Edexcel A Level Maths: Pure

Your notes

8.2 Further Integration

Contents

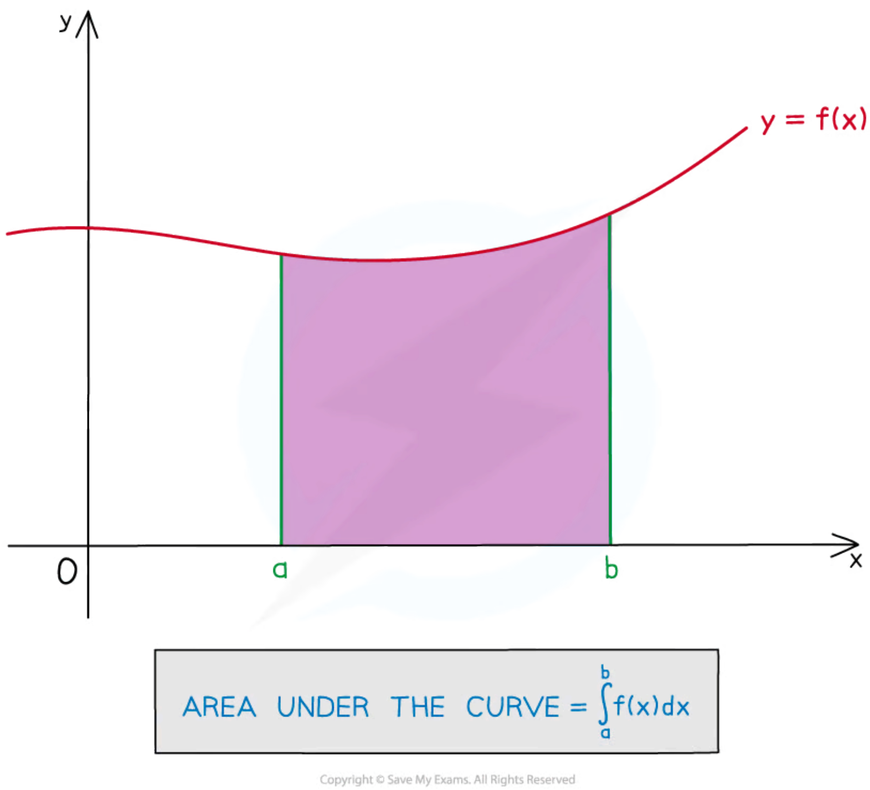
8.2.1 Integration as the limit of a sum

Your notes

Integration as the limit of a sum

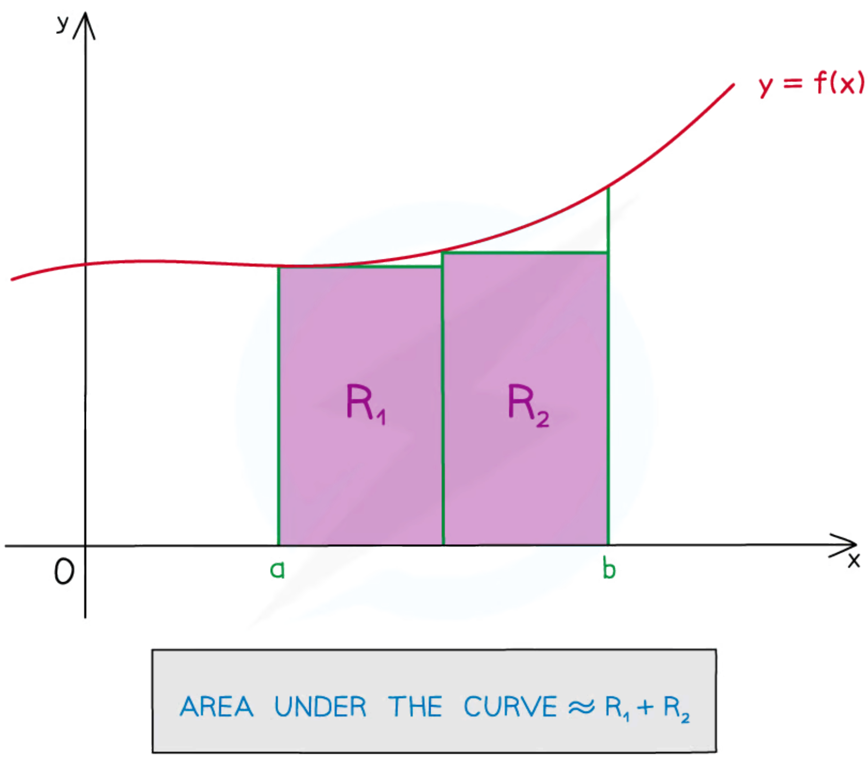
Finding the area under a curve

- Definite integration allows us to find the area under a curve



- An **estimate** for the area under the curve is the sum of the rectangular areas

Your notes



- If the number of rectangles increases and their width decreases, the estimate is more accurate

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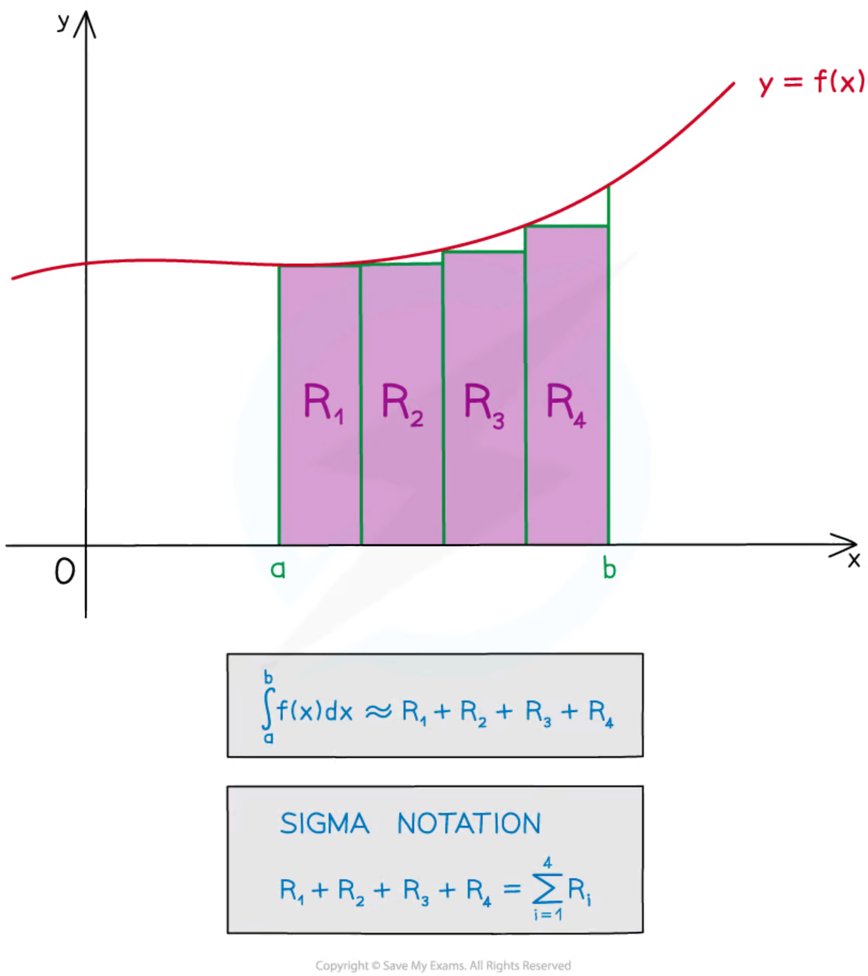
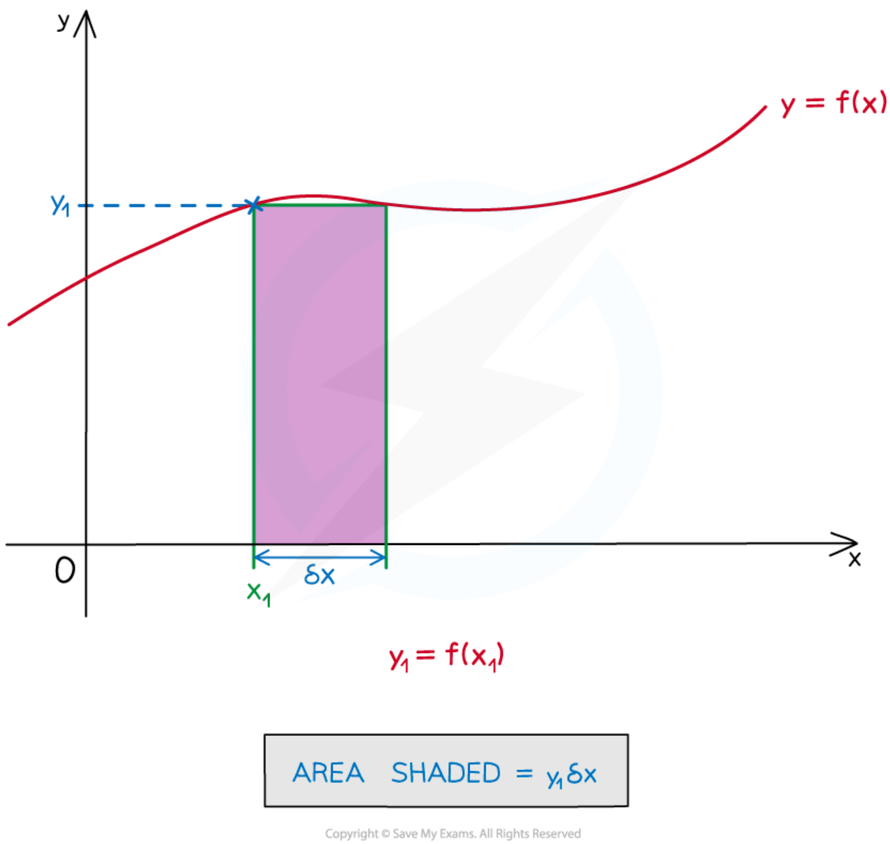


- The sum of the rectangle areas will have a limit, however small they get
  - The sum will become closer and closer to the area under the curve
  - This is called the **limit of the sum**

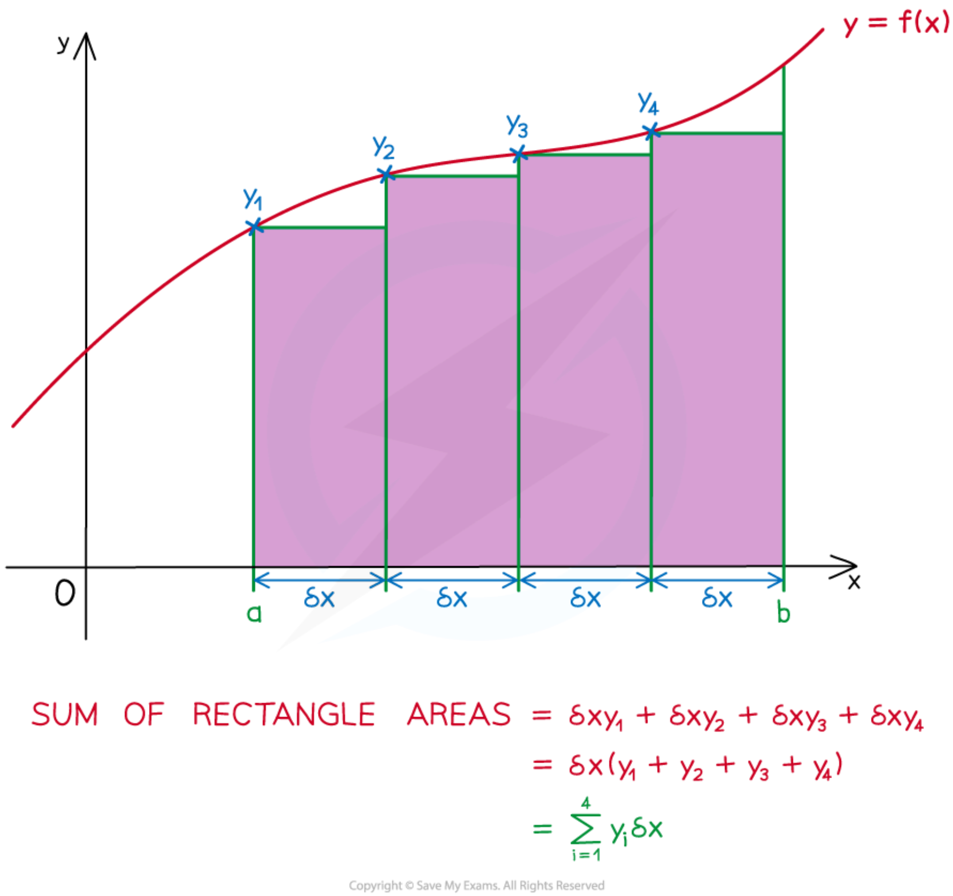
What is integration as the limit of a sum?

Your notes



- The width of a rectangle can be considered as a small increase along the **x**-axis
- This is denoted by $\boldsymbol{\delta x}$
- The height (length) will be the **y**-coordinate at $\mathbf{x_1}$ – ie $\mathbf{f(x_1)}$ (rather than $\mathbf{f(x_1+\delta x)}$)
- If we use four of these small rectangles between a and b we get

Your notes



- As more rectangles are used …
  - … **δx** gets smaller and smaller, ie **δx → 0**
  - … **n**, the number of rectangles, gets bigger and bigger, ie **n → ∞**
  - … the sum of the area of the rectangles becomes closer to the area under the curve

Your notes

AS $\delta x \to 0$ (AND $n \to \infty$)

SUM OF THE AREA OF RECTANGLES $\rightarrow$ AREA UNDER THE CURVE

$$\sum_{i=1}^{n} y_i \delta x \rightarrow \int_a^b f(x)\,dx$$

... THEREFORE ...

$$\boxed{\lim_{\delta x \to 0} \sum_{x=a}^{b} f(x)\delta x = \int_a^b f(x)\,dx}$$

- This is the meaning of **integration as the limit of a sum**

How do questions use integration as the limit of a sum?

- STEP 1 Recognise the notation
- STEP 2 Convert to a definite integral
- STEP 3 Find the value of the integral

Your notes

e.g. CALCULATE $\lim_{\delta x \to 0} \sum_{x=2}^{5} (3x^2 - 4)\delta x$

STEP 1 — RECOGNISE THE NOTATION

$f(x) = 3x^2 - 4$

UPPER LIMIT, $b = 5$

LOWER LIMIT, $a = 2$

STEP 2 — CONVERT TO AN INTEGRAL

$$\lim_{\delta x \to 0} \sum_{x=2}^{5} (3x^2 - 4)\delta x = \int_2^5 (3x^2 - 4)\,dx$$

STEP 3 — FIND THE VALUE OF THE INTEGRAL

$$\int_2^5 (3x^2 - 4)\,dx = \left[x^3 - 4x\right]_2^5$$

$$= (5^3 - 4 \times 5) - (2^3 - 4 \times 2)$$

$$= 105 - 0$$

$$= 105$$

UNITS MAY BE NEEDED IF QUESTION HAD REFERRED TO AREA

Your notes

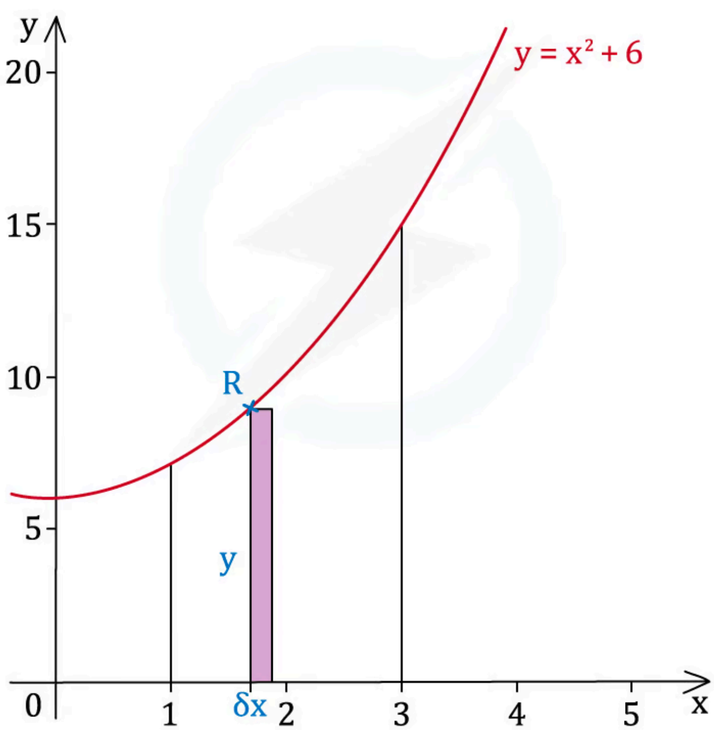
Worked example

A point $R(x,\ y)$ lies on the curve.

The rectangle shown has width δx and height y.

Calculate

$$\lim_{\delta x \to 0} \sum_{x=1}^{3} (x^2 + 6)\delta x$$



Your notes

CONVERT TO A DEFINITE INTEGRAL...

$$\lim_{\delta x \to 0} \sum_{x=1}^{3} (x^2 + 6)\delta x = \int_1^3 (x^2 + 6)\,dx$$

... AND FIND ITS VALUE

$$= \left[\frac{x^3}{3} + 6x\right]_1^3$$

$$= (9 + 18) - \left(\frac{1}{3} + 6\right)$$

$$= \frac{62}{3}$$

8.2.2 Integrating Other Functions (Trig, ln & e etc)

Your notes

Integrating Other Functions (Trig, ln & e etc)

Isn't integration just the reverse of differentiation?

- Yes, but remember "+c", the constant of integration ...
  - ... unless finding a definite integral
- Recognising common results helps to make integration easier (See Differentiating Other Functions)

$$y = e^{kx} \qquad \frac{dy}{dx} = ke^{kx}$$

$$y = \ln kx \qquad \frac{dy}{dx} = \frac{1}{x}$$

NOTE: NO k

$$y = \sin x \qquad \frac{dy}{dx} = \cos x$$

$$y = \cos x \qquad \frac{dy}{dx} = -\sin x$$

$$y = \tan x \qquad \frac{dy}{dx} = \sec^2 x$$

How do I integrate exponentials (e^x)?

$$\int e^x \, dx = e^x + c$$

c IS THE CONSTANT OF INTEGRATION

- The gradient of $\mathbf{e^{kx}}$ is $\mathbf{ke^{kx}}$

- ie **$y = e^{kx}$**, **$dy/dx = ke^{kx}$**
- The reverse applies when integrating
- This is an example of **reverse chain rule**

$$\int ke^{kx}\,dx = e^{kx} + c$$

Your notes

Integrating $\frac{1}{x}$

$$\int \frac{1}{x}\,dx = \ln|x| + c$$

MODULUS OF x
x>0 IN lnx BUT NOT IN $\frac{1}{x}$

- Remember $\frac{1}{x} = x^{-1}$
  - The method for integrating powers does **not** apply if the power is -1

Integrating sin and cos

$$\int \sin x\,dx = -\cos x + c$$

$$\int \cos x\,dx = \sin x + c$$

- Note the minus in the integral of **sin x**
- The integral of **tan x** is **ln|sec x| + c**

Your notes

Examiner Tip

- Make sure you have a copy of the formula booklet during revision but don't try to remember everything in the formula booklet.
- However, do be familiar with the **layout** of the formula booklet – you'll be able to locate quickly whatever you are after, and you do not want to be searching every line of every page!
- For formulae you think you have remembered, use the booklet to double-check.

Your notes

Worked example

Find the following integrals

(a) $\int 3e^{3x}\,\mathrm{d}x$

(b) $\int_{-6}^{-3} \frac{1}{x}\,\mathrm{d}x$

(c) $\int \sin x\,\mathrm{d}x$

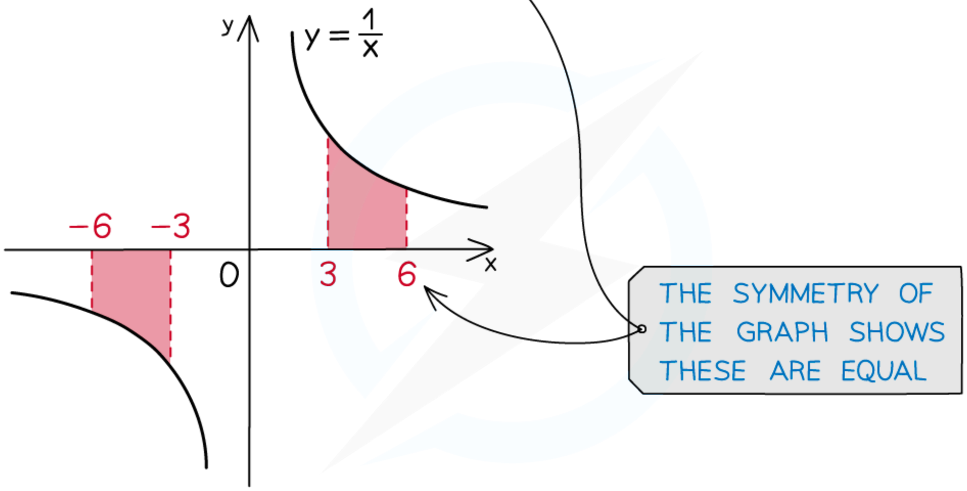
a) $\int 3e^{3x}dx = e^{3x} + c$

IN THE FORM ke^{kx}

Your notes

b) $\int_{-6}^{-3} \frac{1}{x}\,dx = \left[\ln|x|\right]_{-6}^{-3}$

$= \left[\ln x\right]_{3}^{6}$



$= \ln 6 - \ln 3$

$= \ln \frac{6}{3}$

$= \ln 2$

c) $\int \sin x\,dx = -\cos x + c$

BEWARE OF THE MINUS!

Your notes

8.2.3 Reverse Chain Rule

Reverse Chain Rule

What is the chain rule?

- The Chain Rule is a way of differentiating two (or more) functions

FOR $y = f(u)$, WHERE $u = g(x)$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

- In many simple cases the above formula/substitution is not needed

1. $y = (3x)^5$

$$\frac{dy}{dx} = 5(3x)^4 \times 3$$

$$\frac{dy}{dx} = 15(3x)^4$$

2. $y = e^{5x}$

$$\frac{dy}{dx} = e^{5x} \times 5$$

$$\frac{dy}{dx} = 5e^{5x}$$

ALSO FROM $e^{kx} \longrightarrow ke^{kx}$

3. $y = \sin^4 2x$

$$\frac{dy}{dx} = 4\sin^3 2x \times \cos 2x \times 2$$

$$\frac{dy}{dx} = 8\cos 2x \sin^3 2x$$

CHAIN RULE APPLIED TWICE

- The same can apply for the reverse – integration

Integrating with reverse chain rule

1. $\int 3\cos 3x \, dx = \sin 3x + c$

$$\boxed{\begin{array}{l} \frac{d}{dx}[\sin 3x] = 3\cos 3x \\ \text{BY CHAIN RULE} \end{array}}$$

2. $\int 8e^{8x} \, dx = e^{8x} + c$

$$\boxed{\frac{d}{dx}[e^{8x}] = 8e^{8x}}$$

3. $\int 10(2x+1)^4 \, dx = (2x+1)^5 + c$

$$\boxed{\begin{aligned} \frac{d}{dx}\left[(2x+1)^5\right] &= 5(2x+1)^4 \times 2 \\ &= 10(2x+1)^4 \end{aligned}}$$

- In more awkward cases it can help to write the numbers in before integrating
- STEP 1: Spot the 'main' function
- STEP 2: 'Adjust' and 'compensate' any numbers/constants required in the integral
- STEP 3: Integrate and simplify

Your notes

1. $\int \cos 3x \, dx$

STEP 1: SPOT THE "MAIN" FUNCTION

cos – WHICH WOULD COME FROM sin

CHAIN RULE SAYS TO MULTIPLY BY THE DERIVATIVE OF 3x, WHICH IS 3

THERE IS NO 3 IN THE INTEGRAL SO I NEED TO ADJUST AND COMPENSATE FOR IT

STEP 2: "ADJUST" AND "COMPENSATE" ANY NUMBERS REQUIRED IN THE INTEGRAL

$$= \frac{1}{3}\int 3\cos 3x \, dx$$

3cos3x IS WHAT I'D EXPECT TO SEE FROM DIFFERENTIATING sin3x USING CHAIN RULE

STEP 3: INTEGRATE AND SIMPLIFY

$$= \frac{1}{3}\sin 3x + c$$

Your notes

Examiner Tip

- If in doubt you can always use a substitution.
- Differentiation is easier than integration so if stuck try the opposite, eg. **sin** and **cos** are linked (remember that minus!) so if integrating a **sin** function, start by differentiating the corresponding **cos** function.
- Lastly, check your final answer by differentiating it.

Your notes

Worked example

Find the following integrals

(a) $\int \cos 3x \, dx$

(b) $\int 5e^{6x} \, dx$

(c) $\int (2x+5)^3 \, dx$

a) $\int \cos 3x \, dx$

STEP 1: cos IS THE MAIN FUNCTION

STEP 2: • "ADJUST": 3
• "COMPENSATE": $\frac{1}{3}$

$= \frac{1}{3}\int 3\cos 3x \, dx$

STEP 3: 3cos3x IS THE DIFFERENTIAL OF sin3x BY CHAIN RULE

$= \frac{1}{3}\sin 3x + c$

Your notes

b) $\int 5e^{6x}\,dx$

STEP 1: e^{6x} IS THE MAIN FUNCTION

$$= 5 \times \frac{1}{6}\int 6e^{6x}\,dx$$

CONSTANT

STEP 2: • "ADJUST": 6
• "COMPENSATE": $\frac{1}{6}$

$$= \frac{5}{6}e^{6x} + c$$

STEP 3: $\frac{d}{dx}[e^{6x}] = 6e^{6x}$

c) $\int (2x+5)^3\,dx$

STEP 1: $(\ldots)^3$ IS THE MAIN FUNCTION

$$= \frac{1}{4} \times \frac{1}{2}\int 4 \times 2 \times (2x+5)^3\,dx$$

STEP 2: • "ADJUST": 4 FROM POWERS
2 FROM CHAIN RULE
• "COMPENSATE": $\frac{1}{4}$ AND $\frac{1}{2}$

$$= \frac{1}{8}(2x+5)^4 + c$$

STEP 3: $\frac{d}{dx}[(2x+5)^4] = 4(2x+5)^3 \times 2 = 8(2x+5)^3$

Your notes

- AFTER SOME PRACTICE, YOU MAY FIND STEP 2 IS NOT NEEDED.
- BUT DO USE IT ON MORE AWKWARD QUESTIONS.
- ACCURACY IS THE MOST IMPORTANT THING.

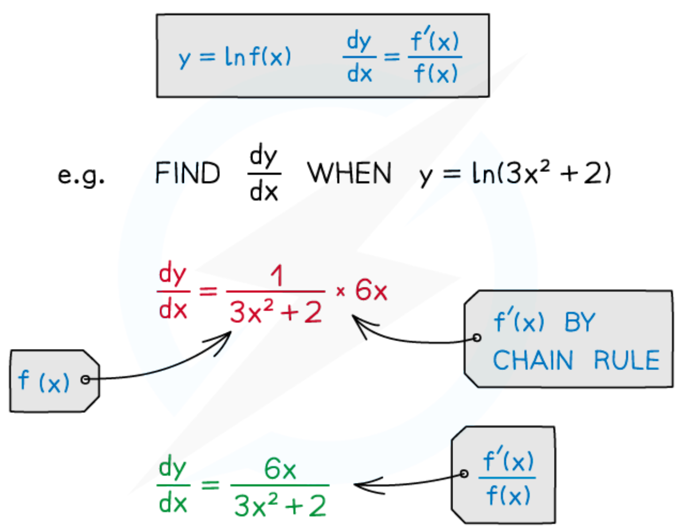
8.2.4 f'(x)/f(x)

Your notes

f'(x)/f(x)

Integrating fractions

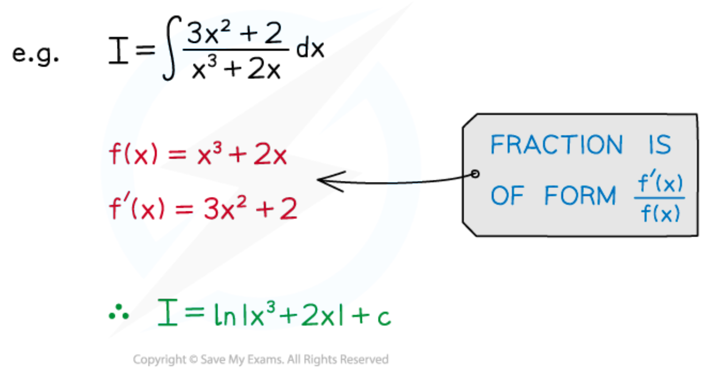
- The technique for integrating fractions depends on the type of fraction
- For **polynomial denominators** see Integration using Partial Fractions
- If $\dfrac{dy}{dy} = \dfrac{1}{x}$ then **y = ln |x| + c** – see Integrating Other Functions
- The type of fraction dealt with here is a specific case of Reverse Chain Rule



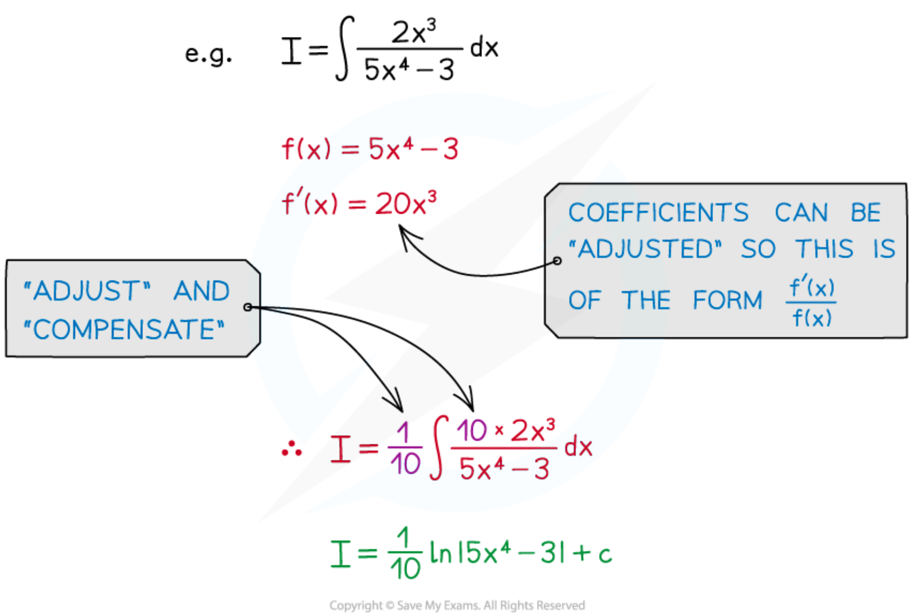
How do I integrate $\dfrac{f'(x)}{f(x)}$?

Your notes

- "The top is 'almost' the derivative of the bottom"
  - 'almost' here meaning 'a multiple of' (see below)
- The integral will involve **ln |f(x)|** - ie **ln** of the bottom
  - Due to reverse chain rule



Why 'almost'?



- There may be coefficients to 'adjust' and 'compensate' for

Examiner Tip

- If you're unsure if the fraction is of the form **f'(x)/f(x)**, differentiate the denominator.
- Compare this to the numerator but you can ignore any **coefficients**.
- If the coefficients do not match then 'adjust' and 'compensate' for them.

Your notes

Your notes

Worked example

Show $\displaystyle\int_{\frac{\pi}{12}}^{\frac{\pi}{2}} \frac{\sin 4x}{1+2\cos 4x}\,dx = \frac{1}{8}\ln\frac{2}{3}$

$$I = \int_{\frac{\pi}{12}}^{\frac{\pi}{2}} \frac{\sin 4x}{1+2\cos 4x}\,dx$$

$$f(x) = 1 + 2\cos 4x$$

$$f'(x) = 2 \times -\sin 4x \times 4$$
$$= -8\sin 4x$$

IGNORING COEFFICIENTS, I IS OF THE FORM $\frac{f'(x)}{f(x)}$

"ADJUST" AND "COMPENSATE"

$$I = -\frac{1}{8}\int_{\frac{\pi}{12}}^{\frac{\pi}{2}} \frac{-8\sin 4x}{1+2\cos 4x}\,dx$$

$$I = \left[-\frac{1}{8}\ln|1+2\cos 4x|\right]_{\frac{\pi}{12}}^{\frac{\pi}{2}}$$

"ln OF BOTTOM"

$$I = \left(-\frac{1}{8}\ln|1+2\cos 2\pi|\right) - \left(-\frac{1}{8}\ln\left|1+2\cos\frac{\pi}{3}\right|\right)$$

Your notes

$$I = \left(-\frac{1}{8}\ln 3\right) - \left(-\frac{1}{8}\ln 2\right)$$

$$I = \frac{1}{8}\left(\ln 2 - \ln 3\right)$$

SWAPPED TERMS SO FIRST IS POSITIVE

$$\therefore\ I = \frac{1}{8}\ln\frac{2}{3}$$

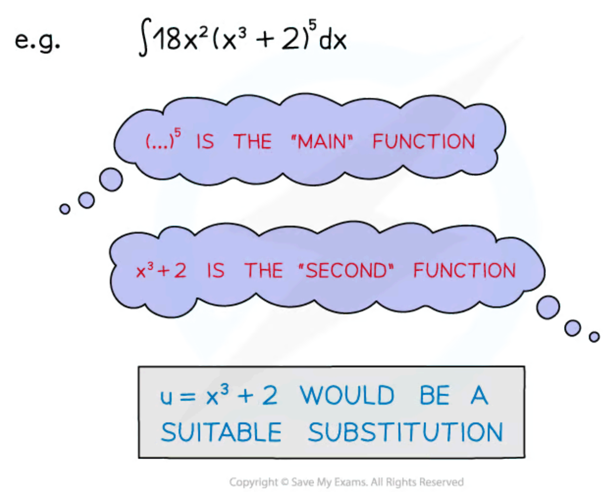
8.2.5 Substitution (Reverse Chain Rule)

Your notes

Substitution (Reverse Chain Rule)

What is integration by substitution?

- Make sure you are familiar with Chain Rule and Reverse Chain Rule
- In more awkward problems it is harder to spot the reverse chain rule
- It is possible to use a substitution
- These kinds of substitutions usually will **not** be given
  - The substitution is deemed 'obvious'



How do I integrate when a substitution is not given?

- Look to substitute the 'second' (rather than the 'main') function
- STEP 1: Determine the substitution
  - What is the 'main' function? 'second' function?
- STEP 2: Differentiate the substitution and rearrange
  - **du/dx** here can be treated like a fraction (eg × **dx** to get rid of fractions)
- STEP 3: Replace all parts of the integral
  - all **x** terms should be replaced with equivalent **u** terms, including **dx**
  - if a definite integral change the limits from **x** to **u** too
- STEP 4: Integrate and either ...
  - ...substitute **x** back in

    or
  - ... evaluate the definite integral using the **u** limits

- STEP 5: Find **c**, the constant of integration, if needed

Your notes

$$I = \int 6u^5 \, du$$

THE RESULT SHOULD BE REALLY EASY TO INTEGRATE

STEP 4: INTEGRATE, SUBSTITUTE x BACK IN

$$I = u^6 + c$$

$$I = (x^3 + 2)^6 + c$$

Your notes

FIND $\int_0^1 15x^2\cos(5x^3)dx$ GIVING YOUR ANSWER TO 3 SIGNIFICANT FIGURES.

$$I = \int_0^1 15x^2\cos(5x^3)dx$$

STEP 1: DETERMINE THE SUBSTITUTION

LET $u = 5x^3$

STEP 2: DIFFERENTIATE AND REARRANGE

$$\frac{du}{dx} = 15x^2$$

$$du = 15x^2 dx$$

STEP 3: REPLACE ALL PARTS OF THE INTEGRAL

$x = 0, \quad u = 0$

$x = 1, \quad u = 5$

CHANGE LIMITS TOO!

Your notes

$$I = \int_0^5 \cos u \, du$$

$\cos(5x^3) = \cos u$

$du = 15x^2 dx$

STEP 4: INTEGRATE AND EVALUATE

$$I = [\sin u]_0^5$$

$$I = \sin 5 - \sin 0$$

RADIANS FOR CALCULUS!

$$I = -0.95892...$$

$$I = -0.959 \quad (3 \text{ sf})$$

Examiner Tip

- A lot of the "work" in these problems happens in your head :
  - what is the 'main' function?
  - what should the substitution be?
  - are there any numbers to 'adjust' and 'compensate' for?
- Be sure that what you write down is clear and easy to follow, and remember that you can check your final answer by differentiating it.

Worked example

Your notes

Your notes

Given that y passes through the point $(1, 2e^3)$, and that

$$y = \int 5x^2 e^{x^3+2}\, dx$$

find y in terms of x.

$$y = \int 5x^2 e^{x^3+2}\, dx$$

STEP 1: DETERMINE THE SUBSTITUTION

$e^{\cdots}$ IS THE MAIN FUNCTION

LET $u = x^3 + 2$

STEP 2: DIFFERENTIATE AND REARRANGE

$$\frac{du}{dx} = 3x^2$$

$$du = 3x^2 dx$$

STEP 3: REPLACE ALL PARTS OF THE INTEGRAL

$$y = 5 \times \frac{1}{3} \int 3x^2 e^u\, dx$$

CONSTANT

"ADJUST" AND "COMPENSATE"

Your notes

$$y = \frac{5}{3}\int e^u \, du$$

DEAD EASY!

STEP 4: INTEGRATE, SUBSTITUTE x BACK IN

$$y = \frac{5}{3}e^u + c$$

$$y = \frac{5}{3}e^{x^3+2} + c$$

STEP 5: FIND c

AT $(1, 2e^3)$ $\quad 2e^3 = \frac{5}{3}e^3 + c$

$$c = \frac{1}{3}e^3$$

$$\therefore \quad y = \frac{5}{3}e^{x^3+2} + \frac{1}{3}e^3$$

8.2.6 Harder Substitution

Your notes

Harder Substitution

What is integration by substitution?

- Make sure you are familiar with Reverse Chain Rule and Substitution (RCR)
- In more difficult questions the substitution will be given to you

e.g. USE THE SUBSTITUTION $u = x + 3$

TO FIND $\int \left(\frac{x}{x+3}\right)^3 dx.$

How do I integrate using a given substitution?

- The process is very much the same as before but expect the algebra to be trickier

Your notes

e.g. USE THE SUBSTITUTION $u = x + 3$

TO FIND $\int \left(\frac{x}{x+3}\right)^3 dx.$

STEP 1: DIFFERENTIATE AND REARRANGE

$u = x + 3$

$\frac{du}{dx} = 1$

$du = dx$

$x = u - 3$

SPOT YOU MAY NEED THIS FOR THE NUMERATOR

STEP 2: REPLACE ALL PARTS OF THE INTEGRAL

$$\int \left(\frac{x}{x+3}\right)^3 dx = \int \left(\frac{u-3}{u}\right)^3 du$$

ALGEBRA MORE AWKWARD!

$$= \int (1 - 3u^{-1})^3 du$$

Your notes

USE BINOMIAL EXPANSION

$$= \int\Big(1 + 3(-3u^{-1}) + 3(-3u^{-1})^2 + (-3u^{-1})^3\Big)du$$

$$= \int(1 - 9u^{-1} + 27u^{-2} - 27u^{-3})du$$

STEP 4: INTEGRATE, SUBSTITUTE x BACK IN

$$= u - 9\ln|u| + \frac{27u^{-1}}{-1} - \frac{27u^{-2}}{-2} + c$$

$$= x + 3 - 9\ln|x+3| - \frac{27}{x+3} + \frac{27}{2(x+3)^2} + c$$

- STEP 1: Differentiate the substitution and rearrange
  - $\frac{du}{dx}$ here can be treated like a fraction (eg × **dx** to get rid of fractions)
- STEP 2: Replace all parts of the integral
  - all **x** terms should be replaced with equivalent **u** terms, including **dx**
  - if a definite integral change the limits from **x** to **u** too
- STEP 3: Integrate and either
  - substitute **x** back in or
  - evaluate the definite integral using the **u** limits
- STEP 4: Find **c**, the constant of integration, if needed

Your notes

Worked example

Use the substitution $u^2 = x^3 + 1$ to find the exact value of

$$\int_1^2 \frac{x^2}{\sqrt{1+x^3}}\,dx$$

$$I = \int_1^2 \frac{x^2}{\sqrt{1+x^3}}\,dx$$

$u^2 = x^3 + 1$

STEP 1: DIFFERENTIATE AND REARRANGE

$2u\frac{du}{dx} = 3x^2$

$2u\,du = 3x^2\,dx$

DEAL WITH THIS LATER

$x^3 = u^2 - 1$

STEP 2: REPLACE ALL PARTS OF THE INTEGRAL

$x = 1, \quad u^2 = 1^3 + 1 = 2, \quad u = \sqrt{2}$

$x = 2, \quad u^2 = 2^3 + 1 = 9, \quad u = 3$

EXACT VALUE REQUIRED AT END

Your notes

"ADJUST" AND "COMPENSATE"

$$I = \frac{1}{3}\int_1^2 \frac{3x^2}{\sqrt{1+x^3}}\,dx$$

$$I = \frac{1}{3}\int_{\sqrt{2}}^3 \frac{2u\,du}{\sqrt{1+(u^2-1)}}$$

$$I = \frac{2}{3}\int_{\sqrt{2}}^3 \frac{u}{\sqrt{u^2}}\,du$$

$$I = \frac{2}{3}\int_{\sqrt{2}}^3 du$$

$$\int_{\sqrt{2}}^3 1\,du$$

STEP 3: INTEGRATE AND EVALUATE

$$I = \frac{2}{3}\left[u\right]_{\sqrt{2}}^3$$

$$I = \frac{6-2\sqrt{2}}{3}$$

8.2.7 Integrating with Trigonometric Identities

Your notes

Integrating with Trigonometric Identities

What are trigonometric identities?

ESSENTIAL TRIGONOMETRIC IDENTITIES

$$\sin^2 x + \cos^2 x \equiv 1$$

$$\tan x \equiv \frac{\sin x}{\cos x}$$

- Some are given in the formulae booklet
  - Be sure to note the difference between the ± and ∓ symbols!

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B} \quad \left(A \pm B \neq \left(k + \frac{1}{2}\right)\pi\right)$$

$$\sin A + \sin B = 2\sin\frac{A+B}{2}\cos\frac{A-B}{2}$$

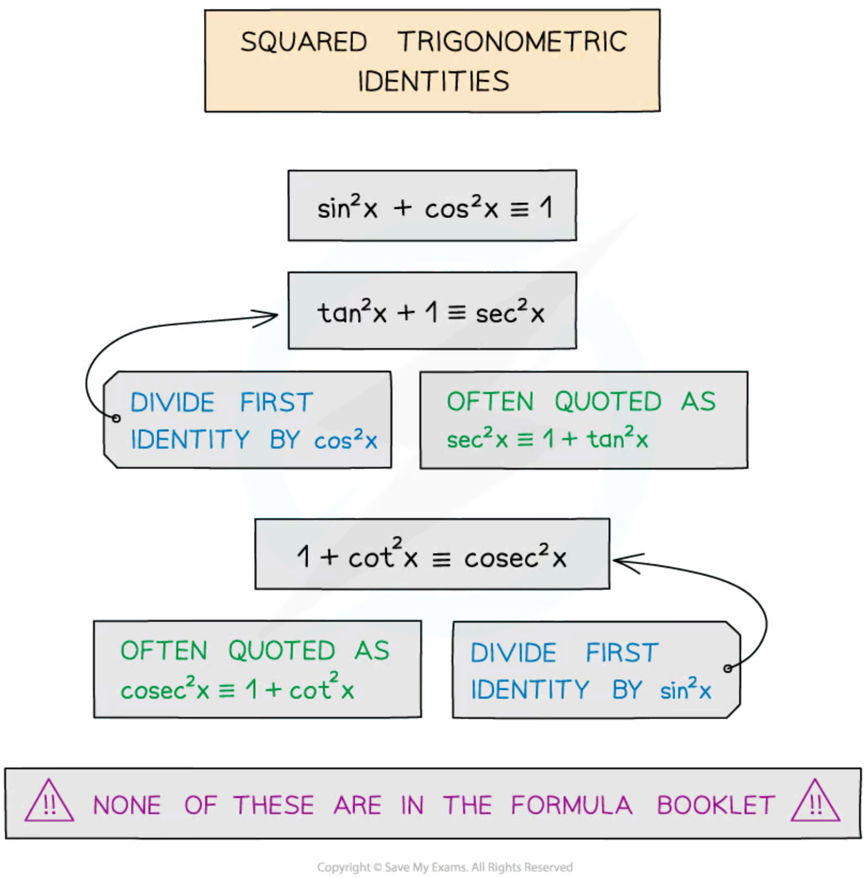
$$\sin A - \sin B = 2\cos\frac{A+B}{2}\sin\frac{A-B}{2}$$

$$\cos A + \cos B = 2\cos\frac{A+B}{2}\cos\frac{A-B}{2}$$

$$\cos A - \cos B = -2\sin\frac{A+B}{2}\sin\frac{A-B}{2}$$

How do I know which trig identities to use?

- There is no set method
- This is a matter of experience, familiarity and recognition
  - Practice as many questions as possible
  - Be familiar with trigonometric functions that can be integrated easily
  - Be familiar with common identities – especially **squared** terms
  - **$\sin^2 x$**, **$\cos^2 x$**, **$\tan^2 x$**, **$\text{cosec}^2 x$**, **$\sec^2 x$**, **$\tan^2 x$** all appear in identities



How do I integrate $\tan^2$, $\cot^2$, $\sec^2$ and cosec^2?

- The **integral** of **$\sec^2 x$** is **$\tan x$ (+c)**
  - This is because the **derivative** of $\tan x$ is $\sec^2 x$
- The **integral** of **$\text{cosec}^2 x$** is **$-\cot x$ (+c)**
  - This is because the **derivative** of $\cot x$ is $-\text{cosec}^2 x$
- The **integral** of **$\tan^2 x$** can be found by using the identity to rewrite $\tan^2 x$ before integrating:
  - $1 + \tan^2 x = \sec^2 x$
- The **integral** of **$\cot^2 x$** can be found by using the identity to rewrite $\cot^2 x$ before integrating:
  - $1 + \cot^2 x = \text{cosec}^2 x$

Your notes

How do I integrate sin and cos?

- For functions of the form **sin kx**, **cos kx** … see Integrating Other Functions
- **sin kx × cos kx** can be integrated using the identity for **sin 2A**
  - **sin 2A = 2sinAcosA**

e.g. $\int \sin x \cos x \, dx$

$\boxed{\sin 2A = 2\sin A \cos A}$

$= \int \frac{1}{2} \sin 2x \, dx$

$= -\frac{1}{2} \times \frac{1}{2} \int -2 \sin 2x \, dx$

"ADJUST" AND "COMPENSATE"

$= -\frac{1}{4} \cos 2x + c$

AN EQUIVALENT ANSWER CAN BE FOUND USING THE FOLLOWING TECHNIQUE

- **$\sin^n$ kx cos kx** or **sin kx $\cos^n$ kx** can be integrated using **reverse chain rule** or **substitution**
- Notice no identity is used here but it looks as though there should be!

Your notes

Your notes

e.g. $\int \sin x \cos^3 x \, dx$

$$= -\frac{1}{4}\int -4\sin x \cos^3 x \, dx$$

"ADJUST" AND "COMPENSATE"

$-\sin x$ IS DERIVATIVE OF $\cos x$

REVERSE CHAIN RULE

$$= -\frac{1}{4}\cos^4 x + c$$

NOTICE NO IDENTITY USED IN THIS QUESTION

- $\sin^2 kx$ and $\cos^2 kx$ can be integrated by using the identity for $\cos 2A$
  - For $\sin^2 A$, $\cos 2A = 1 - 2\sin^2 A$
  - For $\cos^2 A$, $\cos 2A = 2\cos^2 A - 1$

Your notes

e.g. $\int \sin^2 x \, dx$

$$\boxed{\begin{array}{l}\cos 2A = 1 - 2\sin^2 A \\ \text{SO } \sin^2 A = \frac{1}{2}(1 - \cos 2A)\end{array}}$$

$$= \int \frac{1}{2}(1 - \cos 2x)\,dx$$

$$= \int \left(\frac{1}{2} - \frac{1}{2}\cos 2x\right) dx$$

$$= \int \left[\frac{1}{2} - \frac{1}{2} \times \frac{1}{2}(2\cos 2x)\right] dx$$

$$= \frac{1}{2}x - \frac{1}{4}\sin 2x + c$$

How do I integrate tan?

$$\boxed{\int \tan x \, dx = \ln|\sec x| + c}$$

$\int \tan kx \, dx$ IS IN THE FORMULA BOOKLET

- This is a standard result from the formulae booklet

How do I integrate other trig functions?

- The formulae booklet lists many standard trigonometric derivatives and integrals
  - Check both the “Differentiation” and “Integration” sections
  - For integration using the "Differentiation" formulae, remember that the integral of **f'(x)** is **f(x)** !

Your notes

DIFFERENTIATION

$f(x)$	$f'(x)$
$\tan kx$	$k\sec^2 kx$
$\sec kx$	$k\sec kx \tan kx$
$\cot kx$	$-k\,\text{cosec}^2 kx$
$\text{cosec}\,kx$	$-k\,\text{cosec}\,kx \cot kx$

INTEGRATION (+ CONSTANT)

$f(x)$	$\int f(x)dx$
$\sec^2 kx$	$\frac{1}{k}\tan kx$
$\tan kx$	$\frac{1}{k}\ln\lvert\sec kx\rvert$
$\cot kx$	$\frac{1}{k}\ln\lvert\sin kx\rvert$
$\text{cosec}\,kx$	$-\frac{1}{k}\ln\lvert\text{cosec}\,kx + \cot kx\rvert,\quad \frac{1}{k}\ln\left\lvert\tan\left(\frac{1}{2}kx\right)\right\rvert$
$\sec kx$	$\frac{1}{k}\ln\lvert\sec kx + \tan kx\rvert,\quad \frac{1}{k}\ln\left\lvert\tan\left(\frac{1}{2}kx + \frac{1}{4}\pi\right)\right\rvert$

- Experience, familiarity and recognition are important – practice, practice, practice!
- Problem-solving techniques

Your notes

e.g. $$\int\left(\frac{2\sin^2 x}{1+\cos 2x}+1\right)dx$$

- MIXTURE OF x's AND 2x's
- $\sin^2 x$ IS GOOD–COMMON IDENTITY
- $1+\cos 2x$ IS NOT A GOOD DENOMINATOR

$$=\int\left(\frac{2\sin^2 x}{1+(2\cos^2 x-1)}+1\right)dx$$

$$\boxed{\cos 2A \equiv 2\cos^2 A - 1}$$

$$=\int\left(\frac{2\sin^2 x}{2\cos^2 x}+1\right)dx$$

$$=\int(\tan^2 x+1)dx$$

$$=\int\sec^2 x\,dx$$

$$\boxed{\sec^2 x \equiv 1+\tan^2 x}$$

$$=\tan x + c$$

A STANDARD RESULT — BUT UNDER DIFFERENTIATION IN THE FORMULAE BOOKLET

Examiner Tip

- Make sure you have a copy of the formulae booklet during revision.
- Questions are likely to be split into (at least) two parts:
  - The first part may be to show or prove an identity
  - The second part may be the integration
- If you cannot do the first part, use a given result to attempt the second part.

Worked example

Your notes

Your notes

(a) Show that $(1 + \cos 2x)\operatorname{cosec} 2x \equiv \cot x$

(b) Hence, or otherwise, find the exact value of

$$\int_{\frac{\pi}{6}}^{\frac{3\pi}{4}} (1 + \cos 2x)\operatorname{cosec} 2x \, dx \text{ in the form } a\ln b,$$

where a and b are constants to be found.

a) $\text{LHS} \equiv (1 + \cos 2x)\operatorname{cosec} 2x$

- NOTHING GAINED EXPANDING
- cosec IS $\frac{1}{\sin}$ AND cot IS A FRACTION

$$\text{LHS} \equiv \frac{1 + \cos 2x}{\sin 2x}$$

- 2x's AT MOMENT, NEED x's
- IDENTITIES FOR cos2x AND sin2x
- BUT WHICH cos2x?
- cot NEEDS cos ON TOP SO TRY
- $\cos 2x = 2\cos^2 x - 1$

$$\text{LHS} \equiv \frac{1 + (2\cos^2 x - 1)}{2\sin x \cos x}$$

$$\text{LHS} \equiv \frac{\cancel{2}\cos^{\cancel{2}} x}{\cancel{2}\sin x \cancel{\cos x}}$$

$$\text{LHS} \equiv \cot x$$

$$\boxed{\cot x \equiv \frac{\cos x}{\sin x}}$$

Your notes

b) $$\int_{\frac{\pi}{6}}^{\frac{3\pi}{4}} (1+\cos 2x)\operatorname{cosec} 2x\, dx$$

$$\equiv \int_{\frac{\pi}{6}}^{\frac{3\pi}{4}} \cot x\, dx \quad (*)$$

IN FORMULAE BOOKLET

$$= \Big[\ln|\sin x|\Big]_{\frac{\pi}{6}}^{\frac{3\pi}{4}}$$

$$= \left(\ln\left|\sin \frac{3\pi}{4}\right|\right) - \left(\ln\left|\sin \frac{\pi}{6}\right|\right)$$

$$= \ln\frac{1}{\sqrt{2}} - \ln\frac{1}{2}$$

$$\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

$$= \ln 2^{-\frac{1}{2}} - \ln 2^{-1}$$

$$= -\frac{1}{2}\ln 2 + \ln 2$$

$$= \frac{1}{2}\ln 2$$

(*) YOUR CALCULATOR WILL DO THIS BUT NOT GIVE YOU AN EXACT VALUE. YOU CAN SEE YOUR CALCULATOR TO CHECK BOTH DECIMAL ANSWERS.

Your notes

8.2.8 Integration by Parts

Integration by Parts

What is Integration by Parts?

INTEGRATION BY PARTS

$$\int u \frac{dv}{dx}\,dx = uv - \int v \frac{du}{dx}\,dx$$

- For integrating the product of two functions - reverse product rule
- Crucially the product is made from **u** and **dv/dx** (rather than **u** and **v**)
- Alternative notation may be used

ALTERNATIVE NOTATION

FUNCTION NOTATION

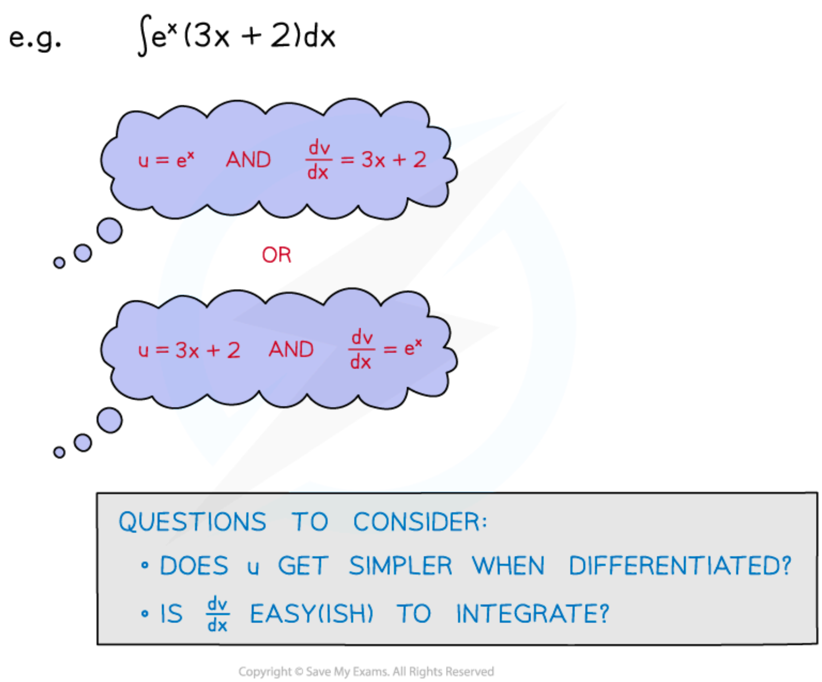
$$\int f(x)g'(x)\,dx = f(x)g(x) - \int g(x)f'(x)\,dx$$

DASH NOTATION

$$\int uv'\,dx = uv - \int vu'\,dx$$

How do I use Integration by Parts?

Your notes



- The hardest part is choosing **u** and **dv/dx** as there is no method for doing so
- **u**, ideally, becomes **simpler** when differentiated but this is not always possible
- **dv/dx** should be a function that can be integrated fairly easily
- Be wary of functions that ‘cycle’/’repeat’ when differentiated/integrated
  - $\mathbf{e^x \to e^x}$
  - $\mathbf{\sin x \to \cos x \to -\sin x \to -\cos x \to \sin x}$

Your notes

e.g. $\quad I = \int e^x(3x+2)\,dx$

STEP 1: CHOOSE u AND v′, FIND u′ AND v

$u = 3x + 2 \qquad v = e^x$

$u' = 3 \qquad v' = e^x$

u′ IS MORE SIMPLE

EASY TO INTEGRATE

STEP 2: APPLY INTEGRATION BY PARTS

$I = (3x+2)e^x - \int e^x \times 3\,dx$

$I = e^x(3x+2) - 3\int e^x\,dx$

STEP 3: "SECOND" INTEGRAL

$I = e^x(3x+2) - 3e^x + c$

+ c FOR INDEFINITE INTEGRATION

STEP 4: SIMPLIFY

$I = e^x(3x + 2 - 3) + c$

$I = e^x(3x-1) + c$

- STEP 1: Choose u and v', find u' and v
- STEP 2: Apply Integration by Parts
  - Simplify anything straightforward
- STEP 3: Do the 'second' integral
  - If an **indefinite** integral remember "**+c**", the **constant of integration**
- STEP 4: Simplify and/or apply limits

What happens if I cannot integrate v × du/dx?

- It is possible integration by parts may need to be applied more than once

e.g. $I = \int x^2 \cos x \, dx$

STEP 1: CHOOSE u AND v′, FIND u′ AND v

$u = x^2 \qquad v = \sin x$

$u' = 2x \qquad v' = \cos x$

SIMPLER AFTER DIFFERENTIATING

"CYCLES" BETWEEN cos AND sin

STEP 2: APPLY INTEGRATION BY PARTS

$$I = x^2 \sin x - \int 2x \sin x \, dx$$

STEP 3: REPEAT STEP 1 AND STEP 2 FOR THE SECOND INTEGRAL

$u = 2x \qquad v = -\cos x$

$u' = 2 \qquad v' = \sin x$

$$I = x^2 \sin x - \left[-2x\cos x - \int -2\cos x \, dx\right]$$

STEP 4: INTEGRATE AND SIMPLIFY

$$I = x^2 \sin x + 2x\cos x - 2\sin x + c$$

$$I = (x^2 - 2)\sin x + 2x\cos x + c$$

"+c"!!

ln x

Your notes

Your notes

$$\boxed{\int \ln x \, dx}$$

$$\int \ln x \, dx = \int 1 \times \ln x \, dx$$

$u = \ln x \qquad v = x$

$u' = \frac{1}{x} \qquad v' = 1$

$$= x \ln x - \int x \times \frac{1}{x} \, dx$$

$$= x \ln x - \int dx$$

$$= x \ln x - x + c$$

$$\boxed{\int \ln x \, dx = x \ln x - x + c}$$

- A classic 'set piece' in almost every A level maths textbook ever written!
- In general, rewriting **f(x)** as **1×f(x)** can be a powerful problem-solving technique
- This could be a question in the exam

How do I find a definite integral using parts?

Your notes

e.g. FIND THE VALUE OF $I = \int_0^1 3xe^{2x}\,dx$

STEP 1: CHOOSE u AND v′, FIND u′ AND v

$u = 3x \qquad v = \frac{1}{2}e^{2x}$ ← REVERSE CHAIN RULE

$u' = 3 \qquad v' = e^{2x}$

STEP 2: APPLY INTEGRATION BY PARTS

$$I = \left[\frac{3}{2}xe^{2x} - \int \frac{3}{2}e^{2x}\,dx\right]_0^1$$

$$I = \left[\frac{3}{2}xe^{2x} - \frac{3}{2} \times \frac{1}{2}\int 2e^{2x}\,dx\right]_0^1$$

STEP 3: DO THE "SECOND" INTEGRAL

$$I = \left[\frac{3}{2}xe^{2x} - \frac{3}{4}e^{2x}\right]_0^1$$

$$I = \left[\frac{3}{4}e^{2x}(2x - 1)\right]_0^1$$

STEP 4: APPLY LIMITS AND SIMPLIFY

$$I = \left(\frac{3}{4}e^2\right) - \left(\frac{3}{4}e^0 \times -1\right)$$

$$I = \frac{3}{4}e^2 + \frac{3}{4}$$

$$I = \frac{3}{4}(e^2 + 1)$$

Your notes

Examiner Tip

- Always think about what an elegant, slick, professional maths solution looks like – solutions normally get more complicated at first but quickly get simpler.
- If your work is continuing to get more complicated, stop and check for an error.
- Try to develop a sense of 'having gone too far down the wrong path'.
- This general advice is useful to remember:
  - Is the second integral harder than the first?
  - Try swapping your choice of **u** and **dv/dx**
  - It is rare to have to apply integration by parts more than twice

Your notes

Worked example

Find $\int x^2 \ln x \, dx$

LET $I = \int x^2 \ln x \, dx$

STEP 1: CHOOSE u AND v′, FIND u′ AND v

NO CHOICE AS CANNOT INTEGRATE ln x EASY

$u = \ln x \qquad v = \frac{1}{3}x^3$

$u' = \frac{1}{x} \qquad v' = x^2$

STEP 2: APPLY INTEGRATION BY PARTS

$I = \frac{1}{3}x^3 \ln x - \int \frac{1}{3}x^3 \times \frac{1}{x} \, dx$

$I = \frac{1}{3}x^3 \ln x - \frac{1}{3}\int x^2 \, dx$

STEP 3: DO THE "SECOND" INTEGRAL

$I = \frac{1}{3}x^3 \ln x - \frac{1}{3}\left(\frac{1}{3}x^3\right) + c$

STEP 4: SIMPLIFY

$I = \frac{1}{9}x^3(3\ln x - 1) + c$

8.2.9 Integration using Partial Fractions

Integration using Partial Fractions

What are Partial Fractions?

- This is the reverse process to adding (or subtracting) fractions
  - The common polynomial denominator is split into factors
- Make sure you are familiar with the notes on Partial Fractions

$$\frac{1}{x^2+3x+2} = \frac{1}{(x+2)(x+1)}$$

$$\frac{1}{(x+2)(x+1)} \equiv \frac{A}{(x+2)} + \frac{B}{(x+1)}$$

$$1 \equiv A(x+1) + B(x+2)$$

$$\text{LET } x=-1, \quad 1 = B, \quad B = 1$$
$$\text{LET } x=-2, \quad 1 = -A, \quad A = -1$$

$$\therefore \quad \frac{1}{x^2+3x+2} = \frac{1}{x+1} - \frac{1}{x+2}$$

How do I integrate using partial fractions?

- Fractions with linear denominators can be integrated (See Integrating Other Functions)

$$\int \frac{1}{x+3}\,dx = \ln|x+3| + c$$

- A fraction with a polynomial denominator (degree 2+) can be integrated by ...
  - ... splitting using partial fractions
  - ... integrating each partial fraction

Your notes

Your notes

e.g. $$\int \frac{3}{x^2 + 5x + 4}\,dx$$

STEP 1: FACTORISE DENOMINATOR, IF NEEDED

$$\frac{3}{x^2 + 5x + 4} = \frac{3}{(x + 4)(x + 1)}$$

STEP 2: EXPRESS AS PARTIAL FRACTIONS

$$\frac{3}{(x + 4)(x + 1)} \equiv \frac{A}{x + 4} + \frac{B}{x + 1}$$

$$3 \equiv A(x + 1) + B(x + 4)$$

LET $x = -1$, $3 = 3B$, $B = 1$

LET $x = -4$, $3 = -3A$, $A = -1$

$$\therefore \frac{3}{x^2 + 5x + 4} = \frac{1}{x + 1} - \frac{1}{x + 4}$$

STEP 3: INTEGRATE EACH TERM

$$\int \frac{3}{x^2 + 5x + 4}\,dx = \int \left(\frac{1}{x + 1} - \frac{1}{x + 4}\right)dx$$

$$= \ln|x + 1| - \ln|x + 4| + c$$

REMEMBER MODULUS WHEN INTEGRATING TO ln

- STEP 1: Factorise denominator, if needed
- STEP 2: Express as partial fractions
- STEP 3: 'Adjust' and 'compensate' each term so it can be integrated (See Reverse Chain Rule)
- STEP 4: Integrate each term (usually involves **ln**) and simplify

Your notes

Examiner Tip

- When there are several parts, read the whole question:
- An early part may be a "show that" involving partial fractions
- A later part may be to integrate the original fraction
- If you can't do, or get stuck on, the partial fractions bit of the question you can still use the "show that" result to help with the integration.

Your notes

Worked example

(a) Fully factorise $4x^3 - 4x^2 + x$

(b) Express $\dfrac{2}{4x^3 - 4x^2 + x}$ in partial fractions

(c) Find $\displaystyle\int \frac{2}{4x^3 - 4x^2 + x}\,dx$

a) $4x^3 - 4x^2 + x = x(4x^2 - 4x + 1)$

$= x(2x - 1)^2$

IF x IS NOT A FACTOR, FACTOR THEOREM AND / OR ALGEBRAIC DIVISION WOULD BE NEEDED TO FACTORISE A CUBIC

LOOKING AHEAD, THIS IS STEP 1 FOR PART c)

Your notes

b) $$\frac{2}{4x^3 - 4x^2 + x} = \frac{2}{x(2x-1)^2}$$ USING a)

$$\equiv \frac{A}{x} + \frac{B}{2x-1} + \frac{C}{(2x-1)^2}$$

SQUARED LINEAR DENOMINATOR

$$2 \equiv A(2x-1)^2 + Bx(2x-1) + Cx$$

LET $x = 0$, $2 = A$, $A = 2$

LET $x = \frac{1}{2}$, $2 = \frac{1}{2}C$, $C = 4$

LET $x = 1$, $2 = A + B + C$

$2 = 2 + B + 4$, $B = -4$

NO VALUES OF x WILL LEAD TO AN EQUATION IN B ONLY

$$\frac{2}{4x^3 - 4x^2 + x} \equiv \frac{2}{x} - \frac{4}{2x-1} + \frac{4}{(2x-1)^2}$$

LOOKING AHEAD, THIS IS STEP 2 FOR PART c)

Your notes

c) $$\int \frac{2}{4x^3 - 4x^2 + x}\,dx = \int \left(\frac{2}{x} - \frac{4}{2x-1} + \frac{4}{(2x-1)^2} \right) dx$$

STEP 1: FACTORISE DENOMINATOR

PART a)

STEP 2: EXPRESS AS PARTIAL FRACTIONS

PART b)

STEP 3: "ADJUST" AND "COMPENSATE"

4 SPLIT INTO 2×2
2 FROM REVERSE CHAIN RULE

4 SPLIT INTO 2×2
2 FROM REVERSE CHAIN RULE
−1 FROM POWERS

$$= \int \left(\frac{2}{x} - \frac{2 \times 2}{2x-1} + \frac{2 \times 2 \times -1}{-1} (2x-1)^{-2} \right) dx$$

STEP 4: INTEGRATE EACH TERM AND SIMPLIFY

$$= 2\ln|x| - 2\ln|2x-1| - 2(2x-1)^{-1} + c$$

"EXTRA" 2 ON TOP

$\frac{2}{-1} = -2$

$$= 2\ln|x| - 2\ln|2x-1| - \frac{2}{2x-1} + c$$

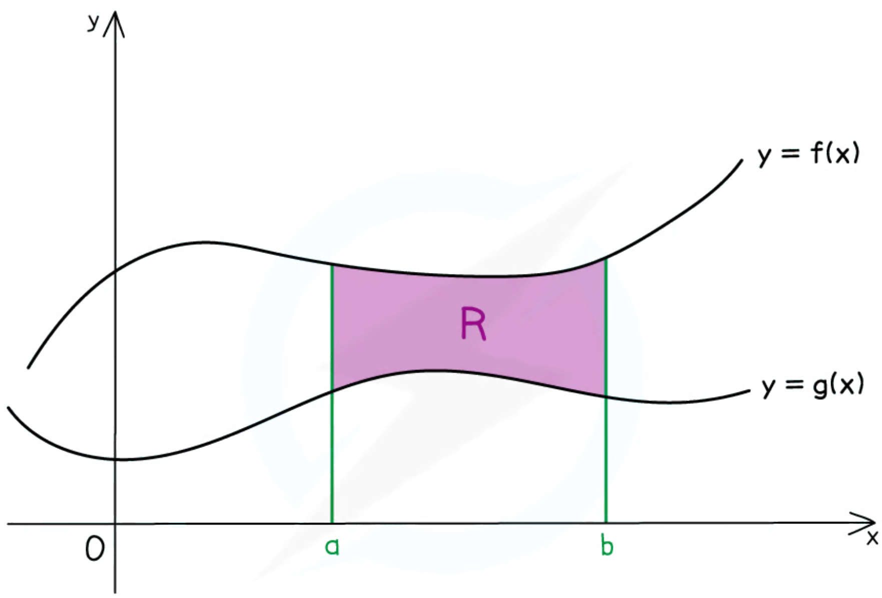
Your notes

8.2.10 Area between 2 curves

Area between 2 curves

What is the area between two curves?

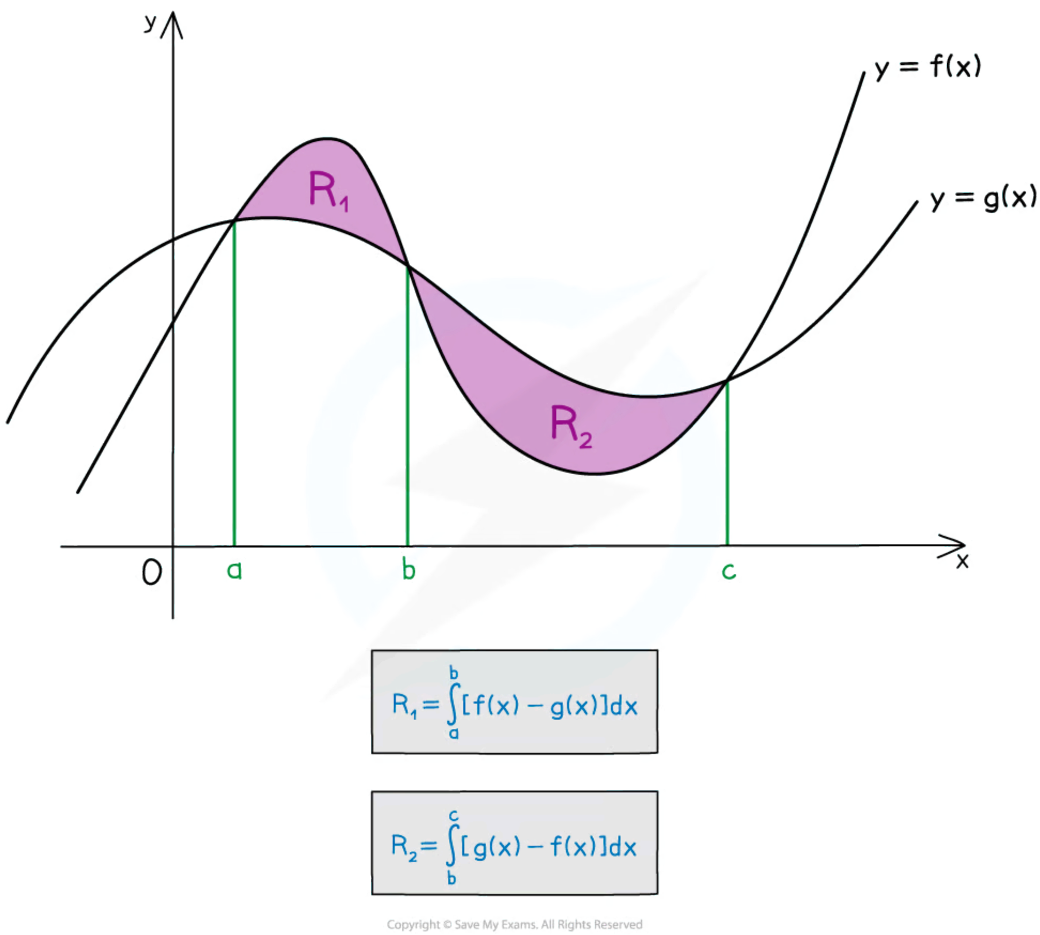
- Ensure you are familiar with ...
    - Area under a curve
    - Area between a line and a curve



$$R = \int_a^b [f(x) - g(x)]\,dx$$

- In general find the definite integral of "upper curve" – "lower curve"
- However this does depend on ...
    - ... the area being found
    - ... if the curves intersect (and cross over)
- The area may have to be split into separate integrals

Your notes

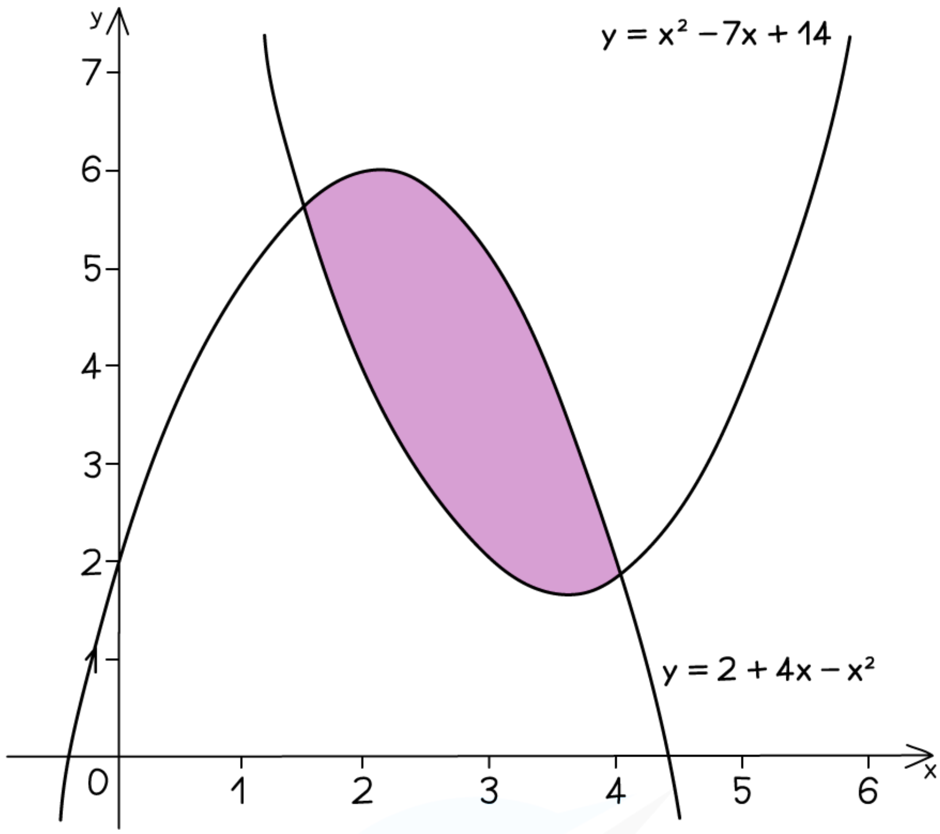


- The points at which curves intersect may need to be calculated

How do I find the area between two curves?

Your notes

e.g. FIND THE SHADED AREA IN THE DIAGRAM BELOW



STEP 1: FIND THE INTERSECTIONS

$x^2 - 7x + 14 = 2 + 4x - x^2$

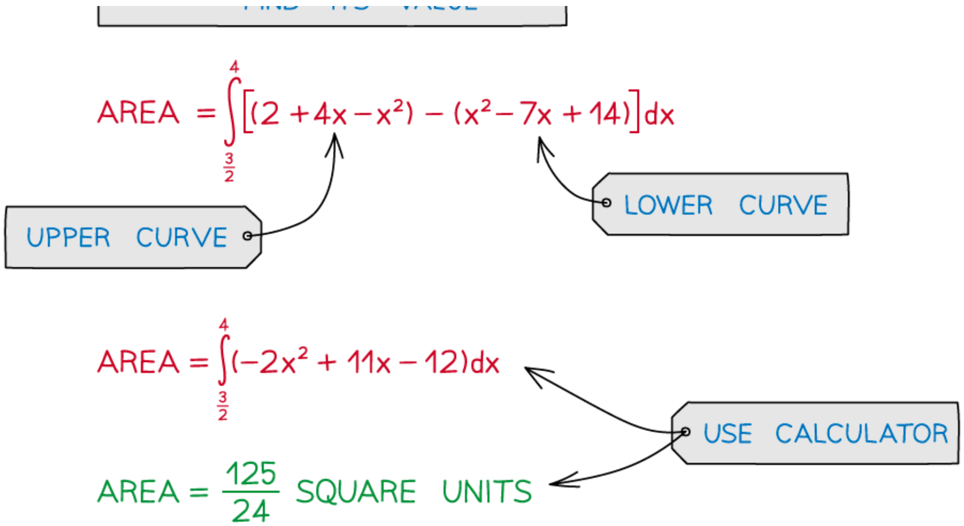
$2x^2 - 11x + 12 = 0$

$(2x - 3)(x - 4) = 0$

$x = \frac{3}{2}, \quad x = 4$

COULD USE CALCULATOR

STEP 2: FORM INTEGRAL AND FIND ITS VALUE



- STEP 1: Find the intersections of the curves if needed
- STEP 2: Form the integral ...
  - ... using the intersections as limits
  - ... "upper curve" – "lower curve" ...
  - ... and find the value of the integral
- STEP 3: Repeat STEP 2 if more than one area needed
- STEP 4: Add areas together

Examiner Tip

- If no diagram is provided sketch one, even if the curves are not accurate.
- Add information to any given diagram as you work through a question.Maximise use of your calculator to save time and maintain accuracy:
  - Solving equations, especially **cubic** equations
  - Finding definite integrals

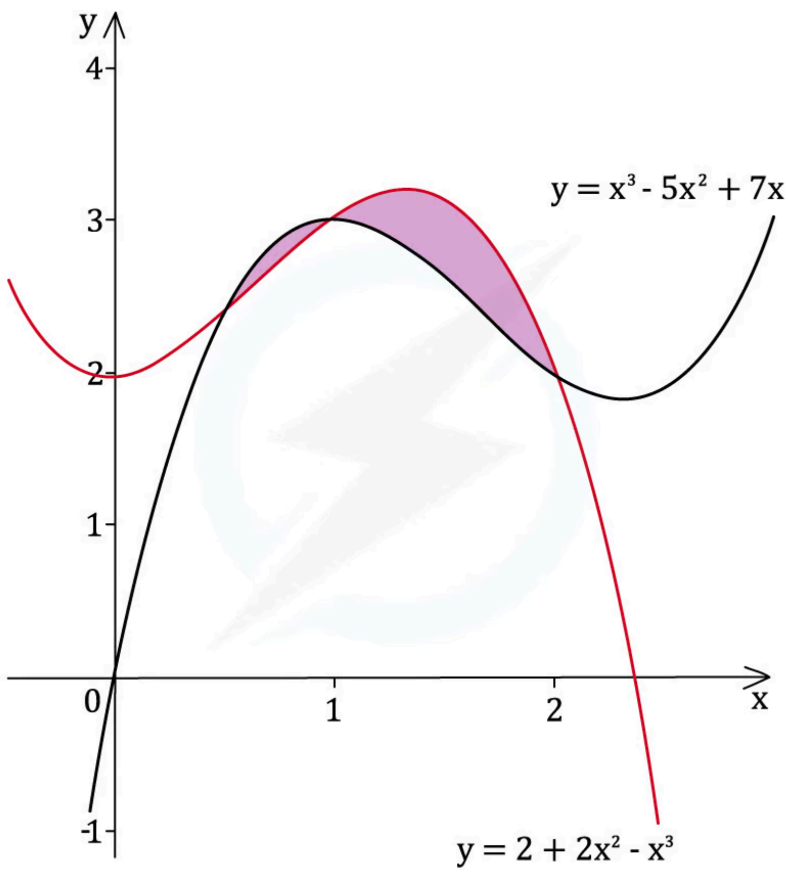
Your notes

Your notes

Worked example

The diagram below shows a sketch of part of the graphs with equations

$y = x^3 - 5x^2 + 7x$ and $y = 2 + 2x^2 - x^3$



Find the shaded area.

Your notes

STEP 1: FIND THE INTERSECTIONS

$x^3 - 5x^2 + 7x = 2 + 2x^2 - x^3$

$2x^3 - 7x^2 + 7x - 2 = 0$ SOLVE USING CALCULATOR

$x = 2, \quad x = 1, \quad x = \frac{1}{2}$

STEP 2: FORM AND FIND INTEGRAL FOR FIRST AREA

$$R_1 = \int_{0.5}^{1} \left[(x^3 - 5x^2 + 7x) - (2 + 2x^2 - x^3)\right] dx$$

$$R_1 = \int_{0.5}^{1} (2x^3 - 7x^2 + 7x - 2) dx$$

USE CALCULATOR

$$R_1 = \frac{5}{96}$$

STEP 3: FORM AND FIND INTEGRAL FOR SECOND AREA

$$R_2 = \int_{1}^{2} \left[(2 + 2x^2 - x^3) - (x^3 - 5x^2 + 7x)\right] dx$$

$$R_2 = \int_{1}^{2} (-2x^3 + 7x^2 - 7x + 2) dx$$

$$R_2 = \frac{1}{3}$$

STEP 4: ADD AREAS TOGETHER

$$\text{SHADED AREAS} = R_1 + R_2 = \frac{5}{96} + \frac{1}{3}$$

SHADED AREA = $\frac{37}{96}$ SQUARE UNITS

Your notes

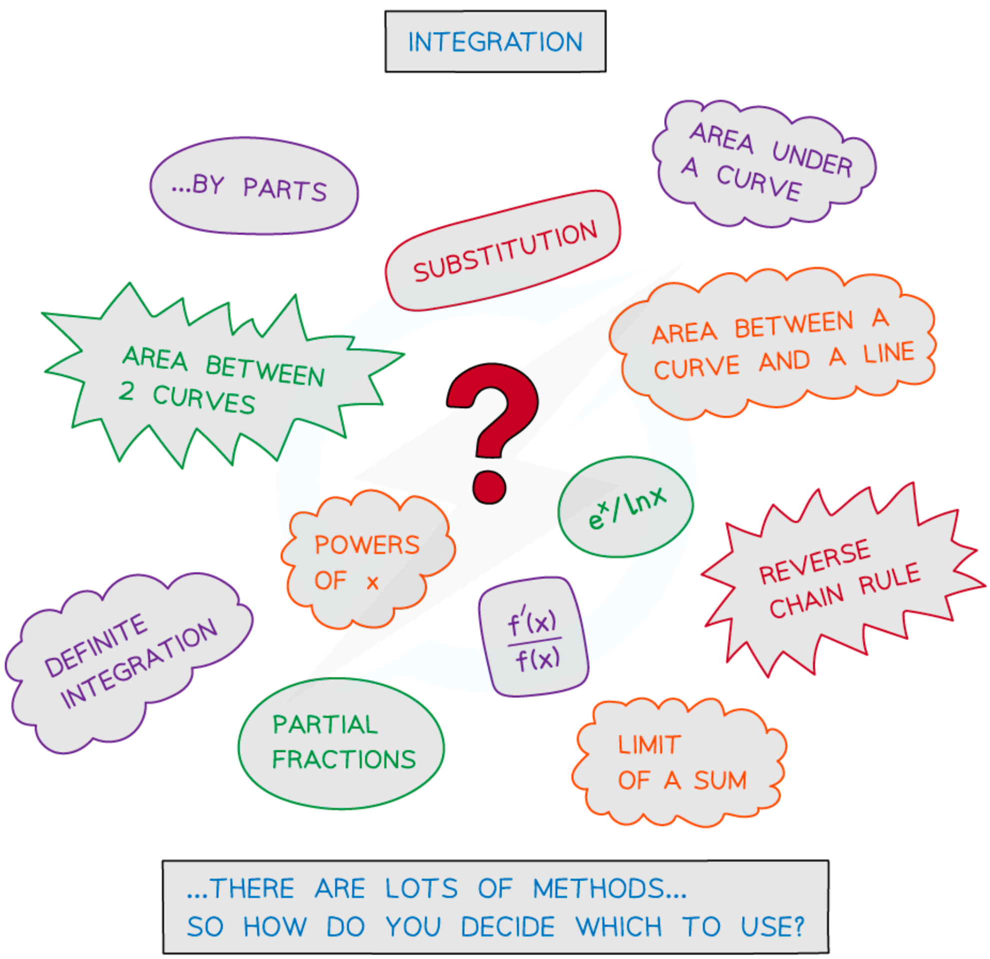
8.2.11 Decision Making

Your notes

Decision Making

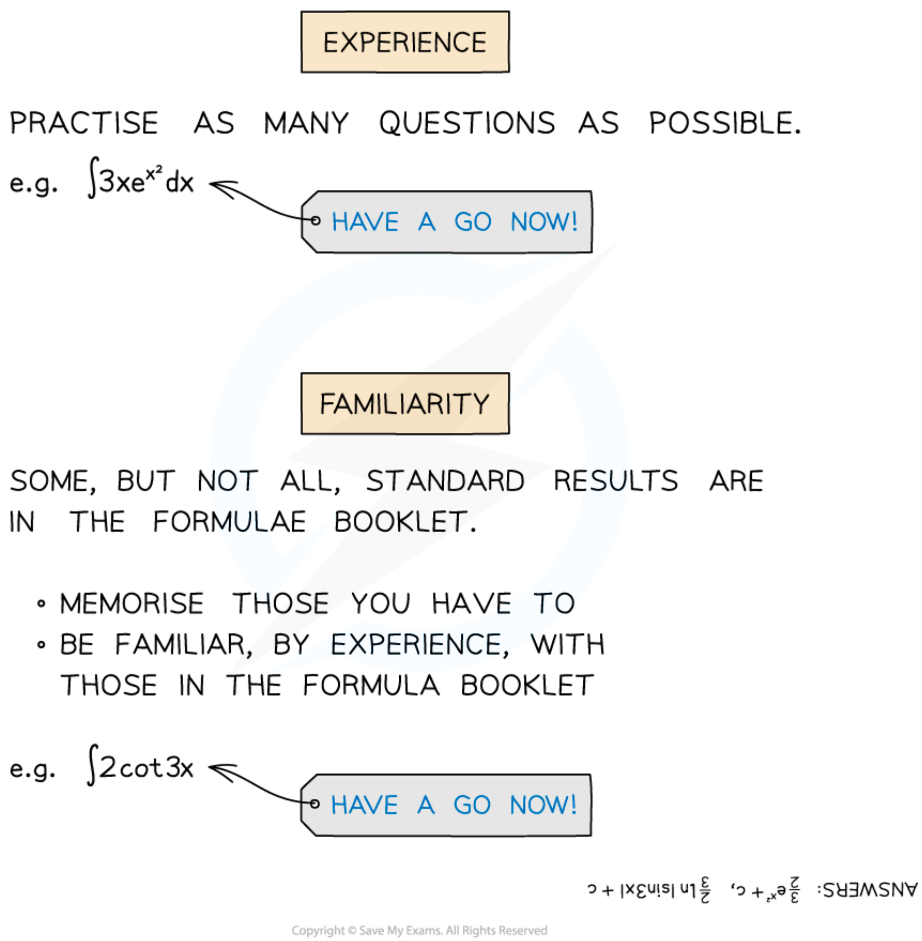
What is meant by integration and decision making?

- The hardest part of integration is deciding which technique to use
- There are lots of them!



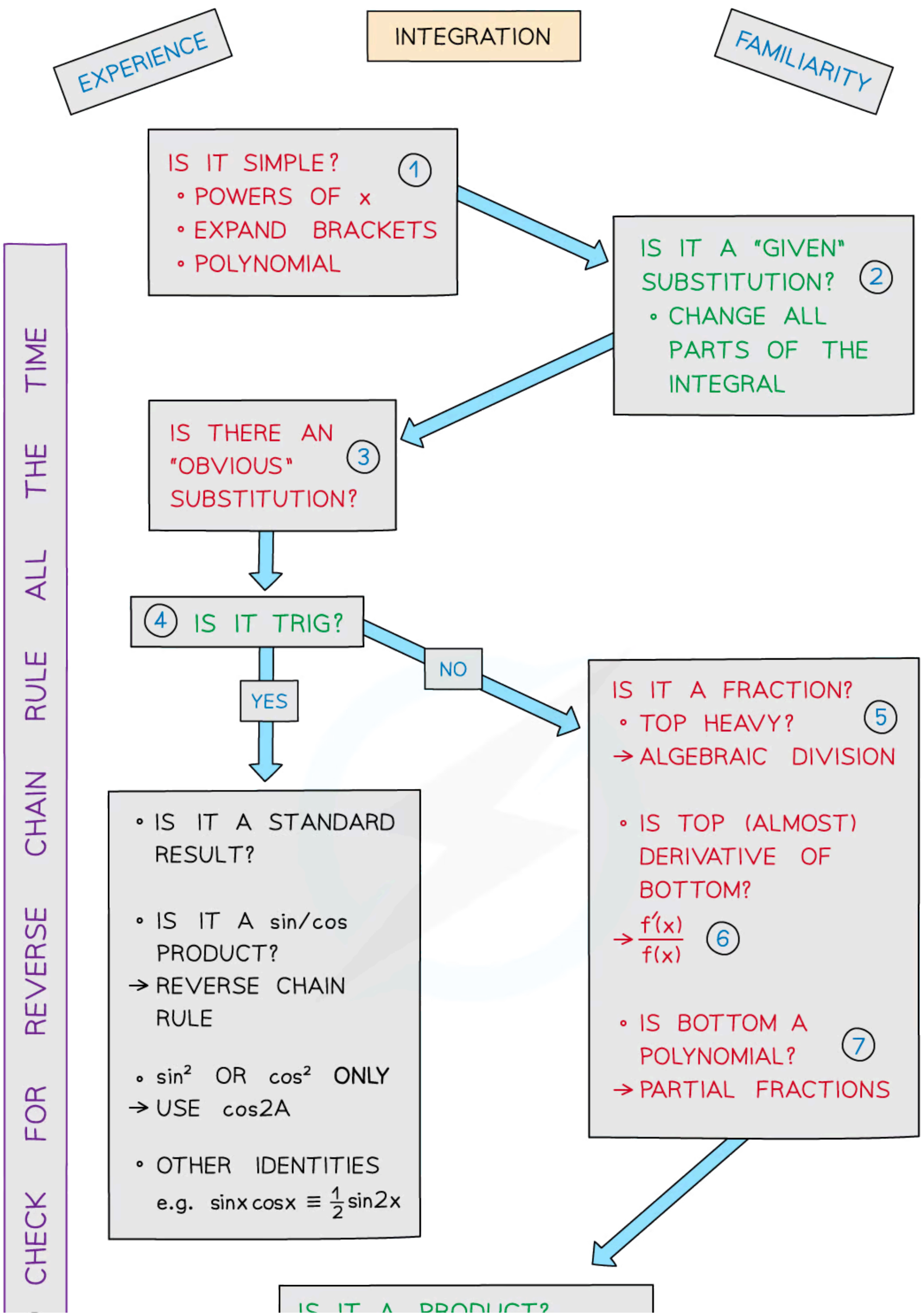
How do I know which integration method to use?

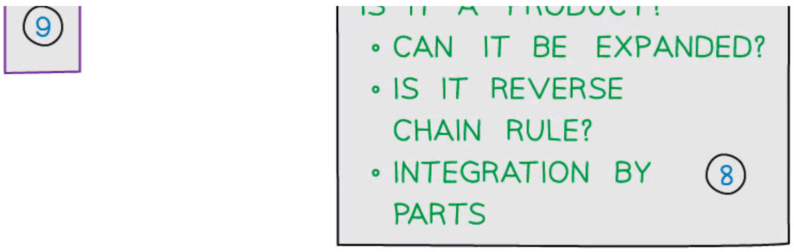
Your notes



- The main two requirements for success with integration are
  - **Experience**
  - **Familiarity**
- Always check for Reverse Chain Rule ...
  - ... particularly after changing part of an integral ...
  - ... by using a trig identity for example

Your notes


EXPERIENCE
INTEGRATION
FAMILIARITY
IS IT SIMPLE? 1
• POWERS OF x
• EXPAND BRACKETS
• POLYNOMIAL
IS IT A "GIVEN" SUBSTITUTION? 2
• CHANGE ALL PARTS OF THE INTEGRAL
IS THERE AN "OBVIOUS" SUBSTITUTION? 3
4 IS IT TRIG?
YES
NO
• IS IT A STANDARD RESULT?
• IS IT A sin/cos PRODUCT?
→ REVERSE CHAIN RULE
• sin² OR cos² ONLY
→ USE cos2A
• OTHER IDENTITIES
e.g. sinx cosx ≡ ½sin2x
IS IT A FRACTION?
• TOP HEAVY? 5
→ ALGEBRAIC DIVISION
• IS TOP (ALMOST) DERIVATIVE OF BOTTOM?
→ f′(x)/f(x) 6
• IS BOTTOM A POLYNOMIAL? 7
→ PARTIAL FRACTIONS
CHECK FOR REVERSE CHAIN RULE ALL THE TIME



Your notes

Key to diagram above:

(1) Powers of x

(2) Substitution

(3) Obvious substitution

(4) Trig

(5) Algebraic Division

(6) $\frac{f'(x)}{f(x)}$

(7) Partial fractions

(8) Integration by Parts

(9) Reverse Chain Rule

Examiner Tip

- If in doubt, try something ... You may gain marks for attempting a suitable substitution.
- You may gain marks from using a trig identity.
- You may gain marks from attempting to simplify or rewrite a fraction.

Your notes

Worked example

Find the following

(a) $\int sin\theta cos\theta \, d\theta$

(b) $\int \frac{3x^2}{2+x^3} \, dx$

(c) $\int 3x\sqrt{x^2+1} \, dx$

a) $\int \sin\theta \cos\theta \, d\theta$

CLEAR TRIG?

$\sin 2A \equiv 2\sin A\cos A$

$= \int \frac{1}{2}\sin 2\theta \, d\theta$

$= \frac{1}{2} \times -\frac{1}{2}\int -2\sin 2\theta \, d\theta$

"ADJUST" & "COMPENSATE"

$= -\frac{1}{4}\cos 2\theta + c$

Your notes

b) $\int \frac{3x^2}{2+x^3}\,dx$

$\frac{f'(x)}{f(x)}$

TOP IS DERIVATIVE OF BOTTOM

$= \ln|2+x^3| + c$

c) $\int 3x\sqrt{x^2+1}\,dx$

THIS IS, IF YOU SPOT IT, REVERSE CHAIN RULE.
LET'S ASSUME THIS WAS NOT SPOTTED...

CHOOSE A SENSIBLE SUBSTITUTION

LET $u = x^2 + 1$

$x = \sqrt{u-1}$ $\qquad \frac{du}{dx} = 2x$

$\frac{1}{2}du = x\,dx$

$$\int 3x\sqrt{x^2+1}\,dx = \int 3u^{\frac{1}{2}} \times \frac{1}{2}du$$

$$= \int \frac{3}{2}u^{\frac{1}{2}}$$

$$= \frac{\frac{3}{2}u^{\frac{3}{2}}}{\frac{3}{2}} + c$$

$$= u^{\frac{3}{2}} + c$$

$$= (x^2+1)^{\frac{3}{2}} + c$$

Your notes