Edexcel A Level Maths: Pure

Your notes

5.3 Trigonometric Equations

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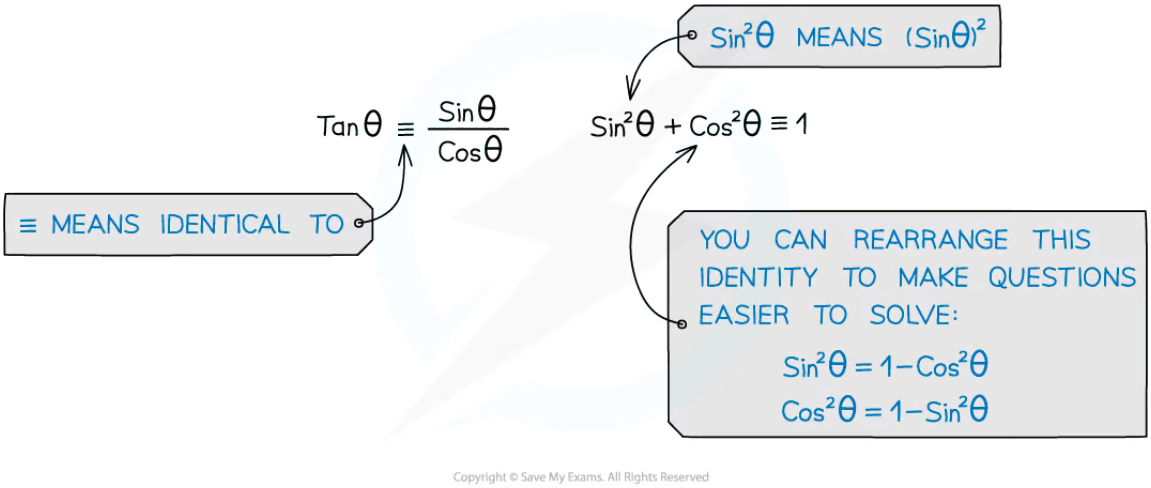
5.3.1 Trigonometry – Simple Identities

Your notes

Trigonometry – Simple Identities

What is a trigonometric identity?

- Trigonometric identities are statements that are true for all values of *x* or theta (θ)
- They are used to help simplify trig equations before solving them
- The first two identities you must know are:



Where do these identities come from?

- Although you don't need to know the proof for these identities it is important to understand where they come from

Your notes

FOR THE FIRST IDENTITY WE KNOW

$$\text{Sin}\theta = \frac{\text{OPP}}{\text{HYP}} \quad \text{AND} \quad \text{Cos}\theta = \frac{\text{ADJ}}{\text{HYP}} \quad \text{AND} \quad \text{Tan}\theta = \frac{\text{OPP}}{\text{ADJ}}$$

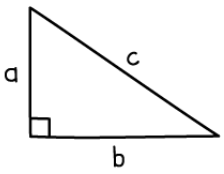
SO IF WE DIVIDE $\text{Sin}\theta$ BY $\text{Cos}\theta$

$$\frac{\text{Sin}\theta}{\text{Cos}\theta} = \frac{\text{OPP} \div \text{HYP}}{\text{ADJ} \div \text{HYP}} = \frac{\text{OPP}}{\text{ADJ}} = \text{Tan}\theta$$

SO

$$\text{Tan}\theta \equiv \frac{\text{Sin}\theta}{\text{Cos}\theta}$$

FOR THE SECOND IDENTITY WE NEED TO REMEMBER PYTHAGORAS' THEOREM



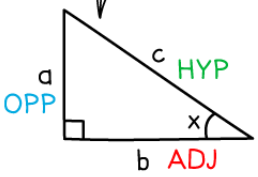
$$a^2 + b^2 = c^2$$

DIVIDING BY c^2

$$\frac{a^2}{c^2} + \frac{b^2}{c^2} = \frac{c^2}{c^2}$$

$$\left(\frac{a}{c}\right)^2 + \left(\frac{b}{c}\right)^2 = 1$$

IF WE RELABEL TRIANGLE FOR TRIG

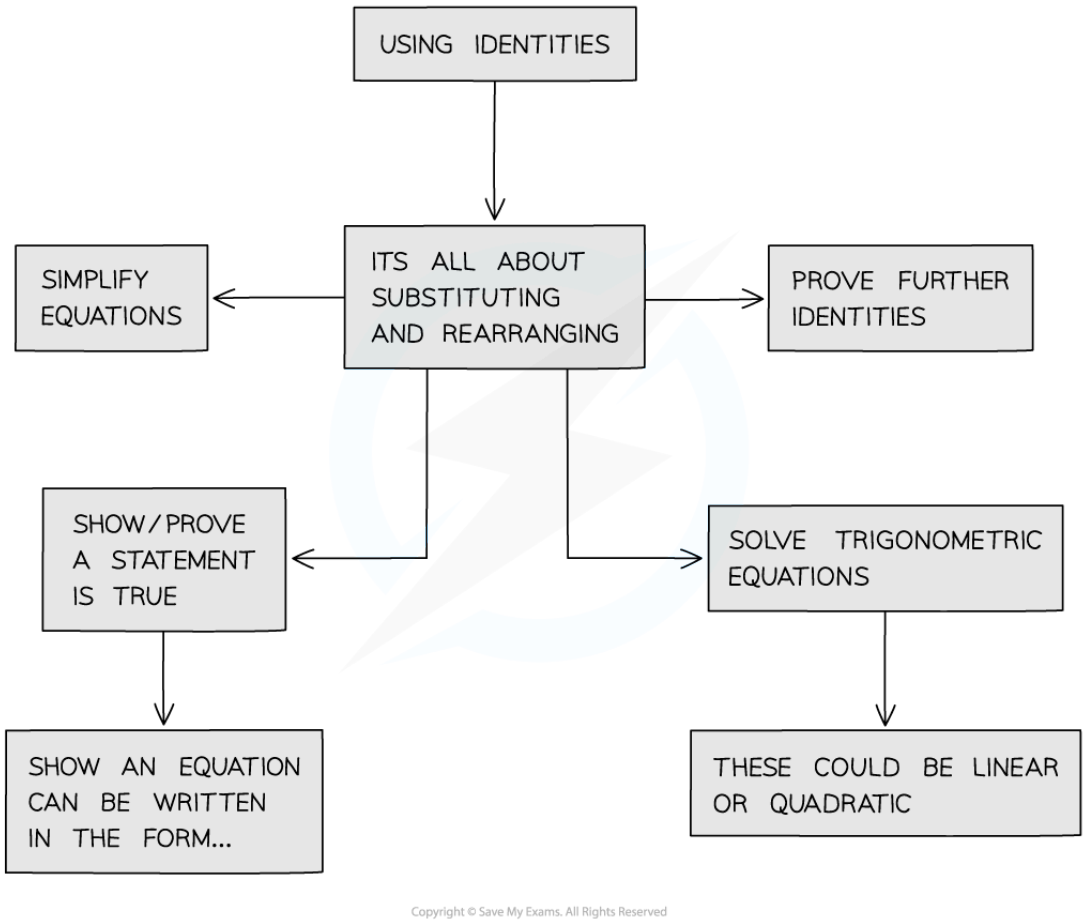


$$\frac{a}{c} = \frac{\text{OPP}}{\text{HYP}} = \text{Sin}\theta \quad \text{AND} \quad \frac{b}{c} = \frac{\text{ADJ}}{\text{HYP}} = \text{Cos}\theta$$

SO $\text{Sin}^2\theta + \text{Cos}^2\theta \equiv 1$

How to use the identities

Your notes



Examiner Tip

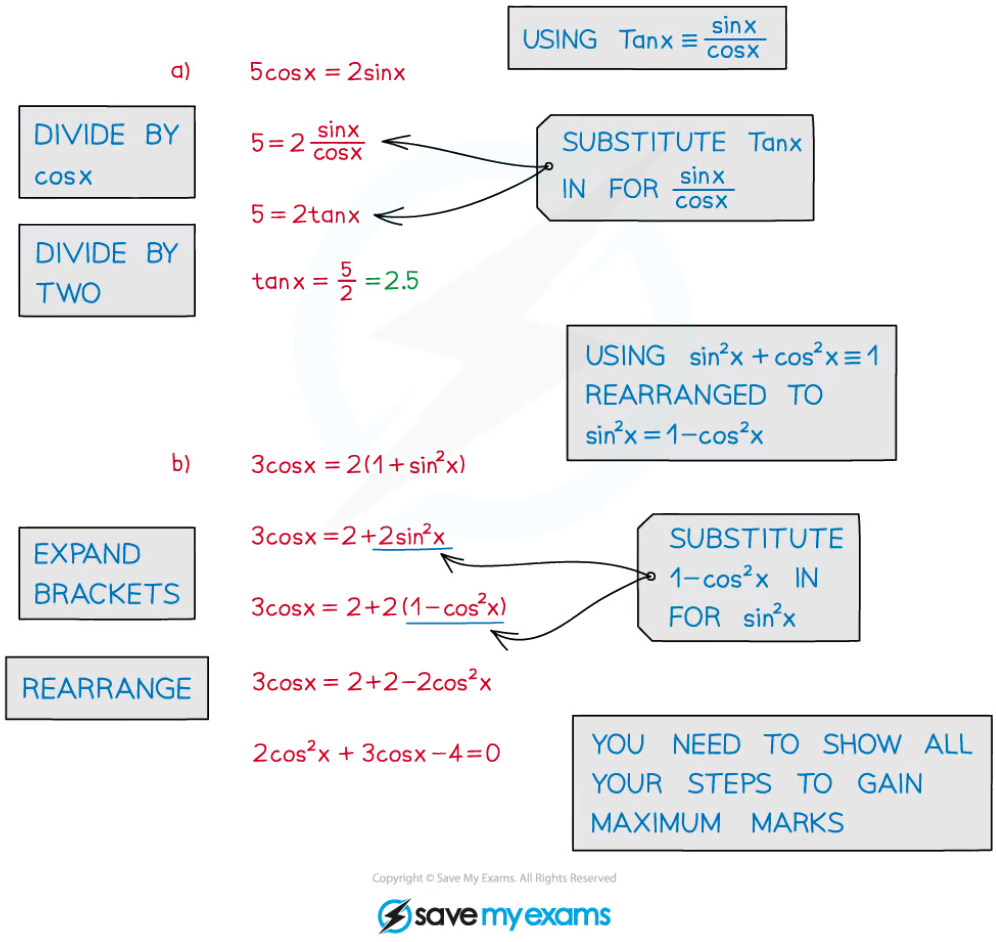
- You'll need to remember these identities as they aren't in the formula booklet.
- If asked to show one thing is identical (≡) another look at what parts are missing – for example, if tan x has gone it must have been substituted.

Worked example

Your notes

a) For an angle x, $5\cos x = 2\sin x$. Find tan x.

b) Show that the equation $3\cos x = 2(1 + \sin^2 x)$ can be written in the form $2\cos^2 x + 3\cos x - 4 = 0$.



5.3.2 Linear Trigonometric Equations

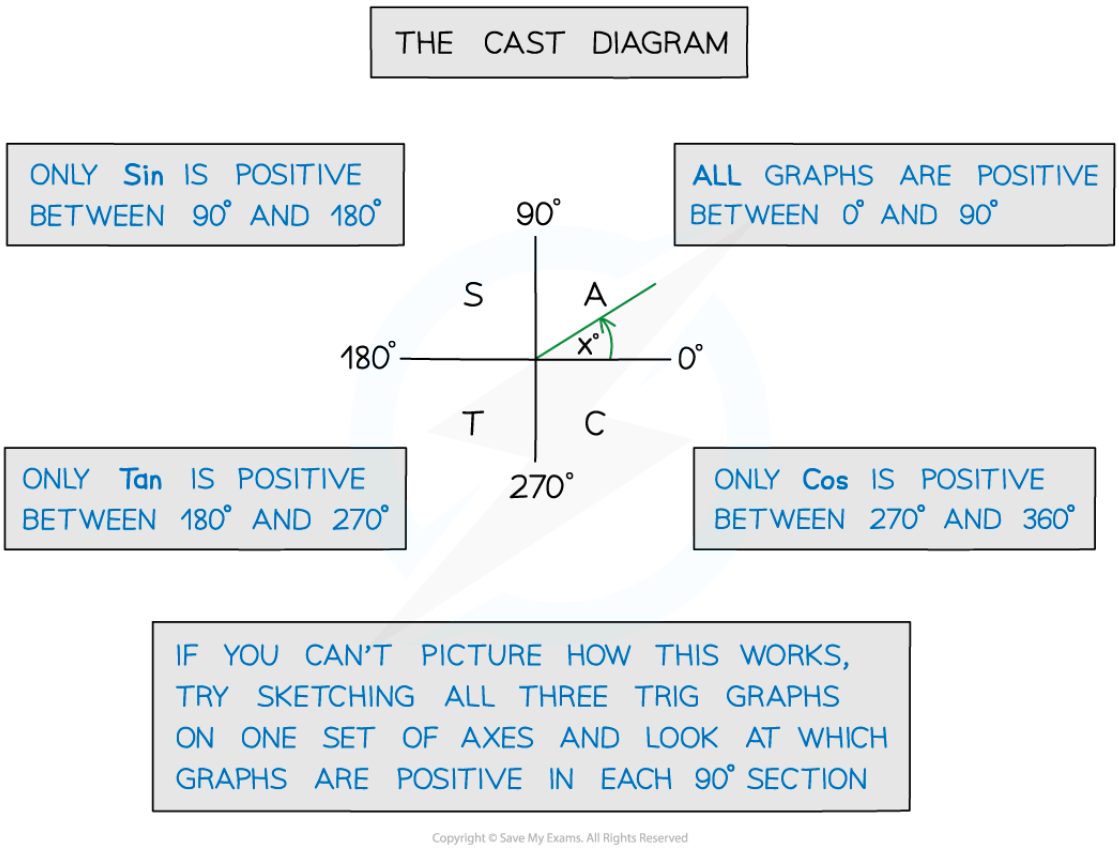
Your notes

Linear Trigonometric Equations

Solving linear trigonometric equations

- You should have already seen two ways to solve linear trig equations
  - By **sketching the graph** (see Graphs of Trigonometric Functions) you can read off all the solutions in a given range (or interval)
  - By using **trigonometric identities** you can simplify harder equations
- Another way to find solutions is by using the CAST diagram which shows where each function has positive solutions
- You may be asked to use **degrees or radians** to solve trigonometric equations
  - Make sure your calculator is in the **correct mode**
  - Remember common angles
    - 90° is ½π radians
    - 180° is π radians
    - 270° is 3π/2 radians
    - 360° is 2π radians

Your notes



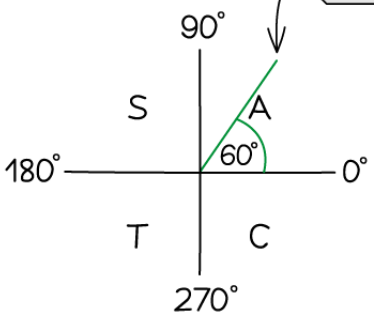
How do I use the CAST diagram?

Your notes

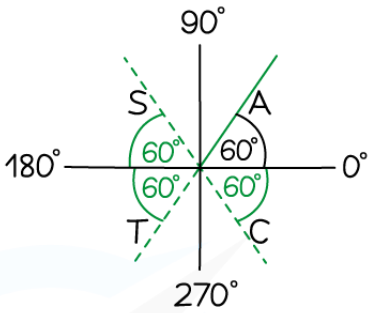
e.g. FIND ALL THE VALUES OF x IN THE INTERVAL $-360^\circ \leqslant x \leqslant 720^\circ$ WHERE $\cos x = \frac{1}{2}$

PRINCIPAL VALUE
$\cos^{-1}\left(\frac{1}{2}\right) = 60^\circ$

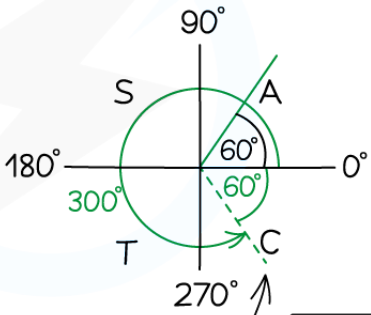
STEP 1: CALCULATE THE **PRINCIPLE VALUE** AND **DRAW IT** INTO THE CAST DIAGRAM STARTING FROM 0°



STEP 2: CONTINUE THE LINE ACROSS THE CAST DIAGRAM MAKING THE **SAME ANGLE** IN ALL **4 QUADRANTS**



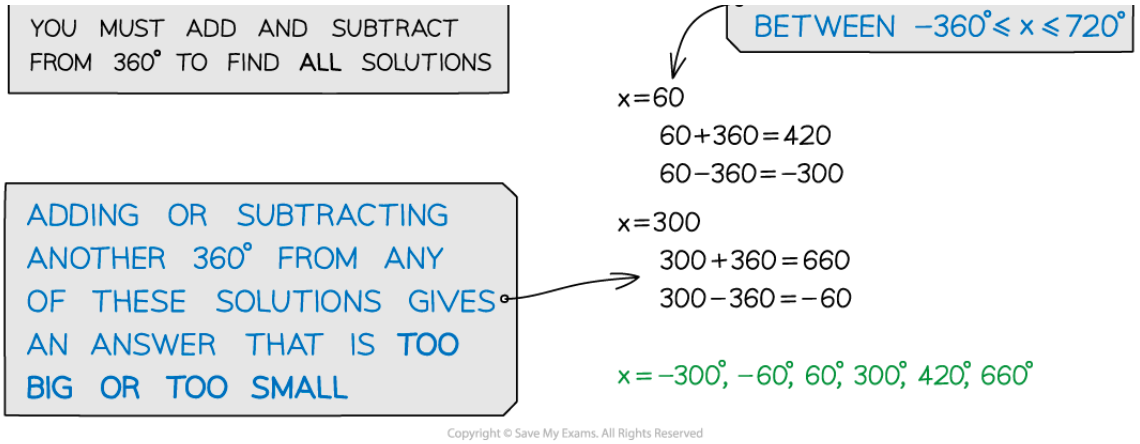
STEP 3: FOCUSING ON THE LINES IN THE **VALID QUADRANTS** WORK OUT ALL POSITIVE SOLUTIONS FOR $0^\circ \leqslant x \leqslant 360^\circ$



SOLUTIONS FOR $\cos x = \frac{1}{2}$ ARE POSITIVE IN THE 1st AND 4th QUADRANTS ONLY
$x = 60^\circ, 300^\circ$

STEP 4: CHECK YOUR QUESTION FOR THE GIVEN **INTERVAL**, IF THIS IS BEYOND $0^\circ \leqslant x \leqslant 360^\circ$

FIND ALL SOLUTIONS



Your notes

What about more complicated trig equations?

- Some trig equations could involve a **function** of *x* or θ (see Transformations of Trigonometric Functions)
- Functions could be in two forms, either y = **sin(θ ± k)** or y = **sin kθ**
- An easy way to solve an equation involving sin, cos or tan of (θ± k) or kθ is by transforming the range of the question

Your notes

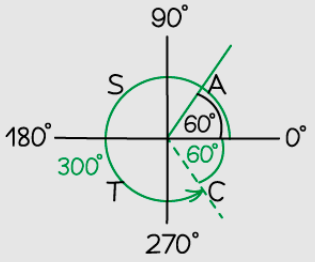
FOR FUNCTIONS OF sin, cos OR tan (θ ± k)

e.g. SOLVE $\cos(\theta-30)=0.5$ FOR $0^\circ \leqslant \theta \leqslant 360^\circ$

-30 ↓ $0^\circ \leqslant \theta \leqslant 360^\circ$ ↓ -30

$-30^\circ \leqslant \theta-30 \leqslant 330^\circ$

USING $Z=\theta-30$
$\cos(\theta-30)=0.5$
BECOMES
$\cos(Z)=0.5$
$Z=\cos^{-1}(0.5)=60$



FOR $-30^\circ \leqslant Z \leqslant 330^\circ$
$Z=60$ (IN RANGE)
$Z=360-60=300$
(IN RANGE)
$Z=300-360=-60$
(NOT IN RANGE)

REMEMBER THESE ARE THE SOLUTIONS FOR $\theta-30$

FOR $-30^\circ \leqslant Z \leqslant 330^\circ$

STEP 1: TRANSFORM THE RANGE FROM θ TO THE NEW FUNCTION

STEP 2: SOLVE THE BASIC FUNCTION TO FIND THE PRINCIPLE VALUE

STEP 3: USE CAST OR SKETCH THE GRAPH TO FIND ALL SOLUTIONS IN THE TRANSFORMED RANGE

STEP 4: CONVERT SOLUTIONS BACK INTO CORRESPONDING SOLUTIONS FOR θ (IF YOU SKETCHED THE TRANSFORMED GRAPH YOU WILL ALREADY HAVE THE CORRECT

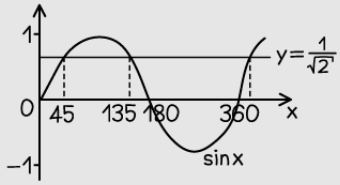
FOR FUNCTIONS OF sin, cos OR tan (kθ)

e.g. SOLVE $\sin(2x)=\frac{1}{\sqrt{2}}$ FOR $0^\circ \leqslant x \leqslant 360^\circ$

$\times 2$ ↓ $0^\circ \leqslant x \leqslant 360^\circ$ ↓ $\times 2$

$0^\circ \leqslant 2x \leqslant 720^\circ$

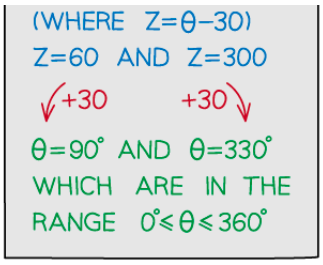
USING $Z=2x$
$\sin(2x)=\frac{1}{\sqrt{2}}$
BECOMES
$\sin(Z)=\frac{1}{\sqrt{2}}$
$Z=\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)=45$

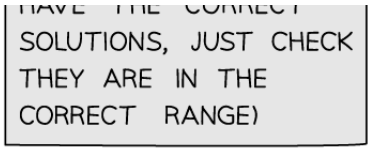


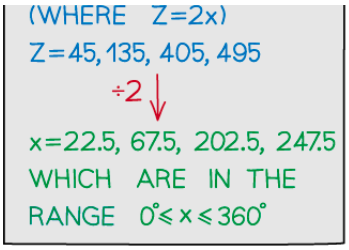
FOR $0^\circ \leqslant Z \leqslant 720^\circ$
$Z=45$ (IN RANGE)
$Z=180-45=135$
(IN RANGE)
$Z=45+360=405$
(IN RANGE)
$Z=135+360=495$
(IN RANGE)
$Z=405+360=765$
(NOT IN RANGE)

REMEMBER THESE ARE THE SOLUTIONS FOR $2x$

FOR $0^\circ \leqslant Z \leqslant 720^\circ$







Your notes

Examiner Tip

- Your calculator will only give you the principal value and you need to find all other solutions for the given interval.
- Also, remember the CAST diagram only gives you some solutions, so again you may need to find more depending on the given interval.
- It is entirely up to you how you solve a trig equation, but some ways are more helpful than others depending on the type of equation you are trying to solve

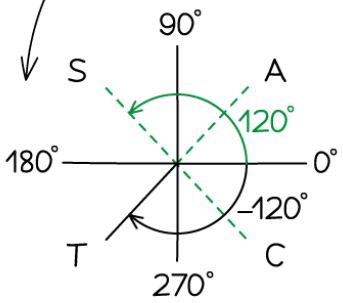
Worked example

Your notes

Solve the equation $2\cos x = -1$, in the range $-180° \leq x \leq 540°$.

Cos x IS NOT POSITIVE IN THE 2nd AND 3rd QUADRANT SO THESE MUST BE THE SOLUTIONS 120° AND 240°

THE SOLUTION IS NEGATIVE SO LOOK FOR WHERE cos x IS NOT POSITIVE



$2\cos x = -1$

$\cos x = -\frac{1}{2}$

$\cos^{-1}\left(-\frac{1}{2}\right) = -120$

PRINCIPLE VALUE IS NEGATIVE SO GO CLOCKWISE FROM 0° THEN READ OFF ANTICLOCKWISE ANGLE (360° − 120° = 240°)

FOR INTERVAL $-180° \leqslant x \leqslant 540°$
SOLUTIONS ARE 120° AND 240°

$120 + 360 = 480$

$120 - 360 = -240$ (TOO SMALL)

$240 + 360 = 600$ (TOO BIG)

$240 - 360 = -120$

WE ALREADY KNOW THIS ONE

± 360 TO EACH SOLUTION FOR MORE POSSIBLE SOLUTIONS

$x = -120, 120, 240, 480$

5.3.3 Quadratic Trigonometric Equations

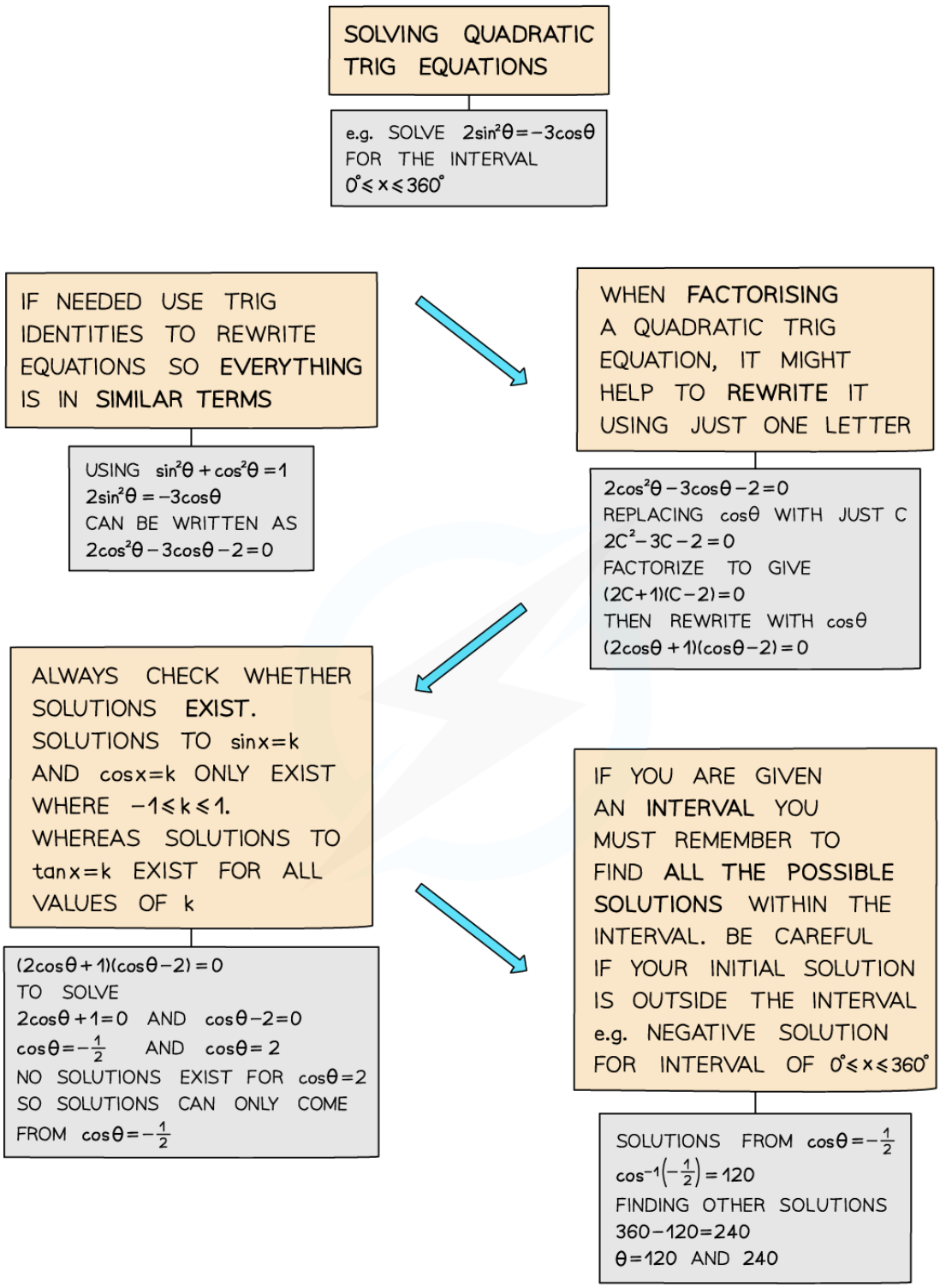

Your notes

Quadratic Trigonometric Equations

Solving quadratic trigonometric equations

- If an equation involves $\sin^2\theta$ or $\cos^2\theta$ then it is a quadratic trigonometric equation
- These can be solved by factorising and/or using **trigonometric identities** (see **Trigonometry – Simple Identities**)
- As a quadratic can result in two solutions, will need to consider whether each solution exists and then find all solutions within a given interval for each
- You may be asked to use **degrees or radians** to solve trigonometric equations
  - Make sure your calculator is in the **correct mode**
  - Remember common angles
    - 90° is ½π radians
    - 180° is π radians
    - 270° is 3π/2 radians
    - 360° is 2π radians

Your notes



Examiner Tip

- Sketch the appropriate sin, cos, tan graph to ensure you find ALL solutions **within** the given interval, and be super-careful if you get a negative solution but have a positive interval.
- For example, for an equation, in the interval $0° \leq x \leq 360°$, with solution $\sin x = -¼$ then $\sin^{-1}(-¼) = -14.5$ (to 1d.p.), which is not between 0 and 360 – by sketching the graph you'll be able to spot the two solutions will be $180 + 14.5$ and $360 - 14.5$.

Your notes

Worked example

Your notes

Your notes

a) Solve $\tan^2 x - 2\tan x = 0$, in the interval $0° \leq x \leq 360°$.
Giving each solution correct to 1 decimal place.

b) i) Show that $11\sin x = 5\cos^2 x + 7$ can be written in the form $5\sin^2 x + 11\sin x - 12 = 0$.

ii) Find all the solutions of the equation $11\sin x = 5\cos^2 x + 7$, in the interval $0° \leq x \leq 360°$.
Giving each solution correct to 1 decimal place.

a) $\tan^2 x - 2\tan x = 0$

FACTORISE

$\tan x(\tan x - 2) = 0$

TWO POSSIBLE SOLUTIONS

$\tan x = 0$

$\tan^{-1}(0) = 0$

$\tan x - 2 = 0$

$\tan x = 2$

$\tan^{-1}(2) = 63.43$

$63.4 + 180 = 243.4$

Tan REPEATS EVERY 180° SO ADD 180° TO FIND ALL SOLUTIONS

$x = 0, 180, 360$ AND $x = 63.4, 243.4$

b) i) $11\sin x = 5\cos^2 x + 7$

USE $\cos^2 x = 1 - \sin^2 x$ TO GET EVERYTHING IN TERMS OF $\sin x$

$11\sin x = 5(1 - \sin^2 x) + 7$

$11\sin x = 5 - 5\sin^2 x + 7$

EXPAND THEN REARRANGE

$5\sin^2 x + 11\sin x - 12 = 0$

Your notes

ii) $5\sin^2 x + 11\sin x - 12 = 0$

$(5\sin x - 4)(\sin x + 3) = 0$

FACTORISE USING $\sin x = s$ TO MAKE IT EASIER $(5s-4)(s+3)$

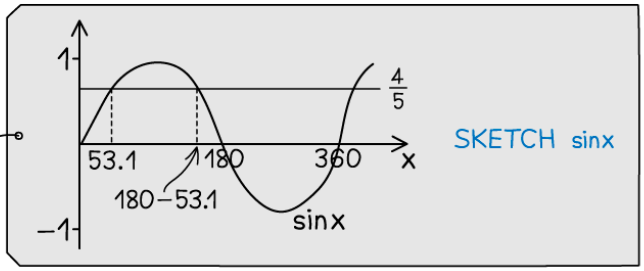
USING $11\sin x = 5\cos^2 x + 7$ WRITTEN IN THE FORM $5\sin^2 x + 11\sin x - 12 = 0$ FROM PART i)

$5\sin x - 4 = 0$ $\quad$ $\sin x + 3 = 0$

$5\sin x = 4$ $\quad$ $\sin x = -3$

$\sin x = \frac{4}{5}$

$\sin x = -3$ HAS NO SOLUTION



SKETCH $\sin x$

ONLY TWO VALID SOLUTIONS FOR GIVEN INTERVAL

$\sin^{-1}\left(\frac{4}{5}\right) = 53.1$

$180 - 53.1 = 126.9$

TWO SOLUTIONS $x = 53.1, 126.9$

5.3.4 Strategy for Trigonometric Equations

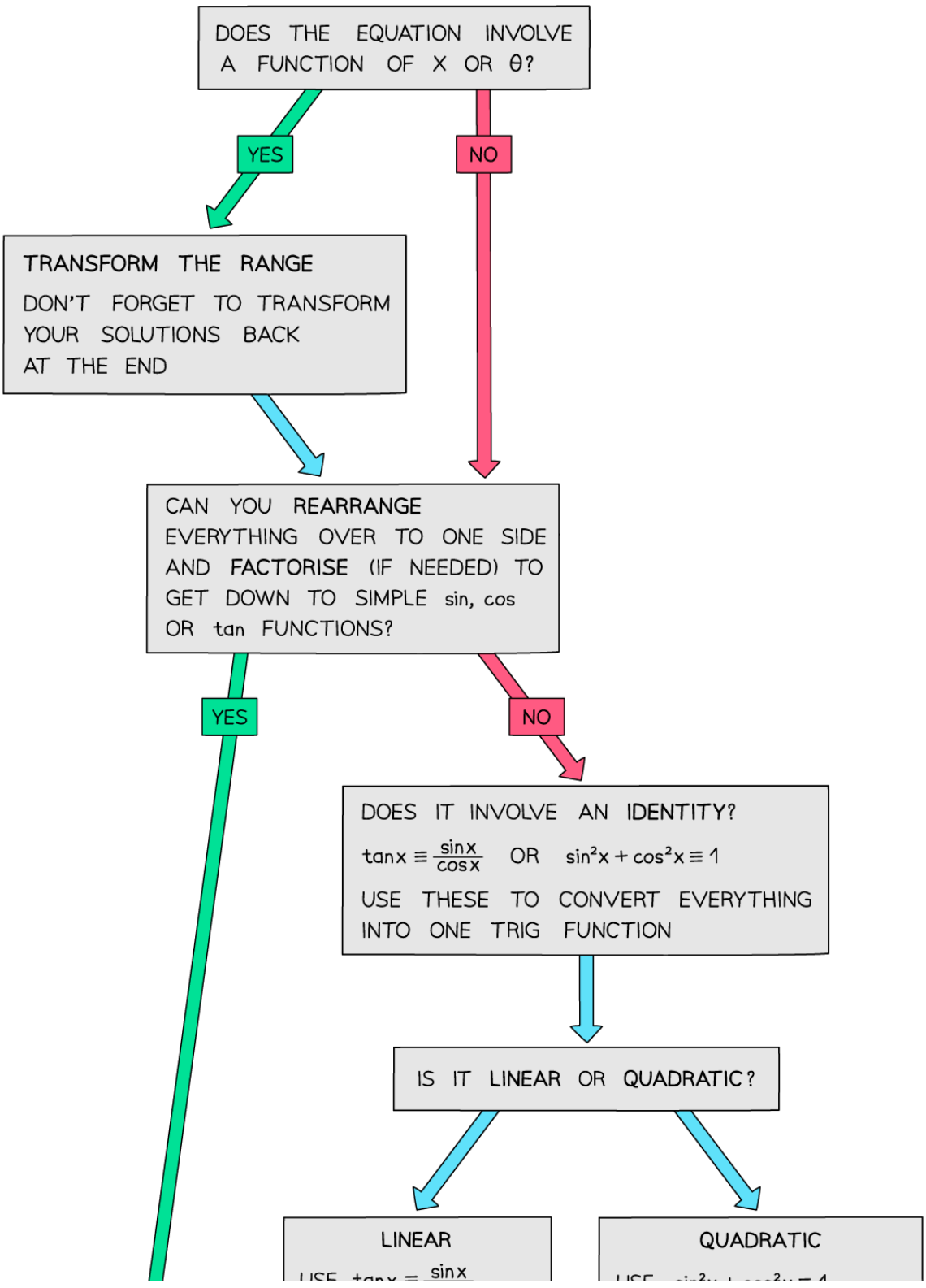

Your notes

Strategy for Trigonometric Equations

How to approach solving trig equations

- You can solve trig equations in a variety of different ways
  - **Sketching a graph** (see **Graphs of Trigonometric Functions**)
  - Using **trigonometric identities** (see **Trigonometry – Simple Identities**)
  - Using the **CAST** diagram (see **Linear Trigonometric Equations**)
  - Factorising **quadratic** trig equations (see **Quadratic Trigonometric Equations**)
- You may be asked to use **degrees or radians** to solve trigonometric equations
  - Make sure your calculator is in the **correct mode**
  - Remember common angles
    - 90° is ½π radians
    - 180° is π radians
    - 270° is 3π/2 radians
    - 360° is 2π radians
- If you're having trouble solving a trig equation, this flowchart might help:

Your notes



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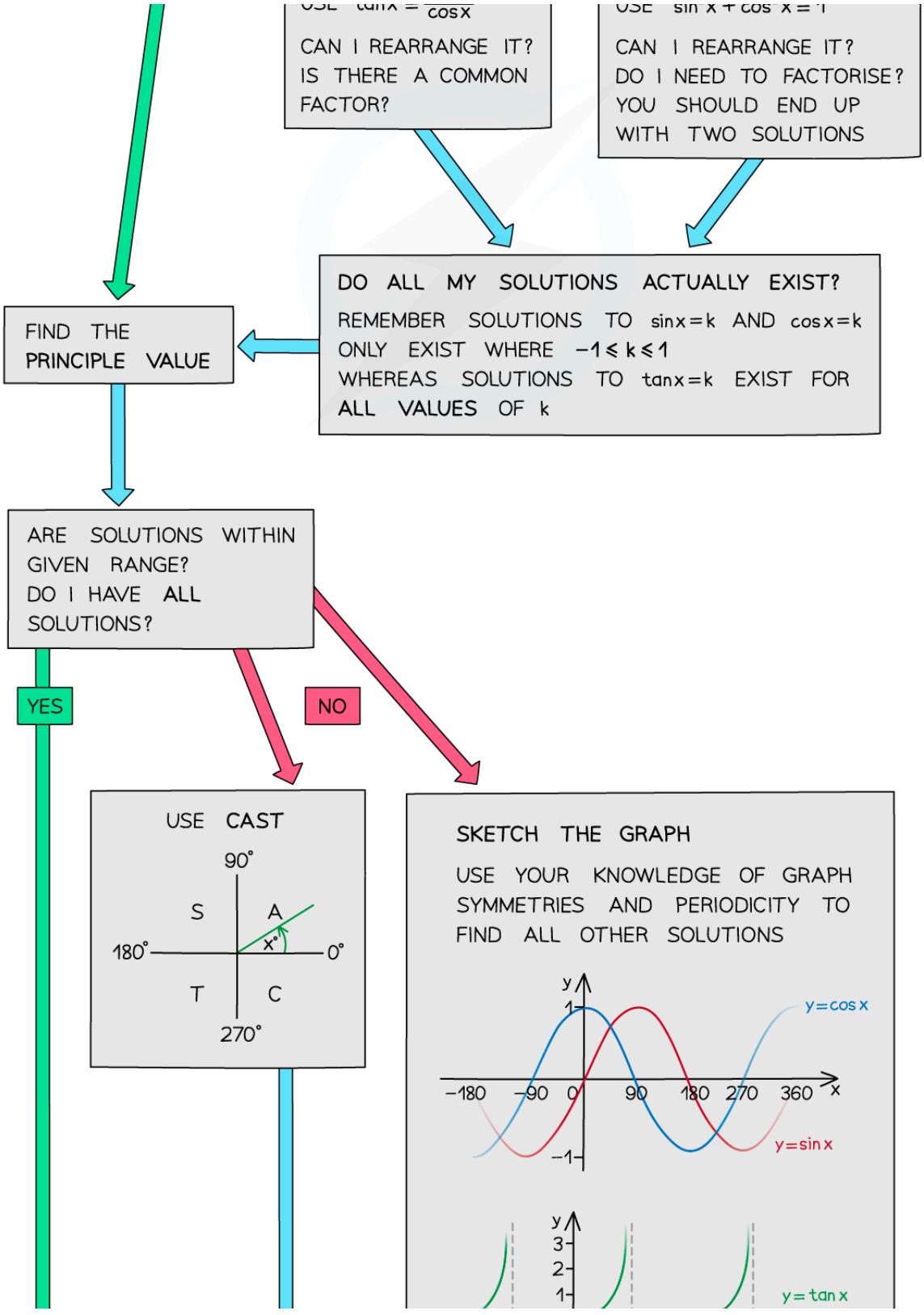
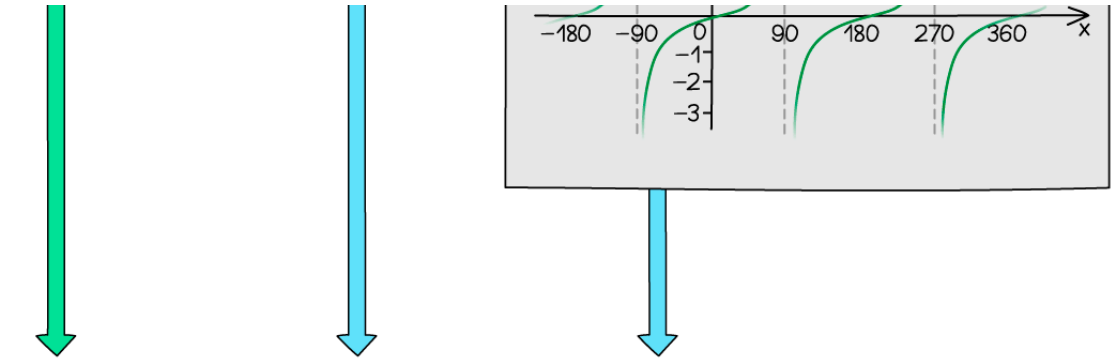




CHECK WHETHER YOU HAVE FOUND **ALL SOLUTIONS** AND THAT THEY ARE **WITHIN THE GIVEN RANGE**
(DON'T FORGET TO TRANSFORM YOUR SOLUTIONS BACK AT THE END IF THEY WERE FROM A TRANSFORMED RANGE)

Your notes

Examiner Tip

- Don't forget to check the function range and ensure you have included all possible solutions.
- If the question involves a function of x or θ, make sure you transform the range first (and ensure you transform your solutions back again at the end!).

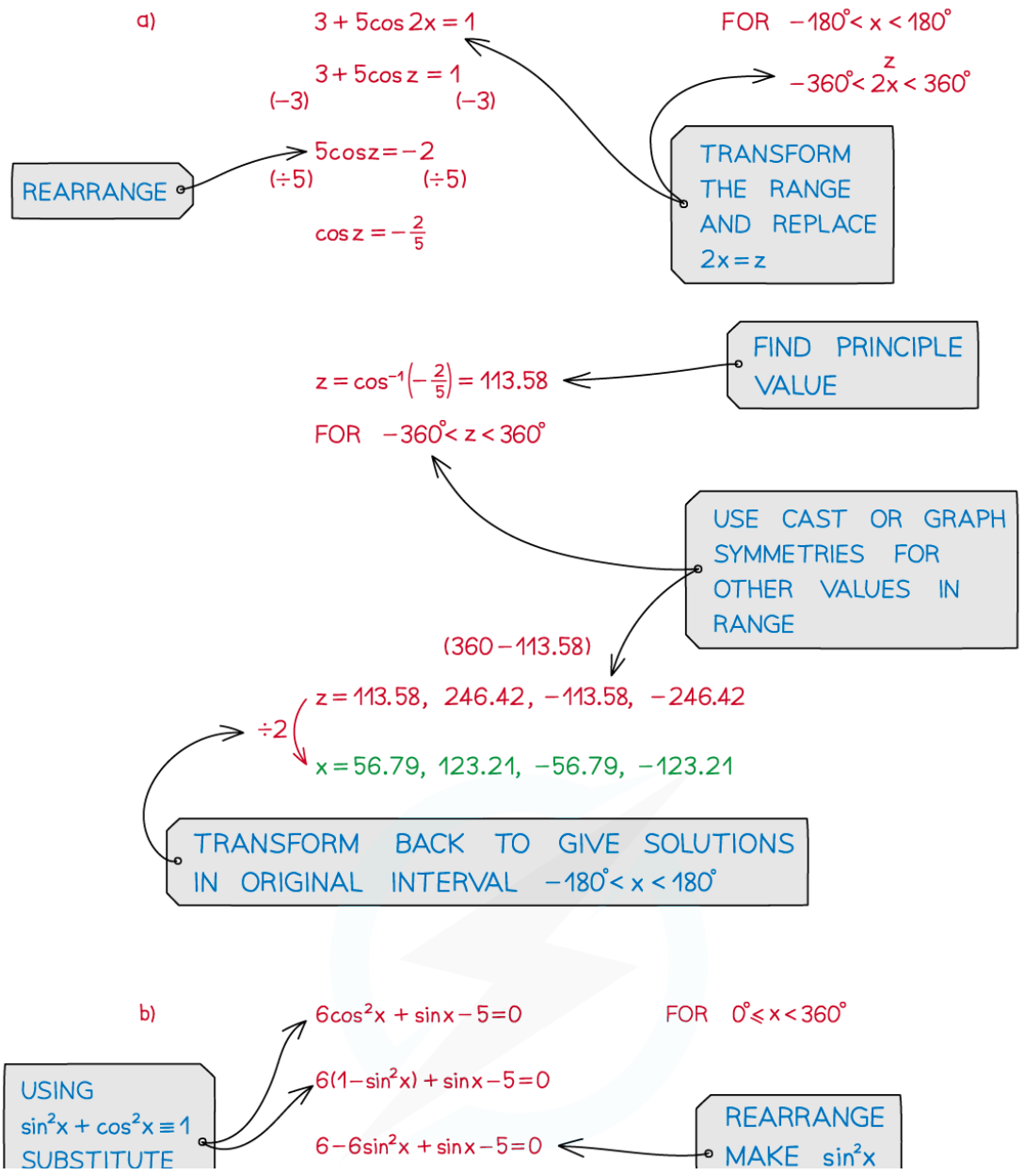
Worked example

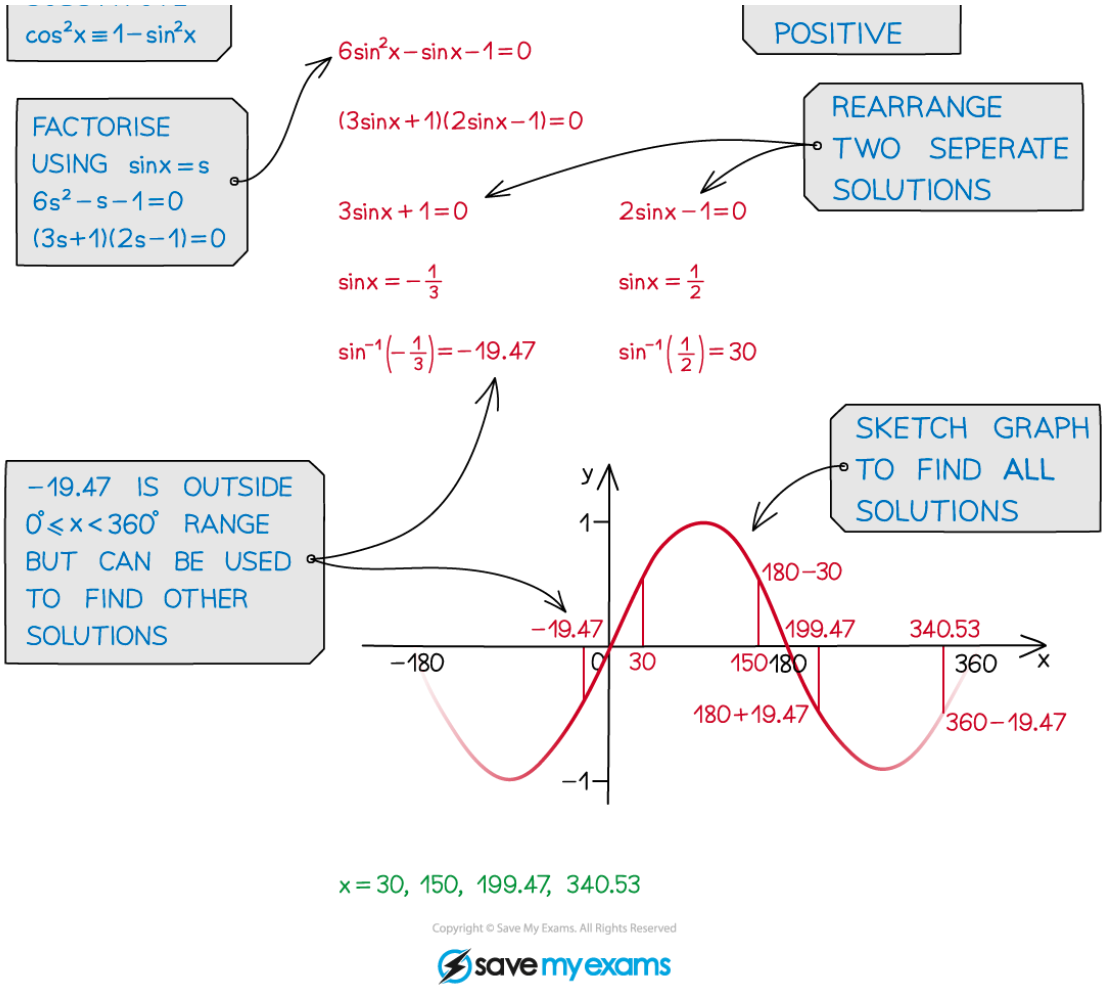
Your notes

Your notes

a) Solve $3 + 5\cos 2x = 1$, for $-180° < x < 180°$.

b) Solve $6\cos^2 x + \sin x - 5 = 0$, for $0 \leq x < 360°$.





$x = 30,\ 150,\ 199.47,\ 340.53$

Your notes