Edexcel A Level Maths: Pure

Your notes

10.2 Modelling involving Numerical Methods

Contents

10.2.1 Numerical Methods in Context


Your notes

Numerical Methods in Context

Numerical Methods

- **Numerical methods** can be used to model a variety of situations, from the motion of planets across the solar system to the depreciation in a cars value or the fluctuation in crime rates over time
- Numerical methods can be used to find solutions for questions when **analytical methods** may not always work or even be possible
- It is important to understand how each method can be applied to different contexts and the different aspects of a question you may be asked about
- Questions in context will often ask for an **interpretation** or **criticism** of certain parts of the problem

Your notes

EASY WAY TO SHOW A ROOT LIES WITHIN AN INTERVAL – BE CAREFUL OF CHANGE OF SIGN FAILURE.
ANY FORMULA OR MODEL CAN BE USED IN CHANGE OF SIGN, BUT SOME MAY NEED COMMON SENSE IDEAS TOO.

a) $f(24) = 48.68$,
$f(25) = -30.96$ (CHANGE OF SIGN)

b) INTERPRETING A SOLUTION MEANS DRAWING ON COMMON SENSE: SOME CARS GET MORE VALUABLE AS THEY GET OLDER

A CARS VALUE IN £'s AFTER x YEARS IS MODELLED BY THE FUNCTION
$f(x) = 17000(0.86)^x - 1000\sin x$
FOR $x > 0$

a) SHOW THAT $f(x)$ HAS A ROOT BETWEEN 24 AND 25

b) EXPLAIN WHY THIS MODEL MAY NOT WORK AS THE CAR GETS OLDER

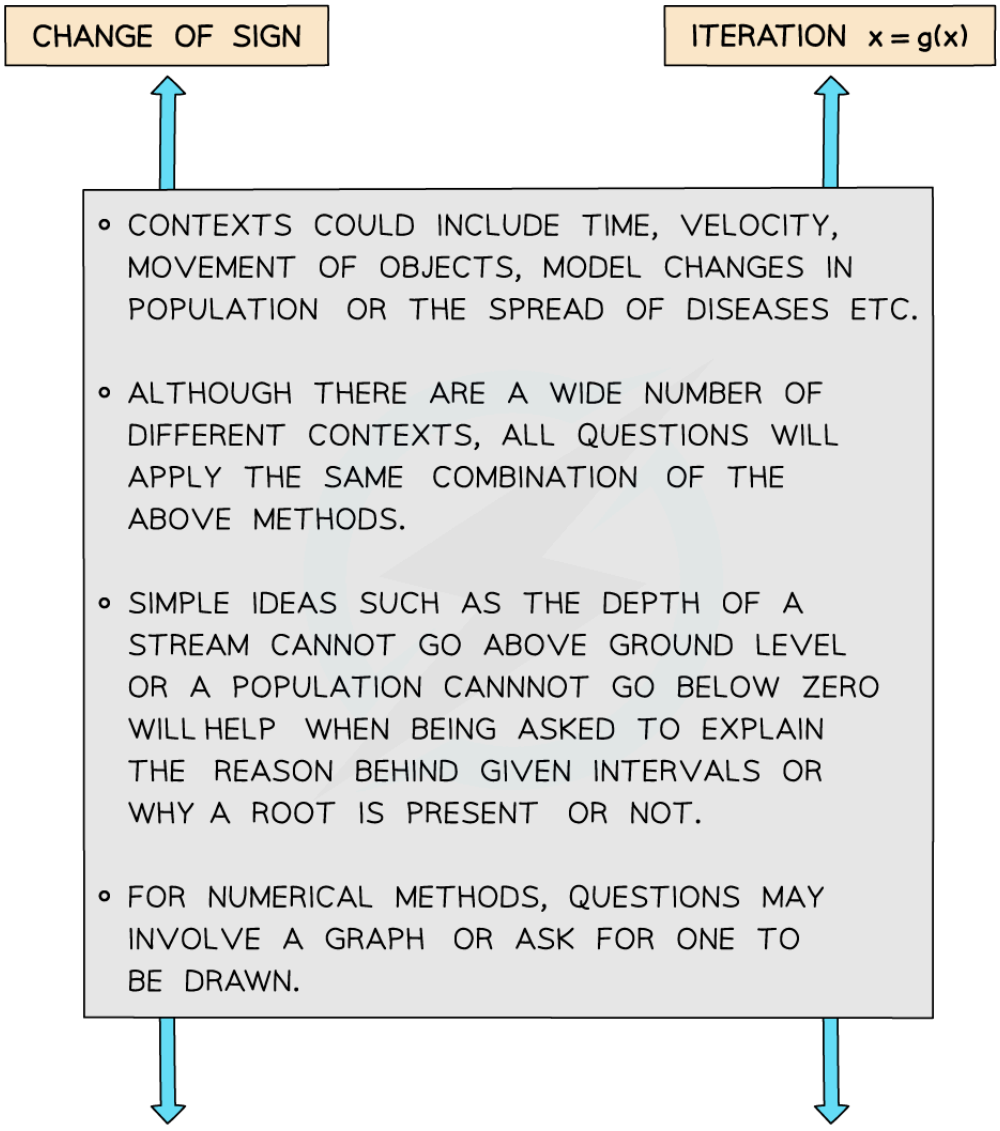
CHANGE OF SIGN

REPEATEDLY MORE ACCURATE SOLUTIONS – MAY BE ASKED TO REARRANGE EQUATION AND/OR SKETCH DIAGRAM
ITERATION IS COMMON IN REAL LIFE SCIENTIFIC MODELLING.

$x_3 = 43.36$ SO TAKES THREE YEARS FOR ANSWER TO DROP BELOW HALF OF THE ORIGINAL POPULATION

THE ITERATIVE FORMULA
$x_{n+1} = \frac{27\sqrt{x_n} + x_n}{0.75e^2}$ IS USED TO MODEL THE DECLINE IN ACTIVE BEE HIVE POPULATION (x MILLION) IN THE UK.
STARTING WITH $x0 = 90$, AFTER HOW MANY YEARS WOULD THE BEE POPULATION MORE THAN HALVE?

ITERATION $x = g(x)$



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- CONTEXTS COULD INCLUDE TIME, VELOCITY, MOVEMENT OF OBJECTS, MODEL CHANGES IN POPULATION OR THE SPREAD OF DISEASES ETC.

Your notes

- ALTHOUGH THERE ARE A WIDE NUMBER OF DIFFERENT CONTEXTS, ALL QUESTIONS WILL APPLY THE SAME COMBINATION OF THE ABOVE METHODS.
- SIMPLE IDEAS SUCH AS THE DEPTH OF A STREAM CANNOT GO ABOVE GROUND LEVEL OR A POPULATION CANNNOT GO BELOW ZERO WILL HELP WHEN BEING ASKED TO EXPLAIN THE REASON BEHIND GIVEN INTERVALS OR WHY A ROOT IS PRESENT OR NOT.
- FOR NUMERICAL METHODS, QUESTIONS MAY INVOLVE A GRAPH OR ASK FOR ONE TO BE DRAWN.

NEWTON–RAPHSON

THE TEMPERATURE $\theta^{\circ}C$ OF A ROOM t HOURS AFTER A HEATING SYSTEM HAS BEEN TURNED ON IS GIVEN BY $\theta = t + 26 - 20e^{-0.5t}$ WITH $t \geqslant 0$.
STARTING AT $x_0 = 1.9$, USE THE NEWTON–RAPHSON METHOD ONCE TO OBTAIN A SECOND APPROXIMATION.
GIVE YOUR ANSWER TO 3 DECIMAL PLACES.

USING DIFFERENTIATION AND FORMULA TO OBTAIN REPEATEDLY MORE ACCURATE SOLUTIONS – BE

NUMERICAL INTEGRATION
TRAPEZIUM RULE

THE SPEED OF A SMALL BOAT WAS RECORDED EVERY FIVE SECONDS, FROM THE TIME IT LEFT THE JETTY IN A HARBOUR, TO THE TIME IT REACHED OPEN WATER.

TIME (s)	0	5	10	15	20	25
SPEED (ms^{-1})	2	5	10	18	28	42

a) USING THIS INFORMATION ESTIMATE THE LENGTH OF THE HARBOUR FROM JETTY TO OPEN WATER.

b) GIVEN THE BOAT ACCELERATED

CAREFUL OF NEWTON–RAPHSON FAILURE

AS LONG AS THE FUNCTION IS DIFFERENTIABLE AND $f'(x) \neq 0$ THE NEWTON–RAPHSON FORMULA CAN BE APPLIED TO ANY CONTEXT

$f(1.9) = 1.9 + 6 - 20e^{-0.5 \times 1.9}$
$f'(t) = 1 + 10e^{-0.5t}$
$f'(1.9) = 1 + 10e^{-0.5 \times 1.9}$
$\alpha_2 = 1.9 - \frac{0.16518}{4.867410} \approx 1.866$

b) GIVEN THE BOAT ACCELERATED SMOOTHLY BEFORE REACHING OPEN WATER, EXPLAIN WHETHER YOUR ANSWER TO PART a) IS AN UNDER ESTIMATE OR AN OVER ESTIMATE.

USE TRAPEZOIDAL STRIPS TO CALCULATE THE ESTIMATE FOR THE AREA UNDER A GRAPH – COULD BE OVER OR UNDERESTIMATE AND ASKED FOR PERCENTAGE ERROR.

THE AREA UNDER THE GRAPH CAN REPRESENT PART OF A COMPOUND MEASURE DEPENDING ON THE UNITS.

a) HERE SPEED AND TIME IN THE QUESTION MEAN AREA CAN BE USED TO CALCULATE TOTAL DISTANCE.
TRAPEZIUM FORMULA = 415m

b) INTERPRETING THE SOLUTION CAN BE TRICKY, BUT USING A DIAGRAM CAN HELP SEE WHETHER YOUR TRAPEZIUMS ARE OVER OR UNDERESTIMATE.
GRAPH WOULD BE CONVEX SO OVERESTIMATE

Your notes

Examiner Tip

- Exam questions will often tell you which method to use
- But beware... numerical methods questions can often combine multiple methods in one large question

Worked example

Your notes

When studying the elliptical orbit of a planet, an astronomer uses the formula $M = E - 0.1\sin E$, to model the angle of the planet at a given time, E radians, against the angle the planet would have moved if it were on a circular orbit, M radians.

In order to predict the correct position of the planet at a given time, the astronomer needs to find the value of E when $M = \frac{\pi}{6}$

a) Show that this value of E is a root of the function $f(x) = x - 0.1\sin x - k$ where k is a constant to be determined.

b) Starting with $x_0 = 0.6$ as a first approximation, use the Newton-Raphson method to obtain a second approximation for the value of E when $M = \frac{\pi}{6}$.

c) By considering change of sign using a suitable interval of f(x), show that your answer to part b) is correct to 3 decimal places.

Your notes

a) $M = E - 0.1\sin E$

$M = \frac{\pi}{6}$ → $\frac{\pi}{6} = E - 0.1\sin E$

E IS A ROOT OF $f(x)$ WHEN $f(E) = 0$ → $f(x) = x - 0.1\sin x - k = 0$

REARRANGE TO MAKE $f(x) = 0$ → $E - 0.1\sin E - \frac{\pi}{6} = 0$

$k = \frac{\pi}{6}$

b) $f(E) = E - 0.1\sin E - \frac{\pi}{6}$

DIFFERENTIATE FOR NEWTON–RAPHSON $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$ → $f'(E) = 1 - 0.1\cos E$

$x_0 = 0.6$

$x_{n+1} = x_n - \frac{x_n - 0.1\sin x_n - \frac{\pi}{6}}{1 - 0.1\cos x_n}$

SUBSTITUTE $x_0 = 0.6$ TO FIND SECOND APPROXIMATION

$$x_1 = 0.6 - \frac{0.6 - 0.1\sin(0.6) - \frac{\pi}{6}}{1 - 0.1\cos 0.6} = 0.5782...$$

$x_1 = 0.5782...$

c) 0.5782... ROUNDS TO 0.578 (3 dp) SO USING BOUNDS $0.5775 \leqslant x < 0.5785$

SUBSTITUTE BOUNDS INTO $f(x)$

$$f(0.5775) = 0.5775 - 0.1\sin 0.5775 - \frac{\pi}{6} = -0.00069...$$

$$f(0.5785) = 0.5785 - 0.1\sin 0.5785 - \frac{\pi}{6} = 0.00022...$$

CHANGE OF SIGN IN INTERVAL MEANS $E = 0.578$ CORRECT TO 3 DECIMAL PLACES


Your notes