Edexcel A Level Maths: Pure

Your notes

9.2 Further Parametric Equations

Contents

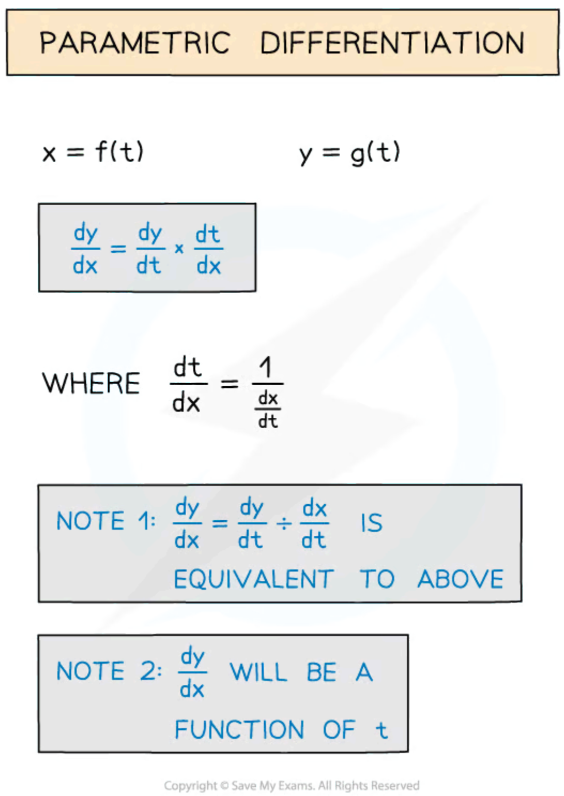
9.2.1 Parametric Differentiation

Your notes

Parametric Differentiation

How do I find dy/dx from parametric equations?

- Ensure you are familiar with Parametric Equations – Basics first



- This method uses the chain rule and the reciprocal property of derivatives
  - $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}t} \times \frac{\mathrm{d}t}{\mathrm{d}x}$
  - $\frac{\mathrm{d}t}{\mathrm{d}x} = 1 \div \frac{\mathrm{d}x}{\mathrm{d}t}$
- Equivalently, $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}t} \div \frac{\mathrm{d}x}{\mathrm{d}t}$

- $\frac{dy}{dx}$ will be in terms of *t* – this is fine
  - Questions usually involve finding gradients, tangents and normals
- The chain rule is always needed when there are three variables or more – see Connected Rates of Change

How do I find gradients, tangents and normals from parametric equations?

- To find a gradient ...
  - STEP 1: Find **dx/dt** and **dy/dt**
  - STEP 2: Find **dy/dx** in terms of **t**

    Using either **dy/dx = dy/dt ÷ dx/dt**

    or **dy/dx = dy/dt × dt/dx** where **dt/dx = 1 ÷ dx/dt**
  - STEP 3: Find the value of **t** at the required point
  - STEP 4: Substitute this value of **t** into **dy/dx** to find the gradient

Your notes

Your notes

e.g. FIND THE GRADIENT OF THE CURVE DEFINED PARAMETRICALLY BY

$$x = 2t - 3$$
$$y = t^3 - 4t + 1$$

AT THE POINT WHERE $x = 5$

STEP 1 FIND $\frac{dx}{dt}$ AND $\frac{dy}{dt}$

$$\frac{dx}{dt} = 2 \qquad \frac{dy}{dt} = 3t^2 - 4$$

STEP 2 FIND $\frac{dy}{dt}$ (IN TERMS OF t)

$$\frac{dy}{dx} = (3t^2 - 4) \times \frac{1}{2}$$

$\frac{dy}{dt}$

$\frac{1}{\frac{dy}{dt}}$

$$\frac{dy}{dx} = \frac{1}{2}(3t^2 - 4)$$

STEP 3 FIND THE VALUE OF t AT THE REQUIRED POINT

AT $x = 5$, $5 = 2t - 3$

$t = 4$

STEP 4 SUBSTITUTE THIS VALUE OF t INTO $\frac{dy}{dx}$ TO FIND THE GRADIENT

AT $x = 5$, $t = 4$, $\frac{dy}{dx} = \frac{1}{2}(3 \times 4^2 - 4)$

$$\frac{dy}{dx} = 22$$

Your notes

- to then go on to find the equation of a tangent ...
  - STEP 5: Find the **y** coordinate
  - STEP 6: Use the gradient and point to find the equation of the tangent

e.g. FIND THE EQUATION OF THE TANGENT TO THE CURVE DEFINED PARAMETRICALLY BY

$$x = 2t - 3$$
$$y = t^3 - 4t + 1$$

AT THE POINT WHERE $x = 5$

TANGENT: "$y - y_1 = m(x - x_1)$"

$m = \frac{dy}{dx}$ AT $x = 5$

FROM STEP 4 → $m = 22$

STEP 5 FIND THE y−COORDINATE

TO FIND THE EQUATION OF A LINE WE NEED THE GRADIENT, m, AND A POINT (x_1, y_1) THE LINE PASSES THROUGH

AT $x = 5$, $t = 4$, $y = 4^3 - 4 \times 4 + 1 = 49$

FROM STEP 3

STEP 6 USE GRADIENT AND POINT TO FIND EQUATION OF TANGENT

$y - 49 = 22(x - 5)$

$y = 22x - 61$

- To find a normal...

- STEP 7: Use perpendicular lines property to find the gradient of the normal $\mathbf{m_1 \times m_2 = -1}$
- STEP 8: Use gradient and point to find the equation of the normal $\mathbf{y - y_1 = m(x - x_1)}$

Your notes

What else may I be asked to do?

- Questions may require use of tangents and normals as per the coordiante geometry sections
  - Find points of intersection between a tangent/normal and **x/y** axes
  - Find areas of basic shapes enclosed by axes and/or tangents/normal
- Find stationary points (**dy/dx = 0**)

Your notes

e.g. FIND THE STATIONARY POINTS ON THE CURVE DEFINED PARAMETRICALLY BY

$$x = 2t - 3$$
$$y = t^3 - 4t + 1$$

$$\frac{dy}{dx} = \frac{1}{2}(3t^2 - 4)$$

FROM EARLIER EXAMPLES

$$\frac{1}{2}(3t^2 - 4) = 0$$

$\frac{dy}{dx} = 0$ AT STARTING POINTS

$$t^2 = \frac{4}{3}$$

$$t = \pm\frac{2}{\sqrt{3}}$$

SAME AS $t = \pm\frac{2\sqrt{3}}{3}$

WORK OUT x & y FROM t

AT $t = \frac{2}{\sqrt{3}}$, $x = \frac{-9 + 4\sqrt{3}}{3}$ $y = \frac{9 - 16\sqrt{3}}{9}$

AT $t = \frac{-2}{\sqrt{3}}$, $x = \frac{-9 - 4\sqrt{3}}{3}$ $y = \frac{9 + 16\sqrt{3}}{9}$

CAREFUL WITH CALCULATION DISPLAY: $-\frac{9 + 4\sqrt{3}}{3}$

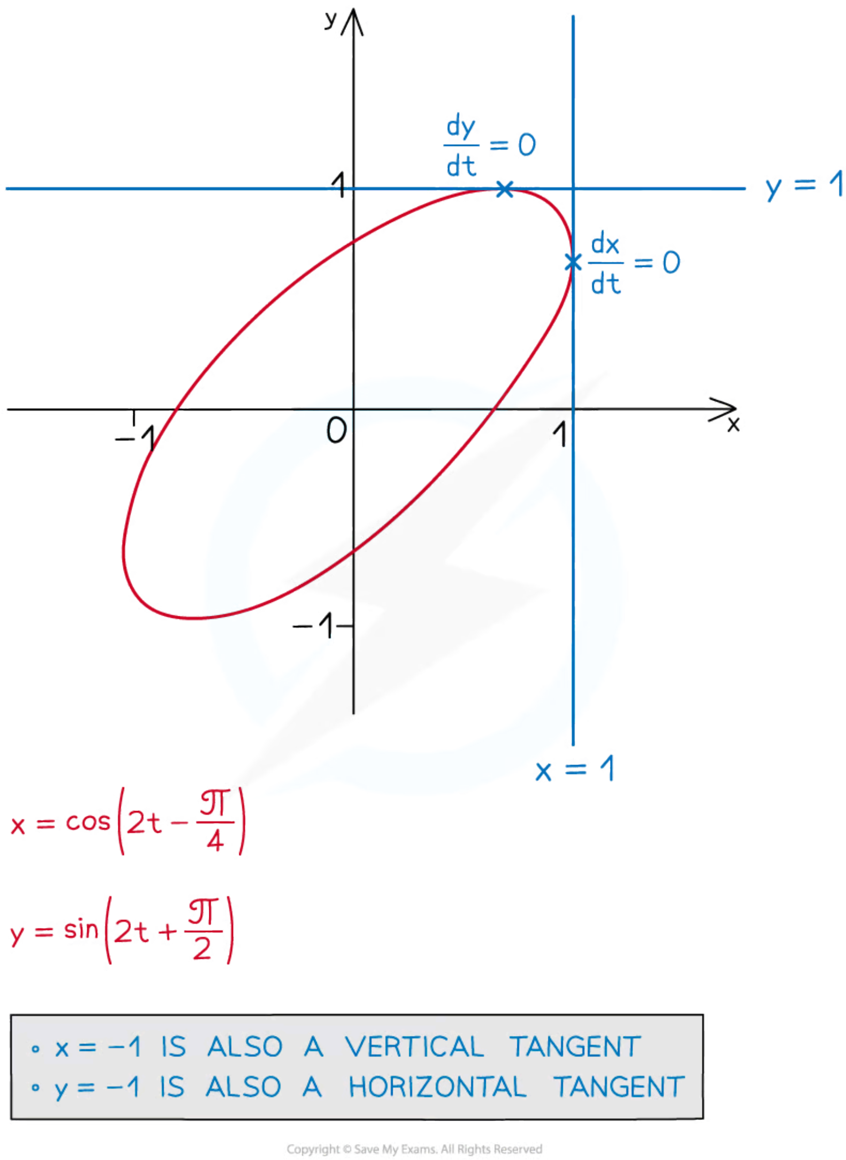
$\therefore$ THE COORDINATES OF THE STATIONARY POINTS ARE

$$\left(\frac{-9 + 4\sqrt{3}}{3}, \frac{9 - 16\sqrt{3}}{9}\right)$$

AND

$$\left(\frac{-9 - 4\sqrt{3}}{3}, \frac{9 + 16\sqrt{3}}{9}\right)$$

- You may also be asked about horizontal and vertical tangents
  - At horizontal (parallel to the **x**-axis) tangents, **dy/dt = 0**
  - At vertical (parallel to **y**-axis) tangents, **dx/dt = 0**



Just for fun ...

- Try plotting the graph from the question below using graphing software
- Plenty of free online tools do this – for example Desmos and Geogebra
- Try changing the domain of **t** to **$-\pi/3 \le t \le \pi/3$**

Your notes

Worked example

Your notes

Your notes

Curve C is defined parametrically by the equations

$$x = 2\cos 3t \text{ and } y = \sin(3t + \frac{\pi}{3})$$

$$\text{where } -\frac{\pi}{3} \leq t \leq 0$$

The line l_1 is the normal to the curve at the point where $x = 1$.

(a) Find the equation of l_1

(b) Find the exact area of the triangle made by l_1 and the coordinate axes

a) l_1: "$y - y_1 = m(x - x_1)$"

STEP 1 $\boxed{\text{FIND } \frac{dx}{dt} \text{ AND } \frac{dy}{dt}}$

$$\frac{dx}{dt} = -6\sin 3t$$

$$\frac{dy}{dt} = 3\cos\left(3t + \frac{\pi}{3}\right)$$

BOTH INVOLVE CHAIN RULE

STEP 2 $\boxed{\text{FIND } \frac{dy}{dx} \text{ (IN TERMS OF t)}}$

$$\frac{dy}{dx} = \frac{3\cos\left(3t + \frac{\pi}{3}\right)}{-6\sin 3t}$$

$$\boxed{\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt}}$$

$$-\cos\left(3t + \frac{\pi}{3}\right)$$

Your notes

$$= \frac{\qquad}{2\sin 3t}$$

STEP 3 FIND THE VALUE OF t AT x = 1

$2\cos 3t = 1$

$\cos 3t = \frac{1}{2}$

$3t = -\frac{\pi}{3}$

$t = -\frac{\pi}{9}$

EXTREME CARE

$-\frac{\pi}{3} \leqslant t \leqslant 0$

$\therefore -\pi \leqslant 3t \leqslant 0$

STEP 4 FIND $\frac{dy}{dx}$ AT $t = -\frac{\pi}{9}$

$$\frac{dy}{dx} = \frac{-\cos(0)}{2\sin\left(-\frac{\pi}{3}\right)} = \frac{\sqrt{3}}{3}$$

STEP 5 FIND THE y-COORDINATE USING THE VALUE OF t

$$y = \sin\left(3 \times \left(-\frac{\pi}{9}\right) + \frac{\pi}{3}\right)$$

$y = \sin(0) = 0$

STEP 6 "$m_1 \times m_2 = -1$"

GRADIENT OF NORMAL, $m = \frac{-1}{\frac{\sqrt{3}}{3}} = -\sqrt{3}$

STEP 7 "$y - y_1 = m(x - x_1)$"

$y - 0 = -\sqrt{3}(x - 1)$

$l_1 : y = -\sqrt{3}x + \sqrt{3}$

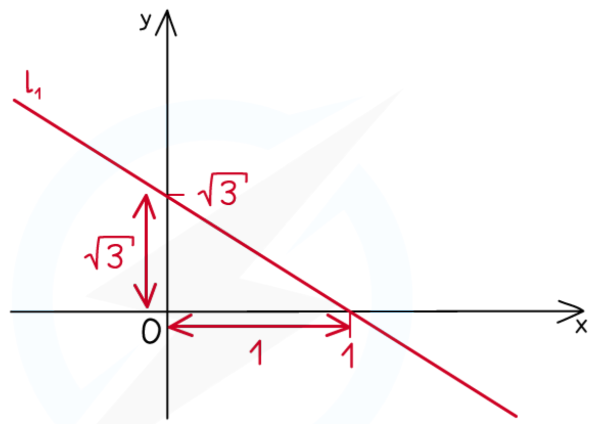
Your notes

b) $y = -\sqrt{3}x + \sqrt{3}$

FIND WHERE LINE INTERCEPTS AXES

AT $x = 0$, $y = \sqrt{3}$

AT $y = 0$, $x = 1$



A SKETCH MAKES THE AREA "EASY TO SEE"

$\text{AREA} = \frac{1}{2} \times 1 \times \sqrt{3}$

$\text{AREA} = \frac{\sqrt{3}}{2}$ SQUARE UNITS

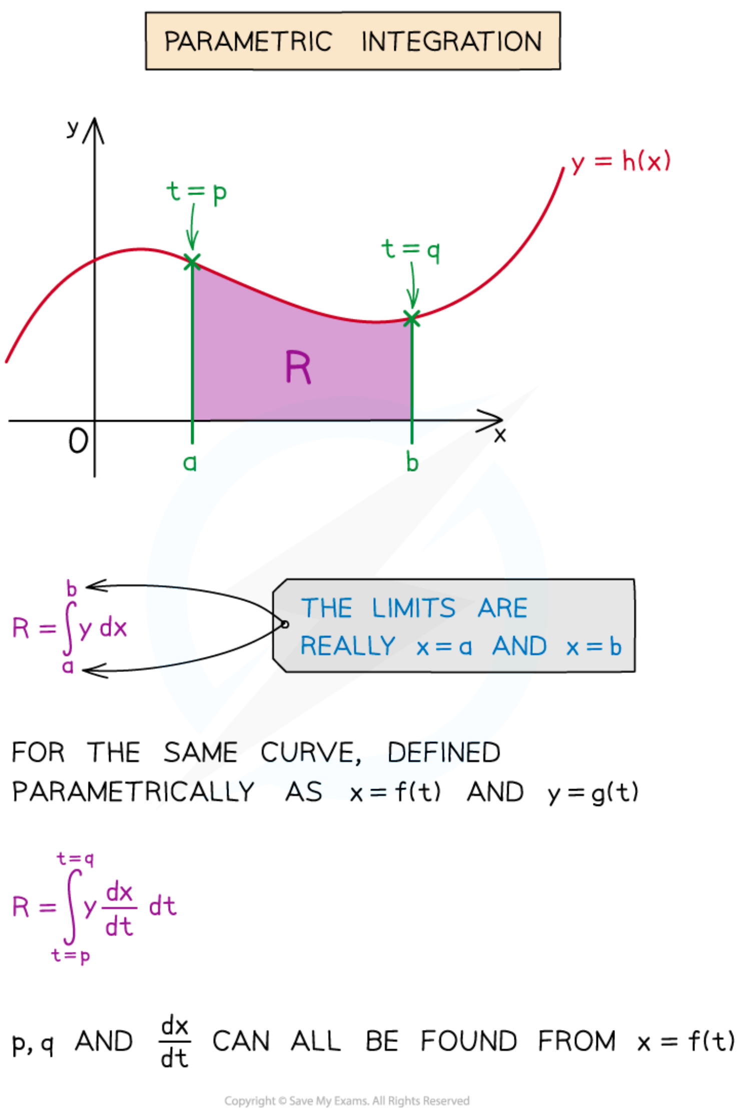
9.2.2 Parametric Integration

Your notes

Parametric Integration

How does integration work with parametric equations?

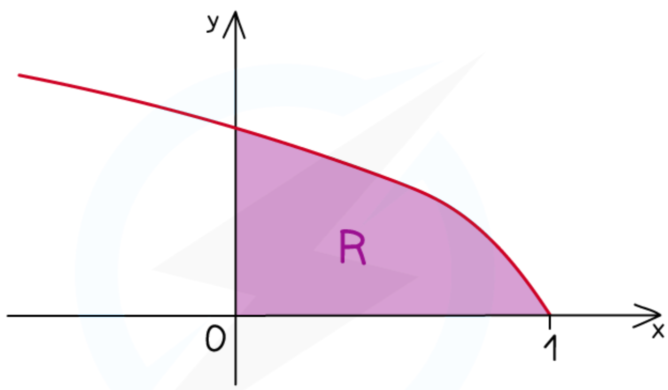
- Integration is used to find the area under a curve where the curve has been defined by parametric equations
  - **x = f(t)**
  - **y = g(t)**



- The key point is to ensure the limits of the integral are changed to the parameter

e.g. THE DIAGRAM BELOW SHOWS A SKETCH OF THE CURVE DEFINED PARAMETRICALLY BY

$$x = \cos 2\theta \qquad 0 < \theta < \pi$$
$$y = \sin\theta$$



FIND THE AREA MARKED R

STEP 1 WRITE DOWN THE INTEGRAL IN GENERAL TERMS OF x AND y

$$R = \int_0^1 y \, dx$$

Your notes

Your notes

STEP 2 | CHANGE BOTH LIMITS FROM x's TO THE PARAMETER

AT $x = 0$, $\cos 2\theta = 0$

$$2\theta = \frac{\pi}{2}$$

$$\theta = \frac{\pi}{4}$$

AT $x = 1$, $\cos 2\theta = 1$

$$2\theta = 0$$

$$\theta = 0$$

STEP 3 | FIND $\frac{dx}{dt}$

$$\frac{dx}{dt} = -2\sin 2\theta$$

STEP 4 | PUT THE INTEGRAL IN TERMS OF THE PARAMETER TOGETHER

CAREFUL WITH LIMITS, KEEP THEM IN THE RIGHT ORDER

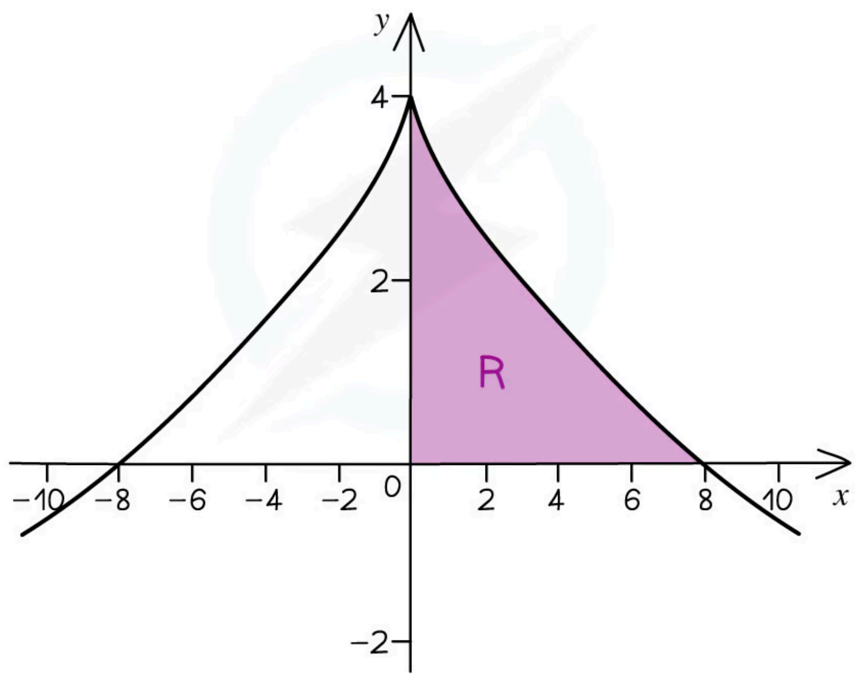
$$R = \int_{\frac{\pi}{4}}^{0} -2\sin\theta \sin 2\theta \, d\theta \qquad (\bullet)$$

Your notes

Worked example

The sketch below shows the graph of a curve defined parametrically by

$$x = t^3 \text{ and } y = (4 - t^2)$$



Find the exact area of the region marked R.

Your notes

STEP 1 WRITE THE INTEGRAL IN TERMS OF x AND y

$$R = \int_0^8 y\,dx$$

STEP 2 CHANGE LIMITS FROM x TO t

AT $x = 0$, $t^3 = 0$
$t = 0$

AT $x = 8$, $t^3 = 8$
$t = 2$

STEP 3 FIND $\frac{dx}{dt}$

$$\frac{dx}{dt} = 3t^2$$

STEP 4 PUT THE INTEGRAL TOGETHER IN t

$$R = \int_0^2 (4 - t^2) \times 3t^2\,dt$$

COULD USE CALCULATOR

$$R = \int_0^2 (12t^2 - 3t^4)\,dt$$

$$R = \left[4t^3 - \frac{3}{5}t^5\right]_0^2$$

$$R = \left(32 - \frac{96}{5}\right) - (0 - 0)$$

$$R = \frac{64}{5} \text{ SQUARE UNITS}$$

Your notes

9.2.3 Modelling with Parametric Equations

Your notes

Modelling with Parametric Equations

What can be modelled with parametric equations?

- A common use for parametric equations is motion in two dimensions
  - **x = f(t)** – horizontal position, dependent on time
  - **y = f(t)** – vertical position, dependent on time
- This is explored in vectors and projectiles

A STONE IS LAUNCHED FROM A CATAPULT AT GROUND LEVEL. RELATIVE TO THE ORIGIN, WHERE THE CATAPULT IS SITUATED, THE POSITION OF THE STONE, AT TIME t SECONDS, IS DEFINED PARAMETRICALLY BY

$$x = t^2 + t$$
$$y = 8t - 2t^2$$

WHERE x AND y ARE MEASURED IN METRES

a) FIND THE POSITION OF THE STONE AFTER 1 SECOND
b) AT WHICH TIMES IS THE STONE AT A HEIGHT OF 6 METRES
c) ASSUMING THE GROUND IS HORIZONTAL, HOW FAR AWAY FROM THE CATAPULT DOES THE STONE FIRST HIT THE GROUND AFTER LAUNCH?
e) i) FIND THE TIME AT WHICH THE STONE IS AT ITS MAXIMUM HEIGHT
ii) FIND THE COORDINATES OF THE STONE WHEN IT IS AS ITS MAXIMUM HEIGHT
f) WHAT CAN YOU DEDUCE ABOUT THE PATH OF THE STONE FROM YOUR ANSWERS ABOVE?

Your notes

a) AT $t = 1$, $x = 1^2 + 1 = 2$

$y = 8 - 2 = 6$

$\therefore$ AFTER 1 SECOND STONE IS AT (2, 6)

b) $\boxed{\text{HEIGHT OF 6m} \rightarrow y = 6}$

$8t - 2t^2 = 6$

$t^2 - 4t + 3 = 0$

$(t-1)(t-3) = 0$

$t = 1, \quad t = 3$

STONE IS AT 6m AFTER
1 SECOND AND 3 SECONDS

c) $\boxed{\text{GRAND LEVEL, HEIGHT OF 0m} \rightarrow y = 0}$

$8t - 2t^2 = 0$

$2t(4 - t) = 0$

AT START STONE ON CATAPULT!

$\cancel{t = 0} \quad t = 4$

THIS IS THE TIME, QUESTION WANTS DISTANCE

$x = 4^2 + 4$

$\therefore$ $x = 20$ m

Your notes

d) i) MAXIMUM HEIGHT – HORIZONTAL TANGENT

$\frac{dy}{dt} = 8 - 4t = 0$ WHEN $t = 2$

$\frac{d^2y}{dt^2} = -4 < 0$ $\therefore$ MAXIMUM

NOT ESSENTIAL UNLESS ASKED FOR

STONE IS AT MAXIMUM HEIGHT AFTER 2 SECONDS

ii) AT $t = 2$, $x = 2^2 + 2 = 6$

$y = 16 - 8 = 8$

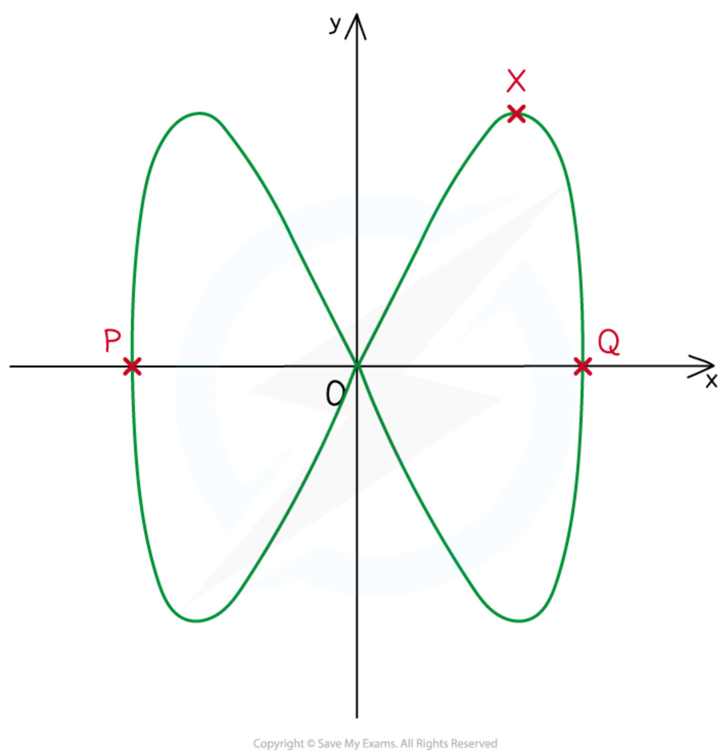
COORDINATES OF STONE AT MAXIMUM HEIGHT IS $(6, 8)$

e) THE STONE DOES NOT FOLLOW A SYMMETRICAL PATH – IF IT DID WE'D EXPECT MAX HEIGHT TO BE AT $x = 10\text{m}$

HALF OF 20 m WHEN IT HITS GROUND

Your notes

Worked example



Your notes

(a) Find the initial position of the car, C

(b) Find the times at which $(0 \leq t \leq 2\pi)$ the car is at the origin

(c) Find the coordinates of the point labelled X on the diagram

(d) At which other times $(0 \leq t \leq 2\pi)$ is C the same distance from the origin as it is when it is at point X?

(e) Find the distance PQ on the diagram

(f) If the restriction $0 \leq t \leq 2\pi$ was removed from the model what could you say about the motion of the model car?

a) INITIAL POSITION, $t = 0$

$x = 2\cos 0 = 2$

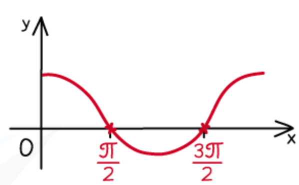
$y = 3\sin 0 = 0$

$\therefore$ INITIAL POSITION OF C IS $(2, 0)$

b) ORIGIN POSITION $(0, 0)$

$2\cos t = 0$

$t = \frac{\pi}{2}, \frac{3\pi}{2}$

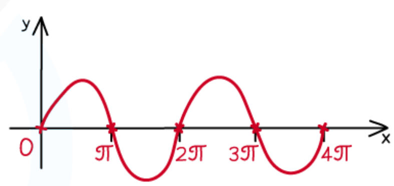


MUST HAVE BOTH x & y EQUAL TO 0

$3\sin 2t = 0$

$2t = 0, \pi, 2\pi, 3\pi, 4\pi$

$t = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$



Your notes

$\therefore$ C IS AT ORIGIN WHEN $t = \frac{\pi}{2}$ AND $t = \frac{3\pi}{2}$ MINUTES

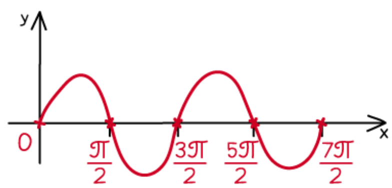
c) AT x, HORIZONTAL TANGENT, $\frac{dy}{dt} = 0$

$$\frac{dy}{dt} = 6\cos 2t$$

$$\cos 2t = 0$$

$$2t = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}$$

$$t = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$$



x AND y ARE BOTH POSITIVE AT x SO THIS MUST BE WHEN $t = \frac{\pi}{4}$

AT $t = \frac{\pi}{4}$, $x = 2\cos\left(\frac{\pi}{4}\right) = \frac{2}{\sqrt{2}} = \sqrt{2}$

$$y = 3\sin\left(\frac{\pi}{2}\right) = 3$$

$\therefore$ $x(\sqrt{2}, 3)$

Your notes

d) THESE WERE FOUND IN PART c)

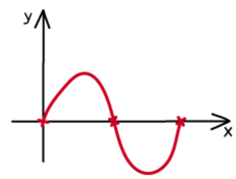
$t = \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$ MINUTES

e) P AND Q ARE VERTICAL TANGENTS, $\frac{dx}{dt} = 0$

$\frac{dx}{dt} = -2\sin t = 0$

$t = 0, \pi, (2\pi)$

SAME AS $t = 0$



AT $t = 0$, $x = 2\cos 0 = 2$

$y = 3\sin 0 = 0$

AT $t = \pi$, $x = 2\cos\pi = -2$

$y = 3\sin 2\pi = 0$

OBVIOUS FROM DIAGRAM BUT WORTH CHECKING

$P(-2, 0)$ $Q(2, 0)$

$\therefore$ $PQ = 4m$

f) FOR $0 \leqslant t \leqslant 2\pi$ THE CAR COMPLETES ONE FULL CIRCUIT OF THE TRACK

IF THIS RESTRICTION IS REMOVED THE CAR WILL CONTINUE TO REPEAT ITS JOURNEY AROUND THE TRACK INDEFINITELY