Integration 11K

1 a $\frac{dP}{dt} = kP$

$\int \frac{1}{P}\, dP = kt + c$

$\ln P = kt + c$

$t = 0, P = 200 \Rightarrow c = \ln 200$

$\ln P - \ln 200 = kt$

$\ln\left(\frac{P}{200}\right) = kt$

$\frac{P}{200} = e^{kt}$

$P = 200e^{kt}$

b $k = 3$: $4000 = 200e^{3t}$

$e^{3t} = 20$

$t = \frac{1}{3}\ln 20 \approx 1$ year

c The population could not increase in size in this way forever due to limitations such as available food or space.

2 $\frac{dM}{dt} = M - M^2$

$\Rightarrow \int \frac{1}{M(1-M)}\, dM = \int 1\, dt$

but $\frac{1}{M(1-M)} \equiv \frac{A}{M} + \frac{B}{1-M}$

$\therefore 1 \equiv A(1-M) + BM$

$M = 0: 1 = 1A, A = 1$

$M = 1: 1 = 1B, B = 1$

$\Rightarrow \int\left(\frac{1}{M} + \frac{1}{1-M}\right) dM = \int 1\, dt$

$\Rightarrow \ln|M| - \ln|1-M| = t + c$

$\Rightarrow \ln\left|\frac{M}{1-M}\right| = t + c$

$\Rightarrow \frac{M}{1-M} = Ae^t$

a $t = 0, M = 0.5 \Rightarrow \frac{0.5}{0.5} = A^0 \Rightarrow A = 1$

$\therefore M = e^t - e^t M \Rightarrow M = \frac{e^t}{1+e^t}$

b $t = \ln 2 \Rightarrow M = \frac{e^{\ln 2}}{1+e^{\ln 2}} = \frac{2}{1+2} = \frac{2}{3}$

c $t \to \infty \Rightarrow M = \frac{1}{e^{-t}+1} \to \frac{1}{1} = 1$

3 a $\frac{dx}{dt} \propto \frac{1}{x^2} \Rightarrow \frac{dx}{dt} = \frac{k}{x^2}$

$\int x^2\, dx = kt$

$\frac{x^3}{3} = kt + c$

$t = 0, x = 1 \Rightarrow c = \frac{1}{3}$

$x^3 = 3kt + 1$

$t = 20, x = 2 \Rightarrow 8 = 60k + 1 \Rightarrow k = \frac{7}{60}$

$x^3 = \frac{7}{20}t + 1$

$x = \sqrt[3]{\frac{7}{20}t + 1}$

b $x = 3 \Rightarrow 27 = \frac{7}{20}t + 1$

$t = \frac{520}{7} = 74.3$ days

So time taken to go from 2 cm to 3 cm is 74.3 – 20 = 54.3 days.

4 a $\frac{dT}{dt} \propto -(T-25)$

$\frac{dT}{dt} = -k(T-25)$

The difference in temperature is $T - 25$. The tea is cooling, so there should be a negative sign. k has to be positive or the tea would be warming.

4 b $\frac{dT}{dt} = -k(T-25)$

$\int \frac{1}{T-25}\, dT = -kt + c$

$\ln|T-25| = -kt + c$

$t = 0, T = 85 \Rightarrow c = \ln 60$

$\frac{T-25}{60} = e^{-kt}$

$t = 10, T = 55 \Rightarrow \frac{30}{60} = e^{-10k}$

$\ln\frac{1}{2} = -10k \Rightarrow k = 0.0693$

$t = 15 \Rightarrow \frac{T-25}{60} = e^{-0.0693\times 15}$

$T = 60\times e^{-0.0693\times 15} + 25 = 46.2\ °\text{C}$ to 1 d.p.

5 a $\frac{dA}{dt} = \frac{A^{\frac{3}{2}}}{10t^2}$

$\int A^{-\frac{3}{2}}\, dA = \int \frac{1}{10t^2}\, dt$

$-\frac{2}{\sqrt{A}} = -\frac{1}{10t} + c$

$t = 1, A = 1 \Rightarrow -2 = -\frac{1}{10} + c$

$c = -\frac{19}{10}$

$-\frac{2}{\sqrt{A}} = -\frac{1}{10t} - \frac{19}{10}$

$\frac{2}{\sqrt{A}} = \frac{1}{10t} + \frac{19}{10} = \frac{1+19t}{10t}$

$\sqrt{A} = \frac{20t}{1+19t}$

$A = \left(\frac{20t}{1+19t}\right)^2$

b As $t \to \infty, A \to \left(\frac{20}{19}\right)^2 = \frac{400}{361}$ from below.

6 a Volume $V = 6000h \Rightarrow \frac{dV}{dh} = 6000$

$\frac{dV}{dt} = 12\,000 - 500h$ as the tub is filling at the rate of $12\,000\ \text{cm}^3/\text{min}$ and losing water at the rate of $500h\ \text{cm}^3/\text{min}$.

$\frac{dh}{dt} = \frac{dh}{dV}\times\frac{dV}{dt} = \frac{1}{6000}(12\,000 - 500h)$

$\frac{dh}{dt} = \frac{1}{60}(120 - 5h)$

$60\frac{dh}{dt} = 120 - 5h$

b $60\frac{dh}{dt} = 120 - 5h$

$\int \frac{60}{120-5h}\, dh = t + c$

$-12\ln(120-5h) = t + c$

$t = 0, h = 6 \Rightarrow c = -12\ln 90$

$-12\ln(120-5h) = t - 12\ln 90$

$h = 10$:

$t = 12\ln 90 - 12\ln 70$

$t = 12\ln\frac{9}{7}$

7 a $\frac{1}{P(10\,000-P)} = \frac{A}{P} + \frac{B}{10\,000-P}$

$1 = A(10\,000-P) + BP$

$P = 0 \Rightarrow A = \frac{1}{10\,000}$

$P = 10\,000 \Rightarrow B = \frac{1}{10\,000}$

$\frac{1}{P(10\,000-P)} = \frac{\frac{1}{10\,000}}{P} + \frac{\frac{1}{10\,000}}{10\,000-P}$

7 b $\frac{dP}{dt} = \frac{1}{20000} P(10000 - P)$

$\frac{1}{10000} \int \left(\frac{1}{P} + \frac{1}{10000 - P} \right) dP = \frac{1}{20000} t + c$

$\frac{1}{10000} \left(\ln|P| - \ln|10000 - P| \right) = \frac{1}{20000} t + c$

$\ln P - \ln|10000 - P| = 0.5t + d$

$t = 0, P = 2500 \Rightarrow d = \ln\left(\frac{2500}{7500} \right) = \ln\frac{1}{3}$

$\ln P - \ln|10000 - P| - \ln\frac{1}{3} = 0.5t$

$\ln\left| \frac{3P}{10000 - P} \right| = 0.5t$

$\frac{3P}{10000 - P} = e^{0.5t}$

$3P = (10000 - P)e^{0.5t}$

$3Pe^{-0.5t} + P = 10000$

$P\left(3e^{-0.5t} + 1\right) = 10000$

$P = \frac{10000}{1 + 3e^{-0.5t}}$

$a = 10000, b = 1, c = 3$

c Maximum number of deer is when

$\frac{dP}{dt} = 0 \Rightarrow P = 0$ or 10000

So maximum population is 10 000.

8 a $\frac{dV}{dt} = 40 - \frac{1}{4}V$

$-4\frac{dV}{dt} = V - 160$

b $\int -\frac{4}{V - 160}\, dV = t + c$

$-4\ln|V - 160| = t + c$

$t = 0, V = 5000 \Rightarrow c = -4\ln 4840$

$4\ln 4840 - 4\ln|V - 160| = t$

$\ln\left| \frac{4840}{V - 160} \right| = \frac{t}{4}$

$\frac{4840}{V - 160} = e^{\frac{t}{4}}$

$\frac{V - 160}{4840} = e^{-\frac{t}{4}}$

$V = 160 + 4840e^{-\frac{t}{4}}$

$a = 160, b = 4840$

c $t \to \infty \Rightarrow V \to 160$, as $e^{-\frac{t}{4}} \to 0$

9 a $\frac{dR}{dt} = -kR$

$\ln|R| = -kt + c$

$t = 0, R = R_0 \Rightarrow c = \ln R_0$

$\ln R - \ln R_0 = -kt$

$\frac{R}{R_0} = e^{-kt}$

$R = R_0 e^{-kt}$

b $t = 5730, R = \frac{R_0}{2} \Rightarrow \frac{R_0}{2} = R_0 e^{-5730k}$

$e^{5730k} = 2$

$5730k = \ln 2$

$k = \frac{1}{5730}\ln 2$

c $R = \frac{R_0}{10} \Rightarrow \frac{R_0}{10} = R_0 e^{-kt}$

$e^{kt} = 10$

$t = \frac{1}{k}\ln 10 = \frac{5730\ln 10}{\ln 2} = 19035$