BOARD EXAMINATION SOLVED QUESTIONS

1. **Find the positive root of the equation $f(x) = \cos x - 3x + 1$ correct upto 3 decimal places using Bisection method.** **[2013/Fall]**

Solution:

$f(x) = \cos x - 3x + 1$

Let initial guess be

$x = 0,\quad f(0) = \cos(0) - 3 \times 0 + 1 = 2 > 0$

$x = 1,\quad f(1) = \cos(1) - 3(1) + 1 = -1.4596 < 0$

So root lies between x = 0 and x = 1

$\therefore\quad x_L = 0$ and $x_U = 1$

Now, first approximated root using bisection method

$$x_N = \frac{x_L + x_U}{2} = \frac{0+1}{2} = 0.5$$

$f(x_N) = 0.3775 > 0$, so now root lies between 0.5 and 1

Remaining iterations are solved in tabular form

Iteration	x_L	$f(x_L) = \cos x_L - 3x_L + 1$	x_U	$f(x_U) = \cos x_U - 3x_U + 1$	x_N	$f(x_N) = \cos x_N - 3x_N + 1$
1	0	2	1	-1.4596	0.5	0.3775
2	0.5	0.3775	1	-1.4596	0.75	-0.5183
3	0.5	0.3775	0.75	-0.5183	0.625	-0.0640
4	0.5	0.3775	0.625	-0.0640	0.5625	0.1584
5	0.5625	0.1584	0.625	-0.0640	0.5937	0.0477
6	0.5937	0.0477	0.625	-0.0640	0.6093	-7.85×10^{-3}
7	0.5937	0.0477	0.6093	-7.85×10^{-3}	0.6015	0.0199
8	0.6015	0.0199	0.6093	-7.85×10^{-3}	0.6054	6.07×10^{-3}
9	0.6054	6.07×10^{-3}	0.6093	-7.85×10^{-3}	0.6073	-7.08×10^{-4}
10	0.6054	6.07×10^{-3}	0.6073	-7.08×10^{-4}	0.6063	2.86×10^{-3}
11	0.6063	2.86×10^{-3}	0.6073	-7.08×10^{-4}	0.6068	1.07×10^{-3}

Here, the value of x_N do not change upto 3 decimal places.

Hence, the positive root of the equation is 0.6068.

NOTE:

Procedure to iterate in programmable calculator:

Let, $A = x_L$, $B = x_U$, $C = x_N$, $D = f(x_L)$, $E = f(x_U)$, $F = f(x_N)$

Step 1: Set the calculator in radian mode.

Step 2: Set the following in calculator as shown:

$$A : B : C = \frac{A+B}{2} : D = \cos A - 3A + 1 : E = \cos B - 3B + 1 :$$

$$F = \cos C - 3C + 1$$

Step 3: Press CALC then,

Enter the value of A? then press =

Enter the value of B? then press =

Step 4: Now press = only, again and again to get the values for the respective row for each column.

Step 5: Update the values when A? and B? is asked again.

Step 6: Go to step 4.

2. Calculate the root of non-linear equation 3x = cos x + 1 using secant method. [2013/Fall]

Solution:

Let, $f(x) = 3x - \cos x - 1$

$x_0 = 0$ and $x_1 = 1$ be two initial guesses

$f(x_0) = -2$ and $f(x_1) = 1.4596$

Then, next approximated root by secant method is given by

$$x_2 = x_1 - \frac{f(x_1)\,(x_1 - x_0)}{f(x_1) - f(x_0)} = 1 - \frac{1.4596\,(1 - 0)}{1.4596 - (-2)} = 0.5781$$

$f(x) = -0.1032$ and now root lies between 1 and 0.5781.

Now, solving other iterations in tabular form as follows,

Iteration	x_{n-1}	$f(x_{n-1}) = 3x_{n-1} - \cos x_{n-1} - 1$	x_n	$f(x_n) = 3x_n - \cos x_n - 1$	$x_{n+1} = x_n - \frac{f(x_n)\,(x_n - x_{n-1})}{f(x_n) - f(x_{n-1})}$	$f(x_{n+1}) = 3x_{n+1} - \cos x_{n+1} - 1$
1	0	-2	1	1.4596	0.5781	-0.1032
2	1	1.4596	0.5781	-0.1032	0.6059	-4.28×10^{-3}
3	0.5781	-0.1032	0.6059	-4.28×10^{-3}	0.6071	-588×10^{-6}
4	0.6059	-4.28×10^{-3}	0.6071	-5.88×10^{-6}	0.6071	5.73×10^{-9}

Here, the value of x_{n+1} do not change up to 4 decimal places. Hence, the root of given non-linear equation is 0.6071.

NOTE:

Procedure to iterate in programmable calculator:

Let, $A = x_{n-1}$, $B = x_n$, $C = x_{n+1}$, $D = f(x_{n-1})$, $E = f(x_n)$, $F = f(x_{n+1})$

Step 1: Set the calculator in radian mode.

Step 2: Set the following in calculator as shown;

$$A : B : D = 3A - \cos A - 1 : E = 3B - \cos B - 1 : C = B - \frac{E(B - A)}{E - D} :$$

$$F = 3C - \cos C - 1$$

Step 3: Press CALC then,

Enter the value of A? then press =

Enter the value of B? then press =

Step 4: Now press = only, again and again to get the values for the respective row for each column.

Step 5: Update the values when A? and B? is asked again.

Step 6: Go to step 4.

3. Find a real roof of the equation $x \log_{10} x = 1.2$ by using Newton Raphson (NR) method such that the root must have error less than 0.0001%. [2013/Fall, 2018/Fall]

Solution:

Let, $f(x) = x \log_{10} x - 1.2$ ---- (1)

Differentiating equation (1) with respect to x.

$$f'(x) = 1 + \log_{10} x \quad \text{---- (2)}$$

From equation (1).

Let the initial guess be

$$x_0 = 1,\ f(x_0) = -1.2,\ f'(x_0) = 1$$

Using Newton Raphson method, next approximated root is

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 1 - \frac{(-1.2)}{1} = 2.2$$

$$f(x_1) = -0.4466$$

Now, continuing process in tabular form

Iteration	x_n	$f(x_n) = x_n \log_{10} x_n - 1.2$	$f'(x_n) = 1 + \log_{10} x_n$	$f(x_{n+1}) = x_n - \frac{f(x_n)}{f'(x_n)}$	$f(x_{n+1}) = x_{n+1} \log_{10} x_{n+1} - 1.2$
1	1	−1.2	1	2.2	−0.4466
2	2.2	−0.4466	1.3424	2.5326	−0.1779
3	2.5326	−0.1779	1.4035	2.6593	−0.0704
4	2.6593	−0.0704	1.4247	2.7087	−0.0277
5	2.7087	−0.0277	1.4327	2.7280	−0.0110
6	2.7280	−0.0110	1.4358	2.7356	-4.39×10^{-3}
7	2.756	-4.39×10^{-3}	1.4370	2.7386	-1.78×10^{-3}
8	2.7386	-1.78×10^{-3}	1.4375	2.7398	-7.37×10^{-4}
9	2.7398	-7.37×10^{-4}	1.4377	2.7403	-3.01×10^{-4}
10	2.7403	-3.01×10^{-4}	1.4377	2.7405	-1.27×10^{-4}
11	2.7405	-1.27×10^{-4}	1.4378	2.7405	-5.03×10^{-5}

Here, the value of x_{n+1} do not change up to 4 decimal places and have error less than 0.0001%. Hence, required root is 2.7405.

NOTE:

Procedure to iterate in programmable calculator:

Let, $A = x_n$, $B = f(x_n)$, $C = f'(x_n)$, $D = x_{n+1}$, $E = f(x_{n+1})$

Step 1: Set the calculator in radian mode.

Step 2: Set the following in calculator as shown;

$$A : B : A \log_{10} A - 1.2 : C = 1 + \log_{10} A : D = A - \frac{B}{C} : E = D \log_{10} D - 1.2$$

Step 3: Press CALC then,

Enter the value of A? then press =

Step 4: Now press = only, again and again to get the values for the respective row for each column.

Step 5: Update the values when A? is asked again.

Step 6: Go to step 4.

4. Solve $f(x) = 3x + \sin x - e^x$ by secant method up to 5th iteration.

[2013/Spring, 2017/Fall]

Solution:

$f(x) = 3x + \sin x - e^x$

Let, $x_0 = 0$ and $x_1 = 1$ be two initial guesses.

$f(x_0) = -1$ and $f(x_1) = 1.1231$

Then, next approximated root by secant method is given by

$$x_2 = x_1 - \frac{f(x_1)(x_1 - x_0)}{f(x_1) - f(x_0)}$$

$$= 1 - \frac{1.1231\,(1 - 0)}{1.1231 - (-1)} = 0.4710$$

$f(x_2) = 0.2651$

Now, solving up to 5th iteration in tabular form as follows

Iteration	x_{n-1}	$f(x_{n-1}) = 3x_{n-1} + \sin x_{n-1} - e^{x_{n-1}}$	x_n	$f(x_n) = 3x_n + \sin x_n - e^{x_n}$	$x_{n+1} = x_n - \frac{f(x_n)(x_n - x_{n-1})}{f(x_n - f(x_{n-1}))}$	$f(x_{n+1}) = 3x_{n+1} + \sin x_{n+1} - e^{x_{n+1}}$
1	0	−1	1	1.1231	0.4710	0.2651
2	1	1.1231	0.4710	0.2651	0.3075	−0.1348
3	0.4710	0.2651	0.3075	−0.1348	0.3626	5.44×10^{-3}
4	0.3075	−0.1348	0.3626	5.44×10^{-3}	0.3604	-5.42×10^{-5}
5	0.3626	5.44×10^{-3}	0.3604	-5.42×10^{-5}	0.3604	-1.84×10^{-10}

Here, the value of x_{n+1} do not change up to 4 decimal places. Hence, the root of the given equation is 0.3604.

NOTE:

Procedure to iterate in programmable calculator:

Let, $A = x_{n-1}$, $B = f(x_{n-1})$, $C = x_n$, $D = f(x_n)$, $E = (x_{n+1})$, $F = f(x_{n+1})$

Step 1: Set the calculator in radian mode.

Step 2: Set the following in calculator:

$$A : C : B = 3A + \sin A - e^A : D = 3C + \sin C - e^C : E = C - \frac{D(C - A)}{D - B} :$$

$$F = 3E + \sin E - e^E$$

Step 3: Press CALC then,

Enter the value of A? then press =

Enter the value of C? then press =

Step 4: Now press = only, again and again to get the values for the respective row for each column.

Step 5: Update the values when A? and C? is asked again.

Step 6: Go to step 4.

5. The equation $\alpha \tan\alpha = 1$ occurs in theory of vibrations. Find one of the positive real roots by using any close-end method, correct to at least three decimal places. [2014/Spring]

Solution:

Let, $f(\alpha) = \alpha \tan \alpha - 1$

Initial guess value be

$\alpha = 0$, $f(0) = -1 < 0$

$\alpha = 1$, $f(1) = 0.5574 > 0$

so, root between $\alpha = 0$ and $\alpha = 1$

$\therefore$ $x_L = 0$ and $x_U = 1$

Now, first approximated root using bisection method as closed end method

$$x_N = \frac{x_L + x_U}{2} = \frac{0+1}{2} = 0.5$$

$$f(x_N) = -0.7268 < 0$$

So root now lies between 0.5 and 1.

Remaining iterations are solved in tabular form.

Iteration	x_L	$f(x_L) = x_L \tan x_L - 1$	x_U	$f(x_U) = x_U \tan x_U - 1$	$x_N = \frac{x_L + x_U}{2}$	$f(x_N) = x_N \tan x_N - 1$
1	0	-1	1	0.5574	0.5	-0.7268
2	0.5	-0.7268	1	0.5574	0.75	-0.3013
3	0.75	-0.3013	1	0.5574	0.875	0.0477
4	0.75	-0.3013	0.875	0.0477	0.8125	-0.1422
5	0.8125	-0.1422	0.875	0.0477	0.8437	-0.0517
6	0.8437	-0.0517	0.875	0.0477	0.8593	-3.28×10^{-3}
7	0.8593	-3.28×10^{-3}	0.875	0.0477	0.8671	0.0217
8	0.8593	-3.28×10^{-3}	0.8671	0.0217	0.8632	9.16×10^{-3}
9	0.8593	-3.28×10^{-3}	0.8632	9.16×10^{-3}	0.8612	2.76×10^{-3}
10	0.8593	-3.28×10^{-3}	0.8612	2.76×10^{-3}	0.8602	-4.25×10^{-4}
11	0.8602	-4.25×10^{-4}	0.8612	2.76×10^{-3}	0.8607	1.16×10^{-3}

Here, the value of x_N do not change up to 3 decimal places.

Hence, the positive real root of the equation is 0.8607.

NOTE:

Procedure to iterate in programmable calculator:

Let, $A = x_L$, $B = f(x_L)$, $C = x_U$, $D = f(x_U)$, $E = x_N$, $F = f(x_N)$

Set the following in calculator:

$$A : C : B = A \tan A - 1 : D = C \tan C - 1 : E = \frac{A+C}{2} : F = E \tan E - 1$$

$\boxed{\text{CALC}}$

6. **Find the root of the equation $f(x) = x^2 - 3x + 2$ in the vicinity of $x = 0$, using Newton Raphson method.** **[2014/Spring]**

Solution:

$$f(x) = x^2 - 3x + 2 \qquad \text{..... (1)}$$

Differentiating equation (1) with respect to x

$$f'(x) = 2x - 3 \qquad \text{..... (2)}$$

Let the initial guess be

$x_0 = 0,\qquad f(x_0) = 0^2 - 3 \times (0) + 2 = 2,\qquad f'(0) = -3$

Using Newton Raphson method, next approximated root is

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 0 - \frac{2}{(-3)} = 0.6667$$

$$f(x_1) = 0.4443$$

Now, continuing process in tabular form.

Iteration	x_n	$f(x_n) = x_n^2 - 3x_n + 2$	$f(x_{n+1}) = x_n - \frac{f(x_n)}{f'(x_n)}$	$f(x_{n+1}) = x_{n+1}^2 - 3x_{n+1} + 2$
1	0	2	0.6667	0.4443
2	0.6667	0.443	0.9332	0.0712
3	0.9332	0.0712	0.9960	4.01×10^{-3}
4	0.9960	4.01×10^{-3}	0.9999	1.00×10^{-4}
5	0.9999	1.00×10^{-4}	0.9999	1.99×10^{-8}

Here, the value of x_{n+1} do not change up to 4 decimal places.

Hence, the root of the equation is 0.9999.

NOTE:

Procedure to iterate in programmable calculator:

Let, $A = x_n$, $B = f(x_n)$, $C = x_{n+1}$, $D = f(x_{n+1})$

Set the following in calculator:

$$A : B := A^2 - 3A + 2 : C = A - \frac{B}{2A - 3} : D = C^2 - 3C + 2$$

$\boxed{\text{CALC}}$

7. **Find the square root of 7 using Newton Raphson method and fixed point iteration method correct up to 4 decimal digit.** **[2014/Spring]**

Solution:

For Newton Raphson method

Let, $x = \sqrt{N}$ or $x^2 - N = 0$

Taking $f(x) = x^2 - N$

We have,

$$f'(x) = 2x$$

Then Newton's formula gives,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^2 - N}{2x_n} = \frac{1}{2}\left(x_n + \frac{N}{x_n}\right)$$

Now, taking N = 7, the above formula becomes

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{7}{x_n}\right)$$

For initial guess, taking approximate value of $\sqrt{7}$

i.e., $\sqrt{7} \approx \sqrt{9} = \sqrt{3^2} = 3$

i.e., we take $x_0 = 3$

Then, $x_1 = \frac{1}{2}\left(x_0 + \frac{7}{x_0}\right) = \frac{1}{2}\left(3 + \frac{7}{3}\right) = 2.6667$

$$x_2 = \frac{1}{2}\left(x_1 + \frac{7}{x_1}\right) = 2.6458$$

$$x_3 = \frac{1}{2}\left(x_2 + \frac{7}{x_2}\right) = 2.6457$$

$$x_4 = \frac{1}{2}\left(x_3 + \frac{7}{x_3}\right) = 2.6457$$

Here, $x_3 = x_4$ upto 4 decimal places

Hence, the value of $\sqrt{7}$ is 2.6457

Now, for fixed point iteration method

$$x^2 = 7$$

$$f(x) = x^2 - 7$$

Differentiating with respect to x,

$$f'(x) = 2x$$

Let initial guess be $x_1 = 3$

$$f(x_1) = 3^2 - 7 = 2$$

Now, $x^2 - 7 = 0$

or, $2x^2 - x^2 = 7$

or, $x = \frac{7 + x^2}{2x}$

$\therefore$ $x_i = \dfrac{\frac{7}{x} + x}{2}$ ..... (1

First iteration

$$x_1 = \frac{\frac{7}{3} + 3}{2} = 2.6666$$

Error = |2.6666 − 3| = 0.3333

Second iteration

$$x_2 = \frac{\frac{7}{2.6666} + 2.6666}{2} = 2.6458$$

Error = |2.6458 − 2.6666| = 0.0208

Third iteration

$$x_3 = \frac{\frac{7}{2.6458} + 2.6458}{2} = 2.6457$$

Error = |2.6457 − 2.6458| = 0.0001

Fourth iteration

$$x_4 = \frac{\frac{7}{2.6457} + 2.6457}{2} = 2.64575$$

Error = |2.64575 − 2.6457| = 0.00005

Here, $x_3 = x_4$ up to 4 decimal places.

Hence, the value of $\sqrt{7}$ is 2.64575

8. **The flux equation of an iron core electric circuit is given by $f(\phi) = 10 - 2.1\,\phi - 0.01\,\phi^3$. The steady state value of flux is obtained by solving the equation $f(\phi) = 0$. By using any close end method, estimate the steady state value of "ϕ" correct to 3 decimal places. [2014/Fall]**

Solution:

$$f(\phi) = 10 - 2.1\,\phi - 0.01\,\phi^3$$

Let initial guess be

$x = \phi = 4,\quad f(4) = 10 - 2.1 \times 4 - 0.01\,(4)^3 = 0.96 > 0$

$x = \phi = 5,\quad f(5) = 10 - 2.1 \times 5 - 0.01 \times 5^3 = -1.75 < 0$

So root lies between x = 4 and x = 5

∴ $x_L = 4$ and $x_U = 5$

Now, first approximated root using bisection method as close end method,

$$x_N = \frac{x_L + x_U}{2} = \frac{4+5}{2} = 4.5$$

$f(x_N) = -0.3612 < 0$ so now root lies between 4 and 4.5

Remaining iterations are solved in tabular form.

Iteration	x_L	$f(x_L) = 10 - 2.1x_L - 0.01x_L^3$	x_U	$f(x_U) = 10 - 2.1x_U - 0.01x_U^3$	$x_N = \frac{x_L + x_U}{2}$	$f(x_N) = 10 - 2.1x_N - 0.01x_N^3$
1	4	0.96	5	−1.75	4.5	−0.3612
2	4	0.96	4.5	−0.3612	4.25	0.3073
3	4.25	0.3073	4.5	−0.3612	4.375	−0.0249
4	4.25	0.3073	4.375	−0.0249	4.3125	0.1417
5	4.3125	0.1417	4.375	−0.0249	4.3437	0.0586
6	4.3437	0.0586	4.375	−0.0249	4.3593	0.0170
7	4.3593	0.0170	4.375	−0.0249	4.3671	-3.78×10^{-3}
8	4.3593	0.1070	4.3671	-3.78×10^{-3}	4.3632	6.63×10^{-3}

9	4.3632	6.63×10^{-3}	4.3671	-3.78×10^{-3}	4.3651	1.55×10^{-3}
10	4.3651	1.55×10^{-3}	4.3671	-3.78×10^{-3}	4.3661	-1.11×10^{-3}
11	4.3651	1.55×10^{-3}	4.3661	-1.11×10^{-3}	4.3656	2.23×10^{-4}
12	4.3656	2.23×10^{-4}	4.3661	-1.11×10^{-3}	4.3658	-3.10×10^{-4}

Here, the value of x_N do not change up to 3 decimal places.

Hence, the steady state value of ϕ is 4.3658

NOTE:
Procedure to iterate in programmable calculator:
Let, $A = x_L$, $B = f(x_L)$, $C = x_U$, $D = f(x_U)$, $E = x_N$, $F = f(x_N)$
Set the following in calculator:

$$A : C : B = 10 - 2.1\,A - 0.01\,A^3 : D = 10 - 2.1\,C - 0.01\,C^3 :$$

$$E = \frac{A + C}{2} : F = 10 - 2.1\,E - 0.01\,E^3$$

$\boxed{\text{CALC}}$

9. Evaluate one of the real roots of the given equation $xe^x - \cos x = 0$ by NR method accurate to at least 4 decimal places. [2014/Fall]

Solution:

Let $f(x) = xe^x - \cos x$ ..... (1)

Differentiating equation (1) with respect to x.

$$f'(x) = x\,e^x + e^x + \sin x \quad \text{..... (2)}$$

From equation (1)

Let the initial guess be

$$x_0 = 0$$

$$f(x_0) = 0e^0 - \cos(0) = -1$$

$$f'(x_0) = 0e^0 + e^0 + \sin(0) = 1$$

Using Newton Raphson method, next approximated root is

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 0 - \frac{(-1)}{1} = 1$$

$$f(x_1) = 2.1779$$

Now, continuing process in tabular form.

Iteration	x_n	$f(x_n) = x_n e^{x_n} - \cos x_n$	$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$	$f(x_{n+1}) = x_{n+1} e^{x_{n+1}} - \cos x_{n+1}$
1	0	-1	1	2.1779
2	1	2.1779	0.6530	0.4603
3	0.6530	0.4603	0.5313	0.0416
4	0.5313	0.0416	0.5179	4.33×10^{-4}
5	0.5179	4.33×10^{-4}	0.5177	-1.74×10^{-4}
6	0.5177	-1.74×10^{-4}	0.5177	-4.90×10^{-7}

Here, the value of x_{n+1} do not change up to 4 decimal places.

Hence, the desired root is 0.5177 of the equation.

> **NOTE:**
> Procedure to iterate in programmable calculator:
> Let, $A = x_N$, $B = f(x_n)$, $C = x_{n+1}$, $D = f(x_{n+1})$
> Set the following in calculator:
> $$A : B = Ae^A - \cos A : C = A - \frac{B}{Ae^A + e^A + \sin A} : D = Ce^C - \cos C$$
> CALC

10. Determine the root of $e^x = x^3 + \cos 25x$ using secant method correct to four decimal place. **[2015/Fall]**

Solution:

Let $f(x) = e^x - x^3 - \cos 25x$

Let $x_0 = 4$ and $x_1 = 5$ be two initial guesses

$f(x_0) = -10.2641$ and $f(x_1) = 22.6254$

Then, next approximated root by secant method is given by

$$x_2 = x_1 - \frac{f(x_1)(x_1 - x_0)}{f(x_1) - f(x_0)}$$

$$= 5 - \frac{22.6254(5-4)}{22.2654 - (-10.2641)} = 4.3210$$

$f(x_2) = -6.1371$

Now, solving other iterations in tabular form as follows

Itn.	x_{n-1}	$f(x_{n-1}) = e^{x_{n-1}} - x_{n-1}^3 - \cos 25x_{n-1}$	x_n	$f(x_n) = e^{x_n} - x_n^3 - \cos 25x_n$	$x_{n+1} = x_n - \frac{f(x_n)(x_n - x_{n-1})}{f(x_n) - f(x_{n-1})}$	$f(x_{n+1}) = e^{x_{n+1}} - x_{n+1}^3 - \cos 25x_{n+1}$
1	4	−10.2641	5	22.6254	4.3120	−6.1371
2	5	22.6254	4.3120	−6.1371	4.4587	−2.2048
3	4.3120	−6.1371	4.4587	−2.2048	4.5409	−0.7681
4	4.4587	−2.2048	4.5409	−0.7681	4.5848	1.5611
5	4.5409	−0.7681	4.5848	1.5611	4.5553	−0.0979
6	4.5848	1.5611	4.5553	−0.0979	4.5570	−0.0112
7	4.5553	−0.0979	4.5570	−0.0112	4.5572	-9.43×10^{-4}
8	4.5570	−0.0112	4.5572	-9.43×10^{-4}	4.5572	3.39×10^{-6}

Here, the value of x_{n+1} do not change up to four decimal place.

Hence, the root of the equation is 4.5572.

> **NOTE:**
> Procedure to iterate in programmable calculator:
> Let, $A = x_{n-1}$, $B = f(x_{n-1})$, $C = x_n$, $D = x_c$, $E = x_{n+1}$, $F = f(x_{n+1})$
> Set the following in calculator:
> $$A : C : B = e^A - A^3 - \cos 25A : D = e^C - C^3 - \cos 25C : E = C - \frac{D(C-A)}{D-B} :$$
> $$F = e^E - E^3 - \cos 25E$$
> CALC

11. The current i in an electric circuit is given by $i = 10\ e^{-x} \sin 2\pi x$ where x is in seconds. Using NR method, find the value of x correct up to 3 decimal places for i = 2 ampere. **[2015/Fall]**

Solution:

Given that;

$$i = 10e^{-x} \sin 2\pi x$$

At i = 2 Ampere

$$2 = 10e^{-x} \sin 2\pi x \quad \text{..... (1)}$$

Let, $f(x) = 10e^{-x} \sin 2\pi x - i$

or, $f(x) = (10e^{-x} \sin 2\pi x) - 2$ for i = 2 amp

Differentiating equation (1) with respect to x,

$$f'(x) = 10\,(e^{-x}\, 2\pi \cos 2\pi x - \sin 2\pi x \cdot e^{-x})$$

$$= 10\ e^{-x}\,(2\pi \cos 2\pi x - \sin 2\pi x) \quad \text{.....(2)}$$

From equation (1),

Let the initial guess be,

$x_0 = 0$, $\quad f(x_0) = 10e^0 \sin 0 - 2 = -2$

Using Newton Raphson method, next approximated root is

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 0 - \frac{(-2)}{62.8318} = 0.0318$$

$$f(x_1) = -0.0773$$

Now, continuing process in tabular form

Iteration	x_n	$f(x_n) = 10e^{-x_n} \sin 2\pi x_n$	$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$	$f(x_{n+1}) = 10e^{-x_{n+1}} \sin 2\pi\, x_{n+1}$
1	0	-2	0.0318	−0.0773
2	0.0318	−0.0773	0.0331	-2.45×10^{-3}
3	0.0331	-2.45×10^{-3}	0.0331	-1.20×10^{-6}

Here, the value of x_{n+1} do not change up to 3 decimal places.

Hence, the value of x is 0.0331 seconds.

12. Solve the equation log x − cos x = 0 correct to three significant digit after decimal using bracketing method. **[2015/Fall]**

Solution:

Let $f(x) = \log x - \cos x$

Let initial guess be

$x = 1$, $\quad f(1) = \log(1) - \cos(1) = -0.5403 < 0$

$x = 2$, $\quad f(2) = \log(2) - \cos(2) = 0.7171 > 0$

so, root lies between x = 1 and x = 2

$\therefore \quad x_L = 1$ and $x_U = 2$

Now, first approximated root using bisection method.

$$x_N = \frac{x_L + x_U}{2} = \frac{1+2}{2} = 1.5$$

$f(x_N) = 0.1053 > 0$ so now root lies between 1 and 1.5.

Remaining iterations are carried out in tabular form

Itn.	x_L	$f(x_L) = \log x_L - \cos x_L$	x_U	$f(x_U) = \log x_U - \cos x_U$	$x_N = \frac{x_L+x_U}{2}$	$f(x_N) = \log x_N - \cos x_N$
1	1	−0.5403	2	0.7171	1.5	0.1053
2	1	−0.5403	1.5	0.1053	1.25	−0.2184
3	1.25	−0.2184	1.5	0.1053	1.375	−0.0562
4	1.375	−0.0562	1.5	0.1053	1.4375	0.0247
5	1.375	−0.0562	1.4375	0.0247	1.4062	−0.0157
6	1.4062	−0.0157	1.4375	0.0247	1.4218	4.39×10^{-3}
7	1.4062	−0.0157	1.4218	4.39×10^{-3}	1.4140	-5.70×10^{-3}
8	1.4140	5.70×10^{-3}	1.4218	4.39×10^{-3}	1.4179	-6.55×10^{-4}
9	1.4179	-6.55×10^{-4}	1.4218	4.39×10^{-3}	1.4198	1.80×10^{-3}
10	1.4179	-6.55×10^{-4}	1.4198	1.8×10^{-3}	1.4188	5.09×10^{-4}
11	1.4179	-6.55×10^{-4}	1.4188	5.09×10^{-4}	1.4183	-1.37×10^{-4}

Here, the value of x_N do not change up to three significant digits after decimal. Hence, the root of the equation is 1.4183.

NOTE:
Procedure to iterate in programmable calculator:
Let, $A = x_L$, $B = f(x_L)$, $C = x_U$, $D = f(x_U)$, $E = x_N$, $F = f(x_N)$
Set the following in calculator:

$$A : C : B = \log A - \cos A : D = \log C - \cos C : E = \frac{A+C}{2} :$$

$$F \log E - \cos E$$

$\boxed{\text{CALC}}$

13. Find the root of the equation x − 1.5 sin x − 2.5 = 0 using Newtòn Raphson method so that relative error is less than 0.01%. [2015/Spring]

Solution:

Let $f(x) = x - 1.5 \sin x - 2.5$ ..... (1)

Differentiating equation (1) with respect to x,

$f'(x) = 1 - 1.5 \cos x$ ..... (2)

From equation (1),

Let the initial guess be

$x_0 = 3$

$f(x_0) = 3 - 1.5 \sin(3) - 2.5 = 0.2883$

$f'(x_0) = 1 - 1.5 \cos(3) = 2.4849$

Now, using Newton Raphson method, next approximated root is

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 3 - \frac{0.2883}{2.4849} = 2.8839$$

$f(x_1) = 1.624 \times 10^{-3}$

Now continuing process in tabular form.

Iteration	x_n	$f(x_n)$ $x_n - 1.5 \sin x_n - 2.5$	$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$	$f(x_{n+1}) = x_{n+1} - 1.5 \sin x_{n+1} - 2.5$
1	3	0.2883	2.8839	1.624×10^{-3}
2	2.8839	1.624×10^{-3}	2.8832	-9.034×10^{-5}
3	2.8832	-9.034×10^{-5}	2.8832	-5.250×10^{-9}

Here, the value of x_{n+1} do not change up to 4 decimal places and relative error is also less than 0.01%. Hence, the root of the equation is 2.8832.

NOTE:

Procedure to iterate in programmable calculator:

Let, $A = x_n$, $B = f(x_n)$, $C = x_{n+1}$, $D = f(x_{n+1})$

Set the following in calculator:

$$A : B = A - 1.5 \sin A - 2.5 : C = A - \frac{B}{1 - 1.5 \cos A} :$$

$$D = C - 1.5 \sin C - 2.5$$

$\boxed{\text{CALC}}$

14. Find the root of the equation $xe^x = \cos x$ using secant method correct to four decimal place. [2015/Spring]

Solution:

Let, $f(x) = xe^x - \cos x$

$x_0 = 0$ and $x_1 = 1$ be the initial guesses

$f(x_0) = 0e^0 - \cos(0) = -1$

$f(x_1) = 1 \times e^1 - \cos(1) = 2.1779$

Then, next approximated root by secant method is given by,

$$x_2 = x_1 - \frac{f(x_1)(x_1 - x_0)}{f(x_1) - f(x_0)} = 1 - \frac{2.1779(1-0)}{2.1779 - (-1)} = 0.3146$$

$f(x_2) = -0.5200$

Now, solving other iterations in tabular form as follows

Itn.	x_{n-1}	$f(x_{n-1}) = x_{n-1} e^{x_{n-1}} - \cos x_{n-1}$	x_n	$f(x_n) = x_n e^{x_n} - \cos x_n$	$x_{n+1} = x_n - \frac{f(x_n)(x_n - x_{n-1})}{f(x_n) - f(x_{n-1})}$	$f(x_{n+1}) = x_{n+1} e^{x_{n+1}} - \cos x_{n+1}$
1	0	-1	1	2.1779	0.3146	-0.5200
2	1	2.1779	0.3146	-0.5200	0.4467	-0.2036
3	0.3146	-0.5200	0.4467	-0.2036	0.5317	0.0429
4	0.4467	-0.2036	0.5317	0.0429	0.5169	-2.60×10^{-3}
5	0.5317	0.0429	0.5169	-2.60×10^{-3}	0.5177	-1.74×10^{-4}
6	0.5169	-2.60×10^{-3}	0.5177	-1.74×10^{-4}	0.5177	4.47×10^{-8}

Here, the value of x_{n+1} do not change up to four decimal places.

Hence, the root of the equation is 0.5177.

NOTE:

Procedure to iterate in programmable calculator:

Let, $A = x_{n-1}$, $B = f(x_{n-1})$, $C = x_n$, $D = f(x_n)$, $E = x_{n+1}$, $F = f(x_{n+1})$

Set the following in calculator:

$$A : C : B = Ae^A - \cos A : D = Ce^C - \cos C : E = C - \frac{D(C-A)}{D-B} :$$

$$F = Ee^E - \cos E$$

CALC

15. Using the bisection method, find the approximate root of the equation $\sin x = \frac{1}{x}$ that lies between x = 1 and x = 1.5 (in radian's). Carry out up to 7th stage. [2013/Spring, 2015/Spring, 2017/Fall]

Solution:

Let $f(x) = \sin x - \frac{1}{x}$

The initial guess be,

$x = 1$, $\quad f(1) = \sin 1 - \frac{1}{1} = -0.1585 < 0$

$x = 1.5$, $\quad f(1.5) = \sin(1.5) - \frac{1}{1.5} = 0.3308 > 0$

As root lies between x = 1 and x = 1.5,

$\therefore \quad x_L = 1$ and $x_U = 1.5$

Now, first approximated root using bisection method,

$$x_N = \frac{x_L + x_U}{2} = \frac{1 + 1.5}{2} = 1.25$$

$f(x_N) = 0.1489 > 0$ so now root lies between 1 and 1.25.

Performing the iterations up to 7th stage in tabular form.

Itn.	x_L	$f(x_L) = \sin x_L - \frac{1}{x_L}$	x_U	$f(x_U) = \sin x_U - \frac{1}{x_U}$	$x_N = \frac{x_L + x_U}{2}$	$f(x_N) = \sin x_N - \frac{1}{x_N}$
1	1	-0.1585	1.5	0.3308	1.25	0.1489
2	1	-0.1585	1.25	0.1489	1.125	0.0133
3	1	-0.1585	1.125	0.0133	1.0625	-0.0676
4	1.0625	-0.0676	1.125	0.0133	1.09375	-0.0259
5	1.09375	-0.0259	1.125	0.0133	1.109375	-5.98×10^{-3}
6	1.109375	-5.98×10^{-3}	1.125	0.0133	1.1171875	3.76×10^{-3}
7	1.109375	-5.98×10^{-3}	1.1171875	3.76×10^{-3}	1.11328125	-1.09×10^{-3}

Thus, the desired approximation to the root carried out up to 7th stage is 1.11328125.

NOTE:
Procedure to iterate in programmable calculator:
Let, $A = x_L$, $B = f(x_L)$, $C = x_U$, $D = f(x_U)$, $E = x_N$, $F = f(x_N)$
Set the following in calculator:

$$A : C : B = \sin A - \frac{1}{A} : D = \sin C - \frac{1}{C} : E = \frac{A + C}{2} : F = \sin E - \frac{1}{E}$$

CALC

16. **Find a real root of the equation $xe^x = 3$ by using any bracketing method correct to three decimal places (Take $x_1 = 1$ and $x_2 = 1.5$).** **[2016/Fall]**

Solution:

Let $f(x) = x e^x - 3$

And, initial guess be the provided value

i.e., $x = 1$, $f(1) = 1e^1 - 3 = -0.2817 < 0$

$x = 1.5$, $f(1.5) = 1.5e^{1.5} - 3 = 3.7225 > 0$

Root lies between $x = 1$ and $x = 1.5$,

$\therefore$ $x_L = 1$ and $x_U = 1.5$

Now, first approximated root using bisection method as bracketing method

$$x_N = \frac{x_L + x_U}{2} = \frac{1 + 1.5}{2} = 1.25$$

$f(x_N) = 1.3629 > 0$ so now root lies between 1 and 1.25.

Remaining iterations are carried out in tabular form.

Itn.	x_L	$f(x_L) = x_Le^{x_L} - 3$	x_U	$f(x_U) = x_Ue^{x_U} - 3$	$x_N = \frac{x_L+x_U}{2}$	$f(x_N) = x_Ne^{x_N} - 3$
1	1	−0.2817	1.5	3.7225	1.25	1.3629
2	1	−0.2817	1.25	1.3629	1.125	0.4652
3	1	−0.2817	1.125	0.4652	1.0625	0.0744
4	1	−0.2817	1.0625	0.0744	1.0312	−0.1077
5	1.0312	−0.1077	1.0625	0.0744	1.0468	−0.0181
6	1.0468	−0.0181	1.0625	0.0744	1.0546	0.0275
7	1.0468	−0.0181	1.0546	0.0275	1.0507	4.63×10^{-3}
8	1.0468	−0.0181	1.0507	4.63×10^{-3}	1.0487	-7.07×10^{-3}
9	1.0487	-7.07×10^{-3}	1.0507	4.63×10^{-3}	1.0497	-1.22×10^{-3}
10	1.0497	-1.22×10^{-3}	1.0507	4.63×10^{-3}	1.0502	1.70×10^{-3}
11	1.0497	-1.22×10^{-3}	1.0502	1.70×10^{-3}	1.0499	-5.21×10^{-5}
12	1.0499	-5.21×10^{-5}	1.0502	1.70×10^{-3}	1.0500	5.33×10^{-4}
13	1.0499	-5.21×10^{-5}	1.0500	5.33×10^{-4}	1.0499	-5.21×10^{-5}
14	1.0499	-5.21×10^{-5}	1.0500	5.33×10^{-4}	1.0499	-5.21×10^{-5}

Here, the value of x_N do not change up to 3 decimal places

Hence, the real root of the equation is 1.0499.

NOTE:

Procedure to iterate in programmable calculator:

Let, $A = x_L$, $B = f(x_L)$, $C = x_U$, $D = f(x_U)$, $E = x_N$, $F = f(x_N)$

Set the following in calculator:

$$A : C : B = Ae^A - 3 : D = Ce^C - 3 : E = \frac{A + C}{2} : F = Ee^E - 3$$

$\boxed{\text{CALC}}$

17. Obtain a real root of the equation sin x + 1 = 2x by using secant method such that the real root must have relative error less than 0.0001.

[2016/Fall]

Solution:

Let $f(x) = \sin x + 1 - 2x$

Let $x_0 = 0$ and $x_1 = 1$ be two initial guesses.

$f(x_0) = 1$ and $f(x_1) = -0.1585$

Then, next approximated root by secant method is given by,

$$x_2 = x_1 - \frac{f(x_1)\,(x_1 - x_0)}{f(x_1) - f(x_0)} = 1 - \frac{-0.1585(1 - 0)}{-0.1585 - 1} = 0.8631$$

$f(x_2) = 0.0336$

Now, solving other iterations in tabular form as follows

Itn.	x_{n-1}	$f(x_{n-1}) = \sin x_{n-1} + 1 - 2x_{n-1}$	x_n	$f(x_n) = \sin x_n + 1 - 2x_n$	$x_{n+1} = x_n - \frac{f(x_n)(x_n - x_{n-1})}{f(x_n) - f(x_{n-1})}$	$f(x_{n+1}) = \sin x_{n+1} + 1 - 2x_{n+1}$
1	0	1	1	-0.1585	0.8631	0.0336
2	1	-0.1585	0.8631	0.0336	0.8870	1.18×10^{-3}
3	0.8631	0.0336	0.8870	1.18×10^{-3}	0.8878	8.51×10^{-5}
4	0.8870	1.18×10^{-3}	0.8878	8.51×10^{-5}	0.8878	4.43×10^{-8}

Here, the value of x_{n+1} do not change up to 4 decimal places and have relative error less than 0.0001.

Hence, the real root of the equation is 0.8878

NOTE:

Procedure to iterate in programmable calculator:

Let, $A = x_{n-1}$, $B = f(x_{n-1})$, $C = x_n$, $D = f(x_n)$, $E = x_{n+1}$, $F = f(x_{n+1})$

Set the following in calculator:

$$A : C : B = \sin A + 1 - 2A : D = \sin C + 1 - 2C : E = C - \frac{D(C - A)}{D - B} :$$

$$F = \sin E + 1 - 2E$$

$\boxed{\text{CALC}}$

18. Find the root of the equation x sin x + cos x = 0 using Newton Raphson's method so that relative error is less than 0.1. **[2016/Fall]**

Solution:

Let, $f(x) = x \sin x + \cos x$ ..... (1)

Differentiating equation (1) with respect to x,

$f'(x) = x \cos x$ ..... (2)

From equation (1)

Let the initial guess be

$x_0 = 2, f(x_0) = 1.4024, f'(x_0) = -0.8322$

Using NR method, next approximated root is,

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 2 - \frac{1.4024}{-0.8322} = 3.6851$$

$f(x_1) = -2.7616$

Now, continuing the process in tabular form.

Itn.	x_n	$f(x_n) = x_n \sin x_n + \cos x_n$	$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$	$f(x_{n+1}) = x_{n+1} \sin x_{n+1} + \cos x_{n+1}$
1	2	1.4024	3.6851	-2.7616
2	3.6851	-2.7616	2.8095	-0.0294
3	2.8095	-0.0294	2.7984	-0.03×10^{-3}
4	2.7984	-0.03×10^{-3}	2.7983	2.26×10^{-4}
5	2.7983	2.26×10^{-4}	2.7983	$7.32\times10\text{-}7$

Here, the value of x_{n+1} do not change up to 4 decimal places. And, relative error is also less than 0.1.

$$\text{Relative error} = \left(\frac{|x_{n+1} - x_n|}{x_{n+1}}\right)$$

$$= \left(\frac{|2.7983 - 2.7984|}{2.7983}\right)$$

$$= 0.003574$$

Hence, the desired root of the equation is 2.7983.

> **NOTE:**
> Procedure to iterate in programmable calculator:
> Let, $A = x_n$, $B = f(x_n)$, $C = x_{n+1}$, $D = f(x_{n+1})$
> Set the following in calculator:
> $$A : B : A \sin A + \cos A : C = A - \frac{B}{A \cos A} : D = C \sin C + \cos C$$
> CALC

19. Using Newton-Raphson method find a root of the equation $xe^x = 2$. **[2016/Spring]**

Solution:

Let, $f(x) = xe^x - 2$ ..... (1)

Differentiating equation (1) with respect to x,

$f'(x) = e^x + x e^x$ ..... (2)

From equation (1),

Let the initial guess be,

$x_0 = 0, \quad f(x_0) = 0e^0 - 2 = -2, \quad f'(x_0) = e^0 + 0e^0 = 1$

Using Newton Raphson method, next approximated root is

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 0 - \frac{-2}{1} = 2$$

$$f(x_1) = 12.7781$$

Now, continuing process in tabular form.

Iteration	x_n	$f(x_n) = x_n e^{x_n} - 2$	$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$	$f(x_{n+1}) = x_{n+1} e^{x_{n+1}} - 2$
1	0	-2	2	12.7781
2	2	12.7781	1.4235	3.9098
3	1.4235	3.9098	1.0349	0.9130
4	1.0349	0.9130	0.8755	0.1012
5	0.8755	0.1012	0.8530	1.71×10^{-3}
6	0.8530	1.71×10^{-3}	0.8526	-2.39×10^{-5}
7	0.8526	-2.39×10^{-5}	0.8526	-1.01×10^{-8}

Here, the value of x_{n+1} do not change up to 4 decimal places.

Hence, a root of the equation is 0.8526.

NOTE:

Procedure to iterate in programmable calculator:

Let, $A = x_n$, $B = f(x_n)$, $C = x_{n+1}$, $D = f(x_{n+1})$

Set the following in calculator:

$$A : B = Ae^A - 2 : C = A - \frac{B}{e^A + Ae^A} : D = Ce^C - e$$

$\boxed{\text{CALC}}$

20. Find a real root of the cos x = 3x − 1, correct to three decimal places, using fixed point method. [2016/Spring]

Solution:

Let, $f(x) = \cos x - 3x + 1 = 0$ ..... (1)

or, $\cos x + 1 = 3x$

or, $x = \frac{1 + \cos x}{3}$

i.e., $g(x) = \frac{1 + \cos x}{3}$ ..... (2)

Let initial guess be $x_0 = 1$ then,

$$|g'(x_0)| = \left|\frac{1}{3}(-\sin x)\right| = \left|\frac{1}{3}(-\sin 1)\right| = 0.2804$$

Here, $|0.2804| < 1$

Then next approximated root by fixed point method is given by,

$$g(x_0) = x_1 = \frac{1 + \cos(1)}{3} = 0.5134$$

Now, continuing the process in tabular form.

Itn.	x_n	$f(x_n) = \cos x_n - 3x_{n+1}$	$x_{n+1} = g(x_n) = \frac{1 + \cos x_n}{3}$	$f(x_{n+1}) = \cos x_{n+1} - 3x_{n+1} + 1$
1	1	-1.45	0.5134	0.3308
2	0.5134	0.3308	0.6236	-0.0590
3	0.6236	-0.0590	0.6039	0.0114
4	0.6039	0.0114	0.6077	-2.13×10^{-3}
5	0.6077	-2.13×10^{-3}	0.6069	7.19×10^{-4}
6	0.6069	7.19×10^{-4}	0.6071	5.88×10^{-6}
7	0.6071	5.88×10^{-6}	0.6071	-1.11×10^{-6}

Here, the value of $g(x_n)$ do not change up to 4 decimal places.

Hence, the real real of the equation is 0.6071.

> **NOTE:**
> Procedure to iterate in programmable calculator:
> Let, $A = x_n$, $B = f(x_n)$, $C = x_{n+1} = g(x_n)$, $D = f(x_{n+1})$
> **Step 1:** Set the calculator in radian mode.
> **Step 2:** Set the following in calculator:
> $$A : B = \cos A - 3A + 1 : C = \frac{1 + \cos A}{3} : D = \cos C - 3C + 1$$
> **Step 3:** Press CALC then,
> Enter the value of A? then press =
> **Step 4:** Now press = only, again and again to get the values for the respective row for each column.
> **Step 5:** Update the values when A? is asked again.
> **Step 6:** Go to step 4.

21. Find a real root of $e^{\cos x} - \sin x - 1 = 0$ correct to 4 decimal places using false position method. **[2017/Spring]**

Solution:

Let, $f(x) = e^{\cos x} - \sin x - 1$

The initial guess be,

$x_L = x_0 = 0$, $f(x_0) = e^{\cos(0)} - \sin(0) - 1 = 1.71828 > 0$

$x_U = x_1 = 1$, $f(x_1) = e^{\cos(1)} - \sin(1) - 1 = -0.12494 < 0$

i.e., Root lies between 0 and 1.

Now, using false position method,

$$x_2 = x_0 - \frac{(x_1 - x_0)\, f(x_0)}{f(x_1) - f(x_0)} = 0 - \frac{(1 - 0) \times 1.71828}{(-0.12494 - 1.71828)} = 0.93221$$

$\therefore$ $f(x_2) = 0.01201$

Since the value of $f(x_2)$ is positive, now root lies between 0.9322 and 1.

Solving other iterations in tabular form as follows;

Itn.	x_L	$f(x_L) = e^{\cos x_L} - \sin x_L - 1$	x_U	$f(x_U) = e^{\cos x_U} - \sin x_U - 1$	$x_N = x_L - \frac{f(x_L)(x_U - x_L)}{f(x_U) - f(x_L)}$	$(x_N) = e^{\cos x_N} - \sin x_N - 1$
1	0	1.71828	1	-0.12494	0.93221	0.01201
2	0.93221	0.01201	1	-0.12494	0.93815	-1.64×10^{-4}
3	0.93221	0.01201	0.93815	-1.64×10^{-4}	0.93806	1.95×10^{-5}
4	0.93806	1.95×10^{-5}	0.93815	-1.64×10^{-4}	0.93806	1.95×10^{-5}

Here, the value of x_N do not change up to 4 decimal places.

Hence, the root of the equation is 0.93806.

NOTE:

Procedure to iterate in programmable calculator:

Let, $A = x_L$, $B = f(x_L)$, $C = x_U$, $D = f(x_U)$, $E = x_N$, $F = f(x_N)$

Step 1: Set the calculator in radian mode.

Step 2: Set the following in calculator:

$$A : C : B = e^{\cos A} - \sin A - 1 : D = e^{\cos C} - \sin C - 1 : E = A - \frac{(C - A)B}{D - B} :$$

$$F = e^{\cos E} - \sin E - 1$$

Step 3: Press CALC then,

Enter the value of A? then press =

Enter the value of C? then press =

Step 4: Now press = only, again and again to get the values for the respective row for each column.

Step 5: Update the values when A? and C? is asked again.

Step 6: Go to step 4.

22. Find the root of the equation $3x = \cos x + 1$ using NR method with the tolerance is 10E – 5. [2017/Spring]

Solution:

Let, $f(x) = 3x - \cos x - 1$ ..... (1)

Differentiating equation (1) with respect to x,

$f'(x) = 3 + \sin x$ ..... (2)

From equation (1),

Let the initial guess be,

$x_0 = 0$, $f(x_0) = 3 \times 0 - \cos 0 - 1 = -2 < 0$

$x_1 = 1$, $f(x_1) = 3 \times 1 - \cos(1) - 1 = 1.4596 > 0$.

So, a root lies between 0 and 1.

Using Newton Raphson method, next approximated root is,

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 0 - \frac{(-2)}{3} = 0.6667$$

$$f(x_1) = 0.2142$$

Now, continuing process in tabular form.

Itn.	x_n	$f(x_n) = 3x_n - \cos x_n - 1$	$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$	$f(x_{n+1}) = 3x_{n+1} - \cos x_{n+1} - 1$
1	0	-2	0.6667	0.2142
2	0.6667	0.2142	0.6075	1.422×10^{-3}
3	0.6075	1.422×10^{-3}	0.6071	-5.88×10^{-6}
4	0.6071	-5.88×10^{-6}	0.6071	-4.53×10^{-9}

Here, the value of x_{n+1} do not change up to 4 decimal places.

Hence, the desired root is 0.6071 of the equation.

NOTE:

Procedure to iterate in programmable calculator:

Let, $A = x_n$, $B = f(x_n)$, $C = x_{n+1}$, $D = f(x_{n+1})$

Set the following in calculator:

$$A : B = 3A - \cos A - 1 : C = A - \frac{B}{3 + \sin A} : D = 3C - \cos C - 1$$

CALC

23. Find the root of $e^x \tan x = 1$ by creating iterative formula of Newton-Raphson method. **[2018/Spring]**

Solution:

Let $f(x) = e^x \tan x - 1$ ..... (1)

Differentiating equation (1) with respect to x,

$f'(x) = e^x (\tan x + \sec^2 x)$ ..... (2)

From equation (1),

Let the initial guess be,

$x_0 = 0$

$f(x_0) = e^0 \tan 0 - 1 = -1$

$f'(x_0) = e^0(\tan 0 + \sec^2 0) = 1$

Using Newton Raphson method, next approximated root is

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 0 - \frac{(-1)}{1} = 1$$

$f(x_1) = 3.2334$

Now, continuing process in tabular form.

Itn.	x_n	$f(x_n) = e^{x_n} \tan x_n - 1$	$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$	$f(x_{n+1}) = e^{x_{n+1}} \tan x_{n+1} - 1$
1	0	-1	1	3.2334
2	1	3.2334	0.7612	1.0396
3	0.7612	1.0396	0.5914	0.2132
4	0.5914	0.2132	0.5357	0.0142
5	0.5357	0.0142	0.5314	3.007×10^{-5}
6	0.5314	3.007×10^{-5}	0.5313	-2.988×10^{-4}
7	0.5313	-2.988×10^{-4}	0.5313	3.311×10^{-8}

Here, the value of x_{n+1} do not change up to 4 decimal places.
Hence, the desired root is 0.5313 of the equation.

NOTE:
Procedure to iterate in programmable calculator:
Let, $A = x_n$, $B = f(x_n)$, $C = x_{n+1}$, $D = f(x_{n+1})$
Set the following in calculator:

$$A : B = e^A \tan A - 1 : C = A - \frac{B}{e^A(\tan A + \sec^2 A)} : D = e^C \tan C - 1$$

$\boxed{\text{CALC}}$

24. Solve $f(x) = xe^x - 1$ by secant method for tolerance value 0.0001.

[2018/Spring]

Solution:

$$f(x) = x\,e^x - 1$$

Let $x_0 = 0$ and $x_1 = 1$ be two initial guesses.
Then, next approximated root by secant method is given by,

$$x_2 = x_1 - \frac{f(x_1)\,(x_1 - x_0)}{f(x_1) - f(x_0)}$$

$$= 1 - \frac{(1.7182)\,(1 - 0)}{1.7182 - (-1)} = 0.3678$$

$$f(x_2) = -0.4686$$

Now, solving other iterations in tabular form as follows,

Itn.	x_{n-1}	$f(x_{n-1}) = x_{n-1}e^{x_{n-1}} - 1$	x_n	$f(x_n) = x_n e^{x_n} - 1$	$x_{n+1} = x_n - \frac{f(x_n)(x_n - x_{n-1})}{f(x_n) - f(x_{n-1})}$	$f(x_{n+1}) = x_{n+1}e^{x_{n+1}} - 1$
1	0	−1	1	1.7182	0.3678	−0.4686
2	1	1.7182	0.3678	−0.4686	0.5032	−0.1677
3	0.3678	−0.4686	0.5032	−0.1677	0.5786	0.0319
4	0.5032	−0.1677	0.5786	0.0319	0.5665	−0.0017
5	0.5786	0.0319	0.5665	−0.0017	0.5671	−0.0001
6	0.5665	−0.0017	0.5671	−0.0001	0.5671	−0.0001

Here, the value of x_{n+1} do not change up to 4 decimal places with the tolerance value of 0.0001.

Hence, the root of the equation is 0.5671.

NOTE:
Procedure to iterate in programmable calculator:
Let, $A = x_{n-1}$, $B = f(x_{n-1})$, $C = x_n$, $D = f(x_n)$, $E = x_{n+1}$, $F = f(x_{n+1})$
Set the following in calculator:

$$A : C : B = Ae^A - 1 : D = Ce^C - 1 : E = C - \frac{D(C - A)}{D - B} : F = Ee^E - 1$$

$\boxed{\text{CALC}}$

25. Using secant method, find a root of the equation $e^x \sin x - x^2 = 0$ correct up to three decimal places. **[2018/Fall]**

Solution:

Let, $f(x) = e^x \sin x - x^2$

and, $x_0 = 2$ and $x_1 = 3$ be two initial guesses.

$f(x_0) = e^2 \sin(2) - 2^2 = 2.7188$ and $f(x_1) = -6.1655$

Then, next approximated root by secant method is given by,

$$x_2 = x_1 - \frac{f(x_1)(x_1 - x_0)}{f(x_1) - f(x_0)}$$

$$= 3 - \frac{-6.1655(3-2)}{-6.1655 - 2.7188}$$

$$= 2.3060$$

$f(x_2) = 2.1246$

Now, solving other iterations in tabular form as follows,

Itn.	x_{n-1}	$f(x_{n-1}) = e^{x_{n-1}} \sin x_{n-1} - x_{n-1}^2$	x_n	$f(x_n) = e^{x_n} \sin x_n - x_n^2$	$x_{n+1} = x_n - \frac{f(x_n)(x_n - x_{n-1})}{f(x_n) - f(x_{n-1})}$	$f(x_{n+1}) = e^{x_{n+1}} \sin x_{n+1} - x_{n+1}^2$
1	2	2.7188	3	-6.1655	2.3060	2.1246
2	3	-6.1655	2.3060	2.1246	2.4838	1.1590
3	2.3060	2.1246	2.4838	1.1590	2.6972	-0.8958
4	2.4838	1.1590	2.6972	-0.8958	2.6041	0.1401
5	2.6972	-0.8958	2.6041	0.1401	2.6166	0.0144
6	2.6041	0.1401	2.6166	0.0144	2.6180	1.43×10^{-4}
7	2.6166	0.0144	2.6180	1.43×10^{-4}	2.6180	-8.70×10^{-7}

Here, the value of x_{n+1} do not change up to three decimal places.

Hence, the root of given equation is 2.6180.

> **NOTE:**
> Procedure to iterate in programmable calculator:
> Let, $A = x_{n-1}$, $B = f(x_{n-1})$, $C = x_n$, $D = f(x_n)$, $E = x_{n+1}$, $F = f(x_{n+1})$
> Set the following in calculator:
> $$A : C : B = e^A \sin A - A^2 : D = e^C \sin C - C^2 : E = C - \frac{D(C-A)}{D-B} :$$
> $$F = e^E \sin E - E^2$$
> $\boxed{\text{CALC}}$

26. Find where the graph of $y = x - 3$ and $y = ln(x)$ intersect using bisection method. Get the intersection value correct to four decimal places. **[2019/Fall]**

Solution:

$y = x - 3$ and $y = ln(x)$

$f(x_1) = x - 3$, $f(x_2) = ln(x)$

In order to intersect,

$$f(x_1) - f(x_2) = 0$$

i.e., $f(x) = x - 3 - \ln(x) = 0$

Let initial guess be,

$x = 1$, $f(1) = 1 - 3 - \ln(1) = -2$

$x = 2$, $f(2) = -1.6991 < 0$

$x = 3$, $f(3) = -1.0986 < 0$

$x = 4$, $f(4) = -0.3862 < 0$

$x = 5$, $f(5) = 0.3905 > 0$

so, root lies between $x = 4$ and $x = 5$.

$\therefore$ $x_L = 4$ and $x_U = 5$

Now, first approximated root using bisection method,

$$x_N = \frac{x_L + x_U}{2} = \frac{4+5}{2} = 4.5$$

$f(x_N) = -0.0040 < 0$ so now root lies between 4.5 and 5.

Remaining iterations are solved in tabular form.

Itn.	x_L	$f(x_L) = x_L - 3 - \ln(x_L)$	x_U	$f(x_U) = x_U - 3 - \ln(x_U)$	x_N	$f(x_N) = x_N - 3 - \ln(x_N)$
1	4	-0.3862	5	0.3905	4.5	-0.0040
2	4.5	-0.0040	5	0.3905	4.75	0.1918
3	4.5	-0.0040	4.75	0.1918	4.625	0.0935
4	4.5	-0.0040	4.625	0.0935	4.5625	0.0446
5	4.5	-0.0040	4.5625	0.0446	4.53125	0.0202
6	4.5	-0.0040	4.53125	0.0202	4.515625	0.0080
7	4.5	-0.0040	4.515625	0.0080	4.5078125	0.0020
8	4.5	-0.0040	4.5078125	0.0020	4.50390625	-0.0010
9	4.50390625	-0.0010	4.5078125	0.0020	4.505859	0.0004
10	4.503906	-0.0010	4.505859	0.0004	4.504882	-0.0002
11	4.504882	-0.0002	4.505859	0.0004	4.505370	-0.0001
12	4.504882	-0.0002	4.505370	0.0001	4.505126	-8.985×10^{-5}
13	4.505126	-8.985×10^{-5}	4.505370	0.0001	4.505248	5.060×10^{-6}
14	4.505126	-8.985×10^{-5}	4.505248	5.060×10^{-6}	4.505187	-4.239×10^{-5}
15	4.505187	-4.239×10^{-5}	4.505248	5.060×10^{-6}	4.5052175	-1.866×10^{-5}
16	4.5052175	-1.866×10^{-5}	4.505248	5.060×10^{-6}	4.505232	-6.804×10^{-6}

Here, the value of x_N do not change up to 4 decimal places.

Hence, the graph of $y = x - 3$ and $y = ln(x)$ intersects at $x = 4.505232$.

NOTE:
Procedure to iterate in programmable calculator:
Let, $A = x_L$, $B = f(x_L)$, $C = x_U$, $D = f(x_U)$, $E = x_N$, $F = f(x_N)$
Set the following in calculator:

$$A : C : B = A - 3 - ln(A) : D = C - 3 - ln(C) : E = \frac{A + C}{2} : F = E - 3 - ln(E)$$

CALC

27. Find value of $\sqrt{18}$ using Newton Raphson method. **[2019/Fall]**

Solution:

Let $x = \sqrt{N}$ or $x^2 - N = 0$

Taking $f(x) = x^2 - N$, we have $f'(x) = 2x$

Then Newton's formula gives,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^2 - N}{2x_n} = \frac{1}{2}\left(x_n + \frac{N}{x_n}\right)$$

Now, taking $N = 18$, the above formula becomes

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{18}{x_n}\right)$$

For initial guess, taking approximate value of $\sqrt{18}$

i.e., $\sqrt{18} = \sqrt{4^2} \doteq \sqrt{16} = 4$

i.e., we take $x_0 = 4$

Then,

$$x_1 = \frac{1}{2}\left(x_n + \frac{18}{x_0}\right) = \frac{1}{2}\left(4 + \frac{18}{4}\right) = 4.25$$

$$x_2 = \frac{1}{2}\left(x_1 + \frac{18}{x_1}\right) = \frac{1}{2}\left(4.25 + \frac{18}{4.25}\right) = 4.2426$$

$$x_3 = \frac{1}{2}\left(x_2 + \frac{18}{x_2}\right) = \frac{1}{2}\left(4.2426 + \frac{18}{4.2426}\right) = 4.2426$$

Here, $x_2 = x_3$ up to 4 decimal places.

Hence, the value of $\sqrt{18}$ is 4.2426.

28. Using secant method, find the zero of function $f(x) = 2x - \log_{10} x - 7$ correct up to three decimal places. **[2019/Spring]**

Solution:

$$f(x) = 2x - \log_{10} x - 7$$

Let, $x_0 = 1$ and $x_1 = 2$ be two initial guesses.

NOTE:
0 is not taken as initial guess because it gives the undetermined value of f(x) at x = 0.

Then, next approximated root by secant method is given by,

$$x_2 = x_1 - \frac{f(x_1)(x_1 - x_0)}{f(x_1) - f(x_0)}$$

$$= 2 - \frac{(-3.3010)(2-1)}{-3.3010-(-5)}$$

$$= 3.9429$$

$$f(x_2) = 0.2899$$

Now, solving other iterations in tabular form as follows

Itn.	x_{n-1}	$f(x_{n-1}) = 2x_{n-1} - \log_{10}x_{n-1} - 7$	x_n	$f(x_n) = 2x_n - \log_{10}x_n - 7$	$x_{n+1} = x_n - \frac{f(x_n)(x_n - x_{n-1})}{f(x_n) - f(x_{n-1})}$	$f(x_{n+1}) = 2x_{n+1} - \log_{10}x_{n+1} - 7$
1	1	-5	2	-3.3010	3.9429	0.2899
2	2	-3.3010	3.9429	0.2899	3.7860	-6.180×10^{-3}
3	3.9429	0.2899	3.7860	-6.180×10^{-3}	3.7892	-1.475×10^{-3}
4	3.7860	-6.180×10^{-3}	3.7892	-1.475×10^{-3}	3.7902	-0.1508
5	3.7892	-1.475×10^{-3}	3.7902	-0.1508	3.7891	-3.360×10^{-4}
6	3.7902	-0.1508	3.7891	-3.360×10^{-4}	3.7890	-5.246×10^{-4}
7	3.7891	-3.360×10^{-4}	3.7890	-5.246×10^{-4}	3.7892	-1.475×10^{-4}

Here, the value of x_{n+1} do not change up to 3 decimal places.

Hence, the zero of function f(x) is 3.7892.

NOTE:

Procedure to iterate in programmable calculator:

Let, $A = x_{n-1}$, $B = f(x_{n-1})$, $C = x_n$, $D = f(x_n)$, $E = x_{n+1}$, $F = f(x_{n+1})$

Set the following in calculator:

$$A : C : B = 2A - \log_{10} A - 7 : D = 2C - \log_{10} C - 7 : E = C - \frac{D(C-A)}{D-B} :$$

$$F = 2E - \log_{10} E - 7$$

$\boxed{\text{CALC}}$

29. Find the root of the equation log x − cos x = 0 correct up to three decimal placed by using N-R method. [2019/Spring]

Solution:

Let, $f(x) = \log x - \cos x$ ..... (1)

Differentiating equation (1) with respect to x,

$$f'(x) = \frac{1}{x} + \sin x \quad \text{..... (2)}$$

From equation (1),

Let the initial guess be,

$$x_0 = 1, f(x_0) = -0.5403, f'(x_0) = 1.8414$$

Using Newton Raphson method, next approximated root is

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 1 - \frac{-0.5403}{1.8414} = 1.2934$$

$f(x_1) = -0.1621$

Now, continuing process in tabular form.

Iteration	x_n	$f(x_n) = \log x_n - \cos x_n$	$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$	$f(x_{n+1}) = \log x_{n+1} - \cos x_{n+1}$
1	1	0.5403	1.2934	-0.1621
2	1.2934	-0.1621	1.3868	-0.0409
3	1.3868	-0.0409	1.4107	-9.97×10^{-3}
4	1.4107	-9.97×10^{-3}	1.4165	-2.46×10^{-3}
5	1.4165	-2.46×10^{-3}	1.4179	-6.55×10^{-4}
6	1.4179	-6.55×10^{-4}	1.4182	-2.67×10^{-4}
7	1.4182	-2.67×10^{-4}	1.4183	-1.37×10^{-4}

Here, the value of x_{n+1} do not change up to 3 decimal places.

Hence, the desired root is 1.4183 of the equation.

NOTE:

Procedure to iterate in programmable calculator:

Let, $A = x_n$, $B = f(x_n)$, $C = x_{n+1}$, $D = f(x_{n+1})$

Set the following in calculator:

$$A : B = \log A - \cos A : C = A - \frac{B}{\sin A + \frac{1}{A}} : D = \log C \cos C$$

CALC

30. Find the positive real root of the equation $\cos x + e^x + x^2 = 3$. Using false position method, correct to 3 decimal places **[2020/Fall]**

Solution:

Let, $(x) = \cos x + e^x \pm x^2 - 3$

The initial guess be,

$x_0 = 0$, $\quad f(x_0) = \cos 0 + e^0 + 0^2 - 3 = -1 < 0$

$x_1 = 1$, $\quad f(x_1) = \cos(1) + e^1 + 1^2 - 3 = 1.2585 > 0$

so, root lies between 0 and 1.

Now, using false position method,

$$x_2 = x_0 - \frac{(x_1 - x_0) f(x_0)}{f(x_1) - f(x_0)}$$

$$= 0 - \frac{(1-0)(-1)}{1.2585 - (-1)} = 0.4427$$

$\therefore \quad f(x_2) = -0.3435$

Since the value of $f(x_2)$ is negative, now root lies between 0.4427 and 1.

Solving other iterations in tabular form as follows,

Itn.	x_L	$f(x_L) = \cos x_L + e^{x_L} + x_L^2 - 3$	x_U	$f(x_U) = \cos x_U + e^{x_U} + x_U^2 - 3$	$x_N = x_L - \frac{f(x_L)(x_U - x_L)}{f(x_U) - f(x_L)}$	$f(x_N) = \cos x_N + e^{x_N} + x_N^2 - 3$
1	0	−1	1	1.2585	0.4427	−0.3435
2	0.4427	−0.3435	1	1.2585	0.5621	−0.0835
3	0.5621	−0.0835	1	1.2585	0.5893	−0.0186
4	0.5893	−0.0186	1	1.2585	0.5952	-4.30×10^{-3}
5	0.5952	-4.30×10^{-3}	1	1.2585	0.5965	-1.12×10^{-3}
6	0.5965	-1.12×10^{-3}	1	1.2585	0.5968	-3.94×10^{-4}

Here, the value of x_N do not change up to three decimal places.

Hence, the positive real root of the equation is 0.5968.

NOTE:

Procedure to iterate in programmable calculator:

Let, $A = x_L$, $B = f(x_L)$, $C = x_U$, $D = f(x_U)$, $E = x_N$, $F = f(x_N)$

Set the following in calculator:

$$A : C : B = \cos A + e^A + A^2 - 3 : D = \cos C + e^C + C^2 - 3 :$$

$$E = A - \frac{(C - A)B}{D - B} : F = \cos E + e^E + E^2 - 3$$

CALC

31. Find the real root of the equation x sin x − cos x = 0 using Newton-Raphson method, correct to 3 decimal places. [2020/Fall]

Solution:

Let, $f(x) = x \sin x - \cos x$ ..... (1)

Differentiating equation (1) with respect to x,

$$f'(x) = x \cos x + \sin x + \sin x \quad \text{..... (2)}$$

$$= x \cos x + 2 \sin x$$

From equation (1),

Let the initial guess be,

$x_0 = 1$, $f(x_0) = 0.3011$, $f'(x_0) = 2.2232$

Using NR method, next approximated root is

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 1 - \frac{0.3011}{2.2232} = 0.8645$$

$$f(x_1) = 8.66 \times 10^{-3}$$

Now, continuing process in tabular form.

Itn.	x_n	$f(x_n) = x_n \sin x_n - \cos x_n$	$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$	$f(x_{n+1}) = x_{n+1} \sin x_{n+i} - \cos x_{n+1}$
1	1	0.3011	0.8645	8.66×10^{-3}
2	0.8645	8.66×10^{-3}	0.8603	-6.97×10^{-5}
3	0.8603	-6.97×10^{-5}	0.8603	-7.02×10^{-8}

Here, the value of x_{n+1} do not change up to 3 decimal places.

Hence, the desired root of the equation is 0.8603.

NOTE:
Procedure to iterate in programmable calculator:
Let, $A = x_n$, $B = f(x_n)$, $C = x_{n+1}$, $D = f(x_{n+1})$
Set the following in calculator:

$$A : B = A \sin A - \cos A : C = A - \frac{B}{A \cos A + 2 \sin A} :$$

$$D = C \sin C - \cos C$$

CALC

32. Find the real root of the equation $x \log_{10} x - 1.2 = 0$ correct to four places of decimal using bracketing method. **[2014/Fall]**

Solution:

Let. $f(x) = x \log_{10} x - 1.2$

Let initial guess be,

$x = 2.5$, $f(2.5) = -0.2051 < 0$

$x = 3$, $f(3) = 0.2313 > 0$

so, root lies between $x = 2.5$ and $x = 3$

$\therefore$ $x_L = 2.5$ and $x_U = 3$

Now, first approximated root using bisection method,

$$x_N = \frac{x_L + x_U}{2} = \frac{2.5 + 3}{2} = 2.75$$

$f(x_N) = 8.16 \times 10^{-3} > 0$ so now root lies between 2.5 and 2.75

Remaining iterations are solved in tabular form.

Itn.	x_L	$f(x_L) = x_L \log_{10} x_L - 1.2$	x_U	$f(x_U) = x_U \log_{10} x_U - 1.2$	$x_N = \frac{x_L + x_U}{2}$	$f(x_N) = x_N \log_{10} x_N - 1.2$
1	2.5	−0.2051	3	0.2313	2.75	8.16×10^{-3}
2	2.5	−0.2051	2.75	8.160×10^{-3}	2.625	−0.0997
3	2.625	−0.0997	2.75	8.16×10^{-3}	2.6875	−0.0461
4	2.6875	−0.0461	2.75	8.16×10^{-3}	2.7187	−0.0191
5	2.7187	−0.0191	2.75	8.16×10^{-3}	2.7343	-5.53×10^{-3}
6	2.7343	-5.53×10^{-3}	2.75	8.16×10^{-3}	2.7421	1.26×10^{-3}
7	2.7343	-5.53×10^{-3}	2.7421	1.26×10^{-3}	2.7382	-2.13×10^{-3}
8	2.7382	-2.13×10^{-3}	2.7421	1.26×10^{-3}	2.7401	-4.76×10^{-4}
9	2.7401	-4.76×10^{-4}	2.7421	1.26×10^{-3}	2.7411	3.95×10^{-4}
10	2.7401	-4.76×10^{-4}	2.7411	3.95×10^{-4}	2.7406	-4.02×10^{-5}
11	2.7406	-4.02×10^{-5}	2.7411	3.95×10^{-4}	2.7408	1.34×10^{-4}
12	2.7406	-4.02×10^{-5}	2.7408	1.34×10^{-4}	2.7407	4.70×10^{-5}
13	2.7406	-4.02×10^{-5}	2.7407	4.70×10^{-5}	2.7406	-4.02×10^{-5}
14	2.7406	-4.02×10^{-5}	2.7407	4.70×10^{-5}	2.7406	-4.022×10^{-5}