Eigenvalue Distribution of Pentadiagonal Doubly Stochastic Matrices

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Doubly Stochastic Matrix, Birkhoff–von Neumann Theorem

A square matrix $A = [a_{ij}] \in \mathbb{R}^{n\times n}$ is called **doubly stochastic** if:

1. **Non-negative entries:** $a_{ij} \geq 0$
2. **Row sums equals to 1:** $\sum_{j=1}^{n} a_{ij} = 1$
3. **Column sums equals to 1:** $\sum_{i=1}^{n} a_{ij} = 1$

Example:

$$\begin{bmatrix} 0.5 & 0.25 & 0.25 \\ 0.25 & 0.5 & 0.25 \\ 0.25 & 0.25 & 0.5 \end{bmatrix}$$

Birkhoff–von Neumann Theorem:

$$A = \sum_{k=1}^{m} \lambda_k P_k, \quad \text{where } \lambda_k \geq 0, \quad \sum_{k=1}^{m} \lambda_k = 1$$

- P_k: permutation matrices
- $m \leq n!$

Perfect–Mirsky Conjecture

Perfect–Mirsky Conjecture: Let Ω_n be the set of all eigenvalues of $n \times n$ doubly stochastic matrices. Then:

$$\Omega_n = \bigcup_{k=1}^{n} \operatorname{conv}\left\{ e^{2\pi i j/k} : j = 0, \ldots, k-1 \right\}$$

- Verified for $n = 2, 3, 4$
- Proven **false** for $n = 5$
- Numerical evidence for $n = 6, \ldots, 11$.
- Still **open** for $n \geq 12$

Implication: Suggests a geometric boundary for eigenvalue distributions.

Problem Statement and Approach

Pentadiagonal doubly stochastic matrices of orders up to 6.

Objective: To analyze the eigenvalue distribution of these matrices in the complex plane as we vary the combinations of permutation matrices used in their construction.

Approach:

- "Out of all $n!$ permutation matrices, we kept only those that fit the pentadiagonal pattern by removing any that have **ones** outside the allowed band—like at positions (1,4) or (4,1) in the 4×4 case."
- Constructed pentadiagonal doubly stochastic matrices as convex combinations of:
  - Pairs of valid permutation matrices.
  - Triples of valid permutation matrices.
  - General combinations ranging from 2 up to all valid permutation matrices.
- Compute and plot the eigenvalues of these matrices in the complex plane.

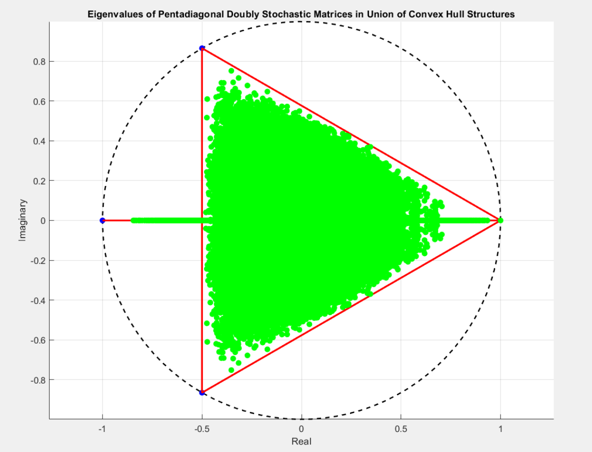
Results – For Order 3

Let ω_n^{pd} denotes the eigenvalue region of pentadiagonal doubly stochastic matrices.

Observation: Let π_k denote the set of k-th roots of unity. Define

$$\omega_3 = \text{conv}(\pi_1) \cup \text{conv}(\pi_2) \cup \text{conv}(\pi_3)$$

For $n = 3$, same as general 3×3 doubly stochastic matrices, i.e., $\omega_3 = \omega_3^{pd}$.



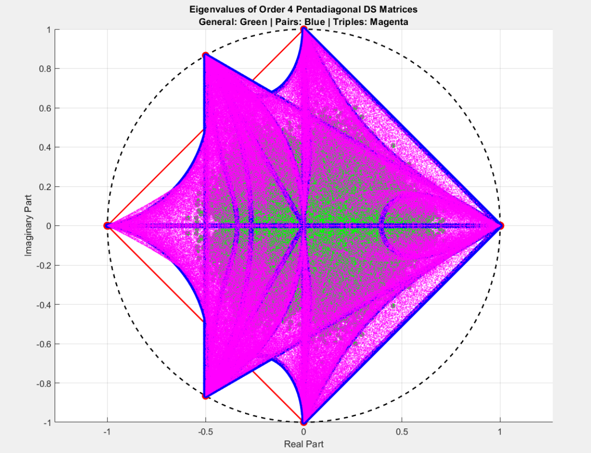
Results – For Order 4

Out of total $4! = 24$ permutations, only **14 are compatible** with the pentadiagonal structure.

Observation: Let π_k denote the set of k-th roots of unity. Define:

$$\omega_4 = \mathrm{conv}(\pi_1) \cup \mathrm{conv}(\pi_2) \cup \mathrm{conv}(\pi_3) \cup \mathrm{conv}(\pi_4)$$

A. $\omega_4^{pd} < \omega_4$

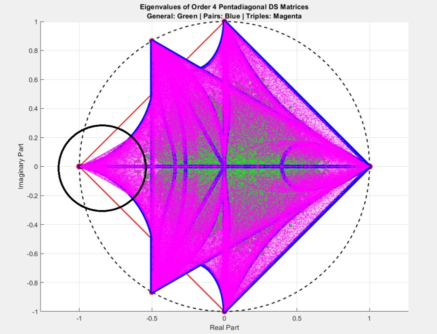


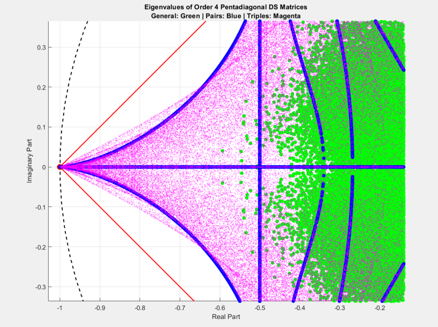
Results - For Order 4

B. Eigenvalues from:
   - Pairs (blue) form the outer spectral boundary.
   - Triples (magenta) and higher combinations (green) lie strictly inside this boundary.

   As combination size increases, eigenvalues tend to concentrate toward the center.

C. Although (B) is generally true, **Figure** shows an exception where triples extend the boundary formed by pairs.





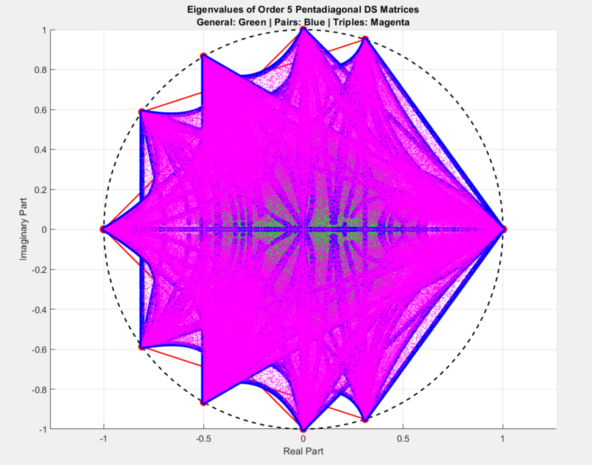
Results – For Order 5

Out of 5! $= 120$ possible permutations, only 31 are compatible with the pentadiagonal structure.

Observation: Let π_k denote the set of k-th roots of unity. Define:

$$\omega_5 = \text{conv}(\pi_1) \cup \text{conv}(\pi_2) \cup \text{conv}(\pi_3) \cup \text{conv}(\pi_4) \cup \text{conv}(\pi_5)$$

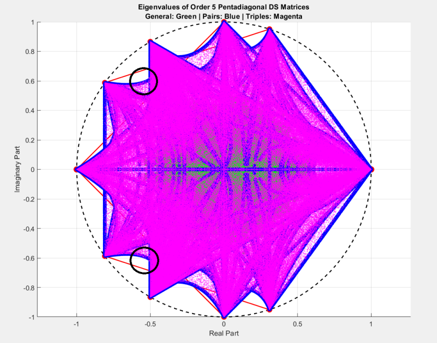
A. $\omega_5^{pd} < \omega_5$

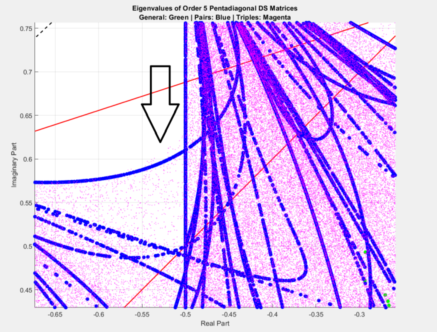


Results - For Order 5

B. Pairs (blue) form the spectral boundary; triples (magenta) and higher combinations (green) lie strictly inside. More combinations result in stronger central clustering.

C. **Figure 2** shows an exception where some eigenvalues from triples stretch beyond the pair-formed boundary.





Results – For Order 6

Out of 6! $=720$ possible permutations, only 73 are compatible with the pentadiagonal structure.

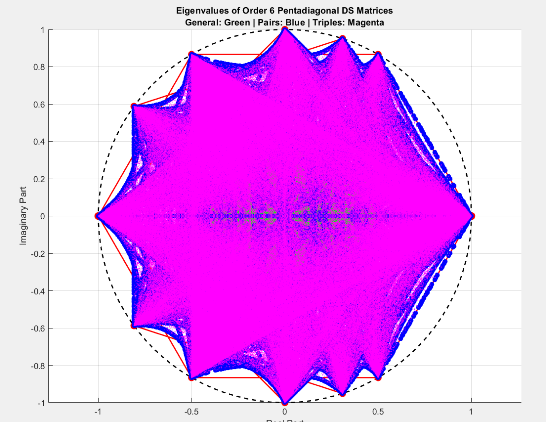
Observation: Let π_k denote the set of k-th roots of unity. Define:

$$\omega_6 = \mathrm{conv}(\pi_1) \cup \mathrm{conv}(\pi_2) \cup \mathrm{conv}(\pi_3) \cup \mathrm{conv}(\pi_4) \cup \mathrm{conv}(\pi_5) \cup \mathrm{conv}(\pi_6)$$

A. $\omega_5^{pd} < \omega_5$

B. Pairs (blue) form the spectral boundary. Triples (magenta) and higher combinations (green) are strictly inside. Increasing the number of combinations increases central clustering.

C. **No exceptions** to result (B) were found in order 6. This trend may persist for $n \geq 7$.



Conclusion

Explored the spectral behavior of pentadiagonal doubly stochastic matrices constructed from convex combinations of permutation matrices. By analyzing eigenvalue distributions across different orders($n = 3$ to 6), a consistent pattern was observed:

For $n \geq 4$, we observed:

- $\omega_n^{pd} < \omega_n$.
- Pairwise combinations generate the **outer boundary** of the spectral region.
- Triples and higher combinations produce eigenvalues **strictly inside**, clustering toward the center.
- Notably, exceptions occurred at $n = 4$ and 5, where some triples extended beyond the boundary formed by pairs. However, this anomaly vanished at $n = 6$, suggesting a stable and exhaustive boundary for higher orders.

These results reveal a geometric-algebraic regularity linking permutation structure to eigenvalue distribution and provide a foundation for future exploration in complex stochastic systems.

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THANK YOU!