Edexcel A Level Maths: Pure

Your notes

7.3 Further Differentiation

Contents

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7.3.1 First Principles Differentiation – Trigonometry

Your notes

First Principles Differentiation – Trigonometry

How do I derive the derivatives of trigonometric functions from first principles?

- Recall that for a function **f(x)** the definition of the **derivative from first principles** (see First Principles Differentiation) is:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

- The derivatives of the trigonometric functions depend on the following **small angle approximations**
  - When **θ** is small (i.e. close to zero) and measured in radians then

$$\sin\theta \approx \theta$$

$$\cos\theta \approx 1 - \frac{1}{2}\theta^2$$

$$\tan\theta \approx \theta$$

- The small angle approximations allow us to produce the following intermediate limit results:

$$\lim_{h\to 0} \frac{\sin h}{h} = 1 \quad \text{and} \quad \lim_{h\to 0} \frac{\cos h - 1}{h} = 0$$

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i.e. $\lim_{h \to 0} \frac{\sin h}{h} = \lim_{h \to 0} \frac{h}{h} = \lim_{h \to 0} 1 = 1$

$\sin h \approx h$ WHEN h IS CLOSE TO ZERO

$\lim_{h \to 0} \frac{\cos h - 1}{h} = \lim_{h \to 0} \frac{1 - \frac{1}{2}h^2 - 1}{h} = \lim_{h \to 0} \left(-\frac{1}{2}h\right) = 0$

$\cos h \approx 1 - \frac{1}{2}h^2$ WHEN h IS CLOSE TO ZERO

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- And those intermediate results allow us to find the derivatives of **sin** and **cos**

1. IF $y = \sin x$ THEN $\frac{dy}{dx} = \cos x$

2. IF $y = \cos x$ THEN $\frac{dy}{dx} = -\sin x$

DON'T FORGET THE MINUS SIGN WITH THE DERIVATIVE OF cos!

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i.e. IF $f(x) = \sin x$ THEN

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$f'(x) = \lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h}$$

$$\sin(A+B) = \sin A \cos B + \cos A \sin B$$

$$= \lim_{h \to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h}$$

$$= \lim_{h \to 0} \left(\sin x \left(\frac{\cos h - 1}{h} \right) + \cos x \left(\frac{\sin h}{h} \right) \right)$$

$$= \sin x (0) + \cos x (1) = \cos x$$

$$\lim_{h \to 0} \frac{\cos h - 1}{h} = 0 \qquad \lim_{h \to 0} \frac{\sin h}{h} = 1$$

SO THE DERIVATIVE OF $\sin x$ IS $\cos x$

IF $f(x) = \cos x$ THEN

$$f'(x) = \lim_{h \to 0} \frac{\cos(x+h) - \cos x}{h}$$

$$\cos(A+B) = \cos A \cos B - \sin A \sin B$$

$$= \lim_{h \to 0} \frac{\cos x \cos h - \sin x \sin h - \cos x}{h}$$

$$= \lim_{h \to 0} \left(\cos x \left(\frac{\cos h - 1}{h} \right) - \sin x \left(\frac{\sin h}{h} \right) \right)$$

$$= \cos x (0) - \sin x (1) = -\sin x$$

SO THE DERIVATIVE OF $\cos x$ IS $-\sin x$

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Derivatives of other trigonometric functions

- The derivative of **tan** is given by the following formula:

$$3. \quad \text{IF} \quad y = \tan x \quad \text{THEN} \quad \frac{dy}{dx} = \sec^2 x$$

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- The easiest way to derive this is to use the **quotient rule** and the derivatives of **sin** and **cos**
- But it can also be derived from first principles using the small angle approximation for **tan** (see the Worked Example)
- The general formulae for the derivatives of the trigonometric functions are:

$$4. \quad \text{IF} \quad y = \sin kx \quad \text{THEN} \quad \frac{dy}{dx} = k\cos kx$$

$$5. \quad \text{IF} \quad y = \cos kx \quad \text{THEN} \quad \frac{dy}{dx} = -k\sin kx$$

$$6. \quad \text{IF} \quad y = \tan kx \quad \text{THEN} \quad \frac{dy}{dx} = k\sec^2 kx$$

DON'T FORGET THE MINUS SIGN WITH THE DERIVATIVE OF cos!

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- These formulae follow from combining the derivatives of the three basic functions with the **chain rule**, but they are worth knowing on their own

Examiner Tip

- Remember that when doing calculus with trigonometric functions you have to measure angles in **radians.**
- The formula for the derivative of **tan *x*** is included in the exam formulae booklet.
- The derivatives of **sin *x*** and **cos *x*** are NOT included in the formula booklet – you have to know them.
- The small angle approximations for **cos *x***, **sin *x*** and **tan *x*** are included in the exam formulae booklet – you don't have to memorise them.
- Be sure to read first principle differentiation exam questions clearly – they will state any results you can treat as 'givens' in your answer.

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Worked example

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a) Use the appropriate trigonometric identity to rewrite $\tan(x+h)$ in terms of $\tan x$ and $\tan h$.

b) Using the small angle approximation $\tan\theta \approx \theta$ when θ is small and measured in radians, show from first principles that the derivative with respect to x of $\tan x$ is $\sec^2 x$.

a) $\tan(x+h) = \dfrac{\tan x + \tan h}{1 - \tan x \tan h}$

$\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$

b) IF $f(x) = \tan x$ THEN

$$f'(x) = \lim_{h \to 0} \frac{\tan(x+h) - \tan x}{h}$$

USE DEFINITION OF DERIVATIVE FROM FIRST PRINCIPLES

$$= \lim_{h \to 0} \frac{\frac{\tan x + \tan h}{1 - \tan x \tan h} - \tan x}{h}$$

SUBSTITUTE RESULT FROM PART a)

$$= \lim_{h \to 0} \frac{\frac{\tan x + \tan h - \tan x(1 - \tan x \tan h)}{1 - \tan x \tan h}}{h}$$

$$= \lim_{h \to 0} \frac{\tan x + \tan h - \tan x + \tan^2 x \tan h}{h(1 - \tan x \tan h)}$$

$$= \lim_{h \to 0} \frac{\tan h(1 + \tan^2 x)}{h(1 - \tan x \tan h)}$$

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BUT WHEN h IS SMALL, $\tan h \approx h$

USE SMALL ANGLE APPROXIMATION

AND $\dfrac{\tan h(1+\tan^2 x)}{h(1-\tan x \tan h)} \approx \dfrac{h(1+\tan^2 x)}{h(1-h\tan x)} = \dfrac{1+\tan^2 x}{1-h\tan x}$

SO AS $h \to 0$

$$\frac{\tan h\,(1+\tan^2 x)}{h(1-\tan x \tan h)} \longrightarrow \frac{1+\tan^2 x}{1-(0)\tan x} = 1+\tan^2 x$$

THEREFORE

$$f'(x) = \lim_{h\to 0} \frac{\tan h(1+\tan^2 x)}{h(1-\tan x \tan h)} = 1+\tan^2 x$$

$$= 1 + \frac{\sin^2 x}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x$$

$\tan x \equiv \dfrac{\sin x}{\cos x}$

$\cos^2 x + \sin^2 x = 1$

$\dfrac{1}{\cos x} \equiv \sec x$

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7.3.2 Differentiating Other Functions (Trig, ln & e etc)

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Differentiating Other Functions (Trig, ln & e etc)

How do I differentiate common functions?

- These are the common results
  - $\frac{\mathrm{d}}{\mathrm{d}x}(x^n) = nx^{n-1}$
  - $\frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{e}^x) = \mathrm{e}^x$
  - $\frac{\mathrm{d}}{\mathrm{d}x}(a^x) = a^x \ln a$ for $a > 0$
  - $\frac{\mathrm{d}}{\mathrm{d}x}(\ln x) = \frac{1}{x}$
  - $\frac{\mathrm{d}}{\mathrm{d}x}(\sin x) = \cos x$
  - $\frac{\mathrm{d}}{\mathrm{d}x}(\cos x) = -\sin x$
  - $\frac{\mathrm{d}}{\mathrm{d}x}(\tan x) = \sec^2 x$
  - $\frac{\mathrm{d}}{\mathrm{d}x}(\cot x) = -\operatorname{cosec}^2 x$
  - $\frac{\mathrm{d}}{\mathrm{d}x}(\sec x) = \sec x \tan x$
  - $\frac{\mathrm{d}}{\mathrm{d}x}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$

How do I differentiate exponentials and logarithms?

- The two basic differentiation formulae are:

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1. IF $y = e^{kx}$, WHERE k IS A REAL CONSTANT, THEN $\frac{dy}{dx} = ke^{kx}$

2. IF $y = \ln x$ THEN $\frac{dy}{dx} = \frac{1}{x}$

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- From those basic formulae are derived these two additional formulae:

3. IF $y = a^x$, WHERE a IS A REAL CONSTANT AND $a>0$, THEN $\frac{dy}{dx} = (\ln a)a^x$

4. IF $y = \ln kx$, WHERE k IS A REAL CONSTANT, THEN $\frac{dy}{dx} = \frac{1}{x}$

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i.e. IF $y = a^x$ THEN

WE CAN WRITE $a = e^{\ln a}$

REMEMBER THIS TRICK! e AND ln ARE INVERSES, SO THEY CANCEL EACH OTHER

THEN $a^x = (e^{\ln a})^x = e^{(\ln a)x}$

LAWS OF INDICES $(a^m)^n = a^{mn}$

SO $y = e^{(\ln a)x}$ AND

$\frac{dy}{dx} = (\ln a)e^{(\ln a)x} = (\ln a)a^x$

IF $y = e^{kx}$, $\frac{dy}{dx} = ke^{kx}$
ln a IS JUST A CONSTANT!

IF $y = \ln kx$ THEN

$y = \ln k + \ln x$

LAWS OF LOGARITHMS $\ln(ab) = \ln a + \ln b$

SO $\frac{dy}{dx} = \frac{1}{x}$

ln k IS A CONSTANT, SO ITS DERIVATIVE IS ZERO

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- And for exponentials more generally:

5. IF $y = a^{kx}$, WHERE a AND k ARE REAL CONSTANT AND $a>0$,

THEN $\frac{dy}{dx} = k(\ln a)a^{kx}$

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- This last formula can be derived from Formula 3 by using the **chain rule**

Your notes

Examiner Tip

- The formulae for some of these derivatives are not given in the formulae booklet – you need to know them.
- The formulae for $\mathbf{e}^{kx}$ and **ln *x*** are the ones you absolutely do need to know .
- The other formulae can be derived from those two as shown above, and remember – the derivative of **ln *kx*** is $\frac{1}{x}$, NOT $\frac{k}{x}$!

Worked example

A curve has the equation $y = e^{-3x} + 3^{\frac{x}{2}} - \ln 5x$.

Find the gradient of the curve at the point with x-coordinate 2.

$$\frac{dy}{dx} = -3e^{-3x} + \frac{1}{2}(\ln 3)3^{\frac{x}{2}} - \frac{1}{x}$$

FORMULA 1

FORMULA 5

FORMULA 4

WHEN $x = 2$,

$$\frac{dy}{dx} = -3e^{-6} + \frac{1}{2}(\ln 3)3^{1} - \frac{1}{2}$$

$$= \frac{1}{2}(3\ln 3 - 1) - 3e^{-6} \qquad (\approx 1.14)$$

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7.3.3 Chain Rule

Your notes

Chain Rule

What is the chain rule?

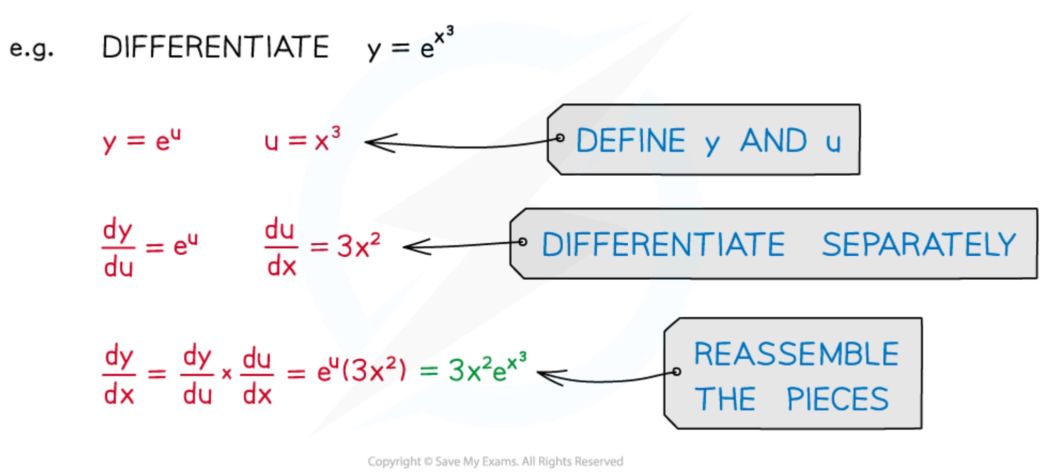
- The chain rule is a formula that allows you to differentiate **composite functions**
- If ***y*** is a function of ***u***, and ***u*** is a function of ***x***, then the **chain rule** tells us that:

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

  - Note that because ***u*** is a function of ***x***, ***y*** is also a function of ***x***
- In **function notation**, if $y = f(g(x))$ then the chain rule can be written as:

$$\frac{dy}{dx} = f'(g(x))g'(x)$$

This allows us to differentiate more complicated expressions:



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e.g. DIFFERENTIATE $y = \ln(7x^2)$

$f(x) = \ln x \qquad g(x) = 7x^2$ DEFINE f(x) AND g(x)

$f'(x) = \frac{1}{x} \qquad g'(x) = 14x$ DIFFERENTIATE SEPARATELY

$f'(g(x)) = \frac{1}{7x^2}$

$\frac{dy}{dx} = f'(g(x))g'(x) = \left(\frac{1}{7x^2}\right)(14x) = \frac{2}{x}$ REASSEMBLE THE PIECES

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What are the special cases of the chain rule?

- It allows us to differentiate a function raised to a power:
  - $\frac{\mathrm{d}}{\mathrm{d}x}((\mathrm{f}(x)^n) = n(\mathrm{f}(x))^{n-1}\mathrm{f}'(x)$

e.g. DIFFERENTIATE $y = (x^2 - 5x + 7)^7$ n=7

$f(x) = x^2 - 5x + 7$

$f'(x) = 2x - 5$

$\frac{dy}{dx} = n(f(x))^{n-1}f'(x) = 7(x^2 - 5x + 7)^6(2x-5)$

$\frac{dy}{dx} = 7(2x-5)(x^2 - 5x + 7)^6$

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- It allows us to differentiate functions that are not given in the form ***y* = f(*x*)**:
  - $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\frac{\mathrm{d}x}{\mathrm{d}y}}$

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e.g. FIND THE VALUE OF $\frac{dy}{dx}$ AT THE POINT (3,1) ON THE CURVE WITH EQUATION $y^3 + 2y^2 = x$

$$\frac{dx}{dy} = 3y^2 + 4y$$

DIFFERENTIATE x WITH RESPECT TO y

$$\frac{dy}{dx} = \frac{1}{\left(\frac{dx}{dy}\right)} = \frac{1}{3y^2 + 4y}$$

INVERT TO FIND $\frac{dy}{dx}$

AT (3,1) $y = 1$ AND

$$\frac{dy}{dx} = \frac{1}{3(1)^2 + 4(1)} = \frac{1}{7}$$

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- There is a useful case which is used to integrate certain functions:
  - $$\frac{d}{dx}(\ln(f(x))) = \frac{f'(x)}{f(x)}$$

Examiner Tip

- The chain rule formulae are not in the exam formulae booklet – you have to know them.
- When using the chain rule be sure to keep your functions straight (ie which function is ***y*** and which is ***u***, or which is ***f*** and which is ***g***).

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Worked example

For the curve with equation $y = 5^{x^4}$, find the gradient of the curve at the point $(1, 5)$.

Give your answer as an exact value.

$y = 5^u \qquad u = x^4$ DEFINE y AND u

$\frac{dy}{du} = (\ln 5)5^u \qquad \frac{du}{dx} = 4x^3$ DIFFERENTIATE SEPARATELY

$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = (\ln 5)5^u(4x^3)$

$\frac{dy}{dx} = (4\ln 5)x^3\left(5^{x^4}\right)$ REASSEMBLE

WHEN $x = 1$,

$\frac{dy}{dx} = (4\ln 5)(1)^3\left(5^{1^4}\right) = 20\ln 5$

THIS COULD BE DONE USING THE FUNCTION NOTATION VERSION OF THE CHAIN RULE, WITH $f(x) = 5^x$ AND $g(x) = x^4$

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7.3.4 Product Rule

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Product Rule

What is the product rule?

- The product rule is a formula that allows you to differentiate a **product of two functions**
- If $y = u \times v$ where u and v are functions of x then the product rule is:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = u\frac{\mathrm{d}v}{\mathrm{d}x} + v\frac{\mathrm{d}u}{\mathrm{d}x}$$

- In **function notation**, if $\mathrm{f}(x) = \mathrm{g}(x) \times \mathrm{h}(x)$ then the product rule can be written as:

$$\mathrm{f}'(x) = \mathrm{g}(x)\mathrm{h}'(x) + \mathrm{h}(x)\mathrm{g}'(x)$$

- The easiest way to remember the product rule is, for $y = u \times v$ where u and v are functions of x:

$$y' = uv' + vu'$$

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e.g. $y = xe^x$ $\quad$ $u = x, \quad v = e^x$

$$\frac{dy}{dx} = \underbrace{(x)}_{u}\underbrace{(e^x)}_{\frac{dv}{dx}} + \underbrace{(e^x)}_{v}\underbrace{(1)}_{\frac{du}{dx}}$$

$$\frac{dy}{dx} = xe^x + e^x$$

$$f(x) = x^3(x^2+7)$$

$g(x) = x^3, \quad h(x) = x^2 + 7$

$$f'(x) = \underbrace{(x^3)}_{g(x)}\underbrace{(2x)}_{h'(x)} + \underbrace{(x^2+7)}_{h(x)}\underbrace{(3x^2)}_{g'(x)}$$

$$f'(x) = 5x^4 + 21x^2$$

COMPARE $\quad f(x) = x^3(x^2+7)$

$f(x) = x^5 + 7x^3$

$f'(x) = 5x^4 + 21x^2$

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Examiner Tip

- The product rule formulae are NOT in the formulae booklet – you need to know them.
- Don't confuse the **product** of two functions with a **composite function:**
  - The **product** of two functions is two functions **multiplied** together
  - A **composite** function is a **function of a function**

e.g. $f(x) = 3x^2$ $\quad g(x) = x - 3$

$$f(x)g(x) = 3x^2(x-3) = 3x^3 - 9x^2 \qquad \boxed{\text{PRODUCT}}$$

$$fg(x) = f(g(x)) = 3(x-3)^2 = 3x^2 - 18x + 27 \qquad \boxed{\text{COMPOSITE FUNCTION}}$$

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- To differentiate composite functions you need to use the **chain rule**

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Worked example

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Differentiate with respect to x:

a) $(5\sin x - 7\cos x)e^{3x+2}$

b) $5x\ln 5x$

a) $u = 5\sin x - 7\cos x$ $\quad$ $v = e^{3x+2}$

$\frac{du}{dx} = 5\cos x + 7\sin x$ $\quad$ $\frac{dv}{dx} = 3e^{3x+2}$

DON'T FORGET MINUS SIGN WHEN YOU DIFFERENTIATE cos

USE CHAIN RULE

$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

$$\frac{dy}{dx} = (5\sin x - 7\cos x)(3e^{3x+2}) + (e^{3x+2})(5\cos x + 7\sin x)$$

$$\frac{dy}{dx} = (22\sin x - 16\cos x)e^{3x+2}$$

b) $u = 5x$ $\quad$ $v = \ln 5x$

$\frac{du}{dx} = 5$ $\quad$ $\frac{dv}{dx} = \frac{1}{x}$

BE CAREFUL WITH THIS ONE! BECAUSE OF THE CHAIN RULE THE DERIVATIVE OF $\ln(ax)$ IS $\frac{1}{x}$ FOR **ANY** CONSTANT a

$$\frac{dy}{dx} = (5x)\left(\frac{1}{x}\right) + (\ln 5x)(5)$$

$$\frac{dy}{dx} = 5 + 5\ln 5x$$

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7.3.5 Quotient Rule

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Quotient Rule

What is the quotient rule?

- The quotient rule is a formula that allows you to differentiate a **quotient of two functions** (ie one function divided by another)
- If $\boldsymbol{y = \frac{u}{v}}$ where ***u*** and ***v*** are functions of ***x*** then the quotient rule is:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{v\frac{\mathrm{d}u}{\mathrm{d}x} - u\frac{\mathrm{d}v}{\mathrm{d}x}}{v^2}$$

- In **function notation**, if $\mathbf{f}(\boldsymbol{x}) = \frac{\mathbf{g}(\boldsymbol{x})}{\mathbf{h}(\boldsymbol{x})}$ then the quotient rule can be written as:

$$\mathrm{f}'(x) = \frac{\mathrm{h}(x)\mathrm{g}'(x) - \mathrm{g}(x)\mathrm{h}'(x)}{(\mathrm{h}(x))^2}$$

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e.g. $y = \dfrac{x^2+1}{x-3}$ ← $u = x^2+1, \quad v = x-3$

$v \qquad \dfrac{du}{dx} \qquad u \qquad \dfrac{dv}{dx}$

$$\frac{dy}{dx} = \frac{(x-3)(2x) - (x^2+1)(1)}{(x-3)^2}$$

← v^2

$$\frac{dy}{dx} = \frac{x^2-6x-1}{(x-3)^2}$$

$f(x) = \dfrac{\sin x}{\cos x}$ ← $g(x) = \sin x, \quad h(x) = \cos x$

$h(x) \qquad g'(x) \qquad g(x) \qquad h'(x)$

$$f'(x) = \frac{(\cos x)(\cos x) - (\sin x)(-\sin x)}{(\cos x)^2}$$

← $h((x))^2$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}$$

$f'(x) = \sec^2 x$ ← $\dfrac{1}{\cos x} = \sec x$

$\dfrac{\sin x}{\cos x} = \tan x$ SO THIS IS THE DERIVATIVE OF $\tan x$

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Examiner Tip

- The quotient rule formula is in the formulae booklet – you don't have to memorise it.
- Be careful using the formula – because of the minus sign in the numerator the order of the functions is important.
- Look out for functions of the form $f(x) = g(x)(h(x))^{-1}$
  - You could differentiate that using a combination of the **chain rule** and the **product rule** (and it can be good practice for you to try it!)
  - But it can also be seen as just a quotient rule question in disguise
    - $g(x)(h(x))^{-1} = \frac{g(x)}{h(x)}$

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Worked example

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Differentiate $f(x) = (\cos 2x)\left(x^2 + 2x + 1\right)^{-1}$

with respect to x.

$$f(x) = \frac{\cos 2x}{x^2 + 2x + 1}$$

REWRITE AS $f(x) = \frac{g(x)}{h(x)}$

$g(x) = \cos 2x$ $\qquad$ $h(x) = x^2 + 2x + 1$

$g'(x) = -2\sin 2x$ $\qquad$ $h'(x) = 2x + 2$

CHAIN RULE – AND REMEMBER MINUS SIGN WHEN DIFFERENTIATING cos

$$f'(x) = \frac{(x^2 + 2x + 1)(-2\sin 2x) - (\cos 2x)(2x + 2)}{(x^2 + 2x + 1)^2}$$

$f'(x) = \frac{h(x)g'(x) - g(x)h'(x)}{(h(x))^2}$

$$f'(x) = \frac{-2\sin 2x(x^2 + 2x + 1) - 2\cos 2x(x + 1)}{(x^2 + 2x + 1)^2}$$

$$= \frac{-2\sin 2x(x + 1)^2 - 2\cos 2x(x + 1)}{(x + 1)^4}$$

$x^2 + 2x + 1 = (x + 1)^2$

$$f'(x) = \frac{-2(\cos 2x + (x + 1)\sin 2x)}{(x + 1)^3}$$

SIMPLIFY IT

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7.3.6 Differentiating Reciprocal and Inverse Trig Functions

Differentiating Reciprocal and Inverse Trig Functions

How do I differentiate the reciprocal trigonometric functions?

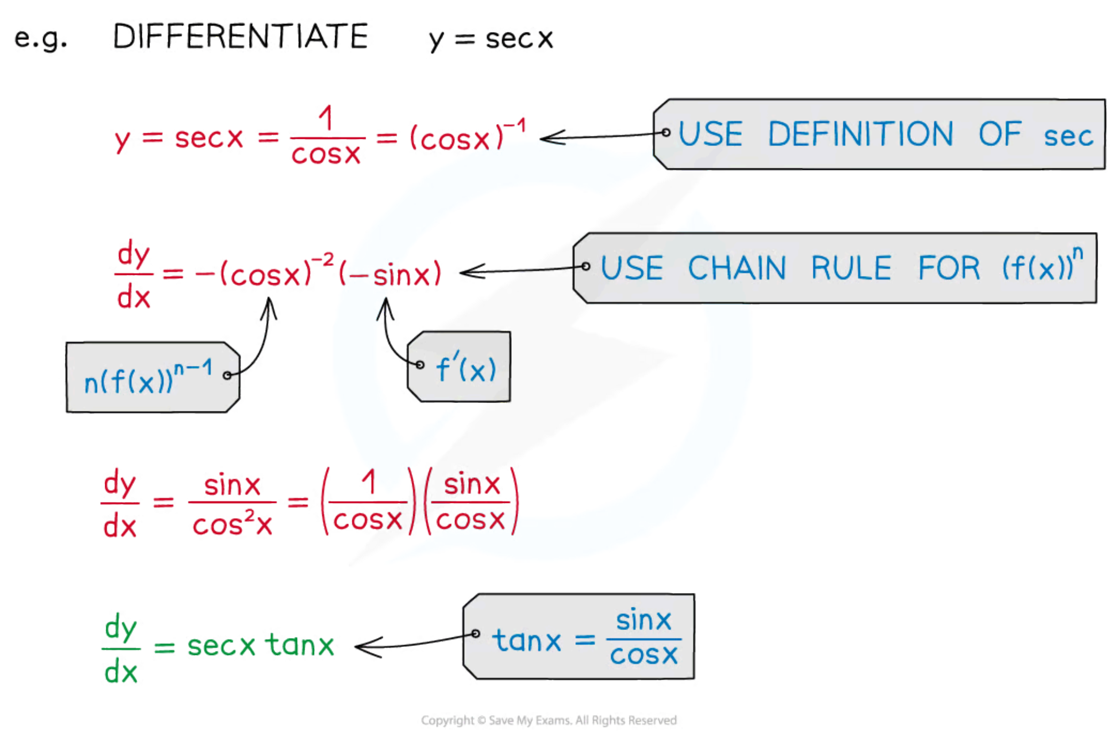
- The formulae for the derivatives of the reciprocal trigonometric functions are:

$$\frac{\mathrm{d}}{\mathrm{d}x}(\sec x) = \sec x \tan x$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(\cot x) = -\operatorname{cosec}^2 x$$

- You can derive the derivatives for **sec**, **cosec**, and **cot** using the **chain rule** and the derivatives of the basic trigonometric functions



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How do I differentiate the inverse trigonometric functions?

- The formulae for the derivatives of the inverse trigonometric functions are:

$$\frac{\mathrm{d}}{\mathrm{d}x}(\arcsin x) = \frac{1}{\sqrt{1 - x^2}}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(\arccos x) = -\frac{1}{\sqrt{1 - x^2}}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(\arctan x) = \frac{1}{1 + x^2}$$

- You can derive the derivatives of **arcsin**, **arccos**, and **arctan** using the reciprocal form of the **chain rule** and the derivatives of the basic trigonometric functions

Your notes

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Your notes

e.g. DIFFERENTIATE $y = \arccos x$

IF $y = \arccos x$
THEN $x = \cos y$

arccos AND cos ARE INVERSES

$$\frac{dx}{dy} = -\sin y$$

DIFFERENTIATE WITH RESPECT TO y

BUT $\sin^2 y + \cos^2 y = 1$

USE TRIG IDENTITY

SO $\sin y = \sqrt{1 - \cos^2 y}$

AND $\cos y = x$
SO $\sin y = \sqrt{1 - x^2}$

$$\frac{dx}{dy} = -\sin y = -\sqrt{1 - x^2}$$

AND $\dfrac{dy}{dx} = \dfrac{1}{\frac{dx}{dy}}$

USE RECIPROCAL FORM OF CHAIN RULE

SO $\dfrac{dy}{dx} = -\dfrac{1}{\sqrt{1 - x^2}}$

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Examiner Tip

- The formulae for the derivatives of **sec *x***, **cosec *x***, and **cot *x*** are in the formulae booklet – you don't need to memorise them
  - However, you should know how to derive the derivatives of **sec**, **cosec**, and **cot** using the chain rule
- The formulae for the derivatives of **arcsin**, **arccos**, and **arctan** are not in the formulae booklet
  - You should know how to derive those derivatives using the reciprocal form of the chain rule

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Worked example

Your notes

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a) Given that $y = 5\mathrm{cosec}^2 x$ find $\dfrac{dy}{dx}$

b) Show that the derivative of $\arctan x$ is $\dfrac{1}{1+x^2}$

a) $y = 5(\mathrm{cosec}\, x)^2$ ← $5(f(x))^2$ USE CHAIN RULE

$\dfrac{dy}{dx} = 10\mathrm{cosec}\, x\,(-\mathrm{cosec}\, x \cot x)$ ← $5 \times 2(f(x))^{2-1} f'(x)$

$\dfrac{dy}{dx} = -10\mathrm{cosec}^2 x \cot x$

b) LET $y = \arctan x$

THEN $x = \tan y$ ← tan AND arctan ARE INVERSES

SO $\dfrac{dx}{dy} = \sec^2 y$ ← DIFFERENTIATE WITH RESPECT TO y

$\dfrac{dx}{dy} = 1 + \tan^2 y$ ← USE TRIG IDENTITY

$\dfrac{dy}{dx} = \dfrac{1}{\frac{dx}{dy}} = \dfrac{1}{1+\tan^2 y}$ ← RECIPROCAL FORM OF CHAIN RULE

BUT $x = \tan y$ SO

$\dfrac{dy}{dx} = \dfrac{1}{1+x^2}$

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Your notes

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