Edexcel A Level Maths: Pure

Your notes

2.2 Quadratics

Contents

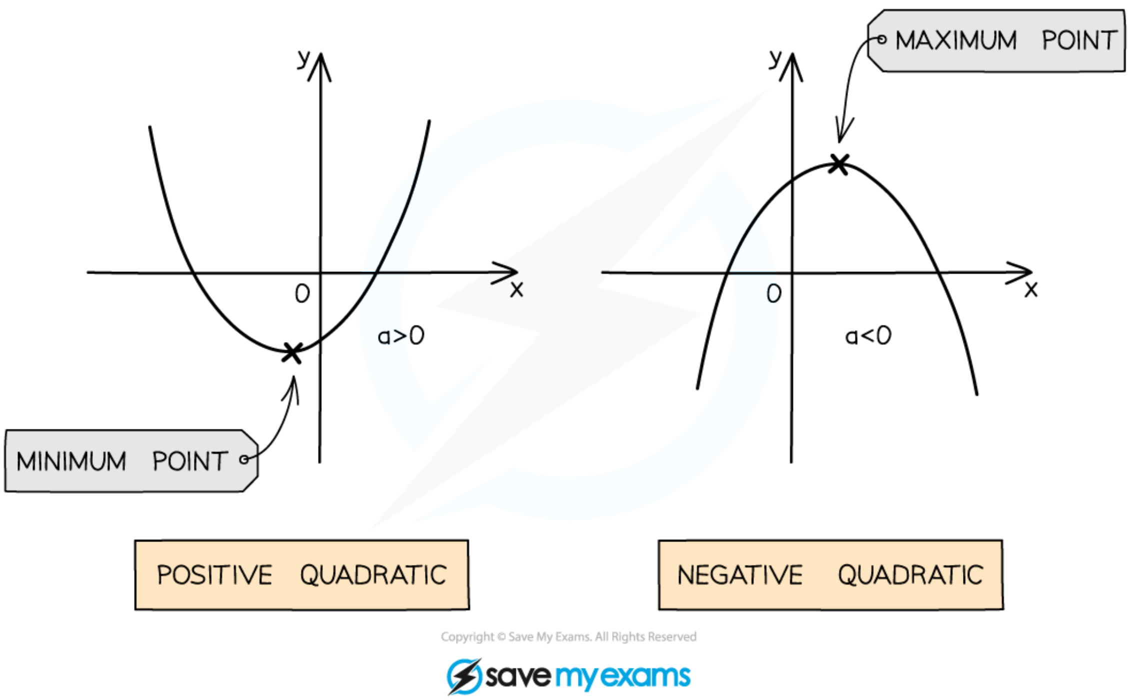
2.2.1 Quadratic Graphs

Your notes

Quadratic Graphs

What are quadratic graphs?

- The general equation of a quadratic graph is $y = ax^2 + bx + c$
- Their shape is called a parabola ("U" shape)
- Positive quadratics have a value of $a > 0$ so the parabola is upright ∪
- Negative quadratics have a value of $a < 0$ so the parabola is upside down ∩



Using quadratic graphs

You need to be able to:

- **sketch** a quadratic graph given an equation or information about the graph
- determine, from the equation, the **axes intercepts**
- **factorise**, if possible, to find the roots of the quadratic function
- find the coordinates of the **turning point** (maximum or minimum)

You may have to rearrange the equation before you can find some of these things

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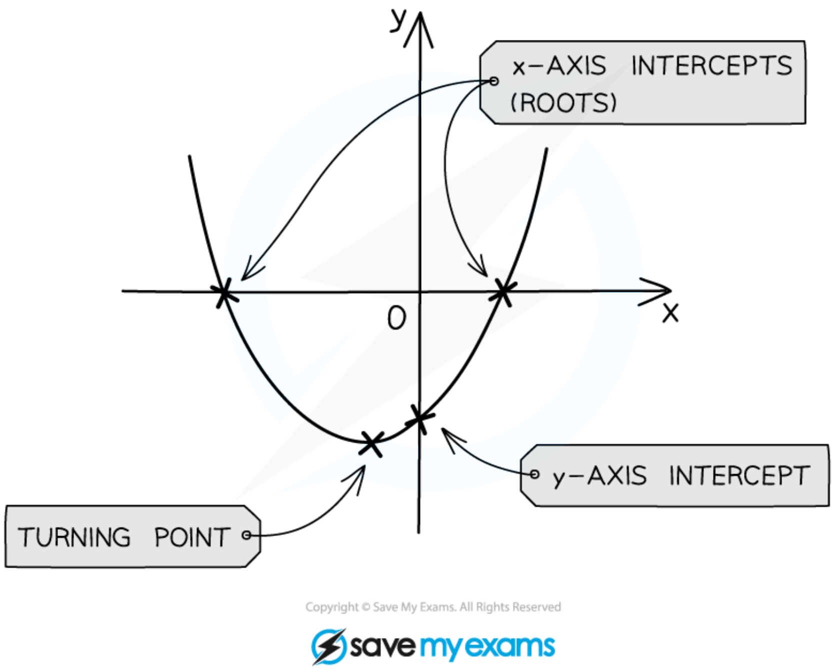


Examiner Tip

- Your calculator may tell you the **roots** of a quadratic function and the coordinates of the turning point
- But don't rely on it – think about how many marks the question is worth and how much method/working you should show
- Remember sometimes you'll need to rearrange an equation into the form $y = ax^2 + bx + c$

Worked example

Your notes

Your notes

Sketch the graph of $y = 2x^2 + 5x - 12$ stating the coordinates of the axes intercepts and the turning point.

$y = 2x^2 + 5x - 12$ POSITIVE QUADRATIC: U-SHAPED

AT $x = 0$, $y = -12$ SET $x=0$ FOR y-AXIS INTERCEPT

$y = (2x-3)(x+4)$

FACTORISE AND SOLVE FOR x-AXIS INTERCEPTS

$y = 0$ AT $x = \frac{3}{2}$ AND $x = -4$

$\therefore$ AXES INTERCEPTS ARE $(0,-12)$, $(-4,0)$ AND $(\frac{3}{2},0)$

$y = 2(x^2 + \frac{5}{2}x) - 12$

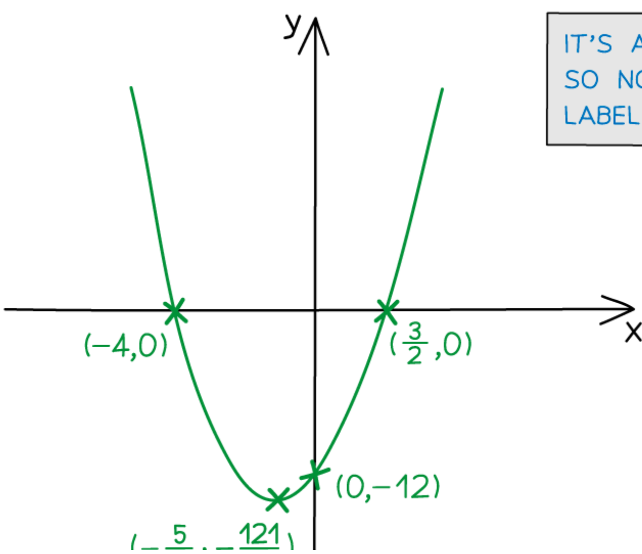
COMPLETE THE SQUARE...

$y = 2\left[(x+\frac{5}{4})^2 - \frac{25}{16}\right] - 12$

$y = 2(x+\frac{5}{4})^2 - \frac{121}{8}$

... TO FIND THE TURNING POINT

$\therefore$ TURNING POINT IS $(-\frac{5}{4}, -\frac{121}{8})$



IT'S A SKETCH SO NO NEED TO LABEL AXES

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2.2.2 Discriminants

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Discriminants

What is a discriminant?

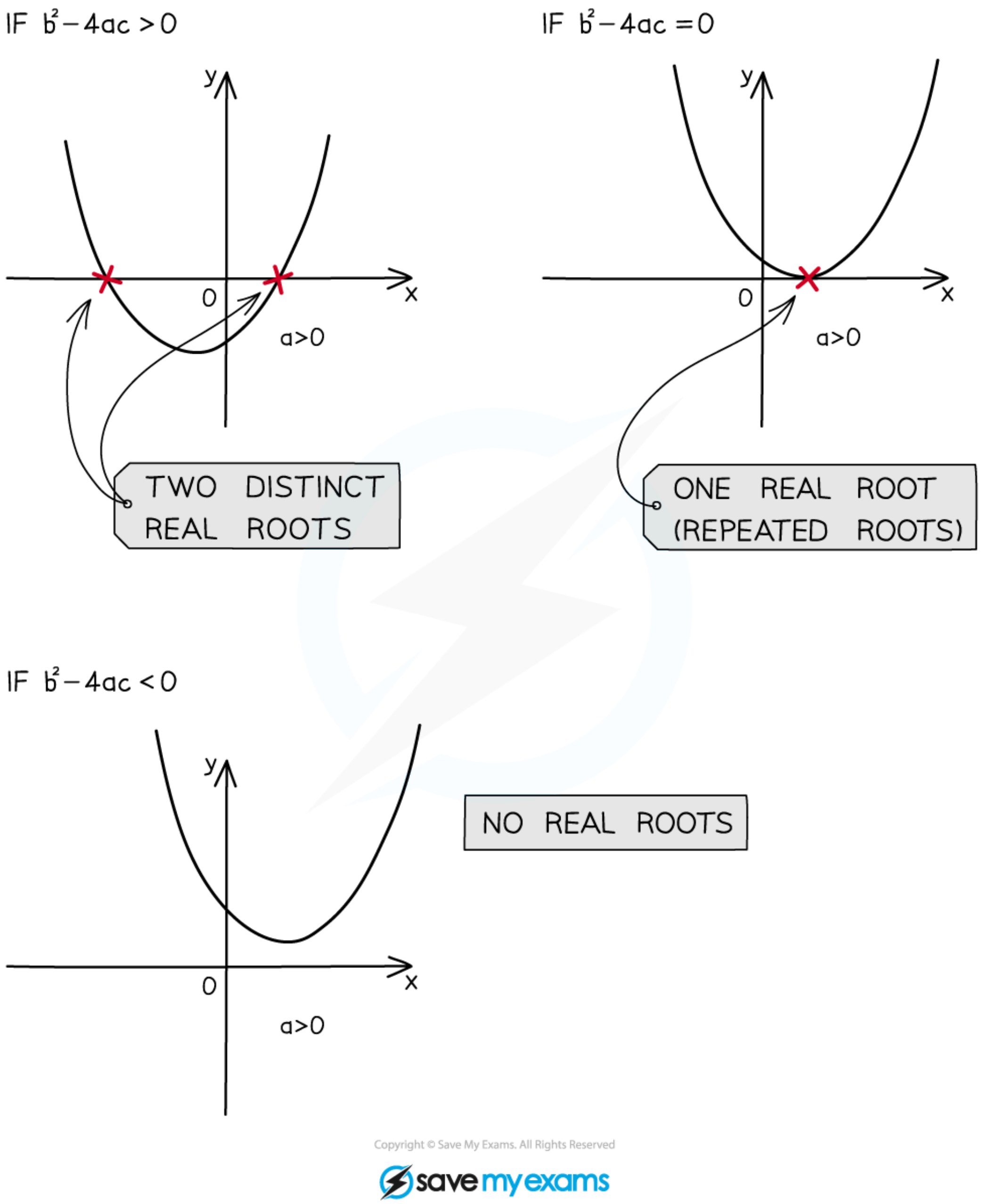
- The discriminant is the part of the quadratic formula that is under the square root sign $b^2 - 4ac$
- It is sometimes denoted by the Greek letter Δ (capital delta)

How does the discriminant affect graphs and roots?

There are three options for the outcome of the discriminant:

- If $b^2 - 4ac > 0$ the quadratic crosses the x-axis **twice** meaning there are **two distinct real roots**
- If $b^2 - 4ac = 0$ the quadratic touches the x-axis **once** meaning there is **one real root** (also called repeated roots)
- If $b^2 - 4ac < 0$ the quadratic **does not cross** the x-axis meaning there are no real roots

Your notes



Discriminant and inequalities

- You need to be able to **set up** and **solve** equations and inequalities (often quadratic) arising from the discriminant
- Sketch the quadratic and decide whether you're looking above or below zero to write your solutions correctly

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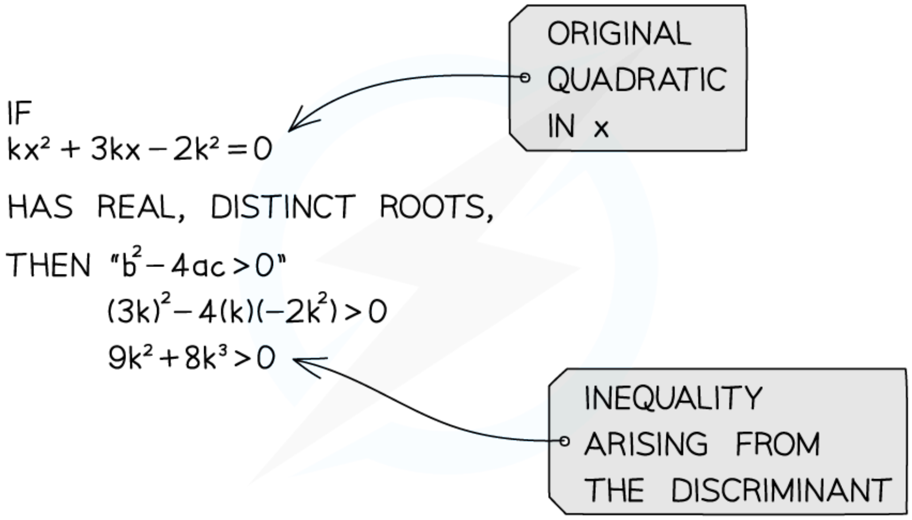


Examiner Tip

- When questions just mention "real roots", the roots could be **distinct** or **repeated** (i.e. they arent talking about complex numbers!)
- In these cases, you only need to worry about solving $b^2 - 4ac \geq 0$
- When solving using inequalities always sketch the quadratic and decide whether you're looking above or below zero to help write your solutions correctly

Your notes

Worked example

Find the values of k for which the quadratic function

$f(x) = 2kx^2 + kx - k + 2$ has real roots.

$f(x) = 2kx^2 + kx - k + 2$

$(k)^2 - 4(2k)(-k+2) \geqslant 0$

"$b^2 - 4ac \geqslant 0$" FOR REAL ROOTS

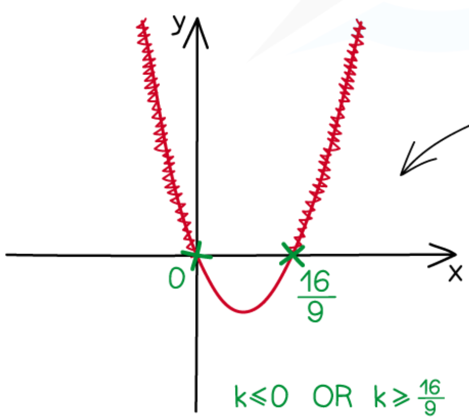
$k^2 - 8k(2-k) \geqslant 0$

$k^2 - 16k + 8k^2 \geqslant 0$

$9k^2 - 16k \geqslant 0$

$k(9k - 16) \geqslant 0$

YOUR CALCULATOR MAY SOLVE THIS...



... BUT A SKETCH IS ESSENTIAL

$k \leqslant 0$ OR $k \geqslant \frac{16}{9}$

2.2.3 Completing the Square

Your notes

Completing the square

What is completing the square?

- Completing the square is another method used to solve quadratic equations
- It simply means writing $y = ax^2 + bx + c$ in the form $y = a(x+p)^2 + q$
- It can be used to help find other information about the quadratic like coordinates of the turning point

How do I complete the square?

The method used will depend on the value of the coefficient of the x^2 term in $y = ax^2 + bx + c$

When **a = 1**

- p is half of the coefficient of b
- q is $c - p^2$

e.g. $y = x^2 + 8x - 2$

STEP 1: "$p = \frac{b}{2}$" $\quad p = \frac{8}{2} = 4$

STEP 2: "$q = c - p^2$" $\quad q = -2 - 4^2 = -18$

STEP 3: WRITE THE ANSWER!

$y = (x+4)^2 - 18$

When **a ≠ 1**

- You first need to take a out as a factor of the x^2 and x terms
- Then continue as above

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NO NEED TO INVOLVE "c"

e.g. $y = 4x^2 + 16x + 5$

STEP 1: FACTOR "a" ON RHS $y = 4[x^2 + 4x] + 5$

STEP 2: WORKING WITH $[x^2 + 4x]$ ONLY, "$p = \frac{b}{2}$"

$p = \frac{4}{2} = 2$

STEP 3: STILL WITH $[x^2 + 4x]$ ONLY, "$q = c - p^2$"

$q = 0 - 2^2 = -4$

STEP 4: EXPAND AND SIMPLIFY $y = 4[(x+2)^2 - 4] + 5$

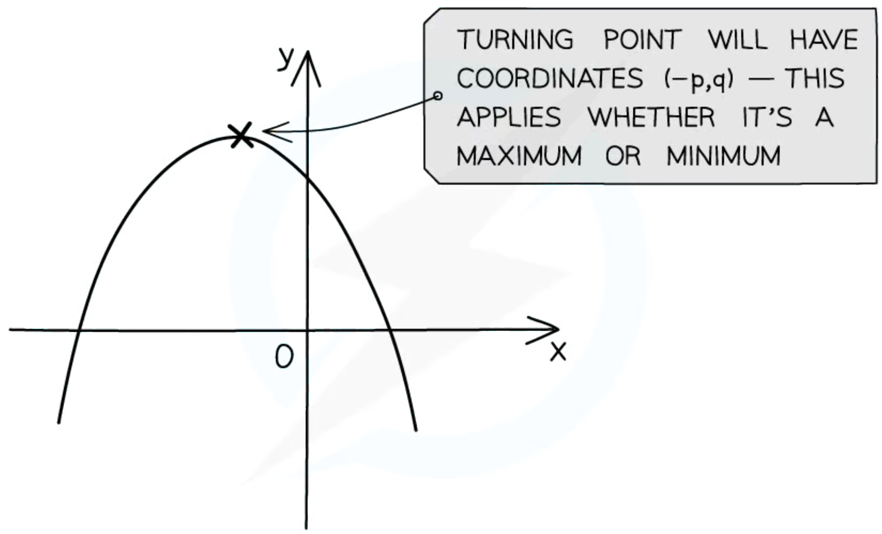
$y = 4(x+2)^2 - 16 + 5$

$y = 4(x+2)^2 - 11$

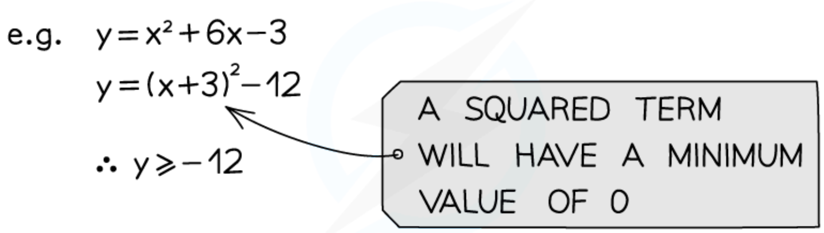
When is completing the square useful?

- Completing the square helps us find the **turning point** on a quadratic graph
- It can also help you **create the equation** of a quadratic when given the turning point

Your notes



- It can also be used to **prove** and/or **show** results using the fact that a squared term will always be greater than or equal to 0



Examiner Tip

- Sometimes the question will explicitly ask you to complete the square
- Sometimes it will even remind you of the form to write it in
- But sometimes it will expect you to spot that completing the square is what you need to do to help with other parts of the question... like finding turning points!

Your notes

Worked example

Your notes

(a) Show $2x^2 + 6x + k > 0$ for $k > \frac{9}{2}$

(b) What can you say about the roots of the equation

$2x^2 + 6x + k = 0$ when $k = \frac{9}{2}$?

(a)

">0" IMPLIES A MINIMUM VALUE — SO COMPLETE THE SQUARE WITH "a≠1"

$2x^2 + 6x + k = 2[x^2 + 3x] + k$

STEP 1: FACTOR "a"

$= 2[(x + \frac{3}{2})^2 - \frac{9}{4}] + k$

STEP 2: "$p = \frac{b}{2}$"

STEP 3: "$q = c - p^2$"

NOTE WITH STEP 2 & STEP 3 YOU ARE ONLY WORKING WITHIN THE SQUARE [ ] BRACKETS

$= 2(x + \frac{3}{2})^2 - \frac{9}{2} + k$

$= 2(x + \frac{3}{2})^2 + k - \frac{9}{2}$

STEP 4: EXPAND AND SIMPLIFY

$2(x + \frac{3}{2})^2 \geqslant 0$ FOR ALL VALUES OF x

$\therefore\ 2(x + \frac{3}{2})^2 + k - \frac{9}{2} > 0$ WHEN $k - \frac{9}{2} > 0$

$k > \frac{9}{2}$

SHOW ALL STAGES IN A "SHOW THAT" QUESTION

(b) WHEN $k=\frac{9}{2}$ THE QUADRATIC BECOMES $2(x+\frac{3}{2})^2$ AND SO THERE WILL ONLY BE ONE ROOT (OR REPEATED ROOTS) TO THE EQUATION $2x^2+6x+k=0$

THE EQUATION IS NOW $2(x+\frac{3}{2})^2=0$

2.2.4 Solving Quadratic Equations

Your notes

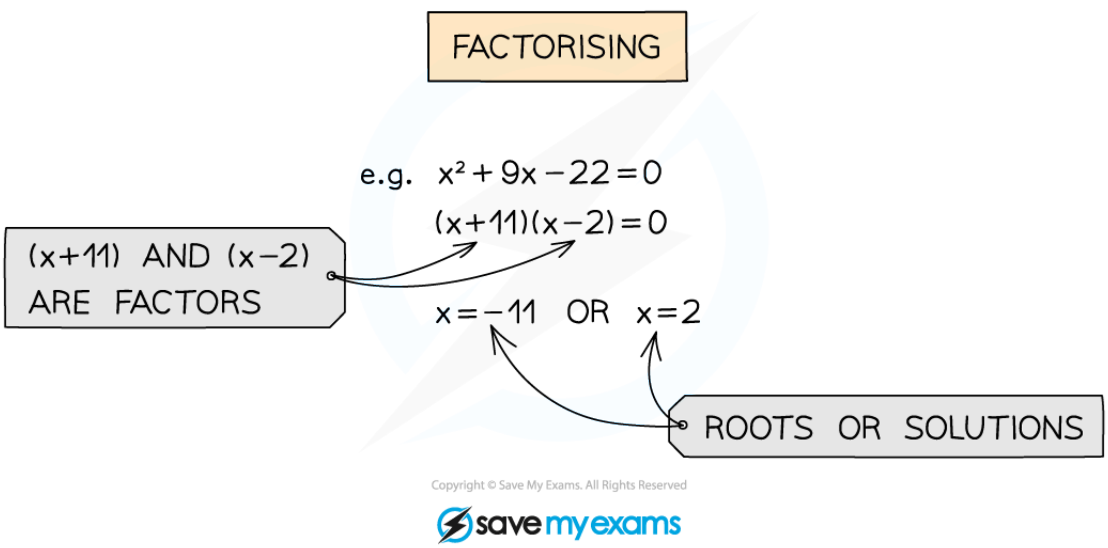
Solving Quadratic Equations

Solving quadratic equations

- We can solve quadratic equations when they are written in the form $ax^2 + bx + c = 0$
- If given an unusual looking equation, try to rearrange it into this form first
- The three ways to solve a quadratic you must know are
  - Factorising
  - Completing the square
  - Quadratic formula

Solving a quadratic equation by factorising

- Factorising is a great way to solve a quadratic quickly but won't work for all quadratics
- If the numbers are simple, try factorising first
- Once factorised, set each bracket to = 0 and solve



Solving a quadratic equation by completing the square

- Completing the square will work for any quadratic
- Make sure you know how to complete the square
- Remember this will help with questions involving turning points too

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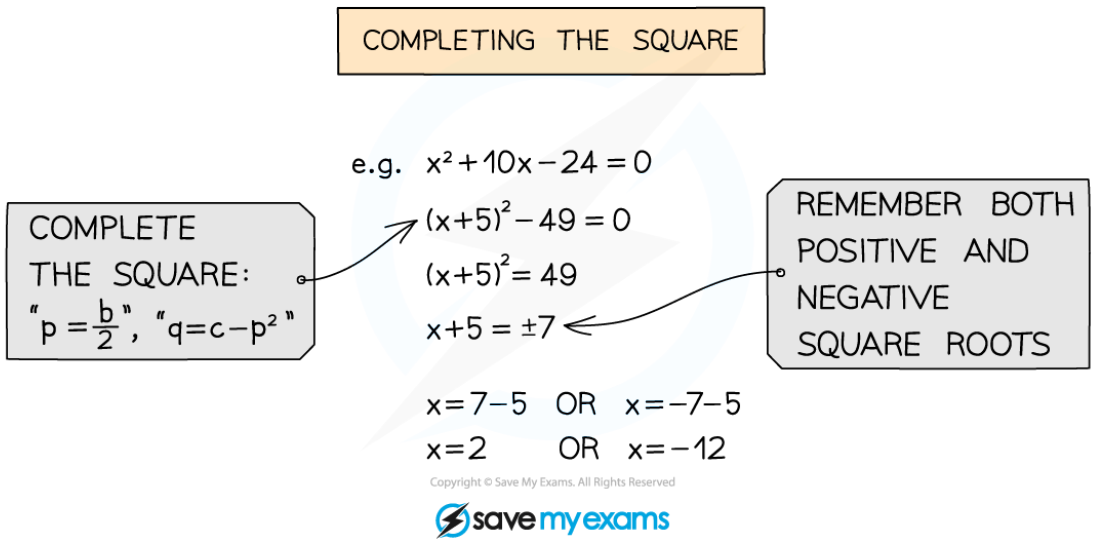
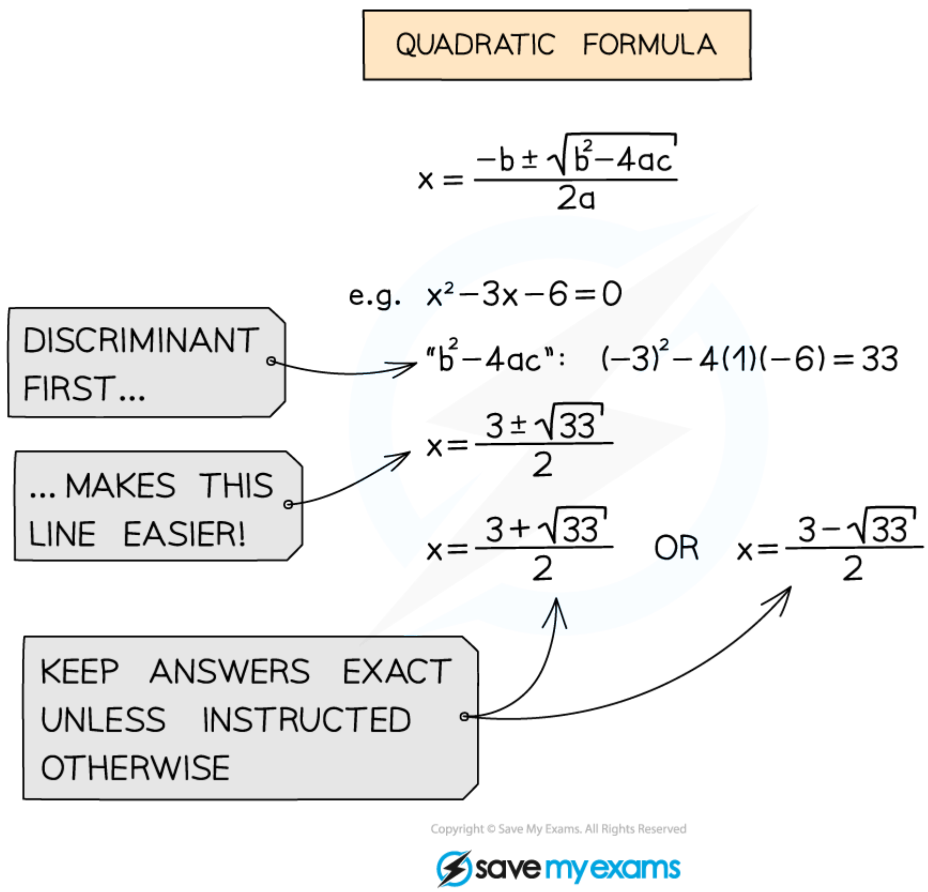


Solving a quadratic equation by the quadratic formula

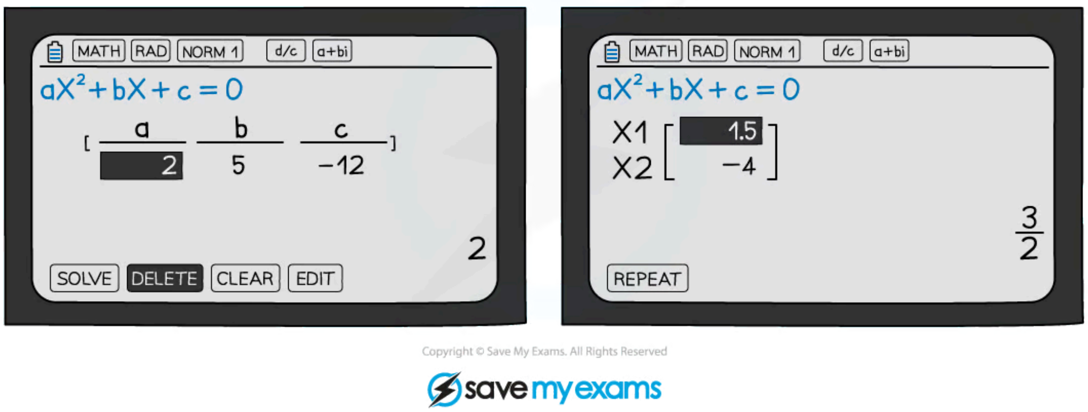
- The quadratic formula might look complicated but it just uses the coefficients a, b and c from the quadratic equation
- The quadratic formula will work for any quadratic

Your notes



Solving a quadratic equation by using a calculator

- Most calculators now have the ability to solve quadratics
- Get used to how your calculator functions work



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- So the solution to the quadratic equation $2x^2 + 5x - 12 = 0$ are **x = 1.5** and **x = -4**

Examiner Tip

- A calculator can be super-efficient but be aware some marks are for method
- There will never be many marks for solving a quadratic at AS/A level
- Use your judgement:
  - is it a “show that” or “prove” question?
  - how many marks?
  - how long is the question?
- Remember the quadratic formula with a song... there are loads of fun ones on YouTube

Your notes

Worked example

Solve the equation $x + 5 = \dfrac{12}{x}$.

$x + 5 = \dfrac{12}{x}$

IT'S UNUSUAL SO TRY REARRANGING.
$\times x$ TO GET RID OF FRACTIONS

$x^2 + 5x = 12$

$x^2 + 5x - 12 = 0$

THIS DOESN'T FACTORISE, COMPLETING THE SQUARE IS NOT "NICE" DUE TO THE 5

WORKING OUT THE DISCRIMINANT FIRST MAKES THINGS EASIER AND TIDY

"$b^2 - 4ac$" $= 5^2 - 4 \times 1 \times (-12)$

$= 25 + 48$

$= 73$

$\therefore\ x = \dfrac{-5 \pm \sqrt{73}}{2}$

$x = \dfrac{-5 + \sqrt{73}}{2}$ $\qquad$ $x = \dfrac{-5 - \sqrt{73}}{2}$

CHECK THESE ON YOUR CALCULATOR

Your notes

2.2.5 Further Solving Quadratic Equations (Hidden Quadratics)

Further Solving Quadratic Equations (Hidden Quadratics)

What are hidden quadratic equations?

- Hidden quadratic equations are quadratics written in terms of a function $f(x)$
- A normal quadratic appears in the form $ax^2 + bx + c = 0$
- Whereas a hidden quadratic appears in the form $a[f(x)]^2 + b[f(x)] + c = 0$
- This might look complicated but it simply means x has been replaced by $f(x)$
- e.g. $\sin^2 x + 2\sin x - 3 = 0$ is just the hidden quadratic of $x^2 + 2x - 3 = 0$ where $f(x) = \sin x$

How to solve hidden quadratic equations

- First rearrange the function into the form $a[f(x)]^2 + b[f(x)] + c = 0$
- Replacing the function and solving the 'normal' quadratic first
- Then substitute the function back into the solutions to solve the original quadratic

Your notes

e.g. $2\cos x = 3 - \cos^2 x \qquad 0° \leqslant x \leqslant 360°$

STEP 1: REARRANGE INTO THE FORM
$a[f(x)]^2 + bf(x) + c = 0$

$\cos^2 x + 2\cos x - 3 = 0$

STEP 2:
- "IGNORE" THE FUNCTION OF x AND SOLVE AS A NORMAL QUDRATIC
- USE A SUBSTITUTION, $y = f(x)$ IF YOU PREFER

$y = \cos x, \quad y^2 + 2y - 3 = 0$
$(y+3)(y-1) = 0$
$y = -3 \quad y = 1$

STEP 3: USE THESE SOLUTIONS, AND THE FUNCTION OF x, TO SOLVE THE ORIGINAL QUESTION

$\cos \cancel{x = -3}$ NO SOLUTIONS

$\cos x = 1$
$x = 0°, 360°$

Your notes

Worked example

Solve the equation $3^{2x+1} = 82 \times 3^x - 27$.

$3^{2x+1} - 82 \times 3^x + 27 = 0$

REARRANGE SO IT "= 0" AND WORK TOWARDS CORRECT FORM

$3^1 \times 3^{2x} - 82 \times 3^x + 27 = 0$

AWKWARD TO SPOT BUT 82×3^x IS THE CLUE

$3(3^x)^2 - 82(3^x) + 27 = 0$

THIS IS NOW IN THE FORM $a[f(x)]^2 + bf(x) + c = 0$

LET $y = 3^x$

$3y^2 - 82y + 27 = 0$

$(3y - 1)(y - 27) = 0$

$y = \frac{1}{3} \qquad y = 27$

YOU DO NOT HAVE TO USE A SUBSTITUTION

$\therefore \; 3^x = \frac{1}{3} \qquad 3^x = 27$

$x = -1 \qquad x = 3$

MAKE SURE YOU FIND x's FOR YOUR FINAL ANSWER