Edexcel A Level Maths: Pure

Your notes

11.1 Vectors in 2 Dimensions

Contents

11.1.1 Basic Vectors

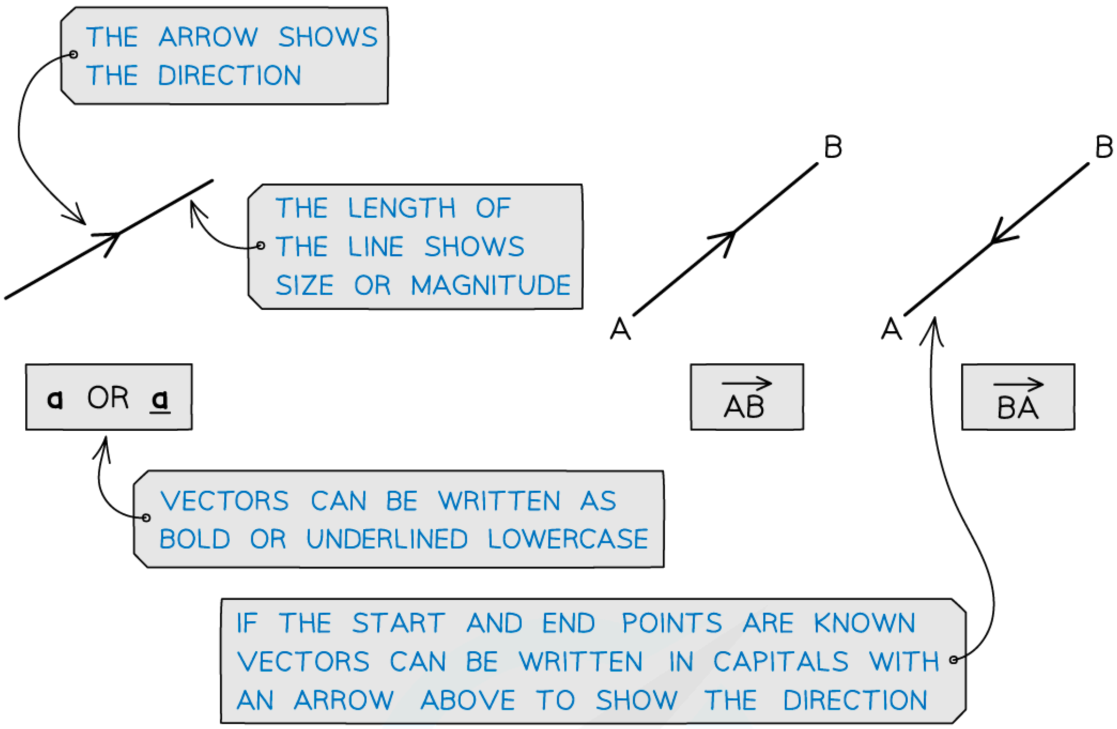
Your notes

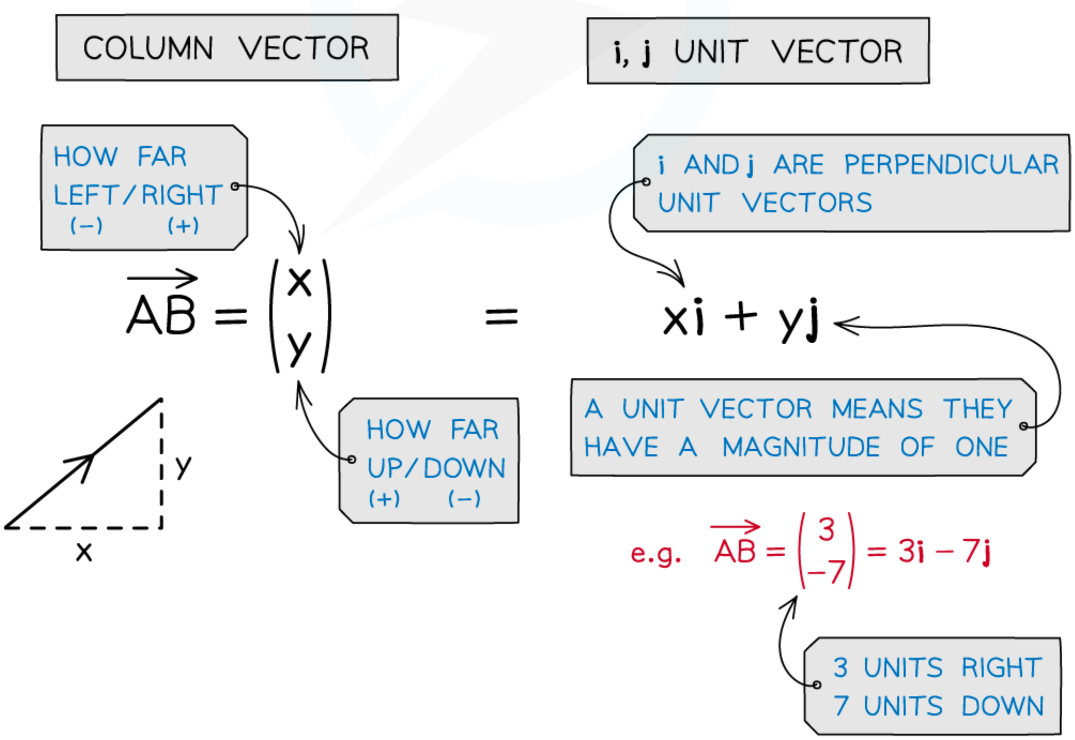
Basic Vectors

What is a vector?

- Vectors represent a movement of a certain **magnitude** (size) in a given **direction**
- You should have already come across **vectors** when translating functions of graphs
- They appear in many contexts of maths including **mechanics** for modelling forces
- Vectors can be represented in different ways such as a **column vector** or as an **i and j unit vector**

Your notes


THE ARROW SHOWS THE DIRECTION
THE LENGTH OF THE LINE SHOWS SIZE OR MAGNITUDE
A
B
A
B
a OR a
AB
BA
VECTORS CAN BE WRITTEN AS BOLD OR UNDERLINED LOWERCASE
IF THE START AND END POINTS ARE KNOWN VECTORS CAN BE WRITTEN IN CAPITALS WITH AN ARROW ABOVE TO SHOW THE DIRECTION


COLUMN VECTOR
i, j UNIT VECTOR
HOW FAR LEFT/RIGHT (−) (+)
i AND j ARE PERPENDICULAR UNIT VECTORS
AB = (x y) = xi + yj
y
x
HOW FAR UP/DOWN (+) (−)
A UNIT VECTOR MEANS THEY HAVE A MAGNITUDE OF ONE
e.g. AB = (3 −7) = 3i − 7j
3 UNITS RIGHT 7 UNITS DOWN

Your notes

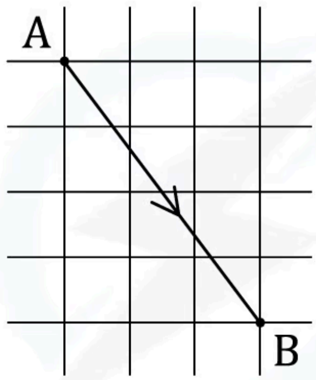
Examiner Tip

- Think of vectors like a journey from one place to another.
- Diagrams can help, if there isn't one, draw one.
- In your exam you can't write in **bold** so should underline your vector notation.

Your notes

Worked example

a) From the diagram below, write the column vector for $\overrightarrow{AB}$.



b) Write the column vector $\begin{pmatrix} -2 \\ 5 \end{pmatrix}$ using the unit vectors **i** and **j**.

Your notes

a) $\overrightarrow{AB} = \begin{pmatrix} 3 \\ -4 \end{pmatrix}$

REMEMBER THE COLUMN VECTOR SIMPLY REPRESENTS THE PATH $\begin{pmatrix} \text{LEFT/RIGHT} \\ \text{UP/DOWN} \end{pmatrix}$ FROM A TO B

b) $\begin{pmatrix} -2 \\ 5 \end{pmatrix} = -2\mathbf{i} + 5\mathbf{j}$

THE VECTOR USING $\mathbf{i}$ AND $\mathbf{j}$

IN YOUR EXAM YOU WOULD WRITE $-2\underline{i} + 5\underline{j}$

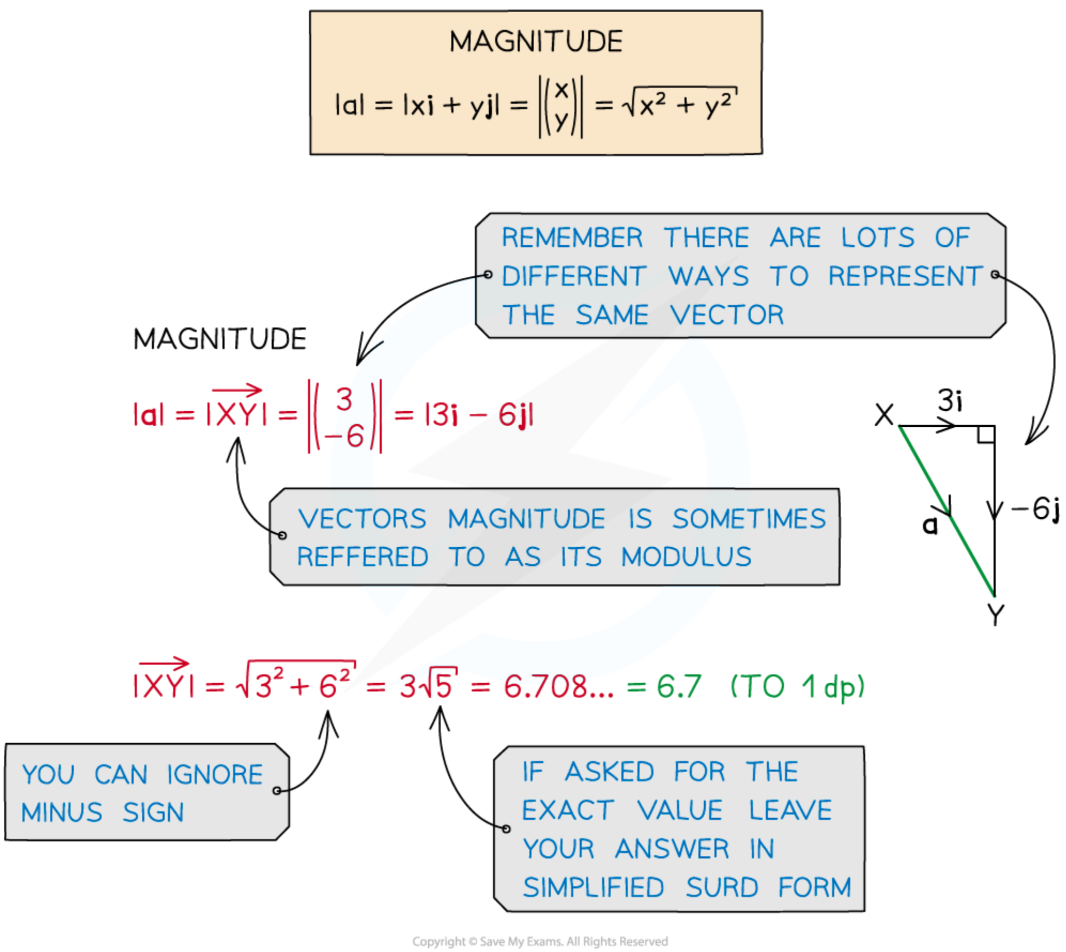
11.1.2 Magnitude & Direction

Your notes

Magnitude & Direction

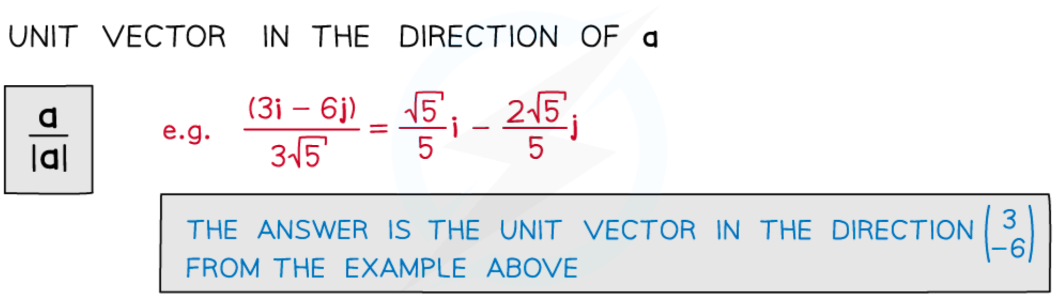
What is the magnitude of a vector?

- The magnitude of a vector is simply its size
- It also tells us the distance between two points
- You can find the magnitude of a vector using Pythagoras' theorem
- The magnitude of a vector $\mathbf{a}$ is written $|\mathbf{a}|$ when typed (or $|\underline{a}|$ when handwritten)



- To work out the **unit vector** in the direction of a given vector
- A **unit vector** has a **magnitude** of 1
  So to find the unit vector of a given vector, divide by its magnitude

Your notes

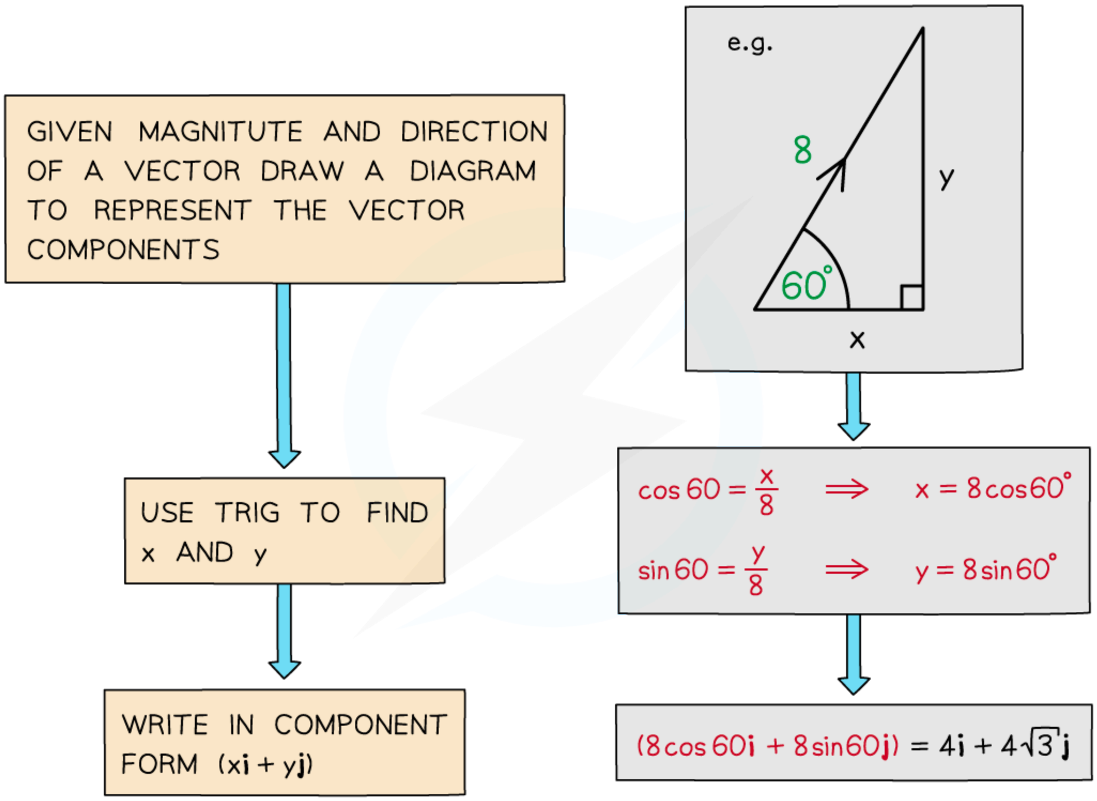


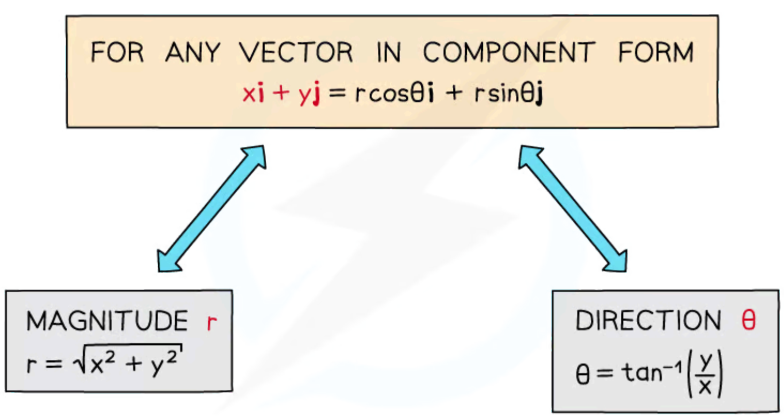
What is the direction of a vector?

- Vectors have opposite direction if they are the same size but opposite signs
  - e.g. if $\mathbf{a}$ or $\overrightarrow{BC}=\begin{pmatrix}-3\\8\end{pmatrix}$ then $-\mathbf{a}$ or $\overrightarrow{CB}=\begin{pmatrix}3\\-8\end{pmatrix}$
- The direction of a vector is what makes it more than just a scalar
  - Eg. two objects with velocities of 7 m/s and -7 m/s are travelling at the **same speed** but in **opposite directions**
- Two vectors are **parallel** if and only if one is a **scalar multiple** of the other
- For real-life contexts such as mechanics, direction can be calculated from a given vector using **trigonometry** (see Right-Angled Triangles)
- It is usually calculated **anticlockwise** from the **positive x-axis** (unless otherwise stated eg. a bearing)

How do I write a vector in component form?

- We have already seen that vectors can be written in different forms
- Component form means writing a vector in terms of **i** and **j components**
- Given the **magnitude** and **direction** of a vector you can work out its components and vice versa


GIVEN MAGNITUTE AND DIRECTION OF A VECTOR DRAW A DIAGRAM TO REPRESENT THE VECTOR COMPONENTS
USE TRIG TO FIND x AND y
WRITE IN COMPONENT FORM $(x\mathbf{i} + y\mathbf{j})$
e.g.
8
y
60°
x
$\cos 60 = \frac{x}{8} \Rightarrow x = 8\cos 60°$
$\sin 60 = \frac{y}{8} \Rightarrow y = 8\sin 60°$
$(8\cos 60\mathbf{i} + 8\sin 60\mathbf{j}) = 4\mathbf{i} + 4\sqrt{3}\mathbf{j}$


FOR ANY VECTOR IN COMPONENT FORM
$x\mathbf{i} + y\mathbf{j} = r\cos\theta\mathbf{i} + r\sin\theta\mathbf{j}$
MAGNITUDE r
$r = \sqrt{x^2 + y^2}$
DIRECTION θ
$\theta = \tan^{-1}\left(\frac{y}{x}\right)$

Examiner Tip

- Diagrams can help, especially when working out direction – if there isn't a diagram, draw one.
- Remember, resolving a vector just means writing it in component form.

Your notes

Worked example

Your notes

? A vector $\overrightarrow{XY} = \begin{pmatrix} -9 \\ 6 \end{pmatrix}$.

a) Find $\left|\overrightarrow{XY}\right|$, giving your answer to 3 significant figures.

b) Write the exact value of the unit vector in the direction of XY, giving your answer in simplified surd form.

a) $|\overrightarrow{XY}| = \sqrt{9^2 + 6^2}$

$= 3\sqrt{13}$

$= 10.8$ (3 sf)

$|\overrightarrow{XY}|$ MEANS FIND THE MAGNITUDE USING PYTHAGORAS

b) USING $\overrightarrow{XY} = \begin{pmatrix} -9 \\ 6 \end{pmatrix}$

$= -9\mathbf{i} + 6\mathbf{j}$

WRITE VECTOR USING **i**, **j** NOTATION

AND $|\overrightarrow{XY}| = 3\sqrt{13}$

$\therefore \dfrac{(-9\mathbf{i} + 6\mathbf{j})}{3\sqrt{13}} = \dfrac{-9}{3\sqrt{13}}\mathbf{i} + \dfrac{6}{3\sqrt{13}}\mathbf{j}$

UNIT VECTOR IN GIVEN DIRECTION $= \dfrac{\mathbf{a}}{|\mathbf{a}|}$

$= \dfrac{-3}{\sqrt{13}}\mathbf{i} + \dfrac{2}{\sqrt{13}}\mathbf{j}$

SIMPLIFYING SURD FORM AS MUCH AS POSSIBLE

$= \dfrac{-3\sqrt{13}}{13}\mathbf{i} + \dfrac{2\sqrt{13}}{13}\mathbf{j}$

WITH RATIONALISED DENOMINATORS

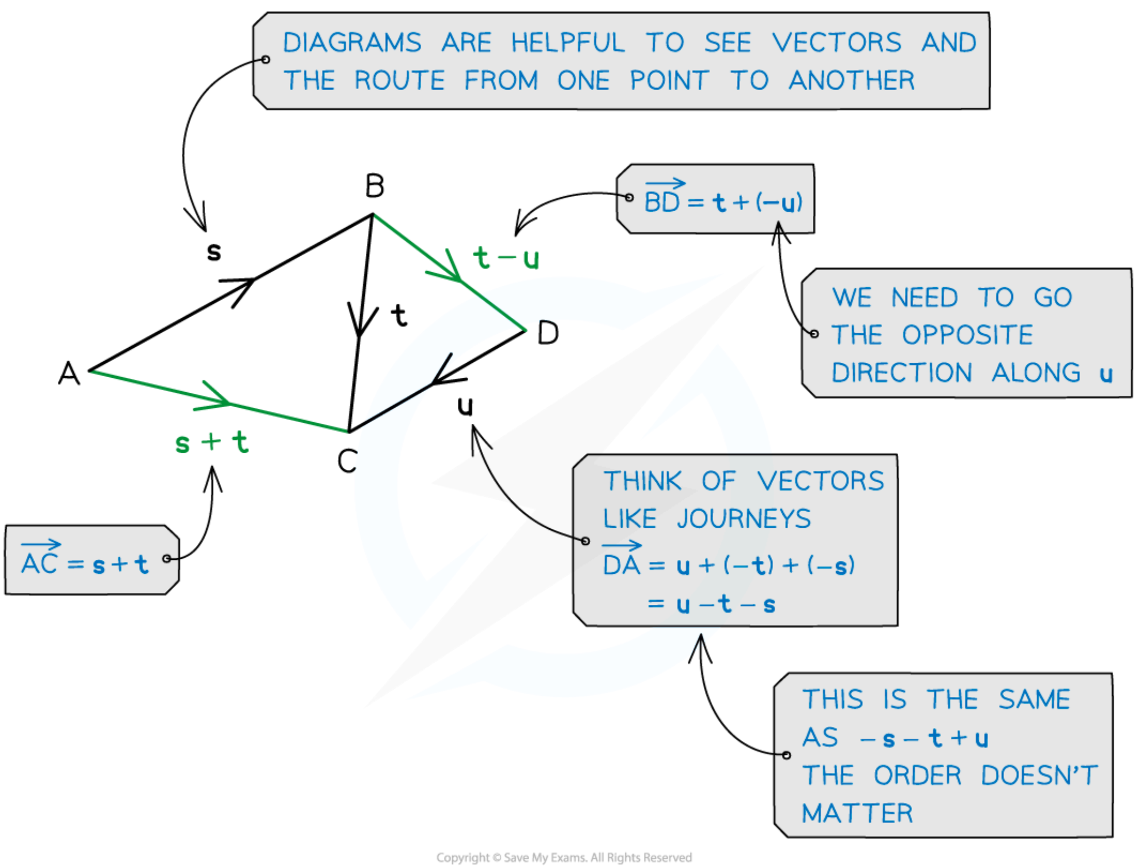
Your notes

11.1.3 Vector Addition

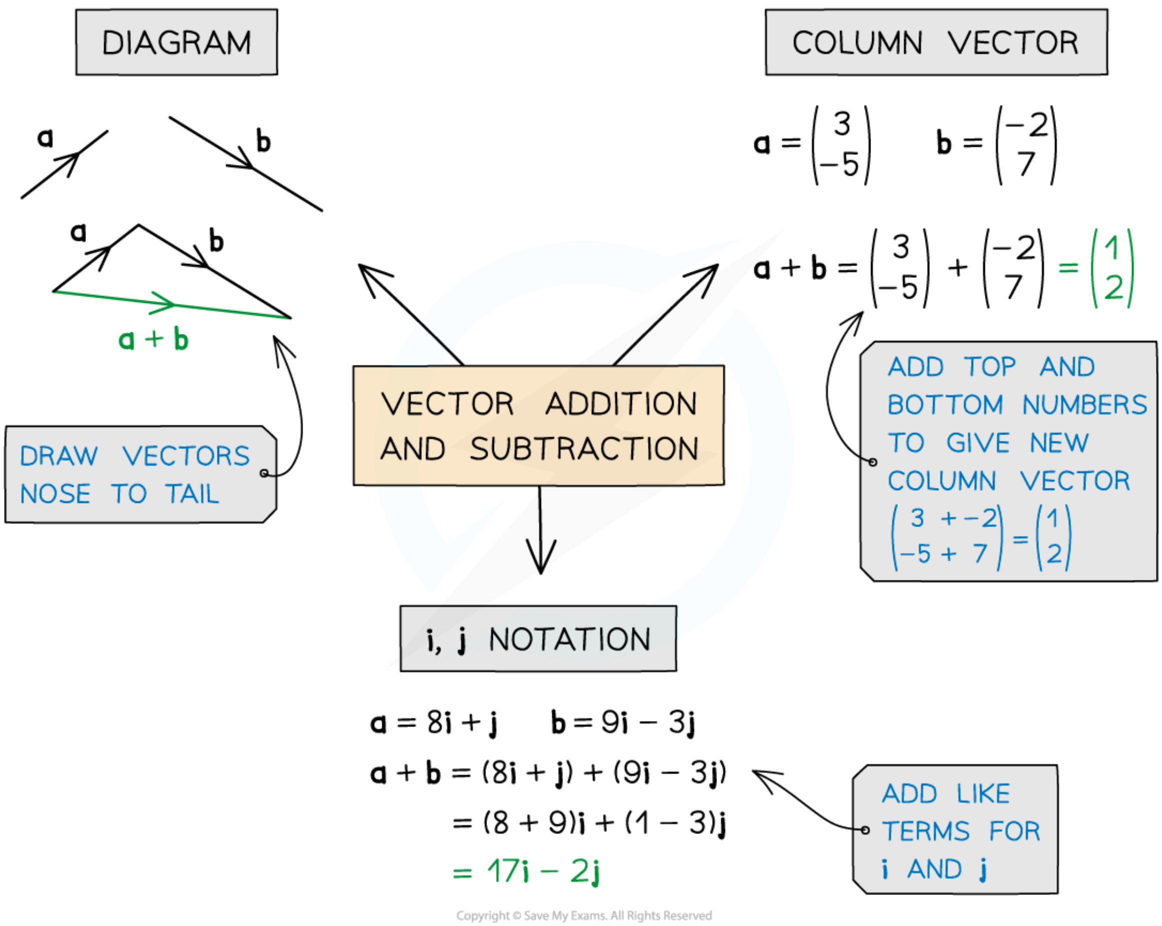
Vector Addition

Vector addition

- **Adding** vectors together lets us describes the **movement between two points**
- The **single** vector this creates is called the **resultant** vector
- **Subtracting** a vector is the same as **adding the negative** vector
- Adding the vectors **PQ** and **QP** gives the zero vector, denoted by a bold zero **0**



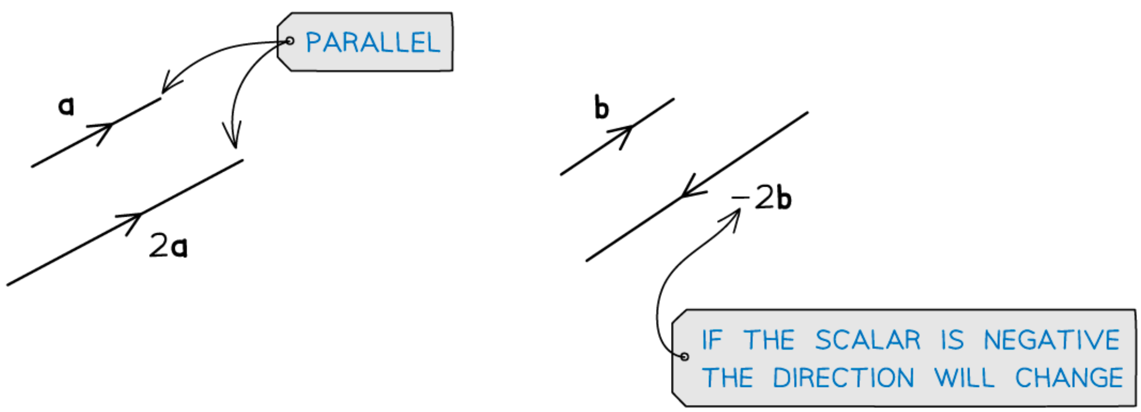
- You can add and subtract vectors in any of the forms previously described:

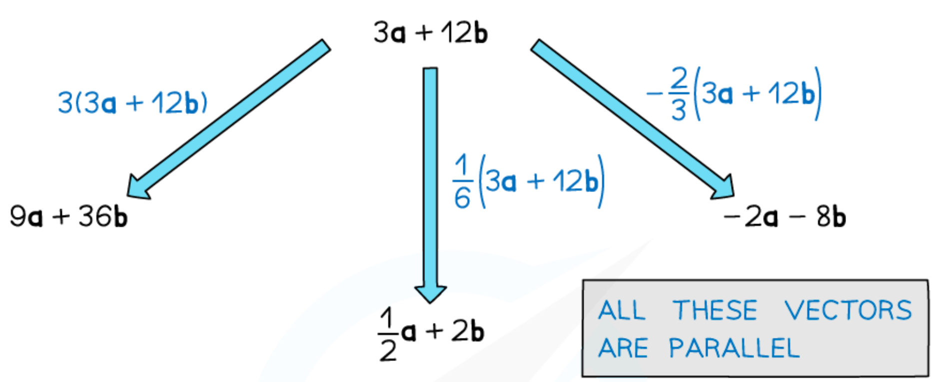


Scalars and parallel vectors

- Multiplying by a positive **scalar** only changes the **size** of a vector, not its **direction**
- Two vectors are **parallel** if and only if one is a **scalar multiple** of the other

Your notes





$\mathbf{a} = \begin{pmatrix} 3 \\ -5 \end{pmatrix}$ $\quad 2\mathbf{a} = 2 \times \begin{pmatrix} 3 \\ -5 \end{pmatrix} = \begin{pmatrix} 6 \\ -10 \end{pmatrix}$

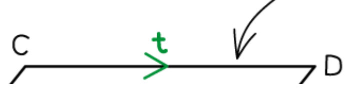
FOR COLUMN VECTORS JUST MULTIPLY TOP AND BOTTOM NUMBERS BY THE SCALAR

$\mathbf{a} = 3\mathbf{i} + 7\mathbf{j}$

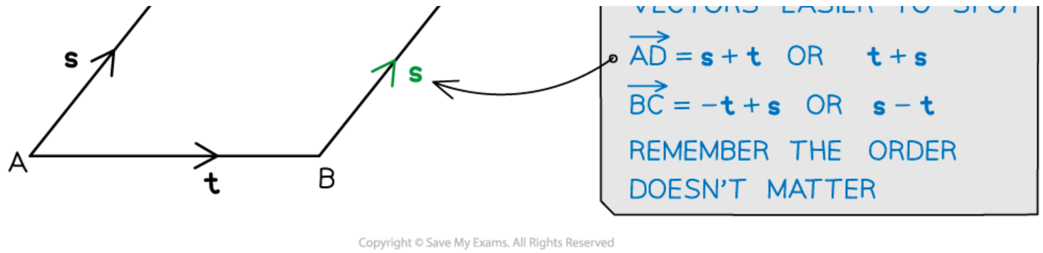
$3\mathbf{a} = 3(3\mathbf{i} + 7\mathbf{j}) = 9\mathbf{i} + 21\mathbf{j}$

FOR **i j** VECTORS JUST MULTIPLY BOTH NUMBERS BY THE SCALAR

ABCD IS A PARALLELOGRAM MEANING $\overrightarrow{AB} = \overrightarrow{CD}$ AND $\overrightarrow{AC} = \overrightarrow{BD}$



LABEL PARALLEL SIDES ON DIAGRAMS, TO MAKE NEW VECTORS EASIER TO SPOT



Your notes

Examiner Tip

- Think of vectors like a journey from one place to another – you may have to take a detour eg. A to B might be A to O then O to B.
- Diagrams can help, so if there isn't one, draw one. If there are any, labelling parallel vectors will help.

Worked example

Your notes

Your notes

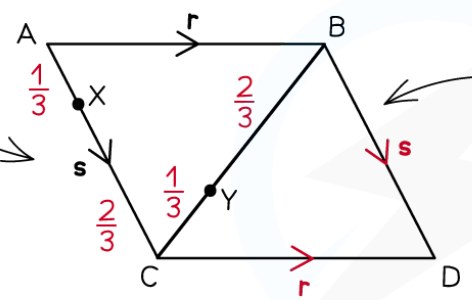
ABCD is a parallelogram with $\overrightarrow{AB} = \mathbf{r}$ and $\overrightarrow{AC} = \mathbf{s}$.

Point X lies on $\overrightarrow{AC}$ such that AX : XC = 1 : 2 and

point Y lies on $\overrightarrow{CB}$ such that CY : YB = 1 : 2

Show that $\overrightarrow{XY}$ is parallel to $\overrightarrow{AD}$.

AX:XC = 1:2 MEANS $AX = \frac{1}{3}\mathbf{s}$ AND $XC = \frac{2}{3}\mathbf{s}$, LABEL THE DIAGRAM TO HELP



DRAW A DIAGRAM TO HELP, LABEL ALL PARALLEL SIDES

$\overrightarrow{AD} = \mathbf{s} + \mathbf{r}$

$\overrightarrow{CB} = \mathbf{r} - \mathbf{s}$

$\overrightarrow{XY} = \frac{2}{3}(\mathbf{s}) + \frac{1}{3}(\mathbf{r} - \mathbf{s})$

$\overrightarrow{XY} = \frac{2}{3}(\overrightarrow{AC}) + \frac{1}{3}(\overrightarrow{CB})$ THINK ABOUT THE SEPARATE PARTS OF THE "JOURNEY"

$\overrightarrow{XY} = \frac{2}{3}\mathbf{s} + \frac{1}{3}\mathbf{r} - \frac{1}{3}\mathbf{s} = \frac{1}{3}\mathbf{r} + \frac{1}{3}\mathbf{s} = \frac{1}{3}(\mathbf{r} + \mathbf{s})$

$\overrightarrow{XY} = \frac{1}{3}(\overrightarrow{AD})$

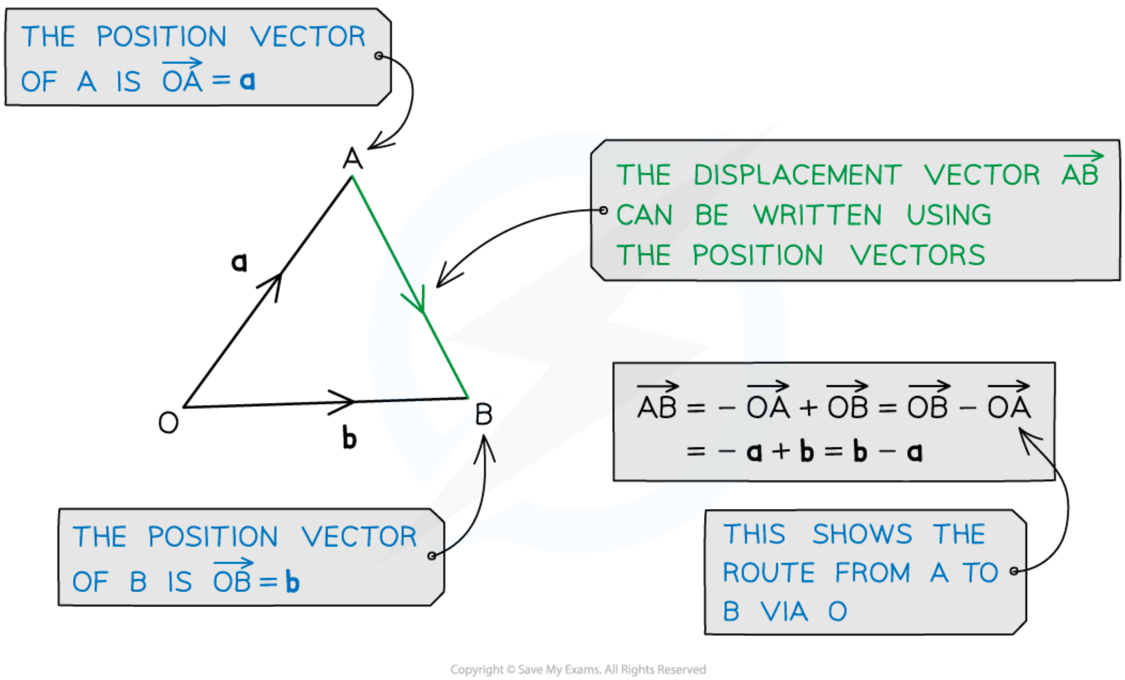
THEY ARE SCALAR MULTIPLES SO MUST BE PARALLEL

11.1.4 Position Vectors

Position Vectors

What is a position vector?

- Position vectors describe the **position of a point** in relation to the origin
- They are different to **displacement vectors** which describe the **direction and distance** between any two points

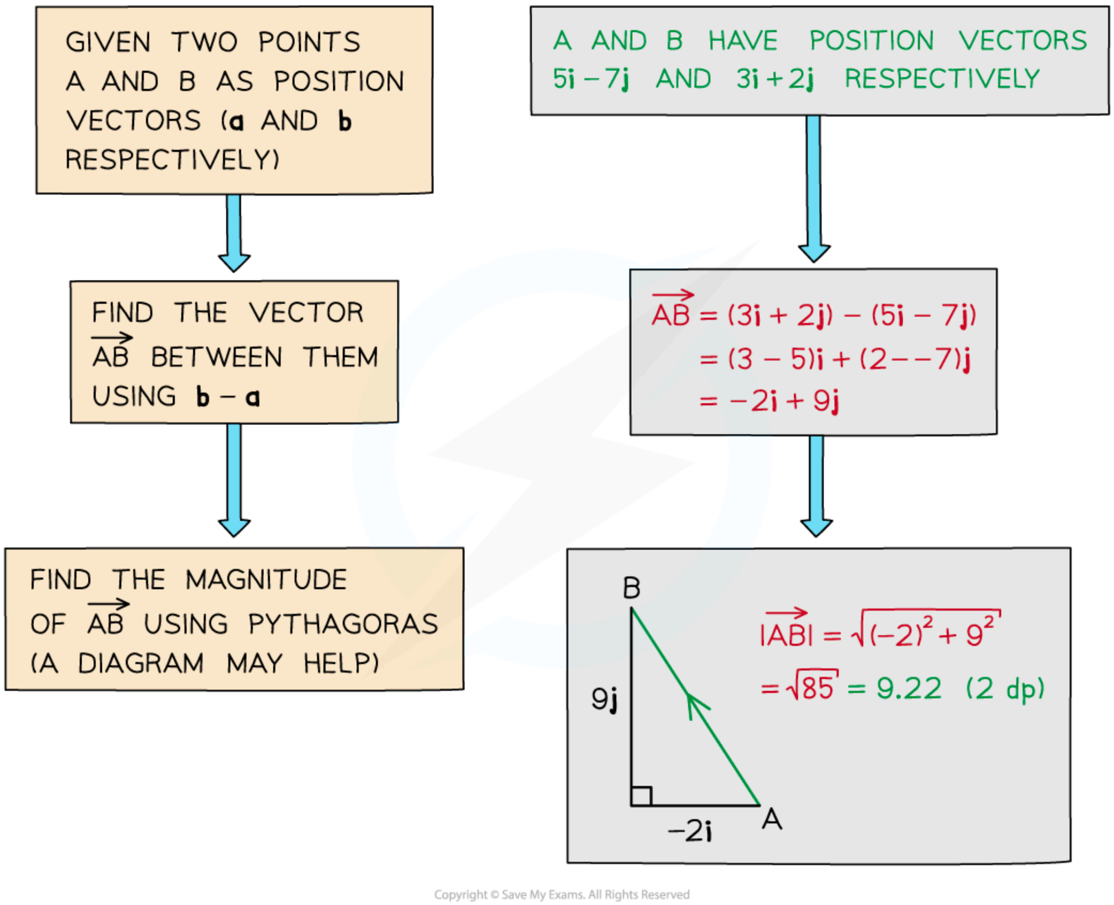


Distance between two points

- The distance between two points is the **magnitude** of the vector between them (see Magnitude Direction)

Your notes

Your notes



How do I find the magnitude of a displacement vector?

- You can use **coordinate geometry** to find **magnitudes** of **displacement vectors** from A to B
  - From the **position vectors** of A and B you know their coordinates
    - If $\mathbf{a} = \overrightarrow{OA} = x_1\mathbf{i} + y_1\mathbf{j} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$, then point A has coordinates (x_1, y_1)
    - If $\mathbf{b} = \overrightarrow{OB} = x_2\mathbf{i} + y_2\mathbf{j} = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$, then point B has coordinates (x_2, y_2)
  - The **distance** between two points is given by $d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$
    - So $|\overrightarrow{AB}| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$
  - For example, if points A and B have position vectors $5\mathbf{i} + 3\mathbf{j}$ and $3\mathbf{i} - 6\mathbf{j}$ respectively

Your notes

- then $\left|\overrightarrow{AB}\right| = \sqrt{(5-3)^2 + (3-(-6))^2} = \sqrt{85} = 9.22\ (3\text{ s.f.})$

- Alternatively, you could find $\left|\overrightarrow{AB}\right|$ by
  - first using $\overrightarrow{AB} = -\overrightarrow{OA} + \overrightarrow{OB}$ to find $\overrightarrow{AB}$ in **vector form**
    - and then calculating its **magnitude** directly
  - See the Worked Example below

Examiner Tip

- Remember if asked for a position vector, you must find the vector all the way from the origin.
- Diagrams can help, if there isn't one, draw one.

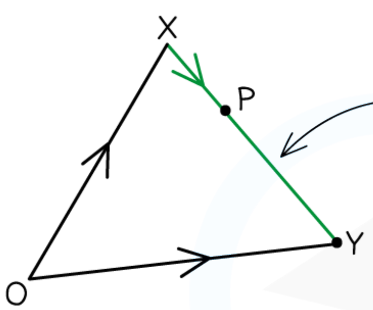
Your notes

Worked example

The points X and Y have position vectors $3\mathbf{i} + 5\mathbf{j}$ and $6\mathbf{i} - 4\mathbf{j}$ respectively.

Point P is on the line XY such that XP : PY = 1 : 2.

Determine the position vector of P.



START BY DRAWING A DIAGRAM TO HELP

FIND VECTORS FOR ANY UNKNOWN PARTS OF THE DIAGRAM

$$\overrightarrow{XY} = \overrightarrow{OY} - \overrightarrow{OX} = (6\mathbf{i} - 4\mathbf{j}) - (3\mathbf{i} + 5\mathbf{j})$$
$$= (6 - 3)\mathbf{i} + (-4 - 5)\mathbf{j}$$
$$\overrightarrow{XY} = 3\mathbf{i} - 9\mathbf{j}$$

XP:PY = 1:2 MEANS P IS $\frac{1}{3}$ OF THE WAY ALONG $\overrightarrow{XY}$

$$\overrightarrow{XP} = \frac{1}{3}(3\mathbf{i} - 9\mathbf{j})$$
$$= \mathbf{i} - 3\mathbf{j}$$

WE ARE ASKED FOR POSITION VECTOR OF P SO MUST START AT THE ORIGIN

$$\overrightarrow{OP} = \overrightarrow{OX} + \overrightarrow{XP} = (3\mathbf{i} + 5\mathbf{j}) + (\mathbf{i} - 3\mathbf{j})$$
$$= (3 + 1)\mathbf{i} + (5 - 3)\mathbf{j}$$
$$= (4\mathbf{i} + 2\mathbf{j})$$

Your notes

11.1.5 Problem Solving using Vectors

Your notes

Problem Solving using Vectors

Problem-solving with vectors

- Vectors can be used to prove two lines are **parallel** (see Vector Addition)
- They can also be used to show points are **collinear** (lie on the same straight line)
- Vectors can be used to find missing vertices of a given shape
- You will need a good understanding of how to divide a line segment into a given ratio

Your notes

VECTORS ARE PARALLEL IF ONE IS A SCALAR **MULTIPLE** OF THE OTHER

$\overrightarrow{AB}$ IS **PARALLEL** TO $\overrightarrow{CD}$

$\overrightarrow{AB} = \mathbf{s}$ $\quad \overrightarrow{CD} = \overrightarrow{CA} + \overrightarrow{AB} + \overrightarrow{BD}$

$= \mathbf{s} - \mathbf{t} + \mathbf{s} + \mathbf{t}$

$= 2\mathbf{s}$

$2\mathbf{s}$ IS A MULTIPLE OF $\mathbf{s}$

$\overrightarrow{CD} = 2\overrightarrow{AB}$

FIND SIMPLIFIED VECTOR FOR BOTH $\overrightarrow{AB}$ AND $\overrightarrow{CD}$ TO PROVE THEY ARE PARALLEL

THREE POINTS P, Q, R ARE **COLLINEAR** IF $\overrightarrow{PQ}$ AND $\overrightarrow{PR}$ ARE PARALLEL

A, X AND D ARE **COLLINEAR**

$\overrightarrow{AX} = \overrightarrow{AC} + \overrightarrow{CX}$

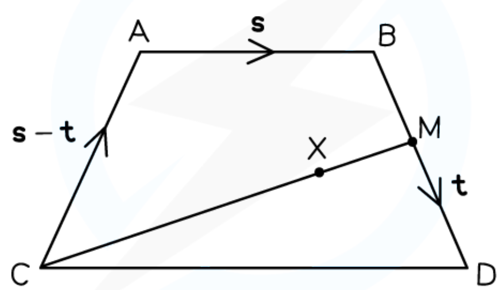
$= \mathbf{t} - \mathbf{s} + \frac{4}{5}(2\mathbf{s} - \frac{1}{2}\mathbf{t})$

$= \frac{3}{5}\mathbf{s} + \frac{3}{5}\mathbf{t}$

$\overrightarrow{AD} = \mathbf{s} + \mathbf{t}$

$\overrightarrow{AX} = \frac{3}{5}\overrightarrow{AD}$ SO $\overrightarrow{AX}$ IS PARALLEL TO $\overrightarrow{AD}$

FIND SIMPLIFIED VECTOR FOR BOTH $\overrightarrow{AX}$ AND $\overrightarrow{AD}$ TO PROVE THEY ARE PARALLEL THEREFORE COLLINEAR



ABCD IS A QUADRILATERAL

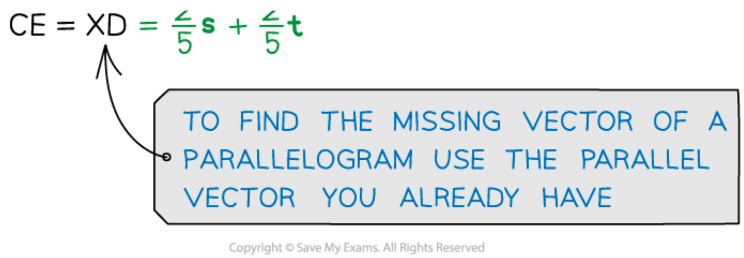
$\overrightarrow{AB} = \mathbf{s}$, $\overrightarrow{BD} = \mathbf{t}$ AND $\overrightarrow{CA} = \mathbf{s} - \mathbf{t}$

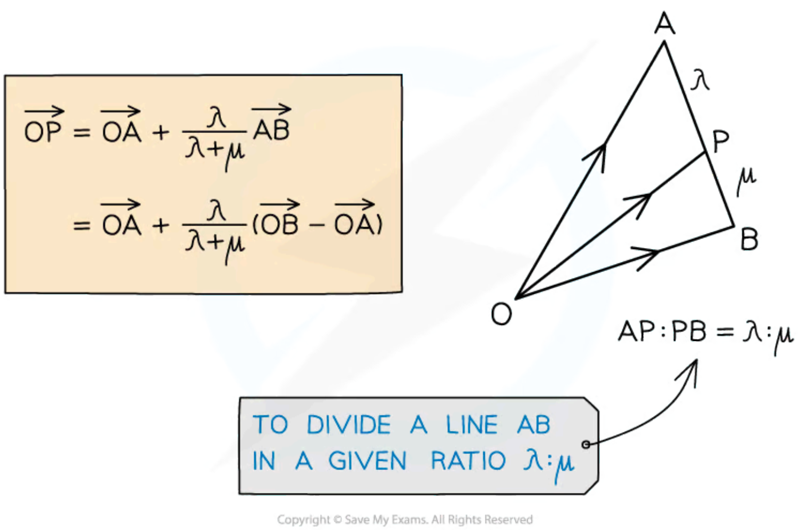
M IS THE MIDPOINT OF BD

X IS A POINT ON CM SUCH THAT CX:XM = 4:1

THE POINT E FORMS A PARALLELOGRAM CXDE

PARALLEL SIDES OF A PARALLELOGRAM HAVE THE **SAME VECTOR**



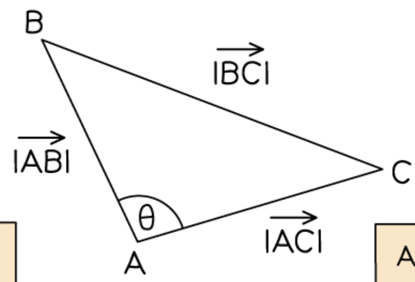


Using trig in vector problems

- When problem-solving with vectors, trigonometry can help us:
  - convert between **component form** and magnitude/direction form (see Magnitude Direction)
  - find the **angle between two vectors** using **Cosine Rule** (see Non-Right-Angled Triangles)
  - find the **area of a triangle** using a variation of **Area Formula** (see Non-Right-Angled Triangles)

Your notes

AFTER IDENTIFYING A TRIANGLE IN 3-D SPACE, APPROACH IT AS YOU WOULD A 2-D VECTOR PROBLEM



ANGLE BETWEEN TWO VECTORS

STEP 1: USE GIVEN VECTORS TO CONSTRUCT A **TRIANGLE** (USUALLY GIVEN TWO SIDES)

STEP 2: FIND THE **VECTOR** FOR THE THIRD SIDE OF THE TRIANGLE (IF NOT GIVEN)

STEP 3: FIND THE **MAGNITUDE** (LENGTH) OF EACH SIDE OF THE TRIANGLE USING **PYTHAGORAS**

STEP 4: USE **COSINE RULE** TO FIND THE ANGLE θ

$$\cos\theta = \frac{b^2 + c^2 - a^2}{2bc}$$

AREA OF A VECTOR TRIANGLE

STEP 1: USE GIVEN VECTORS TO CONSTRUCT A **TRIANGLE** (USUALLY GIVEN TWO SIDES)

STEP 2: FIND THE **VECTOR** FOR THE THIRD SIDE OF THE TRIANGLE (IF NOT GIVEN)

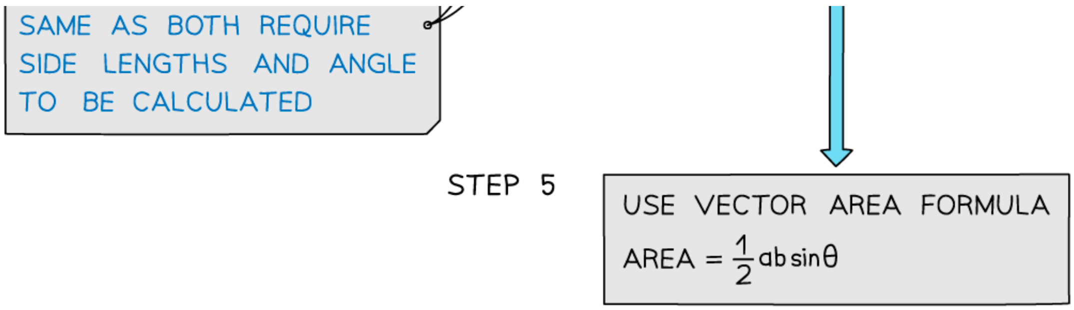
STEP 3: FIND THE **MAGNITUDE** (LENGTH) OF EACH SIDE OF THE TRIANGLE USING **PYTHAGORAS**

STEP 4: USE **COSINE RULE** TO FIND THE ANGLE θ (IF ANGLE NOT GIVEN)

$$\cos\theta = \frac{b^2 + c^2 - a^2}{2bc}$$

THE FIRST FOUR STEPS OF BOTH PROBLEMS ARE THE

Your notes



Examiner Tip

- Think of vectors like a journey from one place to another – you may have to take a detour eg. A to B might be A to O then O to B.
- Diagrams can help, if there isn't one, draw one. For a given diagram labelling all known vectors and quantities will help.

Worked example

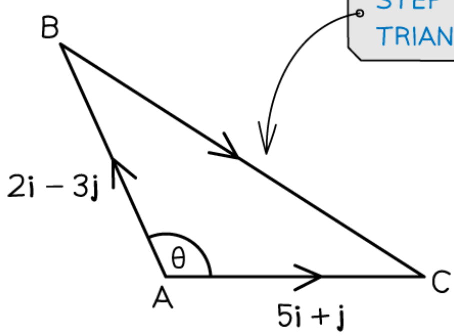
Your notes

Your notes

The vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are 2i – 3j and 5i + j respectively.

Find the area of triangle ABC.

Give your answer to 3 significant figures.



STEP 1: DRAW TRIANGLE

STEP 2: FIND $\overrightarrow{BC}$

$$\overrightarrow{AB} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$$

$$\overrightarrow{AC} = \begin{pmatrix} 5 \\ 1 \end{pmatrix}$$

$$\overrightarrow{BC} = \overrightarrow{AC} - \overrightarrow{AB}$$

$$\begin{pmatrix} 5 \\ 1 \end{pmatrix} - \begin{pmatrix} 2 \\ -3 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

$$|\overrightarrow{AB}| = \sqrt{2^2 + 3^2} = \sqrt{13}$$

$$|\overrightarrow{AC}| = \sqrt{5^2 + 1^2} = \sqrt{26}$$

$$|\overrightarrow{BC}| = \sqrt{3^2 + 4^2} = \sqrt{25} = 5$$

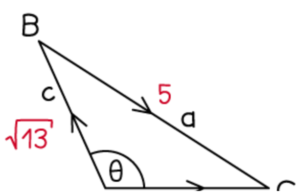
STEP 3: FIND **MAGNITUDE** OF EACH SIDE–LEAVE ANSWERS IN **SURD FORM**

$$\cos\theta = \frac{b^2 + c^2 - a^2}{2bc}$$

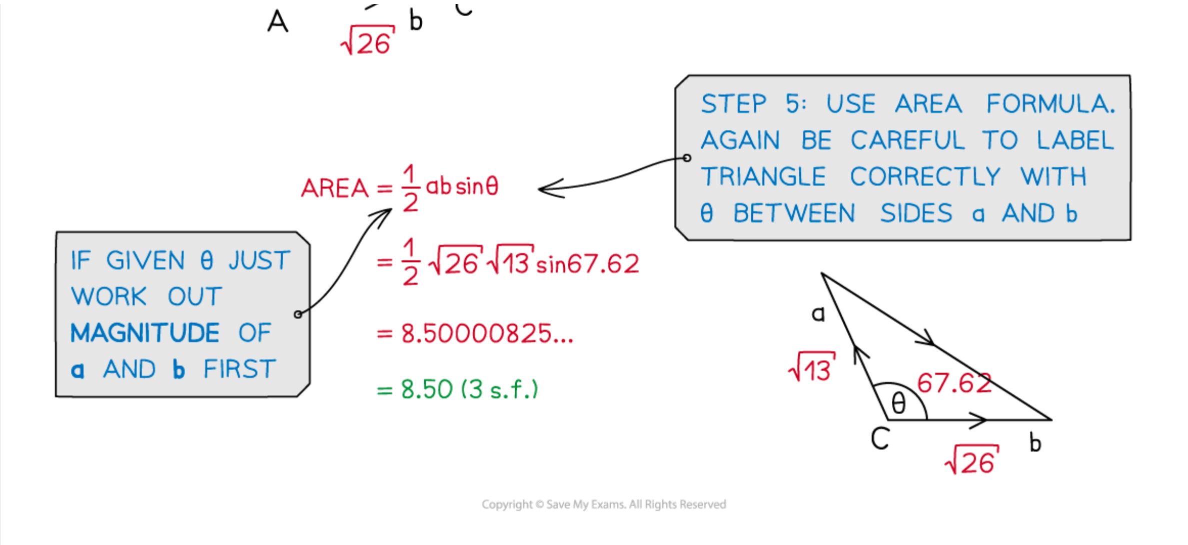
STEP 4: USE **COSINE RULE** TO FIND θ BE CAREFUL TO **LABEL TRIANGLE CORRECTLY** WITH a OPPOSITE θ

$$\cos\theta = \frac{\sqrt{26}^2 + \sqrt{13}^2 - 5^2}{2 \times \sqrt{26} \times \sqrt{13}}$$

$$\cos\theta = \frac{7\sqrt{2}}{26}$$



$$\theta = \cos^{-1}\left(\frac{7\sqrt{2}}{26}\right) = 67.62 \quad (2 \text{ dp})$$


A
$\sqrt{26}$
b
STEP 5: USE AREA FORMULA. AGAIN BE CAREFUL TO LABEL TRIANGLE CORRECTLY WITH θ BETWEEN SIDES a AND b
AREA = $\frac{1}{2}ab\sin\theta$
= $\frac{1}{2}\sqrt{26}\sqrt{13}\sin 67.62$
= 8.50000825...
= 8.50 (3 s.f.)
IF GIVEN θ JUST WORK OUT MAGNITUDE OF a AND b FIRST
a
$\sqrt{13}$
67.62
θ
C
b
$\sqrt{26}$
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