Edexcel A Level Maths: Pure

Your notes

7.2 Applications of Differentiation

Contents

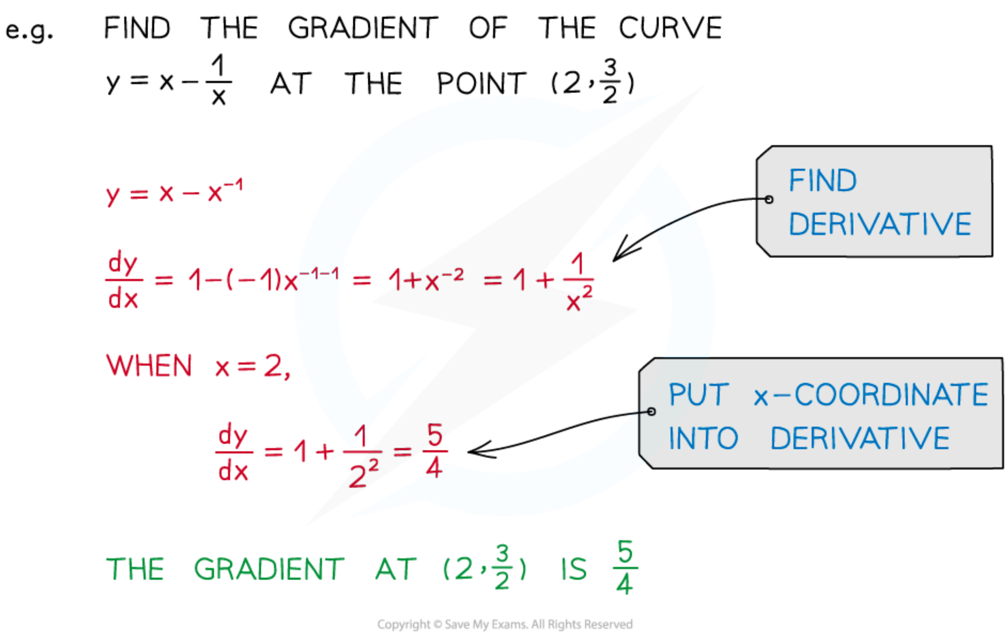
7.2.1 Gradients, Tangents & Normals

Your notes

Gradients, Tangents & Normals

Using the derivative to find the gradient of a curve

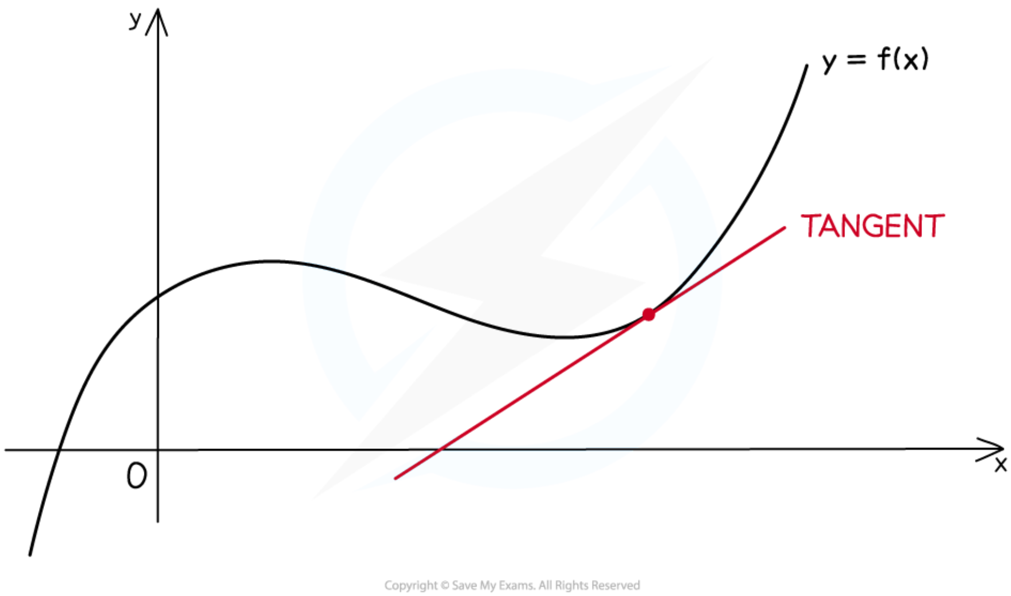
- To find the gradient of a curve ***y*= f(*x*)** at any point on the curve, substitute the ***x*-coordinate** of the point into the derivative **f'(*x*)**



Using the derivative to find a tangent

- At any point on a curve, the **tangent** is the line that goes through the point and has the same gradient as the curve at that point

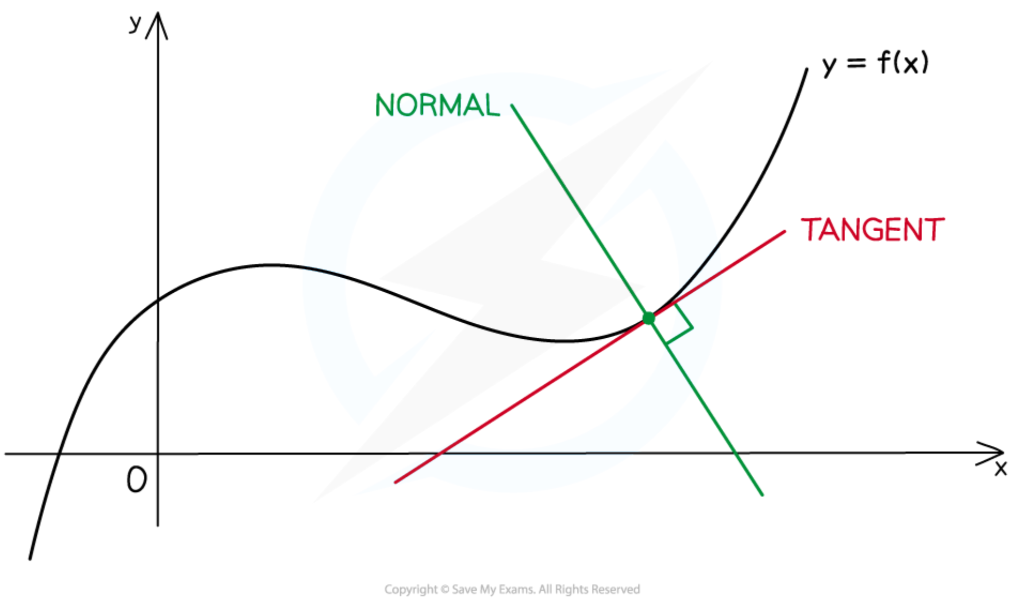
Your notes



- For the curve y = f(x), you can find the equation of the **tangent** at the point (a, f(a)) using
  $y - \mathrm{f}(a) = \mathrm{f}'(a)(x - a)$

Using the derivative to find a normal

- At any point on a curve, the **normal** is the line that goes through the point and is **perpendicular** to the tangent at that point



- For the curve y = f(x), you can find the equation of the **normal** at the point (a, f(a)) using

$$y - \mathrm{f}(a) = -\frac{1}{\mathrm{f}'(a)}(x - a)$$

Your notes

Examiner Tip

- The formulae above are not in the exam formulae booklet, but if you understand what tangents and normals are, then the formulae follow from the equation of a straight line combined with parallel and perpendicular gradients (see Worked Example below).

Worked example

Your notes

Your notes

? The curve C has the equation $y = 7 - 12\sqrt{x}$.

a) Find the equation of the tangent to C at the point where $x = 9$.

b) Find the equation of the normal to C at the same point.

a) WHEN $x = 9$, $y = 7 - 12\sqrt{9} = -29$ FIND y−COORDINATE

$y = 7 - 12x^{\frac{1}{2}}$ $\sqrt{x} = x^{\frac{1}{2}}$

$\frac{dy}{dx} = -12\left(\frac{1}{2}\right)x^{\frac{1}{2}-1} = -6x^{-\frac{1}{2}} = -\frac{6}{\sqrt{x}}$ FIND DERIVATIVE

WHEN $x = 9$,

$\frac{dy}{dx} = -\frac{6}{\sqrt{9}} = -2$ USE DERIVATIVE TO FIND GRADIENT OF TANGENT

SO THE TANGENT AT POINT $(9, -29)$ HAS EQUATION

$y - (-29) = -2(x - 9)$ $y - y_1 = m(x - x_1)$

$y = -2x - 11$

b) THE NORMAL IS PERPENDICULAR TO THE TANGENT, SO THE GRADIENT OF THE NORMAL IS $-\frac{1}{(-2)} = \frac{1}{2}$

AT $(9, -29)$ THE NORMAL HAS EQUATION

$y - (-29) = \frac{1}{2}(x - 9)$

$m_2 = -\frac{1}{m_1}$ FOR PERPENDICULAR LINES

$$y = \frac{1}{2}x - \frac{67}{2}$$

Your notes

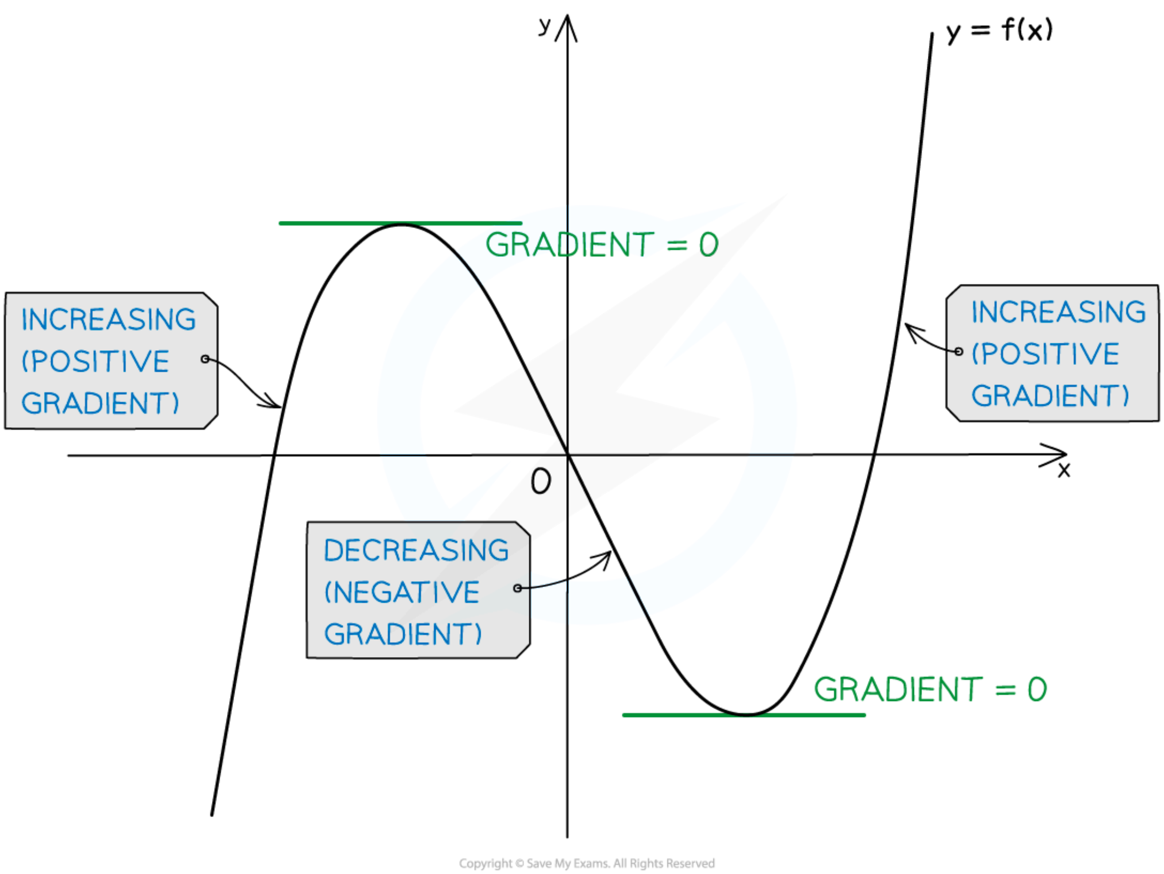
7.2.2 Increasing & Decreasing Functions

Your notes

Increasing & Decreasing Functions

What are increasing and decreasing functions?

- A function **f(*x*)** is **increasing** on an interval **[*a*, *b*]** if **f'(*x*) ≥ 0** for all values of ***x*** such that ***a*< *x* < *b***.
  - If **f'(*x*) > 0** for all ***x*** values in the interval then the function is said to be **strictly increasing**
  - In most cases, on an increasing interval the graph of a function goes **up** as ***x*** increases
- A function **f(*x*)** is **decreasing** on an interval **[*a*, *b*]** if **f'(*x*) ≤ 0** for all values of ***x*** such that ***a* < *x* < *b***
  - If **f'(*x*) < 0** for all ***x*** values in the interval then the function is said to be **strictly decreasing**
  - In most cases, on a decreasing interval the graph of a function goes **down** as ***x*** increases



- To identify the intervals on which a function is increasing or decreasing you need to:
  1. Find the derivative **f'(*x*)**
  2. Solve the inequalities **f'(*x*) ≥ 0** (for increasing intervals) and/or **f'(*x*) ≤ 0** (for decreasing intervals)

Examiner Tip

- On an exam, if you need to show a function is increasing or decreasing you can use either strict ($<$, $>$) or non-strict ($\leq$, $\geq$) inequalities

Your notes

Worked example

Your notes

Your notes

? $f(x) = x^3 + x^2 - x + 2$

a) Find the values of x for which $f(x)$ is an increasing function.

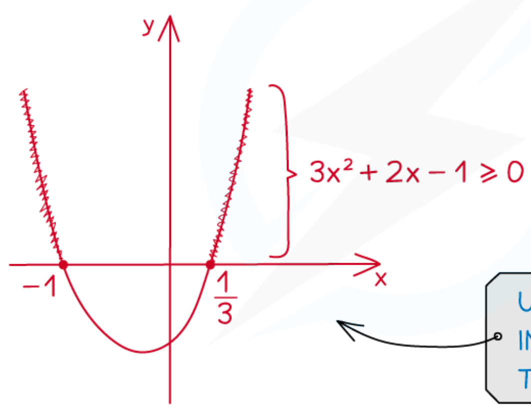
b) Find the values of x for which $f(x)$ is a decreasing function.

a) $f'(x) = 3x^2 + 2x - 1$ ← FIND THE DERIVATIVE

$3x^2 + 2x - 1 = 0$

$(3x-1)(x+1) = 0$

$x = \frac{1}{3}$ OR $x = -1$ ← SOLVE THE QUADRATIC



USE SKETCH TO IDENTIFY INTERVALS SATISFYING THE INEQUALITY

$f(x)$ IS INCREASING WHERE $f'(x) \geqslant 0$

SO $f(x)$ IS INCREASING ON THE INTERVALS $(-\infty, -1]$ AND $[\frac{1}{3}, +\infty)$

b) $f(x)$ IS DECREASING WHERE $f'(x) \leqslant 0$

SO $f(x)$ IS DECREASING
ON THE INTERVAL $[-1, \frac{1}{3}]$

Your notes

7.2.3 Second Order Derivatives

Second Order Derivatives

What is the second order derivative of a function?

- If you differentiate the derivative of a function (ie differentiate the function a second time) you get the **second order derivative** of the function
- For a function y = f(x), there are two forms of notation for the second derivative (or second order derivative)

$$f''(x) \text{ or } \frac{d^2y}{dx^2}$$

  - Note the positions of the power of 2's in the second version

e.g. $f(x) = x^3$

$$f'(x) = 3x^2$$

$$f''(x) = 6x$$

$$y = 7x^2 - 4x + 19$$

$$\frac{dy}{dx} = 14x - 4$$

$$\frac{d^2y}{dx^2} = 14$$

- The second order derivative can also be referred to as simply as the **second derivative**
  - Similarly, the 'regular' derivative can also be referred to as either the **first order derivative** or the **first derivative**
- The second order derivative gives the rate of change of the gradient function (ie of the first derivative) – this will be important for identifying the nature of **stationary points**

Examiner Tip

- When finding second derivatives be especially careful with functions that have negative or fractional powers of x (see Worked Example below).
- Mistakes made with fractions or negative signs can build up as you calculate the derivative more than once.

Your notes

Worked example

Your notes

Your notes

Given that $f(x) = 4 - \sqrt{x} + \frac{3}{\sqrt{x}}$

a) Find $f'(x)$ and $f''(x)$.

b) Evaluate $f''(3)$. Give your answer in the form $a\sqrt{b}$, where b is an integer and a is a rational number.

a) $f(x) = 4 - x^{\frac{1}{2}} + 3x^{-\frac{1}{2}}$ ← REWRITE AS POWERS OF x

$f'(x) = -\left(\frac{1}{2}\right)x^{\frac{1}{2}-1} + 3\left(-\frac{1}{2}\right)x^{-\frac{1}{2}-1}$

$f'(x) = -\frac{1}{2}x^{-\frac{1}{2}} - \frac{3}{2}x^{-\frac{3}{2}}$ ← DIFFERENTIATE ONCE TO FIND $f'(x)$

$f''(x) = -\frac{1}{2}\left(-\frac{1}{2}\right)x^{-\frac{1}{2}-1} - \frac{3}{2}\left(-\frac{3}{2}\right)x^{-\frac{3}{2}-1}$

$f''(x) = \frac{1}{4}x^{-\frac{3}{2}} + \frac{9}{4}x^{-\frac{5}{2}}$ ← DIFFERENTIATE A SECOND TIME TO FIND $f''(x)$

b) $f''(x) = \frac{1}{4x\sqrt{x}} + \frac{9}{4x^2\sqrt{x}}$ ← $x^{\frac{3}{2}} = x\sqrt{x}$ $x^{\frac{5}{2}} = x^2\sqrt{x}$

$f''(3) = \frac{1}{12\sqrt{3}} + \frac{9}{36\sqrt{3}}$

$= \frac{12}{36\sqrt{3}} = \frac{1}{3\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{3}}{9}$

RATIONALISE DENOMINATOR

$f''(3) = \frac{1}{9}\sqrt{3}$

Your notes

7.2.4 Stationary Points & Turning Points

Your notes

Stationary Points & Turning Points

What are stationary points?

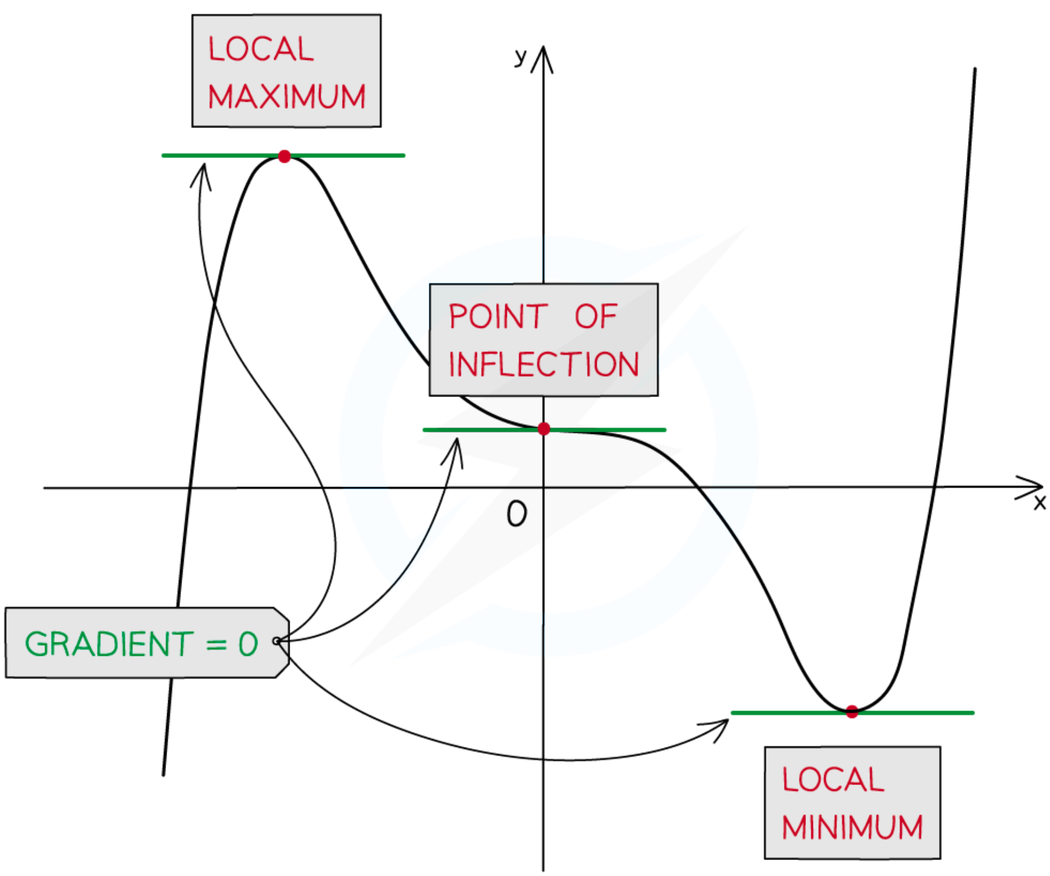
- A stationary point is any point on a curve where the gradient is zero
- To find stationary points of a function **f(*x*)**

  Step 1: Find the first derivative **f'(*x*)**

  Step 2: Solve **f'(*x*) = 0** to find the *x*-coordinates of the stationary points

  Step 3: Substitute those *x*-coordinates into **f(*x*)** to find the corresponding *y*-coordinates

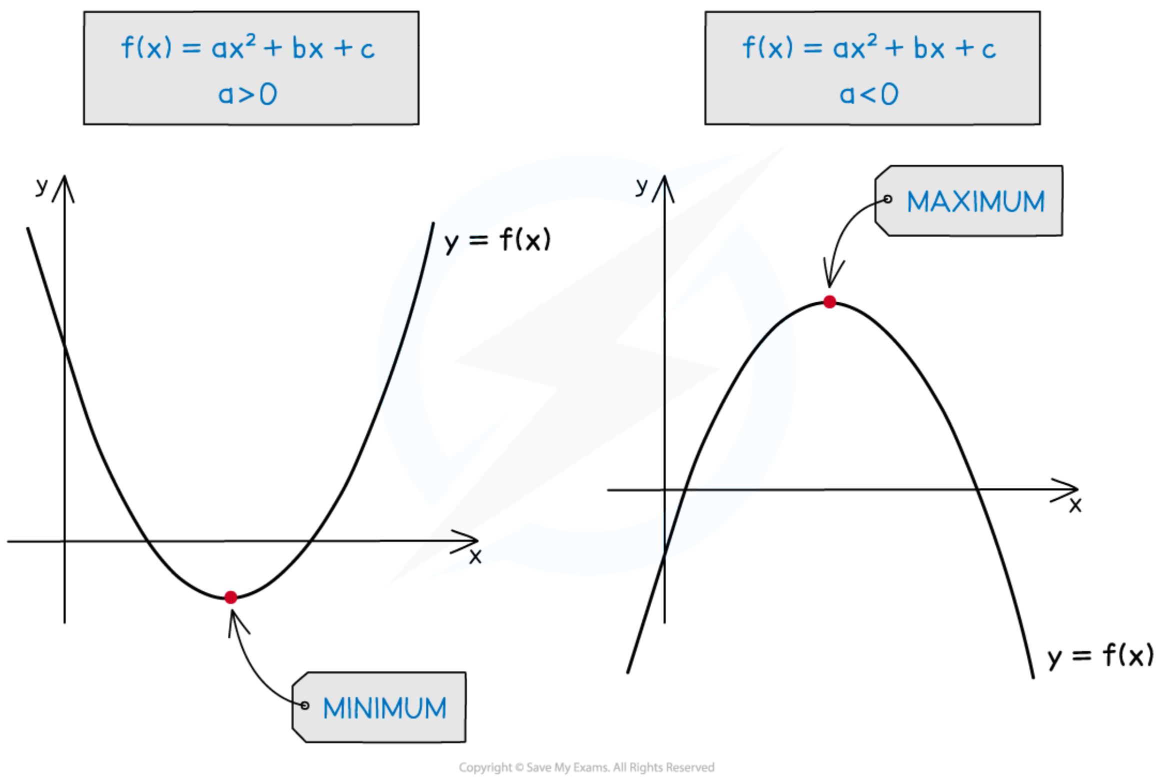
- A stationary point may be either a **local minimum**, a **local maximum**, or a **point of inflection**



Stationary points on quadratics

Your notes

- The graph of a quadratic function (ie a parabola) only has a single stationary point
- For an 'up' parabola this is the **minimum**; for a 'down' parabola it is the **maximum** (no need to talk about 'local' here)



- The ***y*** value of the stationary point is thus the minimum or maximum value of the quadratic function
- For quadratics especially minimum and maximum points are often referred to as **turning points**

How do I determine the nature of stationary points on other curves?

- For a function **f(x)** there are two ways to determine the nature of its stationary points
- **Method A:** Compare the signs of the **first derivative** (positive or negative) a little bit **to either side of the stationary point**
  - (After completing **Steps 1 – 3 above** to find the stationary points)

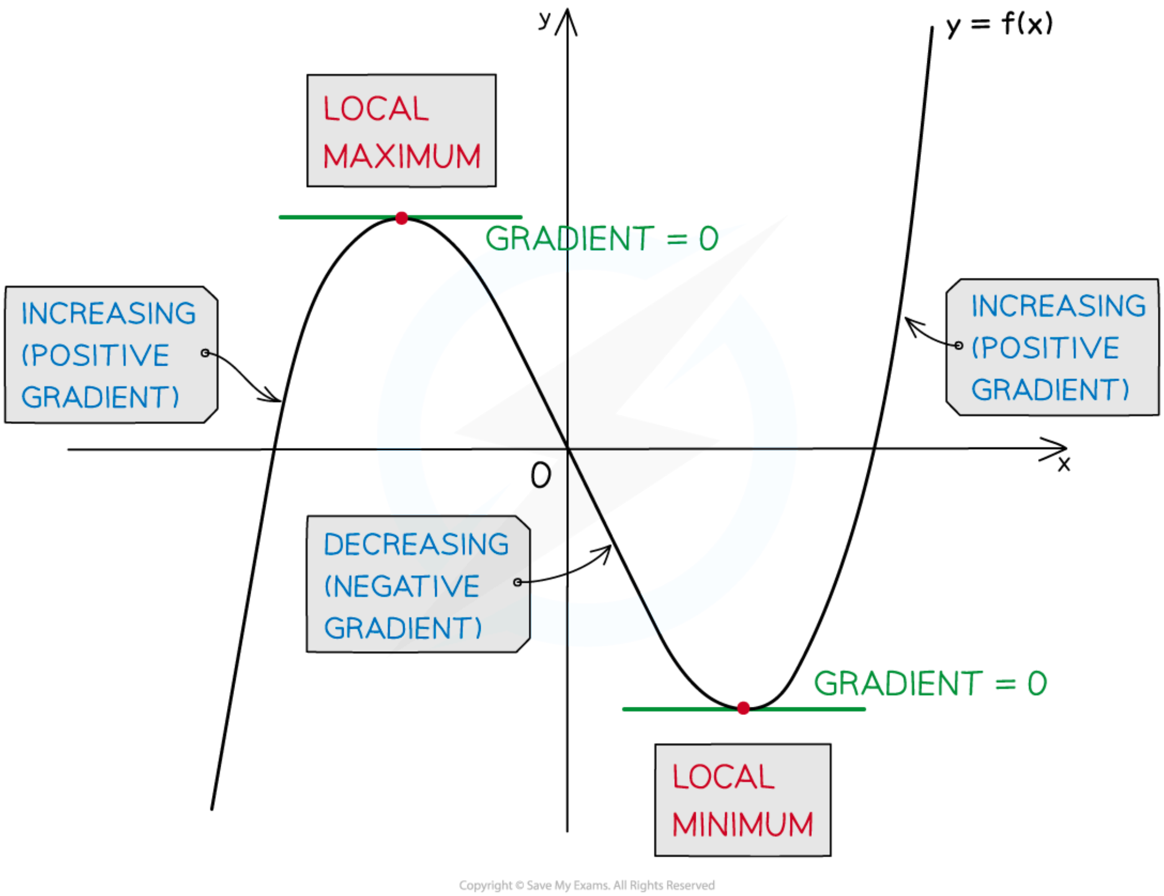
  Step 4: For each stationary point find the values of the first derivative a little bit 'to the left' (ie slightly smaller *x* value) and a little bit 'to the right' (slightly larger *x* value) of the stationary point

Your notes

- CHOOSING x VALUES ONE LESS AND ONE MORE THAN THE x–COORDINATE OF THE STATIONARY POINT WILL USUALLY WORK HERE (SEE THE EXAMPLE AT THE END)
- DO CHOOSE A CONVENIENT POINT WHERE POSSIBLE ($x = 0$ AND $x = 1$ ARE BOTH VERY EASY TO PUT INTO FORMULAS FOR EXAMPLE)
- DO NOT HOWEVER CHOOSE AN x VALUE TO THE RIGHT OR LEFT THAT JUMPS PAST THE x–COORDINATE OF ANOTHER STATIONARY POINT!

Step 5: Compare the **signs** (positive or negative) of the derivatives on the left and right of the stationary point

- If the derivatives are **negative** on the **left** and **positive** on the **right**, the point is a **local minimum**
- If the derivatives are **positive** on the **left** and **negative** on the **right**, the point is a **local maximum**
- If the signs of the derivatives are the **same** on both sides (both positive or both negative) then the point is a **point of inflection**


y
y = f(x)
LOCAL MAXIMUM
GRADIENT = 0
INCREASING (POSITIVE GRADIENT)
INCREASING (POSITIVE GRADIENT)
0
x
DECREASING (NEGATIVE GRADIENT)
GRADIENT = 0
LOCAL MINIMUM
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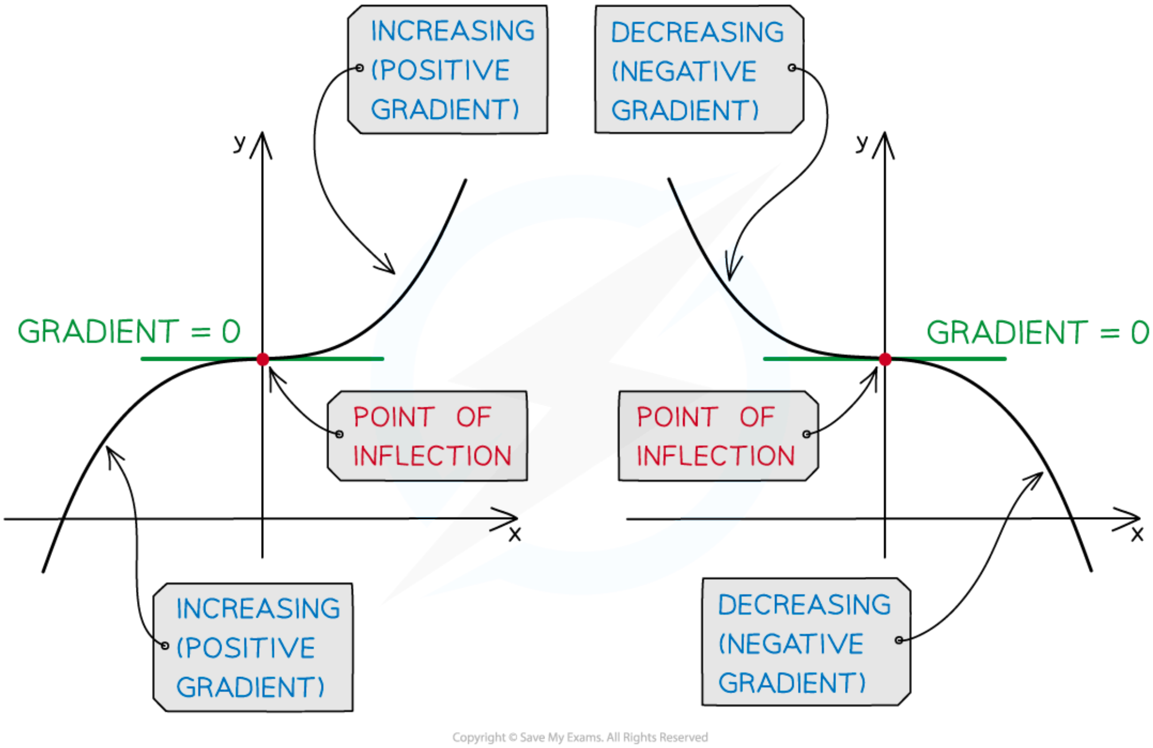
Your notes

Your notes


INCREASING (POSITIVE GRADIENT)
DECREASING (NEGATIVE GRADIENT)
y
y
GRADIENT = 0
GRADIENT = 0
POINT OF INFLECTION
POINT OF INFLECTION
x
x
INCREASING (POSITIVE GRADIENT)
DECREASING (NEGATIVE GRADIENT)
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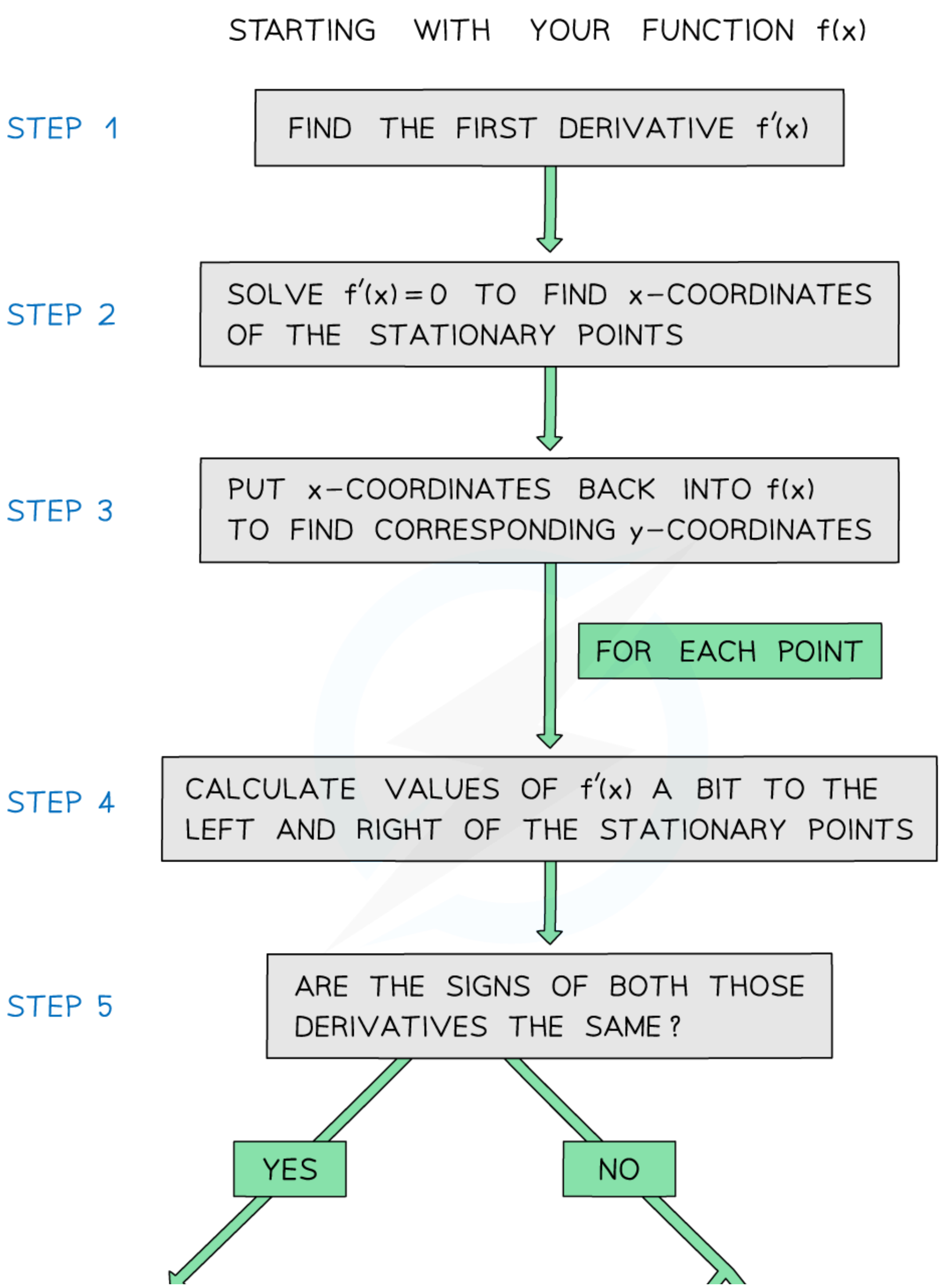
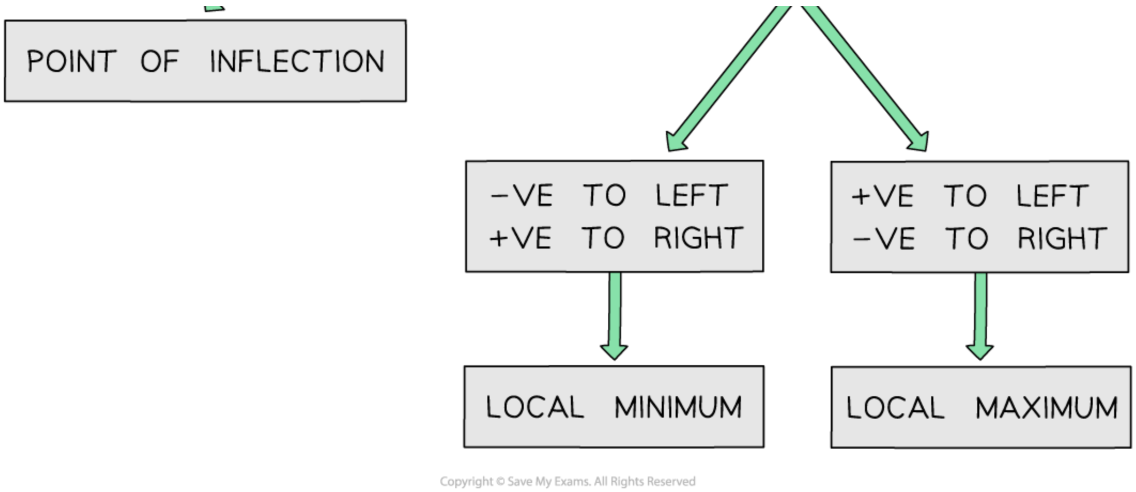
Your notes

METHOD A: DETERMINING NATURE OF STATIONARY POINTS USING FIRST DERIVATIVE





Your notes

Method B: Look at the sign of the **second derivative** (positive or negative) **at the stationary point**

(After completing **Steps 1 – 3 above** to find the stationary points)

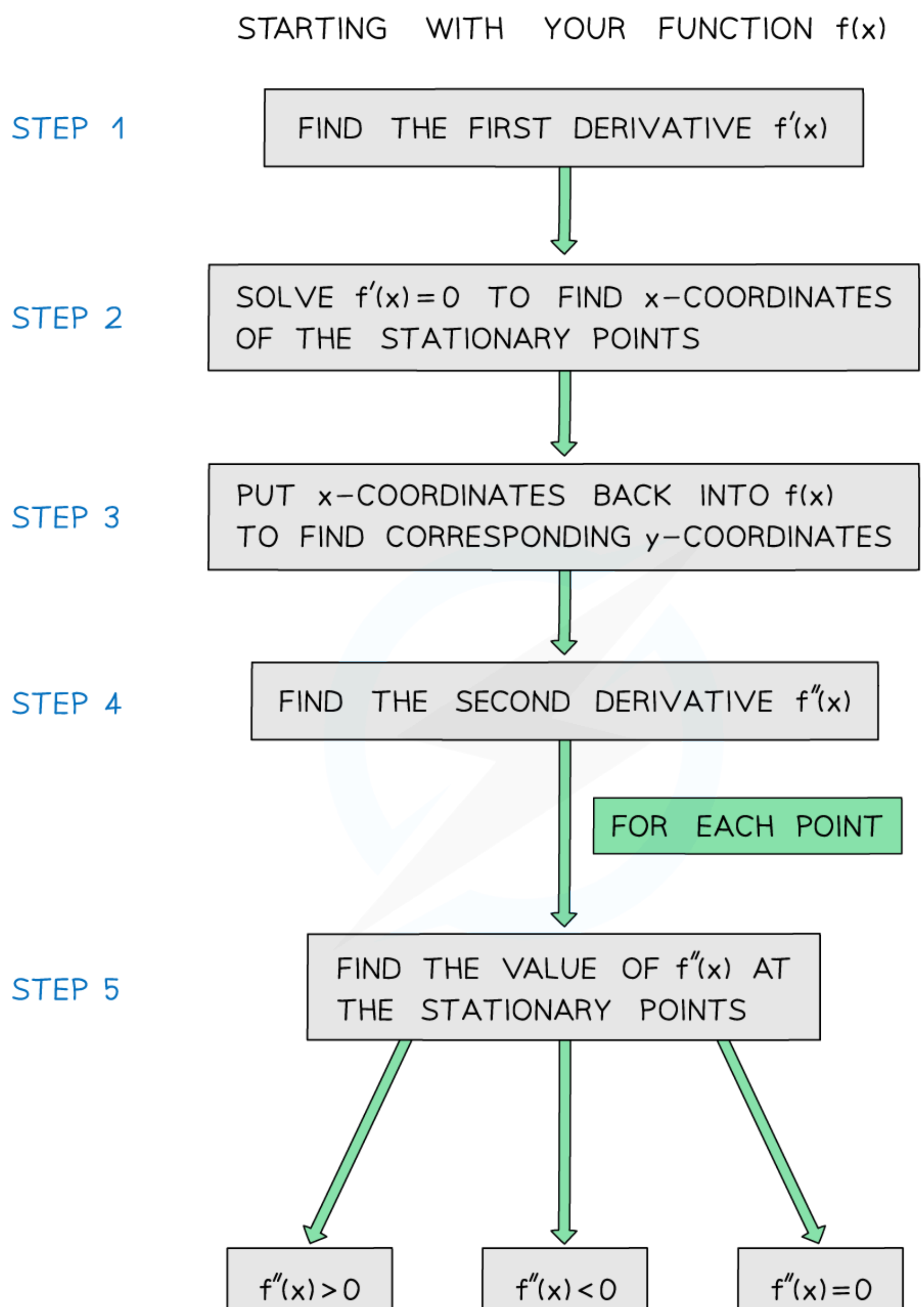
Step 4: Find the second derivative **f''(*x*)**

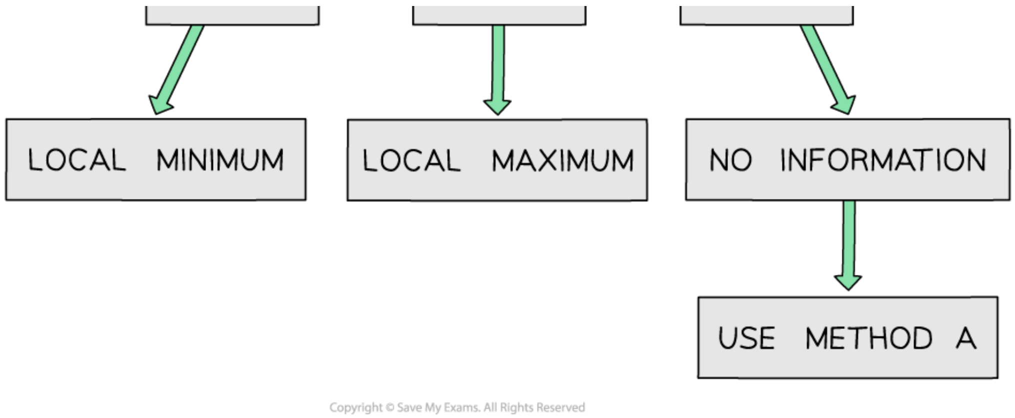
Step 5: For each stationary point find the value of **f''(*x*)** at the stationary point (ie substitute the *x*-coordinate of the stationary point into **f''(*x*)**)

- If **f''(*x*)** is **positive** then the point is a **local minimum**
- If **f''(*x*)** is **negative** then the point is a **local maximum**
- If **f''(*x*)** is **zero** then the point could be a **local minimum**, a **local maximum** OR a **point of inflection** (use Method A to determine which)

Your notes

METHOD B: DETERMINING NATURE OF STATIONARY POINTS USING SECOND DERIVATIVE





Your notes

Examiner Tip

- Usually using the second derivative (Method B above) is a much quicker way of determining the nature of a stationary point.
- However, if the second derivative is zero it tells you nothing about the point.
  - In that case you will have to use Method A (which **always** works – see the Worked Example).

Your notes

Worked example

Your notes

Find the stationary points of $f(x) = x^3(3x^2 - 20)$, and determine the nature of each.

$f(x) = 3x^5 - 20x^3$ ← EXPAND THE BRACKETS

$f'(x) = 15x^4 - 60x^2$ ← STEP 1: FIND FIRST DERIVATIVE

$15x^4 - 60x^2 = 0$ ← STEP 2: SOLVE $f'(x) = 0$

$x^4 - 4x^2 = 0$

$x^2(x^2 - 4) = 0$

$x^2(x+2)(x-2) = 0$

$x = 0, -2, 2$ ← x-COORDINATES OF THE TURNING POINTS

IF $x = 0$, $y = (0)^3(3(0)^2 - 20) = 0$

IF $x = -2$, $y = (-2)^3(3(-2)^2 - 20) = 64$

IF $x = 2$, $y = (2)^3(3(2)^2 - 20) = -64$

STEP 3: FIND CORRESPONDING y-COORDINATES

THE STATIONARY POINTS ARE $(-2, 64)$, $(0,0)$ AND $(2, -64)$

Your notes

START WITH METHOD B

$f''(x) = 60x^3 - 120x$

STEP 4: FIND SECOND DERIVATIVE

$f''(-2) = 60(-2)^3 - 120(-2) = -240 < 0$

$f''(0) = 60(0)^3 - 120(0) = 0$

$f''(2) = 60(2)^3 - 120(2) = 240 > 0$

STEP 5: TEST SECOND DERIVATIVE AT STATIONARY POINTS

(−2,64) IS A LOCAL MAXIMUM

(2,−64) IS A LOCAL MINIMUM

NOTE THAT AT (0,0) THE SECOND DERIVATIVE IS ZERO, AND SO DOESN'T TELL US WHAT THE POINT IS!

SWITCH TO METHOD A

TEST $f'(x)$ AT $x = -1$ AND $x = 1$

STEP 4: COMPARE FIRST DERIVATIVES TO LEFT AND RIGHT OF STATIONARY POINT

$f'(-1) = 15(-1)^4 - 60(-1)^2 = -45 < 0$

$f'(1) = 15(1)^4 - 60(1)^2 = -45 < 0$

STEP 5: INTERPRET RESULT

$f'(x)$ HAS SAME SIGN ON BOTH SIDES OF (0,0)

(0,0) IS A POINT OF INFLECTION

Your notes

7.2.5 Sketching Gradient Functions

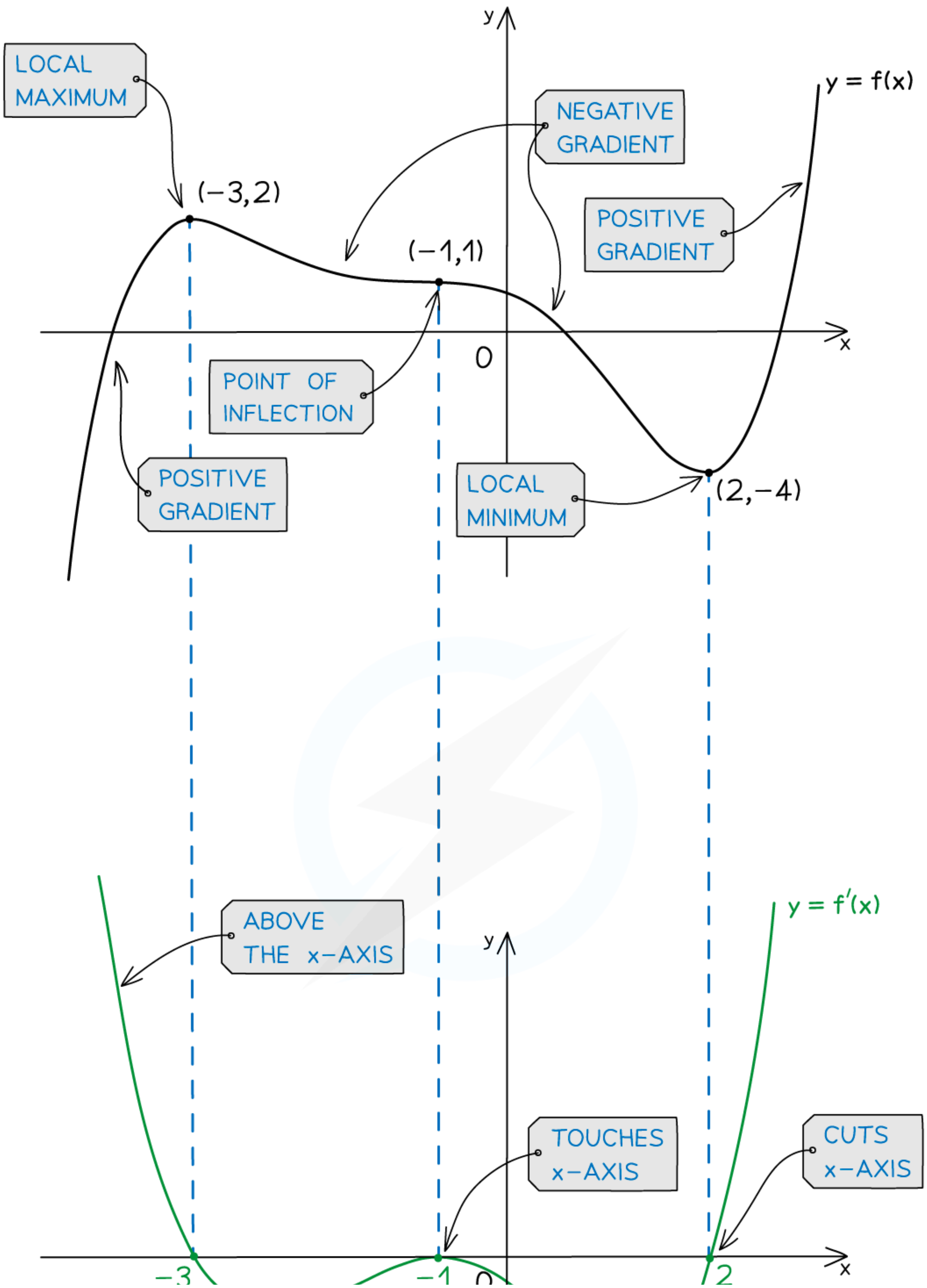
Your notes

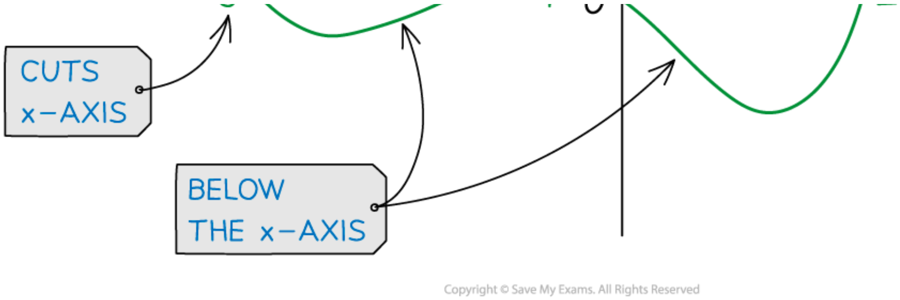
Sketching Gradient Functions

How can I sketch a function's gradient function?

- Using your knowledge of gradients and derivatives you can use the graph of a function to sketch the corresponding gradient function
- The behaviour of a function tells you about the behaviour of its gradient function

Your notes


y
LOCAL MAXIMUM
y = f(x)
NEGATIVE GRADIENT
(−3,2)
POSITIVE GRADIENT
(−1,1)
x
0
POINT OF INFLECTION
POSITIVE GRADIENT
LOCAL MINIMUM
(2,−4)
y = f′(x)
ABOVE THE x−AXIS
y
TOUCHES x−AXIS
CUTS x−AXIS
x
−3
−1
0
2


CUTS
x–AXIS
BELOW
THE x–AXIS
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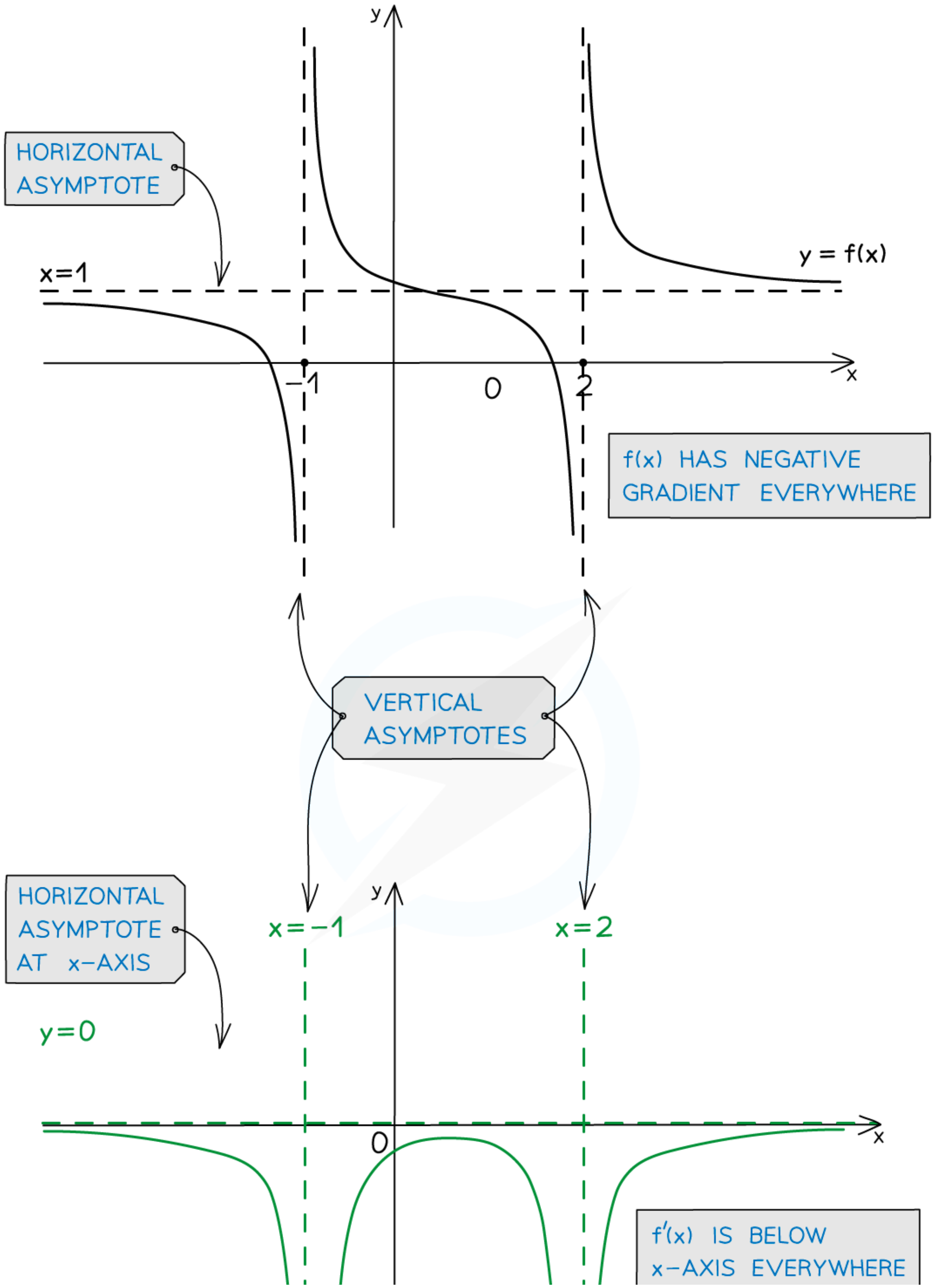
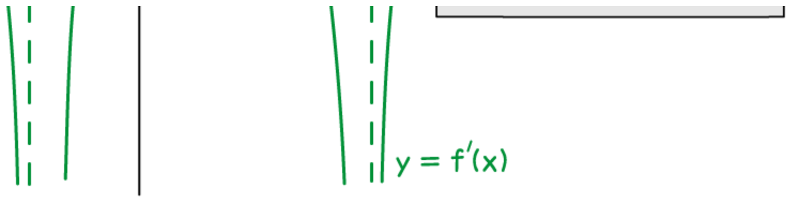
Your notes


HORIZONTAL ASYMPTOTE
x=1
y = f(x)
y
x
−1
0
2
f(x) HAS NEGATIVE GRADIENT EVERYWHERE
VERTICAL ASYMPTOTES
HORIZONTAL ASYMPTOTE AT x−AXIS
y=0
x=−1
x=2
f′(x) IS BELOW x−AXIS EVERYWHERE

Your notes



Examiner Tip

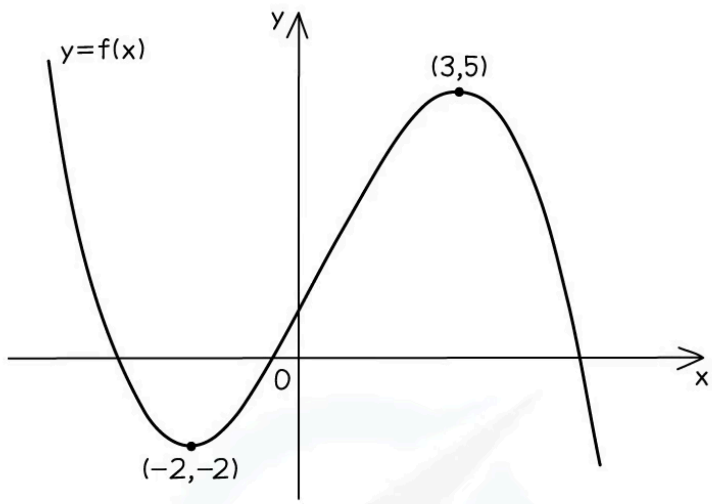
- If **f(*x*)** is a smooth curve then **f'(*x*)** will also be a smooth curve.
- Take what you know about **f'(*x*)** (based on the table above) and then 'fill in the blanks' in between.
- If all you have is the graph of **f(*x*)** you will not be able to specify the coordinates of the *y*-intercept or any stationary points of **f'(*x*).**
- Be careful – points where **f(*x*)** cuts the *x*-axis don't tell you anything about the graph of **f'(*x*)**!

Your notes

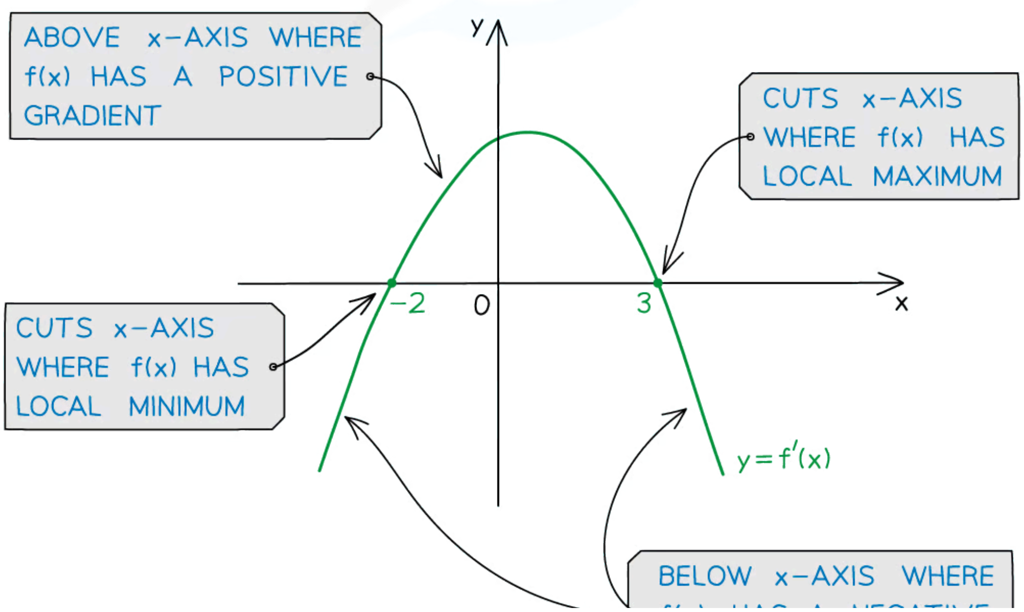
Worked example

Your notes

The following diagram shows the graph of $y = \mathrm{f}(x)$



Sketch the graph of $y = \mathrm{f}'(x)$.




f(x) HAS A NEGATIVE
GRADIENT
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7.2.6 Modelling with Differentiation inc. Optimisation

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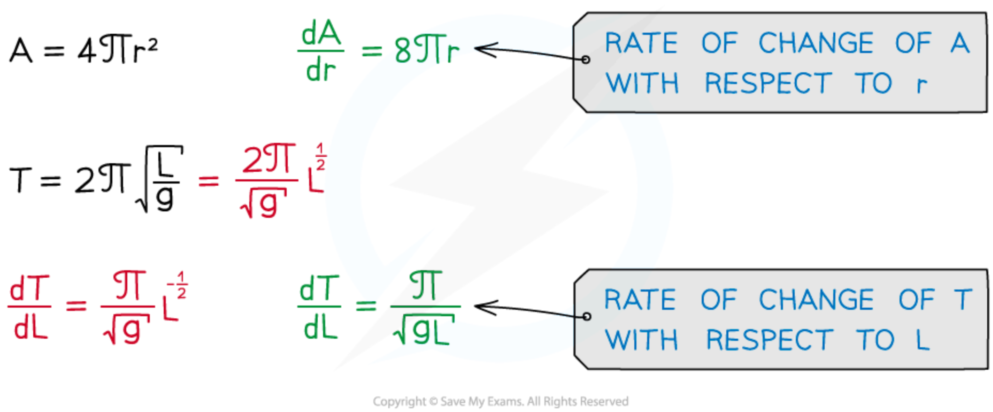
Modelling with Differentiation inc. Optimisation

How can I use differentiation to solve modelling questions?

- Derivatives can be calculated for any variables – not just ***y*** and ***x***
- In every case the derivative is a formula giving the rate of change of one variable with respect to the other variable



- Differentiation can be used to find maximum and minimum points of a function (see Stationary Points)
- Therefore it can be used to solve maximisation and minimisation problems in modelling questions

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e.g. AT TIME t SECONDS THE HEIGHT OF A BALL ABOVE THE GROUND IN METRES IS GIVEN BY FORMULA

$$h = 20t - 5t^2$$

USE DIFFERENTIATION TO FIND THE MAXIMUM HEIGHT REACHED BY THE BALL.

$$\frac{dh}{dt} = 20 - 10t$$

$$20 - 10t = 0$$

THE GRAPH OF $20t - 5t^2$ IS A "DOWN" PARABOLA. THE MAXIMUM VALUE OCCURS WHEN $\frac{dh}{dt} = 0$

$$t = 2$$

WHEN $t = 2$,

$$h = 20(2) - 5(2)^2 = 20 \text{ METRES}$$

Examiner Tip

Exam questions on this topic will often be divided into two parts:

- First a 'Show that...' part where you derive a given formula from the information in the question
- And then a 'Find...' part where you use differentiation to answer a question about the formula

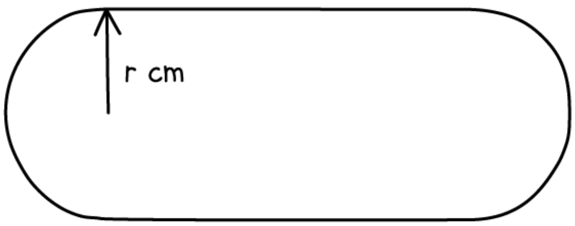
Even if you can't answer the first part you can still use the formula to answer the second part.

Worked example

Your notes

Your notes

A large allotment bed is being designed as a rectangle with a semicircle on each end, as shown in the diagram below.



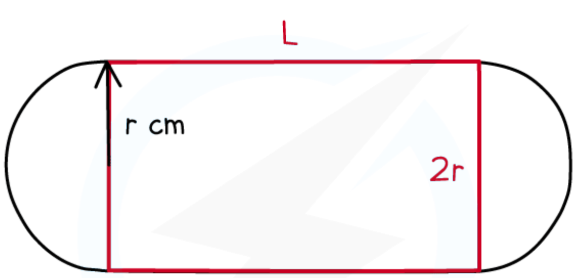
The total area of the bed is to be 100π m^2.

a) Show that the perimeter of the bed is given by the formula $P = \pi\left(r + \dfrac{100}{r}\right)$

b) For what value of r is the rate of change of P with respect to r equal to $-\dfrac{9\pi}{16}$ metres per metre?

c) Find the perimeter of the bed for the value of r found in part (b).

a)



THE WIDTH OF THE RECTANGLE IS 2r AND THE LENGTH IS L. THE AREA OF THE BED IS

Your notes

THE LENGTH IS L. THE AREA OF THE BED IS

$$A = \frac{1}{2}\pi r^2 + 2rL + \frac{1}{2}\pi r^2 = \pi r^2 + 2rL$$

SEMICIRCLE

RECTANGLE

SEMICIRCLE

THE AREA IS 100π, SO

$$\pi r^2 + 2rL = 100\pi$$

USE GIVEN AREA TO WRITE L IN TERMS OF r

$$2rL = 100\pi - \pi r^2$$

$$L = \frac{50\pi}{r} - \frac{\pi}{2}r$$

THE PERIMETER OF THE BED IS

$$P = \pi r + \pi r + 2L = 2\pi r + 2L$$

SEMICIRCLES

$$P = 2\pi r + 2\left(\frac{50\pi}{r} - \frac{\pi}{2}r\right)$$

SUBSTITUTE VALUE FOR L INTO PERIMETER FORMULA

$$= 2\pi r + \frac{100\pi}{r} - \pi r$$

$$= \pi r + \frac{100\pi}{r}$$

$$P = \pi\left(r + \frac{100}{r}\right)$$

Your notes

Your notes

b) $P = \pi r + 100\pi r^{-1}$

$$\frac{dP}{dr} = \pi - 100\pi r^{-2} = \pi\left(1 - \frac{100}{r^2}\right)$$

DIFFERENTIATE P WITH RESPECT TO r

$$\pi\left(1 - \frac{100}{r^2}\right) = -\frac{9\pi}{16}$$

$$1 - \frac{100}{r^2} = -\frac{9}{16}$$

$$\frac{100}{r^2} = \frac{25}{16}$$

$$r^2 = 64 \implies r = \pm 8$$

BECAUSE r IS A LENGTH IT CANNOT BE NEGATIVE

$r = 8$ METRES

c) WHEN $r = 8$

$$P = \pi\left(8 + \frac{100}{8}\right)$$

$P = \dfrac{41\pi}{2}$ METRES $(\approx 64.4 \text{ m})$