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Solving 2D-Poisson equation using modified cubic B-spline differential quadrature method

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ABSTRACT

In this study a modified cubic B-spline differential quadrature method (MCBDQM) is used to solve the two dimensional Poisson equation. Using the cubic B-spline functions, explicit expressions of weighting coefficients for approximation of derivatives are obtained. Examples of two dimensional Poisson equation under Dirichlet and mixed boundary conditions are studied using the method. Comparisons between the results of the method, the results of using Shu's general approach and results of other methods are presented. Good agreement with Shu's general approach and with the other methods is reached in case of Dirichlet boundary condition. The differential quadrature based on modified cubic B-splines is found to be an efficient method to solve the two dimensional Poisson equation.

1. Introduction

Solving the partial differential equation (PDE) using the differential quadrature (DQ) method is receiving special interest as it requires less memory and computational time. The method uses fewer number of grid points for results with an acceptable error [1]. The method also has advantages over other techniques in case of formulating non-uniform grids.

The DQ method depends mainly on converting the PDE into a system of ordinary differential equations (ODEs) or a system of algebraic equations by means of the weighting coefficients. By the use of proper basis function the weighting coefficients obtained for each derivative separately. There are many approaches to calculate the weighting coefficients like Bellman's 1st and 2nd approaches, Quan and Chang's approach, Shu's general approach and spline differential quadrature [1,1–5].

Shu solved the two dimensional Poisson equation with the help of the Shu's general approach to get the weighting coefficients and the used knots are selected according to harmonic discretization method [1]. Mittal and Jain solved the nonlinear Burgers' equation numerically using collocation of modified cubic B-Spline method [6]. Shukla, Tamsir and Srivastava simulated the two dimensional sine-gordon equation by cubic B-Spline DQ method [7]. Zhong solved a fourth order PDE by a quintic B-Spline differential quadrature [4]. Nguyen, Karčiauskas, and Peters compared the numerical results of solving Poisson equation by seven different methods where the DQ was not included among them [8]. Faruk Civan and Sliepcevich used Bellman's 1st approach to solve the two dimensional Poisson equation numerically [9]. Shu and Chew presented generalized DQ method equivalent to highest order finite difference [10]. Arora and Kumar Singh presented the modified cubic B-spline DQ method and used it to solve the Burgers' equation [11]. A spline DQ technique based on quasi-interpolation was introduced by Barrera, González, Ibáñez and Ibáñez [12]. Rajashekhar Reddy, Sridhar Reddy and Ramana Murthy solved Laplace equation by Bi-cubic B-spline collocation solution [13]. Chapra and Canale solved Poisson equation and other important equations by conventional methods in their book Numerical Methods for Engineers (McGraw-Hill, 1988) [14]. DQ method was used to study and solve certain practical applications like Fitzhugh-Nagumo equation [15,16]. Mittal and Rohila solved reaction-diffusion systems by means of MCBDQM [17]. Other DQ techniques were presented to solve some important practical applications efficiently [18,19]. Wu and Zhang introduced quartic B-splines based finite element method to solve the PDE [20]. Two dimensional and three dimensional Poisson equations were solved by different DQ and other numerical methods under mixed and Dirichlet boundary conditions [21–24]. Jia, Zhang, Xu, Zhuang

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and Rabczuk reproduced Kernel Triangular B-spline-based FEM for Solving PDEs [25]. Nguyen-Thanh, Nguyen-Xuan, Bordas and Rabczuk made Isogeometric analysis using polynomial splines over hierarchical T-meshes for two-dimensional elastic solids alternative to standard finite elements due to its flexibility in handling complex geometries [26]. Also Nguyen-Thanh, Zhou, Zhuang, Areias, Nguyen-Xuan, Bazilevs and Rabczuk made Isogeometric analysis of large-deformation thin shells using RHT-splines for multiple-patch coupling [27].

According to the knowledge of the authors, the two dimensional Poisson equation was not solved by the cubic B-spline DQ method until recently [21]. The aim of this paper is to solve the two dimensional Poisson equation by the cubic B-spline DQ method however with introducing the modification proposed by [11]. The results of this method (modified cubic B-Spline based differential quadrature method (MCBDQM)) was compared to the exact solution and with results available in the literature.

2. Methods description

We consider the well-known two dimensional Poisson equation:

$$\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}=f(x,y),\quad x,y\in[a,b] \tag{1}$$

Under Dirichlet and mixed boundary conditions.

The DQ method is a function approximation to the spatial partial derivatives in the PDE by a sum of weights multiplied by the functional values at the discrete knots over domain [a, b]. The 1st and 2nd order partial derivatives with respect to x over domain [a, b] are defined as:

$$U_x(x_i)=\sum_{j=1}^{N}W_{i,j}^{(1)}U(x_j)\quad \text{for } i=1,\cdots,N \tag{2}$$

$$U_{xx}(x_i)=\sum_{j=1}^{N}W_{i,j}^{(2)}U(x_j)\quad \text{for } i=1,\cdots,N \tag{3}$$

where $W_{i,j}^{(1)}$ and $W_{i,j}^{(2)}$ are the weighting coefficients of the 1st and 2nd order partial derivatives respectively [1].

The weighting coefficients for N discrete knots $(x_1,\ x_2,\ x_3\cdots x_N)$ can be calculated using different methods. In this paper we used the modified cubic-B-spline method. In addition we present the general Shu as a method for obtaining the weighting coefficients.

2.1. Modified cubic B-Spline differential quadrature method

In the spline based differential quadrature we use uniform discretization: $x_1 < x_2 < \ldots < x_n$ such that $x_{i+1}-x_i=h$. and the basis functions of the Cubic B-Spline DQ is:

$$\Psi_j(x)=\frac{1}{h^3}\begin{cases}(x-x_{j-2})^3 & x\in(x_{j-2},x_{j-1})\\ (x-x_{j-2})^3-4(x-x_{j-1})^3 & x\in(x_{j-1},x_j)\\ (x_{j+2}-x)^3-4(x_{j+1}-x)^3 & x\in(x_j,x_{j+1})\\ (x_{j+2}-x)^3 & x\in(x_{j+1},x_{j+2})\\ 0 & \textit{Otherwise},\end{cases} \tag{4}$$

where $\{\Psi_0,\ \Psi_1,\cdots,\Psi_{N+1}\}$ are basis on the region [a,b] [11].

Mittal and Jain presented a modification in cubic B-splines basis functions for the problems with Dirichlet boundary conditions to get a diagonally dominant system of equations [6].

In order to get a system of equations that can be represented by a diagonally dominant matrix, the basis function is modified as follows [11]:

$$\left.\begin{aligned}\psi_1(x)&=\Psi_1(x)+2\Psi_0(x)\\ \psi_2(x)&=\Psi_2(x)-\Psi_0(x)\\ \psi_j(x)&=\Psi_j(x)\quad \textit{for } j=3,\cdots,N-2\\ \psi_{N-1}(x)&=\Psi_{N-1}(x)-\Psi_{N+1}(x)\\ \psi_N(x)&=\Psi_N(x)+2\Psi_{N+1}(x)\end{aligned}\right\} \tag{5}$$

The derivative of the basis functions are as follows:

$$\Psi'_j(x)=\frac{1}{h^3}\begin{cases}3(x-x_{j-2})^2 & x\in(x_{j-2},x_{j-1})\\ 3(x-x_{j-2})^2-12(x-x_{j-1})^2 & x\in(x_{j-1},x_j)\\ -3(x_{j+2}-x)^2+12(x_{j+1}-x)^2 & x\in(x_j,x_{j+1})\\ -3(x_{j+2}-x)^2 & x\in(x_{j+1},x_{j+2})\\ 0 & \textit{Otherwise},\end{cases} \tag{6}$$

So the derivative of the modified basis functions can be calculated from the following relations:

$$\left.\begin{aligned}\psi'_1(x)&=\Psi'_1(x)+2\Psi'_0(x)\\ \psi'_2(x)&=\Psi'_2(x)-\Psi'_0(x)\\ \psi'_j(x)&=\Psi'_j(x)\quad \textit{for } j=3,\cdots,N-2\\ \psi'_{N-1}(x)&=\Psi'_{N-1}(x)-\Psi'_{N+1}(x)\\ \psi'_N(x)&=\Psi'_N(x)+2\Psi'_{N+1}(x)\end{aligned}\right\} \tag{7}$$

Taking into consideration that:

$$\psi'_k(x_i)=\sum_{j=1}^{N}W_{i,j}^{(1)}\psi_k(x_j)\quad \text{for } i=1,2,\cdots,N;\ k=1,2,\cdots,N \tag{8}$$

So this leads to the following system of equations:

$$\begin{bmatrix}6 & 1 & & & & & \\ 0 & 4 & 1 & & & & \\ & 1 & 4 & 1 & & & \\ & & 1 & 4 & \ddots & & \\ & & & 1 & \ddots & 1 & \\ & & & & \ddots & 4 & 1 \\ & & & & & 1 & 4 & 0\\ & & & & & & 1 & 6\end{bmatrix}\begin{bmatrix}W_{i,1}^{(1)}\\ W_{i,2}^{(1)}\\ W_{i,3}^{(1)}\\ \vdots\\ \vdots\\ W_{i,N-1}^{(1)}\\ W_{i,N}^{(1)}\end{bmatrix}=\begin{bmatrix}\psi'_1(x_i)\\ \psi'_2(x_i)\\ \psi'_3(x_i)\\ \vdots\\ \vdots\\ \psi'_{N-1}(x_i)\\ \psi'_N(x_i)\end{bmatrix} \tag{9}$$

Solving the system of Eq. (9) leads to the weighting coefficients of the 1st derivatives.

To get the weighting coefficients of 2nd order derivatives:

$$W_{i,j}^{(2)}=2W_{i,j}^{(1)}\left(W_{i,i}^{(1)}-\frac{1}{X_i-X_j}\right)\quad \text{for } i,j=1,2,\cdots,N,\ \text{and } i\neq j \tag{10}$$

and

$$W_{i,i}^{(2)}=-\sum_{j=1,j\neq i}^{N}W_{i,j}^{(2)} \tag{11}$$

2.2. Shu's general approach differential quadrature

The basis functions of the Shu's General Approach DQ [10] are:

$$r_k=\frac{H(x)}{(x-x_k).H^1(x_k)},\quad k=1,2,\cdots,N \tag{12}$$

where

$$H(x)=\prod_{i=1}^{N}(x-x_i) \tag{13}$$

$$H^1(x_k) = \prod_{i=1, i\neq k}^{N} (x_k - x_i) \tag{14}$$

So the weighting coefficients of the 1st derivatives can be calculated from:

$$W_{i,j}^{(1)} = \frac{H^1(x_i)}{(x_i - x_j).H^1(x_j)} \quad \text{for } j\neq i \tag{15}$$

$$W_{i,i}^{(1)} = -\sum\nolimits_{j=1, j\neq i}^{N} W_{i,j}^{(1)} \tag{16}$$

The weighting coefficients of the 2nd derivatives can be calculated using Eqs. (10) and (11).

3. Numerical examples and analysis of results

Different examples for Poisson equation were solved using the MCBDQM and for specific examples using Shu's method also. The method was first applied to Poisson's equation with Dirichlet boundary conditions and then after proved successful, one example with mixed boundary conditions is tested.

As mentioned previously, the results of the MCBDQM method are compared to results of the exact solution in addition to results of different methods/techniques including Shu's method. The examples are presented in different research papers [1,13,14,20–24].

Considering N discrete knots in the x direction and M knots in the y direction, the differential quadrature method is applied to the first four examples, this leads to a set of DQ algebraic equations as follows [1,9]:

$$\sum_{k=1}^{N} W_{i,k}^{(2)} Z_{k,j} + \sum_{k=1}^{M} W_{j,k}^{(2)} Z_{i,k} = f(x,y) \tag{17}$$

which can be rewritten as:

$$\sum_{k=2}^{N-1} W_{i,k}^{(2)} Z_{k,j} + \sum_{k=2}^{M-1} W_{j,k}^{(2)} Z_{i,k} = f(x_i, y_j) - W_{i,1} Z_{1,j} - W_{i,N} Z_{N,j} - W_{j,1} Z_{i,1} - W_{j,M} Z_{i,M} \tag{18}$$

Solving this set of algebraic equations (18) taking into consideration the boundary conditions associated with each example, the numerical solution of the problem is obtained.

The accuracy of the numerical result is measured by four types of errors:

$$\text{root mean square error (RMS)} = L_2 = \sqrt{\frac{\sum_{i,j}(Z_{\text{numerical}} - Z_{\text{Exact}})^2}{MN}} \tag{19}$$

$$\text{Max of the absolute error (MAE)} = L_\infty = \max_{i,j} |Z_{\text{numerical}} - Z_{\text{Exact}}| \tag{20}$$

$$\text{Average value of errors (AVE)} = L = \sum\nolimits_{i,j} (Z_{\text{numerical}} - Z_{\text{Exact}})/(M-2)(N-2) \tag{21}$$

$$\text{Relative Error (RE)} = |Z_{\text{numerical}} - Z_{\text{reference}}|/Z_{\text{reference}} \tag{22}$$

Example 1. The following 2D-Poisson equation (23) is solved over the region $0 \le x, y \le 1$:

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 10(x-1)cos(5y) - 25(x-1)(y-1)\sin(5y), 0 \le x, y \le 1 \tag{23}$$

The equation is subjected to the following boundary conditions:

$$z(0,y) = (1-y)\sin(5y), \quad 0 < y < \tag{24}$$

$$z(1,y) = 0, \quad 0 \le y \le \tag{25}$$

$$z(x,0) = 0, \quad 0 \le x \le 1 \tag{26}$$

$$z(x,1) = 0, \quad 0 \le x \le 1 \tag{27}$$

The exact solution is given in [20] as:

$$z(x,y) = (1-x)(1-y)\sin(5y) \tag{28}$$

Taking mesh of size (5 × 5), the weighting coefficients are calculated by MCBDQM and Shu's general approach differential quadrature method. Tables 1 and 2 show that for 5 × 5 grid points MCBDQM gives better results than Shu's general approach.

Example. 2:. The following 2D-Poisson equation (29) is solved over the region $0 \le x, y \le 1$:

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0, \quad 0 \le x, y \le 1 \tag{29}$$

The equation is subjected to the following boundary conditions:

$$z(0,y) = 0, \quad 0 \le y \le 1 \tag{30}$$

$$z(1,y) = 0, \quad 0 < y < 1 \tag{31}$$

$$z(x,0) = 0, \quad 0 \le x \le 1 \tag{32}$$

$$z(x,1) = 25, \quad 0 \le x \le 1 \tag{33}$$

The exact solution is given in [14] as:

$$z(x,y) = \frac{50}{\pi} \sum_{n=0}^{\infty} \frac{\left(-1^{n+1} + 1\right).\sin(n\pi x).\sinh(n\pi y)}{n.\sinh(n\pi)} \tag{34}$$

Taking the same parameters taken in the 1st example, the weighting coefficients for the same mesh size (5 × 5) are calculated by the same approaches, the MCBDQM and Shu's general approach differential quadrature method.

Tables 3 and 4 show that for 5 × 5 grid points MCBDQM gives better results than Shu's general approach.

Table 1
Absolute error of the results by MCBDQM and Shu's general approach (Mesh Size: 5 × 5).

Point	Exact solution	MCBDQM	Shu's general approach
(0.25,0.25)	0.5338	0.0092	0.0623
(0.25,0.5)	0.2244	0.0045	0.0122
(0.25,0.75)	−0.1072	0.0196	0.0201
(0.5,0.25)	0.3559	0.0078	0.0578
(0.5,0.5)	0.1496	0.0051	0.0146
(0.5,0.75)	−0.0714	0.0184	0.0160
(0.75,0.25)	0.1779	0.0040	0.0323
(0.75,0.5)	0.0748	0.0031	0.0092
(0.75,0.75)	−0.0357	0.0101	0.0081

Table 2
Different types of errors by MCBDQM and Shu's general approach (Mesh Size: 5 × 5).

Type of error	Root mean square error (L_2)	Max of the absolute errors (L_∞)	Average value of errors(L)
MCBDQM	0.0323	0.0196	0.0091
Shu's general approach	0.0971	0.0623	0.0258

Table 3
Absolute error of the results by MCBDQM and Shu's general approach (Mesh Size: 5x5).

Point	Exact solution	MCBDQM	Shu's general approach
(0.25,0.25)	1.6973	0.0103	0.0204
(0.25,0.5)	4.5462	0.0484	0.0734
(0.25,0.75)	10.7928	0.0003	0.0304
(0.5,0.25)	2.3825	0.0133	0.0101
(0.5,0.5)	6.2440	0.0060	0.0060
(0.5,0.75)	13.5042	0.0625	0.1360
(0.75,0.25)	1.6973	0.0103	0.0204
(0.75,0.5)	4.5462	0.0484	0.0734
(0.75,0.75)	10.7928	0.0003	0.0304

Table 4
Different types of errors by MCBDQM and Shu's general approach (Mesh Size: 5x5).

Type of error	Root mean square error (L_2)	Max of the absolute errors (L_∞)	Average value of errors (L)
MCBDQM	0.095017	0.062541	0.022238
Shu's general approach	0.179131	0.13597	0.044509

Example. 3:. The following 2D-Poisson equation (35) is solved over the region $0 \le x, y \le 1$ under the boundary conditions below.

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0, \quad 0 \le x, y \le 1 \tag{35}$$

$$z(0, y) = 25, \quad 0 \le y \le 1 \tag{36}$$

$$z(1, y) = 50, \quad 0 < y < 1 \tag{37}$$

$$z(x, 0) = 100, \quad 0 \le x \le 1 \tag{38}$$

$$z(x, 1) = 75, \quad 0 < x < 1 \tag{39}$$

Considering 5×5 grid points, the results of the proposed method are compared to the results of Shu's general approach and Bi-cubic B-spline collocation method by computing the relative error with respect to the finite element method as shown in Table 5 [13]. Table 5 shows that the modified cubic B-spline method gives the best numerical results for example 3.

Example 4. The following 2D-Poisson equation (40) is solved over the region $0 \le x, y \le 1$ under the boundary conditions below.

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \sin(\pi x).\sin(\pi y) \quad 0 \le x, y \le 1 \tag{40}$$

$$z(x, y) = 0, \quad \textit{at all boundaries} \tag{41}$$

The exact solution shown in Fig. 1 is given in [21–24] as:

$$z(x, y) = -\frac{\sin(\pi x).\sin(\pi y)}{2\pi^2} \quad 0 \le x, y \le 1 \tag{42}$$

Considering 15×15 grid points, comparison between the results of the proposed method and the results of using different methods is shown in Table 6 [21–24].

Table 5
Relative error of the results of different numerical methods.

Point	Finite element 81 × 81 grid points	Bi-cubic B-spline collocation method	MCBDQM	Shu's general approach
(0.25,0.25)	62.489	0.007	0.000176031	0.000176031
(0.25,0.5)	50.131	0.0071	0.004209969	0.007170334
(0.25,0.75)	53.391	0.0054	0.000453258	0.000699396
(0.5,0.25)	74.859	0.0281	0.002685715	0.004668189
(0.5,0.5)	62.496	0.0271	6.40041E−05	6.40041E−05
(0.5,0.75)	63.732	0.027	0.002299177	0.003107791
(0.75,0.25)	71.594	0.0305	0.000128501	0.000731087
(0.75,0.5)	61.263	0.0613	0.002473453	0.003314656
(0.75,0.75)	62.502	0.04	3.1999E-05	3.1999E-05

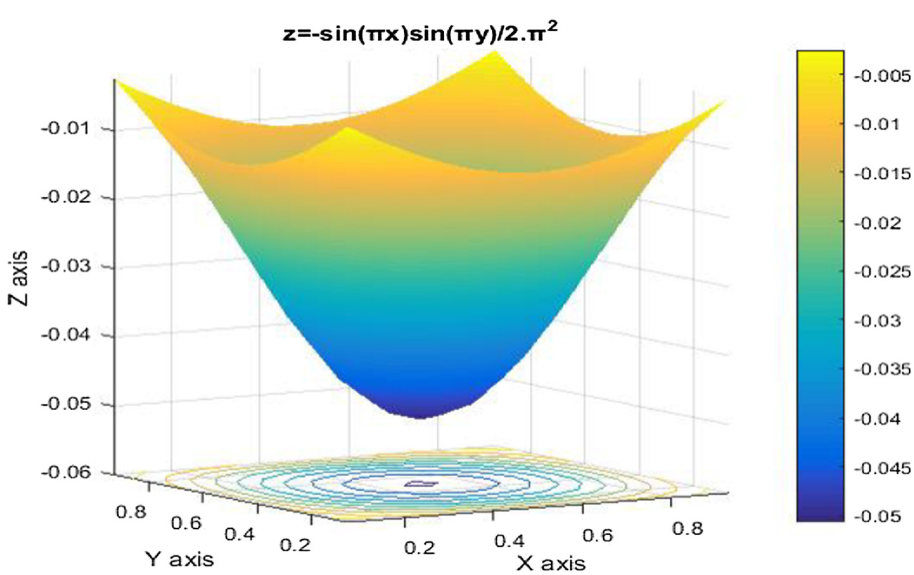

Fig. 1. The exact solution of Example 4.

Table 6
Maximum absolute error according to the method used for Example 4.

Method	MCBDQM (The proposed method)	Spline-based DQM [21]	Haar wavelet method [22]	spectral collocation method based on Haar wavelets [23]	based on quartic B-spline bases[24]
MAE(L_∞)	2.11E−05	1.62E−04	3.08E−04	3.08E−04	2.71E−07

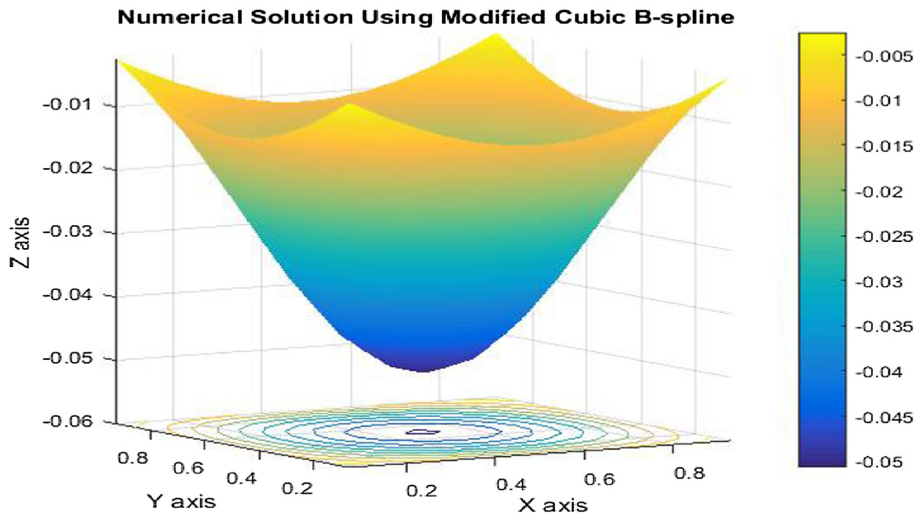

Fig. 2. Numerical solution of Example 4 using modified cubic B-spline with 15 × 15 grid points.

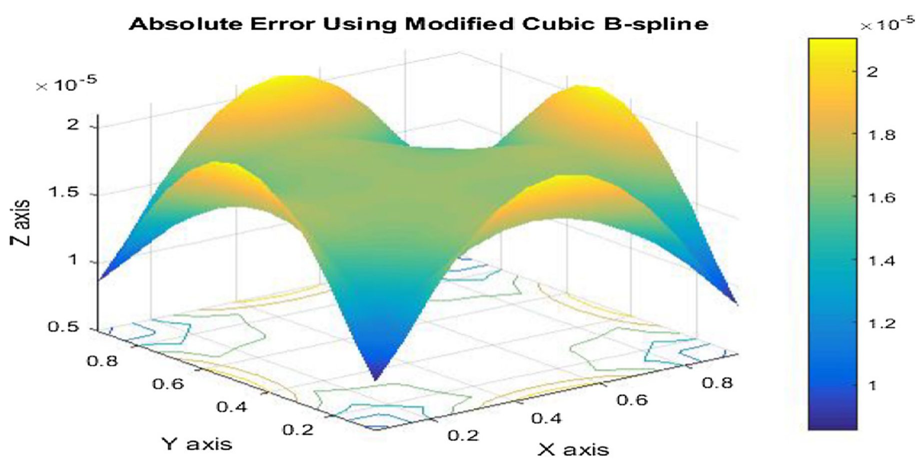

Fig. 3. Absolute error profile of Example 4 using modified cubic B-spline with 15 × 15 grid points.

Table 6 clarifies that the proposed method gives acceptable results for example 4. Numerical error and absolute error are demonstrated in Figs. 2 and 3.

Example 5. The following 2D-Poisson equation (43) is solved over the region $0 \le x, y \le 1$ under the boundary conditions below.

$$\frac{\partial^2 z}{\partial x^2} + \beta^2 \frac{\partial^2 z}{\partial y^2} = 0, \quad 0 \le x, y \le 1 \tag{43}$$

$$z(x, 0) = 0, \quad 0 \le x \le 1 \tag{44}$$

$$z(x, 1) = \sin(\pi x/2), \quad 0 \le x \le 1 \tag{45}$$

$$z(0, y) = 0, \quad 0 < y < 1 \tag{46}$$

$$z_x(1, y) = 0, \quad 0 < y < 1 \tag{47}$$

where β represents the aspect ratio [1,9].

Considering N discrete knots in the x direction and M knots in the y direction, and applying the differential quadrature method to Eqs. (43) and (47), leads to the following set of DQ algebraic equations:

$$\sum_{k=1}^{N} W_{i,k}^{(2)} Z_{k,j} + \beta^2 \sum_{k=1}^{M} W_{j,k}^{(2)} Z_{i,k} = 0 \tag{48}$$

which can be rewritten as:

$$\sum_{k=2}^{N-1} W_{i,k}^{(2)} Z_{k,j} + \beta^2 \sum_{k=2}^{M-1} W_{j,k}^{(2)} Z_{i,k} = -W_{i,1} Z_{1,j} - W_{i,N} Z_{N,j} - \beta^2 W_{j,1} Z_{i,1} - \beta^2 W_{j,M} Z_{i,M} \tag{49}$$

Table 7
Comparison between the obtained results.

Mesh size	Method	5×5	9×9	11×11	15×15
Root mean square error (L_2)	Shu's	1.4356E−07	2.1489E−08	2.2288E−08	2.3245E−08
	MCBDQ	4.9607E−06	4.0724E−06	3.7402E−06	3.2474E−06
Max of the absolute error (L_∞)	Shu's	4.2322E−07	4.9496E−08	5.0222E−08	5.0645E−08
	MCBDQ	1.6849E−05	1.7728E−05	1.7992E−05	1.8070E−05
Average value of errors (L)	Shu's	1.9303E−07	2.4350E−08	2.3682E−08	2.2928E−08
	MCBDQ	5.7533E−06	2.2566E−06	1.7131E−06	1.1451E−06

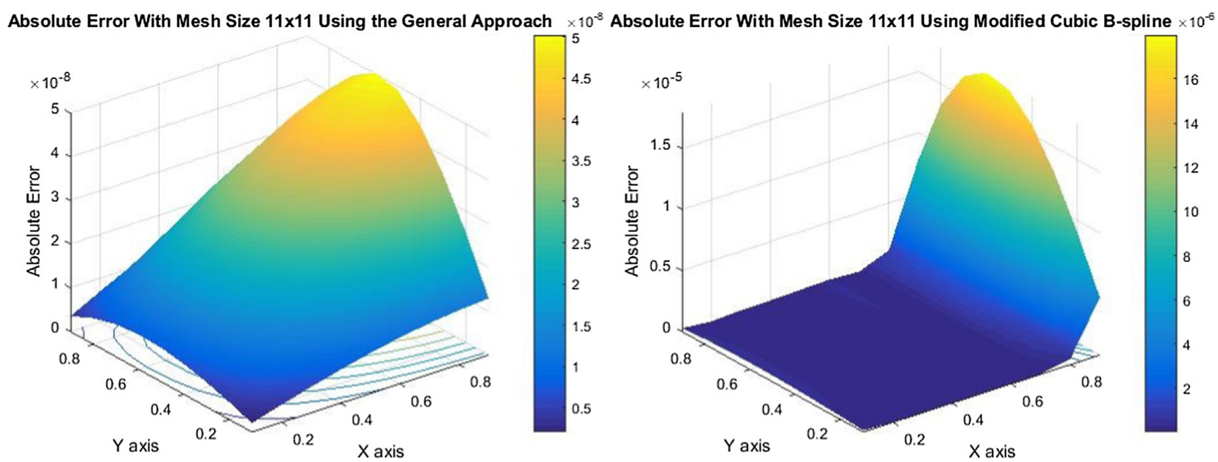

Fig. 4. The absolute error at each grid point of mesh size (11×11).

The Neumann boundary condition (47) can be written as follow [1,9]:

$$\sum_{k=1}^{N} W_{N,k}^{(1)} Z_{k,j} = 0 \quad \text{for} j = 2, 3, \cdots, M-1 \tag{50}$$

Solving the set of algebraic equations (49) and (50), taking into consideration the rest of boundary conditions (44), (45) and (46), the numerical solution of the required problem is developed.

The exact solution of the considered problem is given in [1] as:

$$z(x, y) = \sinh\left(\frac{\pi y}{2\beta}\right) . \frac{\sin\left(\frac{\pi x}{2}\right)}{\sinh\left(\frac{\pi}{2\beta}\right)} \tag{51}$$

For different mesh sizes taken: (5×5, 9×9, 11×11 and 15×15), the weighting coefficients are calculated by MCBDQM and Shu's general approach differential quadrature method (see Table 7).

Samples of the numerical results using constant aspect ratio equals to 50 are tabulated and graphs are plotted in Figs. 3 and 4.

The results show that relatively high error at the right boundary ($x = 1$). This boundary is the only boundary in the form of Neumann boundary condition where the remaining boundaries are Dirichlet. As this method was originally devised for handling the differential equations with Dirichlet boundary conditions, higher error values were expected. Further investigations and studies should be performed to improve the results of the method in case of mixed boundary conditions (see Fig. 5).

4. Conclusion

The modified cubic B-spline differential quadrature method (MCBDQM) is used to solve two dimensional Poisson equation with both Dirichlet and mixed boundary conditions. As per the authors'

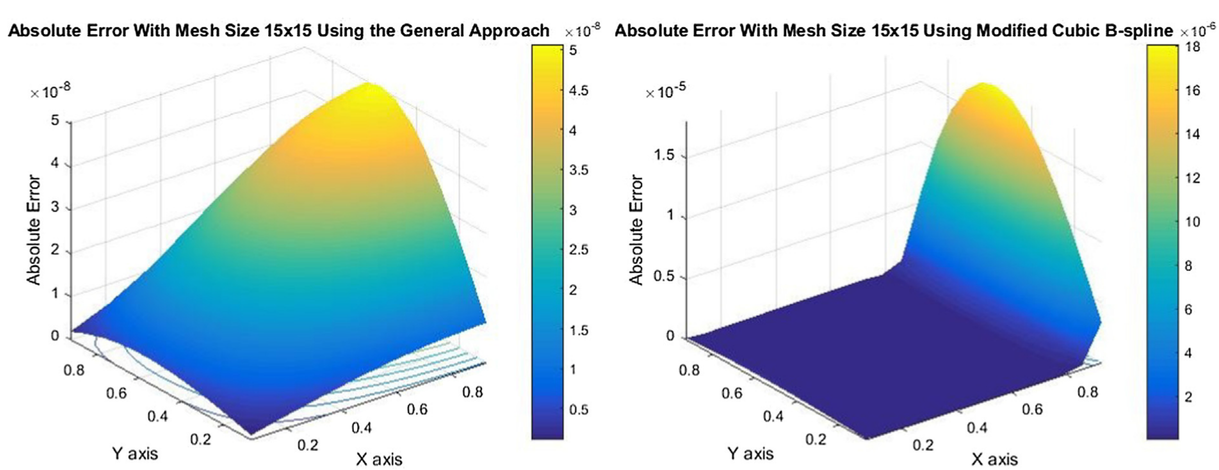

Fig. 5. The absolute error at each grid point of mesh size (15×15).

knowledge this method was not used previously for solving two dimensional Poisson equation. However it is worth noting that the cubic B-spline DQ method was recently proposed to solve Poisson equation [21]. The study did not apply the modification applied in this study.

The modified cubic B-spline method was applied on the Poisson equation with Dirichlet boundary conditions and show good results with small relative error. These results encouraged the authors to extend its use to solve the Poisson equation with mixed boundary conditions. The results for the mixed boundary problem shows relatively small absolute, RMS and average error. The error is relatively high near the Neumann boundary at $(x = 1)$ which indicates that the MCBDQM is better to be applied for Poisson equation with Dirichlet boundary conditions than with mixed boundary conditions.

Further studies are needed to formulate a modification for solving the Poisson's equation with mixed boundary conditions efficiently. In addition, it is further advised to apply the method presented in [24] however with adding the modification addressed in the current study as it is expected to show more improved results.

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