Edexcel A Level Maths: Pure

Your notes

3.2 Circles

Contents

3.2.1 Equation of a Circle

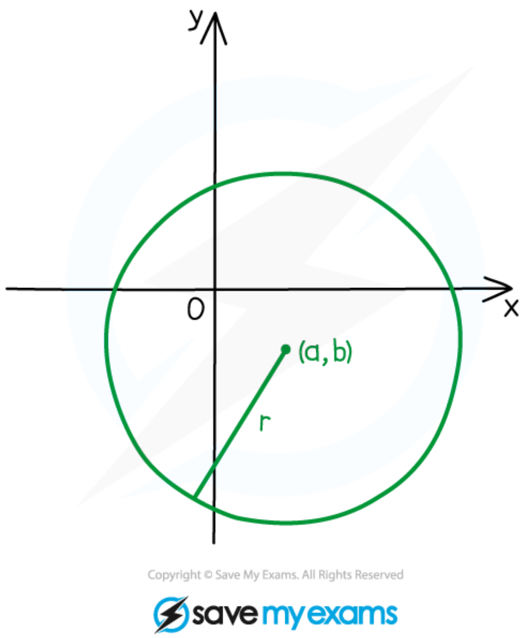
Your notes

Equation of a Circle

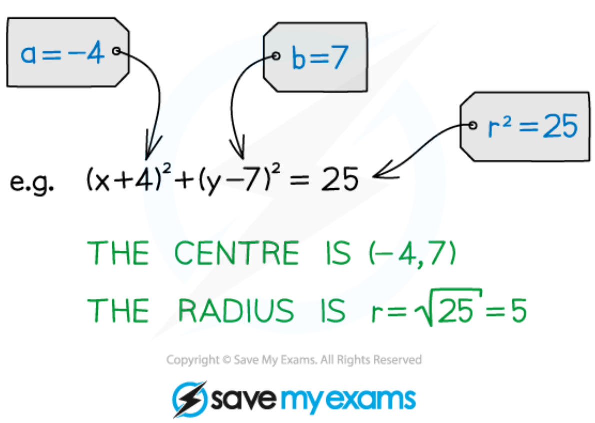
What do I need to know about the equation of a circle?

- A circle with centre **(*a*, *b*)** and radius ***r*** has the equation

$$(x - a)^2 + (y - b)^2 = r^2$$



- You need to be able to find the centre and radius of a circle from its equation



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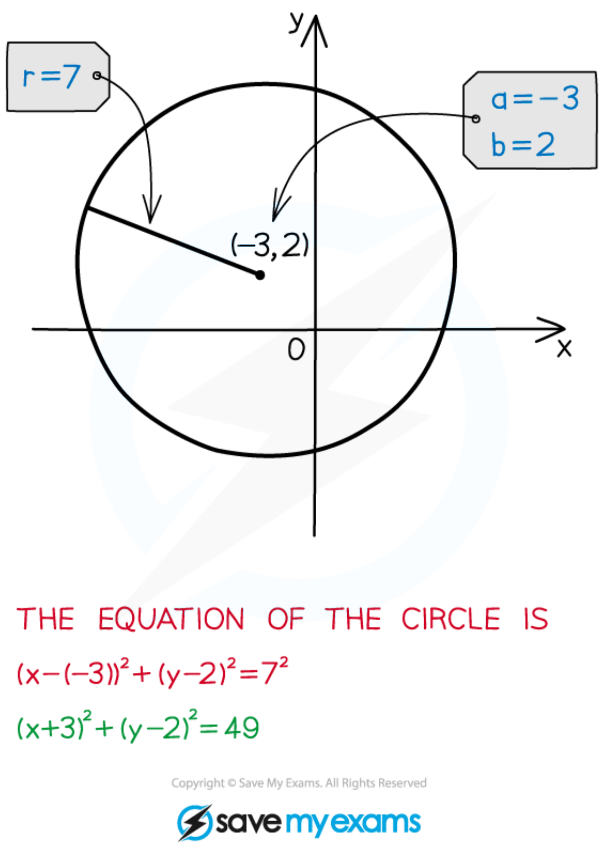
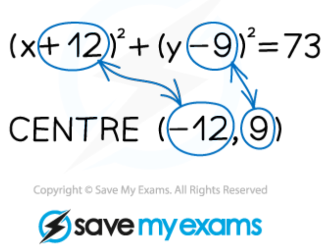
- You need to be able to find the equation of a circle given its centre and radius



Examiner Tip

- Remember that the numbers in the brackets have the opposite signs to the coordinates of the centre.



- Don't forget to take the **square root** of the right-hand side of the equation when finding the radius.

Your notes

Worked example

A circle has centre $(-5, -7)$ and goes through the point $(1, 1)$.

Find the equation of the circle.

THE RADIUS IS THE DISTANCE FROM $(-5,-7)$ TO $(1,1)$

$r = \sqrt{(1-(-5))^2 + (1-(-7))^2}$

$= \sqrt{6^2+8^2} = \sqrt{100} = 10$

THE DISTANCE BETWEEN TWO POINTS (x_1, y_1) AND (x_2, y_2) IS $\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$

THE EQUATION OF THE CIRCLE IS

$(x-(-5))^2 + (y-(-7))^2 = 10^2$

$(x+5)^2 + (y+7)^2 = 100$

$a = -5$
$b = -7$

3.2.2 Finding the Centre & Radius

Your notes

Finding the Centre & Radius

How can I find the centre and radius from any form of the equation of a circle?

- A circle equation in a different form can always be rearranged into $(x - a)^2 + (y - b)^2 = r^2$ form in order to find the centre and radius
- This will often involve **completing the square**

e.g. $x^2 - 6x + y^2 + y - 15 = 0$

$x^2 - 6x = (x-3)^2 - 9$

$y^2 + y = (y + \frac{1}{2})^2 - \frac{1}{4}$

COMPLETING THE SQUARE

SO

$(x-3)^2 - 9 + (y + \frac{1}{2})^2 - \frac{1}{4} - 15 = 0$

$(x-3)^2 + (y + \frac{1}{2})^2 - \frac{97}{4} = 0$

$(x-3)^2 + (y + \frac{1}{2})^2 = \frac{97}{4}$

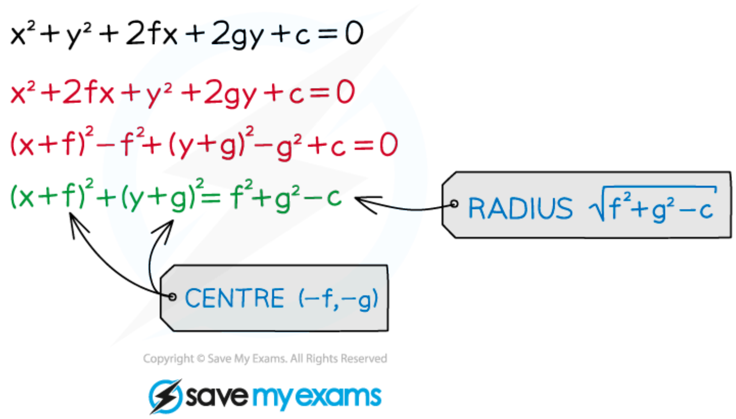
RADIUS $\sqrt{\frac{97}{4}} = \frac{\sqrt{97}}{2}$

CENTRE $(3, -\frac{1}{2})$

Your notes

Examiner Tip

Make sure you are able to rearrange circle equations given in the general form $x^2 + y^2 + 2fx + 2gy + c = 0$



Your notes

Worked example

A circle has the equation

$$x^2 + 2x + y^2 - 6y + k = 0$$

and passes through the point $(5, 12)$.

Find the centre and radius of the circle.

CIRCLE PASSES THROUGH (5,12) SO

$(5)^2 + 2(5) + (12)^2 - 6(12) + k = 0$

$25 + 10 + 144 - 72 + k = 0$

$k = -107$

USE THE COORDINATES OF THE KNOWN POINT TO FIND THE VALUE OF k

EQUATION OF THE CIRCLE IS

$x^2 + 2x + y^2 - 6y - 107 = 0$

$(x+1)^2 - 1 + (y-3)^2 - 9 - 107 = 0$

$(x+1)^2 + (y-3)^2 = 117$

REARRANGE INTO $(x-a)^2 + (y-b)^2 = r^2$ FORM

CENTRE IS $(-1, 3)$

RADIUS IS $\sqrt{117}$

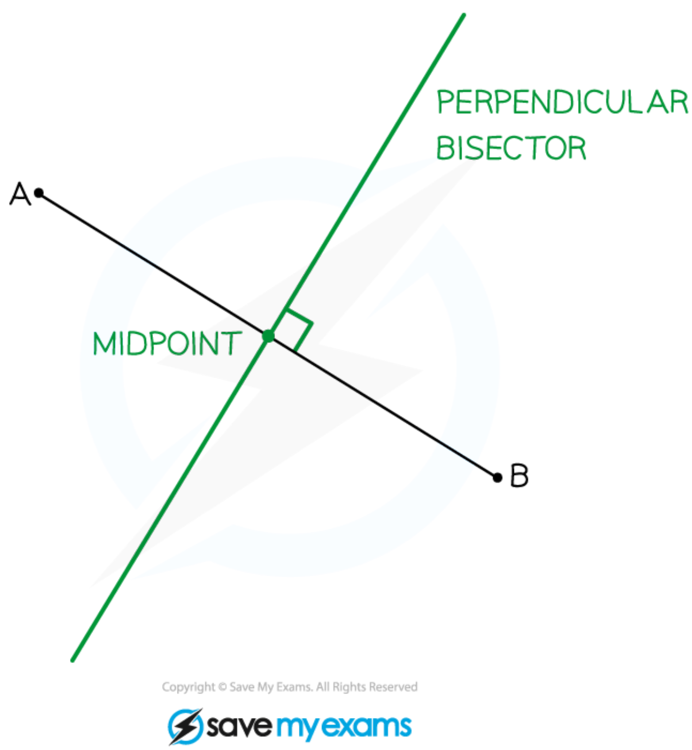
3.2.3 Bisection of Chords

Your notes

Bisection of Chords

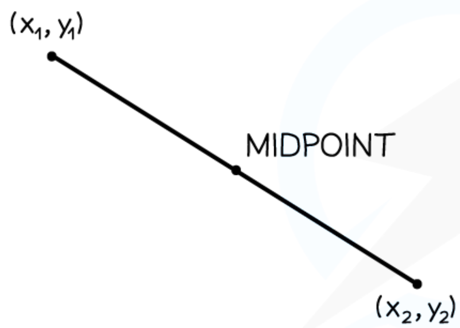
How can I find the equation of a perpendicular bisector?

- The **perpendicular bisector** of a line segment:
  - is perpendicular to the line segment
  - goes through the midpoint of the line segment



- The midpoint and gradient of the line segment between points (x_1, y_1) and (x_2, y_2) are given by the formulae

Your notes



$$\text{MIDPOINT} = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

$$\text{GRADIENT} = \left(\frac{y_2 - y_1}{x_2 - x_1}\right)$$

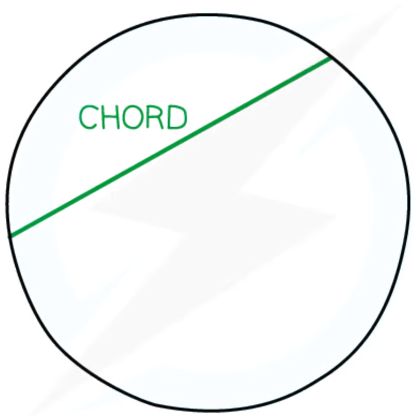
- The gradient of the perpendicular bisector is therefore

$$\text{GRADIENT OF PERPENDICULAR BISECTOR} = -\left(\frac{x_2 - x_1}{y_2 - y_1}\right)$$

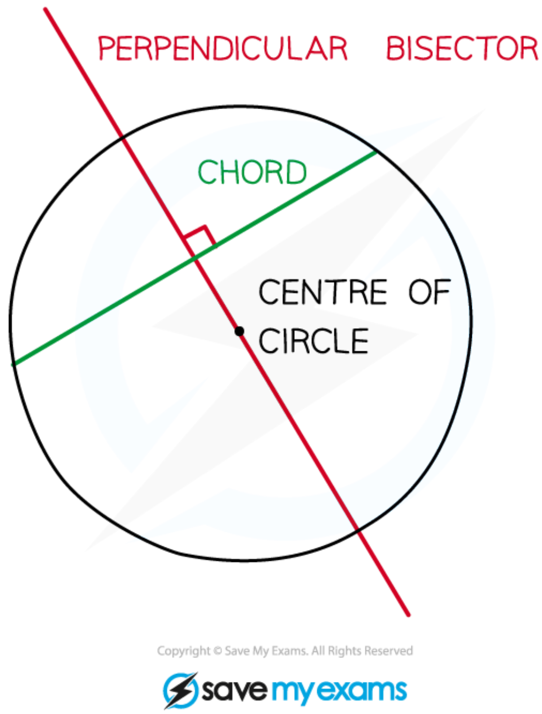
- The equation of the perpendicular bisector is the equation of the line with that gradient through the line segment's midpoint (see Equation of a Straight Line)

How can I use perpendicular bisectors to find the equation of a circle?

- A **chord** of a circle is a straight line segment between any two points on the circle



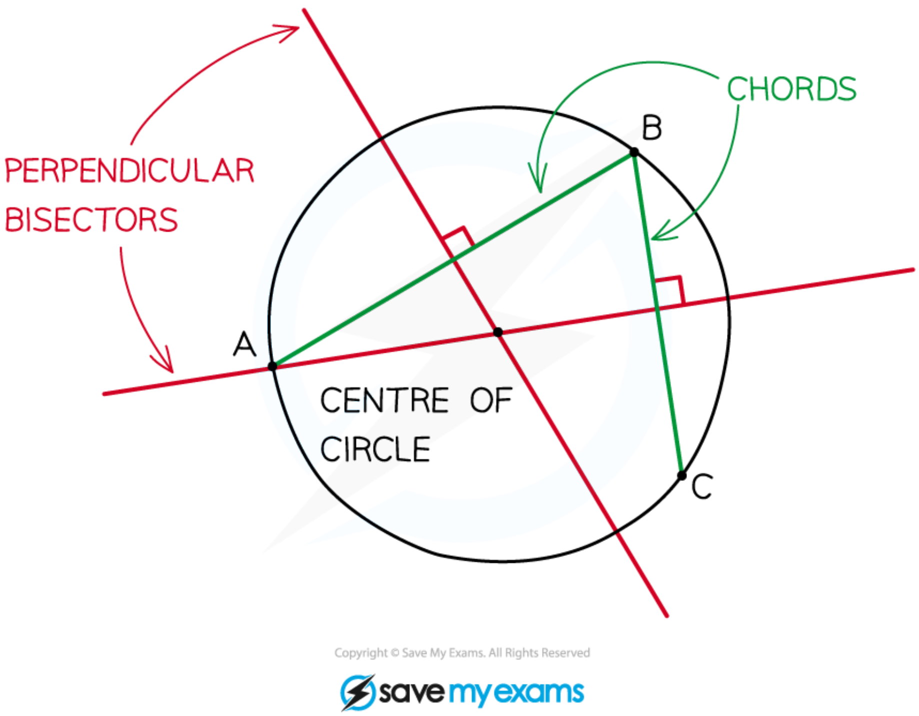
- The perpendicular bisector of a chord always goes through the centre of the circle



- If you know three points on a circle, draw any two chords between them – the perpendicular bisectors of the chords will meet at the centre of the circle

Your notes

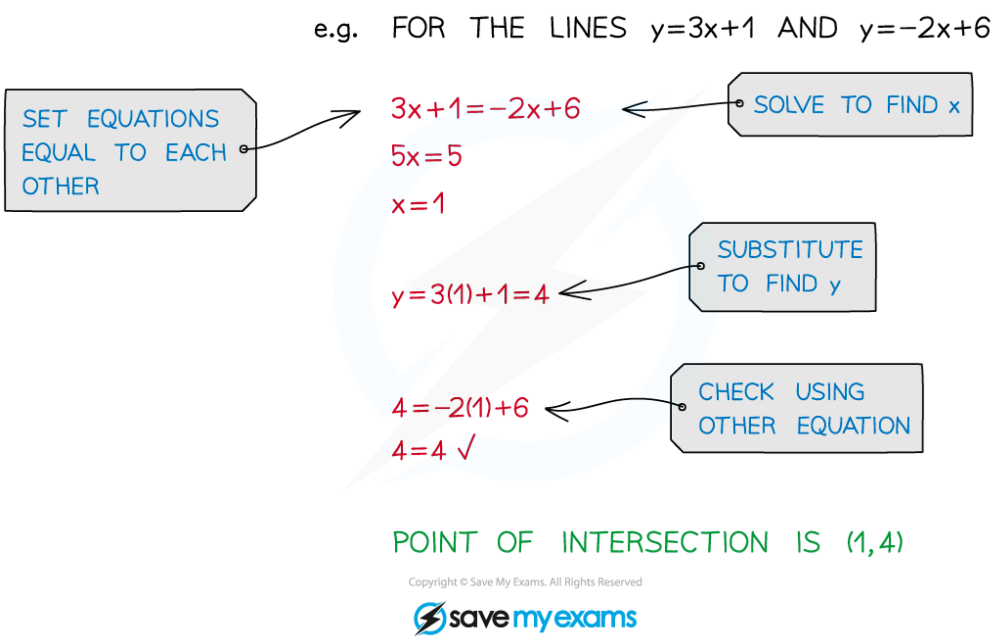
Your notes



- Now that you know the centre of the circle and a point on the circle you can write the equation of the circle

Examiner Tip

- To find the point of intersection of two straight lines, set the equations of the lines equal to each other and solve.



Your notes

Worked example

Your notes

Your notes

The points $A\ (0,\ 17)$, $B\ (8,\ 13)$ and $C\ (-10,\ 7)$ lie on the circumference of a circle. Use the properties of chords and perpendicular bisectors to find the equation of the circle.

PERPENDICULAR BISECTOR OF AB:

$\text{GRADIENT} = -\frac{8-0}{13-17} = 2$ ← $-\left(\frac{x_2-x_1}{y_2-y_1}\right)$

$\text{MIDPOINT OF AB} = \left(\frac{0+8}{2}, \frac{17+13}{2}\right) = (4, 15)$ ← $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$

$y = 2(x-4) + 15 = 2x + 7$ ← USE GRADIENT AND MIDPOINT TO FIND EQUATION OF PERPENDICULAR BISECTOR

PERPENDICULAR BISECTOR OF BC:

$\text{GRADIENT} = -\frac{-10-8}{7-13} = -3$

$\text{MIDPOINT OF BC} = \left(\frac{8+(-10)}{2}, \frac{13+7}{2}\right) = (-1, 10)$

$y = -3(x-(-1)) + 10 = -3x + 7$

POINT OF INTERSECTION OF PERPENDICULAR BISECTORS:

$2x + 7 = -3x + 7$ ← SET EQUATIONS EQUAL

$5x = 0$ ← SOLVE FOR x

Your notes

$x = 0$

$y = 2(0) + 7 = 7$ SUBSTITUTE TO FIND y

$\therefore$ CENTRE OF CIRCLE IS $(0,7)$

EQUATION OF CIRCLE:

$(x-0)^2 + (y-7)^2 = r^2$ $(x-a)^2 + (y-b)^2 = r^2$

$x^2 + (y-7)^2 = r^2$

BUT $(0,17)$ IS ON THE CIRCLE SO

$(0)^2 + (17-7)^2 = r^2$ SUBSTITUTE THE COORDINATES OF ANY ONE POINT INTO THE EQUATION

$100 = r^2$

THE EQUATION OF THE CIRCLE IS

$x^2 + (y-7)^2 = 100$

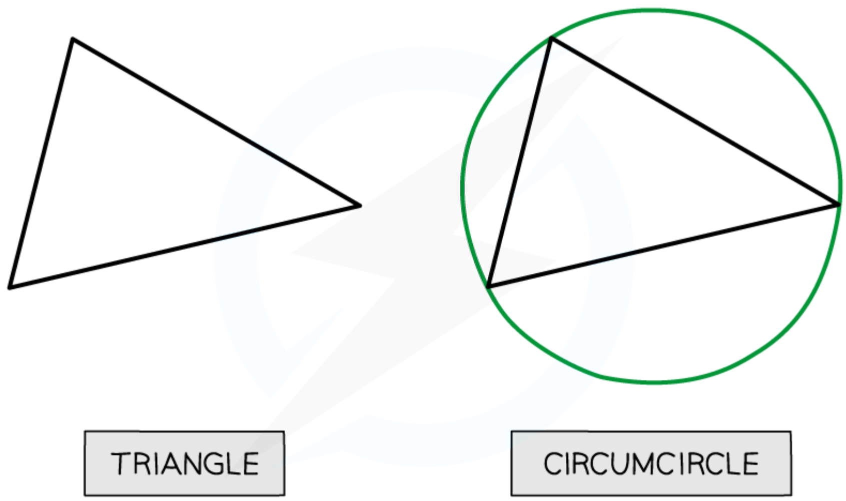
3.2.4 Angle in a Semicircle

Your notes

Angle in a Semicircle

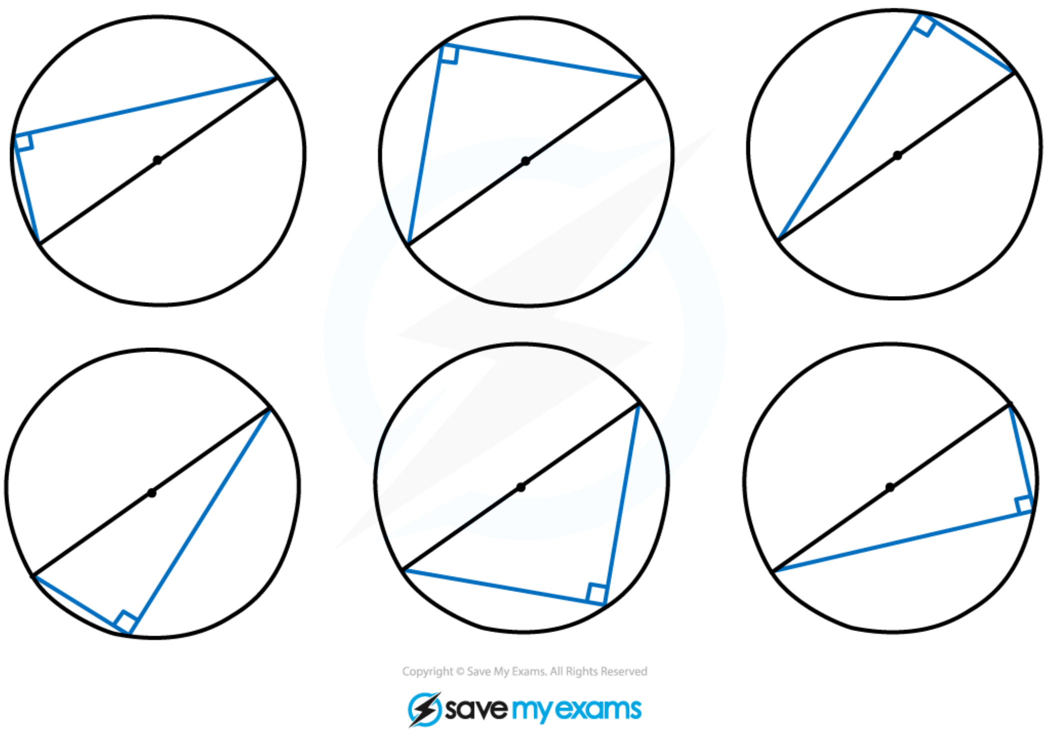
What is the angle in a semicircle property?

- It is always possible to draw a unique circle through the three vertices of a triangle – this is called the **circumcircle** of the triangle



- The **angle in a semicircle property** says that If a triangle is right-angled, then its hypotenuse is a diameter of its circumcircle

Your notes

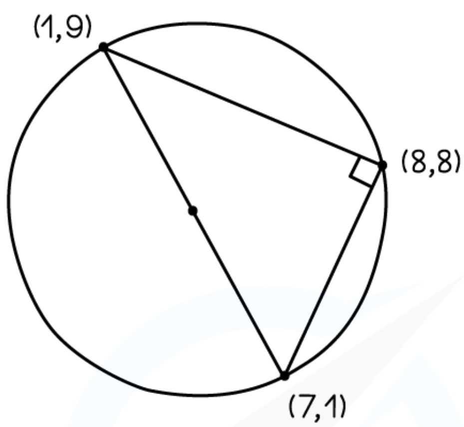


- It also says that **any** angle at the circumference in a semicircle is a right angle

How can I use the angle in a semicircle property to find the equation of a circle?

- Because the hypotenuse of a right-angled triangle is a diameter of the triangle's circumcircle you also know that:
    1. the radius of the circumcircle is half the length of the hypotenuse
    2. the centre of the circumcircle is the midpoint of the hypotenuse

Your notes



CENTRE OF CIRCLE = MIDPOINT OF HYPOTENUSE

$$= \left(\frac{1+7}{2}, \frac{9+1}{2}\right)$$

$$= (4,5)$$

MIDPOINT OF LINE SEGMENT FROM (x_1, y_1) TO (x_2, y_2) IS

$$\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$$

RADIUS $= \frac{1}{2} \times$ LENGTH OF HYPOTENUSE

$$= \frac{1}{2}\left(\sqrt{(7-1)^2+(1-9)^2}\right.$$

$$= \frac{1}{2}\sqrt{100} = 5$$

LENGTH OF LINE SEGMENT FROM (x_1, y_1) TO (x_2, y_2) IS

$$\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$

- Once you know the radius and the centre you can write down the **equation of the circle**

EQUATION OF CIRCLE WITH CENTRE (4,5) AND RADIUS OF 5 IS

$$(x-4)^2+(y-5)^2 = 5^2$$

$$(x-4)^2+(y-5)^2 = 25$$

Examiner Tip

- To show that a triangle is right-angled, show that the lengths of its sides satisfy Pythagoras' theorem.

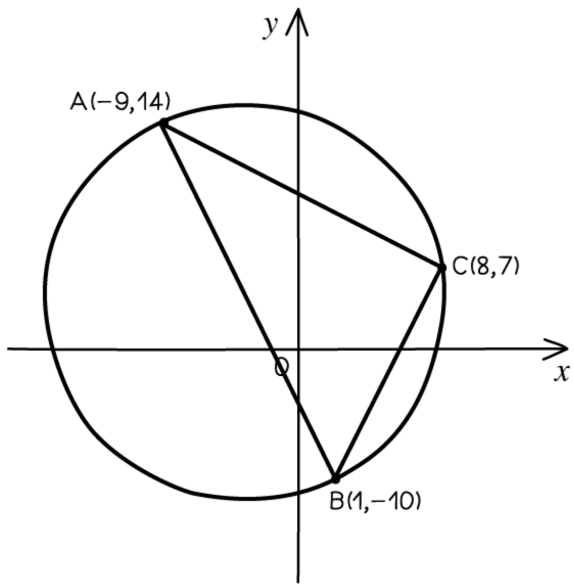
Your notes

Worked example

Your notes

Your notes

The three points $A\ (-9,\ 14)$, $B\ (1,\ -10)$ and $C\ (8,\ 7)$ lie on a circle, as shown in the diagram.



a) Show that AB is a diameter of the circle

b) Find an equation of the circle

a) LENGTHS OF THE SIDES ARE:

$AB^2 = (1-(-9))^2 + (-10-14)^2 = 676$

$AC^2 = (8-(-9))^2 + ((7-14))^2 = 338$

$BC^2 = (8-1)^2 + (7-(-10))^2 = 338$

BECAUSE WE'RE DEALING WITH PYTHAGORAS, IT'S EASIER TO KEEP EVERYTHING SQUARED HERE

$AC^2 + BC^2 = 338 + 338 = 676 = AB^2$

∴ BY PYTHAGORAS ABC IS A RIGHT-ANGLED TRIANGLE, AND AB IS THE HYPOTENUSE

$\therefore$ BY THE ANGLE IN A SEMICIRCLE PROPERTY, AB IS THE DIAMETER OF THE CIRCLE THROUGH A, B AND C

b) LENGTH OF AB $= \sqrt{676} = 26$

RADIUS OF CIRCLE $= \frac{1}{2} \times 26 = 13$

CENTRE OF CIRCLE = MIDPOINT OF AB

$$= \left(\frac{-9+1}{2}, \frac{14+(-10)}{2}\right)$$

$$= (-4, 2)$$

EQUATION OF THE CIRCLE IS

$$(x-(-4))^2 + (y-2)^2 = 13^2$$

$$(x+4)^2 + (y-2)^2 = 169$$

Your notes

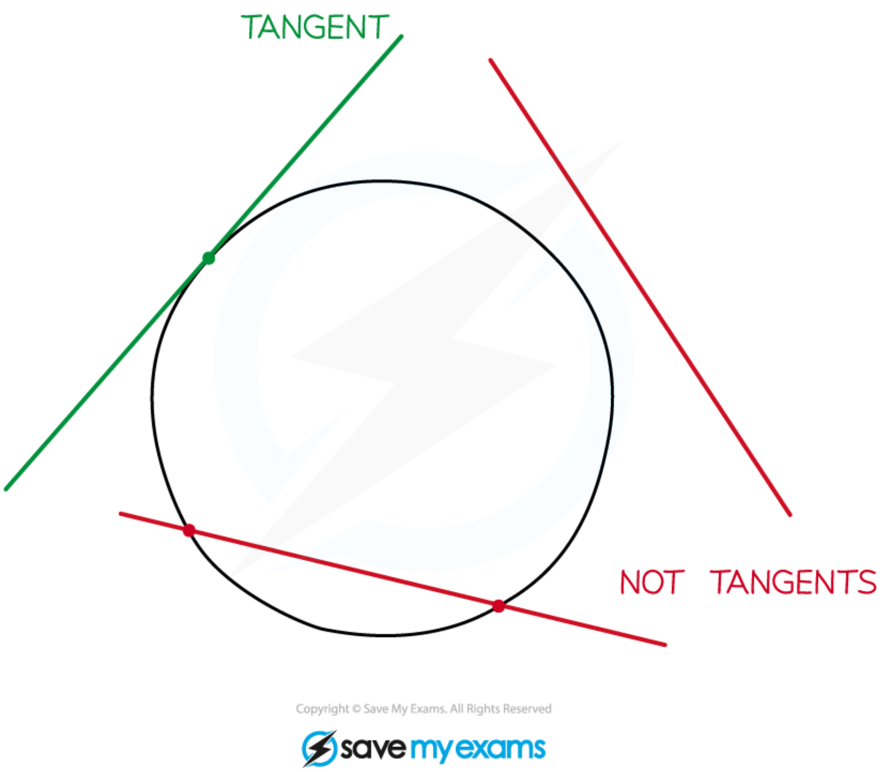
3.2.5 Radius & Tangent

Your notes

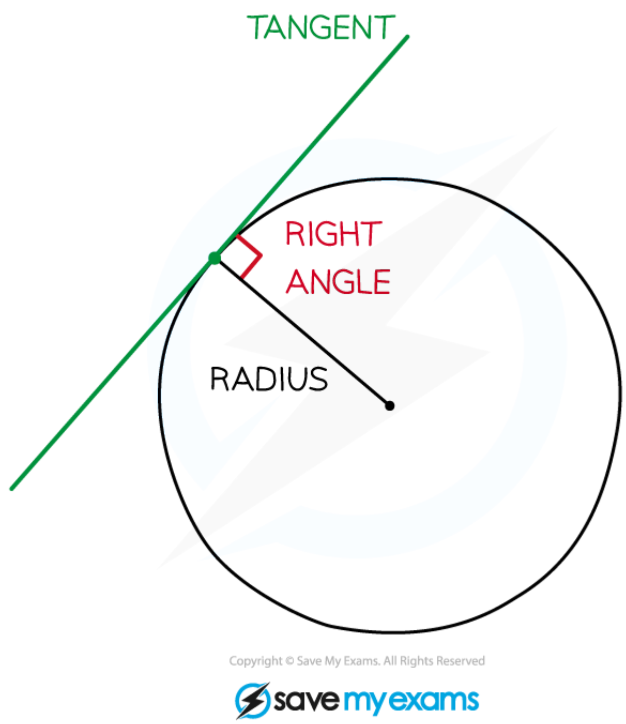
Radius & Tangent

What is the relationship between tangents and radii?

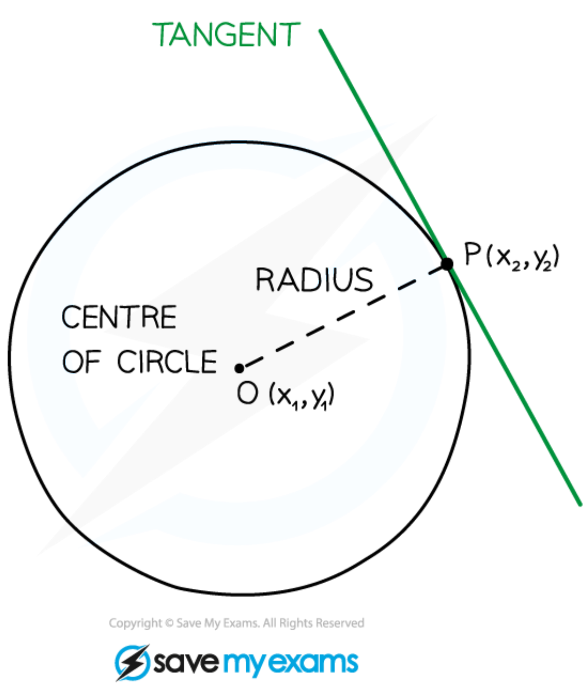
- A **tangent** is a line that **touches** a circle at a single point but doesn't cut across the circle



- A tangent to a circle is perpendicular to the radius of the circle at the point of intersection



How can I find the equation of the tangent line to a circle at a given point?



Your notes

- STEP 1: Find the gradient of the radius ***OP***

$$\text{GRADIENT OF RADIUS} = \frac{y_2 - y_1}{x_2 - x_1}$$

- STEP 2: Find the gradient of the tangent

THE TANGENT IS PERPENDICULAR TO THE RADIUS, SO:

$$\text{GRADIENT OF TANGENT} = -\frac{x_2 - x_1}{y_2 - y_1}$$

- STEP 3: The equation of the tangent is the equation of the line with that gradient that goes through point ***P*** (see Equation of a Straight Line)

Examiner Tip

- If you understand the formula in Step 2 above, you can find the gradient of the tangent without having to find the gradient of the radius first.

Your notes

Worked example

The circle C has equation $(x-5)^2+(y-2)^2=25$.

The point $P(2, 6)$ lies on C.

Find an equation for the tangent to C at point P.

STEP 1: C HAS CENTRE $(5,2)$

GRADIENT OF RADIUS AT $P=\frac{6-2}{2-5}=-\frac{4}{3}$ $\frac{y_2-y_1}{x_2-x_1}$

STEP 2: GRADIENT OF TANGENT AT $P=\frac{3}{4}$ $-\frac{x_2-x_1}{y_2-y_1}$

STEP 3: $y=\frac{3}{4}(x-2)+6$ LINE THROUGH $(2,6)$ WITH GRADIENT $\frac{3}{4}$

$=\frac{3}{4}x-\frac{3}{2}+6$

THE EQUATION OF THE TANGENT IS

$y=\frac{3}{4}x+\frac{9}{2}$