BOARD EXAMINATION SOLVED QUESTIONS

1. **Solve $\frac{dy}{dx} = y - \frac{2x}{y}$, $y(0) = 1$ in the range $0 \le x \le 0.2$ by using (i) Euler's method and (ii) Huen's method. Comment on the results. Take h = 0.2.** **[2013/Fall]**

Solution:

$$\frac{dy}{dx} = y - \frac{2x}{y} \text{ and } y(0) = 1$$

$$\Rightarrow x_0 = 0 \text{ and } y_0 = 1$$

Also, $h = 0.2$, $0 \le x \le 0.2$

i) From Euler's method,

$$f(x_0, y_0) = y_0 - \frac{2x_0}{y_0} = 1 - \frac{2(0)}{1} = 1$$

Now,

$$y_1 = y_{new} = y_0 + hf(x_0, y_0)$$

or, $y_{new} = y_{old} + h\frac{dy}{dx} = 1 + 0.2(1)$

$\therefore$ $y_1 = 1.2$

iii) From Huen's method or modified Euler's method

$$h = 0.2$$

Solving in tabular form

S.N.	x	$\frac{dy}{dx} = y - \frac{2x}{y}$	Mean slope	$y_{new} = y_{old} + h$ (mean slope)
1	0	$1 - \frac{2(0)}{1} = 1$	-	$1 + 0.2 \times 1 = 1.2$
2	0.2	$1.2 - \frac{2(0.2)}{1.2}$ $= 0.8667$	$\frac{1}{2}(1 + 0.8667)$ $= 0.9333$	$1 + 0.2 \times 0.9333 = 1.1866$
3	0.2	$1.1866 - \frac{2(0.2)}{1.1866}$ $= 0.8495$	$\frac{1}{2}(1 + 0.8495)$ $= 0.9247$	$1 + 0.2 \times 0.9247 = 1.1849$

Here the last two values are equal at $y_1 = 1.1849$.

The result from Euler's method is 1.2 and from Huen's method is 1.1849 which shows better result and we prefer Huen's method or modified Euler's method.

2. **Using Runge Kutta method of order 4, solve the equation $\frac{d^2y}{dx^2} = 6xy^2 + y$, $y(0) = 1$ and $y'(0) = 0$ to find $y(0.2)$ and $y'(0.2)$. Take h = 0.2.** **[2013/Fall]**

Solution:

$$\frac{d^2y}{dx^2} - 6xy^2 - y = 0$$

or, $y'' - 6xy^2 - y = 0$ ..... (1)

Also, $y(0) = 1$ and $y'(0) = 0$ and $h = 0.2$

Let, $y' = z = f_1(x, y, z)$ ..... (A)

so, $y'' = z'$, then equation (1) becomes

$z' = 6xy^2 + y = f_2(x, y, z)$ ..... (B)

Given that;

$y(0) = 1 \quad \Rightarrow x_0 = 0, \quad y_0 = 1$

and, $y'(0) = 0 = z_0$

Now, using RK method to find increment value of k and l

$$k_1 = hf_1(x_0, y_0, z_0) \qquad \text{at equation (A)}$$
$$= hf_1(0, 1, 0)$$
$$= 0$$

$$l_1 = hf_2(x_0, y_0, z_0) \qquad \text{at equation (B)}$$
$$= hf_2(0, 1, 0)$$
$$= (6(0)(1)^2 + 1)\, 0.2$$
$$= 0.2$$

Likewise,

$$k_2 = hf_1\left(x_0 + \frac{h}{2},\ y_0 + \frac{k_1}{2},\ z_0 + \frac{l_1}{2}\right)$$
$$= 0.2f_1\left(0 + \frac{0.2}{2},\ 1 + \frac{0}{2},\ 0 + \frac{0.2}{2}\right)$$
$$= 0.2 \times 0.1$$
$$= 0.02$$

$$l_2 = hf_2(0.1, 1, 0.1)$$
$$= 0.2\,[6(0.1)(1)^2 + 1]$$
$$= 0.32$$

$$k_3 = hf_1\left(x_0 + \frac{h}{2},\ y_0 + \frac{k_2}{2},\ z_0 + \frac{l_2}{2}\right)$$
$$= 0.2f_1(0.1, 1.01, 0.16)$$
$$= 0.2 \times 0.16$$
$$= 0.032$$

$$l_3 = hf_2(0.1, 1.01, 0.16)$$
$$= 0.2\,[6(0.1)(1.01)^2 + 1.01] = 0.324$$

$$k_4 = hf_1(x_0 + h, y_0 + k_3, z_0 + l_3)$$
$$= 0.2f_1(0.2, 1.032, 0.324)$$
$$= 0.2 \times 0.324 = 0.064$$

$$l_4 = hf_2(0.2, 1.032, 0.324) = 0.462$$

Now,

$$k = \frac{1}{6}[k_1 + k_4 + 2(k_2 + k_3)]$$

$$= \frac{1}{6}[0 + 0.064 + 2(0.02 + 0.032)]$$

$$= 0.028$$

$$l = \frac{1}{6}[l_1 + l_4 + 2(l_2 + l_3)]$$

$$= \frac{1}{6}[0.2 + 0.462 + 2(0.32 + 0.324)]$$

$$= 0.325$$

Now,

$$y_1 = y_0 + k = 1 + 0.028 = 1.028$$

and, $z_1 = z_0 + l = 0 + 0.325 = 0.325$

are the required answer for $y'(0.2)$ and $y(0.2)$.

3. Use the Runge-Kutta 4th order method to estimate y(0.2) of the following equation with h = 0.1

$$y'(x) = 3x + \frac{1}{2}y,\ y(0) = 1$$ **[2013/Spring]**

Solution:

Given that,

$$y'(x) = 3x + 0.5y$$

and, $y(0) = 1$

→ $x_0 = 0,\ y_0 = 1,\ h = 0.1$

Now, using RK method to find increment on k

$$k_1 = hf(x_0, y_0)$$
$$= 0.1f(0, 1)$$
$$= 0.1[3(0) + 0.5(1)]$$
$$= 0.05$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$
$$= 0.1f\left(0 + \frac{0.1}{2}, 1 + \frac{0.05}{2}\right)$$
$$= 0.1f(0.05, 1.025)$$
$$= 0.0662$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$$
$$= 0.1f(0.05, 1.033)$$
$$= 0.0666$$

$$k_4 = hf(x_0 + h, y_0 + k_3)$$
$$= 0.1f(0.1, 1.0666)$$
$$= 0.0833$$

Now,

$$k = \frac{1}{6}[k_1 + k_4 + 2(k_2 + k_3)]$$

$$= \frac{1}{6}[0.05 + 0.0833 + 2(0.0662 + 0.0666)]$$

$$= 0.0664$$

$\therefore$ $x_1 = x_0 + h = 0 + 0.1 = 0.1$

$y_1 = y_0 + k = 1 + 0.0664 = 1.0664$

Again,

$x_1 = 0.1,\ y_1 = 1.0664,\ h = 0.1$

Then,

$$k_1 = hf(x_1, y_1) = 0.1f(0.1, 1.0664) = 0.0833$$

$$k_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right)$$

$$= 0.1f(0.15, 1.1080)$$

$$= 0.1004$$

$$k_3 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right)$$

$$= 0.1f(0.15, 1.1166)$$

$$= 0.1008$$

$$k_4 = hf(x_1 + h, y_1 + k_3)$$

$$= 0.1f(0.2, 1.1672)$$

$$= 0.1183$$

Now,

$$k = \frac{1}{6}[k_1 + k_4 + 2(k_2 + k_3)]$$

$$= \frac{1}{6}[0.0833 + 0.1183 + 2(0.1004 + 0.1008)]$$

$$= 0.1006$$

So,

$x_2 = x_1 + h = 0.1 + 0.1 = 0.2$

$y_2 = y_1 + k = 1.0664 + 0.1006 = 1.167$

is the required estimated value of $y(0.2)$.

4. **Solve the following equation by Picard's method and estimate y(0.1), y(0.2) and y(1).** **[2013/Spring]**

$$y'(x) = x^2 + y^2,\ y(0) = 0$$

Solution:

Given that;

$y'(x) = x^2 + y^2$, $\quad y(0) = 0$

$x_0 = 0, \quad y_0 = 0$

Now, using Picard's method

$$y = y_0 + \int_{x_0}^{x} f(x, y_0)\,dx = 0 + \int_0^x (x^2 + y^2)\,dx$$

First approximation, put y = 0 in the integrand

$$y_1 = 0 + \int_0^x (x^2 + 0^2)\,dx$$

$$= \int_0^x (x^2)\,dx = \left[\frac{x^3}{3}\right]_0^x = \frac{x^3}{3}$$

Second approximation, put $y = \frac{x^3}{3}$ in the integrand

$$y_2 = 0 + \int_0^x \left[x^2 + \left(\frac{x^3}{3}\right)^2\right] dx$$

$$= \int_0^x \left(x^2 + \frac{x^6}{9}\right) dx$$

$$= \left[\frac{x^3}{3} + \frac{x^7}{63}\right]_0^x$$

$$= \frac{x^3}{3} + \frac{x^7}{63}$$

Further processing of this task is difficult from here so we stop at

$$y_2 = \frac{x^3}{3} + \frac{x^7}{63}$$

Now, using the second approximation and taking

x = 0.1, 0.2 and 1

We have,

$$y(0.1) = \frac{(0.1)^3}{3} + \frac{(0.1)^7}{63} = 0.000033$$

$$y(0.2) = \frac{(0.2)^3}{3} + \frac{(0.2)^7}{63} = 0.0026$$

$$y(1) = \frac{(1)^3}{3} + \frac{(1)^7}{63} = 0.3492$$

5. **Given: $\frac{dy}{dx} = \frac{2x + e^x}{x^2 + xe^x}$; y(1) = 0. Solve for y at x = 1.04, by using Euler's method (take h = 0.01).** **[2014/Fall]**

Solution:

Given that;

$$\frac{dy}{dx} = \frac{2x + e^x}{x^2 + xe^x}$$

$y(1) = 0$, $h = 0.01$

$x_0 = 1$, $y_0 = 0$

Using Euler's method, in tabular form

S.N.	x	y	$\frac{dy}{dx} = \frac{2x + e^x}{x^2 + xe^x}$	$y_{new} = y_{old} + h\frac{dy}{dx}$
1	1	0	1.268	0 + 0.01 (1.268) = 0.0126
2	1.01	0.0126	1.256	0.0126 + 0.01 (1.256) = 0.0251
3	1.02	0.0251	1.244	0.0375
4	1.03	0.0375	1.231	0.0498
5	1.04	0.0498		

Hence the required solution at x = 1.04 for y is 0.0498.

6. **Solve $\frac{dy}{dx} = 1 + xz$, $\frac{dz}{dx} = -xy$ for y(0.6) and z(0.6), given that y = 0, z = 1 at x = 0 by using Heun's method. Assume, h = 0.3.** **[2014/Fall]**

Solution:

$$\frac{dy}{dx} = 1 + xz, \quad x_0 = 0, \quad y_0 = 0, \quad h = 0.3$$

and, $$\frac{dz}{dx} = -xy, \quad x_0 = 0, \quad y_0 = 0, \quad h = 0.3$$

Using Heun's method, solving in tabular form

S.N.	x	$\frac{dy}{dx} + 1 + xz$	Mean slope	$y_{new} = y_{old} + h$ (mean slope)
1	0	1 + (0) (1)	-	0 + 0.3 × 1 = 0.3
2	0.3	1 + (0.3) (1) = 1.3	$\frac{1 + 1.3}{2} = 1.15$	0 + 0.3 × 1.15 = 0.345
3	0.3	1 + (0.3) (0.9865) = 1.295	$\frac{1 + 1.295}{2} = 1.147$	0 + 0.3 × 1.147 = 0.344
4	0.3	1 + (0.3) (0.9847) = 1.295	$\frac{1 + 1.295}{2} = 1.147$	0 + 0.3 × 1.147 = 0.344

Here, the last two values are equal at $y_1 = 0.344$

S.N.	x	$\frac{dy}{dx} = -xy$	Mean slope	$z_{new} = z_{old} + h$ (mean slope)
1	0	(−0) (0)	-	1 + 0.3 × 0 = 1
2	0.3	−(0.3) (0.3) = −0.09	$\frac{0 - 0.09}{2} = -0.045$	1 + 0.3 × −0.045 = 0.9865
3	0.3	−(0.3) (0.344) = −0.103	$\frac{0 - 0.103}{2} = -0.051$	1 + 0.3 × −0.051 = 0.9847
4	0.3	−(0.3) (0.344) = −0.103	$\frac{0 - 0.103}{2} = -0.051$	1 + 0.3 × −0.051 = 0.9847

Here, the last two values are equal at $z_1 = 0.9847$.

NOTE:

Use both tables to use the value of y_{new} and z_{new} to calculate $\frac{dy}{dx}$ and $\frac{dz}{dx}$

Again,

S.N.	x	$\frac{dy}{dx} + 1 + xz$	Mean slope	$y_{new} = y_{old} + h$ (mean slope)
1	0.3	$1 + (0.3)(0.9847)$ $= 1.295$	–	$0.344 + 0.3(1.295)$ $= 0.732$
2	0.6	$1 + (0.6)(0.9538)$ $= 1.572$	$\frac{1.295 + 1.572}{2}$ $= 1.433$	$0.344 + 0.3(1.433)$ $= 0.773$
3	0.6	$1 + (0.6)(0.899)$ $= 1.539$	$\frac{1.295 + 1.539}{2}$ $= 1.417$	$0.344 + 0.3(1.417)$ $= 0.769$
4	0.6	$1 + (0.6)(0.9)$ $= 1.54$	$\frac{1.295 + 1.54}{2}$ $= 1.417$	$0.344 + 0.3(1.417)$ $= 0.769$

Here, the last two values are equal at $y_2 = 0.769$.

S.N.	x	$\frac{dy}{dx} = -xy$	Mean slope	$z_{new} = z_{old} + h$ (mean slope)
1	0.3	$-(0.3)(0.344)$ $= -0.103$	–	$0.9847 + 0.3(-0.103)$ $= 0.9538$
2	0.6	$-(0.6)(0.773)$ $= -0.463$	$\frac{-0.103 - 0.463}{2}$ $= -0.283$	$0.9847 + 0.3(-0.283)$ $= 0.899$
3	0.6	$-(0.6)(0.769)$ $= -0.461$	$\frac{-0.103 - 0.461}{2}$ $= -0.282$	$0.9847 + 0.3(-0.282)$ $= 0.900$
4	0.6	$-(0.6)(0.769)$ $= -0.461$	$\frac{-0.103 - 0.461}{2}$ $= -0.282$	$0.9847 + 0.3(-0.282)$ $= 0.9$

Here, the last two values are equal at $z_2 = 0.9$.

Hence, the required values of $y(0.6) = 0.769$ and $z(0.6) = 0.9$.

7. Using R-K fourth order method, solve the given differential equation $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 3y = 6$, $y(0) = 0$, $y'(0) = 1$ with $h = 0.2$ for $y(0.4)$?

[2014/Spring]

Solution:

Given that;

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 3y = 6$$

or, $y'' + 2y' - 3y = 6$ ..... (1)

Also, $y(0) = 0$, $y'(0) = 1$, $h = 0.2$

$\rightarrow$ $x_0 = 0$, $y_0 = 0$

Let, $y' = z = f_1(x, y, z)$

so, $y'' = z'$ then (1) becomes

$z' = 6 + 3y - 2z = f_2(x, y, z)$

Subject to

$y'(0) = 1$

$\rightarrow$ $z_0 = 1$

Now, using RK method to find increments,

$$k_1 = hf_1(x_0, y_0, z_0)$$
$$= 0.2f_1(0, 0, 1) = 0.2$$
$$l_1 = hf_2(x_0, y_0, z_0)$$
$$= 0.2[6 + 3(0) - 2(1)] = 0.8$$
$$k_2 = hf_1\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right)$$
$$= 0.2f_1(0.1, 0.1, 1.4) = 0.28$$
$$l_2 = hf_2(0.1, 0.1, 1.4)$$
$$= 0.2[6 + 3(0.1) - 2(1.4)] = 0.7$$
$$k_3 = hf_1\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right)$$
$$= 0.2f_1(0.1, 0.14, 1.35) = 0.27$$
$$l_3 = hf_2(0.1, 0.14, 1.35) = 0.744$$
$$k_4 = hf_1(x_0 + h, y_0 + k_3, z_0 + l_3)$$
$$= 0.2f_1(0.2, 0.27, 1.744) = 0.348$$
$$l_4 = hf_2(0.2, 0.27, 1.744) = 0.664$$

Now,

$$k = \frac{1}{6}[k_1 + k_4 + 2(k_2 + k_3)]$$
$$= \frac{1}{6}[0.2 + 0.348 + 2(0.28 + 0.27)] = 0.274$$
$$l = \frac{1}{6}[l_1 + l_4 + 2(l_2 + l_3)]$$
$$= \frac{1}{6}[0.8 + 0.664 + 2(0.7 + 0.744)] = 0.725$$

Then,

$$y_1 = y_0 + k = 0 + 0.274 = 0.274$$
$$z_1 = z_0 + l = 1 + 0.725 = 1.725$$

Again, $x_1 = x_0 + h = 0 + 0.2 = 0.2$

$y_1 = 0.274$

$z_1 = 1.725$

Using RK method to find increment on k and l.

$$k_1 = hf_1(x_1, y_1, z_1)$$
$$= 0.2f_1(0.2, 0.274, 1.725)$$
$$= 0.345$$

$$l_1 = hf_2(x_1, y_1, z_1)$$
$$= 0.2(0.2, 0.274, 1.725)$$
$$= 0.674$$

$$k_2 = hf_1\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}, z_1 + \frac{l_1}{2}\right)$$
$$= 0.2f_1(0.3, 0.4465, 2.062)$$
$$= 0.412$$

$$l_2 = hf_2(0.3, 0.4465, 2.062)$$
$$= 0.643$$

$$k_3 = hf_1\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}, z_1 + \frac{l_2}{2}\right)$$
$$= 0.2f_1(0.3, 0.48, 2.046)$$
$$= 0.409$$

$$l_3 = hf_2(0.3, 0.48, 2.046)$$
$$= 0.669$$

$$k_4 = hf_1(x_1 + h, y_1 + k_3, z_1 + l_3)$$
$$= 0.2f_1(0.4, 0.683, 2.394)$$
$$= 0.478$$

$$l_4 = hf_2(0.4, 0.683, 2.394)$$
$$= 0.652$$

Then,
$$k = \frac{1}{6}[k_1 + k_4 + 2(k_2 + k_3)]$$
$$= \frac{1}{6}[0.345 + 0.478 + 2(0.412 + 0.409)$$
$$= 0.410$$

$$l = \frac{1}{6}[l_1 + l_4 + 2(l_2 + l_3)]$$
$$= \frac{1}{6}[0.674 + 0.652 + 2(0.643 + 0.669)]$$
$$= 0.658$$

Now,

$x_2 = x_1 + h = 0.2 + 0.2 = 0.4$

$\therefore$ $y_2 = y(0.4) = y_1 + k = 0.274 + 0.410 = 0.684$

8. Given the boundary value problem: y" = 6x with y(1) = 2 and y(2) = 9.. Solve it in the interval (1, 2) by using RK method of second order (take, h = 0.5 and guess value = 3.25) **[2014/Spring]**

Solution:

Given that;

$$y'' = 6x \quad \text{..... (1)}$$

$$y(1) = 2$$

$$y(2) = 9$$

$$h = 0.5$$

Let, $y' = z = f_1(x, y, z)$

$$y'' = z'$$

So equation (1) becomes,

$$z' = 6x = f_2(x, y, z)$$

Subjected to

$$y(1) = 2 = z(1)$$

Initial guess value = 3.25

Now, from RK method of second order

Iteration 1:

$$x_0 = 1,\ y_0 = 2,\ z_0 = 2$$

$$k_1 = hf_1(x_0, y_0, z_0) = 0.5f_1(1, 2, 2) = 0.5 \times 2 = 1$$

$$l_1 = hf_2(x_0, y_0, z_0) = 0.5 \times 6 \times 1 = 3$$

$$k_2 = hf_1(x_0 + h, y_0 + k_1, z_0 + l_1) = 0.5f_1(1.5, 3, 5) = 0.5 \times 5 = 2.5$$

$$l_2 = hf_2(1.5, 3, 5) = 0.5 \times 6 \times 1.5 = 4.5$$

Then, $k = \frac{1}{2}(k_1 + k_2) = \frac{1}{2}(1 + 2.5) = 1.75$

$$l = \frac{1}{2}(l_1 + l_2) = \frac{1}{2}(4.5 + 3) = 3.75$$

So,

$$y_1 = y_0 + k = 2 + 1.75 = 3.75$$

$$z_1 = z_0 + l = 2 + 3.75 = 5.75$$

$$x_1 = x_0 + h = 1 + 0.5 = 1.5$$

[illegible]

[illegible]

$$[illegible]$$

$$[illegible]$$

$$[illegible]$$

$$[illegible]$$

$$[illegible]$$

$$[illegible]$$

$$[illegible]$$

$$[illegible]$$

$$[illegible]$$

$$[illegible]$$

$$[illegible]$$

$$[illegible]$$

$$[illegible]$$

$$[illegible]$$

$$[illegible]$$

$$[illegible]$$

$$[illegible]$$

$$[illegible]$$

$$[illegible]$$

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

$$[illegible]$$

$$[illegible]$$

$$[illegible]$$

[illegible]

[illegible]

$$[illegible]$$

$$[illegible]$$

[illegible]

$$[illegible]$$

$$[illegible]$$

$$[illegible]$$

Now, using Euler's method

$$y_1 = y(0.2) = y_0 + h\frac{dy_0}{dx_0} = 0 + 0.2\ (z_0) = 0.2 \times 1 = 0.2$$

$$z_1 = z(0.2) = z_0 + hz'(x_0) \text{ from equation (B)}$$
$$= 1 + 0.2\ (6 + 3y_0 - 2z_0)$$
$$= 1 + 0.2\ [6 + 3(0) - 2(1)] = 1.8$$

Again, $y(0.4) = y_1 + hy'(x_1) = y_1 + h\dfrac{dy_1}{dx_1} = y_1 + h(z_1)$

$$= 0.2 + 0.2(1.8) = 0.56$$

$\therefore$ $y(0.4) = 0.56$ is the required solution.

10. Solve the following differential equation within $0 \le x \le 0.5$ using RK 4th order method. $20\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} - 4y = 5$, $y(0) = 0$, $y'(0) = 0$. Take $h = 0.25$. **[2015/Fall]**

Solution:

Given that;

$$20\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 4y = 5$$

Let, $\dfrac{dy}{dx} = y' = z = f_1\ (x, y, z)$

Then, $\dfrac{d^2y}{dx^2} = y'' = z' = f_2\ (x, y, z)$

or, $20z' + 2z - 4y = 5$

or, $z' = \dfrac{5 - 2z + 4y}{20} = f_2\ (x, y, z)$

Subject to

$$y(0) = 0 \rightarrow x_0 = 0,\ y_0 = 0$$
$$y'(0) = 0 \rightarrow z_0 = 0$$

and, $h = 0.25$

Now, by RK 4th order method

$$k_1 = hf_1\ (x_0, y_0, z_0)$$
$$= 0.25f_1\ (0, 0, 0)$$
$$= 0$$

$$l_1 = hf_2\ (x_0, y_0, z_0)$$
$$= 0.25\left(\frac{5 - 0 + 0}{20}\right)$$
$$= 0.0625$$

$$k_2 = hf_1\left(x_0 + \frac{h}{2},\ y_0 + \frac{k_1}{2},\ z_0 + \frac{l_1}{2}\right)$$
$$= hf_1\ (0.125, 0, 0.03125)$$
$$= 0.25 \times 0.03125$$
$$= 0.00781$$

$$l_2 = hf_2(0.125, 0, 0.03125)$$
$$= 0.25 \times \left(\frac{5 - 2(0.03125) + 4(0)}{20}\right)$$
$$= 0.06171$$
$$k_3 = hf_1\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right)$$
$$= 0.25\,(0.125, 0.0039, 0.0308)$$
$$= 0.0077$$
$$l_3 = hf_2(0.125, 0.0039, 0.0308)$$
$$= 0.0619$$
$$k_4 = hf_1(x_0 + h, y_0 + k_3, z_0 + l_3)$$
$$= 0.25f_1(0.25, 0.0077, 0.0619)$$
$$= 0.0154$$
$$l_4 = hf_2(0.25, 0.0077, 0.0619)$$
$$= 0.0613$$

Then, $$k = \frac{1}{6}[k_1 + k_4 + 2(k_2 + k_3)]$$
$$= \frac{1}{6}[0 + 0.0154 + 2(0.00781 + 0.0077)]$$
$$= 0.0077$$
$$l = \frac{1}{6}[l_1 + l_4 + 2(l_2 + l_3)]$$
$$= \frac{1}{6}[0.0625 + 0.0613 + 2(0.06171 + 0.0619)]$$
$$= 0.0618$$

so, $$x_1 = x_0 + h = 0 + 0.25 = 0.25$$
$$y_1 = y_0 + k = 0 + 0.0077 = 0.0077$$
$$z_1 = z_0 + l = 0 + 0.0618 = 0.0618$$

Again, $$k_1 = hf_1(x_1, y_1, z_1)$$
$$= 0.25f_1(0.25, 0.0077, 0.0618)$$
$$= 0.25 \times 0.0618$$
$$= 0.0154$$
$$l_1 = hf_2(x_1, y_1, z_1)$$
$$= 0.613$$
$$k_2 = hf_1\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}, z_1 + \frac{l_1}{2}\right)$$
$$= 0.25f_1(0.375, 0.0154, 0.0924)$$
$$= 0.0231$$
$$l_2 = hf_2(0.375, 0.0154, 0.0924)$$
$$= 0.609$$

$$k_3 = hf_1\left(x_1 + \frac{h}{2},\ y_1 + \frac{k_2}{2},\ z_1 + \frac{l_2}{2}\right)$$
$$= 0.25f_1\ (0.375, 0.0192, 0.0922)$$
$$= 0.0230$$
$$l_3 = hf_2\ (0.375, 0.0192, 0.0922)$$
$$= 0.0611$$
$$k_4 = hf_1\ (x_1 + h, y_1 + k_3, z_1 + l_3)$$
$$= 0.25f_1\ (0.5, 0.0307, 0.1229)$$
$$= 0.0307$$
$$l_4 = hf_2\ (0.5, 0.0307, 0.1229)$$
$$= 0.0609$$

Now,

$$k = \frac{1}{6}[k_1 + k_4 + 2(k_2 + k_3)]$$
$$= \frac{1}{6}[0.0154 + 0.0307 + 2(0.0231 + 0.0230)]$$
$$= 0.0229$$
$$l = \frac{1}{6}[l_1 + l_4 + 2(l_2 + l_3)]$$
$$= \frac{1}{6}[0.0613 + 0.0609 + 2(0.0609 + 0.0611)]$$
$$= 0.0610$$

Then, $x_2 = x_1 + h = 0.25 + 0.25 = 0.5$

$y_2 = y_1 + k = 0.0077 + 0.0229 = 0.0306$

$z_2 = z_1 + l = 0.0618 + 0.0610 = 0.1228$

11. **Solve the following differential equation within $0 \le x \le 0.5$ using RK 4th order method. $10\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 3y = 5$, $y(0) = 0$, $y'(0) = 0$. Take $h = 0.25$.** **[2015/Spring]**

Solution:

Given that;

$$10\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 4y = 5$$

Let, $\frac{dy}{dx} = y' = z = f_1(x, y, z)$

Then, $\frac{d^2y}{dx^2} = y'' = z' = f_2(x, y, z)$

or,
$$z' = \frac{5 - 2z + 4y}{10} = f_2(x, y, z)$$

Subject to

$$y(0) = 0 \rightarrow x_0 = 0, y_0 = 0$$
$$y'(0) = 0 \rightarrow z_0 = 0$$

and, $h = 0.25$

Now, by RK 4th order method,

$$k_1 = hf_1(x_0, y_0, z_0)$$
$$= 0.25f_1(0, 0, 0)$$
$$= 0$$
$$l_1 = hf_2(x_0, y_0, z_0)$$
$$= 0.25\left(\frac{5 - 2(0)0 + 4(0)}{10}\right)$$
$$= 0.125$$
$$k_2 = hf_1\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right)$$
$$= 0.25f_1(0.125, 0, 0.0625)$$
$$= 0.25 \times 0.0625$$
$$= 0.0156$$
$$l_2 = hf_2(0.125, 0, 0.0625)$$
$$= 0.25 \times \left(\frac{5 - 2(0.0625) + 4(0)}{10}\right)$$
$$= 0.1218$$
$$k_3 = hf_1\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right)$$
$$= 0.25f_1(0.125, 0.0078, 0.0609)$$
$$= 0.0152$$
$$l_3 = hf_2(0.125, 0.0078, 0.0609)$$
$$= 0.1227$$
$$k_4 = hf_1(x_0 + h, y_0 + k_3, z_0 + l_3)$$
$$= 0.25f_1(0.25, 0.0152, 0.1227)$$
$$= 0.0306$$
$$l_4 = hf_2(0.25, 0.0152, 0.1227)$$
$$= 0.1203$$

Now,

$$k = \frac{1}{6}[k_1 + k_4 + 2(k_2 + k_3)]$$
$$= \frac{1}{6}[0 + 0.0306 + 2(0.0156 + 0.0152)]$$
$$= 0.0153$$

$$l = \frac{1}{6}[l_1 + l_4 + 2(l_2 + l_3)]$$

$$= \frac{1}{6}[0.125 + 0.1203 + 2(0.1218 + 0.1227)]$$

$$= 0.1223$$

so, $x_1 = x_0 + h = 0 + 0.25 = 0.25$

$y_1 = y_0 + k = 0 + 0.0153 = 0.0153$

$z_1 = z_0 + l = 0 + 0.01223 = 0.1223$

Again, $k_1 = hf_1(x_1, y_1, z_1)$

$= 0.25f_1(0.25, 0.0153, 0.1223)$

$= 0.0305$

$l_1 = hf_2(0.25, 0.0153, 0.1223)$

$= 0.1204$

$$k_2 = hf_1\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}, z_1 + \frac{l_1}{2}\right)$$

$= 0.25f_1(0.375, 0.0305, 0.1825)$

$= 0.0456$

$l_2 = hf_2(0.375, 0.0305, 0.1825)$

$= 0.1189$

$$k_3 = hf_1\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}, z_1 + \frac{l_2}{2}\right)$$

$= 0.25f_1(0.375, 0.0381, 0.1817)$

$= 0.0454$

$l_3 = hf_2(0.375, 0.0381, 0.1817)$

$= 0.1197$

$k_4 = hf_1(x_1 + h, y_1 + k_3, z_1 + l_3)$

$= 0.25f_1(0.5, 0.0607, 0.242)$

$= 0.0605$

$l_4 = hf_2(0.5, 0.0607, 0.242)$

$= 0.1189$

Then,

$$k = \frac{1}{6}[k_1 + k_4 + 2(k_2 + k_3)]$$

$$= \frac{1}{6}[0.0305 + 0.0605 + 2(0.0456 + 0.0454)]$$

$$= 0.0455$$

$$l = \frac{1}{6}[l_1 + l_4 + 2(l_2 + l_3)]$$

$$= \frac{1}{6}[0.1204 + 0.1189 + 2(0.1189 + 0.1197)]$$

$$= 0.1194$$

Now,

$$x_2 = x_1 + h = 0.25 + 0.25 = 0.5$$
$$y_2 = y_1 + k = 0.0153 + 0.0455 = 0.0608$$
$$z_2 = z_1 + l = 0.1223 + 0.1194 = 0.2417$$

12. Solve the given differential equation by RK 4th order method $y'' - xy' + y = 0$ with initial condition $y(0) = 3$, $y'(0) = 0$ for $y(0.2)$ taking $h = 0.2$ **[2016/Fall]**

Solution:

Given that;

$$y'' - xy' + y = 0 \quad \text{—(1)}$$

Let $y'' = z'$ and $y' = z$

So equation (1) becomes

$$z' - xz + y = 0$$

or, $z' = xz - y$

We have,

$$y' = z = f_1(x, y, z)$$

and, $z' = xz - y = f_2(x, y, z)$

Subject to

$$y(0) = 3 \rightarrow x_0 = 0, y_0 = 3$$
$$y'(0) = 0 \rightarrow z_0 = 0$$

Taking $h = 0.2$.

Now, using RK 4th order method,

$$k_1 = hf_1(x_0, y_0, z_0)$$
$$= 0.2f_1(0, 3, 0)$$
$$= 0.2 \times 0$$
$$= 0$$

$$l_1 = hf_2(x_0, y_0, z_0)$$
$$= 0.2f_2(0, 3, 0)$$
$$= 0.2(0 \times 0 - 3)$$
$$= -0.6$$

$$k_2 = hf_1\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right)$$
$$= 0.2f_1(0.1, 3, -0.3)$$
$$= -0.06$$

$$l_2 = hf_2(0.1, 3, -0.3)$$
$$= -0.606$$

$$k_3 = hf_1\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right)$$
$$= 0.2f_1(0.1, 2.97, -0.303)$$
$$= -0.0606$$

$l_3 = hf_2(0.1, 2.97, -0.303)$
$= -0.6$

$k_4 = hf_1(x_0 + h, y_0 + k_3, z_0 + l_3)$
$= 0.2f_1(0.2, 2.9394, -0.6)$
$= -0.12$

$l_4 = hf_2(0.2, 2.9394, -0.6)$
$= -0.6118$

Now,

$$k = \frac{1}{6}[k_1 + k_4 + 2(k_2 + k_3)]$$

$$= \frac{1}{6}[0 + (-0.12) + 2(-0.06 - 0.0606)]$$

$$= -0.0602$$

$$l = \frac{1}{6}[l_1 + l_4 + 2(l_2 + l_3)]$$

$$= \frac{1}{6}[-0.6 - 0.6118 + 2(-0.606 - 0.6)]$$

$$= -0.6039$$

Then, $x_1 = x_0 + h = 0 + 0.2 = 0.2$

$\therefore$ $y_1 = y(0.2) = y_0 + k = 3 - 0.0602 = 2.9398$

13. **Solve the differential equation y' = x + y using approximate method within $0 \le x \le 0.2$ with initial condition y(0) = 1 and stepsize h = 0.1.** **[2016/Fall]**

Solution:

Given that;

$y' = x + y, \quad 0 \le x \le 0.2$

Subject to

$y(0) = 1$ at $h = 0.1$

$x_0 = 0, y_0 = 1$

Now, using modified Euler's method

Solving in tabular form

S.N.	x	$\frac{dy}{dx} = x + y$	Mean slope	$y_{new} = y_{old} + h$ (mean slope)
1	0	0 + 1	-	$1 + 0.1 \times 1 = 1.1$
2	0.1	0.1 + 1.1 = 1.2	$\frac{1 + 1.2}{2} = 1.1$	$1 + 0.1 \times 1.1 = 1.11$
3	0.1	0.1 + 1.11 = 1.21	$\frac{1 + 1.21}{2} = 1.105$	$1 + 0.1 \times 1.105 = 1.1105$
4	0.1	0.1 + 1.1105 = 1.2105	$\frac{1 + 1.2105}{2} = 1.1052$	$1 + 0.1 \times 1.1052 = 1.1105$

Here, last two values are equal at $y_1 = 1.1105$.

S.N.	x	$\frac{dy}{dx} = x + y$	Mean slope	$y_{new} = y_{old} + h$ [mean slope]
5	0.1	$0.1 + 1.1105 = 1.2105$	–	$1.1105 + 0.1 \times 1.2105 = 1.2315$
6	0.2	$0.2 + 1.2315 = 1.4315$	$\frac{1.2105 + 1.4315}{2} = 1.3210$	1.2426
7	0.2	1.4426	1.3265	1.2431
8	0.2	1.4431	1.3268	1.2431

Here, last two values are equal at $y_2 = 1.2431$.

Hence the required solution within $0 \le x \le 0.2$ are,

$x_0 = 0,\quad y_0 = 1$

$x_1 = 0.1,\quad y_1 = 1.1105$

and, $x_2 = 0.2,\quad y_2 = 1.2431$

14. Employ Taylor's method to obtain approximate value of y at x = 0.2 for the differential equation.

$$y' = 2y + e^x,\ y(0) = 0$$

[2016/Spring]

Solution:

We have,

$$y' = 2y + e^x \quad \text{and} \quad y(0) = 0$$

Then, $y'(0) = 2y(0) + e^0 = 2(0) + 1 = 1$

Now, differentiating successively and substituting $x_0 = 0$ and $y_0 = 0$ we get,

$$y'' = 2y' + e^x \quad , \quad y''(0) = 2y'(0) + e^0 = 2(1) + 1 = 3$$

$$y''' = 2y'' + e^x \quad , \quad y'''(0) = 2y''(0) + 1 = 2(3) + 1 = 7$$

$$y^{iv} = 2y''' + e^x \quad , \quad y^{iv}(0) = 2y'''(0) + 1 = 2(7) + 1 = 15$$

and so on.

Now, putting these values in the Taylor's series. We have,

$$y(x) = y(0) + xy'(0) + \frac{x^2}{2!}y''(0) + \frac{x^3}{3!}y'''(0) + \frac{x^4}{4!}y^{iv}(0) + \ldots\ldots$$

$$= 0 + x(1) + \frac{x^2}{2}(3) + \frac{x^3}{6}(7) + \frac{x^4}{24}(15) + \ldots\ldots$$

$$= x + \frac{3x^2}{2} + \frac{7x^2}{6} + \frac{5}{8}x^4 + \ldots\ldots$$

Hence, $y(0.2) = 0.2 + \frac{3(0.2)^2}{2} + \frac{7(0.2)^3}{6} + \frac{5(0.2)^4}{8} + \ldots\ldots$

$\therefore \quad y(0.2) = 0.2703$

15. **Using Runge-Kutta second order method, solve the differential equation $y'' = xy' - y$; $y(0) = 3$, $y'(0) = 0$ for $x = 0, 0.2, 0.4$.** **[2016/Spring]**

Solution:

Given that;

$$y'' = xy' - y \qquad ..... (1)$$

Let $y' = z$

Then, $y'' = z'$

So equation (1) becomes

$$z' = xz - y = f_2(x, y, z)$$

and, $y' = z = f_1(x, y, z)$

Subject to

$$y(0) = 3 \rightarrow x_0 = 0, \quad y_0 = 3$$

$$y'(0) = 0 \rightarrow z_0 = 0$$

Taking $h = 0.2$

Now, using Runge-Kutta second order method

$$k_1 = hf_1(x_0, y_0, z_0)$$
$$= 0.2f_1(0, 3, 0)$$
$$= 0.2(0)$$
$$= 0$$

$$l_1 = hf_2(x_0, y_0, z_0)$$
$$= 0.2f_2(0, 3, 0)$$
$$= 0.2[0(0) - 3]$$
$$= -0.6$$

$$k_2 = hf_1(x_0 + h, y_0 + k_1, z_0 + l_1)$$
$$= 0.2f_1(0.2, 3, -0.6)$$
$$= -0.12$$

$$l_2 = 0.2f_2(0.2, 3, -0.6)$$
$$= -0.624$$

Then, $k = \frac{1}{2}(k_1 + k_2) = \frac{1}{2}[0 + (-0.12)] = -0.06$

$$l = \frac{1}{2}(l_1 + l_2) = \frac{1}{2}(-0.6 - 0.624) = -0.612$$

and, $x_1 = x_0 + h = 0 + 0.2 = 0.2$

$$y_1 = y_0 + k = 3 + (-0.06) = 2.94$$

$$z_1 = z_0 + l = 0 - 0.612 = -0.612$$

Again, $k_1 = hf_1(x_1, y_1, z_1)$

$$= 0.2f_1(0.2, 2.94, -0.612)$$
$$= -0.1224$$

$$l_1 = hf_2(0.2, 2.94, -0.612)$$
$$= -0.6124$$

$$k_2 = hf_1 (x_1 + h, y_1 + k_1, z_1 + l_1)$$
$$= 0.2f_1 (0.4, 2.8176, -1.2244)$$
$$= -0.2448$$
$$l_2 = hf_2 (0.4, 2.8176, -1.2244)$$
$$= -0.6614$$

Then, $k = \frac{1}{2}(k_1 + k_2) = \frac{1}{2}(-0.1224 - 0.2448) = -0.1836$

$$l = \frac{1}{2}(l_1 + l_2) = \frac{1}{2}(-0.6124 - 0.6614) = -0.6369$$

and, $x_2 = x_1 + h = 0.2 + 0.2 = 0.4$

$$y_2 = y_1 + k = 2.94 - 0.1836 = 2.7564$$
$$z_2 = z_1 + l = -0.612 - 0.6369 = -1.2489$$

16. Solve the differential equation $y' = y + \sin x$ using appropriate method within $0 \le x \le 0.2$ with initial condition $y(0) = 2$ and step size = 0.1.

[2017/Fall]

Solution:

Given that;

$$y' = y + \sin x \quad , \quad 0 \le x \le 0.2$$

and, $y(0) = 2$

$\rightarrow$ $x_0 = 0$, $y_0 = 2$

Taking step size $h = 0.1$

Now, using Euler's method for solving the differential equation. We have,

$$y_{new} = y_{old} + h\frac{dy}{dx} = y_{old} + hf(x, y)$$

Then, $y_1 = y_0 + hf(x_0, y_0)$

$$= 2 + 0.1 [2 + \sin(0)]$$

$\therefore$ $y_1 = 2.2$

$$y_2 = y_1 + hf(x_1, y_1)$$
$$= 2.2 + 0.1 [2.2 + \sin(0.1)]$$

$\therefore$ $y_2 = 2.429$

and, $y_3 = y_2 + hf(x_2, y_2)$

$$= 2.429 + 0.1 [2.429 + \sin(0.2)]$$

$\therefore$ $y_3 = 2.691$

17. Apply RK-4 method to solve y(0.2) for the equation $\frac{d^2y}{dx^2} = x\frac{dy}{dx} - y$ given that $y = 1$ and $\frac{dy}{dx} = 0$ when $x = 0$. (Assume $h = 0.2$)

Solution: **[2017/Fall, 2017/Spring]**

Given that;

$$\frac{d^2y}{dx^2} = x\frac{dy}{dx} - y$$

or, $y'' = xy' - y = 0$

Let, $y' = z$ ..... (1)

Then, $y'' = z'$

So, equation (1) becomes

$z' = xz - y = f_2(x, y, z)$

and, $y' = z = f_1(x, y, z)$

Also,

$x_0 = 0, \quad y_0 = 1, \quad z_0 = 0$

At $h = 0.2$

Now, using RK-4 method

$$\begin{aligned}
k_1 &= hf_1(x_0, y_0, z_0) \\
&= 0.2f_1(0, 1, 0) \\
&= 0.2 \times 0 \\
&= 0
\end{aligned}$$

$$\begin{aligned}
l_1 &= hf_2(0, 1, 0) \\
&= 0.2\,[0(0) - 1] \\
&= -0.2
\end{aligned}$$

$$\begin{aligned}
k_2 &= hf_1\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right) \\
&= 0.2f_1(0.1, 1, -0.1) \\
&= -0.02
\end{aligned}$$

$$\begin{aligned}
l_2 &= hf_2(0.1, 1, -0.1) \\
&= 0.2\,[0.1(-0.1) - 1] \\
&= -0.202
\end{aligned}$$

$$\begin{aligned}
k_3 &= hf_1\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right) \\
&= 0.2f_1(0.1, 0.99, -0.101) \\
&= -0.0202
\end{aligned}$$

$$\begin{aligned}
l_3 &= hf_2(0.1, 0.99, -0.101) \\
&= -0.2
\end{aligned}$$

$$\begin{aligned}
k_4 &= hf_1(x_0 + h, y_0 + k_3, z_0 + l_3) \\
&= 0.2f_1(0.2, 0.979, -0.2) \\
&= -0.04
\end{aligned}$$

$$\begin{aligned}
l_4 &= hf_2(0.2, 0.979, -0.2) \\
&= -0.203
\end{aligned}$$

Now,

$$\begin{aligned}
k &= \frac{1}{6}[k_1 + k_4 + 2(k_2 + k_3)] \\
&= \frac{1}{6}[0 - 0.04 + 2(-0.02 - 0.0202)] \\
&= -0.02006
\end{aligned}$$

$$l = \frac{1}{6}[l_1 + l_4 + 2(l_2 + l_3)]$$

$$= \frac{1}{6}[-0.2 - 0.203 + 2(-0.202 - 0.2)]$$

$$= -0.2011$$

Then, $x_1 = x_0 + h = 0 + 0.2 = 0.2$

$y_1 = y_0 + k = 1 - 0.02006 = 0.9799$

$z_1 = z_0 + l = 0 - 0.2011 = -0.2011$

Hence, $y(0.2) = 0.9799$ is the required solution.

18. Solve the given differential equation by RK 4th order method $y'' - x^2y' - 2xy = 0$ with initial condition $y(0) = 1$ $y'(0) = 0$, for $y(0.1)$ taking $h = 0.1$. **[2018/Fall]**

Solution:

Given that;

$$y'' - x^2y' - 2xy = 0 \quad \text{...(1)}$$

Let, $y' = z$

Then, $y'' = z'$

So, equation (1) becomes

$$z' = x^2z + 2xy = f_2(x, y, z)$$

and, $y' = z = f_1(x, y, z)$

Subject to

$y(0) = 1 \rightarrow x_0 = 0, \quad y_0 = 1$

$y'(0) = 0 \rightarrow z_0 = 0$

Taking $h = 0.1$

Now, using RK-4th method

$$k_1 = hf_1(x_0, y_0, z_0)$$
$$= 0.1f_1(0, 1, 0)$$
$$= 0.1 \times 0$$
$$= 0$$

$$l_1 = hf_2(0, 1, 0)$$
$$= 0.1[0^2(0) + 2(0)(1)]$$
$$= 0$$

$$k_2 = hf_1\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right)$$
$$= 0.1f_1(0.05, 1, 0)$$
$$= 0$$

$$l_2 = hf_2(0.05, 1, 0)$$
$$= 0.01$$

$$k_3 = hf_1\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right)$$
$$= 0.1f_1\,(0.05, 1, 0.005)$$
$$= 0.0005$$
$$l_3 = hf_2\,(0.05, 1, 0.005)$$
$$= 0.010$$
$$k_4 = hf_1\,(x_0 + h, y_0 + k_3, z_0 + l_3)$$
$$= 0.1f_1\,(0.1, 1.0005, 0.010)$$
$$= 0.001$$
$$l_4 = hf_2\,(0.1, 1.0005, 0.010)$$
$$= 0.020$$

Then,
$$k = \frac{1}{6}[k_1 + k_4 + 2(k_2 + k_3)]$$
$$= \frac{1}{6}[0 + 0.001 + 2(0 + 0.0005)]$$
$$= 0.00033$$
$$l = \frac{1}{6}[l_1 + l_4 + 2(l_2 + l_3)]$$
$$= \frac{1}{6}[0 + 0.02 + 2(0.01 + 0.01)]$$
$$= 0.01$$

Now,

$x_1 = x_0 + h = 0 + 0.1 = 0.1$

$\therefore$ $y_1 = y(0.1) = y_0 + k = 1 + 0.00033 = 1.00033$

$\therefore$ $z_1 = z_0 + l = 0 + 0.01 = 0.01$

19. **Solve the differential equation $y' = y - \frac{2x}{y}$ using appropriate method within $0 \le x \le 0.2$ with initial conditions $y(0) = 1$ and step size $h = 0.1$.** **[2018/Fall]**

Solution:

Given that;

$$y' = y - \frac{2x}{y}, \quad 0 \le x \le 0.2$$

and, $y(0) = 1$

$x_0 = 0$, $y_0 = 1$

Step size $= h = 0.1$

Now, using Euler's method

$$f(x_0, y_0) = y_0 - \frac{2x_0}{y_0} = 1 - \frac{2(0)}{1} = 1$$
$$y_1 = y_0 + hf\,(x_0, y_0) = 1 + 0.1 \times (1) = 1.1$$

Again, $f(x_1, y_1) = y_1 - \frac{2x_1}{y_1} = 1.1 - \frac{2(0.1)}{1.1} = 0.918$

$$y_2 = y_1 + hf(x_1, y_1) = 1.1 + 0.1 \times 0.918 = 1.1918$$

Hence, the required solutions are

$\therefore \quad x_0 = 0 \quad , \quad y_0 = 1$

$\therefore \quad x_1 = 0.1 \quad , \quad y_1 = y(0.1) = 1.1$

$\therefore \quad x_2 = 0.2 \quad , \quad y_2 = y(0.2) = 1.1918$

20. Use the Runge-Kutta 4th order to solve $10\frac{dy}{dx} = x^2 + y^2$, $y(0) = 1$ for the interval $0 \le x \le 0.4$ with $h = 0.1$. **[2018/Spring]**

Solution:

Given that;

$$10\frac{dy}{dx} = x^2 + y^2 \; ; \; 0 \le x \le 0.4$$

or, $$y' = \frac{x^2 + y^2}{10}$$

Subjected to

$$y(0) = 1 \rightarrow x_0 = 0, \quad y_0 = 1$$

Taking $h = 0.1$

Now, using Runge-Kutta 4th order method

$$k_1 = hf(x_0, y_0) = 0.1f(0, 1) = 0.1\left(\frac{0^2 + 1^2}{10}\right) = 0.01$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = 0.1f(0.05, 1.005) = 0.0101$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = 0.1f(0.05, 1.00505) = 0.0101$$

$$k_4 = hf(x_0 + h, y_0 + k_3) = 0.1f(0.1, 1.0101) = 0.0103$$

Then, $k = \frac{1}{6}[k_1 + k_4 + 2(k_2 + k_3)]$

$$= \frac{1}{6}[0.01 + 0.0103 + 2(0.0101 + 0.0101)]$$

$$= 0.01011$$

so, $\quad x_1 = x_0 + h = 0 + 0.1 = 0.1$

$\therefore \quad y_1 = y_0 + k = 1 + 0.01011 = 1.01011$

Again, $k_1 = hf(x_1, y_1) = 0.1f(0.1, 1.01011) = 0.0103$

$$k_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right) = 0.1f(0.15, 1.0152) = 0.0105$$

$$k_3 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right) = 0.1f(0.15, 1.0153) = 0.01053$$

$$k_4 = hf(x_1 + h, y_1 + k_3) = 0.1f(0.2, 1.0206) = 0.0108$$

Then, $k = \frac{1}{6}[k_1 + k_4 + 2(k_2 + k_3)] = 0.0105$

so, $x_2 = x_1 + h = 0.1 + 0.1 = 0.2$

$\therefore$ $y_2 = y_1 + k = 1.01011 + 0.0105 = 1.0206$

Again,

$$k_1 = hf(x_2, y_2) = 0.1f(0.2, 1.0206) = 0.0108$$

$$k_2 = hf\left(x_2 + \frac{h}{2}, y_2 + \frac{k_1}{2}\right) = 0.1f(0.25, 1.026) = 0.0111$$

$$k_3 = hf\left(x_2 + \frac{h}{2}, y_2 + \frac{k_2}{2}\right) = 0.1f(0.25, 1.0261) = 0.0111$$

$$k_4 = hf(x_2 + h, y_2 + k_3) = 0.1f(0.3, 1.0317) = 0.0115$$

Then, $k = \frac{1}{6}[k_1 + k_4 + 2(k_2 + k_3)] = 0.0111$

so, $x_3 = x_2 + h = 0.3$

$\therefore$ $y_3 = y_2 + k = 1.0206 + 0.0111 = 1.0317$

Again,

$$k_1 = hf(x_3, y_3) = 0.1f(0.3, 1.0317) = 0.0115$$

$$k_2 = hf\left(x_3 + \frac{h}{2}, y_3 + \frac{k_1}{2}\right) = 0.1f(0.35, 1.0374) = 0.0119$$

$$k_3 = hf\left(x_3 + \frac{h}{2}, y_3 + \frac{k_2}{2}\right) = 0.1f(0.35, 1.0376) = 0.0119$$

$$k_4 = hf(x_3 + h, y_3 + k_3) = 0.1f(0.4, 1.0436) = 0.0124$$

Then, $k = \frac{1}{6}[k_1 + k_4 + 2(k_2 + k_3)] = 0.0119$

so, $x_4 = x_3 + h = 0.4$

$\therefore$ $y_4 = y_3 + k = 1.0317 + 0.0119 = 1.0436.$

21. **Solve the boundary value problem**

$y''(x) = y(x)$,

$y(0) = 0$ and $y(1) = 1.1752$ by shooting method, **[2018/Spring]**

taking $m_0 = 0.8$ and $m_1 = 0.9$

Solution:

Given that;

$m_0 = 0.8$ and $m_1 = 0.9$ be initial guess for $y'(0) = m$

Then, using shooting method,

$y'' = y(x)$, $y(0) = 0$ gives

$y'(0) = m$, $y''(0) = y(0) = 0$

$y'''(0) = y'(0) = m$, $y^{iv}(0) = y''(0) = 0$

$y^{v}(0) = y'''(0) = m$, $y^{vi}(0) = y^{iv}(0) = 0$

and so on.

Putting these values in the Taylor's series. We have,

$$y(x) = y(0) + xy'(0) + \frac{x^2}{2!}y''(0) + \frac{x^3}{3!}y'''(0) + \frac{x^4}{4!}y^{iv}(0) + \ldots\ldots$$

$$= m\left(x + \frac{x^3}{6} + \frac{x^5}{120} + \frac{x^7}{5040} + \ldots\ldots\right)$$

$$\therefore \quad y(1) = m(1 + 0.1667 + 0.0083 + 0.0002 + \ldots\ldots$$

$$= m\,(1.175)$$

For $m_0 = 0.8$, $\quad y(m_0, 1) = 0.85 \times 1.175 = 0.94$

For $m_1 = 0.9$, $\quad y(m_1, 1) = 0.9 \times 1.175 = 1.057$

So, for better approximation of m,

$$m_2 = m_1 - (m_1 - m_0)\frac{y(m_1, 1) - y(1)}{y(m_1, 1) - y(m_0, 1)}$$

$$= 0.9 - (0.1)\frac{1.057 - 1.175}{1.057 - 0.94}$$

$$= 0.9 + 0.10085$$

$$= 1.00085$$

Here, $m_2 = 1.00085$ is closer to the exact value of $y'(0) = 0.996$.

We know solve the initial value problem

$$y''(x) = y(x),\ y(0) = 0,\ y'(0) = m_2$$

Taylor's series solution is given by

$$y(m_2, 1) = m_2\,(1.175) = 1.00085 \times 1.175 = 1.17599$$

Hence, the solution at $x = 1$ is $y = 1.176$ which is close to the exact value of $y(1) = 1.1752$.

22. Use Picard's method to approximate the value of y when $x = 0.1$, $x = 0.2$ and $x = 0.4$, given that $y = 1$ at $x = 0$ and $\frac{dy}{dx} = 1 + xy$ correct to three decimal places. (Use upto second approximation) **[2019/Fall]**

Solution:

Given that;

$$\frac{dy}{dx} = 1 + xy = f(x, y)$$

and, $x_0 = 0$, $\quad y_0 = 1$

Using Picard's method, we have,

$$y = y_0 + \int_{x_0}^{x} f(x, y)\,dx$$

First approximation, put $y = 1$ in the integrand

$$y_1 = 1 + \int_0^x [1 + x(1)]\,dx = 1 + \left[x + \frac{x^2}{2}\right]_0^x = 1 + x + \frac{x^2}{2}$$

Second approximation, put $y = 1 + x + \frac{x^2}{2}$ in the integrand

$$y_2 = 1 + \int_0^x \left[1 + x\left(1 + x + \frac{x^2}{2}\right)\right] dx$$

$$= 1 + \int_0^x \left(1 + x + x^2 + \frac{x^3}{2}\right) dx$$

$$= 1 + \left[x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{8}\right]_0^x$$

$$\therefore \quad y_2 = 1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{8}$$

Now, using the first approximation and taking

$x = 0.1, 0.2, 0.4$

We have,

$$y_1(0.1) = 1 + x + \frac{x^2}{2} = 1 + 0.1 + \frac{(0.1)^2}{2} = 1.105$$

$$y_1(0.2) = 1.06$$

$$y_1(0.4) = 1.24$$

Now, using the second approximation and taking

$x = 0.1, 0.2, 0.4$

We have,

$$y_2(0.1) = 1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{8} = 1.1053$$

$$y_2(0.2) = 1.2228$$

$$y_2(0.4) = 1.5045$$

Also, the exact solution of $y' = 1 + xy$ is e^x

$$y(0) = e^0 = 1$$

$$y(0.1) = e^{0.1} = 1.1051$$

$$y(0.2) = e^{0.2} = 1.221$$

$$y(0.4) = e^{0.4} = 1.492$$

Here, $y(0.1) = 1.105$ is correct upto 3 decimal places.

For $y(0.2)$ using $y(0.1) = 1.105$ as initial value.

First approximation, put $y = 1.105$ in the integrand

$$y_1 = 1.105 + \int_{0.1}^x [1 + x(1.105)] dx$$

$$= 1.105 + \left[x + \frac{x^2}{2}(1.105)\right]_{0.1}^x$$

$$= 1.105 + x + 0.5525x^2 - 0.1 - 0.0055$$

$$= 0.999 + x + 0.5525x^2$$

Second approximation, put $y = 0.999 + x + 0.5525x^2$ in the integrand

$$y_2 = 1.105 + \int_{0.1}^x [1 + x(0.999 + x + 0.5525x^2)] dx$$

$$= 1.105 + \int_{0.1}^{x} [1 + 0.999x + x^2 + 0.5525x^3]dx$$

$$= 1.105 + \left[x + \frac{0.999x^2}{2} + \frac{x^3}{3} + \frac{0.5525x^4}{4}\right]_{0.1}^{x}$$

$$= 1.105 + x + 0.499x^2 + 0.333x^3 + 0.1381x^4 - 0.1$$

$$- \frac{0.999(0.1)^2}{2} - \frac{(0.1)^3}{3} - \frac{0.5525(0.1)^4}{4}$$

$$= 0.999 + x + 0.499x^2 + 0.333x^3 + 0.1381x^4$$

Now, using the second approximation and taking $x = 0.2, 0.4$

We have,

$$\therefore \quad y(0.2) = 0.999 + 0.2 + 0.499\,(0.2)^2 + 0.333\,(0.2)^3 + 0.1381\,(0.2)^4$$

$$= 1.2218$$

$$\therefore \quad y(0.4) = 1.5036$$

Here, $y(0.2) = 1.2218$ is correct upto three decimal places compared to exact solution.

For, $y(0.4)$, using $y(0.2) = 1.2218$ as initial value.

First approximation, put $y = 1.2218$ in the integrand.

$$y_1 = 1.2218 + \int_{0.2}^{x} [1 + x(1.2218)]dx$$

$$= 1.2218 + \left[x + \frac{1.2218x^2}{2}\right]_{0.2}^{x}$$

$$= 1.2218 + x + 0.6109x^2 - 0.2 - 0.0244$$

$$= 0.9974 + x + 0.6109x^2$$

Second approximation, put $y = 0.9974 + x + 0.6109x^2$ in the integrand.

$$y_2 = 1.2218 + \int_{0.2}^{x} [1 + x(0.9974 + x + 0.6109x^2)]\,dx$$

$$= 1.2218 + \left[x + \frac{0.9974x^2}{2} + \frac{x^3}{3} + \frac{0.6109x^4}{4}\right]_{0.2}^{x}$$

$$= 0.9989 + x + \frac{0.9974x^2}{2} + \frac{x^3}{3} + \frac{0.6109x^4}{4}$$

Now, using the second approximation and taking

$x = 0.4$

We have,

$$y(0.4) = 0.9989 + 0.4 + \frac{0.9974}{2}(0.4)^2 + \frac{(0.4)^3}{3} + \frac{0.6109}{4}(0.4)^4$$

$$\therefore \quad y(0.4) = 1.5039$$

Here, $y(0.4) = 1.5039$ is correct upto 3 decimal places.

Thus, $y(0.1) = 1.105$

$y(0.2) = 1.221$

$y(0.4) = 1.503$

Using Runge-Kutta method of second order (RK-2), obtain a solution of the equation y" = y + xy' with initial condition y(0) = 1, y'(0) = 0 to find y(0.2) and y'(0.2), taking h = 0.1. **[2019/Fall]**

Solution:

Given that;

$$y'' = xy' + y \qquad \text{..... (1)}$$

Let, $y' = z$

Then, $y'' = z'$

So equation (1) becomes

$$z' = xz + y = f_2(x, y, z)$$

and, $y' = z + f_1(x, y, z)$

Subject to

$$y(0) = 1 \rightarrow x_0 = 0, \quad y_0 = 1$$

$$y'(0) = 0 \rightarrow z_0 = 0$$

Taking $h = 0.1$

Now, using Runge-Kutta method of second order,

$$k_1 = hf_1 (x_0, y_0, z_0)$$
$$= 0.1f_1 (0, 1, 0)$$
$$= 0.1 \times 0$$
$$= 0$$

$$l_1 = hf_2 (0, 1, 0)$$
$$= 0.1 (0(0) + 1)$$
$$= 0.1$$

$$k_2 = hf_1 (x_0 + h, y_0 + k_1, z_0 + l_1)$$
$$= 0.1f_1 (0.1, 1, 0.1)$$
$$= 0.01$$

$$l_2 = hf_2 (0.1, 1, 0.1)$$
$$= 0.101$$

Then,

$$k = \frac{1}{2}(k_1 + k_2) = \frac{1}{2}(0 + 0.01) = 0.005$$

$$l = \frac{1}{2}(l_1 + l_4) = \frac{1}{2}(0.1 + 0.101) = 0.1005$$

$$x_1 = x_0 + h = 0 + 0.1 = 0.1$$
$$y_1 = y_0 + k = 1 + 0.005 = 1.005$$
$$z_1 = z_0 + l = 0 + 0.1005 = 0.1005$$

Again,

$$k_1 = hf_1 (x_1, y_1, z_1)$$
$$= 0.1f_1 (0.1, 1.005, 0.1005)$$
$$= 0.01$$

$$l_1 = hf_2\,(0.1, 1.005, 0.1005)$$
$$= 0.1015$$
$$k_2 = hf_1\,(x_1 + h, y_1 + k_1, z_1 + l_1)$$
$$= 0.1f_1\,(0.2, 1.015, 0.202)$$
$$= 0.020$$
$$l_2 = hf_2\,(0.2, 1.015, 0.202)$$
$$= 0.1055$$

Then, $k = \frac{1}{2}(k_1 + k_2) = \frac{1}{2}(0.01 + 0.02) = 0.015$

$$l = \frac{1}{2}(l_1 + l_2) = \frac{1}{2}(0.1015 + 0.1055) = 0.1035$$

Hence,

$$x_2 = x_1 + h = 0.1 + 0.1 = 0.2$$

$\therefore$ $y_2 = y_1 + k = 1.005 + 0.015 = 1.02$

$\therefore$ $z_2 = z_1 + l = 0.1005 + 0.1035 = 0.204$

24. Solve the given differential equation by Heun's method $y'' - y' - 2y = 3e^{2x}$ with initial condition $y(0) = 0$, $y'(0) = -2$ for $y(0.2)$ taking $h = 0.1$ **[2019/Spring]**

Solution:

Given that;

$$y'' - y' - 2y = 3e^{2x} \quad \text{---(1)}$$

Let, $y' = z$

Then, $y'' = z'$

So, equation (1) becomes

$$z' - z - 2y = 3e^{2x}$$

and, $z' = z + 2y + 3e^{2x}$

Subject to

$y(0) = 1 \quad \rightarrow x_0 = 0, \quad y_0 = 0$

$y'(0) = -2 \quad \rightarrow z_0 = -2$

Taking $h = 0.1$

Now, using Heun's method or modified Euler's method solving in tabular form.

S.N.	x	y' = z	Mean slope	$y_{new} = y_{old} + h$ (mean ...
1	0	−2	–	$0 + 0.1 \times (-2) = -0.2$
2	0.1	−1.9	$\frac{-2 - 1.9}{2} = -1.95$	$0 + 0.1 \times (-1.95) = -0.195$
3	0.1	−1.882	$\frac{-2 - 1.882}{2} = -1.94$	$0 + 0.1 \times (-1.94) = -0.194$
4	0.1	−1.881	$\frac{-2 - 1.882}{2} = -1.94$	$0 + 0.1 \times (-1.94) = -0.194$

Here, the last two values are equal at $y_1 = -0.194$.

S.N.	x	$z' = z + 2y + 3e^{2x}$	Mean slope	$z_{new} = z_{old} + h$ (mean slope)
1	0	$-2 + 2(0) + 3e^{2(0)} = 1$	–	$-2 + 0.1 \times 1 = -1.9$
2	0.1	$-1.9 + 2(-0.2) + 3e^{2(0.1)} = 1.364$	$\frac{1 + 1.364}{2} = 1.18$	$-2 + 0.1 \times 1.18 = -1.882$
3	0.1	$-1.88 + 2(-0.195) + 3e^{2(0.1)} = 1.394$	$\frac{1 + 1.394}{2} = 1.19$	$-2 + 0.1 \times 1.19 = -1.881$
4	0.1	$-1.88 + 2(-0.194) + 3e^{2(0.1)} = 1.396$	$\frac{1 + 1.396}{2} = 1.19$	$-2 + 0.1 \times 1.19 = -1.881$

Here, the last two values are equal at $z_1 = -1.881$.

Again,

S.N.	x	$y' = z$	Mean slope	$y_{new} = y_{old} + h$ (mean slope)
1	0.1	-1.881	–	$-0.194 + 0.1 \times (-1.881) = -0.382$
2	0.2	-1.741	$\frac{-1.881 - 1.741}{2} = -1.811$	$-0.194 + 0.1 \times (-1.811) = -0.375$
3	0.2	-1.712	$\frac{-1.881 - 1.712}{2} = -1.796$	$-0.194 + 0.1 \times (-1.796) = -0.373$
4	0.2	-1.710	$\frac{-1.881 - 1.710}{2} = -1.795$	$-0.194 + 0.1 \times (-1.795) = -0.373$

Here, the last two values are equal at $y_2 = -0.373$.

S.N.	x	$z' = z + 2y + 3e^{2x}$	Mean slope	$z_{new} = z_{old} + h$ (mean slope)
1	0	$-1.881 + 2(-0.194) + 3e^{2(0.1)} = 1.395$	–	$-1.881 + 0.1(1.395) = -1.741$
2	0.1	$-1.741 + 2(-0.382) + 3e^{2(0.2)} = 1.970$	$\frac{1.395 + 1.970}{2} = 1.682$	$-1.881 + 0.1(1.682) = -1.712$
3	0.1	$-1.712 + 2(-0.375) + 3e^{2(0.2)} = 2.013$	$\frac{1.395 + 2.013}{2} = 1.704$	$-1.881 + 0.1(1.704) = -1.710$
4	0.1	$-1.71 + 2(-0.373) + 3e^{2(0.2)} = 2.019$	$\frac{1.395 + 2.019}{2} = 1.707$	$-1.881 + 0.1(1.707) = -1.710$

Here, the last two values are equal at $z_2 = -1.710$.

Hence, the required solution of $y(0.2) = -0.373$.

Solve $y' = y + e^x$, $y(0) = 0$ for $y(0.2)$ and $y(0.4)$ by RK-4th order method. **[2019/Spring]**

Solution:

Given that;

$$y' = y + e^x$$
$$y(0) = 0 \quad \rightarrow x_0 = 0, \quad y_0 = 0$$

Taking $h = 0.2$

Now, using RK-4th order method

$$k_1 = hf(x_0, y_0) = 0.2f(0, 0) = 0.2(0 + e^0) = 0.2$$
$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = 0.2f(0.1, 0.1) = 0.241$$
$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = 0.2f(0.1, 0.120) = 0.245$$
$$k_4 = hf(x_0 + h, y_0 + k_3) = 0.2f(0.2, 0.245) = 0.293$$

Then,

$$k = \frac{1}{6}[k_1 + k_4 + 2(k_2 + k_3)]$$
$$= \frac{1}{6}[0.2 + 0.293 + 2(0.241 + 0.245)]$$
$$= 0.244$$

so, $x_1 = x_0 + h = 0 + 0.2 = 0.2$

$\therefore$ $y_1 = y_0 + k = 0 + 0.244 = 0.244$

Again,

$$k_1 = hf(x_1, y_1) = 0.2f(0.2, 0.244) = 0.293$$
$$k_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right) = 0.2f(0.3, 0.39) = 0.347$$
$$k_3 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right) = 0.2f(0.3, 0.417) = 0.353$$
$$k_4 = hf(x_1 + h, y_1 + k_3) = 0.2f(0.4, 0.597) = 0.417$$

Then,

$$k = \frac{1}{6}[k_1 + k_4 + 2(k_2 + k_3)]$$
$$= \frac{1}{6}[0.293 + 0.417 + 2(0.347 + 0.353)]$$
$$= 0.351$$

so, $x_2 = x_1 + h = 0.2 + 0.2 = 0.4$

$\therefore$ $y_2 = y_1 + k = 0.244 + 0.351 = 0.595$

Hence, $y(0.2) = 0.244$ and $y(0.4) = 0.595$ are the required solutions.

26. Applying Runge-Kutta fourth order method to find an approximate value of y when x = 0.3 given that: $y' = 2.5y + e^{0.3x}$ with an initial $y(0) = 1$, taking $h = 0.3$ **[2020 Fall]**

Solution:

Given that;

$y' = 2.5y + e^{0.3x}$

$y(0) = 1 \quad \rightarrow x_0 = 0, \quad y_0 = 1$

$h = 0.3$

Now, using Runge-Kutta fourth other method

$k_1 = hf(x_0, y_0)$

$= 0.3f(0, 1)$

$= 0.3\,[2.5(1) + e^{0.3\times 0}]$

$= 1.05$

$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$

$= 0.3f(0.15, 1.525)$

$= 0.3[2.5(1.525) + e^{0.3(0.15)}]$

$= 1.457$

$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$

$= 0.3f(0.15, 1.728)$

$= 1.609$

$k_4 = hf(x_0 + h, y_0 + k_3)$

$= 0.3f(0.3, 2.609)$

$= 2.285$

Then, $k = \frac{1}{6}[k_1 + k_4 + 2(k_2 + k_3)]$

$= \frac{1}{6}[1.05 + 2.285 + 2(1.457 + 1.609)]$

$= 1.577$

Now,

$x_1 = x_0 + h = 0 + 0.3 + 0.3$

$y_1 = y(0.3) = y_0 + k = 1 + 1.577 = 2.577$

27. **Solve the Boundary value problem (BVP) using shooting method by dividing into four sub-interval employing Euler's method.**

$y'' + 2y' - y = x$ **[2020/Fall]**

Subjective to boundary condition y(1) = 2 and y(2) = 4.

Solution:

Given that;

$y' + 2y' - y = x$ ..... (1)

Let $y' = z$

Then, $y'' = z'$

So equation (1) becomes,

$x' + 2z - y = x$

or, $z' = x + y - 2z = f_2(x, y, z)$

and, $y' = z = f_1(x, y, z)$

Subject to

$y(1) = 2 \quad \rightarrow x_0 = 1, \quad y_0 = 2$

Assuming

$y'(1) = 4 \quad \rightarrow z_0 = 4$

And having four subintervals, h = 0.25

Now, using shooting method by employing Euler's method

At, $i = 0, x_0 = 1, y_0 = 2, z_0 = 4, h = 0.25$

$$y_1 = y_0 + hf_1(x_0, y_0, z_0)$$
$$= 2 + 0.25f_1(1, 2, 4)$$
$$= 2 + 0.25 \times 4 = 3$$
$$z_1 = z_0 + hf_2(x_0, y_0, z_0)$$
$$= 4 + 0.25f_2(1, 2, 4)$$
$$= 4 + 0.25(1 + 2 - 2 \times 4)$$
$$= 2.75$$

At, $i = 1, x_1 = x_0 + h = 1.25, y_1 = 3, z_1 = 1.25, h = 0.25$

$$y_2 = y_1 + hf_1(x_1, y_1, z_1)$$
$$= 3 + 0.25f_1(1.25, 3, 2.75)$$
$$= 3 + 0.25(2.75)$$
$$= 3.687$$
$$z_2 = z_1 + hf_2(x_1, y_1, z_1)$$
$$= 2.75 + 0.25f_2(1.25, 3, 2.75)$$
$$= 2.75 + 0.25(1.25 + 3 - 2(2.75)$$
$$= 2.437$$

At, $i = 2, x_2 = 1.5, y_2 = 3.687, z_2 = 2.43, h = 0.25$

$$y_3 = y_2 + hf_1(x_2, y_2, z_2)$$
$$= 3.687 + 0.25f_1(1.5, 3.687, 2.437)$$
$$= 4.296$$
$$z_3 = z_2 + hf_2(1.5, 3.687, 2.437)$$
$$= 2.515$$

At, $i = 3, x_3 = 1.75, y_3 = 4.296, z_3 = 2.515, h = 0.25$

$$y_4 = y_3 + hf_1(x_3, y_3, z_3)$$
$$= 4.296 + 0.25f_1(1.75, 4.296, 2.515)$$
$$= 4.924$$
$$z_4 = z_3 + hf_2(x_3, y_3, z_3)$$
$$= 2.769$$

Here, given $y(2) = 4$

and we obtain $y(2) = y_4 = 4.924$ which is greater than 4.

So, we choose $y'(0) = 1 = z_0$ and carry out the process

At $i = 0, x_0 = 1, y_0 = 2, z_0 = 1, h = 0.25$

$$y_1 = y_0 + hf_1(x_0, y_0, z_0)$$
$$= 2 + 0.25f_1(1, 2, 1)$$
$$= 2.25$$
$$z_1 = z_0 + hf_2(1, 2, 1)$$
$$= 1.25$$

At, $i = 1, x_1 = 1.25, y_1 = 2.25, z_1 = 1.25, h = 0.25$

$$y_2 = y_1 + hf_1(x_1, y_1, z_1)$$
$$= 2.562$$
$$z_2 = z_1 + hf_2(x_1, y_1, z_1)$$
$$= 1.5$$

At, $i = 2, x_2 = 1.5, y_2 = 2.562, z_2 = 1.5, h = 0.25$

$$y_3 = y_2 + hf_1(x_2, y_2, z_2)$$
$$= 2.937$$
$$z_3 = z_2 + hf_2(x_2, y_2, z_2)$$
$$= 1.765$$

At, $i = 3, x_3 = 1.75, y_3 = 2.937, z_3 = 1.765, h = 0.25$

$$y_4 = y_3 + hf_1(x_3, y_3, z_3)$$
$$= 3.378$$
$$z_4 = z_3 + hf_2(x_3, y_3, z_3)$$
$$= 2.054$$

Here, we obtain,

$$y_4 = y(2) = 3.378 \quad \text{at } y'(0) = 1$$

Also, we have,

$$y_4 = y(2) = 4.924 \quad \text{at } y'(0) = 4$$

So for better approximation

$P_1 = y'(0) = 4$, $Q_1 = y(2) = 4.924$

$P_2 = y'(0) = 1$, $Q_2 = y(2) = 3.378$

Then to obtain $y(2) = 4 = Q$

$$P = P_1 + \frac{P_2 - P_1}{Q_2 - Q_1}(Q - Q_1)$$
$$= 4 + \frac{1 - 4}{3.378 - 4.924}(4 - 4.924)$$
$$= 2.206$$

So, now using $y'(0) = 2.206 = z_0$ and continuing the process.

At, $i = 0, x_0 = 1, y_0 = 2, z_0 = 2.206, h = 0.25$

$$y_1 = y_0 + hf_1(x_0, y_0, z_0)$$
$$= 2.551$$
$$z_1 = z_0 + hf_2(x_0, y_0, z_0)$$
$$= 1.853$$

At, $i = 1, x_1 = 1.25, y_1 = 2.551, z_1 = 1.853, h = 0.25$

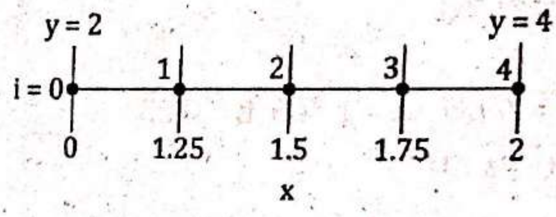
$$y_2 = y_1 + hf_1(x_1, y_1, z_1)$$
$$= 3.014$$
$$z_2 = z_1 + hf_2(x_1, y_1, z_1)$$
$$= 1.876$$

At, $i = 2, x_2 = 1.5, y_2 = 3.014, z_2 = 1.876, h = 0.25$

$$y_3 = y_2 + hf_1(x_2, y_2, z_2)$$
$$= 3.483$$
$$z_3 = z_2 + hf_2(x_2, y_2, z_2)$$
$$= 2.066$$

At, $i = 3, x_3 = 1.75, y_3 = 3.483, z_3 = 2.066, h = 0.25$

$$y_4 = y_3 + hf_1(x_3, y_3, z_3)$$
$$= 3.995$$
$$z_4 = z_3 + hf_2(x_3, y_3, z_3)$$
$$= 2.341$$



Here, we obtain $y_4 = y(2) = 3.995$ which is close to the exact value of $y(2) = 4$. Hence, the solution at $x = 2$ is $y = 3.995$.

28. Write short notes on: Finite differences. **[2020/Fall]**

Solution: See the topic 5.10 'B'.

29. Write short notes on: Picard's iterative formula. **[2020/Fall]**

Solution: See the topic 5.2.

30. Write short notes on: Solution of 2nd order differential equation. **[2016/Fall]**

Solution: See the topic 5.9.

31. Write short notes on: Boundary value problem. **[2017/Spring]**

Solution: See the topic 5.10.

A boundary value problem is a system of ordinary differential equations with solution and derivative values specified at more than one point. Most commonly, the solution and derivatives are specified at just two points (the boundaries) defining a two point boundary value problem. In the field of differential equations, a boundary value problem is a differential equation together with a set of additional constraints, called the boundary conditions. A solution to a boundary value problem is a solution to the differential equation which also satisfies the boundary conditions. Boundary value problem arise in several branches of physics as any physical differential equation will have them.

Boundary value problems are similar to initial value problems. A boundary value problem has conditions specified at the extremes ("boundaries") of the independent variable in the equation whereas on initial value problem has all of the conditions specified at the same value of the independent variable (and that value is at the lower boundary of the domain, thus the term "initial value). A boundary value is a data value that corresponds to a minimum or maximum input, internal or output value specified for a system or component.

32. Write short notes on: algorithm for second order Runge-Kutta (RK-2) method.

[2020/Fall]

Solution:

1. Define function f(x, y)
2. Get values of x_0, y_0, h, x_n

   where, x_0 is starting value of x *i.e.*, x_0, x_n is the value of x for which y is to be determined.
3. If $x = x_n$ then go to step 7

   else

   $k_1 = h \times f(x, y)$

   $k_2 = h \times f(x + h, y + k_1)$
4. Compute $k = \left(\frac{k_1 + k_2}{2}\right)$ and,

   $x = x + h$

   $y = y + k$
5. Display x and y
6. Go to step 3
7. Stop.

33. Write short notes on: Taylor series for solving ordinary differential equations.

[2015/Spring]

Solution: See the topic 5.3.

34. Write short notes on: Algorithm for Euler methods.

[2018/Spring]

1. Define function df(x, y) *i.e.*, dy/dx
2. Get values of x_0, y_0, h, x

   where, x_0 is x_{n+0}

   x_1 is x_{n+1}
3. Assign $x_1 = x_0$ and $y_1 = y_0$
4. If $x_1 > x$, then go to step 7

   else

   Compute $y_1 += h \times df(x_1, y_1)$

   and, $x_1 += h$ *i.e.*, $x_1 = x_1 + h$
5. Display x_1 and y_1
6. Go to step 4.
7. Stop