Edexcel A Level Maths: Pure

Your notes

6.1 Exponential & Logarithms

Contents

6.1.1 Exponential Functions

Your notes

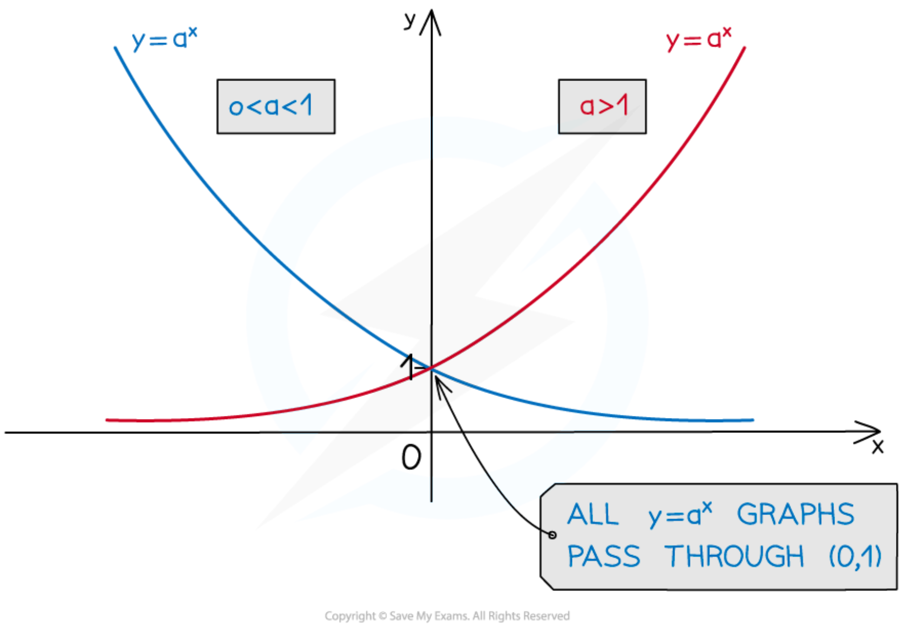
Exponential Functions

Exponential functions

- Exponential functions of the form $y = a^x$ with $a > 0$ are considered at A level

Exponential graphs

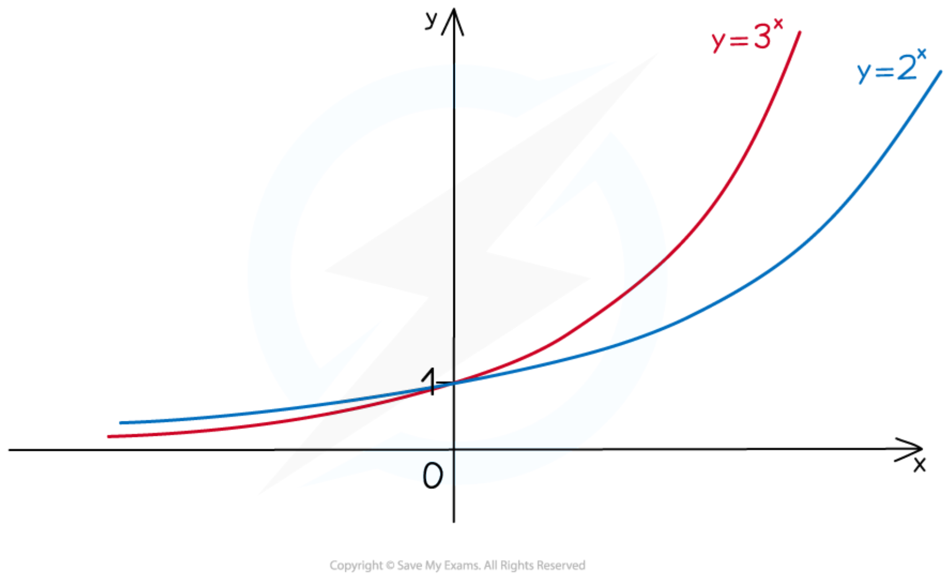
- All graphs of the form $y = a^x$ will pass through (0, 1) because $a^0 = 1$
- The **x**-axis is an **asymptote**



a > 1

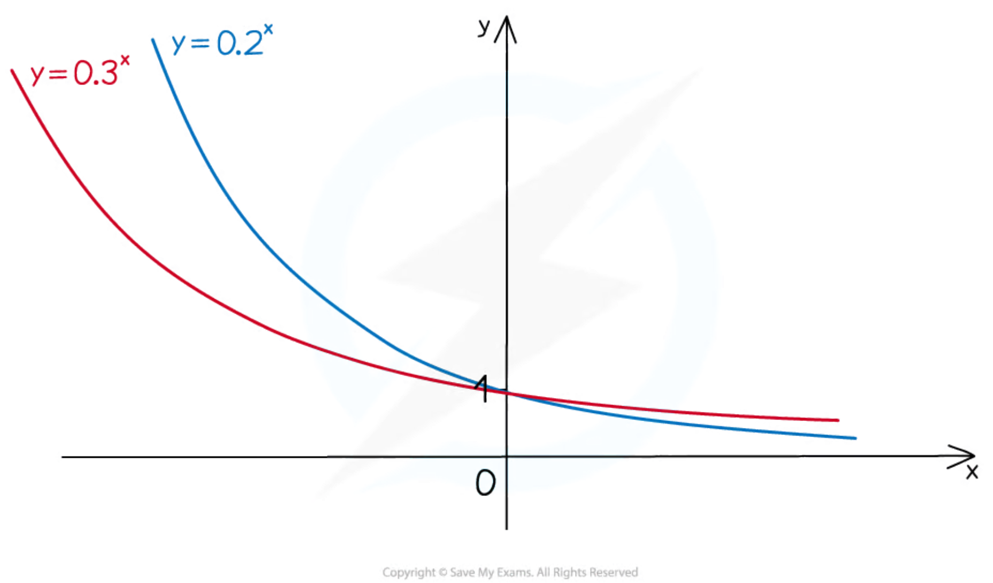
- Where $x < 0$ the higher value of **a** is the "lower" graph
- Where $x > 0$ the higher value of **a** is the "higher" graph
- $a > 1$ is **exponential growth** (see Exponential Growth & Decay)

Your notes



0 < a < 1

- Where **x < 0** the higher value of **a** is the "lower" graph
- Where **x > 0** the higher value of **a** is the "higher" graph
- **0 < a < 1** is **exponential decay**



What about when a = 1?

- You may like to think about why **a = 1** is **not** considered... If a = 1, $y = 1^x = 1$ for all values of x

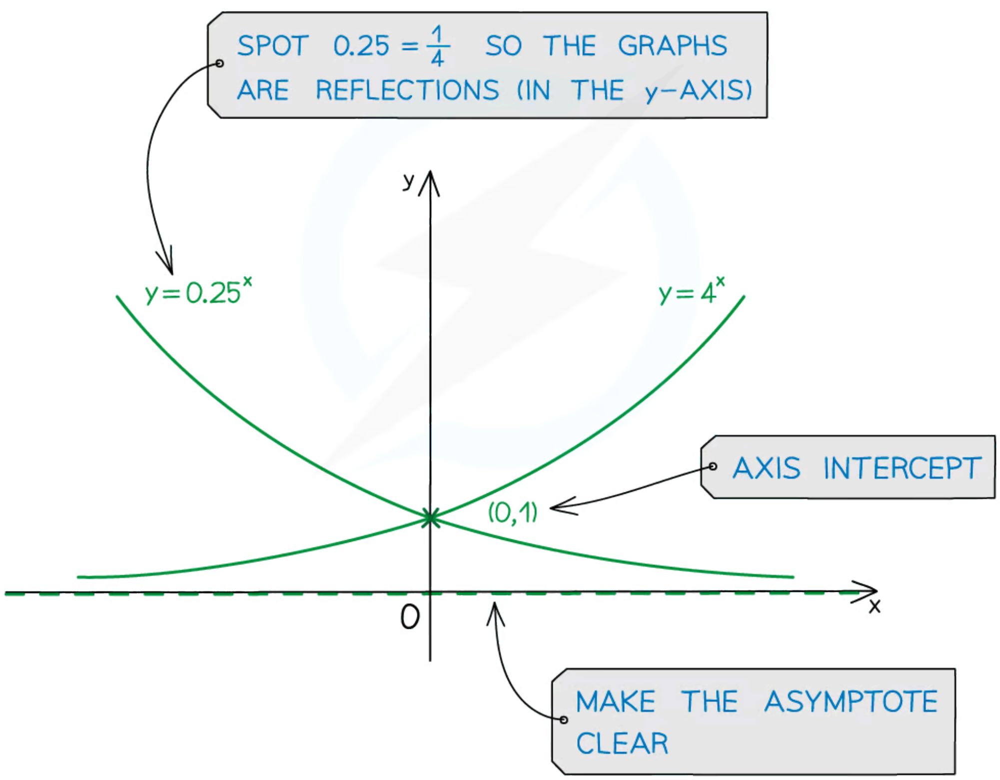
Your notes

Worked example

On the same diagram sketch the graphs of

$y = 4^x$ and $y = 0.25^x$.

Mark any points where the graphs cross the coordinate axes and make any asymptotes clear.

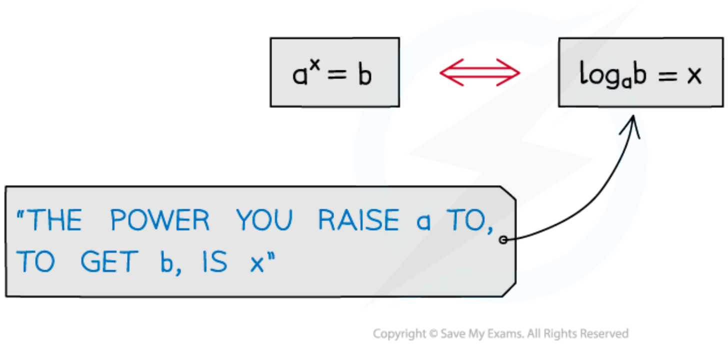


6.1.2 Logarithmic Functions

Your notes

Logarithmic Functions

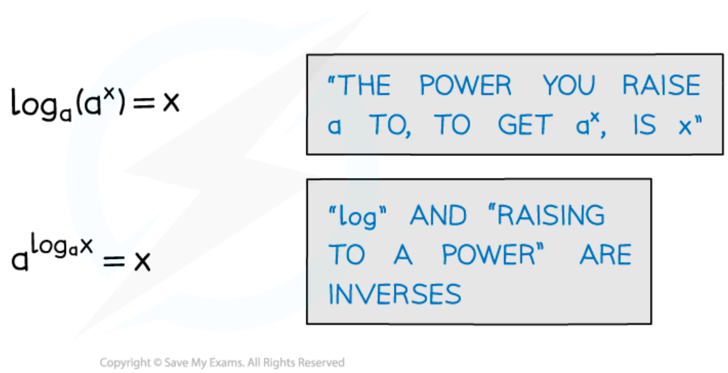
Logarithmic functions



- $a = b^x$ and $\log_b a = x$ are equivalent statements
- $a > 0$
- **b** is called the base
- Every time you write a logarithm statement say to yourself what it means
  - $\log_3 81 = 4$
    "the power you raise 3 to, to get 81, is 4"
  - $\log_p q = r$
    "the power you raise **p** to, to get **q**, is **r**"

Logarithm rules

- A logarithm is the inverse of raising to a power so we can use rules to simplify logarithmic functions



How do I use logarithms?

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e.g. WRITE $\dfrac{2^{\log_3 81} + 3^{\log_{10} 1000}}{5^{\log_5 5}}$ IN THE FORM $\dfrac{a}{b}$

WHERE a AND b ARE INTEGERS TO BE FOUND.

"THE POWER YOU RAISE 3 TO, TO GET 81, IS 4"

$2^{\log_3 81} = 2^4 = 16$

"THE POWER YOU RAISE 10 TO, TO GET 1000, IS 3"

$3^{\log_{10} 1000} = 3^3 = 27$

$\log_5 5 = \log_5 5^1 = 1$

$5^{\log_5 5} = 5^1 = 5$

$$\therefore \frac{2^{\log_3 81} + 3^{\log_{10} 1000}}{5^{\log_5 5}} = \frac{16 + 27}{5}$$

$$= \frac{43}{5}$$

- Recognising the rules of logarithms allows expressions to be simplified e.g. WITHOUT USING A CALCULATOR, EVALUATE $\log_2 64\sqrt{2}$

RECOGNISE 64 AND $\sqrt{2}$ ARE POWERS OF 2

LET $x = \log_2 64\sqrt{2}$

"THE POWER YOU RAISE 2 TO, TO GET $64\sqrt{2}$, IS x"

$2^x = 64\sqrt{2}$

REWRITE THE STATEMENT

$2^x = 2^6 \times 2^{\frac{1}{2}}$

$2^x = 2^{6+\frac{1}{2}}$

USING LAWS OF INDICES

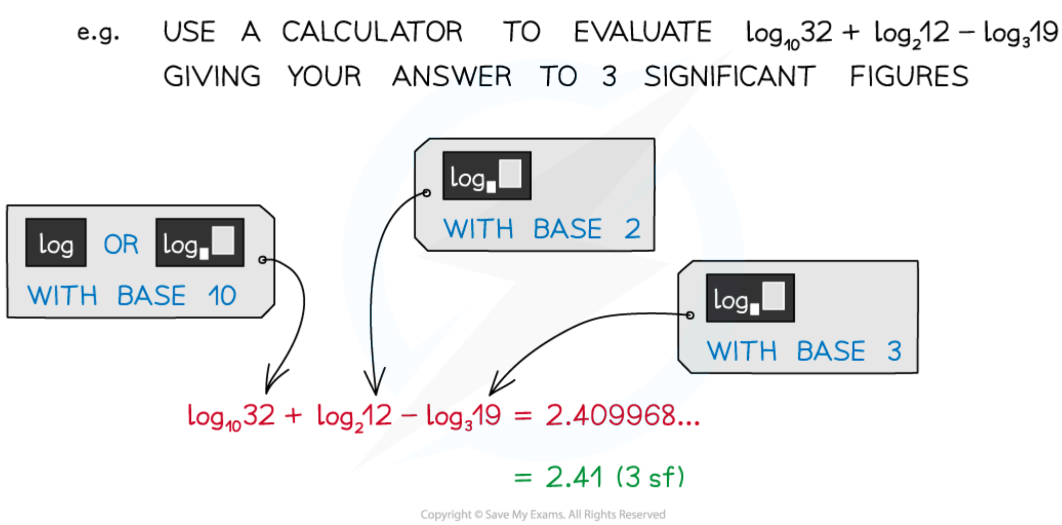
$2^x = 2^{\frac{13}{2}}$

$x = \frac{13}{2}$

$\therefore \log_2 64\sqrt{2} = \frac{13}{2}$

- Recognition of common powers helps in simple cases
  - Powers of 2: $2^0 = 1$, $2^1 = 2$, $2^2 = 4$, $2^3 = 8$, $2^4 = 16$, ...
  - Powers of 3: $3^0 = 1$, $3^1 = 3$, $3^2 = 9$, $3^3 = 27$, $3^4 = 81$, ...
  - The first few powers of 4, 5 and 10 should also be familiar

  For more awkward cases a calculator is needed



- Calculators can have, possibly, three different logarithm buttons

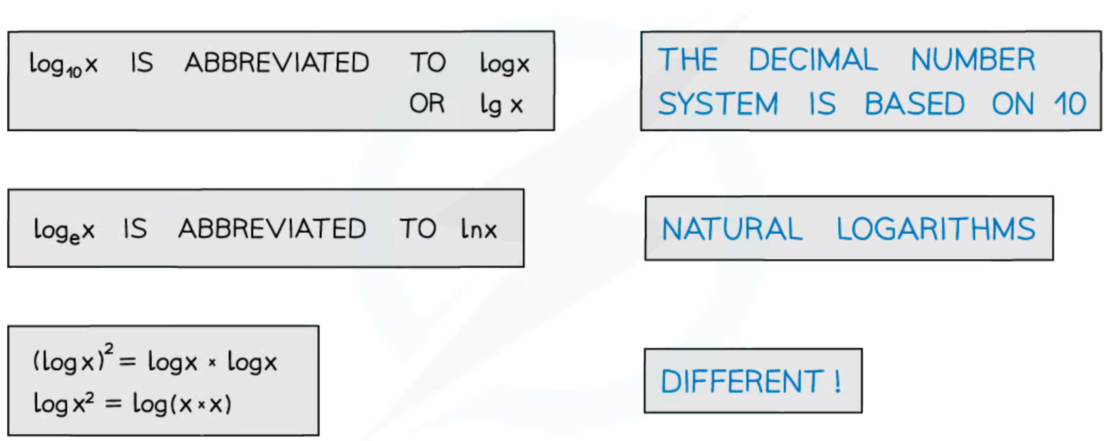
- This button allows you to type in any number for the base

- Natural logarithms (see "e")

Your notes

- Shortcut for base 10 although SHIFT button needed
- Before calculators, logarithmic values had to be looked up in printed tables

Notation



- 10 is a common base
  - $\log_{10} x$ is abbreviated to **$\log x$** or **$\lg x$**
- The value **e** is another common base
  - $\log_e x$ is abbreviated to **$\ln x$**
- $(\log x)^2 \neq \log x^2$

Your notes

Worked example

Solve the equation $2^{2x+3} - 41\left(2^x\right) + 5 = 0$,

giving your answers correct to 3 significant figures.

$2^{2x+3} - 41(2^x) + 5 = 0$ HIDDEN QUADRATIC

$2^3 \times 2^{2x} - 41(2^x) + 5 = 0$

$8(2^x)^2 - 41(2^x) + 5 = 0$ REWRITE

LET $y = 2^x$: $8y^2 - 41y + 5 = 0$ SUBSTITUTION NOT ESSENTIAL

$(8y-1)(y-5) = 0$ USE CALCULATOR TO SOLVE

$y = \frac{1}{8}$ $\quad$ $y = 5$

$2^x = \frac{1}{8}$ $\quad$ $2^x = 5$

SPOT THIS ?!

$x = \log_2 \frac{1}{8}$ $\quad$ $x = \log_2 5$ USE LOGARITHMS

$x = -3$ $\quad$ $x = 2.32192\ldots$

$x = 2.32$ (3sf)

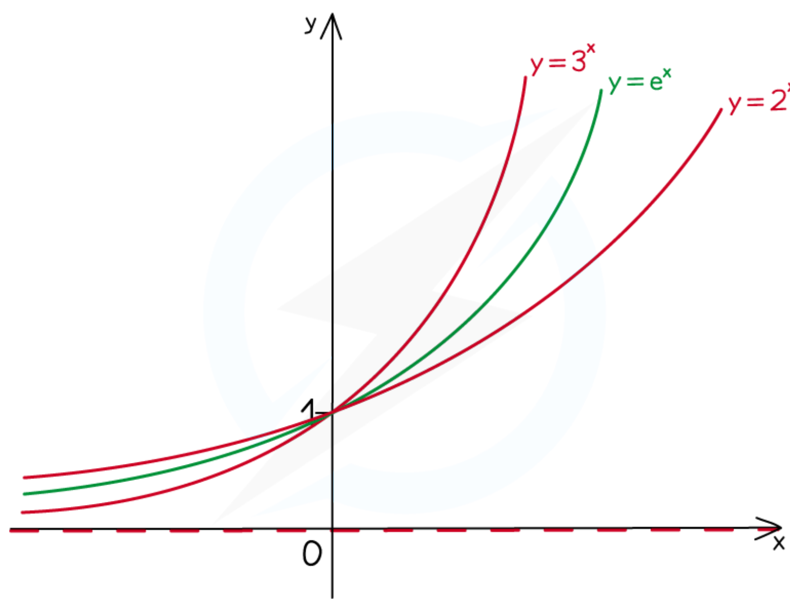
Your notes

6.1.3 "e"

"e" Exponential Function

What is e, the exponential function?

- The exponential function is $y = e^x$
  - **e** is an irrational number
  - $e \approx 2.718$
- As with other exponential graphs $y = e^x$
  - passes through (0, 1)
  - has the x-axis as an asymptote

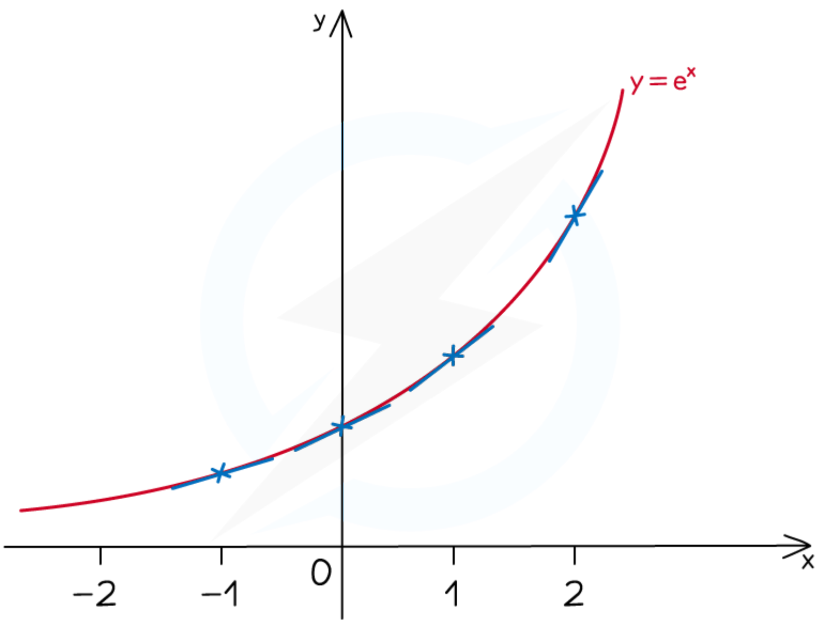


What is the big deal with e?

- $y = e^x$ has the particular property

$$\frac{dy}{dx} = e^x$$

- ie for *every* real number **x**, the gradient of $y = e^x$ is also equal to e^x (see Derivatives of Exponential Functions)



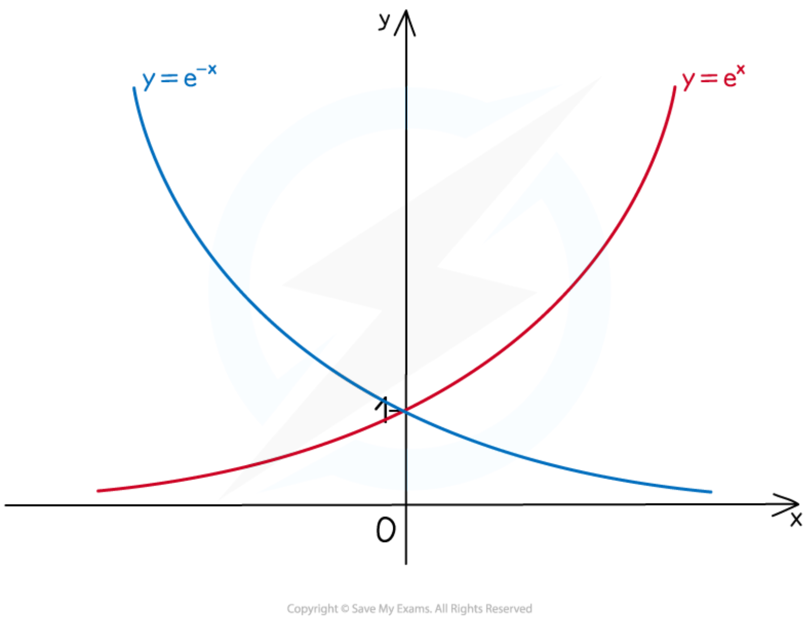
x	y	dy / dx
−1	0.3678...	0.3678...
0	1	1
1	2.7182...	2.7182...
2	7.3890...	7.3890...

Your notes

The negative exponential graph

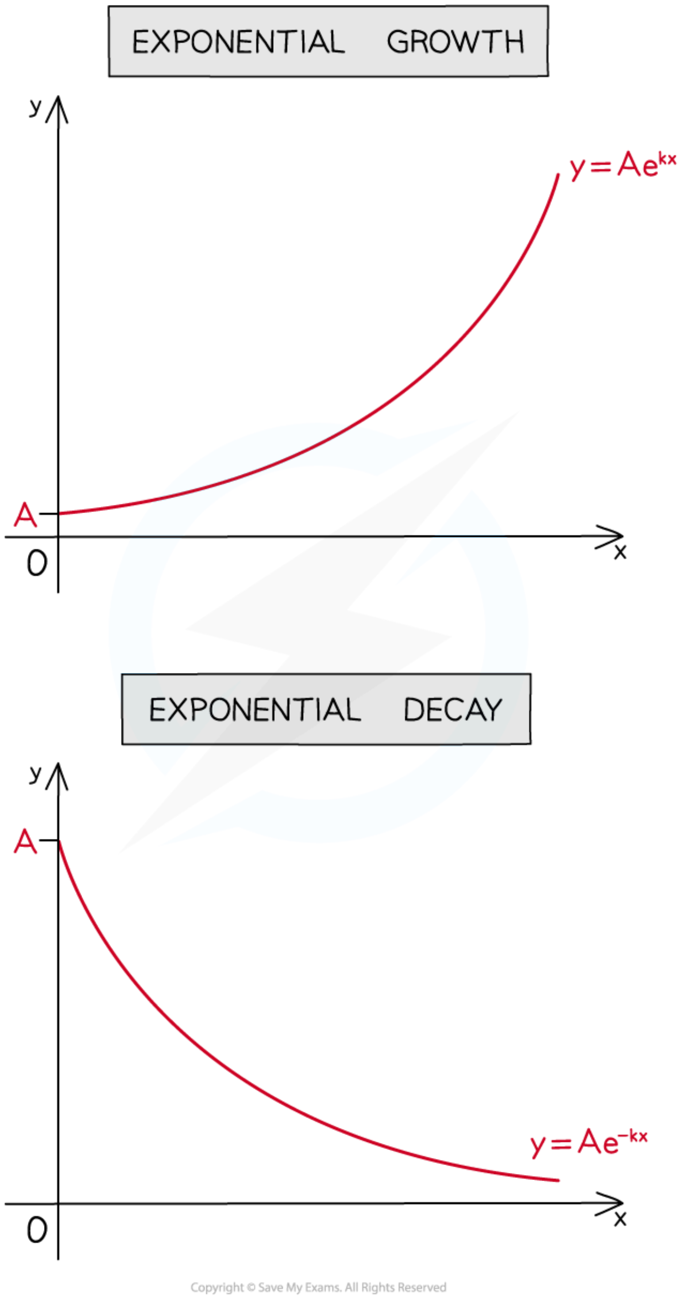
- $\mathbf{y = e^{-x}}$ is a reflection in the **y**-axis of $\mathbf{y = e^x}$
- They are of the form **y = f(x)** and **y = f(−x)** (see Transformations of Functions - Reflections)

Your notes



What is exponential growth and decay?

Your notes



- **$y = Ae^{kx}$** (**$k > 0$**) is exponential growth
- **$y = Ae^{-kx}$** (**$k > 0$**) is exponential decay
- **A** is the initial value
- **k** is a (usually positive) constant

- "-" is used in the equation making clear whether it is growth or decay

Your notes

Your notes

Worked example

(a) On the same diagram sketch the graphs of

(i) $y = e^x$

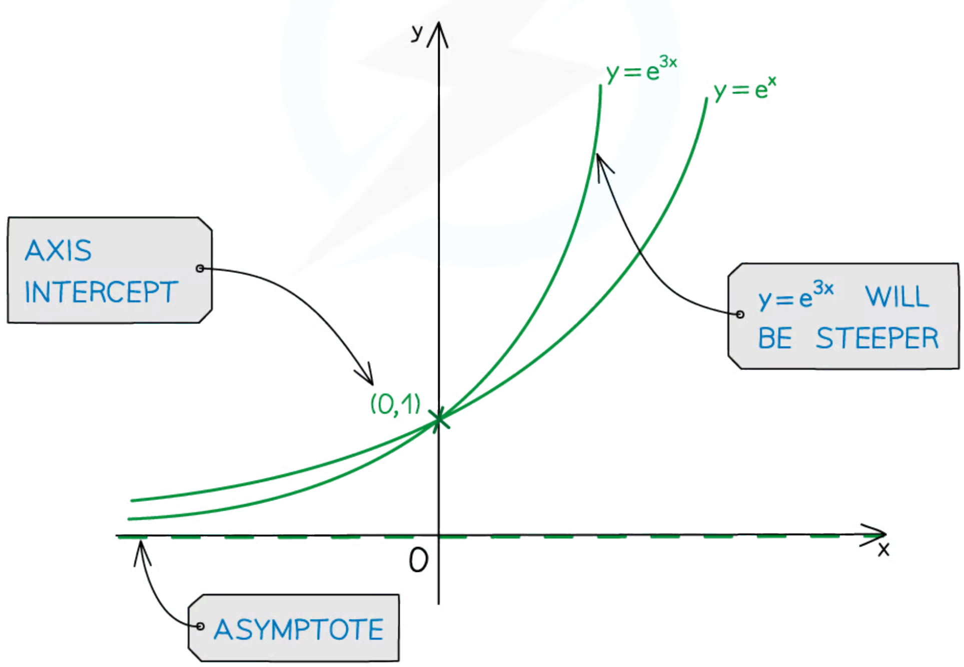
(ii) $y = e^{3x}$

(b) On the same diagram sketch the graphs of

(i) $y = e^{-x}$

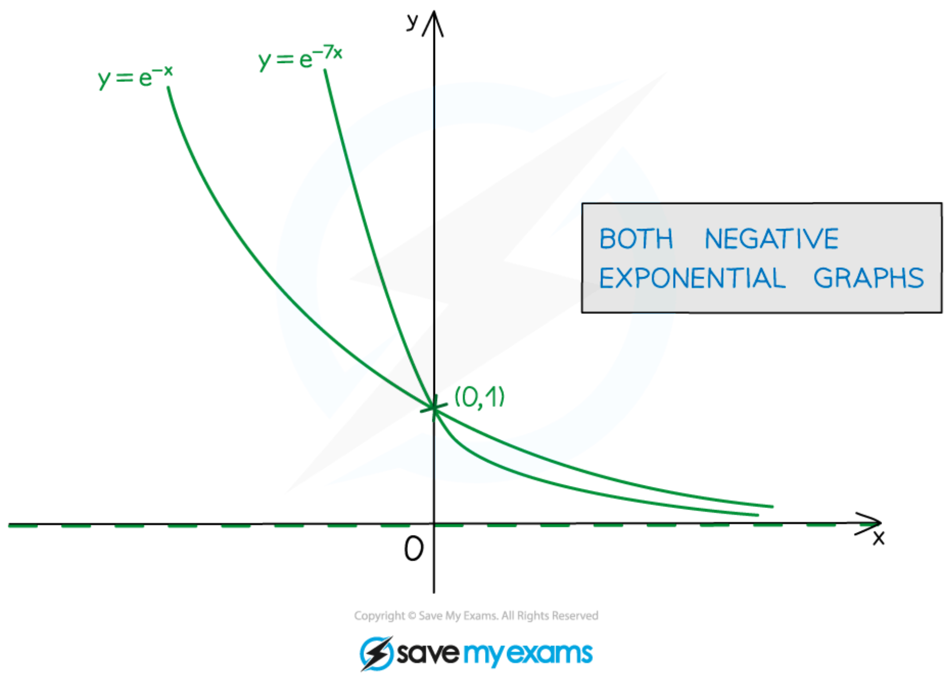
(ii) $y = e^{-7x}$

a) (i) & (ii)



Your notes

b) (i) & (ii)



6.1.4 Derivatives of Exponential Functions

Your notes

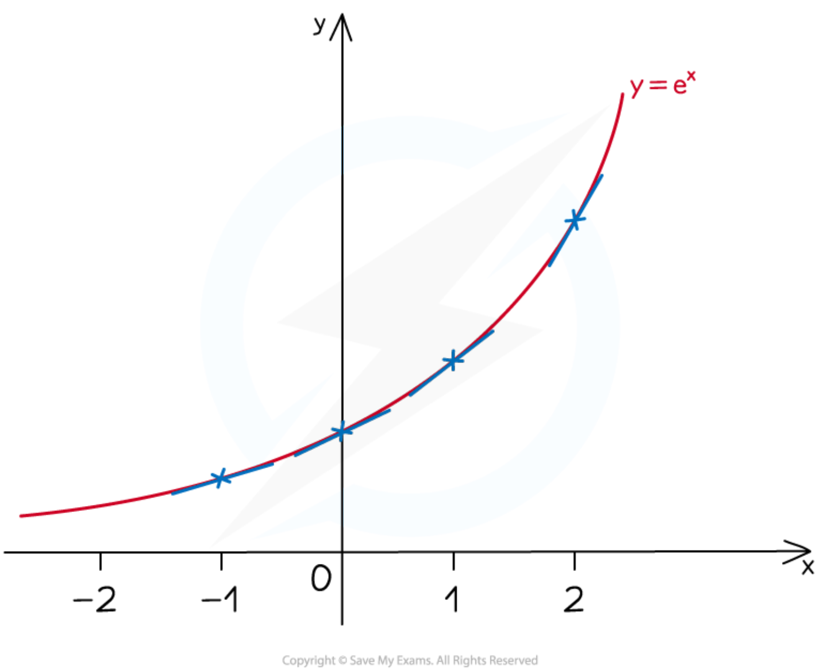
Derivatives of Exponential Functions

What is the derivative of e?

- $\mathbf{y = e^x}$ has the particular property

$$\frac{dy}{dx} = e^x$$

- ie for *every* real number **x**, the gradient of $\mathbf{y = e^x}$ is also equal to $\mathbf{e^x}$ (see Derivatives of Exponential Functions)

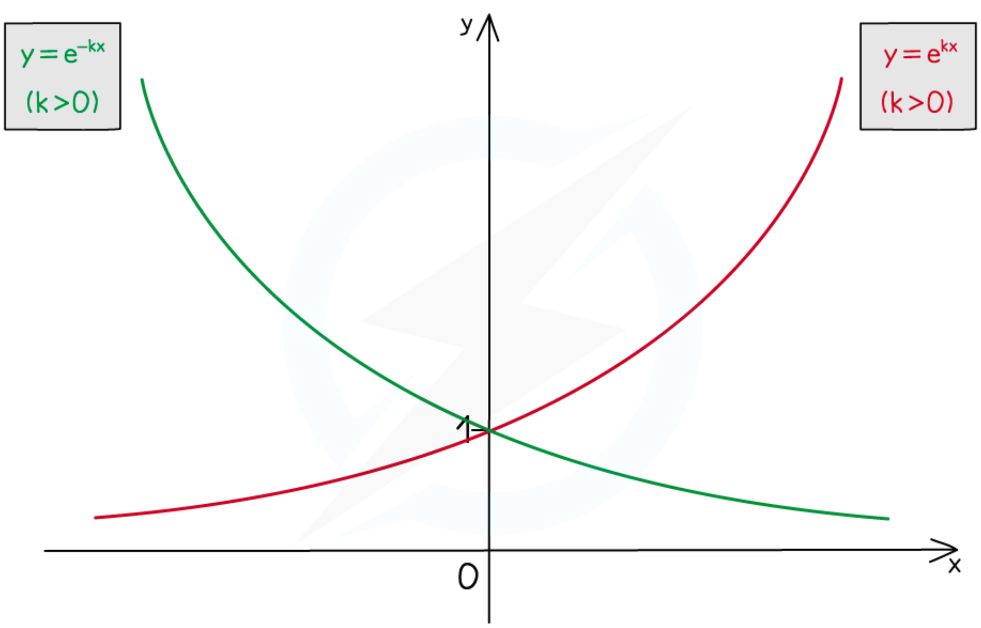


Your notes

x	y	dy / dx
−1	0.3678...	0.3678...
0	1	1
1	2.7182...	2.7182...
2	7.3890...	7.3890...

- **e ≈ 2.718** (see "e")
- Recall that the derivative is the gradient function for a curve (see First Principles Differentiation)

Graphs and derivatives related to e



- The derivative of **$y = e^{kx}$** is
  - $$\frac{dy}{dx} = ke^{kx}$$

- The derivative of $y = e^{-kx}$ is
  - $$\frac{dy}{dx} = -ke^{-kx}$$

Examiner Tip

- Remember that (like **π**) **e** is a **number**.
- Exam questions can ask for answers to be given as exact values in terms of **e** (see the Worked Example below).

Your notes

Your notes

Worked example

Find, in terms of e, the gradient at $x = 3$ for each of the following functions ...

(a) $y = e^x$

(b) $y = e^{5x}$

(c) $y = e^{-x}$

(d) $y = 3e^{2x}$

(e) $y = 5e^{-4x}$

a) $y = e^x$ $\quad \frac{dy}{dx} = e^x$

$\frac{dy}{dx} = y$

AT $x=3$, $\frac{dy}{dx} = e^3$

$\frac{dy}{dx} = e^3$

FINAL ANSWER IN TERMS OF e

Your notes

b) $y = e^{5x}$ $\quad \frac{dy}{dx} = 5e^{5x}$

$\frac{dy}{dx} = ke^{kx}$

AT $x = 3$, $\quad \frac{dy}{dx} = 5e^{5\times3}$

$\frac{dy}{dx} = 5e^{15}$

c) $y = e^{-x}$ $\quad \frac{dy}{dx} = -e^{-x}$

$\frac{dy}{dx} = -ke^{-kx}$

$k = 1$

AT $x = 3$, $\quad \frac{dy}{dx} = -e^{-3}$

$\frac{dy}{dx} = -e^{-3} = -\frac{1}{e^3}$

NOT ESSENTIAL

d) $y = 3e^{2x}$ $\quad \frac{dy}{dx} = 6e^{2x}$

$\frac{dy}{dx} = ke^{kx}$

3 IS A CONSTANT SO IS MULTIPLIED BY k

AT $x = 3$, $\quad \frac{dy}{dx} = 6e^{2\times3}$

$\frac{dy}{dx} = 6e^{6}$

Your notes

e) $y = 5e^{-4x}$ $\quad \frac{dy}{dx} = -20e^{-4x}$

$\frac{dy}{dx} = -ke^{-kx}$

5 IS A CONSTANT

AT $x=3$, $\quad \frac{dy}{dx} = -20e^{-4 \times 3}$

$$\frac{dy}{dx} = -20e^{-12}$$