + u_{2,1} + u_{1,0} + u_{1,2} - 4u_{1,1} = 4(1)^2(1) + 3(1)(1)^2$$

or, $\quad 0 + u_4 + 12.1 + u_1 - 4u_3 = 7$

or, $\quad u_3 = \frac{1}{4}(u_1 + u_4 + 5.7)$

For node u_4, put $i = 2, j = 1$

$$u_{1,1} + u_{3,1} + u_{2,0} + u_{2,2} - 4u_{2,1} = 4(2)^2(1) + 3(2)(1)^2$$

or, $\quad u_3 + 21 + 12.8 + u_2 - 4u_4 = 22$

or, $\quad u_4 = \frac{1}{4}(u_2 + u_3 + 11.8)$

Let the initial guess for u_1, u_2, u_3 and u_4 be zero.

Now, using Gauss Seidel method in tabular form,

Itn.	$u_1 = \frac{1}{4}(u_2 + u_3 - 3)$	$u_2 = \frac{1}{4}(u_1 + u_4 - 14.4)$	$u_3 = \frac{1}{4}(u_1 + u_4 + 5.7)$	$u_4 = \frac{1}{4}(u_2 + u_3 + 11.8)$
1	-0.75	-3.788	1.238	2.312
2	-1.388	-3.369	1.656	2.522
3	-1.178	-3.264	1.761	2.574
4	-1.126	-3.238	1.787	2.587
5	-1.113	-3.231	1.794	2.591
6	-1.109	-3.230	1.796	2.592
7	-1.109	-3.229	1.796	2.592
8	-1.108	-3.229	1.796	2.592

Hence the required values of interior points are

$u_1 = -1.108$

$u_2 = -3.229$

$u_3 = 1.796$

and, $u_4 = 2.592$

NOTE:
Procedure to iterate in programmable calculator:
Let, $A = u_1, B = u_2, C = u_3, D = u_4$
Set the following in calculator;

$$A = \frac{B + C - 3}{4} : B = \frac{A + D - 14.4}{4} : C = \frac{A + D + 5.7}{4} : D = \frac{B + C + 11.8}{4}$$

Now press CALC and enter the initial value of B and C and continue pressing = only for the required number of iterations.

12. Write short notes on: Laplacian equation.
[2013/Fall, 2013/Spring, 2016/Fall, 2016/Spring]

Solution: See the topic 6.2 'C'.

13. Write short notes on; Hyperbolic equations. **[2015/Spring]**

Solution: See the topic 6.2. 'F'.

14. Write short notes on: Laplace method for partial differential.
[2017/Fall, 2018/Fall]

Solution: See the topic 6.2 'C'.

15. Write short notes on: Parabolic equation. **[2017/Spring]**

Solution: See the topic 6.2.'E'.

16. Write short notes on Elliptical equations. **[2017/Spring]**

Solution: See the topic 6.2. 'B'.