Edexcel A Level Maths: Pure

Your notes

8.1 Integration

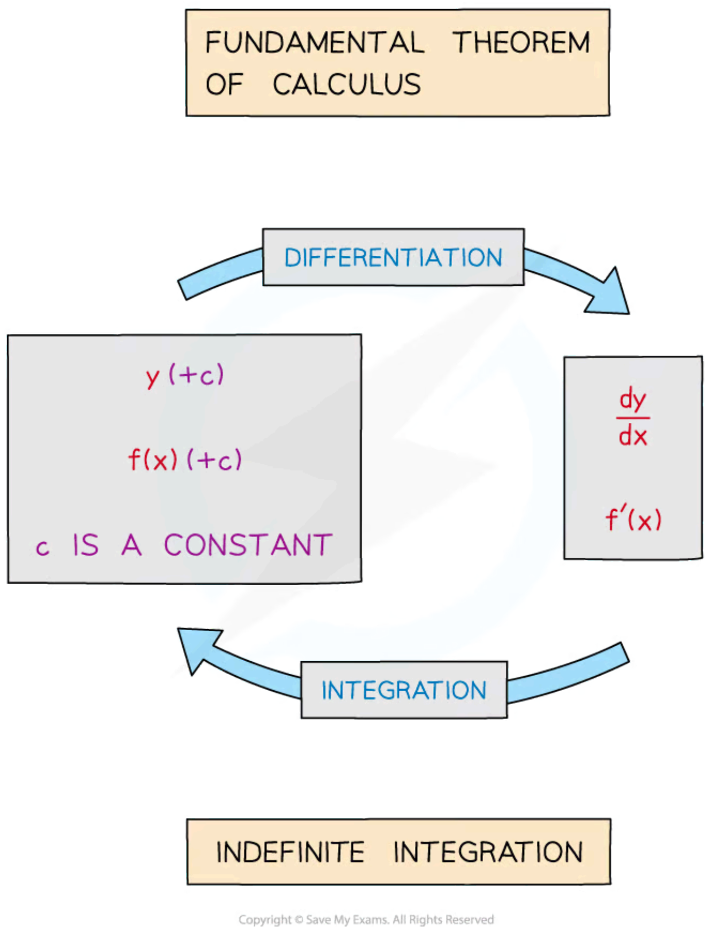
Contents

8.1.1 Fundamental Theorem of Calculus

Your notes

Fundamental Theorem of Calculus

What is the fundamental theorem of calculus?



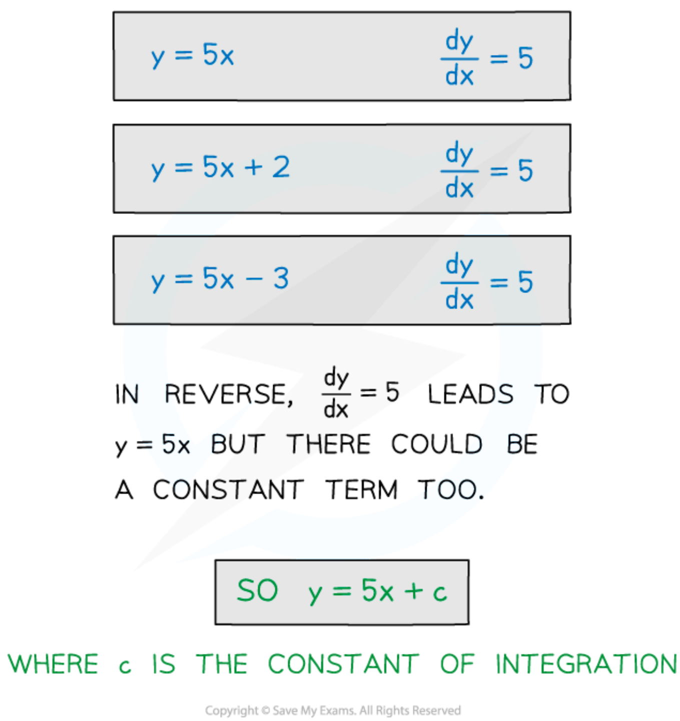
- The Fundamental Theorem of Calculus states that integration is the inverse process of differentiation
  - This form of the Theorem relates to **Indefinite Integration**
  - An alternative version of the Fundamental Theorem of Calculus involves **Definite Integration**

What is "+c" (plus c)?

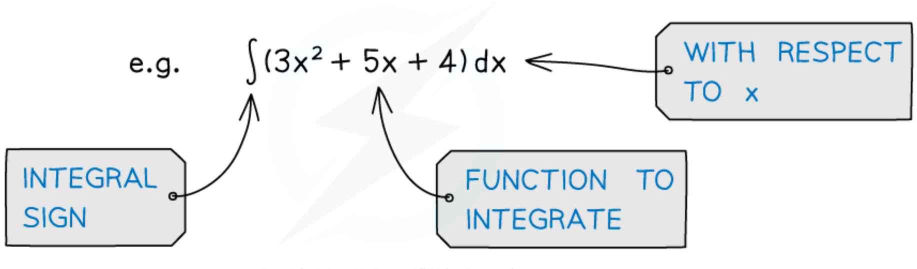
- When differentiating **y**, constant terms 'disappear'

Your notes

  - for constants **y = c**, $\frac{dy}{dx} = 0$
  - graphs of constants are horizontal lines and so have gradient $\left(\frac{dy}{dx}\right)$ of 0
- Integrating $\frac{dy}{dx}$, to get **y**, cannot determine the constant
  - To acknowledge this constant, "**+ c**" is used
  - **c** is called the **constant of integration**



Notation



- $\int$ is the sign for integration
- If it has more than one term the function to be integrated (called the **integrand**) should be in brackets
  - "Integrate" -- "**all** of (...)" -- "with respect to x"
- **dx** means integrate with respect to **x**, any other letter is treated like a number (ie like a constant)

Worked example

(a) Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ for $y = 6x + 3$

(b) Find $\int 6 \, \mathrm{d}x$

a) $\frac{dy}{dx} = 6$

b) $y = 6x + c$

CONSTANT OF INTEGRATION

Your notes

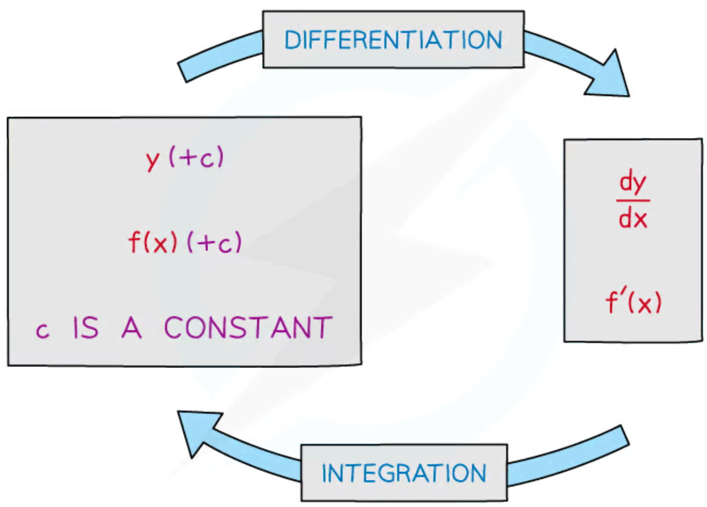
8.1.2 Integrating Powers of x

Your notes

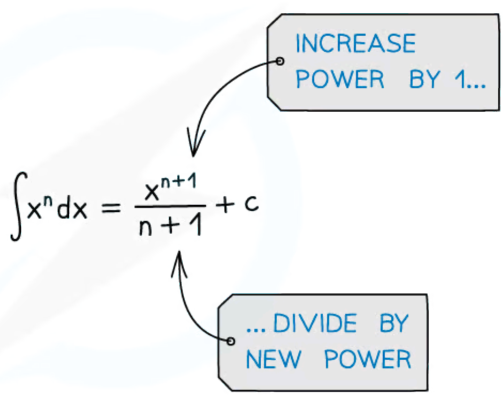
Integrating Powers of x

What is integration?

- Integration is the **inverse** ('opposite') of differentiation



How do I integrate powers of x?



- For each term ...
  - ... increase the power (of x) by 1
  - ... divide by the new power

- This does not apply when the original power is -1
  - the new power would be 0 and division by 0 is undefined

Your notes

Finding the Constant of Integration

How do I find c?

- STEP 1 Rewrite the function into a more easily integrable form
  - Each term needs to be a power of **x** (or a constant)
- STEP 2 Integrate each term and remember "+c"
  - Increase power by 1 and divide by new power
- STEP 3 Substitute the coordinates of a given point in to form an equation in **c**
  - Solve the equation to find **c**

e.g. FIND THE EQUATION OF THE GRAPH THAT PASSES THROUGH (2,6) AND HAS GRADIENT FUNCTION $3x^2(2x-1)$

ANOTHER NAME FOR $\frac{dy}{dx}$

STEP 1 REWRITE IN A MORE EASILY INTEGRABLE FORM

$$\frac{dy}{dx} = 3x^2(2x-1)$$

$$= 6x^3 - 3x^2$$

STEP 2 INTEGRATE...

$$y = \frac{6x^4}{4} - \frac{3x^3}{3} + c$$

...REMEMBERING "+c"

$$y = \frac{3x^4}{2} - x^3 + c$$

Your notes

Your notes

STEP 3 $\boxed{\text{USE THE GIVEN POINT TO FIND } c}$

AT (2,6) $6 = \frac{3}{2}(2)^4 - 2^3 + c$

$c = -10$

$\therefore \quad y = \frac{3}{2}x^4 - x^3 - 10$

Your notes

Worked example

Given $f'(x) = \dfrac{(x+3)^2}{\sqrt{x}}$ and $f(1) = 25$,

find $f(x)$.

STEP 1

$$f'(x) = \frac{x^2 + 6x + 9}{x^{\frac{1}{2}}}$$

REWRITE IN A MORE EASILY INTEGRABLE FORM

$$f'(x) = \frac{x^2}{x^{\frac{1}{2}}} + \frac{6x}{x^{\frac{1}{2}}} + \frac{9}{x^{\frac{1}{2}}}$$

$$f'(x) = x^{\frac{3}{2}} + 6x^{\frac{1}{2}} + 9x^{-\frac{1}{2}}$$

STEP 2

$$f(x) = \frac{x^{\frac{5}{2}}}{\frac{5}{2}} + \frac{6x^{\frac{3}{2}}}{\frac{3}{2}} + \frac{9x^{\frac{1}{2}}}{\frac{1}{2}} + c$$

INTEGRATE EACH TERM

REMEMBER "+c"!

$$f(x) = \frac{2}{5}x^{\frac{5}{2}} + 4x^{\frac{3}{2}} + 18x^{\frac{1}{2}} + c$$

Your notes

STEP 3 $f(1) = \frac{2}{5} + 4 + 18 + c = 25$ $\boxed{\text{FIND } c}$

$c = \frac{13}{5}$

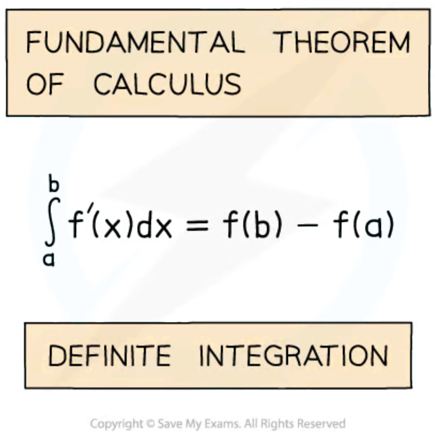
$\therefore\ f(x) = \frac{2}{5}x^{\frac{5}{2}} + 4x^{\frac{3}{2}} + 18x^{\frac{1}{2}} + \frac{13}{5}$

8.1.3 Definite Integration

Definite Integration

What is definite integration?

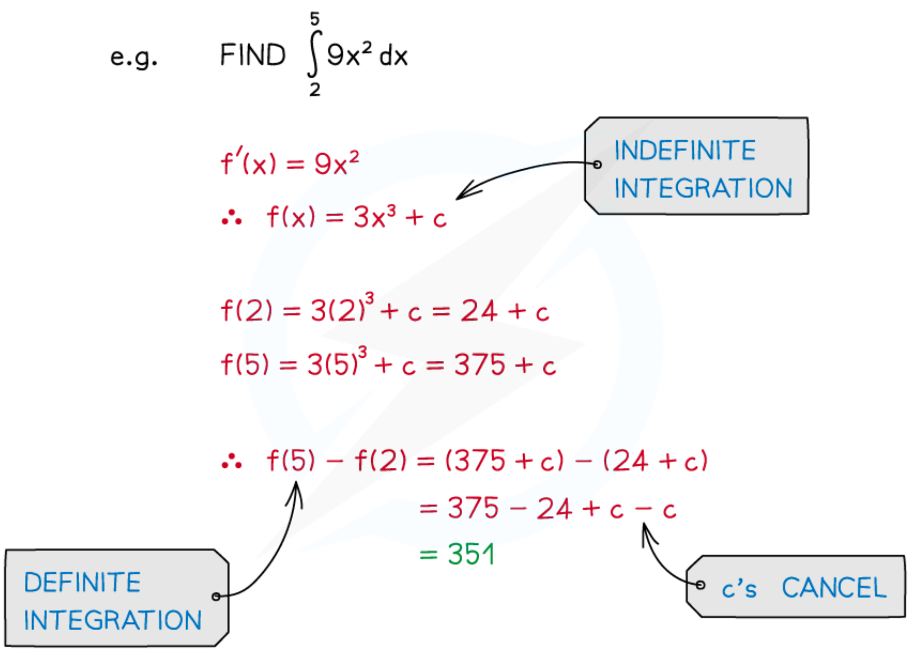
- **Definite Integration** occurs in an alternative version of the Fundamental Theorem of Calculus
- This version of the Theorem is the one referred to by most AS/A level textbooks/websites



- **a** and **b** are called limits
  - **a** is the lower limit
  - **b** is the upper limit
- **f'(x)** is the **derivative** of **f(x)**

What happened to c, the constant of integration?

Your notes



- "**+c**" would appear in both **f(a)** and **f(b)**
  - Since we then calculate **f(b)** – **f(a)** they cancel each other out
  - There would be a "**+c**" from **f(b)** and a **–**"**+c**" from **f(a)**
- So "**+c**" is not included with definite integration

How do I find a definite integral?

- STEP 1: If not given a name, call the integral
  - This saves you having to rewrite the whole integral every time!
- STEP 2: If necessary rewrite the integral into a more easily integrable form
  - Not all functions can be integrated directly
- STEP 3: Integrate without applying the limits
  - Notation: use square brackets [ ] with limits placed after the end bracket
- STEP 4: Substitute the limits into the function and calculate the answer

Your notes

e.g. EVALUATE $\int_1^9 \left(2x^2 + \frac{3}{x^2} - 4\sqrt{x}\right) dx$

STEP 1 NAME THE INTEGRAL, I

LET $I = \int_1^9 \left(2x^2 + \frac{3}{x^2} - 4\sqrt{x}\right) dx$

STEP 2 REWRITE IN A MORE EASILY INTEGRABLE FORM

$I = \int_1^9 \left(2x^2 + 3x^{-2} - 4x^{\frac{1}{2}}\right) dx$

STEP 3 INTEGRATE

$I = \left[\frac{2}{3}x^3 - 3x^{-1} - \frac{8}{3}x^{\frac{3}{2}}\right]_1^9$

- USE SQUARE BRACKETS
- LIMITS AT END

STEP 4 SUBSTITUTE LIMITS AND CALCULATE

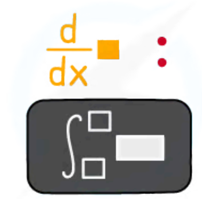
$I = \left(486 - \frac{1}{3} - 72\right) - \left(\frac{2}{3} - 3 - \frac{8}{3}\right)$

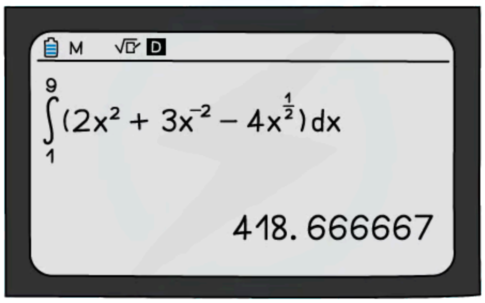
$I = \frac{1256}{3}$

Using a calculator

- Advanced scientific calculators can work out the values of definite integrals
- The button will look similar to:

Your notes





- (Note how the calculator did not return the exact value $\left(\frac{1256}{3}\right)$ of the integral)

Examiner Tip

- Look out for questions that ask you to find an **indefinite** integral in one part (so "**+c**" needed), then in a later part use the same integral as a **definite** integral (where "**+c**" is not needed).

Your notes

Worked example

Find the value of

$$\int_2^4 3x(x^2-2)\,dx$$

Start by expanding the brackets inside the integral

$$\int_2^4 (3x^3-6x)\,dx$$

Integrate as usual (here it's a 'powers of x' integration)

Write the answer in square brackets with the integration limits outside

$$\int_2^4 (3x^3-6x)\,dx = \left[3\left(\frac{x^{3+1}}{3+1}\right)-6\left(\frac{x^{1+1}}{1+1}\right)\right]_2^4$$
$$= \left[\frac{3}{4}x^4-3x^2\right]_2^4$$

Now substitute 4 into that function
And subtract from it the function with 2 substituted in

$$\left[\frac{3}{4}x^4-3x^2\right]_2^4 = \left(\frac{3}{4}(4)^4-3(4)^2\right)-\left(\frac{3}{4}(2)^4-3(2^2)\right)$$
$$=(192-48)-(12-12)$$
$$=144-0$$
$$=144$$

$$\int_2^4 3x(x^2-2)\,dx = 144$$

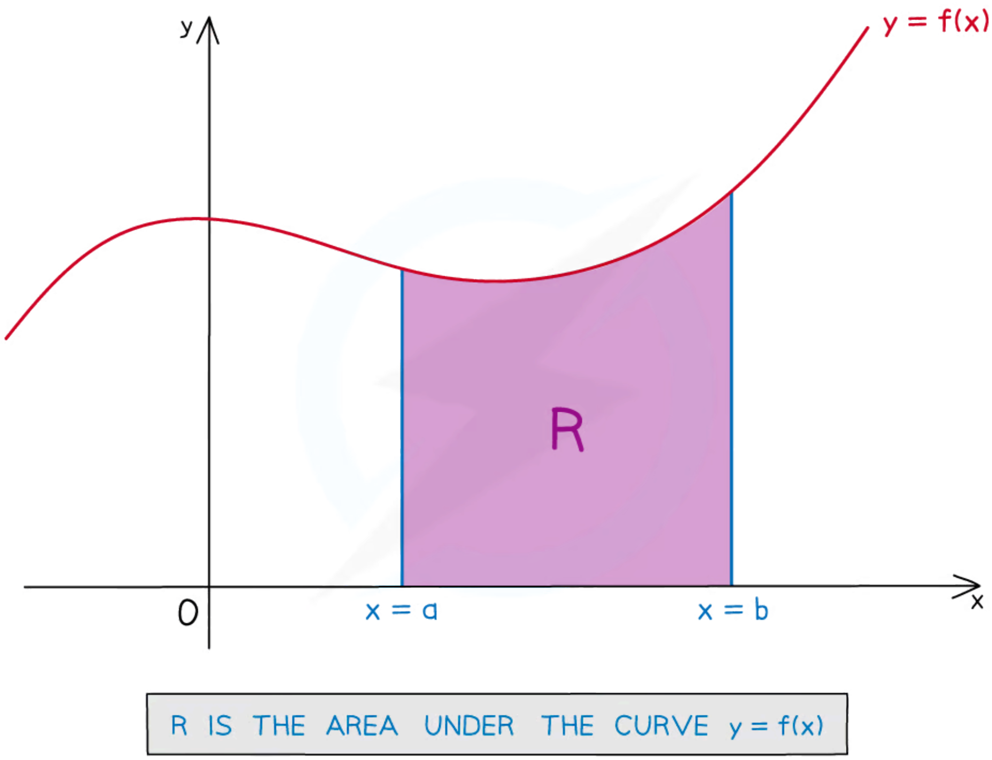
Your notes

8.1.4 Area Under a Curve

Area Under a Curve

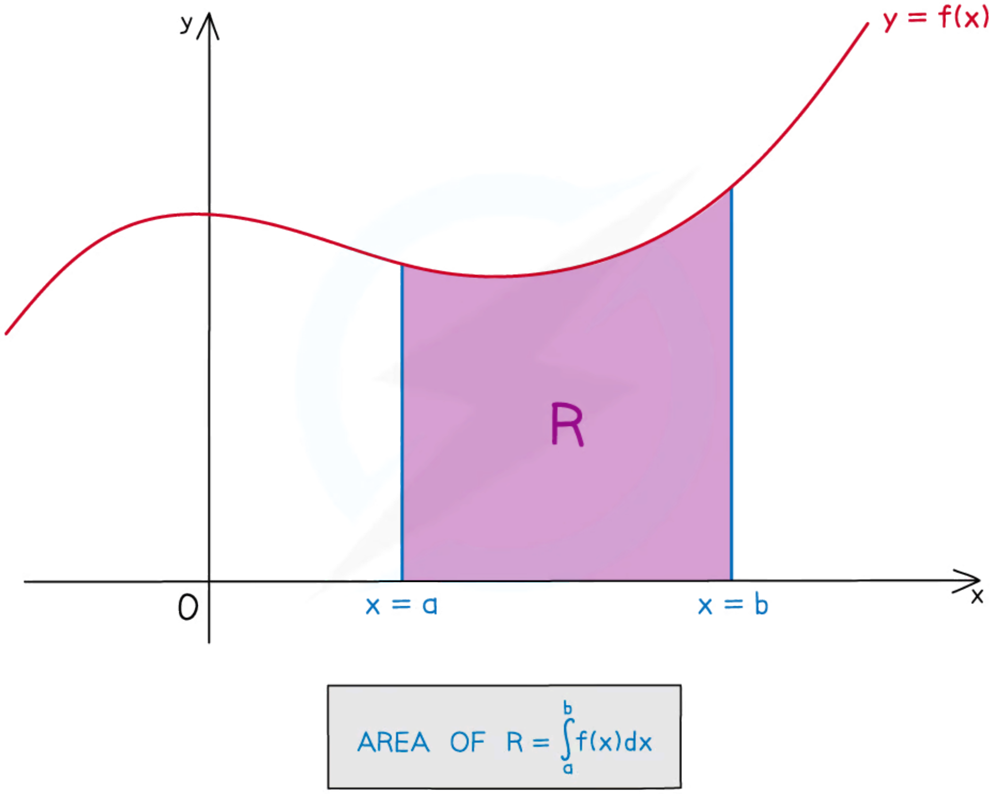
What does area under a curve mean?

- The phrase "area under a curve" refers to the area bounded by ...
  - ... the **x**-axis
  - ... the graph of **y = f(x)**
  - ... the vertical line **x = a**
  - ... the vertical line **x = b**



How do I find the area under a curve?

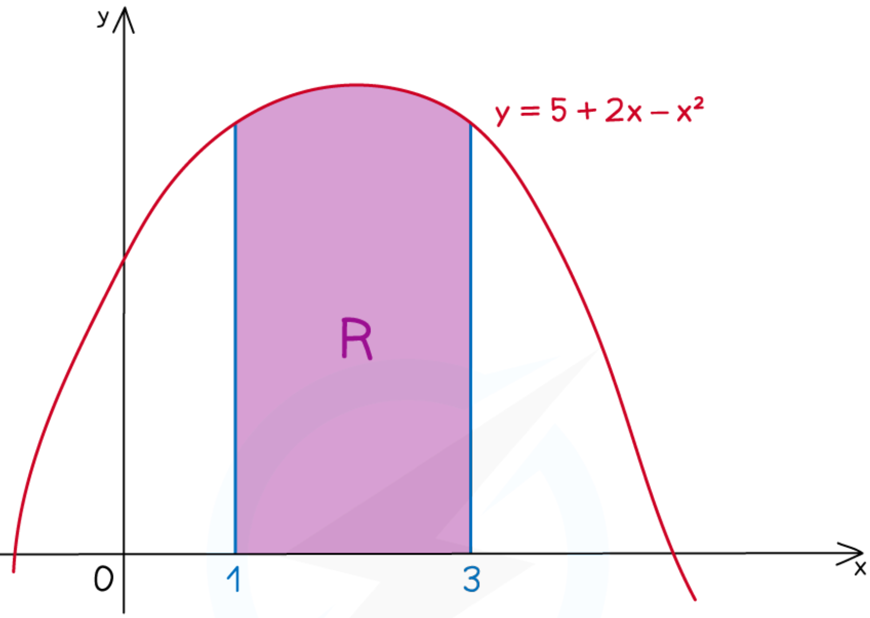
Your notes



- The value from **definite integration** is equal to the "area under a curve"

Your notes

e.g.



FIND THE AREA OF THE SHADED REGION.

$$\text{AREA } = \int_1^3 (5 + 2x - x^2)\,dx$$

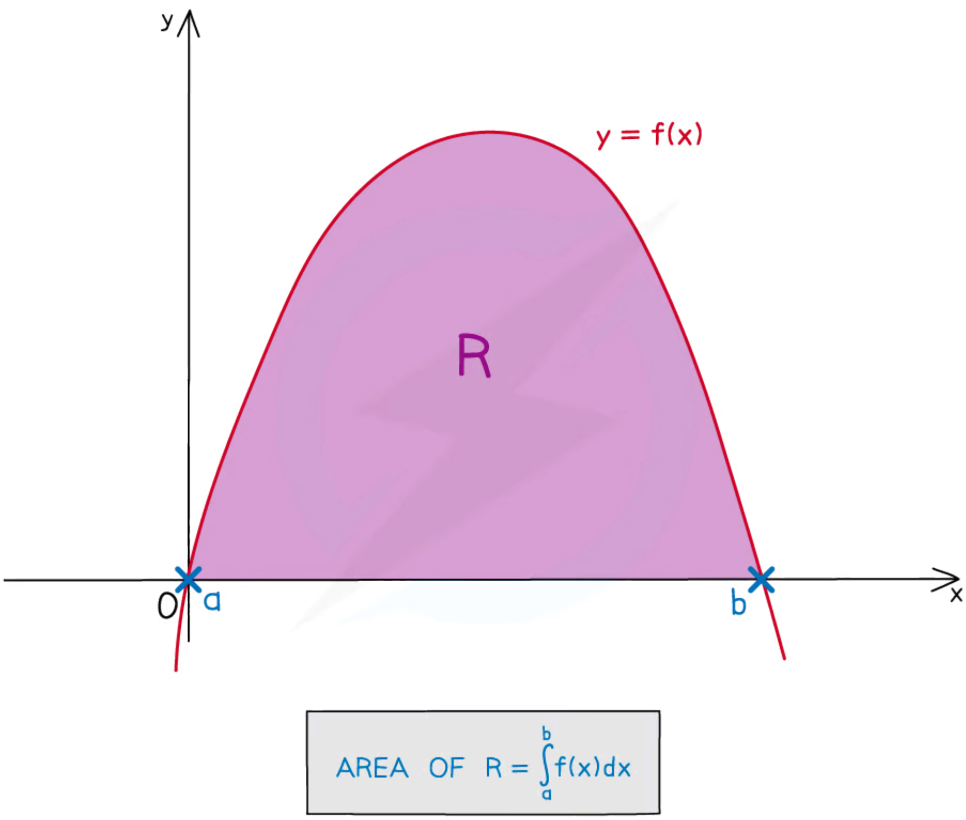
$$= \left[5x + x^2 - \frac{1}{3}x^3\right]_1^3$$

$$= (15 + 9 - 9) - (5 + 1 - \frac{1}{3})$$

$$\text{AREA } = \frac{28}{3} \text{ SQUARE UNITS}$$

What if I am not told the limits?

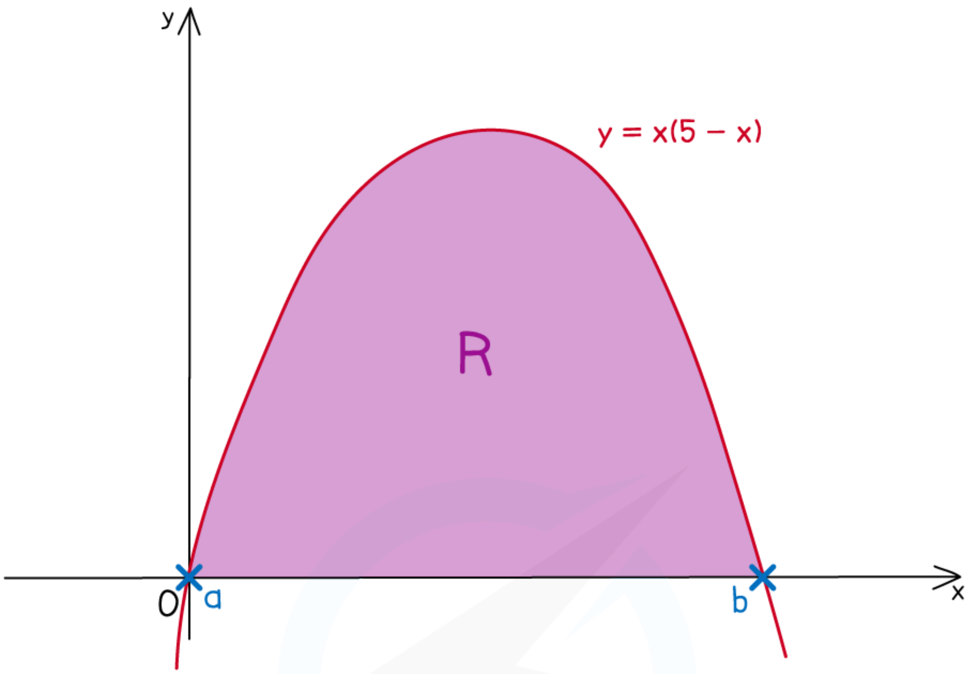
Your notes



- If limits are not provided they will be the **x**-axis intercepts
  - Set **y = 0** and solve the equation to find the **x**-axis intercepts

Your notes

e.g. FIND THE SHADED AREA MARKED R IN THE DIAGRAM BELOW



$x(5-x)=0$ ← SET $y=0$ FOR x–AXIS INTERCEPTS

$x=0 \quad x=5$

$\therefore \quad a=0 \quad b=5$

EXPAND SO "INTEGRATE–ABLE"

AREA OF R $= \int_0^5 (5x - x^2)\,dx$

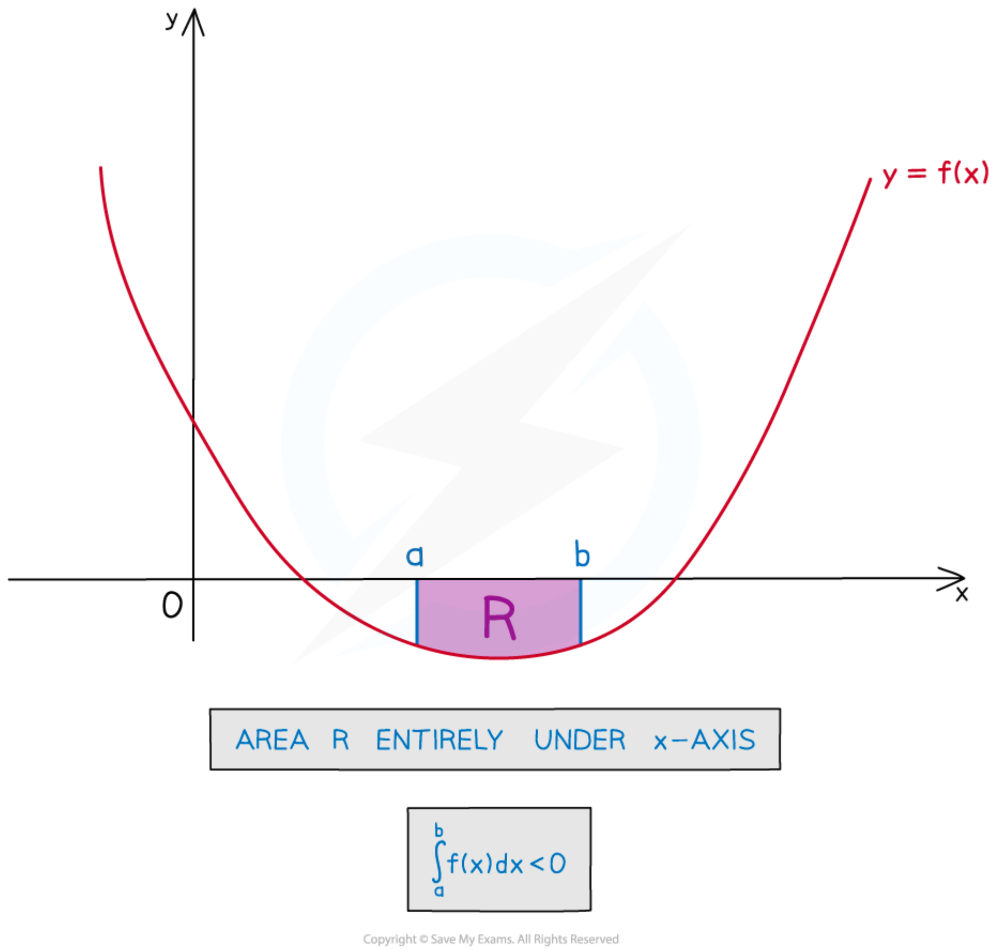
$$= \left[\frac{5x^2}{2} - \frac{x^3}{3}\right]_0^5$$

$$= \frac{125}{2} - \frac{125}{3} - (0-0)$$

AREA OF R $= \frac{125}{6}$ SQUARE UNITS

Your notes

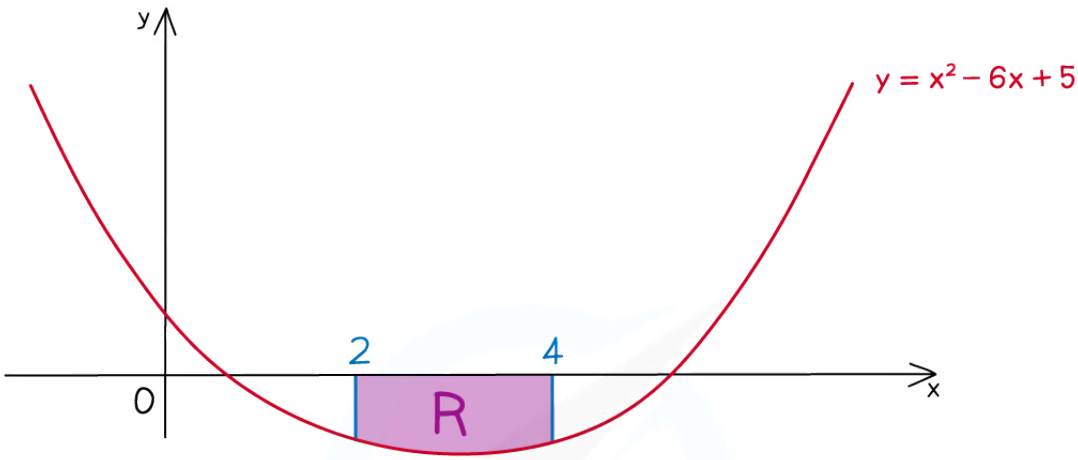
Negative areas



- If the area lies underneath the **x**-axis the value of the integral will be negative
- An **area** cannot be negative, so take the **modulus** of the integral

Your notes

e.g. FIND THE SHADED AREA R IN THE DIAGRAM BELOW



$$\int_2^4 (x^2 - 6x + 5)\,dx = \left[\frac{x^3}{3} - 3x^2 + 5x\right]_2^4$$

WRITE AS AN INTEGRAL

$$= \left(\frac{64}{3} - 48 + 20\right) - \left(\frac{8}{3} - 12 + 10\right)$$

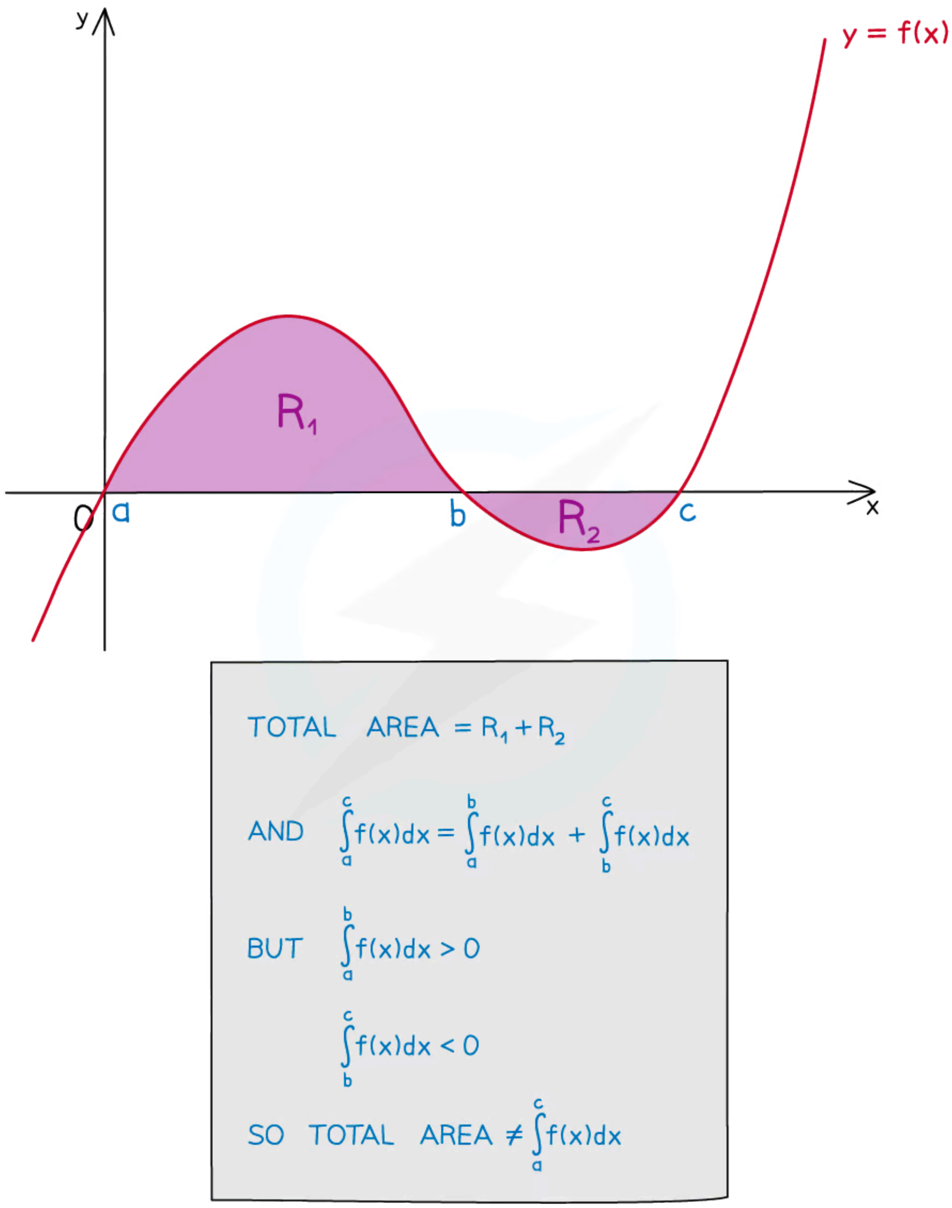
$$= -\frac{22}{3}$$

$\therefore$ AREA OF $R = \frac{22}{3}$ SQUARE UNITS

$$\left|-\frac{22}{3}\right| = \frac{22}{3}$$

What if the area is made up of more than one section?

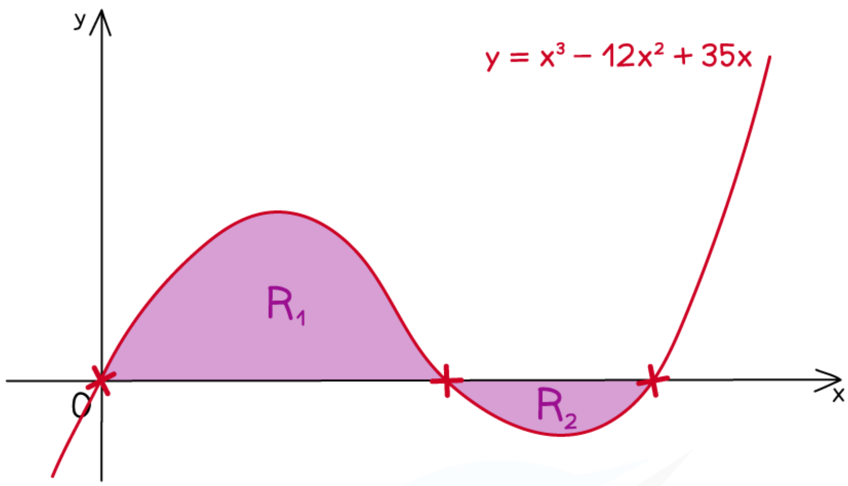
Your notes



- Be careful when one section is above and one section is below the **x**-axis
  - You will need a separate integral for each section, BUT...
  - ...One section's integral will be **negative**
  - ...One section's integral will be **positive**
  - So you'll need to take the **modulus** before adding to find the total area

Your notes

e.g. FIND THE TOTAL OF THE SHADED AREAS



FACTORISE TO SOLVE $f(x) = 0$

$$x^3 - 12x^2 + 35x = x(x^2 - 12x + 35)$$
$$= x(x-7)(x-5) = 0$$

$\therefore$ x-AXIS INTERCEPTS ARE 0, 5 AND 7

$$\int_0^5 (x^3 - 12x^2 + 35x)dx = \frac{375}{4}$$

$$\int_5^7 (x^3 - 12x^2 + 35x)dx = -8$$

FIND INTEGRALS SEPARATELY

USE CALCULATOR ?

$\therefore$ AREA OF $R_1 = \frac{375}{4}$

AREA OF $R_2 = 8$

MODULUS NEEDED

TOTAL SHADED AREA $= \frac{375}{4} + 8 = \frac{407}{4}$ SQUARE UNITS

Examiner Tip

- Add information to any diagram provided in the question, as well as axes intercepts and values of limits.
- Mark and shade the area you're trying to find, and if no diagram is provided, **sketch** one!

Your notes

Worked example

Your notes

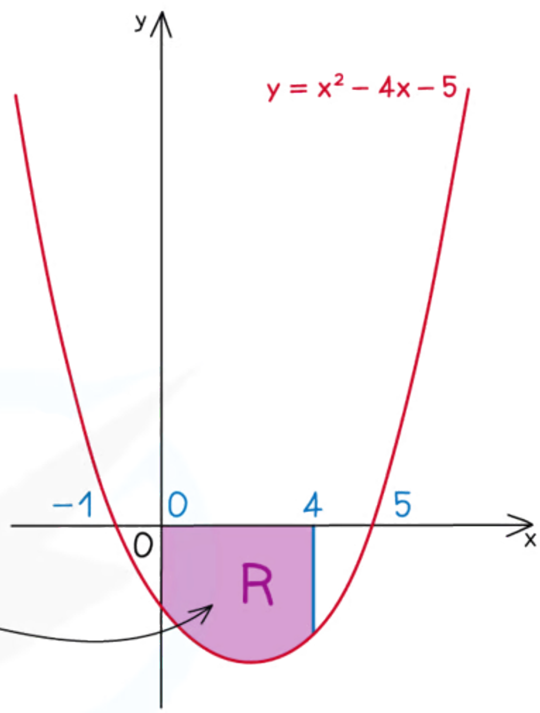
Your notes

Find the area bounded by the curve with equation $y = x^2 - 4x - 5$, the x-axis and the lines $x = 0$ and $x = 4$.

SKETCH AS NO DIAGRAM GIVEN

$x^2 - 4x - 5 = (x-5)(x+1) = 0$

$\therefore$ x–AXIS INTERCEPTS ARE –1 AND 5



EXPECT NEGATIVE AS BELOW x–AXIS

$$\int_0^4 (x^2 - 4x - 5)\,dx = \left[\frac{x^3}{3} - 2x^2 - 5x\right]_0^4$$

WRITE AS AN INTEGRAL

$$= \left(\frac{64}{3} - 32 - 20\right) - (0 - 0 - 0)$$

$$= -\frac{92}{3}$$

$\therefore$ AREA REQUIRED $= \frac{92}{3}$ SQUARE UNITS

MODULUS

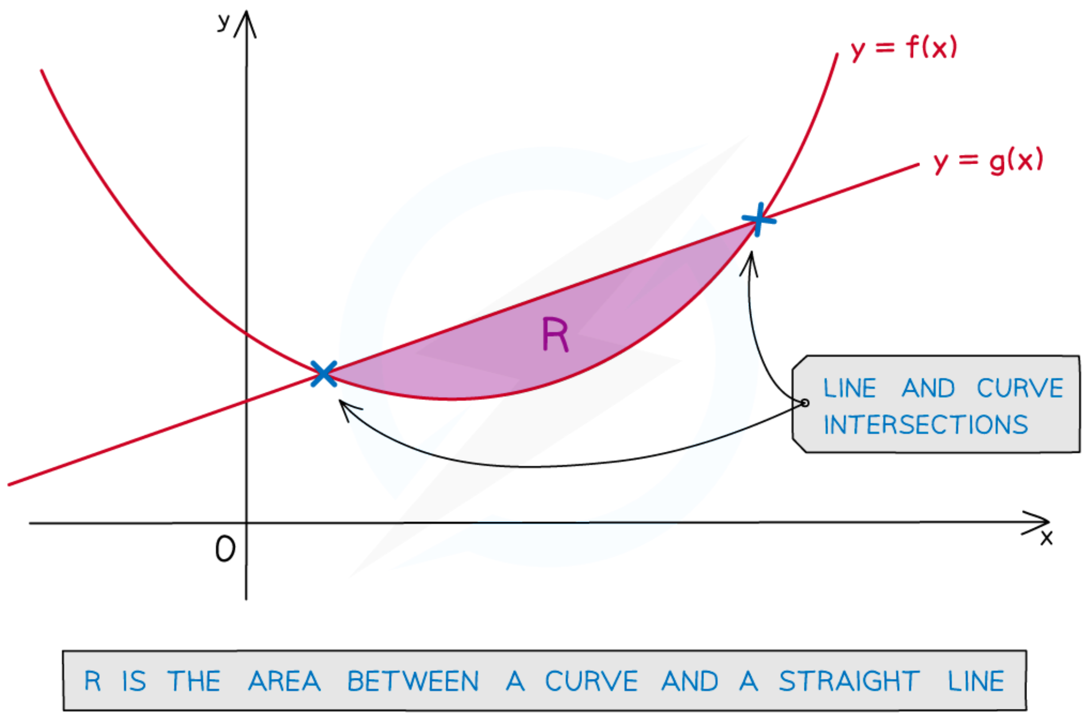
Your notes

Your notes

8.1.5 Area between a curve and a line

Area between a curve and a line

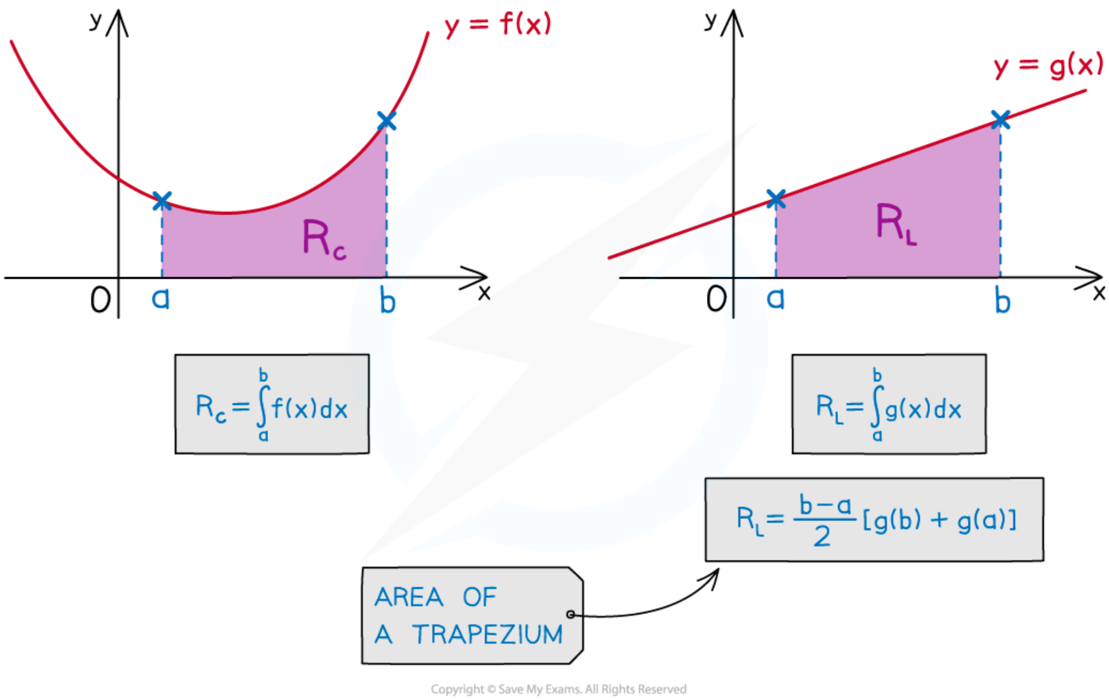
What is the area between a curve and a straight line?



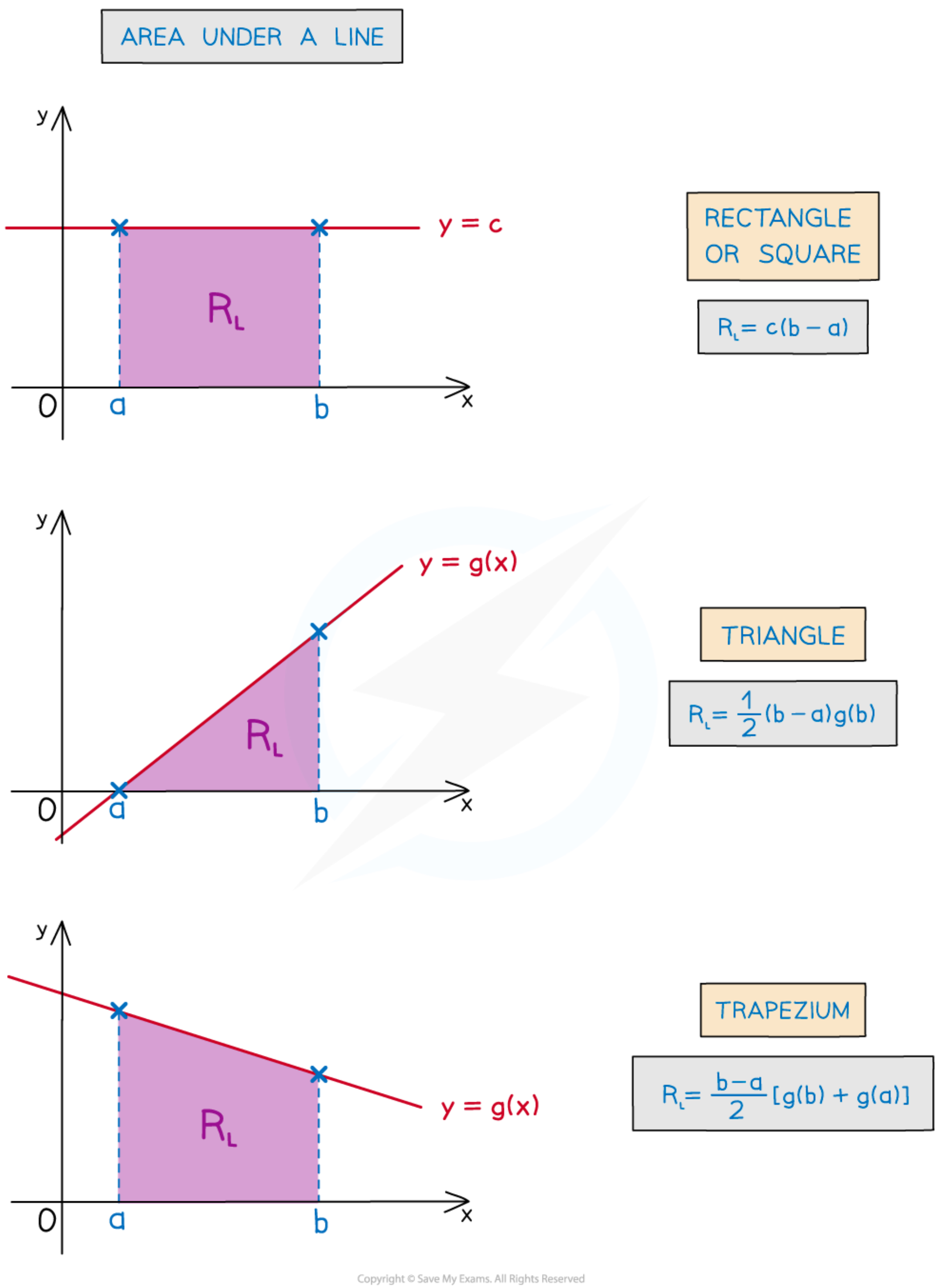
- This is the area enclosed by a curve and a line between their **intersections**
- The intersections may need to be found using simultaneous equations

How do I find the area between a curve and a line?

- The area enclosed will be the **difference** between ...
  - the **area under the curve** and
  - the **area under** the line
- These can be found separately



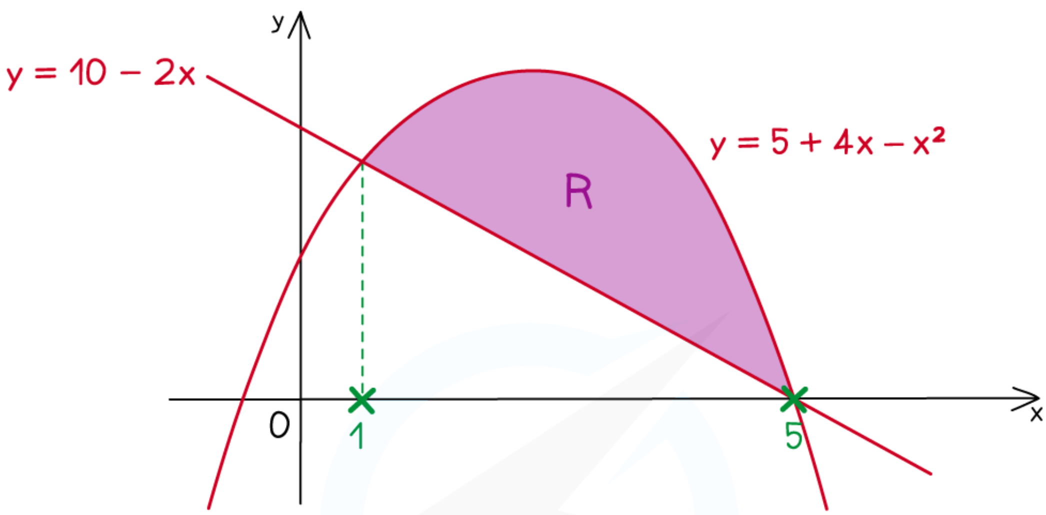
- There may be easier ways to find the area under a line



- Once the areas are found a question can be answered in full

Your notes

e.g. FIND THE AREA MARKED R IN THE DIAGRAM BELOW.



STEP 1 FIND THE INTERSECTIONS OF CURVE AND LINE

$10 - 2x = 5 + 4x - x^2$

$x^2 - 6x + 5 = 0$

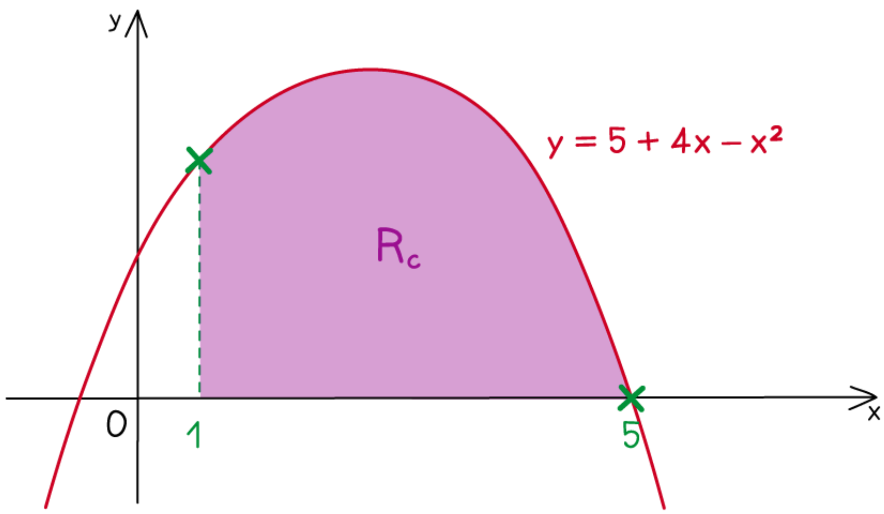
$(x - 5)(x - 1) = 0$

$x = 5, \quad x = 1$

SOLVE SIMULTANEOUSLY

Your notes

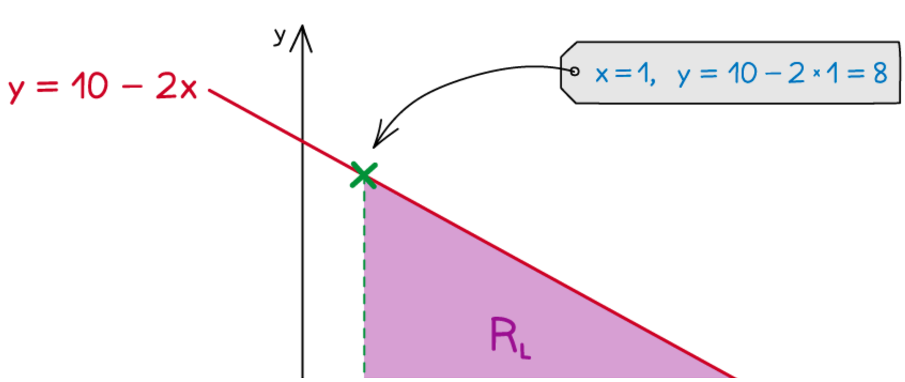
STEP 2 FIND AREA UNDER THE CURVE



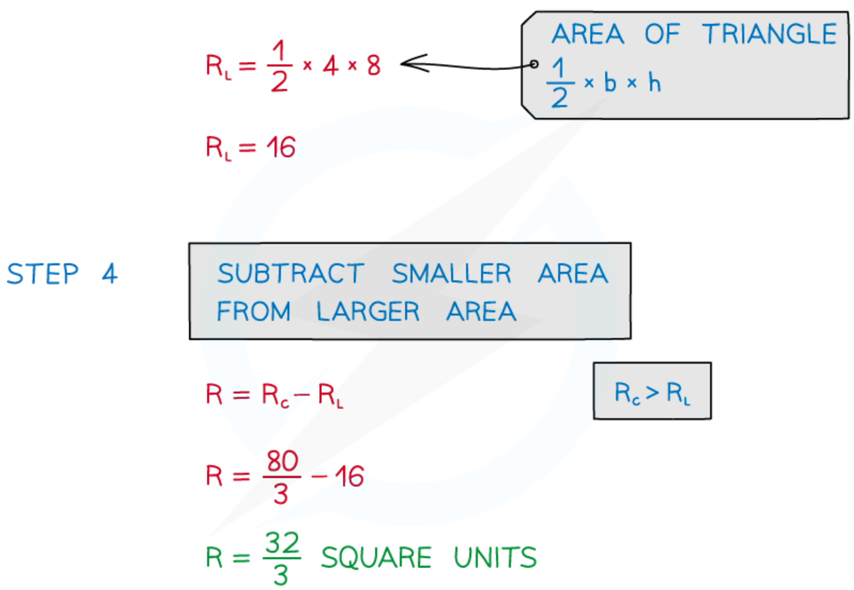
$$R_c = \int_1^5 (5 + 4x - x^2)\,dx$$ DEFINITE INTEGRATION

$$R_c = \frac{80}{3}$$ USE CALCULATOR

STEP 3 FIND AREA UNDER THE LINE



Your notes

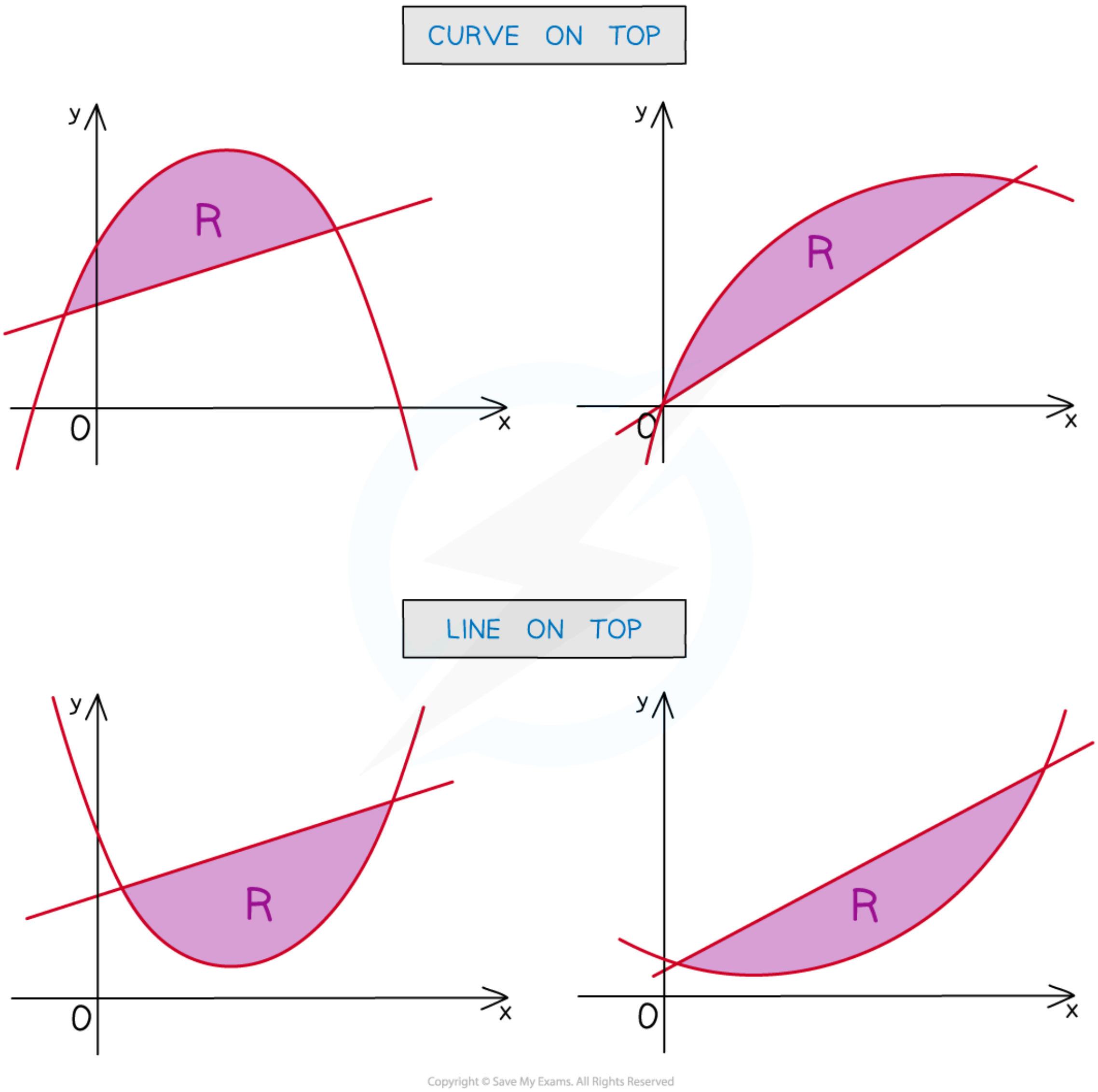


- STEP 1: Find the intersections of the line and the curve
- STEP 2: Find the area under a curve, $\mathbf{R_C}$, using definite integration
- STEP 3: Find the area under a line, $\mathbf{R_L}$, either using definite integration **or** the area formulae for basic shapes
- STEP 4: To find the area, **R**, between the curve and the line **subtract** the **smaller** area **from** the **larger** area
  - If curve on top this will be $\mathbf{R_C - R_L}$
  - If line on top this will be $\mathbf{R_L - R_C}$

Alternative method

- An alternative method allows subtraction of the two functions first
- But be extremely careful as to whether the curve or the line is on top

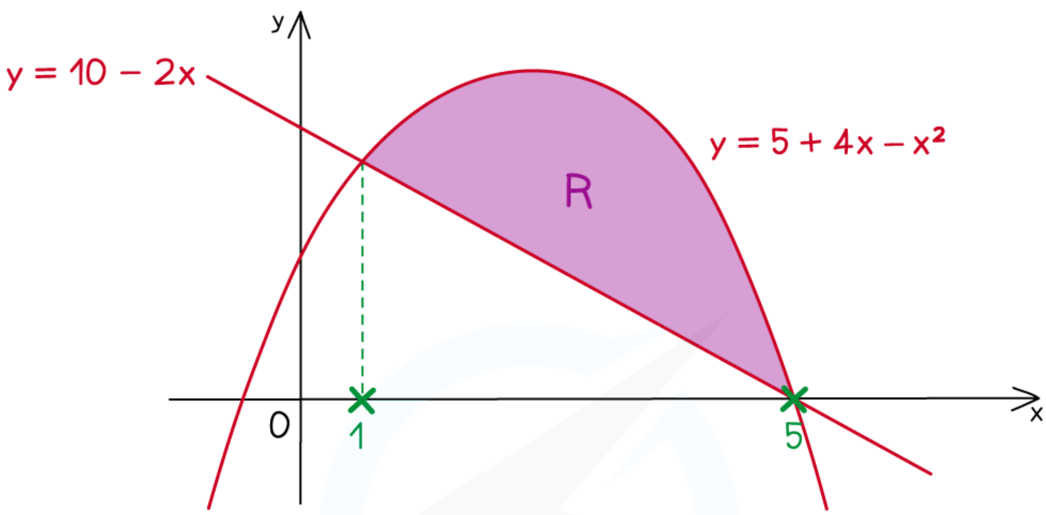
Your notes



- STEP 1: Find the intersections of the line and the curve
- STEP 2: Subtract the functions algebraically
  - Ensure this is done the correct way round!
- STEP 3: Find the area, **R**, between the curve and the line using **definite integration**

Your notes

e.g. FIND THE AREA MARKED R IN THE DIAGRAM BELOW.



STEP 1 FIND THE INTERSECTIONS OF CURVE AND LINE

$10 - 2x = 5 + 4x - x^2$

$x^2 - 6x + 5 = 0$

$(x - 5)(x - 1) = 0$

$x = 5, \quad x = 1$

SOLVE SIMULTANEOUSLY

Your notes

STEP 2 SUBTRACT THE FUNCTIONS

$$5 + 4x - x^2 - (10 - 2x) = -x^2 + 6x - 5$$

CURVE ON TOP

SUBTRACT "ALL" OF LINE

STEP 3 DEFINITE INTEGRATION FINDS R

$$R = \int_1^5 (-x^2 + 6x - 5)\,dx$$

$$R = \frac{32}{3} \text{ SQUARE UNITS}$$

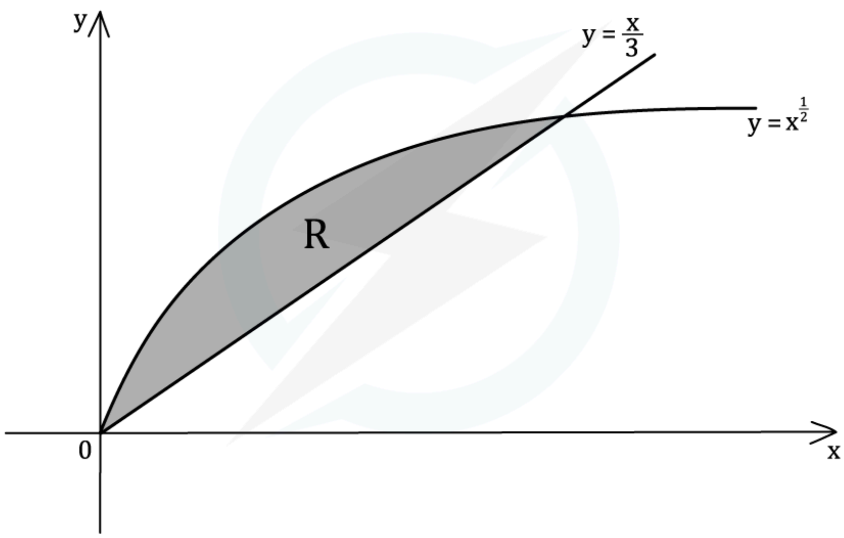
Examiner Tip

- Add information to any diagram provided
- Add axes intercepts, as well as intercepts between lines and curves
- Mark and shade the area you're trying to find
- If no diagram provided, **sketch** one!

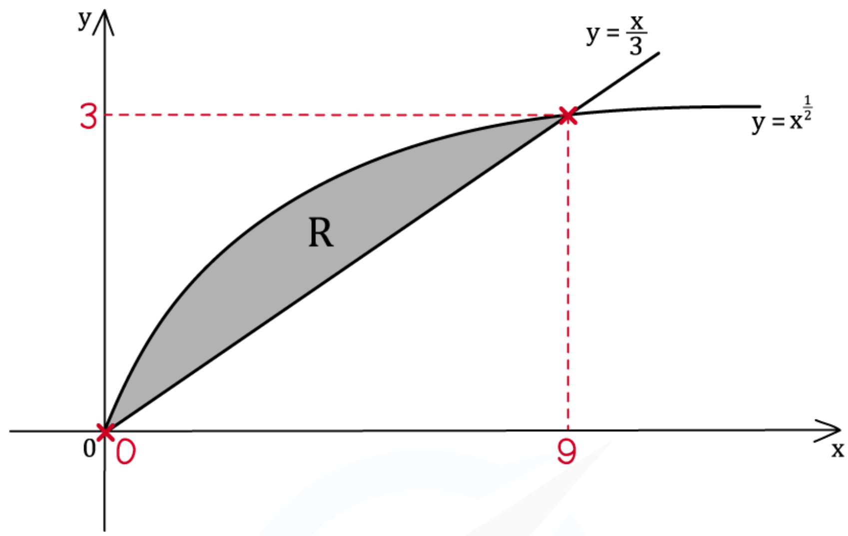
Your notes

Worked example

Find the area indicated by R in the diagram below.



Your notes



STEP 1 FIND THE INTERSECTIONS

$\frac{x}{3} = x^{\frac{1}{2}}$ ← SIMULTANEOUS EQUATIONS

$\frac{x^2}{9} = x$

$x^2 = 9x$ ← QUADRATIC

$x^2 - 9x = 0$

$x(x-9) = 0$

$x = 0$ AND $x = 9$ ← ADD TO DIAGRAM

Your notes

STEP 2 FIND THE AREA UNDER THE CURVE

$$R_c = \int_0^9 x^{\frac{1}{2}}\,dx = \left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^9$$

$$= \left[\frac{2}{3}x^{\frac{3}{2}}\right]_0^9$$

$$= \left(\frac{2}{3}(9)^{\frac{3}{2}}\right) - (0)$$

$$R_c = 18$$

STEP 3 FIND THE AREA UNDER THE LINE

$$R_L = \frac{1}{2} \times 9 \times 3$$

TRIANGLE BASE = 9
HEIGHT = 3

$$R_L = \frac{27}{2}$$

STEP 4 SUBTRACT SMALL FROM LARGE

$$R = R_c - R_L$$

CURVE IS ON TOP

$$R = 18 - \frac{27}{2}$$

$R = \frac{9}{2}$ SQUARE UNITS