Edexcel A Level Maths: Pure

Your notes

11.2 Vectors in 3 Dimensions

Contents

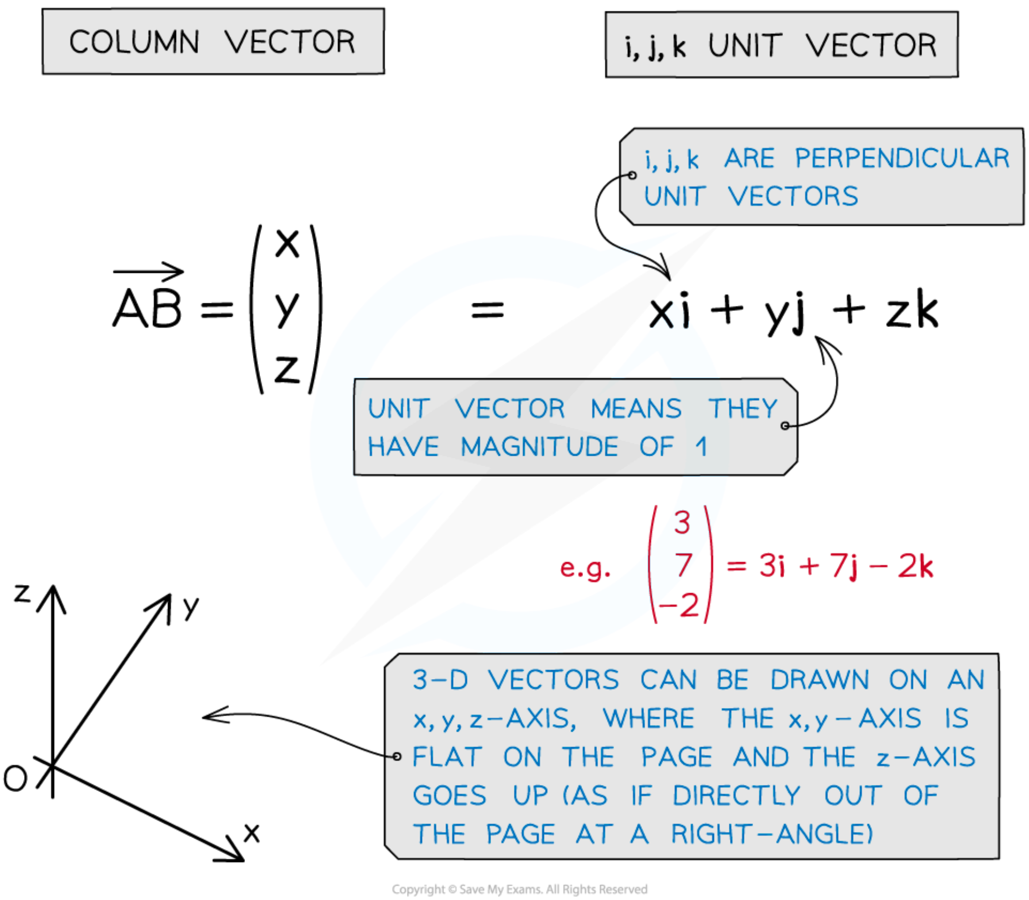
11.2.1 Vectors in 3 Dimensions

Your notes

Vectors in 3 Dimensions

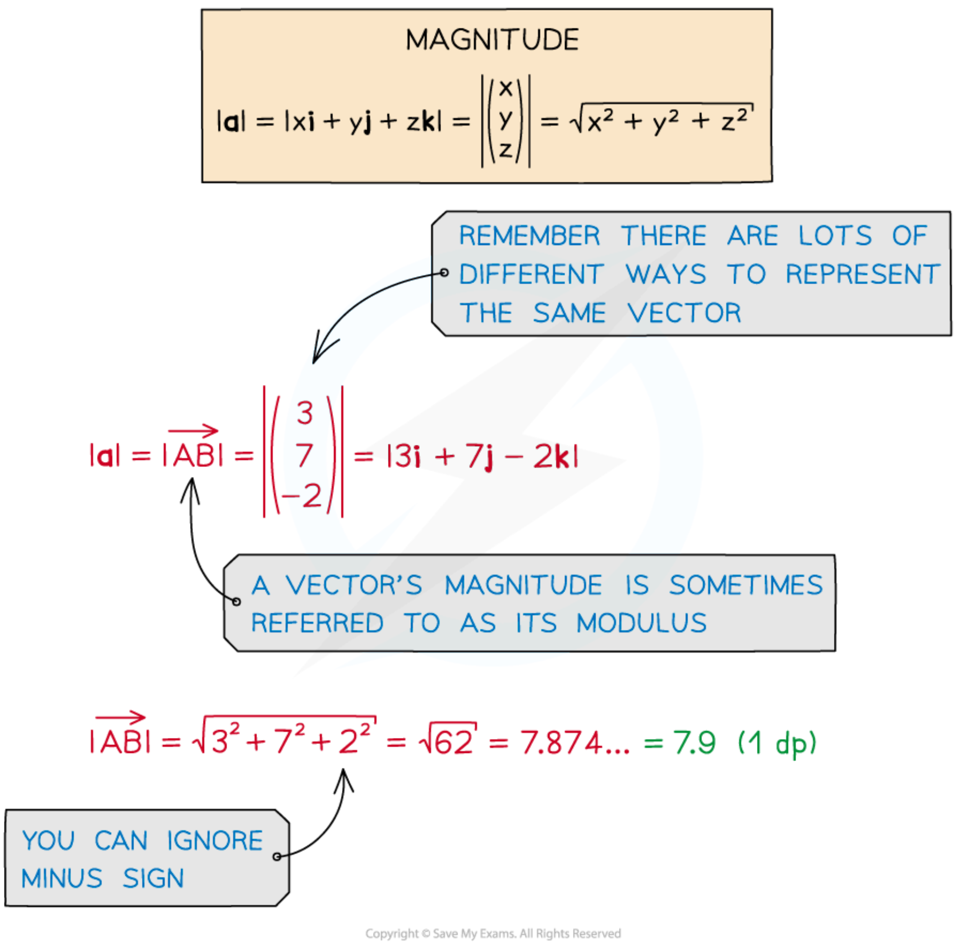
What is a 3-D vector?

- Vectors represent a movement of a certain **magnitude** (size) in a given **direction**
  You should have already come across (2D) **vectors** at AS (see Basic Vectors)
- 3-D vectors describe the **position of a point** in a 3-D space in relation to the origin
- They can be represented in different ways such as a **column vector** or in **i, j, k unit vector form**



Magnitude of a 3-D vector

- The magnitude of a 3-D vector is simply its size
- Like 2-D vectors we can find the **magnitude** using Pythagoras' theorem (see Magnitude Direction)



- For 3-D **position vectors** we can find the distance between two points
- By using the respective co-ordinates we can calculate the magnitude of the vector between them:

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DISTANCE BETWEEN TWO POINTS FROM POSITION VECTORS

(x_1, y_1, z_1) AND $(x_2, y_2, z_2) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$

WHERE x, y, z ARE FROM POSITION VECTORS $x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$

e.g. FIND $|\overrightarrow{AB}|$ FOR POSITION VECTORS A AND B, $5\mathbf{i} + 3\mathbf{j} - 7\mathbf{k}$ AND $3\mathbf{i} - 6\mathbf{j} + 5\mathbf{k}$ RESPECTIVELY

A HAS CO-ORDINATES $(5, 3, -7)$

B HAS CO-ORDINATES $(3, -6, 5)$

$$|\overrightarrow{AB}| = \sqrt{(5-3)^2 + (3 - -6)^2 + (-7-5)^2} = \sqrt{4 + 81 + 144}$$

$$= \sqrt{229} = 15.1 \text{ UNITS (1 dp)}$$

3-D vector addition, scalars, parallel vectors and unit vectors

- 3-D vectors work in the same way as 2-D vectors, just in three dimensions rather than two
- **Vector addition** and subtraction and scalar multiplication can be carried out in exactly the same way, this time involving **i**, **j** and **k** or x, y and z
- 3-D vectors are also **parallel** if one is a **multiple** of the other

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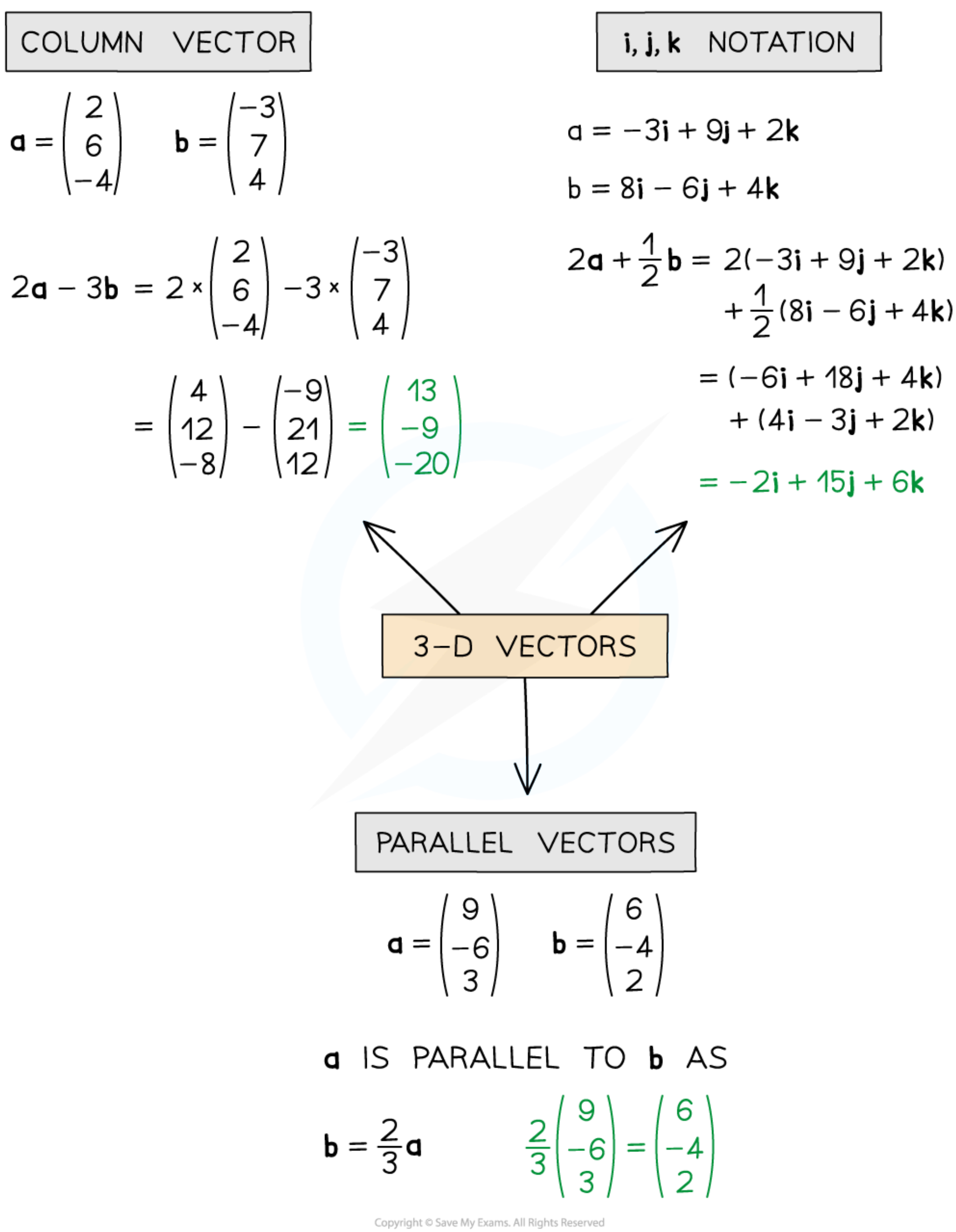

- **Unit vectors** in 3-D are found in exactly the same way as in 2-D

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TO FIND THE **UNIT VECTOR** IN THE SAME DIRECTION AS A GIVEN VECTOR, SIMPLY DIVIDE THE VECTOR BY ITS MAGNITUDE

- THE UNIT VECTOR IN THE SAME DIRECTION AS VECTOR $\overrightarrow{AB}$ IS $\dfrac{\overrightarrow{AB}}{|\overrightarrow{AB}|}$
- THE UNIT VECTOR IN THE SAME DIRECTION AS VECTOR $\mathbf{a}$ IS $\dfrac{\mathbf{a}}{|\mathbf{a}|}$

e.g. FIND THE UNIT VECTOR IN THE SAME DIRECTION AS THE VECTOR $\mathbf{a} = 3\mathbf{i} - 4\mathbf{j} + 5\mathbf{k}$

$$|\mathbf{a}| = \sqrt{3^2 + (-4)^2 + 5^2} = 5\sqrt{2}$$

$$\text{UNIT VECTOR} = \frac{\mathbf{a}}{|\mathbf{a}|} = \frac{3\mathbf{i} - 4\mathbf{j} + 5\mathbf{k}}{5\sqrt{2}} = \frac{1}{5\sqrt{2}}(3\mathbf{i} - 4\mathbf{j} + 5\mathbf{k})$$

Your notes

Worked example

Your notes

Points A and B have position vectors $-\mathbf{i} - 2\mathbf{j} + 5\mathbf{k}$ and $3\mathbf{i} + 7\mathbf{j} - 2\mathbf{k}$ respectively.

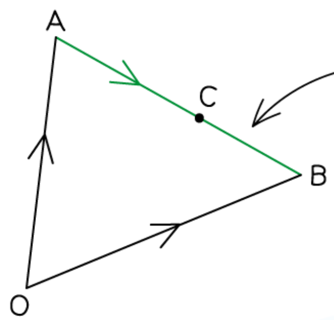
Point C divides the line AB in the ratio 3:2.

a) Find the position vector of C.

b) Calculate the distance of C from the origin.

Give your answer to 2 decimal places.

a)



START BY DRAWING A DIAGRAM TO HELP, DON'T WORRY ABOUT THE 3-D ASPECT, THE VECTORS WILL TAKE CARE OF THAT!

FIND VECTORS FOR ANY UNKNOWN PARTS OF THE DIAGRAM

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = (3\mathbf{i} + 7\mathbf{j} - 2\mathbf{k}) - (-\mathbf{i} - 2\mathbf{j} + 5\mathbf{k})$$
$$= (3 - -1)\mathbf{i} + (7 - -2)\mathbf{j} + (-2 - 5)\mathbf{k}$$
$$= 4\mathbf{i} + 9\mathbf{j} - 7\mathbf{k}$$

$$\overrightarrow{AC} = \frac{3}{5}(4\mathbf{i} + 9\mathbf{j} - 7\mathbf{k})$$
$$= \frac{12}{5}\mathbf{i} + \frac{27}{5}\mathbf{j} - \frac{21}{5}\mathbf{k}$$

RATIO 3:2 MEANS C IS $\frac{3}{5}$ ALONG $\overrightarrow{AB}$

$$\overrightarrow{OC} = \overrightarrow{OA} + \overrightarrow{AC} = (-\mathbf{i} - 2\mathbf{j} + 5\mathbf{k}) + \left(\frac{12}{5}\mathbf{i} + \frac{27}{5}\mathbf{j} - \frac{21}{5}\mathbf{k}\right)$$
$$= \frac{7}{5}\mathbf{i} + \frac{17}{5}\mathbf{j} + \frac{4}{5}\mathbf{k}$$

POSITION VECTOR NEEDS TO START

FROM ORIGIN

b) $|\overrightarrow{OC}| = \left|\frac{7}{5}\mathbf{i} + \frac{17}{5}\mathbf{j} + \frac{4}{5}\mathbf{k}\right|$

DISTANCE = MAGNITUDE

$$|\overrightarrow{OC}| = \sqrt{\left(\frac{7}{5}\right)^2 + \left(\frac{17}{5}\right)^2 + \left(\frac{4}{5}\right)^2} = \frac{\sqrt{354}}{5}$$

$= 3.76$ UNITS (2 dp)

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11.2.2 Problem Solving using 3D Vectors

Problem Solving using 3D Vectors

Problem-solving with 3-D vectors

- 3-D vector problems can be solved using the same principles as **2-D vector problems** (see Problem Solving using Vectors)
- Vectors can be used to prove two lines are **parallel**, to show points are **collinear** (lie on the same straight line) or to find missing vertices of a given shape

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VECTORS ARE PARALLEL IF ONE IS A SCALAR **MULTIPLE** OF THE OTHER

PARALLEL VECTORS

$$\mathbf{a} = \begin{pmatrix} 9 \\ -6 \\ 3 \end{pmatrix} \quad \mathbf{b} = \begin{pmatrix} 6 \\ -4 \\ 2 \end{pmatrix}$$

$\mathbf{a}$ IS PARALLEL TO $\mathbf{b}$ AS

$$\mathbf{b} = \frac{2}{3}\mathbf{a} \qquad \frac{2}{3}\begin{pmatrix} 9 \\ -6 \\ 3 \end{pmatrix} = \begin{pmatrix} 6 \\ -4 \\ 2 \end{pmatrix}$$

THREE POINTS P, Q, R ARE **COLLINEAR** IF $\overrightarrow{PQ}$ AND $\overrightarrow{PR}$ ARE PARALLEL

SHOW THAT POINTS A(1, 2, 3) B(3, 8, 1) AND C(7, 20, −3) ARE COLLINEAR

$$\overrightarrow{AB} = \begin{pmatrix} 3 \\ 8 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 6 \\ -2 \end{pmatrix}$$

$$\overrightarrow{AC} = \begin{pmatrix} 7 \\ 20 \\ -3 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 6 \\ 18 \\ -6 \end{pmatrix}$$

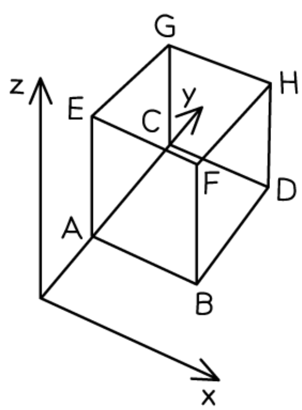
$\overrightarrow{AC} = 3\,\overrightarrow{AB}$ SO $\overrightarrow{AC}$ IS PARALLEL TO $\overrightarrow{AB}$.

$\therefore$ ABC ARE COLLINEAR

USE COORDINATES TO GIVE POSITION VECTORS THEN FIND AB AND AC TO PROVE THEY ARE COLLINEAR

3-D VECTOR PROBLEMS

PARALLEL SIDES OF A SHAPE SUCH AS A CUBOID HAVE THE **SAME VECTOR**



A CUBOID HAS POINTS B AND G SUCH THAT B HAS POSITION VECTOR $5\mathbf{i} + 3\mathbf{j}$ AND G HAS POSITION VECTOR $7\mathbf{j} + 6\mathbf{k}$.

FIND THE POSITION VECTOR OF POINT H IN UNIT VECTOR FORM

$$H = \begin{pmatrix} 5 \\ 7 \\ 6 \end{pmatrix} = 5\mathbf{i} + 7\mathbf{j} + 6\mathbf{k}$$

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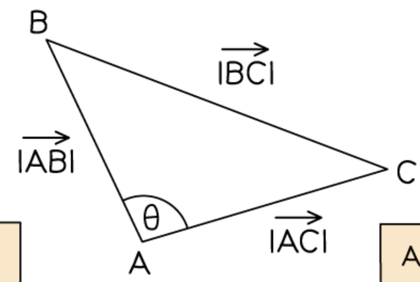
Using trig in 3-D vector problems

- 3-D vector problems can also involve using trigonometry to:
  - find the **angle between two vectors** using **Cosine Rule**
  - find the **area of a triangle** using a variation of **Area Formula**

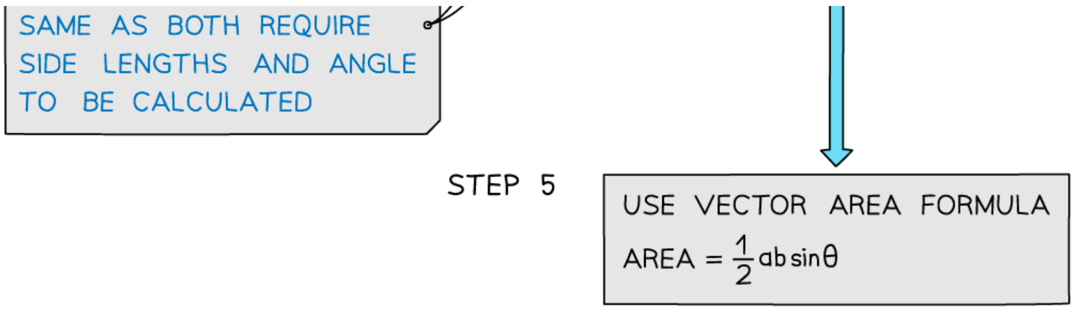
AFTER IDENTIFYING A TRIANGLE IN 3-D SPACE, APPROACH IT AS YOU WOULD A 2-D VECTOR PROBLEM



ANGLE BETWEEN TWO VECTORS		AREA OF A VECTOR TRIANGLE
USE GIVEN VECTORS TO CONSTRUCT A **TRIANGLE** (USUALLY GIVEN TWO SIDES)	STEP 1	USE GIVEN VECTORS TO CONSTRUCT A **TRIANGLE** (USUALLY GIVEN TWO SIDES)
FIND THE **VECTOR** FOR THE THIRD SIDE OF THE TRIANGLE (IF NOT GIVEN)	STEP 2	FIND THE **VECTOR** FOR THE THIRD SIDE OF THE TRIANGLE (IF NOT GIVEN)
FIND THE **MAGNITUDE** (LENGTH) OF EACH SIDE OF THE TRIANGLE USING **PYTHAGORAS**	STEP 3	FIND THE **MAGNITUDE** (LENGTH) OF EACH SIDE OF THE TRIANGLE USING **PYTHAGORAS**
USE **COSINE RULE** TO FIND THE ANGLE θ $\cos\theta = \frac{b^2 + c^2 - a^2}{2bc}$	STEP 4	USE **COSINE RULE** TO FIND THE ANGLE θ (IF ANGLE NOT GIVEN) $\cos\theta = \frac{b^2 + c^2 - a^2}{2bc}$

THE FIRST FOUR STEPS OF BOTH PROBLEMS ARE THE

Your notes



- You can find the angle between a vector and any one of the coordinate axes by using the following formulae

IF θ_x, θ_y AND θ_z ARE RESPECTIVELY THE ANGLES THAT THE VECTOR $\mathbf{a} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ MAKES WITH THE $x-$, $y-$ AND $z-$AXES, THEN

$$\cos\theta_x = \frac{x}{|\mathbf{a}|}$$

$$\cos\theta_y = \frac{y}{|\mathbf{a}|}$$

$$\cos\theta_z = \frac{z}{|\mathbf{a}|}$$

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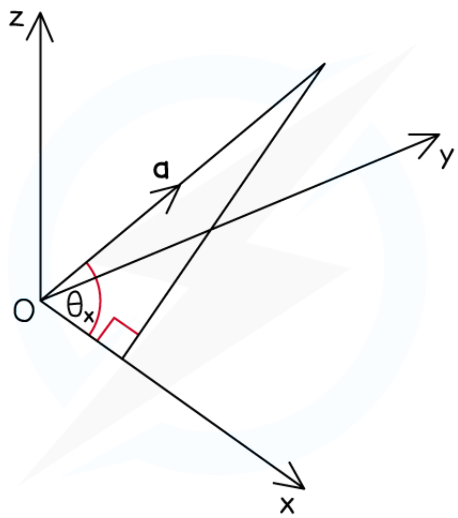
THESE FORMULAE COME FROM APPLYING SOHCAHTOA (SPECIFICALLY CAH) TO A RIGHT-TRIANGLE FORMED BY THE VECTOR AND ONE OF THE COORDINATE AXES.



IN THE DIAGRAM ABOVE FOR EXAMPLE, THE LENGTH OF THE SIDE ADJACENT TO ANGLE θ_x IS SIMPLY THE $\mathbf{x}$ COMPONENT OF THE VECTOR $\mathbf{a} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, WHILE THE LENGTH OF THE HYPOTENUSE IS $|\mathbf{a}|$.

- These formulae are not in the exam formulae book
- However, they may be derived using SOHCAHTOA as shown in the diagram
- The formulae also work for vectors in two dimensions

Examiner Tip

- If there is a diagram, labelling all known vectors and quantities will help, and don't worry about trying to make your diagram 3-D, as long as you label it well.

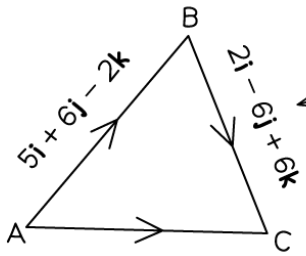
Worked example

Your notes

Your notes

The points A, B and C are such that $\overrightarrow{AB} = 5\mathbf{i} + 6\mathbf{j} - 2\mathbf{k}$ and $\overrightarrow{BC} = 2\mathbf{i} - 6\mathbf{j} + 6\mathbf{k}$.

Show that ABC is an isosceles triangle.



DRAW A DIAGRAM

TO SHOW ABC IS AN **ISOSCELES** TRIANGLE WE MUST SHOW TWO SIDE **LENGHTS** ARE **EQUAL**

FIND ANY **MISSING VECTORS**

$\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$

$= (5\mathbf{i} + 6\mathbf{j} - 2\mathbf{k}) + (2\mathbf{i} - 6\mathbf{j} + 6\mathbf{k})$

$= 7\mathbf{i} + 4\mathbf{k}$

$6\mathbf{j} - 6\mathbf{j} = 0\mathbf{j}$ SO DOES NOT APPEAR

FIND MAGNITUDE OF EACH SIDE

$|\overrightarrow{AB}| = \sqrt{5^2 + 6^2 + 2^2} = \sqrt{65}$

$|\overrightarrow{BC}| = \sqrt{2^2 + 6^2 + 6^2} = \sqrt{76}$

$|\overrightarrow{AC}| = \sqrt{7^2 + 4^2} = \sqrt{65}$

TWO SIDES ARE **EQUAL** LENGTH

$|\overrightarrow{AB}| = |\overrightarrow{AC}|$ THEREFORE TRIANGLE ABC IS ISOSCELES