Edexcel A Level Maths: Pure

Your notes

5.4 Radian Measure

Contents

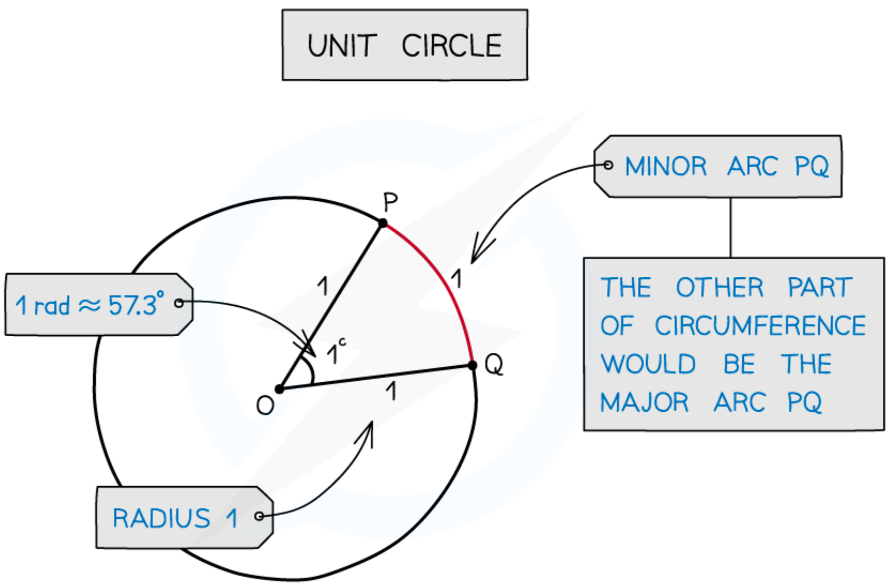
Your notes

5.4.1 Radian Measure

Radian Measure

Radian measure

- **Radians** are an alternative to degrees for measuring angles



- 1 radian is the angle in a sector of radius 1 and arc length 1
- Radians are normally quoted in terms of π
- This leads to
  - **2π = 360°**
  - **π = 180°**
- The symbol for radians is ^{c} but it is more usual to see **rad**
  - Often, when π is involved, no symbol is given as it is obviously in radians
- Radians are used in **trigonometry** (see Exact Values) and **calculus** (First Principles Differentiation – Trigonometry)

How do I change between radians and degrees?

- The most common are below, these are helpful to know

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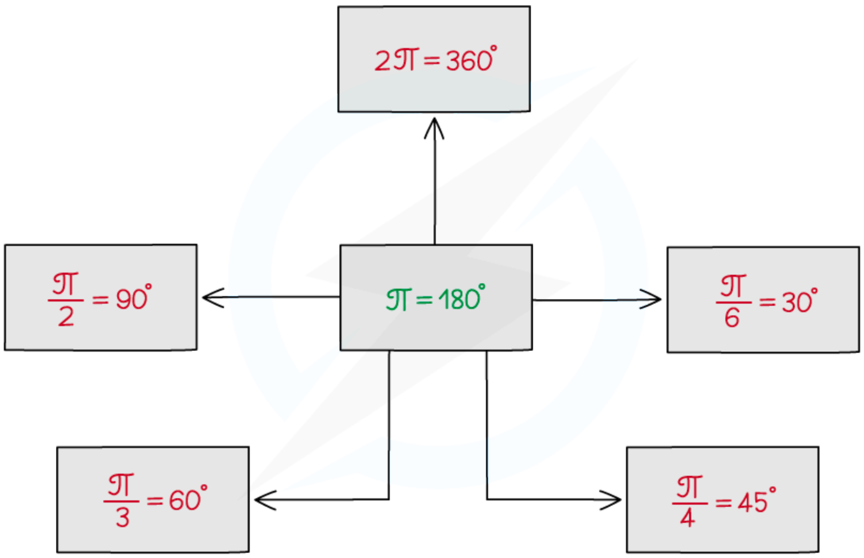
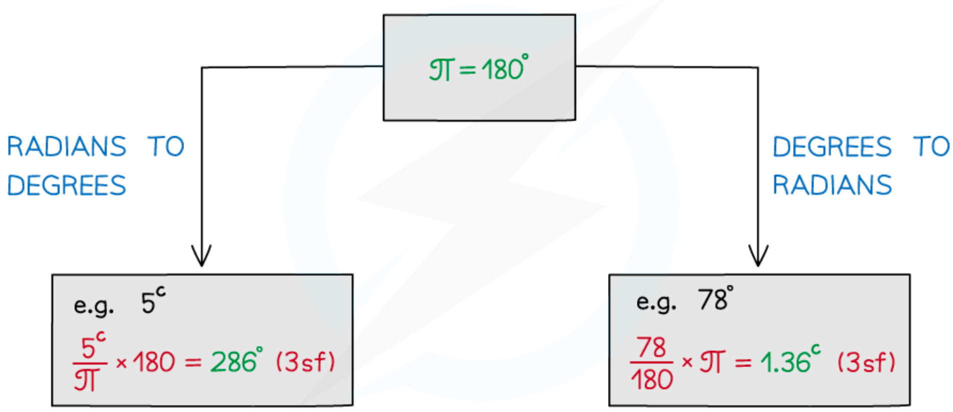


- Multiples of these are quite common too, eg $\frac{3}{4}\pi = 135°$
- For less familiar angles use **π = 180°** to convert



How do I use radians to find the length of an arc?

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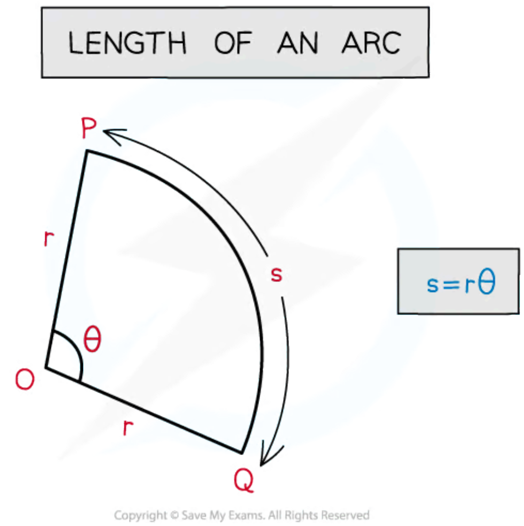
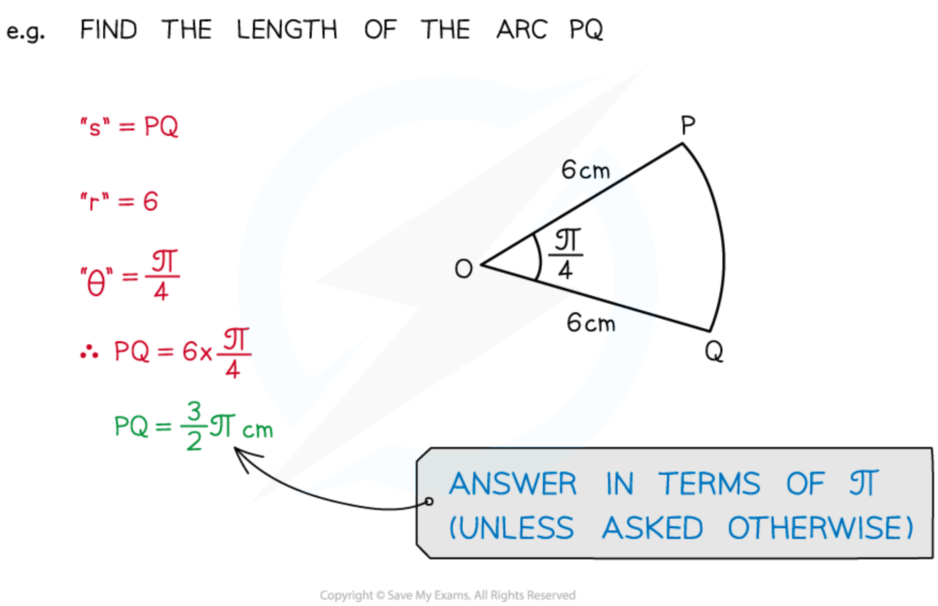


- The length of an arc is **s = rθ**



How do I use radians to find the area of a sector?

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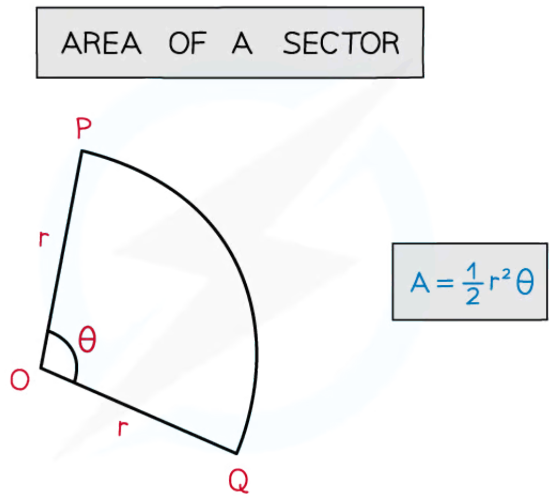
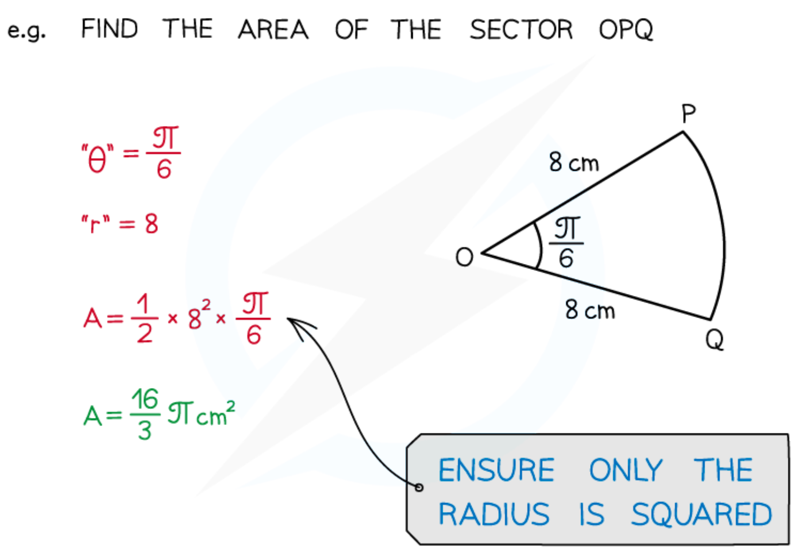


- The area of a sector is **A = ½r²θ**



Solving problems with radians

- Other **angle**, **circle**, **area**, etc skills may be needed
- For example, area of a triangle **"A = ½absinC"**

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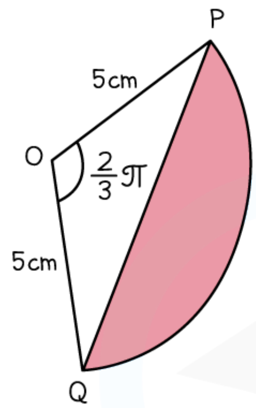
e.g. FIND THE AREA OF THE SHADED SEGMENT



"SEGMENT" =
"SECTOR" − "TRIANGLE"

AREA OF SECTOR OPQ $= \frac{1}{2} \times 5^2 \times \frac{2}{3}\pi = \frac{25}{3}\pi$ "$\frac{1}{2}r^2\theta$"

AREA OF TRIANGLE OPQ $= \frac{1}{2} \times 5 \times 5 \times \sin\frac{2}{3}\pi = \frac{25\sqrt{3}}{4}$ "$\frac{1}{2}ab\sin C$"

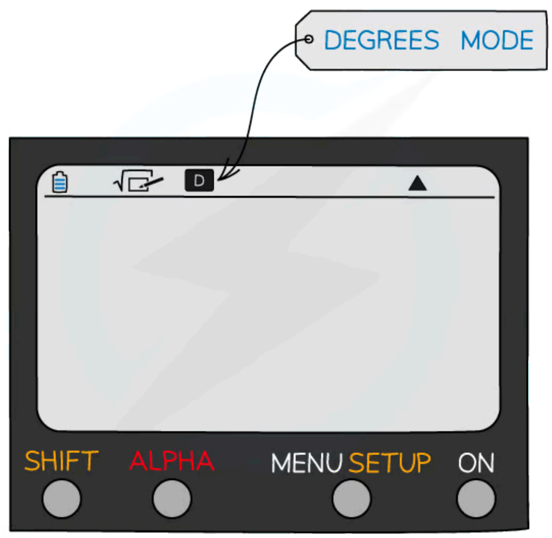
$\therefore$ AREA OF SEGMENT $= \frac{25}{3}\pi - \frac{25\sqrt{3}}{4}$

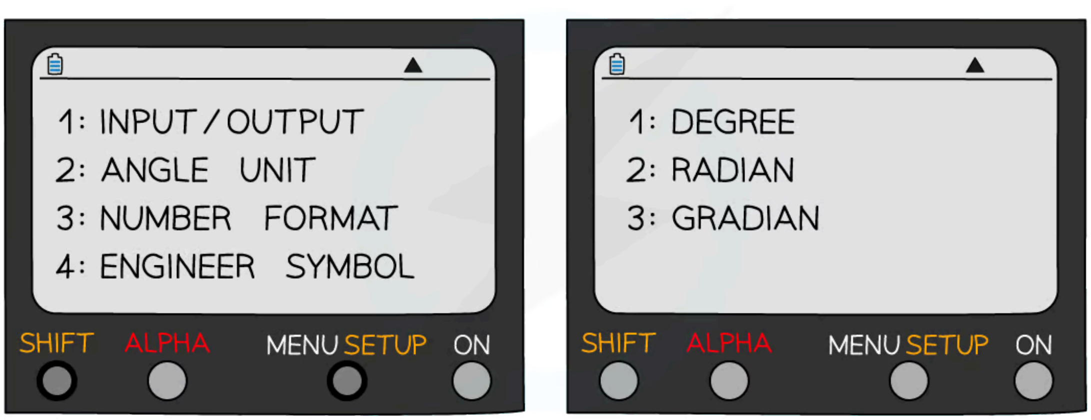
$= \frac{100\pi - 75\sqrt{3}}{12}$ cm^2

Your notes

Examiner Tip

- Exam papers will have a mixture of questions in degrees and radians
- Check the mode (degrees/radians) of your calculator before using it
- Ensure you know how to change the mode on your calculator
- For the Casio fx-991EX Classwiz
  - **SHIFT** then **MENU SETUP**
  - Choose option **2: ANGLE UNIT**
  - Choose either **1: DEGREE** or **2: RADIAN**





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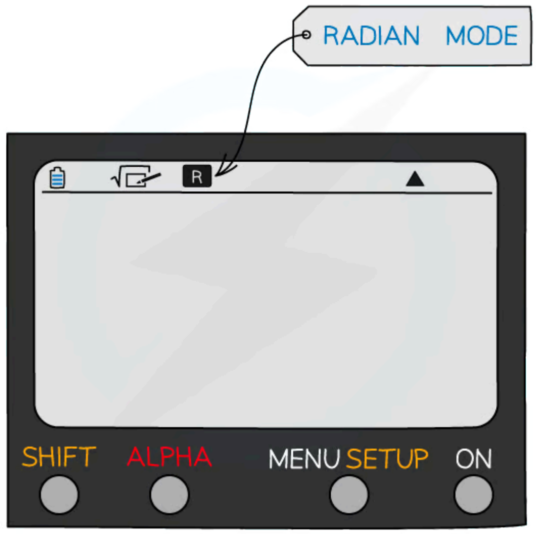

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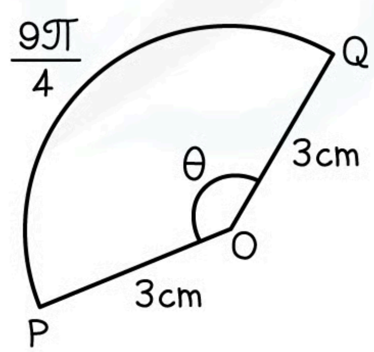
Worked example

In the diagram below OPQ is the sector of a circle with radius 3 cm.

The length of the arc PQ is $\frac{9\pi}{4}$.

The angle at the centre of the circle is θ, measured in radians.

Find the area of the sector OPQ.



Your notes

"$A = \frac{1}{2}r^2\theta$"

"r" $= 3$

$\theta = ?$

USE LENGTH OF ARC TO FIND θ

"$s = r\theta$"

$$\frac{9\pi}{4} = 3\theta$$

$$\theta = \frac{3\pi}{4}$$

$\therefore$ AREA OF SECTOR OPQ $= \frac{1}{2} \times 3^2 \times \frac{3\pi}{4}$

NOW FIND THE AREA USING θ AND r

$$= \frac{27\pi}{8}\ \text{cm}^2$$

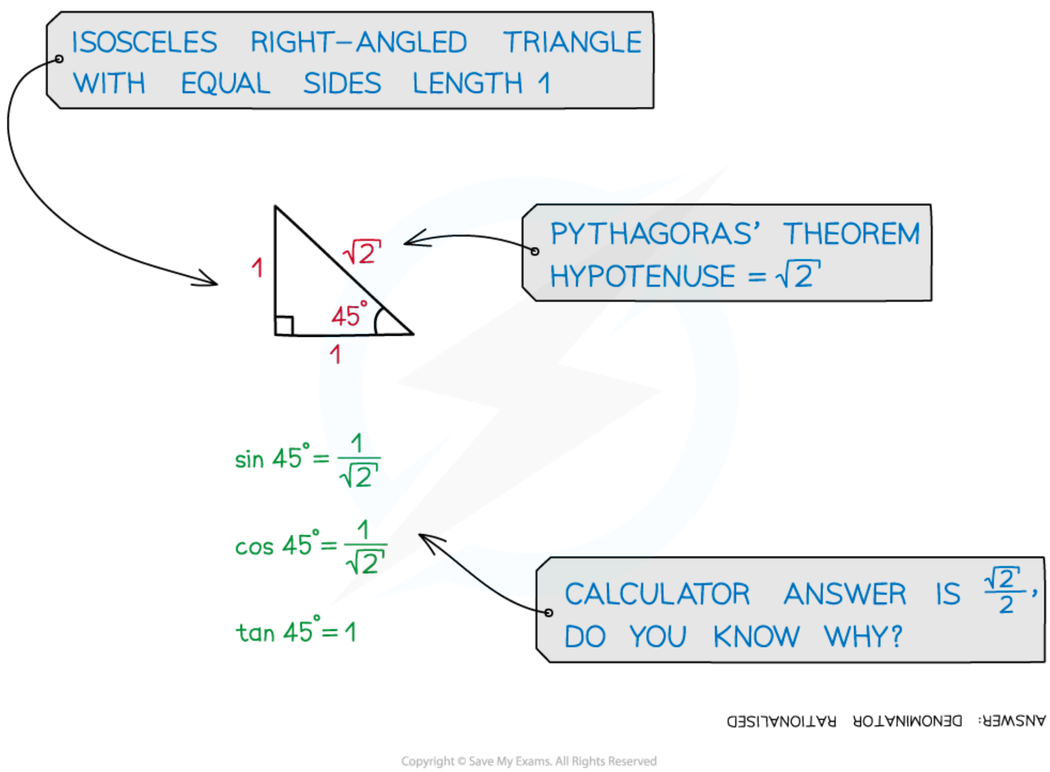
Your notes

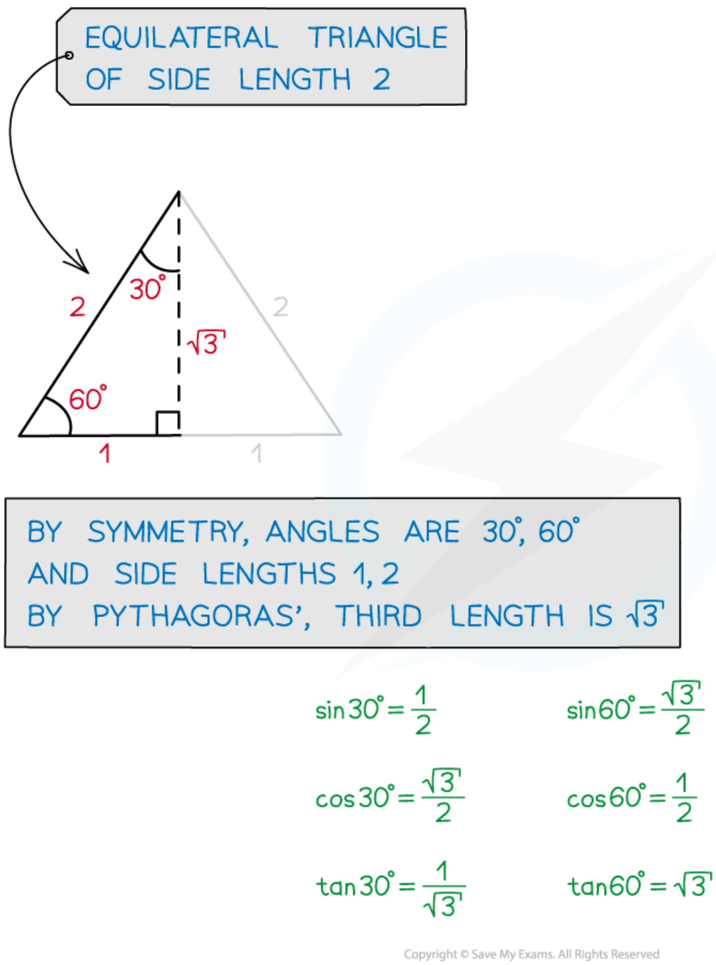
5.4.2 Trigonometry Exact Values

Trigonometry Exact Values

Exact values

- In trigonometry some values of sin, cos and tan are "nice"
- For example **sin 30° = ½**, **tan 45° = 1**
- We can find those associated with the angles **30°**, **45°** and **60°** using SOHCAHTOA on isosceles and equilateral triangles





- **sin**, **cos** and **tan** values for **30°**, **45°** and **60°** should be recalled easily
- Memorise either the actual values or how to work them out from the triangle
- **sin**, **cos** and **tan** for **0°**, **90°**, **180°** should also be very familiar
- They can be recalled using the relevant **trigonometric graph** (see Graphs of Trigonometric Functions)

Exact values and radians

- All of the above applies to radians as well as degrees
- Here is a table of all exact values including **radians** (see Radian Measure)

Your notes

DEGREES	0°	30°	45°	60°	90°	180°	360°
RADIANS	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π	2π
sin	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	0	0
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	-1	1
tan	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	UNDEFINED	0	0

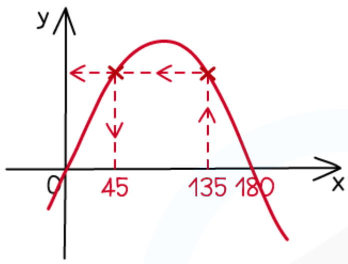
How do I find exact values of other angles?

- Exact **sin**, **cos** and **tan** values of **multiples** of **30°**, **45°**, **60°** can also be found
- This is a combination of recalling the basic values and using the graph

Your notes

e.g. FIND THE EXACT VALUE OF $\sin 135^\circ$

STEP 1



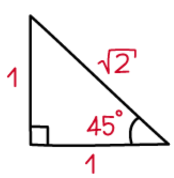
SKETCH THE GRAPH OF $\sin x$

STEP 2

$\sin 135^\circ = \sin 45^\circ$

SYMMETRY OF GRAPH RELATES 135° TO 45°

STEP 3



$\sin 45^\circ = \frac{1}{\sqrt{2}}$

RECALL EXACT VALUE
DRAW DIAGRAM TO HELP

$\therefore \sin 135^\circ = \frac{1}{\sqrt{2}}$

Examiner Tip

- Draw the triangles for **sin**, **cos** and **tan** of **30°**, **45°** and **60°** on the exam paper so that you can then use them as many times as you need throughout the paper.

Worked example

Your notes

Your notes

Write down the exact values of the following:

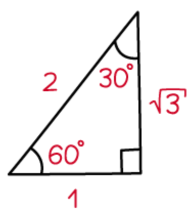
(i) $\sin 30°$

(ii) $\cos\dfrac{\pi}{3}$

(iii) $\tan 135°$

(iv) $\sin\dfrac{4\pi}{3}$

(i)



USE TRIANGLE TO RECALL EXACT VALUE

$\sin 30° = \dfrac{1}{2}$

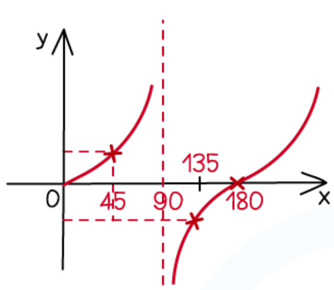
(ii) $\dfrac{\pi}{3} = 60°$

USE DIAGRAM ABOVE

$\cos\dfrac{\pi}{3} = \dfrac{1}{2}$

(iii) STEP 1

SKETCH GRAPH OF $\tan x$



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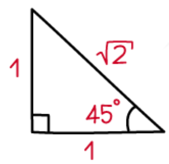
STEP 2 USE SYMMETRY OF GRAPH

$\tan 135^\circ = -\tan 45^\circ$

STEP 3



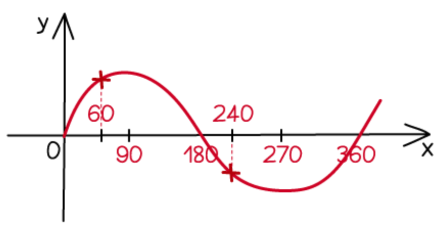
RECALL EXACT VALUE

$\tan 135^\circ = -1$

(iv) $\frac{4\pi}{3} = 4\left(\frac{\pi}{3}\right) = 4 \times 60 = 240^\circ$

CONVERSION NOT ESSENTIAL

STEP 1



SKETCH THE GRAPH OF sinx

STEP 2

$\sin 240^\circ = -\sin 60^\circ$

SYMMETRY

$\sin \frac{4\pi}{3} = -\sin \frac{\pi}{3}$

STEP 3

$\sin \frac{4\pi}{3} = -\frac{\sqrt{3}}{2}$

USING TRIANGLE FROM PART (i)

5.4.3 Small Angle Approximations

Your notes

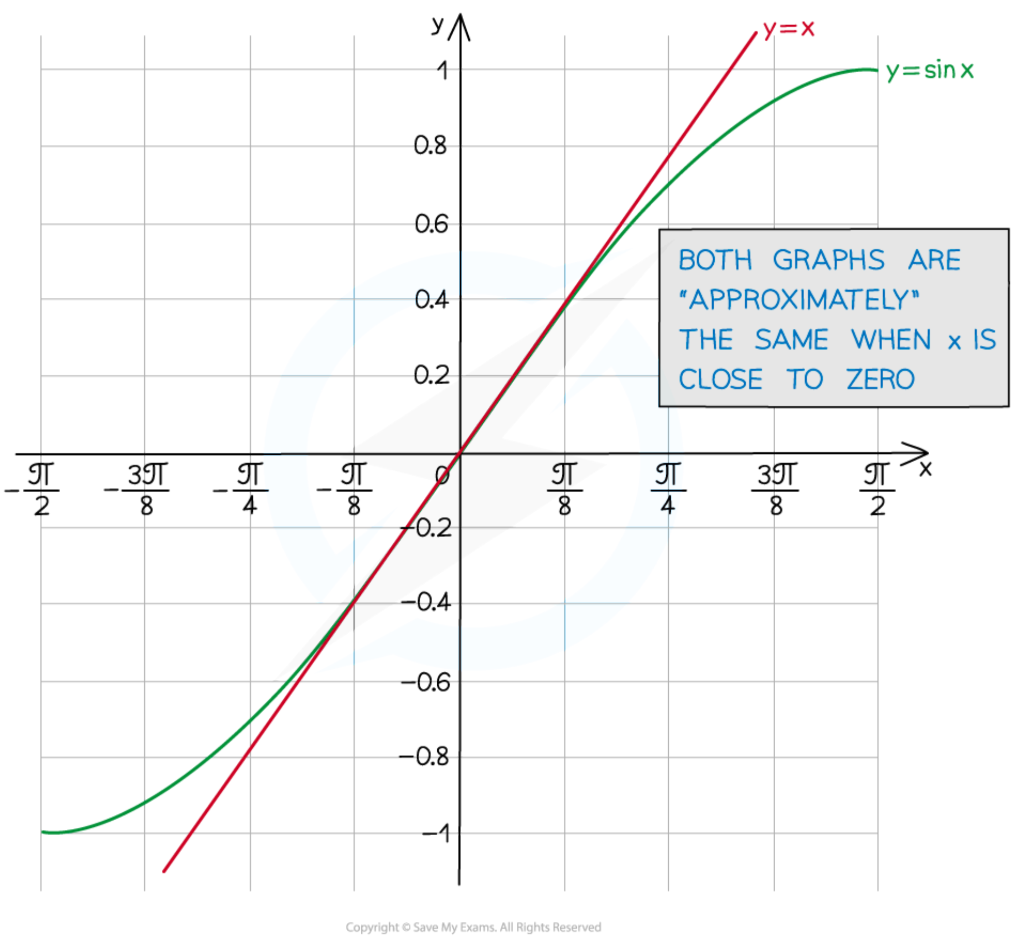
Small Angle Approximations

Small angle approximations

- When an angle measured in **radians** is **very small,** you can **approximate** the value using **small angle approximations**
- These **only** apply when angles are measured in **radians**
- They can be applied to positive **and** negative small angles

What's the small-angle approximation of sin θ?

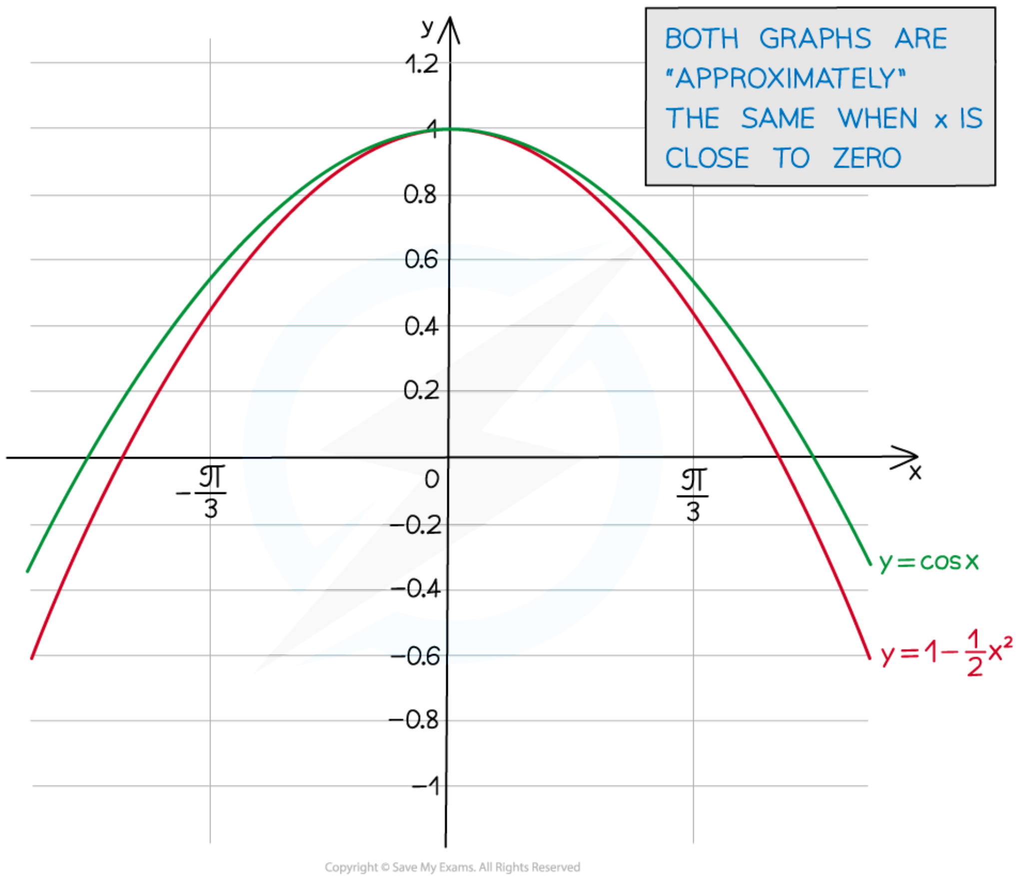
$$\sin\theta \approx \theta$$



Your notes

What's the small-angle approximation of cos θ?

$$\cos\theta \approx 1 - \frac{1}{2}\theta^2$$

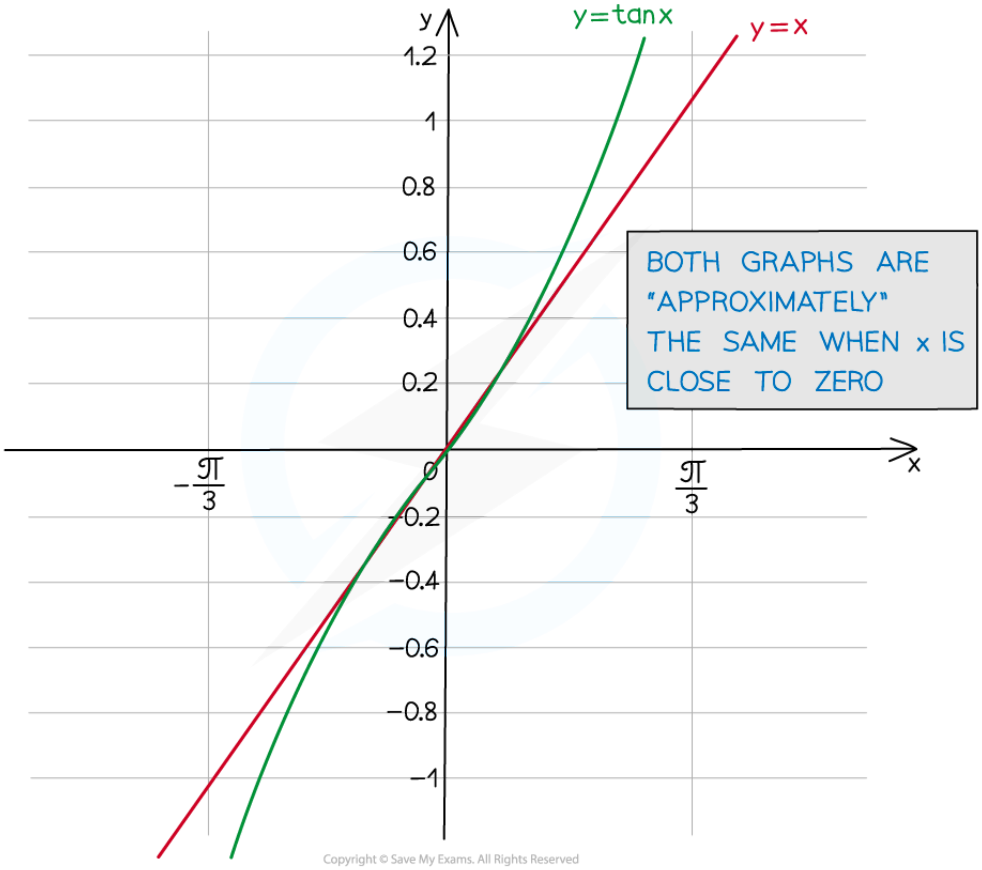


- **y = cos θ** (near zero) is similar to a "negative quadratic" (parabola)

What's the small-angle approximation of tan θ?

$$\tan\theta \approx \theta$$

Your notes



How do I use small angle approximations in solving problems?

- Replace $\sin\theta$, $\cos\theta$ or $\tan\theta$ with the appropriate approximation
- Given angles are often 2θ, 3θ, ...
  - Replace "θ" in the approximation by 2θ, 3θ, ...

Your notes

e.g. GIVEN θ IS SMALL AND MEASURED IN RADIANS, FIND AN APPROXIMATION, IN TERMS OF θ FOR

a) $\cos 2\theta$

b) $\dfrac{\sin^2 3\theta + 2\theta}{\tan\theta}$

$\boxed{\cos\theta \approx 1 - \frac{1}{2}\theta^2}$

a) $\cos 2\theta \approx 1 - \frac{1}{2}(2\theta)^2$ "θ" $= 2\theta$

$= 1 - 2\theta^2$

$\boxed{\sin\theta \approx \theta}$

$\boxed{\tan\theta \approx \theta}$

b) $\dfrac{\sin^2 3\theta + 2\theta}{\tan\theta} \approx \dfrac{(3\theta)^2 + 2\theta}{\theta}$ "θ" $= 3\theta$

$= \dfrac{9\theta^2 + 2\theta}{\theta}$

$= \dfrac{\cancel{\theta}(9\theta + 2)}{\cancel{\theta}}$

$= 9\theta + 2$

- **Binomial expansion** (see **GBE**) may be involved in more awkward expressions

Your notes

e.g. FIND AN EXPRESSION, IN TERMS OF θ FOR $\frac{\sin\theta}{\cos 2\theta}$.

YOU MAY ASSUME θ IS SMALL SUCH THAT TERMS INVOLVING θ^3 OR HIGHER MAY BE IGNORED.

$\boxed{\sin\theta \approx \theta}$ $\boxed{\cos\theta \approx 1-\frac{1}{2}\theta^2}$

$$\frac{\sin\theta}{\cos 2\theta} \approx \frac{\theta}{1-\frac{1}{2}(2\theta)^2}$$

"θ" $= 2\theta$

$$= \frac{\theta}{1-2\theta^2}$$

$$= \theta(1-2\theta^2)^{-1}$$

$$= \theta[1 + (-1)(-2\theta^2) + \ldots]$$

GENERAL BINOMIAL EXPANSION

$$\approx \theta(1-4\theta^2)$$

$$\approx \theta$$

IGNORING θ^3 AND HIGHER

Examiner Tip

- Small angle approximations are given in the formula booklet.
- They can be used in proofs – particularly differentiation from first principles (see **First Principles Differentiation – Trigonometry**).

Worked example

(a) Given θ is sufficiently small, find an approximation, in terms of θ, of $\sin^2 3\theta(1-\cos 2\theta)$

(b) Use your expression to approximate

$$\sin^2\frac{\pi}{2}\left(1-\cos\frac{\pi}{3}\right)$$

a) $\sin 3\theta(1-\cos 2\theta) \approx (3\theta)^2\left[1-\left(1-\frac{1}{2}(2\theta)^2\right)\right]$

$= 9\theta^2(2\theta^2)$

$= 18\theta^4$

"θ" $= 3\theta$

"θ" $= 2\theta$

Your notes

Your notes

b) $3\theta = \frac{\pi}{2}$

$\theta = \frac{\pi}{6}$

CHECK...

$2\theta = \frac{\pi}{3}$

$\theta = \frac{\pi}{6}$

$\therefore \ \sin\frac{\pi}{2}\left(1-\cos\frac{\pi}{3}\right) \approx 18\left(\frac{\pi}{6}\right)^4$

$= \frac{\pi^4}{72}$ (1.35 TO 3sf)

IT'S AN APPROXIMATION ANYWAY SO DECIMALS FINE!