Edexcel A Level Maths: Pure

Your notes

5.6 Compound & Double Angle Formulae

Contents

5.6.1 Compound Angle Formulae

Your notes

Compound Angle Formulae

What are the compound angle formulae?

- There are six **compound angle formulae** (also known as **addition formulae**), two each for **sin**, **cos** and **tan**:
- For **sin** the +/- sign on the left-hand side **matches** the one on the right-hand side

$$\sin(A+B) \equiv \sin A\cos B + \cos A\sin B$$

$$\sin(A-B) \equiv \sin A\cos B - \cos A\sin B$$

- For **cos** the +/- sign on the left-hand side is **opposite to** the one on the right-hand side

$$\cos(A+B) \equiv \cos A\cos B - \sin A\sin B$$

$$\cos(A-B) \equiv \cos A\cos B + \sin A\sin B$$

- For **tan** the +/- sign on the left-hand side **matches** the one in the **numerator** on the right-hand side, and is **opposite to** the one in the **denominator**

$$\tan(A+B) \equiv \frac{\tan A + \tan B}{1 - \tan A\tan B}$$

$$\tan(A-B) \equiv \frac{\tan A - \tan B}{1 + \tan A\tan B}$$

- You can **derive** the tan identity by:
  - Writing $\tan(A+B) \equiv \dfrac{\sin(A+B)}{\cos(A+B)}$
  - Dividing the numerator and denominator by $\cos A\cos B$

Examiner Tip

- All these formulae are in the formulae booklet – you don't have to memorise them.

Worked example

Your notes

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a) Express $\tan(225° - 30°)$ in terms of $\tan 225°$ and $\tan 30°$.

b) Hence show that $\tan 195° = 2 - \sqrt{3}$.

a) $\tan(A-B) \equiv \dfrac{\tan A - \tan B}{1 + \tan A \tan B}$ ← FORMULA 6 ABOVE

$$\tan(225° - 30°) \equiv \frac{\tan 225° - \tan 30°}{1 + \tan 225° \tan 30°}$$

b) $\tan 225° = 1$ $\qquad \tan 30° = \dfrac{\sqrt{3}}{3}$ ← YOU SHOULD KNOW THESE!

$$\tan(195°) = \tan(225° - 30°)$$

$$= \frac{1 - \frac{\sqrt{3}}{3}}{1 + (1)\left(\frac{\sqrt{3}}{3}\right)} = \frac{1 - \frac{\sqrt{3}}{3}}{1 + \frac{\sqrt{3}}{3}}$$

$$\left[\times \frac{3}{3}\right] \quad = \frac{3 - \sqrt{3}}{3 + \sqrt{3}}$$

$$= \frac{3 - \sqrt{3}}{3 + \sqrt{3}} \times \frac{3 - \sqrt{3}}{3 - \sqrt{3}}$$ ← RATIONALISE THE DENOMINATOR

$$= \frac{12 - 6\sqrt{3}}{6}$$

$$= 2 - \sqrt{3}$$

Your notes

5.6.2 Double Angle Formulae

Your notes

Double Angle Formulae

What are the double angle formulae?

- The **double angle formulae** are derived from the 'A+B' versions of the **compound angle formulae**:
  - $\sin(2A) \equiv 2\sin A\cos A$
  - $\cos(2A) \equiv \cos^2 A - \sin^2 A$
    - $\cos(2A) \equiv 2\cos^2 A - 1$
    - $\cos(2A) \equiv 1 - 2\sin^2 A$
  - $\tan(2A) \equiv \dfrac{2\tan A}{1 - \tan^2 A}$

1. $\sin 2A \equiv 2\sin A\cos A$

$\sin(A+B) \equiv \sin A\cos B + \cos A\sin B$

LET $B=A$. THEN

$$\begin{aligned}\sin(2A) &= \sin(A+A)\\ &= \sin A\cos A + \cos A\sin A\\ &= 2\sin A\cos A\end{aligned}$$

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2. $\cos 2A \equiv \cos^2 A - \sin^2 A \equiv 2\cos^2 A - 1 \equiv 1 - 2\sin^2 A$

$\cos(A+B) \equiv \cos A \cos B - \sin A \sin B$

LET $B=A$. THEN

$$\begin{aligned} \cos(2A) &\equiv \cos(A+A) \\ &\equiv \cos A \cos A - \sin A \sin A \\ &\equiv \cos^2 A - \sin^2 A \end{aligned}$$

THE OTHER TWO FORMS COME FROM REARRANGING

$$\sin^2 A + \cos^2 A \equiv 1$$

AND SUBSTITUTING

3. $\tan 2A \equiv \dfrac{2\tan A}{1 - \tan^2 A}$

$\tan(A+B) \equiv \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$

LET $B=A$. THEN

$$\begin{aligned} \tan(2A) &\equiv \tan(A+A) \\ &\equiv \frac{\tan A + \tan A}{1 - \tan A \tan A} \\ &\equiv \frac{2\tan A}{1 - \tan^2 A} \end{aligned}$$

Examiner Tip

- The double angle formulae are **not** included in the formulae booklet – you have to know them.
- If you forget them, you can always derive them from the compound angle formulae (which *are* in the formula booklet) as shown above.

Worked example

Your notes

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a) Show that $7\sin 2\theta - 3\sin\theta = 0$ can be written in the form

$$a\sin\theta\left(b\cos\theta + c\right) = 0$$

stating the values of a, b and c.

b) Hence solve, for $0 \le \theta < 360°$, the equation $7\sin 2\theta - 3\sin\theta = 0$. Give your answers to 1 decimal place.

USE DOUBLE ANGLE FORMULA FOR sin

a) $7\sin 2\theta = 7(2\sin\theta\cos\theta) = 14\sin\theta\cos\theta$

WE CAN REWRITE THE EQUATION AS

$14\sin\theta\cos\theta - 3\sin\theta = 0$

$\sin\theta(14\cos\theta - 3) = 0$

WITH $a = 1$ $b = 14$ $c = -3$

b) $\sin\theta(14\cos\theta - 3) = 0$

SO $\sin\theta = 0$ OR $\cos\theta = \frac{3}{14}$

IF $\sin\theta = 0$

$\theta = 0$ OR $\theta = 180$

YOU SHOULD KNOW THESE!

IF $\cos\theta = \frac{3}{14}$

$\theta = 77.63$ OR $\theta = 360 - 77.63 = 282.37$

Your notes

USE CALCULATOR TO FIND PRINCIPAL VALUE

THEN FIND OTHER VALUES IN THE REQUIRED INTERVAL

THE SOLUTIONS ARE

$\theta = 0^\circ, 77.6^\circ, 180^\circ, 282.4^\circ$ (1 d.p.)

5.6.3 R addition formulae Rcos Rsin etc

Your notes

R addition formulae Rcos Rsin etc

How can I simplify expressions like $a \sin x \pm b \cos x$ or $a \cos x \pm b \sin x$?

- If a and b are positive constants, then
  - $a \sin x + b\cos x$ can be rewritten as $R \sin (x + \alpha)$
  - $a \sin x - b\cos x$ can be rewritten as $R \sin (x - \alpha)$
  - $a \cos x + b\sin x$ can be rewritten as $R \cos (x - \alpha)$
  - $a \cos x - b\sin x$ can be rewritten as $R \cos (x + \alpha)$
- In all four cases $R > 0$ and $0 < \alpha < 90°$, with:

$$a = R\cos\alpha$$

$$b = R\sin\alpha$$

$$R = \sqrt{a^2 + b^2}$$

- To rewrite the expression use the appropriate **compound angle formula** and equate coefficients

Your notes

e.g. $5\cos x - 12\sin x$

THIS IS CASE 4 ABOVE, WITH $a=5$ AND $b=12$

$\cos(x+\alpha) = \cos\alpha\cos x - \sin\alpha\sin x$

EXPAND USING COMPOUND ANGLE FORMULA FOR cos

SO $R\cos(x+\alpha) = R\cos\alpha\cos x - R\sin\alpha\sin x$

COMPARE TO ORIGINAL EXPRESSION:

$R\cos\alpha = 5 \qquad R\sin\alpha = 12$

$R\cos\alpha = a$
$R\sin\alpha = b$

$$\frac{R\sin\alpha}{R\cos\alpha} = \tan\alpha = \frac{12}{5}$$

$$\alpha = \tan^{-1}\left(\frac{12}{5}\right) = 67.38^\circ$$

ALSO $(R\cos\alpha)^2 + (R\sin\alpha)^2 = 5^2 + 12^2$

$R^2\cos^2\alpha + R^2\sin^2\alpha = 25 + 144$

$R^2(\cos^2\alpha + \sin^2\alpha) = 169$

$\cos^2\alpha + \sin^2\alpha = 1$

$R^2 = 169$

$R = 13$

$R = \sqrt{a^2 + b^2}$

$5\cos x - 12\sin x = 13\cos(x + 67.38^\circ)$

$a\cos x - b\sin x = R\cos(x+\alpha)$

Examiner Tip

- There is more than one way to rewrite a given expression, and the question in the exam will tell you which form to use.

Worked example

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a) Express $\sin\theta - 4\cos\theta$ in the form $R\sin(\theta - \alpha)$, where $R > 0$ and $0 < \alpha < 90°$.

b) Hence solve $\sin\theta - 4\cos\theta = 3$ for $0 \leq \theta < 360°$.

COMPOUND ANGLE FORMULA FOR sin

a) $R\sin(\theta - \alpha) = R\cos\alpha\sin\theta - R\sin\alpha\cos\theta$

$R\cos\alpha = 1 \qquad R\sin\alpha = 4$

$R\cos\alpha = a$
$R\sin\alpha = b$

$\dfrac{R\sin\alpha}{R\cos\alpha} = \tan\alpha = 4$

$\alpha = \tan^{-1}(4) = 75.96°$

$R = \sqrt{1^2 + 4^2} = \sqrt{17}$

$R = \sqrt{a^2 + b^2}$

$\sin\theta - 4\cos\theta = \sqrt{17}\sin(\theta - 75.96°)$

b) $\sqrt{17}\sin(\theta - 75.96°) = 3$

$\sin(\theta - 75.96°) = \dfrac{3}{\sqrt{17}}$

USE $\sin^{-1}$ ON YOUR CALCULATOR TO FIND THE PRINCIPAL VALUE OF θ

$(\theta - 75.96°) = 46.69°$

OR $(\theta - 75.96°) = 180 - 46.69 = 133.31°$

THEN FIND OTHER SOLUTIONS IN THE REQUIRED RANGE

THE SOLUTIONS ARE

$\theta = 122.65°$ OR $209.27°$