Unit 2

INFINITE SERIES

Theorem: The necessary condition for the convergence of an infinite series Σu_n, is $\lim_{n\to\infty} u_n = 0$ but not sufficient.

[2014 Spring Q. No. 5(a)]; [2008 Spring Q.No. 5(a); 2001 Q.No. 5(a)]

2013 Spring Q. No. 5(a)

Prove that for infinite series ΣU_n to be convergent it is necessary that $\lim_{n\to\infty} u_n = 0$. By taking suitable example show that the converse may not be true.

Proof: Let Σu_n be a series. Let s_n be the partial sum of the series. Also, suppose that the series is convergent.

So, $\lim_{n\to\infty} s_n = s$. Then $\lim_{n\to\infty} s_{n-1} = s$.

Now, $u_n = s_n - s_{n-1}$

Therefore,

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} (s_n - s_{n-1}) = \lim_{n\to\infty} s_n - \lim_{n\to\infty} s_{n-1} = s - s = 0.$$

Thus, if the series converges then necessarily $\lim_{n\to\infty} u_n = 0$.

But the condition is not sufficient. That is $\lim_{n\to\infty} u_n = 0$ may not imply that the series converges.

Take a series $\Sigma\left(\frac{1}{n}\right)$. Then, $u_n = \frac{1}{n}$. So, $\lim_{n\to\infty}\left(\frac{1}{n}\right) = 0$. Thus, $\lim_{n\to\infty} u_n = 0$.

$$\text{But, } \Sigma\left(\frac{1}{n}\right) = 1 + \frac{1}{2} + \frac{1}{3} + \ldots = 1 + \left(\frac{1}{2}\right) + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \ldots$$

$$> 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) + \ldots$$

$$= 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \ldots$$

This series has no fixed finite value. So, $\Sigma\left(\frac{1}{n}\right)$ is divergent.

Geometric series

Definition: A series $a + ar + ar^2 + ar^3 + \ldots$ (having ratio r) is called a geometric series.

Theorem: The geometric series $a + ar + ar^2 + \ldots$ converges to $\frac{a}{1-r}$ if $|r| < 1$ and diverges if $|r| \geq 1$.

Proof: Let the geometric series is,

$$a + ar + ar^2 + ar^3 + \ldots$$

Let s_n be the partial sum of n-terms of series (1). Then,

$$s_n = a + ar + ar^2 + \ldots + ar^{n-1}$$

So, $rs_n = ar + ar^2 + ar^3 + \ldots + ar^n$.

Then,

$$s_n - rs_n = a - ar^n \Rightarrow (1-r)\, s_n = a(1-r^n)$$

$$\Rightarrow s_n = \frac{a(1-r^n)}{1-r} \text{ for } r \neq 1.$$

Case I: If $|r| < 1$ then r^n decreases as n increases.

And finally, $\lim_{n\to\infty} r^n \to 0$. Therefore, $s_n = \frac{a}{1-r}$ for $|r| < 1$

This shows that the series converges to $\frac{a}{1-r}$ for $|r| < 1$.

Case II: If $|r| > 1$ then r^n increases as n increases. So, $\lim_{n\to\infty} r^n \to \infty$ for very large value of n. So, the series diverges for $|r| > 1$.

Case II: If $|r| = 1$ then $s_n = na \to \infty$ as $n \to \infty$.

That means, the series diverges for $|r| = 1$.

Thus the series converges for $|r| < 1$ and diverges for $|r| \geq 1$.

2011 Fall Q.No. 3(a); 2011 Spring Q.No. 5(a); 2010 Fall Q.No. 5(a); 2006 Spring Q. No. 6(a); 2004 Fall Q.No. 5(a); 2003 Fall Q.No. 6(a); 2002 Q.No. 5(a)

Theorem (P–test):

The infinite series $\Sigma\frac{1}{n^p}$ converges if $p > 1$ and diverges if $p \leq 1$.

Proof: Since,

$$\Sigma\frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \ldots \qquad \ldots\ldots(1)$$

Case I: Let $P > 1$. Then, for $n \geq 1$, $(n+1)^p > n^p \Rightarrow \frac{1}{(n+1)^p} < \frac{1}{n^p}$

Then (1) becomes,

$$\Sigma\frac{1}{n^p} = \frac{1}{1^p} + \left(\frac{1}{2^p} + \frac{1}{3^p}\right) + \left(\frac{1}{4^p} + \frac{1}{5^p} + \frac{1}{6^p} + \frac{1}{7^p}\right) + \frac{1}{8^p} + \ldots$$

$$< \frac{1}{1^p} + \left(\frac{1}{2^p} + \frac{1}{2^p}\right) + \left(\frac{1}{4^p} + \frac{1}{4^p} + \frac{1}{4^p} + \frac{1}{4^p}\right) + \ldots$$

$$< \frac{1}{1^p} + \frac{2}{2^p} + \frac{4}{4^p} + \ldots \qquad \ldots\ldots\ldots (2)$$

The series on the right is a geometric series with ratio $r = \frac{2}{2^p} = \frac{1}{2^{p-1}} < 1$. Therefore, by geometric series, the series on the right of (2), converges. So, the series $\Sigma\frac{1}{n^p}$ converges being less than the convergent series.

Case II: Let $P = 1$. Then (1) gives,

$$\Sigma\frac{1}{n^p} = \Sigma\frac{1}{n} = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \ldots$$

$$> 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) + \ldots$$

$$> 1 + \frac{1}{2} + \frac{2}{4} + \frac{4}{8} + \ldots = 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \ldots$$

This shows that the series on the right diverges.

Therefore $\Sigma\frac{1}{n^p}$ diverges for P = 1 being greater than the convergent series.

Case III: Let P < 1. Then for n > 1, $n^p < n$. $\Rightarrow \frac{1}{n^p} > \frac{1}{n}$

Thus, $\Sigma\frac{1}{n^p} > \Sigma\frac{1}{n}$. By case II, the series $\Sigma\frac{1}{n}$ diverges. So, $\Sigma\frac{1}{n^p}$ diverges for P < 1.

Thus, the series $\Sigma\frac{1}{n^p}$ is converges for P > 1 and diverges for P ≤ 1.

Comparison test:

Let Σu_n and Σv_n are infinite series of positive terms and

(a) if Σv_n converges with (i) $u_n \le v_n$ (ii) $\frac{u_n}{v_n} \le k$ for $k \ge 0$ then Σu_n converges.

(b) if Σu_n diverges with (i) $v_n \le u_n$ (ii) $\frac{u_n}{v_n} \ge k$ for $k > 0$ then Σv_n diverges.

Proof: Let Σu_n and Σv_n are two infinite series of positive terms.

(a) Suppose that Σv_n converges. Then $\Sigma v_n \to s$ for some finite value s.

(i) Also, suppose that, $u_n \le v_n$.
Then $\Sigma u_n \le \Sigma v_n \to s$ as $n \to \infty \Rightarrow \Sigma u_n \to s$ as $n \to \infty$.
Therefore, Σu_n converges.

(ii) Also, suppose that, $\frac{u_n}{v_n} \le k$ for $k \ge 0 \quad \Rightarrow \quad u_n \le kv_n$ for $k \ge 0$

So, $\Sigma u_n \le k\,\Sigma v_n \to ks$ as $n \to \infty$.
This shows that Σu_n converges.

(b) Suppose that Σu_n diverges. So, Σu_n is infinite.

(i) Also, suppose that, $v_n \le u_n$. Then, $\Sigma v_n \le \Sigma u_n$.
This shows that Σv_n has no finite sum. So, Σv_n diverges.

(ii) Also, suppose that $\frac{u_n}{v_n} \ge k$ for some constant k > 0.

$\Rightarrow \quad u_n \ge kv_n \to \Sigma u_n \ge k\Sigma v_n$

Then by (i) Σv_n diverges as Σu_n diverges.

2007 Fall Q.No. 5(a)

Let Σu_n and Σv_n are given series of positive terms. If $\lim_{n\to\infty} \frac{u_n}{v_n} = l$ (where l is any finite number), prove that both of series are either convergent or divergent.

Proof: Let Σu_n and Σv_n are two infinite series of positive terms.

Suppose that Σv_n converges. Then $\Sigma v_n \to s$ for some finite value s.

Also, suppose that, $\lim_{n\to\infty} \frac{u_n}{v_n} = l$ (where l is any finite number)

$\Rightarrow \quad u_n = lv_n \qquad$ as $n \to \infty$.

$\Rightarrow \quad u_n \to ls$ (a finite value) $\qquad$ as $n \to \infty$.

This shows that Σu_n converges.

Next, suppose that Σv_n diverges. Then $\Sigma v_n \to \infty$ as $n \to \infty$.

Also, suppose that, $\lim_{n\to\infty} \frac{u_n}{v_n} = l$ (where l is any finite number)

$\Rightarrow \quad u_n = lv_n \qquad$ as $n \to \infty$.

$\Rightarrow \quad u_n \to \infty \qquad$ as $n \to \infty$.

This shows that Σu_n diverges.

[2009 Fall Q.No. 5(a); 2009 Spring Q.No. 5(a)]

State and prove D'Alembert Ratio test.

D'Alembert Ratio Test

Let Σu_n be a series of positive terms. Also, if there is a positive number N such that for all $n \ge N$,

(i) $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} < 1$ then the series Σu_n converges.

(ii) $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} > 1$ then the series Σu_n diverges.

(iii) $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = 1$ then the test fail.

Proof:

Let Σu_n be a series of positive term. Let, $\frac{u_{n+1}}{u_n} < 1$ then there we can find a number r such that r ≤ 1. Then $\frac{u_{n+1}}{u_n} \le 1$ for n ≥ N (natural number).

(i) Then $\frac{u_2}{u_1} \le r, \frac{u_3}{u_2} \le r, \ldots, \frac{u_n}{u_{n-1}} \le r.$

Now,

$$\Sigma u_n = u_1 + u_2 + \ldots + u_n + \ldots$$
$$= u_1 + \frac{u_2}{u_1}.u_1 + \frac{u_3}{u_2}.\frac{u_2}{u_1}.u_1 + \ldots = u_1\left[1 + \frac{u_2}{u_1} + \frac{u_3}{u_2}.\frac{u_2}{u_1} + \ldots\right]$$
$$\le u_1\,[1 + r + r.r + \ldots]$$
$$= u_1\,[1 + r + r^2 + \ldots]$$

Since the series $1 + r + r^2 + \ldots$ be a geometric series with ratio r < 1. So, it converges.

Thus, Σu_n is converges being less than a convergent series is also convergent.

(ii) Next suppose that $\frac{u_{n+1}}{u_n} > 1$ for all n ≥ N > 0.

Then, $\frac{u_2}{u_1} > 1, \frac{u_3}{u_2} > 1, \ldots, \frac{u_n}{u_{n-1}} > 1$. Therefore, $\frac{u_2}{u_1} > 1 \Rightarrow u_2 > u_1$

And $\frac{u_3}{u_2} > 1 \Rightarrow u_3 > u_2 > u_1 \Rightarrow u_3 > u_1$

Consequently, $u_n > u_{n-1} > u_{n-2} > \ldots > u_3 > u_2 > u_1$.

Now,

$$\Sigma u_n = u_1 + u_2 + u_3 + \ldots \quad > u_1 + u_1 + u_1 + \ldots$$
$$= u_1(1 + 1 + 1 + \ldots).$$

Since the series $1 + 1 + 1 + \ldots$ diverges. So, by comparison test, the series Σu_n is also diverges.

(iii) If $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = 1$ then as the given series, it may converges or diverge.

Take an infinite series $\Sigma\left(\frac{1}{n}\right)$ which is divergent by p-test with p = 1.

For $\Sigma\left(\frac{1}{n}\right)$,

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n} = \lim_{n\to\infty}\frac{n}{n+1} = \lim_{n\to\infty}\frac{1}{1+1/n} = \frac{1}{1+0} = 1.$$

Also, take an infinite series $\Sigma\left(\frac{1}{n^2}\right)$ which is convergent by p-test with p = 2. For $\Sigma\left(\frac{1}{n^2}\right)$,

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n} = \lim_{n\to\infty}\frac{n^2}{(n+1)^2} = \lim_{n\to\infty}\frac{1}{(1+1/n)^2} = \frac{1}{1+0} = 1.$$

Cauchy's Root Test:

[2004 Spring Q.No. 5(a)]

A series Σu_n of positive terms with $\sqrt[n]{u_n} = (u_n)^{1/n} = r$ then

(i) the series is convergent if r < 1. (ii) the series is divergent if r > 1.

(iii) test is inconclusive if r = 1.

2013 Fall Q. No. 2(b)

Given an infinite series Σu_n of positive terms and $(u_n)^{1/n} = r$, then prove that series is convergent if r < 1, the series is divergent if r > 1 and the test inconclusive when r = 1.

Proof: Let Σu_n be an infinite series of positive terms.

Also, let $\sqrt[n]{u_n} = r \Rightarrow u_n = r^n$. Therefore, $u_1 = r$, $u_2 = r^2$, $u_3 = r^3$ and so on.

Now,

$$\Sigma u_n = u_1 + u_2 + u_3 + u_4 + \ldots = r + r^2 + r^3 + r^4 + \ldots$$

This is a geometric series having ratio r.

(i) If r < 1, then the series $r + r^2 + r^3 + \ldots$ converges.

Therefore, Σu_n converges.

(ii) If r > 1, then by geometric series theorem the series $r + r^2 + r^3 + \ldots$ diverges. Therefore, Σu_n diverges.

(iii) If r = 1 then the test fails. That means the series may converge and diverge, as the nature of the given series. For example,

Take two series $\Sigma\frac{1}{n}$ and $\Sigma\frac{1}{n^2}$.

Then clearly, $\lim_{n\to\infty}\left(\frac{1}{n}\right)^{1/n} = 1$ and $\lim_{n\to\infty}\left(\frac{1}{n^2}\right)^{1/n} = 1$.

However, the series $\Sigma\frac{1}{n^2}$ converges by p–test with p > 1 and $\Sigma\frac{1}{n}$ diverges by p–test with p = 1.

Cauchy's Integral Tests:

If f(x) is non–negative, monotonic decreasing function for $x \geq 1$ such that $u_n = f(n)$ for all n, then the series $\sum_{n=1}^{\infty} u_n$ and the integral $\int_1^{\infty} f(x)\,dx$ converges or diverges together.

Proof: Let f(x) is non–negative, monotonic decreasing function for $x \geq 1$ such that $f(n) = u_n$ for all n.

Suppose, $r \geq 1$ such that $r + 1 \geq x \geq r$.

Then,

$$f(r+1) \leq f(x) \leq f(r) \quad \text{being f(x) is monotonic decreasing}$$

$$\Rightarrow \quad u_{r+1} \leq f(x) \leq u_r$$

So that,

$$\int_r^{r+1} u_{r+1}dx \leq \int_r^{r+1} f(x)dx \leq \int_r^{r+1} u_r dx \Rightarrow u_{r+1}[x]_r^{r+1} \leq \int_r^{r+1} f(x)dx \leq u_r[x]_r^{r+1}$$

$$\Rightarrow u_{r+1} \leq \int_r^{r+1} f(x)\,dx \leq u_r$$

Setting r = 1, 2, 3, ……… , n and then adding all the terms then,

$$u_2 + u_3 + u_4 + \ldots + u_{n+1} \leq \int_1^2 f(x)\,dx + \int_2^3 f(x)\,dx + \ldots + \int_n^{n+1} f(x)\,dx$$

$$\leq u_1 + u_2 + \ldots + u_n$$

$$\Rightarrow \sum_{r=1}^{n} u_{r+1} \leq \int_1^{n+1} f(x)\,dx \leq \sum_{r=1}^{n} u_r$$

Let $s_n = \sum_{r=1}^{n} u_r$. Then, $s_{n+1} - u_1 \leq \int_1^{n+1} f(x)\,dx \leq s_n$

So,

$$\lim_{n\to\infty}(s_{n+1} - u_1) \leq \lim_{n\to\infty}\int_1^{n+1} f(x)\,dx \leq \lim_{n\to\infty} s_n.$$

$$\Rightarrow \lim_{n\to\infty}(s_{n+1} - u_1) \leq \int_1^{\infty} f(x)\,dx \leq \lim_{n\to\infty} s_n \quad \ldots\ldots (1)$$

Case I: Suppose that the integral $\int_1^{\infty} f(x)\,dx$ converges.

So, for a fixed value M, $\int_1^{\infty} f(x)\,dx = M$. Then by (1),

$$\lim_{n\to\infty}(s_{n+1} - u_1) \leq M$$

$$\Rightarrow \lim_{n\to\infty} s_{n+1} \le (M + u_1). \quad \text{a fixed and finite value.}$$

This shows that $\lim_{n\to\infty} \sum_{r=1}^{n} u_{r+1}$ has a finite value. So, the series converges.

Thus, the integral and the series converges together.

Case II: Suppose that the integral $\int_1^\infty f(x)\,dx$ diverges.

So, for a fixed value M, $\int_1^\infty f(x)\,dx = M$. Then by (1),

$$\lim_{n\to\infty} (s_{n+1} - u_1) \le M$$

$$\Rightarrow \lim_{n\to\infty} s_{n+1} \le (M + u_1), \quad \text{a fixed and finite value.}$$

This shows that $\lim_{n\to\infty} \sum_{r=1}^{n} u_{r+1}$ has a finite value. So, the series converges.

Case II: Suppose that the integral $\int_1^\infty f(x)\,dx$ diverges. So, $\int_1^\infty f(x)\,dx$ has no finite value

That is, $\int_1^\infty f(x)\,dx = \infty$. Then by (1),

$$\infty = \int_1^\infty f(x)\,dx \le \lim_{n\to\infty} s_n = \lim_{n\to\infty} \sum_{r=1}^{n} u_r$$

This shows that the series $\sum_{r=1}^{\infty} u_r$ is unbounded. Therefore, the series diverges.

Thus, the integral and the series diverges together.

Leibnitz's theorem (Alternative series)

An alternative series, $\sum_{n=1}^{\infty} (-1)^{n+1} u_n = u_1 - u_2 + u_3 - u_4 + \ldots$ in which each term is numerically less than the proceeding term and $\lim_{n\to\infty} u_n = 0$ then the series $\sum_{n=1}^{\infty} (-1)^{n+1} u_n$ converges.

OR [2010 Spring Q.No. 3(a)]

If the terms of an infinite series $u_1 - u_2 + u_3 - u_4 + \ldots \ldots \ldots$ Are alternately positive and negative and each term is numerically less than preceding term. Prove that the series is convergent if $\lim_{n\to\infty} u_n = 0$.

Proof: Let, $\lim_{n\to\infty} u_n = 0$. We have

$$s_{2n} = u_1 - u_2 + u_3 - u_4 + \ldots + u_{2n-1} - u_{2n}$$
$$= (u_1 - u_2) + (u_3 - u_4) + \ldots + (u_{2n-1} - u_{2n}) \quad \ldots\ldots (1)$$

Since, each term of the series

$$\Sigma(-1)^{n+1} u_n = u_1 - u_2 + u_3 - u_4 + \ldots$$

is numerically less than the preceding term. So,

$$u_1 > u_2 > u_3 > \ldots > u_{2n-1} > u_{2n}$$

So, $(u_1 - u_2) > 0, (u_3 - u_4) > 0, \ldots, (u_{2n-1} - u_{2n}) > 0$

Then (1) shows that,

$$s_{2n} > 0 + 0 + 0 \Rightarrow s_{2n} > 0$$

Also,

$$s_{2n} = u_1 - u_2 + u_3 - u_4 + \ldots + u_{2n-1} - u_{2n}$$
$$= u_1 - (u_2 - u_3) - (u_4 - u_5) - \ldots - (u_{2n-2} - u_{2n-1}) - u_{2n}$$
$$= u_1 - [(u_2 - u_3) + (u_4 - u_5) + \ldots + (u_{2n-2} - u_{2n-1})] \quad \ldots\ldots (2)$$

Since we have,

$$u_1 > u_2 > u_3 > \ldots > u_{2n-1} > u_{2n}$$

So, $(u_2 - u_3) > 0, \quad (u_4 - u_5) > 0, \ldots, \quad (u_{2n-2} - u_{2n-1}) > 0.$

Then, $\{(u_2 - u_3) + (u_4 - u_5) + \ldots + (u_{2n-2} - u_{2n-1}) + u_{2n}\} > 0$

Therefore, (2) gives us,

$$s_{2n} < u_1.$$

Thus, $0 < s_{2n} < u_1$ for all n. Therefore, s_{2n} as $n\to\infty$, should have a finite limit being u_1 is a finite value. Let $\lim_{n\to\infty} s_{2n} = s$.

Now,

$$u_{2n+1} = s_{2n+1} - s_{2n}$$

Then,

$$\lim_{n\to\infty} u_{2n+1} = \lim_{n\to\infty} s_{2n+1} - \lim_{n\to\infty} s_{2n}$$

$$\Rightarrow 0 = \lim_{n\to\infty} s_{2n+1} - \lim_{n\to\infty} s_{2n}$$

$$\Rightarrow \lim_{n\to\infty} s_{2n+1} = \lim_{n\to\infty} s_{2n} = s$$

This shows that the finite sum of $\Sigma(-1)^{n+1} u_n$ converges. So, the series converges.

Absolute convergence:

An infinite series Σu_n is called converges absolutely if $\Sigma|u_n|$ converges.

Statement of absolute convergence test:

If a series $\Sigma|u_n|$ converges then the series Σu_n converges.

EXERCISE 2.1

Test the convergent and divergent of the following series.

1. $\frac{1}{2}+\frac{2}{3}+\frac{3}{4}+\frac{4}{5}+\ldots..$

Solution: Given series is $\frac{1}{2}+\frac{2}{3}+\frac{3}{4}+\frac{4}{5}+\ldots..$

The general term of the series is $u_n = \frac{n}{n+1}$

Now,

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{n}{n+1} = \lim_{n\to\infty} \frac{n}{n(1+/n)} = \lim_{n\to\infty}\left(\frac{1}{1+1/n}\right) = \frac{1}{1+0} = 1 \neq 0$$

So, the given series diverges.

2. $\frac{1}{\sqrt{2}}+\frac{2}{\sqrt{5}}+\frac{4}{\sqrt{17}}+\frac{8}{\sqrt{65}}+\ldots+\frac{2^n}{\sqrt{4^n+1}}+\ldots..$

Solution: Given series is,

$$\frac{1}{\sqrt{2}}+\frac{2}{\sqrt{5}}+\frac{4}{\sqrt{17}}+\frac{8}{\sqrt{65}}+\ldots+\frac{2^n}{\sqrt{4^n+1}}+\ldots..$$

Clearly, the general term of the series is, $u_n = \frac{2^n}{\sqrt{4^n+1}}$

Now,

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{2^n}{\sqrt{4^n+1}} = \lim_{n\to\infty} \frac{2^n}{2^n\sqrt{1+1/4^n}} = \lim_{n\to\infty} \frac{1}{\sqrt{1+1/4^n}} = \frac{1}{\sqrt{1+0}} = 1 \neq 0$$

So, the given series is diverges.

3. $\frac{1}{2}+\frac{1+x}{2+x}+\frac{1+2x}{2+2x}+\ldots+\frac{1+nx}{2+nx}+\ldots$

Solution: Given series is

$$\frac{1}{2}+\frac{1+x}{2+x}+\frac{1+2x}{2+2x}+\ldots+\frac{1+nx}{2+nx}+\ldots$$

The general term of the series is, $u_n = \frac{1+nx}{2+nx}$

Now,

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1+nx}{2+nx} = \lim_{n\to\infty} \frac{n(x+1/n)}{n(x+2/n)} = \lim_{n\to\infty}\left(\frac{x+1/n}{x+2/n}\right)$$

$$\Rightarrow \lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{x+0}{x+0} = \lim_{n\to\infty} \frac{x}{x} = \lim_{n\to\infty} (1) = 1 \neq 0$$

This shows that the series diverges.

4. $\Sigma\left(1+\frac{1}{n}\right)^n$ **[2005 Fall– Short]**

Solution: Given series is $\Sigma\left(1+\frac{1}{n}\right)^n$. The general term of the series is, $u_n = (1+1/n)^n$.

Here,

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} (1+1/n)^n = e \neq 0$$

This shows that the series diverges.

5. $\Sigma(1+1/n)^{-n}$

Solution: Given series is $\Sigma(1+1/n)^{-n}$. The general term of the series is, $u_n = (1+1/n)^{-n}$

Here,

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} (1+1/n)^{-n} = \lim_{n\to\infty} \frac{1}{(1+1/n)^n} = \frac{1}{e} \neq 0$$

This shows that the series diverges.

6. $1+4+9+16+\ldots\ldots\ldots$

Solution: Given series is, $1+4+9+16+\ldots$ The general term of the series is, $u_n = n^2$.

Here,

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} n^2 = \infty \neq 0.$$

This shows that the series diverges.

7. $2+\frac{3}{2}+\frac{4}{3}+\frac{5}{4}+\ldots$ **[2002 – Short]**

Solution: Given series is, $2+\frac{3}{2}+\frac{4}{3}+\frac{5}{4}+\ldots$

The general term of the series is, $u_n = \frac{n+1}{n} = \frac{n(1+1/n)}{n} = 1+\frac{1}{n}$

Here,

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} (1+1/n) = 1+0 = 1 \neq 0$$

This shows that the series is diverges.

EXERCISE 2.2

Test the convergent and divergent of the following series.

1. $\frac{1}{1.2.3}+\frac{3}{2.3.4}+\frac{5}{3.4.5}+\ldots$

Solution: Given series is

$$\frac{1}{1.2.3}+\frac{3}{2.3.4}+\frac{5}{3.4.5}+\ldots \qquad \ldots\ldots(i)$$

The general term of the series (i) is, $u_n = \frac{2n-1}{n(n+1)(n+2)}$

Choose $v_n = \frac{1}{n^2}$ then the series Σv_n converges by p–test with $p = 2 > 1$.

Then the series Σu_n converges by comparison test if $\lim_{n\to\infty} \frac{u_n}{v_n}$ has a non-zero finite value.

Here,

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{2n-1}{n(n+1)(n+2)} \times \frac{n^2}{1}$$

$$= \lim_{n\to\infty} \frac{n(2-1/n) \times n^2}{n^3(1+1/n)(1+2/n)}$$

$$= \lim_{n\to\infty} \frac{2-1/n}{(1+1/n)(1+2/n)}$$

$$= \frac{2-0}{(1+0)\,(1+0)} = 2 \text{ (a non-zero finite value)}$$

This shows that Σu_n converges, by comparison test.

2. $\sqrt{\frac{1}{2^3}}+\sqrt{\frac{2}{3^3}}+\sqrt{\frac{3}{4^3}}+\dots$

Solution: Given series is

$$\sqrt{\frac{1}{2^3}}+\sqrt{\frac{2}{3^3}}+\sqrt{\frac{3}{4^3}}+\dots \qquad \text{..............(i)}$$

The general term of the series (i) is,

$$u_n = \sqrt{\frac{n}{(n+1)^3}} = \sqrt{\frac{n}{n^3\,(1+1/n)^3}} = \sqrt{\frac{1}{n^2\,(1+1/n)^3}} = \frac{1}{n}\sqrt{\frac{1}{(1+1/n)^3}}$$

Choose $v_n = \frac{1}{n}$ then by p–test, the series Σv_n diverges being Σv_n is a p–series with $= 1$.

Being Σv_n is divergent, by comparison test the series Σu_n is also divergent if $\lim\limits_{n\to\infty}$ $\frac{u_n}{v_n}$ = a non-zero finite value.

Here,

$$\lim_{n\to\infty}\frac{u_n}{v_n} = \lim_{n\to\infty}\frac{1}{n}\sqrt{\frac{1}{(1+1/n)^3}}\times\frac{n}{1} = \lim_{n\to\infty}\sqrt{\frac{1}{(1+1/n)^3}} = \sqrt{\frac{1}{(1+0)^3}} = 1.$$

which is a non-zero finite value.

Thus, Σu_n diverges. That is, the given series (i) diverges.

3. $\frac{2}{3^2}+\frac{3}{4^2}+\frac{4}{5^2}+\dots\dots\dots$ **[2009 Fall – Short]**

Solution: Given series is,

$$\frac{2}{3^2}+\frac{3}{4^2}+\frac{4}{5^2}+\dots\dots \qquad \text{..........(i)}$$

The general term of the series (i) is,

$$u_n = \frac{n+1}{(n+2)^2} = \frac{n(1+1/n)}{n^2\,(1+2/n)^2} = \frac{(1+1/n)}{n(1+2/n)^2}$$

Choose $v_n = \frac{1}{n}$ the comparing with p–series we get $p = 1$.

Then by p–test, the series Σv_n diverges being $p = 1$.

Since the series Σv_n diverges, then by comparison test the series Σu_n diverges only if $\lim\limits_{n\to\infty}\frac{u_n}{v_n}$ is a non-zero finite value.

Here,

$$\lim_{n\to\infty}\frac{u_n}{v_n} = \lim_{n\to\infty}\frac{(1+1/n)}{n(1+2/n)^2}\times\frac{n}{1} = \lim_{n\to\infty}\frac{(1+1/n)}{(1+2/n)^2} = \frac{1+0}{(1+0)^2} = 1$$

which is a non-zero finite value.

This shows that Σu_n diverges. That is, the given series (i) diverges.

4. $\frac{14}{1^3}+\frac{24}{2^3}+\frac{34}{3^3}+\dots\dots\dots+\frac{4+10n}{n^3}+\dots\dots\dots$

Solution: Given series is,

$$\frac{14}{1^3}+\frac{24}{2^3}+\frac{34}{3^3}+\dots\dots+\frac{4+10n}{n^3}+\dots \qquad \text{.......(i)}$$

The general term of series (i) is,

$$u_n = \frac{4+10n}{n^3}$$

Choose $v_n = \frac{1}{n^2}$ then comparing it with the p–series we get $p = 2$. So that the series Σv_n converges by p–test, being Σv_n is a p–series with $p = 2 > 1$.

Since Σv_n converges then by comparison test the series Σu_n converges if $\lim\limits_{n\to\infty}\frac{u_n}{v_n}$ has a non-zero finite value.

Here,

$$\lim_{n\to\infty}\frac{u_n}{v_n} = \lim_{n\to\infty}\frac{4+10n}{n^3}\times\frac{n^2}{1}$$

$$= \lim_{n\to\infty}\frac{n(4/n+10)}{n} = \lim_{n\to\infty}\left(\frac{4}{n}+10\right) = 0+10 = 10$$

which is a non-zero finite value.

This shows that Σu_n is also convergent.

5. $\frac{3}{4}+\frac{5}{9}+\frac{7}{16}+\frac{9}{25}+\dots$

Solution: Given series is,

$$\frac{3}{4}+\frac{5}{9}+\frac{7}{16}+\frac{9}{25}+\dots$$

The general term of the series is,

$$u_n = \frac{2n+1}{(n+1)^2} = \frac{n(2+1/n)}{n^2(1+1/n)^2} = \frac{(2+1/n)}{n(1+1/n)^2}$$

Choose $v_n = \frac{1}{n}$ then comparing it with the p–series we get $p = 1$. So, by p–test, the series Σv_n is divergent.

Since Σv_n is divergent, then by comparison test, the series Σu_n diverges if $\lim\limits_{n\to\infty}\frac{u_n}{v_n}$ has a non-zero finite value.

Here,

$$\lim_{n\to\infty}\frac{u_n}{v_n} = \lim_{n\to\infty}\frac{2+1/n}{n(1+1/n)^2}\times\frac{n}{1} = \lim_{n\to\infty}\frac{2+1/n}{(1+1/n)^2} = \frac{2+0}{(1+0)^2} = 2$$

which is a non-zero finite value.

This shows that the series Σu_n diverges.

6. $\frac{1}{2}+\frac{2}{3}+\frac{3}{4}+\frac{4}{5}+\frac{5}{6}+\dots$

Solution: Given series is

$$\frac{1}{2}+\frac{2}{3}+\frac{3}{4}+\frac{4}{5}+\frac{5}{6}+\dots$$

whose n^{th} term is

$$u_n = \frac{n}{n+1}$$

Here,

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{n}{n+1} = \lim_{n\to\infty} \frac{1}{1+1/n} = \frac{1}{1+0} = 1 \neq 0$$

This means the given series is divergent.

7. $\frac{101}{3} + \frac{102}{10} + \frac{103}{29} + \ldots + \frac{100+n}{n^3+2} + \ldots$

Solution: Process as Q. 4.

Hint: $u_n = \frac{100+n}{n^3+2} = \frac{n(100/n+1)}{n^3(1+2/n^3)} = \frac{(1+100/n)}{n^2(1+2/n^3)}$

Choose $v_n = \frac{1}{n^2}$ which is a p–series with $p = 2 > 1$.

So, the series Σv_n converges by p–test. Also, Σu_n is convergent.

8. $\Sigma[\sqrt{n^2+1} - n]$ **[2009 Spring]**

Solution: Given series is, $\Sigma[\sqrt{n^2+1} - n]$

The general term of the series is,

$$u_n = \sqrt{n^2+1} - n$$

which is in $\infty - \infty$ form as $n \to \infty$. So, multiply numinator and denominator by its conjugate. Then,

$$u_n = (\sqrt{n^2+1} - n) \times \frac{(\sqrt{n^2+1}+n)}{(\sqrt{n^2+1}+n)} = \frac{n^2+1-n^2}{\sqrt{n^2+1}+n} = \frac{1}{n\left(\sqrt{1+\frac{1}{n^2}}+1\right)}$$

This is a p–series with $p = 1$. Therefore Σv_n diverges, by p–test.

Then by comparison test, the series Σu_n is also diverges being the series Σv_n diverges if $\lim_{n\to\infty} \frac{u_n}{v_n}$ has a non-zero finite value.

Here,

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{1}{n(\sqrt{1+1/n^2}+1)} \times \frac{n}{1}$$

$$= \lim_{n\to\infty} \frac{1}{(\sqrt{1+1/n^2}+1)} = \frac{1}{\sqrt{1+0}+1} = \frac{1}{1+1} = \frac{1}{2}$$

which is non-zero finite value.

This shows that the given series is divergent.

9. $\Sigma[\sqrt{n+1} - \sqrt{n}]$

Solution: Process as Q. 8

Hint: $u_n = \sqrt{n+1} - \sqrt{n}$

Multiply numerator and denominator of u_n by $\sqrt{n+1} + \sqrt{n}$ and then choose $v_n = \frac{1}{\sqrt{n}} = \frac{1}{n^{1/2}}$. Then the series Σv_n is convergent by p–test with $p = \frac{1}{2} < 1$.

So, Σv_n diverges and Σu_n is also divergent.

10. $\Sigma[\sqrt{n^2+1} - n]$

Solution: Process as Q. 8

11. $\Sigma[\sqrt{n^4+1} - \sqrt{n^4-1}]$

Solution: Given series is, $\Sigma(\sqrt{n^4+1} - \sqrt{n^4-1})$

The general term of the series is

$$u_n = \sqrt{n^4+1} - \sqrt{n^4-1}$$

which is in $\infty - \infty$ form as $n\to\infty$. So, multiply numinator and denominator by its conjugate then,

$$u_n = \sqrt{n^4+1} - \sqrt{n^4-1} \times \frac{\sqrt{n^4+1}+\sqrt{n^4-1}}{\sqrt{n^4+1}+\sqrt{n^4-1}}$$

$$= \frac{(n^4+1)-(n^4-1)}{\sqrt{n^4+1}+\sqrt{n^4-1}} = \frac{2}{n^2[\sqrt{1+1/n^4}+\sqrt{1-1/n^4}]}$$

Choose $v_n = \frac{1}{n^2}$. Then the series Σv_n is a p–series with $p = 2 > 1$. So, the series Σv_n converges, by p–test.

Since Σv_n converges then by comparison test the series Σu_n is also converges if $\lim_{n\to\infty} \frac{u_n}{v_n}$ has a non-zero finite value.

Here,

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{2}{n^2[\sqrt{1+1/n^4}+\sqrt{1-1/n^4}]} \times \frac{n^2}{1}$$

$$= \lim_{n\to\infty} \frac{2}{\sqrt{1+1/n^4}+\sqrt{1-1/n^4}} = \frac{2}{\sqrt{1+0}+\sqrt{1-0}} = \frac{2}{1+1} = \frac{2}{2} = 1$$

which is a non-zero finite value.

This shows that the series Σu_n converges.

12. $\Sigma\left(\frac{n}{(a+nb)^2}\right)$

Solution: Given series is, $\Sigma\left(\frac{n}{(a+nb)^2}\right)$

Here, the general term of the series is

$$u_n = \frac{n}{(a+nb)^2} = \frac{n}{n^2(a+b/n)^2} = \frac{1}{n(a+b/n)}$$

Choose $v_n = \frac{1}{n}$. Then Σv_n is a p–series with $p = 1$, so series is divergent by p–test.

Since Σv_n diverges then by comparison test Σu_n diverges if $\lim_{n\to\infty} \frac{u_n}{v_n}$ has a non-zero finite value.

Here,

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{1}{n(a+b/n)} \cdot \frac{n}{1} = \lim_{n\to\infty} \frac{1}{(a+b/n)} = \frac{1}{a+0} = \frac{1}{a}$$

which is a non-zero finite value.

This shows that the series Σu_n diverges.

13. $\Sigma\left(\dfrac{\sqrt{n}}{n^2+1}\right)$

Solution: Given series is, $\Sigma\left(\dfrac{\sqrt{n}}{n^2+1}\right)$

The general term of the series is

$$u_n = \frac{\sqrt{n}}{n^2+1} = \frac{n^{1/2}}{n^2(1+1/n^2)} = \frac{1}{n^{3/2}(1+1/n^2)}$$

Choose $v_n = \frac{1}{n^{3/2}}$. Then the series Σv_n is a p–series with $p = \frac{3}{2} > 1$. So, Σv_n converges by p–test.

Since Σv_n converges then by comparison test, the series Σu_n is also convergent if $\lim_{n\to\infty} \frac{u_n}{v_n}$ has a non-zero finite value.

Here,

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{1}{n^{3/2}(1+1/n^2)} \times \frac{n^{3/2}}{1} = \lim_{n\to\infty} \frac{1}{1+1/n^2)} = \frac{1}{1+0} = 1$$

which is a non-zero finite value.

This shows that Σu_n is convergent.

14. $\Sigma[(n^3+1)^{1/3} - n]$

Solution: Given series is, $\Sigma[(n^3+1)^{1/3} - n]$

The general term of the series is, $u_n = (n^3+1)^{1/3} - n$

The general term is in $\infty - \infty$ form as $n\to\infty$. So, multiply numerator and denominator by $(n^3+1)^{2/3} + n(n^3+1)^{1/3} + n^2$ to make the form as $a^3 - b^3$. Then,

$$u_n = [(n^3+1)^{1/3} - n] \times \left[\frac{(n^3+1)^{2/3} + n(n^3+1)^{1/3} + n^2}{(n^3+1)^{2/3} + n(n^3+1)^{1/3} + n^2}\right]$$

$$= \frac{((n^3+1)^{1/3})^3 - n^3}{(n^3+1)^{2/3} + n(n^3+1) + n^2} = \frac{n^3+1-n^3}{n^2(1+1/n^3)^{2/3} + n^2} = \frac{1}{n^2[(1+1/n^3)^{2/3}+1]}$$

Choose $v_n = \frac{1}{n^2}$. Then Σv_n is a p–series with $p = 2 > 0$. Therefore, the series Σv_n converges by p–test.

Since Σv_n converges. Then by comparison test, the series Σu_n converges if $\lim_{n\to\infty} \frac{u_n}{v_n}$ has a non–zero finite value.

Here,

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{1}{n^2[(1+1/n^3)^{2/3}+1]} \times \frac{n^2}{1}$$

$$= \lim_{n\to\infty} \frac{1}{(1+1/n^3)^{2/3}+1} = \frac{1}{(1+0)^{2/3}+1} = \frac{1}{1+1} = \frac{1}{2}$$

which is a non-zero finite value.

This shows that the series Σu_n converges.

15. $\frac{2}{1^p} + \frac{3}{2^p} + \frac{4}{3^p} + \frac{5}{4^p} + \ldots\ldots\ldots$

Solution: Given series is

$$\frac{2}{1^p} + \frac{3}{2^p} + \frac{4}{3^p} + \frac{5}{4^p} + \ldots\ldots$$

The general term of the series is

$$u_n = \frac{n+1}{n^p} = \frac{n(1+1/n)}{n^p} = \frac{1+1/n}{n^{p-1}}$$

Choose $v_n = \frac{1}{n^{p-1}}$. Then the series Σv_n is p–series. Therefore, by p–test, the series Σv_n converges for $p-1>1 \to p>2$ and diverges for $p-1 \le 1 \Rightarrow p \le 2$.

Since by comparison test we have if $\lim_{n\to\infty} \frac{u_n}{v_n}$ has a non–zero finite value then either both convergent or both divergent.

Here,

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{1+\frac{1}{n}}{n^{p-1}} \times \frac{n^{p-1}}{1} = \lim_{n\to\infty} (1+1/n) = 1+0 = 1$$

which is a non-zero finite value.

Since Σv_n converges for $p > 2$ and diverges for $p \le 2$. So, by comparison test, the series Σu_n converges for $p > 2$ and diverges for $p \le 2$. That is, the given series converges for $p > 2$ and diverges for $p \le 2$.

EXERCISE 2.3

Test the convergent and divergent of the following series.

1. $1 + \frac{2^2}{2!} + \frac{3^2}{3!} + \frac{4^2}{4!} + \ldots\ldots\ldots$

Solution: Given series is,

$$1 + \frac{2^2}{2!} + \frac{3^2}{3!} + \frac{4^2}{4!} + \ldots\ldots\ldots$$

The general term of the series is, $u_n = \frac{n^2}{n!}$

Here,

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{(n+1)^2}{(n+1)!} \times \frac{n!}{n^2}$$

$$= \lim_{n\to\infty} \frac{(n+1)^2 \times n!}{(n+1) \times n! \times n^2} = \lim_{n\to\infty} \frac{n+1}{n^2} = \lim_{n\to\infty} \frac{(1+1/n)}{n} = \frac{1+0}{\infty}$$

$$= 0 < 1.$$

That is, $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = 0 < 1$. Then by D'Alembert test, the given series converges.

2. $1 + \frac{2!}{2^2} + \frac{3!}{3^2} + \frac{4!}{4^2} + \ldots$

Solution: Given series is,

$$1 + \frac{2!}{2^2} + \frac{3!}{3^2} + \frac{4!}{4^2} + \ldots$$

The general term of the series is, $u_n = \frac{n!}{n^2}$

Here,

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n}=\lim_{n\to\infty}\frac{(n+1)!}{(n+1)^2}\times\frac{n^2}{n!}$$

$$=\lim_{n\to\infty}\frac{(n+1)\times n!}{(n+1)^2}\times\frac{n^2}{n!}=\lim_{n\to\infty}\frac{n^2}{n+1}=\lim_{n\to\infty}\frac{n}{1+1/n}=\infty>1$$

That is, $\lim_{n\to\infty}\frac{u_{n+1}}{u_n}>1$. So, by D'Alembert ratio test the given series diverges.

3. $\frac{1}{1+2}+\frac{2}{1+2^2}+\frac{3}{1+2^3}+\ldots\ldots$

Solution: Given series is,

$$\frac{1}{1+2}+\frac{2}{1+2^2}+\frac{3}{1+2^3}+\ldots\ldots$$

The general term of the series is, $u_n=\frac{n}{1+2^n}$

Here,

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n}=\lim_{n\to\infty}\left(\frac{n+1}{1+2^{n+1}}\right)\div\left(\frac{1+2^n}{n}\right)$$

$$=\lim_{n\to\infty}\left(\frac{n+1}{2.2^n+1}\right)\times\left(\frac{1+2^n}{n}\right)$$

$$=\lim_{n\to\infty}\frac{n(1+1/n)}{2^n(2+1/2^n)}\times\frac{2^n(1+1/2^n)}{n}$$

$$=\lim_{n\to\infty}\frac{1+1/n}{2+1/2^n}\times\frac{1+1/2^n}{1}=\left(\frac{1+0}{2+0}\right)\times\left(\frac{1+0}{1}\right)=\frac{1}{2}<1$$

Thus, by D'Alembert ratio test, the given series converges.

4. $\frac{2}{1}+\frac{2.5.8}{1.5.9}+\frac{2.5.8.11}{1.5.9.13}+\ldots\ldots\ldots$

Solution: Given series is,

$$\frac{2}{1}+\frac{2.5.8}{1.5.9}+\frac{2.5.8.11}{1.5.9.13}+\ldots\ldots\ldots$$

The general term of the series is, $u_n=\frac{(3n-1)!}{(4n-3)!}$

Here,

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n}=\lim_{n\to\infty}\frac{(3(n+1)-1)!}{(4(n+1)-3)!}\times\frac{(4n-3)!}{(3n-1)!}$$

$$=\lim_{n\to\infty}\frac{(3n+2)!}{(4n+1)!}\times\frac{(4n-3)!}{(3n-1)!}$$

$$=\lim_{n\to\infty}\frac{(3n+2).(3n+1).3n.(3n-1)!}{(4n+1).4n.(4n-1).(4n-2).(4n-3)!}\times\frac{(4n-3)!}{(3n-1)!}$$

$$=\lim_{n\to\infty}\frac{3(3n+2)(3n+1)}{(4n+1)(4n-1)(4n-2)}$$

$$=\lim_{n\to\infty}\frac{3n^2(3+2/n)(3+1/n)}{(4+1/n)(4-1/n)(4-2/n)}$$

$$=\lim_{n\to\infty}\frac{3(3+2/n)(3+1/n)}{4(4+1n)(4-1/n)(4-2/n)}=\frac{3(3+0)(3+0)}{4(4+0)(4-0)(4-0)}$$

$$=\frac{27}{256}<1$$

This shows that the given series is convergent, by D'Alembert ratio test.

5. $\frac{2!}{3}+\frac{3!}{3^2}+\frac{4!}{3^3}+\ldots\ldots\ldots$

Solution: Given series is,

$$\frac{2!}{3}+\frac{3!}{3^2}+\frac{4!}{3^3}+\ldots\ldots\ldots$$

The general term of the series is, $u_n=\frac{(n+1)!}{3^n}$

Here,

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n}=\lim_{n\to\infty}\frac{((n+1)+1)!}{3^{n+1}}\times\frac{3^n}{(n+1)!}$$

$$=\lim_{n\to\infty}\frac{(n+2)!}{3^n\,3}\times\frac{3^n}{(n+1)!}$$

$$=\lim_{n\to\infty}\frac{(n+2).(n+1)!}{3}\times\frac{1}{(n+1)!}$$

$$=\lim_{n\to\infty}\frac{(n+2)}{3}=\lim_{n\to\infty}\frac{n(1+2/n)}{3}=\infty$$

That is, $\lim_{n\to\infty}\frac{u_{n+1}}{u_n}\sim\infty>1$. Then by D'Alembert ratio test, the given series is divergent.

6. $\left(\frac{1}{3}\right)^2+\left(\frac{1.2}{3.5}\right)^2+\left(\frac{1.2.3}{3.5.7}\right)^2+\left(\frac{1.2.3.4}{3.5.7.9}\right)^2+\ldots\ldots$

Solution: Given series is,

$$\left(\frac{1}{3}\right)^2+\left(\frac{1.2}{3.5}\right)^2+\left(\frac{1.2.3}{3.5.7}\right)^2+\left(\frac{1.2.3.4}{3.5.7.9}\right)^2+\ldots\ldots$$

The general term of the series is. $u_n=\left(\frac{n!}{(2n+1)!}\right)^2$

Here,

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n}=\lim_{n\to\infty}\left[\frac{(n+1)!}{(2(n+1)+1)!}\right]^2\times\left[\frac{(2n+1)!}{n!}\right]^2$$

$$=\lim_{n\to\infty}\left(\frac{(n+1)\,n!}{(2n+3)!}\right)^2\times\left(\frac{(2n+1)!}{n!}\right)^2$$

$$=\lim_{n\to\infty}\frac{(n+1)^2(n!)^2}{[(2n+3)(2n+2)(2n+1)!]^2}\times\frac{[(2n+1)!]^2}{(n!)^2}$$

$$=\lim_{n\to\infty}\frac{(2n+1)^2}{(2n+3)^2(2n+2)^2((2n+1)!)^2}\times\frac{((2n+1)!)^2}{1}$$

$$=\lim_{n\to\infty}\frac{(n+1)^2}{n^2(2+3/n)^2.4(n+1)^2}$$

$$=\lim_{n\to\infty}\frac{1}{4n^2(2+3/n)^2}\sim\frac{1}{\infty}\sim 0$$

That is, $\lim_{n\to\infty}\frac{u_{n+1}}{u_n}\sim 0<1$. This shows that the given series converges, by D'Alembert ratio test.

7. $\Sigma\left(\frac{2^{n-1}}{3^{n+1}}\right)$

Solution: Given series is, $\Sigma\left(\frac{2^{n-1}}{3^{n+1}}\right)$.

The general term of the series is, $u_n = \frac{2^{n-1}}{3^{n+1}}$

Here,

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{2^{(n+1)-1}}{3^{(n+1)+1}} \times \frac{3^{n+1}}{2^{n-1}}$$

$$= \lim_{n\to\infty} \frac{2^n}{3^{n+2}} \times \frac{3^{n+1}}{2^{n-1}}$$

$$= \lim_{n\to\infty} \frac{2^n}{3.3^{n+1}} \times \frac{3^{n+1}}{2^n.2^{-1}} = \lim_{n\to\infty} \frac{1}{3.2^{-1}} = \frac{1}{3.2^{-1}} = \frac{2}{3} = 0.667 < 1$$

Thus, by D'Alembert ratio test, the given series is convergent.

8. $\Sigma\left(\frac{1}{n!}\right)$

Solution: Given series is $\Sigma\left(\frac{1}{n!}\right)$

The general term of the series is, $u_n = \frac{1}{n!}$

Here,

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{1}{(n+1)!} \times \frac{n!}{1}$$

$$= \lim_{n\to\infty} \frac{1}{(n+1).n!} \times n! = \lim_{n\to\infty} \frac{1}{n+1} = \lim_{n\to\infty} \frac{1}{n(1+1/n)} \sim \frac{1}{\infty} \sim 0 < 1.$$

Thus, $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} \sim 0 < 1$. So, by D'Alembert ratio test, the given series is divergent.

9. $\Sigma \frac{n^2(n+1)^2}{n!}$

Solution: Given series is, $\Sigma \frac{n^2(n+1)^2}{n!}$

The general term of the series is, $u_n = \frac{n^2(n+1)^2}{n!}$

Here,

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{(n+1)^2 (n+1+1)^2}{(n+1)!}$$

$$= \lim_{n\to\infty} \frac{(n+1)^2 (n+2)^2}{(n+1)\, n(n-1)\,(n-2)\,(n-3)!}$$

$$= \lim_{n\to\infty} \frac{(n+1)\, n^2\, (1+2/n)^2}{n.n^2\,(1-1/n)\,(1-2/n)\,(n-3)!}$$

$$= \lim_{n\to\infty} \frac{n(1+1/1n)\,(1+2/n)^2}{n(1-1/n)\,(1-2/n)\,(n-3)!} = \frac{(1+0)\,(1+0)^2}{(1-0)\,(1-0)\,\infty} = \frac{1}{\infty} = 0.$$

Thus, $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = 0 < 1$. So, by D'Alembert ratio test the given series is convergent.

10. $\Sigma \frac{n^2}{3^n}$

Solution: Given series is, $\Sigma \frac{n^2}{3^n}$

The general term u_n of the series is, $u_n = \frac{n^2}{3^n}$

Here,

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{(n+1)^2}{3^{n+1}} \times \frac{3^n}{n^2}$$

$$= \lim_{n\to\infty} \frac{(1+1/n)^2}{3} = \lim_{n\to\infty} \frac{(1+1/n)^2}{3} \sim \frac{(1+0)^2}{3} = \frac{1}{3} < 1.$$

This shows that the given series converges, by D'Alembert ratio test.

11. $\Sigma\left(\frac{3^n-2}{3^n+1}\right)$

Solution: Given series is, $\Sigma\left(\frac{3^n-2}{3^n+1}\right)$

The general term of the series is, $u_n = \frac{3^n-2}{3^n+1}$

Here,

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{3^n-2}{3^n+1} = \lim_{n\to\infty} \frac{1-2/3^n}{1+1/3^n} = \frac{1-0}{1+0} = 1 \neq 0.$$

Thus, the given series is divergent.

12. $\Sigma\sqrt{\frac{n^2+a}{2^n+a}}$

Solution: Given series is, $\Sigma\sqrt{\frac{n^2+a}{2^n+a}}$

The general term of the series is, $u_n = \frac{n^2+a}{2^n+a}$

Here,

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \left(\frac{(n^2+1)^2+a}{2^{n+1}+a} \times \frac{2^n+a}{n^2+a}\right)$$

$$= \lim_{n\to\infty} \frac{n^2\,[(1+1/n)^2+a/n^2]}{2^n\,(2+a/2^2)} \times \frac{2^n\,(1+a/2^n)}{n^2\,(1+a/n^2)}$$

$$= \lim_{n\to\infty} \left[\frac{(1+1/n)^2+a/n^2}{2+a/2^2} \times \frac{1+a/2^n}{1+a/n^2}\right] = \frac{(1+0)^2+0}{2+0} \times \frac{1+0}{1+0} = \frac{1}{2}$$

Thus, $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \frac{1}{2} < 1$. Therefore, the given series is convergent, by D'Alembert ratio test.

13. $1 + \frac{2^p}{2!} + \frac{3^p}{3!} + \frac{4^p}{4!} + \ldots\ldots$

Solution: Given series is,

$$1 + \frac{2^p}{2!} + \frac{3^p}{3!} + \frac{4^p}{4!} + \ldots\ldots$$

The general term of the series is, $u_n = \frac{n^p}{n!}$

Here,

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{(n+1)^p}{(n+1)!} \times \frac{n!}{n^p}$$

$$= \lim_{n\to\infty} \frac{n^p\,(1+1/n)^p}{(n+1).\,n!} \times \frac{n!}{n^p} = \lim_{n\to\infty} \frac{(1+1/n)^p}{n(1+1/n)} = \frac{(1+0)^p}{\infty} \sim 0$$

Thus, $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = 0 < 1$. This shows that the given series is convergent by D'Alembert ratio test.

14. $\sum_{n=1}^{\infty} \frac{n^3}{2^n}$

Solution: Given series is, $\sum_{n=1}^{\infty} \frac{n^3}{2^n}$

The general term of the series is, $u_n = \frac{n^3}{2^n}$

Here,

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{(n+1)^3}{2^{n+1}} \times \frac{2^n}{n^3}$$
$$= \lim_{n\to\infty} \frac{n^3(1+1/n)^3}{2^n.2} \times \frac{2^n}{n^3} = \lim_{n\to\infty} \frac{(1+1/n)^3}{2} = \frac{(1+0)^3}{2} = \frac{1}{2}$$

Thus, $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \frac{1}{2} < 1$. Therefore, the given series is convergent by D'Alembert ratio test.

15. $\sum_{n=1}^{\infty} \frac{2^n}{n+1}$

Solution: Given series is, $\sum_{n=1}^{\infty} \frac{2^n}{n+1}$

The general term of the series is, $u_n = \frac{2^n}{n+1}$

Here,

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \left(\frac{2^{n+1}}{n+1+1} \times \frac{n+1}{2^n}\right)$$
$$= \lim_{n\to\infty} \left(\frac{2^n.2}{n+2} \times \frac{n+1}{2^n}\right)$$
$$= \lim_{n\to\infty} \frac{2(1+1/n)}{(1+2/n)} = \lim_{n\to\infty} \frac{2(1+1/n)}{1+2/n} = \frac{2(1+0)}{1+0} = 2.$$

Thus, $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = 2 > 1$. So, the series is divergent, by D'Alembert ratio test.

16. $\sum_{n=1}^{\infty} \left(\frac{-n^2}{2^n}\right)$

Solution: Similar to Q. 14.

17. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3+2}}$

Solution: Given series is, $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3+2}}$

The general term of the series is,

$$u_n = \frac{1}{\sqrt{n^3+2}} = \frac{1}{n^{3/2}\sqrt{1+2/n^3}}$$

Choose $v_n = \frac{1}{n^{3/2}}$. Then the series Σv_n is convergent by p-test with $p = 3/2 > 1$.

Here,

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{1}{\sqrt{1+2/n^3}}$$
$$= \lim_{n\to\infty} \frac{1}{\sqrt{1+0}}$$
$= 1$ which is a non-zero finite value.

So, the given series Σu_n is also convergent by limit comparison test.

18. $\sum_{n=1}^{\infty} \frac{(n+1)(n+2)}{n!}$

Solution: Given series is

$$\sum_{n=1}^{\infty} \frac{(n+1)(n+2)}{n!}$$

The general term of the series is, $u_n = \frac{(n+1)(n+2)}{n!}$

Here,

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \frac{(n+1+1)(n+1+2)}{(n+1)!} \times \frac{n!}{(n+1)(n+2)}$$
$$= \lim_{n\to\infty} \frac{(n+2)(n+3)}{(n+1)n!} \times \frac{n!}{(n+1)(n+2)}$$
$$= \lim_{n\to\infty} \frac{(n+3)}{(n+1)^2}$$
$$= \lim_{n\to\infty} \frac{n(1+3/n)}{n^2(1+1/n)^2} = \lim_{n\to\infty} \frac{1+3/n}{n(1+1/n)^2} = \frac{1+0}{\infty} = 0.$$

As, $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = 0 < 1$. Then by D'Alembert ratio test, the given series is convergent.

19. $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$

Solution: Given series is,

$$\sum_{n=1}^{\infty} \frac{n}{n^2+1}$$

The general term of the series is. $u_n = \frac{n}{n^2+1} = \frac{1}{n(1+1/n^2)}$

Choose, $v_n = \frac{1}{n}$. Then the series Σv_n is divergent by p-test with $p = 1$.

Here,

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{1}{(1+1/n^2)} = \frac{1}{(1+0)} = 1 < 0.$$

Since the series Σv_n is divergent. Therefore the given series Σu_n is also divergent by comparison test.

20. $\sum_{n=1}^{\infty} \frac{(n+3)!}{3!\, n!\, 3^n}$

Solution: Given series is

$$\sum_{n=1}^{\infty} \frac{(n+3)!}{3!\, n!\, 3^n} = \sum_{n=1}^{\infty} \frac{(n+3)!}{6(n!)\, 3^n}$$

The general term of the series is, $u_n = \frac{(n+3)!}{6(n!)\, 3^n}$

Here,

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{(n+1+3)!}{6(n+1)!\, 3^{n+1}} \times \frac{6(n!)\, 3^n}{(n+3)!}$$
$$= \lim_{n\to\infty} \frac{(n+4)!}{(n+1)\, n!\, 3^n . 3} \times \frac{n!\, 3^n}{(n+3)!}$$
$$= \lim_{n\to\infty} \frac{(n+4)(n+3)!}{3(n+1)(n+3)!}$$
$$= \lim_{n\to\infty} \frac{(n+4)}{3(n+1)}$$
$$= \lim_{n\to\infty} \frac{n(1+4/n)}{3n(1+/1n)} = \lim_{n\to\infty} \frac{1+4/n}{3(1+1/n)} = \frac{1+0}{3(1+0)} = \frac{1}{3}.$$

Thus, $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \frac{1}{3} < 1$. Therefore the series is convergent, by D'Alembert ratio test.

21. $\sum_{n=1}^{\infty} \frac{1}{(2n+1)!}$

Solution: Given series is, $\sum_{n=1}^{\infty} \frac{1}{(2n+1)!}$

The general term of the series is, $u_n = \frac{1}{(2n+1)!}$

Here,

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{1}{(2(n+1)+1)!} \times \frac{(2n+1)!}{1}$$
$$= \lim_{n\to\infty} \frac{(2n+1)!}{(2n+3)!} = \lim_{n\to\infty} \frac{(2n+1)!}{(2n+3)(2n+2)(2n+1)!}$$
$$= \lim_{n\to\infty} \frac{1}{(2n+3)(2n+2)} = \frac{1}{\infty} = 0.$$

Thus, $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = 0 < 1$. Therefore the given series is convergent by D'Alembert ratio test.

EXERCISE 2.4

Test the convergent and divergent of the following series.

1. $\sum_{n=1}^{\infty} \frac{1}{(\log n)^n}$

Solution: Given series is, $\sum_{n=1}^{\infty} \frac{1}{(\log n)^n}$

The general term of the series is. $u_n = \frac{1}{(\log n)^n}$

Here,

$$\lim_{n\to\infty} u_n^{1/n} = \lim_{n\to\infty} \left(\frac{1}{(\log n)^n}\right)^{1/n} = \lim_{n\to\infty} \frac{1}{\log n} = \frac{1}{\log(\infty)} = \frac{1}{\infty} = 0$$

Thus, $\lim_{n\to\infty} u_n^{1/n} = 0 < 1$. Therefore, the given series is convergent by Cauchy's radical test.

2. $\Sigma\left(\frac{n+1}{3n}\right)^n$

Solution: Given series is, $\Sigma\left(\frac{n+1}{3n}\right)^n$

The general term of the series is, $u_n = \left(\frac{n+1}{3n}\right)^n$

Here,

$$\lim_{n\to\infty} u_n^{1/n} = \lim_{n\to\infty} \left(\left(\frac{n+1}{3n}\right)^n\right)^{1/n}$$
$$= \lim_{n\to\infty} \frac{n+1}{3n} = \lim_{n\to\infty} \frac{n(1+1/n)}{3n} = \lim_{n\to\infty} \frac{1+1/n}{3} = \frac{1+0}{3} = \frac{1}{3}.$$

Thus, $\lim_{n\to\infty} u_n^{1/n} = \frac{1}{3} < 1$. Therefore, the given series is convergent, by Cauchy's radical test.

3. $\Sigma\left(\frac{n-\log n}{2n}\right)^n$

Solution: Given series is, $\Sigma\left(\frac{n-\log n}{2n}\right)^n$

The general term of the series is, $u_n = \left(\frac{n-\log n}{2n}\right)^n$

Then, $u_n^{1/n} = \frac{n-\log n}{2n} = \frac{1}{2}\left(1 - \frac{\log n}{n}\right)$

Here,

$$\lim_{n\to\infty} u_n^{1/n} = \lim_{n\to\infty} \frac{1}{2}\left(1-\frac{\log n}{n}\right) = \frac{1}{2}(1-0) = \frac{1}{2} < 1.$$

Therefore, by the Cauchy's radical test, the given series is convergent.

4. $\Sigma\left(1+\frac{1}{\sqrt{n}}\right)^{-n^{3/2}}$

Solution: Given series is, $\Sigma\left(1+\frac{1}{\sqrt{n}}\right)^{-n^{3/2}}$

The general term of the series is, $u_n = \left(1+\frac{1}{\sqrt{n}}\right)^{-n^{3/2}}$

Then,

$$u_n^{1/n} = \left[\left(1+\frac{1}{\sqrt{n}}\right)^{-n^{3/2}}\right]^{1/n} = \left(1+\frac{1}{\sqrt{n}}\right)^{-\sqrt{n}}$$

Here,

$$\lim_{n\to\infty} u_n^{1/n} = \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{\sqrt{n}}\right)^{\sqrt{n}}} = \frac{1}{e} \qquad \left[\because \lim_{n\to\infty}\left(1+\frac{1}{x}\right)^x = e\right]$$

$$= \frac{1}{2.71} < 1.$$

Therefore, the given series is convergent by Cauchy's root test.

5. $\Sigma\left(\frac{n}{n+1}\right)^{n^2}$

Solution: Given series is, $\Sigma\left(\frac{n}{n+1}\right)^{n^2}$

The general term of the series is, $u_n = \left(\frac{n}{n+1}\right)^{n^2}$

Then,

$$u_n^{1/n} = \left[\left(\frac{n}{n+1}\right)^{n^2}\right]^{1/n} = \left(\frac{n}{n+1}\right)^n = \left[\frac{n}{n(1+1/n)}\right]^n = \frac{1}{(1+1/n)^n}$$

Here,

$$\lim_{n\to\infty} u_n^{1/n} = \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)^n} = \frac{1}{e} \qquad \left[\because \lim_{n\to\infty}\left(1+\frac{1}{x}\right)^x = e\right]$$

Thus, $\lim_{n\to\infty} u_n^{1/n} = \frac{1}{e} = \frac{1}{2.71} < 1$. Therefore, the given series is convergent, by Cauchy's root test.

6. $\left(\frac{2^2}{1^2} - \frac{2}{1}\right)^{-1} + \left(\frac{3^3}{2^3} - \frac{3}{2}\right)^{-2} + \left(\frac{4^4}{3^4} - \frac{4}{3}\right)^{-3} + \dots\dots$

Solution: Given series is,

$$\left(\frac{2^2}{1^2} - \frac{2}{1}\right)^{-1} + \left(\frac{3^3}{2^3} - \frac{3}{2}\right)^{-2} + \left(\frac{4^4}{3^4} - \frac{4}{3}\right)^{-3} + \dots\dots$$

The general term of the series is

$$u_n = \left[\left(\frac{n+1}{n}\right)^{n+1} - \frac{n+1}{n}\right]^{-n}$$

Then,

$$u_n^{1/n} = \left[\left(\frac{n+1}{n}\right)^{n+1} - \frac{n+1}{n}\right]^{-1} = \left[\left(1+\frac{1}{n}\right)^{n+1} - \left(1+\frac{1}{n}\right)\right]^{-1}$$

$$= \left(1+\frac{1}{n}\right)\left[\left(1+\frac{1}{n}\right)^n - 1\right]^{-1}$$

Here,

$$\lim_{n\to\infty} u_n^{1/n} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)\left[\frac{1}{\left(1+\frac{1}{n}\right)^n - 1}\right]$$

$$= \lim_{n\to\infty}\left(1+\frac{1}{n}\right)\left(\frac{1}{e-1}\right) \qquad \left[\because \lim_{n\to\infty}\left(1+\frac{1}{x}\right)^x = 1\right]$$

$$= (1+0)\frac{1}{e-1} = \frac{1}{e-1} = \frac{1}{2.71-1} = \frac{1}{1.71} < 1.$$

Therefore, the given series is convergent, by Cauchy's radical test.

7. $\Sigma\frac{1}{10^n}$

Solution: Given series is, $\Sigma\frac{1}{10^n}$

The general term of the series is, $u_n = \frac{1}{10^n}$

Here,

$$\lim_{n\to\infty} u_n^{1/n} = \lim_{n\to\infty} u_n^{1/n}\left(\frac{1_n}{10}\right)^{1/n} = \lim_{n\to\infty}\frac{1}{10} = \frac{1}{10} < 1.$$

Therefore, the given series is convergent, by Cauchy's radical test.

8. $\Sigma\left(1+\frac{1}{n}\right)^n$

Solution: Given series is, $\Sigma\left(1+\frac{1}{n}\right)^n$

The general term of the series is, $u_n = \left(1+\frac{1}{n}\right)^n$

Here,

$$\lim_{n\to\infty} u_n^{1/n} = \lim_{n\to\infty}\left[\left(1+\frac{1}{n}\right)^n\right]^{1/n} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right) = 1+0 = 1.$$

So, the test fail.

Here,

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = \lim_{n\to\infty}(e) = e \neq 0.$$

Therefore, the given series is divergent.

9. $\Sigma\left(-\frac{n^2}{2^n}\right)$

Solution: Given series is, $\Sigma\left(-\frac{n^2}{2^n}\right)$

The general term of the series is, $u_n = \left(-\frac{n^2}{2^n}\right)$

Here,

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n} = \lim_{n\to\infty}\left(\frac{-(n+1)^2}{2^{n+1}} \times \frac{2^n}{-n^2}\right)$$

$$= \lim_{n\to\infty}\left(\frac{(1+1/n)^2}{2}\right) = \left(\frac{(1+0)^2}{2}\right) = \frac{1}{2} < 1.$$

Therefore, the given series is convergent, by D'Alembert ratio test.

10. $\Sigma\left(\frac{2^{n+1}}{5^{n+1}}\right)^{1/n}$

Solution: Given series is, $\Sigma\left(\frac{2^{n+1}}{5^{n+1}}\right)^{1/n}$

The general term of the series is, $u_n = \left(\frac{2^{n+1}}{5^{n+1}}\right)^{1/n}$

Here,

$$\lim_{n\to\infty} u_n^{1/n} = \lim_{n\to\infty}\left(\frac{2^{n+1}}{5^{n+1}}\right)^{1/n} = \lim_{n\to\infty}\left(\frac{2}{5}\right)^{1+1/n} = \frac{2}{5} < 1$$

Therefore, the given series is convergent, by radical test.

11. $\sum\limits_{n=1}^{\infty} e^{-2n}$

Solution: Given series is, $\sum\limits_{n=1}^{\infty} e^{-2n}$

The general term of the series is, $u_n = e^{-2n}$

Here,

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n} = \lim_{n\to\infty}\frac{e^{-2(n+1)}}{e^{-2n}} = \lim_{n\to\infty}\frac{e^{-2n}e^{-2}}{e^{-2n}} \lim_{n\to\infty}(e^{-2}) = e^{-2} = \frac{1}{e^2} < 1.$$

Therefore, the given series is convergent by D'Alembert ratio test.

12. $\sum\limits_{n=1}^{\infty}\frac{2^n}{3^n}$

Solution: Similar to Q. 10

13. $\sum\limits_{n=1}^{\infty}\left(-\frac{1}{8^n}\right)$

Solution: Similar to Q. 7

14. $\sum\limits_{n=1}^{\infty}\left(\frac{n}{3n+1}\right)^n$

Solution: Given series is, $\sum\limits_{n=1}^{\infty}\left(\frac{n}{3n+1}\right)^n$

The general term of the series is, $u_n = \left(\frac{n}{3n+1}\right)^n$

Then,

$$u_n^{1/n} = \left[\left(\frac{n}{3n+1}\right)^n\right]^{1/n} = \frac{n}{3n+1} = \frac{1}{3+1/n}$$

Here,

$$\lim_{n\to\infty} u_n^{1/n} = \lim_{n\to\infty}\left(\frac{1}{3+1/n}\right) = \frac{1}{3+0} = \frac{1}{3} < 1.$$

Therefore, the given series is convergent, by Cauchy's radical test.

EXERCISE 2.5

Test the convergent and divergent of the following series.

1. $\sum\limits_{n=1}^{\infty}\left(\frac{1}{2n+3}\right)$

Solution: Given series is, $\sum\limits_{n=1}^{\infty}\left(\frac{1}{2n+3}\right)$

The general term of the given series is, $u_n = \frac{1}{2n+3}$

So, $f(x) = \frac{1}{2x+3}$. This is valid for $x \geq 1$, $p > 0$.

Here, f(x) is positive and monotonic decreasing. So, the Cauchy integral test is applicable.

For $p > 0$,

$$\int_1^{\infty} f(x)\,dx = \int_1^{\infty}\frac{dx}{2x+3} = [\log(2x+3)]_1^{\infty} = \log(\infty) - \log(2+3) = \infty$$

This shows that $\int_1^{\infty} f(x)\,dx$ diverges.

So, by integral test the given series Σu_n diverges for $x \geq 1$.

$$\int_1^{\infty} f(x)\,dx = \int_1^{\infty}\left(\frac{1}{x} - \frac{1}{x+1}\right)dx$$

$$= [\log x - \log(x+1)]_1^{\infty}$$

$$= [\log(\infty) - \log(\infty+1)] - [\log(1) - \log(2)] = \infty.$$

This shows that the given series diverges.

2. $\sum\limits_{n=1}^{\infty}\frac{1}{n(n+1)}$

Solution: Given series is, $\sum\limits_{n=1}^{\infty}\frac{1}{n(n+1)}$

So, $f(x) = \frac{1}{x(x+1)} = \frac{1}{x} - \frac{1}{x+1}$

Now,

$$\int_1^{\infty} f(x)\,dx = \lim_{n\to\infty}\int_1^{n}\left(\frac{1}{x} - \frac{1}{x+1}\right)dx$$

$$= \lim_{n\to\infty}[\log(x) - \log(x+1)]_1^n$$

$$= \lim_{n\to\infty}\left[\log\left(\frac{x}{x+1}\right)\right]_1^n$$

$$= \lim_{n\to\infty}\left[\log\left(\frac{n}{n+1}\right) - \log\left(\frac{1}{2}\right)\right]$$

$$= \lim_{n\to\infty}\log\left(\frac{1}{1+\frac{1}{n}}\right) - \log\left(\frac{1}{2}\right)$$

$$= \log\left(\frac{1}{1+0}\right) - \log\left(\frac{1}{2}\right)$$

$$= \log(1) + \log\left(\left(\frac{1}{2}\right)^{-1}\right) \quad = \log(2) \qquad [\because \log(1) = 0]$$

Since log(2) is a finite value, so the integral $\int_1^{\infty} f(x)\,dx$ converges. This implies the given series is convergent.

3. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$

Solution: Given series is, $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$

The general term of the given series is, $u_n = \frac{1}{\sqrt{n}}$

So, $f(x) = \frac{1}{\sqrt{x}} = x^{-1/2}$. This is valid for $x \geq 1, p > 0$.

Here, for $x \geq 1, p > 0$,

$$\int_1^{\infty} f(x)\,dx = \int_1^{\infty} x^{-1/x}\,dx = \left[\frac{x^{1/2}}{1/2}\right]_1^{\infty} = 2[\sqrt{x}]_1^{\infty} = 2(\sqrt{\infty} - \sqrt{1}) = \infty.$$

This shows that the integral diverges. So, by Cauchy integral test, the given series Σu_n is divergent.

Note: The problem also can be tested by p–test.

4. $\sum_{n=1}^{\infty} \left(\frac{1}{n^2+1}\right)$

Solution: Given series is, $\sum_{n=1}^{\infty} \left(\frac{1}{n^2+1}\right)$

The general term of the given series is, $u_n = \frac{1}{n^2+1}$

So, $f(x) = \frac{1}{x^2+1}$ for $x \geq 1, p > 0$

Here,

$$\int_1^{\infty} f(x)\,dx = \int_1^{\infty} \frac{dx}{x^2+1} = [\tan^{-1} x]_1^{\infty}$$

$$= \tan^{-1}(\infty) - \tan^{-1}(1) = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4} \text{ (a finite value)}$$

This shows that the integral converges. So, by Cauchy integral test, the given series Σu_n is convergent.

5. $\sum_{n=1}^{\infty} \frac{1}{(3+2n)^2}$

Solution: Given series is, $\sum_{n=1}^{\infty} \frac{1}{(3+2n)^2}$

The general term of given series is, $u_n = \frac{1}{(3+2n)^2}$

So, $f(x) = \frac{1}{(3+2n)^2} = (3+2x)^{-2}$ for $x \geq 1, p > 0$

Here,

$$\int_1^{\infty} f(x)\,dx = \int_1^{\infty} (3+2x)^{-2}\,dx = \frac{1}{4}\int_1^{\infty} (x+3/2)^{-2}\,dx$$

$$\Rightarrow \int_1^{\infty} f(x)\,dx = \frac{1}{4}\left[\frac{(x+3/2)^{-1}}{-1}\right]_1^{\infty} = -\frac{1}{4}\left[\frac{1}{x+3/2}\right]_1^{\infty} - \frac{1}{4}\left(\frac{1}{\infty} - \frac{1}{1+(3/2)}\right)$$

$$\Rightarrow \int_1^{\infty} f(x)\,dx = -\frac{1}{4}\left(0 - \frac{2}{5}\right) = \frac{1}{10} \text{ (a finite value)}$$

So, the integral converges. Then by Cauchy integral test, the given series Σu_n converges.

6. $\sum_{n=1}^{\infty} \left(\frac{2n^3}{n^4+3}\right)$

Solution: Given series is, $\sum_{n=1}^{\infty} \left(\frac{2n^3}{n^4+3}\right)$

The general term of given series is, $u_n = \frac{2n^3}{n^4+3}$

So, $f(x) = \left(\frac{2x^3}{x^4+3}\right)$ for $x \geq 1, p > 0$

Here,

$$\int_1^{\infty} f(x)\,dx = \int_1^{\infty} \left(\frac{2x^3}{x^4+3}\right) dx$$

Put $x^4 = u$ then $4x^3\,dx = du$. Also, $x = 1 \Rightarrow u = 1, x \to \infty, \Rightarrow u \to \infty$. Then,

$$= \frac{2}{4}\int_1^{\infty} \frac{du}{u+3}$$

$$= \frac{1}{2}[\log(u+3)]_1^{\infty} = \frac{1}{2}[\log(\infty) - \log(4)] = \frac{1}{2}(\infty - \log(4)) = \infty$$

This shows that the integral diverges. Then by Cauchy integral test, the given series Σu_n diverges.

7. $\sum_{n=1}^{\infty} \frac{n}{(n^2+1)^2}$

Solution: Process as Q. No. 6

8. $\sum_{n=1}^{\infty}\left(\frac{1}{n\sqrt{n^2-1}}\right)$

Solution: Given series is, $\sum_{n=1}^{\infty}\left(\frac{1}{n\sqrt{n^2-1}}\right)$

The general term of given series is, $u_n = \frac{1}{n\sqrt{n^2-1}}$ for $n \geq 2$

So, $f(x) = \frac{1}{x\sqrt{x^2-1}}$ for $x \geq 2$, $p > 0$

Here,

$$\int_1^{\infty} f(x)\,dx = \frac{dx}{x\sqrt{x^2-1}}$$

$$= [\sec^{-1}(x)]_1^{\infty} = \sec^{-1}(\infty) - \sec^{-1}(2) = \frac{\pi}{2} - \frac{\pi}{3} = \frac{\pi}{6}$$

This shows that the given integral is convergent. Then by Cauchy integral test given series Σu_n is convergent.

9. $\sum_{n=1}^{\infty} ne^{-n^2}$

Solution: Given series is, $\sum_{n=1}^{\infty} ne^{-n^2}$

The general term of the series is, $u_n = ne^{-n^2}$ for $n \geq 1$

So, $f(x) = xe^{-x^2}$ for $x \geq 1$, $p > 0$

Here,

$$\int_1^{\infty} f(x)\,dx = \int_1^{\infty} xe^{-x^2}\,dx$$

Put $x^2 = u$ then $2xdx = du$. Also, $x = 1 \Rightarrow u = 1$, $x \to \infty$, $u \to \infty$. Then,

$$= \frac{1}{2}\int_1^{\infty} e^{-u}\,du = \frac{1}{2}\left[\frac{e^{-u}}{-1}\right]_1^{\infty} = -\frac{1}{2}(e^{-\infty} - e^{-1}) = \frac{1}{2e} \text{ (a finite value)}$$

So, the integral converges. The by Cauchy's integral test, the given series converges.

10. $\sum_{n=1}^{\infty} n^2 e^{-n^3}$

Solution: Process as Q. 9

11. $\sum_{n=3}^{\infty} \frac{\log(n)}{n}$

Solution: Given series is, $\sum_{n=3}^{\infty} \frac{\log(n)}{n}$

The general term of the series is, $u_n = \frac{\log(n)}{n}$ for $n \geq 3$

So, $f(x) = \frac{\log(x)}{x}$ for $x \geq 3$, $p > 0$

Here,

$$\int_3^{\infty} f(x)\,dx = \int_3^{\infty} \frac{\log(x)}{x}\,dx$$

Set $\log(x) = u$ then $dx/x = du$.

Also, $x = 3 \Rightarrow u = \log(3)$ and $x \to \infty \Rightarrow u \to \log(\infty) \sim \infty$. Therefore,

$$\int_3^{\infty} f(x)\,dx = \int_{\log(3)}^{\infty} u\,du = \left[\frac{u^2}{2}\right]_{\log(3)}^{\infty} = \frac{1}{2}[\infty - (\log 3)^2] = \infty$$

This shows that the integral diverges. So, by Cauchy integral test, the given series Σu_n diverges.

12. $\sum_{n=1}^{\infty} \frac{\arctan(n)}{1+n^2}$

Solution: Given series is, $\sum_{n=1}^{\infty} \frac{\arctan(n)}{1+n^2}$

The general term of the series is, $u_n = \frac{\arctan(n)}{1+n^2} = \frac{\tan^{-1}(n)}{1+n^2}$ for $n \geq 1$

So, $f(x) = \frac{\tan^{-1}(x)}{1+x^2}$ for $x \geq 1$, $p > 0$

Here,

$$\int_1^{\infty} f(x)\,dx = \int_1^{\infty} \frac{\tan^{-1}(x)}{1+x^2}\,dx$$

Put $\tan^{-1} x = u$ then $\frac{dx}{1+x^2} = du$.

Also, $x = 1 \to u = \tan^{-1}(1) = \frac{\pi}{4}$ and $x \to \infty \Rightarrow u = \tan^{-1}(\infty) = \frac{\pi}{2}$. Then,

$$= \int_{\pi/4}^{\pi/2} u\,du = \left[\frac{u^2}{2}\right]_{\pi/4}^{\pi/2} = \frac{1}{2}\left(\frac{\pi^2}{4} - \frac{\pi^2}{16}\right) = \frac{3\pi^2}{32} \text{ (a finite value).}$$

This shows that the integral converges. Then by Cauchy integral tests, the givens series Σun converges.

13. $\sum_{n=1}^{\infty} \frac{1}{n(\log n)^p}$ for $p > 0$

Solution: Given series is, $\sum_{n=1}^{\infty} \frac{1}{n(\log n)^p}$ for $p > 0$

The general term of the series is, $u_n = \frac{1}{n(\log n)^p}$ for $n \geq 2$, $p > 0$

So, $\quad f(x) = \dfrac{1}{x(\log x)^p} \quad$ for $x \geq 2, p > 0$

Here,

$$\int_2^{\infty} f(x)\,dx = \int_2^{\infty} \frac{dx}{x(\log x)^p}$$

Put $\log(x) = u$ then $\dfrac{dx}{x} = du$. Also, $x = 2 \Rightarrow u = \log 2$, $x = \infty \Rightarrow u = \log(\infty) = \infty$.

Then,

$$= \int_{\log(2)}^{\infty} \frac{du}{u^p} = \left[\frac{u^{-p+1}}{-p+1}\right]_{\log(2)}^{\infty} \qquad \ldots\ldots\ldots (*)$$

For $p > 1$, we get $(p - 1) > 0$. So (*) becomes,

$$\int_2^{\infty} f(x)\,dx = \frac{1}{1-p}\left[\frac{1}{u^{p-1}}\right]_{\log 2}^{\infty} = \frac{1}{1-p}\left(0 - \frac{1}{(\log 2)^{p-1}}\right) \text{ (a finite value)}$$

So, the integral converges for $p > 1$. Then by Cauchy integral test the given series converges for $p > 1$.

And for $p = 1$, then (*) becomes,

$$\int_2^{\infty} f(x)\,dx = \infty.$$

So, the integral diverges and the given series Σu_n diverges, by Cauchy integral test.

EXERCISE 2.6

Test the convergent of the following series.

1. $1 - \dfrac{1}{\sqrt{2}} + \dfrac{1}{\sqrt{3}} - \dfrac{1}{\sqrt{4}} + \ldots\ldots\ldots$ **[2003 Spring]**

Solution: Given series is

$$1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \ldots$$

Thus, the given series is an alternative series.

The general term of the series is, $u_n = |a_n| = \dfrac{1}{\sqrt{n}}$

Here,

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{\sqrt{n}} \sim \frac{1}{\infty} \sim 0.$$

And, $$\frac{u_{n+1}}{u_n} = \frac{\sqrt{n}}{\sqrt{n+1}} = \frac{1}{\sqrt{1+1/n}} < 1$$

$\Rightarrow \quad u_{n+1} < u_n \qquad$ for every n.

That is every positive term of the series is numerically less than the preceding term. Therefore given series Σa_n is convergent, by Leibnitz theorem.

Moreover, if $\Sigma|a_n|$ is convergent then it is absolute convergent otherwise the given series is conditionally convergent.

Here,

$$\Sigma u_n = \Sigma\left|\frac{1}{\sqrt{n}}\right| = \Sigma\left|\frac{1}{n^{1/2}}\right| = \Sigma\frac{1}{n^{1/2}}$$

This is a p–series with $p = \dfrac{1}{2} < 1$. So, the series $\Sigma|a_n|$ diverges by p–test.

This shows that the given series is conditionally convergent.

2. $1 - \dfrac{1}{2\sqrt{2}} + \dfrac{1}{3\sqrt{3}} - \dfrac{1}{4\sqrt{4}} + \ldots\ldots$ **[2010 Fall Short]**

Solution: Given series is,

$$1 - \frac{1}{2\sqrt{2}} + \frac{1}{3\sqrt{3}} - \frac{1}{4\sqrt{4}} + \ldots\ldots$$

This is an alternative series. The general term of the series is, $u_n = |a_n| = \dfrac{1}{n\sqrt{n}}$

Here,

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{n\sqrt{n}} \sim \frac{1}{\infty} \sim 0.$$

And, $$\frac{u_{n+1}}{u_n} = \frac{n\sqrt{n}}{(n+1)\sqrt{n+1}} = \frac{1}{(1+1/n)\sqrt{1+1/n}} < 1$$

$\Rightarrow \quad u_{n+1} < u_n \qquad$ for every n.

That is every positive term of the series is numerically less than the preceding term. This shows that the given series is convergent, by Leibnitz theorem.

Moreover, the series is absolutely convergent if $\Sigma|a_n|$ convergent otherwise it is conditionally convergent.

Here,

$$\Sigma|a_n| = \Sigma\left|\frac{1}{n\sqrt{n}}\right| = \Sigma\left|\frac{1}{n^{3/2}}\right| = \Sigma\frac{1}{n^{3/2}}$$

This is a p–series with $p = 3/2 > 1$. Therefore, the series $\Sigma|a_n|$ converges by p–series.

Thus, the given series is absolutely convergent.

3. $\dfrac{1}{a} - \dfrac{1}{a+b} + \dfrac{1}{a+2b} - \dfrac{1}{a+3b} + \ldots\ldots\ldots$

Solution: Given series is

$$\frac{1}{a} - \frac{1}{a+b} + \frac{1}{a+2b} - \frac{1}{a+3b} + \ldots\ldots\ldots$$

This is an alternative series whose general term is, $\quad u_n = |a_n| = \dfrac{1}{a+nb}$

Here,

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{a+nb} \sim \frac{1}{\infty} \sim 0.$$

And, $$\frac{u_{n+1}}{u_n} = \frac{a+nb}{a+(n+1)b} < 1$$

$\Rightarrow \quad u_{n+1} < u_n \qquad$ for every n.

That is every positive term of the series is numerically less than the preceding term. This shows that the given series is convergent, by Leibnitz theorem.

Moreover, the series is absolutely convergent if $\Sigma|a_n|$ converges otherwise it is conditionally convergent.

Here,

$$\Sigma u_n = \Sigma\left|\frac{1}{a+nb}\right| = \Sigma\left(\frac{1}{a+nb}\right) = \Sigma\left(\frac{1}{n(b+a/n)}\right)$$

Choose $v_n = \frac{1}{n}$. Then Σv_n is a p-series with p = 1. Therefore, Σv_n diverges by p–test.

So, by ratio comparison test series the series $\Sigma|a_n|$ is divergent if $\lim_{n\to\infty} \frac{|a_n|}{v_n}$ has a non-zero finite value.

Here,

$$\lim_{n\to\infty} \frac{|a_n|}{v_n} = \lim_{n\to\infty} \frac{1}{b + a/n} \sim \frac{1}{b+0} = \frac{1}{b}$$

which is a non-zero finite value.

This shows that $\Sigma|a_n|$ diverges, by ratio comparison test. Thus, the given series converges conditionally.

4. $\frac{1}{\log(2)} - \frac{1}{\log(3)} + \frac{1}{\log(4)} - \frac{1}{\log(5)} + \ldots\ldots\ldots$

Solution: Given series is

$$\frac{1}{\log 2} - \frac{1}{\log 3} + \frac{1}{\log 4} - \frac{1}{\log 5} + \ldots\ldots$$

This is an alternative series whose general term is,

$$u_n = |a_n| = \frac{1}{\log(n+1)}$$

Here,

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{\log(n+1)} \sim \frac{1}{\log(\infty)} \sim \frac{1}{\infty} \sim 0.$$

And, $\frac{u_{n+1}}{u_n} = \frac{\log(n+1)}{\log(n+2)} < 1$

$\Rightarrow\ u_{n+1} < u_n$ for every n.

That is every positive term of the series is numerically less than the preceding term.

This shows that the given series is convergent, by Leibnitz theorem.

5. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n!}$

Solution: Given series is, $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n!}$

This is an alternative series whose general term is, $u_n = |a_n| = \frac{1}{n!}$

Here,

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{n!} \sim \frac{1}{\infty!} \sim 0.$$

And, $\frac{u_{n+1}}{u_n} = \frac{n!}{(n+1)!} = \frac{1}{(n+1)} < 1.$

$\Rightarrow\ u_{n+1} < u_n$ for every n.

That is every positive term of the series is numerically less than the preceding term.

This shows that the given series is convergent by Leibnitz theorem.

Moreover, the series is absolutely convergent if $\Sigma|a_n|$ convergent and otherwise it is conditionally convergent.

Here,

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{n!}{(n+1)!} = \lim_{n\to\infty} \frac{1}{(n+1)} = \lim_{n\to\infty} \frac{1}{n(1+1/n)} \sim \frac{1}{\infty} \sim 0 < 1.$$

This shows that the series Σu_n converges by D'Alembert ratio. That is, the series $\Sigma|a_n|$ converges.

This shows that the given series is absolutely convergent.

6. $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{3^{n-2}}$

Solution: Given series is, $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{3^{n-2}}$

This is an alternative series, whose general positive term is,

$$u_n = |a_n| = \frac{1}{3^{n-2}}$$

Here,

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{3^{n-2}}$$

$$= \lim_{n\to\infty} \frac{3^2}{3^n} = 9 \lim_{n\to\infty} \frac{1}{3^n} \sim 9.\frac{1}{3^\infty} \sim 9.\frac{1}{\infty} \sim 9.0 = 0.$$

And, $\frac{u_{n+1}}{u_n} = \frac{3^{n-2}}{3^{n-1}} = \frac{1}{3} < 1.$

$\Rightarrow\ u_{n+1} < u_n$ for every n.

That is every positive term of the series is numerically less than the preceding term.

This shows that the given series is convergent, by Leibnitz theorem.

Moreover, the series is absolutely convergent if $\Sigma|a_n|$ converges and conditionally convergent for otherwise.

Here,

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{3^{n-2}}{3^{n-1}} = \lim_{n\to\infty} \frac{1}{3} = \frac{1}{3} < 1$$

Therefore, the series Σu_n converges by D'Alembert ratio test. So, the series $\Sigma|a_n|$ converges.

Thus, the given series is absolutely convergent.

7. $\sum_{n=1}^{\infty} \frac{n(-1)^{n-1}}{5^n}$

Solution: Given series is, $\sum_{n=1}^{\infty} \frac{n(-1)^{n-1}}{5^n}$

This is an alternative series, whose general positive term is,

$$u_n = |a_n| = \frac{n}{5^n}$$

Here,

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{n}{5^n} \quad \left[\text{in } \frac{\infty}{\infty} \text{ form}\right]$$

$$= \lim_{n\to\infty} \frac{1}{n\,5^{n-1}}$$

$$= 0.$$

And, $\frac{u_{n+1}}{u_n} = \frac{n+1}{5^{n+1}} \times \frac{5^n}{n} = \frac{n+1}{5n} < 1.$

$\Rightarrow \quad u_{n+1} < u_n \quad$ for every n.

That is every positive term of the series is numerically less than the preceding term.

This shows that the given series is convergent, by Leibnitz theorem.

Here,

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \left(\frac{n+1}{5^{n+1}} \times \frac{5^n}{n}\right) = \lim_{n\to\infty} \frac{1+1/n}{5} = \frac{1+0}{5} = \frac{1}{5} < 1$$

Therefore, the series Σu_n converges by D'Alembert ratio test. So, the series $\Sigma|a_n|$ converges.

Thus, the given series is absolutely convergent.

8. $\frac{1}{1+\sqrt{2}} - \frac{1}{1+\sqrt{3}} + \frac{1}{1+\sqrt{4}} - \frac{1}{1+\sqrt{5}} + \ldots\ldots$

Solution: Given series is

$$\frac{1}{1+\sqrt{2}} - \frac{1}{1+\sqrt{3}} + \frac{1}{1+\sqrt{4}} - \frac{1}{1+\sqrt{5}} + \ldots\ldots$$

This is an alternative series, whose general positive term is,

$$u_n = |a_n| = \frac{1}{1+\sqrt{n+1}}$$

Here,

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{1+\sqrt{n+1}} \sim \frac{1}{\infty} \sim 0$$

And, $$\frac{u_{n+1}}{u_n} = \frac{1}{1+\sqrt{n+2}} \times \frac{1+\sqrt{n+1}}{1} = \frac{1+\sqrt{n+1}}{1+\sqrt{n+2}} < 1.$$

$\Rightarrow \quad u_{n+1} < u_n \quad$ for every n.

That is every positive term of the series is numerically less than the preceding term.

This shows that the given series is convergent by Leibnitz theorem.

Moreover, the series Σa_n is absolutely convergent if $\Sigma|a_n|$ converges and is conditionally convergent for otherwise.

Here,

$$u_n = \frac{1}{1+\sqrt{n+1}} = \frac{1}{\sqrt{n}\left((1/\sqrt{n}) + \sqrt{1+(1/\sqrt{n})}\right)}$$

Set $v_n = \frac{1}{\sqrt{n}}$. Then the series Σv_n diverges by p-test.

Here,

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{1}{(1/\sqrt{n}) + \sqrt{1+(1/\sqrt{n})}}$$

$$= \frac{1}{0+\sqrt{1+0}} \text{ ; a finite non-zero value.}$$

Therefore by the limit comparison test the series Σu_n is divergent. That is the series $\Sigma|a_n|$ is divergent.

Thus,

Therefore, the given series converges conditionally.

9. $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n}{n^2+1}$

Solution: Given series is, $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n}{n^2+1}$

This is an alternative series, whose general positive term is,

$$u_n = |a_n| = \frac{n}{n^2+1}$$

Here,

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \left(\frac{n}{n^2+1}\right)$$

$$= \lim_{n\to\infty} \frac{n}{n^2(1+1/n^2)} = \lim_{n\to\infty} \frac{1}{n(1+1/n^2)} = \frac{1}{\infty(1+0)} = 0.$$

And, $$\frac{u_{n+1}}{u_n} = \frac{n+1}{(n+1)^2+1} \times \frac{n^2+1}{n} = \frac{n^3+n^2+n+1}{n^3+2n^2+n} = 1 - \frac{n^2-1}{n^3+2n^2+n} < 1.$$

$\Rightarrow \quad u_{n+1} < u_n \quad$ for every n.

That is every positive term of the series is numerically less than the preceding term.

This shows that the given series is convergent, by Leibnitz theorem.

Moreover, the series is convergent absolutely if $\Sigma|a_n|$ converges and is convergent conditionally for otherwise.

Here,

$$\Sigma u_n = \Sigma\left(\frac{n}{n^2+1}\right) = \Sigma \frac{n}{n^2(1+1/n^2)} = \Sigma \frac{1}{n(1+1/n^2)}$$

Set, $v_n = \frac{1}{n}$. Then the series Σv_n is p–series with p = 1. So, by p–test, the series Σv_n diverges by p–test.

Here,

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{1}{(1+1/n^2)} = \frac{1}{1+0} = 1 \text{ ; which is a finite non-zero value.}$$

Since the series Σv_n diverges and therefore the series Σu_n diverges by limit comparison test. That is, $\Sigma|a_n|$ diverges. So, the given series is conditionally convergent.

10. $\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n)!}$

Solution: Given series is, $\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n)!}$

This is an alternative series, whose general positive term is,

$$u_n = |a_n| = \frac{1}{(2n)!}$$

Here,

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{(2n)!} \sim \frac{1}{\infty!} \sim 0$$

And, $$\frac{u_{n+1}}{u_n} = \frac{(2n)!}{[2(n+1)]!}$$

$$= \frac{(2n)!}{(2n+2)!} = \frac{(2n)!}{(2n+2)(2n+1)(2n)!} = \frac{1}{(2n+2)(2n+1)} < 1.$$

$\Rightarrow \quad u_{n+1} < u_n \quad$ for every n.

That is every positive term of the series is numerically less than the preceding term.

This shows that the given series is convergent, by Leibnitz theorem.

Moreover, the series converges absolutely if $\Sigma|u_n|$ converges and converges conditionally for otherwise.

Here,

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{1}{(2n+2)(2n+1)} \sim \frac{1}{\infty} \sim 0 < 1.$$

Thos shows that Σu_n converges, by D'Alembert ratio test. That is, $\Sigma|a_n|$ converges. So, the given series is absolutely convergent.

12. $\sum_{n=1}^{\infty} (-1)^{n+1}\left(\frac{1}{n^2}\right)$

Solution: Given series is. $\sum_{n=1}^{\infty} (-1)^{n+1}\left(\frac{1}{n^2}\right)$

This is an alternative series, whose general positive term is,

$$u_n = |a_n| = \frac{1}{n^2}$$

Here,

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{n^2} \sim \frac{1}{\infty} \sim 0.$$

And,

$$\frac{u_{n+1}}{u_n} = \frac{n^2}{(n+1)^2} = \frac{n^2}{n^2+2n+1} = 1 - \frac{2n+1}{n^2+2n+1} < 1.$$

$$\Rightarrow \quad u_{n+1} < u_n \qquad \text{for every n.}$$

That is every positive term of the series is numerically less than the preceding term

This shows that the given series converges, by Leibnitz theorem.

Moreover, the series converges absolutely if $\Sigma|u_n|$ converges and converges conditionally for otherwise.

Here,

$$\Sigma|a_n| = \Sigma\left(\frac{1}{n^2}\right)$$

Clearly, the series $\Sigma\left(\frac{1}{n^2}\right)$ is a p–series with $p = 2 > 1$. So, the series converges by p test. That is, $\Sigma|a_n|$ converse. So, the given series is absolutely convergent.

13. $\sum_{n=1}^{\infty} (-1)^{n+1}\left(\frac{\sqrt{n}+1}{n+1}\right)$

14. **Solution:** Same to Q. 2.

15. $\sum_{n=1}^{\infty} (-1)^{n+1}\left(\frac{n}{n^3+1}\right)$ **[2014 Fall Q. No. 3(a)]**

Solution: Given series is, $\sum_{n=1}^{\infty} (-1)^{n+1}\left(\frac{n}{n^3+1}\right)$

This series is an alternative series whose positive general term is. $u_n = \frac{n}{n^3+1}$

Solution is similar to Q. 9.

16. $\sum_{n=1}^{\infty} (-1)^{n}\left(\frac{1}{n+3}\right)$

Solution: Given series is, $\sum_{n=1}^{\infty} (-1)^{n}\left(\frac{1}{n+3}\right)$

This is an alternative series whose general positive term is, $u_n = \frac{1}{n+3}$

Solution is similar to Q. 3.

17. $\sum_{n=1}^{\infty} (-1)^{n+1}\left(\frac{3+n}{5+n}\right)$

Solution: Given series is, $\sum_{n=1}^{\infty} (-1)^{n+1}\left(\frac{3+n}{5+n}\right)$

This is an alternative series, whose general positive term is,

$$u_n = |a_n| = \frac{3+n}{5+n}$$

Here,

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty}\left(\frac{3+n}{5+n}\right) = \lim_{n\to\infty} \frac{n(1+3/n)}{n(1+5/n)} = \frac{1+0}{1+0} = 1 \neq 0.$$

Hence the given series is not convergent.

18. **Solution:** Solution is similar to Q. 9.

19. $\sum_{n=1}^{\infty} (-1)^{n+1}(\sqrt{n+1}-\sqrt{n})$

Solution: Given series is

$$\sum_{n=1}^{\infty} (-1)^{n+1}(\sqrt{n+1}-\sqrt{n})$$

This is an alternative series, whose general positive term is,

$$u_n = |a_n| = \sqrt{n+1}-\sqrt{n})$$

This is in $\infty - \infty$ form as $n \to \infty$. So, multiply nominator and denominator by its conjugate.

Therefore,

$$u_n = \sqrt{n+1}-\sqrt{n} \times \frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+1}+\sqrt{n}} = \frac{(n+1)-n}{\sqrt{n+1}+\sqrt{n}} = \frac{1}{\sqrt{n+1}+\sqrt{n}}$$

Here,

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{\sqrt{n+1}+\sqrt{n}} \sim \frac{1}{\infty} \sim 0.$$

And,

$$\frac{u_{n+1}}{u_n} = \left[\frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+2}+\sqrt{n+1}}\right].$$

Since $\sqrt{n+1} < \sqrt{n+2}$ and $\sqrt{n} < \sqrt{n+1}$. Therefore,

$$(\sqrt{n+1}+\sqrt{n}) < (\sqrt{n+2}+\sqrt{n+1})$$

$$\Rightarrow \left[\frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+2}+\sqrt{n+1}}\right] < 1.$$

$$\Rightarrow \frac{u_{n+1}}{u_n} < 1 \quad \text{for every n.}$$

$$\Rightarrow u_{n+1} < u_n \quad \text{for every n.}$$

That is every positive term of the series is numerically less than the preceding term. This shows that the given series is convergent, by Leibnitz theorem.

Moreover, the series is absolutely convergent if $\Sigma|a_n|$ converges and is conditionally convergent for otherwise.

Here,

$$\Sigma u_n = \Sigma\left(\frac{1}{\sqrt{n+1}+\sqrt{n}}\right) = \Sigma\left[\frac{1}{\sqrt{n}\,(\sqrt{1+(1/n)}+1)}\right]$$

Set $v_n = \frac{1}{\sqrt{n}} = \frac{1}{n^{1/2}}$ then Σv_n is a p-series with $p = \frac{1}{2} < 1$. So, Σv_n is divergent.

Here,

$$\lim_{n\to\infty}\frac{u_n}{v_n} = \lim_{n\to\infty}\left(\frac{1}{\sqrt{1+(1/n)}+1}\right) = \frac{1}{\sqrt{1+0}+1} = \frac{1}{1+1} = \frac{1}{2}$$

which is a non-zero finite value.

This shows that Σu_n is divergent. That is, $\Sigma|a_n|$ is divergent. Therefore, the given series is conditionally convergent.

EXERCISE 2.7

A. Discuss the convergence of the following series.

1. $1 + 2^2x + 3^2x^2 + 4^2x^3 + \ldots\ldots\ldots$

Solution: Given series is

$$1 + 2^2x + 3^2x^2 + 4^2x^3 + \ldots\ldots\ldots$$

This is a power series whose general term is, $u_n = n^2x^{n-1}$.

Here,

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n} = \lim_{n\to\infty}\frac{(n+1)^2x^n}{n^2x^{n-1}} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^2 x = (1+0)\,x = x.$$

By D'Alembert ratio test, the given series is convergent for $|x| < 1$ and is divergent for $|x| > 1$ and further test will tell at $|x| = 1$.

At, $x = 1$, $\quad u_n = n^2$

Here,

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} n^2 = \infty \neq 0.$$

This shows that the series diverges at $x = 1$.

At, $x = -1$, $\quad u_n = (-1)^{n-1}n^2$

This is an alternative series, whose general positive term is

$$w_n = n^2$$

Here,

$$\lim_{n\to\infty} w_n = \lim_{n\to\infty} n^2 = \infty \neq 0.$$

Therefore the given series will diverge at $x = -1$ by Leibnitz test.

This shows that the series diverges at $x = 1$.

Thus, the given series is convergent for $|x| < 1$ and is divergent for $|x| \geq 1$.

2. $1 + \frac{x}{2} + \frac{x^2}{5} + \frac{x^3}{10} + \ldots\ldots + \frac{x^n}{n^2+1} + \ldots\ldots$

Solution: Given series is

$$1 + \frac{x}{2} + \frac{x^2}{5} + \frac{x^3}{10} + \ldots\ldots + \frac{x^n}{n^2+1} + \ldots\ldots$$

This is power series whose general term is, $u_n = \frac{x^{n-1}}{(n-1)^2+1}$

Here,

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n} = \lim_{n\to\infty}\frac{x^n}{n^2+1}\times\frac{(n-1)^2+1}{x^{n-1}}$$

$$= \lim_{n\to\infty}\frac{n^2[(1-1/n)^2+1/n^2]}{n^2(1+1/n^2)}\,x$$

$$= \lim_{n\to\infty}\left[\frac{(1-1/n)^2+1/n^2}{1+1/n^2}\right]x = \left(\frac{(1-0)^2+0}{1+0}\right)x = x.$$

By D'Alembert ratio test, the given series is convergent for $|x| < 1$, is divergent for $|x| > 1$ and further test will tell at $|x| = 1$.

At $x = 1$, $\quad u_n = \frac{1}{(n-1)^2+1} = \frac{1}{n^2\left[(1-(1/n))^2+(1/n^2)\right]}$

Set, $v_n = \frac{1}{n^2}$ then the series Σv_n converges by p-test.

Here,

$$\lim_{n\to\infty}\frac{u_n}{v_n} = \lim_{n\to\infty}\frac{1}{[(1-(1/n))^2+(1/n^2)]} = \frac{1}{[(1-0)^2+0]} = 1.$$

which is a non-zero finite value.

Since the series Σv_n converges and therefore the given series is convergent at $x = 1$ by limit comparison test.

Also, at $x = -1$, $\quad u_n = \frac{(-1)^{n-1}}{(n-1)^2+1}$

This is an alternative series whose general positive term is

$$w_n = \frac{1}{(n-1)^2+1}.$$

Here,

$$\lim_{n\to\infty} w_n = \lim_{n\to\infty}\frac{1}{(n-1)^2+1} = \frac{1}{\infty} = 0.$$

And,

$$\frac{w_{n+1}}{w_n} = \frac{(n-1)^2+1}{n^2+1} = \frac{n^2-2n+2}{n^2+1} = 1 - \frac{2n-1}{n^2+1} < 1.$$

$$\Rightarrow w_{n+1} < w_n \quad \text{for every n.}$$

That is every positive term of the series is numerically less than the preceding term. This shows that the given series is convergent, by Leibnitz theorem.

This shows that the given series is convergent at $x = -1$.

Thus, the given series is convergent for $|x| \leq 1$ and is divergent for $|x| > 1$.

3. $1 + \frac{2x}{5} + \frac{6x^2}{9} + \frac{14x^3}{17} + \ldots\ldots + \frac{(2^n-2)x^{n-1}}{(2^n+1)} + \ldots$

Solution: Given series is,

$$1+\frac{2x}{5}+\frac{6x^2}{9}+\frac{14x^3}{17}+\ldots\ldots+\frac{(2^n-2)x^{n-1}}{(2^n+1)}$$

This is a power series whose general term is,

$$u_n=\left(\frac{2^n-2}{2^n+1}\right)x^{n-1}$$

Here,

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n}=\lim_{n\to\infty}\left(\frac{2^{n+1}-2}{2^{n+1}+1}\right)x^n\times\left(\frac{2^n+1}{2^n-2}\right)\cdot\frac{1}{x^{n-1}}$$

$$=\lim_{n\to\infty}\left[\frac{2^n(2-2/2^n)}{2^n(2+1/2^n)}\times\frac{2^n(1+1/2^n)}{2^n(1-2/2^n)}\times\frac{x^n}{x^{n-1}}\right]$$

$$=\lim_{n\to\infty}\left(\frac{2-2/2^n}{2+1/2^n}\times\frac{1+1/2^n}{1-2/2^n}\right)\cdot x$$

$$\sim\left(\frac{2-0}{2+0}\times\frac{1+0}{1-0}\right)\cdot x\ =\frac{2}{2}x=x$$

Then by D'Alembert ratio test, the given series is convergent for $|x|<1$, is divergent for $|x|>1$ and further test needed at $|x|=1$.

At $x=1$, $\quad u_n=\frac{2^n-2}{2^n+1}=\frac{2^n(1-2/2^n)}{2^n(1+1/2^n)}=\frac{1-2/2^n}{1+1/2^n}$

Here,

$$\lim_{n\to\infty}u_n=\lim_{n\to\infty}\left(\frac{1-2/2^n}{1+1/2^n}\right)\sim\frac{1-0}{1+0}=1\neq 0.$$

So, the given series is divergent at $x=1$.

And at $x=-1$, $\quad u_n=(-1)^{n-1}\left(\frac{2^n-2}{2^n+1}\right)=(-1)^{n-1}\left(\frac{1-2/2^n}{1+1/2^n}\right)$

This is an alternative series whose general positive term is,

$$v_n=\frac{2-2/2^n}{1+1/2^n}$$

Here,

$$\lim_{n\to\infty}v_n=\lim_{n\to\infty}\left(\frac{1-2/2^n}{1+1/2^n}\right)\sim\frac{1-0}{1+0}=1\neq 0.$$

This shows the series diverges at $x=-1$ by Leibnitz test.

Thus, the given series converges for $|x|<1$ and diverges for $|x|\geq 1$.

4. $\displaystyle \frac{x^2}{2}+\frac{2x^3}{3}+\frac{3x^4}{4}+\frac{4x^5}{5}+\ldots$

Solution: Given series is

$$\frac{x^2}{2}+\frac{2x^3}{3}+\frac{3x^4}{4}+\frac{4x^5}{5}+\ldots\ldots$$

This is a power series whose general term is, $\quad u_n=\frac{n\,x^{n+1}}{n+1}$

Solution is similar to Q. 2.

5. $\displaystyle \Sigma\left(\frac{x^n}{3^n n^2}\right)$ **for $x>0$.**

Solution: Given series is, $\quad \Sigma\left(\frac{x^n}{3^n n^2}\right)$ for $x>0$.

This is a power series whose general term is, $\quad u_n=\frac{x^n}{3^n n^2}$

Here,

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n}=\lim_{n\to\infty}\frac{x^{n+1}}{3^{n+1}(n+1)^2}\times\frac{3^n x^2}{x^n}$$

$$=\lim_{n\to\infty}\left(\frac{1}{3(1+1/n)^2}\right)x\ \sim\frac{1}{3(1+0)^2}x\ =\frac{x}{3}$$

By D'Alembert ratio test, the given series is convergent for $\left|\frac{x}{3}\right|<1\Rightarrow|x|<3$ and is divergent for $\left|\frac{x}{3}\right|>1\Rightarrow|x|>3$. And, at $\left|\frac{x}{3}\right|=1\Rightarrow|x|=3$, the further test needed.

Given that $x>0$. So, the. Leibnitz theorem, the given series is convergent for $x>3$ and diverges for $x>3$ and, further test needed at $x=3$.

At $x=3$, $\quad u_n=\frac{3^n}{3^n\cdot n^2}=\frac{1}{n^2}$

This shows that Σu_n is a p–series with $p=2>1$. Then the series is convergent at $x=3$, by p–test.

Thus, the given series is convergent for $x\leq 3$ and is divergent for $x>3$.

6. $\displaystyle \Sigma\left(\frac{x^n}{n}\right)$ **for $x>0$**

Solution: Given series is, $\quad \Sigma\left(\frac{x^n}{n}\right)$ for $x>0$

This is a power series whose general term is, $\quad u_n=\frac{x^n}{n}$

Here,

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n}=\lim_{n\to\infty}\left(\frac{x^{n+1}}{n+1}\times\frac{n}{x^n}\right)$$

$$=\lim_{n\to\infty}\left(\frac{n}{n+1}\right)x\ =\lim_{n\to\infty}\left(\frac{x}{1+1/n}\right)\sim\frac{x}{1+0}=x$$

By Leibnitz theorem, the given series is convergent for $|x|<1$ and is divergent for $|x|>1$. And further test needed for $|x|=1$.

Since $x>0$. So, the given series converges for $x<1$ and diverges for $x>1$. And, further test is needed at $x=1$.

At $x=1$, $\quad u_n=\frac{1^n}{n}=\frac{1}{n}$

This shows that Σu_n is a p–series with $p=1$. So, the series diverges by p–test.

Thus, the given series is convergent for $x<1$ and is divergent for $x\geq 1$.

7. $\displaystyle \Sigma\left(\frac{n}{n^2+1}\right)x^n$ **for $x>0$**

Solution: Given series is, $\quad \Sigma\left(\frac{n}{n^2+1}\right)x^n$ for $x>0$

This is a power series whose general term is, $\quad u_n=\left(\frac{n}{n^2+1}\right)x^n$

Then,

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n}=\lim_{n\to\infty}\left[\left(\frac{n+1}{(n+1)^2+1}\right)x^{n+1}\times\left(\frac{n^2+1}{n}\right)\cdot\frac{1}{x^n}\right]$$

$$= \lim_{n\to\infty}\left[\frac{n(1+1/n)}{n^2[(1+1/n)^2+1/n^2]}\right]\times\frac{n^2(1+1/n^2)}{n}\,x$$

$$= \lim_{n\to\infty}\left[\frac{(1+1/n)(1+1/n^2)}{(1+1/n)^2+1/n^2}\right]x \sim \frac{(1+0)(1+0)}{(1+0)^2+0}\,x = x$$

By D'Alembert ratio test, the given series is convergent for $x < 1$ being $x > 0$ and the series is divergent for $x > 1$ being $x > 0$. And, further test is needed at $x = 1$.

At $x = 1$, $\quad u_n = \dfrac{n}{n^2+1} = \dfrac{1}{n(1+1/n^2)}$

Set $v_n = \dfrac{1}{n}$ then the series Σv_n diverges by p-test.

Here,

$$\lim_{n\to\infty}\frac{u_n}{v_n} = \lim_{n\to\infty}\frac{1}{(1+1/n^2)} = \frac{1}{1+0} = 1$$

which is a non-zero finite value.

Since the series Σv_n diverges and therefore the given series is divergent at $x = 1$ by limit comparison test.

Thus, the given series is convergent for $x < 1$ and diverges for $x \geq 1$.

8. $\Sigma\left(\sqrt{\dfrac{n+1}{n^3+1}}\right)x^n$ **for $x > 0$**

Solution: Given series is, $\quad \Sigma\left(\sqrt{\dfrac{n+1}{n^3+1}}\right)x^n \quad$ for $x > 0$

This is a power series whose general term is, $\quad u_n = \left(\sqrt{\dfrac{n+1}{n^3+1}}\right)x^n$

Here,

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n} = \lim_{n\to\infty}\left[\sqrt{\frac{n+1+1}{(n+1)^3+1}}\cdot x^{n+1}\times\sqrt{\frac{n^3+1}{n+1}}\cdot\frac{1}{x^n}\right]$$

$$= \lim_{n\to\infty}\left[\frac{x}{n}\sqrt{\frac{1+2/n}{(1+1/n)^3+1/n^3}}\times n\sqrt{\frac{1+1/n^3}{1+1/n}}\right]$$

$$= \lim_{n\to\infty} x\left[\sqrt{\frac{1+2/n}{(1+1/n)^3+1/n^3}}\times\sqrt{\frac{1+1/n^3}{1+1/n}}\right]$$

$$\sim x\left[\sqrt{\frac{1+0}{(1+0)^3+0}}\times\sqrt{\frac{1+0}{1+0}}\right]$$

$$= x$$

By D'Alembert ratio test, the given series converges for $x < 1$ and diverges for $x > 1$ being $x > 0$. And further test is needed at $x = 1$.

At $x = 1$,

$$u_n = \sqrt{\frac{n+1}{n^3+1}} = \frac{1}{n}\sqrt{\frac{1+1/n}{1+1/n^3}}$$

Set $v_n = \dfrac{1}{n}$ then the series Σv_n diverges by p-test.

Here,

$$\lim_{n\to\infty}\frac{u_n}{v_n} = \lim_{n\to\infty}\sqrt{\frac{1+1/n}{1+1/n^3}} = \sqrt{\frac{1+0}{1+0}} = 1.$$

which is a non-zero finite value.

Since the series Σv_n diverges and therefore the given series is divergent at $x = 1$ by limit comparison test.

Thus, the given series is convergent for $x < 1$ and is divergent for $x \geq 1$.

9. $x + 2x^2 + 3x^3 + 4x^4 + \ldots\ldots$ **for $x > 0$**

Solution: Given series is,

$$x + 2x^2 + 3x^3 + 4x^4 + \ldots\ldots \quad \text{for } x > 0$$

This is a power series whose general term is, $\quad u_n = n\,x^n$

Here,

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n} = \lim_{n\to\infty}\frac{(n+1)\,x^{n+1}}{n\,x^n} = \lim_{n\to\infty}(1+1/n)x \sim (1+0)x = x$$

By D'Alembert ratio test, the given series converges for $x < 1$ and diverges for $x > 1$ being $x > 0$. And, further test is needed at $x = 1$.

At $x = 1$, $\quad u_n = n$.

Here,

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} n \sim \infty \neq 0.$$

That means, the series is divergent at $x = 1$.

Thus, the series converges for $x < 1$ and diverges for $x \geq 1$.

10. $1 + \dfrac{x}{2} + \dfrac{x^2}{5} + \dfrac{x^3}{10} + \ldots\ldots\ldots + \dfrac{x^n}{n^2+1} + \ldots\ldots\ldots$ **for $x > 0$**

Solution: Similar to Q. 2 with $x > 0$.

11. $\dfrac{x}{1.3} + \dfrac{x^2}{3.5} + \dfrac{x^3}{5.7} + \ldots\ldots$ **for $x > 0$**

Solution: Similar to Q. 6

12. **Solution:** Similar to Q. 8

13. $\dfrac{x}{2.\sqrt{3}} + \dfrac{x^2}{3\sqrt{4}} + \dfrac{x^3}{4\sqrt{5}} + \ldots\ldots$ **for $x > 0$**

Solution: Given series is

$$\frac{x}{2.\sqrt{3}} + \frac{x^2}{3\sqrt{4}} + \frac{x^3}{4\sqrt{5}} + \ldots\ldots \quad \text{for } x > 0$$

This is a power series whose general term is, $\quad u_n = \dfrac{x^n}{(n+1)\sqrt{n+2}}$

Here,

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n} = \lim_{n\to\infty}\left[\frac{x^{n+1}}{(n+1+1)\sqrt{n+1+2}}\times\frac{(n+1)\sqrt{n+2}}{x^n}\right]$$

$$= \lim_{n\to\infty}\left[\frac{x^n.x}{(n+2)\sqrt{n+3}}\times\frac{n\sqrt{n}\,(1+1/n)\sqrt{1+2/n}}{x^n}\right]$$

$$= \lim_{n\to\infty}\frac{x.n\sqrt{n}\,(1+1/n)\sqrt{1+2/n}}{n\sqrt{n}\,.\,(1+2/n)\sqrt{1+3/n}}$$

$$= \lim_{n\to\infty}\frac{x(1+1/n)\sqrt{1+2/n}}{(1+2/)\sqrt{1+3/n}} \sim \frac{x(1+0)\sqrt{1+0}}{(1+0)\sqrt{1+0}} = x$$

By D'Alembert ratio test, the given series is convergent for $x < 1$ and is divergent for $x > 1$ being $x > 0$. And, further test needed for $x = 1$.

At $x = 1$, $\quad u_n = \dfrac{1}{(n+1)\sqrt{n+2}} = \dfrac{1}{n^{3/2}(1+(1/n))\sqrt{1+(2/n)}}$

Set $v_n = \frac{1}{n^{3/2}}$ then the series Σv_n converges by p-test.

Here,

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{1}{(1+(1/n))\sqrt{1+(2/n)}} = \frac{1}{(1+0)\sqrt{1+0}} = 1.$$

which is a non-zero finite value.

Since the series Σv_n converges and therefore the given series is convergent at $x = 1$ by limit comparison test.

This shows that the given series is convergent at $x = 1$.

Thus, the given series is convergent for $x \le 1$ and is divergent for $x > 1$.

B. Find the interval, center and radius of convergence of the following series:

14. $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\ldots$

Solution: Given series is

$$x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\ldots$$

This is an alternative series, whose general positive term is,

$$u_n = |a_n| = \frac{x^n}{n}$$

Here,

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty}\left(\frac{x^{n+1}}{n+1} \times \frac{n}{x^n}\right)$$

$$= \lim_{n\to\infty}\left(\frac{x^n.x}{n(1+1/n)} \times \frac{n}{x^n}\right) = \lim_{n\to\infty}\left(\frac{x}{(1+1/x)}\right) - \frac{x}{1+0} = x$$

D'Alembert ratio test, the given series converges for $|x| < 1$ and diverges for $|x| > 1$ and, further test is needed at $|x| = 1$.

At $x = 1$, $\quad u_n = \frac{1}{n}$

Here,

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{n} - \frac{1}{\infty} - 0.$$

And,

$$\frac{u_{n+1}}{u_n} = \frac{1}{n+1} \times \frac{n}{1} = \frac{1}{1+(1/n)} < 1.$$

$\Rightarrow \quad u_{n+1} < u_n \quad$ for every n.

That is every positive term of the series is numerically less than the preceding term.

This shows that the series is convergent at $x = 1$ by Leibnitz theorem.

And at $x = -1$, $u_n = -\frac{1}{n}$

Here, the series $\Sigma u_n = -\Sigma\left(\frac{1}{n}\right)$ is a p–series with $p = 1$.Then the series diverges by p-test.

Thus, the series is convergent on $(-1, 1]$.

Thus, the interval of convergence is, $\quad -1 < x \le 1$.

And, centre of convergent is,

$$\frac{1+(-1)}{2} = \frac{1-1}{2} = \frac{0}{2} = 0.$$

Also, radius of convergence is,

$$\frac{1-(-1)}{2} = \frac{1+1}{2} = \frac{2}{2} = 1.$$

Thus, the interval is $(-1, 1]$, centre is 0 and radius of convergence is 1.

15. $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\ldots$

Solution: Given series is,

$$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\ldots$$

This is a power series whose general term is, $\quad u_n = \dfrac{x^{n-1}}{(n-1)!}$

Here,

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty}\left(\frac{x^n}{n!} \times \frac{(n-1)!}{x^{n-1}}\right) = \lim_{n\to\infty}\left(\frac{x}{n}\right) - \frac{x}{\infty} - 0 < 1.$$

This shows that the given series is convergent, by D'Alembert ratio test for all finite value of x in $(-\infty, \infty)$. Therefore the interval of convergence is $(-\infty, \infty)$.

And centre of convergence is, $\dfrac{\infty + (-\infty)}{2} = 0$.

Also, radius of convergence is, $\dfrac{\infty - (-\infty)}{2} = \infty$.

Thus, the interval is $(-\infty, \infty)$, centre is 0 and radius is ∞, of convergence of given series.

16. $x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots\ldots$

Solution: Given series is

$$x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots\ldots$$

This is an alternative series, whose general positive term is,

$$u_n = |a_n| = \frac{x^{2n-1}}{2n-1}$$

Here,

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{x^{2(n+1)-1}}{2(n+1)-1} \times \frac{2n-1}{x^{2n-1}}$$

$$= \lim_{n\to\infty}\left(\frac{x^{2n+1}}{2n+1} \times \frac{2n-1}{x^{2n-1}}\right)$$

$$= \lim_{n\to\infty} \frac{x^{2n}.x}{n(2+1/n)} \times \frac{n(2-1/n)}{x^n.x^{-1}}$$

$$= \lim_{n\to\infty} \frac{x^2(2-1/n)}{2+1/n} - \frac{x^2(2-0)}{2+0} = x^2$$

By D'Alembert ratio test, the given series is convergent for $x^2 < 1 \Rightarrow |x| < 1$ and is divergent for $|x| > 1$. And, further test is needed at $|x| = 1$.

At $x = 1$, the given series is.

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots\ldots$$

This is an alternative series, whose general positive term is,

$$v_n = |a_n| = \frac{1}{2n-1}$$

Here,
$$\lim_{n\to\infty} v_n = \lim_{n\to\infty}\left(\frac{1}{2n-1}\right) = \frac{1}{\infty} = 0.$$

And,
$$\frac{v_{n+1}}{v_n} = \frac{1}{2n+1}\times\frac{2n-1}{1} = \frac{2n+1-3}{2n+1} = 1-\frac{3}{2n+1} < 1.$$
$$\Rightarrow u_{n+1} < u_n \quad \text{for every } n.$$

That is every positive term of the series is numerically less than the preceding term.
This shows that the given series converges at $x = 1$.
And at $x = -1$, the given series becomes
$$-1+\frac{1}{3}-\frac{1}{5}+\frac{1}{7}-\ldots\ldots$$
$$\Rightarrow -\left[1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\ldots\ldots\right]$$

This series is convergent as above, by Leibnitz test.
Thus, the given series is convergent for $|x| \le 1$ and diverges for $|x| > 1$.
That is, the series converges on $[-1, 1]$. So, the interval of convergence is $[-1, 1]$.

And, the centre of convergence is, $\frac{1+(-1)}{2} = \frac{1-1}{2} = 0.$

Also, the radius of convergence is, $\frac{1-(-1)}{2} = \frac{1+1}{2} = 1.$

Thus, the interval is $[-1, 1]$, centre is 0 and radius is 1, of the convergence of given series.

17. $x-\frac{x^2}{2^2}+\frac{x^3}{3^2}-\frac{x^4}{4^2}+\ldots\ldots$

Solution: Given series is,
$$x-\frac{x^2}{2^2}+\frac{x^3}{3^2}-\frac{x^4}{4^2}+\ldots\ldots$$
This is an alternative series, whose general positive term is,
$$u_n = |a_n| = \frac{x^n}{n^2}$$
Here,
$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n} = \lim_{n\to\infty}\left[\frac{x^{n+1}}{(n+1)^2}\times\frac{n^2}{x^n}\right] = \lim_{n\to\infty}\frac{x}{(1+1/n)^2} = \frac{x}{(1+0)^2} = x$$
By D–Alembert ratio test the given series is convergent for $|x| < 1$ and diverges for $|x| > 1$. And, further test is needed at $|x| = 1$.
At $x = 1$, the given series is
$$1-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\ldots\ldots$$
This is an alternative series, whose general positive term is,
$$u_n = |a_n| = \frac{1}{n^2}$$
Clearly the series $\Sigma u_n = \Sigma\frac{1}{n^2}$ is a p–series with $p = 2 > 1$. Therefore, the series Σu_n is convergent by p–test.
Therefore, the given series is convergent at $x = 1$.

And, at $x = -1$, the given series is,
$$-1-\frac{1}{2^2}-\frac{1}{3^2}-\frac{1}{4^2}-\ldots\ldots \qquad \ldots\ldots (*)$$
$$= -\left[1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\ldots\ldots\right]$$
The general term of the series $1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\ldots\ldots$ is, $u_n = \frac{1}{n^2}$.

Clearly, the series $\Sigma u_n = \Sigma\left(\frac{1}{n^2}\right)$ is p–series with $p = 2 > 1$. So, the series Σu_n is convergent by p–test. So, the series (*) converges at $x = -1$.
Therefore, the given series is convergent at $x = -1$.
Thus, the given series is convergent for $|x| \le 1$ and is divergent for $|x| > 1$.
That is the interval of convergence given series is, $[-1, 1]$.

And, centre of the convergence of given series is, $\frac{-1+1}{2} = 0.$

Also, radius of convergence of given series is, $\frac{1-(-1)}{2} = \frac{1+1}{2} = 1$

Thus, the interval is $[-1, 1]$, centre is 0 and radius is 1, of convergence of given series.

C. Find the interval of convergence of the power series:

18. $\sum_{n=1}^{\infty}\frac{x^n}{n+4}$

Solution: Given series is, $\sum_{n=1}^{\infty}\frac{x^n}{n+4}$

This is a power series whose general term is, $u_n = \frac{x^n}{n+4}$

Here,
$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n} = \lim_{n\to\infty}\left[\frac{x^{n+1}}{(n+1)+4}\times\frac{n+4}{x^n}\right]$$
$$= \lim_{n\to\infty}\left[\frac{x^n \cdot x}{n(1+5/n)}\times\frac{n(1+4/n)}{x^n}\right]$$
$$= \lim_{n\to\infty}\left(\frac{1+4/n}{1+5/n}\right)x = \left(\frac{1+0}{1+0}\right)x = x$$
By D'Alembert ratio test the given series is convergent for $|x| < 1$ and is divergent for $|x| > 1$. And, further test is needed at $|x| = 1$.

At $x = 1$, $\quad u_n = \frac{1}{n+4} = \frac{1}{n(1+(4/n))}$

Choose $v_n = \frac{1}{n}$ then the series Σv_n is divergent by p-test.

Here, $\quad \lim_{n\to\infty}\frac{u_n}{v_n} = \lim_{n\to\infty}\frac{1}{1+(4/n)} = \frac{1}{1+0} = 1$

which is a non-zero finite value.
Since the series Σv_n is divergent and therefore the series $\Sigma u_n = \Sigma\left(\frac{1}{n+4}\right)$ is also divergent by limit comparison test.

This shows that the given series converges at $x = 1$.

And at $x = -1$, $u_n = \frac{(-1)^n}{n+4}$

This is an alternative series, whose general positive term is,

$$v_n = |u_n| = \frac{1}{n+4}$$

Here,

$$\lim_{n\to\infty} v_n = \lim_{n\to\infty} \frac{1}{n+4} \sim \frac{1}{\infty} \sim 0.$$

And,

$$\frac{v_{n+1}}{v_n} = \frac{1}{n+5} \times \frac{n+4}{1} = \frac{n+5-1}{n+5} = 1 - \frac{1}{n+5} < 1.$$

$$\Rightarrow \quad u_{n+1} < u_n \quad \text{for every } n.$$

That is every positive term of the series is numerically less than the preceding term.

This shows that the given series $\Sigma u_n = \Sigma\frac{(-1)^n}{n+4}$ is convergent at $x = -1$, by Leibniz test.

Thus, the given series is convergent for $|x| < 1$ and at $x = -1$ and diverges for $|x| > 1$ and at $x = 1$.

That is, the interval of convergence of given series is $[-1, 1)$.

19. $\sum_{n=0}^{\infty} \frac{n^2x^n}{2^n}$ **[2009 Spring Q. No. 5(b)]**

Solution: Given series is, $\sum_{n=0}^{\infty} \frac{n^2x^n}{2^n}$

This is a power series whose general term is, $u_n = \frac{n^2x^n}{2^n}$

Here,

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \left[\frac{(n+1)^2 x^{n+1}}{2^{n+1}} \times \frac{2^n}{n^2 x^n}\right]$$

$$= \lim_{n\to\infty} \left[\frac{n^2(1+1/n)^2 x^n . x}{2^n . 2} \times \frac{2^n}{n^2 x^n}\right]$$

$$= \lim_{n\to\infty} \frac{(1+1/n)^2}{2} . x \sim \frac{(1+0)^2}{2} . x = \frac{x}{2}$$

By D'Alembert ratio test, the given series is convergent for $\left|\frac{x}{2}\right| < 1 \Rightarrow |x| < 2$ and diverges for $\left|\frac{x}{2}\right| > 1 \Rightarrow |x| > 2$. And further test is needed at $\left|\frac{x}{2}\right| = 1 \Rightarrow |x| = 2$.

At $x = 2$, $u_n = \frac{n^2 2^n}{2^n} = n^2$.

Here,

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} n^2 \sim \infty \neq 0.$$

So, the given series is divergent at $x = 2$.

And at $x = -2$, $u_n = \frac{n^2(-2)^n}{2^n} = (-1)^n n^2$

As above, the series of positive term of an alternative series $\Sigma v_n = \Sigma n^2$, is divergent. So, the alternative series Σu_n is also divergent at $x = -2$.

Thus, the given series is convergent for $|x| < 2$ and diverges for $|x| \geq 2$. That is, the interval of convergence of given series is $(-2, 2)$.

20. $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{\sqrt{n}}$

Solution: Process is similar to Q. 17.

21. $\sum_{n=2}^{\infty} \left(\frac{n}{n^2+1}\right) x^n$

Solution: Similar to Q. 7.

22. $\sum_{n=1}^{\infty} \left(\frac{n+1}{10^n}\right)(x-4)^n$

Solution: Given series is $\sum_{n=1}^{\infty} \left(\frac{n+1}{10^n}\right)(x-4)^n$

This is a power series whose general term is, $u_n = \left(\frac{n+1}{10^n}\right)(x-4)^n$

Here,

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \left[\left(\frac{n+1+1}{10^{n+1}}\right)(x-4)^{n+1} \times \left(\frac{10^n}{n+1}\right)\left[\frac{1}{(x-4)^n}\right]\right]$$

$$= \lim_{n\to\infty} \left[\frac{n(1+2/n)}{10^n.10} \times (x-4)^n . (x-4) \times \frac{10^n}{n(1+1/n)} \times \frac{1}{(x-4)^n}\right]$$

$$= \lim_{n\to\infty} \frac{(1+2/n)(x-4)}{10(1+1/n)} \sim \frac{(1+0)}{10(1+0)}(x-4) = \frac{x-4}{10}$$

By D'Alembert ratio test, the given series is convergent for $\left|\frac{x-4}{10}\right| < 1 \Rightarrow |x-4| < 10$ and is divergent for $|x-4| > 10$.

And further test needed at $|x-4| = 10$.

At $x - 4 = 10$, $u_n = \frac{n+1}{10^n}(10)^n = (n+1)$.

Then,

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} (n+1) \sim \infty \neq 0.$$

This shows that the given series is divergent at $x = 14$.

Also at $x - 4 = -10$, the general term of given series is

$$u_n = \frac{n+1}{10^n}(-10)^n = (-1)^n(n+1)$$

Here, $\lim_{n\to\infty} u_n = \lim_{n\to\infty} (-1)^n(n+1) \sim \infty \neq 0.$

This means the given series is divergent at $x = 14$ by Leibnitz test.

Then the alternative series is divergent, by above result. Thus, the given series is convergent for $|x-4| < 10$ and is divergent for $|x-4| \geq 10$.

That is, the interval of convergence of given series is

$$-10 < (x-4) < 10 \to -6 < x < 14 \text{ i.e. } (-6, 14).$$

23. $\sum_{n=0}^{\infty} \frac{n!}{100^n} x^n$ 24. $\sum_{n=0}^{\infty} \frac{x^{2n+1}}{(-4)^n}$ 25. $\sum_{n=0}^{\infty} \frac{2^n}{(2n)!} x^{2n}$

Solution: Process is similar to Q. 19.

26. $\sum_{n=0}^{\infty} \frac{3^{2n}}{(n+1)} (x-2)^n$ 27. $\sum_{n=0}^{\infty} \frac{n^2}{2^{3n}} (x+4)^n$ **[2011 Spring Q.No. 5(b)]**

28. $\sum_{n=1}^{\infty} \frac{(-1)^n (2x-1)^n}{n6^n}$ 29. $\sum_{n=1}^{\infty} (-1)^n \frac{3^n}{n!} (x-4)^n$

Solution: Process is similar to Q. 22.

EXERCISE 2.8

A. Use Maclaurin expansion, show that

1. $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots\dots$ **for $|x| < 1$**

Solution: Here, $f(x) = \frac{1}{1-x}$ for $x \neq 1$.

By Maclaurin expansion,

$$f(x) = f(0) = f'(0)\, x + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \dots\dots \quad \dots\dots(i)$$

Here, $f(x) = \frac{1}{1-x}$

Differentiating we get,

$$f'(x) = \frac{1}{(1-x)^2}, \quad f''(x) = \frac{2}{(1-x)^3}, \quad f'''(x) = \frac{3 \times 2}{(1-x)^4}$$

and so on for $|x| < 1$.

At $x = 0$,

$$f(0) = 1, \quad f'(0) = 1, \quad f''(0) = 2, \quad f'''(0) = 3 \quad \text{and so on.}$$

Then (i) becomes,

$$f(x) = 1 + x + \frac{x^2}{2!} \times 2 + \frac{x^3}{3!} \times 3 \times 2 + \dots\dots \quad \text{for } |x| < 1$$

$$\Rightarrow f(x) = 1 + x + x^2 + x^3 + \dots\dots \quad \text{for } |x| < 1.$$

2. $\frac{1}{1+x} = 1 - x + x^2 - x^3 + \dots\dots$ **for $|x| < 1$**

Solution: Similar to Q. 1

3. $e^x = 1 + x + \frac{x^2}{2!} + \dots\dots + \frac{x^n}{n!} + \dots\dots$ **for $|x| < \infty$.** **[2007 Fall Q. No. 5(b) OR]**

Solution: Here, $f(x) = e^x$

By Maclaurin expansion we have,

$$f(x) = f(0) + x\, f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \dots\dots \quad \dots\dots(i)$$

Here, $f(x) = e^x$

Differentiating we get,

$$f'(x) = e^x = f''(x) = f'''(x) = \dots \quad , \forall x$$

At $x = 0$,

$$f(a) = e^0 = 1 = f'(0) = f''(0) = f'''(0) = \dots \quad , \forall x$$

Then (i) becomes,

$$f(x) = 1 + x.1 + \frac{x^2}{2!}.1 + \frac{x^3}{3!} + 1 + \dots\dots \quad , \forall x$$

$$= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots\dots$$

4. $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots\dots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \dots\dots$ **[2002 – Short]**

Solution: Here, $f(x) = \sin x$

By Maclaurin's series we have,

$$f(x) = f(0) + x\, f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \dots\dots \quad \dots\dots(i)$$

Here, $f(x) = \sin x$

Differentiating we get,

$$f'(x) = \cos x, \quad f''(x) = -\sin x, \quad f'''(x) = -\cos x, \quad f^{iv}(x) = \sin x,$$

$$f^{v}(x) = \cos x, \quad f^{vi}(x) = -\sin x, \quad \text{and so on.}$$

Set $x = 0$ then

$$f(0) = 0 = f''(0) = f^{iv}(0) = f^{vi}(0) = \dots$$

$$\text{and } f'(0) = 1, f'''(0) = -1, f^{(v)} = 1, \dots\dots$$

Then (i) becomes,

$$f(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots\dots$$

The general term of the series is, $u_n = (-1)^n \frac{x^{2n-1}}{(2n-1)!}$

Therefore,

$$f(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots\dots + (-1)^n \frac{x^{2n-1}}{(2n-1)} + \dots\dots$$

5. $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots\dots + (-1)^n \frac{x^{2n}}{(2n)!} + \dots\dots$

Solution: Process is similar to Q. 4.

6. $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots\dots + (-1)^{n-1} \frac{x^n}{n} + \dots\dots$ **for $-1 < x \leq 1$**

Solution: Here, $f(x) = \log(1 + x)$ which is valid only for $x > -1$.

By Maclaurin's series expansion we have,

$$f(x) = f(0) = x\, f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \dots\dots \quad \dots(i)$$

Here, $f(x) = \log(1 + x)$

Differentiating we get,

$$f'(x) = \frac{1}{1+x}, \quad f''(x) = \frac{-1}{(1+x)^2}, \quad f'''(x) = \frac{(-1)(-2)}{(1+x)^3} = \frac{(-1)^2 . 2!}{(1+x)^3},$$

$$f^{iv}(x) = \frac{(-1)^2 . 2! (-3)}{(1+x)^4} = \frac{(-1)^3 . 3!}{(1+x)^4}, \quad \text{and so on.}$$

At $x = 0$,

$f(0) = \log(1) = 0$, $\quad f'(0) = 1$, $\quad f''(0) = -1$, $\quad f'''(0) = (-1)^2 . 2!$,

$f^{iv}(0) = (-1)^3 . 3!$ and so on.

Then (i) becomes;

$$f(x) = x.1 + \frac{x^2}{2!}.(-1) + \frac{x^3}{3!}(-1)^2.2! + \frac{x^4}{4!}(-1)^3.3! + \ldots \quad \text{for } x > -1$$

$$= x - \frac{x^2}{2} + \frac{x^3.2!}{3.2!} - \frac{x^4}{4.3!}.3! + \ldots\ldots \quad \text{for } x > -1$$

$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\ldots \quad \text{for } x > -1$$

The general term of the series is,

$$u_n = (-1)^{n+1}\frac{x^n}{n} \quad \text{for } x > -1$$

Therefore,

$$f(x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\ldots \quad \text{for } x > -1$$

7. $\log\left(\frac{1+x}{1-x}\right) = 2\tanh^{-1}x = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \ldots\right)$ **for** $|x| < 1$

Solution: Here, $\quad f(x) = \log\left(\frac{1+x}{1-x}\right)$

Since f(x) is valid only for $\frac{1+x}{1-x} > 0$ and this gives for $-1 < x < 1 \Rightarrow |x| < 1$.

So, $\quad f(x) = \log\left(\frac{1+x}{1-x}\right) \quad$ for $|x| < 1$

By Maclarain expansion we have,

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \ldots \quad \ldots\ldots\ldots\text{(i)}$$

Here,

$$f(x) = \log\left(\frac{1+x}{1-x}\right) = \log(1+x) - \log(1-x) \quad \text{for } |x| < 1$$

Differentiating we get,

$$f'(x) = \frac{1}{1+x} + \frac{1}{1-x}, \qquad f''(x) = \frac{-1}{(1+x)^2} + \frac{1}{(1-x)^2}$$

$$f'''(x) = \frac{(-1)^2.2}{(1+x)^3} + \frac{2}{(1-x)^3},$$

$$f^{iv}(x) = \frac{(-1)^3.2.3}{(1+x)^4} + \frac{2.3}{(1-x)^4} = \frac{(-1)^3.3!}{(1+x)^4} + \frac{3!}{(1-x)^4}$$

$$f^{v}(x) = \frac{(-1)^4.3!.4}{(1+x)^5} + \frac{3!.4}{(1-x)^5} = \frac{(-1)^4.4!}{(1+x)^5} + \frac{4!}{(1-x)^5} \text{ and so on.}$$

At $x = 0$,

$$f(0) = \log\left(\frac{1+0}{1-0}\right) = 0, \quad f'(0) = 1 + 1 = 2, \quad f''(0) = -1 + 1 = 0$$

$$f'''(0) = (-1)^2.2 + 2 = 2! + 2! = 2(2!)$$

$$f^{iv} = (-1)^3.3! + 3! = -3! + 3! = 0$$

$$f^{v}(0) = (-1)^4.4! + 4! = 4! + 4! = 2(4!) \quad \text{and so on.}$$

Therefore (i) becomes,

$$f(x) = 0 + x.2 + 0 + \frac{x^3}{3!}2(2!) + 0 + \frac{x^5}{5!}2(4!) + \ldots\ldots \quad \text{for } |x| < 1$$

$$\Rightarrow f(x) = 2\left[x + \frac{x^3}{3.2!}.2! + \frac{x^5}{5.4!}4! + \ldots\right] \quad \text{for } |x| < 1.$$

$$= 2\left[x + \frac{x^3}{3} + \frac{x^5}{5} + \ldots\right] \quad \text{for } |x| < 1.$$

8. $\tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots\ldots + (-1)^{n-1}\frac{x^{2n-1}}{2n-1} + \ldots$ **for** $|x| \le 1$

Solution: Here, $f(x) = \tan^{-1}x$

Since the tan function is continuous only on the period included origin in $\left(-\frac{\pi}{4}, \frac{\pi}{4}\right)$.

That is, $\tan^{-1}x$ is continuous on $(-1, 1)$.

Therefore, $f(x) = \tan^{-1}x \quad$ for $-1 < x < 1 \Rightarrow |x| < 1$.

Differentiating we get,

$$f'(x) = \frac{1}{1+x^2}, \qquad f''(0) = \frac{-2x}{(1+x^2)^2}$$

$$f'''(x) = \frac{(1+x^2)^2(-2) + 2x.2(1-x^2).2x}{(1+x^2)^4} = \frac{-2-2x^2+8x^2}{(1+x^2)^3} = \frac{-2+6x^2}{(1+x^2)^3}$$

$$f^{iv}(x) = \frac{(1+x)^3\,12x - (-2+6x^2)\,6x\,(1+x^2)^2}{(1+x^2)^6}$$

$$= \frac{6x\,(2+2x^2+2-6x^2)}{(1+x^2)^4} = \frac{6x\,(4-4x^2)}{(1+x^2)^4} = \frac{24(x-x^3)}{(1+x^2)^4}$$

$$f^{v}(x) = \frac{(1+x^2)^4\,24(1-3x^2) - 24(x-x^3)\,8x(1+x^2)^3}{(1+x^2)^8} \quad \text{and so on.}$$

At $x = 0$,

$f(0) = 0$, $\quad f'(0) = 1$, $\quad f''(0) = 0$, $\quad f'''(0) = -2$, $\quad f^{iv}(0) = 0$,

$f^{v}(0) = 24$, and so on.

Therefore (i) becomes,

$$f(x) = 0 + x.1 + 0 + \frac{x^3}{3!}(-2) + 0 + \frac{x^5}{5!}.24 + 0 + \ldots\ldots \quad \text{for } |x| < 1$$

$$\Rightarrow f(x) = x - \frac{x^3}{3.2!}.2! + \frac{x^5}{5.4!}.4! + \ldots\ldots \quad \text{for } |x| < 1$$

$$\Rightarrow f(x) = x - \frac{x^3}{3} + \frac{x^5}{5} + \ldots\ldots \quad \text{for } |x| < 1.$$

B. Assuming the validity of expansion, find expansion of following functions by using Maclaurin expansion.

1. $\log(1 + \sin x)$ **[2006 Spring Q. No. 6(b) OR]**

[2014 Fall Q. No. 3(b) OR; 2014 Spring Q. No. 5(b) OR]

Find expansion of $\log(1 + \sin x)$ as far as the term x^4, by using Maclaurin expansion.

Solution: Here, $\quad f(x) = \log(1 + \sin x)$

Since $\log(0)$ is undefined and $\sin\pi = -1$ gives $\log(0)$.

Here, $\sin x = -1 = \sin\frac{3\pi}{2} \Rightarrow x = \frac{3\pi}{2}$

Therefore, f(x) is valid for $-\frac{\pi}{2} < x < \frac{\pi}{2} \Rightarrow |x| < \frac{\pi}{2}$.

By Maclaurin's expansion we have,

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \ldots\ldots \qquad \ldots\ldots(i)$$

Here, $f(x) = \log(1 + \sin x)$

Differentiating we get,

$$f'(x) = \frac{\cos x}{1 + \sin x}$$

and,

$$f''(x) = \frac{(1 + \sin x)(-\sin x) - \cos^2 x}{(1 + \sin x)^2}$$

$$= \frac{-\sin x - \sin^2 x - \cos^2 x}{(1 + \sin x)^2} = \frac{-1 - \sin x}{(1 + \sin x)^2} = \frac{-1}{1 + \sin x}$$

and,

$$f'''(x) = \frac{\cos x}{(1 + \sin x)^2}$$

and,

$$f^{iv}(x) = \frac{(1 + \sin x)^2(-\sin x) - 2(1 + \sin x)\cos^2 x}{(1 + \sin x)^4}$$

$$= \frac{-\sin x - \sin^2 x - 2\cos^2 x}{(1 + \sin x)^3}$$

$$= \frac{-\sin x - \sin^2 x - 2(1 - \sin 2x)}{(1 + \sin x)^3} = \frac{\sin^2 x - \sin x - 2}{(1 + \sin x)^3}$$

and so on.

At x = 0,

$f(0) = \log(0) = 0$, $f'(0) = 1$, $f''(0) = -1$, $f'''(0) = 1$,

$f^{iv}(0) = -2$, and so on.

Then, (i) becomes,

$$f(x) = x - \frac{x^2}{2!} + \frac{x^3}{3!} - \frac{2x^4}{4!} + \ldots\ldots \qquad \text{for } |x| < \pi$$

$$\Rightarrow f(x) = x - \frac{x^2}{2} + \frac{x^3}{6} - \frac{x^4}{12} + \ldots\ldots \qquad \text{for } |x| < \pi$$

2. secx

Solution: Process is similar to Q. A(4)

3. log(1 + tanx)

Solution: Process is similar to Q. 1

4. e^x secx **[2013 Fall Q. No. 2(a) OR]**

Solution: Let $f(x) = e^x \sec x$

Since secx is continuous on $(-\pi/2, \pi/2)$ including origin.

So, f(x) is valid for $(-\pi/2, \pi/2)$.

By Maclaurin's expansion we have,

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \ldots. \qquad \ldots\ldots(i)$$

Here $f(x) = e^x \sec x$

Differentiating we get,

$$f'(x) = e^x \sec x + e^x \sec x \tan x$$

and,

$$f''(x) = e^x \sec x + 2e^x \sec x \tan x + e^x (\sec x \tan^2 x + \sec^3 x)$$

$$= e^x \sec x + 2e^x \sec x \tan x + e^x \sec x(-1 + \sec^2 x) + e^x \sec^3 x$$

$$= e^x[\sec x + 2\sec x \tan x + \sec^3 x - \sec x + \sec^3 x]$$

$$= 2e^x[\sec x \tan x + \sec^3 x]$$

and,

$$f'''(x) = 2e^x[\sec x \tan x + \sec^3 x] + 2e^x[\sec x \tan^2 x + \sec^3 x + 3\sec^3 x \tan x]$$

$$= 2e^x[\sec x \tan x + \sec^3 x + \sec^3 x - \sec x + \sec^3 x + 3\sec^3 x \tan x]$$

and so on.

At x = 0,

$f(0) = 1$, $f'(0) = 1$, $f''(0) = 2$, $f'''(0) = 4$, and so on.

Therefore (i) becomes,

$$f(x) = 1 + x + \frac{2x^2}{2!} + \frac{4x^3}{3!} + \ldots\ldots \qquad \text{for } |x| < \frac{\pi}{2}$$

5. e^{ax} cosbx

Solution: Let $f(x) = e^{ax}\cos bx$

By Maclaurin's series we have,

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \ldots\ldots\ldots \qquad \ldots\ldots(i)$$

Here,

$$f(x) = e^{ax}\cos bx$$

Differentiating we get,

$$f'(x) = ae^{ax}\cos bx - be^{ax}\sin bx$$

and,

$$f''(x) = a^2e^{ax}\cos bx - abe^{ax}\sin bx - abe^{ax}\sin bx - b^2e^{ax}\cos bx$$

$$= (a^2 - b^2)e^{ax}\cos bx - 2abe^{ax}\sin bx$$

and,

$$f'''(x) = (a^2 - b^2)[ae^{Ax}\cos bx - be^{ax}\sin bx] - 2ab[ae^{ax}\sin bx + be^{ax}\cos bx]$$

$$= e^{ax}[a^3\cos bx - ab^2\cos bx - a^2b\sin bx + b^3\sin bx - 2a^2b\sin bx - 2ab^2\cos bx]$$

$$= e^{ax}[(a^3 - 3ab^2)\cos bx + (b^3 - 3a^2b)\sin bx]$$

and so on.

At x = 0,

$f(0) = 1$, $f'(0) = a$, $f''(0) = a^2 - b^2$, $f'''(0) = a^3 - 3ab^2$ and so on.

Therefore (i) becomes,

$$f(x) = 1 + ax + (a^2 - b^2)\frac{x^2}{2!} + (a^3 - 3ab^2)\frac{x^3}{3!} + \ldots\ldots$$

6. $\log(1 + x + x^2)$

Solution: Let, $f(x) = \log(1 + x + x^2) = \log\left(\frac{1 - x^3}{1 - x}\right) = \log(1 - x^3) - \log(1 - x)$.

By Maclaurin's expansion we have,

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \ldots \qquad \ldots\ldots(i)$$

Here,

$$f(x) = \log(1 - x^3)$$

Differentiating we get,

$$f' = \frac{-2x}{1-x^3} \qquad f''(x) = \frac{(1-x^3)(-2) - (-2x)(-3x^2)}{(1-x^3)^2}$$

$$= \frac{-2 + 2x^3 - 6x^3}{(1-x^3)^2} = \frac{2-4x^3}{(1-x^3)^2}$$

and,

$$f'''(x) = \frac{(1-x^3)^2(-12x^2) - (2-4x^3)\ 2(1-x^3)(-3x^2)}{(1-x^3)^4}$$

$$= \frac{(1-x^3)(-12x^2) - (2-4x^3)\ 2(-3x^2)}{(1-x^3)^3}$$

$$= \frac{-12x^2 + 12x^5 + 12x^2 - 24x^5}{(1-x^3)^3} = \frac{-12x^5}{(1-x^3)^3}$$

and so on.

At $x = 0$,

$f(0) = \log(1) = 0$, $f'(0) = 0$, $f''(0) = 2$, $f'''(0) = 0$ and so on.

Here,

$$f(x) = \log(1 - x)$$

Differentiating we get,

$$f' = \frac{-1}{1-x} = -(1-x)^{-1} \qquad f''(x) = (1-x)^{-2} \text{ and, } f'''(x) = -2(1-x)^{-3}$$

and so on.

At $x = 0$,

$f(0) = \log(1) = 0$, $f'(0) = -1$, $f''(0) = 1$, $f'''(0) = -2$ and so on.

Therefore (i) becomes,

$$f(x) = \left(0 + x(0) + \frac{x^2}{2!}(2) + \ldots\right) - \left(0 - x + \frac{x^2}{2!} - \frac{4x^3}{3!}(2) + \ldots\right)$$

$$\Rightarrow f(x) = x + \frac{x^2}{2} - \frac{2x^3}{3} + \ldots$$

7. $\sin^{-1}x$ and hence $\cos^{-1}x$

Solution: Process similar to Q. No. A. 8.

8. $e^{\sin^{-1}x}$ **[2009 Spring Q.No. 5(a) OR] [2008 Spring Q.No. 5(b) OR]**

Solution: Let $f(x) = e^{\sin^{-1}x}$

By Maclaurin's expansion we have,

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \ldots \qquad \ldots\ldots(i)$$

Here,

$$y = f(x) = e^{\sin^{-1}x}$$

Then, $y' = e^{\sin^{-1}x} \cdot \dfrac{1}{\sqrt{1-x^2}} = \dfrac{y}{\sqrt{1-x^2}}$

$$\Rightarrow (1-x^2)(y')^2 = y^2$$

And, $2y'y''(1-x^2) + (y')^2(-2x) = 2y$ and so on.

At $x = 0$,

$f(0) = y(0) = e^0 = 1$, $f'(0) = y'(0) = \dfrac{y(0)}{\sqrt{1-0}} = 1$

$2y''(0) = 2 \Rightarrow f''(0) = y''(0) = 1$ and so on.

Therefore (i) becomes,

$$f(x) = 1 + x + \frac{x^2}{2!} + \ldots\ldots$$

9. x cosec x

Solution: Let, $f(x) = x \operatorname{cosec} x \qquad \ldots\ldots(i)$

Let $x \operatorname{cosec} x = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \ldots\ldots \qquad \ldots\ldots(ii)$

Since we have,

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots\ldots$$

By (ii)

$$x = \sin x\,[a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \ldots\ldots]$$

$$= \left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots\right][a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + \ldots\ldots]$$

$$= a_0x + a_1x^2 + \left(a_2 - \frac{a_0}{3!}\right)x^3 + \left(a_3 - \frac{a_1}{3!}\right)x^4 + \left(-a_4 - \frac{a_2}{3!} + \frac{a_4}{5!}\right)x^5 + \ldots\ldots$$

Comparing the coefficients of x, x^2, x^3, x^4 and so on.

Then we get,

$a_0 = 1$, $a_1 = 0$, $a_2 - a_0/3! = 0 \Rightarrow a_2 = \dfrac{1}{3!}$, $a_3 - \dfrac{a_1}{3!} = 0 \Rightarrow a_3 = 0$,

$a_4 = \dfrac{a_2}{3!} - \dfrac{a_0}{5!} = \dfrac{1}{36} - \dfrac{1}{120} = \dfrac{7}{360}$ and so on.

Now, (ii) becomes,

$$x \operatorname{cosec} x = 1 + \frac{x^2}{6} + \frac{7x^4}{360} + \ldots\ldots$$

10. $[\sin^{-1}x]^2$

Solution: Let $f(x) = (\sin^{-1}x)^2$

By Maclaurin's series expansion we have,

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \ldots \qquad \ldots\ldots(i)$$

Here,

$$f(x) = (\sin^{-1}x)^2$$

Differentiating we get,

$$f'(x) = \frac{2\sin^{-1}x}{\sqrt{1-x^2}}$$

and, $f''(x) = \dfrac{2}{1-x^2} + 2\sin^{-1}x\left(-\dfrac{1}{2}\right)\left(\dfrac{-2x}{(1-x^2)^{3/2}}\right)$

$$= \frac{2}{1-x^2} + \frac{2x\sin^{-1}x}{(1-x^2)^{3/2}}$$

and,

$$f'''(x) = \frac{-4x}{(1-x^2)^2} + \frac{1}{(1-x^2)^3}$$

$$\left[(1-x^2)^{3/2} \cdot 2\left(\sin^{-1}x + \frac{x}{\sqrt{1-x^2}}\right) - 2x\sin^{-1}x.\ 3.\ (1-x^2)^{1/2}.\ (-2x)\right]$$

$$= \frac{-4x}{(1-x^2)^2} + \frac{2}{(1-x^2)^3}\left[(1-x^2)^{3/2}\left(\sin^{-1}x + \frac{x}{\sqrt{1-x^2}}\right) + 3x^2(1-x^2)^{1/2}\sin^{-1}x\right]$$

At $x = 0$,

$f(0) = 0, \quad f'(0) = 0, \quad f''(0) = 2, \quad f''' = 0$ and so on.

Therefore (i) becomes,

$$f(x) = \frac{x^2}{2!} \times 2 + \frac{x^4}{4!} \times 8 + \ldots\ldots$$

$$\Rightarrow \quad f(x) = x^2 + \frac{x^4}{3} + \ldots\ldots$$

11. x cotx

Solution: Let, $\quad x\cot x = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + \ldots\ldots \quad$ (i)

Since we have,

$$\sin x = x - \frac{x^3}{3!} + \frac{x^4}{5!} - \ldots\ldots \quad \text{and} \quad \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots\ldots$$

Now (i) becomes,

$$x\cos x = \sin x\,(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^3 + a_5x^5 + \ldots\ldots)$$

$$\Rightarrow \quad x\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots\right) = \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots\right)(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + \ldots\ldots)$$

$$\Rightarrow \quad x - \frac{x^3}{2!} + \frac{x^5}{4!} - \ldots\ldots = a_0x + a_1x^2 + \left(a_2 - \frac{a_0}{3!}\right)x^3 + \left(a_3 - \frac{a_1}{3!}\right)x^4 + \left(a_4 - \frac{a_2}{3!} + \frac{a_0}{5!}\right)x^5 + \ldots\ldots$$

Comparing the coefficients of the variables then,

$$a_0 = 1, \quad a_1 = 0, \quad a_2 - \frac{a_0}{3!} = -\frac{1}{2!} \Rightarrow a_2 = \frac{a_0}{6} - \frac{1}{2} = \frac{1}{6} - \frac{1}{2} = \frac{2}{6},$$

$$a_3 - \frac{a_1}{3!} = 0 \Rightarrow a_3 = 0$$

$$a_4 - \frac{a_2}{3!} + \frac{a_0}{5!} = \frac{1}{4!} \Rightarrow a_4 = \frac{a_2}{3!} - \frac{a_0}{5!} + \frac{1}{4!} = -\frac{2}{36} - \frac{1}{12} + \frac{1}{24} = -\frac{1}{45}$$

Therefore (i) becomes,

$$x\cot x = 1 - \frac{x^2}{3} - \frac{x^4}{45} - \ldots\ldots$$

12. log secx **[2012 Fall Q.No. 6(b) OR]**

Solution: Let $\quad f(x) = \log(\sec x) \quad$ for $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

By Maclaurin's expansion we have,

$$f(x) = f(0) = xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \ldots\ldots \quad \ldots\ldots(i)$$

Here, $\quad f(x) = \log(\sec x)$

Differentiating we get,

$$f'(x) = \frac{\sec x \tan x}{\sec x} = \tan x, \quad f''(x) = \sec^2 x, \quad f'''(x) = 2\sec^2 x \tan x,$$

$$f^{iv}(x) = 4\sec^2 x \tan^2 x + 2\sec^4 x$$

$$= 4\sec^2 x(\sec^2 x - 1) + 2\sec^4 x = 6\sec^4 x - 4\sec^2 x,$$

and so on.

At $x = 0$,

$f(0) = 0, \quad f'(0) = 0, \quad f''(0) = 1, \quad f'''(0) = 0, \quad f^{iv}(0) = 2$

and so on.

Therefore (i) becomes,

$$f(x) = 0 + 0 + \frac{x^2}{2!} + 0 + \frac{2x^4}{4!} + \ldots\ldots$$

$$\Rightarrow \quad f(x) = \frac{x^2}{2!} + \frac{2x^4}{4!} + \ldots\ldots$$

13. $(1 + x^2)^{-1/2}$

Solution: Let $\quad f(x) = y = (1 + x^2)^{-1/2}$

By Maclaurin series expansion we have,

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \ldots\ldots \quad \ldots\ldots(i)$$

Here,

$$f(x) + (1 + x^2)^{-1/2}$$

Differentiating we get,

$$y' = f'(x) = -\frac{1}{2}(1 + x^2)^{-3/2}.\ 2x = \frac{-x}{(1 + x^2)^{3/2}}$$

$$\Rightarrow \quad y' = \frac{-x}{y^3} \Rightarrow y'y^3 = -x$$

and,

$$y''y^3 + 3y^2(y')^2 = -1$$

Also,

$$y'''y^3 + 3y''y^2y' + 6y^2y'y'' + 6y(y')^3 = 0$$

$$\Rightarrow \quad y'''y^2 + 9y''y'y + 6(y')^3 = 0 \qquad [\because y \neq 0]$$

and,

$$y^{iv}y^2 + 2y'''y'y + 9y'''y'y + 9(y'')^2y + 9y''(y')^2 + 18(y')^2y'' = 0$$

and so on.

At $x = 0$,

$y(0) = f(0) = 1, \quad y'(0) = f'(0) = 0, \quad y''(0) = f''(0) = -1,$

$y'''(0) = f'''(0) = 0, \quad y^{iv}(0) = f^{iv}(0) = -9 \quad$ and so on.

Therefore, (i) becomes,

$$f(x) = 1 + 0 + \frac{x^2}{2!}(-1) + 0 + \frac{x^4}{4!}(-9) + \ldots\ldots$$

$$\Rightarrow \quad f(x) = 1 - \frac{x^2}{2!} - \frac{9x^4}{4!} + \ldots\ldots$$

14. sinhx and hence coshx

Solution: Let $f(x) = \sinh x$

By Maclaurin series expansion we have,

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \ldots\ldots \quad \ldots\ldots\text{(i)}$$

Here,

$f(x) = \sinh x$

Differentiating we get,

$f'(x) = \cosh x$, $f'''(x) = \sinh x$, $f^{iv}(x) = \cosh x$,

$f''(x) = \sinh x$ and so on.

At $x = 0$,

$f(0) = f''(0) = \sinh 0 = 0 = f^{iv}(0) = f^{vi}(0) = \ldots\ldots\ldots$

and, $f'(0) = \cosh 0 = 1 = f'''(0) = f^{v}(0) = \ldots\ldots\ldots$

Therefore (i) becomes,

$$f(x) = 0 + x\,.\,1 + 0 + \frac{x^3}{3!} + 0 + \frac{x^5}{5!} + 0 + \ldots\ldots$$

$$\Rightarrow\ f(x) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots\ldots\ldots \quad \ldots\ldots\ldots\text{(ii)}$$

Since (ii) gives,

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots\ldots\ldots$$

Differentiating we get,

$$\cosh x = 1 + \frac{3x^2}{3!} + \frac{5x^4}{5!} + \ldots\ldots\ldots$$

$$\Rightarrow\ \cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots\ldots$$

C. Write the Taylor series of the given function at the given point a.

1. $f(x) = e^x$ at $a = 10$

Solution: Let $f(x) = e^x$

By Taylor's series expansion of f(x) is

$$f(x) = f(a) = (x - a)\,f'(a) + \frac{(x-a)^2}{2!}\,.\,f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \ldots. \quad \ldots\text{(i)}$$

Here,

$f(x) = e^x = f'(x) = f''(x) = f^{v}(x) = \ldots\ldots\ldots$

At $x = a = 10$

$f(a) = e^{10}$, $f'(10) = e^{10} = f''(10) = f'''(10) = f^{v}(1) = \ldots\ldots$

Then (i) becomes at $a = 10$,

$$f(x) = e^{10}\left[1 + (x - 10) + \frac{(x-10)^2}{2!} + \frac{(x-10)^3}{3!} + \ldots\ldots\ldots\right]$$

2. $f(x) = \log x$ at $a = 1$

Solution: Let $f(x) = \log(x)$

By Taylor's series expansion of f(x) is

$$f(x) = f(a) + (x - a)\,f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \ldots\ldots \quad \ldots\text{(i)}$$

Here, $f(x) = \log(x)$

Differentiating we get,

$f'(x) = \frac{1}{x} = x^{-1}$, $f''(x) = -x^{-2}$, $f'''(x) = 2!\,x^{-3}$,

$f^{iv}(x) = -2 \times 3x^{-4} = -3!\,x^{-4}$, $f^{v}(x) = 4!\,x^{-5}$ and so on.

At $x = a = 1$,

$f(1) = \log(1) = 0$, $f'(1) = 1$, $f''(1) = -1$, $f'''(1) = 2!$

$f^{iv}(1) = -3!$, $f^{v}(1) = 4!$ and so on.

Therefore (i) becomes at $a = 1$,

$$f(x) = 0 + (x - 1) - \frac{(x-a)^2}{2!} + \frac{(x-1)^3}{3!}\times 2! - \frac{(x-1)^4}{4!}\times 3! + \frac{(x-1)^5}{5!}\times 4! - \ldots\ldots\ldots$$

$$\Rightarrow\ f(x) = (x - 1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \frac{(x-1)^5}{5} - \ldots\ldots\ldots$$

3. $f(x) = \frac{1}{x}$ at $a = -1$

Solution: Process as Q. 2

4. $f(x) = \tan x$ at $a = \pi/4$

Solution: Let $f(x) = \tan x$

By Taylor's series expansion we have,

$$f(x) = f(a) + (x - a)\,f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \ldots\ldots \quad \ldots\text{(i)}$$

Here,

$f(x) = \tan x$

Differentiating we get,

$f'(x) = \sec^2 x$,

$f''(x) = 2\sec^2 x.\tan x = 2(1 + \tan^2 x)\tan x = 2\tan x + 2\tan^3 x$

$f'''(x) = 2\sec^2 x + 6\tan^2 x\sec^2 x = 2\tan^2 x + 2\tan x + 6\tan^4 x + 6\tan^2 x$

$f^{iv}(x) = 24\tan^3 x\sec^2 x + 16\tan x\sec^2 x + 2\sec^2 x$

$= 24\tan^5 x + 24\tan^3 x + 16\tan^3 x + 16\tan x + 2\tan^2 x + 2$

$= 24\tan^5 x + 40\tan^3 x + 2\tan^2 x + 16\tan x + 2$

and so on.

At $x = a = \frac{\pi}{4}$,

$f(a) = 1$, $f'(a) = 1 + 1 = 2$, $f''(a) = 2 + 2 = 4$

$f'''(a) = 6 + 8 + 2 = 16$, $f^{iv}(a) = 24 + 40 + 2 + 16 + 2 = 104$

and so on.

Therefore (i) becomes, at $a = \frac{\pi}{4}$,

$$f(x) = 1 + 2\left(x - \frac{\pi}{4}\right) + 4\frac{(x - \pi/4)^2}{2!} + \frac{16(x - \pi/4)^3}{3!} + \ldots\ldots$$

$$\Rightarrow\ f(x) = 1 + 2\left(x - \frac{\pi}{4}\right) + 2\left(x - \frac{\pi}{4}\right)^2 + \frac{8}{3}\left(x - \frac{\pi}{4}\right)^2 + \ldots\ldots\ldots$$

OTHER IMPORTANT QUESTION FROM FINAL EXAM

2012 Fall Q.No. 6(a)

Examine the convergence of the series

$$\frac{1}{1+\sqrt{2}}+\frac{1}{1+2\sqrt{3}}+\frac{1}{1+3\sqrt{4}}+\ldots\ldots+\frac{1}{1+n\sqrt{n+1}}+\ldots\ldots$$

Solution: See the part of Exercise 2.6 Q. No. 2.

2005 Fall Q.No. 6(a)

Test the convergence of

i. $\frac{2}{3^2}+\frac{3}{4^2}+\frac{4}{5^2}+\ldots\ldots$ ii. $\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+\frac{1}{4.5}+\ldots\ldots$

Solution: (i) See the solution of Q. 3, Exercise 2.2.

(ii) We have given infinite series is

$$\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+\ldots\ldots$$

Its general term is $u_n=\frac{1}{n(n+1)}$

Taking $\lim_{n\to\infty}$ on both side, we get

$$\lim_{n\to\infty} u_n=\lim_{n\to\infty}\frac{1}{n(n+1)}=\lim_{n\to\infty}\frac{1}{n^2(1+1/n)}=\frac{1}{\infty}=0$$

This shows that given infinite series is probably convergent. Let us check comparison test.

Let us choose another infinite series $\Sigma v_n=\Sigma\left(\frac{1}{n^2}\right)$ which is a convergent series by test.

Then,

$$\lim_{n\to\infty}\frac{u_n}{v_n}=\lim_{n\to\infty}\frac{\frac{1}{n(n+1)}}{\frac{1}{n^2}}=\lim_{n\to\infty}\frac{n^2}{n(n+1)}=\lim_{n\to\infty}\left(\frac{1}{1+\frac{1}{n}}\right)=1.$$

$$\Rightarrow \lim_{n\to\infty}\frac{u_n}{v_n}=1$$

which is nonzero and finite quantity.

Thus by comparison test, Σu_n is also convergent series.

2003 Spring Q.No. 5(a)

Show that the necessary condition for an infinite series Σu_n to be convergent is $\lim_{n\to\infty} u_n=0$. Examine the convergence of

$$\frac{1}{1+\sqrt{2}}+\frac{2}{1+2\sqrt{3}}+\frac{3}{1+3\sqrt{4}}+\ldots+\frac{n}{1+n\sqrt{n+1}}+\ldots$$

Solution: First Part: See the theoretical theorem.

Second Part: See the problem part of 2012 Fall.

2002 Q.No. 5(a)

Show that the alternating series $u_1-u_2+u_3-u_4+\ldots$ in which each term is numerically less than the preceding term and $\lim_{n\to\infty} u_n=0$, is convergent. And using it shows that the series $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots$ is convergent.

Solution: First Part: See the theoretical theorem – Leibnitz theorem.

Second Part: See the problem part of 2004 Spring, Short question.

Problems of type to solve the infinite series by using Maclaurin's series

2010 Fall Q.No. 5(b) OR

Find Maclaurin's series of the function $f(x)=\frac{e^x}{e^x+1}$.

Solution: We have, $f(x)=\left(\frac{e^x}{1+e^x}\right)$

By Maclaurin expansion

$$f(x)=f(0)+xf'(0)+\frac{x^2}{2!}+f''(0)+\frac{x^3}{3!}f'''(0)+\ldots\ldots \qquad \ldots\ldots(1)$$

Here,

$$f(x)=\left(\frac{e^x}{1+e^x}\right)\Rightarrow y=\frac{e^x}{(1+e^x)}$$

$$\Rightarrow (1+e^x)y=e^x \qquad \text{and } y(0)=\frac{1}{2}.$$

Differentiating w. r. t. x, we get

$$(1+e^x)y_1+e^xy=e^x \Rightarrow 2y_1(0)+y(0)=1$$

$$\Rightarrow 2y_1(0)=1-\frac{1}{2}=\frac{1}{2}$$

$$\Rightarrow y_1(0)=\frac{1}{4}$$

Again, differentiating, we get

$$(1+e^x)y_2+2e^xy_1+e^xy=e^x \Rightarrow 2y_2(0)+2y_1(0)+y(0)=1$$

$$\Rightarrow 2y_2(0)=1-2y_1(0)-y(0)=1-\frac{2}{4}-\frac{1}{2}$$

$$\Rightarrow y_2(0)=0.$$

Again differentiating, we get

$$(1+e^x)y_3+3e^xy_2+3e^xy_1+e^xy=e^x$$

$$\Rightarrow 2y_3(0)+3y_2(0)+3y_1(0)+y(0)=1$$

$$\Rightarrow 2y_3(0)=1-3y_2(0)-3y_1(0)-y(0)$$

$$=1-3.0-3.\frac{1}{4}-\frac{1}{2}=1-\frac{3}{4}-\frac{1}{2}=\frac{4-3-2}{4}=-\frac{1}{4}.$$

$$\Rightarrow y_3(0)=-\frac{1}{8}$$

Then from equation (1), we get

$$f(x)=\frac{1}{2}+\frac{1}{4}x+0-\frac{1}{8}\frac{x^3}{3!}+\ldots\ldots \qquad \ldots\ldots(2)$$

$$\Rightarrow \frac{e^x}{1+e^x}=\frac{1}{2}+\frac{x}{4}-\frac{x^3}{48}+\ldots\ldots$$

This is the required expansion of $\frac{e^x}{1+e^x}$.

2005 Fall Q.No. 6(b) OR

Expand $f(x) = e^x \sin x$ by using Maclaurin's expansion.

Solution: Let $f(x) = e^x \sin x$

By Maclaurin's series we get,

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \ldots\ldots \qquad \ldots\ldots\ldots(i)$$

Here, $f(x) = e^x \sin x$

Then,

$$f'(x) = e^x \sin x + e^x \cos x = e^x (\sin x + \cos x)$$

$$f''(x) = e^x (\sin x + \cos x) + e^x(\cos x - \sin x)$$

$$= 2e^x \cos x$$

$$f'''(x) = 2e^x \cos x - 2e^x \sin x = 2e^x (\cos x - \sin x)$$

and so on.

At $x = 0$,

$$f(0) = 0,\ f'(0) = 1,\ f''(0) = 2,\ f'''(0) = 2 \text{ and so on.}$$

Then (i) becomes,

$$f(x) = x + \frac{2x^2}{2!} + \frac{2x^3}{3!} + \ldots\ldots$$

2004 Fall Q.No. 5(a) OR; 2002 Q.No. 5(b) OR

Expand $e^{\sin x}$ in power of x by using Maclaurin's infinite series.

Solution: We know by Maclaurin expansion

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \frac{x^4}{4!} f^{iv}(0) + \ldots \qquad \ldots\ldots(1)$$

Here, $f(x) = e^{\sin x} \Rightarrow y = e^{\sin x}$ and $y_1(0) = 1$

Differentiating w. r. t. x, we get

$$y_1 = e^{\sin x} \cos x \Rightarrow y_1 = y \cos x \quad \text{and } y_1(0) = 1$$

Again,

$$y_2 = y_1 \cos x - y \sin x \quad \text{and } y_2(0) = 1$$

And,

$$y_3 = y_2 \cos x - y_1 \sin x - y_1 \sin x - y \cos x$$

$$\Rightarrow y_3 = y_2 \cos x - 2y_1 \sin x - y \cos x \quad \text{and } y_3(0) = y_2(0) - y_1(0)$$

$$\Rightarrow y_3(0) = 1 - 1 = 0.$$

Also we get,

$$y_4 = y_3 \cos x - y_2 \sin x - 2y_2 \sin x - 2y_1 \cos x - y_1 \cos x + y \sin x$$

$$\Rightarrow y_4(0) = y_3(0) - 2y_1(0) - y_1(0)$$

$$= 0 - 2 - 1.$$

$$\Rightarrow y_4(0) = -3.$$

Then from equation (1), we get

$$f(x) = 1 + x + \frac{x^2}{2!} - \frac{3}{4!} x^4 + \ldots\ldots$$

$$\Rightarrow e^{\sin x} = 1 + x + \frac{x^2}{2} - \frac{x^4}{8} - \ldots\ldots$$

This is the required expansion of $e^{\sin x}$.

2003 Fall Q.No. 6(a) OR; 2003 Spring Q.No. 5(b) OR

Expand tan in power of x and hence find the expansion of $\sec^2 x$.

2013 Spring Q. No. 5(b) OR

Expand tanx in the ascending integral power of x using Maclaurin's expansion and hence get the expansion of $\sec^2 x$.

Solution: Let $f(x) = \tan x$.

Then by Maclaurin expansion,

$$f(x) = f(0) + f'(0)\, x + f^{ii}\frac{(0)}{2!} x^2 + f^{iii}\frac{(0)}{3!} x^3 + \ldots\ldots \qquad \ldots\ldots(i)$$

Here, $f(x) = \tan x \Rightarrow f(0) = 0$

Differentiating, we get

$$f'(x) = \sec^2 x \Rightarrow f'(0) = 1$$

And,

$$f^{ii}(x) = 2\sec^2 x \tan x \Rightarrow f^{ii}(0) = 0$$

Also,

$$f^{iii}(x) = 2[2\sec^2 x \tan^2 x + \sec^4 x] \Rightarrow f^{iii}(0) = 2$$

Also,

$$f^{iv}(x) = 2[4\sec^2 x \tan^3 x + 4\tan x \sec^4 x + 4\sec^4 x \tan x)$$

$$= 8[\sec^2 x \tan^3 x + 2\tan x \sec^4 x]$$

$$\Rightarrow f^{iv}(0) = 0$$

Also,

$$f^{v}(x) = 8[2\sec^2 x + 3\sec^4 x \tan^2 x + 2\sec^6 x + 8\sec^4 x \tan^2 x]$$

$$\Rightarrow f^{v}(0) = 16$$

Putting these value in eq^n. (1), we get

$$f(x) = x + x^3 \frac{2}{3!} + x^5 \frac{15}{5!} + \ldots\ldots$$

$$\Rightarrow f(x) = x + \frac{x^3}{3} + \frac{2x^5}{15} + \ldots\ldots$$

This gives, $\tan x = x + \frac{x^3}{3} + \frac{2}{15} x^5 + \ldots\ldots$

Differentiating with respect to x, we get

$$\sec^2 x = 1 + x^2 + \frac{2}{3} x^4 + \ldots\ldots$$

This is the required expansion of $\sec^2 x$.

Problems of type to find the interval, centre and radius of convergence

2012 Fall Q.No. 6(b)

Test the convergence and find the interval and radius of convergence of the power series $\sum_{n=1}^{\infty} \frac{(x-5)^n}{n\,5^n}$.

Solution: The general term of given infinite series is

$$u_n = \frac{(x-5)^n}{n5^n} \text{ and } u_{n+1} = \frac{(x-5)^{n+1}}{(n+1)5^{n+1}}$$

So,

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{(x-5)^{n+1}}{(n+1)5^{n+1}} \cdot \frac{n5^n}{(n-5)^n}$$

$$= \lim_{n\to\infty} \frac{(x-5)}{\left(1+\frac{1}{n}\right)5}$$

$$= \frac{x-5}{5}$$

Then by ratio test the series is convergent for $\left|\frac{x-5}{5}\right| < 1$ and divergent for $\left|\frac{x-5}{5}\right|$ > 1. Further test is necessary for $\left|\frac{x-5}{5}\right| = 1$.

At $\left|\frac{x-5}{5}\right| = 1$ implies $\frac{x-5}{5} = 1$ and $\frac{x-5}{5} = -1$.

At $\frac{x-5}{5} = 1$, given series reduces to $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots\ldots$ with general term $u_n = \frac{1}{n}$ which is divergent by p–test.

At $\frac{x-5}{5} = -1$, given series reduced to $-1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \frac{1}{6} - \frac{1}{7} + \ldots\ldots$ which is a alternating series and each term is less than preceding term and

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty}\left(\frac{1}{n}\right) = 0$$

So it is convergent at $\frac{x-5}{5} = -1$.

Thus we get given series is convergent for

$$-1 \le \frac{x-5}{5} < 1 \Rightarrow \quad -5 \le x - 5 < 5$$

$$\Rightarrow \quad 0 \le x < 10$$

Required interval of convergence is [0, 10).

And the radius of the series is, $r = \frac{10-0}{2} = \frac{10}{2} = 5$

2014 Fall Q.No. 3(b); 2011 Fall Q.No. 3(b)

Find the centre, radius of convergence and interval of convergence of the power series $\sum_{n=0}^{\infty}\frac{(x-5)^n}{n\,5^n}$.

Solution: See the solution of 2012 Fall.

And, centre of the convergence is,

$$C = \frac{0+10}{2} = 5.$$

2010 Fall Q.No. 5(b)

Find the interval, centre and radius of convergence of the power series $\sum_{n=0}^{\infty}\frac{n+1}{10^n}(x-4)^n$.

Solution: See Exercise 2.7, Q. No. 22 for interval of convergence.

And, centre $= \frac{-6+10}{2} = 2$. Also, radius $= \frac{10-(-6)}{2} = \frac{10+6}{2} = 8$.

2010 Spring Q.No. 3(b); 2004 Spring Q.No. 5(b)

Find the radius of convergence and interval of convergence of the series

$$x + \frac{3}{5}x^2 + \frac{8}{10}x^3 + \frac{15}{17}x^4 + \ldots + \frac{n^2-1}{n^2+1}x^n + \ldots$$

Solution: Similar to 2010 Fall.

2009 Fall Q.No. 5(b)

Find the interval of convergence of an infinite series $\sum\left[\frac{n}{2^n}(x+4)^n\right]$.

Solution: Similar to 2011 Spring.

2014 Spring Q. No. 5(b); 2008 Spring Q. No. 5(b)

Find the interval and radius of convergence of power series; $\sum_{n=0}^{\infty}\frac{n+1}{10^n}(x-4)^n$

Solution: See the solution–part of 2010 Fall.

2013 Fall Q.No. 2(a); 2007 Fall Q.No. 5(b)

Find the radius of convergence and interval of convergence of the series $\sum_{x=0}^{\infty}\frac{(x-2)^n}{10^n}$.

Solution: The general term of the series $\sum_{n=0}^{\infty}\frac{(x-2)^n}{10^n}$ is, $u_n = \frac{(x-2)^{n-1}}{10^{n-1}}$

Here,

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n} = \lim_{n\to\infty}\left[\frac{(x-2)^n}{10^n}\times\frac{10^{n-1}}{(x-2)^{n-1}}\right]$$

$$= \lim_{n\to\infty}\left(\frac{x-2}{10}\right) = \frac{x-2}{10}$$

Then by D'Alembert ratio test the series is convergent for $\left|\frac{x-2}{10}\right| < 1$ and is divergent for $\left|\frac{x-2}{10}\right| > 1$. And further test is needed for $\left|\frac{x-2}{10}\right| = 1$.

At $\frac{x-2}{10} = 1 \Rightarrow x - 2 = 10$,

$$u_n = \frac{(10)^{n-1}}{(10)^{n-1}} = 1$$

This shows that at $x - 2 = 10$, $\Sigma u_n = n$. The series is divergent. Thus, the given series is divergent at $\frac{x-2}{10} = 1$.

At $\frac{x-2}{10} = -1 \Rightarrow x - 2 = -10$,

$$u_n = \frac{(-10)^{n-1}}{10^{n-1}} = (-1)^{n-1}$$

So, the series at $\frac{x-2}{10} = -1$ is, $\Sigma u_n = (-n)^{n-1}$

This is an alternative series which is divergent.

Thus, the given series is convergent for

$$\left|\frac{x-2}{10}\right| < 1 \Rightarrow |x-2| < 10$$

$$\Rightarrow -10 < x - 2 < 10 \quad \Rightarrow \quad -8 < x < 12.$$

and is divergent for $\left|\frac{x-2}{10}\right| \geq 1$.

This shows that the internal of convergence of given series is (−8, 12).
And the radius of convergence is

$$\text{radius} = \frac{12-(-8)}{2} = \frac{20}{2} = 10.$$

2006 Spring Q. No. 6(b)

Find the interval of convergence of an infinite series $x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \ldots$

Solution: Similar to 2007 Fall 5(b).

2005 Fall Q.No. 6(b)

Find the radius of convergence and interval of convergence of the infinite series:
$$(x-2) + \frac{(x-2)^2}{4} + \frac{(x-2)^3}{9} + \frac{(x-2)^3}{16} + \ldots\ldots$$

Solution: Similar to 2007 Fall 5(b).

Hint: The general term of given series is, $u_n = \frac{(x-2)^n}{n^2}$.

2004 Fall Q.No. 5(b)

Find the interval and radius of convergence of the series
$$\frac{x}{1.3} + \frac{x^2}{2.5} + \frac{x^3}{3.7} + \ldots + \frac{x^n}{n(2n+1)} + \ldots\ldots$$

Solution: The general term of given series
$$\frac{x}{1.3} + \frac{x^2}{2.5} + \frac{x^3}{3.7} + \ldots\ldots + \frac{x^n}{n(2n+1)} + \ldots\ldots$$
is, $u_n = \frac{x^n}{n(2n+1)}$

Here,
$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{x^{n+1}}{(n+1)(2n+3)} \times \frac{n(2n+1)}{x^n}$$
$$= \lim_{n\to\infty} \frac{x \cdot n(2+1/n)}{n(1+1/n)\,n(2+3/n)} = \frac{2x}{1\times 2} = x.$$

By D'Alembert ratio test the given series is convergent for $|x| < 1$ and divergent for $|x| > 1$. And further test needed for $|x| = 1$.
At $x = 1$,
$$u_n = \frac{1}{n(2n+1)} = \frac{1}{n^2(2+1/n)}$$

Choose $u_n = \frac{1}{n^2}$ then the series Σv_n converges by p–test with $p = 2 > 1$. Also,
$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \left[\frac{1/n^2(2+1)n}{1/n^2}\right] = \lim_{n\to\infty}\left(\frac{1}{2+1/4}\right) - \frac{1}{2} \neq 0$$

Then by comparison test the series $\Sigma u_n = \Sigma \frac{1}{n(2n+1)}$ converges being Σv_n converges.

And at $x = -1$, $u_n = \frac{(-1)^n}{n(2n+1)}$

This is an alternative series. Set, $v_n = \frac{1}{n(2n+1)}$

This is an alternative series. Set, $v_n = \frac{1}{n(2n+1)}$

Here,
$$\lim_{n\to\infty} v_n = \lim_{n\to\infty} \frac{1}{n(2n+1)} - \frac{1}{\infty} - 0$$

This shows that the series $\Sigma u_n = \Sigma \frac{(-1)^n}{n(2n+1)}$ converges, by Leibnitz theorem.
Thus the given series converges for $|x| \leq 1$ and diverges for $|x| > 1$.
So, the interval of convergence of the given series is [−1, 1].
And the radius of convergence is
$$\text{radius} = \frac{1-(-1)}{2} = \frac{2}{2} = 1.$$

Similar questions for practice:

Find the radius, centre and the interval of the convergence of the series.
$\Sigma \frac{(3x+4)^n}{\sqrt{3n+4}}$. **[2013 Spring Q. No. 5(b)]**

Find that the interval and radius of convergence of the power series
$\sum_{n=0}^{\infty} \frac{x^{2n+1}}{n!}$. **[2003 Fall Q.No. 6(b)]**

Find the interval of convergence of the series
$\frac{1}{2}x + \left(\frac{2}{3}\right)^2 x^2 + \left(\frac{3}{4}\right)^3 x^3 + \ldots + \left(\frac{n}{n+1}\right)^n x^n + \ldots$ **[2003 Spring Q.No. 5(b)]**

Find the radius of convergence of $2x + \frac{3x^2}{8} + \frac{4x^3}{27} + \ldots + \frac{(n+1)}{n^3} x^n + \ldots$
[2002 Q.No. 5(b)]

Find the radius and interval of convergence of the series
$\frac{x}{1.2} + \frac{x^2}{2.3} + \frac{x^3}{3.4} + \ldots + \frac{x^n}{n(n+1)} + \ldots\ldots$ **[2002 Q.No. 5(b)]**

Find the radius of convergence and interval of convergence of the power series $\sum_{n=0}^{\infty} \frac{n(x+3)^n}{5^n}$. **[2001 Q.No. 5(b)]**

SHORT QUESTIONS

2014 Fall Q. No. 7(c); 2013 Fall Q. No. 7(d): Show that $\Sigma u_n = \frac{n+3}{4n+3}$ is divergent.

Solution: The general term of given series, $\Sigma u_n = \frac{n+3}{4n+3}$ is
$$u_n = \frac{n+3}{4n+3}$$

Now,
$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \left(\frac{n(1+3/n)}{n(4+3/n)}\right) - \frac{1+0}{4+0} = \frac{1}{4} \neq 0.$$

This shows that given series is divergent.

2014 Spring Q. No. 7(a): Discuss the convergence and divergence of the series $\sum\limits_{n=2}^{\infty} \frac{1}{(\log n)^n}$.

OR 2012 Fall: Test the convergence of the series $\sum \frac{1}{\log n^n}$.

Solution: The general term of the given series $\Sigma\left(\frac{1}{\log n^n}\right)$ is

$$u_n = \frac{1}{\log n^n} = \frac{1}{n \log(n)}$$

Here,

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \left\{\frac{1}{(n+1)\log(n+1)} \times n\log(n)\right\}$$

$$= \lim_{n\to\infty} \frac{1}{n(1+1/n)\log(n) \,.\, \log(1)} \times n\log(n)$$

$$= \lim_{n\to\infty} \frac{1}{\log(1) \,.\, (1+1/n)}$$

$$= \frac{1}{\log(1)(1+0)} = \frac{1}{\log(1)} = \frac{1}{0} = \infty.$$

This shows that the given series is divergent by D'Alembert ratio test.

2011 Fall, 2010 Spring: Test whether the following series is convergent or divergent $\frac{2}{3} + \frac{3}{4} + \frac{4}{5} \ldots\ldots$

Solution: The general term of given series. $\frac{2}{3} + \frac{3}{4} + \frac{4}{5} + \ldots\ldots$ is $u_n = \frac{n+1}{n+2}$

Now,

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \left(\frac{n(1+1/n)}{n(1+2/n)}\right) = \frac{1+0}{1+0} = 1 \neq 0$$

This shows that given series is divergent.

2011 Fall: Test the convergence and divergence of the series $\sum \frac{n^2}{e^n}$.

Solution: We have given infinite series is $\Sigma\left[\frac{n^2}{e^n}\right]$

Its general term, $u_n = \frac{u^n}{e^n}$ and $u_{n+1} = \frac{(n+1)^2}{e^{n+1}}$.

Then,

$$\frac{u_{n+1}}{u_n} = \frac{(n+1)^2}{e^{n+1}} \times \frac{e^n}{n^2}$$

$$\Rightarrow \frac{u_{n+1}}{u_n} = \frac{\left(1+\frac{1}{n}\right)^2}{e}$$

Taking $\lim\limits_{n\to\infty}$ on both sides, we get

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{\left(1+\frac{1}{n}\right)^2}{e} = \frac{1}{e}$$

which is less than one, thus given infinite series is convergent.

2011 Spring: Test the convergence and divergence of $1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \ldots\ldots$

Solution: Given series is $1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \ldots\ldots$

Comparing with $\Sigma\frac{1}{n^p}$, we get $p = \frac{1}{2}$ which is less than one. Thus by definition of p-test, the given series is divergent.

2008 Spring: Test the convergence or divergence of series $\frac{3}{4} + \frac{5}{9} + \frac{7}{16} + \frac{9}{25} + \ldots \frac{(2n+1)}{(n+1)^2} + \ldots\ldots$

Solution: The general term of given series,

$\frac{3}{4} + \frac{5}{9} + \frac{7}{16} + \frac{9}{25} + \ldots\ldots + \frac{(2n+1)}{(n+1)^2} + \ldots\ldots$ is $u_n = \frac{(2n+1)}{(n+1)^2}$

Here,

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{n(2+1/n)}{n^2(1+1/n)^2} = \frac{2+0}{0(1+0)^2} = 0.$$

This shows that the given series is divergent.

2007 Fall: Test whether of the series $\frac{1}{3} + \frac{2}{4} + \frac{3}{5} + \frac{4}{6} + \ldots$ **is convergent?**

Solution: We have given series is. $\frac{1}{3} + \frac{2}{4} + \frac{3}{5} + \frac{4}{6} + \ldots\ldots$

Its general term is, $u_n = \frac{n}{n+2}$

Taking $\lim\limits_{n\to\infty}$ on both sides, we get

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \left(\frac{n}{n+2}\right) = \lim_{n\to\infty} \left[\frac{1}{1+2/n}\right] = \frac{1}{1+\frac{2}{\infty}} = \frac{1}{1+0} = 1 \neq 0.$$

So by definition the given series is divergent series.

2006 Spring: Show that the infinite series $\sum\limits_{n=1}^{\infty} \frac{2n+1}{n^3}$ **Is convergent.**

Solution: The general term of the series $\sum\limits_{n=1}^{\infty} \left(\frac{2n+1}{n^3}\right)$ is $u_n = \frac{2n+1}{n^3}$.

Choose $v_n = \frac{1}{n^2}$ then the series Σv_n converges by p-test with $p = 2 > 1$.

Now,

$$\sum_{n=1}^{\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \left(\frac{2n+1}{n^3} \times n^2\right)$$

$$= \lim_{n\to\infty} \left(\frac{2n+1}{n}\right) = \lim_{n\to\infty} \left(\frac{2+1/n}{1}\right) = 2 + 0 = 2 \neq 0$$

This shows that $\lim_{n\to\infty} \frac{u_n}{v_n}$ has non–zero finite value. So, by comparison test, the given series is convergent because Σv_n is convergent.

2005 Fall: State Leibnitz's test for the convergence of an alternating infinite series.

Solution: See the statement of Leibnitz's theorem.

2004 Fall; 2002: Show that the series Σu_n where $u_n = \frac{n}{n^3+1}$ is a convergent series.

Solution: The general term of the series $\Sigma u_n = \Sigma\left(\frac{n}{n^3+1}\right)$ is

$$u_n = \left(\frac{n}{n^3+1}\right) = \frac{1}{n^2}\left(\frac{1}{1+(1/n^3)}\right).$$

Choose $v_n = \frac{1}{n^2}$. Then the series Σv_n converges by p-test.

Here,

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty}\left(\frac{1}{1+(1/n^3)}\right) = \left(\frac{1}{1+0}\right) = 1.$$

which is a non-zero finite value.

Therefore the given series is convergent by limit comparison test.

2004 Fall: Show that $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots$ is convergent.

Solution: Given series is

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots\ldots \qquad \text{——(i)}$$

The general term of (i) is $\quad u_n = (-1)^{n+1}\left(\frac{1}{2n-1}\right)$

Here,

$$\lim_{n\to\infty} v_n = \lim_{n\to\infty}\left(\frac{1}{2n-1}\right) \sim \frac{1}{\infty} \sim 0.$$

And,

$$\frac{v_{n+1}}{v_n} = \frac{1}{2n+1} \times \frac{2n-1}{1} = \frac{2n+1-2}{2n+1} = 1 - \frac{2}{2n+1} < 1.$$

$\Rightarrow \quad u_{n+1} < u_n \qquad$ for every n.

That is every positive term of the series is numerically less than the preceding term. This shows that the given series is convergent by Leibnitz theorem.

2004 Spring; 2001

Test for the convergence of the series, $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots$

Solution: Similar to 2004 Fall.

2003 Fall: Prove, $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots$ is convergent.

Solution: Similar to 2011 Spring.

2003 Spring: Show that $\Sigma u_n = \sum \frac{n}{1+n^2}$ is divergent.

Solution: Similar to 2004 Fall.

❑❑❑