Your notes

Edexcel A Level Maths: Pure

7.1 Differentiation

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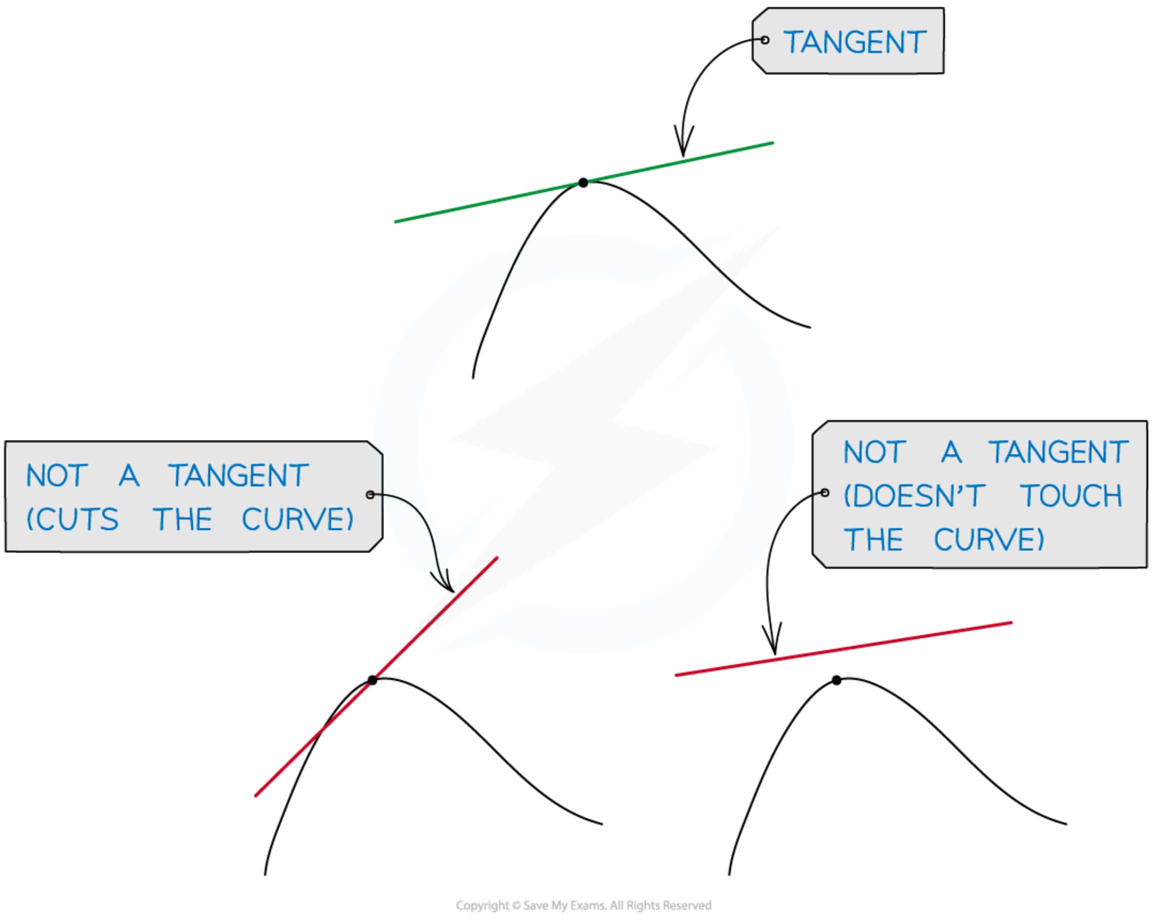
7.1.1 Definition of Gradient

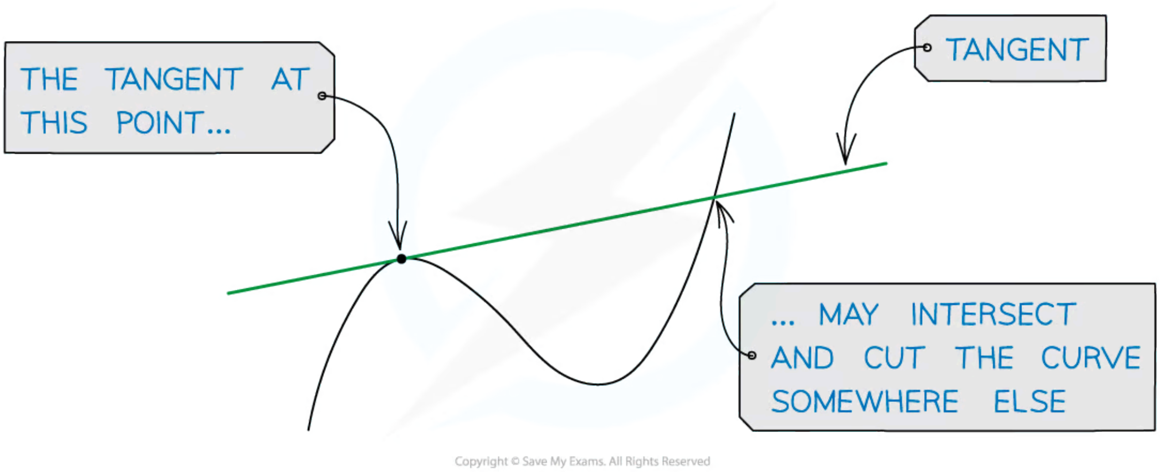
Definition of Gradient

What is the gradient of a curve?

- At a given point the **gradient** of a curve is defined as the gradient of the **tangent** to the curve at that point
- A **tangent** to a curve is a line that just **touches** the curve at one point but doesn't cut the curve at that point



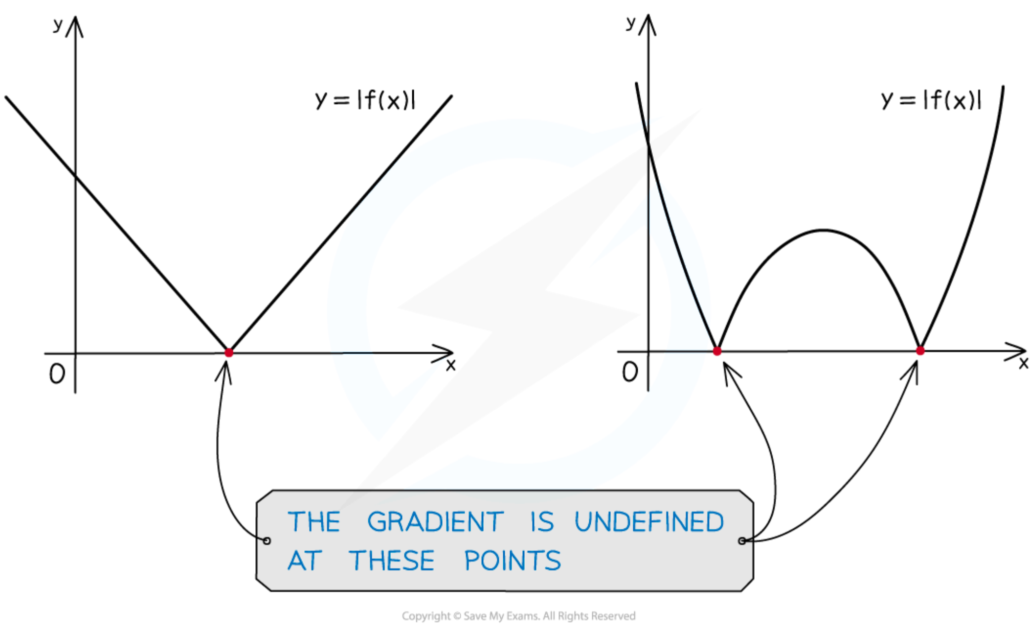
- A tangent *may* cut the curve somewhere else on the curve



- It is only possible to draw **one** tangent to a curve at any given point
- Note that unlike the gradient of a straight line, the gradient of a curve is constantly changing

Examiner Tip

- A tangent only exists at points where a curve is smooth.
- For example, there is no tangent (or gradient) at one of the 'corners' in a modulus function.



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Worked example

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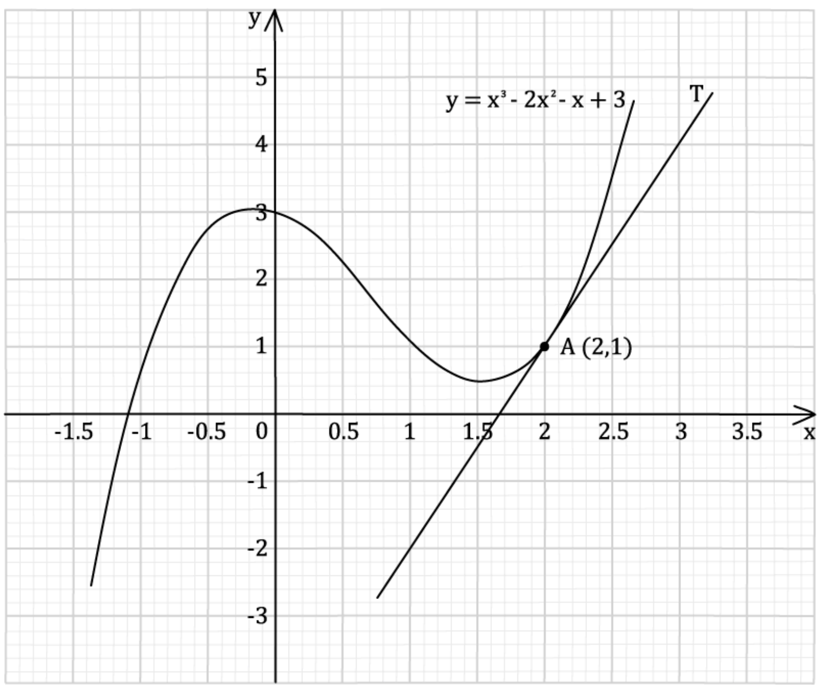
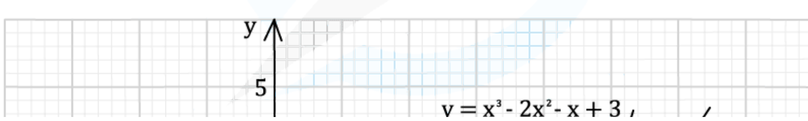
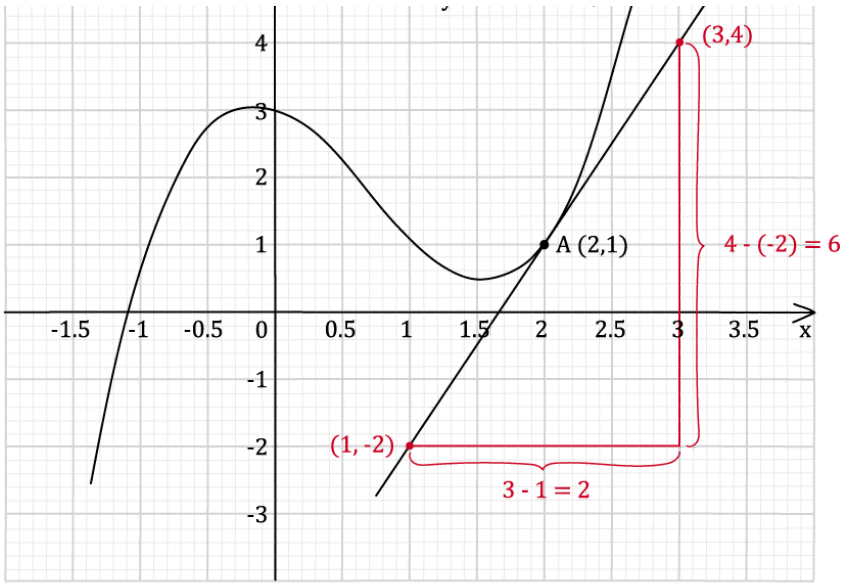
The diagram shows the curve with equation $y = x^3 - 2x^2 - x + 3$. The tangent, T, to the curve at point $A(2, 1)$ is also shown.



Using the diagram, calculate the gradient of the curve at point A.





THE GRADIENT OF THE CURVE AT POINT A IS THE SAME AS THE GRADIENT OF THE TANGENT LINE.

SO

$$\text{GRADIENT} = \frac{\text{CHANGE IN } y}{\text{CHANGE IN } x} = \frac{4-(-2)}{3-1} = \frac{6}{2} = 3$$

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7.1.2 First Principles Differentiation

First Principles Differentiation

What is the derivative or gradient function?

- For a curve **y = f(x)** there is an associated function called the **derivative** or **gradient function**
- The derivative of f(x) is written as f'(x) or $\frac{dy}{dx}$
- The derivative is a formula that can be used to find the gradient of y = f(x) at any point, by substituting the x coordinate of the point into the formula
- The process of finding the derivative of a function is called differentiation
- We differentiate a function to find its derivative

What is differentiation from first principles?

- Differentiation from **first principles** uses the **definition** of the **derivative** of a function ***f(x)***
- The definition is

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

- $\lim_{h \to 0}$ means the '*limit as **h** tends to zero*'
- When $h = 0$, $\frac{f(x+h) - f(x)}{h} = \frac{f(x) - f(x)}{0} = \frac{0}{0}$ which is **undefined**
  - Instead we consider what happens as ***h*** gets closer and closer to zero
- Differentiation **from first principles** means using that definition to show what the **derivative** of a function is

How do I differentiate from first principles?

STEP 1: Identify the function ***f(x)*** and substitute this into the first principles formula

e.g. Show, from first principles, that the derivative of $3x^2$ is $6x$

$f(x) = 3x^2$ so $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{3(x+h)^2 - 3x^2}{h}$

STEP 2: Expand ***f(x+h)*** in the numerator

$$f'(x) = \lim_{h \to 0} \frac{3(x^2 + 2hx + h^2) - 3x^2}{h}$$

$$f'(x) = \lim_{h \to 0} \frac{3x^2 + 6hx + 3h^2 - 3x^2}{h}$$

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STEP 3: Simplify the numerator, factorise and cancel h with the denominator

$$f'(x) = \lim_{h \to 0} \frac{h(6x + 3h)}{h}$$

STEP 4: Evaluate the remaining expression as ***h*** *tends to zero*

$$f'(x) = \lim_{h \to 0} (6x + 3h) = 6x \quad As\ h \to 0,\ (6x + 3h) \to (6x + 0) \to 6x$$

$\therefore$ The derivative of $3x^2$ is $6x$

Examiner Tip

- Most of the time you will not use first principles to find the derivative of a function (there are much quicker ways!). However, you can be asked on the exam to demonstrate differentiation from first principles.
- Make sure you can use first principles differentiation to find the derivatives of kx, kx^2 and kx^3 (where k is a constant).

Worked example

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Prove, from first principles, that the derivative of $5x^3$ is $15x^2$.

STEP 1: FOR $f(x) = 5x^3$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{5(x+h)^3 - 5x^3}{h}$$

EXPAND $(x+h)^3$

STEP 2:

$$= \lim_{h \to 0} \frac{5(x^3 + 3x^2h + 3xh^2 + h^3) - 5x^3}{h}$$

$$= \lim_{h \to 0} \frac{5x^3 + 15x^2h + 15xh^2 + 5h^3 - 5x^3}{h}$$

STEP 3:

$$= \lim_{h \to 0} \frac{15x^2h + 15xh^2 + 5h^3}{h}$$

$$= \lim_{h \to 0} 15x^2 + 15xh + 5h^2$$

TAKE THE LIMIT AS $h \to 0$

STEP 4: AS $h \to 0$

$$(15x^2 + 15xh + 5h^2) \longrightarrow (15x^2 + 15x(0) + 5(0)^2) = 15x^2$$

$\therefore$ THE DERIVATIVE OF $5x^3$ IS $15x^2$

TO GET FULL MARKS, MAKE SURE YOU KEEP WRITING $\lim_{h \to 0}$ RIGHT UP TO THE END!

7.1.3 Differentiating Powers of x

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Differentiating Powers of x

How do I differentiate expressions involving powers of *x*?

- Powers of ***x*** are differentiated according to the following formulae:

IF n IS ANY REAL NUMBER, AND a IS A CONSTANT, THEN

1. FOR $f(x) = x^n$, $f'(x) = nx^{n-1}$

   FOR $y = x^n$, $\frac{dy}{dx} = nx^{n-1}$

2. FOR $f(x) = ax^n$, $f'(x) = anx^{n-1}$

   FOR $y = ax^n$, $\frac{dy}{dx} = anx^{n-1}$

$f(x)$ AND $\frac{dy}{dx}$ ARE BOTH WAYS OF WRITING THE DERIVATIVE, AND THE PAIRS OF FORMULAE UNDER EACH NUMBER ABOVE ARE ENTIRELY EQUIVALENT. WE TEND TO TALK ABOUT $f(x)$ WHEN WE HAVE OUR "FUNCTIONS" HAT ON, AND ABOUT $y=\ldots$ WHEN WE HAVE OUR "GRAPHS" HAT ON – BUT ULTIMATELY THEY ARE JUST DIFFERENT NOTATIONS FOR THE SAME THING.

- Differentiating terms with positive integer powers of ***x*** is straightforward

$$f(x) = x^2 \longrightarrow f'(x) = 2x^{2-1} = 2x$$

$$f(x) = 5x^3 \longrightarrow f'(x) = 5 \times 3x^{3-1} = 15x^2$$

$$y = -4x^5 \longrightarrow \frac{dy}{dx} = -4 \times 5x^{5-1} = -20x^4$$

$$y = \frac{2}{3}x^7 \longrightarrow \frac{dy}{dx} = \frac{2}{3} \times 7x^{7-1} = \frac{14}{3}x^6$$

- But negative and fractional powers of ***x*** can also be differentiated this way

$$f(x) = 2x^{-1} \longrightarrow f'(x) = 2 \times (-1)x^{-1-1} = -2x^{-2}$$

$$f(x) = 5\sqrt{x} = 5x^{\frac{1}{2}} \longrightarrow f'(x) = 5 \times \frac{1}{2}x^{\frac{1}{2}-1} = \frac{5}{2}x^{-\frac{1}{2}} = \frac{5}{2\sqrt{x}}$$

$$\sqrt{x} = x^{\frac{1}{2}}$$

$$y = \frac{4}{5}x^{\frac{2}{3}} \longrightarrow \frac{dy}{dx} = \frac{4}{5} \times \frac{2}{3}x^{\frac{2}{3}-1} = \frac{8}{15}x^{-\frac{1}{3}}$$

$$y = \frac{1}{x^4} = x^{-4} \longrightarrow \frac{dy}{dx} = -4x^{-4-1} = -4x^{-5} = -\frac{4}{x^5}$$

$$\frac{1}{x^p} = x^{-p}$$

- Don't forget these two special cases:

IF a IS CONSTANT, THEN

3. FOR $f(x) = ax$, $f'(x) = a$

   FOR $y = ax$, $\frac{dy}{dx} = a$

4. FOR $f(x) = a$, $f'(x) = 0$

   FOR $y = a$, $\frac{dy}{dx} = 0$

- These allow you to differentiate constants and linear terms in ***x***

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$$f(x) = 3 \longrightarrow f'(x) = 0$$

$$f(x) = -5x \longrightarrow f'(x) = -5$$

$$y = \sqrt{7} \longrightarrow \frac{dy}{dx} = 0$$

$$y = x \longrightarrow \frac{dy}{dx} = 1$$

- If the expression involves a sum or difference of terms just differentiate one term at a time

$$f(x) = 3x^2 - 5x + 3$$

$$f'(x) = 6x - 5$$

THE DERIVATIVE OF 3 (A CONSTANT) IS ZERO

$$y = x^4 - \sqrt{x} + \frac{4}{x}$$

$$\frac{dy}{dx} = 4x^3 - \frac{1}{2\sqrt{x}} - \frac{4}{x^2}$$

IN GENERAL IF $y = f(x) + g(x) + h(x) + \ldots$

THEN $\frac{dy}{dx} = f'(x) + g'(x) + h'(x) + \ldots$

Examiner Tip

- Take extra care when differentiating negative and fractional powers of ***x*** as mishandling negative signs and fractions is a common way to lose marks in these sorts of questions.

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Worked example

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$f(x) = \dfrac{8}{q\sqrt{x}} + \sqrt{x}$, where q is a real constant and $x > 0$.

Given that $f'(3) = -\dfrac{\sqrt{3}}{18}$, find the value of q.

$$f(x) = \frac{8}{q}x^{-\frac{1}{2}} + x^{\frac{1}{2}}$$

$\sqrt{x} = x^{\frac{1}{2}}$ $\quad \dfrac{1}{\sqrt{x}} = x^{-\frac{1}{2}}$

$$f'(x) = -\frac{1}{2} \times \frac{8}{q}x^{-\frac{3}{2}} + \frac{1}{2} \times x^{-\frac{1}{2}}$$

$$= \frac{1}{2}x^{-\frac{1}{2}} - \frac{4}{q}x^{-\frac{3}{2}}$$

$$= \frac{1}{2\sqrt{x}} - \frac{4}{q\,x\sqrt{x}}$$

$x^{\frac{3}{2}} = x\sqrt{x}$

$$f'(3) = \frac{1}{2\sqrt{3}} - \frac{4}{3q\sqrt{3}} = \frac{3q-8}{6q\sqrt{3}}$$

REWRITE AS SINGLE FRACTION WITH COMMON DENOMINATOR

$$\text{SO} \quad \frac{3q-8}{6q\sqrt{3}} = -\frac{\sqrt{3}}{18}$$

$$3q-8 = -\frac{\sqrt{3}}{18}\left(6q\sqrt{3}\right) = -q$$

$$4q = 8$$

$$q = 2$$

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