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Unified and extended trigonometric B-spline DQM for the numerical treatment of three-dimensional wave equations

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ABSTRACT

In this work, we propose a new differential quadrature method (DQM) based on the Unified and Extended Trigonometric B-spline (UETB-spline) functions for the numerical approximation of 3D wave equations. The UETB -spline functions are modified and then utilized in DQM to compute the weighting coefficients of spatial derivatives. After inverting the obtained coefficients, we acquire systems of ODEs that are solved by the SSP-$RK_{5,4}$ scheme. Some numerical examples are considered to examine the accuracy and proficiency of the proposed approach. The obtained solutions give excellent agreement with analytical solutions. Moreover, the analysis for the rate of convergence (ROC) is performed. Computational complexity exhibits that the approach is not complex in the view of the computational cost. All calculations have been done using Dev-C++ 6.3 version and graphs have been plotted using MATLAB 2015b software.

1. Introduction

The highly accurate numerical methods, to approximate wave equations, are a demanding task, particularly, when 3D computational domain is involved. These methods help for eliminating the errors by the computation of the underlying mathematical as well as physical models. So, it is essential to propose suitable accurate and stable numerical algorithms to handle 3D computational domain. First, an operator compact implicit scheme has been developed in [1,2] to approximate wave equations. Baccouch and Temimi [3] presented and analyzed a new time–space ultra-weak discontinuous Galerkin (DG) finite element method (FEM) for the 1D wave equation. The proposed method has been used to discretize both space as well as temporal derivatives. Mohanty et al. [4] have established 4th order, in both space and time, three-level implicit difference methods to integrate 3D nonlinear hyperbolic equations. In [5], the authors have extended their method to integrate quasi-linear hyperbolic equations of order 2. They have proved that the methods presented in [4,5] are conditionally stable. Based on off-step discretization. Mohanty and Gopal [6] proposed a high accurate numerical technique of order (h^4, t^2) to solve 3D nonlinear wave equation.

Titarev and Toro [7] presented the extension of the ADER methodology to multi-dimensional nonlinear hyperbolic systems which is the 4th order accurate in both space as well as time. Shukla et al. [8] and Tamsir et al. [9] presented exponential modified cubic B-spline (Expo-MCB-DQM) and hyperbolic B-spline DQM to approximate 3D nonlinear wave equations, respectively. Zhang et al. [10] purported an improved EFG technique, while Shivanian [11] proposed element free Galerkin (EFG) and Meshless local Petrov-Galerkin (MLPG) techniques to approximate aforementioned equations. To approximate the variable coefficients viscous wave equation, Wang et al. [12] proposed a one-step mesh free method which is based on RBF. Yağmurlu and Karakaş [27,28] used trigonometric cubic B-spline collocation method to approximate equal width and modified equal width equations, respectively.

The DQM can be referred to [13–19] to solve the differential equations. The DQM based on various functions viz. sinc functions, Lagrange interpolation, cubic B-spline, exponential, extended and trigonometric B-spline functions is presented in [20–25]. The authors of [29,30] proposed DQM based on quintic B-spline functions to find numerical solutions of the modified Korteweg-de Vries (MKdV) and complex MKdV equations. Başhan [31] proposed a DQM based on fifth-order quintic B-spline functions to approximate the modified Kawahara equation. Başhan [32] obtained solitary wave, undular-bore and wave-maker solutions of the cubic, quartic and quintic generalized equal width wave equation. In [33], the author used a hybrid DQM to study the regularized long-wave equation (RLW). The authors of [34] proposed a mixed

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Table 1

The values of $\widehat{B}_{i,4}$ and $\widehat{B}_{i,4}^{'}$ at x_i.

	x_{i-2}	x_{i-1}	x_i	x_{i+1}	x_{i+2}
$\widehat{B}_{i,4}(x)$	0	$\widehat{\lambda}_1$	$\widehat{\lambda}_2$	$\widehat{\lambda}_1$	0
$\widehat{B}_{i,4}^{'}(x)$	0	$\widehat{\lambda}_3$	0	$\widehat{\lambda}_4$	0

Table 2

Comparison the proposed method and methods available in literature with $h = 0.1$ and $\Delta t = 0.01$ at different values of t.

t	UETB-spline DQM	EFG method [11]	MLPG method [11]	Expo-MCB-DQM [8]
0.1	3.849e-07	1.3615e-01	6.3890e-04	1.013e-06
0.2	3.339e-07	1.1087e-01	1.6210e-03	1.666e-06
0.3	3.023e-07	9.0318e-02	2.0694e-03	1.725e-06
0.4	2.791e-07	7.5552e-02	1.8515e-03	1.498e-06
0.5	2.460e-07	6.1133e-02	1.4064e-03	1.196e-06
0.6	2.001e-07	5.0760e-02	1.1202e-03	9.059e-07
0.7	1.490e-07	4.2763e-02	8.7629e-04	7.061e-07
0.8	1.053e-08	3.4162e-02	5.7628e-04	5.566e-07
0.9	8.039e-08	3.0724e-02	7.7790e-04	4.758e-07
1.0	7.367e-08	2.5621e-02	8.6382e-04	4.417e-07

Table 3

The rate of convergence in space variable with $\eta = 0.0000001$ and $\Delta t = 0.01$ at $t = 1$ for Example 1.

Grid size	RMS errors	ROC
$6 \times 6 \times 6$	4.08E−07	–
$12 \times 12 \times 12$	5.62E−08	2.86
$16 \times 16 \times 16$	2.90E−08	2.30
$20 \times 20 \times 20$	1.83E−08	2.06
$24 \times 24 \times 24$	1.30E−08	1.87

method based on low-order modified cubic B-spline functions to approximate the solitary wave, undular bore and boundary-forced problems for RLW equation.

Moreover, Mirzaee and Alipour [35,36] used cubic B-spline collocation method to solve the stochastic fractional integro-differential equation and stochastic quadratic integral equations, respectively. In [37], they used Quintic B-spline collocation method to approximate n-dimensional stochastic Itô-Volterra integral equations. In [38], a method based on bicubic B-spline interpolation, Gauss-Legendre quadrature formula, and two-dimensional Itô approximation is used to approximate 2D weakly singular stochastic integral equation. Mirzaee et al. [39–42] introduced composite methods based on radial basis functions and finite difference to approximate stochastic Sine-Gordon, fractional Sine-Gordon and coupled fractional Sine-Gordon, and two-dimensional fractional sine-Gordon equations, respectively.

B-splines have numerous remarkable properties and are the basis of the vector space generated by the spline with minimal support and with respect to a certain degree of smoothness and domain partitions. Existing splines, such as cubic B-splines, hyperbolic B-splines and usual polynomial B-splines, are all special circumstances of UE-splines. Recently, Wang and Fang [43] proposed polynomial, hyperbolic, and trigonometric UE-spline functions. Alinia and Zarebnia [44] presented a technique based on a new kind of tension B-spline function to approximate Burgers-Huxley equation.

Influenced by the above-mentioned work, we propose an UETB-spline DQM to solve the 3D wave equations, numerically. The primary persistence of this paper is to exhibit an accurate, efficiently easy, and stable method to solve the hyperbolic PDEs. We take into consideration,

$$\frac{\partial^2 v}{\partial t^2} + (\alpha_1 - \beta f_1(v))\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} - \alpha_2 v + f_2(x, y, z, t), \ \ (x, y, z) \in \Omega, \tag{1}$$

subject to

$$v(x, y, z, 0) = \phi_1 \ and \ \frac{\partial v}{\partial t}(x, y, z, 0) = \phi_2 \tag{2}$$

and with boundary conditions

$$\left.\begin{array}{l} v(0, y, z, t) = \varphi_1, \ v(1, y, z, t) = \varphi_2, \ v(x, 0, z, t) = \varphi_3 \\ v(x, 1, z, t) = \varphi_4, \ v(x, y, 0, t) = \varphi_5, \ v(x, y, 1, t) = \varphi_6 \end{array}\right\} \tag{3}$$

where $\Omega = \{(x, y, z) : x, y, z \in [0, 1]\}$ is the problem domain. The functions f_1, f_2, ϕ_1, ϕ_2 and $\varphi_i, \ i = 1$ to 6 are known while $v = v(x, y, z, t)$

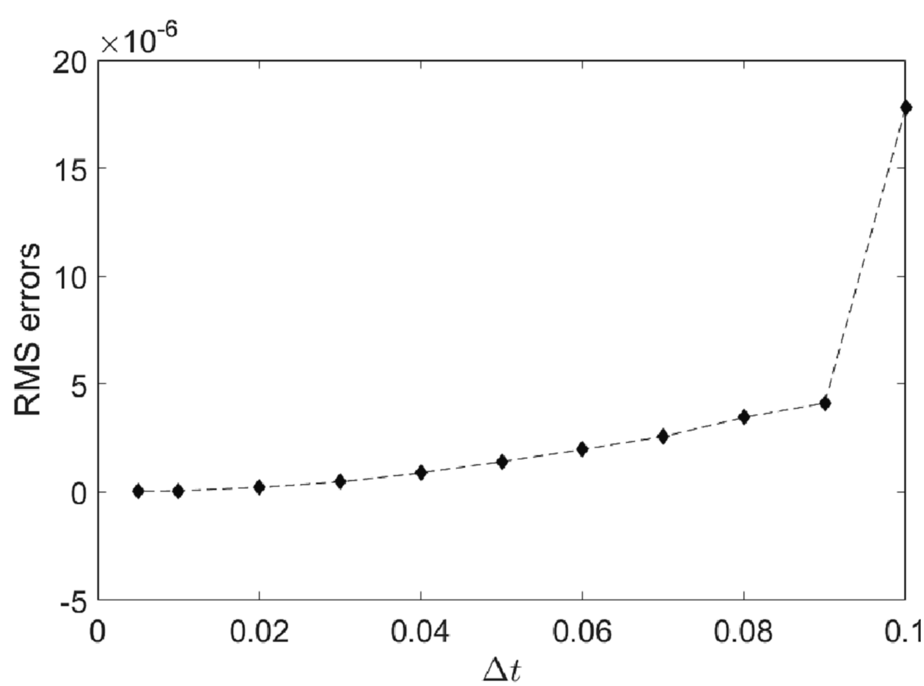

Fig. 1. The RMS errors vs Δt with Grid size $12 \times 12 \times 12$ and $\eta = 0.0000001$ at $t = 1$.

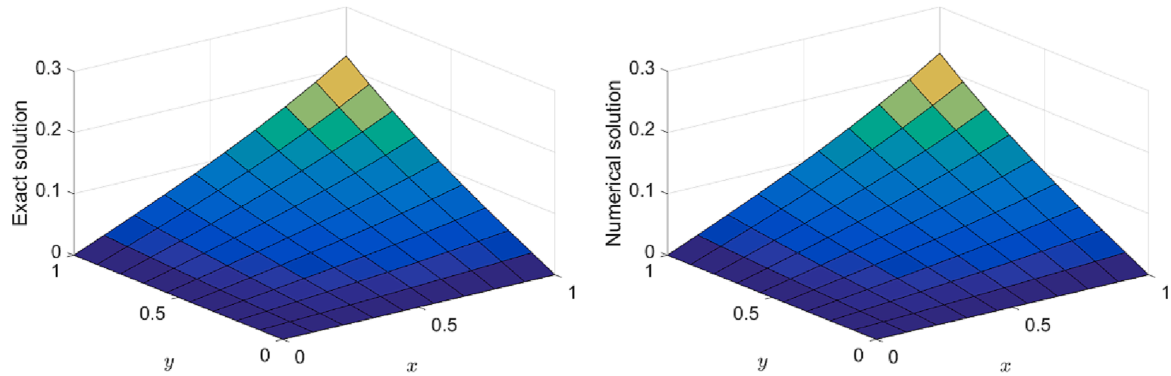

Fig. 2. Surface plots of exact and numerical solutions with $\eta = 0.0000001$, $h = 0.1$ and $\Delta t = 0.01$ at $t = 1$ for Example 1.

is to be resolved. The expressions α_1, α_2, β are real constants.

2. UETB-spline basis functions and discretization of the problem

This section studies the problem (1)–(3) when α_1, α_2, β, f_1, f_2 are given and v is to be estimated. First, we discuss the UE-spline and cubic UETB-spline basis functions.

First, we divide $\Omega = \{(x, y, z) : x, y, z \in [0, 1]\}$ into meshes of equal length as $h_x = \frac{1}{N_x - 1}$, $h_y = \frac{1}{N_y - 1}$ and $h_z = \frac{1}{N_z - 1}$. Throughout the paper, we represent $v_{ijk} = v\left(x_i, y_j, z_k, t\right)$, $i = 1, 2, \ldots, N_x$, $j = 1, 2, \ldots, N_y$ and $k = 1, 2, \ldots, N_z$.

In DQM, v_{xx} , v_{yy} and v_{zz} are approximated as follows:

$$\frac{\partial^l v_{ijk}}{\partial x^l} = \sum_{s=1}^{N_x} a_{is}^{(l)} v_{rjk}, \quad i = 1, 2, \ldots, N_x \tag{4}$$

$$\frac{\partial^l v_{ijk}}{\partial y^l} = \sum_{s=1}^{N_y} b_{js}^{(l)} v_{isk}, \quad j = 1, 2, \ldots, N_y \tag{5}$$

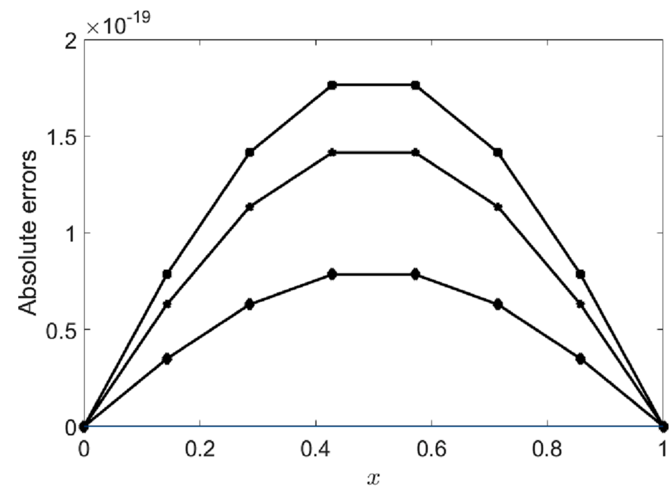

Fig. 3. The 2D error plots for $\eta = 5$ and $\Delta t = 0.01$ at $t = 1$ for Example 2.

Table 4
Comparison of the proposed method and methods available in literature with $h = 0.1$ and $\Delta t = 0.01$ at various times.

t	UETB-spline DQM	EFG method [11]	MLPG method [11]	Expo-MCB-DQM [8]
0.1	5.596e-06	1.653265e + 00	2.777931e-03	5.667e-06
0.2	9.426e-06	1.005632e + 00	8.477482e-03	9.701e-06
0.3	1.169e-05	9.786343e-01	1.352534e-02	1.231e-05
0.4	1.338e-05	7.456237e-01	1.583307e-02	1.512e-05
0.5	1.678e-05	6.213675e-01	1.550351e-02	1.824e-05
0.6	2.044e-05	4.354421e-01	1.367202e-02	2.222e-05
0.7	2.364e-05	1.345213e-01	1.052578e-02	2.570e-05
0.8	2.636e-05	9.973233e-02	6.216680e-03	2.866e-05
0.9	2.867e-05	7.132423e-02	5.280951e-03	3.117e-05
1.0	3.061e-05	6.124572e-02	2.276681e-03	3.329e-05

Table 5
The rate of convergence with $\eta = 5$ and $\Delta t = 0.01$ at $t = 1$ for Example 2.

Space grid size	RMS errors	ROC	CPU Time (sec)
$6 \times 6 \times 6$	5.22E-04	–	0.031
$12 \times 12 \times 12$	2.07E-05	4.65	0.281
$16 \times 16 \times 16$	5.49E-06	4.61	0.734
$20 \times 20 \times 20$	1.96E-06	4.62	1.68
$24 \times 24 \times 24$	8.39E-07	4.65	3.18
$28 \times 28 \times 28$	4.08E-07	4.68	5.63
$32 \times 32 \times 32$	2.17E-07	4.73	9.48
$36 \times 36 \times 36$	1.24E-07	4.75	15.32

Table 6
The comparison of the proposed method and methods available in the literature with $h = 0.1$ and $\Delta t = 0.01$ at various times.

t	UETB-spline DQM	EFG method [11]	MLPG method [11]	Expo-MCB-DQM [8]
0.1	2.212e-07	1.436e-03	8.903e-05	2.887e-07
0.2	8.399e-07	3.867e-03	9.910e-05	1.257e-06
0.3	1.442e-06	5.034e-03	1.590e-04	2.944e-06
0.4	2.108e-06	7.655e-03	3.777e-04	5.348e-06
0.5	2.975e-06	9.200e-03	4.781e-04	8.787e-06
0.6	4.159e-06	1.034e-02	6.416e-04	1.361e-05
0.7	5.688e-06	3.280e-02	8.809e-04	2.029e-05
0.8	9.749e-06	5.233e-02	9.279e-04	2.918e-05
0.9	1.240e-05	6.072e-02	1.059e-04	4.049e-05
1.0	1.565e-05	7.545e-02	1.529e-03	5.432e-05

Table 7
The rate of convergence with $\eta = 5$ and $\Delta t = 0.01$ at $t = 1$ for Example 3.

Space grid size	RMS errors	ROC	CPU Time (Sec)
$6 \times 6 \times 6$	8.36e-04	–	0.029
$12 \times 12 \times 12$	3.39e-05	4.62	0.276
$16 \times 16 \times 16$	8.99e-06	4.61	0.722
$20 \times 20 \times 20$	3.20e-06	4.63	1.71
$24 \times 24 \times 24$	1.37e-06	4.65	3.27
$28 \times 28 \times 28$	6.68e-07	4.66	5.47
$32 \times 32 \times 32$	3.56e-07	4.69	9.17
$36 \times 36 \times 36$	2.03e-07	4.77	14.9

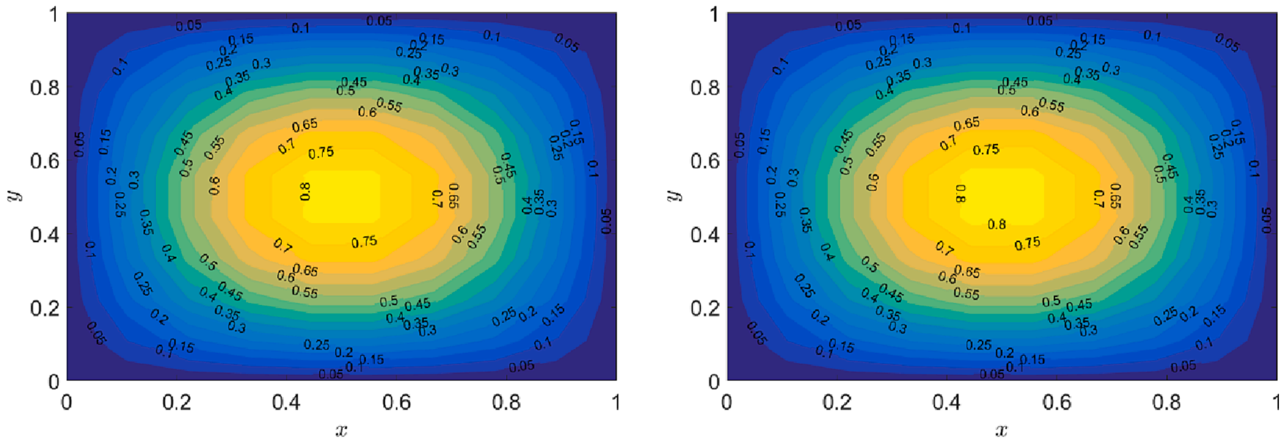

Fig. 4. Contour plots of numerical (left) and exact solutions (right) with $\eta = 5$, $h = 0.1$ and $\Delta t = 0.01$ at $t = 1$ for Example 3.

$$\frac{\partial^l v_{ijk}}{\partial z^l} = \sum_{s=1}^{N_z} c_{ks}^{(l)} v_{ijs}, \quad k = 1, 2, \ldots, N_z \tag{6}$$

where $a_{ij}^{(l)}$, $b_{js}^{(l)}$, $c_{ks}^{(l)}$ are the weighting coefficients relating to $\frac{\partial^l v_{ijk}}{\partial x^l}$, $\frac{\partial^l v_{ijk}}{\partial y^l}$ and $\frac{\partial^l v_{ijk}}{\partial z^l}$ at time t, respectively.

The unified and extended spline (UE-spline) functions of order $r = 2$ are given as [43–46]:

$$\widehat{B}_{i,2}(x) = \frac{1}{\sin(\eta h)} \begin{cases} \sin(\eta(x - x_{i-2})), & [x_{i-2}, x_{i-1}), \\ \sin(\eta(x_i - x)), & [x_{i-1}, x_i), \\ 0, & \text{otherwise}, \end{cases} \tag{7}$$

$$\eta = \sqrt{\overline{\eta}_i \overline{\eta}_j} \in \mathbb{R}$$

The recursive formula for $r \geqslant 3$ is given as [43–46]:

$$\widehat{B}_{i,r}(x) = \int_{-\infty}^{x} \left(\delta_{i,r-1} \widehat{B}_{i,r-1}(\xi) - \delta_{i+1,r-1} \widehat{B}_{i+1,r-1}(\xi) \right) d\xi \tag{8}$$

$$\delta_{i,r} = \left(\int_{-\infty}^{\infty} \widehat{B}_{i,r}(\xi) d\xi \right)^{-1}, i = 0, \pm 1, \ldots$$

Also, for $\widehat{B}_{i,m}(\xi) = 0$, we take

$$\int_{-\infty}^{x} \delta_{i,r} \widehat{B}_{i,r}(\xi) d\xi = \begin{cases} 1, & x \geqslant x_{i+r-2}, \\ 0, & x < x_{i+r-2}. \end{cases} \tag{9}$$

The cubic UETB-spline functions of order $r = 4$ for $0 < \eta \leqslant \pi / h$ are given as [46]:

where $\widehat{\gamma} = \frac{1}{2h(1 - \cos(\eta h))}$. The cubic UETB-spline basis functions $\{\widehat{B}_{0,4}, \widehat{B}_{1,4}, \ldots, \widehat{B}_{N_x,4}, \widehat{B}_{N_x+1,4}\}$ form a basis over $\Omega = \{(\xi, \zeta, \varsigma) : \xi, \zeta, \varsigma \in [0, 1]\}$. The values of cubic UETB-spline functions together with derivatives at the knots are given in Table 1, where $\widehat{\lambda}_1 = \widehat{\gamma}(h - \frac{\sin(\eta h)}{\eta})$, $\widehat{\lambda}_2 = \widehat{\gamma}(-2h\cos(\eta h) + \frac{2\sin(\eta h)}{\eta})$, $\widehat{\lambda}_3 = \widehat{\gamma}(\cos(\eta h) - 1)$ and $\widehat{\lambda}_4 = -\widehat{\gamma}(\cos(\eta h) - 1)$.

Now, we modify the cubic UETB-spline functions as follows:

$$\left. \begin{array}{c} \widetilde{B}_{1,4}(x) = \widehat{B}_{1,4}(x) + 2\widehat{B}_{0,4}(x) \\ \widetilde{B}_{2,4}(x) = \widehat{B}_{2,4}(x) - \widehat{B}_{0,4}(x) \\ \widetilde{B}_{i,4}(x) = \widehat{B}_{i,4}(x) \text{for } i = 3, 4, \ldots, N_x - 3 \\ \widetilde{B}_{N_x-2,4}(x) = \widehat{B}_{N_x-2,4}(x) - \widehat{B}_{N_x,4}(x) \\ \widetilde{B}_{N_x,4}(x) = \widehat{B}_{N_x,4}(x) + 2\widehat{B}_{N_x+1,4}(x) \end{array} \right\}, \tag{11}$$

where, the modified set of cubic UETB-spline functions $\left\{ \widetilde{B}_{i,4}(x), i = 1, 2, \ldots, N_x \right\}$ form a basis over Ω. Now, for $l = 1$, we have

$$\widetilde{B}_{k,4}^{'}(x_i) = \sum_{j=1}^{N_x - 1} a_{ij}^{(1)} \widetilde{B}_{k,4}(x_i), \text{ for } i, k = 1, 2, \ldots, N_x, \tag{12}$$

Using the Table 1 and Eqs. (10) and (11) in (12), we get a system of linear equations of order $(N_x - 1) \times (N_x - 1)$ as follows:

$$\widehat{B}_{i,4}(x) = \widehat{\gamma} \begin{cases} (x - x_{i-2}) - \dfrac{\sin(\eta(x_{i-2} - x))}{\eta}, & x \in [x_{i-2}, x_{i-1}) \\ (x_i - x) + (2x_{i-1} - x)\cos(\eta h) - \dfrac{2\sin(\eta(x_{i-1} - x)) + \sin(\eta(x_i - x))}{\eta}, & x \in [x_{i-1}, x_i) \\ (x - x_i) - (2x_{i+1} - x)\cos(\eta h) + \dfrac{\sin(\eta(x_i - x)) + 2\sin(\eta(x_{i+1} - x))}{\eta}, & x \in [x_i, x_{i+1}) \\ (x_{i+2} - x) - \dfrac{\sin(\eta(x_{i+2} - x))}{\eta}, & x \in [x_{i+1}, x_{i+2}) \\ 0, & \text{otherwise} \end{cases} \tag{10}$$

$$\begin{bmatrix} 2\widehat{\lambda}_1+\widehat{\lambda}_2 & \widehat{\lambda}_1 & 0 & & & & \\ 0 & \widehat{\lambda}_2 & \widehat{\lambda}_1 & & & & \\ & \widehat{\lambda}_1 & \widehat{\lambda}_2 & \widehat{\lambda}_1 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & \widehat{\lambda}_1 & \widehat{\lambda}_2 & \widehat{\lambda}_1 & \\ & & & & \widehat{\lambda}_1 & \widehat{\lambda}_2 & 0 \\ & & & & 0 & \widehat{\lambda}_1 & 2\widehat{\lambda}_1+\widehat{\lambda}_2 \end{bmatrix} \begin{bmatrix} a_{i1}^{(1)} \\ a_{i2}^{(1)} \\ a_{i3}^{(1)} \\ \vdots \\ \vdots \\ a_{i(N_x-1)}^{(1)} \\ a_{iN_x}^{(1)} \end{bmatrix} = \begin{bmatrix} \widetilde{B}_{1,4}^{'}(x_i) \\ \widetilde{B}_{2,4}^{'}(x_i) \\ \widetilde{B}_{3,4}^{'}(x_i) \\ \vdots \\ \vdots \\ \widetilde{B}_{N_x-1,4}^{'}(x_i) \\ \widetilde{B}_{N_x,4}^{'}(x_i) \end{bmatrix} \tag{13}$$

$$\widetilde{B}_{1,4}^{'}(x_1)=2\widehat{\lambda}_4,\ \widetilde{B}_{2,4}^{'}(x_1)=2\widehat{\lambda}_3,\ \widetilde{B}_{j,4}^{'}(x_1)=0,\ \text{for } j=3,4,\dots,N_x,$$

$$\widetilde{B}_{1,4}^{'}(x_2)=\widehat{\lambda}_4,\ \widetilde{B}_{2,4}^{'}(x_2)=0,\ \widetilde{B}_{3,4}^{'}(x_1)=\widehat{\lambda}_3,\ \widetilde{B}_{4,4}^{'}(x_1)=0,\dots,\ \widetilde{B}_{N_\xi,4}^{'}(x_1)=0,$$
$$\vdots$$
$$\widetilde{B}_{1,4}^{'}(x_2)=0,\dots,\ \widetilde{B}_{(N_x-3),4}^{'}(x_2)=0, \widetilde{B}_{(N_x-2),4}^{'}(x_1)=\widehat{\lambda}_4, \widetilde{B}_{(N_x-1),4}^{'}(x_1)=0,\ \widetilde{B}_{N_x,4}^{'}(x_1)=\widehat{\lambda}_3,$$

$$\widetilde{B}_{j,4}^{'}(x_2)=0,\ \text{for } j=1,\dots,N_x-2,\ \text{and } \widetilde{B}_{(N_x-1),4}^{'}(x_1)=2\widehat{\lambda}_4,\ \widetilde{B}_{N_x,4}^{'}(x_1)=2\widehat{\lambda}_3$$

The above system is solved by using Thomas algorithm. Now, we compute the weighting coefficients $a_{jk}^{(2)}$ utilizing the formula given by Shu [19] as:

$$\begin{cases} a_{jk}^{(2)}=2\left(a_{jk}^{(1)}a_{jj}^{(1)}-\dfrac{a_{jk}^{(1)}}{x_j-x_k}\right),\ \text{for } k\neq j \text{ and } j=1,2,\dots,N_x-1, \\ a_{jj}^{(2)}=-\displaystyle\sum_{k=1,k\neq j}^{N_x}a_{jk}^{(2)},\ \text{for } k=j. \end{cases} \tag{14}$$

Now, using $v_t=\widehat{v}$, $v_{tt}=\widehat{v}_t$, and replacing the approximated v_{xx}, v_{yy} and v_{zz} by cubic UETB-spline DQM, Eqs. (1) and (2) becomes

$$\frac{dv_{ijk}}{dt}=\widehat{v}_{ijk} \tag{15}$$

and

$$\frac{d\widehat{v}_{ijk}}{dt}=\sum_{s=1}^{N_x-1}a_{is}^{(2)}v_{sjk}+\sum_{s=1}^{N_y-1}b_{js}^{(2)}v_{isk}+\sum_{s=1}^{N_z-1}c_{ks}^{(2)}v_{ijs}-\big(\alpha_1-\beta f_1(v_{ijk})\big)\widehat{v}_{ijk}-\alpha_2 v_{ijk}+\overline{P}_{ijk} \tag{16}$$

$i=0,1,\cdots,N_x,\ j=0,1,\cdots,N_y,\ k=0,1,\cdots,N_z$

where

$$\overline{P}_{ijk}=a_{i1}^{(2)}v_{1jk}+a_{iN_x}^{(2)}v_{N_xjk}+b_{j1}^{(2)}v_{i1k}+b_{jN_y}^{(2)}v_{iN_yk}+c_{k1}^{(2)}v_{ij1}+c_{kN_z}^{(2)}v_{ijN_z}+f_2$$

Finally, the SSP-RK$_{5,4}$ scheme [26] is employed to solve the above-mentioned systems.

3. Numerical results and discussion

This section considers three examples to verify the accuracy and efficiency of the proposed approach.

Example 1. We study Eq. (1) for $\alpha_1=\alpha_2=2$ and $\beta=0$ with the exact solution $v(x,y,z,t)=e^{-2t}\sinh x\,\sinh y\,\sinh z$.

The f_2 is defined correspondingly to the exact solution and the problem (1). We choose $h=0.05$, 0.1 and $\Delta t=0.01$. Table 2 compares the existing results with the results obtained by the proposed method for $h=0.05$, 0.1 and $\Delta t=0.01$ with $\eta=0.0000001$ at different t while Table 3 shows the ROC with respect to space variable at $t=1$. It is observed that the results obtained by the proposed method are more precise than given in [8,11] and are second order accurate in space. Fig. 1 shows the RMS errors with respect to Δt with Grid size $12\times12\times12$ and $\eta=0.0000001$ at $t=1$. It is seen that the RMS errors decreasing as Δt decreases. Fig. 2 demonstrates the behaviors of exact and numerical solutions with $\eta=0.0000001$, $h=0.1$ and $\Delta t=0.01$ at $t=1$ for Example 1.

Example 2. In this example, we study the Equation (1) with $\alpha_1=\beta=\widehat{\theta}$, $\alpha_2=0$, $f_1(v)=v^2$ including the exact solution $v(x,y,z,t)=\sin\pi x\,\sin\pi y\,\sin\pi z\,e^{-\widehat{\theta}t}$.

The function f_2 is defined correspondingly to the exact solution and the problem (1). For numerical calculation, the parameters $h=0.05$, 0.1, $\Delta t=0.01$, and$\widehat{\theta}=3$ are chosen. Table 4 compares of the obtained results with those obtained by Expo-MCB-DQM [8], EFG [11] and MLPG [11] methods by means of RMS errors. It can be seen that the proposed method provides improved results than the results produced in [16,17]. Table 5 shows the ROC together with CPU time (in sec) with respect to space variable for $\eta=5$ and $\Delta t=0.01$ at $t=1$. It is studied that the proposed method is fourth order accurate in space for this example. Fig. 3 shows the two-dimensional absolute errors plot for $\eta=5$ and $\Delta t=0.01$ at $t=1$. This figure guarantees that the accuracy of the proposed method is ($\approx10^{-19}$ absolute errors) for this example.

Example 3. Finally, the Equation (1) is considered with $\alpha_1=\alpha_2=0$, $\beta=-2$ and $f_1(v)=v$ with the exact solution $v(x,y,z,t)=\sin\pi x\,\sin\pi y\,\sin\pi z\,\sin t$.

The function f_2 is chosen appropriately. We take parameters: $h=0.05$ and 0.1, $\Delta t=0.01$. The results obtained by proposed method are compared with the results obtained by Expo-MCB-DQM [8], EFG [11] and MLPG [11] methods by means of RMS errors and indicated in Table 6. It can be seen that the proposed method provides more accurate results than those of [16,17]. Table 7 shows the ROC together with CPU time (in sec) with respect to space variable for $\eta=5$ and $\Delta t=0.01$ at $t=1$. It is observed that the method is fourth order accurate in space for this example also. Fig. 4 shows the contour plots of numerical as well as exact solutions with $\eta=5$, $h=0.1$ and $\Delta t=0.01$ at $t=1$.

3.1. Computational complexity

It is easy to go over the computational complexity of the UETB-spline DQM for 3D wave equations as there are two major portions in the computational effort. The first point is to estimate the weighting coefficients by employing Thomas algorithm which utilizes $3N$ multiplications, $3N$ subtractions, and $(2N+1)$ divisions. So, Thomas algorithm utilizes $(8N+1)$ elementary arithmetic operations. Consequently, this algorithm takes $O(n)$ operations.

The other point is the SSP-RK$_{5,4}$ scheme which is used for solving the system of ODEs. In the view of the computational cost, the cost of the SSP-RK$_{5,4}$ scheme, and the usual ODE's solvers is identical. Hence, the proposed method is not more complex in the view of the computational cost.

4. Conclusions

In this work, we approximated 3D wave equations by means of a new DQM based on the UETB-spline functions composed with the SSP-RK$_{5,4}$ scheme. The numerical examples illustrate that the proposed method delivers more accurate results than those discoursed in [8,11]. The investigation for the rate of convergence (ROC) is also executed which shows that the method is second order accurate for Example 1 while fourth order accurate for Examples 2 and 3. The change in CPU times and RMS errors for Examples 1, 2, and 3 are illustrated in Tables 2, 4, and 6 respectively over the time interval [0, 1] for various grid points. One can observe that CPU times increase as the grid points increase for all the problems. The proposed method produces exceedingly accurate results and is proficient in managing huge-scale handling problems. Also, this method is economically easy-to-employ for the solution of hyperbolic PDEs. Furthermore, the complexity analysis demonstrates that the method is not complex in the view of the computational cost.

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Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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