Edexcel A Level Maths: Pure

Your notes

7.4 Further Applications of Differentiation

Contents

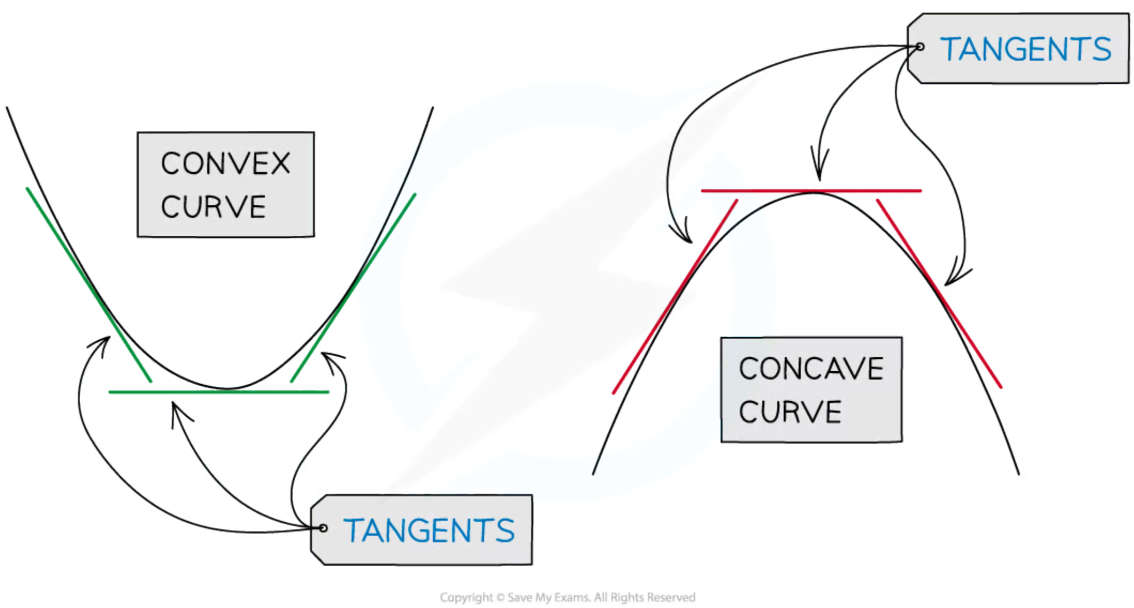
7.4.1 Applications of the Second Derivative

Your notes

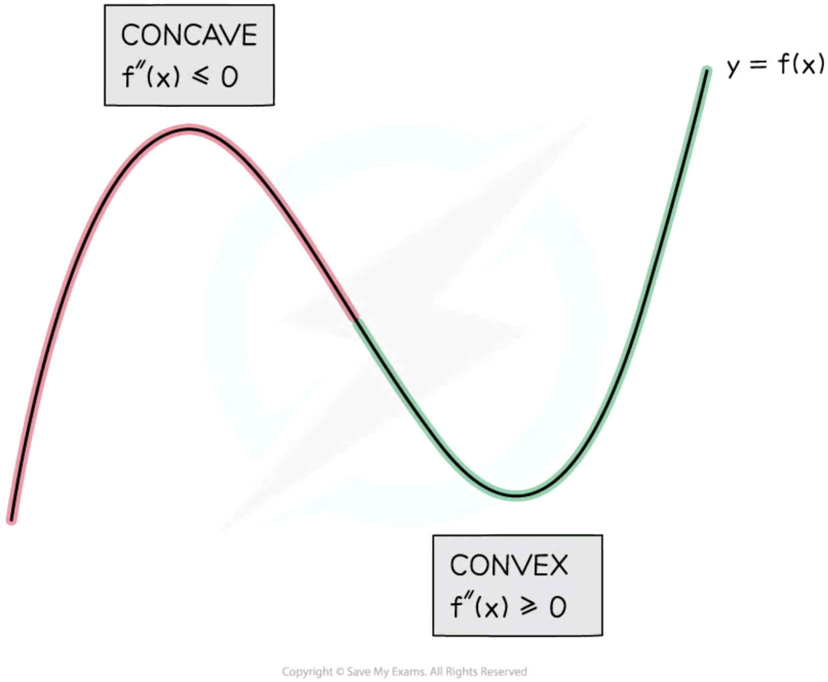
Applications of the Second Derivative

What applications of the second derivative do I need to know?

- The **second order derivative** (or simply **second derivative**) is encountered at AS level
  - At AS level second derivatives are used to help determine the nature of a stationary point
- At A level you need to be able to use the second derivative to determine if a function is **convex** or **concave** on a given interval
  - Where a function is **convex** its graph 'curves up' and its tangent lines lie below the graph
  - Where a function is **concave** its graph 'curves down' and its tangent lines lie above the graph



- It's the sign of the second derivative (+ or -) that's important:
  - A function **f(*x*)** is **convex** on a given interval if and only if **f''(*x*) ≥ 0** for every value of ***x*** in the interval
  - A function **f(*x*)** is **concave** on a given interval if and only if **f''(*x*) ≤ 0** for every value of ***x*** in the interval



Your notes

Examiner Tip

- To get full marks in an exam your working needs to include the condition for a function to be convex or concave.

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Worked example

Find the interval on which the function

$f(x) = 2x^3 - 6x^2 + 13x - 7$ is convex.

$f'(x) = 6x^2 - 12x + 13$ ← FIND FIRST DERIVATIVE

$f''(x) = 12x - 12$ ← FIND SECOND DERIVATIVE

f(x) IS CONVEX WHEN $f''(x) \geq 0$ ← STATE CONDITION FOR FUNCTION TO BE CONVEX

$12x - 12 \geq 0$ ← SOLVE INEQUALITY

$12x \geq 12$

$x \geq 1$

f(x) IS CONVEX ON THE INTERVAL $[1, \infty)$

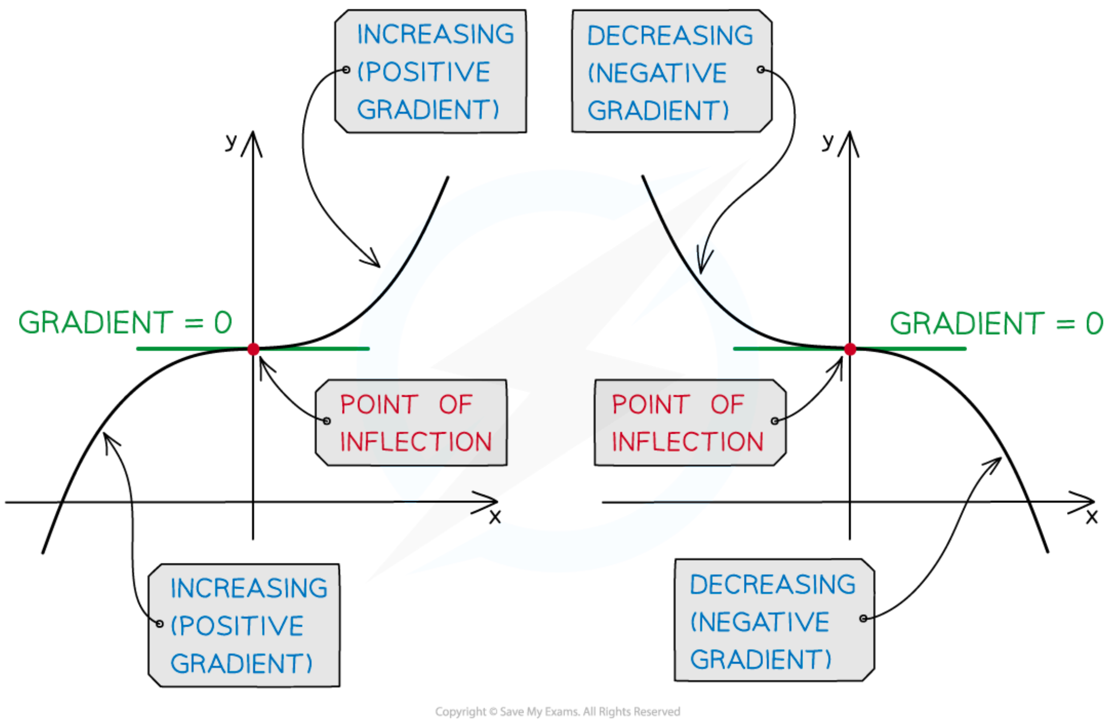
Your notes

7.4.2 Points of Inflection

Points of Inflection

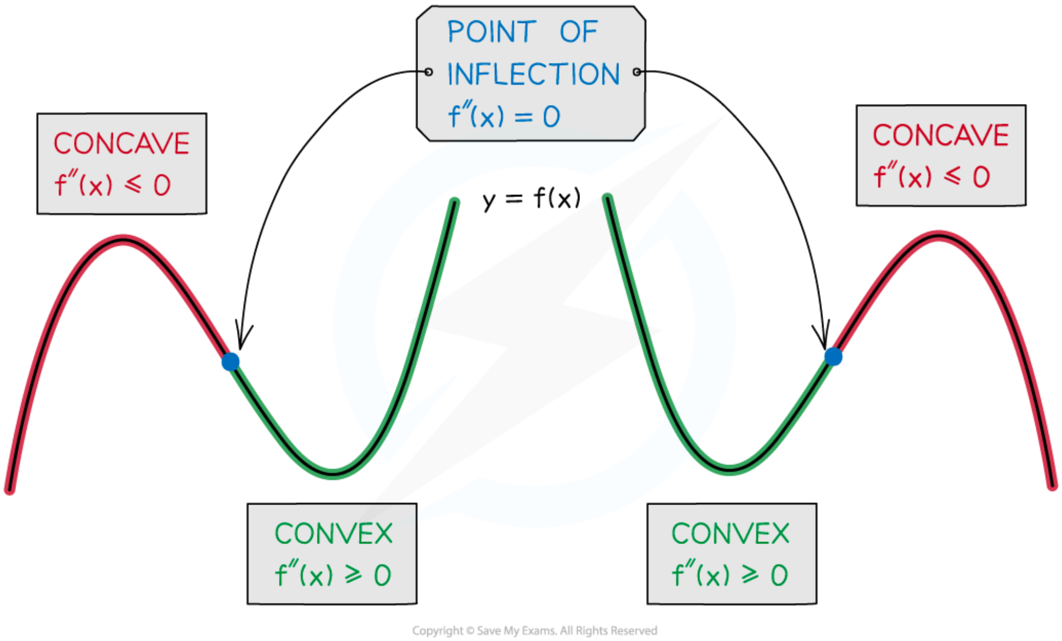
What is a point of inflection?

- At AS level you encountered points of inflection when discussing **stationary points**
  - When the sign of the first derivative (ie of the gradient) is the same on both sides of a stationary point, then the stationary point is a point of inflection



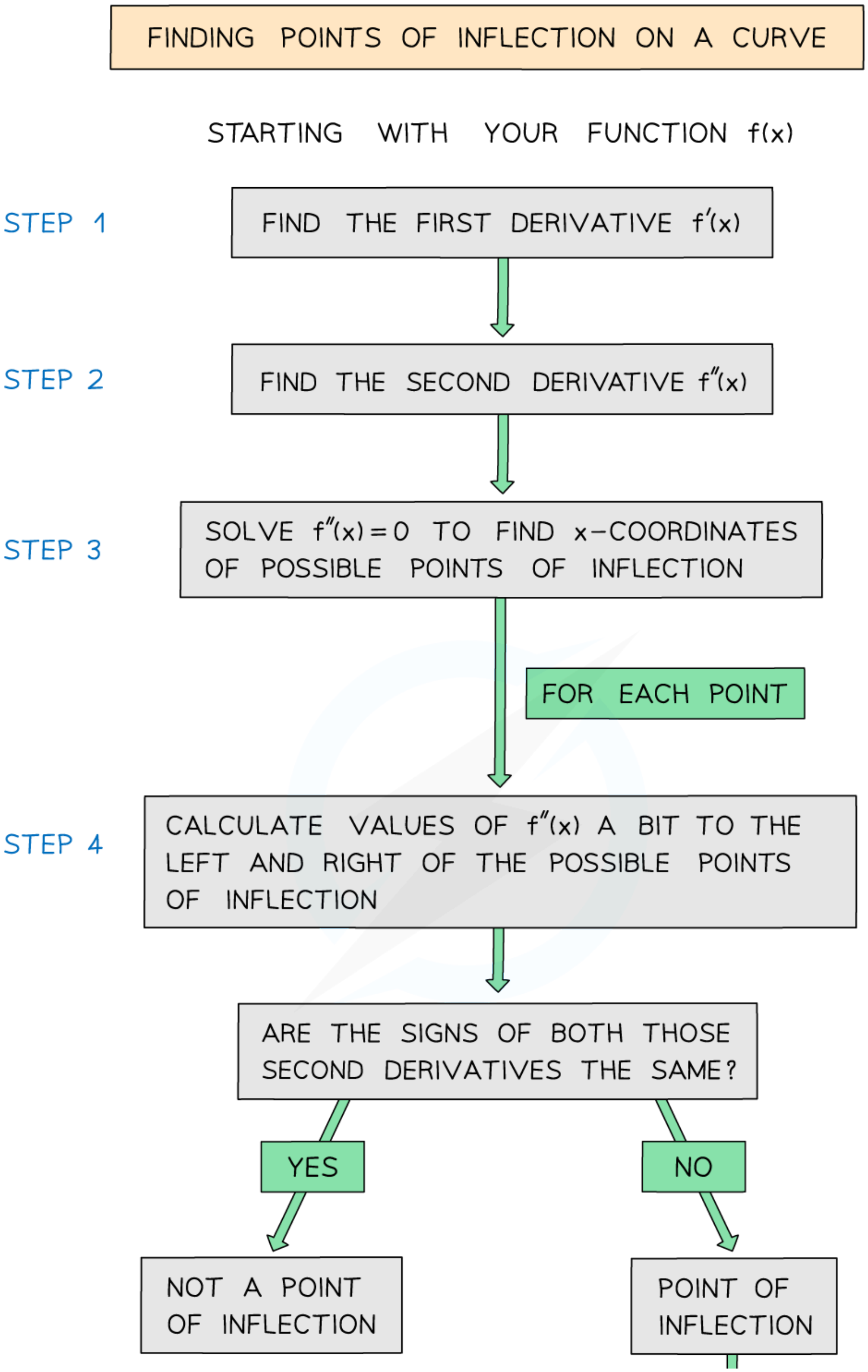
- A point of inflection does not have to be a stationary point however
- A **point of inflection** is any point at which a curve changes from being convex to being concave
  - This means that a **point of inflection** is a point where the **second derivative changes sign** (from positive to negative or vice versa)

Your notes



- To find the points of inflection of a curve with equation $y = f(x)$:

Your notes


FINDING POINTS OF INFLECTION ON A CURVE
STARTING WITH YOUR FUNCTION f(x)
STEP 1
FIND THE FIRST DERIVATIVE f′(x)
STEP 2
FIND THE SECOND DERIVATIVE f″(x)
STEP 3
SOLVE f″(x) = 0 TO FIND x-COORDINATES OF POSSIBLE POINTS OF INFLECTION
FOR EACH POINT
STEP 4
CALCULATE VALUES OF f″(x) A BIT TO THE LEFT AND RIGHT OF THE POSSIBLE POINTS OF INFLECTION
ARE THE SIGNS OF BOTH THOSE SECOND DERIVATIVES THE SAME?
YES
NO
NOT A POINT OF INFLECTION
POINT OF INFLECTION

Your notes

STEP 5

PUT THE x–COORDINATES INTO f(x) TO FIND CORRESPONDING y–COORDINATES

WHEN CHOOSING TEST VALUES OF x 'TO THE LEFT OF' (i.e. A LITTLE BIT LESS THAN) AND 'TO THE RIGHT OF' (i.e. A LITTLE BIT MORE THAN) THE x COORDINATE OF A POSSIBLE POINT OF INFLECTION

- **DO** CHOOSE A CONVENIENT POINT WHERE POSSIBLE ($x = 0$ AND $x = 1$ ARE BOTH VERY EASY TO PUT INTO FORMULAS FOR EXAMPLE)
- **DO NOT** HOWEVER CHOOSE AN x VALUE TO THE LEFT OR RIGHT THAT JUMPS PAST THE x–COORDINATE OF ANOTHER POSSIBLE POINT OF INFLECTION!

Examiner Tip

- Remember – the first derivative (ie the gradient) does NOT have to be zero at a point of inflection!

Worked example

Your notes

Your notes

Find any point(s) of inflection for the curve with equation $y = e^x(2x^2 + 5x + 4)$.

In your answer, give coordinates accurate to 3 decimal places.

STEP 1: $f(x) = e^x(2x^2 + 5x + 4)$

FIND FIRST DERIVATIVE

$f'(x) = e^x(2x^2 + 5x + 4) + e^x(4x + 5)$ ← PRODUCT RULE

$f'(x) = e^x(2x^2 + 9x + 9)$

STEP 2: $f''(x) = e^x(2x^2 + 9x + 9) + e^x(4x + 9)$

FIND SECOND DERIVATIVE

PRODUCT RULE AGAIN

$f''(x) = e^x(2x^2 + 13x + 18)$

STEP 3: $e^x(2x^2 + 13x + 18) = 0$

SOLVE $f''(x) = 0$

$2x^2 + 13x + 18 = 0$ $(e^x > 0$ FOR ALL $x)$

$(2x + 9)(x + 2) = 0$

$x = -4.5$ $x = -2$

STEP 4: WHEN $x = -5$,
$f''(x) = e^{-5}(2(-5)^2 + 13(-5) + 18)$
$= 3e^{-5} > 0$

CHECK SIGN OF $f''(x)$ TO LEFT AND RIGHT OF EACH POINT

Your notes

WHEN $x = -3$,
$f''(x) = e^{-3}(2(-3)^2 + 13(-3) + 18)$
$= -3e^{-3} < 0$

WHEN $x = 0$,
$f''(x) = e^{0}(2(0)^2 + 13(0) + 18)$
$= 18 > 0$

$f''(x)$ CHANGES SIGN AT BOTH POINTS,
SO BOTH ARE POINTS OF INFLECTION

STEP 5: WHEN $x = -4.5$,
$y = e^{-4.5}(2(-4.5)^2 + 5(-4.5) + 4) = 0.24439...$

FIND y-COORDINATES

WHEN $x = -2$,
$y = e^{-2}(2(-2)^2 + 5(-2) + 4) = 0.27067...$

THE POINTS OF INFLECTION
ARE $(-4.5, 0.244)$ AND $(-2, 0.271)$

7.4.3 Connected Rates of Change

Your notes

Connected Rates of Change

What are connected rates of change?

- In situations involving more than two variables you can use the **chain rule** to connect multiple rates of change into a single equation e.g. THE VOLUME V, OF A CUBE IS INCREASING AT A CONSTANT RATE OF 3 cm^3 PER SECOND. ASSUMING THAT THE SHAPE REMAINS CUBICAL AT ALL TIMES, FIND THE RATE OF CHANGE OF THE SIDE LENGTH, S, IN cm PER SECOND AT THE MOMENT WHEN $S = 5\text{ cm}$.

$$\frac{dV}{dt} = 3\text{ cm}^3/\text{sec}$$

WE KNOW THIS

WE WANT TO KNOW THIS

BY THE CHAIN RULE $\frac{dV}{dt} = \frac{dV}{ds} \times \frac{ds}{dt}$

USE THE CHAIN RULE TO CONNECT WHAT YOU KNOW AND WHAT YOU WANT TO KNOW

FOR A CUBE, $V = s^3$

SO $\frac{dV}{ds} = 3s^2$

USE DIFFERENTIATION TO GET THE 'MISSING' DERIVATIVE FROM THE CHAIN RULE EQUATION

SO $\frac{ds}{dt} = \frac{\frac{dV}{dt}}{\frac{dV}{ds}} = \frac{3}{3s^2} = \frac{1}{s^2}$

REARRANGE

WHEN $s = 5$,

$$\frac{ds}{dt} = \frac{1}{5^2} = \frac{1}{25}\text{ cm}/\text{sec}$$

- Equations involving derivatives (ie rates of changes) are known as **differential equations**
  - These can be solved using methods of integration (see Differential Equations)
  - However setting up the equation from the information given can involve the chain rule and connected rates of change

Your notes

e.g. A RIGHT–ANGLED TRIANGLE HAS A BASE OF **3p** cm AND A HEIGHT OF **2p** cm, WHERE **p** > 0 IS A PARAMETER THAT IS INCREASING OVER TIME. THE RATE OF INCREASE OF **p** IS SUCH THAT THE RATE OF INCREASE OF THE THE AREA OF THE TRIANGLE, **A**, IS PROPORTIONAL TO THE SQUARE ROOT OF **p**. WRITE DOWN A DIFFERENTIAL EQUATION FOR THE RATE OF CHANGE OF **p**.

$\frac{dA}{dt} \propto \sqrt{p}$ SO $\frac{dA}{dt} = k\sqrt{p}$

A IS INCREASING SO $k > 0$

WHERE k IS A POSTITVE CONSTANT

WE KNOW THIS

BY THE CHAIN RULE, $\frac{dA}{dt} = \frac{dA}{dp} \times \frac{dp}{dt}$

WE WANT TO KNOW THIS

USE THE CHAIN RULE TO CONNECT WHAT YOU KNOW AND WHAT YOU WANT TO KNOW

THE AREA OF THE TRIANGLE IS

$A = \frac{1}{2} \times 3p \times 2p = 3p^2$

USE DIFFERENTIATION TO GET THE 'MISSING' DERIVATIVE FROM THE CHAIN RULE EQUATION

SO $\frac{dA}{dp} = 6p$

THEN $\frac{dA}{dt} = \frac{dA}{dp} \times \frac{dp}{dt}$

$k\sqrt{p} = 6p\left(\frac{dp}{dt}\right)$

SUBSTITUTE AND REARRANGE

AND $\frac{dp}{dt} = \frac{k\sqrt{p}}{6p}$

$\frac{dp}{dt} = \frac{k}{6\sqrt{p}}$ (WHERE k IS A POSITIVE CONSTANT)

- Note from the examples above that you will often need to differentiate a formula to get one of the rates of change you need

Examiner Tip

- These problems can involve a lot of letters – be sure to keep track of what they all refer to.
- Be especially sure that you are clear about which letters are **variables** and which are **constants** – these behave very differently when differentiation is involved!

Worked example

Your notes

In a manufacturing process, plastic spheres are produced in such a way that the volume, V, of a sphere increases at a constant rate of 10 cm³ per second. Find the rate of change of the surface area, A, of a sphere at the moment when the surface area is equal to 32π cm².

$\frac{dV}{dt} = 10 \text{ cm}^3/\text{sec}$

WE KNOW THIS

WE WANT TO KNOW THIS

BY THE CHAIN RULE $\frac{dV}{dt} = \frac{dV}{dA} \times \frac{dA}{dt}$

USE THE CHAIN RULE TO CONNECT WHAT YOU KNOW AND WHAT YOU WANT TO KNOW

$V = \frac{4}{3}\pi r^3$ (WHERE r IS THE RADIUS)

WRITE V IN TERMS OF A

$A = 4\pi r^2$

$A^{\frac{3}{2}} = (4\pi r^2)^{\frac{3}{2}} = 8\pi^{\frac{3}{2}} r^3 = 8\pi\sqrt{\pi}\, r^3$

$\frac{8\pi\sqrt{\pi}\, r^3}{6\sqrt{\pi}} = \frac{4}{3}\pi r^3$ SO $V = \frac{1}{6\sqrt{\pi}} A^{\frac{3}{2}}$

AND $\frac{dV}{dA} = \frac{3}{2} \times \frac{1}{6\sqrt{\pi}} A^{\frac{3}{2}-1} = \frac{1}{4\sqrt{\pi}} A^{\frac{1}{2}}$

DIFFERENTIATE TO FIND THE 'MISSING' DERIVATIVE

Your notes

SUBSTITUTE AND REARRANGE

$$\text{SO} \quad \frac{dV}{dt} = \frac{dV}{dA} \times \frac{dA}{dt}$$

$$10 = \frac{1}{4\sqrt{\pi}} A^{\frac{1}{2}} \left(\frac{dA}{dt}\right)$$

$$\text{AND} \quad \frac{dA}{dt} = \frac{40\sqrt{\pi}}{A^{\frac{1}{2}}} = 40\sqrt{\frac{\pi}{A}}$$

WHEN $A = 32\pi$,

$$\frac{dA}{dt} = 40\sqrt{\frac{\pi}{32\pi}} = \frac{40}{\sqrt{32}} = 5\sqrt{2}\ \text{cm}^2/\text{sec}$$