Edexcel A Level Maths: Pure

Your notes

10.1 Solving Equations

Contents

10.1.1 Change of Sign

Your notes

Change of Sign

What does a change of sign mean?

- A sign change between f(a) to f(b) means there must be a root between a and b
- When an equation cannot be solved using the usual **analytical methods**, we can use change of sign to find **approximate solutions**
- We can say a root lies within a particular interval or is correct to a given accuracy
- Bounds can be used to show a root is correct
- Using sign change to find a root is only appropriate for continuous functions in a small interval

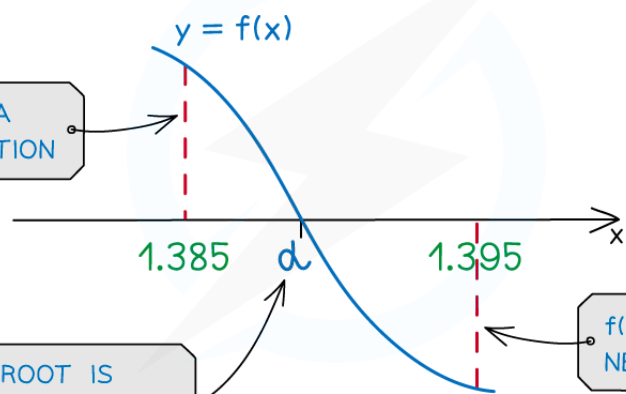
FOR A **CONTINUOUS** FUNCTION f(x)

FIND f(a) AND f(b) BY SUBSTITUTING GIVEN NUMBERS INTO THE FUNCTION
IF THE TWO ANSWERS HAVE **DIFFERENT SIGNS** THERE IS A **ROOT** BETWEEN THEM

e.g. SHOW THAT $f(x) = 27 - 5\tan x$ HAS A ROOT OF 1.39 CORRECT TO 2 DECIMAL PLACES.

THERE IS A ROOT, ie. $f(x) = 0$ BETWEEN 1.385 AND 1.395

f(1.385) GIVES A POSITIVE SOLUTION



f(1.395) GIVES A NEGATIVE SOLUTION

THE ROOT IS SHOWN BY THE SYMBOL α (ALPHA)

TO SHOW A ROOT IS CORRECT, FIND ITS **UPPER AND LOWER BOUND** AND SHOW THERE IS A **SIGN CHANGE** BETWEEN THEM

ALL VALUES IN THIS INTERVAL ROUND TO 1.39, SO $\alpha = 1.39$ TO 2 DECIMAL PLACES (USING BOUNDS $1.385 \leqslant \alpha < 1.395$)

Examiner Tip

- Remember this will only work if the function is continuous and the interval is small enough.
- Sign change questions may be part of bigger numerical methods questions.

Worked example

The graph of function $f(x) = 3xe^x - 5$ crosses the x-axis at the point P (p, 0).

a) Show that $0.7 < p < 0.8$.

b) Show that $p = 0.771$ to 3 decimal places.

$f(x) = 3xe^x - 5$

a) $f(0.7) = 3 \times 0.7e^{0.7} - 5 = -0.771119...$ NEGATIVE

$f(0.8) = 3 \times 0.8e^{0.8} - 5 = 0.34129...$ POSITIVE

SUBSTITUTE VALUES INTO f(x)

CHANGE OF SIGN MEANS $0.7 < p < 0.8$

b) $p = 0.771$ (3dp)

$0.7705 \leqslant p < 0.7715$

FIND UPPER AND LOWER BOUND OF p

$f(0.7705) = -0.0052...$ NEGATIVE

$f(0.7715) = 0.0062...$ POSITIVE

SUBSTITUTE BOUNDS INTO f(x)

SOLUTION MUST BE $p = 0.771$ TO 3dp

Your notes

10.1.2 Change of Sign Failure

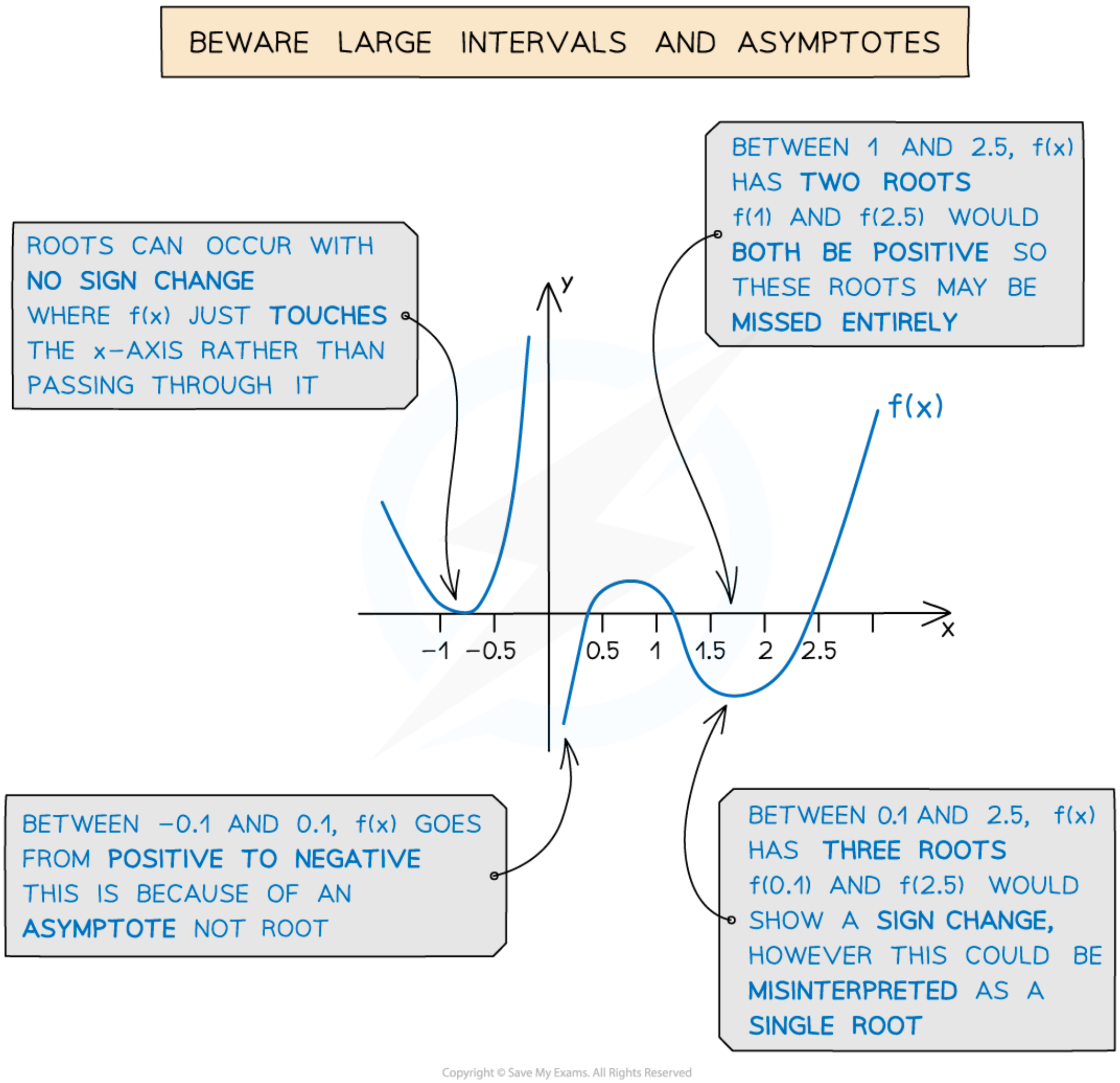
Your notes

Change of Sign Failure

When can the sign change method fail?

- There are certain circumstances where a **change of sign** fails to appropriately reflect the number or existence of roots
- If the interval is **too large** there may be more than one root within it
  - An **even number of roots** mean roots are missed entirely as a sign change is not identified
  - An **odd number of roots (> 1)** may mean not ALL roots are identified
- In a discontinuous a sign change may occur which is not caused by a root but by an asymptote
- If a function **touches the x-axis** but does not pass through it there would be a root but no sign change

Your notes


BEWARE LARGE INTERVALS AND ASYMPTOTES
ROOTS CAN OCCUR WITH NO SIGN CHANGE WHERE f(x) JUST TOUCHES THE x-AXIS RATHER THAN PASSING THROUGH IT
BETWEEN 1 AND 2.5, f(x) HAS TWO ROOTS f(1) AND f(2.5) WOULD BOTH BE POSITIVE SO THESE ROOTS MAY BE MISSED ENTIRELY
y
f(x)
x
−1
−0.5
0.5
1
1.5
2
2.5
BETWEEN −0.1 AND 0.1, f(x) GOES FROM POSITIVE TO NEGATIVE THIS IS BECAUSE OF AN ASYMPTOTE NOT ROOT
BETWEEN 0.1 AND 2.5, f(x) HAS THREE ROOTS f(0.1) AND f(2.5) WOULD SHOW A SIGN CHANGE, HOWEVER THIS COULD BE MISINTERPRETED AS A SINGLE ROOT
Copyright © Save My Exams. All Rights Reserved

Your notes

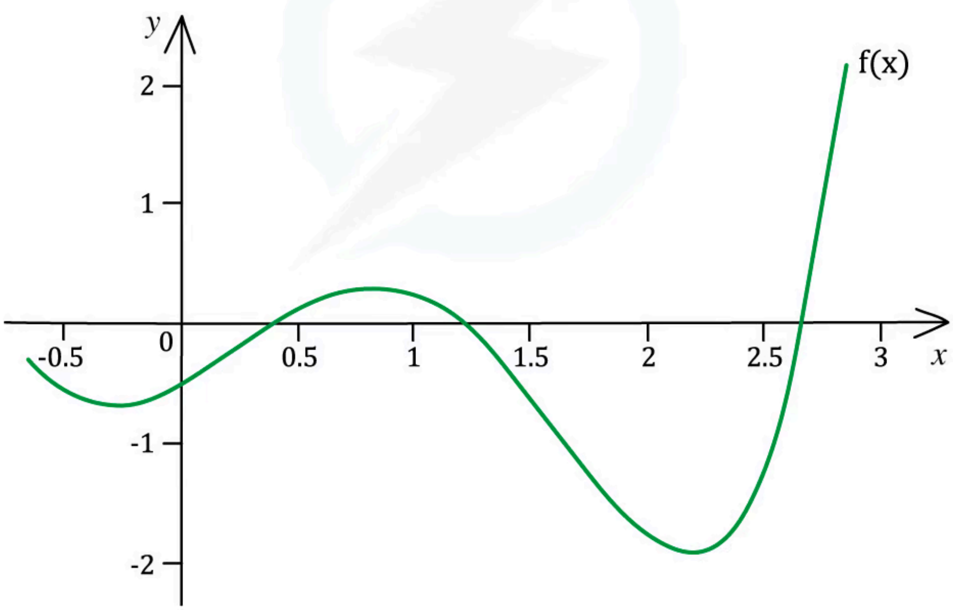
Worked example

The graph shows the function of $f(x) = xe^x - 2x^3 - 0.5$ for $-0.5 \le x \le 3$.

a) State how many roots the equation $xe^x - 2x^3 - 0.5 = 0$ has in the interval $-0.5 \le x \le 3$.

b) Give a reason why using a sign change method to check for a root over the interval $-0.5 \le x \le 3$ will fail.

c) By using an appropriate interval find the first positive root to 1 decimal place for the function $f(x) = xe^x - 2x^3 - 0.5$.



Your notes

a) 3 ROOTS

GRAPH CROSSES x–AXIS 3 TIMES

b) INTERVAL IS TOO BIG, SIGN CHANGE WILL REFLECT 3 ROOTS NOT JUST 1

c) $0.4e^{0.4} - 2(0.4)^3 - 0.5 = -0.031...$

$0.5e^{0.5} - 2(0.5)^3 - 0.5 = 0.074...$

SIGN CHANGE METHOD WORKS BEST FOR SMALL INTERVALS

$0.4 < x < 0.5$

$0.45e^{0.45} - 2(0.45)^3 - 0.5 = 0.23$

$0.4 < x < 0.45$ $\quad x = 0.4$ (1 dp)

SIGN CHANGE BETWEEN 0.4 AND 0.45 MEANS ROOT = 0.4 (1 dp)

10.1.3 x = g(x) Iteration

Your notes

x = g(x) Iteration

What does $x_{n+1} = g(x_n)$ mean?

- When an equation cannot be solved using the usual **analytical methods**, we can still find **approximate solutions** to a certain degree of accuracy
- **Iteration** is one way to do this, by repeatedly using each answer as the new starting value for a function, we can achieve an ever more accurate answer
- Iterations are shown using the notation $x_{n+1} = g(x_n)$
- This is a **recurrence relation** where, starting with a number (x_n), we will get an answer x_{n+1} which we can then reuse in the original function
- Equations need to be rearranged into an **iterative formula** – ie. the form $x = g(x)$

Your notes

REARRANGING

WHEN **REARRANGING AN EQUATION** INTO AN **ITERATIVE FORMULA** g(x), PAY ATTENTION TO THE **"SHOW THAT"** PART OF THE QUESTION TO ENSURE YOU REARRANGE IT INTO THE CORRECT FORMAT

SHOW THAT $x^3 - x^2 - 9 = 0$

CAN BE REARRANGED INTO $x = \sqrt{\dfrac{9}{x-1}}$

$$x^3 - x^2 - 9 = 0$$

SEPARATING LIKE TERMS CAN BE A GOOD PLACE TO START AS 9 IS ON ITS OWN IN THE ITERATIVE FORMULA

$$\Rightarrow x^3 - x^2 = 9$$

FACTORISING OFTEN HELPS, ALSO THE QUESTION HAS A $\sqrt{\text{SQUARE ROOT}}$ SO GETTING x^2 ON ITS OWN IS HELPFUL

$$x^2(x-1) = 9$$

$$x^2 = \frac{9}{x-1}$$

DIVIDING BY $x-1$ THAN SQUARE ROOTING BOTH SIDES LEAVES THE APPROPRIATE ITERATIVE FORMULA

$$x = \sqrt{\frac{9}{x-1}}$$

$x^3 - x^2 - 9 = 0$ ALSO REARRANGES TO $\sqrt{x = x^3 - 9}$ SO BE CAREFUL TO GET THE CORRECT FUNCTION

ITERATION

GIVEN A STARTING VALUE x_0, YOU CAN **SUBSTITUTE** EACH ANSWER INTO THE **ITERATIVE FORMULA.**
THE NOTATION x_1, x_2 ... IS USED TO **TRACK EACH ITERATION.**
ANSWERS MAY **CONVERGE** ON A PARTICULAR NUMBER OR **DIVERGE.**

USING THE ITERATING FORMULA

$x_{n+1} = \sqrt{\dfrac{9}{x_n - 1}}$ STARTING $x_0 = 2$

a) FIND x_1 AND x_2

b) SOLVE $x^3 - x^2 - 9 = 0$ TO 2dp

a) $x_1 = \sqrt{\dfrac{9}{2-1}} = 3$

$x_2 = \sqrt{\dfrac{9}{3-1}} = 2.121...$

SUBSTITUTE x_0 INTO FORMULA AND REPEAT WITH NEW ANSWER

b) $x_3 = 2.833...$

$x_4 = 2.215...$

CONTINUE...

$x = 2.47$ (2 dp)

CONTINUE THIS PROCESS, THE "ANS" BUTTON ON YOUR CALCULATOR IS VERY HELPFUL FOR THIS $\sqrt{\dfrac{9}{ANS - 1}}$

STOP ONCE THE ANSWER IS THE SAME TO TWO DECIMAL PLACES WHEN ROUNDED

Iteration diagrams

- Iterations can be shown on diagrams called **staircase** or **cobweb** diagrams
- These can be drawn by **plotting the graphs** of **y = x** against **y = g(x)** from your iterative formula

Your notes

Your notes

DRAWING **STAIRCASE** AND **COBWEB** DIAGRAMS

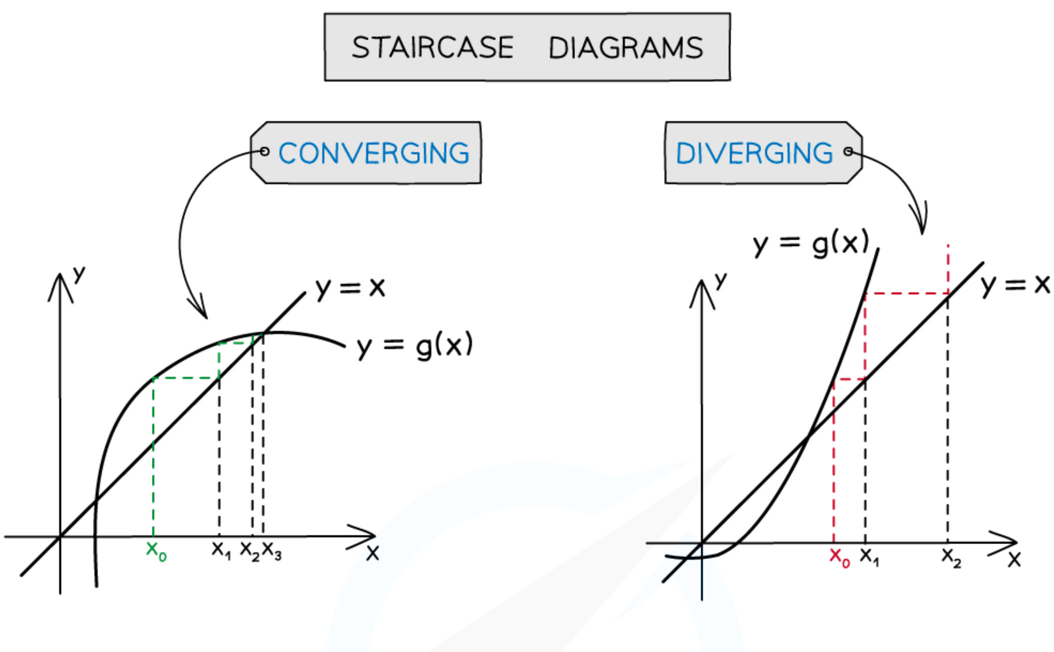
STEP 1: SKETCH GRAPHS OF $y = x$ AND $g(x)$ FROM YOUR ITERATIVE FORMULA (POINTS WHERE THE TWO GRAPHS MEET ARE ROOTS)

STEP 2: MARK x_0 ONTO THE x–AXIS AND DRAW A **VERTICAL LINE** TO MEET THE **CURVE** $y = g(x)$

STEP 3: FROM THIS POINT DRAW A **HORIZONTAL LINE** TO MEET THE **LINE** $y = x$. THIS IS x_1.

STEP 4: CONTINUE BY DRAWING ANOTHER **VERTICAL LINE** TO THE **CURVE** $y = g(x)$ THEN A HORIZONTAL LINE TO THE LINE $y = x$. EACH TIME YOU REPEAT THIS STEP IS **ANOTHER ITERATION.**

STEP 5: IF THE LINES ARE GETTING **CLOSER** TO THE ROOT THE ITERATION IS **CONVERGING,** IF THEY ARE GETTING **FURTHER** AWAY IT IS **DIVERGING.**



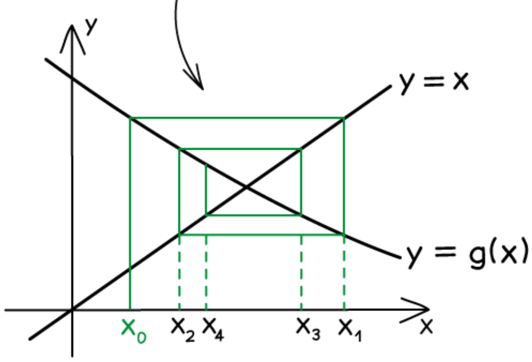
Your notes

FOR **CONVERGING DIAGRAMS**, EACH ITERATION FROM x_0 WILL GET **CLOSER TO THE ROOT** (POINT WHERE THE TWO GRAPHS INTERSECT)
FOR COBWEB DIAGRAMS THE SOLUTIONS WILL ALTERNATE ABOVE AND BELOW THE ROOT BUT THEY WILL STILL BE GETTING CLOSER EACH TIME

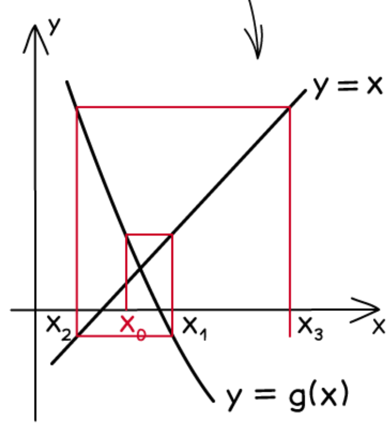
FOR **DIVERGING DIAGRAMS**, EACH ITERATION FROM x_0 WILL GET **FURTHER FROM THE ROOT**

COBWEB DIAGRAMS

CONVERGING



DIVERGING



IN GENERAL THE GRADIENT OF g(x) MUST BE BETWEEN −1 AND 1

IF THE GRADIENT OF g(x) IS **NEGATIVE** AT THE ROOTS IT WILL CREATE A **COBWEB** DIAGRAM

IF THE GRADIENT OF g(x) IS **POSITIVE** AT THE ROOTS IT WILL CREATE A **STAIRCASE** DIAGRAM

Your notes

Examiner Tip

- You must show all your steps when rearranging an equation into an iterative formula
- Working backwards can often be helpful to figure out how an equation has been rearranged but you must write your answer as if you worked forwards
- Use ANS button on your calculator to calculate repeated iterations
- Keep track of your iterations using x_2, x_3... notation
- Iteration may be part of bigger numerical methods questions

Worked example

Your notes

Your notes

a) Show that $3x^2 - x^3 + 3 = 0$ can be written in the form $x = 3 + \frac{3}{x^2}$.

b) Use the iteration formula $x_{n+1} = 3 + \frac{3}{x_n^2}$ and $x_0 = 2$ to find a solution to 2 decimal places.

c) Sketch a diagram to show the convergence of the sequence for $x_1,\ x_2$ and x_3.

a) $3x^2 - x^3 + 3 = 0$

IT CAN BE HARD TO KNOW WHERE TO START, THERE IS NO x^3 IN FINAL ANSWER. BEGIN BY MAKING THAT POSITIVE

$3 = x^3 - 3x^2$

$3 = x^2(x - 3)$

SEPARATE x TERMS TO FACTORISE

WE NEED TO TURN x^3 AND x^2 INTO x^2 AND x SO x^2 MUST BE THE FACTOR

$\frac{3}{x^2} = x - 3$

$3 + \frac{3}{x^2} = x$

REARRANGE UNTIL FINAL ITERATIVE FORMULA IS REACHED (SOMETIMES WORKING BACKWARDS WILL HELP)

b) $x_0 = 2$

$x_1 = 3 + \frac{3}{2^2} = 3.75$

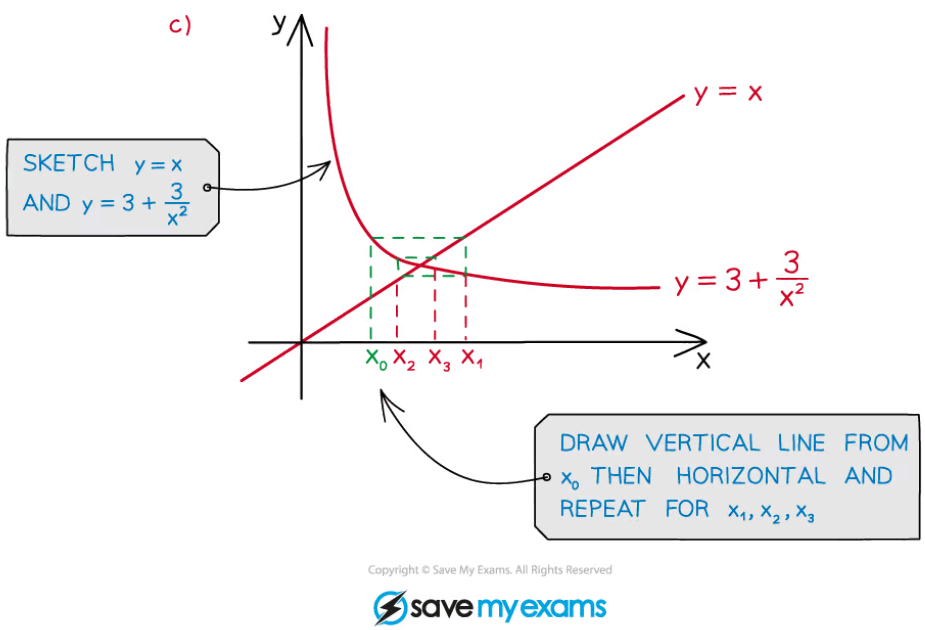
SUBSTITUTE $x = 2$ INTO ITERATIVE FORMULA

$x_2 = 3.213$

USE $3 + \frac{3}{\text{ANS}^2}$ TO CONTINUE THIS UNTIL x is THE SAME TO 2 dp

$x = 3.28$ (2 dp)

Your notes

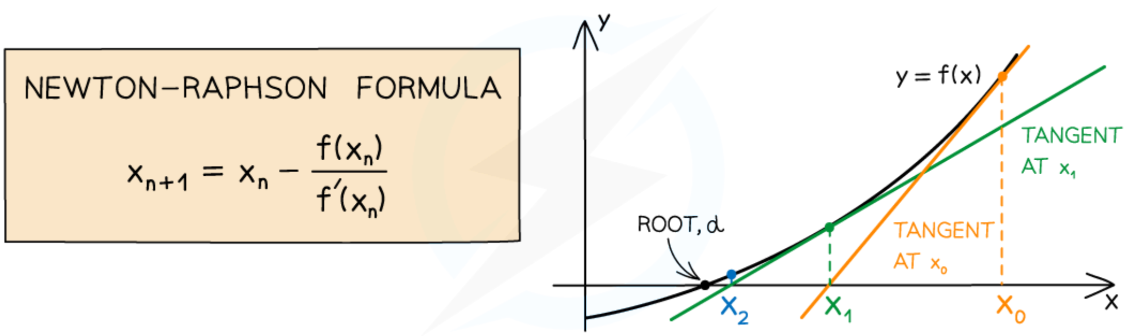

c)
y
x
y = x
$y = 3 + \frac{3}{x^2}$
SKETCH y = x AND $y = 3 + \frac{3}{x^2}$
x_0 x_2 x_3 x_1
DRAW VERTICAL LINE FROM x_0 THEN HORIZONTAL AND REPEAT FOR x_1, x_2, x_3
Copyright © Save My Exams. All Rights Reserved
save my exams

10.1.4 Newton-Raphson

Newton-Raphson

The Newton-Raphson method

- The Newton-Raphson method finds **roots** of equations in the form **f(x) = 0**
- It can be used to find **approximate solutions** when an equation cannot be solved using the usual **analytical methods**
- It works by finding the **x-intercept** of **tangents to f(x)** to get closer and closer to a root



Using the Newton-Raphson method

- The formula for Newton-Raphson uses the same **$x_{n+1} = f(x_n)$ notation** as used in iteration and other recurrence relations
- After using **differentiation** to find f'(x) the formula uses iteration to come to an ever more accurate solution

Your notes

TO USE NEWTON–RAPHSON

1. REARRANGE EQUATION INTO $f(x) = 0$
2. DIFFERENTIATE TO FIND $f'(x)$
3. SUBSTITUTE $f(x)$ AND $f'(x)$ INTO NEWTON – RAPHSON FORMULA
4. SUBSTITUTE x_0 INTO FORMULA TO BEGIN ITERATING
5. REPEAT ITERATION UNTIL ROOT IS FOUND TO APPROPRIATE DEGREE OF ACCURACY

FIND A ROOT OF THE EQUATION $x^2 \ln x = 5$ TO 5 s.f. USING $x_0 = 2$

1. $x^2 \ln x = 5$

   $f(x) = x^2 \ln x - 5 = 0$

2. USING PRODUCT RULE

   LET $u = x^2 \quad \frac{du}{dx} = 2x$

   LET $v = \ln x \quad \frac{dv}{dx} = \frac{1}{x}$

   $f'(x) = \frac{dy}{dx} = x(1 + 2\ln x)$

3. $x_{n+1} = x_n - \frac{x_n^2 \ln x_n - 5}{x_n(1 + 2\ln x_n)}$

4. $x_0 = 2$

   $x_1 = 2 - \frac{2^2 \ln 2 - 5}{2(1 + 2\ln 2)}$

   $= 2.466709...$

5. $x_2 = 2.395369...$

   $x_3 = 2.393518...$

   $x_4 = 2.393517...$

   $x_5 = 2.393517...$

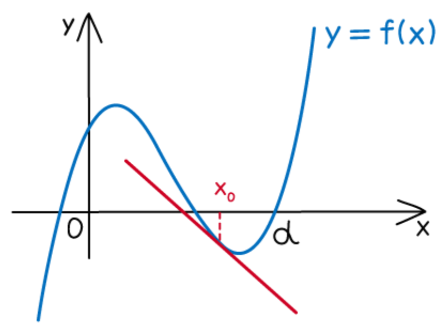
   $x = 2.3935$ (5 sf)

Can the Newton-Raphson method fail?

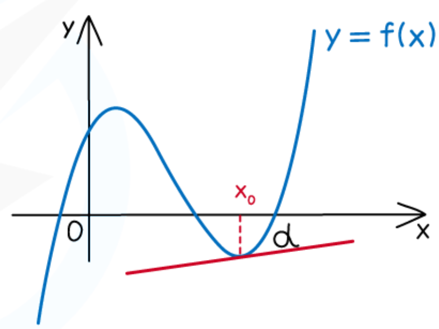
The Newton-Raphson method can fail when:

- the starting value x_0 **is too far away** from the root leading to a **divergent** sequence or a **different root**
- the **tangent gradient** is **too small**, where f'(x) close to 0 leading to a **divergent** sequence or one which **converges very slowly**
- the **tangent** is **horizontal**, where f'(x) = 0 so the tangent will **never meet the x-axis**
- the equation **cannot be differentiated** (or is awkward and time-consuming to do)

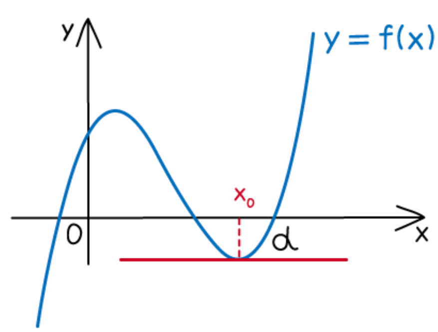
x_0 IS TOO FAR AWAY FROM THE ROOT (α) LEADING TO A DIVERGENT SEQUENCE OR A DIFFERENT ROOT



TANGENT GRADIENT IS TOO SMALL, SO THE POINT OF INTERSECTION IS TOO FAR FROM (α) LEADING TO A DIVERGENT SEQUENCE OR ONE WHICH CONVERGES VERY SLOWLY



TANGENT IS HORIZONTAL SO WILL NEVER MEET THE x-AXIS



Your notes

Examiner Tip

- The formula for the Newton-Raphson method is given in the formula booklet.
- Use ANS button on your calculator to calculate repeated iterations.
- Keep track of your iterations using x_2, x_3... notation.
- Newton-Raphson questions may be part of bigger numerical methods questions.

Your notes

Your notes

Worked example

Your notes

a) Use the Newton-Raphson method to find a root of $x^3 - 2\sqrt{x^3} = 1$ to 4 decimal places, starting with $x_0 = 2$.

b) Explain why $x_0 = 1$ would have been an unsuitable starting value.

a) $x^3 - 2\sqrt{x^3} = 1$

STEP 1 REARRANGE EQUATION INTO $f(x) = 0$

$f(x) = x^3 - 2\sqrt{x^3} - 1 = 0$

$f(x) = x^3 - 2x^{\frac{3}{2}} - 1$

REWRITE AS POWERS TO MAKE DIFFERENTIATION EASIER

STEP 2 DIFFERENTIATE TO FIND $f'(x)$

$f'(x) = 3x^2 - 3x^{\frac{1}{2}}$

$= 3x^2 - 3\sqrt{x}$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

STEP 3 SUBSTITUTE $f(x)$ AND $f'(x)$ INTO NEWTON – RAPHSON FORMULA

$$x_{n+1} = x_n - \frac{x_n^3 - 2\sqrt{x_n^3} - 1}{3x_n^2 - 3\sqrt{x_n}}$$

IT CAN BE EASIER TO WRITE x's FIRST THEN GO BACK AND ADD n's FOR x_n

$x_0 = 2$

STEP 4 SUBSTITUTE x_0 INTO FORMULA TO BEGIN ITERATING

$$x_1 = 2 - \frac{2^3 - 2\sqrt{2^3} - 1}{3(2^2) - 3\sqrt{2}} = 1.82685...$$

$$x_2 = 1.80024...$$

STEP 5 REPEAT ITERATION UNTIL ROOT IS FOUND TO REQUIRED DEGREE OF ACCURACY

$$x_3 = 1.7996...$$

$$\vdots$$

$$x = 1.7996 \quad (4 \text{ dp})$$

USING "ANS" BUTTON

$$ANS - \frac{ANS^3 - 2\sqrt{ANS^3} - 1}{3ANS^2 - 3\sqrt{ANS}}$$

b) $f'(x) = 3x^2 - 3\sqrt{x}$

SUBSTITUTE $x_0 = 1$ INTO $f'(x)$

$f'(1) = 3(1)^2 - 3\sqrt{1} = 0$

$f'(1) = 0$ SO TANGENT WILL NEVER MEET x-AXIS ie. THE NEWTON-RAPHSON METHOD WILL FAIL

Your notes

10.1.5 Trapezium Rule (Numerical Integration)

Your notes

Trapezium Rule (Numerical Integration)

What is the trapezium rule?

- The **trapezium rule** is a numerical method of integration used to find the **approximate area** under a curve
- It can be used when **analytical methods** of **integration** won't work
- It finds an **approximation** of the area by finding the **sum of the areas** of trapeziums beneath the curve
- The **formula** for an **estimate** of the area when splitting it into *n* strips (trapezium) is

$$\int_a^b y\,\mathrm{d}x \approx \frac{1}{2}h\{(y_0 + y_n) + 2(y_1 + y_2 + \ldots + y_{n-1})\} \text{ where } h = \frac{b-a}{n}$$

  - If you have ***n*** **strips** then you need ***n* + 1 *y*-values**

Your notes

YOU DON'T NEED TO REMEMBER THIS BUT YOU NEED TO KNOW HOW IT WORKS

THE TRAPEZIUM RULE:

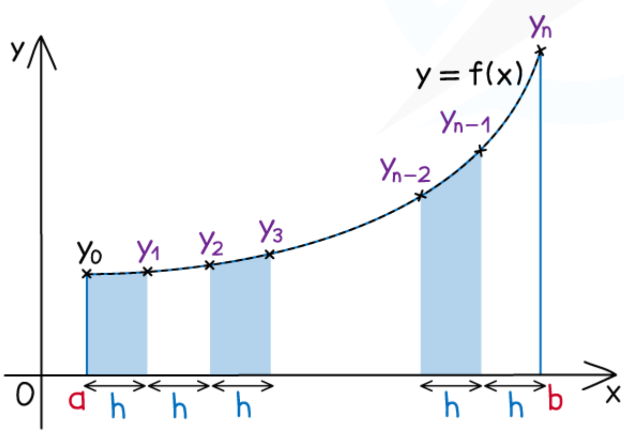
a AND b ARE THE LIMITS OF THE AREA YOU NEED TO FIND $\left(\int_a^b y\,dx\right)$

h = WIDTH OF EACH STRIP

$$\int_a^b y\,dx \approx \frac{1}{2}h\{(y_0 + y_n) + 2(y_1 + y_2 + \dots + y_{n-1})\}, \qquad \text{WHERE } h = \frac{b-a}{n}$$

$y_0, y_1, \dots, y_n$ ARE THE HEIGHTS OF EACH SIDE OF THE TRAPEZIUM

n = NUMBER OF STRIPS OR INTERVALS



STEP 1: FIND THE WIDTH OF EACH STRIP h

STEP 2: USE h TO CALCULATE THE x VALUES x_0, x_1 etc FOR EACH STRIP IN THE GIVEN INTERVAL $[a, b]$ (YOU MAY ALREADY BE GIVEN THESE IN THE QUESTION)

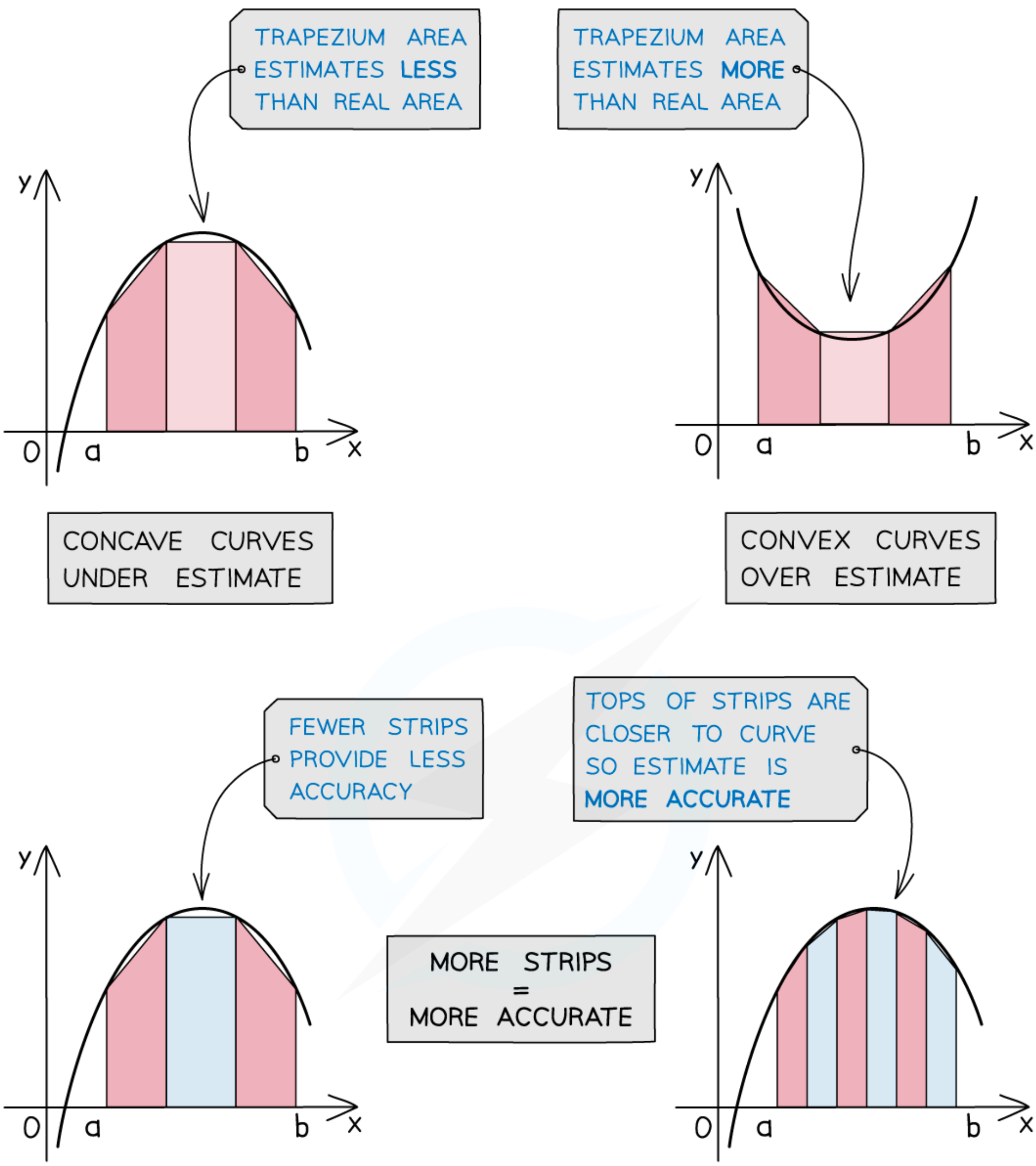
STEP 3: SET UP A TABLE AND USE THE EQUATION TO CALCULATE THE y VALUES y_0, y_1 etc

STEP 4: SUBSTITUTE ALL y VALUES INTO THE FORMULA ALONG WITH h

Underestimate or overestimate?

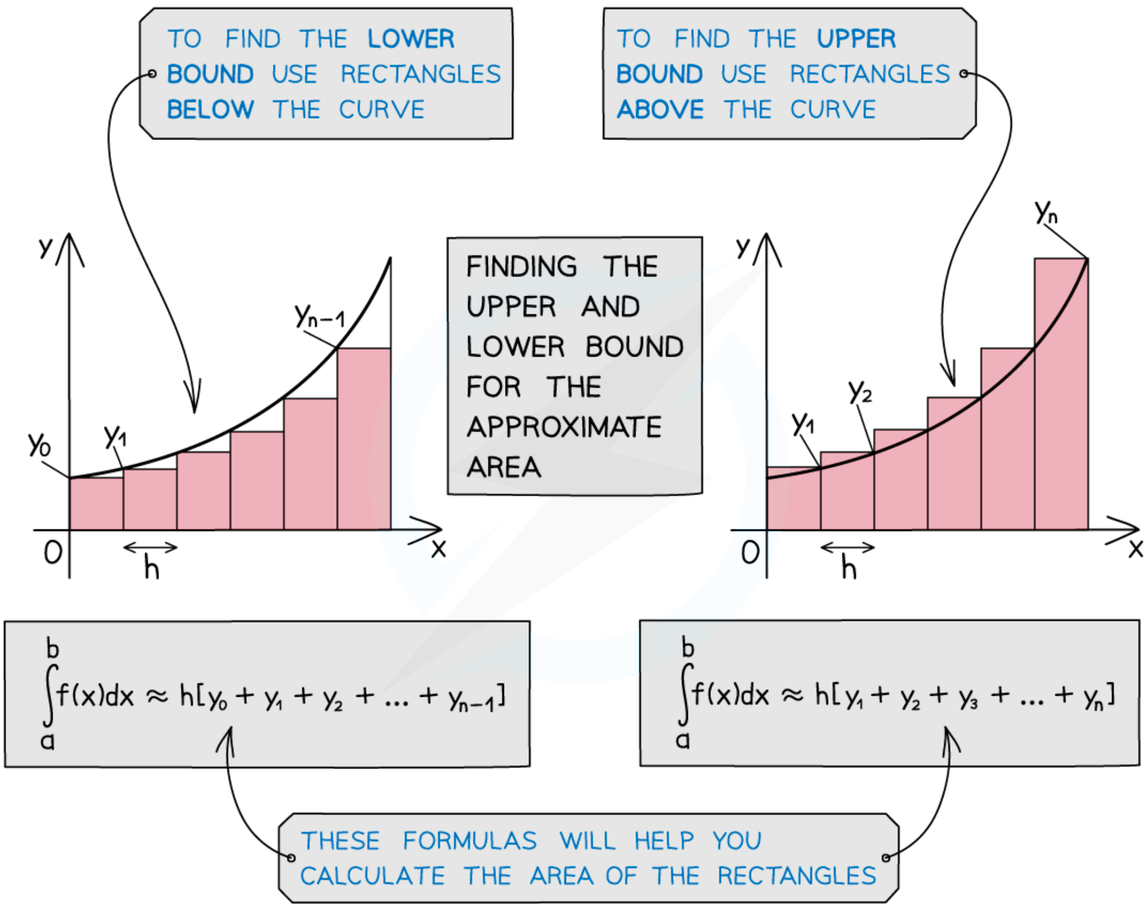
- Depending on the shape of the curve, the trapezium rule could be an **underestimate** or **overestimate**
- The **more trapezium strips** used the **more accurate** the estimate
- Calculating the **percentage error** can tell you how **accurate** your estimate is
- Using **rectangular strips** can find the **upper and lower bounds** of the interval for the trapezium rule

Your notes



$$\text{ABSOLUTE ERROR} = \text{ESTIMATE} - \text{EXACT VALUE}$$

$$\text{PERCENTAGE ERROR} = \frac{\text{ESTIMATE} - \text{EXACT VALUE}}{\text{EXACT VALUE}} \times 100$$



Examiner Tip

The formula for the trapezium rule is given in the formula booklet.

Your notes

Worked example

Your notes

a) Using the trapezium rule, find an approximate value for $\int_0^4 \frac{6x^2}{x^3 + 2} dx$, to 3 decimal places, using n = 4.

b) Given that the exact value of

$$\int_0^4 \frac{6x^2}{x^3 + 2} dx = 6.993 \text{ to 3 decimal places,}$$

calculate the percentage error for a) to 2 decimal places.

a) $h = \frac{4-0}{4} = 1$

STEP 1: FIND $h = \frac{b-a}{n}$ USING $\int_{a=0}^{b=4}$ AND $n = 4$

STEP 2: USING WIDTH $h = 1$, VALUES ARE 0, 1, 2, 3, 4

x	$y = \frac{6x^2}{x^3+2}$
$x_0 = 0$	$y_0 = 0$
$x_1 = 1$	$y_1 = 2$
$x_2 = 2$	$y_2 = 2.4$
$x_3 = 3$	$y_3 = 1.862...$
$x_4 = 4$	$y_4 = 1.454...$

STEP 3: SETUP TABLE AND USE EQUATION TO CALCULATE y VALUES

STEP 4: USING FORMULA FROM FORMULA BOOKLET

$$\int_a^b y\,dx \approx \frac{1}{2}h(y_0 + y_n + 2(y_1 + y_2 + ... + y_{n-1}))$$

SUBSTITUTE ALL VALUES

$$\frac{1}{2} \times 1 \times (0 + 1.454... + 2(2 + 2.4 + 1.862...))$$

$$= \frac{1}{2}(1.454... + 12.524...)$$

$$= 6.9893... = 6.989 \quad (3 \text{ dp})$$

USING

$$\% \text{ ERROR} = \frac{\text{ESTIMATE} - \text{EXACT}}{\text{EXACT}} \times 100$$

b) $$\frac{6.989 - 6.993}{6.993} \times 100$$

$$= -0.\dot{0}5720\dot{0}$$

IGNORE MINUS, YOU COULD DO EXACT−ESTIMATE TO GET RID OFF THIS

$$= 0.06\% \quad (2 \text{ dp})$$

Your notes